Near-Optimal Randomized Exploration for Tabular MDP

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Abstract

We study exploration using randomized value functions in Thompson Sampling (TS)-like algorithms in reinforcement learning. This type of algorithms enjoys appealing empirical performance. We show that when we use 1) a single random seed in each episode, and 2) a Bernstein-type magnitude of noise, we obtain a worst-case $\tilde{O}(H\sqrt{SAT})$ regret bound for episodic time-inhomogeneous Markov Decision Process where $S$ is the size of state space, $A$ is the size of action space, $H$ is the planning horizon and $T$ is the number of interactions. This bound polynomially improves all existing bounds for TS-like algorithms based on randomized value functions, and for the first time, matches the $\Omega(H\sqrt{SAT})$ lower bound up to logarithmic factors. Our result highlights that randomized exploration can be near-optimal, which was previously only achieved by optimistic algorithms. To achieve the desired result, we develop 1) a new clipping operation to ensure both the probability being optimistic and the probability being pessimistic are lower bounded by a constant, and 2) a new recursion formula for the absolute value of estimations errors to analyze the regret.

1 Introduction

This paper concerns learning in tabular Markov Decision Processes (MDP), arguably the most fundamental model for reinforcement learning (RL). Existing algorithms that achieve the near-optimal minimax $\tilde{O}(H\sqrt{SAT})$ regret bound are based on Optimistic in the face of Uncertainty (OFU) [Azar et al., 2017, Zanette and Brunskill, 2019, Dann et al., 2019, Zhang et al., 2020c,a]. Here $S$ is the number of states, $A$ is the number of actions, $H$ is the planning horizon, and $T$ is the total number of interactions between the agent and the environment.

Another broad category is Thompson Sampling(TS)-like methods [Osband et al., 2013, Agrawal and Jia, 2017, Osband et al., 2014]. These algorithms inject (carefully tuned) random noise to value functions to encourage exploration. Empirically, TS-like algorithms have been widely used [Osband et al., 2017, Chapelle and Li, 2011, Burda et al., 2018, Osband et al., 2018]. However, theoretically, the best known worst-case regret bound for TS-like algorithms is $\tilde{O}(H^2S\sqrt{AT})$ [Agrawal et al., 2021], which is far from optimal. In this paper, we introduce a new TS-like algorithm and show it enjoys a near-optimal $\tilde{O}(H\sqrt{SAT})$ worst-case regret bound, thus closing the gap. UCB-type algorithms enjoy well-established theoretical guarantees but suffer from difficult implementation. In practice, people prefer randomized exploration such as Fortunato et al. [2018]. However, how to perform randomized exploration in a principled way and the theoretical foundation of randomized exploration are far from clear. Our work might shed new light on randomized exploration both on the algorithmic side and the theoretical side.

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1This bound is for time-inhomogeneous MDP with each reward bounded by 1 and $T$ is sufficiently large.

2We follow Agrawal et al. [2021] which uses the terminology “TS-like” to summarize this category.
Our Contributions. Our contributions are summarized below:

- We propose a new algorithm, Single Seed Randomization (SSR), which incorporates a crucial algorithmic idea: using a single random seed for the entire episode, in contrast to previous TS-like algorithms which use one seed for each time step. Theoretically, we show, thanks to this new idea, if one uses a Hoeffding-type magnitude of noise, SSR achieves an $\tilde{O}\left(H^{1.5}\sqrt{SAT}\right)$ regret bound, improving upon the best existing result on TS-like algorithm [Agrawal et al., 2021].

- We further design a new Bernstein-type magnitude of noise for our algorithm, and achieve an $\tilde{O}\left(H\sqrt{SAT}\right)$ regret bound, resolving an open problem raised in Agrawal et al. [2021]. To our knowledge, this is the first time that a Bernstein-type bound is used in TS-like algorithms. More importantly, our upper bound matches the $\Omega\left(H\sqrt{SAT}\right)$ minimax lower bound up to logarithmic factors.

Main Challenge and Technique Overview Besides the aforementioned algorithmic ideas (single random seed and Bernstein-type magnitude of noise), we also need additional ideas in analysis to prove the desired regret bound. The main challenge is that unlike UCB-type algorithms, the estimated value in TS-like algorithm, is not an upper bound of the true optimal value, which leads to the failure of directly utilizing their analysis which only need to analyze the one-sided error in estimation. We instead work on the absolute value of the estimation error, whose analysis is more complicated than that for the one-sided error in UCB-type algorithms. Working with absolute value forces us to ensure that both the probability that the estimated value is optimistic and the probability that the estimated value is pessimistic are lower bounded. However, the clipping strategy in existing algorithm cannot maintain pessimism. To tackle with this issue, we develop a new clipping method. Below we list our technical contributions.

1. First, we propose a new clipping strategy to constrain the estimated value function (cf. Eqn. (4)). Previous clipping strategies in [Zanette et al., 2020, Agrawal et al., 2021] are based on uncertainty and can only maintain optimism. Our clipping strategy directly works on the value function, which is similar to those used in UCB-type algorithms [Azar et al., 2017, Jin et al., 2018, Zhang et al., 2020c]. Our clipping strategy can maintain both the optimism and pessimism. In addition, the number of clipping happening can still be bounded.

2. Second, we prove that the single seed randomization ensures that the estimated value function can both be optimistic or pessimistic with constant probability at all states and timesteps. This is stronger than previous randomized exploration algorithms that are only shown to be optimistic at the initial state with constant probability. With this property, we can then bound the difference between the optimal value function and estimated value function from both above and below, which results in a bound on its absolute value. See Section 5.1, Appendix C and Appendix D.

3. Third, we prove a novel recursion argument on the absolute value of the policy estimation error. As mentioned in [Agrawal et al., 2021], the recursion in UCB-type algorithms can not be directly utilized because our estimated value function is not a high-probability upper bound of the true optimal value function. With the bound of absolute value, we are able to prove new recursion formulas and together we can control the policy estimation error. See Section 5.2 and Appendix E.

4. At last, we bound the sum of variance in a novel manner. In [Azar et al., 2017], the UCB-type estimation guarantees that the policy estimation error is always positive so the difference of the variance can be directly bounded. We generalize the argument to the absolute value of the estimation error to bound the sum of variance. See Section 5.4 and Appendix G.

2 Related Work

In this section we review existing provably efficient algorithms for tabular MDP. There is a long list of sample complexity guarantees for tabular MDP [Kearns and Singh, 2002, Brafman and Tennenholtz, 2003, Kakade, 2003, Strehl et al., 2006, Strehl and Littman, 2008, Kolter and Ng, 2009, Bartlett and Tewari, 2009,
Thompson Sampling has been proved to enjoy favorable regret bounds in bandit problems [Lai and Robbins, 1985, Agrawal and Goyal, 2012, Kaufmann et al., 2012, Bubeck and Liu, 2014, Agrawal and Goyal, 2017]. In certain settings, TS-like algorithms can match the worst-case regret bound of UCB-based approaches and achieve nearly minimax optimal regret bounds [Jin et al., 2020, Agrawal et al., 2021, Russo, 2019, Agrawal and Jia, 2019, Domingues et al., 2021, Menard et al., 2021, Li et al., 2021]. The state-of-the-art methods are based on upper confidence bound (UCB) [Azar et al., 2017, Zanette and Brunskill, 2019, Dann et al., 2019, Zhang et al., 2020c, a, Menard et al., 2021, Li et al., 2021]. For the setting considered in this paper where the transition is time-inhomogeneous and the reward is bounded by 1, one can achieve an $\hat{O}(H\sqrt{SAT})$ in the regime where $T$ is sufficiently large.

3 Preliminaries

We consider time-inhomogeneous finite-horizon MDP $M = (H, S, A, P, R, s_1)$, where $|S| = S$ and $|A| = A$. Here, $S = \{1, \ldots, S\}$ is the finite state space, $A = \{1, \ldots, A\}$ is the finite action space. $H$ is the length of an episode. For convenience, we take $s_1$ to be the fixed initial state, although a more general initial distribution will not change the conclusion. $P : S \times A \times \{H\} \rightarrow \Delta(S)$ is the transition function, where if the agent stays at state $s$ and takes action $a$ at time $h$, it transits to state $s'$ with probability $P_{h, s, a}(s') \in [0, 1]$. $R : S \times S \times \{H\} \rightarrow [0, 1]$ is the reward function, where if the agent stays at $s$ and takes action $a$ at time $h$, it will receive reward $r_{h, s, a} \in [0, 1]$ such that $\mathbb{E}[r_{h, s, a}] = R_{h, s, a}$. A deterministic policy for such a MDP is defined as a tuple $\pi = (\pi_1, \ldots, \pi_H)$, where $\pi : S \rightarrow A$. The associated value function at state $s \in S$ and level $h \in \{1, \ldots, H\}$ is recursively defined as

$$V^\pi_h(s) = R_{h, s, \pi_h(s)} + \sum_{s' \in S} P_{h, s, \pi_h(s)}(s') V^\pi_{h+1}(s').$$

For convenience, we set $V^\pi_{H+1} = 0 \in \mathbb{R}^S$. The corresponding optimal value function is $V^*_{h}(s) = \max_{\pi \in \Pi} V^\pi_h(s)$, where $\Pi$ is the set of all possible deterministic policies. For a particular algorithm Alg, let $\pi^k$ denote the policy that Alg employs during episode $k$. Then, the regret of running Alg on MDP $M$ for $K$ episodes is defined as

$$\text{Reg}(M, K, \text{Alg}) = \sum_{k=1}^{K} \left( V^*_1(s^k_1) - V^*_{h+1}(s^k_1) \right).$$

Note that the regret, $\text{Reg}(M, K, \text{Alg})$, is a random variable due to randomness in state transition and the algorithm, Alg. In this paper, we show the regret of our proposed algorithm can be upper bounded with high probability, and the upper bound matches the known lower bound up to logarithmic factors.

To facilitate our later analysis, we introduce some notations for empirical estimation. Let $n_k(h, s, a) = \sum_{i=1}^{k-1} I\{(s^i_n, a^i_n) = (s, a)\}$ be the number of times action $a$ is taken at state $s$ and time $h$ before episode $k$, where $I\{\cdot\}$ is the indicator function. We define

$$\hat{P}^k_{h, s, a}(s') = \frac{\sum_{i=1}^{k-1} I\{(s^i_n, a^i_n) = (s, a)\} r_{h, s^i_n, a^i_n}}{n_k(h, s, a) + 1}$$

and

$$\hat{P}^k_{h, s, a}(s') = \frac{\sum_{i=1}^{k-1} I\{(s^i_n, a^i_n, s^i_{h+1}) = (s, a, s')\}}{n_k(h, s, a) + 1}.$$
Then, define empirical MDP based on our observation and estimation before episode $k$ as the tuple $M^k = (H, S, A, \hat{P}^k, \hat{R}^k, s_1)$. Since $\hat{P}^k_{h,s,a}$ is not a valid distribution over $S$, for being rigorous, we can imagine there is an additional virtual absorbing state that every state will transit to with remaining probability. We further define

$$\hat{P}^k_{h,s,a}(s') = \frac{\sum_{l=1}^{k-1} \mathbb{1}\{(s'_1, a'_1, s'_l) = (s, a, s')\}}{\max\{n_k(h, s, a), 1\}},$$

which will be used as the estimated transition when computing the random value perturbation in the algorithm.

In addition to the above notations, let $\tilde{O}(\cdot), \tilde{\Theta}(\cdot)$ and $\tilde{\Omega}(\cdot)$ be asymptotic notations ignoring all poly-logarithmic terms. For distribution $D \in \Delta^S$ and value function $V \in \mathbb{R}^S$, let $\mathbb{V}(D, V)$ denote the variance of $V$ under distribution $D$, which is defined as $\mathbb{V}(D, V) = \sum_{s \in S} D(s)(V(s) - \langle D, V \rangle)^2$. For constant $a > 0$, we define the corresponding clipping function as $\text{clip}_a(\cdot) = \max\{-a, \min\{a, \cdot\}\}$. Immediately we have $|\text{clip}_a(x)| \leq a$ for any $a > 0$. We introduce the definitions of other notations when used. In appendix, we summarize the notations and definitions used in this paper.

4 Main Results

4.1 Algorithm

The main contribution of this paper is that we show algorithm with randomized value functions can achieve regret that matches the known lower bound $\Omega\left(\sqrt{SAT}\right)$ [Jaksch et al., 2010, Domingues et al., 2021] up to logarithmic factors in tabular setting. To facilitate exploration, this type of algorithms uses random value perturbation instead of deterministic bonus. The algorithm we consider is summarized in Algorithm 1. In our algorithm, SSR, the random perturbation ensures that optimism/pessimism can be obtained with constant probability in each episode. Moreover, randomized value function has its origin from posterior sampling for reinforcement learning (Thompson sampling). The randomized perturbation can be interpreted as approximate sampling from the posterior distribution of the value function on randomized training data [Russo, 2019].

We first give an overview of SSR. In Algorithm 1, the policy used at episode $k$ is computed using the empirical MDP, $M^k = (H, S, A, \hat{P}^k, \hat{R}^k, s_1)$, which is based on observation and estimation before episode $k$. However, instead of directly choosing optimal policy for $M^k$, we add a small random perturbation when computing the value of each state and action pair. To be more precise, at each episode $k$, we first estimate the reward and transition function for each state $s$ and action $a$ based on (2) and (3). Then, we compute the value function for state $s$ and action $a$,

$${\bar{Q}}_{h,k}(s) \leftarrow \hat{R}_{h,s,a}^k + \left(\hat{P}_{h,s,a}^k, \hat{V}_{h+1,k}^k\right) + \sigma_{tv}(h, s, a) \hat{z}_k.$$
Here, $\hat{z}_k \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable sampled once every episode. The magnitude of the perturbation, $\sigma_{ty}^k$, depends on how many samples $n^k(h, s, a)$ we have observed and how confident we are on the estimations $\hat{R}_{h,s,a}^k$ and $\hat{P}_{h,s,a}^k$. We will discuss more about the choice of the magnitude later in this section.

In order to prevent estimated value function from behaving badly, we add a clipping to the value function:

$$\nabla_{h,k}(s) = \text{clip}_{2(H-h+1)} \left( \max_{a \in A} \bar{Q}_{h,k}(s, a) \right) = \max \left\{ -2(H-h+1), \min \left\{ 2(H-h+1), \max_{a \in A} \bar{Q}_{h,k}(s, a) \right\} \right\}. \quad (4)$$

As our analysis will show, this kind of clipping can bound the value function, maintain optimism and pessimism and also guarantee that clipping will not happen for a lot of times. The constant 2 (instead of 1) plays a crucial role because it means the value function grows at an additive rate of 2 from $h = H$ to $h = 1$. If we do not consider the added noise, then the value function should at most grow 1 at each timestep because the reward is at most 1. For our clipping technique, if a clip is triggered, there exists a timestep such that the added noise is more than 1, which is equivalent to a small number of visits (cf. Definition 29 and Lemma 8). As our later analysis will show, the clipping only affects the warm-up phase and will not compromise the long-term performance of the algorithm. Finally, after computing the value function and clipping, SSR chooses the action $a^k_h$ that maximizes $\bar{Q}_{h,k}(s^k_h, a)$ at each time step, $h = 1, \ldots, H$, throughout the episode.

Note that from a Bayesian perspective, when there is no clipping, in Algorithm 1, $\bar{Q}_{h,k}$ follows distribution

$$\bar{Q}_{h,k}(s, a) \mid \nabla_{h+1,k} \sim \mathcal{N} \left( \hat{R}_{h,s,a}^k + \left\langle P_{h,s,a}^k, \nabla_{h+1,k} \right\rangle, (\sigma_{ty}^k(h, s, a))^2 \right).$$

This resembles posterior sampling because when estimating some parameter $\theta^* \sim \mathcal{N}(0, \beta^2)$ based on noisy observation $\theta_1, \ldots, \theta_n \sim \mathcal{N}(\theta, \beta^2)$, the posterior distribution of $\theta^*$ given $\{\theta_i\}_{i=1}^n$ is $\theta^* \mid \{\theta_i\}_{i=1}^n \sim \mathcal{N}(\frac{1}{n+1} \sum_{i=1}^n \theta_i, \frac{\beta^2}{n+1})$. Although exact posterior sampling may not be possible in complex reinforcement learning settings, in SSR, $\sigma_{ty}^k(h, s, a)$ is chosen at scale $\hat{\Theta} \left( \frac{1}{\sqrt{n_k(h, s, a)+1}} \right)$ and therefore can be interpreted as doing approximate posterior sampling. Moreover, SSR can be viewed as a variant of Randomized Least Square Value Iteration (RLSVI). The only differences are clipping and a single random seed used in each episode instead of different random seeds at different tuples $(h, s, a)$. We will discuss more about the choice of the random seed later in this section. We refer to Osband et al. [2017] and Russo [2019] for a more detailed discussion on the relationship among RLSVI, posterior sampling and randomized value function.

In the following paragraphs, we discuss in more details about the three major algorithmic innovations:

**Single Random Seed in Each Episode.** SSR is similar to the algorithms analyzed in Russo [2019] and Agrawal et al. [2021]. The major difference is that in the algorithm we propose, we use a single random seed $\hat{z}_k$ to generate the perturbations for all time steps $h = 1, \ldots, H$ in an episode $k$.

When using different random seeds in an episode, the algorithm can be optimistic in some time step while being pessimistic in others. Then, the effects of the perturbations at different time steps will cancel with each other. As a result, to ensure sufficient exploration, the magnitude of the perturbation has to be large. This issue was also pointed out in Agrawal et al. [2021], Abeille and Lazaric [2017].

A large perturbation magnitude can increase the instability of the algorithm and worsen the algorithm’s performance. When a single random seed is used, a small perturbation magnitude is enough to guarantee that the algorithm is optimistic with constant probability in any episode. We are able to show that using a single random seed can significantly increase the stability of the algorithm and therefore enjoy much smaller regret. Coincidentally, Vaswani et al. [2020] also uses a similar single randomization in bandit problems to build a near-optimal TS-like algorithm and our work can be treated as its natural extension to RL problems.

**Clipping.** To obtain a tight regret bound, the estimated value function needs to be well bounded. In [Russo, 2019], no clipping is used and the estimated value function is at the order of $O(H^{5/2}S)$, which results in suboptimal regret bound. Generally there are two types of clipping methods. The first one is uncertainty-based, i.e. whenever the uncertainty is large then the value is clipped to $H - h + 1$ at timestep $h$ [Zanette et al., 2020, Agrawal et al., 2021]. However this type of clipping cannot maintain pessimism.
which is critical in our analysis. The other kind of clipping is value-based, mostly in UCB-type algorithms [Jin et al., 2019]. These algorithms truncate estimated greater than a certain threshold, i.e. $H - h + 1$ at time step $h$. The problem here is that the number of clipping cannot be bounded because if the true value function is closed to $H - h + 1$ at timestep $h$, the clipping will happen with some constant probability.

Our clipping method leverages both type of clipping methods in the existing literature. Though our clipping is based on the value function, we show that whenever the clip is triggered, the estimation error must be large, which implies that the uncertainty at that state is large. This clipping method inherits the desired properties from both uncertain-based and value-based clipping, i.e. the optimism/pessimism is maintained and the number of clippings can be bounded.

**Magnitude of Perturbation.** A large magnitude of perturbation can encourage exploration, but at the same time increase instability. In our algorithm, the magnitudes are chosen as the smallest values so that the algorithm can be optimistic with constant probability. Since the value function can roughly be bounded by $O(H)$, a naive choice of the perturbation magnitude can be $\Theta(H\sqrt{SAT})$.

To make the magnitude even smaller, inspired by Azar et al. [2017] who showed one can use an (empirical) Bernstein’s inequality to derive a sharp exploration bonus for UCB-based algorithms, we propose a new choice of perturbation magnitude based on Bernstein’s inequality. The Bernstein-based perturbation uses the empirical variance of the value function, which makes it smaller than the Hoeffding-based one mostly, but still maintain optimism with constant probability.

In our paper, we study both types of magnitudes. In particular, we show that the regret of SSR based on Bernstein’s inequality matches the known lower bound $\Omega(H\sqrt{SAT})$. Following are the two choices:

$$\sigma_{Ho}^k(h, s, a) = H \sqrt{\frac{\log(2HSAk^2)}{nk(h, s, a) + 1}} + \frac{H}{nk(h, s, a) + 1},$$

$$\sigma_{Be}^k(h, s, a) = \sqrt{\frac{16V(\tilde{P}_{h, s, a}^k, \nabla_{k,h+1}) \log(2HSAk^2)}{nk(h, s, a) + 1}} + \frac{65H \log(2HSAk^2)}{nk(h, s, a) + 1} + \sqrt{\frac{\log(2HSAk^2)}{nk(h, s, a) + 1}},$$

where subscript “Ho” represents that the perturbation is based on Hoeffding’s inequality and “Be” represents Bernstein’s inequality, correspondingly. To clarify, when subscript “ty” is used, which stands for “type” as a placeholder for “Ho” or “Be”, it means that there is no need to write two copies of expressions for Hoeffding-based and Bernstein-based noise separately.

**4.2 Regret Analysis**

We analyze the regret, defined in (1), of our algorithm SSR using both types of perturbations. Our main theorems are presented in Theorem 1 and 2. The regret bounds we get are high probability bounds and are strictly stronger than expectation bound (by choosing $\delta \leq \frac{1}{\sqrt{T}}$). In particular, Theorem 2 shows SSR with Bernstein-based perturbation can achieve the regret that matches the known lower bound $\Omega(H\sqrt{SAT})$ up to logarithmic factors. We sketch the proof of Theorem 1 and Theorem 2 in Section 5.

**Theorem 1.** If the Hoeffding-type noise (5) is used, then for any MDP $M = (H, S, A, P, R, s_1)$, with probability at least $1 - \delta$, Algorithm 1 satisfies

$$\text{Reg}(M, K, SSR_{Ho}) \leq \tilde{O}(H^{1.5}\sqrt{SAT} + H^4S^2A).$$

In particular, when $T \geq \Omega(H^5SA^3)$, it holds that $\text{Reg}(M, K, SSR_{Ho}) \leq \tilde{O}(H^{1.5}\sqrt{SAT})$. 

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Theorem 2. If the Bernstein-type noise (6) is used, then when \( T \geq \tilde{\Omega}(H^5S^2A) \), for any MDP \( M = (H, S, A, P, R, s_1) \), with probability at least 1 − \( \delta \), Algorithm 1 satisfies
\[
\text{Reg}(M, K, \text{SSR}_\text{Be}) \leq \tilde{O}(H\sqrt{SAT} + H^4S^2A).
\]
In particular, if we further have \( T \geq \tilde{\Omega}(H^6S^3A) \), it then holds that \( \text{Reg}(M, K, \text{SSR}_\text{Be}) \leq \tilde{O}(H\sqrt{SAT}) \).

We give a brief comparison of SSR and other related works. Russo [2019] shows that RLSVI, an algorithm similar to SSR, can achieve \( \tilde{O}(H^{2.5}S^{1.5}\sqrt{AT}) \) regret in expectation over the randomness of MDP and the algorithm. In addition, Zanette et al. [2020] showed a \( \tilde{O}(H^2d^2\sqrt{T}) \) frequentist regret bound of RLSVI with linear function approximation. Converting their bound to the tabular setting directly will give a \( \tilde{O}(H^2S^2A^2\sqrt{T}) \) bound. Our paper closes the gap between those previous bounds and the lower bound in tabular setting. In Theorem 2, we show randomized exploration can be near-optimal.

5 Proof Outline

In this section, we present an outline of Theorem 1 and 2. Since their proofs follow the same framework, we will present an unified outline and explain the individual steps particularly for each case when necessary. The details of complete proof are deferred to the appendix.

We present the first step in Section 5.1, which shows the concentration property of the estimated value functions based on \( \hat{R} \) and \( \hat{P} \) and the perturbations. We show that in most time, the deviation of the estimated functions from the true ones can be bounded by the perturbation magnitude. Therefore, SSR can be both optimistic and pessimistic with constant probability at each episode. Moreover, we show that the perturbation is bounded with high probability.

In Section 5.2, we decompose the regret into the pessimism term and the estimation error term. While the pessimism term studies the regret induced by the algorithm’s underestimation of the optimal value, the estimation error term studies the difference between policy \( \pi^k \)'s estimated value and true value. We bound these two terms separately.

Both pessimism term and estimation error term are reduced to a sum of perturbation magnitude and related lower-order terms in Section 5.3. The key difference between Hoeffding-type and Bernstein-type perturbation is that the latter one requires to bound the sum of variance of the value functions. In Section 5.4, we address this additional technical difficulty by bounding the sum of the variance, which completes the last piece of the proof.

Notations For the ease of exposure, we will use a simplified notations during this sketch. Specifically, let \( x = (h, s, a) \) and \( x^k_h = (h, s^k_h, a^k_h) \).

5.1 Concentration and Optimism/Pessimism

We start by introducing a set of MDPs \( \mathcal{M}^k_{\text{ty}} \) as a confidence set such that the empirical MDP \( \hat{M}^k \) belongs to it with high probability. Finding such a confidence set helps us focus on analysing regret under the event \( \mathcal{C}^k_{\text{ty}} := \{ \hat{M}^k \in \mathcal{M}^k_{\text{ty}} \} \). Specifically, with \( M' = (H, S, A, P', R', s_1) \), we define
\[
\mathcal{M}^k_{\text{ty}} := \left\{ M' : \forall x = (h, s, a), |\text{err}(x)| \leq \sqrt{e^k_{\text{ty}}(x)} \right\},
\]
where \( \text{err}(x) = (R'_x - R_x) + \langle P'_x - P_x, V^*_{h+1} \rangle \) and
\[
\sqrt{e^k_{\text{Ho}}(x)} = \sigma^k_{\text{Ho}}(x), \quad \sqrt{e^k_{\text{Be}}(x)} \approx \sigma^k_{\text{Be}}(x).
\]
Here, since \( \sqrt{c_{by}(x)} \) and \( \sigma_{by}(x) \) are approximately in the same order, we use "\( \approx \)" to avoid technical details. Then, by an appropriate application of Hoeffding’s inequality or Bernstein’s inequality, for both types of perturbation, it is possible to show that
\[
\sum_{k=1}^{\infty} \mathbb{P} \left( (C^k_{by})^c \right) = \sum_{k=1}^{\infty} \mathbb{P} \left( \tilde{M}^k \notin M^k_{by} \right) \leq \frac{\pi^2}{3}.
\]
Since the value function is bounded in \([0, H]\), this inequality tells us that the regret incurred by bad estimation is at most \( O(H) \). To be precise, it holds with high probability
\[
\sum_{k=1}^{K} \mathbb{1} \left\{ (C^k_{by})^c \right\} \left( V^*_{1} - V^{\pi_k}_{1,k} \right)(s^k) \leq O(H).
\]  
(8)

It should be noted that although Gaussian randomness in value function encourages exploration, its actual effect can both be optimistic or pessimistic because Gaussian random variable takes values from the whole real line. Therefore, conceptually, a mechanism that prevents Gaussian noise from behaving badly will help improve the algorithm and clipping is such a mechanism. From a technical viewpoint, clipping effectively bounds the value function and allows tighter concentration arguments.

Specifically, we will use two crucial properties of our clipping method. First, if \( Q_{h,k}(s,a) \geq Q^*_k(s,a) \), \( \forall (s,a) \in \mathcal{S} \times \mathcal{A} \), then we have
\[
\nabla_{h,k}(s) \geq V^*_k(s), \forall s \in \mathcal{S},
\]
and similarly if \( Q_{h,k}(s,a) \leq Q^*_k(s,a) \), \( \forall (s,a) \in \mathcal{S} \times \mathcal{A} \), then we have
\[
\nabla_{h,k}(s) \leq V^*_k(s), \forall s \in \mathcal{S}.
\]
In addition, we can prove that whenever a clip is triggered for \( s^k \), we have \( n_k(h, s^k, a^k) \geq \alpha_k \), where \( \alpha_k = \tilde{O}(H^2) \). The proof is deferred to the appendix. The high level intuition is that the value function is bounded by \( 2(H - h) \) at timestep \( h + 1 \), so the estimated value function without noise \( \sigma^{k}_{by}(h, s, a) \tilde{z}_k \) should be bounded by \( 2(H - h) + 1 \) because the reward at timestep \( h \) is bounded by 1. So if the value function at \( s^k \) is clipped, we have that
\[
|\sigma^{k}_{by}(h, s, a) \tilde{z}_k| \geq 1,
\]
which implies that \( n_k(h, s^k, a^k) \) is small. Moreover, since \( \alpha_k = \tilde{O}(H^2) \), it is possible to show that the total regret incurred by clipping is at most \( \tilde{O}(H^4 \mathcal{S} \mathcal{A}) \), which is a lower-order term when \( T \) is sufficiently large. That is, let \( E^{cum}_{h,k} \) denote the event that there is no clipping during episode \( k \). Then, it holds with high probability that\(^3\)
\[
\sum_{k=1}^{K} \mathbb{1} \left\{ C^k_{by} \cap \left( E^{cum}_{h,k} \right)^c \right\} \left( V^*_1 - V^{\pi_k}_{1,k} \right)(s^k) \leq \tilde{O}(H^4 \mathcal{S} \mathcal{A}).
\]  
(9)

As claimed before, because of the randomness in Gaussian noise, our algorithm \( \text{SSR} \) will encourage exploration and it takes effect when there is no clipping and the estimation is not too bad. In other words, it can be optimistic. However, also because of this randomness, its optimism only holds in a probabilistic sense. In precise, it is possible to show that
\[
\mathbb{P} \left( \nabla_{h,k}(s) \geq V^*_k(s), \forall h \in [H], s \in \mathcal{S} \mid \mathcal{H}^{k-1}_{H}, C^k_{by} \right) \geq C_{by},
\]  
(10)
where \( \mathcal{H}^{k-1}_{H} \) denotes the observations from previous episodes and the value of constant \( C_{by} \) depends on the type of noise we choose. Meanwhile, we can also prove a very similar probabilistic pessimism, which means to have \( \nabla_{h,k}(s) \leq V^*_k(s), \forall h \in [H], s \in \mathcal{S} \) with constant probability. The property of optimism and pessimism will help us upper bound the absolute value of \( V^*_k(s^k) - \nabla_{1,k}(s^k) \), which will be discussed soon.

\(^3\)Technically, this is not precisely how we bound the regret incurred by clipping, but it aligns better with the intuition. Full technical details can be found in Appendix.
We note that the chosen magnitude of noise is close to the confidence width $\sqrt{\kappa h k (x)}$ as specified in equation (7). Therefore, it is intuitive to consider that when $\hat{z}_k$ happens to not be too small, which holds with constant probability for Gaussian, the estimated value function $V_{h,k}^*$ will be optimistic as long as the estimation is not too bad. From this perspective, we can also see the advantage of using single random source in a episode over using independent random sources for all different $(h, s, a)$. The reason is that when independent random sources are used, we need most of them to be not too small for being optimistic, which will hold with small probability since independence makes probability to multiplicate. Therefore, to achieve optimism, a larger magnitude of noise is necessary and it thus contributes to a larger regret at the end.

5.2 Regret Decomposition

Now, given equations (8) and (9), we can see that it only remains to bound $\mathbb{I} \{C_{ly} \cap C_{ly}^{\text{cum}} \} \left( V_{1,k}^* - V_{1,k}^\pi \right) (s_k^l)$ for each episode $k$. Technically, the further defined the good event $G_k$ will help make $\nabla V_{h,k}$ better-behaved. Its precise definition will be given in the appendix. Therefore, it is sufficient to bound $\mathbb{I} \{G_k\} \left( V_{1,k}^* - V_{1,k}^\pi (s_k^l) \right)$, which means to have

$$\text{Reg} (M, K, SSR_{ly}) \leq \mathbb{I} \{G_k\} \left( V_{1,k}^* - V_{1,k}^\pi \right) + \mathbb{I} \{G_k\} \left( V_{1,k}^\pi - V_{1,k}^\pi \right) (s_k^l).$$  

(11)

**Notations** Before proceeding, we need to introduce several more notations. We will use $L$ to denote any poly-logarithmic terms without specifying them precisely. Recall the definition of $\sigma^2_{ly} (x)$ in equations (5) and (6). We define $w^k_{ly} (x) = \sigma^2_{ly} (x) \hat{z}_k$ with $\hat{z}_k \sim \mathcal{N}(0, 1)$ and $\overline{V}_{ly}^k = (H, S, A, \hat{P}, \hat{R}^k + w^k_{ly}, s_1)$. Further, let $w^k_{ly} (x) = -\sigma^2_{ly} (x) L$ such that $\vert \hat{z}_k \vert \leq L$ with high probability and define $\overline{M}_{ly}^k = (H, S, A, \hat{P}, \hat{R}^k + w^k_{ly}, s_1)$. Define $V_{h,k}$ to be the value function obtained by running $\pi^k$ on $\overline{M}_{ly}^k$. Similarly, we let $\overline{w}_{ly} (x) = \sigma^2_{ly} (x) L$ and define $\overline{M}_{ly}^k = (H, S, A, \hat{P}, \hat{R}^k + \overline{w}_{ly}, s_1)$. Define $\overline{V}_{h,k}$ to be the value function obtained by running $\pi^k$ on $\overline{M}_{ly}^k$.

**Remarks** With these definitions, we can see that the estimated value function $V_{h,k}$ is exactly the optimal value function of $\overline{M}_{ly}$. Further, since $G_k$ includes the event $\vert \hat{z}_k \vert \leq L$, under $G_k$, we have $V_{h,k} \leq V_{h,k} \leq \overline{V}_{h,k}$.

5.2.1 The Pessimism Term

Let $C_1 = 1 / \min \{C_{ly}^r, C_{ly} \}$. Due to the help of optimism and pessimism with constant probability, it is possible to show that

$$\mathbb{I} \{G_k\} \left( V_{h,k}^* (s_k^l) - V_{h,k} (s_k^l) \right) \leq \mathbb{I} \{G_k\} C_1 \left( \left\vert \overline{V}_{h,k}^k (s_k^l) - V_{h,k}^\pi (s_k^l) \right\vert + \left\vert V_{h,k}^\pi (s_k^l) - V_{h,k}^\pi (s_k^l) \right\vert \right).$$  

(12)

Different from Zanette et al. [2020] and Agrawal et al. [2021], here we do not resort to an independent copy of the perturbed MDP and give conceptually simpler analysis. This is possible because we resort to both optimism and pessimism. Specifically, let $O_k$ be the event that $V_{h,k} (s) \geq V_{h,k}^* (s), \forall h \in [H], s \in S$. As specified in (10), we know that $\mathbb{P} (O_k \mid H_{ly}^{k-1}, G_k) \leq C_{ly}$ regardless the type of noise used.

The definition of $O_k$ implies $V_{h,k}^* \leq \mathbb{E} [\overline{V}_{h,k} \mid O_k, H_{ly}^{k-1}, G_k]$. Meanwhile, notice that

$$\mathbb{I} \{G_k\} \left( \mathbb{E} [\overline{V}_{h,k} \mid H_{ly}^{k-1}, G_k] - V_{h,k} \right) = \mathbb{I} \{G_k\} \mathbb{P} (O_k \mid H_{ly}^{k-1}, G_k) \left( \mathbb{E} [\overline{V}_{h,k} \mid O_k, H_{ly}^{k-1}, G_k] - V_{h,k} \right) \geq \mathbb{I} \{G_k\} \mathbb{P} (O_k \mid H_{ly}^{k-1}, G_k) \left( \mathbb{E} [\overline{V}_{h,k} \mid O_k, H_{ly}^{k-1}, G_k] - V_{h,k} \right) \geq \mathbb{I} \{G_k\} \mathbb{P} (O_k \mid H_{ly}^{k-1}, G_k) \left( \mathbb{E} [\overline{V}_{h,k} \mid O_k, H_{ly}^{k-1}, G_k] - V_{h,k} \right) \geq \mathbb{I} \{G_k\} \left( \mathbb{E} [\overline{V}_{h,k} \mid O_k, H_{ly}^{k-1}, G_k] - V_{h,k} \right) \leq \mathbb{I} \{G_k\} \left( \mathbb{E} [\overline{V}_{h,k} \mid O_k, H_{ly}^{k-1}, G_k] - V_{h,k} \right) \leq \mathbb{I} \{G_k\} C_1 \left( \mathbb{E} [\overline{V}_{h,k} \mid H_{ly}^{k-1}, G_k] - V_{h,k} \right).$$
Since good event \( G_k \) implies \( \mathbf{V}_{h,k} \leq \mathbf{V}_{h,k} \), the RHS of (13) is non-negative and the RHS of (14) is non-positive. Therefore, we can then conclude

\[
1 \{ G_k \} \left( V^*_h (s^k_h) - V_{h,k} (s^k_h) \right) \leq 1 \{ G_k \} C_1 \left( \mathbb{E} \left[ \mathbf{V}_{h,k} | \mathcal{H}^{H-1}_H, G_k \right] (s^k_h) - \mathbf{V}_{h,k} (s^k_h) \right) 
\]

By using the property of pessimism with constant probability, we can similarly show that

\[
1 \{ G_k \} \left( V^*_h (s^k_h) - V_{h,k} (s^k_h) \right) \geq 1 \{ G_k \} C_1 \left( \mathbb{E} \left[ \mathbf{V}_{h,k} | \mathcal{H}^{H-1}_H, G_k \right] (s^k_h) - \mathbf{V}_{h,k} (s^k_h) \right). \tag{14} \]

5.2.2 The Estimation Error Term

Define \( \mathcal{R}^k_x = \hat{\rho}^k - \rho, \mathcal{P}^k_{x^k} = \langle \hat{p}^k - P_{x^k}, V^*_h \rangle, \mathcal{Q}^k_{s^k, a^k} = \mathbf{V}_{h,k} - V^*_{h,k}, \mathcal{D}^k_{s^k, a^k} = \mathbf{V}_{h,k} - V^*_{h,k}, \) and \( \mathcal{M}_{2,k} = \mathbf{V}_{h,k} - V^*_{h,k} \).

Note that all the above kinds of \( \delta \) are the value difference under the sample policy \( \pi^k \) but different underlying MDP and we call them (policy) estimation error. For simplicity, we will use \( \mathcal{M}_{h,k} \) to denote any martingale difference sequence (MDS) terms at period \( h \), episode \( k \), without specifying them precisely.

Throughout this subsection, we assume the good event \( G_k \) holds. We then bound the estimation error term. First, it is possible to show that

\[
| \mathcal{D}^k_{s^k, a^k} | = | \mathbf{Q}^k_{s^k, a^k} | + | \mathcal{P}^k_{s^k} + \mathcal{R}^k_{x^k} + \omega^k_{x^k} (x^k_1) | + | \mathcal{D}^k_{s^2, a^2} | + \mathcal{B}^k (x^k_1) + \mathcal{M}_{2,k}.
\]

Here, the inequality (i) holds with high probability and it needs to take several steps of algebra. Further, we have \( \mathcal{B}^k (x^k_1) = \left\langle \hat{p}^k_{x^k_1} - P_{x^k_1}, \mathbf{V}_{2,k} - V^*_2 \right\rangle \), which will be bounded in an approach different from previous literature because now \( \mathbf{V} \) does not have high-probability optimism.

Then, we focus on bounding \( \mathcal{B}^k (x^k_1) \) by applying techniques proposed in [Azar et al., 2017]. In particular, we have

\[
\mathcal{B}^k (x^k_1) = \left\langle \hat{p}^k_{x^k_1} - P_{x^k_1}, \mathbf{V}_{2,k} - V^*_2 \right\rangle 
\]

\[
= \sum_{s_2 \in \mathcal{S}} \hat{p}^k_{x^k_1} (s_2) - P_{x^k_1} (s_2) \left| \mathbf{V}_{2,k} (s_2) - V^*_2 (s_2) \right| 
\]

\[
\leq \sum_{s_2 \in \mathcal{S}} \sqrt{\frac{P_{x^k_1} (s_2)}{n_k (x^k_2)}} \left| \mathbf{V}_{2,k} (s_2) - V^*_2 (s_2) \right| 
\]

\[
\leq \sum_{s_2 \in \mathcal{S}} \frac{2 \mathcal{P}_{x^k_1} (s_2)}{H C_1 \left( | \mathcal{D}^k_{2,k} (s_2) | + | \mathcal{D}^k_{2,k} (s_2) | \right)} \quad \text{\textup{(By Eq. (12))}}
\]
Then, a final high-probability regret bound can be obtained by summing each individual term over $k, h$, which is

\[
\|\tilde{\pi}^*_{1,k}(s_1^k)\| \lesssim \|P_{x_1^k}^k + R_{x_1^k}^k + \bar{w}_{ty}(x_1^k)\| + \|\tilde{\pi}^*_{2,k}(s_2^k)\| + C_1 \frac{H}{H} \left( \|\tilde{\pi}^*_{2,k}(s_2^k)\| + |\delta_{2,k}^*(s_2^k)| \right) + \mathcal{M}_{2,k},
\]

We want to emphasize the difference between our method and techniques in [Azar et al., 2017]. In their algorithm, the estimated value is optimistic with high probability, which means $\bar{V}_{2,k}(s_2) - V^*_2(s_2)$ is always positive and leads to a single recursion. We use absolute value to keep quantities positive but the recursion incurs new two terms, $|\delta_{2,k}^*(s_2^k)|$ and $|\bar{\delta}_{2,k}^*(s_2^k)|$. Fortunately it is possible to derive similar recursions for these two terms, which are

\[
\|\bar{\delta}_{1,k}^*(s_1^k)\| \lesssim \|P_{x_1^k}^k + R_{x_1^k}^k + \bar{w}_{ty}(x_1^k)\| + \|\bar{\pi}^*_{2,k}(s_2^k)\| + C_1 \frac{H}{H} \left( \|\bar{\pi}^*_{2,k}(s_2^k)\| + |\bar{\delta}_{2,k}^*(s_2^k)| \right) + \mathcal{M}_{2,k},
\]

\[
\|\bar{\delta}_{1,k}^*(s_1^k)\| \lesssim \|P_{x_1^k}^k + R_{x_1^k}^k + \bar{w}_{ty}(x_1^k)\| + \|\bar{\pi}^*_{2,k}(s_2^k)\| + C_1 \frac{H}{H} \left( \|\bar{\pi}^*_{2,k}(s_2^k)\| + |\bar{\delta}_{2,k}^*(s_2^k)| \right) + \mathcal{M}_{2,k}.
\]

By summing over the above three inequalities, we can get a recursion on $\|\tilde{\delta}_{1,k}^*\| + |\tilde{\delta}_{1,k}^* + |\tilde{\delta}_{1,k}^*|$, which is

\[
\left( \|\tilde{\delta}_{1,k}^*\| + |\tilde{\delta}_{1,k}^* + |\tilde{\delta}_{1,k}^*\| \right) (s_1^k) \lesssim 3 \|P_{x_1^k}^k + R_{x_1^k}^k + \bar{w}_{ty}(x_1^k)\| + \left( 1 + \frac{3C}{H} \right) \left( \|\tilde{\pi}^*_{2,k}\| + |\tilde{\delta}_{2,k}^*| + |\tilde{\delta}_{2,k}^*| \right) (s_2^k) + \mathcal{M}_{2,k},
\]

We can keep unrolling this procedure from $h = 1$ to $h = H$. Then, since $(1 + \frac{3C}{H})^H \leq e^{3C}$, it is possible to show that

\[
\left( \|\tilde{\delta}_{1,k}^*\| + |\tilde{\delta}_{1,k}^* + |\tilde{\delta}_{1,k}^*\| \right) (s_1^k) \lesssim e^{3C} \sum_{h=1}^{H} \left( \sqrt{e_{ty}^k + L\sigma_{ty}^k} \right) (x_h^k) + \sum_{h=1}^{H} \mathcal{M}_{h,k}. \quad (15)
\]

### 5.3 Combining Different Terms

If we add the ignored low-order terms back together with more steps of algebra, then by combining equations (11) and (15) and applying concentration inequalities to MDP $\mathcal{M}_{h,k}$, we can have

\[
\sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \} \left( V^*_{1,k} - V_{1,k}^* \right) (s_1^k) \leq e^{3C} \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{I} \{ \mathcal{G}_k \} \left( \sqrt{e_{ty}^k + L\sigma_{ty}^k} \right) (x_h^k) + \bar{O} \left( H\sqrt{T} + H^4S^2A \right).
\]

Then, a final high-probability regret bound can be obtained by summing each individual terms over $k, h$ separately. It is well-known among literature that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{1}{n_k(x_h^k) + 1}} \leq \bar{O} \left( \sqrt{HSA} \right),
\]

and

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{n_k(x_h^k) + 1} \leq \bar{O} \left( HSA \right).
\]

Recall the definition of $\sigma_{H}^*$ in equation (5) and the fact that $\sqrt{e_{H0}^k} = \sigma_{H0}^k$. By using these two inequalities, the bound in equation (16) can be made explicit if we use Hoeffding-type noise. As a result, it holds with high probability that

\[
\sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \} \left( V^*_{1,k} - V_{1,k}^* \right) (s_1^k) \leq \bar{O} \left( H^{1.5}\sqrt{SAT} \right).
\]
Combining the results in equations (8) and (9), we can conclude if Hoeffding-based noise is used, it holds with high probability that

\[
\text{Reg}(M, K, \text{SSR}_H) \leq \tilde{O} \left( H^{1.5} \sqrt{SAT} + H^4 S^2 A \right).
\]

### 5.4 Bound on Sum of Variance

Analyses become more involved when Bernstein-type noise is used. Specifically, notice that inequalities (17) and (18) cannot be used to bound \( \sum_{k,h} \hat{V}(\hat{P}_{x,k}^h, V_{h,k}^\ast) \) since the behavior of variance can be complicated. Therefore, to obtain a regret bound when Bernstein-type noise is used, we need to do further analysis. Here, we apply some techniques developed in Azar et al. [2017]. However, since for Algorithm 1 the optimism only holds with constant probability, the details of bounding specific terms are quite different.

#### Notations

For the ease of exposure, we will ignore all constants and define

- \( \hat{V}_{h,k}^* = V(\hat{P}_{x,k}^h, V_{h,k}^*) \)
- \( \hat{V}_{h,k} = V(\hat{P}_{x,k}^h, \hat{V}_{h,k}) \)
- \( \hat{V}_{h,k}^\ast = V(\hat{P}_{x,k}^h, \hat{V}_{h,k}^\ast) \)
- \( \hat{V}_{h,k} = V(\hat{P}_{x,k}^h, \hat{V}_{h,k}) \)
- \( \hat{V}_{h,k} = V(\hat{P}_{x,k}^h, \hat{V}_{h,k}) \)
- \( \hat{V}_{h,k} = V(\hat{P}_{x,k}^h, \hat{V}_{h,k}) \)

We further define

\[
U = \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \sqrt{\frac{L}{n_h(x_k^h)} + 1} \left( \hat{V}_{h,k} + \hat{V}_{h,k} \right).
\]

As our detailed analysis in appendix will show, \( U \) is basically the quantity we need to bound in order to get an explicit bound for equation (15) under Bernstein-type noise. By using Cauchy-Schwartz inequality, we can get

\[
U \leq \tilde{O}(HSA) \hat{W}.
\]

Then, the term \( \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \hat{V}_{h,k}^* \) can be decomposed as

\[
\sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \hat{V}_{h,k}^* \]
\[
= \frac{3}{2} \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \hat{V}_{h,k}^* + \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \left( \hat{V}_{h,k}^* - \frac{3}{2} \hat{V}_{h,k}^* \right).
\]

First, the Lemma 8 in Azar et al. [2017] gives us (a) \( \leq \tilde{O} \left( HT + H^2 \sqrt{T} + H^3 \right) \) and with some steps of algebra, it is possible to show that with high probability, we have

\[
(b) \leq \tilde{O} \left( H^3 S^2 A + H^2 \sqrt{T} + H \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \delta_{h,k}^k (s_{h,k}^k) \right).
\]

We can further show similar results for \( \hat{V}_{h,k} \), which leads to

\[
\hat{W} \leq \tilde{O} \left( HT + H^2 \sqrt{T} + H^3 S^2 A \right) + \tilde{O} \left( H \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \left( \delta_{h,k}^k + \sum_{k=1}^K \sum_{h=1}^H 1 \{ \mathcal{G}_k \} \delta_{h,k}^k (s_{h,k}^k) \right) \right).
\]
Therefore, by further applying inequalities (15), (16), (17) and (18), it is possible to show that with high probability, we have

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} 1 \{G_k\} \left( \delta_{h,k}^e + |\delta_{h,k}^e| \right) (s_k^e) \leq \tilde{O} \left( \sqrt{H^3T} + H^5S^2A + HU \right). \tag{21}
\]

By combining equations (20) and (21), we have

\[
\mathcal{W} \leq \tilde{O} \left( HT + H^6S^2A + \sqrt{H^5T} + H^2U \right) \leq \tilde{O} \left( HT + H^2U \right). \tag{When \( T \geq \tilde{\Omega} (H^5S^2A) \)}
\]

Then, by plugging this result into equation (19), we can obtain

\[
U \leq \tilde{O} \left( \sqrt{HSA(HT + H^2U)} \right) \leq \tilde{O} \left( H\sqrt{SAT} + H^{1.5}\sqrt{U} \right).
\]

Now, we can see that \( U \leq \tilde{O} \left( H\sqrt{SAT} \right) \) satisfies this inequality. Therefore, when \( T \geq \tilde{\Omega} (H^5S^2A) \), it holds with high probability that \( U \leq \tilde{O} \left( H\sqrt{SAT} \right) \).

Finally, by plugging this result back into equation (16) and combining results in equations (8) and (9), we can conclude if Bernstein-type noise is used, it holds with high probability that

\[
\text{Reg}(M, K, \text{SSR}_{be}) \leq \tilde{O} \left( H\sqrt{SAT} + H^4S^2A \right),
\]

which is a result that matches the known lower bound when \( T \geq \tilde{\Omega} (H^5S^3A) \).

6 Conclusion

We gave a new TS-like algorithm, SSR, for tabular MDP, which enjoys a near-optimal \( \tilde{O} \left( H\sqrt{SAT} \right) \) regret bound in the time-homogeneous model. Previously, near-optimal regret bounds can only be achieved by optimistic algorithms. Our result also highlights the importance of using a single random seed for the entire episode and using the variance information in tuning the magnitude of noise (cf. Bernstein’s inequality).

One important open problem is whether TS-like algorithm can achieve a horizon-free regret bound in the time-homogeneous model where the transition is the same at different levels [Zanette and Brunskill, 2019, Wang et al., 2020, Zhang et al., 2020a].

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| Symbol | Meaning |
|--------|---------|
| $S$ | The state space |
| $A$ | The action space |
| $S$ | Size of state space |
| $A$ | Size of action space |
| $H$ | The length of horizon |
| $K$ | The total number of episodes |
| $T$ | The total number of steps, $T = HK$ |
| $\pi^k$ | The greedy policy generated in Algorithm 1 at episode $k$ |
| $R_{h,s,a}$ | Expected reward function at $(h, s, a)$ |
| $P_{h,s,a}(s')$ | Transition probability |
| $M$ | Underlying true MDP, $M = (H, S, A, R, P, s_1)$ |
| $n_k(h, s, a)$ | $\sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h) = (s, a)\}r_{l, s_h, a_l}^k$ |
| $\hat{R}_{h,s,a}^k$ | Estimated reward function, $\frac{1}{n_k(h,s,a)+1}\sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h) = (s, a)\}r_{l, s_h, a_l}^k$ |
| $\hat{P}_{h,s,a}(s')$ | Estimated transition probability, $\frac{1}{n_k(h,s,a)+1}\sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h, s_{h+1}^l) = (s, a, s')\}$ |
| $\hat{R}_{h,s,a}(s')$ | Estimated transition probability with a slightly different denominator, $\frac{1}{\max\{n_k(h, s, a), 1\}}\sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h, s_{h+1}^l) = (s, a, s')\}$ |
| $\tilde{M}^k$ | Estimated MDP, $\tilde{M}^k = (H, S, A, \hat{P}, \hat{R}, s_1)$ |
| $\gamma_{ty}^k(h, s, a)$ | $\sigma_{ty}^k(h, s, a)\sqrt{\log(40k^4)}$ |
| $\tilde{z}_k$ | Perturbation’s single random source during episode $k$ from a standard Gaussian, $\tilde{z}_k \sim \mathcal{N}(0, 1)$ |
| $w_{ty}^k(h, s, a)$ | Noise of type “ty”, $w_{ty}^k(h, s, a) = \sigma_{ty}^k(h, s, a)\tilde{z}_k$ |
| $w_{ty}^k(h, s, a)$ | $-\gamma_{ty}^k(h, s, a)$ |
| $\hat{M}_{ty}$ | Perturbed estimated MDP with ty-type noise, $\hat{M}_{ty}^k = (H, S, A, \hat{P}, \hat{R} + w_{ty}^k, s_1)$ |
| $\tilde{M}_{ty}$ | Negatively perturbed MDP, $\tilde{M}_{ty}^k = (H, S, A, \hat{P}, \hat{R} + w_{ty}^k, s_1)$ |
| $\tilde{V}_{h,k}^*$ | Optimal value function for true MDP $M$ |
| $\tilde{V}_{h,k}^*$ | Value function by running policy $\pi^k$ on true MDP $M$ |
| $Q_{h,k}^*$ | $Q$-value function obtained by running policy $\pi^k$ on $\tilde{M}_{ty}^k$ |
| $V_{h,k}$ | Value function obtained by running policy $\pi^k$ on $\tilde{M}_{ty}^k$ with a clipping of threshold $2(H - h + 1)$ |
| $\tilde{V}_{h,k}^*$ | Value function obtained by running policy $\pi^k$ on $\tilde{M}_{ty}^k$ with a clipping of threshold $2(H - h + 1)$ |
| $R_{h,s,a}^k$ | $\tilde{R}_{h,s,a}^k - R_{h,s,a}$ |
| $P_{h,s,a}(s', \pi^k)$ | $\langle \hat{P}_{h,s,a} - P_{h,s,a}, V_{h+1}^* \rangle$ |
| $\mathcal{H}_h^k$ | The historical observations and actions till time $h$ in episode $k$, $\{(s^j_h, a^j_h, r^j_l) : j \leq k \text{ and } l \leq H \text{ if } j < k, \text{ else } l \leq h\}$ |
| $\mathcal{H}_h$ | The historical observations and actions till time $h$ and episode $k$, plus the randomness in episode $k$, $\mathcal{H}_h^k \cup \{\tilde{z}_k\}$ |
| $\nabla (P, V)$ | Variance of $V \in \mathbb{R}^S$ under distribution $P \in \Delta^S$, $\sum_{s \in S} P(s)(V(s) - \langle P, V \rangle)^2$ |
| $\alpha_k$ | $200H^2 \log(2HSAk^2) \log(40k^4)$ |
| $\sigma_{ty}^k(h, s, a)$ | Magnitude of perturbation. $\text{ty} \in \{\text{Ho, Be}\}$ |
| $\sigma_{ty}^k(h, s, a)$ | $\text{ty}$ Reserved subscript for denoting perturbation type, $\text{ty} \in \{\text{Ho, Be}\}$, where “Ho” denotes Hoeffding-type and “Be” denotes Bernstein-type |
| $\sigma_{\text{Ho}}^k(h, s, a) = H \sqrt{\frac{\log(2HSAk^2)}{n_k(h,s,a)+1} + \frac{H}{n_k(h,s,a)+1}}$ |
Lemma 1. Let \( \sigma^k_{be} (h, s, a) \) where the confidence widths are set as

\[
\sigma^k_{be} (h, s, a) = \sqrt{\frac{16V(\hat{P}^k_{h, s, a}, V_{h+1}) \log(2HSA_k^2)}{n_k(h, s, a)+1} + \frac{65H \log(2HSA_k^2)}{n_k(h, s, a)+1} + \sqrt{\frac{\log(2HSA_k^2)}{n_k(h, s, a)+1}}}
\]

\[
\sigma^k_{ho} (h, s, a) = \sqrt{\frac{6V(\hat{P}^k_{h, s, a}, V_{h+1}^*) \log(2HSA_k^2)}{n_k(h, s, a)+1} + \frac{9H \log(2HSA_k^2)}{n_k(h, s, a)+1} + \sqrt{\frac{\log(2HSA_k^2)}{n_k(h, s, a)+1}}}
\]

We also define two events \( E_{1}^k \) and \( E_{2}^k \) as the following:

\[
E_{1}^k = \left\{ |\hat{R}^k_{h, s, a} - R_{h, s, a}| \leq \sqrt{\frac{\log(2HSA_k^2)}{n_k(h, s, a)+1} + \frac{1}{n_k(h, s, a)+1}}, \forall (h, s, a) \right\},
\]

\[
E_{2}^k = \left\{ |\langle \hat{P}^k_{h, s, a} - P_{h, s, a}, V_{h+1}^* \rangle| \leq \sqrt{\frac{6V(\hat{P}^k_{h, s, a}, V_{h+1}^*) \log(2HSA_k^2)}{n_k(h, s, a)+1} + \frac{8H \log(2HSA_k^2)}{n_k(h, s, a)+1}}, \forall (h, s, a) \right\}.
\]

We have the following lemmas about concentration of events.

**Lemma 1.** For fixed \((k, h, s, a)\), let \( n = n_k(h, s, a) \). Then, if \( n \geq 1 \), for any fixed \( \delta > 0 \), we have

\[
P \left( |\hat{R}^k_{h, s, a} - R_{h, s, a}| \geq \frac{\log(2/\delta)}{n+1} + \frac{1}{n+1} \right) \leq \delta.
\]

**Proof:** Let \( \hat{R}^k_{h, s, a} = \frac{1}{n+1} \sum_{i=1}^{n} r(h_{(s,a)}, i) \), where \( r(h_{(s,a)}, i) \sim \mathcal{R}_{h, s, a} \) are i.i.d. samples. By definition of the MDP, we have \( E[r(h_{(s,a)}, i)] = R_{h, s, a} \). Then, notice that

\[
\hat{R}^k_{h, s, a} = \frac{1}{n+1} \sum_{i=1}^{n} r(h_{(s,a)}, i) = \frac{1}{n} \sum_{i=1}^{n} r(h_{(s,a)}, i) - \frac{1}{n} \sum_{i=1}^{n} r(h_{(s,a)}, i).
\]

Since the reward is assumed to be bounded in \([0, 1]\), we have \( \frac{1}{n+1} \sum_{i=1}^{n} r(h_{(s,a)}, i) \leq \frac{1}{n+1} \). Then, for fixed \( \delta > 0 \), we have

\[
P \left( |\hat{R}^k_{h, s, a} - R_{h, s, a}| \geq \frac{\log(2/\delta)}{n+1} + \frac{1}{n+1} \right)
\]
\[ = \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} r_{(h,s,a)} - R_{h,s,a} - \frac{1}{n(n+1)} \sum_{i=1}^{n} r_{(h,s,a)} \right) \geq \sqrt{\frac{\log(2/\delta)}{n+1} + \frac{1}{n+1}} \]
\[ \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} r_{(h,s,a)} - R_{h,s,a} \right) \leq \frac{1}{n+1} + \sqrt{\frac{\log(2/\delta)}{n+1} + \frac{1}{n+1}} \] (By triangle inequality)
\[ \leq \mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} r_{(h,s,a)} - R_{h,s,a} \right) \geq \sqrt{\frac{\log(2/\delta)}{2n}} \] (Since \( n+1 \leq 2n \) for \( n \geq 1 \))
\[ \leq \delta. \] (By standard Hoeffding’s inequality)

**Lemma 2.** For fixed \( (k,h,s,a) \), let \( n = n_k (h,s,a) \) and \( V \in \mathbb{R}^S \) be some non-negative value function such that \( \|V\|_\infty \leq H \). Then, if \( n \geq 1 \), for any fixed \( \delta > 0 \), we have

\[ \mathbb{P} \left( \left\langle \tilde{P}_{h,s,a}^k - P_{h,s,a}, V \right\rangle \geq H \sqrt{\frac{\log(2/\delta)}{n+1} + \frac{H}{n+1}} \right) \leq \delta, \] (27)

\[ \mathbb{P} \left( \left\langle \tilde{P}_{h,s,a}^k - P_{h,s,a}, V \right\rangle \leq \sqrt{\frac{6V \left( \tilde{P}_{h,s,a}^k, V \right) \log(2/\delta)}{n+1} + \frac{8H \log(2/\delta)}{n+1}} \right) \leq \delta. \] (28)

**Proof.** For fixed \( (h,s,a) \), we generate \( n \) i.i.d. samples of \( s_{(h,s,a),i} \sim P_{h,s,a} \) and consider \( V \left( s_{(h,s,a),i} \right) \). Then, by taking \( n_k (h,s,a) = n \), we have

\[ \left\langle \tilde{P}_{h,s,a}^k, V \right\rangle = \frac{1}{n} \sum_{i=1}^{n} V \left( s_{(h,s,a),i} \right) - \frac{1}{n(n+1)} \sum_{i=1}^{n} V \left( s_{(h,s,a),i} \right). \]

The first result in equation (27) can be proved very similarly as Lemma 1 using Hoeffding’s inequality by simply replacing the upper bound of 1 in reward by \( H \).

Then, for second result, we first consider \( n \geq 2 \). For some \( \delta > 0 \), define

\[ b_{(h,s,a),n} = \sqrt{\frac{2V \left( \tilde{P}_{h,s,a}^k, V \right) \log(2/\delta)}{n-1} + \frac{7H \log(2/\delta)}{3(n-1)} + \frac{H}{n+1}}. \]

By noticing that \( F(s) \leq H \) and applying similar technique in proof of Lemma 1, we have

\[ \mathbb{P} \left( \left\langle \tilde{P}_{h,s,a}^k - P_{h,s,a}, V \right\rangle \geq b_{(h,s,a),n} \right) \]
\[ \leq \mathbb{P} \left( \left\langle \tilde{P}_{h,s,a}^k - P_{h,s,a}, V \right\rangle \geq \sqrt{\frac{2V \left( \tilde{P}_{h,s,a}^k, V \right) \log(2/\delta)}{n-1} + \frac{7H \log(2/\delta)}{3(n-1)} + \frac{H}{n+1}} \right) \]
\[ \leq \delta. \] (By Lemma 31, the empirical Bernstein’s inequality)

Then, since \( 3(n-1) \geq n+1 \) when \( n \geq 2 \), we can easily check that

\[ b_{(h,s,a),n} \leq \sqrt{\frac{6V \left( \tilde{P}_{h,s,a}^k, V \right) \log(2/\delta)}{n+1} + \frac{8H \log(2/\delta)}{n+1}}. \]

Finally, since \( \|V\|_\infty \leq H \), when \( n = 1 \), we trivially have

\[ \left\langle \tilde{P}_{h,s,a}^k - P_{h,s,a}, V \right\rangle \leq H \leq \sqrt{\frac{6V \left( \tilde{P}_{h,s,a}^k, V \right) \log(2/\delta)}{n+1} + \frac{8H \log(2/\delta)}{n+1}}. \]
Therefore, we can conclude that
\[
\mathbb{P}\left( \left\| \hat{P}^k_{h,s,a} - P_{h,s,a}, V \right\| \geq \sqrt{\frac{6V(\hat{P}^k_{h,s,a}, V) \log (2/\delta)}{n + 1} + \frac{8H \log (2/\delta)}{n + 1}} \right) \leq \delta.
\]

**Lemma 3.** \( \sum_{k=1}^{\infty} \mathbb{P} \left((\mathcal{E}^1_k)^c\right) \leq \frac{\pi^2}{6} \).

**Proof.** Let \( n = n_k (h, s, a) \). Then, for some fixed \((h, s, a), n \geq 1 \) and \( \delta_n > 0 \), by Lemma 1, we have
\[
\mathbb{P}\left( |\hat{P}^k_{h,s,a} - R_{h,s,a}| \geq \sqrt{\frac{\log (2/\delta_n)}{n + 1} + \frac{1}{n + 1}} \right) \leq \delta_n.
\]

Therefore, by taking \( \delta_n = \frac{1}{HSA^2} \), a union bound will give us
\[
\sum_{n=1}^{\infty} \sum_{h,s,a} \mathbb{P}\left( |\hat{P}^k_{h,s,a} - R_{h,s,a}| \geq \sqrt{\frac{\log (2HSA^2)}{n + 1} + \frac{1}{n + 1}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

Therefore, we have
\[
\sum_{k=1}^{\infty} \mathbb{P}\left( \exists (h, s, a) : n_k (h, s, a) > 0, |\hat{P}^k_{h,s,a} - R_{h,s,a}| \geq \sqrt{\frac{\log (2HSA^2 n_k (h, s, a)^2)}{n_k (h, s, a) + 1} + \frac{1}{n_k (h, s, a) + 1}} \right) \leq \frac{\pi^2}{6}.
\]

Since the MDP is time-inhomogeneous, each \((h, s, a)\) can only be visited at most once during one episode, which implies \( n_k (h, s, a) \leq k \). Therefore, we have
\[
\sqrt{\frac{\log (2HSA^2 n_k (h, s, a)^2)}{n_k (h, s, a) + 1} + \frac{1}{n_k (h, s, a) + 1}} \leq \frac{\log (2HSA^2 k^2)}{n_k (h, s, a) + 1} + \frac{1}{n_k (h, s, a) + 1}
\]
and thus the proof is complete.

**Lemma 4.** \( \sum_{k=1}^{\infty} \mathbb{P} \left((\mathcal{E}^2_k)^c\right) \leq \frac{\pi^2}{6} \).

**Proof.** This proof will be very similar to proof of Lemma 3. In specific, for fixed \((h, s, a)\), let \( n = n_k (h, s, a) \geq 1 \). Then, for any \( \delta_n > 0 \), since \( \|V^*_{h+1}\|_\infty \leq H \), by Lemma 2, we have
\[
\mathbb{P}\left( \left\| \hat{P}^k_{h,s,a} - P_{h,s,a}, V^*_{h+1} \right\| \geq \sqrt{\frac{6V(\hat{P}^k_{h,s,a}, V^*_{h+1}) \log (2/\delta_n)}{n + 1} + \frac{8H \log (2/\delta_n)}{n + 1}} \right) \leq \delta_n.
\]

Therefore, by taking \( \delta_n = \frac{1}{HSA^2} \) and applying a similar union bound argument used in the proof of Lemma 3, we can conclude \( \sum_{k=1}^{\infty} \mathbb{P} \left((\mathcal{E}^2_k)^c\right) \leq \frac{\pi^2}{6} \).

We further define the event \( \mathcal{C}_{k} = \left\{ \hat{M}^k \in \mathcal{M}_{k}^{x} \right\} \). With what we have proved above, it will be straightforward to show the following results about \( \mathcal{C}_{k}^{c} \).

**Lemma 5.** \( \sum_{k=1}^{\infty} \mathbb{P} \left((\mathcal{C}_{k}^{c})^c\right) = \sum_{k=1}^{\infty} \mathbb{P} \left( \hat{M}^k \notin \mathcal{M}_{k}^{x} \right) \leq \frac{\pi^2}{3} \).

**Proof.** We can easily notice \( \mathcal{E}_{k}^1 \cap \mathcal{E}_{k}^2 \Rightarrow \hat{M}^k \in \mathcal{M}_{k}^{x} \), which implies \( \hat{M}^k \notin \mathcal{M}_{k}^{x} \Rightarrow (\mathcal{E}_{k}^1)^c \cup (\mathcal{E}_{k}^2)^c \). The first result then follows straightforwardly by applying Lemma 3 and Lemma 4.
Lemma 6. \( \sum_{k=1}^{\infty} \mathbb{P} \left( (C_{\text{ty}}^k)^c \right) = \sum_{k=1}^{\infty} \mathbb{P} \left( \hat{M}^k \notin M_{\text{ty}}^k \right) \leq \frac{2}{3}. \)

Proof. Similarly, for fixed \((h, s, a)\), we generate \(n\) i.i.d. samples \(s_{(h, s, a), i} \sim P_{h, s, a}\) and \(r_{(h, s, a), i} \sim \mathcal{R}_{h, s, a}\) for \(i = 1, \ldots, n\) respectively. Define \(Y_{(h, s, a), i} = r_{(h, s, a), i} + V_{h+1}^*(\delta_{(h, s, a), i})\) and we have \(\mathbb{E} \left[ Y_{(h, s, a), i} \right] = R_{h, s, a} + (P_{h, s, a}, V_{h+1}^*)\).

By definition of MDP, we know that \(Y_{(h, s, a), i} \leq H\). Thus, we can use an argument similar to the proof of Lemma 1. In specific, let \(n = n_k(h, s, a)\) and for \(\delta_n > 0\), we have
\[
\mathbb{P} \left( \left| \frac{1}{n+1} \sum_{i=1}^{n} Y_{(h, s, a), i} - \mathbb{E} \left[ Y_{(h, s, a), i} \right] \right| \geq H \sqrt{\frac{\log (2/\delta_n)}{n+1} + \frac{H}{n+1}} \right) \\
= \mathbb{P} \left( \left| \hat{R}_{h, s, a} - R_{h, s, a} \right| + \left| \hat{P}_{h, s, a} - P_{h, s, a}, V_{h+1}^* \right| \geq H \sqrt{\frac{\log (2/\delta_n)}{n+1} + \frac{H}{n+1}} \right) \\
\leq \delta_n.
\]

Then, we can take \(\delta_n = \frac{1}{n^2 \alpha n^2}\) and apply a similar union bound argument in used in the proof of Lemma 3. As a result, we can obtain
\[
\sum_{k=1}^{\infty} \mathbb{P} \left( \hat{M}^k \notin M_{\text{ty}}^k \right) \leq \frac{\pi^2}{6} \leq \frac{\pi^2}{3}.
\]

We can also have well-behaved bounds on magnitude of noise and estimated value functions.

**Definition 2.** We define \(w^k_{\text{ty}}(h, s, a) = \sigma^k_{\text{ty}}(h, s, a) \hat{z}_k\) and \(\gamma^k_{\text{ty}}(h, s, a) = \sigma^k_{\text{ty}}(h, s, a) \sqrt{\log(40k^4)}\). We define the event \(\mathcal{E}^w_k\) as
\[
\mathcal{E}^w_k = \left\{ \forall (h, s, a), |w^k_{\text{ty}}(h, s, a)| \leq \gamma^k_{\text{ty}}(h, s, a) \right\}.
\]

**Lemma 7.** \(\sum_{k=1}^{K} \mathbb{P} \left( (\mathcal{E}^w_k)^c \right) \leq \frac{2}{3}\) regardless the type of noise we choose.

Proof. For any \(k \in [K]\), by the tail bound of Gaussian distribution,
\[
\mathbb{P} \left( |\hat{z}_k| \geq \sqrt{\log(40k^4)} \right) \leq 2 \exp \left( -\frac{\log (40k^4)}{2} \right) \leq \frac{2}{k^2}.
\]

Summing over \(k \in [K]\),
\[
\sum_{k=1}^{K} \mathbb{P} \left( (\mathcal{E}^w_k)^c \right) = \sum_{k=1}^{K} \mathbb{P} \left( |\hat{z}_k| \geq \sqrt{\log(40k^4)} \right) \leq \sum_{k=1}^{K} \frac{2}{k^2} \leq \frac{\pi^2}{3}.
\]

Note that this result does not depend on the type of noise we choose.

Now, we define the following good events that hold with high probability and will be used throughout the whole proof.

**Definition 3** (Good events \(\mathcal{G}_k\)). Let \(\mathcal{G}_{k, \text{ty}} = \{C_{\text{ty}}^k \cap \mathcal{E}^w_k\}\).

The subscript “\(\text{ty}\)” will be ignored later since it is clear from the context.

**Definition 4.** We define events \(\mathcal{E}^\text{th}_{h,k}\) and \(\mathcal{E}^\text{cum}_{h,k}\) as
\[
\mathcal{E}^\text{th}_{h,k} = \left\{ n_k(h, s^k_h, a^k_h) \geq \alpha_k \right\}, \quad \mathcal{E}^\text{cum}_{h,k} = \bigcap_{i=1}^{h} \mathcal{E}^\text{th}_{i,k}.
\]

We will show that under events \(\mathcal{E}^w_k, \mathcal{E}^\text{th}_{h,k}\) and \(\hat{M}^k \in M_{\text{ty}}^k\), no clipping happens on \(s^k_h\).
Lemma 8. Assume that $\mathcal{E}_w^k$, $\mathcal{E}_h^k$, and $\hat{M}^k \in \mathcal{M}_{\mathcal{V}}^k$ hold. Then, regardless the type of noise we choose, it holds that

$$|\overline{Q}_{h,k}(s_h^k, a_h^k)| \leq 2(H - h + 1),$$

which immediately tells us that no clipping is triggered for any $(s_h^k, a_h^k)$.

Proof. We have that

$$\overline{Q}_{h,k}(s_h^k, a_h^k) = \hat{R}_{h,s_h^k,a_h^k}^k + \langle \hat{P}_{h,s_h^k,a_h^k}^k, \nabla_{h+1,k} \rangle + \sigma_{\nu}^k(h, s_h^k, a_h^k) \hat{z}_k.$$ 

As we have $|\nabla_{h+1,k}| \leq 2(H - h)$ by clipping and $\hat{R}_{h,s_h^k,a_h^k}^k \in [0, 1]$, we only need to show that $\sigma_{\nu}^k(h, s_h^k, a_h^k) \hat{z}_k \leq 1$. Under event $\mathcal{E}_{w}^k$, we have $|\sigma_{\nu}^k(h, s_h^k, a_h^k) \hat{z}_k|$ is bounded by $\gamma_{\nu}^k(h, s_h^k, a_h^k) = \sigma_{\nu}^k(h, s_h^k, a_h^k) \sqrt{\log(40k^4)}$.

Note that we have $\nabla_{h+1,k}(s) \in [-2H, 2H]$ by clipping for any $s \in S$. Thus, by Lemma 32, we have

$$\forall \langle \hat{P}_{h,s,a}^k, \nabla_{h+1,k} \rangle \leq 4H^2$$ for any $(h, s, a)$.

By the choice of $\alpha_k = 200H^2 \log(2HSAk^2) \log(40k^4)$ and referring to the definitions of $\sigma_{\nu}^k(h, s, a)$ in Equation (6), we can check that

$$\gamma_{\nu}^k(h, s_h^k, a_h^k) = \sigma_{\nu}^k(h, s_h^k, a_h^k) \sqrt{\log(40k^4)}$$

$$= \left( \sqrt{\frac{16\langle \hat{P}_{h,s_h^k,a_h^k}^k, \nabla_{h+1,k} \rangle}{n_k(h, s_h^k, a_h^k) + 1}} + \frac{65H \log(2HSAk^2)}{n_k(h, s_h^k, a_h^k) + 1} + \sqrt{\frac{\log(2HSAk^2)}{n_k(h, s_h^k, a_h^k) + 1}} \right) \sqrt{\log(40k^4)}$$

$$\leq \left( \sqrt{\frac{64H^2 \log(2HSAk^2)}{\alpha_k}} + \frac{65H \log(2HSAk^2)}{\alpha_k} + \frac{\log(2HSAk^2)}{\alpha_k} \right) \sqrt{\log(40k^4)}$$

$$(\text{Event } \mathcal{E}_{w}^k \text{ implies } n_k(h, s_h^k, a_h^k) \geq \alpha_k)$$

$$\leq \sqrt{\frac{64}{200}} + \frac{65}{200H} + \sqrt{\frac{1}{200H^2}}$$

$$\leq 1.$$

Thus, we have $\gamma_{\nu}^k(h, s, a) \leq 1$ and we can similarly check that $\gamma_{\nu}^k(h, s, a) \leq 1$. As a result, we have

$$|\overline{Q}_{h,k}(s_h^k, a_h^k)| \leq 2(H - h + 1),$$

which completes the proof.

C Optimism

Let $\mathcal{H}_h^k$ denote the history trajectory, which is defined as

$$\mathcal{H}_h^k = \{ (s_j^l, a_j^l, r_j^l) : j \leq k \text{ and } l \leq H \text{ if } j < k, \text{ else } l \leq h \}.$$ 

(30)

We will prove that for both types of noise, $\nabla_{h,k}$ is optimistic with constant probability under certain conditions.

C.1 Hoeffding-type Noise

Lemma 9. Condition on history $\mathcal{H}_h^{k-1}$, if $\mathcal{G}_{k,\nu}^k$ holds and Hoeffding-based noise is applied, then $\nabla_{h,k}$ is optimistic with constant probability for any $h \in [H]$. Specifically, we have

$$\mathbb{P}(\nabla_{h,k}(s) \geq V_h^*(s), \forall h \in [H], s \in S | \mathcal{H}_h^{k-1}, \mathcal{G}_{k,\nu}^k) \geq \Phi(1.9) - \Phi(1) := C_{\nu}.$$
Proof. We will show that if \( \hat{z}_k \geq 1 \), then for all \( h \in [H] \) and \( s \in S \), we have \( V_{h,k}(s) \geq V_h^*(s) \). The proof will use induction and the argument is true for \( h = H + 1 \) as \( V_{H+1,k}(s) = V_{H+1}^*(s) = 0 \). Suppose the argument is true for timestep \( h + 1 \) and for timestep \( h \) we have

\[
V_{h,k}(s) = \text{clip}_{2(H-h+1)} \left( \max_{a \in A} \overline{Q}_{h,k}(s, a) \right)
\]

\[
\geq \min \left\{ 2(H-h+1), \max_{a \in A} \overline{Q}_{h,k}(s, a) \right\}
\]

\[
\geq \min \left\{ (H-h+1), \overline{Q}_{h,k}(s, \pi_h^*(s)) \right\}
\]

\[
\geq \min \left\{ (H-h+1), \overline{R}_{h,s,\pi_h^*(s)} + \left( \hat{P}^k_{h,s,\pi_h^*(s)} + \sigma^k_{iy}(h, s, \pi_h^*(s)) \hat{z} \right) \right\}
\]

\[
\geq \min \left\{ (H-h+1), \overline{R}_{h,s,\pi_h^*(s)} + \left( \hat{P}^k_{h,s,\pi_h^*(s)} + \sigma^k_{iy}(h, s, \pi_h^*(s)) \hat{z} \right) \right\} \quad \text{(Inductive hypothesis)}
\]

\[
\geq \min \left\{ (H-h+1), Q_{h}^*(s), \pi_h^*(s) \right\}
\]

\[
\geq V_h^*(s).
\]

Then by induction we have that the optimism is achieved for all \( h \in [H] \) and \( s \in S \) simultaneously. Meanwhile, as stated in Definition 2, we have \( \hat{z}_k \leq \sqrt{\log (40k^4)} \) under event \( E^w_k \) and numerically, \( \sqrt{\log (40k^4)} \geq 1.9 \).

Therefore, the probability that \( \hat{z}_k \geq 1 \) under \( E^w_k \), inferred by \( G_{k,H} \), is at least

\[
P(\hat{z}_k \geq 1 \mid H_{H}^{k-1}, G_{k,H} = 0) = \Phi(1.9) - \Phi(1) \quad \text{since} \quad \Phi(1.9) - \Phi(1) \geq C_{H_0}.
\]

Thus, we can conclude that

\[
P(V_{h,k}(s) \geq V_h^*(s), \forall h \in [H], s \in S \mid H_{H}^{k-1}, G_{k,H_0}) \geq C_{H_0}.
\]

\[
\square
\]

C.2 Bernstein-type Noise

The following proof of optimism applies some techniques used in Zhang et al. [2020a]. We first present a technical lemma.

**Lemma 10.** Let \( f : \Delta^S \times \mathbb{R}_+^S \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) with \( f(p, v, n, L) = \frac{n}{n+1} \langle p, v \rangle + \max \left\{ 4 \sqrt{\frac{\langle p, v \rangle L}{n+1}}, \frac{64HL}{n+1} \right\} \cdot z \) for some constant \( H > 0 \). Then, \( f \) satisfies

(i) \( f(p, v, n, L) \) is non-decreasing in \( v(s) \) for all \( p \in \Delta^S \), \( \|v\|_{\infty} \leq 2H \), \( L > 0 \), \( n \geq 3 \) and \( z \in [-1.5, 1.5] \)

(ii) \( f(p, v, n, L) \geq \frac{n}{n+1} \langle p, v \rangle + \left( 3 \sqrt{\frac{\langle p, v \rangle L}{n+1}} + \frac{8HL}{n+1} \right) \cdot z \) for \( z \in [1, 1.5] \).

(iii) \( f(p, v, n, L) \leq \frac{n}{n+1} \langle p, v \rangle + \left( 3 \sqrt{\frac{\langle p, v \rangle L}{n+1}} + \frac{8HL}{n+1} \right) \cdot z \) for \( z \in [-1.5, 1] \).

**Proof.** It is obvious that \( f(p, v, n, L) \) is continuous in \( v(s) \) and not differentiable at only one point where \( 4 \sqrt{\frac{\langle p, v \rangle L}{n+1}} = \frac{64HL}{n+1} \). Therefore, to prove statement (i), we only need to show that \( \frac{\partial f(p, v, n, L)}{\partial v(s)} \geq 0 \). Specifically, we have

\[
\frac{\partial f(p, v, n, L)}{\partial v(s)} = \frac{n}{n+1} \cdot p(s) + 1 \left\{ 4 \sqrt{\frac{\langle p, v \rangle L}{n+1}} \geq \frac{64HL}{n+1} \right\} \frac{4p(s) (v(s) - \langle p, v \rangle) L}{\sqrt{(n+1) \langle p, v \rangle L}} \cdot z
\]
Here, the inequality (a) above holds because when the condition inside indicator \( \mathbb{1} \{ \cdot \} \) holds, we will have 
\[
\sqrt{(n+1)\mathcal{V}(p,v)L} \leq 16HL.
\]

The inequality (b) holds because \( ||v||_\infty \leq 2H \) and \( v \) is non-negative, which means to have 
\[
\frac{\sqrt{\mathcal{V}(p,v)L}}{4H} \geq -\frac{1}{2}.
\]

The last inequality holds because we have \( n \geq 3 \) and \( z \leq 1.5 \). Therefore, \( f(p,v,n,L) \) is non-decreasing in \( v(s) \).

For the statement (ii), we consider two cases. First, when 
\[
4\sqrt{\frac{\mathcal{V}(p,v)L}{n+1}} \geq \frac{54HL}{n+1}
\]
holds, we have 
\[
\frac{8HL}{n+1} \leq \frac{7z}{2} \sqrt{\frac{\mathcal{V}(p,v)L}{n+1}} \leq f(p,v,n,L).
\]

When 
\[
4\sqrt{\frac{\mathcal{V}(p,v)L}{n+1}} \leq \frac{54HL}{n+1}
\]
holds, we have 
\[
\frac{3}{7} \leq \frac{54HL}{n+1},
\]
which similarly leads to
\[
\frac{n}{n+1} \langle p,v \rangle + \left( 3 \frac{\sqrt{\mathcal{V}(p,v)L}}{n+1} + \frac{8HL}{n+1} \right) \cdot z \leq f(p,v,n,L).
\]

The state (iii) can be shown similarly and thus the proof is complete.

**Lemma 11.** Condition on history \( \mathcal{H}_{H}^{k-1} \), if \( G_{k,\beta} \) holds and Bernstein-based noise is applied, then \( \mathcal{V}_{H,k} \) is optimistic with constant probability for any \( h \in [H] \). Specifically, we have
\[
\mathbb{P} \left( \mathcal{V}_{h,k} (s) \geq V^*_h (s), \forall h \in [H], s \in \mathcal{S} \mid \mathcal{H}_{H}^{k-1}, G_{k,\beta} \right) \geq \Phi (1.5) - \Phi (1) := C_{\beta e}
\]

*Proof.* Similar to what we have discussed in the proof of Lemma 9, under event \( \mathcal{E}^w_k \), we have \( \hat{z}_k \in [1,1.5] \) with probability at least \( \Phi (1.5) - \Phi (1) = C_{\beta e} \). Then, we will show that \( \mathcal{Q}_{h,k}(s,a) \geq Q^*_h (s,a) \) for any \( h \) with arbitrary \( s,a \) and \( \hat{z}_k \in [1,1.5] \). The proof will use induction. For simplicity, let \( L = \log (2HSAk^2) \).

For \( h = H+1 \), the inequality holds trivially because both sides are 0. Then, by assuming \( \mathcal{Q}_{h+1,k}(s,a) \geq Q^*_h (s,a) \) for any \( (s,a) \) such that \( n_k(h,s,a) \geq 3 \), we have
\[
\mathcal{Q}_{h,k}(s,a) = \hat{P}^k_{h,s,a} + \langle \hat{P}^k_{h,s,a}, \mathcal{V}_{h+1,k} \rangle + \sigma^k_{\beta e} (h,s,a) \hat{z}_k
\]
\[
\geq R_{h,s,a} + \langle \hat{P}^k_{h,s,a}, \mathcal{V}_{h+1,k} \rangle + \left( 4 \sqrt{\frac{\mathcal{V}(\hat{P}^k_{h,s,a}, \mathcal{V}_{h+1,k})}{n_k(h,s,a) + 1}} + \frac{64HL}{n_k(h,s,a) + 1} \right) \cdot \hat{z}_k
\]

(Replace \( R_{h,s,a} \) by \( R_{h,s,a} \) through applying event \( \mathcal{E}^l_k \) defined in (25))
\[
\geq R_{h,s,a} + \langle \hat{P}^k_{h,s,a}, \mathcal{V}_{h+1,k} \rangle + \max \left\{ 4 \sqrt{\frac{\mathcal{V}(\hat{P}^k_{h,s,a}, \mathcal{V}_{h+1,k})}{n_k(h,s,a) + 1}} \cdot \hat{z}_k, \frac{64HL}{n_k(h,s,a) + 1} \cdot \hat{z}_k \right\}
\]
\[
\geq R_{h,s,a} + \langle \hat{P}^k_{h,s,a}, \mathcal{V}^*_h \rangle + \max \left\{ 4 \sqrt{\frac{\mathcal{V}(\hat{P}^k_{h,s,a}, \mathcal{V}^*_h)}{n_k(h,s,a) + 1}} \cdot \hat{z}_k, \frac{64HL}{n_k(h,s,a) + 1} \cdot \hat{z}_k \right\}
\]
\[
\geq R_{h,s,a} + \langle \hat{P}^k_{h,s,a}, \mathcal{V}^*_h \rangle + \left( 3 \sqrt{\frac{\mathcal{V}(\hat{P}^k_{h,s,a}, \mathcal{V}^*_h)}{n_k(h,s,a) + 1}} + \frac{8HL}{n_k(h,s,a) + 1} \right) \cdot \hat{z}_k
\]

(By applying statement (ii) of Lemma 10)
\[ \geq R_{h,s,a} + \langle \hat{P}_{h,s,a}^k, V_{h+1}^* \rangle + \frac{6\mathcal{V}(\hat{P}_{h,s,a}^k, V_{h+1}^*)}{n_k(h,s,a) + 1} + \frac{8HL}{n_k(h,s,a) + 1} \] (Since \( \hat{z}_k \geq 1 \))

\[ \geq R_{h,s,a} + \langle P_{h,s,a}, V_{h+1}^* \rangle \]

\[ = Q_h^*(s,a). \]

Here, the above inequality (a) holds by applying inductive hypothesis and statement (i) in Lemma 10. It is applicable because when \( E_k^\alpha \) holds, and by the clipping function, \( \| \nabla_{h+1,k} \|_\infty \leq 2H \). When \( n_k(h,s,a) < 3 \), \( \overline{Q}_{h,k}(s,a) \geq Q_h^*(s,a) \) holds trivially because \( Q_h^*(s,a) \leq H \) by definition. Therefore, the induction is complete.

Now, for arbitrary \((h,s)\), set \( a = \arg\max_{a \in A} Q_{h,k}(s,a) \) and we have

\[ \nabla_{h,k}(s) = \text{clip}_{2(H-h+1)}(\max_{a \in A} Q_{h,k}(s,a)) \]

\[ \geq \min \left\{ 2(H-h+1), \max_{a \in A} Q_{h,k}(s,a) \right\} \]

\[ \geq \min \left\{ (H-h+1), Q_{h,k}(s, \pi_h^*(s)) \right\} \]

\[ \geq \min \left\{ (H-h+1), Q_{h,k}^*(s, \pi_h^*(s)) \right\} \]

\[ \geq V_{h,k}^*(s) \]

\[ \square \]

## D Pessimism

Similar to what we have proved in Section C, in this section we will prove that for both types of noise, \( \nabla_{h,k} \) is pessimistic with constant probability under certain conditions.

### D.1 Hoeffding-type Noise

**Lemma 12.** Condition on history \( \mathcal{H}_H^{k-1} \), if \( \mathcal{G}_{k,\text{Ho}} \) holds and Hoeffding-based noise is applied, then \( \nabla_{h,k} \) is optimistic with constant probability for any \( h \in [H] \). Specifically, we have

\[ \mathbb{P}(\nabla_{h,k}(s) \leq V_h^*(s), \forall h \in [H], s \in \mathcal{S} | \mathcal{H}_H^{k-1}, \mathcal{G}_{k,\text{Ho}}) \geq \Phi(1.9) - \Phi(1) := C_{\text{Ho}}. \]

**Proof.** We will show that if \( \hat{z}_k \leq -1 \), then for all \( h \in [H] \) and \( s \in \mathcal{S} \), we have \( \nabla_{h,k}(s) \leq V_h^*(s) \). The proof will use induction and the argument is true for \( h = H + 1 \) as \( \nabla_{H+1,k}(s) = V_{H+1}^*(s) = 0 \). Suppose the argument is true for timestep \( h+1 \) and we consider timestep \( h \). Set \( a = \arg\max_{a \in A} Q_{h,k}(s,a) \).

\[ \nabla_{h,k}(s) = \text{clip}_{2(H-h+1)}(Q_{h,k}(s,a)) \]

\[ \leq \max \left\{ -(H-h+1), \overline{Q}_{h,k}(s,a) \right\} \]

\[ \leq \max \left\{ -(H-h+1), \hat{R}_{h,s,a}^k + \langle \hat{P}_{h,s,a}^k, \nabla_{h+1,k} \rangle + \sigma_{1y}(h,s,a) \hat{z}_k \right\} \]

\[ \leq \max \left\{ -(H-h+1), \hat{R}_{h,s,a}^k + \langle \hat{P}_{h,s,a}^k, V_{h+1,k}^* \rangle + \sigma_{1y}(h,s,a) \hat{z}_k \right\} \]

\[ \leq \max \left\{ -(H-h+1), R_{h,s,a}^k + \langle P_{h,s,a}^k, V_{h+1,k}^* \rangle \right\} \]

\[ \leq \max \left\{ -(H-h+1), \hat{P}_{h,s,a}^k, V_{h+1,k}^\ast \right\} \]

\[ \leq \max \left\{ -(H-h+1), \hat{M}_h^k \right\} \]

\[ \leq \max \left\{ -(H-h+1), \hat{M}_h^k \right\} \]

\[ \leq \max \left\{ -(H-h+1), \max_{a \in A} Q_h^*(s,a) \right\} \]

\[ \leq V_h^*(s). \]
Then by induction we have that the optimism is achieved for all $h \in [H]$ and $s \in S$ simultaneously. By using an argument similar to the proof of Lemma 9, we can see that when $\hat{z}_k \leq -1$, we have $\nabla h,k (s) \leq V^*_h (s)$ on this hold simultaneously for any $h \in [H], s \in S$. Further as stated in Definition 2, we have $|\hat{z}_k| \leq \sqrt{\log (40k^4)}$ under event $E^w_k$ and numerically, $\sqrt{\log (40k^4)} \geq 1.9$. Therefore, the probability that $\hat{z}_k \leq -1$ under $E^w_k$ is at least

\[
P(\hat{z}_k \leq -1 | \mathcal{H}^{h-1}_H, \mathcal{G}_{k,H_0}) = \frac{\Phi (1.9) - \Phi (1)}{\Phi (1.9) - \Phi (-1.9)} \geq \Phi (1.9) - \Phi (1) = C_{H_0}.
\]

Thus, we can conclude that

\[
P(\nabla h,k (s) \leq V^*_h (s), \forall h \in [H], s \in S | \mathcal{H}^{k-1}_H, \mathcal{G}_{k,H_0}) \geq C_{H_0}.
\]

\[\square\]

**D.2 Bernstein-type Noise**

**Lemma 13.** Condition on history $\mathcal{H}^{h-1}_H$, if $\mathcal{G}_{k,H_0}$ holds and Bernstein-based noise is applied, then $\nabla h,k$ is pessimistic with constant probability for any $h \in [H]$. Specifically, we have

\[
P(\nabla h,k (s) \leq V^*_h (s), \forall h \in [H], s \in S | \mathcal{H}^{k-1}_H, \mathcal{G}_{k,B_e}) \geq C_{B_e}
\]

**Proof.** Similar to what we have discussed in the proof of Lemma 12, under event $E^w_k$, we have $\hat{z}_k \in [-1.5, -1]$ with probability at least $\Phi (1.5) - \Phi (1) = C_{B_e}$. Then, we will show that $\nabla h,k(s,a) \leq Q^*_h(s,a)$ for any $h$ with arbitrary $s,a$ and $\hat{z}_k \in [-1.5, -1]$. The proof will go by induction. For simplicity, let $L = \log (2HS \kappa^2)$.

For $h = H+1$, the inequality holds trivially because both sides are 0. Then, by assuming $\nabla h,k+1(s,a) \leq Q^*_h(s,a)$ for any $(s,a)$ such that $n_k(h,s,a) \geq 3$, we have

\[
\nabla h,k (s,a) = \tilde{R}_{h,s,a}^k + \left( \tilde{\nabla} h,k^k(s,a, \nabla h+1,k) + \sigma_{B_e}^k(h,s,a) \hat{z}_k \right)
\]

\[
\leq R_{h,s,a} + \left( \tilde{\nabla} h,k^k(s,a, \nabla h+1,k) - 4 \left( V \left( \frac{\tilde{\nabla} h,k^k(s,a, \nabla h+1,k) L}{n_k(h,s,a) + 1} + \frac{64HL}{n_k(h,s,a) + 1} \right) \right) \right)
\]

(Replace $\hat{R}_{h,s,a}$ by $R_{h,s,a}$ through applying event $E^k$ defined in (25))

\[
\leq R_{h,s,a} + \left( \tilde{\nabla} h,k^k(s,a, \nabla h+1,k) + \max \left( \frac{V}{n_k(h,s,a) + 1} \left( \frac{64HL}{n_k(h,s,a) + 1} \right) \right) \right)
\]

\[
\leq R_{h,s,a} + \left( \tilde{\nabla} h,k^k(s,a, \nabla h+1,k) - 3 \left( V \left( \frac{\tilde{\nabla} h,k^k(s,a, \nabla h+1,k) L}{n_k(h,s,a) + 1} + \frac{64HL}{n_k(h,s,a) + 1} \right) \right) \right)
\]

(By applying statement (iii) of Lemma 10)

\[
\leq R_{h,s,a} + \left( \tilde{\nabla} h,k^k(s,a, \nabla h+1,k) - \frac{6V}{n_k(h,s,a) + 1} - \frac{8HL}{n_k(h,s,a) + 1} \right)
\]

\[
\leq R_{h,s,a} + \left( \tilde{\nabla} h,k^k(s,a, \nabla h+1,k) - Q^*_h(s,a) \right).
\]

Here, the above inequality (a) holds by applying inductive hypothesis and statement (i) in Lemma 10. It is applicable because when $E^w_k$ holds, by the clipping function, $\frac{\nabla h+1,k}{n_k(h,s,a) + 1} \leq 2H$. When $n_k(h,s,a) < 3$, $\nabla h,k(s,a) \leq Q^*_h(s,a)$ holds trivially because $0 \leq Q^*_h(s,a) \leq H$ by definition. Therefore, the induction is complete.

Now, for arbitrary $(k,h,s)$, set $a = \text{argmax}_{a \in A} \nabla h,k(s,a)$ and we have

\[
\nabla h,k(s) = \text{clip}_{2(H-h+1)} \left( \nabla h,k(s,a) \right)
\]

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\[
\begin{align*}
&\leq \max \left\{ -2(H-h+1), \overline{Q}_{h,k}(s,a) \right\} \\
&\leq \max \left\{ -(H-h+1), \overline{Q}_{h,k}(s,a) \right\} \\
&\leq \max \left\{ -(H-h+1), Q^*_h(s,a) \right\} \\
&\leq \max \left\{ -(H-h+1), Q^*_h(s,\pi^*_h(s)) \right\} \\
&\leq V^*_h(s). 
\end{align*}
\]

### E Regret Decomposition

In this section, we prove the multiple lemmas necessary for bounding the regret. The regret is mainly composed of two terms, the pessimism term and the estimation error term. The pessimism term, \(V^*_h(s_k^\pi) - \overline{V}_{1,k}(s_k^\pi)\), measures how much regret is due to the value the algorithm uses, \(\overline{V}_{1,k}(s_k^\pi)\), is smaller than the true value, \(V^*_h(s_k^\pi)\). The estimation error term, \(\overline{V}_{1,k}(s_k^\pi) - \overline{V}_{1,k}^w(s_k^\pi)\) measure how much regret is due to the value, \(\overline{V}_{1,k}\), does not estimate \(V^*_h(s_k^\pi)\), the true value of the policy \(\pi^k\) accurately.

We first introduce a few definitions key to this section. In this section, we omit \(k\) if it is clear from the context. Let \(a^k_h = \pi^k(s_k^h)\) unless specified otherwise.

**Definition 5.** Let \(\mathcal{P}_{h,s,a}^k = \left\{ \hat{\mathcal{P}}_{h,s,a}^k - P_{h,s,a}, V^*_h(s_{k+1}) \right\} \) and \(\mathcal{R}_{h,s,a}^k = \hat{\mathcal{R}}_{h,s,a}^k - R_{h,s,a}\).

**Definition 6 (\(M^k_{ty}\) and \(\overline{V}_{h,k}\)).** Given history \(\mathcal{H}^k_{H}^{-1}\) (defined in equation (30)), \(\hat{\mathcal{P}}^k\) and \(\hat{\mathcal{R}}^k\), we define \(\overline{w}^k_{ty}(h,s,a) = -\gamma^k_{ty}(h,s,a)\) and \(\overline{V}_{h,k}\) be the value function obtained by running policy \(\pi^k\) on the MDP \(M^k_{ty} = (H, \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}^k, \hat{\mathcal{R}}^k + \overline{w}^k_{ty}, s_k^\pi)\) plus a magnitude clipping with threshold \(2(H-h+1)\).

**Definition 7 (\(\overline{M}^k_{ty}\) and \(\overline{V}_{h,k}\)).** Given history \(\mathcal{H}^k_{H}^{-1}\) (defined in equation (30)), \(\hat{\mathcal{P}}^k\) and \(\hat{\mathcal{R}}^k\), we define \(\overline{w}^k_{ty}(h,s,a) = \gamma^k_{ty}(h,s,a)\) and \(\overline{V}_{h,k}\) be the value function obtained by running policy \(\pi^k\) on the MDP \(\overline{M}^k_{ty} = (H, \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}^k, \hat{\mathcal{R}}^k + \overline{w}^k_{ty}, s_k^\pi)\) plus a magnitude clipping with threshold \(2(H-h+1)\).

Similar to Lemma 8, we can also show that under good event \(\mathcal{G}_k\) and \(\mathcal{E}^{th}_{h,k}\), no clipping happens on \(s_k^h\) for \(\overline{V}_{h,k}(s_k^h)\) and \(\overline{V}_{h,k}(s_k^h)\).

**Lemma 14.** Under the good event \(\mathcal{G}_k\), we have \(\overline{V}_{h,k}(s) \leq \overline{V}_{h,k}(s) \leq \overline{V}_{h,k}(s)\) for all \(h \in [H], s \in \mathcal{S}\).

**Proof.** This is an immediate result by noticing that under good event \(\mathcal{G}_k\), we have \(\overline{w}^k_{ty}(h,s,a) \leq w^k_{ty}(h,s,a) \leq \overline{w}^k_{ty}(h,s,a)\) for all \(h \in [H]\) and \(s \in \mathcal{S}\).

**Definition 8.** Define \(\delta^\pi(s_h), \delta^\pi(s_h), \delta^\pi_h(s_h), \delta^\pi_h(s_h), \delta^\pi_h(s_h)\) and \(\delta^\pi_h(s_h)\) as

\[
\delta^\pi(s_h) = V_h(s_h) - V^*_h(s_h), \\
\delta^\pi_h(s_h) = \overline{V}_h(s_h) - V^*_h(s_h), \\
\delta^\pi_h(s_h) = \overline{V}(s_h) - V^*_h(s_h), \\
\delta^\pi_h(s_h) = V^*_h(s_h) - V^*_h(s_h), \\
\delta^\pi_h(s_h) = \overline{V}_h(s_h) - V^*_h(s_h), \\
\delta^\pi_h(s_h) = \overline{V}(s_h) - V^*_h(s_h).
\]

**Definition 9.** We denote the history trajectory \(\overline{H}^k_h = \mathcal{H}^k_h \cup \{\hat{z}_k\}\). With filtration sets \(\{\overline{H}^k_h\}_{h,k}\), we define the following sequences:

\[
\mathcal{M}_{\delta_h(s_h)}(s_h) = 1\{G_k \cap E^{cum}_{h,k}\} \mathcal{E}^{h+1}_{h,k}(s_h(s_h) - \delta_h(s_h+1))(s_h+1),
\]
where \( \delta \in \{ \delta^x, \delta^y, \delta^z, \delta^\pi, \delta^\delta, \delta \} \). We will show the sequences are martingales in Lemma 22.

Finally, the regret can be decomposed as

\[
\text{Regret} (M, K, \text{SSR}_{ty}) = \sum_{k=1}^{K} \left( V_T^* (s_1^k) - V_{T,k}^\pi (s_1^k) \right)
\]

\[
= \sum_{k=1}^{K} \mathbb{I} \left( C_{ty}^k \right) \left( V_T^* (s_1^k) - V_{T,k}^\pi (s_1^k) \right) + \sum_{k=1}^{K} \mathbb{I} \left( (C_{ty}^k)^c \right) \left( V_T^* (s_1^k) - V_{T,k}^\pi (s_1^k) \right)
\]

\[
= \sum_{k=1}^{K} \mathbb{I} \left( C_{ty}^k \right) \left( V_T^* (s_1^k) - \overline{\pi}_{1,k}(s_1^k) \right) + \sum_{k=1}^{K} \mathbb{I} \left( (C_{ty}^k)^c \right) \left( V_T^* (s_1^k) - \overline{\pi}_{1,k}(s_1^k) \right)
\]

By Lemma 5 and 6, we know that

\[
\mathbb{E} \left[ \sum_{k=1}^{K} \mathbb{I} \left( (C_{ty}^k)^c \right) \right] = \sum_{k=1}^{K} \mathbb{P} \left( (C_{ty}^k)^c \right) \leq \sum_{k=1}^{\infty} \mathbb{P} \left( (C_{ty}^k)^c \right) \leq \frac{\pi^2}{3}.
\]

Therefore, by standard Hoeffding’s inequality, it holds with probability at least \( 1 - \delta \) that

\[
\sum_{k=1}^{K} \mathbb{I} \left( (C_{ty}^k)^c \right) \leq \frac{\pi^2}{3} + \sqrt{\frac{\log(1/\delta)}{2K}}.
\]

Since the value functions of true MDP is bounded in \([0, H]\), with probability at least \( 1 - \delta \), we have

\[
(a) \leq H \sum_{k=1}^{K} \mathbb{I} \left( (C_{ty}^k)^c \right) \leq \frac{\pi^2 H}{3} + H \sqrt{\frac{\log(1/\delta)}{2K}} = \tilde{O}(H).
\]

Further, notice that the good event \( \mathcal{G}_k = C_{ty}^k \cap E_k^w \) and by Lemma 7, we have \( \sum_{k=1}^{\infty} \mathbb{P} \left( (E_k^w)^c \right) \leq \frac{\pi^2}{4} \).

Therefore, we can similarly address the regret incurred by \( (E_k^w)^c \) as the bound for term \( a \). As a result, it will be sufficient to only consider \( \mathbb{I} \{ \mathcal{G}_k \} \left( V_{T,k}^* (s_1^k) - V_{T,k}^\pi (s_1^k) \right) \) when bounding pessimism and estimation error terms. That is, with probability at least \( 1 - \delta \), it holds that

\[
\text{Regret} (M, K, \text{SSR}_{ty}) \leq \sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \} \left( \left| \overline{\pi}_{1,k}(s_1^k) \right| + \left| \overline{\pi}_{1,k}^\pi (s_1^k) \right| \right) + \tilde{O}(H). \tag{31}
\]

Then, we decompose the estimation error term in Section E.2. We decompose the pessimism term in Section E.1. We combine the decomposition of the pessimism term and the estimation error term in Section E.3.

### E.1 Pessimism Term

**Lemma 15.** Let \( C_1 = \max \left\{ \frac{1}{\sqrt{1-h^2-\delta_{1,h}(s_1^k)}}, \frac{1}{\sqrt{1-h^2-\delta_{1,h}(s_1^k)}} \right\} = \frac{1}{\sqrt{(1-h^2-\delta_{1,h}(s_1^k))}} \approx 10.9. \) Then, for any \( h, k, s_1^k \) and the type of noise we used, under the good event \( \mathcal{G}_k \), the following bound holds,

\[
\mathbb{I} \{ \mathcal{G}_k \} \left| \delta_{h,k}(s_1^k) \right| \leq \mathbb{I} \{ \mathcal{G}_k \} C_1 \left( \left| \delta_{h,k}^* (s_1^k) \right| + \left| \delta_{h,k}^\pi (s_1^k) \right| \right). \tag{32}
\]
Proof. Let \( \mathcal{O}_k \) be the event that \( V_{h,k}(s) \geq V_h^*(s) \) simultaneously for all \( s \in S \) and \( h \in [H] \). By Lemma 9 and 11, we know that \( \mathbb{P}(\mathcal{O}_k | H_{H}^{k-1}, G_k) \geq \min \{ \Phi(1.9) - \Phi(1), \Phi(1.5) - \Phi(1) \} = \Phi(1.5) - \Phi(1) \), which means \( \frac{1}{\mathbb{P}(\mathcal{O}_k)} \leq C_1 \) regardless the type of noise used.

The definition of \( \mathcal{O}_k \) implies \( V_h^* \leq \mathbb{E}[V_{h,k} | \mathcal{O}_k, H_{H}^{k-1}, G_k] \). Meanwhile, notice that

\[
1 \{ G_k \} (E[V_{h,k} | H_{H}^{k-1}, G_k] - V_{h,k}) = 1 \{ G_k \} \mathbb{P}(O_k | H_{H}^{k-1}, G_k) (E[V_{h,k} | O_k, H_{H}^{k-1}, G_k] - V_{h,k}) \\
+ 1 \{ G_k \} \mathbb{P}(O_k | H_{H}^{k-1}, G_k) (E[V_{h,k} | O_k, H_{H}^{k-1}, G_k] - V_{h,k})
\]

\[ \geq 1 \{ G_k \} \mathbb{P}(O_k | H_{H}^{k-1}, G_k) (E[V_{h,k} | O_k, H_{H}^{k-1}, G_k] - V_{h,k}) \]

\[ \Rightarrow 1 \{ G_k \} (E[V_{h,k} | O_k, H_{H}^{k-1}, G_k] - V_{h,k}) \leq 1 \{ G_k \} C_1 (E[V_{h,k} | H_{H}^{k-1}, G_k] - V_{h,k}). \] (33)

We can similarly use constant probability pessimism shown in Lemma 12 and 13. In particular, let \( N_k \) be the event that \( \overline{V}_{h,k}(s) \leq V_h^*(s) \) for all \( s \in S \) and \( h \in [H] \). Then, we have

\[
1 \{ G_k \} \left( E[V_{h,k} | H_{H}^{k-1}, G_k] - \overline{V}_{h,k} \right) = 1 \{ G_k \} \mathbb{P}(N_k | H_{H}^{k-1}, G_k) \left( E[V_{h,k} | N_k, H_{H}^{k-1}, G_k] - \overline{V}_{h,k} \right) \\
+ 1 \{ G_k \} \mathbb{P}(N_k | H_{H}^{k-1}, G_k) \left( E[V_{h,k} | N_k, H_{H}^{k-1}, G_k] - \overline{V}_{h,k} \right)
\]

\[ \leq 1 \{ G_k \} \mathbb{P}(N_k | H_{H}^{k-1}, G_k) \left( E[V_{h,k} | N_k, H_{H}^{k-1}, G_k] - \overline{V}_{h,k} \right) \]

\[ \Rightarrow 1 \{ G_k \} \left( E[V_{h,k} | N_k, H_{H}^{k-1}, G_k] - \overline{V}_{h,k} \right) \geq 1 \{ G_k \} C_1 \left( E[V_{h,k} | H_{H}^{k-1}, G_k] - \overline{V}_{h,k} \right). \]

Thus, we have

\[
1 \{ G_k \} \left( V_h^*(s_h^k) - \overline{V}_{h,k}(s_h^k) \right) \geq 1 \{ G_k \} \left( E[V_{h,k} | H_{H}^{k-1}, G_k] (s_h^k) - \overline{V}_{h,k}(s_h^k) \right) \\
\leq 1 \{ G_k \} C_1 \left( E[V_{h,k} | H_{H}^{k-1}, G_k] (s_h^k) - V_{h,k}(s_h^k) \right). \] (34)

Since good event \( G_k \) implies \( V_{h,k} \leq \overline{V}_{h,k} \leq \overline{V}_{h,k} \) by Lemma 14, the RHS of (33) is non-negative and the RHS of (34) is non-positive. Therefore, we can then conclude

\[
1 \{ G_k \} |V_h^*(s_h^k) - \overline{V}_{h,k}(s_h^k)| \\
\leq 1 \{ G_k \} C_1 \left( E[V_{h,k} | H_{H}^{k-1}, G_k] (s_h^k) - V_{h,k}(s_h^k) \right) \\
+ 1 \{ G_k \} C_1 \left( E[V_{h,k} | H_{H}^{k-1}, G_k] (s_h^k) - \overline{V}_{h,k}(s_h^k) \right)
\]

\[ \leq 1 \{ G_k \} C_1 \left( |\overline{V}_{h,k}(s_h^k)| + |\overline{V}_{h,k}(s_h^k)| \right).
\]

\[ \Box \]

### E.2 Estimation Error Term

We first bound the estimation error of \( \overline{V} \), which can be regarded as the optimistic estimate used in UCB-type algorithms. For convenience, we will ignore notation \( \mathbb{1} \{ G_k \} \) in this section since all statements are proved under the good event \( G_k \).
Lemma 16. With probability at least $1 - \delta$, for all $(k, h, s^k_{h+1})$, under the good event $\mathcal{G}_k$ it holds that

\[
\mathbb{I}\left\{ \mathcal{E}^{\text{cum}}_{h,k} \right\} \mathbb{E}_{h,k}(s^k_{h+1}) \leq \mathbb{I}\left\{ \mathcal{E}^{\text{cum}}_{h,k} \right\} \left( \left| \mathbb{P}^{h,k}_{s,h,k} \mathbb{E}(h, s^k_{h+1}, a^k_{h+1}) + \mathcal{M}_{\mathbb{E}(s^k_{h+1})} + \frac{2SH^2L}{n_k(h, s^k_{h}, a^k_{h})} \right| + \mathbb{I}\left\{ \mathcal{E}^{\text{cum}}_{h+1,k} \right\} \right)
\]

where $L = \log(2HS^2AK/\delta)$.

Proof. Since both $\overline{V}$ and $V_{\pi^*}$ are obtained by choosing actions based on policy $\pi^k$ under event $\mathcal{G}_k$, we have

\[
\mathbb{I}\left\{ \mathcal{E}^{\text{cum}}_{h,k} \right\} \mathbb{E}_{h,k}(s^k_{h+1}) = \mathbb{I}\left\{ \mathcal{E}^{\text{cum}}_{h,k} \right\} \left| \overline{V}_{h,k}(s^k_{h+1}) - V_{\pi^*}(s^k_{h+1}) \right|
\]

For the last term, we use Lemma 33 and then for $L = \log(2HS^2AK/\delta)$, with probability at least $1 - \delta$, we have

\[
\mathbb{E}_{h,k}(s^k_{h+1}) \leq \sum_{s_{h+1} \in S} \left| \mathbb{P}^{h,k}_{s,h,k}(s_{h+1}) - P_{h,s^k_{h+1}}(s_{h+1}) \right| \mathbb{E}_{h+1,k}(s_{h+1}) - V_{\pi^*}(s_{h+1})
\]

\[
\leq \sum_{s_{h+1} \in S} \left( \frac{L}{n_k(h, s^k_{h}, a^k_{h})} + \frac{4L}{3n_k(h, s^k_{h}, a^k_{h})} \right) \mathbb{E}_{h+1,k}(s_{h+1})
\]

\[
= \sum_{s_{h+1} : P_{h,s^k_{h+1}}(s_{h+1}) \geq 4LH^2} 2P_{h,s^k_{h+1}}(s_{h+1}) \mathbb{E}_{h+1,k}(s_{h+1}) + \sum_{s_{h+1} : P_{h,s^k_{h+1}}(s_{h+1}) < 4LH^2} \frac{2L}{n_k(h, s^k_{h}, a^k_{h})} \mathbb{E}_{h+1,k}(s_{h+1})
\]

\[
\leq \frac{1}{H} \mathbb{E}_{h+1,k}(s^k_{h+1}) + \mathcal{M}_{\mathbb{E}(s^k_{h+1})} + \frac{2SH^2L}{n_k(h, s^k_{h}, a^k_{h})}
\]

31
\begin{align*}
\leq & \frac{1}{H} \left| \widetilde{\delta}_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| + M_{\pi_{h+1}}(s_{h+1}^k) + \frac{2SH^2L}{n_k(h, s_{h+1}^k, a_{h+1}^k)} \\
\leq & \frac{1 + C_1}{H} \left| \widetilde{\delta}_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + C_1 \frac{1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| + M_{\pi_{h+1}}(s_{h+1}^k) + \frac{2SH^2L}{n_k(h, s_{h+1}^k, a_{h+1}^k)} \\
& \quad \text{(By triangle inequality)}
\end{align*}

Combining the above two arguments, we can prove the argument:

\begin{align*}
1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left| \widetilde{\delta}_{h,k}^{\pi}(s_h^k) \right| & \leq 1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left| \underbrace{\left| \left| \tau_{h,k}^{n,k}_{h,s_h^k, a_h^k} + \tau_{h,k}^{n,k}_{h,s_h^k, a_h^k} + \Theta_{h,k}^{n,k}(h, s_h^k, a_h^k) \right| + M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) + \frac{2SH^2L}{n_k(h, s_{h+1}^k, a_{h+1}^k)} }_{\mathcal{E}_{h,k}^{\text{cum}}} \right| \\
& \quad + 1 \{ \mathcal{E}_{h+1,k}^{\text{cum}} \} \left( \frac{H + 1 + C_1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right) \\
& \quad + 1 \{ \mathcal{E}_{h+1,k}^{\text{cum}} \} \left( \frac{H + 1 + C_1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right).
\end{align*}

Then, the proof is complete by noticing that \( \mathcal{E}_{h+1,k}^{\text{cum}} = \mathcal{E}_{h+1,k}^{\text{th}} \cap \mathcal{E}_{h,k}^{\text{th}} \).

**Lemma 17.** With probability at least \( 1 - \delta \), for all \( (k, h, s_h^k) \), under good event \( \mathcal{G}_k \) it holds that

\begin{align*}
1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left| \widetilde{\delta}_{h,k}^{\pi}(s_h^k) \right| & \leq 1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left| \underbrace{\left| \left| \tau_{h,k}^{n,k}_{h,s_h^k, a_h^k} + \tau_{h,k}^{n,k}_{h,s_h^k, a_h^k} + \Theta_{h,k}^{n,k}(h, s_h^k, a_h^k) \right| + M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) + \frac{2SH^2L}{n_k(h, s_{h+1}^k, a_{h+1}^k)} }_{\mathcal{E}_{h,k}^{\text{cum}}} \right| \\
& \quad + 1 \{ \mathcal{E}_{h+1,k}^{\text{cum}} \} \left( \frac{C_1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{H + 1 + C_1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right) \\
& \quad + 1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left( \frac{C_1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{H + 1 + C_1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right). 
\end{align*}

**Proof.** The proof exactly follows the proof of Lemma 16.

**Lemma 18.** With probability at least \( 1 - \delta \), for all \( (k, h, s_h^k) \), under good event \( \mathcal{G}_k \) it holds that

\begin{align*}
1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left| \widetilde{\delta}_{h,k}^{\pi}(s_h^k) \right| & \leq 1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left| \underbrace{\left| \left| \tau_{h,k}^{n,k}_{h,s_h^k, a_h^k} + \tau_{h,k}^{n,k}_{h,s_h^k, a_h^k} + \Theta_{h,k}^{n,k}(h, s_h^k, a_h^k) \right| + M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) + \frac{2SH^2L}{n_k(h, s_{h+1}^k, a_{h+1}^k)} }_{\mathcal{E}_{h,k}^{\text{cum}}} \right| \\
& \quad + 1 \{ \mathcal{E}_{h+1,k}^{\text{cum}} \} \left( \frac{C_1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{H + 1 + C_1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right) \\
& \quad + 1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left( \frac{C_1}{H} \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \frac{H + 1 + C_1}{H} \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right). 
\end{align*}

**Proof.** The proof exactly follows the proof of Lemma 16.

**Lemma 19.** With probability at least \( 1 - \delta \), for all \( (k, i, s_i^k) \), under good event \( \mathcal{G}_k \) it holds that

\begin{align*}
1 \{ \mathcal{E}_{i,k}^{\text{cum}} \} \left| \left| \underbrace{\left| \widetilde{\delta}_{i,k}^{\pi}(s_i^k) \right| + \widetilde{\delta}_{i,k}(s_i^k)}_{\mathcal{E}_{i,k}^{\text{cum}}} \right| \right| \\
\leq & 3e^{C_1} \left( \sum_{h=1}^{H} \sqrt{\frac{\gamma_{h}^{k}(h, s_h^k, a_h^k)}{n_k(h, s_h^k, a_h^k)}} + \sum_{h=1}^{H} \frac{SH^2L}{n_k(h, s_h^k, a_h^k)} \right) + \sum_{h=1}^{H} \left( 1 + \frac{3C_1}{H} \right)^{h-1} 1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} M_{h,k} \\
& + e^{3C_1} \sum_{h=1}^{H} \left( \mathcal{E}_{h+1,k}^{\text{th}} \right) \left[ \left| \widetilde{\delta}_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \left| \widetilde{\delta}_{h+1,k}(s_{h+1}^k) \right| + \left| \delta_{h+1,k}^{\pi}(s_{h+1}^k) \right| + \left| \delta_{h+1,k}(s_{h+1}^k) \right| \right],
\end{align*}

where \( M_{h,k} = M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) + M_{\pi_{h,k}}(s_h^k) \).

**Proof.** By summing results in Lemma 16, Lemma 17 and Lemma 18, we have

\begin{align*}
1 \{ \mathcal{E}_{h,k}^{\text{cum}} \} \left( \left| \widetilde{\delta}_{h,k}^{\pi}(s_h^k) \right| + \left| \delta_{h,k}^{\pi}(s_h^k) \right| + \left| \delta_{h,k}(s_h^k) \right| \right)
\end{align*}

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\begin{align*}
&\leq \mathbb{I} \{ \mathcal{E}_{h+1,k} \} \left( 1 + \frac{3C_1}{H} \right) \left( |\mathcal{P}_{h+1,k}(s^k_{h+1})| + |\mathcal{R}_{h+1,k}(s^k_{h+1})| + |\mathcal{N}_{h+1,k}(s^k_{h+1})| \right) + 3 \left| \mathcal{P}_{h,s^k,h} + \mathcal{R}_{h,s^k,h} \right|
&+ \left| w_{h}^k(h, s^k_{h}, a^k_{h}) \right| + \left| w_{h}^k(h, s^k_{h}, a^k_{h}) \right| + \left| w_{h}^k(h, s^k_{h}, a^k_{h}) \right| + \frac{6SH^2L}{n_k(h, s^k_{h}, a^k_{h})} + \mathbb{I} \{ \mathcal{E}_{h,k} \} \mathcal{M}_{h,k}
&+ \mathbb{I} \{ \mathcal{E}_{h,k} \cap (\mathcal{E}_{th,h,k}^c) \} \left( 1 + \frac{3C_1}{H} \right) \left( |\mathcal{P}_{h+1,k}(s^k_{h+1})| + |\mathcal{R}_{h+1,k}(s^k_{h+1})| + |\mathcal{N}_{h+1,k}(s^k_{h+1})| \right) \\
&\leq \mathbb{I} \{ \mathcal{E}_{h+1,k} \} \left( 1 + \frac{3C_1}{H} \right) \left( |\mathcal{P}_{h+1,k}(s^k_{h+1})| + |\mathcal{R}_{h+1,k}(s^k_{h+1})| + |\mathcal{N}_{h+1,k}(s^k_{h+1})| \right)
&+ 3\sqrt{e_{th}^k(h, s^k_{h}, a^k_{h})} + \gamma_{h}^k(h, s^k_{h}, a^k_{h}) + \frac{6SH^2L}{n_k(h, s^k_{h}, a^k_{h})} + \mathbb{I} \{ \mathcal{E}_{h,k} \} \mathcal{M}_{h,k}
&+ \mathbb{I} \{ (\mathcal{E}_{th,h,k}^c) \} \left( 1 + \frac{3C_1}{H} \right) \left( |\mathcal{P}_{h+1,k}(s^k_{h+1})| + |\mathcal{R}_{h+1,k}(s^k_{h+1})| + |\mathcal{N}_{h+1,k}(s^k_{h+1})| \right).
\end{align*}

Here, the inequality (i) above holds because of two reasons. Firstly, under event \( \mathcal{G}_k \), we have \( |w_{h}^k(h, s^k_{h}, a^k_{h})| \leq |w_{h}^k(h, s^k_{h}, a^k_{h})| \) and
\[
|\mathcal{P}_{h,s^k,h} + \mathcal{R}_{h,s^k,h}| = \left| \hat{P}_{h,s^k,h} - P_{h,s^k,h}, V_{h+1}^* \right| + \left( \hat{R}_{h,s^k,h} - R_{h,s^k,h} \right) | \quad \text{(By Definition 5)}
\]
\[
\leq \sqrt{e_{th}^k(h, s^k_{h}, a^k_{h})}.
\]
\((\text{Under event } \mathcal{G}_k, \hat{M} \in \mathcal{M}_{h,k}^c)\)

Then, the proof is complete by using this recursion from \( h = i \) to \( h = H \) and utilizing the fact that \( 1 + \frac{3C_1}{H} \geq e^{3C_1} \).

\section*{E.3 Combining Estimation and Pessimism Terms}

\textbf{Lemma 20.} With probability at least \( 1 - \delta \), it holds that
\[
\text{Regret}(M, K, \text{SSR}_{ly}) \leq \mathbb{I} \{ \mathcal{G}_k \} 3C_1 e^{3C_1} K \sum_{h=1}^{H} \left( \sqrt{e_{th}^k(h, s^k_{h}, a^k_{h})} + \gamma_{h}^k(h, s^k_{h}, a^k_{h}) \right) + \tilde{O}(H) + \tilde{O}(S^2A + H\sqrt{T}).
\]

\textbf{Proof.} Recall in equation (31), with probability at least \( 1 - \delta \), we have
\[
\text{Regret}(M, K, \text{SSR}_{ly})
\]
\[
\leq \sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \} \left( |\mathcal{P}_{1,k}(s^k_{1})| + |\mathcal{R}_{1,k}(s^k_{1})| \right) + \tilde{O}(H)
\]
\[
\leq \sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \} C_1 \left( |\mathcal{P}_{1,k}(s^k_{1})| + |\mathcal{R}_{1,k}(s^k_{1})| + |\mathcal{N}_{1,k}(s^k_{1})| \right) + \tilde{O}(H) \quad \text{(By Lemma 15)}
\]
\[
= \sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \cap \mathcal{E}_{h,k}^c \} C_1 \left( |\mathcal{P}_{1,k}(s^k_{1})| + |\mathcal{R}_{1,k}(s^k_{1})| + |\mathcal{N}_{1,k}(s^k_{1})| \right) + \tilde{O}(H)
\]
\[
+ \sum_{k=1}^{K} \mathbb{I} \{ \mathcal{E}_{h,k} \} \left( |\mathcal{P}_{1,k}(s^k_{1})| + |\mathcal{R}_{1,k}(s^k_{1})| + |\mathcal{N}_{1,k}(s^k_{1})| \right) \quad \text{(By Definition 5)}
\]
\[
\leq \sum_{k=1}^{K} \mathbb{I} \{ \mathcal{G}_k \} 3C_1 e^{3C_1} \left( \sum_{h=1}^{H} \sqrt{e_{h}^k(h, s^k_{h}, a^k_{h})} + \sum_{h=1}^{H} \gamma_{h}^k(h, s^k_{h}, a^k_{h}) + \sum_{h=1}^{H} \frac{SH^2L}{n_k(h, s^k_{h}, a^k_{h})} \right)
\]
\[
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \left( 1 + \frac{3C_1}{H} \right) \mathbb{I} \{ \mathcal{G}_k \cap \mathcal{E}_{h,k}^c \} \mathcal{M}_{h,k} + \tilde{O}(H)
\]
\[
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{I} \{ (\mathcal{E}_{h,k}^c) \} \left( |\mathcal{P}_{h,k}(s^k_{h})| + |\mathcal{R}_{h,k}(s^k_{h})| + |\mathcal{N}_{h,k}(s^k_{h})| \right) \quad \text{(By using Lemma 19)}
\]
\( (i) \sum_{k=1}^{K} \sum_{h=1}^{H} \left( e_{\tau_{h}}^{k}(h, s_{h}^{k}, a_{h}^{k}) + \gamma_{\tau_{h}}^{k}(h, s_{h}^{k}, a_{h}^{k}) \right) + O\left(H^{3}S^{2}A + H\sqrt{T}\right) \\
+ \tilde{O}(H) \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{1}\left\{ (\xi_{h,k}^{th})_{c} \right\} \\
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{1}\left\{ (\xi_{h,k}^{th})_{c} \right\} \leq O\left(H^{3}SA\right). \\
\)

**Lemma 21** (Lemma 20 in Agrawal et al. [2021]).

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{1}\left\{ (\xi_{h,k}^{th})_{c} \right\} \leq O\left(H^{3}SA\right).
\]

**Proof.** It holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{1}\left\{ (\xi_{h,k}^{th})_{c} \right\} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{1}\left\{ n_{h}(h, s_{h}^{k}, a_{h}^{k}) \leq \alpha_{k} \right\} \\
\leq \sum_{s \in S} \sum_{a \in A} \sum_{h=1}^{H} \alpha_{k} \\
\leq 200H^{3}SA \log(2HK^{2}A) \log(4K^{4}) \\
= O\left(H^{3}SA\right).
\]

**F.1 Bounds on Individual Terms**

**Lemma 22.** For \( i \in [H] \), the sequences starting from 0 and with difference between two consecutive terms given by \( \mathbb{1}\{\mathcal{G}_{k}\} \mathcal{M}_{h,k} \) for \( h = i, \ldots, H \), \( k = 1, \ldots, K \) are martingales with respect to filtration \( \{\mathcal{H}_{h}\}_{k=1, \ldots, K} \).

Moreover, for any \( \delta' > 0 \), with probability at least \( 1 - \delta' \), for any \( i \in [H] \), the following hold,

\[
\left| \sum_{k=1}^{K} \sum_{h=1}^{H} \left( 1 + \frac{3C_{1}}{H} \right)^{h} \mathbb{1}\{\mathcal{G}_{k} \cap \xi_{h,k}^{cum}\} \mathcal{M}_{h,k} \right| = O\left(H\sqrt{T}\right).
\]

**Proof.** We first show the sequence starting from 0 and with difference between two consecutive terms given by \( \mathbb{1}\{\mathcal{G}_{k} \cap \xi_{h,k}^{cum}\} \left( 1 + \frac{3C_{1}}{H} \right)^{h} \mathcal{M}_{\mathcal{G}_{h,k}}(s_{h}^{k}) \) is a martingale sequence. For any \( h \in \{i, \ldots, H\} \) and \( k \in [K] \),

\[
\mathbb{E}\left[ \mathbb{1}\{\mathcal{G}_{k} \cap \xi_{h,k}^{cum}\} \mathcal{M}_{\mathcal{G}_{h,k}(s_{h}^{k})} \bigg| \mathcal{H}_{h} \right] = \mathbb{E}\left[ \mathbb{1}\{\mathcal{G}_{k} \cap \xi_{h,k}^{cum}\} \left( \langle P_{h,s_{h}^{k},a_{h}^{k}} \mathcal{M}_{\mathcal{G}_{h+1,k}(s_{h+1}^{k})} \rangle - \mathcal{M}_{\mathcal{G}_{h+1,k}(s_{h+1}^{k})} \right) \bigg| \mathcal{H}_{h} \right] = 0.
\]

Similarly, we have \( \mathbb{1}\{\mathcal{G}_{h,k} \cap \xi_{h,k}^{cum}\} \mathcal{M}_{\mathcal{G}_{h,k}(s_{h}^{k})}, \mathbb{1}\{\mathcal{G}_{h,k} \cap \xi_{h,k}^{cum}\} \mathcal{M}_{\mathcal{G}_{h,k}(s_{h}^{k})}, \mathbb{1}\{\mathcal{G}_{h,k} \cap \xi_{h,k}^{cum}\} \mathcal{M}_{\mathcal{G}_{h,k}(s_{h}^{k})} \) are martingale difference sequences. As \( \mathbb{1}\{\mathcal{G}_{h,k} \cap \xi_{h,k}^{cum}\} \mathcal{M}_{h,k} \) is the sum of several martingale difference sequences, it is a martingale difference sequence.
Next, we bound \(| \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{\mathcal{P}_{h,k}^k(s_k^+)} \)|. When \( h = H \), \(| \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{\mathcal{P}_{h,k}^k(s_k^+)} \)| is 0. When \( G_k \) holds, for \( h < H \) and any state \( x \),
\[
\left| \mathcal{P}_{h+1,k}^k(x) \right| = |V_{h+1}(x) - V_{h+1}^\pi(x)| = \left| \langle P_{h+2,x,\pi(x)}, V_{h+2}^\pi \rangle \right| + w_{ty}^k(h + 1, x, \pi(x)) \]
By our choice of \( \alpha_k \), when \( G_k \) holds, \(|w_{ty}^k(h, s, a)| \leq \alpha_k^k(h, s, a) \leq 1 \) for all \( h, s, a \) as shown in Lemma 8. Then, by expanding \( \mathcal{P}_{h+1,k}^k(x) \) recursively from \( h + 1 \) to \( H \), we have
\[
\left| \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{\mathcal{P}_{h,k}^k(s_k^+)} \right| \leq 2H_{ty}^k(h, s, a) \leq 2H.
\]

Similarly, we have the bound on \(| \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{\mathcal{P}_{h,k}^k(s_k^+)} \)|, \( \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{\mathcal{P}_{h,k}^k(s_k^+)} \), \( \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{\mathcal{P}_{h,k}^k(s_k^+)} \), \( \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} \) is bounded by \( 2e^{3C_1}H \). By Azuma-Hoeffding inequality, with probability at least \( 1 - \delta' \), we have
\[
\sum_{k=1}^K \sum_{h=1}^H \left( 1 + \frac{3C_1}{H} \right)^h \left| \{G_k \cap \mathcal{E}_{h,k}^{\text{cum}} \} M_{h,k} \right| \lesssim O \left( \sum_{k=1}^K \sum_{h=1}^H H^2 \right) = O \left( H \sqrt{T} \right).
\]

**F.2 Bounds on Lower-order Terms**

The following two lemmas are standard results in literature and we present their proofs here for completeness.

**Lemma 23.**
\[
\sum_{k=1}^K \sum_{h=1}^H \sqrt{\frac{\log (2HSAk^2)}{n_k(h, s_h^k, a_h^k) + 1}} \lesssim O \left( \sqrt{HSA} \right).
\]

**Proof.** Let \( L = \log (2HSAk^2) \). Then, it can be bounded as
\[
\sum_{k=1}^K \sum_{h=1}^H \sqrt{\frac{\log (2HSAk^2)}{n_k(h, s_h^k, a_h^k) + 1}} \leq \sqrt{L} \sum_{k=1}^K \sum_{h=1}^H \sqrt{\frac{1}{n_k(h, s_h^k, a_h^k) + 1}}
\]
\[
= \sqrt{L} \sum_{h,s,a} \sqrt{n_k(h,s,a)} \sum_{k=1}^K \frac{1}{n_k(h,s,a) + 1}
\]
\[
\leq \sqrt{L} \sum_{h,s,a} \sqrt{n_k(h,s,a)} \frac{1}{\sqrt{n_k(h,s,a)}}
\]
\[
\leq 2\sqrt{L} \sum_{h,s,a} \sqrt{n_k(h,s,a)}
\]
\[
\leq 2\sqrt{L} \cdot \sqrt{HSA} \sum_{h,s,a} n_k(h,s,a) \quad \text{(By Cauchy-Schwartz inequality)}
\]
\[
= O \left( \sqrt{HSA} \right). \quad \text{(Since } \sum_{h,s,a} n_k(h,s,a) = T) \]

**Lemma 24.**
\[
\sum_{k=1}^K \sum_{h=1}^H \frac{\log (2HSAk^2)}{n_k(h, s_h^k, a_h^k) + 1} \lesssim O (HSA).
\]

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Proof. Let \( L = \log (2HSAK^2) \). Then, it can be bounded as

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{\log (2HSAK^2)}{n_k(h, s_h^k, a_h^k) + 1} \leq L \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{n_k(h, s_h^k, a_h^k) + 1}
\]

\[
= L \sum_{h,s,a} \sum_{n=1}^{N} \frac{1}{n+1}
\]

\[
\leq L \sum_{h,s,a} \log (n_K(h, s, a)) \quad \text{(Since } \sum_{n=1}^{N} \frac{1}{n} \leq \log(N) + 1\text{)}
\]

\[
\leq LHS \cdot \max_{h,s,a} \log(n_K(h, s, a))
\]

\[
\leq LHS \log(T)
\]

\[
= \tilde{O} \left( HSA \right).
\]

\[\square\]

G Bounds on Sum of Variance

When we use the Bernstein-type noise, the regret analysis needs to bound the sum of variance. This proof applies some techniques developed in Azar et al. [2017]. However, since our optimism only holds with constant probability instead of deterministically, the details are quite different. For simplicity, we first define

\[
\tilde{\psi}_{h+1,k} = \mathbb{V} \left( \bar{P}_{h,s_h^k,a_h^k}^*, V_{h+1}^* \right), \quad \psi_{h+1,k} = \mathbb{V} \left( P_{h,s_h^k,a_h^k}, V_{h+1} \right),
\]

\[
\psi_{h+1,k} = \mathbb{V} \left( \bar{P}_{h,s_h^k,a_h^k}^*, \psi_{h+1,k} \right), \quad \psi_{h+1,k} = \mathbb{V} \left( P_{h,s_h^k,a_h^k}, \psi_{h+1,k} \right),
\]

\[
U_{h,k,1} = \sqrt{\frac{\psi_{h+1,k} \log (2HSAK^2)}{n_k(h, s_h^k, a_h^k) + 1}}, \quad U_{h,k,2} = \sqrt{\frac{\tilde{\psi}_{h+1,k} \log (2HSAK^2)}{n_k(h, s_h^k, a_h^k) + 1}}.
\]

We will first give a full proof of the bound on sum of variance and then present all the auxiliary lemmas in Section G.1.

**Lemma 25.** Let \( U_{h,k} = U_{h,k,1} + U_{h,k,2} \). For any \( \delta > 0 \), with probability at least \( 1 - \delta \), when \( T \geq \Omega \left( H^5 S^2 A \right) \), it holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \left\{ G_k \right\} U_{h,k} \leq \tilde{O} \left( H \sqrt{SAT} \right).
\]

**Proof.** First, we have

\[
\sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \left\{ G_k \right\} U_{h,k} \leq \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \left\{ G_k \right\} \sqrt{\frac{\log (2HSAK^2)}{n_k(h, s_h^k, a_h^k) + 1} \left( \sqrt{\psi_{h+1,k}^*} + \sqrt{\psi_{h+1,k}} \right)}
\]

\[
\leq \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \left\{ G_k \right\} \sqrt{\frac{\log (2HSAK^2)}{n_k(h, s_h^k, a_h^k) + 1} \cdot \sqrt{2} \sqrt{\psi_{h+1,k}^*} + \psi_{h+1,k}}
\]

\[
\text{(Since } \sqrt{a} + \sqrt{b} \leq \sqrt{2(a + b)} \text{ for } a, b \geq 0\text{)}
\]

\[
\leq \sqrt{2} \left( \sum_{k=1}^{K} \sum_{h=1}^{H-1} \frac{\log (2HSAK^2)}{n_k(h, s_h^k, a_h^k) + 1} \right) \left( \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \left\{ G_k \right\} \left( \sqrt{\psi_{h+1,k}^*} + \psi_{h+1,k} \right) \right)
\]

\[
\text{(By Cauchy-Schwartz inequality)}
\]

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\[ \leq \sqrt{O(HSA)} \left( \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \hat{\psi}_{h+1,k} + \hat{\psi}_{h+1,k} \right) \right), \]  

(35)

where the last inequality above applies Lemma 24.

We will then bound the two sums of variance separately. Specifically, by applying Lemma 26 and Lemma 28, we have with probability at least \(1 - \delta/3\),

\[ \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \hat{\psi}_{h+1,k} = \frac{3}{2} \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \psi_{h+1,k}^* + \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \psi_{h+1,k}^* - \frac{3}{2} \psi_{h+1,k}^* \right) \]

\[ \leq \tilde{O} \left( HT + H^2 \sqrt{T} + H^3 + H^3 S^2 A + H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \delta_{h+1,k}^* \right). \]  

(36)

By similarly applying Lemma 26 and Lemma 29, we have with probability at least \(1 - \delta/3\),

\[ \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \hat{\psi}_{h+1,k} = \frac{3}{2} \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \psi_{h+1,k}^* + \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \psi_{h+1,k}^* - \frac{3}{2} \psi_{h+1,k}^* \right) \]

\[ \leq \tilde{O} \left( HT + H^2 \sqrt{T} + H^3 + H^3 S^2 A + H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \delta_{h+1,k} + |\delta_{h+1,k}(s_{h+1})| \right) \right). \]  

(37)

By combining equations (36) and (37), we have

\[ \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \hat{\psi}_{h+1,k} + \hat{\psi}_{h+1,k} \right) \]

\[ \leq \tilde{O} \left( HT + H^2 \sqrt{T} + H^3 S^2 A + H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \delta_{h+1,k}(s_{h+1}) + |\delta_{h+1,k}(s_{h+1})| \right) \right). \]  

(38)

Then, by referring to definitions of \(e_{Be}^k(h,s_h^k,a_h^k)\) and \(\gamma_{Be}^k(h,s_h^k,a_h^k)\), with probability at least \(1 - \delta/3\), we have

\[ \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \delta_{h+1,k}(s_{h+1}) + |\delta_{h+1,k}(s_{h+1})| \right) \]

\[ \leq \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( |\delta_{h+1,k}(s_{h+1})| + |\delta_{h+1,k}(s_{h+1})| \right) \]

\[ \leq \mathbb{1}\{g_k\} 3C_1 e^{3C_1} H \sum_{k=1}^{K} \sum_{h=1}^{H} \left( \sqrt{\frac{e_{Be}^k(h,s_h^k,a_h^k) + \gamma_{Be}^k(h,s_h^k,a_h^k)}{2}} + \tilde{O} \left( H^3 S^2 A + H^2 \sqrt{T} \right) \right) \]

(By referring to the proof of Lemma 20)

\[ \leq \tilde{O} \left( H^5 S^2 A + H^2 \sqrt{T} + \sqrt{H^3 S A T} + H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} U_{h,k} \right) \]  

(39)

(39)

By plugging equation (39) into equation (38), we can have

\[ \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1}\{g_k\} \left( \hat{\psi}_{h+1,k} + \hat{\psi}_{h+1,k} \right) \]
\[
\leq O \left( HT + H^2 \sqrt{T} + H^3 S^2 A + H^6 S^2 A + H^3 \sqrt{SAT} + H^2 \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \right)
\]

\[
\leq O \left( HT + H^3 \sqrt{SAT} + H^6 S^2 A + H^2 \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \right)
\]

\[
\leq O \left( HT + H^2 \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \right)
\]

(When \( T \geq \Omega (H^5 S^2 A) \))

Now, by plugging the above result into equation (35), when \( T \geq \Omega (H^5 S^2 A) \), it holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \leq O \left( HSA \left( HT + H^2 \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \right) \right)
\]

\[
\leq O \left( H^2 SAT + H^{1.5} \sqrt{\sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k}} \right).
\]

It is easy to check that the above inequality implies \( \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \leq O \left( H^2 SAT \right) \) and thus the proof is complete. \( \square \)

### G.1 Auxiliary Lemmas

The lemmas used for proving Lemma 25 are presented as the following.

**Lemma 26** (Lemma 8 in Azar et al. [2017]). For any \( \delta > 0 \), with probability at least \( 1 - \delta \), it holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{1} \{ \mathcal{G}_k \} U_{h,k} \leq O \left( HT + H^2 \sqrt{T} + H^3 \right).
\]

**Lemma 27.** For any \( \delta > 0 \), with probability at least \( 1 - \delta \), for any \( k \in [K], h \in [H] \), it holds that

\[
\tilde{V}_{h+1,k}^* \leq \frac{3}{2} \tilde{V}_{h+1,k} + \frac{2H^2S \log (2HS^2AK/\delta)}{n_k(h,s_h^k,a_h^k)},
\]

\[
\tilde{V}_{h+1,k} \leq \frac{3}{2} \tilde{V}_{h+1,k} + \frac{2H^2S \log (2HS^2AK/\delta)}{n_k(h,s_h^k,a_h^k)}.
\]

**Proof.** The proof apply some techniques in Zhang et al. [2020a]. Fix some \( \delta > 0 \) and let \( L = \log (2HS^2AK/\delta) \) for simplicity. First, by Lemma 30, for some tuple \((k,h,s,a,s')\), we have

\[
P \left( \tilde{P}_{h,s,a}(s') \geq \frac{3}{2} P_{h,s,a}(s') + \frac{2L}{n_k(h,s,a)} \right)
\]

\[
\leq P \left( \tilde{P}_{h,s,a}(s') - P_{h,s,a}(s') \geq \sqrt{\frac{2P_{h,s,a}(s')L}{n_k(h,s,a)}} + \frac{L}{n_k(h,s,a)} \right) \quad \text{(Since } a + b \geq 2\sqrt{ab} \text{ for } a, b \geq 0\text{)}
\]

\[
\leq \frac{\delta}{HS^2AK}.
\]

Then, a union bound says that its complement holds for any \((k,h,s,a,s')\) with probability at least \(1 - \delta\). Thus, we have

\[
\tilde{V}_{h+1,k}^* = \sum_{s' \in S} \tilde{P}_{h,s_h^k,a_h^k}(s') \left( V_{h+1}(s') - \left( \tilde{P}_{h,s_h^k,a_h^k} \right)^* \right)^2
\]

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\[
\leq \sum_{s' \in S} \hat{P}_{h,s',a_h}^k (s') (V_{h+1}^*(s') - \langle P_h, s_h^k, a_h^k, V_{h+1}^* \rangle)^2
\]

(Since \(E[X]\) is the minimizer of \(\min_x E[(X - x)^2]\))

\[
\leq \sum_{s' \in S} \left( \frac{3}{2} P_{h,s,a}(s') + \frac{2L}{n_k(h,s,a)} \right) (V_{h+1}^*(s') - \langle P_h, s_h^k, a_h^k, V_{h+1}^* \rangle)^2
\]

\[
\leq \frac{3}{2} V_{h+1,k}^* + \frac{2H^2 S \log (2HS^2 AK/\delta)}{n_k(h,s_h^k,a_h^k)}.
\]

For \(\tilde{V}_{h+1,k}\), we just need to follow a similar argument and thus the proof is complete. \(\square\)

**Lemma 28.** For any \(\delta > 0\), with probability at least 1 - \(\delta\), it holds that

\[
\sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \left( \tilde{V}_{h+1,k}^* - \frac{3}{2} V_{h+1,k}^* \right) \leq \tilde{O} \left( H \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \delta_{h+1,k}^* (s_h^k) + H^2 \sqrt{T} + H^3 S^2 A \right)
\]

**Proof.** We begin by applying Lemma 27. Thus, with probability at least 1 - \(\frac{\delta}{2}\), we have

\[
\sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \left( \tilde{V}_{h+1,k}^* - \frac{3}{2} V_{h+1,k}^* \right) \leq \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \frac{3}{2} V_{h+1,k}^* - \frac{3}{2} V_{h+1,k}^* + 4H^2 S \log \left( \frac{4HS^2 AK/\delta}{3n_k(h,s_h^k,a_h^k)} \right) \quad \text{(By Lemma 27)}
\]

\[
\leq \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \frac{3}{2} V_{h+1,k}^* - \frac{3}{2} V_{h+1,k}^* + \tilde{O} \left( H^3 S^2 A \right) \quad \text{(By Lemma 24)}
\]

\[
\leq 3H \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \tilde{E}_{s' \sim P_{h,s',a_h^k}} \left[ (V_{h+1}^*(s'))^2 - (V_{h+1,k}^*(s'))^2 \right] + \tilde{O} \left( H^3 S^2 A \right) \quad \text{(Since } V_{h+1,k}^* \leq V_{h+1}^*)
\]

\[
\leq \tilde{O} \left( H \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \delta_{h+1,k}^* (s_h^k) + H^2 \sqrt{T} + H^3 S^2 A \right) \quad \text{(By Lemma 22)}
\]

The last line above holds because by Lemma 22, with probability at least 1 - \(\frac{\delta}{2}\), we have

\[
\left| \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \tilde{E}_{s' \sim P_{h,s',a_h^k}} \left[ \delta_{h+1,k}^* (s') - \delta_{h+1,k}^* (s_h^k) \right] \right| \leq \tilde{O} \left( H \sqrt{T} \right).
\]

\(\square\)

**Lemma 29.** For any \(\delta > 0\), with probability at least 1 - \(\delta\), it holds that

\[
\sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \left( \tilde{V}_{h+1,k} - \frac{3}{2} V_{h+1,k}^* \right) \leq \tilde{O} \left( H \sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \left| \delta_{h+1,k}^* (s_h^k) \right| + H^2 \sqrt{T} + H^3 S^2 A \right)
\]

**Proof.** Similarly, we begin by applying Lemma 27 and with probability at least 1 - \(\frac{\delta}{4}\), we have

\[
\sum_{k=1}^K \sum_{h=1}^{H-1} 1 \{G_k\} \left( \tilde{V}_{h+1,k} - \frac{3}{2} V_{h+1,k}^* \right) \]

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In this section, we state and prove our two main theorems.

**H Proof of the Main Theorems**

We will bound (a) and (b) separately. For term (a), with probability at least $1 - \frac{\delta}{3}$, we have

$$
\leq \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \frac{3}{2} \mathbb{V}_{h+1,k} - \frac{3}{2} \mathbb{V}_{h+1,k}^\pi \right) + 4H^2 S \log \left( \frac{6HS^2 AK}{\delta} \right) \\
\leq \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \frac{3}{2} \mathbb{V}_{h+1,k} - \frac{3}{2} \mathbb{V}_{h+1,k}^\pi \right) + \tilde{O} \left( H^3 S^2 A \right)
$$

(By Lemma 24)

$$
= 3 \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k})^2 - \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k})^\pi \right) + \tilde{O} \left( H^3 S^2 A \right).
$$

(By definition of variance)

We will bound (a) and (b) separately. For term (a), with probability at least $1 - \frac{\delta}{3}$, we have

(a) $= \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k})^2 - (\mathbb{V}_{h+1,k}^\pi)^2 \right)$

$$
\leq \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k} - \mathbb{V}_{h+1,k}^\pi) (\mathbb{V}_{h+1,k} + \mathbb{V}_{h+1,k}^\pi) \right)
$$

(Since $a^2 - b^2 = (a + b)(a - b)$)

$$
\leq 3H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k} - \mathbb{V}_{h+1,k}^\pi) \right)
$$

(Since $\|\mathbb{V}_{h+1,k}\|_\infty \leq 2H$ under $G_k$)

$$
\leq 3H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left| \delta_{h+1,k}(s^k_{h+1}) \right| + \tilde{O} \left( H^2 \sqrt{T} \right). \quad \text{(By Lemma 22)}
$$

For term (b), with probability at least $1 - \frac{\delta}{4}$, we have

(b) $= \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k} + \mathbb{V}_{h+1,k}) \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k} - \mathbb{V}_{h+1,k}) \right)$

$$
\leq 3H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{P}_{h,s^k_h,a^k_h} (\mathbb{V}_{h+1,k} - \mathbb{V}_{h+1,k}) \right)
$$

$$
\leq 3H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left| \delta_{h+1,k}(s^k_{h+1}) \right| + \tilde{O} \left( H^2 \sqrt{T} \right). \quad \text{(By Lemma 22)}
$$

Therefore, in summary, we have with probability at least $1 - \delta/2$,

$$
\sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left( \mathbb{V}_{h+1,k} - \frac{3}{2} \mathbb{V}_{h+1,k}^\pi \right) \leq \tilde{O} \left( H \sum_{k=1}^{K} \sum_{h=1}^{H-1} \mathbb{I} \{ G_k \} \left| \delta_{h+1,k}(s^k_{h+1}) \right| + H^2 \sqrt{T} + H^3 S^2 A \right)
$$

\(\square\)

**H Proof of the Main Theorems**

In this section, we state and prove our two main theorems.
Theorem 1. If the Hoeffding-type noise is used, then for any MDP \( M = (H, S, A, P, R, s_1) \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), Algorithm 1 satisfies
\[
\text{Reg}(M, K, \text{SSR}_{H0}) \leq \widetilde{O} \left( H^{1.5} \sqrt{SAT} + H^4 S^2 A \right).
\]
In particular, when \( T \geq \Omega \left( H^5 S^3 A \right) \), it holds that \( \text{Reg}(M, K, \text{SSR}_{H0}) \leq \widetilde{O} \left( H^{1.5} \sqrt{SAT} \right) \).

Proof. By using the result of Lemma 20, under Hoeffding-type noise, with probability at least \( 1 - \delta \), we have
\[
\text{Reg}(M, K, \text{SSR}_{H0}) \leq 1 \{ \mathcal{G}_k \} 3C_1 e^{3C_1} \sum_{k=1}^{K} \sum_{h=1}^{H} \left( \sqrt{e_{H0}(h, s_{h}^k, a_{h}^k)} + \gamma_{H0}(h, s_{h}^k, a_{h}^k) \right) + \widetilde{O} \left( H^4 S^2 A + H \sqrt{T} \right)
\]
\[
\leq 6C_1 e^{3C_1} \sum_{k=1}^{K} \sum_{h=1}^{H-1} \left( H \sqrt{\log(2HSAk^2)} + \frac{H}{n_k(h, s, a) + 1} \right) + \widetilde{O} \left( H^4 S^2 A + H \sqrt{T} \right)
\]
\[
= \widetilde{O}(H^{1.5} \sqrt{SAT} + H^4 S^2 A).
\]
Here, the second inequality is from the definitions of \( \sqrt{e_{H0}(h, s_{h}^k, a_{h}^k)} \) and \( \gamma_{H0}(h, s_{h}^k, a_{h}^k) \), and the last step is from Lemma 23 and 24.

\[\square\]

Theorem 2. For Bernstein-type noise and \( T \geq \Omega \left( H^6 S^2 A \right) \), then for any MDP \( M = (H, S, A, P, R, s_1) \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), Algorithm 1 satisfies
\[
\text{Reg}(M, K, \text{SSR}_{Be}) \leq \widetilde{O} \left( \sqrt{S}AT + H^4 S^2 A \right).
\]
In particular, if we further have \( T \geq \Omega \left( H^6 S^3 A \right) \), it then holds that \( \text{Reg}(M, K, \text{SSR}_{Be}) \leq \widetilde{O} \left( \sqrt{S}AT \right) \).

Proof. Similar to the proof of Theorem 1, under Bernstein-type noise, it holds with probability at least \( 1 - \frac{\delta}{2} \) that
\[
\text{Reg}(M, K, \text{SSR}_{Be}) \leq 1 \{ \mathcal{G}_k \} 3C_1 e^{3C_1} \sum_{k=1}^{K} \sum_{h=1}^{H} \left( \sqrt{e_{Be}(h, s_{h}^k, a_{h}^k)} + \gamma_{Be}(h, s_{h}^k, a_{h}^k) \right) + \widetilde{O} \left( H^4 S^2 A + H \sqrt{T} \right)
\]
\[
\leq 6C_1 e^{3C_1} \sum_{k=1}^{K} \sum_{h=1}^{H-1} \left( \sqrt{\log(2HSAk^2)} + \frac{H}{n_k(h, s, a) + 1} \right) + \widetilde{O}(H^4 S^2 A + H \sqrt{T})
\]
\[
= \widetilde{O}(H \sqrt{SAT} + H^4 S^2 A),
\]
where the last step is from Lemma 25.

\[\square\]

I. Technical Lemmas

Lemma 30 (Bennet’s Inequality). Let \( Z_1, \ldots, Z_n \) be i.i.d. random variables bounded in \([0, 1]\). Then, for any \( \delta > 0 \), we have
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} Z_i - \mathbb{E}[Z] \right| \geq \sqrt{\frac{2 \text{Var}(Z) \log(2/\delta)}{n}} + \frac{\log(2/\delta)}{n} \right) \leq \delta.
\]

Lemma 31 (from Maurer and Pontil [2009]). Let \( Z_1, \ldots, Z_n \) with \( n \geq 2 \) be i.i.d. random variables bounded in \([0, H]\). Define \( \bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i \) and \( \bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} (Z_i - \bar{Z})^2 \). Then, for any \( \delta > 0 \), we have
\[
P \left( \left| \mathbb{E}[Z] - \frac{1}{n} \sum_{i=1}^{n} Z_i \right| \geq \sqrt{\frac{2 \bar{Z}_n \log(2/\delta)}{n - 1}} + \frac{7 \log(2/\delta)}{3(n - 1)} \right) \leq \delta.
\]
Lemma 32. Let $X$ be arbitrary random variable bounded in $[a, b]$ for some $a, b \in \mathbb{R}$. Then, we have $\text{Var}(X) \leq \frac{(b-a)^2}{4}$.

Lemma 33. For any $\delta > 0$, with probability at least $1 - \delta$, it holds for all $k, h, s, a, s'$ that

$$|\hat{P}_{h,s,a}^k(s') - P_{h,s,a}(s')| \leq \sqrt{\frac{4P_{h,s,a}(s')(1 - P_{h,s,a}(s')) \log(2HS^2AK/\delta)}{n_k(h, s, a) + 1} + \frac{3\log(2HS^2AK/\delta)}{n_k(h, s, a) + 1}}.$$ 

Proof. Let $\delta' = \frac{\delta}{HS^2AK}$ and fix $(k, h, s, a, s')$ such that $n_k(h, s, a) \geq 1$. Then, we have

$$P \left( |\hat{P}_{h,s,a}^k(s') - P_{h,s,a}(s')| \geq \sqrt{\frac{4P_{h,s,a}(s')(1 - P_{h,s,a}(s')) \log(2/\delta')}{n_k(h, s, a) + 1} + \frac{3\log(2/\delta')}{n_k(h, s, a) + 1}} \right) \leq \delta' = \frac{\delta}{HS^2AK}.$$ 

Then, the proof is complete by taking a union bound over all possible $(k, h, s, a, s')$. 

\[\square\]