Is there a hidden chiral density-wave in the iron-based superconductors?

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Superconductivity in most iron pnictides and iron chalcogenides is found near an instability towards an orthorhombic stripe magnetic state. A remarkable property of the latter is that its melting into the tetragonal paramagnetic phase happens generally in two stages, resulting in a “vestigial” nematic paramagnetic phase that breaks the point-group symmetry. Recent experiments in hole-doped iron arsenides, however, revealed a magnetically ordered ground state that preserves tetragonal symmetry, which is likely either a checkerboard charge-and-spin density wave (CSDW) or a spin-vortex crystal (SVC). Here we show that either of these phases can also melt in two stages, resulting in an intermediate phase with vestigial order, namely: a charge density-wave (CDW) for a CSDW ground state, and a remarkable spin-vorticity density-wave (SVDW) for a SVC ground state. While the CDW has an Ising-like order parameter, the SVDW has a vector chiral order parameter that breaks inversion without breaking time-reversal symmetry, giving rise to Goldstone modes. We propose experimentally detectable signatures of these phases, and point out that their fluctuations can lead to an enhancement of the superconducting transition temperature.

One of the hallmarks of the superconducting state of the iron-based materials is its typical proximity to a magnetically ordered state that displays magnetic stripes, with spins aligned parallel to each other along one in-plane direction and anti-parallel along the other (see Fig. 1b). As a result, this stripe state breaks two distinct symmetries of the high-temperature paramagnetic-tetragonal state: a continuous spin-rotational O(3) symmetry and an Ising-like Z_2 symmetry related to the equivalence of the x and y directions. Magnetic fluctuations present in the paramagnetic state can cause these two symmetries to be broken at different temperatures, giving rise to an intermediate nematic phase that preserves the spin-rotational O(3) symmetry but, as a “vestige” of the stripe order, breaks the tetragonal Z_2 symmetry. Indeed, in the phase diagrams of most iron-based superconductors, the magnetic transition line is found to be closely followed by a structural/nematic transition line at slightly higher temperatures. The corresponding nematic degrees of freedom impact not only the normal state electronic properties but also the onset and gap structure of the superconducting state.

Recently, experiments in the hole-doped pnictides Ba(Fe_{1−x}Mnx)2As2, (Ba_{1−x}Na_x)Fe2As2, and (Ba_{1−x}K_x)Fe2As2 have revealed another type of magnetically ordered state that does not break the tetragonal Z_2 symmetry of the lattice. Neutron scattering experiments showed that its magnetic Bragg peaks are at the same momenta as in the stripe magnetic phase – namely, Q_1 = (π, 0) and Q_2 = (0, π) in the Brillouin zone of the iron square lattice. Consequently, it has been proposed that the tetragonal magnetic state is the realization of one of two possible biaxial (i.e. double-Q) magnetic orders. One possibility is a “charge-and-spin density wave” (CSDW), displaying a non-uniform magnetization which vanishes at the even lattice sites and is staggered along the odd lattice sites (Fig. 1b). The other option is a “spin-vortex crystal” (SVC), in which the magnetization is non-coplanar (but coplanar) and forms spin vortices staggered across the plaquettes. Both CSDW and SVC phases are tetragonal, but have a unit cell four times larger than the paramagnetic phase. Interestingly, in (Ba_{1−x}Na_x)Fe2As2 and (Ba_{1−x}K_x)Fe2As2, the tetragonal magnetic state is observed very close to optimal doping, where superconductivity displays its highest transition temperature. Therefore, understanding the properties of these biaxial tetragonal magnetic phases is important to assess their relevance for the superconductivity.

In this paper, we show that both the CSDW and the SVC magnetic phases support composite order parameters that can condense at temperatures above the onset of magnetic order. As with the nematic phase, these partially ordered phases are paramagnetic, i.e. fluctuations restore the time-reversal symmetry that is broken in the ground state. In contrast to the nematic phase, however, they preserve the point group symmetry of the lattice, but break other symmetries, including translational symmetry. In particular, upon melting the SVC phase, we find a vestigial state that retains the broken translational symmetry related to the fact that half the sites are magnetized and half are not in the CSDW phase. As a result, the charge of the previously magnetized sites is different than the charge of the previously non-magnetized sites, and the corresponding Ising-like composite order parameter, ϕ_{CDW}, describes a checkerboard charge density-wave (CDW) with ordering vector Q_1 + Q_2 = (π, π).

On the other hand, upon melting the SVC ground state, we find a vestigial phase that retains memory of the preferred plane of magnetization (in spin space), and of the staggering of the spin vortices across the plaquettes. In terms of broken symmetries, this phase corresponds to a spin-current density-wave, also known as a triplet d-density wave. The corresponding composite order...
We start with the appropriate low-energy action that describes the three different types of magnetic ground states of the pnictides. We define two magnetic order parameters, \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \), associated with the two ordering vectors \( \mathbf{Q}_1 = (\pi, 0) \) and \( \mathbf{Q}_2 = (0, \pi) \), respectively. Thus, the local spin is given by \( \mathbf{S}(\mathbf{r}) = \sum \mathbf{M}_\mathbf{e}^{i \mathbf{Q} \cdot \mathbf{r}} \). As discussed in Refs. [6, 30, 32, 34], the most general lowest order action that respects the tetragonal and spin-rotational symmetries is given by:

\[
S[\mathbf{M}] = \int (\mathbf{M}_1^2 + \mathbf{M}_2^2) + \frac{u}{2} \int \left( \mathbf{M}_1 \cdot \mathbf{M}_2 \right)^2 - \frac{g}{2} \int \left( \mathbf{M}_1^2 - \mathbf{M}_2^2 \right)^2 + 2w \int \left( \mathbf{M}_1 \cdot \mathbf{M}_2 \right)^2. \tag{1}
\]

For simplicity, we will consider the finite temperature problem, in which case \( S \) is an effective Landau-Ginzburg-Wilson free energy, but the present considerations are easily extended to include quantum fluctuation phenomena at \( T = 0 \) by taking \( S \) to be the appropriate effective action. Here we introduce the notations \( \int_q = \int \frac{d^d q}{(2\pi)^d} \) and \( \int_x = \int d^d x \) where \( \mathbf{q} \) is the momentum and \( \mathbf{x} \) is the position. In the neighborhood of a finite \( T \) magnetic transition, only the small \( q \) properties of the magnetic susceptibility are important, which for quasi-2D systems like the iron pnictides means \( \chi_{ij} \approx r_0 + q_0^2 + J_z \sin^2 \frac{q \cdot \mathbf{r}}{2} \), where \( r_0 \) is the distance to the mean-field magnetic critical point and \( q_0^2 = q_x^2 + q_y^2 \).

The quartic coefficients \( u, g, w \) determine the nature of the magnetic ground state. These are, in turn, sensitive to microscopic considerations. The localized \( J_{1}-J_{2} \) model favors positive \( g \) and \( w \) [28]. On the other hand, itinerant approaches have found parameter regimes in which \( g \) and \( w \) can be either positive or negative [6, 30, 32, 34], as have numerical studies of a multi-orbital Hubbard model in the strong-coupling regime [33]. For \( g > \max(0, -w) \), the energy is minimized by the stripe state shown in Fig. 1, in which either \( \langle \mathbf{M}_1 \rangle = 0 \) (stripes modulated along the \( y \) direction) or \( \langle \mathbf{M}_2 \rangle = 0 \) (stripes modulated along the \( x \) direction). Thus, in addition to breaking the \( O(3) \) spin-rotational symmetry, the magnetic ground state spontaneously breaks a \( Z_2 \) symmetry by selecting one of the two order parameters to be non-zero. Since \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \) are related by a 90° rotation, once this \( Z_2 \) symmetry is broken the tetragonal symmetry of the system is lowered to orthorhombic (see Fig. 2b). A composite Ising-nematic order parameter, living on the bonds of the lattice, can be identified by performing a Hubbard-Stratonovich transformation on the quartic term with coefficient \( g \), yielding \( \langle \phi_{\text{nem}} \rangle = g \langle \mathbf{M}_1^2 - \mathbf{M}_2^2 \rangle \). Because \( Z_2 \) is a discrete symmetry, while spin-rotational \( O(3) \) is a continuous symmetry, a strongly anisotropic three-dimensional system will generically display a vestigial paramagnetic nematic phase where \( \langle \mathbf{M}_1 \rangle = 0 \) but \( \langle \phi_{\text{nem}} \rangle \neq 0 \). Indeed, this behavior has been confirmed by numerical and analytical calculations [3, 8, 40].

For \( g < \max(0, -w) \), the ground state of Eq. (1) is no longer a uniaxial magnetic stripe state, but a biaxial magnetic state with \( \langle \mathbf{M}_1 \rangle^2 = \langle \mathbf{M}_2 \rangle^2 \). Such states preserve tetragonal symmetry. If \( w < 0 \), the energy is minimized by \( \langle \mathbf{M}_1 \rangle \parallel \langle \mathbf{M}_2 \rangle \), which in terms of the local spin
configuration $S(r)$ corresponds to a non-uniform state as depicted in Fig. [Fig]. We identify this state as a charge-and-spin density-wave (CSDW). On the other hand, if $w > 0$, the energy minimization gives $\langle M_1 \rangle \perp \langle M_2 \rangle$, corresponding to a non-collinear, coplanar spin configuration (see Fig. [Fig]). This state is identified as a spin vortex-crystal (SVC). Having established a general low-energy model that captures the magnetic ground-states of the pnictides, we now discuss the structure of the order parameter space and the nature of the transitions in the different cases.

**Composite CDW order.** Consider the charge-and-spin density-wave (CSDW) state favored by $w < 0$ and $g < |w|$: Once the axis of quantization is chosen by spontaneous breaking of the $O(3)$ spin-rotational symmetry, there remains a four-fold ground state degeneracy, corresponding to whether $M_1$ and $M_2$ are parallel or anti-parallel to the chosen axis. As is apparent in Fig. 2b, this corresponds to the breaking of translational symmetry, leading to a four-site unit cell. Notice, however, that the product of two symmetries is preserved, namely, translation by a second-neighbor lattice vector $\mathbf{x} + \mathbf{y}$ followed by time-reversal. Thus, there is an essential $Z_2$ symmetry that interchanges the magnetic and non-magnetic sublattices of the CSDW state, corresponding to a translation by one lattice constant in either $\mathbf{x}$ or $\mathbf{y}$ directions.

To introduce an appropriate order parameter field for this $Z_2$ symmetry, we perform a Hubbard-Stratonovich transformation on the quartic term with coefficient $2w$ in Eq. (1), obtaining $\langle \varphi_{\text{CSDW}} \rangle = 2|w| \langle M_1 \cdot M_2 \rangle$. Clearly, $\varphi_{\text{CSDW}}$ is a scalar that carries momentum $Q_1 + Q_2 = (\pi, \pi)$, i.e. the condensed phase is a charge density-wave that doubles the unit cell, but leaves time-reversal and the tetragonal symmetry of the lattice intact (see Fig. [Fig]). Thus, in real space, the CDW order parameter lives on the lattice sites. The fact that the unit cell decreases from four to two sites upon going from the CSDW to the CDW phase is due to the restoration of time-reversal symmetry, which implies the restoration of the translational symmetry by $\mathbf{x} + \mathbf{y}$. A simple change of variables in Eq. (1), $M_1 \rightarrow 2^{-1/2}[M_1 + M_2]$ and $M_1 \rightarrow 2^{-1/2}[M_1 - M_2]$, interchanges the identities of the two scalar orders, $\varphi_{\text{CSDW}} \leftrightarrow \varphi_{\text{CDW}}$, but leaves the form of $S$ unchanged albeit with $(g, w) \rightarrow -(w, g)$. Thus, the properties of the CDW phase are akin to those of the Ising-nematic phase – in particular, if the system is sufficiently quasi-2D, there must again exist a range of intermediate temperatures where $\langle M_i \rangle = 0$ but $\langle \varphi_{\text{CDW}} \rangle \neq 0$.

**Composite SVDW order.** Consider now the non-collinear spin vortex-crystal (SVC) favored when $w > 0$ and $g < 0$, and characterized by two equal magnitude orthogonal vectors $M_1$ and $M_2$. Upon fixing the direction of $M_1$, which breaks the $O(3)$ spin-rotational symmetry, there remains an additional $O(2)$ symmetry related to choosing $M_2$ in any direction along the plane perpendicular to $M_1$ [Fig]. Thus, the SVC phase can be completely characterized by a pseudo-vector $\varphi_{\text{SVDW}}$ that specifies the ordering plane which contains $M_1$ and $M_2$, and also by the orientation of $M_1$ within that plane. To introduce an order parameter field corresponding to $\varphi_{\text{SVDW}}$, we use simple vector identities to replace the term $w (M_1 \cdot M_2)^2 \rightarrow -w (M_1 \times M_2)^2$ in Eq. (1) with an accompanying renormalization of $g$ and $u$. A Hubbard-Stratonovich transformation of this new term introduces the appropriate pseudo-vector order parameter $\langle \varphi_{\text{SVDW}} \rangle = 2w (M_1 \times M_2)$, which can be identified as a vector chirality [42, 43]. Thus, upon

![Figure 2: The vestigial composite states associated with (a) the stripe state, (b) the CSDW state, and (c) the SVC state. The real-space spins (in gray) and the magnetic order parameters in spin space (red and blue arrows, representing $M_1$ and $M_2$, respectively) should be understood as fluctuating, i.e. $\langle M_i \rangle = 0$ in all cases. In (a), the vestigial state is nematic (unequal blue and red bonds), associated with selecting between $M_1$ and $M_2$ fluctuations in spin space. The original unit cell is shown as a dashed square. In (b), the vestigial state breaks translational symmetry via a checkerboard charge-density-wave (unequal blue and red sites). $M_1$ and $M_2$ are locked to be collinear in spin space. In (c), the vestigial spin-vorticity density-wave state breaks inversion and translational symmetries via a staggered pattern of spin vortices in the center of the plaquettes (unequal blue and red curved arrows). The corresponding spin-current pattern is shown by the green arrows. Because $M_1$ and $M_2$ are locked to be orthogonal in spin space, the residual spin-rotational symmetry is $O(2)$ instead of $O(3)$. Both (b) and (c) preserve tetragonal symmetry, as shown by the dashed-line unit cell.]
approaching the SVC phase from high-temperatures or by melting it, there can be intermediate states where \( \langle \varphi_{\text{SVDW}} \rangle \neq 0 \) but the orientation of \( \mathbf{M}_1 \) is not fixed, \( \langle \mathbf{M}_1 \rangle = 0 \). This chiral paramagnetic state therefore preserves time-reversal symmetry and retains the memory of the staggering pattern of spin vortices along the plaquettes in the SVC phase, and is therefore called a spin-vorticity density-wave (SVDW). In the SVDW state, not only is the translational symmetry lowered by the doubling of the unit cell (since \( \varphi_{\text{SVDW}} \) carries momentum \( \mathbf{Q}_1 + \mathbf{Q}_2 = (\pi, \pi) \)), but also the soft spin fluctuations near the magnetic transition are constrained to lie in the plane defined by \( \varphi_{\text{SVDW}} \) (see Fig. 2c). This means that the transition between the magnetic SVC and the non-magnetic SVDW phases is in the XY (O(2)) universality class, as opposed to the Heisenberg (O(3)) universality class that would apply in the nematic and CDW cases. Furthermore, because \( \varphi_{\text{SVDW}} \) breaks a continuous O(3) symmetry, there are two Goldstone modes in the SVDW phase.

An important issue is whether it is plausible for this intermediate chiral phase with vestigial SVDW order to exist at all. In contrast to the Ising-nematic cases, the Mermin-Wagner theorem does not ensure its existence even in the two-dimensional limit: because both \( \varphi_{\text{SVDW}} \) and \( \mathbf{M}_1 \) break continuous symmetries, \( T_{\text{SVDW}} = T_M = 0 \) in \( d = 2 \). To investigate this issue, we have calculated the phase diagram in the regime of parameters corresponding to a magnetic SVC ground state treating the action in Eq. 1 in the saddle point approximation (details of the calculation are provided in the supplementary material.) We have considered an anisotropic system with small values of the interlayer coupling, \( J_z \). We find that there is a wide range of values of \( u/w \) for which there are two transitions, with an intermediate SVDW phase and a low-temperature SVC phase. However, in this approximation, the transition to the SVDW phase is always first-order. Additional numerical investigations would be desirable to establish whether this transition could become second-order.

Spin rotational symmetry is not an exact symmetry of nature, and indeed most iron pnictides display a sizable spin anisotropy, manifested as a spin-gap of the order of 10 meV in the magnetically ordered state [44–46]. In view of the observation that the ordered moments tend to point parallel to the FeAs plane, the most significant effects of spin-orbit coupling can be captured phenomenologically in Eq. 1 by including an easy-plane anisotropy term \( \kappa (M_{1,2}^2 + M_{2,2}^2) \) with coupling constant \( \kappa > 0 \). The spin rotational symmetry is thus reduced to \( O(2) \) and the SVDW chiral order parameter becomes the pseudo-scalar, \( \xi_{\text{SVDW}} = 2w (\mathbf{M}_1 \times \mathbf{M}_2) \cdot \hat{z} \), which only breaks a discrete chiral \( Z_2 \) symmetry. For such \( O(2) \times Z_2 \) model, it is known from both numerical and analytical investigations that in 2D the \( Z_2 \) symmetry is broken at higher temperatures than the Kosterlitz-Thouless transition of the \( O(2) \) order parameter [47,48], i.e. there is no doubt that there is a vestigial chiral SVDW phase. The extent to which the spin anisotropy is quantitatively significant depends on the (currently unknown) value of the ratio \( \kappa/r_{0,\text{SVDW}} \), where \( r_{0,\text{SVDW}} \) is the value of \( r_0 \) at the SVDW transition temperature \( T_{\text{SVDW}} \).

**Microscopic considerations.** To discuss the experimental manifestations of the vestigial CDW and SVDW states, we investigate their coupling to the low-energy electronic states of the pnictides. We consider a three-band model [6,8] with a circular hole pocket \( \xi_{h,k} \) at the center of the Brillouin zone, and two elliptical electron pockets \( \xi_{e_1,k+Q_1} \) and \( \xi_{e_2,k+Q_2} \) centered at momenta \( \mathbf{Q}_1 = (\pi,0) \) and \( \mathbf{Q}_2 = (0,\pi) \), respectively (see Fig. 3). The magnetic order parameters couple to these electronic states via \( \sum_{k,\alpha\beta} \mathbf{M}_i \cdot \mathbf{c}_{\alpha\beta} \xi_{h,k} \xi_{e_1,k+Q_1} + \mathbf{c}_{e_2,k+Q_2} \xi_{e_2,k+Q_2} \), where the operator \( \mathbf{c}_{\alpha} \) annihilates an electron in band \( \alpha \) with momentum \( \mathbf{k} \) (measured with respect to the center of the pocket) and spin \( \alpha \), and \( \mathbf{c}_{\alpha\beta} \) are the Pauli matrices. We further introduce magnetic and scalar order parameter fields, \( \Delta_S \) and \( \Delta_C \) respectively, with ordering vector \( \mathbf{Q}_1 + \mathbf{Q}_2 = (\pi,\pi) \):

\[
\mathcal{H}_S = \sum_{k,\alpha\beta} \left[ \Delta_S \cdot \mathbf{c}_{\alpha\beta}^{\dagger} \xi_{e_2,k} \xi_{e_2,k+Q_1} + \text{h.c.} \right],
\]

\[
\mathcal{H}_C = \sum_{k,\alpha\beta} \left[ \Delta_C \mathbf{c}_{\alpha\beta}^{\dagger} \xi_{e_2,k} \xi_{e_2,k+Q_2} + \text{h.c.} \right].
\]

Here these fields have real and imaginary parts, \( \Delta_S = \Delta_S^R + i \Delta_S^I \) and \( \Delta_C = \Delta_C^R + i \Delta_C^I \), where the real parts correspond to conventional SDW or CDW orders, while the imaginary parts correspond to spin or charge current orders. By integrating out the electronic degrees of freedom, we obtain the coupling between \( \mathbf{M}_i \) and \( \Delta_S, \Delta_C \) to lowest-order in the action (see supplementary material):
\[ \delta S_{\text{eff}} = \lambda \left[ \Delta_C^2 (M_1 \times M_2) - \Delta_C' (M_1 \cdot M_2) \right] \] (3)

with the coefficient \( \lambda = 4 \int \prod G_{h,k} G_{c_1,k} G_{c_2,k} \), where \( G_{c_0}^{-1} = \iota \omega_n - \xi_{a,k} \) is the corresponding non-interacting Green's function. As expected, the Ising-like order parameter \( \varphi_{\text{CDW}} \propto M_1 \cdot M_2 \) induces a checkerboard-like charge order (see Fig. [2]). As discussed above, such a CDW may appear even before the onset of magnetic CSDW order, \( \langle M_1 \rangle \parallel \langle M_2 \rangle \neq 0 \). On the other hand, the SVDW order parameter \( \varphi_{\text{SVDW}} \propto M_1 \times M_2 \), which may precede the non-collinear SVC state \( \langle M_1 \rangle \perp \langle M_2 \rangle \neq 0 \), is manifested as a spin-current density-wave with propagation vector \((\pi, \pi)\). The SVDW is characterized by a spin current polarized parallel to \( \varphi_{\text{SVDW}} \) and propagating along the bonds of the lattice in a staggered pattern across the square plaquettes (see Fig. [2]). Clearly, the SVDW does not break time-reversal symmetry, but selects a preferred “handedness”.

While the real CDW could in principle be detected experimentally by a probe sensitive to the local charge on the Fe sites, such as STM, detecting a spin-current density-wave would be rather challenging. Alternatively, one can consider the effects of a Zeeman field \( \mathbf{H} \). Despite not coupling to the nematic order parameter \( \varphi_{\text{nem}} \), we find that it couples to both \( \varphi_{\text{CDW}} \) and \( \varphi_{\text{SVDW}} \) in the action via the terms \( \gamma \Delta_C^2 \mathbf{H} \cdot (M_1 \times M_2) \) and \( \gamma (\mathbf{H} \cdot \Delta_C^2) (M_1 \cdot M_2) \), with the same Ginzburg-Landau coefficient \( \gamma = 4 \int \prod \mathbf{G}_{h,k} \mathbf{G}_{c_1,k} \mathbf{G}_{c_2,k} \mathbf{G}_{c_1,k} \mathbf{G}_{c_2,k} - \mathbf{G}_{h,k} \). Therefore, in the presence of a magnetic field, the CDW acquires an SDW component, whereas the spin-current density-wave acquires a charge-current density-wave component analogous to the \( d \)-density-wave discussed in the context of the cuprates [37].

Although these couplings are expected from symmetry considerations [42], our microscopic calculations reveal that the Ginzburg-Landau coefficient \( \gamma \) can be made comparable to \( \lambda \) in Eq. (3) for systems whose Fermi surfaces are not too far from nesting conditions. In this case, even relatively small magnetic fields \( H \) can induce sizable secondary charge-currents. The resulting staggered pattern of circulating orbital currents produces staggered magnetic fields in the center of the square plaquettes, where the pnictogen or chalcogen atoms of the iron superconductors are located. As a result, NMR in the pnictogen or chalcogen sites would be an ideal probe to detect the charge currents and, consequently, the vestigial chiral SVDW state preceding the non-collinear SVC phase. Similarly, the secondary SDW state induced by the magnetic field applied in the vestigial CDW state could also be detected experimentally via neutron scattering or even NMR. Table II summarizes the ground states of the pnictides along their vestigial paramagnetic states.

**Superconductivity.** Finally, we comment on the impact of the vestigial SVDW and CDW states on the superconductivity of the iron pnictides. In particular, in the compounds \((\text{Ba}_{1-x} \text{Na}_x)\text{Fe}_2\text{As}_2\) and \((\text{Ba}_{1-x} \text{K}_x)\text{Fe}_2\text{As}_2\), the tetragonal magnetic ground state is observed near optimal doping [26, 27], where the superconducting \( T_c \) is the largest. Because the magnetic fluctuations associated with the \( M_1 \) and \( M_2 \) order parameters are peaked at momenta \( Q_1 = (\pi, 0) \) and \( Q_2 = (0, \pi) \), they promote a repulsive inter-pocket interaction \( V > 0 \) between the hole and the electron pockets (see Fig. 4). Solution of the corresponding linearized gap equations yields the so-called \( s^+ \) state, where the gap functions have different signs in the electron and in the hole pockets [49]. The transition temperature is given by \( T_c \propto \exp \left( -\frac{1}{\Lambda_0} \right) \), with the leading eigenvalue \( \Lambda_0 = \sqrt{2 N_A N_c} \lambda V \), and \( N_A \) denoting the density of states of band \( a \). The fluctuations of the vestigial tetragonal phases SVDW and CDW, on the other hand, are peaked at the momentum \( Q_1 + Q_2 = (\pi, \pi) \) and promote an attractive inter-pocket interaction \( U < 0 \) between the two electron pockets (see Fig. 4). Solution of the linearized gap equation reveals that the leading eigenstate remains the \( s^+ \) one, but the eigenvalue is enhanced, \( \Lambda = \sqrt{\Lambda_0^2 + \Lambda_U^2} + \Lambda_U \), with \( \Lambda_U = N_A U \). Therefore, fluctuations associated with these vestigial states may enhance the value of \( T_c \) promoted by spin-fluctuations pairing, without affecting the symmetry of the Cooper pair wave-function. Similar conclusions have been found for the combination of pairing promoted by nematic fluctuations (peaked at \( Q = 0 \)) and magnetic fluctuations (peaked at \( Q_1 \) and \( Q_2 \) [23].

**Concluding remarks.** In summary, we showed that both biaxial tetragonal magnetic ground states of the pnictides – the non-uniform CSDW and non-collinear
Table I: Magnetic ground states of the pnictides and their corresponding vestigial states. $M_1$ and $M_2$ are the magnetic order parameters corresponding to the ordering vectors $Q_1 = (\pi, 0)$ and $Q_2 = (0, \pi)$.

| Magnetic ground state | Vestigial state | Broken symmetry | Real space pattern | Physical manifestation |
|-----------------------|----------------|-----------------|-------------------|------------------------|
| Stripe: $\langle M_2 \rangle$ or $\langle M_1 \rangle = 0$ | Nematic: $\langle M_1^2 - M_2^2 \rangle \neq 0$ | Rotational (tetragonal) | Unequal bonds | Orthorhombic distortion |
| CSDW: $\langle M_1 \rangle \parallel \langle M_2 \rangle$ | CDW: $\langle M_1 \cdot M_2 \rangle \neq 0$ | Translational | Unequal sites | Charge density-wave |
| SVC: $\langle M_1 \rangle \perp \langle M_2 \rangle$ | SVDW: $\langle M_1 \times M_2 \rangle \neq 0$ | Translational + chiral | Unequal plaquettes | Spin-current density-wave |

SVC states – can melt in two-stage processes, giving rise to CDW and SVDW vestigial states, respectively. While both preserve the point-group and time-reversal symmetries, but break the translational symmetry of the iron square lattice, only the SVDW state also breaks inversion symmetry by entangling the spin-space handedness to a doubling of the real-space unit cell. Strictly speaking, in the iron superconductors, the hybridization between the puckered pnictogen/chalcogen and the iron atoms already doubles the unit cell of the single-Fe square lattice. In this regard, the CDW and SVDW vestigial states are more rigorously classified as intra-unit-cell orders. Interestingly, the SVDW state is manifested in the electronic degrees of freedom as unconventional density-wave patterns, namely, a spin-current density-wave (in the absence of a uniform magnetic field) and a charge-current density-wave (in the presence of a uniform field). Both types of order have been proposed to be realized in other strongly correlated systems, such as the pseudogap phase of underdoped cuprates [37] and the hidden-order phase of the heavy fermion compound URu$_2$Si$_2$ [50, 51]. While the motivation behind these proposals was mostly phenomenological, our work provides a rather general microscopic mechanism, based on the melting of double-$Q$ magnetic states, that naturally accounts for the existence of such states. Whether this mechanism is directly applicable to those systems – and in particular to URu$_2$Si$_2$ [52], which shares the same crystalline structure as the iron pnictide BaFe$_2$As$_2$ – is an appealing topic for future investigation.

We thank C. Batista, A. Boehmer, A. Chubukov, J. Kang, C. Meingast, R. Osbourn, M. Schuetz, and X. Wang for fruitful discussions. RMF is supported by the U.S. Department of Energy under Award Number de-sc0012336. SAK is supported by the U.S. Department of Energy under Contract No. DE-AC02-76SF00515. EB was supported by the US-Israel Binational Science Foundation, and by an Alon fellowship. We thank the hospitality of the Aspen Center for Physics, where this work was initiated.

[1] K. Ishida, Y. Nakai and H. Hosono, J. Phys. Soc. Japan 78, 062001 (2009); D. C. Johnston, Adv. Phys. 59, 803 (2010); J. Paglione and R. L. Greene, Nature Phys. 6, 645 (2010); P. C. Canfield and S. L. Bud’ko, Annu. Rev. Cond. Mat. Phys. 1, 27 (2010); H. H. Wen and S. Li, Annu. Rev. Cond. Mat. Phys. 2, 121 (2011).
[2] P. Dai, J. Hu, and E. Dagotto, Nature Phys. 8, 709 (2012).
[3] C. Fang, H. Yao, W.-F. Tsai, J. P. Hu and S. A. Kivelson, Phys. Rev. B 77, 224509 (2008).
[4] C. Xu, M. Muller, and S. Sachdev, Phys. Rev. B 78, 020501(R) (2008).
[5] M. D. Johannes and I. I. Mazin, Phys. Rev. B 79, 220510(R) (2009).
[6] I. Eremin and A. V. Chubukov, Phys. Rev. B 81, 024511 (2010).
[7] E. Abrahams and Q. Si, J. Phys.: Condens. Matter 23, 223201 (2011).
[8] R. M. Fernandes, A. V. Chubukov, J. Knolle, I. Eremin and J. Schmalian, Phys. Rev. B 85, 024534 (2012).
[9] S. Liang, A. Mukherjee, N. D. Patel, E. Dagotto, and A. Moreo, Phys. Rev. B 90, 184507 (2014).
[10] L. Nie, G. Tarjus, and S. A. Kivelson, PNAS 111, 7980 (2014).
[11] R. M. Fernandes, A. V. Chubukov, and J. Schmalian, Nature Phys. 10, 97 (2014).
[12] J.-H. Chu, J. G. Analytis, K. De Greve, P. L. McMahon, Z. Islam, Y. Yamamoto, and I. R. Fisher, Science 329, 824 (2010).
[13] T.-M. Chuang, M. P. Allan, J. Lee, Y. Xie, N. Ni, S. L. Bud’ko, G. S. Boebinger, P. C. Canfield, and J. C. Davis, Science 327, 181 (2010).
[14] M. Yi, D. Lu, J.-H. Chu, J. G. Analytis, A. P. Sorini, A. F. Kemper, B. Moritz, S.-K. Mo, R. G. Moore, M. Hashimoto, W. S. Lee, Z. Hussain, T. P. Devereaux, I. R. Fisher, Z.-X. Shen, Proc. Nat. Acad. Sci. 2011 108, 6878 (2011).
[15] J.-H. Chu, H.-H. Kuo, J. G. Analytis, and I. R. Fisher, Science 337, 710 (2012).
[16] S. Kasahara, H. J. Shi, K. Hashimoto, S. Tonegawa, Y. Mizukami, T. Shibauchi, K. Sugimoto, T. Fukuda, T. Terashima, A. H. Nevidomskyy, and Y. Matsuda, Nature 486, 382 (2012).
[17] Y. Gallais, R. M. Fernandes, I. Paul, L. Chauviere, Y.-X. Yang, M.-A. Measson, M. Cazayous, A. Sacuto, D. Colson, and A. Forget, Phys. Rev. Lett. 111, 267001 (2013).
[18] X. Lu, J. T. Park, R. Zhang, H. Luo, A. H. Nevidomsky, Q. Si, and P. Dai, Science 345, 657 (2014).
[19] E. P. Rosenthal, E. F. Andrade, C. J. Arguello, R. M. Fernandes, L. Y. Xing, X. C. Wang, C. Q. Jin, A. J. Millis, and A. N. Pasupathy, Nature Phys. 10, 225 (2014).
[20] C. C. Lee, W. G. Yin, and W. Ku, Phys. Rev. Lett. 103, 267001 (2009).
[21] R. Applegate, R. R. P. Singh, C.-C. Chen, and T. P.
I. SADDLE-POINT EQUATIONS FOR THE SVDW ORDER

We start with the effective action for the magnetic order parameters:

\[ S[M_i] = \int_q \chi_s^{-1} (M_1^2 + M_2^2) + \frac{\mu}{2} \int_x (M_1^2 + M_2^2)^2 - \frac{a}{2} \int_x (M_1^2 - M_2^2)^2 + 2\omega \int_x (M_1 \cdot M_2)^2 \]  

(S1)

where \( \int_q \equiv T \sum_{\omega_n} \int \frac{d^d q}{(2\pi)^d} \) and \( \int_x \equiv \int_0^\beta d\tau \int d^d x \). To proceed, we use the identity:

\[ (M_1 \cdot M_2)^2 = \frac{1}{4} \left( M_1^2 + M_2^2 \right)^2 - \frac{1}{4} \left( M_1^2 - M_2^2 \right)^2 - (M_1 \times M_2)^2 \]  

(S2)
yielding:

\[
S[M_i] = \int q \chi_q^{-1} (M^2_1 + M^2_2) + \frac{(u + w)}{2} \int x (M^2_1 + M^2_2)^2 - \frac{(g + w)}{2} \int x (M^2_1 - M^2_2)^2 - 2w \int x (M_1 \times M_2)^2 \quad (S3)
\]

Hereafter for simplicity we introduce the parameters \( \tilde{g} = g + w \) and \( \tilde{u} = u + w \). Since we are interested in the vestigial phase of the spin vortex-crystal, which has tetragonal symmetry, the nematic order parameter \( \langle M^2_1 \rangle - \langle M^2_2 \rangle \) never condenses, and we can ignore the corresponding quartic term. Introducing the Hubbard-Stratonovich fields corresponding to the other two quadratic terms, we obtain:

\[
e^{-\frac{1}{2}(M^2_1 + M^2_2)^2} = \mathcal{N} \int d\psi e^{\frac{\psi^2}{2} - \psi(M^2_1 + M^2_2)}
\]

\[
e^{2w(M_1 \times M_2) \cdot (M_1 \times M_2)} = \mathcal{N} \int d\varphi_{\text{SVDW}} e^{-\frac{\varphi^2_{\text{SVDW}}}{2w}} + 2\varphi_{\text{SVDW}} (M_1 \times M_2)
\quad (S4)
\]

Here, \( \varphi_{\text{SVDW}} \) is the spin-vorticity density-wave (SVDW) vectorial order parameter whose mean value is given by \( \langle \varphi_{\text{SVDW}} \rangle = 2w \langle (M_1 \times M_2) \rangle \). The field \( \psi \) is not an order parameter, and just renormalizes the magnetic correlation length via \( \langle \psi \rangle = \tilde{u} \langle (M^2_1 + M^2_2) \rangle \), i.e. it corresponds to Gaussian magnetic fluctuations. Thus, the effective action is given by:

\[
S[M, \psi, \varphi_{\text{SVDW}}] = \int q (\chi_q^{-1} + \psi) (M^2_1 + M^2_2) - 2 \int x \varphi_{\text{SVDW}} \cdot (M_1 \times M_2) + \frac{\varphi^2_{\text{SVDW}}}{2w} - \frac{\psi^2}{2\tilde{u}}
\quad (S5)
\]

Approaching the SVDW phase from the paramagnetic state, we can integrate out the magnetic degrees of freedom, yielding an effective action for \( \psi \) and \( \varphi \):

\[
S_{\text{eff}}[\psi, \varphi_{\text{SVDW}}] = \frac{\varphi^2_{\text{SVDW}}}{2w} - \frac{\psi^2}{2\tilde{u}} + \frac{1}{2} \int_q \log \left( \prod \lambda_{i,q} \right)
\quad (S6)
\]

where \( \lambda_{i,q} \) are the eigenvalues of the matrix \( A_{ij} \) corresponding to the Gaussian action in \( M_i \). The Gaussian part of the action can be rewritten in the convenient matrix form:

\[
\begin{pmatrix}
\chi_q^{-1} + \psi & 0 & 0 & 0 & -\varphi_z & \varphi_y \\
0 & \chi_q^{-1} + \psi & 0 & \varphi_z & 0 & -\varphi_x \\
0 & 0 & \chi_q^{-1} + \psi & -\varphi_y & \varphi_x & 0 \\
0 & \varphi_z & -\varphi_y & \chi_q^{-1} + \psi & 0 & 0 \\
-\varphi_z & 0 & \varphi_x & 0 & \chi_q^{-1} + \psi & 0 \\
\varphi_y & -\varphi_x & 0 & 0 & 0 & \chi_q^{-1} + \psi \\
\end{pmatrix}
\begin{pmatrix}
M_1 \\
M_2
\end{pmatrix}
\quad (S7)
\]

Evaluation of the eigenvalues gives:

\[
S_{\text{eff}}[\psi, \varphi_{\text{SVDW}}] = \frac{\varphi^2_{\text{SVDW}}}{2w} - \frac{\psi^2}{2\tilde{u}} + \int_q \log \left[ (\chi_q^{-1} + \psi) (\chi_q^{-1} + \psi + \varphi_{\text{SVDW}}) (\chi_q^{-1} + \psi - \varphi_{\text{SVDW}}) \right]
\quad (S8)
\]

So far our result is exact. To proceed, we employ the saddle-point approximation to determine the equations of state for \( \psi \) and \( \varphi_{\text{SVDW}} \), which corresponds to self-consistently accounting for the Gaussian magnetic fluctuations. The saddle-point equations become:

\[
\frac{\varphi_{\text{SVDW}}}{w} = \int_q \frac{1}{\chi_q^{-1} + \psi - \varphi_{\text{SVDW}}} - \int_q \frac{1}{\chi_q^{-1} + \psi + \varphi_{\text{SVDW}}}
\]

\[
\frac{\psi}{\tilde{u}} = \int_q \frac{1}{\chi_q^{-1} + \psi - \varphi_{\text{SVDW}}} + \int_q \frac{1}{\chi_q^{-1} + \psi + \varphi_{\text{SVDW}}} + \int_q \frac{1}{\chi_q^{-1} + \psi}
\quad (S9)
\]
Since our focus is on the proximity to a finite-temperature magnetic transition, we ignore the spin dynamics and use the low-energy expansion for the spin susceptibility appropriate for anisotropic layered systems:

\[ \chi_q^{-1} = r_0 + q_z^2 + J_z \sin^2 \frac{q_z}{2} \]  

(S10)

where \( r_0 = a(T - T_N) \), \( a > 0 \), \( T_N \) is the mean-field magnetic transition temperature, \( q_z^2 = q_x^2 + q_y^2 \), and \( J_z \) is the inter-layer magnetic coupling. Defining the renormalized magnetic mass:

\[ r = r_0 + \psi \propto \xi^{-2} \]  

(S11)

where \( \xi \) is the magnetic correlation length, we obtain:

\[ \varphi_{\text{SVDW}} = w \left[ \int \frac{1}{q r + q_z^2 + J_z \sin^2 \frac{q_z}{2} - \varphi_{\text{SVDW}}} - \int \frac{1}{q r + q_z^2 + J_z \sin^2 \frac{q_z}{2} + \varphi_{\text{SVDW}}} \right] \]

(S12)

The integrals can be evaluated in a straightforward way (we consider only the \( \omega_n = 0 \) contribution to the sum over Matsubara frequencies, since we are interested in the finite temperature transition):

\[ \int \frac{1}{q r + q_z^2 + J_z \sin^2 \frac{q_z}{2} + a} = \frac{T_N}{4\pi} \int_0^{2\pi} \frac{dq_z}{2\pi} \int_{J_z \sin^2 \frac{q_z}{2} + a}^{\Lambda^2} \frac{dx}{x} \]

\[ = \frac{T_N}{4\pi} \int_0^{2\pi} \frac{dq_z}{2\pi} \ln \left( \frac{\Lambda^2}{J_z \sin^2 \frac{q_z}{2} + a} \right) \]

\[ = \frac{T_N}{2\pi} \ln 2\Lambda - \ln \left( \sqrt{J_z + a} + \sqrt{a} \right) \]  

(S13)

Defining the renormalized critical temperature \( \tilde{r}_0 = a(T - \tilde{T}_N) \) via:

\[ \tilde{r}_0 = r_0 + \frac{3\tilde{u}T_N}{2\pi} \ln \frac{2\Lambda}{\sqrt{J_z}} \]  

(S14)

we obtain the self-consistent equations:

\[ \varphi_{\text{SVDW}} = \frac{wT_N}{2\pi} \ln \frac{\sqrt{J_z + r + \varphi_{\text{SVDW}} + \sqrt{r + \varphi_{\text{SVDW}}}}}{\sqrt{J_z + r - \varphi_{\text{SVDW}}} + \sqrt{r - \varphi_{\text{SVDW}}}} \]  

(S15)

\[ r = \tilde{r}_0 - \frac{\tilde{u}T_N}{2\pi} \ln \left[ \frac{\left( \sqrt{J_z + r + \varphi_{\text{SVDW}}} \right)^2}{\sqrt{J_z + r - \varphi_{\text{SVDW}}} + \sqrt{r - \varphi_{\text{SVDW}}}} \left( \sqrt{J_z + r + \varphi_{\text{SVDW}}} \right)^2 \right]^2 \]

For simplicity, we define the renormalized parameters \( (\tilde{w}, \tilde{u}) \equiv (w, u) \frac{T_N}{2\pi} \) as well as \( \alpha \equiv \frac{u}{w} + 1 \) and \( \tilde{J}_z \equiv J_z / \tilde{w} \). Then the equations can be written as:

\[ \varphi_{\text{SVDW}} = \ln \frac{\sqrt{J_z + r + \varphi_{\text{SVDW}}} + \sqrt{r + \varphi_{\text{SVDW}}}}{\sqrt{J_z + r - \varphi_{\text{SVDW}}} + \sqrt{r - \varphi_{\text{SVDW}}}} \]  

(S16)

\[ r = \tilde{r}_0 - \alpha \ln \left[ \frac{\left( \sqrt{J_z + r + \varphi_{\text{SVDW}}} \right)^2}{\sqrt{J_z + r - \varphi_{\text{SVDW}}} + \sqrt{r - \varphi_{\text{SVDW}}}} \right]^2 \]

(S17)
where \( r, \tilde{r}_0 \), and \( \varphi_{\text{SVDW}} \) were rescaled by \( \tilde{w} \) as well. The SVDW transition temperature can be obtained by linearizing the equations around \( \varphi_{\text{SVDW}} = 0 \). From the first equation, we obtain the correlation length \( r_1 \) at the SVDW transition:

\[
 r_1 = \frac{\sqrt{\tilde{J}_z^2 + 4 - \tilde{J}_z}}{2}
\]  

(S17)

which, when substituted in the second equation, gives the SVDW transition temperature \( \tilde{r}_{0,\text{SVDW}} \):

\[
 \tilde{r}_{0,\text{SVDW}} = \frac{\sqrt{\tilde{J}_z^2 + 4 + \tilde{J}_z} + \sqrt{\tilde{J}_z^2 + 4 - \tilde{J}_z}}{2 \sqrt{\tilde{J}_z}} + 3\alpha \ln \left( \frac{\sqrt{\tilde{J}_z^2 + 4 + \tilde{J}_z} + \sqrt{\tilde{J}_z^2 + 4 - \tilde{J}_z}}{2 \sqrt{\tilde{J}_z}} \right)
\]  

(S18)

The magnetic transition temperature \( \tilde{r}_{0,\text{mag}} \) is signaled by the vanishing of the renormalized magnetic mass, i.e. the lowest eigenvalue of the Eq. \( \llbracket \text{S7} \rrbracket \), \( r - \varphi_{\text{SVDW}} \). Therefore, it takes place when \( r \) reaches the value \( r_2 \) determined implicitly by:

\[
 r_2 = \ln \frac{\sqrt{\tilde{J}_z^2 + 2r_2} + \sqrt{2r_2}}{\sqrt{\tilde{J}_z}}
\]  

(S19)

The magnetic transition temperature is therefore given by:

\[
 \tilde{r}_{0,\text{mag}} = r_2 (1 + \alpha) + \alpha \ln \left( \frac{\sqrt{\tilde{J}_z^2 + r_2} + \sqrt{r_2}}{\sqrt{\tilde{J}_z}} \right)
\]  

(S20)

The SVDW and magnetic transitions are split when \( \tilde{r}_{0,\text{SVDW}} > \tilde{r}_{0,\text{mag}} \). The region in the \( \left( \tilde{w}, \tilde{J}_z \right) \) parameter space where this condition is satisfied corresponds to the shaded area of Fig. 3 in the main text (recall that \( \frac{\bar{w}}{\bar{J}_z} = \alpha - 1 \)).

To determine the character of the SVDW transition, we can expand \( \tilde{r}_0 \) for small \( \varphi_{\text{SVDW}} \). Substituting \( r = r_1 + a\varphi_{\text{SVDW}} \) in the first equation of \( \llbracket \text{S16} \rrbracket \) and expanding for small \( \varphi_{\text{SVDW}} \) gives the coefficient of the quadratic term:

\[
 a = \frac{8 + 3\tilde{J}_z^2}{12\sqrt{\tilde{J}_z^2 + 4}}
\]  

(S21)

Substituting it in the second equation of \( \llbracket \text{S16} \rrbracket \) and collecting the quadratic terms in \( \varphi_{\text{SVDW}} \) yields:

\[
 \tilde{r}_0 (\varphi_{\text{SVDW}}) \approx \tilde{r}_{0,\text{SVDW}} + \left[ \frac{16 + 3\tilde{J}_z^2 (2 + \alpha)}{24\sqrt{\tilde{J}_z^2 + 4}} \right] \varphi_{\text{SVDW}}^2
\]  

(S22)

Therefore, because the coefficient is always positive, the solution with \( \varphi_{\text{SVDW}} \neq 0 \) is achieved at a larger temperature than the solution with \( \varphi_{\text{SVDW}} = 0 \). As a result, the SVDW transition is first-order within the saddle-point approximation, even when it is split from the magnetic transition.

II. MICROSCOPIC DERIVATION OF THE GINZBURG-LANDAU FREE-ENERGY

Our starting point is a 3-band model with a circular hole pocket \( h \) centered at \((0, 0)\) and two elliptical electron pockets \( e_1, e_2 \) centered at \( Q_1 = (\pi, 0) \) and \( Q_2 = (0, \pi) \), respectively. The band dispersions can be conveniently parametrized by \( \llbracket 8 \rrbracket \):

\[
\begin{align*}
\xi_{h, \mathbf{k}} &= -\xi_{\mathbf{k}} = -\frac{k^2}{2m} + \varepsilon_0 \\
\xi_{e_1, \mathbf{k}+Q_1} &= \xi_{\mathbf{k}} = (\delta_0 + \delta_2 \cos 2\theta) \\
\xi_{e_2, \mathbf{k}+Q_2} &= \xi_{\mathbf{k}} = (\delta_0 - \delta_2 \cos 2\theta)
\end{align*}
\]  

(S23)
Here, $\delta_0$ is proportional to the chemical potential and $\delta_2$ to the ellipticity of the electron pockets. The angle $\theta$ is measured relative to the $k_x$ axis. The non-interacting Hamiltonian is therefore given by (hereafter sums over repeated spin indices are implicitly assumed):

$$H_0 = \sum_k \xi_{h,k} c_{h,k \sigma}^\dagger c_{h,k \sigma} + \sum_k \xi_{e1,k} c_{e1,k \sigma}^\dagger c_{e1,k \sigma} + \sum_k \xi_{e2,k} c_{e2,k \sigma}^\dagger c_{e2,k \sigma}$$

(S24)

These electronic states couple to the magnetic order parameters $M_1$ and $M_2$ according to:

$$H_{\text{mag}} = \sum_{k,i} M_i \cdot \left( c_{e1,k \alpha}^\dagger \sigma_{\alpha \beta} c_{h,k \beta} + \text{h.c.} \right)$$

(S25)

In principle, this last term can be obtained via a Hubbard-Stratonovich transformation of the original interaction terms projected into the magnetic channel, as shown in Ref. 8. Here, because we are interested in the higher-order couplings of the action involving the $M_i$ order parameters, we neglect these interaction terms, since they only affect the quadratic terms of the action.

### A. Absence of magnetic field

In the case where there is no external magnetic field, we focus on the two types of fermionic order that couple directly to the SVDW order parameter, $M_1 \times M_2$, and to the CDW order parameter $M_1 \cdot M_2$. Thus, we introduce the $Q_1 + Q_2 = (\pi, \pi)$ spin-current density-wave $\Delta_{\text{SDW}}$ ($\Delta_{\text{CDW}}$ in the notation of the main text) and the checkerboard charge order $\Delta_C$ ($\Delta_{\text{C}}$ in the notation of the main text) defined by:

$$\mathcal{H}_{\text{iSD}} = \sum_k \Delta_{\text{iSD}} \cdot \sigma_{\alpha \beta} \left( c_{e2,k \alpha}^\dagger c_{e1,k \beta} - c_{e1,k \alpha}^\dagger c_{e2,k \beta} \right)$$

$$\mathcal{H}_{\text{CDW}} = \sum_k \Delta_{\text{CDW}} \sigma_{\alpha \beta} \left( c_{e2,k \alpha}^\dagger c_{e1,k \beta} + c_{e1,k \alpha}^\dagger c_{e2,k \beta} \right)$$

(S26)

To proceed, we introduce the 6-dimensional Nambu operator:

$$\hat{\Psi}_k = \begin{pmatrix} c_{h,k \uparrow}^\dagger & c_{h,k \downarrow}^\dagger & c_{e1,k \uparrow}^\dagger & c_{e1,k \downarrow}^\dagger & c_{e2,k \uparrow}^\dagger & c_{e2,k \downarrow}^\dagger \end{pmatrix}$$

(S27)

which allows us to write the fermionic action in the compact form:

$$S = -\int_k \hat{\Psi}_k^\dagger \hat{G}_k^{-1} \hat{\Psi}_k + S_0 \left[ M_i^2 \right]$$

(S28)

In the previous expression, $S_0 \left[ M_i^2 \right]$ corresponds to the terms $M_i^2$ that arise from the decoupling of the fermionic interactions. As we explained above, these terms can be ignored for our purposes. The total Green’s function is given by:

$$\hat{G}_k^{-1} = \left( \hat{G}_k^{(0)} \right)^{-1} - \hat{V}_{\text{mag}} - \hat{V}_{\text{iSD}} - \hat{V}_{\text{CDW}}$$

(S29)

The bare part is:

$$\hat{G}_k^{(0)} = \begin{pmatrix} G_{h,k} & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{h,k} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{e1,k} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{e1,k} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{e2,k} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{e2,k} \end{pmatrix}$$

(S30)
where \( G_{i,k}^{-1} = i\omega_n - \xi_{i,k} \) are the non-interacting single-particle Green’s functions. The interacting parts are:

\[
\hat{V}_{\text{mag}} = \begin{pmatrix}
0 & 0 & -M_{1,z} & -M_{1,x} - iM_{1,y} & -M_{2,z} & -M_{2,x} + iM_{2,y} \\
0 & 0 & -M_{1,z} - iM_{1,y} & M_{1,z} & 0 & 0 \\
-M_{1,x} - iM_{1,y} & M_{1,z} & 0 & 0 & 0 & 0 \\
-M_{2,z} & 0 & 0 & 0 & 0 & 0 \\
-M_{2,x} + iM_{2,y} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(S31)

and:

\[
\hat{V}_{\text{S}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\Delta_{S,z} & i(\Delta_{S,x} - i\Delta_{S,y}) & -i\Delta_{S,z} \\
0 & 0 & i(\Delta_{S,x} + i\Delta_{S,y}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(S32)

as well as:

\[
\hat{V}_{C} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\Delta_C & 0 & 0 \\
0 & 0 & 0 & 0 & -\Delta_C & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(S33)

It is now straightforward to integrate out the fermions and obtain the effective magnetic action:

\[
S_{\text{eff}}[M_1, M_2, \Delta_{IS}, \Delta_C] = -\text{Tr} \ln \left[ 1 - \hat{\Theta}_0 \left( \hat{V}_{\text{mag}} + \hat{V}_{\text{S}} + \hat{V}_{C} \right) \right] \approx \sum_n \frac{1}{n} \text{Tr} \left[ \hat{\Theta}_0 \left( \hat{V}_{\text{mag}} + \hat{V}_{\text{S}} + \hat{V}_{C} \right) \right]^n
\]

(S34)

where, in the last step, we expanded for small \( M_1, M_2 \). Here, \( \text{Tr}(\cdots) \) refers to sum over momentum, frequency and Nambu indices. A straightforward evaluation gives, to leading order in the coupling between \( \Delta_{IS}, \Delta_C \), and \( M_1 \):

\[
S_{\text{eff}}[M_1, M_2, \Delta_S] = S[M_1, M_2] + \lambda \Delta_{IS} \cdot (M_1 \times M_2) - \lambda \Delta_C (M_1 \cdot M_2)
\]

(S35)

with the coefficient:

\[
\lambda = 4 \int \frac{d^2k}{2\pi^2} G_{\psi,k} G_{\varphi_1,k} G_{\varphi_2,k}
\]

(S36)

For perfect nesting, \( \delta_0 = \delta_2 = 0 \), this coefficient vanishes. For a system in proximity to a finite temperature phase transition, expansion in powers of \( \delta_0 \) gives:

\[
\lambda \approx 4\rho_F T \sum_n \int d\xi \frac{1}{(i\omega_n + \xi)(i\omega_n - \xi + \delta_0)\xi} \\
= 8\delta_0 \rho_F T \sum_n \int d\xi \frac{1}{(i\omega_n + \xi)(i\omega_n - \xi)^3} \\
= -\frac{\delta_0}{T} \frac{\zeta(3) \rho_F}{2\pi^2 T}
\]

(S37)

where \( \rho_F \) is the density of states at the Fermi level. Therefore, it is clear that a spin-current density-wave \( \Delta_{IS} \) parallel to \( \varphi_{\text{SVDW}} \) is triggered by the SVDW order parameter, \( \varphi_{\text{SVDW}} \propto M_1 \times M_2 \), whereas a checkerboard charge order \( \Delta_C \) is triggered by the CDW order parameter \( \varphi_{\text{CDW}} \propto M_1 \cdot M_2 \).
B. Non-zero magnetic field

In the presence of a magnetic field, additional types of fermionic order are triggered by the condensation of the SVDW and CDW order parameters. To show that, we first introduce the Zeeman coupling between the uniform field $H$ and the electrons:

$$H_{\text{Zeeman}} = \sum_{k,i} \mathbf{H} \cdot \sigma_{\alpha\beta} c_{i,k\alpha}^\dagger c_{i,k\beta}$$  \hspace{1cm} (S38)

We also introduce the charge-current density-wave $\Delta_{iC}$ ($\Delta''_{iC}$ in the notation of the main text) and the spin density-wave $\Delta_{S}$ ($\Delta''_{S}$ in the notation of the main text) defined by:

$$H_{\Delta_{iC}} = \sum_{k} \Delta_{iC} \delta_{\alpha\beta} \left( c_{e2,k\alpha}^\dagger c_{e1,k\beta} - c_{e1,k\alpha}^\dagger c_{e2,k\beta} \right)$$

$$H_{\Delta_{S}} = \sum_{k} \Delta_{S} \cdot \sigma_{\alpha\beta} \left( c_{e1,k\alpha}^\dagger c_{e1,k\beta} + c_{e1,k\alpha}^\dagger c_{e2,k\beta} \right)$$  \hspace{1cm} (S39)

Following the same steps as in the previous subsection, we obtain the expanded action:

$$S_{\text{eff}}[M_1, M_2, \Delta_{S}, \Delta_{iC}] \approx \sum_{n} \frac{1}{n} \text{Tr} \left[ \hat{G}_0 \left( \hat{V}_{\text{mag}} + \hat{V}_{S} + \hat{V}_{iC} + \hat{V}_{\text{Zeeman}} \right) \right]^n$$  \hspace{1cm} (S40)

where the Nambu-space matrices are given by:

$$\hat{V}_{\text{Zeeman}} = \begin{pmatrix} -H_z & -H_x + iH_y & 0 & 0 & 0 & 0 \\ -H_x - iH_y & H_z & 0 & 0 & 0 & 0 \\ 0 & 0 & -H_z & -H_x + iH_y & 0 & 0 \\ 0 & 0 & H_x - iH_y & h_z & 0 & 0 \\ 0 & 0 & 0 & 0 & -H_z & -H_x + iH_y \\ 0 & 0 & 0 & 0 & H_x - iH_y & H_z \end{pmatrix}$$  \hspace{1cm} (S41)

and:

$$\hat{V}_{iC} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\Delta_{iC} & 0 \\ 0 & 0 & 0 & 0 & 0 & i\Delta_{iC} \\ 0 & 0 & -i\Delta_{iC} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\Delta_{iC} & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (S42)

as well as:

$$\hat{V}_{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta_{S,z} & -\left( \Delta_{S,x} - i\Delta_{S,y} \right) \\ 0 & 0 & 0 & 0 & -\left( \Delta_{S,x} + i\Delta_{S,y} \right) & \Delta_{S,z} \\ 0 & 0 & -\Delta_{S,z} & -\left( \Delta_{S,x} - i\Delta_{S,y} \right) & 0 & 0 \\ 0 & 0 & -\left( \Delta_{S,x} + i\Delta_{S,y} \right) & \Delta_{S,z} & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (S43)

A straightforward evaluation yields, to leading order in the magnetic field:
\[ S_{\text{eff}} = S_{\text{eff}} [H = 0] + \zeta \left[ (H \cdot M_1)^2 + (H \cdot M_2)^2 \right] \]
\[ + \gamma [\Delta_{iC}H \cdot (M_1 \times M_2) + (H \cdot \Delta_S) (M_1 \cdot M_2)] \]
\[ + \eta [(M_1 \cdot H) (M_2 \cdot \Delta_S) + (M_2 \cdot H) (M_1 \cdot \Delta_S)] \]  

where we neglected all isotropic biquadratic terms of the form \( H^2 M_i^2 \). The coefficients are given by:

\[ \zeta = 4 \int_k G_{h,k}^2 G_{e_1,k} \]
\[ \gamma = 4 \int_k G_{h,k} G_{e_1,k} G_{e_2,k} (G_{e_1,k} + G_{e_2,k} - G_{h,k}) \]
\[ \eta = 4 \int_k G_{h,k}^2 G_{e_1,k} G_{e_2,k} \]  

It is useful to perform an expansion around perfect nesting, \( \delta_0 = \delta_2 = 0 \). The coefficients \( \zeta \) and \( \eta \) become identical in this limit:

\[ \zeta = \eta = \frac{\rho_F}{T^2} \left( \frac{7 \zeta (3)}{2 \pi^2} \right) \]  

The fact that \( \zeta > 0 \) implies that the magnetic field induces an easy plane, rather than an easy axis anisotropy. As for the coefficient \( \eta \), it remains zero in all orders in perturbation theory if an infinite bandwidth is assumed. However, keeping the top of the hole pocket \( W \) (or bottom of the electron pocket) throughout the calculation gives:

\[ \gamma \approx \frac{\rho_F}{T^2} \left( \frac{W}{T} \right)^{-2} \]  

The fact that \( \gamma \neq 0 \) implies that, in the presence of a uniform field, the SVDW order parameter \( \varphi_{\text{SVDW}} \propto M_1 \times M_2 \) also triggers a charge-current density-wave \( \Delta_{iC} \), whereas the CDW order parameter \( \varphi_{\text{CDW}} \propto M_1 \cdot M_2 \) triggers a spin density-wave of same period, \( \Delta_S \). Although this was expected by symmetry, here we have microscopic expressions for the corresponding Ginzburg-Landau coefficients. It is interesting then to compare the coefficient \( \gamma \) in Eq. \( (S44) \), which determines the amplitudes of \( \Delta_{iC} \) and \( \Delta_S \), to the coefficient \( \lambda \) in Eq. \( (S35) \), which determines the amplitudes of \( \Delta_{iS} \) and \( \Delta_C \). We find that:

\[ \frac{\gamma H}{\lambda} \approx -2.3 \left( \frac{T^2 H}{W^2 \delta_0} \right) \]  

Therefore, for pnictides whose band dispersions do not deviate strongly from perfect nesting, and whose bandwidths are not too large either, it is conceivable that the two coupling constants \( \gamma H \) and \( \lambda \) will be of similar order for moderate values of the magnetic field \( H \). As a result, the charge-current density-wave and the spin density-wave generated in the presence of the field could be as large as the spin-current density-wave and the charge density-wave generated in the absence of the field.