Phase space analysis of quintessence fields trapped in a Randall–Sundrum braneworld: a refined study

Dagoberto Escobar\(^1\), Carlos R Fadragas\(^2\), Genly Leon\(^{3,4}\) and Yoelsy Leyva\(^5\)

\(^1\) Departamento de Física, Universidad de Camagüey, Camagüey, Cuba
\(^2\) Departamento de Física, Universidad Central de Las Villas, 54830 Santa Clara, Cuba
\(^3\) Departamento de Matemática, Universidad Central de Las Villas, 54830 Santa Clara, Cuba
\(^4\) Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile
\(^5\) División de Ciencias e Ingeniería de la Universidad de Guanajuato, AP 150, 37150 León, Guanajuato, México

E-mail: dagoberto.escobar@reduc.edu.cu, fadragas@uclv.edu.cu, genly@uclv.edu.cu and yoelsy.leyva@fisica.ugto.mx

Received 25 January 2012, in final form 13 July 2012
Published 8 August 2012
Online at stacks.iop.org/CQG/29/175005

Abstract
In this paper, we investigate, from the perspective of dynamical systems, the evolution of a scalar field with an arbitrary potential trapped in a Randall–Sundrum’s braneworld of type II. We consider an homogeneous and isotropic Friedmann–Robertson–Walker brane filled also with a perfect fluid. Center manifold theory is employed to obtain sufficient conditions for the asymptotic stability of the de Sitter solution. We obtain conditions on the potential for the stability of scaling solutions as well for the stability of the scalar-field-dominated solution. We prove the fact that there are not late-time attractors with 5D-modifications (they are saddle like). This fact correlates with a transient primordial inflation. In the particular case of a scalar field with the potential \(V = V_0 e^{-\chi \phi} + \Lambda\), we prove that for \(\chi < 0\) the de Sitter solution is asymptotically stable. However, for \(\chi > 0\), the de Sitter solution is unstable (of saddle type).

PACS numbers: 04.20.−q, 04.20.Cv, 04.20.Jb, 04.50.Kd, 11.25.−w, 11.25.Wx, 95.36.+x, 98.80.–k, 98.80.Bp, 98.80.Cq, 98.80.Jk

1. Introduction
The Randall–Sundrum brane of type-II model (RS2), introduced originally as an alternative mechanism to the Kaluza–Klein compactifications [1], have intensively been studied over the last few years, among other reasons, because of its appreciable cosmological impact in the inflationary scenario [2–4]. The setup of the model starts with the particles of the standard model confined in a four-dimensional hypersurface with positive tension embedded in a five-dimensional bulk with a negative cosmological constant. It is well known that the cosmological field equations on the brane are essentially different from the standard
four-dimensional cosmology [5–7]. In fact, the appearance of a quadratic term of the total energy density in the Friedmann equation is responsible for gravitational modifications at very high energy. The dynamics of the Universe during the quadratic dominant stage have been studied by several authors. In [8, 9], it is shown that the (inverse) power-law potential model allows wide conditions for a successful quintessence scenario in contrast to the exponential potential and k-essence models which do not present favorable scenarios. The stability of the scaling solutions for the case of a power-law potential model in the presence of a perfect fluid with arbitrary barotropic index γ was developed in [10]. This study was extended by [11] to a generalized background $H^2 \propto \rho^n_T$ for an arbitrary n (the RS2 case is recovery when $n = 2$). Another interesting and recent feature of this scenario is that the fate of the cosmic expansion can be modified if the energy density of some matter component grows as the expansion proceeds [12].

Several astrophysical observations, such as type Ia supernovae [13–15], large scale structure [16] and cosmic microwave background [17–19], strongly confirm that our universe currently experiences an accelerated expansion phase. Several models, based on RS2 framework, have been proposed in order to deal with these feature of our universe. One approach for explaining the accelerated expansion is the modified Chaplygin gas [20]. For these models, it was showed that the Universe follows a power-law expansion around the critical points. Another important approach consists in adding a self-interacting scalar field to the matter inventory in the brane [21, 8, 22–24]. Scalar fields arise, in a natural way, in particle physics and they can act as candidates for dark energy playing the roles of quintessence, phantom, quintom, tachyonic field, etc [25].

The dynamical behavior of the scalar field coupled with a barotropic fluid in a spatially flat Friedmann–Robertson–Walker (FRW) universe has been studied by many authors; see, e.g., [26–32]. A natural generalization of [26] is to include higher dimensional behavior (RS2 scenario). This program was carried out in [33] where the corresponding scalar-field potentials were found which lead to an attractor-scaling solution for several energy-density modifications to the Friedmann equation; for the RS2 framework, the potential $V \propto \text{cosech}^2(A\phi)$ was found. The dynamics of a scalar field with constant and exponential potentials were investigated in [34]. These results were extended to a wider class of self-interaction potentials in [35] by the use of a method proposed in [29] supporting the idea that this scenario modifies gravity only at very high energy/short scales (UV modifications only) having an appreciable impact on primordial inflation but does not affecting the late-time dynamics of the Universe. In this paper, we make a step forward with respect to the previous studies by exploring more deeply the dynamics in the phase space associated with this scenario around both hyperbolic and non-hyperbolic critical points. The last subject cannot be consistently studied with the help of linear analysis, but using the center manifold theory. Our claim is that the more interesting solution is the non-hyperbolic critical points, in particular the de Sitter critical points.

In this paper, we employ the center manifold theory for obtaining sufficient conditions for the asymptotic stability of the de Sitter solution and for proving that here are not late-time attractors with 5D-modifications. This fact correlates with a transient primordial inflation. Also, we provide conditions on the potential for the stability of scaling solutions and also for the stability of the scalar-field-dominated solution.

---

6 This kind of modification does not appear if the energy density of the matter content in the brane dilutes with the cosmic expansion as occurs with the common matter sources, e.g., quintessence scalar field, radiation, dust, etc.

7 At early times, where the quadratic energy term dominates, this potential behave as a (inverse) power-law potential being consistent with the previous analysis [8, 9].

8 A word of caution: this claim is not in general true, specially if the energy density of the matter trapped in the brane increases at late times [12].
The rest of the paper is organized as follows. In section 2, we give the essential details of the RS model and deals with the dynamical system analysis; these details also include the center manifold study. We explore the dynamics of a scalar field with an exponential potential plus a cosmological constant trapped in the brane using the previous results in section 3. Section 4 is devoted to the physical discussion of the previous results, while the conclusions are given in section 4.

2. Dynamical systems analysis of the FRW brane

In this section, we will focus our attention on a braneworld model where a scalar field, with an arbitrary self-interaction potential, is trapped on a RS2 brane. In the flat FRW metric, the field equations read [36–39]

\[ H^2 = \frac{1}{3} \rho_T \left( 1 + \frac{\rho_T}{2\lambda} \right) + \frac{2U}{\lambda}, \]

\[ 2\dot{H} = -\left( 1 + \frac{\rho_T}{\lambda} \right) (\dot{\phi}^2 + \gamma \rho_m) - \frac{4U}{\lambda}, \]

\[ \dot{\rho}_m = -3\gamma H \rho_m, \]

\[ \dot{\phi} + \partial_\phi \ln V = -3H\phi, \]

where we have used the RS fine-tuning condition, i.e. we neglect the cosmological constant term ($\Lambda_4 = 0$), $\rho_T = \rho_\phi + \rho_m$, $\lambda$ is the brane tension, $\gamma$ is the barotropic index of the background fluid and $V$ is the scalar-field self-interaction potential. $U(t) = \frac{C}{\alpha_m}$ is the dark radiation term that arises from a nonvanishing bulk Weyl tensor, with $C$ being a constant parameter related with black hole mass in the bulk; if the bulk is AdS–Schwarzschild, $C \neq 0$ [37]. When the black hole mass vanishes, the bulk geometry reduces to AdS, and $C = 0$ [7, 39]. In the following, we will study the latter case, i.e. we will not consider here the dark radiation term. Here and throughout, we use $\partial_\phi$ to denote the derivative with respect to $\phi$.

From the Friedmann equation (1), it is deduced how the brane effects modify the early-time dynamics: at high energy ($\rho_T \gg \lambda$) this equation reduces to $H \propto \rho_T$. At late times, due to the expansion rate, the energy density of the matter trapped in the brane dilutes ($\rho_T \ll \lambda$), and the standard 4D TGR behavior is recovered, leading to $H \propto \sqrt{\rho_T}$.

Having presented the cosmological equations, our purpose now is to define a dynamical system from (1)–(4) in order to examine all possible cosmological behaviors. We know that dynamical system techniques provide one of the better tools for obtaining useful information about the evolution of a wide class of cosmological models9. In order to take advantage of this tool, we introduce the Hubble-normalized variables

\[ x = \frac{\dot{\phi}}{\sqrt{6H}}, \quad y = \frac{V}{3H^2}, \quad \Omega_\lambda = \frac{\rho_\lambda}{6\lambda H^2}, \]

the new temporal variable $\tau = \int H dt$, and the additional dynamical (non-compact) variable $s$ given by

\[ s = -\partial_\phi \ln V(\phi), \]

which is a function of the scalar field.

For the scalar potential treatment, we proceed following [29]. Let us define the scalar function as

\[ f = \Gamma - 1, \quad \Gamma = \frac{V'}{V^2}. \]

9 See, e.g., the seminal work [26], and [40, 41].
system of ordinary differential equations (ODE): cases possible. In table 1, the functions where the comma denotes the derivative with respect to $\tau$.

Using (12), the energy condition $0 \leq x^2 + y + \Omega_\lambda - 1 \leq 1$. Therefore, in general, the treatment of general classes of potentials using an ‘$f$-deviser’ is possible. In table 1, the functions $f(s)$ for some usual quintessence potentials are tabulated. Cases (a)–(c) have been studied in $\Omega_2$ branes in [35].

Bearing this in mind, and using the variables (5)–(6), we deduce the following autonomous system of ordinary differential equations (ODE):

$$x' = \sqrt{\frac{3}{2}}y - 3x + \frac{3(\Omega_\lambda + 1)(y - 2)x^3 + 3y(\Omega_\lambda + 1)(y + \Omega_\lambda - 1)}{2(\Omega_\lambda - 1)}x,$$

$$y' = \frac{3y(\Omega_\lambda + 1)}{(\Omega_\lambda - 1)}[(y - 2)x^2 + y(\Omega_\lambda + y - 1)] - \sqrt{6}xys,$$

$$\Omega'_\lambda = 3\Omega_\lambda[(y - 2)x^2 + y(\Omega_\lambda + y - 1)],$$

$$s' = -\sqrt{6}x^2y f(s),$$

where the comma denotes the derivative with respect to $\tau$.

From the Friedmann equation (1), we have the following relation:

$$\Omega_m = 1 - x^2 - y - \Omega_\lambda.$$  

Using (12), the energy condition $0 \leq \Omega_m \leq 1$ can be written as

$$0 \leq x^2 + y + \Omega_\lambda - 1.$$  

From the definition of $\Omega_\lambda$ and the Friedmann equation, we obtain the useful relation

$$\frac{\rho_\lambda}{\lambda} = \frac{2\Omega_\lambda}{1 - \Omega_\lambda}.$$  

From (16), it follows that the invariant set $\Omega_\lambda = 1$ corresponds to cosmological solutions where $\rho_\lambda \gg \lambda$ (corresponding to the formal limit $\lambda \to 0$). Therefore, they are associated with high-energy regions, i.e. to cosmological solutions in a neighborhood of the initial singularity [10].

Due to the classic nature of model, it is not appropriate for describing the dynamics near the initial singularity, where quantum effects appear. However, from the mathematical viewpoint, this region ($\Omega_\lambda = 1$) is reached asymptotically. In fact, as some numerical integrations corroborate, there exists an open set of orbits in the phase interior that tends to the boundary $\Omega_\lambda = 1$ as $\tau \to -\infty$. Therefore, for mathematical motivations, it is common to attach the boundary

### Table 1. Explicit forms of $f(s)$ for some self-interaction potentials. To homogenize the notations, we use units in which $\kappa^2 = 8\pi G = 1$.

| Label | Potential | $f(s)$ | Reference |
|-------|-----------|--------|-----------|
| (a)   | $V = V_0 \sinh^{-\nu} \chi \phi$ | $\frac{1}{2} - \frac{x^2}{y}$ | [42, 43] |
| (b)   | $V = V_0[(\cosh(\chi \phi) - 1)^\nu]$ | $-\frac{1}{2} + \frac{x^2}{y}$ | [44] |
| (c)   | $V = \frac{V_0}{\Gamma_1(\nu + 2\beta \phi)}$ | $\frac{1}{2} + \frac{y}{x}$ | [45] |
| (d)   | $V = V_0 \sinh^{-\nu} \chi \phi + \Lambda$ | $-1 - \frac{y}{x}$ | [46] |
| (e)   | $V = \frac{V_0 \sin^2 \chi \phi}{2 \phi}$ | $\frac{x^2 + \Lambda_{\phi} + \sqrt{x^2 + \Lambda_{\phi}}}{\Lambda_{\phi}}$ | [47, 48] |
| (f)   | $V = V_0[\exp + e^{y\phi}]$ | $\frac{(x^2 + \Lambda_{\phi})^3}{\Lambda_{\phi}^2}$ | [49] |

Since $\Gamma$ is a function of the scalar field $\Gamma(\phi)$ (see definition (7)), also is the variable $s = S(\phi)$. Assuming that the inverse of $S$ exists, we have $\phi = S^{-1}(s)$. Thus, one can obtain the relation $\Gamma = \Gamma(S^{-1}(s))$, and finally, the scalar-field potential can be parameterized by a function $f(s)$. Thus, in general, the treatment of general classes of potentials using an ‘$f$-deviser’ is possible. In table 1, the functions $f(s)$ for some usual quintessence potentials are tabulated. Cases (a)–(c) have been studied in $\Omega_2$ branes in [35].

10 See [50, 30] for a classical treatment of cosmological solutions near the initial singularity.
Strictly speaking, the system admits one more curve of critical points with coordinates $\Omega_3 = 1$ to the phase space. On the other hand, the points (with $\Omega_3 = 0$) are associated with the standard 4D behavior ($\rho_T \ll \lambda$ or $\lambda \to \infty$) and correspond to the low-energy regime.

From definition (5) and from the restriction (13), and taking into account the previous statements, it is enough to investigate to the flow of (8)–(11) defined in the phase space

$$\Psi = \{(x, y, \Omega) : 0 \leq x^2 + y + \Omega_3 \leq 1, \quad -1 \leq x \leq 1, \\
0 \leq y \leq 1, 0 \leq \Omega_3 \leq 1\} \times \{s \in \mathbb{R}\}. \tag{15}$$

Some cosmological parameters like the equation-of-state parameter of the scalar matter $\omega_\phi = \frac{p_\phi}{\rho_\phi}$ and the deceleration parameter $q = -\left(1 + \frac{\dot{H}}{H^2}\right)$ can be re-expressed as the functions of new variables as follows:

$$\omega_\phi = \frac{x^2 - y}{x^2 + y}, \quad \Omega_\phi = x^2 + y, \tag{16}$$

$$q = \left(\frac{1 + \Omega_3}{1 - \Omega_3}\right) \left[3x^2 + \frac{3y}{2}(1 - x^2 - y - \Omega_3)\right] - 1. \tag{17}$$

### 2.1. Critical points

The system (8)–(11) admits the curves of critical points $P_1, P_2, P_3$; the critical points $P_6^\pm$ and $P_7$; and the classes of critical points $P_6^\pm, P_1$ and $P_5$ parameterized by $s^2$ satisfying $f(s^2) = 0$. In Table 2, the location, existence conditions and some basic observables of these critical points\(^{11}\) are tabulated. The critical points $P_1 - P_6^\pm$ always exist; point $P_5$ exists for $s^2 \geq 3y$, whereas point $P_6^\pm$ exists for $s^2 \leq 6$ with $f(s^2) = 0$.

Now let us comment on the stability of the first-order perturbations of (8)–(11) near the critical points displayed in Table 2. Let us comment briefly on their physical interpretation.

The line of critical points $y = 1 - \Omega_3$, called $P_3$ represents solutions with 5D-corrections, since, in general, $\Omega_3 \neq 0$. From the relationship between $y$ and $\Omega_3$, the fact that this solution is dominated by the potential energy of the scalar field $\rho_T = V(\phi)$ follows, i.e. it is de Sitter-like solution ($\omega_\phi = -1$). In this case, the Friedmann equation can be expressed as

$$3H^2 = V\left(1 + \frac{V}{2\lambda}\right). \tag{18}$$

\(^{11}\) Strictly speaking, the system admits one more curve of critical points with coordinates $x \in [-1, 1], y = -\frac{x^2 - 2}{y}, \Omega_3 = 1, x = 0$, but since the energy condition (13) is not satisfied, we omit it from the analysis.
Table 3. Eigenvalues for the critical points of the equations system (8)–(11). We use the notation 
\[ \beta_{\pm} = \frac{1}{4} \left( y - 2 \pm \sqrt{(2 - y) \left( \frac{24\gamma^2}{\Omega_1^2} - 9\gamma + 2 \right)} \right). \]

| \( P_i \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) |
|---|---|---|---|---|
|\( P_1 \) | 0 | 0 | \(-3\) | \(-3\gamma\) |
|\( P_2 \) | \(-3\gamma\) | \(3\gamma\) | \(\frac{1}{4}(y - 2)\) |
|\( P_3 \) | \(-6\) | \(6\) | \(6\gamma\) |
|\( P_5 \) | 0 | 0 | \(-3\) | \(-3\gamma\) |
|\( P_{n+} \) | \(-6\) | \(6 - 3\gamma\) | \(6 \mp \sqrt{6}s^* \) | \(\mp \sqrt{6}(s^*)^2 f'(s^*)\) |
|\( P_7 \) | \(-3\gamma\) | \(\beta_{\pm}\) | \(-3y^2 s^* f'(s^*)\) |
|\( P_8 \) | \(\frac{1}{2}(s^2 - 6)\) | \(s^2 - 3\gamma\) | \(-s^2\) | \(-s^2 f'(s^*)\) |

In the early universe, where \( \lambda \ll V \), the expansion rate of the universe for the RS model differs from the general relativity predictions

\[ \frac{H_{RS}}{H_{GR}} = \sqrt{\frac{V}{2\lambda}}. \]  

(19)

\( P_1 \) admits a 2D stable manifold \( M_2 \). Due the importance of de Sitter solutions in the cosmological context, in section 2.1.2, we explicitly calculate their center manifold proving that this critical point is locally asymptotically unstable.

Point \( P_2 \) represents a matter-dominated solution (\( \Omega_\Lambda = 1 \)). Although it is non-hyperbolic, it behaves like a saddle point in the space of phase of the RS model, since they have a nonempty stable and unstable manifolds (see table 3).\(^{12}\)

The critical point \( P_3 \) is located at the boundary \( \Omega_\Lambda = 1 \) of the phase space region (15). From the physical viewpoint, this solution represents the Big Bang singularity (\( \rho_T \to \infty \)). The eigenvalues for \( P_3 \) are displayed in table 3. They were calculated for orbits contained completely in the invariant set \( x = y = 0 \) and by taking the limit as \( \Omega_\Lambda \to 1 \). For orbits outside the above invariant set, we cannot make the above limit process since the system is not of class \( C^1 \) at \( \Omega_\Lambda = 1 \). However, several numerical integrations suggest that this solution is, indeed, the past attractor.

The critical points \( P_{\pm}^4 \) are the solutions dominated by the kinetic energy of the scalar field and they represent solutions with a standard behavior (\( \Omega_\Lambda = 0 \)). These critical points are non-hyperbolic. However, they behave as saddle-like points in the space of phase because of the instability in the eigendirection associated with a positive eigenvalue and the stability of an eigendirection associated with a negative eigenvalue.

The critical point \( P_5 \) is a particular case of \( P_1 \) when (\( \Omega_\Lambda = 0 \)). They represent a solution dominated by the potential energy of the scalar field. Indeed, it is a late attractor of Sitter provided \( f(0) > 0 \).\(^{13}\)

The stability analysis of the critical points \( P_{\pm}^6 \), \( P_7 \) and \( P_8 \) is a little more complicated task since the eigenvalues of the linearization matrix do depend on the function \( f(s) \), their zeros, \( s = s^* \), and the value of the first derivative at \( s = s^* \).

The critical points \( P_{\pm}^6 \) are the solutions dominated by the kinetic energy of the scalar field and represent transient states (saddle points in the phase space) in the evolution of the universe for \( y < 2 \). For \( y = 2 \), point \( P_6^+ \) is non-hyperbolic; the stable manifold is 3D provided

\[ s^* > \sqrt{6} \]  

(20)

\(^{12}\) Strictly speaking, the concept of saddle point is not applicable to non-hyperbolic critical points.

\(^{13}\) We have arrived at this conclusion by making the stability analysis of its center manifold (see section 2.1.1 for an explicit computation).
and \( f'(s^*) > 0 \); otherwise, the stable manifold is of dimension less than 3. Similarly, for \( \gamma = 2 \), point \( P_6^* \) is non-hyperbolic; and their stable manifold is 3D provided
\[
s' < -\sqrt{6}
\]
and \( f'(s^*) < 0 \); otherwise, the stable manifold is of dimension less than 3.

The critical points \( P_7 \) are non-hyperbolic for \( s^* \in \left[-\sqrt{3y}, \sqrt{3y}\right] \) or \( s^* f'(s^*) = 0 \) or \( \gamma = 2 \). Points \( P_8 \) represent scalar-field-dominated solutions (\( \Omega_\phi = 1 \)), which are non-hyperbolic provided \( s'^2 \in [0, 3\gamma, 6] \) or \( f'(s^*) = 0 \).

Having presented the eigenvalues of the Jacobian matrix for the critical points \( P_7 \) and \( P_8 \) in Table 3, we straightforwardly formulate the following results.

The sufficient conditions for the asymptotic stability of the matter-scalar-field scaling solution \( (P_7) \) are

1. \( 0 \leq \gamma < 2, s^* < -\sqrt{3y} \) and \( f'(s^*) < 0 \), or
2. \( 0 \leq \gamma \leq 2, s^* > \sqrt{3y} \) and \( f'(s^*) > 0 \).

The sufficient conditions for the asymptotic stability of the scalar-field-dominated solution \( (P_8) \) are either

1. \( 0 \leq \gamma < 2, -\sqrt{3y} < s^* < 0 \) and \( f'(s^*) < 0 \), or
2. \( 0 \leq \gamma \leq 2, 0 < s^* < \sqrt{3y} \) and \( f'(s^*) > 0 \).

### 2.1.1. Dynamics of the center manifold of \( P_5 \)

The solution \( P_5 \) is a particular case of \( P_1 \), which can be extended to being a late-time de Sitter attractor without 5D-corrections (\( \Omega_\Lambda = 0 \)). To analyze its stability, we carry out a detailed stability study of their center manifold using the center manifold theory [32].

Introducing the new variables
\[
x_1 = s, \quad x_2 = \Omega_\Lambda, \quad y_1 = x - \frac{s}{\sqrt{6}}, \quad y_2 = y + \Omega_\Lambda - 1,
\]
and Taylor expanding the evolution equations for the new variables (22), we obtain the vector field
\[
x'_1 = -x_1^2 (x_1 + \sqrt{6} y_1) f(0) + O(4),
\]
\[
x'_2 = \frac{1}{2} x_2 (x_1^2 + 2 \sqrt{6} y_1 x_1 + 6 y_1^2) (y - 2) + 3 x_2 y_2 y + O(4),
\]
\[
y'_1 = -3 y_1 + \frac{1}{2} (-\sqrt{6} x_1 (2 x_2 + y_2 (y - 2)) - 6 y_1 y_2 y)
\]
\[
\quad + \frac{1}{24} (\sqrt{6} (y - 4 f(0) + 2) x_1^3 + 6 y_1 (y - 3 y + 4 f(0) + 6) x_1^2
\]
\[
\quad - 6 \sqrt{6} (3 (y - 2) y_1^2 + 2 x_2 y_2 y_1) x_1
\]
\[
\quad - 36 ((y - 2) y_1 + 2 x_2 y_2 y_1)) + O(4),
\]
and
\[
y'_2 = -\frac{y_1 x_1^2}{2} + (x_2 - \frac{y_2 y}{2}) x_1^2 - \sqrt{6} y_1 (y - 1) x_1 + \sqrt{6} y_1 (x_2 + y_2 y_2) x_1 - 3 y_2 y
\]
\[
\quad - 3 y_2 ((y - 2) y_1^2 + 2 x_2 y_2 y_1) - 3 ((y - 2) y_1^2 + y_2^2 y) + O(4),
\]
where \( O(4) \) denotes error terms of fourth order in the vector norm.

Accordingly to the center manifold theorem, the local center manifold of the origin for the vector field (23)–(26) is given by the graph
\[
W_{bc}(0) = \{(x_1, x_2, y_1, y_2) : y_1 = F(x_1, x_2), \]
\[
y_2 = G(x_1, x_2), x_1^2 + x_2^2 < \delta\},
\]
where \( \delta > 0 \) is a small enough real value.
Deriving each one of the functions in (27) with respect to τ, one can obtain the system of quasi-linear partial differential equations
\[ y_1' = \frac{\partial F}{\partial x_1} y_1 - \frac{\partial F}{\partial x_2} y_2' = 0 \] (28)
\[ y_2' = \frac{\partial G}{\partial x_1} y_1 - \frac{\partial G}{\partial x_2} y_2' = 0. \] (29)
Since we have used Taylor expansions up to third order for obtaining the system (23)–(26), we must seek a solution for (28)–(29) in the following form (see [40, 41, 32]):
\[ F(x_1, x_2) = a_1 x_1^3 + a_2 x_1^2 + a_3 x_1 x_2 + a_4 x_2^2 + a_5 x_1 x_2 + a_6 x_2^2 + a_7 x_1^2 + \mathcal{O}(4) \] (30)
\[ G(x_1, x_2) = b_1 x_1^3 + b_2 x_1^2 + b_3 x_1 x_2 + b_4 x_1 x_2 + b_5 x_1 x_2 + b_6 x_2^2 + b_7 x_1^2 + \mathcal{O}(4), \] (31)
as \( x_i \to 0 \), where \( \mathcal{O}(4) \) is an error term of fourth order in the vector norm. Substituting expressions (30) and (31) into equations (28)–(29), and comparing terms of equal powers, we obtain that the non-null coefficients in the above expressions (30) and (31) are
\[ a_1 = \frac{f(0)}{3\sqrt{6}}, \quad a_2 = -\frac{1}{\sqrt{6}}, \quad b_2 = -\frac{1}{6}, \quad b_3 = \frac{1}{3}, \] (32)
i.e.
\[ y_1 = F(x_1, x_2) = \frac{x_1 x_2}{\sqrt{6}} + \frac{x_1^3 f(0)}{3\sqrt{6}} + \mathcal{O}(4), \]
\[ y_2 = G(x_1, x_2) = -\frac{x_2}{6} + \frac{x_1^2 x_2}{3} + \mathcal{O}(4). \] (33)
Thus, the dynamics on the center manifold are given by
\[ x_1' = -x_1^3 f(0) + \mathcal{O}(4), \] (34)
\[ x_2' = -x_1^2 x_2 + \mathcal{O}(4). \] (35)
Neglecting the error terms, and introducing the coordinate transformation \( u_1 = x_1 \), the system (34)–(35) reduces to the simpler form
\[ u_1' = -2u_1^2 f(0), \] (36)
\[ x_2' = -u_1 x_2, \] (37)
where the region of physical interest is \( u_1 \geq 0, x_2 \geq 0 \).

Observe that the dynamics on the center manifold, governed by (36)–(37), depend on the value \( f(0) \). If either \( f(0) = 0 \) or \( f \) is singular at the origin, the system (36)–(37) does not represent correctly the dynamics of the center manifold. In such a case, we must incorporate higher order terms in the scheme, increasing the problem complexity. Thus, we assume that \( f(0) \) is a real number, such that \( f(0) \neq 0 \).

According to the center manifold theorem, the stability analysis of \( P_3 \) is reduced to the analysis of the stability of the origin of the system (36)–(37). For this analysis, we resort to numerical investigation. In figure 1, several orbits contained in the physical region \( u_1 \geq 0, x_2 \geq 0 \) are displayed. Observe that the axis are invariant sets. For \( f(0) > 0 \) (see figure 1(a)), there is an open set of orbits that converge to the origin as time goes forward; thus, the origin is asymptotic stable for initial conditions in a vicinity of the origin whenever \( f(0) > 0 \). From the asymptotic stability of the origin of (34)–(35), it follows that, for \( f(0) \), the center manifold of \( P_3 \) is locally asymptotic stable and, hence, also the solution \( P_3 \) of the system (8)–(11). Therefore, \( P_3 \) with \( f(0) > 0 \) corresponds to a late-time de Sitter attractor. This result
for RS2 brane cosmology is in a perfect agreement with the standard four-dimensional TGR framework.

2.1.2. Dynamics of the center manifold of $P_1$. In this section, we investigate the stability of the curve of critical points $P_i$ for $0 < \Omega_0 < 1$. This solutions correspond to a de Sitter expansion with 5D-corrections. According to the RS2 model, this solution cannot behave like a late-time attractor since 5D-corrections are typically for the high energy (early universe) regimes and not for low-energy (universe late) regimes. If we can prove that this solution is of saddle type, we can correlate this behavior with a transient inflationary stage for the universe. In order to verify our claim, we appeal to the center manifold theory.

Let us consider an arbitrary critical point with coordinates $(x = 0, y = 1 - u_c, \Omega_0 = u_c, s = 0)$ located at $P_1$. 

---

Figure 1. Phase space of the system (36)–(35) for (a) $f(0) = 1$ and (b) $f(0) = -0.1$. 

---
In order to prepare the system (8)–(11) for the application of the center manifold theorem, we introduce the coordinate change
\[ u_1 = -\frac{s(u_c - 1)}{\sqrt{6}}, \quad u_2 = -\Omega_i - u_c(y + \Omega_i - 2), \]
\[ v_1 = (u_c + 1)(y + \Omega_i - 1), \quad v_2 = \frac{s(u_c - 1)}{\sqrt{6}} + x. \] (38)

Then, we Taylor expand the system \( u_1', u_2', v_1', v_2' \) in a neighborhood of the origin with an error of order \( O(4) \).

According to the center manifold theorem, the local center manifold of the origin for the resulting vector field is given by the graph
\[ W_{loc}^c(0) = \{(u_1, u_2, v_1, v_2) : v_1 = F_1(u_1, u_2), v_2 = G_1(u_1, u_2), u_1^2 + u_2^2 < \delta \} \] (39)
for \( \delta > 0 \) being a small enough real value.

The functions \( F_1 \) and \( G_1 \) in definition (39) are a solution of a system of quasi-linear differential equations analogous to (28)–(29). This system should be solved with an error of order \( O(4) \), obtaining the functional dependence:
\[ v_1 = \frac{2u_1^2u_2(u_c + 1)}{u_c - 1} - u_2^2(u_c + 1), \]
\[ v_2 = \frac{2(u_1^3u_c - u_1^3f(0))}{u_c - 1} - \frac{u_1u_2}{u_c - 1}. \] (40)

Then, the dynamics on the center manifold are given by
\[ u_1' = \frac{6u_1^3f(0)}{u_c - 1} + O(4), \] (41)
\[ u_2' = 6u_1u_2 + \frac{6u_2(1 - 3u_c)u_1^2}{u_c - 1} + O(4). \] (42)

In the same way as for \( P_5 \), the dynamics of the system (41)–(42) depend on the values of \( f(0) \). We assume that \( f(0) \in \mathbb{R} \setminus \{0\} \). Otherwise, it is required to include higher order terms in the Taylor expansion, increasing the numerical complexity.

In figures 2(a) and (b), some orbits are displayed in the phase space of the system (41)–(42) for the choices \( f(0) = 2 \) and \( u_c = 0.5 \), and \( f(0) = -2 \) and \( u_c = 0.5 \), respectively. The origin of coordinates is locally asymptotically unstable (of saddle type) irrespective of the sign of \( f(0) \). Henceforth, the center manifold of \( P_1 \) is locally asymptotic unstable (saddle type) for \( f(0) \neq 0 \), and also is \( P_1 \).

The physical interpretation of this result is that there are not late-time attractors with 5D-modifications. These types of corrections are the characteristics of the early universe. In this sense, the cosmological solution associated with the critical \( P_1 \) correlates with the primordial inflation.

### 3. Exponential potential

The objective of this section is to illustrate our analytical results for the exponential potential:
\[ V(\phi) = V_0 e^{-\chi \phi} + \Lambda. \] (43)

This potential has widely been investigated in the literature. It was studied for quintessence models in [46], where it is considered as a negative cosmological constant \( \Lambda \). In our case of
interest, we assume \( \Lambda \geq 0 \) to avoid dealing with negative values of \( y \). However, we can apply our procedure by permitting negative values for \( y \) for the case \( \Lambda < 0 \). The dark energy models with the exponential potential and negative cosmological constant were baptized as quinstant cosmologies. They were investigated in [51] using an alternative compactification scheme. The asymptotic properties of a cosmological model with a scalar field with the exponential potential have been investigated in the context of general relativity in [29, 26], and in the context of the RS braneworlds in [52, 34]. In both cases, the pure exponential potential (\( \Lambda = 0 \)) was studied. Potentials of exponential orders at infinity were studied in the context of scalar–tensor theories and conformal \( F(R) \) theories in [30, 32].
We comment that the procedure introduced in previous sections is fairly general and can be applied to others potentials as those given in table 1. We continue with the exponential potential for simplicity. Also, we consider a pressure-less (dust) background, i.e. $\gamma = 1$.

The function $f(s)$ corresponding to the potential (43) is given by

$$f(s) = -1 - \frac{\chi}{s}. \quad (44)$$

The zero of this function is

$$s^* = -\chi, \quad f'(s^*) = \frac{\chi}{s^*} = \frac{1}{\chi}. \quad (45)$$

Observe that for the potential (43), $s^* f'(s^*) < 0$. Thus, the only relevant late-time attractor should be the de Sitter solution. In fact, the critical points $P_7$ of table 2 are reduced to the single point

$$P_7 = \left( -\sqrt{\frac{3}{2}} \frac{1}{\chi}, -\frac{3}{2\chi^2}, 0, \chi \right), \quad (46)$$

which represents a saddle point in the phase space. The critical points $P_8$ are reduced to the single one:

$$P_8 = \left( -\sqrt{\frac{3}{2}} \frac{1}{\chi}, -\frac{3}{2\chi^2}, 0, -\chi \right). \quad (47)$$

This point represents a scalar-field-dominated solution ($\Omega_\phi = 1$). It is a saddle point in the phase space. Observe that all the trajectories in the phase space always emerge from the point $(x, y, \Omega_\lambda) = (0, 0, 1)$.

In figure 3, we present, for different choices of the free parameters, two numerical integrations which suggest that $P_5$ is a de Sitter late-time attractor with a standard 4D behavior ($\Omega_\lambda = 0$). However, in order to prove this claim, we need to use the center manifold theory. Although for the potential (43) the result in the appendix does not apply since $f(0) = -\text{sgn}(\chi)\infty$ is not a real number, we can use the same procedure as mentioned in section 3 to obtain the center manifold of $P_5$ by setting from the beginning the functional form of $f(s)$ given in (44).

For $P_5$, the graph of the center manifold is given, up to an error term $O(4)$, by

$$y_1 = \left( \frac{x_1^2 - 1}{3} \frac{x_1}{\sqrt[3]{6}} + \left( \frac{1}{3} \sqrt[3]{3} \frac{x_2^2 - 1}{\sqrt[3]{6}} \right) x_1^2 - \frac{x_2 x_1}{\sqrt{6}},
\quad (48)$$

where we have introduced the variables given in (22).

The dynamics on the center manifold are governed by the equations

$$x_1' = \left( 1 - \frac{x_2^2}{3} \right) x_1^2 + \chi (1 - x_2) x_1^2 + O(4), \quad (49)$$

$$x_2' = -x_1^2 x_2 + O(4), \quad (50)$$

defined in the phase plane $x_1 \in \mathbb{R}, x_2 \in [0, 1]$.

In general, the system (8)-(11) is not invariant under the change $(s, x) \to (-s, -x)$ unless $f(s)$ is an even function, i.e. $f(-s) = f(s)$. However, in the particular case of the potential (43), we observe from (44) that the system (8)-(11) is invariant under the discrete symmetry $(s, x, \chi) \to (-s, -x, -\chi)$. It is easy to show that for the exponential case the symmetry
Figure 3. Some orbits in the projection $(x, y, \Omega_1)$ of the phase space for (8)–(11) and potential $V(\phi) = V_0 e^{-\chi \phi} + \Lambda$ for the choices (a) $(\gamma, \chi) = (1, 0.5)$ and (b) $(\gamma, \chi) = (1, 100)$.

\[(s, x, \chi) \rightarrow (-s, -x, -\chi)\] is induced by the discrete symmetry $(\phi, \chi) \rightarrow (-\phi, -\chi)$. On the other hand, for the potential (43), the function $s(\phi)$ is given by

$$s(\phi) = \frac{V_0 \chi}{V_0 + e^{\phi} \Lambda}.$$ 

Assuming $V_0 > 0$ and $\Lambda > 0$, we have $\text{sgn}(s) = \text{sgn}(\chi)$. As we see, the function $s(\phi)$ is not defined for all the real values, i.e. the region of physical interest is restricted to one semiplane $s \leq 0$ (or in the degenerate case $\chi = 0$ to the hyperplane $s = 0$) depending of the sign of $\chi$. That is, $s$ is either positive, negative or zero, for $\chi$ either positive, negative or zero, provided $V_0 > 0$ and $\Lambda > 0$. This fact limits the application of the above discrete symmetry. In figures 4(a) and (b), some orbits of the flow of (49)–(50) for the choices $\chi = +0.5$ and $\chi = -0.5$ are displayed, respectively. In both cases, the $x_2$-axis is invariant so the orbits cannot cross through it, i.e. the orbits with $s(0) \leq 0$ at the initial time remain in the respective semiplane all the time. Also, the vector field displayed in the right (resp. left) semiplane of
Figure 4. Some orbits of the flow of (49)–(50) for the choices (a) $\chi = 0.5$ and (b) $\chi = -0.5$. In both cases, the $x_2$-axis is invariant. In (a) the physical region is the semiplane $x_1 > 0$; for orbits with $s(0) > 0$, the origin is unstable to perturbations in the $s$-direction. In (b) the physical region is the semiplane $x_1 < 0$. Thus, for orbits with $s(0) < 0$, the origin (hence $P_5$) is asymptotically stable.

Figure 4(a) is the same vector field displayed in the left (resp. right) semiplane of figure 4(b). Thus, the discrete symmetry $(s, x, \chi) \rightarrow (-s, -x, -\chi)$ is preserved. However, relative to the axis $x_2$, the vector field in the right semiplane of figure 4(a) points away from the origin, and the vector field in the left semiplane of figure 4(b) points toward the origin. From the physical viewpoint, for the choice $\chi = 0.5$, the physical region is the right semiplane $x_1 = s > 0$. Thus, the physical orbits depart from the origin as the time goes forward. This means that the origin is unstable. For the choice $\chi = -0.5$, the physical region is the left semiplane $x_1 = s < 0$; thus, the physical orbits approach the origin as the time goes forward.

Summarizing, for $V_0 > 0$ and $\Lambda > 0$, we have $\text{sgn}(s) = \text{sgn}(\chi)$. Thus, for the potentials (43) with $\chi < 0$, the de Sitter solution ($P_5$) is asymptotically stable. This fact is self-consistent with the condition $f(0) = -\text{sgn}(\chi)\infty > 0$, i.e. the sufficient condition for the stability of
the de Sitter solution derived in section 2.1.1. However, for the potentials (43) with \( \chi > 0 \), the de Sitter solution is unstable (of saddle type) to perturbations in the \( s \)-direction. In this case, the late-time solution is given by the asymptotic configuration \( x = 0, y = 1, \Omega_1 = 0, \phi \to -\infty \).

4. Results and discussion

The main results of this paper can be summarized as follows. The critical point \( P_3 = (0, 0, 1) \) represents a Big Bang singularity. According to our numerical integrations in figures 3, we observe that all the trajectories in the phase space, but a measure zero set, emerge from the vicinity of this point. This result agrees with the previous results obtained in [35]. In [35], the authors use another coordinate system that is equivalent, except diffeomorphisms, to the system used in this paper.

In the particular case of a scalar field with the potential \( V = V_0 e^{-x\phi} + \Lambda \) trapped in the brane, we have proved that for \( \chi < 0 \) the de Sitter solution (\( P_5 \)) is asymptotically stable. However, for \( \chi > 0 \), the origin, i.e. \( P_5 \), is unstable (of saddle type) and the late-time solution is given by the asymptotic configuration \( x = 0, y = 1, \Omega_1 = 0, \phi \to -\infty \). This class of potentials contains the previously studied potentials in [34] with \( \Lambda = 0 \). Thus, our present results generalize those in [34].

In the general case, for potentials satisfying \( f(0) \in \mathbb{R} \), we have the following results. By an explicit computation of the center manifold of \( P_1 \) and of \( P_5 \), we prove the following.

- \( P_1 \) is locally asymptotic unstable (of saddle type) irrespective of the sign of \( f(0) \in \mathbb{R} \setminus \{0\} \). This feature is corroborated in figure 2.
- \( P_5 \) is locally asymptotically stable for \( f(0) > 0 \) and unstable (of saddle type) for \( f(0) < 0 \). This result is illustrated in figures 1 and 4.

The solutions dominated by the kinetic energy of the scalar field \( P_{4\pm} \) and \( P_{6\pm} \) behave like saddle-type solutions. This is a main difference with respect to the standard 4D theory where this type of solutions are always past attractors.

In this general case, the possible late-time attractors are as follows.

- The standard 4D de Sitter solution \( P_3 \) (\( a_0 = -1 \)) whenever \( f(0) > 0 \).
- The matter-scalar-field scaling solution \( P_1 \) (\( \Omega_\phi \sim \Omega_m \)). The sufficient conditions for its asymptotic stability are \( s^* < -\sqrt{3\gamma}, f'(s^*) < 0 \) or \( s^* > \sqrt{3\gamma}, f'(s^*) > 0 \).
- The scalar-field-dominated solution \( P_8 \) (\( \Omega_\phi = 1 \)). The sufficient conditions for its asymptotic stability are \( -\sqrt{3\gamma} < s^* < 0, f'(s^*) < 0 \) or \( 0 < s^* < \sqrt{3\gamma}, f'(s^*) > 0 \).

5. Conclusions

In this paper, we have investigated the phase space of the Randall–Sundrum (RS) braneworlds models with a self-interacting scalar field trapped in the brane with an arbitrary potential. From our numerical experiments, we claim that \( P_3 \) is associated with the Big Bang singularity type. The numerical investigation suggests that it is always the past attractor in the phase space of the RS cosmological models.

Using the center manifold theory, we have obtained sufficient conditions for the asymptotic stability of the de Sitter solution.

We have obtained conditions on the potential for the stability of the scaling solutions and also for the stability of the scalar-field-dominated solution.

We have proved, using the center manifold theory and numerical investigation, that there are not late-time attractors with 5D-modifications since they are always saddle like. This fact correlates with a transient primordial inflation.
In the particular case of a scalar field with the potential \( V = V_0 e^{-\chi\phi} + \Lambda \), we have proved that for \( \chi < 0 \) the de Sitter solution is asymptotically stable. However, for \( \chi > 0 \), the de Sitter solution is unstable (of saddle type).

Acknowledgments

This work was partially supported by PROMEP, DAIP, and by CONACyT, México, under grant 167335; by MECESUP FSM0806, from ministerio de Educación de Chile; and by the National Basic Science Program (PNCB) and Territorial CITMA Project (no 1115), Cuba. DE, CRF and GL wish to thank the MES of Cuba for partial financial support of this investigation. YL is grateful to the Departamento de Física and the CA de Gravitación y Física Matemática for their kind hospitality and their joint support for a postdoctoral fellowship. The authors wish to thank two anonymous referees for their useful comments and for bringing our attention to several references.

References

[1] Randall L and Sundrum R 1999 An alternative to compactification Phys. Rev. Lett. 83 4690–3
[2] Hawkins R M and Lidsey J E 2001 Inflation on a single brane: exact solutions Phys. Rev. D 63 041301
[3] Huey G and Lidsey J E 2001 Inflation, brane worlds and quintessence Phys. Lett. B 514 237–25
[4] Huey G and Lidsey J E 2002 Inflation and brane worlds: degeneracies and consistencies Phys. Rev. D 66 043514
[5] Binetruy P, Deffayet C and Langlois D 2000 Nonconventional cosmology from a brane universe Nucl. Phys. B 565 269–87
[6] Binetruy P, Deffayet C, Ellwanger U and Langlois D 2000 Brane cosmological evolution in a bulk with cosmological constant Phys. Lett. B 477 285–91
[7] Bowcock P, Charmousis C and Gregory R 2000 General brane cosmologies and their global space-time structure Class. Quantum Grav. 17 4745–64
[8] Maeda K-i 2001 Brane quintessence Phys. Rev. D 64 123525
[9] Mizuno S, Maeda K-i and Yamamoto K 2003 Dynamics of scalar field in a brane world Phys. Rev. D 67 023516
[10] Savchenko N Yu and Toporensky A V 2003 Scaling solutions on a brane Class. Quantum Grav. 20 2553–62
[11] Tsujikawa S and Sami M 2004 A unified approach to scaling solutions in a general cosmological background Phys. Lett. B 603 113–23
[12] Garcia-Salcedo R, Gonzalez T, Moreno C and Quiros I 2011 Randall–Sundrum brane cosmology: modification of late-time cosmic dynamics by exotic matter Class. Quantum Grav. 28 105017
[13] Riess A G et al 2007 New Hubble Space Telescope discoveries of type Ia supernovae at \( z \geq 1 \): narrowing constraints on the early behavior of dark energy Astrophys. J. 659 98–121
[14] Davis T M et al 2007 Scrutinizing exotic cosmological models using ESSENCE supernova data combined with other cosmological probes Astrophys. J. 666 716–25
[15] Wood-Vasey W M et al 2007 Observational constraints on the nature of the dark energy: first cosmological results from the ESSENCE supernova survey Astrophys. J. 666 694–715
[16] Tegmark M et al 2004 Cosmological parameters from SDSS and WMAP Phys. Rev. D 69 103501
[17] Jarosik N et al 2011 Seven-year Wilkinson microwave anisotropy probe (WMAP) observations: sky maps, systematic errors and basic results Astrophys. J. Suppl. 192 14
[18] Larson D et al 2011 Seven-year Wilkinson microwave anisotropy probe (WMAP) observations: power spectra and WMAP-derived parameters Astrophys. J. Suppl. 192 16
[19] Komatsu E et al 2011 Seven-year Wilkinson microwave anisotropy probe (WMAP) observations: cosmological interpretation Astrophys. J. Suppl. 192 18
[20] Rudra P, Biswas R and Deb Nath U 2012 Dynamics of modified chaplygin gas in brane world scenario: phase plane analysis Astrophys. Space Sci. 339 53–64
[21] Gonzalez-Diaz P F 2000 Quintessence in brane cosmology Phys. Lett. B 481 353–9
[22] Majumdar A S 2001 From brane assisted inflation to quintessence through a single scalar field Phys. Rev. D 64 083503
[23] Nunes N J and Copeland E J 2002 Tracking quintessential inflation from brane worlds Phys. Rev. D 66 043524
[24] Sami M and Dadhich N 2004 Unifying brane world inflation with quintessence TSPU Vestn. 44N7 25–36
[25] Copeland E J, Sami M and Tsujikawa S 2006 Dynamics of dark energy Int. J. Mod. Phys. D 15 1753–936
[26] Copeland E J, Liddle A R and Wands D 1998 Exponential potentials and cosmological scaling solutions Phys. Rev. D 57 4686–90
[27] Aguirregabiria J M and Lazkoz R 2004 Tracking solutions in tachyon cosmology Phys. Rev. D 69 123502
[28] Lazkoz R, Leon G and Quirós I 2007 Quintom cosmologies with arbitrary potentials Phys. Lett. B 649 103–10
[29] Fang W, Ying L, Zhang K and Hui-Qing L 2009 Exact analysis of scaling and dominant attractors beyond the exponential potential Class. Quantum Grav. 26 155005
[30] Leon G 2009 On the past asymptotic dynamics of non-minimally coupled dark energy Class. Quantum Grav. 26 035008
[31] Leon G, Silveira P and Fadragas C R 2012 Phase-space of flat Friedmann–Robertson–Walker models with both a scalar field coupled to matter and radiation Classical and Quantum Gravity: Theory, Analysis and Applications (New York: Nova Science) (arXiv:1009.0689 [gr-qc])
[32] Leon G and Fadragas C R 2011 Cosmological Dynamical Systems (Saarbrücken: Lambert Academic)
[33] Copeland E J, Lee S-J, Lidsey J E and Mizuno S 2005 Generalised cosmological scaling solutions Phys. Rev. D 71 023526
[34] Gonzalez T, Matos T, Quiros I and Vazquez-Gonzalez A 2009 Self-interacting scalar field trapped in a Randall–Sundrum braneworld: the dynamical systems perspective Phys. Lett. B 676 161–7
[35] Leyva Y, Gonzalez D, Gonzalez T, Matos T and Quiros I 2009 Dynamics of a self-interacting scalar field trapped in the braneworld for a wide variety of self-interaction potentials Phys. Rev. D 80 044026
[36] Langlois D 2003 Brane cosmology: an introduction Prog. Theor. Phys. Suppl. 148 181–212
[37] Brax P and van de Bruck C 2003 Cosmology and brane worlds: a Review Class. Quantum Grav. 20 R201–32
[38] Langlois D 2004 Cosmology of brane-worlds arXiv:astro-ph/0403579
[39] Maartens R 2004 Brane world gravity Living Rev. Rel. 7 7
[40] Coley A 2003 Dynamical Systems and Cosmology (Astrophysics and Space Science Library vol 291) (Dordrecht: Kluwer)
[41] Tavakol R 2005 Introduction to dynamical systems Dynamical Systems in Cosmology (Cambridge: Cambridge University Press) p 360
[42] Sahni V and Starobinsky A A 2000 The case for a positive cosmological Lambda term Int. J. Mod. Phys. D 9 373–444
[43] Urena-Lopez L A and Matos T 2000 A new cosmological tracker solution for quintessence Phys. Rev. D 62 081302
[44] Sahni V and Wang L-M 2000 A new cosmological model of quintessence and dark matter Phys. Rev. D 62 103517
[45] Zhou S-Y 2008 A new approach to quintessence and solution of multiple attractors Phys. Lett. B 660 7–12
[46] Cardenas R, Gonzalez T, Leyva Y, Martin O and Quirós I 2003 A model of the universe including dark energy accounted for by both a quintessence field and a (negative) cosmological constant Phys. Rev. D 67 083501
[47] Brax P and Martin J 1999 Quintessence and supergravity Phys. Lett. B 468 40–5
[48] Brax P and Martin J 2000 The robustness of quintessence Phys. Rev. D 61 103502
[49] Barreiro T, Copeland Edmund J and Nunes NJ 2000 Quintessence arising from exponential potentials Phys. Rev. D 61 127301
[50] Foster S 1998 Scalar field cosmologies and the initial space-time singularity Class. Quantum Grav. 15 3485–504
[51] Leon G, Leyva Y, Saridakis E N, Martin O and Cardenas R 2010 Falsifying field-based dark energy models Dark Energy: Theories, Developments and Implications (New York: Nova Science) (arXiv:0912.0542 [gr-qc])
[52] Goheer N and Dunsby P K S 2003 Exponential potentials on the brane Phys. Rev. D 67 103513