Unitary irreducible representations of SL(2, \mathbb{C})
in discrete and continuous SU(1,1) bases

Florian Conrady
Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada

Jeff Hnybida
Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada and
Department of Physics, University of Waterloo, Waterloo, Ontario, Canada

We derive the matrix elements of generators of unitary irreducible representations of SL(2, \mathbb{C}) with respect to basis states arising from a decomposition into irreducible representations of SU(1,1). This is done with regard to a discrete basis diagonalized by \( J^3 \) and a continuous basis diagonalized by \( K^1 \), and for both the discrete and continuous series of SU(1,1). For completeness we also treat the more conventional SU(2) decomposition as a fifth case. The derivation proceeds in a functional / differential framework and exploits the fact that state functions and differential operators have a similar structure in all five cases. The states are defined explicitly and related to SU(1,1) and SU(2) matrix elements.

I. INTRODUCTION

Unitary irreducible representations of SL(2, \mathbb{C}) play a central role in loop and spin foam quantum gravity \([1, 2]\). States in these representations are used as quanta of a field theory that generates spacetime \([3–5]\). More precisely, spacetime appears in the form of cell complexes that are dual to Feynman diagrams. The states of the unitary irreducible representations of the Lorentz group describe 2–cells and propagate along the lines of diagrams. They interact at vertices to form 4–cells and thereby give rise to the cell complex. The quantum numbers are spins and encode the geometry of 2–cells. A given assignment of spins to all 2–cells is a spin foam. The perturbative expansion results in a sum over cell complexes (Feynman diagrams) and geometries (quantum numbers) and can be seen as a version of Wheeler’s spacetime foam.

Recent years saw considerable progress in this approach to quantum gravity. It was understood how states have to be constrained to reflect the geometry of a 2–cell correctly \([6–9]\). The link between quantum states and classical geometry was greatly clarified through the use of coherent states \([10]\). Remarkably, the simplest possible interaction between these quanta leads to amplitudes that are closely related to Regge geometry \([11–15]\).

In order to encode the full structure of a Lorentzian geometry, one requires both spacelike and timelike 2–cells. As was shown by the authors recently, the latter are implemented by

*Electronic address: fconrady@perimeterinstitute.ca
†Electronic address: jhnybida@perimeterinstitute.ca
certain irreps in the SU(1,1) decomposition of SL(2, C) irreps [14, 17]. The constraints on these irreps were obtained by constructing coherent states that mimic the properties of classical timelike 2–cells. It was required, in particular, that expectation values of SL(2, C) generators behave like classical bivectors of a timelike triangle. It turned out that the usual eigenstates of $J^3$ are not suitable to build such coherent states. Instead we had to use eigenstates of the generator $K^1$ [18] and compute the associated expectation values. In [14] these expectation values were stated without proof. One of the aims of this paper is to present a derivation for these results.

There are several ways to arrive at the action of generators on states of unitary irreps of SL(2, C). The first approach is algebraic and matrix elements are inferred by repeated use of the SL(2, C) commutation relations.

This is the method by which Gelfand and Naimark determined the matrix elements of SL(2, C) generators in the SU(2) decomposition (see [20]). The same matrix elements were derived independently by Harish–Chandra [21]. Sciarrino & Toller [22] and Delbourgo et al. [23] investigated SL(2, C) matrix elements from the perspective of the method of induced representations. They deduced expressions for matrix elements of finite SL(2, C) transformations and for transition functions between SU(2) and SU(1,1) states. By continuing from this point one could also determine infinitesimal expressions. Another possibility is to start from the realization in terms of homogeneous functions of two complex variables (see [24, 25, 26]) and to evaluate the matrix elements of finite transformations by explicit integration. For SU(2) this was done by Strom [26] and Duc & Van Hieu [27]. Finally, one can take the differential approach—represent generators as differential operators and act on state functions with them. This is the method by which Mukunda derived the matrix elements of SL(2, C) generators in the SU(1,1) decomposition, for eigenstates of $J^3$ and integer spin [28].

This is also the strategy followed in the present paper. While based on the same method, our results extend those of ref. [28] in several ways. In addition to the usual basis diagonalized by $J^3$, we compute the matrix elements for a basis of $K^1$ eigenstates\(^1\). Since $K^1$ generates a noncompact subgroup of SU(1,1), this basis is labelled by continuous eigenvalues. Furthermore, we present both the treatment of the multiplicative and derivative part of the operator and find certain corrections to Mukunda’s result\(^2\). We also clarify the definition of the state functions by relating them directly to the $D$–functions of SU(1,1) and SU(2). By means of suitable parametrizations we are able to highlight the common structure present in differential operators and state functions for different choices of basis. Thus, we can reduce the derivation to one main equation and treat several cases at once. With the inclusion of the canonical SU(2) basis we deal in total with five cases that are listed in table (I).

The article is organized as follows. In section I we briefly review basic facts about representations of SL(2, C), SU(2) and SU(1,1) that are needed to understand the rest of the paper. In section II we give explicit definitions of the basis states used to define matrix elements of the SL(2, C) representation. The main result of the paper is stated in sec. IV: the matrix elements of SL(2, C) generators in the SU(1,1) decomposition—in a discrete and continuous basis and for both discrete and continuous series. The derivation of the matrix elements is explained, in some detail, in sec. V. This section also refers to the appendix, where we provide further details on the parametrization of the groups (sec. A), the definition of the Bargmann functions (sec. B) and on the derivation of the main equation of the paper.

\(^1\) As mentioned before, this is the type of states needed to represent timelike quantum triangles.

\(^2\) In [28] the proof is given for the multiplicative term and the result for the total operator is only stated.
Table I: Cases treated in this paper, listed according to group, series and diagonal operator. In the last line, eigenvalues of $K^1$ occur with multiplicity 2.

(see [3]). We conclude with a brief summary and discussion (see [VI]).

II. REPRESENTATION THEORY OF $\text{SL}(2, \mathbb{C})$, $\text{SU}(2)$ AND $\text{SU}(1,1)$

In the defining representation, $\text{SL}(2, \mathbb{C})$ has the generators $J^i = \sigma^i / 2, K^i = i \sigma^i / 2, i = 1, 2, 3$, with commutation relations

\[
[J^i, J^j] = i \epsilon^{ijk} J^k, \quad [J^i, K^j] = i \epsilon^{ijk} K^k, \quad [K^i, K^j] = -i \epsilon^{ijk} K^k. \tag{1}
\]

The subgroups $\text{SU}(2)$ and $\text{SU}(1,1)$ are generated by $J^1, J^2, J^3$ and $J^3, K^1, K^2$ respectively. $\vec{J}$ and $\vec{K}$ transform as vectors under $\text{SU}(2)$. For $\text{SU}(1,1)$ an analogous role is played by the vectors $\vec{F} \equiv (J^3, K^1, K^2)$ and $\vec{G} \equiv (K^3, -J^1, -J^2)$, which transform as Minkowski vectors under $\text{SU}(1,1)$ [28].

Unitary irreps of $\text{SL}(2, \mathbb{C})$ are labelled by pairs of numbers $(\rho, n)$, $\rho \in \mathbb{R}$, $n \in \mathbb{Z}_+$, which are related to the two Casimirs $C_1$ and $C_2$:

\[
C_1 = 2 \left( \vec{J}^2 - \vec{K}^2 \right) = \frac{1}{2} (n^2 - \rho^2 - 4), \tag{2}
\]

\[
C_2 = -4 \vec{J} \cdot \vec{K} = n \rho. \tag{3}
\]

The representation space $\mathcal{H}_{(\rho, n)}$ consists of functions $F : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ with the homogeneity property

\[
F(\alpha z_1, \alpha z_2) = \alpha^{i \rho / 2 + n / 2 - 1} \alpha^{* i \rho / 2 - n / 2 - 1} F(z_1, z_2) \quad \forall \alpha \in \mathbb{C} \setminus \{0\}. \tag{4}
\]

The representation acts on these functions by

\[
D^{(\rho, n)}(g)F(z_1, z_2) = F(az_1 + cz_2, bz_2 + dz_2), \quad g \in \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{C}). \tag{5}
\]

The inner product of $\mathcal{H}_{(\rho, n)}$ is constructed from the $\text{SL}(2, \mathbb{C})$–invariant 2–form

\[
\omega = \frac{i}{2} (z_2 dz_1 - z_1 dz_2) \wedge (\overline{z_2} d\overline{z_1} - \overline{z_1} d\overline{z_2}). \tag{6}
\]
For homogeneous functions $F_1, F_2$ of the type (1), the 2–form $F_1^* F_2 \omega$ is invariant under $(z_1, z_2) \rightarrow (\lambda z_1, \lambda z_2), \lambda \in \mathbb{C}\{0\}$. Thus, $F_1^* F_2 \omega$ projects to a 2–form under $\pi : \mathbb{C}^2 \{0\} \rightarrow \mathbb{C}P^1$ and one can specify the inner product by

$$\langle F_1 | F_2 \rangle \equiv \int_{\mathbb{C}P^1} \pi(F_1^* F_2 \omega).$$

(7)

Since $\omega$ is SL(2, $\mathbb{C}$)–invariant, the representation is unitary w.r.t. this inner product. Equation (4) can be equivalently expressed in terms of sections of the bundle $\mathbb{C}^2 \{0\} \rightarrow \mathbb{C}P^1$. In particular, when choosing the section $z \mapsto (z, 1)$, one obtains the integral

$$\langle F_1 | F_2 \rangle = \int dx \, dy \, F_1^*(z, 1) F_2(z, 1), \quad z = x + iy.$$  

(8)

The unitary irreps of SU(2) and SU(1,1) can be built from eigenstates $|j m\rangle$ of $J^3$:

$$J^3 |j m\rangle = m |j m\rangle, \quad \langle j m | j m'\rangle = \delta_{mm'}. \quad (9)$$

In the case of SU(2), the irreps are labelled by the Casimir $\tilde J^2$:

$$\tilde J^2 |j m\rangle = j(j + 1)|j m\rangle, \quad \text{where } j = k/2, k \in \mathbb{N}_0. \quad (10)$$

The representation space $D_j$ of spin $j$ consists of states with $m = -j, \ldots, j$. The raising and lowering operators are given by

$$J^\pm = J^1 \pm i J^2, \quad J^\pm | j m\rangle = (j \pm m + 1)(j \mp m) | j m \pm 1\rangle. \quad (11)$$

Unitary irreps of SU(1,1) have the Casimir $Q = \tilde F^2 = (J^3)^2 - (K^1)^2 - (K^2)^2$,

$$Q |j m\rangle = j(j + 1)|j m\rangle, \quad (12)$$

and split into two classes, the discrete series and the continuous series. For the discrete series, the spin $j$ assumes negative values $j = -k/2, k \in \mathbb{N}$. Irreps of the positive (negative) discrete series are denoted by $D_j^\pm$ and consist of states $|j m\rangle$ with eigenvalues $m = -j, -j + 1, -j + 2, \ldots$ and $m = j, j - 1, j - 2, \ldots$ respectively. In the case of the continuous series, the spin $j$ is complex and the Casimir has a continuous spectrum:

$$Q |j m\rangle = j(j + 1)|j m\rangle, \quad \text{where } j = -\frac{1}{2} + is, \quad 0 < s < \infty. \quad (13)$$

Irreps of this series are denoted by $C_s^\epsilon$. The allowed values for $m$ are either

$$m = 0, \pm 1, \pm 2, \ldots \quad \text{or} \quad m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, \quad (14)$$

and the label $\epsilon = 0, \frac{1}{2}$ designates these two possibilities. In both the discrete and continuous series raising and lowering is achieved by the operators

$$F^\pm = F^2 \mp i F^1, \quad F^\pm | j m\rangle = \sqrt{(m \pm j \pm 1)(m \mp j)} | j m \pm 1\rangle. \quad (15)$$

As an alternative to the $|j m\rangle$ basis one can use eigenstates of $K^1$ (see [18] for details and [19] for early work in this direction):

$$K^1 | j \lambda \sigma\rangle = \lambda | j \lambda \sigma\rangle, \quad \langle j \lambda' \sigma' | j \lambda \sigma\rangle = \delta(\lambda' - \lambda)\delta_{\sigma' \sigma}. \quad (16)$$
Since $K^1$ generates a noncompact subgroup, these eigenstates are not normalizable. In the continuous series there occurs a two-fold degeneracy of the spectrum which is labelled by the additional index $\sigma = 0, 1$. In analogy to (15) one may define “raising” and “lowering” operators

\[ F^\pm = F^0 \mp F^2, \quad F^\pm | j \lambda \sigma \rangle = i(\pm j \pm 1 - i\lambda) | j (\lambda \pm i) (\sigma + 1 \mod 2) \rangle. \] (17)

The shift $\lambda \pm i$ to complex eigenvalues follows from the commutation relations, but how is this consistent with $K^1$ being a self–adjoint operator? The answer is related to the fact that eigenvectors of $K^1$ are described as elements of a dual space $D'$ in a Gelfand triple $D \subset \mathcal{H} \subset D'$. The operator $K^1$ is self–adjoint in the Hilbert space $\mathcal{H}$ and has the generalized eigenvectors $| j \lambda \sigma \rangle \in D'$ with real eigenvalues $\lambda$. However, when extended to $D'$, the operator $K^1$ has eigenvectors for all complex $\lambda$ and the state $F^\pm | j \lambda \sigma \rangle \in D'$ is an example of such an eigenvector.

### III. BASIS STATES FOR UNITARY IRREPS OF $\text{SL}(2, \mathbb{C})$

#### A. SU(2) and SU(1,1) decomposition

Clearly, every unitary irrep of $\text{SL}(2, \mathbb{C})$ defines a representation of its subgroups SU(2) and SU(1,1). However, these representations are reducible. As a result, the Hilbert space $\mathcal{H}_{(\rho,n)}$ splits into a direct sum of irreps of SU(2), or a direct sum of irreps of SU(1,1) [24]. The SU(2) decomposition is given by the following isomorphism and completeness relation:

\[ \mathcal{H}_{(\rho,n)} \simeq \bigoplus_{j=\frac{n}{2}}^{\infty} D_j, \quad \mathbb{I}_{(\rho,n)} = \sum_{j=\frac{n}{2}}^{\infty} \sum_{m=-j}^{j} |\Psi_j m \rangle \langle \Psi_j m |. \] (18)

The states $|\Psi_j m \rangle$ form the so–called canonical basis of $\mathcal{H}_{(\rho,n)}$. For fixed spin $j$ and $m = -j, \ldots, j$, they span a subspace of $\mathcal{H}_{(\rho,n)}$ that is isomorphic to $D_j$. The SU(1,1) reduction can be written as

\[ \mathcal{H}_{(\rho,n)} \simeq \left( \bigoplus_{j<\frac{1}{2}}^{\frac{-n}{2}} D_j^+ \oplus \int_0^{\infty} ds \mathcal{C}_s^\epsilon \right) \oplus \left( \bigoplus_{j<-\frac{1}{2}}^{\frac{-n}{2}} D_j^- \oplus \int_0^{\infty} ds \mathcal{C}_s^\epsilon \right). \] (19)

The precise meaning of this statement is encoded in the completeness relation

\[ \mathbb{I}_{(\rho,n)} = \sum_{\tau=\pm 1} \left\{ \sum_{\substack{j<\frac{1}{2} \quad \tau m = -j}}^{\frac{-n}{2}} \sum_{m=-j}^{j} |\Psi_j^\tau m \rangle \langle \Psi_j^\tau m | + \int_0^{\infty} ds \mu_\epsilon(s) \sum_{\pm m = \epsilon} |\Psi_j^\tau m \rangle \langle \Psi_j^\tau m | \right\}. \] (20)

Here, the states $|\Psi_j^\tau m \rangle$, $-\frac{n}{2} \leq j < -\frac{1}{2}$, and $|\Psi_j^\tau m \rangle$, $j = -\frac{1}{2} + i\epsilon$, correspond to states $|j m \rangle$ of the discrete and continuous series respectively. The sum over $j$ extends over values such that $j - n/2$ is integral. Moreover, $\epsilon$ has a value such that $\epsilon - n/2$ is an integer. The measure factor $\mu_\epsilon(s)$ depends on the specific choice of normalization for the states $|\Psi_j^\tau m \rangle$ and will be given below.
The decompositions (18) and (19) can be derived from the homogeneity property (4) and the Plancherel decomposition of SU(2) and SU(1,1) respectively. Due to (4) every function $F$ in $\mathcal{H}_{(\rho,n)}$ can be equivalently described by a function $f$ of SU(2) via

$$F(z_1, z_2) = \sqrt{\pi} \left( |z_1|^2 + |z_2|^2 \right)^{i\rho/2-1} f(u(z_1, z_2)), \quad u(z_1, z_2) = \frac{1}{\sqrt{|z_1|^2 + |z_2|^2}} \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}. \quad (21)$$

Thus, the Hilbert space $\mathcal{H}_{(\rho,n)}$ is isomorphic to a subspace of $L^2(SU(2))$. Under this isomorphism the inner product (8) turns into

$$\langle f_1 | f_2 \rangle = \int du \, f_1^*(u) f_2(u), \quad (22)$$

where $du$ denotes the normalized Haar measure on SU(2). Alternatively, the functions $F$ can be characterized by pairs $(f^+, f^-)$ of functions $f^\tau : SU(1,1) \to \mathbb{C}, \tau = \pm 1$, via

$$F(z_1, z_2) = \sqrt{\pi} \left( |\tau z_1|^2 - |\tau z_2|^2 \right)^{i\rho/2-1} f^\tau(v^\tau(z_1, z_2)), \quad \tau = \begin{cases} 1, & |z_1| > |z_2|, \\ -1, & |z_1| < |z_2|, \end{cases} \quad (23)$$

where we choose

$$v^\tau(z_1, z_2) = \begin{cases} \frac{1}{\sqrt{|z_1|^2-|z_2|^2}} \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}, & \tau = 1, \\ \frac{1}{\sqrt{|z_2|^2-|z_1|^2}} \begin{pmatrix} z_2 & \overline{z}_1 \\ z_1 & \overline{z}_2 \end{pmatrix}, & \tau = -1. \end{cases} \quad (24)$$

As a result, $\mathcal{H}_{(\rho,n)}$ is isomorphic to a subspace of $L^2(SU(1,1)) \oplus L^2(SU(1,1))$ with the inner product

$$\langle (f_1^+, f_1^-) | (f_2^+, f_2^-) \rangle = \sum_{\tau=\pm 1} \int dv \, (f_1^\tau(v))^* f_2^\tau(v). \quad (25)$$

The measure $dv$ is specified in appendix A.

Functions of SU(2) or SU(1,1) can be expanded in matrix elements

$$D_{x'x}(j \rho) \equiv \langle x' | D^j(g) | x \rangle, \quad (26)$$

where $x'$ and $x$ label appropriate basis states. When applied to the functions $f$ on SU(2), this leads to the decomposition (18) with states given by the basis functions

$$\Psi_{jm}(u) = \sqrt{2j + 1} D_{n/2m}^j(u). \quad (27)$$

Similarly, the SU(1,1) decomposition results in (19) with states represented by functions

$$\Psi_{jx}^\tau(v) = \begin{cases} \sqrt{2j + 1} \left( D_{n/2x}^j(v), 0 \right), & \tau = 1, \\ \sqrt{2j + 1} \left( 0, D_{-n/2x}^j(v) \right), & \tau = -1, \end{cases} \quad j \neq -1. \quad (28)$$
The label $x$ is $m$ if we use a basis of $J^3$ eigenstates. When employing $K^1$ eigenstates, one has $x = \lambda$ for the discrete series and $x = \lambda \sigma$ for the continuous series. The label $x'$ is set to $\pm n/2$ corresponding to the state $|j \pm n/2\rangle$. The choice of normalization (28) determines the measure factor $\mu_\epsilon(s)$ in (19) to be

$$
\mu_\epsilon(s) = \begin{cases} 
-i \tanh(\pi s), & \epsilon = 0, \\
-i \coth(\pi s), & \epsilon = 1/2.
\end{cases}
$$

The irrep of spin $j = -1/2$ represents a special case that does not appear in the Plancherel decomposition. However, since it will come up in calculations below, we define the associated state

$$
\Psi_{jx}^\tau(v) \equiv \begin{cases} 
(D_{n/2x}^j(v), 0), & \tau = 1, \\
(0, D_{-n/2x}^j(v)), & \tau = -1.
\end{cases}
$$

B. Explicit expressions for basis functions

In order to derive the action of generators on states in sec. [IV] we need explicit expressions for the state functions (27) and (28). Altogether we will encounter five different cases, depending on the group, the choice of basis states and the series (see table I). When dealing with the associated $D$–functions we will exploit the fact that they all share a similar structure. In each case, the $D$–function can be built from the expression

$$
F_{m'm}^j(z) = (1 - z)^{(m' + m)/2} z^{(m' - m)/2} \, _2F_1(-j + m', j + m' + 1, m' - m + 1; z)
$$

where $_2F_1$ denotes Gauss’ hypergeometric function. The full $D$–functions are obtained by including normalization factors, phases and a suitable parametrization of $z$.

Let us start with the group SU(2). When using the parametrization (A1) the $D$–function of SU(2) reduces to the Wigner $d$–function via

$$
D_{m'm}^j(u) = e^{im'\psi} d_{m'm}^j(\theta) e^{im\varphi}.
$$

In the case $m' \geq m, m' + m \geq 0$, the $d$–function has the explicit form

$$
d_{m'm}^j(\theta) = \frac{1}{(m' - m)!} N_{m'm}^j F_{m'm}^j(z(\theta))
$$

where

$$
z(\theta) \equiv \frac{1}{2} (1 - \cos \theta)
$$

and $F_{m'm}^j(z)$ is the function defined in (31) [29]. The normalization factor $N_{m'm}^j$ can be written in several ways:

$$
N_{m'm}^j = \left[ \prod_{l=0}^{m'-m-1} (j + m' - l)(j - m - l) \right]^{1/2}
$$

$$
= \frac{(j + m')!(j - m)!}{(j + m)! (j - m')!} \frac{1}{\Gamma(j + m + 1) \Gamma(j - m + 1)} = \frac{\Gamma(j + m' + 1) \Gamma(j - m + 1)}{\Gamma(j + m + 1) \Gamma(j - m' + 1)}
$$
In table II, the Wigner functions \( d_{m',m}^j(\theta) \) obey symmetries that allow one to infer its values for general \( m', m \) from those for \( m' \geq m, m' + m \geq 0 \). For example, for \( m' < m \) and \( m' + m \leq 0 \), \( d_{m',m}^j = (-1)^{m'-m} d_{m',-m}^j(\theta) \). Similar rules apply to the Bargmann function \( b_{m',m}^j(t) \).

The expressions for other values of \( m' \) and \( m \) follow from table II.

Next consider the case of SU(1,1) with a basis of \( J^3 \) eigenstates. Under the parametrization (A3) we have, for both the discrete and continuous series, that

\[
D_{m'm}(v) = e^{im'\psi} b_{m'm}^j(t) e^{im'\varphi}
\]

where \( b_{m'm}^j(t) \) is an SU(1,1) analog of the Wigner \( d \)-function. The explicit form of the \( b \)-functions was determined by Bargmann [30]. For \( m' \geq m, m' + m \geq 0 \), these can be written as

\[
b_{m'm}^j(t) = \sqrt{(-1)^{m'-m}} d_{m'm}^j(it) = \frac{1}{(m'-m)!} \tilde{N}_{m'm}^j F_{m'm}^j(z(it))
\]

where the normalization factor is given by

\[
\tilde{N}_{m'm}^j = \left[ \frac{\Gamma(m' + j + 1)\Gamma(m' - j)}{\Gamma(m + j + 1)\Gamma(m - j)} \right]^{1/2}.
\]

The other cases are obtained from table II. In appendix B it is shown that this definition is indeed equivalent to the one provided by Bargmann.

Finally, we come to SU(1,1) and a basis of \( K^1 \) eigenstates. According to eq. (28), we need expressions for \( D \)-functions in “mixed” bases, where the left state belongs to the discrete \( J^3 \) basis and the right state is from the continuous basis diagonal in \( K^1 \). This case was worked out by Lindblad [31], using previous results by Lindblad and Nagel [18]. For the discrete series and the parametrization (A5),

\[
D_{m'\lambda}(v) = e^{im'\psi} d_{m'\lambda}^j(t) e^{i\lambda u}
\]

and for \( m \geq -j \)

\[
d_{m'\lambda}^j(t) = N_{m'\lambda}^j F_{m',-\lambda}^j(z(-t))
\]

Here, \( z \) is parametrized by

\[
z(t) = \frac{1}{2}(1 - i \sinh t)
\]
and the normalization factor is defined by
\[ N^j_m \equiv \frac{\sqrt{2}}{\pi} 2^{-j-2} S^j_m R^j_{m\lambda}, \quad S^j_m \equiv \frac{[\Gamma(m-j)\Gamma(m+j+1)]^{\frac{1}{2}}}{\Gamma(m+j+1)}, \tag{43} \]
and
\[ R^j_{m\lambda} \equiv \frac{\Gamma(j+1+i\lambda)\Gamma\left(\frac{-j-i\lambda}{2}\right)\Gamma\left(\frac{-j+1+i\lambda}{2}\right)}{\Gamma(m-j)\Gamma(-m+1+i\lambda)}. \tag{44} \]
The \(d\)-function for \(m \leq j\) results from
\[ d^j_{m\lambda}(t) = d^j_{-m\lambda}(-t). \tag{45} \]

For the continuous series, one has
\[ D^j_{m\lambda\sigma}(v) = e^{im\varphi} d^j_{m\lambda\sigma}(t) e^{i\lambda u} \tag{46} \]
with the \(d\)-function
\[ d^j_{m\lambda\sigma}(t) = S^j_m \left[ T^j_{m\lambda\sigma} F^j_{m,-i\lambda}(z(t)) - (-1)^\sigma T^j_{-m\lambda\sigma} F^j_{-m,-i\lambda}(z(-t)) \right]. \tag{47} \]
The factor \(S^j_m\) is specified as in (43) and
\[ T^j_{m\lambda\sigma} = \frac{2^{j-1}\Gamma(-j+i\lambda)}{i^\sigma \sin \left[ \frac{\pi}{2}(-j+\sigma-i\lambda) \right] \Gamma(-m-j)\Gamma(m+1+i\lambda)}. \tag{48} \]
The above formulas are identical to Lindblad’s except for sign switches due to differing conventions (\(t \to -t\) and \(\lambda \to -\lambda\)).

**IV. MATRIX ELEMENTS OF SL(2, C) GENERATORS**

In this section we state our results—the matrix elements of SL(2, C) generators in discrete and continuous bases of SU(1,1). To save space we write down only the formula for one of the generators outside SU(1,1). In the case of the \(J^3\) basis, we choose the generator \(K^3\) and in the case of the \(K^1\) basis we select \(J^1\). Given the matrix elements of \(K^3\) (or \(J^1\)), the entire set of matrix elements can then be readily computed from the commutation relations (1) and the known action of generators of SU(1,1) (see eqns. (9), (15) (16) and (17)). For completeness we also include the result for the subgroup SU(2). The derivation of the different cases is presented in section V. Each of the subsequent formulas has been checked numerically for a number of parameter values.

Let us define coefficients
\[ A_j = \frac{\rho n}{4j(j+1)}, \quad C_j = \frac{\sqrt{n^2/4 - j^2 - m^2 - j^2}}{j\sqrt{2j-1}\sqrt{2j+1}}, \tag{49} \]
For the canonical SU(2) decomposition the action of \(K^3\) is well–known \[20, 25\] and we state it here for the explicit choice of states given in eq. (27). For \(j \neq 0, \frac{1}{2}\), one has
\[ K^3 |\Psi_{jm}\rangle = \left[\rho/2 + i(j+1)\right]C_{j+1} |\Psi_{j+1m}\rangle - m A_j |\Psi_{jm}\rangle + (\rho/2 - i j) C_j |\Psi_{j-1m}\rangle \tag{50} \]
In the case of \( j = 1/2 \), the third term on the right-hand side is absent and for \( j = 0 \) the second and third term are absent. This formula differs slightly from the one in [20], since the algebraic derivation assumes a suitable choice of phase in the states, dependent on \( \rho \) and \( j \), so that \( \rho/2 + i(j + 1) \) becomes \( \sqrt{(\rho/2)^2 + (j + 1)^2} \) etc.\(^3\)

Next consider the SU(1,1) decomposition w.r.t. \( J^3 \) eigenstates. For the discrete series \( j \leq -2 \) and the continuous series, \( K^3 \left| \Psi_j^{\tau \lambda} \right> = \frac{1}{2} \rho mn \partial_j \left| \Psi_j^{\tau \lambda} \right> |_{j=-1} + \frac{1}{2} (\rho/2 - i)mn \left| \Psi_{j-1}^{\tau \lambda} \right> + \frac{1}{\sqrt{3}} \tau (\rho/2 + i) \sqrt{n^2/4 - 1} \sqrt{m^2 - 1} \left| \Psi_{-2}^{\tau \lambda} \right> . \)

The equation for \( j = -3/2 \) is special, since then \( j + 1 = -1/2 \), leading to the state \((30)\) outside the Plancherel decomposition. In this case, the denominator \((j + 1)\sqrt{2j + 1} \sqrt{2j + 3} \) in \( C_{j+1} \) has to be replaced by \((j + 1)\sqrt{2j + 1} \). Similar “boundary” effects occur for \( j = -1 \), where the \( K^3 \) action can be cast in the form

\[
K^3 \left| \Psi_{j}^{\tau \lambda} \right> = \frac{1}{2} \rho mn \partial_j \left| \Psi_j^{\tau \lambda} \right> |_{j=-1} + \frac{1}{\sqrt{3}} \tau (\rho/2 + i) \sqrt{n^2/4 - 1} \sqrt{m^2 - 1} \left| \Psi_{-2}^{\tau \lambda} \right> .
\]

As before, the action on the state \( j = -3/2 \) results in a state \( j + 1 = -1/2 \), with the factor \((j + 1)\sqrt{2j + 1} \sqrt{2j + 3} \) in \( C_{j+1} \) substituted by \((j + 1)\sqrt{2j + 1} \). For brevity, we do not spell out the case \( j = -1 \), which produces a formula similar to eq. \((52)\). On continuous series

\[
K^3 \left| \Psi_j^{\tau \lambda} \right> = \left[ \rho/2 + i(j + 1) \right] B_j B_{j+1} \tilde{C}_{j+1} \left| \Psi_{j+1}^{\tau \lambda} \right> + \lambda A_j \left| \Psi_j^{\tau \lambda} \right> - \frac{1}{2} (\rho/2 - i) \sqrt{\lambda^2 + j^2} B_j B_{j-1} \tilde{C}_j \left| \Psi_{j-1}^{\tau \lambda} \right> .
\]

---

\(^3\) See the remarks on choice of phase in Tung’s textbook [32], sec. 10.3.3 and appendix VII.

\(^4\) For integer spin, this case was previously derived by Mukunda [28]. We find, however, minor discrepancies with our result, which can be traced back to parts of the proof that were not presented in [28] (see appendix C). For example, \((\rho - i(k - 1))\) in eq. \((3.19)\) [28] should be replaced by \((\rho + i(k - 1))\).
states $J^1$ yields, for $n \neq 0$,
\[ J^1 \left| \Psi_{j\lambda\sigma} \right> = -\frac{1}{2} \left[ \rho/2 + i(j + 1) \right] \left[ (j + 1)^2 + \lambda^2 \right] \tilde{C}_{j+1} \left| \Psi_{j+1\lambda\sigma'} \right> + \lambda A_j \left| \Psi_{j\lambda\sigma} \right> \]
\[ - 2 \left( \rho/2 - ij \right) \tilde{C}_j \left| \Psi_{j-1\lambda\sigma'} \right>, \]
where $\sigma' = \sigma + 1 \mod 2$. When $n = 0$, the right-hand side comes with an additional factor $\tau$ in front of the $j + 1$ term.

In all of the previous equations, $\Delta j = \pm 1, 0$, in accordance with the Wigner–Eckart theorem and the fact that $K^3$ and $J^1$ are components of vector operators. Since the vectors $\vec{F} = (J^3, K^1, K^2)$ and $\vec{G} = (K^3, -J^1, -J^2)$ transform as Minkowski vectors under SU(1,1), the associated Clebsch–Gordan coefficients correspond to the coupling of unitary SU(1,1) irreps with the non–unitary SU(1,1) irrep of spin 1 (see [33]).

V. DERIVATION OF MATRIX ELEMENTS

In this section, we outline how the matrix elements of the previous section were obtained. The first step consists in expressing the generators of SL(2, $\mathbb{C}$) as differential operators of the relevant subgroup (SU(1,1) or SU(2)). In doing so we employ parametrizations that are adapted to the choice of basis states (either $J^3$ or $K^1$ eigenstates). Once the differential operators are determined, we apply them to the state functions defined in sec. III B. More precisely, we act with $K^3$ on the basis of $J^3$ eigenstates and with $J^1$ on the basis diagonalized by $K^1$. The resulting states are decomposed with respect to the original basis, thus giving us the matrix elements of $K^3$ and $J^1$ respectively.

A. Generators as differential operators

In order to derive the differential operators associated to SL(2, $\mathbb{C}$) generators, we start from the definition of the representation (5) and combine it with the relation (23) between functions $F$ of $\mathbb{C}^2\setminus\{0\}$ and functions $f^\tau$ of SU(1,1) to get the finite transformation of $f^\tau$: \[
D^{(\rho,n)}(g)f^\tau(v^\tau) = (\tau|av_1 + cv_2|^2 - \tau|bv_1 + dv_2|^2)^{i\rho/2 - 1} f^\tau(v^\tau \cdot g) \quad (56)
\]
where
\[
v^\tau \equiv \begin{cases} \left( \begin{array}{c} v_1 \ v_2 \\ \overline{v}_2 \ \overline{v}_1 \end{array} \right), & \tau = 1, \\
\left( \begin{array}{c} \overline{v}_2 \ \overline{v}_1 \\ v_1 \ v_2 \end{array} \right), & \tau = -1,
\end{cases}
\]
and
\[
\begin{align*}
\overrightarrow{v}_1 \cdot g &= \frac{av_1 + cv_2}{(\tau|av_1 + cv_2|^2 - \tau|bv_1 + dv_2|^2)^{1/2}}, \\
\overrightarrow{v}_2 \cdot g &= \frac{bv_1 + dv_2}{(\tau|av_1 + cv_2|^2 - \tau|bv_1 + dv_2|^2)^{1/2}}.
\end{align*}
\quad (57)
\]
From this we obtain the corresponding infinitesimal operators via
\[ J^i f^\tau = -i\partial_\epsilon \left[ D^{(\rho,n)}(a_i(\epsilon)) f^\tau(v^\tau) \right] \bigg|_{\epsilon=0}, \quad K^i f^\tau = -i\partial_\epsilon \left[ D^{(\rho,n)}(b_i(\epsilon)) f^\tau(v^\tau) \right] \bigg|_{\epsilon=0}. \] (58)

Here, \( a_i(\epsilon) \) and \( b_i(\epsilon) \) stand for the group elements generated by \( J^i \) and \( K^i \), as defined in appendix A. Analogous formulas hold for the SU(2) case.

When the generator resides in su(1,1), the transformation (56) is the natural action of SU(1,1) on functions of SU(1,1) and the associated differential operator can be determined by standard methods (see e.g. [34]). When the generator lies outside of su(1,1) (like \( K^3 \)), the prefactor in (56) leads to a multiplicative term and the infinitesimal transformation of the function’s argument turns into a linear combination of SU(1,1) generators. By substituting the differential expressions for these, one arrives at the differential operator for the SL(2, \( \mathbb{C} \)) generator.

Let us list the results for the different cases and parametrizations. In the case of the subgroup SU(2), where we use a \( J^3 \) basis and the parametrization (A1), we have
\[ J^3 = -i\partial_\varphi, \] (59)
\[ J^\pm = i e^{\pm i\varphi} \left( \cot \theta \partial_\varphi \mp i\partial_\theta - \frac{1}{\sin \theta} \partial_\psi \right), \] (60)
\[ K^3 = -\left( \rho/2 + i \right) \cos \theta - i \sin \theta \partial_\theta. \] (61)

For SU(1,1) with a \( J^3 \) basis and parametrization (A3),
\[ J^3 = -i\partial_\varphi \mathbb{1}, \] (62)
\[ F^\pm = \pm e^{\pm i\varphi} \left( i \coth t \partial_\varphi \mp i\partial_t - \frac{1}{\sinh t} \partial_\psi \right) \mathbb{1}, \] (63)
\[ K^3 = \left[ - \left( \rho/2 + i \right) \cosh t - i \sinh t \partial_t \right] \sigma_3. \] (64)

Since states are represented by pairs of SU(1,1) functions, the differential operators come in the form of \( 2 \times 2 \) matrices. Finally, for SU(1,1) and a basis of \( K^1 \) eigenstates, we coordinatize the group as in (A5), so that
\[ K^1 = -i\partial_u \mathbb{1}, \] (65)
\[ F^\pm = i e^{\pm u} \left( \tanh t \partial_u \pm \partial_t - \frac{1}{\cosh t} \partial_\varphi \right) \mathbb{1}, \] (66)
\[ J^1 = \left[ - \left( \rho/2 + i \right) \sinh t - i \cosh t \partial_t \right] \sigma_3. \] (67)

In the above parametrizations, \( J^3 \) and \( K^1 \) are given by a single derivative, and the corresponding SL(2, \( \mathbb{C} \)) counterparts \( K^3 \) and \( J^1 \) have a particularly simple form as well. Note that we use differential operators on the group, which is parametrized by three variables, while Mukunda works with quotient spaces of the group, which have only two coordinates [28].
B. Action on state functions

Our next task is to apply the differential operators $K^3$ and $J^1$ on the state functions specified in sec. III B. This step is facilitated by the fact that, in their respective parametrizations, the operators and states for the different cases are all of a similar form.

In each case, the selected SL(2, C) generator ($K^3$ or $J^1$) depends only on one of the three coordinates of the group, so that it acts only on the $d$– or $b$–function within the $D$–function. It is convenient to express this operator in terms of the variable $z$ which was used earlier when defining the $d$– and $b$–functions (see eqns. (33), (38), (41) and (47)). In fact, for all cases, the operator is essentially of the form

$$\hat{O} \equiv (\rho/2 + i)(1 - 2z) + 2iz(1 - z)\partial_z.$$  \hfill (68)

For SU(2), we have $K^3 = -\hat{O}$, for SU(1,1) in the $J^3$–adapted parametrization, we find $K^3 = -\hat{O} \sigma_3$, and for SU(1,1) in the $K^1$–adapted coordinates $J^1 = i\hat{O} \sigma_3$. The $d$– and $b$–functions, on the other hand, are all given by linear combinations of the function $F_{j'm'}^{j} (z)$ in eq. (31). Thus, the problem is essentially reduced to finding the action of the operator $\hat{O}$ on the function $F_{j'm'}^{j} (z)$.

This action can be determined from rather lengthy manipulations of hypergeometric functions which we delegate to appendix C. The result is that for $j \neq -1$

$$\hat{O} F_{m'm}^{j} = \left[ \rho/2 + i(j + 1) \right] C_{m'm}^{j+1} F_{m'm}^{j+1} + \frac{1}{2} \rho C_{m'm}^{j} F_{m'm}^{j} + (\rho/2 - ij)C_{m'm}^{j-1} F_{m'm}^{j-1},$$  \hfill (69)

with the coefficients given by

$$C_{m'm}^{j+1} = \frac{(j + m' + 1)(j - m + 1)}{(j + 1)(2j + 1)},$$  \hfill (70)

$$C_{m'm}^{j} = \frac{m'm}{j(j + 1)},$$  \hfill (71)

$$C_{m'm}^{j-1} = \frac{(j - m')(j + m)}{j(2j + 1)}.$$  \hfill (72)

This is the central equation from which the matrix elements for all cases follow. The equation for the special value $j = -1$ arises from the limit $j \to -1$ of eq. (69), which yields

$$\lim_{j \to -1} \hat{O} F_{m'm}^{j} (z) = -\rho m'm \partial_j F_{m'm}^{j} (z) \bigg|_{j = -1}$$

$$+ \frac{1}{2} \left[ \rho m'm - \rho(m' - m) + 2im'm \right] F_{m'm}^{-1} (z)$$

$$- (\rho/2 + i)(m' + 1)(m - 1)F_{m'm}^{-2} (z).$$  \hfill (73)

C. Treatment of normalization factors

Once eq. (69) is established, it remains to include the normalization factors in the calculation in order to obtain the action of the generators on $D$–functions and states. Since the
normalization is spin dependent, the coefficients $C_{m'm}^{j+1}$ and $C_{m'm}^{j-1}$ in [39] have to be adjusted by additional factors that compensate for the change from $j$ to $j+1$ or $j-1$ in the states.

We will not present the complete derivation of these factors, but comment on some of the subtleties. In the derivation for SU(2), we can first consider the values $m' \geq m$ and $m' + m \geq 0$, in which case the definition (33) of the $d$–function applies. The result is (50). For the other cases, we use table [1] and the property that $m$ and $m'$ appear either as $\sqrt{m'^2 - j^2} \sqrt{m^2 - j^2}$ or $mm'$ in [51], which are invariant under $m' \leftrightarrow m$ and $(m',m) \rightarrow (-m',-m)$. Hence the matrix elements have the same form as in the first case.

For the discrete series of SU(1,1) in the $J^3$ basis, the calculation is analogous except for a sign factor under the square root in $N_{m'm}^j$ and a factor $\tau$ from the $\sigma_3$ of the differential operator. The former is compensated by the fact that $\sqrt{m'^2 - j^2} \sqrt{m^2 - j^2}$ switches sign as we go from SU(2) to SU(1,1) and the $\tau$ is cancelled in the $j$ term, since $m' = \tau n/2$. When dealing with the continuous series of SU(1,1) in the $J^3$ basis, square roots have to be treated with particular care, since $j$ is complex and, in general, $\sqrt{z_1} \sqrt{z_2} \neq \sqrt{z_1 z_2}$ for complex numbers $z_1$ and $z_2$. Consider the case $m' \geq m$ for which (33) holds. We first note that

$$m'-m-1 \prod_{l=0}^{m'-m-1} (j+m'-l)(m-j+l) = m'-m-1 \prod_{l=0}^{m'-m-1} (m-j+l)(m+j+1+l)$$  \hspace{1cm} (74)

Since $(m-j+l)^* = (m+j+1+l)$, this implies that

$$\left[ m'-m-1 \prod_{l=0}^{m'-m-1} (j+m'-l)(m-j+l) \right]^\frac{1}{2} = m'-m-1 \prod_{l=0}^{m'-m-1} (j+m'-l)^\frac{1}{2}(m-j+l)^\frac{1}{2}. \hspace{1cm} (75)$$

Furthermore, one can show that for all $j = -\frac{1}{2} + is$, $s > 0$, and $m$,

$$|\arg(m+j+1) + \arg(m-j-1)| < \pi, \hspace{1cm} (76)$$

$$|\arg(m+j) + \arg(m-j)| < \pi, \hspace{1cm} (77)$$

and

$$|\arg(m+j+1) - \arg(m-j-1)| \begin{cases} < \pi, & m \geq \frac{1}{2}, \\ = \pi, & m = 0, \end{cases} \hspace{1cm} (78)$$

$$|\arg(m-j) - \arg(m+j)| \begin{cases} < \pi, & m \geq \frac{1}{2}, \\ = -\pi, & m = 0. \end{cases} \hspace{1cm} (79)$$

This allows us to write

$$\sqrt{m+j+1} \sqrt{m-j-1} = \sqrt{m^2 - (j+1)^2}, \quad \sqrt{m+j} \sqrt{m-j} = \sqrt{m^2 - j^2}, \hspace{1cm} (80)$$

and for $m \geq \frac{1}{2}$ ($m = 0$),

$$\frac{\sqrt{m+j+1}}{\sqrt{m-j-1}} = \sqrt{\frac{m+j+1}{m-j-1}}, \quad \frac{\sqrt{m-j}}{\sqrt{m+j}} = \pm \sqrt{\frac{m-j}{m+j}}. \hspace{1cm} (81)$$
With the help of equations (73) and (80) one arrives at the final formula (51). The case \( m' < m \) follows from table II.

When coming to the discrete series in the \( K^1 \) basis, we observe that for \( m \geq -j \)
\[
S_j^m = \sqrt{m^2 - (j + 1)^2} S_{j+1}^m, \quad S_j^m = \frac{S_j^m}{\sqrt{m^2 - j^2}}.
\] (82)

We first derive eq. (54) for \( \tau = 1 \), for which \( m = n/2 \geq -j \). Then, the case \( \tau = -1 \) is inferred from inspection of eq. (45): \( m \) goes to \( -m \) in the coefficients and the change \( t \to -t \) amounts to a sign for all three terms; the substitution \( m \to -m \) has no net effect, however, since also \( m = \tau n/2 \), and the sign from \( t \to -t \) is cancelled by the sign from \( \sigma_3 \) in \( J^1 \). Thus, the coefficients have the same form as for \( \tau = 1 \).

In the case of the continuous series, we use \( \Gamma(m-j)\Gamma(m+j+1) \geq 0 \) together with eqns. (80) and (81) to obtain
\[
S_j^m = \epsilon_{j+1} \sqrt{m^2 - (j + 1)^2} S_{j+1}^m, \quad S_j^m = \epsilon_j \frac{S_{j-1}^m}{\sqrt{m^2 - j^2}}.
\] (83)

where \( \epsilon_{j+1} = \pm 1 \) for \( m \geq 0 \) \((m \leq -\frac{1}{2})\) and \( \epsilon_j = \pm 1 \) for \( m \geq \frac{1}{2} \) \((m \leq 0)\). To derive the matrix elements we need furthermore the identities
\[
T_{m\lambda\sigma}^j = \frac{i}{2} \frac{(-j + i\lambda - 1)}{(-m - j - 1)} T_{m\lambda\sigma}^j, \quad T_{m\lambda\sigma}^j = -i \frac{2(-m - j)}{-j + i\lambda} T_{m\lambda\sigma}^j.
\] (84)

Given eq. (55) for \( \tau = 1 \), we deduce the \( \tau = -1 \) component by noting the following. The sign from \( \sigma_3 \) in \( J^1 \) produces an overall sign for all three terms. When \( m = \tau n/2 < 0 \), we get another sign for the \( j \) term, and a sign for the \( j + 1 \) and \( j - 1 \) term from relation (83). Therefore, the coefficients are identical to those for \( \tau = 1 \). On the other hand, if \( n = 0 \) and hence \( m = \tau n/2 = 0 \), eq. (83) gives only a sign for the \( j - 1 \) term. In this case, there remains a sign change for the \( j + 1 \) term.

VI. SUMMARY AND DISCUSSION

In this paper, we have determined the matrix elements of generators in SU(1,1) decompositions of unitary irreducible representations of SL(2, \( \mathbb{C} \)). By extending and building on previous work by Mukunda [28], we derived these matrix elements for both the discrete basis diagonal in \( J^3 \) and the continuous basis of \( K^1 \) eigenstates, and in each case for the discrete and continuous series. By identifying the common structure of differential operators and states across different bases, the problem was reduced to one main equation. Basis–specific differences appeared in the treatment of normalization factors.

As explained in the introduction, unitary representations of SL(2, \( \mathbb{C} \)) and its states are central elements in the spin foam approach to quantum gravity. It was shown by us in ref. [16] that coherent states of the SU(1,1) reduction represent quantum states of timelike 2–cells. For this, we used the matrix elements of these states, anticipating the proof of

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To be precise, states in the \( K^1 \) basis and continuous series correspond to timelike 2–cells, while states in the \( J^3 \) basis and discrete series implement spacelike 2–cells.
the present paper. It is also likely that these matrix elements will be relevant for future calculations in spin foam and loop quantum gravity.

Finally, this paper could be useful for anybody interested in the reduction of SL(2, \mathbb{C}) representations, since it collects some of the know-how that is dispersed over references from more than 40 years ago.

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Appendix A: Parametrization of $\text{SL}(2, \mathbb{C}), \text{SU}(2)$ and $\text{SU}(1,1)$

In this section, we state our conventions for parametrizations and measures on $\text{SL}(2, \mathbb{C}), \text{SU}(2)$ and $\text{SU}(1,1)$. The one–parameter subgroups of $\text{SL}(2, \mathbb{C})$ are parametrized as follows:

\[ J_1 = \frac{1}{2} \sigma_1, \quad a_1(\psi) = e^{i\psi J_1} = \begin{pmatrix} \cos(\psi/2) & i \sin(\psi/2) \\ i \sin(\psi/2) & \cos(\psi/2) \end{pmatrix} \]

\[ J_2 = \frac{1}{2} \sigma_2, \quad a_2(\theta) = e^{i\theta J_2} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \]

\[ J_3 = \frac{1}{2} \sigma_3, \quad a_3(\varphi) = e^{i\varphi J_3} = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \]

\[ K_1 = \frac{i}{2} \sigma_1, \quad b_1(u) = e^{iu K_1} = \begin{pmatrix} \cosh(u/2) - \sinh(u/2) \\ -\sinh(u/2) & \cosh(u/2) \end{pmatrix} \]

\[ K_2 = \frac{i}{2} \sigma_2, \quad b_2(t) = e^{it K_2} = \begin{pmatrix} \cosh(t/2) & i \sinh(t/2) \\ -i \sinh(t/2) & \cosh(t/2) \end{pmatrix} \]

\[ K_3 = \frac{i}{2} \sigma_3, \quad b_3(\delta) = e^{i\delta K_3} = \begin{pmatrix} e^{-\delta/2} & 0 \\ 0 & e^{\delta/2} \end{pmatrix} \]

For elements $u$ of $\text{SU}(2)$ we use the parametrization

\[ u = e^{i\psi J_1^3} e^{i\theta J_2^3} e^{i\varphi J_3^3}, \quad 0 \leq \psi < 4\pi, \quad 0 \leq \theta < \pi, \quad -\pi \leq \varphi < \pi. \quad (A1) \]

In these coordinates, the normalized Haar measure takes the form

\[ du = \frac{1}{(4\pi)^2} \sin \theta \, d\psi \, d\theta \, d\varphi. \quad (A2) \]
For SU(1,1) elements $v$ we adopt two kinds of parametrizations. When using a $J^3$ basis, we employ

$$v = e^{i\psi J^3} e^{itK^2} e^{i\varphi J^3}, \quad 0 \leq \psi < 4\pi, \quad 0 \leq t < \infty, \quad -\pi \leq \varphi < \pi,$$

(A3)

together with the measure

$$dv = \frac{1}{(4\pi)^2} \sinh t \, d\psi \, dt \, d\varphi.$$  

(A4)

In the case of $K^1$ eigenstates, we use instead the following parametrization given in [31],

$$v = e^{i\varphi J^3} e^{itK^2} e^{iuK^1}, \quad 0 \leq \varphi < 4\pi, \quad 0 \leq t, u < \infty,$$

(A5)

for which the measure (A4) reads

$$dv = \frac{1}{(4\pi)^2} \cosh t \, d\varphi \, dt \, du.$$  

(A6)

Appendix B: Representation functions of SU(1,1)

We verify below that the definition of the $D$–functions in eqns. (37) and (38) is equivalent to the original expressions derived by Bargmann ([30], see also [35]). Let us write a general SU(1,1) element as

$$v = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}. \quad (B1)$$

For the discrete series, Bargmann gives the following definitions of the $D$–functions.

For $m', m \geq -j,$

$$D^j_{m'm}(v)$$

$$= \left\{ \begin{array}{ll}
\Theta_{m'm} \alpha^{*(m'+m)} \beta^{m'-m} F(-j - m, j - m + 1, m' - m + 1, -|\beta|^2), & m' \geq m \\
(-1)^{m-m'} \Theta_{m'm} \alpha^{*(m'+m)} \beta^{m-m'} F(-j - m', j - m' + 1, m - m' + 1, -|\beta|^2), & m' < m
\end{array} \right. \quad (B2)$$

$$\Theta_{m'm} = \frac{1}{(m' - m)! \left[ (m + j)! (m - j - 1)! \right]^{1/2}} \left[ (m' + j)! (m' - j - 1)! \right]^{1/2}$$

For $m', m \leq -j,$

$$D^j_{m'm}(v)$$

$$= \left\{ \begin{array}{ll}
\Theta_{m'm} \alpha^{m'+m} \beta^{m'-m} F(-j + m', j + m' + 1, m' - m + 1, -|\beta|^2), & m' \geq m, \\
(-1)^{m-m'} \Theta_{mm'} \alpha^{m'+m} \beta^{m-m'} F(-j + m, j + m + 1, m - m' + 1, -|\beta|^2), & m' < m
\end{array} \right. \quad (B3)$$

$$\Theta_{m'm} = \frac{1}{(m' - m)! \left[ (m' + j)! (m' - j - 1)! \right]^{1/2}} \left[ (-m + j)! (-m - j - 1)! \right]^{1/2}$$
By using the parametrization \( [A3] \),
\[
\alpha = e^{i(\psi + \varphi)/2} \cosh(t/2), \quad \beta = i e^{i(\psi - \varphi)/2} \sinh(t/2), \tag{B2}
\]
and identity 2.9 (2) from Bateman/Erdelyi \([36]\),
\[
F(a, b, c; z) = (1 - z)^c F(c - b, c - a, c; z), \tag{B3}
\]
one can rewrite these \( D \)-functions in the form stated in eqns. \( [37] \) and \( [38] \) of the main text. In the case of the continuous series, Bargmann defines
\[
D_{m'm}^j(v)
\]
\[
= \Theta_{m'm} \alpha^{m' + m \beta^{m' - m}} F(j + m' + 1, -j + m', m' - m + 1, -|\beta|^2), \quad m' \geq m,
\]
\[
( -1)^{m - m'} \Theta_{m'm} \alpha^{m' + m \beta^{m - m'}} F(j + m + 1, -j + m, m - m' + 1, -|\beta|^2), \quad m' < m,
\]
\[
\Theta_{m'm} = \frac{1}{(m' - m)!} \prod_{k=1}^{m'-m} \left[ \frac{1}{4} + s^2 + (m + k)(m + k - 1) \right]^{1/2}, \quad m' \geq m.
\]
By eq. \( [74] \) this is equivalent to eqns. \( [37] \) and \( [38] \).

Appendix C: Derivation of main equation

In this section, we derive the main equation for the determination of the matrix elements (eq. \( [39] \)). The proof consists of two parts, corresponding to the multiplicative and derivative part of the operator \( \hat{O} \) respectively. An explicit treatment of the multiplicative term has been given in the appendix of \([28]\), so we will only quote the result:
\[
(1 - 2z) F^j_{m'm}(z) = C_{m'm}^{j+1} F^j_{m'm}(z) + C_{m'm}^j F^j_{m'm}(z) + C_{m'm}^{j-1} F^j_{m'm}(z) \tag{C1}
\]
The coefficients are the ones defined in eqns. \( [70] \)–\( [72] \). The statement of the full equation \( [39] \) and the proof for the derivative part were omitted in \([28]\). We will provide this derivation now. In the following, the hypergeometric function \( _2F_1 \) is abbreviated by \( F \).

The starting point is identity 2.8 (27) in Bateman/Erdelyi \([36]\): 
\[
(c - 1) z^{c-2} (1 - z)^{a+b-c-1} F(a - 1, b - 1; c - 1; z) = \frac{d}{dz} \left[ z^{c-1} (1 - z)^{a+b-c} F(a, b; c; z) \right] \tag{C2}
\]
Setting
\[
a = -j + m', \quad b = j + m' + 1, \quad c = m' - m + 1, \tag{C3}
\]
one can write the function \( F^j_{m'm} \) as
\[
F^j_{m'm}(z) = \frac{1}{(c - 1)!} z^{(c-1)/2} (1 - z)^{(a+b-c)/2} F(a, b; c; z). \tag{C4}
\]
Then, eq. 2.8 (27) implies
\[
z^{-(c-1)/2} (1 - z)^{-(a+b-c)/2} \frac{d}{dz} \left[ z^{(c-1)/2} (1 - z)^{(a+b-c)/2} F^j_{m'm}(z) \right] \tag{C5}
\]
\[
= (c - 1) z^{-1} (1 - z)^{-1} z^{(c-1)/2} (1 - z)^{(a+b-c)/2} F(a - 1, b - 1; c - 1; z)/(c - 1)!. \tag{C6}
\]
From here a brief calculation leads to
\[
\frac{d}{dz} F^j_{m'm}(z) = \frac{1}{2} m F^j_{m'm}(z) - \frac{1}{2} m' (1 - 2z) F^j_{m'm}(z) + (m' - m) \frac{1}{(m' - m)!} (1 - z)^{(m' + m)/2} z^{(m' - m)/2} F(a - 1, b - 1; c - 1; z). \tag{C7}
\]

The aim is to decompose the right-hand side into functions \(F(a - 1, b + 1; c; z)\), \(F(a, b; c; z)\) and \(F(a + 1, b - 1; c; z)\) which will give the \(j + 1\), \(j\) and \(j - 1\) term of the final equation. To decompose \(F(a - 1, b - 1; c - 1; z)\) suitably we use several identities from Bateman/Erdelyi. From 2.8 (35) it follows that
\[
F(a - 1, b - 1; c - 1; z) = \frac{c - a}{c - 1} F(a - 1, b - 1; c; z) + \frac{a - 1}{c - 1} F(a, b - 1; c; z). \tag{C8}
\]

Identity 2.8 (33) implies that
\[
F(a - 1, b - 1; c; z) = \frac{c - a - b + 1}{c - b} F(a - 1, b; c; z) + \frac{a - 1}{c - b} (1 - z) F(a, b; c; z). \tag{C9}
\]

From 2.8 (32) we get
\[
F(a - 1, b; c; z) = \frac{b}{b - a + 1} F(a - 1, b + 1; c; z) - \frac{a - 1}{b - a + 1} F(a, b; c; z). \tag{C10}
\]

Likewise, one obtains
\[
F(a, b - 1; c; z) = \frac{b - 1}{b - a - 1} F(a, b; c; z) - \frac{a}{b - a - 1} F(a + 1, b - 1; c; z). \tag{C11}
\]

We start from (C8) and plug in (C9) and (C11), giving us
\[
F(a - 1, b - 1; c - 1; z)
= \frac{c - a}{c - 1} \left[ \frac{c - a - b + 1}{c - b} F(a - 1, b; c; z) + \frac{a - 1}{c - b} (1 - z) F(a, b; c; z) \right]
+ \frac{a - 1}{c - 1} \left[ \frac{b - 1}{b - a - 1} F(a, b; c; z) - \frac{a}{b - a - 1} F(a + 1, b - 1; c; z) \right]. \tag{C12}
\]

By inserting (C10), we arrive at
\[
F(a - 1, b - 1; c - 1; z)
= \frac{(c - a)(c - a - b + 1)b}{(c - 1)(c - b)(b - a + 1)} F(a - 1, b + 1; c; z)
- \frac{(a - 1)a}{(c - 1)(b - a - 1)} F(a + 1, b - 1; c; z)
+ \frac{a - 1}{c - 1} \left[ \frac{(c - a - b + 1)(c - a)}{(c - b)(b - a + 1)} + \frac{c - a}{2(c - b)} + \frac{b - 1}{b - a - 1} \right] F(a, b; c; z)
+ \frac{(c - a)(a - 1)}{2(c - 1)(c - b)} (1 - 2z) F(a, b; c; z). \tag{C13}
\]
When inserting the values (C3) this assumes the form

\[ F(a - 1, b - 1; c - 1; z) \]

\[ = - \frac{(j - m + 1)(-m' - m + 1)(j + m' + 1)}{2(m' - m)(j + m)(j + 1)} F(a - 1, b + 1; c; z) \]  
(C14)

\[ - \frac{(-j + m' - 1)(-j + m')}{2(m' - m)j} F(a + 1, b - 1; c; z) \]  
(C15)

\[ + \frac{-j + m' - 1}{2(m' - m)} \left[ \frac{(-m' - m + 1)(j - m + 1)}{(j + m)(j + 1)} - \frac{j - m + 1}{j + m} + \frac{j + m'}{j} \right] F(a, b; c; z) \]  
(C16)

\[ - \frac{(j - m + 1)(-j + m' - 1)}{2(m' - m)(j + m)} (1 - 2z) F(a, b; c; z). \]  
(C17)

Having obtained a formula for \( F(a - 1, b - 1; c - 1; z) \), we can proceed with eq. (C4), namely,

\[ z(1 - z) \partial_z F_{m'm}^j(z) = - \frac{(j - m + 1)(-m' - m + 1)(j + m' + 1)}{2(j + m)(j + 1)} F_{m'm}^{j+1}(z) \]

\[ - \frac{(-j + m' - 1)(-j + m')}{2j} F_{m'm}^{j-1}(z) \]

\[ + \frac{1}{2} \left\{ m + (-j + m' - 1) \left[ \frac{(-m' - m + 1)(j - m + 1)}{(j + m)(j + 1)} - \frac{j - m + 1}{j + m} + \frac{j + m'}{j} \right] \right\} F_{m'm}^j(z) \]

\[ + \frac{1}{2} \left[ -m' - \frac{(j - m + 1)(-j + m' - 1)}{j + m} \right] (1 - 2z) F_{m'm}^j(z). \]  
(C18)

The final step is to insert \((1 - 2z)F_{m'm}^j(z)\) from eq. (C11), which results in

\[ z(1 - z) \partial_z F_{m'm}^j(z) \]

\[ = \left\{ - \frac{(j - m + 1)(-m' - m + 1)(j + m' + 1)}{2(j + m)(j + 1)} \right\} F_{m'm}^{j+1}(z) \]

\[ + \frac{1}{2} \left\{ m + (-j + m' - 1) \left[ \frac{(-m' - m + 1)(j + m' + 1)(j - m + 1)}{(j + 1)(2j + 1)} \right] \right\} F_{m'm}^j(z) \]

\[ + \frac{1}{2} \left\{ -\frac{(-j + m' - 1)(-j + m')}{2j} + \frac{1}{2} \left[ -m' - \frac{(-j + m' - 1)(j + m' + 1)}{j + m} \right] \right\} F_{m'm}^{j-1}(z) \]

\[ + \frac{1}{2} \left\{ m + (-j + m' - 1) \left[ \frac{(-m' - m + 1)(j - m + 1)}{(j + m)(j + 1)} - \frac{j - m + 1}{j + m} + \frac{j + m'}{j} \right] \right\} \]

\[ + \left[ -m' - \frac{(j - m + 1)(-j + m' - 1)}{j + m} \right] \frac{m'm}{j(j + 1)} F_{m'm}^j(z). \]

Simplification yields

\[ z(1 - z) \partial_z F_{m'm}^j(z) = \frac{1}{2} j C_{m'm}^{j+1} F_{m'm}^{j+1}(z) - \frac{1}{2} C_{m'm}^{j} F_{m'm}^{j}(z) - \frac{1}{2} (j + 1) C_{m'm}^{j-1} F_{m'm}^{j-1}(z). \]  
(C19)
By combining this with the multiplicative part we arrive at eq. in the main part of the article.

[1] C. Rovelli, “Quantum Gravity”, Cambridge University Press, Cambridge (2004).
[2] T. Thiemann, “Modern canonical quantum general relativity”, Cambridge University Press, Cambridge (2007).
[3] M. Reisenberger, C. Rovelli, Spin foams as Feynman diagrams, [arXiv:gr-qc/0002083].
[4] M.P. Reisenberger and C. Rovelli, Spacetime as a Feynman diagram: The connection formulation, Class.Quant.Grav. 18 121 (2001), [arXiv:gr-qc/0002095].
[5] D. Oriti, The group field theory approach to quantum gravity, [arXiv:gr-qc/0607032].
[6] J.W. Barrett, L. Crane, Relativistic spin networks and quantum gravity, J.Math.Phys. 39, 3296 (1998), [arXiv:gr-qc/9709028].
[7] J. Engle, E. Livine, R. Perreira and C. Rovelli, LQG vertex with finite Immirzi parameter, Nucl.Phys. B799, 136 (2008), [arXiv:0711.0146 [gr-qc]].
[8] L. Freidel, K. Krasnov, A New Spin Model for 4d Gravity, Class.Quant.Grav. 25, 125018 (2008), [arXiv:0708.1595 [gr-qc]].
[9] E.R. Livine, S. Speziale, Consistently Solving the Simplicity Constraints for Spinfoam Quantum Gravity, Europhys.Lett. 81, 50004 (2008), [arXiv:0708.1915 [gr-qc]].
[10] E.R. Livine, S. Speziale, A new spinfoam vertex for quantum gravity, Phys.Rev. 76D, 084028 (2007), [arXiv:0705.0674 [gr-qc]].
[11] F. Conrady, L. Freidel, Path integral representation of spin foam models of 4d gravity, Class.Quant.Grav. 25, 245010 (2008), [arXiv:0806.4640 [gr-qc]].
[12] F. Conrady, L. Freidel, On the semiclassical limit of 4d spin foam models, Phys.Rev. D78, 104023 (2008), [arXiv:0809.2280 [gr-qc]].
[13] J.W. Barrett, R.J. Dowdall, W.J. Fairbairn, H. Gomes, F. Hellmann, Asymptotic analysis of the EPRL four-simplex amplitude, J.Math.Phys. 50, 112504 (2009), [arXiv:0902.1170 [gr-qc]].
[14] J.W. Barrett, R.J. Dowdall, W.J. Fairbairn, F. Hellmann, R. Pereira, Lorentzian spin foam amplitudes: graphical calculus and asymptotics, [arXiv:0907.2440 [gr-qc]].
[15] F. Conrady, L. Freidel, Quantum geometry from phase space reduction, J.Math.Phys. 50, 123510 (2009), [arXiv:0902.0351 [gr-qc]].
[16] F. Conrady, J. Hnybida, A spin foam model for general Lorentzian 4–geometries, [arXiv:1002.1959 [gr-qc]].
[17] F. Conrady, Spin foams with timelike surfaces, Class.Quant.Grav. 27, 155014 (2010), [arXiv:1003.5652 [gr-qc]].
[18] G. Lindblad, B. Nagel, Continuous bases for unitary irreducible representations of SU(1,1), Ann. De L.T.H.P., Sec. A. 13, 27-56 (1970).
[19] N. Mukunda, Unitary representations of the group O(2,1) in an O(1,1) basis, J.Math.Phys. 8, 2210 (1967).
[20] M.A. Naimark, “Linear representations of the Lorentz group”, Pergamon Press, 1964.
[21] Harish–Chandra, Infinite irreducible representations of the Lorentz group, Proc.Roy.Soc. A189, 372 (1947).
[22] A. Sciarrino, M. Toller, Decomposition of the Unitary Irreducible Representations of the Group SL(2C) Restricted to the Subgroup SU(1,1), J.Math.Phys. 8, 1252-1265 (1967).
[23] R. Delbourgo, K. Koller, P. Mahanta, On transformations between SL(2,C) representations.
Nuovo Cim. **52A**, 1254 (1967).

[24] W. Ruhl, “Lorentz group and harmonic analysis”, W.A. Benjamin, New York, (1970).

[25] M. Carmeli, “Group theory and general relativity”, McGraw-Hill, New York (1977).

[26] S. Strom, *A note on the matrix elements of a unitary representation of the homogeneous Lorentz group*, Arkiv Fysik **33**, 465 (1967).

[27] D.V. Duc, N. Van Hieu, *On the theory of unitary representations of the SL(2,C) group*, Ann.Inst. Henri Poincare, **6**, 17–37 (1967).

[28] N. Mukunda, *Unitary Representations of the Homogeneous Lorentz Group in an O(2,1) Basis*, J.Math.Phys. **9**, 50 (1968).

[29] M. Andrews, J. Gunson, *Complex angular momenta and many–particle states. I. Properties of local representations of the rotation group*, J.Math.Phys. **5**, 1391 (1964).

[30] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Annals Math. **48**, 568 (1947).

[31] G. Lindblad, *Eigenfunction expansions associated with unitary irreducible representations of SU(1,1)*, Phys.Scripta **1**, 201 (1970).

[32] W.K. Tung, “Group Theory In Physics”, Singapore, World Scientific (1985).

[33] H. Ui, *Clebsch-Gordan formulas of the SU(1,1) group*, Prog.Theor.Phys. **44**, 689 (1970).

[34] M.S. Byrd, *Differential geometry on SU(3) with applications to three state systems*, J.Math.Phys. **39**, 6125-6136 (1998).

[35] W.J. Holman III, L.C. Biedenharn Jr., *Complex angular momenta and the groups SU(1, 1) and SU(2)*, Ann.Phys. **39**, 1–42 (1966).

[36] H. Bateman, A. Erdelyi, “Higher Transcendental Functions”, Vol. 1, McGraw-Hill, 1953.