BARGMANN-FOCK PERCOLATION IS NOISE SENSITIVE

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ABSTRACT. We show that planar Bargmann-Fock percolation is noise sensitive under the Ornstein-Uhlenbeck process. The proof is based on the randomized algorithm approach introduced by Schramm and Steif ([SS10]) and gives quantitative polynomial bounds on the noise sensitivity of crossing events for Bargmann-Fock.

A rather counter-intuitive consequence is as follows. Let $F$ be a Bargmann-Fock Gaussian field in $\mathbb{R}^3$ and consider two horizontal planes $P_1, P_2$ at small distance $\varepsilon$ from each other. Even though $F$ is a.s. analytic, the above noise sensitivity statement implies that the full restriction of $F$ to $P_1$ (i.e. $F|_{P_1}$) gives almost no information on the percolation configuration induced by $F|_{P_2}$.

As an application of this noise sensitivity analysis, we provide a Schramm-Steif based proof that the near-critical window of level line percolation around $\ell_c = 0$ is polynomially small. This new approach extends earlier sharp threshold results to a larger family of planar Gaussian fields.

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1. Main results

1.1. Bargmann-Fock percolation is noise sensitive. The planar Bargmann-Fock field is the smooth Gaussian field $f$ on $\mathbb{R}^2$ defined by the following covariance kernel

$$\forall x, y \in \mathbb{R}^2, \text{Cov} [f(x), f(y)] = \exp(-\frac{1}{2} |x - y|^2).$$

It can be realized as the following random entire function

$$f(x) = f(x_1, x_2) = e^{-\frac{1}{2} |x|^2} \sum_{i,j \in \mathbb{N}} \frac{a_{i,j}}{\sqrt{i!j!}} x_1^i x_2^j, \quad (1.1)$$

where $(a_{i,j})_{i,j}$ are i.i.d standard Gaussians.

The percolation model induced by the level sets of smooth Gaussian fields has been studied extensively these last few years (see [BG17a, BM18, BG17b, RV17a, RV17b, MV18, Riv19]) and is believed to behave like Bernoulli percolation for a large family of planar Gaussian fields (see [Wei84, BS02, BDS06, BS07]).
the case of the Bargmann-Fock field - and for a wide class of Gaussian fields, see Subsection 1.2 - it is known that this percolation model exhibits a sharp phase transition at the critical level $\ell_c = 0$ in the following sense. If $\ell \in \mathbb{R}$ and if $Q$ be a quad (i.e. a Jordan domain of $\mathbb{R}^2$ whose boundary $\partial Q$ is piecewise smooth with two distinguished disjoint segments on $\partial Q$), we let $\text{Cross}_\ell(Q)$ be the event that there is a continuous path included in $Q \cap \{f > -\ell\}$ that connects the two distinguished segments of $\partial Q$. Then,

- Theorem 1.1 of [BG17a]: for every quad $Q$, there exists $c = c(Q) \in ]0, 1[$ such that, for every $s > 0$, $c \leq \mathbb{P}[\text{Cross}_0(sQ)] \leq 1 - c$;
- Theorem 1.8 of [RV17b]: for every $\ell > 0$ and every quad $Q$, there exists $c = c(\ell, Q) > 0$ such that, for every $s > 0$, $\mathbb{P}[\text{Cross}_{-\ell}(sQ)] \leq e^{-cs}$ and $\mathbb{P}[\text{Cross}_\ell(sQ)] \geq 1 - e^{-cs}$.

As explained in [BG17a, RV17b], this implies that, if $\ell \leq 0$, a.s. there is no unbounded component in $\{f > -\ell\}$ (this was first proved in [Ale96]) whereas if $\ell > 0$, a.s. there is a unique unbounded component in $\{f > -\ell\}$. In the present paper, we prove that the model is noise sensitive at the critical level $\ell_c = 0$. Let us explain what it means. By analogy with the model of dynamical percolation introduced in [HPS97], it is natural to consider the following dynamics we shall call dynamical Bargmann-Fock model:

$$f(t, x) = f(t, x_1, x_2) = e^{-\frac{1}{2}|x|^2} \sum_{i,j} \frac{a_{i,j}(t)}{\sqrt{i!j!}} x_i^i x_2^j,$$

where $(t \mapsto a_{i,j}(t))_{i,j}$ are independent Ornstein-Uhlenbeck processes with invariant measure $\mathcal{N}(0, 1)$ i.e. $da_{i,j}(t) = -\frac{1}{2}a_{i,j}(t)dt + dB_{i,j}(t); a_{i,j}(0) \sim \mathcal{N}(0, 1)$.

Our main result states that Bargmann-Fock percolation is sensitive to a polynomially small noise. This result is analogous to the quantitative noise sensitivity result for Bernoulli percolation obtained by Schramm and Steif [SS10].

**Theorem 1.1** (See Theorem 1.8 below for a more general statement). There exists $\alpha > 0$ such that, for any quad $Q$, there exists $c = c(Q) > 0$ such that, for every sequence $(t_n)_{n \geq 1}$ that satisfies $t_n \geq n^{-\alpha}$, we have

$$\text{Cov}[1_{f(0) \in \text{Cross}_0(nQ)}, 1_{f(t_n) \in \text{Cross}_0(nQ)}] \leq cn^{-\alpha}.$$ 

Thanks to the Wiener chaos expansion of the $L^2$ functional $1_{f \in \text{Cross}(nQ)}$, we will extract from the above theorem the following counter-intuitive property for the 3D Bargmann-Fock field. Let $F$ be a 3D Bargman-Fock field i.e. the smooth Gaussian field on $\mathbb{R}^3$ with covariance

$$\forall x, y \in \mathbb{R}^3, \text{Cov}[F(x), F(y)] = \exp\left(-\frac{1}{2}|x - y|^2\right).$$

Note that the restriction of $F$ to any plane is a planar Bargmann-Fock field. The following proposition states that the restriction of $F$ to a horizontal plane gives almost no information about the percolation properties in a plane at small distance $\varepsilon$ from this plane. Note that, on the contrary, knowing $F$ restricted to a slide $\mathbb{R}^2 \times [t, t + \varepsilon]$ (or to any open non-empty subset of $\mathbb{R}^3$) freezes the whole field by analytical rigidity.
Proposition 1.2. For every $t \in \mathbb{R}$, let $P(t) = \mathbb{R}^2 \times \{t\}$. There exists $\alpha > 0$ such that, for any quad $Q$, there exists $c = c(Q) > 0$ such that the following holds: for every sequence $(t_n)_{n \geq 1}$ that satisfies $t_n \geq n^{-\alpha}$, we have

$$\mathbb{E} \left[ \int udW \int vdW \right] = \int uv.$$

Let $q : \mathbb{R}^2 \to \mathbb{R}$ be a $L^2$ function. Most of the time, we will ask that $q$ satisfies some of the following conditions listed below.

Condition 1.3 (Symmetry and regularity). The function $q$ is not identically zero, for every $x \in \mathbb{R}^2$, $q(x) = q(-x)$, and $q$ is symmetric under reflection in the $x$-axis and rotation by $\pi/2$ about the origin. Moreover, $q$ is $C^3$ and there exist $c > 0$ and $\varepsilon > 0$ such that, for every multi-index $\alpha$ such that $|\alpha| \leq 3$, $|\partial^\alpha q(x)| \leq c|x|^{-(1+\varepsilon)}$. Furthermore, the support of the Fourier transform of $q$ contains an open set.

Condition 1.4 (Weaker than Condition 1.3). The function $q$ is $C^2$ and there exist $c, \delta > 0$ such that, for every multi-index $\alpha$ such that $|\alpha| \leq 2$, we have

$$\forall x \in \mathbb{R}^2, |\partial^\alpha q(x)| \leq c|x|^{-(1+\delta)}.$$

Condition 1.5. The function $q$ satisfies

$$q \star q \geq 0.$$

Condition 1.6 (Polynomial decay; depends on a parameter $\beta > 0$). There exist $c > 0$ such that, for every multi-index $\alpha$ such that $|\alpha| \leq 1$, $|\partial^\alpha q(x)| \leq c|x|^{-\beta}$.

Assume that $q$ satisfies Conditions 1.3, 1.5 and 1.6 for some $\beta > 2$. Let $f$ be the planar Gaussian field

$$x \in \mathbb{R}^2 \mapsto f(x) = q \star W(x) = \int_{\mathbb{R}^2} q(x-y)dW_y.$$

Notice that the covariance of $f$ is $(x, y) \in (\mathbb{R}^2)^2 \mapsto q \star q(x-y)$. Moreover, one can show (see for instance Subsection 3.2 of [MV18]) that Condition 1.3 implies that there exists a modification of $f$ which is a.s. $C^2$. In the rest of the paper, we work with this $C^2$ modification (N.B the weaker Condition 1.4 only implies the property that there exists a modification of $f$ which is continuous, see for instance the appendix). Note that the Bargmann-Fock field can be realized by choosing $q(x) = (2/\pi)^{1/4}e^{-|x|^2}$ (which satisfies all the above conditions, and for any $\beta > 0$).

\footnote{Note that this is a direct consequence of $q$ not identically equal to 0 when $q$ is $L^1$, which will be case most of the time for us since we will often assume that Condition 1.6 holds for some $\beta > 2.$}
We extend our above noise sensitivity result to the dynamical processes $t \mapsto f(t)$ defined as follows. Let $(W_t(dx))_{t \geq 0}$ be a planar white noise driven by an Ornstein-Uhlenbeck dynamics. More precisely, we consider a centered Gaussian process $(\int u \, dW_t)_{u,t}$ indexed by $(u, t) \in L^2(\mathbb{R}^2) \times \mathbb{R}_+$ with covariance
\[ E \left[ \int u \, W_t \int v \, W_s \right] = e^{-|t-s|} \int uv. \]
For every $t \geq 0$, let $f(t) = q \ast W_t$.

As shown in the appendix, if we assume that $q$ satisfies Condition 1.4, then there exists a modification of $(x, t) \mapsto f(t, x)$ that is continuous. In the following, we consider such a modification. Note that, in the case of the Bargmann-Fock field, this dynamics is the same as the dynamics from Subsection 1.1.

**Remark 1.7.** For any $t \geq 0$, one may realize the joint coupling $(f(0), f(t))$ as follows:
\[ f(t) = e^{-\frac{1}{2}t} f(0) + \sqrt{1 - e^{-t}} \tilde{f}, \]
where $\tilde{f}$ is an independent copy of $f(0)$.

The following theorem generalizes Theorem 1.1.

**Theorem 1.8.** The content of Theorem 1.1 extends to any $q$ satisfying Conditions 1.3, 1.5 and 1.6 for some $\beta > 2$ (with constants $c$ and $\alpha$ that may depend on $q$).

In Section 4, we will prove a sharp threshold result by relying on the above noise sensitivity result as well as an idea originating from [BKS99] which requires the analysis carried in Section 3. Before stating our result, here is a short overview of the current “state of the art” on the phase-transition of planar Gaussian fields. Assume that that $q$ satisfies Conditions 1.3, 1.5 and 1.6 for some $\beta > 2$. Then,

- Theorem 1.11 of [MV18] (see [BG17a, BM18, RV17a] for the same result with stronger assumptions on $\beta$). For every quad $Q$, there exists $c = c(Q, q) \in [0, 1]$ such that, for every $s > 0$, $c \leq P[\text{Cross}_s(0)] \leq 1 - c$;
- Theorem 1.7 of [Riv19] (see [MV18] for the same result with the less general assumption $q \geq 0$ instead of Condition 1.5: $q \ast q \geq 0$). For every $\ell > 0$ and every quad $Q$, there exists $c = c(\ell, Q, q) > 0$ such that, for every $s > 0$, $P[\text{Cross}_s(0)] \leq e^{-cs}$ and $P[\text{Cross}_s(sQ)] \geq 1 - e^{-cs}$;
- Theorem 1.15 of [MV18]. If we assume furthermore that $q \geq 0$, then the near-critical window is polynomially small in the sense that there exists $\alpha = \alpha(q) > 0$ such that, for every quad $Q$, $1 - P[Cross_{s^{1-\alpha}}(sQ)]$ and $P[Cross_{s^{1-\alpha}}(sQ)]$ go to 0 as $s \to +\infty$.

In the present paper, we generalize the above results by combining in some sense the conclusions of [MV18, Theorem 1.15] with those of [Riv19, Theorem 1.7]. I.e. i) we obtain a new proof of the sharp threshold result from [Riv19] (see Section 4), and ii) we obtain that the near-critical window is polynomially small (see Theorem 1.9 below) when $q \ast q \geq 0$ (rather than $q \geq 0$).

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2Actually, the results from [Riv19] are more general in the sense that they require less regularity conditions on $q$. However, in the present paper, our focus is more on positivity conditions and conditions about the speed of decay of $q$. 
Theorem 1.9. If $q$ satisfies Conditions 1.3, 1.5. and 1.6 for some $\beta > 2$, then there exists $\alpha = \alpha(q) > 0$ such that, for every quad $Q$, $1 - \mathbb{P}[\text{Cross}_{s-\alpha}(sQ)]$ and $\mathbb{P}[\text{Cross}_{s-\alpha}(sQ)]$ go to 0 as $s \to +\infty$.

1.3. $L^2$ versus $L^1$ methods. In this subsection, we wish to compare briefly the methods to prove sharpness results for such models from the present paper and [RV17b, MV18, Riv19]. In [RV17b, MV18], the main intermediate result is the proof that the following quantity goes to $+\infty$ uniformly in $\ell \in \mathbb{R}$

$$\frac{d\mathbb{P}[\text{Cross}_\ell(nQ)]}{dt} \times \frac{1}{\mathbb{P}[\text{Cross}_\ell(nQ)] (1 - \mathbb{P}[\text{Cross}_\ell(nQ)]).} \quad (1.2)$$

In [RV17b] and [MV18], discretization procedures are used in order to apply respectively a KKL type theorem and the OSSS inequality. If $\text{Cross}_\varepsilon^r(nQ)$ is the discrete version of $\text{Cross}_\ell(nQ)$, it was roughly obtained respectively in [RV17b] and [MV18] that (1.2) was less than $O(1/(\sqrt{\log(n)}\varepsilon))$ and $O(n^{-c}/\varepsilon)$. In particular, these estimates are useless in the limit $\varepsilon \to 0$ so it was necessary to be very quantitative on the discretization procedure.

In [Riv19], Rivera uses a Talagrand inequality - which is an analogue of the KKL theorem - and obtains an estimate (not on (1.2) but on another suitable quantity) uniform in $\varepsilon$. The fact that this estimate behaves well when one passes to the limit may come from the fact that Talagrand’s inequality is an inequality on the $L^2$ norm of the gradient (while KKL type inequalities are estimates on the $L^1$ norm of the gradient), which may correspond more to the Gaussian setting. In the present paper, we also obtain an estimate uniform in $1/N$ (which has to be interpreted as the discretization mesh $\varepsilon$), see Step 4 in the proof of Theorem 1.1 written in Section 2. Similarly, we can interprete this by noting that the Schramm-Steif theorem (see Subsection 1.4) is an $L^2$ estimate whereas the OSSS inequality - which is also an estimate involving randomized algorithms - is an $L^1$ estimate.

Let us finally note, in the case of planar percolation models, that KKL and Talagrand type inequalities give a logarithmic upper bound on the size of the near-critical window while the OSSS and Schramm-Steif inequalities give sharper polynomial upper-bounds.

1.4. The main tool: the Schramm-Steif randomized algorithm theorem. The main tool of our proof is the Schramm-Steif randomized algorithm theorem [SS10]. Let us recall this result. We refer to [GS14] for more details on Boolean functions and noise sensitivity. Let $n \in \mathbb{N}^*$ and consider the hypercube $\Omega_n = \{-1, 1\}^n$. We equip $\Omega_n$ with the uniform probability measure $\mathbb{P}$ and we consider the Fourier-Walsh basis $(\chi_S)_{S \subseteq \{1, \ldots, n\}}$ which is the orthonormal basis of $L^2(\Omega_n, \mathbb{P})$ defined by

$$\chi_S(\sigma) = \prod_{i \in S} \sigma_i.$$
Every function \( g : \Omega_n \to \mathbb{R} \) can be decomposed in a unique way as
\[
g = \sum_{S \subseteq \{1, \ldots, n\}} \hat{g}(S) \chi_S.
\]
If the dynamics \( t \mapsto \sigma(t) \) is defined by sampling a configuration \( \sigma(0) \sim P \) and by resampling each bit independently at rate 1, then we have
\[
\text{Cov}[g(\sigma(0)), g(\sigma(t))] = \sum_{S \neq \emptyset} \hat{g}(S)^2 e^{-t|S|}.
\]
Let us recall that a sequence of Boolean function \( g_n : \Omega_{m_n} \to \{0, 1\} \) for some sequence \((m_n)_{n \in \mathbb{N}}\) that goes to \(+\infty\) is said noise sensitive ([BKS99]) if
\[
\text{Cov}[g_n(\sigma(0)), g_n(\sigma(t))] \to_{n \to +\infty} 0.
\]
Let us say that noise sensitivity has been proved for discrete percolation [BKS99, SS10, GPS10] and for some continuous percolation models such as the Poisson Boolean model [ABGM14] and Voronoi percolation [AGMT16, AB18]. In these two last works, Schramm-Steif approach is central. In order to state Schramm-Steif theorem, we need to recall what is a (randomized) algorithm. If \( g : \Omega_n \to \mathbb{R} \), a randomized algorithm that determines \( g \) is a procedure that asks the values of the bits \( i \in \{1, \ldots, n\} \) step by step where at each step the algorithm can ask for the value of one or several bits and the choice of the new bit to ask is based on the values of the bits previously queried. The first bit may be random. We also ask that the algorithm stops once \( g \) is determined. The revealment of the algorithm is the supremum on every bit \( i \) that \( i \) required by the algorithm. The revealment \( \delta(g) \) of \( g \) is the infimum of the revealments of all the algorithms that determine \( g \).

**Theorem 1.10 ([SS10])**. For every \( g : \Omega_n \to \mathbb{R} \) and every \( k \in \mathbb{N}^* \), we have
\[
\sum_{S : |S|=k} \hat{g}(S)^2 \leq k \delta(g) \int g^2 dP.
\]
We will use this theorem as follows: in Section 2, we will approximate the white noise by a discrete white noise with \( \pm 1 \) bits and we will observe that running the above dynamics on the bits of the discrete white noise is the same - in the limit - as the Ornstein-Uhlenbeck dynamics from Section 1.2. Applying Schramm and Steif theorem to the \( \pm 1 \) bits and estimating the revealment thanks to [MV18] will give the result. Actually, in order to define a suitable algorithm, we will have to work with a truncated (i.e. finite range) version of our field.

1.5. A motivation: exceptional times and exceptional planes. Our initial motivation in studying the noise sensitivity of Bargmann-Fock percolation was not our above application to sharp thresholds (Theorem 1.9) but rather to establish the existence of **exceptional times** for different natural dynamics on Bargmann-Fock percolation on \( \mathbb{R}^2 \) which are listed below. In each case, as we explain below we still miss at least one key ingredient in order to prove the existence of exceptional times.

i) Ornstein-Uhlenbeck dynamics on Bargmann-Fock. We already considered this dynamics above. Using the above notations, recall it can be
defined as $f(t, x_1, x_2) = e^{-\frac{1}{2}|x|^2} \sum_{i,j \in \mathbb{N}} a_{i,j}(t) x_i^1 x_2^j$ or as $f(t) = q \ast W_t$ where $q(x) = (2/\pi)^{1/4} e^{-|x|^2}$.

From [BG17a], it is known that for any fixed $t$, a.s. there are no infinite connected component in $C_t := \{x \in \mathbb{R}^2, f(t, x) > 0\}$. An exceptional time in this case is a random time $t$ where an infinite component arises for the level set $C_t = C_t(f(t))$. By analogy with site-percolation on the triangular grid, we conjecture that the following happens.

**Conjecture 1.11.** A.s. under the O.U. dynamics, exceptional times do exist for $t \mapsto f(t)$. Furthermore, if $E \subseteq \mathbb{R}_+$ denotes the random set of such exceptional times, a.s. the Hausdorff dimension of $E$ is $\frac{67}{72}$ (N.B. for dynamical percolation on the triangular lattice, it was shown in [GPS10] that the dimension exceptional times is a.s. $\frac{31}{36}$).

We are far from being able to prove this conjecture. Here is why: already for the classical dynamical percolation model on the square lattice $\mathbb{Z}^2$, it is not known up to now how to prove the existence of exceptional times using the randomized algorithms techniques from [SS10] (the only proof for $\mathbb{Z}^2$ is provided in [GPS10] and would not extend easily to Bargmann-Fock). In fact the situation is worse for Bargmann-Fock than on $\mathbb{Z}^2$: indeed, in order to define a suitable algorithm on the white noise bits, we will have to work with a finite range version of the Bargmann-Fock field. Let $m_n = C \sqrt{\log(n)}$ with $C$ very large. With techniques from [MV18] (see Subsection 3.4 therein), by truncating $q$ the Bargmann-Fock field can be approximated on a quad $nQ$ by a $m_n$-dependent field $f_{\text{trunc}}$, but we cannot obtain less spatial dependencies. As a result, the bound we can get on the revealment is not as good as for $\mathbb{Z}^2$ because, if one wants to reveal $f_{\text{trunc}}(x)$ then one has to reveal all the bits of the white noise at distance $m_n$ from $x$. Moreover, one does not have separation of arms tools for Bargmann-Fock percolation. Because of these two reasons, it looks out of reach at the moment to prove exceptional times for this dynamical model.

Let us now briefly explain why we expect a $67/72$-Hausdorff dimension instead of the classical one $31/36$. There are two ways to see where the difference comes from. 1) If one is looking for an upper-bound on the Hausdorff dimension, then one may proceed exactly as in [SS10] by dividing the unit-interval of times $[0, 1]$ into $\varepsilon^{-1}$ intervals of length $\varepsilon$. Then, on each of these intervals, one tries to have an upper bound on the probability that this interval contains an exceptional time. Usually one proceeds by relying on an easy stochastic domination. In this case though, one cannot hope to stochastically dominate $\bigcup_{0 \leq u \leq \varepsilon} \{x \in \mathbb{R}^2 : f(u, x) > 0\}$ by $\{x \in \mathbb{R}^2, f(0, x) > -\lambda(\varepsilon)\}$ for some small and well-chosen $\varepsilon \mapsto \lambda(\varepsilon)$ (this is due to the fact that at large distances there will be arbitrary large fluctuations). Yet, Appendix A suggests that for any $a > 0$, $\lambda(\varepsilon) := \varepsilon^{1/2-a}$ would give an “almost” such stochastic domination. If one believes that later non-trivial fact plus the believed same universal behaviour for the near-critical Gaussian percolation process $\ell \mapsto \{x, f(0, x) > -\ell\}$, then we obtain that an $\varepsilon$-interval of time should contain an exceptional time with probability at most $\varepsilon^{1/2 \times 5/36 + o(1)}$ which implies our expected bound. 2) The second reason is that by a close inspection of
\[ \frac{d}{dt} \mathbb{P}[f(0), f(t) \in \text{Cross}_0(RQ)] \] for some quad \( Q \), it appears\(^4\) that decorrelation should happen when \( t^{1/2} \approx R^2 \alpha_4^{BF}(R) \) - where \( \alpha_4^{BF}(R) \) is the probability of the 4-arm event from scale 1 to scale \( R \) for the Bargmann-Fock field - which by universality is also believed to be of order \( R^{-3/4} \). This computation suggests that the dynamical correlation length for this O.U. dynamics will be \( t^{-2/3+o(1)} \) (instead of the classical \( t^{-4/3} \)). This also suggests that the Hausdorff dimension of exceptional times should indeed be \( 1 - 2/3 \times 5/48 \).

ii) Exceptional planes for Bargmann-Fock field in \( \mathbb{R}^3 \). Consider now a 3D Bargmann-Fock field \( F \) on \( \mathbb{R}^3 \), and for each \( t \in \mathbb{R} \), let \( f^{\text{hor}}(t) \) be the two-dimensional Bargmann-Fock field obtained by restricting \( F \) to the horizontal plane \( \{(x, y, t)\}_{x,y \in \mathbb{R}} \). We are interested in the following (non-Markovian) dynamics:

\[ t \mapsto f^{\text{hor}}(t). \]

It is easy to check that for every \( t \in \mathbb{R} \) the joint coupling \((f^{\text{hor}}(0), f^{\text{hor}}(t))\) can be realized as follows:

\[ f^{\text{hor}}(t) = e^{-t^2} f^{\text{hor}}(0) + \sqrt{1 - e^{-t^2}} \tilde{f}^{\text{hor}}, \]

where \( \tilde{f}^{\text{hor}} \) is an independent copy of \( f^{\text{hor}}(0) \). In particular, we see here that this dynamics is locally much slower than the above O.U. dynamics on Bargmann-Fock. Despite this slowing down, we claim that a proof of the above conjecture would imply the following one:

**Conjecture 1.12.** A.s. the set \( \mathcal{E}^{\text{hor}} \) of exceptional times for \( t \mapsto f^{\text{hor}}(t) \) is non-empty. Furthermore, a.s. the Hausdorff dimension of \( \mathcal{E}^{\text{hor}} \) is \( 31/36 \) (N.B. same as for the triangular lattice).

(Note this would imply in particular the existence of an infinite 0-level cluster for \( F \), Bargmann-Fock field on \( \mathbb{R}^3 \).)

Let us explain briefly why we expect this smaller dimension compared to the O.U. case and why this should follow from a proof of Conjecture 1.11. As explained in [SS10] for standard dynamical percolation, the value of the Hausdorff dimension follows directly from the knowledge of the “two-point function”, \( \mathbb{P}[0 \leftrightarrow_0 \partial B_R \text{ and } 0 \leftrightarrow_0 \partial B_R] \), where \( \omega_t \) is a dynamical percolation process. In the case of the triangular grid, this two-point function is shown in [GPS10] to behave as \( t^{-5/36+o(1)} \alpha_1(R)^2 \), where \( \alpha_1(R) = \mathbb{P}[0 \leftrightarrow \partial B_R] \). From the above discussion in i), we believe that the two-point function for O.U. dynamics should instead behave as \( t^{-5/72+o(1)} \alpha_1(R)^2 \). If so, not only it would prove Conjecture 1.11 but also, thanks to the above identity for the joint coupling \((f^{\text{hor}}(0), f^{\text{hor}}(t))\), it would imply

\[ \mathbb{P}[0 \leftrightarrow f^{\text{hor}}(0) \partial B_R \text{ and } 0 \leftrightarrow f^{\text{hor}}(t) \partial B_R] = (t^2)^{-5/72+o(1)} \alpha_{1^{\text{BF}}}(R)^2 = t^{-5/36+o(1)} \alpha_{1^{\text{BF}}}(R)^2. \]

As in [SS10, GPS10], this estimate would readily imply Conjecture 1.12.

\(^4\)The study of such a derivative in terms of pivotal events can be extracted from works of Piterbarg [Pit12], see also the more recent approach in Section 2 of [RV17a] or the continuum analogue in [BMR18].
Finally, in order to detect interesting exceptional times, let us point out that one may also try to integrate further on the angle of planes in the spirit of [BS98].

**Notations.** We use the following notations for $\sigma$-algebras: $\mathcal{F}$ is the usual $\sigma$-algebra on the set of continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ ($\mathcal{F}$ is generated by the functions $u \mapsto u(x)$ for every $x \in \mathbb{R}^2$). Moreover, for every subset $D$ of $\mathbb{R}^2$, we let $\mathcal{F}_D$ denote the $\sigma$-algebra generated by $u \mapsto u(x)$ for every $x \in D$.

Finally, we denote by $O(1)$ a positive bounded function, by $\Omega(1)$ a positive function bounded away from 0 and by $\Theta(1)$ a positive function bounded away from 0 and $+\infty$.

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2. **Proof of noise sensitivity.**

In this Section, we prove Theorem 1.8 (and as a byproduct Theorem 1.1 which is a particular case). We will rely on Schramm and Steif randomized algorithm theorem and on the estimates from [MV18] (Sections 3 and 4 but not Section 5 where another randomized algorithm approach is used based on the OSSS inequality rather than [SS10]). Let $q$ satisfying Conditions 1.3, 1.5 and 1.6 for some fixed $\beta > 2$, let $f$ be the $C^2$ random Gaussian field $q \ast W$, and let $Q$ be a quad. The proof is divided into the following steps.

**Step 1.** We first observe that, by linearity and Remark 1.7, we have the following useful rewriting of $(f(0), f(t))$,

\[
\begin{align*}
    f(0) &= q \ast W \\
    f(t) &= e^{-t/2}q \ast W + \sqrt{1-e^{-t}}q \ast \overline{W}
\end{align*}
\]

where $\overline{W}$ is an independent copy of the white noise $W$.

**Step 2.** In [MV18], the following local approximation of the field $f$ is introduced in order to have spatial independency: for any radius $r \geq 1$,

\[
f_r(x) := q_r \ast W,
\]

where $q_r(z) = \chi_r(z)q(z)$ and $\chi_r : \mathbb{R}^2 \to [0, 1]$ is a smooth approximation of $1_{|\cdot| > r}$. More precisely, we ask that $\chi_r$ is smooth, isotropic, that for every $k \geq 1$, the $k^{th}$ derivatives of $\chi_r$ are uniformly bounded, and that

\[
\chi_r(x) = 1 \text{ if } |x| \leq r/2 - 1/4; \chi_r(x) = 0 \text{ if } |x| \geq r/2.
\]

(Note that either $q_r$ is identically equal to 0 or satisfies Conditions 1.3, 1.5 and 1.6 since $q$ does.) In our setting, we are interested in connection events at large scale $n$ (such as $\{f \in \text{Cross}_0(nQ)\}$). It will be convenient at these scales to rely on the approximation

\[
f \approx f_{n^\gamma} \text{ if } \gamma > \frac{1}{\beta - 1}.
\]
This approximation is robust for any monotonic event as can be seen from the following proposition.

**Proposition 2.1** (Proposition 3.11 and Corollary 3.7 of [MV18]). For every \( \gamma > 0 \), there exist \( c = c(q, \beta, \gamma) \) such that, for every \( R \geq 2 \), every \( D \) subset of \( \mathbb{R}^2 \) with diameter less than \( R \), and every monotonic event \( A \in \mathcal{F}_D \), we have

\[
P \left[ \{ f \in A \} \Delta \{ f_{R'} \in A \} \right] \leq c \log(R) R^{1+\gamma(1-\beta)}.
\]

Let us fix some exponent \( \gamma \in ]0,1[ \) such that \( 1+\gamma(1-\beta) < 0 \), which is possible since \( \beta > 2 \). Thanks to Proposition 2.1, it is enough for us to prove the following estimate for some \( \alpha > 0 \) and \( c > 0 \)

\[
\text{Cov}[1_{f_{n\gamma}}(0) \in \text{Cross}_0(nQ), 1_{f_{n\gamma}(t_n) \in \text{Cross}_0(nQ)}] \leq cn^{-\alpha}
\]
as soon as \( t_n \leq n^{-\alpha} \).

**Step 3.** We now proceed to a further approximation step, where one approximates the Gaussian white noise \( W(dx) \) using independent Bernoulli variables. This second approximation step is reminiscent to the definition of \( f_x^\varepsilon \) in [MV18] except that we rely on Bernoulli variables here instead of Gaussian variables. As such we are as close as we may from the setup in [SS10].

Let \( N \geq 1 \) be an integer and let us consider the following discrete white noise on \( \mathbb{R}^2 \)

\[
W^N(x) := N \sum_{v \in \frac{1}{N} \mathbb{Z}^2} \sigma_v 1_{x \in v+[1/N,1/N]^2},
\]

where the random variables \( \sigma_v \) are independent and \( \mathbb{P}[\sigma_v = 1] = \mathbb{P}[\sigma_v = -1] = 1/2 \). We thus define

\[
f^N_{n\gamma} := q_{n\gamma} \star W^N.
\]

Notice that the indicator function \( 1_{f^N_{n\gamma} \in \text{Cross}_0(nQ)} \) is nothing but a Boolean function defined on a hypercube \( \Omega_{n,N} = \{-1, 1\}^{\Theta(n^2N^2)} \). (N.B. \( \Theta(n^2N^2) \) comes from the fact that there are of order \( n^2N^2 \) Bernoulli bits in the \( n^2 \)-neighbourhood of the rescaled quad \( nQ \).) Another important remark at this stage is that if we let each Bernoulli variable \( \sigma_v \) evolve according to a rate 1 Poisson Point process - i.e. \( t \mapsto \sigma_v(t) \) switches its state independently at rate 1 for each \( v \in \frac{1}{N} \mathbb{Z}^2 \) - then it induces a dynamics \( t \mapsto W^N_t \) which is such that, for every \( t \geq 0 \),

\[
(W^N_0, W^N_t) \xrightarrow{\text{law}} (W_0, W_t),
\]

where \( W_t = e^{-t/2}W_0 + \sqrt{1-e^{-t}} \tilde{W}, \tilde{W} \) is a white noise independent of \( W \), and where the convergence in law holds in \( H^{-1-\varepsilon}(S) \times H^{-1-\varepsilon}(S) \) for any square \( S \subset \mathbb{R}^2 \) and any \( \varepsilon > 0 \).\(^5\) Let us end this step by showing the following consequence of (2.1) where, for every \( n \), \( O_n \) is some square that contains the \( n^2 \)-neighbourhood of \( nQ \).

\(^5\) This classical fact follows easily by using Rellich-Kondrachov theorem for tightness, by relying on the fact that the eigenfunctions \( \Psi_n \) of the Laplacian on squares (with eigenvalues \( \lambda_n = \Theta(n) \)) are uniformly bounded, by relying on the following expression for the norm on \( H^s(S) \): \( ||u||^2_{H^s(S)} = \sum_n < u, \Psi_n >^2 \lambda_n^s \), and by noting that the law of a random variable \( h \) in \( H^s(S) \) is determined by the joint law of the Gaussian variables \( \{ h, \Psi_n > \}_n \).
Lemma 2.2. For every $t \in \mathbb{R}_+$ and every $n$, we can couple $(W_0^N, W_t^N)_{N \in \mathbb{N}}$ and $(W_0, W_t)$ such that a.s. the following holds

$$
\|f_{n,\gamma}(0) - f_{n,\gamma}^N(0)\|_{\infty, O_n} \xrightarrow{N \to \infty} 0 \quad \text{and} \quad \|f_{n,\gamma}(t) - f_{n,\gamma}^N(t)\|_{\infty, O_n} \xrightarrow{N \to \infty} 0 .
$$

Note that Lemma 2.2 implies that

$$\text{Cov}[1_{f_{n,\gamma}^N(0) \in \text{Cross}_t(nQ)}, 1_{f_{n,\gamma}(t) \in \text{Cross}_t(nQ)}] \xrightarrow{N \to \infty} \text{Cov}[1_{f_{n,\gamma}^N(1) \in \text{Cross}_t(nQ)}, 1_{f_{n,\gamma}(1) \in \text{Cross}_t(nQ)}]. \quad (2.2)
$$

Proof of Lemma 2.2. One difference here with [MV18] is that the discrete white noise field $W^N$ is less naturally coupled to $W$. Let $t \in \mathbb{R}_+$. By (2.1) and by relying on Skorokhod’s representation theorem, one may couple $(W_0^N, W_t^N)_{N \in \mathbb{N}}$ and $(W_0, W_t)$ on the same probability space so that, for every $n$,

$$
\|W_0^N - W_0\|_{H^{-1-\varepsilon}(O_n)} \to 0; \quad \|W_t^N - W_t\|_{H^{-1-\varepsilon}(O_n)} \to 0 .
$$

Now, using the fact that for any $C^2$ function $h$ with compact support included in $O_n$ we have

$$
\langle W^N - W, h \rangle \leq \|W^N - W\|_{H^{-2}(O_n)} \|h\|_{H^2(O_n)},
$$

we readily conclude (by considering, for any $x$, the function $h(y) = h_x(y) := g_{n,\gamma}(x - y)$).

Step 4. Let $g_{n,N} : \Omega_{n,N} \to \{0, 1\}$ be the indicator function of the crossing event of the quad $nQ$ for $f_{n,\gamma}^N$. Schramm and Steif (Theorem 1.10) show that:

$$
\text{Cov}[1_{f_{n,\gamma}^N(0) \in \text{Cross}_t(nQ)}, 1_{f_{n,\gamma}(t) \in \text{Cross}_t(nQ)}] = \text{Cov}[g_{n,N}(\sigma(0)), g_{n,N}(\sigma(t))]
$$

$$
= \sum_{S \neq \emptyset} \hat{g}_{n,N}(S) e^{-4|S|}
$$

$$
\leq \sum_{k \geq 1} ke^{-4k} \delta(g_{n,N}),
$$

where $\delta(g_{n,N})$ is the revealment of $g_{n,N}$. Note that, by (2.2), it is now sufficient to show that there exists some $\delta_n$ that decay at least polynomially fast and such that, for every $n$, $\delta(g_{n,N})$ converge to $\delta_n$ as $N \to +\infty$. Let us prove this. Let us first define a (randomized) algorithm that determines $g_{n,N}$. In this definition, “discovering” a region of the plane means that we reveal all the bits $\sigma_v$ at distance less than $n^{\gamma}/2$ from this region. Since $g_{n,\gamma}(x) = 0$ for every $x \in \mathbb{R}_+$ such that $|x| \geq n^{\gamma}/2$, this gives us the value of $f_{n,\gamma}^N(x)$ for every $x$ in this region. Let us first define the algorithm in the case where $Q = [0, 1]$ and where the left and right sides are the two distinguished segments. Choose uniformly at random some $k \in \{1, \ldots, [n^{1-\gamma}]\}$ and discover the segment $\{kn^\gamma\} \times \mathbb{R} \cap nQ$. Then, discover all the $1 \times 1$ squares of the grid $\mathbb{Z}^2$ (for instance) that contain a point that is connected to $\{kn^\gamma\} \times \mathbb{R}$ by a path included in the intersection of the region already explored and of $\{f_{n,\gamma}^N > 0\} \cap nQ$. Stop the algorithm when all the connected components of $\{f_{n,\gamma}^N > 0\} \cap nQ$ that intersect $\{kn^\gamma\} \times \mathbb{R}$ have been discovered. Note that this algorithm determines the crossing event. Let $\text{Arm}_0(r, R)$ denote the event that there is a positive continuous path included in $[-R, R]^2 \setminus -r, r]^2$.
that crosses this annulus. There exists $c > 0$ such that the revealment of the above algorithm is less than or equal to

$$cn^{\gamma-1} \sum_{k=1}^{\lfloor n^{-\gamma} \rfloor} \mathbb{P} \left[ f_{n^\gamma}^N \in \text{Arm}_0(cn^{\gamma}, (k-1)n^{\gamma}/c) \right].$$

By Lemma 2.2, the above converges to

$$cn^{\gamma-1} \sum_{k=1}^{\lfloor n^{-\gamma} \rfloor} \mathbb{P} \left[ f_{n^\gamma} \in \text{Arm}_0(cn^{\gamma}, (k-1)n^{\gamma}/c) \right].$$

Remember that - in order to prove Theorem 1.8 - it is sufficient to show that the above decays at least polynomially fast in $n$. By Proposition 2.1, it is enough to show that $\mathbb{P} \left[ f \in \text{Arm}_0(r, R) \right]$ decays at least polynomially fast in $(r/R)$. This is given by Theorem 4.7 of [MV18].

Let us end this section by explaining how to generalize this to any quad $Q$. Let $\eta, \eta'$ be the two distinguished segments of $Q$. First note that there exist $h = h(Q) > 0$ and $m \in \mathbb{N}$ such that there exist $h \times h$ squares of the grid $\mathbb{Z}^2 S_1, \ldots, S_m$ that satisfy i) $S_1$ intersects $\eta$ but not $\eta'$, ii) $S_m$ intersects $\eta'$ but not $\eta$, iii) all the $S_i$’s are distinct and $S_{i+1}$ shares a side with $S_i$ for every $i \in \{1, \ldots, m-1\}$. We then run exactly the same algorithm as in the case $Q = [0, 1]$ but by replacing the line $\{k\} \times \mathbb{R}$ by the $\Theta(n^{1-\gamma})$ lines included in $n \cup_i S_i$ depicted in Figure 1. Now, the arguments are exactly the same as in the case $Q = [0, 1]$. This ends the proof of Theorem 1.8.

3. Proof of Proposition 1.2.

In this section, we prove Proposition 1.2, which is specific to the Bargmann-Fock field. However, it will be a consequence of the following more general result (and of Theorem 1.1).
Proposition 3.1. Let $q$ be a function that satisfies Condition 1.4 and remember that $f(t) = q \ast W_t$. For every event $A \in \mathcal{F}$, we have

$$\text{Var}[\mathbb{P}[f(t) \in A | f(0)]] = \text{Var}[\mathbb{P}[f(t) \in A | W_0]] = \text{Cov}[1_{f(0) \in A}, 1_{f(2t) \in A}].$$

Proof of Proposition 1.2 using Proposition 3.1. Consider a 3D Bargmann-Fock field $F$. For any $t \in \mathbb{R}$, recall $f_{\text{hor}}(t, \cdot) := F(\cdot, t)$ is the restricted planar Bargmann-Fock field to $\mathbb{R}^2 \times \{t\}$. As observed in Section 1.5, for every $t \in \mathbb{R}$ the joint coupling $(f_{\text{hor}}(0), f_{\text{hor}}(t))$ can be realized as follows:

$$f_{\text{hor}}(t) = e^{-\frac{t^2}{2}} f_{\text{hor}}(0) + \sqrt{1 - e^{-t^2}} \tilde{f}_{\text{hor}},$$

where $\tilde{f}_{\text{hor}}$ is an independent copy of $f_{\text{hor}}(0)$. Together with Proposition 3.1 (and Remark 1.7), this implies that

$$\text{Var}[\mathbb{P}[f_{\text{hor}}(t) \in A | f_{\text{hor}}(0)]] = \text{Cov}[1_{f(0) \in A}, 1_{f(2t) \in A}].$$

In particular, Proposition 1.2 is now a direct consequence of Theorem 1.1. \hfill \Box

The rest of this section is devoted to the proof of Proposition 3.1. As we shall explain below, it seems one cannot avoid a spectral proof here. Let us start by recalling the simpler case of Boolean functions. We use the notations from Subsection 1.4. If $g : \{-1, 1\}^n \to \{0, 1\}$ is a Boolean function, then we have the following useful identity: $\mathbb{E}[g(\sigma(t)) | \sigma(0)] = \sum \tilde{g}(S) e^{-t|S|} \chi_S(\sigma(0))$, which leads to

$$\text{Var}[\mathbb{E}[g(\sigma(t)) | \sigma(0)]] = \sum_{S \neq 0} \tilde{g}(S)^2 e^{-2t|S|}.$$

If we combine the above with the facts recalled in Subsection 1.4, we obtain that

$$\text{Var}[\mathbb{E}[g(\sigma(t)) | \sigma(0)]] = \text{Cov}[g(\sigma(2t)), \sigma(0)],$$

which is the discrete analogue of Proposition 3.1. These spectral identities show that proving noise sensitivity in terms of covariance implies the seemingly stronger fact that the whole knowledge of the initial condition $\sigma(0)$ almost says nothing on the event $\{g(\sigma(t)) = 1\}$. At this stage, as the proof of Theorem 1.1 proceeds by approximation,

$$1_{f \in A} \approx g_N(\sigma)$$

where $g_N$ is a Boolean function on $\Omega_N = \{-1, 1\}^{\Theta(N^2)}$, it seems natural to conclude by approximation, using that we probably have

$$\text{Var}[\mathbb{E}[g_N(\sigma(t)) | \sigma(0)]] \to \text{Var}[\mathbb{P}[f(t) \in A | f(0)]].$$

But conditional expectations are in general not continuous functions of the conditioning. Because of this, we argue differently as below. Before writing the proof, let us explain why Theorem 1.1 and Proposition 1.2 seem counter-intuitive even from the Fourier point of view.

Remark 3.2. Remember that $F$ is a 3D Bargann-Fock and let $q(x) = q(x_1, x_2, x_3) = (2/\pi)^{1/4} e^{-|x|^2}$. As in Section 2, we can approximate $F$ by a field $F^N_r = q_r \ast W^N$ where $W^N$ is now a (3D) discrete white noise and $q_r$ is a truncation of $q$ (for some suitable $r \leq n$). Let $P(t) = \mathbb{R}^2 \times \{t\}$, let $Q$ be a quad, and
let $g = g_{r,n,N} : \{-1,1\}^{\Theta(n^3N^3)} \to \{0,1\}$ be the Boolean function such that $g_{r,n,N}(\sigma) = 1_{(F^N_{r})_{t(0)\in \text{Cross}_0(nQ)}}$. Then we can show that

$$\text{Cov}[1_{(F^N_{r})_{t(0)\in \text{Cross}_0(nQ)}}, 1_{(F^N_{r})_{t(l)\in \text{Cross}_0(nQ)}}] = \sum_S \hat{g}(S)\hat{g}(S_l)$$

where $S_l$ is $S$ translated by $(0,0,-t)$ and where the sum is on subsets of a 3D grid of size $\Theta(n^3N^3)$. It does not seem obvious at all at first sight that the above is small when $n$ is large. Indeed, it does not seem clear why the above does not behave like the following analogous 2D quantity: consider a planar Bargmann-Fock field, some $x_0 \in \mathbb{R}^2$ and let $h = h_{r,n,N} : \{-1,1\}^{\Theta(n^2N^2)} \to \{0,1\}$ be the Boolean function corresponding to the crossing of $nQ$ by the planar analogue $f^N_r$ of $F^N_r$. Then,

$$\text{Cov}[1_{f^N_r \in \text{Cross}_0(nQ)}, 1_{f^N_r(-+x_0) \in \text{Cross}_0(nQ)}] = \sum_S \hat{h}(S)\hat{h}(S_{x_0})$$

where $S_{x_0}$ is $S$ translated by $x_0$ and where the sum is on subsets of a 2D grid of size $\Theta(n^2N^2)$. The above does not go to zero; $1_{f^N_r \in \text{Cross}_0(nQ)}$ and $1_{f^N_r(-+x_0) \in \text{Cross}_0(nQ)}$ are highly correlated!

**Proof of Proposition 3.1.** First note that (for instance by Remark 1.7) the distribution of $f(t)$ conditioned on $W_0$ is the same as the distribution of $f(t)$ conditioned on $f(0)$. This implies the first equality. Let us now prove the second equality. In our present continuous setting, we may apply the exact same idea as in the discrete by relying on the appropriate spectral identities. Here we shall use the fact that our events can be seen as functionals in $L^2(\sigma(W(dx)))$ and use **Wiener Chaos expansion** in $L^2(\sigma(W(dx)))$ which is the good analogue of the discrete Fourier expansion. We may thus write for every event $A \in \mathcal{F}$ (recall $f := q \ast W$):

$$1_{f \in A} = \sum_{k=0}^{\infty} \int_{(\mathbb{R}^2)^k} h_k(z_1, \ldots, z_k) dW(z_1) \ldots dW(z_k)$$

where each $h_k \in L^2((\mathbb{R}^2)^k)$ and

$$\mathbb{P}[f \in A] = \sum_{k=0}^{\infty} \int_{(\mathbb{R}^2)^k} h_k^2(z_1, \ldots, z_k) dz_1 \ldots dz_k.$$
Even though these identities are classical, let us say a few words on their proofs. (a) First, let us write $H_k(t) = \int_{(\mathbb{R}^2)^k} h_k(z_1, \ldots, z_k)dW_t(z_1) \ldots dW_t(z_k)$ and let us note that the identities
\[
\text{Cov}[H_k(0), H_k(t))] = e^{-tk} \int_{(\mathbb{R}^2)^k} h_k^2(z_1, \ldots, z_k)dz_1 \ldots dz_k;
\]
\[
\text{Var}[E[H_k(t) | W(0)]] = e^{-tk} \int_{(\mathbb{R}^2)^k} h_{n,k}^2(z_1, \ldots, z_k)dz_1 \ldots dz_k
\]
are direct consequences of the definition of multiple integrals with respect to the white noise in the case of special simple functions (and of density arguments), see for instance Subsection 2 of [CSZ16]. (b) Second, let us note that a direct use of Fubini’s theorem would not be sufficient here to exchange sum and expectation. One way to proceed is as follows. If $X, Y$ are any two variables in $L^2(\sigma(W(dx)))$, their cut-off Wiener chaos expansions (below order $n$, say) $X_n, Y_n$ are such that $X_n \xrightarrow{L^2} X$ and $Y_n \xrightarrow{L^2} Y$. Now, simply by linearity of the expectation, we have that $\mathbb{E}[X_nY_n]$ is the finite sum over $k \in \{0, \ldots, n\}$ of the corresponding $k$-fold integrals. Using the fact that $X_nY_n \xrightarrow{} XY$ in $L^1$ and therefore that $\mathbb{E}[X_nY_n] \rightarrow \mathbb{E}[XY]$, this justifies why one can interchange sum and expectation.

4. Noise sensitivity implies sharp threshold

In this section, we explain how one can combine our noise sensitivity result together with the analysis in Section 3 to prove results about the phase transition. In particular, we prove Theorem 1.9. We begin with the following result.

**Proposition 4.1.** Let $q$ be a function that satisfies Condition 1.4. Also, let $(A_n)_{n \in \mathbb{N}}$ be a sequence of events in $\mathcal{F}$ and let $(t_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers. If
\[
\text{Cov}[1_{f(0) \in A}, A_{f(t_n) \in A}] \xrightarrow{n \to +\infty} 0
\]
then for every $(s_n)_{n \in \mathbb{N}}$ sequence of positive numbers that satisfies $t_n = o(s_n)$, the number of times that $1_{f(0) \in A}$ switches from 0 to 1 between 0 and $s_n$ goes to $+\infty$ in probability as $n \rightarrow +\infty$.

**Proof.** The proof is exactly the same as the analogous result from [BKS99] for dynamical Bernoulli percolation i.e. this is a rather direct consequence of Proposition 3.1 and of the Markov property of the dynamics $t \mapsto f(t)$. We refer to Section 8 of Chapter I of [GS14] for the proof. \hfill \Box

**Proof of Theorem 1.9.** For every $\alpha > 0$, let $M(\alpha)$ be the supremum of $|f(t,x) - f(0,x)|$ for $x \in nQ$ and $t \in [0, n^{-\alpha}]$. By Proposition 4.1 and Theorem 1.8, there exists $\alpha > 0$ such that, with high probability, there exists $t \in [0, n^{-\alpha}]$ such that $f(t) \in \text{Cross}_0(nQ)$. Note that - since the crossing events are increasing - this implies that with high probability $f(0) \in \text{Cross}_{M(\alpha)}(nQ)$. Hence, it is sufficient to show that with high probability, $M(\alpha)$ is polynomially small in $n$. By Lemma A.1 (applied to $a = 1/2$) and an union bound on the $1 \times 1$ boxes of the grid $\mathbb{Z}^2$ that intersect $nQ$, we have
\[
\mathbb{P}[M(\alpha) \geq O(1)n^{-\alpha/4}] \leq O(1)n^2 \exp(-\Omega(1)n^{-\alpha/2}).
\]
This completes the proof. □

Let us note that in [MV18] (Section 6) we have shown that, if one assumes that \( q \) satisfies Conditions 1.3 and 1.6 for some \( \beta > 2 \) and if

\[
\forall \ell > 0, \mathbb{P}[f \in \text{Cross}_\ell(nQ)] \xrightarrow{n \to +\infty} 1
\]

for the quad \( Q = [0, 2] \times [0, 1] \) whose distinguished sides are the left and right sides, then we even have that the convergence is **exponentially fast**. As a result, Theorem 1.9 gives a randomized algorithm proof of the recent sharpness result from [Riv19] which we have stated in Subsection 1.2 (in order to prove the results in the case \( \ell < 0 \), one can use the results at \( \ell > 0 \) and the duality property of the model, see for instance Lemma A.9 of [RV17a]).

**Appendix A. An estimate for a dynamical planar Gaussian field**

Let \((W_t(dx))_{t \geq 0}\) be the dynamical planar white noise defined in Subsection 1.2 and let \( f(t, \cdot) = q \ast W_t \) for some \( q: \mathbb{R}^2 \to \mathbb{R} \) that satisfies Condition 1.4 for some \( c, \delta > 0 \). So, there exist \( c, \delta > 0 \) such that, for every \( x \in \mathbb{R}^2 \) and every multi-index \( \alpha \) with \(|\alpha| \leq 2\), we have

\[
|\partial^\alpha q(x)| \leq c(1 \vee \|x\|)^{-(1+\delta)}.
\] (A.1)

Below, the constants in the \( \Omega(1) \) and \( O(1) \)'s only depend on \( c \) and \( \delta \) except if we add some subscript. We can (and do) consider a modification of \((t, x) \mapsto f(t, x)\) that is continuous in \( t \) and \( x \) (see the application of Kolmogorov’s continuity theorem below). The goal of this appendix is to prove the following lemma.

**Lemma A.1.** Let \( h \in [0, 1] \) and \( a \in [0, 1] \). Then,

\[
\mathbb{P}\left[ \sup_{t \in [0, h], x \in [0, 1]^2} |f(t, x) - f(0, x)| \geq O_a(1)h^{(1-a)/2}\right] \leq O_a(1) \exp(-\Omega_a(1)h^{-a}).
\]

**Remark A.2.** Note that the lemma is sharp is the sense that

\[
\mathbb{P}\left[ \sup_{t \in [0, h], x \in [0, 1]^2} |f(t, x) - f(0, x)| \geq h^{(1+a)/2}\right]
\]

goes to 1 as \( h \) goes to 0, for any \( a > 0 \). Actually, this is even true pointwise: for every \( x \), the typical order of \( |f(t, x) - f(0, x)| \) is \( \sqrt{t} \).

To prove Lemma A.1, we use the two following classical results that we state for an a.s. continuous centered Gaussian process \((X(t, x))_{t \in [0, h], x \in [0, 1]^2}\) on \([0, h] \times [0, 1]^2\). Let \( M := \sup_{t \in [0, h], x \in [0, 1]^2} X(t, x) \).

**Theorem A.3** (Dudley’s theorem). Consider the following pseudo-metric on \([0, h] \times [0, 1]^2\)

\[
d((t, x), (s, y)) = \sqrt{\mathbb{E}[(X(t, x) - X(s, y))^2]}.
\]

Also, let \( \Delta \) be the \( d \)-diameter of \([0, h] \times [0, 1]^2\) and, for every \( \varepsilon > 0 \), let \( N(d, \varepsilon) \) denote the minimal number of \( d \)-balls of radius \( \varepsilon \) required to cover \([0, h] \times [0, 1]^2\). Then,

\[
\mathbb{E}[M] \leq 24 \int_0^{\Delta} \sqrt{\log(N(d, \varepsilon))} \, d\varepsilon.
\]
Theorem A.4 (Borell-Tsirelson-Ibragimov-Sudakov (BTIS) inequality; Theorem 2.9 of [AW09]). We have

$$
\mathbb{P} \left[ M \geq u + \mathbb{E} [M] \right] \leq 2 \exp \left( -\frac{u^2}{\sup_{t \in [0,h], x \in [0,1]^2} \mathbb{E} [X(t,x)^2]} \right).
$$

Proof of Lemma A.1. We will apply the above theorems to the field $$X(t,x) = f(t,x) - f(0,x)$$. First note that, by the BTIS inequality applied to $$u = h^{(1-a)/2}$$, it is sufficient - in order to prove the lemma - to show that (a) $$\mathbb{E} [M] \leq O(\sqrt{h})$$ and (b) $$\mathbb{E} [(f(t,x) - f(0,x))^2] \leq O(h)$$. In order to prove (a) and (b), we are going to show that, for every $$s,t \in [0,h]$$ and $$x,y \in [0,1]^2$$,

$$
\mathbb{E} \left[ (f(t,x) - f(0,x) - (f(s,y) - f(0,y)))^2 \right] \leq O(1) \left( |t-s| + h |x-y|^2 \right).
$$

(A.2)

Let us first explain why (A.2) implies the estimates (a) and (b). This implies readily (b). The fact that this also implies (a) is a consequence of Dudley’s theorem. Indeed, first note that (A.2) implies that $$\Delta \leq O(\sqrt{h})$$.

Let $$\epsilon \in [0,\Delta]$$ and note that (by (A.2)), for every $$\epsilon \leq \sqrt{h}$$, the set $$[0,\epsilon^2] \times [0,\epsilon/\sqrt{h}]^2 \subset [0,h] \times [0,1]^2$$ is included in a $$d$$-ball of radius $$\leq O(1)\epsilon$$. Moreover, one can cover $$[0,h] \times [0,1]^2$$ by using $$\leq O(1) h^2/\epsilon^4$$ sets obtained by translating $$[0,\epsilon^2] \times [0,\epsilon/\sqrt{h}]^2$$. As a result,

$$
N(d, \epsilon) \leq O(1) h^2/\epsilon^4,
$$

and thus, by Dudley’s theorem,

$$
\mathbb{E} [M] \leq 24 \int_0^{O(\sqrt{h})} \sqrt{\log(O(h^2/\epsilon^4))} d\epsilon \leq O(\sqrt{h}).
$$

This ends the proof of Lemma A.1 provided that we prove (A.2). Let us end this appendix by showing this variance estimate. Let $$x,y \in [0,1]^2$$ and $$s,t \in [0,h]$$. We have

$$
\mathbb{E} \left[ (f(t,x) - f(0,x) - (f(s,y) - f(0,y)))^2 \right] = \mathbb{E} [(f(t,x) - f(0,x))^2] + \mathbb{E} [(f(s,y) - f(0,y))^2]
$$

$$
-2 \mathbb{E} [(f(t,x) - f(0,x))(f(s,y) - f(0,y))]
$$
\[ (2 - 2e^{-t}) \int q^2 + (2 - 2e^{-s}) \int q^2 \]
\[-2(e^{-|t-s|} + 1 - e^{-s} - e^{-t}) \int q(x - z)q(y - z)dz \]
\[= (4 - 2e^{-t} - 2e^{-s}) \int q^2 - (2e^{-|t-s|} + 2 - 2e^{-s} - 2e^{-t}) \int q(z)q(z + y - x)dz \]
\[= (4 - 2e^{-t} - 2e^{-s}) \int q(z)(q(z) - q(z + y - x))dz \]
\[+ (2 - 2e^{-|t-s|}) \int q(z)q(z + y - x)dz. \]

Let us first note that
\[ 0 \leq 4 - 2e^{-t} - 2e^{-s} \leq O(h); \quad 0 \leq 2 - 2e^{-|t-s|} \leq O(|t - s|) \]
and
\[ \left| \int q(z)q(z + y - x)dz \right| \leq O(1). \]

Let us estimate \( \int q(z)(q(z) - q(z + y - x))dz \). By Taylor’s theorem and (A.1), for every \( z \in \mathbb{R}^2 \) and every multi-index \( \alpha \) such that \(|\alpha| = 2\), there exists a function \( R_{\alpha} : \mathbb{R}^2 \to \mathbb{R} \) such that \( ||R_{\alpha}||_\infty \leq O(|z \vee 1|^{1+\delta}) \) and \( q(z) - q(z + y - x) \) equals
\[ \partial^{(1,0)} q(z) (x_1 - y_1) + \partial^{(0,1)} q(z) (x_2 - y_2) + \sum_{\alpha \in \mathbb{N}^2, \alpha_1 + \alpha_2 = 2} R_{\alpha} (x - y) \prod_{i=1}^{2} (x_i - y_i)^{\alpha_i}. \]

By using that \( \int q \partial^{(1,0)} q = \int q \partial^{(0,1)} q = 0 \) (by integration by parts), we obtain that
\[ \left| \int q(z) (q(z) - q(z + y - x))dz \right| \leq O(1) \sum_{\alpha \in \mathbb{N}^2, \alpha_1 + \alpha_2 = 2} \prod_{i=1}^{2} (x_i - y_i)^{\alpha_i} \leq O(|x - y|^2). \]

This implies the desired estimate (A.2) and completes the proof. \( \square \)

Let us conclude this appendix by explaining why \( (t, x) \mapsto f(t, x) \) is a.s. continuous. A similar computation for the variance gives that, for every \( t, s \in [0, 1] \) and \( x, y \in [0, 1]^2 \),
\[ \mathbb{E} \left[ (f(t, x) - f(s, y))^2 \right] \leq O \left( |t - s| + |x - y|^2 \right), \]
so for every \( p > 0 \) we have
\[ \mathbb{E} \left[ (f(t, x) - f(s, y))^p \right] \leq O_p(1) \left( (|t - s| + |x - y|^2)^{p/2} \right). \]

In particular, if we apply apply Kolmogorov’s continuity theorem to some \( p > 2 \) (and if we use transitivity), we obtain that there exists a modification of \((t, x) \mapsto f(t, x)\) that is a.s. continuous.
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