DEGENERATIONS TO SECANT CUBIC HYPERSURFACES AND LIMITING HODGE STRUCTURE

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ABSTRACT. The secant variety of the Veronese surface is a singular cubic fourfold. The degeneration of Hodge structures of one-parameter degenerations to this secant cubic fourfold is a key ingredient for B. Hassett and R. Laza in studying the moduli space of cubic fourfolds via the period mapping. We generalize some of their results to the cubic hypersurface that is the secant variety of a Severi variety. Specifically, we study the limit mixed Hodge structures associated to one-parameter degenerations to the secant cubic hypersurface. Considering S. Usui's partial compactification of a period domain for Hodge structures of general weights, we apply the limit mixed Hodge structure to characterize a local extension of the period map for the corresponding cubic hypersurfaces.

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INTRODUCTION

The secant variety $X_0 := Sec(S)$ of the Veronese surface $S \subset \mathbb{P}^5$ is a cubic fourfold singular along $S$. It is a semi-stable cubic fourfold that plays an important role in the study of moduli space of cubic fourfolds. In the paper [24], R. Laza showed that the period map for cubic fourfolds induces an isomorphism

$$\mathcal{F}^s \xrightarrow{\sim} \Gamma \backslash (\mathcal{D} - \mathcal{H}_2),$$

where $\mathcal{F}^s$ is the moduli space of cubic fourfolds with at worst simple singularities, $\mathcal{D}$ is the period domain for cubic fourfolds, $\Gamma$ is an arithmetic subgroup of $\text{Aut}(\mathcal{D})$, and $\mathcal{H}_2$ is a $\Gamma$-invariant arrangement of hyperplanes of discriminant two. Consider one-parameter families of smooth cubic fourfolds that degenerate to the secant cubic fourfold $X_0$. According to the theory of limiting Hodge structure by W. Schmid
and J. Steenbrink [30, 31], one can associate a canonical mixed Hodge structure to a one-parameter degeneration to $X_0$. B. Hassett proved that the limit mixed Hodge structure is pure and a special Hodge structure of discriminant two, see [13, §4.4]. It reflects that the divisor $\Gamma \setminus \mathcal{H}_2$, which parametrizes the Hodge structures of degree two $K3$ surfaces, can be viewed as the limit period points of smooth cubic fourfolds that degenerate to the secant cubic fourfold. In the present paper, we aim to study the degeneration of Hodge structures arising from deformations of a cubic hypersurface obtained as the secant variety of a Severi variety.

The Veronese surface is a so-called Severi variety. F. Zak proved that the secant variety $\text{Sec}(S)$ of a nondegenerate nonsingular projective variety $S \subset \mathbb{P}^{m+1}$ covers the ambient space $\mathbb{P}^{m+1}$ if $\dim S = n > \frac{2(m-1)}{3}$. The Severi variety is the variety $S^n \subset \mathbb{P}^{m+1}$ having the maximal dimension $\dim S = n = \frac{2(m-1)}{3}$ among those with degenerate secant $\text{Sec}(S) \neq \mathbb{P}^{m+1}$. The complete classification of the Severi varieties is also given by F. Zak [34, Thm. 4.7]. Up to projective equivalence, there are only four Severi varieties:

1. $n = 2$, $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, the Veronese surface;
2. $n = 4$, $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$, the Segre fourfold;
3. $n = 8$, $\text{Gr}(2,6) \hookrightarrow \mathbb{P}^{14}$, the Plücker embedding of the Grassmannian of lines in $\mathbb{P}^5$;
4. $n = 16$, $E_6 \hookrightarrow \mathbb{P}^{26}$, the Cartan variety (or the Cayley plane $\mathbb{O}\mathbb{P}^2$), given by a minimal irreducible representation of the algebraic group $E_6$.

It is known that, like the case of the Veronese surface, the secant variety of a Severi variety is always a cubic hypersurface in $\mathbb{P}^{m+1}$, which is singular along $S$, see [34, Thm. 2.4].

Let $F$ be the equation of the secant cubic $\text{Sec}(S)$, and $G$ be a nonsingular cubic equation such that $G|_S$ cuts out a smooth hypersurface $V$ in $S$. The equation $F + tG = 0$ with $t$ the coordinate of the open unit disk $\Delta$ presents a one-parameter degeneration to $\text{Sec}(S)$. Following Mumford’s semistable reduction theorem, we can associate a specific semistable degeneration $f : X \to \Delta$ to the one-parameter degeneration. Denote by $H^m_{\text{lim}}$ the middle cohomology of a smooth cubic hypersurface in $\mathbb{P}^{m+1}$. Our main theorem characterizes the limit mixed Hodge structure on $H^m_{\text{lim}}$ associated to $f$.

**Theorem 0.1.** Let $S^n \subset \mathbb{P}^{m+1}$ be a Severi variety with $\dim S > 2$. Then the monodromy transformation of the semistable degeneration $f$ has order two, and the associated monodromy weight filtration is of the form

\[(1) \quad 0 \subset W_{m-1} \subset W_m \subset W_{m+1} = H^m_{\text{lim}},\]

with $\dim W_{m-1} = 1$. The canonical polarized Hodge structure on the subquotient $\text{Gr}^W_m := W_m/W_{m-1}$ determined by the mixed Hodge structure is isomorphic to the middle cohomology of the hypersurface $V \subset S$. The Hodge structure on $\text{Gr}^W_{m-1}$ (resp. $\text{Gr}^W_{m+1}$) is a Tate twist of weight $\frac{-m}{2}$ (resp. $\frac{1+m}{2}$).

For $\dim S > 2$ the mixed Hodge structure on $H^m_{\text{lim}}$ is not pure any more, which is different from the case of cubic fourfolds. But still, the degenerating Hodge structure on the main subquotient $\text{Gr}^W_m$ is closely related to the hypersurface $V$ in $S$ defined by the degeneration. Such pattern similarly appears in the case of cubic fourfolds.
The main tool to prove Theorem 0.1 is the Clemens-Schmid exact sequence attached to a semistable degeneration. The construction of the specific semistable degeneration \( f : X \to \Delta \) follows B. Hassett’s treatment for the secant cubic fourfold \([13, \S 4.4]\). It is essential to compute the cohomology of a quadric fibration as one component in the central fiber of the semistable degeneration. For this purpose we generalize A. Beauville’s computation of the cohomology of quadric fibrations over \( \mathbb{P}^2 \), cf. \([2]\).

The degeneration \( f : X \to \Delta \) defines a period map on the punctured disk \( \Delta^* \). Griffiths generally conjectured the existence of a partial compactification of any period domain such that any period map defined on \( \Delta^* \) can be continued across the puncture. For the bounded symmetric domain, the Baily-Borel compactification \([1, 3]\) serves as such a role. For general cases there are a series of research \([18, 19, 21]\) considering the construction of such a compactification in terms of mixed Hodge structures or nilpotent orbit cones. In this paper, we employ a particular partial compactification, introduced by S. Usui \([33]\). Our purpose is using the result of Theorem 0.1 to study local extension property of the period map on the moduli space of cubic hypersurfaces.

Suppose that \( \mathcal{F} \) is the GIT compactification of the moduli space of smooth cubic hypersurfaces in \( \mathbb{P}^{m+1} \), and \( \omega \in \mathcal{F} \) is the point representing the secant cubic \( \text{Sec}(S) \). Consider Kirwan’s blow-up of \( \mathcal{F} \) at \( \omega \). Let \( \mathcal{M} \) be the exceptional divisor. Through Luna’s slice theorem \([28\text{ p. 198}]\), we can prove

**Proposition 0.2.** The exceptional divisor \( \mathcal{M} \) can be identified with certain GIT-quotient of hypersurfaces in the Severi variety \( S \subset \mathbb{P}^{m+1} \) cut off by cubics in \( \mathbb{P}^{m+1} \).

By the construction of the blowing up, a generic point in \( \mathcal{M} \) corresponds to a semistable degeneration of a one-parameter deformation of the secant cubic, also a period map

\[ \varphi : \Delta^* \to \Gamma \backslash \mathcal{D} \]

where \( \Gamma \backslash \mathcal{D} \) is the global period domain for Hodge structures of smooth cubic hypersurfaces in \( \mathbb{P}^{m+1} \). Let \( \overline{\Gamma \backslash \mathcal{D}} \) denote Usui’s partial compactification by adding suitable boundary components \( \mathcal{B}(W_*) \) defined by the specific weight filtrations \( W_* \) of the form \([1]\). We realize \( \mathcal{B}(W_*) \) indeed parametrizes Hodge structures on \( Gr^W_m \) of the type \( H^{n-1}(V) \) as in Theorem 0.1. Let \( \overline{\varphi} : \Delta \to \overline{\Gamma \backslash \mathcal{D}} \) be the extended map. Then we show that the limit period point \( \overline{\varphi}(0) \in \mathcal{B}(W_*) \) is exactly the Hodge structure on \( Gr^W_m \) determined by the limit mixed Hodge structure of \( \varphi \). In conclusion, we have

**Theorem 0.3.** Under Kirwan’s blowing up, the rational period map \( \mathcal{P} : \mathcal{F} \to \overline{\Gamma \backslash \mathcal{D}} \) can be (generically) continued across the exceptional divisor \( \mathcal{M} \) such that the image of \( \mathcal{M} \) is contained in the boundary component \( \mathcal{B}(W_*) \). Moreover, the extended map \( \mathcal{P}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{B}(W_*) \) is exactly the period map for the hypersurfaces that \( \mathcal{M} \) parametrizes.

As for the organization of this paper, the first section is a review of basic notions and properties for limit mixed Hodge structures and the Clemens-Schmid exact sequence. In Section 2 we describe the explicit semistable degeneration of the deformation of secant cubic hypersurface, and prove Theorem 0.1 (=Theorem 2.4). The third section is devoted to the cohomology of quadric fibrations with certain mild degenerations. The local extension property of the period map is discussed.
in Section 4, where the proof of Proposition 0.2=Corollary 4.3 and Theorem 0.3=Theorem 4.4 are given. The $SL_2$-orbit theory is indispensable to study the extension property. For convenience we add the appendix A to collect notions and results of the $SL_2$-orbit that needed.

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1. Degeneration of Hodge structures

1.1. Limiting mixed Hodge structures. Let $\pi: X^* \to \Delta^*$ be a family of smooth projective varieties over the punctured disk $\Delta^*$. Fix a non-negative integer $m$. Let $H^m_{\mathbb{Q}}$ be the vector space that is isomorphic to the cohomology group $H^m(X, \mathbb{Q})$ of any fiber $X$ of the family $\pi$. Let $D$ be the corresponding classifying space of polarized Hodge structures on $H$. The variation of polarized Hodge structures of the family $\pi$ defines a period map

$$\phi: \Delta^* \to \Gamma\backslash D$$

where the monodromy group $\Gamma$ is generated by the monodromy transformation $T: H \to H$ of the family $\pi$.

**Theorem 1.1.** [22, Monodromy theorem] The monodromy transformation $T$ is quasi-unipotent, and the index of the unipotency is at most $m$, i.e., there exists an integer $k$ such that $(T^k - 1)^{m+1} = 0$.

Let $t$ be the coordinate on $\Delta^*$. By taking the base change $t \mapsto t^k$, we may assume $T$ is unipotent. Define the nilpotent map $N$ to be

$$N := \log(T) = -\sum_{n=1}^{m} \frac{(I - T)^n}{n}.$$

Note that the logarithm of $T$ is well-defined since the index of the unipotency of $T$ is finite. In particular, the index of $T$ is equal to the index of $N$. Consider the universal covering

$$e: \mathfrak{h} \to \Delta^*, \quad e(z) = e^{2\pi iz}$$

by the upper half plane $\mathfrak{h}$. Since $\mathfrak{h}$ is simply connected, the period map $\phi$ is lifted to

$$\begin{array}{ccc}
\mathfrak{h} & \xrightarrow{\tilde{\phi}} & D \\
\downarrow e & & \downarrow \\
\Delta^* & \xrightarrow{\phi} & \Gamma\backslash D
\end{array}$$

The lifting $\tilde{\phi}$ satisfies $\tilde{\phi}(z + 1) = T\tilde{\phi}(z)$ for $z \in \mathfrak{h}$. Set the map $\tilde{\psi}: \mathfrak{h} \to \tilde{D}$ by

$$\tilde{\psi}(z) := \exp(-zN)\phi(z)$$

where $\tilde{D}$ is the compact dual of $D$. We have

$$\tilde{\psi}(z + 1) = \exp(-(z + 1)N)\phi(z + 1) = \exp(-zN)T^{-1}T\tilde{\phi}(z) = \tilde{\psi}(z).$$

It implies that $\tilde{\psi}$ descends to a single-valued map $\psi: \Delta^* \to \tilde{D}$. Cornalba and Griffiths [8] proved that $\psi$ extends across the puncture to a map $\psi: \Delta \to \tilde{D}$. The
filtration $\psi(0) \in \mathcal{D}$ is called the limiting Hodge filtration and is usually denoted by

$$\{F^p_n\} := \lim_{\text{Im}(z) \to \infty} \exp(-zN)\tilde{\phi}(z) \in \mathcal{D}.$$  

The description of the monodromy weight filtration is a matter of linear algebra as follows.

**Proposition 1.2 (30 Lem. 6.4).** Let $H$ be a finite dimensional linear space over a field of characteristic zero. Let $N : H \to H$ be a linear map with $N^{m+1} = 0$. Then there is a unique increasing filtration $W(N)$

$$0 \subset W_0 \subset \cdots \subset W_m = H$$

such that

1. $N(W_l) \subset W_{l-2}$,
2. the map $N^l : Gr^{W}_{m+l} \to Gr^{W}_{m-l}$ is an isomorphism for all $l \geq 0$.

For any $l \geq 0$, define the primitive part $P_{m+l} \subset Gr^W_{m+l}H$ to be the kernel of

$$N^{l+1} : Gr^W_{m+l}H \to Gr^W_{m-l-2}H,$$

and set $P_{m-l} = 0$. Then there is the decomposition of Lefschetz type

$$Gr^W_k H \cong \bigoplus_i N^i(P_{k+2i}), i \geq \max(m-k,0)$$

If $N$ is an infinitesimal isometry of a nondegenerate form $S$ on $H$, that is,

$$S(Nu,v) + S(u,Nv) = 0, \ \forall u, v \in H,$$

then $W_k = W_{2m-l-1}$. Moreover, the bilinear form $S_l := S(\cdot, N^l \cdot)$ is nondegenerate on $Gr^W_{m+l}$, and the bilinear form $S_l := S((N^l)^{-1} \cdot, \cdot)$ is nondegenerate on $Gr^W_{m+l}$.

The following is another form of decomposition on the subquotient $Gr^W_k$ for $k \leq m$, which will be useful in the proof of Theorem 2.4.

**Lemma 1.3.** Let $K$ denote the kernel of $N : H \to H$, and let $Gr^W_k K$ be $(W_k \cap K)/(W_{k-1} \cap K)$. If $k \leq m$, we have the decomposition

$$Gr^W_k H \cong \bigoplus_{a=0}^{\lfloor \frac{k}{2} \rfloor} Gr^W_{k-2a}(K)$$

**Proof.** By abuse of notation we denote $N$ for the induced map $N : Gr^W_i H \to Gr^W_{i-2}H$. We claim that $Gr^W_i K = Ker(N)$ when $i \leq m$.

The inclusion $Gr^W_i K \subseteq Ker(N)$ is obvious. Assume that $\tilde{v} \in Ker(N)$. Choose a representative $v \in W_iH$. Then $N(v) \in W_{i-3}H$. The property (2) on the weight filtration in Proposition 1.2 implies that $N : W_{i-1}H \to W_{i-3}H$ is surjective. So there exists an element $u \in W_{i-1}H$ such that $N(u) = N(v)$. Then we have $u - v \in W_iH \cap K$, and thus $\tilde{v} = [u - v] \in Gr^W_i K$.

For $k \leq m$ the decomposition (4) can be written as

$$Gr^W_k H \cong \bigoplus_{a=0}^{\lfloor \frac{k}{2} \rfloor} N^{m-k+a}P_{2m-k+2a}$$

Via the isomorphism

$$N^{m-k+2a} : Gr^W_{2m-k+2a} \to Gr^W_{k-2a}$$
the primitive part $P_{2m-k+2a}$ is isomorphic to the kernel of

$$N : Gr^W_{k-2a} H \rightarrow Gr^W_{k-2a-2} H,$$

which is $Gr^W_{k-2a} K$. Then our decomposition follows. □

Let $H$ be a vector space over $\mathbb{Q}$, let $m$ be an integer, and let $S$ be a nondegenerate bilinear form on $H$ such that $S(u,v) = (-1)^m S(v,u)$.

**Definition 1.4.** [5, Def. 2.26] A polarized mixed Hodge structure on $H$ consists of a mixed Hodge structure $(W,F)$ and an infinitesimal isometry $N$ of $S$ such that

1. $N^{m+1} = 0$;
2. $W$ is the monodromy weight filtration $W(N)$;
3. $N F_p \subset F_{p-1}$;
4. $S(F_p, F^{m-p+1}) = 0$;
5. the Hodge structure on the primitive part $P_{m+1}$ is polarized by the form $S(\cdot, N^l \cdot)$.

Return back to the family $\pi : X^* \rightarrow \Delta^*$. Let $H^m_{\text{lim}}$ denote the vector space of the cohomology group of a fiber of $\pi$ with the natural polarization. Let $F_\infty$ be the limiting Hodge filtration defined by (3). Through Proposition 1.2 and the Monodromy theorem, the nilpotent map $N : H^m_{\text{lim}} \rightarrow H^m_{\text{lim}}$ associated to the family $\pi$ defines the monodromy weight filtration $W(N)$

$$0 \subset W_0 \subset \cdots \subset W_{2m} = H^m_{\text{lim}}.$$

As a consequence of the nilpotent orbit theorem, Schmid showed

**Theorem 1.5** ([30, Thm. 6.16]). The two filtrations $W(N)$ and $F_\infty$ determine a polarized mixed Hodge structure on $H^m_{\text{lim}}$. The nilpotent map $N$ is a morphism of mixed Hodge structures of type $(-1, -1)$.

### 1.2. Clemens-Schmid exact sequence

We say a family $f : X \rightarrow \Delta$ is a degeneration if $X$ is a smooth variety, $f$ is a proper and flat morphism that is smooth over the punctured disk $\Delta^*$. The degeneration $f$ is called semistable if the central fiber $X_0$ is a reduced divisor in $X$ with simple normal crossing. Mumford's semistable reduction theorem [20, §II] states that any degeneration over the unit disk can be brought into a semistable form after a finite base change ramified at the origin and a birational modification on the central fiber.

Let $f : X \rightarrow \Delta$ be a semistable degeneration. The limit mixed Hodge structure of the smooth family $\pi : X^* \rightarrow \Delta^*$ can be characterized by the central fiber $X_0$ using the Clemens-Schmid exact sequence.

The cohomology of the central fiber $X_0$ carries a canonical mixed Hodge structure, which involves combinatorial data of irreducible components of $X_0$. Let $\{X_i\}$ be the irreducible components of $X_0$. Set

$$X^{[p]} := \bigsqcup_{i_0 < \cdots < i_p} X_{i_0} \cap \cdots \cap X_{i_p}$$

to be the disjoint union of the codimension $p$ stratum of $X_0$. There is a spectral sequence

$$E_1^{p,q} := H^q(X^{[p]}, \mathbb{Q}) \Rightarrow H^m(X_0, \mathbb{Q}), \quad p + q = m,$$

with the first differential map

$$d_1 : H^q(X^{[p]}, \mathbb{Q}) \rightarrow H^q(X^{[p+1]}, \mathbb{Q})$$
induced by the natural combinatorial boundary map \( \tau_p : \mathcal{X}^{[p+1]} \to \mathcal{X}^{[p]} \). One can put a weight filtration

\[ W_k := \bigoplus_{q \leq k} E_1^{*,q}. \]

on the spectral sequence, which induces a weight filtration on \( H^m(\mathcal{X}_0, \mathbb{Q}) \):

\[ 0 \subset W_0 H^m(\mathcal{X}_0, \mathbb{Q}) \subset \cdots \subset W_m H^m(\mathcal{X}_0, \mathbb{Q}) = H^m(\mathcal{X}_0, \mathbb{Q}). \]

**Proposition 1.6.** The spectral sequence \( (\text{6}) \) degenerates at \( E_2 \). As a result, the \( k \)-th subquotient \( G_{k,-}^W H^m(\mathcal{X}_0, \mathbb{Q}) \) is isomorphic to the \( E_2 \)-term \( E_{2,-k}^m \).

For a simple normal crossing divisor, the strata \( \mathcal{X}^{[p]} \) are smooth. Then the cohomology group of \( \mathcal{X}^{[p]} \) carries a canonical Hodge structure. Through the spectral sequence \( (\text{6}) \) it induces a decreasing (Hodge) filtration on \( H^m(\mathcal{X}_0, \mathbb{Q}) \). Together with the above weight filtration it determines the canonical mixed Hodge structure on \( H^m(\mathcal{X}_0, \mathbb{Q}) \). Denote by \( H^\ast \) (resp. \( H_\ast \)) the cohomology (resp. homology) group \( H^\ast(\mathcal{X}_0, \mathbb{Q}) \) (resp. \( H_\ast(\mathcal{X}_0, \mathbb{Q}) \)). One can associate a mixed Hodge structure to the homology group \( H_\ast \) by duality. Precisely, the weight filtration on \( H_\ast \) is defined by

\[ W_{-k} H_\ast := \text{Ann}(W_{k-1} H^m) = \{ \alpha \in H_\ast \mid (\alpha, W_{k-1} H^m) = 0 \}. \]

It is easy to check that \( G_{k,-}^W H_\ast \cong (G_{k,-}^W H^m)^\ast \).

Suppose that the semistable degeneration \( f : \mathcal{X} \to \Delta \) is of relative dimension \( n \). Let \( \mathcal{X}_i \) be a smooth fiber of \( f \). Deligne’s local invariant cycle theorem asserts any monodromy invariant class on \( H^m(\mathcal{X}_i, \mathbb{Q}) \) is a global class on \( \mathcal{X}^\ast \), i.e., there is the exact sequence

\[ H^m(\mathcal{X}_i, \mathbb{Q}) \xrightarrow{i^\ast} H^m(\mathcal{X}_0, \mathbb{Q}) \xrightarrow{N} H^m(\mathcal{X}_0, \mathbb{Q}) \]

where \( i : \mathcal{X}_i \hookrightarrow \mathcal{X} \) is the natural inclusion. For a semistable degeneration the total space \( \mathcal{X} \) admits a deformation retraction \( r : \mathcal{X} \to \mathcal{X}_0 \), which induces the isomorphisms

\[ r^\ast : H^\ast(\mathcal{X}_0) \cong H^\ast(\mathcal{X}), r_\ast : H_\ast(\mathcal{X}) \cong H_\ast(\mathcal{X}_0). \]

Using the \textit{Wang sequence} for the smooth family \( \mathcal{X}^\ast \to \Delta^\ast \), one can extend the above sequence to the Clemens-Schmid exact sequence

\[ H^{m-2}_m(\mathcal{X}_0) \xrightarrow{\alpha} H^m(\mathcal{X}_0) \xrightarrow{i^\ast} H^m(\mathcal{X}_0) \xrightarrow{N} H^m(\mathcal{X}_0) \xrightarrow{\beta} H_{2n-m}(\mathcal{X}_0) \to 0. \]

The morphisms \( \alpha \) and \( \beta \) are induced by the Poincaré duality.

**Theorem 1.7** ([27 §3]). The morphism \( \alpha, \beta, N, \beta \) are morphisms of mixed Hodge structures of weight \( (n+1, n+1), (0, 0), (-1, -1), (-n, -n) \).

**Corollary 1.8.** Denote by \( H^m_\lim \) (resp. \( H^\ast_\lim, H_\ast \)) the vector space \( H^\ast(\mathcal{X}_i) \) (resp. \( H^\ast(\mathcal{X}_0), H_\ast(\mathcal{X}_0) \)) in the exact sequence \( (\text{7}) \). Suppose \( k > 0 \). Then the \( k \)-th iterated nilpotent map \( N^k : H^m_\lim \to H^m_\lim \) is zero if and only if \( W_{m-k} H^m = 0 \).

**Proof.** The property (2) in Proposition [1.2] implies that

\[ W_{m-k} H^m_\lim = 0 \text{ if and only if } N^k = 0, \forall \ 0 < k \leq m. \]

Let \( K \) be the kernel of the nilpotent map \( N : H^m_\lim \to H^m_\lim \). We claim that

\[ W_{m-k} H^m_\lim = 0 \text{ if and only if } W_{m-k} \cap K = 0. \]

It suffices to show the left hand side follows the right hand side. Assume that \( W_{m-k} H^m_\lim \neq 0 \) and \( W_{m-j} H^m_\lim \subset W_{m-k} H^m_\lim \) the smallest nonzero weight subspace. This yields the contradiction \( 0 \neq W_{m-j} H^m_\lim \subset W_{m-k} \cap K. \)
2. Degenerations to secant cubic hypersurfaces

Let \( S \subset \mathbb{P}^{m+1} \) be the Severi variety of dimension \( d = \frac{2}{3} (m - 1) \). The secant variety \( \text{Sec}(S) \) is a cubic hypersurface in \( \mathbb{P}^{m+1} \). Let us consider an one-parameter degeneration \( \mathcal{X} \to \Delta \) to the secant cubic \( X_0 := \text{Sec}(S) \) defined by the equation

\[
F + tG = 0, \quad t \in \Delta.
\]

Here \( F \) is the defining equation of \( \text{Sec}(S) \), and \( G \) is a cubic equation on \( \mathbb{P}^{m+1} \) satisfying

- the cubic defined by \( G \) transversally intersects along the smooth locus of \( X_0 \);
- the hypersurface \( V \subset S \) cut out by the equation \( G|_S \) is smooth.

**Proposition 2.1.** A semistable reduction \( \mathcal{X}' \to \Delta' \) is obtained from the degeneration \( \mathcal{X} \to \Delta \) by the following two steps: (1) Taking the base change \( \Delta' \to \Delta \) by \( t = u^2 \); (2) Blowing up \( \mathcal{X} \times \Delta \to \Delta' \) along the subvariety \( S \) in the central fiber.

The central fiber of the semistable reduction \( \mathcal{X}' \to \Delta' \) consists of two irreducible components

- the blowup \( \overline{X}_0 \) of the secant cubic \( X_0 \) along \( S \);
- the exceptional divisor \( E \) of the blowup \( \mathcal{X}' \).

We need the following result on Severi varieties for the degeneration \( \mathcal{X}' \to \Delta' \) being semistable.

**Lemma 2.2.** Let \( \overline{\text{Sec}(S)} \) be the blowup of \( \text{Sec}(S) \) along the singular locus \( S \), and let \( E_0 \) be the exceptional divisor. Then \( \pi : E_0 \to S \) is a family of smooth quadrics.

**Proof.** The secant variety \( \text{Sec}(S) \) is double along \( S \), i.e., \( \text{mult}_x(\text{Sec}(S)) = 2 \) for \( x \in S \). Then \( \pi : E_0 \to S \) is a quadric bundle. We aim to show \( \pi \) is a smooth morphism.

For the hypersurface \( \text{Sec}(S) \) we consider the Gauss map

\[
\gamma : \text{Sec}(S) \dashrightarrow \mathbb{P}^{m+1}, \quad p \mapsto T_p\text{Sec}(S)
\]

where \( T_p\text{Sec}(S) \) is the hyperplane in \( \mathbb{P}^{m+1} \) tangent to \( \text{Sec}(S) \) at a smooth point \( p \). Obviously the rational map \( \gamma \) is not defined on the singular locus \( S \). The conormal variety \( C \) of \( \text{Sec}(S) \) is the closure of the graph

\[
\{(p, [H]) \in (\text{Sec}(S) - S) \times \mathbb{P}^{m+1} \mid T_p\text{Sec}(S) = H\}.
\]

of the map \( \gamma \). The dual variety \( \text{Sec}(S)^* \) of \( \text{Sec}(S) \) is defined to be the image of \( C \) in \( \mathbb{P}^{m+1} \) via the projection \( pr : C \to \mathbb{P}^{m+1} \). It is well-known that for the Severi variety \( S \) the dual variety \( \text{Sec}(S)^* \) is isomorphic to \( S \) in \( \mathbb{P}^{m+1} \), see [24, §III, Thm. 2.4].
For any $[\hat{T}_p \mathcal{S}] \in \mathcal{S}(\mathcal{S})^*$, the fiber of the projection $pr : \mathcal{C} \to \mathbb{P}^{m+1}$ over $[\hat{T}_p \mathcal{S}]$ is the *secant cone*
\[
\Sigma_p := \{ u \in \mathcal{S}(\mathcal{S}) - S \mid \hat{T}_u \mathcal{S} = \hat{T}_p \mathcal{S} \}.
\]
For the Severi variety $S$ the secant cone $\Sigma_p$ is a $(\frac{d}{2} + 1)$-dimensional linear subspace in $\mathcal{S}(\mathcal{S})$, see [25] Thm. 2.1. Therefore $p : \mathcal{C} \to \mathcal{S}(\mathcal{S})^*$ is a projective bundle with rank $\frac{d}{2} + 2$. As a result, $\mathcal{C}$ is a resolution of the Gauss map obtained by blowing up $\mathcal{S}(\mathcal{S})^*$ along the singular locus $S$.

The exceptional divisor $E_0$ is the restriction of $\mathcal{C}$ to $S$. Consider the induced morphism $pr|_{E_0} : E_0 \to \mathcal{S}(\mathcal{S})^*$. The preimage of a point $[\hat{T}_p \mathcal{S}] \in \mathcal{S}(\mathcal{S})^*$ is the intersection
\[
Q_p := \Sigma_p \cap S,
\]
which is called the *secant locus* of the point $p$. By [25] §1a, the secant locus $Q_p$ consists of the points $x \in S$ such that the join line $(x, p)$ is secant to $S$. It is a smooth quadric in $\Sigma_p \cong \mathbb{P}^{\frac{d}{2}+1}$. For the quadric bundle $\pi : E_0 \to S$ and any $x \in S$, we have
\[
\pi^{-1}(x) = \{ (x, [\hat{T}_p \mathcal{S}]) \mid x \in Q_p \}.
\]
In other words, $\pi^{-1}(x)$ parametrizes the secant quadrics in $S$ passing through the point $x$. It follows from the result [24] §IV, Prop. 3.1 that $\pi^{-1}(x)$ is isomorphic to a smooth quadric with dimension $\frac{d}{2}$.

Proof of Proposition 2.1 We first prove the total space $X'$ is smooth, then show the central fiber of $X' \to \Delta'$ is a normal crossing divisor.

Under the base change $t = u^2$ the family $X \times_\Delta \Delta'$ is defined by the equation $F + u^2G = 0$. Let $(z_1, \ldots, z_{m+1}, u)$ be a local coordinate of $\mathbb{P}^{m+1}_{\Delta'}$, and $f$ (resp. $g$) the local equation of $F$ (resp. $G$). We have the derivatives
\[
\begin{align*}
& (1) \quad \frac{\partial (f + u^2g)}{\partial z_i} = \frac{\partial f}{\partial z_i} + u^2 \frac{\partial g}{\partial z_i}, \quad 1 \leq i \leq m + 1; \\
& (2) \quad \frac{\partial (f + u^2g)}{\partial u} = 2ug.
\end{align*}
\]
We claim the singularity of $X \times_\Delta \Delta'$ supports at the central fiber. Otherwise, assume that $p$ is a singular point with $u(p) \neq 0$. We see from the derivative that $g(p) = f(p) = 0$. Hence $p \in X \cap X_0$ where $X$ is the smooth hypersurface defined by $G$. The derivatives $\frac{\partial f}{\partial z_i}(p)$ are not all equal to zero. Then the equality
\[
\frac{\partial f}{\partial z_i}(p) + u(p)^2 \frac{\partial g}{\partial z_i}(p) = 0
\]
implies that $p$ is a nonsingular point of $X_0$, and $X$ is tangent to $X_0$ at $p$, which contradicts our assumptions on the degeneration [5]. Moreover, for $u(p) = 0$, the point $p$ is singular if and only if $\frac{\partial f}{\partial z_i}(p) = 0$ for all $i$. Therefore the singular locus of $X \times_\Delta \Delta'$ is the Severi variety $S$.

Note that $X'$ is the proper transform of $X \times_\Delta \Delta'$ in the blow up of $\mathbb{P}^{m+1}_{\Delta'}$ along the smooth subvariety $S$. To verify the smoothness of $X'$, we look at the blowing up locally. Let $\mathbb{D}_p \subset \mathbb{C}^d$ be a polydisc around a point $p \in S$ with a local chart $(x_1, \ldots, x_d)$. Extend it to a polydisc $\mathbb{D}_{m+2} \subset \mathbb{C}^{m+2}$ around $p \in \mathbb{P}^{m+1}_{\Delta'}$ with a local chart $(x_1, \ldots, x_d, y_1, \ldots, y_{\frac{d}{2}+2}, u)$ such that the local embedding of the Severi variety $S \subset \mathbb{P}^{m+1}$ can be arranged as
\[
(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 0, \ldots, 0).
\]
The blow up of $\mathbb{D}_{m+2}$ along $\mathbb{D}_d$ is the closed subvariety
$$
\widetilde{\mathbb{D}}_{m+2} \subset \mathbb{D}_{m+2} \times \mathbb{P}[W_1, \ldots, W_{d+2}, T]
$$
defined by the following equations
$$
y_jW_i - y_iW_j = 0, \quad y_iT - uW_i = 0, \quad 1 \leq i, j \leq \frac{d}{2} + 2.
$$

Denote by $f$ and $g$ the local equations of $F$ and $G$ under the coordinate $(x_1, \ldots, x_d, y_1, \ldots, y_{d+2})$. Since $\text{Sec}(S)$ is double along the singular locus $S$ the local equation $f$ can be written as
$$
f = Q + \sum_{i=1}^d x_i Q_i + H
$$
where $Q, Q_i, H$ are homogeneous polynomials in $\mathbb{C}[y_1, \ldots, y_{d+2}]$ with $\deg Q = \deg Q_i = 2, \deg H = 3$. Under the local chart $(x_1, \ldots, x_d)$, the exceptional divisor $E_0$ of the blow up $X_0$, as a quadric bundle over $S$, can be presented by the family of quadratic forms
$$
Q + \sum_{i=1}^d x_i Q_i.
$$

It follows from Lemma 2.2 that the quadratic forms $Q + \sum_{i=1}^d a_i Q_i$ are non-degenerate for $(a_i) \in \mathbb{D}_d$. In particular, $Q$ is non-degenerate. We may assume that the smooth divisor $V \subset S$ cut out by $G$ is given by the equation $x_d = 0$. Therefore we can write $g$ to be
$$
g = x_d + g'
$$
where $g'$ has no monomials of $(x_1, \ldots, x_d)$.

Let $U_i \subset \widetilde{\mathbb{D}}_{m+2}$ be the open subset with $W_i \neq 0$. The restriction of $\mathfrak{X}$ to $U_i$ is read off from the pullback
$$
\pi^*(f + u^2 g)|_{U_i} = y_i^2 \tilde{f}|_{U_i}
$$
where $\pi : \widetilde{\mathbb{D}}_{m+2} \to \mathbb{D}_{m+2}$ is the natural projection. Here $V(y_i = 0)$ is the exceptional divisor, and $\tilde{f}|_{U_i}$ defines the proper transform $\mathfrak{X}' \cap U_i$. Set $w_j = \frac{W_j}{W_i}, t = \frac{T}{W_i}$. We have
$$
\tilde{f}|_{U_i} = Q(..., w_{i-1}, 1, w_{i+1}, ...) + \sum x_i Q_i(..., w_{i-1}, 1, w_i, ..., ) + \text{higher terms}.
$$
Recall that $Q$ is a non-degenerate quadratic form. Hence $Q(..., w_{i-1}, 1, w_{i+1}, ...)$ contains a non-trivial linear term, and thus $\mathfrak{X}$ is smooth in $U_i$. On the open subset $V \subset \widetilde{\mathbb{D}}_{m+2}$ with $T \neq 0$ we set $w_i = \frac{W_i}{W_d}$, and have
$$
\pi^*(f + u^2 g)|_V = u^2 (Q(w_1, \ldots, w_d) + x_d + \text{higher terms}).
$$
Then $\mathfrak{X}' \cap V$ defined by the equation $x_d + Q + \{\text{higher terms}\} = 0$ is smooth. It concludes the total space $\mathfrak{X}'$ is smooth. The central fiber of $\mathfrak{X}' \to \Delta'$ has two irreducible components:

1. the proper transform $X_0$ of $X_0$;
2. the exceptional divisor $E$ of $\mathfrak{X}'$. 

Lemma 2.2 shows the proper transform $\overline{X}_0$ is isomorphic to the conormal variety of $X_0$, which is non-singular. Suppose that $\mathcal{E}$ is the exceptional divisor of the blow up of $\mathbb{P}^{n+1}$ at $s$. Then $E$ is the intersection of $\mathcal{E}$ and $X$. Both $\mathcal{E}$ and $X$ are smooth divisors of the blow up. They transversally intersect along $E$. Hence $E$ is a smooth divisor of $X$. In addition, $E$ intersects with $\overline{X}_0$ along the exceptional divisor $E_0$ of $\overline{X}_0$. It concludes the central fiber of $X'$ is a normal crossing divisor.

The following corollary is significant to compute the cohomology of the exceptional divisor $E$ in $\mathbb{P}^d$.

**Corollary 2.3.** The exceptional divisor $E$ is a family of $(\frac{d}{2} + 1)$-dimensional quadrics over $S$. The discriminant locus of the quadric bundle is $V$. The singular fiber $E_s$ for any $s \in V$ is a quadric cone with a single vertex.

**Proof.** The exceptional divisor $E$ is the projective normal cone of $S$ to $X \times_\Delta X'$. Applying the analysis of the local equation $f + u^2 g$ in the proof of Proposition 2.1, we can locally present the normal cone by the family of quadratic forms $Q + \sum_{i=1}^d x_i Q_i + u^2 x_d$ that parametrized by the coordinate $(x_1, \ldots, x_d, u)$ of the polydisc $\mathbb{D}_{d+1}$. By the proof of Proposition 2.1, we conclude that the quadratic forms $Q + \sum_{i=1}^d a_i Q_i + u^2 a_d$ is non-degenerate if and only if $a_d \neq 0$. Therefore for $s \in S \setminus V$, i.e., $x_d \neq 0$, the projective normal cone at $s$ is a smooth quadric with the maximal rank $\frac{d}{2} + 2$. For $s \in V$, i.e., $x_d = 0$, the projective normal cone at $s$ is a quadric cone with the rank $\frac{d}{2} + 1$. □

Suppose that $S \subset \mathbb{P}^{m+1}$ is a Severi variety of dimension $d = \frac{2(m-1)}{3}$ with $d > 2$. By the classification of Severi varieties we recall that $d = 4, 8, 16$ and $m = 7, 13, 25$ respectively. Let $f : X \to \Delta'$ be the semistable degeneration to the secant cubic $Sec(S)$ discussed in Proposition 2.1, and let $H^m_{\lim}$ denote the $\mathbb{Q}$-coefficients Betti cohomology of a smooth fiber of $f$ with the associated nilpotent map $N : H^m_{\lim} \to H^m_{\lim}$.

**Theorem 2.4.** Let $V$ be the hypersurface in $S$ cut off by any smooth fiber of the one-parameter degeneration $\mathbb{P}^d$. The nilpotent map $N$ has index 1, i.e., $N \neq 0, N^2 = 0$. Moreover, the limit mixed Hodge structure on $H^m_{\lim}$ satisfies

1. $Gr^{W}_{m-k} H^m_{\lim} = 0$ if $k > 1$ or $k < -1$;
2. $Gr^W_{m-k} H^m_{\lim}$ is isomorphic to the Tate twist $\mathbb{Q}(\frac{m-k}{2})$ if $k = \pm 1$;
3. $Gr^W_{m-k} H^m_{\lim} \cong H^{d-1}(V, Q)$ is an isomorphism of polarized Hodge structures.

Before giving the proof of the theorem let us look at the following observation on the Segre fourfold, which initiates the work of this article.

**Example 2.5.** Let $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ be the Segre fourfold, and $X$ be a generic cubic sevenfold in $\mathbb{P}^8$. It is known that the Hodge structure of a smooth cubic sevenfold is of Calabi-Yau type, cf. [17]. Precisely the Hodge numbers are
$$h^{7,0} = h^{6,1} = 0, h^{5,2} = 1, h^{4,3} = 84.$$
Consider the $(3,3)$-hypersurface $V := X \cap (\mathbb{P}^2 \times \mathbb{P}^2)$. The adjunction formula
\[ K_V \cong (K_{\mathbb{P}^2 \times \mathbb{P}^2} + V)|_V = (\mathcal{O}(-3,-3) + \mathcal{O}(3,3))|_V = \mathcal{O}_V \]
asserts $V$ is Calabi-Yau. By the Lefschetz hyperplane theorem
\[ H^i(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}) \xrightarrow{\sim} H^i(Y, \mathbb{Z}), \forall i \leq 2 \]
we have $H^i(V, \mathcal{O}_V) = 0$ for $0 \leq i \leq 2$. The Hodge diamond of $V$ turns out to be
\[
\begin{array}{cccccc}
  & & h^0 & & & \\
  & h^1 & & h^2 & & h^3 \\
  h^2 & & 0 & & 0 & \\
  h^3 & & 1 & & 83 & & 83 & & 1 \\
\end{array}
\]
The Hodge numbers $h^{1,2} = h^{2,1} = 83$ can be calculated by the Euler characteristic of $V$ which is equal to the top Chern class $c_3(T_V)$ of the tangent bundle $T_V$. Through the exact sequence
\[ 0 \to T_V \to T_{\mathbb{P}^2 \times \mathbb{P}^2}|_V \to N_V/\mathbb{P}^2 \times \mathbb{P}^2 \cong \mathcal{O}(3,3) \to 0 \]
we get $c(T_V) \cdot c(N_V/\mathbb{P}^2 \times \mathbb{P}^2) = c(T_{\mathbb{P}^2 \times \mathbb{P}^2})|_V$. It follows that
\[
\sum_i c_i(V)(1 + 3(H_1 + H_2)) = (1 + p_1^*c_1(\mathbb{P}^2) + p_1^*c_2(\mathbb{P}^2))(1 + p_2^*c_1(\mathbb{P}^2) + p_2^*c_2(\mathbb{P}^2))|_V.
\]
where $H_i$ is the pullback of the hyperplane class $H \subset \mathbb{P}^2$ along the $i$-th projection $p_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$. As a result,
\[
c_2(V) = c_1(E) \cdot c_1(F) + c_2(E) + c_2(F) = 9H_1H_2 + 3H_1^2 + 3H_2^2;
\]
\[
c_3(V) = 9(H_1^2H_2 + H_1H_2^2) - 3c_2(V)(H_1 + H_2).
\]
Then the degree of $c_3(V)$ is equal to $-162$, and $h^{1,2} = h^{2,1} = 83$. From the numerical analysis, we expect, as well as the case of the Veronese surface and the cubic fourfolds, the limiting Hodge structure of the degeneration of cubic sevenfolds to the secant cubic should be the Hodge structure of the Calabi-Yau threefold $V$.

Proof of Theorem 2.4. Let $X_0$ denote the central fiber of the semistable degeneration $f : X' \to \Delta'$. Recall that the simple normal crossing divisor $X_0$ has two irreducible components $X_0$ and $E$ that intersects along $E_0$, see Proposition 2.1. Consider the Clemens-Schmid exact sequence (cf. (7))
\[
\cdots \to H_{m+2}(X_0) \xrightarrow{\alpha} H^m(X_0) \xrightarrow{\beta} H^m_{\lim} \xrightarrow{\gamma} H^m_{\lim} \to H_m(X_0) \to \cdots
\]
for the semistable degeneration $f$. Denote by $H^*$ (resp. $H_*$) the cohomology group $H^*(X_0)$ (resp. $H_*(X_0)$). To prove $N^2 = 0$, it suffices to show $W_{m-2}H^m = 0$ by Corollary 1.8. Recall the spectral sequence
\[ E_1^{p,q} = H^q(\mathcal{X}^{[p]}, \mathbb{Q}) \Rightarrow H^m(\mathcal{X}_0). \]
For $p > 1$ the term $E_1^{p,q}$ vanishes since $\mathcal{X}_0$ has only two components. Note that $Gr_{m-k}^W H^m \cong E_2^{k,m-k}$, hence for $k > 1$ we obtain $Gr_{m-k}^W H^m = 0$ and thus $W_{m-2}H^m = 0$, which asserts $N^2 = 0$.

Since $N^2 = 0$ the subspace $W_{m-2}H^m_{\lim} = 0$, which leads to $Gr_i^W H^m_{\lim} = 0$ for $i < m - 1$, as well as $i > m - 1$ by symmetry. The strictness of the morphisms in the Clemens-Schmid exact sequence yields the long exact sequence
\[
\to Gr_{m-k-2}^W H^m_{m+2} \xrightarrow{\alpha} Gr_{m-k}^W H^m \xrightarrow{\beta} Gr_{m-k}^W H^m_{\lim} \xrightarrow{\gamma} Gr_{m-k-2}^W H^m_{\lim} \to
\]
on the graded weight spaces. For $k \geq 0$ we have seen $Gr_{m-k-2}^W H^m_{\lim} = 0$. Moreover, we claim that for $k \geq 0$ the map $\alpha$ is zero, which will assert

$$Gr_{m-k}^W H^m \cong Gr_{m-k}^W H^m_{\lim}, \quad k \geq 0.$$ 

It follows from Proposition 1.6 that

$$Gr_{m-k-2}^W H^m_{\lim} \cong (Gr_{m+k+2}^W H^{m+2})^* \cong (E_{2}^{-k,m+k+2})^*.$$ 

It is direct to see $E_{2}^{-k,m+k+2} = 0$ for $k > 1$ from the definition of the spectral sequence. For $k = 0$ we have

$$(E_{2}^{0,m+2})^* \cong \text{coker}(H_{m+2}(E_0) \to H_{m+2}(X_0) \oplus H_{m+2}(E)).$$

Recall from Lemma 2.2 that $X_0$ is a projective bundle over the dual variety $X_0^*$ of the secant cubic $X_0$, and $E_0$ is a smooth family of quadric bundles over $X_0^*$. Note that the dimension of $X_0^*$ is even since the dual variety of the secant cubic $X_0$ is the cokernel of the following map

$$\rho: H^{m-1}(X_0) \oplus H^{m-1}(E) \to H^{m-1}(E_0).$$

By the same reason as above $H^{m}(X_0) = H^{m}(E_0) = 0$ then $Gr_{m}^W H^m = H^m(E)$. We latter prove that

$$H^{m}(E, Q) \cong H^{d-1}(V, Q)$$

as an isomorphism of polarized Hodge structures in Proposition 3.4.

The graded piece $Gr_{m-1}^W H^m$ is the cokernel of the following map

$$(9) \quad \rho : H^{m-1}(X_0) \oplus H^{m-1}(E) \to H^{m-1}(E_0).$$

For the smooth quadric bundle $\pi : E_0 \to S$, the Leray spectral sequence

$$E_{2}^{pq} := H^p(S, R^q\pi_\ast Q) \Rightarrow H^{p+q-1}(E_0), \quad p + q = m - 1$$

degenerates at $E_2$. Note that the base space $S$ is simply connected. The trivial local system $R^q\pi_\ast Q$ is the constant sheaf $H^q(F)$ where $F$ is the fiber of $\pi$. Then it gives rise to a (non-canonical) decomposition

$$\bigoplus_{p+q=m-1} H^p(S) \otimes H^q(F) \cong H^{m-1}(E_0).$$

The cohomology classes of $S$ are algebraic since $S$ is a homogeneous space. Hence the cohomology classes of $E_0$ are algebraic, and the Hodge structure of $H^{m-1}(E_0)$ is isomorphic to $\mathbb{Q}(\frac{1}{m})^r$ where $r$ is the rank of $H^{m-1}(E_0)$. Moreover, we show the quotient Hodge structure $\text{Coker}(\rho)$ is isomorphic to $\mathbb{Q}(\frac{1}{m})$. The proof is given in the next lemma because it is a bit lengthy to include it here. 

**Lemma 2.6.** Let $S \subset \mathbb{P}^{m+1}$ be the Severi variety of dimension $d = 4, 8, 16$. Let $F$ be any fiber of the quadric bundle $\pi : E_0 \to S$ in Lemma 2.2. Then the quotient of the map

$$\rho : H^{m-1}(X_0) \oplus H^{m-1}(E) \to H^{m-1}(E_0)$$

is isomorphic to the Tate twist $\mathbb{Q}(\frac{1}{m})$ which can be generated by one algebraic class in $H^d(S) \otimes H^\frac{d}{2}(F) \subset H^{m-1}(E_0)$.
Proof: The Severi variety $S$ is a homogeneous space. The cohomology of $S$ are generated by the Schubert-type classes. The fiber $F$ is a smooth quadric of even dimension $\frac{d}{2}$. The Fano scheme of the $\frac{d}{2}$-planes contained in $F$ has two connected components. Let $\lambda_1, \lambda_2$ be the generators of $H^{\frac{d}{2}}(F)$ that represented by two $\frac{d}{2}$-planes in the different components. The goal is to show that the quotient of the class $\sigma$ where $\sigma$ represents any $\frac{d}{2}$-dimensional Schubert-type class of $S$ and $i = 1$ or $2$.

We split the proof into several steps. Based on the decomposition

$$\bigoplus_{p+q=m-1} H^p(S) \otimes H^q(F) \cong H^{m-1}(E_0),$$

the first step is to prove that for $q \neq \frac{d}{2}$ the direct summand $H^q(S) \otimes H^{\frac{d}{2}}(F)$ is contained in the image of $\rho$. The classes $\{\sigma \otimes \lambda_i\}$ forms a basis of $H^q(S) \otimes H^{\frac{d}{2}}(F)$ where $\sigma$ is a Schubert-type class of $S$, and $\lambda_1, \lambda_2$ is the two generators of $H^{\frac{d}{2}}(F)$. The second step shows that any two basis elements are linearly dependent in the quotient $\text{Coker}(\rho)$. The last step assures the class $\sigma \otimes \lambda_i$ is non-trivial in $\text{Coker}(\rho)$.

**Step 1.** The quadric bundle $\pi : E_0 \to S$ is a hyperplane section of the quadric bundle $f : E \to S$. Denote by $i : E_0 \to E$ be the inclusion. Recall that $V$ is the discriminant locus of $f$. Let $U := S \setminus V$ be the open complement, let $\mathcal{Y} := f^{-1}(V)$ be the family of singular quadrics of $f$, and let $\mathcal{W} := \mathcal{Y} \cap E_0$ be the restriction. Consider the following commutative diagram of Gysin sequences

$$\begin{array}{ccc}
H^{m-3}(\mathcal{Y}) & \longrightarrow & H^{m-1}(E) \\
\downarrow & & \downarrow \pi^* \downarrow \\
H^{m-3}(\mathcal{W}) & \longrightarrow & H^{m-1}(E_0) \\
\end{array}$$

(10)

Let $\alpha \otimes \eta \in H^p(S) \otimes H^q(F)$ with $q \neq \frac{d}{2}$. Let $\alpha|_U \otimes \eta$ be the restriction to $E_0|_U$. It is easy to see the restriction map

$$i^* : R^q f|_{U,Q} \to R^q \pi|_{U,Q}$$

is an isomorphism if $q \neq \frac{d}{2}$. Hence $\alpha|_U \otimes \eta$ lifts to an class $\alpha|_U \otimes \eta' \in H^{m-1}(E|_U)$, which is also algebraic. Taking the closure of $\alpha|_U \otimes \eta'$ in $E$, we obtain an algebraic class $Z \in H^{m-1}(E)$. By the exactness, the class $[Z]|_{E_0} - \alpha \otimes \eta$ comes from a class on $\mathcal{W}$. In order to prove $\alpha \otimes \eta$ is contained in the image of $i^*$, it suffices to show the left arrow

$$H^{m-3}(\mathcal{Y}) \to H^{m-3}(\mathcal{W})$$

is surjective. Let $\Sigma \subset \mathcal{Y}$ be the subset of the unique singular vertex in each fiber of the family $\mathcal{Y}$. The $V$-relative projection $p_\Sigma$ along $\Sigma$ maps $\mathcal{Y}$ to a smooth family $Q$ of $\frac{d}{2}$-dimensional quadrics over $V$. The projection $p_\Sigma$ identifies $Q$ with $\mathcal{W}$ since $\mathcal{W} \cap \Sigma = \emptyset$. It implies the restriction map $H^*(\mathcal{Y}) \to H^*(\mathcal{W})$ is surjective.

**Step 2.** Note that $\mathcal{W}$ is a smooth family of quadrics over $V$. The Gysin map

$$H^{m-3}(\mathcal{W}) \to H^{m-1}(E_0)$$

restricts to the following map

$$\iota_{\mathcal{W}*} \otimes id : H^{d-2}(V) \otimes H^\frac{d}{2}(F) \to H^d(S) \otimes H^\frac{d}{2}(F).$$
Note that \( V \) is an ample divisor of \( S \). By the Lefschetz hyperplane theorem, the image of the Gysin map \( i_V^* \) equals to the image of the cup-product

\[
\cup[V] : H^{d-2}(S) \to H^d(S).
\]

Let us look into the image case by case.

- **\( S := \mathbb{P}^2 \times \mathbb{P}^2 \)**, the cohomology \( H^2(S) \) has two generators \( H_1, H_2 \) where \( H_i \) is the hyperplane section on the \( i \)-th factor. The image of \( i_V^* \) is generated by \( \langle H_1(H_1 + H_2), H_2(H_1 + H_2) \rangle \). Therefore, for two distinct elements \( \sigma, \sigma' \in \{ H_1^2, H_2^2, H_1 H_2 \} \), the classes \( \sigma \otimes \lambda_i \) and \( \sigma' \otimes \lambda_i \) are linearly dependent in \( \text{Coker}(\rho) \).

- **\( S := \text{Gr}(2, 6) \)**, the cohomology \( H^6(S) \) is generated by the Schubert classes \( \{ \sigma_3, \sigma_2, 1 \} \). The class \( [V] = 3\sigma_1 \), and the cohomology \( H^8(S) \) is generated by the Schubert classes \( \{ \sigma_4, \sigma_3, \sigma_2, 2 \} \). By the Pieri’s formula \([11, \S 4.3]\) on the intersection products of Schubert classes, we have

\[
\sigma_3 \cdot \sigma_1 = \sigma_4 + \sigma_3, \quad \sigma_2, 1 \cdot \sigma_1 = \sigma_3 + \sigma_2, 2.
\]

Then we can see for distinct \( \sigma, \sigma' \in \{ \sigma_4, \sigma_3, 1, \sigma_2, 2 \} \) the classes \( \sigma \otimes \lambda_i \) and \( \sigma' \otimes \lambda_i \) are linearly dependent in \( \text{Coker}(\rho) \).

- **\( S := \mathbb{O} \mathbb{P}^2 \)**, we find \( \dim H^{14}(S) = 2 \) and \( \dim H^{16}(S) = 3 \). Moreover, the calculus on the Schubert classes of \( S \) is the same as the case \( \mathbb{G}(1, 5) \). For the details we refer to \([16, \S 3]\).

Now let us deal with the map \( j_{E_0}^* : H^{m-1}(\overline{X_0}) \to H^{m-1}(E_0) \). Consider the blow-up diagram

\[
\begin{array}{ccc}
E_0 & \xrightarrow{j_{E_0}^*} & \overline{X_0} \\
\downarrow{\pi} & & \downarrow{\epsilon} \\
S & \xrightarrow{i_S} & X_0.
\end{array}
\]

The map

\[
j_{E_0}^* + \epsilon^* : H^{m-3}(E_0) \oplus H^{m-1}(X_0) \to H^{m-1}(\overline{X_0}).
\]

is surjective. Then the image of \( j_{E_0}^* \) equals to the image of

\[
j_{E_0}^* j_{E_0} + \pi^* i_S^* : H^{m-3}(E_0) \oplus H^{m-1}(X_0) \to H^{m-1}(E_0).
\]

The component \( H^{m-1}(X_0) \) maps into the component \( H^m(S) \otimes H^0(F) \). The composition \( j_{E_0}^* j_{E_0} + \pi^* i_S^* \) is the cup product map \( \cup|E_0||E_0 \). Let \( \mathcal{O}_\pi(1) \) be the canonical line bundle of the quadric bundle \( \pi \). We have

\[
\mathcal{O}(E_0)|E_0 = \mathcal{O}_\pi(-1).
\]

The first Chern class \( \eta := c_1(\mathcal{O}_\pi(1)) \) can be viewed as a global section of the local system \( R^2 \pi_* \mathbb{Q} \). Furthermore, the section \( \eta^* \) induces a canonical isomorphism \( \mathbb{Q} \to R^2 \pi_* \mathbb{Q} \) for all \( i \neq \frac{4}{2} \). For \( i = \frac{4}{2} \), the local system \( R^2 \pi_* \mathbb{Q} \) has rank 2 with the two (local) generators \( \lambda_1, \lambda_2 \). We have \( \eta^* \mathcal{O} = \mathbb{Q} \cdot (\lambda_1 + \lambda_2) \). Therefore the image of

\[
\cup|E_0||E_0 : R^2 \pi_* \mathbb{Q} \to R^4 \pi_* \mathbb{Q}
\]

is generated by \( \lambda_1 + \lambda_2 \). Passing to the cohomology, the image of the cup-product \( \cup|E_0||E_0 \) on the component \( H^d(S) \otimes H^d(F) \) is generated by the classes \( \sigma \otimes (\lambda_1 + \lambda_2), \sigma \in H^d(S) \). Hence

\[
\sigma \otimes (\lambda_1 + \lambda_2) \equiv 0 \mod \text{Im}(\rho).
\]
Step 3. Through the previous discussions, we immediately see that the class \( \sigma \otimes \lambda_1 \), or equivalent \( \sigma \otimes \lambda_2 \) is not contained in the image of \( H^{m-1}(X_0) \). It is also not contained in the image of \( H^{m-1}(E) \). Otherwise, the commutative diagram (10) asserts there exists a class in \( H^{m-1}(E|_U) \) that restricts to \( \sigma|_U \otimes \lambda_1 \in H^{m-1}(E_0|_U) \). However, it is not true because the image of \( i^*: R^2 f_* \mathbb{Q} \to R^2 \pi_* \mathbb{Q} \) is generated by \( \langle \lambda_1 + \lambda_2 \rangle \).

The Example 2.5 shows that the Hodge structure of a smooth cubic sevenfold and the associated Calabi-Yau threefold are closely related. Now we can see it is a consequence of Theorem 2.4 which deduces analogous results for the higher dimensional Severi varieties as well. To be precise, we have

**Corollary 2.7.** Let \( \mathcal{O}(1) \) denote the canonical polarizations of the Severi varieties

\[
\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \ Gr(2, 6) \subset \mathbb{P}^{14}, \ \mathbb{OP}^2 \subset \mathbb{P}^{26}.
\]

Let \( V \in |\mathcal{O}(3)| \) be a smooth divisor. For \( \mathbb{P}^2 \times \mathbb{P}^2 \), the Hodge numbers of \( V \) are

\[
h^{3,0} = 1, h^{2,1} = \binom{9}{3} - 1.
\]

For \( Gr(2, 6) \), the Hodge numbers of \( V \) are

\[
h^{7,0} = 0, h^{6,1} = 1, h^{5,2} = \binom{15}{3}, h^{4,3} = \binom{15}{6} - 1.
\]

For \( \mathbb{OP}^2 \), the Hodge numbers of \( V \) are

\[
h^{15,0} = h^{14,1} = h^{13,2} = 0,
\]

\[
h^{12,3} = 1, h^{11,4} = \binom{27}{3}, h^{10,5} = \binom{27}{6}, h^{9,6} = \binom{27}{9}, h^{8,7} = \binom{27}{12} - 1.
\]

**Proof.** Let \( X \subset \mathbb{P}^{m+1} \) be a smooth cubic \( m \)-fold that cut out \( V \). For an odd integer \( m \), we have

\[
h^{p,m-p}(X) = \frac{m + 2}{2m + 1 - 3p},
\]

see [15, Remark 1.17]. More explicitly, when \( m = 7, 13, 25 \), the non-zero Hodge numbers on \( H^m(X) \) are

1. \( m=7 \), \( h^{5,2} = h^{2,5} = 1, h^{4,3} = h^{3,4} = 84; \)
2. \( m=13 \), \( h^{9,4} = h^{4,9} = 1, h^{8,5} = h^{5,8} = \binom{15}{3}, h^{7,6} = h^{6,7} = \binom{15}{6}; \)
3. \( m=25 \), \( h^{17,8} = h^{8,17} = 1, h^{16,9} = h^{9,16} = \binom{27}{3}, h^{15,10} = h^{10,15} = \binom{27}{6}, h^{14,11} = h^{11,14} = \binom{27}{9}, h^{13,12} = h^{12,13} = \binom{27}{12}. \)

Consider the degeneration (8) to the secant cubic and the limit mixed Hodge structure on \( H^m_{\lim} \). It follows from Theorem 2.4 that \( H^{d-1}(V, \mathbb{Q}) \cong G^W_m H^m_{\lim} \). Let \( F^p \subset H^m(X, \mathbb{C}) \) be the Hodge filtration, and let \( F^p_{\infty} \subset H^m_{\lim} \) be the limiting Hodge filtration. By (8) we have

\[
\dim F^p = \dim F^p_{\infty}.
\]

By abuse of notation we denote by \( h^{p,q} \), \( 0 \leq p, q \leq m \) the virtual Hodge numbers of the mixed Hodge structure on \( H^m_{\lim} \). There is

\[
\dim H^{p,m-p}(X) = \dim F^p_{\infty} / F^{p+1}_{\infty} = \sum_{i=0}^{m} h^{p,i}.
\]
The non-trivial weights of $H^m_{\lim}$ are $m-1$, $m$, $m+1$, and $Gr^W_m$ (resp. $Gr^W_{m+1}$) is isomorphic to the Tate twist $\mathbb{Q}(\frac{m}{2})$ (resp. $\mathbb{Q}(\frac{m+1}{2})$). Therefore we have

1. $\dim F^k_\infty / F^{k+1}_\infty = h^{k,m-k}$, $k \neq \frac{m-1}{2}, \frac{m+1}{2}$,

2. $\dim F^m_\infty / F^{m+1}_\infty = h^{m+1,m-1} + h^{m-1,m+1} = h^{m+1,m-1} + 1$,

3. $\dim F^{m+1}_\infty / F^{m+2}_\infty = h^{m+1,m-1} + h^{m+1,m+1} = h^{m+1,m-1} + 1$.

Then the Hodge structure on $Gr^W_m H^m_{\lim}$ follows.

**Remark 2.8.** All four Severi varieties are homogeneous spaces. The Hodge structure of a smooth hypersurface of a homogeneous space may be calculated by the generalized Jacobi-ring.

3. **Cohomology of quadric fibrations**

The goal of this section is to study the cohomology of the quadric fibrations appeared in Corollary 2.3 and the main theorem. We can work on the quadric fibrations over a general base space in the following set-up.

Let $S$ be a simply connected smooth projective variety with even dimension $d$, and let $V$ be a smooth ample divisor of $S$. Suppose that

- $H^*(S, \mathbb{Z})$ are torsion-free, and $H^k(X, \mathbb{Z}) = 0$ for all odd $k$;
- $f : \mathcal{X} \to S$ is a quadric fibration of relative dimension $2n - 1$ whose discriminant divisor is $V$, and every singular fiber $\mathcal{X}_t$ for $t \in V$ is a quadric cone with corank one.

Notice that the Severi varieties and the quadric fibrations we deal with satisfy the assumption.

**Lemma 3.1.** Let $(S, V)$ be the pair of a smooth projective variety and a ample divisor in the above set-up. Denote by $U = S - V$ the open complement. Then we have

1. $H^*(V, \mathbb{Z})$ are torsion-free,
2. $H^k(U, \mathbb{Z}) = 0$ if $k > d$ or $k$ is odd.

**Proof.** The first assertion is deduced from the Lefschetz hyperplane theorem and the universal coefficient theorem for the cohomology of $V$.

The open subset $U$ is a smooth affine variety with complex dimension $d$. By Morse theory $U$ is homotopic to a CW-complex of real dimension at most $d$. Then $H^k(U, \mathbb{Z}) = 0$ if $k > d$.

To compute the cohomology of odd degree $k < d$, we consider the localization long exact sequence

$$\ldots \to H^k_{\mathcal{X}}(S, \mathbb{Z}) \to H^k(S, \mathbb{Z}) \to H^k(U, \mathbb{Z}) \to H^{k+1}_{\mathcal{X}}(S, \mathbb{Z}) \to \ldots$$

By our assumption $H^k(S, \mathbb{Z}) = 0$ for odd $k$. Then for odd $k$ the group $H^k(U, \mathbb{Z})$ is the kernel of the Gysin map

$$H^{k+1}_{\mathcal{X}}(S, \mathbb{Z}) \to H^{k+1}(S, \mathbb{Z}).$$

Through the Thom isomorphism $H^{k+1}_{\mathcal{X}}(S, \mathbb{Z}) \cong H^{k-1}(V, \mathbb{Z})$ and the isomorphism $H^{k-1}(S, \mathbb{Z}) \cong H^{k-1}(V, \mathbb{Z})$ by the Lefschetz hyperplane theorem, we identify the Gysin map to the composition

$$H^k(S, \mathbb{Z}) \xrightarrow{\gamma} H^k(V, \mathbb{Z}) \to H^{k+1}(S, \mathbb{Z})$$
which is the cup-product map of the class $[V]$. Since $H^*(S, \mathbb{Z})$ is torsion-free, the hard Lefschetz theorem asserts the cup-product map is injective. Therefore $H^k(U, \mathbb{Z}) = 0$ for odd $k < d$. □

**Corollary 3.2.** Let $f_U : X_U \to U$ be a smooth family of quadrics of odd dimension. Then the cohomology $H^k(X_U, \mathbb{Z})$ vanishes if $k$ is odd.

**Proof.** Consider the Leray spectral sequence

$$E_2^{p,q} := H^p(U, R^q f_U^* \mathbb{Z}) \Rightarrow H^{p+q}(X_U, \mathbb{Z}).$$

If $q$ is odd, the sheaf $R^q f_U^* \mathbb{Z}$ is zero. If $q$ is even, the sheaf $R^q f_U^* \mathbb{Z}$ is isomorphic to the constant sheaf $\mathbb{Z}$. It follows from the above Lemma that $H^p(U, R^q f_U^* \mathbb{Z}) = 0$ if $p + q$ is odd. Hence our assertion follows. □

For the quadric fibration $f : X \to S$, we have a closed embedding $e : V \hookrightarrow X$ such that the image $e(t)$ for $t \in V$ is the unique singular point of $f^{-1}(t)$. Let $\epsilon : \tilde{X} \to X$ be the blowing up along the smooth center $e(V)$. Let $Y := f^{-1}(V)$ be the family of singular quadrics over $V$, and let $\tilde{Y}$ be the proper transform of $Y$. Consider the composition $h : \tilde{Y} \to Y \to V$. Each fiber $h^{-1}(t)$ for $t \in V$ is the blow-up of the quadric cone $X_t$ along the unique singular point. Hence $h^{-1}(t)$ is a $\mathbb{P}^1$-bundle over a $(2n-2)$-dimensional smooth quadric. It follows that $h$ factors through a smooth family $Q$ of $(2n-2)$-dimensional quadrics over $V$. Let us write

$$h := q \circ p : \tilde{Y} \to Q \to V$$

where $p : \tilde{Y} \to Q$ is a $\mathbb{P}^1$-bundle. We can summarize the paragraph by the following diagram

(11)

Let $\pi_q : \mathcal{F}_{n-1}(Q/V) \to V$ be the relative Hilbert scheme of $(n-1)$-planes contained in the fibers of $q$. For a smooth quadric of even dimension, the variety of the maximal linear subspaces is the disjoint union of two components. Hence every fiber of the morphism $\pi$ has two disjoint components. Consider the Stein factorization

$$\mathcal{F}_{n-1}(Q/V) \to \tilde{V} \to V.$$  

The finite map $\pi : \tilde{V} \to V$ is an étale double covering.

**Proposition 3.3.** Keep the above notations. There is an isomorphism

$$\phi : H^{d-1}(\tilde{V}, \mathbb{Z}) \cong H^{2n+d-3}(Q, \mathbb{Z})$$

such that

1. $\phi$ is an isomorphism of Hodge structures of type $(n-1, n-1)$

$$\phi(H^{p,q}(\tilde{V})) = H^{p+n-1,q+n-1}(Q), \quad \forall \ p + q = d - 1;$$
for any \( a, b \in H^{d-1}(\tilde{V}, \mathbb{Z}) \), we have

\[
\int_{Q} \phi(a) \cup \phi(b) = \begin{cases} 
\langle a, b \rangle, & \text{if } n \text{ is odd;} \\
\langle a, \iota^* b \rangle, & \text{if } n \text{ is even}
\end{cases}
\]

where \( \iota^* \) is the natural involution on \( \tilde{V} \), and \( \langle \ , \rangle \) is the intersection pairing on \( \tilde{V} \).

**Proof.** We consider the Leray spectral sequence for the smooth family \( q \)
\[(12) \quad E_{r}^{i,j} := H^i(V, R^j q_* \mathbb{Z}) \Rightarrow H^{i+j}(Q, \mathbb{Z}).\]

By a result in [10, EXP. XII, thm. 3.3], there is an isomorphism
\[(13) \quad u : \pi_* \mathbb{Z} \simto R^{2n-2} q_* \mathbb{Z}.\]

This isomorphism can be seen via the above Stein factorization. Then \( R^j q_* \mathbb{Z} \) is isomorphic to
\[
\begin{cases}
0, & j \text{ odd; } \\
\mathbb{Z}, & j \neq 2n - 2 \text{ even; } \\
\pi_* \mathbb{Z}, & j = 2n - 2.
\end{cases}
\]

Recall the integer \( d \) is even. It is easy to see \( E_{2}^{d-1,2n-2} \) is the only non-zero term among \( \{E_{r}^{i,j}\} \) for \( i + j = 2n + d - 3 \). We claim the Leray spectral sequence \((12)\) degenerates at \( E_2 \). Therefore
\[
H^{d-1}(\tilde{V}, \mathbb{Z}) \cong E_{2}^{d-1,2n-2} = E_{\infty}^{d-1,2n-2} = H^{2n+d-3}(Q, \mathbb{Z}).
\]

On the one hand, Deligne’s degeneration theorem asserts the Leray spectral sequence \((12)\) with \( \mathbb{Q} \)-coefficients degenerates at \( E_2 \), i.e. \( d_r \otimes \mathbb{Q} = 0, r \geq 2 \). On the other hand, it follows from Lemma 3.1 that for \( j \neq 2n - 2 \) the terms \( E_{r}^{i,j} = H^i(V, R^j q_* \mathbb{Z}) \) are torsion-free. For \( j = 2n - 2 \), note that \( V \) is simply connected and thus \( \tilde{V} = V \cup V \), hence \( E_{2}^{1,2n-2} = H^i(\tilde{V}, \mathbb{Z}) \) is torsion-free as well. As a result, we obtain \( d_r = 0, r \geq 2 \) and the spectral sequence \((12)\) degenerates at \( E_2 \).

The proof of the other two properties on \( \phi \) is the same as the one given by Beauville. For the reader’s convenience we sketch the arguments.

The first property deduces from Borel’s result on fiber bundles [14, app. 2, thm. 2.1]. Let \( F \) denote the fiber of \( q \). Let \( \mathbf{H}(F) \) be the holomorphic vector bundle \( R^j q_* \mathcal{C} \otimes \mathcal{O}_V \) on \( V \), and let \( \mathbf{H}_s^{r,s}(F), r + s = j \) be the canonical Hodge subbundles. The complexified Leray spectral sequence of \((12)\) admits a canonical grading
\[
\sum_{p+q-i+j, p,q \geq 0} p,q E_{r}^{i,j} = E_{r}^{i,j}
\]
of type \( (p,q) \) with the differential \( d_r \) maps \( p,q E_{r}^{i,j} \) to \( p,q+1 E_{r+1,j}^{i+j} \). Moreover, the grading of type \( (p,q) \) converges to the Hodge component \( H^{p,q}(Q) \). On the \( E_2 \)-page, the grading \( p,q E_{r}^{i,j} \) has the Künneth decomposition
\[
p,q E_{2}^{i,j} \cong \sum_{t \geq 0} H_{\mathcal{O}}^{i-t}(V, \mathbf{H}_s^{p-t,q-i+t}(F)).
\]

In our case, the isomorphism \[13\] shifts the weights of Hodge structures by \( 2n - 2 \). Hence it is direct to verify that
\[
\phi(H^{p,q}(\tilde{V})) = H^{p+n-1,q+n-1} E_{2}^{d-1,2n-2} = H^{p+n-1,q+n-1}(Q).
\]
Now let us prove the second property. Under the isomorphism
\[ H^{d-1}(V, R^{2n-2}q_*Z) \to H^{2n+d-3}(Q, Z), \]
the intersection form on \( H^{2n+d-3}(Q, Z) \) corresponds to the cup-product
\[ \cup : R^{2n-2}q_*Z \otimes R^{2n-2}q_*Z \to R^{4n-4}q_*Z \rightarrow \mathbb{Z}. \]
The intersection form on \( H^{d-1}(\tilde{V}, Z) \) corresponds to the cup-product
\[ (\ , \ ) : \pi_*Z \otimes \pi_*Z \to \pi_*Z \rightarrow \mathbb{Z}. \]

We illustrate the relation between the two cup-products \( \cup \) and \( (\ , \ ) \) under the isomorphism (13) in the following manner (cf. [10, EXP. XII, thm. 3.3 (iii)]):

Recall the variety of \((n-1)\)-planes in the smooth quadric \( F \) has two connected components. Any two \((n-1)\)-planes \( \Lambda, \Lambda' \subset F \) are in the same component if and only if
\[ \dim \Lambda \cap \Lambda' \equiv (n-1) \mod 2. \]
Let \( \Lambda_1, \Lambda_2 \subset F \) be two maximal linear subspaces in different components representing the generators of \( H^{2n-2}(F) \). The above dimension congruence concludes the intersection relation
\[
\begin{cases}
\Lambda_1^2 = \Lambda_2^2 = 1, & \text{if } n \text{ is odd;} \\
\Lambda_1^2 = \Lambda_2^2 = 0, & \text{if } n \text{ is even.}
\end{cases}
\]
Let \( U \) be an étale local chart of \( V \) such that \( U \times V \tilde{V} = U \cup U \). Write \( a, b \in H^{d-1}(U, \pi_*Z) \) as
\[ a = \alpha_1 + \alpha_2, b = \beta_1 + \beta_2 \]
where \( \alpha_i, \beta_i \) are classes supported on the different pieces of \( U \times V \tilde{V} \). Locally we can write
\[ u(\alpha_i) = \alpha_i \otimes \Lambda_i, u(\beta_i) = \beta_i \otimes \Lambda_i \in H^{d-1}(U, R^{2n-2}q_*Z). \]
It follows from the intersection relation on \( \Lambda_1 \) and \( \Lambda_2 \) that
\[ u(a) \cup u(b) = \begin{cases} 
\alpha_1 \cup \beta_1 + \alpha_2 \cup \beta_2, & \text{if } n \text{ is odd;} \\
\alpha_1 \cup \beta_2 + \alpha_2 \cup \beta_1, & \text{if } n \text{ is even.}
\end{cases} \]
Note that the involution \( \iota^* \) on \( \tilde{V} \) exchanges two pieces of \( U \times V \tilde{V} \). Therefore
\[ u(a) \cup u(b) = \begin{cases} 
(a,b), & \text{if } n \text{ is odd;} \\
(a,\iota^*b), & \text{if } n \text{ is even.}
\end{cases} \]

The above proposition and the diagram (11) defines an homomorphism of Hodge structure
\[ \tilde{\psi} : H^{d-1}(\tilde{V}, Z) \to H^{2n+d-1}(\mathcal{X}, Z) \]
via the composition
\[ H^{d-1}(\tilde{V}) \overset{\iota^*}{\to} H^{2n+d-3}(Q) \overset{p_*}{\to} H^{2n+d-3}(\tilde{\mathcal{Y}}) \overset{\tilde{\iota}}{\to} H^{2n+d-1}(\tilde{\mathcal{X}}) \overset{\iota^*}{\to} H^{2n+d-1}(\mathcal{X}). \]

**Proposition 3.4.** (1) The homomorphism \( \tilde{\psi} : H^{d-1}(\tilde{V}, Z) \to H^{2n+d-1}(\mathcal{X}, Z) \) is surjective.
(2) The intersection form \( \langle \ , \ \rangle \) on \( H^{d-1}(\tilde{V}, \mathbb{Z}) \) satisfies
\[
\int_X \tilde{\psi}(a) \cup \tilde{\psi}(b) = (-1)^n \langle a, b - t^*b \rangle, \forall a, b \in H^{d-1}(\tilde{V}, \mathbb{Z})
\]
where \( t^* \) is the natural involution on \( H^{d-1}(\tilde{V}, \mathbb{Z}) \).

Proof. Let \( U = S - V \), let \( \mathcal{X}_U := f^{-1}(U) \) be the family of smooth quadrics. Consider the localization long exact sequence
\[
\rightarrow H^{2n+d-1}_{\mathcal{Y}}(\mathcal{X}, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}_U, \mathbb{Z}) \rightarrow .
\]
By Lemma \( \ref{lemma} \) the cohomology \( H^{2n+d-1}(\mathcal{X}_U, \mathbb{Z}) \) vanishes. Hence the Gysin map
\[
H^{2n+d-3}(\mathcal{Y}, \mathbb{Z}) \cong H^{2n+d-1}_{\mathcal{Y}}(\mathcal{X}, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}, \mathbb{Z})
\]
is surjective, which implies the composition
\[
\epsilon_*j_* : H^{2n+d-3}(\tilde{\mathcal{Y}}, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}, \mathbb{Z})
\]
is surjective.

Let \( E \) be the exceptional divisor of the blow-up \( \mathcal{X} \). Then \( E \cap \tilde{\mathcal{Y}} \) is the exceptional divisor of the proper transform \( \tilde{\mathcal{Y}} \) of \( \mathcal{Y} \). As shown before, every fiber of \( \mathcal{Y} \rightarrow V \) is a quadric cone of dimension \( 2n - 1 \) with a single vertex, and the proper transform \( \tilde{\mathcal{Y}} \) is the blowing up along the collection of the vertex of every quadric cone. Therefore \( E \cap \tilde{\mathcal{Y}} \) is a family of smooth quadric of dimension \( 2n - 2 \) over \( V \), which is isomorphic to \( Q \) via the projection \( p \). The isomorphism gives a section \( k : Q \hookrightarrow \mathcal{Y} \) of \( p \) that fits into the following commutative diagram
\[
\begin{array}{ccc}
Q & \xrightarrow{i} & E \\
\downarrow{k} & & \downarrow{i} \\
\tilde{\mathcal{Y}} & \xrightarrow{j} & \mathcal{X}.
\end{array}
\]
The canonical line bundle \( \mathcal{O}_p(1) \) for the \( \mathbb{P}^1 \)-bundle \( p : \tilde{\mathcal{Y}} \rightarrow Q \) is isomorphic to \( \mathcal{O}(E)|_{\tilde{\mathcal{Y}}} \). Then the above diagram implies the decomposition
\[
H^{2n+d-3}(\tilde{\mathcal{Y}}, \mathbb{Z}) \cong p^*H^{2n+d-3}(Q, \mathbb{Z}) \oplus k_*H^{2n+d-5}(Q, \mathbb{Z}).
\]
We claim the direct summand \( k_*H^{2n-1}(Q, \mathbb{Z}) \) is annihilated by the map \( \epsilon_*j_* \). By diagram \( \ref{diagram} \) we have \( \epsilon_*j_*k_* = \epsilon_*i_*l_* \). The map \( \epsilon_*i_*l_* \) factors through
\[
\epsilon_*i_* : H^{2n+d-3}(E, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}, \mathbb{Z}).
\]
Denote by \( h \in H^2(E) \) the divisor class \( c_1(\mathcal{O}_p(1)) \), and \( \pi : E \rightarrow V \) the projection. The cohomology of \( E \) has the decomposition
\[
H^{2n+d-3}(E, \mathbb{Z}) \cong \bigoplus_{i=0}^{2n-1} \pi^*H^{2n+d-3-2i}(V, \mathbb{Z}) \cdot [h^i].
\]
Note that \( \pi_*h^i \neq 0 \) unless \( i = 2n - 1 \), and \( H^{d-1-2n}(V, \mathbb{Z}) = 0 \) by Lemma \( \ref{lemma} \). Hence the map \( \epsilon_*i_* \) is zero. Then the map
\[
\epsilon_*j_*p^* : H^{2n+d-3}(Q, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}, \mathbb{Z})
\]
is surjective. In conclusion, the morphism \( \psi : H^{d-1}(\tilde{V}, \mathbb{Z}) \rightarrow H^{2n+d-1}(\mathcal{X}, \mathbb{Z}) \) is surjective.

Let \( a, b \in H^{d-1}(\tilde{V}, \mathbb{Z}) \), and \( \phi(a) = x, \phi(b) = y \). We have
\[
\langle \epsilon_*j_*p^*x, \epsilon_*j_*p^*y \rangle = \langle j_*p^*x, \epsilon^*j_*p^*y \rangle.
\]
Let $i : E \hookrightarrow \hat{X}$ denote the inclusion of the exceptional divisor $E$ of the blowing up $\epsilon : \hat{X} \to \mathcal{X}$, and $\pi : E \to V$ the projective normal bundle $N$ of the smooth subvariety $\epsilon(V)$ in $\mathcal{X}$. It follows from Fulton’s key formula \cite{12} prop. 6.7 for blow-up. The cohomology class $\epsilon^*\epsilon_*p^*y$ can be expressed as

$$
\epsilon^*\epsilon_*p^*y = j_*p^*y + i_* \left( \sum_{r=0}^{2n-2} 2n \cdot \pi^*(\gamma_{2n-2-r} \cdot i^*j_*p^*y) \right),
$$

where

$$
\gamma_s = h^s + h^{s-1} \cdot \pi^*c_1(N) + \ldots + \pi^*c_s(N).
$$

is a codimension $s$ algebraic cycle in $A^s(E)$. For the first term $j_*p^*y$ we have

$$
\langle j_*p^*x, j_*p^*y \rangle = p^*x \cdot p^*y \cdot [\hat{Y}] / Y.
$$

Denote by $D_V$ the divisor class of $V$ in $\text{Pic}(\mathcal{S})$. Then the divisor class of the proper transform $\hat{Y}$ in $\text{Pic}(\hat{X})$ is

$$
\epsilon^*f^*D_V - 2[E] = [\hat{Y}].
$$

It implies $\langle j_*p^*x, j_*p^*y \rangle$ is equal to

$$
(p^*(x \cdot y) \cdot (j^*\epsilon^*f^*D_V - 2k, 1)) = p^*(x \cdot y \cdot q^*D_V | V) - 2(x \cdot y) = -2(x \cdot y).
$$

Now we deal with the second term. Let us set

$$
P_r := \langle j_*p^*x, i_* (h^r \cdot \pi^*\pi_*(\gamma_{2n-2-r} \cdot i^*j_*p^*y)) \rangle
$$

$$
= (i^*j_*p^*x \cdot h^r \cdot \pi^*\pi_*(\gamma_{2n-2-r} \cdot i^*j_*p^*y))
$$

The cartesian diagram \cite{10} deduces that $l^*i^*j_*p^* = l_*$. It follows that

$$
P_r = l_*x \cdot h^r \cdot \pi^*\pi_*(\gamma_{2n-2-r} \cdot l_*y)
$$

$$
= x \cdot l^*h^r \cdot q^*q_*(l^*\gamma_{2n-2-r} \cdot y).
$$

The degree of the cohomology class $q_*(l^*\gamma_{2n-2-r} \cdot y)$ is

$$
2(2n - 2 - r) + 2n + d - 3 - 2(2n - 2) = 2n + d - 3 - 2r.
$$

Note that the number $2n + d - 3 - 2r$ is odd. The Lefschetz hyperplane theorem asserts $H^{2n+d-1-2r}(V, \mathbb{Z}) = 0$ unless $2n + d - 3 - 2r = d - 1$. Hence $P_r$ is possibly non-zero if and only if $r = n - 1$. Recall

$$
\gamma_{n-1} = h^{n-1} + h^{n-2} \cdot \pi^*c_1(N) + \ldots + \pi^*c_{n-1}(N)
$$

Then $q_*(l^*\gamma_{n-1} \cdot y) = q_*(l^*h^{n-1} \cdot y)$. It follows that

$$
\int_X \psi(a) \cup \psi(b) = -2(\phi(a), \phi(b)) + (\phi(a) \cdot l^*h^{n-1} \cdot q^*q_*(l^*h^{n-1} \cdot \phi(b))).
$$

Combining the results of Lemma 3.5 and Proposition 3.3 (2) we obtain

$$
\int_X \psi(a) \cup \psi(b) = \begin{cases} 
(a, -b + i^*b), & n \text{ odd}; \\
(a, b - i^*b), & n \text{ even}. 
\end{cases}
$$

\begin{lemma}
Set $l^*h = \eta \in H^2(Q)$. For all $b \in H^{d-1}(\hat{V})$, we have

$$
\phi(b + i^*b) = \eta^{n-1} \cdot q^*q_*(\eta^{n-1} \cdot \phi(b))
$$

\end{lemma}

\begin{proof}
See \cite{2} lem. 2.4
\end{proof}
Corollary 3.6. The surjective map \( \hat{\psi} \) induces an isomorphism
\[ \psi : H^{d-1}(V, \mathbb{Z}) \rightarrow H^{2n+d-1}(X, \mathbb{Z}) \]
of polarized Hodge structures up to a sign. To be precise, let \( (\ , \ ) \) denote the intersection form on \( H^{d-1}(V) \) and \( H^{2n+d-1}(X) \) respectively. For any \( x, y \in H^{d-1}(V, \mathbb{Z}) \), we have
\[ (\psi(x), \psi(y)) = (-1)^n(x, y). \]

Proof. The assertion (2) of Proposition 3.4 implies the following exact sequence
\[ 0 \rightarrow \text{Ker}(1 - \iota^*) \rightarrow H^{d-1}(\tilde{V}, \mathbb{Z}) \rightarrow H^{2n+d-1}(X, \mathbb{Z}) \rightarrow 0. \]
Recall that \( V \) is simply connected. Then the étale double cover \( \tilde{V} \) is the disjoint union \( V \sqcup \tilde{V}, \) and the involution \( \iota^* \) on \( \tilde{V} \) exchanges two disjoint pieces. The quotient group \( H^{d-1}(\tilde{V}, \mathbb{Z})/(\iota^*) \) is isomorphic to \( H^{d-1}(V, \mathbb{Z}) \) via the map
\[ H^{d-1}(\tilde{V}, \mathbb{Z})/(\iota^*) \rightarrow H^{d-1}(V, \mathbb{Z}), \quad (z_1, z_2) \mapsto z_1 - z_2. \]
Therefore it induces an isomorphism \( \psi : H^{d-1}(V, \mathbb{Z}) \rightarrow H^{2n+d-1}(X, \mathbb{Z}). \)

Let \( x, y \in H^{d-1}(V, \mathbb{Z}) \), and \( a, b \in H^{d-1}(\tilde{V}, \mathbb{Z}) \) that map to \( x, y \) respectively. We will show
\[ (\psi(x), \psi(y)) = (\tilde{\psi}(a), \tilde{\psi}(b)) = (-1)^n(x, y). \]
By Proposition 3.4 (2) it suffices to prove
\[ (a, b - \iota^*b) = (x, y). \]
Firstly it is independent of the choice of the classes \( a \) and \( b \). In fact, suppose any other \( b' \in H^{d-1}(\tilde{V}, \mathbb{Z}) \) maps to \( y \). Then \( b - b' \) is \( \iota^* \)-invariant, which implies \( \langle a, b - \iota^*b \rangle = \langle a, b' - \iota^*b' \rangle \). Similarly it is independent of the choice of \( a \) because
\[ \langle a, b - \iota^*b \rangle = \langle a - \iota^*a, b \rangle. \]
Hence we can assume \( a = (x, 0), b = (y, 0) \). It is direct to verify \( \langle a, b - \iota^*b \rangle = (x, y). \) \( \square \)

For the rest of the section we prove
\[ H^{2n+d-3}(X, \mathbb{Z}) \cong H^{2n+d+1}(X, \mathbb{Z})^* = 0. \]
Let \( \varphi : P \rightarrow S \) be the projective bundle associated to a quadric fibration \( f : Q \rightarrow S \).
Let \( i : Q \hookrightarrow P \) be the natural inclusion over \( S \). We define
\[ (R^k f_*)_v := \text{coker}(i^* : R^k \varphi_* \mathbb{Z} \rightarrow R^k f_* \mathbb{Z}). \]
On the level of cohomology we define
\[ H^k(Q, \mathbb{Z})_v := \text{coker}(i^* : H^k(P, \mathbb{Z}) \rightarrow H^k(Q, \mathbb{Z})). \]
In the habilitation \[29], J. Nagel introduced a modified Leray spectral sequence \( E^\bullet(f)_v \) defined as the quotient of the the homomorphism
\[ i^* : E^\bullet(\varphi) \rightarrow E^\bullet(f) \]
of Leray spectral sequences with respect to \( \varphi \) and \( f \). To be precise, the term \( E_{r,q}^p(f)_v \) is defined to be the cokernel of \( i^* : E_{r,q}^p(\varphi) \rightarrow E_{r,q}^p(f) \). Using the mixed Hodge structures on Leray spectral sequence, Nagel showed that it forms a spectral sequence whose \( E_2 \)-page is
\[ E_2^{p,q}(f)_v := H^p(S, (R^q f_* \mathbb{Z})_v), \]
and it converges to \( H^{p+q}(Q, \mathbb{Z})_v. \)
Lemma 3.7. Let $f : X \to S$ be the quadric fibration of relative dimension $2n - 1$ in our setting, and $V$ be the discriminant divisor of $f$. Denote by $i : V \to S$ the closed immersion, and by $j : U := S \setminus V \to S$ the open embedding of the open complement. The sheaf $(R^{2n} f_*)_v$ fits into the following exact sequence

$$0 \to j_! \mathbb{Z}[2] \to (R^{2n} f_*)_v \to i_* L \to 0$$

where $L$ is a rank one local system on $V$. In addition, $(R^q f_*)_v = 0$ if $q < 2n$, and $(R^q f_*)_v = \mathbb{Z}/2\mathbb{Z}$ for even $q > 2n$.

Proof. The sheaf $(R^q f_*)_v$ is constructible with respect to the degeneracy loci of the quadric fibration $f$. The local system on each stratum is fully determined by the corank of the quadrics parametrized the stratum.

Suppose a quadric $Q \subset \mathbb{P}^{2n}$ has corank $s$. The stalk of the constructible sheaf $(R^q f_*)_v$ at the point $[Q]$ is the primitive quotient $H^q(Q, \mathbb{Z})_v$:

$$H^q(Q, \mathbb{Z})_v \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{even } q > 2n - 1 + s; \\ \mathbb{Z}, & \text{even } q = 2n - 1 + s; \\ 0, & \text{even } q < 2n - 1 + s \text{ or odd } q. \end{cases}$$

For the quadric fibration $f$ the possible coranks of fibers are 0 and 1. By the characterization of stalks we immediately conclude $(R^q f_*)_v = 0$ if $q < 2n$, and $(R^q f_*)_v = \mathbb{Z}/2\mathbb{Z}$ for even $q > 2n$.

For $q = 2n$ we consider the natural square diagrams

$$Y \xrightarrow{g} X \xleftarrow{X_U} X_U$$

and the canonical exact sequence

$$0 \to j_! j^* (R^{2n} f_*)_v \to (R^{2n} f_*)_v \to i_* i^* (R^{2n} f_*)_v \to 0.$$  

By the proper base change theorem, it is easy to verify the quotient $(R^k f_*)_v$ is invariant under any base change. Namely for any $S$-scheme $u : T \to S$ the base change map

$$u^* (R^k f_*)_v \to (u^* R^k f_*)_v$$

is an isomorphism. Therefore we obtain the exact sequence

$$0 \to j_! (R^{2n} f_*)_v \to (R^{2n} f_*)_v \to i_* (R^{2n} g_* Z)_v \to 0.$$  

Then it suffices to show $(R^{2n} f_*)_v \cong \mathbb{Z}/2\mathbb{Z}$ and $(R^{2n} g_* Z)_v \cong L$.

Note that $f_0$ is a smooth family of $(2n - 1)$-dimensional quadrics. By passage to the stalk we can see $(R^{2n} f_*)_v$ is a local system of the constant group $\mathbb{Z}/2\mathbb{Z}$. The family $g$ of singular quadrics is not a smooth morphism. Let $\Sigma \subset Y$ be the locus of the unique singular point in each fiber. Consider the diagram

$$Y \setminus \Sigma \xrightarrow{k} Y \xleftarrow{\epsilon} \Sigma \xrightarrow{g} V \xrightarrow{h} \Sigma.$$  

Applying the derived functor $Rg_*$ to the canonical exact sequence

$$0 \to k_* \mathbb{Z} \to Z \to e_* \mathbb{Z} \to 0,$$
resolves the period map at the point $\omega$ in the GIT compactification of $M$ corresponding to the objects that is the monodromy group. The induced period map $D$ in the derived category $\omega$ is smooth. By passage to the stalk, we conclude that $(R^{2n}g_*Z)_v$ is a local system of rank one.

**Corollary 3.8.** The cohomology $H^{2n+d-3}(\mathcal{X}, \mathbb{Z})$ is zero.

**Proof.** In our setting, the base space $S$ is even dimensional. Then the odd degree cohomology of the projective bundle $P$ vanishes. Hence

$$H^{2n+d-3}(\mathcal{X}) = H^{2n+d-3}(\mathcal{X})_v.$$ 

We use Nagel’s spectral sequence

$$E_2^{p,q}(f)_v := H^p(S, (R^q f_*Z)_v) \Rightarrow H^{2n+d-3}(\mathcal{X})_v.$$ 

By the description of $(R^q f_*Z)_v$ in the preceding Lemma, the only non-trivial $E_2$-term that degenerates to $H^{2n+d-3}(\mathcal{X})_v$ is

$$E_2^{d-3, 2n}(f)_v = H^{d-3}(S, (R^{2n}f_*Z)_v).$$

Applying the derived functor $R\Gamma(S, -)$ to the exact sequence [17] we obtain the long exact sequence

$$\cdots \rightarrow H^{d-3}_c(U, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{d-3}(S, (R^{2n}f_*Z)_v) \rightarrow H^{d-3}(V, \mathbb{L}) \rightarrow \cdots$$

The Poincaré duality asserts $H^{d-3}_c(U, \mathbb{Z}/2\mathbb{Z}) \cong H_{d+3}(U, \mathbb{Z}/2\mathbb{Z})$. Recall that $U$ is a smooth affine variety of dimension $d$. Then $U$ is homotopic to a CW-complex of real dimension $\leq d$ by the Morse theory. Hence $H_i(U, \mathbb{Z}/2\mathbb{Z}) = 0$ for $i > d$.

The category of local systems on $V$ is equivalent to the category of monodormy representations of $\pi_1(V)$. By our hypothesis $V$ is simply connected. Therefore the rank one local system $L$ is isomorphic to the constant group $\mathbb{Z}$. Lemma [3.1] asserts $H^{d-3}(V, \mathbb{Z}) = 0$. As a result, we have $H^{d-3}(S, (R^{2n}f_*Z)_v) = 0$.  

4. Extension of period mappings

Let $\mathcal{F}$ be the moduli space of smooth cubic hypersurfaces in $\mathbb{P}^N$. Let $\overline{\mathcal{F}}$ be the GIT compactification of $\mathcal{F}$, which is obtained as the GIT quotient by the action of the reductive group $SL_{N+1}$ on the parameter space $\mathbb{P}(H^0(\mathbb{P}^N, O_{\mathbb{P}^N}(3)))$. Consider the period map

$$\mathcal{P}: \mathcal{F} \rightarrow \Gamma \backslash \mathcal{D},$$

where $\mathcal{D}$ is the local period domain for smooth cubic hypersurfaces in $\mathbb{P}^N$ and $\Gamma$ is the monodromy group. The induced period map $\overline{\mathcal{F}} \rightarrow \Gamma \backslash \mathcal{D}$ is singular at the point $\omega := [Sec(S)]$. Let us consider the Kirwan blow-up $\overline{\mathcal{F}}_{\text{kir}}$ of $\overline{\mathcal{F}}$ at the point $\omega$, and let $\mathcal{M}$ be the exceptional divisor. We show that the blowing up partially resolves the period map at the point $\omega$ in the following sense:

There exists a partial compactification $\overline{\Gamma \backslash \mathcal{D}}$ of the period domain $\Gamma \backslash \mathcal{D}$. The period map $\overline{\mathcal{F}}_{\text{kir}} \rightarrow \overline{\Gamma \backslash \mathcal{D}}$ extends holomorphically over the generic point of the exceptional divisor $\mathcal{M}$ such that the image of $\mathcal{M}$ lies in boundary component of $\overline{\Gamma \backslash \mathcal{D}}$. Furthermore, the extension map on $\mathcal{M}$ can be identified with the period map corresponding to the objects that $\mathcal{M}$ parametrizes.
4.1. Kirwan’s blow-up and the exceptional divisor. We firstly look into the exceptional divisor $\mathcal{M}$. Denote $P := \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(3)))$ and $G := SL_{N+1}$. Kirwan’s blow-up for the GIT quotient $P//G$ at the point $\omega$ is illustrated in the following manner. Let $x \in P$ represents the secant cubic $Sec(S)$ over the point $\omega \in \mathcal{F}$. Let $G_x \subset G$ be the stabilizer of $x$. Luna’s slice theorem [28, p. 198] describes the étale local structure around the point $\omega \in \mathcal{F}$ as the quotient space $\mathcal{N}_x//G_x$. Here $\mathcal{N}_x$ is the normal bundle of the closed orbit $G \cdot x$ in $P$ at the point $x$. Via the étale local topology, the exceptional divisor $\mathcal{M}$ is isomorphic to the GIT quotient $\mathbb{P}(\mathcal{N}_x)//G_x$.

For the case of the secant cubic fourfold, it has been shown in [23, §4.1.1] that the normal bundle $\mathcal{N}_x$ can be identified with the space of plane sextic curves. Via the Veronese embedding $v_2(\mathbb{P}^2) = S$, a plane sextic on $\mathbb{P}^2$ is a section of the line bundle $\mathcal{O}_S(3)$ on $S$. The stabilizer group $G_x \cong SL_3$ naturally acts on $S \cong \mathbb{P}^2$.

We realize the analogous descriptions of the normal bundle $\mathcal{N}_x$ and the stabilizer $G_x$ occur for higher dimensional Severi varieties. It involves the characterization of the Severi variety $S \subset \mathbb{P}^N$ and the secant variety $Sec(S)$ by an irreducible representation $H \to Aut(W)$ of a (semi-)simple algebraic groups $H$ and a vector space $W \cong \mathbb{C}^{N+1}$. Let us recall the following

1. $H = SL(3, \mathbb{C})$ acts on the space $W$ of symmetric $3 \times 3$-matrices. The action is induced from the standard representation of $SL(3, \mathbb{C})$ on $\mathbb{C}^3$;
2. $H = SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$ acts on the space $W$ of $3 \times 3$-matrices. The action is induced from the standard representation of $SL(3, \mathbb{C})$ on $\mathbb{C}^3$;
3. $H = SL(6, \mathbb{C})$ acts on the space $W$ of skew-symmetric $6 \times 6$-matrices. The action is induced from the standard representation of $SL(6, \mathbb{C})$ on $\mathbb{C}^6$;
4. $H = E_6$ acts on the space $W$ as the 27-dimensional Jordan algebra of Hermitian $3 \times 3$-matrices over the Cayley algebra $\mathcal{O}$.

In terms of the representation, the Severi variety and the secant variety can be regarded as $H$-orbits in the projective space $\mathbb{P}(W)$.

Lemma 4.1. Let $x \in P$ represents the secant cubic $Sec(S)$ of the Severi variety $S \subset \mathbb{P}^N$. The stabilizer subgroup $G_x \subset G := SL(N+1, \mathbb{C})$ is isomorphic to the corresponding algebraic group $H$ of $S$ described above.

Proof. Assume $g \in G_x$ is a linear automorphism of the secant cubic $Sec(S)$. The Severi variety $S$ is the singular part of $Sec(S)$. Hence $g|_S$ is a linear automorphism of $S$ as well. For the classical cases $S = \mathbb{P}^2, \mathbb{P}^2 \times \mathbb{P}^2$, and $Gr(2, 6)$, it is easy to see that the linear automorphism of matrices preserving the determinant is exactly the group $SL(3, \mathbb{C}), SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$, and $SL(6, \mathbb{C})$. For the exceptional group $E_6$, the result can be found in [26].

Lemma 4.2. Let $S \subset \mathbb{P}^N$ be the Severi variety, and let $\mathcal{O}_S(1)$ be the induced ample line bundle. The above normal bundle $\mathcal{N}_x$ is isomorphic to the space of global sections of $\mathcal{O}_S(3)$.

Proof. There is a natural restriction map

$$\text{Sym}^3 W = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(3)) \to H^0(S, \mathcal{O}_S(3)).$$

Let $T_x\mathcal{O}_x$ be the tangent space of the orbit $\mathcal{O}_x := G \cdot x \subset P$ at the point $x$. Our assertion will follow if the sequence

$$0 \to T_x\mathcal{O}_x \to \text{Sym}^3 W/\mathbb{C} \cdot x \to \Gamma(\mathcal{O}_S(3))$$
is exact at the middle and

\[(\dim \text{Sym}^3 W - 1) - \dim T_x O_x = \dim \Gamma(O_S(3)).\]

Let \( f \) be the equation of the secant cubic hypersurface \( x \). The tangent space \( T_x O_x \) is the subspace of \( \text{Sym}^3 W/\mathbb{C} \cdot x \) generated by the Jacobian ideal

\[J_f = \langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N} \rangle\]

of the equation \( f \). It is known that the Severi variety \( S \) is cut out by the differentials \( \{ \frac{\partial f}{\partial x_i} \}_{0 \leq i \leq N} \). Therefore a cubic polynomial \( q \) restricts to zero on \( S \) if and only if \( q \) is generated by the Jacobian ideal \( J_f \), which implies the exactness. Consider the stabilizer \( G_x \) of \( x \). The orbit \( O_x \sim G/G_x \). Then we have

\[\dim T_x O_x = \dim G - \dim G_x.\]

It follows from Lemma [12] that \( G_x \) is the (semi-)simple algebraic group \( H \) associated to the Severi variety \( S \). Hence it suffices to verify (18) case by case as follows

1. \( n = 4, N = 8 \). Then \( G = SL_9 \) and \( G_x = SL_3 \times SL_3 \). We have

\[\dim \text{Sym}^3 W = \begin{pmatrix} 8 + 3 \\ 3 \end{pmatrix} = 165, \quad \dim G - \dim G_x = 64.\]

It is easy to compute \( \dim \Gamma(O_S(3)) = \dim \Gamma(O_{F \times F}(3, 3)) = 100; \)

2. \( n = 8, N = 14 \). Then \( G = SL_{15} \), \( G_x = SL_6 \). We have

\[\dim \text{Sym}^3 W = \begin{pmatrix} 17 \\ 3 \end{pmatrix} = 680, \quad \dim G - \dim G_x = 189.\]

By Borel-Weil-Bott’s theory, the vector space \( \Gamma(O_S(3)) = \Gamma(O_{Gr(2,6)}(3)) \) is an irreducible \( \mathfrak{sl}_6 \)-module that corresponds to a fundamental weight of \( \text{Sym}^3 W \). Using Weyl’s character formula we obtain \( \dim \Gamma(O_{Gr(2,6)}(3)) = 490; \)

3. \( n = 16, N = 26 \). Then \( G = SL_{27} \) and \( G_x = E_6 \). We have

\[\dim \text{Sym}^3 W = \begin{pmatrix} 29 \\ 3 \end{pmatrix} = 3654, \quad \dim G - \dim G_x = 650.\]

Again the space \( \Gamma(O_S(3)) = \Gamma(O_{OP^2}(3)) \) is an irreducible \( E_6 \)-module that corresponds to a highest weight of \( \text{Sym}^3 W \). Weyl’s character formula implies \( \dim \Gamma(O_{OP^2}(3)) = 3003 \), see [26].

Corollary 4.3. Let \( S \subset \mathbb{P}^N \) be the Severi variety. Let \( \omega \) represent the secant cubic \( \text{Sec}(S) \) in the moduli space \( \mathcal{T} \). The exceptional divisor \( \mathcal{M} \) of Kirwan’s blow-up of \( \mathcal{T} \) at the point \( \omega \) is isomorphic to the GIT quotient \( \mathbb{P}(|O_S(3)|)/H \) of cubic sections of the line bundle \( O_S(3) \) on \( S \), where \( H \) is the corresponding algebraic group of \( S \).

When \( S = \mathbb{P}^2 \), the exceptional divisor \( \mathcal{M} \) parametrizes plane sextic curves. When \( S = \mathbb{P}^2 \times \mathbb{P}^2 \), \( \mathcal{M} \) parametrizes Calabi-Yau threefolds shown in Example 2.5. For \( S = G(2,6) \) or \( O\mathbb{P}^2 \), the cubic section of \( O_S(3) \) is a Fano variety.
4.2. Boundary components and extension theorem. The goal of this subsection is to prove the following statement

**Theorem 4.4.** The period map \( \mathcal{P} : \mathcal{F}^\text{vir} \rightarrow \Gamma \backslash \mathcal{D} \) extends holomorphically over the generic points of the exceptional divisor \( \mathcal{M} \) into a partial compactification \( \overline{\Gamma \backslash \mathcal{D}} \). Moreover, the restriction of the extension map \( \mathcal{P} \) to \( \mathcal{M} \) coincides with the period map for the cubic sections of \( \mathcal{O}_S(3) \) that \( \mathcal{M} \) parametrizes.

It is sufficient to study the extension locally. Let \( \Delta^n \subset \mathcal{F}^\text{vir} \) be an open polygon with local coordinate \( (z_1, \ldots, z_n) \) such that the smooth divisor \( \mathcal{M} \cap \Delta^n \) is defined by \( z_1 = 0 \). Consider the local period map

\[ \varphi : \Delta^* \times \Delta^{n-1} \rightarrow \Gamma \backslash \mathcal{D}. \]

For a generic point \((0, w) \in \mathcal{M} \cap \Delta^n\), the one-variable period map \( \varphi_w : \Delta^* \rightarrow \Gamma \backslash \mathcal{D} \) corresponds to the semi-stable model \( \mathfrak{X} \rightarrow \Delta \) of an one-parameter degeneration \( F + t G \) as in (8). Here the cubic polynomial \( G \) can be chosen such that \( G|_S \in \Gamma(\mathcal{O}_S(3)) \) represents the point \((0, w) \in \mathcal{M}\). It follows from our main theorem 2.4 that the monodromy transformation of \( \varphi_w \) has order 2, and the nilponent logarithm \( N \in \mathfrak{g}_Q \) (cf. (2)) induces the monodromy weight filtration

\[ 0 \subset W_{k-1} \subset W_k \subset W_{k+1} = H_{\lim} \]

which further satisfies

\[ \dim \text{Im}(N) = \dim W_{k-1} = 1. \]

We found the extension of period maps on the punctured disk with such specific monodromy weight filtration has been investigated by Usui [33], which is analogous to Cattani and Kaplan’s work [7] of the extension of period maps for variations of Hodge structures of weight two. We will show through the following that our statement is an immediate consequence of Theorem 1.7

Let \( V \) be a \( \mathbb{Q} \)-vector space, \( k \) an odd integer, \( S \) a non-degenerate anti-symmetric form on \( V \), \( \{ h^{p,q} \} \) a collection of non-negative integers with \( p + q = k \). Let \( D \) be the classifying space of polarized Hodge structures of type \((V, h^{p,q}, S, k)\). Suppose that

\[ \phi : \Delta^* \rightarrow \Gamma \backslash \mathcal{D} \]

is any period map whose nilpotent logarithm \( N \in \mathfrak{g}_Q \) has order 2, and satisfies the condition (\(*\)). Let \( F \) be the limit Hodge filtration of \( \phi \). Theorem 2.4 asserts a mixed Hodge structure \((W(N), F)\) polarized by \( N \). It is easy to see the graded Hodge type of \((W(N), F)\) is invariant.

**Lemma 4.5.** Let \( W \) be the monodromy weight filtration on \( V \) defined by any rational nilpotent operator \( N \) of order 2 that satisfies the condition (\(*\)). Suppose that \((W, F)\) is a mixed Hodge structure on \( V \). Let \( \{ p^{a,b}_{\lambda} \}_{a+b=\lambda} \) be the induced primitive Hodge numbers on the primitive part \( P_\lambda \subset Gr^W_\lambda, k - 1 \leq \lambda \leq k + 1 \). Then we have

- \( p^{a,b}_{\lambda} = h^{a,b}, a \neq \frac{k-1}{2}, \frac{k+1}{2}, \lambda = k; \)
- \( p^{a,b}_{\lambda} = h^{a,b} - 1, a = \frac{k-1}{2} \text{ or } \frac{k+1}{2}, \lambda = k; \)
- \( p^{a,b}_{\lambda} = 1, a = b = \frac{k-1}{2} \text{ or } a = b = \frac{k+1}{2}. \)

**Proof.** Note that in our case \( P_\lambda = Gr^W_\lambda \). Since \( \dim W_{k-1} = 1 \) the induced Hodge structure on \( P_{k-1} = W_{k-1} \) must be of the Tate twist \( \mathbb{Q}(-\frac{k-1}{2}) \). Then the Hodge
number $p_{a,b}^\lambda = 1$ for $a = b = \frac{k-1}{2}$. By duality the same reason holds as well for $P_{k+1}$. Note that

$$\dim F^a/F^{a+1} = h^{a,b} = \sum_{j=b}^{a} p_{a,j}.$$ 

Therefore if $\lambda = k$ the primitive Hodge number $p_{a,b}^\lambda = \begin{cases} h_{a,b}, & a \neq \frac{k-1}{2}, \frac{k+1}{2}; \\ h_{a,b} - 1, & a = \frac{k-1}{2}, \frac{k+1}{2}. \end{cases}$

The partial compactification of $\Gamma \setminus D$ is constructed by adding boundary components that relates to the weight filtration \cite{19} and the collection of integers $p := \{p_{a,b}^\lambda\}$. Following Cattani and Kaplan’s method \cite{7}, Usui described the action of the arithmetic group $\Gamma$ on the boundary components, as well as the Satake topology and complex structure on the compactification. We will recall the boundary components and the limit of of the period map $\phi$. The contents of the Satake topology and complex structure on the compactification can be found in \cite{32, 33}.

Let $W_{-1} \subset V_\mathbb{R}$ be an $S$-isotropic subspace with $\dim W_{-1} = 1$, $W_0 := W_{-1}^\perp$ the annihilator of $W_{-1}$ relative to $S$. Let $\tilde{S}$ denote the non-degenerate form on $W_0/W_{-1}$ induced by $S$. Choose a polarizing bilinear form $\psi$ on $W_{-1}$. Two such forms are considered to be equivalent if they are different up to a positive multiplicative constant. Denote by $\rho$ the collection of primitive Hodge numbers $\{p_{a,b}^\lambda\}$. Given the data $(W_{-1}, p, \psi)$, the associated boundary component $B(W_{-1}, p, \psi) = B^k(W_{-1}, p) \times B^{k-1}(W_{-1}, p, \psi)$ is defined by

1. the classifying space $B^k(W_{-1}, p)$ of $\tilde{S}$-polarized Hodge structures on the quotient $W_0/W_{-1}$ of type $\{p_{a,b}^\lambda\}$
2. the classifying space $B^{k-1}(W_{-1}, p, \psi)$ of $\psi$-polarized Hodge structures on $W_{-1}$ of type $\{p_{a,b}^{a-1}\}$.

The boundary bundle with respect to $(W_{-1}, p)$ is the disjoint union of boundary components

$$B(W_{-1}, p) := \bigsqcup_\psi B(W_{-1}, p, \psi)$$

where $\psi$ runs over all equivalence class of polarizing forms on $W_{-1}$. A boundary bundle is rational if the subspace $W_{-1}$ is defined over $\mathbb{Q}$. A boundary component $B(W_{-1}, p, \psi)$ is rational if the subspace $W_{-1}$ and the form $\psi$ are defined over $\mathbb{Q}$. Denote by $D^{**} \subset D^*$ the union of all rational boundary components, and the union of all rational boundary bundles, respectively. The action of the arithmetic group $\Gamma$ on the extend set $D^*$ introduces a Satake topology on $\Gamma \setminus D^*$. The partial compactification $\Gamma \setminus D$ is set to be the quotient space $\Gamma \setminus D^{**}$ which inherits the Satake topology on $\Gamma \setminus D^*$.

Remark 4.6. (1) $\Gamma \setminus D^{**}$ and $\Gamma \setminus D^*$ are locally compact and Hausdorff, $\Gamma \setminus D \subset \Gamma \setminus D^{**}$ is open and dense.

(2) Note that $\dim W_{-1} = 1$. Then the equivalence class of the polarizing form $\psi$ is unique. Hence the boundary component coincides with the boundary bundle $D^{**} = D^*$. In addition, the boundary component $B(W_{-1}, p, \psi) = B^k(W_{-1}, p)$ can be identified with the classifying space $B^k(W_{-1}, p)$ of polarized Hodge structures on $W_0/W_{-1}$. 

Theorem 4.7. Let \( \phi : \Delta^* \to \Gamma \setminus D \) be a period map such that the nilpotent logarithm \( N \) has order 2, and satisfies the condition \((*)\). Then \( \phi \) extends holomorphically over the puncture to a map \( \overline{\phi} : \Delta \to \Gamma \setminus D^{**} \). Let \( F \) be the limit Hodge filtration \( \mathfrak{m} \) of \( \overline{\phi} \), and \( (W(N), F) \) be the limit mixed Hodge structure. Then the graded Hodge structure \( F(Gr^W_k) \) exactly represents the limit \( \overline{\phi}(0) \) in the boundary component \( B(W, p, N) \).

Proof. The first conclusion is proved by Usui \[33\] Thm. 5.1. The illustration of the second conclusion uses the \( SL(2) \)-orbits developed through Cattani, Kaplan and Schmid \[6, 50\]. Appendix A includes some basics of \( SL(2) \)-orbits that we need.

Consider the monodromy weight filtration (cf. \[17\] ) of the nilpotent map \( N \). \( W_k \) is a rational \( S \)-isotropic subspace. \( W_k \) follows from Proposition \[1, 2\] that \( S(N^{-1}, \cdot) \) is a polarization on \( W_{k-1} \). Therefore it corresponds to a rational boundary component \( B(W, p, N) := B(W_{k-1}, p, \psi) \) where \( \psi := S(N^{-1}, \cdot) \) and \( p = \{ p_{a,b} \} \). Note from Remark \[4.4\] that \( B(W, p, N) \) can be regarded as the classifying space \( D(Gr^W_k) \) of \( S \)-polarized Hodge structures of type \( p \) on \( Gr^W_k \).

Let \( \tilde{\phi} : \mathfrak{h} \to D \) be the lifting of \( \phi \) to the upper half plane. Let \( F \) be the limit Hodge filtration \( \mathfrak{m} \) associated to \( \phi \), and \( \theta : z \to \exp(zN) \cdot F \) be the nilpotent orbit. By the nilpotent orbit theorem \( \theta(z) \) approximates asymptotically to \( \tilde{\phi}(z) \). Schmid’s \( SL(2) \)-orbit theorem asserts a rational \( SL(2) \)-orbit \( (\rho, r) \), that is, a representation

\[ \rho : SL(2, \mathbb{C}) \to G_{\mathbb{C}} \]

defined over \( \mathbb{Q} \) whose differential

\[ \rho_* : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{g}_{\mathbb{C}} \]

is horizontal at the reference point \( r \in D \) and \( \rho_*(\mathfrak{n}_-) = N \), such that \( \theta(z) \) can be approximated by the holomorphic and horizontal embedding

\[ \tilde{\rho} : \mathbb{P}^1 \to D \]

corresponding to \( (\rho, r) \). Hence \( \tilde{\phi} \) is approximated by \( \tilde{\rho} \) as well. Let \( F_r \) be the corresponding Hodge filtration of \( r \in D \). Theorem \[A.3\] concludes that \( (W(N), e^{-IN}F_r) \) is an \( S \)-polarized mixed Hodge structure. Since \( N(W_{\lambda_i}) \subset W_{\lambda_{i-2}} \) the action of \( N \) on the quotient \( Gr^W_{\lambda_i} \) is zero. Hence the Hodge structures on \( Gr^W_{\lambda_i} \) induced by the filtrations \( e^{-IN}F_r \) and \( F_r \) are the same. One assigns the boundary point

\[ b(\rho, r) := (\psi, F_r(Gr^W_{\lambda_i})) \in B(W, p, N) \]

to the \( SL(2) \)-orbit \( (\rho, r) \). In \[32, A.4.1\], \[7, 6.5\] it is proved that in the Satake topology on \( D^{**} \)

\[ \lim_{\Im z \to \infty} \tilde{\phi}(z) = r_{\infty} \cdot b(\rho, r) \]

for some \( r_{\infty} \in \exp \mathfrak{c} \) with \( \mathfrak{c} = \text{Ker(ad}_R N) \cap \text{Im(ad}_R N) \), and the element \( r_{\infty} \) acts trivially on the graded weight spaces \( Gr^W_{\lambda_i} \), which asserts

\[ \overline{\phi}(0) = b(\rho, r) \in B(W, p, N). \]

A consequence \[50\] Cor. 6.21 of the \( SL_2 \)-orbit theorem shows that the limit Hodge filtration \( F \) and the reference filtration \( F_r \) induces the same graded polarized Hodge structures on \( Gr^W_{\lambda_i} \). Hence

\[ \overline{\phi}(0) = F(Gr^W_k) = F_r(Gr^W_k). \]

in \( B(W, p, N) = D(Gr^W_k) \).  \(\square\)
Appendix A. SL(2)-orbits and mixed Hodge structures

Let \((V_Q, V^{p,q}, S)\) denote a Hodge structure of weight \(k\) with the polarization form \(S\). Denote by \(\mathfrak{g}_R\) the Lie algebra of the orthogonal group \(G_R = \text{Aut}(V, S)\). The Hodge decomposition \(\{V^{p,q}\}\) induces a Hodge structure of weight zero on \(\mathfrak{g}_R\) in the following manner
\[
\mathfrak{g}^{r,-r} := \{X \in \mathfrak{g}_C \mid X(V^{p,q}) \subset V^{p+1-r,n-r}, \forall p, q\}.
\]

**Example A.1.** Consider weight one Hodge structures on \(\mathbb{C}^2\) polarized by the standard symplectic form. \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\). The orthogonal group \(G_R\) is \(\text{SL}(2, \mathbb{R})\) and the classifying space \(D\) identifies to the upper-half plane \(\mathfrak{h}\). Let \(i \in \mathfrak{h}\) corresponds to the Hodge decomposition

\[
H^{1,0} = \mathbb{C} \cdot i e_1 + e_2, H^{0,1} = \mathbb{C} \cdot e_1 + i e_2.
\]

Then the induced weight zero Hodge structure on \(\mathfrak{sl}_2\mathbb{R}\) is

\[
(\mathfrak{sl}_2)^{-1} = \mathbb{C} \cdot \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, (\mathfrak{sl}_2)^{1} = \mathbb{C} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (\mathfrak{sl}_2)^{-1,i} = \mathbb{C} \cdot \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}.
\]

Denote \(Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\), \(X_+ = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}\), \(X_- = \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\).

**Definition A.2.** Fix a polarized Hodge structure \(\{V^{p,q}\}\) on \(V\). A Lie algebra homomorphism \(\rho_\ast : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{g}_C\) is called horizontal at \(\{V^{p,q}\}\) if it is a morphism of real-Hodge structures of type \((0,0)\), i.e. \(\rho_\ast(X_-) \in \mathfrak{g}^{-1,1}\).

A Lie algebra homomorphism \(\rho_\ast : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{g}_C\) horizontal at a reference point \(r := \{V^{p,q}\} \in D\) lifts to a group homomorphism

\[
\rho : \text{SL}(2, \mathbb{C}) \to G_C
\]

of complex Lie groups such that \(\rho(\text{SL}(2, \mathbb{R})) \subset G_R\). The group \(\text{SL}(2, \mathbb{C})\) (resp. \(G_C\)) acts transitively on \(\mathbb{P}^1\) (resp. \(\tilde{D}\)). Then \(\rho\) induces a holomorphic, horizontal, equivariant embedding

\[
\tilde{\rho} : \mathbb{P}^1 \to \tilde{D}
\]

given by

\[
\tilde{\rho}(g \cdot i) = \rho(g) \cdot r, \forall g \in \text{SL}(2, \mathbb{C})
\]

Note that the upper-half plane \(\mathfrak{h}\) is the \(G_R\)-orbit in \(\mathbb{P}^1\). Then \(\tilde{\psi}(\mathfrak{h}) \subset \tilde{D}\). We call such a pair \((\rho, r)\) an \(\text{SL}(2)\)-orbit.

To a nilpotent element \(N \in \mathfrak{gl}(V_C)\) with \(N^{k+1} = 0\), one can associate the monodromy weight filtration \(W(N)\), cf. [12]. Set \(W_l := W(N)_{l+k}\) denote the weight filtration shift by \(k\). Then the weight filtration \(W_\ast\) satisfies

1. \(N(W_l) \subset W_{l-2}\);
2. \(W_l' : Gr_1^W \cong Gr_{l-1}^W\).

A grading of the weight filtration \(W_\ast\) that is compatible with \(N\) is a decomposition \(V = \bigoplus_i H_i\) of subspaces \(H_i\) such that

\[
W_l = \bigoplus_{\lambda \leq l} H_\lambda, N(H_l) \subset H_{l-2}.
\]
Such a grading corresponds to a semisimple element $Y \in \mathfrak{gl}(V_C)$ such that $H_\lambda$ is the $\lambda$-eigenspace of $Y$. It implies $[Y, N] = -2N$. Let

$$ n_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, n_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} $$

be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. The pair $(N, Y)$ induces a Lie algebra homomorphism

$$ \rho_\ast : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{gl}(V_C) $$

with $\rho_\ast(n_-) = N, \rho_\ast(y) = Y$. Conversely, for any $\mathfrak{sl}_2$-representation $\rho_\ast$ with $\rho_\ast(n_-) = N$, the semisimple element $Y = \rho_\ast(y)$ has integral eigenvalues. Then the eigenspaces $H_\lambda(Y)$ forms a grading of the weight filtration $W_\bullet$ that is compatible with $N$.

Lemma A.3. [6, Prop. 2.8] Let $\mathcal{L}(W, N)$ denote the set of gradings of $W$ that is compatible with $N$. Set $c := \ker(ad_{\mathfrak{g}} N) \cap \im(ad_{\mathfrak{g}} N)$.

(1) There is a one-to-one correspondence

$$ \{ \rho_\ast : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{gl}(V_C) \mid \rho_\ast(n_-) = N \} \to \mathcal{L}(W, N) $$

given by $\rho \to \rho_\ast(y)$

(2) The group $\exp(c)$ acts simply and transitively on $\mathcal{L}(W, N)$.

If $S$ is the polarization form on $V$ and $N \in \mathfrak{g}_C$, it is easy to check $Y \in \mathfrak{g}_C$. Then the homomorphism $\rho_\ast$ factor through $\mathfrak{g}_C$.

Let $(W(N), F)$ be a mixed Hodge structure on $V$. Consider Deligne’s generalized Hodge decomposition [9]

$$ V = \bigoplus_{a, b} I^{a, b} $$

such that

$$ W_l = \bigoplus_{a+b \leq l} I^{a, b}, F_p = \bigoplus_{a \geq p} I^{a, b} $$

where the subspaces $I^{p, q}$ is defined to be

$$ I^{a, b} := F_p \cap W_{a+b} \cap (F_b \cap W_{a+b} + \sum_{j \geq 1} F_{b-j} \cap W_{a+b-j-1}). $$

The complex conjugate on $I^{p, q}$ satisfies

$$ I^{p, q} = \overline{I^{q, p}} + \bigoplus_{a < p, b < q} I^{a, b}. $$

We say the mixed Hodge structure $(W, F)$ splits over $\mathbb{R}$ if $I^{p, q} = \overline{I^{q, p}}$, in which case $I^{p, q} = F_p \cap \overline{F_q} \cap W_{p+q}$.

Proposition A.4. [6, Prop. 2.20] For any mixed Hodge structure $(W, F)$ on $V$, there exists a unique operator

$$ \delta \in L^{-1, -1}_\mathbb{R}(W, F) := \{ T \in \text{End}_\mathbb{R}(V) \mid T(I^{p, q}) \subset \bigoplus_{a < p, b < q} I^{a, b} \} $$

such that $(W, e^{-i\delta} F)$ splits over $\mathbb{R}$. 
Suppose that \((W(N), F)\) is an \(\mathbb{R}\)-split polarized mixed Hodge structure. The generalized Hodge decomposition gives a canonical grading of \(W := W(N)[k]\) with 
\[
H_l = \bigoplus_{p+q=k+l} I^{p,q}.
\]

The corresponding semisimple element \(Y\) acts on \(V\) by 
\[
Y(u) = (p + q - k) \cdot u \quad \text{for} \quad u \in I^{p,q}.
\]

Note that the eigenspace \(H_l\) is defined over \(\mathbb{R}\) if \((W(N), F)\) is \(\mathbb{R}\)-split. Then \(Y \in \mathfrak{g}_\mathbb{R}\) and the corresponding homorphism \(\rho^* : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{g}_\mathbb{C}\) is defined over \(\mathbb{R}\). The following is a remarkable connection between \(\mathbb{R}\)-split polarized mixed Hodge structures and \(SL_2\)-orbits.

**Theorem A.5.** Let \((W(N), F)\) be a polarized mixed Hodge structure, split over \(\mathbb{R}\). Then the canonical homomorphism \(\rho_* : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{g}_\mathbb{C}\) is horizontal at \(e^{iN}F \in D\). Conversely, if a Lie algebra homomorphism \(\rho_* : \mathfrak{sl}_2 \mathbb{C} \to \mathfrak{g}_\mathbb{C}\) is horizontal at a reference point \(r \in D\). Denote by \(N = \rho_*(n_-)\), and \(F_r\) the corresponding Hodge filtration of \(r\). Then \((W(N), e^{-iN}F_r)\) is a polarized mixed Hodge structure, split over \(\mathbb{R}\).

**Proof.** See [5, Prop. 3.9], [6, Lem. 3.12] or [4, Prop. 2.18] \qed

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