Two classes of explicitly solvable sextic equations

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Abstract

The generic monic polynomial of sixth degree features 6 a priori arbitrary coefficients. We show that if these 6 coefficients are appropriately defined—in two different ways—in terms of 5 arbitrary parameters, then the 6 roots of the corresponding polynomial can be explicitly computed in terms of radicals of these parameters. We also report the 2 constraints on the 6 coefficients of the polynomial implied by the fact that they are so defined in terms of 5 arbitrary parameters; as well as the explicit determination of these 5 parameters in terms of the 6 coefficients of the sextic polynomial.

1 Introduction

The task of computing the roots of a given polynomial has been a fundamental problem and a significant engine of progress in mathematics. An important breakthrough occurred about five centuries ago, with the discovery of a technique to find the roots of a generic polynomial of third degree (for a terse description of this development see, for instance, the item "Cubic equation" in Wikipedia). The second major progress occurred about two centuries ago and was due to Paolo Ruffini, Niels Henrick Abel and Évariste Galois: it was the proof that the zeros of a generic polynomial of any degree $N > 4$ cannot be represented—in terms of the coefficients of that polynomial—by a formula involving only radicals: this breakthrough opened the way to what is now called "Galois theory" (see for instance the item "Galois Theory" in Wikipedia). This development of course does not exclude that there exist specific polynomials of arbitrary degree $N$ all roots of which can be explicitly computed in terms of elementary functions: for instance consider the monic polynomial of arbitrary degree 6,

$$P_N(z) = \prod_{n=1}^{N} (z - z_n) = z^N + \sum_{m=0}^{N-1} (c_m z^m),$$

(1a)

where we now imagine the $N$ roots $z_n$ to be a priori arbitrarily assigned (hence to be known), and the $N$ coefficients $c_m$ to be then computed in terms of them—as easily implied by the simultaneous validity of the two expressions of the polynomial (1a) as a product and a sum, see for instance the item "Vieta’s formulas" in Wikipedia. But of course any attempt to invert the Vieta’s formulas in order to obtain the $N$ zeros $z_n$ from the $N$ coefficients $c_m$ would eventually require the solution of an algebraic equation of degree $N$.

Nevertheless the identification of classes of polynomials of degree $N > 4$—defined by assigning their $N$ coefficients $c_m$, see (1a)—which do allow their $N$ zeros to be computed by radicals is an interesting mathematical topic. For instance a rather recent example—based on Galois theory—of such an endeavour for polynomials of sixth degree is provided by the paper (1) (easily reachable via Google). An analogous endeavour—but based on more elementary mathematics—is reported in the present paper, where two classes of sextic monic polynomials are identified, which allow the computation of their 6 zeros by radicals from their 6 coefficients, provided these are defined by explicitly
provided formulas in terms of 5 arbitrary parameters. These findings are relatively trivial in the context of Galois theory and are presumably already implied by the results reported in [1]; but are obtained below by more elementary means, and they are also much simpler, as indicated by the fact that each of the explicit formulas written in the present paper require only one line to be displayed, while several of those displayed in the paper [1] takes one or more pages.

Hereafter the generic monic polynomial of sixth degree is defined as follows:

\[ P_6(x) = x^6 + \sum_{n=0}^{5} (c_n x^n), \]  

(2a)

with the 6 coefficients \( c_n, n = 0, 1, 2, 3, 4, 5 \) a priori arbitrary (except for the conditions mentioned below); and \( z_{\lambda \mu} \) are the 6 roots of this polynomial,

\[ P_6(z_{\lambda \mu}) = 0, \quad \lambda = 1, 2, \quad \mu = 1, 2, 3. \]  

(2b)

2 Results: first model

Proposition 2-1. Assume that the 6 coefficients \( c_n \) of the sextic polynomial (2a) may be expressed as follows in terms of the 5 arbitrary parameters \( a_0, a_1, a_2, b_0, b_1 \):

\[ c_5 = 2a_2, \]  

(3a)

\[ c_4 = 2a_1 + (a_2)^2, \]  

(3b)

\[ c_3 = 2a_0 + 2a_1 a_2 + b_1, \]  

(3c)

\[ c_2 = (a_1)^2 + (2a_0 + b_1) a_2, \]  

(3d)

\[ c_1 = (2a_0 + b_1) a_1, \]  

(3e)

\[ c_0 = (a_0)^2 + a_0 b_1 + b_0. \]  

(3f)

Then the 6 roots \( z_{\lambda \mu} \) of the sextic polynomial (2a) are explicitly given, in terms of the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \), by the following definitions: the 6 numbers \( z_{\lambda \mu} \) are the 3 roots (with \( \lambda = 1, 2 \) and \( \mu = 1, 2, 3 \)) of the following 2 cubic equation in \( z \),

\[ z^3 + a_2 z^2 + a_1 z + a_0 = y_{\lambda}, \quad \lambda = 1, 2, \]  

(4a)

where \( y_{\lambda} \) is one of the 2 roots of the following quadratic equation in \( y \),

\[ y^2 + b_1 y + b_0 = 0. \]  

(4b)

Remark 2-1. The quadratic respectively cubic equations (4b) respectively (4a) can of course be solved explicitly:

\[ y_{\lambda} = \left\{ -b_1 + (-1)^{\lambda} \sqrt{(b_1)^2 - 4b_0} \right\} / 2, \quad \lambda = 1, 2, \]  

(5)

while the 3 roots \( z_{\lambda \mu} \) of the cubic equation (4a) are given by the well-known "Cardano" formulas (see again the item "Cubic equation" in Wikipedia). The resulting explicit formula expressing the 6 zeros \( z_{\lambda \mu} \) in terms of the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \) involves—in a nested way—only square and cubic roots; it is of course a bit more complicated than the Cardano formulas, as the reader who takes the trouble—indeed, an easy task—to write it out shall easily find out.

But, as indicated above, the task of obtaining—from the assignment of a number of parameters—a corresponding set of both the coefficients of a polynomial and its zeros may be a relatively easy task. Less trivial is the task to assign a priori the \( N \) coefficients \( c_n \) of a monic polynomial of degree \( N \) (see [1a]) and to then find a (generally smaller) number of parameters which determine—as it were, a posteriori—via explicit formulas both these preassigned \( N \) coefficients \( c_n \) and the \( N \) zeros of the corresponding monic polynomial, as well as the explicit formulas displaying the corresponding constraints implied by these assignments on the \( N \) coefficients \( c_n \).
The following proposition provides—for \( N = 6 \)—such findings, which complement those reported in Proposition 2-1.

**Proposition 2-2.** If the 6 parameters \( c_n \) are expressed in terms of the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \) by the 6 formulas (3), then the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \) are themselves expressed as follows in terms of the 6 coefficients \( c_n \):

\[
\begin{align*}
a_2 &= (c_5/2) , \\
a_1 &= \left[ c_4 - (a_2)^2 \right] / 2 = \left[ 4c_4 - (c_5)^2 \right] / 8 , \\
2a_0 + b_1 &= c_3 - 2a_1a_2 = \left[ c_2 - (a_1)^2 \right] / a_2 = c_1/a_1 , \\
b_0 &= c_0 - (a_0 + b_1)a_0 ;
\end{align*}
\]

and the 6 coefficients \( c_n \) satisfy the following 2 constraints:

\[
\begin{align*}
c_1 &= a_1(2a_0 + b_1) = \left[ 4c_4 - (c_5)^2 \right] \left( c_3 - \left[ 4c_4 - (c_5)^2 \right] c_5/8 \right) / 8 , \\
c_2 &= a_2c_3 + a_1 \left[ a_1 - 2(a_2)^2 \right] = c_3c_5/2 + \left[ 4c_4 - (c_5)^2 \right] \left[ 4c_4 - 5(c_5)^2 \right] / 64 .
\end{align*}
\]

Note that the 2 parameters \( a_2 \) and \( a_1 \) are given explicitly in terms of the coefficients \( c_4 \) and \( c_5 \) by the eqs. (6a) and (6d); while the 3 equalities (6c) imply (via (6a) and (6d)) the 2 constraints (7) on the 6 coefficients \( c_n \)—which express explicitly the 2 coefficients \( c_1 \) and \( c_2 \) in terms of the 3 coefficients \( c_3, c_4, c_5 \). Then—once these 2 constraints are satisfied—the 3 equalities (6e), together with eq. (6d), provide the explicit determination of the parameters \( b_0 \) and \( b_1 \) in terms of the parameters \( c_n \) and of the parameter \( a_0 \)—this easy task amounts to solving a system of 2 linear equations—while \( a_0 \) remains as a free parameter; this freedom might be used to simplify all the above formulas, for instance by assuming that \( a_0 \) vanishes or that \( a_0 = -b_1/2 \) implying \( c_1 = 0 \) (see (6f)), but of course at the cost of decreasing the generality of these findings.

Proofs of Propositions 2-1 and 2-2 are provided in the Appendix.

## 3 Results: second model

The findings reported in this Section 3 are analogous, but different, from those reported in Section 2; accordingly, the variables and parameters used in this Section 3 are different from those having the same name in Section 2, although they play analogous roles.

We only report below these new findings, without detailing their derivation; which is quite analogous to that described above and below (see Section 2 and Appendix A), and may be recommended as an interesting exercise for the enterprising reader (clue: compare the eqs. (3) to the eqs. (4), see below).

**Proposition 3-1.** Assume that the 5 coefficients \( c_n \) may be expressed as follows in terms of the 5 arbitrary parameters \( a_0, a_1, a_2, b_0, b_1 \):

\[
\begin{align*}
c_5 &= 3b_1 , \\
c_4 &= a_2 + 3 \left[ b_0 + (b_1)^2 \right] , \\
c_3 &= \left[ 2a_2 + 6b_0 + (b_1)^2 \right] b_1 , \\
c_2 &= a_1 + a_2 \left[ 2b_0 + (b_1)^2 \right] + 3b_0 \left[ b_0 + (b_1)^2 \right] , \\
c_1 &= a_1b_1 + 2a_2b_0b_1 + 3(b_0)^2 b_1 , \\
c_0 &= a_0 + a_1b_0 + a_2(b_0)^2 + (b_0)^3 .
\end{align*}
\]

Then the 6 roots \( z_{\lambda\mu} \) (with \( \lambda = 1, 2 \) and \( \mu = 1, 2, 3 \)) of the sextic polynomial (8a) are explicitly given, in terms of the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \), by the following definitions: \( z_{\lambda\mu} \) is one of the 2 roots of the following 3 quadratic equations in \( z \),

\[
\begin{align*}
z^2 + b_1z + b_0 &= y_\mu , & \mu &= 1, 2, 3 ,
\end{align*}
\]
where \( y_\mu \) is one of the 3 roots of the following cubic in \( y \):

\[
y^3 + a_2y^2 + a_1y + a_0 = 0 .
\] (9b)

**Proposition 3-2.** If the 6 parameters \( c_n \) are expressed in terms of the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \) by the 6 formulas (8), then the 5 parameters \( a_0, a_1, a_2, b_0, b_1 \) are themselves expressed as follows in terms of the 6 coefficients \( c_n \):

\[
b_1 = c_5/3 ,
\] (10a)

\[
a_2 = c_4 - (c_5)^2 / 3 - 3b_0
\] (10b)

\[
a_1 = c_2 - 2 \left[ 3c_4 - 9b_0 - (c_5)^2 \right] \left[ 18b_0 + (c_5)^2 \right] / 27 - b_0 \left[ 9b_0 + (c_5)^2 \right] / 3 ,
\] (10c)

\[
a_0 = c_0 - a_1b_0 - a_2(b_0)^2 - (b_0)^3 ;
\] (10d)

while the 6 coefficients \( c_n \) are required to satisfy the following 2 *constraints*:

\[
27c_3 - 18c_4c_5 + 5(c_5)^3 = 0 ,
\] (11a)

\[
c_1 = \left[ 27c_2 - 3c_4(c_5)^2 + (c_5)^4 \right] c_5/81 .
\] (11b)

Note that in this case we wrote out *explicitly*—in terms of the parameters \( c_n \)—the expressions of the 3 parameters \( b_1, a_2, a_1 \), and that an *explicit* expression of the parameter \( a_0 \) is also implied by (10d) via (10c) and (10b); while the parameter \( b_0 \) remains as a *free* parameter (the same role played by the parameter \( a_0 \) in Proposition 2-2). ■

### 4 Outlook

Obvious generalizations of the approach employed in this paper may be used to get analogous results for polynomials of degree \( N > 6 \), especially whenever \( N = 2^{p_1}3^{p_2} \) with \( p_1 \) and \( p_2 \) arbitrary nonnegative integers (for instance for \( N = 8 \) or \( N = 9 \)); but then the number of restrictions on the coefficients of these polynomials for the applicability of this approach shall of course grow as \( N \) grows.

### 5 Appendix

In this Appendix we prove the results reported in Section 2.

To derive the results reported in Proposition 2-1 all one needs to do is to replace \( y \) in eq. (11b) by the expression \( z^3 + b_2z^2 + b_1z + b_0 \) (see (1a)), expand the resulting expression in powers of \( z \), and identify the coefficients of the resulting *sextic* equation in \( z \) with the coefficients \( c_n \), see (2a).

The results reported in Proposition 2-2 are easy consequences of the formulas (3). It is plain that the 3 eqs. (6a), (6b) respectively (6d) are implied by the 3 eqs. (3a), (3b) respectively (3d). Next, it is easily seen that the 3 equalities (6c) are implied by the 3 eqs. (3c), (3d) and (3e). And the fact that these 3 equalities imply the 2 *constraints* (7) is then obvious, as well as the fact that they allow the *explicit* computation of the parameters \( b_0 \) and \( b_1 \) while leaving \( a_0 \) as a free (undetermined) parameter; as mentioned in Proposition 2-2, which is thereby proven.

### References

[1] T. R. Hagedorn, ”General formulas for solving solvable sextic equations”, J. Algebra 233, 704-757 (2000).