Conservation laws in the 1f7/2 shell model of 48Cr

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Conservation laws in the 1f7/2 shell model of 48Cr found in numeric studies by Escuderos, Zamick, and Bayman [A. Escuderos, L. Zamick, and B. F. Bayman, arXiv:nucl-th/0506050 (2005)] and me [K. Neergård, Phys. Rev. C 90, 014318 (2014)] are explained by symmetry under particle-hole conjugation and the structure of the irreps of the symplectic group Sp(4). A generalization is discussed.

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I. INTRODUCTION

In a recent article, I analyzed the ground states of the nuclei 48Cr, 88Ru, and 92Pa calculated in the 1f7/2 or 1g9/2 shell model with effective interactions from the literature in terms of irreps of the symplectic group Sp(2j + 1), where j is the single-nucleon angular momentum [1]. A surprising observation emerged from this study: In 48Cr, independently of the interaction, the interaction matrix elements between the irrep vt = 61 and other irreps vanishes within the numeric accuracy. Here, v denotes the seniority, and t the reduced isospin [2], and I have introduced a notation to be used in this article: For example, vt = 61 is shorthand for (v, t) = (6, 1). In [1], I had to leave this observation unexplained; it is explained below.

The phenomenon turns out to be related to another one observed in the literature, by Escuderos, Zamick, and Bayman [3]. In calculations of the entire spectrum of 48Cr in the 1f7/2 shell model with a certain interaction, these authors found that states with opposite parities of the interaction matrix elements between the irrep vt = 61 and Bayman [3]. In calculations of the entire spectrum of 48Cr found in numeric studies by Escuderos, Zamick, and Bayman [A. Escuderos, L. Zamick, and B. F. Bayman, arXiv:nucl-th/0506050 (2005)] and me [K. Neergård, Phys. Rev. C 90, 014318 (2014)] are explained by symmetry under particle-hole conjugation and the structure of the irreps of the symplectic group Sp(4). A generalization is discussed.

II. CLASSIFICATION OF STATES IN A j SHELL

Helmers has shown that the configuration space V of the system of all possible numbers of fermions of different kinds τ = 1, …, k in a j shell, or more generally, in any single-fermion space of even dimension 2Ω = 2j + 1, is composed of different single irreps of Sp(2Ω) × Sp(2k), where Sp(2d) is the symplectic group in 2d dimensions [7]. [Helmers considers, equivalently, the unitary subgroup USp(2Ω) × USp(2k).] The irreps of Sp(2d) are associated with Young frames with at most d rows [8]. In the decomposition of V, the irreps of Sp(2Ω) have Young frames with at most k columns, and the irreps of Sp(2k) have Young frames with at most Ω rows, and they are combined so that the lengths of the ith row in the Sp(2Ω) frame and the (Ω + 1 – i)th column in the Sp(2k) frame add up to k. Equivalently, the lengths of the ith column in the Sp(2Ω) frame and the (k + 1 – i)th row in the Sp(2k) frame add up to Ω.

Due to the even dimension of the single-fermion space, its basic states can be combined in Ω pairs, which I label (m, ˜m). In a j shell, m can be taken as the magnetic quantum number restricted to m > 0, and ˜m may label the time reversed state so that with usual phase conventions [9], |m⟩ = (−)j+m|m⟩. Helmers pairs |m⟩ with (−)j−m|−m⟩ = |−m⟩. The present choice will prove its advantage in later considerations of time reversal. A basis for the representation of the Sp(2Ω) infinitesimal algebra is formed by the operators

\[ A_{mm'} = \sum_{τ} (a_{mτ}^* a_{m'τ} - a_{m'τ}^* a_{mτ}), \]
\[ B_{mm'} = B_{m'm} = \sum_{τ} (a_{mτ}^* a_{mτ} + a_{mτ}^* a_{m'τ}), \]  
\[ B_{m'm}^\dagger = B_{m'm}^\dagger, \]  

where \( a_{mτ} \) and \( a_{mτ}^\dagger \) annihilate a fermion in the states |mτ⟩ and |−mτ⟩. A basis for the representation of the Sp(2k) infinitesimal algebra is formed by the operators

\[ N_{ττ'} = \sum_{m} (a_{mτ}^* a_{mτ'} - a_{mτ'}^* a_{mτ}), \]
\[ P_{ττ'} = P_{ττ'} = \sum_{m} (a_{mτ}^* a_{mτ'} + a_{mτ'}^* a_{mτ}), \]  
\[ P_{ττ'}^\dagger = P_{ττ'}^\dagger. \]
Notice $N_{r\tau} = n_r - \Omega$, where $n_r$ is the number of fermions of kind $\tau$. It is easily verified that each operator (1) commutes with each operator (2).

The best known case is that of identical fermions, $k = 1$. There the $Sp(2\Omega)$ Young frame has only one column, whose length is the seniority $v$ [4, 5]. The infinitesimal algebra of $Sp(2k) = Sp(2)$ is isomorphic to the angular momentum algebra. This is Kerman’s quasispin algebra [10] with basic operators

$$S^0 = \frac{1}{2}(n - \Omega), \quad S^- = \frac{1}{2}P, \quad S^+ = \frac{1}{2}P^1,$$

(3)

where the index $\tau = 1$ is omitted for convenience. With $S^2 = S(S + 1)$, where $S$ is the vector with complex coordinates (3), the length of the single row of the $Sp(2)$ Young frame is $2S$. The general rule for the correspondence of the $Sp(2\Omega)$ and $Sp(2k)$ Young frames gives the well known relation [11]

$$S = \frac{1}{2}(\Omega - v).$$

(4)

For $k > 1$, we have

$$\mathcal{V} = \bigotimes_\tau \mathcal{V}_\tau,$$

(5)

where $\mathcal{V}_\tau$ is the space of states where only fermions of kind $\tau$ are present. Acting within each such space we have groups $Sp(2\Omega)_\tau$ and $Sp(2)_\tau$ with infinitesimal algebras spanned by

$$A_{\tau,mm'} = a^\dagger_{m\tau} a_{m\tau'} - a^\dagger_{m\tau'} a_{m\tau},$$

$$B_{\tau,mm'} = B_{\tau,m'm} = a^\dagger_{m\tau} a_{m\tau'} + a^\dagger_{m\tau'} a_{m\tau},$$

(6)

$$B^\dagger_{\tau,mm'} = B^\dagger_{\tau,m'm},$$

and

$$S^0_{\tau} = \frac{1}{2}(n_r - \Omega), \quad S^-_{\tau} = \frac{1}{2}P_{\tau}, \quad S^+_{\tau} = \frac{1}{2}P^1_{\tau}.$$

(7)

Their irreps can be labeled by seniorities $v_\tau$ and quasispins $S_\tau$, respectively, related by (4) separately for each $\tau$, that is,

$$S_\tau = \frac{1}{2}(\Omega - v_\tau).$$

(8)

Any pair of operators (6) and (7) with different $\tau$ evidently commute.

Particularly relevant for nuclear physics is the case $k = 2$ with the two kinds of fermions being neutrons and protons. I denote in this case the values of $\tau$ accordingly by $n$ and $p$. Its $Sp(2\Omega)$ Young frames have at most two columns and give rise to Flowers’s classification of the states in $\mathcal{V}$ with given number of nucleons and isospin by a seniority $v$ and a reduced isospin $t$ [2]. This seniority $v$ is the total number of cells of the Young frame, and $2t$ is the difference of its column lengths. The infinitesimal algebra of $Sp(2k) = Sp(4)$ is isomorphic to that of the proper orthogonal group $SO(5)$ and may be better known by this name [12]. Its dimension is ten. The four basic operators $N_{r\tau\tau'}$ combine to $n - 2\Omega$, where $n = n_n + n_p$ is the number of nucleons, and coordinates of the isospin $T$, the three linearly independent basic operators $P_{\tau\tau'}$, to coordinates of a pair annihilation isovector $P$, and the three linearly independent basic operators $P^1_{\tau\tau'}$, to coordinates of a pair creation isovector $P^1$.

I label the $Sp(4)$ irreps in the conventional notation for Young frames (or partitions) by $[\lambda \mu]$, where

$$[\lambda \mu] = [\Omega - \frac{1}{2}t, \pm t]$$

(9)

are the row lengths of the Young frame. The operators $S^0_\tau$ form a basis for a Cartan subalgebra of the infinitesimal algebra. Adopting this basis I refer to the eigenvalues of $(S^0_\tau, S^0_p)$ that occur in a given representation as the weights of this representation. For the irrep $[\lambda \mu]$ they obey [13]

$$\frac{1}{2}(\lambda + \mu) + S^0_n + S^0_p \text{ is integral,}$$

$$|S^0_n| \leq \frac{1}{2}\lambda \text{ for } \tau = n, p,$$

(10)

$$|S^0_n \pm S^0_p| \leq \frac{1}{2}(\lambda + \mu).$$

Figure 1 shows this range in a generic case.

Because Sp$(2)_n$ commutes with Sp$(2)_p$, the Sp$(4)$ irrep $[\lambda \mu]$ is the direct sum of irreps of Sp$(2)_n \times$ Sp$(2)_p$. I shall show that this direct sum includes exactly once each tensor product of Sp$(2)_n$ and Sp$(2)_p$ irreps with quasispins $S_n$ and $S_p$ such that $(S_n, S_p)$ is a weight, and

$$|S_n - S_p| \leq t \leq S_n + S_p.$$ 

(11)
The border of this range is shown as a rectangle in Fig. 1. It follows that a basis for the Sp(4) irrep is labeled uniquely by the quantum numbers $S_n, S_p, S_0^n$, and $S_0^p$. Each such basic vector is associated with an irrep of Sp(2Ω) in $\mathcal{V}$ whose states have the quantum numbers $v$ and $t$ corresponding by (9) to $\lambda$ and $\mu$, as well as the quantum numbers $v_1, v_2, n_1$, and $n_2$ corresponding by (8) and (7) to $S_n, S_p, S_0^n$, and $S_0^p$.

To prove the assertion, I consider characters. Weyl derived a general formula for the characters of Sp(2d) [14]. We need the Sp(2) characters

$$\chi(\lambda)(u) = \frac{u^{\lambda+1} - u^{-\lambda-1}}{u - u^{-1}},$$  \hspace{1cm} (12)

where $u$ and $u^{-1}$ are the eigenvalues of the symplectic transformation, and the Sp(4) characters

$$\chi_{\lambda\mu}(u, v) = \frac{u^{\lambda+2} - u^{-\lambda-2} u^{\mu+1} - u^{-\mu-1} u^{\lambda+2} - u^{-\lambda-2} u^{\mu+1} - u^{-\mu-1}}{u^2 - u^{-2}, u^2 - u^{-2}},$$  \hspace{1cm} (13)

where $u, v, u^{-1}$, and $v^{-1}$ are the eigenvalues of the symplectic transformation. Because each matrix in Sp(2$p$) as a subgroup of Sp(4) is the direct sum of the matrices from Sp(2$n$) and Sp(2$p$), what must be proven is

$$\chi_{\lambda\mu}(u, v) = \sum_{S_n, S_p} \chi_{2S_n, \lambda}(u) \chi_{2S_p, \mu}(v),$$  \hspace{1cm} (14)

with summation over $[\lambda\mu]$ weights ($S_n, S_p$) obeying (11).

It takes a straightforward calculation to verify this algebraic identity.

III. PARTICLE-HOLE CONJUGATION

Condon and Shortley [15] and Racah [16] introduced the concept of particle-hole conjugation in a shell in a context of atomic physics. It is noticed already in this early work that symmetry under particle-hole conjugation gives rise to special conservation laws in systems where the shell is half filled. I offer a derivation which is more general and seems simpler than ones I have found in the literature and clearly displays the few required assumptions.

In a $j$ shell, particle-hole conjugation may be defined [17] as a unitary transformation determined within an unimportant phase factor by

$$a_{m\tau} \rightarrow a_{m\tau}^\dagger, \quad a_{m\tau} \rightarrow -a_{m\tau}^\dagger$$  \hspace{1cm} (15)

for any $m$ and $\tau$. This implies

$$n_\tau \rightarrow 2\Omega - n_\tau.$$  \hspace{1cm} (16)

For the nuclear system with $k = 2$, the complex coordinates $T^\alpha$ of the isospin vector $T$ transform by

$$T^0 \rightarrow -T^0, \quad T^\pm \rightarrow T^\mp,$$  \hspace{1cm} (17)

so that

$$T^2 \rightarrow T^2, \hspace{1cm} (18)$$

that is, particle-hole conjugation commutes with $T$.

The transformations (15) are easily seen to be realized by the transformation of any operator $X$ by

$$X \rightarrow U^\dagger X U$$  \hspace{1cm} (19)

in terms of coordinates $S^\tau$ of

$$S = \sum_\tau S_\tau,$$  \hspace{1cm} (21)

where $S_\tau$ is the vector with the complex coordinates (7). The transformation (20) is the rotation by the angle $-\pi$ about the $y$ axis in quasispin space, so [9]

$$U|S_1 S_0^0 \ldots S_k S_k^0\rangle = (-1)^{\sum_\tau S_\tau} |S_1 (-S_1^0) \ldots S_k (-S_k^0)\rangle.$$  \hspace{1cm} (22)

In particular, the state $|S_0 \ldots S_0\rangle$, where the shell is half filled for all $\tau$, is an eigenstate of $U$ with eigenvalue $(-1)^{k\Omega - \sum_\tau n_\tau}/2$.

Now consider a Hamiltonian

$$H = \sum_\tau \epsilon_\tau n_\tau + \sum_{\tau_1, \tau_2} \frac{1}{2(1 + \delta_{\tau_1 \tau_2})} \langle a_1 a_2^\dagger v_{\tau_1 \tau_2} | a_1^\dagger a_2 | a_2 a_1^\dagger v_{\tau_1^\dagger \tau_2^\dagger} \rangle,$$  \hspace{1cm} (23)

with a convention to be followed from now on that a variable denoted by a Greek letter takes any value available to $m$ or $\bar{m}$, where $m$ is the magnetic quantum number restricted to $m > 0$, and $|\bar{m}\rangle$ is the time-reversed of the state $|m\rangle$. This choice of $|m\rangle$ and $|\bar{m}\rangle$ implies that $U$ commutes with the angular momentum vector. The interaction matrix element is assumed symmetric in the particles 1 and 2,

$$\langle a_1 a_2 | v_{\tau_1 \tau_2} | a_1^\dagger a_2^\dagger \rangle = \langle a_2 a_1 | v_{\tau_2 \tau_1} | a_1^\dagger a_2^\dagger \rangle,$$  \hspace{1cm} (24)

and antisymmetrized for $\tau_1 = \tau_2$, that is,

$$\langle a_1 a_2 | v_{\tau_1} | a_1^\dagger a_2^\dagger \rangle = -\langle a_1 a_2 | v_{\tau_1} | a_2 a_1^\dagger \rangle,$$  \hspace{1cm} (25)

and the interaction is supposed to be rotationally invariant and invariant under time reversal. The rotational invariance implies

$$\sum_\beta \langle a_\beta | v_{\tau_1 \tau_2} | a_\beta^\dagger \rangle = \delta_{\alpha \alpha'} 2\Omega \bar{v}_{\tau_1 \tau_2}$$  \hspace{1cm} (26)
with
\[ \tilde{v}_{\tau_1 \tau_2} = \frac{1}{(2\Omega)^2} \sum_{\alpha, \beta} (\alpha | v_{\tau_1 \tau_2} \rangle | \alpha \rangle), \tag{27} \]
and by the time-reversal invariance we have
\[ \langle \alpha_1 \alpha_2 | v_{\tau_1 \tau_2} | \alpha'_1 \alpha'_2 \rangle = \langle \alpha'_1 \alpha'_2 | v_{\tau_1 \tau_2} | \alpha_1 \alpha_2 \rangle \tag{28} \]
with \(|\tilde{\alpha}\rangle = |\tilde{\beta}\rangle\) and \(-|m\rangle\) for \(\alpha = m\) and \(\tilde{m}\). A simple calculation by means of (15) then shows that by particle-hole conjugation,
\[ H \rightarrow H + 2\Omega \sum_{\tau} \epsilon_{\tau} + \frac{1}{2} (2\Omega)^2 \sum_{\tau, \tau'} \tilde{v}_{\tau, \tau'} n_{\tau}, \tag{29} \]
In a space with definite \(n_{\tau}\) the terms in this expression after the first one are constant. Therefore the transformation \(U\) maps each eigenstate of \(H\) in such a space to an eigenstate of \(H\) in the image of that space whose energy differs from the original one only by these constant terms. Particularly in the midshell space with \(S^2 = 0\) for all \(\tau\), the Hamiltonian \(H\) commutes with \(U\) so that \((-1)^{\sum_{\tau} S_\tau} = (-1)^{\frac{1}{2}(\Omega + \sum_{\tau} v_{\tau})} / 2\) is conserved. Because the factor \((-1)^{\frac{1}{2}\Omega}\) is a constant, this is the conservation law observed by Escuderos, Zamick, and Bayman [3].

IV. CONSERVATION LAWS IN \(v\) AND \(t\)
Now assume, in the nuclear case with \(k = 2\), that \(H\) is charge invariant. I shall show how it then follows from the conservation law just derived that in the subspace of a \(j = 7/2\) shell with \(n_n = n_p = \Omega = 4\) and angular momentum and isospin \(I = T = 0\), the \(\text{Sp}(2\Omega)\) irrep \(vt = 61\) is conserved. The \(\text{Sp}(2\Omega)\) irreps contained in this subspace are \(vt = 00, 40, 61,\) and \(80\) [2]. The weight diagrams of the corresponding \(\text{Sp}(4)\) irreps are shown in Fig. 2. Their ranges (11) are seen to lie on single lines. In the case of \(vt = 61\), this line slopes downwards and \(S_n + S_p = \text{constant} = 1\). For \(t = 0\), the line is \(S^0_n = S^0_p\), so \(S_n + S_p = 2S_n\). This increases from zero in steps of one, but if \(S_n\) is half-integral, the \(\text{Sp}(2)_n\) irrep has no state with \(S^0_n = 0\), that is, \(n_n = \Omega\). Therefore only integral \(S_n\) contribute, and \(S_n + S_p\) is even. Because the \(\text{Sp}(2\Omega)\) irreps \(vt = 00, 40,\) and \(80\) thus have only even \(S_n + S_p\), and \(vt = 61\) only odd \(S_n + S_p\), it follows that the latter is conserved. The weights \((S_n, S_p)\) with integral coordinates are indicated by rings in Fig. 2.
When, more generally, can such a situation occur? First of all, by the first condition (10), for \((0, 0)\) to be a weight of the \(\text{Sp}(4)\) irrep, \((\lambda + \mu) / 2\), and therefore \(t = (\lambda - \mu) / 2\), must be integral. If \(S_n + S_p\) is to have constant parity for integral \(S_n\) and \(S_p\) within a given \(\text{Sp}(4)\) irrep, no \((S_n, S_p)\) with integral \(S_n\) and \(S_p\) must have a neighbor in the horizontal or vertical direction. This is satisfied for \(t = 0\), that is, \(\lambda = \mu\). The weight diagram then has the shape of the upright square, possibly shrunk to a point, encountered above for \(vt = 00, 40,\) and \(80\), and the range (11) shrinks to the line segment from the center to the upper right corner of this square. For integral \(S_n\) and \(S_p\), the sum \(S_n + S_p\) is even. If \(t > 0\) then \((t - 1, 1)\) is a possible value of \((S_n, S_p)\). Then \((t, 1)\) must not be a weight, which requires \(t = (\lambda + \mu) / 2, \) or \(\mu = 0\). Equivalently, \(v/2 + t = \Omega\). The weight diagram then has the shape of the tilted square encountered above for \(vt = 61\), and the range (11) shrinks to the upper right side of this square. In this case, \(S_n + S_p = t\).
In the subspace of \(V\) with \(n_n = n_p = \Omega\), angular momentum \(I\), and isospin \(T\), the states with a definite parity of \(S_n + S_p\) thus carry definite irreps of \(\text{Sp}(2\Omega)\) when all such irreps present have either \(t = 0\) or \(v/2 + t = \Omega\). Using Flowers’s classification of a basis for \(V\) by (the unitary subgroups of) the chain \(\text{GL}(2\Omega) \supset \text{Sp}(2\Omega) \supset \text{SL}(2)\), for \(j = 3/2 - 7/2\), where \(\text{GL}(2\Omega)\) and \(\text{SL}(2)\) are the general and special linear groups, and the infinitesimal algebra of the latter is the algebra of angular momentum [2], one can list for these \(j\) all such subspaces, along with the classification of a basis by \(v\) and \(t\). This is done in Tables I and II. The Hamiltonian does not mix the states in the last but one and last columns. When \(T > 0\), this is true not only for \(n_n = n_p\), that is, \(t^0 = 0\), but holds in the entire isospin multiplet (which has \(n = 2\Omega\) because \(\text{Sp}(2\Omega)\) commutes with \(T\). The reader may check that for each \(I\) and \(T\) in Table II, the dimensions of the subspaces with even and odd \(S_n + S_p\), that is, even and odd \((v_n + v_p)/2\), concur with those found numerically by Escuderos, Zamick, and Bayman [3].
For \(j = 3/2\), all \(\text{Sp}(2\Omega)\) irreps present for \(n = 2\Omega = 4\) have either \(t = 0\) or \(v/2 + t = \Omega\). It is seen from Table I that all eigenstates of \(H\) then have definite \(v\) and \(t\). This is known already to hold for other reasons for any \(n\) [18]. For \(j = 5/2\), the only \(\text{Sp}(2\Omega)\) irrep present for \(n = 2\Omega = 6\) other than those in Table I is \(vt = 21\). For \(j = 7/2,\) the only \(\text{Sp}(2\Omega)\) irreps present for \(n = 2\Omega = 8\) other than those in Table II are \(vt = 21\) and 41.

V. SUMMARY
The irreps of the symplectic group \(\text{Sp}(4)\), which is the commutator group of the group \(\text{Sp}(2\Omega) = \text{Sp}(2) + 1\) of symplectic transformations of the single-nucleon states in the space \(V\) of any numbers of nucleons of both kinds in a \(j\) shell [7], were shown to be the direct sum of different single irreps of \(\text{Sp}(2)_n \times \text{Sp}(2)_p\), where \(\text{Sp}(2)_n\) and \(\text{Sp}(2)_p\) are Kerman’s quasispin groups [10] for neutrons and protons. A rule for the range of \(\text{Sp}(2)_n \times \text{Sp}(2)_p\) irreps in this decomposition is given in (11) and its surrounding text. The theory of particle-hole conjugations was reviewed, and conservation of the parity of \((v_n + v_p)/2\) in the system with \(\Omega\) valence nucleons of both kinds, where \(v_n\) and \(v_p\) are the neutron and proton seniorities [4, 5],
FIG. 2. Weight diagrams of the $Sp(4)$ irreps corresponding for $j = 7/2$ to the $Sp(2\Omega) = Sp(8)$ irreps $vt = 00, 40, 80,$ and $61$. Rings indicate weights $(S_n, S_p)$ with integral coordinates.

| $j$ | $I$ | $T$ | $vt$ with even $S_n + S_p$ | $vt$ with odd $S_n + S_p$ |
|-----|-----|-----|-----------------------------|----------------------------|
| 3/2 | 0   | 0   | 00                          |                            |
|     | 0   | 2   | 40                          | 21                        |
|     | 1   | 1   | 20                          |                            |
|     | 2   | 0   | 40                          | 21                        |
|     | 2   | 1   | 20                          |                            |
| 5/2 | 1   | 0   | 00                          | 40                        |
|     | 0   | 3   | 40                          |                            |
|     | 1   | 1   | 20                          | 41                        |
|     | 1   | 2   | 20                          |                            |
|     | 2   | 0   | 40                          | 41                        |
|     | 3   | 0   | 20 60^2                     | 41^2                      |
|     | 3   | 1   | 40                          | 41^2                      |
|     | 3   | 2   | 20                          |                            |
|     | 4   | 0   | 60                          | 41                        |
|     | 5   | 0   | 20 60                       | 41                        |
|     | 5   | 1   | 40                          | 41                        |
|     | 5   | 2   | 20                          |                            |
|     | 6   | 0   | 60                          | 41                        |
|     | 6   | 1   | 40^2                        | 41                        |
|     | 7   | 0   | 60                          | 41                        |
|     | 7   | 1   | 40                          | 41                        |
| 8   | 1   | 40                          |                            |
| 9   | 0   | 60                          |                            |

TABLE I. For each subspace of $V$ with $n_n = n_p = \Omega$ and given $I$ and $T$ containing only states from $Sp(2\Omega)$ irreps with $t = 0$ or $v/2 + t = \Omega$ the classification of a basis by such irreps is indicated. Irreps with even and odd $S_n + S_p$ are listed in separate columns. $j = 3/2$ and $5/2$.

TABLE II. As Table I for $j = 7/2$.

| $j$ | $I$ | $T$ | $vt$ with even $S_n + S_p$ | $vt$ with odd $S_n + S_p$ |
|-----|-----|-----|-----------------------------|----------------------------|
| 7/2 | 0   | 0   | 00                          | 40 80                      |
|     | 0   | 1   | 80                          | 61^2                       |
|     | 0   | 2   | 00                          | 40^2 80                    |
|     | 0   | 4   | 00                          |                            |
|     | 1   | 0   | 80                          | 61^2                       |
|     | 1   | 3   | 20                          |                            |
|     | 3   | 0   | 40^2 80^2                   | 61^5                       |
|     | 3   | 3   | 20                          |                            |
|     | 5   | 0   | 40^2 42 80^3                | 61^6                       |
|     | 5   | 3   | 20                          |                            |
|     | 7   | 0   | 40^2 80^3                   | 61^6                       |
|     | 7   | 3   | 20                          |                            |
|     | 9   | 0   | 40^3 42 80^4                | 61^6                       |
|     | 9   | 3   | 40 80^3                     | 61^4                       |
|     | 10  | 0   | 40^4 80^3                   | 61^4                       |
|     | 11  | 0   | 80^2                        | 61^2                       |
|     | 12  | 0   | 40 80^2                     | 61^2                       |
|     | 12  | 1   | 60^2                        | 61^2                       |
|     | 12  | 2   | 40                          |                            |
|     | 13  | 0   | 80                          | 61                        |
|     | 13  | 1   | 60^2                        | 61                        |
|     | 14  | 0   | 80                          | 61                        |
|     | 14  | 1   | 61                          |                            |
|     | 15  | 1   | 60                          |                            |
|     | 16  | 0   | 80                          |                            |

derived in a simple manner. This derivation only requires that the Hamiltonian is the sum of a one- and a two-body term and invariant under rotations and time reversal. The conservation of the parity of $(v_n + v_p)/2$ was observed by Escuderos, Zamick, and Bayman in calculations for $^{48}$Cr in the $1f_{7/2}$ shell model [3].
these results, I explained the conservation of the Sp(2Ω) irrep \((v, t) = (6, 1)\), where \(v\) is the seniority, and \(t\) the reduced isospin \([2]\), for angular momentum and isospin \(I = T = 0\) found in my own calculations of this kind \([1]\). Using Flowers’s classification of a basis for \(V\) by the chain \(GL(2Ω) \supset Sp(2Ω) \supset SL(2)\) for \(j = 3/2–7/2\) \([2]\), I listed for these \(j\) all the cases of angular momentum \(I\) and isospin \(T\) where an analogous mechanism takes effect, along with the Sp(2Ω) classification of a basis in each case. For \(j = 7/2\), this list explains in each such case the dimensions of the subspaces with definite parities of \((v_n + v_p)/2\) observed in the calculations by Escuderos, Zamick, and Bayman.

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