Example of a stable wormhole in general relativity

K. A. Bronnikov, L. N. Lipatova, I. D. Novikov and A. A. Shatskiy

We study a static, spherically symmetric wormhole model whose metric coincides with that of the so-called Ellis wormhole but the material source of gravity consists of a perfect fluid with negative density and a source-free radial electric or magnetic field. For a certain class of fluid equations of state, it has been shown that this wormhole model is linearly stable under both spherically symmetric perturbations and axial perturbations of arbitrary multipolarity. A similar behavior is predicted for polar nonspherical perturbations. It thus seems to be the first example of a stable wormhole model in the framework of general relativity (at least without invoking phantom thin shells as wormhole sources).

1 Introduction

The stability of any static or stationary object under small perturbations is a necessary condition for its steady existence in the Universe. Traversable Lorentzian wormholes, being a subject of substantial attention in the modern research in general relativity and its extensions, are not an exception, and much effort has been applied to their stability studies, see, e.g., [1–10]. Stable wormhole models have been obtained in some generalized theories of gravity (see, e.g., [11]), but, to our knowledge, in general relativity such examples have not been found by now (at least without invoking phantom thin shells as wormhole sources).

The simplest (zero-mass) wormhole model with the metric

\[ ds^2 = dt^2 - dx^2 - r^2(x)[d\theta^2 + \sin^2 \theta d\varphi^2], \]
\[ r^2(x) = q^2 + x^2 \]

and a material source in the form of a massless scalar field with negative energy (a phantom scalar field), obtained in [12, 13] and also discussed by Morris and Thorne [14], turned out to be unstable, contrary to a conclusion of [1] (where not all possible perturbations were considered) and according to later, more complete studies [2–4, 8, 15] which also allowed for nonzero wormhole masses. As follows from [5], inclusion of an electric or magnetic charge (the corresponding exact wormhole solutions in general relativity and scalar-tensor gravity are known from [12]) does not stabilize wormholes supported by a phantom scalar.

Evidently, the stability properties of different configurations with the same metric but with other kinds of matter should also be different, depending on the particular dynamics of the material source. Accordingly, a wormhole with the same metric (1) but another material source, a radial magnetic field and phantom dust with negative mass density, was studied in [8], and it turned out to be stable under all spherical perturbations except for inertial radial motion of dust particles. The unstable mode grows slowly enough, linearly in time [9]. It was also shown [10] that nonlinear perturbations of this model lead to shell-crossing singularities.

It was later conjectured that the instabilities of this model can be damped by introducing an additional parameter, related to a nonzero pressure proportional to a deflection from the background static configuration. This created a hope to construct a completely stable model in which the unstable mode would be absent and where any shell crossing would be prevented by repulsive hydrodynamic forces.

Following this idea, in this paper we study the stability of a wormhole model with the metric (1) and a matter source in the form of a radial monopole magnetic (or electric) field and a perfect phantom fluid whose equation of state is close to that of dust. The electromagnetic field has no source (“a charge without charge” according to Wheeler [16]), but its lines of force extend from one spatial infinity to the other.\(^1\) In the static

\(^1\) Similar configurations with phantom dust were previously considered as possible nonsingular classical particle
configuration, whose stability is under study, the fluid energy density $\varepsilon$ is negative, and its absolute value is twice the energy density of the magnetic field. The pressure $p$ is absent in the static case (phantom dust) but grows proportionally to the difference of the perturbed density from its static value.

A tentative study, indicating the stability of such a wormhole under spherical perturbations, was performed in [18], but the perturbation mode with a changing throat radius was not considered there. In the present paper we will prove the stability of a class of such models under all spherical and axial nonspherical perturbations. We will also speculate on the behavior of polar nonspherical perturbations and conclude that they should also be stable. If this conclusion is correct, it seems to be the first example of a stable wormhole model in general relativity.

2 The static model and perturbations

The background static metric (1) is well known to be a solution to the Einstein equations with a source in the form of a massless phantom scalar field [12, 13]. The same metric is also a solution to the Einstein equations with a composite source consisting of neutral phantom dust with the energy density of matter, and for perturbations of gravitating systems it has been discussed, in particular, by Chandrasekhar [19]. So we can use the perturbed metric in the general axially symmetric form

$$ds^2 = e^{2\nu}dt^2 - e^{2\mu}dx^2 - e^{2\phi}d\phi^2 - e^{2\psi}(d\varphi - \sigma dt - q_x dx - q_\theta d\theta)^2,$$

where $\nu, \mu, \phi, \psi, q_x, q_\theta$ are functions of $t, x, \theta$. The background static configuration is characterized by

$$e^{2\nu} = e^{2\mu} = e^{2\phi} = 1, \quad e^{2\psi} = r^2 = x^2 + q^2, \quad \sigma = q_x = q_\theta = 0. \quad (4)$$

The perturbed electromagnetic field tensor may contain any small additions $\delta F_{\mu\nu}(t, x, \theta)$ to the background $F_{\mu\nu}$ described above. As to the matter content, we assume it in the form of a perfect fluid with the SET in its standard form

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu}. \quad (5)$$

In the background configuration, $\varepsilon$ is given in (2), and $p = 0$. For the perturbed configuration we put, following [18],

$$8\pi \varepsilon = -\frac{2q^2}{(q^2 + x^2)^2} + f(t, x, \theta), \quad (6)$$

$$8\pi p = h(x)f(t, x, \theta). \quad (7)$$

Thus $f$ characterizes the density perturbation of matter, and $h(x)$ determines its equation of state.

Considering small perturbations of the above configuration, we can restrict ourselves to axially symmetric perturbations, independent of the azimuthal angle $\varphi$, because the possible $\varphi$ dependence of the form $e^{i\nu \varphi}$ does not affect the perturbation equations. (A similar phenomenon is well known for the equations of quantum mechanics, and for perturbations of gravitating systems it has been discussed, in particular, by Chandrasekhar [19].) So we can use the perturbed metric in the general axially symmetric form

$$ds^2 = e^{2\nu}dt^2 - e^{2\mu}dx^2 - e^{2\phi}d\phi^2 - e^{2\psi}(\sigma dt - q_x dx - q_\theta d\theta)^2, \quad (3)$$

where $\nu, \mu, \phi, \psi, q_x, q_\theta$ are functions of $t, x, \theta$. The background static configuration is characterized by

$$e^{2\nu} = e^{2\mu} = e^{2\phi} = 1, \quad e^{2\psi} = r^2 = x^2 + q^2, \quad \sigma = q_x = q_\theta = 0. \quad (4)$$

The perturbed electromagnetic field tensor may contain any small additions $\delta F_{\mu\nu}(t, x, \theta)$ to the background $F_{\mu\nu}$ described above. As to the matter content, we assume it in the form of a perfect fluid with the SET in its standard form

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu}. \quad (5)$$

In the background configuration, $\varepsilon$ is given in (2), and $p = 0$. For the perturbed configuration we put, following [18],

$$8\pi \varepsilon = -\frac{2q^2}{(q^2 + x^2)^2} + f(t, x, \theta), \quad (6)$$

$$8\pi p = h(x)f(t, x, \theta). \quad (7)$$

Thus $f$ characterizes the density perturbation of matter, and $h(x)$ determines its equation of state.

The 4-velocity has the form $u_\mu = (1, 0, 0, 0) = u^0$ in the background, but it may acquire small spatial components $u_i$ in the perturbed configuration.

Small perturbations of the static background split into two classes, polar and axial perturbations, depending on their symmetry with respect to reflection $\varphi \mapsto -\varphi$, and these classes can be studied independently of each other (see, e.g., [19]).

Polar perturbation, which are even at $\varphi \mapsto -\varphi$, are characterized by nonzero increments $\delta \nu, \delta \psi, \delta \mu_x, \delta \mu_\theta$ as well as those of $F_{tx}, F_{t\theta}$ and $F_{x\theta}$ and nonzero velocity components $u^x$ and $u^\theta$.
Axial perturbations, which are odd at $\varphi \mapsto -\varphi$, involve nonzero perturbations $\sigma$, $q_x$, $q_\theta$, $\delta F_{\mu\varphi}$ and the velocity component $u^\varphi$.

Each class of perturbations can evidently contain a nonzero increment $f(t, x, \theta)$ of matter density.

In what follows, we will study the monopole modes of polar perturbations, which do not violate spherical symmetry, and general axial perturbations of the above background. A full consideration of nonspherical polar perturbations, which is technically more complicated, is postponed for the future.

3 Stability of the spherical mode

In the case of spherically symmetric perturbations, the only dynamic mode is related to a possible motion of matter particles while the gravitational and electromagnetic variables are excluded using the Einstein and Maxwell equations. This calculation has been performed for the wormhole under study in [18], resulting in the following equation:

$$
\dot{\eta} - h(x)\eta'' + \left[-h' + \frac{2h}{r^2x}\right]\eta' + U(x)\eta = 0,
$$

$$
U(x) \equiv \frac{12h x^2 + 3h - 3h' x r^2 + 1}{r^4},
$$

(8)

where the perturbation $\eta(x)$ is defined by the relations $e^{2\mu x} = r^2 e^{2\eta}$, $e^{2\psi} = r^2 e^{2\eta} \sin^2 \theta$. Here, without loss of generality, we have passed on to dimensionless variables by putting formally $x = 1$ (lengths are now measured in units of the wormhole throat radius), so that now $r = 1 + x^2$. It is assumed $h \geq 0$, $h(\pm \infty) = \text{const} > 0$, and the potential $U(x)$ then tends to zero as $x \to \pm \infty$.

We separate the variables putting $\eta \sim e^{i\omega t}$ and write down the equation for a separate mode with the frequency $\omega$:

$$
\dot{h}(x)\eta'' + \left[h' - \frac{2h}{r^2x}\right]\eta' + [\omega^2 - U(x)]\eta = 0.
$$

(9)

Let us assume $U(x) > 0$ and consider unstable modes with imaginary frequencies, $\omega^2 < 0$. Denote $\Omega^2(x) := -\omega^2 + U(x) > 0$. Then evidently the physically admissible asymptotics at $x \to \pm \infty$ have the form $\eta(x) \propto e^{\pm i\omega x}$. Suppose without loss of generality that at such an asymptotic, as $x \to -\infty$, we have $\eta > 0$, then at large negative $x$ we inevitably have $\eta' > 0$, $\eta'' > 0$.

A physically admissible solution $\eta(x)$, whatever be its behavior at finite $x$, should return to zero at large $x$, hence it should have a maximum at some $x = x_0$, at which $\eta = \eta_0 > 0$, $\eta' = 0$, $\eta'' < 0$.

It is clear that such a maximum is impossible at $x_0 \neq 0$, where the coefficient at $\eta'$ in Eq. (9) is finite, and the equation leads to $\eta'' > 0$ at a point where $\eta' = 0$.

At $x_0 = 0$ the situation is different due to the singularity $\sim 1/x$ in the coefficient at $\eta'$. Consider a solution near $x_0 = 0$ as a Taylor series

$$
\eta(x) = \eta_0 + \frac{1}{2}\eta_2 x^2 + \ldots,
$$

(10)

corresponding to a minimum if $\eta_2 > 0$ and to a maximum if $\eta_2 < 0$.

Assuming $h(0) = h_0 \neq 0$, Eq. (9) in the order $O(1)$ leads to the equality $\eta_2 = -\Omega^2(0)\eta_0/h_0 < 0$, so that a maximum is possible, hence an unstable mode of perturbations is also possible.

Let us assume $h(x) = ax^n + o(x^n)$, $a > 0$, $n > 0$ at small $x$, then in the senior order of magnitude in $x$ Eq. (9) gives

$$
a(n - 1)\eta_2 x^n - \Omega^2(0)\eta_0 = 0.
$$

(11)

At $n \neq 1$, under the above assumptions $\Omega^2(0) > 0$, $\eta_0 > 0$, this equality makes a contradiction, which (provided $U(x) > 0$) proves the nonexistence of unstable modes of our system with $\omega^2 < 0$.

It is not hard to verify, in particular, that the condition $U(x) > 0$ holds at all $x$ if

$$
h(x) = ax^n/r^n,
$$

(12)

and $n$ is an even integer from 2 to 14 (if $h(x)$ is even, $U(x)$ is even as well).

One should also consider a possible zero mode, $\omega = 0$, at which the perturbation can linearly grow with time. Then the physically admissible asymptotic behavior of the solution to (9) is a decay by a power law instead of an exponential,\(^3\) $\eta \sim |x|^{-k}$, $k > 0$. However, the further reasoning completely repeats that for $\omega^2 < 0$ with the same result.

We conclude that our background configuration is stable under spherically symmetric perturbations for matter with the equation of state involving the function (12).

\(^3\)As $x \to \pm \infty$, under the above assumptions, $h \to h_*$ and $U \approx U_*/x^2$, where $0 < h_* \leq 1$ and $U_* > 0$. Then the substitution $\eta \sim |x|^{-k}$ in (9) leads to $k(k + 1) = U_*/h_*$. 
It should be noted that this result has been obtained without bringing Eq. (9) to the canonical form and without dividing the \( x \) axis into parts, unlike [18]; moreover, we have included a mode with a nonzero perturbation of the throat radius, the most “dangerous” one as regards the instability.

4 Axial perturbations

As described in Section 2, axial perturbations include (a) perturbations of matter density and pressure as well as the velocity directed along the azimuthal angle \( \varphi \); (b) perturbations of the metric (3), including nonzero \( \sigma, q_x, q_\theta \), while perturbations of \( \nu, \psi, \mu_x, \mu_\theta \) are zero; (c) perturbations of the electromagnetic field \( F_{\varphi \mu} \). It is easy to see that a small coordinate transformation \( \varphi \to \varphi + \delta \varphi(t, x, \theta) \) makes it possible to turn to zero any small velocity field \( v^\varphi = d\varphi/dt \), directed along \( \varphi \). Therefore, without loss of generality, we can assume that the fluid is in its comoving reference frame, and the velocity field has the form

\[
 u^\nu = (e^{-\nu}, 0, 0, 0); \quad u_\mu = (e^\nu, 0, 0, \sigma e^{\psi - \nu}). \tag{13}
\]

Consider the fluid equations of motion in terms of its SET \( T^\nu_{\mu} \). \( \nabla_\nu T^\nu_{\mu} = 0 \). It is sufficient for us to consider the generalized continuity equation \( u^\nu \nabla_\nu T^\nu_{\mu} = 0 \), which now reads

\[
 [(\rho + p) e^{\nu + \mu_x + \mu_\theta}] = e^{\psi + \mu_x + \mu_\theta} \dot{\rho}.
\]

Since the quantities \( \psi, \mu_x, \mu_\theta \) are not perturbed, this equation leads to the equality \( (\rho + p) = \dot{\rho} \), that is, \( \dot{\rho} = 0 \), so that the density (hence also the pressure) have only static perturbations, which can be neglected since we are only interested in dynamic perturbation modes. Thus matter is not perturbed, and we can restrict ourselves to perturbations of free (though mutually interacting) electromagnetic and gravitational fields. Such a problem has been considered in a general form in [21] for an arbitrary static, spherically symmetric background configuration. It has been shown there that, after separating the time variable with the factor \( e^{i\omega t} \) and the angular variable \( \theta \) using the appropriate Gegenbauer functions, the dynamic perturbations can be described by two radial functions \( H_1(x) \) (responsible for electromagnetic perturbations) and \( H_2(x) \) (for the gravitational ones), which obey the equations [21]

\[
 \frac{d^2 H_1}{dx^2} + \omega^2 H_1 = \frac{\Delta}{r^4} \left( L^2 + 2 + \frac{4q^2}{R^2} \right) H_1 + \frac{2qL\Delta}{r^5} H_2, \tag{15}
\]

\[
 \frac{d^2 H_2}{dx^2} + \omega^2 H_2 = \left( -\frac{r_{xx}}{r} + \frac{2r_x^2}{r^2} + L^2 \frac{\Delta}{r^4} \right) H_2 + \frac{2qL\Delta}{r^5} H_1, \tag{16}
\]

where \( r^2 = e^{2\mu_x} \), \( \Delta = r^2 e^{2\psi} \), \( L^2 = (\ell - 1)(\ell + 2) \), \( r_x = \partial r/\partial x \), \( r_{xx} = \partial^2 r/\partial x^2 \), \( \ell \) is the multipolarity order, and \( x \) is the “tortoise” radial coordinate defined by the condition \( g_{tt} = -g_{xx} \).

In our case of the metric (1), the coordinate \( x \) satisfies this condition, and substitution of this metric into Eqs. (15) and (16) leads to the coupled oscillator equations

\[
 H_1'' + \omega^2 H_1 = V_{11} H_1 + V_{12} H_2, \quad H_2'' + \omega^2 H_2 = V_{21} H_1 + V_{22} H_2, \tag{17}
\]

where

\[
 V_{11} = \frac{1}{r^4} \left[ r^2 (L^2 + 2) + 4q^2 \right],
\]

\[
 V_{22} = \frac{1}{r^4} \left[ r^2 (L^2 + 2) + q^2 (L^2 - 1) \right],
\]

\[
 V_{12} = V_{21} = \frac{2qL}{r^3}. \tag{18}
\]

For these equations to admit decoupling, it is necessary and sufficient that the matrix \( (V_{ab}) \) have an eigenvector independent of \( x \) (see, e.g., [22]). Such a decoupling made it possible to obtain two separate wave equations for perturbations of the Reissner–Nordsrtröm solution [19], but in our case this method does not work since the eigenvectors of the matrix \( V_{ab} \) can be written as

\[
 \left\{ 7q^2 \pm \sqrt{49q^4 + 16L^2q^2r^2 + 4Lqr} \right\}. \tag{19}
\]

and none of them is a multiple of a constant vector, except for the case \( L = 0 \) (that is, \( \ell = 1 \)) where \( V_{12} = V_{21} = 0 \) and the equations are already decoupled.

If the equations are not decoupled, a sufficient condition for \( \omega^2 > 0 \) under zero boundary conditions at both infinities is that the matrix \( V_{ab} \) is nonnegative-definite at each \( x \), which for a \( 2 \times 2 \) matrix reduces to the requirements that its trace and determinant are nonnegative (see the
Appendix). One can directly verify that this is indeed the case for $\ell \geq 2$, hence all such modes are stable.

At $\ell = 1$, $V_{12} = V_{21} = 0$, and (17) are two separate equations, the first one with the manifestly positive potential $V_{11}$ guaranteeing stability of the corresponding mode, and the other one with

$$V_{22} = (2x^2 - q^2)/r^4,$$

containing a potential well near $x = 0$. This potential was considered by Armendaris-Picon in [1], and it was shown that the ground state of this problem with zero boundary conditions ($H_2(\pm \infty) = 0$) corresponds to $\omega = 0$. It leads to $\dot{H_2} = 0$, hence $H_2$ can linearly grow with time, indicating an instability of the background configuration. It has been shown, however, in [1] (where perturbations of the same metric (1) were considered, though in the presence of a scalar field but without an electromagnetic one), that the axial gravitational perturbations with $\ell = 1$ are actually a pure gauge and can be annihilated by a suitable coordinate transformation. It is quite a natural observation from a physical viewpoint since the dynamic (wave) degrees of freedom of the gravitational field of a compact source begin with quadrupole modes ($\ell = 2$) while dipole perturbations should be stationary. This result applies to our system because the gravitational degree of freedom is decoupled from the electromagnetic one in the dipole mode.

We conclude that our wormhole model is linearly stable under all axial perturbations.

5 Conclusion

We have shown that, in general relativity, it is possible to construct a static traversable Lorentzian wormhole which is stable under all spherical and axial perturbations. The latter turn out to be independent of the equation of state assumed for the matter supporting the wormhole (specifically, on the particular choice of $h(x)$).

The experience of stability studies for different static, spherically symmetric configurations shows that nonspherical polar perturbations behave qualitatively in the same way as their axial counterparts because, for both kinds of perturbations, the wave equations with nonzero multipolarities $\ell$ possess effective potentials with positive “centrifugal” terms [1, 19]. Therefore if a model is really unstable, the instability will most probably manifest itself in spherical modes where $\ell = 0$.

Nevertheless, to complete the stability study of our model, nonspherical polar perturbations should be studied, and we are planning to consider them in the near future. The most probable result of such a study must show that the present model is (to our knowledge, at least for distributed systems as opposed to thin-shell wormholes) the first example of a stable wormhole model in general relativity.

Appendix. Coupled wave equations: a sufficient condition for stability

We present this elementary and well-known proof for completeness. Consider a set of coupled equations of the form

$$\ddot{\vec{y}} + \omega^2 \vec{y} = V(x)\vec{y}, \quad x \in \mathbb{R},$$

where $\omega = \text{const}$, $V = (V_{ab}(x))$ is an $n \times n$ matrix with $x$-dependent elements, and $\vec{y} = (y_1(x), y_2(x), \ldots, y_n(x))$ is a column of $n$ unknown functions of $x$, which are assumed to be square-integrable, so that, in particular, $y_a \to 0$ as $x \to \pm \infty$. The prime denotes $d/dx$.

Consider $\vec{y}$ as a vector in $n$-dimensional Euclidean space with the usual scalar product. Let us scalarly multiply Eq. (A.1) by $\vec{y}$ from the left and integrate over $\mathbb{R}$ to obtain (omitting the limits near the integral sign)

$$\int \vec{y} \ddot{\vec{y}} dx + \omega^2 \int \vec{y}^2 dx = \int \vec{y} (V \dot{\vec{y}}) dx.$$  \hspace{1cm} (A.2)

The first integral can be rewritten as

$$\int (\vec{y} \vec{y}^t)' dx - \int \vec{y}^2 dx = - \int \vec{y}^t \dot{\vec{y}}^2 dx,$$

since the first term here is directly integrated and vanishes due to the boundary condition. As a result, we obtain the following expression for $\omega^2$:

$$\omega^2 = \frac{1}{\int \vec{y}^2 dx} \left[ \int \vec{y}^2 dx + \int \vec{y} (V \dot{\vec{y}}) dx \right].$$  \hspace{1cm} (A.3)

We assume that the solution $\vec{y}$ is nontrivial, hence both the denominator and the first term in the square brackets are nonzero. Therefore a sufficient condition for $\omega^2 > 0$ is that

$$\vec{y} (V \dot{\vec{y}}) \equiv V_{ab}(x)y_a(x)y_b(x) \geq 0$$  \hspace{1cm} (A.4)
at all values of \( x \), or, in other words, that this quadratic form is nonnegative-definite at all \( x \).

For \( n = 1 \) this reduces to \( V \geq 0 \), the well-known sufficient condition of only positive energy states in a one-dimensional quantum-mechanical system.

For \( n = 2 \) this requirement leads to two conditions: the trace \( V_{11} + V_{22} \geq 0 \) and \( \det(V_{ab}) \geq 0 \).

Acknowledgments

The authors are thankful to the participants of seminars at the Astro Space Center of Lebedev Physical Institute of RAN and at the Sternberg Astrophysical Institute (Moscow) as well as to Roman Konoplya, Valentina Kolybasova and Milena Skvortsova for helpful discussions and remarks.

The work has been supported in part by RFBR Projects 13-02-00757-a, 12-02-00276-a, 11-02-00244-a, 11-02-12168-ofi-m-2011, Scientific School 2915.2012.2 (Formation of the Large-Scale Structure of the Universe and Cosmological Processes), the Program “Non-stationary Phenomena in Objects of the Universe 2012”, and the Program “Scientific and Pedagogical Personnel of Innovative Russia 2009–2013’, Federal Goal-Oriented Program 16.740.11.0460.

References

[1] C. Armendariz-Picon, Phys. Rev. D 65, 104010 (2002); gr-qc/0201027.
[2] H. Shinkai and S. A. Hayward, Phys. Rev. D 66, 044005 (2002); gr-qc/0205041.
[3] J. A. Gonzalez, F. S. Guzman and O. Sarbach, Class. Quantum Grav. 26, 015010 (2009); ArXiv: 0806.0608.
[4] J. A. Gonzalez, F. S. Guzman and O. Sarbach, Class. Quantum Grav. 26, 015011 (2009); ArXiv: 0806.1370.
[5] J. A. Gonzalez, F. S. Guzman and O. Sarbach, Phys. Rev. D 80, 024023 (2009); ArXiv: 0906.0420.
[6] K. A. Bronnikov and S. Grinyok, Grav. Cosmol. 7, 297 (2001); gr-qc/0201083.
[7] K. A. Bronnikov and S. Grinyok, in: Inquiring the Universe, Festschrift in honor of Prof. Mario Novello, ed. by J. M. Salim et al. (Frontiers Group, 2003), p. 33–53; gr-qc/0205131.
[8] A. Doroshkevich, J. Hansen, I. Novikov, and A. Shatskiy, Int. J. Mod. Phys. D 18, 1665 (2009); ArXiv: 0812.0702.
[9] I. D. Novikov, Astron. Reports 53 (12), 1079 (2009).
[10] O. Sarbach and T. Zannias, Phys. Rev. D 81, 047502 (2010); ArXiv: 1001.1202.
[11] P. Kanti, B. Kleihaus, and J. Kunz, Phys. Rev. Lett. 107, 271101 (2011); ArXiv: 1108.3003.
[12] K. A. Bronnikov, Acta Phys. Pol. B 4, 251 (1973).
[13] H. G. Ellis, J. Math. Phys. 14, 104 (1973).
[14] M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988).
[15] K. A. Bronnikov, J. C. Fabris, and A. Zhidenko, EuroPhys J. C 71 (11), 1791 (2011); ArXiv: 1109.6576.
[16] J. A. Wheeler, Phys. Rev. 97, 511 (1955).
[17] K. A. Bronnikov, V. N. Melnikov, G. N. Shikin, and K. P. Staniukovich, Ann. Phys. (NY) 118, 84 (1979).
[18] I. D. Novikov and A. A. Shatskiy, JETP 114 (5), 801 (2012); ArXiv: 1201.4112.
[19] S. Chandrasekhar, The Mathematical Theory of Black Holes (Clarendon, Oxford University Press, New York, 1983).
[20] A. A. Shatskii, I. D. Novikov, and N. S. Kardashev, Physics-Uspekhi 51, 457 (2008).
[21] K. A. Bronnikov, R. Konoplya, and A. Zhidenko, Phys. Rev. D 86, 024028 (2012); Arxiv: 1205.2224.
[22] V. Ferrari, M. Pauri, and F. Piazza, Phys. Rev. D 63, 064009 (2001); gr-qc/0005125.