On Local Martingale Deflators and Market Portfolios

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Received: date / Accepted: date

Abstract. The supermartingale (local martingale) deflator is a strictly positive multiplier transforming value processes of admissible portfolios into supermartingales (into local martingales). It may happen that its reciprocal is a numéraire, i.e. the value process of a portfolio understood as a dynamically rebalanced basket of assets traded on the market. In such a case, by abuse of language, we use the terminology traded supermartingale (local martingale) deflator. The traded supermartingale deflator is always unique. The Takaoka theorem claims that in the very general model of financial market, with a finite-dimensional semimartingale price process $S$ describing the dynamics of basic securities, the $NAA_1$-property (called also $NUPBR$-property) is equivalent to the existence of a local martingale deflator. On the other hand, by the Karatzas–Kardaras theorem the $NAA_1$-property is equivalent to the existence of the traded supermartingale deflator. In this paper we show that though the traded local martingale deflator may not exist under original probability $P$, the $NAA_1$-property implies that in arbitrary small neighborhood one can find a probability measure $\tilde{P} \sim P$ under which the traded local martingale deflator does exist. This result (available previously only for one risky asset model) is in the striking resemblance with the Delbaen–Schachermayer theorem on the existence of an equivalent $\sigma$-martingale measure. Our arguments are based heavily on this theorem as well as on the Karatsas–Kardaras theorem. To keep
the presentation self-contained we provide the proof of a "reduced" version of latter, using some new ideas. An inspection of this proof shows that if the Lévy measures of $S$ are concentrated in finite number of points, then $NAA1$ implies the existence of the traded local martingale for the original probability measure $P$. A simple proof of the Delbaen–Schachermayer theorem is given in the appendix.

**Keywords** Asymptotic arbitrage · Market viability · Fundamental Theory of Asset Pricing · Numéraire · Local martingale deflator · $\sigma$-Martingale

**Mathematics Subject Classification (2010)** 91B70 · 60G44

**JEL Classification** C60 · G13

1 Introduction

In the modern theory of financial markets one of the central questions is the existence of a market portfolio. The concept is very simple in its nature. In any market, the values of assets can be expressed in units of a selected one, called the numéraire. E.g., the oil prices are traditionally quoted in the US dollars, the values of assets in EU markets are given in euros. For a long time, the gold was used as the numéraire in many markets etc. In a financial market one can take as the numéraire any dynamically rebalanced basket of the basic traded assets which has a positive value. The market portfolio can be defined as a basket such that the prices of all traded assets expressed in the units of the latter are local martingales. We call market index the value process of the market portfolio.

Another important concept of the theory is local martingale deflator, i.e. a process whose product with the price process of any asset yields in a local martingale. A supermartingale deflator (appeared in [2] under the name of numéraire portfolio) transforms price processes into supermartingales. So, the reciprocal of the market index is a local martingale deflator. Of course, the reciprocal of a local martingale deflator may not be a market index, i.e. the value process of a basket of traded assets, see an example in [31]. If the deflator is a reciprocal of the value process, it is called tradable.

A classical arbitrage theory already provides some information on the existence of local martingale deflators in very general models. In the case of a frictionless market with a finite number of basic assets whose price processes are locally bounded semimartingales, the No Free Lunch property (NFL) implies the existence of a local martingale deflator. Indeed, in this case the Kreps–Yan theorem says that there is an equivalent local martingale measure and the product of the density process of latter on the price process is a local martingale with respect to the original measure. Thus, the density process is a local martingale deflator. One can deduce from the result of Delbaen and Schachermayer on the existence of the equivalent $\sigma$-martingale measure that the NFL-property (equivalent to the NFLVR-property, which, in turns is
equivalent to the simultaneous fulfillment of $NA$ and $NAA_1$, see Section 2.2 implies the existence of a local martingale deflator without the assumption on local boundedness.

In the fundamental paper [21], between other important results, it was shown, for the general semimartingale models with finite number of basic securities that the following properties are equivalent:

$(a)$ $NAA_1$ ($NUPBR$, No-Unbounded Profits with Bounded Risk, in the terminology of [21]);

$(b)$ the *numéraire portfolio* (i.e. a tradable supermartingale deflator) does exist.

In the recently published note by Takaoka and Schweizer [31], based on the Takaoka manuscript circulating from 2009, a new important equivalence was added, namely, that the $NAA_1$-property, No Asymptotic Arbitrage of the 1st Kind, is equivalent to the existence of the local supermartingale deflator (a strict $\sigma$-martingale density in the terminology of [31]) does exist. Note that for continuous semimartingales, amongst local martingale deflectors always there is one whose reciprocal is a value process of some portfolio, see [3]. Already mentioned example of [31] shows that a discontinuous price process may admit local martingale deflectors but non of them is tradable, that is the reciprocal of a value process. It is worthy to mention that arguments of [31] used the change of numéraire techniques and a clever reduction to the Delbaen–Schachermayer FTAP, that is the criterion for the NFLVR. On the other hand, it was shown in [22] that the existence of a market portfolio is the fact that can be used to get the FTAP in a somewhat simpler way. That is why it is quite desirable to get an alternative proof of the existence and, in the present paper, we get one using the technics of local characteristics which happened to be very efficient in a number of related problems, see, e.g., [11], [18], [21], etc.

The problem of supermartingale deflectors (and with closely related concepts of log-optimal portfolios and optimal growth rate portfolios) was intensively studied under various level of generality, see [1], [2], [7], [28], [9] and references therein. The reader might be confused by the variety of terminology used in the literature and we add to this confusion by calling *numéraires* positive value processes of self-financing portfolios (by an abuse of language this label was attached to reciprocal of supermartingale deflectors). The concepts of asymptotic arbitrage of the 1st and 2nd kind was introduced in [12] in the context of large financial markets modeled by a sequence of filtered probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n)$ on which are given semimartingales $S^n = (S^n_t)_{t \leq T^n}$, interpreted as price processes, see also [20], [13], [25] etc. These concepts happen to be interesting also for the “stationary” case where there is no dependence of $n$, i.e. in the conventional setting of the classical arbitrage theory and its extension to markets with transaction costs. The relevance of $NAA_1$ to the deflator problems seems to be not understood and this notion was reinvented as $BK$ property, [11], $NUPBR$, [21], $NA_1$, [23].

In the paper [23] it was proven that in the one-asset semimartingale model the $NAA_1$-property holds if and only if in any neighborhood of the reference probability there is an equivalent probability measure under which there exists
a market portfolio. Unfortunately, to the moment it is not clear whether one can use the method of

In this note we extend this result (in which non-trivial is the "only if" part) to the multi-asset case (Theorem 2.10). We show also that in the case where the Lévy measures of the price processes are concentrated in a finite number of points the market portfolio exists under the original probability measure (Theorem 8.3).

To obtain the claims we prove that the NAA1-property implies the existence of a supermartingale traded numéraire assuming that the function $|x|^2 \wedge |x|$ is integrable with respect to the Lévy measures of $S$, the condition which is trivially fulfilled when the latter are concentrated in finite number of points and also for special semimartingales. Of course, this result in not new: it is obtained in [21] without additional assumption and in a larger generality (in particular, with constraints on portfolio). The reason to give a complete proof here are the following: under the mentioned assumption the arguments are more transparent and lead to an explicit formula of the portfolio. Namely, the portfolio value process (i.e. the inverse of the supermartingale deflator) is the stochastic exponential $E(g \cdot S)$ where $g = (g_t)$ is the process such that $g(t, \omega)$ solves a finite-dimensional convex maximization problems. The existence of solutions are ensured by the N.A-property in the class of strategies with non-negative value processes which is weaker than NAA1. To establish that under NAA1 the process $g$ is $S$-integrable we provide new arguments based on checking the Shiryaev–Cherny criterion with help of a kind of laws of large numbers for sequences of stochastic integrals.

A thorough examination of the maximization problems allows us to conclude that in the case where the Lévy measures are concentrated in a finite number of points the procedure leads directly to a market portfolio. In the general case $E^{-1}(g \cdot S)$ is only a supermartingale deflator but it happens to be a local martingale deflator with respect to an equivalent probability measure $\bar{P}$ which can be chosen arbitrary close (in the total variation distance) to the original one. The proof of this fact is based on the observation the the ratio of positive value processes can be represented as a value process corresponding to a market with a semimartingale price process $\bar{S}$ for which, by the choice of the numéraire, the original probability is a separating measure. It remains to apply the Delbaen–Schachermayer theorem and take as $\bar{P}$ the equivalent $\sigma$-martingale measure for $\bar{S}$.

The structure of the paper is the following. In Section 2 we formulate the general semimartingale setting. In Section 2.2 we recall some basic facts from the arbitrage theory aiming to relate the concept of NAA1 with those of NA and NFLVR in the general context of value processes. Section 3 contains necessary preliminaries from stochastic calculus for semimartingales. In Section 3.2 we write a representation of the ratio of stochastic exponentials (i.e. of the ratio of the numéraire processes) and provide the main ideas on the existence of the traded semimartingale deflator postponing the technical realization to Sections 4, 5 and 6. In Section ?? we give another representation of the ratio of stochastic exponentials, namely, as a single stochastic exponential but with
respect to another integrator and deduce from this representation, making use the theorem on the existence of an equivalent \( \sigma \)-martingale measure, the existence the market portfolio. In Section 5 we deduce from the Cherny–Shiryaev criterion the \( S \)-integrability of the process solving pointwise maximization problems of the type analyzed in Section 7. As the tool we use laws of large numbers for a sequences of stochastic integrals with truncated integrands proven in Section B. In Section A we provide a proof of the existence of an equivalent \( \sigma \)-martingale measure slightly simplified with respect to that given in [11]. In Section 8 we show that in the case of discrete Lévy measures the maximum of the optimization problems is attained in the interior points of the domains and this fact ensures that there is a market portfolio under the original probability measures.

2 Framework and Results

2.1 Basic Definitions

Let \((\Omega, \mathcal{F}, F, P)\) be a stochastic basis satisfying the usual conditions and let \(\mathcal{S}\) be the space of semimartingales \(X\) defined on a finite interval \([0, T]\) and starting from zero. We fix in \(\mathcal{S}\) a convex subset \(X^1\) of processes \(X \geq -1\) containing the zero process. Note that \(\lambda X^1 \subseteq X^1\) when \(\lambda \in [0, 1]\). We suppose that a stronger property is fulfilled, namely, for every \(\lambda \geq 0\) and \(X \in X^1\) the process \(\lambda X \in X^1\), if \(\lambda X \geq -1\).

Put \(\mathcal{X} := \text{cone}X^1 = \mathbb{R}^+X^1\).

In the context of mathematical finance the elements of \(\mathcal{X}\) are interpreted as admissible value processes (corresponding to admissible self-financing strategies) starting from zero initial capital; the elements of \(\mathcal{X}^\lambda := \lambda X^1\) are called \(\lambda\)-admissible.

"Standard" model. A typical example is the model where we a given a \(d\)-dimensional semimartingale \(S\) and \(X^1\) is the set of stochastic integrals \(H \cdot S\) where \(H\) is \(S\)-integrable and \(H \cdot S \geq -1\). Though our results deal with the standard model, the basic definitions and their relations with concepts of the arbitrage theory is natural to discuss in a more natural framework. In particular, we want to avoid at this stage the stochastic integration theory.

Define the set of strictly 1-admissible processes \(X^1_2 \subseteq X^1\) composed of \(X \in X^1\) such that \(X, X^- \geq -1\).

The sets \(x + X^1, x + X^1_2\), etc., \(x \in \mathbb{R}\), have obvious sense. We are particularly interested in the set \(1 + X^1_2\). Its elements are positive value processes corresponding with the initial capital equal to unit with trajectories bounded away from zero. Such processes are called traded numéraires.

Definition 2.1. The family \(\mathcal{X}\) has NAA1-property (No Asymptotic Arbitrage of the 1st Kind) if for any sequence of reals \(x^n \downarrow 0\) and any sequence of value processes \(X^n \in \mathcal{X}\) such that \(x^n + X^n \geq 0\) we have \(\limsup_n P(x^n + X^n_T \geq 1) = 0\).
Lemma 2.2 The following properties are equivalent

(a) NAA1;
(b) the set \( \{X_T : X \in \mathcal{X}_1^0\} \) is \( P \)-bounded;\(^1\) (BK or NUPBR);
(c) \( \left( \bigcap_{\varepsilon > 0} \{x + X_T : X \in \mathcal{X}^x\} - L_{0}^{0}\right) \cap L_{1}^{0} = \{0\} \) (NA1).

Proof. (a) \( \rightarrow \) (b) If (b) fails, i.e. \( P(1 + \tilde{X}_n^\varepsilon \geq n) \geq \varepsilon > 0 \) for a sequence of \( \tilde{X}_n \in \mathcal{X}_2^0 \), then we have a violation of NAA1 with \( n^{-1} + n^{-1} \tilde{X}_n^\varepsilon \).

(b) \( \rightarrow \) (c) Since

\[
(1/2)\{X_T : X \in \mathcal{X}_1^1\} = \{X_T : X \in \mathcal{X}_1^{1/2}\} \subseteq \{X_T : X \in \mathcal{X}_1^0\},
\]

the sets \( \{X_T : X \in \mathcal{X}_1^0\} \) and \( \{X_T : X \in \mathcal{X}_1^1\} \) are \( P \)-bounded simultaneously. If (c) fails, there are \( \xi \in L^0 \setminus \{0\} \) and a sequence \( X_n \in \mathcal{X}_1^1\) such that \( 1/n + X_n \geq \xi \). Then the sequence \( nX_n^\varepsilon \) is \( P \)-unbounded.

(c) \( \rightarrow \) (a) If (a) does not hold, then there exist sequences \( x^n \downarrow 0 \) and \( X_n \geq -x^n \) such that \( P(x^n + X_n^\varepsilon \geq 1) \geq 2\varepsilon > 0 \). Recall that any sequence of random variables bounded from below contains a subsequence converging in Cesaro sense a.s. with its further subsequences (von Weizsäcker theorem, see \[32\] or \[17\], 5.2.3). We may assume without loss of generality that already for \( \xi^n := x^n + X_n^\varepsilon \) the sequence \( \xi^n := (1/n) \sum_{i=1}^n \xi_i \) converges to some \( \xi \neq 0 \). Note that \( \xi \neq 0 \). Indeed,

\[
\varepsilon(1 - P(\xi^n \geq \varepsilon)) \geq \frac{1}{n} \sum_{i=1}^n E\xi^i I_{(\xi^n < \varepsilon)} \geq \frac{1}{n} \sum_{i=1}^n E\xi^i I_{(\xi^i \geq 1, \xi^n < \varepsilon)}
\]

\[
\geq \frac{1}{n} \sum_{i=1}^n P(\xi^i \geq 1, \xi^n < \varepsilon) \geq \frac{1}{n} \sum_{i=1}^n (P(\xi^i \geq 1) - P(\xi^n \geq \varepsilon))
\]

\[
\geq 2\varepsilon - P(\xi^n \geq \varepsilon).
\]

It follows that \( P(\xi^n \geq \varepsilon) \geq \varepsilon/(1 - \varepsilon) \). Thus,

\[
E(\xi \land 1) = \lim_n E(\xi^n \land 1) \geq \varepsilon^2/(1 - \varepsilon) > 0.
\]

So, there is \( a > 0 \) such that \( P(\xi \geq 2a) > 0 \) Using the Egorov theorem on uniform convergence of sequences converging a.s., one can find a measurable set \( \Gamma \subseteq \{\xi \geq a\} \) with \( P(\Gamma) > 0 \) on which \( x^n + X_n \geq a \) for all sufficiently \( n \geq n_0 \). But this means that the random variable \( aI_\Gamma \neq 0 \) can be super-replicated starting with arbitrary small initial capital in contradiction with the assumed property (c). \( \square \)

Definition 2.3 A process \( Z > 0 \) with \( Z_0 = 1 \) is supermartingale deflator (respectively, local martingale deflator) if \( Z(1 + X) \) is a supermartingale (respectively, a local martingale) for each \( Z \in \mathcal{X}_1^1 \).

\(^1\) We use the abbreviation "\( P \)-bounded" for boundedness in probability instead of confusing "bounded in \( L^0\)"; for most common metrics in \( L^0 \) the whole space is bounded.
A local martingale bounded from below is a supermartingale by the Fatou lemma, so that a local martingale deflator is, automatically, a supermartingale one.

It is rather obvious that the existence of a supermartingale deflator implies NAA1. Indeed, if \( Z \) is a supermartingale deflator, then \( E Z_T (1 + X_T) \leq 1 \) for every \( X \in \mathcal{X}^1 \). The boundedness in \( L^1 \) implies the boundedness in probability (due to the Chebyshev inequality). Thus, the set \( \{ Z_T (1 + X_T) : X \in \mathcal{X}^1 \} \) is \( P \)-bounded and so is the set \( \{ 1 + X_T : X \in \mathcal{X}^1 \} \).

**Definition 2.4** An element \( V \) of \( 1 + \mathcal{X}^1 \) is called *traded supermartingale numéraire* (respectively, *traded local martingale numéraire*) if \( \frac{1}{V} \) is a supermartingale deflator (respectively, local martingale deflator).

Abusing language, we add the adjective “traded” to denote the deflator \( V \) whose reverse \( 1/V \) is a traded numéraire.

Recall that the traded supermartingale deflator (hence, the traded local martingale deflator), if exists, is unique. Indeed, the ratio of two strictly positive processes starting from unit and its reciprocal are, simultaneously, supermartingales only if this ratio is identically equal to unit. To see this, note that the function \( x \mapsto x^{-1} \) is strictly convex and decreasing on \([0, \infty)\). So, for a supermartingale \( X > 0 \) with \( X_0 = 1 \) we have that

\[
E X_t^{-1} \geq (E X_t)^{-1} \geq 1
\]

where the first inequality is strict except for the case where the random variable \( X_t^{-1} \) is equal to a constant (a.s.). On the other hand, if \( X^{-1} \) is a supermartingale, then \( E X_t^{-1} \leq 1 \). This is consistent with the above only if \( X = 1 \) (a.s.).

**Comments on terminology:**

The NAA1-property has a clear financial meaning. It appeared first with this name in the paper [12] in a much more general context of large financial markets where the notion of the No Asymptotic Arbitrage of the 2nd Kind was also introduced. The importance of the property equivalent to the set \( K_0^1 := \{ X_T : X \in \mathcal{X}^1 \} \) is \( P \)-bounded was already clear in [5]. The concept was isolated in [11] where it was referred to as the BK-property. The relations with NAA1 was also overlooked in further studies where it appeared under the name NUPBR (No Unbounded Profit with Bounded Risk) in [21], etc. Though it is obviously equivalent to NAA1 (in the simplest version of the latter) but in some cases it is more convenient to use.

Having in mind that NAA1, as defined here, is a particular (“stationary”) case of a more general concept introduced in [12] and intensively studied in the literature (see, e.g., [26], [13], [25]), we suggest to keep this label for all above reformulations.
2.2 NA, NAA1, and NFLVR

In this subsection we recall, following [11], some well-known results explaining the relations between NA, NAA1, and NFLVR.

The model has concatenation property if for any \(X, X' \in \mathcal{X}^1\) and any bounded predictable processes \(H, G \geq 0\) such that \(HG = 0\) and \(\tilde{X} := H \cdot X + G \cdot X' \geq -1\), the process \(\tilde{X} \in \mathcal{X}^1\).

Define the convex sets \(K_0 := \{X_T : X \in \mathcal{X}\}, C := (K_0 - L^\infty_0) \cap L^\infty\), and denote by \(\bar{C}, \bar{C}^*,\) and \(C^*\) the norm closure, the sequential weak* closure, and weak* closure of \(C\) in \(L^\infty\).

The properties NA, NFLVR, NFLBR, and NFL mean, respectively, that \(C \cap L^\infty_\mathcal{F} = \{0\}\), \(C \cap L^\infty_\mathcal{H} = \{0\}\), \(\bar{C}^* \cap L^\infty_\mathcal{F} = \{0\}\), and \(\bar{C}^* \cap L^\infty_\mathcal{H} = \{0\}\).

Consecutive inclusions induce the hierarchy of these properties:

\[
C \subseteq \bar{C} \subseteq \bar{C}^* \subseteq C^*
\]

NA \(\iff\) NFLVR \(\iff\) NFLBR \(\iff\) NFL.

Remark. Suppose that \(\mathcal{X}\) has the concatenation property and \(\hat{P}\) is a separating measure. In such a case, if \(X\) is bounded and \(H = \pm \xi I_{[s,t]}\) where \(\xi \in L^\infty_{\mathcal{F}_s}\) then \(H \cdot X \in \mathcal{X}\). It follows that the process \(X\) is a \(\hat{P}\)-martingale. If \(X\) is locally bounded then \(X\) is a local \(\hat{P}\)-martingale. Applying the Fatou lemma we get from here that all locally bounded elements of \(\mathcal{X}^1\) are \(\hat{P}\)-supermartingales.

Lemma 2.5 Let \(X \in \mathcal{X}\). If NA holds, then \(X \in \lambda \mathcal{X}^1\) with \(\lambda = \|X_T\|_{\infty}\).

Proof. If \(P(X_s < -\lambda) > 0\), then the process

\[
\tilde{X}_t := I_{\{X_s < -\lambda\}} I_{[s,T]} \cdot X_t = I_{\{X_s < -\lambda, t \geq s\}} (X_t - X_s)
\]

belongs to \(\mathcal{X}\), \(\tilde{X}_T \geq 0\), and \(P(\tilde{X}_T > 0) > 0\) in violation of NA. \(\square\)

Lemma 2.6 NFLVR \(\iff\) NA & NAA1.

Proof. \((\Rightarrow)\) The NA-propety follows trivially. If NAA1 fails we can find \(X^n \in \mathcal{X}^1\) such that for the sequence \(\xi_n := n^{-1} X^n_T\) we have \(P(\xi^n \geq 1) \geq \alpha > 0\). Note that the random variables \(\xi_n \land 1\) belong to the convex set \(C\). Using the von Weizsäcker (or Komlós) theorem on subsequences, we construct a sequence \(\xi_n^n\) converging a.s. to \(\xi \geq 0\). Then \(E\xi = \lim E\xi_n \geq \alpha\). Hence, \(P(\xi > 0) = \alpha\). By the Egorov theorem there is \(\Gamma\) such that \(P(\Gamma) > 1 - \alpha/2\) and \(E \xi_n I_\Gamma - \xi I_\Gamma \to 0\) in \(L^\infty\) where \(P(\xi I_\Gamma > 0) \geq \alpha/2\).

\((\Leftarrow)\) If NFLVR fails, there are a sequence \(\xi_n \in C\) and \(\xi \geq 0\) such that \(P(\xi > 0) > 0\) and \(\|\xi_n - \xi\|_{\infty} \leq n^{-1}\). By definition, \(\xi_n \leq \eta_n = X^n_T\) where \(X^n \in \mathcal{X}^1\).
Obviously, \( \| \eta^n_n \|_{\infty} \leq n^{-1} \) and, since the NA-property holds, \( nX^n_n \in \mathcal{X}^1 \) in virtue of the lemma above. By the von Weizsäcker theorem we may assume that \( \eta_n \to \eta \) a.s. Since \( P(\eta > 0) > 0 \), the sequence \( nX^n_T \in K^1_0 \), tending to infinity with positive probability, violates NAA1. 

Simple examples showing that the properties NFLVR, NA, and NAA1 are all different can be found in [8].

Recall that \( S \) is a Fréchet space with the quasinorm 

\[
D(X) := \sup \{ E1 \wedge |H \cdot X_T| : H \text{ is predictable, } |H| \leq 1 \}.
\]

**Theorem 2.7** Suppose that the concatenation property holds and \( \mathcal{X}^1 \) is closed in \( S \). Then under the NFLVR condition \( C = \bar{C}^* \) and, as a corollary, we have that 

\[
\text{NFLVR} \iff \text{NFLBR} \iff \text{NFL} \iff \text{ESM}.
\]

For the “standard model” where the processes \( X \) are of the form \( X = H \cdot S \), where \( S \) is a \( d \)-dimensional semimartingale and \( H \) runs through \( L(S) \) (the latter symbol denotes the space for which the stochastic integral is defined), the hypothesis of closedness follows from the Mémin theorem, [27]. So, the hypotheses of Theorem 2.7 are fulfilled. In the ”standard” model the process \( S \) under separating measure is a martingale, if \( S \) is bounded, and a local martingale, if \( S \) is locally bounded. Without assuming the local boundedness of \( S \), we have only the following:

**Theorem 2.8** In any neighborhood of a separating measure there is a probability measure under which \( S \) is a \( \sigma \)-martingale.

Note that the hypotheses of Theorem 2.7 make it applicable for some classes of financially interesting models, e.g., models involving infinite many securities or models where some components of portfolio strategies \( H \) are positive, i.e. on the corresponding assets shortselling constraints are imposed.

All the mentioned results for the standard model are obtained in [5] and [6], see also [11] for a short proof of the theorem on the existence of a \( \sigma \)-martingale measure based on the theory of Hellinger processes.

The following result, which is a particular case of Theorem 1.7 from [24], provides a criterion relating NAA1 and the existence of a supermartingale deflator.

**Theorem 2.9** Suppose that the set \( 1 + \mathcal{X}^1 \) is fork-convex, i.e. for every moment \( s \in [0, T] \), an \( \mathcal{F}_s \)-random variable with values in \( [0, 1] \), \( X \in 1 + \mathcal{X}^1 \) and strictly positive \( X' \), \( X'' \in 1 + \mathcal{X}^1 \) the process 

\[
XI_{[0,s]} + (\alpha(X_s/X'_s)X'_s + (1 - \alpha)(X_s/X''_s)X''_s)I_{[s,T]}
\]

belongs to \( 1 + \mathcal{X}^1 \). Then the NAA1-property holds if and only if there exists the traded supermartingale deflator.
2.3 Main Results

The main results of our paper are the following:

**Theorem 2.10** Let $X^1 = \{ H \cdot S : H \cdot S \geq -1, H \in L(S) \}$ where $S$ is a $d$-dimensional semimartingale. Suppose that the NAA1 property holds. Then in any neighborhood of $P$ there is a probability measure $P' \sim P$ under which the traded local martingale deflator does exist.

**Theorem 2.11** Let $X^1 = \{ H \cdot S : H \cdot S \geq -1, H \in L(S) \}$ where $S$ is a $d$-dimensional semimartingale such that its Lévy measures are concentrated in finite number of points. Then the NAA1 property holds if and only if there exists the traded local martingale deflator.

Theorem 2.10 implies as an easy corollary the Takaoka theorem

**Theorem 2.12** Let $X^1 = \{ H \cdot S : H \cdot S \geq 1, H \in L(S) \}$ where $S$ is a $d$-dimensional semimartingale. Then the NAA1-property is equivalent to the existence of the local martingale deflator.

Proof. Let $V' = H' \cdot S$ be the traded local martingale deflator with respect to $P'$ and let $\rho$ be the density process of $P'$ with respect to $P$, i.e. $\rho_t = E(dP'/dP|F_t)$. Recall that a process $M$ is a local martingale with respect to $P'$ if and only if $\rho M$ is a local martingale with respect to $P$. It follows that $Z := \rho/V'$ is a local martingale deflator under the measure $P$. Thus, the implication $\Rightarrow$ is a direct consequence of the previous theorem. Since a local martingale deflator is a supermartingale deflator, the implication $\Leftarrow$ of the theorem, as we already noted, is obvious.

3 Traded Numérais and Stochastic Exponentials

3.1 Local Characteristics of a Semimartingale

Here we summarize some basic facts and make some reductions allowing us to work under (slightly) simplifying hypotheses.

Let $(B^h, C, \nu)$ be the triplet predictable characteristics of the semimartingale $S$ written in the canonical form

$$S = S_0 + S^c + xh \ast (\mu - \nu) + \bar{h} \ast \mu + B^h.$$ 

Here $h$ denotes the truncation function, $\bar{h} := 1 - h$; we assume throughout the paper that $h(x) := I_{|x| \leq 1}$.

It is convenient to work with a “local form” of the triplet. One can always choose a predictable increasing cádlág process $A$ with $A_0 = 0$ and $A_T \leq 1$ such that

$$B^h = b^h \cdot A, \quad \nu(dt, dx) = dA_t K_t(dx),$$ 

$b^h$ is predictable, $K_t(dx) = K_{\omega,t}(dx)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \bar{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ with $\int (|x|^2 \wedge 1) K_{\omega,t}(dx) < \infty$; if $\Delta A_t(\omega) > 0$ then we have
Let us consider the probability density $S$ increments the measures $K$ instead of $\int Y(x)K_{\omega,t}(dx)$ and omit $\omega$ or $\omega, t$ to alleviate formulae. As usual, $S_t^\varepsilon = \sup_{s \leq t}|S_s|$. By analogy with the theory of processes with independent increments the measures $K_{\omega,t}$ are referred to as the Lévy measures.

Lemma 3.1 Let $\varepsilon > 0$. Then there exists a probability measure $\hat{P} \sim P$ with the bounded density $d\hat{P}/dP$ such that $||\hat{P} - P|| \leq \varepsilon$, and $|x|^2 * \mu_T, S_T^2, C_T^2, \var{T}B^b$ belong to $L^1(\hat{P})$.

Proof. Let us consider the probability density $Z^n_T := c_n(1 + n^{-1}U_T)^{-1}$ where the random variable $U_T := |x|^2 * \mu_T + S_T^2 + C_T^2 + \var{T}B^b$ and $c_n$ is a normalizing constant. The measures $P^n = Z^n_T P$ for sufficiently large $n$ meet the requirements. \qed

Note that the NAA11-property is invariant under an equivalent change of the probability measure. On the other hand, if $P' \sim P$ with the density process $\rho$ and $Z'$ is a local martingale deflator with respect to $P'$, then $\rho Z'$ is a local martingale deflator with respect to $P'$.

Thus, if we want to prove the existence of a local martingale deflator, is sufficient to find it under a “suitably” chosen equivalent probability measure which can be taken arbitrary close to the objective probability. According to Lemma 3.1 above, we may assume without loss of generality that $S$ is a special semimartingale of the form

$$S = S_0 + S^c + x * (\mu - \nu) + B$$

(3.1)

where $S^c \in \mathcal{M}^{2,c}$, $\langle X^c \rangle = c \cdot A, B = b \cdot A, b = b^b + K(x1_{|x|>1})$, the compensator $\nu(dt,dx) = K_1(dx)dB_t$ with $E|x|^2 * \nu_T < \infty$, and the process $S^d := x * (\mu - \nu)$ belongs to $\mathcal{M}^2$.

Comment on stochastic integration:

The construction of stochastic integral with respect to a semimartingale for non locally bounded integrands is rather involved. By definition, a semimartingale is an (adapted cadlag) process $S$ admitting the decomposition $S = M + A$ where $M$ is a local martingale and $A$ is a process of bounded variation. It seems that the natural definition of the set $L(S)$ of integrands for $S$ could be the intersections of the sets of predictable processes $L_{mar}(M)$ and $L_{Leb}(A)$, integrable correspondingly, in the sense of integration with respect to a local martingale $M$ and Lebesgue integrable with respect to $A$. The complication arises because the decomposition is not unique and the intersection depends on the chosen one. The most common definition is the following: a predictable process $H \in L(S)$ if there exists a decomposition $S = M^H + A^H$ such that $H \in L_{mar}(M^H) \cap L_{Leb}(A^H)$; in such a case $H \cdot S = H \cdot M^H + H \cdot A^H$. One should take care about seemingly innocent manipulations with stochastic integrals with respect to semimartingales: even in the case $S^c = 0$, the condition $H \in L(S)$ does not imply that $H$ is integrable with respect to the components of the canonical decomposition and the additivity of the integral is ensured.
3.2 Ratio of Stochastic Exponentials: The First Representation.

Let $X$ be a real-valued semimartingale with $\Delta X > -1$ and let $\mathcal{E}(X)$ be the corresponding stochastic exponential, i.e. the solution of the linear equation $Z = 1 + Z \cdot X$ which solution can be expressed by the Doléan-Dade formula

$$
\mathcal{E}(X) = \exp \left\{ X - X_0 - \frac{1}{2} \langle X^c \rangle + \sum_{s \leq t} [\ln(1 + \Delta X_s) - \Delta X_s] \right\},
$$

(3.2)

Note that $\mathcal{E}^{-1}(X) = \mathcal{E}(\tilde{X})$ where

$$
\tilde{X} = -X + \langle X^c \rangle + \sum_{s \leq t} \frac{(\Delta X_s)^2}{1 + \Delta X_s}.
$$

(3.3)

Indeed, $X + \tilde{X} + [X, \tilde{X}] = 0$ and it remains to use the following simple identity known as the Yor product formula

$$
\mathcal{E}(X) \mathcal{E}(\tilde{X}) = \mathcal{E}(X + \tilde{X} + [X, \tilde{X}]).
$$

For a semimartingale $X$ with $X_0 = 1$ and such that $X, X_\leq > 0$ we have that

$$
X = 1 + (X_\leq X_{-1}) \cdot X = 1 + X_\leq \cdot (X_{-1} \cdot X)
$$

and, hence, such a process admits the representation $X = \mathcal{E}(X_{-1} \cdot X)$. In particular, for the standard model where $X = 1 + H \cdot S$ we have that $X = \mathcal{E}(f \cdot S)$ where $f = X_{-1} H$.

Thus, for the standard model the set of numéraires $1 + X^1 \cdot$ coincides with the set of stochastic exponentials $\{ \mathcal{E}(f \cdot S) : f \in L(S), f \Delta S > -1 \}$.

The ratio $\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S)$ of two stochastic exponentials can be transformed, using the expression for the inverse and the Yor formula:

$$
\frac{\mathcal{E}(f \cdot S)}{\mathcal{E}(g \cdot S)} = \mathcal{E}(f \cdot S) \mathcal{E} \left( -g \cdot S + \langle g \cdot S^c \rangle + \sum_{s \leq t} \frac{(g_s \Delta S_s)^2}{1 + g_s \Delta S_s} \right) = \mathcal{E}(R).
$$

(3.4)

where

$$
R = (f - g) \cdot S + \langle g \cdot S^c, g \cdot S^c \rangle - (f - g) \frac{g_x}{1 + g_x} x * \mu.
$$

(3.5)

To make clear the idea, we assume for a moment that $f$ and $g$ are bounded processes, $K(|x|^2) \cdot A_T < \infty$, and $K(|g_x|x|(1 + g_x)) \cdot A_T < \infty$. It is easy to see that in this case we can, using the representation of $S$ and regrouping terms, arrive to the identity

$$
R = M + F(f, g) \cdot A,
$$

(3.6)

where

$$
M := (f - g) \cdot S^c + (f - g)x * (\mu - \nu) - \frac{g_x}{1 + g_x} (f - g)x * (\mu - \nu) \in \mathcal{M}_{loc},
$$
\[ F(f, g) := (f - g)(b - cg) - K \left( (f - g)x \frac{gx}{1 + gx} \right). \]  

(3.7)

From the representation (3.6) it is clear that if \( F(f, g) \leq 0 \text{ m-a.e.} \), then \( R \) is local supermartingale and \( \mathcal{E}(R) \) is a local supermartingale. For unbounded processes \( f \) and \( g \) we cannot guaranty that the above manipulations are legitimate: though both are integrable with respect to \( S \) but might not be integrable with respect to the summands \( x * (\mu - \nu) \) and \( B \) in (3.1). Nevertheless, we have the following result:

**Lemma 3.2** Suppose that \( f, g \in L(S), f \Delta S > -1, g \Delta S > -1. \) If \( F(f, g) \leq 0 \text{ m-a.e.} \), then \( Z := \mathcal{E}(R) \) is a supermartingale. If \( F(f, g) = 0 \text{ m-a.e.} \), then \( \mathcal{E}(R) \) is a local martingale and supermartingale.

**Proof.** Integrating the predictable process \( \theta := 1/(1 + |f| + |g|) \) taking values in \([0, 1]\) with respect to \( R \) and rearranging terms we obtain the representation

\[ \theta \cdot R = \theta \cdot M + \theta F(f, g) \cdot A \]

where \( \theta F(f, g) \leq 0 \text{ m-a.e.} \), and

\[ \theta \cdot M := \theta(f - g) \cdot S' + \theta(f - g)x * (\mu - \nu) - \theta \frac{gx}{1 + gx} (f - g)x * (\mu - \nu) \in \mathcal{M}_{loc}. \]

Thus, \( R \) is a \( \sigma \)-supermartingale with \( \Delta R > -1 \) and, therefore, \( R \) is a local supermartingale, see [7]. But in such a case \( \mathcal{E}(R) \) is a supermartingale. If \( \theta F(f, g) = 0 \), then the process \( R \) is a local martingale and so is \( \mathcal{E}(R) \). \( \square \)

### 3.3 The Existence of the Traded Supermartingale Deflator.

**Theorem 3.3** Suppose that \( K(|x|^2 \wedge |x|) < \infty \text{ m-a.s.} \) If the NAA1-property holds, then there exists \( g \in L(S), g \Delta S > -1, \) such that the process \( \mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S) \) is a supermartingale for every \( f \in L(S), f \Delta S > -1. \)

**Strategy of the Proof.** We take as \( g \) a predictable process such that for every \((\omega, t)\) (except \( m \)-null set) the value \( g = g(\omega, t) \) is the solution of the following maximization problems:

\[ \Psi(g) := bg - \frac{1}{2} \left| 1^{1/2} |g| \right|^2 - K(gx - \ln(1 + gx)) \to \max, \]

\[ K(gx \leq -1) = 0. \]

Of course, it is not obvious that such \( g \) does exist and belongs to \( L(S) \). The proof consists from two independent parts. The first, easier, part is to show that NAA1-property (in fact, even a weaker property) guarantees the existence of solutions \( g(\omega, t) \) of the above maximization problems forming a predictable process and for these solutions we have, “instantaneously”, as the first order necessary conditions for the maximum, the bounds \( F(f, g) \leq 0 \text{ m-a.e.} \) for every \( f \in \mathbb{R}^d \). The second, technically, more difficult part, is to show that if NAA1
the predictable process holds, then any process \( g \) satisfying the above bounds is \( S \)-integrable.

The arguments on the existence under \( NAA_1 \) are provided in Section 4 where we show also that \( F(v, g) \leq 0 \) for any \( v \in \mathbb{R}^d \) such that \( K(vx) \leq -1 \) = 0. In Section 6 we check that the predictable process \( g \) belongs to \( L(S) \).

Combining these facts with the above lemma we get the claim of the theorem. □

3.4 Ratio of Stochastic Exponentials: The Second Representation.

Let \( g \in L(S) \), \( g \Delta S > -1 \). Let us define the semimartingale

\[
\bar{S} = S - cg \cdot A - \sum_{s \leq t} \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \Delta S_s.
\]

Since \( \Delta \bar{S} = \Delta S/(1 + g \Delta S) \) we have that

\[
S = \bar{S} + cg \cdot A + \sum_{s \leq t} (g_s \Delta S_s) \Delta \bar{S}_s.
\]

**Lemma 3.4** \( L(S) = L(\bar{S}) \).

**Proof.** Let \( f \in L(S) \). Then

\[
|(f, cg)| \cdot AT \leq \frac{1}{2} |c^{1/2} f|^2 \cdot AT + \frac{1}{2} |c^{1/2} g|^2 \cdot AT < \infty,
\]

\[
\sum_{s \leq t} \frac{|g_s \Delta S_s f_s \Delta S_s|}{1 + g_s \Delta S_s} \leq \frac{1}{2} \sum_{s \leq t} \frac{|f_s \Delta S_s|^2}{1 + g_s \Delta S_s} + \frac{1}{2} \sum_{s \leq t} \frac{|g_s \Delta S_s|^2}{1 + g_s \Delta S_s} < \infty.
\]

Thus, \( L(S) \subseteq L(\bar{S}) \). To show the opposite inclusion, take \( f \in L(\bar{S}) \). The conditions \( g \in L(S) \) and \( f \in L(\bar{S}) \) imply \( f \) and \( g \) are integrable with respect to \( S^c = \bar{S}^c \), i.e. that \( |c^{1/2} g|^2 \cdot AT < \infty \) and \( |c^{1/2} f|^2 \cdot AT < \infty \). So, as above we have that \( |(f, cg)| \cdot AT \). Since also

\[
\sum_{s \leq t} |(g_s \Delta S_s)(f_s \Delta \bar{S}_s)| \leq \frac{1}{2} \sum_{s \leq t} |g_s \Delta S_s|^2 + \frac{1}{2} \sum_{s \leq t} |f_s \Delta \bar{S}_s|^2 < \infty,
\]

we get that \( f \in L(S) \), i.e. the inclusion \( L(\bar{S}) \subseteq L(S) \) holds. □

Fix \( g \in L(S) \) with \( g \Delta S > -1 \).

**Lemma 3.5** We have the identity

\[
\{ h \in L(\bar{S}) : h \Delta \bar{S} > -1 \} = \{ f \in L(S) : f \Delta S > -1 \} - g. \tag{3.8}
\]

If \( f \in L(S) \), \( f \Delta S > -1 \), then

\[
\frac{\mathcal{E}(f : S)}{\mathcal{E}(g : S)} = \mathcal{E}((f - g) \cdot \bar{S}). \tag{3.9}
\]
Proof. Let \( h := f - g \) belongs to the set in the rhs of (3.8). Then \( h \in L(S) = L(\bar{S}) \) and
\[
h \Delta S = (f - g) \Delta S = \frac{(f - g) \Delta S}{1 + g \Delta S} = \frac{1 + f \Delta S}{1 - g \Delta S} - 1 > -1.
\]
That is, \( f - g \) belongs to the set in the lhs of (3.8). On the other hand, let \( h \) belongs to the lhs of (3.8). Then \( f := h + g \) belongs to \( L(S) \). Substituting the expression \( \Delta S = \Delta \bar{S}/(1 - \Delta \bar{S}) \) we get that
\[
f \Delta S = (h + g) \Delta \bar{S} = \frac{(h + g) \Delta \bar{S}}{1 - g \Delta \bar{S}} = \frac{1 + h \Delta \bar{S}}{1 - g \Delta \bar{S}} - 1 > -1.
\]
Therefore, \( h \) belongs to the rhs of (3.8).

The formula (3.9) follows directly from (3.4), (3.5), and the definition of \( \bar{S} \). ✷

Remark. The statement of Lemma 3.5 implies that
\[
1 + X_1^2(\bar{S}) = \mathcal{E}^{-1}(g \cdot S)(1 + X_1^2(S)).
\] (3.10)

It is easy to deduce from here that
\[
1 + X_1^2(\bar{S}) = \mathcal{E}^{-1}(g \cdot S)(1 + X_1^2(S)).
\] (3.11)

Indeed, let \( 1 + H \cdot \bar{S} \geq 0 \). Then \( 1 + (1/2)H \cdot \bar{S} > 0 \) and, in virtue of (3.10), there is \( \bar{H} \) in \( L(S) \) such that \( 1 + \bar{H} \cdot S > 0 \) and \( \mathcal{E}(g \cdot S)(2 + \bar{H} \cdot S) = 2(1 + \bar{H} \cdot S) \).

It follows that
\[
\mathcal{E}(g \cdot S)(1 + H \cdot \bar{S}) = \mathcal{E}(g \cdot S)(1 + (2H - \mathcal{E}(g \cdot S)) \cdot S) = 1 + X_1^2(S).
\]
Thus, we have the inclusion "\( \subseteq \)". Since \( \mathcal{E}(-g \cdot \bar{S}) = \mathcal{E}^{-1}(g \cdot S) \), by same arguments work in the proof of the opposite inclusion.

3.5 Existence of the Traded Local Martingale Deflator: the Proof of Main Theorem.

Proof Theorem 2.10. Due to Lemma 3.4 we may assume without loss of generality that \( K(|x|^2) \cdot A < \infty \). According to Theorem 3.2 there is \( g \in L(S) \) such that \( g \Delta S > -1 \) such that the ratio of stochastic exponentials \( \mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S) \) is a supermartingale for every \( f \in L(S) \), \( f \Delta S > -1 \). It follows from the above lemma that \( \mathcal{E}(h \cdot \bar{S}) \) is a supermartingale for any \( h \in L(\bar{S}) \), \( h \Delta \bar{S} > -1 \). Thus, \( EH \cdot \bar{S}_T \leq 0 \) for every \( H \in L(\bar{S}) \) such that \( H \cdot S > 0 \) and \( H \cdot S_- > 0 \). It follows that \( EH \cdot \bar{S}_T \leq 0 \) for any admissible integrand of \( \bar{S} \). Hence, \( P \) is the separating measure for \( \bar{S} \) and, by virtue of Theorem A.1 in any neighborhood of \( P \) there exists a \( \sigma \)-martingale measure \( \tilde{P} \). With respect to \( \tilde{P} \) the stochastic exponential \( \mathcal{E}(h \cdot \bar{S}) \) is a \( \sigma \)-martingale, hence, a local martingale. So, the process \( \mathcal{E}(g \cdot S) \) is the market index with respect to \( \tilde{P} \). ✷
Remark. To apply the Theorem 3.2 we need to make first a change to probability measure. Making a direct reference to results of [21] or [24] we can get a more precise result: the NAA1 condition implies that the traded supermartingale deflator $E(g \cdot S)$ does exist and in any neighborhood of $P$ there is a probability measure $\tilde{P}$ under which $E(g \cdot S)$ is a traded local martingale deflator.

4 Supermartingale Numeraire Portfolios

The property NAA1 implies NA in the class of strategies with non-negative value processes, i.e. there is no $H \in L(S)$ such that $H \cdot S \geq 0$ but $H \cdot S_T \neq 0$.

Define the set-valued mappings

$$(\omega, t) \mapsto N_{\omega, t} := \{ v \in \mathbb{R}^d : K_{\omega, t}(vx) \neq 0, \ c_t(\omega)v = 0, \ vb_t(\omega) = 0 \}$$

and

$$(\omega, t) \mapsto J_{\omega, t} := \{ v \in \mathbb{R}^d : K_{\omega, t}(vx) < 0, \ c_t(\omega)v = 0, \ vb_t(\omega) \geq K_{\omega, t}(vx) \} \setminus N_{\omega, t}.$$ 

The graphs of these mappings are $\bar{P} \otimes B(\mathbb{R}^d)$-measurable.

Lemma 4.1 If NAA1, then $m(\text{dom } J^k) = 0$.

Proof. Let $H$ be a $\bar{P} \otimes B(\mathbb{R}^d)$-measurable selector of the set-valued mapping

$$(\omega, t) \mapsto J^k_{\omega, t} := J_{\omega, t} \cap \{ v : |v| \leq r \}$$

extended by the value $0 \in \mathbb{R}^d$ outside the set $\text{dom } J^k$. Then $H \cdot S \geq 0$. Indeed,

$$EHxI_{\{Hx < 0\}} \ast \mu_T = EHxI_{\{Hx < 0\}} \ast \nu_T = EHxI_{\{Hx < 0\}} \cdot A_T = 0$$

and, therefore,

$$H \cdot S = HxI_{\{Hx > n^{-1}\}} \ast \mu + HxI_{\{Hx \leq n^{-1}\}} \ast (\mu - \nu) + (bx - K(HxI_{\{Hx > n^{-1}\}})) \cdot A.$$ 

Note that the first and the third terms in the right-hand are non-negative, while, by the Doob inequality,

$$E\sup_{t \leq T} (HxI_{\{Hx \leq n^{-1}\}} \ast (\mu - \nu)t)^2 \leq 4E(HxI_{\{Hx \leq n^{-1}\}} \ast (\mu - \nu)t)^2 \leq 4E|x|^2 I_{\{Hx \leq n^{-1}\}} \ast \nu_T \rightarrow 0, \ n \rightarrow \infty.$$ 

Thus, $H \cdot S \geq 0$. If $m(\text{dom } J^k) > 0$, then either

$$EHx \ast \mu_T = EHx \ast \nu_T = EHxI_{\{Hx > 0\}} \cdot A_T > 0,$$

or

$$E(bx - K(Hx)) \cdot A_T > 0.$$ 

It follows that $H \cdot S_T > 0$ in contradiction with NAA1. So, $m(\text{dom } J^k) = 0$ implying that $m(\text{dom } J) = 0$. \[\square\]
Let us consider the set-valued mapping
\[(ω, t) ↦ F_{ω, t} := \{ v^0 ∈ R^d : v^0 ∈ \text{argmax}_D \Psi_{ω, t}(v) \}\]
where
\[D_{ω, t} := \{ v ∈ R^d : K_{ω, t}(vx) ≤ -1 \} = 0 \]
and
\[Ψ_{ω, t}(v) := b(t)v - \frac{1}{2} |c|^{1/2}(ω)v^2 - K_{ω, t}(vx - \ln(1 + vx)) \].

This mapping is $\bar{P} ⊗ B(R^d)$-measurable and has, due to Proposition 7.2, non-empty sections modulo $m$-null set. Take its measurable selector $g$. According to the inequality (7.4) (with $v = 0$ and $v^0 = g$ we have that
\[-g(b - cg) - K(\frac{(gx)^2}{1 + gx}) ≤ 0 \quad m\text{-a.e.}
\]
This bound holds also when $g$ is replaced by $g^n := gI_{\{|g| ≤ n\}}$, i.e. (5.1) holds. In virtue of ?? the processes $E^{-1}(g^n · S)$ are supermartingales. The NAA1 condition ensures that $E(g^n · S)$ the sequence is $P$-bounded and, in virtue of Proposition 6.2, so is the sequence $g^n · S$. Thus, the hypotheses of Proposition 6.2 are fulfilled and $g ∈ L(S)$.

5 Boundedness in Probability of Stochastic Exponentials

For a scalar semimartingale $X$ such that $X_0 = 1$ and $X, X_- > 0$ we define the stochastic logarithm as the semimartingale $L(X) := X^{-1} · X$. Note that $ΔL(X) > -1$. It easily seen that $X = E(L(X))$ and $R = L(E(R))$ for every semimartingale $R$ with $R = 0$ and $ΔR > -1$.

Let $\mathcal{R}$ be a set of real-valued semimartingales $R$ with $R_0 = 0$, $ΔR > -1$. We denote by $R_T$ and $E_T(R)$ the sets of terminal random variables $R_T$ and $E_T(R)$ of processes $R ∈ \mathcal{R}$. We write that $\mathcal{R}$ is $P$-bounded from above if the set of random variables $\sup_{s ≤ T} R_s$ is $P$-bounded. In the same spirit we define for set of semimartingales the notions ”$P$-bounded from below” and ”$P$-bounded”.

Since
\[E_t(R) = \exp \left\{ R_t - \frac{1}{2} (R^c)_t + \sum_{s ≤ t} [\ln(1 + ΔR_s) - ΔR_s] \right\}, \quad (5.1)\]
we have the inequality $E(R) ≤ e^R$ and, therefore, if the set of process $\mathcal{R}$ is $P$-bounded, so is the set $E(\mathcal{R}) := \{ E(R) : R ∈ \mathcal{R} \}$. Converse, in general, may not be true but one can prove the following result of independent interest (cf. Lemma A.4 in [21]).

**Proposition 5.1** Let $\mathcal{R}$ be a set of real-valued semimartingales $R$ such that $R_0 = 0$, $ΔR > -1$, and $E^{-1}(R)$ is a supermartingale. Put $Z := L(E^{-1}(R))$. Let us introduce the following conditions:

(a) $\mathcal{R}$ is $P$-bounded;
(a′) \( \mathcal{R} \) is \( P \)-bounded from above;
(a′′) \( \mathcal{R}^T \) is \( P \)-bounded;
(b) \( \mathcal{E}_T(\mathcal{R}) \) is \( P \)-bounded;
(c) \( \mathcal{E}(\mathcal{R}) \) is \( P \)-bounded;
(d) \( \mathcal{Z} \) is \( P \)-bounded;
(d′) \( \mathcal{Z} \) is \( P \)-bounded from below.

Then

\[ (a) \Leftrightarrow (a′) \Leftrightarrow (a′′) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (d′). \]

**Proof.** The implications \((a) \Rightarrow (a′), (a) \Rightarrow (a′′), (c) \Rightarrow (b), (d) \Rightarrow (d′)\) are trivial and \((a′′) \Rightarrow (b)\) in virtue of the bound \( \mathcal{E}_T(\mathcal{R}) \leq e^{R_T} \).

\((a′) \Rightarrow (a)\) Since \( \mathcal{E}^{-1}(R) \) is a supermartingale when \( R \in \mathcal{R} \), we have by the Kolmogorov inequality that

\[ P\left( \inf_t \mathcal{E}_t(R) \leq n^{-1} \right) = P\left( \sup_t \mathcal{E}^{-1}_t(R) \geq n \right) \leq n^{-1}. \]

It follows that the set \( \ln \mathcal{E}(\mathcal{R}) \) is \( P \)-bounded from below. But \( R \geq \ln \mathcal{E}(\mathcal{R}) \).

So, the set \( \mathcal{R} \) is always \( P \)-bounded from below under the assumption of the lemma.

\((b) \Rightarrow (c)\) If \((c)\) fails, there are \( \varepsilon > 0 \) and \( R^n \in \mathcal{R} \) such that \( \mathcal{E}_{\tau^n}(R^n) \geq n \) and \( P(\tau^n < T) \geq \varepsilon \). Using the abbreviation \( M^n := \mathcal{E}^{-1}(R^n) \) we have, applying the Chebyshev inequality and the supermartingale property, that

\[
P(M^n_T \geq n^{-1/2}) = P(M^n_T / M^n_{\tau^n} \geq n^{-1/2}/M^n_{\tau^n}, \tau^n < T) + P(M^n_T \geq n^{-1/2}, \tau^n = T)
\]

\[
\leq P(M^n_T / M^n_{\tau^n} \geq n^{1/2}) + P(\tau^n = T)
\]

\[
\leq n^{-1/2} + P(\tau^n = T) \leq 1 - \varepsilon/2
\]

for all \( n \) sufficiently large. Thus, \( P(\mathcal{E}_T(R^n) > n^{1/2}) \geq \varepsilon/2 \) in contradiction with \((b)\).

\((d′) \Rightarrow (d)\) Take arbitrary \( \varepsilon > 0 \). The set \( \mathcal{Z} \) being bounded from below, there is \( N_0 > 0 \) such that

\[
\sup_{Z \in \mathcal{Z}} P\left( \inf_t Z_t \leq -N + 1 \right) \leq \varepsilon \quad \forall N \geq N_0.
\]

Omitting the dependence on \( N \) we define the stopping time

\[ \tau_Z := \inf\{t \geq 0 : Z_t \leq -N + 1 \}. \]

Then \( P(\tau_Z < T) \leq \varepsilon \). Since \( \Delta Z > -1 \), the local supermartingale \( Z^{\tau_Z} \), being bounded from below, is a supermartingale, and, by the Kolmogorov inequality (applied to the supermartingale \( Z^{\tau_Z} + N \geq 0 \)) we have:

\[
P\left( \sup_t Z_t \geq N/\varepsilon \right) \leq P(\tau_Z < T) + P\left( \sup_t Z^{\tau_Z}_t \geq N/\varepsilon \right) \leq \varepsilon + 1/(1 + 1/\varepsilon) \leq 2\varepsilon.
\]

It follows that \( \mathcal{Z} \) is also \( P \)-bounded from above, i.e. \((d)\) holds.

\((c) \Rightarrow (d′)\) Note

\[
\{\mathcal{E}(R) : R \in \mathcal{R}\} = \{\exp\{-\ln \mathcal{E}(Z)\} : Z \in \mathcal{Z}\}
\]

(5.2)
and \( \ln \mathcal{E}(Z) \leq Z \). Since \( \mathcal{E}(\mathcal{R}) \) is \( P \)-bounded, so is the set \( \{ e^{-Z} : Z \in \mathcal{Z} \} \), and \((d')\) holds.

(c) \( \Rightarrow \) (a) For \( Z = \mathcal{L}(\mathcal{E}^{-1}(\mathcal{R})) \) we have, using the formula for the inverse of the stochastic exponential, that

\[
Z = -R + \langle R^c \rangle + \sum_{s \leq T} \frac{(\Delta R_s)^2}{1 + \Delta R_s},
\]

(5.3)

On the other hand,

\[
\ln \mathcal{E}(Z) = -\ln \mathcal{E}(R) = -R + \frac{1}{2} \langle R^c \rangle + \sum_{s \leq T} (\Delta R_s - \ln(1 + \Delta R_s)).
\]

Hence,

\[
Z - \ln \mathcal{E}(Z) = \frac{1}{2} \langle R^c \rangle + \sum_{s \leq T} \left( \ln(1 + \Delta R_s) - \frac{\Delta R_s}{1 + \Delta R_s} \right).
\]

(5.4)

We shown already that \((c)\) ensures that the set \( \mathcal{Z} \) is \( P \)-bounded, and, in virtue of (5.2), the set \( \{ \ln \mathcal{E}(Z) : Z \in \mathcal{Z} \} \) is \( P \)-bounded from below. So, the set of random variables

\[
\Gamma_1 := \left\{ \frac{1}{2} \langle R^c \rangle_T + \sum_{s \leq T} \left( \ln(1 + \Delta R_s) - \frac{\Delta R_s}{1 + \Delta R_s} \right), R \in \mathcal{R} \right\}
\]

is \( P \)-bounded. The property \((a)\) follows from (5.3) because the set \( \Gamma_1 \) and the set

\[
\Gamma_2 := \left\{ \langle R^c \rangle_T + \sum_{s \leq T} \frac{(\Delta R_s)^2}{1 + \Delta R_s}, R \in \mathcal{R} \right\}
\]

are \( P \)-bounded simultaneously, the fact requiring some comments. Of course, the \( P \)-boundedness of \( \Gamma_2 \) implies the \( P \)-boundedness of \( \Gamma_1 \) because

\[
\varphi(y) := \ln(1 + y) - \frac{y}{1 + y} \leq \psi(y) := \frac{y^2}{1 + y}, \quad y > -1.
\]

More surprising is the converse implication needed in the proof. To check it, suppose that \( \Gamma_1 \) is \( P \)-bounded. Then the set

\[
\left\{ \langle R^c \rangle_T + \sum_{s \leq T} \frac{(\Delta R_s)^2 I_{\{\Delta R_s \leq 2\}}}{1 + \Delta R_s}, R \in \mathcal{R} \right\}
\]

is \( P \)-bounded (due to the inequality \( \varphi(y) \geq (1/4)\psi(y) \) for \( y \in [-1, 2] \)). Using the bound \( \sup \varphi(\Delta R_s) \leq \sum \varphi(\Delta R_s) \), we infer that \( \{ \sup_{s \leq T} I_{\{\Delta R_s \leq 2\}} \ln(1 + \Delta R_s), R \in \mathcal{R} \} \) is a \( P \)-bounded set implying that also \( \{ \sup_{s \leq T} \Delta R_s I_{\{\Delta R_s > 2\}}, R \in \mathcal{R} \} \) is \( P \)-bounded. Noting that

\[
\sup_{s \leq T} (\Delta R_s)^2 I_{\{\Delta R_s \leq 2\}} \sum_{s \leq T} \left( \ln(1 + \Delta R_s) - \frac{\Delta R_s}{1 + \Delta R_s} \right) I_{\{\Delta R_s \leq 2\}} \geq \sum_{s \leq T} (\Delta R_s)^2 \left( 1 - \frac{\Delta R_s}{1 + \Delta R_s} \right) I_{\{\Delta R_s > 2\}}
\]
and the set of random variables in the left-hand side of this inequality when \( R \) runs \( \mathcal{R} \) is \( P \)-bounded we get that the set
\[
\left\{ \sum_{s \leq T} \frac{(\Delta R_s)^2}{1 + \Delta R_s} I_{\{\Delta R_s > 2\}}, \ R \in \mathcal{R} \right\}
\]
is \( P \)-bounded and so is \( I_2 \). \( \square \)

6 Integrability

Recall the criterion of the integrability of a vector-valued process with respect to a vector-valued semimartingale, see (29) and (10).

**Proposition 6.1** A predictable process \( H \in L(S) \) if and only if
\[
\left( |c^{1/2}g|^2 + K(|gx|^2 \wedge 1) + |gb^h - K(gxI_{\{|x| \leq 1, \ |gx| > 1\}})| \right) \cdot A_T < \infty.
\]
In the case where \( S \) is such that \( K(gx \leq -1) = 0 \ m\text{-}a.e. \) and \( K(|x|I_{\{|x| > 1\}}) < \infty \), the coefficient \( b^h = b - K(gxI_{\{|x| > 1\}}) \) and the last relation can be written as
\[
\left( |c^{1/2}g|^2 + K(|gx|^2 \wedge 1) + |gb - K(gx(1 - I_{\{|x| \leq 1, \ |gx| \leq 1\}}))| \right) \cdot A_T < \infty.
\]

**Proposition 6.2** Let \( g \) be a \( d \)-dimensional process such that \( K(gx \leq -1) = 0 \ m\text{-}a.e. \) and let \( g^n := gI_{\{|g| \leq n\}} \). Suppose that for every \( n \geq 1 \)
\[
-g^n(b - cg^n) + K\left( \frac{(g^n x)^2}{1 + g^n x} \right) \leq 0 \ m\text{-}a.e. \tag{6.1}
\]
and the sequence of random variables \( g^n \cdot S_T \) is bounded in probability. Then \( g \in L(S) \).

Proof. Using the decomposition
\[
S = S_0 + S^c + xI_{\{|x| \leq 1\}} * (\mu - \nu) + xI_{\{|x| > 1\}} * \mu + b^h \cdot A
\]
and taking into account that
\[
b^h g^n + K(g^n xI_{\{|x| > 1\}}) = bg^n \geq |c^{1/2}g^n|^2 + K\left( \frac{(g^n x)^2}{1 + g^n x} \right),
\]
we get that
\[
g^n \cdot S_T \geq \sum_{j=1}^{5} I^n_j
\]
where
\[ I_n^1 := g_n \cdot S_n^2 + |c^{1/2}g_n|^2 \cdot A_T, \]
\[ I_n^2 := g_n^3 x I_{|x| \leq 1} \cdot \mu \cdot x + K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| \leq 1} \right) \cdot A_T, \]
\[ I_n^3 := g_n^3 x I_{|x| \leq 1} \cdot \mu \cdot x + K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| \leq 1} \right) \cdot A_T, \]
\[ I_n^4 := g_n^3 x I_{|x| > 1} \cdot \mu \cdot x + K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| > 1} \right) \cdot A_T, \]
\[ I_n^5 := g_n^3 x I_{|x| > 1} \cdot \mu \cdot x + K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| > 1} \right) \cdot A_T. \]

1. In virtue of Lemma [B.1] the sequence \( I_n^1 \) diverges to \( +\infty \) a.s. on the set
\[ \{|e^{1/2}g|^2 \cdot A_T = \infty\} \]
and is bounded on its component.

2. Since
\[ K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| \leq 1} \right) \cdot A_T \geq \frac{1}{2} K \left( (g_n^3 x)^2 I_{|x| \leq 1} \right) \cdot A_T \]
\[ \geq \frac{1}{2} g_n^3 x I_{|x| \leq 1} \cdot (\mu - \nu) \cdot A_T, \]
we infer, again from Lemma [B.1] that the sequence \( I_n^2 \) diverges to \( +\infty \) a.s. on the set
\[ \{(g_n x)^2 I_{|x| \leq 1} \cdot \nu_T = \infty\} \]
and is bounded on its complement.

3. Rearranging terms we rewrite \( I_n^3 \) as follows:
\[ I_n^3 = I_n^3,1 + I_n^3,2 \]
where
\[ I_n^3,1 := g_n^3 x I_{|x| \leq 1, 1 < g_n \leq 2} \cdot (\mu - \nu) \cdot A_T \]
\[ I_n^3,2 := g_n^3 x I_{|x| \leq 1, g_n > 2} \cdot \mu \cdot x - K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| \leq 1} \right) \cdot A_T. \]

Note that
\[ (1/3) K \left( (g_n^3 x)^2 I_{|x| \leq 1, 1 < g_n \leq 2} \right) \cdot A_T \leq K \left( (g_n^3 x)^2 / (1 + g_n^3 x) I_{|x| \leq 1, 1 < g_n \leq 2} \right) \cdot A_T \]
\[ \leq (1/2) K \left( (g_n^3 x)^2 I_{|x| \leq 1, 1 < g_n \leq 2} \right) \cdot A_T, \]
and we infer from Lemma [B.1] that the sequence \( I_n^3,1 \) diverges to \( +\infty \) a.s. on the set \( \{K \left( I_{|x| \leq 1} \right) \cdot A_T = \infty\} \) and is bounded on its complement. Also
\[ I_n^3,2 \geq 2 I_{|x| \leq n, 1 < g_n \leq 2} \cdot \mu \cdot x - K \left( I_{|g| \leq n} I_{|x| \leq 1, g_n > 2} \right) \cdot A_T \]
and Lemma [B.2] implies that the sequence \( I_n^3,2 \) diverges to \( +\infty \) a.s. on the set
\[ \{K \left( I_{|x| \leq 1} \right) \cdot A_T = \infty\} \]
and is bounded from below on its complement.
4. The sequence $I^n_4$ is bounded from below and diverges to $+\infty$ a.s. on the set 
\[ \{ K((gx)^2 I_{\{|x|>1, |gx| \leq 1\}}) \cdot A_T = \infty \}. \]
This follows from the estimates
\[ |g^n x I_{\{|x|>1, |gx| \leq 1\}} \ast (\mu - \nu)_T | \leq I_{\{|x|>1\}} \ast (\mu + \nu)_T < \infty \]
and
\[ K\left((g^n x)^2/(1 + g^n x) I_{\{|x|>1, |gx| \leq 1\}}\right) \cdot A_T \geq \frac{1}{2} K\left((g^n x)^2 I_{\{|x|>1, |gx| \leq 1\}}\right) \cdot A_T. \]

5. Note that the sequence $|I^n_5|$ is bounded by a finite random variable. Indeed,
\[ |I^n_5| \leq g^n x I_{\{|x|>1, |gx| > 1\}} \ast \mu_T + K\left(g^n x/(1 + g^n x) I_{\{|x|>1, |gx| > 1\}}\right) \cdot A_T \]
where the first integral is dominated by the integral $gx I_{\{|x|>1, |gx| > 1\}} \ast \mu_T$ (which is just a finite sum) while the second is dominated by $K\left(I_{\{|x|>1, |gx| > 1\}}\right) \cdot A_T < \infty$.

Combining the above facts we obtain that the inequality (6.1) implies that
\[ |c^{1/2}g|^2 \cdot A_T + K\left((gx)^2 \land 1\right) \cdot A_T < \infty. \]
Now we check that
\[ |gb - K\left(gx(1 - I_{\{|x| \leq 1, |gx| \leq 1\}})\right)| \cdot A_T < \infty \]
or, equivalently, that
\[ |K\left(gx/(1 + gx) - gx I_{\{|x| \leq 1, |gx| \leq 1\}}\right)| \cdot A_T < \infty, \quad (6.2) \]
By above
\[ K\left((gx)^2/(1 + gx) I_{\{|x| \leq 1, |gx| \leq 1\}}\right) \cdot A_T \leq \frac{1}{2} K\left((gx)^2 I_{\{|gx| \leq 1\}}\right) \cdot A_T < \infty \]
and
\[ K\left(gx/(1 + gx) I_{\{|gx|>1\}}\right) \cdot A_T \leq K\left(I_{\{|gx|>1\}}\right) \cdot A_T < \infty. \]
Also
\[ K\left(gx/(1 + gx) I_{\{|x|>1, 0 \leq |gx| \leq 1\}}\right) \cdot A_T \leq K\left(I_{\{|x|>1\}}\right) \cdot A_T < \infty, \]
Finally,
\[ K\left(|gx|/(1 + gx) I_{\{|x|>1, -1 \leq |gx| \leq 0\}}\right) \cdot A_T < \infty, \]
because in the opposite case the sequence $I^n_4$ will diverges to infinity on the set of positive probability.

These observations show that (6.2) holds and, therefore, $H \in L(S). \quad \square$
7 Maximization Problem

We suppose that $K(|x|^2 \land |x|) < \infty$.

Let us consider the convex set $D := \{ v \in \mathbb{R}^d : vx > -1 \text{-a.e.} \}$ with the closure $\bar{D} = \{ v \in \mathbb{R}^d : vx \geq -1 \text{-a.e.} \}$. In particular, if $K = 0$, then $D = \mathbb{R}^d$.

Since $y - \ln(1 + y) \geq 0$ for $y \geq -1$, the concave function

$$
\Psi(v) := bv - \frac{1}{2}|c^{1/2}v|^2 - K(vx - \ln(1 + vx)).
$$

(7.1)

is well defined on $\bar{D}$ though it may take the value $-\infty$ even on $D$.

Let us consider the linear subspace $N := \{ v \in \mathbb{R}^d : vx = 0 \text{-a.e.}, cv = 0, vb = 0 \}$ and the cone $J := \{ v \in \mathbb{R}^d : vx \geq 0 \text{-a.e.}, cv = 0, vb \geq K(vx) \} \setminus N$.

Note that $\Psi^0 := \sup_{v \in D \setminus N^+} \Psi(v) = \sup_{v \in D} \Psi(v) \geq \Psi(0) = 0$.

Let $u, v \in D$ be such that $\Psi(u), \Psi(v) > -\infty$. The concave function

$$
\lambda \mapsto \Psi((1 - \lambda)u + \lambda v) = \Psi(u + \lambda(v - u)), \quad \lambda \in [0, 1]
$$

is finite and has the right derivatives on $[0, 1]$. In particular, its right derivative at zero is the directional derivative $D\Psi(u; v - u)$ of $\Psi$ at $u$ in the direction $v - u$. If one can differentiate under the sign of the integral in (7.1), then

$$
D\Psi(u; v - u) = (v - u)\Psi'(u) = (v - u)\left(b - cu - K\left(x \frac{ux}{1 + ux}\right)\right).
$$

The well-known sufficient condition for such an operation: the derivatives of the integrand at the with respect to the parameter are bounded by an integrable function. In the general, the directional derivative dominates the right-hand side of the above formula. Indeed, by the Jensen inequality applied for the convex function $\phi(u) := y - \ln(1 + y)$ we have that

$$
\frac{\phi((1 - \lambda)ux + \lambda vx) - \phi(ux)}{\lambda} \leq \phi(vx) - \phi(ux) \leq \phi(vx).
$$

Since $\Psi(v) > -\infty$, the function $x \mapsto \phi(vx)$ is integrable and the Fatou lemma can be applied:

$$
\limsup_{\lambda \downarrow 0} \frac{K(\phi(ux + \lambda(v - u)x) - \phi(ux))}{\lambda} \leq K\left((v - u)x \frac{ux}{1 + ux}\right).
$$

Thus,

$$
D\Psi(u; v - u) \geq (v - u)(b - cu) - K\left((v - u)x \frac{ux}{1 + ux}\right).
$$

(7.2)
Lemma 7.1 \( J = \{ v \in \mathbb{R}^d : vx \geq 0 \} \cap N \). 

Proof. If \( vx \geq 0 \) \( K \)-a.e., then we can integrate under the sign of the integral and

\[
D\Psi(av; -av) = -a\left(bv - a|c^{1/2}v|^2 - K\left(vx - \frac{vx}{1 + avx}\right)\right).
\]

The inclusion \( \subseteq \) is obvious. On the other hand, the expression in the brackets is greater or equal to zero for arbitrary large \( a \) only if \( cv = 0 \) and \( bv \geq K(vx) \).
\( \square \)

Proposition 7.2 Let \( J = \emptyset \). Then there is \( v^0 \in D \cap N^\perp \) such that

\[
\Psi(v^0) = \sup_{v \in D} \Psi(v) < \infty. \tag{7.3}
\]

For any point \( v^0 \in D \) at which the supremum above is attained

\[
F(v, v^0) := (v - v^0)(b - cv^0) - K\left((v - v^0)x - \frac{v^0x}{1 + v^0x}\right) \leq 0 \quad \forall v \in D. \tag{7.4}
\]

Proof. For the case \( K = 0 \) the claim is obvious. Let \( v_n \in D \cap N^\perp \) form a sequence such that \( \Psi(v_n) \to \Psi^0 \). If this sequence contains a bounded subsequence, we may assume that the latter converges to some point \( v^0 \in D \cap N^\perp \).

By the Fatou lemma

\[
\liminf_n K(v_n x - \ln(1 + v_n x)) \geq K(\liminf_n (v_n x - \ln(1 + v_n x))) = K(v^0 x - \ln(1 + v^0 x))
\]

and, therefore, \( \lim_n \Psi(v_n) \leq \Psi(v^0) \), i.e. the supremum is attained at \( v^0 \). Since \( \Psi^0 \geq 0 \), we have that \( v^0 \in D \cap N^\perp \).

Let us check that the sequence \( |v_n| \) does not converge to infinity. Indeed, if it is the case, we may assume that the normalized sequence \( \tilde{v}_n := v_n/|v_n| \) converges to some point \( \tilde{v} \) with \( |\tilde{v}| = 1 \). Since \( v_n x \geq -1 \) implies that \( \tilde{v}_n x \geq -1/|v_n| \), we have that \( \tilde{v}_n x \geq 0 \) \( K \)-a.e.

Without loss of generality we may assume that the function \( \lambda \mapsto \Psi(\lambda v_n) \) defined on \([0, 1]\) attains its maximum at \( \lambda = 1 \) (otherwise we replace \( v_n \) by the point \( v'_n = \lambda_n v_n \) where \( \lambda_n \) is the point where this function attains its maximum). Thus, for arbitrary \( a > 0 \) we have that \( D(a\tilde{v}_n; -a\tilde{v}_n) \leq 0 \) when \( |v_n| \geq a \). Hence,

\[
b\tilde{v}_n - a|c^{1/2}\tilde{v}_n|^2 - K\left(\tilde{v}_n x - \frac{\tilde{v}_n x}{1 + a\tilde{v}_n x}\right) \geq 0
\]

implying that

\[
b\tilde{v} - a|c^{1/2}\tilde{v}|^2 - K\left(\tilde{v} x - \frac{\tilde{v} x}{1 + a\tilde{v} x}\right) \geq 0
\]

and we conclude by the lemma above that \( \tilde{v} \in J \) in contradiction with the assumption \( J = \emptyset \).

At any point \( v^0 \in D \) where \( \Psi \) attains its maximum we have that \( D\Psi(v^0, v - v^0) \leq 0 \) for every \( v \in D \) and, therefore, \( \text{(7.4)} \) holds in virtue of \( \text{(7.2)} \). \( \square \)
8 Discrete Lévy Measures

Let $K$ be a linear combination of Dirac measures, i.e. the unit masses placed at points $x_i \in \mathbb{R}^d \setminus \{0\}$, $i = 1, ..., N$. In symbols,

$$K(dx) = \sum_{i=1}^{N} a_i \delta_{x_i}(dx).$$

Let us consider the concave function

$$\Psi(v) = bv - \frac{1}{2} |c|^{1/2} v^2 - \sum_{i=1}^{N} a_i [x_i v - \ln(1 + x_i v)].$$

defined on the open convex polyhedron $D := \{v \in \mathbb{R}^d : vx_i > -1, \ i = 1, ..., N\}$. Its derivative (gradient) is given by the formula

$$\Psi'(v) = b - cv - \sum_{i=1}^{N} a_i \frac{x_i v}{1 + x_i v}.$$  

If $\Psi$ attains its minimum at some point $v_0$, then its derivative at this point is zero, i.e. $\Psi'(v_0) = 0$.

Proposition 8.1 Suppose that for any $v$ such that $vc \neq 0$ we have the inequalities

$$\sum_{i=1}^{N} a_i x_i v \geq bv \text{ if } \min_i x_i v \geq 0 \text{ and } \sum_{i=1}^{N} a_i x_i v \leq bv \text{ if } \max_i x_i v \leq 0$$

and these inequalities are strict except the case where all $vx_i$ are zero. Then the function $\varphi$ attains its minimum on $D$ and the equation $\Psi(v)$ has a solution.

The proof is based on the following general result on a behavior of concave functions.

Lemma 8.2 Let $\varphi : D \to \mathbb{R}$ be a concave function defined on an open subset $D$ containing the origin. For any $v \in \mathbb{R}^d$ we consider the function $f(\lambda; v) := \varphi(\lambda v)$ defined on the interval $\lambda v := \{\lambda : \lambda v \in D\}$. Suppose that $\varphi(0) = 0$ and for each $v$ the function $f(\lambda; v)$ tends to $-\infty$ as $\lambda \to \lambda_v$ or $\lambda \to \lambda^v$. Then $\varphi$ attains its maximum on $D$.

Proof. If sup $\varphi(v) = 0$, there is nothing to prove. Suppose that there exists $v \in D$ such that $a := \varphi(v_0) > 0$. Let us show that the set $D_a := \{v \in D : \varphi(v) \geq a\}$ is bounded. Take $r > 0$ such that $\{v : |v| \leq r\} \subset D$. Suppose that $D$ is not bounded i.e. there is a sequence of $v_n \in D_a$ with $|v_n| \to \infty$. Without loss of generality, we may assume that $rv_n/|v_n| \to v$. By the growth assumption, if $f(\lambda, v)$ is not equal to zero, there is $\lambda > 0$ such that $\lambda v \in D$ and $\varphi(\lambda v) \leq -1$. On the other hand, since $\varphi(0) = 0$, $\varphi(v_n) \geq a$, and the function $\varphi$ is concave, we have the bound $\varphi(\lambda r v_n/|v_n|) \geq 0$ when $\lambda r/|v_n| \leq 1$. The concave function $\varphi$ is continuous on its effective domain, i.e. on $D$. Thus, $\varphi(\lambda v) \geq 0$. A contradiction.

Moreover, $D_a$ is closed. We need to consider only the situation where $v_n \in D_a$ and $v_n \to v$ with $v \in \partial D$. Let $U \subset D$ be an open neighborhood of the
origin. For any \( \lambda \in [0, 1] \) we have that \( \lambda v + (1 - \lambda)U \subset \hat{D} \). It follows that the point \( \lambda v \) belongs to the interior of \( D \), that is \( \lambda v = 1 \). We can take \( \lambda \) sufficiently close to unit to ensure that \( f(\lambda v) \leq -1 \) while due to concavity \( \varphi(\lambda v_n) \geq \lambda \varphi(v_n) \geq 0 \) and we get a contradiction with the continuity of \( \varphi \) at the point of \( \lambda v \). The supremum of \( \varphi \) on \( D \) coincides with the supremum on the set \( D_0 \) where it is attained since the latter set is compact and the function \( \varphi \) on it is continuous. \( \square \)

**Proof of Proposition 8.1.** We analyze the behavior of the function \( f(\lambda; v):= \varphi(\lambda v) \) of the variable \( \lambda \) in dependence of \( v \). We shall use below the notations \( y_i := x_i v, \underline{y} := \min\{y_i \colon y_i \neq 0\} \), and \( \bar{y} := \max\{y_i \colon y_i \neq 0\} \).

Case 1: \( cv \neq 0 \) (i.e. \( |c^{1/2}v| > 0 \)). If all \( y_i \) are equal to zero, then \( |\lambda v, \lambda v| = -\infty, \infty \). If \( \bar{y} \) is strictly positive, then \( |\lambda v, \lambda v| = -1/\bar{y}, \infty \). If \( \bar{y} \) is strictly negative, then the interval \( |\lambda v, \lambda v| = -\infty, -1/\bar{y} \). If \( y < 0, \bar{y} > 0 \), then \( |\lambda v, \lambda v| = -1/\bar{y}, -1/\bar{y} \). In all cases the function \( f(\lambda; v) \) converges to minus infinity at the extremities of the interval.

Case 2: \( cv = 0 \) but not all \( x_i v = 0 \). Again, if \( y < 0 \) and \( \bar{y} > 0 \), then the interval \( |\lambda v, \lambda v| = -1/\bar{y}, -1/\bar{y} \). If \( \bar{y} \) is strictly positive, then \( |\lambda v, \lambda v| = -1/\bar{y}, \infty \). Obviously, \( f(\lambda; v) \) tends to minus infinity at the left extremity of the interval. But the assumption of the theorem implies that \( \sum a_i y_i \geq bv \) and the equality is possible only if both sides are equal to zero, hence, all \( y_i = 0 \). Thus, we have a strict inequality ensuring that \( f(\lambda; v) \) tends to minus infinity as \( \lambda \to \infty \). If \( \bar{y} < 0 \), the situation is symmetric.

Case 3: \( cv = 0 \) and all \( x_i v = 0 \). Due to the assumption \( vb = 0 \) and the function \( f(\lambda; v) = 0 \). Thus, we cannot apply directly the lemma. Considering the linear subspace \( L := \{v \colon cv = 0, x_i v = 0, i = 1, \ldots, N\} \) and noticing that \( \varphi(v + w) = \varphi(w) \) for any \( v \in L \), we reduce the problem to a search of the minimum of the restriction of \( \varphi \) to \( D \cap L_1^\perp \). \( \square \)

Now we can prove the main result of this section.

**Theorem 8.3** Suppose that \( K_{\omega, t}(dx) \) is a finite combination of the Dirac measures for all \( (\omega, t) \) except a \( m \)-null set. Then the condition NAA1 implies the existence of the traded local martingale deflator.

**Proof.** The condition NAA1 ensures that for all \( (\omega, t) \) except a \( m \)-null set the assumption of the theorem is fulfilled and therefore one can find solutions of the equations \( q_i(\omega) \). Moreover, there are such solutions that the mapping \( (\omega, t) \mapsto g(\omega) \) is predictable. By virtue of the integrability of the solution, we have that \( g \in L(S) \) and the process \( E(g \cdot S) \) is a reciprocal of a local martingale deflator. \( \square \)

**A Existence of Equivalent \( \sigma \)-Martingale Measures**

**Theorem A.1** Suppose that \( P \) is a separating measure. Then there exists \( \tilde{P} \sim P \) such that \( S \) is a \( \sigma \)-martingale with respect to \( \tilde{P} \).
Proof. We start with the construction of the density process for \( \hat{P} \).

**Lemma A.2** Let \( \varepsilon > 0 \) and let \( Y : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}_+ \setminus \{0\} \) be a \( \hat{P} \)-measurable function such that the following conditions are satisfied \( \hat{P} \otimes A \)-a.e.:

(a) \( K_t(|Y - 1|) \leq \varepsilon/2 \),
(b) \( I_{\{\Delta > 0\}} K_t(Y - 1) = 0 \).

Then the process \( Z := \mathcal{E}(Y - 1) \ast (\mu - \nu) \) is a strictly positive uniformly integrable martingale and \( \hat{P} := Z_T \hat{P} \) is a probability measure such that the total variation distance \( ||\hat{P} - P|| \leq \varepsilon \). The triplet \( (\hat{B}, \hat{C}, \hat{Y}) \) of predictable characteristics of \( S \) with respect to \( \hat{P} \) has the form: \( \hat{B} = B + (Y - 1)xI_{\{|x| \leq 1\}} \ast \nu \), \( \hat{C} = C \), \( \hat{\nu} = Y \nu \).

Proof. Note that

\[
|Y - 1| \ast \nu_T = \int_{[0,T]} K_t(|Y - 1|) dA_T \leq (\varepsilon/2)A_T \leq \varepsilon/2
\]

in view of (b). The process \( M := (Y - 1) \ast (\mu - \nu) \) is a martingale. In virtue of (b)

\[
\Delta M_t = \int (Y(t,x) - 1)\mu(\{t\}, dx) = K_t(Y - 1)\Delta A_t > -1.
\]

Thus, \( Z = \mathcal{E}(M) \) is a strictly positive local martingale of bounded variation satisfying the linear equation \( Z = 1 + Z \cdot M \). Since

\[
E \sup_{t \leq T} |Z_t - 1| = E \sup_{t \leq T} |Z_-(Y - 1) \ast (\mu - \nu)| \leq EZ_-(Y - 1) \ast (\mu + \nu)_{T}
\]

\[
= 2EZ_-(Y - 1) \ast \nu_T = 2EZ_T|Y - 1| \ast \nu_T \leq \varepsilon/2,
\]

the process \( Z \) is uniformly integrable martingale and the total variation distance \( ||\hat{P} - P|| = E|Z_T - 1| \leq \varepsilon/2 \). The form of the triplet of predictable characteristics of \( S \) follows from the Girsanov theorem. \( \square \)

Let denote by \( \hat{\mathbb{R}}^d \) the one-point compactification of \( \mathbb{R}^d \). Let \( C(\hat{\mathbb{R}}^d) \) denote the compact space of continuous functions on \( \hat{\mathbb{R}}^d \) equipped by the uniform norm and the Borel \( \sigma \)-algebra \( \mathcal{B}(C(\hat{\mathbb{R}}^d)) \) and let \( Y = Y(\mathbb{R}^d) \) be its subset formed by the strictly positive continuous functions. We define, for every \( (\omega,t) \), the convex sets

\[
\Gamma^+_{\omega,t} := \left\{ Y \in Y : K_t(|x|) \ast Y < 2, K_t(|Y - 1|) \leq 2, I_{\{\Delta > 0\}} K_t(Y - 1) = 0 \right\},
\]

\[
\Gamma^-_{\omega,t} := \left\{ Y \in Y : K_t\left(|xY - xI_{\{|x| \leq 1\}}|\right) < 2, b_t + K_t\left(xY - xI_{\{|x| \leq 1\}}\right) = 0 \right\}.
\]

The graphs of the set-valued mappings \( (\omega,t) \mapsto \Gamma^+_{\omega,t} \) and \( (\omega,t) \mapsto \Gamma^-_{\omega,t} \) are \( \mathcal{P} \otimes \mathcal{B}(C(\hat{\mathbb{R}}^d)) \)-measurable sets (they are intersections of level sets of functions \( \mathcal{P} \)-measurable in \( (\omega,t) \) and continuous in \( Y \), hence, \( \mathcal{P} \otimes \mathcal{B}(C(\hat{\mathbb{R}}^d)) \)-measurable).

**Proposition A.3** The semimartingale \( S \) is a \( \sigma \)-martingale if and only if

\[
K_t(|x|) < \infty, \quad b_t + K_t\left(xI_{\{|x| > 1\}}\right) = 0 \quad m-a.e.
\]
The crucial element of the proof of the theorem is the following

**Proposition A.4** \( m(\{(\omega, t) : \Gamma_{\omega,t}^\infty \cap \Gamma_{\omega,t} \neq \emptyset\}) = 0 \).

With this the claim of the theorem follows easily. Applying the measurable selection theorem to the set-valued mapping \((\omega, t) \mapsto \Gamma_{\omega,t}^\infty \cap \Gamma_{\omega,t} \) we find a \( \mathcal{P} \)-measurable function \((\omega, t) \mapsto Y(\omega, t, .) \) with values in \( C([\Omega]) \) and such that \( Y(\omega, t, .) \in \Gamma_{\omega,t}^\infty \cap \Gamma_{\omega,t} \) m-a.e. Note that the mapping \((\omega, t, x) \mapsto Y(\omega, t, x)\) from \( \Omega \times [0, T] \times \mathbb{R}^d \) into \( \mathbb{R}^+ \setminus \{0\} \) is measurable with respect to the \( \sigma \)-algebra \( \mathcal{P} = \mathcal{P} \otimes \mathcal{B}^d \) (due to continuity in \( x \)). Using Lemma [A.2] we define the new probability measure \( \bar{P} \) under which the local characteristics of \( S \) are as follows:

\[
\bar{b}_t = b_t + K_t((Y - 1)x I_{\{|x| \leq 1\}}, \bar{c} = c, K_t(dx) = Y(t, x)K_t(dx).
\]

The criterion of Proposition [A.3] is fulfilled for \( \bar{P} \).

**Proof of Proposition A.4.** Let us consider first that \( d = 1 \). Fix \((\omega, t)\). On the set \( \Gamma_{\omega,t}^\infty \) we have that \( K_t(|xY - xI_{\{|x| \leq 1\}}|) < \infty \) and the affine mapping \( \Psi_{\omega,t} \mapsto K_t(xY - xI_{\{|x| \leq 1\}}) \) is well-defined and its image \( \Psi_{\omega,t}(\Gamma_{\omega,t}^\infty) \) is a convex set, hence, an interval. So, in the considered scalar case we need to check that the point \(-b_t\) belongs to this interval except of the \( m \)-null set.

Let us define the predictable processes \( r := \sup\{x : K(|-\infty, x|) = 0\} \) and \( R := \sup\{x : |x, \infty| = 0\} \). Note that

\[
EI_{\{|r| = -n\}}x^-I_{\{x \leq -n\}} * \nu_T = EI_{\{|r| = -n\}}x^-I_{\{x \leq -n\}} * \nu_T
\]

Thus, the finite increasing process \( I_{\{|r| = -n\}}x^-I_{\{x < -1\}} * \mu \) is locally bounded as having jumps do not exceeding \( n \). The processes \( I_{\{|r| = -n\}} \cdot M^c, I_{\{|r| = -n\}} \cdot M^d, \) and \( I_{\{|r| = -n\}} \cdot |b| \cdot A \) are also locally bounded. If all the mentioned processes would be bounded, the process \( I_{\{|r| = -n\}} \cdot S \) would be bounded from below, and we could use the hypothesis of the theorem implying that \( EI_{\{|r| = -n\}} \cdot S \leq 0 \), i.e.

\[
EI_{\{|r| = -n\}}x^-I_{\{x < -1\}} * \mu_T - EI_{\{|r| = -n\}}x^-I_{\{x < -1\}} * \nu_T + EI_{\{|r| = -n\}}b \cdot A_T \leq 0.
\]

It follows that

\[
EI_{\{|r| = -n\}}x^+I_{\{x > 1\}} * \mu_T - EI_{\{|r| = -n\}}x^-I_{\{x < 1\}} * \nu_T < \infty.
\]

This implies, in particular, that \( I_{\{|r| = -n\}}K(|x|) < \infty \) m-a.e. In the general case, using the localization, we may conclude that the increasing processes \( I_{\{|r| = -n\}}xI_{\{x > 1\}} * \mu \) and \( I_{\{|r| = -n\}}xI_{\{x > 1\}} * \nu \) are locally integrable. Applying the similar arguments to the integrand \( I_{\{|r| = -n\}}I_D \) where \( D \in \mathcal{P} \), we infer that

\[
I_{\{|r| = -n\}}(K(xI_{\{|x| > 1\}}) + b) \leq 0 \quad m-a.e.
\]

It follows that

\[
K(|x|I_{\{|x| > 1\}}) < \infty, \quad K(xI_{\{|x| > 1\}}) + b \leq 0 \quad m-a.e. \text{ on the set } \{r > -\infty\}.
\]
Arguing in the same way with the integrand $-I_{(R < n)}$ we obtain that
\[ K(\lfloor x \rfloor I_{\lfloor x \rfloor > 1}) < \infty, \quad K(x I_{\lfloor |x| > 1 \rfloor}) + b \geq 0 \quad \text{m-a.e. on the set } \{ R < \infty \}. \]

This means that, modulo $m$-null subset, we have the following properties:

- on $\{ r > -\infty \}$ the constant function $1 \in \Gamma^c$ and $-b \geq \Psi(1) = K(x I_{\lfloor |x| > 1 \rfloor})$;
- on $\{ R < \infty \}$ the constant function $1 \in \Gamma^c$ and $-b \geq \Psi(1) = K(x I_{\lfloor |x| > 1 \rfloor})$.

Thus, on the intersections of these sets $-b = \Psi(1)$. The conclusion of the proposition in the case of $d = 1$ from the following (purely deterministic) assertion.

**Lemma A.5** If $R = \infty$, then the interval $\Psi(\Gamma^c)$ is unbounded from above.

**Proof.** Let $K_n(dx) := I_{(x > n)}(x)K(dx)$. For $\gamma > 0$ define the set $W_{n,\gamma}$ of strictly positive functions $W \in C([n, \infty])$ such that $W(n) = 1$, $xW(x) \to 0$ as $x \to \infty$, and $K_n(W) = \gamma$. Then for any $N > 0$ there exists $W_N \in W_{n,\gamma}$ such that $K_n(xW) \geq N$. Indeed, take a continuous function $V > 0$ such that $V(n) = 1$, $K_n(V) < \infty$ and $K_n(xV) = \infty$. Choose $A > n$ such that $K_n([n, A]) > 0$ and $\gamma_1 := K_n(V I_{(x > A)}) < \gamma/2$. Take $B > A$ such that $K_n(xV I_{(A < x \leq B)}) \geq N$. For sufficiently large $p$ we have that $\gamma_2 := K_n(I_{(x > B)} V (B) e^{p(B-x)}) \leq \gamma/2$. Put

\[ W_N := f I_{(n < x \leq A)} + V I_{(A < x \leq B)} + V(B) e^{p(B-x)} I_{(x > B)} \]

where $f$ is a strictly positive continuous function on $[n, A]$ with $f(n) = 1$, $f(A) = V(A)$, and $K_n(f I_{(n < x \leq A)}) = \gamma_1$ and $K_n(xW_N) \geq N$. Take $W_N \in W_{n,\gamma}$ with $\gamma = K(\mathbb{R} \setminus [-n, n])$, and $K_n(xW_N) \geq N$. Then

\[ Y_N(x) := e^{\gamma(x + n)} I_{(x > n)} + I_{\lfloor |x| \leq n \rfloor} + W_N(x) I_{(x > n)} \in \Gamma^c \]

and $\Psi(Y_N) \to \infty$ as $N \to \infty$. \[ \square \]

The vector case is reduced to the scalar one. Indeed, the sets

\[ \Xi_{\omega,t} := \Psi_{\omega,t}(\Gamma_{\omega,t}) + b_t(\omega) \subseteq \mathbb{R}^d \]

are convex and $\{ (\omega, t, x) : x \in \Xi_{\omega,t} \} \in \mathcal{P} \otimes \mathcal{B}_t$. By the measurable version of the separation theorem, there is a predictable process $l$ with values in $\mathbb{R}^d$ such that, outside a $m$-negligible set, $\|l_{\omega,t}\| = 1$ and $l_{\omega,t}x < 0$ for every $x \in \Xi_{\omega,t}$ if $0 \notin \Xi_{\omega,t}$, and $l_{\omega,t}x = 0$, otherwise. We use the superscript $l$ to denote objects related to the scalar semimartingale $S^l := l \cdot S$. It is easily seen that $\nu'(\omega, dt, dx) = K^l_{\omega,t}(dx) dA_t(\omega)$ with $K^l_{\omega,t}(dx) = (K_{\omega,t,l_{\omega,t}})(dx)$ and

\[ B^l = lb \cdot A + K(I_{\lfloor |x| \leq 1 \rfloor} - I_{\lfloor |x| \leq 1 \rfloor}) \cdot A, \]

see [10], IX.5.3: $P$ is a separating measure for $S^l$. We have proved that for every fixed $(\omega, t)$ outside of a $m$-negligible set the equation $\Psi_{\omega,t}(Y) = -b_t(\omega)$
Remark Theorem A.1 is due to Delbaen and Schachermayer [6]. The proof given here is borrowed, with some simplifications, from [11]. The argument to get the bound for the total variation is the same as in the paper [14]. It allows us to avoid references to the Hellinger processes ([15], [16]) used in [11]. It seems that the criterion of Proposition A.3 appeared for the first time in [11], see also [20], [10].

B Laws of Large Numbers

The classical law of large numbers for locally square integrable martingale $M$ is a theorem asserting that ratio $M_T/(1 + \langle M \rangle_T)$ tends to zero a.s. as $T \to \infty$ on the set where $\langle M \rangle_\infty$ and tends to $M_\infty/(1 + \langle M \rangle_\infty)$ on the complement. Here we present some simple results in the same spirit for sequences of stochastic integrals with truncated integrands where not the time horizon but the level of truncation tends to infinity.

Let $J$ be a $d$-dimensional locally square integrable martingale ($J \in \mathcal{M}^2_\text{loc}$ in standard notations) with the quadratic characteristics $\langle J \rangle = q \cdot A$. Let $H$ be a $d$-dimensional predictable process. Put $\Gamma_\infty := \{|q^{1/2}H|^2 \cdot A_T = \infty\}$ and define the scalar locally square integrable martingale $M^n := H^n \cdot J$ where $H^n := HI_{\{|H| \leq n\}}$. Then

$$L^n := 1 + \langle M^n \rangle = 1 + |q^{1/2}H^n|^2 \cdot A.$$

Lemma B.1 The sequence of random variables $M^n_T/L^n_T \to 0$ in probability on the set $\Gamma_\infty$ and is $P$-bounded on the set $\Gamma_\infty^c$.

Proof. Put $X^n := (L^n)^{-3/4} \cdot M^n = (L^n)^{-3/4}H^n \cdot J$. Note that $(L^n)^{-3/2}L^n_T \leq 2$. Indeed, using the change of variable formula for $F(x) := 2x^{-1/2}$, the finite increment formula, and the monotonicity of $F$ we get that

$$F(L^n_T) - F(1) = F'(L^n_T) \cdot L_T + \sum_{s \leq T} (F(L^n_s) - F(L^n_{s-}) - F'(L^n_s) \Delta L^n_s) \leq F'(L^n_T) \cdot L^n_T$$

implying the claimed bound. Thus, $X^n \in \mathcal{M}^2$ and, by the Doob inequality,

$$E \sup_{s \leq T} |X^n_s|^2 \leq 4E[X^n_T]^2 = 4E(L^n)^{-3/2} \cdot \langle M^n \rangle_T = 4E(L^n)^{-3/2} \cdot L^n_T \leq 8. \quad (B.1)$$

That is, $\sup_{s \leq T} |X^n_s|$ is bounded in $L^2$, hence, in probability.

The positive process $U^n := (L^n)^{3/4}$ is of bounded variation and

$$X^n_T U^n_T = X^n_T \cdot U^n_T + U^n_T \cdot X^n_T = X^n_T \cdot U^n_T + M^n_T.$$

This implies that

$$\frac{M^n_T}{(L^n_T)^{3/4}} = X^n_T - \frac{1}{U^n_T}X^n_T \cdot U^n_T.$$
It follows that \(|M^n_T/(L^n_T)^{3/4}| \leq 2 \sup_{s \leq T} |X^n_s|\). Therefore, the sequence of ratios \(M^n_T/(L^n_T)^{3/4}\) is \(P\)-bounded. Since \(L^n_T \to 1 + |q^{1/2}H^2| \cdot A_T\), the assertion of the lemma becomes obvious. □

Let \(N^n\) be a sequence of counting processes with the compensators of the form \(\tilde{N}^n = I_{\{G \leq n\}} \cdot \tilde{N}\) where \(\tilde{N}\) is a predictable increasing càdlàg process and \(G \geq 0\) is a predictable process. Let \(\Theta_\infty := \{\tilde{N}_T = \infty\}\) and let \(R^n_T := 1 + \tilde{N}^n_T\).

**Lemma B.2** The sequence of random variables \(N^n_T/R^n_T \to 1\) in probability on the set \(\Theta_\infty\) and is \(P\)-bounded on the set \(\Theta^c_\infty\).

**Proof.** Put \(M^n := I_{\{G \leq n\}} \cdot (N^n - \tilde{N}^n)\). Then the process \(M^n \in \mathcal{M}^2_{\text{loc}}\) and \(\langle M^n \rangle = (1 - \Delta \tilde{N}^n) \cdot \tilde{N}^n\). Exactly in the same way as in the proof of the previous lemma but replacing the \(L^n\) by \(R^n\) we obtain that \(M^n_T/(R^n_T)^{3/4}\) is \(P\)-bounded (the only change is in (B.1) where the second equality should be replaced by the inequality). This implies the claim. □

**Acknowledgements** The research of Yuri Kabanov is funded by the grant 14.12.31.0007.

**References**

1. Algoet P., Cover T. Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *Ann. Probab.*, 16 (1988), 2, 876-898.
2. Becherer D. The numéraire portfolio for unbounded semimartingales, *Finance and Stochastics*, 5 (2001), 327-341.
3. Choulli T., Stricker C. Deux applications de Galtchouk–Kunita–Watanabe decomposition. Séminaire de Probabilités, XXX, Lect. Notes Math., 1626 (1996) 12-23.
4. Choulli T., Deng J., Ma J. The fundamental theorem of utility maximization and numéraire portfolio. Preprint.
5. Delbaen F., Schachermayer W. A general version of the fundamental theorem of asset pricing. *Math. Annalen*, 300 (1994), 463–520.
6. Delbaen F., Schachermayer W. The Fundamental Theorem of Asset Pricing for unbounded stochastic processes. *Math. Annalen*, 312 (1998), 215–250.
7. Goll Th., Kallsen J. A complete explicit solution to the log-optimal portfolio problem. *Ann. Appl. Probab.*, 13 (2003), 2, 395-816.
8. Herdegen M., Herrmann S. A class of strict local martingales. Preprint, 2014.
9. Hulley H., Schweizer M. On minimal market models and minimal martingale measures. In: C. Chiarella and A. Novikov (eds.), “Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen”, Springer, 2010, 35-51.
10. Jacod J., Shiryaev A.N. *Limit Theorems for Stochastic Processes*. Second edition, Springer, Berlin–Heidelberg–New York, 2010.
11. Kabanov Yu.M. On the FTAP of Kreps–Delbaen–Schachermayer. *Statistics and Control of Random Processes. The Liptser Festschrift. Proceedings of Steklov Mathematical Institute Seminar*, World Scientific, Singapore, 1997, 191–203.
12. Kabanov Yu.M., Kramkov D. O. Large financial markets: asymptotic arbitrage and contiguity. *Probab. Theory and Its Applications*, 39 (1994) 1, 222–229.
13. Kabanov Yu.M., Kramkov D.O. Asymptotic arbitrage in large financial markets. *Finance and Stochastics*, 2 (1998), 2, 143–172.
14. Kabanov Yu.M., Liptser R.Sh. On convergence in variation of the distributions of multivariate point processes. *Z. Wahrsch. Verw. Gebiete*, 63 (1983).
15. Kabanov Yu.M., Liptser R.Sh., Shiryaev A.N. Absolute continuity and singularity of locally absolute continuous probability distributions. Part I: *Mat. Sbornik* 107 (1978) 3, 364–415; part II: *Mat. Sbornik* 108 (1979), 1, 32–61. English translation: *Math. USSR Sbornik*, 35 (1979), 5, 631–680; 36 (1980), 1, 31–58.
16. Kabanov Yu., Liptser R.Sh., Shiryaev A.N. On the variation distance for probability measures defined on a filtered space. *Probab. Theory and Related Fields*, 71 (1986), 19–36.
17. Kabanov Yu., Safarian M. *Markets with Transaction Costs. Mathematical Theory*. Springer, Berlin–Heidelberg–New York, 2010.
18. Kabanov Yu.M., Stricker Ch. On equivalent martingale measures with bounded densities. *Séminaire de Probabilités*, XXXV, Lect. Notes Math., 1755 (2001), 139-148.
19. Kabanov Yu.M., Stricker Ch. Remarks on the true no-arbitrage property. *Séminaire de Probabilités*, XXXVIII, Lect. Notes Math., 1857 (2005), 186–194.
20. Kallsen J. σ-Localization and σ-martingales. *Probab. Theory and Its Applications*, 48 (2004) 1, 152163
21. Karatzas I., Kardaras C. The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, 9 (2007), 4, 447-493.
22. Kardaras C. Finitely additive probabilities and the fundamental theorem of asset pricing. In: Contemporary quantitative finance, Essays in Honour of Eckhard Platen. 1934, Springer, Berlin–Heidelberg–New York, 2010.
23. Kardaras C. Market viability via absence of arbitrages of the first kind. *Finance and Stochastics*, 16 (2012), 4, 651-667.
24. Kardaras C. On the closure in the Emery topology of semimartingale wealth-process sets. *Ann. Appl. Probab.*, 23 (2013), 4, 1355-1376.
25. Klein, I. A fundamental theorem of asset pricing for large financial markets. *Math. Finance*, 10 (2000), 4, 443-458.
26. Klein I., Schachermayer W. Asymptotic arbitrage in non-complete large financial markets. *Probab. Theory and Its Applications*, 41 (1996), 4, 927–934.
27. Mémin J. Espaces de semimartingales et changement de probabilité. *Z. Wahrsch. Verw. Gebiete*, 52 (1980), 1, 9–39.
28. Rokhlin D.B. On the existence of an equivalent supermartingale density for a fork-convex family of stochastic processes. *Mat. Zametki*, 87 (2010), 4, 59-603.
29. Shiryaev A.N., Cherny A.S. On stochastic integrals up to infinity and predictable criteria for integrability. *Séminaire de Probabilités XXXVIII, Lecture Notes Math.*, 1857 (2005), 165-185.
30. Stricker C., Yan J.A. Some remarks on the optional decomposition theorem. *Séminaire de Probabilités*, XXXII, Lecture Notes Math., 1686 (1998), 56-66.
31. Takaoka K., Schweizer M. A note on the condition of no unbounded profit with bounded risk. *Finance and Stochastics*, 18 (2014), 2, 393-405.
32. von Weizsäcker H. Can one drop $L^1$-boundedness in Komlós subsequence theorem? *Amer. Math. Monthly*, 111 (2004), 10, 900–903.