Lorentz Invariance of Entanglement

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Abstract

We study the transformation of maximally entangled states under the action of Lorentz transformations in a fully relativistic setting. By explicit calculation of the Wigner rotation, we describe the relativistic analog of the Bell states as viewed from two inertial frames moving with constant velocity with respect to each other. Though the finite dimensional matrices describing the Lorentz transformations are non-unitary, each single particle state of the entangled pair undergoes an effective, momentum dependent, local unitary rotation, thereby preserving the entanglement fidelity of the bipartite state. The details of how these unitary transformations are manifested are explicitly worked out for the Bell states comprised of massive spin 1/2 particles and massless photon polarizations. The relevance of this work to non-inertial frames is briefly discussed.

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**I. INTRODUCTION**

Entanglement of bipartite quantum states forms a vital resource for many quantum information processing protocols, including quantum teleportation, cryptography, computation and clock synchronization. According to the principle of special relativity the physics involved in utilizing such states should not depend on the arbitrary inertial coordinate system from which the states are observed. Therefore we should expect the states to transform unitarily from one inertial frame to another. This is clearly the case for rotations. However, from the famous theorem by Wigner [1] the finite dimensional representations of Lorentz boosts are non-unitary. At first glance, it is then not immediately obvious where unitarity arises in the case of boosts. The resolution to this apparent dilemma arises from the fact that in relativistic quantum mechanics the creation and annihilation operators, as well as the associated mode functions, for the quantum field that creates a given state transform under Lorentz transformations (LTs) by local unitary spin-$j$ representations of the 3D rotation group [2]. Key to these transformations is the representation of the Wigner rotation $W$, which is a rotation in the rest frame of the particle, that leaves the rest momentum invariant. The purpose of this work is to review the role the Wigner rotation plays in restoring unitarity in the transformations between relativistic single and multi-particle states. In particular, through explicit calculation of the Wigner rotation we describe the observation of the entangled Bell states from two inertial frames moving with constant velocity with respect to each other. The details are worked out in a fully relativistic framework for the two important cases of spin entangled and photon polarization entangled Bell states, which occur most often in quantum information processing protocols. The end results of these calculations will be that under Lorentz transformations, each constituent particle of the relativistic generalization of the Bell states will undergo an effective, momentum dependent, local unitary rotations, which will therefore preserve the entanglement fidelity of the bipartite state.

For the purpose of concreteness we consider the symmetric Bell state in the center of momentum frame $S$, with the two constituent particles $A$ (Alice) and $B$ (Bob) travelling along the $\pm z$ direction. For the symmetric Bell state $\beta_{00}^{(1/2)} = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$, composed of spin $1/2$ electrons with the quantization axis along $z$, we will show that an observer $S'$ travelling with constant velocity with respect to $S$ will observe a rotation of the spins in the
direction of boost, at an angle less than the direction of the new spatial momentum. For the photon polarization entangled state \( \beta_0^{(1)}(\{HH\} + |VV\})/\sqrt{2} \), where \( H \) and \( V \) represent horizontal and vertical polarizations, we will find \( S' \) observes a rotation of the plane of polarization, tilted towards the direction of boost, and perpendicular to the new observed momentum. Though these two cases are analogous, the explicit details are different due to the form of the little group \( \mathbb{ISO}(2) \) which governs the invariance of the rest momentum for massive and massless particles. For the massive electrons, the little group governing the Wigner rotation is \( \mathbb{SO}(3) \), the ordinary group of 3D rotations. For the massless photons, the little group is \( \mathbb{ISO}(2) \), the Euclidean group of rotations and translations in the 2D plane perpendicular to the propagation direction. Things are a little more complicated in the case of massless particles, since the little group in this case can induce gauge transformations in the 4-potentials. In order to ensure that the 4-potentials transform unitarily under boosts, we adopt the procedure of Han et al. which reduces to the choice of a particular gauge in which the photon polarization vectors lie in the same plane as the electric and magnetic fields. Though some generality is lost by gauge-fixing, the explicit unitarity for the representations of the Lorentz boosts is sufficient gain for most all quantum optical information processing applications.

The organization of this paper is as follows. In Section II we review the formalism of quantum fields in Minkowski space, the representations of the Lorentz transformations and Wigner’s little group. In Section III specialize our discussion to the case of the electron Bell state \( \beta_{00}^{(1/2)} \) and work out the Wigner rotation and transformed state for a representative boost in a direction orthogonal to particle’s momentum. In Section IV we repeat the previous calculations for the photon Bell state \( \beta_{00}^{(1)} \). Here we make special note of the work by Han et al. which shows how a LT on the polarization vectors, preceeded by a gauge transformation leads to a pure rotation, which is finite dimensional and unitary. Both the gauge transformation and the rotation are elements of the little group for photons. Finally, in the last section we summarize our results and comment on their relevance to the discussion of entanglement in non-inertial, accelerated frames.
II. QUANTUM FIELDS IN MINKOWSKI SPACE

For our discussion of quantum fields in Minkowski space, we follow the text by Weinberg [2] and (for ease of reference) adopt his notation, metric signature and index ordering. As such, Greek indices \( \mu, \nu \), etc. run over the four spacetime coordinates labels \( \{1, 2, 3, 0\} \) with \( x^0 \) the time component. Latin indices \( i, j, k \), etc. run over the three spatial coordinates labels \( \{1, 2, 3\} \). The spacetime metric \( \eta_{\mu \nu} \) is diagonal with elements \( \{1, 1, 1, -1\} \). Four-vectors are in un-boldfaced type while spatial vectors are boldfaced. For e.g. the 4-momentum for particle of mass \( m \) is given by \( p^\mu = (p^1, p^2, p^3, p^0) = (p, p^0) \), with norm \( p^\mu p_\mu = p^2 - (p^0) = -m^2 \). We use natural units where \( \hbar = c = 1 \), and occasionally include explicit factors of \( c \) for clarity.

A. Single Particle States

Single particle quantum states are classified by their transformation under the inhomogeneous Lorentz group, or Poincaré group, consisting of homogeneous Lorentz transformations (rotations and boosts) \( \Lambda \) and translations \( b \) ([2], Chapter 2). A general Poincaré transformation relates the coordinates \( x^\mu \) in an inertial frame \( S \) to those of another inertial frame \( S' \) with coordinates \( x'^\mu \) via

\[
x'^\mu \equiv T(\Lambda, b)x^\mu = \Lambda^\mu _{\nu}x^\nu + b^\mu. \tag{1}
\]

For future reference, we denote the transformed 4-momentum as \( p' \rightarrow \Lambda p \) and its 3-vector spatial momentum as \( p_\Lambda \). A product of Lorentz transformations satisfies the composition rule

\[
T(\bar{\Lambda}, \bar{b}) T(\Lambda, b) = T(\bar{\Lambda}\Lambda, \bar{b} + b). \tag{2}
\]

Single particle quantum states are denoted by \( \Psi_{p, \sigma} \) where \( p \) labels the 4-momenta and \( \sigma \) labels all other degrees of freedom. For our purposes, we may concentrate on the spin degree of freedom; spin for massive particles and helicity for massless particles. The state-vectors \( \Psi_{p, \sigma} \) have the property \( P^\mu \Psi_{p, \sigma} = p^\mu \Psi_{p, \sigma} \), where \( P^\mu \) is the momentum operator and \( p^\mu \) is its eigenvalue. A Poincaré transformation \( T(\Lambda, a) \) induces a linear unitary transformation on the vectors in the physical Hilbert space of states via

\[
\Psi \rightarrow U(\Lambda, b)\Psi. \tag{3}
\]
The unitary operators $U(\Lambda, b)$ satisfy the same composition rule as in Eq. (2) (with $T$ replaced by $U$). The commutation relations for the Poincaré algebra [2] tell us that under translations the state-vectors transform as $U(1, b)\Psi_{p,\sigma} = e^{-ip \cdot b} \Psi_{p,\sigma}$. Under homogeneous Lorentz transformations (LTs) $\Lambda$, the state-vector $\Psi_{p,\sigma}$ with momentum $p$ must transform to a linear combination of the state-vectors $\Psi_{\Lambda p,\sigma}$ with momentum $\Lambda p$, i.e.

$$U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p) \Psi_{\Lambda p,\sigma'}.$$  \hfill (4)

The matrix $C_{\sigma'\sigma}$ can be chosen to be block diagonal in the index $\sigma$, with each block forming an irreducible representation of the inhomogeneous Lorentz group.

1. Massive Particles

Consider for the moment the case of massive particles, $p^2 < 0$. We can always choose some standard 4-momentum $k^\mu$ (usually taken in the particle’s rest frame) and express any $p^\mu$ of this class by

$$p^\mu = L^\mu_{\nu}(p) k^\nu$$  \hfill (5)

or

$$p = L(p) k,$$

where $L^\mu_{\nu}(p)$ is some standard Lorentz transformation that depends on $p$ and takes $k \to p$. We can then define the state-vectors $\Psi_{p,\sigma}$ in terms of standard momentum states $\Psi_{k,\sigma}$ as

$$\Psi_{p,\sigma} \equiv N(p)U(L(p))\Psi_{k,\sigma'},$$  \hfill (6)

where $N(p)$ is a normalization factor which Weinberg conventionally takes as $N(p) = \sqrt{k^0/p^0}$. Now the importance of the Wigner rotation can be seen to arise as follows. Using the fact $U(L_1)U(L_2) = U(L_1 L_2)$ where $L_1$ and $L_2$ are arbitrary LTs, we have upon acting on Eq. (3) with an arbitrary LT, $U(\Lambda)$

$$U(\Lambda)\Psi_{p,\sigma} = N(p)U(\Lambda L(p))\Psi_{k,\sigma'}$$

$$= N(p)U(L(\Lambda p)) \left[ U(L^{-1}(\Lambda p)\Lambda L(p)) \right] \Psi_{k,\sigma'}$$

$$\equiv N(p)U(L(\Lambda p))U(W(\Lambda, p))\Psi_{k,\sigma'}.$$  \hfill (7)

In the second line of Eq. (7) we have inserted the identity matrix in the form of $L(\Lambda p) L^{-1}(\Lambda p)$ in the argument of $U$ and have defined the *Wigner rotation* as the product of LTs in the argument of $U$ in the square brackets:

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p) \Lambda L(p).$$  \hfill (8)
That $W$ is a rotation can be seen as follows from Eq. (5). Operating from right to left, $L(p)$ takes the standard momentum $k$ to $L(p)k = p$. The LT, $\Lambda$ takes $p$ to $\Lambda p$. The final LT, $L^{-1}(\Lambda p)$ takes $\Lambda p$ back to $k$. Thus $W$ belongs to the subgroup of the homogeneous Lorentz group that leaves $k^\mu$ invariant:

$$W_{\nu}^\mu k^\nu = k^\mu. \quad (9)$$

This subgroup is called (Wigner’s) little group. The end product of all this is that we can rewrite Eq.(4) as

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{(\Lambda p)}^0 \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))\Psi_{\Lambda p,\sigma'}, \quad (10)$$

where $D(W)$ furnishes a representation of the little group element $W$.

For massive particles $p^2 = -m^2 < 0, p^0 > 0$, the standard momentum can be chosen as $k^\mu = mc(0, 0, 0, 1)$ and the little group is the usual group of ordinary rotations in 3D, $SO(3)$. In this case the $D^{(j)}_{\sigma'\sigma}(W(\Lambda, p))$ form the usual spin-$j$ representations of the rotation group.

The standard boost in the direction $\hat{p} \equiv p/|p|$ with rapidity $\eta$ defined by the relations

$$\cosh \eta = \sqrt{p^2 + m^2}/m, \quad \sinh \eta = |p|/m \quad (11)$$

is given by

$$L_i^j(\eta) = \delta_{ij} + (\cosh \eta - 1) \hat{p}_i \hat{p}_j, \quad L_0^0(\eta) = \sinh \eta \hat{p}_i, \quad L_0^i(\eta) = \cosh \eta. \quad (12)$$

The boost in Eq.(12) can always be written in the form

$$L(p) = R(\hat{p})B_z(|p|)R^{-1}(\hat{p}) \quad (13)$$

where $R(\hat{p})$ is a rotation that takes the $z$-axis into $p$ by first rotating about the $y$-axis by an angle $\theta$ and then about the $z$-axis by an angle of $\phi$, and $B_z(|p|)$ is a pure boost in the $z$-direction. The unitary representation of $R(\hat{p})$ on the Hilbert space is given by $U(R(\hat{p})) = e^{i\phi J_z}e^{i\theta J_x}$. Finally, if the LT, $\Lambda$ is a pure arbitrary 3D rotation $\mathcal{R}$, then $W(\Lambda, p) \equiv \mathcal{R}$ for all $p$.

2. Massless Particles

For massless particles $p^2 = 0, p^0 > 0$, the standard momentum can be taken to be $k^\mu = (0, 0, 1, 1)$. The little group $W(\Lambda, p)$ which leaves this $k$ invariant (i.e. satisfies Eq.(4))
is the group $ISO(2)$ which consists of rotations $R_z(\theta)$ about the $z$-axis by an angle $\theta$ and 2D translations $S(\alpha, \beta)$ in the $x-y$ plane with displacements vector $(\alpha, \beta, 0, 0)$. The Wigner rotation can be expressed as a product of this rotation and translation as

$$W(\theta, \alpha, \beta) = S(\alpha, \beta)R_z(\theta).$$  \hfill (14)

This leads to a representation of $D(W)$ as $[2]$

$$D_{\sigma'\sigma}(W) = e^{i\theta}\delta_{\sigma'\sigma},$$  \hfill (15)

where $\sigma$ labels the possible helicity states of the particle. In the case of photons, $\sigma = \pm 1$ corresponding to states of right and left circularly polarization. Instead of Eq.(10), we now have the transformation of state-vectors under a homogeneous Lorentz transformation given by

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{(\Lambda p)^0\frac{p^0}{p^0}} e^{i\theta(\Lambda, p)} \Psi_{\Lambda p,\sigma},$$  \hfill (16)

where $\theta(\Lambda, p)$ is defined by Eq.(14). Due to the gauge freedom in the electromagnetic field, there is still work that needs to performed to construct the Wigner rotation for photons and compute the angle $\theta$ in Eq.(16). We take this up in Section IV.

B. Multi-Particle States

The generalization of the single particle states to many particle states is relatively straightforward but notationally cumbersome. We denote a multi-particle state vector by $\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\ldots}$ where $p_i$ labels the momentum, $\sigma_i$ is the spin $z$-component (or helicity for massless particles), and $n_i$ is a species label for the $i$th particle. In keeping with the notation of $[2]$, $\Phi$ could refer to either free particle states $\Psi$, or 'In' and 'Out' scattering states. We will be concerned only with free particle states, but will retain Weinberg’s notation of using $\Phi$ for the state-vectors and $U(\Lambda) \rightarrow U_0(\Lambda)$ to denote the representations of Lorentz transformations on the Hilbert space of states. From now own we will be concerned mainly with proper orthochronous LTs.

A multi-particle state transforms as the direct product of single particles states. Considering massive particles for the time being, we can write the transformation of a multi-particle state under a proper orthochronous inhomogeneous Lorentz transformation $U_0(\Lambda, b)$ as

$$U_0(\Lambda, b)\Phi_{p_1,\sigma_1,n_1;p_2,\sigma_2,n_2;\ldots} = \exp \left(-ib_{\mu}(p_1^\mu + p_2^\mu + \cdots)\right)$$
The 0-particle state $\Phi_0$ is the Lorentz invariant vacuum with normalization of unity, $(\Phi_0, \Phi_0) = 1$, where the parentheses denote the inner product on the Hilbert space. The 1-particle state is denoted by $\Phi_q$, where we use the shorthand notation $q = (p, \sigma, n)$ to represent the relevant quantum numbers. This has norm $(\Phi_{q'}, \Phi_q) = \delta(q' - q) \equiv \delta(p' - p)\delta_{\sigma'\sigma}\delta_{n'n}$. The 2-particle state $\Phi_{q'q}$ is physically equivalent to the state $\Phi_{qq'}$ so we must take its norm to be $(\Phi_{q'q'}\Phi_{q_1q_2}) = \delta(q'_1 - q_1)\delta(q'_2 - q_2) \pm \delta(q'_2 - q_1)\delta(q'_1 - q_2)$, where the $-$ is taken if both particles are fermions and + otherwise. The general $N$-particle state $\Phi_{q_1q_2...q_N}$ is taken to have norm $(\Phi_{q'q'...q'_{M'}}\Phi_{q_1q_2...q_N}) = \delta_{NM}\sum\delta_p\prod\delta(q_i - q'_i)$ where the sum is over all signed permutations of the integers $\{1, 2, \ldots, N\}$.

The above multi-particle states can be produced by the action of the creation operator $a^\dagger(q)$ which adds a particle with quantum numbers $q$ to the front of the list of particles in the state, $a^\dagger(q)\Phi_{q_1q_2...q_n} = \Phi_{qq_1q_2...q_n}$. The general $N$-particle state can be produced from the vacuum by acting upon it with $N$ creation operators

$$a^\dagger(q_1)a^\dagger(q_2)\ldots a^\dagger(q_N)\Phi_0 = \Phi_{q_1q_2\ldots q_N}.$$  

Our main point of interest is the observation that in order for the state in Eq.(18) to transform properly, i.e. in accordance with Eq.(17), the creation operator must satisfy the transformation rule

$$U_0(\Lambda, b)a^\dagger(p\sigma n)U_0^{-1}(\Lambda, b) = e^{-i(\Lambda p)b}\sqrt{(\Lambda p)^0/p^0}\times \sum_{\sigma'} D^{(j_n)}_{\sigma'\sigma}(W(\Lambda, p))a^\dagger(p_{\Lambda}\sigma'n),$$  

where $j_n$ is the spin of the $n$th particle species. (For massless particles, the $D$ Eq.(19) must be replaced by that in Eq.(13)).

We can now create quantum fields $\psi_l(x) = \psi_l^+(x) + \psi_l^-(x)$ where the $\pm$ indicates the positive and negative frequency field operators and $l$ is the field index label, e.g. $l = \{1, 2, 3, 4\}$ for a spin-1/2 Dirac bispinor representing the electron-positron field, and $l \rightarrow \mu = \{1, 2, 3, 0\}$ for spin-1 electromagnetic 4-potential field. The positive frequency annihilation field $\psi_l^+(x)$ and negative frequency creation field $\psi_l^-(x)$ are given by

$$\psi_l^+(x) = \sum_{\sigma n} \int d^3p u_l(x; p, \sigma, n) a(p, \sigma, n),$$  

where
\[ \psi^-(x) = \sum \int d^3p \, v_l(x; p, \sigma, n) u^\dagger(p, \sigma, n), \] (21)

where the mode functions \( u_l(x; p, \sigma, n) \) and \( v_l(x; p, \sigma, n) \) are chosen so that under LTs each field is multiplied by a position-independent matrix

\[
U_0(\Lambda, b) \psi^+_l(x) U_0^{-1}(\Lambda, b) = \sum D_{l' l}(\Lambda^{-1}) \psi^+_l(\Lambda x + b),
\] (22)

\[
U_0(\Lambda, b) \psi^-_l(x) U_0^{-1}(\Lambda, b) = \sum D_{l' l}(\Lambda^{-1}) \psi^-_l(\Lambda x + b).
\] (23)

Here the \( D_{ll'}(\Lambda) \) are matrices which form a block diagonal representation of the LTs for the fields, with each block containing irreducible representations.

If we now form \( U_0(\Lambda, b) \psi^+_l(x) U_0^{-1}(\Lambda, b) \) from Eq.(21) and use the adjoint of Eq.(19), we obtain consistency with Eq.(22) and Eq.(23) if the following transformation of the mode functions holds for massive particles (see [2] for the details)

\[
\sum_{\sigma'} u_{l^*}(p \Lambda, \sigma, n) D_{\sigma \sigma'}(W(\Lambda, p)) = \sqrt{p^0 / (\Lambda p)^0} \sum_l D_{vl}(\Lambda) u_l(p, \sigma, n),
\] (24)

and

\[
\sum_{\sigma'} v_{l^*}(p \Lambda, \sigma, n) D_{\sigma \sigma'}^{(s)}(W(\Lambda, p)) = \sqrt{p^0 / (\Lambda p)^0} \sum_l D_{vl}(\Lambda) v_l(p, \sigma, n).
\] (25)

and for massless particles (from Eq.(16))

\[
u(p \Lambda, \sigma, n) e^{i \sigma \theta(\Lambda, p)} = \sqrt{p^0 / (\Lambda p)^0} \sum_l D_{vl}(\Lambda) u_l(p, \sigma, n)
\] (26)

and

\[
u(p \Lambda, \sigma, n) e^{-i \sigma \theta(\Lambda, p)} = \sqrt{p^0 / (\Lambda p)^0} \sum_l D_{vl}(\Lambda) v_l(p, \sigma, n)
\] (27)

In obtaining Eq.(24)-Eq.(27) we have used the fact that under pure translations, \( U_0(1, b) \) one can deduce that the mode functions must take the form \( u_l(x; p, \sigma, n) = (2\pi)^{-3/2} e^{ip \cdot x} u_l(p, \sigma, n) \) and \( v_l(x; p, \sigma, n) = (2\pi)^{-3/2} e^{-ip \cdot x} v_l(p, \sigma, n) \).

We can interpret Eq.(24) as follows. Recall that \( u_l(p, \sigma, n) \) forms a column vector of field components characterized by a momentum \( p \), spin or helicity \( \sigma \) and species index \( n \) which we temporarily denote as \( \vec{u}(p, \sigma, n) \). For electrons, \( \vec{u} \) has four bispinor components and \( \sigma = \pm 1/2 \) denote spin up or down along some quantization axis. For photons, \( \vec{u} \) has four spacetime components \( (l' \rightarrow \mu) \) and \( \sigma = \pm 1 \) denote states of right and left circularly polarization. Under a Lorentz transformation \( U(\Lambda, b) \), \( \vec{u}(p, \sigma, n) \) is transformed to a new
vector $\vec{u}'(p, \Lambda, \sigma, n)$. Eq.(24) tells us that we can compute the transformed vector $\vec{u}'(p, \Lambda, \sigma, n)$ in two ways. Up to a normalization factor, the right hand side of Eq.(24) indicates that (in matrix notation) we can compute $\vec{u}'(p, \Lambda, \sigma, n) = D(\Lambda)\vec{u}(p, \sigma, n)$ for a fixed spin or helicity index $\sigma$, i.e. by transforming the field components according to $D(\Lambda)$. The left hand side of Eq.(24) states, that for a fixed field component $l'$, we can re-write $\vec{u}'(p, \Lambda, \sigma, n)$ as a linear combination of the spin/helicity mode functions with momentum $p\Lambda$ with coefficients given by the spin-$j_n$ matrix representations $D_{\sigma'\sigma}^{(j_n)}(W)$ of the Wigner rotation $W$. It is in this later case that we see that a Lorentz transformation induces a momentum dependent, local unitary rotation of the spin components of each particle in a multi-particle state. Each constituent single particle state is transformed at most into a superposition of spin states with the transformed momentum $p\Lambda$. Such local unitary rotations cannot effect the entanglement fidelity of the multi-particle state. We shall give explicit examples in the next section.

Though we will mainly be concerned in this paper with the transformation of states, for completeness we list the corresponding transformations of the creation and annihilation operators in the new inertial frame ($x' = \Lambda x + b$). These can re-expressed from Eq.(19) (using the unitarity of the rotation matrices $D_{\sigma'\sigma}^{(j_n)}(W)$) as

$$U_0(\Lambda, b) a(p, \sigma, n) U_0^{-1}(\Lambda, b) = e^{i(\Lambda p) \cdot b} \sqrt{(\Lambda p)^0 / p^0} \sum_{\sigma'} D_{\sigma'\sigma}^{(j_n)}(W^{-1}(\Lambda, p)) a(p, \Lambda, \sigma', n),$$

and

$$U_0(\Lambda, b) a^\dagger(p, \sigma, n) U_0^{-1}(\Lambda, b) = e^{-i(\Lambda p) \cdot b} \sqrt{(\Lambda p)^0 / p^0} \sum_{\sigma'} D_{\sigma'\sigma}^{(j_n)*}(W^{-1}(\Lambda, p)) a^\dagger(p, \Lambda, \sigma', n).$$

In the next two sections we will specialize the results of the Lorentz transformation rules for the mode functions Eq.(24) and Eq.(26), to the specific cases of 2-qubit spin entangled states and 2-qubit polarization entangled states.

III. ELECTRONS: SPIN 1/2 FIELDS

The spin 1/2 Dirac field is given by

$$\psi_1(x) = \sum_{\sigma} \int d^3 p \left[ u_1(p, \sigma) e^{ip \cdot x} a(p, \sigma) + v_1(p, \sigma) e^{-ip \cdot x} a^\dagger(p, \sigma) \right],$$

(30)
where we have dropped the species label \( n \). Here \( a(p, \sigma) \) annihilates a particle in the (Dirac bi-)spinor state \( u_l(p, \sigma) \), corresponding to an electron with momentum \( p \) with spin \( \sigma = \pm 1/2 \) along a quantization axis, which we shall take as the \( z \)-axis. The charge conjugate creation operator (needed to conserve electric charge) \( a^c(p, \sigma) \) creates antiparticles in the spinor state \( v_l(p, \sigma) \). Since \( e^{ip \cdot x} = e^{i(-Et + p \cdot x)} \), the factor \( e^{-ip \cdot x} \) associated with the antiparticle state \( v_l(p, \sigma) \) implies that it can also be interpreted as a negative energy solution with negative momentum.

The mode functions of momentum \( p \) and spin \( \sigma = \pm 1/2 \) are given by Lorentz transformations

\[
u_l(p, \sigma) = \sqrt{\frac{mc}{p^0}} D(L(p)) u(0, \sigma), \quad v_l(p, \sigma) = \sqrt{\frac{mc}{p^0}} D(L(p)) v(0, \sigma) \quad (31)
\]
of their rest frame values which are taken to be \( \mathbb{4} \)

\[
u(0, 1/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \nu(0, -1/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad (32)
\]

\[
u(0, 1/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \nu(0, -1/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \quad (33)
\]

The mode functions \( u(p, \sigma) \) and \( v(p, \sigma) \) are eigenvectors of \( -ip^\mu \gamma_\mu \) with eigenvalues +1 and −1 respectively, i.e.

\[
(ip^\mu \gamma_\mu + m)u(p, \sigma) = 0, \quad (-ip^\mu \gamma_\mu + m)v(p, \sigma) = 0, \quad (34)
\]

so that the field Eq.\( (30) \), satisfies the Dirac equation

\[
(\gamma^\mu \partial_\mu + m)\psi(x) = 0. \quad (35)
\]

In the above we have used the notation of Wienberg \( \mathbb{4} \) to define the gamma matrices as follows:

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu, \quad \gamma^0 \equiv -i\alpha_4 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma \equiv -i\alpha = -i \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad (36)
\]
where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the usual $2 \times 2$ Pauli matrices. The above is the chiral representation in which $\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$ (which commutes with each of the $\gamma^\mu$) is diagonal,

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

For an infinitesimal Lorentz transformation with parameters $\omega^\mu_\nu$, written in the form

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu,$$  \hspace{1cm} (37)

the induced unitary transformation $D(\Lambda)$ of the spinors is given by

$$D(\Lambda) = 1 + i\frac{\omega}{2} \gamma^\mu \gamma^\nu.$$  \hspace{1cm} (38)

Here the generators of the Lorentz transformations on the spinors is given by $\gamma^\mu \gamma^\nu = -\frac{i}{4} \gamma^\mu \gamma^\nu$. In the chiral representation, these matrices take the explicit form

$$\mathcal{J}^{ij} = \frac{1}{2} \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \hspace{1cm} \mathcal{J}^{i0} = \frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix},$$  \hspace{1cm} (39)

where the matrices on the left generate rotations and the matrices on the right generate boosts. Note that the generators $\mathcal{J}^{ij}$ are Hermitian so that rotations are represented by unitary matrices. However, the generators are $\mathcal{J}^{i0}$ are anti-Hermitian and therefore pure boosts are not represented by unitary matrices. This follows from the well known theorem that all finite dimensional representations of boost matrices are non-unitary [1].

The relativistic two-particle state $\Phi(\beta^{(1/2)}_{00})$ associated with the non-relativistic spin-entangled Bell state $\beta^{(1/2)}_{00} = (|\uparrow \uparrow \rangle + |\downarrow \downarrow \rangle)/\sqrt{2}$ is given by

$$\Phi(\beta^{(1/2)}_{00}) \equiv \frac{1}{\sqrt{2}} \left( \Phi_{p,1/2;-p,1/2} + \Phi_{p,-1/2;-p,-1/2} \right) \hspace{1cm} (40)$$

$$= \frac{1}{\sqrt{2}} \left( u_A(p,1/2) \otimes u_B(-p,1/2) + u_A(p,-1/2) \otimes u_B(-p,-1/2) \right).$$  \hspace{1cm} (41)

The state $\Phi(\beta^{(1/2)}_{00})$ represents two particles $A$ (Alice) and $B$ (Bob) travelling in opposite directions (which we take to be the $z$-direction) with equal and opposite momenta $p$ in a superposition of products states of both spins up and both spins down, along a quantization axis which, without loss of generality, we also take as the $z$-axis. There are two things to note here. First, if we had made a unitary transformation $U$ given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$  \hspace{1cm} (42)
and defined new rest frame spinors $\tilde{u}(0, \sigma) = \mathcal{U} u(0, \sigma)$ and $\tilde{v}(0, \sigma) = \mathcal{U} v(0, \sigma)$ we’d find $\tilde{u}(0, 1/2) = (1, 0, 0, 0)$ and $\tilde{u}(0, -1/2) = (0, 1, 0, 0)$ while $\tilde{v}(0, 1/2) = (0, 0, 0, 1)$ and $\tilde{v}(0, -1/2) = (0, 0, -1, 0)$. The upper two ”large” components of $\tilde{u}(0, \sigma)$ correspond to familiar 2-spinor components which we associate with the non-relativistic states $| \uparrow \rangle$ and $| \downarrow \rangle$ while the lower two ”small” components of $\tilde{u}(0, \sigma)$ are of order $|p| c/E$, and are typically neglected in the non-relativistic theory. This is true even when we consider the boosted states $\tilde{u}(p, \sigma)$. In the chiral representation used here, the states $u$ and $v$ are just rotated versions of $\tilde{u}$ and $\tilde{v}$. Thus in the limit of small velocities, the state $\Phi(\beta^{(1/2)}_{00}) \rightarrow \beta^{(1/2)}_{00}$.

Secondly, the two-particle state $\Phi(\beta^{(1/2)}_{00})$ involves only the single-particle states $u(p, \sigma)$, and not the anti-particle states $v(p, \sigma)$. This occurs because the positive and negative energy states $u$ and $v$ transform among themselves separately and do not mix with each other under proper LTs, as well as under spatial inversions \textsuperscript{[4]}. The factor $e^{ip \cdot x}$ associated with $u$ is future-directed in the light cone in $p$ space and the factor $e^{-ip \cdot x}$ associated with $v$ is past-directed. Since $p \cdot x$ is a Lorentz invariant, the positive and negative energy states remain distinct, and hence do not mix.

Our two-particle state $\Phi(\beta^{(1/2)}_{00})$ transforms as superposition of direct product states according to Eq.(17), so it is enough for us to consider the Lorentz transformation of the of the single particle state $u(p, \sigma)$. Our goal is to find the Wigner rotation $W(\Lambda, p)$ Eq.(8), associated with an arbitrary Lorentz boost $\Lambda$ of the state $u(p, \sigma)$. $W$ is a rotation that keeps the standard momentum $k^\mu = mc(0, 0, 0, 1)$ invariant, Eq.(4). Without loss of generality we take $p^\mu = L(p)^\mu_{\nu} k^\nu$, with $L(p)$ a standard boost given by Eq.(12), along the $z$-axis with rapidity $\eta$:

$$L(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \eta & \sinh \eta \\ 0 & 0 & \sinh \eta & \cosh \eta \end{pmatrix}, \quad p^\mu = L(p)^\mu_{\nu} k^\nu = mc \begin{bmatrix} 0 \\ 0 \\ \sinh \eta \\ \cosh \eta \end{bmatrix}$$ (43)

Recall that the rows and column of $L(p)$ and $p^\mu$ are labelled by indices $(1, 2, 3, 0)$. In Eq.(13) we have made a boost to a coordinate system $S'$ travelling in the $-z$ direction with velocity given by $\tanh(-\eta) = v/c$ so that in $S'$ the particle, initially at rest in $S$ with momentum $k^\mu$ and state $u(0, \sigma)$, will be observed to have velocity $v/c$ in the $+z$ direction with state $u(p, \sigma)$ (where $|p| = \gamma_v m v$ with $\gamma_v \equiv (1 - v^2/c^2)^{-1/2}$).
For a Lorentz transformation in the ±z direction, \( W \) in Eq.(8) trivially reduces to the identity matrix (0 angle rotation), since two boosts in the same direction are equivalent to a single boost along the same direction. Thus, as observed from either Alice’s or Bob’s rest frame, the state remains unaltered. Therefore, without loss of generality, we will consider a boost \( \Lambda \) in the \( x \) direction with rapidity \( \omega \) corresponding to a LT to a frame travelling along the \(-x\) direction with velocity \(-v_x\) such that \( \tanh(-\omega) = v_x/c \):

\[
\Lambda = \begin{pmatrix}
\cosh \omega & 0 & 0 & \sinh \omega \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \omega & 0 & 0 & \cosh \omega \\
\end{pmatrix},
\]

(44)

with

\[
(\Lambda p)^\mu = \Lambda^\mu_\nu p^\nu = mc \begin{pmatrix}
\sinh \omega \cosh \eta \\
0 \\
\sinh \eta \\
\cosh \omega \cosh \eta \\
\end{pmatrix} \equiv mc \begin{pmatrix}
\sin \theta \sinh \xi \\
0 \\
\cos \theta \sinh \xi \\
\cosh \xi \\
\end{pmatrix}.
\]

(45)

In Eq.(45) we have introduced the polar angle \( \theta \) which \( \mathbf{p}_\Lambda \) makes with respect to the \( z \) axis in the \( xz \) plane, and the rapidity \( \xi \) by the relations

\[
\tan \theta = \frac{\sinh \omega}{\tanh \eta} = (\mathbf{p}_\Lambda)_1/(\mathbf{p}_\Lambda)_3, \\
cosh \xi = \cosh \omega \cosh \eta = E_{\mathbf{p}_\Lambda} / mc^2, \\
\sinh \xi = \sqrt{\cosh^2 \omega \cosh^2 \eta - 1} = |\mathbf{p}_\Lambda| / mc.
\]

(46) (47) (48)

We now want to construct the standard boost Lorentz transformation \( L^{-1}(\Lambda \mathbf{p}) \) such that \( L(\mathbf{p}) \) takes \( k \rightarrow \Lambda \mathbf{p} \) directly from rest. From Eq.(12) we identify \((\hat{\mathbf{p}}_\Lambda)_1 = \sin \theta \) and \((\hat{\mathbf{p}}_\Lambda)_3 = \cos \theta \) and the rapidity as \( \xi \) appropriate for \( L(\mathbf{p}) \). For \( L^{-1}(\mathbf{p}) \) we let \( \theta \rightarrow \theta + \pi \) (a LT in the reverse direction) thereby obtaining

\[
L^{-1}(\mathbf{p}) = \begin{pmatrix}
1 + (\cosh \xi - 1) \sin^2 \theta & 0 & (\cosh \xi - 1) \sin \theta \cos \theta & -\sin \theta \sinh \xi \\
0 & 1 & 0 & 0 \\
(\cosh \xi - 1) \sin \theta \cos \theta & 0 & 1 + (\cosh \xi - 1) \sin^2 \theta & -\cos \theta \sinh \xi \\
-\sin \theta \sinh \xi & 0 & -\cos \theta \sinh \xi & \cosh \xi \\
\end{pmatrix},
\]

(49)

A brute force calculation reveals that indeed, \( L^{-1}(\mathbf{p})^{\mu} (\mathbf{p})^\nu = k^{\mu} \). A quick way to see this is to note that the 4th column of \( L(\mathbf{p}) \) (obtained from Eq.(13) by letting \((\sin \theta, 0, \cos \theta) \rightarrow \)
\(- \sin \theta, 0, -\cos \theta\) is just \((A_p)/mc \) given by Eq.(43). Since \(L^{-1}(A_p) L(A_p) = I\) by construction, \(L^{-1}(A_p)\) acting on the 4th column of \(L(A_p)\) produces \(k^\mu = mc(0, 0, 0, 1)\).

In order to calculate the Wigner rotation \(W(A, p) = L^{-1}(A_p) \Lambda L(p)\), we need the product of the matrices \(\Lambda L(p)\):

\[
\Lambda L(p) = \begin{pmatrix}
\cosh \omega & 0 & \sinh \omega \sinh \eta & \sinh \omega \cosh \eta \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \eta & \sinh \eta \\
\sinh \omega & 0 & \cosh \omega \sinh \eta & \cosh \omega \cosh \eta
\end{pmatrix}. \tag{50}
\]

To check that \(W\) represents a pure rotation we consider a spatial vector \(z^\mu \equiv (0, 0, 1, 0)\) in the rest frame and compute its transformation under \(W\). For a pure rotation we must have

\[
(Wz)^\mu = W^\mu_\nu z^\nu \equiv \begin{pmatrix}
\sin \Omega_p \\
0 \\
\cos \Omega_p \\
0
\end{pmatrix}, \quad \text{where} \quad z^\mu \equiv \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}. \tag{51}
\]

Equation (51) represents a pure rotation about the \(y\) axis by an angle \(\Omega_p\), since the two pure boosts in Eq.(50) both occur in the \(xz\) plane. Noting that \(\Lambda L(p)z\) is the third column of Eq.(50), a straightforward, but tedious calculation of \(L^{-1}(A_p)\) times this vector, using the definitions of \(\xi\) and \(\theta\) from Eq.(46) yields

\[
(Wz)^\mu = \begin{pmatrix}
\sinh \eta \sinh \omega / (1 + \cosh \omega \cosh \eta) \\
0 \\
(\cosh \omega + \cosh \eta) / (1 + \cosh \omega \cosh \eta) \\
0
\end{pmatrix}, \tag{52}
\]

allowing us to identify the Wigner rotation angle \(\Omega_p\) by

\[
\tan \Omega_p = \frac{\sinh \eta \sinh \omega}{\cosh \omega + \cosh \eta} \equiv \frac{\sinh \eta \tanh \eta}{\cosh \omega + \cosh \eta} \tan \theta. \tag{53}
\]

From Eq.(53) we can infer that for all values of \(\eta\) and \(\omega\) associated with boosts \(L(p)\) in the \(z\) direction and with \(\Lambda\) in the \(x\) direction, respectively we have

\[
\Omega_p < \theta, \quad 0 \leq \eta, \omega < \infty, \tag{54}
\]

where \(\theta\) is the angle that \(p_\Lambda\) makes with \(p\) (see Fig.(1)).
FIG. 1: Effect of a boost $\Lambda$ in x direction on the electron spinors $u(\pm p, \sigma)$. In the frame $S$, the (blue) electrons (a) $u(\pm p, 1/2)$, (b) $u(\pm p, -1/2)$ are travelling in the $\pm z$ direction with momentum $\pm p$ with spins aligned or anti-aligned along the quantization axis $z$. The figures show the (red) electrons (a) $u(\pm p_\Lambda, 1/2)$, (b) $u(\pm p_\Lambda, -1/2)$ as observed in a frame $S'$ travelling along the $-x$ direction with respect to $S$ with velocity $v_x/c$. As observed by $S'$, the momentum $\pm p_\Lambda$ of the electrons rotates by an angle $\pm \theta$ about the $+y$ axis (pointing out of the plane of the page) where $+\theta$ is a counter clockwise rotation. However, the direction of spin is observed by $S'$ to rotate by an angle $\pm \Omega p$, in the same sense as $\theta$, but of lesser magnitude.

We are now ready to describe the effect of this Wigner rotation on the transformations of the spinor $u(p, \sigma)$ according to Eq.(24). First we need a representation of $u(p, \sigma)$ for arbitrary $p$. This is given by the formula Eq.(31) using the rest frame spinors in Eq.(32). From Eq.(37) - Eq.(39) we have the spinor representation $D(L(p))$ of a standard boost $L(p)$. 

16
of rapidity $\zeta$

$$D(L(p)) = e^{i/2 \sigma \omega_i} = \exp \left[ -\frac{\zeta}{2} \begin{pmatrix} \sigma \cdot \hat{p} & 0 \\ 0 & -\sigma \cdot \hat{p} \end{pmatrix} \right]$$

$$= \cosh \frac{\zeta}{2} \begin{pmatrix} 1 - \hat{p}_3 \tanh \frac{\zeta}{2} & -\hat{p}_- \tanh \frac{\zeta}{2} & 0 & 0 \\ -\hat{p}_+ \tanh \frac{\zeta}{2} & 1 + \hat{p}_3 \tanh \frac{\zeta}{2} & 0 & 0 \\ 0 & 0 & 1 + \hat{p}_3 \tanh \frac{\zeta}{2} & \hat{p}_- \tanh \frac{\zeta}{2} \\ 0 & 0 & \hat{p}_+ \tanh \frac{\zeta}{2} & 1 - \hat{p}_3 \tanh \frac{\zeta}{2} \end{pmatrix}. \quad (55)$$

In Eq. (53), $L(p)$ is a coordinate Lorentz transformation to a frame $S'$ moving with velocity $v/c = |p|c/E_p = \tanh(-\zeta)$ such that from $S'$ the particle at rest in frame $S$ is observed to have velocity $v/c$. The vector $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ is a unit vector in the direction of $p$ with $\hat{p}_i \equiv \hat{p}_1 \pm i\hat{p}_2$ and $\omega_i = \hat{p}_i$. In terms of the transformed momenta $\hat{p}$ and energy $E_p$, we have the following relations

$$\cosh \zeta = \frac{E_p}{mc^2}, \quad -\sinh \zeta = \frac{|p|}{mc}, \quad -\tanh \zeta = \frac{v}{c}$$

$$\cosh \frac{\zeta}{2} = \sqrt{\frac{E_p + mc^2}{2mc^2}}, \quad -\tanh \frac{\zeta}{2} = \frac{|p|c}{E_p + mc^2}. \quad (56)$$

Taking into account that $p^0 = mc \cosh \zeta$ so that $\sqrt{mc/p^0} = \sqrt{mc^2/E_p}$, Eq. (34) yields

$$u(p, 1/2) = \frac{\cosh \zeta}{\sqrt{2 \cosh \zeta}} \begin{pmatrix} 1 - \hat{p}_3 \tanh \frac{\zeta}{2} \\ -\hat{p}_+ \tanh \frac{\zeta}{2} \\ 1 + \hat{p}_3 \tanh \frac{\zeta}{2} \\ \hat{p}_+ \tanh \frac{\zeta}{2} \end{pmatrix}, \quad u(p, -1/2) = \frac{\cosh \zeta}{\sqrt{2 \cosh \zeta}} \begin{pmatrix} -\hat{p}_- \tanh \frac{\zeta}{2} \\ 1 + \hat{p}_3 \tanh \frac{\zeta}{2} \\ \hat{p}_- \tanh \frac{\zeta}{2} \\ 1 - \hat{p}_3 \tanh \frac{\zeta}{2} \end{pmatrix}. \quad (57)$$

The content of Eq. (24) is that under a Lorentz transformation $\Lambda$ taking $p \to \Lambda p$ the transformed spinors (right hand side of Eq. 24) can be re-written as a Wigner rotation of the spinors $u(p, \sigma)$ (left hand side of Eq. 24), the later of which can be obtained from Eq. (7) by a substitution of $\hat{p} \to \hat{p}_\Lambda$ with the appropriate redefinition $\cosh \zeta \to E_{p,\Lambda}/mc^2$. With the Wigner angle $\Omega_p$ in hand, the rotation matrices on the left hand side of Eq. (24) are given by [3]

$$D_{\sigma' \sigma}^{(j_\Lambda)}(W(\Lambda, p)) = \begin{pmatrix} \cos(\Omega_p/2) - \sin(\Omega_p/2) \\ \sin(\Omega_p/2) \cos(\Omega_p/2) \end{pmatrix}, \quad (58)$$
with the rows and columns of the matrix in Eq.(58) labelled by \( \sigma = (1/2, -1/2) \). Thus in
matrix notation we can write Eq.(24) as

\[
u'(p, \frac{1}{2}) \equiv \sqrt{\frac{p^0}{\Lambda p^0}} D(\Lambda) u(p, \frac{1}{2}) = \cos \left( \frac{\Omega p}{2} \right) u(p_{\Lambda}, \frac{1}{2}) + \sin \left( \frac{\Omega p}{2} \right) u(p_{\Lambda}, -\frac{1}{2}), \quad (59)
\]

\[
u'(p, -\frac{1}{2}) \equiv \sqrt{\frac{p^0}{\Lambda p^0}} D(\Lambda) u(p, -\frac{1}{2}) = -\sin \left( \frac{\Omega p}{2} \right) u(p_{\Lambda}, \frac{1}{2}) + \cos \left( \frac{\Omega p}{2} \right) u(p_{\Lambda}, -\frac{1}{2}).\quad (60)
\]

Note that for \( u(-p, \sigma) \) the standard boost in Eq.(43) is performed in the opposite direction
(coordinate transformation to a frame moving along the +z axis with velocity \( v/c \)) so that
we simply change the sign of the rapidity \( \eta \rightarrow -\eta \). This leads to the following sign changes

\[
p \rightarrow -p \Rightarrow \theta \rightarrow -\theta, \quad \Omega_{-p} = -\Omega_p.\quad (61)
\]

In Fig.(1) we illustrate the transformation of the product states \( u_A(p, 1/2) \otimes u_B(-p, 1/2) \)
and \( u_A(p, -1/2) \otimes u_B(-p, -1/2) \) appearing as terms in \( \Phi(\beta_{\frac{1}{2}}) \), which correspond to the
non-relativistic product states \( |\uparrow_A, \uparrow_B\rangle \) and \( |\downarrow_A, \downarrow_B\rangle \), respectively. The effect of the Lorentz
boost \( \Lambda \) is to rotate \( p \rightarrow p_{\Lambda} \) through and angle \( \theta \) defined by Eq.(46). The orientations of
the spins with respect to the quantization axis \( z \) are rotated by the momentum dependent
Wigner angle \( \Omega_p \) defined in Eq.(53), such that \( \Omega_p < \theta \). The rotation is counter-clockwise
for particles momentum \( p \) and clockwise for particles with momentum \( -p \).

IV. PHOTONS: SPIN 1 FIELDS

The massless spin 1 photon field is given by

\[
a^\mu(x) = \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_{\sigma = \pm 1} \left[ e^\mu(p, \sigma) e^{ip \cdot x} a(p, \sigma) + e^\mu(p, \sigma)^* e^{-ip \cdot x} a^\dagger(p, \sigma) \right], \quad (62)
\]

where \( a^\dagger(p, \sigma) \) creates photons in \( \sigma = \pm 1 \) helicity states (right and left circular polarization)
\( e^\mu(p, \sigma) \). Since Eq.(62) has the form of a 4-vector field the gauge independent representation
\( D(\Lambda) \) of Lorentz transformation \( \Lambda \) is given by the LT itself [2], i.e.

\[
x'^\mu = \Lambda_{\mu}^\nu x^\nu \Rightarrow e^\mu(p, \sigma) \equiv D(\Lambda)^\mu_\nu e^\nu(p, \sigma) = \Lambda_{\mu}^\nu e^\nu(p, \sigma). \quad (63)
\]

However, as is well known, \( a^\mu(x) \) cannot be a pure 4-vector field since the electromagnetic
field has only two degrees of freedom. Thus we have a 4-vector field with a gauge freedom.
Matters are also complicated by the fact that while rotations of 4-vectors are represented
by finite dimension unitary matrices, the finite dimension matrices representing boosts are non-unitary. The question at hand is can one find a finite dimensional unitary representation for the transformation of the polarization vectors? This was answered by Han et al [3] who showed that by first pre-multiplying a polarization vector by a matrix \( D \) in the little group appropriate for photons, and then applying the boost, the net effect is a pure spatial rotation of \( \epsilon^\mu(p, \sigma) \). This procedure essentially reduces to a choice of a particular gauge (described below) in which there are only two photon polarization vectors [6] which always lie in the plane perpendicular to the photon’s momentum. This choice of gauge consistent with most common definitions of polarization vectors found in the quantum optics literature. In the following we follow gauge-fixing choice of [3], and afterwards return to make connection with the gauge independent transformation equation for massless particles as given by Weinberg, Eq.(16).

Our ultimate goal is to describe the effect of Lorentz boost on the 2-qubit polarization entangled state \( \beta_{00}^{(1)} = (|HH\rangle + |VV\rangle)/\sqrt{2} \) which is given by

\[
\Phi(\beta_{00}^{(1)}) \equiv \frac{1}{\sqrt{2}}(\Phi_{p,1;-p,1} + \Phi_{p,-1;-p,-1}) \\
= \frac{1}{\sqrt{2}}(\epsilon^\mu_A(p, 1) \otimes \epsilon^\mu_B(-p, 1) + \epsilon^\mu_A(p, -1) \otimes \epsilon^\mu_B(-p, -1)).
\]

Here we have again taken Alice and Bob to be travelling along the \( z \) axis with equal and opposite momentum \( p \). We let the single-particle horizontal polarization state \( |H\rangle \) be represented by the positive helicity state \( \epsilon^\mu(p, +1) \equiv \epsilon^\mu_+(p) \), and the vertical polarization state \( |V\rangle \) by \( \epsilon^\mu(p, -1) \equiv \epsilon^\mu_-(p) \). Again, it is enough to consider the transformation of the single particle polarization state \( \epsilon^\mu(p, \sigma) \). For the first calculation we choose the photon momentum \( p \) in \( S \) to lie along the \( z \) axis, the direction of the standard momentum, and consider a boost along the \( x \) direction. The net result of this calculation will be the simple result, that in the boosted frame, an observer \( S' \) travelling along the \(-x\) axis will observe a tilting of the plane of polarization towards the \(+x\) axis, (see Fig.(2)a). We then generalize this calculation to an arbitrary LT (not necessarily a boost only) for a photon of momentum \( p \) along an arbitrary direction in \( S \). We show that for the observer \( S' \), the triad of 3-vectors \( (e_x(p), e_y(p), p) \) is rigidly rotated to the triad \( (e_x(p_A), e_y(p_A), p_A) \), where \( e_x(p) \) is the 3-vector portion of \( (\epsilon^\mu_+(p) + \epsilon^\mu_+(p))/\sqrt{2} \) and \( e_y(p) \) is the 3-vector portion of \(-i(\epsilon^\mu_+(p) - \epsilon^\mu_+(p))/\sqrt{2} \) (see Fig.(2)b).

We begin by considering a photon travelling in the \(+z\) direction in the local inertial frame
FIG. 2: (a) Effect of pure-boost $\Lambda = B_x(\omega)$ in $x$ direction on the polarization vector $\epsilon_x(p)$ given by the 3-vector portion of $(\epsilon^\mu_+(p) + \epsilon^\mu_-(p))/\sqrt{2}$. In a frame $S$ the photon (blue) is propagating in the $+z$ with momentum $p$ with the orthogonal polarization vector $\epsilon_x(p)$. In the frame $S'$, travelling in the $-x$ direction with respect to $S$ with velocity $v_x/c$, the photon (red) is observed to have momentum $p_\Lambda$ inclined at a polar angle $\theta$ in the $+xz$ plane. The plane of polarization of $S$ is observed by $S'$ to rotate to a new plane at angle $\theta$ with respect to $S$. (b) Effect of an arbitrary Lorentz transformation $\Lambda$. In frame $S$ (blue) the photon has momentum $p$, not necessarily along the $z$ direction, with orthogonal polarization vector $\epsilon_x(p)$. In a frame $S'$, related to $S$ by a Lorentz transformation $\Lambda$, the triad $(\epsilon_x(p), \epsilon_y(p), p)$ in $S$ is observed to be rigidly rotated to the triad $(\epsilon_x(p_\Lambda), \epsilon_y(p_\Lambda), p_\Lambda)$. 

S. We take as the standard momentum $k^\mu = (0, 0, 1, 1)$ such that $k^\mu k_\mu = 0$. An arbitrary 4-potential has the form $A^\mu(x) = A^\mu \exp \left(ik(z - ct)\right)$ with $A^\mu = (A^1, A^2, A^3, A^0) = (A, A^0)$. As stated in Section II, the little group for photons is $ISO(2)$ (often called $E(2)$), the Euclidean group of rotations and translations in the polarization plane perpendicular to the
momentum of the photon. The generator of rotations $J_3$ is given by

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(66)
called the helicity operator. The generators $A$ and $B$ for translations in the plane of polarization are given by $A = J_2 + K_1$ and $B = -J_1 + K_2$ with $[J_3, A] = iB$, $[J_3, B] = -iA$ and $[A, B] = 0$, where $J_i$ is a rotation about the $i$th axis and $K_i$ is a pure boost along the $i$th direction [2, 7]. The operators $A$ and $B$ generate translations and their particular form need not concern us here. However, we note that they are responsible for inducing gauge transformations of the 4-potentials, $A' = A + \partial\chi$. We take as our 4-potentials $A$ eigenstates of $J_3$ namely

$$J_3 \epsilon_\pm^\mu(k) = \pm \epsilon_\pm^\mu(k), \quad \epsilon_\pm^\mu(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \\ 0 \end{pmatrix}. \quad (67)$$

In order for a $A^\mu$ to be a proper 4-potential (i.e. represent a physical polarization 4-vector) it must satisfy the following two properties: (1) $A^0 = 0$ and (2) $p \cdot A = 0$. These two conditions are equivalent to the combined effect of the Lorentz condition

$$\frac{\partial}{\partial x^\mu} A^\mu(x) = p^\mu A^\mu(x) = 0, \quad (68)$$

and the transversality condition

$$\nabla \cdot A(x) = 0, \quad \text{or} \quad p \cdot A = 0. \quad (69)$$

The first of these conditions Eq.(68) is a Lorentz invariant statement, the second Eq.(69), is not. Han et al. refers to these two conditions as the helicity gauge.

From the form of $k^\mu$ and $\epsilon_\pm^\mu(k)$ in Eq.(67), a standard boost $L(p)$ in the $z$ direction with rapidity $\eta$, given by Eq.(43), will change the momentum to $p^\mu = L(p)^\mu_\nu k^\nu = (0, 0, k, k)$ with $k \equiv |p| = (\cosh \eta + \sinh \eta)$ but will leave $\epsilon_\pm^\mu(p) \equiv \epsilon_\pm^\mu(k)$ invariant. This last statement is obvious, since vectors perpendicular to the direction of a pure boost are unaltered. Thus, as in the case for massive spin 1/2 particles, the state observed from either Alice’s or Bob’s
frame of reference, is unaltered. Therefore, we can again consider, without loss of generality, a boost $\Lambda$ in the $x$ direction given by Eq. (14) i.e. a transformation to a frame $S'$ moving in the $-x$ direction with velocity $v_x/c$ such $\tanh(-\omega) = v_x/c$. In the frame $S'$, the photon, originally travelling in the $+z$ direction in frame $S$ will be observed to be travelling in the $+xz$ plane. Under $\Lambda$, $p \to \Lambda p$ with

$$
(Ap)^{\mu} = \Lambda^{\mu}_{\nu} p^\nu = k \begin{bmatrix} \sinh \omega & 0 & 1 \\ 0 & 1/cosh \omega & 0 \\ \cosh \omega & 0 & 1 \end{bmatrix} \equiv k \begin{bmatrix} \sin \theta \\ 0 \\ \cos \theta \end{bmatrix}.
$$

(70)

In Eq. (70) we have defined the polar rotation angle $\theta$ that $p_\Lambda$ makes with $p$ by factoring out $|p_\Lambda| = k \cosh \omega$ and defining

$$
\sin \theta \equiv \tanh \omega, \quad \cos \theta \equiv 1/cosh \omega, \quad \tan \theta = \sinh \omega.
$$

(71)

Consider $\tilde{\epsilon}_\pm \equiv \Lambda \epsilon_\pm(p) = k (\cosh \omega, \pm i, 0, \sinh \omega)$. Although it satisfies the transversality condition $(Ap)^{\mu} \tilde{\epsilon}_{\pm \mu}(p) = 0$, it fails to be a valid 4-potential since $\tilde{\epsilon}_{\pm 0}(p) \neq 0$.

In order to calculate the Wigner rotation $W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p)$ we note the standard boost $L(p)$ which takes $k \to p$ can in general be written as

$$
L(p) = R(\tilde{p}) B_z(|p|),
$$

(72)

where $R(\tilde{p})$ is a pure rotation that takes the $z$ axis into $\tilde{p}$. For an a momentum in an arbitrary direction $\tilde{p} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ we can take $R(\tilde{p}) = R_z(\phi) R_y(\theta)$, where $R_y(\theta)$ is a rotation about the $y$ axis taking $(0, 0, 1)$ to $(\sin \theta, 0, \cos \theta)$, followed by a rotation about the $z$ axis by the angle $\phi$, taking the intermediate direction to $\tilde{p}$. $B_z(|p|)$ is a boost in the $z$ direction taking the standard momentum $k$ of unit magnitude $|k| = 1$ to magnitude $|p|$, given by

$$
B_z(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (u^2 + 1)/2u & (u^2 - 1)/2u \\ 0 & 0 & (u^2 - 1)/2u & (u^2 + 1)/2u \end{bmatrix}.
$$

(73)

In terms of a rapidity $\xi$ (see Eq. (13) with $\eta \to \xi$), we have $(u^2 + 1)/2u = \cosh \xi$ or $u = \cosh \xi + \sinh \xi$. In addition, we define the polarization vector $\epsilon_\pm(\tilde{p})$ for arbitrary momentum.
\( \mathbf{p} \) in terms of standard polarization vector \( \epsilon^\mu_\pm(\mathbf{k}) \) of Eq.(67) by

\[
\epsilon^\mu_\pm(\mathbf{p}) \equiv L(p) \epsilon^\mu_\pm(\mathbf{k}) = R(\hat{\mathbf{p}}) B_z(|\mathbf{p}|) \epsilon^\mu_\pm(\mathbf{k}) = R(\hat{\mathbf{p}}) \epsilon^\mu_\pm(\mathbf{k}),
\]

(74)

where the last equality follows since \( \epsilon^\mu_\pm(\mathbf{k}) \) is left invariant by boosts along the \( z \) direction.

For the particular case we have chosen to consider, i.e. \( \mathbf{p} = |\mathbf{p}| \hat{\mathbf{k}} \) along the \( z \) direction, we have

\[
L(\mathbf{p}) \equiv B_z(k), \text{ with } k = |\mathbf{p}|.
\]

With our Lorentz transformation taken to be \( \Lambda \equiv B_x(\omega) \)

Eq.(44), we can compute \( L(\Lambda \mathbf{p}) \) by substituting \( \mathbf{p}_\Lambda \) for \( \mathbf{p} \) in Eq.(72), with

\[
|\mathbf{p}_\Lambda| = k \cosh \omega.
\]

Using the angle \( \theta \) defined in Eq.(70) which \( \Lambda \mathbf{p} = L(\Lambda \mathbf{p}) \) makes with the \( z \) axis, we have

\[
L(\Lambda \mathbf{p}) = R_y(\theta) B_z(|\mathbf{p}_\Lambda|) \text{ with }
\]

\[
R_y(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(75)

Note that we can write \( B_z(|\mathbf{p}_\Lambda|) = B_z(|\mathbf{p}_\Lambda|/|\mathbf{p}|) B_z(|\mathbf{p}|) = B_z(\cosh \omega) L(p) \) so that we can write \( (\Lambda \mathbf{p}) \) in two equivalent forms:

\[
(\Lambda \mathbf{p}) = B_x(\omega) \mathbf{p} = R_y(\theta) B_z(\cosh \omega) \mathbf{p} = L(\Lambda \mathbf{p}) k.
\]

(76)

Collecting these results we have

\[
W(\Lambda, \mathbf{p}) = L^{-1}(\Lambda \mathbf{p}) \Lambda L(p)
\]

\[
= L^{-1}(p) \left[ B_z^{-1}(\cosh \omega) R_z^{-1}(\theta) B_z(\omega) \right] L(p)
\]

\[
\equiv L^{-1}(p) D^{-1}(\omega) L(p),
\]

(77)

where we have defined

\[
D(\omega) \equiv B_z^{-1}(\omega) R_z(\theta) B_z(\cosh \omega).
\]

(78)

A trivial rearrangement of second equality of Eq.(76) shows that

\[
D(\omega)^\mu_\nu p^\nu = p^\mu,
\]

(79)

so that \( D(\omega) \) is a member of the little group of \( p \), i.e. LTs which leave \( p \) (as opposed to \( k \)) invariant. Eq.(73) also arises from a rearrangement of the defining property of the Wigner
by construction, we can also transform $p \rightarrow (\Lambda p)$ via the product of matrices $\Lambda D(\omega)$ acting on $p$. Similarly, if we precede the action of $\Lambda$ on $\epsilon^\mu_\pm(p)$ by $D(\omega)$ we find [3]

$$
\epsilon^\mu_\pm(p) = \Lambda D(\omega)\epsilon^\mu_\pm(p)
= (\Lambda \Lambda^{-1}) R_y(\theta) \left(B_z(\omega) \epsilon^\mu_\pm(p)\right)
= R_y(\theta) \epsilon^\mu_\pm(p)
= \epsilon^\mu_\pm(p_\Lambda)
$$

where we have used the fact that $B_z(\cosh \omega) \epsilon^\mu_\pm(p) = \epsilon^\mu_\pm(p)$. Since $\theta = \tan^{-1}(\sinh \omega)$ is the polar angle $(\Lambda p)$ makes with respect to $p$, the net effect of the transformation is just a rotation of the plane of polarization by the angle $\theta$ (see Fig.4a). Thus we have $\epsilon^\mu_\pm(p, \sigma) \equiv \epsilon^\mu_\pm(p_\Lambda)$, i.e. the polarization vector appropriate for a photon with momentum in the direction $p_\Lambda$ (see Eq.(74)). Note that $\epsilon^\mu_\pm(p_\Lambda)$ is a valid 4-potential in the helicity gauge since $\epsilon^0_\pm(p_\Lambda) = 0$ and $(\Lambda p)^\mu \epsilon_{\pm\mu}(p) = 0$ as required by Eq.(68) and Eq.(69).

The salient point here is that the representation $D(\Lambda)$ of the Lorentz transformation $\Lambda$ as given by $\Lambda D(\omega)$ induces the unitary rotation $R_y(\theta)$ on the polarization vector $\epsilon^\mu_\pm(p)$ by the Wigner angle $\theta$. This derives from Eq.(76) which states that $\Lambda p$ can be reached in two ways from $p$: first by the direct action of $\Lambda = B_x$ on $p$, and second by a boost $B_z$ along the $z$ direction with rapidity $\cosh \omega$ acting on $p$, followed by a Wigner rotation about the $y$ axis by the angle $\theta$. These parameters are related by $|p_\Lambda|/|p| = \cosh \omega$ and $\tan \theta = \sinh \omega$.

We can now generalize the above arguments to an arbitrary Lorentz transformation $\Lambda$, which is not necessarily a pure boost, and for momentum $p$ in $S$ which lies along an arbitrary direction. Thus we take $p = L(p) k$ with $L(p)$ given by Eq.(72) and $\Lambda$ arbitrary. The key ingredient is to find $D$ which leaves $p$ invariant. We begin by generalizing Eq.(73)

$$(\Lambda p) = L(\Lambda p) k$$

rotation Eq.(9), using the expression for $W$ in Eq.(77) and $p = L(p) k$. We also note $D(\omega)$ induces gauge transformations when acting on 4-potentials:

$$
\bar{\epsilon}^\mu_\pm(p) \equiv D(\omega)^\mu_\nu, \epsilon^\mu_\pm(p) = (1, \pm i, -\tanh \omega, -\tanh \omega)
= \epsilon^\mu_\pm(p) - k^\mu \tanh \omega
$$

(as can be shown by direct matrix multiplication) which can be associated with a gauge function $\chi = (ct - z) k^\mu \tanh \omega$.
\[ \Lambda p = R(\hat{p}_\Lambda) B_z(|\hat{p}_\Lambda|) L^{-1}(p) p \]
\[ \Rightarrow p = \Lambda^{-1} R(\hat{p}_\Lambda) B_z(|\hat{p}_\Lambda|) L^{-1}(p) p \]
\[ \equiv D p, \quad (82) \]

where we have used \( L(\Lambda p) = R(\hat{p}_\Lambda) B_z(|\hat{p}_\Lambda|) \) and \( k = L^{-1}(p) p \). By using this definition of \( L(\Lambda p) \) and pre-multiplying by unity in the form of \( L^{-1}(p) L(p) \) we obtain
\[ W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) \]
\[ = L^{-1}(p) D^{-1} L(p) \quad (83) \]

Then defining the representation \( D(\Lambda) \) of the LT \( \Lambda \) as
\[ D(\Lambda) = \Lambda D, \quad (84) \]
as opposed to just \( D(\Lambda) = \Lambda \), we have its action upon \( \epsilon^\mu_{\pm}(p) \) given by
\[ \epsilon^\mu_{\pm}(p) \equiv \Lambda D \epsilon^\mu_{\pm}(p) \]
\[ = \left( \Lambda \Lambda^{-1} \right) R(\hat{p}_\Lambda) \left( B_z(\hat{p}_\Lambda) | L^{-1}(p) \epsilon^\mu_{\pm}(p) \right) \]
\[ = R(\hat{p}_\Lambda) \epsilon^\mu_{\pm}(k) \]
\[ \equiv \epsilon^\mu_{\pm}(p_\Lambda), \quad (85) \]

where the last equality follows from Eq.(74). The last two lines of Eq.(85) leads to the transformation
\[ \epsilon^\mu_{\pm}(p_\Lambda) = D(\Lambda) \epsilon^\mu_{\pm}(p) = R(\hat{p}_\Lambda) R^{-1}(\hat{p}) \epsilon^\mu_{\pm}(p) \quad (86) \]

which is explicitly unitary. As depicted in Fig.(2)b, the transformation in Eq.(86) rigidly rotates the triad of 3-vectors \((e_x(p), e_y(p), p)\) into the triad \((e_x(p_\Lambda), e_y(p_\Lambda), p_\Lambda)\).

To make connection with Weinberg’s transformation equation Eq.(26), we note that in Eq.(81) and Eq.(86) there are no explicit phase factors of \( \exp(i \sigma \theta) \). This results from the (helicity) gauge-fixing convention of Han et al which represents the LT \( D(\Lambda) \) acting on the 4-potentials as \( \Lambda D \). In essence, the gauge transformations induced by \( \Lambda \) are undone by the \( p \)-little group element \( D \). The price one pays for ensuring explicit unitary representations for boosts is that \( D(\Lambda) \) must be represented by \( \Lambda D \).

This is in contrast to Weinberg’s gauge invariant representation of \( D(\Lambda) \) by \( \Lambda \) itself, Eq.(83). In the local Lorentz frame of the photon, the Wigner rotation is represented in a
gauge invariant manner by the product of a translation $S(\alpha, \beta)$ and a rotation $R_z(\theta)$, Eq. (14). Acting upon $\epsilon^\mu_{\pm}(k)$, the rotation about the $z$ axis produces the phase factor $\exp(i \sigma \theta)$ appearing in Eq. (23). However, the action of the translation $S(\alpha, \beta)$ induces gauge transformations on the 4-potential, so that in general the transformed potential contains a non-zero time component $\epsilon^0_{\pm}$ and is no longer a helicity state. For general momentum $p$ the transformed 4-potential is given by $\Lambda^\mu\nu \epsilon^\nu_{\pm}(p)$ plus gauge induced components parallel to $p^\mu$ (see discussion in [2], p249-251). All this stems from the requirement that $\epsilon^0_{\pm}(p)$ and hence the quantum field operator $a^0$ vanish in all Lorentz frames, ensuring that the field $a^\mu$ cannot be a true 4-vector field. Of course, the gauge invariant physical electric and magnetic fields are not affected by such considerations. However, it is the polarization vectors that are found in quantum optics to be most useful in representing the state of the system. At the minor cost of losing some generality by gauge fixing, one gains explicit unitarity in the representations of boosts by finite dimensional matrices.

V. SUMMARY AND DISCUSSION

In non-relativistic quantum mechanics, the only kinematic transformations of reference frames we are allowed to consider are translations and rotations, which are explicitly unitary. In relativistic quantum mechanics, we must also consider Lorentz boosts, which when represented by finite dimensional matrices are explicitly non-unitary. In spite of this, each single particle state in a multi-particle state undergoes an effective, momentum dependent, local unitary rotation under Lorentz boosts governed by the little group element $W$ which leaves the appropriate standard momentum $k$ invariant. For massive spin $1/2$ particles, the standard momentum (in the particle’s rest frame) is $k^\mu = mc(0, 0, 0, 1)$ and the little group is $SO(3)$, the group of ordinary rotations in 3D. Even though $W$ itself is not unitary, its $3 \times 3$ ($x, y, z$)-block acts as an effective rotation matrix (since the components $W^i_t$ need not be zero). For a pure boost taking momentum $p$ into $p_\Lambda$, the spin of the transformed particle is rotated by the Wigner angle $\Omega_p$, which is in the same sense, but less in magnitude than the polar angle $\theta$ which $p_\Lambda$ makes with $p$. For massless photons, the little group is $ISO(2)$, the group of rotations and translations in the plane perpendicular to the standard momentum $k^\mu = (0, 0, 1, 1)$. Though $W$ itself is not unitary, in a gauge invariant description of the states, its $2 \times 2$ ($x, y$)-block acts as effective rotation matrix (since components outside
this block are not necessarily zero). By fixing the choice of gauge the transformation which takes the polarization vector $\epsilon_\pm^\mu(p)$ to $\epsilon_\pm^\mu(p_\Lambda)$ can be made explicitly unitary, i.e. a $4 \times 4$ rotation matrix. In this case the triad of 3-vectors $(\epsilon_x(p), \epsilon_y(p), p)$ in one inertial frame is observed to be rigidly rotated to the triad $(\epsilon_x(p_\Lambda), \epsilon_y(p_\Lambda), p_\Lambda)$ in another inertial frame. Since a Lorentz transformation of a (massive or massless) multi-particle state acts as a direct product, each constituent single particle state is transformed at most into a superposition of spin or helicity states with the appropriate transformed momenta. Consequently, tracing out over one state in maximally entangled bipartite state will still produce a maximally mixed density matrix for the reduced state. The entanglement fidelity is not effected by the Lorentz transformation. In this work we explicitly demonstrated the above considerations for the relativistic generalization of a symmetric Bell state comprised of electrons, and of photons, for arbitrary strength Lorentz boosts.

The case of entanglement for accelerated observers poses a whole host of new problems. Consider first, for example the situation depicted in Fig.(3)a in which Bob (red worldline) is moving with momentum $p$ in the $z$ direction relative to a stationary Alice (blue worldline). At the event $P$ let Alice and Bob share an entangled state $\Phi$, described by Alice as

$$\Phi = \frac{1}{\sqrt{2}} \left( u_A(0, 1/2) \otimes u_B(p, 1/2) + u_A(0, -1/2) \otimes u_B(p, -1/2) \right).$$

If Alice has some other single particle state $\Psi$ which she wishes to teleport to Bob, she can perform the usual procedure of mixing $\Psi$ with her portion of $\Phi$ and transmit the result of her Bell measurement to Bob along a classical channel, depicted in Fig.(3)a as a light signal emitted at the event $Q$. If we consider the teleportation from an inertial frame in which Bob is at rest, the situation is symmetric and Bob observes the same state $\Phi$ except now the momentum for his particle is zero and for Alice it is $-p$. Since we are boosting along the direction of motion of Bob, this is the trivial case of zero Wigner rotation, so the spins are unaltered. Therefore the entanglement fidelity of $\Phi$ is unaffected, as we would expect.

The situation is very different if Bob is not travelling at constant velocity. Consider Fig.(3)b in which Bob (red worldline) is undergoing constant acceleration $a$, while Alice (blue worldline) again remains stationary. Bob’s coordinates $(z_B, t_B)$ are related to Alice’s coordinates $(z_A, t_A)$ by

$$z_A = z_B \cosh at_B, \quad t_A = z_B \sinh at_B.$$
FIG. 3: (a) Minkowski diagram for the case of Alice (blue) stationary and Bob (red) travelling at constant velocity. Alice and Bob share an entangled state $\Phi$ at the event $P$ (see text). Alice can complete the teleportation protocol by sending classical signals to Bob at a representative event $Q$. The entanglement fidelity of the state $\Phi$ is unaltered if viewed from either Alice’s or Bob’s rest frame. (b) Alice (blue) is again stationary, but Bob (red) undergoes constant acceleration. The light-like lines $H^-_B$ and $H^+_B$ form past and future particle horizon corresponding to Bob’s proper times $t_B = -\infty$ and $t_B = +\infty$ respectively. At the event $Q$ Alice crosses $H^+_B$ (in her finite proper time $t_A$), and can no longer communicate with Bob. Bob, however, can still send signals to Alice across $H^+_B$. The status of the entanglement fidelity of the state $\Phi$ is unclear.

In these Rindler coordinates, Bob moves on a hyperbola of constant $z_B$, crossing lines of his proper time $t_B$, which are straight (dotted red) lines emanating from the origin $O$. At event $P$ Alice and Bob again share the entangled state $\Phi$, and Alice wishes to teleport her state $\Psi$ to Bob. Bob’s world is very different from Alice’s since he perceives that he is moving through a thermal bath of radiation at the Unruh temperature $T_U = \hbar a/2\pi k_B c$, where $k_B$ is Boltzman’s constant. Since Alice is in an inertial Lorentz frame, she perceives no such Unruh radiation. In fact it is unclear how states would transform between Alice’s and Bob’s
reference frame since they each employ inequivalent quantization schemes [8]. Alice follows the usual quantization scheme in Minkowski spacetime, as discussed in this paper, and her states are built up from the Minkowski vacuum $|0\rangle_M$ by the usual Minkowski creation and annihilation operators $a_M^\dagger$ and $a_M$, such that $a_M |0\rangle_M = 0$. The right and left Rindler wedges $z_A > 0, z_A > |t_A|$ and $z_A < 0, |z_A| > |t_A|$ labelled $I$ and $II$ respectively in Fig.(3)b, each support complete, and distinct quantization schemes. This results in operators $a_I^\dagger, a_I$ and $a_{II}^\dagger, a_{II}$ and vacua $|0\rangle_I$ and $|0\rangle_{II}$ in region $I$ and $II$ respectively, inequivalent to each other and to $|0\rangle_M$. The Rindler Hamiltonian $H_R$ which annihilates $|0\rangle_M$ and generates time translations with respect to Bob’s proper time $t_B$ is given by $H_R = H_I - H_{II}$, where for a fixed mode $H_I \sim a_I^\dagger a_I$ and $H_{II} \sim a_{II}^\dagger a_{II}$. The Minkowski vacuum through which Bob moves is described by a product over modes of maximally entangled two-mode squeezed states, comprised of superpositions of Fock states of the form $|n\rangle_I \otimes |n\rangle_{II}$ for each mode. (Note: a particle in the right Rindler wedge is correlated with an antiparticle in the left Rindler wedge with opposite spatial momentum, and visa versa). However, since Bob lives in region $I$, he describes his physics in terms of states constructed solely from the operators $a_I^\dagger, a_I$. In addition, Bob is causally disconnect from region $II$, with the light-like lines $H_-$ and $H_+$ in Fig.(3)b acting as his past ($t_B = -\infty$) and future ($t_B = +\infty$) particles horizons. Thus by tracing the maximally entangled state $|0\rangle_M \langle 0|$ over region $II$ states, Bob describes the Minkowski vacuum by a maximally mixed, thermal reduced density matrix. A particle detector carried by Bob will observe the unusual behavior of excitation of the detector accompanied by the emission of a Minkowski particle, i.e. a particle registered by an inertial detector [9, 10].

In examining Fig.(3)b one sees that Alice’s last communication with Bob is at the event $Q$ where she crosses Bob’s future particle horizon $H_+$. This occurs at $t_B = \infty$ with respect to Bob’s proper time, yet at some finite proper time with respect to Alice’s inertial frame. Clearly at this stage the teleportation protocol cannot continue. More importantly is the observation that Bob can still communicate with Alice (say by photons) after she crosses $H_+$, but Alice can no longer communicate with Bob. In this asymmetric situation, with states described by different quantization schemes, it is not at all apparent if the entanglement fidelity of the shared state $\Phi$ is preserved. These considerations are the subject of a future publication.

It is worthwhile to note that by the equivalence principle, the situation considered above
can essentially be considered as the local Lorentz description of a static observer around a black hole. In the case of constant acceleration in Minkowski space the Unruh radiation ultimately stems from the force that is keeping Bob in the state of constant acceleration. At fixed position outside a black hole, the static observer must accelerate to stay in place and experiences a thermal flux of Hawking radiation analogous (though different) to the Unruh radiation in Minkowski space. In both cases the presence of a horizon plays a central role in the resulting radiation that is perceived. The question of entanglement across the horizon, and whether or not unitary evolution still holds when a pure state falls behind the horizon and is apparently converted into pure thermal radiation is still actively debated under the name of the "black hole information loss" problem [11].

It is tantalizing to contemplate whether Unruh and/or Hawking radiation might be derived from a quantum information theoretic point of view. As a heuristic consideration, note that the infinitesimal work \( \Delta W \), performed on a massive particle over its Compton wavelength \( \lambda_c \) (the particle’s characteristic length over which we could consider it to be co-moving with a given inertial frame of constant velocity for a time \( \Delta t_A = \lambda_c/v(t_A) \)) is given by \( \Delta W = Fdx = (ma) (h/mc) = (2\pi)^2 k_BT_U \). Up to a numerical factor this is the energy associated with the thermal bath that Bob perceives as he accelerates through the Minkowski vacuum. This energy is the source of such processes discussed above whereby a detector carried by Bob observes an excitation accompanied by an emission of a Minkowski particle. By Landauer’s erasure principal [12] there is an energy, and hence an entropy cost to erase information. Might this energy absorbed by Bob be considered as going into the erasure of the correlations in the pure state density matrix \( |0\rangle_M \langle 0| \) for the Minkowski vacuum through which he is accelerating, resulting in an entropy increase whose net effect is to create the thermal vacuum which he perceives? In addition, can the loss, in principle, of access to a quantum communication resource such as teleportation, when Alice crosses Bob’s future horizon \( H_+ \), be thought of in terms of erasure of information, and an increase in entropy which is maximized by a thermal mixed state? These considerations will be explored in a future publication.
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31