Palindromic length of words and morphisms in class $\mathcal{P}$

Petr Ambrož, Ondřej Kadlec, Zuzana Masáková, Edita Pelantová

$^a$FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Praha 2, Czech Republic

Abstract

We study the palindromic length of factors of infinite words fixed by morphisms of the so-called class $\mathcal{P}$ introduced by Hof, Knill and Simon. We show that it grows at most logarithmically with the length of the factor. For the Fibonacci word and the Thue-Morse word we provide estimates on the constants of the growth. We also construct an infinite word rich in palindromes for which the palindromic length grows as $\sqrt{n}$.

Keywords: palindromic length; class $\mathcal{P}$; palindromic richness.

1. Introduction

In this paper we show a connection between two conjectures concerning palindromes in languages of infinite words. A palindrome is a word which reads the same forward as backward, such as eye or kayak. An impulse to formulate the first conjecture comes from paper by Hof, Knill and Simon [16], where they studied infinite words generated by primitive morphisms. They defined a certain class of morphisms, called class $\mathcal{P}$, by requiring that the image of any letter $a$ in the alphabet is in the form $a \mapsto pq_a$, where $p$ and $q_a$ are palindromes, $p$ being common to every $a$. They showed that fixed points of morphisms in this class contain an infinite number of palindromes. They asked whether any palindromic fixed point of a primitive substitution arises using such a morphism. This question was eventually turned into a conjecture (called later the HKS conjecture), however, due to a certain vagueness of the original question several versions of this conjecture have been considered (for details see Introduction in [13]). Validity of HKS conjecture was

*Corresponding author

Email addresses: petr.ambroz@fjfi.cvut.cz (Petr Ambrož),
kadleond@fjfi.cvut.cz (Ondřej Kadlec), zuzana.masakova@fjfi.cvut.cz (Zuzana Masáková), edita.pelantova@fjfi.cvut.cz (Edita Pelantová)
proven for binary words [25] and for words fixed by a morphism of certain types [18, 19].

In 2013, Frid, Puzynina and Zamboni [12] introduced the palindromic length of a finite word \( w \), denoted by \(|w|_{\text{pal}}\), as the minimal number of palindromes whose concatenation is equal to \( w \). They conjectured that if there is a constant \( K \) such that the palindromic length of every factor in an infinite word \( u \) is bounded by \( K \), then \( u \) is eventually periodic. Formally, defining for a given infinite word \( u \) the function \( \text{PL}_u : \mathbb{N} \to \mathbb{N} \) by

\[
\text{PL}_u(n) := \max\{|w|_{\text{pal}} : w \text{ is a factor of length } n \text{ in } u\},
\]

the conjecture is that if \( u \) is not eventually periodic, then \( \limsup_{n \to \infty} \text{PL}_u(n) = +\infty \). The authors of [12] proved the conjecture for infinite words which do not contain an \( r \)-power for any positive integer \( r \). In particular, the conjecture is true for any aperiodic fixed point of a primitive morphism, as such fixed points have bounded powers [20]. Later, Frid [11] showed that Sturmian words have unbounded palindromic length even if they contain unbounded powers. Palindromic length of Sturmian words is studied also in [2]. It is shown that \( \text{PL}_u \) can grow arbitrarily slowly. For other infinite words besides Sturmian words and bounded-repetition words the conjecture of \( \limsup_{n \to \infty} \text{PL}_u(n) = +\infty \) remains open.

We study palindromic length of factors of fixed points of primitive morphisms. Here, as we have stated above, the palindromic length in unbounded, whenever the fixed point is not eventually periodic. The main results of this contribution are formulated as Proposition [9] and Theorem [10]. We prove that if the HKS conjecture is valid then for any primitive morphism \( \varphi \) there is a constant \( K > 0 \) such that the palindromic length of every factor \( w \) in the language of \( \varphi \) is less than or equal to \( K \ln |w| \). We also provide a method of estimating the constant \( K \).

For the case of the Fibonacci word \( f \) our computations suggest that \( \limsup_{n \to \infty} \frac{\text{PL}_f(n)}{\ln n} \leq \frac{2}{3\ln \tau} \), where \( \tau = \frac{1}{2}(1 + \sqrt{5}) \) is the golden ratio. We prove an upper bound on \( \limsup_{n \to \infty} \frac{\text{PL}_u(n)}{\ln n} \) also for the Thue-Morse word. The estimates are given in Section [4]. Let us mention that a lower bound on \( \limsup \frac{\text{PL}_u(n)}{\ln n} \) is not known even for the Fibonacci word. Frid [10] conjectures that \( \limsup_{n \to \infty} \frac{\text{PL}_f(n)}{\ln n} \geq \frac{1}{3\ln \tau} \).

The Fibonacci word \( f \) belongs among the so-called rich words (or full words) introduced in [8, 6]. An infinite word is rich if each of its finite factors is rich, i.e., contains as many palindromes as possible. Intuitively, the palindromic length of of such words should grow slowly. We demonstrate that this is not necessarily the case. In Section [5] we use finite rich words
introduced by Guo, Shallit and Shur [14] to construct an infinite word $u$ for which $\text{PL}_u(n) \geq c \sqrt{n}$ for some positive constant $c$.

Section 6 is devoted to further computer experiments on the Thue-Morse word $t$ whose language contains besides palindromes also infinitely many antipalindromes. We introduce the combined pal-antipal length and compare its growth to the growth of $\text{PL}_t$.

In the final section we formulate several open questions concerning factorization to palindromes and/or antipalindromes.

2. Preliminaries

Let $A$ be a finite set called alphabet, its elements are called letters. A word $w = w_1 \cdots w_n$ (over $A$) is a finite sequence of elements in $A$, its length (the number of its elements) is denoted by $|w| = n$. The notation $|w|_a$ is used for the number of occurrences of the letter $a$ in $w$. The empty word – unique word of length zero – is denoted by $\varepsilon$. The concatenation of words $v = v_1 \cdots v_k$ and $w = w_1 \cdots w_l$ is $vw = v \cdot w = v_1 \cdots v_k w_1 \cdots w_l$. The set of all finite words over $A$ equipped with the operation concatenation of words is a free monoid, denoted by $A^*$. For a word $w = w_1 \cdots w_n$ we define its mirror image as $\overline{w} = w_n \cdots w_1$. A word $w$ is called palindrome if $w = \overline{w}$. The palindromic length of a word $w$, denoted by $|w|_{\text{pal}}$, is the smallest number $K$ of palindromes $p_1, \ldots, p_K$ such that $w = p_1 \cdots p_K$, i.e., the minimal number of palindromes whose concatenation is equal to $w$. For convenience, we define $|\varepsilon|_{\text{pal}} = 0$.

An infinite sequence of letters $u = (u_i)_{i \geq 1}$ in $A$ is called infinite word. The set of all infinite words over $A$ is denoted $A^\mathbb{N}$. The word $u \in A^\mathbb{N}$ is said to be eventually periodic if it is of the form $u = vz\omega$, where $v, z \in A^*$, $z \neq \varepsilon$ and $z\omega = zzz \cdots$.

A factor of a (finite of infinite) word $w$ is a finite word $v$ such that $w = w_1vw_2$ for some words $w_1, w_2$. If $w_1 = \varepsilon$ then $v$ is called a prefix of $w$, if $w_2 = \varepsilon$ then $v$ is called a suffix of $w$. The set of all factors of an infinite word $u$, called the language of $u$, is denoted by $\mathcal{L}(u)$. Let $v_p$ be a prefix of a word $v$, that is, there is a word $x$ such that $v = v_p x$. Then we define $v_p^{-1} v = v_p^{-1} x = v$. Similarly, let $v_s$ be a suffix of a word $v$, that is, there is a word $y$ such that $v = yv_s$. Then we define $v v_s^{-1} = v \cdot v_s^{-1} = y$.

Let $w$ be a finite word over the alphabet $A = \{a_1, \ldots, a_r\}$. Then the Parikh vector of $w$ is the vector, denoted $\vec{V}(w)$, whose $i$-th element is the
number of occurrences of $a_i$ in $w$, i.e.,

$$\vec{V}(w) = \begin{pmatrix} |w|_{a_1} \\ \vdots \\ |w|_{a_r} \end{pmatrix}. $$

Obviously, $|w| = 1 \cdot \vec{V}(w)$, where $1 = (1 1 \cdots 1)$.

A morphism of the free monoid $A^*$ is a map $\varphi : A^* \to A^*$ such that $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in A^*$. A morphism of $A^*$, where $A = \{a_1, \ldots, a_r\}$, is called primitive if there is a constant $K$ such that $\varphi^K(a_i)$ contains $a_j$ for every $i, j \in \{1, \ldots, r\}$. The action of the morphism $\varphi$ is naturally extended to infinite words by concatenation, in particular, we have

$$\varphi(u_0u_1u_2 \cdots) = \varphi(u_0)\varphi(u_1)\varphi(u_2) \cdots$$

An infinite word $u$ is called a fixed point of the morphism $\varphi$ if $u = \varphi(u)$. Clearly, a morphism $\psi$ can have several different fixed points, however, if $\psi$ is primitive then all its fixed points have the same language, denoted $L(\psi)$.

Let $A = \{a_1, \ldots, a_r\}$ and let $\psi$ be a morphism of $A^*$. The incidence matrix of $\psi$ is the $r \times r$ matrix $M_\psi$ given by $[M_\psi]_{ij} = |\psi(a_j)|_{a_i}$. The incidence matrix of $\psi$ can be used to compute the Parikh vector of the image of a word $w$ under $\psi$ by

$$\vec{V}(\psi(w)) = M_\psi \cdot \vec{V}(w). \quad (1)$$

### 3. Morphisms in class $P$

**Definition 1.** A primitive morphism $\psi : A^* \to A^*$ belongs to class $P$ if there is a palindrome $p \in A^*$ such that for each $a \in A$

$$\psi(a) = pq_a, \quad \text{where } q_a \in A^* \text{ is a palindrome.} \quad (2)$$

**Example 2.** The Fibonacci morphism $\varphi_F : a \mapsto ab, b \mapsto a$ belongs to class $P$; Equation (2) is fulfilled for $p = a, q_a = b, q_b = \varepsilon$.

**Example 3.** The Thue-Morse morphism $\varphi_{TM} : a \mapsto ab, b \mapsto ba$ does not belong to class $P$, however, its square $\varphi_{TM}^2 : a \mapsto abba, b \mapsto baab$ does ($p = \varepsilon, q_a = abba, q_b = baab$).

The following simple observation is due to Hof, Knill and Simon [16].

**Observation 4.** Let $\psi$ be a primitive morphism in the form (2) and let $u$ be a fixed point of $\psi$. Then
i) if $w \in \mathcal{L}(\psi)$ then $\psi(w)p \in \mathcal{L}(\psi)$,

ii) if $w$ is a palindrome then $\psi(w)p$ is a palindrome.

By repeated application of Observation 4, one obtains the following corollary, which was first shown in [16].

**Corollary 5.** The language of a fixed point of a morphism in class $\mathcal{P}$ contains infinitely many palindromes.

In fact, as was noticed in [1], the same statement as Corollary 5 is valid for fixed points of morphisms that are not in class $\mathcal{P}$ by themselves, but some of their conjugates is, see Definition 6 below. The reason for that is that languages of infinite words fixed by conjugated primitive morphisms coincide.

**Definition 6.** Morphisms $\psi_1, \psi_2 : A^* \to A^*$ are said to be conjugated, denoted by $\psi_1 \sim \psi_2$, if there is a word $w \in A^*$ such that either $\psi_1(a)w = w\psi_2(a)$ for every $a \in A$ or $w\psi_1(a) = \psi_2(a)w$ for every $a \in A$.

The proof of the following fact can be found for example in [1].

**Proposition 7.** Let $\psi_1$ be a primitive morphism and let $\psi_2$ be conjugated with $\psi_1$. Then $\mathcal{L}(\psi_1) = \mathcal{L}(\psi_2)$.

In [1] the authors also make the observation that any morphism of class $\mathcal{P}$ is conjugated to a morphism of the form (2) where the palindrome $p$ is either the empty word or a single letter. From now on, in view of Proposition 7, we will only consider morphisms in class $\mathcal{P}$ of this form. The following lemma shows how palindromic length of a finite word changes under application of such a morphism.

**Lemma 8.** Let $\psi : A^* \to A^*$ be a morphism in class $\mathcal{P}$ in the form (2).

i) If $p = \varepsilon$, then $|\psi(w)|_{\text{pal}} \leq |w|_{\text{pal}}$ for every $w \in A^*$.

ii) Suppose $|p| = 1$. If $|w|_{\text{pal}}$ is even then $|\psi(w)|_{\text{pal}} \leq |w|_{\text{pal}}$, otherwise $|\psi(w)|_{\text{pal}} \leq |w|_{\text{pal}} + 1$.

iii) If $|\psi(w)|_{\text{pal}} = |w|_{\text{pal}} + 1$ then $|\psi^2(w)|_{\text{pal}} \leq |\psi(w)|_{\text{pal}}$.

**Proof.** i) Let $w = p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are palindromes. Then by Observation 4 $\psi(w) = \psi(p_1) \cdot \psi(p_2) \cdots \psi(p_k)$ is a concatenation of $\psi(w)$ into $k$ palindromes.
ii) Let \( w = p_1 \cdots p_{2k} \), where \( p_1, \ldots, p_{2k} \) are palindromes. Then by Observation 3

\[
\psi(w) = \psi(p_1) \cdot p^{-1} \psi(p_2) \cdot \psi(p_3) \cdot p^{-1} \psi(p_4) \cdots \psi(p_{2k-1}) \cdot p^{-1} \psi(p_{2k})
\]

is a concatenation of \( \psi(w) \) into \( 2k \) palindromes \( q_1, \ldots, q_{2k} \). If \( w = p_1 \cdots p_{2k+1} \), where \( p_1, \ldots, p_{2k+1} \) are palindromes then \( \psi(w) \) can be factorized into \( 2k + 2 \) palindromes similarly:

\[
\psi(w) = \psi(p_1) \cdot p^{-1} \psi(p_2) \cdots \psi(p_{2k-1}) \cdot p^{-1} \psi(p_{2k}) \cdot p_p \psi(p_{2k+1}).
\]

iii) By i) and ii), \( |\psi(w)|_{\text{pal}} = |w|_{\text{pal}} + 1 \) may happen only if \( |w|_{\text{pal}} \) is odd. Then \( |\psi(w)|_{\text{pal}} \) is even and thus \( |\psi^2(w)|_{\text{pal}} \leq |\psi(w)|_{\text{pal}} \).

The above lemma states that by applying a morphism \( \psi \) of class \( \mathcal{P} \) to a word, palindromic length can increase by at most one, and this happens only at alternating iterations of the morphism \( \psi \). With this knowledge, we can find an estimate on the growth of the palindromic length \( PL_u(n) \).

**Proposition 9.** Let \( \psi : A^* \to A^* \) be a morphism in class \( \mathcal{P} \) such that for each \( a \in A \) it holds that \( \psi(a) = pq_a \), where \( p \in \{\varepsilon\} \cup A \) and \( q_a \) is a palindrome. Let us denote

\[
C := \max\{|x|_{\text{pal}} : \exists a \in A, x \text{ is a proper prefix of } q_a\}.
\]

Then for a fixed point \( u \) of \( \psi \) we have

\[
\limsup_{n \to \infty} \frac{PL_u(n)}{\ln n} \leq \frac{2C + \frac{3}{2}|p|}{\ln \Lambda}.
\]

where \( \Lambda \) is the dominant eigenvalue of the incidence matrix of \( \psi \).

**Proof.** First realize that under our assumptions, the set \( \{|x|_{\text{pal}} : \exists a \in A, x \text{ is a proper prefix of } q_a\} \) is non-empty. Otherwise, \( \psi(a) = p \) for every letter \( a \), and as \( |p| \leq 1 \), the morphism \( \psi \) is not primitive (and thus not in class \( \mathcal{P} \)). Therefore the constant \( C \) is well defined. Moreover, note that it is enough to consider only prefixes (in the definition of \( C \)) since \( q_a \) are palindromes and thus if \( x \) is a suffix of \( q_a \) then \( \overleftarrow{x} \) is its prefix and \( |x|_{\text{pal}} = |\overleftarrow{x}|_{\text{pal}} \).

Consider a fixed point \( u \) of \( \psi \). If it is eventually periodic, then by [12], the palindromic length of its factors is bounded and the statement of the
proposition is trivially valid. Assume that \( \psi \) has an aperiodic fixed point. Since \( \psi \) is a primitive morphism, every sufficiently long factor of \( u \) has a uniquely determined preimage [21]. More precisely, there exists \( n_0 \in \mathbb{N} \) such that for each \( v \in \mathcal{L}(u), |v| > n_0 \), there are factors \( x, v', y \) of \( u \) such that \( v = x\psi(v')y \), where \( v' \neq \varepsilon \), \( x \) is a proper suffix of \( \psi(a) \) and \( y \) is a proper prefix of \( \psi(b) \) for some letters \( a, b \in A \), cf. Figure 1.

![Figure 1: Illustration of the construction of preimage of \( v \) in the proof of Proposition 9.](image)

Obviously, \( |v|_{\text{pal}} \leq |x|_{\text{pal}} + |\psi(v')|_{\text{pal}} + |y|_{\text{pal}} \). By definition of \( C \), we have \( |x|_{\text{pal}} \leq C \), since \( \psi(a) = pq_a \) we have \( |y|_{\text{pal}} \leq C + |p| \). Using Lemma 8, \( |\psi(v')|_{\text{pal}} \leq |v'|_{\text{pal}} + \delta \), where \( \delta = 1 \) if \( |v'|_{\text{pal}} \) is odd and \( |p| = 1 \), and \( \delta = 0 \) otherwise. Together, we obtain

\[
|v|_{\text{pal}} \leq 2C + |p| + |v'|_{\text{pal}} + \delta.
\]

If \( |v'| > n_0 \) we apply the same procedure to \( v' \). In this way, for a given \( v \in \mathcal{L}(\psi) \) we create a sequence \( v = v^{(0)}, v^{(1)}, \ldots, v^{(k)} \) such that for each \( i = 1, 2, \ldots, k \) we have

\[
|v^{(i-1)}|_{\text{pal}} \leq 2C + |p| + |v^{(i)}|_{\text{pal}} + \delta_i,
\]

\[
|v^{(i-1)}| \geq |\psi(v^{(i)})|,
\]

and \( |v^{(k)}| \leq n_0 \). From (5) we get

\[
|v|_{\text{pal}} \leq 2kC + n_0 + |p| \left( k + \left\lceil \frac{k}{2} \right\rceil \right),
\]

where we used that \( |v^{(k)}| \leq n_0 \), and iii) of Lemma 8.

On the other hand, (6) implies that

\[
|v| \geq |\psi^k(v^{(k)})| = 1 \cdot M_{\psi}^k \cdot \bar{V}(v^{(k)}).
\]

Since \( \psi \) is a primitive morphism, by the Perron-Frobenius theorem, the dominant eigenvalue \( \Lambda \) of the matrix \( M_{\psi} \) is positive and strictly greater
that absolute values of all other eigenvalues of $M_\psi$. Denote by $R$ the non-singular matrix such that $RM_\psi R^{-1}$ is in the Jordan canonical form, i.e., it is block diagonal. Notably, the block corresponding to the eigenvalue $\Lambda$ is of dimension $1 \times 1$. Without loss of generality, let it be the first block on the diagonal of $RM_\psi R^{-1}$. Then necessarily,

$$\lim_{k\to\infty} \frac{1}{\Lambda^k} M_\psi^k = R \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} R^{-1}.$$ 

Combining with (8), we have that

$$|v| \geq |\psi^k(v(k))| = c_1\Lambda^k(1 + o(1)).$$

Putting the latter estimate together with (7),

$$\frac{|v|_{\text{pal}}}{\ln |v|} \leq \frac{2kC + n_0 + |p|(k + \left\lceil \frac{k}{2} \right\rceil)}{\ln c_1 + k \ln \Lambda + \ln(1 + o(1))}.$$ 

If $|v| = n$ tends to infinity then $k$ tends to infinity as well. The validity of the proposition follows.

Proposition 9 provides an upper estimate on the palindromic length $\text{PL}_u(n)$ for any fixed point $u$ of any morphism of class $\mathcal{P}$. The result is valid independently of the size of the alphabet. Reducing our consideration to binary infinite words, we recall the result of Bo Tan [25]. He shows that any binary morphism $\varphi$ producing a fixed point with infinitely many palindromes is either itself conjugated to a morphism in class $\mathcal{P}$, or this can be said about its second iterate $\varphi^2$. This allows us to formulate a summarizing corollary to our Proposition 9.

Theorem 10. Let $u$ be a fixed point of a primitive morphism over a binary alphabet. Then there is a constant $K > 0$ such that either

$$\text{PL}_u(n) \leq K \ln n \quad \text{for all } n \in \mathbb{N},$$

or

$$\text{PL}_u(n) \geq Kn \quad \text{for all } n \in \mathbb{N}.$$
Proof. Let $\mathcal{L}(u)$ contain only finitely many palindromes. Then obviously for every factor $w$ of length $n$, we have $|w|_{\text{pal}} \geq \frac{1}{c} n$, where $c$ is the length of the longest palindrome in $\mathcal{L}(u)$. Thus $\text{PL}_u(n)$ grows at least linearly.

On the other hand, if $\mathcal{L}(u)$ contains infinitely many palindromes then it coincides with the language $\mathcal{L}(\psi)$ for some morphism $\psi$ in class $\mathcal{P}$ (cf. [25]) and thus we can employ Proposition [9].

Note that on morphisms over an alphabet with more than two letters, one cannot prove a result as strong as that of Bo Tan [25]. A counterexample was given on a ternary morphism in [17].

4. Fibonacci and Thue-Morse words

Let us provide an upper bound on the constant $K$ of Theorem [10] for the Fibonacci word $f$ and for the Thue-Morse word $t$.

The Fibonacci word. The Fibonacci morphism $\varphi_F : a \mapsto ab, b \mapsto a$ belongs to class $\mathcal{P}$ (cf. Example [2]). Since its fixed point, the Fibonacci word $f$, is a Sturmian word, it follows from the result by Frid [11] that

$$\limsup_{n \to \infty} \frac{\text{PL}_f(n)}{\ln n} \leq 3 \ln \tau,$$

(9)

The Fibonacci word $f$ is also the fixed point of $\varphi_F^3 : a \mapsto abaab, b \mapsto aba$. Consider morphism $\psi : a \mapsto ababa, b \mapsto aba$. Taking $w = aba$, we see that

$$\varphi_F^3(a)w = abaababa = w\psi(a), \quad \varphi_F^3(b)w = ababa = w\psi(b),$$

which by Definition [6] means that $\varphi_F^3 \sim \psi$. According to Proposition [7] we have $\mathcal{L}(\varphi_F^3) = \mathcal{L}(\psi)$. Let us apply Proposition [9] to $\psi$. Obviously $p = \varepsilon$ and $C = 2$. The dominant eigenvalue of $\mathbf{M}_\psi$ is $\tau^3$. Therefore

$$\limsup_{n \to \infty} \frac{\text{PL}_f(n)}{\ln n} \leq \frac{4}{3 \ln \tau},$$

(10)

which gives a better estimate than (9).

If we use $\psi^2 : a \mapsto ababaababaababaababa, b \mapsto ababaabaababa$, which also fixes the Fibonacci word, we have $p = \varepsilon, C = 3,$ and $\Lambda = \tau^6$. This
improves the constant in estimate \[(10)\] to \(\frac{1}{\ln \tau}\). Making similar considerations for \(\psi^k, k \leq 13\), we obtain that \(K \leq \frac{2(k+1)}{3k\ln \tau}\). This makes us conjecture that

\[
\limsup_{n \to \infty} \frac{\text{PL}_f(n)}{\ln n} = \frac{2}{3\ln \tau}.
\]

Let us remark that Frid [10] investigated the palindromic length only of prefixes of the Fibonacci word. She conjectures that the prefix \(w(k)\) (of the Fibonacci word) whose length written in the Zeckendorf numeration system is \((|w(k)|)_F = (100)^{2k-1}101\) has \(|w(k)|_{\text{pal}} = 2k+1\). Should this conjecture be valid, it would imply that

\[
\limsup_{n \to \infty} \frac{\text{PL}_f(n)}{\ln n} \geq \frac{1}{3\ln \tau}.
\]

The Thue-Morse word. Let us consider the Thue-Morse word \(t\), i.e., the fixed point of the morphism \(\varphi_{TM}^2 : a \mapsto abba, b \mapsto baab\) (cf. Example 3). Similarly to the case of the Fibonacci word we are interested in the constant \(K\) where \(\limsup_{n \to \infty} \frac{\text{PL}_t(n)}{\ln n} \leq K\).

Applying Proposition 9 to \(\varphi_{TM}^2\), where \(\Lambda = 4, C = 2, p = \epsilon\), we get \(K \leq \frac{4}{1\ln 4}\). Further iteration of the procedure in Proposition 9 with \(\varphi_{TM}^{2k}\) for \(k \leq 13\) show that \(K \leq (3 + \frac{1}{k})\frac{1}{1\ln 4}\). This leads us to conjecture that

\[
\limsup_{n \to \infty} \frac{\text{PL}_t(n)}{\ln n} = \frac{3}{\ln 4}.
\]

5. Words rich in palindromes

Intuitively, a word containing many palindromic factors should have small palindromic length. Recall that Droubay et al. [8] found out that a finite word \(w\) contains at most \(|w| + 1\) different palindromic factors. If this bound is attained, the word \(w\) is called rich (in palindromes). An infinite word \(u\) is called rich if each of its factors is rich. The difference between the upper bound \(|w| + 1\) and actual number of palindromes in \(w\) is called the (palindromic) defect of \(w\) and denoted by \(D(w)\), see [6].

Example 11. Word \(w_1 = abababaab\) contains 9 palindromic factors: \(\epsilon, a, b, aa, bb, aba, bab, abba, baab\), and thus \(w_1\) is rich. On the other hand, the word \(w_2 = bbabababb\) contains only 8 palindromic factors: \(\epsilon, a, b, aa, bb, aba, bab, baab\), and thus \(w_2\) is not rich. And so the Thue-Morse word \(t\) is not rich, since \(w_2\) is one of its factors. The defect of \(w_2\) is \(D(w_2) = 1\). It follows from the results of [4] that \(\sup_{w \in \mathcal{L}(t)} D(w) = +\infty\).
Some of the rich words have small palindromic length. For example, the palindromic length of morphic Sturmian words such as the Fibonacci word \( f \) grows at most logarithmically (cf. Theorem 10). Nevertheless, quite surprisingly, richness of a word does not imply that its palindromic length be small. We will demonstrate this fact on finite words \( w(k) = \text{abaabb} \cdots a^k b^k \), \( v(k) = \text{abbaabbb} \cdots a^{2k-1} b^{2k} \).

In all these words the sequences of runs of \( a \) and of runs of \( b \) are monotone, and therefore \( w(k) \) and \( v(k) \) are rich for all \( k \in \mathbb{N} \) by a result of Guo et al. [14].

**Proposition 12.** For every \( k \in \mathbb{N} \) we have \( |w(k)|_{\text{pal}} = k + 1 \) and \( |v(k)|_{\text{pal}} = k + 1 \).

**Proof.** A palindromic factor in \( w(k) \) has one of the following forms \( a^i \), \( b^i \), \( a^i b^j a^i \), \( b^i a^j b^i \) for some \( i, j \in \mathbb{N} \). Obviously, a minimal factorization will contain as many as possible blocks of the 3rd and 4th type. One can use at most \( k - 1 \) such palindromes (since there are \( 2k - 1 \) runs of zeroes and ones in \( w(k) \), excluding the outer ones) plus at least two palindromes of the type \( a^i \) or \( b^i \) to cover whole \( w(k) \). Examples of such minimal palindromic factorizations follows.

\[
 w(6) = \text{aba} \cdot \text{abba} \cdot \text{aa} \cdot \text{bbbaaa} \cdot \text{baaaaab} \cdot \text{bbbaaaabbb} \cdot \text{bb} = \\
 = a \cdot \text{baab} \cdot \text{baaab} \cdot \text{bbbaaaabbb} \cdot \text{bbbaaaabbb} \cdot \text{bbb}.
\]

The same consideration holds for \( v(k) \).

With the use of the above proposition and a result on rich words derived in [23], we can summarize the following information about palindromic length of finite rich words.

**Corollary 13.** There are constants \( c > 0, d > 1 \) such that

i) \( |w|_{\text{pal}} \geq c \sqrt{|w|} \) for infinitely many finite rich words \( w \),

ii) \( |w|_{\text{pal}} \leq d \frac{|w|}{\ln|w|} \) for every rich word \( w \).

**Proof.** i) By definition, the length of the word \( w(k) \) is \( |w(k)| = k(k + 1) \). Using Proposition [12], we have \( |w(k)|_{\text{pal}} = k + 1 > \sqrt{k(k + 1)} \). Similarly, we can use the words \( v(k) \).

ii) This is the statement of Theorem 3 in [23].
The finite words \( w(k) \) (\( v(k) \) resp.), \( k \in \mathbb{N} \), allow one to define infinite rich words.

**Proposition 14.** There exists an infinite word \( u \) rich in palindromes for which
\[
\sqrt{n} - 3 \leq \mathrm{PL}_u(n) \leq \sqrt{n} + 4 \quad \text{for every } n \in \mathbb{N}.
\]

**Proof.** It suffices to define the infinite word \( u \) as the word having the prefix \( w(k) \) for every \( k \in \mathbb{N} \).

Let \( u \) be a prefix of \( u \) of length \( n \), then we show that
\[
\sqrt{n} - 3 \leq |u|_{\text{pal}} \leq \sqrt{n} + 4. \tag{11}
\]

Let \( k \) be such that
\[
k(k + 1) < n \leq (k + 1)(k + 2). \tag{12}
\]

Then

- since \( u \) has prefix \( w(k) \), we have \( u = w(k)z \), where \( z = a^l \) or \( z = a^{k+1}b^l \) for some \( l \leq k + 1 \). Thus \( |z|_{\text{pal}} \leq 2 \),
- \( w(k+1) \) has prefix \( u \) and thus \( w(k+1) = uv \), where (analogously to the previous case) \( |v|_{\text{pal}} \leq 2 \).

Lemma 6 from [24] states that \( |x|_{\text{pal}} \leq |y|_{\text{pal}} + |xy|_{\text{pal}} \). Using this lemma for \( x = w(k) \) and \( y = z \) and then for \( x = u \) and \( y = v \) we get
\[
k - 1 = |w(k)|_{\text{pal}} - 2 \leq |u|_{\text{pal}} \leq |w(k+1)|_{\text{pal}} + 2 = k + 4.
\]

Since \( (12) \) implies \( k < \sqrt{n} < k + 2 \) the estimate \( (11) \) follows.

Analogously to the proof of Proposition 12, the maximum of the set \( \{|u|_{\text{pal}} : |u| = n, u \in \mathcal{L}(u)\} \) is reached on the prefixes of \( u \). Therefore
\[
\sqrt{n} - 3 \leq \mathrm{PL}_u(n) \leq \sqrt{n} + 4. \tag{13}
\]

**6. Combined pal-antipal length**

Let us reconsider the Thue-Morse word. Besides infinitude of palindromes, it contains also infinitely many the so-called antipalindromes [4]. These words have been considered in a wider context under the name \( f \)-pseudo-palindromes (or \( f \)-palindromes) already in [3, 7, 15]. We use the name antipalindrome in accordance with [13].
A finite word \( w = w_1 \cdots w_n \) over a binary alphabet \( \{a, b\} \) is an antipalindrome, if \( w = E(w_n)E(w_{n-1}) \cdots E(w_1) \), where \( E \) exchanges the letters, \( E(a) = b, E(b) = a \). Obviously, an antipalindrome is always of even length. A finite word thus need not to be factorizable into only antipalindromes.

An extension of the question on palindromic length could be on the factorization of a given finite word into the smallest possible number of factors which are either palindromes or antipalindromes. For that purpose, we have adapted the simple quadratic algorithm for minimal palindromic factorization given in [9]. We have computed the palindromic and combined pal-antipal length for the prefixes of the Thue-Morse word \( t \) of length up to \( 10^6 \), see graph of \( PL_t(n)/\ln n \) in Figure 2.

In Tables 1 and 2 we include some of the results of our computations comparing the palindromic length \( |w|_{\text{pal}} \) and the combined pal-antipal length \( |w|_{\text{cpal}} \) of the prefixes \( |w| \) of the Thue-Morse word \( t \).

Our computations suggest that including both palindromes and antipalindromes into the factorization of Thue-Morse word, one can reduce the number of factors by half (and not more), see Table 1. On the other hand, we suppose that there exist prefixes of arbitrary palindromic length whose combined pal-antipal length as big as the palindromic one, see Table 2. Other research could be done in this direction.

7. Open problems

The palindromic length of finite and infinite words has been introduced in 2013 [12]. Since then, several groups of authors focused on the design of fast algorithms for computing the minimal palindromic factorization, see
| $w$ | $|w|_{\text{pal}}$ | $|w|_{\text{cpal}}$ |
|-----|----------------|----------------|
| 2   | 2              | 1              |
| 10  | 4              | 2              |
| 118 | 6              | 3              |
| 630 | 8              | 4              |
| 7542| 10             | 5              |

| $w$ | $|w|_{\text{pal}}$ | $|w|_{\text{cpal}}$ |
|-----|----------------|----------------|
| 40310| 12            | 6              |
| 482678| 14           | 7              |
| 2579830| 16         | 8              |
| 30891382| 18        | 9              |

Table 1: Combined pal-antipal length $|w|_{\text{cpal}}$ of prefixes $w$ of the Thue-Morse word $t$ can be up to twice smaller than their palindromic length $|w|_{\text{pal}}$. For $1 \leq n \leq 9$, we display the first occurrences of the case when $|w|_{\text{pal}} = 2|w|_{\text{cpal}} = 2n$.

| $w$ | $|w|_{\text{pal}}$ | $|w|_{\text{cpal}}$ |
|-----|----------------|----------------|
| 1   | 1              | 1              |
| 3   | 2              | 2              |
| 11  | 3              | 3              |
| 43  | 4              | 4              |
| 171 | 5              | 5              |
| 683 | 6              | 6              |
| 2731| 7              | 7              |
| 10923| 8           | 8              |
| 43691| 9           | 9              |
| 174763| 10         | 10             |
| 699051| 11         | 11             |
| 2796203| 12        | 12             |
| 11184811| 13        | 13             |
| 44739243| 14        | 14             |

Table 2: The prefixes of the infinite word $t$ on which considering antipalindromes does not improve the palindromic length. For each $1 \leq n \leq 14$, we display the first occurrences of the case when $|w|_{\text{pal}} = |w|_{\text{cpal}} = n$ happens.

e.g. [9] [22] [5]. On the other hand, an analytic study of the palindromic length is still in its beginnings. Let us formulate several open questions which we consider of interest.

i) When studying palindromic length of the Fibonacci and Thue-Morse word, we have conveniently considered a power of the morphism that could be conjugated to a morphism in which the image of every letter is a palindrome, i.e., $a \mapsto q_a$ for every $a \in A$. According to our knowledge, question on determining for which morphisms such a power exists, has not been considered yet.

ii) In the study of the growth of $\text{PL}_u(n)$ we provide a method of finding
an upper bound on the constant $K$, in the estimate $\text{PL}_u(n) \leq K \ln n$, which is valid for any fixed point $u$ of any morphism in class $\mathcal{P}$. According to our knowledge, no methods for giving a lower bound on $\text{PL}_u(n)$ have been mentioned in the literature. So far, only Frid [10] has focused on finding a lower bound on the palindromic length. Her study is specific for the Fibonacci word $f$. She states a conjecture describing the prefixes $w$ of $f$ having palindromic length $|w|_{\text{pal}}$ strictly bigger than all the shorter prefixes of $f$.

iii) The validity of the conjectured lower bound of Frid [10] would imply that $\limsup_{n \to \infty} \text{PL}_f(n) / \ln n \geq 1 / (3 \ln \tau)$. Our computations (cf. Section 4) suggest that $\limsup$ should have a bigger value. This is probably caused by the fact that Frid only considers the palindromic length of prefixes of the Fibonacci word. It may be the case that bigger palindromic length is achieved on factors that are not prefixes of $f$. We do not have candidates for such factors. It should be mentioned that Saarela [24] shows equivalence between the unboundedness of the palindromic length when taken over the factors and considering only the prefixes. This, however, does not mean that the growth of the function dependingly on $n$ should be equal.

iv) In Proposition 14, we give an infinite rich word whose palindromic length grows with $n$ at least as $\sqrt{n}$. The infinite word is however not uniformly recurrent. All the other considered classes of palindromic uniformly recurrent words have palindromic length bounded by $K \ln n$. Does there exist a uniformly recurrent infinite word $u$ such that $\text{PL}_u(n) \geq c \sqrt{n}$?

v) HKNS conjecture was formulated in view of characterization of morphisms providing fixed points with infinitely many palindromes. It is not obvious which morphisms generate fixed points that besides infinitely many palindromes contain also arbitrarily long antipalindromes, as it is the case of the Thue-Morse morphism.

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