MATHEMATICAL ANALYSIS OF MEMORY EFFECTS AND THERMAL RELAXATION IN NONLINEAR SOUND WAVES ON UNBOUNDED DOMAINS

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Abstract. Motivated by the propagation of nonlinear sound waves through relaxing hereditary media, we study a nonlocal third-order Jordan–Moore–Gibson–Thompson acoustic wave equation. Under the assumption that the relaxation kernel decays exponentially, we prove local well-posedness in unbounded two- and three-dimensional domains. In addition, we show that the solution of the three-dimensional model exists globally in time, while the energy of the system decays polynomially.

1. Introduction

Nowadays ultrasound waves are an indispensable tool in medicine, commonly used in imaging and non-invasive treatments of various disorders [4, 8, 21, 28]. Because of the high amplitude-to-frequency ratio that ultrasonic waves are likely to have, nonlinear effects can often be observed in their propagation. This necessitates a deeper understanding of the nonlinear acoustic models and their analytical properties.

Our work is particularly motivated by nonlinear sound waves in relaxing media that exhibit memory effects. These relaxation processes can occur when there are inhomogeneities in the propagation region; for example, through excitation of molecular degrees of freedom or some impurity effects in the fluid; cf. [23, Chapter 1]. In such cases, the pressure-density state equation is not satisfied exactly but up to a term that involves the history of the process.

Additionally, classical models of nonlinear acoustics, such as the Westervelt and Kuznetsov equation, are known to exhibit parabolic-like behavior with an infinite speed of propagation [12, 22]. To avoid this paradox, we can replace the Fourier temperature law by the Maxwell–Cattaneo law during the derivation, resulting in a third-order acoustic equation with a finite propagation speed [11].

We investigate here such a third-order nonlinear acoustic model with a memory term. In particular, we are concerned with its behavior in terms of global solvability and energy decay in the whole $\mathbb{R}^n$. In bounded domains, it is known that the exponential decay of the relaxation kernel directly influences how the energy of the system decays [17]. The situation in the whole space $\mathbb{R}^n$ is different. As it turns out, although our memory kernel will decay exponentially, the solution at most decays polynomially.

We organize the paper as follows. We begin by discussing the modeling aspects and setting our problem in Section 2. Section 3 contains the necessary theoretical preliminaries. In Section 4 we formally derive several energy estimates for our problem.
rewritten as a first-order evolution equation. Section 5 is dedicated to proving short-time well-posedness of the problem. In Section 6, we prove that in $\mathbb{R}^3$ the solution exists globally in time. Finally, in Section 7, we show the energy of the system decays polynomially with time.

2. Problem setting and modelling

In nonlinear acoustic, the Kuznetsov equation is one of the classical models. It is given by

\begin{equation}
\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t = \left( \frac{1}{c^2} \frac{B^2}{2A} (\psi_t)^2 + |\nabla \psi|^2 \right)_t,
\end{equation}

where $\psi = \psi(x,t)$ represents the acoustic velocity potential for $x \in \mathbb{R}^n$ and $t > 0$; see [16]. The equation (2.1) can be obtained as an approximation of the governing equations of fluid mechanics by means of asymptotic expansions in powers of small parameters; see [5, 15, 16]. The constants $c > 0$ and $\delta > 0$ are the speed and the diffusivity of sound, respectively. The ratio $B/A$ indicates the nonlinearity of the equation of state for the given medium. Typical values of these parameters in different media can be found in, e.g., [15, 23]. If we can neglect local nonlinear effects and assume

\[ |\nabla \psi|^2 \approx \frac{1}{c^2} \psi_t^2 , \]

we arrive at the Westervelt equation in the potential form

\begin{equation}
\psi_{tt} - c^2 \Delta \psi - \delta \Delta \psi_t = \left( \frac{1}{c^2} \frac{B^2}{2A} + 1 \right) (\psi_t)^2 ;
\end{equation}

cf. [32]. After solving equation (2.1) or (2.2) for the acoustic velocity potential, we can compute the acoustic pressure as $u = \varrho \psi_t$, where $\varrho$ denotes the mass density of the medium.

In the derivation of these models, the classical Fourier law of heat conduction is used in the equation for the conservation of energy. It is, however, well-known that the Fourier law predicts an infinite speed of heat propagation: any thermal disturbance at one point has an instantaneous effect elsewhere in the medium [20]. To overcome this drawback, the Maxwell–Cattaneo law can be used instead. Introducing this law of heat conduction in the derivation of (2.2) leads to a third-order equation given by

\begin{equation}
\tau \psi_{ttt} + \psi_{ttt} - c^2 \Delta \psi - b \Delta \psi_t = \left( \frac{1}{c^2} \frac{B^2}{2A} + 1 \right) (\psi_t)^2 ;
\end{equation}

cf. [11]. This nonlinear equation is often referred to as the Jordan–Moore–Gibson–Thompson (JMGT) equation. Here $\tau > 0$ stands for the relaxation time. The constant $b > 0$ is given by

\begin{equation}
b = \delta + \tau c^2 .
\end{equation}

Additionally, it is well-known that relaxation processes play an important role in high-frequency waves in fluids and gases. If relaxation occurs, acoustic pressure can depend on the density at all prior times. Such a process, therefore, introduces a memory term into the state equation. This motivates us to consider the general nonlocal JMGT equation in the form of

\begin{equation}
\tau \psi_{ttt} + \alpha \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t + \int_0^t g(s) \Delta \psi(t-s) \, ds = \left( k \psi_t^2 \right)_t .
\end{equation}
The function \( g \) denotes the relaxation memory kernel related to the particular relaxation mechanism. The constant \( k \in \mathbb{R} \) indicates the nonlinearity of the model and \( \alpha > 0 \) the friction. Equation (2.5) is here considered with the following initial data:

\[
\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad \psi_{tt}(x, 0) = \psi_2(x),
\]

whose regularity will be specified in the theorems below.

2.1. Memory kernel. Throughout the paper, we make the following standard assumptions on the relaxation kernel; cf. \([7, \text{Section 1}]\).

**Assumption 1.** The memory kernel is assumed to satisfy the following conditions:

(G1) \( g \in W^{1,1}(\mathbb{R}^+) \) and \( g' \) is almost continuous on \( \mathbb{R}^+ = (0, +\infty) \).

(G2) \( g(s) \geq 0 \) for all \( s > 0 \) and

\[
c^2_0 := c^2 - \int_0^{\infty} g(s) \, ds > 0.
\]

(G3) There exists \( \zeta > 0 \), such that the function \( g \) satisfies the differential inequality given by

\[
g'(s) \leq -\zeta g(s)
\]

for every \( s \in (0, \infty) \).

(G4) It holds that \( g'' \geq 0 \) almost everywhere.

In relaxing media, the memory kernel typically has the exponential form

\[
g(s) = mc^2 \exp\left(-s/\tau\right),
\]

where \( m \) is the relaxation parameter; see \([23, \text{Chapter 1}] \) and \([17, \text{Section 1}] \). The value of \( m \) is small, so the condition \((2.7)\), equivalent to \( m < \tau \), is easily satisfied. With this choice of the kernel, we have

\[
g'(s) \leq -g(s);
\]

i.e., we can take \( \zeta = 1 \). We see also that for \( \tau \rightarrow 0^+ \), the kernel tends to zero and we are formally in the regime of the Westervelt equation, as expected.

**Figure 1.** The fading relaxation kernel
In smooth bounded domains, exponential decay of the memory kernel $g$ leads to the exponential decay of the energy of the system; cf. [17, Theorem 1.4]. This changes when waves propagate in the whole space $\mathbb{R}^n$. Even though our memory kernel decays exponentially, the solution will decay at most polynomially.

The optimality of the decay in $\mathbb{R}^n$ is usually measured with respect to the decay rate of the heat kernel: the solution of $u_t - \Delta u = 0$ with initial data being the delta distribution. For the heat equation in bounded domains, the solution decays exponentially fast, however, in the whole space $\mathbb{R}^n$ the solution (i.e., its energy norm) decays at most polynomially with the rate $(1 + t)^{-n/4}$ provided that the initial datum is in $L^1(\mathbb{R}^n)$; see [9]. This decay rate of the heat energy is, in fact, optimal because we can explicitly deduce it from the form of the heat kernel.

2.2. Previous work. The JMGT equation and its linearization have been a subject of extensive study. The linearization of this equation without memory is given by

$$\tau \psi_{ttt} + \alpha \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = 0. \hspace{1cm} (2.8)$$

This equation is known as the Moore–Gibson–Thompson equation, although, as mentioned in [3], this model originally appears in the work of Stokes [30]. Interestingly, equation (2.8) also arises in viscoelasticity theory under the name of standard linear model of viscoelasticity; see [10] and references given therein.

If $b = 0$ in equation (2.8), there is no semigroup associated with the linear dynamics; see [13]. For $b > 0$, the linear dynamics is described by a strongly continuous semigroup, which is exponentially stable if

$$ab - \tau c^2 > 0. \hspace{1cm} (2.9)$$

If $ab = \tau c^2$, the energy is conserved.

The linear model associated with the JMGT equation with memory (2.5) in the pressure form reads as

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t + \int_0^t g(s) \Delta u(t-s) \, ds = 0. \hspace{1cm} (2.10)$$

recall that the pressure and potential are connected via $u = \rho \psi_t$. In [13], a generalization of this equation is studied in smooth bounded domains with a memory term in the form of $\int_0^t g(s) \Delta z(t-s) \, ds$, where $z$ is one of the three functions: $z = u$, $z = u_t$, or $z = u + u_t$. It turns out that if the memory kernel $g$ decays exponentially, the same holds for the solution, provided that the critical condition (2.9) holds. This result is extended in [18] by allowing the memory kernel to satisfy a more general decay property.

The critical case where $ab = \tau c^2$ and $\int_0^\infty g(s) \, ds > 0$ is investigated in [7] with a general strictly positive self-adjoint linear operator $A$ instead of $-\Delta$. The linearized problem is exponentially stable if and only if $A$ is a bounded operator. In the case of an unbounded operator $A$, the corresponding energy decays polynomially with the rate $1/t$ for regular initial data.

Taking the quadratic nonlinear effects into account leads to the nonlinear JMGT equation of Westervelt type given in (2.3). Without memory effects, it is analyzed in [14] in terms of existence and regularity of solutions on bounded smooth domains.
Moreover, it is shown that its solution converges weakly to the solution of the Wester-velt equation in the limit $\tau \to 0$.

The JMGT equation with memory is investigated in [17] on regular bounded domains, expressed in terms of the acoustic pressure $u$. There it is proven that with suitable adjustment of the memory kernel, solutions exist globally for sufficiently small and regular initial data. With exponentially decaying memory kernel these solutions exhibit exponential decay rates.

Due to the lack of Poincaré’s inequality, the analysis of nonlinear acoustic models is more delicate in $\mathbb{R}^n$. Nevertheless, the linearized problem (2.10) with and without memory is well-understood; see [2, 27]. The nonlinear JMGT equation (2.3) is also known to have solutions globally in time in $\mathbb{R}^3$ in non-hereditary media; cf. [29].

3. Theoretical preliminaries and notation

We collect here several theoretical results that will be helpful in the later proofs.

3.1. The past-history framework. Following [7], we use the so-called past history framework of Dafermos [6] to transform our problem into an evolution one. We then introduce the auxiliary past-history variable $\eta(t, s) = \eta_t(s)$ for $t \geq 0$, defined as

\[
\eta_t(s) = \begin{cases} 
\psi(t) - \psi(t - s), & 0 < s \leq t, \\
\psi(t), & s > t.
\end{cases}
\]

By setting $\eta^0(x, s) = \psi_0(x)$, the JMGT equation (2.5) then transforms into the following problem:

\[
\begin{aligned}
\tau \psi_{ttt} + \alpha \psi_{tt} - b \Delta \psi_t - c_g^2 \Delta \psi - \int_0^\infty g(s) \Delta \eta_t(s) \, ds &= 2k \psi_t \psi_{tt}, \\
\eta^0_t(x, s) &= \psi_l(x, t),
\end{aligned}
\]

where we recall that the modified speed of sound squared $c_g^2$ is defined in (2.7). The problem is supplemented with the initial data (2.6).

Note that from the second equation in (3.2) we get (3.1) via Duhamel’s formula, assuming that we set $\eta^0 = \psi_0$; see also [7, Remark 3.3]. Therefore, we can obtain equation (2.5) from (3.2). Indeed, it is enough to check that

\[
\int_0^\infty g(s) \Delta \eta_t(s) \, ds = \int_0^t g(s) \Delta \eta_t(s) \, ds + \int_t^\infty g(s) \Delta \eta_t(s) \, ds
\]

\[
= \int_0^t g(s) \Delta (\psi(t) - \psi(t - s)) \, ds + \int_t^\infty g(s) \Delta \psi(t) \, ds
\]

\[
= (c^2 - c_g^2) \Delta \psi - \int_0^t g(s) \Delta \psi(t - s) \, ds.
\]

3.2. Setting $\alpha = 1$. From this point on, we set $\alpha = 1$. We may do so without the loss of generality since we can always re-scale other coefficients in the equation. The critical condition (2.9) then reads as

\[
b > \tau c^2,
\]
which, having in mind relation (2.4), means that we need the sound diffusivity \( \delta \) to be positive. In other words, we are assuming our medium to be non-inviscid. For our well-posedness result, we will also require that \( \tau c^2 > \tau c_g^2 \), which is equivalent to assuming that \( \int_0^\infty g(s) \, ds > 0 \).

3.3. **Functional spaces.** For future use, we introduce here the weighted \( L^2 \)-spaces, 
\[
L^2_{\tilde{g}} = L^2_{\tilde{g}}(\mathbb{R}^+, L^2(\mathbb{R}^n)).
\]
We will have three types of weights: \( \tilde{g} \in \{ g, -g', g'' \} \). The space is endowed with the inner product 
\[
(\eta^t, \tilde{\eta}^t)_{L^2_{\tilde{g}}} = \int_0^t \tilde{g}(s) (\eta^t(s), \tilde{\eta}^t(s))_{L^2(\mathbb{R}^n)} \, ds
\]
and with the following norm:
\[
\|\eta^t\|_{L^2_{\tilde{g}}}^2 = \int_0^t \tilde{g}(s) \|\eta^t(s)\|_{L^2}^2 \, ds,
\]
for \( \eta^t, \tilde{\eta}^t \in L^2_{\tilde{g}} \). We can then further introduce the spaces 
\[
H^m_{\tilde{g}} = \{ \eta^t \in L^2_{\tilde{g}} : D^s \eta^t \in L^2_{\tilde{g}}, \forall |a| \leq m \}, \quad m \in \{1, 2 \}.
\]
The infinitesimal generator of the right-translation \( C_0 \)-semigroup on \( L^2_{\tilde{g}} \) is given by the linear operator \( T \):
\[
T \eta^t = -\eta^t_s \quad \text{with} \quad D(T) = \{ \eta^t \in L^2_{\tilde{g}} : \eta^t_s \in L^2_{\tilde{g}}, \eta^t(0) = 0 \},
\]
where the index \( s \) denotes the distributional derivative with respect to the variable \( s > 0 \); cf. [2, 7].

3.4. **Auxiliary inequalities.** Throughout the paper, we often use the Ladyzhenskaya inequality for functions \( f \in H^1(\mathbb{R}^n) \), with \( n \in \{2, 3\} \), given by
\[
\|f\|_{L^4} \leq C_n \|f\|_{L^2}^{1-n/4} \|\nabla f\|_{L^2}^{n/4},
\]
where the constant \( C_n > 0 \) depends on \( n \). Furthermore, we frequently rely also on this particular case of the Gagliardo–Nirenberg interpolation inequality [24, 25]:
\[
\|\nabla u\|_{L^4} \leq C_n \|\nabla u\|_{L^2}^{1-n/4} \|\nabla^2 u\|_{L^2}^{n/4}.
\]
We need the following estimate as well:
\[
\|\nabla (uv)\|_{L^2} \leq C(\|u\|_{L^\infty} \|\nabla v\|_{L^2} + \|v\|_{L^4} \|\nabla u\|_{L^4}).
\]
The next technical estimate will be employed when deriving the decay rate of the energy of our system.

**Lemma 3.1** (see Lemma 3.5 in [27]). Let \( n \geq 1 \) and \( t \geq 0 \). Then the following estimate holds:
\[
\int_0^1 r^{n-1} e^{-r^2 t} dr \leq C(n) (1 + t)^{-n/2}.
\]
We state here one more useful inequality that will be crucial in our energy arguments.
Lemma 3.2 (see Lemma 3.7 in [31]). Let $M = M(t)$ be a non-negative continuous function satisfying the inequality

$$M(t) \leq C_1 + C_2 M(t)^\kappa,$$

in some interval containing $0$, where $C_1$ and $C_2$ are positive constants and $\kappa > 1$. If $M(0) \leq C_1$ and

$$C_1C_2^{1/(\kappa-1)} < (1 - 1/\kappa)^{-1/(\kappa-1)},$$

then in the same interval

$$M(t) < \frac{C_1}{1 - 1/\kappa}.$$

3.5. Notation. Throughout the paper, the constant $C$ denotes a generic positive constant that does not depend on time, and that may take different values of different occasions. We use $x \lesssim y$ to denote $x \leq Cy$.

4. Energy estimates

In this section, we formally derive several energy estimates for our problem that we will rely on later. We begin by rewriting our equation (3.2) as a first-order in time system. To this end, we introduce the functions

$$v = \psi_t \quad \text{and} \quad w = \psi_{tt},$$

which leads to the following system of equations:

$$
\begin{align*}
\psi_t &= v, \\
v_t &= w, \\
\tau w_t &= -w + c_2^2 \Delta \psi + b \Delta v + \int_0^\infty g(s) \Delta \eta(s) \, ds + 2kwv, \\
\eta_t &= v - \eta_s,
\end{align*}
$$

with the initial data

$$\left(\psi, v, w, \eta\right)|_{t=0} = (\psi_0, \psi_1, \psi_2, \psi_0).$$

By using the notation

$$\Psi = (\psi, v, w, \eta^t)^T,$$

and setting $\Psi_0 = \Psi(0)$, we can convert our problem into an initial value problem for a first-order abstract evolution equation. Indeed, $\Psi$ satisfies

$$
\begin{align*}
\frac{d}{dt} \Psi(t) &= A\Psi(t) + F(\Psi), \quad t > 0, \\
\Psi(0) &= \Psi_0,
\end{align*}
$$

where the operator $A$ is defined as

$$A \begin{bmatrix} \psi \\ v \\ w \\ \eta^t \end{bmatrix} = \begin{bmatrix} v \\ w \\ -\frac{1}{\tau}w + \frac{c_2^2}{\tau} \Delta \psi + \frac{b}{\tau} \Delta v + \frac{1}{\tau} \int_0^\infty g(s) \Delta \eta^t(s) \, ds \\ v + T\eta^t \end{bmatrix}.$$
The nonlinear term in (4.3) is given by

\begin{equation}
F(\Psi) = \frac{2k}{r} [0,\ 0,\ vw,\ 0]^T.
\end{equation}

Going forward, our work plan is to introduce the mapping

\[ \mathcal{T}(\Phi) = \Psi, \]

where \( \Psi \) solves the inhomogeneous linear problem

\begin{align*}
\begin{cases}
\partial_t \Psi - A\Psi = F(\Phi), \\
\Psi_{t=0} = \Psi_0
\end{cases}
\end{align*}

on a suitably defined ball in a Banach space and employ the contraction principle on \( \mathcal{T} \). The unique fixed-point is then the solution to our nonlinear problem.

As a preparation, we first derive several energy estimates for (4.1) which are uniform in time and thus crucial in later proving global existence.

### 4.1. Functional setting

In order to formulate our results, we introduce the Hilbert spaces

\begin{equation}
\mathcal{H}^{s+1} = H^{s+1}(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \times H_{-g'}^{s+1}(\mathbb{R}^n),
\end{equation}

for \( s \in \{0, 1\} \) and \( n \in \{2, 3\} \). It is known that the homogeneous Sobolev space \( \dot{H}^1(\mathbb{R}^n) \) is a Hilbert space if and only if \( n > 2 \); see [1, Proposition 1.34]. For \( n = 3 \), we can therefore work with the Hilbert spaces

\begin{equation}
\mathcal{H}^1 = \dot{H}^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times \dot{H}_{-g'}^1(\mathbb{R}^3),
\end{equation}

as well as

\begin{equation}
\mathcal{H}^2 = \{ \psi : \psi \in \dot{H}^1(\mathbb{R}^3), \Delta \psi \in L^2(\mathbb{R}^3) \} \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times \dot{H}_{-g'}^1(\mathbb{R}^3).
\end{equation}

We intend to work with these spaces to show global well-posedness in \( \mathbb{R}^3 \).

#### Energy functionals

We then define the energy of first order by

\begin{equation}
\mathcal{E}_1[\Psi] = \|\nabla (\psi + \tau v)\|_{L^2}^2 + \|v + \tau w\|_{L^2}^2 + \|\nabla \eta\|_{L^2,-g'}^2.
\end{equation}

We also introduce the energy functional of second order as follows:

\begin{equation}
\mathcal{E}_2[\Psi] = \|\Delta (\psi + \tau v)\|_{L^2}^2 + \|\nabla (v + \tau w)\|_{L^2}^2 + \|\Delta v(t)\|_{L^2}^2 + \|\Delta \eta\|_{L^2,-g'}^2.
\end{equation}

Thus, for \( \Psi \in \mathcal{H}^2 \), we have the norm

\[ \|\Psi\|_{\mathcal{H}^2} = (\mathcal{E}_1[\Psi] + \mathcal{E}_2[\Psi] + \|w\|_{L^2}^2)^{1/2}, \]

whereas for \( \Psi \in \mathcal{H}^2 \), the norm is given by

\[ \|\Psi\|_{\mathcal{H}^2} = (\|\psi\|_{L^2}^2 + \mathcal{E}_1[\Psi] + \mathcal{E}_2[\Psi] + \|w\|_{L^2}^2)^{1/2} . \]

For \( \Psi \in C([0,T]; \mathcal{H}^2) \), we can introduce here the energy semi-norm by

\begin{equation}
|\Psi|_{E(t)} = \sup_{0 \leq \sigma \leq t} (\mathcal{E}_1[\Psi](\sigma) + \mathcal{E}_2[\Psi](\sigma) + \|w(\sigma)\|_{L^2}^2)^{1/2} .
\end{equation}
The corresponding dissipation semi-norm is given by

\[ |\Psi|_{D(t)} \]

(4.11)

\[ = \left\{ \int_0^t \left( \|\nabla v(\sigma)\|_{L^2}^2 + \|\nabla \eta(\sigma)\|_{L^2,-g'}^2 + \mathcal{E}_2(\sigma) + \|w(\sigma)\|_{L^2}^2 \right) d\sigma \right\}^{1/2}. \]

We can easily see here one of the difficulties in the analysis of the JMTG equation in \( \mathbb{R}^n \), which is that in general, we do not have direct control over \( \|\psi(t)\|_{L^2} \) because of the lack of Poincaré’s inequality.

4.2. Derivation of the estimates. We derive the energy estimates under the assumption that a sufficiently smooth solution \( \Psi = (\psi, v, w, \eta)^T \) of our system (4.1) with initial conditions (4.2) exists on some time interval \( [0,T] \). In particular, we assume that \( |\Psi|_{L^2(\mathbb{R}^n)} < \infty \). The estimates below will then be rigorously justified in Section 5.

To simplify the notation that involves the nonlinear term \( 2kvw \) in the system, we also introduce the functionals \( R^{(1)} \) and \( R^{(2)} \) as

\[ R^{(1)}(\varphi) = 2k(vw, \varphi)_{L^2}, \quad R^{(2)}(\varphi) = 2k(\nabla(vw), \nabla \varphi)_{L^2}, \]

where \( \varphi \) stands for various test functions that we use in the proofs.

Our main goal now is to derive an estimate in the form

\[ |\Psi|_{E(t)}^2 + |\Psi|_{D(t)}^2 \leq |\Psi|_{E(0)}^2 + \sum_i \int_0^t |R^{(1)}(\varphi_i)| d\sigma + \sum_j \int_0^t |R^{(2)}(\varphi_j)| d\sigma \]

\[ \leq |\Psi|_{E(0)}^2 + |\Psi|_{E(t)}|\Psi|_{D(t)}^2 \]

for all \( t \in [0,T] \). On account of Lemma (4.2), this inequality together with a bootstrap argument yields

\[ |\Psi|_{E(t)} + |\Psi|_{D(t)} \lesssim |\Psi|_{E(0)} \]

provided that \( |\Psi|_{E(0)} \) is small enough. The hidden constant does not depend on time, and so the above estimates allow us to continue the solution to \( T = \infty \).

4.3. Lower-order estimates. In order to formulate our results and following [9], we introduce here the weighted lower-order energy of first order at time \( t \geq 0 \) as

\[ E_1(t) = \frac{1}{2} \left[ c_g^2 \|\nabla(\psi + \tau v)\|_{L^2}^2 + \tau \|v + \tau w\|_{L^2}^2 + \tau \|\nabla \eta\|_{L^2,-g'}^2 + \|\nabla \eta\|_{L^2,g'}^2 + 2\tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v ds \, dx \right]. \]

We remark that the last term in (4.13) has an undefined sign, but we will show that the other terms in the energy functional can absorb it. In fact, \( E_1 \) is equivalent to the energy \( \mathcal{E}_1 = \mathcal{E}_1(\Psi) \), introduced in (4.8).

Lemma 4.1. Assume that \( b \geq \tau c_g^2 > \tau c_g^2 \). There exist positive constants \( C_1 \) and \( C_2 \), such that

\[ C_1 \mathcal{E}_1(t) \leq E_1(t) \leq C_2 \mathcal{E}_1(t), \]

for all \( t \geq 0 \).
Proof. The statement follows by [7, Lemma 3.1]; we include the proof here for completeness. To show (4.14), we first have by Young’s inequality
\[
2\tau \int_{\mathbb{R}^n} \int_0^{\infty} g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx \leq \frac{\tau^2(c^2 - c_g^2)}{\varepsilon + 1} \|\nabla v\|^2_{L^2} + (\varepsilon + 1) \int_0^{\infty} g(s) \|\nabla \eta(s)\|^2_{L^2} \, ds.
\]
for every \(\varepsilon > 0\). By using assumption (G3) on the relaxation kernel \(g\), we then have
\[
2\tau \int_{\mathbb{R}^n} \int_0^{\infty} g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx \geq -\frac{\tau^2(c^2 - c_g^2)}{\varepsilon + 1} \|\nabla v\|^2_{L^2} - \|\nabla \eta\|^2_{L^2,g} - \frac{\varepsilon}{\xi} \int_0^{\infty} (-g'(s)) \|\nabla \eta(s)\|^2_{L^2} \, ds.
\]
Since \(b - \tau c_g^2 > \tau(c^2 - c_g^2)\), by reducing \(\varepsilon\), we obtain
\[
E_1(t) \geq \frac{1}{2} \left[ c_g^2 \|\nabla (\psi + \tau v)\|^2_{L^2} + \|v + \tau w\|^2_{L^2} + (\tau - \varepsilon/\xi) \|\nabla \eta\|^2_{L^2,-g'} + \tau^2 \varepsilon/(c^2 - c_g^2)/(1 + \varepsilon) \|\nabla v\|^2_{L^2} \right].
\]
Consequently, the left-hand side inequality in (4.14) holds. The right-hand side inequality follows analogously. \(\square\)

The next step is to derive a lower-order energy estimate for \(E_1\).

**Proposition 4.1.** Let \((\psi, v, w, \eta)\) be a smooth solution of the system (4.1) with initial data (4.2). Then the following estimate holds:
\[
\frac{d}{dt} E_1(t) + (b - \tau c^2) \|\nabla v(t)\|^2_{L^2} + \frac{1}{2} \|\nabla \eta\|^2_{L^2,-g'} \leq |R^{(1)}(v + \tau w)|
\]
for all \(t \geq 0\), where the functional \(R^{(1)}\) is defined in (4.12).

Proof. Looking at the definition (4.13) of the energy \(E_1\), we begin by obtaining an expression for \(\frac{1}{2} c_g^2 \frac{d}{dt} \|\nabla (\psi + \tau v)\|^2_{L^2}\). It’s clear that
\[(\psi + \tau v)_t = v + \tau w.\]
Multiplying the above equation by \(-c_g^2 \Delta (\psi + \tau v)\) and integrating over \(\mathbb{R}^n\) gives the identity
\[
\frac{c_g^2}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla (\psi + \tau v)|^2 \, dx
= \tau c_g^2 |\nabla v|^2 + c_g^2 \int_{\mathbb{R}^n} \nabla \psi \cdot \nabla v \, dx + \tau^2 c_g^2 \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx + \tau c_g^2 \int_{\mathbb{R}^n} \nabla w \cdot \nabla \psi \, dx.
\]
To tackle the time derivative of the second term in the energy (4.13), we then multiply the second equation in the system (4.1) by \(-\tau(b - \tau c_g^2)\Delta v\) and integrate over \(\mathbb{R}^n\). By doing so, we obtain
\[
\frac{1}{2} \tau(b - \tau c_g^2) \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla v|^2 \, dx = \tau(b - \tau c_g^2) \int_{\mathbb{R}^n} \nabla w \cdot \nabla v \, dx.
\]
To handle the term $\frac{1}{2} \frac{d}{dt} ||v + \tau w||^2_{L^2}$, we add the second equation in the system (4.1) to the third one. Then the $w$ terms cancel out and we have

$$ (v + \tau w)_t = b \Delta v + c_g^2 \Delta \psi + \int_0^\infty g(s) \Delta \eta(s) \, ds + 2kw. $$

Multiplying the above equation by $v + \tau w$ and integrating over $\mathbb{R}^n$ yields

$$ \frac{1}{2} \frac{d}{dt} ||v + \tau w||^2_{L^2} = -b \int_{\mathbb{R}^n} |\nabla v|^2 \, dx - \tau b \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx - c_g^2 \int_{\mathbb{R}^n} \nabla \psi \cdot \nabla v \, dx
$$

$$ - c_g^2 \tau \int_{\mathbb{R}^n} \nabla \psi \cdot \nabla w \, dx - \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx
$$

$$ - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla w \, ds \, dx + R^{(1)}(v + \tau w). $$

We can further transform the first term on the right that contains the memory kernel by using the fact that $\eta_t + \eta_s = v$. Indeed, we have

$$ - \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx = - \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla \eta_t(s) \, ds \, dx
$$

$$ - \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla \eta_s(s) \, ds \, dx. $$

Integrating by parts with respect to $s$ in the second term on the right leads to

$$ - \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx = - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) |\nabla \eta(s)|^2 \, ds \, dx
$$

$$ + \frac{1}{2} \int_{\mathbb{R}^n} \int_0^\infty g'(s) |\nabla \eta(s)|^2 \, ds \, dx; $$

noting that the boundary terms vanish; cf. [26]. Hence, we get

$$ - \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx = - \frac{1}{2} \frac{d}{dt} ||\nabla \eta||^2_{L^2, g} - \frac{1}{2} ||\nabla \eta||^2_{L^2, -g'}. $$

Similarly, using the relation $\eta_{tt} + \eta_{ts} = w$ results in

$$ - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla w \, ds \, dx
$$

$$ = - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla \eta_{tt}(s) \, ds \, dx - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla \eta_{ts}(s) \, ds \, dx. $$

Then, by integrating by parts with respect to $s$, we have

$$ - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla w \, ds \, dx
$$

$$ = - \tau \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla \eta_t(s) \, ds \, dx + \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta_t(s) \cdot \nabla \eta_t(s) \, ds \, dx
$$

$$ - \tau \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta_t(s) \cdot \nabla \eta_s(s) \, ds \, dx + \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta_t(s) \cdot \nabla \eta_s(s) \, ds \, dx. $$
By inserting the derived identities into (4.19), we infer
\[
\frac{1}{2} \frac{d}{dt} \left( \| v + \tau w \|_{L^2}^2 + \| \nabla \eta \|_{L^2 \cdot g}^2 + 2\tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx \right)
= -b \| \nabla v \|_{L^2}^2 - \tau b \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx - c_g^2 \int_{\mathbb{R}^n} \nabla \psi \cdot \nabla v \, dx
- c_g^2 \tau \int_{\mathbb{R}^n} \nabla \psi \cdot \nabla w \, dx - \frac{1}{2} \| \nabla \eta(s) \|_{L^2, g'}^2 + \tau \int_{\mathbb{R}^n} \int_0^t g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx
+ R^{(0)}(v + \tau w).
\]

By adding also equation (4.17) to the above expression, we infer
\[
\frac{1}{2} \frac{d}{dt} \left( E_1(t) - \tau \| \nabla \eta \|_{L^2, g'}^2 \right) = - \left( b - \tau c_g^2 \right) \| \nabla v \|_{L^2}^2 - \frac{1}{2} \| \nabla \eta(s) \|_{L^2, g'}^2
+ \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx + 2k(vw, v + \tau w)_{L^2}.
\]

To further transform the memory term on the right, we can substitute \( \eta = v - \eta_s \).
This action leads to
\[
\frac{1}{2} \frac{d}{dt} \left( E_1(t) - \tau \| \nabla \eta \|_{L^2, g'}^2 \right) = - \left( b - \tau c_g^2 \right) \| \nabla v \|_{L^2}^2 - \frac{1}{2} \| \nabla \eta(s) \|_{L^2, g'}^2
+ \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla (v - \eta_s(s)) \cdot \nabla v \, ds \, dx + R^{(1)}(v + \tau w).
\]

Integrating once by parts with respect to \( s \) in the memory term yields
\[
\frac{1}{2} \frac{d}{dt} \left( E_1(t) - \tau \| \nabla \eta \|_{L^2, g'}^2 \right) = - \left( b - \tau c_g^2 \right) \| \nabla v \|_{L^2}^2 - \frac{1}{2} \| \nabla \eta(s) \|_{L^2, g'}^2
+ \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \nabla \eta(s) \cdot \nabla v \, ds \, dx + R^{(1)}(v + \tau w).
\]

We then again use the same trick of substituting \( v = \eta_k + \eta_s \), which results in
\[
\frac{1}{2} \frac{d}{dt} \left( E_1(t) - \tau \| \nabla \eta \|_{L^2, g'}^2 \right) = - \left( b - \tau c_g^2 \right) \| \nabla v \|_{L^2}^2 - \frac{1}{2} \| \nabla \eta(s) \|_{L^2, g'}^2
+ \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \nabla \eta(s) \cdot \nabla \eta_k \, ds \, dx
+ \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \nabla \eta(s) \cdot \nabla \eta_s \, ds \, dx + R^{(1)}(v + \tau w).
\]

Finally, integrating by parts once again with respect to \( s \) in the second memory term on the right leads to
\[
\frac{1}{2} \frac{d}{dt} E_1(t) = - \left( b - \tau c_g^2 \right) \| \nabla v \|_{L^2}^2 - \frac{1}{2} \| \nabla \eta \|_{L^2, g'}^2 - \frac{\tau}{2} \| \nabla \eta \|_{L^2, g'}^2 + R^{(1)}(v + \tau w),
\]
which immediately yields the desired result. \( \square \)

4.4. Higher-order estimates. Next we analogously define the energy of the second order at time \( t \geq 0 \) as
\[
E_2(t) = \frac{1}{2} \left[ c_g^2 \| \Delta (\psi + \tau v) \|_{L^2}^2 + \tau (b - \tau c_g^2) \| \Delta v \|_{L^2}^2 + \| \nabla (v + \tau w) \|_{L^2}^2 + \right.
\]
\[
\left. + \tau \| \Delta \eta \|_{L^2, g'}^2 + \| \Delta \eta \|_{L^2, g'}^2 + 2\tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta(s) \, ds \, dx \right].
\]
Observe that the last term above has an undefined sign; nevertheless, the other terms in the energy functional can absorb it. In fact, the functional $E_2$ is equivalent to $E_2 = E_2[\Psi]$, which we introduced in (4.9).

**Lemma 4.2.** Assume that $b \geq \tau c^2 > \tau c_g^2$. Then there exist positive constants $C_1$ and $C_2$, such that

$$C_1 E_2(t) \leq E_2(t) \leq C_2 E_2(t),$$

for all $t \geq 0$.

**Proof.** The proof follows the same steps as proof of Lemma 4.1. We omit the details here.

We move onto deriving a higher-order energy estimate for $E_2$, analogous to the one of Proposition 4.1.

**Proposition 4.2.** Let $(\psi, v, w, \eta)$ be a smooth solution of the system (4.1) with initial data (4.2). Then the following inequality holds:

$$(4.21) \quad \frac{d}{dt} E_2(t) + (b - \tau c^2) \|\Delta v\|_{L^2}^2 + \frac{1}{2} \|\Delta \eta\|_{L^2}^2 \leq \left| R^{(2)}(v + \tau w) \right|,$$

for all $t \geq 0$, where the functional $R^{(2)}$ is defined in (4.12).

**Proof.** The proof follows by testing our problem with convenient test functions. Looking at the definition (4.20) of the higher-order energy, we first need to tackle the time derivative of the term $\frac{1}{2} c^2_g \|\Delta(\psi + \tau v)(t)\|_{L^2}^2$. Clearly,

$$\Delta(\psi + \tau v)_t = \Delta(v + \tau w).$$

Multiplying the above equation by $\Delta(\psi + \tau v)$ and integrating over $\mathbb{R}^n$ results in

$$(4.22) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\Delta(\psi + \tau v)|^2 dx = \tau \int_{\mathbb{R}^n} \Delta w \Delta \psi dx + \tau^2 \int_{\mathbb{R}^n} \Delta w \Delta v dx + \int_{\mathbb{R}^n} \Delta v \Delta \psi dx + \tau \int_{\mathbb{R}^n} |\Delta v|^2 dx.$$

Next we work with the time derivative of $\|\Delta v(t)\|_{L^2}^2$. By applying the Laplacian to the second equation of the system (4.1), multiplying the resulting expression by $-\tau(b - \tau c_g^2)\Delta v$, integrating over $\mathbb{R}^n$, and using integration by parts, we find

$$(4.23) \quad \frac{1}{2} \tau(b - \tau c_g^2) \frac{d}{dt} \int_{\mathbb{R}^n} |\Delta v|^2 dx = \tau(b - \tau c_g^2) \int_{\mathbb{R}^n} \Delta w \Delta v dx.$$

To handle the time derivative of the third term in (4.20), we apply the operator $\Delta$ to (4.18) we get (in the sense of distribution)

$$ (\Delta(v + \tau w))_t = b \Delta^2 v + c_g^2 \Delta^2 \psi + \int_0^\infty g(s) \Delta^2 \eta(s) ds + 2k \Delta(vw). $$
We multiply the above equation by $-(v + \tau w)$ and integrate over $\mathbb{R}^n$, yielding

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla (v + \tau w)|^2 \, dx + b \int_{\mathbb{R}^n} |\Delta v|^2 \, dx
$$

(4.24)

$$
= - b \tau \int_{\mathbb{R}^n} \Delta v \Delta w \, dx - c_\theta^2 \int_{\mathbb{R}^n} \Delta \psi \Delta (v + \tau w) \, dx
$$

$$
- \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta (v + \tau w) \Delta \eta(s) \, ds \, dx + R^{(2)}(v + \tau w).
$$

By summing up (4.24) + (4.23) + $c_\theta^2$ (4.22), we obtain

$$
\frac{1}{2} \frac{d}{dt} \left[ \|\nabla (v + \tau w)\|_{L^2}^2 + \tau (b - \tau c_\theta^2) \|\Delta v\|_{L^2}^2 + c_\theta^2 \|\Delta (\psi + \tau v)\|_{L^2}^2 \right]
$$

$$
+ (b - \tau c_\theta^2) \|\Delta v\|_{L^2}^2
$$

(4.25)

$$
= - \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta (v + \tau w) \Delta \eta(s) \, ds \, dx + R(\Delta (v + \tau w))
$$

$$
= - \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta(s) \, ds \, dx - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta w \Delta \eta(s) \, ds \, dx
$$

$$
+ R^{(2)}(v + \tau w).
$$

We next want to further transform the first two terms on the right-hand side. By taking the Laplacian of the last equation in (4.11), we obtain $\Delta v = \Delta \eta_t + \Delta \eta_s$. We can then use this relation to find that

$$
\int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta(s) \, ds \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) |\Delta \eta(s)|^2 \, ds \, dx
$$

$$
+ \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta_s \Delta \eta(s) \, ds \, dx.
$$

By integrating by parts with respect to $s$ in the last term, we infer

$$
\int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta(s) \, ds \, dx = \frac{1}{2} \frac{d}{dt} \|\Delta \eta\|_{L^2, \theta}^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g'(s) |\Delta \eta(s)|^2 \, ds \, dx
$$

(4.26)

$$
= \frac{1}{2} \frac{d}{dt} \|\Delta \eta\|_{L^2, \theta}^2 - \frac{1}{2} \|\Delta \eta\|_{L^2, \theta}^2.
$$

To tackle the second memory term on the right in equation (4.25), we can use the relation $\Delta w = \Delta \eta_{tt} + \Delta \eta_{ts}$, which holds in the sense of distribution. Doing so yields

$$
\tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta w \Delta \eta(s) \, ds \, dx
$$

$$
= \tau \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta_t \Delta \eta(s) \, ds \, dx
$$

$$
- \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta_t (s) \Delta \eta_t (s) \, ds \, dx - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta_s (s) \Delta \eta_t (s) \, ds \, dx
$$

$$
+ \tau \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta_t (s) \Delta \eta_s (s) \, ds \, dx - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta_s (s) \Delta \eta_t (s) \, ds \, dx.$$
Since \( v = \eta_t + \eta_s \), we further have
\[
\tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta(s) \, ds \, dx
\]
(4.27)
\[= \frac{d}{dt} \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta(s) \, ds \, dx - \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta_t(s) \, ds \, dx.\]

Consequently, from (4.25), (4.26) and (4.27), we have
\[
\frac{d}{dt}(E_2(t) - \frac{\tau}{2} \| \Delta \eta \|^2_{L^2,-g'}) + (b - \tau c_2^2) \| \Delta v \|^2_{L^2} + \frac{1}{2} \| \Delta \eta \|^2_{L^2,-g'}
\]
\[= \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \Delta v \Delta \eta(s) \, ds \, dx + R^{(2)}(v + \tau w).\]

By using the fact that
\[
\tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v \Delta \eta_t(s) \, ds \, dx = \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta v (\Delta v - \Delta \eta_s(s)) \, ds \, dx
\]
\[= \tau (c^2 - c_1^2) \| \Delta v \|^2_{L^2} + \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \Delta v \Delta \eta(s) \, ds \, dx,
\]
we find that
\[
\frac{d}{dt}(E_2(t) - \frac{\tau}{2} \| \Delta \eta \|^2_{L^2,-g'}) + (b - \tau c_2^2) \| \Delta v \|^2_{L^2} + \frac{1}{2} \| \Delta \eta \|^2_{L^2,-g'}
\]
(4.28)
\[= \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \Delta v \Delta \eta(s) \, ds \, dx + R^{(2)}(v + \tau w).
\]

The term on the right-hand side of (4.28) can be written as, by using the fact that
\[\Delta v = \Delta \eta + \Delta \eta_s,\]
\[\tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \Delta v \Delta \eta(s) \, ds \, dx
\]
\[= \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \Delta \eta(s) \Delta \eta_t(s) \, ds \, dx + \tau \int_{\mathbb{R}^n} \int_0^\infty g'(s) \Delta \eta(s) \Delta \eta_s \, ds \, dx
\]
\[= -\frac{\tau}{2} \frac{d}{dt} \| \Delta \eta \|^2_{L^2,-g'} - \frac{\tau}{2} \| \Delta \eta \|^2_{L^2,g'},\]
where we integrated by parts with respect to \( s \) in the second term. By plugging this identity into (4.28), we deduce (4.21). This finishes the proof of Proposition 4.2. \( \square \)

In order to capture the dissipation of the terms \( \| \Delta (\psi + \tau v) \|_{L^2} \) and \( \| \nabla (v + \tau w) \|_{L^2} \), we introduce two functionals \( F_1 \) and \( F_2 \) as
\[
F_1(t) = \int_{\mathbb{R}^n} \nabla (\psi + \tau v) \cdot \nabla (v + \tau w) \, dx, \quad F_2(t) = -\tau \int_{\mathbb{R}^n} \nabla v \cdot \nabla (v + \tau w) \, dx,
\]
everywhere in time; see also [29]. We prove their properties in the following two lemmas.

**Lemma 4.3.** Let \( (\psi, v, w, \eta) \) be a smooth solution of the system (4.11) with initial data (4.12). For any \( \epsilon_0, \epsilon_1 > 0 \), it holds
\[
\frac{d}{dt} F_1(t) + (c_2^2 - \epsilon_0 - (c^2 - c_1^2) \epsilon_1) \| \Delta (\psi + \tau v) \|^2_{L^2}
\]
(4.29)
\[\leq \| \nabla (v + \tau w) \|^2_{L^2} + C(\epsilon_0) \| \Delta v \|^2_{L^2} + C(\epsilon_1) \| \Delta \eta \|^2_{L^2,g'} + |R^{(2)}(\psi + \tau v)|.\]
Proof. We first compute the derivative of the functional $F_1$ as

\[
\frac{d}{dt} F_1(t) = - \int_{\mathbb{R}^n} \Delta (\psi + \tau v)(v + \tau w)_t \, dx - \int_{\mathbb{R}^n} (\psi + \tau v)_t \Delta (v + \tau w) \, dx.
\]

We clearly have to further transform the two terms on the right-hand side. Recall that $(v + \tau w)_t = b \Delta v + c_g^2 \Delta \psi + \int_0^\infty g(s) \Delta \eta(s) \, ds + 2kvw$.

Multiplying this equation by $-\Delta (\psi + \tau v)$ and integrating over $\mathbb{R}^n$ leads to

\[
\frac{d}{dt} F_1(t) = - \int_{\mathbb{R}^n} (c_g^2 \Delta \psi + b \Delta v)(\Delta \psi + \tau \Delta v) \, dx
\]

\[
= - \int_{\mathbb{R}^n} (c_g^2 \Delta \psi + b \Delta v)(\Delta \psi + \tau \Delta v) \, dx
\]

\[
- \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta(s) (\Delta \psi + \tau \Delta v) \, ds \, dx + R^{(2)}(\psi + \tau v).
\]

We can conveniently rearrange the first term on the right as

\[
\int_{\mathbb{R}^n} (c_g^2 \Delta \psi + b \Delta v)(\Delta \psi + \tau \Delta v) \, dx
\]

\[
= - \frac{c_g^2}{2} \|\Delta (\psi + \tau v)\|^2_{L^2} + (b - \tau c_g^2) \int_{\mathbb{R}^n} \Delta v \Delta (\psi + \tau v) \, dx.
\]

The second term on the right in (4.30) can be written as

\[
- \int_{\mathbb{R}^n} (\psi + \tau v)_t \Delta (v + \tau w) \, dx = - \int_{\mathbb{R}^n} (v + \tau w)_t \Delta (v + \tau w) \, dx = \|\nabla (v + \tau w)\|^2_{L^2}.
\]

By adding together (4.31) and the above identity, and then integrating by parts in space, we obtain

\[
\frac{d}{dt} F_1(t) + c_g^2 \int_{\mathbb{R}^n} |\Delta (\psi + \tau v)|^2 \, dx
\]

\[
= \int_{\mathbb{R}^n} |\nabla (v + \tau w)|^2 \, dx - (b - \tau c_g^2) \int_{\mathbb{R}^n} \Delta v (\Delta \psi + \tau \Delta v) \, dx
\]

\[
+ R^{(2)}(\psi + \tau v) - \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta(s) (\Delta \psi + \tau v) \, ds \, dx.
\]

Applying Young’s inequality results in (4.29) for any $\epsilon_0, \epsilon_1 > 0$. □

We next prove an important energy property of the functional $F_2$.

**Lemma 4.4.** Let $(\psi, v, w, \eta)$ be a smooth solution of the system (4.1) with initial data (4.2). For any $\epsilon_2, \epsilon_3 > 0$, we have

\[
\frac{d}{dt} F_2(t) + (1 - \epsilon_3) \|\nabla (v + \tau w)\|^2_{L^2}
\]

\[
\leq \epsilon_2 \|\Delta (\psi + \tau v)\|^2_{L^2} + C(\epsilon_3, \epsilon_2)(\|\Delta v\|^2_{L^2} + \|\nabla v\|^2_{L^2}) + \frac{1}{2} \|\nabla \eta\|^2_{L^2,g} + |R^{(2)}(\tau v)|,
\]

where the functional $R^{(2)}$ is defined in (4.12).
Proof. We can express the derivative of the functional $F_2$ as

$$
\begin{align*}
\frac{d}{dt} F_2(t) &= \tau \int_{\mathbb{R}^n} v_t \Delta (v + \tau w) \, dx + \tau \int_{\mathbb{R}^n} \Delta v (v + \tau w) \, dx \\
&= \tau \int_{\mathbb{R}^n} w \Delta (v + \tau w) \, dx + \tau \int_{\mathbb{R}^n} \Delta v (v + \tau w) \, dx,
\end{align*}
$$

(4.33)

where the second line follows from $v_t = w$. To further transform the second term on the right, we multiply equation (4.18) by $\tau_c$ and

$$
\begin{align*}
\tau \int_{\mathbb{R}^n} (v + \tau w) \, \Delta v \, dx \\
&= \int_{\mathbb{R}^n} \left( \tau c_g^2 \Delta (\psi + \tau v) + \tau (b - \tau c_g^2) \Delta v + (v + \tau w) \right) \\
&\quad - (v + \tau w) \, \Delta v \, dx + \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta(s) \Delta v \, ds \, dx + R^{(2)}(\tau v).
\end{align*}
$$

By plugging this identity into (4.33), we obtain

$$
\begin{align*}
\frac{d}{dt} F_2(t) + \int_{\mathbb{R}^n} |\nabla (v + \tau w)|^2 \, dx \\
&= \tau c_g^2 \int_{\mathbb{R}^n} \Delta (\psi + \tau v) \Delta v \, dx + \tau (b - \tau c_g^2) \int_{\mathbb{R}^n} |\Delta v|^2 \, dx \\
&\quad + \int_{\mathbb{R}^n} \nabla (v + \tau w) \cdot \nabla v \, dx + \tau \int_{\mathbb{R}^n} \int_0^\infty g(s) \Delta \eta(s) \Delta v \, ds \, dx + R^{(2)}(\tau v).
\end{align*}
$$

By additionally applying Young’s inequality with $\varepsilon_2, \varepsilon_3 > 0$, we arrive at the final estimate (4.32). □

4.5. The Lyapunov functional. We are now ready to introduce the Lyapunov functional $\mathcal{L}$ as

$$
\mathcal{L}(t) = L_1(E_1(t) + E_2(t) + \varepsilon \tau \|w\|_{L^2}^2) + F_1(t) + L_2 F_2(t),
$$

(4.34)

for $t \geq 0$. The positive constants $L_1$ and $L_2$ should be sufficiently large and the constant $\varepsilon > 0$ small enough; we will make them more precise below.

This Lyapunov functional can be made equivalent to $\mathcal{E}_1 + \mathcal{E}_2 + \|w\|_{L^2}^2$, where the energies $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in (4.8) and (4.8), respectively. We prove this statement next.

Lemma 4.5. Let $b \geq \tau c^2 > \tau c_g^2$. There exist positive constants $C_1$ and $C_2$, such that

$$
C_1(\mathcal{E}_1(t) + \mathcal{E}_2(t) + \|w\|_{L^2}^2) \leq \mathcal{L}(t) \leq C_2(\mathcal{E}_1(t) + \mathcal{E}_2(t) + \|w\|_{L^2}^2),
$$

(4.35)

for all $t \geq 0$, provided that the constant $L_1$ in the Lyapunov functional (4.34) is chosen large enough.

Proof. To derive (4.35), we are missing the bounds on $F_1$ and $F_2$. We can estimate these terms in the Lyapunov functional as follows:

$$
|F_1(t)| \leq \|\nabla (\psi + \tau v)\|_{L^2} \|\nabla (v + \tau w)\|_{L^2} \lesssim E_1(t) + E_2(t),
$$

and

$$
|F_2(t)| \leq \tau \|\nabla v\|_{L^2} \|\nabla (v + \tau w)\|_{L^2} \lesssim E_1(t) + E_2(t)
$$

as

$$
\begin{align*}
\mathcal{L}(t) &= \mathcal{E}_1(t) + \mathcal{E}_2(t) + \|w\|_{L^2}^2 \\
&\quad + \tau_1 \int_0^t \int_{\mathbb{R}^n} \nabla (\psi + \tau v) \cdot \nabla v \, dx \, ds + \tau_2 \int_0^t \int_{\mathbb{R}^n} \nabla (v + \tau w) \cdot \nabla w \, dx \, ds \\
&\quad + \tau_3 \int_0^t \int_{\mathbb{R}^n} \nabla (v + \tau w) \cdot \nabla (v + \tau w) \, dx \, ds + \tau_4 \int_0^t \int_{\mathbb{R}^n} \nabla (v + \tau w) \cdot \nabla (v + \tau w) \, dx \, ds.
\end{align*}
$$

(4.36)

for $\tau_1, \tau_2, \tau_3, \tau_4 > 0$ small enough; we will make them more precise below.
for all $t \geq 0$. Hence, there exists $C^* = C^*(\tau, c_g^2, b, L_2) > 0$ such that

$$|L(t) - L_1(E_1(t) + E_2(t) + \varepsilon \tau \|w\|_{L^2}^2)| \leq C^*(E_1(t) + E_2(t) + \varepsilon \tau \|w\|_{L^2}^2).$$

Choosing $L_1$ large enough so that

$$L_1 > C^*(\tau, c_g^2, b, L_2)$$

leads to the estimates given in (4.35).

We next derive an energy bound for the Lyapunov functional.

**Proposition 4.3.** Let $b > \tau c_g^2 > \tau c_1^2$. There exist a constant $L_1 > 0$ large enough and a constant $\varepsilon > 0$ small enough such that the Lyapunov functional, defined in (4.34), satisfies

$$\frac{d}{dt} L(t) + \|\nabla v(t)\|_{L^2}^2 + \|\nabla \eta\|_{L^2, -g'}^2 + \delta_2[\Psi](t) + \|w(t)\|_{L^2}^2$$

$$\leq |R^{(1)}(v + \tau w)| + |R^{(2)}(v + \tau w)| + |R^{(1)}(w)| + |R^{(2)}(\psi + \tau v)| + |R^{(2)}(\tau v)|,$$

for all $t \in [0, T]$, where the functionals $R^{(1)}$ and $R^{(2)}$ are defined in (4.12), and the energy $\delta_2$ in (4.9).

**Proof.** To derive the desired estimate, we have to get a bound on $\frac{d}{dt}\|w\|_{L^2}^2$ first. By multiplying the third equation in the system (1.1) by $w$ and integrating over $\mathbb{R}^n$, we infer

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \tau |w|^2 \, dx + \int_{\mathbb{R}^n} |w|^2 \, dx$$

$$\leq C \left( \|\Delta \psi\|_{L^2} + \|\Delta v\|_{L^2} + \|\Delta \eta\|_{L^2, g} \right) \|w\|_{L^2} + |R^{(1)}(w)|.$$

By applying Young’s inequality to the first term on the right, we obtain

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \frac{1}{2} \|w\|_{L^2}^2$$

$$\leq C \left( \|\Delta (\psi + \tau v)\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta \eta\|_{L^2, g}^2 \right) + |R^{(1)}(w)|.$$

Collecting previously derived bounds in the form of (4.15) + (4.21) + $2\varepsilon (4.38)$, we get

$$\frac{d}{dt} \left( E_1(t) + E_2(t) + \varepsilon \tau \|w\|_{L^2}^2 \right) + (b - \tau c_2^2)(\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2)$$

$$+ \varepsilon \|w\|_{L^2}^2 + \frac{1}{2} \|\nabla \eta\|_{L^2, -g'}^2 + \frac{1}{2} \|\Delta \eta\|_{L^2, -g'}^2$$

$$\leq 2C\varepsilon (\|\Delta (\psi + \tau v)\|_{L^2}^2 + \|\Delta v\|_{L^2}^2) + \|\Delta \eta\|_{L^2, g}^2$$

$$+ |R^{(1)}(v + \tau w)| + |R^{(2)}(v + \tau w)| + 2\varepsilon |R^{(1)}(w)|.$$

Note that the first term on the left in the brackets is equal to $L_1^{-1}(L(t) - F_1(t) - L_2 F_2(t))$. Taking into account Lemmas 4.3 and 4.3 as well as assumption (G3) on the memory
where \( \Lambda_0, \gamma \) are independent of \( \epsilon > 0 \). This outcome can be achieved as follows: we pick \( \epsilon > 0 \) small enough such that
\[
\epsilon_0 \leq \frac{c_g^2}{1 + (c^2 - c_g^2)}, \quad \text{and} \quad \epsilon < \frac{b - \tau c^2}{2C}.
\]
Afterwards, we take \( L_2 \) large enough such that
\[
L_2 > \frac{1}{1 - \epsilon_3}.
\]
Once \( L_2 \) and \( \epsilon_0 \) are fixed, we select \( \epsilon_2 > 0 \) small enough such that
\[
\epsilon_2 < \frac{c_g^2 - \epsilon_0(1 + (c^2 - c_g^2))}{L_2}.
\]
Keeping in mind the assumption \( b > \tau c^2 \), we take \( L_1 \) large enough such that condition (4.36) holds together with
\[
L_1 \geq \max \left\{ \frac{C(\epsilon_0) + L_2 C(\epsilon_2, \epsilon_3)}{b - \tau c^2}, \frac{2\Lambda_0}{\zeta} \right\}.
\]
Finally, we decrease \( \epsilon > 0 \) additionally so that
\[
\epsilon < \min \left( \frac{L_1 (b - \tau c^2) - C(\epsilon_0) - C(\epsilon_3, \epsilon_2) L_1}{2CL_1}, \frac{L_1/2 - \Lambda_0/\zeta}{2CL_1} \right).
\]
Consequently, we obtain the desired estimate (4.37).

Now, by integrating estimate (4.37) over the time interval \((0, \sigma)\) for \( \sigma \in (0, t) \) and then taking the supremum over time, we obtain
\[
|\Psi|^2_{L^2(t)} + |\Psi|^2_{D(t)} \lesssim |\Psi|^2_{L^2(0)} + \int_0^t \left\{ |R^{(1)}(v + \tau w)| + |R^{(2)}(v + \tau w)| + |R^{(1)}(w)| + |R^{(2)}(\psi + \tau w)| + |R^{(2)}(\tau v)| \right\} \, d\sigma,
\]
where we have additionally exploited the equivalence of the Lyapunov functional and \( \epsilon_1 + \epsilon_2 + \|w\|^2_{L^2} \) given in (4.35).
4.6. Estimates of the right-hand side terms. To finalize the energy bound, we have to estimate the remaining \(R^{(1)}\) and \(R^{(2)}\) terms. We wish to bound each of these terms by \(|\Psi|_{E(t)}|\Psi|_{D(t)}^2\) multiplied by some positive constant \(C\) that is independent of \(t\). The estimates are split into two lemmas.

**Lemma 4.6.** Let \(\Psi = (\dot{v}, v, w, \eta)\) be a smooth solution of the system \((4.11)\) with initial data \((4.2)\). For all \(t \in [0, T]\), it holds

\[
|R^{(1)}(v + \tau w)(\sigma)| \leq 2|k||v||L_2|\|v\|_{L_2}^2 + 2\tau|k||v||L_2|\|w\|_{L_2}^2.
\]

where the functional \(R^{(1)}\) is defined in \((4.12)\) and the energy semi-norms \(|\cdot|_{E(t)}\) and \(|\cdot|_{D(t)}\) in \((4.10)\) and \((4.11)\), respectively.

**Proof.** By employing Hölder’s inequality, we can proceed as follows:

\[
|R^{(1)}(v + \tau w)| = 2k \int_{\mathbb{R}^n} v w(v + \tau w) \, dx \leq 2|k||v||L_2|\|v\|_{L_2}^2 + 2\tau|k||v||L_2|\|w\|_{L_2}^2.
\]

We can then rely on the Ladyzhenskaya interpolation inequality \((3.3)\). We thus have for the first term on the right

\[
2|k||v||L_2|\|v\|_{L_2}^2 \lesssim \|v\|_{L_2}^2\|v\|_{L_2}^{2(1-n/4)}\|\nabla v\|_{L_2}^{n/2} \lesssim \|v\|_{L_2}^2(\|v\|_{L_2}^2 + \|\nabla v\|_{L_2}^2),
\]

where we have also employed Young’s inequality in the second line. Similarly, the second term can be estimated as

\[
2\tau|k||v||L_2|\|w\|_{L_2}^2 \lesssim \|v\|_{L_2}^2\|w\|_{L_2}^{2(1-n/4)}\|\nabla w\|_{L_2}^{n/2} \lesssim \|v\|_{L_2}^2(\|w\|_{L_2}^2 + \|\nabla w\|_{L_2}^2).
\]

Altogether, this strategy yields

\[
\int_0^t |R^{(1)}(v + \tau w)(\sigma)| \, d\sigma \lesssim \sup_{0 \leq \sigma \leq t} \|v(\sigma)\|_{L_2}^2 \int_0^t (\|v(\sigma)\|_{L_2}^2 + \|\nabla v(\sigma)\|_{L_2}^2) \, d\sigma + \sup_{0 \leq \sigma \leq t} \|v(\sigma)\|_{L_2}^2 \int_0^t (\|v(\sigma)\|_{L_2}^2 + \|\nabla w(\sigma)\|_{L_2}^2) \, d\sigma.
\]

By additionally using the fact that

\[
\|v(t)\|_{L_2} \leq \|v(t)\|_{L_2} + \|\nabla v(t)\|_{L_2},
\]

\[
\|\nabla w(t)\|_{L_2} \leq \frac{1}{\tau}\|\nabla v(t)\|_{L_2} + \frac{1}{\tau}\|\nabla (v + \tau w)(t)\|_{L_2},
\]

for all \(t\), we find that the first term on the left in \((4.40)\) can be bounded by \(|\Psi|_{E(t)}|\Psi|_{D(t)}^2\) up to a constant. The second term can be estimated directly by noting that

\[
|R^{(1)}(w)| \lesssim 2|k||v||L_4|\|w\|_{L_4} \|w\|_{L_2} \\lesssim \|v\|_{L_2}^{1-n/4}\|\nabla v\|_{L_2}^{n/4}\|w\|_{L_2} \|w\|_{L_2} \lesssim (\|v\|_{L_2} + \|\nabla v\|_{L_2})(\|w\|_{L_2}^2 + \|\nabla w\|_{L_2}^2),
\]

and recalling the above bounds on \(\|v(t)\|_{L_2}\) and \(\|\nabla w(t)\|_{L_2}\). \(\square\)
Lemma 4.7. Let $\Psi = (\psi, v, w, \eta)$ be a smooth solution of the system (4.1) with initial data (4.2). Then it holds

\begin{equation}
\int_0^t |R^{(2)}(v + \tau w)(\sigma)| \, d\sigma + \int_0^t |R^{(2)}(\psi)(\sigma)| \, d\sigma \leq |\Psi|_{E(t)}^2 + |\Psi|_{D(t)}^2,
\end{equation}

for all $t \geq 0$, where the functional $R^{(2)}$ is defined in (4.12) and the energy semi-norms $| \cdot |_{E(t)}$ and $| \cdot |_{D(t)}$ in (4.10) and (4.11), respectively.

Proof. We only estimate the first term on the left in (4.42), the second and third one can be bounded analogously. By applying Hölder’s inequality, we obtain

\begin{equation} |R^{(2)}(v + \tau w)| \leq 2|k||w|_{L^4}||\nabla v||_{L^4}||\nabla (v + \tau w)||_{L^2} + 2|k||v||_{L^\infty}||\nabla w||_{L^2}||\nabla (v + \tau w)||_{L^2} \end{equation}

for all times. For the first term on the right, we can then use the the Ladyzhenskaya interpolation inequality (3.3) in two- and three-dimensions to obtain

\begin{align*}
2|k||w|_{L^4}||\nabla v||_{L^4}||\nabla (v + \tau w)||_{L^2} & \lesssim ||w||_{L^2}^{1-n/4}||\nabla w||_{L^2}^{n/4}||\nabla v||_{L^2}^{1-n/4}||\nabla^2 v||_{L^2}^{n/4}||\nabla (v + \tau w)||_{L^2}.
\end{align*}

From here, by employing Young’s inequality and the bound (4.11) for $||\nabla w||_{L^2}$, we have

\begin{align*}
2|k||w|_{L^4}||\nabla v||_{L^4}||\nabla (v + \tau w)||_{L^2} & \lesssim (||w||_{L^2} + ||\nabla w||_{L^2})(||\nabla (v + \tau w)||_{L^2}^2 + ||\nabla v||_{L^2}^2 + ||\Delta v||_{L^2}^2)
\lesssim (||w||_{L^2} + ||\nabla v||_{L^2} + ||\nabla (v + \tau w)||_{L^2})(||\nabla (v + \tau w)||_{L^2}^2 + ||\nabla v||_{L^2}^2 + ||\Delta v||_{L^2}^2).
\end{align*}

The second term on the right in (4.43) we can estimate as follows:

\begin{align*}
2|k||v||_{L^\infty}||\nabla w||_{L^2}||\nabla (v + \tau w)||_{L^2} & \lesssim (||v||_{L^2} + ||\nabla v||_{L^2} + ||\Delta v||_{L^2})||\nabla w||_{L^2}||\nabla (v + \tau w)||_{L^2}
\lesssim (||v + \tau w||_{L^2} + ||v||_{L^2} + ||\nabla v||_{L^2} + ||\Delta v||_{L^2})||\nabla w||_{L^2}||\nabla (v + \tau w)||_{L^2}.
\end{align*}

Consequently, we can deduce that

\begin{equation}
\int_0^t |R^{(2)}(v + \tau w)(\sigma)| \, d\sigma \lesssim \sup_{\sigma \in [0,t]} (||v + \tau w||_{H^1} + ||w||_{L^2} + ||\nabla v||_{L^2} + ||\Delta v||_{L^2})
\times \int_0^t (||\nabla (v + \tau w)||_{L^2}^2 + ||\nabla v||_{L^2}^2 + ||\Delta v||_{L^2}^2) \, d\sigma,
\end{equation}

from which the first estimate in (4.42) follows. □

Altogether, our previous considerations allow us to conclude that if a smooth solutions of the system (4.11) with initial data (4.2) exists on $[0, T]$, it must satisfy the estimate

\begin{equation}
|\Psi|_{E(t)} + |\Psi|_{D(t)} \lesssim |\Psi|_{E(0)} + |\Psi|_{D(t)} |\Psi|_{D(t)}^2, \quad t \in [0, T].
\end{equation}

We next deal with the issue of existence of such a solution.
5. Local solvability of the JMGT equation with memory

In this section, we rely on the Banach fixed-point theorem to show the local well-posedness of our problem in \( \mathbb{R}^n \), where \( n \in \{2,3\} \).

5.1. Linear local existence theory. We begin by extending a linear existence result from [2] to allow for the possibility of having a source term. We recall how the Hilbert space \( \mathcal{H} \) is defined in (4.5) and additionally introduce the domain of the operator \( \mathcal{A} \) as

\[
D(\mathcal{A}) = H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times (H^2_{-g'}(\mathbb{R}^n) \cap D(\mathcal{T})).
\]

We can now state a well-posedness result for a linearization of our problem.

**Proposition 5.1.** Assume that \( b > \tau \epsilon^2 > \tau \epsilon_g^2 \) and let the final time \( T > 0 \) be given. Furthermore, assume that \( (\psi_0, \psi_1, \psi_2) \in H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \) and that a source term is given by

\[
F = [0,0,f,0]^T \in C^1([0,T]; \mathcal{H}^1) \cap C([0,T]; D(\mathcal{A})),
\]

where \( n \in \{2,3\} \). Then the initial-value problem

\[
\begin{cases}
\partial_t \Psi - \mathcal{A} \Psi = F, \\
\Psi_{t=0} = \Psi_0
\end{cases}
\]

has a unique solution \( \Psi \in C^1([0,T]; \mathcal{H}^1) \cap C([0,T]; D(\mathcal{A})) \). This solution satisfies the following energy estimate:

\[
(5.1) \quad |\Psi|^2_{\mathcal{E}(t)} + |\Psi|^2_{\mathcal{D}(t)} \lesssim |\Psi_0|^2_{\mathcal{E}(0)} + \|f\|^2_{L^2(0,T; H^1(\mathbb{R}^n))}, \quad t \in [0,T],
\]

with the energy semi-norms \( |\cdot|_{\mathcal{E}(t)} \) and \( |\cdot|_{\mathcal{D}(t)} \) defined in (4.10) and (4.11), respectively.

**Proof.** The existence and regularity in the case \( F = 0 \) follow by [2, Corollary 2.6]. The proof is based on the operator \( \mathcal{A} \) being the infinitesimal generator of a \( C_0 \) semigroup of contraction on \( \mathcal{H}^1 \). The general case \( F \neq 0 \) follows by relying on standard semigroup results; see, for example, [33, Theorem 2.4.1 and Corollary 2.4.1].

We can derive the estimate by employing similar energy arguments to the ones of Section 4, where now the functionals \( R^{(1)} \) and \( R^{(2)} \) are given by

\[
R^{(1)}(\varphi) = \tau(f, \varphi)_{L^2}, \quad R^{(2)}(\varphi) = \tau(\nabla f, \nabla \varphi)_{L^2}.
\]

This approach first leads to

\[
|\Psi|^2_{\mathcal{E}(t)} + |\Psi|^2_{\mathcal{D}(t)} \lesssim |\Psi_0|^2_{\mathcal{E}(0)} + |\Psi|_{\mathcal{E}(t)}^2 \lesssim \|f\|_{L^1(0,T; H^1(\mathbb{R}^n))}, \quad t \in [0,T].
\]

An application of Young’s inequality then results in (5.1). \( \quad \Box \)

5.2. Short-time existence for the nonlinear problem. By relying on Proposition 5.1, we can prove that a unique solution to our problem exists for a sufficiently small final time horizon.

**Theorem 5.1.** Let \( b > \tau \epsilon^2 > \tau \epsilon_g^2 \). Assume that \( (\psi_0, \psi_1, \psi_2) \in H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \), where \( n \in \{2,3\} \). Then there exists a final time

\[
T = T(|\Psi_0|^2_{\mathcal{E}(0)}, \|\psi_0\|_{L^2})
\]
such that the problem given by (4.1), (4.2) has a unique solution
\[ \Psi = (\psi, v, w, \eta) \in X = C^1([0, T]; H^1) \cap C([0, T]; D(A)). \]
Moreover, the solution satisfies the following energy inequality:
\[ |\Psi(t)|^2_{\mathcal{E}(t)} + |\Psi(t)|^2_{\mathcal{D}(t)} \lesssim |\Psi(0)|^2_{\mathcal{E}(0)} + |\Psi(t)|^2_{\mathcal{E}(t)} \quad t \in [0, T]. \]

**Proof.** We intend to prove the statement by using the Banach fixed-point theorem, following standard techniques in nonlinear acoustics; see, e.g., [14, 17, 29]. We first need to introduce a suitable mapping.

As already announced, for a given \( \Phi = (\psi^\phi, v^\phi, w^\phi, \eta^\phi)^T \) in an appropriately chosen ball \( \mathcal{B}_L \), we consider the mapping \( \mathcal{T} : \Phi \mapsto \Psi \), where \( \Psi \) solves the following inhomogeneous linear problem:
\[
\begin{aligned}
\partial_t \Psi - A \Psi &= \mathcal{F}(\Phi), \\
\Psi_{t=0} &= \Psi_0,
\end{aligned}
\]
with the functional \( \mathcal{F} \) defined in (4.4). To choose a suitable space for \( \Phi \), we expect based on the linear existence theory and our previous energy arguments that it is a subspace of \( C([0, T]; H^2) \), where \( H^2 \) is defined in (4.5). Additionally, in order to use Proposition 5.1, we need to have \( \mathcal{F}(\Phi) \in C^1([0, T]; H^1) \cap C([0, T]; D(A)) \). This condition is equivalent to
\[ v^\phi w^\phi \in C^1([0, T]; L^2(\mathbb{R}^n)) \cap C([0, T]; H^1(\mathbb{R}^n)). \]

Motivated by this, we introduce the ball
\[ \mathcal{B}_L = \{ \Phi = (\psi^\phi, v^\phi, w^\phi, \eta^\phi)^T \in C([0, T]; H^2) : \|
\Phi\|_{C(H^2)} + \sup_{t \in [0, T]} \|v^\phi(t)\|_{H^1} + \sup_{t \in [0, T]} \|w^\phi(t)\|_{L^2} \leq L, \quad \Phi(0) = \Psi_0 \}. \]

The associated norm is given by
\[
\|
\Phi\|_{\mathcal{B}_L} = \|
\Phi\|_{C(H^2)} + \sup_{t \in [0, T]} \|v^\phi(t)\|_{H^1} + \sup_{t \in [0, T]} \|w^\phi(t)\|_{L^2} = |\Psi|_{\mathcal{E}(T)}^2 + \sup_{t \in [0, T]} \|v^\phi(t)\|_{L^2} + \sup_{t \in [0, T]} \|w^\phi(t)\|_{L^2}.
\]

The radius \( L \geq |\Psi_0|_{\mathcal{E}_0} + |\psi_0|_{L^2}^2 \) of the ball will be conveniently chosen as large enough below. The set \( \mathcal{B}_L \) is a closed subset of a complete metric space \( C([0, T]; H^2) \) with the metric induced by the norm \( \|
\cdot\|_{\mathcal{B}_L} \). This set is non-empty for sufficiently large \( L \) thanks to the linear existence result from Proposition 5.1.

We split the rest of the proof into two parts: proving that \( \mathcal{T} \) is a self-mapping and proving its contractivity.

**The self-mapping property.** We focus first on proving that \( \mathcal{T}(\mathcal{B}_L) \subset \mathcal{B}_L \). Take \( \Phi \in \mathcal{B}_L \). We know that \( \mathcal{F}(\Phi) = [0, f, 0]^T \), where \( f = \frac{2k}{T}v^\phi w^\phi \). We can directly check that
\[
\begin{aligned}
\sup_{t \in [0, T]} (\|f(t)\|_{L^2} + \|f(t)\|_{L^2}) &
\lesssim \sup_{t \in [0, T]} \left( \|v^\phi(t)\|_{L^2} \|w^\phi(t)\|_{L^2} + \|v^\phi(t)\|_{L^4} \|w^\phi(t)\|_{L^4} + \|v^\phi(t)\|_{L^6} \|w^\phi(t)\|_{L^6} \right).
\end{aligned}
\]
Therefore, we immediately have
\[ \|F\|_{C^1(H^1)} + \|F\|_{C(D(A))} = \|f\|_{C^1(L^2)} + \|f\|_{C(H^1)} \lesssim \|\Phi\|_{B_L}^2 < +\infty. \]

By taking into account also the regularity assumptions on the initial data, we conclude that problem (5.3) has a unique solution \( \Psi \in X = C^1([0, T]; H) \cap C([0, T]; D(A)) \) on account of Proposition 5.1. Thus our mapping is well-defined and it maps \( B_L \) into the space \( X \).

To show \( \|\Psi\|_{B_L} \leq L \), we rely on the energy estimate (5.2) from Proposition 5.1. We have
\[ |\Psi(0)|_{L^2}^2 + |\Psi(t)|_{L^2}^2 \lesssim |\Psi(0)|_{L^2}^2 + \|\psi\|_{L^2}^2 \lesssim T^2 |\Psi(0)|_{L^2}^2 + T^2 L^4. \]

By observing that \( \psi(t) = \int_0^t \psi(s) \, ds + \psi_0 \), we find that
\[ \sup_{t \in [0, T]} \|\psi(t)\|_{L^2}^2 \lesssim T^2 |\Psi(0)|_{L^2}^2 + \|\psi_0\|_{L^2}^2 \lesssim T^2 |\Psi(0)|_{L^2}^2 + T^2 L^4 + \|\psi_0\|_{L^2}^2. \]

Moreover, we have the identity
\[ \sup_{t \in [0, T]} \|w_2(t)\|_{L^2}^2 = \sup_{t \in [0, T]} \left| -\frac{1}{\tau} w(t) + \frac{c^2}{\tau} \Delta \psi(t) + \frac{b}{\tau} \Delta v(t) + \frac{1}{\tau} \int_0^\infty g(s) \Delta \eta(s) \, ds \right|^2, \]
which implies
\[ \sup_{t \in [0, T]} \|w_2(t)\|_{L^2}^2 \lesssim |\Psi(0)|_{L^2}^2 + T^2 L^4. \]

We also know that
\[ \sup_{t \in [0, T]} \|\tau_1(t)\|_{H^1}^2 = \sup_{t \in [0, T]} \|\tau_2(t)\|_{H^1}^2 \lesssim |\Psi(0)|_{L^2}^2. \]

 Altogether, there exists a positive constant \( C_* \) such that
\[ \|\Psi\|_{B_L}^2 \leq C_*(T^2 + 1)(|\Psi(0)|_{L^2}^2 + \|\psi_0\|_{L^2}^2 + T^2 L^4). \]

We can then choose the final time \( T \) small enough and the radius \( L \) large enough so that \( \Psi \in B_L \). Indeed, for \( L_0^2 = |\Psi(0)|_{L^2}^2 + \|\psi_0\|_{L^2}^2 \), we have
\[ \|\Psi\|_{B_L}^2 \leq C_* L_0^2 + C_* T^2 (L_0^2 + L^4 + T^2 L^4). \]

We first fix \( L \) large enough such that
\[ C_* L_0^2 \leq \frac{L^2}{2}. \]

Once \( L \) is fixed, we can choose \( T > 0 \) small enough such that
\[ T^2 \leq \min \left( 1, \frac{L^2}{2C_*(L_0^2 + 2L^4)} \right). \]

By doing so, we obtain
\[ \|\Psi\|_{B_L}^2 \leq L^2. \]

Therefore, we conclude that \( \mathcal{T}(\Phi) \in B_L \) for this choice of the radius \( L \) and the final time \( T \).
**Contractivity.** To show contractivity, we take $\Phi, \Phi^* \in B_L$ and

$$T(\Phi) = \Psi \quad \text{and} \quad T(\Phi^*) = \Psi^*.$$ 

We have

$$T(\Phi) - T(\Phi^*) = \Psi - \Psi^*.$$ 

We can see the difference $W = \Psi - \Psi^*$ as a solution of the inhomogeneous problem with zero initial data:

$$\begin{cases}
\partial_t W - AW = F(\Phi) - F(\Phi^*), \\
W|_{t=0} = 0.
\end{cases}$$ 

Let $\Phi^* = (\psi^*, v^*, w^*, \eta^*)^T$. The right-hand side of the above problem is given by

$$F(\Psi) - F(\Phi) = \frac{2k}{\tau}(0, 0, v^* w^* - v^* w^*, 0)^T.$$ 

Then by relying on the energy bound (5.2) from Proposition 5.1, we directly obtain the estimate

$$\begin{aligned}
|W|_{L^2(t)}^2 + |W|_{D(t)}^2 &\lesssim \|v^* w^* - v^* w^*\|_{L^1 H^1}^2 \\
&\lesssim \|(v^* - v^*) w^* + v^* (w^* - w^*)\|_{L^1 H^1}^2.
\end{aligned}$$

From here we have

$$|W|_{L^2(t)}^2 + |W|_{D(t)}^2 \lesssim T^2 \left(|\Phi|_{L^2(t)}^2 + |\Phi^*|_{L^2(t)}^2\right) |\Phi - \Phi^*|_{L^2(t)}^2 \lesssim T^2 L^2 \|\Phi - \Phi^*|_{L^2(t)}^2.$$ 

Denote $W = (\psi - \psi^*, v - v^*, w - w^*, \eta - \eta^*)$. Similarly to before, we can derive the bound

$$\sup_{t \in [0, T]} \|\psi(t) - \psi^*(t)\|_{L^2}^2 \lesssim T^2 |\Psi - \Psi^*|_{L^2(t)}^2 = T^2 |W|_{L^2(t)}^2,$$

as well as the estimate

$$\sup_{t \in [0, T]} \|v(t) - v(t)\|_{H^1}^2 + \sup_{t \in [0, T]} \|w(t) - w(t)\|_{L^2}^2 \lesssim |W|_{L^2(t)}^2.$$ 

Altogether, we have

$$|W|_{B_L}^2 \lesssim T^2 (1 + T^2) \left(|\Phi|_{L^2(t)}^2 + |\Phi^*|_{L^2(t)}^2\right) \|\Phi - \Phi^*|_{B_L}^2$$

$$\lesssim T^2 (1 + T^2) L^2 \|\Phi - \Phi^*|_{B_L}^2.$$ 

Therefore, we can guarantee that the mapping $T$ is strictly contractive by reducing the final time $T$. An application of Banach’s fixed-point theorem then yields a unique solution $\Psi = \Phi \in B_L$. 
Unique solvability in $X$. Since $\mathcal{T}$ maps $\mathcal{B}_L$ into $X$, we conclude that, in fact,

$$\Psi \in X = C^1([0,T]; \mathcal{H}^1) \cap C([0,T]; D(A)).$$

It remains to prove that uniqueness holds also in this space. Take $\Psi \in C^1([0,T]; \mathcal{H}^1) \cap C([0,T]; D(A))$ and $\Psi^* \in C^1([0,T]; \mathcal{H}^1) \cap C([0,T]; D(A))$. Then similarly to (5.8), we can show that

$$|\Psi(t) - \Psi^*(t)|^2 \lesssim \|(v - v^*)w + v^*(w - w^*)\|^2_{L^1 H^1}$$

$$\lesssim T \int_0^t \left( |\Psi(s)|^2_{E^1} + |\Psi^*(s)|^2_{E^1} \right) |\Psi(s) - \Psi^*(s)|^2_{E^2} \, ds$$

$$\lesssim T \int_0^t L^2 |\Psi(s) - \Psi^*(s)|^2_{E^2} \, ds$$

for $t \in [0,T]$, where the semi-norm is given by

$$|\Psi(t)|^2_{E^2} = \varepsilon_1[\Psi](t) + \varepsilon_2[\Psi](t) + \|w(t)\|^2_{L^2},$$

for energies $\varepsilon_1$ and $\varepsilon_2$ defined in (4.8) and (4.9), respectively. Then by Gronwall’s inequality we have $|\Psi(t) - \Psi^*(t)|_{E^1} = 0$. By combining this with the fact that $\psi(t) - \psi^*(t) = \int_0^t (\psi_1(s) - \psi^*_1(s)) \, ds$, we obtain $(\Psi - \Psi^*)(t) = 0$ at all times $t \in [0,T]$. This concludes the proof. \(\square\)

Due to the hard restriction (5.7) on final time, we cannot expect to get the global solvability of the JMGT equation based on this result. The main issue is that we had to use the estimate

$$\|\psi(t)\|_{L^2} \lesssim \sqrt{T}\|\psi_1\|_{L^2 L^2} + \|\psi_0\|_{L^2}$$

to control $\|\psi(t)\|_{L^2}$ because we do not have Poincaré’s inequality at our disposal. A way of resolving this problem is to consider acoustic velocity potentials in homogeneous spaces $\dot{H}^1(\mathbb{R}^n)$. However, this means that we have to restrict our setting to $n > 2$ to work in Hilbert spaces.

6. Global solvability in $\mathbb{R}^3$

To achieve global solvability, we first have to modify the local existence result by working with acoustic potentials in $\dot{H}^1(\mathbb{R}^n)$.

As already mentioned, the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$ is a Hilbert space if and only if $n > 2$; see [1, Proposition 1.34]. For this reason, we restrict ourselves to the physically most relevant setting $n = 3$ to show global well-posedness and later suitable energy decay.

We recall how the Hilbert space $\dot{H}^1$ is defined in (4.6) and also introduce the domain of the operator $A$ as

$$D(A) = \{ \psi : \psi \in \dot{H}^1(\mathbb{R}^3), \Delta \psi \in L^2(\mathbb{R}^3) \} \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times \{ \eta \in \dot{H}^1_{-g'}(\mathbb{R}^3) \cap D(T), \Delta \eta \in L^2_{-g'}(\mathbb{R}^3) \}.$$  \hfill (6.1)

We first restate the linear existence result in $\mathbb{R}^3$ using the homogeneous Sobolev spaces.
Proposition 6.1. Let \( b > \tau c^2 > \tau c_g^2 \) and let the final time \( T > 0 \) be given. Assume that \( (\psi_0, \psi_1, \psi_2) \in \{ \psi_0 \in H^1(\mathbb{R}^3) : \Delta \psi_0 \in L^2(\mathbb{R}^3) \} \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \) and that the source term is given by

\[
F = [0, 0, f, 0]^T \in C^1([0, T]; \mathcal{H}^1) \cap C([0, T]; \mathcal{D}(\mathcal{A})).
\]

Then the linear initial-value problem

\[
\begin{cases}
\partial_t \Psi - \mathcal{A} \Psi = F, \\
\Psi_{t=0} = \Psi_0
\end{cases}
\]

has a unique solution \( \Psi \in C^1([0, T]; \mathcal{H}^1) \cap C([0, T]; \mathcal{D}(\mathcal{A})) \). Moreover, the following estimate holds:

\[
||\Psi||_{\mathcal{L}(t)}^2 + ||\Psi||_{\mathcal{D}(t)}^2 \lesssim ||\Psi_0||_{\mathcal{L}(0)}^2 + ||f||_{L^1(\mathcal{H}^1)}^2, \quad t \in [0, T],
\]

where \( || \cdot ||_{\mathcal{L}(t)} \) and \( || \cdot ||_{\mathcal{D}(t)} \) are defined in (4.10) and (4.11), respectively.

Proof. The proof follows the same steps of the proof of Proposition 5.1, based on the operator \( \mathcal{A} \), with \( \mathcal{D}(\mathcal{A}) \) defined in (6.1), being the infinitesimal generator of a \( C_0 \) semigroup of contraction on \( \mathcal{H}^1 \). \( \square \)

Next we can re-state the nonlinear local existence result in \( \mathbb{R}^3 \).

Theorem 6.1. Let \( b > \tau c^2 > \tau c_g^2 \) and assume that \( (\psi_0, \psi_1, \psi_2) \in \{ \psi_0 \in \dot{H}^1(\mathbb{R}^3), \Delta \psi_0 \in L^2(\mathbb{R}^3) \} \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \).

Then there exists a final time \( T = T(||\Psi_0||_{\mathcal{L}(0)}) \)

such that problem (4.11), (4.12) has a unique solution

\[
\Psi = (\psi, v, w, \eta) \in C^1([0, T]; \mathcal{H}^1) \cap C([0, T]; \mathcal{D}(\mathcal{A})).
\]

The solution of the problem satisfies the energy estimate

\[
||\Psi||_{\mathcal{L}(t)}^2 + ||\Psi||_{\mathcal{D}(t)}^2 \lesssim ||\Psi_0||_{\mathcal{L}(0)}^2 + ||\Psi||_{\mathcal{L}(t)}^2 ||\Psi||_{\mathcal{D}(t)}^2, \quad t \in [0, T].
\]

Proof. The proof follows along the same lines as before, with the difference that now \( || \cdot ||_{\mathcal{L}(T)} \) defines a norm in \( C([0, T]; \mathcal{H}^2) \), where the Hilbert space \( \mathcal{H}^2 \) is defined in (4.7). We can, therefore, define the ball in \( C([0, T]; \mathcal{H}^2) \) as

\[
\mathcal{B}_L = \{ \Phi = (\psi^\phi, v^\phi, w^\phi, \eta^\phi) \in C([0, T]; \mathcal{H}^2) : ||\Phi||_{\mathcal{B}_L} \leq L, \; \Phi(0) = \Psi_0 \}.
\]

but this time supplemented with the norm

\[
||\Phi||_{\mathcal{B}_L} = ||\Phi||_{\mathcal{L}(T)} + ||\Phi||_{\mathcal{D}(T)} + \sup \limits_{t \in [0, T]} ||v^\phi(t)||_{H^1} + \sup \limits_{t \in [0, T]} ||w^\phi(t)||_{L^2}.
\]

When proving that \( \mathcal{T}(\mathcal{B}_L) \subset \mathcal{B}_L \), the bound (5.6) changes to

\[
||\Psi||_{\mathcal{B}_L} \leq C_\star (||\Psi_0||_{\mathcal{L}(0)}^2 + T^2 L^4).
\]

We can thus guarantee that \( ||\Psi||_{\mathcal{B}_L} \leq L \) by choosing the radius large enough and then the final time small enough so that

\[
C_\star ||\Psi_0||_{\mathcal{L}(0)}^2 \leq \frac{1}{2} L^2, \quad T^2 \leq \frac{1}{2C_\star L^2}.
\]
The rest of the proof follows along the same lines as before. We omit the details here.

From (6.3), it is intuitively clear that we can increase \( T \) with smaller data. We prove this next.

**Theorem 6.2.** Let \( b > \tau c^2 > \tau c_g^2 \) and assume that

\[
(\psi_0, \psi_1, \psi_2) \in \{\psi_0 : \psi_0 \in \dot{H}^1(\mathbb{R}^3), \Delta \psi_0 \in L^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\}.
\]

Then there exists small \( \delta > 0 \) such that if

\[
|\Psi_0|^2_{E(0)} \leq \delta,
\]

then problem (4.1), (4.2) has a global solution \( \Psi \in \{\Psi = (\psi,v,w,\eta)^T : \Psi \in C([0,\infty); \dot{H}^2), (v,w) \in C^1((0,\infty); H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))\}.\)

**Proof.** Because of the term \(-b\Delta_t u\) in equation (2.5) and the type of nonlinearity in the model, we can prove the global existence without using the decay of the linearized problem. Let \( T > 0 \) be the maximal time of local existence given by Theorem 6.1. Our goal is to prove by a continuity argument that the norm

\[
|||\Psi|||_{(0,t)} = ||\Psi||_{E(t)} + |\Psi|_{D(t)}
\]

is uniformly bounded for all time if the initial energy \( |\Psi_0|^2_{E(0)} \) is sufficiently small. Note that thanks to estimates (5.4) and (5.5), we know that

\[
||\Psi||_{bL(0,t)} \lesssim ||\Psi||_{E(t)} + |\Psi|_{D(t)} = |||\Psi|||_{(0,t)},
\]

where the norm \( \| \cdot \|_{bL(0,t)} \) is defined as in (6.2), only with the time interval \([0,T]\) replaced by \([0,t]\). Theorem 6.1 provides us with the energy bound

\[
||\Psi||^2_{E(t)} + |\Psi|^2_{D(t)} \lesssim ||\Psi||^2_{E(0)} + ||\Psi||_{E(t)}|\Psi|^2_{D(t)}, \quad t \in [0,T].
\]

This implies that for all \( t \in [0,T], \)

\[
|||\Psi|||_{(0,t)} \leq ||\Psi_0||_{E(0)} + C|||\Psi|||^{3/2}_{(0,t)},
\]

On account of Lemma 3.2, the above inequality implies that there exists a positive constant \( C \), independent of \( t \), such that

\[
|||\Psi|||_{(0,t)} \leq C.
\]

This uniform bound guarantees that our local solution can be continued to \( T = \infty \). □

Accordingly, the JMGT equation in hereditary media with initial data (6.4) admits a unique solution \( \psi \) such that

\[
\psi, \psi_t \in C((0,\infty); H^2(\mathbb{R}^3)) \cap C^1((0,\infty); \dot{H}^1(\mathbb{R}^3)),
\]

\[
\psi_t, \psi_{tt} \in C((0,\infty); H^1(\mathbb{R}^3)) \cap C^1((0,\infty); L^2(\mathbb{R}^3)).
\]
7. Decay rates for the JMGT equation in three-dimensional domains

We next wish to see if and how the solution of (2.5) decays with time. To answer these questions, we first need to derive new decay estimates for \( v = \psi_t \) in the linearized model.

7.1. Decay estimates for the linearized system. The corresponding linear problem is given by the system

\[
\begin{cases}
\psi_t = v, \\
v_t = w, \\
\tau w_t = -w + c_2^2 \Delta \psi + b \Delta v + \int_0^\infty g(s) \Delta \eta(s) \, ds, \\
\eta_t = v - \eta_s,
\end{cases}
\]

supplemented with the same initial data (4.2). To formulate the result, we introduce the vector

\[
U = (v + \tau w, \nabla (\psi + \tau v), \nabla v),
\]

and the corresponding initial vector \( U_0 = (\psi_1 + \tau \psi_2, \nabla (\psi_0 + \tau \psi_1), \nabla \psi_1) \). The decay rates for \( U \) are given by the following two results.

**Lemma 7.1** (see Theorem 3.1 in [2]). Let \( s \geq 0 \) be an integer and assume that \( U_0 \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) \), where \( n \in \mathbb{N} \). Moreover, assume that \( b > \tau c^2 \). Then, for any \( 0 \leq j \leq s \), it holds that

\[
\|\nabla^j U(t)\|_{L^2} \lesssim (1 + t)^{-n/4 - j/2}\|U_0\|_{L^1} + e^{-\lambda t}\|\nabla^j U_0\|_{L^2},
\]

where \( \lambda \) is a positive constant, independent of \( t \).

The estimate above does not directly yield a decay rate for \( \|\nabla^j \psi_t\|_{L^2} = \|\nabla^j v\|_{L^2} \), which we need to prove the decay rate of the nonlinear equation. However, we can obtain it through the bound

\[
\|\nabla^j v\|_{L^2} \lesssim \|\nabla^j (v + \tau w)\|_{L^2} + \|\nabla^j w\|_{L^2}
\]

and (7.1) if we have a decay rate for \( \|w\|_{L^2} \). This rate is the result of the next proposition. Note that for the nonlinear problem we actually only need \( j \in \{0,1\} \).

**Proposition 7.1.** Let the assumptions of Lemma 7.1 hold with \( s \geq 1 \) and let \( w_0 \in H^s(\mathbb{R}^n) \). Then, for any \( n \in \mathbb{N} \) and any \( 0 \leq j \leq s - 1 \), we have

\[
\|\nabla^j w(t)\|_{L^2} \lesssim (\|\nabla^j w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla^{j+1} U_0\|_{L^2})(1 + t)^{-n/2 - j/2},
\]

provided that the thermal relaxation \( \tau > 0 \) is sufficiently small.

**Proof.** For proving the above estimate, we need to employ the decay rates of the Fourier transform of the solution; cf. [2]. Recall how the low-order energy \( \hat{E}_1 \) is defined in (4.13). We then define

\[
\hat{E}_1(\xi, t) = \mathcal{F}(E_1(x, t)),
\]

where \( \mathcal{F} \) stands for the Fourier transform and we denote the variable dual to \( x \) by \( \xi \). Then the following estimate holds:

\[
\hat{E}_1(\xi, t) \lesssim \hat{E}_1(\xi, 0) \exp (-\lambda |\xi|^2 \tau t)
\]
Thus we know that which directly leads to

We can then apply the differential version of Gronwall’s inequality to arrive at

Thus we know that

By plugging in estimate (7.4) for \( \hat{\psi} \), we obtain

We can then apply the differential version of Gronwall’s inequality to arrive at

which directly leads to

To further bound the right side, we can use the fact that

So, assuming that the thermal relaxation is small enough in the sense of \( \tau < \frac{1}{\lambda} \), it holds

Altogether for small \( \tau > 0 \), we obtain

We can use the fact that

where \( \hat{\Psi}(\xi, t) = F(\Psi(x, t)) \); see [2, Lemma 4.3]. By applying Plancherel’s theorem and (7.5) at \( t = 0 \), we find

\[
\| \nabla^j w(t) \|_{L^2} = \int_{\mathbb{R}^n} |\xi|^j |\hat{\Psi}(\xi, t)|^2 d\xi
\]
for any \( j \geq 0 \). The second term on the right-hand side of estimate (7.6) can be split into
\[
\int_{\mathbb{R}^n} |\xi|^{2(j+1)} \exp \left(-\frac{|\xi|^2}{1+|\xi|^2} \right) |\hat{U}(\xi,0)|^2 d\xi
\]
\[= \int_{|\xi| \leq 1} |\xi|^{2(j+1)} \exp \left(-\frac{|\xi|^2}{1+|\xi|^2} \right) |\hat{U}(\xi,0)|^2 d\xi
\]
\[+ \int_{|\xi| > 1} |\xi|^{2(j+1)} \exp \left(-\frac{|\xi|^2}{1+|\xi|^2} \right) |\hat{U}(\xi,0)|^2 d\xi.\]  

We can then use the fact that
\[
|\xi|^2 \leq \begin{cases} \frac{1}{2} |\xi|^2, & \text{if } |\xi| \leq 1, \\ \frac{1}{2}, & \text{if } |\xi| \geq 1. \end{cases}
\]

Concerning the first integral on the right in (7.7), by exploiting the inequality
\[
\int_{0}^{1} r^{n-1} e^{-r^2 t} dr \leq C(n)(1+t)^{-n/2},
\]
given in Lemma 3.1 together with (7.8), we find that
\[
\int_{|\xi| \leq 1} |\xi|^{2j} \exp \left(-\frac{|\xi|^2}{1+|\xi|^2} \right) |\hat{U}(\xi,0)|^2 d\xi \leq \|\hat{U}_0\|_{L^\infty}^2 \int_{|\xi| \leq 1} |\xi|^{2(j+1)} \exp \left(-\frac{1}{2} \frac{|\xi|^2}{1+|\xi|^2} \right) d\xi
\]
\[\lesssim (1+t)^{-\frac{n}{2} - j}\|U_0\|_{L^1}^2.\]

On the other hand, in the high-frequency region where \(|\xi| \geq 1\), we have
\[
\int_{|\xi| \geq 1} |\xi|^{2j} \exp \left(-\frac{|\xi|^2}{1+|\xi|^2} \right) |\hat{U}(\xi,0)|^2 d\xi \leq e^{-\frac{1}{2} t} \int_{|\xi| \geq 1} |\xi|^{2(j+1)} |\hat{U}(\xi,0)|^2 d\xi
\]
\[\leq e^{-\frac{1}{2} t} \|\nabla^{j+1} U_0\|_{L^2}^2.\]

By plugging the above two estimates into inequality (7.6), we finally obtain
\[
\|\nabla^j w(t)\|_{L^2} \lesssim e^{-\frac{1}{2} t} \|\nabla^j w_0\|_{L^2} + C(1+t)^{-\frac{n}{4} - \frac{j}{2} - \frac{j}{4}} \|U_0\|_{L^1(\mathbb{R}^n)} + e^{-\frac{1}{2} t} \|\nabla^{j+1} U_0\|_{L^2}.
\]

This implies estimate (7.3) holds for large \( t \), thus completing the proof of Proposition 7.1.

Now we are ready to prove the decay rate for \( v = \psi_t \).

**Lemma 7.2.** Let the assumptions of Proposition 7.1 hold. Then, for any \( n \in \mathbb{N} \) and any \( 0 \leq j \leq s - 1 \), we have
\[
\|\nabla^j v(t)\|_{L^2} \lesssim (\|\nabla^j w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla^j U_0\|_{H^1})(1+t)^{-\frac{n}{4} - \frac{j}{4}}.
\]

Moreover, assuming \( U_0 \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3) \) and \( w_0 \in H^2(\mathbb{R}^3) \), it holds
\[
\|v(t)\|_{L^\infty} \lesssim (\|w_0\|_{H^2} + \|U_0\|_{L^1} + \|U_0\|_{H^3})(1+t)^{-\frac{n}{4}}.
\]
Proof. The estimate \((7.9)\) is a result of combining the bounds \((7.1)\), \((7.3)\), and estimate \((7.2)\). To prove the second estimate, we use the Gagliardo–Nirenberg interpolation inequality in the form of

\[
\|v\|_{L^\infty} \leq C \left\| \nabla^2 v \right\|_{L^2}^{\frac{n}{2}} \left\| v \right\|_{L^2}^{1 - \frac{n}{2}}.
\]

(7.11)

Taking into account estimate \((7.9)\) then immediately yields \((7.10)\). \(\square\)

To prove the decay rate for the nonlinear problem, we have to use the bound \((7.10)\) in the \(L^\infty\) norm. This means that we need the initial data to be more regular than what we had for the global solvability.

7.2. Decay estimates for the nonlinear problem. We are now ready to prove decay estimates for the solution to the nonlinear problem.

**Theorem 7.1.** Let \(b > \tau c^2 > \tau c_g^2\) and \(n = 3\). Assume that the initial data satisfy \((6.4)\). Moreover, assume that \(U_0 \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)\) and \(w_0 \in H^2(\mathbb{R}^3)\) and suppose that

\[
\Lambda_0 = \|u_0\|_{L^2} + \|U_0\|_{L^1} + \|U_0\|_{H^1}
\]

is small enough. Then, the global solution of \((3.2)\) satisfies the following decay rates:

\[
\left\| \nabla^j U(t) \right\|_{L^2} \lesssim \Lambda_0 (1 + t)^{-\frac{n}{2} - \frac{j}{2}}, \quad j = 0, 1,
\]

\[
\left\| v(t) \right\|_{L^2} \lesssim \Lambda_0 (1 + t)^{-\frac{n}{4}}, \quad \left\| w(t) \right\|_{L^2} \lesssim \Lambda_0 (1 + t)^{-\frac{n}{4} - \frac{1}{2}}.
\]

**Proof.** Let \(\Psi = (\psi, v, w, \eta)^T\) be the global solution of our system according to Theorem \(6.2\). We introduce here the norm given by the lower-order energy \((4.8)\),

\[
\|\Psi(t)\|_{\mathcal{E}_1} = \left\| \nabla(\psi + \tau \psi_t)(t) \right\|_{L^2}^2 + \left\| (v + \tau \psi_t)(t) \right\|_{L^2}^2 + \left\| \nabla \psi_t(t) \right\|_{L^2}^2 + \left\| \nabla \eta \right\|_{L^{2, -g'}}^2.
\]

(7.12)

We have

\[
\|U(t)\|_{L^2} \leq \|\Psi(t)\|_{\mathcal{E}_1} \quad \text{and} \quad \|\nabla U(t)\|_{L^2} \leq \|\nabla \Psi(t)\|_{\mathcal{E}_1}.
\]

Motivated by the decay estimates for the linearized problem obtained in Lemma \(7.1\) and Proposition \(7.1\), we define

\[
\mathcal{M}(t) = \sup_{0 \leq \sigma \leq t} \left[ (1 + \sigma)^{n/4} \left\| U(\sigma) \right\|_{L^2} + (1 + \sigma)^{n/4 + 1/2} \left\| \nabla U(\sigma) \right\|_{L^2} \right.
\]

\[
+ (1 + \sigma)^{n/4} \left\| v(\sigma) \right\|_{L^2} + (1 + \sigma)^{3/4 + 1/2} \left\| w(\sigma) \right\|_{L^2} \bigg].
\]

(7.13)

Keeping in mind the \(L^\infty\) bound \((7.10)\) for \(v\), we also introduce the quantity

\[
M_0(t) = \sup_{0 \leq \sigma \leq t} (1 + \sigma)^{3n/8} \left\| v(\sigma) \right\|_{L^\infty}.
\]

(7.14)

By using the inequality \((7.11)\), we deduce that

\[
M_0(t) \lesssim \mathcal{M}(t).
\]

(7.15)

The reason for taking the exponent \(3n/8\) in \((7.14)\) instead of \(n/2\) is to make sure that the inequality above holds. The resulting slow decay of \(\|v\|_{L^\infty}\) is a consequence of the slow decay of \(\|\nabla^2 v\|_{L^2}\) given by \((1 + t)^{-n/4-1/2}\). Despite this, we can still prove that the vector \(U(t)\) decays as fast as in the linear equation thanks to the fast decay of \(\|w\|_{L^2}\) given in \((7.3)\).
Our next aim is to show that $\mathcal{M}(t)$ is bounded uniformly in $t$ if $\Lambda_0$, defined in (7.12), is small enough. We can formally write the solution to our problem as

$$\Psi(t) = e^{tA}_{\Psi} + \int_0^t e^{(t-r)A} F(\Psi)(r) \, dr.$$  

From here we directly estimate

$$\|\nabla^j U(t)\|_{L^2} = \|\nabla^j (t)\|_{\delta_1} \leq \|\nabla^j e^{tA}_{\Psi} \|_{\delta_1} + \int_0^t \left\|\nabla^j e^{(t-r)A} F(\Psi)(r)\right\|_{\delta_1} \, dr$$

$$= \|\nabla^j e^{tA}_{\Psi} \|_{\delta_1} + \int_0^{t/2} \left\|\nabla^j e^{(t-r)A} F(\Psi)(r)\right\|_{\delta_1} \, dr \quad + \int_{t/2}^t \left\|\nabla^j e^{(t-r)A} F(\Psi)(r)\right\|_{\delta_1} \, dr$$

for $j \in \{0, 1\}$. By applying the linear decay rate (7.1) from Lemma 7.1, we have

$$\|\nabla^j e^{tA}_{\Psi} \|_{\delta_1} \lesssim (1 + t)^{-n/4-j/2} \left(\|\Psi_0\|_{L^1} + \|\nabla^j \Psi_0\|_{L^2}\right)$$

$$\lesssim (1 + t)^{-n/4-j/2} \left(\|U_0\|_{L^1} + \|\nabla^j U_0\|_{L^2}\right),$$

with $j \in \{0, 1\}$. In the second inequality above we have used the fact that $\eta(x, t = 0, s) = \psi_0(x)$.

We need to estimate the remaining two integrals on the right side of (7.16). To estimate the first one, we have by using the linear estimate (7.1) and Duhamel’s principle,

$$\int_0^{t/2} \left\|\nabla^j e^{(t-r)A} F(\Psi)(r)\right\|_{\delta_1} \, dr \lesssim \int_0^{t/2} (1 + t - r)^{-n/4-j/2} \|F(\Psi)(r)\|_{L^1} \, dr$$

$$+ \int_0^{t/2} e^{-(t-r)} \|\nabla^j F(\Psi)(r)\|_{L^2} \, dr,$$

where $F$ is defined as in (4.14). We have by employing Hölder’s inequality,

$$\|F(\Psi)(t)\|_{L^1} \lesssim \|v w\|_{L^1} \lesssim \|v\|_{L^2} \|w\|_{L^2} \lesssim \|v\|_{L^2}^2 + \|w\|_{L^2}^2.$$  

By using the above estimate and recalling the definition of $\mathcal{M}$ in (4.13), we have

$$\int_0^{t/2} (1 + t - r)^{-n/4-j/2} \|F(\Psi)(r)\|_{L^1} \, dr$$

$$\lesssim \mathcal{M}^2(t) \int_0^{t/2} (1 + t - r)^{-n/4-j/2} (1 + r)^{-n/2} \, dr$$

$$\lesssim \mathcal{M}^2(t) \int_0^{t/2} (1 + t)^{-n/4-j/2} (1 + r)^{-n/2} \, dr.$$

We can further bound the integral on the right, leading to

$$\int_0^{t/2} (1 + t - r)^{-n/4-j/2} \|F(\Psi)(r)\|_{L^1} \, dr$$

$$\lesssim \mathcal{M}^2(t)(1 + t)^{-n/4-j/2} \int_0^{t/2} (1 + r)^{-n/2} \, dr \lesssim \mathcal{M}^2(t)(1 + t)^{-n/4-j/2},$$
because \( n > 2 \). To estimate \( \int_0^{t/2} e^{-(t-r)} \| \nabla^j \mathcal{F}(\Psi)(r) \|_{L^2} \, dr \), we distinguish the cases \( j = 0 \) and \( j = 1 \). First for \( j = 0 \), we have

\[
\| \mathcal{F}(\Psi)(t) \|_{L^2} \lesssim \| v \|_{L^\infty} \| w \|_{L^2} \lesssim M_0(t)(1 + t)^{-3n/8} \mathcal{M}(t)(1 + t)^{-n/4 - 1/2} \\
\lesssim M_0(t) \mathcal{M}(t)(1 + t)^{-5n/8 - 1/2} \lesssim M_0(t) \mathcal{M}(t)(1 + t)^{-3n/4},
\]

because \( n \leq 4 \). For \( j = 1 \), we have by using (3.3) and (3.4)

\[
\| \nabla \mathcal{F}(\Psi)(t) \|_{L^2} \lesssim \| \nabla (vw) \|_{L^2} \\
\lesssim \| \nabla v \|_{L^4} \| w \|_{L^4} + \| v \|_{L^\infty} \| \nabla w \|_{L^2} \\
\lesssim \| \nabla v \|_{L^2}^{1-4/5} \| \nabla^2 v \|_{L^2}^{4/5} \| w \|_{L^2}^{1-4/5} \| \nabla w \|_{L^2}^{4/5} + \| v \|_{L^\infty} \| \nabla w \|_{L^2} \\
\lesssim \| \nabla v \|_{L^2}^{1-4/5} \| \nabla U \|_{L^2}^{4/5} \| w \|_{L^2}^{1-4/5} \| \nabla w \|_{L^2}^{4/5} + \| v \|_{L^\infty} \| \nabla w \|_{L^2}.
\]

Keeping in mind how \( M_0 \) is defined in (7.14), we have from above

\[
\| \nabla \mathcal{F}(\Psi)(t) \|_{L^2} \lesssim (1 + t)^{-5n/8 - 1/2} \mathcal{M}^2(t) + M_0(t) \mathcal{M}(t)(1 + t)^{-5n/8 - 1/2} \\
\lesssim (1 + t)^{-5n/8 - 1/2} (\mathcal{M}^2(t) + M_0(t) \mathcal{M}(t)).
\]

Consequently, by combining the above bound with (7.20), we deduce that

\[
\int_0^{t/2} e^{-(t-r)} \| \nabla^j \mathcal{F}(\Psi)(r) \|_{L^2} \, dr \lesssim (1 + t)^{-5n/8 - j/2} (\mathcal{M}^2(t) + M_0(t) \mathcal{M}(t))
\]

for \( j \in \{0, 1\} \). The integral \( \int_{t/2}^t \| \nabla^j e^{(t-r)A} \mathcal{F}(\Psi)(r) \|_{E_1} \, dr \) is estimated by applying the linear decay rate given in (7.11) with \( j = 1 \), but using \( \mathcal{F}(\Psi)(t) \) instead of \( U_0 \). By doing so, we obtain

\[
\int_{t/2}^t \| \nabla^j e^{(t-r)A} \mathcal{F}(\Psi)(r) \|_{E_1} \, dr = \int_{t/2}^t \| \nabla e^{(t-r)A} \mathcal{F}(\Psi)(r) \|_{L^2} \, dr \\
\lesssim \int_{t/2}^t (1 + t - r)^{-\frac{n}{2} - \frac{1}{2}} \| \mathcal{F}(\Psi)(r) \|_{L^1} \, dr \\
+ \int_{t/2}^t e^{-\lambda(t-r)/2} \| \nabla \mathcal{F}(\Psi)(r) \|_{L^2} \, dr.
\]

On the other hand, we have by applying (7.18) and recalling the definition of \( \mathcal{M} \) in (7.13),

\[
\| \mathcal{F}(\Psi)(t) \|_{L^1(\mathbb{R}^n)} \lesssim \mathcal{M}^2(t)(1 + t)^{-n/2}.
\]

Thus, we can derive the bound

\[
\int_{t/2}^t (1 + t - r)^{-\frac{n}{2} - \frac{1}{2}} \| \mathcal{F}(\Psi)(r) \|_{L^1} \, dr \lesssim \mathcal{M}^2(t) \int_{t/2}^t (1 + t - r)^{-\frac{n}{2} - \frac{1}{2}} (1 + r)^{-\frac{n}{2}} \, dr \\
\lesssim \mathcal{M}^2(t)(1 + t/2)^{-\frac{n}{2}} \int_{t/2}^t (1 + t - r)^{-\frac{n}{2} - \frac{1}{2}} \, dr.
\]
Because \( n > 2 \), then we know that
\[
\int_{t/2}^{t} (1 + t - r)^{-\frac{n}{2} - \frac{1}{2}} \| F(\Psi)(r) \|_{L^1} \, dr
\]
(7.23)
\[
\lesssim \mathcal{M}^2(t) (1 + t/2)^{-\frac{n}{2} - \frac{1}{2}} \int_{0}^{t/2} (1 + r)^{-\frac{n}{2} - \frac{1}{2}} \, dr \lesssim \mathcal{M}^2(t)(1 + t)^{-\frac{n}{4} - \frac{1}{2}}.
\]
Furthermore, we have by using the bound (7.21) that
\[
\int_{t/2}^{t} e^{-\lambda(t-r)/2} \| \nabla \mathcal{F}(\Psi)(r) \|_{L^2} \, dr \lesssim (1 + t)^{-5n/8-1/2}(\mathcal{M}^2(t) + M_0(t)\mathcal{M}(t)).
\]
Therefore, by combining estimates (7.17), (7.19), (7.22), (7.23), and the above inequality, we infer
\[
\| \nabla^j U(t) \|_{L^2} \lesssim (1 + t)^{-n/4-j/2} (\| U_0 \|_{L^1} + \| \nabla U_0 \|_{L^2})
\]
(7.24)
\[
+ \mathcal{M}^2(t)(1 + t)^{-n/4-j/2} + (1 + t)^{-n/4-1/2-j/2} M_0(t)\mathcal{M}(t)
\]
for \( n = 3 \) and \( j \in \{0,1\} \). At this point we also need an estimate of \( \| w \|_{L^2} \). Recalling the energy bound we obtained in (4.38), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \tau |w|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |w|^2 \, dx \lesssim \| \nabla U \|_{L^2}^2 + |R^{(1)}(w)|.
\]
By applying Gronwall’s inequality, we deduce that
\[
\| w(t) \|_{L^2}^2 \lesssim \| w_0 \|_{L^2}^2 \exp \left( -\frac{1}{2} t \right) + \int_{0}^{t} \| \nabla U(s) \|_{L^2}^2 \exp \left( -\frac{1}{2} (t - s) \right) \, ds
\]
(7.25)
\[
+ \int_{0}^{t} |R^{(1)}(w)(s))| \exp \left( -\frac{1}{2} (t - s) \right) \, ds.
\]
We need to further estimate the two integrals on the right. By making use of the bound (7.24) with \( j = 1 \), we have
\[
\int_{0}^{t} \| \nabla U(s) \|_{L^2}^2 \exp \left( -\frac{1}{2} (t - s) \right) \, ds
\]
\[
\lesssim (1 + t)^{-n/4-1/2} (\| U_0 \|_{L^1} + \| \nabla U_0 \|_{L^2})
\]
\[
+ \mathcal{M}^2(t)(1 + t)^{-n/4-1/2} + (1 + t)^{-n/4-1} M_0(t)\mathcal{M}(t).
\]
Concerning the second integral on the right in (7.25), we find
\[
|R^{(1)}(w)(t)| \lesssim \| v w \|_{L^1} \lesssim \| v \|_{L^\infty} \| w \|_{L^2} \lesssim M_0(t)\mathcal{M}^2(t) (1 + t)^{-3n/8}(1 + t)^{-n/2-1}
\]
\[
\lesssim M_0(t)\mathcal{M}^2(t)(1 + t)^{-\left(\frac{2n}{8}+1\right)}.
\]
This inequality immediately yields
\[
\int_{0}^{t} |R^{(1)}(w)(s))| \exp \left( -\frac{1}{2} (t - s) \right) \, ds \lesssim M_0(t)\mathcal{M}^2(t)(1 + t)^{-\left(\frac{2n}{8}+1\right)}.
\]
Consequently, we deduce from above that
\[
\|w(t)\|_{L^2} \lesssim \|w_0\|_{L^2} \exp\left(-\frac{1}{2\tau}t\right) + (1 + t)^{-n/4-1/2} \left(\|U_0\|_{L^1} + \|\nabla U_0\|_{L^2}\right)
+ \mathcal{M}^2(t)(1 + t)^{-n/4-1/2}
+ \sqrt{M_0(t)\mathcal{M}(t)}(1 + t)^{-n/4-1/2},
\]
which further implies that
\[
\|w(t)\|_{L^2} \lesssim \left(\|w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla U_0\|_{L^2}\right)(1 + t)^{-n/4-1/2}
+ \left(\mathcal{M}^2(t) + \sqrt{M_0(t)\mathcal{M}(t)} + M_0(t)\mathcal{M}(t)\right)(1 + t)^{-n/4-1/2}.
\]
By also using the fact that \(\|v\|_{L^2} \lesssim \|w\|_{L^2} + \|U\|_{L^2}\), together with estimates (7.24) and (7.26), we obtain
\[
\|v(t)\|_{L^2} \lesssim \left(\|w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla U_0\|_{H^1}\right)(1 + t)^{-n/4}
+ \left(\mathcal{M}^2(t) + \sqrt{M_0(t)\mathcal{M}(t)} + M_0(t)\mathcal{M}(t)\right)(1 + t)^{-n/4}.
\]
By collecting (7.24), (7.26) and (7.27) and recalling the definition of \(\mathcal{M}(t)\) in (7.13), we find
\[
\mathcal{M}(t) \lesssim \|w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla U_0\|_{H^1} + \mathcal{M}^2(t) + \sqrt{M_0(t)\mathcal{M}(t)} + M_0(t)\mathcal{M}(t).
\]
By relying on (7.15), we deduce that
\[
\mathcal{M}(t) \lesssim \|w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla U_0\|_{L^2} + \mathcal{M}^{3/2}(t) + \mathcal{M}^2(t).
\]
This last estimate together with Lemma 3.2 implies that
\[
\mathcal{M}(t) \leq C,
\]
provided that \(\|w_0\|_{L^2} + \|U_0\|_{L^1} + \|\nabla U_0\|_{L^2}\) is small enough. This step completes the proof of Theorem 7.1 \(\square\)

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