The Darboux Transformation and N-Soliton Solutions of Coupled Cubic-Quintic Nonlinear Schrödinger Equation on a Time-Space Scale

Huanhe Dong, Chunming Wei, Yong Zhang, Mingshuo Liu and Yong Fang *

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China; donghuanhe@sdust.edu.cn (H.D.); 202082150035@sdust.edu.cn (C.W.); 201881501056@sdust.edu.cn (Y.Z.); liumingshuo@sdust.edu.cn (M.L.)
* Correspondence: fangyong@sdust.edu.cn

Abstract: The coupled cubic-quintic nonlinear Schrödinger (CQNLS) equation is a universal mathematical model describing many physical situations, such as nonlinear optics and Bose–Einstein condensate. In this paper, in order to simplify the process of similar analysis with different forms of the coupled CQNLS equation, this dynamic system is extended to a time-space scale based on the Lax pair and zero curvature equation. Furthermore, Darboux transformation of the coupled CQNLS dynamic system on a time-space scale is constructed, and the N-soliton solution is obtained. These results effectively combine the theory of differential equations with difference equations and become a bridge connecting continuous and discrete analysis.

Keywords: coupled cubic-quintic nonlinear Schrödinger equation; time-space scales; Darboux transformation; N-soliton solution

1. Introduction

In recent years, many useful methods have been applied to obtain solutions of integrable systems, such as Darboux transformation [1–5], inverse scattering transformation [6], bilinear transformation [7], and Backlund transformation [8]. Darboux transformation, originating from the work of Darboux in 1882 on the Sturm–Liouville equation, is a powerful method for constructing solutions for integrable systems. The basic idea of Darboux transformation is to construct the solution of integrable equations by solutions, which is called the eigenfunctions, of the linear partial differential equation associated with its Lax pair. Lots of literature can be found in this regard. In Ref. [9], the authors develop Darboux’s idea to solve linear and nonlinear partial differential equations arising in soliton theory. In Ref. [1–5], various approaches have been proposed to construct a Darboux transformation for nonlinear partial differential equations, such as operator factorization, gauge transformation, and loop group transformation.

The time scale was introduced in 1988 by Stefan Hilger, and its main purpose is to unify continuous and discrete analysis [10], which is further researched and developed by Bohner and Peterson [11,12]. This theory is a powerful tool to unify various types of time-variable forms and simplifies the process of similar analysis on time scales with different forms, and can better solve complex models that contain multiple situations, such as continuous and discrete. Therefore, it is used widely in various fields due to unification and extension. For example, in biology, functional connectivity patterns can be studied on multiple time scales by simulating the neural dynamics of large scale inter-regional connection networks in the macaque cortex [13–15]. In physics, the fixed point index theory is used to establish a double fixed point theorem for a completely continuous operator in Banach space, and then an application of the two-point conjugate boundary problem is discussed [16–18]. In economics, the time scales model provides information for a problem for not evenly spaced intervals, for which the standard continuous and discrete models do not [19,20].
The coupled cubic-quintic nonlinear Schrödinger (CQNLS) equation is introduced as the following form
\[
\begin{align*}
q_t &= i\rho_1 q^3 r^2 - 2\rho_1 q q_x r - 2\rho_1 q^2 r_x - 2i q^2 r + iq_{xx}, \\
r_t &= -i\rho_1 q^5 r^2 - 2\rho_1 q r r_x - 2\rho_1 q x r^2 + 2qr^2 - ir_{xx},
\end{align*}
\]
where \(\rho_1\) is a real parameter [21,22]. This equation has been one of the universal mathematical models in the field of nonlinear science, which is applied widely into optics [23,24], Bose–Einstein condensation [25,26], and other fields [27]. Many scholars obtain soliton solutions of this equation through different methods, such as Darboux transformation, backscattering, and Backlund transformation [28,29]. In Ref. [28], \(N\)-order rogue wave solutions of coupled CQNLS equation are obtained by generalized Darboux transformation. The dynamics of its general first and second-order rogue waves are further discussed and illustrated. In Ref. [29], the soliton solution of the coupled CQNLS equation is obtained through bilinear Backlund transformation; this result has important applications for the ultrashort optical pulse propagation in non-Kerr media. In this paper, we extend the coupled CQNLS equation on a time-space scale based on the Lax pair and zero curvature [30–33]. This result simplifies the process of similar analysis with different forms and builds a bridge between differential equations and difference equations. Furthermore, this equation can be used in complex models with both discrete and continuous applications.

The structure of this article is as follows. Some preliminaries about the time-space scale are devoted in Section 2. The coupled CQNLS equation on a time-space scale is derived in Section 3. Darboux transformation of this equation on a time-space scale is constructed, and its \(N\)-soliton solution is obtained in Section 4. Lastly, the conclusion is given in Section 5.

2. Preliminaries

In this section, some definitions of the time-space scale are introduced first [34–37].

**Definition 1.** A time-space scale is any non-empty closed subset of the real number \(\mathbb{R}\), and it has topological and sequential relations induced by \(\mathbb{R}\).

**Definition 2.** Assuming \(T\) and \(X\) are time and space scales, for \((t, x) \in T \times X\), the forward jump operators are respectively defined as
\[
\sigma : T \to T, \quad \rho : X \to X,
\]
\[
\sigma(t) = \inf\{s \in T : s > t\}, \quad \rho(x) = \inf\{y \in X : y > x\}.
\]
for \(x \in X\), the backward jump operator \(\beta(x) : X \to X\) is defined as
\[
\beta(x) = \rho^{-1}(x) := \sup\{y \in X : y < x\}.
\]

**Definition 3.** The \(\nabla\)–derivative related to time and space variables is defined as
\[
\nabla_t f(t,x) = \lim_{p \to \mu(t)} \frac{f(t,x) - f^p(t,x)}{p},
\]
\[
\nabla_x f(t,x) = \lim_{q \to \nu(x)} \frac{f(t,x) - f^q(t,x)}{q},
\]
where the grayscale functions are defined as
\[
\mu : T \to [0, +\infty), \quad \nu : X \to [0, \infty),
\]
\[
\mu(t) = t - \sigma(t), \quad \nu(x) = x - \rho(x).
\]
Note that,
\[ f^\sigma(t, x) := f(\sigma(t), x) = f(t, x) - \mu(t) \nabla f(t, x), \]
\[ f^\rho(t, x) := f(t, \rho(x)) = f(t, x) - \nu(x) \nabla f(t, x). \]

**Definition 4.** Exponential function on a time scale is defined in the following form
\[ e_\alpha(x, x_0) := \exp \left( \int_{x_0}^{x} \xi_{\alpha(s)}(\alpha(s)) \Delta s \right), \]
where \( \alpha : \mathbb{X} \rightarrow \mathbb{C} \) is a given function and
\[ \xi_{0}(z) := \frac{1}{h} \log(1 + zh), h > 0, \xi_{0}(z) := z. \]

This definition applies to the \( \nu \)-regressive functions \( \alpha = \alpha(x) \), i.e., those satisfying
\[ 1 + \nu(x)\alpha(x) \neq 0. \]

Such functions are usually called regressive.
In the constant discrete case \( (\mathbb{X} = \varepsilon \mathbb{Z}, \varepsilon = \text{const}) \), with \( \alpha = \text{const} \), we have
\[ e_\alpha(x) = (1 + \alpha \varepsilon)^{\frac{x}{\varepsilon}}, \]
and in the case \( \mathbb{X} = \mathbb{R} \), we have
\[ e_\alpha(x) = \exp \int_{0}^{x} \alpha(\tau) d\tau. \]

**Property 1.** The properties associated with \( \nabla \)-derivatives are as follows
\[ \begin{cases} 
\nabla_1 f(t, x)g(t, x) = f^\sigma(t, x) \nabla g(t, x) + \nabla_1 f(t, x)g(t, x), \\
\nabla_\rho f(t, x)g(t, x) = f^\rho(t, x) \nabla x g(t, x) + \nabla x f(t, x)g(t, x).
\end{cases} \]

3. The Coupled CQNLS Equation on a Time-Space Scale

In order to obtain the coupled CQNLS equation on a time-space scale, the \( \nabla \)-dynamical system is considered as follows
\[ \begin{cases} 
\nabla x \varphi(t, x) = U(t, x) \varphi(t, x), \\
\nabla_\rho \varphi(t, x) = V(t, x) \varphi(t, x),
\end{cases} \]
with
\[ U(t, x) = \begin{pmatrix} -i\lambda - \frac{i}{2} \rho_1 qr & i \lambda + \frac{q}{2} \rho_1 qr \\ r & -i \end{pmatrix}, \]
\[ V(t, x) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \]
where \( A, B, C \) are functions which contain spectral parameters \( \lambda \) and potential functions \( q, r. \)

According to the compatibility condition \( \nabla_{x\rho} \varphi = \nabla_{x\lambda} \varphi \), the zero curvature equation on a time-space scale is derived to
\[ U^\rho V + \nabla_\rho U - V^\rho U - \nabla x V = 0. \]

By putting \( U(t, x), V(t, x) \) into Equation (2), these equations are obtained
\[ \begin{cases} 
i\lambda(A^\rho - A) + \frac{1}{2} \rho_1(qr)^\rho A + q^\rho C - \frac{1}{2} \rho_1 \nabla_1(qr) + \frac{1}{2} \rho_1 qr A^\rho - r B^\rho - \nabla_\rho A = 0, \\
-i\lambda(B^\rho + B) - \frac{1}{2} \rho_1(qr)^\rho B - q^\rho C + \nabla_1(qr) - q A^\rho - \frac{1}{2} \rho_1 qr B^\rho - \nabla x B = 0, \\
i\lambda(C + C^\rho) + r^\rho A + \frac{1}{2} \rho_1(qr)^\rho C + \nabla_1(r) + \frac{1}{2} \rho_1 qr C^\rho + r A^\rho - \nabla x C = 0, \\
-i\lambda(A - A^\rho) + r^\rho B - \frac{1}{2} \rho_1(qr)^\rho A + \frac{1}{2} \rho_1 \nabla_1(qr) - q C^\rho + \frac{1}{2} \rho_1 qr A^\rho + \nabla x A = 0.
\end{cases} \]
\( A, B, C \) are taken as quadratic polynomials of \( \lambda \)

\[
A = \sum_{i=0}^{2} a_i \lambda^i, \quad B = \sum_{i=0}^{2} b_i \lambda^i, \quad C = \sum_{i=0}^{2} c_i \lambda^i. \tag{4}
\]

Take \( a_2 = -2i \). By putting (4) into (3) and comparing coefficients of \( \lambda \), the relations are obtained as follows

\[
\begin{align*}
b_2 &= 0, \quad c_2 = 0, \\
b_1 &= -b_0^p + iq^r a_2 + iqa_2^p, \\
c_1 &= -c_0^p + ir^a a_2 + ira_2^p, \\
a_1 &= \frac{1}{2} \nabla_x^{-1}(-r^p b_1 + q c_1 + q^r c_1 - r b_1^p), \\
b_0 &= -b_0^p + iq^r a_1 + iqa_1^p - \frac{1}{2} \rho_1qr b_1^p - \frac{1}{2} \rho_1(qr)^p b_1 + i \nabla_x b_1, \\
c_0 &= -c_0^p + ir^a a_1 + ira_1^p - \frac{1}{2} \rho_1qr c_1^p - \frac{1}{2} \rho_1(qr)^p c_1 + i \nabla_x c_1,
\end{align*}
\]  

(5)

and

\[
\begin{align*}
\nabla_1 q &= q^r a_0 + qa_0^r + \frac{1}{2} \rho_1qr b_0^p + \frac{1}{2} \rho_1(qr)^p b_0 + \nabla_x b_0, \\
\nabla_1 r &= -r^p a_0 - ra_0^p - \frac{1}{2} \rho_1qr c_0^p - \frac{1}{2} \rho_1(qr)^p c_0 + \nabla_x c_0.
\end{align*}
\]  

(6)

According to the derivative rule on a time-space scale, \( a_i, b_i, c_i (i = 0, 1) \) are obtained as follows

\[
\begin{align*}
a_1 &= 0, \\
b_1 &= 2(2 - v(x) \nabla_x)^{-1}(q + qr), \\
c_1 &= 2(2 - v(x) \nabla_x)^{-1}(r + r^r), \\
b_0 &= 2M_1(q + qr), \\
c_0 &= 2M_1(r + r^r), \\
a_0 &= 2M_2M_1|M_4 + M_3|,
\end{align*}
\]  

(7)

with

\[
\begin{align*}
M_1 &= \left[(2 - v(x) \nabla_x)^{-2}\left(i \nabla_x - \frac{1}{2} \rho_1 qr(1 - v(x) \nabla_x) - \frac{1}{2} \rho_1(qr)^p \right) \right], \\
M_2 &= \left[2 \nabla_x + iq^r r^p(1 - r^r) + i |p_1 r(1 - v(x) \nabla_x)(q - qr)^p \right]^{-1}, \\
M_3 &= \frac{1}{2} \rho_1^2(qr)^p qr - \frac{1}{2} \rho_1^2(qr)^p qr^p - \rho_1 \nabla_x(2qr + qr + q^r) + qr r^r + q^r r - qr^r, \\
M_4 &= (1 - v(x) \nabla_x) \left(\frac{1}{2} \rho_1^2 q^r r^p - \frac{1}{2} \rho_1^2 q^r qr + qr^r - q^r r \right).
\end{align*}
\]

Then, Equation (6) is the coupled CQNLS dynamical system on a time-space scale, where \( a_0, b_0, c_0 \) are defined by Equation (7). Next, several special cases of the coupled CQNLS dynamical system will be obtained.

Case I

Consider the case \( T \times X = R \times R \). We get \( \mu(t) = v(x) = 0 \). Then, (7) can be converted to

\[
\begin{align*}
a_1 &= 0, \\
b_1 &= 2q, \\
c_1 &= 2r, \\
b_0 &= -\rho_1 q^2 r + iq_x, \\
c_0 &= -\rho_1qr^2 - ir_x, \\
a_0 &= i\rho_1 q^2 r^2 + \frac{1}{2} \rho_1 q r^2 - \frac{1}{2} \rho_1 q r x - iqr,
\end{align*}
\]
System (6) is transformed into the coupled CQNLS equation, i.e.,

\[
\begin{aligned}
q_t &= i\rho_1^2 q^3 r^2 - 2\rho_1 q x r - 2\rho_1 q^2 r^2 - 2iq^2 r + iq_{xx}, \\
r_t &= -i\rho_1^2 q^3 r^2 - 2\rho_1 qr x r - 2\rho_1 q^2 r^2 + 2iq^2 r - ir_{xx}.
\end{aligned}
\]

Case II

Consider the case \( \mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{Z} \). We get \( \mu(t) = 0, \nu(x) = 1 \). Then,

\[
\begin{aligned}
f^{\sigma}(x,t) &= f(x,t), \\
f^\sigma(x,t) &= Ef(x,t) = f(x,t) - (1-E)f(x,t),
\end{aligned}
\]

where \( E \) is the shift operator. By calculation, Equation (7) can be converted to

\[
\begin{aligned}
a_1(x) &= 0, \\
b_1(x) &= 4(1+E)^{-1}q(x), \\
c_1(x) &= 4(1+E)^{-1}r(x), \\
b_0(x) &= (1+E)^{-2}(-\frac{1}{2}\rho_1 Eq(x)r(x) - \frac{1}{2}\rho_1 q(x)r(x) + i - iE)4q(x), \\
c_0(x) &= (1+E)^{-2}(-\frac{1}{2}\rho_1 Eq(x)r(x) - \frac{1}{2}\rho_1 q(x)r(x) + i - iE)4r(x), \\
a_0(x) &= (1+E)^{-2}2i\rho_1 |q|^2 q(x)r(x)(1 + E) - 2i(1-E)|q|^2 q(x)r(x).
\end{aligned}
\]

Equation (6) can be transformed into

\[
\begin{aligned}
qu_t(x) &= (1+E)^{-2}[i(1+E)\rho_1 q(x)r(x) + 2(1-E)][(1+E)\rho_1 q(x)r(x) + 2i(1-E)]q(x), \\
rq_t(x) &= -((1+E)^{-2}[i(1+E)\rho_1 q(x)r(x) + 2(1+E)][(1+E)\rho_1 q(x)r(x) - 2i(1-E)]r(x),
\end{aligned}
\]

which is a semi-discrete coupled CQNLS equation.

4. Darboux Transformation of CQNLS Equation on a Time-Space Scale

In this section, the generalized Darboux transformation on a time-space scale will be constructed, and the \( N \)-soliton solution of the CQNLS equation will be obtained.

Firstly, \( U, V \) of the CQNLS equation are rewritten as follows

\[
\begin{aligned}
U &= -i\lambda c_3 + Q - \frac{1}{2}i\rho_1 Q^2 c_3, \\
V &= -2\lambda^2 c_3 + B_1 \lambda + a_0 c_3 + B_0,
\end{aligned}
\]

with \( Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \), \( B_1 = \begin{pmatrix} 0 & b_1 \\ c_1 & 0 \end{pmatrix} \), \( B_0 = \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix} \), \( c_3 \) is the Pauli matrix.

Proposition 1. The Lax pair of the CQNLS equation is invariant under the following Darboux transformation

\[
\begin{aligned}
\varphi[1] &= T[1] \varphi = (\lambda I - S) \varphi, \\
q[1] &= q - is_{12} - is_{12}^*,
\end{aligned}
\]

where

\[
S = HAH^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_1^*),
\]

\( H \) satisfies

\[
\begin{aligned}
\nabla_x H &= -ic_3 H \Lambda + QH - \frac{1}{2}i\rho_1 Q^2 c_3 H, \\
\nabla_t H &= -2ic_3 H \Lambda^2 + B_1 H \Lambda + a_0 c_3 + B_0 H.
\end{aligned}
\]

Then, \( \varphi[1] \) satisfies the linear spectrum problem as follows

\[
\begin{aligned}
\nabla_x \varphi[1] &= U[1] \varphi[1], \\
\nabla_t \varphi[1] &= V[1] \varphi[1],
\end{aligned}
\]
where
\[
U[1] = -i\lambda c_3 + Q[1] - \frac{1}{2}i\varphi_1 Q[1]^2 c_3,
\]
\[
V[1] = -2i\lambda^2 c_3 + B_1[1]\lambda + a_0[1]c_3 + B_0[1],
\]
\[
Q[1] = Q + i\varphi_3 + iS^\phi c_3, 
B_1[1] = \begin{pmatrix}
0 & b_1[1] \\
c_1[1] & 0
\end{pmatrix},
B_0 = \begin{pmatrix}
0 & b_0[1] \\
c_0[1] & 0
\end{pmatrix}.
\]  

Proof. Assume a gauge transformation
\[
\varphi[1] = T[1]\varphi, \quad T[1] = T_0 + T_1\lambda,
\]
where
\[
T_0 = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad T_1 = \begin{pmatrix}
a_{11} & b_{11} \\
c_{11} & d_{11}
\end{pmatrix}.
\]

By substituting Equation (12) into Equation (1), the constraint relation on the x part is obtained
\[
\nabla T[1] + T[1]^\dagger U - U[1]T[1] = 0.
\] (13)

Substituting Equations (8), (11), and (12) into Equation (13) and comparing the coefficients of \(\lambda\), we get
\[
\begin{align*}
\nabla a & = \frac{i}{2}i\varphi_1 q^* b^\phi + q^* d^\phi - q c^\phi, \\
\nabla b & = \frac{i}{2}i\varphi_1 q^* c^\phi + q^* a^\phi + \frac{1}{2}i\varphi_1 q[1]q^*[1]c + \frac{1}{2}i\varphi_1 q[1]q^*[1]b - q[1]d, \\
\nabla c & = -\frac{1}{2}i\varphi_1 q^* a^\phi + q^* c^\phi + \frac{1}{2}i\varphi_1 q[1]q^*[1]c + q[1]a - q[1]q^* b, \\
\nabla d & = -\frac{1}{2}i\varphi_1 q^* b^\phi + q^* d^\phi - \frac{1}{2}i\varphi_1 q[1]q^*[1]b + q[1]a + \frac{1}{2}i\varphi_1 q[1]q^*[1]d + q[1]c.
\end{align*}
\] (14)

Taking \(a_{11} = d_{11} = 1\), we obtain
\[
q[1] = q + ib^\phi + ib, \\
q[1]^* = q^* + ic^\phi + ia.
\] (15)

Then, matrix \(S\) is constructed as
\[
S = \begin{pmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{pmatrix} = -T_0.
\]

Then,
\[
T[1] = \lambda I - S, \\
q[1] = q - is_{12} - is_{12}^\phi.
\]

From the constant term of Equation (13), we obtain
\[
\nabla S = -S^\phi Q + \frac{1}{2}i\varphi_1 S^\phi Q^2 c_3 + QS - iS^2 c_3 - iS^\phi S c_3 + \frac{1}{2}i\varphi_1 Q^2 S c_3.
\] (16)
If $H$ is a solution of Equation (1), then $H$ satisfies
\[ \nabla_x H = -i\omega_3 H\Lambda + Q H - \frac{1}{2} i \rho_1 Q^2 \sigma_3 H, \]
(17)
where $\Lambda = diag(\lambda, \lambda)$. Then \[ S = H\Lambda H^{-1}, \]
(18)
then,
\[ \nabla_x S = \nabla_x \left( H\Lambda H^{-1} \right) \]
\[ = -i\omega_3 S^2 + QS + \frac{1}{2} i \rho_1 Q^2 \sigma_3 \]
\[ - i\sigma_3 S \sigma_3 - S^\rho Q + \frac{1}{2} i \rho_1 S^\rho Q^2 \sigma_5. \]
In addition, the gauge transformation (9) satisfies another constraint relation concerning 1 part
\[ \nabla_i T[1] + T[1]^T V - V[1] T[1] = 0. \]
(19)
By substituting Equation (12) into Equation (19), the following formula is derived
\[ -\nabla_i S + \left( \lambda I - S^\rho \right) V = V[1] \left( \lambda I - S \right), \]
(20)
where $V$ and $V[1]$ are obtained from Equations (8) and (11). By comparing coefficients of $\lambda$, Equations (21)–(24) are obtained as follows
\[ B_1 + 2i S^\rho \sigma_3 = 2i\omega_3 S + B_1[1], \]
(21)
\[ a_0 \sigma_3 + B_0 - S^\rho B_1 = -B_1[1] S + a_0[1] \sigma_3 + B_0[1], \]
(22)
\[ \nabla_i S + S^\rho \left( a_0 \sigma_3 + B_0 \right) = \left( a_0[1] \sigma_3 + B_0[1] \right) S. \]
(23)
when $T[1] = \lambda I - S$, $q[1] = q - is_{12} - is_{12}^\rho$, Equations (21)–(24) are constant. The proof is completed. Having the explicit form of the Darboux transformation, we are ready to construct the exact solutions of the CQNLS equation.

Matrix $H$ and $\Lambda$ are constructed as
\[ H = \begin{pmatrix} \psi_1 & \phi_1^* \\ \phi_1 & -\psi_1^* \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \]
where the column vector $(\psi_1, \phi_1)^T$ is a set of solutions to the Lax pair (1) when $\lambda = \lambda_1$, column vector $(\phi_1^*, -\psi_1^*)^T$ is a set of solution when $\lambda = \lambda_1^*$. Then one soliton solution will be obtained as follows.

First, rewrite Equation (1) as
\[ \begin{cases} \nabla_x \varphi[0] = \begin{pmatrix} -i\omega_1 + \frac{1}{2} \rho_1 q[0] q[0]^* - q[0]^* & \frac{1}{2} \rho_1 q[0] q[0]^* \\ -q[0]^* & i\omega_1 - \frac{1}{2} \rho_1 q[0] q[0]^* \end{pmatrix} \varphi[0], \\ \nabla \varphi[0] = \begin{pmatrix} -2i\omega_2^2 + a_0[0] & b_1[0] \lambda + b_0[0] \\ c_1[0] \lambda + c_0[0] & 2i\omega_2^2 - a_0[0] \end{pmatrix} \varphi[0], \end{cases} \]
(24)
Then,
\[
\varphi[1] = T[1] \varphi[0] = (\lambda I - S[0]) \varphi[0] = \begin{pmatrix} \lambda - s_{11}[0] & -s_{12}[0] \\ -s_{21}[0] & \lambda - s_{22}[0] \end{pmatrix} \varphi[0],
\]
\[
q[1] = q[0] - is_{12}[0] - is_{12}[0]^\rho = q[0] + i(\lambda_1 - \lambda_1^*) \frac{\psi_1[0]^\ast \psi_1[0]^\ast}{\psi_1[0]^2 + \psi_1[0]^\ast 2} + i(\lambda_1 - \lambda_1^*) \frac{\psi_1[0]^\ast \psi_1[0]^\ast}{\psi_1[0]^2 + \psi_1[0]^\ast 2}.
\]
where
\[
S[0] = \frac{1}{\psi_1[0]^2 + \psi_1[0]^\ast 2} \begin{pmatrix} -\lambda_1 |\psi_1[0]|^2 - \lambda_1^* |\psi_1[0]|^2 - (\lambda_1 - \lambda_1^*) \psi_1[0] \psi_1[0]^\ast & -\lambda_1 |\psi_1[0]|^2 - \lambda_1^* |\psi_1[0]|^2 \end{pmatrix}.
\]
\[
\varphi_1[0] = \begin{pmatrix} \psi_1[0] \\ \phi_1[0] \end{pmatrix}
\]
is a solution to (24) when \( \lambda = \lambda_1 \). At the same time, \( \varphi[1] \) satisfies
\[
\nabla_x \varphi[1] = \begin{pmatrix} -i\lambda + \frac{1}{2} \rho_1 q[1]^\ast q[1] \psi_1[0] \psi_1[0]^\ast + \frac{q[1]}{2} & -q[1]^\ast \\ -q[1]^\ast & -i\lambda - \frac{1}{2} \rho_1 q[1]^\ast q[1] \psi_1[0] \psi_1[0]^\ast \end{pmatrix} \begin{pmatrix} \varphi[1] \\ \varphi[1] \end{pmatrix},
\]
\[
\nabla_t \varphi[1] = \begin{pmatrix} a_2[1] \lambda^2 + a_0[1] & b_1[1] \lambda + b_0[1] \\ c_1[1] \lambda + c_0[1] & -d_2[1] \lambda^2 - a_0[1][1] \end{pmatrix} \begin{pmatrix} \varphi[1] \\ \varphi[1] \end{pmatrix}.
\]

Taking “seed solution” \( q = 0 \), we derive
\[
\psi_1 = e^{-i\lambda} (x,0) e^{-2i\lambda^2} (t,0),
\]
\[
\phi_1 = e_{i\lambda} (x,0) e^{2i\lambda^2} (t,0),
\]
\[
\psi_1^\ast = [1 - i\lambda v(x)] e^{-i\lambda} (x,0) e^{-2i\lambda^2} (t,0),
\]
\[
\phi_1^\ast = [1 + i\lambda v(x)] e_{i\lambda} (x,0) e^{2i\lambda^2} (t,0).
\]

Then, one soliton solution is obtained
\[
q[1] = q[0] + i(\lambda_1 - \lambda_1^*) \frac{M_1}{M_1 + M_2} + i(\lambda_1 - \lambda_1^*) \frac{[1 - i\lambda v(x)]^2 M_1}{[1 - i\lambda v(x)]^2 M_1 + [1 + i\lambda v(x)]^2 M_2},
\]
with
\[
M_1 = e^{-2i\lambda} (x,0) e^{-4i\lambda^2} (t,0),
\]
\[
M_2 = e^{2i\lambda} (x,0) e^{4i\lambda^2} (t,0).
\]

Next, the second Darboux transformation is constructed and two soliton solutions are obtained in a similar way
\[
\varphi[2] = T[2] \varphi[1] = (\lambda I - S[1]) \varphi[1] = \begin{pmatrix} \lambda - s_{11}[1] & -s_{12}[1] \\ -s_{21}[1] & \lambda - s_{22}[1] \end{pmatrix} \varphi[1] = T[2] T[1] \varphi[0],
\]
\[
q[2] = q[1] - is_{12}[1] - is_{12}[1]^\rho = q[1] + i(\lambda_2 - \lambda_2^*) \frac{\psi_2[1]^\ast \psi_2[1]^\ast}{\psi_2[1]^2 + \psi_2[1]^\ast 2} + i(\lambda_2 - \lambda_2^*) \frac{\psi_2[1]^\ast \psi_2[1]^\ast}{\psi_2[1]^2 + \psi_2[1]^\ast 2},
\]
where
\[
\varphi_2[1] = \begin{pmatrix} \psi_2[1] \\ \phi_2[1] \end{pmatrix} = \begin{pmatrix} \lambda_2 - s_{11}[1] & -s_{12}[1] \\ -s_{21}[1] & \lambda_2 - s_{22}[1] \end{pmatrix} \begin{pmatrix} \psi_2 \\ \phi_2 \end{pmatrix}.
\]
Fractal Fract. 2022, 6, 12
9 of 13

φ₂[1] is a solution to Equation (26) when λ = λ₂ and (ψ₂, φ₂)ᵀ is a solution to (24) when λ = λ₂. φ[2] satisfies

\[
\begin{align*}
\nabla_x φ[2] &= \left( -iλ + \frac{1}{2}ρ₁ q[2]q[2]^* - q[2]^* \right) φ[2], \\
\nabla_t φ[2] &= \left( a₂[2]λ^2 + a₀[2] \right) φ[2] + \left( b₁[2]λ + b₀[2] \right) \left( c₁[2]λ + c₀[2] \right) φ[2]. 
\end{align*}
\]

Then, the N-soliton solution is obtained

\[
φ[N] = T[N]φ[N - 1] \\
= (λI - S[N - 1])φ[N - 1] \\
= \left( \begin{array}{cc}
λ - s₁₁[N - 1] - s₁₂[N - 1] \\
−s₂₁[N - 1] & \lambda - s₂₂[N - 1]
\end{array} \right) φ[N - 1] \\
= T[N] \cdots T[3]T[2]T[1]φ[0],
\]

\[
q[N] = q[N - 1] + i(λN - λN^*) \frac{ψ_N[N - 1]φ[N - 1]^*}{ψ_N[N - 1]^2 + φ_N[N - 1]^2} + i(λN - λN^*) \frac{ψ_N'[N - 1]φ_N'[N - 1]^*}{ψ_N'[N - 1]^2 + φ_N'[N - 1]^2}
\]

\[
= q[0] + i \sum_{j=1}^{N} (λ_j - λ_j^*) \frac{ψ_j[j - 1]φ_j[j - 1]^*}{ψ_j[j - 1]^2 + φ_j[j - 1]^2} + i \sum_{j=1}^{N} (λ_j - λ_j^*) \frac{ψ_j'[j - 1]φ_j'[j - 1]^*}{ψ_j'[j - 1]^2 + φ_j'[j - 1]^2}.
\]

Finally, three different cases of Darboux transformation are discussed, their N-soliton solutions are given separately.

Case I

Considering the case T × X = R × R, we have

\[
\begin{align*}
\frac{df}{dt} &= f(t, x), \\
\frac{df^*}{dt} &= f(t, x).
\end{align*}
\]

Equation (15) can be simplified into

\[
\begin{align*}
q[1] &= q + 2ib, \\
q[1]^* &= q^* + 2ic. 
\end{align*}
\]

when q[0] = 0 and λ = ζ + iη, ζ and η are real constants, we derive

\[
\begin{align*}
φ₁ = e^{(η - iζ)x + [4ζη - 2i(ζ^2 - η^2)]t}, \\
φ₂ = e^{-(η - iζ)x - [4ζη - 2i(ζ^2 - η^2)]t}.
\end{align*}
\]

Then one-soliton solution of the CQNLS equation is obtained

\[
q[1] = \frac{-2i}{|ψ₁|^2 + |ψ₂|^2} (λ₁ - λ₁^*)ψ₁ψ₂^* = 2ηe^{-2iζx - 4i(ζ^2 - η^2)} \text{sech}(2ηx + 8ζηt)
\]

(29)

The dynamic of the one-soliton solution is presented in Figure 1.

Then, the N-soliton solution can be obtained

\[
q[N] = q[N - 1] + 2i(λN - λN^*) \frac{ψ_N[N - 1]φ_N[N - 1]^*}{ψ_N[N - 1]^2 + φ_N[N - 1]^2} = 2i \sum_{j=1}^{N} (λ_j - λ_j^*) \frac{ψ_j[j - 1]φ_j[j - 1]^*}{ψ_j[j - 1]^2 + φ_j[j - 1]^2}.
\]
Case II

Considering the case $T \times \mathbb{X} = \mathbb{R} \times \mathbb{C}$, we have

$$
\mu(t) = 0, \quad \nu(x) = \left\{ \begin{array}{ll}
0 & x = \frac{\lambda}{3^{n+1}} + \frac{1}{3^{n+1}} \in \mathbb{L}, \\
\frac{1}{3^{n+1}} & x \in \mathbb{C} \setminus \mathbb{L},
\end{array} \right.
$$

where $\mathbb{C}$ is a Cantor set and $\mathbb{L}$ is a set that contains the left discrete elements of $\mathbb{C}$,

$$
\mathbb{L} = \left\{ \sum_{k=1}^{n} \frac{e_k}{3^k} + \frac{1}{3^{n+1}} : n \in \mathbb{N}, \ e_k \in \{0, 2\}, \ 1 \leq k \leq n \right\},
$$

where $\lambda = \zeta + i\eta$, one-soliton solution is obtained by the first iteration.

$$
q[1] = \left\{ \begin{array}{ll}
-2\eta\sum_{N_1}^{N_1 + N_2} - \frac{1}{3^n} [1 - 3^{-n-1}(i\xi - \eta)]^2 N_1 \sum_{N_2}^{N_1} [1 - 3^{-n-1}(i\xi - \eta)]^2 N_2, & x \in \mathbb{L}, \ t \in \mathbb{R}, \\
2\eta e^{-2i\xi x - 4i(\zeta^2 - \eta^2)} \text{sech}(2\eta x + 8\xi \eta t), & x \in \mathbb{C} \setminus \mathbb{L}, \ t \in \mathbb{R},
\end{array} \right. \tag{30}
$$

with

$$
N_1 = \left\{ e^{-2i(\xi - \eta)}(x, 0)^2 - 4i(\zeta^2 + 2i\eta - \eta^2)(t, 0), \\
N_2 = \left\{ e^{2i(\xi - \eta)}(x, 0)^2 - 4i(\zeta^2 + 2i\eta - \eta^2)(t, 0)
\right. \right.
$$

Then, the $N$-soliton solution is obtained

$$
q[N] = \left\{ \begin{array}{ll}
N_3, & x \in \mathbb{L}, \ t \in \mathbb{R}, \\
N_4, & x \in \mathbb{C} \setminus \mathbb{L}, \ t \in \mathbb{R},
\end{array} \right. \tag{31}
$$

where

$$
N_3 = i \sum_{j=1}^{N} (\lambda_j - \lambda_j^*) \left( \psi_j[j - 1|j - 1] \right)^* \left( |\psi_j[j - 1]|^2 + |\phi_j[j - 1]|^2 \right)^*,
$$

$$
N_4 = i \sum_{j=1}^{N} (\lambda_j - \lambda_j^*) \left( \psi_j[j - 1] \right)^* \left( |\psi_j[j - 1]|^2 + |\phi_j[j - 1]|^2 \right)^*.
$$

Case III

Considering the case $T \times \mathbb{X} = \mathbb{R} \times \mathbb{K}_p$, we have

$$
\mu(t) = 0, \\
\nu(x) = \left\{ \begin{array}{ll}
0, & x = 0, \\
(1 - p^{-1})x, & x = p^k \in p\mathbb{Z}.
\end{array} \right.
$$

where $p > 1$, $p\mathbb{Z} = \{ p^k : k \in \mathbb{Z} \}$ and $\mathbb{K}_p = p\mathbb{Z} \cup \{ 0 \}$.

By the iteration of the Darboux transformation, when $\lambda = \zeta + i\eta$, one-soliton solution can be presented as

$$
q[1] = \left\{ \begin{array}{ll}
-2\eta \sum_{x \in \{0, p^k\}} \left( \psi_j[j - 1] - 3^{-n-1} \nabla_x \psi_j[j - 1] \right) \left( \phi_j[j - 1] - 3^{-n-1} \nabla_x \phi_j[j - 1] \right) \left( \psi_j[j - 1]|^2 + |\phi_j[j - 1]|^2 \right)^*, & x \in p\mathbb{Z}, \ t \in \mathbb{R}, \\
2\eta e^{-2i\xi x - 4i(\zeta^2 - \eta^2)} \text{sech}(2\eta x + 8\xi \eta t), & x = 0, \ t \in \mathbb{R},
\end{array} \right. \tag{31}
$$

where

$$
N_5 = \sum_{x \in \{0, p^k\}} (\eta - i\xi) x - 2i(\xi^2 + 2i\xi \eta - \eta^2),
$$

$$
N_6 = \sum_{x \in \{0, p^k\}} (i\xi - \eta) x + 2i(\xi^2 + 2i\xi \eta - \eta^2).
$$
Then, the N-soliton solution is obtained

\[
q[N] = \begin{cases} 
   i \sum_{i=1}^{N} \left( \lambda_i - \lambda_i^+ \right) \frac{\psi_i[j-1] \psi_i[j-1]^*}{|\psi_i[j-1]|^2 + |\psi_i[j-1]^*|^2} + i \sum_{i=1}^{N} \left( \lambda_i - \lambda_i^+ \right) M_{13}, & x \in p^2, \ t \in \mathbb{R}, \\
   2i \sum_{i=1}^{N} \left( \lambda_i - \lambda_i^+ \right) \frac{\psi_i[j-1] \psi_i[j-1]^*}{|\psi_i[j-1]|^2 + |\psi_i[j-1]^*|^2}, & x = 0, \ t \in \mathbb{R}, 
\end{cases}
\]

where

\[
M_{13} = \frac{\left[ \psi_i[j-1] - (1 - p^{-1}) x \nabla_x \psi_i[j - 1] \right] \left[ \psi_i[j-1] - (1 - p^{-1}) x \nabla_x \psi_i[j - 1] \right] \left| \psi_i[j-1] - (1 - p^{-1}) x \nabla_x \psi_i[j - 1] \right|^2}{|\psi_i[j-1]|^2 + |\psi_i[j-1]|^2 + |\psi_i[j-1]^*|^2 + |\psi_i[j-1]^*|^2}.
\]

Figure 1. One-soliton solution with \( \xi = 0.825, \eta = 0.512 \).

5. Conclusions

In this paper, starting from the \( \nabla \)-dynamical system, the specific form of the coupled CQNLS equation on a time-space scale is derived by the zero curvature equation and different forms of this equation in continuous time scales and discrete time scale are discussed separately. In addition, the Darboux transformation of the coupled CQNLS equation on a time-space scale is elaborately constructed; its one, two, and N-soliton solutions are further investigated. These results effectively combine the theory of differential equations with the theory of difference equations, which can simplify the process of similar analysis on different time and space scales, and can better solve complex models that include continuous and discrete situations.

The extension to arbitrary time or space scales provides access to a wider range of nonlinear integrable dynamic equations. By taking the seed solution \( q = 0, \lambda = \xi + i\eta \), one-solution of the CQNLS equation is obtained on three different time-space scales \( (X = \mathbb{R}, X = \mathbb{C}, \text{and} \ X = \mathbb{K}_p) \). In one case, the exact solution (29) and its dynamic figure are obtained when \( x \in \mathbb{R} \). In the other cases, when \( x \in \mathbb{C}\backslash\mathbb{L} \) and \( x = 0 \), exact solutions (30) and (31) are obtained and are similar to Equation (29). Nevertheless, when \( x \in \mathbb{L} \) and \( x \in p^2 \), the structures of solutions are more complicated at discontinuity points. Due to the limitations of the computer, it was difficult to obtain their dynamic figures at this stage. We will find the most effective way to reduce structures of solutions on \( \mathbb{C} \) and \( \mathbb{K}_p \), then extend the nonlocal symmetry reduction [38] to a time-space scale, which is the focus of our future work.

Author Contributions: Conceptualization, H.D., Y.Z. and C.W.; methodology, H.D.; software, Y.Z.; validation, H.D., Y.Z. and Y.F.; formal analysis, M.L.; investigation, Y.Z. and M.L.; writing—original draft preparation, Y.Z. and C.W.; project administration, H.D. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 11975143, 12105161, 61602188), Natural Science Foundation of Shandong Province (Grant No. ZR2019QD018), CAS Key Laboratory of Science and Technology on Operational Oceanography.
Acknowledgments: The authors would like to express their thanks to the editors and the reviewers for their kind comments to improve our paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zhang, H.Q.; Tian, B.; Xu, T.; Li, H.; Zhang, C.; Zhang, H. Lax pair and Darboux transformation for multi-component modified korteweg-de vries equations. J. Phys. A Math. Theor. 2008, 41, 1–13. [CrossRef]
2. Doktorov, E.V.; Leble, S.B. A Dressing Method in Mathematical Physics; Springer: Berlin/Heidelberg, Germany, 2007.
3. Guo, B.; Ling, L.; Liu, Q.P. Nonlinear Schrödinger equation: Generalized Darboux transformation and rogue wave solutions. Phys. Rev. E 2012, 85, 026607. [CrossRef] [PubMed]
4. Bagrov, V.G.; Samsonov, B.F. Darboux transformation of the Schrödinger equation. Phys. Part. Nucl. 1997, 28, 374–397. [CrossRef]
5. Xu, S.; He, J.; Wang, L. The Darboux transformation of the derivative nonlinear Schrödinger equation. J. Phys. A Math. Theor. 2011, 44, 6629–6636. [CrossRef]
6. Debnath, L. Solitons and the Inverse Scattering Transform. SIAM Rev. Soc. Ind. Appl. Math. 1981, 9, 426–533.
7. Matsuno, Y. The N-soliton solution of a two-component modified nonlinear Schrödinger equation. Appl. Phys. Lett. 2011, 375, 3090–3094. [CrossRef]
8. Miki, W.; Heiji, S.; Kimiaki, K. Relationships among Inverse Method, Backlund Transformation and an Infinite Number of Conservation Laws. Prog. Orthod. 1975, 53, 419–436.
9. Matveev, V.B.; Salle, M.A. Darboux Transformations and Solitons; Springer: Berlin, Germany, 1991.
10. Hilger, S. Analysis on Measure Chains-A Unified Approach to Continuous and Discrete Calculus. Results Math. 1990, 18, 18–56. [CrossRef]
11. Bohnen, M.; Peterson, A. Dynamic Equations on Time Scales An Introduction with Applications; Birkhauser: Boston, MA, USA, 2001.
12. Agarwal, R.P.; Băleanu, D.; Onur, O.; Tunç, E. Stability of stationary states in the cubic nonlinear Schrödinger equation: Applications to the Bose-Einstein condensate. Phys. Scr. 2012, 85, 015002. [CrossRef]
13. Christiansen, F.B.; Christiansen, R. Theories of Populations in Biological Communities; Springer: Berlin, Germany, 1997.
14. Manore, C.A.; Hyman, J.M.; Bergsman, L.D. A mathematical model for the spread of west nile virus in migratory and resident birds. Math. Biosci. Eng. 2017, 13, 401–424.
15. Peng, Y.; Xiang, X.; Jiang, Y. Nonlinear dynamics and optimal control problems on time scales. Esaim. Contr. Optim. Calc. Var. 2010, 17, 654–681. [CrossRef]
16. Zhang, H.; Li, Y. Existence of Positive Periodic Solutions For Functional Differential Equations With Impulse Effects On Time Scales. Commun. Nonlinear. Sci. Numer. Simul. 2019, 14, 19–26. [CrossRef]
17. Benoist, Y.; Foulon, P.; Labourie, F. Double solutions of impulsive dynamic boundary value problems on time scale. J. Differ. Equ. Appl. 2002, 8, 345–356. [CrossRef]
18. Attici, F.M.; Biles, D.C.; Lebedinsky, A. An application of time scales to economics. Commun. Nonlinear. Sci. Numer. Simul. 2006, 43, 718–726. [CrossRef]
19. Ramsey, J.B.; Lampart, C. Decomposition of economic relationships by timescale using wavelets: Money and income. Macroecon.Dyn. 1998, 2, 49–71. [CrossRef]
20. Albuch, L.; Malomed, B.A. Transitions between symmetric and asymmetric solitons in dual-core systems with cubic-quintic nonlinearity. Math. Comput. Simulat. 2007, 74, 312–322. [CrossRef]
21. Shah, W.R.; Qi, F.H.; Guo, R.; Xue, Y.S.; Wang, P.; Tian, B. Conservation laws and solitons for the coupled cubic-quintic nonlinear Schrödinger equations in nonlinear optics. Phys. Scr. 2012, 85, 015002. [CrossRef]
22. Azzouzi, E.; Triki, H.; Mezghiche, K.; Akhrif, A.E. Solitary wave solutions for high dispersive cubic-quintic nonlinear Schrödinger equation. Chaos. Solitons Fract. 2009, 39, 1304–1307. [CrossRef]
23. Triki, H.; Wazwaz, A.M. Soliton solutions of the cubic-quintic nonlinear Schrödinger equation with variable coefficients. Rom. J. Phys. 2016, 61, 360–366.
24. Kengne, E.; Vaillancourt, R.; Malomed, B.A. Bose-Einstein condensates in optical lattices: The cubic-quintic nonlinear Schrödinger equation with a periodic potential. J. Phys. B At. Mol. Opt. 2008, 41, 205202. [CrossRef]
25. Carr, L.D.; Kutz, J.N.; Reinhardt, W.P. Stability of stationary states in the cubic nonlinear Schrödinger equation: Applications to the Bose-Einstein condensate. Phys. Rev. E 2001, 63, 066604. [CrossRef] [PubMed]
26. Gagnon, L.; Winternitz, P. Exact solutions of the cubic and quintic nonlinear Schrödinger equation for a cylindrical geometry. Phys. Rev. A. 1989, 39, 296. [CrossRef]
27. Zhang, Y.; Nie, X. J.; Zha, Q. L. Rogue wave solutions for the coupled cubic-quintic nonlinear Schrödinger equations in nonlinear optics. J. Am. Math. Soc. 2014, 378, 191–197. [CrossRef]
28. Wang, P.; Tian, B. Symbolic computation on soliton dynamics and Backlund transformation for the generalized coupled nonlinear Schrödinger equations with cubic-quintic nonlinearity. J. Mod. Opt. 2012, 9, 1786–1796. [CrossRef]
30 Hamanaka, M. Noncommutative Solitons and Integrable Systems. Physics 2005, 861, 175–198.
31 Takao, K. Soliton Equations Extracted from the Noncommutative Zero-Curvature Equation. Prog. Theor. Phys. 2001, 105, 1045–1057.
32 Tu, G.Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. J. Phys. A 1989, 22, 2375.
33 Krichever, I. Vector bundles and Lax equations on algebraic curves. Commun. Math. Phys. 2001, 229, 229–269. [CrossRef]
34 Ma, W.X.; Xu, X. Positive and Negative Hierarchies of Integrable Lattice Models Associated with a Hamiltonian Pair. Int. J. Theor. Phys. 2004, 43, 219–235. [CrossRef]
35 Hovhannisyan, G. On Dirac equation on a time scale. J. Math. Phys. 2011, 52, 1967–1981. [CrossRef]
36 Liu, M.S.; Dong, H.; Fang, Y.; Zhang, Y. Lie symmetry analysis of burgers equation and the euler equation on a time scale. Symmetry 2019, 12, 10. [CrossRef]
37 Anderson, D.R.; Bullock, J.; Erbe, L.; Peterson, A.; Tran, H.N. Nabla dynamic equations on time scales. Discret. Appl. Math. 2003, 13, 1–47.
38 Ablowitz, M.J.; Musslimani, Z.H. Integrable nonlocal nonlinear schrdinger equation. Phys. Rev. Lett. 2013, 110, 064105. [CrossRef] [PubMed]