Efficient two-sample functional estimation and the super-oracle phenomenon

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Abstract

We consider the estimation of two-sample integral functionals, of the type that occur naturally, for example, when the object of interest is a divergence between unknown probability densities. Our first main result is that, in wide generality, a weighted nearest neighbour estimator is efficient, in the sense of achieving the local asymptotic minimax lower bound. Moreover, we also prove a corresponding central limit theorem, which facilitates the construction of asymptotically valid confidence intervals for the functional, having asymptotically minimal width. One interesting consequence of our results is the discovery that, for certain functionals, the worst-case performance of our estimator may improve on that of the natural ‘oracle’ estimator, which is given access to the values of the unknown densities at the observations.

1 Introduction

This paper concerns the estimation of two-sample density functionals of the form

\[ T = T(f, g) := \int_{\mathcal{X}} f(x)\phi(f(x), g(x), x) \, dx, \]  

where \( \mathcal{X} := \{ x \in \mathbb{R}^d : f(x) > 0, g(x) > 0 \} \), based on independent \( d \)-dimensional random vectors \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \), where \( X_1, \ldots, X_m \) have density \( f \) and \( Y_1, \ldots, Y_n \) have density \( g \). The interest in the estimation of such functionals arises from many applications: for instance, many divergences such as the Kullback–Leibler divergence, total variation and Hellinger distances (or more generally, all \( \varphi \)-divergences) are of this form. The estimation
of such divergences is important for two-sample testing (Wornowizki and Fried, 2016), registration problems in image analysis (Hero et al., 2002) and generative adversarial networks (Nowozin, Cseke and Tomioka, 2016), to name just a few examples. Moreover, functionals such as $\int_{\mathbb{R}^d} fg$ occur naturally in estimating $L_2$-distances between densities (Singh, Sriperumbudur and Póczos, 2018). Of course, we can regard the problem of estimation of one-sample density functionals

$$H(f) := \int \{x : f(x) > 0\} f(x) \psi(f(x), x) \, dx,$$

which include Shannon and Rényi entropies, as a special case.

Motivated by these applications, the estimation of the two-sample functional (1) (or closely related quantities) has received considerable attention in the literature recently (e.g. Krishnamurthy et al., 2014; Kandasamy et al., 2015; Singh and Póczos, 2016; Singh, Sriperumbudur and Póczos, 2018; Moon et al., 2018). Naturally, the one-sample version of the problem, and special cases of it, have been highly-studied subjects over several decades (e.g. Kozachenko and Leonenko, 1987; Bickel and Ritov, 1988; Birgé and Massart, 1995; Laurent, 1996; Beirlant et al., 1997; Leonenko, Pronzato and Savani, 2008; Biau and Devroye, 2015; Han et al., 2018; Berrett, Samworth and Yuan, 2019). It turns out that many functionals of interest involve functions $\phi$ in (1) that are non-smooth as their first or second arguments approach zero, or functions $\psi$ in (2) that are non-smooth as their first argument vanishes. For instance, for the Shannon entropy, $\psi(y, x) = -\log y$, while the Rényi entropy of order $\kappa$ is essentially equivalent to $\psi(y, x) = y^{\kappa-1}$, which is non-smooth as $y \to 0$ when $\kappa \in (0, 1)$.

To avoid problems caused by this lack of smoothness, many of the aforementioned authors assume that the density $f$ is bounded away from zero on its (compact) support $X$. In that case, efficient estimators can sometimes be obtained; to give just one example, when $f$ is also $s$-Hölder smooth on $\{x : f(x) > 0\}$ with $s > d/4$, Laurent (1996) obtained an estimator $\hat{H}_m$ satisfying

$$m \mathbb{E}[\{\hat{H}_m - H(f)\}^2] \to \int_X f \log^2 f - H(f)^2.$$

The limit in (3) is also obtained by the oracle estimator $H_m^* := -m^{-1} \sum_{i=1}^m \log f(X_i)$, and is optimal in a local asymptotic minimax sense (Ibragimov and Khas’minskii, 1991; Laurent, 1996).

However, the assumption that the density $f$ is bounded away from zero on its support is made purely for mathematical convenience; it assumes away the essential difficulty of the problem caused by the non-smoothness and rules out many standard densities of common interest. In the related problem of density estimation, it is known that, depending on the
loss function and the smoothness of the densities considered, optimal rates of convergence can be very different when densities with unbounded support are allowed (Donoho et al., 1996; Juditsky and Lambert-Lacroix, 2004; Goldenshluger and Lepski, 2014).

It is therefore of great interest to understand the ways in which low density regions interact with the potential non-smoothness of the functional to determine what is achievable in terms of the behaviour of estimators. Previous works in this direction have tended to focus on specific functionals and on rates of convergence (e.g. Tsybakov and van der Meulen, 1996; Han et al., 2018). By contrast, in this work we are interested in general classes of functionals, and in conditions under which efficient estimation remains feasible. Our estimators will be deterministically weighted versions of preliminary estimators based on nearest neighbour distances. To set the scene, for integers $k_X \in \{1, \ldots, m - 1\}$ and $k_Y \in \{1, \ldots, n\}$, write $\rho(d, i, X)$ for the (Euclidean) distance between $X_i$ and its $k_X$th nearest neighbour in the sample $\{X_1, \ldots, X_m\} \setminus \{X_i\}$, and write $\rho(d, i, Y)$ for the distance between $X_i$ and its $k_Y$th nearest neighbour in the sample $\{Y_1, \ldots, Y_n\}$. The starting point for the construction of our estimators is the approximation

$$f(X_i)V_d\rho(d, i, X) \approx k_X/m,$$

where $V_d := \pi^{d/2}/\Gamma(1 + d/2)$ denotes the $d$-dimensional Lebesgue measure of the unit Euclidean ball in $\mathbb{R}^d$; this arises by comparing the number of points in a ball of radius $\rho(d, i, X)$ about $X_i$ with a local constant approximation to the probability content of the same ball. This motivates the initial estimator

$$\tilde{T}_{m,n} = \tilde{T}_{m,n,k_X,k_Y} := \frac{1}{m} \sum_{i=1}^{m} \phi\left(\frac{k_X}{mV_d\rho(d, i, X)}, \frac{k_Y}{nV_d\rho(d, i, Y)}, X_i\right).$$

(4)

Restricting attention for simplicity of exposition to the one-sample analogue $\tilde{T}_m = \tilde{T}_{m,k}$ of (4) that simply replaces $\phi(\cdot, \cdot, \cdot)$ with $\psi(\cdot, \cdot, \cdot)$ and $k_X$ with $k$, it has long been known in the special case of the Shannon entropy functional that one should debias $\tilde{T}_m$ by replacing $k$ with $e^{\Psi(k)}$, where $\Psi(\cdot)$ denotes the digamma function (Kozachenko and Leonenko, 1987). This amounts to adding $\log k - \Psi(k)$ to the original estimator. Ganguly et al. (2018) argued that for general two-sample functionals, the estimator (4) can be debiased to leading order via an implicit inverse Laplace transform, and showed that this has an explicit expression in certain examples. A subtle question concerns the issue of whether to apply our weights to the original estimators (4) or their debiased versions. We address this by using fractional
calculus techniques to provide an explicit expression for the leading order remaining bias of the debiased estimators. We conclude that, in general, the gain from the fact that fewer non-zero weights are required to obtain an efficient estimator when applying these weights to the debiased estimator is outweighed by the added complication of the resulting estimator. However, in special cases such as the Kullback–Leibler and Rényi divergences, where the correct explicit debiasing terms are available, the weighting scheme simplifies and we advocate applying the weights to the debiased estimator.

Returning to the general case, our final estimators $\hat{T}_{m,n}$ are based on weighted averages of estimators of the form $\tilde{T}_{m,n,k_X,k_Y}$ for different choices of $k_X$ and $k_Y$; such estimators are attractive because they generalise easily to multivariate cases (unlike, for example, estimators based on sample spacings), and because they are straightforward to compute. Our first main result (Theorem 2 in Section 2), reveals that the dominant asymptotic contribution to the squared error risk of $\hat{T}_{m,n}$ is of the form $v_1/m + v_2/n$ as $m, n \to \infty$, uniformly over appropriate classes of densities $f, g$, functions $\phi$ and choices of weights, for certain variance functionals $v_1 = v_1(f, g)$ and $v_2 = v_2(f, g)$ given in (9) below. Theorem 11 in Section 2 complements this by establishing that $v_1$ and $v_2$ are optimal in a local asymptotic minimax sense. We therefore conclude that, under the conditions of Theorem 2, the estimators $\hat{T}_{m,n}$ are efficient.

In addition to studying the efficiency of our estimators $\hat{T}_{m,n}$, it is also highly desirable to be able to derive their asymptotic distributions; such a result could be used, for instance, to obtain an asymptotically valid confidence interval for $T$. Despite the fact that the summands in our estimator are dependent, for the special case of the one-sample Shannon entropy functional, it is straightforward to derive the asymptotic normality of the weighted nearest neighbour estimator, as it is well approximated by the efficient, ‘oracle’ estimator $-m^{-1} \sum_{i=1}^{m} \log f(X_i)$. However, for general functionals, the natural oracle estimator may not be efficient, as explained in the next paragraph; this means that deriving the asymptotic distribution of $\hat{T}_{m,n}$ in such cases remains a significant challenge. In our second main result (Theorem 3 in Section 2), we show how the problem can be reexpressed in a form where we can apply the central limit theorem of Baldi and Rinott (1989) for dependent random variables for which the degrees of the nodes in the pairwise dependency graph are controlled. Thus, the estimators $\hat{T}_{m,n}$ are indeed asymptotically normal under appropriate conditions.

As a byproduct of our efficiency analysis, we uncover a curious phenomenon that can occur for certain functionals; for ease of exposition here, we focus on the Rényi-type functional

$$H_\kappa := \int_{\mathbb{R}^d} f(x)^\kappa \, dx,$$
with $\kappa \in (1/2, 1)$. Given access to $\{f(X_i) : i = 1, \ldots, m\}$, the natural oracle estimator in this setting is

$$H_m^* := \frac{1}{m} \sum_{i=1}^{m} f(X_i)^{\kappa-1}. $$

Surprisingly, we find that there exists an estimator $\hat{H}_m$ and general classes $\mathcal{F}$ of densities for which

$$\lim_{m \to \infty} \sup_{f \in \mathcal{F}} \mathbb{E}_f \left\{ \left( \frac{\hat{H}_m - H_\kappa}{H_m^* - H_\kappa} \right)^2 \right\} = \kappa^2 < 1. $$

(5)

We refer to this as the super-oracle phenomenon. It is important to note that this is very different from the phenomenon of superefficiency, as occurs with, e.g., the Hodges estimator (Lehmann and Casella, 1998, Example 6.2.5). There, in the case of scalar parameter estimation, asymptotic improvement in mean squared error risk is possible at a set of fixed parameter values, which form a Lebesgue null set (Le Cam, 1953; van der Vaart, 1997). Moreover, and more importantly from our perspective, the superefficient asymptotic behaviour is necessarily accompanied by worse finite-sample performance in a neighbourhood of points of superefficiency, so that any apparent improvement is really an artefact of the pointwise asymptotic regime considered. By contrast, in (5), the supremum is taken inside the limit, so that the super-oracle improvement for large $m$ can be considered as genuine.

The remainder of the paper is organised as follows: in Section 2, we present our main results on the asymptotic squared error risk and asymptotic normality of our general two-sample functional estimators. Section 3 is devoted to understanding the bias of these estimators and a discussion of the potential benefits of debiasing them before computing our weighted averages, while Section 4 considers their variance properties. In Section 5, we describe the super-oracle phenomenon in greater detail, and in Section 6 we present a local asymptotic minimax lower bound that illustrates the asymptotic optimality of our estimators and justifies referring to them as efficient. Our main theoretical arguments are given in Section 7, as well as various auxiliary results and bounds on remainder terms.

We end this section by introducing some notation used throughout the paper. For $m \in \mathbb{N}_0$, we write $[m] := \{0, 1, \ldots, m\}$. If $A$ is a vector, matrix or array, we write $\|A\|$ for its Euclidean vectorised norm. For $x \in \mathbb{R}^d$ and $r \geq 0$, let $B_x(r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ denote the closed Euclidean ball or radius $r$ about $x$. For vectors $a$ and $b$ of the same dimension, we write $a \circ b$ for their Hadamard product. If $Z$ is a random variable, we write $\mathcal{L}(Z)$ for its law. We write $Z := (0, \infty)^2 \times \mathbb{R}^d$. For a smooth function $\phi : Z \to \mathbb{R}$, $z = (u, v, x) \in Z$ and $j, l \in \mathbb{N}$, we write $\phi_{jl}(z) := \frac{\partial^{j+l} \phi}{\partial u^j \partial v^l}$. We also use multi-index notation for derivatives, so that,
for a sufficiently smooth density \( f^* \) on \( \mathbb{R}^d \), \( x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d \), \( t \in \mathbb{N} \) and a multi-index 
\( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \) with \( |\alpha| := \sum_{j=1}^d \alpha_j = t \), we write \( \partial^\alpha f^* := \frac{\partial^t f^*}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \). For \( \alpha > 0 \) and a density \( f^* \) on \( \mathbb{R}^d \), we write \( \mu_\alpha(f^*) := \int_{\mathbb{R}^d} \|x\|^{\alpha} f^*(x) \, dx \) and \( \|f^*\|_\infty := \sup_{x \in \mathbb{R}^d} f^*(x) \). For \( r \in [0, \infty) \) and \( x \in \mathbb{R}^d \), we also define \( h_{x,f^*}(r) := \int_{B_x(r)} f^*(y) \, dy \) and, for \( s \in [0, 1) \), let \( h_{x,f^*}^{-1}(s) := \inf\{r \geq 0 : h_{x,f^*}(r) \geq s\} \). Recall that, for \( a, b > 0 \), the beta function is defined by \( B_{a,b} := \int_0^1 t^{a-1}(1-t)^{b-1} \, dt \) and define also the corresponding density \( B_{a,b}(s) := s^{a-1}(1-s)^{b-1}/B_{a,b} \) for \( s \in (0, 1) \).

## 2 Main results

Let \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \) be independent \( d \)-dimensional random vectors, with \( X_1, \ldots, X_m \) having density \( f \) and \( Y_1, \ldots, Y_n \) having density \( g \), both with respect to Lebesgue measure on \( \mathbb{R}^d \). We consider the estimation of the functional \( T(f, g) \) in (1).

Before we can state our main theorems on the asymptotic risk and normality of our functional estimators, we need quite a bit of preparatory work. This will consist of definitions of the classes of densities and functionals over which our results will hold, the definitions of our weighted nearest neighbour estimators and the corresponding classes of allowable weights, as well as various parameters that will play a role in the statements of our results.

Starting with our classes of densities, for \( \beta > 0 \), a density \( f \) on \( \mathbb{R}^d \), and for \( x \in \mathbb{R}^d \) with \( f(x) > 0 \) and such that \( f \) is \( \beta := \lceil \beta \rceil - 1 \)-times differentiable at \( x \), we define

\[
M_{f,\beta}(x) := \inf \left\{ M \geq 1 : \max_{t \in \mathbb{N}} \left( \frac{\|f^{(t)}(x)\|}{f(x)} \right)^{1/t} \right. \\
\left. \sup_{y,z \in B_x(\{2d^{1/2}M\}^{-1})} \left( \frac{\|f^{(\beta)}(z) - f^{(\beta)}(y)\|}{f(x)} \right) \left( \frac{z - y}{\beta - \beta} \right)^{1/\beta} \right\};
\]

otherwise, we set \( M_{f,\beta}(x) := \infty \). The quantity \( M_{f,\beta}(x) \) measures the smoothness of derivatives of \( f \) in neighbourhoods of \( x \), relative to \( f(x) \) itself, but does not require \( f \) to be smooth everywhere. For instance, if \( f \) is the uniform density on the unit ball \( B_0(1) \), then \( M_{f,\beta}(x) = \{2d^{1/2}(1-\|x\|)\}^{-1} \) for \( \|x\| < 1 \). Now, for \( \theta = (\alpha, \beta, \lambda, C) \in (0, \infty)^4 \), let

\[
\mathcal{G}_{d,\theta} := \left\{ f \in \mathcal{F}_d : \mu_\alpha(f) \leq C, \|f\|_\infty \leq C, \int_X f(x) \left( \frac{M_{f,\beta}(x)}{f(x)} \right)^\lambda \, dx \leq C \right\}.
\]

Thus, in addition to requiring a moment assumption and a bounded density, the classes \( \mathcal{G}_{d,\theta} \)
also impose an integrability condition on our local measure of smoothness; it is an attractive feature that this condition comes in an integral form, as opposed to requiring a boundedness condition on \( M_{f,\beta}(x) \), for instance. This integrability condition is our primary tool for avoiding the assumption that the density is bounded away from zero on its support (see the discussion in the Introduction). While Tsybakov and van der Meulen (1996) and Berrett, Samworth and Yuan (2019) made first steps in this direction in the context of Shannon entropy estimation, the former of these works, which focused on the case \( d = 1 \), required a strictly positive density on the whole real line; the latter relaxed this condition a little, but made extremely stringent requirements on the behaviour of the density \( f \) in neighbourhoods of points \( x_0 \in \mathbb{R}^d \) with \( f(x_0) = 0 \). In particular, no Beta\((a, a)\) density was allowed, for any \( a > 0 \), and the only densities having points \( x_0 \) with \( f(x_0) = 0 \) that were shown to belong to their classes involved all derivatives also vanishing at \( x_0 \). By contrast, Proposition 1 below shows that a multivariate spherically symmetric generalisation of a Beta\((a, b)\) density belongs to \( G_{d,\theta} \) for suitable \( \theta \in (0, \infty)^4 \), provided only that \( a, b \geq 1 \) (though in fact the requirements of our Theorem 2 on efficiency would actually also need \( b > d - 1 \) for this family).

**Proposition 1.** Fix \( a, b \in [1, \infty) \), and let \( f \) denote the density on \( \mathbb{R}^d \) given by

\[
f(x) = C_{d,a,b} \|x\|^{a-1}(1 - \|x\|)^{b-1} 1_{\{\|x\| \leq 1\}},
\]

where \( C_{d,a,b} := \frac{\Gamma(a+b+d-1)}{dV_d(a+d-1)\Gamma(b)} \). Then for any \( \alpha, \beta > 0 \) and any \( \lambda \in (0, b/(b + d - 1)) \), there exists \( C_0 > 0 \), depending only on \( \alpha, \beta \) and \( \lambda \), such that \( f \in G_{d,\alpha,\beta,\lambda,C} \) for any \( C \geq C_0 \).

From Proposition 1 we also see that discontinuous densities may also belong to \( G_{d,\theta} \) for suitable \( \theta \in (0, \infty)^4 \); in particular, the \( U[-1, 1] \) density belongs to \( G_{d,\alpha,\beta,\lambda,C} \) for any \( \alpha, \beta > 0 \), \( \lambda \in (0, 1) \) and \( C \geq 1/(1 - \lambda) \). We also remark that, similar to Berrett, Samworth and Yuan (2019), all Gaussian densities belong to \( G_{d,\theta} \) for any \( \alpha, \beta > 0, \lambda \in (0, 1) \) and sufficiently large \( C > 0 \), and multivariate-\( t \) densities with \( \nu \) degrees of freedom belong to \( G_{d,\theta} \) for any \( \alpha \in (0, \nu) \), any \( \beta > 0, \lambda \in (0, \nu/(\nu + d)) \) and \( C > 0 \) sufficiently large.

To define our main class of densities, then, for \( \Theta = (0, \infty)^6 \) and \( \vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta \), let \( M_{\beta}(x) \equiv M_{f,g,\beta}(x) := M_{f,\beta}(x) \vee M_{g,\beta}(x) \) and set

\[
\mathcal{F}_{d,\theta} := \left\{ (f, g) \in G_{d,\alpha,\beta,\lambda_1,\lambda_2,C} \times \mathcal{F}_d : \mu_{C^{-1}}(g) \leq C, \|g\|_{\infty} \leq C, \int_{\mathcal{X}} f(x)^2 g(x)^{-\gamma} dx \leq C, \int_{\mathcal{X}} f(x) \left[ \left\{ \frac{M_{\beta}(x)^d}{f(x)} \right\}^{\lambda_1} + \left\{ \frac{M_{\beta}(x)^d}{g(x)} \right\}^{\lambda_2} \right] dx \leq C \right\}.
\]
To understand the first integrability condition in $\mathcal{F}_{d,\vartheta}$, we first note that, in the estimation of many two-sample functionals of interest, such as $\varphi$-divergences, it is natural to assume that the probability measure induced by $f$ is absolutely continuous with respect to the probability measure induced by $g$. We can think of the condition $\int_X f(x)^2 g(x)^{-\gamma} \, dx \leq C$ as imposing a quantitative absolute continuity requirement. The choice of the exponents is made so that we can bound $v_2$ defined in (9) below.

We now discuss the class of functionals that we consider, by imposing conditions on the function $\phi$ in (1). Let $\Xi := [0, \infty)^2 \times (0, \infty)^2 \times (1, \infty)$, and for $\xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi$, let $\Phi \equiv \Phi(\xi)$ denote the class of functions $\phi : Z \to \mathbb{R}$ for which

(i) writing $\beta_1^* := [\beta_1^*] - 1$ and $\beta_2^* := [\beta_2^*] - 1$, the partial derivative $\phi_{\beta_1^*, \beta_2^*}$ exists; moreover, for all $\epsilon = (\epsilon_1, \epsilon_2, 0) \in (-L^{-1}, L^{-1})^2 \times \{0\}$ and $z = (u, v, x) \in Z$, we have

$$\left| \phi(z + \epsilon \circ z) - \sum_{l_1 = 0}^{\beta_1^*} \sum_{l_2 = 0}^{\beta_2^*} \frac{(u \epsilon_1)^l_1 (v \epsilon_2)^l_2}{l_1! l_2!} \phi_{l_1, l_2}(z) \right| \leq L (u^{-\kappa_1} \vee u^L) (v^{-\kappa_2} \vee v^L) (|\epsilon_1|^{\beta_1^*} + |\epsilon_2|^{\beta_2^*});$$

(ii) for all $z = (u, v, x) \in Z$, $l_1 \in [\beta_1^*]$ and $l_2 \in [\beta_2^*]$, we have

$$\left| u^{l_1} v^{l_2} \phi_{l_1, l_2}(z) \right| \leq L (u^{-\kappa_1} \vee u^L) (v^{-\kappa_2} \vee v^L);$$

(iii) as $r \searrow 0$ we have

$$\sup_{\epsilon \in \{0\}^2 \times B_0(r)} \sup_{z \in Z} \left\{ \left| \frac{\phi(z + \epsilon)}{\phi(z)} - 1 \right|, \left| \frac{\phi_{10}(z + \epsilon)}{\phi_{10}(z)} - 1 \right|, \left| \frac{\phi_{01}(z + \epsilon)}{\phi_{01}(z)} - 1 \right| \right\} \to 0.$$

Condition (i) controls the Hölder regularity of $\phi$ of order $\beta_1^*$ in its first argument and $\beta_2^*$ in its second argument, with weaker regularity required for both small and large values of these first and second arguments. This latter aspect will allow us to include functionals such as the Kullback–Leibler and Rényi divergences, for which the corresponding $\phi$ is non-smooth as the densities approach zero. Condition (ii) is a growth condition on $\phi$ and its partial derivatives of order up to $\beta_1^*$ in the first argument and up to $\beta_2^*$ in the second. Condition (iii) is most easily understood in the special (but important) case where $\phi(u, v, x) = w(x) \bar{\phi}(u, v)$, for a weight function $w$ and another function $\bar{\phi}$. In that case, the condition amounts to asking that

$$\sup_{\epsilon \in (-r, r)} \sup_{x \in \mathbb{R}^d} \left| \frac{w(x + \epsilon)}{w(x)} - 1 \right| \to 0 \quad (6)$$

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as \( r \to 0 \). Examples of classes of functionals satisfying these conditions include \( \phi(u, v, x) = w(x) \log(u/v) \) and \( \phi(u, v, x) = w(x)(u/v)^{\kappa-1} \), for \( \kappa \in \mathbb{R} \) and any bounded weight function \( w \) satisfying (6). More generally, for the weighted \( \varphi \)-divergence functional with \( \phi(u, v, x) = w(x)\varphi(v/u) \), it is straightforward to express conditions (i), (ii) and (iii) in terms of conditions on \( w \) and \( \varphi \). As a final example, the functional \( \int_{\mathbb{R}^d} fg \) corresponds to the choice \( \phi(u, v, x) = v \), from which we see that \( \phi \in \Phi(0, 0, \beta, \beta, 1 + 1/\beta) \), for every \( \beta > 0 \).

We now introduce the class of weights that we consider for our estimators. To this end, for \( k, I \in \mathbb{N} \) and \( c \in (0, 1) \), define

\[
W_{I,c}^{(k)} := \left\{ w = (w_1, \ldots, w_k) \in \mathbb{R}^k : \sum_{j=1}^k w_j = 1 \text{ and } w_j = 0 \text{ for } j < ck, \right. \\
\left. \sum_{j=1}^k j^{2l-1}w_j = 0 \text{ for } (l, i) \in \left(\left[\frac{d}{2}\right] \times [I]\right) \setminus \{(0,0)\}\right\}. \tag{7}
\]

Fixing \( \xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi \) and \( c \in (0, 1) \), and for \( w_X \in W_{[\beta_1^*/2],c}^{(k_X)} \) and \( w_Y \in W_{[\beta_2^*/2],c}^{(k_Y)} \), we can now define our weighted functional estimators as

\[
\hat{T}_{m,n} = \hat{T}_{m,n}^{w_X,w_Y} := \sum_{j_X=1}^{k_X} \sum_{j_Y=1}^{k_Y} w_{X,j_X} w_{Y,j_Y} \hat{T}_{m,n,k_X,k_Y}. \tag{8}
\]

Note that the constraint on the support of \( w_X \) ensures that all component indices with non-zero weight are of the same order as \( k_X \), with the corresponding property also holding for \( w_Y \). Once this is satisfied, and given appropriate choices of \( k_X, k_Y \), the remaining two constraints in (7) will ensure that the bias of \( \hat{T}_{m,n} \) is asymptotically negligible.

For \( x \in \mathbb{R}^d \), it will be convenient to use the shorthand \( \phi_x := \phi(f(x), g(x), x), (f\phi_{10})_x := f(x)\phi(f(x), g(x), x) \) and \( (f\phi_{01})_x := f(x)\phi_{01}(f(x), g(x), x) \). Our result on the asymptotic risk of \( \hat{T}_{m,n} \) will be expressed in terms of

\[
v_1 = v_1(f, g) := \text{Var}(\phi_{X1} + (f\phi_{10})_{X1}) \quad \text{and} \quad v_2 = v_2(f, g) := \text{Var}((f\phi_{01})_{Y1}). \tag{9}
\]

Moreover, fixing \( d \in \mathbb{N}, \vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta \) and \( \xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi \), we will impose requirements on various derived parameters. In particular, it will also be convenient.
to define
\[
\zeta := \frac{\kappa_1}{\lambda_1} + \frac{\kappa_2}{\lambda_2} + \frac{d(\kappa_1 + \kappa_2)}{\gamma}\tag{10}
\]
\[
\tau_1 := 1 - \max\left(\frac{d}{2\beta}, \frac{d}{2(2 \wedge \beta) + d}, \frac{d}{4\beta_1}, \frac{1}{2\lambda_1(1 - \zeta)}\right)
\]
\[
\tau_2 := 1 - \max\left(\frac{d}{2\beta}, \frac{d}{2(2 \wedge \beta) + d}, \frac{d}{4\beta_2}, \frac{1}{2\lambda_2(1 - \zeta)}\right)
\]
\[
\gamma_* := \frac{(2\alpha + d)(1 + 2\kappa_2)}{2\alpha + d - 2(\alpha + d)\kappa_1}.
\]

Finally, then, we are in a position to state our first main result, on the asymptotic squared error risk of \(\widehat{T}_{m,n}\):

**Theorem 2.** Fix \(d \in \mathbb{N}_0\), \(\vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta\) and \(\xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi\). Assume that \(\zeta < 1/2\), that \(\tau_1 > 1/\beta_1^*\), that \(\tau_2 > 1/\beta_2^*\), and that \(\gamma > \gamma_* > 0\). Let \((k_X^1), (k_Y^1)\), \((k_X^U)\) and \((k_Y^U)\) be deterministic sequences of positive integers satisfying \(\min(k_X^1, m^{1/\beta_1^*}, k_Y^1, n^{1/\beta_2^*}) \to \infty\) and \(\max(k_X^U m^{-(r_1 - \epsilon)}, k_Y^U n^{-(r_2 - \epsilon)}) \to 0\) for some \(\epsilon > 0\). Then for each \(c \in (0,1)\), each \(w_X = w_X^{(k_X)} \in \mathcal{W}_{[2/2,c]}\) with \(\|w_X\|_1 \leq 1/c\), and each \(w_Y = w_Y^{(k_Y)} \in \mathcal{W}_{[2/2,c]}\) with \(\|w_Y\|_1 \leq 1/c\), we have

\[
\sup_{\phi \in \Psi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,\vartheta}} \max_{k_X \in \{k_X^1, \ldots, k_X^U\}} \max_{k_Y \in \{k_Y^1, \ldots, k_Y^U\}} \left| \mathbb{E}_{f,g} \left\{ (\widehat{T}_{m,n} - T)^2 \right\} - \frac{v_1}{m} - \frac{v_2}{n} \right| = o\left(\frac{1}{m} + \frac{1}{n}\right)
\]

as \(m, n \to \infty\).

Theorem 2 follows immediately from combining Proposition 5 in Section 3 with Proposition 9 in Section 4, which elucidate the asymptotic bias and variance of \(\widehat{T}_{m,n}\) respectively. We therefore defer a description of the main ideas of our proofs until after the statements of these results.

To study the asymptotic normality of \(\widehat{T}_{m,n}\), we impose a stronger condition on the pair \((f,g)\): for \(\vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta\), let

\[
\mathcal{F}_{d,\vartheta} := \left\{ (f,g) \in \mathcal{F}_{d,\vartheta} : \int_X f(x)^4 g(x)^{4\alpha + 3d} dx \leq C, v_1 \geq C^{-1}, v_2 \geq C^{-1} \right\}.
\]

The exponent of \(g(x)\) is chosen so that, by Hölder’s inequality, \(\mathcal{F}_{d,\vartheta} \subseteq \mathcal{F}_{d,\vartheta'}\) for all \(\theta' = (\alpha, \beta, \lambda_1, \lambda_2, \gamma', C')\) with \(\gamma' < \gamma\) and \(C' > 0\) sufficiently large. This additional condition provides control over the fourth moment of \(\widehat{T}_{m,n}\), which allows us to apply the crucial central limit theorem of Baldi and Rinott (1989).
To explain the lower bounds on \(v_1\) and \(v_2\) in (11), consider the setting in which \(\phi(u,v,x) = \varphi(v/u)\), as is the case with \(\varphi\)-divergences. Then, writing \(W := g(X_1)/f(X_1)\) and \(Z := g(Y_1)/f(Y_1)\) we have that

\[
v_1 = \text{Var} (\varphi(W) - W\varphi'(W)) \quad \text{and} \quad v_2 = \text{Var} (\varphi'(Z)).
\]

Now, if \(f = g\) then we have \(v_1 = v_2 = 0\), and \(\hat{T}_{m,n}\) will converge to \(T\) at a faster rate than \(m^{-1/2} + n^{-1/2}\) (with a potentially non-normal limiting distribution). Thus, in order to state uniform results on the asymptotic normality of \(\hat{T}_{m,n}\), we work over a class of densities for which \(v_1\) and \(v_2\) are bounded below. For two random variables \(X\) and \(Y\) with distribution functions \(F\) and \(G\) (where for later convenience we allow \(X\) and \(Y\) to take values in the extended real line), let

\[
d_K(\mathcal{L}(X), \mathcal{L}(Y)) := \sup_{t \in \mathbb{R}} |F(t) - G(t)|
\]
denote the Kolmogorov distance between the distributions of \(X\) and \(Y\).

**Theorem 3.** Suppose that the conditions of Theorem 2 hold. Writing

\[
\gamma_1^* := \frac{6\kappa_2(2\alpha + d)}{3\alpha + d - 6\kappa_1(\alpha + d)}, \quad \tau_1^* := 1 - \frac{3\alpha + 3d}{3\alpha + 3d + \{3\alpha + d - 6\kappa_1(\alpha + d)\}(1 - \gamma_1^*/\gamma)}
\]

\[
\gamma_2^* := \frac{3(1 + 2\kappa_2)(2\alpha + d)}{2\{4\alpha + 3d - (1 + 3\kappa_1)(\alpha + d)\}}, \quad \tau_2^* := 1 - \frac{1}{1 + (1 + 2\kappa_2)(\gamma/\gamma_2^* - 1)},
\]
suppose also that \(\min(\gamma_1^*, \gamma_2^*) > 0\), that \(\gamma > \max(\gamma_1^*, \gamma_2^*)\), and that, for some \(\epsilon > 0\), we have \(\max(k_X^U m^{-(\gamma_1^* - \epsilon)}, k_Y^U n^{-(\gamma_2^* - \epsilon)}) \to 0\). Then

\[
\sup_{\phi \in \Phi} \sup_{(f,g) \in \mathcal{F}_{d,\delta}} \max_{k_X \in \{k_X^1, \ldots, k_X^N\}, \ k_Y \in \{k_Y^1, \ldots, k_Y^N\}} d_K\left(\mathcal{L}\left(\frac{\hat{T}_{m,n} - T}{\{m^{-1}v_1 + n^{-1}v_2\}^{1/2}}\right), N(0,1)\right) \to 0
\]
as \(m, n \to \infty\).

To simplify the following discussion of the proof of Theorem 3, we introduce the shorthand

\[
\hat{f}_{(k_X),i} := \frac{k_X}{mV_{d\rho_{(k_X),i} d_X}} \quad \text{and} \quad \hat{g}_{(k_Y),i} := \frac{k_Y}{nV_{d\rho_{(k_Y),i} d_Y}}
\]

for \(i \in \{1, \ldots, m\}, k_X \in \{1, \ldots, m - 1\} \) and \(k_Y \in \{1, \ldots, n\} \). The proof is aided by the interesting observation (which can be derived from the calculations in the proof of our variance
expansion in Proposition 9) that under our conditions,

\[ \tilde{T}_{m,n} - \mathbb{E}(\tilde{T}_{m,n}) = \tilde{T}_m^{(1)} - \mathbb{E}(\tilde{T}_m^{(1)}) + \tilde{T}_n^{(2)} - \mathbb{E}(\tilde{T}_n^{(2)}) + o_p(m^{-1/2} + n^{-1/2}), \]

where

\[ \tilde{T}_m^{(1)} := \frac{1}{m} \sum_{i=1}^{m} \phi(\hat{f}(k_X),i, g(X_i), X_i) \quad \text{and} \quad \tilde{T}_n^{(2)} := \mathbb{E}\{\phi(f(X_1), \hat{g}(k_Y), 1, X_1) \mid Y_1, \ldots, Y_n\}. \]

Here, the crucial point is that \( \tilde{T}_m^{(1)} \) depends only on \( X_1, \ldots, X_m \) and \( \tilde{T}_n^{(2)} \) depends only on \( Y_1, \ldots, Y_n \).

Theorem 3 also facilitates the construction of asymptotically valid confidence intervals of asymptotically minimal width, provided we can find consistent estimators of \( v_1 \) and \( v_2 \). To this end, define

\[ \hat{V}_{m,n}^{(1),1} := \frac{1}{m} \sum_{i=1}^{m} \min\left\{ \phi(\hat{f}(k_X),i, \hat{g}(k_Y),i, X_i) + \hat{f}(k_X),i, \phi_0(\hat{f}(k_X),i, \hat{g}(k_Y),i, X_i) \right\}^2, \log m, \log n \]

\[ \hat{V}_{m,n}^{(1),2} := \hat{T}_{m,n} + \frac{1}{m} \sum_{i=1}^{m} \hat{f}(k_X),i, \phi_0(\hat{f}(k_X),i, \hat{g}(k_Y),i, X_i) \]

\[ \hat{V}_{m,n}^{(2),1} := \frac{1}{m} \sum_{i=1}^{m} \min\left\{ \hat{f}(k_X),i, \hat{g}(k_Y),i, \phi_0(\hat{f}(k_X),i, \hat{g}(k_Y),i, X_i)^2, \log m, \log n \right\} \]

\[ \hat{V}_{m,n}^{(2),2} := \frac{1}{m} \sum_{i=1}^{m} \hat{g}(k_Y),i, \phi_0(\hat{f}(k_X),i, \hat{g}(k_Y),i, X_i), \]

as well as \( \hat{V}_{m,n}^{(1)} := \max\{\hat{V}_{m,n}^{(1),1} - (\hat{V}_{m,n}^{(1),2})^2, 0\} \) and \( \hat{V}_{m,n}^{(2)} := \max\{\hat{V}_{m,n}^{(2),1} - (\hat{V}_{m,n}^{(2),2})^2, 0\} \). It turns out that \( \hat{V}_{m,n}^{(1)} \) and \( \hat{V}_{m,n}^{(2)} \) satisfy the consistency we seek, so, writing \( z_q \) for the \( (1 - q) \)th quantile of the standard normal distribution and

\[ I_{m,n,q} := \left[ \hat{T}_{m,n} - z_q/2 \{m^{-1}\hat{V}_{m,n}^{(1)} + n^{-1}\hat{V}_{m,n}^{(2)}\}^{1/2}, \hat{T}_{m,n} + z_q/2 \{m^{-1}\hat{V}_{m,n}^{(1)} + n^{-1}\hat{V}_{m,n}^{(2)}\}^{1/2} \right], \]

we have the following result.

**Theorem 4.** Suppose that the conditions of Theorem 3 hold. Then

\[ \sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{\alpha,\theta}} \max_{k_X \in \{k_X^1, \ldots, k_X^L\}} \max_{k_Y \in \{k_Y^1, \ldots, k_Y^L\}} d_K\left( \mathcal{L}\left( \frac{\hat{T}_{m,n} - T}{\{m^{-1}\hat{V}_{m,n}^{(1)} + n^{-1}\hat{V}_{m,n}^{(2)}\}^{1/2}} \right), N(0, 1) \right) \to 0 \]
as \( m, n \to \infty \). In particular,

\[
\sup_{q \in (0,1)} \sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,\delta}} \max_{k_X \in \{k_X^1, \ldots, k_X^U\}, k_Y \in \{k_Y^1, \ldots, k_Y^U\}} \left| \mathbb{P}(I_{m,n,q} \geq T(f,g)) - (1-q) \right| \to 0
\]

as \( m, n \to \infty \).

3 Bias

3.1 Bias of the naive estimator

Here we state a result on the bias of the estimator (4). It is in fact an immediate consequence of a more general statement, given as Proposition 12 in the online supplement, which considers a wider range of choices of \( k_X \) and \( k_Y \).

**Proposition 5.** Fix \( d \in \mathbb{N} \), \( \vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta \) and \( \xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi \). Assume that \( \zeta < 1/2 \), that \( \tau_1 > 1/\beta_1^* \) and that \( \tau_2 > 1/\beta_2^* \). Suppose further that \( \min(k_X^{-1/\beta_1^*}, k_Y^{-1/\beta_2^*}) \to \infty \) and that there exists \( \epsilon > 0 \) with \( \max(k_X^{-\tau_1+\epsilon}, k_Y^{-\tau_2+\epsilon}) \to 0 \). Then for each \( i_1, i_2 \in [[d/2]-1] \), \( j_1 \in [[\beta_1^*/2]] \) and \( j_2 \in [[\beta_2^*/2]] \), we can find coefficients \( \lambda_{i_1,i_2,j_1,j_2} \equiv \lambda_{i_1,i_2,j_1,j_2}(d,f,g,\phi) \), with the properties that \( \lambda_{0,0,0,0} = T(f,g) \), that

\[
\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,\delta}} |\lambda_{i_1,i_2,j_1,j_2}| < \infty,
\]

and that

\[
\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,\delta}} \sup_{k_X \in \{k_X^1, \ldots, k_X^U\}, k_Y \in \{k_Y^1, \ldots, k_Y^U\}} \left| \mathbb{E}_{f,g}(\tilde{T}_{m,n}) - \sum_{i_1,i_2=0}^{[d/2]-1} \sum_{j_1=0}^{[\beta_1^*/2]} \sum_{j_2=0}^{[\beta_2^*/2]} \frac{\lambda_{i_1,i_2,j_1,j_2}}{k_X^{n_i} k_Y^{n_j}} \left( \frac{k_X}{m} \right)^{\frac{2i}{n}} \left( \frac{k_Y}{n} \right)^{\frac{2j}{n}} \right| = o(m^{-1/2} + n^{-1/2})
\]

as \( m, n \to \infty \).

Proposition 5 provides conditions on the classes of densities and functionals under which we can give a uniform asymptotic expansion of the bias of \( \tilde{T}_{m,n} \), up to terms of negligible order. This expansion also holds uniformly over a range of values of \( k_X \) and \( k_Y \), which can be chosen adaptively (i.e. without knowledge of the parameters of the underlying densities) to satisfy the conditions of the theorem, e.g. by setting \( k_X = m^{1/\beta_1^*} \log m \) and \( k_Y = n^{1/\beta_2^*} \log n \).
As revealed by Corollary 6 below, Proposition 5 allows us to form weighted versions of the estimators $\tilde{T}_{m,n,k_X,k_Y}$, for different choices of $k_X$ and $k_Y$, so as to cancel the dominant terms in the expression for the bias of the naive estimator. Indeed, it was this result that motivated our choice of the class of weights that we consider in Theorem 2.

**Corollary 6.** Suppose that the conditions of Proposition 5 hold. Then for each $c \in (0,1)$, each $w_X = w^{(k_X)}_X \in W^{(k_X)}_{d/2,c}$ with $\|w_X\|_1 \leq 1/c$, and each $w_Y = w^{(k_Y)}_Y \in W^{(k_Y)}_{d/2,c}$ with $\|w_Y\|_1 \leq 1/c$, we have

$$\sup_{\phi \in \Phi(d)} \sup_{(f,g) \in F_{d,\theta}} \sup_{k_X \in \{k^*_1, \ldots, k^*_V\}} \sup_{k_Y \in \{k^*_1, \ldots, k^*_V\}} \left| E_{\phi}(\tilde{T}_{m,n}^{w_X,w_Y}) - T(f,g) \right| = o(m^{-1/2} + n^{-1/2})$$

as $m, n \to \infty$.

In order to gain intuition about the level of smoothness of the functional required in Corollary 6, it is helpful to consider the following (favourable) case: if our assumptions hold for all $\alpha, \beta, \gamma, \lambda_2 > 0$ and all $\lambda_1 < 1$, then it suffices that $\beta_1^* > \max\{1 + d/4, 2(1 - \kappa_1)\}$, that $\beta_2^* > 1 + d/4$, and that $\kappa_1 < 1/2$.

The key idea of our bias proofs is a truncation argument that partitions $\mathcal{X}$ as $\mathcal{X}_{m,n} \cup (\mathcal{X} \setminus \mathcal{X}_{m,n})$, where

$$\mathcal{X}_{m,n} := \left\{ x \in \mathcal{X} : \frac{f(x)}{M_\beta(x)^d} \geq \frac{k_X \log m}{m}, \frac{g(x)}{M_\beta(x)^d} \geq \frac{k_Y \log n}{n} \right\}.$$

By Lemma 16, we have that $f$ and $g$ are uniformly well-approximated in a relative sense, over balls of an appropriate radius, by their values at the centres of these balls; more precisely, for every $\vartheta \in \Theta$, and writing $A := (16d)^{1/(\beta - 2)}$ and $r_0(x) := \{AM_\beta(x)\}^{-1}$,

$$\sup_{(f,g) \in F_{d,\theta}} \sup_{y \in B_x(r_0(x))} \left| \frac{f(y)}{f(x)} - 1 \right| \left| \frac{g(y)}{g(x)} - 1 \right| \leq \frac{1}{2}.$$

In particular, this means that

$$\inf_{x \in \mathcal{X}_{m,n}} h_{x,f}(r_0(x)) \geq \frac{V_dr_X \log m}{2A^4m} \text{ and } \inf_{x \in \mathcal{X}_{m,n}} h_{x,g}(r_0(x)) \geq \frac{V_dr_Y \log n}{2A^4n} \quad (12)$$

whenever $(f,g) \in F_{d,\theta}$. Thus for each $x \in \mathcal{X}_{m,n}$, it is the case that with high probability, the $k_X$ nearest neighbours of $x$ among $X_1, \ldots, X_m$, as well as the $k_Y$ nearest neighbours of $x$ among $Y_1, \ldots, Y_n$, lie in $B_x(r_0(x))$. Moreover, the functions $h_{x,f}(\cdot)$ and $h_{x,g}(\cdot)$ can be
approximated by a Taylor expansion on $[0, r_0(x)]$. Since $h_{X_1,f}(\rho_X(i),X)|X_i \sim \text{Beta}(k, m-k)$ and $h_{X_1,g}(\rho_Y(i),Y)|X_i \sim \text{Beta}(k, n+1-k)$, these facts, in combination with (12), allow us to deduce a stochastic expansion for $\rho_X(i,X)$ and $\rho_Y(i,Y)$ in terms of powers of the relevant beta random variables. The contribution to the bias from the region $X_{m,n}$ can then be computed by a Taylor expansion of $\phi$ and using exact formulae for moments of beta random variables. For $x \in X \setminus X_{m,n}$, we have no guarantees about the proximity of the $k_X$ nearest neighbours of $x$ among $X_1, \ldots, X_m$, nor the $k_Y$ nearest neighbours of $x$ among $Y_1, \ldots, Y_n$; however,

$$P(X_1 \in X \setminus X_{m,n}) \leq C\left\{ \left( \frac{k_X \log m}{m} \right)^{\lambda_1} \sqrt{\left( \frac{k_Y \log n}{n} \right)^{\lambda_2}} \right\},$$

so the integrability conditions in our classes $F_{d,\vartheta}$ allow us to control the contribution to the bias from this region.

### 3.2 Bias of an alternative debiased estimator

As mentioned in the introduction, building on the original debiasing idea of Kozachenko and Leonenko (1987), Ganguly et al. (2018) proposed a debiasing technique for the naive estimator $\tilde{T}_{m,n}$ of a general two-sample functional. The initial goal of this subsection is to use fractional calculus techniques to give an informal study of the remaining bias of these resulting estimators, with a view to addressing the question of whether to apply our weighting scheme to the naive estimator (4) or that of Ganguly et al. (2018).

For simplicity we will focus on the one-sample setting in (2), where the function $\psi$ does not depend on its second argument, though all of the calculations have analogues in the two-sample setting and with additional $x$ dependence. Suppose that there exists a sequence of differentiable functions $(\psi_k)$ for which

$$\psi(u) = \int_0^\infty e^{-s} s^{k-1} \frac{k^\psi_k}{s} ds$$

for all $u \in (0, \infty)$; examples in the cases of Shannon and Rényi entropies will be given below. We will consider the debiased estimator of $H(f)$ given by

$$\tilde{H}_m := \frac{1}{m} \sum_{i=1}^m \psi_k(\hat{f}_k(i)).$$

Then, under regularity conditions on $f$ and $\psi_k$, since $m\text{Beta}(k, m-k)$ can be approximated
by a $\Gamma(k, 1)$ random variable, we have that

$$
\mathbb{E}\bar{H}_m = \int_X f(x) \int_0^1 \psi_k \left( \frac{k}{mV_d h_{x,f}(s)^d} \right) B_{k,m-k}(s) \, ds \, dx
$$

$$
\approx \int_X f(x) \int_0^1 \left\{ \psi_k \left( \frac{k f(x)}{m s} \right) - \frac{k f(x)}{m s} V_d f(x) h_{x,f}(s)^d \right\} B_{k,m-k}(s) \, ds \, dx
$$

$$
\approx \int_X f(x) \int_0^1 \psi_k \left( \frac{k f(x)}{t} \right) + \frac{kt^{2/d-1}}{2(d+2)V_d n^{2/d} f(x)^{2/d}} \psi_k \left( \frac{k f(x)}{t} \right) \frac{e^{-t k^{-1}}}{\Gamma(k)} \, dt \, dx
$$

$$
= H(f) + \frac{1}{2(d+2)V_d n^{2/d}} \int_X \frac{\Delta f(x)}{f(x)^{2/d-1}} \int_0^\infty e^{-t k^{2/d-2}} \frac{\psi_k \left( \frac{k f(x)}{t} \right)}{\Gamma(k)} \, dt \, dx. \quad (14)
$$

In order to understand the behaviour of the dominant bias term on the right-hand side of (14), for $\alpha \in [0, 1]$ define the operator $D^\alpha$ by

$$
(D^\alpha g)(u) := -\frac{1}{\Gamma(1-\alpha)} \int_u^\infty \frac{g'(s)}{(s-u)^\alpha} \, ds.
$$

This is closely related to the Caputo fractional derivative (Kilbas et al., 2006, Section 2.4). Then, with $g(s) = e^{-\lambda s}$ for some $\lambda \in (0, \infty)$, we have that

$$
(D^\alpha g)(u) = \frac{1}{\Gamma(1-\alpha)} \int_u^\infty \frac{\lambda e^{-\lambda s}}{(s-u)^\alpha} \, ds = \lambda^\alpha e^{-\lambda u} = \lambda^\alpha g(u).
$$

From (13) we can see that

$$
\frac{\Gamma(k-1)}{u^{k-1}} \psi'(u) = u^{-(k-1)} \int_0^\infty e^{-t k^{k-2}} \psi_k \left( \frac{k u}{t} \right) \, dt = \int_0^\infty e^{-s u} s^{k-2} \psi_k \left( \frac{k u}{s} \right) \, ds. \quad (15)
$$

When $d \geq 3$, we can apply the operator $D^{2/d}$ to both sides of (15) to simplify the inner integral in our expression for the dominant bias term in (14) as follows:

$$
\frac{1}{\Gamma(k-1)} \int_0^\infty e^{-t k^{k+2/d-2}} \psi_k \left( \frac{k u}{t} \right) \, dt = \frac{u^{k+2/d-1}}{\Gamma(k-1)} \int_0^\infty e^{-s u} s^{k+2/d-2} \psi_k \left( \frac{k u}{s} \right) \, ds
$$

$$
= -\frac{u^{k+2/d-1}}{\Gamma(1-2/d)} \int_u^\infty \frac{d}{ds} \left( \psi'(s) / s^{k-1} \right) \, ds = \frac{u^{k+2/d-1}}{\Gamma(1-2/d)} \int_u^\infty \frac{(k-1)s^{-k} \psi'(s) - s^{-k} \psi''(s) \, ds}{(s-u)^{2/d}}
$$

$$
= \frac{\Gamma(k+2/d-1)}{\Gamma(k-1)} \int_0^1 B_{1-2/d,k+2/d-1}(s) \left\{ \psi'(\frac{u}{1-s}) - \frac{u}{(k-1)(1-s)} \psi'' \left( \frac{u}{1-s} \right) \right\} \, ds. \quad (16)
$$

For Shannon and Rényi entropies, both $\psi'$ and $\psi''$ are constant multiples of functions $g$
with the property that \( g(xy) = g(x)g(y) \) for any \( x,y \in (0, \infty) \). In these cases, the leading order bias separates into a coefficient depending only on \( d, n \) and \( f \) and a factor which is a function of \( k \). Using weights, this leading order bias may be cancelled out, and it can be seen that, when \( f \) is sufficiently regular, the next term is of order \( k^{4/d} / n^{4/d} \). However, the only continuous functions \( g \) with this property are \( g(x) = x^a \) for some \( a \in \mathbb{R} \) (e.g. Dieudonné, 1969, (4.3.7), p. 86). If the term in braces in (16) is separable for all values of \( k \) then both \( u \mapsto \psi'(u) \) and \( u \mapsto u \psi''(u) \) must be separable individually, and so \( \psi'(u) \propto u^a \) for some \( a \in \mathbb{R} \). Thus the Shannon and Rényi entropies are the only functionals with this property.

In general, all that can be said is that this term in the bias can be expanded as a series of the form \( k^{2/d} / n^{2/d} (c_0 + c_1 k^{-1} + c_2 k^{-2} + \ldots) \). For larger values of \( d \), to cancel out sufficient bias that the resulting estimator is efficient, the weighting scheme is then marginally simpler than the weighting scheme for the naive estimator, and the analysis is significantly more complicated.

Despite the general conclusion of our discussion in the previous paragraph, returning to the two-sample functional setting, we now show that in the special case of the Kullback–Leibler and Rényi divergence functionals, the debiasing scheme described above significantly simplifies the weighting scheme, while facilitating the same conclusions regarding efficiency. To this end, for the Kullback–Leibler divergence, we define the following class of weight vectors:

\[
\mathcal{W}_{c}^{(k),KL} := \left\{ w = (w_1, \ldots, w_k) \in \mathbb{R}^k : \sum_{j=1}^{k} w_j = 1 \text{ and } w_j = 0 \text{ for } j < ck, \right. \\
\left. \sum_{j=1}^{k} \frac{\Gamma(j + 2l/d)}{\Gamma(j)} w_j = 0 \text{ for } l \in \left[\lceil d / 2 \rceil - 1 \right] \setminus \{0\} \right\}.
\]

The analogue of the Kozachenko–Leonenko debiased estimator is

\[
\tilde{D}_{m,n} := \frac{1}{m} \sum_{i=1}^{m} \log \left( \frac{e^{\psi(k_X)}}{m_{\rho^d(k_X),i,X}} \frac{n_{\rho^d(k_Y),i,Y}}{e^{\psi(k_Y)}} \right) = \tilde{T}_{m,n} + \Psi(k_X) - \log k_X - \Psi(k_Y) + \log k_Y
\]

(Ganguly et al., 2018). If the weighted estimator \( \hat{D}_{m,n}^{w_X,w_Y} \) is then formed as in (8) then the following theorem elucidates its asymptotic bias. Since this result uses very similar (in fact, somewhat simpler) arguments to those in Proposition 12, its proof, together with that of Proposition 8 below, is omitted for brevity.

**Proposition 7.** Fix \( d \in \mathbb{N} \), let \( \vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta \) and let \( \phi(u, v, x) = \log(u/v) \).
Assume that

$$\tau_1 = 1 - \max\left(\frac{d}{2\beta^+}, \frac{1}{2\lambda_1}\right) > 0 \quad \text{and} \quad \tau_2 = 1 - \max\left(\frac{d}{2\beta^+}, \frac{1}{2\lambda_2}\right) > 0,$$

and that there exists $\epsilon > 0$ such that $\max(k_X^U m^{-\tau_1+\epsilon}, k_Y^U n^{-\tau_2+\epsilon}) \to 0$. Then for each $c \in (0, 1)$, each $w_X = w_X^{(k_X)} \in W_{c(k_X),KL}$ with $\|w_X\|_1 \leq 1/c$, and each $w_Y = w_Y^{(k_Y)} \in W_{c(k_Y),KL}$ with $\|w_Y\|_1 \leq 1/c$, we have

$$\sup_{(f,g) \in F_{d,\vartheta}} \sup_{k_X \in \{1, \ldots, k^U_X\}} \sup_{k_Y \in \{1, \ldots, k^U_Y\}} \left| \mathbb{E}_{f,g}(\tilde{D}_{m,n}^{w_X,w_Y}) - T(f,g) \right| = o(m^{-1/2} + n^{-1/2})$$

as $m, n \to \infty$.

Since $\tilde{D}_{m,n}$ is simply a deterministic translation of $\tilde{T}_{m,n}$, our variance results in Section 4 continue to hold, so the corresponding efficiency result for $\tilde{D}_{m,n}^{w_X,w_Y}$ is immediate.

When estimating the Rényi integral $\int_X f^\vartheta g^{-(\kappa-1)}$, for $b \in \mathbb{R}$ and $c > 0$, we define

$$W_{b,c}^{(k)} := \left\{ w = (w_1, \ldots, w_k) \in \mathbb{R}^k : \sum_{j=1}^k w_j = 1 \text{ and } w_j = 0 \text{ for } j < ck, \right. \left. \sum_{j=1}^k \frac{\Gamma(j-b+2l/d)}{\Gamma(j-b)} w_j = 0 \text{ for } l \in \left[\lfloor d/2 \rfloor - 1 \right] \setminus \{0\} \right\}.$$

The corresponding debiased estimator is

$$\tilde{D}_{m,n} := \frac{1}{m} \sum_{i=1}^m \frac{\Gamma(k_X)\Gamma(k_Y)}{\Gamma(k_X-\kappa+1)\Gamma(k_Y+\kappa-1)} \left( \frac{n\rho^d_{(k_Y),i,Y}}{m\rho^d_{(k_X),i,X}} \right)^{\kappa-1}$$

(Ganguly et al., 2018). If the weighted estimator $\tilde{D}_{m,n}^{w_X,w_Y}$ is again formed as in (8) then the following result provides the corresponding bias guarantee.

**Proposition 8.** Fix $d \in \mathbb{N}$, let $\vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta$ and let $\phi(u,v,x) = (u/v)^{\kappa-1}$ for some $\kappa \in (1/2, \infty)$. With $\zeta$ is defined as in (10) with $\kappa_1 = (\kappa-1)_-$ and $\kappa_2 = (\kappa-1)_+$, assume that $\zeta < 1/2$,

$$\tau_1 = 1 - \max\left(\frac{d}{2\beta^+}, \frac{1}{2\lambda_1(1-\zeta)}\right) > 0 \quad \text{and} \quad \tau_2 = 1 - \max\left(\frac{d}{2\beta^+}, \frac{1}{2\lambda_2(1-\zeta)}\right) > 0.$$
Suppose further that there exists \( \epsilon > 0 \) such that \( \max(k_X^U m^{-\tau_1+\epsilon}, k_Y^U n^{-\tau_2+\epsilon}) \to 0 \). Then for each \( c \in (0,1) \), each \( w_X = w_X^{(k_X)} \in \mathcal{W}_{\kappa-1,c}^{(k_X)} \) with \( \|w_X\|_1 \leq 1/c \), and each \( w_Y = w_Y^{(k_Y)} \in \mathcal{W}_{1-\kappa,c}^{(k_Y)} \) with \( \|w_Y\|_1 \leq 1/c \), we have

\[
\sup_{(f,g) \in \mathcal{F}_{d,\vartheta}} \sup_{\begin{array}{l} k_X \in \{1, \ldots, k_X^U \} \\ k_Y \in \{1, \ldots, k_Y^U \} \end{array}} \left| \mathbb{E}_{f,g}(\hat{D}_{m,n}^{w_X,awY}) - T(f,g) \right| = o(m^{-1/2} + n^{-1/2})
\]

as \( m, n \to \infty \).

In this case, with \( k_X^I \) and \( k_Y^I \) defined as in Theorem 2, we have

\[
\sup_{\begin{array}{l} k_X \in \{k_X^I, \ldots, k_X^U \} \\ k_Y \in \{k_Y^I, \ldots, k_Y^U \} \end{array}} \left| \frac{\hat{D}_{m,n} - 1}{T_{m,n}} \right| = \sup_{\begin{array}{l} k_X \in \{k_X^I, \ldots, k_X^U \} \\ k_Y \in \{k_Y^I, \ldots, k_Y^U \} \end{array}} \left| \frac{k_X^{1-\kappa}\Gamma(k_X)k_Y^{-1}\Gamma(k_Y)}{\Gamma(k_X - \kappa + 1)\Gamma(k_Y + \kappa - 1)} - 1 \right| \to 0,
\]

so we can again deduce an efficiency result for \( \hat{D}_{m,n}^{w_X,awY} \).

4 Variance

The following result provides the main asymptotic variance expansion for our weighted estimators.

**Proposition 9.** Fix \( d \in \mathbb{N} \), \( \vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta \) and \( \xi = (\kappa_1, \kappa_2, \beta_1^* \gamma_1, \beta_2 \gamma_2, L) \in \Xi \). Assume that \( \zeta < 1/2 \), that \( \tau_1 > 1/\beta_1^* \), that \( \tau_2 > 1/\beta_2^* \), and that \( \gamma > \gamma_\ast \). Let \( (k_X^I), (k_Y^I), (k_X^U) \) and \( (k_Y^U) \) be deterministic sequences of positive integers satisfying \( \min(k_X^I / \log^5 m, k_Y^I \log^5 n) \to \infty \) and \( \max(k_X^I m^{-\tau_1-\epsilon}, k_Y^U n^{-\tau_2-\epsilon}) \to 0 \) for some \( \epsilon > 0 \). Then for each \( c \in (0,1) \), each \( w_X = w_X^{(k_X)} \in \mathcal{W}_{\beta_1^*/2,c}^{(k_X)} \) with \( \|w_X\|_1 \leq 1/c \), and each \( w_Y = w_Y^{(k_Y)} \in \mathcal{W}_{\beta_2/2,c}^{(k_Y)} \) with \( \|w_Y\|_1 \leq 1/c \), we have

\[
\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,\vartheta}} \max_{\begin{array}{l} k_X \in \{k_X^I, \ldots, k_X^U \} \\ k_Y \in \{k_Y^I, \ldots, k_Y^U \} \end{array}} \left| \text{Var}_{f,g}(\hat{D}_{m,n}^{w_X,awY}) - \frac{v_1}{m} - \frac{v_2}{n} \right| = o\left( \frac{1}{m} + \frac{1}{n} \right)
\]

as \( m, n \to \infty \).

Comparing the conditions of Proposition 9 with those of Corollary 6, we see that there is an additional lower bound on \( \gamma \) here, which ensures that \( v_2 \) is finite.

The proof of Proposition 9 is significantly more complicated than those of the bias proofs in Section 3, primarily owing to the need to consider the joint distribution of nearest neigh-
bour distances around two different points, $X_1$ and $X_2$, say. These have an intricate dependence structure because, for instance, $X_1$ may be one of the five nearest neighbours of $X_2$, but not vice-versa. To describe our main strategy for approximating the $\text{Var}_{f,g}(\hat{T}_{m,n}^{w_X,w_Y})$, we write $\hat{T}_{m,n}^{w_X,w_Y} = m^{-1} \sum_{i=1}^{m} \hat{T}_{m,n}^{(i)}$ as shorthand, so that

$$\text{Var}_{f,g}(\hat{T}_{m,n}^{w_X,w_Y}) = \frac{1}{m} \text{Var}_{f,g}(\hat{T}_{m,n}^{(1)}) + \frac{m-1}{m} \text{Cov}_{f,g}(\hat{T}_{m,n}^{(1)}, \hat{T}_{m,n}^{(2)}).$$

(17)

Using similar techniques to those employed in Section 3, it can be shown that $\text{Var}_{f,g}(\hat{T}_{m,n}^{(1)}) \to \text{Var}_f \phi_{X_1}$. For the covariance term in (17), we first condition on $X_1$ and $X_2$. It turns out that this term can be further decomposed into a sum of two terms, representing the contributions from the events on which $X_1$ and $X_2$ either share or do not share nearest neighbours. Observe that if $\|X_1 - X_2\| > \left\{ \frac{k_X}{mV_d} \left( 1 + \frac{\log^{1/2} m}{k_X^{1/2}} \right) \right\}^{1/d} \left\{ f(X_1)^{-1/d} + f(X_2)^{-1/d} \right\} =: R(X_1, X_2)$, say, then, with high probability, $X_1$ and $X_2$ do not share any of their $k_X$ nearest neighbours among $X_3, \ldots, X_m$. This means that the random vector $(h_{X_1}(\rho(k_X),1,X), h_{X_2}(\rho(k_X),2,X), 1 - h_{X_1}(\rho(k_X),1,X) - h_{X_2}(\rho(k_X),2,X))$ has approximately the same distribution as $(Z_1, Z_2, Z_3)$, say, where $(Z_1, Z_2, Z_3) \sim \text{Dirichlet}(k_X, k_X, m - 2k_X - 1)$. Writing $\| \cdot \|_{TV}$ for the total variation norm on signed measures, we can then exploit the facts that

$$\|\mathcal{L}(Z_1, Z_2) - \text{Beta}(k_X, m - k_X) \otimes \text{Beta}(k_X, m - k_X)\|_{TV} = O(k_X/m)$$

and

$$\frac{\hat{f}(k_X,1)}{f(X_1)} = 1 + O_p(k_X^{-1/2})$$

(18)

to show that the contribution to the covariance from this region is $O(1/m)$ (where in fact we also determine the leading constant). On the other hand,

$$\mathbb{P}\left[\left\{ \|X_1 - X_2\| \leq R(X_1, X_2) \right\} \cap \{X_1 \in X_{m,n}\} \right] = O(k_X/m),$$

and this, together with (18) again, allows us to demonstrate that the contribution to the covariance from this region due to the nearest neighbour distances among $X_3, \ldots, X_m$ is also

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$O(1/m)$ (with a different leading constant). The terms arising from the nearest neighbour distances of $Y_1, \ldots, Y_n$ from $X_1$ and $X_2$ can be handled similarly, and their contributions can be shown to be $O(1/n)$. Combining these dominant terms results in the expansion

$$\text{Cov}_{f,g}(\hat{T}_{m,n}^{(1)}, \hat{T}_{m,n}^{(2)}) = \frac{2}{m} \text{Cov}_f(\phi_{X_1}, (f \phi_{10})_{X_1}) + \frac{1}{m} \text{Var}_f((f \phi_{10})_{X_1}) + \frac{v_2}{n} + o\left(\frac{1}{m} + \frac{1}{n}\right),$$

and the conclusion follows.

### 5 The super-oracle phenomenon

Given access to \{f(X_1), \ldots, f(X_m), g(X_1), \ldots, g(X_m)\}, the natural oracle estimator of the functional $T(f,g)$ in (1) is given by

$$T^*_m := \frac{1}{m} \sum_{i=1}^m \phi(f(X_i), g(X_i), X_i).$$

This is unbiased, and moreover, $m^{1/2}(T^*_m - T) \overset{d}{\to} N(0, \sigma^2)$, where $\sigma^2 = \sigma^2(f,g) := \text{Var}_f \phi_{X_1}$.

The following result, which is an immediate consequence of Theorem 2, compares the asymptotic worst-case squared error risks of $\hat{T}_{m,n}$ and $T^*_m$. We first define a slight modification of the class $\mathcal{F}_{d,\vartheta}$, by setting

$$\mathcal{F}^*_{d,\vartheta} := \{(f,g) \in \mathcal{F}_{d,\vartheta} : \min(v_1, v_2) \geq C^{-1}\}.$$

**Theorem 10.** Assume the conditions of Theorem 2. Then

$$\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}^*_{d,\vartheta}} \max_{\{k_X \in \{k^{(1)}_X, \ldots, k^{(U)}_X\} \}} \frac{\mathbb{E}_{f,g}\{(\hat{T}_{m,n} - T)^2\}}{\mathbb{E}_f\{(T^*_m - T)^2\}} \cdot \frac{\sigma^2/m}{v_1/m + v_2/n} \rightarrow 1$$

as $m, n \to \infty$.

To understand the implications of this theorem, consider the case where $n$ is at least of the same order as $m$, so that $A := \limsup_{n \to \infty} m/n \in [0, \infty)$. If $\sigma^2/(v_1 + Av_2) > 1$, then the worst-case risk of $\hat{T}_{m,n}$ is asymptotically better than that of $T^*_m$, and we have an illustration of the super-oracle phenomenon. The one-sample functional (2) corresponds to $A = 0$, and we saw in the Introduction that the Rényi-type functional $\int_{\mathbb{R}^d} f(x)^\kappa \, dx$ with $\kappa \in (1/2, 1)$
provides such an example. In general, the phenomenon occurs if and only if

\[ 2\text{Cov}_f(\phi_{X_1}, (f\phi_{10})_{X_1}) < -\text{Var}_f(f\phi_{10})_{X_1} - Av_2. \]

6 A local asymptotic minimax lower bound

Before we can state our local asymptotic minimax result we require some further assumptions on the function \( \phi \). Extending our previous notation, for a smooth function \( \phi : Z \to \mathbb{R}, z = (u, v, x) \in Z \) and \( l_1, l_2, l_3 \in \mathbb{N} \), we write \( \phi_{l_1l_2l_3}(z) := \frac{\partial^{l_1+l_2+l_3}\phi}{\partial u^{l_1}\partial v^{l_2}\partial x^{l_3}} \). For \( \xi = (\kappa_1, \kappa_2, \beta^*_1, \beta^*_2, L) \in \Xi \) and \( \beta^*_3 > 0 \) let \( \Phi(\xi, \beta^*_3) \) denote the subset of \( \Phi(\xi) \) consisting of those \( \phi \) for which

(i) writing \( \beta^-_3 := [\beta^*_3] - 1 \), the partial derivative \( \phi_{\beta^-_1, \beta^-_2, \beta^-_3} \) exists; moreover, for all \( z = (u, v, x) \in Z, l_1 \in [\beta^*_1] \) and \( l_3 \in [\beta^*_3] \) we have

\[
\max_{l_2 \in [\beta^*_2]} \frac{u^{l_1}v^{l_2}|\phi_{l_1l_2l_3}(z)|}{|\phi(z) + u\phi_{10}(z)|} > 1 \leq L
\]

(ii) for all \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in (-L^{-1}, L^{-1})^2 \times \mathbb{R}^d, z = (u, v, x) \in Z, l_1 \in [\beta^*_1] \) and \( l_3 \in [\beta^*_3] \), we have

\[
\frac{u^{l_1}v^{l_2} \phi_{l_1l_2l_3}(z + \epsilon) - \phi_{l_1l_2l_3}(z)}{|\phi(z) + u\phi_{10}(z)|} \leq L(\epsilon^1_1/u^1_l \beta^-_3 - l_1 \wedge 1) + |\epsilon^2_2/u^2_l \beta^-_3 - l_2 \wedge 1| + \|\epsilon^3_3\| \beta^-_3 - l_3 \wedge 1)
\]

for all \( l_2 \in [\beta^*_2] \);

\[
\frac{u^{l_1}v^{l_2} \phi_{l_1l_2l_3}(z + \epsilon) - \phi_{l_1l_2l_3}(z)}{(u|\phi_{01}(z)|)} \leq L(\epsilon^1_1/u^1_l \beta^-_3 - l_1 \wedge 1) + |\epsilon^2_2/u^2_l \beta^-_3 - l_2 \wedge 1| + \|\epsilon^3_3\| \beta^-_3 - l_3 \wedge 1)
\]

for all \( l_2 \in [\beta^*_2] \). \( \wedge 1 \)

To understand these conditions it is instructive to consider the case of \( \varphi \)-divergences, for which \( \phi(u, v, x) = \varphi(v/u) \) for some function \( \varphi \). Here, (i) reduces to requiring that

\[
\sup_{w > 0} \left\{ \max_{l \in [m]} \frac{u^l |\varphi(l)(w)|}{|\varphi(w) - w\varphi'(w)|} \bigg|> 1 \max_{l \in [m] \setminus \{0\}} \frac{u^{l-1} |\varphi(l)(w)|}{|\varphi'(w)|} \bigg| \right\} < \infty,
\]

and a similar reduction holds for (ii). This is satisfied for the Kullback–Leibler divergence and all Rényi divergences. Moreover, when \( \phi(u, v, x) = v \), we have \( \phi \in \Phi(0, 0, \beta, \beta, 1+1/\beta, \beta) \) for every \( \beta > 0 \).
Now fix \((f, g) \in \mathcal{F}_d^2\) and \(\phi : \mathcal{Z} \to \mathbb{R}\) and define the functions
\[
\begin{align*}
h_1(x) &:= \phi_x + (f \phi_{10})_x - \mathbb{E}\{\phi_{X_1} + (f \phi_{10})_{X_1}\} \\
h_2(x) &:= (f \phi_{01})_x - \mathbb{E}\{(f \phi_{01})_{Y_1}\}.
\end{align*}
\]

This enables us to define, for each \(t = (t_1, t_2) \in \mathbb{R}^2\), the densities
\[
f_{t_1}(x) := c_1(t_1)K(t_1h_1(x))f(x) \quad \text{and} \quad g_{t_2}(x) := c_2(t_2)K(t_2h_2(x))g(x),
\]
where \(K(t) := 1/2 + (1 + e^{-4t})^{-1}\) and \(c_1(\cdot), c_2(\cdot)\) are normalising constants. Our choice of \(K\) is made so that \(K(0) = K'(0) = 1\), that \(K\) is smooth, and that \(K\) is bounded above and below by positive constants. Now, for each \(t = (t_1, t_2) \in \mathbb{R}^2\) we define the sequence of probability measures \((P_{n,t})\) on \(\mathbb{R}^{(m+n) \times d}\) so that \(P_{n,t}\) has density \(f_{m-1/2t_1} \otimes g_{n-1/2t_2}\) (here we think of \(m\) as a function of \(n\)). It turns out that the family \(\{P_{n,t} : t \in \mathbb{R}^2\}\) constitutes a least favourable parametric sub-model for this estimation problem. For an arbitrary probability measure \(P\) on \(\mathbb{R}^{(m+n) \times d}\), we write \(\mathbb{E}_P\) to denote expectation over \((X_1, \ldots, X_m, Y_1, \ldots, Y_n)^T \sim P\) and say that \((T_{m,n})\) is an estimator sequence if \(T_{m,n} : \mathbb{R}^{(m+n) \times d} \to \mathbb{R}\) is a measurable function for each \(m, n \in \mathbb{N}\).

We are now in a position to state our local asymptotic minimax lower bound.

**Theorem 11.** Fix \(d \in \mathbb{N}\), \(\vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta\), \(\xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi\) and \(\beta_3^* > 0\).

Let \(m = m_n\) be any sequence of positive integers such that \(m/n \to A\) for some \(A \in [0, \infty]\), let \((f, g) \in \mathcal{F}_d\), let \(\phi \in \Phi(\xi, \beta_3^*)\) and let \((T_{m,n})\) be any estimator sequence. Then, writing \(\mathcal{I}\) for the set of finite subsets of \(\mathbb{R}^2\), we have that
\[
\sup_{I \in \mathcal{I}} \liminf_{n \to \infty} \max_{t \in I} \mathbb{E}_{P_{n,t}} \left[ \frac{(T_{m,n} - T(f, g))^2}{m^{-1}v_1(f, g) + n^{-1}v_2(f, g)} \right] \geq 1.
\]

Moreover, there exists \(t_0 = t_0(d, \vartheta, \xi, \beta_3^*) > 0\) such that, for any \(t_1, t_2 \in (-t_0, t_0)\), we have \((f_{t_1} \otimes g_{t_2}) \in \mathcal{F}_{d, \tilde{\vartheta}}\), where \(\tilde{\vartheta} = (\alpha, \tilde{\beta}, \lambda_1, \lambda_2, \gamma, C/t_0)\) and \(\tilde{\beta} := \min\{\beta, (1 \land \beta)(\beta_1^* - 1), (1 \land \beta)(\beta_2^* - 1), \beta_3^*\}\).

This lower bound justifies the claim that suitably chosen versions of our weighted nearest neighbour estimator (8) are efficient over these classes of densities and functionals.

We remark that Theorem 11 may be extended to broader classes of loss functions, namely those that have closed, convex, symmetric sub-level sets; see van der Vaart and Wellner (1996, Theorem 3.11.5) for details. Finally, Theorem 3 allows us to extend our results on the worst-
case risk of our estimators $\tilde{T}_{m,n}$ to $\ell_q$-losses with $q \in (0, 2)$. The combination of these results implies that our estimators are asymptotically optimal for these $\ell_q$ losses too; we omit formal statements for brevity.

7 Appendix

Throughout our proofs we will use the notation

$$u_{x,s} := \frac{k_X}{mV_d h_{x,f}(s)^d} \quad \text{and} \quad v_{x,t} := \frac{k_Y}{nV_d h_{x,g}(t)^d},$$

for $x \in X$ and $s, t \in (0, 1)$. Moreover, since many of our error terms will depend on $k_X$, $k_Y$, $f$, $g$ and $\phi$ (as well as $q$, in Theorem 4), we adopt the convention, without further comment, that all of these error bounds hold uniformly over the relevant sets as claimed in the statements of the results. In addition, when we write $a \lesssim b$, we mean that there exists $C > 0$, depending only on the parameters $d, \vartheta$ and $\xi$ of the problem, such that $a \leq Cb$. It will be convenient throughout to assume that $m, n \geq 3$.

7.1 Proof of Proposition 1

Proof of Proposition 1. First, we have that $\mu_\alpha(f) \leq 1$ and $\|f\|_\infty \leq C_{d,a,b}$, and it remains to bound the function $M_{f,\beta}(\cdot)$ for each $\beta > 0$. Writing $g(r) := C_{d,a,b} r^{a-1} (1 - r)^{b-1} \mathbb{1}_{\{r \leq 1\}}$, so that $f(x) = g(\|x\|)$ we may see by induction that

$$\sup_{r \in (0,2/3)} r^{-(a-t-1)}|g^{(t)}(r)| < \infty \quad \text{and} \quad \sup_{r \in (1/3,1)} r^{-(b-t-1)}|g^{(t)}(r)| < \infty$$

for any $t \in \mathbb{N}$. Moreover, for any $t \in \mathbb{N}$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with $|\alpha| = t$, we have that

$$\sup_{x \in B_0(2/3)} \|x\|^t |\partial^\alpha \|x\|| < \infty \quad \text{and} \quad \sup_{x \in B_0(1) \setminus B_0(1/3)} |\partial^\alpha \|x\|| < \infty.$$ 

Using these facts we have that

$$|\partial^\alpha f(x)| \lesssim_{d,t,a,b} \frac{f(x)}{\|x\| \|x\|^t (1 - \|x\|)^t}.$$ 

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for any $t \in \mathbb{N}$. Now, writing $\underline{\beta} := \lfloor \beta \rfloor - 1$ and fixing $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with $|\underline{\alpha}| = \underline{\beta}$, if $y, z \in B_x(\|x\|(1 - \|x\|)/8)$ then we have for any some $w$ on the line segment between $x$ and $y$ that

$$\left| \partial^{\underline{\alpha}} f(z) - \partial^{\underline{\alpha}} f(y) \right| \leq d^{1/2} \|z - y\| \|f^{(\underline{\beta} + 1)}(w)\|$$

$$\lesssim_{d,\beta,a,b} \|z - y\| w^{a-1-(\beta+1)}(1 - \|w\|)^{b-1-(\beta+1)}$$

$$\lesssim_{d,\beta,a,b} \|f(y)\| z - y \|^{\beta-\beta} \frac{\|z - y\|^{\beta+1-\beta}}{\|x\|^2(1 - \|x\|)^{\beta+1}}$$

It follows that $M_{f,\beta}(x) \lesssim_{d,\beta,a,b} \|x\|^{-1}(1 - \|x\|)^{-1}$. Therefore, for any $\lambda \in (0, b/(b + d - 1))$, we have

$$\int_{B_0(1)} f(x) \left\{ \frac{M_{f,\beta}(x)^d}{f(x)} \right\}^\lambda \, dx \lesssim_{d,\beta,a,b} \int_{B_0(1)} \|x\|^{a-1}(1 - \|x\|)^{b-1} \left\{ \frac{1}{\|x\|^{a+d-1}(1 - \|x\|)^{b+d-1}} \right\}^\lambda \, dx$$

$$= dV_d \int_0^1 r^{a+d-2-\lambda(a+d-1)}(1 - r)^{b-1-\lambda(b+d-1)} \, dr < \infty,$$

as claimed.

\[ \Box \]

### 7.2 Proof of Proposition 5 on asymptotic bias

The following general result on the bias of the naive estimator $\tilde{T}_{m,n}$ yields Proposition 5 as an immediate consequence.

**Proposition 12.** Fix $d \in \mathbb{N}$, $\varnothing = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta$ and $\zeta = (\kappa_1, \kappa_2, \beta_1, \beta_2, L) \in \Xi$. Let $k_X^\lambda \leq k_X^\beta \leq k_Y^\beta$ be deterministic sequences of positive integers such that $k_X^\lambda / \log m \to \infty$, $k_Y^\lambda / \log n \to \infty$, $k_X^\beta = O(m^{1-\epsilon})$ and $k_Y^\lambda = O(n^{1-\epsilon})$ for some $\epsilon > 0$. Suppose that $\zeta < 1$. Then for each $i_1, i_2 \in \lfloor [d/2] - 1 \rfloor$, $j_1 \in \lfloor [\beta_1^*/2] \rfloor$ and $j_2 \in \lfloor [\beta_2^*/2] \rfloor$, we can find $\lambda_{i_1,i_2,j_1,j_2} \equiv \lambda_{i_1,i_2,j_1,j_2}(d, f, g, \phi)$, with the properties that $\lambda_{0,0,0,0} = T(f, g)$,

$$\sup_{\varnothing \in \Phi(\zeta)} \sup_{(f,g) \in \mathcal{F}_{d,\varnothing}} |\lambda_{i_1,i_2,j_1,j_2}| < \infty,$$
and that, for every \( \epsilon > 0 \),

\[
\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,a}} \left| \mathbb{E}_{f,g}(\tilde{T}_{m,n}) - \sum_{i_1,\ldots,i_d = 0}^{[d/2]-1} \sum_{j_1,j_2 = 0}^{[\beta_1/2]} \sum_{j_2 = 0}^{[\beta_2/2]} \lambda_{i_1,i_2,j_1,j_2} \left( \frac{k_{X}}{m} \right)^{2i_1/d} \left( \frac{k_{Y}}{n} \right)^{2i_2/d} k_{X}^{-j_1} k_{Y}^{-j_2} \right| \leq O\left( \max\left\{ \left( \frac{k_{X}}{m} \right)^{\beta_1/2}, \left( \frac{k_{Y}}{n} \right)^{\beta_2/2} \right\} \right)
\]

as \( m, n \to \infty \), uniformly for \( k_X \in \{k_X^1, \ldots, k_X^L\} \) and \( k_Y \in \{k_Y^1, \ldots, k_Y^U\} \).

**Proof.** Define

\[
a_{m,X}^\pm := 0 \lor \left( \frac{k_{X}}{m} \pm \frac{3k_{X}^{1/2} \log^{1/2} m}{m} \right) \land 1, \quad a_{n,Y}^\pm := 0 \lor \left( \frac{k_{Y}}{n} \pm \frac{3k_{Y}^{1/2} \log^{1/2} n}{n} \right) \land 1,
\]

let \( \mathcal{X}_{m,n} := [a_{m,X}^-, a_{m,X}^+] \), \( \mathcal{I}_{n,Y} := [a_{n,Y}^-, a_{n,Y}^+] \), and set

\[
\mathcal{X}_{m,n} := \left\{ x \in \mathcal{X} : \frac{f(x)}{M_{\beta}(x)^d} \geq \frac{k_{X} \log m}{m}, \frac{g(x)}{M_{\beta}(x)^d} \geq \frac{k_{Y} \log n}{n} \right\}.
\]

To begin our bias calculation, we observe that, conditionally on \( X_1 \), we have \( h_{X_1,f}(\|X_{j} - X_1\|) \sim U[0, 1] \) for \( j \in \{2, \ldots, n\} \), and it follows that

\[
(f(k_X), g(k_Y), 1) \bigg| \ X_1 \overset{d}{=} \left( \frac{k_{X}}{mV_{d}h_{X_{1},1}^{-1}(B_{1})^d}, \frac{k_{Y}}{nV_{d}h_{X_{1},1}^{-1}(B_{2})^d} \right) \bigg| X_1,
\]

where \( B_1 \sim \text{Beta}(k_X, m - k_X) \) and \( B_2 \sim \text{Beta}(k_Y, n + 1 - k_Y) \) are independent. Moreover, we may write, for example,

\[
\frac{u_{x,s}}{f(x)} - 1 = \frac{k_{X}}{ms} - 1 + \frac{s}{V_{d}f(x)h_{x,f}^{-1}(s)^d} - 1 + \left( \frac{k_{X}}{ms} - 1 \right) \left( \frac{s}{V_{d}f(x)h_{x,f}^{-1}(s)^d} - 1 \right), \quad (19)
\]

and use Lemma 15 to expand \( V_{d}f(x)h_{x,f}^{-1}(s)^d/s \) in powers of \( s^{2/d} \). Since the \( \text{Beta}(k, n - k) \) distribution concentrates around its mean at rate \( k^{-1/2} \) in an approximately symmetric way, we will also see later that for every \( a \in \mathbb{R} \), we have an asymptotic expansion of the form

\[
\left( \frac{n}{k} \right)^a \int_{0}^{1} s^a \left( \frac{k}{ns} - 1 \right)^j B_{k,n-k}(s) \, ds = c_1 k^{-[j/2]} + c_2 k^{-[j/2]-1} + \ldots + O(n^{-1}),
\]

(26)
provided that \( k = k_n \to \infty \) and \( k/n \to 0 \) as \( n \to \infty \). These facts mean that for remainder terms \( R_1, \ldots, R_4 \) to be bounded below and functions \( c_{i_1,i_2,j_1,j_2}(x) \) to be specified later we may write

\[
\mathbb{E} \tilde{T}_{m,n} = \int_{X} f(x) \int_{0}^{1} \int_{0}^{1} \phi(u_{x,s}, v_{x,t}, x) B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx \\
= \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{m,Y}} \phi(u_{x,s}, v_{x,t}, x) B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx + R_1 \\
= \sum_{l_1=0}^{[d/2]-1} \sum_{l_2=0}^{[d/2]-1} \frac{1}{l_1!l_2!} \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{m,Y}} \left( \frac{u_{x,s}}{f(x)} - 1 \right)^{l_1} \left( \frac{v_{x,t}}{g(x)} - 1 \right)^{l_2} \\
\times f(x)^{l_1} g(x)^{l_2} \phi_{l_1l_2}(f(x), g(x), x) B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx + R_1 + R_2 \\
= \sum_{i_1,j_1=0}^{[d/2]-1} \sum_{i_2,j_2=0}^{[d/2]-1} \sum_{i_1,i_2,j_1,j_2} \lambda_{i_1,i_2,j_1,j_2} \left( \frac{kX}{m} \right)^{2i_1/d} \left( \frac{kY}{n} \right)^{2i_2/d} \left( \frac{kX}{X} \right)^{j_1} \left( \frac{kY}{Y} \right)^{j_2} \\
\times B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx + R_1 + R_2 + R_3 + R_4. \tag{20}
\]

It now remains to bound each of the remainder terms.

To bound \( R_1 \): Since we are assuming that \( \zeta < 1 \), we may apply Lemma 20 to see that

\[
\int_{X} f(x) \int_{0}^{1} \int_{0}^{1} \left( 1 - \mathbb{1}_{s \in I_{m,X}} \mathbb{1}_{t \in I_{n,Y}} \right) \phi(u_{x,s}, v_{x,t}, x) \\
\times B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx = o(m^{-4} + n^{-4}). \tag{21}
\]

Let \( \mathcal{X}_{m,f} := \{ x : f(x) M_\beta(x)^{-d} \geq kX \log m/m \} \), let \( \mathcal{X}_{n,g} := \{ x : g(x) M_\beta(x)^{-d} \geq kY \log n/n \} \), and consider the decomposition \( \mathcal{X}_{m,n}^c = \mathcal{X}_{m,f}^c \cup \mathcal{X}_{n,g}^c \). Using Lemma 18 and Lemma 19 we
have that
\[
\int_{X_{m,f}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \phi(u_{x,s}, v_{x,t}, x) B_{k_{X,m-k_{X}}(s)} B_{k_{Y,n+1-k_{Y}}(t)} \, ds \, dt \, dx \\
\lesssim \inf_{a > 0} \left( \frac{k_{X}}{m} \log m \right)^{a} \int_{X} f(x) \left\{ \frac{M_{\beta}(x)^{a}}{f(x)} \right\}^{a+\kappa_{1}} \left\{ \frac{M_{\beta}(x)^{a}}{g(x)} \right\}^{\kappa_{2}} \left( 1 + \|x\| \right)^{d(\kappa_{1}+\kappa_{2})} \, dx \\
= O \left( \left( \frac{k_{X}}{m} \right)^{\lambda_{1}(1-\zeta)-\epsilon} \right)
\]
(22)
for every \( \epsilon > 0 \). With a similar bound over \( X_{n,g} \) we conclude that
\[
\int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \phi(u_{x,s}, v_{x,t}, x) B_{k_{X,m-k_{X}}(s)} B_{k_{Y,n+1-k_{Y}}(t)} \, ds \, dt \, dx \\
= O \left( \max \left\{ \left( \frac{k_{X}}{m} \right)^{\lambda_{1}(1-\zeta)-\epsilon}, \left( \frac{k_{Y}}{n} \right)^{\lambda_{2}(1-\zeta)-\epsilon} \right\} \right)
\]
(23)
for every \( \epsilon > 0 \). From (21) and (23), we deduce that
\[
R_{1} = O \left( \max \left\{ \left( \frac{k_{X}}{m} \right)^{\lambda_{1}(1-\zeta)-\epsilon}, \left( \frac{k_{Y}}{n} \right)^{\lambda_{2}(1-\zeta)-\epsilon}, \frac{1}{m^{4}}, \frac{1}{n^{4}} \right\} \right).
\]

To bound \( R_{2} \): We first observe that, by (19) and Lemma 15, we have that
\[
\epsilon_{m,n} := \sup_{(f,g) \in \mathcal{F}_{d,0}} \max_{k_{X} \in \{k_{X}^{1}, \ldots, k_{X}^{U}\}} \sup_{k_{Y} \in \{k_{Y}^{1}, \ldots, k_{Y}^{U}\}} \left\{ \sup_{x \in X_{m,f}} \sup_{s \in I_{m,X}} \left| \frac{u_{x,s}}{f(x)} - 1 \right| \vee \sup_{x \in X_{n,g}} \sup_{t \in I_{n,Y}} \left| \frac{v_{x,t}}{g(x)} - 1 \right| \right\} \rightarrow 0.
\]
(24)
Now, for \( t \in [0, 1] \) we have that \( h(t) := t - \log(1+t) \geq t^{2}/4 \). Thus, letting \( B \sim \text{Beta}(k, n-k) \), whenever \( \frac{3a^{1/2} \log^{1/2} n}{k^{1/2}} \leq 1 \) and \( \frac{k^{1/2} + 3a^{1/2} \log^{1/2} n}{n^{1/2}} \leq 2^{1/2} - 1 \) we may integrate the Beta tail bound
in Lemma 17 to see that

\[
\int_0^1 \left| \frac{ns}{k} - 1 \right|^\alpha B_{k,n-k}(s) \, ds = \alpha \int_0^{n/k} y^{\alpha-1} \mathbb{P} \left( \left| B - \frac{k}{n} \right| \geq \frac{ky}{n} \right) \, dy
\]

\[
\leq 2\alpha k^{-\alpha/2} \int_0^{k^{-1/2}n} u^{\alpha-1} \left\{ \exp \left( -kh \left( \frac{n^{1/2}k^{-1/2}u}{n^{1/2} + k^{1/2} + u} \right) \right) + \exp \left( -nh \left( \frac{u}{n^{1/2} + k^{1/2} + u} \right) \right) \right\} \, du
\]

\[
\leq 4\alpha k^{-\alpha/2} \int_0^{3a^{1/2} \log^{1/2} n} u^{\alpha-1} e^{-u^2/s} \, du + \frac{4n^\alpha}{k^\alpha} \exp \left( -\frac{9\alpha \log n}{8} \right)
\]

\[
\leq \frac{2^{3(\alpha-1)/2} \alpha \Gamma(\alpha/2)}{k^{\alpha/2}} + \frac{4}{k^\alpha}.
\]  

(25)

Next, by Lemma 18, we have for any \( \tau \geq 0 \) that

\[
\left( \frac{k_X}{m} \right)^\tau \int_{\mathcal{X}_{m,n}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^\tau \, dx
\]

\[
\leq \inf_{\alpha>0} \left( \frac{k_X}{m} \right)^{\tau-a} \int_{\mathcal{X}} f(x) \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^{\tau+\kappa_1-a} \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^{\kappa_2} \, dx
\]

\[
= O \left( \max \left\{ \left( \frac{k_X}{m} \right)^\tau, \left( \frac{k_X}{m} \right)^{\lambda_1(1-\zeta)-\epsilon} \right\} \right)
\]  

(26)

for all \( \epsilon > 0 \). Analogously,

\[
\left( \frac{k_Y}{n} \right)^\tau \int_{\mathcal{X}_{m,n}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^\tau \, dx = O \left( \max \left\{ \left( \frac{k_Y}{n} \right)^\tau, \left( \frac{k_Y}{n} \right)^{\lambda_2(1-\zeta)-\epsilon} \right\} \right)
\]  

(27)

for any \( \tau \geq 0 \) and \( \epsilon > 0 \). Now, since \( \phi \in \Phi \) and by (19), (24), (25), (26), (27) and Lemma 15 we have, for \( m, n \) large enough that \( \epsilon_{m,n} < L^{-1} \), that

\[
|R_2| \leq L \int_{\mathcal{X}_{m,n}} f(x) \left\{ f(x)^{-\kappa_1} \lor f(x)^L \right\} \left\{ g(x)^{-\kappa_2} \lor g(x)^L \right\}
\]

\[
\times \left( \int_{I_{m,n}} \int_{I_{n,Y}} B_{k_X,m-k_X}(s)B_{k_Y,n+1-k_Y}(t) \left\{ \left| \frac{u_{x,s}}{f(x)} - 1 \right|^{\beta_1} + \left| \frac{v_{x,t}}{g(x)} - 1 \right|^{\beta_2} \right\} \, ds \, dt \right) \, dx
\]

\[
\leq \int_{\mathcal{X}_{m,n}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} \left( \int_{I_{m,n}} \int_{I_{n,Y}} B_{k_X,m-k_X}(s)B_{k_Y,n+1-k_Y}(t) \right)
\]

\[
\times \left[ \left| \frac{k_X}{ms} - 1 \right|^{\beta_1} + \left| \frac{k_Y}{nt} - 1 \right|^{\beta_2} + \left\{ s M_\beta(x)^d f(x) \right\}^{\frac{2\alpha}{d} \beta_1} + \left\{ t M_\beta(x)^d g(x) \right\}^{\frac{2\alpha}{d} \beta_2} \right] \, ds \, dt \, dx
\]

\[
= O \left( \max \left\{ \left( \frac{k_X}{m} \right)^{\beta_1/2}, \left( \frac{k_Y}{n} \right)^{\beta_2/2}, \left( \frac{k_X}{m} \right)^{\lambda_1(1-\zeta)-\epsilon}, \left( \frac{k_Y}{n} \right)^{\lambda_2(1-\zeta)-\epsilon} \right\} \right).
\]  

29
To bound $R_3$: By (19) and Lemma 15, when $l_1 > 0$ we have expansions of the form
\[
\left| \left( \frac{u_{x,t}}{f(x)} - 1 \right) - \sum_{i=0}^{[d/2]-1} \sum_{j=0}^{l_1} b_{i,j}(x) s^{2i/d} \left( \frac{k}{m s} - 1 \right)^j \right| \lesssim \left\{ \frac{s M_{\beta}(x)^d}{f(x)} \right\}^{\frac{d}{2} \wedge 1},
\]
with $|b_{i,j}(x)| \lesssim \{M_{\beta}(x)^d/f(x)\}^{2i/d}$ and $b_{0,0} = 0$. A similar expansion can also be written for $(v_{x,t}/g(x) - 1)^2$. Using these two expansions it can be seen that $c_{i_1,i_2,j_1,j_2}$ can be chosen in (20) with $|c_{i_1,i_2,j_1,j_2}(x)| \lesssim f(x)^{-\kappa_1} g(x)^{-\kappa_2} \{M_{\beta}(x)^d/f(x)\}^{2i_1/d} \{M_{\beta}(x)^d/g(x)\}^{2i_2/d}$, with $c_{0,0,0,0}(x) = \phi(f(x), g(x), x)$, and, using (26) and (27), with
\[
|R_3| \lesssim \int_{x_{n,n}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} \left\{ \left( \frac{k X M_{\beta}(x)^d}{m f(x)} \right)^{\frac{d}{2} \wedge 1} + \left( \frac{k Y M_{\beta}(x)^d}{n g(x)} \right)^{\frac{d}{2} \wedge 1} \right\} \, dx
= O\left( \max\left\{ \left( \frac{k X}{m} \right)^{\frac{d}{2} \wedge 1}, \left( \frac{k X}{m} \right)^{x_1(1-\zeta)-\epsilon}, \left( \frac{k Y}{n} \right)^{\frac{d}{2} \wedge 1}, \left( \frac{k Y}{n} \right)^{x_2(1-\zeta)-\epsilon} \right\} \right).
\]

To bound $R_4$: Whenever $a \in \mathbb{R}$ is fixed, we have an asymptotic series of the form
\[
\frac{\Gamma(m+a)}{k_X^a \Gamma(m)} \int_0^1 s^a B_{k_X,m-k_X}(s) \, ds = \frac{\Gamma(k_X+a)}{\Gamma(k_X) k_X^a} = 1 + c_1 k_X^{-1} + c_2 k_X^{-2} + \ldots. \tag{28}
\]
On the other hand, arguing similarly to (25), for fixed $j \in \mathbb{N}$ we have the bound
\[
\left( \frac{m}{k_X} \right)^a \int_0^1 s^a \left| \frac{k X}{m s} - 1 \right|^j B_{k_X,m-k_X}(s) \, ds
\leq 2^{j-1} \left( \frac{m}{k_X} \right)^a \int_0^1 s^{-j} \left\{ s - \frac{k X + a - j}{m + a - j} \left| j \right| + \left| \frac{k X + a - j}{m + a - j} - \frac{k X}{m} \right| \right\} B_{k_X,m-k_X}(s) \, ds
= \frac{2^{j-1} m^a \Gamma(k_X+a-j) \Gamma(m)}{k_X^a \Gamma(k_X) \Gamma(m+a-j)} \left\{ \int_0^1 s - \frac{k X + a - j}{m + a - j} \left| j \right| B_{k_X,a-j,m-k_X}(s) \, ds + \left| \frac{k X + a - j}{m + a - j} - \frac{k X}{m} \right|^j \right\}
= O(k_X^{-j/2}) \tag{29}.
\]
Moreover, by Lemma 17, letting $B \sim \text{Beta}(k_X + a - j, m - k_X)$ we have that
\[
\left( \frac{m}{k_X} \right)^a \int_{[0,1]\setminus I_{m,X}} s^a \left| \frac{k X}{m s} - 1 \right|^j B_{k_X,m-k_X}(s) \, ds \lesssim \int_{[0,1]\setminus I_{m,X}} \left| \frac{m s}{k_X} - 1 \right|^j B_{k_X+a-j,m-k_X}(s) \, ds
\leq \mathbb{P} \left( \left| \frac{m B}{k_X} - 1 \right| \geq \frac{3 \log^{1/2} m}{k_X^{1/2}} \right) + \left( \frac{m}{k_X} \right)^j \mathbb{P} \left( \left| \frac{m B}{k_X} - 1 \right| \geq 1 \right) = o(m^{-4}). \tag{30}
\]
With the similar expression in terms of $k_Y$ and $n$, we now conclude from (28), (29) and (30)
that we have an asymptotic expansion of the form

\[
\int_{I_{m,X}} \int_{I_{n,Y}} s^{2i_1} t^{2i_2} \left( \frac{k_X}{ms} - 1 \right)^{j_1} \left( \frac{k_Y}{nt} - 1 \right)^{j_2} B_{k_X,m-k_X}(s)B_{k_Y,n+1-k_Y}(t) \, ds \, dt \\
= \left( \frac{k_X}{m} \right)^{2i_1} \left( \frac{k_Y}{n} \right)^{2i_2} \left\{ \sum_{r=[j_1/2]}^{\infty} c_r k_X^r + O(m^{-1}) \right\} \left\{ \sum_{r=[j_2/2]}^{\infty} d_r k_Y^{-r} + O(n^{-1}) \right\}. \tag{31}
\]

Now for fixed \( i_1, i_2 \in [\lfloor d/2 \rfloor - 1] \) with \( \frac{k_1+2i_1/d}{\lambda_1} + \frac{k_2+2i_2/d}{\lambda_2} \geq 1 \), we have by Lemma 18 that

\[
\int_{X_{m,n}} f(x) |c_{i_1,i_2,j_1,j_2}(x)| \, dx \leq \int_{X_{m,n}} f(x) \left\{ \frac{M_{\beta}(x)}{f(x)} \right\}^{\lambda_1 + \frac{2i_1}{d}} \left\{ \frac{M_{\beta}(x)}{g(x)} \right\}^{\lambda_2 + \frac{2i_2}{d}} \, dx \\
\leq \min \left\{ \inf_{a>0} \left( \frac{k_X}{m} \right)^{-a} \int_X f(x) \left\{ \frac{M_{\beta}(x)}{f(x)} \right\}^{\lambda_1 + \frac{2i_1}{d}} \left\{ \frac{M_{\beta}(x)}{g(x)} \right\}^{\lambda_2 + \frac{2i_2}{d}} \, dx, \right. \\
\inf_{a>0} \left( \frac{k_Y}{n} \right)^{-a} \int_X f(x) \left\{ \frac{M_{\beta}(x)}{f(x)} \right\}^{\lambda_1 + \frac{2i_1}{d}} \left\{ \frac{M_{\beta}(x)}{g(x)} \right\}^{\lambda_2 + \frac{2i_2}{d}} \, dx \right. \\
= O \left( \min \left\{ \left( \frac{k_X}{m} \right)^{\lambda_1 + \frac{2i_1}{d}} \left( \frac{k_Y}{n} \right)^{\lambda_2 + \frac{2i_2}{d}} \right\} \right) \\
= O \left( \left( \frac{k_X}{m} \right)^{-\frac{2i_1}{d}} \left( \frac{k_Y}{n} \right)^{-\frac{2i_2}{d}} \max \left\{ \left( \frac{k_X}{m} \right)^{\lambda_1 - \frac{2i_1}{d}}, \left( \frac{k_Y}{n} \right)^{\lambda_2 - \frac{2i_2}{d}} \right\} \right), \tag{32}
\]

for all \( \epsilon > 0 \), where the final inequality can be established by considering the cases \( \left( \frac{k_X}{m} \right)^{\lambda_1} \geq \left( \frac{k_Y}{n} \right)^{\lambda_2} \) and \( \left( \frac{k_X}{m} \right)^{\lambda_1} < \left( \frac{k_Y}{n} \right)^{\lambda_2} \) separately. For such \( i_1, i_2 \) we set \( \lambda_{i_1,i_2,j_1,j_2} = 0 \) for all \( j_1, j_2 \).

When, instead, \( \frac{k_1+2i_1/d}{\lambda_1} + \frac{k_2+2i_2/d}{\lambda_2} < 1 \), we again consider these two cases separately, use the decomposition \( X_{m,n}^c = (X_{m,f}^c \cap X_{n,g}^c) \cup (X_{m,f}^c \cap X_{n,g}) \cup (X_{m,f} \cap X_{n,g}^c) \) and apply Lemma 18 to
\[
\int_{\mathcal{C}_{m,n}} f(x) \left| c_{i_1,i_2,j_1,j_2}(x) \right| dx \lesssim \int_{\mathcal{C}_{m,f} \cap \mathcal{C}_{n,g}} f(x) \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^{\frac{21}{4}} \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^{\frac{21}{4}} + O \left( \left( \frac{k_X}{m} \right)^{-\frac{21}{4}} \left( \frac{k_Y}{n} \right)^{-\frac{21}{4}} \right) \left\{ \lambda_1(1-\epsilon) \wedge \left( \frac{k_Y}{n} \right)^{\lambda_2(1-\epsilon)} \right\} \]
\[
\leq \min \left\{ \inf_{a>0} \left( \frac{k_X \log m}{m} \right)^a \int_{\mathcal{C}} f(x) \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^{\frac{21}{4}} \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^{\frac{21}{4}} dx, \right. \\
\quad \left. \quad \quad \inf_{a>0} \left( \frac{k_Y \log n}{n} \right)^a \int_{\mathcal{C}} f(x) \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^{\frac{21}{4}} \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^{\frac{21}{4}} dx \right) \\
\quad + O \left( \left( \frac{k_X}{m} \right)^{-\frac{21}{4}} \left( \frac{k_Y}{n} \right)^{-\frac{21}{4}} \left\{ \lambda_1(1-\epsilon) \wedge \left( \frac{k_Y}{n} \right)^{\lambda_2(1-\epsilon)} \right\} \right) \right)
\]
\[= O \left( \left( \frac{k_X}{m} \right)^{-\frac{21}{4}} \left( \frac{k_Y}{n} \right)^{-\frac{21}{4}} \left\{ \lambda_1(1-\epsilon) \wedge \left( \frac{k_Y}{n} \right)^{\lambda_2(1-\epsilon)} \right\} \right) \tag{33} \]
for all \( \epsilon > 0 \). It follows from (31), (32) and (33) that
\[
R_4 = O \left( \max \left\{ k_X^{-\frac{[2\gamma/2]-1}{4}}, k_Y^{-\frac{[2\gamma/2]-1}{4}}, \left( \frac{k_X}{m} \right)^{\lambda_1(1-\epsilon)}, \left( \frac{k_Y}{n} \right)^{\lambda_2(1-\epsilon)}, \frac{1}{m}, \frac{1}{n} \right\} \right),
\]
and this concludes the proof. \( \square \)

### 7.3 Proof of Proposition 9 on asymptotic variance

**Proof of Proposition 9.** We initially consider the unweighted estimator \( \tilde{T}_{m,n} \), deferring the extension to the weighted estimator \( \tilde{T}_{m,n}^{w, w'} \) to the end of the proof. We start by writing
\[
\text{Var}(\tilde{T}_{m,n}) = \frac{1}{m} \text{Var} \left( \phi(\tilde{f}_{(k_X),1}, \tilde{g}_{(k_Y),1}, X_1) \right) + \left( 1 - \frac{1}{m} \right) \text{Cov} \left( \phi(\tilde{f}_{(k_X),1}, \tilde{g}_{(k_Y),1}, X_1), \phi(\tilde{f}_{(k_X),2}, \tilde{g}_{(k_Y),2}, X_2) \right). \tag{34}
\]
Taking $X_{m,n}, I_{m,X}$ and $I_{n,Y}$ as defined in the proof of Proposition 12, and letting $S_1, S_2$ and $S_3$ be error terms, we now write

$$
\mathbb{E} \phi^2(\hat{f}(k_1),1, \hat{g}(k_2),1, X_1)
= \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \phi^2(u_{x,s}, v_{x,t}, x) B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx + S_1
= \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \phi^2(\frac{kX f(x)}{ms}, \frac{kY g(x)}{nt}, x) B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx + S_1 + S_2
= \mathbb{E} \{ (\phi X_1)^2 \} + \sum_{j=1}^{3} S_j.
$$

We show in Section 7.7 that

$$
\sum_{j=1}^{3} S_j = O \left( \max \left\{ \left( \frac{kX}{m} \right)^{\lambda_1(1-2\zeta) - \epsilon}, \left( \frac{kY}{n} \right)^{\lambda_2(1-2\zeta) - \epsilon}, \left( \frac{kX}{m} \right)^{\frac{2\Delta_0}{\alpha}}, \left( \frac{kY}{n} \right)^{\frac{2\Delta_0}{\alpha}}, kX^{-1/2}, kY^{-1/2} \right\} \right)
$$

for every $\epsilon > 0$. Using Proposition 12 we can now see that

$$
\sup_{\phi \in \Phi(f,g) \in F_{d,s}} \max \left| \frac{1}{m} \mathbb{V} \phi(\hat{f}(k_1),1, \hat{g}(k_2),1, X_1) - \mathbb{V} \phi X_1 \right| = o \left( \frac{1}{m} \right). \quad (35)
$$

We now turn to the second term in (34). Let $F_{m,n,x,y} : [0,1]^4 \rightarrow [0,1]$ denote the conditional distribution function of

$$
(h_{x,f}(\rho(k_1),1,x), h_{y,f}(\rho(k_2),2,x), h_{x,g}(\rho(k_1),1,y), h_{y,g}(\rho(k_2),2,y))|X_1 = x, X_2 = y.
$$

Moreover, for $s_1, s_2, t_1, t_2 \in [0,1]$ such that $s_1 + s_2 \leq 1$ and $t_1 + t_2 \leq 1$ define

$$
G_{m}^{(1)}(s_1, s_2) := \int_{0}^{s_1} \int_{0}^{s_2} B_{kX,kX,m-2kX-1}(u_1, u_2) \, du_1 \, du_2
$$

$$
G_{n}^{(2)}(t_1, t_2) := \int_{0}^{t_1} \int_{0}^{t_2} B_{kY,kY,n-2kY+1}(v_1, v_2) \, dv_1 \, dv_2
$$

$$
G_{m,n}(s_1, s_2, t_1, t_2) := G_{m}^{(1)}(s_1, s_2) G_{n}^{(2)}(t_1, t_2),
$$

so that for $s_1, s_2, t_1, t_2$ and $x, y$ such that $\|x-y\| > \max(h_{x,f}^{-1}(s_1) + h_{y,f}^{-1}(s_2), h_{x,g}^{-1}(t_1) + h_{y,g}^{-1}(t_2))$ we have that $F_{m,n,x,y}(s_1, s_2, t_1, t_2) = G_{m,n}(s_1, s_2, t_1, t_2)$. We will also use the shorthand
\[ h(s_1, s_2, t_1, t_2) := \phi(u_{x,s_1}, v_{x,t_1}, x)\phi(u_{y,s_2}, v_{y,t_2}, y) \quad \text{and} \]
\[ H_m^{(1)}(s_1, s_2) := G_m^{(1)}(s_1, s_2) - \int_0^{s_1} \int_0^{s_2} B_{kX,m-kX}(u_1)B_{kX,m-kX}(u_2) \, du_1 \, du_2 \]
\[ H_n^{(2)}(t_1, t_2) := G_n^{(2)}(t_1, t_2) - \int_0^{t_1} \int_0^{t_2} B_{kY,n+1-kY}(v_1)B_{kY,n+1-kY}(v_2) \, dv_1 \, dv_2 \]
\[ H_{m,n}(s_1, s_2, t_1, t_2) := H_m^{(1)}(s_1, s_2)G_n^{(2)}(t_1, t_2) + G_m^{(1)}(s_1, s_2)H_n^{(2)}(t_1, t_2) - H_m^{(1)}(s_1, s_2)H_n^{(2)}(t_1, t_2). \]

With this newly-defined notation, we now have
\[ \text{Cov}\left(\hat{f}(kX),1, \hat{g}(kY),1, X_1, \phi(\hat{f}(kX),2, \hat{g}(kY),2, X_2)\right) \]
\[ = \int_{X \times \hat{X}} f(x)f(y) \int_{[0,1]^4} h(s_1, s_2, t_1, t_2) \left\{ dF_{m,n,x,y}(s_1, s_2, t_1, t_2) \right. \]
\[ - B_{kX,m-kX}(s_1)B_{kX,m-kX}(s_2)B_{kY,n+1-kY}(t_1)B_{kY,n+1-kY}(t_2) \, ds_1 \, ds_2 \, dt_1 \, dt_2 \bigg\} \, dx \, dy \]
\[ = \int_{X \times \hat{X}} f(x)f(y) \int_{T_{m,n}^2} \int_{T_{n,Y}^2} h(s_1, s_2, t_1, t_2) \left\{ d(F_{m,n,x,y} - G_{m,n})(s_1, s_2, t_1, t_2) \right. \]
\[ + dH_{m,n}(s_1, s_2, t_1, t_2) \bigg\} \, dx \, dy + o(m^{-2} + n^{-2}), \quad (36) \]

where the bound on the final term follows from the fact that \( \zeta < 1/2 \), Lemma 20 and Cauchy–Schwarz. We first study the second term in this expansion. The intuition behind the following expansion is that, when \( X_1 \) and \( X_2 \) do not share nearest neighbours, the dependence between \( (\hat{f}(kX),1, \hat{g}(kY),1) \) and \( (\hat{f}(kX),2, \hat{g}(kY),2) \) is relatively weak, and we may expand the functions \( \phi, h_{x,f}^{-1}, h_{x,g}^{-1} \) as in the proof of Proposition 12 and approximate integrals. We therefore make use of the shorthand
\[ h^{(1)}(s_1, s_2, t_1, t_2) := \left\{ \phi(f(x), v_{x,t_1}, x) + \left(\frac{kX}{ms_1} - 1\right)f(x)\phi_{10}(f(x), v_{x,t_1}, x) \right\} \]
\[ \times \left\{ \phi(f(y), v_{y,t_2}, y) + \left(\frac{kX}{ms_2} - 1\right)f(y)\phi_{10}(f(y), v_{y,t_2}, y) \right\} \]
\[ h^{(2)}(s_1, s_2, t_1, t_2) := \left\{ \phi(u_{x,s_1}, g(x), x) + \left(\frac{kY}{nt_1} - 1\right)g(x)\phi_{01}(u_{x,s_1}, g(x), x) \right\} \]
\[ \times \left\{ \phi(u_{y,s_2}, g(y), y) + \left(\frac{kY}{nt_2} - 1\right)g(y)\phi_{01}(u_{y,s_2}, g(y), y) \right\} \]

for linearised versions of \( h \). We also write, for example,
\[ (h \, dH_m^{(1)} dG_n^{(2)})(s_1, s_2, t_1, t_2) := h(s_1, s_2, t_1, t_2) \, dH_m^{(1)}(s_1, s_2) \, dG_n^{(1)}(t_1, t_2). \]
Writing $T_1$, $T_2$ and $T_3$ for error terms, we therefore have

\[
\begin{align*}
\int_{X^2} f(x)f(y) & \int_{I_{m,x}}^{I_{n,y}} (h \, dH_{m,n})(s_1, s_2, t_1, t_2) \, dx \, dy \\
= \int_{X^2} f(x)f(y) & \int_{I_{m,x}}^{I_{n,y}} (h \, dH_{m}^{(1)} \, dG_{n}^{(2)})(s_1, s_2, t_1, t_2) \, dx \, dy \\
& + \int_{X^2} f(x)f(y) \int_{I_{m,x}}^{I_{n,y}} (h \, B_{k,x,m-k,x} B_{k,x,m-k,x} \, dH_{n}^{(2)})(s_1, s_2, t_1, t_2) \, ds_1 \, ds_2 \, dx \, dy + T_1 \\
= \int_{X^2} f(x)f(y) & \int_{I_{m,x}}^{I_{n,y}} (h^{(1)} \, dH_{m}^{(1)} \, dG_{n}^{(2)})(s_1, s_2, t_1, t_2) \, dx \, dy + T_1 + T_2 \\
& + \int_{X^2} f(x)f(y) \int_{I_{m,x}}^{I_{n,y}} (h^{(2)} \, B_{k,x,m-k,x} B_{k,x,m-k,x} \, dH_{n}^{(2)})(s_1, s_2, t_1, t_2) \, ds_1 \, ds_2 \, dx \, dy \\
= -\frac{1}{m} \int_{X^2} f(x)f(y) & \int_{I_{n,y}}^{I_{n,y}} \left\{ 2f(x)\phi_{10}(f(x), v_{x,t_1}, x)\phi(f(y), v_{y,t_2}, y) \\
& + f(x)\phi_{10}(f(x), v_{x,t_1}, x)f(y)\phi_{10}(f(y), v_{y,t_2}, y) \right\} \, dG_{n}^{(2)}(t_1, t_2) \, dx \, dy \\
& - \frac{1}{n} \int_{X^2} f(x)f(y) \int_{I_{m,x}}^{I_{n,y}} g(x)\phi_{01}(u_{x,s_1}, g(x), x)g(y)\phi_{01}(u_{y,s_2}, g(y), y) \times B_{k,x,m-k,x}(s_1)B_{k,x,m-k,x}(s_2) \, ds_1 \, ds_2 \, dx \, dy + T_1 + T_2 + T_3 \\
& = -\frac{2}{m}E\{f\phi_{10}(X_1)\}E(\phi_{X_1}) - \frac{1}{m} \left\{ E(f\phi_{10}(X_1) \right\}^2 - \frac{1}{n} \left\{ E(g\phi_{01}(X_1) \right\}^2 + T_1 + T_2 + T_3 \\
& + o(m^{-1} + n^{-1}), \quad (37)
\end{align*}
\]

where the bound on the final term follows from (24), Lemma 14, Lemma 17 and tail bounds similar to (22). We show in Section 7.7 that

\[
\sum_{j=1}^{3} T_j = O \left( \max \left\{ \left( \frac{k_X}{m} \right)^{1+\lambda_1(1-\zeta)-\epsilon} \frac{k_X^{1+2\lambda_\zeta}}{m^{1+2\zeta}} \left( \frac{k_X}{m} \right)^{1+2(2\lambda_\zeta)} \log \frac{m}{mk_X^{1/2}}, \right. \\
left. \left( \frac{k_Y}{n} \right)^{1+\lambda_2(1-\zeta)-\epsilon} \frac{k_Y^{1+2\lambda_\zeta}}{n^{1+2\zeta}} \left( \frac{k_Y}{n} \right)^{1+2(2\lambda_\zeta)} \log \frac{n}{nk_Y^{1/2}} \right) \right) + o(m^{-1} + n^{-1}). \quad (38)
\]
We now consider the contribution of the first term in (36). In Section 7.7, we show that

\[ U_0 := \int_{X \times X_{m,n}} f(x) f(y) \int_{I_{m,x}}^{I_{m,x}} h(s_1, s_2, t_1, t_2) d(F_{m,n,x,y} - G_{m,n})(s_1, s_2, t_1, t_2) \, dx \, dy = O\left( \max\left\{ \left( \frac{k}{m} \right)^{2\lambda_1(1-\epsilon)-\epsilon}, \left( \frac{k}{n} \right)^{2\lambda_2(1-\epsilon)-\epsilon} \right\} \right), \]  

(39)

so that we may restrict attention to \( x \in X_{m,n} \), in which case \( F_{m,n,x,y} - G_{m,n} \) is only non-zero when \( x \) and \( y \) are close and we may approximate \( f(y) \approx f(x) \) and \( g(y) \approx g(x) \). Let

\[ p^{(1)} := \int_{B_x(h^{-1}_x(s_1)) \cap B_y(h^{-1}_y(s_2))} f(w) \, dw \quad \text{and} \quad p^{(2)} := \int_{B_x(h^{-1}_x(t_1)) \cap B_y(h^{-1}_y(t_2))} g(w) \, dw, \]

and let

\[ (N^{(1)}_1, N^{(1)}_2, N^{(1)}_3, N^{(1)}_4) \sim \text{Multi}(m - 2; s_1 - p^{(1)}; s_2 - p^{(1)}; 1 - s_1 - s_2 + p^{(1)}) \]

\[ (N^{(2)}_1, N^{(2)}_2, N^{(2)}_3, N^{(2)}_4) \sim \text{Multi}(n; t_1 - p^{(2)}; t_2 - p^{(2)}; 1 - t_1 - t_2 + p^{(2)}). \]

Now set

\[ F^{(1)}_{m,x,y}(s_1, s_2) := \mathbb{P}(N^{(1)}_1 + N^{(1)}_3 \geq kX - 1_{\{\|x-y\| \leq h^{-1}_x(s_1)\}}, N^{(1)}_2 + N^{(1)}_4 \geq kX - 1_{\{\|x-y\| \leq h^{-1}_y(s_2)\}}) \]

\[ F^{(2)}_{n,x,y}(t_1, t_2) := \mathbb{P}(N^{(2)}_1 + N^{(2)}_3 \geq kY, N^{(2)}_2 + N^{(2)}_4 \geq kY), \]

so that \( F_{m,n,x,y}(s_1, s_2, t_1, t_2) = F^{(1)}_{m,x,y}(s_1, s_2) F^{(2)}_{n,x,y}(t_1, t_2) \). We use the decomposition

\[ F_{m,n,x,y} - G_{m,n} = F^{(1)}_{m,x,y} F^{(2)}_{n,x,y} - G^{(1)}_{m} G^{(2)}_{n} = (F^{(1)}_{m,x,y} - G^{(1)}_{m})(F^{(2)}_{n,x,y} - G^{(2)}_{n}) \]

\[ + (F^{(1)}_{m,x,y} - G^{(1)}_{m}) G^{(2)}_{n} + G^{(1)}_{m} (F^{(2)}_{n,x,y} - G^{(2)}_{n}), \]

(42)

so that each term is of product form and involves at least one of the marginal errors. We will see that the first term is asymptotically negligible, while the second and third terms can be studied through the normal approximation given in Lemma 22. For a general distribution function \( F \), for \( a_- \leq a_+ \) and for a smooth \( h : [a_-, a_+]^2 \to \mathbb{R} \) with first partial derivatives \( h_{10}, h_{01} \) and mixed second partial derivative \( h_{11} \), we will use the formula

\[
\int_{[a_-, a_+]^2} (h \, dF)(u, v) - \int_{a_-}^{a_+} \int_{a_-}^{a_+} (h_{11} F(u, v)) \, du \, dv \\
= \int_{a_-}^{a_+} \left[ (h_{10} F)(u_-, a_+) - (h_{10} F)(u, a_+) \right] du + \int_{a_-}^{a_+} \left[ (h_{01} F)(a_-, v) - (h_{01} F)(a_+, v) \right] dv \\
+ (h F)(a_+, a_+) + (h F)(a_-, a_-) - (h F)(a_+, a_-) - (h F)(a_-, a_+). \]

(43)
We now deal with each of the three terms on the right-hand side of (42) in turn, starting with $F = F^{(1)} F^{(2)} = (F^{(1)}_{m,x,y} - G^{(1)}_m)(F^{(2)}_{n,x,y} - G^{(2)}_n)$. For remainder terms $U_1$, $U_2$ and $U_3$ to be bounded later, we write

$$\int_{X \times X} f(x) f(y) \int_{I_{m,X}} \int_{I_{n,Y}} (h \, dF)(s_1, s_2, t_1, t_2) \, dx \, dy$$

$$= \int_{X \times X} f(x) f(y) \int_{I_{m,X}} \int_{I_{n,Y}} \left( h_{0011} \left( dF^{(1)} \right) (s_1, s_2, t_1, t_2) \right) dt_1 \, dt_2 \, dx \, dy + U_1$$

$$= \int_{X \times X} f(x) f(y) \int_{I_{n,Y}} F^{(2)}(t_1, t_2) \left\{ \int_{I_{m,X}} (h_{1111} F^{(1)})(s_1, s_2, t_1, t_2) \, ds_1 \, ds_2 \\ - \int_{I_{m,X}} (h_{1011} F^{(1)})(s_1, a_{m,X}^+, t_1, t_2) \, ds_1 - \int_{I_{m,X}} (h_{0111} F^{(1)})(a_{m,X}^+, s_2, t_1, t_2) \, ds_2 \right\} dt_1 \, dt_2 \, dx \, dy + U_1 + U_2$$

$$= \sum_{j=1}^{3} U_j. \quad (44)$$

We show in Section 7.7 that

$$\sum_{j=1}^{3} U_j = O \left( \max \left\{ \frac{1}{m^2}, \frac{1}{n^2}, \frac{\log^2 m}{mk_X}, \frac{\log^2 n}{nk_Y} \right\} \right). \quad (45)$$

We next consider $F = F^{(1)} F^{(2)} = (F^{(1)}_{m,x,y} - G^{(1)}_m)G^{(2)}_n$, and recall from Lemma 22 that $\alpha_z = V_d^{-1} \mu_d(B_0(1) \cap B_z(1))$, that

$$\Sigma = \begin{pmatrix} 1 & \alpha_z \\ \alpha_z & 1 \end{pmatrix},$$

and the definitions of the normal distribution functions $\Phi_{I_2}$ and $\Phi_{\Sigma}$. Then, for remainder terms $U_4, U_5, U_6$ to be bounded below, we use the change of variables $y = x + (\frac{k}{mV_d f(x)})^{1/d} z$}

37
and the approximation \( \frac{\partial}{\partial v} \phi(u,v,x,t,\mathbf{a}) \approx s^{-1} f(x) \phi_0(f(x),g(x),x) \) to write

\[
\int_{X \times X_n} f(x) f(y) \int_{I_{m,Y}} \int_{I_{m,Y}} (h \, dF)(s_1, s_2, t_1, t_2) \, dx \, dy
\]

\[
= \int_{X \times X_n} f(x) f(y) \int_{I_{m,Y}} \int_{I_{m,Y}} B_{ky,kY} - 2ky + 1(t_1, t_2) \left\{ \int_{I_{m,X}} (h_{1100} F^{(1)}) (s_1, s_2, t_1, t_2) \, ds_1 \, ds_2 \right\}
\]

\[
- \int_{I_{m,X}} (h_{1000} F^{(1)}) (s_1, a_{m,X}^+, t_1, t_2) \, ds_1 + \int_{I_{m,X}} (h_{0100} F^{(1)}) (a_{m,X}^+, s_2, t_1, t_2) \, ds_2 \right\} \, dt_1 \, dt_2 \, dx \, dy + U_4
\]

\[
= \frac{1}{m V_d} \int_{X \times X_n} f(x) \int_{R^d} \left\{ (f \phi_0)^2 \int_{R^2} (\Phi_\Sigma - \Phi_{I_2})(u_1, u_2) \, du_1 \, du_2 + 2(f \phi_0)_{x} \phi_x \mathbb{1}_{\{||z|| \leq 1\}} \right\} \, dz \, dx
\]

\[
+ U_4 + U_5
\]

\[
= \frac{1}{m V_d} \int_{X \times X_n} f(x) \left\{ (f \phi_0)^2 x \int_{R^2} \alpha_z \, dz + 2V_d(f \phi_0)_{x} \phi_x \right\} \, dx + U_4 + U_5
\]

\[
= \frac{1}{m} \mathbb{E}\{ (f \phi_0)^2 \chi_1 \} + \frac{2}{m} \mathbb{E}\{ (f \phi_0) \chi_1 \phi_x \} + \sum_{j=4}^{6} U_j.
\]

We show in Section 7.7 that

\[
\sum_{j=4}^{6} U_j = O \left( \frac{1}{m} \max \left\{ \frac{\log^2 m}{k_X^{1/2}}, \frac{\log^2 n}{k_Y^{1/2}}, \log^2 m \left( \frac{k_X}{m} \right)^{1/2}, \left( \frac{k_Y}{n} \right)^{1/2}, \left( \frac{k_Y}{n} \right)^{1/2}, \left( \frac{k_Y}{n} \right)^{1/2} \right\} \right)
\]

(47)

for every \( \epsilon > 0 \). The final term in (42) can be approximated by writing \( F = F^{(1)} F^{(2)} = G_m^{(1)} (F_{n,x,y}^{(2)} - G_n^{(2)}) \), using the changes of variables \( y = x + \left( \frac{k_Y}{n V_d g(x)} \right)^{1/d} z, t_i = n^{-1} (k_Y + k_Y v_i) \) for \( i = 1, 2 \) and using the approximation \( \frac{\partial}{\partial v} \phi(u,v,x,t,\mathbf{a}) \approx t^{-1} g(x) \phi_0(f(x),g(x),x) \) to write

\[
\int_{X \times X_n} f(x) f(y) \int_{I_{m,X}} \int_{I_{m,Y}} (h \, dF)(s_1, s_2, t_1, t_2) \, dx \, dy
\]

\[
= \int_{X \times X_n} f(x) f(y) \int_{I_{m,X}} \int_{I_{m,Y}} (h_{0011} dG_m^{(1)} F^{(2)})(s_1, s_2, t_1, t_2) \, dt_1 \, dt_2 \, ds_1 \, ds_2 + U_7
\]

\[
= \frac{1}{n V_d} \int_{X \times X_n} f(x)^2 g(x)^{-1} \int_{R^d} (g \phi_0)^2 \int_{R^2} (\Phi_\Sigma - \Phi_{I_2})(v_1, v_2) \, dv_1 \, dv_2 \, dz \, dx + U_7 + U_8
\]

\[
= \frac{1}{n} \int_{X \times X_n} g(x) (f \phi_0)_{x}^2 \, dx + U_7 + U_8.
\]

(48)
Now, by Lemma 18, we have that
\[
\int_{\mathcal{X}} \left| g(x)(f\phi_0)^2_x \right| \, dx \leq \int_{\mathcal{X}} f(x)^{2-2\kappa_1} g(x)^{1-2\kappa_2} \, dx
\]
\[
= \int_{\mathcal{X}} f(x)^{2-2\kappa_1} \left\{ \frac{f(x)^2 g(x)^{-\gamma}}{\gamma} \right\}^{\frac{1+2\kappa_2}{\gamma}} \, dx
\]
\[
\leq C^{\frac{1+2\kappa_2}{\gamma}} \left\{ \int_{\mathcal{X}} f(x)^{1-\frac{2\kappa_1}{1+2\kappa_2}} \, dx \right\}^{\frac{1-1+2\kappa_2}{\gamma}} \lesssim 1
\]
since \( 2 - \frac{2\kappa_1}{1-(1+2\kappa_2)/\gamma} > \frac{d}{\alpha+d} \), which holds because we assumed that \( \gamma > \gamma_* \). Hence
\[
\int_{\mathcal{X}_{m,n}} g(x)(f\phi_0)^2_x \, dx \to E\{(f\phi_0)^2_{Y_1}\} \quad \text{(49)}
\]
as \( m, n \to \infty \). We show in Section 7.7 that
\[
U_7 + U_8 = O \left( \frac{1}{n} \max \left\{ \frac{\log^{5/2} n}{k_Y^{1/2}}, \log^2 n \left( \frac{k_Y}{n} \right)^{(1)\beta/d}, \left( \frac{k_Y}{n} \right)^{\lambda_2(\gamma^{-1} - \gamma^{-1}) - \epsilon}, \frac{\log^2 m}{k_X^{1/2}}, \left( \frac{k_X}{m} \right)^{(2\lambda)\beta/d}, \left( \frac{k_X}{m} \right)^{\lambda_1(\gamma^{-1} - \gamma^{-1}) - \epsilon} \right\} \right). \quad \text{(50)}
\]
It now follows from (34), (35), (36), (37), (38), (39), (44), (45), (46), (47), (48), (49) and (50) that
\[
\text{Var}(\tilde{T}_{m,n}) = \frac{1}{m} \left[ \text{Var}(\phi_{X_1}) - 2E\{(f\phi_{10})_{X_1}\}E(\phi_{X_1}) - \{E(f\phi_{10})_{X_1}\}^2 + E\{(f\phi_{00})_{Y_1}\}^2 + \frac{1}{n} \left[ E\{(f\phi_{00})_{Y_1}\} - \{E(g\phi_{00})_{X_1}\}^2 \right] + o(m^{-1} + n^{-1}) \right.
\]
\[
= \frac{v_1}{m} + \frac{v_2}{n} + o(m^{-1} + n^{-1}).
\]
For the general, weighted case, we rely on the decomposition
\[
\text{Var}(\tilde{T}_{m,n}) = \sum_{j_x, j_y} \sum_{j_x, j_y} w_{X,j_x} w_{X,j_y} w_{Y,j_x} w_{Y,j_y}
\]
\[
\times \left\{ \frac{1}{m} \text{Cov}(\phi(\hat{f}_{(j_x),1}, \hat{g}_{(j_y),1}, X_1), \phi(\hat{f}_{(j_x),1}, \hat{g}_{(j_y),1}, X_1)) \right\}
\]
\[
+ \left( 1 - \frac{1}{m} \right) \text{Cov}(\phi(\hat{f}_{(j_x),1}, \hat{g}_{(j_y),1}, X_1), \phi(\hat{f}_{(j_x),2}, \hat{g}_{(j_y),2}, X_2)) \right\} \quad \text{(51)}
\]
Now, for example, when $\ell_X > j_X$, we have

$$(h_{x,f}(\rho(j_X),1,X), h_{x,f}(\rho(\ell_X),1,X), 1 - h_{x,f}(\rho(\ell_X),1,X))\mid X_1 = x \sim \text{Dir}(j_X, \ell_X - j_X, m - \ell_X),$$

and it may therefore be deduced similarly to the arguments leading to (35) that

$${\frac{\max_{j_X, \ell_X: w_X, j_X, w_X, \ell_X \neq 0, j_Y, \ell_Y: w_Y, j_Y, w_Y, \ell_Y \neq 0} \left| \text{Cov}(\phi(\hat{f}(j_X),1, \hat{g}(j_Y),1, X_1), \phi(\hat{f}(\ell_X),1, \hat{g}(\ell_Y),1, X_1)) - \text{Var}(\phi_{X_1}) \right|}{\to 0.} \quad (52)$$

The second term on the right-hand side of (51) is handled using relatively small modifications of the arguments used to study the covariance term in (34). These modifications are required to account for the fact that the $k_X$ that appears twice in the covariance term in (34) is now replaced with $j_X$ and $\ell_X$ (with similar changes to $k_Y$). Thus, for instance, the joint conditional distribution function of

$$(h_{x,f}(\rho(j_X),1,X), h_{y,f}(\rho(\ell_Y),2,X), h_{x,g}(\rho(j_Y),1,Y), h_{g,Y}(\rho(\ell_Y),2,Y))\mid X_1 = x, X_2 = y,$$

is now given by

$$F_{m,n,x,y}(s_1, s_2, t_1, t_2)$$

$$:= \mathbb{P}(N_1^{(1)} + N_3^{(1)} \geq j_X - 1_{\{||x-y|| \leq h_{x,f}(s_1)\}}, N_2^{(1)} + N_3^{(1)} \geq \ell_X - 1_{\{||x-y|| \leq h_{x,f}(s_2)\}})$$

$$\times \mathbb{P}(N_1^{(2)} + N_3^{(2)} \geq j_Y, N_2^{(2)} + N_3^{(2)} \geq \ell_Y).$$

Following the arguments through reveals that

$${\frac{\max_{j_X, \ell_X: w_X, j_X, w_X, \ell_X \neq 0, j_Y, \ell_Y: w_Y, j_Y, w_Y, \ell_Y \neq 0} \left| \text{Cov}(\phi(\hat{f}(j_X),1, \hat{g}(j_Y),1, X_1), \phi(\hat{f}(\ell_X),2, \hat{g}(\ell_Y),2, X_2)) - \frac{v_1 - \text{Var}(\phi_{X_1})}{m} - \frac{v_2}{n} \right|}{o(m^{-1} + n^{-1}).} \quad (53)$$

Finally, then, we can deduce from (51), (52) and (53), and using our hypotheses on $\|w_X\|_1$ and $\|w_Y\|_1$, that

$$\text{Var}(\hat{T}_{m,n}) - \frac{v_1}{m} - \frac{v_2}{n} = o\left(\left(\frac{1}{m} + \frac{1}{n}\right)\|w_X\|_1^2\|w_Y\|_1^2\right) = o\left(\frac{1}{m} + \frac{1}{n}\right),$$

as required.
7.4 Proofs of Theorems 3 and 4 on asymptotic normality and confidence intervals

The proof of Theorem 3 relies on the following result, which, together with Lemmas 15 and 20, shows that we may partition (minor modifications of) $\mathcal{X}_{m,f}$ and $\mathcal{X}_{n,g}$ into small pieces such that the $k$-nearest neighbour distances of points in distant pieces are roughly independent.

**Proposition 13.** Let $f \in \mathcal{F}_d$ be $\beta := (\lceil \beta \rceil - 1)$-times differentiable. Then there exists $n_0 = n_0(d, \beta)$ such that, for all $n \geq n_0$ and $k \in [3, n/\log n)$, we can find a partition $\{C_j : j = 1, \ldots, V_n\}$ of $\mathcal{X}_n := \{x : f(x)M_{f,\beta}(x) - d \geq kn^{-1}\log^2 n\}$ and points $\{x_j : j = 1, \ldots, V_n\}$ in $\widetilde{\mathcal{X}}_n := \{x : f(x)M_{f,\beta}(x) - d \geq kn^{-1}\log^{7/4} n\}$ satisfying the following properties for each $j = 1, \ldots, V_n$:

(i) we have $C_j \subseteq B_{x_j} \left(3 \left(\frac{k\log n}{nV_d(x_j)}\right)^{1/d}\right)$;

(ii) we have $\left|\left\{j' = 1, \ldots, V_n : \text{dist}(C_j, C_{j'}) \leq 4 \left(\frac{k\log n}{nV_d(x_j)}\right)^{1/d}\right\}\right| \leq 2^{2+4d}\log n$.

**Proof of Proposition 13.** Let $\{x_j : j = 1, \ldots, V_n\}$ be a Poisson process on $\widetilde{\mathcal{X}}_n$ with intensity function $n f(\cdot)/k$, and let $P$ denote the corresponding Poisson random measure. Writing $\text{sargmin}(S)$ for the smallest element of an ordered set $\text{argmin}(S)$, we may partition $\mathcal{X}_n$ into the associated (random) Voronoi cells $\{C_j : j = 1, \ldots, V_n\}$, where $C_j := \{x \in \mathcal{X}_n : \text{sargmin}_{j'=1,\ldots,V_n} \|x - x_{j'}\| = j\}$. We proceed by showing that, for $n$ and $k$ sufficiently large, there is an event of positive probability on which $\{C_j : j = 1, \ldots, V_n\}$ and $\{x_j : j = 1, \ldots, V_n\}$ satisfy (i) and (ii), and we therefore deduce the existence of such a partition. First, let $z_1, \ldots, z_N \in \mathcal{X}_n$ be such that

$$\|z_i - z_j\| \geq h^{-1}_{z_j,f}(k/n) + h^{-1}_{z_j,f}(k/n) =: r(z_i, z_j)$$

for all $i \neq j$, and such that $\sup_{x \in \mathcal{X}_n} \min_{j=1,\ldots,N} \|x - z_j\| r(x, z_j) < 1$. (We can construct this set inductively: first, choose $z_1 \in \mathcal{X}_n$ arbitrarily. If the second condition is not satisfied once $z_1, \ldots, z_N$ have been defined, then there exists $x \in \mathcal{X}_n$ such that $\|x - z_j\| \geq r(x, z_j)$ for all $j = 1, \ldots, N$ and we can set $z_{N+1} := x$.) For all $i \neq j$, the intersection $B_{z_i}(h^{-1}_{z_i,f}(k/n)) \cap B_{z_j}(h^{-1}_{z_j,f}(k/n))$ has Lebesgue measure zero and thus

$$1 \geq \sum_{j=1}^{N} h_{z_j,f}(h^{-1}_{z_j,f}(k/n)) = \frac{Nk}{n}.$$
In particular, $N \leq n/k$.

We now show that if $x \in \mathcal{X}_n$ is such that $\|x - z\| < r(x, z)$ for some $z \in \{z_1, \ldots, z_N\} \subseteq \mathcal{X}_n$ then $f(x) \approx f(z)$. Suppose initially that $r_2 := \{M_{f, \beta}(z) d \log n\}^{-1/d} \leq \|x - z\| < r(x, z)$.

Then, writing $r_1 := \|x - z\| - r_2/2$, writing $\bar{x}$ for the point on the line segment between $x$ and $z$ such that $\|\bar{x} - z\| = r_2$ and writing $I(s) := \int_0^s B_{(d+1)/2,1/2}(t) \, dt$, we have by Lemma 16 that, for $n \geq n_0(d, \beta)$ sufficiently large,

$$
\int_{B_x(r_1)} f(w) \, dw \geq \int_{B_x(r_1) \cap B_z(r_2)} f(w) \, dw \geq \frac{f(z)}{2} \mu_d(B_x(r_1) \cap B_z(r_2)) \\
\geq \frac{f(z)}{2} \mu_d(B_x(r_2/2) \cap B_z(r_2)) = \frac{V_d(z)}{2} \left\{ \left( \frac{r_2}{2} \right)^d I(15/16) + r_2^d I(15/64) \right\} \\
\geq \frac{V_d}{2^{d+1}} I(15/16) \frac{k \log n}{n} \geq \frac{k}{n}.
$$

It follows, by Lemma 15 and the fact that $z \in \mathcal{X}_n$, that there exists $n_1 = n_1(d, \beta) \geq n_0$, such that for $n \geq n_1$,

$$
\|x - z\| \leq r_1 + h^{-1}_{z, f}(k/n) \leq r_1 + 2 \left( \frac{k}{nV_d(z)} \right)^{1/d} \leq r_1 + \frac{r_2}{4} = \|x - z\| - \frac{r_2}{4},
$$

which is a contradiction. Thus, for $n \geq n_1$ we have that $\|x - z\| \leq r_2$. In particular, by Lemma 16, for $x, z \in \mathcal{X}_n$ with $\|x - z\| < r(x, z)$, and for $n \geq n_1$, we have that

$$
\left| \frac{f(x)}{f(z)} - 1 \right| \leq \frac{8d^{1/2}}{\log(1+\beta/d) n}. \quad (54)
$$

To establish (i), first we define the event

$$
\Omega_0 := \bigcap_{j=1}^N \left\{ P \left\{ B_{z_j} \left( h_{z_j, f}^{-1}(k \log n/n) \right) \right\} \geq 1 \right\}.
$$

By Lemmas 15 and 16 and very similar arguments to those leading up to (80), there exists $n_2 = n_2(d, \beta) \geq n_1$ such that $B_{z_j} \left( h_{z_j, f}^{-1}(k \log n/n) \right) \subseteq \tilde{\mathcal{X}}_n$ for all $n \geq n_2$ and $j = 1, \ldots, V_n$.

Then, for $n \geq n_2$ we have that

$$
\mathbb{P}(\Omega_0^c) \leq N \exp \left( - \frac{n k \log n}{n} \right) \leq \frac{1}{k^k}.
$$

Let $j \in \{1, \ldots, V_n\}$ be given, and suppose that $x \in C_j$. Let $z$ be in our covering set such
that \(\|x - z\| < r(x, z)\) and, on the event \(\Omega_0\), let \(j' \in \{1, \ldots, V_n\}\) be such that \(\|x_{j'} - z\| \leq h_{z,j}^{-1}(k \log n/n)\). By (54), Lemma 15 and Lemma 16, there exists \(n_3 = n_3(d, \beta) \geq n_2\) such that, for \(n \geq n_3\), we have that \(h_{z,j}^{-1}(k \log n/n) \leq \frac{3}{2} (\frac{k \log n}{n V_d f(z)})^{1/d}\) and hence that

\[
\|x_{j'} - x\| \leq \|x_{j'} - z\| + \|z - x\| < h_{z,j}^{-1}(k \log n/n) + h_{z,j}^{-1}(k/n) + h_{z,j}^{-1}(k/n)
\leq 2 \left(\frac{k \log n}{n V_d f(x_{j'})}\right)^{1/d}.
\]

If \(j' = j\) then we are done, so suppose instead that \(\|x - x_j\| \leq \|x - x_{j'}\|\). Then

\[
\|x_j - x_{j'}\| \leq 2\|x - x_{j'}\| \leq 4 \left(\frac{k \log n}{n V_d f(x_{j'})}\right)^{1/d}
\]

so we can use Lemma 16 to argue that \(f(x_j) \approx f(x_{j'})\). In particular, there exists \(n_4 = n_4(d, \beta) \geq n_3\) such that, for \(n \geq n_4\) we have that

\[
\|x - x_j\| \leq \|x - x_{j'}\| \leq 2 \left(\frac{k \log n}{n V_d f(x_{j'})}\right)^{1/d} \leq 3 \left(\frac{k \log n}{n V_d f(x_{j})}\right)^{1/d}.
\]

So, for \(n \geq n_4\), we have that (i) holds on \(\Omega_0\).

Now, by Lemma 16, there exists \(n_5 = n_5(d, \beta) \geq n_4\) such that, for \(n \geq n_5\) we have that \(\frac{9}{k} h_{z,j} f(16(\frac{k \log n}{n V_d f(z_j)})) \leq 2^{1+4d} \log n\) for all \(j \in \{1, \ldots, N\}\), and hence, by Bennett’s inequality, that the event

\[
\Omega_1 := \bigcap_{j=1}^N \left\{ P\left\{ B_{z_j} \left(16 \left(\frac{k \log n}{n V_d f(z_j)}\right)^{1/d}\right) \leq 2^{2+4d} \log n \right\} \right\}
\]

satisfies

\[
\mathbb{P}(\Omega_1^c) \leq N \exp \left(-\frac{(2^{2+4d} \log n - 2^{1+4d} \log n)^2}{2^{3+4d} \log n}\right) \leq \frac{n}{k} \exp(-2^{4d-1} \log n) \leq \frac{1}{k}.
\]

Now, on \(\Omega_0\), if \(\text{dist}(C_j, C_{j'}) \leq 4 \left(\frac{k}{n V_d f(x_j)}\right)^{1/d}\) then we must have

\[
\|x_j - x_{j'}\| \leq 4 \left(\frac{k}{n V_d f(x_j)}\right)^{1/d} + 3 \left(\frac{k \log n}{n V_d f(x_j)}\right)^{1/d} + 3 \left(\frac{k \log n}{n V_d f(x_{j'})}\right)^{1/d}.
\]

(55)

Using Lemma 15, there exists \(n_6 = n_6(d, \beta) \geq n_5\) such that \(\|x_j - x_{j'}\| \leq 6h_{z,j}^{-1}(k \log n/n) + 6h_{z,j, f}^{-1}(k \log n/n)\) for \(n \geq n_6\) and hence, by a very similar argument to that leading up to (54),
that $|f(x_j')/f(x_j) - 1| \leq 8d^{1/2} \log^{-1/2}(1^\beta)/(2d) n$ for $n \geq n_6$. Thus, writing $z_j^*$ for an element of our covering set with $\|x_j - z_j^*\| < r(x_j, z_j^*)$, there exists $n_7 = n_7(d, \beta) \geq n_6$ such that, on $\Omega_0 \cap \Omega_1$, for all $n \geq n_7$ we have that

$$\left| \left\{ j' \in V_n : \text{dist}(C_j, C_{j'}) \leq 4\left(\frac{k}{nV_d f(x_j)}\right)^{1/d} \right\} \right| \leq \left| \left\{ j' \in V_n : \|x_j - x_j'\| \leq 8\left(\frac{k \log n}{nV_d f(x)}\right)^{1/d} \right\} \right|$$

$$\leq \left| \left\{ j' \in V_n : \|z_j^* - x_j\| \leq 16\left(\frac{k \log n}{nV_d f(x_j)}\right)^{1/d} \right\} \right|$$

$$\leq 2^{2+4d} \log n$$

for all $j \in V_n$. This establishes that, for $n \geq n_7$, with probability at least $1 - 2/k$ we have that both (i) and (ii) hold. Thus, since $k \geq 3$, there is a positive probability of both (i) and (ii) holding simultaneously and we can deduce the existence of the required partition. 

**Proof of Theorem 3.** We start by linearising our unweighted estimator. Consider

$$E_{m,n} := \frac{1}{m} \sum_{i=1}^{m} \left\{ \phi(\hat{f}_{(kX),i}, \hat{g}_{(kY),i}, X_i) - \phi(\hat{f}_{(kX),i}, g(X_i), X_i) - \phi(f(X_i), \hat{g}_{(kY),i}, X_i) + \phi_{X_i} \right\}$$

$$= \frac{1}{m} \sum_{i=1}^{m} \phi^* (\hat{f}_{(kX),i}, \hat{g}_{(kY),i}, X_i)$$

with $\phi^*(u, v, x) := \phi(u, v, x) - \phi(u, g(x), x) - \phi(f(x), v, x) + \phi(f(x), g(x), x)$. This is of the same form as the estimators we have already considered, and we have $\phi^*(f(x), g(x), x) \equiv \phi_{10}^*(f(x), g(x), x) \equiv 0$. Therefore, by very similar arguments to those used in the proof of Proposition 9, we have that $\text{Var}(E_{m,n}) = o(m^{-1} + n^{-1})$. Define

$$X_{m,f} := \left\{ x : f(x)M_\beta(x)^{-d} \geq \frac{k_x \log^2 m}{m} \right\} \quad \text{and} \quad X_{n,g} := \left\{ x : g(x)M_\beta(x)^{-d} \geq \frac{k_y \log^2 n}{n} \right\}.$$ 

Recalling the definitions of $A_i^X$ and $A_i^Y$ in (74), we may now use Lemma 20 and very similar
techniques to those used to bound $S_1, T_1, U_0$ in the proof of Proposition 9 to write

$$
\tilde{T}_{m,n} - \mathbb{E}\tilde{T}_{m,n} = \frac{1}{m} \sum_{i=1}^{m} \left[ \mathbb{1}_{A_i^X} \mathbb{1}_{\{X_i \in \mathcal{X}_{m,f}\}} \phi(f(k_i), g(X_i), X_i) + \mathbb{1}_{A_i^Y} \mathbb{1}_{\{X_i \in \mathcal{X}_{n,g}\}} \left\{ \phi(f(X_i), \hat{g}(k_Y), X_i) - \phi_X \right\} \right]
+ \mathbb{E}(E_{m,n} - \tilde{T}_{m,n}) + \alpha_p(m^{-1/2} + n^{-1/2})
= \tilde{T}^{(1)}_m - \mathbb{E}\tilde{T}^{(1)}_m + \tilde{T}^{(2)}_m - \mathbb{E}\tilde{T}^{(2)}_m + \alpha_p(m^{-1/2} + n^{-1/2}).
$$

Defining $\tilde{T}^{(2)}_n := \mathbb{E}(\tilde{T}^{(2)}_m|Y_1, \ldots, Y_n)$, we can now use Lemma 15 to write

$$
\text{Var}(\tilde{T}^{(2)}_m - \tilde{T}^{(2)}_n) = \mathbb{E}\left[ \text{Var}(\tilde{T}^{(2)}_m - \tilde{T}^{(2)}_n|Y_1, \ldots, Y_n) \right]
= \frac{1}{m} \mathbb{E}\left[ \text{Var}\left( \mathbb{1}_{A_i^Y} \mathbb{1}_{\{X_i \in \mathcal{X}_{n,g}\}} \left\{ \phi(f(X_i), \hat{g}(k_Y), 1) - \phi_X \right\} \bigg| Y_1, \ldots, Y_n \right) \right]
\lesssim \frac{1}{m} \int_{\mathcal{X}_{n,g}} f(x)^{1-2\kappa_1} \mathbb{E}\left( \left\{ \left( \frac{k_Y M^{(d)}_{\beta}(x)\lambda(x)}{ng(x)} \right)^{2(2\alpha,\beta)} + \frac{\log n}{k_Y} \right\} dx = o(1/m). \right.
$$

Thus, we have

$$
\tilde{T}_{m,n} - \mathbb{E}\tilde{T}_{m,n} = \tilde{T}^{(1)}_m - \mathbb{E}\tilde{T}^{(1)}_m + \tilde{T}^{(2)}_m - \mathbb{E}\tilde{T}^{(2)}_m + \alpha_p(m^{-1/2} + n^{-1/2}).
$$

Noting that $\tilde{T}^{(1)}_m$ depends only on $X_1, \ldots, X_m$ and $\tilde{T}^{(2)}_n$ depends only on $Y_1, \ldots, Y_n$ (so they are independent), we now proceed to establish the asymptotic normality of these two random variables separately, and then the result will follow.

Our strategy is to use the partition of Proposition 13 to express each of $\tilde{T}^{(1)}_m$ and $\tilde{T}^{(2)}_n$ as a sum of terms indexed by the vertices of a dependency graph, where the maximum degree of this dependency graph, as well as certain moments, are controlled. This allows us to apply the central limit theorem of Baldi and Rinott (1989) for such sums.

Letting $\{C_j : j \in \{1, \ldots, V_m\}\}$ denote a partition of $\mathcal{X}_{m,f}$ as in the statement of Proposition 13, and writing $\mathcal{X}_{m,f}^{(j)} := C_j \cap \mathcal{X}_{m,f}$, for $j = 1, \ldots, V_m$ define

$$
W_j := \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}_{A_i^X} \mathbb{1}_{\{X_i \in \mathcal{X}_{m,f}^{(j)}\}} \phi(f(k_i), g(X_i), X_i)
$$

so that $\tilde{T}^{(1)}_m = \sum_{j=1}^{V_m} W_j$. For $j, j' = 1, \ldots, V_m$, write $j \sim j'$ if $W_j$ and $W_{j'}$ are dependent. Because we are working on the events $A_i^X$, by Lemma 16 and property (i) of the partition
and arguing as after (55), there exists $m_0 = m_0(d, \vartheta)$ such that for $m \geq m_0$, we can only have $j \sim j'$ if

$$
\text{dist}(C_j, C_{j'}) \leq \sup_{x \in \mathcal{X}^{(j)}_m, f} h^{-1}_x(a^+_{m,x}) + \sup_{x' \in \mathcal{X}^{(j')}_{m,f}} h^{-1}_x(a^+_{m,x})
\leq \sup_{x \in \mathcal{X}^{(j)}_m, f} \left( \frac{3k_X}{2mv_d f(x)} \right)^{1/d} + \sup_{x' \in \mathcal{X}^{(j')}_{m,f}} \left( \frac{3k_X}{2mv_d f(x')} \right)^{1/d} \leq 4 \left( \frac{k_X}{mv_d f(x_j)} \right)^{1/d},
$$

where $\{x_j : j = 1, \ldots, V_m\}$ are the points associated to our partition given in Proposition 13. By property (ii) of our partition, then, for each $j = 1, \ldots, V_m$, we have $|\{j' : j' \sim j\}| \leq 2^{2+4d} \log m$. For $j = 1, \ldots, V_m$ and $p \in \mathbb{N}$, we write $L_j^{(p)}$ for the number of connected subsets of $\{1, \ldots, V_m\}$ (with edge relations defined by $\sim$) of cardinality at most $p$ containing $j$. Then

$$
L_j^{(p)} \leq 2^{(p-1)(2+4d)} \log^{p-1} m
$$

for $p = 3, 4$. Since $\gamma > \gamma_1^* > 4\kappa_2$, we have by Hölder’s inequality that for $p = 3, 4$, any $a > 0$ and any $\epsilon \in (0, \gamma - p\kappa_2)$,

$$
\int X |f(x)|^{1+a-p\kappa_1} g(x)^{-p\kappa_2} dx \leq \left\{ \int X \frac{f(x)^2}{g(x)^{\gamma-\epsilon}} dx \right\}^{\frac{p\kappa_2}{\gamma-\epsilon}} \left\{ \int X f(x)^{1+a-p\kappa_1-2p\kappa_2/(\gamma-\epsilon)} dx \right\}^{1-\frac{p\kappa_2}{\gamma-\epsilon}}.
$$

Thus, whenever $a > 1 + p\kappa_1 - \frac{2a+3}{\alpha+3}(1 - p\kappa_2/\gamma)$ we have by Lemma 18 that

$$
\sup_{(f,g) \in \mathcal{F}_{d,\vartheta}} \int X |f(x)|^{1+a-p\kappa_1} g(x)^{-p\kappa_2} dx < \infty. \quad (56)
$$

Now, by Lemma 16 and property (i) of our partition, for any $j = 1, \ldots, V_m$ we have

$$
\sup_{x \in C_j} \max \left\{ \left| \frac{f(x)}{f(x_j)} - 1 \right|, \left| \frac{g(x)}{g(x_j)} - 1 \right| \right\} \leq 8d^{1/2} \left\{ \frac{3^d k_X M_\beta(x_j)^d \log m}{mv_d f(x_j)} \right\}^{1/\beta} \leq \frac{8d^{1/2} 3^{1/\beta}}{(V_d \log^{3/4} m)^{1/\beta}}. \quad (57)
$$

Moreover, by very similar methods to those used in the proof of Proposition 9, we may see that

$$
\text{Var}(\hat{T}_m^{(1)}) = \text{Var} \left( \frac{1}{m} \sum_{i=1}^{m} 1_{A_i} 1_{X_i \in \mathcal{X}_{m,f}} \phi(\hat{f}_{(k_x)}; g(X_i), X_i) \right) = \frac{v_1}{m} + o \left( \frac{1}{m} \right). \quad (58)
$$

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Hence, using (56), (57), (58) and the facts that \( \mathbb{P}(X_1 \in \mathcal{X}_{m,f}^{(j)}) \lesssim k_X m^{-1} \log m \) and \( v_1 \geq C^{-1} \), we have that for \( p = 3, 4 \),

\[
\frac{1}{\text{Var}^{p/2}(\tilde{T}_m^{(1)})} \sum_{j=1}^m L_j^{(p)} \mathbb{E}\{|W_j - \mathbb{E}W_j|^p\} \lesssim m^{p/2} \log^{p-1} m \sum_{j=1}^m \mathbb{E}\left\{ \left| \frac{1}{m} \sum_{i=1}^m 1_{A_X} 1_{(X_i \in \mathcal{X}_{m,f}^{(j)})} \phi(f_{X,i}, g(X_i), X_i) \right|^p \right\} \lesssim m^{p/2} \log^{p-1} m \sum_{j=1}^m f(x_j)^{-p\kappa_1} g(x_j)^{-p\kappa_2} m^{-p} \{ m^p \mathbb{P}(X_1 \in \mathcal{X}_{m,f}^{(j)})^p + m \mathbb{P}(X_1 \in \mathcal{X}_{m,f}^{(j)}) \} \lesssim k_X^p \log^{2p-2} m \inf_{a > 0} \left( \frac{k_X}{m} \right)^{a - 1} \int_{\mathcal{X}} f(x)^{1+a-p\kappa_1} g(x)^{-p\kappa_2} dx \to 0,
\]

since \( \gamma > \gamma_2^* \) and \( k_X^p m^{-(\tau_1^*-\epsilon)} \to 0 \) for some \( \epsilon > 0 \). It now follows from Theorem 1 of Baldi and Rinott (1989) that

\[
d_K \left( \mathcal{L} \left( \frac{m^{1/2} \{ \tilde{T}_m^{(1)} - \mathbb{E}\tilde{T}_m^{(1)} \} / v_1^{1/2} }{N(0, 1)} \right), N(0, 1) \right) \to 0.
\]

We now take a similar approach to establish the asymptotic normality of \( \tilde{T}_n^{(2)} \). Letting \( \{C_j : j = 1, \ldots, V_n\} \) denote a partition of \( \mathcal{X}_{n,g} \) as in the statement of Proposition 13, we may write \( \rho(k_{Y,Y}(x)) := \|Y(k_{Y,Y}(x)) - x\|, A_Y := \{x : h_{x,g}(k_{Y,Y}(x)) \in \mathcal{I}_{n,Y} \}, \mathcal{X}_{n,g}^{(j)} := C_j \cap \mathcal{X}_{n,g} \), and

\[
W_j := \int_{\mathcal{X}_{n,g}^{(j)} \cap A_Y} f(x) \{ \phi(f(x), k_Y/Y(x)^d, x) - \phi_x \} dx,
\]

so that \( \tilde{T}_n^{(2)} = \sum_{j=1}^{V_n} W_j \). By properties (i) and (ii) of our partition we again have that \( L_j^{(p)} \lesssim \log^{p-1} n \), as above. Moreover, using Lemma 18 and the fact that \( \gamma > \gamma_2^* \) we have for \( p = 3, 4 \) that

\[
\sup_{(f,g) \in \mathcal{F}_{d,0}} \int_{\mathcal{X}} f(x)^{p-p\kappa_1} g(x)^{-\{(p-1)+a-p\kappa_2\}} dx < \infty \quad (59)
\]

when \( a > p - 1 + p\kappa_2 - \frac{4a + 3d - (4 - p + p\kappa_1)(\alpha + d)}{2a + d} \gamma \). Recall the definition of the conditional distribution function \( F_{n,x,y}^{(2)} \) from the proof of Proposition 9. By Lemma 17 and a similar but
Now, using an analogous statement to that in (57), using (59), (60) and the facts that \( \Pr(Y_1 \in X_{j}) \leq k_Y n^{-1} \log n \) and that \( v_2 \geq C^{-1} \), we have for \( p = 3, 4 \) that

\[
\frac{1}{\text{Var}^{p/2}(\tilde{T}_n^{(2)})} \sum_{j=1}^{V_n} L_j^{(p)} \mathbb{E}\{|W_j - \mathbb{E}W_j|^p\} \\
\leq n^{p/2} \log^{p-1} n \int_{\chi_{n,j}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} dx \\
\leq \frac{k_Y^{p-1} \log^{2(p-1)} n}{n^{p/2-1}} \int_{\chi_{n,j}} f(x)^{p-p\kappa_1} g(x)^{-(p-1)-p\kappa_2} dx \\
\leq k_Y^{p/2} \log^{2(p-1)} n \inf_{a > 0} \left( \frac{k_Y}{n} \right)^{\frac{p}{2}-a} \int_{\chi} f(x)^{p-p\kappa_1} g(x)^{-(p-1)+a-p\kappa_2} dx \to 0
\]

since \( \gamma > \gamma_2^* \) and \( k_Y n^{-(\gamma_2^* - \epsilon)} \to 0 \) for some \( \epsilon > 0 \). By Theorem 1 of Baldi and Rinott (1989) we now have that

\[
\text{d}_K\left( \mathcal{L}\left( \frac{n^{1/2} \{ \tilde{T}_n^{(2)} - \mathbb{E}\tilde{T}_n^{(2)} \}}{v_2^{1/2}} \right), N(0, 1) \right) \to 0.
\]

For our weighted estimator \( \hat{T}_{m,n} \), we can define weighted analogues \( \hat{T}_{m}^{(1)} \) and \( \hat{T}_{n}^{(2)} \) of \( \tilde{T}_m^{(1)} \) and \( \tilde{T}_n^{(2)} \) and deduce that

\[
\hat{T}_{m,n} - \mathbb{E}(\hat{T}_{m,n}) = \hat{T}_{m}^{(1)} - \mathbb{E}(\hat{T}_{m}^{(1)}) + \hat{T}_{n}^{(2)} - \mathbb{E}(\hat{T}_{n}^{(2)}) + o_p(m^{-1/2} + n^{-1/2}),
\]

(61)
where
\[
d_{K}\left(\mathcal{L}\left(\frac{m^{1/2}\{\hat{T}_m^{(1)} - \mathbb{E}\hat{T}_m^{(1)}\}}{v_1^{1/2}}, N(0, 1)\right) + d_{K}\left(\mathcal{L}\left(\frac{n^{1/2}\{\hat{T}_n^{(2)} - \mathbb{E}\hat{T}_n^{(2)}\}}{v_2^{1/2}}, N(0, 1)\right) \to 0. \right. \right) \tag{62}
\]

If \( W, X, Y, Z \) are independent random variables it can be seen by simple conditioning arguments that
\[
d_{K}(\mathcal{L}(W + X), \mathcal{L}(Y + Z)) \leq d_{K}(\mathcal{L}(W), \mathcal{L}(Y)) + d_{K}(\mathcal{L}(X), \mathcal{L}(Z)). \tag{63}
\]
Thus, by (61), (62), (63) and Corollary 6, we may write
\[
\widehat{Z}_{m,n} := \frac{\hat{T}_{m,n} - T}{\{m^{-1}v_1 + n^{-1}v_2\}^{1/2}} = Z_{m,n}^{*} + W_{m,n},
\]
where \( d_{K}(\mathcal{L}(Z_{m,n}^{*}), N(0, 1)) \to 0 \) and \( W_{m,n} = o_{p}(1) \). Thus, for any \( \epsilon > 0 \),
\[
d_{K}(\widehat{Z}_{m,n}, N(0, 1)) \leq \sup_{x \in \mathbb{R}}|P(\widehat{Z}_{m,n} \leq x, |W_{m,n}| \leq \epsilon) - \Phi(x)| + P(|W_{m,n}| > \epsilon)
\]
\[
\leq \sup_{x \in \mathbb{R}} \max\{P(Z_{m,n}^{*} \leq x + \epsilon) - \Phi(x), \Phi(x) - P(Z_{m,n}^{*} \leq x - \epsilon)\} + 2P(|W_{m,n}| > \epsilon)
\]
\[
\leq d_{K}(Z_{m,n}^{*}, N(0, 1)) + \frac{\epsilon}{(2\pi)^{1/2}} + 2P(|W_{m,n}| > \epsilon),
\]
so the result follows.

**Proof of Theorem 4.** The main task is to establish the consistency of \( \hat{V}_{m,n}^{(1)} \) and \( \hat{V}_{m,n}^{(2)} \). For the first of these, let \( \delta := (1/2 - \zeta)/(2\zeta) \) so that \( \zeta < 1/(2(1 + \delta)) \) and hence
\[
\mathbb{E}\left[\{(\hat{\phi}_{X_1} + (f\hat{\phi}_{10})X_1\right)^{2(1+\delta)}\] \lesssim \int_\mathcal{X} f(x)^{1-2(1+\delta)\kappa_1}g(x)^{-2(1+\delta)\kappa_2} dx \lesssim 1, \tag{64}
\]
by Lemma 18. Using this and Lemmas 14, 17 and 18, and writing \( \tilde{\phi}(u, v, x) := \{\phi(u, v, x) + \)

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\( u \phi_{10}(u, v, x) \}^2 \) and \( b_{m,n} := \log m \wedge \log n \), we have that

\[
\begin{aligned}
&\left| \mathbb{E} \hat{V}_{m,n}^{(1)} - \int_X f(x) \{ \phi_x + (f \phi_{10})_x \}^2 \, dx \right| \\
&= \left| \int_X f(x) \int_0^1 \int_0^1 \left[ \min \{ \tilde{\phi}(u_x, v_x, t, x), b_{m,n} \} - \tilde{\phi}_x \right] B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx \right| \\
&= \left| \int_X f(x) \int_0^1 \int_0^1 \left[ \min \{ \tilde{\phi}(u_x, v_x, t, x), b_{m,n} \} - \tilde{\phi}_x \right] \\
&\quad \times B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx \right| + O \left( \frac{1}{(\log m \wedge \log n)^\delta} \right)
\end{aligned}
\]

\[
\leq \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \left| \tilde{\phi}(u_x, v_x, t, x) - \tilde{\phi}_x \right| B_{kX,m-kX}(s) B_{kY,n+1-kY}(t) \, ds \, dt \, dx
\]

\[
\leq \int_{X_{m,n}} f(x)^{1-2\kappa} g(x)^{-2\kappa} \left\{ \frac{\log^{1/2} m}{k_{X}^{1/2}} + \frac{\log^{1/2} n}{k_{Y}^{1/2}} + \left( \frac{k_{X} M_{\beta}(x)^d}{m f(x)} \right)^{2\lambda_2} + \left( \frac{k_{Y} M_{\beta}(x)^d}{n g(x)} \right)^{2\lambda_2} \right\} \, dx
\]

\[
+ O \left( b_{m,n} \max \left\{ \left( \frac{k_{X} \log m}{m} \right)^{\lambda_1}, \left( \frac{k_{Y} \log n}{n} \right)^{\lambda_2} \frac{1}{m^{1/4}}, \frac{1}{n^{1/4}}, \frac{1}{b_{m,n}^{1+\delta}} \right\} \right) = o(1).
\]

Now, for \( i = 1, \ldots, m \), write \( \xi_i := \min \{ \tilde{\phi}(f_{(k_X)}), \tilde{\phi}(g_{(k_Y)}), X_i, b_{m,n} \} \), \( \xi^*_i := \min \{ \tilde{\phi}_X, b_{m,n} \} \) and

\[
\tilde{X}_{i,n} := \left\{ x : \frac{f(x)}{M_{\beta}(x)^d} \geq \frac{k_{X}^{1/2}}{m^{1/2}}, \frac{g(x)}{M_{\beta}(x)^d} \geq \frac{k_{Y}^{1/2}}{n^{1/2}} \right\}.
\]
Then, by Cauchy–Schwarz,

\[
\text{Var}(\hat{V}_{m,n}^{(1)}) = \frac{1}{m} \text{Var}(\xi_1) + 2 \left( 1 - \frac{1}{m} \right) \text{Cov}(\xi_1 - \xi_1^*, \xi_2^*) + \left( 1 - \frac{1}{m} \right) \text{Cov}(\xi_1 - \xi_1^*, \xi_2 - \xi_2^*) \\
\leq \frac{b_{m,n}^2}{m} + 2b_{m,n}\mathbb{E}\{(\xi_1 - \xi_1^*)^2\}\]^{1/2} + \mathbb{E}\{(\xi_1 - \xi_1^*)^2\}
\leq 4b_{m,n}\left[\mathbb{E}\{1_{A_1^X \cap A_1^Y}1_{\{X_1 < \tilde{X}_{m,n}\}}(\xi_1 - \xi_1^*)^2\}\right]^{1/2} + \mathbb{E}\{1_{A_1^X \cap A_1^Y}1_{\{X_1 < \tilde{X}_{m,n}\}}(\xi_1 - \xi_1^*)^2\}
+ O\left(b_{m,n}^2 \max\left\{\frac{1}{m}, \frac{1}{n^2}, \left(\frac{kX}{m}\right)^{\lambda_1/4}, \left(\frac{KY}{n}\right)^{\lambda_2/4}\right\}\right)
\leq b_{m,n}\left[\int_{\tilde{X}_{m,n}} \frac{f(x)^{1-2\kappa_1}}{g(x)^{2\kappa_2}} \left\{ \frac{\log m}{kX}, \frac{\log n}{kY}, \left(\frac{kX}{m}\right)^{2\lambda_1}, \left(\frac{kY}{n}\right)^{2\lambda_2} \right\} \right]^{1/2}
+ O\left(b_{m,n}^2 \max\left\{\frac{1}{m}, \frac{1}{n^2}, \left(\frac{kX}{m}\right)^{\lambda_1/4}, \left(\frac{KY}{n}\right)^{\lambda_2/4}\right\}\right)
= O\left(b_{m,n}^2 \max\left\{\frac{\log^{1/2} m}{kX^{1/2}}, \frac{\log^{1/2} n}{kY^{1/2}}, \left(\frac{kX}{m}\right)^{2\lambda_1}, \left(\frac{kY}{n}\right)^{2\lambda_2} \right\}\right) = o(1).
\]

By very similar arguments to those employed in the proof of Proposition 12 we have that
\[
\mathbb{E}(\hat{V}_{m,n}^{(1),2}) - \int_{X} f(x)\{\phi_x + (f \phi_{10})_x\} \, dx = o(1).
\]
By Proposition 9 we have that \(\text{Var}(\hat{T}_{m,n}) = o(1)\). Since \(\zeta < 1/2\), the summands in \(\hat{V}_{m,n}^{(1)} - \hat{T}_{m,n}\) are square integrable and, writing \(\xi_i := \hat{f}_{(kX),i} \phi_{10}(\hat{f}_{(kX),i} \hat{g}_{(kY),i}, X_i)\) and \(\xi_i^* := (f \phi_{10})_i\), we have by Cauchy–Schwarz again that

\[
\text{Var}\left(\frac{1}{m} \sum_{i=1}^{m} \xi_i\right) \leq \frac{1}{m} \text{Var}(\xi_1) + 2\text{Var}^{1/2}(\xi_2)\text{Var}^{1/2}(\xi_1 - \xi_1^*) + \text{Var}(\xi_1 - \xi_1^*) = o(1).
\]

Combining our bounds on expectations and variances we have now established that, for any \(\epsilon > 0\),

\[
\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,\delta}} \max_{kX \in \{kX_1, \ldots, kX_1^\prime\}, kY \in \{kY_1, \ldots, kY_1^\prime\}} \mathbb{P}(|\hat{V}_{m,n}^{(1)} - v_1| \geq \epsilon) \rightarrow 0.
\]

(65)

Now suppose that \(\delta, \epsilon > 0\) are small enough that \(\frac{\delta}{\gamma - \epsilon} + \frac{\delta}{\gamma - \epsilon} < \gamma/\gamma_* - 1\). Then we have that \((1 + 2\kappa_2)(1 + \delta) < \gamma - \epsilon\), that \(2 + \delta - 2(1 + \delta)(\kappa_1 + \frac{1+2\kappa_2}{\gamma - \epsilon}) > d(1 - \frac{(1+2\kappa_2)(1+\delta)}{\gamma - \epsilon})/(\alpha + d)\),
and therefore, by Lemma 18, that

\[
\sup_{(f,g) \in \mathcal{F}_{d,0}} \int_X f(x) \left\{ f(x)^{1-2\kappa_1} g(x)^{-1-2\kappa_2} \right\}^{1+\delta} \, dx \\
\leq \sup_{(f,g) \in \mathcal{F}_{d,0}} \left\{ \int_X f(x)^2 g(x)^{-(\gamma-\epsilon)} \, dx \right\}^{\frac{(1+2\kappa_2)(1+\delta)}{\gamma-\epsilon}} \left\{ \int_X f(x)^2 \left( \frac{1}{1-(1+2\kappa_2)/(1+\delta)} \right)^{1-(1+2\kappa_2)(1+\delta)} \, dx \right\} < \infty.
\]

Hence, by analogous calculations to those carried out earlier in this proof, we have for any \( \epsilon > 0 \) that

\[
\sup_{\phi \in \Phi(\xi)} \sup_{(f,g) \in \mathcal{F}_{d,0}} \max_{k_X \in \{k_X^1,\ldots,k_X^c\}} \max_{k_Y \in \{k_Y^1,\ldots,k_Y^c\}} \mathbb{P}(|\hat{V}_{m,2} - v_2| \geq \epsilon) \to 0. \tag{66}
\]

To conclude the proof, given \( \epsilon > 0 \), define the event \( B_\epsilon := \left\{ \max(|\hat{V}_{m,1}/v_1 - 1|, |\hat{V}_{m,2}/v_2 - 1|) \leq \epsilon \right\} \), and define the shorthand

\[
\hat{Z} := \frac{\hat{T}_{m,n} - T}{\{m^{-1}\hat{V}_{m,1} + n^{-1}\hat{V}_{m,2}\}^{1/2}} \quad \text{and} \quad Z^* := \frac{\hat{T}_{m,n} - T}{\{m^{-1}v_1 + n^{-1}v_2\}^{1/2}}.
\]

For all \( \epsilon \in (0, 1/2) \) we have that

\[
d_K(\mathcal{L}(\hat{Z}), N(0,1)) \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}(\hat{Z} \leq z) - \mathbb{P}(Z^* \leq z) \right| + d_K(\mathcal{L}(Z^*), N(0,1)) \\
\leq \sup_{z \in \mathbb{R}} \left\{ \left| \mathbb{P}(Z^* \leq (1+\epsilon)z) - \mathbb{P}(Z^* \leq z) \right| \lor \left| \mathbb{P}(Z^* \leq (1-\epsilon)z) - \mathbb{P}(Z^* \leq z) \right| \right\} \\
+ d_K(\mathcal{L}(Z^*), N(0,1)) + 2\mathbb{P}(B_\epsilon^c) \\
= \sup_{z \in \mathbb{R}} \left| \mathbb{P}(Z^* \leq (1+\epsilon)z) - \mathbb{P}(Z^* \leq z) \right| + d_K(\mathcal{L}(Z^*), N(0,1)) + 2\mathbb{P}(B_\epsilon^c) \\
\leq 2\epsilon \sup_{z \in \mathbb{R}} \frac{|z| e^{-z^2/8}}{(2\pi)^{1/2}} + 3d_K(\mathcal{L}(Z^*), N(0,1)) + 2\mathbb{P}(B_\epsilon^c). \tag{67}
\]

The first conclusion of Theorem 4 now follows from (65), (66) and (67). The second conclusion is an immediate consequence of the first.

\[
\Box
\]

### 7.5 Proof of Theorem 11 on the local asymptotic minimax lower bound

**Proof of Theorem 11.** To prove the first part of the result we check the conditions of, and apply, Theorem 3.11.5 of *van der Vaart and Wellner (1996)*, and therefore borrow some of
their terminology. Define the Hilbert space $H := \mathbb{R}^2$ with inner product $\langle (t_1, t_2), (t'_1, t'_2) \rangle_H := t_1 t'_1 v_1(f, g) + t_2 t'_2 v_2(f, g)$. We first claim that our sequence of experiments is asymptotically normal. That is to say, for independent normal random variables $Z_1 \sim N(0, v_1)$ and $Z_2 \sim N(0, v_2)$, if we define the iso-Gaussian process $\{\Delta_t = t_1 Z_1 + t_2 Z_2 : t = (t_1, t_2) \in H\}$ we claim that

$$\log \frac{dP_{n,t}}{dP_{n,0}} = \Delta_{n,t} - \frac{1}{2} \|t\|^2_H$$

with $\Delta_{n,t} \overset{d}{\to} \Delta_t$ for each fixed $t \in H$. For $\delta > 0$ sufficiently small that $\int_X f(x)|h_1(x)|^{2+\delta} dx < \infty$ (cf. (64)), since $K(0) = K'(0) = K''(0) = 1$, we have that

$$\left| c_1(t_1)^{-1} - 1 - \frac{t_1^2}{2} v_1 \right|$$

$$= \left| \left( \int_{|t_1 h_1(x)| \leq 1} f(x) \left\{ K(t_1 h_1(x)) - 1 - t_1 h_1(x) - \frac{t_1^2}{2} h_1(x)^2 \right\} dx \right) \right|$$

$$\leq \frac{1}{6} \sup_{w \in [-1,1]} |K''(w)| \int_{|t_1 h_1(x)| \leq 1} f(x)|t_1 h_1(x)|^3 dx$$

$$+ \left\{ 2 \sup_{w \in \mathbb{R}} |K(w)| + 1 + \frac{1}{2} \right\} \int_{|t_1 h_1(x)| > 1} f(x)|t_1 h_1(x)|^{2+\delta} dx = o(t_1^2)$$

as $t_1 \to 0$, with a similar calculation holding for $c_2(t_2)^{-1}$ since $\int g|h_2|^{2+\delta} < \infty$ for $\delta > 0$ sufficiently small. Therefore, for each fixed $t = (t_1, t_2) \in H$ we have

$$\log \frac{dP_{n,t}}{dP_{n,0}} = \sum_{i=1}^{m} \log \frac{f_{m^{-1/2} t_1}(X_i)}{f(X_i)} + \sum_{j=1}^{n} \log \frac{g_{n^{-1/2} t_2}(Y_j)}{g(Y_j)}$$

$$= \sum_{i=1}^{m} \log K \left( \frac{t_1 h_1(X_i)}{m^{1/2}} \right) + m \log c_1(m^{-1/2} t_1) + \sum_{j=1}^{n} \log K \left( \frac{t_2 h_2(Y_j)}{n^{1/2}} \right) + n \log c_2(n^{-1/2} t_2)$$

$$= \frac{t_1}{m^{1/2}} \sum_{i=1}^{m} h_1(X_i) + \frac{t_2}{n^{1/2}} \sum_{j=1}^{n} h_2(Y_j) - \frac{1}{2} \|t\|^2_H + o_p(1) \overset{d}{\to} \Delta_t - \frac{1}{2} \|t\|^2_H,$$

as claimed.

We now show that our sequence of parameters $\kappa_n(t) := T(f_{m^{-1/2} t_1}, g_{n^{-1/2} t_2})$ is regular, in that there exists a continuous linear map $\hat{\kappa} : H \to \mathbb{R}$ and a sequence $(r_n)$ of real numbers such that

$$r_n \{ \kappa_n(t) - \kappa_n(0) \} \to \hat{\kappa}(t)$$

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for each $t \in H$. Indeed, for any fixed $t = (t_1, t_2) \in H$ we have

$$
\kappa_n(t) - \kappa_n(0) = \int_{\mathcal{X}} f_{m^{-1/2}t_1}(x) \phi(f_{m^{-1/2}t_1}(x), g_{n^{-1/2}t_2}(x), x) \, dx - \int_{\mathcal{X}} f(x) \phi_x \, dx
$$

$$
= \int_{\mathcal{X}} f(x) \left\{ K \left( t_1 h_1(x) / m^{1/2} \right) \phi \left( f \left( t_1 h_1(x) / m^{1/2} \right), K \left( t_2 h_2(x) / n^{1/2} \right) g(x), x \right) - \phi_x \right\} \, dx
+ o(m^{-1/2} + n^{-1/2})
$$

$$
= \int_{\mathcal{X}} f(x) \left[ t_1 h_1(x) / m^{1/2} \left\{ \phi_x + (f \phi_{10})x \right\} + t_2 h_2(x) / n^{1/2} (g \phi_{01})x \right] \, dx + o(m^{-1/2} + n^{-1/2})
= \frac{t_1 v_1}{m^{1/2}} + \frac{t_2 v_2}{n^{1/2}} + o(m^{-1/2} + n^{-1/2}).
$$

We may therefore take

$$
r_n = (m^{-1} v_1 + n^{-1} v_2)^{-1/2} \quad \text{and} \quad \kappa(t_1, t_2) = \frac{t_1 v_1 + A^{1/2} t_2 v_2}{(v_1 + A v_2)^{1/2}}
$$

to conclude that our sequence of parameters $\kappa_n$ is regular.

The adjoint $\kappa^* : \mathbb{R} \to H$ of $\kappa$ is given by

$$
\kappa^*(b^*) = \left( \frac{b^*}{(v_1 + A v_2)^{1/2}}, \frac{A^{1/2} b^*}{(v_1 + A v_2)^{1/2}} \right)
$$

as this satisfies $\langle \kappa^*(b^*), t \rangle_H = b^* \kappa(t)$ for all $b^* \in \mathbb{R}$ and $t \in H$. Since $\| \kappa^*(b^*) \|_H^2 = (b^*)^2$ for all $b^* \in \mathbb{R}$, we may therefore take $G \sim N(0, 1)$ and apply Theorem 3.11.5 of van der Vaart and Wellner (1996) to deduce that for any estimator sequence $T_{m,n},$

$$
\sup_{t \in I} \lim_{n \to \infty} \inf_{i \in I} \mathbb{E}_{P_{n,t}} \left\{ \frac{(T_{m,n} - T)^2}{m^{-1} v_1 + n^{-1} v_2} \right\} \geq \mathbb{E}(G^2) = 1.
$$

This concludes the proof of the first part of Theorem 11.

We now turn to the second part of the Theorem 11. Since $k : \mathbb{R} \to [1/2, 3/2]$ we have that $f(x)/3 \leq f_t(x) \leq 3 f(x)$ and $g(x)/3 \leq g_t(x) \leq 3 g(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ and, to establish the result, it remains to show that $\max\{M_{f_t, \beta}(x), M_{g_t, \beta}(x)\} \lesssim M_\beta(x)$ for $t \leq 1$, say. For ease of presentation, we first prove this in the case $\beta \in (0, 1]$. When $x \in \mathcal{X}$ and
$y, z \in B_x\left(\{2d^{1/2}M_\beta(x)\}^{-1}\right)$, we have that

$$\frac{|f_t(z) - f_t(y)|}{f_t(x)} = \frac{|K(th_1(z))f(z) - K(th_1(y))f(y)|}{K(th_1(x))f(x)}$$

$$\leq \frac{3}{f(x)}|f(z) - f(y)| + \frac{2f(y)}{f(x)}|K(th_1(z)) - K(th_1(y))|$$

$$\leq 3\{M_\beta(x)||z - y||\}^\beta + 4|K(th_1(z)) - K(th_1(y))|.$$

(68)

Additionally,

$$|h_1(z) - h_1(y)| \leq |\phi_z - \phi_y| + |f(z)|(\phi_{10})z - (\phi_{10})y| + |(\phi_{10})_y||f(z) - f(y)|$$

$$\leq L(1 \vee |\phi_x + (f\phi_{10})_y|)\left(1 + \left(\frac{f(z)}{f(y)}\right)\right)\left\{3\left|\frac{f(z)}{f(y)} - 1\right|^{|\beta_1 - 1|\Lambda_{1}} + \left|\frac{g(z)}{g(y)} - 1\right|^{|\beta_2 - 1|\Lambda_{1}} + ||z - y||^{|\beta_3|}\right\}$$

$$\lesssim (1 + |h_1(y)|)\{M_\beta(x)||z - y||\}^\beta.$$  

(69)

In particular, there exists $c = c(d, \vartheta, \xi)$ such that, whenever $||z - y||M_\beta(x) \leq c$, we have $|h_1(z) - h_1(y)| \leq \max(1, |h_1(y)| \wedge |h_1(z)|)/2$. Writing $L_{t,y,z}$ for the line segment between $th_1(y)$ and $th_1(z)$, and using the fact that $\sup_{w \in \mathbb{R}}\{1 + |w||K'(w)| < \infty$, we now have for $z, y$ such that $||z - y||M_\beta(x) \leq c$ that

$$|K(th_1(z)) - K(th_1(y))| \leq t|h_1(z) - h_1(y)| \sup_{w \in L_{t,y,z}}|K'(w)|$$

$$\lesssim \frac{(1 + t|h_1(y)| \wedge |h_1(z)|)}{1 + \inf_{w \in L_{t,y,z}}|w|}\{M_\beta(x)||z - y||\}^\beta \lesssim \{M_\beta(x)||z - y||\}^\beta.$$  

(70)

From (68), (69) and (70), we deduce that $M_{f_t, \tilde{\beta}}(x) \lesssim M_\beta(x)$. Moreover, when $y, z \in B_x\left(\{2d^{1/2}M_\beta(x)\}^{-1}\right)$, we have that

$$|h_2(z) - h_2(y)| \leq f(z)|\phi_{01})z - (\phi_{01})_y| + ||(\phi_{01})_y||f(z) - f(y)|$$

$$\leq \frac{f(z)}{f(y)}(1 \vee f(y))(\phi_{01})_y|\left\{3\left|\frac{f(z)}{f(y)} - 1\right|^{|\beta_1 - 1|\Lambda_{1}} + \left|\frac{g(z)}{g(y)} - 1\right|^{|\beta_2 - 1|\Lambda_{1}} + ||z - y||^{|\beta_3|}\right\}$$

$$+ ||(\phi_{01})_y||f(z) - f(y)|$$

$$\lesssim (1 + |h_2(y)|)\{M_\beta(x)||z - y||\}^\beta.$$  

It now follows by very similar arguments to those in (70) above that $M_{y, \tilde{\beta}}(x) \lesssim M_\beta(x)$.

We now extend these arguments to cover the $\beta > 1$ case. For a multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \bar{\beta} := \lceil \beta \rceil - 1$, we have that $\partial^\alpha\{K(th_1(x))\}$ can be written as a finite sum of terms of
the form

\[ t^\ast((\partial^{\alpha_1}h_1)\cdots(\partial^{\alpha_r}h_1))(x)K^{(r)}(\tilde{t}h_1(x)) \quad (71) \]

where \( r \in \mathbb{N}_0 \) satisfies \( r \leq |\alpha| \), and the multi-indices \( \alpha^{(1)}, \ldots, \alpha^{(r)} \in \mathbb{N}_0^d \) satisfy \( |\alpha^{(1)}| + \ldots + |\alpha^{(r)}| = |\alpha| \). Moreover, for any \( j = 1, \ldots, r \), we have that \( \partial^{\alpha^{(j)}}h_1 \) is a finite sum of terms of the form

\[ (\partial^{\beta^{(1)}}f)\cdots(\partial^{\beta^{(l)}}f)(\partial^{\gamma^{(1)}}g)\cdots(\partial^{\gamma^{(l_2-1)}}g)(x)\phi_{l_1l_2l_3}(f(x), g(x), x), \quad (72) \]

where \( l_1, l_2, l_3 \in \mathbb{N}_0 \) satisfy \( l_1 + l_2 + l_3 \leq |\alpha^{(j)}| + 1 \), and where moreover the multi-indices \( \beta^{(1)}, \ldots, \beta^{(l_1)}, \gamma^{(1)}, \ldots, \gamma^{(l_2)} \in \mathbb{N}_0^d \) satisfy \( |\beta^{(1)}| + \ldots + |\beta^{(l_1)}| + |\gamma^{(1)}| + \ldots + |\gamma^{(l_2)}| + l_3 = |\alpha^{(j)}| \). Using the fact that \( \sup_{w \in \mathbb{R}}(1 + |w|^\gamma)|K^{(r)}(w)| < \infty \) for any \( r \in \mathbb{N} \) and assumption (i) in the definition of \( \tilde{\Phi} \), we therefore have the bounds

\[ |\partial^{\alpha^{(j)}}h_1(x)| \lesssim M_\beta(x)^{\max(|\alpha^{(j)}|, 1 + |h_1(x)|)} \quad \text{and} \quad |\partial^{\alpha^{(j)}}K(th_1(x))| \lesssim M_\beta(x)^{|\alpha^{(j)}|}. \]

It follows that, for any multi-index \( \alpha \) with \( |\alpha| \leq \tilde{\beta} \) we have that

\[ \left| \frac{\partial^{\alpha}f_t(x)}{f_t(x)} \right| \lesssim M_\beta(x)^{|\alpha|}. \]

Since, for any multi-index \( \alpha \) with \( |\alpha| \leq \tilde{\beta} \), we have that \( \partial^{\alpha}h_2 \) is a finite sum of terms of the form

\[ (\partial^{\beta^{(1)}}f)\cdots(\partial^{\beta^{(l_1+1)}}f)(\partial^{\gamma^{(1)}}g)\cdots(\partial^{\gamma^{(l_2-1)}}g)(x)\phi_{l_1l_2l_3}(f(x), g(x), x), \]

we deduce by similar arguments that \( |\partial^{\alpha}g_t(x)| \lesssim g_t(x)M_\beta(x)^{|\alpha|} \) for any multi-index \( \alpha \) with \( |\alpha| \leq \tilde{\beta} \). Now we have for any \( l_1 \in [\beta^\ast_1], l_2 \in [\beta^\ast_2], l_3 \in [\beta^\ast_3] \) and \( y, z \in B_x(\{2d^{1/2}M_\beta(x)\}^{-1}) \) that

\[ |\phi_{l_1l_2l_3}(f(z), g(z), z) - \phi_{l_1l_2l_3}(f(y), g(y), y)| \]

\[ \lesssim f(y)^{-l_1}g(y)^{-l_2}(1 \vee |h_1(y)|)\left\{ \left| \frac{f(z)}{f(y)} - 1 \right|^{(\beta^\ast_1-l_1)\wedge 1} + \left| \frac{g(z)}{g(y)} - 1 \right|^{(\beta^\ast_2-l_2)\wedge 1} + \|z - y\|^{|(\beta^\ast_3-l_3)\wedge 1|} \right\} \]

\[ \lesssim f(y)^{-l_1}g(y)^{-l_2}(1 \vee |h_1(y)|)\{M_\beta(x)\|z - y\|^{\min\{1, \beta^\ast_1-l_1, \beta^\ast_2-l_2, \beta^\ast_3-l_3\}}. \]

It follows from this, together with the representation (72) and Lemma 16 that, for any
multi-index $\alpha$ with $|\alpha| \leq \tilde{\beta}$, we have that

$$|\partial^{\alpha} h_1(z) - \partial^{\alpha} h_1(y)| \lesssim M_{\beta}(x)^{|\alpha|} (1 + |h_1(y)|) \{ M_{\beta}(x) \| z - y \| \}^{\min\{1,\beta - \tilde{\beta}, \beta_1^*, \beta_2^* - \tilde{\beta}, \beta_3^* - \tilde{\beta}\}}$$

$$\lesssim M_{\beta}(x)^{|\alpha|} (1 + |h_1(y)|) \{ M_{\beta}(x) \| z - y \| \}^{\tilde{\beta} - \tilde{\beta}}.$$

By a similar argument to (70), and using (71) and the fact that $\sup_{w \in \mathbb{R}} (1 + |w|^r) |K^{(r)}(w)| < \infty$ for any $r \in \mathbb{N}$, we can now see that, for any multi-index $\alpha$ with $|\alpha| \leq \tilde{\beta}$,

$$|\partial^{\alpha} \{ K(t h_1(z)) \} - \partial^{\alpha} \{ K(t h_1(y)) \}| \lesssim M_{\beta}(x)^{|\alpha|} \{ M_{\beta}(x) \| z - y \| \}^{\tilde{\beta} - \tilde{\beta}}.$$

Using Lemma 16 it then follows that, for any multi-index $\alpha$ with $|\alpha| = \tilde{\beta}$, we have

$$|\partial^{\alpha} f_t(z) - \partial^{\alpha} f_t(y)| \lesssim f_t(x) M_{\beta}(x)^{\tilde{\beta}} \{ M_{\beta}(x) \| z - y \| \}^{\tilde{\beta} - \tilde{\beta}} = f_t(x) M_{\beta}(x)^{\tilde{\beta}} \| z - y \|^{\tilde{\beta} - \tilde{\beta}},$$

and so $M_{f_t, \tilde{\beta}}(x) \lesssim M_{\beta}(x)$, as required. Similarly, $M_{g_t, \tilde{\beta}}(x) \lesssim M_{\beta}(x)$, and this completes the proof. \hfill \Box

### 7.6 Auxiliary lemmas

**Lemma 14.** Suppose that $\phi \in \Phi(\xi)$ for some $\xi = (\kappa_1, \kappa_2, \beta_1^*, \beta_2^*, L) \in \Xi$ with $\min(\beta_1^*, \beta_2^*) > 2$. Then

$$\max\{ u|\phi_{10}(z + \epsilon \circ z) - \phi_{10}(z)|, v|\phi_{01}(z + \epsilon \circ z) - \phi_{01}(z)| \}$$

$$\leq 2 \max\{ (1 - L^{-1})^{-\kappa_1}, 2^L \} \max\{ (1 - L^{-1})^{-\kappa_2}, 2^L \} L \| \epsilon \| (u^{-\kappa_1} \lor u^L)(v^{-\kappa_2} \lor v^L)$$

for all $\epsilon = (\epsilon_1, \epsilon_2, 0) \in (-L^{-1}, L^{-1})^2 \times \{0\}$ and $z = (u, v, x) \in \mathcal{Z}$.

**Proof of Lemma 14.** By condition (ii) of the definition of the class $\Phi$, for each $z \in \mathcal{Z}$, the Hessian matrix

$$H(z) := \begin{pmatrix} u^2 \phi_{20}(z) & uv \phi_{11}(z) \\ uv \phi_{11}(z) & v^2 \phi_{02}(z) \end{pmatrix}$$

satisfies $\| H(z) \|_{op} \leq 2L(u^{-\kappa_1} \lor u^L)(v^{-\kappa_2} \lor v^L)$. Now, fixing $z \in \mathcal{Z}$, the function $g : [0, 1] \to \mathbb{R}$ given by $g(t) := u\phi_{10}(z + t\epsilon \circ z)$ is differentiable with $g'(t) = H(z + t\epsilon \circ z)(\epsilon_1, \epsilon_2)^T$. Thus,
by the mean value theorem,
\[
|u(\phi_{10}(z+c) - \phi_{10}(z))| = |g(1) - g(0)| \\
\leq 2 \max\{(1 - L^{-1})^{-\kappa_1}, 2^L\} \max\{(1 - L^{-1})^{-\kappa_2}, 2^L\} L\|v\|((1^{-\kappa_1} \lor u^L)(v^{-\kappa_2} \lor v^L)).
\]
A similar calculation with \(\phi_{01}\) completes the proof.

**Lemma 15.** Fix \(f \in \mathcal{F}_d\) and \(\beta \in (0, \infty)\), and let \(S_n \subseteq (0, 1), X_n \subseteq \mathbb{R}^d\) be such that
\[
a_n := \sup_{s \in S_n} \sup_{x \in X_n} \frac{s M_{f, \beta}(x)^d}{V_d f(x)} \to 0.
\]
Then there exists \(n_* = n_*(d, \beta, (a_n)) \in \mathbb{N}\), coefficients \(b_l(x)\) and \(A = A(d, \beta, (a_n)) \in (0, \infty)\) such that, for all \(n \geq n_*\), \(s \in S_n\) and \(x \in X_n\), we have
\[
\left| V_d f(x) h_{x,f}^{-1}(s)^d - \sum_{l=0}^{[\beta/2]-1} b_l(x)s^{1+2l/d} \right| \leq A s M_{f, \beta}(x)^d f(x)^{\beta/d}.
\]
Moreover, \(b_0(x) = 1\) and \(|b_l(x)| \leq A \{M_{f, \beta}(x)^d / f(x)\}^{2l/d}\).

**Proof of Lemma 15.** By a Taylor expansion, for \(r \leq \{2d^{1/2}M_{f, \beta}(x)\}^{-1}\) we have that
\[
\left| h_{x,f}(r) - V_d r^d f(x) - \sum_{l=1}^{[\beta/2]-1} r^{d+2l} c_l(x) \right| \lesssim_{\beta, d} r^d f(x) \{M_{f, \beta}(x)r\}^\beta \tag{73}
\]
for some coefficients \(c_l(\cdot)\) satisfying \(|c_l(x)| \lesssim_{\beta, d} f(x) M_{f, \beta}(x)^{2l}\). In particular,
\[
\left| \frac{h_{x,f}(r)}{V_d r^d f(x)} - 1 \right| \lesssim_{\beta, d} \{M_{f, \beta}(x)r\}^{2\wedge \beta}.
\]
Thus there exists \(C = C(d, \beta) > 0\) such that \(|h_{x,f}(r) / V_d r^d f(x) - 1| \leq 1/2\) whenever \(r \leq \{CM_{f, \beta}(x)\}^{-1}\).
Setting \(r = \left(\frac{2s}{V_d f(x)}\right)^{1/d}\) we have
\[
r C M_{f, \beta}(x) = 2^{1/d} C \left\{ \frac{s M_{f, \beta}(x)^d}{V_d f(x)} \right\}^{1/d} \leq (2a_n)^{1/d} C \to 0.
\]
So, for \(n\) large enough that \(2a_n)^{1/d} C \leq 1\), we have \(h_{x,f}(\left(\frac{2s}{V_d f(x)}\right)^{1/d}) \geq s\), so \(h_{x,f}^{-1}(s) \leq \left(\frac{2s}{V_d f(x)}\right)^{1/d}\) for all \(x \in X_n\) and \(s \in S_n\). Now, since \(M_{f, \beta}(x) h_{x,f}^{-1}(s) \leq \left(\frac{2s M_{f, \beta}(x)^d}{V_d f(x)}\right)^{1/d} \leq
(2a_n)^{1/d} \to 0$, we may substitute $r = h_{x,f}^{-1}(s)$ into (73) to see that

$$\frac{s}{V_d f(x) h_{x,f}^{-1}(s)^d} - \left[ \sum_{l=1}^{\lfloor \beta/2 \rfloor - 1} c_l(x) h_{x,f}^{-1}(s)^2 \right] \lesssim_{\beta,d,(a_n)} \left\{ \frac{s M_{f,\beta}(x)^d}{f(x)} \right\}^{\beta/d}. $$

This expansion can be inverted to yield the desired result by substituting this bound into itself and expanding functions of the form $r \mapsto r^{2l/d}$ about $r = 1$.

**Lemma 16.** Fix $f \in \mathcal{F}_d$ and $\beta \in (0, \infty)$, and suppose that $\max\{\|y - x\|, \|z - x\|\} \leq (8d)^{1/(\beta - \beta)} M_{f,\beta}(x)^{-1}$. Then, for multi-indices $t \in \mathbb{N}_0^d$ with $|t| \leq \beta$, we have that

$$\left| (\partial^t f)(z) - (\partial^t f)(y) \right| \leq 8d^{1/2} M_{f,\beta}(x)^{\min(\beta,|t|) + 1} f(x) \|z - y\|^{\min(1,\beta - |t|)}. $$

**Proof.** First, if $|t| = \beta$ then we simply have that

$$\left| (\partial^t f)(z) - (\partial^t f)(y) \right| \leq \|f^{(\beta)}(z) - f^{(\beta)}(y)\| \leq M_{f,\beta}(x)^{\beta} f(x) \|z - y\|^{\beta - \beta},$$

and the claim holds. Henceforth assume that $|t| \leq \beta - 1$ and $\beta \geq 1$. Writing $\|\cdot\|$ here for the largest absolute entry of an array, writing $L_{yz}$ for the line segment between $y$ and $z$, and arguing inductively we have that

$$\left| \partial^t f(z) - \partial^t f(y) \right| \leq \|z - y\| \sup_{w \in L_{yz}} \|\nabla \partial^t f(w)\|$$

$$\leq \|z - y\| \|f^{(|t|+1)}(x)\| + 8d \|z - y\| M_{f,\beta}(x)^{\min(\beta,|t|+2)} f(x) \|y - x\|^{\min(1,\beta - |t| - 1)}$$

$$+ d^{1/2} \|z - y\| \sup_{w \in L_{yz}} \|f^{(|t|+1)}(w) - f^{(|t|)+1}(y)\|$$

$$\leq 2\|z - y\| M_{f,\beta}(x)^{|t|+1} f(x) + d^{1/2} \|z - y\| \sup_{w \in L_{yz}} \|f^{(|t|+1)}(w) - f^{(|t|)+1}(y)\|$$

$$\leq 2 \sum_{l=1}^{\beta - |t|} d^{l-1/2} \|z - y\|^{l} M_{f,\beta}(x)^{|t|+l} f(x) + d^{\beta - |t|} \|z - y\|^{\beta - |t|} \sup_{w \in L_{yz}} \|f^{(2)}(w) - f^{(2)}(y)\|$$

$$\leq 2 \sum_{l=1}^{\beta - |t|} d^{l-1/2} \|z - y\|^{l} M_{f,\beta}(x)|t|+l f(x) + d^{\beta - |t|} \|z - y\|^{\beta - |t|} M_{f,\beta}(x)^{\beta} f(x)$$

$$\leq 2 M_{f,\beta}(x)^{|t|+1} f(x) \|z - y\| \left[ \left\{ 1 - d^{1/2} \|z - y\| M_{f,\beta}(x) \right\}^{-1} + d^{1/2} \left\{ d^{1/2} \|z - y\| M_{f,\beta}(x) \right\}^{\beta - |t| - 1} \right]$$

$$\leq 8d^{1/2} M_{f,\beta}(x)^{|t|+1} f(x) \|z - y\|,$$

as required. 

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The following lemma presents a tail bound for a Beta(a, b - a) random variable that is convenient to apply in settings where \( a > 0 \) is large and \( a/b \) is small.

**Lemma 17.** Suppose \( b > a > 0 \) and \( B \sim \text{Beta}(a, b - a) \). Writing \( h(t) := t - \log(1 + t) \) we have that

\[
\Pr\left( \left| B - \frac{a}{b} \right| \geq \frac{a^{1/2}u}{b} \right) \leq 2 \exp\left( -ah\left( \frac{a^{-1/2}b^{1/2}u}{b^{1/2} + a^{1/2} + u} \right) \right) + 2 \exp\left( -bh\left( \frac{u}{b^{1/2} + a^{1/2} + u} \right) \right)
\]

for all \( u \in [0, \infty) \).

**Proof.** Our proof relies on concentration inequalities for gamma random variables, which we establish now. For \( a > 0 \), letting \( \Gamma_a \sim \Gamma(a, 1) \) we have by a Chernoff bound that for \( t \geq 0 \),

\[
\Pr\left( \frac{\Gamma_a - a}{a} \geq t \right) \leq \inf_{\lambda \geq 0} e^{-\lambda t} \left( 1 - \frac{\lambda}{a} \right)^{-a} = \exp(-at + a \log(1 + t)) = e^{-h(t)}.
\]

Similarly, for \( t \in [0, 1) \) we have that

\[
\Pr\left( \frac{\Gamma_a - a}{a} \leq -t \right) \leq \inf_{\lambda > 0} e^{\lambda t} \left( 1 + \frac{\lambda}{a} \right)^{-a} = \exp(at + a \log(1 - t)) = e^{-h(-t)} \leq e^{-h(t)},
\]

and thus, for all \( t \geq 0 \), we have that \( \Pr(|\Gamma_a - a| \geq at) \leq 2e^{-h(t)} \). Now, for independent random variables \( \Gamma_a \sim \Gamma(a, 1) \) and \( \Gamma_{b-a} \sim \Gamma(b - a, 1) \) we have that \( \Gamma_a/(\Gamma_a + \Gamma_{b-a}) \sim \text{Beta}(a, b) \), and so for \( t \geq 0 \) and \( \epsilon \in (0, 1) \) we have that

\[
\Pr\left( \left| B - \frac{a}{b} \right| \geq t \right) = \Pr\left( \left| \frac{\Gamma_a - a}{\Gamma_a + \Gamma_{b-a}} + \frac{a}{b} \left( \frac{b}{\Gamma_a + \Gamma_{b-a}} - 1 \right) \right| \geq t \right)
\]

\[
\leq \Pr\left( \left| \frac{a}{b} \left( \frac{b}{\Gamma_a + \Gamma_{b-a}} - 1 \right) \right| \geq \epsilon t \right) + \Pr\left( \left| \frac{\Gamma_a - a}{b} \right| \geq \frac{(1 - \epsilon)t}{1 + \epsilon tb/a} \right)
\]

\[
\leq \Pr\left( \left| \frac{\Gamma_a + \Gamma_{b-a} - b}{b} \right| \geq \frac{\epsilon tb}{a + \epsilon tb} \right) + \Pr\left( \left| \frac{\Gamma_a - a}{a} \right| \geq \frac{(1 - \epsilon)tb}{a + \epsilon tb} \right).
\]

Choosing \( \epsilon = a^{1/2}/(a^{1/2} + b^{1/2}) \) and writing \( t = a^{1/2}u/b \) we may now see that

\[
\Pr\left( \left| B - \frac{a}{b} \right| \geq \frac{a^{1/2}u}{b} \right) \leq \Pr\left( \left| \frac{\Gamma_a + \Gamma_{b-a} - b}{b} \right| \geq \frac{u}{a^{1/2} + b^{1/2} + u} \right) + \Pr\left( \left| \frac{\Gamma_a - a}{a} \right| \geq \frac{a^{-1/2}b^{1/2}u}{a^{1/2} + b^{1/2} + u} \right)
\]

\[
\leq 2 \exp\left( -bh\left( \frac{u}{b^{1/2} + a^{1/2} + u} \right) \right) + 2 \exp\left( -ah\left( \frac{a^{-1/2}b^{1/2}u}{b^{1/2} + a^{1/2} + u} \right) \right),
\]

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as required. \qed

**Lemma 18.** Fix \( d \in \mathbb{N} \) and \( \vartheta = (\alpha, \beta, \lambda_1, \lambda_2, \gamma, C) \in \Theta. \) Suppose that \( a, b, c \in [0, \infty) \) are such that \( \frac{a}{\lambda_1} + \frac{b}{\lambda_2} + \frac{c}{\alpha} < 1. \) Then

\[
\sup_{(f,g) \in \mathcal{F}_d, \theta} \int_X f(x) \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^a \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^b (1 + \|x\|)^c \, dx < \infty.
\]

Moreover, whenever \( a \in (d/(\alpha + d), 1] \) we have that

\[
\sup_{f \in \mathcal{F}_d: \mu_{\alpha}(f) \leq C} \int_X f(x)^a \, dx < \infty.
\]

**Proof.** By the generalised Hölder inequality (e.g. Folland, 1999, Chapter 6, Exercise 31), we have that

\[
\int_X f(x) \left\{ \frac{M_\beta(x)^d}{f(x)} \right\}^a \left\{ \frac{M_\beta(x)^d}{g(x)} \right\}^b (1 + \|x\|)^c \, dx = \mathbb{E} \left[ \left\{ \frac{M_\beta(X_1)^d}{f(X_1)} \right\}^a \left\{ \frac{M_\beta(X_1)^d}{g(X_1)} \right\}^b (1 + \|X_1\|)^c \right]
\]

\[
\leq \mathbb{E} \left[ \left\{ \frac{M_\beta(X_1)^d}{f(X_1)} \right\}^{\lambda_1/a} \mathbb{E} \left[ \left\{ \frac{M_\beta(X_1)^d}{g(X_1)} \right\}^{\lambda_2/b} \right] \mathbb{E} \left[ (1 + \|X_1\|)^{1-a/\lambda_1-b/\lambda_2} \right] \right]^{1-a/\lambda_1-b/\lambda_2} \leq C^{a/\lambda_1+b/\lambda_2} \mathbb{E}\left\{ (1 + \|X_1\|)^a \right\}^{1-a/\lambda_1-b/\lambda_2} \leq C^{a/\lambda_1+b/\lambda_2} \{2^a(1 + C)\}^{1-a/\lambda_1-b/\lambda_2},
\]

as required. When \( a = 1 \) the second part of the result is trivial. For \( a \in (d/(\alpha + d), 1) \) we have \( a/(1-a) > d/\alpha \) and we may use Hölder’s inequality to see that

\[
\int_X f(x)^a \, dx \leq \left\{ \int_X (1 + \|x\|)^a f(x) \, dx \right\}^a \left\{ \int_{\mathbb{R}^d} (1 + \|x\|)^{-a(1-a)} \, dx \right\}^{1-a}
\]

\[
\leq \left\{ (1 + C) \max(1, 2^{a-1}) \right\}^a \left\{ \frac{V_d \Gamma(d+1) \Gamma(\frac{a}{1-a})}{\Gamma(\frac{a}{1-a})} \right\}^{1-a},
\]

also as required. \qed

**Lemma 19.** Fix \( f \in \mathcal{F}_d \) with \( \max(\|f\|, \mu_{\alpha}(f)) \leq C \) and \( \beta \in (0, \infty). \) Then for all \( x \in X \) and \( s \in (0, 1), \)

\[
\left( \frac{s}{CV_d} \right)^{1/d} \leq h_{x,f}(s) \leq \left( \frac{2s}{V_d f(x)} \right)^{1/d} \left[ 1 + (16d)^{(1-\beta)/\beta} M_{f,\beta}(x) \left\{ \|x\| + \left( \frac{C}{1-s} \right)^{1/\alpha} \right\} \right].
\]
Proof. The lower bound is immediate on noting that

\[ h_{x,f}(r) \leq CV_d r^d. \]

For the upper bound, by Lemma 16, if \( \|y - x\| \leq \{(16d)^{1/(\beta - \beta)} M_{f,\beta}(x)\}^{-1} \), then we have that

\[ \left| \frac{f(y)}{f(x)} - 1 \right| \leq 8d^{1/2} M_{f,\beta}(x)^{1/\beta} \|y - x\|^{1/\beta} \leq \frac{1}{2}. \]

Thus, whenever \( r \leq \{(16d)^{1/(\beta - \beta)} M_{f,\beta}(x)\}^{-1} \) we have that

\[ \frac{1}{2} V_d r^d f(x) \leq h_{x,f}(r) \leq \frac{3}{2} V_d r^d f(x). \]

Now, by the triangle and Markov’s inequalities, for every \( s \in (0, 1) \),

\[ \mathbb{P}\left( \|X_1 - x\| > \|x\| + \left( \frac{C}{1 - s} \right)^{1/\alpha} \right) \leq \mathbb{P}\left( \|X_1\| > \left( \frac{C}{1 - s} \right)^{1/\alpha} \right) \leq 1 - s, \]

so that

\[ h_{x,f}^{-1}(s) \leq \|x\| + \left( \frac{C}{1 - s} \right)^{1/\alpha}. \]

Hence,

\[ h_{x,f}^{-1}(s) \leq \left( \frac{2s}{V_df(x)} \right)^{1/d} \left( 1 + \left( 16d \right)^{1/(\beta - \beta)} M_{f,\beta}(x) \left\{ \|x\| + \left( \frac{C}{1 - s} \right)^{1/\alpha} \right\} \right), \]

as required.

The following lemma shows that we may restrict our main attention to the events

\[ A^X_i := \left\{ h_{X_i,f}(\rho^d_{(k_X)_i,X}) \in \mathcal{I}_{m,X} \right\}, \quad A^Y_i := \left\{ h_{X_1,g}(\rho^d_{(k_Y)_i,Y}) \in \mathcal{I}_{n,Y} \right\}, \quad (74) \]

for \( i = 1, \ldots, n \).

Lemma 20. Fix \( d \in \mathbb{N}, \theta \in \Theta, (\kappa_1, \kappa_2, L) \in [0, \infty)^3 \) and suppose that

\[ \frac{\kappa_1}{\lambda_1} + \frac{\kappa_2}{\lambda_2} + \frac{d(\kappa_1 + \kappa_2)}{\alpha} < 1. \]
Let $k_X^L \leq k_X^U$, $k_Y^L \leq k_Y^U$ be deterministic sequences of positive integers such that $k_X^L / \log m \to \infty$, $k_Y^L / \log n \to \infty$, $k_X^U / m \to 0$ and $k_Y^U / n \to 0$. Then

$$\max_{k_X \in \{k_X^L, \ldots, k_X^U\}} \sup_{k_Y \in \{k_Y^L, \ldots, k_Y^U\}} \mathbb{E} \left[ \max \left\{ \hat{f}_{(k_X),1}^L, \hat{f}_{(k_X),1}^{-k_1}, f(X_1)^{-k_1} \right\} \max \left\{ \hat{g}_{(k_Y),1}^L, \hat{g}_{(k_Y),1}^{-k_2}, g(X_1)^{-k_2} \right\} \left( 1 - \mathbf{1}_{A^2} \mathbf{1}_{A^2} \right) \right] = o(m^{-4} + n^{-4})$$

as $m, n \to \infty$.

**Proof of Lemma 20.** Given $a > -\min(k_X, k_Y)$, $b > -\min(m - k_X, n + 1 - k_Y)$ define

$$\Delta_{a,b}^{(1)} := \int_{[0,1] \setminus \mathcal{I}_{m,X}} B_{k_X + a, m - k_X + b} (s) \, ds,$$  
$$\Delta_{a,b}^{(2)} := \int_{[0,1] \setminus \mathcal{I}_{n,Y}} B_{k_Y + a, n + 1 - k_Y + b} (t) \, dt.$$

By Lemma 17 we have that

$$\max_{k_X \in \{k_X^L, \ldots, k_X^U\}} \sup_{k_Y \in \{k_Y^L, \ldots, k_Y^U\}} \max(\Delta_{a,b}^{(1)}, \Delta_{a,b}^{(2)}) = o(m^{-9(1-\epsilon)/2} + n^{-9(1-\epsilon)/2})$$

for any fixed $A \geq 0$ and $\epsilon > 0$. Now, by Lemma 19, we have that

$$\mathbb{E} \left[ \max \left\{ \hat{f}_{(k_X),1}^L, \hat{f}_{(k_X),1}^{-k_1}, f(X_1)^{-k_1} \right\} \max \left\{ \hat{g}_{(k_Y),1}^L, \hat{g}_{(k_Y),1}^{-k_2}, g(X_1)^{-k_2} \right\} \left( 1 - \mathbf{1}_{A^2} \mathbf{1}_{A^2} \right) \right] \lesssim \int_X f(x) \int_0^1 \int_0^1 \max \left\{ \left( \frac{k_X}{ms}, \left( \frac{ms M_\beta(x)^{d}(1 + \|x\|^d)^{\kappa_1}}{k_X f(x)(1-s)^{d/\alpha}} \right) \right), f(x)^{-k_1} \right\} \times \max \left\{ \left( \frac{k_Y}{nt}, \left( \frac{nt M_\beta(x)^{d}(1 + \|x\|^d)^{\kappa_2}}{k_Y g(x)(1-t)^{d/\alpha}} \right) \right), g(x)^{-k_2} \right\} \times \max \left\{ \mathbf{1}_{\{s \in \mathcal{I}_{m,X}\}}, \mathbf{1}_{\{t \in \mathcal{I}_{n,Y}\}} \right\} B_{k_X, m - k_X} (s) B_{k_Y, n + 1 - k_Y} (t) \, ds \, dt \, dx$$

$$\lesssim \max \left( \Delta_{-L,0}^{(1)}, \Delta_{0,0}^{(1)}, \Delta_{-L,0}^{(2)}, \Delta_{0,0}^{(2)} \right) \times \int_X f(x) \left( \frac{M_\beta(x)^d}{f(x)} \right)^{\kappa_1} \left( \frac{g(x)^d}{M_\beta(x)} \right)^{\kappa_2} (1 + \|x\|)^{d(\kappa_1 + \kappa_2)} \, dx$$

$$\lesssim (m^{-17/4} + n^{-17/4}) \int_X f(x) \left( \frac{M_\beta(x)^d}{f(x)} \right)^{\kappa_1} \left( \frac{g(x)^d}{M_\beta(x)} \right)^{\kappa_2} (1 + \|x\|)^{d(\kappa_1 + \kappa_2)} \, dx$$

The conclusion follows immediately on appealing to Lemma 18. \qed
Lemma 21. Let $a, b, c \in \mathbb{R}$ be any fixed constants, and let $k^L \leq k^U$ be deterministic sequences of positive integers such that $k^L \to \infty$ and $k^U/n \to 0$ as $n \to \infty$. Then

$$\int_0^1 \int_0^1 |B_{j+a,t+b,n+c-j-l}(s, t) - B_{j+a,n-j}(s)B_{t+b,n-l}(t)| \, ds \, dt \leq \frac{(jl)^{1/2}}{n} \{1 + o(1)\}$$

as $n \to \infty$, uniformly for $j, l \in \{k^L, \ldots, k^U\}$.

Proof. In the following bound we make use the standard asymptotic expansions

$$\log \Gamma(z) = z \log z - z - \frac{1}{2} \log \left(\frac{z}{2\pi}\right) + \frac{1}{12z} + O\left(\frac{1}{z^3}\right)$$

$$\Psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + O\left(\frac{1}{z^4}\right)$$

as $z \to \infty$. Using these expansions, by Lemma 17 and Pinsker’s inequality we have that

$$\int_0^1 \int_0^1 |B_{j+a,t+b,n+c-j-l}(s, t) - B_{j+a,n-j}(s)B_{t+b,n-l}(t)| \, ds \, dt$$

$$\leq \left\{2 \int_0^1 \int_0^{1-t} B_{j+a,t+b,n+c-j-l}(s, t) \log \left(\frac{B_{j+a,t+b,n+c-j-l}(s, t)}{B_{j+a,n-j}(s)B_{t+b,n-l}(t)}\right) \, ds \, dt \right\}^{1/2}$$

$$= 2^{1/2} \left[ \log \left(\frac{\Gamma(n + a + b + c)\Gamma(n-j)\Gamma(n-l)}{\Gamma(n+c-j-l)\Gamma(n+a)\Gamma(n+b)}\right) + (n - c - 1)\Psi(n + a + b + c) \right.$$

$$- (n - j - 1)\Psi(n + b + c - j) - (n - l - 1)\Psi(n + a + c - l)$$

$$+ (n + c - j - l - 1)\Psi(n + c - j - l) \left. \right]^{1/2}$$

$$= \frac{(jl)^{1/2}}{n} \{1 + o(1)\}$$

as $n \to \infty$, uniformly for $j, l \in \{k^L, \ldots, k^U\}$. \hfill \Box

The following lemma provides bounds on the normal approximation to relevant multinomial distributions.

Lemma 22. Fix $f \in \mathcal{F}_d$ and $\beta \in (0, 1]$, and let $k^L \leq k^U$ be deterministic sequences of positive integers satisfying $k^L/\log n \to \infty$ and $k^U/n^{-1}\log n \to 0$. For $k \in \{k^L, \ldots, k^U\}$ define $X_n := \{x : f(x)M_{f,\beta}(x)^{-d} \geq kn^{-1}\log n\}$. For $j, l \in \mathbb{N}$ and $z \in \mathbb{R}^d$ define $y \equiv y_{z,j} :=$
\[ x + \left(\frac{j}{N_{ad}(x)}\right)^{1/d} z, \quad \alpha_z(r) := V_d^{-1} \mu_d(B_0(1) \cap B_z(r)), \text{ and} \]

\[ \Sigma := \left(\frac{1}{(j/l)^{1/2} \alpha_z((l/j)^{1/4})} \right). \]

For \( s, t \in (0, 1), j, l \in \mathbb{N} \) and \( x, z \in \mathbb{R}^d \) let \( p_{\gamma} := \int_{B_\gamma(h^{-1}_{x,f}(s)) \cap B_{\gamma}(h^{-1}_{y,f}(t))} f(w) \, dw \), define \((N_1, N_2, N_3, N_4) \sim \text{Multi}(n; s - p_{\gamma}, t - p_{\gamma}, p_{\gamma}, 1 - s - t + p_{\gamma})\), \((M_1, M_2, M_3) \sim \text{Multi}(n; s, t, 1 - s - t)\), and

\[ F(s, t) := F_{n,x,z}^{(j), (s, t)} := \mathbb{P}(N_1 + N_3 \geq j, N_2 + N_3 \geq l) \]

\[ G(s, t) := G_{n,x,z}^{(j), (s, t)} := \mathbb{P}(M_1 \geq j, M_2 \geq l). \]

Then, given \( c \in (0, 1) \) and writing \( \Phi_V \) for the distribution function of the bivariate normal distribution with mean zero and covariance matrix \( V \), there exists \( A = A(d, \beta, c, (k^1), (k^U)) \) such that

\[
\max \left\{ \left| F(s, t) - \Phi_V \left( \frac{ns - j}{j^{1/2}}, \frac{nt - l}{l^{1/2}} \right) \right|, \left| G(s, t) - \Phi_V \left( \frac{ns - j}{j^{1/2}}, \frac{nt - l}{l^{1/2}} \right) \right| \right\}
\leq A \min \left\{ 1, \frac{1}{\|z\|} \left( \log^{1/2} n \right) \left( k^{M_\beta}(x)^d \right)^{\beta/d} \left( \frac{kM_\beta(x)^d}{nf(x)} \right)^{\beta/d} \right\}
\]

for all \( k \in \{k^1, \ldots, k^U\} \), for all \( j, l \in \mathbb{N} \) such that \( ck \leq j, l \leq k \), for all \( x \in \mathcal{X}_n \), for all \( s, t \in (0, 1) \) such that \( j^{-1/2} |ns - j| \lor l^{-1/2} |nt - l| \leq 3 \log^{1/2} n \), and for all \( 0 < \|z\| \leq (\frac{n_{ad}(x)}{j})^{1/d} \{h^{-1}_{x,f}(s) + h^{-1}_{y,f}(t)\} \).

**Proof.** We present here the approximation for \( F(s, t) \), the approximation for \( G(s, t) \) being similar but much simpler. Let \( X_1, \ldots, X_n \) \text{iid} \( f \) and for \( i = 1, \ldots, n \) and \( k, j, l, x, s, t, z \) in the specified ranges, define \( Y_i := (1_{\{\|x_i - x\| \leq h^{-1}_{x,f}(s)\}}, 1_{\{\|x_i - y\| \leq h^{-1}_{y,f}(t)\}})^T \),

\[ V := \text{Cov}(Y_1) = \begin{pmatrix} s(1 - s) & p_{\gamma} - st \\ p_{\gamma} - st & t(1 - t) \end{pmatrix} \quad \text{and} \quad Z_i := V^{-1/2}(Y_i - (s, t)^T). \]

Then by the Berry–Esseen theorem of Götze (1991) we have

\[
\left| \mathbb{P}(N_1 + N_3 \geq j, N_2 + N_3 \geq l) - \Phi_V (ns - j, nt - l) \right| \lesssim n^{-1/2} \mathbb{E}(\|Z_1\|^2). \quad (75)
\]

In order to control the right hand side of this bound, we will require that bounds on \( p_{\gamma} \). Writing
\(\alpha_z\) for \(\alpha_z((l/j)^{1/d})\), we have

\[
\left| \frac{np_{\gamma}}{j} - \alpha_z \right| \leq \frac{n}{j} |p_{\gamma} - f(x)\mu_d(B_x(h_{x,f}^{-1}(s)) \cap B_y(h_{y,f}^{-1}(t)))| + \frac{n}{j} f(x)\mu_d(B_x(h_{x,f}^{-1}(s)) \cap B_y(h_{y,f}^{-1}(t))) - \alpha_z \right|
\]

\[
\leq \frac{n}{j} M_{x,\beta}(x)^{\beta} f(x) \int_{B_x(h_{x,f}^{-1}(s)) \cap B_y(h_{y,f}^{-1}(t))} \|w - x\|^\beta \, dw + \frac{1}{V_d} \mu_d \left( B_0 \left( \left( \frac{nV_d f(x)h_{x,f}^{-1}(s)}{j} \right)^{1/d} \right) \cap B_z \left( \left( \frac{nV_d f(x)h_{y,f}^{-1}(t)}{j} \right)^{1/d} \right) \right) - \alpha_z \right|
\]

\[
\leq \frac{n}{k} M_{x,\beta}(x)^{\beta} f(x)h_{x,f}^{-1}(s)^{d+\beta} + \left| nV_d f(x)h_{x,f}^{-1}(s)^d - 1 \right| \left( \frac{nV_d f(x)h_{y,f}^{-1}(t)^d}{l} - 1 \right) \leq \frac{\{kM_{x,\beta}(x)^d\}}{nf(x)} \beta/d + \log^{1/2} n \frac{nV_d f(x)}{k^{1/2}},
\]

(76)

where the final bound follows by Lemma 15 and similar arguments to those in (78) and (79) in the bounds on \(U_0\) below. We will also need to bound \(s + t - 2p_{\gamma}\) below. If \(h_{y,f}^{-1}(t) \geq h_{x,f}^{-1}(s)\) then, by the mean value theorem and Lemma 15,

\[
\mu_d(B_y(h_{y,f}^{-1}(t)) \cap B_x(h_{x,f}^{-1}(s))^c) \geq \mu_d(B_y(h_{x,f}^{-1}(s)) \cap B_x(h_{x,f}^{-1}(s))^c) = V_dh_{x,f}^{-1}(s)^d \int_{\left( \frac{1}{4nV_d f(x)} \right)^2} B_{d+1} \frac{1}{4} (\xi) d\xi \geq h_{x,f}^{-1}(s)^d \left( \frac{\|x - y\|^d}{h_{x,f}^{-1}(s)} \right) \geq K(\|z\| \wedge 1). n f(x). \]

A similar argument applies with \((x, s)\) and \((y, t)\) swapped and so we have

\[
s + t - 2p_{\gamma} = \int_{B_y(h_{y,f}^{-1}(t)) \cap B_x(h_{x,f}^{-1}(s))^c} f(w) \, dw + \int_{B_x(h_{x,f}^{-1}(s)) \cap B_y(h_{y,f}^{-1}(t))^c} f(w) \, dw \geq f(x) \{ \mu_d(B_y(h_{y,f}^{-1}(t)) \cap B_x(h_{x,f}^{-1}(s))^c) + \mu_d(B_x(h_{x,f}^{-1}(s)) \cap B_y(h_{y,f}^{-1}(t))^c) \} \geq \frac{K(\|z\| \wedge 1)}{n}. \]

We will also use a lower bound on \(|V| := \det(V)\) when \(\|z\| \geq 1\). Note that with \(e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^d\), when \(\|z\| \geq 1\) we have that \(\alpha_z = \alpha_{\|z\|e_1} \leq \alpha_{e_1}\). If \(l/j \geq (3/2)^d\) then

\[
\frac{j \alpha_{e_1}^2}{l} \leq \frac{j \alpha_{e_1}^2}{l} \leq \frac{j}{l} \leq \left( \frac{2}{3} \right)^{d/2} < 1.
\]

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However if \( l/j < (3/2)^d \) then

\[
\frac{j\alpha^2}{l} \leq \alpha^2 < V_d^{-2}\mu_d(B_0(1) \cap B_{e_1}((3/2)^{1/d}))^2 < 1.
\]

Thus there exists \( c_d \in (0, 1) \) such that \( j\alpha^2/l \leq c_d \) whenever \( \|z\| \geq 1 \). Thus, by (76), we have that

\[
|V| = st(1 - s)(1 - t) - (p_{\ell} - st)^2 \geq \frac{(1 - c_d)jl}{n^2}(1 + o(1)),
\]

uniformly over \( \|z\| \geq 1 \). Similar to (36), (37) and (38) in the supplement of Berrett, Samworth and Yuan (2019), and splitting up into cases \( \|z\| < 1 \) and \( \|z\| \geq 1 \) where necessary, we have that

\[
p_{\ell}\left\| V^{-1/2} \begin{pmatrix} 1 - s \\ 1 - t \end{pmatrix} \right\|^3 \leq p_{\ell} \min \left\{ \frac{s + t}{|V|}, \frac{1}{p_{\ell} - st} \right\}^{3/2} \lesssim (n/k)^{1/2},
\]

\[
(1 - s - t + p_{\ell})\left\| V^{-1/2} \begin{pmatrix} s \\ t \end{pmatrix} \right\|^3 = (1 - s - t + p_{\ell})\left\{ \frac{st(s + t - 2p_{\ell})}{|V|} \right\}^{3/2} \lesssim (k/n)^{3/2}.
\]

Likewise,

\[
(s - p_{\ell})\left\| V^{-1/2} \begin{pmatrix} 1 - s \\ -t \end{pmatrix} \right\|^3 \leq (s - p_{\ell})t^{3/2}|V|^{-3/2}
\]

\[
= (s - p_{\ell})\frac{3}{2}\left\{ (s + t - 2p_{\ell})(p_{\ell} - st + \frac{(s - p_{\ell})(t - p_{\ell})}{s + t - 2p_{\ell}}) \right\}^{-3/2} \lesssim \left( \frac{n}{k\|z\|} \right)^{1/2},
\]

with a similar bound holding for \( (t - p_{\ell})\left\| V^{-1/2} \begin{pmatrix} -s \\ 1 - t \end{pmatrix} \right\|^3 \). Thus

\[
n^{-1/2}\mathbb{E}\|Z_3\|^3 \lesssim (k\|z\|)^{-1/2},
\]

which in combination with (75) provides a bound on the difference between \( F(s, t) \) and \( \Phi_{nV}(ns - j, nt - l) \). Next, similar to the displayed equation above (39) in the supplement of
Berrett, Samworth and Yuan (2019), we have

\[
\Phi_n \left( n s - j, n t - l \right) - \Phi_{\Sigma} \left( j^{-1/2} (n s - j), l^{-1/2} (n t - l) \right) \leq \min \left\{ 1, 2 \left\| \Sigma^{-1/2} \left( \frac{n s (1-s)}{j}, \frac{n (p - st)}{j^{1/2} l^{1/2}} - \frac{j^{1/2} l^{-1/2} \alpha z}{n (1-1)} - 1 \right) \Sigma^{-1/2} \right\} \\
\lesssim \left\{ (1 - j^{-1/2} l^{-1/2} \alpha z)^{-1} + \left( 1 + j^{-1/2} l^{-1/2} \alpha z \right)^{-1} \right\} \left\{ \frac{\log^{1/2} n}{k^{1/2}} + \frac{j^{1/2}}{l^{1/2}} \left| \frac{np}{j} - \alpha z \right| \right\} \\
\lesssim \frac{1}{\|z\|} \left\{ \frac{\log^{1/2} n}{k^{1/2}} + \left( \frac{k M_{f,\beta}(x)^d}{nf(x)} \right)^{\beta/d} \right\}
\]

as required.

\[\square\]

### 7.7 Bounds on remainder terms in the proof of Proposition 9

To bound \(S_1\): Since \(\zeta < 1/2\) we may apply Lemma 20 to see that

\[
S_{11} := \int_X f(x) \int_0^{x_1} \int_0^{x_2} \max \{ \mathbb{1}_{s \notin I_{m,X}}, \mathbb{1}_{t \notin I_{n,Y}} \} \phi^2(u, v, x) \times B_{k_{\lambda}, m-k_X}(s)B_{k_{\lambda}, n+1-k_Y}(t) ds dt dx = o(m^{-4} + n^{-4}).
\]

By Lemma 19 we have that for every \(\epsilon > 0\),

\[
|S_{12}| := |S_1 - S_{11}|
\]

\[
= \left| \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \phi^2(u, v, x) B_{k_{\lambda}, m-k_X}(s)B_{k_{\lambda}, n+1-k_Y}(t) ds dt dx \right|
\]

\[
\lesssim \int_{X_{m,n}} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} M_{\beta}(x)^{2d(\kappa_1 + \kappa_2)} (1 + \|x\|)^{2d(\kappa_1 + \kappa_2)} dx
\]

\[
= O \left( \max \left\{ \left( \frac{k_X}{m} \right)^{\lambda_1(1-2\zeta) - \epsilon}, \left( \frac{k_Y}{n} \right)^{\lambda_2(1-2\zeta) - \epsilon} \right\} \right),
\]

where the final bound holds by Lemma 18, as in the bound on \(R_1\).
To bound $S_2$: Using Lemma 15 we now have that

$$|S_2| = \left| \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \left\{ \phi^2 \left( \frac{k_x f(x)}{m s}, \frac{k_y g(x)}{n t}, x \right) - \phi^2 \left( \frac{k_x f(x)}{m s}, \frac{k_y g(x)}{n t}, x \right) \right\} B_{k_{X,m-k_X}}(s)B_{k_{Y,n+1-k_Y}}(t) \ ds \ dt \ dx \right|$$

$$\lesssim \int_{X_{m,n}} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \int_{I_{m,X}} \int_{I_{n,Y}} \left\{ \int_{V_{d,f}} \frac{s}{h_{x,f}^{1,1}(s)} - 1 \right\} B_{k_{X,m-k_X}}(s)B_{k_{Y,n+1-k_Y}}(t) \ ds \ dt \ dx$$

$$\lesssim \int_{X_{m,n}} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \left\{ \left( \frac{k_x M_{\beta}(x)^d}{m f(x)} \right)^{2\beta \frac{1}{d}} + \left( \frac{k_y M_{\beta}(x)^d}{n g(x)} \right)^{2\beta \frac{1}{d}} \right\} \ dx$$

$$= O \left( \max \left\{ \left( \frac{k_x}{m} \right)^{2\beta \frac{1}{d}}, \left( \frac{k_x}{m} \right)^{\lambda_1(1-2\xi) - \epsilon}, \left( \frac{k_y}{n} \right)^{2\beta \frac{1}{d}}, \left( \frac{k_y}{n} \right)^{\lambda_2(1-2\xi) - \epsilon} \right\} \right)$$

for all $\epsilon > 0$, where for the final bound we use Lemma 18 as in (26) and (27).

To bound $S_3$: Using Lemma 20 and Lemma 18 we may write

$$|S_3| = \left| \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \phi^2 \left( \frac{k_x f(x)}{m s}, \frac{k_y g(x)}{n t}, x \right) B_{k_{X,m-k_X}}(s)B_{k_{Y,n+1-k_Y}}(t) \ ds \ dt \ dx \right|$$

$$\lesssim \left| \int_{X_{m,n}} f(x) \int_{I_{m,X}} \int_{I_{n,Y}} \left\{ \phi^2 \left( \frac{k_x f(x)}{m s}, \frac{k_y g(x)}{n t}, x \right) - \phi_x^2 \right\} \right|$$

$$\times B_{k_{X,m-k_X}}(s)B_{k_{Y,n+1-k_Y}}(t) \ ds \ dt \ dx \right| + \int_{X_{m,n}} f(x) \phi_x^2 \ dx + o(m^{-4} + n^{-4})$$

$$\lesssim \left( k_x^{-1/2} + k_y^{-1/2} \right) \int_{X_{m,n}} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \ dx + \int_{X_{m,n}} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \ dx + o(m^{-4} + n^{-4})$$

$$= O \left( \max \left\{ k_x^{-1/2}, k_y^{-1/2}, \left( \frac{k_x}{m} \right)^{\lambda_1(1-2\xi) - \epsilon}, \left( \frac{k_y}{n} \right)^{\lambda_2(1-2\xi) - \epsilon} \right\} \right)$$,

for every $\epsilon > 0$.

To bound $T_1$: We first consider

$$T_{11} := \left( \int_{X^2} - \int_{X_{m,j}^2} \right) \int_{I_{m,X}^2} \int_{I_{n,Y}^2} (h \, dH_m^{(1)} \, dG_n^{(2)})(s_1, s_2, t_1, t_2) \ ds \ dy.$$
By symmetry we may write
\[ T_{11} = T_{111} + 2T_{112}, \]
where
\[
T_{111} := \int_{X_m \times X_m} f(x) f(y) \int_{\mathcal{I}_m \times X_m} \int_{\mathcal{I}_m \times X_m} (h \, dH_m^{(1)} \, dG_n^{(2)}) (s_1, s_2, t_1, t_2) \, dx \, dy
\]
and
\[
T_{112} := \int_{X_m \times X_m} f(x) f(y) \int_{\mathcal{I}_m \times X_m} \int_{\mathcal{I}_m \times X_m} (h \, dH_m^{(1)} \, dG_n^{(2)}) (s_1, s_2, t_1, t_2) \, dx \, dy.
\]

Using Lemma 18 and Lemma 19 as in the bounds on \( S_1 \), and using Lemma 21 we have that
\[
|T_{111}| \lesssim \frac{k_X}{m} \left\{ \int_{X_m \times X_m} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} M_\beta(x)^{d(\kappa_1+\kappa_2)} (1 + \|x\|)^{d(\kappa_1+\kappa_2)} \, dx \right\}^2 = O \left( \frac{k_X}{m} \frac{1+2\lambda_1(1-\zeta)-\epsilon}{1+2\lambda_1(1-\zeta)-\epsilon} \right)
\]
for all \( \epsilon > 0 \). We now turn to \( T_{112} \), and similarly write
\[
|T_{112}| = \frac{k_X}{m} \left| \int_{X_m \times X_m} f(x) f(y) \int_{\mathcal{I}_m \times X_m} \int_{\mathcal{I}_m \times X_m} (h \, dH_m^{(1)} \, dG_n^{(2)}) (s_1, s_2, t_1, t_2) \, dx \, dy \right|
\lesssim \frac{k_X}{m} \int_{X_m \times X_m} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} \int_{\mathcal{I}_m \times X_m} \int_{\mathcal{I}_m \times X_m} (h \, dH_m^{(1)} \, dG_n^{(2)}) (s_1, s_2, t_1, t_2) \, dx \, dy
\]
\[
= O \left( \frac{k_X}{m} \frac{1+\lambda_1(1-\zeta)-\epsilon}{1+\lambda_1(1-\zeta)-\epsilon} \right)
\]
for all \( \epsilon > 0 \). Combining our bounds on \( T_{111} \) and \( T_{112} \) we have that
\[
T_{11} = O \left( \frac{k_X}{m} \frac{1+\lambda_1(1-\zeta)-\epsilon}{1+\lambda_1(1-\zeta)-\epsilon} \right)
\]
for all \( \epsilon > 0 \). We can develop analogous bounds on
\[
T_{12} := \int_{X^2} - \int_{X^2 \times g} \int_{\mathcal{I}_m \times X_m} \int_{\mathcal{I}_m \times X_m} (h \, B_{k_X,m-k_X} B_{k_X,m-k_X} \, dH_n^{(2)}) (s_1, s_2, t_1, t_2) \, ds_1 \, ds_2 \, dx \, dy
\]
to conclude that
\[
T_1 = T_{11} + T_{12} = O \left( \max \left\{ \frac{k_X}{m} \frac{1+\lambda_1(1-\zeta)-\epsilon}{1+\lambda_1(1-\zeta)-\epsilon}, \frac{k_Y}{n} \frac{1+\lambda_2(1-\zeta)-\epsilon}{1+\lambda_2(1-\zeta)-\epsilon} \right\} \right)
\]
for all \( \epsilon > 0 \).

To bound \( T_2 \): Here we use the notation

\[
L_x^1(s,t) := \phi(f(x), v_{x,t}, x) + \left( \frac{k_X}{mS} - 1 \right) f(x) \phi_{10}(f(x), v_{x,t}, x)
\]

for a linearised version of \( \phi(u_{x,s}, v_{x,t}, x) \), so that

\[
h^{(1)}(s_1, s_2, t_1, t_2) = L_x^1(s_1, t_1)L_y^1(s_2, t_2).
\]

Again we write \( T_2 = T_{21} + T_{22} \), with

\[
T_{21} := \int_{X_{m,f}^2} f(x) \int_{T_{m,x}^2} \int_{T_{n,y}^2} \{ h - h^{(1)} \} dH_m^{(1)} dG_n^{(2)}(s_1, s_2, t_1, t_2) \, dx \, dy
\]

\[
= \int_{X_{m,f}^2} f(x) \int_{T_{m,x}^2} \int_{T_{n,y}^2} \left[ \{ \phi(u_{x,s_1}, v_{x,t_1}, x) - L_x^1(s_1, t_1) \} \{ \phi(u_{y,s_2}, v_{y,t_2}, y) - L_y^1(s_2, t_2) \} \right.
\]

\[
+ 2L_x^1(s_1, t_1) \{ \phi(u_{y,s_2}, v_{y,t_2}, y) - L_y^1(s_2, t_2) \} \} dH_m^{(1)}(s_1, s_2) dG_n^{(2)}(t_1, t_2) \, dx \, dy
\]

\[
=: T_{211} + T_{212}
\]

and \( T_{22} := T_2 - T_{21} \) having a similar expression. Now

\[
|T_{211}| \lesssim \frac{k_X}{m} \left[ \int_{X_{m,f}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2 M_\beta(x)} \rho_2(1 + \|x\|)^{\rho_2} \left\{ \left( \frac{k_X M_\beta(x)}{m f(x)} \right)^{2\alpha \beta} + \frac{\log m}{k_X} \right\} \, dx \right]^2
\]

\[
= O\left( \frac{k_X}{m} \max \left\{ \left( \frac{k_X}{m} \right)^{2(2\alpha \beta)} , \left( \frac{k_X}{m} \right)^{2\lambda_1(1-\zeta)-\epsilon} , \frac{\log^2 m}{k_X^2} \right\} \right)
\]

for every \( \epsilon > 0 \). When bounding \( T_{212} \) we first integrate over \( s_1 \) using the facts that

\[
\int_0^1 \{ B_{k_X,m-k_X-1}(s_1, s_2) - B_{k_X,m-k_X}(s_1)B_{k_X,m-k_X}(s_2) \} \, ds_1
\]

\[
= \frac{m-1}{m-k_X-1} B_{k_X,m-k_X-1}(s_2) \left( s_2 - \frac{k_X}{m-1} \right)
\]

and

\[
\int_0^1 \frac{k_X}{ms_1} \{ B_{k_X,m-k_X-1}(s_1, s_2) - B_{k_X,m-k_X}(s_1)B_{k_X,m-k_X}(s_2) \} \, ds_1
\]

\[
= \frac{k_X(m-2)}{m(k_X-1)} B_{k_X,m-k_X-2}(s_2) \left\{ 1 - \frac{(m-1)^2}{(m-k_X-1)(m-k_X-2)}(1-s_2)^2 \right\}
\]

\[
= B_{k_X,m-k_X-2}(s_2) \left\{ 2 \left( \frac{k_X}{m-2} - s_2 \right) + O \left( \frac{k_X^2}{m^2} + \frac{1}{m} \right) \right\},
\]
uniformly for \( s_2 \in \mathcal{I}_{m,X} \). Using (25) and the fact that \( k_X^{3/2}/m \to 0 \) we can now see that

\[
|T_{212}| \leq \frac{k_X^{1/2}}{m} \int_{X_{m,f}} f(y)^{1-\kappa_1} g(y)^{-\kappa_2} M_\beta(y)^{\delta_2} (1 + \|y\|)^{\delta_2} \left\{ \left( \frac{k_X M_\beta(y)^d}{mf(y)} \right)^{2\beta \delta} + \frac{\log m}{k_X} \right\} dy \\
= O\left( \frac{k_X^{1/2}}{m} \max\left\{ \left( \frac{k_X}{m} \right)^{2\beta \delta}, \frac{k_X}{m} \lambda_1 (1-\kappa_1) - \epsilon, \frac{\log m}{k_X} \right\} \right)
\]

for every \( \epsilon > 0 \). Combining our bounds on \( T_{211} \) and \( T_{212} \) we therefore have that

\[
|T_{21}| = O\left( \max\left\{ \left( \frac{k_X}{m} \right)^{1+\lambda_1 (1-\kappa_1) - \epsilon}, \frac{\log m}{mk_X^{1/2}}, \left( \frac{k_X}{m} \right)^{2\beta \delta}, \frac{(k_X)^{1+2(2\beta \delta)}}{m} \right\} \right)
\]

for every \( \epsilon > 0 \). By analogous arguments we can show that

\[
|T_{22}| = O\left( \max\left\{ \left( \frac{k_Y}{n} \right)^{1+\lambda_2 (1-\kappa_2) - \epsilon}, \frac{\log n}{nk_Y^{1/2}}, \left( \frac{k_Y}{n} \right)^{2\beta \delta}, \frac{(k_Y)^{1+2(2\beta \delta)}}{n} \right\} \right),
\]

for every \( \epsilon > 0 \), and this concludes the bound on \( T_2 \).

To bound \( T_3 \): Here we integrate out \((s_1, s_2)\) in the \( X_{m,f} \) term and \((t_1, t_2)\) in the \( X_{n,g} \) term. Now

\[
\int_0^1 \int_0^{1-s_1} h^{(1)}(s_1, s_2, t_1, t_2) B_{k_X,k_X,m-2k_X-1}(s_1, s_2) ds_1 ds_2 \\
- \int_0^1 \int_0^{1-t_1} h^{(1)}(s_1, s_2, t_1, t_2) B_{k_X,m-k_X}(s_1) B_{k_X,m-k_X}(s_2) ds_1 ds_2 \\
= f(x)\phi_{10}(f(x), v_{x,t_1}, x) f(y)\phi_{10}(f(y), v_{y,t_2}, y) \\
\times \left\{ \int_0^1 \int_0^{1-s_1} \left( \frac{k_X}{ms_1} - 1 \right) \left( \frac{k_X}{ms_2} - 1 \right) B_{k_X,k_X,m-2k_X-1}(s_1, s_2) ds_1 ds_2 \\
- \int_0^1 \int_0^{1-t_1} \left( \frac{k_X}{ms_1} - 1 \right) \left( \frac{k_X}{ms_2} - 1 \right) B_{k_X,m-k_X}(s_1) B_{k_X,m-k_X}(s_2) ds_1 ds_2 \right\} \\
+ f(x)\phi_{10}(f(x), v_{x,t_1}, x) f(y)\phi_{10}(f(y), v_{y,t_2}, y) \int_0^1 \frac{k_X}{ms_1} \{ B_{k_X,m-k_X-1}(s_1) - B_{k_X,m-k_X}(s_1) \} ds_1 \\
+ \phi(f(x), v_{x,t_1}, x) f(y)\phi_{10}(f(y), v_{y,t_2}, y) \int_0^1 \frac{k_X}{ms_2} \{ B_{k_X,m-k_X-1}(s_2) - B_{k_X,m-k_X}(s_2) \} ds_2 \\
= -\frac{k_X}{(k_X - 1)m} \left\{ k_X(3m - 5) (k_X - 1)m - 2 \right\} f(x)\phi_{10}(f(x), v_{x,t_1}, x) f(y)\phi_{10}(f(y), v_{y,t_2}, y) \\
+ \left\{ f(x)\phi_{10}(f(x), v_{x,t_1}, x) f(y)\phi_{10}(f(y), v_{y,t_2}, y) + \phi(f(x), v_{x,t_1}, x) f(y)\phi_{10}(f(y), v_{y,t_2}, y) \right\} \right].
\]
The contribution from the $X_{n,g}$ term is simpler because the marginals of the $B_{k_y,n-ky+1}$ density are equal to $B_{k_y,n-ky+1}$, and we have

$$\int_0^1 \int_0^{1-t_1} h^{(2)}(s_1, s_2, t_1, t_2) B_{k_y,n-2ky+1}(t_1, t_2) \, dt_1 \, dt_2$$

$$- \int_0^1 \int_0^1 h^{(2)}(s_1, s_2, t_1, t_2) B_{k_y,n-ky+1}(t_1) B_{k_y,n-ky+1}(t_2) \, dt_1 \, dt_2$$

$$= g(x) \phi_{01}(u_{x,s_1}, g(x), x) g(y) \phi_{01}(u_{y,s_2}, g(y), y) \left\{ \int_0^1 \int_0^{1-t_1} \frac{k_{k_y}^2}{n^2 t_1 t_2} B_{k_y,n-2ky+1}(t_1, t_2) \, dt_1 \, dt_2 \right\}$$

$$- \int_0^1 \int_0^1 \frac{k_{k_y}^2}{n^2 t_1 t_2} B_{k_y,n-ky+1}(t_1) B_{k_y,n-ky+1}(t_2) \, dt_1 \, dt_2 \} \right\}$$

$$= - \frac{k_{k_y}^2}{(k_Y - 1)^2 n} g(x) \phi_{01}(u_{x,s_1}, g(x), x) g(y) \phi_{01}(u_{y,s_2}, g(y), y).$$

The error $T_3$ is the error in, for example, $k_{k_y}^2(k_Y - 1)^{-2} n^{-1} \approx n^{-1}$, together with the contribution from $(s_1, s_2) \notin T_{n,X}$ and $(t_1, t_2) \notin T_{n,Y}$, and we can use Lemma 17 to see that

$$T_3 = o(m^{-1} + n^{-1}).$$

To bound $U_0$: We write $r_{m,x,y}^{(1)} := h_{x,f}^{-1}(a_{m,x}^+) + h_{y,f}^{-1}(a_{m,x}^+)$ and $r_{m,x,y}^{(2)} := h_{x,g}^{-1}(a_{n,Y}^+) + h_{y,g}^{-1}(a_{n,Y}^+)$ as shorthand. For $s_1, s_2 \leq a_{m,X}^+$ and $t_1, t_2 \leq a_{n,Y}^+$ we have $F_{m,n,x,y}(s_1, s_2, t_1, t_2) = G_{m,n}(s_1, s_2, t_1, t_2)$ unless $\|y - x\| \leq \max\{r_{m,x,y}^{(1)}, r_{m,x,y}^{(2)}\}$. Here we will present bounds in the case $\|y - x\| \leq r_{m,x,y}^{(1)}$, but the other case follows using very similar arguments. First, by using Lemma 18 and Lemma 19, we have that

$$\int_{X_{m,n}} f(x) \sup_{s \in I_{m,X}, t \in I_{n,Y}} |\phi(u_{x,s}, v_{x,t}, x)| \, dx$$

$$\leq \int_{X_{m,n}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} M_{\beta} \phi(x)^{d(\kappa_1+\kappa_2)} (1 + \|x\|)^{d(\kappa_1+\kappa_2)} \, dx$$

$$= O\left( \max\left\{ \left( \frac{k_X}{m} \right)^{\lambda_1(1-\zeta)-\epsilon}, \left( \frac{k_Y}{m} \right)^{\lambda_2(1-\zeta)-\epsilon} \right\} \right), \quad (77)$$

for every $\epsilon > 0$, and we proceed by showing that, since $x$ and $y$ are close, the contribution from $X_{m,n} \times X$ behaves similarly to the contribution from $X_{m,n} \times X_{m,n}$, which can be bounded by the square of the final bound in (77). It suffices to consider $(x, y) \in X_{m,n} \times X_{m,n}$, as the contribution from $X_{m,n} \times X_{m,n}$ is more straightforward.

By a very similar argument to that used to establish (54) in the proof of Proposition 13, we
have that \( \|x - y\| \leq \{M_\beta(y)^d \log^{1/2} m\}^{-1/d} \) for \( m \) sufficiently large, and hence, by Lemma 16, that
\[
|f(x)/f(y) - 1| \leq 8d^{1/2} \{M_\beta(y)\|y - x\|\}^{1/\beta} \leq 1/2,
\]
and in particular \( f(x) \geq f(y)/2 \). Thus, again using Lemma 16, we have that
\[
\max_{t=1,\ldots,m} \left( \frac{\|f(t)(x)\|}{f(x)} \right)^{1/t} \leq 2(1 + d^{\beta/2})M_\beta(y). \tag{78}
\]
In addition,
\[
\sup_{w,z \in B_x(\{4d^{1/2}M_\beta(y)\}^{-1})} \frac{\|f^{(\beta)}(z) - f^{(\beta)}(w)\|}{\|z - w\|^2 - \beta f(w)} \leq \sup_{w,z \in B_y(\{2d^{1/2}M_\beta(y)\}^{-1})} \frac{\|f^{(\beta)}(z) - f^{(\beta)}(w)\|}{\|z - w\|^2 - \beta f(w)} \leq M_\beta(y)^2, \tag{79}
\]
and so we have that \( M_\beta(x) \leq 2(1 + d^{\beta/2})M_\beta(y) \). Using this fact and the previously established fact that \( f(x) \geq f(y)/2 \), we may apply Lemma 15 to see that in fact
\[
\|x - y\| \leq r^{(1)}_{m,x,y} \lesssim \left( \frac{k_\lambda \log m}{m f(y)} \right)^{1/d}.
\]
Using Lemma 16 we also have that \( g(x) \geq g(y)/2 \), and therefore that
\[
\max\left\{ \frac{f(y)M_\beta(y)^{-d}}{f(x)M_\beta(x)^{-d}}, \frac{g(y)M_\beta(y)^{-d}}{g(x)M_\beta(x)^{-d}} \right\} \leq 2^{d+1}(1 + d^{\beta/2})^d. \tag{80}
\]
Since \( x \in \mathcal{X}^c_{m,n} \), we have now established that
\[
\min\left\{ \frac{mf(y)M_\beta(y)^{-d}}{k_X \log m}, \frac{ng(y)M_\beta(y)^{-d}}{k_Y \log n} \right\} \leq 2^{d+1}(1 + d^{\beta/2})^d.
\]
Applying the same bounds as we would for \( \mathcal{X}^c_{m,n} \), as in (77), we can now see that
\[
U_0 = O\left( \max\left\{ \left( \frac{k_\lambda}{m} \right)^{2 \lambda_1 (1 - \zeta) - \epsilon}, \left( \frac{k_\lambda}{n} \right)^{2 \lambda_2 (1 - \zeta) - \epsilon} \right\} \right),
\]
for every \( \epsilon > 0 \), as claimed.
To bound $U_1$: By Lemma 17 we have that

$$\max_{t_1 \in \{a_{n,Y}^+,a_{n,Y}^-\}} \sup_{t_2 \in [0,1]} |F^{(2)}_{n,x,y} - G^{(2)}_n||(t_1, t_2) \vee \max_{t_2 \in \{a_{n,Y}^+,a_{n,Y}^-\}} \sup_{t_1 \in [0,1]} |F^{(2)}_{n,x,y} - G^{(2)}_n||(t_1, t_2) = o(n^{-4}). \quad (81)$$

In order to use this to bound $U_1$, corresponding to the right-hand side of (43), we must first develop bounds on the derivatives of $h$. Writing $S_x(r) := \{y \in \mathbb{R}^d : \|x - y\| = r\}$ and $d\text{Vol}_S$ for the associated volume element we have by Lemma 16 that for $r \leq \{(8d)^{1/(\beta - 2)}M_\beta(x)\}^{-1}$,

$$\left| \frac{h'_x,f(r)}{dV_d d^{-1} f(x)} - 1 \right| = \left| \frac{1}{dV_d d^{-1} f(x)} \int_{S_x(r)} \{f(y) - f(x)\} d\text{Vol}_S(y) \right| \lesssim \left\{rM_\beta(x)\right\}^{2\lambda_\beta},$$

with a similar bound holding for $h'_{x,g}(r)$. Using Lemma 15, for $x \in \mathcal{X}_{m,n}$, we have that $\max\{M_\beta(x)^d h_{x,f}^{-1}(a_{m,\mathcal{X}}^+)^d, M_\beta(x)^d h_{x,g}^{-1}(a_{n,Y}^+)^d\} \lesssim 1/\log m \to 0$ and so we have, by Lemma 14, that

$$\left| \frac{\partial}{\partial s} \phi(u_{x,s}, v_{x,t}, x) + \frac{k_X f(x)}{m^2 s^2} \phi_{10}\left(\frac{k_X f(x)}{m^2 s^2}, \frac{k_Y g(x)}{nt}, x\right) \right| \leq \left| -\frac{k_X d}{m V_d h_{x,f}^{-1}(s)^{d+1} h'_{x,f}(h_{x,f}^{-1}(s))} \phi_{10}(u_{x,s}, v_{x,t}, x) + \frac{k_X f(x)}{m^2 s^2} \phi_{10}\left(\frac{k_X f(x)}{m^2 s^2}, \frac{k_Y g(x)}{nt}, x\right) \right|$$

$$\leq \left| \frac{k_X d}{m V_d h_{x,f}^{-1}(s)^{d+1} h'_{x,f}(h_{x,f}^{-1}(s))} \left| \phi_{10}(u_{x,s}, v_{x,t}, x) - \phi_{10}\left(\frac{k_X f(x)}{m^2 s^2}, \frac{k_Y g(x)}{nt}, x\right) \right| \right|$$

$$\lesssim \left| \frac{k_X f(x)}{m^2 s^2} f(x)^{-\kappa_1} g(x)^{-\kappa_2} \left\{ \left(\frac{s M_\beta(x)^d}{f(x)}\right)^{(2\lambda_\beta)/d} + \left(\frac{t M_\beta(x)^d}{g(x)}\right)^{(2\lambda_\beta)/d} \right\}\right|$$

$$\lesssim s^{-1} f(x)^{-\kappa_1} g(x)^{-\kappa_2}, \quad (82)$$

uniformly for $x \in \mathcal{X}_{m,n}$, $s \in \mathcal{I}_{m,X}$ and $t \in \mathcal{I}_{n,Y}$. In particular, we have that

$$\left| \frac{\partial}{\partial s} \phi(u_{x,s}, v_{x,t}, x) \right| \lesssim s^{-1} f(x)^{-\kappa_1} g(x)^{-\kappa_2}, \quad (83)$$

uniformly for $x \in \mathcal{X}_{m,n}$, $s \in \mathcal{I}_{m,X}$ and $t \in \mathcal{I}_{n,Y}$. Analogous arguments also reveal that $\frac{\partial}{\partial t}\phi(u_{x,s}, v_{x,t}, x) \lesssim t^{-1} f(x)^{-\kappa_1} g(x)^{-\kappa_2}$, uniformly for $x \in \mathcal{X}_{m,n}$, $s \in \mathcal{I}_{m,X}$ and $t \in \mathcal{I}_{n,Y}$. Moreover, since $x \in \mathcal{X}_{m,n}$ and $\|y - x\| \leq \max\{r_{m,x,y}^{(1)}, r_{n,x,y}^{(2)}\}$, we may argue as we did leading up to (80) to obtain similar bounds on $\frac{\partial}{\partial s}\phi(u_{x,s}, v_{y,t}, y)$ and $\frac{\partial}{\partial t}\phi(u_{y,s}, v_{y,t}, y)$. Thus, using (43) and (81), we find that $U_1 = o(n^{-4})$. 

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To bound $U_2$: Again using Lemma 17, we have that
\[
\max\left\{ \sup_{s_1 \in [0,1]} |F^{(1)}_{m,x,y} - G^{(1)}_m|(s_1, a^-_{m,X}), \sup_{s_2 \in [0,1]} |F^{(1)}_{m,x,y} - G^{(1)}_m|(a^-_{m,X}, s_2), \right. \\
\left. |F^{(1)}_{m,x,y} - G^{(1)}_m|(a^+_{m,X}, a^+_{m,X}) \right\} = o(m^{-4}). \quad (84)
\]

By similar arguments to those used in the bound on $U_1$ we have that
\[
\left| \frac{\partial^2}{\partial s \partial t} \phi(u_{x,s}, v_{x,t}, x) \right| = \frac{k_X d}{mV_d h_x^{-1}(s)^{d+1} h_x'(s)} nV_d h_x^{-1}(t)^{d+1} h_x'(h_x^{-1}(t)) |\phi_{11}(u_{x,s}, v_{x,t}, x)| \\
\lesssim s^{-1} t^{-1} f(x)^{-\kappa_1} g(x)^{-\kappa_2},
\]
uniformly for $x \in \mathcal{X}_{m,n}$, $s \in I_{m,X}$ and $t \in I_{n,Y}$; moreover, the same bound also holds for $\frac{\partial^2}{\partial s \partial t} \phi(u_{y,s}, v_{y,t}, y)$. We may therefore use (43), (82) and (84) to conclude that $U_2 = o(m^{-4})$.

To bound $U_3$: By Lemma 17, we have that
\[
F^{(1)}_{m,x,y}(s_1, a^+_{m,X}) - G^{(1)}_m(s_1, a^+_{m,X}) = \frac{B_{k_X, m-k_X}(s_1)}{m-1} \mathbb{I}_{\|x-y\| \leq h_x^{-1}(s_1)} + o(m^{-4}), \quad (86)
\]
uniformly for $x \in \mathcal{X}_{m,n}$, $\|y - x\| \leq r^{(1)}_{m,x,y}$ and $s \in I_{m,X}$, with an analogous statement holding for $F^{(1)}_{m,x,y}(a^-_{m,X}, s_2) - G^{(1)}_m(a^-_{m,X}, s_2)$. Now, combining this statement with our bounds on the derivatives of $h$ in (83) and (85), and applying the bounds $|F^{(1)}(s_1, s_2)| \leq \mathbb{I}_{\|y-x\| \leq r^{(1)}_{m,x,y}}$ and $|F^{(2)}(t_1, t_2)| \leq \mathbb{I}_{\|y-x\| \leq r^{(2)}_{m,x,y}}$, we may write
\[
|U_3| \lesssim \int_{X \times X_{m,n}} f(x)^{1-\kappa_1} g(x)^{-\kappa_2} f(y)^{1-\kappa_1} g(y)^{-\kappa_2} \mathbb{I}_{\|y-x\| \leq \min\{r^{(1)}_{m,x,y}, r^{(2)}_{m,x,y}\}} \\
\times \left( \log m \log n \right) \frac{k_X d}{k_X k_Y} + \frac{\log n}{k_X k_Y} + \frac{\log^{1/2} m \log n}{m^{2} k_X^{1/2} k_Y} \, dx \, dy \\
\lesssim \frac{\log m \log n}{k_X k_Y} \int_{X \times X_{m,n}} f(x)^{2-2\kappa_1} g(x)^{-2\kappa_2} \mathbb{I}_{\|y-x\| \leq \min\{r^{(1)}_{m,x,y}, r^{(2)}_{m,x,y}\}} \, dx \, dy \\
\lesssim \frac{\log m \log n}{k_X k_Y} \int_{X_{m,n}} f(x)^{2-2\kappa_1} g(x)^{-2\kappa_2} \min\left\{ \frac{k_X}{m f(x)}, \frac{k_Y}{n g(x)} \right\} \, dx.
\]
Since $\min(m, n) \geq 3$, if $m \geq n$, then $m^{-1} \log m \leq n^{-1} \log n$ and therefore
\[
|U_3| \lesssim \frac{\log m \log n}{k_X k_Y} \int_{X} \frac{k_X}{m} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \, dx \lesssim \frac{\log^2 n}{nk_Y}.
\]
Similarly, if \( n \geq m \) then
\[
|U_3| \lesssim \frac{\log m \log n}{k_X k_Y} \int_{\mathcal{X}} \frac{k_Y}{n} f(x) (2-2\kappa_1 g(x) - 2(1-2\kappa_2) dx \lesssim \frac{\log^2 m}{mk_X}.
\]

Putting these two statements together,
\[
U_3 = O\left( \max \left\{ \frac{\log^2 m}{mk_X}, \frac{\log^2 n}{nk_Y} \right\} \right),
\]
which establishes (45).

To bound \( U_4 \): Using (43), (82) and (84) we have that \( U_4 = o(m^{-4}) \).

To bound \( U_5 \): We first bound the contribution to \( U_5 \) from the discontinuous parts of \( F_{m,x,y}^{(1)} \), arising due to the indicator functions in (41). Recalling the definition of the multinomial random vector \((N_{1}^{(1)}, N_{2}^{(1)}, N_{3}^{(1)}, N_{4}^{(1)})\) in (40), we have that
\[
0 \leq F_{m,x,y}^{(1)}(s_1, s_2) - \mathbb{P}(N_{1}^{(1)} + N_{3}^{(1)} \geq k_X, N_{2}^{(1)} + N_{3}^{(1)} \geq k_X)
\leq \mathbb{P}(N_{1}^{(1)} + N_{3}^{(1)} = k_X - 1) + \mathbb{P}(N_{2}^{(1)} + N_{3}^{(1)} = k_X - 1)
= \left( \frac{m-2}{k_X-1} \right) s^{k_X-1}(1-s)^{m-k_X-1} + \left( \frac{m-2}{k_X-1} \right) t^{k_X-1}(1-t)^{m-k_X-1}
\leq \frac{2}{(2\pi k_X)^{1/2}} (1 + o(1)),
\]
uniformly for \( x \in \mathcal{X}_{m,n}, \|y - x\| \leq r_{m,x,y}^{(1)} \) and \((s_1, s_2) \in I_{m,X}^2\), and we will see is of no larger order than the error in the normal approximation for the continuous part. Now, writing \( y = x + \{\frac{k_Y}{mV_d f(x)}\}^{1/d} z \), define
\[
U_{51} := \int_{\mathcal{X} \times \mathcal{X}_{m,n}} f(x)f(y) \int_{I_{n,Y}^2} B_{k_Y,k_Y,n-2k_Y+1}(t_1, t_2)
\left[ \int_{I_{m,X}^2} h_{1100} \left\{ F^{(1)}(s_1, s_2) - (\Phi_{s_1} - \Phi_{t_2}) \left( ms_1 - k_X, ms_2 - k_X \right) \right\} ds_1 ds_2 
- \int_{I_{m,X}^2} h_{1000} \left\{ F^{(1)}(s_1, a_{m,X}^+) - \frac{B_{k_X,m-k_X}(s_1)}{m-1} 1_{\{\|z\| \leq 1\}} \right\} ds_1 
- \int_{I_{m,X}^2} h_{0100} \left\{ F^{(1)}(a_{m,X}^+, s_2) - \frac{B_{k_X,m-k_X}(s_2)}{m-1} 1_{\{\|z\| \leq 1\}} \right\} ds_2 \right] dt_1 dt_2 dx dy.
\]
By Lemma 15 we have that

\[
\int_X \int_{I_m,x} \frac{1}{m s} B_k x, m - k x (s) \left| \mathbb{1}_{\|y - x\| \leq h^{-1}_x(s)} - \mathbb{1}_{\|z\| \leq 1} \right| \, ds \, dy
\]

\[
\lesssim \frac{1}{k x} \int_{\mathbb{R}^d} \left( \mathbb{1}_{\{ \frac{k x}{m} \}^d < \|y - x\| \leq h^{-1}_x(a_m, x)} + \mathbb{1}_{h^{-1}_x(a_m, x) < \|y - x\| \leq \frac{k x}{m} \}^d/4 \right) \, dy
\]

\[
\leq \frac{\nu}{k x} \left\{ h^{-1}_x f(a_m, x)^d - h^{-1}_x f(a_m, x)^d \right\} \lesssim \frac{1}{m f(x)} \left\{ \log^{1/2} m \left( \frac{1}{k x} \frac{k x M_\beta(x)^d}{m f(x)} \right)^{(1/\beta)/d} + \frac{k x M_\beta(x)^d}{m f(x)} \right\},
\]

uniformly for \( x \in X_{m,n} \). Using this bound together with Lemma 22, (82) and (86) we may say that

\[
|U_{51}| \lesssim \int_{X_{m,n}} \left[ \frac{k x}{m} \log^2 \left( \frac{a_{m,x}}{a_{n,y}} \right) \int_{B_0(2)} \min \left\{ 1, \frac{1}{\|z\|} \left( \log^{1/2} m \left( \frac{1}{k x} \frac{k x M_\beta(x)^d}{m f(x)} \right)^{(1/\beta)/d} \right) \right\} \right] dz
\]

\[
+ \int_{X_{m,n}} \left\{ \frac{\log^{1/2} m}{k x} \left( \frac{k x M_\beta(x)^d}{m f(x)} \right)^{(2/\beta)/d} \right\} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \, dx
\]

\[
= O \left( \frac{1}{m} \max \left\{ \log^{5/2} m, \log^2 m \left( \frac{k x}{m} \right)^{(1/\beta)/d}, \left( \frac{k x}{m} \right)^{(1/\beta)/d} \right\} \right),
\]

for every \( \epsilon > 0 \). In bounding \( U_5 \) it therefore remains to approximate the derivatives of \( h \) using (82) and to bound the contribution from the tails of the \( t_1, t_2, s_1, s_2 \) integrals. By Lemma 17 and standard normal tail bounds the error from these tail contributions is \( o(m^{-4}) \),
and so, using (82),

\[
|U_{52}| := |U_5 - U_{51}| \lesssim \frac{1}{m} \int_{X_{m,n}} \frac{f(x)^{1-2\kappa_1}}{g(x)^{2\kappa_2}} \left\{ \frac{k x M_\beta(x)^d}{m f(x)} \right\}^{(2/\beta)/d} + \left( \frac{k y M_\beta(x)^d}{n g(x)} \right)^{(2/\beta)/d}
\]

\[
+ \int_{X_{m,n}} \left\{ \frac{\log^{1/2} m}{k x} + \frac{\log^{1/2} n}{k y} \right\} \left\{ \frac{k x}{m} \right\}^{2\kappa} \left( \frac{k y}{n} \right)^{2\kappa} \lambda_1(1-2\kappa_1) \epsilon \right\} \right) \right) \right),
\]

for every \( \epsilon > 0 \).

To bound \( U_6 \): Using Lemma 18 we have that

\[
|U_6| \lesssim \frac{1}{m} \int_{X_{m,n}} f(x)^{1-2\kappa_1} g(x)^{-2\kappa_2} \, dx = O \left( \frac{1}{m} \max \left\{ \left( \frac{k x}{m} \right)^{\lambda_1(1-2\kappa_1) \epsilon}, \left( \frac{k y}{n} \right)^{\lambda_2(1-2\kappa_1) \epsilon} \right\} \right),
\]

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for every $\epsilon > 0$. This establishes (47).

To bound $U_7$: Analogously to our bounds on $U_1$, we may use (43), (81) and (82) to show that $U_7 = o(n^{-4})$.

To bound $U_8$: Let $a \in (0, \gamma_*^{-1} - \gamma^{-1})$. Then we also have that $a < 1 - (1 + 2\kappa_2)/\gamma$ and

$$\tau := \frac{2 - 2\kappa_1 - \frac{2(1 + 2\kappa_2)}{\gamma} - a}{1 - \frac{1 + 2\kappa_2}{\gamma} - a} > \frac{d}{\alpha + d}.$$

It follows from Hölder’s inequality and Lemma 18 that

$$\sup_{(f,g) \in \mathcal{F}_{d,\theta}} \int_X f(x)^{2 - 2\kappa_1} g(x)^{1 - 2\kappa_2} \left[ \left\{ \frac{M_\beta(x)^d}{f(x)} \right\} \lambda_1 + \left\{ \frac{M_\beta(x)^d}{g(x)} \right\} \lambda_2 \right]^a dx$$

$$\leq C \frac{1 + 2\kappa_2 + a}{\gamma} \sup_{(f,g) \in \mathcal{F}_{d,\theta}} \left\{ \int f(x)^{\tau} dx \right\}^{1 - \frac{1 + 2\kappa_2 - a}{\gamma}} < \infty.$$

Thus, using Lemma 22, (43), (81) and (82), and the change of variables $y = x + (\frac{ky}{nV_2(x)})^{1/d} z$, we have that

$$|U_8| := \left| \int_X \int_{X \times X_{m,n}} f(x) f(y) \int_{T_{m,X}^2} B_{k_X, k_{X,m}, 2k_X - 1}(s_1, s_2) \int_{T_{n,Y}^2} h_{0011}(s_1, s_2, t_1, t_2) \times \left\{ F(2)(t_1, t_2) - (\Phi_{t_1} - \Phi_{t_2}) \left( \frac{nt_1 - ky}{k_1^{1/2}}, \frac{nt_2 - ky}{k_1^{1/2}} \right) \right\} dt_1 dt_2 ds_1 ds_2 dx dy \right|$$

$$\lesssim \frac{ky}{n} \log \left( \frac{a_{n,y}}{a_{n,Y}} \right) \int_{X_{m,n}} f(x)^{2 - 2\kappa_1} g(x)^{-1 - 2\kappa_2} \times \int_{B_0(2)} \min \left\{ 1, \frac{1}{\|z\|} \left( \frac{\log^{1/2} n}{k_Y^{1/2}} + \left( \frac{ky M_\beta(x)^d}{ng(x)} \right)^{(1 + \beta)/d} \right) \right\} dz dx$$

$$\lesssim \frac{\log^2 n}{n} \int_{X_{m,n}} f(x)^{2 - 2\kappa_1} g(x)^{-1 - 2\kappa_2} \left\{ \frac{\log^{1/2} n}{k_Y^{1/2}} + \left( \frac{ky M_\beta(x)^d}{ng(x)} \right)^{(1 + \beta)/d} \right\} dx$$

$$= O \left( \frac{\log^2 n}{n} \max \left\{ \frac{\log^{1/2} n}{k_Y^{1/2}}, \left( \frac{ky}{n} \right)^{(1 + \beta)/d}, \left( \frac{ky}{n} \right)^{\lambda_2(\gamma_*^{-1} - \gamma^{-1}) - \epsilon} \right\} \right),$$

for all $\epsilon > 0$. As with $U_5$ we now define $U_{82} := U_8 - U_{81}$ and note that to bound $U_{82}$ we need to control the tails of $s_1, s_2, t_1, t_2$ integrals and our approximations to the derivatives of $h$. 79
By (82) and Lemma 17 we have that

\[ |U_{82}| \lesssim \frac{1}{n} \int_{\chi_{m,n}} f(x)^{2-2\kappa_1} g(x)^{-1-2\kappa_2} \times \left\{ \left( \frac{k_X M_\beta(x)^d}{m f(x)} \right)^{(2\lambda\beta)/d} + \left( \frac{k_Y M_\beta(x)^d}{ng(x)} \right)^{(2\lambda\beta)/d} \frac{\log^{1/2} m}{k_X^{1/2}} + \frac{\log^{1/2} n}{k_Y^{1/2}} + m^{-2} \right\} dx \]

= \( O \left( \frac{1}{n} \max \left\{ \log^{1/2} n, \log^{1/2} m, \left( \frac{k_X}{m} \right)^{(2\lambda\beta)/d}, \left( \frac{k_Y}{m} \right)^{(2\lambda\beta)/d}, \left( \frac{k_X}{n} \right)^{\lambda_1(\gamma^*-1-\gamma^{-1})-\epsilon}, \left( \frac{k_Y}{n} \right)^{\lambda_2(\gamma^*-1-\gamma^{-1})-\epsilon}, m^{-2} \right\} \) \),

for every \( \epsilon > 0 \). This establishes (50), and therefore concludes the proof.

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