Waring problems and the Lefschetz properties

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Abstract
We study three variations of the Waring problem for homogeneous polynomials, concerning the Waring rank, the border rank and the cactus rank of a form. We show how the Lefschetz properties of the associated algebra affect them. We construct new families of wild forms, that is, forms whose cactus rank, of schematic nature, is bigger than the border rank, defined geometrically.

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Introduction
The Waring problem, in number theory, is the search, for each exponent $k$, for the minimum $s$ such that every positive integer can be decomposed as a sum of at most $s$ perfect $k$th powers. In analogy, the algebraic Waring problem asks about the minimum $s$ such that any homogeneous polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]_d$, of degree $d$, can be decomposed as a sum of at most $s$ $d$th powers of linear forms.

The Waring problem for polynomials is a classical subject in commutative algebra and algebraic geometry. It goes back to Sylvester, which solves it for binary forms in [40, 41]. An explicit decomposition for a given polynomial is hard to find. For monomials, there is a decomposition given in [11, 12], but this decomposition sometimes is not the minimal one. The Waring problem was solved for generic forms by Alexander and Hirschowitz in [1–3]. There are several applications of Waring problems in computational and applied mathematics (see [8, 14]).
We are interested in three variants of the Waring problem, focusing our attention on special forms. We work over the complex numbers. Let \( f \in R = \mathbb{C}[x_0, \ldots, x_n] \) be a degree \( d \) form. We consider these notions of rank for a form \( f \):

(i) The Waring rank of \( f \) is its algebraic-arithmetic rank: it is the minimum \( s = \text{wrk}(f) \) such that \( f \) can be decomposed as a sum of \( d \)-th powers of \( s \) linear forms.

(ii) The border rank of \( f \) is its rank of geometric flavor: it is the minimum \( s = \text{rk}(f) \) with the property that the class of \( f \) in \( \mathbb{P}(R_d) \), where \( R_d = \mathbb{C}[x_0, \ldots, x_n] \), belongs to the \( s \)-th secant variety of the Veronese image \( V_d(\mathbb{P}^n) \subset \mathbb{P}(R_d) \). It is equivalent to say that there is a one parameter family of forms \( f_t \), having Waring rank \( s \), and \( f = \lim_{t \to 0} f_t \).

(iii) The cactus rank of \( f \) is its rank of schematic nature: it is the minimum \( s = \text{cr}(f) \) for which there is a 0-dimensional scheme \( K \), of length \( s \), \( K \subset V_d(\mathbb{P}^n) \subset \mathbb{P}(R_d) \), so that \([f] \in <K>\), with \(<K>\) denoting the schematic linear span of \( K \).

It follows that \( \text{rk}(f) \leq \text{wrk}(f) \) and \( \text{cr}(f) \leq \text{wrk}(f) \), while in general \( \text{cr}(f) \) and \( \text{rk}(f) \) are incomparable (see [6, 26, 27]). The cactus rank was introduced in [26] where it was called scheme length (see also [27, Definition 5.1, p.135]). The name cactus rank was introduced in [9, 36]. We are interested in special forms for which these notions of rank do not coincide. Simple examples where \( \text{cr}(f) < \text{rk}(f) \) are known. On the other hand very few examples are known satisfying \( \text{cr}(f) > \text{rk}(f) \), they are called wild forms (see [10, 25]). According to the best of our knowledge, the first example of wild form appears in [10]. This example inspires [25] to construct a family of wild forms. In [10, 25], the main ingredient to obtain wild forms is the vanishing of the Hessian.

In this work, we show how the failure of the Lefschetz properties of the algebra associated with the form, via Macaulay–Matlis duality, is deeply connected to the wildness. The failure of the Lefschetz properties is equivalent to the degeneracy of a mixed Hessian matrix, which is a broad generalization of the vanishing of the classical Hessian. The degeneracy of this matrix produces an element in the saturation of the Macaulay inverse system. It gives a lower bound for the cactus rank. A natural class of forms having degenerated mixed Hessians are bi-graded forms generalizing classical work of Gordan–Noether on forms with vanishing Hessian. We were able to control the osculating space of the Veronese at these points. Hence we get an upper bound for the border rank. Together with the lower bound on the cactus rank, this upper bound is our strategy to construct infinite series of wild forms. For the classical Gordan–Noether, see [37, Chapter 7] and [18, Appendix A] for an overview, see [22, 35] for the classical theory and see [13, 17, 19, 32] for a modern approach.

The strong Lefschetz property (SLP) is an algebraic abstraction introduced by Stanley in [39] for standard graded Artinian algebras. The so-called hard Lefschetz Theorem on the cohomology of smooth projective complex varieties inspired it (see [30] and [37, Chapter 7]). Graded Artinian algebras satisfying Poincaré duality are of particular interest since they are models for cohomology rings. These algebras can be characterized as Artinian Gorenstein algebras, AG algebras for short (see [33]). The choice of algebras satisfying Poincaré duality is natural in many categories where the Lefschetz properties have been introduced over the years. From the geometric perspective, Lefschetz properties were studied for projective varieties (see [30, 37]),
arithmetic hyperbolic manifolds (see [5]) and Shimura varieties (see [24]). In combinatorics, they were introduced in the context of simplicial complexes by Stanley in [38, 39] and used in [4, 20, 29] just to cite some results. In representation theory, the Lefschetz properties were posed for coinvariant rings of Coxeter groups [34]. Lefschetz properties also appear in poset context, related with the Sperner property (see [23, 39]).

From Macaulay–Matlis duality, we know that every AG algebras over a field of characteristic zero can be presented as a quotient of a polynomial ring by the annihilator of a single form. The main tools to understand the Lefschetz properties for AG algebras are the Higher Hessian matrix, introduced in [33], that controls the strong Lefschetz property and the mixed Hessian matrix, introduced in [21]. Mixed Hessians generalize the previous notion of higher Hessian and control also the weak Lefschetz property. Our first result is a factorization of the mixed Hessian matrix of a form in a power sum decomposition of a form, see Lemma 2.1. We use this decomposition to give a criterion of maximality of its rank (see Proposition 2.4) and WLP (see Corollary 2.5). As a Corollary, we obtain an inequality between the border rank and the Waring rank of certain forms (see Corollary 2.6). In [26, 27], the authors used power sum decomposition to study AG algebras and vice versa.

We study the border rank of a class of bi-graded forms closely related to the classical works of Gordan–Noether and Perazzo on forms with vanishing Hessian, for a detailed account on the subject, see [18]. In Proposition 3.2 we give an upper bound for the border rank of these forms. To get this bound we study the osculating spaces of some points in a rational normal curve (see Proposition 3.2 and Corollary 3.3).

The main results of this work are Theorems 3.7 and 3.14 and their Corollaries, that produce new series of wild forms (see Corollaries 3.12, 3.17 and 3.19). In [10, 25], the authors studied wild forms of minimal border rank. In [25] they proved that a concise form with minimal border rank is wild if and only if it has vanishing Hessian. We construct classes of wild forms whose border rank is not minimal and also classes whose Hessian is non-vanishing. Since we get an upper bound for the border rank of a class of forms with degenerated mixed Hessian, our strategy was to find a lower bound for the cactus rank in the same philosophy of [10, 25]. As it has been noticed before in [10, 25], a natural ingredient to find a lower bound for the cactus rank is to show that it is strictly bigger than the Hilbert function in degree one of the associated AG algebra. Generalizing this idea, we look for an element in the saturation, in degree $k$, of the ideal generated by the graded parts of degree $k$ of the Macaulay dual of $f$.

We introduce $k$-concise forms, that is, those whose associated algebra has maximal Hilbert function up to degree $k$. To get a lower bound to the cactus rank, we impose that the form is $k$-concise and has degenerate mixed Hessian. In this case, we understand the connection between the degeneracy of the mixed Hessian and the existence of an element in the saturation of the annihilator of $f$. We use it to construct series of wild forms whose classical Hessian is not vanishing (see Corollaries 3.17 and 3.19). We also construct series of wild forms using Gordan–Noether polynomials whose border rank is not minimal (see Corollary 3.12).
1 Preliminaries

1.1 Artinian Gorenstein algebras and Lefschetz properties

Let $\mathbb{K}$ be a field of char($\mathbb{K}$) = 0 and let $A = \bigoplus_{i=0}^{d} A_i$ be an graded Artinian $\mathbb{K}$-algebra with $A_d \neq 0$. We say that $A$ is standard graded if $A_0 = \mathbb{K}$ and $A$ is generated in degree 1 as algebra. The Hilbert function of $A$ can be described by the vector $\text{Hilb}(A) = (h_0, h_1, \ldots, h_d)$, where $h_i = \dim A_i$. The Hilbert vector, $\text{Hilb}(A)$ is unimodal if it has no valleys, that is, there exists $k$ for which $h_0 = 1 \leq h_1 \leq \ldots \leq h_k \geq h_{k+1} \geq \ldots \geq h_d$.

Definition 1.1 A standard graded algebra $A$ is Poincaré duality algebra if $h_d = 1$ and the restriction of the multiplication of the algebra in complementary degree, that is, $A_i \times A_{d-i} \to A_d \simeq \mathbb{K}$ is a perfect paring for $i = 0, 1, \ldots, d$ (see [33]). These algebras can be characterized as graded Artinian Gorenstein algebras, AG algebras for short, and their Hilbert functions are symmetric vectors of length $d$, that is $h_k = h_{d-k}$.

Macaulay–Matlis duality produces standard graded Artinian Gorenstein duality algebras. Let us recall a differential version of the construction. Let $f \in R = \mathbb{K}[x_0, x_1, \ldots, x_n]$ be a form of degree $\deg(f) = d \geq 1$ and let $Q = \mathbb{K}[X_0, X_1, \ldots, X_n]$ be the polynomial ring associated to $R$ in such a way that we consider $X_i = \frac{\partial}{\partial x_i}$ and $Q$ acts in $R$ by differentiation. We consider $R$ as a $Q$ module. We define the annihilator ideal for $f$:

$$\text{Ann}(f) = \{ \alpha \in Q | \alpha(f) = 0 \} \subset Q.$$  

The homogeneous ideal $\text{Ann}(f)$ of $Q$ is also called Macaulay dual of $f$. We set

$$A = \frac{Q}{\text{Ann}(f)}.$$  

$A$ is a standard graded Artinian Gorenstein $\mathbb{K}$-algebra with $A_j = 0$ for $j > d$ and $A_d \neq 0$ (see [33, Section 1.2]). We assume, without loss of generality, that $(\text{Ann}(f))_1 = 0$.

The theory of inverse systems gives us the converse. A proof of this result can be found in [33, Theorem 2.1].

Theorem 1.2 (Double annihilator theorem of Macaulay) Let $R = \mathbb{K}[x_0, x_1, \ldots, x_n]$ and $Q = \mathbb{K}[X_0, X_1, \ldots, X_n]$ as above. Let $A = \bigoplus_{i=0}^{d} A_i = Q/I$ be an Artinian standard graded $\mathbb{K}$-algebra. Then $A$ is Gorenstein if and only if there exists $f \in R_d$ such that $A \simeq \frac{Q}{\text{Ann}(f)}$.

Definition 1.3 With the previous notation, let $A = \bigoplus_{i=0}^{d} A_i = Q/I$ be an Artinian Gorenstein $\mathbb{K}$-algebra with $I = \text{Ann}(f)$, $I_1 = 0$ an $A_d \neq 0$. In this case, the form is called concise. The socle degree of $A$ is $d$ which coincides with the degree of the
form \( f \). By abuse of notation, we say that the codimension of \( A \) is the codimension of the ideal \( I \subset Q \), that is, \( \text{codim} A = n + 1 \).

We now recall the so-called Lefschetz properties for a standard graded Artinian Gorenstein \( \mathbb{K} \)-algebra.

**Definition 1.4** Let \( A = \bigoplus_{i=0}^{d} A_i \) be a standard graded Artinian Gorenstein \( \mathbb{K} \)-algebra.

(i) We say that \( A \) has the strong Lefschetz property (SLP) if there is \( L \in A_1 \) such that the \( \mathbb{K} \)-linear multiplication maps \( \bullet L^k : A_i \to A_{i+k} \) are of maximal rank any \( i, k \).

(ii) We say that \( A \) has the strong Lefschetz property in the narrow sense (SLPN) if there is \( L \in A_1 \) such that the \( \mathbb{K} \)-linear multiplication maps \( \bullet L : A_i \to A_{d-i} \) are isomorphisms for \( i = 1, \ldots, \lfloor \frac{d}{2} \rfloor \).

(iii) We say that \( A \) has the weak Lefschetz property (WLP) if there is \( L \in A_1 \) such that the \( \mathbb{K} \)-linear multiplication maps \( \bullet L : A_i \to A_{i+1} \) are of maximal rank for \( i = 0, \ldots, d \).

For standard graded Artinian Gorenstein algebras SLP and SLPN are equivalent (see [33]) It is evident that SLP implies WLP.

To finish the preliminaries we recall a criteria for SLP and WLP. Let \( A = \mathbb{Q} / \text{Ann}(f) \) be a AG \( \mathbb{K} \)-algebra of socle degree \( d \). Let \( L = a_0 x_0 + \ldots + a_n x_n \in A_1 \). The map \( \bullet L^{d-k} : A_k \to A_1 \), for \( k < l \leq \frac{d}{2} \), has maximal rank if and only if the mixed Hessian matrix \( \text{Hess}_{(d-k)}(a_0, \ldots, a_n) \) has maximal rank. In particular, we get the following:

(1) (Strong Lefschetz Hessian criterion, [33, 42]) \( L \) is a strong Lefschetz element of \( A \) if and only if \( \text{hess}_{(k)}(a_0, \ldots, a_n) \neq 0 \) for all \( k = 0, 1, \ldots, [d/2] \).

(2) (Weak Lefschetz Hessian criterion) \( L \in A_1 \) is a weak Lefschetz element of \( A \) if and only if either \( d = 2q + 1 \) is odd and \( \text{hess}_{(q)}(a_0, \ldots, a_n) \neq 0 \) or \( d = 2q \) is even and \( \text{Hess}_{(d-k)}(a_0, \ldots, a_n) \) has maximal rank.
1.2 Waring rank, border rank, and Cactus rank

Let \( f \in R = \mathbb{C}[x_0, \ldots, x_n]_d \) be a form. Any expression of the form \( f = l_1^d + \cdots + l_k^d \), where \( l_1, \ldots, l_k \) are linear forms in \( R \), will be called a power sum decomposition of \( f \).

**Definition 1.7** The Waring rank of \( f \) over \( \mathbb{C} \) is the least number of terms in a power sum decomposition of \( f \), we denote it by \( \text{wrk}(f) \).

In [40, 41], Sylvester determined the Waring rank for generic binary forms. We summarize these results in the following theorem.

**Theorem 1.8** (Sylvester) The Waring rank of a generic polynomial \( f \in \mathbb{C}[x, y]_d \) is \( \left\lceil \frac{d-1}{2} \right\rceil \).

In [1–3], Alexander and Hirschowitz described the Waring rank for a generic form.

**Theorem 1.9** (Alexander–Hirschowitz) A generic form \( f \in \mathbb{C}[x_0, \ldots, x_n]_d \) has Waring rank \( \text{wrk}(f) = \left\lceil \frac{n+d}{n+1} \right\rceil \), except for:

(i) \((n, 2)\), in this case \( \text{wrk}(f) = n + 1 \);
(ii) \((n, d) = (2, 4), (3, 4), (4, 3), (4, 4)\), in this case \( \text{wrk}(f) = \left\lceil \frac{n+d}{n+1} \right\rceil + 1 \).

We consider a more geometric standpoint in the sequel. Given a power sum decomposition \( f = l_1^d + \cdots + l_k^d \), where \( l_1, \ldots, l_k \) are linear forms on \( R \), let \( P_i \in \mathbb{P}^n \) be the dual points whose coordinates are the coefficients of the linear form \( l_i \). By using Macaulay–Matlis duality, we will identify the ideal of points of \( \Gamma = \{ P_1, \ldots, P_s \} \), \( I_\Gamma \) with an ideal in \( Q \). Under this identification, we have the following useful lemma whose proof can be found in [27, Lemma 1.31].

**Lemma 1.10** Apolarity Lemma A form \( f \in R_d \) can be decomposed as

\[ f = l_1^d + \cdots + l_k^d \]

with \( l_i \) pairwise linearly independent linear forms if and only if \( I_\Gamma \subset \text{Ann}(f) \).

**Definition 1.11** Let \( X \subset \mathbb{P}^n \) be a projective variety. The \( s \)th secant variety of \( X \) is

\[ S^s(X) = \bigcup_{p_1, \ldots, p_s \in X} \langle p_1, \ldots, p_s \rangle \subset \mathbb{P}^n. \]

Consider the ring \( R = \mathbb{C}[x_0, \ldots, x_n] \) and let \( R_d \) be its graded part of degree \( d \).

**Definition 1.12** The map \( V_d : \mathbb{P}(R_1) \to \mathbb{P}(R_d) \) is the morphism given by \( V_d([l]) = [l^d] \). Its image is called the Veronese variety \( V_d(\mathbb{P}^n) \).

**Definition 1.13** Let \( f \in R_d \) and \( p = [f] \in \mathbb{P}(R_d) \) be the corresponding point. The border rank of \( f \) is the minimal integer \( s = \text{rk}(f) \) such that \( p \in S^s(V_d(\mathbb{P}(R_d))) \).
Notice that \( \text{rk}(f) = s \) means that \([f] \) is a limit of forms with Waring rank \( s \).

We recall the following helpful result about the border rank of monomials.

**Theorem 1.14** [31, Theorem 11.2] If \( e_0 \geq e_1 \geq \ldots \geq e_n \), then

\[
\text{rk}(x_0^{e_0}x_1^{e_1} \ldots x_n^{e_n}) \leq (e_1 + 1) \ldots (e_n + 1).
\]

**Definition 1.15** Let \( f \in R_d \) and \( p = [f] \in \mathbb{P}(R_d) \) be the corresponding point. The **cactus rank** of \( f \) is the minimal integer \( s = \text{cr}(f) \) such that there is a length \( s \) 0-dimensional scheme \( K \subset V_d(\mathbb{P}(R_d)) \) such that \( p \in \langle K \rangle \), where \( \langle K \rangle \) is the linear span of \( K \).

**Definition 1.16** Let \( f \in R_d \) and \( p = [f] \in \mathbb{P}(R_d) \) be the corresponding point. The **smoothable rank** of \( f \) is the minimal integer \( s = \text{sr}(f) \) such that there is a a length \( s \) smoothable 0-dimensional scheme \( K \subset V_d(\mathbb{P}(R_d)) \) such that \( p \in \langle K \rangle \), where \( \langle K \rangle \) is the linear span of \( K \).

**Remark 1.17** It is clear, from the definitions that \( \text{wrk}(f) \geq \text{rk}(f) \) and that \( \text{cr}(f) \leq \text{sr}(f) \). See [6, 26, 27] for a detailed discussion about the relations among various notions of the rank of a form. We know that:

\[
\text{rk}(f) \leq \text{sr}(f) \leq \text{wrk}(f).
\]

Moreover, \( \text{cr}(f) \) and \( \text{rk}(f) \) are incomparable. For instance, in [7] there are examples of forms for which \( \text{cr}(f) < \text{rk}(f) \). On the other hand, in [10, 25] and in the present work we give examples of forms for which \( \text{rk}(f) < \text{cr}(f) \). These forms are called wild forms and they are a central part of our work.

**Example 1.18** In [10] the authors showed that the form \( f = xu^2 + y(u + v)^2 + vz^2 \in \mathbb{C}[x, y, z; u, v] \) has

\[
\text{wrk}(f) = 9, \; \text{rk}(f) = 5 \; \text{and} \; \text{sr}(f) = \text{cr}(f) = 6.
\]

Moreover, in [25], the authors showed that inequality \( \text{rk}(f) < \text{cr}(f) \) is a consequence of two properties of \( f \):

(i) \( f \) has minimal border rank, that is, \( \text{rk} = h_1 = 5 \);

(ii) \( \text{hess}_f = 0 \).

Concise cubic forms with vanishing Hessian were studied by Perazzo in [35] and revisited in [19]. In \( \mathbb{C}[x, y, z; u, v] \) there is only one concise cubic form with vanishing Hessian up to projective transformations.

## 2 Hessian matrices of a form in a power sum decomposition

Let \( R = \mathbb{C}[x_0, \ldots, x_n] \) be a polynomial ring and \( Q = \mathbb{C}[X_0, \ldots, X_n] \) be the associated ring. Consider \( Q \) acting in \( R \) by differentiation. Let \( f \in R_d \) be a form and let
\( A(f) = Q / \text{Ann}(f) \) be the associated AG algebra. Consider a power sum decomposition for \( f \).

\[
f = l_1^d + l_2^d + \ldots + l_s^d.
\]

We are considering \( s \geq \text{wrk}(f) \). In this way, we are considering decompositions that are not necessarily a Waring decomposition.

Let \( \{\alpha_1, \ldots, \alpha_{h_k}\} \) be a standard monomial basis of the \( \mathbb{C} \)-vector space \( A_k \), and \( \{\beta_1, \ldots, \beta_{h_l}\} \) be a standard monomial basis of the \( \mathbb{C} \)-vector space \( A_{d-l} \) for some \( k < l \leq d - k \) and \( k \in \{1, \ldots, \lfloor \frac{d}{2} \rfloor\} \). Assume that \( h_k \leq h_l \).

For any linear form \( l_r = \sum_{t=1}^n a_{tr} x_t \) and for any \( \alpha_j = \prod_{t=1}^n X_t^{e_{tj}} \) we have:

\[
\alpha_j (l_r^d) = \frac{d!}{(d-k)!} l_r^{d-k} \prod_{t=1}^n a_{tr}^{e_{tj}}.
\] (1)

We define \( w^{(k)}_{jr} = \prod_{t=1}^n a_{tr}^{e_{tj}} \in \mathbb{C} \) for \( j = 1, \ldots, h_k \). For any \( \beta_i = \prod_{k=1}^n X_t^{f_{ti}} \) set \( w^{(d-l)}_{ir} = \prod_{k=1}^n a_{tr}^{f_{ti}} \) with \( i = 1, \ldots, h_l \). Using Eq. 1, we get:

\[
\beta_i \alpha_j (l_r^d) = \frac{d!}{(l-k)!} l_r^{d-l} w^{(d-l)}_{ir} w^{(k)}_{jr}.
\] (2)

Consider the matrices \( W_k = [w^{(k)}_{jr}]_{h_k \times s} \), \( W_{d-l} = [w^{(d-l)}_{ir}]_{h_l \times s} \) and \( D_{k,l} = \text{Diag}(l_1^{l-k}, l_2^{l-k}, \ldots, l_s^{l-k}) \). Sometimes we omit the index \( (k) \) if it is clear in the context, especially when \( l = d - k \).

**Lemma 2.1** With the previous notations, we get:

(1)

\[
\text{Hess}_{f}^{(d-l,k)} = \frac{d!}{(l-k)!} [W_{d-l}]_{h_l \times s} [D_{k,l}]_{s \times s} [W_k]_{s \times h_k}.
\]

(2)

\[
\text{Hess}_{f}^{k} = \frac{d!}{(d-2k)!} [W_k]_{h_k \times s} [D_{k,d-k}]_{s \times s} [W_k]_{s \times h_k}.
\]
**Proof** By definition \( \text{Hess}_{f}^{(d-l,k)} = (\beta_{l} \alpha_{j}(f))_{h_{l} \times h_{k}} \). Hence,

\[
\text{Hess}_{f}^{(d-l,k)} = (\beta_{l} \alpha_{j}(f))_{h_{l} \times h_{k}} = \frac{d!}{(l-k)!} \sum_{r=1}^{l-k} w_{ir}^{(d-l)} w_{ir}^{(k)}_{h_{l} \times h_{k}}
\]

\[
= \frac{d!}{(l-k)!} \left[ \sum_{r=1}^{l-k} l_{r}^{(l-k)} w_{ir}^{(d-l)} w_{ir}^{(k)} \right]_{h_{l} \times h_{k}}
\]

\[
= \frac{d!}{(l-k)!} \left[ \begin{array}{cccc}
w_{i1}^{(d-l)} & \cdots & w_{i1}^{(d-l)} \\
\vdots & \ddots & \vdots \\
w_{is}^{(d-l)} & \cdots & w_{is}^{(d-l)}
\end{array} \right]_{h_{l} \times s} \left[ \begin{array}{c}
l_{1}^{(l-k)} \cdots 0 \\
\vdots \\
l_{s}^{(l-k)}
\end{array} \right]_{s \times h_{k}}
\]

\[
\left[ \begin{array}{c}
w_{11}^{(k)} \cdots w_{11}^{(k)} \\
\vdots \\
w_{h_{k}1}^{(k)} \cdots w_{h_{k}1}^{(k)}
\end{array} \right]_{h_{l} \times s} \left[ \begin{array}{c}
w_{1s}^{(k)} \cdots w_{1s}^{(k)} \\
\vdots \\
w_{h_{k}s}^{(k)} \cdots w_{h_{k}s}^{(k)}
\end{array} \right]_{s \times h_{k}}
\]

\end{equation}

\[\square\]

**Remark 2.2** Sylvester proved in [40] that \( \text{wrk}(f) \geq \text{rk}(\text{Hess}_{f}^{k}) \) for \( k = \lfloor \frac{d}{2} \rfloor \) (see also [16, Corollary 3.5]). If \( A = A(f) \) has the SLP, it implies that \( s \geq h_{k} \) for all \( k \).

We take the natural exact sequence

\[0 \to I_{k} \to Q_{k} \to A_{k} \to 0.\]

We consider the ring \( Q \) as a polynomial ring. In this context we identify \( \mathbb{P}^{n} = \mathbb{P}(Q_{1}), \mathbb{P}^{(k \cdot n) - 1} = \mathbb{P}(Q_{k}) \) and \( \mathbb{P}^{h_{k} - 1} = \mathbb{P}(A_{k}) \). Consider the Veronese map \( \mathcal{V}_{k} : \mathbb{P}^{n} \to \mathbb{P}^{(\frac{n+k}{k} - 1)} \) given by \( \mathcal{V}_{k}(L) = L^{k} \) and let \( \pi : \mathbb{P}^{(\frac{n+k}{k} - 1)} \to \mathbb{P}^{h_{k} - 1} \) be the projection with center \( I_{k} \). We get the following diagram:

\[
\mathbb{P}^{n} \xrightarrow{\mathcal{V}} \mathbb{P}^{(k \cdot n) - 1} \xrightarrow{\pi} \mathbb{P}^{h_{k} - 1}
\]

We call the map \( \mathcal{V}'_{k} : \mathbb{P}^{n} \to \mathbb{P}^{h_{k} - 1} \) the relative Veronese (see [15]).

**Proposition 2.3** Consider \( f = l_{1}^{d} + \cdots + l_{s}^{d} \in R = \mathbb{K}[x_{0}, \ldots, x_{n}] \) with \( d > 2k, k > 1 \) and \( h_{k} = \dim A_{k} \). Let \( P_{i} = l_{i}^{1} \) be the point that is dual of the hyperplane defined by \( l_{i} \). Consider the relative Veronese map \( \mathcal{V}'_{k} : \mathbb{P}^{n} \to \mathbb{P}^{h_{k} - 1} \). Then,

\[W_{k} = \{ [\mathcal{V}'_{k}(P_{1})] : \cdots : [\mathcal{V}'_{k}(P_{s})] \}_{h_{k} \times s}.
\]

Moreover, \( W_{k} \) has maximal rank.
Note that $A_k$ has a monomial basis. Let $\alpha = X_{c_0}^{1}X_{1}^{c_1} \ldots X_{N}^{c_N} \in A_k$, $c_0 + \cdots + c_n = k$, and we denote $l_r = (a_{1r}x_1 + \cdots + a_{nr}x_n)$, hence:

$$\alpha_{i_1 \ldots i_k}(l_r^t) = \frac{d!}{(d-k)!} l_r^{d-k} a_{i_1}^{c_{i_1}} \ldots a_{i_k}^{c_{i_k}}.$$ 

Hence all the entries of the $r$th column of $W$ are of the form $w_{i_1 \ldots i_r} = a_{i_1}^{c_{i_1}} \ldots a_{i_k}^{c_{i_k}}$.

The maximality of the rank follows from the apolarity Lemma 1.10. In fact, if the rank of $W_k$ drops, then, the linear span of the image of $\Gamma_1$ by the relative Veronese, $\Gamma_1' \subset P^{dh_k-1}$ must be contained in a hyperplane $H$, that is, $<\Gamma'> \subset H \subset P^{dh_k-1}$. It means that there is degree $k$ form $\alpha \in A_k$ in the ideal of the points $P_i$, but $I_{\Gamma} \subset \text{Ann}(f) = I$. The result follows.

**Proposition 2.4** Consider the decomposition of the Hessian matrix:

$$\text{Hess}_f^{(d-l,k)} = \frac{d!}{(l-k)!} [W_{d-l}]_{h_l \times s}[D_{k,l}]_{s \times s}[W_k]^t_{s \times h_k}.$$ 

Assuming that $s \geq h_l \geq h_k$ we get:

(1) If $s = h_k = h_l$, then $\det(\text{Hess}_f^{(d-l,k)}) \neq 0$.

(2) If $s > h_l$, then

$$\dim (\text{Im}(D_{k,l}W_k^t) \cap \text{Ker}(W_{d-l})) = \dim (D_{k,l}W_k^t \left( \text{Ker}(\text{Hess}_f^{(d-l,k)}) \right)) = \dim \left( \text{Ker}(\text{Hess}_f^{(d-l,k)}) \right).$$

Moreover, the following conditions are equivalent:

(a) $\text{rank}(\text{Hess}_f^{(d-l,k)})$ is maximal;

(b) $\dim (\text{Im}(D_{k,l}W_k^t) \cap \text{Ker}(W_{d-l})) = h_l - h_k$;

(c) $\dim (\text{Im}(W_k^t) \cap \text{Ker}(W_{d-l}D_{k,l})) = h_l - h_k$.

**Proof** Consider the decomposition of $\text{Hess}_f^{(d-l,k)} = \frac{d!}{(l-k)!} [W_{d-l}]_{h_l \times s}[D_{k,l}]_{s \times s}$ $[W_k]^t_{s \times h_k}$ described by a diagram of $\mathbb{L}$-vector spaces, with $\mathbb{L} = \mathbb{C}(x)$. Recall that $D_{k,l}$ is an $\mathbb{L}$-linear isomorphism.

$$\text{Hess}_f^{(d-l,k)}: \mathbb{L}^{h_k} \rightarrow \mathbb{L}^{h_l}$$

$$W_k^t \downarrow \mathbb{L}^s \quad \uparrow W_{d-l} \quad \downarrow D_{k,l} \mathbb{L}^s$$

(1) If $s = h_k = h_l$, then $\text{Hess}_f^{(d-l,k)}$ is a square matrix. Since $\det(D) \neq 0$, the result follows immediately from the decomposition formula.
(2) It is easy to check that \( \text{Im}(D_{k,l}W^t_k) \cap \text{Ker}(W_{d-l}) = D_{k,l}W^t_k \left( \text{Ker} \left( \text{Hess}_{f}^{(d-l,k)} \right) \right) \).

Since \( W^t_k \) and \( D_{k,l} \) are injective, we have

\[
\dim \left( \text{Im}(D_{k,l}W^t_k) \cap \text{Ker}(W_{d-l}) \right) = \dim \left( D_{k,l}W^t_k \left( \text{Ker} \left( \text{Hess}_{f}^{(d-l,k)} \right) \right) \right) = \dim \left( \text{Ker} \left( \text{Hess}_{f}^{(d-l,k)} \right) \right)
\]

\( \text{Hess}_{f}^{(d-l,k)} \) has maximal rank if and only if \( \dim(\text{Ker}(\text{Hess}_{d-l,k} f)) = h_l - h_k \). Therefore, the equivalence \((a) \iff (b) \iff (c)\) follows.

**Corollary 2.5** Let \( f \in \mathbb{R}^d \) be a form and let \( A \) be the associated AG algebra. Suppose that \( f \) has a power sum decomposition with \( s = h_k \) for some \( k \leq d/2 \). Then

\[ \text{hess}_{f}^{k} \neq 0. \]

In particular, if \( d = 2q + 1 \) and \( k = q \), then \( A \) has the WLP.

**Proof** If follows from Proposition 2.4 and from the Hessian criteria Theorem 1.6. \( \square \)

**Corollary 2.6** Let \( f \in \mathbb{R}^d \) be a concise homogeneous form and \( A = A(f) \) be the associated algebra. If the border rank of \( f \) satisfies \( \text{rk}(f) = \dim A_k \) and \( \text{hess}_{f}^{k} = 0 \), then

\[ \text{wrk}(f) > \text{rk}(f). \]

In particular, all concise forms of minimal border rank and vanishing Hessian have rank greater than its border rank.

**Proof** We get that \( \text{wrk}(f) \geq \text{rk}(f) \). If equality holds true, we define \( r = \text{wrk}(f) \). We get a limit \( \lim_{t \to 0} l_i(t) = l_i \in A_1 \) satisfying

\[ f = \lim_{t \to 0} \sum_{i=1}^{r} l_i(t)^d = \sum_{i=1}^{r} \lim_{t \to 0} l_i(t)^d = \sum_{i=1}^{r} l_i^d \]

On the other hand, if \( \text{wrk}(f) = r \), then, by Corollary 2.5, \( \text{hess}_{f}^{k} \neq 0. \) \( \square \)

The following result is a straightforward generalization of [18, Proposition 2.1]. It will be helpful in the following sections.

**Proposition 2.7** Let \( R = \mathbb{K}[x_0, \ldots, x_n; u_1, \ldots, u_m] \) be a bi-graded polynomial ring in \( m + n + 1 \) variables and \( Q = \mathbb{K}[X_0, \ldots, X_n; U_1, \ldots, U_m] \) be the associated ring, acting on \( R \) by differentiation. Let \( f \in \mathbb{R}^d \) be a form of degree \( d \), \( k < d/2 \) an integer and \( A = A(f) = Q/\text{Ann}(f) \). Consider \( k < l < d - k \) and \( h_k \leq h_l \). Suppose that there exists \( s \) monomials \( \beta_1, \beta_2, \ldots, \beta_s \in Q_k \setminus \mathbb{K}[U_1, \ldots, U_m]_k \) linearly independent in \( A_k \) such that \( \beta_j(f) \in \mathbb{K}[u_1, \ldots, u_m] \). If \( s > (m+l-k)_k \), then the Hessian matrix \( \text{Hess}_{f}^{(d-l,k)} \) is degenerated.
**Proof** The proof is the same of [18, Proposition 2.1]. Indeed, it is enough to find a null submatrix of $\text{Hess}_{f}^{(d-l, k)}$ with $s$ columns. Let $\alpha_i \in A_{d-k}$ be the classes of differential operators $\alpha_i \in Q_k \setminus \mathbb{K}[U_1, \ldots, U_m]_{d-k}$, then $\alpha_i(\beta_j)(f) = 0$ and the result follows.

**Corollary 2.8** Let $f \in C \left[ x_1, \ldots, x_n; u, v \right]_{(k, d-k)}$ be a bi-homogeneous form of bi-degree $(k, d-k)$ with $1 \leq k \leq d-k$. If $f = \sum_{i=1}^{s} g(x)h(u, v)$ with $s > \left( m+d-k-1 \right)$, then the Hessian matrix $\text{Hess}_{f}^{(d-l, k)}$ is degenerated.

**Proof** The result follows from Proposition 2.7. We obtain $\beta_i \in Q_k[X]$ such that $\beta(g) = 1$.

### 3 Wild forms

We define wild forms following [10, 25].

**Definition 3.1** We say that a form $f \in R_d$ is wild if

$$rk(f) < sr(f).$$

We know that $sr(f) \geq cr(f)$. Since the smoothable rank is harder to estimate, we produce wild forms showing that $rk(f) < cr(f)$. Our strategy is to find an upper bound $a$ to $rk(f)$ which is also a lower bound to $cr(f)$, that is:

$$rk(f) \leq a < cr(f).$$

The following result is a generalization of [10, Proposition 2.6].

**Proposition 3.2** Let $X \subset \mathbb{P}^N$ be a projective variety of dimension $\dim X = n$ and let $p_1, \ldots, p_r \in X$ be smooth points. Suppose that $\dim < p_1, \ldots, p_r > \leq r - 1$. Then

$$< T_{p_1}^k X, \ldots, T_{p_r}^k X > \subset S^r T^{k-1} X \subset S^{kr} X.$$

**Proof** Since $T^{k-1} X \subset S^k X$, we have $S^r (T^{k-1} X) \subset S^r (S^k X) \subset S^{kr} (X)$. Take points $q_1, \ldots, q_r \in \mathbb{C}^{N+1}$ such that $[q_i] = p_i$, for $i = 1, \ldots, r$. Since $\dim < p_1, \ldots, p_r > \leq r - 1$, we can suppose that $q_1 + \ldots + q_r = 0$. Let $v$ be an arbitrary point of $< T_{p_1}^k X, \ldots, T_{p_r}^k X >$. We can write $v = v_1 + \ldots + v_r$ with $v_i \in T_{p_i}^k X$. Let $\alpha_i(t) \subset T^{k-1} X$ be a curve such that $\alpha_i(0) = p_i$ and $\alpha_i'(0) = v_i$. It is possible since the vectors of $T_{p_i}^k X$ belongs to the tangent cone of $T^{k-1} X$ in $p_i$. Define the curve $\alpha(t) = \frac{1}{t} \sum_{i=1}^{r} \alpha_i(t)$. Note that $[\alpha(t)] \in S^r (T^{k-1}(X))$. Therefore:
\[ S^r(T^k(X)) \ni [\alpha(0)] = \left[ \lim_{t \to 0} \frac{1}{t} \sum_{i=1}^{r} (\alpha_i(t) - q_i) \right] = \left[ \sum_{i=1}^{r} \lim_{t \to 0} \frac{\alpha_i(t) - \alpha_i(0)}{t} \right] = \left[ \sum_{i=1}^{r} v_i \right] = [v]. \]

The next result is a generalization of [25, Lemma 6.1].

**Corollary 3.3** Let \( f \in \mathbb{C}[x_1, \ldots, x_n; u, v]_{(k,d-k)} \) be a bi-homogeneous form of bi-degree \((k, d-k)\) with \(1 \leq k \leq d-k\). The border rank of \( f \) satisfies:

\[ \text{rk}(f) \leq k(d + 2). \]

**Proof** Since \( \dim \mathbb{C}[u, v]_d = d + 1 \), let \( l_0^d, \ldots, l_d^d \in \mathbb{C}[u, v]_d \) be a basis. It is easy to see that \( f = \sum_{i=0}^{d} f_i(x)l_i^{d-k} \). Let \( l_{d+1} \in \mathbb{C}[u, v] \) be an arbitrary linear form. The points \( p_0 = [l_0^d], \ldots, p_d = [l_d^d], p_{d+1} = [l_{d+1}^d] \in V_d(\mathbb{P}^1) = X \) are linearly dependent, that is, \( \dim < p_0, \ldots, p_{d+1} > \leq d + 1 \). Therefore, by Proposition 3.2,

\[ < T^k_{p_0}X, \ldots, T^k_{p_{d+1}}X > \subseteq S^{k(d+2)}X. \]

As long as \([f] \in < T^k_{p_0}X, \ldots, T^k_{p_{d+1}}X > \subseteq S^{k(d+2)}X\), the result follows. \( \Box \)

### 3.1 \( k \)-concise wild forms with vanishing Hessian

**Definition 3.4** Consider \( f \in R_d \) and \( A = Q/I \) with \( I = \text{Ann}(f) \). We say that \( f \in R_d \)

\( k \)-concise, with \( d \geq 2k + 1 \), if \( I_j = 0 \) for \( j = 1, 2, \ldots, k \). It is equivalent to say that the Hilbert function of \( A \) is maximal up to degree \( k \), that is, \( h_j = \binom{n+j}{j} \) for \( j = 0, \ldots, k \). As usual, \( 1 \)-concise forms are called concise.

The following lemma is a generalization for higher Hessians of an idea contained in the proof of [25][Theorem 3.5] for the case of classical Hessians.

**Lemma 3.5** Let \( f \in R_d \) be a concise form and \( A = A(f) = Q/I \) be the associated AG algebra. Suppose that \( h_k \leq h_{d-s} \) and \( k + s \leq d \). If \( \text{Hess}^{(k,s)}_f \) is degenerated, then exists \( \alpha \in I^s_{\text{sat}} \setminus I_k \).

**Proof** We are considering \( \text{Hess}^{(k,s)}_f \) as a matrix in \( R \). By the Hessian criteria 1.6, for each \( L \in A_1 \), the map \( \bullet L^{d-s-k} : A_k \to A_{d-s} \) is represented by \( \text{Hess}^{(k,s)}_f(L^\perp) \).

Therefore, there is a universal polynomial in the kernel of \( \text{Hess}^{(k,s)}_f \) such that its image \( \alpha \in A_k \) belongs the kernel of \( \bullet L^{d-s-k} \) for every \( L \in A_1 \), that is \( L^{d-s-k} \alpha \in I_{d-s} \). In particular, \( X_i^{d-k-s} \alpha \in I_{d-s} \) for \( i = 0, \ldots, n \), that is, \( \alpha \in I^s_{\text{sat}} \setminus I_k \). \( \Box \)
Lemma 3.6  Let $f \in R_d$ be a $k$–concise form with $2k \leq d$ and let $I = \text{Ann}(f) \subset Q$. Let $J = (I_{d-k}) \subset Q$ be the ideal generated by the degree $d-k$ part of $I$. If $J_{l}^\text{sat} \neq 0$ for some $l \leq k$, then

$$\text{cr}(f) > h_k = \binom{n+k}{k}.$$ 

Proof  Let $A = Q/I$ and $h_i = \dim A_i$. Since $A$ is Gorenstein, we get $h_k = h_{d-k}$, by Poincaré duality. Let $B = Q/J$ and $h'_i = \dim B_i$. We know that $h_k = h'_k = \binom{n+k}{k}$ and $h'_{d-k} = h_{d-k}$, therefore:

$$h'_k = h_k = h_{d-k} = h'_{d-k}.$$ 

Let $K \subset I = \text{Ann}(f)$ be a saturated ideal satisfying the definition of cactus rank for $f$. Precisely, the zero dimensional scheme $X$ defined by $K$ has length $\text{cr}(f)$ and $f \in <X>$. We know that the Hilbert function of $Q/K$ is non decreasing and stabilizes in the constant polynomial $\ell(K) = \text{cr}(f) \in \mathbb{N}$. Suppose that $\text{cr}(f) \leq h_k$. Thus,

$$\dim(Q/K)_{d-k} \leq \text{cr}(f) \leq h_k = h'_{d-k} = \dim(Q/J)_{d-k}.$$ 

On the other hand, $K_{d-k} \subset I_{d-k}$, hence

$$\dim(Q/K)_{d-k} \geq \dim(Q/J)_{d-k}.$$ 

Therefore we get

$$\dim(Q/K)_{d-k} = \dim(Q/J)_{d-k}.$$ 

Which gives us $K_{d-k} = J_{d-k}$, that is $J \subset K$, since $J$ is generated in degree $d-k$. Then, we get $J_{l}^\text{sat} \subset K_{l}^\text{sat} = K$, since $K$ is saturated. Since $f$ is $k$–concise and $K \subset I$, we have

$$J_{l}^\text{sat} = K_l = I_l = 0,$$

for all $l \leq k$. By hypothesis, for some $l \leq k$ we have $J_{l}^\text{sat} \neq 0$, this implies $K_l \neq 0$. It is a contradiction, therefore, $\text{cr}(f) > h_k$. $\square$

Theorem 3.7  Let $f \in R_d$ be a $k$–concise homogeneous form, with $2k \leq d$. If $\text{hess}_f = 0$, then

$$\text{cr}(f) > \binom{n+k}{k}.$$ 

In particular, if $\text{rk}(f) \leq \binom{n+k}{k}$, then $f$ is wild.
Proof Since \( \text{hess}_f = 0 \), by Lemma 3.5, we know that \( I^{sat} \) contains a linear form \( \alpha \). With the same notation of Lemma 3.6, \( \alpha^l \in J^{sat}_l \). The result follows from Lemma 3.6.

\( \square \)

The following corollary is one of the main results of [25] (see [25, Theorem 3.5]).

**Corollary 3.8** Let \( f \in R_d \) be a concise form with minimal border rank. If \( \text{hess}_f = 0 \), then \( f \) is wild.

**Proof** Minimal border rank means \( \text{rk}(f) = h_1 \). Since \( f \) is 1-concise and \( \text{hess}_f = 0 \), by Theorem 3.7, we get \( \text{cr}(f) > h_1 \).

\( \square \)

In the sequel, we construct examples of wild forms with vanishing Hessian whose border rank is not minimal (see also [25] for other examples).

**Example 3.9** Consider the forms \( f_d \in \mathbb{C}[x, y, z; u, v]_{d-1} \) with \( d \geq 3 \), given by

\[
f_d = (xu^d + yu^{d-1}v + zv^d)^{d-1}.
\]

Using Macaulay2 we know that \( f_d \) is \((d-1)\)-concise for \( d \leq 17 \). In this case, by Theorem 3.7, \( \text{cr}(f) > \left(\frac{d+3}{4}\right) \). The \( f_d = g^{d-1} \) is a \((d-1)\)th power of a form \( g = xu^d + yu^{d-1}v + zv^d \) that we know it has vanishing Hessian. Indeed, by Gordan–Noether criteria, since the partial derivatives of \( g \) are algebraically dependent, we have \( \text{hess}_d = 0 \). For more details about Gordan–Noether theory see [22] and [13, §2.3]. Moreover, the choice of \( g \) was in such a way that its polar image has degree \( d \). In fact

\[
g_x^{d-1}g_z = g_y^d.
\]

If the polar degree was lower than \( d \), then \( f_d \) could be not \((d-1)\)-concise. On the other hand, by Proposition 3.2, we get \( \text{rk}(f) \leq (d-1)(d^2 + 1) \). It is easy to check that \( f_{17} \) is a wild form with non minimal border rank. Indeed, \( \text{cr}(f_{17}) > h_{16} = 4845 \) and \( \text{rk}(f) \leq 4640 \).

The following example is related to the original approach of Gordan–Noether (see [22] and [13, §2.3]).

**Definition 3.10** Let \( R = \mathbb{C}[x_0, \ldots, x_l; u, v] \) with natural bi-grading. For \( l = 1, \ldots, t - m \), let \( q_l = x_0M_{l0} + \ldots + x_lM_{lt} \in R_{(1, e-1)} \) be a generic forms with vanishing Hessian (see [13, §2.3]). Let \( d = \mu e \) and let \( P_\mu(z_1, \ldots, z_s) \) be a generic form of degree \( \mu \). A generic GN hypersurface of type \((t + 2, t, m, e)\) and degree \( d \) is defined by:

\[
f = P_\mu(q_1, \ldots, q_{t-m}).
\]

**Example 3.11** Consider a generic GN polynomial of type \((t + 2, t, t - 2, e)\), and degree \( d = 4e \), it means that there are two forms with vanishing Hessian, \( q_1, q_2 \in \mathbb{C}[x_0, x_1, \ldots, x_l; u, v]_{(1, e)} \) and a generic quartic polynomial \( P(z_1, z_2) \) such that \( f = \ldots \)
$P(q_1, q_2)$. By the genericity of $q_1, q_2$ and $P$, $f$ is 2-concise. By [13, Proposition 2.9], $\text{hess}_f = 0$. For $s = 28$ and $e = 30$, we get:

$$\text{cr}(f) = 496 > 488 = \text{rk}(f).$$

Let $P(z_1, z_2)$ be a generic quartic polynomial, let $q_i \in \mathbb{C}[x_0, x_1, \ldots, x_s; u, v]_{(1, e)}$ be a generic forms with vanishing Hessian with $e = 2\lceil \frac{s}{2} \rceil$ and let $f = P(q_1, q_2)$ be a generic GN polynomial of type $(t + 2, t, t - 2, e)$ and degree $d = 4e$.

**Corollary 3.12** With the previous notation, let $f = P(q_1, q_2)$ be a generic GN polynomial of type $(t + 2, t, t - 2, e)$ and degree $d = 4e$. If $s \geq 28$, then $f$ is wild.

**Proof** The genericity of $q_1, q_2$ and $P$ implies that $f$ is 2-concise. In fact, by Sylvester Theorem 1.8, $P = l_1^4 + l_2^4$ and we write $q_1 = x_0 M_0 + \ldots + x_t M_t$ and $q_2 = x_0 N_0 + \ldots + x_t N_t$, to simplify the notation. We get

$$X_i X_j(f) = 12(M_i M_j q_1^2 + N_i N_j q_2^2).$$

Suppose that $\sum c_{ij} X_i X_j(f) = 0$, then, using the bi-grading, we get $\sum c_{ij} M_i M_j = 0$ and $\sum c_{ij} N_i N_j = 0$, which implies $c_{ij} = 0$. By [13, Proposition 2.9], $\text{hess}_f = 0$. From Proposition 3.2,

$$\text{rk}(f) \leq 4(4(e + 1) + 2) = 16e + 40.$$  

By Theorem 3.7,

$$\text{cr}(f) > \binom{s + 4}{2}.$$  

For $s \geq 28$,

$$\text{cr}(f) > \text{rk}(f).$$

\[\square\]

### 3.2 $k$-concise wild forms with degenerated mixed Hessian

In this section, we construct wild forms with non-vanishing Hessian. We produce examples of wild forms whose first Hessian is non-vanishing.

**Lemma 3.13** Let $f \in R_d$ be a $k$-concise form with $2k \leq d$. Let $I = \text{Ann}(f) \subset Q$ and $A = Q/I$. Suppose that $\text{Hilb}(A)$ is unimodal. Let $J = (I_{\leq d-k}) \subset Q$ be the ideal generated by the graded parts of degree $\leq d - k$ of $I$. If $J_{\geq l}^{\text{sat}} \neq 0$ for some $l \leq k$, then

$$\text{cr}(f) > h_k = \binom{n + k}{k}.$$
Proof Since $A$ is Gorenstein, $h_k = h_{d-k} = \binom{n+k}{k}$. Let $B = Q/J$ and $h'_i = \dim B_i$, we get that $h'_k = \binom{n+k}{k}$, since $I_k = J_k = 0$ by the $k$-conciseness of $f$. For $s \in \{k+1, \ldots, d-k\}$ we get $h'_s = h_s$. Notice that $h_k \leq h_s$, since $\text{Hilb}(A)$ is unimodal.

Let $K \subset I = \text{Ann}(f)$ be a saturated ideal satisfying the definition of cactus rank for $f$. In this case the zero dimensional scheme $X$ defined by $K$ has length $\text{cr}(f)$ and $f \in < X >$. We know that the Hilbert function of $Q/K$ is non decreasing and stabilizes in the constant polynomial $\ell(K) = \text{cr}(f) \in \mathbb{N}$. Suppose that $\text{cr}(f) \leq h_k$. For any $s \in \{k+1, \ldots, d-k\}$, we get

$$\dim(Q/K)_s \leq \text{cr}(f) \leq h_k \leq h_s = \dim(Q/J)_s.$$ 

On the other hand, $K_s \subset I_s$, hence

$$\dim(Q/K)_s \geq \dim(Q/J)_s.$$ 

Therefore

$$\dim(Q/K)_s = \dim(Q/J)_s.$$ 

Which gives us $K_s = J_s$, that is $J \subset K$, since $J$ is generated in degree $\{k+1, \ldots, d-k\}$. Then $J^{\text{sat}} \subset K^{\text{sat}} = K$, since $K$ is saturated. Since $f$ is $k-$concise and $K \subset I$, we have

$$J^{\text{sat}}_l = K_l = I_l = 0,$$

for all $l \leq k$. It is a contradiction. Therefore $\text{cr}(f) > h_k$. \hfill $\Box$

Theorem 3.14 Let $f \in R_d$ be a $k$-concise homogeneous form with $2k \leq d$ and let $l, s$ be an integers such that $l \leq k \leq s$ and $s + l \leq d$. Let $I = \text{Ann}(f)$ and $A = Q/I$ and suppose that $\text{Hilb}(A)$ is unimodal. Suppose that $\text{Hess}(f)_{l,s}$ is degenerated, or equivalently, for a generic $L \in A_1$, the map $\bullet L : A_l \to A_{d-s}$ is not injective. Then:

$$\text{cr}(f) > \binom{n+k}{k}.$$

In particular, if $\text{rk}(f) \leq h_k$, then $f$ is wild.

Proof Since $A$ is Gorenstein, $h_k = h_{d-k} = \binom{n+k}{k}$. Let $J = (I_{d-k})$ be the ideal generated by the pieces of $I$ in degree $\leq d-k$. Let $B = Q/J$ and $h'_i = \dim B_i$, we get that $h'_k = \binom{n+k}{k}$ and $h'_{d-k} = h_{d-k}$. By hypothesis we have

$$h_l = h'_l \leq h_k = h'_k \leq h_s = h'_s = h_{d-s} = h'_{d-s}.$$ 

By hypothesis $s \geq k$, therefore, $d-s \leq d-k$, which implies $I_{d-s} = J_{d-s}$. By Lemma 3.5, there is $\gamma \in I^{\text{sat}}_l = J^{\text{sat}}_l$. The result follows from Lemma 3.13. \hfill $\Box$
The first example of a form with a vanishing second Hessian whose Hessian is non-vanishing was given by Ikeda in [28], see also [18, 33] for further discussions. In [25, Example 3.9], this example was treated with an independent approach. They also proved that the associated algebra is wild.

**Example 3.15** Let \( f = xu^3v + yuv^3 + x^2y^3 \in \mathbb{C}[x, y; u, v]_5 \). Let \( A = Q / \text{Ann}(f) \), we get

\[
\text{Hilb}(A) = (1, 4, 10, 10, 4, 1).
\]

Therefore \( f \) is 2-concise. We know that \( \text{hess}_2^2 f = 0 \). By Proposition 3.2, \( \text{rk}(f) \leq 7 \). By Theorem 1.14, \( \text{rk}(x^2y^3) = 3 \), then \( \text{rk}(f) \leq 10 \). By Theorem 3.14 we get that \( \text{cr}(f) > 10 \), therefore \( f \) is wild.

In [18, Theorem 2.3], we generalized the example of Ikeda introducing a series of forms with vanishing Hessian of order \( k \). They are called exceptional polynomials of order \( k \) and degree \( d \).

**Example 3.16** Let \( f = xu^5v + yu^3v^3 + zuv^5 + \sum_{i=1}^{6} l_7 \in \mathbb{C}[x, y, z; u, v] \) with \( l_i \in \mathbb{C}[x, y, z] \) be a generic linear forms. We checked, using Macaulay2, that \( f \) is 2-concise and that the Hilbert vector of the algebra is unimodal. By Theorem [18, Theorem 2.3], \( \text{hess}_2^2 f = 0 \), which can also be checked directly. By Proposition 3.2,

\[
\text{rk}(xu^5v + yu^3v^3 + zuv^5) \leq 9.
\]

Hence, \( \text{rk}(f) \leq 15 \). By Theorem 3.14, \( \text{cr}(f) > 15 \). Therefore, \( f \) is wild.

The following corollary generalizes the idea of the previous example.

**Corollary 3.17** Let \( f \in \mathbb{C}[x_1, \ldots, x_n; u, v]_{d+2} \) be a exceptional form of degree \( d + 2 \) with \( d = 2n - 1 > 3 \) given by:

\[
f = x_1u^d v + x_2u^{d-2}v^3 + \ldots + x_nuv^d + h
\]

with \( h = \sum_{i=1}^{(d+1)} l_i^{d+2} \in \mathbb{C}[x_1, \ldots, x_n] \) where \( l_i \) are generic linear forms. Then \( f \) is wild.

**Proof** For such exceptional form, it is easy to see that if \( h \in \mathbb{C}[x_1, \ldots, x_n]_{d+2} \) is 2-concise, then \( f \) is 2-concise. The Hilbert vector of the associated AG algebra is

\[\text{Hilb}(A) = (1, 4, 10, 10, 4, 1)\]

Therefore \( f \) is 2-concise. We know that \( \text{hess}_2^2 f = 0 \). By Proposition 3.2, \( \text{rk}(f) \leq 7 \). By Theorem 1.14, \( \text{rk}(x^2y^3) = 3 \), then \( \text{rk}(f) \leq 10 \). By Theorem 3.14 we get that \( \text{cr}(f) > 10 \), therefore \( f \) is wild.
unimodal (see [18]). Since \( h = \sum_{i=1}^{\binom{n+1}{2}} l_i^{d+2} \) and \( l_1 \in \mathbb{C}[x_1, \ldots, x_n]_1 \) are generic, then it is 2-concise. By [18, Theorem 2.3], \( \text{hess}_f^2 = 0 \). By Proposition 3.2

\[
\text{rk}(f) \leq (d + 2) + 2 + \text{rk}(h) \leq 2n + 3 + \binom{n+1}{2} = \binom{n+3}{2}.
\]

Since \( h_2 = \binom{n+3}{2} \), then by Theorem 3.14, \( \text{cr}(f) > \binom{n+3}{2} \). The result follows. 

In [18], we generalized for higher Hessians some classical constructions of forms with vanishing Hessians tracing back to Gordan–Noether and Perazzo’s counterexamples to Hesse’s claim.

**Example 3.18** Consider \( M_i \in \mathbb{C}[x, y, z]_4 \) with \( i = 0, \ldots, 14 \), be all the quartic monomials in 3 variables and let

\[
f = \sum_{i=0}^{14} M_i u^{14-i} v^i \in \mathbb{C}[x, y, z; u, v]_{18}.
\]

We checked, using Macaulay2, that \( f \) is 4-concise. By Proposition 2.8, \( \text{hess}_f^4 = 0 \). By Theorem 3.14, \( \text{cr}(f) > \binom{4+4}{4} = 140 \). By Proposition 3.2, \( \text{rk}(f) \leq 4(18 + 2) = 80 \). Thus \( f \) is wild.

Generalizing this example, we have the following.

**Corollary 3.19** Let \( M_i \in \mathbb{C}[x_0, \ldots, x_n]_k \) with \( i = 0, \ldots, b-1 \) be all the monomials of degree \( k \), where \( b = \binom{n+k}{k} \). Let

\[
f = \sum_{i=0}^{b-1} M_i u^{b-i} v^i \in \mathbb{C}[x, y, z; u, v]_{b-1+k}.
\]

If \( \binom{n+k+2}{k} > k[(k+1) + \binom{n+k}{k}] \), then \( f \) is wild.

**Proof** We want to show that \( f \) is \( k \)-concise, that is, \( h_k = \binom{n+k+2}{k} \). Consider the decomposition of \( A_k \) given by the bi-grading of \( f \):

\[
A_k = A_{(k,0)} \oplus \cdots \oplus A_{(i,k-i)} \oplus \cdots \oplus A_{(0,k)}.
\]

By the choice of all the monomials in both variables, we obtain

\[
\dim A_{(i,k-i)} = \dim A_{(0,k-i)} \dim A_{(i,0)} = (k-i+1)\binom{n+i}{i}.
\]

Therefore

\[
\dim A_k = \sum_{i=0}^{k} (k-i+1)\binom{n+i}{i} = \binom{n+k+2}{k}.
\]
By Proposition 2.8, $\text{hess}_f^k = 0$. By Proposition 3.2, $\text{rk}(f) \leq k[k + b - 1 + 2] = k[(k + 1) + \binom{n+k}{k}]$. The result follows from Theorem 3.14. 

**Remark 3.20** The numerical condition $\binom{n+k+2}{k} > k[(k + 1) + \binom{n+k}{k}]$ is satisfied for $n \geq 3$ and $k > (n + 2)^2$.

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