EQUIVARIANT EHRHART THEORY

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ABSTRACT. Motivated by representation theory and geometry, we introduce and develop an equivariant generalization of Ehrhart theory, the study of lattice points in dilations of lattice polytopes. We prove representation-theoretic analogues of numerous classical results, and give applications to the Ehrhart theory of rational polytopes and centrally symmetric polytopes. We also recover a character formula of Procesi, Dolgachev, Lunts and Stembridge for the action of a Weyl group on the cohomology of a toric variety associated to a root system.

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1. INTRODUCTION

Let $G$ be a finite group acting linearly on a lattice $M'$ of rank $n$, and let $P$ be a $d$-dimensional $G$-invariant lattice polytope. Let $M$ be a translation of the intersection of the affine span of $P$ and $M'$ to the origin, and consider the induced representation $\rho : G \to GL(M)$ (see Section 4). If $\chi_{mP}$ denotes the permutation

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character associated to the action of $G$ on the lattice points in the $m^{th}$ dilate of $P$, and $R(G)$ denotes the ring of virtual characters of $G$, then we introduce virtual characters \( \{ \varphi_i \}_{i \in \mathbb{N}} \) determined by the equation

\[
\sum_{m \geq 0} \chi_{mP} t^m = \frac{\varphi[t]}{(1 - t) \det[I - pt]} \quad \text{in} \quad R(G)[[t]],
\]

where $\varphi[t] = \varphi[P,G;t] = \sum_i \varphi_i t^i$. These virtual characters naturally appear when one studies the action of a finite group on the cohomology of an invariant hypersurface in a toric variety [36]. If we restrict to the action of the trivial group, then $L(m) = \chi_{mP}$ restricts to the \textbf{Ehrhart polynomial} of $P$, and $\varphi[t]$ restricts to the \textbf{$h^*$-polynomial} of $P$. Our goal is to establish and exploit an equivariant generalization of Ehrhart theory, the study of Ehrhart polynomials and $h^*$-polynomials of lattice polytopes (see, for example, [1], [18], [25, Chapter 12]).

We first consider the permutation characters \( \{ \chi_{mP} \}_{m \geq 0} \). For any positive integer $m$, let $\chi^*_{mP}$ denote the permutation character corresponding to the action of $G$ on the interior lattice points $\text{Int}(mP) \cap M'$ in $mP$. The theorem below is due to Ehrhart in the case when $G = 1$ [12].

\textbf{Theorem} (Theorem 5.7). \textit{The function $L(m) = \chi_{mP} \in R(G)$ is a quasi-polynomial in $m$ of degree $d$ and period dividing the exponent of $G$. Moreover, $L(0)$ is the trivial character, and $(-1)^d L(-m) = \chi^*_{mP} \cdot \det(\rho)$ for any positive integer $m$.}

As an application, for any positive integer $m$, let $f_{P/G}(m)$ (respectively $\tilde{f}_{P/G}(m)$) denote the number of $G$-orbits of $mP \cap M'$ (respectively $\text{Int}(mP) \cap M'$). Similarly, let $\tilde{f}_{P/G}(m)$ (respectively $\tilde{f}^0_{P/G}(m)$) denote the number of $G$-orbits of $mP \cap M'$ (respectively $\text{Int}(mP) \cap M'$) whose isotropy subgroup is contained in \( \{ g \in G \mid \det(\rho(g)) = 1 \} \). By computing multiplicities of the trivial character and $\det(\rho)$ in $\chi_{mP}$, we deduce the following corollary.

\textbf{Corollary} (Corollary 5.8). \textit{With the notation above, $f_{P/G}(m)$ and $\tilde{f}_{P/G}(m)$ are quasi-polynomials in $m$ of degree $d$, with leading coefficient $\frac{\text{vol}_{mP}}{|\rho|}$ and period dividing the exponent of $G$. Moreover, $f_{P/G}(0) = \tilde{f}_{P/G}(0) = 1$, and $(-1)^d f_{P/G}(-m) = \tilde{f}_{P/G}(m)$ and $(-1)^d \tilde{f}_{P/G}(m) = f^0_{P/G}(m)$ for any positive integer $m$.}

For example, if $G = \text{Sym}_n$ acts on $\mathbb{Z}^n$ by permuting coordinates, and $P$ is the standard simplex with vertices \( \{ e_1, \ldots, e_n \} \), then $f_{P/G}(m)$ equals the number of partitions of $m$ with at most $n$ parts, and $\tilde{f}_{P/G}(m)$ equals the number of partitions of $m$ with $n$ distinct parts. In this case, the reciprocity result above is a classical result on partitions [35, Theorem 4.5.7].
With the notation of the above theorem, we may write
\[ L(m) = L_d(m)m^d + L_{d-1}(m)m^{d-1} + \cdots + L_0(m), \]
where \( L_i(m) \in R(G) \) is a periodic function in \( m \). We prove that the leading coefficient equals \( L_d = L_d(m) = \text{vol}_G \chi_{\text{st}}, \) where \( \chi_{\text{st}} \) is the character associated to the standard representation of \( G \) (Corollary 5.9). The latter fact may also be deduced from the work of Howe in [20] (Remark 5.10). We give a complete description of the second leading coefficient \( L_{d-1}(m) \), and show that its period divides 2 (Corollary 5.11). Moreover, if \( \varphi[t] \) is a polynomial, then we prove that \( L_{d-1} = L_{d-1}(m) \) is independent of \( m \) (Remark 5.12).

We next consider the power series \( \varphi[t] \). We show that \( \varphi[t](g) \) is a rational function in \( t \) that is regular at \( t = 1 \) (Lemma 6.3), and give a complete description of the corresponding rational class function \( \varphi[1] \) (Proposition 6.4). The degree \( s \) of \( P \) is the degree of \( h^*(t) \), and the codegree of \( P \) equals \( l = d + 1 - s \).

**Corollary** (Corollary 6.6). With the notation above,
\[ \sum_{m \geq 1} \chi_m^* t^m = \frac{t^{d+1} \varphi[t^{-1}]}{(1 - t) \det[I - \rho t^*]). \]
In particular, if \( \varphi[t] \) is a polynomial, then \( \varphi[t] \) has degree \( s \) and \( \varphi_s = \chi^*_P \).

We deduce that \( \varphi[t] = t^s \varphi[t^{-1}] \) if and only if \( lP \) is a translate of a reflexive polytope (Corollary 6.9), generalizing a result of Stanley in the case when \( G = 1 \) [31, Theorem 4.4]. We also describe the behavior of \( \{\chi_mP\}_{m \geq 0} \) and \( \varphi[t] \) under the operations of direct product and direct sum, and prove an equivariant generalization of a theorem of Braun [7, Theorem 1]. More specifically, we prove that if \( P \) is a \( G \)-invariant reflexive polytope and \( Q \) is an \( H \)-invariant lattice polytope containing the origin in its interior, then \( \varphi_{P \oplus Q}[t] = \varphi_P[t] \cdot \varphi_Q[t] \) (Proposition 6.12).

We next consider the delicate question of when \( \varphi[t] \) is a polynomial. In Section 7, we provide distinct criterion that guarantee either that \( \varphi[t] \) is a polynomial (Lemma 7.1), or is not a polynomial (Lemma 7.3). We say that \( \varphi[t] \) is **effective** if each virtual character \( \varphi_i \) is a character. Clearly, if \( \varphi[t] \) is effective, then \( \varphi[t] \) is a polynomial. We prove that \( \varphi_1 \) is a character (Corollary 6.7), and if \( P \) is a simplex (i.e. \( P \) has \( d + 1 \) vertices), then we show that the \( \varphi_i \) are explicit permutation representations (Proposition 6.1).

We offer the following conjecture. If \( Y \) denotes the toric variety corresponding to \( P \) with corresponding ample, torus-invariant line bundle \( L \), then one may ask whether \( (Y, L) \) admits a \( G \)-invariant hypersurface that is **non-degenerate** in the sense of Khovanskii [19]. We refer the reader to Section 7 for details.
**Conjecture** (Conjecture 12.1). With the notation above, the following conditions are equivalent

- $(Y, L)$ admits a $G$-invariant non-degenerate hypersurface,
- $\varphi[t]$ is effective,
- $\varphi[t]$ is a polynomial.

The fact that the first condition implies the second condition is Theorem 7.7, and is proved in [36] by realizing $\varphi_{i+1} \cdot \det(\rho)$ as the character associated to the action of $G$ on the $i$th graded piece of the Hodge filtration on the primitive part of the middle cohomology (with compact support) of a $G$-invariant non-degenerate hypersurface. In fact, this result provided the initial motivation for this project. The following corollary can be deduced from this result using Bertini’s theorem [17, Corollary 10.9].

**Corollary.** (Theorem 7.7) Let $\Gamma(Y, L)^G \subseteq \Gamma(Y, L)$ denote the sub-linear system of $G$-invariant global sections of $L$. If $\Gamma(Y, L)^G$ is base point free, then $\varphi[t]$ is effective.

One easily deduces the following useful combinatorial criterion for effectiveness.

**Corollary** (Corollary 7.10). If every face $Q$ of $P$ with $\dim Q > 1$ contains a lattice point that is $G_Q$-fixed, where $G_Q$ denotes the stabilizer of $Q$, then $\varphi[t]$ is effective.

In particular, if $\dim P = 2$ and $P$ contains a $G$-fixed lattice point, then $\varphi[t]$ is effective (Corollary 7.12), and if the order of $G$ divides $m$, then $\varphi_{mP}[t]$ is effective (Corollary 7.14).

Finally, we consider applications and examples of this theory. Firstly, we prove that if $P$ contains the origin and the fan over its faces can be refined to a smooth, $G$-invariant fan $\Delta$ such that the primitive integer vectors of the rays of $\Delta$ coincide with the non-zero vertices of $P$, then $\varphi[t]$ coincides with the character of the representation of $G$ on the cohomology $H^*(X, \mathbb{C})$ of the associated toric variety $X = X(\Delta)$ (Proposition 8.1). If $\Delta$ is the Coxeter fan associated to a root system, then we recover a formula of Procesi, Dolgachev and Lunts, and Stembridge [40, Theorem 1.4] for the character associated to the action of the Weyl group on $H^*(X, \mathbb{C})$ (Corollary 8.4). These characters have been studied by Procesi [30], Stanley [34, p. 529], Dolgachev, Lunts [11], Stembridge [40, 39] and Lehrer [23]. In the type $A$ case, we show that we may also realize $\varphi[t]$ from the action of $G = \text{Sym}_d$ on the hypercube $P = [0, 1]^d$ (Lemma 9.3). In this case, we use results of Stembridge in [39] to give an explicit description of $\varphi[t]$ in terms of marked tableaux (Proposition 9.7).
Observe that whenever $\varphi[t]$ is effective, we obtain a refinement of the $h^*$-polynomial by considering dimensions of isotypic components of $\varphi[t]$. In the case of the hypercube above, we recover Stembridge’s refinement of the Eulerian numbers (Remark 9.8).

Secondly, observe that the characters $\{\chi_{mP}\}_{m \geq 0}$ encode the Ehrhart theory of the rational polytopes $P_g = \{u \in P \mid g \cdot u = u\}$ for all $g \in G$. More specifically, $\chi_{mP}(g) = f_{P_g}(m) := \#(mP_g \cap M')$ (Lemma 5.2), where $f_{P_g}(m)$ is a quasi-polynomial called the Ehrhart quasi-polynomial of $P_g$. When $P$ is a simplex, we deduce a formula for the generating series of $f_{P_g}(m)$ (Proposition 6.1). A pseudo-integral polytope is a rational polytope whose Ehrhart quasi-polynomial is a polynomial (see the work of De Loera, McAllister and Woods [10, 24]), and we apply this formula to construct new pseudo-integral polytopes in all dimensions in Section 10.

Lastly, it follows from Corollary 7.10 that if $G = \mathbb{Z}/2\mathbb{Z}$ and $P$ is a centrally symmetric polytope, then $\varphi[t]$ is effective. In Section 11 we show that this fact is equivalent to the lower bounds $h^*_i \geq \binom{i}{i}$ on the coefficients of the $h^*$-polynomial of a centrally symmetric polytope, that were proved by Bey, Henk and Wills in [5, Remark 1.6]. We give an explicit description of $\varphi[t]$ for all non-singular, centrally symmetric, reflexive polytopes (Proposition 11.4), using a classification result of Klyachko and Voskresenski˘ı [22].

We end the introduction with a brief outline of the contents of the paper. In Section 2 and Section 3 we recall some basic facts about Ehrhart theory and representation theory respectively. In Section 4 we reduce to the case when $\dim P + 1 = \dim M'_R$, and provide the setup for the rest of the paper. In Section 5 and Section 6 we prove our results on the representations $\{\chi_{mP} \mid m \geq 0\}$ and $\varphi[t]$ respectively. In Section 7 we give criterion to determine whether or not $\varphi[t]$ is effective, and in Section 8 we give a geometric interpretation of $\varphi[t]$ for a special class of polytopes. In Section 9 and Section 11 we present examples when $P$ is a hypercube and a centrally symmetric polytope respectively. In Section 10 we demonstrate how our results can be applied to compute the Ehrhart quasi-polynomials of certain rational polytopes. Finally, in Section 12 we present some open questions and conjectures.

Notation and conventions. All representations will be defined over $\mathbb{C}$. We often identify a representation $\chi$ with its associated character and write $\chi(g)$ for the evaluation of the character of $\chi$ at $g \in G$. If $V$ is a $\mathbb{Z}$-module, then we write $V_R = V \otimes_{\mathbb{Z}} \mathbb{R}$ and $V_C = V \otimes_{\mathbb{Z}} \mathbb{C}$.

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2. Ehrhart theory

In this section, we recall some basic facts about Ehrhart theory, and refer the reader to \cite{1} and \cite{18} for introductions to the subject.

Let $M$ be a lattice and let $Q \subseteq M_R$ be a rational $d$-dimensional polytope. The **denominator** of $Q$ is the smallest positive integer $m$ such that $mQ$ is a lattice polytope. If we let $f_Q(m) = \#(mQ \cap M)$ for any positive integer $m$, then a classical result of Ehrhart \cite{12} asserts that $f_Q(m)$ is a quasi-polynomial of degree $d$, called the **Ehrhart quasi-polynomial** of $Q$. That is, there exists a positive integer $l$ and polynomials $g_0(m), \ldots, g_{l-1}(m)$ such that $f_Q(m) = g_i(m)$ whenever $m \equiv i \pmod{l}$. The minimal choice of such $l$ is called the **period** of the quasi-polynomial $f_Q(m)$. Ehrhart proved that $f_Q(0) = 1$, and that the period of $f_Q(m)$ divides the denominator of $Q$. Moreover, if we set $f_Q^0(m) = \#(\text{Int}(mQ) \cap M)$ for any positive integer $m$, where $\text{Int}(mQ)$ denotes the interior of $mQ$, then

$$f_Q(-m) = (-1)^d f_Q^0(m).$$

The latter result is known as **Ehrhart reciprocity**. The **index** $\text{ind}(Q)$ of $Q$ is the smallest positive integer $m$ such that the affine span of $mQ$ contains a lattice point. With the notation above, the polynomial $g_i(m)$ is a polynomial of degree $d$ with leading coefficient equal to the volume $\text{vol}(Q)$ of $Q$ if $\text{ind}(Q)$ divides $i$, and is identically zero otherwise. Here $\text{vol}(Q)$ equals the Euclidean volume of $Q \subseteq \text{aff}(Q)$ with respect to the lattice $\text{aff}(Q) \cap M$, where $\text{aff}(Q)$ denotes the affine span of $Q$. Alternatively, if we write

$$f_Q(m) = c_d(m)m^d + c_{d-1}(m)m^{d-1} + \cdots + c_0(m),$$

where $c_i(m)$ is a periodic function in $m$, then $c_0(0) = 1$, and

$$c_d(m) = \begin{cases} 
\text{vol}(Q) & \text{if } \text{ind}(Q) | m \\
0 & \text{otherwise}.
\end{cases}$$

If $Q$ is a lattice polytope, then $c_i(m) = c_i$ is constant, and $c_{d-1}$ may be interpreted as half the (normalized) surface area of $Q$. Here the **normalized surface area** of a facet $F$ of $Q$ equals the Euclidean volume of $F \subseteq \text{aff}(F)$ with respect to the lattice $\text{aff}(F) \cap M$. 
Let \( P \subseteq M_{\mathbb{R}} \) be a \( d \)-dimensional lattice polytope. After possibly replacing \( M_{\mathbb{R}} \) with the affine span of \( P \), we may and will assume that \( M \) has rank \( d \). It follows from the above result that \( f_P(m) \) is a polynomial of degree \( d \), called the Ehrhart polynomial of \( P \). By a routine argument, it follows that its generating series has the form

\[
\sum_{m \geq 0} f_P(m) t^m = \frac{h^*(t)}{(1-t)^{d+1}},
\]

where \( h^*(t) = h^*_P(t) = \sum_{i=0}^{d} h^*_i t^i \) is a polynomial of degree at most \( d \) with integer coefficients, called the \( h^* \)-polynomial of \( P \). Alternative names in the literature include \( \delta \)-polynomial and Ehrhart \( h \)-polynomial. Ehrhart reciprocity translates into the following equality

\[
\sum_{m \geq 1} f^*_P(m) t^m = t^{d+1} h^*(t-1)/(1-t)^{d+1}.
\]

Observe that \( h^*_0 = 1, h^*_1 = \#(P \cap M) - d - 1 \) and \( h^*_d = \#(\text{Int}(P) \cap M) \). Since \( P \) has at least \( d + 1 \) vertices, we conclude that

\[
0 \leq h^*_d \leq h^*_1.
\]

In fact, Stanley used the theory of Cohen Macauley rings to prove that the coefficients \( h^*_i \) are non-negative integers \([32]\). A combinatorial proof was later given by Betke and McMullen in \([4]\). The degree \( s \) of \( P \) is defined to be the degree of \( h^*(t) \), and the codegree \( l \) of \( P \) is defined by \( l = d + 1 - s \). Ehrhart reciprocity implies that the codegree can be interpreted as \( l = \min\{m \mid \text{Int}(mP) \cap M \neq \emptyset\} \), and the leading coefficient of \( h^*(t) \) is given by \( h^*_s = \#(\text{Int}(lP) \cap M) \).

The polytope \( P \) is reflexive if the origin is its unique interior lattice point, and every non-zero lattice point in \( M \) lies in the boundary of \( mP \) for some positive integer \( m \). The following theorem of Stanley was proved using commutative algebra, while a combinatorial proof was recently given by the author in \([38, \text{Corollary 2.18}]\).

**Theorem 2.1.** \([31, \text{Theorem 4.4}]\) If \( P \) is a lattice polytope of degree \( s \) and codegree \( l \), then the following are equivalent

- \( f^*_P(m) = f_P(m-l) \) for \( m \geq l \),
- \( h^*_P(t) = t^s h^*_P(t^{-1}) \),
- \( lP \) is a translate of a reflexive polytope.

**Remark 2.2.** Let \( \triangle \) be a smooth, \( d \)-dimensional fan in \( M_{\mathbb{R}} \), and let \( |\triangle| \) denotes the support of \( \triangle \). Let \( \psi : |\triangle| \to \mathbb{R} \) be the piecewise linear function with respect to \( \triangle \) that has value 1 at the primitive lattice points of the rays of \( \triangle \). If \( P = \{ v \in M_{\mathbb{R}} \mid \)
\( \psi(v) \leq 1 \) is convex, then the \( h^* \)-polynomial of \( P \) is equal to the \( h \)-polynomial of \( \Delta \). That is,

\[
h^*_P(t) = h_\Delta(t) = \sum_{i=0}^{\Delta} f_i t^{i}(1-t)^{d-i},
\]

where \( f_i \) equals the number of cones in \( \Delta \) of dimension \( i \). Note that if \( |\Delta| = M_\mathbb{R} \), then \( P \) is reflexive. Also, one can define the \( h^* \)-polynomial of \( P \) even if \( P \) is not convex, and the above equality holds.

3. Representation theory of finite groups

In this section, we recall some basic facts about the representation theory of finite groups over the complex numbers. We refer the reader to [15] and [21] for an introduction to the subject and proofs of the statements below.

If \( G \) is a finite group, then a (complex) representation of \( G \) is a finite-dimensional complex vector space \( V \) with a linear action \( \rho : G \to GL(V) \) of \( G \). We say that \( V \) is irreducible if it contains no non-trivial \( G \)-invariant subspaces. Every representation is isomorphic to a direct sum of irreducible representations. If \( W \) is an irreducible representation and \( V \cong \bigoplus V_i \), where each \( V_i \) is irreducible, then the multiplicity of \( W \) in \( V \) is the number of irreducible representations \( V_i \) isomorphic to \( W \).

The representation ring \( R(G) \) is defined to be the quotient of the free abelian group generated by isomorphism classes of \( G \)-representations by the \( \mathbb{Z} \)-submodule generated by relations of the form \( V \oplus W - V - W \). Addition (respectively multiplication) of classes of representations is given by taking direct sums (respectively tensor products) of representations. Elements of \( R(G) \) are called virtual representations, and we let \( 1 \in R(G) \) denote the class of the trivial representation of \( G \). An element of \( R(G) \) is an effective representation if is equal to the class of a representation \( V \) of \( G \).

The character \( \chi : G \to \mathbb{C} \) associated to a representation \( V \) is the function \( \chi(g) = \text{tr}(\rho(g)) \), where \( \text{tr} \) denotes the trace function. A character of a representation is a class function i.e. a function from \( G \) to \( \mathbb{C} \) that is constant on conjugacy classes. Addition and multiplication in \( \mathbb{C} \) gives the set of all class functions \( \mathbb{C}_{\text{class}}(G) \) the structure of a \( \mathbb{C} \)-algebra. The \( \mathbb{C} \)-algebra homomorphism from \( R(G) \otimes_{\mathbb{Z}} \mathbb{C} \) to \( \mathbb{C}_{\text{class}}(G) \), taking a representation to its character, is an isomorphism. The vector space \( \mathbb{C}_{\text{class}}(G) \) admits a Hermitian inner product

\[
\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta}(g),
\]
where $|G|$ denotes the order of $G$, and $\overline{a}$ denotes the complex conjugate of $a \in \mathbb{C}$. The characters of the irreducible representations of $G$ form an orthonormal basis of $C_{\text{class}}(G)$. In the remainder of the paper, we will often identify a representation with its character.

If $\bigwedge^m V$ and $\text{Sym}^m V$ denote the exterior and symmetric powers of $V$ respectively, then we have the following (well-known) equality in $R(G)[[t]]$.

**Lemma 3.1.** Let $G$ be a finite group and let $V$ be an $r$-dimensional representation. Then

$$\sum_{m \geq 0} \text{Sym}^m V t^m = \frac{1}{1 - Vt + \bigwedge^2 Vt^2 - \cdots + (-1)^r \bigwedge^r Vt^r}.$$  

Moreover, if an element $g \in G$ acts on $V$ via a matrix $A$, and if $I$ denotes the identity $r \times r$ matrix, then both sides equal $\frac{1}{\det(I - tA)}$ when the associated characters are evaluated at $g$.

**Proof.** The following simple proof was related to me by John Stembridge. If an element $g \in G$ acts on $V$, then, since $g$ has finite order, we may assume, after a change of basis, that $g$ acts via a diagonal matrix $(\lambda_1, \ldots, \lambda_r)$. Then both sides of the equation equal $\frac{1}{(1-\lambda_1 t) \cdots (1-\lambda_r t)}$ when evaluated at $g$. \hfill \Box

If $H$ is a subgroup of $G$, with group algebra $\mathbb{C}[H]$, and $W$ is an $H$-representation, then the induced representation $\text{Ind}_H^G W$ is the $G$-representation $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. If $W'$ is a representation of $G$, then we let $\text{Res}_H^G W'$ denote the restriction of $W'$ to an $H$-representation. Frobenius reciprocity states that for a $G$-character $\chi$ and an $H$-character $\varphi$,

$$\langle \text{Ind}_H^G \chi, \varphi \rangle = \langle \chi, \text{Res}_H^G \varphi \rangle.$$  

If $G$ acts transitively on a set $S$, then the associated isotropy group $H$ is the subgroup of $G$ that fixes a given $s$ in $S$, and is well-defined up to conjugation. The corresponding permutation representation is isomorphic to the induced representation $\text{Ind}_H^G 1$ of the trivial representation of $H$. We immediately deduce the following lemma.

**Lemma 3.2.** Suppose $G$ acts on a set $S$, and let $\chi$ denote the corresponding permutation character. Then $\chi(g)$ equals the number of elements of $S$ fixed by $g$ in $G$, and if $\lambda : G \to \mathbb{C}$ is a 1-dimensional representation, then the multiplicity of $\lambda$ in $\chi$ is equal to the number of $G$-orbits of $S$ whose isotropy subgroup is contained in the subgroup $\lambda^{-1}(1)$ of $G$. 
Example 3.3 (The symmetric group). If \( G = \text{Sym}_d \) denotes the symmetric group on \( d \) letters, then the irreducible representations \( \chi^\lambda \) of \( G \) are indexed by partitions \( \lambda \) of \( d \). For example, \( \chi^{(d)} \) is the trivial representation, \( \chi^{(1^d)} \) is the sign representation, and \( \chi^{(d-1,1)} \) is the reflection representation corresponding to the standard action of \( G \) on \( \mathbb{C}^d / \mathbb{C}(1, \ldots, 1) \). More generally, the hook partitions \( \chi^{(d-r,1^r)} = \bigwedge^r \chi^{(d-1,1)} \) correspond to exterior powers of the reflection representation.

4. The setup

Recall from the introduction that \( G \) is a finite group acting linearly on a lattice \( M' \cong \mathbb{Z}^n \), and \( P \) is a \( d \)-dimensional \( G \)-invariant lattice polytope. In this section, we explain how one can always reduce to the case when \( M' = M \oplus \mathbb{Z} \) for some lattice \( M \) of rank \( d \) and \( P \subseteq M \times 1 \). We also show that one may equivalently consider \( d \)-dimensional lattice polytopes in lattices of rank \( d \), that are \( G \)-invariant ‘up to translation’. The setup deduced at the end of the section will be used throughout the paper.

Observe that the affine span \( W \) of \( P \) in \( M' \) is \( G \)-invariant. If we fix a lattice point \( \overline{u} \in W \cap M' \), then \( M := W \cap M' - \overline{u} \) has the structure of a lattice of rank \( d \) and \( G \) acts linearly on \( M \) via

\[ g \cdot (u - \overline{u}) = gu - g\overline{u} = (gu - g\overline{u} + \overline{u}) - \overline{u}, \]

for all \( g \in G \) and \( u \in W \cap M' \). Regarding \( P \) as a lattice polytope in \( M' \), we see that \( P \) is invariant under \( G \) ‘up to translation’. That is, if we set consider the function \( w : G \to M \) defined by \( w(g) = g\overline{u} - \overline{u} \), then \( w(1) = 0 \), \( w(gh) = w(g) + g \cdot w(h) \), and if we identify \( P \) with the lattice polytope \( P - \overline{u} \) in \( M \), then \( g \cdot P = P - w(g) \) in \( M \) for all \( g \in G \).

Conversely, assume that \( G \) acts linearly on a \( d \)-dimensional lattice \( M' \), and \( P \) is a \( d \)-dimensional lattice polytope that is invariant under \( G \) ‘up to translation’. That is, assume there exists a function \( w : G \to M \) satisfying \( w(1) = 0 \) and \( w(gh) = w(g) + g \cdot w(h) \), and such that \( g \cdot P = P - w(g) \) for all \( g \in G \). Then \( G \) acts linearly on the lattice \( M' = M \oplus \mathbb{Z} \) as follows: \( g \cdot (u, \lambda) = (g \cdot u - \lambda w(g), \lambda) \) for any \( g \in G \) and \( (u, \lambda) \in M' \). If we identify \( P \) with the lattice polytope \( P \times 1 \) in \( M' \), then \( P \) is invariant under the action of \( G \). Note that we recover the original linear action of \( G \) on \( M \) and the induced action on \( P \) ‘up to translation’ via the action of \( G \) on \( M \times 0 \subseteq M' \) and \( P \times 0 \) respectively. Moreover, the complex \( G \)-representation \( (M')_\mathbb{C} \) is isomorphic to \( M_\mathbb{C} \oplus \mathbb{C} \), where \( \mathbb{C} \) denotes the trivial representation.

The preceding discussion motivates the following setup:
Let $G$ be a finite group acting linearly on a lattice $M' = M \oplus \mathbb{Z}$ of rank $d+1$ such that the projection $M' \to \mathbb{Z}$ is equivariant with respect to the trivial action of $G$ on $\mathbb{Z}$. Let $P \subseteq M_\mathbb{R} \times 1$ be a $G$-invariant, $d$-dimensional lattice polytope.

By identifying $M$ with $M \times 0$, we regard $M$ as a lattice with a linear $G$-action $\rho : G \to \text{GL}(M)$, and consider the corresponding complex $G$-representation $M_\mathbb{C}$.

We often identify $P$ with the lattice polytope \( \{ u \in M_\mathbb{R} | u \times 1 \in P \} \) in $M_\mathbb{R}$, that is $G$-invariant ‘up to translation’.

5. Equivariant Ehrhart theory

The goal of this section is to study the permutation characters \( \{ \chi_{mP} \}_{m \geq 0} \), and establish equivariant analogues of Ehrhart’s original results (see Section 2). Throughout the paper, we often identify representations with their characters.

We will continue with the setup of Section 4 above. We often abuse notation, and consider $P$ as a polytope in $M_\mathbb{R}$. Recall that for any positive integer $m$, $\chi_{mP}$ (respectively $\chi^*_{mP}$) denotes the complex permutation representation induced by the action of $G$ on the lattice points $mP \cap M$ (respectively Int($mP$) \cap M), and $\chi_{mP}$ denotes the trivial representation when $m = 0$.

Consider the following rational polytopes.

Definition 5.1. For any $g \in G$, let $P_g = \{ u \in P | g \cdot u = u \} \subseteq M_\mathbb{R}$.

The following simple lemma provides the motivation to consider these polytopes. Recall from Section 2 that $f_{P_g}(m) = \#(mP_g \cap M)$ is the Ehrhart quasi-polynomial of $P_g$, and $f_{P_g}^\circ(m) = \#(\text{Int}(mP_g) \cap M)$ for $m \geq 1$.

Lemma 5.2. For any positive integer $m$, $\chi_{mP}(g) = f_{P_g}(m)$ and $\chi^*_{mP}(g) = f_{P_g}^\circ(m)$. Also, $\chi_{mP}(g) = f_{P_g}(m) = 1$ when $m = 0$.

Proof. Since $\chi_{mP}$ is a permutation representation, $\chi_{mP}(g)$ is equal to the number of lattice points in $mP$ fixed by $g$ (Lemma 3.2). The latter is equal to $f_{P_g}(m)$. The rest of the lemma follows similarly.

Let $M^g$ denote the subspace of $M_\mathbb{R}$ fixed by $g$. Note that, if we fix an isomorphism $M \cong \mathbb{Z}^d$, then $g$ acts on $M$ via an integer-valued matrix $A$, and dim $M^g$ equals the number of times 1 occurs (with multiplicity) as an eigenvalue of $A$.

Lemma 5.3. With the notation above, dim $P_g = \text{dim } M^g$.

Proof. If $(M')^g$ denotes the linear subspace of $M'_\mathbb{R}$ fixed by $g$, then, by definition, $P_g$ is the rational polytope $P \cap (M')^g$. Note that $(M')^g$ intersects $M_\mathbb{R} \times 1$ in an
affine subspace of dimension \( \dim M^g \), and hence we only need to show that \( (M')^g \cap \text{Int}(P) \neq \emptyset \). On the other hand, if \( \{v_i \mid i \in I\} \) denotes the vertices of \( P \), then \( \frac{1}{|I|} \sum_{i \in I} v_i \) is a \( G \)-invariant point in the interior of \( P \). \( \square \)

If we fix an element \( g \in G \), then \( g \) permutes the set of vertices \( \{v_i \mid i \in I\} \) of \( P \). Let \( I/g \) denote the set of orbits of \( I \) under the action of \( g \), and, for each orbit \( \iota \in I/g \), let \( v_\iota = \frac{1}{|\iota|} \sum_{i \in \iota} v_i \) be the corresponding rational point in \( P_g \). Recall that \( P \) is a simplex if it has precisely \( d + 1 \) vertices.

**Lemma 5.4.** With the notation above, \( P_g \) is the convex hull of \( \{v_\iota \mid \iota \in I/g\} \). In particular, if \( g^r \) fixes the vertices of \( P \), then \( rP_g \) is a lattice polytope. Moreover, if \( P \) is a simplex, then \( P_g \) is a simplex with (distinct) vertices \( \{v_\iota \mid \iota \in I/g\} \).

**Proof.** Any element \( w \in P \) can be written in the form \( w = \sum_{i \in I} \lambda_i v_i \) for some \( \lambda_i \geq 0 \) satisfying \( \sum_{i \in I} \lambda_i = 1 \). If \( w \in P_g \) and \( g^r \) fixes the vertices of \( P \), then

\[
w = \frac{1}{r} \sum_{i=0}^{r-1} g^i w = \sum_{\iota \in I/g} \frac{1}{|\iota|} \sum_{i \in \iota} \lambda_i \sum_{i \in I} v_i = \sum_{\iota \in I/g} (\sum_{i \in \iota} \lambda_i) v_\iota.
\]

Since \( \sum_{\iota \in I/g} (\sum_{i \in \iota} \lambda_i) = 1 \), this proves the first statement. Observing that \( |\iota| \) divides \( r \), we obtain the second statement. Finally, if \( P \) is a simplex, then the vertices \( \{v_i \mid i \in I\} \) of \( P \) form a basis of \( M'_C \). Hence we may identify the representation \( M'_C \) with the permutation representation induced by the action of \( G \) on \( \{v_i \mid i \in I\} \). In particular, the dimension of the subspace \( (M')^g \) of \( M'_C \) fixed by \( g \) is precisely the number of orbits \( \iota \in I/g \). By Lemma 5.3, we conclude that \( \dim P_g + 1 \) equals the number of \( g \)-orbits of vertices of \( P \), and the result follows. \( \square \)

Recall that \( G \) acts on \( M \) via \( \rho : G \to GL(M) \).

**Lemma 5.5.** With the notation above, the function \( g \mapsto (-1)^{d - \dim P_g} \) equals the representation \( \det(\rho) \).

**Proof.** If \( g \) acts on \( M \) via an integer-valued matrix \( A \), then the eigenvalues of \( A \) are roots of unity, and hence

\[
\det(tI - A) = (t - 1)^a (t + 1)^b \prod_\zeta (t - \zeta)(t - \overline{\zeta}),
\]

for some complex roots of unity \( \zeta \). Comparing constant terms on both sides yields \((-1)^d \det(\rho(g)) = (-1)^a \). The result now follows from Lemma 5.3. \( \square \)

**Remark 5.6.** In fact, the above proof holds provided that \( \rho : G \to GL(M_{\mathbb{R}}) \) is a real representation.
We are now ready to prove an equivariant analogue of Ehrhart’s results in [12]. Recall that the exponent of G is the smallest positive integer N such that \( g^N = 1 \) for all \( g \in G \), and that the denominator of a rational polytope \( Q \) is the smallest positive integer \( m \) such that \( mQ \) is a lattice polytope.

**Theorem 5.7.** Consider the function \( L(m) = \chi_{mP} \in R(G) \) for any non-negative integer \( m \). Then \( L(m) \) is a quasi-polynomial in \( m \) of degree \( d \) and period dividing the exponent of \( G \), and, for any positive integer \( m \), \((-1)^d L(-m) = \chi_{mP}^* \cdot \det(\rho)\).

**Proof.** By Lemma 5.2, \( \chi_{mP}(g) = f_{P_g}(m) \) for any non-negative integer \( m \), and, by Lemma 5.4 and Ehrhart’s results (Section 2), \( f_{P_g}(m) \) is a quasi-polynomial of degree \( \dim P_g \) with period dividing the exponent of \( G \). Hence, by Ehrhart reciprocity and Lemma 5.2 the character of \((-1)^d L(-m)\) evaluated at \( g \) equals

\[
(-1)^{d-\dim P_g} (-1)^{\dim P_g} f_{P_g}^*(m) = (-1)^{d-\dim P_g} f_{P_g}^*(m) = (-1)^{d-\dim P_g} \chi_{mP}^*(g).
\]

The result now follows from Lemma 5.5. \( \square \)

For any positive integer \( m \), Lemma 5.2 implies that \( f_{P/G}(m) = \langle \chi_{mP}, 1 \rangle \) (respectively \( \bar{f}_{P/G}(m) = \langle \chi_{mP}^*, 1 \rangle \)) equals the number of \( G \)-orbits of \( mP \cap M \) (respectively \( \text{Int}(mP) \cap M \)). Similarly, \( \bar{f}_{P/G}(m) = \langle \chi_{mP}, \det(\rho) \rangle \) (respectively \( \bar{f}_{P/G}(m) = \langle \chi_{mP}^*, \det(\rho) \rangle \)) equals the number of \( G \)-orbits of \( mP \cap M \) (respectively \( \text{Int}(mP) \cap M \)) whose isotropy subgroup is contained in \( \{ g \in G \mid \det(\rho(g)) = 1 \} \).

**Corollary 5.8.** With the notation above, \( f_{P/G}(m) \) and \( \bar{f}_{P/G}(m) \) are quasi-polynomials in \( m \) of degree \( d \), with leading coefficient \( \frac{\chi_{mP}}{[0]} \) and period dividing the exponent of \( G \). Moreover, \( f_{P/G}(0) = \bar{f}_{P/G}(0) = 1 \), and \((-1)^d f_{P/G}(-m) = \bar{f}_{P/G}(m) \) and \((-1)^d \bar{f}_{P/G}(-m) = f_{P/G}(m) \) for any positive integer \( m \).

**Proof.** We apply Theorem 5.7 to the inner products \( \langle \chi_{mP}, 1 \rangle \) and \( \langle \chi_{mP}, \det(\rho) \rangle \), using the fact that \( \chi_{mP} \) is the trivial representation when \( m = 0 \), and the fact that \( \langle \chi_{mP}^* \cdot \det(\rho), 1 \rangle = \langle \chi_{mP}^*, \det(\rho) \rangle \). The statement about the leading coefficients of \( f_{P/G}(m) \) and \( \bar{f}_{P/G}(m) \) follows from Corollary 5.9 below. \( \square \)

With the notation of Theorem 5.7, we may write

\[
L(m) = L_d(m)m^d + L_{d-1}(m)m^{d-1} + \cdots + L_0(m),
\]

where \( L_i(m) \in R(G) \) is a periodic function in \( m \) with period dividing the exponent of \( G \). Observe that \( L_0(0) \) is the trivial representation. Below we give an explicit description of the two leading terms of this quasi-polynomial. Recall that the standard representation \( \chi_{\text{st}} \) of \( G \) is the permutation representation induced by the action of
$G$ on itself by left multiplication. Since $g \neq 1$ has no fixed points with respect to this action, the character of $\chi_{st}$ is given by

\begin{equation}
\chi_{st}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise}. \end{cases}
\end{equation}

It is a standard fact that if $\{\chi_1, \ldots, \chi_r\}$ denote the irreducible representations of $G$, then $\chi_{st} = \sum_{i=1}^{r} \chi_i(1) \chi_i$.

**Corollary 5.9.** With the notation above, the leading coefficient $L_d(m) = L_d$ is independent of $m$ and given by $L_d = \frac{\text{vol} P}{|G|} \chi_{st}$. In particular, the multiplicity of a fixed irreducible representation $\chi$ in $\chi_{mP}$ is a quasi-polynomial in $m$ of degree $d$ with leading coefficient $\frac{\chi(1) \text{vol} P}{|G|}$.

**Proof.** By Lemma 5.2, $\chi_{mP}(g) = f_{Pg}(m)$ is a quasi-polynomial of degree strictly less than $d$ unless $g = 1$, in which case, the leading term is $\text{vol}(P)m^d$ (see Section 2). The first statement now follows from (1). The second statement is immediate from the fact that the multiplicity $\langle \chi_{st}, \chi \rangle$ of an irreducible representation $\chi$ in $\chi_{st}$ is equal to its dimension $\chi(1)$.

**Remark 5.10.** If $C$ denotes the cone over $P$, then $\chi_{mP}$ may be viewed as the representation of $G$ on the $m^{th}$ graded piece of the semi-group algebra $R = \mathbb{C}[C \cap M']$. From this perspective, the above corollary may be viewed as a special case of the results of Howe in [20]. We refer the reader to the work of Paolletti [27, Theorem 1] for a similar result on equivariant volumes of big line bundles on smooth, projective complex varieties.

For any $g \in G$, recall that the index $\text{ind}(P_g)$ of $P_g$ is the smallest positive integer $m$ such that the affine span of $mP_g$ contains a lattice point. By Lemma 5.4, the index of $P_g$ divides the order of $g$. If $g$ acts on $M$ via the integer-valued matrix $A$, then $g$ is a reflection if all but one of the eigenvalues of $A$ is equal to 1 (the other eigenvalue is necessarily $-1$). By Lemma 5.3, $g$ is a reflection if and only if $\dim P_g = d - 1$. Let $s(P)$ denote the (normalized) surface area of $P$ (see Section 2).

**Corollary 5.11.** With the notation above, the second leading coefficient $L_{d-1}(m)$ is periodic in $m$ with period dividing 2, and is given by the (virtual) character

\begin{equation}
L_{d-1}(m)(g) = \begin{cases} \frac{s(P)}{2} & \text{if } g = 1 \\ \text{vol} P_g & \text{if } g \text{ is a reflection and } m \equiv 0 \mod \text{ind}(P_g) \\ 0 & \text{otherwise}. \end{cases}
\end{equation}

In particular, $L_{d-1}(m)$ is independent of $m$ if and only if every reflection fixes a point in $M \times 1$. 
Proof. By Lemma 5.2 and the previous discussion, $\chi_{mP}(g) = f_{P_g}(m)$ is a quasi-polynomial of degree strictly less than $d - 1$ unless $g = 1$ or $g$ is a reflection. In the latter case, $g^2 = 1$ and Lemma 5.4 implies that $2P_g$ is a lattice polytope. Hence, $\text{ind}(P_g) \in \{1, 2\}$, and the first two statements now follow from basic properties of Ehrhart quasi-polynomials (see (1) in Section 2 and the surrounding discussion). The final statement follows from the observation that if $g$ is a reflection, then $\text{ind}(P_g) = 1$ if and only if $g$ fixes a point in $M \times 1$. □

Remark 5.12. Recall from the introduction that the generating series of $L(m)$ can be written in the form

$$\sum_{m \geq 0} L(m) t^m = \frac{\varphi[t]}{(1-t) \det(I-\rho t)},$$

for some power series $\varphi[t] \in R(G)[[t]]$. It follows from Corollary 5.11 and Lemma 7.3 below that if $\varphi[t]$ is a polynomial, then $L_{d-1} = L_{d-1}(m)$ is independent of $m$.

6. An equivariant analogue of the $h^*$-polynomial

The goal of this section is to study the power series $\varphi[t]$ of virtual representations introduced in the introduction, that may be viewed as an equivariant analogue of the $h^*$-polynomial of a lattice polytope.

We will continue with the notation of Section 4 and Section 5. That is, $G$ acts linearly on the lattice $M' = M \oplus \mathbb{Z}$ of rank $d + 1$, and $P \subseteq M_{\mathbb{R}} \times \mathbb{1}$ is a $G$-invariant, $d$-dimensional lattice polytope. If $R(G)$ denotes the representation ring of $G$ and $\rho : G \to GL(M)$, then we may write

$$\sum_{m \geq 0} \chi_{mP} t^m = \frac{\varphi[t]}{(1-t) \det(I-\rho t)},$$

for some power series $\varphi[t] = \varphi_{P,G}[t] = \sum_{i \geq 0} \varphi_i t^i \in R(G)[[t]]$, where, by Lemma 3.1

$$\det(I-\rho t) = 1 - M(t) + \sum_{i=2}^{2} M(t)^2 - \cdots + (-1)^{d} \sum_{i=d}^{d} M(t)^d.$$

We first give an explicit description of $\varphi[t]$ when $P$ is a simplex. Recall that $P$ is a simplex if it has precisely $d + 1$ vertices $\{v_0, \ldots, v_d\}$. In this case, we define

$$\text{Box}(P) = \{v \in M' \mid v = \sum_{i=0}^{d} a_i v_i \text{ for some } 0 \leq a_i < 1\},$$

and let $\text{Box}(P)^g$ denote the elements of $\text{Box}(P)$ fixed by $g \in G$. Let $u : M' = M \oplus \mathbb{Z} \to \mathbb{Z}$ denote projection onto the second coordinate.
Proposition 6.1. With the notation above, if $P$ is a simplex, then $\varphi_i$ is the permutation representation induced by the action of $G$ on $\{v \in \text{Box}(P) \mid u(v) = i\}$. In particular,

$$\sum_{m \geq 0} f_P(g)(m)t^m = \frac{\sum_{v \in \text{Box}(P)_g} t^{u(v)}}{(1-t) \det(I - \rho(g)t)},$$

and the multiplicity of the trivial representation (respectively $\det(\rho)$) in $\varphi_i$ equals the number of $G$-orbits of $\{v \in \text{Box}(P) \mid u(v) = i\}$ (respectively the number of $G$-orbits of $\{v \in \text{Box}(P) \mid u(v) = i\}$ whose isotropy subgroup is contained in $\{g \in G \mid \det(\rho(g)) = 1\}$).

Proof. Since the vertices $\{v_0, \ldots, v_d\}$ of $P$ form a basis of $M'_C$, we may identify the representation $M'_C$ with the permutation representation induced by the action of $G$ on $\{v_0, \ldots, v_d\}$. Hence we may identify $\text{Sym}^* M'_C$ with the graded permutation representation of $G$ on $\{\sum_{i=0}^d b_i v_i \mid b_i \in \mathbb{Z}_{\geq 0}\}$, where the latter set is graded by projection $M' = M \oplus \mathbb{Z} \to \mathbb{Z}$ onto the second coordinate. Similarly, let $V$ denote the graded permutation representation induced by the action of $G$ on $\text{Box}(P)$. If $C$ denotes the cone over $P$, then every lattice point $v$ in $C$ can be uniquely written as a sum $v = w + \sum_{i=0}^d b_i v_i$, for some $w \in \text{Box}(P)$ and $b_i \in \mathbb{Z}_{\geq 0}$. It follows that the tensor product $\text{Sym}^* M'_C \otimes V$ is the graded permutation representation induced by the action of $G$ on $C \cap M'$. The first statement now follows since Lemma 3.1 implies that $\sum_{m \geq 0} \text{Sym}^m M'_C t^m = \frac{1}{(1-t) \det(I - \rho)}$. The second statement follows immediately by evaluating characters at $g \in G$ using Lemma 5.2 and the final statement follows from Lemma 5.2.

The following remark can be useful for producing examples.

Remark 6.2. The pyramid $\text{Pyr}(P)$ of $P$ is the convex hull in $M'_R$ of $P$ and the origin. One may verify that $\text{Pyr}(P)$ is a $(d+1)$-dimensional, $G$-invariant lattice polytope with $\varphi_P[t] = \varphi_{\text{Pyr}(P)}[t]$.

Lemma 6.3. For any $g$ in $G$, $\varphi[t](g)$ is a rational function in $t$ that is regular at $t = 1$.

Proof. By Lemma 5.2 $\sum_{m \geq 0} \chi_m g(t)m^m = \sum_{m \geq 0} f_P(g)(m)t^m$, and the latter generating series is a rational function with a pole of order at most $\dim P_g + 1$ at $t = 1$ [2 Theorem 4]. Hence Lemma 5.3 implies that $(1-t) \det(I - \rho(g)t) \sum_{m \geq 0} \chi_m g(t)m^m$ is regular at $t = 1$. 

□
It follows from the lemma above that we may consider the rational class function \( \varphi[1] \). Recall that \( M^g \) denotes the subspace of \( M_\mathbb{R} \) fixed by \( g \), and let \( (M^g)^\perp \) denote the orthogonal subspace in \( M_\mathbb{R} \). Recall that the index \( \text{ind}(Q) \) of a rational polytope \( Q \) is the smallest positive integer \( m \) such that the affine span of \( mQ \) contains a lattice point (see Section 2).

**Proposition 6.4.** With the notation above,

\[
\varphi[1](g) = \frac{\dim(P_g)! \text{vol}(P_g) \det(I - \rho(g))_{(M^g)^\perp}}{\text{ind}(P_g)}.
\]

In particular, \( \varphi[1] \) takes non-negative values.

**Proof.** It follows from Lemma 5.2 and Section 2 that if \( N = \dim P_g \) and \( r = \text{ind}(P_g) \), then

\[
\chi_m P(g) = f_{P_g}(m) = c_N(m) m^N + c_{N-1}(m) m^{N-1} + \cdots + c_0(m),
\]

where \( c_i(m) \) is a periodic function in \( m \), and

\[
c_N(m) = \begin{cases} 
\text{vol}(P_g) & \text{if } r|m \\
0 & \text{otherwise} 
\end{cases}
\]

It follows from [2, Theorem 4] that \( \sum_{m \geq 0} c_i(m) m^i t^m \) is a rational function in \( t \) with a pole at \( t = 1 \) of order at most \( N \) for \( i < N \). Also,

\[
\sum_{m \geq 0} c_N(m) m^N t^m = \text{vol}(P_g) \sum_{r | m} m^N t^m = r^N \text{vol}(P_g) \sum_{m \geq 0} m^N (t^r)^m.
\]

Here \( \sum_{m \geq 0} m^N t^m = \frac{tA(N; t)}{(1-t)^{N+1}} \), where \( tA(N; t) \) is the \( N \)th Eulerian polynomial, and \( A(N; 1) = N! \) (cf. Section 9). It follows from Lemma 5.3 that \( \varphi[1](g) \) equals

\[
[(1-t)^{N+1} \det(I - \rho(g)t)_{(M^g)^\perp} \sum_{m \geq 0} \chi_m P(g) t^m]|_{t=1} =
\]

\[
[(1-t)^{N+1} \det(I - \rho(g)t)_{(M^g)^\perp} r^N \text{vol}(P_g) \frac{t^r A(N; t^r)}{(1-t^r)^{N+1}}]|_{t=1},
\]

and the first statement follows. The second statement follows since the complex eigenvalues of \( \rho(g) \) come in conjugate pairs \( \{e^{\pm i\alpha} | \alpha \in \mathbb{R}\} \), and \( (1-e^{i\alpha})(1-e^{-i\alpha}) = 2 - 2 \cos \alpha \geq 0 \). \( \square \)

**Remark 6.5.** Observe that if \( \varphi[t] \) is a polynomial, then \( \varphi[1] \) is an integral-valued virtual character, and the right hand side of Proposition 6.4 is a (non-negative) integer. We conjecture that this holds in general (Conjecture 12.3).
The result below is a consequence of the equivariant generalization of Ehrhart reciprocity in Theorem 5.7. Recall from Section 2 that the codegree of \( P \) is \( l = \min\{m \in \mathbb{Z}_{>0} \mid \text{Int}(mP) \cap M \neq \emptyset\} \) and the degree \( s = d + 1 - l \) of \( P \) is the degree of \( h^*(t) \).

**Corollary 6.6.** With the notation above,

\[
\sum_{m \geq 1} \chi_{mP}^* t^m = \frac{t^{d+1} \varphi[t^{-1}]}{(1-t) \det(I - \rho(t))}.
\]

In particular, if \( \varphi[t] \) is a polynomial then the degree of \( \varphi[t] \) is equal to the degree of \( P \), and \( \varphi_s = \chi_{sP}^* \). In this case, the multiplicity of the trivial representation (respectively \( \det(\rho) \)) in \( \varphi_s \) equals the number of \( G \)-orbits of \( \text{Int}(lP) \cap M \) (respectively the number of \( G \)-orbits of \( \text{Int}(lP) \cap M \) whose isotropy subgroup is contained in \( \{g \in G \mid \det(\rho(g)) = 1\} \)).

**Proof.** By Lemma 5.2, \( \chi_{mP}^*(g) = f_{P_g}^0(m) \) for any positive integer \( m \). By Ehrhart Reciprocity (see Section 2), \( f_{P_g}^0(m) = (-1)^{\dim P_g} f_{P_g}(-m) \) for any positive integer \( m \), and, by Proposition 4.2.3 in [35], \( \sum_{m \geq 1} f_{P_g}(-m) = \frac{-\varphi[t^{-1}](g)}{(1-t) \det(I - \rho(g)t^{-1})} \). Hence

\[
\sum_{m \geq 1} f_{P_g}^0(m) t^m = \frac{(-1)^{\dim P_g-1} \varphi[t^{-1}](g)}{(1-t^{-1}) \det(I - \rho(g)t^{-1})} = \frac{(-1)^{d - \dim P_g} t^{d+1} \varphi[t^{-1}](g)}{\det \rho(g)(1-t) \det(I - \rho(g)^{-1} t)}.
\]

Since the eigenvalues of \( \rho(g) \) are roots of unity and \( \rho(g) \) is integer-valued, \( \det(I - t \rho(g)^{-1}) = \det(I - t \rho(g)) = \det(I - t \rho(g)) \). Moreover, by Lemma 5.5, \( \det \rho(g) = (-1)^{d - \dim P_g} \). We conclude that

\[
\sum_{m \geq 1} f_{P_g}^0(m) t^m = \frac{t^{d+1} \varphi[t^{-1}](g)}{(1-t) \det(I - \rho(g)t)},
\]

as desired. The second statement is immediate, and the final statement follows from Lemma 3.2.

If \( V \) and \( W \) are virtual representations of \( G \), then we write \( V \geq W \) if \( V - W \) is an effective representation.

**Corollary 6.7.** With the notation above, \( \varphi_0 = 1 \) and \( \varphi_1 \) is an effective representation. Moreover, if \( \varphi[t] \) is a polynomial, then \( \varphi_1 \geq \varphi_d \geq 0 \).

**Proof.** Since \( \chi_{mP} \) is the trivial representation when \( m = 0 \), it follows from the definitions that \( \varphi_0 = 1 \) and \( \varphi_1 = \chi_P - M'_C \). Let \( W \) denote the permutation representation of \( G \) acting on the vertices \( \{v_i \mid i \in I\} \) of \( P \), and let \( W' \subseteq W \) be the \( G \)-submodule consisting of all relations satisfied by the vectors \( \{v_i \mid i \in I\} \) in \( M'_C \). Since \( P \) is
$d$-dimensional, the vertices of $P$ span $M'_R$ as a vector space, and we have an isomorphism of $G$-representations $W/W' \cong M'_C$. Since $W \oplus \chi_P^*$ is a subrepresentation of $\chi_P$ by definition, we conclude that $\chi_P - M'_C - \chi_P^*$ is an effective representation. Finally, Corollary 6.6 implies that if $\varphi[t]$ is a polynomial, then $\varphi_d = \chi_P^*$, and the result follows.

The example below demonstrates that we cannot hope that $\varphi_i$ is effective for $i \geq 2$, even if $h_i^* > 0$.

**Example 6.8.** Let $G = \mathbb{Z}/2\mathbb{Z}$ with generator $\tau$, and let $\chi : G \to \mathbb{C}$ be the linear character sending $\tau$ to $-1$, so that the irreducible representations of $G$ are $\{1, \chi\}$. Let $P$ be the convex hull of $(0,0)$, $(3,0)$, $(0,3)$ and $(3,3)$, and consider the action of $G$ on $M = \mathbb{Z}^2$ given by

$$\tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

Then, with the notation of Section 4, $P$ is $G$-invariant ‘up to translation’, and one computes that $\varphi_2 = 5 - \chi$. Observe that in this example $\varphi[t]$ is not a polynomial by Lemma 7.3.

Recall that a $d$-dimensional lattice polytope $P$ in $M$ is **reflexive** if the origin is the unique interior lattice point of $P$ and every non-zero lattice point in $M$ lies in the boundary of $mP$ for some positive integer $m$. We have the following equivariant version of Theorem 2.1. Recall that the degree $s$ of $P$ is the degree of $h^*(t)$ and the codegree of $P$ is $l = d + 1 - s$.

**Corollary 6.9.** With the notation above, if $P$ is a $G$-invariant lattice polytope of degree $s$ and codegree $l$, then the following are equivalent

- $\varphi[t] = t^s \varphi[t^{-1}]$,
- $\chi^*_{mP} = \chi_{(m-l)P}$ for $m \geq l$,
- $f_P^*(m) = f_P(m-l)$ for $m \geq l$,
- $h_P^*(t) = t^s h_P^*(t^{-1})$,
- $lP$ is a translate of a reflexive polytope.

**Proof.** The third, fourth and fifth conditions are equivalent by Theorem 2.1 and the second condition clearly implies the third. The fact that the first two conditions are equivalent is a formal consequence of Corollary 6.6. If $lP$ is a translate of a reflexive polytope, then let $v$ denote the unique (and hence $G$-invariant) interior lattice point in $lP$. For $m \geq l$, the equality $f_P^*(m) = f_P(m-l)$ implies that $\text{Int}(mP) \cap M' = (m-l)P \cap M' + v$, and hence $\chi^*_{mP} = \chi_{(m-l)P}$.  

□
Remark 6.10. Example 7.6 demonstrates that $\phi(t)$ is not necessarily a polynomial when $P$ is reflexive.

It is clear from the definitions that $\phi_i(g) \in \mathbb{Z}$ for all $g \in G$. Moreover, if $\phi(t)$ is a polynomial, then Corollary 6.6 implies that the leading term $\phi_s$ is a permutation representation, and hence $\phi_s(g)$ is a non-negative integer for all $g \in G$. The example below demonstrates that one can not expect that $\phi_i(g) \in \mathbb{Z}_{\geq 0}$ in general.

Example 6.11. Let $G = \mathbb{Z}/6\mathbb{Z}$ with generator $\sigma$, and let $\chi: G \to \mathbb{C}$ be the linear character sending $\sigma$ to $\zeta_6 = e^{\frac{2\pi i}{6}}$. Then the irreducible characters of $G$ are $\{1, \chi, \ldots, \chi^5\}$. Consider the representation of $G$ on $M = \mathbb{Z}^2$ via

$$\sigma \mapsto A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

and let $P$ be the (reflexive) lattice polytope with vertices $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$. One computes that $\phi[t] = 1 + (1 + \chi^2 + \chi^3 + \chi^4)t + t^2$, and $\phi[t](\sigma) = 1 - t + t^2$.

In fact, $P$ is a non-singular reflexive polytope in the sense that the vertices of each facet of $P$ form a basis of $M$, and the fact that $\phi[t]$ is a polynomial is guaranteed by Corollary 7.12 (and Proposition 8.1).

Observe that the character $\phi[1] = 3 + \chi^2 + \chi^3 + \chi^4$ has non-negative values. In fact, $\phi[1]$ is isomorphic to a permutation representation with isotropy subgroups $\{1, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}$.

If $H$ is a finite group acting on a lattice $N$ via $\rho': H \to GL(N)$, and $Q$ is an $H$-invariant lattice polytope, then the direct product $P \times Q = \{(p,q) \mid m \in P, n \in Q\}$ and the direct sum $P \oplus Q = \text{conv}\{P \times 0, 0 \times Q\}$ are $(G \times H)$-invariant lattice polytopes. We may and will regard a $G$-representation (respectively an $H$-representation) as a $(G \times H)$-representation via the projection of $G \times H$ onto $G$ (respectively $H$).

Proposition 6.12. With the notation above, $\chi_{m(P \times Q)} = \chi_m P \cdot \chi_m Q$, and if $P$ is a reflexive polytope and $Q$ contains the origin in its interior, then $\phi_{P \oplus Q}[t] = \phi_P[t] \cdot \phi_Q[t]$.

Proof. The first statement is clear since the lattice points of $m(P \times Q)$ are $\{(p,q) \mid p \in mP \cap M, q \in mQ \cap N\}$. For the second statement, it follows from Braun’s proof of [7, Theorem 1] that the lattice points of $m(P \oplus Q)$ are $\{(p,q) \mid p \in \partial(kP) \cap M, q \in (m-k)Q \cap N, 0 \leq k \leq m\}$, and hence $\chi_{m(P \oplus Q)} = \chi_m Q + \sum_{k=1}^{m}(\chi_{kP} - \chi_{(k-1)P}) \cdot \chi_{(m-k)Q}$. 

We compute
\[
\varphi_{P \oplus Q}[t] = \sum_{m \geq 0} \chi_{m(P \times Q)} t^m =
\sum_{m \geq 0} \chi_m P \cdot \chi_m Q =
\sum_{m \geq 0} \chi_m Q + \sum_{k=1}^m (\chi_{kP} - \chi_{(k-1)P} \cdot \chi_{m-k} Q) t^m =
(1-t) \sum_{m \geq 0} \chi_m P t^m \sum_{m \geq 0} \chi_m Q t^m =
\frac{\varphi_P[t]}{1-t} \cdot \frac{\varphi_Q[t]}{1-t} =
\varphi_P[t] \cdot \varphi_Q[t].
\]

\[
(1-t) \sum_{m \geq 0} \chi_m P t^m = (1-t) \sum_{m \geq 0} \chi_m Q t^m =
(1-t) \sum_{m \geq 0} \chi_m P t^m = (1-t) \sum_{m \geq 0} \chi_m Q t^m =
(1-t) \sum_{m \geq 0} \chi_m P t^m = (1-t) \sum_{m \geq 0} \chi_m Q t^m =
\varphi_P[t] \cdot \varphi_Q[t].
\]

\[\square\]

7. Effectiveness of representations

The goal of this section is to provide criterion to determine whether the power
series \(\varphi[t]\) is effective and whether it is a polynomial. We continue with the notation
of Section 4, Section 5 and Section 6.

Recall that the generating series of \(\{\chi_m P\}_{m \geq 0}\) can be written in the form
\[
\sum_{m \geq 0} \chi_m P t^m = \frac{\varphi_P[t]}{1-t} = \frac{\varphi_P[t]}{1-t} \det(I - \rho t),
\]
for some power series \(\varphi_P[t] \in R(G)[[t]]\). We begin with a criterion that guarantees
that \(\varphi_P[t]\) is a polynomial.

**Lemma 7.1.** If \(P_g\) is a lattice polytope for all \(g\) in \(G\), then \(\varphi[t]\) is a polynomial.

**Proof.** By Lemma 5.2 and Lemma 5.3,
\[
\sum_{m \geq 0} \chi_m P t^m = \frac{\varphi_P[t]}{1-t} = \frac{\varphi_P[t]}{1-t} \det(I - \rho t),
\]
where \(\det(I - \rho t) = (1-t)^{\dim P_g - 1} \det(I - \rho t)\) (cf. Proposition 6.4), and
hence \(\varphi[t] = h_{P_g}^*(t) \det(I - \rho(t))\) (see Section 2).

**Remark 7.2.** It follows from Lemma 5.4 and the lemma above that \(\varphi_m P[t]\) is a polynomial if the exponent of \(G\) divides \(m\).

We next provide a negative result. Recall that the index of a rational polytope \(Q\)
is the smallest positive integer \(m\) such that the affine span of \(mQ\) contains a lattice
point (see Section 2).

**Lemma 7.3.** With the notation above, fix an element \(g\) in \(G\), and let \(r\) denote
the index of \(P_g\). If \(rP_g\) is a lattice polytope, and \(\dim P_g > \frac{d-r+1}{r}\), then \(\varphi[t]\) is not
a polynomial. In particular, if there exists a reflection that doesn’t fix a point in
\(M \times 1\), then \(\varphi[t]\) is not a polynomial.
Proof. Recall from Section 2 that \( f_{\rho g}(m) = 0 \) unless \( r|m \). Hence, by Lemma 7.3,

\[
\sum_{m \geq 0} f_{\rho g}(m)t^m = \frac{h^*_r(t^r)}{(1 - t)^{\dim P_g + 1}} = \frac{h^*_r(t^r)\det(I - \rho(g)t)_{(M^g)\perp}}{(1 - t)\det(I - \rho(g)t)(1 + t + \cdots + t^{r-1})^{\dim P_g + 1}},
\]

where \( \det(I - \rho(g)t) = (1 - t)^{\dim P_g} \det(I - \rho(g)t)_{(M^g)\perp} \) (cf. Proposition 6.4). If \( \zeta \) is an \( r^{th} \) root of unity, then \( \zeta \) is not a root of \( h^*_r(t^r) \) (in fact, \( h^*_r(1) \) is equal to \((\dim P_g)!\) times the volume of \( rP_g \)). Hence if \( \varphi[t] \) is a polynomial, then \((1 + t + \cdots + t^{r-1})^{\dim P_g + 1} \) divides \( \det(I - \rho(g)t)_{(M^g)\perp} \). However, \( \det(I - \rho(g)t)_{(M^g)\perp} \) has degree \( d - \dim P_g < (r - 1)(\dim P_g + 1) \), a contradiction.

If \( g^2 = 1 \), then \( 2P_g \) is a lattice polytope by Remark 7.4 below. If \( g \) is a reflection that doesn’t fix a point in \( M \times 1 \), then \( r = 2 \), \( \dim P_g = d - 1 \), and the second statement follows. \( \square \)

**Remark 7.4.** With the notation of the above lemma, if \( g^r \) acts trivially on \( M \), then \( rP_g \) is a lattice polytope by Lemma 5.4.

**Example 7.5.** Let \( G = \mathbb{Z}/2\mathbb{Z} \) with generator \( \tau \), and consider the representation of \( G \) on \( M = \mathbb{Z}^3 \) (cf. Example 9.1) via

\[
\tau \mapsto A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

If \( P \) is the square with vertices \((0, 0, 1), (1, 0, 1), (0, 1, 1) \) and \((1, 1, 1) \), then \( P \) is \( G \)-invariant. Since \( \tau \) is a reflection with no fixed points in \( M \times 1 \), Lemma 7.3 implies that \( \varphi[t] \) is not a polynomial. Similarly, one can show that \( \varphi_{mP}[t] \) is not a polynomial when \( m \) is odd. By taking the convex hull of \( P \) and the origin, Remark 6.2 implies that one obtains a lattice polytope with a \( G \)-fixed point such that \( \varphi[t] \) is not a polynomial.

**Example 7.6.** We present an example to show that \( \varphi[t] \) need not be a polynomial when \( P \) contains a \( G \)-fixed interior lattice point. As in the previous example, let \( G = \mathbb{Z}/2\mathbb{Z} \) with generator \( \tau \), and consider the representation of \( G \) on \( M = \mathbb{Z}^3 \) via

\[
\tau \mapsto A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Let \( P \) be the \( G \)-invariant, 3-dimensional lattice polytope with vertices \( \pm(0, 0, 1), \pm(1, 0, 1), \pm(0, 1, 1) \) and \( \pm(1, 1, 1) \), and observe that the origin is a \( G \)-fixed lattice point in the interior of \( P \). In fact, \( P \) is reflexive and has \( h^* \)-polynomial \( h^*(t) = 

1 + 5t + 5t^2 + t^3 \) (cf. Corollary [6.9]). If \( M^T \) denotes the lattice fixed by \( \tau \), then we have an isomorphism \( M^T \cong \mathbb{Z}^2 \), \((1,0,2) \mapsto e_1, (0,1,0) \mapsto e_2 \). Under this isomorphism, \( \mathcal{P}_\tau \) corresponds to the rational polytope with vertices \( \pm \frac{3}{2} \) and \( \pm (\frac{3}{2} + e_2) \). One computes that \( 2 \mathcal{P}_\tau \) is a 2-dimensional lattice polytope with \( h^*_Q(t) = 1 + 6t + t^2 \) and \( f_{\mathcal{P}_\tau}(m) = f_{2\mathcal{P}_\tau}([\frac{3}{2}]) \), and hence

\[
\sum_{m \geq 0} f_{\mathcal{P}_\tau}(m)t^m = \frac{(1 + t)h^*_Q(t^2)}{(1 - t^2)^3} = \frac{1 + 6t^2 + t^4}{(1 - t) \det(I - tA)(1 + t)}.
\]

We conclude that if \( \chi : G \to \mathbb{C} \) denotes the linear character sending \( \tau \) to \(-1\), then

\[
\varphi[t] = \frac{1 + 3t + 8t^2 + 3t^3 + t^4}{1 + t} + \frac{t(3 + 2t + 3t^2)}{1 + t}\chi.
\]

We say that \( \varphi[t] \) is \textit{effective} if all the virtual representations \( \varphi_i \) are effective representations. Note that if \( \varphi[t] \) is effective, then \( \varphi[t] \) is a polynomial. For example, if \( G = 1 \), then \( \varphi[t] = h^*(t) \) is a polynomial with non-negative coefficients, and if \( P \) is a simplex, then Proposition [6.1] implies that the representations \( \varphi_i \) are effective. For the remainder of the section we will provide criterion that guarantee that \( \varphi[t] \) is effective.

We briefly recall some basic facts about toric varieties, and refer the reader to [14] and [41] for details. The lattice polytope \( P \) determines a complex, projective \( d \)-dimensional toric variety \( Y = Y_P \) and an ample line bundle \( L \) on \( Y \). The toric variety \( Y \) is a disjoint union of tori \( T_Q \) of dimension \( \dim Q \), indexed by the faces \( Q \) of \( P \). A section \( s \in \Gamma(Y,L) \) determines a hypersurface \( X = X(s) \) in \( Y \), and we say that \( X \) is \textit{non-degenerate} if \( X \cap T_Q \) is a smooth (possibly empty) hypersurface in \( T_Q \) for all \( Q \subseteq P \). Non-degenerate hypersurfaces were first studied by Khovanskii [19], and, recently, have been extended to the notion of a \textit{Schön} subvariety of a torus by Televé [41].

Equivalently, a hypersurface \( X^o = \{ f = \sum_{u \in \mathcal{P} \cap M} a_u \chi^u = 0 \} \subseteq T = T_P \) is non-degenerate with Newton polytope \( P \) if \( \{ f_Q = \sum_{u \in Q \cap M} a_u \chi^u = 0 \} \subseteq T \) is a smooth (possibly empty) hypersurface in \( T \) for all \( Q \subseteq P \). One can show that these two notions of non-degenerate coincide. That is, \( X = X^o \) is non-degenerate if and only if \( X^o = X \cap T \) is non-degenerate. The key point in proving this equivalence is the fact that

\[
\{ f_Q = 0 \} \cong (X \cap T_Q) \times (\mathbb{C}^*)^{d - \dim Q}.
\]

Recall that \( G \) acts on \( M \) and leaves \( P \) invariant ‘up to translation’. There is an induced action of \( G \) on \( Y \) via toric morphisms satisfying \( g^*L \cong L \) for all \( g \in G \). Let
\( \Gamma(Y, L)^G \subseteq \Gamma(Y, L) \) denote the sub-linear system of \( G \)-invariant sections of \( L \). The following result is proved in \cite{36} using the theory of mixed Hodge structures.

**Theorem 7.7.** \cite{36} With the notation above, if there exists a \( G \)-invariant non-degenerate hypersurface with Newton polytope \( P \), then \( \varphi[t] \) is effective. In particular, this assumption holds if the linear system \( \Gamma(Y, L)^G \) on \( Y \) is base point free.

**Remark 7.8.** The second statement of this theorem is an easy application of Bertini’s theorem (see Corollary 10.9 and Remark 10.9.2 in \cite{17}). In order to compute examples, we give the following condition that, using (5), one can show is equivalent to the condition that \( \Gamma(Y, L)^G \) is base point free: for each face \( Q \subseteq P \), the linear system

\[
\{ \sum_{u \in Q \cap M} a_u \chi^u | a_u = a_{u'} \text{ if } u, u' \text{ lie in the same } G_Q\text{-orbit} \}
\]

on \( T \) is base point free, where \( G_Q \) denotes the stabilizer of \( Q \).

**Remark 7.9.** Continuing with the notation of Remark 7.8 above, observe that in order to guarantee the existence of a \( G \)-invariant non-degenerate hypersurface, it is enough to check condition (6) holds for faces \( Q \subseteq P \) with \( \dim Q > 1 \). Indeed, if \( Q \) is a vertex, then the condition holds automatically since the non-zero elements of the linear system define the empty hypersurface in \( T \). If \( \dim Q = 1 \), then we claim that an element (and hence a non-empty open subset) of the linear system (6) defines a smooth hypersurface in \( T \). This follows since \( T_Q \cong \mathbb{C}^* = \text{Spec } \mathbb{C}[t^{+1}] \), and there exists an isomorphism \( T \cong \mathbb{C}^* \times (\mathbb{C}^*)^{d-1} \) such that (6) is isomorphic to a sub-linear system of \( \{ \sum_{i=0}^s a_i t^i = 0 \} \). The claim follows since the polynomial \( \{ \sum_{i=0}^s a_i t^i = 0 \} \) has \( s \) distinct roots.

The following corollary is immediate from the above two remarks.

**Corollary 7.10.** If every face \( Q \) of \( P \) with \( \dim Q > 1 \) contains a lattice point that is \( G_Q \)-fixed, where \( G_Q \) denotes the stabilizer of \( Q \), then \( \varphi[t] \) is effective.

**Remark 7.11.** The existence of a \( G \)-fixed lattice point is not necessary for \( \varphi[t] \) to be effective. For example, if \( G = \text{Sym}_{d+1} \) acts on \( \mathbb{Z}^{d+1} \) via the standard representation and \( P \) is the convex hull of the basis vectors \( e_0, \ldots, e_d \), then \( P \) has no \( G \)-fixed lattice points and \( \varphi[t] = 1 \) by Proposition 6.1.

We have the following two applications.

**Corollary 7.12.** If \( \dim P = 2 \) and \( P \) contains a \( G \)-fixed lattice point, then \( \varphi[t] \) is effective.
Proof. This follows immediately from Corollary [7.10] □

Remark 7.13. The above corollary is false if dim $P > 2$ by Example [7.5] (and Example [7.6]).

The corollary below should be compared with Remark [7.2].

Corollary 7.14. If the order of $G$ divides $m$, then $\varphi_mP[t]$ is effective.

Proof. As in the proof of Lemma [5.4] fix a face $Q$ of $P$ and let $\{v_i \mid i \in \iota\}$ denote a $G_Q$-orbit of $Q \cap M$. The rational point $v_\iota = \frac{1}{|\iota|} \sum_{i \in \iota} v_i$ is $G_Q$-fixed and $mv_\iota$ is a lattice point if $|\iota|$ divides $m$. Finally note that $|\iota|$ divides $|G_Q|$ that divides $|G|$, and apply Corollary [7.10] □

Example 7.15. In Example [7.5] $G = \mathbb{Z}/2\mathbb{Z}$, dim $P = 2$, and we see that $\varphi_mP[t]$ is effective if and only if $m$ is an even, positive integer.

Example 7.16. We say that the action of $G$ on $P$ is proper if for any proper face $Q$ of $P$ and any element $g$ in the stabilizer $G_Q$ of $Q$, $g$ fixes $Q$ pointwise (cf. [30, Section 1]). In this case, if $P$ contains the origin then one verifies that $P_g$ is a lattice polytope for all $g$ in $G$, and $\varphi[t]$ is effective by Corollary [7.10]. For an example, we refer the reader to Corollary [8.4].

8. Group actions on cohomology

In this section we consider a class of polytopes for which $\varphi[t]$ is effective and has a natural geometric interpretation. We continue with the notation of Section 4, Section 5 and Section 6.

Let $\Delta$ be a smooth, $G$-invariant, $d$-dimensional fan in $M_\mathbb{R}$, and let $X = X(\Delta)$ denote the associated toric variety. Note that $X$ has no odd cohomology, and the action of $G$ on $X$ induces a representation of $G$ on $H^*(X; \mathbb{C})$. If $|\Delta|$ denotes the support of $\Delta$, then let $\psi : |\Delta| \to \mathbb{R}$ be the piecewise linear function with respect to $\Delta$ that has value 1 at the primitive lattice points of the rays of $\Delta$. Recall that $P$ is reflexive if the origin is its unique interior lattice point, and every non-zero lattice point in $M$ lies in the boundary of $mP$ for some positive integer $m$.

Proposition 8.1. With the notation above, if $P = \{v \in M_\mathbb{R} \mid \psi(v) \leq 1\}$ is convex, then $P$ is a $G$-invariant lattice polytope and $\varphi_i$ is isomorphic to the $G$-representation $H^{2i}(X; \mathbb{C})$. In particular, if $|\Delta| = M_\mathbb{R}$, then $P$ is reflexive, and the multiplicities of a fixed irreducible representation in the representations $\varphi_i$ form a symmetric, unimodal sequence.
Proof. We refer the reader to [8, Section 1] and [28] for basic facts on the equivariant cohomology of toric varieties. Let $\mathbb{C}[M]^\Delta$ denote the deformed group ring of $\Delta$. It has a $\mathbb{C}$-vector space basis $\{x^v \mid v \in \Delta \cap M\}$ and multiplication

$$x^v \cdot x^w = \begin{cases} x^{v+w} & \text{if } v, w \text{ lie in a common cone in } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

The action of $G$ on the $T$-fixed points of $X$ induces (via GKM localization) an action of $G$ on the $T$-equivariant cohomology ring $H^*_T X$. In particular, there is a natural $G$-equivariant, graded isomorphism $\mathbb{C}[M]^\Delta \cong H^*_T(X; \mathbb{C})$ (in fact, both sides are naturally isomorphic to the complex Stanley-Reisner ring of $\Delta$). Here $\mathbb{C}[M]^\Delta$ inherits a natural grading from $\psi : \Delta \to \mathbb{R}$ (see the above discussion), and we consider $H^{2m}_T X$ to have degree $m$. In particular, the representation of $G$ on $H^{2m}_T(X; \mathbb{C})$ is isomorphic to $\chi_{mP} - \chi_{(m-1)P}$ for all positive integers $m$.

There is a natural $G$-invariant graded isomorphism $H^*_T X \cong H^*(X; \mathbb{C}) \otimes \text{Sym}^* N_C$. Here, if we fix a basis for $M$ and $g \in G$ acts on $M$ via an integer matrix $A$, then $g$ acts on $N$ via the inverse transpose of $A$. If $\{\lambda_i\}$ denote the eigenvalues of $A$, then the eigenvalues of $A^{-1}$ are the conjugates $\{\overline{\lambda}_i\}$. Since $A$ is integer valued, we conclude that $A$ and the inverse transpose of $A$ have the same eigenvalues and hence we have an isomorphism of $G$-representations $M_C \cong N_C$. We conclude that we have the following equality in $R(G)[[t]]$

$$(1 - t) \sum_{m \geq 0} \chi_{mP} t^m = \sum_{i=0}^d H^{2i}(X; \mathbb{C}) t^i \cdot \sum_{m \geq 0} \text{Sym}^m M_C t^m,$$

and the result follows by Lemma 3.1. The second statement is well-known and follows from the fact that the Hard Lefschetz theorem holds when $X$ is projective, and the observation that taking the cap product with a hyperplane class commutes with the action of $G$ on the cohomology of $X$ (see, for example, [33, p. 64]). Finally, if $|\Delta| = M_\mathbb{R}$ and $P$ is convex, then $P$ is reflexive by definition. \hfill $\Box$

Remark 8.2. In the above proposition, the assumption that $P = \{v \in M_\mathbb{R} \mid \psi(v) \leq 1\}$ is convex is not essential. That is, with the notation above, one can easily extend the definition of $\varphi[t]$ to the ‘star-shaped complex’ $P = \{v \in M_\mathbb{R} \mid \psi(v) \leq 1\}$, and then the proof of Proposition 8.1 holds verbatim.

If $R$ is a reduced, crystallographic root system of rank $d$, then the hyperplanes orthogonal to the roots of $R$ determine a smooth, projective, $d$-dimensional fan $\Delta_R$...
in the weight lattice $M$ of $R$, called the **Coxeter fan** of $R$. The associated toric variety $X_R = X_R(\Delta_R)$ is the normalization of the closure of a generic torus orbit in the homogeneous space $G/B$ associated to $R$, and the induced action of the Weyl group $W$ on the cohomology $H^*(X_R; \mathbb{C})$ has been studied by Procesi [30], Stanley [34] p. 529, Dolgachev, Lunts [11], Stembridge [40, 39] and Lehrer [23].

**Remark 8.3.** Using an intricate case-by-case argument, Stembridge gave a description of the $W$-representation $H^*(X_R; \mathbb{C})$ as an explicit (ungraded) permutation representation [40], and proved that $\langle \sum_{i=0}^{d} H^{2i}(X_R; \mathbb{C}) t^i, 1 \rangle = (1+t)^d$ (see the proof of Lemma 3.2 in [40]). Lehrer proved that the sign representation of $W$ does not appear in $H^*(X_R; \mathbb{C})$ [23, Theorem 3.5 (iii)], and gave a formula for the multiplicity of the reflection representation $M_\mathbb{C}$ in $H^{2i}(X_R; \mathbb{C})$ [23, Corollary 4.6].

With the notation above, let $G = W$ and consider the action $\rho : W \to GL(M)$. Let $\psi : M_{\mathbb{R}} \to \mathbb{R}$ be the piecewise linear function with respect to $\Delta_R$ that has value 1 at the primitive lattice points of the rays of $\Delta_R$, and let $P = P_R = \{ v \in M_{\mathbb{R}} \mid \psi(v) \leq 1 \}$. The action of $W$ on $P$ is proper in the sense of Example 7.16 and for every $w$ in $W$, $\Delta_R$ restricts to a non-singular fan $\Delta^w_R$ in the subspace $M^w_{\mathbb{R}}$ of $M_{\mathbb{R}}$ fixed by $w$ (see Section 2 in [40]). Moreover, $P_w = \{ v \in M^w_{\mathbb{R}} \mid \psi^w(v) \leq 1 \}$, where $\psi^w = \psi|_{M^w_{\mathbb{R}}}$ is the piecewise linear function that has value 1 at the primitive lattice points of the rays of $\Delta^w_R$.

We recover the character formula for the graded $W$-representation $H^*(X_R; \mathbb{C})$ due to Procesi, Dolgachev and Lunts, and Stembridge [40, Corollary 1.6].

**Corollary 8.4.** With the notation above, $\varphi[t] = \sum_{i=0}^{d} H^{2i}(X_R; \mathbb{C}) t^i$, and

$$
\varphi[t](w) = \frac{h_{\Delta^w_R}(t) \det(1 - \rho(w)t)}{(1-t)^{\dim M^w_{\mathbb{R}}}}.
$$

**Proof.** The fact that $\varphi[t] = \sum_{i=0}^{d} H^{2i}(X_R; \mathbb{C}) t^i$ follows immediately from Proposition 8.1 (and Remark 8.2). By Remark 2.2 and Lemma 5.2

$$
\frac{\varphi[t](w)}{(1-t) \det(1 - \rho(w)t)} = \sum_{m \geq 0} \chi_m P(w) t^m = \sum_{m \geq 0} f_{P_w}(m) t^m = \frac{h_{\Delta^w_R}(t)}{(1-t)^{\dim P_w+1}}.
$$

**Remark 8.5.** With the notation above, if $R$ is irreducible, then $P = \{ v \in M_{\mathbb{R}} \mid \psi(v) \leq 1 \}$ is convex if $R$ has type $A, B, C$ or $D$. On the other hand, as explained to the author by Dave Anderson, $P$ is not convex when $R = G_2$.

**Remark 8.6.** If $\{ s_1, \ldots, s_d \}$ denotes a set of simple reflections in $W$, then the length $l(w)$ of an element $w$ in $W$ is the minimum length of any factorization $w = s_{i_1} \cdots s_{i_r}$,
and the descent set of $w$ is $D(w) = \{ i \mid l(ws_i) < l(w) \}$. It is a theorem of Björner [6, Theorem 2.1] that

$$h_{\Delta_R}(t) = \sum_{w \in W} t^{|D(w)|}.$$  

In particular, if $R = A_d$, then $th_{\Delta_R}(t)$ is the $d$th Eulerian polynomial (cf. Section 9).

We refer the reader to [40] for explicit computations of the characters $\varphi[t]$.

**Example 8.7.** If $R = A_d$, then $W = \text{Sym}_{d+1}$ acts on $M = \mathbb{Z}^{d+1}/\mathbb{Z}(1, \ldots, 1)$ by permuting coordinates, and $P$ is the reflexive polytope with vertices given by the images of $\{e_{i_1} + \cdots + e_{i_l} \mid i_1 < \cdots < i_l, 1 \leq l \leq d - 1\}$ in $\mathbb{Z}^{d+1}$. We consider this example in more detail in Section 9 (cf. Remark 9.4).

**Example 8.8.** If $R = B_d$ (or $R = C_d$), then $W$ is the group of signed permutations, and the image of $W \to GL_d(\mathbb{Z})$ consists of all matrices with entries in $\{0, \pm 1\}$, and precisely one non-zero element in each row and column. In this case, $W$ may be interpreted as the full symmetry group of the hypercube $P = [-1, 1]^d$.

### 9. Symmetries of the hypercube

The goal of this section is to explicitly describe the equivariant Ehrhart theory of the $d$-dimensional hypercube. We continue with the notation of Section 4, Section 5 and Section 6.

Let $M = \mathbb{Z}^d$ and let $P = [0, 1]^d$. The Ehrhart polynomial and $h^*$-polynomial of $P$ are well-known. That is, $f_P(m) = (m + 1)^d$ and $h^*(t) = A(d; t) = \sum_{i=0}^{d-1} A(d, i)t^i$, where $A(d, i)$ is the number of permutations of $d$ elements with $i$ descents (see, for example, [1, p. 28]). The polynomial $tA(d, t)$ is the $d$th Eulerian polynomial and its coefficients are called Eulerian numbers.

The full symmetry group of $P$ is the Coxeter group of type $B_d$ consisting of all signed permutations of $d$ elements (cf. Example 8.8). More precisely, $B_d$ acts faithfully on $\mathbb{C}^d$ and its image in $GL(d)$ consists of all matrices with entries in $\{0, \pm 1\}$, and precisely one non-zero element in each row and column. In particular, we may view $\text{Sym}_d$ as a subgroup of $B_d$. With the notation of Section 4, observe that $B_d$ preserves the lattice $M$ and leaves $P$ invariant ‘up to translation’. Note that the diagonal matrix $(-1, 1, \ldots, 1)$ does not fix any lattice points, and hence Lemma 7.3 implies that $\varphi[t]$ is not a polynomial. We demonstrate this failure explicitly below for the square.

**Example 9.1.** Let $M = \mathbb{Z}^2$ and let $P$ be the convex hull of $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$. The group $G = B_2 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$ may be viewed as the
subgroup of Sym$_4$ generated by $\sigma = (1234)$ and $\tau = (12)(34)$. It has 5 conjugacy classes $\{1\}$, $\{\sigma, \sigma^3\}$, $\{\sigma^2\}$, $\{\tau, \tau \sigma^2\}$ and $\{\tau \sigma, \tau \sigma^3\}$. The group $G$ acts linearly on $M$ via

$$\sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

with corresponding 1-dimensional representation $\det(\rho)$. If $\epsilon$ denotes the restriction of the sign representation of Sym$_4$ to $G$, then

$$\varphi[t] = 1 + \epsilon \cdot \det(\rho)t + (1 - \epsilon \cdot \det(\rho)) \frac{t^2}{1 + t}.$$  

The subgroup Sym$_2 \subseteq B_2$ consists of the identity element and $\tau \sigma = (13)$. Observe that if we restrict to Sym$_2$, then $\varphi[t] = 1 + t$.

**Remark 9.2.** If one considers the corresponding action of $B_d$ on $2P$, then $\varphi[t]$ is effective and has an explicit geometric interpretation (Example 8.8).

For the remainder of the section, we consider the action of the symmetric group $G = \text{Sym}_d \subseteq B_d$. In this case, $P$ is invariant under the action of $G$, and $\rho : G \to \text{GL}(M)$ is the standard representation of the symmetric group [4]. If $g \in G$ has cycle type $(\mu_1, \ldots, \mu_r)$, then $P_g$ is isomorphic to an $r$-dimensional cube, and hence

$$\sum_{m \geq 0} f_{P_g}(m)t^m = \frac{A(r; t)}{(1 - t)^{r+1}} = \frac{A(r; t) \prod_i(1 + t + \cdots + t^{\mu_i-1})}{(1 - t) \prod_i(1 - t^{\mu_i})}.$$  

We conclude that

$$(7) \quad \varphi[t](g) = A(r; t) \prod_i(1 + t + \cdots + t^{\mu_i-1}).$$

Recall from Section 8 that the Coxeter fan $\Delta_{A_{d-1}}$ is the fan determined by the hyperplanes associated with the root system $A_{d-1}$, and Sym$_d$ acts on the cohomology $H^*(X_{A_{d-1}}; \mathbb{C})$ of the associated $(d-1)$-dimensional toric variety $X_{A_{d-1}} = X_{A_{d-1}}(\Delta_{A_{d-1}})$.

**Lemma 9.3.** With the notation above, the representation $\varphi_i$ is isomorphic to the representation of Sym$_d$ on $H^{2i}(X_{A_{d-1}}; \mathbb{C})$. In particular, the multiplicities of a fixed irreducible representation in the representations $\varphi_i$ form a symmetric, unimodal sequence.

**Proof.** This follows by comparing (7) with Corollary 8.4 using the fact that if $g \in G$ has cycle type $(\mu_1, \ldots, \mu_r)$, then $\dim M^g = r$ and $\det(I - \rho(g)t) = \prod_i(1 - t^{\mu_i})$. □
Remark 9.4. As explained to the author by Kalle Karu, Lemma 9.3 has the following geometric explanation. The \( d! \) smooth cones
\[
\left\{ \left( e_{w(1)}, e_{w(1)} + e_{w(2)}, \ldots, e_{w(1)} + e_{w(2)} + \cdots + e_{w(d)} \right) \mid w \in \text{Sym}_d \right\}
\]
form a smooth fan \( \Sigma \) supported on \( (\mathbb{R}^d_{\geq 0})^d \), and Proposition 8.1 implies that \( \varphi_i \) is isomorphic to the representation \( H^2_i(X(\Sigma); \mathbb{C}) \) on the cohomology of the associated toric variety \( X(\Sigma) \). Moreover, the projection \( \mathbb{R}^d \to \mathbb{R}^d/\mathbb{R}(1, \ldots, 1) \) is a morphism of fans from \( \Sigma \) to \( \triangle_{A_{d-1}} \) (see Example 8.7), and induces an \( \text{Sym}_d \)-equivariant isomorphism \( H^*(X(\Sigma)) \cong H^*(X_{A_{d-1}}) \).

Remark 9.5. The fact that \( \varphi[t] = t^{d-1} \varphi[t^{-1}] \) also follows the observation that \( 2P \) is a reflexive polytope and Corollary 6.9.

In fact, Stembridge proves that the representation \( H^*(X_{A_{d-1}}; \mathbb{C}) \) is an explicit graded permutation representation [39]. We briefly recall his decomposition of \( H^*(X_{A_{d-1}}; \mathbb{C}) \) into isotypic components, and refer the reader to [39, Section 4] for more details. Recall from Example 3.3 that the irreducible representations \( \chi^\lambda \) of \( \text{Sym}_d \) are indexed by partitions \( \lambda \) of \( d \). If \( D_\lambda \) denotes the Young diagram of \( \lambda \), then a tableau \( T \) with shape \( \lambda \) is a function \( T : D_\lambda \to \mathbb{N} \) such that \( T \) is weakly increasing along rows and strictly increasing down columns. If \( m_j(T) \) equals the number of times \( j \) appears in \( T \), then we say that \( T \) is admissible if \( S^+(T) := \{ j \in \mathbb{Z}_{>0} \mid m_j(T) > 0 \} = \{1,2,\ldots,k\} \) for some \( k \in \mathbb{N} \). A marked tableau is a pair \((T,f)\), where \( T \) is an admissible tableau and \( f : S^+(T) \to \mathbb{N} \) satisfies \( 1 \leq f(k) < m_j(T) \). The index of a marked tableau is \( \text{ind}(T,f) = \sum_{j \in S^+} f(j) \), and if \( \lambda = (d) \), then the pair \((T,0)\), where \( T : D_\lambda \to \mathbb{N} \) is the zero function, is a marked tableau of index zero. Observe that if \( m_j(T) = 1 \) for some \( j \in S^+(T) \), then there are no marked tableaux \((T,f)\). For example, the marked tableaux corresponding to partitions of 2, with indices 0 and 1 respectively, are
\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
\]
and the marked tableaux corresponding to partitions of 3, with indices 0, 1, 1, 2 and 1 respectively, are
\[
\left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right)
\]
Let \( P_\lambda(t) = \sum_{i=0}^{d-1} p_{i,\lambda} t^i \), where \( p_{i,\lambda} \) denotes the multiplicity of \( \chi^\lambda \) in \( H^{2i}(X_{A_{d-1}}; \mathbb{C}) \).
Theorem 9.6. [39, Theorem 4.2] With the notation above, for any partition \( \lambda \) of \( d \),

\[
P_{\lambda}(t) = \sum_{(T,f)} t^{\text{ind}(T,f)},
\]

where \((T,f)\) is summed over all marked tableaux of shape \( \lambda \).

Stembridge also used the above theorem to give a combinatorial proof that the coefficients of \( P_{\lambda}(t) \) are symmetric and unimodal. For \( d \leq 2r \), observe that there are no marked tableaux of shape \( \lambda = (d-r,1^r) \), and hence the corresponding irreducible representations \( \chi^{(d-r,1^r)} = \chi^{(d-1,1)} \) do not appear in \( H^*(X_{A_{d-1}}; \C) \). In particular, the sign representation \( \chi^{(1^d)} \) does not appear for \( d \geq 2 \) (cf. Remark 8.3). It is a result of Stembridge that \( P_{(d)}(t) = (1 + t)^d \) [40, Lemma 3.2], and a result of Lehrer that \( P_{(d-1,1)}(t) = (d-2)t(1+t)^{d-3} \) [23, Theorem 4.5].

We summarize the results of this discussion in the following proposition.

Proposition 9.7. If \( G = \text{Sym}_d \) acts on the \( M = \Z^d \) via the standard representation and \( P = [0,1]^d \), then \( \varphi_{i} \) is isomorphic to the representation of \( \text{Sym}_d \) on \( H^2(X_{A_{d-1}}; \C) \), where \( X_{A_{d-1}} \) is the \((d-1)\)-dimensional smooth, projective toric variety associated to the Coxeter complex of \( A_{d-1} \). Moreover, \( \varphi_{i} \) is a permutation representation, and if we write

\[
\varphi[t] = \sum_{|\lambda| = d} P_{\lambda}(t) \chi^\lambda,
\]

where \( \{ \chi^\lambda \mid |\lambda| = d \} \) are the irreducible representations of \( \text{Sym}_d \), then the coefficients of \( P_{\lambda}(t) = t^{d-1}P_{\lambda}(t^{-1}) \) are unimodal, and

\[
P_{\lambda}(t) = \sum_{(T,f)} t^{\text{ind}(T,f)},
\]

where \((T,f)\) is summed over all marked tableaux of shape \( \lambda \). In particular, \( P_{(d)}(t) = (1 + t)^d \) and \( P_{(d-1,1)}(t) = (d-2)t(1+t)^{d-3} \).

For example, \( \varphi[t] = 1 + t \) when \( d = 2 \), \( \varphi[t] = 1 + (2 + \chi^{(2,1)})t + t^2 \) when \( d = 3 \), and \( \varphi[t] = 1 + (3 + 2\chi^{(3,1)} + \chi^{(2,2)})t + (3 + 2\chi^{(3,1)} + \chi^{(2,2)}))t^2 + t^3 \) when \( d = 4 \).

Remark 9.8. If \( \lambda \) is a partition of \( n \), then an (admissible) tableau \( T \) of shape \( \lambda \) is semi-standard if \( S^+(T) = \{1, \ldots, n\} \), and it is well known that \( \dim \chi^\lambda \) equals the number of semi-standard tableaux of shape \( \lambda \). In particular, by considering dimensions of representations in Proposition 9.7, one obtains Stembridge’s refinement of the Eulerian numbers [39].
10. Applications to rational polytopes

In this section, we present an example to demonstrate how one can use the theory developed in the previous sections to explicitly describe the Ehrhart theory of certain classes of rational polytopes. We continue with the notation of Section 4, Section 5 and Section 6.

Recall that the characters \( \{ \chi_m P \}_{m \geq 0} \) encode the Ehrhart theory of the rational polytopes \( P_g = \{ u \in P \mid g \cdot u = u \} \) for all \( g \) in \( G \). More specifically, by Lemma 5.2, \( \chi_m P(g) = f_{P_g}(m) \), where \( f_{P_g}(m) \) is the Ehrhart quasi-polynomial of \( P_g \). In the case when \( P \) is a simplex, we have deduced an explicit formula for the generating series of \( f_{P_g}(m) \) (Proposition 6.1).

A rational polytope is a pseudo-integral polytope or PIP if its Ehrhart quasi-polynomial is a polynomial. The first examples of rational PIP’s in arbitrary dimension were found by De Loera and McAllister in [10]. This work was extended later by McAllister and Woods in [24], who found PIP’s with arbitrary denominator (see Section 2) in all dimensions. Below we use our techniques to construct a new family of PIP’s.

Let \( G = \text{Sym}_{2n} \) and consider the natural action of \( G \) on \( M = \mathbb{Z}^{2n}/\mathbb{Z}(1, \ldots, 1) \). That is, \( M_C = \chi^{(2n-1,1)} \) is the quotient of the standard representation (4) by a 1-dimensional invariant subspace. Let \( P \) be the standard reflexive simplex with vertices given by the images of the standard basis vectors \( e_1, \ldots, e_{2n} \) in \( \mathbb{Z}^{2n} \). It is well known that \( h^*(t) = 1 + t + \cdots + t^{2n-1} \), and hence Proposition 6.1 implies that \( \varphi[t] = 1 + t + \cdots + t^{2n-1} \). By Proposition 6.1 if \( g \in G \) has cycle type \((\mu_1, \ldots, \mu_r)\), then \( P_g \) is isomorphic to the \((r-1)\)-dimensional simplex with vertices given by the images of

\[
\frac{e_1 + \cdots + e_{\mu_1}}{\mu_1}, \frac{e_{\mu_1} + \cdots + e_{\mu_1 + \mu_2}}{\mu_2}, \ldots, \frac{e_{\mu_1 + \cdots + \mu_{r-1} + 1} + \cdots + e_{\mu_1 + \cdots + \mu_r}}{\mu_r},
\]

and

\[
\sum_{m \geq 0} f_{P_g}(m) t^m = \frac{1 + t + \cdots + t^{2n-1}}{\prod_i (1 - t^{\mu_i})}.
\]

For example, \( g = (12 \ldots n) \) has cycle type \((n, 1^n)\), and hence \( P_g \) is the \( n \)-dimensional rational polytope with denominator \( n \) and vertices given by the images of \( e_1 + \cdots + e_n \), \( e_{n+1}, \ldots, e_{2n} \).

Moreover,

\[
\sum_{m \geq 0} f_{P_g}(m) t^m = \frac{1 + t + \cdots + t^{2n-1}}{(1 - t^n)(1 - t)^n} = \frac{1 + t^n}{(1 - t)^{n+1}}.
\]
In particular, $P_g$ is a PIP with Ehrhart quasi-polynomial $f_{P_g}(m) = \binom{m+n}{n} + \binom{m}{n}$.

For $n \geq 2$, note that the coefficient of $t$ in the numerator $1 + t^n$ is strictly less than the coefficient of $t^n$, and hence the inequality (3) implies that $f_{P_g}(m)$ is not the Ehrhart polynomial of a $n$-dimensional lattice polytope.

11. Centrally symmetric polytopes

The goal of this section is to explicitly describe the equivariant Ehrhart theory of centrally symmetric polytopes. We continue with the notation of Section 4, Section 5 and Section 6.

Let $P$ be a $d$-dimensional lattice polytope in a lattice $M$ of rank $d$. Let $G = \mathbb{Z}/2\mathbb{Z}$ with generator $\sigma$, and let $\chi : G \rightarrow \mathbb{C}$ be the linear character sending $\sigma$ to $-1$, so that the irreducible representations of $G$ are $\{1, \chi\}$. The polytope $P$ is centrally symmetric if it is $G$-invariant with respect to the action of $G$ on $M$ in which $\sigma$ acts via the diagonal matrix $A = (-1, \ldots, -1)$.

First observe that the origin is an interior lattice point of $P$, and hence Corollary 7.10 implies that $\varphi[\chi] \equiv 0$. On the other hand, we claim that the polynomial $\varphi[\chi]$ is determined by the $h^*$-polynomial of $P$. Indeed, observe that $P_{\sigma} = \{0\}$, and hence, by Lemma 5.2

$$\sum_{m \geq 0} \chi_m P(\sigma) t^m = \sum_{m \geq 0} f_{P_g}(m) t^m = \frac{1}{1-t} = \frac{(1 + t)^d}{(1-t) \det(I-tA)}.$$}

It easily follows that

$$\varphi = \frac{h^*_i + \binom{d}{i}}{2} + \frac{h^*_i - \binom{d}{i}}{2} \chi.$$}

In particular, observe that the effectiveness of the representations $\varphi_i$ is equivalent to the lower bound $h^*_i \geq \binom{d}{i}$. The latter lower bound was proved by Bey, Henk and Wills in [5, Remark 1.6]. As they observe, the lower bound may be deduced from results of Betke and McMullen [4, Theorem 2] and Stanley [33] (cf. [5, Remark 1.6]), and equality is achieved when $P$ is the $d$-dimensional cross-polytope (see the discussion below). Hence we deduce an alternative proof of this lower bound. In conclusion, we have established the following result.

**Corollary 11.1.** With the notation above, if $G = \mathbb{Z}/2\mathbb{Z}$ and $P$ is a centrally symmetric polytope, then $\varphi[\chi]$ is effective, and the multiplicity of the trivial representation in $\varphi_i$ is at least $\binom{d}{i}$.
Similarly, the representations $\chi_{mP}$ are determined by the Ehrhart polynomial $f_P(m)$. Indeed, $L(m) = \chi_{mP}$ is the polynomial

$$L(m) = \frac{f_P(m) + 1}{2} + \frac{f_P(m) - 1}{2} \chi.$$ 

In order to compute some explicit examples, we recall the classification of non-singular, centrally symmetric, reflexive polytopes by Klyachko and Voskresenski˘ı in [22]. We note that these results have been extended by Ewald [13], Casagrande [9] and Nill [26]. Recall that a $d$-dimensional lattice polytope $P$ in $\mathbb{R}^d$ is reflexive if the origin is the unique interior lattice point of $P$ and every non-zero lattice point in $M$ lies in the boundary of $mP$ for some positive integer $m$. We say that $P$ is non-singular if, furthermore, the vertices of each facet of $P$ form a basis for $M$.

Let $\{e_1, \ldots, e_d\}$ be a lattice basis for $M$, and let $V(2k,d)$ be the lattice polytope with vertices $\{\pm e_i, \pm(e_1 + \cdots + e_{2k}) \mid 1 \leq i \leq d\}$ for $0 \leq 2k \leq d$. The polytope $V(0,d)$ is called the $d$-dimensional cross-polytope and, if $d$ is even, then $V(d,d)$ is called the Klyachko-Voskresenski˘ı polytope or KV-polytope.

**Theorem 11.2.** [22] The $d$-dimensional, non-singular, reflexive, centrally symmetric lattice polytopes are (up to unimodular transformation) precisely the polytopes $V(2k,d)$ for $0 \leq 2k \leq d$.

By Proposition 8.1 if $P$ is a non-singular, reflexive centrally symmetric polytope, then $\varphi[t]$ is a polynomial describing the representations of $\mathbb{Z}/2\mathbb{Z}$ on the graded pieces of the cohomology of the toric variety associated to the fan over the faces of $P$. In particular, the multiplicities of a fixed irreducible representation in the representations $\varphi_i$ form a symmetric, unimodal sequence. By [8], the unimodality of these sequences is equivalent to the inequalities

$$h_i^* - h_{i-1}^* \geq \binom{d}{i} - \binom{d}{i-1},$$

for $0 \leq i \leq \frac{d}{2}$, which may also be deduced from [33].

Our next goal is to compute the representations explicitly in this case. Since $V(2k,d)$ may be regarded as the free sum of $V(2k,2k)$ and $d-2k$ copies of the interval $[-1,1]$, it follows from Proposition 8.12 that

$$\varphi_{V(2k,d)}[t] = \varphi_{V(2k,2k)}[t](1 + t)^{d-2k}.$$ 

By the above discussion, it remains to compute the $h^*$-polynomial of a KV-polytope. If $\triangle$ denotes the fan over the faces of $V(2k,2k)$, then Remark 2.2 implies that

$$h_{V(2k,2k)}^*(t) = h_\triangle(t) = \sum_{i=0}^{2k} f_{i-1} t^i (1 - t)^{2k-i},$$
where \( f_{i-1} \) equals the number of faces of \( V(2k, 2k) \) of dimension \( i - 1 \), and \( f_{-1} = 1 \). It follows from Section 3 in [9] that \( f_{i-1} = \binom{2k+1}{i}2^i \) for \( 0 \leq i \leq k \), and \( \sum_i h_i^* = f_{d-1} = (2k + 1)\binom{2k}{k} \). For example, \( h_{V(2,2)}^*(t) = t^2 + 4t + 1 \) and \( h_{V(4,4)}^*(t) = t^4 + 6t^3 + 16t^2 + 6t + 1 \).

For any \( d \), let \( T_d(t) = \sum_{i=0}^{d} T(d, i) t^i \) be the symmetric polynomial of degree \( d \) with \( T(d, i) \) equal to the coefficient of \( t^i \) in \( \sum_{j=0}^{d} \binom{d+1}{j}2^i t^j (1-t)^{d-j} \) for \( 0 \leq i \leq \frac{d}{2} \). In particular, if \( d = 2k \), then \( T_d(t) = h_{V(2k,2k)}^*(t) \) and \( T_d(1) = (2k + 1)\binom{2k}{k} \). Here we set \( \binom{2k}{k} = 1 \) if \( k = 0 \). The following lemma shows that the coefficients \( T(d, i) \) may be interpreted as partial sums of rows of Pascal’s triangle.

**Lemma 11.3.** With the notation above, if \( i < \frac{d}{2} \), then

\[
T(d, i) = T(d - 1, i) + T(d - 1, i - 1),
\]

and if \( d = 2k \), then

\[
T(2k, k) = 2T(2k - 1, k - 1) + \binom{2k}{k}.
\]

In particular, \( T(d, i) = \sum_{j=0}^{i} \binom{d+1}{j} \) for \( 0 \leq i \leq \frac{d}{2} \).

**Proof.** For \( 0 \leq i < \frac{d}{2} \), \( T(d, i) \) is equal to the coefficient of \( t^i \) in

\[
\sum_{j=0}^{d} \binom{d}{j} + \binom{d}{d-j} 2^i t^j (1-t)^{d-j}.
\]

\[
= (1-t) \sum_{j=0}^{d-1} \binom{d}{j} 2^i t^j (1-t)^{d-1-j} + 2t \sum_{j=0}^{d-1} \binom{d}{j} 2^i t^j (1-t)^{d-1-j}
\]

\[
= (1+t) \sum_{j=0}^{d-1} \binom{d}{j} 2^i t^j (1-t)^{d-1-j}.
\]

Hence \( T(d, i) = T(d - 1, i) + T(d - 1, i - 1) \), as desired. It follows that if \( d \) is odd, then \( T_d(1) = 2T_{d-1}(1) \), and if \( d = 2k \) is even, then \( T_{2k}(1) = 2T_{2k-1}(1) - 2T(2k - 1, k - 1) + T(2k, k) \). On the other hand, by the above discussion, if \( d = 2k \), then \( T_{2k}(1) = (2k + 1)\binom{2k}{k} \), and we deduce the equality

\[
T(2k, k) = 2T(2k - 1, k - 1) + (2k + 1)\binom{2k}{k} - 4(2k - 1)\binom{2k-2}{k-1}.
\]

One verifies that the latter two terms sum to \( \binom{2k}{k} \). The final statement now follows by a simple induction argument. \( \square \)
Using the above lemma and induction, we deduce the following expressions for the $h^*$-polynomials of the KV-polytopes

$$h^*_V(2k, 2k)(t) = \sum_{i=0}^{k} \binom{2k + 1}{i} (t^i + \cdots + t^{2k-i}) = \sum_{i=0}^{k} \binom{2i}{i} t^i (1 + t)^{2(k-i)}.$$

We summarize the results of this discussion in the following proposition.

**Proposition 11.4.** The $d$-dimensional, non-singular, reflexive, centrally symmetric lattice polytopes are (up to unimodular transformation) precisely the polytopes $V(2k, d)$ for $0 \leq 2k \leq d$, with $h^*$-polynomials

$$h^*_V(2k, d)(t) = \sum_{i=0}^{k} \binom{2i}{i} (1 + t)^{d-2i},$$

and

$$\varphi[t] = \frac{h^*_V(2k, d)(t) + (1 + t)^d}{2} + \frac{h^*_V(2k, d)(t) - (1 + t)^d}{2} \chi.$$

12. OPEN QUESTIONS AND CONJECTURES

We end the paper by presenting some open problems and directions for future research. We continue with the notation of Section 4, Section 5, and Section 6.

Recall that the power series $\varphi[t]$ is effective if all the virtual representations $\varphi_i$ are effective representations. The main open problem is to characterize when $\varphi[t]$ is effective. We have seen that if the toric variety $Y$ and ample line bundle $L$ associated to the $G$-invariant lattice polytope $P$ admit a $G$-invariant non-degenerate hypersurface, then $\varphi[t]$ is effective, and, in particular, $\varphi[t]$ is a polynomial (Theorem 7.7). We offer the following conjecture.

**Conjecture 12.1.** With the notation above, the following conditions are equivalent

- $(Y, L)$ admits a $G$-invariant non-degenerate hypersurface,
- $\varphi[t]$ is effective,
- $\varphi[t]$ is a polynomial.

Observe that the equivalence of the last two conditions in the above conjecture holds in dimension 2 by Corollary 6.7.

We have seen that if $P$ is a simplex (Proposition 6.1), or if $G = \text{Sym}_d$ and $P = [0, 1]^d$ (Proposition 9.7), then the representations $\varphi_i$ are permutation representations. When $\varphi[t]$ is effective, the ungraded character $\varphi[1] = \sum_i \varphi_i$ has non-negative integer values (Proposition 6.4), and if $G$ is cyclic of prime order, then this guarantees that $\varphi[1]$ is a permutation representation. Moreover, Stembridge proved that the $\varphi[1]$ is a permutation representation in the example of Corollary 8.3.
(Remark 8.3). On the other hand, \( \varphi_1 \) need not be a permutation representation, even when \( \varphi[t] \) is effective (Example 6.11).

**Conjecture 12.2.** If \( \varphi[t] \) is effective, then \( \varphi[1] \) is a permutation representation.

If \( \varphi[t] \) is not a polynomial, then one can still define a rational character \( \varphi[1] \) (Lemma 6.3). We conjecture that \( \varphi[1] \) in fact takes (non-negative) integer values (cf. Proposition 6.4).

**Conjecture 12.3.** For any \( g \) in \( G \),

\[
\varphi[1](g) = \frac{\dim(P_g)! \vol(P_g) \det(I - \rho(g))_{(M^g)^\perp}}{\ind(P_g)}
\]

is a non-negative integer.

We also offer the following conjecture on the appearance of the trivial representation.

**Conjecture 12.4.** If \( \varphi[t] \) is a polynomial and \( h_i^* > 0 \), then the trivial representation occurs with non-zero multiplicity in \( \varphi_i \).

It may also be interesting to consider the above conjecture in the special case when \( P \) contains a \( G \)-fixed lattice point in its interior (in this case, \( h_i > 0 \) for \( 0 \leq i \leq d \)).

The conjecture holds in all examples presented in this paper. In particular, it holds when \( P \) is a simplex by Proposition 6.1, and when \( \dim P = 2 \) by Corollary 6.6 and Corollary 6.7.

Lastly, we list a number of possible directions for future research.

- Resolve the conjectures and questions above when \( G = \mathbb{Z}/2\mathbb{Z} \).
- Consider the lower degree coefficients \( L_i(m) \) in Section 5. Can one say something about their minimal periods, even in the case when \( P \) is a simplex?
- What can one say about the asymptotic behavior of \( \varphi_{mP}[t] \) for \( m \) sufficiently large and divisible (cf. Corollary 7.14)?
- What can be deduced from the equivariant Riemann-Roch formula [16, Appendix I]? This was suggested to the author by Roberto Paoletti.
- Can one develop a natural equivariant version of weighted Ehrhart theory [37]? 
- Describe the equivariant Ehrhart theory of the permutahedron (cf. [29]).
- Compute the representations \( \varphi[t] \) when \( G \) is the full symmetry group of the polytopes appearing in Proposition 11.4.
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