A Finite Rank Bundle over $J$-Holomorphic map Moduli Spaces

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Abstract

We study a finite rank bundle $F$ over a neighborhood of $J$-Holomorphic map Moduli Spaces, prove the exponential decay of the derivative of the gluing maps for $F$ with respect to the gluing parameter.

1 Introduction and Preliminary

In [10], [6] and [2] the authors introduced a finite rank bundle $F$ over a neighborhood of $J$-Holomorphic map Moduli Spaces. This bundle plays an important role in the study of the Gromov-Witten theory and the relative Gromov-Witten theory. In this paper we study some local analysis properties of this bundle.

Let $(M, \omega, J)$ be a closed $C^\infty$ symplectic manifold of dimension $2m$ with $\omega$-tame almost complex structure $J$, let $(\Sigma, j, y)$ be a smooth Riemann surface of genus $g$ with $n$ marked points with $n > 2 - 2g$. We fix a local coordinate system $\psi : U \to A$ for the Teichmüller space $T_{g,n}$, where $U \subset T_{g,n}$ is an open set. Let $a_o = (j_o, y_o) \in A$, $u : \Sigma \to M$ be a $(j_o, J)$-holomorphic map. Then $F$ can be viewed locally as a bundle over $A \times W^{k,2}(\Sigma, u^*TM)$, denoted by $\bar{F}$. In [4] we study the smoothness of $\bar{F}$.

In [4] and [6] we study the gluing theory for $\bar{F}$. Let $(\Sigma, j, y, q)$ be a marked nodal Riemann surface with one nodal point $q$. We write $\Sigma = \Sigma_1 \cup \Sigma_2$. Let $u = (u_1, u_2)$, where $u_i : \Sigma_i \to M$ is a $(j_i, J)$-holomorphic map. We glue $\Sigma$ and $u$ at $q$ with gluing parameter $(r, \tau) := (r)$ to get $\Sigma(\tau)$ and pregluing map $u(\tau) : \Sigma(\tau) \to M$. We have a gluing map from $\bar{F}|_u$ to $\bar{F}|_{u(\tau)}$. We prove the exponential decay of the derivatives of the gluing maps with respect to the gluing parameter.

1.1 Metrics on $\Sigma$

Let $(\Sigma, j, y)$ be a smooth Riemann surface of genus $g$ with $n$ marked points. In this paper we assume that $n > 2 - 2g$, and $(g, n) \neq (1, 1), (2, 0)$. It is well-known that there is a unique complete hyperboloc metric $g_0$ in $\Sigma \setminus \{ y \}$ of constant curvature $-1$ of finite volume, in the given conformal class $j$ (see [12]). Let $\mathbb{H} = \{ \zeta = \lambda + \sqrt{-1} \mu | \mu > 0 \}$ be the half upper plane with the Poincare metric

$$g_0(\zeta) = \frac{1}{(Im(\zeta))^2}d\zeta d\bar{\zeta}.$$ 

Let

$$D = \{ \zeta \in \mathbb{H} | Im(\zeta) \geq 1 \}$$

$$\zeta \sim \zeta + 1$$

be a cylinder, and $g_0$ induces a metric on $D$, which is still denoted by $g_0$. Let $z = e^{2\pi i \zeta}$, through which we identify $D$ with $D(e^{-2\pi}) := \{ z | |z| < e^{-2\pi} \}$. An important result is that for any punctured point $y_i$ there exists a neighborhood $O_i$ of $y_i$ in $\Sigma$ such that

$$(O_i \setminus \{ y_i \}, g_0) \cong (D(e^{-2\pi}) \setminus \{ 0 \}, g_0),$$
For every node \( p \) marked nodal surfaces with another metric \( g \) (resp. \( O \) moreover, all \( O \)'s are disjoint with each other. Then we can view \( D_{g_i}(e^{-2\pi}) \) as a neighborhood of \( y_i \) in \( \Sigma \) and \( z \) is a local complex coordinate on \( D_{g_i}(e^{-2\pi}) \) with \( z(y_i) = 0 \). For any \( c > 0 \) denote
\[
D(c) = \bigcup D_{g_i}(c), \quad \Sigma(c) = \Sigma \setminus D(c).
\]

Let \( g' = dzd\bar{z} \) be the standard Euclidean metric on each \( D_{g_i}(e^{-2\pi}) \). We fix a smooth cut-off function \( \chi(|z|) \) to glue \( g_0 \) and \( g' \), we get a smooth metric \( g \) in the given conformal class \( j \) on \( \Sigma \) such that
\[
g = \begin{cases} 
  g_0 & \text{on } \Sigma \setminus D(e^{-2\pi}), \\
  g' & \text{on } D\left(\frac{1}{2}e^{-2\pi}\right).
\end{cases}
\]

Let \( g^c = ds^2 + d\theta^2 \) be the cylinder metric on each \( D_{g_i}(e^{-2\pi}) \), where \( z = e^{s+2\pi\sqrt{-1}\theta} \). We also define another metric \( g^\circ \) on \( \Sigma \) as above by glue \( g_0 \) and \( g^c \), such that
\[
g^\circ = \begin{cases} 
  g_0 & \text{on } \Sigma \setminus D(e^{-2\pi}), \\
  g^c & \text{on } D\left(\frac{1}{2}e^{-2\pi}\right).
\end{cases}
\]

The metric \( g \) (resp. \( g^\circ \)) can be generalized to marked nodal surfaces in a natural way. Let \((\Sigma, j, y)\) be a marked nodal surfaces with \( \epsilon \) nodal points \( p = (p_1, \cdots, p_\epsilon) \). Let \( \sigma : \tilde{\Sigma} = \bigcup_{\nu=1}^\epsilon \Sigma_\nu \to \Sigma \) be the normalization. For every node \( p_i \) we have a pair \( \{a_i, b_i\} \). We view \( a_i, b_i \) as marked points on \( \tilde{\Sigma} \) and define the metric \( g_\nu \) (resp. \( g^\circ_\nu \)) for each \( \Sigma_\nu \). Then we define
\[
g := \bigoplus_{\nu} g_\nu, \quad g^\circ := \bigoplus_{\nu} g^\circ_\nu.
\]

### 1.2 Teichmüller space

Denote by \( \mathcal{J}(\Sigma) \subset \text{End}(T\Sigma) \) the manifold of all \( C^\infty \) complex structures on \( \Sigma \). Denote by \( \text{Diff}^+(\Sigma) \) the group of orientation preserving \( C^\infty \) diffeomorphisms of \( \Sigma \), by \( \text{Diff}_0^+(\Sigma) \) the identity component of \( \text{Diff}^+(\Sigma) \). \( \text{Diff}^+(\Sigma) \) acts on \( \mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta) \) by
\[
\phi(j, y) = (\{d\phi_x\}^{-1}J_{\phi(x)}d\phi_x, \phi^{-1}y)
\]
for all \( \phi \in \text{Diff}^+(\Sigma), x \in \Sigma \), where \( \Delta \subset \Sigma^n \) denotes the fat diagonal. Put
\[
P := \mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta).
\]

The orbit spaces are
\[
\mathcal{M}_{g,n} = (\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)) / \text{Diff}^+(\Sigma), \quad T_{g,n} = (\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)) / \text{Diff}_0^+(\Sigma).
\]

\( \mathcal{M}_{g,n} \) is called the Deligne-Mumford space, \( T_{g,n} \) is called the Teichmüller space.

Consider the principal fiber bundle \( \text{Diff}_0^+(\Sigma) \to P \to T_{g,n} \) and the associated fiber bundle
\[
\pi_T : Q := P \times_{\text{Diff}_0^+(\Sigma)} \Sigma \to T_{g,n}.
\]

The following result is well-known (cf [13]):
Lemma 1.1. Suppose that \( n + 2g \geq 3 \). Then for any \( \gamma_0 = [(j_0, y_0)] \in T_{g,n} \) and any \((j_o, y_o) \in \mathbb{P} \) with 
\[ \pi_T(j_0, y_0) = \gamma_0 \]
there is an open neighborhood \( A \) of zero in \( \mathbb{C}^{3g-3+n} \) and a local holomorphic slice \( \iota = (\iota_0, \ldots, \iota_n) : A \to \mathbb{P} \) such that 
\[ u_0(o) = j_0, \quad \iota_i(o) = y_{i\alpha}, \quad i = 1, \ldots, n, \]  
\[ (1.1) \]
and the map 
\[ A \times \text{Diff}_0(\Sigma) \to \mathbb{P} : (a, \phi) \mapsto (\phi^*(\iota_0(a)), \phi^{-1}(\iota_1(a)), \ldots, \phi^{-1}(\iota_n(a))) \]
is a diffeomorphism onto a neighborhood of the orbit of \((j_o, y_o)\).

From the local slice we have a local coordinate chart on \( U \) and a local trivialization on \( \pi_T^{-1}(U) \):
\[ \psi : U \to A, \quad \Psi : \pi_T^{-1}(U) \to A \times \Sigma, \]  
\[ (1.2) \]
where \( U \subset T_{g,n} \) is a open set. We call \((\psi, \Psi)\) in \([12]\) a local coordinate system for \( Q \). Suppose that we have two local coordinate systems
\[ (\psi, \Psi) : (O, \pi_T^{-1}(O)) \to (A, A \times \Sigma), \]  
\[ (1.3) \]
\[ (\psi', \Psi') : (O', \pi_T^{-1}(O')) \to (A', A' \times \Sigma). \]  
\[ (1.4) \]
Suppose that \( O \cap O' \neq \emptyset \). Let \( W \) be a open set with \( W \subset O \cap O' \). Denote \( V = \psi(W) \) and \( V' = \psi'(W) \). Then ( see \([13]\))

Lemma 1.2. \( \psi' \circ \psi^{-1}|_V : V \to V' \) and \( \Psi' \circ \Psi^{-1}|_V : V \times \Sigma \to V' \times \Sigma \) are holomorphic.

1.3 J-holomorphic maps

Let \((M, \omega, J)\) be a closed \( C^\infty \) symplectic manifold of dimension \( 2m \) with \( \omega \)-tame almost complex structure \( J \), where \( \omega \) is a symplectic form. Then there is a Riemannian metric 
\[ G_J(v, w) := < v, w >_J := \frac{1}{2} (\omega(v, Jw) + \omega(w, Jv)) \]  
\[ (1.5) \]
for any \( v, w \in TM \). Following \([5]\) we choose the complex linear connection 
\[ \tilde{\nabla}_X Y = \nabla_X Y - \frac{i}{2} J(\nabla_X J) Y \]
induced by the Levi-Civita connection \( \nabla \) of the metric \( G_J \).

Let \((\Sigma, j, y)\) be a marked nodal Riemann surface of genus \( g \) with \( n \) marked points. Let \( \sigma : \tilde{\Sigma} = \sum_{\nu=1}^n \Sigma_\nu \to \Sigma \) be the normalization. Let \( u : \Sigma \to M \) be a smooth map. Here and later we say a map ( or section ) is smooth we mean that it is a continuous map such that, restricting to every \( \Sigma_\nu \), it is smooth. The map \( u \) is called a \((j, J)\)-holomorphic map if, restricting to each \( \Sigma_\nu \), \( du \circ j = J \circ du \). Alternatively
\[ \tilde{\partial}_{j,J}(u) := \frac{1}{2} (du + J(du) du \circ j) = 0. \]
\[ (1.6) \]
Given \( A \in H_2(M, \mathbb{Z}) \). Let \( u : \Sigma \to M \) be \((j_0, J)\)-holomorphic map with \( u([\Sigma]) = A \). Set \( b_0 = (s_0, u) \), \( s_0 = (j_0, y_0) \). Let \( A = A_1 \times A_2 \times \ldots \times A_k \) be a local coordinate system of complex structures on \( \Sigma \) such that \( s_0 \in A \). Denote by \( j_s \) the complex structure corresponding to \( s = (j, y) \in A \). Let \( \alpha \) be a small constant such that \( 0 < \alpha < 1 \). For any section \( h \in C^\infty(\Sigma; u^*TM) \) and section \( \eta \in C^\infty(\Sigma, u^*TM \otimes \wedge^1_J T^*\Sigma) \) and given integer \( k > 4 \) we define the norms \( \|h\|_{j, k, 2, \alpha} \) and \( \|\eta\|\gamma, k-1, 2, \alpha \) ( see \([3]\) ). Denote by \( W^{k,2,\alpha}(\Sigma; u^*TM) \) and
$W^{k,2,\alpha}(\Sigma, u^*TM \otimes \Lambda^{0,1}_j T^*\Sigma)$ the complete spaces with respect to the norms $\|h\|_{j,k,2,\alpha}$ and $\|\eta\|_{j,k-1,2,\alpha}$ respectively. We can also define $\mathcal{W}_u^{k,2,\alpha}(\Sigma; u^*TM)$ as in [3]. Let

$$\tilde{B} = \{ u \in W^{k,2,\alpha}(\Sigma, M) \mid u_*([\Sigma]) = A \}.$$  

For fixed $s_o$, restricting to each $\Sigma_\nu$, $\tilde{B}$ is an infinite dimensional Banach manifold. Let $\delta > 0$, $\rho > 0$ be two small numbers. Denote

$$\tilde{O}_{s_o}(\delta, \rho) := \{ (s, v) \in A \times \tilde{B} \mid d_A(s, s) < \delta, \|h\|_{j,k,2,\alpha} < \rho \},$$

where $v = \exp_{t_u} h$, $d_A$ is the distance function induced by the Weil-Petersson metric on the Deligne-Mumford space $\mathcal{M}_{g,n}$.

## 2 A finite rank bundle and weighted norms

### 2.1 A finite rank bundle

We slightly deform $\omega$ to get a rational class $[\omega^*]$. By taking multiple, we can assume that $[\omega^*]$ is an integral class on $M$. Therefore, it is the Chern class of a complex line bundle $L$ over $M$. Let $i$ be the complex structure on $L$. We choose a Hermitian metric $G_L$ and the associate unitary connection $\nabla^L$ on $L$.

Let $(\Sigma, j, y)$ be a marked nodal Riemann surface of genus $g$ with $n$ marked points. Let $u : \Sigma \rightarrow M$ be a $\mathcal{W}_u^{k,2,\alpha}$ map. We have a complex line bundle $u^*L$ over $\Sigma$ with complex structure $u^*i$ and unitary connection $u^*\nabla^L$. Put $b = (s, u)$, $s = (j, y)$. The unitary connection $u^*\nabla^L$ splits into $u^*\nabla^L := u^*\nabla^L_{(0,1)} \oplus u^*\nabla^L_{(0,1)}$. We can define the spaces $\mathcal{W}_u^{k,2,\alpha}(\Sigma, u^*L)$ and $W^{k-1,2,\alpha}(\Sigma, u^*L \otimes \Lambda^{0,1}_j T^*\Sigma)$ as in [3] (see also section 2.2).

Denote

$$D^L := u^*\nabla^L_{(0,1)} : \mathcal{W}_u^{k,2,\alpha}(\Sigma, u^*L) \rightarrow W^{k-1,2,\alpha}(\Sigma, u^*L \otimes \Lambda^{0,1}_j T^*\Sigma).$$

One can check that

$$D^L(f\xi) = \bar{\partial} j f \otimes \xi + f \cdot D^L\xi.$$  

$D^L$ determines a holomorphic structure on $u^*L$, for which $D^L$ is an associated Cauchy-Riemann operator (see [8, 9]). Then $u^*L$ is a holomorphic line bundle.

Let $\Sigma$ be a smooth Riemann surface. Let $\{V\}$ be a covering of $\Sigma$ such that each $V \subset \Sigma$ is a trivializing open set of $u^*L$. $D^L$ becomes in each $V$

$$D^L f = \frac{\partial}{\partial z} f + a_V f.$$  

(2.1)

Consider the PDE

$$\frac{\partial}{\partial \bar{z}} f + a_V f = 0.$$  

(2.2)

We can find a nonvanishing solution $e_V$ of (2.2). (see [8, 9]). Then $\{e_V\}$ define a holomorphic structure on $u^*L$ such that $D^L$ is $\bar{\partial}_j$.

Now let $\Sigma$ be nodal Riemann surface. For every smooth component $\Sigma_\nu$, we have a holomorphic structure on $u^*L$ over $\Sigma_\nu$. Suppose that $p$ is a node of $\Sigma_1$ and $\Sigma_2$. We choose nonvanishing solutions $e_{V_i}$ of (2.2), where $V_i \subset \Sigma_i$. Since (2.2) is a linear equation, we can choose $e_{V_i}$ such that $e_{V_i}(p) = e_{V_2}(p)$. Then we have a holomorphic structure on $u^*L$ over $\Sigma$.
Let $\lambda_{(\Sigma, j)}$ be the dualizing sheaf of meromorphic 1-form with at worst simple pole at the nodal points and for each nodal point $p$, say $\Sigma_1$ and $\Sigma_2$ intersects at $p$,

$$\text{Res}_p(\lambda_{(\Sigma_1, j_1)}) + \text{Res}_p(\lambda_{(\Sigma_2, j_2)}) = 0.$$ 

Let $\Pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal curve. Let $\lambda$ be the relative dualizing sheaf over $\overline{\mathcal{C}}_g$, the restriction of $\lambda$ to $(\Sigma, j)$ is $\lambda_{(\Sigma, j)}$.

Set $\Lambda_{(\Sigma, j)} := \lambda_{(\Sigma, j)} \left( \sum_{i=1}^n y_i \right)$. $\Lambda |_{\mathcal{E}_{g,n}}$ is a line bundle over $\mathcal{E}_{g,n}$. Let $(\psi, \Psi) : (O, \pi_1^{-1}(O)) \to (A, A \times \Sigma)$ be a local coordinate systems, where $O \subset T_{g,n}$ is an open set. $\Lambda$ induces a line bundle over $A \times \Sigma$, denoted by $\Lambda$. Then $\tilde{L} |_b := \mathcal{P}^* \Lambda \otimes u^*L$ is a holomorphic line bundle over $\Sigma$, where $\mathcal{P}$ denote the forgetful map. We have a Cauchy-Riemann operator $\bar{\partial}_b$. Then $H^0(\Sigma, \tilde{L} |_b)$ is the ker $\bar{\partial}_b$. Here the $\bar{\partial}$-operator depends on the complex structure $j$ on $\Sigma$ and the bundle $u^*L$, so we denote it by $\bar{\partial}_b$.

If $\Sigma_\nu$ is not a ghost component, there exist a constant $h_o > 0$ such that

$$\int_{u|_{\Sigma_\nu}} \omega^* > h_o.$$ 

Therefore, $c_1(u^*L)(\Sigma_\nu) > 0$. For ghost component $\Sigma_\nu$, $\lambda_{(\Sigma, j)} \left( \sum_{i=1}^n y_i \right)$ is positive. So for any $b = (s, v) \in \tilde{O}_{\nu_0}(\delta, \rho)$ by taking the higher power of $\tilde{L} |_b$, if necessary, we can assume that $\tilde{L} |_b$ is very ample. Hence, $H^1(\Sigma, \tilde{L} |_b) = 0$. Therefore, $H^0(\Sigma, \tilde{L} |_b)$ is of constant rank ( independent of $b \in \tilde{O}_{\nu_0}(\delta, \rho)$). We have a finite rank bundle $\mathcal{F}$ over $\tilde{O}_{\nu_0}(\delta, \rho)$, whose fiber at $b = (j, y, v) \in \tilde{O}_{\nu_0}(\delta, \rho)$ is $H^0(\Sigma, \tilde{L} |_b)$.

**Remark 2.1.** In [10], [6] and [2] the authors constructed a finite rank bundle $\mathcal{F}$ over a neighborhood of $J$-Holomorphic map Moduli Spaces. In this paper we study the local analysis properties, so we only give the local construction here.

### 2.2 Weighted norms

Let $(V, z)$ be a local coordinate system on $\Sigma$ around a nodal point ( or a marked point) $q$ with $z(q) = 0$. Let $b = (s, u) \in \tilde{O}_{\nu_0}(\delta_0, \rho_0)$ and $e$ be a local holomorphic section of $u^*L|_V$ with $\|e\|_{G^L}(q) \neq 0$ for $q \in V$. Then for any $\phi \in \mathcal{F}|_b$ we can write

$$\phi|_V = f \left( \left( \frac{dz}{z} \otimes e \right)^p \right), \quad \text{where } f \in \mathcal{O}(V), \quad p \in \mathbb{Z}. \quad (2.3)$$

In terms of the holomorphic cylindrical coordinates $(s, t)$ defined by $z = e^{s+2\pi \sqrt{-1}t}$ we re-written (2.3) as

$$\phi(s, t)|_V = f(s, t) \left( (ds + 2\pi \sqrt{-1}dt \otimes e)^p \right),$$

where $f(z) \in \mathcal{O}(V)$. It is easy to see that $|f(s, t) - f(-\infty, t)|$ uniformly exponentially converges to 0 with respect to $t \in S^1$ as $|s| \to \infty$.

The metrics $G^L$ and $g^\infty$ together induce a metric $G$ on $\tilde{L}$. We define weighted norms for $C^\infty_c(\Sigma, \tilde{L}|_b)$ and $C^\infty_c(\Sigma, \tilde{L}|_b \otimes \Lambda_j^{0,1} T^* \Sigma)$. Fix a positive function $W$ on $\Sigma$ which has order equal to $e^{\alpha|s|}$ on each end of $\Sigma_i$, where $\alpha$ is a small constant such that $0 < \alpha < 1$. For any $\zeta \in C^\infty_c(\Sigma, \tilde{L}|_b)$ and any section $\eta \in C^\infty_c(\Sigma, \tilde{L}|_b \otimes \Lambda_j^{0,1} T^* \Sigma)$ we define the norms

$$\left\| \zeta \right\|_{j,k,2,\alpha} = \left( \int_{\Sigma} e^{2\alpha|s|} \sum_{i=0}^k |\nabla^i \zeta|^2 dv_{\text{col}}(\Sigma) \right)^{1/2}, \quad (2.4)$$
First we recall a fact about the exponential map on a compact Riemannian manifold. Remark 10.5.5). There are two smooth families of endomorphisms $W_{k-1,2,\alpha}^k$.

Then for $\tilde{\mathcal{F}}$ on top strata $F$ on top strata.

We choose $R_0$ so large that $u_t(\{s_i \geq \frac{R}{2}\})$ lie in $O_{u_t}(q)$ for any $r > R_0$. In this coordinate system we identify $\tilde{T}_b M$ with $T_{u_t(q)} M$ for all $x \in O_{u_t}(q)$. With respect to the base $(e \otimes \frac{dz}{z})^p$ for $\tilde{L}_b$ we have a local trivialization. Any $\zeta_0 \in \tilde{L}_b(q)$ may be considered as a vector field in the coordinate neighborhood. We fix a smooth cutoff function $g$:

$$g(s) = \begin{cases} 1, & \text{if } |s| \geq \tilde{d} \\ 0, & \text{if } |s| \leq \frac{\tilde{d}}{2} \end{cases}$$

where $\tilde{d}$ is a large positive number. Put $\hat{\zeta}_0 = g\zeta_0$.

Then for $\tilde{d}$ large enough $\hat{\zeta}_0$ is a section in $C^\infty(\Sigma; \tilde{L}_b)$ supported in the tube $\{s, t||s| \geq \frac{\tilde{d}}{2}, t \in S^1\}$. Denote

$$\mathcal{W}^{k,2,\alpha}(\Sigma; \tilde{L}_b) = \left\{\zeta + \hat{\zeta}_0|\zeta \in \mathcal{W}^{k,2,\alpha}(\Sigma; \tilde{L}_b), \zeta_0 \in \tilde{L}_b(q)\right\}.$$  

We define weighted Sobolev norm on $\mathcal{W}^{k,2,\alpha}$ by

$$\|\zeta + \hat{\zeta}_0\|_{\mathcal{W}^{j,k,2,\alpha}} = \|\zeta\|_{j,k,2,\alpha} + |\zeta_0|,$$

where $|\zeta_0| = [G(\zeta_0, \xi_0)_\alpha]\frac{\tilde{d}}{2}$.

Let $b = (s, u)$. We define a Cauchy-Riemann operator

$$D\tilde{L}_b : \mathcal{W}^{k,2,\alpha}(\Sigma; \tilde{L}_b) \rightarrow \mathcal{W}^{k-1,2,\alpha}(\Sigma; \tilde{L}_b \otimes \wedge_{j}^{0,1} T^* \Sigma) \text{ by}$$

$$D\tilde{L}_b(f(k \otimes e)^p) = (\tilde{\partial} f)(k \otimes e)^p + (f)(k \otimes D^L e) \otimes (k \otimes e)^p - 1,$$  

where $k$ is a local frame field of $\tilde{\Lambda}$, $f(k \otimes e)^p \in \mathcal{W}^{k,2,\alpha}(\Sigma; \tilde{L}_b)$.

With respect to the holomorphic structure $\{e_{\nu}\}$ we have $D\tilde{L}_b = \tilde{\partial}_b$. The linearized operator of $D\tilde{L}_b$ is also $\tilde{\partial}_b$.

## 3 Smoothness of $\tilde{F}$ on top strata

Let $(\Sigma, j, y)$ be a smooth Riemann surface of genus $g$ with $n$ marked points. Let $b_0 = (a_o, u_o) = (j_o, y_o, u_o)$, $b = (a, u), u = \exp_{u_o} h, b \in \tilde{Q}_{b_0}(\delta_0, \rho_0)$.

### 3.1 Smoothness of $\tilde{F}(h, \xi)$

First we recall a fact about the exponential map on a compact Riemannian manifold $M$ (see [5], Page 362, Remark 10.5.5). There are two smooth families of endomorphisms

$$E_i(p, \xi) : T_p M \rightarrow T_{\exp_p \xi} M, \quad i = 1, 2,$$
that are characterized by the following property. Let $\gamma : \mathbb{R} \to M$ be any smooth path in $M$ and $v(t) \in T_{\gamma(t)}M$ be any smooth vector field along this path then the derivative of the path $t \to \exp_{\gamma(t)}(v(t))$ is given by the formula

$$
\frac{d}{dt} \exp_{\gamma}(v) = E_1(\gamma, v) \dot{\gamma} + E_2(\gamma, v) \nabla_t v,
$$

where $\dot{\gamma} = \frac{d\gamma}{dt}$. We have

$$
E_1(p, 0) = E_2(p, 0) = Id : T_pM \to T_pM, \quad \forall p \in M,
$$

and $E_i(p, \xi)$ are uniformly invertible for sufficiently small $\xi$. Since $M$ is compact, there exists a constant $\epsilon$ such that for any $p \in M$ and $\xi \in T_pM$ with $|\xi|_{T_pM} \leq \epsilon$, $E_i(p, \xi)$ are uniformly invertible.

Given $x \in M$ and $\zeta \in T_xM$ we define two linear maps

$$
E_x(\zeta) : T_xM \xrightarrow{T_{\exp_x(\zeta)}M} L_{\exp_x(\zeta)}
$$

and

$$
\Psi_x(\zeta) : T_xM \times T_xM \to L_{\exp_x(\zeta)}
$$

Choose a local coordinate system $x_1, \ldots, x_{2m}$ on $M$, denote $\frac{\partial}{\partial x_i} = \partial_{x_i}$. Let $\xi$ be a smooth section of the bundle $L$ over a neighborhood $U_\alpha$ of $u_\alpha(\Sigma)$. Let $u = \exp_{u_\alpha} h$. Let $\Phi^{L}_{u_\alpha,u}$ be the parallel transport with respect to the connection $\nabla^L$, along the geodesics $s \to \exp_{u_\alpha}(sh)$. $\Phi^{L}_{u_\alpha,u}$ induce two isomorphisms

$$
W^{k,2,\alpha}(\Sigma, u^*_\alpha L) \to W^{k,2,\alpha}(\Sigma, u^* L), \quad W^{k-1,2,\alpha}(\Sigma, u^*_\alpha L \otimes \wedge^1 \Sigma) \to W^{k-1,2,\alpha}(\Sigma, u^* L \otimes \wedge^1 \Sigma),
$$

still denote them by $\Phi^{L}_{u_\alpha,u}$. Denote $u_t = \exp_{u_\alpha}(h + th')$. We calculate $(u^*_t \nabla^L)_t (\Phi^{L}_{u_\alpha,u} u^*_\alpha \xi)|_{t=0}$. By definition, for any $p \in \Sigma$,

$$
(u^*_t \nabla^L)_t (\Phi^{L}_{u_\alpha,u} u^*_\alpha \xi)(p)|_{t=0} = \nabla^L_t (\Phi^{L}_{u_\alpha,u} \xi) \circ u_t|_{t=0} (p) = \Psi_{u_\alpha}(h; h', \xi)(u(p)).
$$

Since $L$ and $\nabla^L$ are smooth on $M$, $\Psi_{u_\alpha}(h; h', \xi) = \nabla^L_{E_{u_\alpha}(h)h'}(\Phi_{u_\alpha,u} \xi)$ and $\Psi_{u_\alpha}(0; h', \xi) = 0$, there is a constant $C > 0$ independent of $p$ such that

$$
|\Psi_{u_\alpha}(h; h', \xi)|_{u(p)} \leq C|h(p)||h'(p)||(|\xi| + |\nabla \xi|)|_{u(p)}, \quad (3.1)
$$

when $\|h\|_{k,2} \leq \epsilon$. If no danger of confusion we denote $(u^*_t \nabla^L)_t$ by $\nabla^L_t$. Then we have

$$
\|\nabla^L_t (\Phi^{L}_{u_\alpha,u} u^*_\alpha \xi)|_{t=0}\|_{C^0} \leq C\|h\|_{C^0} \|h'\|_{C^0} \|\xi\|_{C^1(U_\alpha)}
$$

for some constant $C > 0$. By the Sobolev embedding Theorem we have

$$
\left\|\nabla^L_t (\Phi^{L}_{u_\alpha,u} u^*_\alpha \xi)|_{t=0}\right\|_{k,2} \leq C\|h\|_{k,2} \|h'\|_{k,2}
$$

where $C > 0$ is a constant depending on $\|\xi\|_{C^{k+1}(U_\alpha)}$, the Sobolev constant and the metric of $M$. Then the operator

$$
\nabla^L_{E_{u_\alpha}(h)}(\Phi^{L}_{u_\alpha,u}) \xi : W^{k,2}(\Sigma, u^*_\alpha TM) \to W^{k-1,2,\alpha}(\Sigma, u^* L)
$$

is a bounded linear operator. For any $l \in \mathbb{Z}^+$, denote $t = (t_1, \ldots, t_l)$, $u_t = \exp_{u_\alpha}(h + \sum_{i=1}^l t_i h_i)$ and

$$
T^l(h; h_1, \ldots, h_l) \xi = \nabla^L_{t_1} \cdots \nabla^L_{t_l} (\Phi^{L}_{u_\alpha,u} u^*_\alpha \xi)|_{t=0}.
$$

A direct calculation gives us

$$
|T^l(h; h_1, \ldots, h_l) \xi| \leq C \Pi_{i=1}^l |h_i(p)|. \quad (3.3)
$$
By the same way as above we can show that

\[ T^l(h; \cdots) : W^{k,2}(\Sigma, u^*_o TM) \times \cdots \times W^{k,2}(\Sigma, u^*_o TM) \times W^{k,2,\alpha}(\Sigma, u^* L) \to W^{k,2,\alpha}(\Sigma, u^* L) \]

is a bounded linear operator with respect to \( h_1, \cdots, h_l \).

Now we calculate \( \nabla^L_t(D^L_t(\Phi_{u_o, u^*_o}(\cdot))) \). Let \( \partial_o \) be a section of \( T\Sigma \), denote \( (u^* \nabla^L_t)\partial_o = \nabla^L_t \partial_o \). Then, by the definition of curvature and \([\partial_o, \partial_o] = 0\), we have

\[ \nabla^L_t \nabla^L_t \Phi_{u_o, u^*_o}(\cdot) |_{t=0} = \nabla^L_t \nabla^L_t \Phi_{u_o, u^*_o}(\cdot) |_{t=0} + R(\partial_o, \partial_o) \Phi_{u_o, u^*_o}(\cdot) |_{t=0} \]

By (3.2) we get

\[ R(\partial_o, \partial_o) \Phi_{u_o, u^*_o}(\cdot) |_{t=0}(p) \]

\[ = \frac{1}{2} \| \nabla^L_t \nabla^L_t \Phi_{u_o, u^*_o}(\cdot) |_{t=0} \| \leq C \| \Phi_{u_o, u^*_o}(\cdot) \|_{k,2}. \]

Since curvature \( R \) is a tensor, we have

\[ \| \nabla^L_t \nabla^L_t \Phi_{u_o, u^*_o}(\cdot) |_{t=0} \| \leq C \| h' \|_{k,2}. \]

Let \( u_t \) be as above. One can check that

\[ \| \nabla^L_t \nabla^L_t \Phi_{u_o, u^*_o}(\cdot) |_{t=0} \| \leq C \| h' \|_{k,2}. \]

Define

\[ \tilde{T}^l(h; \cdots) \nabla^L_t : W^{k,2}(\Sigma, u^*_o TM) \times \cdots \times W^{k,2}(\Sigma, u^*_o TM) \times W^{k,2,\alpha}(\Sigma, u^* L) \to W^{k-1,2,\alpha}(\Sigma, u^* L \otimes L^1 T\Sigma) \]

by

\[ \tilde{T}^l(h; h_1, \cdots, h_l) \nabla^L_t \Phi_{u_o, u^*_o}(\cdot) |_{t=0}. \]

We can show that \( \tilde{T}^l(h; \cdots) \nabla^L_t \) is a bounded linear operator with respect to \( h_1, \cdots, h_l \). Define

\[ \mathcal{F} : W^{k,2}(\Sigma, u^*_o TM) \times W^{k,2,\alpha}(\Sigma, u^*_o L) \to W^{k-1,2,\alpha}(\Sigma, u^*_o L \otimes L^1 T\Sigma) \]

by

\[ \mathcal{F}(h, \xi) = (\Phi_{u_o, u^*_o})^{-1} D^L_{\alpha} \Phi_{u_o, u^*_o} \xi. \]

Lemma 3.1. \( \mathcal{F}(h, \xi) \) is a smooth map.

Proof. Note that \( L \) has finite rank and for any fixed \( h \), \( \mathcal{F}(h, \xi) \) is a linear map. The key point is to prove the smoothness of \( \mathcal{F}(h, \xi) \) with respect to \( h \). Since both \( T^l(h; \cdots) \) and \( \tilde{T}^l(h; \cdots) \) are bounded linear operators for any \( l, \ell \in \mathbb{Z}^+ \), the smoothness of \( \mathcal{F}(h, \xi) \) follows. □
For any $j_0 \in J(\Sigma)$ near $j_0$ we can write $j_0 = (I + H)j_0(I + H)^{-1}$ where $H \in T_{j_0} J(\Sigma)$. We define two maps

\[ \tilde{\Psi}_{j_0,j_0}: \tilde{L}|_{\Sigma} \times J(\Sigma) \to \tilde{L}|_{\Sigma} \times J(\Sigma) \]

by

\[ \tilde{\Psi}_{j_0,j_0}(\eta) = \frac{1}{2}(\Phi_{a,a}^\Lambda \eta - \Phi_{a,a}^\Lambda \eta \cdot j_0) \quad \tilde{\Psi}_{j_0,j_0}(\omega) = \frac{1}{2}(\Phi_{a,a}^\Lambda \omega - \Phi_{a,a}^\Lambda \omega \cdot j_0) \]

Note that

\[ u^* i(k \otimes e_a) = k \otimes u^* i(e_a) \]

\[ \Phi_{a,a}(k \otimes e_u) = \Phi_{a,a}(k) \otimes e_u \]

We have $u^* i \circ \Phi_{a,a}^\Lambda = \Phi_{a,a}^\Lambda u^* i$. Since $u^* i \eta = -\eta j_0$ and $u^* i \omega = -\omega j_0$ for any

\[ \eta \in \tilde{L}|_{\Sigma} \times J(\Sigma) \] \[ \omega \in \tilde{L}|_{\Sigma} \times J(\Sigma) \]

One can check that $u^* i \tilde{\Psi}_{j_0,j_0}(\eta) = -\tilde{\Psi}_{j_0,j_0}(\eta) j_0$ and $u^* i \tilde{\Psi}_{j_0,j_0}(\omega) = -\tilde{\Psi}_{j_0,j_0}(\omega) j_0$. Then $\tilde{\Psi}_{j_0,j_0}$ and $\tilde{\Psi}_{j_0,j_0}$ are well defined. The proof of the following lemma is similar to the proof of Lemma 7.3 in [3], we omit it here.

**Lemma 3.2.** Both $\tilde{\Psi}_{j_0,j_0}$ and $\tilde{\Psi}_{j_0,j_0}$ are isomorphisms when $|H|$ small enough.

Set

\[ P_{b_0}^{\tilde{L}} = \tilde{\Psi}_{j_0,j_0} \circ \Phi_{a,a}^{\tilde{L}} \]

We consider the map

\[ \mathcal{F}: A \times W^{k,2}(\Sigma, u_0^* TM) \times W^{k,2}(\Sigma, \tilde{L}|_{\Sigma}) \to W^{k-1,2}(\Sigma, \tilde{L}|_{\Sigma}) \]

defined by

\[ \mathcal{F}(a, h, \xi) = P_{b_0}^{\tilde{L}} \circ D_{b_0}^{\tilde{L}} \circ (P_{b_0}^{\tilde{L}})^{-1} \xi \]

**Lemma 3.3.** The following hold.

1. $\frac{\partial}{\partial x} \mathcal{F}(a, 0, 0, \lambda \xi) |_{\lambda=0} = D_{\xi} \mathcal{F}|_{a_0}(\xi) = \tilde{\Psi}_{j_0,j_0}(a_0) \circ D_{b_0}^{\tilde{L}}|_{a_0} \circ (\tilde{\Psi}_{j_0,j_0})^{-1}(\xi)$

2. $\mathcal{F}$ is smooth functional of $(a, h, \xi)$. 

**Proof.** (1) is obtained by a direct calculation. Since $\tilde{\Lambda}$ is a smooth finite rank bundle over $A$, by Lemma 3.1 we obtain (2). □

**Lemma 3.4.** In the local coordinate system $A$ the bundle $\tilde{\mathcal{L}}$ is smooth. Furthermore, for any base $\{ e_a \}$ of the fiber at $b_0$ we can get a smooth frame fields $\{ e_a(a, h) \}$ for the bundle $\tilde{\mathcal{L}}$ over $\tilde{O}_{b_0}(\delta_{a_0}, \rho_0)$.

**Proof.** Note that $D_{\xi} \mathcal{F}|_{b_0} = D_{\tilde{L}}^{\tilde{L}}|_{b_0}$. It is a Fredholm operator with $\text{coker} D_{\tilde{L}}^{\tilde{L}}|_{b_0} = 0$ (because of $H^1(\Sigma, \tilde{L}|_{b_0}) = 0$). There is a right inverse $Q_{b_0}^{\tilde{L}}$ of $D_{b_0}^{\tilde{L}}|_{b_0}$. Now we view $a$ and $h$ as parameters. It is easy to check that the conditions of the implicit function theorem (Theorem 7.1 Theorem 7.2) hold. Then there exist $\delta_{a_0} > 0$, $\rho_0 > 0$ and a small neighborhood $O$ of $0 \in \ker D_{\tilde{L}}^{\tilde{L}}|_{b_0}$ and a unique smooth map

\[ f^{\tilde{L}}: \tilde{O}_{b_0}(\delta_{a_0}, \rho_0) \times O \to W^{k-1,2}(\Sigma, \tilde{L}|_{\Sigma}) \]

such that for any $\zeta \in O$ and any $b \in \tilde{O}_{b_0}(\delta_{a_0}, \rho_0)$

\[ D_{b_0}^{\tilde{L}} \circ (P_{b_0}^{\tilde{L}})^{-1}(\zeta + Q_{b_0}^{\tilde{L}} \circ f^{\tilde{L}}(\zeta)) = 0. \]


We get the smoothness of $\tilde{F}$ in $\tilde{O}_{b_o}(\delta_o, \rho_o)$. Furthermore, choosing a base $\{e_\alpha\}$ of the fiber at $b_o$, we get a smooth frame fields $\{e_\alpha(a, h)\}$ by Theorem 7.2. We complete the proof. □

Let $(\Sigma, j, y)$ be a smooth Riemann surface of genus $g$ with $n$ marked points, $u : \Sigma \to M$ be a $C^\infty$ map. Denote $b = (j, y, u)$. For any $\phi \in Diff^+(\Sigma)$ denote

$$b' = (j', y', u') = \phi \cdot (j, y, u) = (\phi^*j, \phi^{-1}y, \phi^*u).$$

Then

$$(u')^*i = \phi^*(u^*i), \quad (u')^*\nabla^L = \phi^*(u^*\nabla^L). \tag{3.5}$$

Let $\phi \in Diff^+(\Sigma)$. For any section $\xi \in L$ we have

$$(\phi \cdot (u^*\xi))^p = (\phi^* \circ u^*\xi)^p = ((u \circ \phi)^*\xi)^p = ((u')^*\xi)^p \tag{3.6}$$

and for any $f(z)(dz)^p$

$$\phi \cdot f(z)(dz)^p = f(\phi^{-1}(w))[d(\phi^{-1}(w))]^p, \tag{3.7}$$

where $w = \phi(z)$. We have the following lemma

**Lemma 3.5.** $(\phi \cdot \tilde{L})|_{b'} = \phi^*(\tilde{L}|_b)$, $D\tilde{L}|_{b'}(\phi^*\xi) = \phi^*(D\tilde{L}|_b(\xi))$ for any $\xi \in \tilde{L}|_b$.

**Proof.** The first inequality follows from (3.6) and (3.7). For any $f(k \otimes e_a)^p \in \tilde{L}|_b$, we have

$$D\tilde{L}|_b(k \otimes e_a)^p = \partial(f \cdot (k \otimes e_a)^p + pf \cdot k \otimes D\tilde{L}|_b(e_a) \otimes (k \otimes e_a)^p)^{p-1}.$$ 

By $\phi \cdot f(z, \bar{z}) = f(\phi^{-1}(w), \phi^{-1}(w))$, we get

$$\partial(\phi(f(z, \bar{z}))) = \frac{\partial f}{\partial z}(\phi^{-1}(w), \phi^{-1}(w))\frac{\partial \phi^{-1}(w)}{\partial w}d\bar{w} = \frac{\partial f}{\partial z}(\phi^{-1}(w), \phi^{-1}(w))d\phi^{-1}(w).$$

Similar (3.7) we have

$$\nabla(\phi(f(z, \bar{z}))) = \phi \cdot \left[\frac{\partial f}{\partial z}(z, \bar{z})d\bar{z}\right] = \frac{\partial f}{\partial z}(\phi^{-1}(w), \phi^{-1}(w))d\phi^{-1}(w).$$

It follows that $\phi \cdot (\bar{\partial}f) = \bar{\partial}(\phi \cdot f)$. Since $D\tilde{L}|_b = (u^*\nabla^L)^{0,1}$ and $\phi$ is holomorphic, by (3.5) we have

$$D\tilde{L}|_{b'} = ((u')^*\nabla^L)^{0,1} = (\phi^* (u^*\nabla^L))^{0,1} = \phi^* (\nabla^L)^{0,1} = \phi^* D\tilde{L}|_b.$$ 

Then the second inequality follows from the first inequality. □

**Remark 3.6.** Let $G_{b_o}$ be the isotropy group at $b_o$. By Lemma 3.5, $D\tilde{L}$ is $G_{b_o}$-equivariant and $G_{b_o}$ acts on $ker D\tilde{L}|_{b_o}$. We may choose a $G_{b_o}$-equivariant right inverse $Q_{b_o}^\tilde{L}$. In fact, let $Q_{b_o}^\tilde{L}$ be a right inverse of $D\tilde{L}|_{b_o}$, we define

$$Q_{b_o}^\tilde{L}(\eta) = \frac{1}{|G_{b_o}|} \sum_{\phi \in G_{b_o}} \phi^{-1} \cdot \tilde{Q}_{b_o}(\phi \cdot \eta).$$

Then, for any $\phi' \in G_{b_o}$, we have

$$Q_{b_o}^\tilde{L}(\phi' \cdot \eta) = \frac{1}{|G_{b_o}|} \sum_{\phi \in G_{b_o}} \phi^{-1} \cdot \tilde{Q}_{b_o}(\phi \cdot \phi' \cdot \eta) = \frac{1}{|G_{b_o}|} \sum_{\phi \in G_{b_o}} \phi' \cdot (\phi')^{-1} \phi^{-1} \cdot \tilde{Q}_{b_o}(\phi \cdot \phi' \cdot \eta) = \phi' \cdot Q_{b_o}(\eta).$$

By uniqueness, it follows that $f^\tilde{L}$ is $G_{b_o}$-equivariant. So we have a $G_{b_o}$-equivariant version of Lemma 3.4. In particular, for any base $\{e_\alpha\}$ of the fiber at $b_o$ we can get a smooth $G_{b_o}$-equivariant frame fields $\{e_\alpha(a, h)\}$ for the bundle $\tilde{F}$ over $\tilde{O}_{b_o}(\delta_o, \rho_o)$. 

10
Remark 3.7. Note that what Lemma 3.4 claim is the smoothness in a local coordinate system \((\psi, \Psi) : (O, \pi^{-1}_T(O)) \to (\mathbf{A}, \mathbf{A} \times \Sigma)\). If we choose another local coordinate system \((\psi', \Psi') : (O', \pi^{-1}_T(O')) \to (\mathbf{A}', \mathbf{A'} \times \Sigma)\) we have \(u' = u \circ d\varphi^{-1}_{a}\) where \(\varphi = \Psi' \circ \Psi^{-1}\). If \(u\) is only \(W^{k,2}\) map, the coordinate transformation is not smooth. Nevertheless the Lemma 3.4 is still very useful in the study of the smoothness of top strata of virtual neighborhood, as we have a PDE here, we can use the standard elliptic estimates to get the smoothness of \(u\) (see [3]).

4 Gluing

4.1 Pregluing for maps

Let \(\Sigma = \Sigma_1 \wedge \Sigma_2, j = (j_1, j_2), y = (y_1, y_2), q = (q_1, q_2)\),

\[
(\Sigma = \Sigma_1 \wedge \Sigma_2, j = (j_1, j_2), y = (y_1, y_2), q = (q_1, q_2)),
\]

where \((\Sigma_i, j_i, y_i, q_i)\) are smooth Riemann surfaces, \((j_i, y_i) \in \mathbf{A}_i, i = 1, 2\). We say that \(q_1, q_2\) are paired to form \(q\). Assume that \((\Sigma_i, j_i, y_i, q_i)\) is stable, i.e., \(n_i + 2g_i + 1 \geq 3, i = 1, 2\). We choose metric \(g_i\) on each \(\Sigma_i\) as in [11]. Let \(z_i\) be the cusp coordinates around \(q_i, z_i(q_i) = 0, i = 1, 2\). Let

\[
z_1 = e^{-s_1 - 2\pi \sqrt{-1} t_1}, \quad z_2 = e^{s_2 + 2\pi \sqrt{-1} t_2},
\]

\((s_i, t_i)\) are called the cusp holomorphic cylindrical coordinates near \(q_i\). In terms of the cusp holomorphic cylindrical coordinates we write

\[
\bar{\Sigma}_1 := \Sigma_1 \setminus \{q_1\} \cong \Sigma_{10} \cup \{[0, \infty) \times S^1\}, \quad \bar{\Sigma}_2 := \Sigma_2 \setminus \{q_2\} \cong \Sigma_{20} \cup \{(-\infty, 0) \times S^1\}.
\]

Here \(\Sigma_{10} \subset \Sigma_i, i = 1, 2\), are compact surfaces with boundary. Put \(\bar{\Sigma} = \Sigma \setminus \{q_1, q_2\} = \bar{\Sigma}_1 \cup \bar{\Sigma}_2\). We introduce the notations

\[
\Sigma_i(R_0) = \Sigma_i \cup \{(s_i, t_i) | |s_i| \leq R_0\}, \quad \Sigma(R_0) = \Sigma_1(R_0) \cup \Sigma_2(R_0).
\]

For any gluing parameter \((r, \tau)\) with \(r \geq R_0\) and \(\tau \in S^1\) we construct a surface \(\Sigma_{(r)}\) with the gluing formulas:

\[
s_1 = s_2 + 2r, \quad t_1 = t_2 + \tau.
\]

(4.1)

where we use \((r)\) to denote gluing parameters.

Let \(b_0 = (a_o, u), a_o = (\Sigma, j, y, q), u = (u_1, u_2)\), where \(u_1 : \Sigma_i \to M\) are are \((j_i, J)\)-holomorphic maps with \(u_1(q) = u_2(q)\). We will use the cusp holomorphic cylinder coordinates to describe the construction of \(u_{(r)} : \Sigma_{(r)} \to M\). We choose local normal coordinates \((x^1, \ldots, x^{2m})\) in a neighborhood \(O_{u(q)}\) of \(u(q)\) and choose \(R_0\) so large that \(u([|s_i| \geq \frac{r}{2}])\) lie in \(O_{u(q)}\) for any \(r > R_0\). We glue the map \((u_1, u_2)\) to get a pregluing maps \(u_{(r)}\) as follows. Set

\[
u_{(r)} = \begin{cases} u_1 & \text{on } \Sigma_{10} \cup \{(s_1, t_1) | 0 \leq s_1 \leq \frac{r}{2}, t_1 \in S^1\} \\
u_1(q) = u_2(q) & \text{on } \{(s_1, t_1) | \frac{3r}{4} \leq s_1 \leq \frac{5r}{4}, t_1 \in S^1\} \\
u_1 & \text{on } \Sigma_{20} \cup \{(s_2, t_2) | 0 \geq s_2 \geq -\frac{r}{2}, t_2 \in S^1\} \end{cases}
\]
To define the map \( u(r) \) in the remaining part we fix a smooth cutoff function \( \beta : \mathbb{R} \to [0, 1] \) such that

\[
\beta(s) = \begin{cases} 
1 & \text{if } s \geq 1 \\
0 & \text{if } s \leq 0 
\end{cases}
\]  

(4.2)

and \( \sqrt{1 - \beta^2} \) is a smooth function, \( 0 \leq \beta'(s) \leq 4 \) and \( \beta^2(\frac{1}{2}) = \frac{1}{2} \). We define

\[
u(r) = u_1(q) + \left( \beta \left( 3 - \frac{4s_1}{r} \right) (u_1(s_1, t_1) - u_1(q)) + \beta \left( \frac{4s_1}{r} - 5 \right) (u_2(s_1 - 2r, t_1 - \tau) - u_2(q)) \right).
\]

4.2 Pregluing for \( \tilde{\mathcal{F}} \)

Let \( b_0 = (a_0, u), u = (u_1, u_2), u : \Sigma_i \to M \) are \((j_i, J)\)-holomorphic maps, \( i = 1, 2 \). We choose \( \{e_V\} \) as in (4.2) such that \( \tilde{L}_{|b_0} \) is a holomorphic lie bundle. Then \( D\tilde{L}_{|b_0} = \tilde{\partial}_{j,u} \). Recall that with respect to the base \((\frac{D}{\tilde{z}} \otimes e)^p \) for \( \tilde{L}_{|b} \) we have a local trivialization.

Denote

\[
\beta_{1;R}(s_1) = \beta \left( 1, \frac{r - s_1}{R} \right), \quad \beta_{2;R}(s_2) = \sqrt{1 - \beta^2 \left( \frac{1, s_2 + r}{R} \right)},
\]

where \( \beta \) is the cut-off function defined in (4.2). Then we have

\[
\beta_{2;R}(s_1 - 2r) = 1 - \beta^2 \left( \frac{1, s_1 - r}{R} \right) = 1 - \beta_{1;R}(s_1).
\]

(4.3)

For any \( \eta \in C^\infty(\Sigma_{(r)}; \tilde{L}_{|b_{(r)}} \otimes \Lambda^0_{J_{\alpha}} T\Sigma_{(r)}) \), let

\[
\eta_i(p) = \begin{cases} 
\eta & \text{if } p \in \Sigma_{i0} \cup \{|s_i| \leq r - 1\} \\
\beta_{i=2}(s_i)\eta(s_i, t_i) & \text{if } p \in \{r - 1 \leq |s_i| \leq r + 1\} \\
0 & \text{otherwise.}
\end{cases}
\]

If no danger of confusion we will simply write \( \eta_i = \beta_{i=2}\eta \). Then \( \eta_i \) can be considered as a section over \( \Sigma_i \).

Define

\[
\|\eta\|_{r, k-1, 2, \alpha} = \|\eta_1\|_{\Sigma_{1, j_1, k-1, 2, \alpha}} + \|\eta_2\|_{\Sigma_{2, j_2, k-1, 2, \alpha}}.
\]

(4.4)

We now define a norm \( \| \cdot \|_{r, k, 2, \alpha} \) on \( C^\infty(\Sigma_{(r)}; \tilde{L}_{|b_{(r)}}) \). For any section \( \zeta \in C^\infty(\Sigma_{(r)}; \tilde{L}_{b_{(r)}}) \) denote

\[
\zeta_0 = \int_{S_{1}} \zeta(r, t)dt,
\]

\[
\zeta_1(s_1, t_1) = (\zeta - \zeta_0)(s_1, t_1) \cdot \beta_{1=2}(s_1), \quad \zeta_2(s_2, t_2) = (\zeta - \zeta_0)(s_2, t_2) \cdot \beta_{2=2}(s_2).
\]

We define

\[
\|\zeta\|_{r, k, 2, \alpha} = \|\zeta_1\|_{\Sigma_{1, j_1, k, 2, \alpha}} + \|\zeta_2\|_{\Sigma_{2, j_2, k, 2, \alpha}} + |\zeta_0|.
\]

(4.5)

Denote the resulting completed spaces by \( W^{k-1, 2, \alpha}(\Sigma_{(r)}; \tilde{L}_{|b_{(r)}} \otimes \Lambda^0_{J_{\alpha}} T\Sigma_{(r)}) \) and \( W^{k, 2, \alpha}(\Sigma_{(r)}; \tilde{L}_{|b_{(r)}}) \) respectively.

In terms of the cusp holomorphic cylinder coordinates we may write

\[
D\tilde{L}_{|b_{(r)}} = \tilde{\partial}_{j_0} + E_{b_{(r)}},
\]

where \( \tilde{\partial}_{j_0} = \frac{1}{2} \left( \frac{\partial}{\partial t} + \sqrt{-1} \frac{\partial}{\partial s} \right), \left( \frac{\partial}{\partial t} \right) = j_0 \frac{\partial}{\partial s} \) and

\[
E_{b_{(r)}} = \frac{p}{2} \left( \sum \frac{\partial u_i^{(r)}}{\partial s} + \sqrt{-1} \sum \frac{\partial u_i^{(r)}}{\partial t} \right) \left( \nabla_{\partial z_j} e_{u_{(r)}}, e_{u_{(r)}} \right).
\]

(4.6)
In fact, for any \( f(k \otimes e_{u(r)})^p \in \tilde{L}|_{b(r)} \), by (2.6) we have

\[
D\tilde{L}|_{b(r)} (f(k \otimes e_{u(r)})^p) \left( \frac{\partial}{\partial s} \right) = \tilde{\partial}_s f(k \otimes e_{u(r)})^p + pf(k \otimes D^L e_{u(r)}) \left( \frac{\partial}{\partial s} \right) (k \otimes e_{u(r)})^{p-1}.
\]  

(4.7)

On the other hand, using \( D^L = \frac{1}{2} \left( \nabla^L + u^* L \cdot \nabla^L : j_o \right) \), we obtain that

\[
D^L e_{u(r)} \left( \frac{\partial}{\partial s} \right) = \frac{1}{2} \left( \nabla^L (e_{u(r)}) \left( \frac{\partial}{\partial s} \right) + u^* \nabla^L (e_{u(r)}) \left( \frac{\partial}{\partial t} \right) \right)
\]

\[
= \frac{1}{2} \nabla C(t ; \eta_{u(r)}) \left( \frac{\partial u^*_\eta}{\partial s} \right) + \sqrt{-1} \left( \frac{\partial u^*_\eta}{\partial t} \right)
\]

\[
= \frac{1}{2} \left( \nabla C(t ; \eta_{u(r)}) \left( \frac{\partial u^*_\eta}{\partial s} \right) \right) - \left( \frac{\partial u^*_\eta}{\partial s} \right)
\]

where we used the fact that \( L \) is a line bundle. Substituting (4.8) into (4.7) we get (4.6).

Note that \( u_i(s_i, t_i) \) exponentially converges to 0, with higher-order derivatives, as \( s_i \to \infty, i = 1, 2 \). We have

\[
E^L_{b(r)} \bigg|_{s_i \leq \frac{T}{2}} = 0, \quad \sum_{p+q=d} \left| \frac{\partial^j E^L_{b(r)}}{\partial s^j \partial t^q} \right| \to 0,
\]

(4.9)

for \( i = 1, 2, \forall d \geq 0 \), exponentially and uniformly in \( t_i \) as \( r \to \infty \).

For any \( b = (a, v) \) with \( v = \exp_{u(r)} (h_r) \), denote \( e_v = P^\tilde{L}_{b(r), b} e_{u(r)} \). We have

\[
(P^\tilde{L}_{b(r), b})^{-1} \circ D\tilde{L}_{b(r), b} e_{u(r)} = \tilde{\partial}_a + E^\tilde{L}_b,
\]

where \( \tilde{\partial}_a = \frac{\partial}{\partial a} + \sqrt{-1} j_a \frac{\partial}{\partial s} \) and

\[
E^\tilde{L}_b = \frac{p}{2} \sum \left( \frac{\partial \delta^j}{\partial s} + \sqrt{-1} \left( j_a \frac{\partial}{\partial s} \right) (v^j) \right) \frac{((P^\tilde{L}_{b(r), b})^{-1} \nabla^L e_v, e_{u(r)})}{(e_{u(r)}, e_{u(r)})}.
\]

(4.10)

It is easy to check that

\[
\|D\tilde{L}_{b(r)} - (P^\tilde{L}_{b(r), b})^{-1} \circ D\tilde{L}_{b(r), b}\| \leq C(|a - a_o| + ||h||_{k, 2, \alpha, r}).
\]

(4.11)

Given \( \eta \in W^{k-1, 2, \alpha}(\Sigma; W^r \tilde{L}|_{b(r)}) \otimes \wedge_{Ja}^{0,1} T\tilde{\Sigma}(r) \) denote

\[
(Q^\tilde{L}_{b_o})^{-1}(\eta) = (\zeta_{\eta}(s_1, s_2, t_2)) = (\beta_{2,2}(s_2)\eta(s_1, t_1), \beta_{2,2}(s_2)\eta(s_1, t_1)),
\]

where \( Q^\tilde{L}_{b_o} \) is a right inverse of \( D\tilde{L}_{b_o} \). Define

\[
\left( Q^\tilde{L}_{b(r)} \right)'(\eta) := (\zeta_{\eta}(s_1, s_2, t_2)) = (\beta_{2,2}(s_1)\eta(s_1, t_1) + \beta_{2,2}(s_2)(s_1 - s_2, t_1 - t_2))).
\]

Lemma 4.1. For any \( \eta \in W^{k-1, 2, \alpha}(\Sigma; W^r \tilde{L}|_{b(r)}) \otimes \wedge_{Ja}^{0,1} T\tilde{\Sigma}(r) \) we have

\[
D\tilde{L}_{b(r)} \circ \left( Q^\tilde{L}_{b(r)} \right)'(\eta) - \eta = \sum (\tilde{\partial}_{\beta_{i,r}}) \zeta_i + \sum \beta_{i,r} E^\tilde{L}_{b(r)} \zeta_i
\]

(4.12)

\[\right.++(\sum (\beta_{i,r} \beta_{i,2} - 1) \eta.\]
Proof: It is obvious that
\[ D_{b_{1}\cap} \circ (Q_{b_{1}})^{r} = \eta \quad \text{for} \quad |s_{i}| \leq \frac{r}{2}, \]  
(4.13)

It suffices to calculate the left hand side in the annulus \( \{ \frac{r}{2} \leq |s_{i}| \leq \frac{3r}{2} \} \). By choosing \( r \) large enough we may assume that \( \{ \frac{r}{2} \leq |s_{i}| \leq \frac{3r}{2} \} \subset \Sigma \setminus \Sigma (R_{0}) \). Note that in this annulus
\[ D_{b_{1}} = \partial_{j_{0},u}, \quad D_{b_{1}} \partial_{i} = D_{b_{1}} |_{\Sigma} \eta_{i}, \quad \beta_{1,r} D_{b_{1}} \partial_{i} \eta_{i} + \beta_{2,r} D_{b_{1}} \partial_{i} \eta_{i} = \sum_{\frac{i=r}{2}}^{2} \beta_{i,r} \beta_{i,2r}. \]

By a direct calculation we get (4.12). □

Lemma 4.2. \( D_{b_{1}} \) is surjective for \( r \) large enough. Moreover, there is a right inverse \( Q_{b_{1}} \) such that
\[ \| Q_{b_{1}} \| \leq C \]
(4.14)
for some constant \( C > 0 \) independent of \( r \).

Proof: We first show that
\[ \left\| Q_{b_{1}}^{r} \right\| \leq C, \]  
(4.15)
\[ \| D_{b_{1}} \circ (Q_{b_{1}})^{r} - \text{Id} \| \leq \frac{2}{3} \]  
(4.16)
for some constant \( C > 0 \) independent of \( r \). Since \( 0 \leq \beta_{1,r} \leq 1 \) we have
\[ |(\zeta_{r})_{0}| \leq e^{\alpha r} \max_{t_{s} \in S_{i}} |e^{\alpha r} \zeta_{r}(r, t)| \leq e^{\alpha r} \max_{t_{s} \in S_{i}} |e^{\alpha r} \zeta_{r}(r, t)| \]
\[ \leq Ce^{\alpha r} \sum_{i=1,2} \| e^{r} |s_{i}| \zeta_{r}(s_{i}, t_{i}) |_{r-1|s_{i}|r} \| k_{2} \leq Ce^{\alpha r} \sum_{i=1,2} \| \zeta_{i} \| k_{2,\alpha}, \]
(4.17)
where we used the Sobolev embedding theorem in the third inequality. By \( \| Q_{b_{1}}^{r} \| \leq C \) and the definition of \( \| \cdot \| k_{2,\alpha,\gamma} \), we have
\[ \| \zeta_{r} \| k_{2,\alpha,\gamma} = \sum_{\beta_{1,2}} \| \beta_{1,2} (\zeta_{r} - \hat{\zeta}_{r})_{0} \| k_{2,\alpha} + |(\zeta_{r})_{0}| \]
\[ \leq 2(C + 1) \| \zeta_{r} \| k_{2,\alpha} \leq C \| (\eta_{1}, \eta_{2}) \| k_{2,\alpha} \leq C \| \eta \| k_{2,\alpha,\gamma} \]
where we used (4.17) in the second inequality. Then (4.15) follows.

We prove (4.16). It follows from (4.12) that
\[ \left\| D_{b_{1}} \circ (Q_{b_{1}})^{r} \right\| \eta_{-r} \right\| k_{-1,2,\alpha,\gamma} \leq \frac{C}{r} \| \eta \| k_{-1,2,\alpha,\gamma} \]
\[ \leq \left( \frac{C}{r} + \frac{1}{2} \right) \| \eta \| k_{-1,2,\alpha,\gamma}, \]
where we used \( \frac{1}{2} \leq \sum_{\beta_{1,2}} \beta_{1,2} \leq \sqrt{2}, \quad \sum_{\beta_{1,2}} \beta_{1,2} \| E_{b_{1}} \| | \Sigma (r/2) | = 0 \), \( \sum_{i=1}^{2} | E_{b_{1}} | \leq C e^{\alpha r} \) in (4.17) in the first inequality, and used \( \| Q_{b_{1}}^{r} \| \leq C \) in the last inequality. Then (4.16) follows when \( r \) large enough.

The estimate (4.16) implies that \( D_{b_{1}} \circ (Q_{b_{1}})^{r} \) is invertible, and a right inverse \( Q_{b_{1}}^{r} \) of \( D_{b_{1}} \) is given by
\[ Q_{b_{1}}^{r} = \left( Q_{b_{1}}^{r} \right)^{r} \left[ D_{b_{1}} \circ (Q_{b_{1}})^{r} \right]^{-1}. \]  
(4.19)
Then the Lemma follows. □

For any \( \zeta + \hat{\zeta}_0 \in \ker D\bar{L}|_{b_o} \), we set

\[
\zeta(r) = \beta_{1;r}(s_1)\zeta_1(s_1, t_1) + \beta_{2;r}(s_1 - 2r)\zeta_2(s_1 - 2r, t_1 - r) + \hat{\zeta}_0,
\]

(4.20)

Define \( I_{(r)}^\bar{L} : \ker D\bar{L}|_{b_o} \to \ker D\bar{L}|_{b(r)} \) by

\[
I_{(r)}^\bar{L}(\zeta + \hat{\zeta}_0) = \zeta(r) - Q_{b(r)}^{\bar{L}} \circ D\bar{L}|_{b(r)}(\zeta(r)).
\]

(4.21)

**Lemma 4.3.** \( I_{(r)}^\bar{L} : \ker D\bar{L}|_{b_o} \to \ker D\bar{L}|_{b(r)} \) is an isomorphism for \( r \) large enough, and

\[
\| I_{(r)}^\bar{L} \| \leq C
\]

for some constant \( C > 0 \) independent of \( r \).

**Proof.** Let \( \zeta + \hat{\zeta}_0 \in \ker D\bar{L}|_{b_o} \) with \( I_{(r)}^\bar{L}(\zeta + \hat{\zeta}_0) = 0 \). By (4.21) and (4.14), we have

\[
\| \zeta(r) \|_{k,2,\alpha,r} = \left\| I_{(r)}^\bar{L}(\zeta + \hat{\zeta}_0) - \zeta(r) \right\|_{k,2,\alpha,r} \leq C\| D\bar{L}|_{b(r)}(\zeta(r)) \|
\]

for some constant \( C > 0 \). A direct calculation gives us

\[
D\bar{L}|_{b(r)}(\zeta(r)) = \sum_{i=1}^{2} \beta_{i;r}(s_1)\beta_{i;r}(s_2)\zeta_1(s_1, t_1) + \sum_{i=1}^{2} (\partial \beta_{i;r})\zeta_i + \sum_{i=1}^{2} \beta_{i;r} E_{b(r)}^{\bar{L}}(\zeta_i + E_{b(r)}^{\bar{L}} \hat{\zeta}_0).
\]

(4.22)

Since \( E_{b(r)}^{\bar{L}}|_{\Sigma(r/2)} = E_{b_o}^{\bar{L}}|_{\Sigma(r/2)} \), by \( D\bar{L}|_{b_o}(\zeta + \hat{\zeta}_0) = \bar{\partial}_{j_o}(\zeta + \hat{\zeta}_0) = 0 \), we have

\[
\| \zeta(r) \|_{k,2,\alpha,r} \leq \frac{C}{r} \left( \| \zeta \|_{k,2,\alpha} + |\hat{\zeta}_0| \right)
\]

(4.23)

for some constant \( C > 0 \).

Let \( \epsilon' \in (0, 1) \) be a constant. By Lemma 7.3 we can choose \( R \) large enough such that

\[
\| \zeta \|_{|s_1| \geq 2R} \leq \epsilon' \left( \| \zeta \|_{k,2,\alpha} + |\hat{\zeta}_0| \right).
\]

Therefore

\[
\| \zeta(r) \|_{k,2,\alpha,r} \geq \| \zeta \|_{|s_1| \leq 2R} \| \zeta \|_{k,2,\alpha} + |\hat{\zeta}_0| \geq (1 - \epsilon')(\| \zeta \|_{k,2,\alpha} + |\hat{\zeta}_0|),
\]

(4.24)

for \( r > 4R \). Then (4.23) and (4.24) give us \( \zeta = 0 \) and \( \hat{\zeta}_0 = 0 \). Hence \( I_{(r)}^\bar{L} \) is injective.

Since \( H^0(\Sigma, \bar{L}|_{b_o}) \) and \( H^0(\Sigma, \bar{L}|_{b(r)}) \) have the same dimension, the Lemma follows. □

### 4.3 Equivariant Gluing

Let \( b_o \) be as in (2.2) and (4.4). Assume that \( (\Sigma_i, y_i, q) \) is stable. Let \( G_{b_o} = (G_{b_1}, G_{b_2}) \) be the isotropy group at \( b_o \), thus,

\[
G_{b_o} = \{ \phi = (\phi_1, \phi_2) | \ \phi_1 \inDiff f^+(\Sigma_i), \ \phi_1^*(j_i, y_i, q, u_i) = (j_i, y_i, q, u_i) \}.
\]

Obviously, \( G_{b_o} \) is a subgroup of \( G_{a_o} \).

It is easy to check that the operator \( D\bar{L}|_{b_o} \) is \( G_{b_o} \)-equivariant. Then we may choose a \( G_{b_o} \)-equivariant right inverse \( Q_{b_o}^{\bar{L}} \). \( G_{b_o} \) acts on \( \ker D\bar{L}|_{b_o} \) in a natural way. Put

\[
\ker D\bar{L}|_{b_o} = \ker D\bar{L}|_{b_o}/G_{b_o}.
\]
Note that we used the cusp holomorphic cylinder coordinates \((s_i, t_i)\) on \(\Sigma_i\) near \(q\) to do gluing in \([\mathsection 2.2]\) and \([\mathsection 4.4]\). Since the cut-off function \(\beta(s)\) depends only on \(s\), \(G_{b_0}\) acts on \(\tilde{O}_{b_0}(\delta, \rho)\).

Denote \(a_{(r)} = (\Sigma_{(r)}, j, y)\) and \(b_{(r)} = (a_{(r)}, u_{(r)})\). Denote by \(G_{b_{(r)}}\) the isotropy group at \(b_{(r)}\), \(G_{b_{(r)}}\) acts on \(\ker D^L_{b_{(r)}}\) in a natural way. Put

\[
\ker D^L|_{b_{(r)}} = \ker D^L_{b_{(r)}}/G_{b_{(r)}}.
\]

It is easy to see that \(G_{b_{(r)}}\) is a subgroup of \(G_{b_0}\) and can be seen as rotation in the gluing part. Then the gluing map is a \(\frac{|G_{b_0}|}{|G_{b_{(r)}}|}\)-multiple covering map. Since \(\beta_{1,r}\) is independent of \(\tau\), \(Q_{b_{(r)}}^L\) is \(G_{b_{(r)}}\)-equivariant. By the definition of \(Q_{b_{(r)}}^L\) and the \(G_{b_{(r)}}\)-equivariance of \(D^L|_{b_{(r)}}\), we conclude that \(Q_{b_{(r)}}^L\) is \(G_{b_{(r)}}\)-equivariant. By the uniqueness, \(f^L\) is \(G_{b_{(r)}}\)-equivariant. Then we have

**Lemma 4.4.** (1) \(\tilde{I}^L_{(r)} : \ker D^L|_{b_{(r)}} \longrightarrow \ker D^L_{b_{(r)}}\) is a \(\frac{|G_{b_0}|}{|G_{b_{(r)}}|}\)-multiple covering map.

(2) \(\tilde{I}^L_{(r)}\) induces a isomorphism \(I^L_{(r)} : \ker D^L|_{b_{(r)}} \longrightarrow \ker D^L_{b_{(r)}}\).

### 4.4 Pregluing several nodes

The above estimates can be generalized to gluing several nodes. Let \((\Sigma, j, y)\) be a marked nodal Riemann surface of genus \(g\) with \(n\) marked points. Suppose that \(\Sigma\) has \(e\) nodal points \(q = (q_1, \cdots, q_e)\) and \(t\) smooth components. For each node \(q_i\) we can glue \(\Sigma\) and \(u\) at \(q_i\) with gluing parameters \(r = ((r_1, \tau_1), \cdots, (r_e, \tau_e))\) to get \(\Sigma_{(r)}\) and \(u_{(r)}\). The operators \(D^L_{b_0}\) and \(D^L_{b_{(r)}}\) are \(G_{b_0}\)-equivariant and \(G_{b_{(r)}}\)-equivariant respectively. We may choose a \(G_{b_0}\)-equivariant right inverse \(Q^L_{b_0}\) and \(G_{b_{(r)}}\)-equivariant right inverse \(Q^L_{b_{(r)}}\). \(G_{b_0}\) (resp. \(G_{b_{(r)}}\)) acts on \(\ker D^L_{b_0}\) (resp. \(\ker D^L_{b_{(r)}}\)) in a natural way. Put

\[
\ker D^L|_{b_0} = \ker D^L_{b_0}/G_{b_0}, \quad \ker D^L_{b_{(r)}} = \ker D^L_{b_{(r)}}/G_{b_{(r)}}.
\]

By the same methods as in \([\mathsection 2.2]\), \([\mathsection 4.4]\) and \([\mathsection 4.3]\) we can prove

**Lemma 4.5.** (1) \(\tilde{I}^L_{(r)} : \ker D^L|_{b_{(r)}} \longrightarrow \ker D^L_{b_{(r)}}\) is a \(\frac{|G_{b_0}|}{|G_{b_{(r)}}|}\)-multiple covering map for \(r_{i}, 1 \leq i \leq e, \text{ large enough, and } ||\tilde{I}^L_{(r)}|| \leq C\) for some constant \(C > 0\) independent of \(r\).

(2) \(\tilde{I}^L_{(r)}\) induces a isomorphism \(I^L_{(r)} : \ker D^L|_{b_{(r)}} \longrightarrow \ker D^L_{b_{(r)}}\).

For fixed \(r\) we consider the family of maps:

\[
\mathcal{F}_{(r)} : \mathbb{A} \times W^{k,2,\alpha}(\Sigma_{(r)}, \nu_{(r)} T M) \times W^{k,2,\alpha}(\Sigma_{(r)}, \tilde{L}_{b_{(r)}}) \rightarrow W^{k-1,2,\alpha}(\Sigma_{(r)}, \Lambda^{0,1} T \Sigma_{(r)} \otimes \tilde{L}_{b_{(r)}})
\]

defined by

\[
\mathcal{F}_{(r)}(s, h, \xi) = P^L_{b_{(r)}} \circ D^L|_b \circ (P^L_{b_{(r)}})^{-1} \xi,
\]

where \(b = ((r), s, v_r)\) and \(v_r = \exp_{u_{(r)}} h\). By implicit function theorem (Theorem \(7.11\), Theorem \(7.2\)), there exist \(\delta > 0, \rho > 0\) and a small neighborhood \(\tilde{O}_{(r)}\) of \(0 \in \ker D^L|_{u_{(r)}}\) and a unique smooth map

\[
\tilde{f}^L_{(r)} : \tilde{O}_{b_{(r)}}(\delta, \rho) \times \tilde{O}_{(r)} \rightarrow W^{k-1,2,\alpha}(\Sigma_{(r)}, \Lambda^{0,1} T \Sigma_{(r)} \otimes \tilde{L}_{b_{(r)}})
\]

such that for any \((b, \zeta) \in \tilde{O}_{b_{(r)}}(\delta, \rho) \times \tilde{O}_{(r)}\)

\[
D^L|_b \circ (P^L_{b_{(r)}})^{-1} \left( \zeta + Q^L_{b_{(r)}} \circ \tilde{f}^L_{s, h_{(r)}}(\zeta) \right) = 0.
\]
Together with Lemma [4.5] and \( I^L_{(r)} \), we have gluing map

\[
\text{Gl}u^L_{(r)} : \text{F} \big|_{[b_o]} \to \text{F} \big|_{[b]} \quad \text{for any } b \in O_{(b)}(\delta, \rho)
\]

defined by

\[
\text{Gl}u^L_{(r)}(\zeta) := \left( (P^L_{b_{(r)}})^{-1} \left( \begin{array}{c} i_{\zeta} + Q^L_{b_{(r)}} \circ f^L_{s,h_{(r)}} f_r \left( \begin{array}{c} \zeta \end{array} \right) \end{array} \right) \right) \quad \forall [\zeta] \in \text{F} \big|_{[b_o]}.
\]

Given a frame \( e_\alpha(z) \) on \( \tilde{F} \big|_{b_o}, 1 \leq \alpha \leq \text{rank } \tilde{F} \), as Remark [3.6] we have a \( G_{b_o} \)-equivariant frame field

\[
e_\alpha((r), s, h)(z) = (P^L_{b_{(r)}})^{-1} \left( \left( \begin{array}{c} i_{\zeta} + Q^L_{b_{(r)}} \circ f^L_{s,h_{(r)}} f_r \left( \begin{array}{c} \zeta \end{array} \right) \end{array} \right) \right)(z)
\]

over \( D^*_{R_0}(0) \times \tilde{O}_{b_o}((\delta, \rho_{b_o})) \), where \( z \) is the coordinate on \( \Sigma \), and

\[
D^*_{R_0}(0) := \bigcup_{i=1} \{(r, \tau) | R_0 < r < \infty, \tau \in S^1 \}.
\]

For any fixed \( (r) \), \( e_\alpha \) is smooth with respect to \( s, h \) over \( \tilde{O}_{b_o}((\delta, \rho_{b_o})) \).

### 4.5 Gluing \( J \)-holomorphic maps

We recall some results in [3]. Let \( b_o = (a_o, u) \) and \( u \) be a \((j_o, J)\)-holomorphic map. The domain \( \Sigma \) of elements of \( M^1 \) are marked nodal Riemann surfaces. Suppose that \( \Sigma \) has nodes \( p_1, \ldots, p_t \) and marked points \( y_1, \ldots, y_n \). We choose local coordinate system \( A \) and define a pregluing map \( u_{(r)} : \Sigma_{(r)} \to M \) as in [4.4]. Set

\[
t_i = e^{-2r_i - 2\pi i}, \quad |r| = \min\{r_1, \ldots, r_t\}, \quad b_{(r)} := (a_o, (r), u_{(r)}).
\]

Let \( K \) be a \( N \)-dimensional linear space. Let

\[
i : K \times A \times W^{k,2,\alpha}(\Sigma(R_o), (u |_{\Sigma(R_o)})^*TM) \to W^{k-1,2,\alpha}(\Sigma(R_o), (u |_{\Sigma(R_o)})^*TM \otimes \lambda_{j_o}^0 T^*Sigma(R_o))
\]

be a smooth map such that \( D_v + di(\kappa, s, v |_{\Sigma(R_o)}) \) is surjective for any \((\kappa, b) \in K \times O_{b_o}(R, \delta, \rho), \) where \( b = (s, (r), v), v = \exp_{u_{(r)}} h \) and \( O_{b_o}(R, \delta, \rho) = \cup_{|r| \geq R} O_{b_o}(\delta, \rho) \).

Define a thickened Fredholm system \((K \times O_{b_o}(R, \delta, \rho), K \times E |_{O_{b_o}(R, \delta, \rho)}, S)\) with

\[
S(\kappa, b) = \delta_{j_o} v + i(\kappa, b).
\]

The following lemma is proved in [3].

**Lemma 4.6.** For \( |r| > R_0 \) there is an isomorphism \( I_{(r)} : \ker DS_{(\kappa_o, b_o)} \to \ker DS_{(\kappa_o, h_{(r)})} \).

For fixed \( (r) \) we consider the family of maps:

\[
F_{(r)} : K \times A \times W^{k,2,\alpha}(\Sigma_{(r)}, u^*_r TM) \to W^{k-1,2,\alpha}(\Sigma_{(r)}, (u^*_r TM \otimes \lambda_{j_o}^0 T^*Sigma_{(r)}),
\]

\[
F_{(r)}(\kappa, s, h) = \Psi_{j_o, b_o} \Phi_{u_{(r)}}(h)^{-1} \left( \delta_{j_o} v + i(\kappa, b) \right),
\]

where \( b = (s, (r), v), v = \exp_{u_{(r)}} h \) and

\[
\Psi_{j_o, b_o} \Phi_{u_{(r)}}(h)^{-1} : W^{k-1,2,\alpha}(\Sigma_{(r)}, v^*TM \otimes \lambda_{j_o}^0 T^*Sigma_{(r)}) \to W^{k-1,2,\alpha}(\Sigma_{(r)}, u^*_r TM \otimes \lambda_{j_o}^0 T^*Sigma_{(r)}).
\]

17
By implicit function theorem (Theorem 7.1, Theorem 7.2), there exist \( \delta > 0, \rho > 0, R > 0 \), a small neighborhood \( O(r) \) of \( 0 \in \ker DS|_{b(r)} \) and a unique smooth map

\[
f_r : \mathbf{A} \times O(r) \to W^{k-1,2,\alpha}(\Sigma(r), u_r^*TM \otimes \wedge^{0,1}T\Sigma(r))
\]

such that for any \((\kappa, s, h) \in \mathbf{A} \times O(\varepsilon) \) and \(|r| > R\),

\[
S(\kappa, b) = 0. \tag{4.28}
\]

Let \((s_l^i, t_l^i), l = 1, 2\) be the cylinder coordinates near the node \( q_i \). Set

\[
V_i := \bigcup_{l=1}^2 \{ (s_l^i, t_l^i) \in \Sigma \mid \frac{\pi}{4} \leq |s_l^i| \leq \frac{3\pi}{4}\}.
\]

Let \( \pi : K \times W^{k,2,\alpha}_{r}(\Sigma_{(\varepsilon)}, u_0^*TM) \to W^{k,2,\alpha}_{r}(\Sigma, u^*TM) \) be the projection. Denote

\[
\begin{align*}
Glu_{s,(r)}(\kappa, \xi) &= I(r)(\kappa, \xi) + Q_{(\kappa, b(\varepsilon))} \circ f_{s,(r)} \circ I_r(\kappa, \xi), \\
Glu_{r}(\kappa, \xi) &= I_r(\kappa, \xi) + Q_{(\kappa, b(\varepsilon))} \circ f_{s,(r)} \circ I_r(\kappa, \xi).
\end{align*}
\]

In [3] we proved

**Theorem 4.7.** There exists positive constants \( C, d, R_0 \) such that for any \((\kappa, \xi) \in \ker DS_{(\kappa, b_0)}\) with \( \|(\kappa, \xi)\| < d \), and any \( X_i \in \{ \frac{\partial}{\partial \psi_i}, \frac{\partial}{\partial \rho_i}\} \), \( i = 1, \ldots, \varepsilon \), the following estimate hold

\[
\begin{align*}
\|X_i \left( \text{Glu}_{s,(r)}^* (\kappa, \xi) \right) \|_{k-2,2,\alpha} &\leq Ce^{-\varepsilon (\varepsilon - 5\alpha)^{\frac{d_2}{4}}} , \\
\|X_iX_j \left( \text{Glu}_{s,(r)}^* (\kappa, \xi) \right) \|_{k-2,2,\alpha} + \|X_i \left( \text{Glu}_{s,(r)}^* (\kappa, \xi) \right) \|_{V_i} \|_{k-2,2,\alpha} &\leq Ce^{-\varepsilon (\varepsilon - 5\alpha)^{\frac{r_i + r_j}{4}}},
\end{align*}
\]

\( 1 \leq i \neq j \leq \varepsilon \), for any \( s \in \bigotimes_{i=1}^\varepsilon O_i \).

5 **Smoothness of \( \text{Glu}_{r}^L(e_\alpha) \mid \Sigma(R_0) \)**

We have shown in [4,4] that for any fixed \( (r) \), \( \text{Glu}_{r}^L \) is smooth with respect to \( s, h \) over \( \tilde{O}_{b_0}(\delta_0, \rho_0) \). In this section we discuss the smoothness with respect to \( (r) \), \( s, h \). To this end we need to fix a Riemann surface \( \Sigma_{(R_0)} \).

We first consider gluing one node case. Let \( \alpha_{(r)} : [0, 2r] \to [0, 2R_0] \) be a smooth increasing function satisfying

\[
\alpha_{(r)}(s) = \begin{cases} 
\frac{R_0}{2} + \frac{R_0}{2r - R_0} (s - R_0/2) & \text{if } s \in [0, R_0/2 - 1] \\
2r + 2R_0 & \text{if } s \in [0, 2r - R_0/2 + 1, 2r]
\end{cases}
\]

Set \( \alpha_{(r)} : [-2r, 0] \to [-2R_0, 0] \) by \( \alpha_{(r)}(s) = -\alpha_{(r)}(-s) \). We can define a map \( \varphi_{(r)} : \Sigma_{(r)} \to \Sigma_{(R_0)} \) as follows:

\[
\varphi_{(r)} = \begin{cases} 
p, & p \in \Sigma(R_0/4), \\
(\alpha_{(r)}(s_1), t_1), & (s_1, t_1) \in \Sigma_{(r)} \setminus \Sigma(R_0/4).
\end{cases}
\]

Obviously, \( \varphi_{(r)}^{-1}(y) = y \). For any \( s_1 \in [0, 2r] \) and \( s_2 \in [-2r, 0] \), we have

\[
s_1 = s_2 + 2r \iff \alpha_{(r)}(s_1) = \alpha_{(r)}(s_2) + 2R_0. \tag{5.1}
\]

Then we obtain a family of Riemann surfaces \( \left( \Sigma_{(R_0)}, (\varphi_{(r)}^{-1})^*j_r, \varphi_{(r)}^{-1}(y) \right) \).
Denote \( u^0_{(r)} := u_{(r)} \circ \varphi_r^{-1} \). The \( \varphi_r^{-1} \) induce an isomorphism

\[
(\varphi_r^{-1})^* : W^{k,2,\alpha}(\Sigma_{(r)}, u^0_{(r)} TM) \rightarrow W^{k,2,\alpha}(\Sigma_{(r)}, (u^0_{(r)})^* TM).
\]

For any \( h \in W^{k,2,\alpha}(\Sigma_{(r)}, (u_{(r)})^* TM), \) denote \( v = \exp u_{(r)}(h) \), we have map \( \varphi^*_r v : \Sigma_{(r)} \rightarrow M. \) There exists a family of functions \( \hat{h}_{(r)} \in W^{k,2,\alpha}(\Sigma_{(r)}, (u_{(r)})^* TM) \) such that \( u^0_{(r)} = \exp u_{(r)}(\hat{h}_{(r)}) \). It is easy to check that \( \hat{h}_{(r)} \) is a smooth family of functions and for any \( l \in \mathbb{Z}^+ \),

\[
\left\| \hat{h}_{(r)} \right\|_{C^l(\Sigma_{(r)})} \leq C(r,l), \tag{5.2}
\]

for some constant \( C(r,l) > 0 \) depending only on \( r \) and \( l \). Denote

\[
j_r^0 = (\varphi_{(r)}^{-1})^* j_r, \quad b_{(r)} := (j_{(r)} u_{(r)}), \quad b^0_{(r)} := (j_r^0, u^0_{(r)}), \quad b := (j_r^0, v).
\]

Let \((s, t)\) be the holomorphic coordinates on \(\Sigma_{(r)} \setminus \Sigma_{(R_0)}\) such that \( j_r^0(\frac{\partial}{\partial s^0}) = \frac{\partial}{\partial s^0}, j_r(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial t} \). Denote \((s^0, t^0) = \varphi_r(s, t)\). Then we have

\[
 j^0_r \frac{\partial}{\partial s^0} = \frac{1}{\varphi_r'(s)} \frac{\partial}{\partial s^0}, \quad j^0_r \frac{\partial}{\partial t^0} = -\varphi_r'(s) \frac{\partial}{\partial s^0} \quad \text{in } \Sigma_{(r)} \setminus \Sigma_{(R_0)/4}. \tag{5.3}
\]

Then for any \( \eta \in W^{k-1,2,\alpha}(\Sigma_{(R_0)}), v^* \tilde{L} \otimes \Lambda_{j_r}^0 T^* \Sigma_{(R_0)} \) and \( p \in \Sigma_{(R_0)}, \Psi_{j_r, j_{(r)}} \eta(p) \) is a smooth family of isomorphisms. Since \( M, u_{(r)} \) and \( v \) are smooth, \( \Phi_{u_{(r)}, v} \) is also a smooth family of isomorphisms. It follows that \( \Phi_{b_{(r)}, v} \) is a smooth family of isomorphisms. In particular, \( \Phi_{b_{(r)}, v} \) is smooth with respect to \( r \).

We have the operator

\[
D_{\tilde{L}}|_{b_{(r)}} : W^{k,2}(\Sigma_{(r)}, L|_{b_{(r)}}) \rightarrow W^{k-1,2}(\Sigma_{(r)}, L|_{b_{(r)}}),
\]

Using (5.3) one can easily check that

\[
(\varphi_{(r)}^{-1})_* D_{\tilde{L}}|_{b_{(r)}} = D_{\tilde{L}}|_{b_{(r)}}.
\]

We define \( Q_{\tilde{L}}^r_{b_{(r)}} : W^{k-1,2}(\Sigma_{(R_0)}, L|_{b_{(r)}}) \otimes \Lambda_{j_r}^0 T^* \Sigma_{(R_0)} \rightarrow W^{k,2}(\Sigma_{(R_0)}, L|_{b_{(r)}}) \) by

\[
Q_{\tilde{L}}^r_{b_{(r)}} \eta_{b_{(r)}}^0 = (\varphi_{(r)}^{-1})^* \left( Q_{\tilde{L}}^r (\varphi_{(r)}^* \eta_{b_{(r)}}^0) \right).
\]

We define \( Q_{\tilde{L}}^r_{b_{(r)}} : W^{k-1,2}(\Sigma_{(R_0)}, L|_{b_{(r)}}) \otimes \Lambda_{j_r}^0 T^* \Sigma_{(R_0)} \rightarrow W^{k,2}(\Sigma_{(R_0)}, L|_{b_{(r)}}) \) by

\[
Q_{\tilde{L}}^r_{b_{(r)}} = Q_{\tilde{L}}^r_{b_{(r)}} \left[ D_{\tilde{L}}|_{b_{(r)}} Q_{\tilde{L}}^r_{b_{(r)}} \right]^{-1}.
\]

We can also define \( \tilde{I}_{\tilde{L}}^r_{b_{(r)}} : Ker D_{\tilde{L}}|_{b_{(r)}} \rightarrow Ker D_{\tilde{L}}|_{b_{(r)}} \) by

\[
\tilde{I}_{\tilde{L}}^r_{b_{(r)}}(\zeta) = (\varphi_{(r)}^{-1})^* (I_{(r)}^r(\zeta)).
\]

It is easy to see that

\[
C(k, \alpha, r)^{-1}\| Q_{\tilde{L}}^r_{b_{(r)}} \| \leq \| Q_{\tilde{L}}^r_{b_{(r)}} \| \leq C(k, \alpha, r)\| Q_{\tilde{L}}^r_{b_{(r)}} \|
\]

where \( C(k, \alpha, r) \) is a constant depending only on \( k, \alpha \) and \( r \).

Denote

\[
D_{\tilde{L}} = F_{b_{(r)}b_{(r)}} \circ D_{\tilde{L}}|_{b_{(r)}} \circ \left[ F_{b_{(r)}b_{(r)}} \right]^{-1}, \quad Q_{\tilde{L}} = F_{b_{(r)}b_{(r)}} \circ Q_{\tilde{L}}^r_{b_{(r)}} \circ \left[ F_{b_{(r)}b_{(r)}} \right]^{-1},
\]
\[ Q^L = P^{\tilde{L}}_{b'(r),b(R_0)} \circ Q^L_{b'(r)} \circ \left[ P^{\tilde{L}}_{b'(r),b(R_0)} \right]^{-1}. \]

For any \( \eta \in W^{k-1,2}(\Sigma(R_0), L|_{b(R_0)} \otimes \wedge^1_{J_{R_0}} T\Sigma(R_0)) \) denote \( \eta^o_{\phi} = \left[ P^{\tilde{L}}_{b'(r),b(R_0)} \right]^{-1} \eta \) and \( \eta_r = \varphi_r^* \eta^o_{\phi} \). Let \((\eta_1, \eta_2) = (\beta_{1,2}(s_1) \eta_r(s_1, t_1), \beta_{1,2}(s_2) \eta_r(s_2, t_2))\). Denote \((h_1, h_2) = Q^L_{b'(r)}(\eta_1, \eta_2)\). Since \((s_1^0, t_1^0) = \varphi(s_1, t_1)\), we have

\[
Q^L_{b'(r)} \circ \left[ P^{\tilde{L}}_{b'(r),b(R_0)} \right]^{-1} \eta = Q^L_{b'(r)} \circ \eta^o_{\phi} = (\varphi_r^{-1})^* \left( Q^L_{b'(r)}(\eta_r) \right)
= \beta_{1,r} \cdot \varphi_r^{-1}(s_1^0)h_1 \cdot \varphi_r^{-1}(s_1^0, t_1^0) + \beta_{2,r} \cdot \varphi_r^{-1}(s_1^0 - 2R_0)h_1 \cdot \varphi_r^{-1}(s_1^0 - 2R_0, t_1^0 - 2R_0).
\]

Since \( P^{\tilde{L}}_{b'(r),b(R_0)} \) is a smooth, we have

\[
\| \nabla^L_{\phi} Q^L_{b'} \eta \|_{k,2} \leq C(k, \alpha, r)\| \eta \|_{k-1,1,2}.
\]

Similarly, we obtain that

\[
\| \nabla^L_{\phi} D^{\tilde{L}}(\zeta) \|_{k-1,2} \leq C(k, \alpha, r)\| \zeta \|_{k+1,2}, \quad \| \nabla^L_{\phi} Q^L_{b'} \eta \|_{k,2} \leq C(k, \alpha, r)\| \eta \|_{k-1,1,2}, \quad (5.4)
\]

\[
\| \nabla^L_{\phi} \left( P^{\tilde{L}}_{b'(r),b(R_0)} \circ I^{\tilde{L}}_{b'(r)} \right)(\zeta') \|_{k,2} \leq C(k, \alpha, r)\| \zeta' \|_{k+1,2}, \quad \forall \zeta' \in \text{Ker} D^L_{b'} \quad (5.5)
\]

The above estimates can be generalized to gluing several nodes.

We can define \( \varphi(r), b'(r) \) and \( b(R_0) \) as above. We define a map

\[
\mathcal{F} : D^L_{R_0}(0) \times A \times W^{k,2}(\Sigma(R_0), u'|_{R_0} TM) \times \mathcal{W}^{k,2,\alpha}(\Sigma, \tilde{L}|_{b_0}) \to W^{k-1,2,\alpha}(\Sigma(R_0), \tilde{L}|_{b(R_0)} \otimes \wedge^1_{J_{R_0}} T\Sigma(R_0))
\]

by

\[
\mathcal{F}((r), s, h, \zeta) = P^{\tilde{L}}_{b(b'(r))} \circ D_{b'} \circ (P^{\tilde{L}}_{b(b'(r))})^{-1} I_{b'(r)}(\zeta),
\]

where \( b = (\Sigma(R_0), (r), s, v), s(j_o, y) = 0 \) and \( v = \exp u|_{R_0}(h) \). By the same argument as in Lemma \( 3.3 \), we see that \( \mathcal{F} \) is a smooth function. There exists a family smooth function \( h_r(r) \) such that \( u^o_r = \exp u|_{R_0}(h_r(r)) \).

Obviously, when \( \| h - h_r(r) \|_{k,2} \) small we have

\[
\mathcal{F}((r), s, h, \zeta) = P^{\tilde{L}}_{b(b'(r))} (P^{\tilde{L}}_{b(b'(r))})^{-1} (\varphi_r^{-1})^* \left( \mathcal{F}(r)(s, h', I_{h_r(r)}(\zeta)) \right),
\]

where \( h' = (\exp u^o_r \circ (\exp u|_{R_0}(h) \circ \varphi(r)) \). Then by \( 4.26 \) and the uniqueness of the implicit function we have

\[
D^{\tilde{L}} \circ (P^{\tilde{L}}_{b(b'(r))})^{-1} \left( I_{b'(r)}(\zeta) + Q^{\tilde{L}}_{b'(r)} \circ I_{s,h,b'(r)}(\tilde{L}_{b'(r)}(\zeta)) \right) = 0
\]

as \( \| s \| \) and \( \| h \|_{k,2} \) small. Since

\[
D \mathcal{F}(r)(s, h', 0)(\zeta_1) = \mathcal{F}(r)(s, h', \zeta_1), \quad D \mathcal{F}(r)(s, h', 0)(0) = 0,
\]

we have an explicit formula for \( f^{\tilde{L}}_{a,h',r}(r) \) ( see \( 7.6 \)) in the proof of Theorem \( 7.2 \):

\[
f^{\tilde{L}}_{a,h',r}(r) \circ I_{r}(\zeta) = \mathcal{F}(r)(0, 0, \mathcal{H}^{-1}(I_{r}(\zeta))
\]

where and \( \mathcal{H} \) is defined by

\[
\mathcal{H}(x) := x + Q^{\tilde{L}}_{r}(\mathcal{F}(r)(s, h', x) - \mathcal{F}(r)(0, 0, x)).
\]
It follows that
\[ f_{a,h,b_0}^L \circ I_{b_0}^L(\zeta) = f_{b_0}^L(\rho_{b_0}(P_{b_0}(R)))^{-1} F((r), 0, \hat{h}(r), (I_{b_0}^L)^{-1} \circ \mathcal{H}_0^{-1} \circ (I_{b_0}^L(\zeta))) \]
where
\[ \mathcal{H}_0(x) := x + Q_{b_0}^L \left( (P_{b_0}(R))^{-1} F((r), s, h, x) - (P_{b_0}(R))^{-1} F((r), 0, \hat{h}(r), x) \right). \]

Choose \( \delta, \rho \) small and \( |r| \) big enough. By (5.4) and \( \nabla_r \mathcal{H}_0^{-1} = -\mathcal{H}_0^{-1} \circ (\nabla_r \mathcal{H}_0) \circ \mathcal{H}_0^{-1} \), one can check that
\[ \|\nabla_r^l \left( f_{b_0}^L \circ f_{a,h,b_0}^L \circ I_{b_0}^L \right)(\zeta) \|_{k-1,2} \leq C\|\zeta\|_{k+l,2,\alpha}, \]
where \( \nabla_r = \nabla_{r_1} \cdots \nabla_{r_l} \) with \( \sum_{i=1}^l l_i = l \). Then we have for any \( \zeta \in Ker \mathcal{D}^L|_{b_0}, \)
\[ \left\| \nabla_r \left[ f_{b_0}^L \circ I_{b_0}^L \right](\zeta) \right\|_{k,2} \leq C\|\zeta\|_{k+l,2,\alpha}. \]

On the other hand, since \( u \) is smooth and \( \mathcal{D}^L|_{b_0} = \partial u \zeta = 0 \), by the standard elliptic estimate we have
\[ \|\zeta\|_{k+l,2,\alpha} \leq C\|\zeta\|_{k,2,\alpha}. \]

Hence \( Glu_{s_0(r),h_0}^L(e_0) \circ \varphi_r^{-1} \) is a smooth family. We have proved

**Lemma 5.1.** There exists positive constants \( d, R \) such that for any \( \zeta \in Ker \mathcal{D}^L|_{b_0} \), \( h \in W^{k,2,\alpha}(\Sigma(R_0), (u(R_0))^* TM) \) with
\[ \|\zeta\|_{W^{k,2,\alpha}} \leq d, \quad \|h - \hat{h}(r)\| \leq d, \quad |r| \geq R, \]
\((\varphi_r^{-1})^*(Glu_{s_0(r),h_0}^L(e_0))\) is smooth with respect to \((s, (r), h)\) for any \( e_0 \in Ker \mathcal{D}^L|_{b_0} \), where \( h' = (\exp_{u_0}^L \circ (\exp_{u_0(R_0)}^L (h)) \circ \varphi_r). \) In particular \( Glu_{s_0(r),h_0}^L(e_0) \mid_{\Sigma(R_0)} \) is smooth.

**6 Estimates of derivatives with respect to gluing parameters**

In this section we prove the following theorem.

**Theorem 6.1.** Let \( l \in \mathbb{Z}^+ \) be a fixed integer. Let \( u : \Sigma \rightarrow M \) be a \((j, J)\)-holomorphic map. Let \( c \in (0, 1) \) be a fixed constant. For any \( 0 < \alpha < \frac{1}{1000} \), there exists positive constants \( C, d, R \) such that for any \( \zeta \in Ker \mathcal{D}^L|_{b_0}, \)
\((\kappa, \xi) \in Ker \mathcal{D} h_{(\kappa, \xi), b_0} \) with
\[ \|\zeta\|_{W^{k,2,\alpha}} \leq d, \quad \|(\kappa, \xi)\| \leq d, \quad |r| \geq R, \]
restricting to the compact set \( \Sigma(R_0) \), the following estimate hold.
\[ \left\| X_i \left( Glu_{s_0(h_0)}^L((r)) \right) \right\|_{C^l(\Sigma(R_0))} \leq C_1 e^{-(\epsilon - 5\alpha)\frac{r}{4}}, \quad (6.1) \]
\[ \left\| X_i X_j \left( Glu_{s_0(h_0)}^L((r)) \right) \right\|_{C^l(\Sigma(R_0))} \leq C_2 e^{-(\epsilon - 5\alpha)\frac{r + r}{4}} \quad (6.2) \]
for any \( X_i \in \left\{ \frac{\partial}{\partial r_i}, \frac{\partial}{\partial r_j} \right\}, i = 1, \cdots, c, s \in \bigotimes_{i=1}^c O_i \) and any \( 1 \leq i \neq j \leq c. \)
6.1 Some operators

It is important to estimate the derivative of the gluing map with respect to \( r \). To this end we need to take the derivative \( \frac{\partial}{\partial r} \) for \( Q^L_{\mathcal{B}(r)} \) and other operators. Note that both \( Q^L_{\mathcal{B}(r)} \) and \( f^L_{\mathcal{B}(r)} \) are global operators, so we need a global estimate. On the other hand, since the domain \( \Sigma_{(r)} \) depends on \( r \), in order to make the meaning of the derivative \( \frac{\partial}{\partial r} \) for these operators clear we need transfer all operators defined over \( \Sigma_{(r)} \) into a family of operators defined over \( \overset{\circ}{\Sigma}_1 \cup \overset{\circ}{\Sigma}_2 \), depending on \( r \). To simplify notations we will denote

\[
W^{k,2,\alpha}_u = W^{k,2,\alpha}(\Sigma, \tilde{L}|_{b_0}), \quad W^{k,2,\alpha}_u = W^{k,2,\alpha}(\Sigma, \tilde{L}|_{b_0}), \quad L^{k-1,2,\alpha}_u = W^{k-1,2,\alpha}(\Sigma, \tilde{L}|_{b_0} \otimes \Lambda_{\alpha}^{0,1} T^* \Sigma).
\]

\[
W^{k,2,\alpha}_{r,u(r)} = W^{k,2,\alpha}_{(r), \tilde{L}|_{b(r)}}, \quad L^{k-1,2,\alpha}_{r,u(r)} = W^{k-1,2,\alpha}_{(r), \tilde{L}|_{b(r)} \otimes \Lambda_{\alpha}^{0,1} T^* \Sigma_{(r)}}.
\]

We first define three maps

\[
H_r : L^{k-1,2,\alpha}_{r,u(r)} \to L^{k-1,2,\alpha}_u, \quad P_r : L^{k-1,2,\alpha}_u \to L^{k-1,2,\alpha}_{r,u(r)}, \quad \phi_r : W^{k,2,\alpha}_u \to W^{k,2,\alpha}_{r,u(r)}
\]

as following. Given \( \eta \in L^{k-1,2,\alpha}_{r,u(r)} \) define

\[
H_r \eta = (\beta_{1,2}(s_1) \eta(s_1, t_1), \beta_{2,2}(s_2) \eta(s_2, t_2)),
\]

where \( \eta(s_i, t_i) \) is the expression of \( \eta \) in terms the coordinates \((s_i, t_i)\). Given \((\eta_1, \eta_2) \in L^{k-1,2,\alpha}_u \) define

\[
P_r(\eta_1, \eta_2) = \begin{cases} 
\eta_1 & \text{if } p \in \Sigma(r/2) \\
\beta_{1,2}(s_1) \eta_1(s_1, t_1) + \beta_{2,2}(s_2 - 2 r) \eta_2(s_2 - 2 r, t_1 - \tau) & \text{if } p \in \Sigma(r) \setminus \Sigma(r/2).
\end{cases}
\]

If no danger of confusion we will denotes \((6.3)\) by \( P_r(\eta_1, \eta_2) = \sum \beta_i \eta_i \). Given \((\zeta_1 + \hat{\zeta}_0, \zeta_2 + \hat{\zeta}_0) \in W^{k,2,\alpha}_u \) with supp \( \zeta_i \subset \Sigma(3r/2), \) define

\[
\phi_r \left( \zeta_1 + \hat{\zeta}_0, \zeta_2 + \hat{\zeta}_0 \right) \bigg|_{\Sigma(r/2)} = \left( \zeta_1 + \hat{\zeta}_0 \right) (s_1, t_1) \bigg|_{\Sigma(r/2)},
\]

\[
\phi_r \left( \zeta_1 + \hat{\zeta}_0, \zeta_2 + \hat{\zeta}_0 \right) \bigg|_{\frac{r}{2} \leq s_1 \leq \frac{3r}{2}} = \left( \zeta_1 (s_1, t_1) + \zeta_2 (s_1 - 2r, t_1 - \tau) + \hat{\zeta}_0 \right) \bigg|_{\frac{r}{2} \leq s_1 \leq \frac{3r}{2}}.
\]

By \((4.3)\) one can check that

\[
P_r H_r = 1d, \quad H_r P_r(\eta_1, \eta_2) = (\tilde{\zeta}_1, \tilde{\zeta}_2).
\]

where

\[
\tilde{\zeta}_1 = \beta_{1,2}(\beta_{1,2}(s_1, t_1) + \beta_{2,2}(s_2 - 2r, t_1 - \tau)),
\]

\[
\tilde{\zeta}_2 = \beta_{2,2}(\beta_{1,2}(s_2 + 2r, t_2 + \tau) + \beta_{2,2}(s_2, t_2)).
\]

In particular, \( H_r \) is injective and \( P_r \) is surjective.

Next we introduce the following three operators

\[
\left( Q^L_{\mathcal{B}(r)} \right)^* : L^{k-1,2,\alpha}_{r,u(r)} \to W^{k,2,\alpha}_u, \quad \left( Q^L_{\mathcal{B}(r)} \right)^* : L^{k-1,2,\alpha}_{r,u(r)} \to W^{k,2,\alpha}_u, \quad \left( f^L_{\mathcal{B}(r)} \right)^* : \ker \tilde{D}|_{b_0} \to W^{k,2,\alpha}_u \).
\]

Given \( \eta \in L^{k-1,2,\alpha}_{r,u(r)} \), denote

\[
(\zeta_1, \zeta_2) = Q^L_{\mathcal{B}(r)} H_r \eta.
\]

Set

\[
(\zeta^*_r = (\beta_{1,r}(s_1) \zeta_1(s_1, t_1), \beta_{2,r}(s_2) \zeta_2(s_2, t_2)) \in W^{k,2,\alpha}_u.
\]

(6.6)
Define
\[
(Q_{b(r)}^L)^* \eta = \zeta_r, \quad (Q_{b(r)}^L)^* = (Q_{b(r)}^L)^* \left( D_{b(r)}^L Q_{b(r)}^L \right)^{-1}.
\] (6.7)

Then we have maps
\[
(Q_{b(r)}^L)^* P_r : L_u^{k-1,2,\alpha} \rightarrow W_u^{k,2,\alpha}, \quad (Q_{b(r)}^L)^* P_r : L_u^{k-1,2,\alpha} \rightarrow W_u^{k,2,\alpha}.
\]

For any \( \zeta + \hat{\zeta}_0 \in \text{ker} D_{b,0}^L \), where \( \zeta = (\zeta_1, \zeta_2) \in W_u^{k,2,\alpha} \), we set
\[
\zeta_r^* = \left( \zeta_1 \beta_{1:r} + \hat{\zeta}_0, \quad \zeta_2 \beta_{2:r} + \hat{\zeta}_0 \right).
\] (6.8)

Define
\[
(I_{r(r)}^L)^*(\zeta + \hat{\zeta}) = \zeta_r^* - (Q_{b(r)}^L)^* D_{b(r)}^L \circ \phi_r \zeta_r^*.
\] (6.9)

By the definition we have
\[
I_{r(r)}^L = \phi_r \circ (I_{r(r)}^L)^*, \quad Q_{b(r)}^L = \phi_r \circ (Q_{b(r)}^L)^*.
\]

Define an operator \( X : L_u^{k-1,2,\alpha} \rightarrow L_r^{k-1,2,\alpha} \) by
\[
X(\eta_1, \eta_2) = D_{b(r)}^L Q_{b(r)}^L P_r(\eta_1, \eta_2) - P_r(\eta_1, \eta_2).
\]

Using \( E_{u, s}^L = 0 \), one can check that
\[
X(\eta_1, \eta_2) = \sum (\tilde{\beta} \beta_{i:r}) h_i + \sum \beta_{i:r} E_{u, s}^L h_i + \left( \sum \beta_{i:r} \beta_{i;2} - 1 \right) \sum \beta_{i;2} \eta_i,
\]
where \( (h_1, h_2) = Q_{b(r)}^L H_r P_r(\eta_1, \eta_2) \). Obviously, \( \text{supp} X(\eta_1, \eta_2) \subset \{ \frac{r}{2} \leq |s_i| \leq \frac{3r}{2} \} \).

Let \( b = (a, v) \in \tilde{O}_{b, \alpha}(\delta_{\alpha}, \rho_0) \), where \( v = \text{exp}_{u, r}(h) \). We define
\[
Gl u_{a, h, (r)}^L := \left( I_{r(r)}^L \right)^* + (Q_{b(r)}^L)^* \circ f_{a, h, (r)}^L \circ \left( I_{r(r)}^L \right)^*: \bar{F}_{b, 0} \rightarrow W_u^{k,2,\alpha}(\Sigma, \bar{L}_{b, 0}).
\]

This definition can be extended to the gluing several nodes case in a natural way:
\[
Gl u_{a, h, (r)}^L := \left( I_{r(r)}^L \right)^* + (Q_{b(r)}^L)^* \circ f_{a, h, (r)}^L \circ \left( I_{r(r)}^L \right)^*: \bar{F}_{b, 0} \rightarrow W_u^{k,2,\alpha}(\Sigma, \bar{L}_{b, 0}).
\]

It is easy to see that, restricting to \( \Sigma(0) \), we have \( Gl u_{a, h, (r)}^L(\zeta) = P_{b, h, (r)}^L \circ Gl u^L_{a, h, (r)}(\zeta) \) for any \( \zeta \in D_{b, 0}^L \).

### 6.2 Estimates of the first derivatives

Let \( \eta = (\eta_1, \cdots, \eta_k) \in L_u^{k-1,2,\alpha} \). Denote
\[
D^i_j(R_0) = \left\{ (s^i_1, t^i_2) \in \Sigma \mid |s^i_1| \geq R_0 \right\}, \quad D^i(R_0) = \bigcup_{i=1}^n D^i_j(R_0).
\]

Denote \( h_{(r)} = \pi \circ Gl u_{a, (r)}(\kappa, \xi), \quad h'_{(r)} = \pi \circ Gl u^*_{a, (r)}(\kappa, \xi) \) and \( v_{(r)} = \text{exp}_{u, (r)}(h_{(r)}) \). Set
\[
\beta_{1,i,R}(s^i_1) = \beta \left( \frac{1}{2} + \frac{r^i - s^i_1}{R} \right), \quad \beta_{2,i,R}(s^i_2) = \sqrt{1 - \beta^2 \left( \frac{1}{2} - \frac{s^i_2 + r^i}{R} \right)}.
\]

To simplify notations we denote
\[
D := D_{b(r)}, \quad Q := Q_{b(r)}^L, \quad I^* := \left( I_{r(r)}^L \right)^*, \quad f = f_{r(r)}^L, \quad Q' := \left( Q_{b(r)}^L \right)^*, \quad P = P_{b(r)}^L, \quad E = E_{b(r)}^L, \quad (Q')^* := \left( Q_{b(r)}^L \right)^{f*}, \quad Q^* := \left( Q_{b(r)}^L \right)^{r*}.
\]

The following Lemmas can be proved by the same method and word-by-word as in [3], we omit them.
Lemma 6.2. For any \((\eta_1, \eta_2) \in L^{k-1, 2, \alpha}_u\), the following estimates hold:

\[
(1) \| (Q')^* P_r(\eta_1, \eta_2) \|_{k, 2, \alpha} \leq C \left( e^{-(c-\alpha)\bar{T}} \sum_{|s_i| \leq \bar{T}} \| \eta_i \|_{s_i \leq r+1} \| k-1, 2, \alpha + \sum_{i=1}^{2} \| \eta_i \|_{s_i \leq r+1} \| k-1, 2, \alpha } \right),
\]

\[
(2) \left\| \frac{\partial}{\partial r} ((Q')^* P_r)(\eta_1, \eta_2) \right\|_{k-1, 2, \alpha} \leq C \left( e^{-(c-\alpha)\bar{T}} \sum_{|s_i| \leq \bar{T}} \| \eta_i \|_{s_i \leq r+1} \| k-1, 2, \alpha + \sum_{i=1}^{2} \| \eta_i \|_{s_i \leq r+1} \| k-1, 2, \alpha } \right),
\]

\[
(3) \left\| \frac{\partial}{\partial r} (H_r(DQ')^{-1} P_r)(\eta_1, \eta_2) \right\|_{k-1, 2, \alpha} \leq C \left( e^{-(c-\alpha)\bar{T}} \| \eta_1 \|_{s_1 \leq \bar{T}} \| k-1, 2, \alpha + \| \eta_1 \|_{s_2 \leq \bar{T}} \| k-1, 2, \alpha } \right),
\]

\[
(4) \left\| H_r(DQ')^{-1} P_r(\eta_1, \eta_2) \right\|_{k-1, 2, \alpha} \leq C \left( e^{-(c-\alpha)\bar{T}} \| \eta_1 \|_{s_1 \leq \bar{T}} \| k-1, 2, \alpha + \| \eta_1 \|_{s_2 \leq \bar{T}} \| k-1, 2, \alpha } \right),
\]

\[
(5) \left\| \frac{\partial}{\partial r} (H_r P_r)(\eta_1, \eta_2) \right\|_{k-1, 2, \alpha} \leq C \sum_{i=1}^{2} \left\| \eta_i \right\|_{s_i \leq \bar{T}} \| k-1, 2, \alpha + \| \eta_i \|_{s_i \leq \bar{T}} \| k-1, 2, \alpha } \right).
\]

Lemma 6.3. There exists a constant \(C > 0\), independent of \(r\), such that for any \(h + \hat{h}_0 \in kerD|_{h_0}\):

\[
\left\| \frac{\partial}{\partial r} \nu^*(h + \hat{h}_0) \right\|_{k-1, 2, \alpha} \leq C \left\| h_i \right\|_{s_i \leq \bar{T}} \| k-1, 2, \alpha + C e^{(c-\alpha)\bar{T}} \| \hat{h}_0 \|. \quad (6.10)
\]

Denote \(\nu_{(r)} = P^{-1} \circ Glv_{h_0}^{L_r}(\cdot)\), and \(\nu^*_{(r)} := \text{Gl}v_{h_0}^{L_r}(\cdot)\). Obviously, \(\nu_{(r)} = \phi_r(\nu^*_{(r)})\)

\[
\nu_{(r)} = I_r(\zeta) + Q \circ f \circ I_r(\zeta), \quad \nu^*_{(r)} = I^*_r(\zeta) + Q^* \circ f \circ I_r(\zeta). \quad (6.11)
\]

Set \(h^* = (h_1\beta_1 + h_0, h_2\beta_2 + h_0)\). Since \(u\) (resp. \(v\)) is a \((j_0, J)\) (resp. \((j_0, J)\)) holomorphic map, we have

\[
\sum_{i+j=d} \left\| \frac{\partial^{i+j} E}{\partial s^i \partial t^j} \right\| + \sum_{i+j=d} \left\| \frac{\partial^{i+j} E^L}{\partial s^i \partial t^j} \right\| \leq C d e^{-c|s|}, \quad R_0 \leq |s| \leq r. \quad (6.12)
\]

Taking derivative \(D\) on \(\nu_{(r)}\) we have

\[
f \circ I_{(r)}(\zeta) = \partial_{\nu_{(r)}}(\nu_{(r)}) + E \nu_{(r)}. \quad (6.13)
\]

On the other hand, by \(D^L h(P \nu_{(r)}) = 0\) we have

\[
\partial_{\nu_{(r)}}(\nu_{(r)}) + P^{-1} \left( \nabla \nu_{(r)}(P(\nu_{(r)}) + E^L h(P \nu_{(r)}) \right) = 0. \quad (6.14)
\]

By the exponential decay of \(u_{(r)}\) and \(v_{(r)}\) we have

\[
\left\| \nabla \nu_{(r)}(P) \right\| \leq C (|du_{(r)}| + |dv_{(r)}|) \leq C d e^{-c|s|}, \quad R_0 \leq |s| \leq r. \quad (6.15)
\]

By \((6.12)\) and \((6.15)\) we conclude that \(\nu_{(r)}\) satisfies the assumption of Lemma \(7.4\) in Appendix. Then Lemma \(7.4\) gives us

\[
\left\| \nu_{(r)} \|_{\xi \leq |s| \leq \bar{T}} \right\|_{k-1, 2, \alpha} \leq C e^{-(c-\alpha)\bar{T}}, \quad (6.16)
\]

It follows from \((6.13)\) and \((6.16)\) that

Lemma 6.4.

\[
\left\| H_r f \circ I_{(r)}(\zeta) \|_{|s| \geq \bar{T}} \right\|_{k-1, 2, \alpha} \leq C e^{-(c-\alpha)\bar{T}} (1 + \| \zeta \|_{k, 2, \alpha}), \quad \forall r \geq 8R_0. \quad (6.17)
\]
Similar Lemma 4.7 in [3], we have

**Lemma 6.5.** There exists a constant $C > 0$ such that for any $\zeta \in \ker D\tilde{L}_b$ we have

$$\left\| \frac{\partial}{\partial s_i} \nu_{(r),i} \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - 5\alpha)\frac{\zeta}{r}} (\|\zeta\|_{k,2,\alpha} + 1).$$

An estimate similar to Lemma 4.8 in [3] can be proved:

**Lemma 6.6.**

$$\left\| H_r \circ D\phi_r \left( \frac{\partial}{\partial r} \nu_{(r)}^* \right) \right\|_{k-1,2,\alpha} \leq C \left( d \left\| \frac{\partial}{\partial r} \nu_{(r)}^* \right\|_{k-1,2,\alpha} + e^{-(\epsilon - 5\alpha)\frac{\zeta}{r}} \right). \quad (6.18)$$

**Proof.** We estimate $\left\| \beta_{1:2} D\phi_r \left( \frac{\partial}{\partial r} \nu_{(r)}^* \right) \right\|_{k-1,2,\alpha}$. The estimates of $\left\| \beta_{2:2} D\phi_r \left( \frac{\partial}{\partial r} \nu_{(r)}^* \right) \right\|_{k-1,2,\alpha}$ is the same. As in [3] we construct two smooth family $\tilde{u}(r), \tilde{h}(r)$, depending on $(r)$, defined over $\Sigma_1$ as follows:

$$\tilde{u}(r) = \begin{cases} u(r), & \text{in } \Sigma_1(r + 1), \\ u_1(q) + \beta(r + 2 - s_1)(u_1(s_1, t_1) - u_1(q)), & \text{if } s_1 \geq r + 1 \end{cases} \quad (6.19)$$

$$\tilde{h}(r) = \begin{cases} h(r), & \text{in } \Sigma_1(r + 1), \\ \beta(r + 2 - s_1)h(r), & \text{if } s_1 \geq r + 1 \end{cases}. \quad (6.20)$$

Set $\tilde{v}(r) = \exp_{\tilde{u}(r)}(\tilde{h}(r)), \tilde{b} = (s, \tilde{v}(r))$ and $\tilde{b}(r) = (s, \tilde{u}(r))$. We can define $\tilde{v}(r)$ as the definition of $\tilde{h}(r)$. So the meaning of $\frac{\partial \tilde{u}}{\partial \nu_{(r)}}, \frac{\partial \tilde{h}}{\partial \nu_{(r)}}$ and $\nabla_{\tilde{v}} \tilde{v}(r)$ is clear. Set

$$\Lambda_r := P^{-1} \circ D\tilde{L} \circ P(\nu_{(r)}), \quad \tilde{\Lambda}_r := P^{-1} \circ D\tilde{L} \circ P(\tilde{v}(r)).$$

Obviously, $\Lambda_r = 0$ and $\Lambda_r|_{\Sigma_{r+1}} = \tilde{\Lambda}_r|_{\Sigma_{r+1}}$. We calculate $\frac{\partial}{\partial r} (\beta_{1:2} \Lambda_r)$:

$$\frac{\partial}{\partial r} (\beta_{1:2} \Lambda_r) = \frac{\partial}{\partial r} (\beta_{1:2} \tilde{\Lambda}_r) = \beta_{1:2} P^{-1} \left[ \nabla_r \left( D\tilde{L} \circ P \right) (\tilde{v}(r)) + D\tilde{L} \circ P(\nabla_r \tilde{v}(r)) \right]. \quad (6.21)$$

Using Theorem 4.7 we have

$$\left\| \beta_{1:2} \nabla_r \left( D\tilde{L} \circ P \right) (\tilde{v}(r)) \right\|_{k,2,\alpha} \leq C e^{-(\epsilon - 5\alpha)\frac{\zeta}{r}}. \quad (6.22)$$

Restricting in $\Sigma_1(r + 1)$ we have

$$\nabla_r \tilde{v}(r) = \phi_r(\nabla \nu_{(r)}^*) - 2\nabla_{s_2} \nu_{(r),2},$$

where $\nu_{(r)}^* = \left( \nu_{(r),1}, \nu_{(r),2} \right)$. By Lemma 6.5 we have

$$\left\| \nabla_{s_2} (P\nu_{(r),2}) \right\|_{\Sigma_{r+1}} \leq C e^{-(\epsilon - \alpha)\frac{\zeta}{r}} \left( \| P\nu_{(r),2} \|_{k,2,\alpha} + 1 \right).$$

Applying the exponential decay of $u_{(r)}$ and $v_{(r)}$, we get

$$\left\| \nabla_{s_2} (\nu_{(r),2}) \right\|_{k,2,\alpha} \leq C e^{-(\epsilon - \alpha)\frac{\zeta}{r}} \left( \| \nu_{(r),2} \|_{k,2,\alpha} + 1 \right). \quad (6.23)$$

Then Lemma follows from (4.11), (6.21), (6.22) and (6.23).
Proof of (6.1). By the definition of $\nu_{(r)}^r$, we have
\[
\frac{\partial}{\partial r} \nu_{(r)}^r = \frac{\partial}{\partial r} I^*(\zeta) + \frac{\partial}{\partial r} (Q^* P_r) H_r f I_r(\zeta) + Q^* P_r \frac{\partial}{\partial r} (H_r f I_r(\zeta)).
\] (6.24)

Then multiplying $H_r D \phi_r$ on both sides of (6.24) we get
\[
H_r D \phi_r \frac{\partial \nu_{(r)}^r}{\partial r} = H_r D \phi_r \frac{\partial I^*(\zeta)}{\partial r} + H_r D \phi_r \frac{\partial}{\partial r} (Q^* P_r) H_r f I_r(\zeta) + H_r D \phi_r \frac{\partial}{\partial r} (H_r f I_r(\zeta)).
\]

It follows together with (6.18) and Theorem 4.7 that
\[
\left\| H_r P_r \frac{\partial (H_r f I_{(r)}(\zeta))}{\partial r} \right\|_{k-2,2,\alpha} \leq C \left[ \left\| \frac{\partial \nu_{(r)}^r}{\partial r} \right\|_{k-1,2,\alpha} + e^{-\frac{(\zeta-5a)^2}{4}} \right] + (A) + (B),
\] (6.25)

where
\[
(A) = \left\| H_r D \phi_r \left( \frac{\partial}{\partial r} I^*(\zeta) \right) \right\|_{k-2,2,\alpha}, \quad (B) = \left\| H_r D \phi_r \left( \frac{\partial}{\partial r} (Q^* P_r) \circ H_r f (I_r(\zeta)) \right) \right\|_{k-2,2,\alpha}.
\]

For any $(h_1, h_2)$ with $\text{supp } h_i \subset \Sigma(R_0) \cup \{|s_i| \leq \frac{3r}{2} \}$ we have
\[
\beta_{1,2} D \phi_r(h_1, h_2) = \beta_{1,2} \partial \beta_{s_r}(h_1 + h_2) + \beta_{1,2} E(h_1 + h_2).
\] (6.26)

Then
\[
\left\| H_r D \phi_r(h_1, h_2) \right\|_{k-1,2,\alpha} \leq C \left\| (h_1, h_2) \right\|_{k,2,\alpha}.
\] (6.27)

Taking the derivation $\frac{\partial}{\partial r}$ of (6.26) we obtain
\[
\left\| \frac{\partial}{\partial r} (\beta_{1,2} D \phi_r)(h_1, h_2) \right\|_{k-2,2,\alpha} \leq C \left[ \left\| h_2 \right\|_{\zeta \leq s_1 \leq r+1} + e^{-\frac{(\zeta-5a)^2}{4}} + \|D \phi_r(h_1, h_2)\|_{r-1 \leq s_1 \leq r+1} \right].
\] (6.28)

Since $H_r D \phi_r I_{(r)}^*(\zeta) = 0$, we have $H_r D \phi_r \frac{\partial I_{(r)}^*}{\partial r}(\zeta) = \frac{\partial H_r D \phi_r}{\partial r} I_{(r)}^*(\zeta)$. Then
\[
(A) \leq C e^{-\frac{(\zeta-5a)^2}{4}}.
\] (6.29)

Since
\[
\frac{\partial}{\partial r} (Q^* P_r) = \frac{\partial}{\partial r} ((Q')^* P_r) \circ (H_r D Q')^{-1} P_r + (Q')^* P_r \circ \frac{\partial}{\partial r} (H_r D Q')^{-1} P_r
\]
by (1), (2), (3), (4) of Lemma 6.2 we get
\[
\left\| \frac{\partial}{\partial r} (Q^* P_r)(\eta_1, \eta_2) \right\|_{k-1,2,\alpha} \leq C \left( e^{-\frac{(\zeta-5a)^2}{4}} \sum \left\| \eta_i \right\|_{s_i \leq r+1} \left\| \nu_{(r)}^r \right\|_{k-1,2,\alpha} + \sum_{i=1}^{2} \left\| \eta_i \right\|_{\zeta \leq s_i \leq r+1} \right).
\] (6.30)

It follows from Lemma 6.3, (6.27) and (6.30) that
\[
(B) \leq C e^{-\frac{(\zeta-5a)^2}{4}}.
\] (6.31)

Note that $H_r P_r \frac{\partial}{\partial r} (H_r f I_{(r)}(\zeta)) + \frac{\partial}{\partial r} (H_r P_r) H_r f I_{(r)}(\zeta) = \frac{\partial}{\partial r} ((H_r f I_{(r)})(\zeta))$. Then (6.29), (6.31), (5) of Lemma 6.2 together with (6.25) gives
\[
\left\| \frac{\partial}{\partial r} (H_r f I_{(r)}(\zeta)) \right\|_{k-1,2,\alpha} \leq C e^{-\frac{(\zeta-5a)^2}{4}} + C \left\| \frac{\partial}{\partial r} (\nu_{(r)}^r) \right\|_{k-1,2,\alpha}.
\] (6.32)
Substituting this into (6.24), and using (6.30), Lemma 6.3, Lemma 6.4 we conclude that

\[ \left\| \frac{\partial}{\partial r} \nu^*_r \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - 5\alpha)/2} \tag{6.33} \]

when \( d \) small. Since \( \nu^*_r \) is a \((j_\alpha, J)\) holomorphic map, by the standard elliptic estimates we have (6.1).

Repeating the all arguments in this section, one can prove that there exists a constant \( C > 0 \) such that

\[ \left\| \frac{\partial}{\partial r} \nu^*_r \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - 5\alpha)/2} (d + 1) \tag{6.34} \]

for any \( \zeta \in \ker \tilde{D}_{b_\alpha}^L \).

### 6.3 Estimates of the second derivatives

We can define \( H_r \) and \( P_r, \cdots \) as before. Let \( \xi = Q_{b_\alpha}^L H_r P_r \eta \) and \( \eta^i_l = \eta \big|_{D^l_i(R_0)} \), \( l = 1, 2 \). Obviously

\[ H_r P_r \eta \big|_{D^l_i(R_0)} = \left( \frac{\sum_{i=1}^{2} \beta_{i,1,2} \eta^i}{} \right) \, \left( \frac{\sum_{i=1}^{2} \beta_{i,2,1} \eta^i}{} \right) \tag{6.35} \]

Set \( W_l^i = \{ (s_l^i, t_l^i) \} \). It is easy to see that for any \( 1 \leq i \neq j \leq \epsilon \), and \( l = 1, 2 \),

\[ \text{supp} \frac{\partial E}{\partial r_i} \subset V_i, \quad \frac{\partial \beta_{i,j,r}^i}{\partial \xi_r} = 0, \quad \text{supp} \frac{\partial H_r P_r}{\partial r_i} \subset V_i, \quad \text{supp} \frac{\partial \beta_{i,j,r}^i}{\partial \xi_r} \subset V_i. \tag{6.37} \]

It follows that

\[ \left\| \frac{\partial}{\partial r_i} \xi \right\|_{k-1,2,\alpha} \leq C \left\| \frac{\partial}{\partial r_i} H_r P_r (\eta) \right\|_{k-2,2,\alpha} \leq C \| \eta \|_{V_i} \| \beta \|_{k-2, \alpha} \tag{6.38} \]

Let \( \xi^i_l = \xi \big|_{D^l_i(R_0)} \), \( l = 1, 2 \). Then \( \xi^1_l, \xi^2_l \) is the restriction of \( \xi \) near the node \( q_i \). Since \( D_{b_\alpha}^L \frac{\partial}{\partial r_i} \xi = \frac{\partial}{\partial r_i} (H_r P_r (\eta)) \), by Lemma 7.4 and (6.37) we have for any \( j \neq i \)

\[ \sum_{l=1}^{2} \left\| \frac{\partial}{\partial r_i} \xi \right\|_{W_l^i} \leq C e^{-(\epsilon - \alpha)/2} \left\| \frac{\partial}{\partial r_i} H_r P_r (\eta) \right\|_{k-2,2,\alpha} \leq C e^{-(\epsilon - \alpha)/2} \| \eta \|_{V_i} \| \beta \|_{k-2, \alpha} \tag{6.39} \]

In the following we assume that \( 1 \leq i \neq j \leq \epsilon \). It is easy to see that

\[ \frac{\partial^2 \xi}{\partial r_i \partial r_j} \right|_{D^l} = 0, \quad (Q^\star P_r \eta ) \big|_{D^l} = (\beta_{1,l,r}^i \xi^1_l, \beta_{2,l,r}^i \xi^2_l), \quad \forall \, 1 \leq l \leq \epsilon. \tag{6.39} \]

Taking the derivative \( \frac{\partial}{\partial r_i} \) and \( \frac{\partial^2}{\partial r_i \partial r_j} \) of \( (Q^\star P_r) \), by (6.36), (6.37), (6.39) and \( (Q^\star P_r) \big|_{D^l_i(R_0)} = \xi \), we obtain

\[ \frac{\partial}{\partial r_j} (Q^\star P_r) \big|_{D^l_i(R_0)} = \frac{\partial^2}{\partial r_i \partial r_j} (Q^\star P_r) \big|_{D^l_i(R_0)} = \frac{\partial^2}{\partial r_i \partial r_j} \left( \frac{\partial \xi^1_l}{\partial r_i}, \frac{\partial \xi^2_l}{\partial r_i} \right) \big|_{D^l_i(R_0)} \],

and \( \text{supp} \frac{\partial^2}{\partial r_i \partial r_j} (Q^\star P_r) \big|_{D^l_i(R_0)} \subset V_i \cup V_j \). By (6.38) we get

\[ \sum_{l=1}^{2} \left\| \frac{\partial}{\partial r_j} ((Q^\star P_r) \eta) \right\|_{W_l^i} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} ((Q^\star P_r) \eta) \right\|_{k-2,2,\alpha} \leq C e^{-(\epsilon - \alpha)/2} \| \eta \|_{V_i} \| \beta \|_{k-2, \alpha} + C \epsilon \]
A direct calculation gives us

\[
H_T \circ DQ' \circ P_T|_{D'} = \left( \beta_{1,1,2} \left( \frac{\partial}{\partial \eta}\left( \sum_{\ell=1}^{2} \beta_{\ell,i,r} \xi^\ell_i \right) + E \sum_{\ell=1}^{2} \beta_{\ell,i,r} \xi^\ell_i \right), \beta_{2,1,2} \left( \frac{\partial}{\partial \eta}\left( \sum_{\ell=1}^{2} \beta_{\ell,i,r} \xi^\ell_i \right) + E \sum_{\ell=1}^{2} \beta_{\ell,i,r} \xi^\ell_i \right) \right).
\]

It follows from \(H_T DQ' P_T|_{\Sigma \cup D'(r_i/2)} = 1\), \(H_T DQ' P_T|_{\Sigma \cup D'(r_i/2)} = 0\), (6.37) and \(\frac{\partial E}{\partial r_i}|_{V_i} = 0\) that

\[
\sup \frac{\partial}{\partial r_i} (H_T DQ' P_T) \subset \bigcup_{j=1}^{\epsilon} V_j, \quad \sup \frac{\partial^2}{\partial r_i \partial r_j} (H_T DQ' P_T) \subset V_j \cup V_i.
\]

(6.41)

Taking the derivative \(\frac{\partial}{\partial r_i}\) and \(\frac{\partial^2}{\partial r_i \partial r_j}\) of \(H_T (DQ')_r\), using (6.36), (6.37), (6.38) and (6.39) one can easily check that

\[
\sum_{l=1}^{2} \left\| \frac{\partial}{\partial r_j} (H_T DQ' P_T) (\eta) \right\|_{W^k_{1,2,\alpha}} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} (H_T DQ' P_T) (\eta) \right\|_{k-2,2,\alpha} \leq C e^{-\frac{(\epsilon-\alpha)r_i}{4}} \|\eta\|_{k-1,2,\alpha} + C e^{-\frac{(\epsilon-\alpha)r_i}{4}} \|\eta\|_{k-1,2,\alpha}.
\]

(6.42)

Note that

\[
\frac{\partial}{\partial r_i} (H_T (DQ')_r) \circ H_T (DQ')^{-1}_r + H_T (DQ')_r \circ \frac{\partial}{\partial r_i} (H_T (DQ')^{-1}_r) = \frac{\partial H_T P_T}{\partial r_i}.
\]

Multiplying \(H_T (DQ')^{-1}_r\) on the both sides, by

\[
\frac{\partial}{\partial r_i} (H_T (DQ')^{-1}_r) = H_T P_T \frac{\partial}{\partial r_i} (H_T (DQ')^{-1}_r) + \frac{\partial H_T P_T}{\partial r_i} H_T (DQ')^{-1}_r,
\]

we have

\[
\frac{\partial}{\partial r_i} (H_T (DQ')^{-1}_r) = H_T (DQ')^{-1}_r \frac{\partial H_T P_T}{\partial r_i} + \frac{\partial H_T P_T}{\partial r_i} H_T (DQ')^{-1}_r - H_T (DQ')^{-1}_r \circ \frac{\partial}{\partial r_i} (H_T (DQ')_r) \circ H_T (DQ')^{-1}_r.
\]

(6.43)

Using (4), (5) of Lemma6.2, (6.36) to the first term, and (6.36) to the second term, applying (4) of Lemma6.2 and (6.42) to the last term we have

\[
\sum_{l=1}^{2} \left\| \frac{\partial}{\partial r_i} (H_T (DQ')^{-1}_r) \right\|_{W^k_{1,2,\alpha}} \leq C e^{-\frac{(\epsilon-\alpha)r_i}{4}} \|\eta\|_{k-1,2,\alpha}.
\]

(6.44)

Taking derivative \(\frac{\partial}{\partial r_j}\) of (6.43), by (6.36) we get

\[
\frac{\partial^2}{\partial r_i \partial r_j} (H_T (DQ')^{-1}_r) = \frac{\partial}{\partial r_j} (H_T (DQ')_r) \frac{\partial H_T P_T}{\partial r_i} + \frac{\partial H_T P_T}{\partial r_i} \frac{\partial}{\partial r_j} (H_T (DQ')_r) - \frac{\partial}{\partial r_j} (H_T (DQ')^{-1}_r) \circ \frac{\partial}{\partial r_i} (H_T (DQ')_r) \circ H_T (DQ')^{-1}_r - H_T (DQ')^{-1}_r \circ \frac{\partial^2}{\partial r_i \partial r_j} (H_T (DQ')_r) \circ H_T (DQ')^{-1}_r - H_T (DQ')^{-1}_r \circ \frac{\partial^2}{\partial r_i \partial r_j} (H_T (DQ')_r) \circ H_T (DQ')^{-1}_r.
\]

By (3), (4) of Lemma6.2, (6.36), (6.44) and (6.42) one can check that

\[
\left\| \frac{\partial^2}{\partial r_i \partial r_j} (H_T (DQ')^{-1}_r) (\eta) \right\|_{k-2,2,\alpha} \leq C e^{-\frac{(\epsilon-\alpha)r_i}{4}} \|\eta\|_{k-1,2,\alpha} + C e^{-\frac{(\epsilon-\alpha)r_i}{4}} \|\eta\|_{k-1,2,\alpha}.
\]

(6.45)
By (6.40), (6.45) and
\[
\frac{\partial^2 Q^* P_r}{\partial r_i \partial r_j} = \frac{\partial^2 (Q^*)^* P_r}{\partial r_i \partial r_j} \circ H_r (DQ')^{-1} P_r + (Q^*)^* P_r \frac{\partial^2}{\partial r_i \partial r_j} \left( H_r (DQ')^{-1} P_r \right) \\
+ \frac{\partial (Q^*)^* P_r}{\partial r_i} \circ \frac{\partial}{\partial r_j} \left( H_r (DQ')^{-1} P_r \right) + \frac{\partial (Q^*)^* P_r}{\partial r_j} \circ \frac{\partial}{\partial r_i} \left( H_r (DQ')^{-1} P_r \right),
\]
we have
\[
\sum_{l=1}^2 \left\| \frac{\partial}{\partial r_j} (Q^* P_r) (\eta) \right\|_{W^*_{k-1,2,\alpha}} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} (Q^* P_r) (\eta) \right\|_{k-2,2,\alpha}
\leq C e^{-\frac{\gamma - \alpha}{4} r_i + r_j} \| \eta \|_{k-1,2,\alpha} + C e^{-\frac{\gamma - \alpha}{4} r_i + r_j} \| \eta \|_{k-1,2,\alpha}.
\]
\[(6.46)\]
Since for any \(\zeta + \hat{\zeta}_0 \in Ker D^L_{b_0}\)
\[
D \phi_r (\zeta^*_r) = \sum (\partial \beta_{\zeta,r}) \zeta_i + \sum \beta_{\zeta,r} (E - E_{\alpha}) (\zeta_i + \hat{\zeta}_0),
\]
we have
\[
supp H_r D \phi_r (\zeta^*_r) \subset \bigcup_i V_i, \quad supp \frac{\partial}{\partial r_i} \left( H_r D \phi_r (\zeta^*_r) \right) \subset V_i, \quad \frac{\partial^2}{\partial r_i \partial r_j} \left( H_r D \phi_r (\zeta^*_r) \right) = 0.
\]
Since \(I^*_r (\zeta + \hat{\zeta}_0) = (Id - Q^* P_r \circ H_r D \circ \phi_r (\zeta^*_r), (6.28)\) and \(6.46), we have
\[
\sum_{l=1}^2 \left\| \frac{\partial}{\partial r_j} I^*_r (\zeta + \hat{\zeta}_0) \right\|_{W^*_{k-1,2,\alpha}} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} I^*_r (\zeta + \hat{\zeta}_0) \right\|_{k-2,2,\alpha}
\leq C e^{-\frac{\gamma - \alpha}{4} r_i + r_j} \| \zeta + \hat{\zeta}_0 \|_{W^*_{k,2,\alpha}}.
\]
\[(6.47)\]
Note that, restricting in \(V_i, i \neq j\)
\[
\tilde{\nabla}_i \tilde{\nu}_r (\tau) = \phi_r \tilde{\nabla}_i \tilde{\nu}_r (\tau), \quad \tilde{\nabla}_i \tilde{h}_r (\tau) = \phi_r \tilde{\nabla}_i \tilde{h}_r (\tau), \quad \frac{\partial E}{\partial r_j} = 0.
\]
Similar \((6.18),\) by Theorem \(4.7\) we can prove that
\[
\sum_{l=1}^2 \left\| H_r \circ D \phi_r \circ \frac{\partial}{\partial r_j} \nu_r^* \right\|_{W^*_{k-1,2,\alpha}} \leq Cd \sum_{l=1}^2 \left\| \frac{\partial}{\partial r_j} \nu_r^* \right\|_{W^*_{k-1,2,\alpha}} + C e^{-\frac{\gamma - \alpha}{4} r_i + r_j}.
\]
\[(6.48)\]
Using \((6.40), (6.42), (6.48),\) and the same argument as in \(13,\) we have
\[
\sum_{l=1}^2 \left\| \frac{\partial}{\partial r_i} \nu_r^* \right\|_{W^*_{k-1,2,\alpha}} + \sum_{l=1}^2 \left\| \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} \nu_r^* \right\|_{W^*_{k-1,2,\alpha}} \leq C e^{-\frac{\gamma - \alpha}{4} r_i + r_j}.
\]
\[(6.49)\]
By \((6.49),\) Theorem \(4.7\) the Cauchy-Schwarz inequality and the same argument of \((6.18),\) we have
\[
\left\| H_r D \phi_r \circ \frac{\partial^2}{\partial r_i \partial r_j} \nu_r^* \right\|_{k-2,2,\alpha} \leq C \left[ d \left\| \frac{\partial^2}{\partial r_i \partial r_j} \nu_r^* \right\|_{k-2,2,\alpha} + e^{-\frac{\gamma - \alpha}{4} r_i + r_j} \right].
\]
\[(6.50)\]
Taking the derivative \(\frac{\partial^2}{\partial r_i \partial r_j} \) of \(\nu_r^* \) and multiplying \(H_r D \phi_r\) on both sides we get
\[
H_r D \phi_r \circ \frac{\partial^2}{\partial r_i \partial r_j} (\nu_r^*)
= H_r D \phi_r \circ \frac{\partial^2}{\partial r_i \partial r_j} (I^*_r (\zeta)) + H_r D \phi_r \circ \frac{\partial^2}{\partial r_i \partial r_j} (Q^* P_r) \circ H_r (DQ')^{-1} P_r + H_r (DQ')^{-1} P_r \frac{\partial^2}{\partial r_i \partial r_j} (H_r f I_r (\zeta))
+ H_r D \phi_r \circ \frac{\partial}{\partial r_j} (Q^* P_r) \circ H_r (DQ')^{-1} P_r + H_r D \phi_r \circ \frac{\partial}{\partial r_i} (Q^* P_r) \circ H_r (DQ')^{-1} P_r.
\]
Moreover, if using Lemma 6.2 and Theorem 4.7, we get
\[ \partial \frac{\partial (H_r f(I_r(\zeta)))}{\partial r_j} = \partial \frac{\partial (H_r P_r)}{\partial r_j} \circ H_r f(I_r(\zeta)) + H_r P_r \frac{\partial}{\partial r_j} (H_r f(I_r(\zeta))), \]
using Lemma 6.2 and Theorem 4.7 we get
\[ \left\| \frac{\partial (Q^* P_r)}{\partial r_i} \frac{\partial (H_r f(I_r(\zeta)))}{\partial r_j} \right\|_{k-2,2,\alpha} \leq C e^{-(\epsilon - 5\alpha) \frac{r^i + r_j}{2}}. \] (6.51)

Then using Theorem 4.7 and repeating the proof of (6.32) we have
\[ \left\| \frac{\partial^2 (H_r f(I_r(\zeta)))}{\partial r_i \partial r_j} \right\|_{k-2,2,\alpha} \leq C e^{-(\epsilon - 5\alpha) \frac{r^i + r_j}{2}} + C d \left\| \frac{\partial^2 \nu^r}{\partial r_i \partial r_j} \right\|_{k-2,2,\alpha}. \] (6.52)

By the definition of \( \nu^r \), we have
\[ \frac{\partial^2 \nu^r}{\partial r_i \partial r_j} = \frac{\partial^2 \nu^r}{\partial r_i \partial r_j} + \frac{\partial^2 (Q^* P_r)}{\partial r_i \partial r_j} \circ H_r f(I_r(\zeta)) + Q^* P_r \frac{\partial^2 (H_r f(I_r(\zeta)))}{\partial r_i \partial r_j} + \frac{\partial (Q^* P_r)}{\partial r_i} \frac{\partial (H_r f(I_r(\zeta)))}{\partial r_j} + \frac{\partial (Q^* P_r)}{\partial r_j} \frac{\partial (H_r f(I_r(\zeta)))}{\partial r_i}. \]

Applying (6.47) to the first term, (6.46) to the second term, (6.52) to the third term, and (6.51) to the last two term we can obtain the estimate of (6.2).

7 Appendix

7.1 Implicit function theorem

We can generalize Theorem A.3.3 and Proposition A.3.4 in [5] to the case with parameters by the same method.

**Theorem 7.1.** Let \((A, \| \cdot \|_A), (X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be Banach spaces, \(U \subset X\) be open sets and \(V \subset A\). \(U \subset X\) be open sets and \(F : V \times U \to Y\) be a continuously differentiable map. For any \((a, x) \in V \times U\) define
\[ D_a F(a, x)(g) = \frac{d}{dt} F(a + tg, x) \big|_{t=0}, \quad D_x F(a, x)(h) = \frac{d}{dt} F(a, x + th) \big|_{t=0}, \quad \forall \ g \in A, \ h \in X. \]
Suppose that \(D_x F(a_0, x_0)\) is surjective and has a bounded linear right inverse \(Q_{(a_0, x_0)} : Y \to X\) with \(\|Q_{(a_0, x_0)}\| \leq C\) for some constant \(C > 0\). Choose a positive constant \(\delta > 0\) such that
\[ \| D_x F(a, x) - D_x F(a_0, x_0) \| \leq \frac{1}{2C}, \quad \forall \ x \in B_\delta(x_0, X), \ a \in B_\delta(a_0, A). \] (7.1)
where \(B_\delta(a_0, A) = \{a \in A \mid \|a - a_0\|_A \leq \delta\}, B_\delta(x_0, X) = \{x \in X \mid \|x - x_0\|_X \leq \delta\}\). Suppose that \(x_1 \in X\) and \(a \in B_\delta(a_0, A)\) satisfies
\[ \| F(a, x_1) \|_Y < \frac{\delta}{4C}, \quad \| x_1 - x_0 \|_X \leq \frac{\delta}{8}. \] (7.2)
Then there exists a unique \(x \in X\) such that
\[ F(a, x) = 0, \quad x - x_1 \in im Q, \quad \| x - x_0 \|_X \leq \delta, \quad \| x - x_1 \|_X \leq 2C\| F(a, x_1) \|_Y. \] (7.3)
Moreover, if \(\| F(a_0, x_0) \|_Y \leq \frac{\delta}{4C}\), there exist a constant \(\delta' > 0\) and a unique family differential map \(f_a : ker D_x F(a_0, x_0) \to Y\) such that for any \((a, x) \in F^{-1}(0) \cap (B_{\delta'}(a_0, A) \times B_{\delta'}(x_0, X))\), we have
\[ F(a, x) = 0 \iff x = x_0 + \zeta + Q_{(a_0, x_0)} \circ f_a(\zeta), \quad \zeta \in ker D_x F(a_0, x_0) \] (7.4)
Using Theorem 7.1, we can obtain the smoothness of implicit function.

**Theorem 7.2.** Let $F$ satisfies the assumption of Theorem 7.1. If $F : V \times U \rightarrow Y$ is of class $C^\ell$, where $\ell$ is a positive integer, then there exists a constant $\delta' > 0$ such that $F^{-1}(0)|_{B_\delta'(a_o, A) \times B_\delta'(x_o, X)}$ is $C^\ell$ manifold, and $\xi \rightarrow x_o + \xi + Q \circ f_a(\xi)$ is a $C^\ell$-chart of $F^{-1}(0)|_{B_\delta'(a_o, A) \times B_\delta'(x_o, X)}$. In particular, 

$$\|D_a (x_o + \xi + Q(a_o,x_o) \circ f_a(\xi))\| \leq C,$$  

where $C > 0$ is a constant depending only on $C_1, C, \delta', \|f_a\|$ and $\|D^2_{a\xi} F(a, x_o)\|$. 

**Proof.** Since $F(a, x)$ satisfies the assumption of Theorem 7.1, $F^{-1}(0)|_{\{a\} \times B_{\delta_1}(x_o, X)}$ is a smooth manifold. We only need consider the smoothness of $F^{-1}(0)$ with respect to $a$.

By the same argument in the proof of Theorem A.3.3 in [5], we have a explicit formula for $f_a$ 

$$f_a(\xi) = D_x F(a_o, x_o) \circ \phi_a^{-1}(\xi + x_o) - D_x F(a_o, x_o)(x_o),$$

where $\phi_a$ is defined by 

$$\phi_a(x) := x + Q(a_o,x_o) (F(a, x) - D_x F(a_o, x_o)(x - x_o)).$$

We choose $\delta'$ small such that in $B_{\delta'}(a_o, A) \times B_{\delta'}(x_o, X)$, 

$$|\phi_a(x) - I| \leq \frac{1}{2}. \quad (7.7)$$

Then by the smoothness of $F$ and 

$$\frac{\partial}{\partial a} \phi_a^{-1}(x) = -\phi_a^{-1} \circ \frac{\partial \phi_a}{\partial a} \circ \phi_a^{-1}(x),$$

we conclude that $f_a$ is a smooth function of $(a, x)$. It follows that the zero set of $F$ is smooth for $a$ and holds.

### 7.2 Exponential decay in tube

By the same method as in [3], we can prove the following lemmas

**Lemma 7.3.** Let $\eta \in L^{k-1,2,\alpha}_{w} \cap H^{k,2,\alpha}$ and $h + \hat{h}_0 \in W^{k,2,\alpha}_{w}$ be a solution of $D^L|_{b}(h + \hat{h}_0) = \eta$ over $\Sigma \setminus \Sigma(R_0)$. Suppose that, for any $p, q \geq 0$, 

$$\left| \frac{\partial^{p+q} E_{b,c}^{L}}{\partial s^p \partial t^q} \right| \leq C_{p,q} e^{-c|s|}, \quad \forall |s| \geq R_0, \quad l = 1, 2 \quad (7.8)$$

for some constant $C_{p,q} > 0$. Then for any $0 < \alpha < \frac{2}{5}$, there exists a constant $C > 0$ such that for any $R > \max\{R_0, \tilde{R}\}$ and $R' > 2 + R$ 

$$\|h\|_{s_1 \geq R'} k_{2,\alpha} \leq C \left( (e^{-(e-\alpha)(R'-R)} + e^{-(e-\alpha)R})\|h + \hat{h}_0\|_{W^{k,2,\alpha}} + \|\eta\|_{s_1 \geq R} k_{1-2,\alpha} \right) \quad (7.9)$$

In particular, if $D^L|_{b}$ has a bounded right inverse $Q_b : L^{k-1,2,\alpha}_{w} \rightarrow W^{k,2,\alpha}_{w}$. Let $h = Q_b \eta$ be a solution of $D^L|_{b}(h) = \eta$ over $(R_0, \infty) \times S^1$. Then there exists a constant $C' > 0$ independent of $r$ such that 

$$\|h\|_{s_1 \geq R'} k_{2,\alpha} \leq C' \left( (e^{-(e-\alpha)(R'-R)} + e^{-(e-\alpha)R})\||\eta||_{k-1,2,\alpha} + \|\eta\|_{s_1 \geq R} k_{1-2,\alpha} \right). \quad (7.10)$$
Lemma 7.4. Let \( h + \hat{h}_0 \in W^{k,2,\alpha}_{r,u(c)} \) be a solution of \( \mathcal{D}L|_b(h + \hat{h}_0) = 0 \) over \( \Sigma_{(r)} \setminus \Sigma(R_0) \). Suppose that, for any \( p, q \geq 0 \),
\[
\left| \frac{\partial^{p+q} \mathcal{P}^L_{h}}{\partial s^p \partial t^q} \right| \leq C_{p,q} e^{-c \min(s_1, 2lr - s_1)} , \quad \forall |s_i| \geq R_0, \ l = 1, 2 \tag{7.11}
\]
for some constant \( C_{p,q} > 0 \). Then for any \( 0 < \alpha < \frac{c}{2} \), there exists a constant \( C > 0 \) such that for any \( R > \max\{R_0, \bar{d}\} \) and \( R' > 2 + R \)
\[
\| h \|_{R' \leq s_1 \leq 2lr - R'} \leq C (e^{-c\alpha(R'-R)} + e^{-(c\alpha)R}) \| h + \hat{h}_0 \|_{W^{k,2,\alpha}} \tag{7.12}
\]

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