Improved Estimates for $G_2$-structures on the Generalised Kummer Construction

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14th February 2022

Abstract

The resolution of the $G_2$-orbifold $T^7/\Gamma$, where $\Gamma$ is a suitably chosen finite group, admits a 1-parameter family of $G_2$-structures with small torsion $\varphi'$, obtained by gluing in Eguchi-Hanson spaces. It was shown in [Joy96b] that $\varphi'$ can be perturbed to torsion-free $G_2$-structures $\tilde{\varphi}'$ for small values of $t$. Using norms adapted to the geometry of the manifold we give an alternative proof of the existence of $\tilde{\varphi}'$. This alternative proof produces the estimate $\|\tilde{\varphi}' - \varphi'\|_{C^0} \leq c t^{5/2}$. This is an improvement over the previously known estimate $\|\tilde{\varphi}' - \varphi'\|_{C^0} \leq c t^{1/2}$. As part of the proof, we show that Eguchi-Hanson space admits a unique (up to scaling) harmonic form with decay, which is a result of independent interest.

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1 Introduction

The first compact examples of Riemannian manifolds with holonomy equal to $G_2$ were constructed in [Joy96b] by resolving an orbifold of the form $T^7/\Gamma$, where $\Gamma$ is a finite group of isometries of $T^7$. This was done by constructing $G_2$-structures with small torsion, and subsequently perturbing them to torsion-free $G_2$-structures. This perturbation made use of a general existence result for torsion-free $G_2$-structures that holds on all 7-manifolds. An immediate question is: how far away is the torsion-free $G_2$-structure from the $G_2$-structure with small torsion? This is important in applications, such as the construction of associative submanifolds and $G_2$-instantons. We give a partial answer to this question by proving an improved estimate for the difference between the torsion-free $G_2$-structure and the one with small torsion. The main result is Theorem 4.58.

Theorem. Choose $\alpha \in (0, 1)$ and $\beta \in (-1, 0)$ both close to 0. Let $N_t$ be the resolution of $T^7/\Gamma$ from Eq. (1.5) and $\varphi^t \in \Omega^2(N_t)$ the $G_2$-structure with small torsion from Eq. (1.7). There exists $c > 0$ independent of $t$ such that the following is true: for $t$ small enough, there exists $\eta^t \in \Omega^2(N_t)$ such that $\widetilde{\varphi} = \varphi^t + d\eta^t$ is a torsion-free $G_2$-structure, and $\eta^t$ satisfies

$$\|\eta^t\|_{C^{\alpha/2, \beta/4}} \leq ct^{7/2 - \beta}. $$

In particular,

$$\|\widetilde{\varphi} - \varphi^t\|_{L^\infty} \leq ct^{3/2} \quad \text{and} \quad \|\widetilde{\varphi} - \varphi^t\|_{C^{\alpha/2}} \leq ct^{5/2 - \alpha/2} \quad \text{as well as} \quad \|\widetilde{\varphi} - \varphi^t\|_{C^{1, \alpha/2}} \leq ct^{3/2 - \alpha/2}. $$

Here, the norm $\| \cdot \|_{C^{\alpha, \beta}}$ is a weighted Hölder norm. The norms in the last line of the theorem are ordinary, unweighted norms. The group $\Gamma$ is a finite group acting through $G_2$-involutions on $T^7$. In [Joy96b, Joyool], the estimate $\|\widetilde{\varphi} - \varphi\|_{L^\infty} \leq ct^{t/2}$ was shown. In this sense, the estimates from Theorem 4.58 are an improvement. The theorem hinges on an estimate for the inverse of the Laplacian acting on 2-forms on the resolution of $T^7/\Gamma$. The crucial idea necessary for obtaining this estimate is to split 2-forms into a part that is harmonic on the 4-dimensional fibres orthogonal to the singular set of $T^7/\Gamma$, and a rest. The 4-dimensional fibres are subsets of Eguchi-Hanson space $X_{\text{EH}}$, and the proof of Theorem 4.58 uses detailed knowledge of the harmonic forms on $X_{\text{EH}}$. The space $X_{\text{EH}}$ admits a harmonic 2-form $v_1$ that can be written down explicitly and comes from rescaling the metric. In Theorem 3.26 we denote the Laplacian on $X_{\text{EH}}$ acting on $p$-forms by $\Delta_{p, \Omega(1)}$, and we prove that $v_1$ is essentially the only form with decay.

Theorem. For $\lambda \in (-4, 0)$, the $L^2_{2, \lambda}$-kernels of $\Delta_{p, \Omega(1)}$ acting on $p$-forms of different degrees are the same as the $L^2$-kernels, namely:

$$\text{Ker}(\Delta_{\Omega(1)} : L^2_{2, \lambda}(\Lambda^2(X_{\text{EH}})) \to L^2_{0, \lambda - 2}(\Lambda^2(X_{\text{EH}}))) = \langle v_1 \rangle,$$$$
$$\text{Ker}(\Delta_{\Omega(1)} : L^2_{2, \lambda}(\Lambda^p(X_{\text{EH}})) \to L^2_{0, \lambda - 2}(\Lambda^p(X_{\text{EH}}))) = 0 \text{ for } p \neq 2.$$

Here $L^2_{2, \lambda}(\Lambda^p(X_{\text{EH}}))$ denote the usual weighted Sobolev spaces on asymptotically conical manifolds. They consist of, roughly speaking, $L^2$-sections with 2 weak derivatives that decay like $r^\lambda$ as $r \to \infty$, where $r$ is a radius function.
2 Background

2.1 Definition of the Eguchi-Hanson Space

The singularities of the $G_2$-orbifolds we are interested in are locally modelled on $\mathbb{R}^3 \times \mathbb{C}^2 / \{ \pm 1 \}$. In order to resolve these singularities, we study the resolution of the point singularity of $\mathbb{C}^2 / \{ \pm 1 \}$, called the Eguchi-Hanson space. Standard references for this space are [Joyoo Section 7.2] and [Danoj]. We begin by defining the Eguchi-Hanson space and the Eguchi-Hanson metrics, which are a 1-dimensional family of Hyperkähler metrics, controlled by a parameter $k \in \mathbb{R}_{\geq 0}$. For $k > 0$ we get a metric on a smooth 4-manifold (this is point one of the following proposition), and for $k = 0$ we get the standard metric on $\mathbb{H} / \{ \pm 1 \}$ or equivalently $\mathbb{C}^2 / \{ \pm 1 \}$ (this is point two of the following proposition).

Proposition 2.1. Let $r$ be a coordinate on the $\mathbb{R}_{\geq 0} \cdot$-factor of $\mathbb{R}_{\geq 0} \times \text{SO}(3)$. Let

$$\eta^1 = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \eta^2 = 2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \eta^3 = 2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

and denote the dual basis extended to left-invariant 1-forms on $\text{SO}(3)$ by the same symbols. For $k \geq 0$, let $f_k : \mathbb{R}_{>0} \times \text{SO}(3) \to \mathbb{R}_{>0}$ be defined by $f_k(r) = (k + r^2)^{1/4}$ and set

$$dt = f^{-1}_k(r) \, dr, \quad e^1(r) = r f^{-1}_k(r) \eta^1, \quad e^2(r) = f_k(r) \eta^2, \quad e^3(r) = f_k(r) \eta^3.$$

Define $\omega_1^{(k)}$, $\omega_2^{(k)}$, $\omega_3^{(k)} \in \Omega^2(\mathbb{R}_{>0} \times \text{SO}(3))$ to be

$$\omega_1^{(k)} = dt \wedge e^1 + e^2 \wedge e^3, \quad \omega_2^{(k)} = dt \wedge e^2 + e^3 \wedge e^1, \quad \omega_3^{(k)} = dt \wedge e^3 + e^1 \wedge e^2,$$

and denote by $g^{(k)}$ the metric on $\mathbb{R}_{>0} \times \text{SO}(3)$ that makes $(dt, e^1, e^2, e^3)$ an orthonormal basis.

1. If $k > 0$, consider the copy of $\text{SO}(2)$ in $\text{SO}(3)$ defined by $\{ \exp(s \cdot \eta^1) : s \in \mathbb{R} \}$, defining a right action of $\text{SO}(2)$ on $\text{SO}(3)$. Denote by $V = \mathbb{R}^2$ the standard representation of $\text{SO}(2)$. Define $\Psi : \text{SO}(3) \times \mathbb{R}_{>0} \to \text{SO}(3) \times V$ as $\Psi(g, r) = (g, (r, 0))$. Denote

$$X_{EH} = \text{SO}(3) \times_{\text{SO}(2)} V.$$

Then $\Psi$ induces a smooth injective map $\hat{\Psi} : \text{SO}(3) \times \mathbb{R}_{>0} \to X_{EH}$ that is a diffeomorphism onto its image, and the forms $\hat{\Psi}_*(\omega_1^{(k)})$ can be extended to smooth 2-forms on all of $X_{EH}$. Furthermore, $\hat{\Psi}_*(g^{(k)})$ can also be extended to a metric on all of $X_{EH}$, and $(X_{EH}, \hat{\Psi}_*(g^{(k)}))$ is a Hyperkähler manifold.

2. If $k = 0$: parametrise the quaternions as $x_0 + x_1 i + x_2 j + x_3 k$ with $x_0, x_1, x_2, x_3 \in \mathbb{R}$, embed $S^3 \subset \mathbb{H}$ as the unit sphere, and fix the identification $\phi : S^3 / \{ \pm 1 \} \to \text{SO}(3)$ that maps $x$ onto the map $y \mapsto x \cdot y \cdot x^{-1}$, where we use $S^3 / \{ \pm 1 \} \subset \mathbb{H} / \{ \pm 1 \}$ and $\cdot$ denotes quaternionic multiplication, for $x \in S^3 / \{ \pm 1 \} \subset \mathbb{H} / \{ \pm 1 \}$. Denote

$$\Phi : \text{SO}(3) \times \mathbb{R}_{>0} \to \mathbb{H} / \{ \pm 1 \}$$

$$(x, t) \mapsto t \cdot \phi^{-1}(x).$$

Then $\Phi^* \omega_i = \omega_i^{(0)}$ for $i \in \{ 1, 2, 3 \}$ and $\Phi^* g = g^{(0)}$, where $g, \omega_1, \omega_2, \omega_3 \in \Omega^2(\mathbb{H})$ are the metric and standard Hyperkähler triple on $\mathbb{H}$.
By slight abuse of notation, we will denote the extensions of \( \omega_i^{(k)} \) for \( i \in \{1, 2, 3\} \) and \( g_{(k)} \) to \( X_{EH} \) in the case \( k > 0 \) by the same symbol, suppressing the pushforward under \( \Psi \).

**Proof.** For \( k > 0 \): the fact that \( \omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, g_{(k)} \) can be extended to all of \( X_{EH} \) was proven, for example, in [LM17] Section 2.4. One checks using a direct computation that \( \omega_i^{(k)} \) for \( i \in \{1, 2, 3\} \) is closed and [Hit87] Lemma 6.8 implies that \( \omega_i^{(k)} \) is also parallel for \( i \in \{1, 2, 3\} \). Both the symplectic forms and the metric are defined using the same orthonormal basis, which proves that they are compatible. The case \( k = 0 \) is a direct calculation.

**Remark 2.3.** A possible point of confusion is that the function \( r : X_{EH} \rightarrow \mathbb{R} \) is approximately the squared distance to the bolt \( SO(3) \times_{SO(2)} \{0\} \) of \( X_{EH} \), so it is not a radius function.

The Hyperkähler structure on \( X_{EH} \) also has the important property that it approximates the flat Hyperkähler structure on \( \mathbb{H} \) for large values of \( r \). The following definition makes this notion precise, and Proposition 2.6 states that the Hyperkähler structure on \( X_{EH} \) does indeed have this property.

**Definition 2.4 (Definition 7.2.1 in [Joyoo]).** Let \( G \) be a finite subgroup of \( \text{Sp}(1) \), and let \( (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{g}) \) be the Euclidean Hyperkähler structure on \( \mathbb{H} \), and let \( \sigma : \mathbb{H}/G \rightarrow [0, \infty) \) the radius function on \( \mathbb{H}/G \). We say that a Hyperkähler 4-manifold \( (X, \omega_1, \omega_2, \omega_3, g) \) is asymptotically locally Euclidean (ALE) asymptotic to \( \mathbb{H}/G \), if there exists a compact subset \( S \subset X \) and a map \( \pi : X \setminus S \rightarrow \mathbb{H}/G \) that is a diffeomorphism between \( X \setminus S \) and \( \{x \in \mathbb{H}/G : \sigma(x) > R\} \) for some \( R > 0 \), such that

\[
\nabla^k (\pi_*(g) - \hat{g}) = O(\sigma^{-4-k}) \quad \text{and} \quad \nabla^k (\pi_*(\omega_i) - \hat{\omega}_i) = O(\sigma^{-4-k})
\]

as \( \sigma \rightarrow \infty \), for \( i \in \{1, 2, 3\} \) and \( k \geq 0 \), where \( \nabla \) is the Levi-Civita connection of \( \hat{g} \).

**Proposition 2.6 (Example 7.2.2 in [Joyoo]).** The 1-form \( r_1^{(k)} \in \Omega^1(X_{EH} \setminus SO(3) \times_{SO(2)} \{0\}) \) given by \( r_1^{(k)} = (f_k^2 - f_0^2) \eta^1 \) satisfies \( \omega_1^{(k)} - \omega_0^{(k)} = dr_1^{(k)} \) and for any \( l \in \mathbb{Z} \)

\[
\nabla^l r_1^{(k)}|_{g_0} = O(r^{-3-l}),
\]

where \( \nabla \) denotes the Levi-Civita connection of \( g_0 \). Furthermore, \( \omega_2^{(k)} - \omega_2^{(0)} = 0 \), and \( \omega_3^{(k)} - \omega_3^{(0)} = 0 \). In particular, \( (X_{EH}, \omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, g_{(k)}) \) is ALE asymptotic to \( \mathbb{H}/\{\pm 1\} \).

**Remark 2.8.** By definition, \( X_{EH} \) is an associated bundle over \( SO(3)/SO(2) = S^2 \). In fact, \( X_{EH} \) is diffeomorphic to the total space of \( T^*S^2 \), which itself is diffeomorphic to \( T^*\mathbb{CP}^1 \). It is a folklore result that \( (X_{EH}, J^{(k)}) \) is biholomorphic to \( T^*\mathbb{CP}^1 \) for all \( k > 0 \), which in turn is the blowup of \( \mathbb{C}^2/\{\pm 1\} \) in the origin, see e.g. [Dan99] p. 17 for the statement. We thus have a blowup map \( \rho : X_{EH} \rightarrow \mathbb{C}^2/\{\pm 1\} \).

### 2.2 \( G_2 \)-structures

In this section we collect standard facts about \( G_2 \)-geometry needed later.

**Definition 2.9 (Definition 10.1.1 in [Joyoo]).** Let \( (x_1, \ldots, x_7) \) be coordinates on \( \mathbb{R}^7 \). Write \( dx_{ij} \) for the exterior form \( dx_i \wedge dx_j \wedge \cdots \wedge dx_7 \). Define \( \varphi_0 \in \Omega^3(\mathbb{R}^7) \) by

\[
\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}.
\]
The subgroup of $GL(7, \mathbb{R})$ preserving $\varphi_0$ is the exceptional Lie group $G_2$. It also fixes the Euclidean metric $g_0 = dx_1^2 + \cdots + dx_7^2$, the orientation on $\mathbb{R}^7$, and $\varphi_0 \in \Omega^4(\mathbb{R}^7)$.

On $\mathbb{H}$ with coordinates $(y_0, y_1, y_2, y_3)$ we have the three symplectic forms $\omega_1, \omega_2, \omega_3$ given as

$$\omega_1 = dy_0 \wedge dy_1 + dy_2 \wedge dy_3, \quad \omega_2 = dy_0 \wedge dy_2 - dy_1 \wedge dy_3, \quad \omega_3 = dy_0 \wedge dy_3 + dy_1 \wedge dy_2.$$ 

Identify $\mathbb{R}^7$ with coordinates $(x_1, \ldots, x_7)$ with $\mathbb{R}^3 \oplus \mathbb{H}$ with coordinates $((x_1, x_2, x_3), (y_1, y_2, y_3, y_4))$. Then we have for $\varphi_0, \ast \varphi_0$ from Definition 2.9

$$\varphi_0 = dx_{123} - \sum_{i=1}^3 dx_i \wedge \omega_i, \quad \ast \varphi_0 = \text{vol}_4 - \sum_{(i,j,k)=(1,2,3)} \omega_i \wedge dx_{jk}. \quad (2.11)$$

This linear algebra statement easily extends to product manifolds in the following sense: if $X$ is a Hyperkähler 4-manifold, and $\mathbb{R}^3$ is endowed with the Euclidean metric, then $\mathbb{R}^3 \times X$ has a $G_2$-structure. The $G_2$-structure is given by the same formula as in the flat case, namely Eq. (2.11), after replacing $(\omega_1, \omega_2, \omega_3)$ with the triple of parallel symplectic forms defining the Hyperkähler structure on $X$.

**Definition 2.12.** Let $M$ be an oriented 7-manifold. A principal subbundle $Q$ of the bundle of oriented frames with structure group $G_2$ is called a $G_2$-structure. Viewing $Q$ as a set of linear maps from tangent spaces of $M$ to $\mathbb{R}^7$, there exists a unique $\varphi \in \Omega^3(M)$ such that $Q$ identifies $\varphi$ with $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ at every point.

Such $G_2$-structures are in 1-1 correspondence with 3-forms on $M$ for which there exists an oriented isomorphism mapping them to $\varphi_0$ at every point. We will therefore also refer to such 3-forms as $G_2$-structures.

Let $M$ be a manifold with $G_2$-structure $\varphi$. We call $\nabla \varphi$ the torsion of a $G_2$-structure $\varphi \in \Omega^3(M)$. Here, $\nabla$ denotes the Levi-Civita induced by $\varphi$ in the following sense: we have $G_2 \subset SO(7)$, so $\varphi$ defines a Riemannian metric $g$ on $M$, which in turn defines a Levi-Civita connection. As a shorthand, we also use the following notation: write $\Theta(\varphi) = \ast \varphi$, where "$\ast$" denotes the Hodge star defined by $g$. Using this, the following theorem gives a characterisation of torsion-free $G_2$-manifolds:

**Theorem 2.15** (Propositions 10.1.3 and 10.1.5 in [Joy].) Let $M$ be an oriented 7-manifold with $G_2$-structure $\varphi$ with induced metric $g$. The following are equivalent:

(i) $\text{Hol}(g) \subseteq G_2$,

(ii) $\nabla \varphi = 0$ on $M$, where $\nabla$ is the Levi-Civita connection of $g$, and

(iii) $d\varphi = 0$ and $d\Theta(\varphi) = 0$ on $M$.

*If these hold then $g$ is Ricci-flat.*

Later on, we will investigate perturbations of $G_2$-structures and analyse how they affect the torsion. To this end, we will use the following estimates for the map $\Theta$ defined before:
Proposition 2.14 (Proposition 10.3.5 in Joy96 and eqn. (21) of part I in Joy96b). There exists $\epsilon > 0$ and $c > 0$ such that whenever $M$ is a 7-manifold with $G_2$-structure $\varphi$ satisfying $d\varphi = 0$, then the following is true. Suppose $\chi \in C^\infty(\Lambda^3 T^*M)$ and $|\chi| \leq \epsilon$. Then $\varphi + \chi$ is a $G_2$-structure, and
\[
\Theta(\varphi + \chi) = * \varphi - T(\chi) - F(\chi),
\]
where “*” denotes the Hodge star with respect to the metric induced by $\varphi$, $T : \Omega^3(M) \to \Omega^4(M)$ is a linear map (depending on $\varphi$), and $F$ is a smooth function from the closed ball of radius $\epsilon$ in $\Lambda^3 T^*M$ to $\Lambda^4 T^*M$ with $F(0) = 0$. Furthermore,
\[
|F(\chi)| \leq c |\chi|^2,
\]
\[
|d(F(\chi))| \leq c \{|\chi|^2 |d^* \varphi| + |\nabla_\chi| |\chi|\},
\]
\[
[d(F(\chi))]_\alpha \leq c \{ |\chi| |\nabla\chi| |\chi| \}
\]
\[
[|\nabla(F(\chi))|]_{C^{0,\alpha}} \leq c \{ |\chi|^2 |\nabla\varphi| + |\nabla_\chi| |\chi| \}
\]
\[
[|\nabla(F(\chi))|]_{C^{0,\alpha}} \leq c \{ |\chi|^2 |\nabla_\varphi| + |\nabla_\chi| |\chi| \}\]
\[
 as well as
\]

Here, $|\cdot|$ denotes the norm induced by $\varphi$, $\nabla$ denotes the Levi-Civita connection of this metric, and $[\cdot]_{C^{0,\alpha}}$ denotes the unweighted Hölder semi-norm induced by this metric.

Finally, the landmark result on the existence of torsion-free $G_2$-structures is the following theorem. It first appeared in Joy96b part I, Theorem A, and we present a rewritten version in analogy with [JK21 Theorem 2.7].

Theorem 2.16. Let $\alpha, K_1, K_2, K_3$ be any positive constants. Then there exist $\epsilon \in (0, 1]$ and $K_4 > 0$, such that whenever $0 < t \leq \epsilon$, the following holds.

Let $M$ be a compact oriented 7-manifold, with $G_2$-structure $\varphi$ with induced metric $g$ satisfying $d\varphi = 0$. Suppose there is a closed 3-form $\psi$ on $M$ such that $d^* \varphi = d^* \psi$ and

(i) $||\psi||_{C^0} \leq K_1 t^{\alpha}$, $||\psi||_{L^2} \leq K_1 t^{7/2-\alpha}$, and $||\psi||_{L^4} \leq K_1 t^{-1/2+\alpha}$.

(ii) The injectivity radius $\text{inj}(g)$ satisfies $\text{inj} \geq K_2 t$.

(iii) The Riemann curvature tensor $Rm$ of $g$ satisfies $||Rm||_{C^0} \leq K_3 t^{-2}$.

Then there exists a smooth, torsion-free $G_2$-structure $\tilde{\varphi}$ on $M$ such that $||\tilde{\varphi} - \varphi||_{C^0} \leq K_4 t^{\alpha}$ and $[\tilde{\varphi}] = [\varphi]$ in $H^3(M, \mathbb{R})$. Here all norms are computed using the original metric $g$.

The main purpose of Section 4 will be to prove an improved existence theorem, specialised to the resolution of $T^7/T$. This will be achieved in Theorem 4.56.

3 Harmonic Forms with Decay on the Eguchi-Hanson Space

The aim of this section is to prove Theorem 3.26. That is, to prove that there is only one harmonic form on Eguchi-Hanson space that decays at infinity, up to scaling. We will achieve this
using the techniques of Lockhart and McOwen (cf. [LM85, Loc87]), which give a description of the harmonic forms on asymptotically conical manifolds, depending on information about harmonic forms on the asymptotic cone. To this end, we begin by studying the asymptotic cone of Eguchi-Hanson space $\mathbb{X}_{EH}$, namely the cone over $SO(3)$.

### 3.1 Harmonic Forms on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$

In this section, we will list homogeneous harmonic forms on $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ with decay. Because $(\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ is the cone over $SO(3)$, we will see that such forms correspond to eigenforms on $SO(3)$, and we will therefore review the spectral decomposition of the Laplacian on $S^3$ and $SO(3)$.

We begin by defining cones and homogeneous forms on them.

**Definition 3.1.** For a Riemannian manifold $(\Sigma, g_\Sigma)$, the Riemannian manifold $C(\Sigma) = \Sigma \times \mathbb{R}_{>0}$ endowed with the metric $g_C = dr^2 + r^2 g_\Sigma$ is called the Cone over $\Sigma$.

**Definition 3.2.** Let $\lambda \in \mathbb{R}$. Then $\gamma \in \Omega^k(C(\Sigma))$ is called homogeneous of order $\lambda$ if there exist $\alpha \in \Omega^{k-1}(\Sigma), \beta \in \Omega^k(\Sigma)$ such that

$$\gamma = r^{\lambda+k} \frac{dr}{r} \wedge \alpha + \beta.$$ 

**Remark 3.3.** For $t \in \mathbb{R}_{>0}$ denote by $(\cdot t) : C(\Sigma) \to C(\Sigma)$ the dilation map given by $(\cdot t)(r, \sigma) = (tr, \sigma)$ for $(r, \sigma) \in C(\Sigma)$. Then, if $\gamma \in \Omega^k(C(\Sigma))$ is homogeneous of order $\lambda$, we have $(\cdot t)^*|\gamma|_{g_C} = t^2 |\gamma|_{g_C}$.

Homogeneous harmonic forms do not exist for all orders and we make the following definition:

**Definition 3.4.** For a cone $C = C(\Sigma)$, denote by $\Delta_{k,C}$ the Laplacian acting on $k$-forms on $C$. The set

$$\mathcal{D}_{\Delta_{k,C}} = \{\lambda \in \mathbb{R} : \exists \gamma \in \Omega^k(C), \gamma \neq 0, \text{ homogeneous of order } \lambda \text{ with } \Delta_{k,C} \gamma = 0\}$$

is called the set of critical rates of $\Delta_{k,C}$.

It will turn out that critical rates are intimately related to harmonic forms on Eguchi-Hanson space. This is the content of the next subsection and we will see the set $\mathcal{D}_{\Delta_{k,C}}$ appear again there. The purpose of the current subsection is to describe $\mathcal{D}_{\Delta_{k,C}(SO(3))}$ and $\mathcal{D}_{\Delta_{k,C}(SU(2))}$, which is achieved in Proposition 3.5. We prepare the proposition by putting some results for harmonic forms on Riemannian cones in place:

**Lemma 3.5** (Lemma A.1 in [FHN20]). Let $\gamma = r^{\lambda+k} \left( \frac{dr}{r} \wedge \alpha + \beta \right)$ be a $k$-form on $C(\Sigma)$ homogeneous of order $\lambda$. For every function $u = u(r)$ we have $\Delta (u \gamma) = r^{\lambda+k-2} \left( \frac{dr}{r} \wedge A + B \right)$, where

$$A = u (\Delta \alpha - (\lambda + k - 2)(\lambda + n - k) \alpha - 2d^* \beta) - r \dot{u} (2\lambda + n - 1) \alpha - r^2 \ddot{u} \alpha,$$

$$B = u (\Delta \beta - (\lambda + n - k - 2)(\lambda + k) \beta - 2\alpha) - r \dot{u} (2\lambda + n - 1) \beta - r^2 \ddot{u} \beta.$$

**Theorem 3.6** (Theorem A.2 in [FHN20]). Let $\gamma = r^{\lambda+k} \left( \frac{dr}{r} \wedge \alpha + \beta \right)$ be a harmonic $k$-form on $C(\Sigma)$ homogeneous of order $\lambda$. Then $\gamma$ decomposes into the sum of homogeneous harmonic forms $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where $\gamma_i = r^{\lambda+k} \left( \frac{dr}{r} \wedge \alpha_i + \beta_i \right)$ satisfies the following conditions.
(i) \( \beta_1 = 0 \) and \( \alpha_1 \) satisfies \( d\alpha_1 = 0 \) and \( \Delta\alpha_1 = (\lambda + k - 2)(\lambda + n - k)\alpha_1 \).

(ii) \((\alpha_2, \beta_2) \in \Omega^{k-1}_{\text{coexact}} \times \Omega^k_{\text{exact}}\) satisfies the first-order system

\[
d\alpha_2 = (\lambda + k)\beta_2, \quad d^*\beta_2 = (\lambda + n - k)\alpha_2.
\]

In particular, if \( (\alpha_2, \beta_2) \neq 0 \) then \( \lambda + k \neq 0 \neq \lambda + n - k \) and the pair \( (\alpha_2, \beta_2) \) is uniquely determined by either of the two factors, which is a coexact/exact eigenform of the Laplacian with eigenvalue \((\lambda + k)(\lambda + n - k)\).

(iii) \((\alpha_3, \beta_3) \in \Omega^{k-1}_{\text{coexact}} \times \Omega^k_{\text{exact}}\) satisfies the first-order system

\[
d\alpha_3 + (\lambda + n - k - 2)\beta_3 = 0 = d^*\beta_3 + (\lambda + k - 2)\alpha_3.
\]

In particular, if \( (\alpha_3, \beta_3) \neq 0 \) then \( \lambda + k - 2 \neq 0 \neq \lambda + n - k - 2 \) and the pair \( (\alpha_3, \beta_3) \) is uniquely determined by either of the two factors, which is a coexact/exact eigenform of the Laplacian with eigenvalue \((\lambda + k - 2)(\lambda + n - k - 2)\).

(iv) \( \alpha_4 = 0 \) and \( \beta_4 \) satisfies \( d^*\beta_4 = 0 \) and \( \Delta\beta_4 = (\lambda + n - k - 2)(\lambda + k)\beta_4 \).

The decomposition \( \gamma = y_1 + y_2 + y_3 + y_4 \) is unique, except when \( \lambda = \frac{-m^2}{2} \); in that case forms of type (ii) and (iii) coincide, and there is a unique decomposition \( \gamma = y_1 + y_2 + y_4 \).

The previous proposition relates harmonic forms on the cone \( C(\text{SO}(3)) \) to eigenforms of the Laplacian on \( \text{SO}(3) \). The group \( \text{SO}(4) \) acts via pullback on complex-valued differential forms on \( S^3 \), and it turns out that the decomposition of this action into irreducible components gives the spectral decomposition for the Laplacian on \( S^3 \). This is made precise in the following two theorems, and as \( S^3 \) is a double cover of \( \text{SO}(3) \), we will get the spectral decomposition of the Laplacian on \( \text{SO}(3) \) from them.

**Theorem 3.7** (Theorem B in [Fol89]). The complex-valued \( L^2 \)-functions and 1-forms on \( S^3 \) decompose into the following irreducible \( \text{SO}(4) \)-invariant subspaces:

\[
\Omega^0(S^3, \mathbb{C}) = \bigoplus_{m=1}^{\infty} \Phi_{0,m}, \\
\Omega^1(S^3, \mathbb{C}) = \bigoplus_{m=1}^{\infty} (\Phi_{1,m} \oplus \Phi_{1,m}^- \oplus \Psi_{1,m}) .
\]

Here, \( \Phi_{0,m}, \Phi_{1,m}, \Phi_{1,m}^-, \Psi_{1,m} \) are defined as follows: denote by \( j : S^3 \to \mathbb{R}^4 \) the inclusion map and define \( z_1 = x_1 + ix_2, z_2 = x_3 + ix_4 \), and \( \partial r = \sum_{j=1}^{4} x_j \partial x_j \). Then let

\[
\Phi_{0,m} = j^* \mathcal{F}_{0,m+1}, \quad \text{where } \mathcal{F}_{0,m+1} \text{ is the smallest } \text{SO}(4) \text{-inv. space containing } z_1^{m-1},
\]

\[
\Phi_{1,m} = j^* \mathcal{F}_{1,m}, \quad \text{where } \mathcal{F}_{1,m} \text{ is the smallest } \text{SO}(4) \text{-inv. space containing } z_1^{m-1} \partial r (dz_1 \wedge dz_2).
\]

\[
\Phi_{1,m}^- = j^* \mathcal{F}_{1,m}^-, \quad \text{where } \mathcal{F}_{1,m}^- \text{ is the smallest } \text{SO}(4) \text{-inv. space containing } z_1^{m-1} \partial r (dz_1 \wedge d\overline{z}_2).
\]

\[
\Psi_{1,m} = j^* \mathcal{F}_{1,m}, \quad \text{where } \mathcal{F}_{1,m} \text{ is the smallest } \text{SO}(4) \text{-inv. space containing } z_1^{m-1} dz_1.
\]

**Theorem 3.8** (Theorem C in [Fol89]). Let \( \Phi_{0,m}, \Phi_{1,m}, \Phi_{1,m}^-, \Psi_{1,m} \) as in Theorem 3.7. Then \( \Phi_{0,m}, \Phi_{1,m} \oplus \Phi_{1,m}^-, \text{ and } \Psi_{1,m} \text{ are eigenspaces for the Laplacian with eigenvalues } m(m+2), (m+1)^2, \text{ and } m(m+2) \) respectively.
Corollary 3.9. Let $S^3$ be endowed with the round metric and $SO(3) = S^3/\{\pm 1\}$ be endowed with the quotient metric.

1. Then, the spectrum of the Laplacian $\Delta_{0,SO(3)}$ acting on real-valued $L^2$-functions on $SO(3)$ is:

$$\text{Spec}(\Delta_{0,SO(3)}) = \{k(k + 2) : k \in \mathbb{Z}_{\geq 0}, k \text{ even} \} = \{0, 8, 24, \ldots \}.$$ 

2. The smallest eigenvalue of the Laplacian $\Delta_{1,SO(3)}$ acting on real-valued 1-forms with coefficients in $L^2$ on $SO(3)$ is 4 and has multiplicity 6.

Proof of Corollary 3.9

1. This follows from Theorems 3.7 and 3.8 and the fact that functions in the space $\Phi_{0,m}$ defined in Theorem 3.7 are invariant under the antipodal map $(-1) : S^3 \to S^3$ if and only if $m$ is even.

2. By Theorem 3.8 the smallest eigenvalue of the Laplacian acting on complex-valued 1-forms on $S^3$ is 3. We see from the explicit description of the eigenspace that the eigenvectors are not invariant under the antipodal map. Thus, the eigenvalue 3 does not occur on $SO(3)$.

The next smallest eigenvalue is 4. It is realised, and it remains to check the dimension of its eigenspace: for the complex vector spaces defined in Theorem 3.7 we have $\Phi_{1,1} \simeq (\Lambda^2)^C$ and $\Phi_{1,-1} \simeq (\Lambda^2)^C$, the complexification of (anti-)self-dual constant forms on $\mathbb{R}^4$.

Here is how to see that $\Phi_{1,1} \simeq (\Lambda^2)^C$, the other isomorphism is analogous. We have

$$dz_1 \wedge dz_2 = dx_{13} - dx_{24} + i dx_{23} + i dx_{14}.$$ 

The element $g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in SO(4)$ sends this to $-dx_{13} + dx_{24} + i dx_{23} + i dx_{14}$, so the smallest $SO(4)$-invariant space containing $\omega$ must also contain the self-dual form $dx_{13} - dx_{24} = \frac{1}{2}(\omega - g\omega)$. Because $\Lambda^2$ is irreducible, this $SO(4)$-invariant space must contain all of $(\Lambda^2)^C$. Contracting with the radial vector field $\partial r$ and restricting to $S^3$ are $SO(4)$-equivariant operations, one checks that the result is non-zero, and therefore $\Phi_{1,1} \simeq (\Lambda^2)^C$.

Altogether, $\Phi_{1,1}$ and $\Phi_{1,-1}$ are representations of $SO(4)$ of complex dimension 3. They consist of 1-forms on $S^3$ that are invariant under the antipodal map, which proves the claim.

$\Box$

We can now combine the results about harmonic forms on $C(SO(3))$ with the spectral decomposition of the Laplacian on $SO(3)$ to find the critical rates for the Laplacian on $C(SO(3))$:
Proposition 3.10.

1. There are no harmonic 1-forms on \((\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}\) which are homogeneous of order \(\lambda\) for \(\lambda \in [-2, 0)\). In other words \(D_{\lambda_1,\lambda_2}(\mathbb{C}^2\setminus\{0\})/\{\pm 1\} \cap [-2, 0) = \emptyset\).

2. There is a six-dimensional space of harmonic 2-forms on \((\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}\) which are homogeneous of order \(-2\).

   There are no harmonic 2-forms on \((\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}\) which are homogeneous of order \(\lambda\) for \(\lambda \in (-2, 0)\).

Proof. It follows from point two in Proposition 2.1 that \(C(\text{SO}(3))\) and \((\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}\) are isometric as Riemannian manifolds and we prove the statements on \(C(\text{SO}(3))\).

1. Let \(\lambda \in [-2, 0)\) and assume there exists a harmonic homogeneous 1-form of order \(\lambda\) on \(C(\text{SO}(3))\). We show that the 1-form must vanish by showing that forms satisfying any of the cases (i), (ii), (iii), and (iv) from Theorem 3.6 are zero. Using the notation from the theorem, we get the following:

   (i) In this case, \(\Delta \alpha_1 = (\lambda - 1)(\lambda + 3)\alpha_1\). For \(\lambda \in [-2, 0)\), the factor \((\lambda - 1)(\lambda + 3)\) is negative, so our assumption implies that \(\alpha_1\) is a closed 0-form that is an eigenform of \(\Delta_{\text{SO}(3)}\) for a negative eigenvalue, which implies \(\alpha_1 = 0\) by Corollary 3.9.

   (ii) In this case, \(\beta_2\) is an exact 1-form with \(\Delta_{\text{SO}(3)}\beta_2 = (\lambda + 1)(\lambda + 3)\beta_2\). We have \((\lambda + 1)(\lambda + 3) < 8\) for \(\lambda \in [-2, 0)\), and therefore \(\beta_2 = 0\) as in case (i).

   (iii) In this case, \(\beta_3\) is an exact 1-form with \(\Delta_{\text{SO}(3)}\beta_3 = (\lambda + 1)(\lambda - 3)\beta_3\), and \(\beta_3 = 0\) follows as before.

   (iv) In this case, \(\beta_4\) is a co-closed 1-form with \(\Delta_{\text{SO}(3)}\beta_4 = (\lambda + 1)^2\beta_4\). For \(\lambda \in [-2, 0)\), we have \((\lambda + 1)^2 < 3\), and because of Corollary 3.9 this implies \(\beta_4 = 0\).

2. Let \(\lambda \in [-2, 0)\). Going through the cases (i), (ii), (iii), and (iv) from Theorem 3.6 we will find that there are six linearly independent harmonic homogeneous 2-forms of order \(-2\) in case (iii), but no other harmonic homogeneous forms. Using the notation from the theorem, we get the following:

   (i) In this case, we get a 1-form that is an eigenform of the Laplacian on \(\text{SO}(3)\) for the eigenvalue \(\lambda(\lambda + 2) < 0\), which must be 0 by Corollary 3.9.

   (ii) In this case, we get a 1-form that is an eigenform of the Laplacian on \(\text{SO}(3)\) for the eigenvalue \((\lambda + 2)^2 < 4\), which must be 0 by Corollary 3.9.

   (iii) In this case, we get a 1-form that is an eigenform of the Laplacian on \(\text{SO}(3)\) for the eigenvalue \(\lambda^2\). There are six of these by Corollary 3.9 for \(\lambda = -2\) and none for \(\lambda \in (-2, 0)\). In the case of \(\lambda = -2\) all six eigenforms give rise to harmonic 2-forms of order \(\lambda = -2\) on \(C(\text{SO}(3))\).

   (iv) In this case, we get a 2-form \(\beta_4\) that is an eigenform of the Laplacian on \(\text{SO}(3)\) for the eigenvalue \((\lambda + 2)^2 < 4\). The Hodge dual *\(\beta_4\) is then a 1-form that is an eigenform for the same eigenvalue, which must be 0 by Corollary 3.9.

□
For an application later we will not only need to know how many harmonic homogeneous forms there are, but also how many harmonic homogeneous forms with log(r) coefficients there are. Often, these two notions coincide, and the following proposition asserts that this is also the case in our setting.

**Definition 3.11.** Let Σ be a connected Riemannian manifold and $C = C(\Sigma)$ its cone. For $\lambda \in \mathbb{R}$, define

$$\mathcal{K}(\lambda)_{\lambda p C(\Sigma)} = \left\{ \gamma = \sum_{j=0}^{m} (\log r)^{j} \gamma_{j} \text{ for } m \geq 0, \gamma_{j} \in \Omega^{p}(C(\Sigma)), \text{ such that } \Delta_{p C(\Sigma)} \gamma = 0, \text{ where each } \gamma_{j} \text{ is homogeneous of order } \lambda \right\}.$$ 

**Proposition 3.12.** Let $\gamma = \sum_{j=0}^{m} (\log r)^{j} \gamma_{j} \in \mathcal{K}(-2)_{\lambda p C(\Sigma)}$, then $\gamma_{j} = 0$ for $j > 0$.

**Proof.** Write $\gamma_{j} = r^{\lambda+k} \left( \frac{dr}{r} \wedge \alpha_{j} + \beta_{j} \right)$. Then, by Lemma 3.15 for $j \geq 1$,

$$\Delta (\log(r)^{j} \gamma_{j}) = r^{-2} \left( \frac{dr}{r} \wedge A + B \right),$$

where

$$A = \log(r) \left( (\Delta \alpha_{j} - 2 d^{*} \beta_{j}) + 2j \log(r)^{-1} \alpha_{j} - j(j-1) \log(r)^{-2} \alpha_{j} \right),$$

$$B = \log(r) \left( (\Delta \beta_{j} - 2 d \alpha_{j}) + 2j \log(r)^{-1} \beta_{j} - j(j-1) \log(r)^{-2} \beta_{j} \right).$$

Here, the terms $\Delta \alpha_{j} - 2 d^{*} \beta_{j}$ and $\Delta \beta_{j} - 2 d \alpha_{j}$ vanish, because $\alpha_{j}$ is coexact and satisfies $2 \beta_{j} = d \alpha_{j}$, and $\beta_{j}$ is exact and satisfies $d^{*} \beta_{j} = 2 \alpha_{j}$ according to the discussion of point 2 of Proposition 3.10. The term $\Delta \gamma$ is a polynomial in $\log(r)$, and the condition $\Delta \gamma = 0$ prescribes that all coefficients of that polynomial vanish. Assume that $m > 0$ and check the coefficient of $\log(r)^{m-1}$: Eq. (3.13) implies that $\alpha_{m} = 0$ and Eq. (3.14) implies that $\beta_{m} = 0$, i.e. $\gamma_{m} = 0$. Repeating the argument, we find that $\gamma_{m-1} = 0, \gamma_{m-2} = 0, \ldots, \gamma_{2} = 0, \gamma_{1} = 0$, which is what we wanted to show. \qed

### 3.2 Harmonic Forms on Eguchi-Hanson Space

In the previous section we looked at certain harmonic forms on $(\mathbb{C}^{2} \setminus \{0\})/\{\pm 1\}$. The Eguchi-Hanson space $X_{EH}$ is asymptotic to the cone $(\mathbb{C}^{2} \setminus \{0\})/\{\pm 1\}$, and we can say a great deal about harmonic forms on $X_{EH}$ just from knowing the harmonic forms on $(\mathbb{C}^{2} \setminus \{0\})/\{\pm 1\}$. This is a consequence of the work of Lockhart and McOwen (cf. [LM85],[Loc87]) and will be the content of this section.

We will want statements about harmonic forms in certain weighted Hölder spaces. These spaces are defined in the following:

**Definition 3.15.** Define the weight functions

$$w : X_{EH} \to \mathbb{R}_{\geq 0}, \quad w(x) \mapsto 1 + |\rho(x)|,$$

$$w : X_{EH} \times X_{EH} \to \mathbb{R}_{\geq 0}, \quad (x, y) \mapsto \min\{w(x), w(y)\}.$$ 

Let $U \subset X_{EH}$. For $\alpha \in (0,1), \beta \in \mathbb{R}, k \in \mathbb{N}$, and $f \in \Omega^{k}(X_{EH})$ define the weighted Hölder norm
of $f$ via

\[ [f]_{C^\alpha_p(U)} := \sup_{x, y \in U} w(x, y)^{\alpha - \beta} \frac{|f(x) - f(y)|}{d_{g(1)}(x, y)^\alpha}. \]

\[ ||f||_{L^\infty(U)} := \left\| w^{-\beta} f \right\|_{L^\infty(U)}, \]

\[ ||f||_{C^{\alpha_p}_{p_j}(U)} := \sum_{j=0}^k ||\nabla^j f||_{L^\infty(U)} + \left[ \nabla^j f \right]_{C^\alpha_{p_j}(U)}. \]

The term $f(x) - f(y)$ in the first line denotes the difference between $f(x)$ and the parallel transport of $f(y)$ to the fibre $\Omega^k(X_{EH})|_x$ along one of the shortest geodesics connecting $x$ and $y$. When $U$ is not specified, take $U = X_{EH}$.

Throughout the article we will set $\beta$ to be a negative number. Informally, an element in the $C^{\alpha_p}_{p_j}$ Hölder space decays like $d_{g(1)}(\cdot, \rho^{-1}(0))^\beta$, as $d_{g(1)}(\cdot, \rho^{-1}(0)) \to \infty$.

We will now make the meaning of $X_{EH}$ being asymptotic to a cone precise.

**Definition 3.16.** Let $\Sigma$ be a connected Riemannian manifold and $C = C(\Sigma)$ be its cone with cone metric $g_C$. A Riemannian manifold $(M, g_M)$ is called asymptotically conical with cone $C$ and rate $\nu < 0$ if there exists a compact subset $L \subset M$, a number $R > 0$, and a diffeomorphism $\phi : (R, \infty) \times \Sigma \to M \setminus L$ satisfying

\[ |\nabla^k (\phi^* (g_M) - g_C)|_{g_C} = O(\rho^{-k}) \text{ for all } k \geq 0 \text{ as } \rho \to \infty. \]

Here, $\nabla$ denotes the Levi-Civita connection with respect to $g_C$ and $\phi : (0, \infty) \times \Sigma \to (0, \infty)$ is the projection onto the first component.

The following is then a consequence of Proposition 2.6.

**Proposition 3.17.** The Eguchi-Hanson space $X_{EH}$ endowed with the metric $g(1)$ is asymptotically conical with cone $C = C(SO(3))$ and rate $\nu = -4$.

We then have the following results about harmonic forms in $L^2$ on Eguchi-Hanson space:

**Lemma 3.18.**

1. We have $H^2_{\text{sing}}(X_{EH}) = H^2_{\text{dRham}}(X_{EH}) = \mathbb{R}$. For $k > 0$ define $v_k \in \Omega^2(X_{EH})$ to be

\[ v_k := f_k(r)^{-6} r \wedge \eta^1 - f_k(r)^{-2} \eta^2 \wedge \eta^3 \]

(3.19) and endow $X_{EH}$ with the metric $g(1)$. Then $v_k \in L^2(\Lambda^2(X_{EH}))$, $\Lambda_{g(1)} v_k = 0$, $[v_k]$ generates $H^2_{\text{dRham}}(X_{EH})$, and $v_k$ is the unique element in $L^2(\Lambda^2(X_{EH})) \cap [v_k]$ satisfying $\Lambda_{g(1)} v_k = 0$. Moreover, $v_1 \in C^{2,\alpha}_{-4}(\Lambda^2(X_{EH}))$. Away from the exceptional orbit $\rho^{-1}(0) \approx S^2$, we have that

\[ v_k = d\lambda_k, \text{ where } \lambda_k = -f_k(r)^{-2} \eta^1. \]

2. The $L^2$-kernels of $\Lambda_{g(1)}$ acting on forms of different degrees are as follows:

\[ \text{Ker}(\Lambda_{g(1)} : L^2(\Lambda^3(X_{EH})) \to L^2(\Lambda^3(X_{EH}))) = \langle v_k \rangle, \]

\[ \text{Ker}(\Lambda_{g(1)} : L^2(\Lambda^p(X_{EH})) \to L^2(\Lambda^p(X_{EH}))) = 0 \text{ for } p \neq 2. \]

For $k = 1$ and $\beta \in [-4, -2)$ they coincide with the $C^{2,\alpha}_{-\beta}$-kernels.
Proof.

1. We have that $X_{\text{EH}} = T^*S^2$ as smooth manifolds, therefore $H^2_{\text{sing}}(X_{\text{EH}}) = \mathbb{R}$. On smooth manifolds $H^2_{\text{sing}}(X_{\text{EH}}) = H^2_{\text{dR}}(X_{\text{EH}})$ by de Rham’s Theorem.

One checks with a direct computation that $v_k$ from Eq. (3.39) is closed and anti-self-dual, and therefore co-closed. The equality $v_k = d\lambda_k$ follows from a direct computation as well.

For $k = 0$, Eq. (3.19) still defines an element $v_0 \in \Omega^2(\mathbb{C}^2/\{\pm 1\} \setminus \{0\})$. One checks through direct calculation that $v_0 \in C^2_{-4}(\Delta^2(\mathbb{C}^2/\{\pm 1\}))$. Using the fact that $X_{\text{EH}}$ is asymptotically locally Euclidean (cf. Proposition 2.6), one gets the Hölder estimate on $X_{\text{EH}}$. Furthermore, $C^2_{-4} \subset L^\infty \subset L^2$, so $v_k$ is an element in $L^2(\Delta^2(\mathbb{C}^2/\{\pm 1\}))$.

By Poincaré duality, we have $H^2_{\text{dR}}(X_{\text{EH}}) = H^0_{\text{sing}}(X_{\text{EH}}) = \mathbb{R}$, where $H^2_{\text{dR}}(X_{\text{EH}})$ denotes the de Rham cohomology with compact support. [Loc87] Example (o.15) and [Loc87] Theorem (7.9) give that the map

$$H^2(X_{\text{EH}}) := \{\xi \in L^2(\Lambda^2 T^*X_{\text{EH}}) : d\xi = d^*\xi = 0\} \rightarrow \text{Im}(H^2_{\text{dR}}(X_{\text{EH}}) \hookrightarrow H^2_{\text{dR}}(X_{\text{EH}}))$$

$$\xi \mapsto [\xi]$$

is an isomorphism. Thus $[v_k]$ generates $H^2_{\text{dR}}(X_{\text{EH}})$ and $v_k \in [v_k]$ is the unique element in $L^2(\Lambda^2(X_{\text{EH}})) \cap [v_k]$ satisfying $d\lambda_k = 0$, $d^*\lambda_k = 0$.

It remains to check that $v_k$ is also the unique element in $L^2(\Lambda^2(X_{\text{EH}})) \cap [v_k]$ satisfying $\Lambda_{g_{(1)}} v_k = 0$. This holds, because the equations $\Lambda_{g_{(1)}} v_k = 0$ and $(d + d^*) v_k = 0$ are equivalent by the same integration by parts argument as in the compact case.

2. The first line is a restatement of the previous point. The other lines are [Loc87] Example (o.15) with proof in [Loc87] Theorem (7.9).

The $L^2$-kernels coincide with the $C^2_{-\beta}$-kernels, as $C^2_{-\beta}(\Lambda^p(X_{\text{EH}}))$ embeds into $L^2(\Lambda^p(X_{\text{EH}}))$ for $\beta < -2$ and the explicit description of the $L^2$-kernels shows that all kernel elements are actually in $C^2_{-\beta}(\Lambda^p(X_{\text{EH}}))$ for $\beta \geq -4$.

□

Remark 3.20. Note that $v_k$ from the lemma cannot have compact support by the unique continuation property for elliptic equations. We only have that $[v_k]$ contains a form of compact support.

The previous lemma makes statements about the $L^2$-kernels of the Laplacian on $X_{\text{EH}}$ acting on $p$-forms. Using the results from the previous section about harmonic forms on $\mathbb{C}^2/\{\pm 1\}$, we can rule out additional harmonic forms even in some of the weighted Hölder spaces that do not embed into $L^2$. The key proposition that will be proved throughout the rest of this section is the following:

Proposition 3.21. For $\beta \in (-4, 0)$, the kernels of the $\Lambda_{g_{(1)}}$ acting on forms in $C^2_{-\beta}$ of different degrees are as follows:

$$\text{Ker}(\Lambda_{g_{(1)}} : C^2_{-\beta}(\Lambda^2(X_{\text{EH}})) \rightarrow C^0_{-\beta}(\Lambda^2(X_{\text{EH}}))) = \langle v_1 \rangle.$$  

$$\text{Ker}(\Lambda_{g_{(1)}} : C^2_{-\beta}(\Lambda^p(X_{\text{EH}})) \rightarrow C^0_{-\beta}(\Lambda^p(X_{\text{EH}}))) = 0 \text{ for } p \neq 2.$$
The connection between the Laplacian on Eguchi-Hanson space and its cone is described in the following results taken from [KL20 Section 4] which were developed in [LM85, Loc87]. The theory works for a much bigger class of operators, but we will only reproduce it for the Laplacian here.

**Definition 3.22.** Let \( M \) be asymptotically conical and let the notation be as in Definition 3.16. Denote by \( \rho : C(\Sigma) \to \mathbb{R}_{\geq 0} \) the radius function, and use the same symbol to denote a map from \( M \) to \( \mathbb{R}_{\geq 0} \) that agrees with \( \phi, Q \) on \( \phi(\mathbb{R}, \infty) \subset M \). Let \( E \) be a vector bundle with metric and metric connection \( \nabla \) over \( M \). Then, for \( 1 > p > \infty, I \geq 0, \lambda \in \mathbb{R} \) denote by \( L^p_{I, \lambda} \) the completion of \( C^\infty_{cs}(E) \) with respect to the norm

\[
||y||_{L^p_{I, \lambda}} = \left( \sum_{j=0}^{I} \int_M |q^{-\lambda+j} \nabla^j y|^p Q^{-4 \lambda} \text{vol}_M \right)^{1/p}.
\]

The space \( L^p_{I, \lambda} \) is called the \( L^p \)-Sobolev space with \( I \) derivatives and decay faster than \( \lambda \).

**Theorem 3.23** (Theorem 4.10 in [KL20]). For \( \lambda \in \mathbb{R}, \) denote by \( \Delta_{p, g(\lambda)} : L^q_{2, \lambda}(\Lambda^p(X_{EH})) \to L^q_{0, \lambda-2}(\Lambda^p(X_{EH})) \) the Laplacian of the metric \( g(\lambda) \) acting on \( p \)-forms. Then, \( \text{Ker} \Delta_{p, g(\lambda)} \) is invariant under changes of \( \lambda \), as long as we do not hit any critical rates. That is, if the interval \([\lambda, \lambda']\) is contained in the complement of \( \mathcal{D}_{p, \Delta(C^2(\mathbb{Z}_+(0)/\{\pm 1\}))} \), then

\[
\text{Ker} \left( \Delta_{p, g(\lambda)} : L^q_{2, \lambda}(\Lambda^p(X_{EH})) \to L^q_{0, \lambda-2}(\Lambda^p(X_{EH})) \right) = \text{Ker} \left( \Delta_{p, g(\lambda')} : L^q_{2, \lambda'}(\Lambda^p(X_{EH})) \to L^q_{0, \lambda'-2}(\Lambda^p(X_{EH})) \right).
\]

**Proposition 3.24** (Theorem 4.20 in [KL20]). Let \( \lambda_1 < \lambda_2 \) such that \( \mathcal{K}(\lambda_i)_{\Delta_{p, C(\lambda)}} = 0 \) for \( i \in \{1, 2\} \). Then, the maps

\[
\Delta_{p, g(\lambda_1)} : L^2_{I, \lambda_1}(\Lambda^p(X_{EH})) \to L^2_{I, \lambda_1-2}(\Lambda^p(X_{EH}))
\]

and \( \Delta_{p, g(\lambda_2)} : L^2_{I, \lambda_2}(\Lambda^p(X_{EH})) \to L^2_{I, \lambda_2-2}(\Lambda^p(X_{EH})) \)

are Fredholm and the difference in their indices is given by

\[
\text{ind} \left( \Delta_{p, g(\lambda_1)} : L^2_{I, \lambda_1} \right) - \text{ind} \left( \Delta_{p, g(\lambda_2)} : L^2_{I, \lambda_2} \right) = \sum_{\lambda \in \mathcal{D}_{p, \Delta(C^2(\mathbb{Z}_+(0)/\{\pm 1\})) \cap (\lambda_1, \lambda_2)}} \text{dim} \mathcal{K}(\lambda)_{\Delta_{p, C(\lambda)}} (3.25)
\]

Combining everything, we get the following characterisation of harmonic forms with decay:

**Theorem 3.26.** For \( \lambda \in (-4, 0) \), the \( L^2_{2, \lambda} \)-kernels of \( \Delta_{p, g(\lambda)} \) acting on \( p \)-forms of different degrees are the same as the \( L^2 \)-kernels, namely:

\[
\text{Ker}(\Delta_{g(\lambda)} : L^2_{2, \lambda}(\Lambda^2(X_{EH})) \to L^2_{0, \lambda-2}(\Lambda^2(X_{EH}))) = \langle \psi_1 \rangle,
\]

\[
\text{Ker}(\Delta_{g(\lambda)} : L^2_{2, \lambda}(\Lambda^p(X_{EH})) \to L^2_{0, \lambda-2}(\Lambda^p(X_{EH}))) = 0 \text{ for } p \neq 2.
\]

**Proof.** 0-forms and 4-forms: it follows from the maximum principle that every harmonic function that decays at infinity must vanish. The Hodge star is an isomorphism between 0-forms and 4-forms that commutes with the Laplacian, so the statement for 0-forms implies that statement for 4-forms.
1-forms and 3-forms: the kernel of the Laplacian is zero for rate $-2$ by the second point of Lemma 3.18. By the first point of Proposition 3.10, there are no critical rates in the interval $[-2, 0)$. So, Theorem 3.23 implies the claim for 1-forms. As above, we get the statement for 3-forms by using the Hodge star.

2-forms: by Proposition 3.10, the only critical rate in $[-2, 0)$ is $-2$. The kernel of the Laplacian on 2-forms stays the same for rates $\lambda \in (-4, -2)$ by Lemma 3.18. By Theorem 3.23, the dimension of the kernel of the Laplacian acting on 2-forms with decay $\lambda \in (-4, 0)$ may therefore only change at $\lambda = -2$. We know from Propositions 3.12 and 3.24 that the index increases by 6 when crossing the critical rate $\lambda = -2$. We will now check that the dimension of the cokernel decreases by 6, which implies that the dimension of the kernel does not change.

The dual space of $L^2_{0,-\lambda}$ is $L^2_{0,-4-\lambda}$. Therefore, the cokernel of $\Delta_0^{\eta_1} : L^2_{2,-2}(\Lambda^2(X_{EH})) \to L^2_{2,-4}(\Lambda^2(X_{EH}))$ is isomorphic to the kernel of the adjoint operator $\Delta_0^{\eta_2} = \Delta_0^{\eta_1} : L^2_{2,0}(\Lambda^2(X_{EH})) \to L^2_{2,-2}(\Lambda^2(X_{EH}))$. Here we used that elements in the cokernel of $\Delta_0^{\eta_1}$ are smooth by elliptic regularity, so it does not matter how many derivatives we demand for sections acted on by the adjoint operator.

We now explicitly write down six linearly independent harmonic forms in $L^2_{2,0}(\Lambda^2(X_{EH}))$: three of them are the (self-dual) Kähler forms $\omega_1^{(1)}$, $\omega_2^{(1)}$, and $\omega_3^{(1)}$ defined in Proposition 2.11. Analogously, we can define three harmonic anti-self-dual forms with respect to $g(k)$ for all $k > 0$. To this end, extend $\eta^1, \eta^2, \eta^3 \in \mathfrak{so}(3)$ from Proposition 2.11 to right-invariant forms on $SO(3)$, denoted by $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3$. These forms satisfy $d\tilde{\eta}_1 = -\tilde{\eta}_2 \wedge \tilde{\eta}_3$ etc. In analogy to Proposition 2.11 define

\[ \hat{\epsilon}^1(r) = rf_k^{-1}(r)\tilde{\eta}_1^1, \quad \hat{\epsilon}^2(r) = f_k(r)\tilde{\eta}_2^2, \quad \hat{\epsilon}^3(r) = f_k(r)\tilde{\eta}_3^3 \]

and

\[ \hat{\omega}_1^{(k)} = dt \wedge \hat{\epsilon}^1 - \hat{\epsilon}^2 \wedge \hat{\epsilon}^3, \quad \hat{\omega}_2^{(k)} = dt \wedge \hat{\epsilon}^2 - \hat{\epsilon}^3 \wedge \hat{\epsilon}^1, \quad \hat{\omega}_3^{(k)} = dt \wedge \hat{\epsilon}^3 - \hat{\epsilon}^1 \wedge \hat{\epsilon}^2. \]

One checks through computation that $\hat{\omega}_i^{(k)}$ are closed and anti-self-dual, and therefore harmonic. A priori, they are defined on $\mathbb{R}_{>0} \times SO(3)$, and it remains to check that they extend to all of $X_{EH}$. We have $\hat{\omega}_2^{(k)} = dt(r\tilde{\eta}_2)$ and $\hat{\omega}_3^{(k)} = dt(r\tilde{\eta}_3)$, where $r\tilde{\eta}_2$ and $r\tilde{\eta}_3$ are well-defined 1-forms on all of $X_{EH}$, because they vanish as $r \to 0$. Therefore, $\hat{\omega}_2^{(k)}$ and $\hat{\omega}_3^{(k)}$ are well-defined on $X_{EH}$.

We have that $\hat{\omega}_1^{(k)} = rf_k^{-1}(r)dr \wedge \hat{\eta}_1^1 - f_k^{-2}(r)\tilde{\eta}_2^2 \wedge \tilde{\eta}_3^3$, where the first summand vanishes as $r \to 0$, and the second summand is a multiple of the volume form on $SO(3) \times SO(2) \{0\} \simeq S^2$ pulled back under the projection

\[ SO(3) \times_{SO(2)} V \to SO(3) \times_{SO(2)} V \]

\[ (g, x) \mapsto (g, 0). \]

Thus $\hat{\omega}_1^{(k)}$ is also defined on all of $X_{EH}$. The forms $\eta^1, \eta^2, \eta^3, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3$ are linearly independent which implies that $\omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, \hat{\omega}_1^{(k)}, \hat{\omega}_2^{(k)}, \hat{\omega}_3^{(k)}$ are linearly independent.

Last, note that for each $g \in SO(3)$ we can express $\tilde{\eta}_i^j(g)$ as a linear combination of $\eta_i^j(g)$. Each $\eta_i^j$ decays like $r^{1/2}$ as $r \to \infty$, which shows that the $\hat{\omega}_i^{(k)}$ have the same decay as the Hyperkähler triple $\omega_i^{(k)}$, which is covariant constant. Thus, we have that $\omega_i^{(1)}, \hat{\omega}_i^{(1)} \in L^2_{2,\epsilon}(\Lambda^2(X_{EH}))$, but $\not\in L^2_{2,\epsilon}(\Lambda^2(X_{EH}))$ for all $\epsilon > 0$ and $i \in \{1, 2, 3\}$.
Therefore, the dimension of the cokernel of $\Lambda_{y_{(1)}} : L^2_{2,\lambda}(\Lambda^2(X_{\text{EH}})) \to L^2_{2,\lambda+2}(\Lambda^2(X_{\text{EH}}))$ changes by six when crossing the critical rate $\lambda = -2$, and the dimension of the kernel stays the same.

Proposition 3.21 is now an immediate consequence of Theorem 3.26.

Proof of Proposition 3.21 For $\epsilon > 0$ we have that $G^2,_{\beta-\epsilon}$ is embedded in $L^2_{\beta}$, so the claim follows from Theorem 3.26.

4 Torsion-Free $G_2$-Structures on the Generalised Kummer Construction

In the two articles [Joy96b], Joyce constructed the first examples of manifolds with holonomy equal to $G_2$. One starts with the flat 7-torus, which admits a flat $G_2$-structure. A quotient of the torus by maps preserving the $G_2$-structure still carries a flat $G_2$-structure, but has singularities. The maps are carefully chosen, so that the singularities are modelled on $T^3 \times \mathbb{C}^2/\{\pm 1\}$. By the results of Section 3, $T^3 \times \mathbb{C}^2/\{\pm 1\}$ has a family of resolutions $T^3 \times X_{\text{EH}} \to T^3 \times \mathbb{C}^2/\{\pm 1\}$ depending on one real parameter, where $X_{\text{EH}}$ denotes the Eguchi-Hanson space, and the parameter defines the size of a minimal sphere in $X_{\text{EH}}$. We can define a smooth manifold by gluing these resolutions over the singularities in the quotient of the torus. The product manifold $T^3 \times X_{\text{EH}}$ carries the product $G_2$-structure from Eq. (2.11). That means we have two torsion-free $G_2$-structures on our glued manifold: one coming from flat $T^7$, and the product $G_2$-structure near the resolution of the singularities. We will interpolate between the two to get one globally defined $G_2$-structure. This will no longer be torsion-free, but it will have small enough torsion in the sense of Theorem 2.10. This is the argument that was used in [Joy96b] to prove the existence of a torsion-free $G_2$-structure, and the construction of this $G_2$-structure with small torsion is the content of Section 4.1.

Sections 4.2 to 4.4 give an alternative proof of the existence of a torsion-free $G_2$-structure on this glued manifold.

4.1 Resolutions of $T^7/\Gamma$

We briefly review the generalised Kummer construction as explained in [Joy96b]. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$, endowed with the flat $G_2$-structure $\varphi_0$ from Definition 2.9. Let $\alpha, \beta, \gamma : T^7 \to T^7$ defined by

$$\alpha : (x_1, \ldots, x_7) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),$$
$$\beta : (x_1, \ldots, x_7) \mapsto \left(-x_1, \frac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7\right),$$
$$\gamma : (x_1, \ldots, x_7) \mapsto \left(\frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4, -x_5, x_6, -x_7\right).$$

Denote $\Gamma = \langle \alpha, \beta, \gamma \rangle$. The next lemmata collect some information about the orbifold $T^7/\Gamma$: 

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Lemma 4.2 (Section 2.1 in part I, [Joy96b]). \( \alpha, \beta, \gamma \) preserve \( \varphi_0 \), we have \( \alpha^2 = \beta^2 = \gamma^2 = 1 \), and \( \alpha, \beta, \gamma \) commute. We have that \( \Gamma \simeq \mathbb{Z}_2^3 \).

Lemma 4.3 (Lemma 2.1.1 in part I, [Joy96b]). The elements \( f(y, \gamma), \alpha \beta, \) and \( \alpha \beta \gamma \) of \( \Gamma \) have no fixed points on \( T^7 \). The fixed points of \( \alpha \) in \( T^7 \) are 16 copies of \( T^3 \), and the group \( \langle \beta, \gamma \rangle \) acts freely on the set of 16 3-tori fixed by \( \alpha \). Similarly, the fixed points of \( \beta, \gamma \) in \( T^7 \) are each 16 copies of \( T^3 \), and the groups \( \langle \alpha, \gamma \rangle \) and \( \langle \alpha, \beta \rangle \) act freely on the sets of 16 3-tori fixed by \( \beta, \gamma \) respectively.

Lemma 4.4 (Lemma 2.1.2 in part I, [Joy96b]). The singular set \( L \) of \( T^7 / \Gamma \) is a disjoint union of 12 copies of \( T^3 \). There is an open subset \( U \) of \( T^7 / \Gamma \) containing \( L \), such that each of the 12 connected components of \( U \) is isometric to \( T^3 \times \left( B_7^4 / \{ \pm 1 \} \right) \), where \( B_7^4 \) is the open ball of radius \( \xi \) in \( \mathbb{R}^4 \) for some positive constant \( \xi \) (\( \xi = 1/9 \) will do).

We now define a compact 7-manifold \( M \), which can be thought of as a resolution of the orbifold \( T^7 / \Gamma \), and a one-parameter family of closed \( G_2 \)-structures \( \varphi(t) \) thereon. We can choose an identification \( U \simeq L \times \left( B_7^4 / \{ \pm 1 \} \right) \) such that we can write on \( U \)

\[
\varphi_0 = \delta_1 \wedge \delta_2 \wedge \delta_3 - \sum_{i=1}^3 \omega_i \wedge \delta_i, \quad \ast \varphi_0 = -\frac{1}{2} \omega_1 \wedge \omega_1 - \sum_{(i,j,k) = (1,2,3)} \omega_i \wedge \delta_j \wedge \delta_k,
\]

where \( \delta_1, \delta_2, \delta_3 \) are covariant constant orthonormal 1-forms on \( L \), and \( \omega_1, \omega_2, \omega_3 \) are the Hyperkähler triple from Section 2.2.

As before, denote by \( X_{\text{EH}} \) the Eguchi-Hanson space and by \( \rho : X_{\text{EH}} \to \mathbb{C}^2 / \{ \pm 1 \} \) the blowup map from Remark 2.8. Define \( \tilde{r} := |\rho| : X_{\text{EH}} \to \mathbb{R}_{\geq 0} \). For \( t \in (0,1) \), let \( \hat{U} := \hat{U}_t := L \times \{ x \in X_{\text{EH}} : \tilde{r}(x) < \xi t^{-1} \} \). Define

\[
N_t := \left( (T^7 / \Gamma) \setminus L \cup \hat{U} \right) / \sim,
\]

where for \( x = (x_h, x_v) \in U \subset L \times \mathbb{C}^2 / \{ \pm 1 \} \) and \( y = (y_h, y_v) \in \hat{U} \subset L \times X_{\text{EH}} \) we have \( x \sim y \) if \( x_h = y_h \) and \( t \cdot \rho(y_v) = x_v \). The smooth manifold \( N_t \) also comes with a natural projection map \( \pi : N_t \to T^7 / \Gamma \) induced by \( \rho \), and we extend \( \tilde{r} \) to a map on all of \( N_t \) via

\[
\tilde{r} : N_t \to \mathbb{R}_{\geq 0},
\]

\[
x \mapsto \begin{cases} 
|\rho(x)| & \text{if } x \in \hat{U}, \\
\xi t^{-1} & \text{otherwise.}
\end{cases}
\]

Write \( r_t := t \tilde{r} \) and choose a non-decreasing function \( \chi : [0, \xi] \to [0, 1] \) such that \( \chi(s) = 0 \) for \( s \leq \xi/4 \) and \( \chi(s) = 1 \) for \( s \geq \xi/2 \), and set

\[
\tilde{\omega}_i := \omega_i^{(1)} - d\left( \chi(r_t) r_i^{(1)} \right),
\]

(4.6)

The \( r_i^{(1)} \) were defined in Proposition 2.6 and are the difference between the flat Hyperkähler triple on \( \mathbb{C}^2 / \{ \pm 1 \} \) and the Hyperkähler triple \( (\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}) \) on \( X_{\text{EH}} \). On \( \hat{U} \) we have \( \tilde{\omega}_i = \omega_i \) where \( r_t > \xi/2 \), and \( \tilde{\omega}_i = \omega_i^{(1)} \) where \( r_t < \xi/4 \). Now define a 3-form \( \varphi \in \Omega^3(M) \) and a 4-form
\( \partial^t \in \Omega^4(N_t) \) as follows: on \( (T^r / \Gamma) \setminus U \subset N_t \), set \( \varphi^t = \varphi \) and \( \partial^t = \ast \varphi \). On \( \tilde{U} \subset L \times X_{\text{EH}} \) let

\[
\varphi^t := \delta_1 \wedge \delta_2 \wedge \delta_3 - t^2 \sum_{i=1}^{3} \tilde{\omega}_i \wedge \delta_i,
\]

\[
\partial^t := t^4 \frac{1}{2} \tilde{\omega}_1 \wedge \tilde{\omega}_1 - t^2 \sum_{(i,j,k)=\{1,2,3\}} \tilde{\omega}_i \wedge \delta_j \wedge \delta_k.
\]

(4.7) and cyclic permutation

This definition mimics the product situation explained in Section 2.2. For small \( t \), the 3-form \( \varphi^t \) is a \( G_2 \)-structure and therefore induces a metric \( g^t \). Both \( \varphi^t \) and \( \partial^t \) are closed forms, so if \( \ast \varphi^t = \partial^t \), then \( \varphi^t \) would be a torsion-free \( G_2 \)-structure by Theorem 2.13. However, this does not hold, and \( \varphi^t \) is not a torsion-free \( G_2 \)-structure. The following 3-form \( \psi^t \) is meant to measure the torsion of \( \varphi^t \):

\[
\ast \psi^t = \Theta(\varphi^t) - \partial^t.
\]

(4.9)

Its crucial properties are:

**Lemma 4.10.** Let \( \psi^t \in \Omega^3(M) \) as in Eq. (4.9). There exists a positive constant \( c \) independent of \( t \) such that

\[
d^* \psi^t = d^* \varphi^t, \quad \|\psi^t\|_{C^2, \alpha} \leq ct^4,
\]

where the Hölder norm is defined with respect to the metric \( g^t \) and its induced Levi-Civita connection.

**Proof.** The equality \( d^* \psi^t = d^* \varphi^t \) follows from Eq. (4.9) and the fact that \( \partial^t \) is closed.

The operator \( \ast \) is parallel, so the covariant derivative \( \nabla_X \) and \( \ast \) commute for every vector field \( X \) on \( N_t \), therefore it suffices to estimate \( \ast \psi^t \) rather than \( \psi^t \). Write \( \varphi_{X_{\text{EH}} \times L}^{(t)} := \delta_1 \wedge \delta_2 \wedge \delta_3 - t^2 \sum_{i=1}^{3} \omega_1^{(1)} \wedge \delta_i \) for the product \( G_2 \)-structure on \( X_{\text{EH}} \times L \) and denote the induced metric, which is the product metric, by \( g_{X_{\text{EH}} \times L}^{(t)} \). Recall the linear map \( T \) and the non-linear map \( F \) from Proposition 2.14, satisfying \( \Theta(\varphi + \xi) = \ast \varphi - T(\xi) - F(\xi) \) for a \( G_2 \)-structure \( \varphi \) and a small deformation \( \xi \). Using this notation, we get:

\[
\Theta(\varphi^t) - \partial^t = \Theta \left( \varphi_{X_{\text{EH}} \times L}^{(t)} - t^2 \delta_1 \wedge d \left( \chi(r_t) \Gamma_1^{(1)} \right) \right)
\]

\[
- \ast \left( \varphi_{X_{\text{EH}} \times L}^{(t)} - t^2 \delta_1 \wedge d \left( \chi(r_t) \Gamma_1^{(1)} \right) \right)
\]

\[
= T \left( \Gamma_2 \delta_1 \wedge d \left( \chi(r_t) \Gamma_1^{(1)} \right) \right) - F \left( -t^2 \Gamma_2 \wedge d \left( \chi(r_t) \Gamma_1^{(1)} \right) \right)
\]

\[
+ t^2 \delta_2 \wedge \delta_3 \wedge d \left( \chi(r_t) \Gamma_1^{(1)} \right).
\]

Here we used the equality \( \omega_1^{(k)} - \omega_1 = d \Gamma_1^{(k)} \) from Proposition 2.6 in the first step and the definition of \( T \) and \( F \) in the second step.

Note that \( \Theta(\varphi^t) - \partial^t \) is supported on \( \{ x \in M : (\xi / 4) t^{-1} < r < (\xi / 2) t^{-1} \} \). Therefore, by Eq. (2.7),

\[
\left| t^2 \left( \Gamma_1^{(1)} \right) \right|_{L^2(\tilde{g}_{11})} \leq \left| t^2 \left( d \chi(r_t) \right) \Gamma_1^{(1)} \right|_{L^2(\tilde{g}_{11})} + \left| t^2 \chi(r_t) \, d \Gamma_1^{(1)} \right|_{L^2(\tilde{g}_{11})}
\]

\[
\leq c \left| \Gamma_1^{(1)} \right|_{L^2(\tilde{g}_{11})} + c \left| t^2 \chi(r_t) \, d \Gamma_1^{(1)} \right|_{L^2(\tilde{g}_{11})}
\]

\[
= tO(\tilde{r}^{-2}) + O(\tilde{r}^{-4}) \leq ct^4.
\]
4.2 The Laplacian on $\mathbb{R}^3 \times X_{EH}$

In the next section we will prove an estimate for the Laplacian on 2-forms on $N_r$. We will use a blowup argument to essentially reduce the analysis on $N_r$ to the analysis on $T^3/\Gamma$ and $\mathbb{R}^3 \times X_{EH}$. In this section we will cite a general result for uniformly elliptic operators on product manifolds $\mathbb{R}^n \times Y$ from [Wal13], where $Y$ is a Riemannian manifold, and use this to find that harmonic 2-forms on $\mathbb{R}^3 \times X_{EH}$ are wedge products of parallel forms on $\mathbb{R}^3$ and harmonic forms on $X_{EH}$.

**Definition 4.11** (Definition 2.75 in [Wal13]). A Riemannian manifold $Y$ is said to be of bounded geometry if it is complete, its Riemannian curvature tensor is bounded from above and its injectivity radius is bounded from below. A vector bundle over $Y$ is said to be of bounded geometry if it has trivialisations over balls of fixed radius such that the transition functions and all of their derivatives are uniformly bounded. We say that a complete oriented Riemannian manifold $X$ has subexponential volume growth if for each $x \in X$ the function $r \mapsto \text{vol}(B_r(x))$ grows subexponentially, i.e., $\text{vol}(B_r(x)) = o(\exp(cr))$ as $r \to \infty$ for every $c > 0$.

**Lemma 4.12** (Lemma 2.76 in [Wal13]). Let $E$ be a vector bundle of bounded geometry over a Riemannian manifold $Y$ of bounded geometry and with subexponential volume growth, and suppose that $D : C^\infty(Y, E) \to C^\infty(Y, E)$ is a uniformly elliptic operator of second order whose coefficients and their first derivatives are uniformly bounded, that is non-negative, i.e., $\langle Da, a \rangle \geq 0$ for all $a \in W^{2, 1}(Y, E)$, and formally self-adjoint. Let $p : \mathbb{R}^n \times Y \to Y$ be the projection onto the second component and $a \in C^\infty(\mathbb{R}^n \times Y, p^*E)$ such that

$$(\Delta_{\mathbb{R}^n} + p^*D) a = 0$$

and $||a||_{L^\infty}$ is finite, then $a$ is constant in the $\mathbb{R}^n$-direction, that is $a(x, y) = a(y)$. Here, $\Delta_{\mathbb{R}^n}$ acts on a section $a \in C^\infty(\mathbb{R}^n \times Y, p^*E)$ by using the identification $C^\infty(\mathbb{R}^n \times Y, p^*E) = C^\infty(\mathbb{R}^n, C^\infty(Y, E))$.

**Corollary 4.13.** Let $Y$ be a manifold of bounded geometry and with subexponential volume growth. If $a \in \Omega^2(\mathbb{R}^3 \times Y)$ satisfies $||a||_{L^\infty} < \infty$ and

$$\Delta_{\mathbb{R}^3 \oplus g(1)} a = 0,$$

then $a$ is a sum of terms of the form $a_1 \wedge a_2$, where $a_1 \in \Omega^k(\mathbb{R}^3)$ is parallel, and $a_2 \in \Omega^l(Y)$ satisfies $\Delta_{g(1)} a_2 = 0$.

**Proof.** We can view the vector bundle of 2-forms over $\mathbb{R}^3 \times Y$ as a pullback bundle pulled back from $Y$ via

$$\Lambda^2(\mathbb{R}^3 \times Y) \cong p^* \left( \Lambda^2(Y) \oplus \Lambda^1(Y) \otimes \Lambda^1(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3) \right)$$

where $\Lambda^k(\mathbb{R}^3)$ denotes the trivial vector bundle over $\mathbb{R}^3$ whose fibre at each point is $\Lambda^k(\mathbb{R}^3)$. Under this identification, $\Delta_{\mathbb{R}^3 \times Y} = \Delta_{\mathbb{R}^3} + p^* (\Delta_Y + \Delta)$, where $\Delta$ is the canonical Laplacian on trivial vector bundles.

So, if $a \in \Omega^2(\mathbb{R}^3 \times Y)$ with $||a||_{L^\infty} < \infty$ and $\Delta_{\mathbb{R}^3 \oplus g(1)} a = 0$, then $a$ is the pullback of a section of $\Lambda^2(Y) \oplus \Lambda^1(Y) \otimes \Lambda^1(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3)$ over $Y$ which is in the kernel of $\Delta_Y + \Delta$ by Lemma 4.12.
Elements in the kernel of $\Delta_Y + \Delta$ over $Y$ are of the form $a_1 \wedge a_2$, where $a_1 \in \Omega^k(\mathbb{R}^3)$ is harmonic, and $a_2 \in \Omega^l(Y)$ satisfies $\Delta g_{(1)} a_2 = 0$. Bounded harmonic $k$-forms on $\mathbb{R}^3$ can be identified with tuples of harmonic functions on $\mathbb{R}^3$ which are constant by the maximum principle. This means that the bounded harmonic $k$-forms are parallel which proves the claim. \hfill \Box

4.3 The Laplacian on $N_t$

We now move on to the heart of the argument: an operator bound for the inverse of the Laplacian on $N_t$. The Laplacian on 2-forms has a kernel of dimension $b^2(N_t)$, so we can only expect such a bound for forms which are not in the kernel. Standard elliptic theory would give an estimate for forms orthogonal to the kernel. This estimate would depend on the gluing parameter $t$, but we want a uniform estimate, i.e. an estimate independent of $t$. Proving such an estimate is the content of this section.

**Stating the estimate** We first define weighted Hölder norms analogous to the previous sections. These norms have the following two important properties: far away from $L$, they are uniformly equivalent to ordinary Hölder norms, and near $L$ they are uniformly equivalent to the weighted Hölder norms on $\mathbb{R}^3 \times X_{EH}$, after applying a rescaling map.

**Definition 4.14.** For $t \in (0, 1)$ define the weight functions

$$w_t : N_t \to \mathbb{R}_{>0}$$

$$x \mapsto t + r_t,$$

$$w_{\mathbb{R}^3 \times \mathbb{R}^4} : \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R}_{>0}$$

$$(x, y) \mapsto |y|,$$

$$w_{\mathbb{R}^3 \times X_{EH}} : \mathbb{R}^3 \times X_{EH} \to \mathbb{R}_{>0}$$

$$x \mapsto 1 + \frac{r}{r_t}$$

and for $k \in \mathbb{N}$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$ the weighted Hölder norms $||| \cdot |||_{c^k_{\beta, \alpha}}$ on $N_t$ and $||| \cdot |||_{c^k_{\beta, \alpha}}$ on $\mathbb{R}^3 \times \mathbb{R}^4$ and $\mathbb{R}^3 \times X_{EH}$ respectively as in Definition 3.15.

We now define a way to decompose elements $a \in \Omega^2(N_t)$ into a component $\pi_t a$ that looks like $v_1 \in \Omega^2(X_{EH})$ from Eq. (2.19) on every fibre $\{y\} \times X_{EH} \subset T^3 \times X_{EH}$, and a remainder, denoted by $\rho_t a$. The reason for this is the following: the Laplacian on $\text{Im} \pi_t$ is approximately the Laplacian on $L$, and its inverse has operator norm of order $O(1)$ uniformly in $t$ as a map $c^{2,\alpha}_{\beta,t}(\Lambda^2(N_t)) \to c^{0,\alpha}_{\beta,t}(\Lambda^2(N_t))$. Notice that the weight does not change when applying the Laplacian. On $\text{Im} \rho_t$, it will turn out that the Laplacian has operator norm of order $O(1)$ uniformly in $t$ as a map $c^{2,\alpha}_{\beta,t}(\Lambda^2(N_t)) \to c^{2,\alpha}_{\beta,2t}(\Lambda^2(N_t))$. Here the weight changed in the same way as it did on the non-compact asymptotically conical space $X_{EH}$, cf. Section 5.2. In order to prove an estimate of the form $||a|| \leq c |||\Delta a|||$ we will define norms that incorporate these two different scaling behaviours in this section. The idea is taken from [Wal17].

Let $\widetilde{v} \in \Omega^2(X_{EH})$ be harmonic and with unit $L^2$-norm with respect to the norm $g_{(1)}$ on $X_{EH}$. As a shorthand, write $\chi_t := \chi(2r_t)$. Define $\pi_t : \Omega^2(N_t) \to \Omega^2(L)$ via

$$(\pi_t a)(y) := \langle a|_{\{y\} \times X_{EH}}, \chi_t v \rangle_{L^2,\mathbb{R}^3 \times g_{X_{EH}}}$$

for $y \in L$, (4.16)
where $v \in \Omega^2(X_{EH})$ is a multiple of $\nabla$ satisfying $\langle \chi_t v, \chi_t v \rangle_{L^2, r^2 g_{X_{EH}}} = 1$. This is equivalent to $\langle \chi_t v, \chi_t v \rangle_{L^2 g_{X_{EH}}} = 1$, i.e. in the metric $g_{X_{EH}}$ rather than $r^2 g_{X_{EH}}$, because the $L^2$-norm on 2-forms is a conformal invariant. Define $\iota_t : \Omega^0(L) \to \Omega^2(N_t)$ via

\[ (\iota_t g) := \chi_t \cdot p^*_L g \cdot p^*_X \nu. \quad (4.17) \]

where $g \in \Omega^0(L)$, and $p_L : L \times X_{EH} \to T^3$, $p_{X_{EH}} : L \times X_{EH} \to X_{EH}$ are projection maps. As written, $(\iota_t g)$ is an element in $\Omega^2(L \times X_{EH})$, but because $\text{supp}(\iota_t g) \subset \hat{U}$, we can view it as an element in $\Omega^2(N_t)$. Then

\[ \pi_t \iota_t g = g \text{ for all } g \in \Omega^0(L). \quad (4.18) \]

Last, define $\overline{\pi}_t := \iota_t \pi_t$, as well as $\rho_t := 1 - \overline{\pi}_t$.

**Proposition 4.19.** For all $k \in \mathbb{N}$ and $\beta > -4$ there exists $c > 0$ independent of $t$ such that for all $g \in \Omega^0(L)$ we have that

\[ \| \iota_t g \|_{C^k_{\beta, t}} \leq ct^{-2-\beta} \| g \|_{C^k_{\beta, t}}. \quad (4.20) \]

**Proof.** For the $L^\infty$-norm we have that

\[ \| p^*_L g \cdot p^*_X \nu \|_{L^\infty_{t^2 - g_{X_{EH}}}} \leq c \| p^*_L g \cdot p^*_X \nu \cdot (t + \hat{r})^4 \|_{L^\infty g_{R^3 \otimes T^2 g_{X_{EH}}}} \leq c \| p^*_L g \cdot p^*_X \nu \cdot (t + \hat{r})^4 t^4 \|_{L^\infty g_{R^3 \otimes g_{X_{EH}}}} \leq ct^2 \| p^*_L g \|_{L^\infty_{t^2}} \]

where we used that $v = O(\hat{r}^{-4})$ and therefore

\[ \| v \cdot \hat{r}^4 \|_{L^\infty g_{X_{EH}}} \leq c, \quad (4.21) \]

in the last step. For $\beta > -4$ we have that $\| \chi_t \|_{L^\infty_{t^2 - \beta}} \leq ct^{-4-\beta}$, which proves the claim for the weighted $L^\infty$-norm. The proof for higher derivatives is analogous. \(\square\)

**Proposition 4.22.** For all $k \in \mathbb{N}$, $\beta < 0$ there exists $c > 0$ independent of $t$ such that for all $a \in \Omega^2(N_t)$ we have that

\[ \| \pi_t a \|_{C^k_{\beta, t}} \leq t^{2+\beta-a} \| a \|_{C^k_{\beta, t}}. \quad (4.23) \]

**Proof.** We first estimate the $L^\infty$-part, i.e. $\| \pi_t a \|_{L^\infty}$. To this end

\[
|\pi_t a(x)| \leq \int_{\{x \in X_{EH} : \hat{r}(x) \leq t^{-1} \hat{r}\}} |a|^2 g_{X_{EH}} \cdot |v|^2 g_{X_{EH}} \, \text{vol}_{g_{X_{EH}}} \\
\leq t^2 \|a\|_{L^\infty_{t^2 - \beta}} \int_{X_{EH}} (t + \hat{r})^\beta \cdot |v|^2 g_{X_{EH}} \, \text{vol}_{g_{X_{EH}}} \\
\leq ct^{2+\beta} \|a\|_{L^\infty_{t^2 - \beta}} \int_{X_{EH}} (1 + \hat{r})^\beta \cdot (1 + \hat{r})^{-4} \, \text{vol}_{g_{X_{EH}}} \\
\leq ct^{2+\beta} \int_0^\infty (1 + \hat{r})^{-4+\beta} \cdot \hat{r}^3 \, d\hat{r} \\
\leq ct^{2+\beta} \|a\|_{L^\infty_{t^2 - \beta}},
\]

where $\text{supp}(\pi_t a) \subset \hat{U}$, because $\text{supp}(\pi_t a) \subset \hat{U}$.
where in the second step we used the definition of $||\cdot||_{L^\infty_{t\beta(t)}}$ and switched from measuring in $t^2g_{X_{EH}}$ to measuring in $g_{X_{EH}}$ which introduces the factor of $t^2$; in the third step we used $|v|_{g_{X_{EH}}} \leq c(1 + r)^{-4}$; in the fourth step we used polar coordinates to switch from integrating over $X_{EH}$ to integrating over $[0, \infty)$. The estimates for the Hölder norm, derivatives, and for other weights are proved analogously.

We are now ready to define the composite norms which weigh the $\pi_t$ and $\rho_t$ components differently.

**Definition 4.24.** For $\alpha \in (0, 1)$ and $\beta \in (-1, 0)$ let

$$||a||_{X_t} := ||\rho_t a||_{C^\alpha_{\beta,t}} + t^{-3/2} ||\pi_t a||_{C^{2,\alpha}} ,$$

$$||a||_{\mathcal{Y}_t} := ||\rho_t a||_{C^{\alpha,\beta-2,t}} + t^{-3/2} ||\pi_t a||_{C^{0,\alpha}} .$$

In the following, we will always assume that $\alpha$ and $\beta$ are close to 0. The most restrictive estimate in which this fact is used is Eq. (4.15). For concreteness, one may choose $\alpha = 1/16$ and $\beta = -1/16$.

**Definition 4.25 (Approximate kernel).** Let $C_1, \ldots, C_{12}$ be the connected components of $\hat{\mathcal{U}}$ and let $\chi_{C_i}$ be the characteristic function of the set $C_i$. Then define the *approximate kernel of $\Delta$ on $N_t$* to be

$$\mathcal{K} := \{(1 - \chi_t)\pi^* a : a \in \text{Ker } \Delta_{T^7/\Gamma}\} \oplus \text{span } \left(\chi_t \cdot \mathfrak{p}_{X_{EH}} \cdot \chi_{C_i}\right)_{i=1, \ldots, 12} ,$$

where $\pi : N_t \to T^7/\Gamma$ is the projection map from the previous section.

**Proposition 4.26.** There exists $c$ independent of $t$ such that for all $a \in \Omega^2(N_t)$, $a \perp \mathcal{K}$ we have

$$||a||_{X_t} \leq c ||\Delta a||_{\mathcal{Y}_t} . \quad (4.27)$$

The proof of this proposition will extend over the rest of the section. This linear estimate perpendicular to the approximate kernel is one thing we need. The following proposition states that by restricting to the orthogonal complement of $\mathcal{K}$ we are not forgetting about any important 2-forms — the image of the Laplacian remains the same when restricted to this orthogonal complement.

**Proposition 4.28.** The operator

$$\Delta : \mathcal{K}^\perp \to \text{Im } \Delta$$

is surjective, where $\text{Im } \Delta$ denotes the image of the Laplacian on all of $\Omega^2(N_t)$.

**Proof.** Step 1: Show that the $L^2$-orthogonal projection $q : \text{Ker } \Delta_{N_t} \to \mathcal{K}$ is an isomorphism.

Assume there exists $0 \neq a \in \Omega^2(N_t)$ with $\Delta a = 0$ such that $q(a) = 0$, i.e. $a \perp \mathcal{K}$. Then $\Delta a \neq 0$ by Proposition 4.26 which is a contradiction. Now note $\dim(\text{Ker } \Delta_{N_t}) = b^0(L) + b^2(T^7/\Gamma) = 12 + \hat{k}$, which is proved using the Künneth formula (see [JK21 Proposition 6.1]). By construction, $\dim(\mathcal{K}) = 12 + \hat{k}$, so $q$ is a surjective linear map between vector spaces of the same dimension, and therefore injective.
Step 2: Check \( \text{Im} \left( \Delta |_{\mathcal{K}^+} \right) = \text{Im} \Delta \).

It suffices to check that \( \text{Im} \Delta \subset \text{Im} \left( \Delta |_{\mathcal{K}^+} \right) \). Let \( y \in \text{Im} \Delta \), and \( \Delta x = y \). Denote the \( L^2 \)-orthogonal projection onto \( \mathcal{K} \) by \( \text{proj}_{\mathcal{K}} \). Let

\[
  z := q^{-1}(\text{proj}_{\mathcal{K}}(-x)).
\]

Then \( \Delta(x + z) = y \), and \( \text{proj}_{\mathcal{K}}(x + z) = 0 \) because of \( \text{proj}_{\mathcal{K}} \circ q^{-1} = \text{Id} \), i.e. \( x + z \perp \mathcal{K} \) which completes the proof. \( \square \)

Comparison with the Laplacian on \( L \) The embedding \( i_t : \Omega^0(L) \to \Omega^2(N_t) \) is defined using a cut-off of \( \nabla \in \Omega^2(X_{\text{EH}}) \). If not for this cut-off, we would have that \( \Delta_{i_t} = i_t \Delta \), where we use the symbol \( \Delta \) to denote the Laplacian on \( N_t \) as well as the Laplacian on \( L \). In our actual situation, we still have that \( \Delta \) and \( i_t \) nearly commute, and that is the content of the following proposition.

**Proposition 4.29.** For any \( \beta \leq 0 \) there exists \( c > 0 \) independent of \( t \) such that for all \( g \in \Omega^0(L) \) we have

\[
  \|(\Delta_{i_t} - i_t \Delta)g\|_{C^{\beta,2} L} \leq ct^2 \|g\|_{C^{2,0} L}. \tag{4.30}
\]

**Proof.** Define the map \( \tilde{i}_t : \Omega^0(L) \to \Omega^2(T^3 \times X_{\text{EH}}) \) via

\[
  \tilde{i}_t(g) = p_{L}^* g \cdot p_{X_{\text{EH}}}^* \nabla,
\]

where \( \nabla \in \Omega^2(X_{\text{EH}}) \) is harmonic and has unit \( L^2 \)-norm with respect to \( g_{X_{\text{EH}}} \). Then

\[
  (\Delta_{\tilde{i}_t} - \tilde{i}_t \Delta)g = 0. \tag{4.31}
\]

We aim to estimate

\[
  (\Delta_{i_t} - i_t \Delta)g = (\Delta_{i_t} - \Delta_{\tilde{i}_t})g + (\Delta_{\tilde{i}_t} - \tilde{i}_t \Delta)g + (\tilde{i}_t \Delta - i_t \Delta)g.
\]

We begin by estimating I, where it will be convenient to estimate on two regions separately:

\[
  \begin{align*}
  \Omega_1 & := \{ x \in L \times X_{\text{EH}} : \tilde{r}(x) \leq t^{-1} \zeta / 8 \}, \\
  \Omega_2 & := \{ x \in L \times X_{\text{EH}} : t^{-1} \zeta / 8 \leq \tilde{r}(x) \leq t^{-1} \zeta / 4 \}.
  \end{align*}
\]

Then

\[
  \|I\|_{C^{\beta,0} L} \leq \|(\tilde{i}_t - i_t)g\|_{C^{\beta,0} L}
  = \left\| p_{L}^* g \cdot p_{X_{\text{EH}}}^* (\nabla - \nabla) \right\|_{C^{\beta,0} L}
  \leq \left\| p_{L}^* g \cdot p_{X_{\text{EH}}}^* (\nabla - \nabla) \right\|_{C^{\beta,0} (\Omega_1)} + \left\| p_{L}^* g \cdot (\nabla - \nabla) \right\|_{C^{\beta,0} (\Omega_2)}.
\]

We will estimate the two summands separately. The first summand is defined on the region \( \Omega_1 = \{ x \in L \times X_{\text{EH}} : \tilde{r}(x) \leq t^{-1} \zeta / 8 \} \) where neither \( \nabla \) nor \( v \) is cut off. We have that

\[
  |v(x) - \nabla(x)|_{C^0 g_{X_{\text{EH}}}} \leq ct^2 \text{ for } x \in X_{\text{EH}} \text{ with } \tilde{r}(x) \leq t^{-1} \zeta / 8
\]

(4.33)
for the following reason: \( (v, v)_{L^2, t^2g_{XH}} = 1 \) by definition, thus
\[
\langle \chi_t, \chi_t \rangle_{L^2, t^2g_{XH}} \geq \langle \nabla_t^2 \rangle_{L^2, t^2g_{XH}} - \int_{\{x \in XH : \hat{r}(x) \geq t^{-1}/8 \}} |\nabla_t^2 \rangle_{L^2, t^2g_{XH}} \ vol_t^2 g_{XH} \\
\geq 1 - \int_{t^{-1}/8}^{\infty} (1 + r)^{-3} dr \geq 1 - ct^4.
\]
If \( \hat{r}(x) \leq t^{-1}/8 \) we have that \( v(x) = \nabla_t / \langle \chi_t, \chi_t \rangle_{L^2, t^2g_{XH}} \) because the cut-off is applied where \( \hat{r}(x) > t^{-1}/8 \). This implies, at the point \( x \),
\[
|v - \nabla_t^2g_{XH}| \leq \left| v \left( 1 - \frac{1}{\langle \chi_t, \chi_t \rangle_{L^2, t^2g_{XH}}} \right) \right| \leq \left| \nabla_t \right| \frac{t^4}{1 - t^4} \leq t^{-2} \left| \nabla_t \right| \frac{t^4}{1 - t^4} \leq ct^2.
\]
Using this for our estimate of the first summand of \( I \), we obtain:
\[
\| \hat{p}_v^* g \cdot \hat{p}_v^* \|_{C^{2, \alpha}(\Omega_1)} \leq t^2 \| \hat{p}_v^* \|_{C^{2, \alpha}} \| g \|_{C^{2, \alpha}}.
\]
For the second summand we get:
\[
\| \hat{p}_v^* g \cdot \chi_t \hat{p}_v^* \|_{C^{2, \alpha}(\Omega_2)} \leq \| \hat{p}_v^* \|_{C^{2, \alpha}} \| \chi_t \hat{p}_v^* \|_{C^{2, \alpha}(\Omega_2)} \leq \| \hat{p}_v^* \|_{C^{2, \alpha}} |\chi|_{C^{2, \alpha}} \cdot |1|_{C^{2, \alpha}(\Omega_2)} \left( |\nabla|_{C^{2, \alpha}(\Omega_2)} + |\nabla^2|_{C^{2, \alpha}(\Omega_2)} \right) \leq ct^2 \| g \|_{C^{2, \alpha}}
\]
where in the last step we used \( |1|_{C^{2, \alpha}(\Omega_2)} \leq c \), which holds because far away from \( L \), the weight function \( w_{\beta+4,t} \) is uniformly bounded. We also used
\[
|\nabla|_{L^2 g_{XH}} = t^{-2} |\nabla|_{g_{XH}} \leq ct^{-2} (1 + t)^{-4} \leq ct^2 (t + t^4)^{-4} \leq ct^2 \| \nabla \|_{L^2 g_{XH}}
\]
Together with Eq. (4.33), this shows that \( |v|_{L^2 g_{XH}} \leq ct^2 \) on \( \Omega_2 \).

Altogether \( ||I||_{C^{2, \alpha}} \leq ct^2 \| g \|_{C^{2, \alpha}} \). Furthermore, \( II = 0 \) because of Eq. (4.33). Lastly, III is estimated like I, which shows the claim. \( \square \)

The goal of this section is to prove an estimate for the operator norm of the inverse of the Laplacian with respect to the norms \( ||| \cdot |||_{X} \) and \( ||| \cdot |||_{Y} \). The purpose of these norms is to essentially split the problem into an estimate on \( \text{Im} \pi_t \) and remainder. The following proposition contains the estimate on \( \text{Im} \pi_t \).

**Proposition 4.35.** There exists \( c > 0 \) independent of \( t \) such that for \( t \) small enough and for all \( g \in \Omega^0(L) \) satisfying \( g \perp \text{Ker} \Delta_L \), we have that
\[
\| g \|_{C^{2, \alpha}} \leq c \| \pi_t \Delta_t g \|_{C^{0, \alpha}}.
\]

**Proof.** We have
\[
\| g \|_{C^{2, \alpha}} \leq c \| \Delta g \|_{C^{0, \alpha}} = c \| \pi_t \Delta g \|_{C^{0, \alpha}} \leq c \| \pi_t \Delta g \|_{C^{0, \alpha}} + c \| \pi_t \Delta g - \pi_t^2 \Delta g \|_{C^{0, \alpha}} \leq c \| \pi_t \Delta g \|_{C^{0, \alpha}} + ct^{2-\alpha} \| g \|_{C^{2, \alpha}}.
\]
where we used elliptic regularity for the operator $\Delta$ on $L$ in the first step, and Propositions $4.22$ and $4.29$ in the last step. At this point, the last summand $ct^{2-\alpha} ||g||_{C^{1,\alpha}}$ can be absorbed into the left hand side for $t$ small enough. \hfill \Box

The model operator on $\mathbb{R}^3 \times X_{EH}$

**Definition 4.37.** For $j \in \{1, \ldots, 12\}$ let $C_j'$ be a connected component of $\hat{U}$, but made slightly smaller, explicitly

$$C_j' := C_j \cap \{ (x_h, x_a) \in L \times X_{EH} : \hat{r}(x_a) \leq t^{-1} \zeta/4 \}.$$

For $\beta \in \mathbb{R}$ let

$$s_{j, \beta, t} : \Omega^2(N_t) \to \Omega^2(\mathbb{R}^3 \times \{ x \in X_{EH} : \hat{r}(x) \leq t^{-1} \zeta/4 \})$$

$$a \mapsto t^{-\beta-2}(p \circ (\cdot, t), \text{Id})^* \left( a|_{C_j'} \right),$$

where $p : \mathbb{R}^3 \to T$ denotes the quotient map.

Then:

**Lemma 4.38.** For $j \in \{1, \ldots, 12\}, \beta \in \mathbb{R}$ we have that for all $a \in \Omega^2(\mathbb{R}^3 \times X_{EH})$ we have

$$||s_{j, \beta, t} a||_{C^{1,\alpha}_\beta} = ||a||_{C^{1,\alpha}_\beta(C_j')}, \text{ and}$$

$$(s_{j, \beta-2, t} \Delta_{N_t} a - \Delta_{g_{R^3} \oplus g_{(1)}} s_{j, \beta, t} a)|_{C_j'} = 0.$$

Here $\Delta_{g_{R^3} \oplus g_{(1)}}$ denotes the Laplacian on $\mathbb{R}^3 \times X_{EH}$ with respect to the metric $g_{R^3} \oplus g_{(1)}$.

**Proof.** The map $((-t) \circ p, \text{Id}) : C_j' \to \mathbb{R}^3 \times \{ x \in X_{EH} : \hat{r}(x) \leq t^{-1} \zeta/4 \}$ pulls back the metric $t^2(g_{R^3} \oplus g_{(1)})$ to the metric induced by $g'$. The extra factor $t^{-\beta-2}$ cancels out the factor $t^2$ when changing the metric from $t^2(g_{R^3} \oplus g_{(1)})$ to $g_{R^3} \oplus g_{(1)}$ on 2-forms and cancels out the factor $t^\beta$ coming from the definition of $||| \cdot |||_{C^{1,\alpha}_\beta}$. \hfill \Box

**Estimate of $\rho_t a$.** Recall the projection $\pi_t$ onto the fibrewise harmonic part from Eq. (4.16) and its complement $\rho_t$. In Propositions 4.35 we essentially proved an estimate for the inverse of the Laplacian on $\text{Im} \pi_t$. In order to get an estimate with respect to $||| \cdot |||_{\pi_t}$ and $||| \cdot |||_{\theta_t}$ we need to estimate the inverse of the Laplacian on $\text{Im} \rho_t$. The two operators satisfy $\pi_t \rho_t = 0$, so the following proposition implies an estimate for the inverse of the Laplacian for elements $a \in \text{Im} \rho_t \subset \Omega^2(N_t)$.

**Proposition 4.39.** Write $\mathcal{K}' := \{(1 - \chi_t)a : a \in \text{Ker} \Delta_{T^2/T} \} \subset \Omega^2(N_t)$. Then there exists $c > 0$ independent of $t$ such that for $a \in \Omega^2(N_t)$ satisfying $a \perp \mathcal{K}'$ we have

$$||a||_{C^{1,\alpha}_\beta} \leq c \left( ||\Delta a||_{C^{0,\alpha}_\beta - 2, \epsilon} + ||\pi_t a||_{L^\infty_{\beta, \epsilon}} \right). \tag{4.40}$$

**Proof.** The Schauder estimate

$$||a||_{C^{0,\alpha}_\beta} \leq c \left( ||\Delta a||_{C^{0,\alpha}_\beta - 2, \epsilon} + ||a||_{L^\infty_{\beta, \epsilon}} \right) \tag{4.41}$$
can be derived as in [Wal17 Proposition 8.15]. It then suffices to show that there exists \( c \) such that

\[
||a||_{L^2_{\bar{\mu},\tau}} \leq c \left( ||\Delta a||_{L^{a,\alpha}_{\bar{\mu},\tau}} + ||\bar{\pi}_\tau a||_{L^1_{\bar{\mu},\tau}} \right).
\]  
(4.42)

Assume Eq. (4.42) is false, then there exist \( t_i \to 0 \), \( a_i \in \Omega^2(N_{t_i}) \) satisfying \( a_i \perp \mathcal{K}' \), and \( x_i \in N_{t_i} \) such that

\[
||a||_{L^{a,\alpha}_{\bar{\mu},t_i}} \leq c, \quad w_{\bar{\mu},t_i}(x_i) a_i(x_i) = 1, \quad \text{and} \quad ||\Delta a_i||_{L^{\alpha,\alpha}_{\bar{\mu},2,2,t_i}} \to 0, \quad ||\bar{\pi}_\tau a_i||_{L^0_{\bar{\mu},t_i}} \to 0.
\]  
(4.43)

Here, we got ||\( a ||_{L^{a,\alpha}_{\bar{\mu},t_i}} \leq c \) from Eq. (4.41). Without loss of generality we can assume to be in one of three following cases, and we will arrive at a contradiction in each of them.

**Case 1:** the sequence \( x_i \) concentrates on one ALE space, i.e. \( t_i^{-1}r_{t_i}(x_i) \to c < \infty \) (see Fig. 1).

\[
\mathbb{R}^3 \times X
\]

Figure 1: Blowup analysis near the associative is reduced to the analysis of the Laplacian on \( \mathbb{R}^3 \times X_{EH} \).

By passing to a subsequence and translating in the \( \mathbb{R}^3 \)-direction if necessary, we can assume that \( x_i \) concentrates near one fixed connected component of \( L \). Let \( C_j \subset L \times X_{EH} \) be the connected component \( \hat{U} \) containing an accumulation point of the sequence \( x_i \). Define \( \tilde{a}_i := s_{\bar{\mu},t_i} a_i \in \Omega^2(\mathbb{R}^3 \times \{x \in X_{EH} : \check{r}(x) \leq t_i^{-1} \xi/4\}) \) and let \( \check{x}_i \) be a lift from \( C_j \) to \( \mathbb{R}^3 \times X_{EH} \). The new 2-form \( \tilde{a}_i \) then satisfies

\[
||\tilde{a}_i||_{L^{a,\alpha}_{\bar{\mu},t_i}} \leq c, \quad (1 + \check{r}(\check{x}_i))^{-\beta} |\tilde{a}_i(\check{x}_i)| \geq c, \quad \text{and} \quad ||\Delta \tilde{a}_i||_{L^{\alpha,\alpha}_{\bar{\mu},2,2,t_i}} \to 0,
\]

which follows from Lemma [4.38]. Now the weight function no longer has \( t_i \) in it and distances and tensors are measured using the metric \( \hat{g}_{\mathbb{R}^3} \oplus \hat{g}(1) \).

By the assumption of case 1, we have \( \check{r}(\check{x}_i) \to c < \infty \). By passing to a subsequence we can assume that \( \check{x}_i \) converges, so write \( x'^* := \lim_{i \to \infty} \check{x}_i \in \mathbb{R}^3 \times X_{EH} \). Using the Arzelà-Ascoli theorem and a diagonal argument, we can extract a limit \( a^* \in \Omega^2(\mathbb{R}^3 \times X_{EH}) \) of the sequence \( \tilde{a}_i \) satisfying:

\[
||a^*||_{L^2_{\tilde{\mu}}} \leq c, \quad \text{and} \quad (4.44)
\]

\[
\Lambda_{\hat{g}_{\mathbb{R}^3} \oplus \hat{g}_{(1)}} a^* = 0, \quad \text{and} \quad (4.45)
\]

\[
(1 + \check{r}(x'^*))^{-\beta} |a^*(x'^*)| > c.
\]  
(4.46)

By Corollary [4.13] (applied to the case \( \mathbb{R}^3 \times X_{EH} \)), we have that \( a^* \) is independent of the \( \mathbb{R}^3 \)-direction. By Proposition [4.21], the only harmonic forms on \( X_{EH} \) that decay like \( \check{r}^{\beta} \) are multiples of \( v_1 \). Thus \( a^* \) is the pullback of a multiple of \( v_1 \) under the projection \( \rho_{X_{EH}} : \mathbb{R}^3 \times X_{EH} \to X_{EH} \).
Because $\left\| \nabla_i a_i \right\|_{L^\infty_{\mu_i}} \to 0$, we have that $a^*$ is perpendicular to $\nabla_\mu$ on every $\{y\} \times X_{EH} \subset \mathbb{R}^3 \times X_{EH}$.

Here is how to see this in detail: let $y \in L$, then we calculate on $\{y\} \times X_{EH}$:

$$\langle a^*, \nabla \rangle = \langle a^*, \nabla - \chi_t \nu \rangle + \langle a^* - \overline{a}_i, \chi_t \nu \rangle + \langle \overline{a}_i, \chi_t \nu \rangle = I + II + III. \quad (4.47)$$

Here,

$$\left| I \right| \leq \left| \langle a^*, \nabla - \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \leq t^{-1}/8\}} + \left| \langle a^*, \nabla - \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \geq t^{-1}/8\}},$$

where we have for the first summand

$$\left| \langle a^*, \nabla - \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \leq t^{-1}/8\}} \leq \int_0^{t^{-1}/8} \left| a^* \right|_{g_{t(i)}} \cdot \left| \nabla - \chi_t \nu \right|_{g_{t(i)}} r^3 \, dr \leq c \int_0^{t^{-1}/8} r^\beta t^4 r^3 \, dr \leq ct^{-\beta} \to 0.$$

Here we used Eq. (4.44) and Eq. (4.33) (after changing from $\cdot |r^3 g_{X_{EH}}$ to $\cdot |g_{X_{EH}}$) in the second step. For the second summand we find

$$\left| \langle a^*, \nabla - \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \geq t^{-1}/8\}} \leq c \int_{-\frac{t^{-1}}{8}}^{\infty} r^{\beta} r^{-4} r^3 \, dr \leq ct^{-\beta} \to 0,$$

where we used $\nabla = O(\hat{r}^{-4})$ and $\nu = O(\hat{r}^{-4})$ in the first step.

In order to estimate $II$, let $l > 0$. Then

$$\left| II \right| \leq \left| \langle a^* - \overline{a}_i, \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \geq l\}} + \left| \langle a^* - \overline{a}_i, \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \leq l\}},$$

and we find for the first summand

$$\left| \langle a^* - \overline{a}_i, \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \geq l\}} \leq c \left( \left\| a^* \right\|_{L^\infty_{t_l}} + \left\| \overline{a}_i \right\|_{L^\infty_{t_l}} \right) \int_l^{\infty} r^{\beta - 4 + 3} \, dr \leq cl^\beta$$

for a constant $c$ independent of $l$. For the second summand we have

$$\left| \langle a^* - \overline{a}_i, \chi_t \nu \rangle \right|_{\{x \in X_{EH}; \xi(x) \leq l\}} \leq \left\| a^* - \overline{a}_i \right\|_{L^\infty_{t_l} (\{x \in X_{EH}; \xi(x) \leq l\})} \cdot \int_0^{l} r^{\beta - 4 + 3} \, dr \leq c \left\| a^* - \overline{a}_i \right\|_{L^\infty_{t_l} (\{x \in X_{EH}; \xi(x) \leq l\})} \to 0$$

as $i \to \infty$ by definition of $a^*$. Last,

$$\left| III \right| = t^{-2-\beta} \left| \left( \pi \partial_t a_i \right) (y) \right| = t^{-2-\beta} \left| \left( \pi \partial_t \pi_i \partial_t a_i \right) (y) \right| \leq c \left\| \overline{\nabla} a_i \right\|_{L^\infty_{t_l}} \to 0,$$

where we used Proposition 4.22 for the estimate.

Altogether we see that, by taking $\lim_{i \to \infty}$ in Eq. (4.47), we have that $\langle a^*, \nabla \rangle \leq cl^\beta$, where the constant $c$ was independent of $l$. This is true for any $l > 0$, therefore $\langle a^*, \nabla \rangle = 0$. But this is a contradiction to Eq. (4.46).

**Case 2:** the sequence $x_i$ concentrates on the regular part, i.e. $r_{\xi}(x_i) \to c > 0$ (see Fig. 2).

Using the Arzelà-Ascoli theorem and a diagonal argument, we can extract a limit $a^* \in \Omega^2(T^2/\Gamma \setminus L)$. Denote, furthermore, $\lim_{i \to \infty} x_i = x^*$. We have $|a^*| < c \cdot d(\cdot, L)^\beta$, so we have that $a^*$ is
By the unique continuation property for elliptic PDEs, the inner product
\[ \langle a^*, (1 - \chi(2d(\cdot, L))) \cdot a_i \rangle_{T^2/I} = \lim_{i \to \infty} \langle a_i, (1 - \chi_i(\cdot)) \cdot \pi^* a_i \rangle_{N_i} = 0. \] (4.48)

By the unique continuation property for elliptic PDEs, the inner product
\[ \langle \cdot, (1 - \chi) \circ (2d(\cdot, L)) \cdot \rangle \]
is non-degenerate on harmonic forms. The 2-form \( a^* \) is a harmonic form that is orthogonal to all harmonic forms with respect to this inner product, therefore \( a^* = 0 \). But this contradicts \( a^*(x^+) > c \).

**Case 3:** the sequence \( x_i \) concentrates on the neck region, i.e. \( \tilde{r}(x_i) \to \infty \), but \( r_i(x_i) \to 0 \) (see Fig. 3).

Define \( \tilde{x_i} \in \Omega^2(\mathbb{R}^3 \times X_{EH}) \) and \( \tilde{x_i} \in \mathbb{R}^3 \times X_{EH} \) as in case 1. In this case, we have that \( |\rho(\tilde{x_i})| \to \infty \).

In order to be able to obtain a limit of this sequence, let \( R_i \to \infty \) be a sequence such that \( R_i / |\rho(\tilde{x_i})| \to 0 \). Cutting out the exceptional locus of the Eguchi-Hanson space, we can consider \( \{(x_i, x_0) \in \mathbb{R}^3 \times X_{EH} : R_i \leq |\rho(x_0)| \leq \zeta i^{-1}\} \) as a subset of \( \mathbb{R}^3 \times C^2 / \{\pm 1\} \). On \( \mathbb{R}^3 \times C^2 / \{\pm 1\} \), we have the rescaling map \( (-|\rho(\tilde{x_i})|) \).
We now define:
\[
\tilde{a}_i := (\cdot |\rho(\tilde{x}_i)|^2 \left( \frac{a_i}{|x_i|} \right) \cdot |\rho(\tilde{x}_i)|^{-2-\beta} \\
eq \Omega^2(\mathbb{R}^3 \times \{x \in X_{EH} : R_i/|\rho(\tilde{x}_i)| \leq |\rho(x)| \leq \zeta t_i^{-1}/|\rho(\tilde{x}_i)|\}),
\]

(4.49)
\[
\tilde{x}_i := \tilde{x}_i/|\rho(\tilde{x}_i)|.
\]

This sequence satisfies
\[
\left\| \tilde{a}_i \right\|_{L^2_{\rho}} \leq c \quad \text{and} \quad \left| \tilde{a}_i(\tilde{x}_i) \right| > c.
\]

(4.50)

The data \(\tilde{a}_i\) and \(\tilde{x}_i\) are defined on (subsets of) \(\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}\). We use the same symbols to denote their pullbacks under the quotient map \(\mathbb{C}^2 \rightarrow \mathbb{C}^2/\{\pm 1\}\).

As before, we extract a \(C^{2,\alpha/2}_{loc}\)-limit \(a^* \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}^4 \setminus \{0\}) \) satisfying
\[
\Delta_{\mathbb{R}^7} a^* = 0, \quad \text{and} \quad ||a^*||_{L^p_{\rho}(\mathbb{R}^3 \times \mathbb{R}^4)} \leq c.
\]

We see as in case 2 that \(a^*\) defines a distribution on all of \(\mathbb{R}^7\), and is smooth by elliptic regularity on all of \(\mathbb{R}^7\).

We also get an \(L^\infty\)-bound for \(a^*\) as follows: away from \(\mathbb{R}^3 \times \{0\}\), this is given by Eq. (4.50). To see that \(a^*\) does not blow up in the \(\mathbb{R}^3\)-direction near \(\mathbb{R}^3 \times \{0\}\), consider any \(y \in \mathbb{R}^3 \times \{0\}\). Let \(1 < p < -4/\beta\), then \(||a^*||_{L^p(B_i(y))} \leq c\), independent of \(y\), by Eq. (4.50). So, by elliptic regularity \(||a^*||_{L^p_{\rho}(\mathbb{R}^3(y))} \leq c\) for any \(m \in \mathbb{N}\), and by the Sobolev embedding we have \(||a^*||_{L^\infty} \leq c\), where all of these estimates were independent of \(y\).

By Corollary (4.25) (applied to \(\mathbb{R}^3 \times \mathbb{R}^4\)), \(a^*\) is constant in the \(\mathbb{R}^3\) direction. The limit \(a^*\) is therefore the pullback of a harmonic, bounded form of \(\mathbb{R}^4\), and must thus vanish, which is a contradiction to the second part of Eq. (4.50).

**Cross-term estimates** We have now established uniform estimates for the inverse of \(\Delta\) on \(\text{Im} \; \pi_i\) and \(\text{Im} \; \rho_i\). As it stands, it could happen that the operator norm of \(\rho_i \Delta \pi_i\) or \(\pi_i \Delta \rho_i\) is very big. It will turn out in our proof of Proposition 4.26 that in such a case one would be unable to deduce anything about the inverse of the operator norm of \(\Delta\) with respect to \(\left|\cdot\right|_{\mathcal{X}_i}\) and \(\left|\cdot\right|_{\mathcal{Y}_i}\). Fortunately, it turns out that the operator norms of \(\rho_i \Delta t_i\) (and therefore \(\rho_i \Delta \pi_i\), because \(\pi_i = t_i \pi_i\)) and \(\pi_i \Delta \rho_i\) are small, which is the content of the following proposition.

**Proposition 4.51.** There exists \(c > 0\) independent of \(t\) such that for all \(g \in \Omega^0(L)\) and for all \(a \in \Omega^2(N_t)\) we have
\[
\left| \rho_i \Delta t_i g \right|_{C^\alpha}_{\beta,t} \leq c t^{2-\alpha} \left|g\right|_{C^\alpha} \text{ if } \beta < 0,
\]
\[
\left| \pi_i \Delta \rho_i a \right|_{C^\alpha}_{\beta,t} \leq c t^{2+2\beta-2\alpha} \left|\rho_i a\right|_{C^\alpha}_{\beta,t} \text{ if } -2 < \beta < 0.
\]

(4.52)

(4.53)
Proof. We first prove Eq. (4.52). We have \( p_t a = 0 \) and therefore

\[
||p_t \Delta t g||_{L^2} = ||p_t (\Delta t g - t_1 \Delta t g)||_{L^2} \\
\leq ||\Delta t g - t_1 \Delta t g||_{L^2} + ||t_1 \pi_t (\Delta t g - t_1 \Delta t g)||_{L^2} \\
\leq ||\Delta t g - t_1 \Delta t g||_{L^2} + ct^{-2} ||\pi_t (\Delta t g - t_1 \Delta t g)||_{L^2} \\
\leq ||\Delta t g - t_1 \Delta t g||_{L^2} + ct^{-2} ||\Delta t g - t_1 \Delta t g||_{L^2} \\
\leq ct^{-2} ||g||_{L^2},
\]

where we used Proposition 4.19 in the third step, Proposition 4.22 in the fourth step, and Proposition 4.29 in the last step.

Now to prove Eq. (4.53): assume without loss of generality that \( a = p_t a \). Define

\[
\tilde{\pi}_t : \Omega^2 (T^3 \times X_{EH}) \rightarrow \Omega^0 (L) \\
(\tilde{\pi}_t a)(x) := (a, \nabla) e^g_{N\text{EH}}.
\]

The difference between \( \tilde{\pi}_t \) and \( \pi_t \) is that they use \( \nabla \) and \( \nabla v \) in their definition, respectively; \( \nabla \) is not cut off, \( \nabla v \) is, and both are rescaled to have unit norm. It suffices to prove the claim for \( a \in \Omega^2 (N) \) which is supported near \( L \). We can view such \( a \) as an element in \( \Omega^2 (T^3 \times X_{EH}) \) and apply \( \tilde{\pi}_t \) to it. Also define \( \tilde{\pi}_t : \Omega^0 (L) \rightarrow \Omega^2 (T^3 \times X_{EH}) \) as \( \tilde{\pi}_t (g) = p_t^* \pi_t e^g_{N\text{EH}} \). Then \( \tilde{\pi}_t \) is Id and we also define \( \tilde{\pi}_t := 1 - \tilde{\pi}_t \).

We have \( \tilde{\pi}_t \Delta = \Delta \tilde{\pi}_t \), thus \( \tilde{\pi}_t a = 0 \Rightarrow \tilde{\pi}_t \Delta a = 0 \), and therefore \( \tilde{\pi}_t \Delta \tilde{\pi}_t = 0 \). Hence

\[
\pi_t \Delta \rho_t a = (\pi_t - \tilde{\pi}_t) \Delta \rho_t a + \pi_t \Delta (\rho_t - \tilde{\rho}_t) a + \pi_t \Delta (1 - \tilde{\rho}_t) \tilde{\rho}_t a.
\]

We first estimate I:

\[
\langle \pi_t \Delta \rho_t a, \nabla - \nabla v \rangle_{L^2} \leq ct^{4+\beta} \int_0^{\xi/8} \left( \int_r^{1/8} (1 + r)^{-2+4\beta} r^3 dr \right) ||\Delta \rho_t a||_{L^2} dr \\
\leq ct^{4+\beta} ||\rho_t a||_{L^2} ||(1 + r)^{-2} r^3 dr \leq ct^{4+\beta} ||\rho_t a||_{L^2} ||(1 + r)^{-2} r^3 dr \\
+ ct^{\beta} \int_0^{\infty} ||\rho_t a||_{L^2} ||(1 + r)^{-2+2\beta-4} r^3 dr \leq ct^{\beta} ||\rho_t a||_{L^2} ||(1 + r)^{-2+2\beta-4} r^3 dr
\]

Here we applied Eq. (4.33) on the region \( \{ x \in X_{EH} : \tilde{\rho}(x) \leq 0 \} \) and we used

\[
||\nabla - \nabla v||_{L^2} \leq ||\nabla||_{L^2} + ||\nabla v||_{L^2} \leq c(t + \tilde{t})^{-2} t^2
\]
on the region \( \{ x \in X_{EH} : \tilde{\rho}(x) \geq \xi^{-1}/8 \} \). Thus

\[
||(\pi_t - \tilde{\pi}_t) \Delta \rho_t a||_{L^2} \leq ct^{2+\beta} ||\rho_t a||_{L^2}
\]

and the \( C^{0,\alpha} \)-estimate follows analogously.
For estimating II we need the estimate
\[ \| \pi_t a \|_{C^{\alpha,\beta}} \leq t^{2-\alpha - k} \| a \|_{C^{\alpha,\beta}} \] (4.54)
which is proved like Proposition 4.22. Then
\[
\| \pi_t \Delta \left( (1 - t \pi_t) - \pi_t \right) a \|_{C^{\alpha,\beta}} = \| \pi_t \Delta (t_t \pi_t - t_t \pi_t) a \|_{C^{\alpha,\beta}} \\
\leq ct^{-\alpha} \| \Delta (t_t \pi_t - \pi_t) a \|_{C^{\alpha,\beta}_{\beta-2}} \\
\leq ct^{-\alpha} \left( \| \Delta (t_t \pi_t - \pi_t) a \|_{C^{\alpha,\beta}_{\beta-2}} + t^2 \| (t_t \pi_t - \pi_t) a \|_{C^{\alpha,\beta}} \right) \\
\leq ct^{-\alpha} (1 + t^2) \| (t_t \pi_t - \pi_t) a \|_{C^{\alpha,\beta}} \\
\leq ct^{-\alpha} (1 + t^2) t^2 \| a \|_{C^{\alpha,\beta}_{\beta-2}} \\
\leq ct^{-\alpha} \| a \|_{C^{\alpha,\beta}_{\beta-2}}
\]
where in the first estimate we used Eq. (4.54), in the second estimate we used Proposition 4.29, in the third estimate we used the estimate for the operator norm of \( t_t \) from Proposition 4.19, and in the fourth estimate we did the same calculation as when estimating I and we again used \(-2 < \beta < 0\). In the last step we used the assumption that \( a = \pi_t a \).

It remains to estimate III. We find
\[
\| \pi_t \Delta \left( (1 - t_t \pi_t) - \pi_t \right) a \|_{C^{\alpha,\beta}} = \| \pi_t \Delta (t_t \pi_t - \pi_t) \pi_t a \|_{C^{\alpha,\beta}} \\
\leq ct^{-\alpha+\beta} \| \Delta (t_t \pi_t - \pi_t) \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} \\
\leq ct^{-\alpha+\beta} \left( \| (t_t \pi_t - \pi_t) \Delta \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} + t^2 \| \pi_t a \|_{C^{\alpha,\beta}} \right) \\
\leq ct^{-\alpha+\beta} \| (t_t \pi_t - \pi_t) \Delta \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} + t^2 \| \pi_t a \|_{C^{\alpha,\beta}} \\
\leq ct^{-\alpha+\beta} \| \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} + t^2 \| \pi_t a \|_{C^{\alpha,\beta}}
\]
where we used Eq. (4.54) in the second step, and \( \pi_t \Delta = \Delta \pi_t \) together with Proposition 4.29 in the third step. Here we find for the first summand
\[
ct^{-\alpha+\beta} \| (t_t - \pi_t) \Delta \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} \leq ct^{-\alpha+\beta} \| \chi t_t \pi_t - \pi_t \|_{C^{\alpha,\beta}_{\beta-2}} \cdot \| p^*_L \Delta \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} \cdot \| 1 \|_{C^{\alpha,\beta}_{\beta-2}} \\
\leq ct^{-\alpha+\beta} \cdot t^2 \cdot \| \Delta \pi_t a \|_{C^{\alpha,\beta}} \\
\leq ct^{2-\alpha+2\beta} \cdot \| \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}}
\]
where we used Eqs. (4.33) and (4.34) in the second step; we used \( \| p^*_L \Delta \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} = \| \Delta \pi_t a \|_{C^{\alpha,\beta}} \) which holds because \( p^*_L \Delta \pi_t a \) is constant in the Eguchi-Hanson direction, so the derivative in the \( C^{0,\alpha}_{\beta-2} \)-norm is just a derivative in the \( L \)-direction; in the last step we used Eq. (4.54). For the second summand we have
\[
t^2 \| \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} \leq t^2 \| \pi_t a \|_{C^{\alpha,\beta}_{\beta-2}} \\
\]
by Eq. (4.54), which proves the claim. \(\Box\)
Proof of Proposition 4.26

By definition, $||a||_{\mathcal{X}_t} = ||\rho_t a||_{C^{2,\alpha}_{\beta,t}} + t^{-3/2} ||\pi_t a||_{C^{2,\alpha}}$. We treat the first summand first:

$$||\rho_t a||_{C^{2,\alpha}_{\beta,t}} \leq ||\Delta \rho_t a||_{C^{0,\alpha}_{\beta-2,t}} \leq \left( ||\nabla \Delta \rho_t a||_{C^{0,\alpha}_{\beta-2,t}} + ||\rho_t \Delta a||_{C^{0,\alpha}_{\beta-2,t}} + ||\rho_t \Delta \pi_t a||_{C^{0,\alpha}_{\beta-2,t}} \right),$$

where we used Proposition 4.39 in the first step and in the second step used $1 = \pi_t + \rho_t$ twice. Here, the first summand satisfies

$$||\nabla \Delta \rho_t a||_{C^{0,\alpha}_{\beta-2,t}} \leq t^{-\beta} ||\pi_t \Delta \rho_t a||_{C^{0,\alpha}},$$

$$\leq t^{\beta+2-2\alpha} ||\rho_t a||_{C^{2,\alpha}_{\beta,t}},$$

where we used Proposition 4.22 in the first step, and Eq. (4.52) in the second step. The remaining term can be absorbed into the left hand side of Eq. (4.27).

For the third summand we get from Eq. (4.52) that

$$||\rho_t \Delta \pi_t a||_{C^{2,\alpha}_{\beta-2,t}} \leq ct^{2-\alpha} ||\pi_t a||_{C^{2,\alpha}},$$

which can be absorbed into the left hand side of Eq. (4.27) if $\alpha$ is sufficiently small. Regarding the $\pi_t$-term, we find that

$$t^{-3/2} ||\pi_t a||_{C^{2,\alpha}} \leq t^{-3/2} ||\pi_t \Delta \pi_t a||_{C^{0,\alpha}} \leq t^{-3/2} \left( ||\pi_t \Delta a||_{C^{0,\alpha}} + ||\pi_t \Delta \pi_t a||_{C^{0,\alpha}} \right),$$

where we used Proposition 4.35 in the first step and $1 = \pi_t + \rho_t$ in the second step. Here we have for the last summand

$$t^{-3/2} ||\pi_t \Delta \rho_t a||_{C^{0,\alpha}} \leq t^{-3/2} t^{2+2\beta-2\alpha} ||\rho_t a||_{C^{2,\alpha}_{\beta,t}},$$

which can be absorbed into the left hand side of Eq. (4.27). The remaining terms, i.e. the ones that have not been absorbed into the left hand side of Eq. (4.27), exactly sum up to $||\Delta a||_{\mathcal{Y}_t}$, which proves the claim.

\[\square\]

4.4 The Existence Theorem

We will now prove the theorem which guarantees the existence of a torsion-free $G_2$-structure when starting from a $G_2$-structure with small torsion.

Theorem 4.56. Assume there exists $c > 0$ such that $\psi^t \in \Omega^3(N_t)$ satisfies $d^* \psi^t = d^* \psi$ and

$$\left\| d^* \psi^t \right\|_{\mathcal{Y}_t} \leq ct^4,$$

$$\left\| \psi \right\|_{C^{0,\alpha}_{0,t}} \leq ct^4.$$

Then, for small $t$, there exists $\eta^t \in \Omega^2(N_t)$ such that $\psi^t + d\eta$ is a torsion-free $G_2$-structure and

$$\left\| \eta^t \right\|_{\mathcal{X}_t} \leq ct^4.$$
To ease notation, we write $\varphi = \varphi^t$, $\psi = \psi^t$, and $\eta = \eta^t$ throughout the proof.

**Proof.** We will construct $\eta \in \Omega^2(N_t)$ satisfying

$$
\Delta \eta = d^* \psi + d^*(f \psi) + * d(F(d\eta)), \quad \text{where } f = \frac{7}{3} (\varphi, d\eta).
$$

(4.57)

Set $\eta_0 = 0$ and, if $\eta_{j-1} \in \Omega^2(N_t)$ is given, let $\eta_j \in \Omega^2(N_t)$ be such that

$$
\Delta \eta_j = d^* \psi + d^*(f_{j-1} \psi) + * d(F(d\eta_{j-1})), \quad \text{where } f_{j-1} = \frac{7}{3} (\varphi, d\eta_{j-1}),
$$

and such that $\eta_j \perp K$. This is well-defined, i.e. such $\eta_j$ exists, because $\text{Im } d^* \subset \text{Im } \Delta$ and restricting $\Delta$ to $K^\perp$ does not change its image by Proposition [4.28]. We aim to show by induction that $\|\eta_j\|_{X_j} \leq ct^4$. For $j = 0$ this is true by definition, and we will now derive the estimate for $j > 0$.

By definition of $\eta_j$ together with Proposition [4.26] we have that

$$
\|\eta_j\|_{X_j} \leq c \|\Delta \eta_j\|_{\mathfrak{q}_j}
$$

$$
\leq c \left( \|d^* \psi\|_{\mathfrak{q}_j} + \|d^*(f_{j-1} \psi)\|_{\mathfrak{q}_j} + \|* d(F(d\eta_{j-1}))\|_{\mathfrak{q}_j} \right)
$$

$$
= c (I + II + III).
$$

By assumption we have $I = \|d^* \psi\|_{\mathfrak{q}_j} \leq ct^4$.

Now to estimate II:

$$
\|d^*(f_{j-1} \psi)\|_{\mathfrak{q}_j} \leq \|df_{j-1} d^*\psi\|_{\mathfrak{q}_j} + \|f_{j-1} d^* \psi\|_{\mathfrak{q}_j} = II.A + II.B.
$$

Here

$$
II.A = \|\rho_t (df_{j-1} d^*\psi)\|_{C^{\alpha,0}_{\beta-2,t}} + t^{-3/2} \|\pi_t (df_{j-1} d^*\psi)\|_{C^{\alpha,0}}
$$

$$
\leq (t^{-\alpha} + t^{-3/2 - \alpha + \beta}) \|df_{j-1} d^*\psi\|_{C^{\alpha,0}_{\beta-2,t}}
$$

$$
\leq (t^{-\alpha} + t^{-3/2 - \alpha + \beta}) \|df_{j-1}\|_{C^{\alpha,0}_{\beta-2,t}} \|\psi\|_{C^{\alpha,0}}
$$

$$
\leq ct^4,
$$

where for the first estimate we used Propositions [4.19] and [4.22], and for the last estimate we used the induction hypothesis $\|\eta_{j-1}\|_{X_{j-1}} \leq ct^4$, which implies $\|df_{j-1}\|_{C^{\alpha,0}_{\beta-2,t}} \leq ct^{7/2}$, together with the assumption $\|\psi\|_{C^{\alpha,0}_{\beta-2,t}} \leq ct^4$. The estimate $II.B \leq ct^4$ is derived analogously.

It remains to estimate III:

$$
III = \|\rho_t (* d(F(d\eta_{j-1})))\|_{C^{\alpha,0}_{\beta-2,t}} + t^{-3/2} \|\pi_t (* d(F(d\eta_{j-1})))\|_{C^{\alpha,0}} = III.A + III.B.
$$

The summand III.A is estimated as

$$
III.A \leq ct^{-\alpha} \|* d(F(d\eta_{j-1}))\|_{C^{\alpha,0}_{\beta-2,t}},
$$
where we first estimate the $L^\infty$-part of the $C^{0,\alpha}$-norm. Namely, by Proposition 2.14

$$
\| \pi_t (F(d\eta_{j-1})) \|_{L^\infty_{\beta-2}} \leq c \| d\eta_{j-1} \|_{L^\infty_{\beta-1}} \| \nabla d\eta_{j-1} \|_{L^\infty_{\beta-2}} t^{-1+\beta} \\
+ c \| d\eta_{j-1} \|_{L^2_{\beta-1}} \| d^* \psi \|_{L^\infty_{\beta-2}} t^{-2+2\beta} \\
\leq ct^4.
$$

The $[\cdot]_{C^{0,\alpha}}$-part is estimated analogously. To estimate $\text{III.B} = t^{-3/2} \| \pi_t (F(d\eta_{j-1})) \|_{C^{0,\alpha}}$, we again estimate the $L^\infty$-part first. Fix some $y \in L$ and compute $\pi_t (F(d\eta_{j-1})) (y)$ by computing an integral over $X_{EH} \approx \{ y \} \times X_{EH} \subset L \times X_{EH}$. By Proposition 2.14 we have

$$
\| \pi_t (F(d\eta_{j-1})) \| \leq \left\{ \begin{array}{c} \| \pi_t (F(d\eta_{j-1})) \cdot \chi_t v \|_{L^2_{\gamma \times \gamma}} \\
\end{array} \right.
$$

Here,

$$
\text{III.B.1} \cdot t^{3/2} = c \int_{X_{EH}} |d(\pi_t \eta_{j-1} + \rho_t \eta_{j-1})| \cdot |\nabla d(\pi_t \eta_{j-1} + \rho_t \eta_{j-1})| \cdot |\chi_t v| \|\|v\|\|_{L^2_{\gamma \times \gamma}}
$$

$$
\leq c \int_0^\infty \left( (t+r)^{-7} \| \pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla d\pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \right) \left( (t+r)^{-4} t^3 \right) \, dr \\
\leq c \| \pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla \pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \leq c t^{2-14/3}
$$

$$
+ c \int_0^\infty \left( (t+r)^{2\beta-3} \| \rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla d\rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \right) \left( (t+r)^{-4} t^3 \right) \, dr \\
\leq c \| \rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla \rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \leq c t^{2/4}
$$

$$
+ c \int_0^\infty \left( (t+r)^{-5} \| \pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla d\pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \right) \left( (t+r)^{-4} t^3 \right) \, dr \\
\leq c \| \pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla \pi_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \leq c t^{4+21/4}
$$

$$
+ c \int_0^\infty \left( (t+r)^{-5} \| \rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla d\rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \right) \left( (t+r)^{-4} t^3 \right) \, dr \\
\leq c \| \rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \| \nabla \rho_t \eta_{j-1} \|_{C^{0,\alpha}_{\beta-3}} \leq c t^{4+21/4}
$$

$$
\leq c \left( t^{2-14/3} - t^{-7} t^2 + t^{2-4} t^{2\beta-3} t^2 + 2 t^{4+3/2} t^{2\beta-3} t^2 \right) \\
\leq c t^6,
$$

thus $\text{III.B.1} \leq c t^4$. The part $\text{III.B.2}$ and the $C^{0,\alpha}$-parts of $\text{III.B.1}$ and $\text{III.B.2}$ are estimated analogously. Altogether, this gives $\text{III} \leq c t^4$. 

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The sequence $\eta_j$ satisfies
\[
\|\eta_j\|_{C^{2,\alpha}/\beta_j} \leq \|\rho_j\eta_j\|_{C^{2,\alpha}/\beta_j} + \|\Pi_j\eta_j\|_{C^{2,\alpha}/\beta_j}
\leq \|\eta_j\|_{X_\epsilon} + t^{-2-\beta+3/2} \|\eta_j\|_{X_\epsilon}
\leq ct^{7/2-\beta}.
\]

As usual, the constant $c$ is independent of $t$, but in particular independent of $j$. Thus, there exists, up to a subsequence, a $C^{2,\alpha}/2$-limit $\lim_{j \to \infty} \eta_j =: \eta$ by the Arzelà–Ascoli theorem. This limit solves Eq. (4.57) and satisfies
\[
\|\eta\|_{C^{2,\alpha}/2} \leq ct^{7/2-\beta}.
\]

By [Joy96a][Theorem 10.3.7], $\varphi + d\eta$ is a torsion-free $G_2$-structure, which proves the claim. □

Taking everything together, this gives us:

**Theorem 4.58.** Let $N_t$ be the resolution of $T^7/\Gamma$ from Eq. (4.5) and $\varphi^t \in \Omega^2(N_t)$ the $G_2$-structure with small torsion from Eq. (4.7). There exists $c > 0$ independent of $t$ such that the following is true: for $t$ small enough, there exists $\eta^t \in \Omega^1(N_t)$ such that $\tilde{\varphi} = \varphi^t + d\eta^t$ is a torsion-free $G_2$-structure, and $\eta^t$ satisfies
\[
\|\eta^t\|_{C^{2,\alpha}/2} \leq ct^{7/2-\beta}.
\]

In particular,
\[
\|\tilde{\varphi} - \varphi^t\|_{L^\infty} \leq ct^{5/2} \quad \text{and} \quad \|\tilde{\varphi} - \varphi^t\|_{C^{0,\alpha}/2} \leq ct^{5/2-\alpha/2} \quad \text{as well as} \quad \|\tilde{\varphi} - \varphi^t\|_{C^{3,\alpha}/2} \leq ct^{3/2-\alpha/2}.
\]

**Proof.** By Lemma 4.10 we have that $\|\psi\|_{C^{2,\alpha}/2} \leq ct^4$. Combined with Propositions 4.10 and 4.22 we also have $\|\psi\|_{\eta_t} \leq ct^4$. Thus, Theorem 4.56 can be applied, which gives the claim. □

**Remark 4.59.** The power $7/2 - \beta$ in Theorem 4.58 can be improved to $4 - \epsilon$ for any $\epsilon \in (0, 1)$ by defining the norms $\|\cdot\|_{X_\epsilon}$ and $\|\cdot\|_{\eta_t}$ with a factor of $t^{-\kappa}$ instead of $t^{-3/2}$ for $\kappa \in (0, 2)$ close to 2.

**Remark 4.60.** In [Joy96a], compact manifolds with holonomy $\text{Spin}(7)$ were constructed. In the simplest case, one constructs $\text{Spin}(7)$-structures with small torsion by gluing together the product $\text{Spin}(7)$-structure on $T^3 \times X_{\text{EH}}$ to the flat $\text{Spin}(7)$-structure on $T^3$. This gluing construction is analogue to the definition of the $G_2$-structure in Eq. (4.7). In contrast to the $G_2$-situation, however, Joyce’s theorem about the existence of torsion-free $\text{Spin}(7)$-structures cannot immediately be applied, because the torsion of the glued structure is too big. He overcame this problem by constructing a correction of the glued structure by hand which has smaller torsion, to which the existence theorem can be applied. The same can be done in the $G_2$ case. In fact, one gets a correction in the $G_2$-case from the $\text{Spin}(7)$-case by considering the $\text{Spin}(7)$-orbifold $T^7/\Gamma \times S^3$. Using this corrected structure, one would get even better control over the difference between glued structure and torsion-free structure than what is known from Theorem 4.58.
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