Periodic self-similar wave maps coupled to gravity

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Abstract

We continue our studies of spherically symmetric self-similar solutions in the SU(2) sigma model coupled to gravity. Using mixed numerical and analytical methods we show existence of an unstable periodic solution lying at the boundary between the basins of two generic attractors.

1 Introduction

This is the third paper in a series aimed at understanding the structure of self-similar spherically symmetric wave maps coupled to gravity. In the first two papers [1, 2] we showed that for small values of the coupling constant there exists a countable family of solutions that are analytic below the Cauchy horizon of the central singularity. In this paper we wish to elaborate on the analysis of a periodic self-similar solution whose existence was only briefly mentioned in [1]. We tried to make this paper self-contained mathematically but we refer the reader to [1] for the discussion of the physical background of the problem and to [3] for more on the role of self-similar solutions in gravitational collapse.

2 Setup

We showed in [1] that under the assumptions of spherical symmetry, equivariance and self-similarity the Einstein equations coupled to the SU(2) sigma field reduce to the following system of autonomous ordinary differential equations for three functions $W(x), A(x)$, and $F(x)$

\[
W' = -1 + \alpha(1 - W^2)F'^2, \quad (1)
\]

\[
A' = -2\alpha WF'^2, \quad (2)
\]

\[
(AF')' = \frac{\sin(2F)}{W^2 - 1}, \quad (3)
\]

subject to the constraint

\[
1 - A - 2\alpha \sin^2 F + \alpha AF'^2(W^2 - 1) = 0. \quad (4)
\]

Physically the functions $W$ and $A$ parametrize the metric, the function $F$ parametrizes the SU(2) sigma field and $\alpha$ is the dimensionless coupling constant ($\alpha = 0$ corresponds
to no gravity). We are interested in solutions of equations (1)-(4) starting at say \( x = 0 \) with the following initial conditions (as explained in [1] these conditions ensure regularity of solutions at the past light cone of the singularity)

\[
F(0) = \frac{\pi}{2}, \quad F'(0) = b, \quad W(0) = 1, \quad A(0) = 1 - 2\alpha,
\]

where \( b \) is a free parameter (since the system has reflection symmetry \( F \to -F \) we may take \( b > 0 \) without loss of generality). The value \( A(0) \) follows from the constraint (4). In what follows we shall refer to solutions of equations (1)-(4) satisfying the initial conditions (5) as \( b \)-orbits. We showed in [2] that for \( \alpha < \frac{1}{2} \), \( b \)-orbits exist locally and are analytic in \( b \) and \( x \).

It follows immediately from (1)-(4) that a \( b \)-orbit can be continued as long as \( |W| < 1 \) (since then \( A \) is bounded away from zero). However, if \( W \) hits \( \pm 1 \) at some \( x \), then the solution becomes singular. Now we shall show that generic \( b \)-orbits become singular in finite time. Throughout the paper we assume that \( \alpha < \frac{1}{2} \).

We first show that if \( b \) is small, then \( W \) tends to -1 for some finite \( x_A \). To see this, let \( f = (F - \pi/2)/b \). Then, in the limit \( b \to 0 \), equations (1)-(3) reduce to

\[
W' = -1 + \alpha(1 - W^2)b^2f'^2 \to -1, \quad (6)
\]

\[
A' = -2\alpha AW b^2f'^2 \to 0, \quad (7)
\]

\[
(Af')' = -\frac{\sin(2bf)}{b(W^2 - 1)} \to -\frac{2f}{W^2 - 1}, \quad (8)
\]

with the initial conditions

\[
f(0) = 0, \quad f'(0) = 1, \quad W(0) = 1, \quad A(0) = 1 - 2\alpha.
\]

The limiting equations (6) and (7) are solved by \( W = 1 - x \) and \( A = 1 - 2\alpha \). Substituting these solutions into (8) we get the equation

\[
(1 - 2\alpha)f'' + \frac{2f}{x(x - 2)} = 0, \quad (10)
\]

whose solution (which can be obtained in closed form) behaves as \( f(x) \sim (1 - 2\alpha) + (2 - x)\ln(2 - x) \) for \( x \to 2 \). Thus, the term \( (1 - W^2)f'^2 \) in (6) stays bounded so, by uniform convergence on compact intervals, the solutions with sufficiently small \( b \) will tend to \( W = -1 \).

Next, we show that if \( b \) is large then \( W \) tends to +1 and \( A \to 0 \) for some finite \( x_B \). This time we define the variables

\[
\xi = b^2x, \quad h(\xi) = b^2(1 - W(x)), \quad s(\xi) = b(F(x) - \frac{\pi}{2}).
\]

Then, in the limit \( b \to \infty \), equations (1)-(3) reduce to (where now prime is \( d/d\xi \))

\[
h' = 1 - \alpha h \left( 2 - \frac{h}{b^2} \right) s'^2 \to 1 - 2\alpha hs'^2, \quad (12)
\]

\[
A' = -2\alpha A \left( 1 - \frac{h}{b^2} \right) s'^2 \to -2\alpha As'^2, \quad (13)
\]

\[
(As')' = \frac{\sin(2s/b)}{b\xi(2 - h/b^2)} \to 0, \quad (14)
\]
with the initial conditions
\[ h(0) = 0, \quad A(0) = 1 - 2\alpha, \quad s(0) = 0, \quad s'(0) = 1. \]  

It follows from (14) and (15) that, in the limit \( b \to \infty \), \( As' = 1 - 2\alpha \). Plugging this into equations (12) and (13), and using (15), we get the limiting solution

\[ A(\xi) = (1 - 2\alpha)\sqrt{1 - 4\alpha \xi}, \quad b(\xi) = \frac{1}{2\alpha} \sqrt{1 - 4\alpha \xi} \left(1 - \sqrt{1 - 4\alpha \xi}\right). \]  

This solution becomes singular at \( \xi = 1/4\alpha \). By uniform convergence on compact intervals, we conclude that for solutions of equations (1)-(4) with large \( b \) (and nonzero \( \alpha \), the function \( W(x) \) attains a minimum and then tends to 1 at some \( x \to x_B \), while the function \( A(x) \) drops to zero at \( x_B \).

To summarize, \( b \)-orbits tend in finite time to \( W = -1 \) if \( b \) is small, and to \( W = +1 \) if \( b \) is large. In what follows, we shall refer to these two kinds of solutions as to type \( A \) and type \( B \) orbits, respectively. We show next that the sets of type \( A \) and type \( B \) orbits are open.

**Lemma 1.** If \( W(x) > 0 \) and \( A(x) < 1/2 - \alpha \) for some \( x > 0 \) then the orbit is of type \( B \), i.e., there is a finite \( x_0 \) such that \( \lim_{x \to x_0} W(x) = 1 \). Moreover, \( \lim_{x \to x_0} A(x) = 0 \).

**Proof:** Substituting equation (4) in (1) we get

\[ W' = -2 + \frac{1 - 2\alpha \sin^2 F}{A} > -2 + \frac{1 - 2\alpha}{A}. \]  

Thus, if \( A(x) < 1/2 - \alpha \) then \( W'(x) > 0 \) so if \( W(x) > 0 \) then \( W \) remains positive. But then by equation (2) \( A \) decreases which implies by (17) that \( W' \) remains positive (bounded away from zero in fact) and hence \( W \) must hit \( +1 \) in finite time. To prove the second part of the lemma, note that by equations (1) and (2) we have (using the abbreviation \( V = 1 - W^2 \))

\[ \left(\frac{V}{A}\right)' = \frac{2W}{A}. \]  

Assume that \( A(x_0) > 0 \). Then \( (V/A)(x_0) = 0 \) and since \( (V/A)(x) > 0 \) for \( x < x_0 \) we get a contradiction. Hence \( A(x_0) = 0 \).

**Corollary.** Type \( B \) orbits are open.

**Proof:** If the \( b_0 \)-orbit is of type \( B \) then by lemma 1, \( A(x, b_0) < 1/2 - \alpha \) and \( W(x, b_0) > 0 \) for some \( x > 0 \). Hence for nearby \( b \) we also have \( A(x, b) < 1/2 - \alpha \) and \( W(x, b) > 0 \) and thus, again by lemma 1, the \( b \)-orbit is of type \( B \).

**Proposition 1.** Type \( A \) orbits are open.

**Proof:** First, note that if the orbit is of type \( A \) and \( W(x) \geq 0 \) then \( A(x) \geq 1/2 - \alpha \) (since otherwise the orbit would be of type \( B \) by lemma 1). But for \( W < 0 \), by equation (2) \( A' > 0 \), hence \( A(x) > A(x_0) \geq 1/2 - \alpha \) for \( x \geq x_0 \) where \( x_0 \) is the point at which \( W(x_0) = 0 \). Thus, \( A > 1 - 2\alpha \) for type \( A \) orbits. Now, let \( b_0 \)-orbit be of type \( A \) and consider a nearby \( b \)-orbit. By continuity, there is a point \( x_1 \) such that \( W(x_1, b) \) is close to \(-1\), \( W(x_1, b) < 0 \), and \( A(x_1, b) \) is greater than, say, \( 1/2 - \alpha \). First we show that such orbits have \( W'(x, b) < 0 \) for all \( x > x_1 \). To see this, notice that from equations (1)-(3)

\[ W'' = -2\alpha F'A^{-1} \left(\alpha AWF''(W^2 - 1) - WA F' + \sin(2F)\right), \]
hence at the first zero of $W'(x, b)$ after $x_1$ we have
\[
W'' \bigg|_{W'=0} = \frac{1}{AV}(4AV \pm 2\sqrt{\alpha V} \sin 2F).
\] (19)

The numerator is negative because $A > 1/2 - \alpha$ and $W$ is close to $-1$ while the denominator is always positive, hence $W'' < 0$ which is a contradiction. Thus, $W'(x, b) < 0$ and $\lim_{x \to x_1} W(x, b)$ exists (if the orbit stays in the region). Now we show that $W(x_2, b) = -1$ for any such orbit for some $x_2 > x_1$. To prove this assume that $\lim_{x \to \infty} W(x) = \bar{W} \geq -1$.

Integrating equation (1) we get
\[
\int_{x_1}^{x} W' \, dx = x_1 - x + \alpha \int_{x_1}^{x} VF'^2 \, dx,
\] (20)

which gives a contradiction as $x \to \infty$ because the last integral in (20) is finite by equation (2) (remember that $A$ is bounded from below).

We now know that both type A and type B orbits are open so there must be orbits that are not type A or type B, that is, orbits that stay in $W^2 < 1$ for all $x$. Call these type C orbits. We note that type C orbits are defined for all $x \geq 0$ since $A(x) > 1/2 - \alpha$ for all $x$.

**Proposition 2.** For any type C orbit we have
\[
\liminf_{x \to \infty} W(x) \leq 0 \quad \text{and} \quad \limsup_{x \to \infty} W(x) \geq 0.
\] (21)

**Proof:** Suppose that there is an $x_1$ such that $W(x) \leq -L < 0$ for $x > x_1$ (this is equivalent to $\limsup_{x \to \infty} W(x) < 0$). Then from equation (18)
\[
\left(\frac{V}{A}\right)' = \frac{2W}{A} \leq \frac{-L}{A} \leq -L
\]
for $x > x_1$ and hence $(V/A) < 0$ for some $x_2 > x_1$ which is a contradiction since $(V/A) \geq 0$ for all $x$.

Similarly, suppose that there is an $x_1$ such that $W(x) \geq L > 0$ for $x > x_1$ (this is equivalent to $\liminf_{x \to \infty} W(x) > 0$). From lemma 1 we have that $A(x) > 1/2 - \alpha$ for $x > x_1$. Thus
\[
\left(\frac{V}{A}\right)' = \frac{2W}{A} \geq 2L
\]
for $x > x_1$ which is a contradiction since by lemma 1 $(V/A) \leq 2/(1 - 2\alpha)$ for all $x$.

From Proposition 2 we see that type C orbits must oscillate at infinity about $W = 0$ (unless $\lim W = 0$).

Once we know that type C orbits exist we turn to their numerical construction. Numerics indicates that the structure of type C orbits is rather complicated for large $\alpha$. In this paper we restrict our attention to small values $\alpha \leq 0.42$, where the structure is simple. Namely for each given $\alpha$ there is a single critical value $b^*(\alpha)$ such that $b$-orbits tend to the attractor $A$ (resp. $B$) if $b < b^*$ (resp. $b > b^*$) and the $b^*$-orbit is of type C. In other words, the $b^*$-orbit is a separatrix lying between two generic attractors $A$ and $B$. In the next section we give numerical and analytical arguments that the $b^*$-orbit is asymptotically periodic.
3 Numerical solution

A straightforward way to determine the critical value $b^*$ is to take two values $b_A$ and $b_B$ leading to attractors $A$ and $B$ respectively and then fine-tune to $b^*$ by bisection. This procedure yields a pair of $b$ that are within a distance $\epsilon$ from $b^*$ (where $\epsilon$ is limited by machine precision). Such marginally critical $b$-orbits exhibit a transient periodic behavior before eventually escaping toward $W = \pm 1$ (see Figures 1 and 2).

![Figure 1: The function $W(x)$ for two marginally critical $b$-orbits for $\alpha = 0.38$: the type A solution with $b = b^* - \epsilon$ (dotted line) and the type B solution with $b = b^* + \epsilon$ (dashed-dotted line), where $\epsilon = 10^{-17}$. Superimposed (solid line) is the periodic solution constructed by the straddle-orbit method.](image)

![Figure 2: The projection on the $(A,W)$ plane of the same solution as in Fig. 1. The periodic solution is seen as the unstable limit cycle.](image)
This suggests that the system has an unstable periodic solution and the $b^*$-orbit belongs to its basin of attraction. In other words, the value $b^*$ corresponds to the intersection of the line of initial data ($W = 1, F = 0, F' = b$) with the 2-dimensional stable manifold of the periodic solution. In fact, if we take any two points $P_A$ and $P_B$ in the phase space which lead to attractors $A$ and $B$, respectively, and perform bisection we obtain the same asymptotically periodic solution. This indicates that the stable manifold of the periodic solution is the boundary between the basins of attractors $A$ and $B$.

Since the periodic solution is unstable and numerically it is impossible to set initial conditions exactly on the stable manifold, we cannot obtain too many cycles of the periodic solution. Although in our case this is not a serious difficulty because the positive Lyapunov exponents are not large (see Figure 7 below), we would like to remark in passing that using so called straddle-orbit method due to Battelino et al. one can pursue the unstable periodic orbit in principle forever. This procedure, which can be viewed as a series of bisections, goes as follows. At initial time we choose two points $P_A(x = 0)$ and $P_B(x = 0)$ which lead to different attractors $A$ and $B$ and perform bisection until the distance between the iterates $P_A(0)$ and $P_B(0)$ is less a prescribed $\delta$. Next we integrate the equations numerically starting from the current $P_A(0)$ and $P_B(0)$ until the distance between the trajectories exceeds $\delta$. When this happens at some time $x$ we stop the integration, assign the points $P_A(x)$ and $P_B(x)$ as current representatives and repeat the bisection. Iterating this procedure one can progressively construct a trajectory staying within a distance $\delta$ from the codimension one stable manifold. The numerical solutions obtained by this method are shown in Figures 3, 4, and 5.

![Figure 3: The profiles of the periodic solution for $\alpha = 0.38$.](image)
Using the fact that non-critical $b$-orbits become singular in finite time we can easily compute the positive Lyapunov exponent $\lambda$ of the periodic solution. To this end consider a marginally critical $b$-orbit with $b = b^* - \epsilon$. Such an orbit approaches the periodic solution, stays close to it for some time and eventually escapes along the unstable manifold to crash at a point $x_A$ where $W(x_A) = -1$. Therefore we can write

$$x_A = x_{\text{approach}} + x_{\text{periodic}} + x_{\text{escape}},$$

where $x_{\text{approach}}$, $x_{\text{periodic}}$, and $x_{\text{escape}}$ denote the lengths of respective intervals of evolution (we say that the solution “escapes” if its distance from the periodic attractor exceeds a prescribed value). During the periodic interval the distance between the $b$-orbit with $b = b^* - \epsilon$ and the periodic solution grows at the rate proportional to $\epsilon \exp(\lambda x)$, hence $x_{\text{periodic}} \sim (-1/\lambda) \ln \epsilon$. This implies that the number of cycles $n$ during this interval

Figure 4: The phase portrait of the periodic solution for $\alpha = 0.38$.

Figure 5: The phase portraits of periodic solutions for different values of the coupling constant $\alpha$ ranging from 0.01 to 0.42. As $\alpha \to 0$ the loop shrinks to zero and $F' \to \infty$. 
behaves as \( n \sim (-1/\lambda T) \ln \epsilon \), where \( T \) is the period of the periodic solution. The length of the escape interval does not depend on the number of cycles but only on the phase of a cycle at which the escape from the periodic solution takes place, hence \( x_{\text{escape}} \sim f(\ln \epsilon) \), where \( f \) is a periodic function with period \( \lambda T \). Summarizing, we have

\[
x_A \approx -\frac{1}{\lambda} \ln \epsilon + f(\ln \epsilon) + \text{const.}
\]

(23)

The numerical verification of this formula is shown in Figure 6. Using (23) we calculated the dependence of \( \lambda \) on the coupling constant \( \alpha \) - the result is shown in Figure 7.

Figure 6: For \( \alpha = 0.2 \), the locus of the point of crash \( x_A \) is plotted as the function of the logarithmic distance from the critical value \( \ln \epsilon \). The fit to the formula (23) gives \( \lambda = 2.029 \). The period of the wiggles, corresponding to the function \( f(\ln \epsilon) \), is equal to 2.887 in agreement with the predicted value \( \lambda T \) (where \( T = 1.418 \) was calculated independently from equation (34)).

Figure 7: The positive Lyapunov exponent \( \lambda \) of the periodic solution as a function of the coupling constant \( \alpha \).
4 Perturbation series

In order to construct periodic solutions we consider equations (1)-(4) with initial conditions

\[ F(0) = 0, \quad F'(0) = c, \quad W(0) = 0, \quad A(0) = (1 + c^2)^{-1}, \]  

where \( c \) is a free parameter and the value \( A(0) \) follows from the constraint (4). We claim that for sufficiently small \( \alpha \) there is a unique \( c \) such that \( F(T) = \pi, F'(T) = F'(0), W(T) = W(0), A(T) = A(0) \) for some \( T > 0 \). Since the system is invariant under the shift \( F \to F + \pi \), we call such solution periodic. Now we shall construct the periodic solution in a perturbative way using the Poincaré-Lindstedt method [6].

We define the new variable \( y = \frac{\omega x}{\sqrt{\alpha}} \) where \( \omega \) is the unknown in advance frequency. We remark that the rescaling of the independent variable by \( \sqrt{\alpha} \) is essential in order to have a well-defined limit for \( \alpha \to 0 \) while the rescaling by \( \omega \) is introduced for convenience in order to have the fixed period \( 2\pi \). In terms of \( y \) equations (1)-(3) transform to

\[ \omega W' = \beta [-1 + \omega^2 (1 - W^2) F'^2], \]  

\[ A' = -2\omega \beta AW F'^2, \]  

\[ \omega^2 (AF')' = \frac{\beta^2 \sin(2F)}{W^2 - 1}. \]

and the constraint (4) becomes

\[ 1 - A - 2\beta^2 \sin^2 F + \omega^2 AF'^2 (W^2 - 1) = 0. \]

We consider these equations on the interval \( 0 \leq y \leq 2\pi \) with the boundary conditions

\[ F(0) = 0, \quad F(2\pi) = \pi, \quad W(0) = 0, \quad A(0) = A_0, \]

where the value of the constant \( A_0 \) follows from the constraint (28). We seek solutions in the form of a power series in \( \beta \)

\[ W(y, \beta) = \sum_{k=0}^{\infty} \beta^k W_k(y), \quad A(y, \beta) = \sum_{k=0}^{\infty} \beta^k A_k(y), \quad F(y, \beta) = \sum_{k=0}^{\infty} \beta^k F_k(y). \]

The key idea of the Poincaré-Lindstedt method is to expand the frequency in the power series

\[ \omega(\beta) = \sum_{k=0}^{\infty} \beta^k \omega_k. \]

and to solve for the coefficients \( \omega_k \) by demanding that the solution contains no secular terms. Thus, we substitute (30) and (31) into (25)-(28), group the terms according to powers of \( \beta \) and require that the coefficients of each power of \( \beta \) vanish separately. In the lowest order \( O(1) \) we get

\[ W_0(y) = 0, \quad A_0(y) = \left( 1 + \frac{\omega_0^2}{4} \right)^{-1}, \quad F_0(y) = \frac{y}{2}. \]

where \( \omega_0 \) is yet undetermined. In the next order we get the equation \( \omega_0 W'_1 = (-1 + \omega_0^2/4) \), so to avoid a secular term we need to have \( \omega_0 = 2 \). Then, all \( O(\beta) \) terms are zero and in the order \( O(\beta^2) \) we get

\[ W_2(y) = 0, \quad A_2(y) = -\frac{1}{2}, \quad F_2(y) = \frac{1}{2} \sin(y). \]
Iterating this procedure with the help of Mathematica we calculated the perturbation series up to order \(O(\beta^{23})\). For example, up to order \(O(\beta^8)\) we have

\[
\omega(\beta) = 2 - \frac{\beta^4}{2} + \frac{\beta^6}{2} - \frac{49}{32} \beta^8 + O(\beta^{10}),
\]

\[
W(y, \beta) = \sin(y) \beta^3 + \frac{\sin(2y)}{4} \beta^5 + \frac{25 \sin(y) - 5 \sin(2y) + \sin(3y)}{16} \beta^7 + O(\beta^9),
\]

\[
A(y, \beta) = \left(1 - \frac{\beta^2}{2} + \frac{-2 + 4 \cos(y)}{8} \beta^4 + \frac{-4 - 8 \cos(y) + 5 \cos(2y)}{16} \beta^6 \right) + \frac{-60 + 204 \cos(y) - 102 \cos(2y) + 52 \cos(3y)}{384} \beta^8 + O(\beta^{10}),
\]

\[
F(y, \beta) = \left(\frac{y}{2} + \frac{\sin(y)}{2} \beta^2 + 16 \sin(2y) \beta^4 + \frac{81 \sin(y) - 21 \sin(2y) + \sin(3y)}{96} \beta^6 + \frac{1656 \sin(y) + 900 \sin(2y) - 616 \sin(3y) + 9 \sin(4y)}{4608} \beta^8 + O(\beta^{10})\right).
\]

We recall that the “physical” frequency is equal to \(\frac{\omega}{\beta}\) so it diverges as \(\beta\) tends to zero (while the amplitude of oscillations goes to zero). In this sense the periodic solution is nonperturbative even though we constructed it by the perturbation technique.

For small values of \(\beta\) the perturbation expansion converges quickly to the periodic solution constructed numerically (see Figure 8). As \(\beta\) grows the convergence becomes slower and we need to take many terms in the perturbation series to approximate well the numerical solution (see Figure 9). The fact that two independent ways of constructing the periodic solution agree, makes us feel confident that the periodic solution does in fact exist.

Figure 8: For \(\alpha = 0.1\) we plot the numerical periodic solution and superimpose the perturbation series (35). Even at this low order the agreement is very good.
Figure 9: For $\alpha = 0.38$ we plot the numerical periodic solution and superimpose the perturbation series in different orders. As the order increases the perturbation series slowly approaches the numerical solution.

5 Final remarks

We showed above that for small values of the coupling constant $\alpha$, the critical $b^*(\alpha)$-orbit is asymptotically periodic as $x \to \infty$. In the preceding papers [1,2] we showed that for a generic value of $\alpha$, the $b^*(\alpha)$-orbit evolved backwards in $x$ becomes singular as $x \to -\infty$ (which corresponds to the singularity at the center). However, there exist isolated values of $\alpha$ (called $\alpha_n$, $n = 0, 1, ...$) for which the $b^*(\alpha)$-orbit is regular as $x \to -\infty$. Combining this with the result obtained above, we conclude that for a finite set of isolated values $\alpha_n$ (satisfying $\alpha_n < 0.42$) the Einstein-wave map equations admit self-similar solutions that are regular at the center and asymptotically periodic outside the past light cone.

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