BISMUT-EINSTEIN METRICS ON COMPACT COMPLEX SURFACES

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ABSTRACT. We establish a Bochner-type formula expressed in terms of Bismut-Ricci curvature for pluriclosed metrics. Using this formula, we can directly prove that a Bismut-Einstein metric with non-zero Einstein constant is Kähler-Einstein, without the help of classification of compact complex surfaces. Moreover, for a pluriclosed metric with zero Einstein constant, we can prove that it must be Bismut-Ricci flat.

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1. INTRODUCTION

Bismut-Ricci curvature plays an important role in pluriclosed flow, which is introduced by Streets and Tian[11, 12]. As Ricci flow or Kähler-Ricci flow, Einstein-type metrics (if exists) can generate solutions to flow by a rescaling which is only depended on time. This is the simplest case of solitons (see e.g., [8, 9, 13]). So studying the property of Bismut-Einstein metric is helpful for understanding pluriclosed flow. We begin with the definition of Bismut-Einstein metrics.
**Definition 1.1.** A metric $\omega$ is Bismut-Einstein if and only if there exists a real number $\lambda$ such that

$$\rho^{1,1}(\omega) = \lambda \omega.$$ 

We call $\lambda$ the Einstein constant.

Here $\rho^{1,1}$ is the $(1,1)$-part of Bismut-Ricci form, which will be defined explicitly in section 2.1. The Bismut-Ricci form $\rho$ is invariant under rescaling, so we can assume $\lambda \in \{-1, 0, 1\}$ without loss of generality.

We can write Bismut-Ricci form as $\rho = \rho^{1,1} + \rho^{2,0} + \bar{\partial} \rho^{2,0}$ for it is real. Since $\rho$ is closed, we have

$$\partial \rho^{2,0} = 0, \quad \bar{\partial} \rho^{1,1} + \bar{\partial} \rho^{2,0} = 0.$$ 

by comparing the bidegree of $d\rho$. So a Bismut-Einstein metric with non-zero Einstein constant is automatically pluriclosed. In fact, in such a case, the metric can be extended to a Hermitian-symplectic form, which is a real closed 2-form with positive definite $(1,1)$-part. On the one hand, with the help of the classification of compact complex surfaces (see e.g., [11]), Streets and Tian [11] prove that only Kähler surfaces admit Hermitian-symplectic forms and Streets [10] give a classification of gradient solitons to the pluriclosed flow on surfaces. On the other hand, it have been proved that Hermitian-symplectic structures are preserved by pluriclosed flow. And in dimension 2, the author [17] used a monotonicity formula to prove that, for global solutions with Hermitian-symplectic initial data, the limitation (if exists) must be Kähler. For this reason, a natural question is whether we can study Hermitian-symplectic structures without the help of classification of compact complex surfaces? In this paper, for Bismut-Einstein metrics, we can directly prove

**Theorem 1.2.** Assume a compact complex surface $(M^4, J)$ admitting a Bismut-Einstein metric $\omega$ with Einstein constant $\lambda$.

(1) If $\lambda \neq 0$, then $\omega$ is Kähler-Einstein;
(2) If $\lambda = 0$ and $\omega$ is pluriclosed, then $\omega$ is Bismut-Ricci flat.

Here Bismut-Ricci flat means $\rho(\omega) = 0$.

To prove Theorem 1.2, we establish a Bochner-type formula expressed in terms of Bismut-Ricci curvature and obtain some vanishing results (see section 4). In particular, on compact surfaces we can obtain a slightly stronger vanishing result, which plays an important role in the proof of Theorem 1.2.

**Theorem 1.3.** Given a compact complex surface with a pluriclosed metric $(M^4, J, \omega)$.

(1) If $\rho^{1,1}$ is non-negative definite, then either $\omega$ is Kähler or $H^{p,0}_\partial(M; \mathbb{C})$ is trivial for $1 \leq p \leq 2$;
(2) If Bismut scalar curvature $r = \text{tr}_{\omega} \rho^{1,1}$ is non-negative, then either $\omega$ is Kähler or $H^{2,0}_\partial(M; \mathbb{C})$ is trivial.

Here is an outline of the rest. In section 2, we recall some basic notions which will be used later. In section 3, we do some computations about several Laplacians, which may be different without the Kähler assumption. In section 4, we establish a Bochner formula for special forms and get some vanishing results from it. In section 5, we give an interesting observation about the $(2,0)$-part of Bismut-Ricci form in dimension 2, and use it to complete the proof of Theorem 1.2. In section 6, we do some discussions on Bismut-Einstein metrics...
in high dimensional cases, and we obtain a weak version of Theorem 1.2.

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2. Preliminary

In this section, we give a quick review of some basic notions which will be used later.

2.1. Bismut connection and Bismut-Ricci form. Given a complex manifold \((M^{2n}, J)\) with a Hermitian metric \(g\). There exists a unique real connection \(\nabla^B\) such that

\[
\nabla^B g = 0, \quad \nabla^B J = 0, \quad B(x, y, z) + B(z, y, x) = 0,
\]

in which

\[
B(x, y, z) = g(\nabla_x y - \nabla_y x - [x, y], z)
\]

is the tensor induced by torsion operator of Bismut connection. We call this unique connection Bismut connection. Notice that \(B\) is a real 3-form. And \(B\) is closed if and only if the metric is pluriclosed.

We define the Bismut-Ricci form by

\[
\rho(x, y) = \sqrt{-1} \sum_{m=1}^{n} R^B(x, y, Z_m, \overline{Z}_m).
\]

where \(R^B(\cdot, \cdot, \cdot, \cdot)\) is the curvature tensor corresponding to Bismut connection and \(\{Z_1 \cdots Z_n\}\) is an arbitrary unitary basis. Notice that \(\rho\) is a closed real 2 form (see e.g., [3]). We can use Hodge operator to rewrite it as

\[
\rho_1^1(\omega) = -\partial \partial^* \omega - \bar{\partial} \bar{\partial}^* \omega - \sqrt{-1} \partial \bar{\partial} \log \det g
\]

\[
\rho^{2,0}(\omega) = -\bar{\partial} \bar{\partial}^* \omega
\]

in which \(\rho = \rho^{1,1} + \rho^{2,0} + \rho^{2,0}\) and \(\omega = g(J\cdot, \cdot)\).

2.2. Chern connection without Kähler assumption. In this subsection we review some basic facts of Chern connection \(\nabla\) without Kähler assumption. In such a case, it is not Levi-Civita connection. And the torsion tensor is

\[
H(x, y, z) = g(\nabla_x y - \nabla_y x - [x, y], z).
\]

In local coordinates, the Christoffel symbol of Chern connection is

\[
\Gamma_{ij}^s = g^{ts} \partial_t g_{ji}
\]

So we have

\[
H_{ijt} = H(\partial_i, \partial_j, \partial_t) = H_{ij}^s g_{st} = \partial_t g_{jt} - \partial_j g_{it}.
\]

Notice that

\[
\partial \omega = \sqrt{-1} \partial \partial g_{jt} dz^i \wedge d\bar{z}^j \wedge d\bar{z}^t = \frac{\sqrt{-1}}{2} H_{ij} dz^i \wedge d\bar{z}^j \wedge d\bar{z}^t
\]

Here we rewrite \(\partial \omega\) such that its coefficients are skew-symmetric. So if we regard \(\partial \omega\) as a tensor, then we have \(\partial \omega = \sqrt{-1} H\).
The Ricci form of Chern connection are defined by
\[ \text{Ric}(x, y) = \sqrt{-1} \sum_{m=1}^{n} R(x, y, Z_m, \overline{Z_m}) \]

where \( R(\cdot, \cdot, \cdot, \cdot) \) is the curvature tensor corresponding to Chern connection and \( \{Z_1, \cdots, Z_n\} \) is an arbitrary unitary basis.

We can define a real \((1, 1)\)-form \( R_n \) by another way of contraction
\[ R_n(x, y) = \sqrt{-1} \sum_{m=1}^{n} R(Z_m, \overline{Z_m}, x, y). \]

For Kähler metric, \( R_n \) is precisely the Ricci form. In general, they are different since Chern connection has torsion. An interesting fact is that \( R_n \) is always elliptic meanwhile Ricci form \( \text{Ric} \) is not elliptic in general.

When the metric is pluriclosed, there is a relationship between \( R_n \) and the \((1, 1)\)-part of Bismut-Ricci form \( \rho^{1,1} \) (see e.g., [11, 12]). More explicitly, we have
\[ \rho^{1,1} = R_n - \sqrt{-1} H^2 \]

where \( \sqrt{-1} H^2 \) is a real \((1, 1)\)-form defined by
\[ \sqrt{-1} H^2 = \sqrt{-1} g^{js} g^{kl} H_{jsk} \overline{H}_{jl} dz^i \wedge d\overline{z}^j. \]

**Remark 2.1.** We can choose a coordinates such that \( g_{ij}(x) = \delta_{ij} \) and \( \sqrt{-1} H^2(x) \) is diagonal at a fixed point \( x \). Then we have
\[ \sqrt{-1} H^2(x) = \sqrt{-1} \sum_{i=1}^{n} \delta_{st} \delta_{ik} H_{jsk} \overline{H}_{jl} dz^i \wedge d\overline{z}^i = \sqrt{-1} \sum_{s,k} |H_{jsk}|^2 dz^i \wedge d\overline{z}^i. \]

So \( \sqrt{-1} H^2 \) is non-negative.

### 2.3. Lefschetz-type operator and its adjoint.

For the convenience of use later, we recall the Lefschetz-type operator in this subsection. For a form \( \gamma \), we can define the Lefschetz-type operator \( L_{\gamma} \) by
\[ L_{\gamma} \alpha = \gamma \wedge \alpha. \]

And the adjoint operator \( L_{\gamma}^* \) with respect to the pointwise inner product \((\cdot, \cdot)\) is defined by
\[ (L_{\gamma}^* \alpha, \beta) = (\alpha, \gamma \wedge \beta). \]

Notice that in the case of \( \gamma = \omega \), those operators defined above are the classical Lefschetz operator \( L \) and \( \Lambda \). We compute the local expression of Lefschetz-type operator in a special case which will be used later.

**Lemma 2.2.** Given a \((p, 1)\)-form
\[ \alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^k \]
satisfying \( \alpha_{i_1 \cdots i_p} + \alpha_{i_p \cdots i_1} = 0 \) and a \((1, 0)\)-form \( \gamma = \gamma_i d\overline{z}^i \). We have
\[ L_{\gamma}^* \alpha = \frac{1}{p!} (-1)^p g^{kl} \alpha_{i_1 \cdots i_p} \gamma_l dz^{i_1} \wedge \cdots \wedge dz^{i_p}. \]
Proof. Choose an arbitrary \((p, 0)\)-form \(\beta = \beta_{j_1, \ldots, j_p} dz^{j_1} \wedge \cdots \wedge dz^{j_p}\). Assume
\[
L^*_{\gamma} \alpha = \frac{1}{p!} \eta_{i_1 \ldots i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}.
\]
We have
\[
(L^*_{\gamma} \alpha, \beta) = g^{i_1 \ldots i_p} \eta_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p}
\]
and
\[
\bar{\gamma} \wedge \beta = (-1)^p \beta_{j_1 \ldots j_p} \eta_i dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^l
\]
By definition,
\[
(L^*_{\gamma} \alpha, \beta) = (\alpha, \bar{\gamma} \wedge \beta) = (-1)^p g^{i_1 \ldots i_p} \eta_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p}
\]
Then we get
\[
g^{i_1 \ldots i_p} \eta_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p} = (-1)^p g^{i_1 \ldots i_p} \eta_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p}
\]
So we obtain
\[
\eta_{i_1 \ldots i_p} = (-1)^p g^{i_1 \ldots i_p} \alpha_{i_1 \ldots i_p} \gamma_l
\]
for \(\beta\) is arbitrary. \(\square\)

3. Some Calculations about Laplacians

Without the Kähler assumption, there are several different Laplacians. In this section, we will show the relationship between them when we apply them on special forms.

Let us begin with the definition of several Laplacians.

**Definition 3.1.** For a tensor \(A\), we can define the Chern Laplacian \(\Delta A\) and the conjugation Chern Laplacian \(\overline{\Delta} A\) by
\[
\Delta A = g^{qp} \nabla_p \nabla_q A, \quad \overline{\Delta} A = g^{qp} \nabla_q \nabla_p A.
\]
And for a form \(\alpha\) we can define the \(\overline{\partial}\)-Laplacian and \(\partial\)-Laplacian by
\[
\Delta_\beta \alpha = \overline{\partial}^* \partial \alpha + \overline{\partial} \partial^* \alpha, \quad \Delta_\partial \alpha = \partial^* \partial \alpha + \partial \partial^* \alpha.
\]
Firstly, we give some lemmas often used in the computation of \(\overline{\partial}\)-Laplacian and \(\partial\)-Laplacian.

**Lemma 3.2.** Given a \((p, q + 1)\)-form
\[
\alpha = \frac{1}{p! (q + 1)!} \alpha_{i_1 \ldots i_p j_1 \ldots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \wedge dz^k
\]
satisfying
\[
\alpha_{\ldots i_u \ldots} + \alpha_{\ldots i_v \ldots} = 0, \quad \alpha_{\ldots j_u \ldots} + \alpha_{\ldots j_v \ldots} = 0.
\]
We have
\[
\overline{\partial}^* \alpha = \frac{1}{p! q!} \eta_{i_1 \ldots i_p j_1 \ldots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}
\]
in which
\[
\eta_{i_1 \ldots i_p j_1 \ldots j_q} = (-1)^{p+q+1} g^{kl} \left( \nabla_l \alpha_{i_1 \ldots i_p j_1 \ldots j_q} + \alpha_{i_1 \ldots i_p j_1 \ldots j_q} H^s_{lm} + \frac{1}{2} g^{is} \sum_{m=1}^q (\alpha_{i_1 \ldots i_p j_1 \ldots j_q} H_{s,lm}) \right).
\]
Proof. We can check it in a local coordinates neighborhood, because we can choose the test form $\beta$ such that it is supported in a coordinates neighborhood. Assume

$$\beta = \beta_{a_1 \cdots a_p b_1 \cdots b_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge dz^{b_1} \wedge \cdots \wedge dz^{b_q}.$$  

We have

$$(\bar{\partial}^* \alpha, \beta)_2 = \int_M (\bar{\partial}^* \alpha, \beta) \det g$$

$$= \int_M \eta_{i_1 \cdots i_p j_1 \cdots j_q} \beta_{a_1 \cdots a_p b_1 \cdots b_q} g^{a_1 i_1} \cdots g^{a_p i_p} g^{j_1 b_1} \cdots g^{j_q b_q} \det g.$$  

Here we use $(\cdot, \cdot)$ and $(\cdot, \cdot)$ to denote $L^2$ inner product and pointwise inner product, respectively. By direct computation, we have

$$\bar{\partial} \beta = (-1)^{p+q} \partial_l \beta_{a_1 \cdots a_p b_1 \cdots b_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge dz^{b_1} \wedge \cdots \wedge dz^{b_q}$$

and then

$$(\bar{\partial}^* \alpha, \beta)_2 = (\alpha, \bar{\partial} \beta)_2 = \int_M (\alpha, \bar{\partial} \beta) \det g$$

$$= (-1)^{p+q} \int_M \alpha_{i_1 \cdots i_p j_1 \cdots j_q} \partial_l \beta_{a_1 \cdots a_p b_1 \cdots b_q} g^{a_1 i_1} \cdots g^{a_p i_p} g^{j_1 b_1} \cdots g^{j_q b_q} \det g$$

$$= (-1)^{p+q+1} \int_M \beta_{a_1 \cdots a_p b_1 \cdots b_q} \partial_l \left( \alpha_{i_1 \cdots i_p j_1 \cdots j_q} g^{a_1 i_1} \cdots g^{a_p i_p} g^{j_1 b_1} \cdots g^{j_q b_q} \det g \right).$$

According to the arbitrariness of $\beta$, we obtain

$$\eta_{i_1 \cdots i_p j_1 \cdots j_q} g^{a_1 i_1} \cdots g^{a_p i_p} g^{j_1 b_1} \cdots g^{j_q b_q} \det g = (-1)^{p+q+1} \partial_l \left( \alpha_{i_1 \cdots i_p j_1 \cdots j_q} g^{a_1 i_1} \cdots g^{a_p i_p} g^{j_1 b_1} \cdots g^{j_q b_q} \det g \right)$$

which means

$$(-1)^{p+q+1} \eta_{c_1 \cdots c_p d_1 \cdots d_q} = g^{kl} \partial_l \alpha_{c_1 \cdots c_p d_1 \cdots d_q} + g^{kl} \sum_{m=1}^p \alpha_{c_1 \cdots c_m d_1 \cdots d_q} g_{cm a_m} \partial_l g^{a_m i_m}$$

$$+ \bar{g}^{kl} \sum_{m=1}^q \alpha_{c_1 \cdots c_p d_1 \cdots d_q} g_{bm a_m} \partial_l g^{a_m i_m}$$

$$+ \alpha_{c_1 \cdots c_p d_1 \cdots d_q} \partial_l \bar{g}^{kl} + \bar{g}^{kl} \alpha_{c_1 \cdots c_p d_1 \cdots d_q} g^{ls} \partial_l g^{s l}.$$  

Recall the derivative formula of inverse matrix

$$\partial_l g^{\bar{i} i} = -g^{\bar{i} p} g^{\bar{i} j} \partial_l g_{pq} = -g^{\bar{i} p} \Gamma^i_{lp}.$$  


Changing some subscripts, we get

\[ (-1)^{p+q+1} \eta_{c_1 \cdots c_p d_1 \cdots d_q} = g^{kl} \partial_l \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} - g^{kl} \sum_{m=1}^p \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} \Gamma^{s}_{lcm} \]

\[ - g^{kl} g^{ts} \sum_{m=1}^q \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} \partial_t \partial_s g_{sdm} \]

\[ - \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} g^{kl} g^{ts} \partial_s \partial_l g_{sf} + \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} \partial_l g_{sdm} \]

Notice that

\[ - g^{kl} g^{ts} \sum_{m=1}^q \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} \partial_t \partial_s g_{sdm} = - \frac{1}{2} g^{kl} g^{ts} \sum_{m=1}^q \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} (\partial_t g_{sdm} - \partial_s g_{dtm}) \]

\[ = \frac{1}{2} g^{kl} g^{ts} \sum_{m=1}^q \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} H_{sldm} \]

So we obtain

\[ (-1)^{p+q+1} \eta_{c_1 \cdots c_p d_1 \cdots d_q} = g^{kl} \nabla_l \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} + \frac{1}{2} g^{kl} g^{ts} \sum_{m=1}^q \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} H_{sldm} \]

\[ + g^{kl} \alpha_{c_1 \cdots c_p d_1 \cdots d_q k} H_{ls}^s \]

The proof is completed. \( \square \)

Similarly, we have

**Lemma 3.3.** Given a \((p + 1, q)\)-form

\[ \alpha = \frac{1}{(p + 1)!q!} \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_q} \]

satisfying

\[ \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} + \alpha_{i_2 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} = 0, \quad \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} + \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} = 0. \]

We have

\[ \partial^* \alpha = \frac{1}{plq!} \eta_{1 \cdots p} \bar{j}_1 \cdots \bar{j}_q \partial^{i_1} \wedge \cdots \wedge \partial^{i_p} \wedge \partial^j \lbar{1} \cdots \partial^j \lbar{q} \]

in which

\[ \eta_{1 \cdots p} \bar{j}_1 \cdots \bar{j}_q = - g^{kl} \left( \nabla_k \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} + \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \overline{H}_{ks}^s - \frac{1}{2} g^{ts} \sum_{m=1}^q \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \overline{H}_{ktlm} \right). \]

**Remark 3.4.** In Lemma 3.2 when \(q = 0\), we do not have the last term in the expression of \(\eta_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}\). In other words, the expression of \((p, 1)\)-form is

\[ \eta_{1 \cdots p} = (-1)^{p+1} g^{kl} \left( \nabla_l \alpha_{i_1 \cdots i_p \bar{k}} + \alpha_{i_1 \cdots i_p \bar{k}} \overline{H}_{ls}^s \right). \]

This is also valid for Lemma 3.3 when \(p = 0\).

**Remark 3.5.** Applying Lemma 3.2 to \(\omega\), we have

\[ \partial^* \omega = \sqrt{-1} H_{ls}^s dz^i. \]
3.1. Laplacians act on functions. Assume $f$ is a smooth function. By definition, we have
\[ \Delta f = \text{tr}_{\omega}(\sqrt{-1} \partial \bar{\partial} f) = \text{tr}_{\omega}(\sqrt{-1} \partial_{ij} f dz^i \wedge d\bar{z}^j) = g^{pq} \partial_p \partial_q f \]
For $\bar{\partial}$-Laplacian, we have
\[ \Delta_{\bar{\partial}} f = \bar{\partial} \bar{\partial}^* f + \bar{\partial}^* \bar{\partial} f = \bar{\partial}^* \bar{\partial} f \]
Applying Lemma 3.2 to $\bar{\partial} f$ and noticing Remark 3.5, we get
\[ \Delta_{\bar{\partial}} f = -g^{kl} \nabla_k (\bar{\partial} f)_l - g^{kl} (\bar{\partial} f)_l H^k_{ls} \]
\[ = -g^{kl} \partial_k f - \sqrt{-1} g^{kl} (\bar{\partial} f)_l (\bar{\partial}^* \omega)_k \]
\[ = -\Delta f + \sqrt{-1} (\bar{\partial} f, \bar{\partial}^* \omega) \]
Similarly, applying Lemma 3.3, we have
\[ \Delta_{\partial} f = -g^{kl} \nabla_k (\partial f)_l - g^{kl} (\partial f)_l H^k_{ls} \]
\[ = -g^{kl} \partial_k f - \sqrt{-1} g^{kl} (\partial f)_l (\partial^* \omega)_k \]
\[ = -\Delta f - \sqrt{-1} (\partial f, \partial^* \omega) \]
Thus we prove
Proposition 3.6. For a smooth function $f$, we have
\[ -\Delta_{\bar{\partial}} f = \Delta f - \sqrt{-1} (\bar{\partial} f, \bar{\partial}^* \omega) \]
\[ -\Delta_{\partial} f = \Delta f + \sqrt{-1} (\partial f, \partial^* \omega) \]
Remark 3.7. From Proposition 3.6 we know that if $\bar{\partial}^* \omega = 0$, then $-\Delta_{\bar{\partial}} f = -\Delta_{\partial} f = \Delta f$ for all functions. Notice that
\[ \bar{\partial}^* \omega = 0 \iff - \ast \partial \ast \omega = 0 \iff \partial \omega^{n-1} = 0 \]
So Chern Laplacian and $\bar{\partial}$-Laplacian are the same when acts on functions if and only if the corresponding metric is balanced, i.e., $d\omega^{n-1} = 0$,
Recall that
\[ \int_M \Delta_{\partial} f dV = (\Delta_{\partial} f, 1)_2 = (\bar{\partial}^* \bar{\partial} f, 1)_2 = (\bar{\partial} f, \bar{\partial} 1)_2 = 0 \]
For Chern Laplacian, we have
Proposition 3.8. If the metric is Gauduchon, i.e., $\partial \bar{\partial} \omega^{n-1} = 0$, then
\[ \int_M \Delta f dV = 0 \]
for all functions.
Proof. By definition, we have
\[ \int_M \Delta f dV = \int_M \text{tr}_{\omega}(\sqrt{-1} \partial \bar{\partial} f) dV = \int_M (\sqrt{-1} \partial \bar{\partial} f, \omega) dV \]
\[ = (\sqrt{-1} \partial \bar{\partial} f, \omega)_2 = (\sqrt{-1} f, \bar{\partial}^* \partial^* \omega)_2 \]
Notice
\[ \bar{\partial}^* \partial^* \omega = 0 \iff \ast \partial \ast \omega = 0 \iff \partial \omega^{n-1} = 0. \]
So we complete our proof. □

Remark 3.9. Gauduchon\textsuperscript{[6]} proves that any complex manifold admits metrics satisfying \(\bar{\partial}\bar{\partial}u^{n-1} = 0\). And in the case of surface, this is precisely pluriclosed metric.

3.2. Laplacians act on \((p,0)\)-forms. For the purpose of use later, we give the relationship between those Laplacians when we apply them on \((p,0)\)-forms. Before we state our results, let us do some explanations to Chern Laplacian when it acts on forms.

In local coordinates, we can write a \((p,q)\)-form \(\beta\) as
\[
\beta = \frac{1}{p!q!}\beta_{i_1\cdots i_p\bar{j}_1\cdots \bar{j}_q}dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.
\]
where the subscripts are skew-symmetric. Equivalently, if we regard it as a skew-symmetric tensor, we have
\[
\beta = \beta_{i_1\cdots i_p\bar{j}_1\cdots \bar{j}_q}dz^{i_1} \otimes \cdots \otimes dz^{i_p} \otimes d\bar{z}^{j_1} \otimes \cdots \otimes d\bar{z}^{j_q}.
\]
Recall that the Chern Laplacian of a skew-symmetric tensor is also skew-symmetric. So we can define Chern Laplacian operator on differential forms.

Now we give our calculation results.

Proposition 3.10. Given a \((p,0)\)-form \(\alpha\), we have
\[
\Delta_{\bar{\partial}}\alpha = -\Delta\alpha + (-1)^p\sqrt{-1}L^*_{\bar{\partial}\omega} \bar{\partial}\alpha.
\]

Here we use the notation of Lefschetz-type operator \(L^*_{(i)}(\cdot)\) which is defined in section 2.3.

Proof. In local coordinates, we assume
\[
\alpha = \frac{1}{p!}\alpha_{i_1\cdots i_p}dz^{i_1} \wedge \cdots \wedge dz^{i_p}, \quad \alpha_{\cdots i_u\cdots i_u} + \alpha_{\cdots i_u\cdots i_u} = 0.
\]
By definition,
\[
\bar{\partial}\alpha = \frac{1}{p!}(-1)^p\partial_k\alpha_{i_1\cdots i_p}dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^k.
\]
Applying Lemma 3.2, we obtain
\[
\bar{\partial}^* \bar{\partial} \alpha = \frac{1}{p!}(g^{kl}\nabla_l \partial_k \alpha_{i_1\cdots i_p} + g^{kl}\bar{\partial}_k \alpha_{i_1\cdots i_p}H^*_t\alpha_i dz^{i_1} \wedge \cdots \wedge dz^{i_p})
\]
From Lemma 2.2 and Remark 3.5 we have
\[
-\frac{1}{p!}g^{kl}\partial_k \alpha_{i_1\cdots i_p}H^*_t\alpha_i dz^{i_1} \wedge \cdots \wedge dz^{i_p} = \sqrt{-1}\frac{1}{p!}g^{kl}\partial_k \alpha_{i_1\cdots i_p}(\bar{\partial}^*\omega)\bar{\partial}\alpha_i dz^{i_1} \wedge \cdots \wedge dz^{i_p} = (-1)^p\sqrt{-1}L^*_{\bar{\partial}\omega} \bar{\partial}\alpha.
\]
So
\[
\Delta_{\bar{\partial}}\alpha = -\Delta\alpha + (-1)^p\sqrt{-1}L^*_{\bar{\partial}\omega} \bar{\partial}\alpha.
\]

□

4. A Bochner Type Formula

In this section, we give a Bochner type formula in terms of Bismut-Ricci curvature for pluriclosed metric.
4.1. Chern Laplacian and conjugation Chern Laplacian. Firstly, we give a relationship between Chern Laplacian and conjugation Chern Laplacian using the form $R_n$, which is another type contraction of curvature tensor. In local coordinates, we assume that

$$R_n = \sqrt{-1} Q_{ij} dz^i \wedge d\bar{z}^j,$$

in which

$$Q_{ij} = g^{kl} R_{tkij}$$

and $R_{tkij}$ is the curvature tensor corresponding to Chern connection. So $Q_{ij}$ is a Hermitian metric in local coordinates. In other words, we have $Q_{ij} = Q_{ji}$ and $R_n$ is a real $(1,1)$-form by definition.

For ease of notations, we give a definition first.

**Definition 4.1.** Given a tensor $A = A^{i_1 \cdots i_p}_{j_1 \cdots j_q} \in T^{p,0} \otimes T^{q,0}$ and a $(1,1)$-form $S = \sqrt{-1} S_{ij} dz^i \wedge dz^j$. We define a tensor $S[A]$ by

$$(S[A])^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \sum_{m=1}^{p} g^{im} S_{st} A^{i_1 \cdots i_{m-1} s \cdots i_p}_{j_1 \cdots j_q} - \sum_{m=1}^{q} g^{is} S_{jm} A^{i_1 \cdots i_1 \cdots i_{m-1} p \cdots i_q}_{j_1 \cdots j_q}$$

Then we have

**Lemma 4.2.** For any tensor $A \in T^{p,0} \otimes T^{q,0}$, we have

$$\overline{\Delta} A = \Delta A - R_n[A].$$

**Proof.** By definition, we have

$$(\overline{\Delta} A)^{i_1 \cdots i_p}_{j_1 \cdots j_q} = g^{kl} \nabla_k \nabla_l A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$$

$$= g^{kl} \nabla_l A^{i_1 \cdots i_p}_{j_1 \cdots j_q} - g^{kl} \left( R(\partial_l, \partial_k) A \right)^{i_1 \cdots i_p}_{j_1 \cdots j_q}.$$

Notice

$$g^{kl} \left( R(\partial_l, \partial_k) A \right)^{i_1 \cdots i_p}_{j_1 \cdots j_q} = g^{kl} \left\{ \sum_{m=1}^{p} R_{tk8}^{im} A^{i_1 \cdots i_{m-1} s \cdots i_p}_{j_1 \cdots j_q} - \sum_{m=1}^{q} R_{tkjm}^{is} A^{i_1 \cdots i_p}_{j_1 \cdots j_{m-1} s \cdots j_q} \right\}$$

$$= \sum_{m=1}^{p} g^{kl} g^{im} R_{tk8}^{is} A^{i_1 \cdots i_{m-1} s \cdots i_p}_{j_1 \cdots j_q} - \sum_{m=1}^{q} g^{kl} g^{is} R_{tkjm}^{ik} A^{i_1 \cdots i_p}_{j_1 \cdots j_{m-1} s \cdots j_q}$$

$$= \sum_{m=1}^{p} g^{im} Q_{st} A^{i_1 \cdots i_{m-1} s \cdots i_p}_{j_1 \cdots j_q} - \sum_{m=1}^{q} g^{is} Q_{jm} A^{i_1 \cdots i_p}_{j_1 \cdots j_{m-1} s \cdots j_q}$$

$$= \sum_{m=1}^{p} Q_{im} A^{i_1 \cdots i_{m-1} s \cdots i_p}_{j_1 \cdots j_q} - \sum_{m=1}^{q} Q_{jm} A^{i_1 \cdots i_p}_{j_1 \cdots j_{m-1} s \cdots j_q}.$$

So we complete the proof. $$\square$$
Additionally, if we assume the metric is pluriclosed, then we can give a formula using the 
(1, 1)-part of Bismut-Ricci form and the torsion tensor of metric.

More precisely, we have

**Lemma 4.3.** If the metric is pluriclosed, then for any tensor $A \in T^{p,0} \otimes T^{q,0}$, we have

$$\nabla A = \Delta A - (\rho^{1,1} + \sqrt{-1}H^2)[A]$$

**Proof.** Just recall that when the metric is pluriclosed, we have

$$\rho^{1,1} = R_n - \sqrt{-1}H^2.$$ 

$\square$

**Remark 4.4.** We would like to give a remark on the computation of $S[\alpha]$ when $\alpha$ is a $(p, 0)$-form. Here we regard $\alpha$ as a tensor whose subscripts are skew-symmetric. Assume $S = \sqrt{-1}S_{ij}dz^i \wedge d\bar{z}^j$ is a real $(1, 1)$-form and

$$\alpha = \frac{1}{p!}\alpha_{i_1...i_p}dz^{i_1} \wedge \cdots \wedge dz^{i_p} = \alpha_{i_1...i_p}dz^{i_1} \otimes \cdots \otimes dz^{i_p}$$

satisfying $\alpha_{i_1...i_p} + \alpha_{i_1...i_p} = 0$. By definition, we have

$$(S[\alpha])_{i_1...i_p} = - \sum_{m=1}^{p} S^{s}_{im}\alpha_{i_1...s...i_p} = - \sum_{m=1}^{p} g^{ts}S_{im}t\alpha_{i_1...s...i_p}.$$  

For convenience, we choose a coordinates such that $g_{ij}(x) = \delta_{ij}$ and $S(x) = \sqrt{-1}\lambda_idz^i \wedge d\bar{z}^i$ for an arbitrary fixed point $x$. Then we have

$$(S[\alpha](x))_{i_1...i_p} = - \sum_{m=1}^{p} \lambda_{im}\alpha_{i_1...i_p}.$$  

In particular, when $p = n$, which is the complex dimension of the manifold. We have

$$(S[\alpha](x))_{1...n} = - \sum_{m=1}^{n} \lambda_{m}\alpha_{1...n} = -(\text{tr}_\omega S)(x)\alpha_{1...n}$$

So we obtain

$$S[\alpha] = -(\text{tr}_\omega S)\alpha.$$  

for $x$ is arbitrary.

**4.2. Bochner formula.** In this subsection, we will give a Bochner formula using another 
type Ricci form $R_n$. And as we have shown, if the metric is pluriclosed, this formula can be expressed in terms of Bismut-Ricci form.

For convenience, in this subsection we regard a form as a skew-symmetric tensor whose 
subscripts are skew-symmetric. Firstly, we recall some notations which will be used later.

**Definition 4.5.** Given a tensor $A \in T^{p,0}$. We denote its gradient by

$$\nabla A = \nabla_i A_{i_1...i_p}dz^i \otimes dz^{i_1} \otimes dz^{i_p}$$

$$\nabla A = \nabla_k A_{i_1...i_p}dz^k \otimes dz^{i_1} \otimes dz^{i_p}$$

Next, we recall the first eigenvalue of a real $(1, 1)$-form.
Definition 4.6. Let $S = \sqrt{-1}S_{ij}dz^i \wedge dz^j$ be a real $(1,1)$-form. The first eigenvalue function of $S$ (with respect to metric $\omega$) is defined by

$$\lambda_1(S)(x) = -\sqrt{-1} \min_{0 \neq X \in T_x^0M} \frac{S(x)(X, \bar{X})}{(X, \bar{X})}.$$  

Remark 4.7. For a fixed point $x$, we can choose a coordinates such that $g_{ij}(x) = \delta_{ij}$ and $S(x) = -\nabla \lambda_1 dz^i \wedge dz^j$. By direct computation, we get that $\lambda_1(S)(x) = \min \{\lambda_1, \ldots, \lambda_n\}$. So if $S$ is non-negative (positive), then $\lambda_1(S)$ is a non-negative (positive) function.

Now we can state the Bochner formula.

Theorem 4.8. Let $\alpha$ be a $(p,0)$-form. We have

$$\Delta|\alpha|^2 = (\Delta \alpha, \alpha) + (\alpha, \Delta \alpha) + |\nabla \alpha|^2 + |\overline{\nabla} \alpha|^2 - (\alpha, \text{Rn}[\alpha]).$$

We would like to give some explanations first. Here we regard forms as skew-symmetric tensors. So the inner product we used are the inner product of tensors, which differs from the inner product of forms by a constant coefficient. And notice that $\nabla \alpha$ is not a form anymore. But $\overline{\nabla} \alpha$ can also be regarded as a $(p,1)$-form. Actually, it just $\partial \alpha$ up to a sign.

Proof. Assume $\alpha = \alpha_{i_1 \ldots i_p}$. By direct computation, we get

$$\Delta|\alpha|^2 = g^{kl} \nabla_l \nabla_k (g^{ji_1} \ldots g^{ji_p} \alpha_{i_1 \ldots i_p} \alpha_{j_1 \ldots j_p})$$

Using Lemma 4.2 we obtain

$$\Delta|\alpha|^2 = (\Delta \alpha, \alpha) + |\nabla \alpha|^2 + |\overline{\nabla} \alpha|^2 + (\alpha, \Delta \alpha) - (\alpha, \text{Rn}[\alpha]).$$

According to Lemma 4.3, we have

Theorem 4.9. If the metric is pluriclosed, then for a $(p,0)$-form $\alpha$, we have

$$\Delta|\alpha|^2 = (\Delta \alpha, \alpha) + (\alpha, \Delta \alpha) + |\nabla \alpha|^2 + |\overline{\nabla} \alpha|^2 - (\alpha, \rho^{1,1}[\alpha]) - (\alpha, (\sqrt{-1}H^2)[\alpha])$$

4.3. Some vanishing results. Using the Bochner formula established in section 4.2 we can obtain some vanishing results. Firstly, we have

Proposition 4.10. Given a compact complex manifold $(M^{2n}, J)$. If $M$ admits a metric such that $\text{Rn}(\omega) > 0$, then the Dolbeault cohomology $H^p_0(M; \mathbb{C})$ is trivial for $1 \leq p \leq n$.

Proof. Let $\alpha$ be a $(p,0)$ $\bar{\partial}$-harmonic form, i.e. $\Delta \bar{\partial} \alpha = 0$. Then we have $\bar{\partial} \alpha = \bar{\partial}^* \alpha = 0$. From Proposition 3.10 we have

$$\Delta \alpha = -\Delta \bar{\partial} \alpha + (-1)^p \sqrt{-1} L_{\bar{\partial}^* \omega} \bar{\partial} \alpha = 0.$$
Applying Theorem 4.8, we get
\[ \Delta |\alpha|^2 = |\nabla \alpha|^2 + |\overline{\nabla} \alpha|^2 - (\alpha, R_n[\alpha]). \]
We will compute the function \((\alpha, R_n[\alpha])(x)\) in coordinates such that \(g_{ij}(x) = \delta_{ij}\) and \(R_n(x) = \sqrt{-1} \lambda_i dz^i \wedge d\overline{z}^i\). From Remark 4.4 and Remark 4.7, we know that
\[-(\alpha, R_n[\alpha])(x) = -\sum_{i_1 \cdots i_p} \alpha_{i_1 \cdots i_p} (R_n[\alpha])_{i_1 \cdots i_p} = \sum_{i_1 \cdots i_p} (\sum_{m=1}^p \lambda_i) \alpha_{i_1 \cdots i_p} \overline{\alpha_{i_1 \cdots i_p}} \]
\[ \geq \lambda_s(R_n)(x) \sum_{i_1 \cdots i_p} \alpha_{i_1 \cdots i_p} \overline{\alpha_{i_1 \cdots i_p}} \]
\[ = \lambda_s(R_n)(x) \cdot |\alpha|^2(x) \]
where \(\lambda_s(R_n)\) is strictly positive by assumption. Assume \(|\alpha|^2\) achieves the maximum at \(x_M\). Then by the maximum principle, we have
\[ 0 \geq \Delta |\alpha|^2(x_M) = |\nabla \alpha|^2(x_M) + |\overline{\nabla} \alpha|^2(x_M) - (\alpha, R_n[\alpha])(x_M) \]
\[ \geq |\nabla \alpha|^2(x_M) + |\overline{\nabla} \alpha|^2(x_M) + \lambda_s(R_n)(x_M) \cdot |\alpha|^2(x_M) \]
\[ \geq 0 \]
So we obtain \(|\alpha|^2(x_M) = 0\), which means \(\alpha = 0\). And we complete our proof for \(H^{p,0}_\partial(M; \mathbb{C}) \simeq \ker \Delta_\partial|_{\Lambda^{p,0}}\). \(\square\)

In particular, for \(H^{n,0}_\partial(M; \mathbb{C})\), we only need the Chern scalar curvature \(s = \text{tr}_\omega R_n\) to be positive. Formally, we have

**Proposition 4.11.** Given a compact complex manifold \((M^{2n}, J)\). If \(M\) admits a metric such that the Chern scalar curvature \(s = \text{tr}_\omega R_n\) is positive, then the Dolbeault cohomology \(H^{n,0}_\partial(M; \mathbb{C})\) is trivial.

**Proof.** From Remark 4.4, we have
\[-(\alpha, R_n[\alpha]) = (\alpha, (\text{tr}_\omega R_n)\alpha) = s|\alpha|^2.\]
for \((n,0)\)-form \(\alpha\). Then the argument is similar to Proposition 4.10 and we omit it. \(\square\)

If we assume the metric is Gauduchon, i.e., \(\partial \overline{\partial} \omega^{n-1} = 0\), then we can weaken the condition on curvature slightly.

**Proposition 4.12.** Given a compact complex manifold \((M^{2n}, J)\). If \(M\) admits a Gauduchon metric such that \(R_n(\omega) \geq 0\) and
\[ \int_M \lambda_s(R_n)dV > 0, \]
then the Dolbeault cohomology \(H^{p,0}_\partial(M; \mathbb{C})\) is trivial for \(1 \leq p \leq n\).

**Proof.** Let \(\alpha\) be a \(\overline{\partial}\)-harmonic \((p,0)\)-form. Integrating the Bochner formula in Theorem 4.8 on \(M\) and applying Proposition 3.8, we get
\[ 0 = \|\nabla \alpha\|^2 + \|\overline{\nabla} \alpha\|^2 + \int_M -(\alpha, R_n[\alpha])dV \]
where \( \| \cdot \| \) denotes the \( L^2 \) norm. Notice that \(- (\alpha, R_n[\alpha])\) is a non-negative function for \( R_n \geq 0 \). So we obtain
\[
\| \nabla \alpha \|^2 = \| \bar{\nabla} \alpha \|^2 = \int_M - (\alpha, R_n[\alpha])dV = 0,
\]
which means \( \nabla \alpha = \bar{\nabla} \alpha = 0 \). By direct computation, we get
\[
\partial_i |\alpha|^2 = \nabla_i (\alpha, \alpha) = (\nabla_i \alpha, \alpha) + (\alpha, \nabla_i \alpha) = 0
\]
Similarly, we obtain \( \partial_i |\alpha|^2 = 0 \). So \( |\alpha|^2 \) is a constant function on \( M \). Meanwhile, we have
\[
0 = \int_M - (\alpha, R_n[\alpha])dV \geq \int_M \lambda_* (R_n)|\alpha|^2 dV = |\alpha|^2 \int_M \lambda_* (R_n)dV
\]
By assumption, we get \( |\alpha|^2 = 0 \). And the proof is completed.

Similarly, for \( H^{n,0}_\bar{\partial}(M; \mathbb{C}) \) we only need assumptions on the Chern scalar curvature \( s \).

**Proposition 4.13.** Given a compact complex manifold \((M^{2n}, J)\). If \( M \) admits a Gauduchon metric such that \( s \geq 0 \) and \( \int_M \rho_1, \rho dV > 0 \), then the Dolbeault cohomology \( H^{n,0}_\bar{\partial}(M; \mathbb{C}) \) is trivial.

**Proof.** We only give a sketch of the proof for it is similar to Proposition 4.12. Firstly, we prove that \( |\alpha|^2 \) is a constant function using a similar argument. Then we can use the equation \(- (\alpha, R_n[\alpha]) = s|\alpha|^2\) to prove that \( |\alpha|^2 \) is actually 0.

When the metric is pluriclosed, we can express \( R_n \) in terms of Bismut-Ricci form and the torsion tensor. Because \( \sqrt{-1}H^2 \) is non-negative, Bismut-Ricci, precisely the \((1,1)\)-part of Bismut-Ricci form can give obstructions to Dolbeault cohomology \( H^{p,0}_\bar{\partial}(M; \mathbb{C}) \). Formally, we prove

**Theorem 4.14.** Given a compact complex manifold with a pluriclosed metric \((M^{2n}, J, \omega)\).

(1) If \( \rho^{1,1} \) is strictly positive definite, then \( H^{p,0}_\bar{\partial}(M; \mathbb{C}) \) is trivial for \( 1 \leq p \leq n \);

(2) If Bismut scalar curvature \( r = tr_\omega \rho^{1,1} \) is positive, then \( H^{n,0}_\bar{\partial}(M; \mathbb{C}) \) is trivial.

**Proof.** Recall
\[
R_n = \rho^{1,1} + \sqrt{-1}H^2
\]
and notice \( \sqrt{-1}H^2 \geq 0 \) (see Remark 2.1). Then we can obtain (1) and (2) from Proposition 4.13 and Proposition 4.11 respectively.

5. **Bismut-Einstein Metrics on Surfaces**

5.1. **A vanishing theorem for surfaces.** In section 4.3, we give several vanishing results under the assumption that \( R_n \) or \( tr_\omega R_n \) is positive. And we have shown that those assumptions can be given by Bismut-Ricci form when the metric is pluriclosed. On the other hand, we show that those assumptions can be weakened when the metric is Gauduchon. Fortunately, in the cases of complex surfaces, Gauduchon metric is exactly pluriclosed metric. So we can obtain a stronger version of Theorem 4.14 in dimension 2.

We begin with an observation in dimension 2.
Lemma 5.1. In dimension 2, we have
\[ \sqrt{-1}H^2 = \frac{1}{2}|H|^2\omega. \]

Proof. We prove it in a coordinates such that \( g_{i\bar{j}}(x) = \delta_{ij} \). By direct computation, we have
\[ \sqrt{-1}H^2(x) = \sqrt{-1}\delta^{ts}\delta^{kl}H_{isk}\overline{H}_{jtl}dz^i \wedge d\bar{z}^j = \sqrt{-1}\sum_{s,k} H_{isl}H_{jtk}dz^i \wedge d\bar{z}^j \]
\[ = \sqrt{-1}\sum_{k} (H_{1\bar{k}}\overline{H}_{1\bar{k}} dz^1 \wedge d\bar{z}^1 + H_{2\bar{k}}\overline{H}_{2\bar{k}} dz^2 \wedge d\bar{z}^2) \]
\[ = \sum_{k} |H_{1\bar{k}}|^2 \sqrt{-1}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2) \]
\[ = \frac{1}{2}|H|^2(x)\omega(x) \]
The second row is because of \( H_{ij\bar{k}} + H_{ji\bar{k}} = 0 \).

Next, we give another lemma which will be used later.

Lemma 5.2. Given a tensor \( A \in T^{p,0}M \). We have \( \omega[A] = -pA \).

Proof. Assume \( A = A_{i_1...i_p} \). By definition,
\[ (\omega[A])_{i_1...i_p} = -\sum_{m=1}^{p} g^{is}g_{imt}A_{i_1...i_{m-1}s...i_p} = -\sum_{m=1}^{p} \sum_{s} \delta_{im}sA_{i_1...i_{m-1}s...i_p} \]
\[ = -\sum_{m=1}^{p} A_{i_1...i_{m-1}s} = -pA_{i_1...i_p}. \]

Now we can give the proof of Theorem 1.3.

Proof of Theorem 1.3. Let \( \alpha \) be a \( \bar{\partial}\)-harmonic \((p,0)\)-form. Integrating the Bochner formula in Theorem 4.9 and applying Proposition 3.8, we get
\[ 0 = \|\nabla\alpha\|^2 + \|\nabla\alpha\|^2 + \int_M -(\alpha, \rho^{1,1}[\alpha])dV + \int_M -(\alpha, (\sqrt{-1}H^2)[\alpha])dV. \]
Firstly, we assume \( \rho^{1,1} \geq 0 \). In such a case, all terms in the right-hand side is zero for they are non-negative. Through a discussion similar to Proposition 4.12 we get that \( |\alpha|^2 \) is a constant function. Then applying Lemma 5.1 and Lemma 5.2, we get
\[ 0 = \int_M -(\alpha, (\sqrt{-1}H^2)[\alpha])dV = \int_M -(\alpha, \frac{1}{2}|H|^2(\omega[\alpha]))dV \]
\[ = \int_M (\alpha, |H|^2\alpha)dV = \int_M |H|^2|\alpha|^2dV \]
\[ = |\alpha|^2 \int_M |H|^2dV. \]
The last row is because \( |\alpha|^2 \) is a constant function. So either \( |\alpha|^2 = 0 \) or \( \int_M |H|^2dV = 0 \). The second case is Kähler for \( |\partial\omega|^2 = |H|^2 = 0 \). Thus we prove case (1) in Theorem 1.3.
To prove case (2), we assume \( r = \text{tr}_\omega \rho^{1,1} \geq 0 \). Let \( \alpha \) be a harmonic \((2,0)\)-form. Through the same discussion we obtain that \( |\alpha|^2 \) is a constant function. From Remark 4.4, we get

\[
0 = \int_M (\alpha, (\text{tr}_\omega \sqrt{-1} H^2) \alpha) dV = \int_M (\alpha, \frac{1}{2} |H|^2 (\text{tr}_\omega \alpha) \alpha) dV = \int_M |H|^2 |\alpha|^2 dV
\]

That means either \( \omega \) is Kähler or \( H^2, 0 \bar{\partial} \partial (M; \mathbb{C}) = 0 \). So we complete our proof. \( \square \)

5.2. The \((2,0)\)-part of Bismut-Ricci form. In this subsection, we introduce an observation about the \((2,0)\)-part of Bismut-Ricci form \( \rho^{2,0} \) of Bismut-Einstein metrics in dimension 2. This observation plays an important role in the proof of Theorem 1.2.

Assume \( \omega \) is a Bismut-Einstein metric with Einstein constant \( \lambda \), i.e., \( \rho^{1,1}(\omega) = \lambda \omega \). Firstly, we consider the case \( \lambda = 0 \). In such a case, \( \rho^{2,0} \) is closed since \( \rho = \rho^{2,0} + \rho^{2,0} \) is a closed form.

In the case of \( \lambda \neq 0 \), we have

\[
0 = \partial \rho^{1,1} + \bar{\partial} \rho^{2,0} = \lambda \partial \omega + \bar{\partial} \rho^{2,0}
\]

and \( \partial \rho^{2,0} = 0 \). On the other hand, recall that

\[
\rho^{2,0} = -\partial \bar{\partial}^* \omega.
\]

In dimension 2, it becomes

\[
\rho^{2,0} = -\partial \bar{\partial}^* \omega = \partial * \partial * \omega = \partial * \partial \omega = -\frac{1}{\lambda} \partial * \bar{\partial} \rho^{2,0} = -\frac{1}{\lambda} \partial * \bar{\partial} \rho^{2,0} = \frac{1}{\lambda} \bar{\partial} \partial \rho^{2,0}
\]

The third equality is because of \( * \omega = \omega \) in dimension 2. The fifth equality uses the fact that \( *\eta = \eta \) for arbitrary \((2,0)\)-form in dimension 2, which is an application of Lefschetz theorem to primitive form (see e.g., [16]).

In conclusion, we observe that

**Proposition 5.3.** Given a compact complex surface \((M^4, J)\) with a Bismut-Einstein metric satisfying \( \rho^{1,1} = \lambda \omega \). We have

\[
\Delta_{\bar{\partial}} \rho^{2,0} = \lambda \rho^{2,0}
\]

5.3. Bismut-Einstein metric on surfaces. In this subsection, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Without loss of generality, we only prove it for three cases \( \lambda \in \{-1, 0, 1\} \). For convenience, we denote \( \phi = \rho^{2,0} \).

**Case I** \( \lambda = -1 \). From Proposition 5.3, we know in such a case

\[
\Delta_{\bar{\partial}} \phi = -\phi.
\]

By direct computation, we have

\[
(\phi, \phi)_2 = -\Delta_{\bar{\partial}} \phi, \phi)_2 = -(\bar{\partial} \partial \phi, \phi)_2 = -(\bar{\partial} \phi, \bar{\partial} \phi)_2.
\]

where \((\cdot, \cdot)_2\) is the \(L^2\) inner product. So we obtain \( \rho^{2,0} = \phi = 0 \). Notice

\[
\partial \omega = -\partial \rho^{1,1} = \bar{\partial} \rho^{2,0} = 0.
\]
which means that $\omega$ is Kähler.

**Case II** $\lambda = 0$. Using Proposition 5.3 again, in this case we have

$$\Delta \bar{\partial} \rho^{2,0} = 0.$$ 

In other words, $\rho^{2,0}$ is a $\bar{\partial}$-harmonic $(2,0)$-form. Applying Theorem 1.3, we obtain that $\rho^{2,0} = 0$. So $\omega$ is Bismut–Ricci flat.

**Case III** $\lambda = 1$. Proposition 5.3 tells us

$$\Delta \bar{\partial} \phi = \phi.$$ 

Recall that in this case the metric is automatically pluriclosed, so Proposition 3.8 can be applied. Applying Bochner formula Theorem 4.9 to $\phi$ and integrating it on the manifold, we get

$$0 = (\Delta \phi, \phi) + (\partial, \Delta \phi) + \|\nabla \phi\|^2 + \|\bar{\nabla} \phi\|^2 - \int_M (\phi, (\rho^{1,1} + \sqrt{-1}H^2)[\phi])dV. \quad (5.1)$$

Using Proposition 3.10, we get

$$\Delta \phi, \phi) = -\Delta \bar{\partial} \phi, \phi + \sqrt{-1}(L^*_{\bar{\partial} \omega} \bar{\partial} \phi, \phi)$$

By the definition of Lefschetz-type operator $L^*_{\bar{\partial} \omega} (\cdot)$ (see section 2.3),

$$\Delta \bar{\partial} \phi, \phi) = \int_M (L^*_{\bar{\partial} \omega} \bar{\partial} \phi, \phi)dV = \int_M (\bar{\partial} \phi, \partial \omega \wedge \phi)dV = (\bar{\partial} \phi, \partial \omega \wedge \phi)$$

By direct computation, we obtain

$$(\bar{\partial} \phi, \partial \omega \wedge \phi) = -\int_M \bar{\partial} \omega \wedge \bar{\partial} \phi = -\int_M \bar{\partial} \phi \wedge \bar{\partial} \phi = 0$$

The second row is because of $\partial \omega = \partial \rho^{1,1} = -\bar{\partial} \rho^{2,0} = -\bar{\partial} \phi$. And the last row uses the fact that $\bar{\partial} \omega \wedge \bar{\partial} \omega = 0$, which holds because $\bar{\partial} \omega$ is an odd degree form. So we obtain

$$(\Delta \phi, \phi) = -\Delta \bar{\partial} \phi, \phi = -\|\phi\|^2.$$ 

Similarly, we have

$$(\phi, \Delta \phi) = -\|\phi\|^2.$$ 

On the other hand, applying Lemma 5.2, we have

$$-\int_M (\phi, \rho^{1,1}[\phi])dV = -\int_M (\phi, \omega[\phi])dV = 2 \int_M (\phi, \phi)dV = 2\|\phi\|^2.$$ 

Then equation (5.1) becomes

$$0 = \|\nabla \phi\|^2 + \|\bar{\nabla} \phi\|^2 + \int_M -(\phi, (\sqrt{-1}H^2)[\phi])dV.$$
Notice that \(-(\phi, (\sqrt{-1}H^2)[\phi])\) is a non-negative function. So we obtain \(\|\nabla \phi\|^2 = \|\nabla \phi\|^2 = 0\). In particular \(\bar{\partial} \phi = 0\) by the definition of \(\nabla \phi\) (see Definition 4.5). We get \(\partial \omega = -\bar{\partial} \phi = 0\), which means \(\omega\) is Kähler.

So we complete the proof since Bismut connection is just Chern connection for Kähler metrics. 

\[\square\]

6. More Discussion on Bismut-Einstein Metrics

In this section, we would like to do some discussions on Bismut-Einstein metrics in high dimensional cases.

6.1. An observation about \(\rho^{2,0}\). We would like to introduce an interesting observation about the \((2,0)\)-part of Bismut-Einstein metrics, which is a generalization of Proposition 5.3 in dimension 2.

Let us begin with an observation given by the authors of [4, 7]. For the convenience of use later, here we would like to give a direct proof by computing in local coordinates.

Lemma 6.1. If we regard \(\rho^{2,0}\) as a tensor whose subscripts are skew-symmetric, then we can use Chern connection to express it as

\[
\rho^{2,0} = -\sqrt{-1} \text{div} H,
\]

where \((\text{div} H)_{ij} = -g^{kl} \nabla_i H_{jk}\).

Proof. Noticing Remark 3.5 we have

\[
\bar{\partial}^* \omega = \sqrt{-1} H_{is} dz^i.
\]

By definition,

\[
\rho^{2,0} = -\partial \bar{\partial}^* \omega = -\frac{1}{2} \left( \partial_i (\bar{\partial}^* \omega)_j - \partial_j (\bar{\partial}^* \omega)_i \right) dz^i \wedge dz^j
\]

By direct computation,

\[
\partial_i H^s_{js} = \partial_i (g^t s H^t_{js}) = \partial_i H^t_{js} g^t s - g^t p g^q s \partial_i g_{pq} H^t_{js}
\]

Recalling the definition of \(H^t_{js}\) in section 2.2, we get

\[
\partial_i H^s_{js} = g^t s (\partial_i g_{st} - \partial_i g_{sj}) - \Gamma^p_{is} \Gamma^s_{jp} + \Gamma^p_{is} \Gamma^s_{pj}
\]

Then

\[
\partial_i H^s_{js} - \partial_j H^s_{is} = -g^t s \partial_s H^t_{ij} - \Gamma^p_{js} \Gamma^s_{pi} + \Gamma^p_{is} \Gamma^s_{pj}
\]

\[
= -g^t s \partial_s H^t_{ij} + \Gamma^p_{ps} H^p_{sj} + \Gamma^p_{sj} H^p_{is}
\]

\[
= -g^t s \partial_s H^t_{ij} + g^p s \Gamma^p_{sj} H^p_{ij} + \Gamma^p_{sj} H^p_{is}
\]

\[
= -g^t s \nabla_s H^t_{ij}.
\]

\[\square\]
Now we give an observation about $\rho^{2,0}$ of Bismut-Einstein metrics.

**Proposition 6.2.** Given a Bismut-Einstein metric $\omega$ with Einstein constant $\lambda$, then we have

$$\Delta \rho^{2,0} = -\lambda \rho^{2,0}. $$

**Proof.** Firstly, in the case of $\lambda = 0$. Notice that $\rho^{2,0}$ is closed for $\rho = \rho^{2,0} + \overline{\rho^{2,0}}$ is closed. Then we get $\Delta \rho^{2,0} = 0$ because of $\bar{\partial}^* \rho^{2,0} = \bar{\partial} \rho^{2,0} = 0$. Applying Lemma 3.10, we obtain $\Delta \rho^{2,0} = 0$.

From now on, we assume $\lambda \neq 0$. Then we have

$$\bar{\partial} \rho = \frac{1}{\lambda} \bar{\partial} \rho^{1,1} = -\frac{1}{\lambda} \bar{\partial} \rho^{2,0}$$

In local coordinates, we assume

$$\rho^{2,0} = \frac{\sqrt{-1}}{2} B_{ij} dz^i \wedge dz^j, \quad B_{ij} + B_{ji} = 0$$

Then equation (6.1) is

$$\bar{\partial} \lambda = \frac{1}{\lambda} \bar{\partial} B_{ij} = -\frac{1}{\lambda} \bar{\partial} B_{ij}$$

or equivalently,

$$H_{ijk} = -\frac{1}{\lambda} \bar{\partial} B_{ij}$$

Applying Lemma 6.1, we obtain

$$B_{ij} = g^{kl} \nabla_l H_{ijk} = -\frac{1}{\lambda} g^{kl} \nabla_l \bar{\partial} B_{ij} = -\frac{1}{\lambda} \Delta B_{ij}$$

And we complete the proof. \qed

### 6.2. Bismut-Einstein metrics in high dimensional cases.

In this section we will give a weak version of Theorem 1.2.

**Theorem 6.3.** Assume a compact complex manifold $(M^{2n}, J)$ admitting a Bismut-Einstein metric $\omega$ with Einstein constant $\lambda$.

1. If $\lambda = 0$ and $\omega$ is a pluriclosed metric such that $\sqrt{-1} H^2 > 0$, then $\omega$ is Bismut-Ricci flat;
2. If $\lambda \neq 0$, then $\sqrt{-1} H^2$ can not be strictly positive everywhere;
3. If $\lambda \neq 0$ and $\omega$ is Gauduchon, then $\omega$ is Kähler-Einstein.

**Proof.** Notice that in the case of $\lambda \neq 0$, $\omega$ is automatically pluriclosed. For ease of notations, we denote $\phi = \rho^{2,0}$. Applying Theorem 4.9 and Proposition 6.2, we get

$$\Delta |\phi|^2 = -2\lambda (\phi, \phi) + |\nabla \phi|^2 + |\bar{\nabla} \phi|^2 - (\phi, \rho^{1,1}[\phi]) - (\phi, (\sqrt{-1} H^2)[\phi])$$

Since $\rho^{1,1} = \lambda \omega$, we have

$$-(\phi, \rho^{1,1}[\phi]) = -\lambda (\phi, \omega[\phi]) = 2\lambda (\phi, \phi)$$

The last equality is because of Lemma 5.2. Meanwhile, recalling Remark 4.4, we have

$$-(\phi, (\sqrt{-1} H^2)[\phi]) \geq \lambda_*(\sqrt{-1} H^2) \cdot (\phi, \phi).$$
\[ \lambda_*(\sqrt{-1}H^2) \] is the first eigenvalue of \( \sqrt{-1}H^2 \) (see Definition 4.6). It is non-negative definite by definition. Combining equations (6.2) and (6.3), we obtain
\[
(6.5) \quad \Delta|\phi|^2 = |\nabla \phi|^2 + |\nabla \phi|^2 - (\phi, (\sqrt{-1}H^2)[\phi]).
\]
Firstly, we would like to prove part (1). Because the manifold is compact, we can assume that \( |\phi|^2(x) \) achieves the maximum at point \( x_M \). Applying maximum principle and noticing (6.4), we have
\[
0 \geq \Delta|\phi|^2(x_M) = |\nabla \phi|^2(x_M) + |\nabla \phi|^2(x_M) - (\phi, (\sqrt{-1}H^2)[\phi])(x_M)
\geq |\nabla \phi|^2(x_M) + |\nabla \phi|^2(x_M) + \lambda_*(\sqrt{-1}H^2)(x_M) \cdot |\phi|^2(x_M)
\]
Then we obtain \( |\phi|^2(x_M) = 0 \) since \( \lambda_*(\sqrt{-1}H^2)(x_M) > 0 \). That means \( \rho_0^2 = \phi = 0 \), i.e., \( \omega \) is Bismut-Ricci flat. So we prove part (1).

For part (2), we first assume it is not true. That means \( \sqrt{-1}H^2 \) is strictly positive at everywhere. By the same argument as above, we obtain \( \rho_0^2 = 0 \). Then we have
\[
\partial \omega = \frac{1}{\lambda} \partial \rho^{1,1} = -\frac{1}{\lambda} \bar{\partial} \rho_0^{2,0} = 0
\]
So \( \omega \) is Kähler and \( \sqrt{-1}H^2 = 0 \). This is a contradiction. Thus part (2) is valid.

Finally, we prove part (3). Because \( \omega \) is Gauduchon in such a case, Proposition 3.8 can be applied. Integrating both sides of equation (6.4), we get
\[
0 = ||\nabla \phi||^2 + ||\bar{\nabla} \phi||^2 + \int_M - (\phi, (\sqrt{-1}H^2)[\phi])dV
\]
The right-hand side is non-negative due to (6.4). So we obtain
\[
\partial \omega = -\bar{\partial} \rho_0^{2,0} = -\bar{\partial} \phi = \nabla \phi = 0.
\]
which means \( \omega \) is Kähler. And we complete the proof. \( \square \)

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