Relativistic Dissipative Hydrodynamics: A Minimal Causal Theory

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Abstract

We present a new formalism for the theory of relativistic dissipative hydrodynamics. Here, we look for the minimal structure of such a theory which satisfies the covariance and causality by introducing the memory effect in irreversible currents. Our theory has a much simpler structure and thus has several advantages for practical purposes compared to the Israel-Stewart theory (IS). It can readily be applied to the full three-dimensional hydrodynamical calculations. We apply our formalism to the Bjorken model and the results are shown to be analogous to the IS.
I. INTRODUCTION

The ideal hydrodynamical description for the dynamics of hot and dense matter achieved in RHIC experiments works amazingly well, particularly for the behavior of collective flow parameters. Together with other signals, the success of the approach is considered as the indication of the emergence of a new state of strongly interacting matter, the plasma of quarks and gluons (QGP). The comparison between RHIC and SPS results shows that this new state of matter is formed at the very early stage of the relativistic heavy ion collisions for RHIC energies. The nature of the QGP seems rather a strongly interacting fluid (sQGP) than a ideal free parton gas \[1\] which still flows like an ideal-fluid. This idea, that the QGP behaves as real ideal fluid, raised an interesting perspective, since the viscosity for the strong coupling limit of 4D conformal theory obtained from the supersymmetric string theory in an (Anti de Sitter) AdS space found to be very small \[2\]. On the other hand, Hirano and Gyulassy argue that this is due to the entropy density of the QGP which is much larger than that of the hadronic phase \[3\].

At any rate, we know that there still exist several open problems in the interpretation of data in terms of the hydrodynamical model \[4\]. These questions require careful examination to extract quantitative and precise information on the properties of QGP. In particular, we should study the effect of dissipative processes on the collective flow variables. Several works have been done in this direction \[5\]. However, strictly speaking, a quantitative and consistent analysis of the viscosity within the framework of relativistic hydrodynamics has not yet been done completely. This is because the introduction of dissipative phenomena in relativistic hydrodynamics casts difficult problems, both conceptual and technical. Initially Eckart, and later, Landau-Lifshitz introduced the dissipative effects in relativistic hydrodynamics in a covariant manner \[6, 7\]. It is, however, known that their formalism leads to the problem of acausality, that is, a pulse signal propagates with infinite speed. Thus, relativistic covariance is not a sufficient condition for a consistent relativistic dissipative dynamics \[8, 9\].

To cure this problem, relativistic hydrodynamics in the framework of extended thermodynamics was developed by Müller \[10\] and later by Israel and Stewart \[11, 12, 13\]. From the kinetic point of view \[14\], this formalism corresponds to the extension of equilibrium thermodynamics to include the second order moments of kinetic variables. This is the reason why this theory is usually referred to as the second order theory. However, this theory is too
general and complex, introducing many unknown parameters from the point of the present QCD dynamics. Moreover, the irreversible currents are treated as independent variables in addition to the usual hydrodynamical degrees of freedom (the velocity field plus densities of extensive quantities) and should be obtained by solving highly coupled partial differential equations. These features make the application of the theory to practical problems very difficult.

On the other hand, even if we overcome the technical difficulties to apply the complete theory to the analysis of relativistic heavy ion reactions, it is very unlikely that we can extract such a detailed information from the experimental data at present moment. This is because there exist many uncertainties of the hydrodynamical approach itself such as the initial condition, finite size correction, event-by-event fluctuations, particle decoupling process etc. However, there exists some interesting questions which requires a correct treatment of the viscosity in a realistic 3D simulations, such as a possible generation of shock wave and its propagation in the QGP triggered by a jet. Analysis of such phenomena will certainly give important information of new, genuine hydrodynamical properties of the QGP. Therefore, what we need now is not a very general theory of relativistic dissipative hydrodynamics with so many unknown parameters but one with the minimum number of parameters necessary to preserve fundamental principles such as causality and entropy production.

As a matter of fact, it is known that the Israel-Stewart theory (IS) is not the unique approach to relativistic dissipative hydrodynamics. To authors’ knowledge, there is at least one other causal theory called the divergence type. In this work, we present a very simple alternative theory which satisfies the minimal conditions mentioned above. We show that the causality problem of the Landau-Lifshitz formalism can be solved by introducing a memory effect. This memory effect is characterized by the relaxation time $\tau_R$, so that our theory introduces only one additional parameter to the usual bulk viscosity, shear viscosity and thermal conduction coefficients of the Navier-Stokes equation. Our theory recovers the relativistic Navier-Stokes equation in the limit of vanishing values of this relaxation time.

As described later more in detail, our approach has a fundamental advantage from the practical point of view in addition to its physical simplicity. The dissipative terms are explicitly given by the integral of the independent variables of the usual ideal hydrodynamics.

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Thus, the implementation of our method to the existing ideal hydro-codes is straightforward, particularly, to those based on the local Lagrangian coordinate system such as SPheRIO [21, 22].

The present paper is organized as follows. In the next section, we briefly review the problem of the acausal propagation in the diffusion equation and the method to cure this problem in terms of the memory effect, which leads to the so-called telegraphist’s equation. In the Sec. III, we analyse the structure of the Landau-Lifshitz theory of relativistic dissipative hydrodynamics and introduce the memory effect to solve the acausal problem due to its parabolic nature. We thus obtain the dissipative hydrodynamical equations with the minimum number of parameters which satisfies causality. In Sec. IV, we discuss the problem of entropy production in our formalism. In Sec.V, we apply our equation to the Bjorken solution, and compare with the previous analyses [23, 24, 25, 26, 27]. In Sec. VI, we summarize our results and discuss possible immediate applications.

II. DIFFUSION EQUATION AND ACAUSALITY

The fundamental problem of the first order theory like the Navier-Stokes theory comes from the fact that the diffusion equation is parabolic. This means that the velocity of signal propagation is infinite. This problem was first addressed by Cattaneo [28] in the case of heat conduction. He discussed that the problem of acausal propagation in usual diffusion equation,

$$\frac{\partial}{\partial t} n = \zeta \nabla^2 n,$$

(1)

can be cured by the introduction of the second order time derivative as

$$\tau_R \frac{\partial^2 n}{\partial t^2} + \frac{\partial n}{\partial t} = \zeta \nabla^2 n,$$

(2)

thus converting the parabolic equation to a hyperbolic one. Here, $\zeta$ and $\tau_R$ are a diffusion constant and a relaxation time, respectively. Eq. (2) is sometimes called as telegraphist’s equation. For a suitable choice of the parameters $\tau_R$ and $\zeta$, we can recover the causal propagation of diffusion process. In fact, the maximum velocity of the signal propagation governed by this equation is shown to be [29]

$$v_{\text{max}} = \sqrt{\frac{\zeta}{\tau_R}}.$$
To be consistent with the special relativity, we should have
\[ \tau_R \geq \frac{\zeta}{c^2}. \] (4)
The diffusion equation, Eq. (1), corresponds to \( \tau_R = 0 \) and hence \( v_{\text{max}} \to \infty \).

The physical origin of above telegraphist’s equation can be understood as follows. The diffusion process is a typical relaxation process for a conserved quantity. Thus, it should satisfy the equation of continuity,
\[ \frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0. \] (5)
In non-equilibrium thermodynamics, the irreversible current \( \vec{j} \) is assumed to be proportional to a thermodynamic force,
\[ \vec{j} = -L \vec{F}, \] (6)
where the Onsager coefficient \( L \) is, in general, a function of thermodynamic quantities. Here, we assume it to be constant for the sake of illustration. When \( n \) is a number density, the thermodynamic force is given by the gradient of \( n \) because of Fick’s law,
\[ \vec{F} = \nabla n. \] (7)
Substituting Eq. (6) into Eq. (5), we get the diffusion equation (1).

Fick’s law tells us that the above diffusion process is induced by inhomogeneous distribution. In Eq. (6), the space inhomogeneity immediately gives rise to the irreversible current. However, this is a very idealized situation. In general, the generation of irreversible currents has a time delay. Thus, we may think of memory effects within the linear response of the system. Phenomenologically, this can be done by introducing the following memory function \[30\],
\[ G(t, t') = \begin{cases} \frac{1}{\tau_R} e^{-(t-t')/\tau_R}, & t \geq t' \\ 0, & t < t' \end{cases} \] (8)
= 0, \quad t < t' \] (9)
where \( \tau_R \) is a constant (relaxation time) and rewrite Eq. (6) as
\[ \vec{j} = -\int_{-\infty}^{t} G(t, t') L \vec{F}(t') \, dt'. \] (10)
In the limit of $\tau_R \to 0$, we have $G(t, t') \to \delta(t - t')$ so that the original equation (6) is recovered [31]. The time derivative of the irreversible current (10) leads to

$$\frac{\partial \vec{j}}{\partial t} = -\frac{1}{\tau_R} L \vec{F}(t) - \frac{1}{\tau_R} \vec{j}.$$  

This is called the Maxwell-Cattaneo type equation. Substituting into the equation of continuity (5), we arrive at

$$\frac{\partial^2 n}{\partial t^2} = \nabla \cdot \left( \frac{1}{\tau_R} \zeta F(t) + \frac{1}{\tau_R} \vec{j} \right) = \frac{1}{\tau_R} \left( -\frac{\partial n}{\partial t} + \zeta \nabla^2 n \right),$$

which is exactly Eq. (2).

We here showed the problem of acausality in the diffusion equation, which is based on non-equilibrium thermodynamics (the first order theory), and how this can be solved by introducing the memory function. For more microscopic derivation of telegraphist’s equation and foundation of the second order theory, see Refs. [9, 30, 32, 33]. In particular, it was recently shown that the macroscopic equation of motion, which is obtained by using systematic coarse-grainings from the Heisenberg equation of motion, is not the diffusion equation but telegraphist’s equation [32].

Till now, we have emphasized the importance of the memory effect in the view of causality, but there exists another important reason to introduce the memory effect. It was pointed out that the diffusion equation also contradicts with sum rules associated with conservation laws. This can also be solved by replacing the diffusion equation with telegraphist’s equation [32]. Relativistic dissipative hydrodynamics should be consistent with QCD, where there are conserved quantities and associated sum rules. In this sense, the presence of memory effect should be inevitable.

### III. MINIMAL CAUSAL HYDRODYNAMICS

Eckart and Landau-Lifshitz derived the relativistic dissipative hydrodynamics following non-equilibrium thermodynamics as discussed in the preceding section [6, 7]. Their theories are just the covariant versions of the Navier-Stokes equation and the corresponding equations still continue to be parabolic. Therefore, they do not satisfy causality and some modification should be required. In the IS and the divergence type theory, the definition
of the entropy four-flux is generalized and, to satisfy the second law of thermodynamics, modified thermodynamic forces are obtained. In this section, we propose another approach, where the problem of acausality is solved by introducing the memory effect like Eq. (10).

For this purpose, let us first analyse briefly the structure of the Landau-Lifshitz theory (LL). The hydrodynamical equation of motion is written as the conservation of the energy-momentum tensor,

$$\partial_{\mu} T^{\mu\nu} = 0,$$

(12)

together with the conservation of a quantity, for example, the baryon number,

$$\partial_{\mu} N^{\mu} = 0.$$

(13)

In the LL, it is assumed that thermodynamic relations are valid in the local rest frame of the energy-momentum tensor. The energy-momentum tensor is expressed as

$$T^{\mu\nu} = \varepsilon u^{\nu} - P^{\mu\nu} (p + \Pi) + \pi^{\mu\nu},$$

(14)

where, $\varepsilon$, $p$, $u^{\mu}$, $\Pi$ and $\pi^{\mu\nu}$ are respectively the energy density, pressure, four velocity of the fluid and bulk and shear viscous stresses. In the LL, the velocity field is defined in such a way that the energy current vanishes in the local rest frame, $u^{\mu} \rightarrow (1, 0, 0, 0)$. In this local rest frame, it is assumed that the equation of state and thermodynamical relations are valid as if the fluid were in equilibrium. As usual, we write

$$u^{\mu} = \left( \begin{array}{c} \gamma \\ \gamma \vec{v} \end{array} \right),$$

where $\gamma$ is the Lorentz factor and

$$u^{\mu} u_{\mu} = 1.$$ 

The tensor $P^{\mu\nu}$ is the projection operator to the space orthogonal to $u^{\mu}$ and given by

$$P^{\mu\nu} = g^{\mu\nu} - u^{\mu} u^{\nu},$$

which has the following trace property,

$$\sum_{\lambda=0}^{D} P_{\lambda} = D.$$

(15)

Here, $D$ is the number of spatial dimensions.
In the LL, the current for the conserved quantity (e.g., net baryon number) takes the form

\[ N^\mu = n u^\mu + \nu^\mu, \]  

(16)

where \( \nu^\mu \) is the heat conduction part of the current. For the irreversible currents, we require the constraints

\[ u_\mu \pi^{\mu\nu} = 0, \]  

(17)

and

\[ u_\mu \nu^\mu = 0. \]  

(18)

These constraints permit us to interpret, respectively as the energy and net baryon number densities in the local rest frame. In fact, from Eq. (18), in the rest frame, we have

\[ N^\mu \rightarrow \begin{pmatrix} n \\ \vec{\nu} \end{pmatrix}, \]

so that \( n \) is the net baryon number density in the local rest frame. It is noteworthy that, different from the ideal case (or Eckart case), \( N^0 \) is not related with the net baryon number density \( n \) in any frame by the Lorentz contraction, but

\[ N^0 = n \gamma + (\vec{v} \cdot \vec{v}) \gamma. \]

With these irreversible currents, of course, the entropy is not conserved. Instead, from Eqs. (12) and (13) with the constraints Eqs. (17) and (18), we have

\[ \partial_\mu (sv^\mu - \alpha \nu^\mu) = \frac{1}{T} (-P^{\mu\nu}\Pi + \pi^{\mu\nu}) \partial_\mu u_\nu - \nu^\mu \partial_\mu \alpha, \]  

(19)

where \( \alpha = \mu/T \) with temperature \( T \) and chemical potential \( \mu \). Landau-Lifshitz identifies the term

\[ \sigma^\mu = sv^\mu - \alpha \nu^\mu, \]  

(20)

as the entropy four-flux. The r.h.s. of Eq. (19) is the source term for entropy production.

In non-equilibrium thermodynamics, it is interpreted that entropy production is the sum of the products of thermodynamic forces and irreversible currents. Thus, we can define the scalar, vector and tensor thermodynamic forces,

\[ F = \partial_\alpha u^\alpha, \quad F_\mu = \partial_\mu \alpha, \quad F_{\mu\nu} = \partial_\mu u_\nu. \]
respectively. To satisfy the second law of thermodynamics, we assume that the entropy production is positive,
\[ \frac{1}{T} (-P^{\mu\nu}\Pi + \pi^{\mu\nu}) \partial_\mu u_\nu - \nu^\mu \partial_\mu \alpha \geq 0. \]  (21)

To maintain this algebraic positive definiteness, the most general irreversible currents are given by linear combinations of the thermodynamic forces with the coefficients appropriately chosen. However, if we accept the Curie (symmetry) principle which forbids the mixture of different types of thermodynamic forces \[34\], the irreversible currents are given by
\[ \Pi = -\zeta F = -\zeta \partial_\alpha u^\alpha, \]
\[ \pi_{\mu\nu} = P_{\mu\alpha\beta} \tilde{\pi}^{\alpha\beta} = \eta P_{\mu\alpha\beta} F^{\alpha\beta} = \eta P_{\mu\alpha\beta} \partial^\alpha u^\beta, \]
\[ \nu_\mu = P_{\mu\nu} \tilde{\nu}^\nu = -\kappa P_{\mu\nu} F^{\nu\nu} = -\kappa P_{\mu\nu} \partial^\nu \alpha, \]  (22)

where \( \zeta, \eta \) and \( \kappa \) are bulk viscosity, shear viscosity and thermal conductivity coefficients, respectively. Here, \( P^{\mu\alpha\beta} \) is the double symmetric traceless projection,
\[ P^{\mu\alpha\beta} = \frac{1}{2} (P^{\mu\alpha} P^{\nu\beta} + P^{\mu\beta} P^{\nu\alpha} - \frac{1}{D} P^{\mu\nu} P^{\alpha\beta}), \]  (23)

and we have introduced the quantities \( \tilde{\pi}^{\alpha\beta} \) and \( \tilde{\nu}^\nu \) which correspond respectively to the shear tensor and irreversible current before the projection. They are in general not orthogonal to \( u^\mu \) so the projection operators are necessary to satisfy the constraints Eqs. (17) and (18).

Eqs. (22) are the prescription of the LL. As mentioned, the LL leads to the acausal propagation of signal. So we should modify these equations to satisfy the relativistic causality principle. The basic point is that the equations of the LL form a parabolic system and we have to convert it to the hyperbolic one. However, at this moment, the generalization of these equation in order to obtain hyperbolic equations is rather self-evident. We introduce the memory function in each irreversible currents, Eq. (22),
\[ \Pi (\tau) = - \int_{-\infty}^{\tau} d\tau' G (\tau, \tau') \zeta \partial_\alpha u^\alpha (\tau'), \]
\[ \tilde{\pi}^{\mu\nu} (\tau) = \int_{-\infty}^{\tau} d\tau' G (\tau, \tau') \eta \partial^\mu u^\nu (\tau'), \]
\[ \tilde{\nu}^\mu (\tau) = - \int_{-\infty}^{\tau} d\tau' G (\tau, \tau') \kappa \partial^\mu \alpha (\tau'), \]  (24)

where \( \tau = \tau (\vec{r}, t) \) is the local proper time. As before, the shear tensor \( \pi^{\mu\nu} \) and the irreversible
current \( \nu^\mu \) are then given by the projection of these integrals as

\[
\tilde{\pi}^{\mu\nu} = P^{\mu\alpha\beta} \pi_{\alpha\beta}^{\nu} (\tau), \\
\nu^\mu = P^{\mu\nu} \tilde{\nu}_\nu (\tau).
\]  

(25)

When we start with the finite initial time, say, \( \tau_0 \), the above integrals should read

\[
\Pi (\tau) = - \int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \zeta \partial_\alpha u^\alpha (\tau') + e^{-(\tau - \tau_0)/\tau_R} \Pi_0, \\
\tilde{\pi}^{\mu\nu} (\tau) = \int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \eta \partial^\mu u^\nu (\tau') + e^{-(\tau - \tau_0)/\tau_R} \tilde{\pi}^{\mu\nu}_0, \\
\tilde{\nu}^\mu (\tau) = - \int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \kappa \partial^\mu \alpha (\tau') + e^{-(\tau - \tau_0)/\tau_R} \tilde{\nu}^\mu_0,
\]  

(26), (27), (28)

where \( \Pi_0, \tilde{\pi}^{\mu\nu}_0 \) and \( \tilde{\nu}^\mu_0 \) are initial conditions given at \( \tau_0 \). The bulk viscosity can be determined by \( T^{\mu\nu} (\tau_0) \) and \( N^\mu (\tau_0) \) as

\[
\Pi_0 = \frac{1}{D} (\varepsilon_0 - T^\mu_\mu (\tau_0)) - p_0.
\]

As for \( \tilde{\pi}^{\mu\nu}_0 \) and \( \tilde{\nu}^\mu_0 \) we observe that they are the projected part of \( T^{\mu\nu} (\tau_0) \) and \( N^\mu (\tau_0) \), respectively,

\[
\nu^\mu_0 = N^\mu (\tau_0) - n_0 u^\mu_0 = P^\mu_\nu N^\nu (\tau_0), \\
\tilde{\pi}^{\mu\nu}_0 = T^{\mu\nu} (\tau_0) - (\varepsilon_0 + p_0 + \Pi_0) u^\mu_0 u^\nu_0 + (p_0 + \Pi_0) g^{\mu\nu} = P^{\mu\nu}_{\alpha\beta} T_{\alpha\beta} (\tau_0).
\]

This is equivalent to set up the following initial conditions,

\[
\tilde{\nu}^\mu_0 = N^\mu (\tau_0) - n_0 u^\mu_0, \\
\tilde{\pi}^{\mu\nu}_0 = T^{\mu\nu} (\tau_0) - (\varepsilon_0 + p_0 + \Pi_0) u^\mu_0 u^\nu_0 + (p_0 + \Pi_0) g^{\mu\nu}.
\]

The initial four-velocity \( u^\mu_0 \) is determined as the time-like eigenvector of the matrix \( T^{\mu\nu} (\tau_0) \) with eigenvalue \( \varepsilon_0 \). Using the equation of state together with \( n_0 = u_\mu N^\mu \), we can determine the pressure \( p_0 \).

In Eqs. (24), we have used the same memory function \( G \) and consequently a common relaxation time \( \tau_R \) for the bulk and shear viscosities and heat conduction. We could have used different relaxation times for each irreversible current and this would not alter the basic structure of our theory. However, here we stay with a common relaxation time for all of them for the sake of simplicity. We consider the situation where the time scales of
the microscopic degrees of freedom are well separated from those of the macroscopic ones. Then, the effect of the differences of the microscopic relaxation times should not be much relevant in the dynamics described in the macroscopic time scale. Thus we just represent these microscopic time scales in terms of one relaxation time $\tau_R$.

The integral expressions (24) are equivalent to the following differential equations,

$$\Pi = -\zeta \partial_\alpha u^\alpha - \tau_R \frac{d\Pi}{d\tau},$$
$$\tilde{\pi}^{\mu\nu} = \eta \partial^\mu u^\nu - \tau_R \frac{d\tilde{\pi}^{\mu\nu}}{d\tau},$$
$$\tilde{\nu}^{\mu} = -\kappa \partial^\mu \alpha - \tau_R \frac{d\tilde{\nu}^{\mu}}{d\tau},$$

(29)

where

$$\frac{d}{d\tau} = u^\mu \partial_\mu,$$

is the total derivative with respect to the proper time. In a practical implementation of our theory, we may solve the above differential equations together with the other hydrodynamical part. In this case, if we wish, we may add also the terms which violates the Curie principle, without any extra difficulties. The above equations, after the projection (25), can be compared to the corresponding equations in the simplest version of the IS, which are obtained phenomenologically based on extended thermodynamics,

$$\Pi_{IS} = -\frac{1}{3} \zeta_{IS} \left( \partial_\alpha u^\alpha + \beta_0 \frac{d\Pi_{IS}}{d\tau} - \alpha_0 \partial_\alpha \nu^\alpha \right),$$
$$\pi_{IS}^{\mu\nu} = 2\eta_{IS} P^{\mu\alpha\beta} \left( \partial^\mu u_\beta - \beta_2 \frac{d\pi_{IS}^{\mu\nu}}{d\tau} - \alpha_1 \partial_\alpha (\nu_{IS})_\beta \right),$$
$$\nu_{IS}^{\mu} = -\kappa_{IS} P^{\mu\nu} \left( \frac{n}{\varepsilon + P} \partial_\nu \alpha + \beta_1 \frac{d\nu_{IS}}{d\tau} + \alpha_0 \partial_\nu \Pi_{IS} + \alpha_1 \partial_\alpha (\pi_{IS})^\alpha_{\nu} \right),$$

(30)

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ and $\beta_2$ are constants. Note that the definitions of parameters $\eta, \zeta$ and $\kappa$ are different from that of the IS. Eqs.(29) and (30) have similar aspects, in particular, if we include the terms which violate the Curie principle and introduce the 3 different relaxation times. However, our equation is not a special case of the IS. In the IS, the projection operators, which are necessary to satisfy the orthogonality conditions (17) and (18), are included in the differential equations themselves. Thus, it is not possible to derive our equations from the IS. For example, in our theory, we can write down the differential equation of the heat conduction $\nu^\mu$ by using Eq. (29) as follows,

$$\nu^\mu = -\kappa P^{\mu\nu} \partial_\nu \alpha - \tau_R \frac{d\nu^\mu}{d\tau} + \frac{dP^{\mu\nu}}{d\tau} \tilde{\nu}_\nu.$$
The last term of the above equation do not appear in the IS.

In practice, the differential equations for the irreversible currents can be solved together with the equations of motion of the usual hydrodynamic variables in a coupled way. In the case of the IS, due to the presence of projection operators, these irreversible current should be determined simultaneously with the equations for acceleration of the fluid. Thus, in the general 3D case, \((14 \times 14)\) matrix inversion is required in each time step. In our case, due to the memory effect integral, the equation for the acceleration of fluid can be determined only from the past values of the irreversible currents, so that we need the inversion of at most \((3 \times 3)\) matrix for the acceleration of the fluid. The time derivatives for the irreversible currents are decoupled from the normal hydrodynamic variables.

In spite of these differences, our equations are found to be still hyperbolic in the linearized form. When we consider the propagation of small perturbations on the homogeneous and static background, the projection operator turns out to be a constant matrix. Therefore we can easily see that our linearized equation of motion has the same structure as the IS with \(\alpha_0 = \alpha_1 = 0\). Thus the speed of pulse propagation is finite as discussed by Hiscock-Lindblom \[35, 36, 37\].

In our case, we can explicit the space component of the four-divergence of the shear tensor \(\pi^\mu\nu\) can be written as

\[
\partial_\mu \pi^{\mu i} = -\sigma \frac{d}{d\tau} \left[ u_\alpha \tilde{\pi}^{(\alpha i)} + \left\{ \frac{1}{\gamma} u_\alpha \tilde{\pi}^{(\alpha 0)} - \frac{2}{3} \left( u_\alpha u_\beta \tilde{\pi}^{(\alpha \beta)} \right) \right\} \right] + d\pi^{(\mu i)}(\tau) \partial_\mu \tau - \nabla \cdot (u^i u_\alpha (\tilde{\pi}^{\alpha} - \tilde{\pi}^{(\alpha 0)}) \vec{v}) + \frac{1}{3} \partial^i (P_{\alpha \beta} \tilde{\pi}^{(\alpha \beta)}) + \frac{1}{3} \sigma \frac{d}{d\tau} u^i \tilde{\pi}_i^{\alpha} , \tag{31}
\]

where we have introduced the symmetric tensor,

\[
\tilde{\pi}^{(\alpha \beta)} = \frac{1}{2} (\tilde{\pi}^{\alpha \beta} + \tilde{\pi}^{\beta \alpha}) ,
\]

and used the three-vector notation,

\[
\tilde{\pi}^\alpha = \begin{pmatrix}
\tilde{\pi}^{(\alpha 1)} \\
\tilde{\pi}^{(\alpha 2)} \\
\tilde{\pi}^{(\alpha 3)} 
\end{pmatrix} .
\]

Furthermore, for the sake of later convenience we have introduced the “reference density” \(\sigma = \sigma (t, \vec{r})\) defined by

\[
\frac{1}{\sigma} \frac{d}{d\tau} \sigma = -\partial_\mu u^\mu . \tag{32}
\]
This will play an important role for the smoothed particle hydrodynamics (SPH) formulation \[21, 22\]. Note that, in general for any function \( f = f(t, \vec{r}) \) we have the identity,

\[
\partial_\mu (f u^\mu) = \sigma \frac{d}{d\tau} \left( \frac{f}{\sigma} \right).
\] (33)

The equation of motion finally has the form,

\[
d\frac{1}{\sigma} \left[ \left\{ \varepsilon + p + \Pi - \frac{1}{\gamma} u_\alpha \tilde{\pi}^{(\alpha \ 0)} + \frac{1}{3} \tilde{\tau}^\alpha + \frac{2}{3} (u_\alpha u_\beta \tilde{\pi}^{(\alpha \beta)}) \right\} u^i - u_\alpha \tilde{\pi}^{(\alpha \ i)} \right] = -\frac{1}{\sigma} \partial_i \left( p + \Pi + \frac{1}{3} (P^{\alpha \beta} \tilde{\pi}^{(\alpha \beta)}) \right) - \frac{1}{\sigma} \frac{d \tilde{\pi}^{(\mu \ i)}}{d\tau} \partial_\mu \tau + \frac{1}{\sigma} \nabla \cdot \left( u^i u_\alpha \left( \tilde{\pi}^\alpha - \tilde{\pi}^{(\alpha \ 0)} \tilde{v} \right) \right). \] (34)

Note that the right-hand side of the above equation does not contain the acceleration. From this equation, we can determine the time derivative of \( u^i, i = 1, 2, 3 \).

The time component is equivalent to the energy (entropy) equation,

\[
\partial_\mu (s u^\mu) = \frac{1}{T} (-P^{\mu \nu} \Pi + \pi^{\mu \nu}) \partial_\mu u_\nu + \alpha \partial_\mu \nu^\mu.
\] (35)

so that using Eq. (33), we have

\[
\frac{d}{d\tau} \left( \frac{s}{\sigma} \right) = \frac{1}{s T} (-P^{\mu \nu} \Pi + \pi^{\mu \nu}) \partial_\mu u_\nu + \alpha \partial_\mu \nu^\mu. \] (36)

The conservation of net baryon number is written as

\[
\frac{d}{d\tau} \left( \frac{n}{\sigma} \right) = -\frac{1}{\sigma} \partial_\alpha \nu^\alpha.
\] (37)

Eqs. (32,33,36,37) specify the time evolution of \( u^i, s, n \) and \( \sigma \). These equations constitute a closed system with the use of the equation of state and thermodynamic relations, \( \mu = \partial \varepsilon / \partial n \) and \( T = \partial \varepsilon / \partial s \), and also the integral expressions, Eq. (21) for \( \Pi, \tilde{\pi}^{\mu \nu} \) and \( \tilde{\nu}^\mu \) (and their projected tensors, \( \pi^{\mu \nu} \) and \( \nu^\mu \)).

From the practical point of view, our approach has several advantages to the IS in addition to its simple physical structure. Although the IS invokes the information from the kinetic theory, it is not probable that we can determine the parameters from the relativistic heavy ion processes quantitatively at the present moment. From the theoretical point of view, one could try to fix these parameters by using the Boltzmann equation approach to QCD. However, the Boltzmann equation is applicable to describe the behavior of a gas and if the QGP is a strongly interacting fluid, then such an approach will fail. Rather, we prefer to keep the simplest physical structure of the Navier-Stokes theory preserving causality.
To accomplish this, we simply introduced the memory effect characterized by the relaxation time. All irreversible currents are then expressed by Eq. (24). We only need the past values of the independent variables of the usual ideal fluid dynamics to advance in time. The integral expressions (24) are easy to be evaluated when we use the Lagrangian coordinate system. Because of the simple form of the dissipative terms, it is straightforward to incorporate these equations to the realistic full three-dimensional hydro-code such as SPheRIO [21, 22].

Until now, we have considered that the relaxation time $\tau_R$ is constant. However in practical problems, it is a function of thermodynamical variables. Then the memory function should be generalized as

$$G(\tau, \tau') \rightarrow \frac{1}{\tau_R(\tau')} e^{-\int_{\tau}^{\tau'} \frac{1}{\tau_R(\tau''')} \mathrm{d}\tau'''}.$$ (38)

Even in this case, Eqs. (24) and (34) are still valid.

IV. ENTROPY PRODUCTION

It should be emphasized that our theory is not a simplified version of the IS but there exists an essential difference for the treatment of the entropy production term. The IS requires the general algebraic form of the non-negative definite expression for entropy production following non-equilibrium thermodynamics. In our approach, we have relaxed this condition, that is, the expression Eq. (21) for the entropy production

$$\frac{1}{T} (-P_{\mu\nu}\Pi + \pi_{\mu\nu}) \partial_\mu u_\nu - \nu^\mu \partial_\mu \alpha \geq 0,$$

does not guarantee algebraically owing to the non-locality in time contained in $\Pi$, $\pi_{\mu\nu}$ and $\nu^\mu$ through Eq. (24). This might seem to be dangerous. However, strictly speaking, the increase of entropy is essentially a concept in the equilibrium thermodynamics and the requirement of positiveness should apply only to thermal equilibrium states. As a matter of fact, it was recently shown that the entropy absorption process can occur in the non-equilibrium processes of mesoscopic systems [38]. In our approach, we are dealing with a fluid element which is out of equilibrium, interacting with the neighboring elements. Therefore, within the relaxation time, its entropy content may increase or decrease depending on the dynamics and its time scales. Thus, the requirement of the algebraic positive definiteness irrespective of any field configuration seems to be too restrictive. The requirement of non-negative entropy production may be relaxed for far from equilibrium states. In our case, apart from the
projection operators, the expression for entropy production has the form

\[ Q(\tau) = f(\tau) \frac{1}{\tau_R} \int^{\tau} d\tau' e^{-(\tau-\tau')/\tau_R} f(\tau') , \quad (39) \]

where \( f \) is one of \( \partial_\mu u_\nu \) or \( \partial_\mu \alpha \). For small \( \tau_R \), we may expand \( f(\tau') \) near \( \tau \),

\[ f(\tau') = f(\tau) - (\tau - \tau') \frac{df(\tau)}{d\tau} + \cdots , \quad (40) \]

and we have

\[ Q(\tau) = f(\tau) \left[ f(\tau) - \tau_R \frac{df(\tau)}{d\tau} + O(\tau_R^2) \right] . \quad (41) \]

Thus, as far as

\[ \left| \tau_R \frac{df(\tau)}{d\tau} \right| < |f(\tau)| , \quad (42) \]

the positiveness of the entropy is ensured. The l.h.s. of the above equation is the amount of variation of \( f(\tau) \) within a small time interval \( \tau_R \). Thus, the above condition shows that if the time evolution of the system is not too violent (the change of field values within the relaxation time is less than its value), then the local entropy production is not negative. For the example discussed below, we can show explicitly the positive definiteness of entropy production on our formulation.

In the above, we considered the relaxation time as constant just for the illustration. The similar conclusion can be derived when the variation of the relaxation time is not so violent.

V. ONE-DIMENSIONAL SCALING SOLUTION

To see how the above scheme works, let us apply it to the one dimensional scaling solution of the Bjorken model. This has been studied already in the framework of the IS \[23, 24, 25,\]...
The components of the irreversible currents is written down explicitly as

\[ \Pi (\tau) = - \int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \frac{\zeta}{\tau} + \tau_R(\tau_0) G(\tau, \tau_0) \Pi(\tau_0), \quad (43) \]

\[ \Omega (\tau) = - \int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \frac{\eta}{\tau} + \tau_R(\tau_0) G(\tau, \tau_0) \Omega(\tau_0), \quad (44) \]

\[ \Phi (\tau) = - \int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \kappa \frac{d\alpha}{d\tau} + \tau_R(\tau_0) G(\tau, \tau_0) \Phi(\tau_0), \quad (45) \]

\[ \left( \tilde{\pi}_{\mu\nu} \right) = - \begin{pmatrix} - \sinh^2 y & \sinh y \cosh y \\ \sinh y \cosh y & - \cosh^2 y \end{pmatrix} \Omega(\tau), \quad (46) \]

\[ \left( \tilde{\nu}_\mu \right) = \begin{pmatrix} \cosh y \\ - \sinh y \end{pmatrix} \Phi(\tau), \quad (47) \]

where we have used the hyperbolic variables,

\[ t = \tau \cosh y, \quad x = \tau \sinh y, \]

and used the scaling ansatz in \( y \) (that is, there is no \( y \) dependence in thermodynamic variables). \( \Pi(\tau_0), \Omega(\tau_0) \) and \( \Phi(\tau_0) \) are initial values for \( \Pi(\tau), \Omega(\tau) \) and \( \Phi(\tau) \). We obtain

\[ \pi^{\mu\nu} = P^{\mu\alpha\nu\beta} \tilde{\pi}_{\alpha\beta} = - \frac{2\Omega}{3} P^{\mu\nu}, \quad (48) \]

\[ \nu^{\mu} = P^{\mu\nu} \tilde{\nu}_{\nu} = 0. \quad (49) \]

As we see, in the one-dimensional case, if \( \zeta \) and \( \eta \) are proportional as functions of thermodynamic quantities such as \( T \) and \( \mu \), then the bulk and shear viscosity terms are not independent, and

\[ \Pi \propto \Omega. \]

However, when \( \zeta \) and \( \eta \) have, in general, different dependence on the thermodynamic quantities, the two viscosities act differently.

The time component of the divergence of \( T^{\mu\nu} \) gives

\[ \frac{d}{d\tau} \epsilon(\tau) + \frac{\epsilon + p + \Pi}{\tau} + \frac{2 \Omega}{3 \tau} = 0. \quad (50) \]

The equation for the space component is automatically satisfied by the scaling ansatz showing its consistency. The entropy production rate is calculated to be

\[ \partial_\mu(su^\mu - \alpha \nu^\mu) = - \frac{1}{T} \frac{1}{\tau} \left( \Pi + \frac{2}{3} \Omega \right). \quad (51) \]

Since \( \Pi \) and \( \Omega \) are negative definite, the entropy production is positive definite.
A. Solutions

When $\zeta = \zeta_0$, $\eta = \eta_0$ and $\tau_R$ are constant, then we can obtain analytic expression for the proper energy density. We obtain

$$\Omega = \frac{\eta_0}{\zeta_0} \Pi = -\frac{\eta_0}{\tau_R} e^{-\frac{\tau}{\tau_R}} \left[ \text{Ei} \left( -\frac{\tau}{\tau_R} \right) - \text{Ei} \left( -\frac{\tau_0}{\tau_R} \right) + E_0 \right], \quad (52)$$

where

$$\text{Ei} (-x) = \int_{x}^{\infty} \frac{e^{-t}}{t} dt,$$

is the exponential integral, and $E_0$ is a constant which should be determined from the initial condition for $\Pi$ (or $\Omega$). For relativistic ideal gas,

$$P = \frac{\varepsilon}{3},$$

we get

$$\frac{d\varepsilon}{d\tau} + \frac{4\varepsilon}{3\tau} + \left( \frac{2\eta_0}{3\zeta_0} + 1 \right) \frac{\Pi (\tau)}{\tau} = 0, \quad (53)$$

so that for $E_0 = 0$,

$$\varepsilon (\tau) = \varepsilon_0 \left( \frac{\tau_0}{\tau} \right)^{4/3} \left[ 1 - \frac{1 - 4\eta_0/3\zeta_0}{\varepsilon_0^{4/3}} \int_{\tau_0}^{\tau} dt t^{1/3} \Pi (t) \right], \quad (54)$$

where the integral containing the exponential function can still be evaluated analytically. The temperature is determined from the energy density as

$$\varepsilon = \sigma_{SB} T^4,$$

where $\sigma_{SB}$ is the Stephan-Boltzmann constant.

On the other hand, a typical estimate from the kinetic theory shows that the shear viscosity $\eta$ is proportional to the entropy density $s$, $\eta = bs$, where $b$ is a constant. Following Ref. [26], we choose $b = 1.1$. Furthermore, we use the relaxation time

$$\tau_R = \frac{3\eta IS}{4p} = \frac{3\eta}{8p}, \quad (55)$$

Here, it should be noted that our definition of $\eta$ is twice of other papers [23, 24, 25, 26, 27]. The effect of the bulk viscosity has not been discussed in previous papers. We analogously assume that the bulk viscosity has the similar $s$ dependence, $\zeta = as$. For a baryon free relativistic gas, $s$ is related as the energy density as

$$s = C\varepsilon^{3/4},$$

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so that the equation for the energy density becomes

$$
\tau \frac{d^2 \varepsilon}{d\tau^2} + \left( \frac{7}{3} + \frac{\tau}{\tau_R} \right) \frac{d\varepsilon}{d\tau} + \frac{1}{\tau_R} \left( \frac{4\varepsilon}{3} - C' \varepsilon^{3/4} \frac{1}{\tau} \right) = 0,
$$

(56)

were

$$
C' = \left( a + \frac{2}{3} b \right) C.
$$

The above equation is the same as the equation derived in Ref. [26]. It should be noted that this coincidence is due to the specific property of this particular model. In Ref. [26], the above equation is obtained under the assumption of no acceleration condition,

$$
\frac{d}{d\tau} u^\mu = 0,
$$

which is automatically satisfied in the scaling solution. It should be noted that in the IS, this condition is not satisfied in general, and one has to solve the original equation of the IS without using the above condition. However, our theory does not require such a condition at all to be applied. Eq.(56) can be solved for the initial condition,

$$
\varepsilon = \varepsilon (\tau_0),
$$

and

$$
\left. \frac{d\varepsilon}{d\tau} \right|_{\tau = \tau_0} = -\frac{1}{\tau_0} \left[ \frac{4}{3} \varepsilon_0 + \Pi (\tau_0) + \frac{2}{3} \Omega (\tau_0) \right].
$$

(57)

Now, we show our numerical results. To compare the previous works, we ignore the bulk viscosity. In Fig. 1, we show the energy density \( \varepsilon \) obtained by solving Eq.(56) as function of proper time \( \tau \). As the initial condition, we set \( \varepsilon (\tau_0) = 1 \text{ GeV/fm}^3 \), \( \Pi (\tau_0) = \Omega (\tau_0) = 0 \) at the initial proper time \( \tau_0 = 0.1 \text{ fm/c} \). The first two lines from the top represents the results of the LL. The next two lines shows the results of our theory. The last line is the result of ideal hydrodynamics. For the solid lines, we calculated with the viscosity and relaxation time which depend on temperature (57). Initially, the effect of viscosity is small because of the memory effect, the behavior of our theory is similar to that of ideal hydrodynamics. After the time larger than the relaxation time, the memory effect is not effective anymore and the behavior is similar to the result of the LL. As we have mentioned, the behavior of our theory is the same as the result obtained in Ref. [26] in this case. For the dashed lines, we calculated with the constant viscosity and relaxation time, \( \eta = \eta (\varepsilon_0) \) and \( \tau_R = \tau_R (\varepsilon_0) \). In this case, the viscosity is constant so that the heat production stays longer and has a smaller slope as function of time asymptotically.
Sometimes the emergence of the initial heat-up in the LL (the dashed curve in Fig.1) is interpreted as an intrinsic problem of the first order theory. However, such behavior can also appear even in the second order theory. In Fig. 2, we set \( \Pi(\tau_0) = \zeta(\tau_0)/\tau_0 \) and \( \Omega(\tau_0) = \eta(\tau_0)/\tau_0 \) as the initial conditions. In particular, the initial heat-up also appear in the second order depending on the initial condition for the irreversible currents (see Fig. 2). Therefore, this heat-up is not the problem of the first order theory but rather the specific property of the scaling ansatz. This was already pointed out by Muronga. The physical reason for this heat-up is due to the use of the Bjorken solution for the velocity field. In this case, the system acts as if an external force is applied to keep the velocity field as a given function of \( \tau \). Thus, depending on the relative intensity of the viscous terms compared to the pressure, the external work converted to the local heat production can overcome the temperature decrease due to the expansion.

VI. SUMMARY AND CONCLUDING REMARKS

One of the most important questions in the relativistic heavy ion physics is to determine the effect of dissipation in flow dynamics of plasma of quarks and gluons. However, a consistent calculation of relativistic dissipative hydrodynamics is not trivial at all. Some important questions such as propagation of shock wave in the hot QCD matter require a careful treatment of the dissipative processes. The Israel-Stewart theory and the divergence type theory contain many parameters difficult to be determined from QCD point of view and also it is technically difficult to be implemented on practical calculations in its full form [26, 27, 39, 40].

From the practical point of view, it is not desirable to deal with many of such parameters, since the hydrodynamical approach to the relativistic heavy ion reaction processes itself has already many uncertainties [4]. In this sense, the IS is not easy to apply at the present stage of the analysis of heavy ion physics data.

Furthermore, when we deal with numerical solutions, we still encounter ambiguities. Because dissipations and relaxation times, required to reproduce the bulk properties of fluid, are not necessarily the same as those determined from the microscopic theory such as Lattice QCD calculations. In a practical numerical calculation, the truncation of hydrodynamic degrees of freedom to the discredited variables introduces a natural cut-off frequencies, and
the elimination of those degrees of freedom such as small scale vorticities or turbulences will appear as effective viscosities [41]. Furthermore, the standard numerical method for dealing with shock phenomena is to introduce the pseudo-viscosity, first invented by von Neumann and Richtmyer [42]. It is basically the bulk viscosity depending on the size of the hydro cell. For relativistic shock motion, we need to incorporate such mechanism to the hydrodynamical code. Considering these aspects, it is desirable to study the effect of dissipation with the minimum number of parameters which characterize the physical processes involved and is still consistent with the framework of relativity.

In this paper, we proposed an alternative approach to this question, different from the IS. We start from the physical analysis of the irreversible currents according to the Landau-Lifshitz theory. Then, the irreversible currents are given by integral expressions which take into account the relaxation time. In this way, causality is recovered and at the same time a simple physical structure of the LL is preserved. In our approach, only one additional parameter was introduced, the relaxation time, $\tau_R$. The resulting equation of motion then becomes hyperbolic and causality can be restored [8]. Naturally, causality depends on the choice of the values of the parameters including the relaxation time.

More specifically, we verified that the linearized equation of motion for small perturbations in the homogeneous, static background coincides with Hiscock-Lindblom [35, 36, 37] except for the coupling among the different irreversible currents. These couplings are not included in our theory considering the Curie principle. Of course the Curie principle is believed to be valid in the regime of the first order theory and in the second order regime these couplings might be present. However, the existence of the Curie principle may imply that these couplings are small compared with the direct terms.

In addition to its simplicity compared to the IS, our formalism has several practical advantages. First, the number of independent variables is kept the same as those of the ideal hydrodynamics. The irreversible currents are expressed explicitly as memory integrals of these independent variables. In extended (irreversible) thermodynamics, the currents are treated as extended thermodynamic variables and thermodynamic relations are modified [9, 43]. Thus, in the IS, we should take the modification of thermodynamic relations into account but in our case the usual thermodynamical relations should be used. Second, in the IS, the projection operators enters in the equations of motion in a complicated way so it is not trivial to extract the standard form to apply the conventional method of the time
integration. In our case, this does not occur. Instead, we need just some additional integrals associated with the local comoving coordinates. Finally, our simplified structure permits to include all of the irreversible currents, such as bulk and shear viscosities and thermal conduction at the same time without any difficulties.

The essential difference of our formalism from the IS is the expression for entropy production. In the IS, the entropy production is required to be positive definite algebraically. Thus, the integral form like our formulation is not possible even neglecting some coupling terms. We relaxed this condition, that is, the positiveness of entropy production is required only for the hydrodynamical motion with time scales longer than the relaxation time. For extremely violent change of variables, the instantaneous entropy for a hydrodynamic cell would not necessarily be positive definite.

We have applied our theory to the case of the one-dimensional scaling solution of Bjorken and obtained the analogous behavior of previous analysis. In this case we can prove explicitly the positiveness of entropy production. We showed the time evolution of the temperature. As expected, our theory gives the same result of Ref. [26], because the no-acceleration condition used in Ref. [26] is automatically satisfied in this model. Note that our theory is applicable to more general case where the acceleration is important.

The transport coefficients contained in relativistic dissipative hydrodynamics should in principle be calculated from QCD. In the first order theory, it is known that the transports coefficients can be calculated by the Kubo formula [44]. However, this formula does not gives the transport coefficients in the second order theory, as is shown in Ref. [32] explicitly and the corresponding corrections should be evaluated.

Our theory is particularly adequate to be applied to the hydro-code such as SPheRIO which is based on the Lagrangian coordinate system [21, 22]. Implementation of the present theory to the full three-dimensional hydrodynamics is now in progress.

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FIG. 1: The time evolution of the energy density. The dashed curves correspond to the calculations with the constant viscosity and relaxation time. The first two lines from the top represent the results of the LL. Next two lines shows the results of our theory. The last line is the result of ideal hydrodynamics.
FIG. 2: The time evolution of energy density with the different initial conditions from Fig. 1. The dashed and short dashed lines represent the result of the LL and our theory, respectively. For comparison, our result of Fig. 1 is shown, again (ideal $T^\mu_\nu(\tau_0)$). The last line from the top is the result of ideal hydrodynamics. In this case, the energy heat-up is observed even in our theory.