Odd-even rule for zero-bias tunneling conductance in coupled Majorana wire arrays

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A semiconducting nanowire with strong Rashba coupling and in proximity of a superconductor hosts Majorana edge modes. An array of such nanowires with inter-wire coupling gives an approximate description of a two-dimensional topological superconductor, where depending on the strength of the magnetic field and the chemical potential, a rich phase diagram hosting trivial and different types of non-trivial phases can be achieved. Here, we theoretically consider such a two-dimensional assembly of spin-orbit coupled superconducting nanowires and calculate the collective tunneling conductance between normal electrodes and the wires in the topological regime. When the number of wires in the assembly is \(N\), as a consequence of the way the Majorana bonding and anti-bonding states form, we find that \(N\) conductance peaks symmetric about the bias \(V = 0\) appear, for even \(N\). When \(N\) is odd, a ZBCP also appears. Such an assembly can be realized by standard nanofabrication techniques where individual nanowires can be turned \(ON\) or \(OFF\) by using mechanical switch (or local top gating) to make \(N\) either even or odd – thereby switching the ZBCP \(OFF\) or \(ON\), respectively. Hence, our results can be used to realize and detect topological superconductivity efficiently, unambiguously and in a controlled manner.

In the mean field approximation, these two Hamiltonians

\[
H = H^\parallel + H^\perp
\]

where \(H^\parallel\) is the Hamiltonian describing intra-wire dynamics while and \(H^\perp\) describes the inter-wire dynamics. In the mean field approximation, these two Hamiltonians can be expressed as:

\[
H^\parallel = -t_x \sum_{i,\delta,\sigma} [c_i^{\dagger} c_{i+\delta,\sigma} c_{i,\sigma}] - \mu \sum_{i,\sigma} c_i^{\dagger} c_{i,\sigma} - V_z \sum_i [c_i^{\dagger} c_{i+1,\sigma} + h.c.] + \Delta \sum_i [c_i^{\dagger} c_{i+1,\sigma} c_{i+1,\sigma}] + \frac{i\alpha}{2} \sum_{i,\delta} [c_i^{\dagger} \sigma_y c_{i+\delta,\sigma} + h.c.]
\]

\[
H^\perp = -t_z \sum_{i,\delta,\sigma} [c_i^{\dagger} c_{i+\delta,\sigma} c_{i,\sigma}] + \frac{i\beta}{2} \sum_{i,\delta} [c_i^{\dagger} \sigma_x c_{i+\delta,\sigma} + h.c.]
\]
where $c_{i\sigma}^\dagger(c_{i\sigma})$ is the fermionic operator that creates (annihilates) a particle at lattice site $i = \{ix, iy\}$ with spin $\sigma$. $\delta_x$ and $\delta_y$ represent the nearest neighbor lattice vectors in $x$ and $y$ directions respectively. $t_x$ gives the hopping matrix element along the wires (chosen to be along $x$), $\alpha$ is the Rashba spin-orbit coupling, $\mu$ is the chemical potential, $t_\perp$ is the weak inter-wire hopping matrix element, $\beta$ is the weak spin-orbit coupling term between the wires in transverse direction and that is linked to inter-wire hopping. It is understood that the effective inter-wire coupling between the wires can be significant only when the distance between the wires is less than a coherence length in the superconductor underneath.

The Hamiltonian (1) can be written in the momentum space as:

$$H = \frac{1}{2} \sum_{k} \Psi_k^\dagger h(k) \Psi_k : k = (k_x, k_y)$$

$$h(k) = \epsilon_k \tau_z + \alpha \sin(k_x) \tau_y \sigma_y - V_z \sigma_z + \Delta \tau_z + \beta \sin(k_y) \tau_x \sigma_x$$

where $\epsilon_k = -2t_x \cos(k_x) - 2t_\perp \cos(k_y) - \mu$, $\sigma$ and $\tau$ are Pauli matrices in spin space and particle-hole basis respectively. The uncoupled wires ($t_\perp = 0; \beta = 0$) can be tuned into two phases: (a) trivial phase, when $V_z < \sqrt{\Delta^2 + (\mu + 2t_x)^2}$ and (b) topological phase, when $V_z > \sqrt{\Delta^2 + (\mu + 2t_x)^2}$. Here, $\mu + 2t_x$ is the chemical potential measured w.r.t bottom of the conduction band. We have chosen $t_x = t$ as the energy unit for our calculations.

Now, the Hamiltonian can be written as:

$$H = \sum_j \int dk_x [\epsilon_j(k_x) \Psi_{k_x,j}^\dagger \Psi_{k_x,j} + \alpha \sin(k_x) \Psi_{k_x,j}^\dagger \sigma_y \Psi_{k_x,j} \tau_y]$$

$$-V_z \Psi_{k_x,j}^\dagger \sigma_z \Psi_{k_x,j} + \Delta \Psi_{k_x,j}^\dagger (i \sigma_y) \Psi_{-k_x,j} + h.c.]$$

$$+ \sum_j \int dk_x [-t_\perp \Psi_{k_x,j}^\dagger \Psi_{k_x,j+1} + i \beta \Psi_{k_x,j}^\dagger \sigma_x \Psi_{k_x,j+1} + h.c.]$$

where $\epsilon_j(k_x) = -2t_x \cos(k_x) - \mu$.

From this, the spectrum of the system is obtained as:

$$E^2 = V_z^2 + \Delta^2 + \epsilon^2 + |\gamma_k|^2 \pm 2 \sqrt{(V_z \Delta)^2 + (V_z^2 + |\gamma_k|^2) \epsilon^2_k}$$

where $\gamma_k = \alpha \sin(k_x) i + \beta \sin(k_y)$.

The energy dispersion as a function of $k_x$ is calculated for the trivial case when $V_z = 0$ for an assembly of five uncoupled (Figure 1 (b)) and weakly coupled (Figure 1 (d)) wires. For $V_z = 0.85t$ the system is in topological regime. In this regime, the dispersion of five uncoupled (Figure 1 (c)) and weakly coupled wires (Figure 1(e)) are also shown. Periodic boundary conditions along $x$ direction and open boundary conditions along $y$ direction are enforced that makes the momentum along $x$ a good quantum number. The results presented in Figure 1 are consistent with the earlier calculations on coupled Majorana wires [31].

In Figure 2, we show the variation of the energy spectra with the Zeeman field $V_z$. Figure 2 (a) and 2 (b) show the variation for an assembly of 3 and 4 uncoupled wires respectively while Figure 2 (c) and 2 (d) show the variation for an assembly of 3 and 4 coupled wires respectively, in the topological regime. When the wires don’t have any inter-wire coupling ($t_\perp = 0; \beta = 0$), as expected, it is seen that for $V_z > \Delta$ ($\Delta = 0.6t$ in this case), we obtain zero energy states indicating the topological regime. Now, when there is non-zero interwire coupling ($t_\perp \neq 0; \beta \neq 0$), the spectra get modified due to mixing of states and the topological regime is achieved for a higher value of Zeeman field ($V_z > 0.75t$ in this case). Furthermore, it is also observed that for three wires, states at $E = 0$ appear along with states that emerge at finite energy near zero energy symmetric about $E = 0$. On the other hand, for a 4-wire assembly no zero energy states
can be seen but 4 peaks around $E = 0$ are seen. We will later see that these lead to odd-even rule in the conductance which is the focal theme of this article.

Now, we design a thought tunneling experiment to investigate the possible role of inter-wire coupling in transport through the aforementioned Majorana wire assembly. A schematic representation of the said set-up is shown in Figure 3 (a). A normal lead is attached to one end of each semi-infinite semiconducting nanowire with strong Rashba coupling proximitized by s-wave superconductor in the presence of an external applied magnetic field. The interface between the metal electrode and each wire falls in the tunneling regime of transport. The normal lead has the same Hamiltonian (equation 2) as the nanowire except for the superconducting term ($\Delta$). Also, chemical potential of the normal lead is chosen greater than superconducting gap so that the normal lead remains topologically trivial. The tunnel barrier at the interface of each N-S junction is modelled by adding an additional onsite energy of strength $10t$ on one end (left) of each wire. We have assumed semi-infinite nanowires for our calculations to exclude finite size effects [33].

The zero temperature tunneling conductance of the array is calculated using $S$-matrix method. By computing the reflection matrix at the N-S junction, tunneling conductance can be found; where the reflection matrix ($r$) is expressed in terms of electron and hole scattering channels at energy $E$ as:

$$r = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix} \quad (7)$$

where $r_{ee}$ ($r_{eh}$) refers to the normal (Andreev) reflection submatrix. For $N$ conducting channels in the lead the differential conductance can be evaluated using the
The most important feature here is the emergence of a Majorana bound state which contributes a conductance of $2e^2/h$.

We have employed KWANT, a numerical transport package in Python, to calculate the components of the reflection matrix.

Figure 3 (b) shows the differential conductance vs applied bias $V$, when there is no inter-wire coupling in the non-topological (trivial) regime. The differential conductance resembles the zero temperature density of states of a conventional superconductor with coherence peaks appearing at induced superconducting gap energy $\Delta$. As expected, the magnitude of conductance at $\pm \Delta$ increases monotonically with increase in number of wires. Unlike the trivial uncoupled case, enhancement of magnitude of conductance at $\pm \Delta$ is not monotonic in number of wires. Figure 3 (c) depicts the differential conductance when there is inter-wire coupling in the non-topological (trivial) case for $N$ number of wires. The differential conductance peaks, symmetric about bias $V$, increases in magnitude with increase in number of wires. Figure 3 (d) shows the plot of conductance vs applied bias $V$ for uncoupled topological phase where each Majorana bound state contributes a conductance of $2e^2/h$ to the total differential conductance. Upto this part, the number of wires only contributes to the overall scaling of the absolute magnitude of the differential conductance. However, more interesting physics emerges when weak coupling between the wires is also introduced in the topological regime.

As it can be seen in Figure 4, in the topological regime, turning on weak coupling between the wires has resulted in $N$ conductance peaks, symmetric about bias $V=0$. The most important feature here is the emergence of a ZBCP when $N$ is odd (Figure 4 (b), 4 (d), 4 (f) and 4 (h)). On the other hand, for even values of $N$ (Figure 4 (a), 4 (c), 4 (e), 4 (g) and 4 (i)), ZBCP doesn’t appear. This is due to Majoranas at the edges of wires hybridizing into bonding and anti-bonding orbitals when coupled. While, in even number of cases all the Majoranas are coupled pairwise leading to only satellite peaks in the conductance (around $V=0$), in the case of odd number of wires one Majorana is left unpaired that contributes a conductance of $2e^2/h$ at zero bias.

It has been earlier proposed that ZBCP in tunneling experiments can be a smoking gun signature of Majorana modes. However, as it is also known, a ZBCP can originate due to other possible factors as well. That makes ZBCP based detection of Majorana modes somewhat ambiguous. In this context, our proposed experimental scheme is very important. In this scheme, the confirmation of Majorana is not based on the appearance of ZBCP alone, but our calculations provide a detailed scheme based on the number of transport-active nanowires in a single device where depending on whether the number of wires is odd or even, the ZBCP can be switched ON or switched OFF respectively, in a controlled fashion. Hence, the proposed scheme would provide direct, unambiguous signature of Majorana bound states and topological superconductivity.

We thank Subhro Bhattacharjee and Tamnoy Das for fruitful discussions. DR thanks DST INSPIRE for financial support. GS acknowledges financial support from Swarnajayanti Fellowship awarded by the Department of Science and Technology, Govt. of India (grant number: DST/SJF/PSA-01/2015-16) to work on the topological phases of matter.

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