Fidelity balance in quantum operations

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I derive a tight bound between the quality of estimating the state of a single copy of a $d$-level system, and the degree the initial state has to be altered in course of this procedure. This result provides a complete analytical description of the quantum mechanical trade-off between the information gain and the quantum state disturbance expressed in terms of mean fidelities. I also discuss consequences of this bound for quantum teleportation using nonmaximally entangled states.

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As a general rule, the more information is obtained from an operation on a quantum system, the more its state has to be altered. This heuristic statement was first exemplified by the Heisenberg microscope gedanken-experiment [1], where the spatial resolution of the apparatus was shown to scale inversely with the uncertainty of the momentum transferred during the observation. Presently, the disturbance caused by the information gain has become an issue of practical significance, as it underlies the security of quantum key distribution [2].

The balance between the information gain and the state disturbance attracts currently a lot of interest, particularly in the context of quantum cryptography [3]. Information theory provides a selection of concepts to quantify both the information gain and the state disturbance expressed in terms of the following inequality:

$$\sqrt{F - \frac{1}{d+1}} \leq \sqrt{G - \frac{1}{d+1}} + \sqrt{(d-1)\left(\frac{2}{d+1} - G\right)}.$$  

(1)

I also show that this inequality cannot be further improved, i.e. there exist quantum operations saturating the equality sign.

The most general strategy that can be applied to the particle has the form of a trace-preserving operation described by a set of operators $A_r$, where $r = 1, \ldots, N$. These operators satisfy the completeness relation:

$$\sum_{r=1}^{N} A_r^\dagger A_r = \mathbb{1}.$$  

(2)

The classical information gained from this operation is given by the index $r$, which is subsequently used to estimate the initial state of the particle. The outcome $r$ of the operation performed on a state $|\psi\rangle$ is obtained with the probability $\langle \psi|A_r^\dagger A_r|\psi\rangle$. This corresponds to the following conditional transformation of the quantum state:

$$|\psi\rangle \rightarrow \frac{A_r|\psi\rangle}{\sqrt{\langle \psi|A_r^\dagger A_r|\psi\rangle}}.$$  

(3)

We shall measure the resemblance of the transformed state to the original one using the squared modulus of the scalar product, equal $|\langle \psi|A_r|\psi\rangle|^2/\langle \psi|A_r^\dagger A_r|\psi\rangle$. Summation of this expression over $r$ with the weights $\langle \psi|A_r^\dagger A_r|\psi\rangle$, and integration over all possible input states...
$|\psi\rangle$, yields the complete expression for the mean operation fidelity $F$:

$$F = \int d\psi \sum_{r=1}^{N} |\langle \psi|\hat{A}_r|\psi\rangle|^2. \quad (4)$$

Here the integral $\int d\psi$ over the space of pure states is performed using the canonical measure invariant with respect to the group unitary transformations on the state vectors of the particle.

Given the outcome $r$ of the operation, we can make a guess $|\psi_r\rangle$ what the state originally was. The quality of this guess, assuming that the initial state was $|\psi\rangle$, can be quantified with the help of the overlap $|\langle \psi_r|\psi\rangle|^2$. The mean estimation fidelity $G$ is given by the average of this expression over all outcomes $r$ with the probability distribution $\langle \psi|\hat{A}_r\hat{A}_r^\dagger|\psi\rangle$, and by integration over states $|\psi\rangle$:

$$G = \int d\psi \sum_{r=1}^{N} \langle \psi|\hat{A}_r\hat{A}_r^\dagger|\psi\rangle |\langle \psi_r|\psi\rangle|^2. \quad (5)$$

We will start derivation of the trade-off between the fidelities $F$ and $G$ by evaluating the integrals over $|\psi\rangle$. For this purpose, let us introduce in Eq. (3) two decompositions of unity in a certain orthonormal basis $|i\rangle$:

$$F = \sum_{r=1}^{N} \sum_{i,j=0}^{d-1} \langle \psi|\hat{A}_r^\dagger|\psi\rangle \langle \hat{A}_r|j\rangle \langle j|\psi\rangle$$

$$= \sum_{r=1}^{N} \sum_{i,j=0}^{d-1} \langle i|\hat{A}_r\hat{A}_r^\dagger|j\rangle$$

where by $\hat{M}_{ij}$ we have denoted the following integrals of projectors on the states $|\psi\rangle$ $|\psi\rangle$:

$$\hat{M}_{ij} = \int d\psi \langle \psi|i\rangle \langle j|\psi\rangle |\psi\rangle = \frac{1}{d(d+1)} (\delta_{ij} \hat{1} + |i\rangle \langle j|). \quad (6)$$

The second explicit form of the operators $\hat{M}_{ij}$ has been derived in Ref. 3. This formula allows us to simplify the expression for the mean operation fidelity $F$ to the form:

$$F = \frac{1}{d(d+1)} \left( \sum_{i=0}^{d-1} \sum_{r=1}^{N} \langle i|\hat{A}_r\hat{A}_r^\dagger|i\rangle + \sum_{r=1}^{N} \sum_{i=0}^{d-1} \langle i|\hat{A}_r^\dagger|i\rangle \right)^2$$

$$= \frac{1}{d(d+1)} \left( d + \sum_{r=1}^{N} \text{Tr}(\hat{A}_r^\dagger)^2 \right)^2 \quad (8)$$

Let us now consider the estimation fidelity $G$. The guess $|\psi_r\rangle$ can be represented as a result of a certain unitary transformation $\hat{U}_r$ acting on a reference state, which we will take for concreteness to be $|0\rangle$:

$$|\psi_r\rangle = \hat{U}_r|0\rangle \quad (9)$$

Using this representation, and changing the integration measure in Eq. (3) according to $|\psi\rangle \to \hat{U}_r|\psi\rangle$, we can evaluate the integral over $|\psi\rangle$:

$$G = \sum_{r=1}^{N} \int d\psi |\langle 0|\psi\rangle|^2 \langle \psi|\hat{U}_r^\dagger\hat{A}_r\hat{A}_r^\dagger\hat{U}_r|\psi\rangle$$

$$= \sum_{r=1}^{N} \text{Tr}(\hat{U}_r^\dagger\hat{A}_r\hat{A}_r^\dagger\hat{U}_r M_{00}) \quad (10)$$

Inserting the explicit form of the operator $M_{00} = (\hat{1} + |0\rangle \langle 0|)/[d(d+1)]$ yields:

$$G = \frac{1}{d(d+1)} \left( \sum_{r=1}^{N} \text{Tr}(\hat{U}_r^\dagger\hat{A}_r\hat{A}_r^\dagger\hat{U}_r) + \sum_{r=1}^{N} \langle 0|\hat{U}_r^\dagger\hat{A}_r\hat{U}_r|0\rangle \right)$$

$$= \frac{1}{d(d+1)} \left( d + \sum_{r=1}^{N} \langle \psi_r|\hat{A}_r\hat{A}_r^\dagger|\psi_r\rangle \right) \quad (11)$$

This expression provides directly a recipe for optimal assignment of guesses $|\psi_r\rangle$ to outcomes of the operation: each of the components $\langle \psi_r|\hat{A}_r\hat{A}_r^\dagger|\psi_r\rangle$ in the sum over $r$ is maximized if $|\psi_r\rangle$ is the eigenvector of $\hat{A}_r\hat{A}_r^\dagger$ corresponding to its maximum eigenvalue. Consequently, the maximum value of the mean estimation fidelity $G$ for a given operation $\{\hat{A}_r\}$ can be written as:

$$G = \frac{1}{d(d+1)} \left( d + \sum_{r=1}^{N} \|\hat{A}_r\|^2 \right) \quad (12)$$

where the operator norm is defined in the standard way:

$$\|\hat{A}_r\| = \sup_{|\varphi\rangle} \sqrt{\langle \varphi|\hat{A}_r^\dagger\hat{A}_r|\varphi\rangle}. \quad (13)$$

In order to relate the fidelities $F$ and $G$ to each other, let us consider a polar decomposition of the operators $\hat{A}_r$:

$$\hat{A}_r = \hat{V}_r \hat{D}_r \hat{W}_r \quad (14)$$

where $\hat{V}_r$ and $\hat{W}_r$ are unitary, and $\hat{D}_r$ is a semi-positive definite diagonal matrix:

$$\hat{D}_r = \sum_{i=0}^{d-1} \lambda_i |i\rangle \langle i|, \quad (15)$$

with the diagonal elements put in a decreasing order: $\lambda_0 \geq \ldots \geq \lambda_{d-1} \geq 0$. We will first show that only the diagonal matrices $\hat{D}_r$ are relevant to the trade-off. Indeed, the modulus of the trace of the matrix $\hat{A}_r$ appearing in Eq. (8) is bounded by:
The expression for the estimation fidelity written in terms of $\lambda_i^r$ takes the form:

$$G = \frac{1}{d(d+1)} \left( d + \sum_{r=1}^{N} (\lambda_i^r)^2 \right).$$

In addition, the trace of the completeness condition given in Eq. (2) yields the following constraint on $\lambda_i^r$:

$$\sum_{r=1}^{N} \sum_{i=0}^{d-1} (\lambda_i^r)^2 = d. \quad (19)$$

To complete the proof of the inequality (1), it is convenient to introduce vector notation. Let us define $d$ real vectors $v_i = (\lambda_1^i, \ldots, \lambda_d^i)$, where the index $i$ runs from 0 to $d-1$. Sums over $r$ appearing in Eqs. (17) and (18) can be written as:

$$f = \sum_{r=1}^{N} \left( \sum_{i=0}^{d-1} \lambda_i^r \right)^2 = \sum_{i,j=0}^{d-1} v_i \cdot v_j \quad (20)$$

$$g = \sum_{r=1}^{N} (\lambda_i^0)^2 = |v_0|^2 \quad (21)$$

where the dot denotes the scalar product, and $| \cdot |$ is the standard quadratic norm. The completeness condition (2) for the operation $\{\hat{A}_r\}$ written in the vector notation takes the form

$$\sum_{i=0}^{d-1} |v_i|^2 = d. \quad (22)$$

Let us now suppose that the vector $v_0$ is fixed. The estimation fidelity is then given by $G = (d + |v_0|^2)/[d(d+1)]$. What is the maximum operation fidelity $F$ that can be achieved with this constraint? The answer to this question is provided by an application of the Schwarz inequality to Eq. (20):

$$f \leq \sum_{i,j=0}^{d-1} |v_i|^2 \leq \sqrt{\sum_{i=0}^{d-1} |v_i|^2} = \sqrt{\frac{d-g}{d-1}} \quad (23)$$

We have excluded here from the sum over $i$ the norm of the vector $v_0$ which is fixed and equal to $\sqrt{d}$. The sum of the norms of the remaining vectors can be estimated using the inequality between the arithmetic and quadratic means:

$$\frac{1}{d-1} \sum_{i=1}^{d-1} |v_i|^2 \leq \frac{1}{d-1} \sum_{i=1}^{d-1} \frac{|v_i|^2}{|v_0|^2} = \sqrt{\frac{d-g}{d-1}}, \quad (24)$$

where we have evaluated the sum $\sum_{i=0}^{d-1} |v_i|^2$ using Eq. (23). Inserting this bound into Eq. (23) we finally obtain the inequality

$$f \leq \left( \sqrt{d} + \sqrt{d-1} \right)^2 \quad (25)$$

which expressed in terms of the fidelities $F$ and $G$ takes the form of Eq. (1).

The necessary and sufficient conditions for a quantum operation to reach the equality sign can be most easily formulated in the vector notation. The Schwarz inequality (23) becomes equality if all the vectors $v_0, \ldots, v_{d-1}$ are collinear. Furthermore, equation sign in Eq. (24) holds if and only if $|v_1| = \ldots = |v_{d-1}|$. It is straightforward to see that an exemplary operation satisfying these conditions for a given estimation fidelity $G = (1 + g/d)/(d + 1)$ is defined by:

$$\hat{A}_r = \sqrt{\frac{g}{d}}|r-1\rangle\langle r-1| + \sqrt{\frac{d-g}{d(d-1)}} (1 - |r-1\rangle\langle r-1|)$$

(26)

where the index $r$ runs from 1 to $d$, and the projectors $|r-1\rangle\langle r-1|$ are constructed using any orthonormal basis. This confirms the inequality (1) is indeed a tight one and cannot be further improved.

A simple transformation of Eq. (1) shows that the quantum mechanically allowed region for the fidelities $F$ and $G$ is bounded by a quadratic curve, which turns out to be a fragment of an ellipse given by the equation:

$$(F - F_0)^2 + d^2(G - G_0)^2 + 2(d-2)(F - F_0)(G - G_0) = \frac{d-1}{(d+1)^2} \quad (27)$$

with $F_0 = (d + 2)/(2d + 2)$ and $G_0 = 3/(2d + 2)$. The shape of the region for several values of $d$ is depicted in Fig. 1.
The balance between the operation and estimation fidelities derived in this Letter has interesting consequences in quantum teleportation based on nonmaximally entangled states. If two parties share a pure bipartite state of the Schmidt form $|\text{tele}\rangle = \sum_{k=0}^{d-1} \mu_k |k\rangle \otimes |k\rangle$, then the maximum teleportation fidelity attainable using this state is given by [8]:

$$F_{\text{tele}} = \frac{1 + \left(\sum_{k=0}^{d-1} \mu_k\right)^2}{d + 1}.$$  \hspace{1cm} (28)

Furthermore, for a nonmaximally entangled state the measurement performed during the teleportation protocol reveals some information on the teleported state. This information can be converted into an estimate for the initial state, whose maximum average fidelity has been shown to equal [6]:

$$G_{\text{tele}} = \frac{1 + \mu_0^2}{d + 1}.$$  \hspace{1cm} (29)

where $\mu_0$ denotes the largest Schmidt coefficient for the state $|\text{tele}\rangle$. As the procedure of teleportation can be viewed as a special case of a quantum operation, the bound (28) applies as well to the pair of fidelities $F_{\text{tele}}$ and $G_{\text{tele}}$. Consequently, for a given teleportation fidelity $F_{\text{tele}}$, the maximum value of the estimation fidelity is achieved for the state $|\text{tele}\rangle$ satisfying the condition $\mu_1 = \ldots = \mu_{d-1} = \sqrt{(1 - \mu_0^2)/(d-1)}$. This condition defines a class of pure bipartite states which are optimal from the point of view of the trade-off between the teleportation fidelity and the estimation fidelity.

In conclusion, I have obtained a tight bound for the fidelities describing the quality of estimating the state of a single copy of a $d$-level particle, and the degree the initial state has to be changed during this operation. This result seems to be one of very few cases, when the trade-off between the information gain and the state disturbance can be derived in a closed analytical form.

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FIG. 1. Rescaled bound for the operation fidelity $F$ versus the estimation fidelity $G$, plotted for $d = 2$ (solid line), $d = 4$ (dashed line), and $d = 8$ (dotted line).