The shadows of Schwarzschild black hole with halos

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Abstract

We have studied the shadows of Schwarzschild black hole with a halo containing quadrupolar and octopolar terms. We found the radius of light rings \(r_{LR}\) in the space-time of Schwarzschild black hole with halo only depends on the quadrupole strength \(Q\), and light ring doesn’t exist for \(Q\) is larger than a critical \(Q_c\), both unstable and stable light rings exist for \(Q_c < Q < 0\), only one unstable light ring exists for \(Q > 0\). The black hole shadow is oblate when the quadrupole strength \(Q\) is larger than zero, and it is prolate when \(Q\) is less than zero. Black hole shadow shifts upward when the octopolar strength \(O\) is less than zero, and shifts downward when \(O\) is larger than zero. From the observable width \(W\), height \(H\), oblateness \(K\) and distortion parameter \(\delta\) of black hole shadow, one can determine the quadrupole strength \(Q\) and the octopolar strength \(O\) of Schwarzschild black hole with halo. Black hole shadow is always a circle for the observers with the inclination angle \(\theta_{obs} = 0\), and becomes bigger with the increase of \(Q\) or \(O\). Our results show that the quadrupolar and octopolar terms yield a series of interesting patterns for the shadow of a Schwarzschild black hole with halo.

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I. INTRODUCTION

In 2019, Event Horizon Telescope (EHT) Collaboration captured the first image of the supermassive black hole in the center of the giant elliptical galaxy M87, which opens a new era in the fields of astrophysics and black hole physics. Nowadays, more and more researchers have devoted themselves to the study of black hole shadows. Black hole shadow is the dark region in the center of black hole image, which corresponds to the light rays captured by black hole. Due to the fingerprints of the geometry around the black hole could be reflected in the shape and size of black hole shadow, the research of black hole shadows plays a vital role in the study of black holes and verification of various gravity theories. It is a perfect black disk for Schwarzschild black hole shadow, and it gradually becomes a D-shaped silhouette with the increase of spin parameter for Kerr black hole shadow. In the space-time of a Kerr black hole with Proca hair and a Konoplya-Zhidenko rotating non-Kerr black hole, the cusp silhouette of black hole shadows emerge.

The self-similar fractal structures appear in the black hole shadow for a rotating black hole with scalar hair, a Majumdar-Papapetrou binary black hole system, Bonnor black diholes with magnetic dipole moment, and a non-Kerr rotating compact object with quadrupole mass moment. Many other black hole shadows with other parameters in various theories of gravity have been recently investigated in Refs. It is hope that these information imprinted in black hole shadow can be captured in the future astronomical observations including the upgraded Event Horizon Telescope and BlackHoleCam to study black holes and verify various gravity theories.

It is widely believed that a massive halo, ring or other shell-like distributions of matter could be concentrated around black hole in the galactic center. The superposition of a black hole and exterior matter will reorganize the space-time structure, and bring about a huge change to black hole shadow. We researched the shadows of a Schwarzschild black hole surrounded by a Bach-Weyl ring, and found black hole shadow becomes a “8” type shaped silhouette, and possesses self-similar fractal structures originating from the chaotic lensing. P. V. P. Cunha et al researched the shadows of a black hole surrounded by a heavy Lemos-Letelier accretion disk, and found black hole shadow becomes more prolate with the increase of the accretion disk mass. In this paper we consider a solution of the Einstein equations that represents the superposition of a Schwarzschild black hole with a halo containing quadrupolar and octopolar contributions. W. M. Vieira et al detected the timelike geodesic orbits of test particles in the space-time of Schwarzschild black hole with halo, and found both the quadrupolar and octopolar terms could bring about chaotic motion. The nonlinear superposition of
Schwarzschild black hole and halo also would break the integrability of the null geodesic motions of photons. It is still an open issue how the non-integrable of photon motion affect the black hole shadows. In this paper, we will study the shadows of Schwarzschild black hole with halos and then probe the effects of the quadrupolar and octopolar terms on black hole shadow.

The paper is organized as follows. In section II, we briefly review the space-time of Schwarzschild black hole with halo, and reveal the influences of the quadrupolar and octopolar terms on the geometry of black hole. In Section III, we present numerically the shadows of Schwarzschild black hole with halos, analyse the new features of black hole shadow arising from the quadrupolar and octopolar terms of halo. Finally, we end the paper with a summary.

II. THE SPACE-TIME OF SCHWARZSCHILD BLACK HOLE WITH HALO

The space-time of Schwarzschild black hole with halo is a vacuum static and axially symmetric space-time, so it can be described by the Weyl metric

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}(d\rho^2 + dz^2) + \rho^2 e^{-2\nu}d\phi^2,$$

where $\nu$ and $\lambda$ only are the functions of $\rho$ and $z$. The Einstein equations reduce to

$$\nu,\rho \rho + \frac{\nu,\rho}{\rho} + \nu,\rho = 0,$$

$$\lambda,\rho = \rho(\nu,\rho)^2 - \rho(\nu,\rho)^2, \quad \lambda,\rho = 2\rho\nu,\rho\nu,\rho.$$ 

The function $\nu(\rho, z)$ satisfies the Laplace equation and behaves like the gravitational potential in the Newtonian theory, thus it can be superposed linearly. However, the function $\lambda(\rho, z)$ does not own such a property of linear superposition. In the solution of Schwarzschild black hole with halo, the functions $\nu$ and $\lambda$ for the whole system can be written as $\nu = \nu_{Schw} + \nu_{halo}$ and $\lambda = \lambda_{Schw} + \lambda_{halo} + \lambda_{int}$, respectively. The functions $\nu_{Schw}, \lambda_{Schw}$ are the solution of Schwarzschild space-time described as

$$\nu_{Schw} = \frac{1}{2} \ln \frac{d_1 + d_2 - 2M}{d_1 + d_2 + 2M},$$

$$\lambda_{Schw} = \frac{1}{2} \ln \frac{(d_1 + d_2)^2 - 4M^2}{4d_1d_2},$$

where $M$ is the mass of Schwarzschild black hole, and $d_{1,2} = \sqrt{\rho^2 + (z \pm M)^2}$. The functions $\nu_{halo}, \lambda_{halo}$ are the solution of halo structure, and $\lambda_{int}$ represents the interaction between black hole and halo.

In the Schwarzschild coordinates, the space-time of Schwarzschild black hole with halo can be described by
the metric

\[ ds^2 = -(1 - \frac{2M}{r})e^{(\nu_Q + \nu_O)} dt^2 + e^{(\lambda_Q + \lambda_O + \lambda_{QO} - \nu_Q - \nu_O)} \left[ (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\theta^2 \right] + e^{-(\nu_Q + \nu_O)} r^2 \sin^2 \theta d\phi^2, \]  

where

\[ \nu_Q = (Q/3)(3u^2 - 1)(3v^2 - 1), \]  
\[ \nu_O = (O/5)uv(5u^2 - 3)(5v^2 - 3), \]  
\[ \lambda_Q = -4Q(1 - u^2) + (Q^2/2)[9u^4v^4 - 10u^4v^2 - 10u^2v^4 + 12u^2v^2 + u^4 + v^4 - 2u^2 - 2v^2 + 1], \]  
\[ \lambda_O = -2O[3u^2v - 3u^2v^3 + v^3 - \frac{9}{5}u^2 + \frac{4}{5}] + 2O[\frac{75}{8}u^6v^6 - \frac{117}{8}u^6v^4 - \frac{117}{8}u^4v^6 + \frac{45}{8}u^6v^2 + \frac{45}{8}u^2v^6 + \frac{189}{8}u^4v^4 - \frac{387}{40}u^2v^4 - \frac{387}{40}u^2v^2 - \frac{891}{200}u^2v^2 - \frac{3}{8}u^6 + \frac{3}{8}u^6 + \frac{27}{40}u^4 + \frac{27}{40}u^4 - \frac{81}{200}u^2 + \frac{81}{200}v^2 + \frac{21}{200}], \]  
\[ \lambda_{QO} = 2QO[9u^5v^5 - 12u^5v^3 + 3u^5v - 12u^3v^5 + \frac{84}{5}u^3v^3 - \frac{24}{5}u^3v + 3uv^5 - \frac{24}{5}uv^3 + \frac{9}{5}uv], \]  

and

\[ u = \frac{1}{2M} \left[ \sqrt{\rho^2 + (z + M)^2} + \sqrt{\rho^2 + (z - M)^2} \right], \]  
\[ v = \frac{1}{2M} \left[ \sqrt{\rho^2 + (z + M)^2} - \sqrt{\rho^2 + (z - M)^2} \right]. \]

The transformation between the Schwarzschild coordinates \((r, \theta)\) and the Weyl coordinates \((\rho, z)\) is:

\[ \rho = \sqrt{r(r - 2M)} \sin \theta, \quad z = (r - M) \cos \theta. \]

The exterior halo is a multipolar structure containing quadrupolar and octopolar terms, and \(Q\) and \(O\) are the quadrupole and octopole strengths respectively. The metric will reduce to Schwarzschild solution for the strengths \(Q = O = 0\). The octopolar term \((O \neq 0)\) will break the reflection symmetry with respect equatorial plane. The existence of halo does not change the position of the event horizon, it is still at \(r = 2M\). To illustrate what would happen to the shape of the event horizon under the influences of the quadrupolar and octopolar terms, we can compute the equatorial circumference \(C_e\) and the polar circumference \(C_p\) of the event horizon, which are given by

\[ C_e = \int_0^{2\pi} \sqrt{g_{\phi\phi}} d\phi \bigg|_{r=2M, \theta = \frac{\pi}{2}} = \int_0^{2\pi} 2Me^{-\frac{\pi}{2}\nu_Q} d\phi, \]  
\[ C_p = 2 \int_0^{\pi} \sqrt{g_{\theta\theta}} d\theta \bigg|_{r=2M} = 2 \int_0^{\pi} 2Me^{\frac{\pi}{2}+\nu_Q} (\lambda_Q + \lambda_O + \lambda_{QO} - \nu_Q - \nu_O) d\theta. \]

One can find the equatorial circumference \(C_e\) of the event horizon is independent of the octopole strength \(O\).

The surface area \(A\) of the event horizon is given by

\[ A = \int_0^{\pi} \int_0^{2\pi} \sqrt{g_{\phi\phi}g_{\theta\theta}} d\theta d\phi \bigg|_{r=2M} = \int_0^{\pi} \int_0^{2\pi} 4M^2 \sin \theta e^{\frac{\pi}{2}+\nu_Q} (\lambda_Q + \lambda_O + \lambda_{QO} - \nu_Q - \nu_O) d\theta d\phi. \]
In Fig. (a), one can find the equatorial circumference $C_e$ of the event horizon increases as the quadrupole strength $Q$ increases, and it is smaller than the event horizon circumference $(4\pi M)$ of Schwarzschild black hole ($Q = O = 0$) for $Q < 0$, is bigger than $4\pi M$ for $Q > 0$. In Fig. (b) and (c), one can find the polar circumference $C_p$ and the surface area $A$ of the event horizon first decrease and then increase with the increase of the quadrupole strength $Q$. What’s more, the polar circumference $C_p$ or the surface area $A$ with $O$ and with $−O$ have the same value. The polar circumference $C_p$ and the surface area $A$ of the event horizon increase as $|O|$ increases, which indicates the octopolar term will make $C_p$ and $A$ of event horizon larger.

The Hamiltonian $\mathcal{H}$ of a photon propagation along null geodesics in the space-time of Schwarzschild black hole with halo can be described as:

$$\mathcal{H} = g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + V_{\text{eff}} = 0,$$

where the effective potential $V_{\text{eff}}$ is defined as

$$V_{\text{eff}} = g^{tt}E^2 + g^{\phi\phi}L_z^2 = E^2(g^{tt} + g^{\phi\phi}\eta^2).$$

$E$ and $L_z$ are two constants of motion for the null geodesics motion, i.e., energy and the $z$-component of the angular momentum, so the impact parameter $\eta = L_z/E$ is also a constant for the photon motion.

The unstable spherical photon orbits are very important to determine the boundary of black hole shadow. Now, let us study the spherical photon orbits in the equatorial plane also known as light rings. The light rings must satisfy

$$\theta = \pi/2, \quad V_{\text{eff}} = 0, \quad \frac{\partial V_{\text{eff}}}{\partial r} = 0.$$
Moreover, when \( \partial^2 V_{\text{eff}}/\partial r^2 < 0 \), the light ring is an unstable circular photon orbit determining the black hole shadow boundary in the equatorial plane. When \( \partial^2 V_{\text{eff}}/\partial r^2 > 0 \), the light ring is stable. Solving the equations (15), we found the radius of light rings \( r_{LR} \) only depends on the quadrupole strength \( Q \), which are shown in Fig.2. One can find there is a critical value of the quadrupole strength \( Q_c \approx -0.0209445 \) for light rings in this figure. Light ring doesn’t exist in the space-time of Schwarzschild black hole with halo for \( Q < Q_c \); both unstable and stable light rings exist for \( Q_c < Q < 0 \); only one unstable light ring exists for \( Q > 0 \). In Fig.3 we show the effective potential \( V_{\text{eff}}(r, \frac{\pi}{2}) \) of the photon motions with different \( \eta \) for the quadrupole strength \( Q = -0.05, -0.01, 0.02 \). From the Hamiltonian \( \mathcal{H} \), one can infer that the effective potential \( V_{\text{eff}} \) can’t be larger than 0 for the photon motion. For the quadrupole strength \( Q = -0.05 \), light ring doesn’t exist because there is no point for \( \partial V_{\text{eff}}/\partial r = 0 \) when \( V_{\text{eff}} = 0 \). Moreover, one can find the maximum radial coordinate \( r \) photons can reach in the equatorial plane is smaller for bigger impact parameter \( \eta \) in Fig.3(a). For \( Q = -0.01 \), there are two points for \( \partial V_{\text{eff}}/\partial r = 0 \) when \( V_{\text{eff}} = 0 \), one is unstable light ring marked by “×”, and one is stable light ring marked by “◦” in Fig.3(b). One can find the photons with impact parameter \( |\eta| > |\eta_{us}| \) can’t reach the event horizon from outside the unstable light ring, where \( \eta_{us} = \pm 4.99 \) is the impact parameter of the unstable light ring. So the unstable light ring determines the boundary of black hole shadow. What’s more, the photons with the impact parameter \( |\eta| > |\eta_s| \) can’t exist outside the unstable light ring in the equatorial plane, where \( \eta_s = \pm 5.80 \) is the impact parameter of the stable light ring. For \( Q = 0.02 \), there is an unstable light ring marked by “×” in Fig.3(c). Only the photons with impact parameter \( |\eta| < 5.55 \) can reach the event horizon from outside the unstable light ring, where \( \pm 5.55 \) is the impact parameter \( \eta \) of the unstable light ring. The photons with impact parameter \( |\eta| > 5.55 \) can’t reach the event horizon from outside the unstable light ring, but they could exist outside the unstable light ring, which is similar with the case of Schwarzschild black hole.

From the plot of the effective potential \( V_{\text{eff}} \) of the photon motions in Fig.3 one can find the halo structure brings a huge influence on the space-time of Schwarzschild black hole with halo. But in real astrophysical situations, the halo structure could be considered as a perturbation of a black hole, couldn’t cause so much change for the space-time. In order to get close to the actual observed shadows of Schwarzschild black hole with halo in the next section, after research we prefer to set the quadrupole strength \( Q \) on the order of \( 10^{-4} \), and to set the quadrupole strength \( \mathcal{O} \) on the order of \( 10^{-6} \).
FIG. 2: The plot of the radius of light rings $r_{LR}$ with the quadrupole strength $Q$.

FIG. 3: The effective potential $V_{eff}(r, \pi/2)$ of the photon motions with different $\eta$ for the quadrupole strength $Q = -0.05, -0.01, 0.02$ in the space-time of Schwarzschild black hole with halo.

III. THE SHADOWS CASTED BY SCHWARZSCHILD BLACK HOLE WITH HALOS

Now, we study the shadows of Schwarzschild black hole with halos through the backward ray-tracing technique [12–24]. We assume that the static observer is locally at $(r_{obs}, \theta_{obs})$ in zero-angular-moment-observers (ZAMOs) reference frame [8], and evolved light rays from the observer backward in time. The shadow of black hole is composed by the light rays falling down into the event horizon of black hole. The coordinates of a photon’s image in observer’s sky can be expressed as [12–24]

$$x = -r_{obs} \sqrt{1 - \frac{2M}{r} L_z} \left( \frac{\sqrt{1 - \frac{2M}{r} L_z}}{r} \right)^{-\frac{1}{2}} |(r_{obs}, \theta_{obs})|, $$

$$y = r_{obs} \sqrt{r(r - 2M) \theta} \left( \frac{\sqrt{r(r - 2M) \theta}}{r} \right)^{-\frac{1}{2}} |(r_{obs}, \theta_{obs})|. $$

(16)
In Fig. 4, we show the influence of the quadrupole term on black hole shadow with the quadrupole strength $Q = -2 \times 10^{-4}, -1 \times 10^{-4}, 0, 1 \times 10^{-4}, 2 \times 10^{-4}$, and the octopolar strength $O = 0$. Here we set $M = 1$ and the static observer at $r_{\text{obs}} = 50$ with the inclination angle $\theta_{\text{obs}} = \pi/2$. In this paper, the background sphere light source we set is same as the celestial sphere in Ref. [18]. One can find the shadow of Schwarzschild black hole with halo is oblate for the quadrupole strength $Q < 0$, and it becomes more oblate as $Q$ decreases. But for $Q > 0$, the black hole shadow becomes more prolate as $Q$ increases. With the increase of $|Q|$, the Einstein ring, the white ring around black hole shadow, is torn into an Einstein cross. One can find the shadow of Schwarzschild black hole only with the quadrupole structure is symmetric about the equatorial plane. In Fig. 5, we show the influence of the octopolar term on black hole shadow with the quadrupole strength $Q = 0$, and the octopolar strength $O = -2 \times 10^{-6}, -1 \times 10^{-6}, 0, 1 \times 10^{-6}, 2 \times 10^{-6}$. For the octopolar strength $O < 0$, one can find the shadow of Schwarzschild black hole with halo shifts upward as $O$ decreases. But for $O > 0$, the black hole shadow shifts downward as $O$ increases. One can find the octopolar term break the reflection symmetry with respect equatorial plane of Schwarzschild black hole shadow, but the black hole shadow with $O$ and with $-O$ are symmetrical to each other about the equatorial plane. In Fig. 6, we show the shadows of Schwarzschild black hole with the halo with the quadrupole strength $Q = -2 \times 10^{-4}, -1 \times 10^{-4}, 0, 1 \times 10^{-4}, 2 \times 10^{-4}$, and the octopolar strength $O = -1 \times 10^{-6}$. Black hole shadow also becomes more oblate as the quadrupole strength $Q$ decreases for $Q < 0$, and becomes more prolate as $Q$ increases for $Q > 0$; meanwhile, black hole shadow shifts upward under the effect of the octopolar term for $O = -1 \times 10^{-6}$.

Black hole in the universe could be perturbed by a halo with quadrupole and octopolar terms, we hope researchers could determine the quadrupole strength $Q$ and octopolar strength $O$ by observation of black hole shadow. So we estimated several observable numerical values of black hole shadow, then studied the effect of the strengths $Q$ and $O$ on these observable values. To characterize the shadow of Schwarzschild black hole with halo, we should first introduce four important points for shadow: the leftmost point $(x_l, y_l)$, the...
FIG. 5: The shadows of Schwarzschild black hole with the halo with the quadrupole strength $Q = 0$, and the octopolar strength $O = -2 \times 10^{-6}, -1 \times 10^{-6}, 0, 1 \times 10^{-6}, 2 \times 10^{-6}$. Here we set $M = 1$ and the static observer at $r_{obs} = 50$ with the inclination angle $\theta_{obs} = \pi/2$.

FIG. 6: The shadows of Schwarzschild black hole with the halo with the quadrupole strength $Q = -2 \times 10^{-4}, -1 \times 10^{-4}, 0, 1 \times 10^{-4}, 2 \times 10^{-4}$, and the octopolar strength $O = -1 \times 10^{-6}$. Here we set $M = 1$ and the static observer at $r_{obs} = 50$ with the inclination angle $\theta_{obs} = \pi/2$.

rightmost point $(x_r, y_r)$, the topmost point $(x_t, y_t)$ and the bottommost point $(x_b, y_b)$, shown in Fig. 7, where the coordinates $(x, y)$ are the celestial coordinates in observer’s sky. So we can define black hole shadow’s observable values: the width $W = (x_r - x_l)/R_s$, the height $H = (y_t - y_b)/R_s$ and the oblateness $K = W/H$, where $R_s$ is the radius of Schwarzschild black hole shadow $(Q = O = 0)$. Fig. 8 shows the varieties of the width $W$, the height $H$ and the oblateness $K$ of black hole shadow with the quadrupole strength $Q$ for different octopolar strength $O$. In Fig. 8(a), one can find the width $W$ of black hole shadow decreases as $Q$ increases, and almost all the width $W$ are larger than 2 for $Q < 0$, almost all $W$ are less than 2 for $Q > 0$. It indicates that the quadrupole term stretches Schwarzschild black hole shadow along the horizontal direction for $Q < 0$, and squeezes the shadow along the horizontal direction for $Q > 0$. In addition, one can find the octopolar strength $O$ barely has effect on the width $W$ of black hole shadow. In Fig. 8(b), one can find the height $H$ of black hole shadow increases as the quadrupole strength $Q$ increases, but the change in height is much small comparing with the change in width. So the change of oblateness $K$ of black hole shadow with the quadrupole strength $Q$ is almost the same to the change of width $W$. The black hole shadow is oblate ($K > 1$) for $Q < 0$ and is prolate ($K < 1$) for $Q > 0$. We can infer that the quadrupole term mainly affects the width of black hole shadow from Fig. 8. Moreover, the height $H$ of black hole shadow increases as the octopolar strength
$|O|$ increases for the fixed $Q$. But we find the main effect of the octopolar strength $O$ is to move black hole shadow up and down from Fig.5. So we define the center of black hole shadow as $(x_c, y_c) = \left( \frac{x_l + x_r}{2}, \frac{y_t + y_b}{2} \right)$, and make use of $Y_c(y_c/R_s)$ to describe the shifting away from the equatorial plane. In Fig.9(a), we show the varieties of $Y_c$ of black hole shadows with the octopolar strength $O$ for different quadrupole strength $Q$. One can find the deviation $|Y_c|$ increases as the octopolar strength $|O|$ increases, and $Y_c > 0$ for $O < 0$, $Y_c < 0$ for $O > 0$. What’s more, the deviation $|Y_c|$ is bigger for bigger $Q$. Unfortunately, in the actual observation of black hole shadows we can’t determine the coordinates of shadow center $(x_c, y_c)$; thus we can’t estimate the value of $O$ by $Y_c$. The octopolar term, meanwhile, cause a slight distortion along the vertical direction in the black hole shadow, make $y_c$ is not equal to $y_l$ or $y_r$. So we can define a distortion parameter $\delta = (y_c - y_l)/R_s$ to describe the distortion caused by the octopolar term, which is shown in Fig.7. Fig.9(b) shows the varieties of the distortion parameter $\delta$ with the octopolar strength $O$ for different quadrupole strength $Q$. One can find the distortion parameter $|\delta|$ increases as the octopolar strength $|O|$ increases, and $\delta < 0$ for $O < 0$, $\delta > 0$ for $O > 0$. In addition, the distortion parameter $|\delta|$ is smaller for bigger $Q$. But observing the distortion parameter $\delta$ need much higher resolution for astronomical telescope, we hope Event Horizon Telescope and BlackHoleCam could observe such slight distortion of black hole shadow in the future.

![Fig. 7: The leftmost point $(x_l, y_l)$, the rightmost point $(x_r, y_r)$, the topmost point $(x_t, y_t)$, the bottommost point $(x_b, y_b)$ and the center $(x_c, y_c)$ of black hole shadow. There is a slight distortion along the vertical direction caused by the octopolar term in the black hole shadow, $y_c$ is not equal to $y_l$ or $y_r$. The distortion parameter $\delta$ can be defined as $(y_c - y_l)/R_s$, where $R_s$ is the radius of Schwarzschild black hole shadow ($Q = O = 0$).](image.png)

In Figs.10 and 11 we present the shadows of Schwarzschild black hole with halos for the observer inclination angle $\theta_{obs} = 0$ and $\pi/4$ respectively. From Fig.10 one can find black hole shadows are always circle for different
FIG. 8: The varieties of the width $W$, the height $H$ and the oblateness $K$ of black hole shadow with the quadrupole strength $Q$ for different octopolar strength $O$.

FIG. 9: The varieties of $Y_c$ and the distortion parameter $\delta$ of black hole shadow with the octopolar strength $O$ for different quadrupole strength $Q$.

$Q$ and $O$ when $\theta_{obs} = 0$. But the interesting thing is black hole shadow becomes bigger with the increase of $Q$ or $O$. Comparing to Schwarzschild black hole shadow (Figs.10(e) with $Q = O = 0$), one can find the negative $Q$ or the negative $O$ will make black hole shadow shrink, and the positive $Q$ or the positive $O$ will make black hole shadow expand. For the observer inclination angle $\theta_{obs} = \pi/4$ (Fig.11), black hole shadow not only becomes more prolate but also shifts upward as the quadrupole strength $Q$ increases for the fixed octopolar strength $O$. Unlike the case that black hole shadow shifts downward with the increase of $O$ when $\theta_{obs} = \pi/2$, black hole shadow shifts upward as $O$ increases when $\theta_{obs} = \pi/4$. 
IV. SUMMARY

We have studied the shadows of superposition of Schwarzschild black hole with halos. The exterior halo is a multipolar structure containing quadrupolar and octopolar terms. We found the radius of light rings $r_{LR}$ only depends on the quadrupole strength $Q$, and light ring doesn’t exist in the space-time of Schwarzschild black hole with halo when $Q$ is larger than a critical $Q_c$; both unstable and stable light rings exist for $Q_c < Q < 0$;
only one unstable light ring exists for $Q > 0$. The shadow of Schwarzschild black hole with halo becomes more oblate as the quadrupole strength $Q$ decreases for $Q < 0$, and becomes more prolate as $Q$ increases for $Q > 0$. Black hole shadow shifts upward with the decrease of octopolar strength $O$ for $O < 0$, and shifts downward with the increase of $O$ for $O > 0$. From the observable width $W$, height $H$, oblateness $K$ and distortion parameter $\delta$ of black hole shadow, one can determine the quadrupole strength $Q$ and the octopolar strength $O$ of Schwarzschild black hole with halo. The width $W$ of black hole shadow decreases as $Q$ increases, and
is practically independent of the octopolar strength \( O \). The height \( H \) of black hole shadow increases as \( Q \) or \( O \) increases, but the change in height is much small comparing with the change in width. So the change of oblateness \( K \) of black hole shadow with the quadrupole strength \( Q \) is almost the same to the change of width \( W \). The deviation \( Y_c \) of black hole shadow decreases as the octopolar strength \( O \) increases, and \(|Y_c|\) is bigger for bigger \( Q \). the distortion parameter \( \delta \) increases as the octopolar strength \( O \) increases, and \(|\delta|\) is smaller for bigger \( Q \). Black hole shadow is always a circle and becomes bigger as \( Q \) or \( O \) increases for the observers with the inclination angle \( \theta_{\text{obs}} = 0 \). Black hole shadow not only becomes more prolate but also shifts upward as the quadrupole strength \( Q \) increases for the observer inclination angle \( \theta_{\text{obs}} = \pi/4 \). But black hole shadow shifts upward as \( O \) increases for \( \theta_{\text{obs}} = \pi/4 \), which is the opposite of the case of \( \theta_{\text{obs}} = \pi/2 \). Our results show that the quadrupolar and octopolar terms yield a series of interesting patterns for the shadow of a Schwarzschild black hole with halo.

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