Modified MODE for Increasing the Maximum Number of Sources in DOA Estimation

Shohei Hamada, Koichi Ichige
Department of Electrical and Computer Engineering, Yokohama National University, Yokohama-shi, 240-8501, Japan.
a) koichi@ynu.ac.jp

Abstract: This paper presents a modified version of the method of direction estimation (MODE) that can estimate a greater number of sources than the original MODE. It is well-known that $M$-element arrays can basically estimate up to $(M-1)$ direction of arrivals (DOAs), whereas MODE can only estimate up to $\frac{M}{2}$ DOAs because of its error-sensitive computation procedure. We propose a modified version of MODE that can estimate up to $(M - 1)$ DOAs. The performance of the proposed method was evaluated through computer simulation.

Keywords: direction of arrival estimation, MODE, array signal processing, number of sources

Classification: Antennas and Propagation

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*Preliminary version of this paper has been presented in [6].
1 Introduction

Direction of arrival (DOA) estimation is an effective tool for digital beamforming or signal separation, and plays an important role in radar, sonar, and indoor and outdoor wireless communication systems [1]. Maximum likelihood (ML) [2] is considered as a solid method that can estimate the DOA of correlated signals without spatial smoothing preprocessing. The expectation-maximization (EM) algorithm [3] and the space-alternating generalized EM (SAGE) algorithm [4] are representative ML algorithms but often come at a high computational cost due to their iterative optimization.

The method of direction estimation (MODE) [5] is a polynomial-based ML method that has been specially developed for uniform linear arrays (ULAs). It accurately estimates the DOAs of correlated signals at very low computational cost. However, MODE has an inherent problem in that it can estimate only up to \(\frac{M}{2}\) DOAs in the case of \(M\)-element ULAs because of its error-sensitive computation procedure. This is in contrast to most of the other DOA estimation methods, which can estimate up to \((M - 1)\) DOAs.

In this paper, we present a modified version of MODE that can estimate up to \((M - 1)\) DOAs while preserving high DOA estimation accuracy. In a prior work [6], we modified MODE by utilizing the peak-search of the primary eigenvector beam pattern instead of the null-search in the original MODE. While this worked effectively, the computational cost increased due to the spectrum peak search. This paper further develops the method in [6] so as to reduce the computational cost by taking a polynomial root finding approach instead of using spectrum peak search. The performance of the proposed method is evaluated through computer simulation.

2 Preliminaries

2.1 Signal Model

Assume that \(K\) far-field narrowband and coherent (or highly correlated) signals are received by an \(M\)-element ULA under an additive white Gaussian noise (AWGN) environment, where the number of signals \(K\) is given or estimated in advance. The array input vector \(\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_M(t)]\) can be written as

\[
\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t),
\]

where \(\mathbf{A}\) is the array steering matrix, \(\mathbf{s}(t) = [s_1(t), s_2(t), \ldots, s_K(t)]\) is the incident signal vector, \(s_k(t)\) is the complex amplitude of the \(k\)-th incident signal, and \(\mathbf{n}(t) = [n_1(t), n_2(t), \ldots, n_M(t)]\) is the noise vector [1]. The covariance matrix \(\mathbf{R}_{xx}\) of the array input \(\mathbf{x}(t)\) can be written as

\[
\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}^H(t)] = \mathbf{A}\mathbf{A}^H + \sigma_n^2\mathbf{I},
\]
where $E[\cdot]$ is an expectation operator, $(\cdot)^H$ is a Hermitian conjugate, $S = E[s(t)s^H(t)]$ is the signal covariance matrix, $\sigma^2$ is the noise power, and $I$ is an $M \times M$ identity matrix.

### 2.2 DOA Estimation by MODE [5]

MODE is one of the ML-based DOA estimation methods specialized for ULA. The target cost function to be minimized is written as

$$F_{ML} = \text{Tr}[R_{xx}] - \text{Tr}[A(A^H A)^{-1} A^H R_{xx}],$$

where $\text{Tr}[\cdot]$ denotes the matrix trace operator. The first term on the right-hand side in (3) becomes constant, so the DOA estimation by MODE can be replaced by the minimization problem of the second term on the right-hand side:

$$F_{MODE} = \text{Tr}[A(A^H A)^{-1} A^H R_{xx}] = b^H H^H H b = |Hb|^2,$$

where $b = [b_0, b_1, \ldots, b_K]$ is the coefficient vector of a polynomial $b(z) = b_0z^K + b_1z^{K-1} + \cdots + b_K$ whose roots

$$z_k = e^{-j2\pi d\sin\theta_k}, \quad k = 1, 2, \ldots, K,$$

are corresponding to DOAs $\{\theta_k\}_{k=1}^K$, while $d$ is the inter-element spacing and $\lambda$ is the wavelength of the incident signals. The matrix $H$ is a $K(M-K) \times (K+1)$ matrix calculated by using the eigendecomposition of the array covariance matrix $R_{xx}$ [5].

This minimization problem deals with eigenvectors of the $(K+1) \times (K+1)$ matrix $H^H H$ in (4). The coefficient vector $b$ can be determined after several iterations of updating $b$ by

$$b = \arg\min_b F_{MODE}.$$

The last step of DOA estimation is solving the polynomial equation $b(z) = 0$ and obtaining the roots $\{z_k\}_{k=1}^K$. Then the DOAs $\{\theta_k\}_{k=1}^K$ can be estimated by (5).

### 3 Modified MODE

#### 3.1 Analysis of Rank Shrinkage in MODE

Our analysis shows that MODE does not work well in the case of more than $M^2/2$ [6] because the rank of the matrix $H^H H$ shrinks and becomes smaller than $K$ when $K > M^2/2$. The reason for the rank shrinkage can be briefly explained as follows. The matrix $H$ can be written as

$$H = [S_1, \ldots, S_K]^T,$$

using the $(M-K) \times (K+1)$ matrix $S_k$:

$$S_k = \begin{bmatrix}
(\hat{e}_k)_{K+1} & (\hat{e}_k)_K & \cdots & (\hat{e}_k)_1 \\
(\hat{e}_k)_{K+2} & (\hat{e}_k)_{K+1} & \cdots & (\hat{e}_k)_2 \\
\vdots & \vdots & \vdots & \vdots \\
(\hat{e}_k)_{M-1} & (\hat{e}_k)_{M-2} & \cdots & (\hat{e}_k)_{M-K-1} \\
(\hat{e}_k)_M & (\hat{e}_k)_{M-1} & \cdots & (\hat{e}_k)_{M-K}
\end{bmatrix}. $$

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The matrix element \((\hat{e}_k)_i\) in (8) denotes the \(i\)-th element of the vector \(\hat{e}_k\), where \(\hat{e}_k\) is given by

\[
[\hat{e}_1, \cdots, \hat{e}_K] = E_S(\Lambda_S - \sigma^2 I)^{1/2}.
\] (9)

The matrices \(E_S\) and \(\Lambda_S\) in (9) are respectively called signal subspace matrix and signal eigenvalue matrix, which are obtained by decomposing the covariance matrix \(R_{xx}\) into

\[
R_{xx} = E_S\Lambda_S E_S^H + E_N\Lambda_N E_N^H,
\] (10)

where the first and second terms of the right-hand side respectively denote the signal and noise components.

We assume coherent (or highly correlated) \(K\) sources and therefore the incident signal subspace becomes rank one, i.e., the elements of vector \(\hat{e}_1\) become much larger than those of \(\hat{e}_2, \ldots, \hat{e}_K\). This means that the matrix \(H\) is numerically approximated as

\[
H \simeq [S_1, 0, \cdots, 0]^T,
\] (11)

where \(0\) denotes the \((M-K) \times (K+1)\) zero matrix. It is obvious from (11) that the rank of matrix \(H\) shrinks to \((M-K)\).

### 3.2 Analysis of Angular Spectrum

Here we analyze the behavior of the angular spectrum \(S_i\) that corresponds to the \(i\)-th largest eigenvalue \(\lambda_i\) of the matrix \(H^H H\). The \(i\)-th angular spectrum \(S_i\) can be written as

\[
S_i(\theta) = |a^H(\theta)v_i|,
\] (12)

where the \((K+1) \times 1\) vector \(v_i\) is the eigenvector corresponding to the \(i\)-th largest eigenvalue \(\lambda_i\) of the matrix \(H^H H\), and \(a(\theta) = [a_0(\theta), a_1(\theta), \ldots, a_K(\theta)]\) is the angular vector whose elements are given by

\[
a_k(\theta) = e^{-j\frac{2\pi kd}{\lambda} \sin \theta}, \quad k = 0, 1, \ldots, K.
\] (13)

Figure 1 shows the behavior of the angular spectrum \(S_i\) as a function of the angle \(\theta\) in cases of \(M = 8\) and \(K = 4, 5\), where the yellow lines indicate true DOA directions. We see from Fig. 1(a) that the angular spectrum \(S_5\) corresponding to the minimum (5th) eigenvalue has nulls to all the DOAs when \(K = 4\). In contrast, we see from Fig. 1(b) that the angular spectrum \(S_6\) corresponding to the minimum (6th) eigenvalue cannot direct nulls to the DOAs when \(K = 5\).

Here we emphasize that the angular spectrum \(S_1\) corresponding to the maximum (1st) eigenvalue can effectively direct mainlobe peaks to DOAs even when \(K = 5\). Therefore, the DOAs would be accurately estimated if we can develop a modified version of MODE using the angular spectrum \(S_1\) corresponding to the maximum eigenvalue when \(K > \frac{M}{2}\).
3.3 Proposed Approach

As stated in Subsection 3.2, the DOAs can be estimated by either (a) the original MODE when $K \leq \frac{M}{2}$ or (b) on the basis of the peak-search of the eigenvector spectrum corresponding to the maximum eigenvalue when $K > \frac{M}{2}$. As a whole, the proposed DOA estimation formula is given by

$$\hat{\theta} = \begin{cases} \arg \min_{\theta} S_{\text{min}}(\theta), & K \leq \frac{M}{2}, \\ \arg \max_{\theta} S_{\text{max}}(\theta), & K > \frac{M}{2}, \end{cases}$$

(14)

where $\hat{\theta}$ is the estimated DOA, and $S_{\text{min}}(\theta)$ and $S_{\text{max}}(\theta)$ denote the angular spectrum corresponding to the minimum and maximum eigenvalues of the matrix $H^H H$, respectively.

Recall that the original MODE estimates DOAs by solving

$$S_{\text{min}} = |a^H(\theta)v_1| = 0,$$

(15)

which is equivalent to finding the roots of the polynomial

$$v_1^H z z^H v_1 = 0,$$

(16)

where $z = [1, z, z^2, \ldots, z^K]$.

In contrast, the angular spectrum $S_1(\theta)$ takes the maximum value of 0 dB (= 1 in linear scale) in the case of $K > \frac{M}{2}$. Therefore, we also estimate DOAs in the case of $K > \frac{M}{2}$ by solving

$$S_{\text{max}} = |a^H(\theta)v_K| = 1,$$

(17)

which is equivalent to finding the roots of the polynomial

$$v_K^H z z^H v_K = 1.$$

(18)

In this manner, we can estimate DOAs not by spectrum peak search but by polynomial root finding. Therefore, the computational cost of the proposed method is completely the same as that of the original MODE.
Fig. 2. Behavior of RMSE as a function of the number of sources, when $M = 6$ or 8 elements ULA, $K = 2$ to $(M - 1)$ coherent sources, 128 snapshots, SNR 20 dB.

4 Numerical Examples

In this section, we report our evaluation of the performance of the proposed DOA estimation method through computer simulation. All the following simulation results are the average of 100 Monte-Carlo trials.

Figures 2(a) and 2(b) respectively show the performance of DOA estimation in the case of $M = 6$ and 8 with that of MODE and Root-MUSIC method [7] which uses the spatial smoothing preprocessing of two subarrays. In this simulation, the number of sources $K$ was changed from 2 to $(M - 1)$. We see in the figures that the estimation error by the proposed method became much smaller than that of the original MODE when $K > \frac{M}{2}$. Therefore, the results of Fig. 2 suggest that the modified MODE works well for any values of $K$ and $M$.

The estimation success rate is shown in Fig. 3. Note that we define ”success” as a case where the DOA estimation error becomes smaller than 1 degree. We see from the figure that the success rate of the original MODE became smaller when $K > \frac{M}{2}$. In contrast, the proposed method achieved the success rate of 100% for any number of sources $K$, even in cases of greater than $\frac{M}{2}$.

5 Concluding Remarks

In this paper, we presented a modified version of MODE that can accurately estimate up to $(M - 1)$ DOAs. The method in [6] was further developed so as to reduce the computational cost while preserving the DOA estimation performance, and gives much better DOA estimation accuracy than the original MODE and Root-MUSIC method in the case of $K > \frac{M}{2}$.

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Fig. 3. DOA estimation success rate as a function of the number of sources, when $M = 8$ elements ULA, $K = 2$ to $(M - 1)$ coherent sources, 128 snapshots, SNR 20 dB.

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