Nonsingular (Vertex-Weighted) Block Graphs*

Ranveer Singh†
Cheng Zheng‡
Naomi Shaked-Monderer§
Abraham Berman¶

May 7, 2019

Abstract

A graph $G$ is nonsingular (singular) if its adjacency matrix $A(G)$ is nonsingular (singular). In this article, we consider the nonsingularity of block graphs, i.e., graphs in which every block is a clique. Extending the problem, we characterize nonsingular vertex-weighted block graphs in terms of reduced vertex-weighted graphs resulting after successive deletion and contraction of pendant blocks. Special cases where nonsingularity of block graphs may be directly determined are discussed.

Key words. Block, Block graph, Nonsingular graph, Vertex-weighted graph

AMS Subject Classifications. 15A15, 05C05.

1 Introduction

In 1957 Collatz and Sinogowitz proposed the problem of characterizing nonsingular graphs, i.e, graphs whose adjacency matrix is nonsingular [17]. This problem is of much interest in various branches of science, in particular quantum chemistry, Hückel molecular orbital theory [7, 10] and social networks theory [11]. Significant work was done towards a solution to this problem for special classes of undirected graphs, such as trees, unicyclic and bicyclic graphs [4, 6, 15, 8, 9, 16, 12, 1, 3, 19, 14, 13]. In particular, a tree is nonsingular if and only if it has a perfect matching [5]. Block graphs are a natural generalization of trees. A block in a graph is a maximal connected subgraph with no cut-vertex. A block graph is a graph in which each block is a clique (i.e., a complete subgraph), see [18, p. 15], [2]. In this article we study nonsingularity of block graphs.

It turns out that in order to characterize nonsingular block graphs, it is useful to consider vertex-weighted graph. A vertex-weighted graph is a pair $(G, x)$, where $G = (V(G), E(G))$ is a simple graph with vertex set $V(G) = \{1, \ldots, n\}$, edge set $E(G)$, and $x \in \mathbb{R}^n$ is a vector of vertex weights, $x_i$ is the weight of vertex $i$. A graph $G$ is the vertex-weighted block graph $(G, o)$, where $o$ is the zero vector. The adjacency matrix $A(G, x)$ of $(G, x)$ is given by

$$A(G, x) = A(G) + \text{diag}(x),$$

where $\text{diag}(x)$ is a diagonal matrix whose $i$-th diagonal entry is $x_i$. If $(G, x)$ is a vertex-weighted graph, and $H$ is a subgraph of $G$, we denote by $x^H$ the restriction of the vector $x$ to the vertices of $H$. We refer to $(H, x^H)$ as a subgraph of $(G, x)$, and if $H$ is a component of $G$ we refer to $(H, x^H)$ as a component of $(G, x)$.

A vertex-weighted block graph $(G, x)$ is nonsingular (singular) if $A(G, x)$ is nonsingular (singular). In Section 2, we give a necessary and sufficient condition for a vertex-weighted block graph to be singular in terms of its reduced graphs resulting after successive contraction and deletion of pendant blocks. We then,
in Section 3, present several families of nonsingular block graphs. In Section 4, we show that replacing edge blocks by paths of even order preserve nonsingularity/singularity.

The following terms and notations are used in the paper. A graph $G$ is a coalescence (at the vertex $v$) of two disjoint graphs $G_1$ and $G_2$ if it is attained by identifying a vertex $v_1 \in V(G_1)$ and a vertex $v_2 \in V(G_2)$, merging the two vertices into a single vertex $v$. We use $J, j, O, o, w$ to denote an all-ones matrix, an all-ones column vector, a zero matrix, a zero column vector and a $(0,1)$-vector of suitable order, respectively. The standard basis vectors in $\mathbb{R}^n$ are denoted by $e_1, \ldots, e_n$. A clique on $n$ vertices is denoted by $K_n$. If $Q$ is a subgraph of $G$, then $G \setminus Q$ denotes the induced subgraph of $G$ on the vertex subset $V(G) \setminus V(Q)$. If $Q$ consists of a single vertex $v$ we will write $G \setminus v$ for $G \setminus Q$. The determinant of a graph $G$ is $\det(G) = \det(\Lambda(G))$. For a nonzero $\alpha \in \mathbb{R}$, we use in this paper the following notation:

$$\alpha^{1/2} = \begin{cases} \sqrt{\alpha} & \text{if } \alpha > 0 \\ i\sqrt{\lvert\alpha\rvert} & \text{if } \alpha < 0. \end{cases}$$

For a diagonal matrix $D$ with nonzero real diagonal entries, $D^{1/2}$ and $D^{-1/2}$ are interpreted accordingly.

## 2 Characterizing nonsingular vertex-weighted block graphs

We start with a complete characterization of nonsingular vertex-weighted complete graphs, and some implications for vertex-weighted graphs that have a pendant block which is a clique. Note that elementary row and column operations do not change the rank of a matrix, and we use this fact in checking the singularity of $\Lambda(G, x)$. In particular, simultaneous permutations of rows and columns of $\Lambda(G, x)$ do not change the rank, thus in checking whether a vertex-weighted block graph $(G, x)$ is singular or not we may relabel the vertices of $G$, and reorder $x$ accordingly, as convenient.

**Theorem 2.1.** Let $x \in \mathbb{R}^n$.

1. If exactly one of $x_1, \ldots, x_n$ is equal to 1, then $(K_n, x)$ is nonsingular.
2. If at least two of $x_1, \ldots, x_n$ are equal to 1, then $(K_n, x)$ is singular.
3. If $x_i \neq 1$, $i = 1, \ldots, n$, let
   $$t(x) = \sum_{i=1}^{n} \frac{1}{1 - x_i}. \quad (1)$$
   then
   (a) $(K_n, x)$ is nonsingular if and only if $t(x) \neq 1$.
   (b) if $(K_n, x)$ is singular, then for any vector $y \in \mathbb{R}^{n+1}$ such that $y_i = x_i$, $i = 1, \ldots, n$, and $y_{n+1} \neq 1$, the graph $(K_{n+1}, y)$ is nonsingular.
4. If $x_i \neq 1$ for $i = 1, \ldots, n$ and $(K_n, x)$ is nonsingular, then any matrix $M$ of the form
   $$M = \begin{bmatrix} A(K_n, x) & j & O^T \\ j^T & \alpha & w^T \\ O & w & B \end{bmatrix},$$
   can be transformed, using elementary row and column operations, to the following matrix
   $$\begin{bmatrix} A(K_n, x) & o \\ o^T & \alpha + \gamma(K_n, x) & O^T \\ O & w & B \end{bmatrix},$$
   where
   $$\gamma(K_n, x) = \begin{cases} \frac{t(x)}{t(x) - 1} & \text{if } x_i \neq 1, i = 1, \ldots, n, \\ -1 & \text{if exactly one of } x_1, \ldots, x_n \text{ is equal to } 1. \end{cases} \quad (2)$$
Proof. Let $D = \text{diag}(j - x)$. Then

$$A(K_n, x) = J - D.$$  

1. Without loss of generality, let $x_1 = 1$. By subtracting the first row from the next $n - 1$ rows we get that $A(K_n, x)$ is row-equivalent to the matrix

$$
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
0 & x_2 - 1 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & x_n - 1
\end{bmatrix},
$$

whose determinant $\prod_{i=2}^{n}(x_i - 1)$ is nonzero.

2. In this case two rows (or columns) are equal.

3. (a) Denote

$$p = D^{-1/2}j.\\tag{3}$$

Then

$$J - D = D^{1/2}(D^{-1/2}JD^{-1/2} - I)D^{1/2} = D^{1/2}(D^{-1/2}jj^TJ^{-1}D^{-1/2} - I)D^{1/2} = D^{1/2}(pp^T - I)D^{1/2} \\tag{4}$$

Thus $J - D = A(K_n, x)$ is nonsingular if and only if $pp^T - I$ is nonsingular. Since the eigenvalues of $pp^T$ are $p^Tp$ and 0, the eigenvalues of the matrix $pp^T - I$ are $pp^T - 1$ and $-1$. Thus $pp^T - I$, and $J - D$, are nonsingular if and only if $p^Tp \neq 1$. As $p_i = (1 - x_i)^{-1/2}$,

$$p^Tp = \sum_{i=1}^{n} \frac{1}{1 - x_i} = t(x),$$

and $(K_n, x)$ is nonsingular if and only if $t(x) \neq 1$.

(b) If $t(x) = 1$, then if $y_{n+1} \neq 1$,

$$t(y) = t(x) + \frac{1}{1 - y_{n+1}} \neq 1,$$

and $(K_{n+1}, y)$ is nonsingular by part 3(a); and if $y_{n+1} = 1$, $(K_{n+1}, y)$ is nonsingular by part 1 of the theorem.

4. (a) Let $x_i \neq 1$, $i = 1, \ldots, n$. For every $p \in \mathbb{R}^n$,

$$(I - pp^T)(I + spp^T) = I + (s - 1 - sp^Tp)pp^T.$$  

Therefore if $p^Tp \neq 1$,

$$(I - pp^T)^{-1} = I + \frac{1}{1 - p^Tp}pp^T.$$  

Thus if $A(K_n, x) = J - D$ is invertible, where $D = \text{diag}(j - x)$, then by (4) above,

$$(A(K_n, x))^{-1} = (J - D)^{-1} = -D^{-1/2}(I + \frac{1}{1 - p^Tp}pp^T)D^{-1/2}.$$  

Hence

$$j^T(A(K_n, x))^{-1}j = -j^TD^{-1/2}(I + \frac{1}{1 - p^Tp}pp^T)D^{-1/2}j$$

$$= -p^T \left( I + \frac{1}{1 - p^Tp}pp^T \right) p = \frac{p^Tp}{p^Tp - 1}.$$
Let
\[
P = \begin{bmatrix} I & -A(K_n, x)^{-1}j & O \\ o^T & 1 & o^T \\ O^T & o & I \end{bmatrix}.
\]

Then
\[
P^T M P = \begin{bmatrix} A(K_n, x) & o & O^T \\ o^T & \alpha + \gamma(K_n, x) & w^T \\ O & w & B \end{bmatrix},
\]
where
\[
\gamma(K_n, x) = -j^T A(K_n, x)^{-1}j = -\frac{p^T p}{p^T p - 1} = -\frac{t(x)}{t(x) - 1}.
\]

(b) When exactly one of \(x_1, \ldots, x_n\) is equal to 1, we may assume without loss of generality that \(x_1 = 1\). If in
\[
M = \begin{bmatrix} A(K_n, x) & j & O^T \\ j^T & \alpha & w^T \\ O & w & B \end{bmatrix}
\]
we subtract the first column from column \(n + 1\), and then the first row from row \(n + 1\), we get the following matrix:
\[
\begin{bmatrix} A(K_n, x) & o & O^T \\ o^T & \alpha + \gamma(K_n, x) & w^T \\ O & w & B \end{bmatrix},
\]
where \(\gamma(K_n, x) = -1\).

\begin{remark}
In part 4 of Theorem 2.1, note the following special cases for \(n \geq 2\):
\begin{enumerate}
\item If \(x_i < 1\) for every \(1 \leq i \leq n\), and \(x_i = 0\) for at least one \(1 \leq i \leq n\), then \(t(x) > 1\). Hence in this case \(\gamma(K_n, x) < -1\).
\item If \(x\) is a zero vector, \(A(K_n, o)^{-1} = -I + \frac{1}{n-1}J\) is a matrix with all diagonal elements equal to \(-\frac{n-2}{n-1}\) and all off diagonal elements equal to \(\frac{1}{n-1}\). In this case we get that
\[\gamma(K_n, o) = -\frac{n}{n-1}.\]
\end{enumerate}
\end{remark}

\begin{remark}
Part 2 of Theorem 2.1 may be generalized: If a vertex-weighted block graph \((G, x)\) has a block \((B, x^B)\) such that \(x_i = x_j = 1\) for two non-cut-vertices \(i \neq j\), then \((G, x)\) is singular.

For a block \((B, x^B)\) of a vertex-weighted block graph \((G, x)\), we denote by \(\bar{x}^B\) the sub-vector of \(x^B\) consisting of the entries corresponding to the non-cut-vertices in \((B, x^B)\). If \(x_i \neq 1\) for every non-cut-vertex in \(B\), we define
\[\tau_{(G,x)}(B, x^B) = t(\bar{x}^B).\]
We simplify the notation to \(\tau(B)\) when no confusion may arise.

We now define two operations on \((G, x)\) using its pendant blocks.
\end{remark}

\begin{definition}
1. **PB-deletion.** Let \((B, x^B)\) be a pendant block such that \(\bar{x}_i^B \neq 1\) for every \(i\), and \(\tau(B) = 1\). A **PB-deletion of** \((B, x^B)\) is the operation of deleting all the vertices of \(B\) and the corresponding entries of the weights vector \(x\), yielding a subgraph \((H, x^H)\), where \(H = G \setminus B\).

2. **PB-contraction.** Let \((B, x^B)\) be a pendant block of \((G, x)\) with a cut-vertex \(k\), such that either exactly one entry in \(\bar{x}^B\) is 1, or \(\bar{x}_i^B \neq 1\) for every \(i\) and \(\tau(B) \neq 1\). A **PB-contraction of** \((B, x^B)\) is the operation of merging all the vertices of \((B, x^B)\) to the cut vertex \(k\), deleting the entries of \(\bar{x}^B\) from \(x\), and adding the weight \(\gamma(B, x^B)\) to \(x_k\), where
\[\gamma(B, x^B) = \begin{cases} -1 & \text{if exactly one entry in } \bar{x}^B \text{ is } 1, \\ -\frac{\tau(B)}{\tau(B)-1} & \text{if no entry in } \bar{x}^B \text{ is } 1. \end{cases}\]
\end{definition}
Note that when \((G, x)\) is a vertex-weighted block graph, both PB-deletion and PB-contraction generate a vertex-weighted block graph. Also, PB-deletions may disconnect a connected vertex-weighted block graph, but PB-contractions preserve connectivity.

**Lemma 2.2.** Let \((B, x^B)\) be a pendant block of \((G, x)\) such that \(x_i \neq 1\) for every non-cut-vertex in \(B\), and \(\tau(B) = 1\). Let \((H, x^H)\) be obtained from \((G, x)\) by PB-deletion of \((B, x^B)\). Then \((G, x)\) is singular if and only if \((H, x^H)\) is singular.

**Proof.** Without loss of generality we may assume that the vertices of \(B\) are \(\{1, \ldots, k\}\), and \(k\) is the cut-vertex. Then

\[
A(G, x) = \begin{bmatrix}
A_1 & j & O^T \\
j^T & x_k & w^T \\
o & w & A_2
\end{bmatrix},
\]

where \(A_1 = A(K_{k-1}, x^B)\) and \(A_2 = A(H, x^H)\). Any nonzero minor on the first \(k\) rows and some \(k\) columns, cannot have a zero column, cannot have more than one column of the form \(e_k\), and cannot consist of the first \(k - 1\) columns and a column of the form \(e_k\), since \(A_1\) is singular. Thus every such nonzero minor includes the \(k\)-th column, and any nonzero minor that does not include all the first \(k - 1\) columns has a zero complementary minor. Hence the Laplace expansion of \(\det A(G, x)\) along the first \(k\) rows yields

\[
\det A(G, x) = \det \begin{bmatrix}
A_1 & j \\
j^T & x_k
\end{bmatrix} \det A_2 = \det A(B, x^B) \det A(H, x^H).
\]

(see also [16, Lemma 2.3].)

By part 3(b) of Theorem 2.1, \(A(B, x^B)\) is nonsingular. Thus \((G, x)\) is nonsingular if and only if \((H, x^H)\) is nonsingular.

**Lemma 2.3.** Let \((B, x^B)\) be a pendant block of \((G, x)\) such that either \(x_i^B \neq 1\) for \(i\) of \(B\) and \(\tau(B) \neq 1\), or exactly one entry in \(x^B\) is 1. Let \((H, y)\) be obtained from \((G, x)\) by a PB-contraction of \((B, x^B)\). Then \((G, x)\) is singular if and only if \((H, y)\) is singular.

**Proof.** Without loss of generality we may assume that the vertices of \(B\) are \(\{1, \ldots, k\}\), and \(k\) is the cut-vertex. Then

\[
A(G, x) = \begin{bmatrix}
A_1 & j & O^T \\
j^T & x_k & w^T \\
o & w & A_2
\end{bmatrix},
\]

where \(A_1 = A(K_{k-1}, x^B)\). If either \(x_i \neq 1\) for every non-cut-vertex \(i\) of \(B\) and \(\tau(B) \neq 1\), or exactly one entry in \(x^B\) is 1, the matrix \(A_1\) is nonsingular by part 3(a) and part 1 of Theorem 2.1. By part 4 of that theorem, \(A(G, x)\) is similar to the matrix

\[
\begin{bmatrix}
A_1 & o & O^T \\
o^T & x_k + \gamma & w^T \\
o & w & A_2
\end{bmatrix},
\]

where

\[
\gamma = \begin{cases} 
-1 & \text{if exactly one entry in } x^B \text{ is 1}, \\
-\frac{\tau(B)}{\tau(B)-1} & \text{if no entry in } x^B \text{ is 1}.
\end{cases}
\]

Hence \((G, x)\) is nonsingular if and only if

\[
A(H, y) = \begin{bmatrix}
x_k + \gamma & w^T \\
w & A_2
\end{bmatrix}
\]

is nonsingular.

**Remark 3.** Note that PB-deletion and PB-contraction may be used for any vertex-weighted graph \((G, x)\) which has a pendant block \((B, x^B)\), where \(B\) is a clique, and the proper conditions on \(x^B\) are satisfied. Lemmas 2.2 and 2.3 hold in this case too.
Definition 2. Reduced vertex-weighted block graph. A vertex-weighted block graph \((H, y)\) is a reduced vertex-weighted block graph of the vertex-weighted block \((G, x)\) if it is obtained from \((G, x)\) by a finite number of PB-deletions and PB-contractions.

Lemmas 2.2 and 2.3 imply that if \((H, y)\) is a reduced vertex-weighted block graph of \((G, x)\), then \((G, x)\) is nonsingular if and only if \((H, y)\) is nonsingular. We can now prove the main theorem.

Theorem 2.4. A vertex-weighted block graph \((G, x)\) is singular if and only if there exists a reduced vertex-weighted block graph \((H, y)\) that has one of the following:

1. A component \((B, y^B)\), where \(B\) is a clique and \(y_i \neq 1\) for every vertex \(i\) and \(\tau(B) = 1\)

2. A block \((B, y^B)\) for which at least two entries of \(y^B\) are equal to 1.

Proof. If \((H, y)\) is a reduced vertex-weighted block graph of \((G, x)\), and \((H, y)\) satisfies 1 or 2, then \((H, y)\) is singular by part 3(a) of Theorem 2.1 or by Remark 2, respectively. By Lemmas 2.2 and 2.3 this implies that \((G, x)\) is singular.

Now suppose no reduced vertex-weighted block graph of \((G, x)\) satisfies 1 or 2. Perform PB-deletions and PB-contractions on \((G, x)\) until a reduced graph \((H, y)\) is obtained, for which no further PB-deletion or PB-contraction is possible. As \((H, y)\) cannot be further reduced, and does not satisfy 2, it does not have any pendant blocks. That is, each of its components is of the form \((B, y^B)\), where \(B\) is a clique. Since 1 and 2 are not satisfied, either \(y_i = 1\) for exactly one vertex \(i\) of \(B\), or \(y_i \neq 1\) for every vertex \(i\) of \(B\) and \(\tau(B) \neq 1\). Hence by Theorem 2.1, each component of \((H, y)\) is nonsingular, and so is \((G, x)\). 

We conclude the section with two of examples of families of vertex-weighted block graphs, where nonsingularity may be easily checked (without actually reducing the vertex-weighted block graph).

Theorem 2.5. Let \((G, x)\) be a vertex-weighted block graph that satisfies the following two properties:

(a) \(x_i \neq 1\) for every vertex \(i\).

(b) \(x_i < 1\) for every cut-vertex \(i\).

(c) For every block \((B, x^B)\) of \((G, x)\), \(\tau(B) > 1\).

Then \((G, x)\) is nonsingular.

Proof. We show that such \((G, x)\) may be reduced by PB-contractions to a vertex-weighted clique satisfying (a) and (b). Since such a reduced graph is nonsingular by Theorem 2.1, this will complete the proof.

It suffices to show that if \((B, x^B)\) is a pendant block of \((G, x)\) satisfying (a)–(c), then \((B, x^B)\) may be PB-contracted and the resulting vertex-weighted block graph will also satisfy (a)–(c).

Let \(k\) be the cut vertex of a pendant block \(B\) of \(G\). By (a)–(c), this pendant block may be PB-contracted. The resulting vertex-weighted block graph \((H, y)\) satisfies \(y_i = x_i \neq 1\) for every vertex \(i\) of \(H\) other than \(k\), and \(y_k = x_k - \frac{\tau(B)}{\tau(B)+1} \). As \(\tau(B) > 1\), \(y_k = x_k - \frac{\tau(B)}{\tau(B)+1} < x_k < 1\). Also, for every block \((C, y^C)\) of \((H, y)\), if \(k\) is not a vertex in \(C\), or \(k\) is a cut-vertex in \(C\), then clearly \(\tau(H,y)(C, y^C) = \tau(G,x)(C, x^C) > 1\). If \(k\) is a non-cut-vertex of \(C\) in \((H, y)\), \(\tau(H,y)(C, y^C) = \tau(G,x)(C, y^C) + \frac{1}{1-y_k} > 1\), since \(y_k < 1\).

Theorem 2.6. Let \((G, x)\) be a vertex-weighted block graph, that satisfies the following three properties:

(a) \(x_i < 1\) for every vertex \(i\).

(b) Each block \(B\) of \(G\) has at least 3 vertices.

(c) For every block \((B, x^B)\) of \((G, x)\), there exists \(i\) such that \(x^B_i = 0\).

Then \((G, x)\) is nonsingular.
Proof. Note that if \((G, x)\) consists of a single block satisfying (a)–(d), then \((G, x)\) is nonsingular: If \(G = K_m\), where \(m \geq 3\) and, without loss of generality, \(x_1 = 0\),

\[
\tau(G) \geq \sum_{i=1}^{m} \frac{1}{1 - x_i} = 1 + \sum_{i=2}^{m} \frac{1}{1 - x_i} > 1
\]

by (a), and thus \((G, x)\) is nonsingular by Theorem 2.1.

If \((G, x)\) has a pendant block \((B, x_B)\), this block may be PB-contracted since \(\tau(B) > 1\) by the first part of Remark 1. As in the previous theorem, the resulting \((H, y)\) also satisfies (a)–(c). Such \((G, x)\) may be reduced by successive PB-contractions to a single vertex-weighted block satisfying (a)–(c), and is therefore nonsingular.

Remark 4. The two families of vertex-weighted block graphs in Theorems 2.5 and 2.6 are not mutually exclusive, but none of these families fully contains the other.

However, a block graph \(G = (G, o)\) satisfies the conditions of Theorem 2.5 if and only if each block of \(G\) has two non-cut-vertices. A block graph \(G\) satisfies the conditions of Theorem 2.6 if and only if each block of \(G\) has at least three vertices, at least one of which is a non-cut-vertex. That is, the family of block graphs satisfying Theorem 2.6 contains all the block graphs satisfying Theorem 2.5.

There are nonsingular block graphs that do not satisfy the requirements in Theorem 2.5. An example of one such graph is given in Figure 1d (see Theorem 3.3).

3 Some classes of nonsingular block graphs

In this section we use Theorem 2.4 to identify some families of nonsingular block graphs. First we name the graphs discussed at the end of the previous section.

Definition 3. \(B^3_1\) block graph. A block graph is a \(B^3_1\) block graph if each block has at least three vertices, at least one of which is a non-cut-vertex.

From Theorem 4 and Remark 4 we deduce the following.

Theorem 3.1. Every \(B^3_1\) block graph is nonsingular

We observe that using Theorem 2.4 one obtains a new proof the following known result.

Theorem 3.2. Let a graph \(F\) be a forest on \(n\) vertices. Then \(F\) is nonsingular if and only if it has a perfect matching.

Proof. Let \(F\) be a forest, and let \((B, o_B)\) be any pendant edge in \((F, o)\). Then \(\tau(B) = 1\) and \((B, o_B)\) may be PB-deleted, yielding a forest \((G, o_G)\). Note that \(F\) has a perfect matching if and only if \(G\) has a perfect matching: if the deleted pendant edge is \(\{u, v\}\), with \(v\) the cut-vertex, then in the PB-deletion all edges incident with \(v\) are deleted. Thus if \(G\) has a perfect matching, adding \(\{u, v\}\) to this matching yields a perfect matching of \(F\). And if \(F\) has a perfect matching, \(\{u, v\}\) has to be one of the edges in the matching, and removing it yields a perfect matching of \(G\).

Given a forest \(F\), reduce \((F, o)\) as much as possible by PB-deletions, until you get a forest \((H, o_H)\) that has no pendant edges. Each component of \((H, o)\) is either an edge, or a singleton. Then \((H, o_H)\) is nonsingular if and only if no component of \(H\) is a singleton. By the above, \(F\) has a perfect matching if and only if \(H\) has a perfect matching, and by Theorem 2.4 \(F\) is nonsingular if and only if \(H\) is.

Next we consider block graphs of a special construction.

Theorem 3.3. Let \(G\) be a block graph consisting of a block \(K_n, n \geq 2\), to which at each vertex \(i = 1, \ldots, n\), \(k_i\) blocks of orders \(m^i_1, \ldots, m^i_{k_i}\) each greater than 2 are attached. Then \(G\) is nonsingular if and only if

\[
\sum_{i=1}^{n} \frac{1}{1 + \sum_{j=1}^{k_i} \left(\frac{m^i_j - 1}{m^i_j - 2}\right)} \neq 1.
\]
Proof. Successively perform PB-contraction of each pendant block of $(G, o)$. Then $(G, o)$ is reduced to a vertex-weighted block graph $(K_n, x)$. By Remark 1, \( x_i = -\sum_{j=1}^{k_i} \left( \frac{m_i^{j-1}-1}{m_i^{j-2}} \right) \). As \( x_i \neq 1, i = 1, \ldots, n \), we get that
\[
\tau(K_n, x) = \sum_{i=1}^{n} \frac{1}{1 + \sum_{j=1}^{k_i} \left( \frac{m_i^{j-1}-1}{m_i^{j-2}} \right)}.
\]
The result follows by Theorem 2.4.

A special case of Corollary 3.3, where the result is simplified is the following. Let \( n \geq 2, m \geq 3, k \geq 1 \) be three integers. We define a family of block graph using these three integers. Let us coalesce \( k \) pendant \( K_m \) blocks at each vertex of \( K_n \). We call the resulting graph an \((n, m, k)\)-block graph. As an example the \((4, 4, 2)\)-block graph is shown in Figure 2a. In the case of \((n, m, k)\)-block graphs the necessary and sufficient condition for nonsingularity in Corollary 3.3 becomes simple:

**Corollary 3.4.** For \( n \geq 1, m \geq 3, k \geq 1 \), an \((n, m, k)\)-block graph is singular if and only if
\[
\left( \frac{m-1}{m-2} \right) k = n - 1.
\]

Another special case of Theorem 3.3 is the case that \( n = 2 \).

**Corollary 3.5.** Let \( G \) be a block graph consisting of a block \( K_2 \), to which at each of the two vertices some blocks of order greater than 2 each are attached. Then \( G \) is nonsingular.

**Proof.** This follows from Theorem 3.3 for \( n = 2 \), as
\[
\sum_{i=1}^{2} \frac{1}{1 + \sum_{j=1}^{k_i} \left( \frac{m_i^{j-1}-1}{m_i^{j-2}} \right)} < \sum_{i=1}^{2} \frac{1}{2} = 1.
\]

Next we consider the following construction.
Definition 4. A **tree of block graphs**. Let \( T \) be a tree on \( k \) vertices, and let \( G_1, \ldots, G_k \) be block graphs. For every edge \( e = \{i, j\} \) of \( T \), choose a vertex \( u_e \) of \( G_i \) and \( v_e \) of \( G_j \), and connect \( u_e \) and \( v_e \) by an edge. The resulting graph \( G \) is a block graph, and we call such graph a **tree of** \( G_1, \ldots, G_k \). We refer to each of the edges \( \{u_e, v_e\} \) in \( G \) as a **skeleton edge**, and to the vertices \( u_e \) and \( v_e \) as **skeleton vertices**. The graph \( G_i \) is considered **pendant** in the tree of \( G_1, \ldots, G_k \) if the vertex \( i \) is pendant in \( T \).

The first result on a tree of block graphs generalizes Corollary 3.5.

**Theorem 3.6.** Let \( T \) be a tree with \( n \) vertices \( i = 1, \ldots, n \), and let \( d(i) \) be the degree of vertex \( i \) in \( T \). Let \( G \) be the graph obtained by coalescing \( k_i \) cliques \( K_{m_i^1}, \ldots, K_{m_i^k} \), each of order at least 3, at each vertex \( i \) of \( T \). If

\[
\sum_{j=1}^{k_i} \frac{m_j^i - 1}{m_j^i - 2} > d(i)
\]

for every \( i \), then \( G \) is nonsingular.

**Proof.** By PB-contractions of all pendant blocks in \((G, o)\) we obtain the reduced vertex-weighted tree \((T, x)\), where

\[
x_i = \sum_{j=1}^{k_i} \frac{m_j^i - 1}{m_j^i - 2}.
\]

If \(|x_i| > d(i)\) for every \( i \), then \( A(T, x) \) is a strictly diagonal dominant matrix, and therefore nonsingular. The result now follows from Theorem 2.4. \( \square \)

Next consider trees of \( B_1^3 \) block graphs.

**Theorem 3.7.** Let \( G \) be a tree of \( B_1^3 \) block graphs \( G_1, \ldots, G_k \), in which

(a) no two skeleton edges share a vertex,

(b) there is at least one non-cut vertex in any block that has at 3 vertices or more.

Then \( G \) is nonsingular.

**Proof.** For such \( G \), consider weight vectors \( x \) with the following three properties:

1. \( x_i < 1 \) for every \( i \).
2. \( x_i = 0 \) for any skeleton vertex.
3. For any block \( B \) of \( G \) with at least three vertices \( \bar{x}_i^B = 0 \) for at least one vertex \( i \).

We show, by induction on \( k \), that if \( G \) is as in the theorem, and a weight vector \( x \) for \( G \) satisfies 1–3, then \((G, x)\) is nonsingular. (As the weight vector \( o \) satisfies 1–3, this will prove the theorem.) For \( k = 1 \), this holds by Theorem 2.6. Suppose the result holds for any such vertex-weighted tree of \( k \) \( B_1^3 \) block graphs, and let \( G \) be a tree of \( B_1^3 \) block graphs \( G_1, \ldots, G_k \) that satisfies (a) and (b), and \( x \) is...
a weight vector for \( G \), satisfying 1–3. Without loss of generality, \( G_1 \) is pendant in \( G \). Let \( u \in V(G_1) \) and \( v \in V(G \setminus G_1) \) be skeleton vertices. Then

\[
A(G, x) = \begin{bmatrix}
A_1 & w_1 & o & OT \\
w_1^T & 0 & 1 & o^T \\
o^T & 1 & 0 & w_2^T \\
o & o & w_2 & A_2
\end{bmatrix},
\]

where \( w_1 \) and \( w_2 \) are (0,1)-column vectors, and \( \begin{bmatrix} A_1 & w_1 \end{bmatrix} \) is \( A(G_1, x^{G_1}) \), and \( \begin{bmatrix} 0 & w_2^T \\
o & A_2 \end{bmatrix} \) is the adjacency matrix of \((G \setminus G_1, x^{G_1})\). As each block graph has at least two pendant blocks, we may perform subsequent PB-contractions of blocks in \( G_1 \), leaving the block containing the skeleton vertex in \( G_1 \) to last. After these contractions, the remaining block \((B, b)\) satisfies (a)–(c) of Theorem 2.6. Moreover, \( b_i = 0 \) at least one non-cut-vertex \( i \) of \( B \). Thus \( \tau(B) > 1 \), and we may contract it also. The adjacency matrix of the resulting vertex-weighted graph is

\[
\begin{bmatrix}
\gamma & 1 & o^T \\
1 & 0 & w_2^T \\
o & w_2 & A_2
\end{bmatrix},
\]

where \( \gamma < -1 \) by part 1 of Remark 1. The pendant edge of this graph has \( \tau = \frac{1}{w_2} < 1 \) and may be PB-contracted, resulting in a weight of \( \alpha = -\frac{\gamma}{w_2} = -\frac{1}{\gamma} < 1 \) to the vertex \( v \). The resulting vertex-weighted graph is \((H, y)\), where \( H = G \setminus G_1 \) is a tree of \( G_2, \ldots, G_k \), and \( y_i = x_i < 1 \) for every vertex except \( v \), whose weight is \( \alpha < 1 \). Note that \( v \) is not a skeleton vertex in \( H \) (due to the assumption that in \( G \) no two skeleton edges share a vertex). Thus \( y \) satisfies 1–3, and by the induction hypothesis \((H, y)\) is nonsingular. By Theorem 2.4 so is \((G, x)\).

None of the two conditions (a) and (b) in Theorem 3.7 may be dropped — see examples in Figure 2.

**Theorem 3.8.** Let \( G \) be a block graph, in which each block has at least two non-cut-vertices. Then any graph \( G' \) obtained by coalescing edges at some of the cut vertices of \( G \) is nonsingular.

**Proof.** By PB-deletion of the coalesced pendant edges, the cut vertices at which they were coalesced are also deleted. The resulting graph is a subgraph of \( G \), whose components are \( B_1^2 \) block graphs, and is thus nonsingular, implying nonsingularity of \( G' \).

Starting with a graph like \( G' \) of Theorem 3.8, and some nonsingular graphs, we can construct another nonsingular tree of block graphs.

**Theorem 3.9.** Let \( G \) be a block graph, in which each block has at least two non-cut-vertices. Let \( G' \) is obtained as in Theorem 3.8 by coalescing edges at different cut vertices \( v_1, \ldots, v_s \), and let \( W_1, \ldots, W_s \) be nonsingular block graphs, \( s \leq k \). Let \( T \) be a star graph \( K_{1,s} \). The tree of block graphs of \( \gamma, W_1, \ldots, W_s \) obtained by choosing \( u_i \in V(W_i), i = 1, \ldots, s \), and letting the skeleton edges be \( \{u_i, v_i\}, i = 1, \ldots, s \), is nonsingular.

**Proof.** PB-delete each of the \( k \) pendant edges. In the resulting graph each component is either a \( B_1^2 \) block graph, or a graph like the one in Theorem 3.8, or one of \( W_1, \ldots, W_s \). Thus each component is nonsingular, and so is \( G \).

### 4 Replacing edge blocks by even order paths

We prove here some results on the determinant of a graph obtained by coalescing two graphs, or combining them by a bridge. These results will imply ways to construct more nonsingular block graphs from known block graphs.

Most of the results in this section are based on [16, Lemma 2.3], restated here for simple graphs with no vertex weights. In this lemma, \( \phi(G) = \det(A(G) - \lambda I) \) denotes the characteristic polynomial of the graph \( G \).
Lemma 4.1. [16] Let $G$ be a coalescence of $G_1$ and $G_2$ at a vertex $v$. Then

$$\phi(G) = \phi(G_1) \times \phi(G \setminus G_1) + \phi(G_2) \times \phi(G \setminus (G_1 \setminus v)) + \lambda \times \phi(G_1 \setminus v) \times \phi(G \setminus G_1),$$

Using this lemma, we deduce the following.

Lemma 4.2. If $G$ is a coalescence of $G_1$ and $G_2$ at a vertex $v$, then

$$\det(G) = \det(G_1) \det(G_2 \setminus v) + \det(G_1 \setminus v) \det(G_2).$$

Proof. Obtain $\det(G)$ by substituting $\lambda = 0$ in $\phi(G)$ in Lemma 4.1. This yields

$$\det(G) = \det(G_1) \det(G_2 \setminus v) + \det(G_1 \setminus v) \det(G_2).$$

Corollary 4.3. If a graph $G$ has a pendant edge $\{u, v\}$ with $v$ the cut vertex, then $\det(G \setminus v) = -\det(G)$.

Proof. In this case, $G$ is the coalescence of $G_1 = G \setminus v$ and $G_2$ consisting of the edge $\{u, v\}$. It is easy to see that $\det(G_2) = -2$ and $\det(G_2 \setminus v) = 0$, the result follows.

Corollary 4.4. A coalescence of any two singular graphs is singular.

Proof. Let $G$ be coalescence of singular graphs $G_1$ and $G_2$. As $\det(G_1) = \det(G_2) = 0$, $\det(G) = 0$.

Note that a coalescence of nonsingular graphs may be singular: e.g., the coalescence of two edges results in a singular tree. More generally, we have the following corollary of Lemma 4.2.

Corollary 4.5. If $G$ is any graph, and two pendant edges are coalesced with it at the same vertex $v$, then the resulting graph $G'$ is singular.

Proof. In Lemma 4.2 let $G_1$ be the coalescence of one of the pendant edges with $G$, and $G_2$ the second pendant edge. Then $G_2 \setminus v$ is a singleton, and $G_1 \setminus v$ has a singleton component, thus $\det(G_2 \setminus v) = \det(G_1 \setminus v) = 0$, implying that $\det(G') = 0$.

Another way to combine two graphs $G_1$ and $G_2$ into a larger graph, is be adding an edge between a vertex of $G_1$ and a vertex of $G_2$. From Lemma 4.4 we get the following.

Lemma 4.6. Let $G_1$ and $G_2$ be two disjoint graphs. If we add an edge $\{v_1, v_2\}$, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then the resulting graph $G'$ is singular if and only if

$$\det(G_1) \det(G_2) = \det(G_1 \setminus v_1) \det(G_2 \setminus v_2).$$

Proof. Let $G$ be the resulting graph. Let $e_{v_1v_2}$ denote the graph consisting of one edge between the vertices $v_1$ and $v_2$ and $G'$ the graph, which is the coalescence of $e_{v_1v_2}$ and the graph $G_2$. Note that $G' \setminus v_1 = G_2$, $\det(e_{v_1v_2}) = -1$ and $\det(v_1) = 0$. 11
Then using Corollary 4.4 twice, first for $G$ with the cut vertex $v_1$, and then for $G'$ and the cut vertex $v_2$, we get that

$$\det(G) = \det(G_1) \det(G_2) + \det(G_1 \setminus v_1) \det(G')$$

$$= \det(G_1) \det(G_2) + \det(G_1 \setminus v_1)(\det(e_{v_1 v_2}) \det(G_2 \setminus v_2) + \det(v_1) \det(G_2))$$

$$= \det(G_1) \det(G_2) - \det(G_1 \setminus v_1) \det(G_2 \setminus v_2).$$

If $G_1 \setminus v_1$ or $G_2 \setminus v_2$ is a null graph then the determinant by convention is equal to 1. Thus $G$ is nonsingular if and only if

$$\det(G_1) \det(G_2) = \det(G_1 \setminus v_1) \det(G_2 \setminus v_2).$$

\[ \tag*{\Box} \]

**Lemma 4.7.** Let $G_1$ and $G_2$ be two graphs. Let $G$ be the graph obtained by adding a path $P$ of order $k$ between a vertex $v_1$ of $G_1$ and a vertex $v_2$ of $G_2$. Then

1. If the order $k$ of $P$ is odd, $G$ is nonsingular if the coalescence of $G_1$ and $G_2$ by identifying $v_1$ and $v_2$ is nonsingular.

2. If the order $k$ of $P$ is even, $G$ is nonsingular if the graph $G'$ obtained by connecting $v_1$ and $v_2$ by a single edge is nonsingular.

**Proof.** For given graphs $G_1$ and $G_2$, let us denote by $G^{(k)}$ the graph obtained by adding a path $P$ between the vertex $v_1$ of $G_1$ and the vertex $v_2$ of $G_2$. It suffices to show that for every $k \geq 3$, $G^{(k)}$ is nonsingular if and only if $G^{(k-2)}$ is nonsingular. Let $k \geq 3$. Choose a vertex $v$ on the path of order $k$ between $v_1$ and $v_2$, whose distance from each of the two end vertices is at least 1. Let $P'$ be the part of the path $P$ connecting $v_1$ to $v$ (including), $P''$ the part of $P$ connecting $v$ and $v_2$. Let $v'_1$ be the neighbor of $v$ in $P'$, and $v'_2$ the neighbor of $v$ in $P''$. Finally, let $G'_1 = G_1 \cup P'$ and $G'_2 = G_2 \cup P''$, $G''_1 = G'_1 \setminus v$ and $G''_2 = G'_2 \setminus v$. Note that the coalescence of $G''_1$ and $G''_2$ by identifying $v'_1$ and $v'_2$ results in a $G^{(k-2)}$. By Lemma 4.2,

$$\det(G^{(k)}) = \det(G'_1) \det(G'_2 \setminus v) + \det(G'_1 \setminus v) \det(G'_2)$$

$$= \det(G'_1) \det(G'_2) + \det(G''_1) \det(G''_2).$$

By Corollary 4.3,

$$\det(G''_1 \setminus v'_1) = -\det(G'_1) \quad \text{and} \quad \det(G''_2 \setminus v'_2) = -\det(G'_2).$$

Combining that with Lemma 4.2 applied to the coalescence of $G''_1$ and $G''_2$ yields

$$\det(G^{(k-2)}) = \det(G''_1) \det(G''_2 \setminus v'_2) + \det(G''_1 \setminus v'_1) \det(G''_2)$$

$$= -\det(G''_1) \det(G''_2) - \det(G'_1) \det(G'_2).$$

That is, $\det(G^{(k)}) = -\det(G^{(k-2)})$. \[ \tag*{\Box} \]

**Lemma 4.8.** Let $G$ be any graph with a pendant edge $\{u, v\}$, where $v$ is the cut vertex. Let $G'$ be obtained by coalescing $G \setminus u$ with a nonsingular tree $T$ at the vertex $v$. Then $G'$ is nonsingular if and only if $G$ is nonsingular.

**Proof.** Let $v$ be the coalescence vertex. Successively PB-delete pendant edges of $T$, until exactly a pendant edge at the vertex $v$ is left. This is possible, since in all the steps up to the last, there are at least two pendant edges, one with both ends different than $v$.

Coalescing a block graph with a tree, and combining block graphs by coalescence or by an edge yields block graphs. Thus the results of this section imply the following for block graphs.

**Remark 5.** Let $G$ be a block graph.

- If $G$ has two pendant edges at the same cut vertex, then $G$ is singular (Corollary 4.5).

- The coalescence of two singular block graphs is singular (Corollary 4.4).
• If $G$ has a block, which is an edge $e$, then replacing this edge by a path of even order results in a block graph $G'$, which is nonsingular if and only if $G$ is nonsingular (by Lemma 4.7, if $e$ is a bridge, or Lemma 4.8, if $e$ is a pendant edge.) In particular, this holds for any tree of block graphs may be thus extended without affecting its singularity/nonsingularity, and for the graphs in Theorems 3.8 and 3.9.

• Pendant edges the pendant edges may also be replaced by nonsingular trees without affecting singularity/nonsingularity (e.g., in Theorems 3.8 and 3.9, see Figure 3b).

References

[1] RB Bapat. A note on singular line graphs. Bull. Kerala Math. Assoc, 8(2), 2011.

[2] RB Bapat and Souvik Roy. On the adjacency matrix of a block graph. Linear and Multilinear Algebra, 62(3):406–418, 2014.

[3] Avi Berman, Shmuel Friedland, Leslie Hogben, Uriel G Rothblum, and Bryan Shader. An upper bound for the minimum rank of a graph. Linear Algebra and its Applications, 429(7):1629–1638, 2008.

[4] Dragoš M Cvetkovic, Michael Doob, and Horst Sachs. Spectra of graphs, volume 87 of pure and applied mathematics, 1980.

[5] Dragoš M Cvetković and Ivan M Gutman. The algebraic multiplicity of the number zero in the spectrum of a bipartite graph. Matematički vesnik, 9(56):141–150, 1972.

[6] Stanley Fiorini, Ivan Gutman, and Irene Sciriha. Trees with maximum nullity. Linear algebra and its applications, 397:245–251, 2005.

[7] Ivan Gutman and Bojana Borovicanin. Nullity of graphs: an updated survey. Selected topics on applications of graph spectra, Math. Inst., Belgrade, pages 137–154, 2011.

[8] Ivan Gutman and Irene Sciriha. On the nullity of line graphs of trees. Discrete Mathematics, 232(1-3):35–45, 2001.

[9] Shengbiao Hu, Tan Xuezhong, and Bolian Liu. On the nullity of bicyclic graphs. Linear Algebra and its Applications, 429(7):1387–1391, 2008.

[10] Shyi-Long Lee and Chiuping Li. Chemical signed graph theory. International journal of quantum chemistry, 49(5):639–648, 1994.

[11] Jure Leskovec, Daniel Huttenlocher, and Jon Kleinberg. Signed networks in social media. In Proceedings of the SIGCHI conference on human factors in computing systems, pages 1361–1370. ACM, 2010.

[12] Milan Nath and Bhaba Kumar Sarma. On the null-spaces of acyclic and unicyclic singular graphs. Linear Algebra and its Applications, 427(1):42–54, 2007.

[13] Irene Sciriha. On the construction of graphs of nullity one. Discrete Mathematics, 181(1-3):193–211, 1998.

[14] Irene Sciriha. A characterization of singular graphs. Electronic Journal of Linear Algebra, 16(1):38, 2007.

[15] Ranveer Singh and Ravindra B Bapat. B-partitions, determinant and permanent of graphs.

[16] Ranveer Singh and RB Bapat. On characteristic and permanent polynomials of a matrix. Spec. Matrices, 5:97–112, 2017.

[17] Lothar Von Collatz and Ulrich Sinogowitz. Spektren endlicher grafen. In Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, volume 21, pages 63–77. Springer, 1957.

[18] Douglas Brent West. Introduction to graph theory, volume 2. Prentice hall Upper Saddle River, 2001.
[19] Tan Xuezhong and Bolian Liu. On the nullity of unicyclic graphs. *Linear algebra and its applications*, 408:212–220, 2005.