Implementation of SWAP test for two unknown states in photons via cross-Kerr nonlinearities under decoherence effect

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We present an optical scheme for a SWAP test (controlled swap operation) that can determine whether the difference between two unknown states (photons) using cross-Kerr nonlinearities (XKNLs). The SWAP test, based on quantum fingerprinting, has been widely applied to various quantum information processing (QIP) schemes. Thus, for a reliable QIP scheme, it is important to implement a scheme for a SWAP test that is experimentally feasible. Here, we utilize linearly and nonlinearly optical (XKNLs) gates to design a scheme for a SWAP test. We also analyze the efficiency and the performance of nonlinearly optical gates in our scheme under the decoherence effect and exhibit a technique employing quantum bus beams and photon-number-resolving measurements to reduce the effect of photon loss and dephasing caused by the decoherence effect. Consequently, our scheme, which is designed using linearly optical devices and XKNLs (nonlinear optics), can feasibly operate the nearly deterministic SWAP test with high efficiency, in practice.

The development of quantum technology has been explosive. Beyond basic quantum cryptography and quantum computing, new fields such as quantum machine learning1–4, quantum communication5–10, advanced quantum computing11–13, and quantum fingerprinting14 have been proposed. A main technique in this area is a SWAP test (controlled swap operation)14–17. The SWAP test can determine with certainty whether two unknown states are different18–20. Basically, the SWAP test involves a Fredkin gate. The Fredkin gate is a representative multi-qubit gate and has one control qubit and two target qubits for swap operation with each other, resulting from the state of the control qubit. Recently, methods have been proposed to implement a linearly optical SWAP test15–17,21,22. In addition, methods for implementing a SWAP test based on nonlinear optics have also been proposed23–25.

Also, two kinds of SWAP tests have been proposed: First, destructive SWAP test is equivalent with Hong-Ou-Mandel (HOM) effect using Mach-Zehnder interferometer15. No ancillary photon (qubit) is needed to perform the SWAP test, which can determine with certainty whether two unknown states are different. However, after performed the destructive SWAP test between two unknown states, they cannot maintain pre-measured (two unknown) states by directly applying the measurement to unknown states. In this case, we can only obtain information whether two unknown states are different or not. On the other hand, nondestructive SWAP test17,22–25 has ancillary system (photon or qubit) for measurement. This SWAP test can be directly applicable from Fredkin gate, which performs the controlled swap operation. And it’s possible to determine whether the difference between two unknown states to conduct the measurement into the ancillary system. Also, if two unknown states are same, two unknown states can be maintained because of no direct measurement regarding to those. By this advantage, although nondestructive SWAP test has difficulties (using linear17,21 or nonlinear optics23–25) to experimentally implement in practice, it is an essential element, and can be directly applied to quantum information processing schemes, such as quantum machine learning1–4, quantum communication5–10, advanced quantum computing11–13, and quantum fingerprinting14).

To realize a SWAP test using nonlinear optics, the interaction of cross-Kerr nonlinearity (XKNL) can be experimentally implemented in practice. The interactions of XKNL between photons and Kerr media are utilized.

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as a quantum non-demolition measurement, in which the indirect measurement of an ancillary (probe) system is applied to a photon-probe system to acquire the form of the quantum state. Thus, many multi-qubit operations or quantum information processing (QIP) schemes have utilized the XKNL interaction between photons, such as in quantum-controlled gates or computations, quantum communications, and the generation and measurement of quantum entanglement. However, the output state from nonlinearly optical gates using XKNLs evolves into a mixed state (decreasing fidelity) because of the decoherence effect (caused by photon loss and dephasing), which consistently occurs in the interaction between photons and Kerr media. Recently, methods that can decrease the decoherence effect have been studied that employ photon-number-resolving (PNR) measurement and quantum bus (qubus) beams with a coherent state having a strong amplitude (probe beam) to reduce the decoherence effect.

In this paper, we present an optical scheme for the SWAP test, which is based on quantum fingerprinting to certainly determine whether two unknown states are different using nonlinearly optical (path-parity and path-merging) gates and a linearly optical gate (HOM gate). For this assessment (certainty difference in two unknown states), our SWAP test scheme utilizes weak XKNLs, qubus (coherent state) beams, and PNR measurements for path-parity and path-merging gates, and also the HOM effect to design a HOM gate using a Mach-Zehnder interferometer. Then, we show the high efficiency and the reliable performance of nonlinearly optical (path-parity and path-merging) gates in our SWAP test through analysis of the fidelities of the output states against the decoherence effect (photon loss and dephasing) when increasing the amplitude of the coherent state (probe beams), in practice. Consequently, our SWAP test scheme can feasibly be experimentally implemented with high efficiency and reliable performance, and it is robust against the decoherence effect, as determined by our analysis of nonlinearly optical gates that employ weak XKNLs, qubus beams, and PNR measurements with a strong coherent state.

**Scheme of SWAP test via XKNLs and linearly optical effect**

First, we introduce the concept of a SWAP test (controlled swap operation) to determine whether two unknown states (|ψ⟩ and |φ⟩) are different. Figure 1 shows a schematic SWAP test and a theoretical SWAP test, consisting of two controlled-NOT (CNOT) gates (two-qubit operation) and one Toffoli gate (three-qubit operation). The two SWAP tests in Fig. 1 are equivalent in terms of the two output states. Let us assume that the input states are |ψ⟩_A ≡ α|0⟩_A + β|1⟩_A, |φ⟩_B ≡ δ|0⟩_B + λ|1⟩_B (two unknown states: we want to distinguish), and |0⟩_C (control qubit: ancillary qubit), as described in Fig. 1. After the input state |ψ⟩_A|φ⟩_B ⊗ |0⟩_C passes the sequential operations (Hadamard gates, and controlled swap gates (or CNOT and Toffoli gates in the circuit of the SWAP test]), the result state, pre-measurement, will be given by

\[ |ψ⟩_A|φ⟩_B ⊗ |0⟩_C \xrightarrow{\text{Hadamard}} |ψ⟩_A|φ⟩_B \otimes (|0⟩_C + |1⟩_C)/\sqrt{2} \]

controlled swap (or CNOT→Toffoli→CNOT)

\[-(|ψ⟩_A|φ⟩_B ⊗ |0⟩_C + |φ⟩_A|ψ⟩_B ⊗ |1⟩_C)/\sqrt{2} \]

\[-(1(|ψ⟩_A|φ⟩_B) ⊗ |0⟩_C + (|ψ⟩_A|φ⟩_B − |φ⟩_A|ψ⟩_B) ⊗ |1⟩_C)/2. \]
When the ancillary qubit, C, is measured, we can determine that two unknown states, A and B, are identical or not, according to Eq. 1. If the two unknown states are the same (ψϕ = ψAψB), the result state is $|0⟩_C$ with probability 1 because the result state is $ψϕ ⊗ |0⟩_C$. In another case, $ψϕ ≠ ψAψB$, the probabilities of the result state in $|0⟩_C$ and $|1⟩_C$ of the ancillary qubit are $φψ + (1/2)$ and $φψ - (1/2)$, respectively. Thus, if the result of the ancillary qubit is $|1⟩_C$, we can be convinced that two unknown states are different. Consequently, we can determine the result of the difference in the two unknown states with reliability through the SW AP test, in principle.

To determine the performance property of nonlinearly optical (path-parity and path-merging) gates using XKNLs, we introduce the Hamiltonian, $H_{Kerr}$, of the XKNLS effect ($H_{Kerr} = ℏχ N_1 N_2$ for $N_i$: photon number operator, and $χ$: strength of nonlinearity in a Kerr medium). The unitary operation of the XKNL is expressed as $U_{Kerr}(n_1, n_2) = e^{iθ_{Kerr}/2}$, where $θ = χ t$ is the magnitude of the conditional phase shift caused by XKNL, and $t$ is the interaction time in a Kerr medium.

From now on, we propose an optical scheme of the SW AP test to be implemented using XKNLs (nonlinear optics) and the HOM effect (linear optics), as described in Fig. 2. We assume two unknown states (A and B) of photons, and an ancillary photon (C: control qubit), as $ψA = +ψHV A A$ and $ψB = +ψHV B B$, and $|R⟩_C$, where the circular polarization (|$R⟩$: right, $|L⟩$: left) are related to the linear polarization (|$H⟩$: horizontal, $|V⟩$: vertical) with the relationship $|R⟩ = (|H⟩ + |V⟩)/\sqrt{2}$ and $|L⟩ = (|H⟩ - |V⟩)/\sqrt{2}$. As described in Fig. 2, after this input state, $|ψA⟩^{|R⟩}_A ⊗ |ψB⟩_B ⊗ |R⟩_C$, passes through two BSs and a CPBS, the state, $|Φ1⟩_{ABC}$, is transformed as

$$|Φ1⟩_{ABC} = \frac{1}{2}(ψA^{|R⟩}_A^{|R⟩} + ψA^{|R⟩}_A^{|L⟩} + ψA^{|L⟩}_A^{|R⟩} + ψA^{|L⟩}_A^{|L⟩}) ⊗ (|H⟩_C^{|R⟩} + |V⟩_C^{|L⟩}).$$

Then, two photons (A and B) in this state, $|Φ1⟩_{ABC}$, are injected to the path-parity gate (1) using XKNLs, qubits beams, and PNR measurement, as described in Fig. 3. After the operation on the state $|Φ1⟩_{ABC}$ in Fig. 3, the state, $|Ψ2⟩_{ABC}^{|R⟩}_{pre}$ (pre-measurement) from path-parity gate (1) is expressed as

Figure 2. Schematic plot of SWAP test (controlled swap gate): This scheme consists of two path-parity gates (1 and 2) and two path-merging gates (1 and 2) using XKNLs, and an HOM gate using the HOM effect with linearly optical devices. As a result of the outcome of measurement of photon C (ancillary photon), this scheme (SWAP test) can make a comparison to determine if two unknown states of photons (A and B) are different. Multi-qubit gates via XKNLs are utilized in our SWAP test for experimental realization.
where \( i \equiv n \sin \theta \), and \( n \) is the photon number. When the PNR measurement \([\text{for precisely measuring photon number, we use the quantum non-demolition detection}^{28,29,31,41}\) is applied in the coherent state (probe beam) of path b, if the outcome is dark detection, \( 0_{p,b} \), the output state, \( |\Phi_2\rangle_{CAB} \), is acquired as

\[
|\Phi_2\rangle_{CAB} = \frac{1}{\sqrt{2}} \left( |H\rangle_C |\varphi\rangle_A |\varphi\rangle_B |0\rangle_B + |V\rangle_C |\varphi\rangle_A |\varphi\rangle_B |1\rangle_B \right).
\]

Also, if the result is the state \( n_{p,b} (n \neq 0) \), the output state can be transformed to the state \( |\Phi_2\rangle_{CAB} \) (dark detection) by feed-forward \([\text{PS, and path switch: APPENDIX (B)}]\) in terms of the result (photon number) on path b. Subsequently, the states of photons (A and B) on path 2 in the state \( |\Phi_2\rangle_{CAB} \) will be exchanged (swapped) to the state \( |\Phi_3\rangle_{CAB} \) after passing through the HOM gate, as follows:

\[
|\Phi_3\rangle_{CAB} = \frac{1}{\sqrt{2}} \left( |H\rangle_C |\varphi\rangle_A |\varphi\rangle_B |0\rangle_B + |V\rangle_C |\varphi\rangle_A |\varphi\rangle_B |1\rangle_B \right).
\]

where the HOM gate (linear optics) using the HOM effect\(^{54}\), in Fig. 3, performs the swap operation. Consequently, the output state \( |\Phi_3\rangle_{CAB} \) on path 2 is given by

\[
|\Phi_3\rangle_{CAB} = \frac{1}{\sqrt{2}} \left( |H\rangle_C |\varphi\rangle_A |\varphi\rangle_B |0\rangle_B + |V\rangle_C |\varphi\rangle_A |\varphi\rangle_B |1\rangle_B \right).
\]

Then, three photons (A, B, and C) in this state, \( |\Phi_3\rangle_{CAB} \), pass through path-parity gate (2) using XKNLs, qubus beams, and PNR measurement, as described in Fig. 4. After the operation, shown in Fig. 4, of path-parity gate (2) on the state \( |\Phi_3\rangle_{CAB} \), the state \( |\Phi_4\rangle_{CAB} \) (pre-measurement), is given by

\[
|\Phi_4\rangle_{CAB} = \frac{1}{\sqrt{2}} \left( |H\rangle_C |\varphi\rangle_A |\varphi\rangle_B |0\rangle_B + |V\rangle_C |\varphi\rangle_A |\varphi\rangle_B |1\rangle_B \right).
\]

Fig. 3. [path-parity gate (1)] - This gate consists of weak XKNLs, qubus beams, and PNR measurement. After PNR measurement in a qubus beam (path b), the feed-forward process is applied to photon B as a result of the outcome of PNR measurement. The output state from this gate is transformed to the form that has the same paths (the sorted paths) of photons A and B. [HOM gate] - This gate is composed of a Mach-Zehnder interferometer with a \( \pi \)–phase shifter and employs the HOM effect\(^{54}\). The output states (photons A and B) from this gate are swapped by passing through two BSs and a \( \pi \)–phase shifter.
When the PNR measurement is applied in the coherent state (probe beam) of path b, if the outcome is dark detection, $|0\rangle_b$, the output state, $|\Phi_4\rangle_{CAB}$, is acquired as

$$|\Phi_4\rangle_{CAB} = \frac{1}{\sqrt{2}} (|\psi_1\rangle_A |\varphi_1\rangle_B + |\varphi_1\rangle_A |\psi_1\rangle_B) \otimes |2\rangle_C \otimes H |0\rangle_b. \tag{5}$$

Also, if the result is the state $|n\rangle_b$ ($n \neq 0$), the output state can be transformed to the state $|\Phi_4\rangle_{CAB}$ (dark detection) by feed-forward (PS, SF, and path switch) with regard to the result (photon number $n$) on path b. Then, after the photon C in the state $|\Phi_4\rangle_{CAB}$ passes through PBS in path-parity gate (2), the output state, $|\Phi_5\rangle_{ABC}$, is expressed as

$$|\Phi_5\rangle_{ABC} = \frac{1}{\sqrt{2}} (|\psi_1\rangle_A |\varphi_1\rangle_B \otimes |H\rangle_C^2 + |\varphi_1\rangle_A |\psi_1\rangle_B \otimes |V\rangle_C^2). \tag{6}$$

Subsequently, for the merging paths (1 and 2) of photon A in Eq. 6, the state $|\Phi_5\rangle_{ABC}$ passes through path-merging gate (1). After the operation, shown in Fig. 4, of path-merging gate (1) on the state $|\Phi_5\rangle_{ABC}$, the state, $|\Phi_6\rangle_{ABC}^{\text{pre-measurement}}$, is given by

$$|\Phi_6\rangle_{ABC}^{\text{pre-measurement}} = \frac{1}{\sqrt{2}} (|\psi_1\rangle_A |\varphi_1\rangle_B \otimes |H\rangle_C^2 + |\varphi_1\rangle_A |\psi_1\rangle_B \otimes |V\rangle_C^2) \otimes |\alpha\rangle^0_p |0\rangle^0_p + e^{-\frac{i (\cos \theta)^2}{2}} \sum_{n=0}^{\infty} (\frac{\sin \theta}{\sqrt{n!}})^2 |\varphi_1\rangle^n_p |\psi_1\rangle^n_p \otimes |\alpha\rangle^0_p |n\rangle^0_p. \tag{7}$$

According to the result of PNR measurement in the coherent state (probe beam) of path b, the output state is obtained as $|\Phi_6\rangle_{ABC} = (|\psi_1\rangle_A |\varphi_1\rangle_B \otimes |H\rangle_C^2 + |\varphi_1\rangle_A |\psi_1\rangle_B \otimes |V\rangle_C^2)$ without feed-forward or with feed-forward (PF and path switch). Also, as described in Fig. 4, the state $|\Phi_6\rangle_{ABC}$ will be expressed as $|\Phi_7\rangle_{ABC}^{\text{pre-measurement}}$ (pre-measurement) after the path-merging gate (2) regarding photon B, as follows:

$$|\Phi_7\rangle_{ABC}^{\text{pre-measurement}} = \frac{1}{\sqrt{2}} (|\psi_1\rangle_A |\varphi_1\rangle_B \otimes |H\rangle_C^2 + |\varphi_1\rangle_A |\psi_1\rangle_B \otimes |V\rangle_C^2) \otimes |\alpha\rangle^0_p |0\rangle^0_p + e^{-\frac{i (\cos \theta)^2}{2}} \sum_{n=0}^{\infty} (\frac{\sin \theta}{\sqrt{n!}})^2 |\varphi_1\rangle^n_p |\psi_1\rangle^n_p \otimes |\alpha\rangle^0_p |n\rangle^0_p. \tag{8}$$

Then, through the PNR measurement and feed-forward (PF and path switch) in path-merging gate (2), the output state is given by
\[ |\Phi_{\gamma}^{\text{ABC}}\rangle = \frac{1}{\sqrt{2}} (|\psi_{\alpha}^{\Lambda}| \varphi_{b}^{1} \otimes |H_{C}^{1}\rangle + |\varphi_{\alpha}^{\Lambda}| \psi_{b}^{1} \otimes |V_{C}^{1}\rangle). \quad (9) \]

From the input state, \(|\psi_{\alpha}^{\Lambda}| \varphi_{b}^{1} \otimes |R_{C}^{1}\rangle\), this output state, \(|\Phi_{\gamma}^{\text{ABC}}\rangle\), in Eq. 9 is transformed by passing the non-linearly and nearly optical gates (path-parity, path-merging, and HOM gates). Finally, the final state, \(|\Phi_{\gamma}^{\text{ABC}}\rangle\), is the same as the output state of the SWAP test in Fig. 1 after CPBS operates on photon C of the output state, \(|\Phi_{\gamma}\rangle_{\text{ABC}}\) as follows:

\[ -|\Phi_{\gamma}^{\text{ABC}}\rangle_{\text{CPBS}} = \frac{1}{\sqrt{2}} (|(|\psi_{\alpha}^{\Lambda}| \varphi_{b}^{1} + |\varphi_{\alpha}^{\Lambda}| \psi_{b}^{1}) \otimes |R_{C}\rangle + (|\psi_{\alpha}^{\Lambda}| \varphi_{b}^{1} - |\varphi_{\alpha}^{\Lambda}| \psi_{b}^{1}) \otimes |L_{C}\rangle). \quad (10) \]

Consequently, we can determine that two unknown states, A and B, are identical or not through the final state \(|\Phi_{\gamma}^{\text{ABC}}\rangle\), in Eq. 10, which is generated by our optical scheme in Fig. 2. In our schematic SWAP test, the non-linearly optical gates (two path-parity and two path-merging gates) are critical components for implementing the SWAP test. Thus, to ensure the high efficiency of these gates, the error probabilities (\(P_{\text{err}}^{\text{PP}}\): path-parity gate and \(P_{\text{err}}^{\text{PM}}\): path-merging gate) can be estimated by the probability to measure \(|0\rangle_{b}^{2}\) (dark detection) in \(|\pm \alpha \sin \theta\rangle_{b}\) on path b of the qubus beams (Figs 3 and 4), as follows:

\[ P_{\text{err}}^{\text{PP}} = P_{\text{err}}^{\text{PM}} = \frac{1}{2} \exp (-\alpha^{2} \sin^{2} \theta) \approx \frac{1}{2} \exp (-\alpha^{2} 2^{2}), \quad (11) \]

where \(2^{2} \sin^{2} \theta \approx 2^{2} \theta^{2}\) for \(\alpha \gg 1 \theta \ll 1\) and. If the parameters \((\alpha: \text{amplitude of coherent state and } \theta: \text{magnitude of conditional phase shift})\) are fixed as \(\alpha^{2} 2^{2} = 2.5\), the error probabilities (\(P_{\text{err}}^{\text{PP}}\) and \(P_{\text{err}}^{\text{PM}}\)) can be acquired as \(P_{\text{err}}^{\text{PP}} = P_{\text{err}}^{\text{PM}} < 10^{-3}\). Moreover, when we increase the amplitude of the coherent state or magnitude of the conditional phase shift in non-linear optical gates, the error probabilities (\(P_{\text{err}}^{\text{PP}}\) and \(P_{\text{err}}^{\text{PM}}\)) can approach zero.

So far, we have presented an optical scheme to implement a SWAP test using non-linearly optical gates (XKNLs, qubus beams, and PNR measurement) and a linearly optical gate (HOM gate) to determine if two unknown states are identical or not. However, because of the use of XKNLs in our scheme, the decoherence effect (photon loss and dephasing), which can induce the evolution of a quantum pure state into a mixed state, occurs in non-linearly optical gates (path-parity and path-merging gates) when our scheme is experimentally realized in practical optical fibers. Thus, we propose a method for the non-linearly optical gates (via XKNLs, qubus beams, and PNR measurement) to obtain robustness against the decoherence effect.

**Analysis of path-parity and path-merging gates under decoherence effect**

The non-linearly optical (path-parity and path-merging) gates consist of the interactions of XKNLs, qubus beams (coherent state), and PNR measurements and are essential components for implementing the proposed SWAP test (controlled swap operation) scheme. However, in optical fibers, photon loss (increasing error probability) and dephasing (decreasing the fidelity of the output state) occur because of the decoherence effect when non-linearly optical (path-parity and path-merging) gates are implemented in our SWAP test scheme, in practice. Thus, we need to analyze the efficiency (related to photon loss) and performance (related to dephasing) of nonlinearly optical gates, using XKNL, under the decoherence effect, and we also should demonstrate path-parity and path-merging gates, in our scheme, having high efficiency and high fidelity (performance) against the decoherence effect by the utilization of a coherent state with a large amplitude.

We introduce the solution of the master equation, which can describe the open quantum system (nonunitary operation), for analysis of the decoherence effect in a Kerr medium, as follows:

\[ \frac{\partial \rho(t)}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \gamma \left( a\rho a^{\dagger} + a^{\dagger} a \rho + \rho a^{\dagger} a \right), \quad (12) \]

where \(\gamma\), \(\langle = \beta/\chi\rangle\), and \(a^{\dagger}(a)\) are the energy decay rate, the interaction time, and the creation (annihilation) operator. The solution of the master equation can be written as \(\rho(t) = \exp \left( \int_{0}^{t} \frac{\partial \rho(s)}{\partial s} ds \right)\).

For application in the analysis of non-linearly optical (path-parity and path-merging) gates, we show the process model of the interaction of XKNLs and the decoherence effect (photon loss and dephasing) using the solution from the master equation (Eq. 12). We assume that the initial state (photon-probe system) is \(|H\rangle \otimes |\alpha\rangle\), and the interaction of XKNL (conditional phase shift: \(U_{\text{XKNL}}(H) |\alpha\rangle \rightarrow |H\rangle |\alpha e^{\beta}\rangle\)) can be operated on the probe beam (coherent state) if the control photon’s polarization is \(H\) (horizontal). After the interaction of XKNL, \(\hat{X}_{\gamma}\), and the decoherence effect, \(\hat{D}_{\gamma}\), which can be described as \(\hat{D}_{\gamma} |\alpha\rangle |\beta\rangle = \exp[-(1 - e^{-\gamma})(1 - e^{-\theta})]|H\rangle \otimes |\alpha \rangle |\beta\rangle\), for interaction time \(t = \beta/\gamma\), the output state can be represented by the solution of the master equation, as follows:

\[ (\hat{D}_{\gamma} \hat{X}_{\gamma}) |H\rangle \otimes |\alpha\rangle |\beta\rangle = \exp[-(\alpha^{2}(1 - e^{-\gamma})(1 - e^{-\theta}))]|H\rangle \otimes |\Lambda_{\gamma} \alpha e^{\beta} \rangle |\Lambda_{\gamma} \beta\rangle, \quad (13) \]

where \(\Lambda_{\gamma} = e^{-\gamma t}\) and \(\alpha^{2}\) is the rate of remaining photons resulting from photon loss. The coefficient on the right hand side in Eq. 14 is the coherent parameter, which can quantify the degree of dephasing. Note that the operation of the decoherence effect, \(\hat{D}_{\gamma}\), occurs with the interaction of XKNL, \(\hat{X}_{\gamma}\), in this process. For a good approximation of...
the process model of the interaction of XKNLs and the decoherence effect, we can take an arbitrarily small time, \( \Delta t = (t/N) \), for the interaction of XKNL between photons and probe beam in a Kerr medium. Finally, equation 14 can be transformed to the process model26,27,32 to analyze the efficiency and performance of nonlinearly optical (path-parity and path-merging) gates, as follows:

\[
\langle \hat{D}_{\Delta t} \hat{X}_M \rangle^N |H\rangle \langle V| \otimes |\alpha\rangle \langle \alpha | = \exp \left[ -\alpha^2 (1 - e^{-\gamma \Delta t}) \sum_{n=1}^{N} e^{-\gamma \Delta t(n-1)} (1 - e^{i\theta n \Delta t}) \right] |H\rangle \langle V| \otimes |A_1 \alpha e^{i\theta} \rangle \langle A_1 \alpha |,
\]

(14)

where \( \hat{D}_{\Delta t} \hat{X}_M \) is the process model of the interaction of XKNLs and the decoherence effect, we can take an arbitrarily small time, \( \Delta t = (t/N) \), and \( \alpha \in \mathbb{R} \). Also, an optical fiber, in which the nonlinearly optical gate using XKNLs is realized, of approximately 3000 km is required to acquire the magnitude of the phase shift, \( \theta = \pi \), of the XKNL56,57. For analysis of the efficiency and performance of nonlinearly optical gates, based on the process model (Eqs 13 and 15) under the decoherence effect, we use commercial fibers56,57 with a signal loss of 0.364 dB/km (\( \chi/\gamma = 0.0125 \)) and pure silica core fibers57 with a signal loss of 0.15 dB/km (\( \chi/\gamma = 0.0303 \)), representing current technology.

**Path-parity gates (1 and 2).** When the path-parity gates (1 and 2) are implemented in an optical fiber56,57, we should consider how the decoherence effect (photon loss and dephasing) affects the efficiency and performance of the output states. Thus, the output states (|\( \Phi_2 \rangle_{CABP} \) and |\( \Phi_3 \rangle_{CABP} \) in Eq. 7) of path-parity gates (1 and 2) will be modified into the form of a density matrix, as a result of the decoherence effect, as follows:

\[
\rho_{CABP}^2 = \rho_{CABP}^4 = \frac{1}{4} \begin{pmatrix} 1 & KC_1 & LF_1 & MC_1 \\ KC_1^* & 1 & OC_1 & LF_1 \\ LF_1^* & OC_1^* & 1 & MC_1 \\ OC_1^* & LF_1^* & MC_1^* & 1 \end{pmatrix},
\]

(15)

where we define the bases of \( \rho_{CABP}^2 \) and \( \rho_{CABP}^4 \) from top to bottom and left to right by the output state of Eqs 3 and 8, as follows:

\[
\rho_{CABP}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} (|H\rangle_{C}^i |\psi_{1}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{1}^2 |\alpha_{b}^0 \rangle^0, \\ (|H\rangle_{C}^i |\psi_{2}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{2}^2 |\alpha_{b}^0 \rangle^0, \\ (|V\rangle_{C}^i |\psi_{1}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{1}^2 |\alpha_{b}^0 \rangle^0, \\ (|V\rangle_{C}^i |\psi_{2}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{2}^2 |\alpha_{b}^0 \rangle^0, \end{pmatrix},
\]

\[
\rho_{CABP}^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} (|H\rangle_{C}^i |\psi_{1}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{1}^2 |\alpha_{b}^0 \rangle^0, \\ (|H\rangle_{C}^i |\psi_{2}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{2}^2 |\alpha_{b}^0 \rangle^0, \\ (|V\rangle_{C}^i |\psi_{1}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{1}^2 |\alpha_{b}^0 \rangle^0, \\ (|V\rangle_{C}^i |\psi_{2}^j,_{\alpha}^k \rangle_{b}^0, & \Lambda_{2}^2 |\alpha_{b}^0 \rangle^0, \end{pmatrix},
\]

(16)

where \( \Lambda = e^{-N \theta / \gamma} \) is the rate of remaining photons resulting from photon loss. Regarding the above equations (15 and 16), the forms of the two output states (|\( \Phi_2 \rangle_{CABP} \) and |\( \Phi_3 \rangle_{CABP} \) are identical, Eq. 15, but have different basis sets, Eq. 16. Also, using the process model (Eq. 14), the coherent parameters (C, O, L, K, and M) in Eq. 15 are given by

\[
C = \exp \left[ -\alpha^2 (1 - e^{-\gamma \Delta t}) \sum_{n=1}^{N} e^{-\gamma \Delta t(n-1)} (1 - e^{i\theta n \Delta t}) \right],
\]

\[
O = \exp \left[ -\alpha^2 (1 - e^{-\gamma \Delta t}) (1 - e^{i\theta}) \sum_{n=1}^{N} e^{-\gamma \Delta t(n-1)} \right],
\]

\[
L = \exp \left[ -\alpha^2 (1 - e^{-\gamma \Delta t}) \sum_{n=1}^{N} e^{-\gamma \Delta t(n-1)} (1 - e^{i\theta n \Delta t}) \right],
\]

\[
M = \exp \left[ -\alpha^2 (1 - e^{-\gamma \Delta t}) \sum_{n=1}^{N} e^{-\gamma \Delta t(n-1)} (1 - e^{i\theta (n-1) \Delta t}) \right],
\]

\[
K = \exp \left[ -\alpha^2 (1 - e^{-\gamma \Delta t}) \sum_{n=1}^{N} e^{-\gamma \Delta t(n-1)} (1 - e^{i\theta n \Delta t}) \right],
\]

(17)
where $\tilde{X}_t = (\tilde{X}_t^{\hat{X}})_{\chi}$, and $\chi = t = \chi N \Delta t = N \Delta \theta$ for small time, $\Delta t = (t/N)$, and $\alpha \in \mathbb{R}$. We can quantify the degree of dephasing to evolve a pure state into a mixed state using the coherent parameters in Eq. 15.

First, for the analysis of the efficiency of the path-parity gate, we fix the parameter value, $\alpha \theta = \alpha \chi t = 2.5$, for $P_{\text{err}}^{\text{PP}} < 10^{-3}$ (which is the error probability, Eq. 11, without the decoherence effect), and assume that the path-parity gate is operated in optical fibers having signal losses of 0.364 db/km ($\chi/\gamma = 0.0125$) and 0.15 dB/km ($\chi/\gamma = 0.0303$). Figure 5 represents the modified error probability, $P_{\text{err}}^{\text{PP}}$, of the output state, $\rho_{\text{CAB-P}}^{2}$ or $\rho_{\text{CAB-P}}^{4}$, and the rate, $\Lambda_i^4$, of the remaining photons in the probe beam against the decoherence effect caused by optical fibers having signal losses of 0.364 db/km ($\chi/\gamma = 0.0125$) and 0.15 dB/km ($\chi/\gamma = 0.0303$). Because of the decoherence effect, the error probability, $P_{\text{err}}^{\text{PP}}$, of the output state, $\rho_{\text{CAB-P}}^{2}$ or $\rho_{\text{CAB-P}}^{4}$, is modified to

$$P_{\text{err}}^{\text{PP}} \approx \exp[- \Lambda_i^4 \cdot \alpha^2 \theta^2/2] = \exp[- e^{-2 \alpha \chi t} \cdot \alpha^2 \theta^2/2],$$

where $\Lambda = e^{-2 \alpha \chi t}$ (the rate of remaining photons) with $\alpha \theta = \alpha \chi t = 2.5$, and the signal loss of 0.364 db/km ($\chi/\gamma = 0.0125$) and 0.15 dB/km ($\chi/\gamma = 0.0303$), depending on the optical fibers. When increasing the amplitude of the coherent state (probe beam), the error probability, $P_{\text{err}}^{\text{PP}}$, can be decreased, and also the rate, $\Lambda_i^4$, of remaining photons can approach 1 with reliable PNR measurement, as described in Fig. 5. In addition, the values of the rate, $\Lambda_i^4$, of remaining photons and the error probability, $P_{\text{err}}^{\text{PP}}$, with respect to the signal loss rates of optical fibers and the amplitude of coherent states ($100 \leq \alpha \leq 80000$), are listed in the Table of Fig. 5. Consequently, by our analysis (using the process model, Eq. 14), the values in the Table clearly show that the path-parity gate can obtain high efficiency, $P_{\text{err}}^{\text{PP}} < 10^{-3}$ and a high rate of remaining photons, $\Lambda_i^4 \rightarrow 1$, with fixed $\alpha \theta = \alpha \chi t = 2.5$ in optical fibers when we employ a coherent state with a strong amplitude, $\alpha > 80000$(probe beam) under the decoherence effect.

Second, for the analysis of the performance of the path-parity gate under the decoherence effect, we should consider the values of coherent parameters, which can quantify the amount of evolution of the pure state into the mixed state, in Eq. 15, and we also should calculate the fidelities between the density matrices ($\rho_{\text{CAB-P}}^{2}$ and $\rho_{\text{CAB-P}}^{4}$ in Eq. 15) and output states ($|\Phi_{2}^{\text{CAB-P}}\rangle_{\chi}$ in Eq. 3, and $|\Phi_{4}^{\text{CAB-P}}\rangle_{\chi}$ in Eq. 7) in optical fibers having signal losses of 0.364 db/km ($\chi/\gamma = 0.0125$) and 0.15 dB/km ($\chi/\gamma = 0.0303$). If we consider the ideal case (without the decoherence effect: the output states in Sec. 2), all of the absolute values of the off-diagonal terms in the output states, $|\langle \Phi_{2}\rangle_{\chi}|_{\text{CAB-P}}$ and $|\langle \Phi_{4}\rangle_{\chi}|_{\text{CAB-P}}$ (i.e., $|\langle \Phi_{2}\rangle_{\chi}|_{\text{CAB-P}}$ and $|\langle \Phi_{4}\rangle_{\chi}|_{\text{CAB-P}}$ the form of the density matrix), of the path-parity gates are 1. This means that the output states are maintained in the pure states. However, the nonlinearly optical gates cannot avoid the decoherence effect when they are implemented in practice. This effect finally induces the pure state to evolve into the mixed state (classical state) by the dephasing of coherent parameters. To analyze this process, we apply the forms of density matrices ($\rho_{\text{CAB-P}}^{2}$ and $\rho_{\text{CAB-P}}^{4}$ in Eq. 15), which consider the coherent
and $\Phi \otimes \rho_{\text{CAB}}$ according to the amplitude of the coherent state with $\alpha = \alpha \chi t = 2.5$ and $N = 10^4$ in optical fibers. Using our process model (Eq. 14), the absolute values of coherent parameters will approach 1 with increasing amplitude of the coherent state ($\alpha > 8000$) in optical fibers with signal losses of 0.364 dB/km ($\chi/\gamma = 0.0125$) and 0.15 dB/km ($\chi/\gamma = 0.0303$).

Figure 6. The graph represents the differences in the absolute values of coherent parameters (off-diagonal terms) in $\rho_{\text{CAB}}^2$ and $\rho_{\text{CAB}}^4$ according to the amplitude of the coherent state with $\alpha \chi t = 2.5$ and $N = 10^4$ in optical fibers.68,69. Using our process model (Eq. 14), the absolute values of coherent parameters will approach 1 according to the amplitude of the coherent state with $\alpha \chi t = 2.5$ and $N = 10^4$ in optical fibers. Using our process model (Eq. 14), the absolute values of coherent parameters will approach 1 according to the amplitude of the coherent state with $\alpha \chi t = 2.5$ and $N = 10^4$ in optical fibers. Using our process model (Eq. 14), the absolute values of coherent parameters will approach 1 according to the amplitude of the coherent state with $\alpha \chi t = 2.5$ and $N = 10^4$ in optical fibers.

The fidelity, $F_{\text{PP}}$, between the output states ($\rho_{\text{CAB}}^2$ and $\rho_{\text{CAB}}^4$) without the decoherence effect and the density matrices ($\rho_{\text{CAB}}^2$ and $\rho_{\text{CAB}}^4$) under the decoherence effect is given by

$$F_{\text{PP}} \equiv \left| \text{Tr} \left[ \rho_{\text{CAB}}^2 \rho_{\text{CAB}}^4 \right] \right| = \left| \text{Tr} \left[ \rho_{\text{CAB}}^2 \rho_{\text{CAB}}^4 \right] \right| = \frac{1}{2} \left| \text{Tr} \left[ \rho_{\text{CAB}}^4 \rho_{\text{CAB}}^2 \right] \right| = \frac{1}{2} \left| \text{Tr} \left[ \rho_{\text{CAB}}^4 \rho_{\text{CAB}}^2 \right] \right|$$

where $C$, $O$, $L$, $K$, and $M$ are the coherent parameters in Eq. 17. As described in Fig. 7, we can confirm the high fidelities ($F_{\text{PP}} > 0.9$) of the output states when utilizing the strong amplitude of the coherent state ($\alpha > 10^4$). The various values of fidelities and the required magnitude of conditional phase shifts ($\theta = \chi t$), according to the amplitudes of the coherent state with $\alpha \chi t = 2.5$ and $N = 10^4$, are summarized in the Table in Fig. 7. From this result (using the strong coherent state), we can obtain two advantages for reliable performance of path-parity gates: (1) high fidelity – According to our process model, the coherent parameters in output states, $\rho_{\text{CAB}}^2$ and $\rho_{\text{CAB}}^4$, approach 1 to maintain pure states. Specifically, we can avoid the evolution into mixed states induced by dephasing of coherent parameters; (2) feasible implementation – The magnitude of the conditional phase shift in nature is tiny, $\theta \approx 10^{-18.59}$, although it can be increased by electromagnetically induced transparency, $\theta \approx 10^{-24.60}$. By our analysis, the magnitude of the conditional phase shift is required to be small with fixed $\alpha \chi t = 2.5$ and $N = 10^4$ when increasing the amplitude of the coherent state (i.e., if $\alpha = 80000$ in the optical fiber with signal loss of $0.15 \text{ dB/km}$, then $F_{\text{PP}} = 0.999$ and $\theta \approx 3.12 \times 10^{-3}$, as listed in the Table of Fig. 7). Thus, when we employ the strong coherent state (probe beam), path-parity gates are feasible to experimentally realize in practice because of the small conditional phase shift.

Path-merging gates (1 and 2). We should also consider the effect of decoherence in the path-merging gates (1 and 2) on the efficiency and performance of the output states. The output states ($\rho_{\text{ABC}}^4$ in Eq. 7, and $\rho_{\text{ABC}}^6$ in Eq. 8) of the path-merging gates (1 and 2) should be modified by the decoherence effect as follows:

$$\rho_{\text{ABC}}^6 = \frac{1}{2} \begin{pmatrix} 1 & C^2 \\ 1 & 1 \end{pmatrix}$$

where we define the bases of $\rho_{\text{ABC}}^6$ and $\rho_{\text{ABC}}^4$ from top to bottom and left to right by the output state of Eqs 8 and 9, as follows:
With the fixed parameters $\alpha \theta = \alpha \chi t = 2.5$ and $N = 10^3$ in optical fibers (with signal losses of 0.364 dB/km and 0.15 dB/km), the fidelities, $F_{PP}$, of the output states in path-parity gates are represented in diagrams depending on the amplitudes of the coherent state ($\alpha = 100, 10^4$). The values of fidelities and magnitude of conditional phase shifts are shown in the Table as calculated using our process model (Eq. 14) with $\alpha \theta = \alpha \chi t = 2.5$ and $N = 10^3$ in optical fibers. If the amplitude of the coherent state increases, the fidelities increase ($F_{PP} \rightarrow 1$) and the magnitude of conditional phase shifts decrease ($\theta \rightarrow \chi t \rightarrow$ small), indicating reliable performance of the path-parity gates.

\[
P_{ABC\otimes P}^6 = \begin{cases} \frac{1}{\sqrt{2}} (|\psi \rangle_{A} |\phi \rangle_{B} |\Lambda \rangle_{C} + |\phi \rangle_{A} |\psi \rangle_{B} |\Lambda \rangle_{C}) \otimes |\alpha \rangle_{P} |\theta \rangle_{P}^b, \\
\frac{1}{\sqrt{2}} (|\psi \rangle_{A} |\phi \rangle_{B} |\Lambda \rangle_{C} - |\phi \rangle_{A} |\psi \rangle_{B} |\Lambda \rangle_{C}) \otimes |\alpha \rangle_{P} |\theta \rangle_{P}^b, \\
\end{cases}
\]

\[
P_{ABC\otimes P}^7 = \begin{cases} \frac{1}{\sqrt{2}} (|\psi \rangle_{A} |\phi \rangle_{B} |\Lambda \rangle_{C} + |\phi \rangle_{A} |\psi \rangle_{B} |\Lambda \rangle_{C}) \otimes |\alpha \rangle_{P} |\theta \rangle_{P}^b, \\
\frac{1}{\sqrt{2}} (|\psi \rangle_{A} |\phi \rangle_{B} |\Lambda \rangle_{C} - |\phi \rangle_{A} |\psi \rangle_{B} |\Lambda \rangle_{C}) \otimes |\alpha \rangle_{P} |\theta \rangle_{P}^b, \\
\end{cases}
\]

Regarding these equations (20 and 21), the two output states $|\phi \rangle_{2} = |\phi \rangle_{ABC\otimes P}$ and $|\phi \rangle_{7} = |\phi \rangle_{ABC\otimes P}$ have the same form (density matrix, Eq. 20) while having different basis sets, Eq. 21. In density matrices, $\rho_{ABC\otimes P}$ and $\rho_{ABC\otimes P}$, the coherent parameter, $\chi$, which can quantify the dephasing, is given in Eq. 17, where $\chi = \chi N \Delta t = N \Delta \theta$ for small time, $\Delta t \approx t/N$, and $\alpha \in \mathbb{R}$.

First, for the analysis of the efficiency of the path-merging gate, comparing the error probability, $P_{err}$, in Eq. 11, without the decoherence effect, we should recalculate the error probability, $P_{err}$, of the output state, $\rho_{ABC\otimes P}$ and $\rho_{ABC\otimes P}$, including photon loss, as follows:

\[
P_{err} \approx \exp(-\Lambda_1 \cdot \alpha^2 \theta^2/2) = \exp(-e^{-\chi t} \cdot \alpha^2 \theta^2/2),
\]

where $\Lambda_1 = e^{-\chi t/2}$ (the rate of remaining photons) with $\alpha \theta = \alpha \chi t = 2.5$, and signal losses of 0.364 dB/km ($\chi / \gamma = 0.0125$) and 0.15 dB/km ($\chi / \gamma = 0.0303$), depending on the optical fibers. In Fig. 8 and the Table therein, as the amplitude of the coherent state in path-merging gates increases, we can confirm the decreasing error probability, $P_{err}$, and the increasing rate of remaining photons, $\Lambda_1 \rightarrow 1$. Consequently, as with the path-parity gates (1 and 2), the values in the Table in Fig. 8 show that high efficiency, $P_{err} < 10^{-3}$ and a high rate of remaining photons $\Lambda_1 \rightarrow 1$, with fixed $\alpha \theta = \alpha \chi t = 2.5$ in optical fibers can be acquired, through our analysis (Eq. 14), using a coherent state with strong amplitude, $\alpha > 80000$ (probe beam), under the decoherence effect.

Second, for the analysis of the performance of the path-merging gate under the decoherence effect, we should analyze the absolute value of the coherent parameter, $|C|^2$, in $\rho_{ABC\otimes P}$ and $\rho_{ABC\otimes P}$ (shortly $\rho_{6 or 7}$), and the fidelities, $F_{PM}$, in optical fibers having signal losses of 0.364 dB/km ($\chi / \gamma = 0.0125$) and 0.15 dB/km ($\chi / \gamma = 0.0303$). As described in Fig. 9, the absolute values of the coherent parameter, $|C|^2$, increase to maintain the output states ($\rho_{6 or 7}$) in pure states (elimination of dephasing) by the strong coherent state under the decoherence effect, in
practice (optical fibers). This result, suggesting that a strong coherent state should be utilized for the reduction of dephasing, is the same as the result of path-parity gates by our analysis. Also, in the diagrams and Table of Fig. 9, the fidelity, \( F_{PM} \), of the density matrices (\( \rho \otimes \rho \) or \( \rho \otimes \rho \) or \( \rho \otimes \rho \)): under the decoherence effect) is calculated as

\[
F_{PM} = \sqrt{\langle \Phi_1 | \rho_{ABC\otimes P} | \Phi_1 \rangle} = \sqrt{\langle \Phi_2 | \rho_{ABC\otimes P} | \Phi_2 \rangle} = \frac{1}{\sqrt{2}} \sqrt{1 - |C|^2}. \tag{23}
\]

Finally, for the reliable performance (high fidelity, and weak XKNL: small magnitude of conditional phase shift) of path-merging gates, we should increase the amplitude of the coherent state for \( \alpha = 2.5 \) and \( N = 10^4 \) when experimentally implemented path-merging gates under the decoherence effect, as described in Fig. 9.

Consequently, according to our analysis (the process model based on the master equation), we demonstrate that the utilization of the strong (increasing amplitude) coherent state in nonlinearly optical gates (path-parity and path-merging gates in our SW AP test) will bring about high efficiency (small error probabilities) and reliable performance (robustness: high fidelities, and feasibility: weak XKNLs) with respect to the decoherence effect.

Conclusions

We presented an optical scheme for the SWAP test (controlled swap operation), via nonlinearly optical (path-parity and path-merging) gates and a linearly optical (HOM) gate, to definitely determine whether the difference between two unknown states in Sec. 2. We also demonstrated a method, which should utilize a strong coherent state according to our analysis, to obtain high efficiency (low error probability) and reliable performance (high fidelity) in nonlinearly optical gates under the decoherence effect, in Sec. 3. Therefore, the proposed scheme (SWAP test via weak XKNLs, qubus beams, and PNR measurements) has the following advantages:

1. When presented with the question of whether two unknown states are equal or not, the SWAP test can determine with certainty whether two unknown states are different in various QIP schemes (quantum communications: quantum authentication, quantum signature, and quantum computation: quantum machine learning, and Fredkin gate). Thus, we proposed a deterministic (determination of difference between two unknown states) and feasible (experimental implementation) scheme for the SWAP test using weak XKNLs, qubus beams, and PNR measurements.

2. In this paper, we demonstrated that nonlinearly optical (path-parity and path-merging) gates, which are designed using XKNLs, qubus beams, and PNR measurement, should employ a coherent state with a strong amplitude to obtain high efficiency (low error probability) and reliable performance (high fidelity) according to our analysis using the process model in Sec. 3. In the previous work\(^{28-30} \), which have proposed the various nonlinearly optical gates (including to path-parity and path-merging gates), for quantum information processing schemes, the affection of the decoherence effect, in practice, have been overlooked.
Compared with these works\textsuperscript{28–30}, we analyzed the decoherence effect by master equation, and derived the method, using strong coherent state, to reduce photon loss and dephasing (decoherence). Thus, when our scheme for the SW AP test is experimentally realized, it will be robust against the decoherence effect (photon loss and dephasing).

(3) Through the analysis in Sec. 3, we showed that our scheme (nonlinearly optical gates) require the small magnitude of the conditional phase shift ($\theta$), as described in Figs 7 and 9, because the conditional phase shift from Kerr media is extreme small\textsuperscript{59}, and difficult to increase by electromagnetically induced transparency\textsuperscript{43,60}. But our gates, compared with the former works\textsuperscript{23,24,28–30}, can obtain the high efficiency and reliable performance with tiny magnitude of conditional phase shift by utilizing the strong coherent state (for the reduction of decoherence effect), according to our analysis in Sec. 3. Therefore, when we employ the strong coherent state (probe beam), path-parity and path-merging gates are feasible to experimentally realize in practice because of the small conditional phase shift.

(4) In our scheme, the designed nonlinearly optical gates employ qubus beams and the strategy of PNR measurement. Therefore, we employed only positive conditional phase shifts ($\theta$) by XKNL in path-parity and path-merging gates. Kok in ref.\textsuperscript{61} showed that it is generally not possible to change the sign of the conditional phase shift ($-\theta$). Thus, our nonlinearly optical gates using only positive conditional phase shifts ($\theta$) with qubus beams and PNR measurement are more feasible than other nonlinearly optical gates\textsuperscript{26,27,32,39,40} that use the negative conditional phase shift ($-\theta$).

(5) As for a minor issue, because PNR measurements are applied on the probe beam of path b in all nonlinearly optical gates, the probe beam of path a can be recycled for other nonlinearly optical gates (if desired) for a more efficient implementation.

Consequently, we demonstrate that our scheme for the SWAP test to determine whether the difference between unknown states, using weak XKNLs, qubus beams, and PNR measurements, can be experimentally realized and is immune to the decoherence effect in optical fibers.

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**Acknowledgements**

This work was supported by the ICT R&D programs of MSIP/IITP (Grant No. B0101-16-1355), the KIST research program (Grant No. 2E29580), and the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2019R1A2C1006167), and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MIST) (No. 2019R1A2C2006381).

**Author Contributions**

Min-Sung Kang, and Jino Heo conceived the main idea. Min-Sung Kang wrote the manuscript. Min-Sung Kang, and Jino Heo calculated the main calculations. Jino Heo, Seong-Gon Choi, Sung Moon, and Sang-Wook Han analyzed the results. Jino Heo supervised the whole project. All authors reviewed the manuscript.

**Additional Information**

**Supplementary information** accompanies this paper at https://doi.org/10.1038/s41598-019-42662-4.

**Competing Interests:** The authors declare no competing interests.

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