Effective Spin Foam Models for Four-Dimensional Quantum Gravity

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A number of approaches to four-dimensional quantum gravity, such as loop quantum gravity and holography, situate areas as their fundamental variables. However, this choice of kinematics can easily lead to gravitational dynamics peaked on flat spacetimes. We show that this is due to how regions are glued in the gravitational path integral via a discrete spin foam model. We introduce a family of ‘effective’ spin foam models that incorporate a quantum area spectrum, impose gluing constraints as strongly as possible, and leverage the discrete general relativity action to specify amplitudes. These effective spin foam models avoid flatness in a restricted regime of the parameter space. The simpler effective spin foam models will be useful for numerical studies and in the investigation of coarse graining and renormalization.

Symplectic, metrical quantization. Envisioning the geometry of spacetime as dynamically evolving founded the revolutionary insights of general relativity (GR) that have resulted in direct measurements of gravitational time dilation, bending of starlight, and gravitational waves. However, this revolution remains incomplete. We still do not know how to fully characterize an evolving quantum spacetime geometry.

The quantization of spacetime geometry is an interplay between its symplectic and metrical aspects. In three dimensions alignment between these two facets of geometry allows construction of a discrete, simplicial path integral formulation of quantum gravity, the Ponzano-Regge model [1]. In this model, spacetime is decomposed into a large collection of tetrahedra that are glued along a subset of edges with matched lengths. The metrical and symplectic aspects of this geometry nicely align: the lengths encode the intrinsic metric and the dihedral angles of the tetrahedra encode the extrinsic geometry and these two sets of variables are canonically conjugated to each other [2, 3]. In the Euclidean signature case the angles are compact, which leads to discrete spectra for the lengths.

In 4D the situation is more subtle, and there is some tension between the symplectic and the metrical aspects. In a space-time split, the metric has two natural discretizations: (i) the lengths of edges, and (ii) the extrinsic curvature angles, which are defined on 2D faces. These variables are not canonically conjugate.

If the lengths are taken as fundamental, then the conjugate variables are contractions of the curvature angles with certain area-length derivatives [4]. If the curvature angles are taken to be fundamental, the conjugate variables are the face areas—whose quantization should then give a discrete area spectrum. Indeed, these variables arise naturally in connection reformulations of GR, like Loop Quantum Gravity (LQG), where the connection encodes the extrinsic curvature [5].

Symplectic geometry is of paramount importance in quantization and so the spin foam models [6]—discrete geometry path integrals—of LQG focus on this second set of ‘curvature-area’ variables. This focus has led to a rich set of results, in particular discrete area and volume spectra [7–10]. Area variables also play a central role in holography [11, 12], in particular for the reconstruction of geometry from entanglement [13, 14]. Discrete area spectra are key in many approaches to black hole entropy counting [15–18]. Nonetheless, there is a certain tension between this choice of curvature-area variables and the dynamics of GR: the area variables need to be constrained to avoid a suppression of curvature.

This possibility of a flat dynamics has arisen in the semiclassical analysis of spin foam amplitudes [19–25]. The semiclassical analysis allows only limited information on how strongly the constraints are imposed, this is one reason why the so-called flatness problem has yet to receive a satisfactory resolution [25]. Indeed, we will see that the discreteness of the areas prevents a sharp imposition of these constraints.

Here we tackle directly the question of whether a discrete, locally independent, area spectrum is consistent with the dynamics of GR. To this end we propose a family of ‘effective’ models that (a) incorporate a discrete area spectrum, (b) impose the constraints between the areas as strongly as allowed by the LQG Hilbert space structure, and (c) use—more directly than current spin foam models—a discretized GR action for the amplitudes.

These effective models allow us to show that the flatness problem can be overcome, but to do so also imposes certain restrictions involving the discretization scale, curvature per triangle, and the Barbero–Immirzi parameter, which controls the area spectral gap. Future work will show whether this is sufficient to ensure general relativistic dynamics in the continuum limit.

Discrete, locally-independent areas. We study a path integral for 4D quantum gravity regulated by a triangulation of spacetime. We work with quantum amplitudes for Euclidean signature simplices, leaving the Lorentzian case to future work. Our key assumption is that the ar-
neas have a discrete, prescribed spectrum. Further, we will take these area eigenvalues to be independent, more precisely (apart from triangle inequities) the measured values in the kinematical Hilbert space will not depend on the state away from the measured triangle.

The particular area spectrum we work with is

\[ a(j) = \gamma \ell_P^2 \sqrt{j(j+1)} \sim \gamma \ell_P^2 (j + 1/2), \tag{1} \]

where \( j \) is a half-integer (spin label), \( \ell_P = \sqrt{8\pi \hbar G/c^3} \) is the Planck length, \( \gamma \) is the dimensionless Barbero-Immirzi parameter, and \( \sim \) indicates the large-\( j \) asymptotic limit. We focus on the equispaced asymptotic spectrum.

This form for the area spectrum was first established in LQG [7–10], but discrete area spectra have been also discussed in the context of black hole spectroscopy [15].

Before taking up the path integral, we review the use of area variables in simplicial discretization of GR. These discretizations were first considered by Regge [26] and used length variables. A wide array of reformulations have been considered [27–32], and we use descriptive adjectives to capture the variables used in each form. The change from length to area variables turns out to be far more subtle than one might expect. A treatment in the more transparent context of Regge calculus will illuminate the issues before discussing the path integral.

**Actions for discretized GR.** In Length Regge Calculus (LRC) one substitutes the metric by lengths \( l \) assigned to the edges \( e \) of a triangulation. The \( l \) determine the triangle areas \( A_t(l) \) and the 4D (internal) dihedral angles \( \theta_e^\sigma(l) \) in 4-simplices \( \sigma \). Varying the LRC action

\[
S_{\text{LRC}} = \sum_{t \in \text{bdry}} \pi A_t(l) + \sum_{t \in \text{blk}} 2\pi A_t(l) - \sum_{\sigma} \sum_{t \supset \sigma} A_t(l) \theta^\sigma_e(l) \\
= \frac{1}{2} \sum_{t \in \text{bdry}} S^t_1(l) + \sum_{t \in \text{blk}} S^t_2(l) + \sum_{\sigma} S^\sigma_2(l) \tag{2}
\]

with respect to the bulk lengths leads to the dynamics

\[
\sum_{t \supset e} \frac{\partial A_t(l)}{\partial l_e} \epsilon_t(l) = 0, \quad \text{with} \quad \epsilon_t = \frac{2\pi}{2} - \sum_{\sigma \ni t} \theta^\sigma_e(l), \tag{3}
\]

where the deficit angle \( \epsilon_t \) is a measure of the curvature concentrated on the triangle \( t \). In the limit of a very fine triangulation one recovers Einstein’s equations [33].

Here we will employ a formulation in terms of area variables. A 4-simplex \( \sigma \) has 10 edges and 10 areas, one can thus locally invert the 10 functions \( A_t(l) \) that give the simplex’s areas in terms of its lengths. We will denote the resulting functions \( L^\sigma_a(a) \), where \( a \) collectively signifies the 10 areas associated to \( \sigma \).

We, thus, define the Area Regge Calculus (ARC) action [27, 28, 32], whose value on configurations with \( a_i = A_t(l) \) agrees with the LRC action

\[
S_{\text{ARC}} = \frac{1}{2} \sum_{t \in \text{bdry}} S^t_1(a) + \sum_{t \in \text{blk}} S^t_2(a) + \sum_{\sigma} S^\sigma_2(a), \tag{4}
\]

where \( S^e_1(a) = 2\pi a_i \) and \( S^\sigma_2(a) = S^\sigma_2(L^\sigma_a(a)) \). Strikingly, freely varying the bulk areas one finds the equations of motion \( \epsilon_i = 0 \), which impose flatness.

Despite these equations of motion, the theory features propagating degrees of freedom, which are, however, of a non-geometrical nature [32]. These arise because the number of matching conditions, when gluing two 4-simplices, differ between LRC and ARC. For this gluing we need to identify the data of the shared tetrahedron. As it has six edges and four triangles we must match six pairs of lengths in LRC, but only four pairs of areas in ARC.

This mismatch can be resolved by introducing 3D dihedral angles \( \Phi_{e,\tau}^\sigma \) associated to edges \( e \) in the tetrahedron \( \tau \). These angles are determined by the lengths of the tetrahedron, and can also be expressed as functions of the areas \( a_i \) of the 4-simplex \( \sigma \). They allow us to introduce two constraints per bulk tetrahedron

\[
C_{i}^{\sigma,\tau}(a) = \Phi_{e_i}^\tau(a) - \Phi_{e_i}^\tau(a) \quad i = 1, 2, \tag{5}
\]

where \((e_1, e_2)\) is any choice of a pair of non-opposite edges in \( \tau \). Together with matching of the four areas, the two matching conditions (5) ensure that the geometries imposed on \( \tau \) by \( \sigma \), on the one hand, and by \( \sigma' \) on the other, agree. Varying the ARC action (4) on the corresponding constraint hypersurface gives equations of motion equivalent to LRC. The constraints (5) involve pairs of 4-simplices, and this makes the specification of free boundary data difficult.

This can be alleviated by introducing auxiliary variables that allow one to localize the constraints onto pairs of tetrahedra. Indeed, as the constraints feature 3D dihedral angles, it is natural to introduce these as explicit variables \( \phi^\tau_i, i = 1, 2 \). We demand for each pair \((\tau, \sigma)\) with \( \tau \subset \sigma \) the new constraints

\[
C_{i}^{\sigma,\tau}(\phi, a) = \phi_{e_i}^\tau(a) - \Phi_{e_i}^\tau(a) \quad i = 1, 2 \tag{6}
\]

This imposes the constraints (5) for each bulk tetrahedron and adds for each boundary tetrahedron two dihedral angles as boundary data as well as two constraints.

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1 These functions also depend on a discrete parameter that accounts for the multiple roots that appear in the inversion of \( A_t(l) \). This discrete parameter appears as a summation variable for the (constrained) Area Regge path integral. To ease notation we will suppress this parameter.
In contrast to (5) the constraints (6) localize onto 4-simplices. This allows path integral amplitudes that factorize over the 4-simplices.

An even more local reformulation of the constraints isolates the conditions on the 3D boundary data. It uses the matching for the geometry of a triangle $t$ induced by the neighbouring tetrahedra $\tau$ and $\tau'$, respectively. This geometry is specified by three variables, in addition to the area matching we need two constraints
\[ G^{\tau,\tau'}_k(\phi, a) = \alpha^{t,\tau}_k(\phi, a) - \alpha^{t,\tau'}_k(\phi, a) \quad k = 1, 2, \quad (7) \]
where $\alpha^{t,\tau}_v$ denotes the 2D angles at two vertices $v_1, v_2$ of $t$, determined by the geometric data of $\tau$. Imposing the constraints (7) for all 10 pairs of neighbouring tetrahedra $(\tau, \tau')$ in a simplex $\sigma$ is equivalent to imposing the constraints (6) for all 5 tetrahedra in $\sigma$ [31].

The original form of Area Angle Regge Calculus (AARC) [31] featured the constraints (7). These specify in concrete terms the enlargement of the LQG phase space [34, 35] as compared to the LRC phase space [4]. Armed with these understandings, we take up the path integral.

**Path integral.** To incorporate a discrete area spectrum (1), we employ the Constrained Area Regge formulation, and thus sum over spin labels $j_t$:
\[ Z = \sum_{\{j_t\}} \mu(j) \prod_t A_t(j) \prod_{\sigma} A_\sigma(j) \prod_{\tau \in \text{blk}} G^{\tau,\sigma}_t(j) \quad (8) \]
Here
\[ A_t = \exp(\gamma X \pi(j_t + \frac{1}{2})) \quad (9) \]
is the weight for the bulk (with $X = 2$) and boundary ($X = 1$) triangles. The simplex amplitude is
\[ A_\sigma = \exp \left(-\gamma \sum_{t \in \sigma} (j_t + \frac{1}{2}) \theta_1^t(j) \right) T_\sigma(j), \quad (10) \]
where $T_\sigma(j) = 1$, if the lengths defined by the areas satisfy the generalized triangle inequalities, and is vanishing otherwise. The precise form of the measure factor $\mu(j)$ will not be important for the discussion here.

The factors $G^{\tau,\sigma}_t(j)$ implement the constraints (5), and are therefore crucial for imposing the dynamics of LRC instead of ARC.

However, imposing the constraints (5) sharply, i.e. setting $G^{\tau,\sigma}_t(j) = 1$ if the constraints are satisfied, and $G^{\tau,\sigma}_t(j) = 0$ otherwise, leads to a severe problem: As we allow only discrete (asymptotically equispaced) values for the areas, the constraints (5) constitute diophantine conditions for the spin labels. These can only be satisfied for a very small set of labels with accidental symmetries, e.g. if all 10 pairs of labels match. The resulting reduction in the density of states prevents a suitable semiclassical limit.

One way out is to weaken the constraints (5), e.g. by allowing a certain error interval. But, one has to navigate between Scylla—reducing too much the density of states—and Charybdis—imposing a dynamics which does not match GR.

Here we will take guidance from loop quantum gravity. The associated phase space includes areas $a_t$ and 3D dihedral angles $\phi^t_{\tau'}$ as variables [34]. Crucially, the 3D dihedral angles at two non-opposite edges $(e_1, e_2)$ in a tetrahedron $\tau$ do not Poisson commute
\[ h \{\phi^t_{e_1}, \phi^t_{e_2}\} = \ell^t_{\gamma \gamma} \sin \alpha^{t,\tau}_v = \sin \alpha^{t,\tau}_v (j_t + \frac{1}{2}), \quad (11) \]
where $\alpha^{t,\tau}_v$ is the angle between $(e_1, e_2)$. This non-commutativity arises as the geometry of a tetrahedron is encoded into the set of normals to its triangles, which are then quantized as (non-commuting) angular momentum operators [9, 38–40].

Respecting the uncertainty relations resulting from (11), we can impose the constraints only weakly. To achieve an as-strong-as-possible imposition we will employ coherent states in the angle variables. There are different constructions available for tetrahedral states [7] that are coherent in the two degrees of freedom encoding the 3D dihedral angles, but are eigenstates for the area operators [42–45]. We will denote such states $K_\tau(\phi_1, \phi_2; \Phi_1, \Phi_2)$, where $(\phi_1, \phi_2)$ are the arguments of the wave functions and $(\Phi_1, \Phi_2)$ are the angles on which the wave function is peaked. With the associated measure $d\mu_K(\phi_1, \phi_2)$ we define
\[ Z' = \sum_{\{j_t\}} \mu(j) \int_{\tau} d\mu_K(\phi) \prod_t A_t(j) \prod_{\sigma} A'_\sigma(j, \phi) \quad (12) \]
where the new simplex amplitude is given by
\[ A'_\sigma(j, \phi) = A_\sigma(j) \prod_{\tau \in \sigma} K_\tau(\phi_{e_1}^t; \Phi^t_{e_1}(\sigma)). \quad (13) \]

3 There are (10) redundancies between the (20) constraints (7) associated to a simplex.
4 The squared volumes of the various sub-simplices of $\sigma$ and of $\sigma$ itself, as defined by the appropriate Caley-Menger determinants, has to be non-negative.
5 It can be specified by requiring a discrete remnant of (approximate) diffeomorphism invariance [36].
6 This Poisson bracket (with $\gamma = 1$) also appears in the Kapovich-Millson phase space for linkages [37], which can be used to describe the space of shapes of tetrahedra with fixed areas [9, 38, 43]. The non-commutativity of the angles is inherited by the lengths [34, 41].
7 The associated Hilbert space is $H_{\{j_t\}} = \text{Inv}(\otimes^t_{i=1} V_{j_i})$, where $V_j$ is a spin-$j$ representation space for $SU(2)$ and $\text{Inv}$ denotes invariance under the global $SU(2)$ action.
8 Here we assume that the tetrahedra have an outward orientation. Changing the orientation leads to a complex conjugated $K_\tau$. 
Integrating out the dihedral angles for the bulk tetrahedra we regain—modulo boundary contributions—a path integral of the form (8) where now

\[ G^\sigma_\tau (j) = \langle K_\tau (\cdot; \Phi_{i,\tau}^\sigma (j)) | K_\tau (\cdot; \Phi_{i,\tau}^\sigma (j)) \rangle. \] (14)

This inner product is peaked on the matching conditions (5) and provides a precise sense in which these conditions are weakly imposed.

To further simplify the models we can approximate \( G^\sigma_\tau \) by, e.g., Gaussians in the angles \( \Phi_{i,\tau}^\sigma \). To count the number of configurations not suppressed by the \( G^\sigma_\tau \) factor we approximate it with a Heaviside function. That is, we allow \( (\Phi_{i,\tau}^\sigma, \Phi_{i,\tau}^\sigma) \) to mismatch by as much as

\[ \sigma(\Phi) = \beta \sqrt{\int_0^{\pi} \frac{\sin \alpha_{u,\tau}}{a_t} \beta \sqrt{\sin \alpha_{u,\tau}} \left( \frac{j_t}{2} + \frac{1}{2} \right)}. \] (15)

Here we have introduced a parameter \( \beta \) which can be tuned between (unconstrained) Area Regge dynamics and a sharp imposition of the matching constraints.

We have also used this criterion to determine the number of allowed configurations on a complex of two glued 4-simplices, see Appendix A. The configuration we have studied is symmetry reduced with 3 length and 3 area parameters on each of its simplices. For the two glued simplices there are 4 length and 5 area parameters. The number of length configurations for the two glued simplices, with area values \( A_1 \in \{ \frac{1}{2}, 1, \cdots, N \} \), scales as \( N^{1.01 \times p} \), where \( p = 3 \) for \( \beta = 0 \) (the shape matching constraints hold exactly), and \( p = 5 \) for \( \beta = \infty \) (the shape matching conditions need not hold). A scaling with \( N^4 \) arises for \( \beta \approx 0.15 \). We also considered just one (symmetry reduced) simplex with \( p = 2, 3 \) and 4 lengths and area parameters. The number of configurations scales with \( N^{1.01 \times p} \). This test suggests that the weakened matching condition (15) does lead to a reasonable number of configurations.

Relation to spin foams. Spin foams arise from a discretized \( SO(4) \)-gauge formulation of GR [6]. The main object is a simplex amplitude [46] depending on the spin labels \( j_t \) and, for the more recent models [47], on intertwiner labels associated to the tetrahedra. A number of key works have shown that in the limit of large spins the simplex amplitude includes saddle points peaked on the cosine of the Regge action [48–50]. The cosine results from a sum over orientation, in addition there are further saddle points describing degenerate simplex configurations. In practice the large spin limit is already obtained with spin values around \( j = 10 \).

The simplex amplitudes require, however, a huge effort for their numerical evaluation [51], and this has hindered deeper insight into the dynamics of spin foams, including a resolution of the flatness problem [19–25]. Other open questions include whether summing over orientations or including degenerate configurations prevent a suitable semiclassical dynamics [25, 53].

Here we rather propose to test a key assumption of LQG, namely a Hilbert space describing independent area variables with a discrete (asymptotically equidistant) spectrum. As we have argued, this allows only a weak imposition of the shape matching constraints. It is not clear whether such a weak imposition is consistent with a (semiclassical) gravitational dynamics. To tackle this question, we need workable amplitudes. Thus we propose to use, instead of the involved spin foam simplex amplitudes, the exponentiated (Area) Regge action\(^{10} \) together with a mechanism to impose the shape matching constraints. If it becomes clear that such a model leads to a gravitational dynamics, one can go to more complicated versions, and e.g. study the effects of including a sum over orientations.

In the following we will elaborate more on a possible relation of our proposed family of models to various spin foam models. Note that the large \( j \) limit of the spin foam amplitudes reveals only a limited amount of information. For example, one finds that for the EPRL-FK models the saddle point conditions include the shape matching equations (7) for non-degenerate configurations [48–50]. However, it is not known how weakly or strongly these constraints are imposed [25]. Another possible source for flatness problems is the imposition of the closure (Gauß) constraints [19–25]. Here we disregard possible issues with the Gauß constraints and assume that the shape matching constraints are as strongly implemented as allowed by the LQG kinematics.

The first spin foam amplitude, known as the Barrett-Crane (BC) model [46], featured only a sum over areas (no sum over 3D dihedral angles). In this model amplitudes factorize over simplices and thus cannot include a gluing factor \( G^\sigma_\tau \), as in our proposal (8). It is therefore conjectured that the BC model describes the dynamics of ARC [27, 28, 32]. Thus, including the factors (14) can be seen as correcting the BC model\(^{11} \).

A newer class of models [47], known as EPRL-FK, include a summation over area and angle variables. Crucially the boundary Hilbert space for these models is the LQG Hilbert space. With the assumptions outlined above for these spin foam models, we conjecture that

\(^{10} \) In 3D the sum over orientations still allows for a semiclassical and continuum limit, which reproduces continuum GR [52].

\(^{11} \) Another possibility is to not take the exponentiated area Regge action as the simplex amplitude \( A_\tau \), but an action resulting from a gauge-reduced SU(2) BF theory, which involves areas and 3D dihedral angles and leads to a topological theory.

\(^{12} \) This would also lead to a boundary Hilbert space coinciding with the LQG Hilbert space (with \( \gamma = 1 \)).
our model (12) describes the behavior of these models for larger spins, if sums over orientations and degenerate configurations can be ignored.

A special feature for our model (12) is that it includes an integration over two dihedral angles per tetrahedron. These two dihedral angles are encoded into only one quantum number (e.g. if one uses a spin network basis). This is why the coherent states are crucial: using a Segal-Bargmann (like) transform one can change the amplitudes and integration from two variables to one quantum number per tetrahedron. This will then allow a more direct comparison with the EPRL-FK amplitudes.

Instead of a gauge formulation, one can also employ a higher gauge formulation to study gravity [54–56]. A related topological state sum model [56, 57] features an amplitude factor given by the cosine of the Regge action (without having to take a semiclassical limit). But the model sums over both (discrete) area variables and (continuous) length variables. Constraining the areas to be functions of the lengths one does obtain a formulation of gravity. However, insisting on discrete areas leads to the same problem as discussed here [55], namely a drastic reduction in the density of states. In fact, a canonical analysis [58] reveals that the corresponding constraint system is, like the shape matching constraints, second class.

On the flatness problem. We now take up the question of whether the constraints are implemented sufficiently strongly to avoid flatness. We consider a first test case consisting of a triangulation where we can control the scale for the bulk area variable and the bulk curvature through the boundary data. Specifically we consider a complex consisting of three 4-simplices sharing a (bulk) triangle. There are no bulk edges, thus no bulk variables to sum over in LRC, and the (bulk) deficit angle is determined by the boundary lengths. Nonetheless in ARC, there is one bulk variable to sum over, which imposes a vanishing deficit angle for the internal triangle.

The shape matching constraints restrict the (effective) summation range for the area variable and the question arises as to whether this restriction is sufficient to allow for a non-vanishing expectation value for the bulk deficit angle.

We will be applying only scaling arguments and approximate the imposition of the constraints with Gaussians. From (15) we see that the deviation for the 3D dihedral angle scales with $\sigma(\Phi) \sim 1/\sqrt{j}$, where we assume that the boundary areas have approximately equal values determined by the spin value $j$.

The deviation $\sigma(j_{blk})$ for the (bulk) spin labels and the deficit angle $\epsilon$ will scale with

$$\sigma(j_{blk}) \sim \left[ \frac{\partial \Phi(j_{blk})}{\partial j_{blk}} \right]^{-1} \times \sigma(\Phi) \sim j \times \frac{1}{\sqrt{j}} = \sqrt{j},$$

$$\sigma(\epsilon) \sim \left[ \frac{\partial \epsilon(j_{blk})}{\partial j_{blk}} \right] \times \sigma(j_{blk}) \sim \frac{1}{j} \times \sqrt{j} = \frac{1}{\sqrt{j}}. \quad (16)$$

As angles are invariant under global rescaling, we can choose boundary data that induce a given deficit angle $\epsilon$, and then choose a sufficiently large scale $j$, so that the $\epsilon = 0$ value is outside the deviation interval. Thus by going to sufficiently large spins $j$, the constraint part of the amplitudes can peak sharply on non-vanishing curvature values. Note that the deviation $\sigma(\epsilon)$ as a function on the spin scale $j$ is independent of the Barbero-Immirzi parameter $\gamma$.

![FIG. 1: The $G$ function (dashed), which imposes the matching conditions weakly, and the real part of the product of the amplitude factors $A_\ell$ and $A_s$ as functions (solid graphs) of the bulk spin $j_{blk}$ for various $\gamma$. Here, the $G$ function peaks on a curvature value $\epsilon \approx 0.5$. Larger $\gamma$’s lead to a more oscillatory behaviour. This example is described in more detail in Appendix B.](image)

The oscillatory behavior resulting from the variation of the action over the $\sigma(j_{blk})$ interval should also be considered, see Fig. 1. Having a highly oscillatory amplitude, the expectation value for the deficit angle will average out to some value different from the one in LRC. As the corresponding contribution to the LRC path integral is rather given by a fixed value of the amplitude, we demand that

$$\sigma(S_{A}) = \frac{1}{\ell_p} \frac{\partial(S_{AR})}{\partial j_{blk}} \times \sigma(j_{blk}) \sim \gamma \epsilon \sqrt{j} \sim O(1). \quad (17)$$

Thus, whereas the scaling for the deficit angle (16) requires a choice of larger $j$, (17) demands that with growing $j$ we choose smaller $\gamma$. These expectations are confirmed by an explicit example, see Appendix B.

To distinguish a small $\epsilon$ from a vanishing $\epsilon$ we also need—due to $\sigma(\epsilon) \sim 1/\sqrt{j}$—a scaling with $|\epsilon| \sim 1/\sqrt{j}$. Thus choosing smaller $\gamma$, which makes the area spectrum denser, allows for a larger range of accessible curvature angles.

We can also interpret (17) as a bound on the curvature per triangle $\epsilon \lesssim 1/(\gamma \sqrt{j})$, which—uncharacteristically—decreases with increasing $j$, the discretization scale.

We have considered the simplest triangulation that differentiates between LRC and ARC. As we only employed scaling arguments, the conclusions apply also for larger triangulations. For larger triangulations, however, the scale set by the boundary spins will not determine a unique scale for the bulk spins that lead to significant am-
plitude contributions. Larger triangulations must, therefore, be studied explicitly. In future work we will investigate examples including bulk edges and bulk vertices. Finally, to make definite conclusions on the continuum limit it will be necessary to see how the implementation of the constraints changes under refining and coarse graining. The models proposed here simplify this task considerably.

Discussion. Area operators are central in a number of approaches to 4D quantum gravity, notably LQG and holography. Discrete area spectra are a key result of LQG and crucial for various black hole entropy countings [15–18]. To achieve a quantum dynamics that reproduces GR constraints between the areas need to hold. This is, however, hindered if areas have an asymptotically equispaced spectrum and are (kinematically) locally independent.

The imposition of these constraints is pivotal in spin foam quantization. This leads to highly involved amplitudes, which has so far prevented a satisfactory resolution of key dynamical questions, most pressing whether the models suppress curvature excitations. Here we proposed a class of effective models, with a transparent encoding of the dynamics and much more amenable for numerical investigations. In these models the constraints are imposed as strongly as allowed by the LQG Hilbert space structure, from which the discrete, locally independent, area spectra result. We emphasize ‘locally independent’ for the following reason: strong imposition of the constraints (that is, first solving the constraints classically and then quantizing the reduced phase space) should also lead to a discrete area spectrum. This follows from Bohr’s correspondence principle, as the areas are also conjugated to (dihedral) angles on the reduced phase space. However, the Dirac brackets, which define the canonically conjugated pairs, have a non-local structure [34, 58] and one would expect a reflection of this non-locality in the resulting Hilbert space.

Insisting on the local structure of the (kinematical) Hilbert space and a prescribed area spectrum we can impose the constraints only weakly. Whether such a weak imposition of second class constraints leads to the correct dynamics is not understood (even in much simpler models than gravity) and should be further tested. In particular, for spin foam models, a too weak imposition of the constraints could lead to suppression of curvature.

Using the effective spin foam models we have found that for triangulations in which the scale for the areas can be controlled, curvature is not necessarily suppressed. This result comes with restrictions connecting the average area $a \sim \gamma_j^2 \gamma_j$, the Barbero-Immirzi parameter $\gamma$, and the curvature $\epsilon_t$ per triangle. The peakedness of the constraints on a given curvature value does improve with growing spin $j$, as $1/\sqrt{j}$, but is independent of $\gamma$. And, to avoid a highly oscillatory behaviour of the amplitudes over the regime allowed by the constraints, we need $\gamma \sqrt{j} \epsilon_t = O(1)$. Not surprisingly, this last condition prefers small $\gamma$, and hence a small spacing in the area spectrum. Furthermore, it can be seen as a bound on curvature, one which is more stringent for larger spins.

In our example, we need large spin values (and correspondingly small $\gamma$) to obtain an expectation value for the deficit angle that approximates well the classical value. This justifies our focus on ‘effective’ models, where we replace the full spin foam simplex amplitude with its large spin asymptotics, given by the cosine (replaced here with the exponential) of the Regge action. It has been argued in [59], that a double scaling limit that takes $\gamma$ small and spins $j$ large, with $\gamma j$ fixed, reproduces the Length Regge equations of motion. Here, we find also that $\gamma$ should be small and $j$ large, but that we need for the combination $\gamma \sqrt{j} \epsilon_t$ to be of order one. Such a combination, and the related bound on curvature has also been identified in [60], based on a generalized stationary phase analysis of the EPRL-FK amplitudes. Using much simpler inputs, we have shown that such a bound does not depend on specific choices for the spin foam amplitudes. The reason for this bound is rather tied to the LQG Hilbert space and the area spectrum it leads to. On this Hilbert space the shape matching constraints are non-commutative and can therefore be imposed only weakly.

The conclusions for the expectation value of the deficit angle hold in general, but assume that we can control the scale of bulk spin and deficit angles, e.g. via the choice of boundary data. This is not necessarily the case for larger triangulations. Moreover, to understand the continuum limit, we would have to investigate how these arguments are impacted by a coarse graining and renormalization process [61]. The investigation of corresponding continuum actions [62], in which the geometricity (simplicity) constraints are also imposed only weakly, might elucidate how these constraints behave under renormalization.

The effective models presented here will make the study of the coarse graining and renormalization flow [61] much more feasible than for the full spin foam models [47] and will help to establish whether loop quantum gravity and spin foams allow for a satisfactory continuum limit.

Appendix A: Counting of length configurations

We consider a triangulation with certain edge lengths chosen to be equal and then compute the number of allowed edge length solutions given locally independent discrete asymptotically equidistant area spectra. To start with we consider one 4-simplex with vertices (12345) and $p = 2, 3, 4$ length parameters. For $p = 2$, we set $l_{ij} = x$ and $l_{i5} = y$, where $i, j = 1, 2, 3, 4$. For the $p = 3$ case we choose: $l_{ij} = x, l_{mn} = y$ and $l_{im} = z$ where $i, j, m, n = 1, 2, 3$ and $l_{i5} = y$ and $l_{45} = z$ where $i, j = 1, 2, 3$. We count all edge length solutions where the triangle
areas take discrete values $A_t \in \{\frac{1}{2}, 1, \cdots, N\}$ for $N \in \mathbb{N}$. The left panel of Figure 2 shows a semilog plot of the number of length solutions for a simplex having $p = 2, 3, 4$ length parameters. The number of length configurations scale as $N^{1.03p} \approx N^p$.

We also consider a gluing of two simplices with vertices $\sigma = (12345)$ and $\sigma' = (12346)$. For the shared tetrahedron we allow two parameters $u$ for the edges (12) and (34) and $v$ for the remaining four edges. All four areas of the tetrahedron therefore agree, and we are left with one area parameter $a = A(u, v)$. Here $A(x_1, x_2, x_3)$ denotes the area of a triangle with edge lengths $(x_1, x_2, x_3)$. For the simple $\sigma'$ we introduce additional edge lengths $w$ for edges (i5) with $i = 1, 2, 3, 4$. This introduces two more area parameters $b = A(u, w, w)$ and $c = A(v, w, w)$, giving us three length and three area parameters for $\sigma$. We make the same kind of choices for $\sigma'$, that is, $w'$ gives the length of the edges (i6) leading to area parameters $b' = A(u, w', w')$ and $c' = A(v, w', w')$. After gluing the complex has four length parameters $(u, v, w, w')$ and five area parameters $(a, b, c, b', c')$.

We proceed to count the number of configurations with all areas valued in $\{\frac{1}{2}, 1, \cdots, N\}$, and which have a maximum deviation (15) for the pairs $(\Phi^{\sigma}_{\epsilon_i}, \Phi^{\sigma'}_{\epsilon_i})$ of 3d dihedral angles in the shared tetrahedron. The right panel of Fig. 2 shows the results for various choices of the parameters $\beta$. For $\beta = 0$, where shape matching is imposed exactly, we find a scaling $N^{3}$. This is explained by the fact that requiring exact shape matching forces $w = w'$, and thus we have only three parameters. Not imposing the shape matching conditions, we find a scaling $N^{5}$ reflecting the five area parameters for the two glued simplices. For $\beta \approx 0.15$ we find a scaling of $N^{4}$, see Fig. 2.

Appendix B: Triangulations with three and with six 4-simplices

Take three 4-simplices with vertices $(12345), (12356)$ and $(13456)$ respectively, and glue these around the shared triangle (135). Here all edges and all but the triangle (135) are in the boundary. Thus we have one bulk triangle and no bulk edge.

We will assume some lengths to be equal, so that we have overall only three length parameters: $x = l_{ij}, y = l_{mn}$ and $z = l_{im}$, where $i, j = 1, 3, 5$ and $m, n = 2, 4, 6$. Correspondingly, we have three area parameters $a = A(x, x, x)$, $b = A(x, z, z)$ and $c = A(y, z, z)$ where $A(x_1, x_2, x_3)$ denotes the area of a triangle with lengths $(x_1, x_2, x_3)$.

Note that with this special choice of boundary data the boundary areas $(b, c)$ do not determine the boundary lengths $(x, y, z)$. To do so one also needs the bulk area $a$. In Area Angle Regge calculus one has also 3D dihedral angles as boundary data. With the given symmetry reduction, all boundary tetrahedra have the same geometry, determined by edge lengths $(z, y, z, z, z, z)$. We can choose a pair of non-opposite edges, both with length $z$. Due to our choice of symmetric boundary data, the 3D dihedral angles $\phi_z$ for the $z$-edges are all the same—thus we have boundary data $(b, c, \phi_z)$. These determine a bulk deficit angle $\epsilon_a(b, c, \phi_z)$.

The matching conditions (5) for the bulk tetrahedra are all satisfied due to our symmetry reduction. Thus, if we start from the AARC path integral (12), and integrate out the bulk 3D dihedral angles, we will just obtain a multiplicative factor, given by the norm of the coherent states $K_f(\cdot, \Phi)$.

We can now consider the AARC path integral with boundary, which, after integrating out the bulk 3D angles, involves only a summation over one spin $j_a$.

Alternatively, we can take two such complexes consisting of three 4-simplices each, and glue these so that we obtain a triangulation of $S^4$. After integrating out all 3D dihedral angles we will have four area parameters, the areas $b, c$ and the bulk areas $a$ and $a'$ from the two complexes respectively. We will compute the expectation value for the deficit angle $\langle \epsilon_a \rangle$—while keeping the areas $(a', b, c)$ fixed. Classically, i.e. with sharp shape-matching constraints and fixed $(a', b, c)$, these data determine the deficit angles $\epsilon_{a'}$ and $\epsilon_a$ with $\epsilon_a = \epsilon_{a'}$.

The summation for the path integral thus involves only the bulk area parameter $a$. There are two contributions to the amplitudes: the exponential of the (Area) Regge action, as well as the inner product $G(a, a')$ between the coherent states, which impose the matching constraints $\Phi_{\epsilon_a}(a, b, c) = \Phi_{\epsilon_{a'}}(a', b, c)$. (If we consider the path integral with boundary this factor is given by the coherent state itself, peaked on $\Phi_{\epsilon_a}(a)$.) We approximate the factor arising from these inner products between the coherent
states by
\[ G(a, a') = \exp \left( - \frac{9}{2\sigma^2(\Phi)} (\Phi_a(a, b, c) - \Phi_{a'}(a', b, c))^2 \right) \]
with
\[ \sigma^2(\Phi) = \frac{1}{2} \frac{\sin \alpha(a, b, c)}{j_0 + 1/2} + \frac{1}{2} \frac{\sin \alpha(a', b, c)}{j_0 + 1/2}. \]  
(18)
where \( \sin \alpha(a, b, c) = 2b/2^2(a, b, c) \) with \( Z(a, b, c) \) the length of a \( z \)-edge in the complex with areas \( (a, b, c) \).
The factor 9 in the exponential arises because we have 9 boundary tetrahedra and therefore 9 inner products.

For the computation of the expectation value \( \langle \epsilon_a \rangle(a', b, c) \) we use
\[ \langle \epsilon_a \rangle(a', b, c) = \frac{1}{Z} \sum \epsilon_a G(a, a') \prod_t A_t \prod_{\sigma} A_{\sigma} \]  
(19)
with
\[ Z = \sum_{j_a} G(a, a') \prod_t A_t \prod_{\sigma} A_{\sigma} \]  
(20)
and \( A_t \) and \( A_{\sigma} \) defined in (9) and (10) above.

The resulting expectation values are shown in Tables I and II. Here we have set \( j_b = j_c = j \). Thus the pair \( (j, j_{a'}) \) determine the scale as well as the deficit angle \( \epsilon_{a'} \). Classically we have \( \epsilon_a = \epsilon_{a'} \). To reproduce this result for the expectation value we need a sufficiently large scale \( j \) and a sufficiently small value for the Barbero-Immirzi parameter, in particular if we consider data leading to a small deficit angle.

| \( (j + \frac{1}{2}, j_{a'} + \frac{1}{2}, \epsilon_{a'}) \) | \( \gamma = 0.01 \) | \( \gamma = 0.1 \) | \( \gamma = 0.5 \) |
|-----------------|---------------|---------------|---------------|
| \( (30, 38.5, 0.5) \) | 0.78 - 0.03i | 0.68 - 0.26i | 0.17 - 0.32i |
| \( (100, 128, 0.54) \) | 0.62 - 0.062i | 0.55 - 0.19i | 0.17 - 0.27i |
| \( (300, 384, 0.54) \) | 0.57 - 0.02i | 0.51 - 0.17i | 0.16 - 0.25i |
| \( (1000, 1280, 0.54) \) | 0.55 - 0.01i | 0.50 - 0.16i | 0.16 - 0.24i |

TABLE I: Expectation value for the deficit angle \( \epsilon_a \) with classical value \( \approx 0.5 \).

| \( (j + \frac{1}{2}, j_{a'} + \frac{1}{2}, \epsilon_{a'}) \) | \( \gamma = 0.01 \) | \( \gamma = 0.1 \) | \( \gamma = 0.5 \) |
|-----------------|---------------|---------------|---------------|
| \( (30, 40, 0.08) \) | 0.39 - 0.02i | 0.33 - 0.15i | 0.03 - 0.14i |
| \( (100, 133.5, 0.06) \) | 0.14 - 0.01i | 0.13 - 0.05i | 0.03 - 0.06i |
| \( (300, 400, 0.08) \) | 0.11 - 0.00i | 0.09 - 0.03i | 0.03 - 0.05i |
| \( (1000, 1335, 0.06) \) | 0.07 - 0.00i | 0.06 - 0.02i | 0.02 - 0.03i |

TABLE II: Expectation value for the deficit angle \( \epsilon_a \) with classical value \( \approx 0.07 \).

In this example the averaging of the deficit angle with the \( G(a, a') \) factor (but without the \( A_t \) and \( A_{\sigma} \) factors) tends to over-estimate the curvature angle. This is due to a certain asymmetry in the example that partially originates with the generalized triangle inequalities, which restrict \( a \leq b \leq c \). The oscillatory behavior of the \( A_t \) and \( A_{\sigma} \) factors tends to average out the expectation values—more so for larger Barbero-Immirzi parameter \( \gamma \), which leads to more oscillations over the interval where \( G(a, a') \) is sufficiently large, see Fig. 3 and Fig. 1 (in the main text). Note that the expectation values do have imaginary contributions. These arise as the \( G(a, a') \) factor peaks away from the stationary point of the action (where \( \epsilon_a = 0 \)), so the imaginary parts do not average out. As the imaginary contributions are sourced by the oscillatory behaviour of the amplitudes, they grow with \( \gamma \). Having imaginary contributions on the order of the real contributions indicates that the regime is unreliable, even if the (real part of the) expectation value happens to be near the classical value.

\[ \text{FIG. 3: The } G(a, a') \text{ factor (dashed) and the real part of the product of the amplitudes } A_t \text{ and } A_{\sigma} \text{ as a function of } j_a \text{ for } \epsilon_{a'} \sim 0.07 \text{ and different } \gamma \text{-values.} \]

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