On Optimization over Tail Distributions

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We investigate the use of optimization to compute bounds for extremal performance measures. This approach takes a non-parametric viewpoint that aims to alleviate the issue of model misspecification possibly encountered by conventional methods in extreme event analysis. We make two contributions towards solving these formulations, paying especial attention to the arising tail issues. First, we provide a technique in parallel to Choquet’s theory, via a combination of integration by parts and change of measures, to transform shape constrained problems (e.g., monotonicity of derivatives) into families of moment problems. Second, we show how a moment problem cast over infinite support can be reformulated into a problem over compact support with an additional slack variable. In the context of optimization over tail distributions, the latter helps resolve the issue of non-convergence of solutions when using algorithms such as generalized linear programming. We further demonstrate the applicability of this result to problems with infinite-value constraints, which can arise in modeling heavy tails.

Key words: distributionally robust optimization, tail modeling, monotonicity, semi-infinite programming, probability

1. Introduction

This paper investigates optimization problems in the form

\[
\sup_{F} E_F[h(X)] \quad \text{subject to} \quad F \in \mathcal{U}
\]

where \(E_F[\cdot]\) is the expectation over the random variable \(X \in \mathbb{R}\) distributed under \(F\). The feasible region \(\mathcal{U}\) on \(F\) encodes information about \(X\). We are especially interested in \(X\) that has unbounded support, with the function \(h(\cdot)\) depending heavily on the tail of \(X\).

The motivation in studying (1) is to apply it to bound performance measures arising in extreme event analysis. The latter has profound importance in risk management in business, engineering, environmental sciences and other disciplines. In conventional studies, extreme event analysis entails understanding the tail behaviors of random variables or data. A common approach is to use extreme value theory (EVT) that asymptotically
justifies fitting data with certain parametric distributions. For instance, the Fisher-Tippett-
Gnedenko theorem implies that, under proper regularity conditions (i.e., maximum domain
of attraction) and normalization, the maximum of an iid sample converges in distribution to
a general extreme value distribution (GEV) [Fisher and Tippett (1928), Gnedenko (1943)].
The Pickands-Balkema-de Haan Theorem states that the excess losses (i.e., the overshoots
of data above a large threshold) converge to a generalized Pareto distribution (GPD)
[Balkema and De Haan (1974), Pickands III (1975)] as the threshold grows, which leads to
the well-known peak-over-threshold (POT) method [Leadbetter (1991)]. Many approaches
have been suggested to estimate tail parameters based on these results (e.g. Smith (1985),
Hosking and Wallis (1987), Hill et al. (1975), Davis and Resnick (1984), Davison and Smith
(1990)), as well as other generalizations; see, e.g., Embrechts et al. (2013a, 2005) for some
excellent reviews of these methods.

In this paper, we study (1) to devise an alternate approach to these conventional meth-
ods in estimating extremal quantities. Formulation (1) is reasoned from the perspective of
robust optimization [Ben-Tal et al. (2009), Bertsimas et al. (2011)] and unlike EVT, this
approach can be nonparametric. The set $\mathcal{U}$, which is known as the uncertainty or the ambi-
guity set, can incorporate for instance shape information like monotonicity and convexity
(prior belief) and moment-type information (estimated from tail data). When $\mathcal{U}$ is chosen
in a statistically correct manner, the optimal value of (1) will give a confidence upper
bound on the true quantity of interest $E[h(X)]$. Similar claims hold for the lower bound
when the maximization is replaced by a minimization. This alternate approach to EVT
is motivated from some documented challenges in using asymptotic approximation due to
the scarcity of tail data. For example, the POT method needs to choose a threshold to
define the “tail” portion of data. When the threshold is too small, there exists bias in using
the GPD fit; on the other hand, when the threshold is large, there can be a lack of data in
the tail portion which leads to a high variance estimate. Thus simultaneously minimizing
bias and variance can be difficult in some cases and, moreover, can be very sensitive to the
underlying distribution (e.g., Embrechts et al. (2013a), p.193). On the other hand, when
calibrating $\mathcal{U}$ in (1), one can, in some sense, afford to choose a smaller cutoff threshold
to improve the overall bias-variance trade-off, by putting mild prior (but nonparametric)
information. This discussion can be found in Lam and Mottet (2017), and we will elaborate
further in Section 2.
In this paper, we make two contributions towards solving (1), paying especial attention to the tail issues of $X$. First, if $\mathcal{U}$ contains monotonicity-based constraints, we provide a technique via integration by parts and change of measures to transform the optimization problem into one that only contains moment constraints. The use of integration by parts is in parallel to so-called Choquet’s theory (e.g., Popescu (2005), Van Parys et al. (2015)) that expresses convex classes of probability distributions as mixtures of their extreme points, but more elementary and is generalized from the work of Lam and Mottet (2017) that focuses only on tail convexity. Our development using change of measures shows, in addition, that in general there can be more than one equivalent moment problems in the considered context, where some of them can be computationally more advantageous than the others and thus allows more flexibility when using numerical solvers. Second, we provide a methodology to reformulate a moment problem over infinite support into one over compact supports, by paying the price of an additional slack variable. In the situation of infinite support, a moment-constrained optimization problem may not possess an optimal measure, and techniques such as generalized linear programming (or cutting-plane procedure) may fail to converge, in particular because of “masses” that escape to infinity. The reduction of such problems into ones with compact support ensures the existence of an optimal solution and thus resolves the potential numerical issues. This result is particularly relevant in our use of (1) since the $h(\cdot)$ we consider depends on the infinite-support tail of $X$. We also demonstrate how even infinite-value moment constraints can be handled under this framework; these constraints arise when one imposes $X$ to be heavy-tailed (e.g., Pareto).

We conclude this introduction with a brief review of related literature in robust optimization (RO). Pioneered by Ben-Tal and Nemirovski (1998), El Ghaoui et al. (1998), it considers decision-making when some parameters in the constraints or objectives are uncertain or nosily observed. It aims to obtain solutions that optimizes the worst-case scenario, among all possibilities of the parameter values within a so-called uncertainty set or ambiguity set (e.g., Bertsimas et al. (2011)). The formulation studied in this paper is closely related to what is known as distributionally robust optimization (DRO), where the uncertain parameter refers to a probability distribution (e.g., Delage and Ye (2010), Wiesemann et al. (2014)). In extreme event analysis, such an approach has been used in finding worst-case bounds for copula models Embrechts and Puccetti (2006), Puccetti and
Rüschendorf (2013), Wang and Wang (2011), Dhara et al. (2017). Numerical procedures (e.g., the rearrangement algorithm Puccetti and Rüschendorf (2012), Embrechts et al. (2013b)) have also been studied. In the DRO literature, Delage and Ye (2010), Goh and Sim (2010), Bertsimas and Popescu (2005) study moment constraints such as mean and second moments. Bertsimas et al. (2013) studies constraints motivated from test statistics such as Kolmogorov-Smirnov tests and $\chi^2$-test. Hanasusanto et al. (2017), Popescu (2005), Van Parys et al. (2015), Li et al. (2016) study the use of unimodal shape information. Our work is closely related to Lam and Mottet (2017) that considers convexity constraints and utilizes an integration by parts technique which we generalize. Other types of constraints include the use of statistical distances such as $\phi$-divergences Ben-Tal et al. (2013) and the Wasserstein metric Esfahani and Kuhn (2015). In particular, Dey and Juneja (2012), Glasserman and Xu (2014), Atar et al. (2015) use Renyi divergence to capture uncertainty in heavy-tail models, and Blanchet and Murthy (2016) studies the worst-case behavior among distributions within a neighborhood surrounding a GEV model. Finally, Bandi et al. (2015) studies robust bounds for systems that are potentially driven by heavy-tailed variates, under a deterministic RO framework.

The remainder of the paper is as follows. Section 2 overviews our formulation and the statistical implications. Section 3 presents our results on solving the formulation, including the transformation into families of moment problems, removal of redundant constraints and reformulation into problems with compactly supported domains. Section 4 shows a numerical example to illustrate our resulting procedure. Section 5 concludes the paper. The Appendix documents all our technical proofs and an auxiliary algorithm.

2. Motivation of the Formulation and Statistical Implications

We start with a target extremal quantity in the form $E[h(X)]$ where $h : \mathbb{R} \to \mathbb{R}$ is measurable and $X \in \mathbb{R}$ is a random variable with distribution function $F$. The $h$ function corresponds to the decision-making task at hand and is stated by the user. Elementary examples include level-crossing probabilities where $h(x) = I(x \geq c)$, excess-mean where $h(x) = (x - c)I(x \geq c)$, or entropic type measure where $h(x) = e^{-\theta x}I(x \geq c)$, for some large $c$.

In estimating $E[h(X)]$, we split the data into a portion that is above a chosen threshold $a$ (the “tail” portion) and below $a$ (the “non-tail” portion). To facilitate discussion, we
assume further that \( h(x) = 0 \) for \( x < a \), so that \( h(\cdot) \) only depends on the tail portion. This assumption is justified from our focus on extremal quantities; on the other hand, if it is not satisfied, one can always separate the estimation problem into \( E[h(X); X < a] \) and \( E[h(X); X \geq a] \), where the former can be handled using standard statistical tools (e.g., empirical estimation).

Next we represent the statistical uncertainty of the tail portion via constraints. We consider the following three types. In our exposition, we denote \( \mathbb{N} = \{0, 1, 2, \ldots \} \), \( \mathbb{R}^+ \) as the non-negative real line, \( F^{(j)}_+ \) and \( F^{(j)} \) as the \( j \)-th order right derivative and the \( j \)-th order derivative of a function \( F \) respectively (assuming they exist).

**Moments:** Consider \( F \) such that

\[
E_F[g_j(X)] \leq \gamma_{j,1}, \ j \in J_1
\]  

(2)

where \( \gamma_{j,1} \in \mathbb{R} \), \( g_j : [a, \infty) \to \mathbb{R} \) are some user-specified moment functions, such as \( g_j(x) = I(x \geq a) \), \( g_j(x) = (x - a)I(x \geq a) \) or \( g_j(x) = (x - a)^2I(x \geq a) \), and \( J_1 \) is a finite index set. Obviously the constraints (2) include equality as well as lower bounds by suitably defining the \( g_j \) functions.

In our framework we will enforce the constraints \( \gamma_{0,1} \leq F(a) = E_F[I(X \geq a)] \leq \bar{\gamma}_{0,1} \), where \( 0 \leq \gamma_{0,1} \leq \bar{\gamma}_{0,1} \leq 1 \), so that \( g_1(x) = I(x \geq a) \) and \( g_2(x) = -I(x \geq a) \) are always included.

For modeling tail, using moment constraints alone can be very conservative. Classical results in moment problems stipulate that the worst-case distribution subject to only moment constraints are typically finitely supported, with the number of support points bounded by the number of constraints [Winkler (1988)]. This does not capture the shape of tail distributions reasonably encountered in practice. In many cases, one may be able to safely conjecture that the tail has decreasing density, yet this information is not captured with moment conditions.

**Monotonicity:** To incorporate shape information, we consider including assumptions on the monotonicity of \( F \) or its derivatives. In the literature, this is known as monotonicity of order \( D \) (e.g., [Pestana and Mendonça (2001), Van Parys et al. (2015)]). Denote \( \mathcal{P}^D[a, \infty) \) as the set of all probability distribution functions that are \( D - 1 \) times differentiable, and the \( D \)-th order right derivative exists and is finite and monotone on \([a, \infty)\). We impose

\[
F \in \mathcal{P}^D[a, \infty)
\]  

(3)
where $D \in \mathbb{N}$ is user-specified. Obviously, (3) holds with $D = 0$ minimally by the definition of $F$. Assumption (3) contains several implicit information, as shown by:

**Lemma 1.** For any $D \in \mathbb{N} \setminus \{0\}$, we have:
1. For any $F \in \mathcal{P}^D[a, \infty)$, $\lim_{x \to \infty} F^{(D)}_+(x) = 0$.
2. The set of distribution functions $\mathcal{P}^D[a, \infty)$ is non-increasing with respect to $D$ in the sense:
   $$\mathcal{P}^D[a, \infty) \subset \mathcal{P}^{D-1}[a, \infty) \subset \ldots \subset \mathcal{P}^0[a, \infty)$$
3. For any $F \in \mathcal{P}^D[a, \infty)$ and $j \in \{1, \ldots, D\}$, the function $(-1)^{j+1} F^{(j)}_+$ exists and is non-negative and non-increasing.

Lemma 1 is a small variation of the remarks made in Pestana and Mendonça (2001) p. 320; we present this variation (and provide the proof in Appendix EC.1) since it is needed for our subsequent discussion. Part 1 of Lemma 1 characterizes the asymptotic behavior of the $D$th-order right derivative to converge to 0 as $x \to \infty$. Part 2 further stipulates that monotonicity of a derivative in the distribution tail implies tail monotonicity of any of its derivatives that are of lower order. To conclude, Part 3 specifies that the direction of monotonicity happens in an “alternating” manner with respect to the order of derivatives. For $D = 1$, $F^{(1)}_+$ can only be non-increasing (instead of non-decreasing) on $[a, \infty)$. This means the tail density exists and is non-increasing. For $D = 2$, $F^{(2)}_+$ can only be non-decreasing (instead of non-increasing). The tail density is then convex instead of concave (the latter can be easily seen to be impossible for a tail density). And so forth for higher $D$.

The main purpose of Assumption (3) is to reduce the conservativeness in capturing tail information using only moment constraints. In practice, however, one would likely be able to visually check the assumption up to at most $D = 2$, which can be done by, e.g., assessing the pattern of a density or density derivative estimate.

**Distributional information at the cutoff threshold:** One can also impose bounds on the derivatives of $F$ at the cutoff threshold

$$\gamma_{j,2} \leq (-1)^{j+1} F^{(j)}_+(a) \leq \bar{\gamma}_{j,2}, \ j \in J_2$$

where $0 \leq \gamma_{j,2} \leq \bar{\gamma}_{j,2} < \infty$ and $J_2$ is a finite set of positive integers. In fact, we will choose $J_2$ to be the empty set if $D = 0$ in (3) and a finite subset of $\{1, \ldots, D\}$ otherwise. This
is because without monotonicity conditions on \( F_+(j) \), bounding their respective values at \( x = a \) has no direct effect on the distributional behavior beyond \( a \). Note that one may also incorporate bounds on positions other than \( a \), but we leave out this option in the current work.

**Optimization formulation:** Putting together (2), (3) and (4), we consider the general formulation

\[
\sup_{F} E_F[h(X)] \quad \text{subject to} \quad E_F[g_j(X)] \leq \gamma_{j,1} \quad \text{for all } j \in J_1 \\
\gamma_{j,2} \leq (-1)^{j+1} F_+(j)(a) \leq \gamma_{j,2} \quad \text{for all } j \in J_2 \\
F \in \mathcal{P}^D[a, \infty)
\]

Using existing terminology, we say that (5) is consistent if it has a feasible solution, and solvable if it has an optimal solution. In either case, the optimal objective value of (5) takes values in \( \mathbb{R} \cup \{+\infty\} \). When (5) is inconsistent, we set its optimal value as \( -\infty \).

We next present an immediate statistical implication in using (5). Let \( F_{\text{true}} \) be the true distribution generating a data set (in the frequentist sense). We have:

**Theorem 1.** Suppose that \( \gamma_{j,1}, \gamma_{j,2} \) and \( \gamma_{j,2} \) are calibrated from data such that

\[
P_{\text{data}} \left( E_{F_{\text{true}}}[g_j(X)] \leq \gamma_{j,1}, j \in J_1 \text{ and } \gamma_{j,2} \leq (-1)^{j+1} F_{\text{true}}(j)(a) \leq \gamma_{j,2}, j \in J_2 \right) \geq 1 - \alpha
\]

where \( P_{\text{data}} \) denotes the probability generated from the data. Then, if \( F_{\text{true}} \in \mathcal{P}^D[a, \infty) \), we have

\[
P_{\text{data}}(E_{F_{\text{true}}}[h(X)] \leq Z^*) \geq 1 - \alpha
\]

where \( Z^* \) is the optimal value of optimization problem (5).

**Proof of Theorem 1:** If \( F_{\text{true}} \) lies in the feasible region of (5), then \( E_{F_{\text{true}}}[h(X)] \leq Z^* \) by the definition of \( Z^* \). Hence under the assumption \( F_{\text{true}} \in \mathcal{P}^D[a, \infty) \), we have

\[
P_{\text{data}} \left( E_{F_{\text{true}}}[g_j(X)] \leq \gamma_{j,1}, j \in J_1 \text{ and } \gamma_{j,2} \leq (-1)^{j+1} F_{\text{true}}(j)(a) \leq \gamma_{j,2}, j \in J_2 \right) \leq P_{\text{data}}(E_{F_{\text{true}}}[h(X)] \leq Z^*)
\]

which concludes the theorem. \( \square \)
Theorem 1 is a direct application of the statistical argument in DRO (see, e.g., Bertsimas et al. (2014)). It suggests to calibrate $\gamma_{j,1}, \gamma_{j,2}, \gamma_{j,2}$ such that (6) holds. In the rest of this section, we discuss some examples to motivate our investigation in Section 3:

**Example 1 (Monotonic tail estimation).** Consider estimating the tail interval probability $P(b \leq X \leq \bar{b})$ where $b < \bar{b}$ are some large numbers. Choose a cutoff threshold $a$ that separates the tail portion of the data, with $a < b$. We impose the assumption that the tail density exists and is non-increasing on $[a, \infty)$. This can be assessed, for instance, by plotting the density estimate (see Section 4). Find the 95% normal confidence interval for $\bar{F}(a)$, given by $[\gamma_{0,1}, \gamma_{0,1}]$ where $0 \leq \gamma_{0,1} \leq \gamma_{0,1} \leq 1$. Then, Theorem 1 implies that the optimization

$$\sup_{F} P_{F}(b \leq X \leq \bar{b})$$

$$\gamma_{0,1} \leq F(a) \leq \gamma_{0,1}$$

$$F \in \mathcal{P}^{1}[a, \infty)$$

(8)

provides a 95% confidence upper bound for the true $P(b \leq X \leq \bar{b})$, under the assumption that the density of $X$ is non-increasing on $[a, \infty)$. Optimization (8) can be seen to bear a simple solution, given by assigning the maximally allowed probability mass on $[a, \infty)$, which is $\gamma_{0,1}$, uniformly over the range $[a, \bar{b}]$. This gives an optimal value $\gamma_{0,1}(\bar{b} - b)/(\bar{b} - a)$.

**Example 2 (Monotonic tail estimation with density estimate).** Continue with Example 1, this time adding an estimate for the density (assumed to exist) at $a$, namely $f(a) = F_{+}^{(1)}(a)$. This involves bootstrapping the kernel estimate at $a$ to obtain a 95% confidence interval for $f(a)$. Suppose one does this jointly with the estimation of $\bar{F}(a)$ that is Bonferroni-corrected, then the optimization

$$\sup_{F} P_{F}(b \leq X \leq \bar{b})$$

$$\gamma_{0,1} \leq \bar{F}(a) \leq \gamma_{0,1}$$

$$\gamma_{1,2} \leq F_{+}^{(1)}(a) \leq \gamma_{1,2}$$

$$F \in \mathcal{P}^{1}[a, \infty)$$

(9)

also gives a 95% confidence upper bound for the true $P(b \leq X \leq \bar{b})$. This is under the assumption that the density of $X$ is non-increasing on $[a, \infty)$ and the bootstrap calibration of $\gamma_{1,2}, \gamma_{1,2}$ is valid.
The optimal value of (9) can be built from that of (8). The upper bound on the density \( f(a) \) is the maximum possible density of \( X \) for \( X \geq a \). If the height of the maximally allocated uniform density from the solution of (8) is within the range \([\gamma_{1,2}, \overline{\gamma}_{1,2}]\), then this uniform density is optimal for (9). If the height is larger than \( \overline{\gamma}_{1,2} \), the solution for (9) becomes the uniform density that has height \( \overline{\gamma}_{1,2} \) starting from position \( a \). The latter holds provided that \( \overline{\gamma}_{1,2}(\overline{b} - a) \geq \gamma_{0,1} \) and \( \overline{\gamma}_{1,2} \geq \gamma_{0,1} / (\overline{b} - a) \), which gives an optimal value

\[
\min \left\{ \frac{\gamma_{0,1}}{\overline{b} - a}, \overline{\gamma}_{1,2} \right\} (\overline{b} - \overline{b})
\]  

(10)

Otherwise, the program is inconsistent. Note that (10) is at most \( \gamma_{0,1}(\overline{b} - \overline{b}) / (\overline{b} - a) \), the optimal value of (8), if one ignores the Bonferroni adjustment. This illustrates the effect in reducing conservativeness by adding extra constraints. Of course, if one adds too many constraints, then the simultaneous estimation issue can become more prominent.

**Example 3 (Monotonic tail estimation with density and moment information).**

Continue with Example 2. Suppose one makes the further assumption that the first moment of \( X \) is finite (which can be assessed by exploratory tools such as the maximum-sum ratio; e.g., Embrechts et al. (2013a) Chapter 6). One can find the 95% normal confidence interval for \( E[(X - a)_+] \), say given by \([\gamma_{1,1}, \overline{\gamma}_{1,1}]\). Suppose that Bonferroni correction is made. Then

\[
\sup_{F} P_{F}(b \leq X \leq \overline{b})
\]

\[
\gamma_{0,1} \leq \overline{F}(a) \leq \gamma_{0,1}
\]

\[
\gamma_{1,1} \leq E_{F}[(X - a)_+] \leq \overline{\gamma}_{1,1}
\]

\[
\gamma_{1,2} \leq F^{(1)}_{+}(a) \leq \overline{\gamma}_{1,2}
\]

\[
F \in \mathcal{P}[a, \infty)
\]

(11)

gives a 95% confidence upper bound for the true \( P(b \leq X \leq \overline{b}) \). This is under the assumption that the density of \( X \) is non-increasing on \([a, \infty)\).

Note that unlike Examples 2 and 3, the solution for (11) is more involved. Section 3 is devoted to a general solution scheme that includes solving (11).
Example 4 (Convex tail estimation). Continue with Example 3. Suppose we now make the additional assumption that the density of $X$ is convex. Hence

$$
\begin{align}
\sup_F P_F(b \leq X \leq \bar{b}) \\
\underline{\gamma}_{0,1} &\leq \bar{F}(a) \leq \overline{\gamma}_{0,1} \\
\underline{\gamma}_{1,1} &\leq E_F[(X - a)_+] \leq \overline{\gamma}_{1,1} \\
\underline{\gamma}_{1,2} &\leq F^{(1)}_+(a) \leq \overline{\gamma}_{1,2} \\
F &\in \mathcal{P}^2[a, \infty)
\end{align}
$$

(12)

gives a 95\% confidence upper bound for the true $P(b \leq X \leq \bar{b})$, under the assumption that the density of $X$ is convex on $[a, \infty)$.

All the example formulations above do not require parametric assumptions on the data, which aim to alleviate the model bias issue and distinguish from the conventional EVT-based methods. On the other hand, the constructed bounds are potentially conservative as they rely on a worst-case calculation. In fact, the smaller the $a$ one chooses, the more sizable is the tail portion of the data which typically gives more “flexibility” to generate a higher optimal value in (5). Instead of a bias-variance trade-off in the case of using GPD, our approach has a conservativeness-variance trade-off. Depending on the risk management purpose, one may place correctness a priority over conservativeness or vice versa. The next section presents our results on solving optimization (1) which covers all the posited example formulations.

3. Results on the Properties and Solutions of the Formulation

We present our main results on the properties and solution structure of optimization problem (5). This consists of two parts. Section 3.1 presents the reformulation of (5) into families of moment-constrained problems. Section 3.2 investigates the reduction of infinite-support moment problems into compact-support ones, including those with infinite-value moment constraints. After these, Section 3.3 shows how one can numerically solve the reduced formulation.

3.1. Reduction to Moment Problems via Integration by Parts and Change of Measures

To start our discussion, we introduce $Q(\mathcal{C})$ as the collection of all bounded non-negative distribution functions on $\mathcal{C} \in \mathbb{R}$, where a distribution function is defined as a function that
is non-decreasing and right-continuous on \( C \) (but not necessarily bounded by 1, as in the case of probability distributions). Note that a distribution function as defined is a Stieltjes function of a bounded measure on \( C \) equipped with the Borel \( \sigma \)-algebra (e.g., Durrett (2010)). Correspondingly, we define \( Q^D(C) \) as the collection of all bounded non-negative distribution functions on \( C \) that are differentiable up to order \( D - 1 \) and have monotone \( D \)-th order right derivatives. To avoid ambiguity, any distribution function on \( C \) is defined to take value 0 on \( \mathbb{R} \setminus C \).

We first redefine the decision variables in (5) to be in the space of \( Q^D[a, \infty) \):

**Lemma 2.** Suppose \( \underline{\gamma}_{0,1} \leq \bar{\gamma}(a) \leq \bar{\gamma}_{0,1} \) is included in the first set of constraints in (5), where \( 0 \leq \underline{\gamma}_{0,1} \leq \bar{\gamma}_{0,1} \leq 1 \). Then (5) can be replaced by

\[
\sup_F \int h dF \quad \text{subject to} \quad \int g_j dF \leq \gamma_j, \quad \text{for all } j \in J_1
\]

\[
\underline{\gamma}_{j,2} \leq (-1)^{j+1} F_j^+(a) \leq \bar{\gamma}_{j,2} \quad \text{for all } j \in J_2
\]

\[
F \in Q^D[a, \infty)
\]

**Proof of Lemma 2.** The lemma follows immediately by checking that \( \underline{\gamma}_{0,1} \leq \bar{\gamma}(a) \leq \bar{\gamma}_{0,1} \) enforces the required properties of probability distributions missing in the definition of \( Q^D[a, \infty) \). \( \square \)

This subsection discusses how Program (13) can be reformulated into a generalized moment problem in the form

\[
\sup_P \int H dP \quad \text{subject to} \quad \int G_j dP \leq \gamma_j, \quad \text{for all } j \in J
\]

\[
P \in Q(\mathbb{R}^+)
\]

where \( H : \mathbb{R}^+ \to \mathbb{R} \) and \( G_j : \mathbb{R}^+ \to \mathbb{R} \) are measurable functions, \( \gamma_j \in \mathbb{R} \) and \( J \) is a finite index set. Program (14) resembles the classic moment-constrained optimization except that the decision variable represents a measure that does not necessarily add up to one. This familiar form allows the adoption of existing optimization routines, as we will discuss in the sequel. The add-up-to-one constraints could be missing in our reformulation because our available mass, i.e., \( \bar{\gamma}(a) \), could be specified in an interval rather than set to be a constant (e.g., 1).
To precisely describe the $H$ and $G_j$ functions, we introduce some further definitions. Define the function $u_a$ as the shift operator $u_a(x) = x + a$ for all $x \in \mathbb{R}^+$. In addition, let $g^{(-d)}(x)$ be the $d$-th order anti-derivative recursively defined as $g^{(-d)}(x) = \int_0^x g^{(-d+1)}(u)du$ and $g^{(0)} \equiv g$. Note that by definition $g^{(-d)}(0) = 0$ for any $d > 0$.

We introduce Theorem 2 which requires the following assumption:

**Assumption 1.** In program (13), the functions $h$ and $g_{j,1}$ for $j \in J_1$ are locally integrable and are either bounded from above or below.

**Theorem 2 (Equivalence with a Family of Moment Problems).** Let $D \in \mathbb{N} \setminus \{0\}$. Denote $Z^*$ as the optimal value of program (13), with $J_2 \subset \{1, \ldots, D\}$. Under Assumption 1, we have

$$Z^* = \sup_P \int H dP$$

subject to

$$\int G_{j,1} dP \leq \gamma_{j,1} \text{ for all } j \in J_1$$

$$\gamma_{j,2} \leq \int G_{j,2} dP \leq \tau_{j,2} \text{ for all } j \in J_2$$

$$P \in Q(\mathbb{R}^+)$$

where for all $x \in \mathbb{R}^+$,

- $H(x) = x^{J-D}(h \circ u_a)^{(-D)}(x)$
- $G_{j,1}(x) = x^{J-D}(g_{j} \circ u_a)^{(-D)}(x)$, for all $j \in J_1$
- $G_{j,2}(x) = \frac{x^{J-j}}{(D-j)!}$, for all $j \in J_2$

for any integer $J \in \{0, \ldots, D\}$ (we suppress the dependence of $H$, $G_{j,1}$ and $G_{j,2}$ on $J$ for convenience). In addition, if program (15) is solvable with solution $P^*$, then $F^*$ defined via

$$D!F^*(x+a) = \int u^J \left(1 - (1 - x/u)^D I(u > x)\right) dP^*(u) \text{ for all } x \in \mathbb{R}$$

is an optimal solution of program (13).

Theorem 2 is proved by a sequential application of integration by parts and the use of monotonicity to control the tail decay rate of $F$ and its derivatives, by generalizing the techniques in Lam and Mottet (2017). The flexibility in choosing $J$ comes from a change of measure argument, where the decision variable, i.e., the measure of $P$ can be re-expressed as another measure with a likelihood ratio adjustment. As far as we know, using changes of measure to reformulate moment problems is new in the literature, and offers some benefits as we will explain below. Details of the derivation are shown in Appendix EC.1.
A natural choice of \( J \) is to set \( J = 0 \) if \( J_2 \) is empty and \( J = \max\{j \in J_2\} \) otherwise. Such a choice of \( J \) also ensures that \( G_{j,2}(x) = 1/(D - J)! \) for all \( x \in \mathbb{R}^+ \), and thus we have the constraint \( \sum_{j \in J} (D - J)! \gamma_{j,2} \leq \int dP \leq (D - J)! \gamma_{J,2} \) so that the distribution function \( P \) in (15) has bounded mass. Furthermore, when \( J_2 \) is not empty, the feasible set of (15) can be further restricted to \( P(\mathbb{R}^+) \). This is because the functions \( H, G_{j,1} \) and \( G_{j,2} \) are by construction equal to 0 at \( x = 0 \), so that we can always add an arbitrary mass at 0 to reach the upper bound \( (D - J)! \gamma_{J,2} \). This in turn deduces that upon proper normalization of the measure we can impose the constraint that \( \int dP = 1 \).

Another advantage of the above choice of \( J \) is that if \( h \) and \( g_j \) are polynomials, then the reformulation with such a choice of \( J \) will also give rise to polynomial forms for \( H, G_{j,1} \) and \( G_{j,2} \). This class of problems can be more susceptible to specialized solution techniques (e.g., semidefinite programming, though this is not the focus of this paper).

Theorem 2 also reveals the optimality structure of (13) in relation to the derivative-based constraints. It is well-known in the theory of moment problems (under non-negative measures) that it suffices to consider \( P \) in (15) that is piecewise constant, i.e., \( P \) corresponds to a finite-support distribution (e.g., Rogosinski (1958)). Therefore, with (16), we deduce that it suffices in (13) to consider linear combinations of distributions in the form \( (1 - (1 - x/u)D I(u > x))u^J \) for some \( u \), i.e.,

\[
F(x + a) = \sum_{i=1}^N p_i x_i^J \left(1 - (1 - x/x_i)D I(x_i > x)\right), \quad \text{for } x > 0 \quad (17)
\]

where \( p_i, x_i \geq 0 \). In other words, either (13) is solvable with a solution in the form (17), or there exists a sequence of solutions in the form (17) whose evaluated objective values in (13) converge to the optimal objective value \( Z^* \). The number \( N \) in (17) is at most the total number of linearly independent functions in the set \( \{(G_{j,1})_{j \in J_1}, (G_{j,2})_{j \in J_2}, 1\} \). This representation is consistent with the notion of generating sets studied in Popescu (2005), where in our case \( u^J \left(1 - (1 - x/u)D I(u > x)\right) \) can be viewed as a generating set. Popescu (2005), however, focuses on constraints on the whole distribution and as such, the weights in that context must be probability weights.

In the special case \( J_2 = \emptyset, J_1 = \{0, 1, 2\} \), and \( G_{j,1}(x) = (x - a)^j_+ \) for all \( j \in J_1 \), by setting \( J = 0 \) in Theorem 2 we arrive at the result given in Theorem 2.1 in Van Parys et al. (2015) in their considered case of univariate \( D \)-monotone distributions. We close this subsection by depicting the specific case where the moments are powers of the overshoot variable:
Corollary 1. If the first set of constraints in program \([13]\) is replaced by

\[
\gamma_{j,1} \leq \int (x - a)^j_+ dF(x) \leq \pi_{j,1} \quad \text{for all } j \in J_1
\]

where \(-\infty \leq \gamma_{j,1} \leq \pi_{j,1} < \infty\), then the conclusion of Theorem \([3]\) holds with the first set of constraints in \([15]\) replaced by

\[
\gamma_{j,1} \leq \int G_{j,1} dP \leq \pi_{j,1} \quad \text{for all } j \in J_1
\]

where for all \(j \in J_1\), \(G_{j,1}(x) = \Gamma(j + 1)/\Gamma(j + D + 1)x^{j+1}\), with \(J\) set to be any integer in \(\{0, \ldots, D\}\), and \(\Gamma(\cdot)\) is the standard Gamma function.

Proof of Corollary \([4]\): The Corollary trivially follows from Theorem \([2]\) with functions \((x - a)^j_+\) and \(-(x - a)^j_+\) both put into the set of \(g_{j,1}(x)\)’s. \(\square\)

3.2. Reduction to Compactly Supported Moment Problems

Our next result shows how one can reduce the moment problem \([14]\) into one whose measures in consideration (i.e., the decision variable) take domain on a compact set. The reason we pursue such a reduction is its requirement to adopt the generalized linear programming technique \cite{GobernaLopez1998}, which sequentially looks for new support points and updates the probability distributions (more details in Section 3.3). Note that \([14]\) admits feasible measures on the whole non-negative real line (a consequence that our problem focuses on the tail region). As a result, instead of possessing an optimal measure, there may only exist a sequence of measures, whose values converging to the optimal, that possess masses gradually moving to \(\infty\) (i.e., such a sequence of measures does not converge weakly; see, e.g., \cite{LamMottet2017}). This violates the sufficiency conditions needed for carrying out the generalized linear programming procedure (Theorem 11.2 in \cite{GobernaLopez1998}) and may potentially deem the procedure non-convergence. In contrast, under the reformulation with compactly supported feasible measures, there always admits an optimal solution (with a finite number of support points) and such an algorithmic issue can be avoided.

We would need strong duality to substantiate our results in this subsection. Assume that
Assumption 2. Suppose program (14) has a representation in the form

\[
\begin{align*}
\sup_P & \quad \int HdP \\
\text{subject to} & \quad \int \tilde{G}_jdP \leq \tilde{\gamma}_j \quad \text{for all } j \in \tilde{J} \\
& \quad \int \tilde{G}_jdP = \tilde{\gamma}_j \quad \text{for all } j \in \tilde{J}' \\
& \quad P \in Q(\mathbb{R}^+) \tag{18}
\end{align*}
\]

where \(\tilde{G}_j : \mathbb{R}^+ \rightarrow \mathbb{R}\) are measurable functions, \(\tilde{\gamma}_j \in \mathbb{R}\) and \(\tilde{J}, \tilde{J}'\) are finite index sets, such that there exists \(P \in Q(\mathbb{R}^+)\) with \(\int \tilde{G}_jdP < \tilde{\gamma}_j\) for all \(j \in \tilde{J}\), and \((\tilde{\gamma}_j)_{j \in \tilde{J}}'\) is in the interior of the set

\[
\left\{ \left( \int \tilde{G}_jdP \right)_{j \in \tilde{J}'} : P \in Q(\mathbb{R}^+) \right\}
\]

Assumption 2 is a Slater-type condition for moment problems. Similar assumptions have been documented in, e.g., Bertsimas and Popescu (2005), Popescu (2005), Shapiro (2001), Karlin and Studden (1966), Smith (1995). Under Assumption 2, strong duality holds for (14):

Theorem 3. Suppose program (14) is consistent and denote \(Z^*\) its optimal objective value. If Assumption 2 is satisfied, then strong duality holds for (14), i.e. \(Z^*\) is equal to

\[
\begin{align*}
\inf_y & \quad \sum_{j \in J} y_j \tilde{\gamma}_j \\
\text{subject to} & \quad \sum_{j \in J} y_j G_j(u) \geq H(u) \quad \text{for all } u \in \mathbb{R}^+ \\
& \quad y_j \geq 0 \quad \text{for all } j \in J \tag{19}
\end{align*}
\]

The proof is a direct application of Lagrangian duality for optimization with both inequality and equality constraints, depicted in Chapter 8, Problem 7 in Luenberger (1997), together with standard weak duality as argued in, e.g., Section 3.1 in Smith (1995).

Next, we exclude some trivial scenarios and redundant constraints. We introduce the collection of index sets \(J(x)\) defined as

\[
J(x) = \left\{ i \in J : \exists u_n \in \text{supp}(G_i) \text{ s.t. } \limsup_{u_n \to x} \frac{H(u_n)}{G_i(u_n)} \geq 0 \text{ and } \limsup_{u_n \to x} \frac{G_j(u_n)}{|G_i(u_n)|} \leq 0 \text{ for all } j \in J \right\} \tag{20}
\]

where \(x \in \mathbb{R}^+ \cup \{\infty\}\) is a placeholder, and \(\text{supp}(G_i)\) denotes the support of the function \(G_i\), i.e \(\text{supp}(G_i) = \{u \in \mathbb{R}^+ | G_i(u) \neq 0\}\).

Theorem 4 (Removal of Redundant Constraints). Given any fixed \(x \in \mathbb{R}^+ \cup \{\infty\}\). Suppose Assumption 2 holds. Denote \(Z^*\) as the optimal value of (14). Then the following statements hold:
1. If $\mathcal{J} = \mathcal{J}(x)$ and $\sup\{H(u) | u \in \mathbb{R}^+\} > 0$, then $Z^* = \infty$.
2. If $\mathcal{J} = \mathcal{J}(x)$ and $\sup\{H(u) | u \in \mathbb{R}^+\} \leq 0$, then $Z^* = 0$.
3. If $\mathcal{J}(x) \subseteq \mathcal{J}$, then

$$Z^* = \sup_P \int HdP \quad \text{subject to} \quad \int G_j dP \leq \gamma_j \quad \text{for all } j \in \mathcal{J} \setminus \mathcal{J}(x)$$

(21)

The set $\mathcal{J}(x)$ is the set of redundant constraints. Identifying it can screen out the trivial cases (Cases 1 and 2 in Theorem 4) and reduce the number of constraints (Case 3). We also note that the choice of $x$ in applying Theorem 4 is self-consistent, in the sense that choosing any $x$ gives rise to valid results and, moreover, one can apply the theorem sequentially on different $x$’s.

When $H$ and $G_i$’s are continuous at $x$, then the definition of $\mathcal{J}(x)$ can be reduced to having the inequalities hold for the ratios evaluated at $x$ (by merely considering $u_n = x$). Definition (20), however, is more general as it covers discontinuous cases and the case where $x = \infty$.

Appendix EC.2 provides the proof of Theorem 4, which relies on analyzing the allowable asymptotic behaviors of solution sequences in relation to the behaviors of $H$ and $G_j$ around $x$.

We have the following simplification in the case where all the constraint functions in (14) have both lower and upper bounds, which can be derived using the definition of $\mathcal{J}(x)$:

**Lemma 3.** Assume (14) can be expressed as

$$\sup_P \int HdP \quad \text{subject to} \quad \underline{\gamma}_j \leq \int \tilde{G}_j dP \leq \overline{\gamma}_j \quad \text{for all } j \in \tilde{\mathcal{J}}$$

(22)

for some $\tilde{G}_j : \mathbb{R}^+ \to \mathbb{R}$ and finite index set $\tilde{\mathcal{J}}$, where $-\infty < \underline{\gamma}_j \leq \overline{\gamma}_j < \infty$. Then $\mathcal{J}(x)$ defined in (20) is empty for all $x \in \mathbb{R}^+ \cup \{\infty\}$.

Our main result in this subsection is to demonstrate how a slack variable $s$ can be introduced to encode the case where some mass “escapes” to $\infty$. This allows us to reduce the search space of (21) to compact-support distributions when some regularity conditions are met.

To this end, when $\mathcal{J} \setminus \mathcal{J}(\infty)$ is not empty, we make the following assumptions.
Assumption 3. For some $M \in \mathcal{J} \setminus \mathcal{J}(\infty)$ and $u$ large enough, the function $G_M(u)$ is bounded away from 0 and $\limsup_{u \to \infty} |G_j(u)/G_M(u)| < \infty$ for all $j \in \mathcal{J} \setminus \mathcal{J}(\infty)$.

Assumption 4. For some $M \in \mathcal{J} \setminus \mathcal{J}(\infty)$, the limit
\[
\lambda_{j,M} = \lim_{u \to \infty} \frac{G_j(u)}{|G_M(u)|}
\] is well-defined (on the extended real line) for all $j \in \mathcal{J} \setminus \mathcal{J}(\infty)$.

Assumptions 3 and 4 ensure the limits of ratios of $G_j(x)$ and $G_M(x)$ are well-defined as $x \to \infty$, which is needed to handle the situation of escaping mass. Next, we impose some mild regularity conditions on $G_j$ and $H$:

Assumption 5. For all $j \in \mathcal{J} \setminus \mathcal{J}(\infty)$, the functions $G_j$ are lower semi-continuous and bounded on any compact set of $\mathbb{R}^+$.

Assumption 6. The function $H$ is upper semi-continuous and bounded on any compact set of $\mathbb{R}^+$.

Lastly, we assume the following non-degeneracy condition for at least one of the $G_j$’s:

Assumption 7. There exists some $j \in \mathcal{J} \setminus \mathcal{J}(\infty)$ such that $\inf_{x \in \mathbb{R}^+} G_j(x) > 0$.

Assumption 7 can be ensured to satisfy in formulation (15) by using $J = \max\{j \in \mathcal{J}_2\}$ in Theorem 2 (i.e., through a particular change of measure on the decision variable as in its proof) so that $G_{J,2} = 1/(D - J)!$ for all $x \in \mathbb{R}^+$.

Theorem 5 (Slack Variable to Encode Escaping Mass). Suppose (14) is consistent with optimal value $Z^*$. We have:

1. If $\mathcal{J} = \mathcal{J}(\infty)$ and $\sup\{H(u)|u \in \mathbb{R}^+\} > 0$, then $Z^* = \infty$.
2. If $\mathcal{J} = \mathcal{J}(\infty)$ and $\sup\{H(u)|u \in \mathbb{R}^+\} \leq 0$, then $Z^* = 0$.
3. If $\mathcal{J}(\infty) \subset \mathcal{J}$ and Assumption 3 holds, then
   (a) If $\lambda_M := \limsup_{u \to \infty} H(u)/|G_M(u)| = \infty$ then $Z^* = \infty$.
   (b) Otherwise, if Assumptions 4, 5 hold, then there is some $C \in \mathbb{R}^+$ such that $Z^* < \infty$ and
\[
Z^* = \sup_{P,s} \int H \, dP + \lambda_M s
\]
subject to $\int G_j dP + \lambda_{j,M} s \leq \gamma_j$ for all $j \in \mathcal{J} \setminus \mathcal{J}(\infty)$
\[
s \geq 0
\]
\[
s = 0 \text{ if } \lambda_M = -\infty
\]
\[
P \in \mathcal{Q}_N[0,C]
\]
where \( N \) is the number of linearly independent functions in the collection \( \{ (G_j)_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)}, 1 \} \), and \( \mathcal{Q}_N[0, C] \) is the set of all distribution functions on \([0, C]\) that are piecewise constant, right-continuous with at most \( N \) jumps. In particular, program (24) is solvable.

Theorem 5 is proved by tracking the limits of all the possible sequences of weights and support points in a finite-support measure that can tend to the optimal value.

Our next result specializes to handle interval-type power function constraints:

**Corollary 2.** Consider Program (15) where the first set of constraints is replaced by

\[
\gamma_{j,1} \leq \int G_{j,1} dP \leq \overline{\gamma}_{j,1} \quad \text{for all } j \in \mathcal{J}_1
\]

\( G_{j,1} \) is defined as in Corollary 1, and \( 0 \leq \gamma_{j,1} \leq \overline{\gamma}_{j,1} < \infty \). Set \( J \) in Program (15) to be 0 if \( J_2 \) is empty and \( \max\{ j \in J_2 \} \) otherwise. Denote \( Z^* \) as the optimal value, \( M = \max\{ j \in J_1 \} \), \( \lambda_M = \limsup_{x \to \infty} \frac{H(x)}{|G_{M,1}(x)|} \), and \( \delta_{j,M} \) as the Kronecker delta function. In addition, assume that (15) is consistent, and that there exists \( P \in \mathcal{Q}(\mathbb{R}^+) \) such that \( \gamma_{j,i} < \int G_{j,i} dP < \overline{\gamma}_{j,i} \) for all \( j \) and \( i \) such that \( \gamma_{j,i} < \overline{\gamma}_{j,i} \), and \( \left( \left( \int_{J_1} G_{j,1} dQ \right)_{j \in J_1}, \left( \int_{J_2} G_{j,2} dQ \right)_{j \in J_2} \right) \) is an interior point of

\[
\left\{ \left( \left( \int_{J_1} G_{j,1} dQ \right)_{j \in J_1}, \left( \int_{J_2} G_{j,2} dQ \right)_{j \in J_2} \right) : Q \in \mathcal{Q}(\mathbb{R}^+) \right\}
\]

with \( J_i = \{ j \in J_1 | \gamma_{j,i} = \overline{\gamma}_{j,i} \} \) and \( i \in \{1, 2\} \). Then

1. If \( \lambda_M = \infty \), then \( Z^* = \infty \).

2. If \( \lambda_M < \infty \) and \( h \) is upper semi-continuous when \( D = 0 \), then \( Z^* < \infty \) and there exists some \( C \in \mathbb{R}^+ \) such that

\[
Z^* = \sup_{P,s} \int H dP + \lambda_M s \\
\text{subject to } \gamma_{j,1} \leq \int G_{j,1} dP + \delta_{j,M} s \leq \overline{\gamma}_{j,1} \quad \text{for all } j \in J_1 \\
\gamma_{j,2} \leq \int G_{j,2} dP \leq \overline{\gamma}_{j,2} \quad \text{for all } j \in J_2 \\
s \geq 0 \\
s = 0 \text{ if } \lambda_M = -\infty \\
P \in \mathcal{Q}_N[0, C]
\]

is solvable and \( N = |J_1| + |J_2| \).

In Theorem 5 and Corollary 2, the non-trivial cases of the optimization formulation, namely (24) and (26), are now posited as moment problems with compact support. With
this formulation we can apply generalized linear programming (discussed in the next section) without running into the numerical issue of having \( n \)

From Theorem 4, we also derive Corollary 3 below. This last result is of interest as it handles, under some regularity conditions, programs in the form

$$\sup_P \int H dP$$
subject to $$\int G_j dP \leq \gamma_j \quad \text{for all } j \in J$$
$$\int |G| dP = \infty$$
$$P \in Q(\mathbb{R}^+)$$  \hspace{1cm} (27)

where \( G: \mathbb{R}^+ \to \mathbb{R} \) is a measurable function on \( \mathbb{R}^+ \). Program (27) extends the scope of (14) by including an infinite-value constraint. Such formulation can arise in practice when the tail decay is estimated to have infinite moments (e.g., Pareto-type tail, which can be assessed by the ratio-of-maximum-and-sum method; Section 6.2.6 in Embrechts et al. (2013a)). For example, from Corollary 3(1) and 3(3), we know that the following optimization with infinite constraint

$$\sup_P \int x I(x \geq b) dP(x)$$
subject to $$\int xdP(x) \leq \gamma$$
$$\int x^2 dP(x) = \infty$$
$$P \in Q(\mathbb{R}^+)$$  \hspace{1cm} (28)

(for some given \( b \)) is the same as

$$\sup_P \int x I(x \geq b) dP(x)$$
subject to $$\int xdP(x) \leq \gamma$$
$$P \in Q(\mathbb{R}^+)$$  \hspace{1cm} (29)

**Corollary 3 (Infinite-value Constraints).** Consider Program (14), denote \( Z^* \) as its optimal objective value, and assume the program is consistent and that Assumption 3 holds. In addition, let \( G \) be a not identically 0 and measurable function satisfying \( \limsup_{u \to \infty} |G(u)/G_M(u)| = \infty \) when \( J(\infty) \not\subset J \). If any one of the following statements holds:

1. \( Z^* = \infty \) and \( \liminf_{u \to \infty} |G(u)/H(u)| > 0 \)
2. \( Z^* = \infty, \liminf_{u \to \infty} H(u)/|G(u)| > 0 \), and \( \inf_{x \in \mathbb{R}^+} G_j(x) > 0 \) for some \( j \in J \)
3. \( Z^* < \infty \) and \( \limsup_{u \to \infty} |G(u)/H(u)| = \infty \)

then (14) and (27) have the same optimal objective value \( Z^* \).
3.3. A Generalized Linear Programming Procedure

With the results in Sections 3.1 and 3.2, we apply generalized linear programming to solve problems in the form (14). Our procedure is shown in Algorithm 1, which is an adaptation of Algorithm 11.4.1 in Goberna and López (1998) (setting $\varepsilon_k = 0$ and $|S_k| = 1$ for all $k \in \mathbb{N}$). This procedure relies on the sufficiency to search for distributions that have finite support. It iteratively searches for the optimal support points by looking for the next point that has the highest current “reduced cost” via solving a “subproblem” (i.e., the point not already in the set of considered support points that gives the highest rate of improvement by assigning it a mass), and updating the solution via a “master problem” that is a linear program on the existing support points. Compared to the semidefinite programming approach, the generalized linear programming applies to non-polynomial objective functions and constraints, but in our context it requires solving potentially a non-convex one-dimensional search in each iteration.

Under Theorem 5, our formulation is cast over measures with compact support. Theorem 11.2 in Goberna and López (1998) guarantees that if the dual optimization is consistent, then Algorithm 1 generates a sequence of dual multipliers $y_j^{(k)}$ that converges to an optimal dual solution and it does so in a finite number of iterations when $\varepsilon > 0$. When the dual multipliers converge, the value returned by the procedure is also the optimal objective value of (24), up to some tolerance $\varepsilon$ (Section 11.1 Goberna and López (1998)). Note that, in general, a good value of the compact support boundary $C$ is not known a priori. In our experiment, we choose $C$ to be in the tens, which appear to work well.

The initialization step in Algorithm 1 can be done by applying a Phase I procedure described in Algorithm 2 in Appendix EC.3, which finds a feasible solution for (24) provided such solution exists. Algorithm 2 attempts to solve the following program

$$
\begin{align*}
\min_{s,r,P} \quad & r \\
\text{subject to} \quad & -r + \int G_j(x)dP + \lambda_{j,M} s \leq \gamma_j, \forall j \in J \setminus J(\infty) \\
& s, r \geq 0 \\
& s = 0 \text{ if } \lambda_M = -\infty \\
& P \in \mathcal{Q}_N(\mathbb{R}^+) 
\end{align*}
$$

(30)

If the algorithm stops with $(P^*, s^*, r^*)$ such that $r^* = 0$, $(P^*, s^*)$ is a feasible solution of (24). If $r^* > 0$, we conclude that (24) has no feasible solution. Under conditions in Corollary
4.1 in Magnanti et al. (1976), a variant of this Phase I procedure converges in finite steps (even with tolerance level 0).

**Algorithm 1** Computing the optimal value of Program (24) when \( J \setminus J(\infty) \) is not empty

**Inputs:** Provide the parameters \( \gamma_j \) and the functions \( G_j \) for all \( j \in J \setminus J(\infty) \), and the function \( H \). Compute the quantities \( \lambda_{j,M}, j \in J \setminus J(\infty) \) and \( \lambda_M \). Also specify a big number \( C \in \mathbb{R}^+ \) and a tolerance level \( \varepsilon \geq 0 \).

**Exclusion of the trivial scenarios**
- IF \( \lambda_M \) is equal to \( \infty \), then \( Z^* = \infty \)
- ELSE proceed to the next step of the procedure

**Initialization:**
- Find an initial feasible solution in \( Q_L[0,C] \) for Program (24), where \( L \in \{1, \ldots, N\} \) and \( N \) is the number of linearly independent functions in the collection \( \{(G_j)_{j \in J(\infty)}\}_1 \). This can be done using the Phase I algorithm in Appendix EC.3. Denote \( (x_i)_{i \in 1 \ldots L} \) as the support points of the initial feasible solution.

**Procedure:** For each iteration \( k = 0, 1, \ldots \), and given \( (x_i)_{i \in 1 \ldots L+k} \):

1. (Master problem) Solve

\[
Z^k = \sup_{p,s} \sum_{i=1}^{L+k} H(x_i) p_i + \lambda_M s \\
\text{subject to } \sum_{i=1}^{L+k} G_j(x_i) p_i + \lambda_{j,M} s \leq \gamma_j, \quad \forall j \in J \setminus J(\infty) \\
\text{with } s \geq 0 \\
\text{and } s = 0 \text{ if } \lambda_M = -\infty \\
p_i \geq 0 \quad \forall i = 1, \ldots, L+k 
\]

Let \( (p^k, s^k) \) be the optimal solution. Find the dual multipliers \( (y^k_j)_{j \in J \setminus J(\infty)} \) satisfying

\[
\left( \sum_{j \in J \setminus J(\infty)} y^k_j G_j(x^k_i) - H(x^k_i) \right) p^k_i = 0, \quad \forall i = 1, \ldots, L+k \\
\left( \sum_{j \in J \setminus J(\infty)} y^k_j \lambda_{j,M} - \lambda_M \right) s^k = 0 \\
y^k_j \geq 0, \text{ for all } j \in J \setminus J(\infty) 
\]

2. (Subproblem) Find \( x_{L+k+1} \) that minimizes

\[
\rho^k(u) = \sum_{j \in J \setminus J(\infty)} y^k_j G_j(u) - H(u), \quad \text{where } u \in [0,C] 
\]

- SET \( \epsilon^k = \rho^k(x_{L+k+1}) \)
- IF \( \epsilon^k \geq -\varepsilon \), STOP and RETURN \( Z^* = Z^k \) ELSE go back to 1.

### 4. Numerical Example

We demonstrate our results and procedure in Section 3 with a numerical example. Figure 1 is a normalized histogram of 500 observations, each representing an independent realization of the random variable \( X \) with distribution function \( F_X(x) = 1 - x^{-1}e^{-x} \) for all \( x \geq x_0 \).
where $x_0e^{x_0} = 1$. The thick line represents the true probability density function and the dashed lines indicate the values $q_p$ of the theoretical $p^{th}$-percentiles associated with $F_X$, when $p \in \{90, 99, 99.9, 99.99\}$.

![Normalized histogram of 500 iid observations sampled from the probability distribution function $F_X(x) = 1 - x^{-1}e^{-x}$, where $x \geq x_0$ and $x_0e^{x_0} = 1$. The thick line is the true probability density function and the dashed lines indicates the theoretical $p^{th}$-percentiles when $p \in \{90, 99, 99.9, 99.99\}$](image_url)

We test our procedure in estimating $P(X \geq q_p)$ for the set of $p$’s depicted above. We consider the class of optimization formulations

$$
\begin{align*}
\sup_F \quad & P_F(X \geq q) \\
\text{subject to} \quad & \gamma_{j,1} \leq E_F[(X-a)_+] \leq \bar{\gamma}_{j,1} \quad \text{for all } j \in J_1 \\
& \gamma_{j,2} \leq (-1)^{j+1}F^{(j)}_+(a) \leq \bar{\gamma}_{j,2} \quad \text{for all } j \in J_2 \\
& F \in \mathcal{P}_D[a, \infty)
\end{align*}
$$

(33)
where $D$ can be any value in $\{0, \ldots, 5\}$, $J_1$ is a subset of $\{0,1,2,3,4\}$, and $J_2$ is either empty if $D = 0$ or a subset of $\{1, \ldots, \min(D,3)\}$ if $D \geq 1$. We caution that in practice, assessing the validity of the monotonicity assumption for $D > 2$ can be difficult unless in the presence of huge data size. Moreover, here we have assumed the moment exists if it is used, whereas in practice one may want to use the ratio-of-maximum-and-sum method Embrechts et al. (2013a) to assess their finiteness.

To obtain a good choice of $a$ and the reliability of the constraints, we plot the trends of the density and density derivatives in Figure 2. The confidence intervals for various $a$’s are constructed using the bootstrap with 1000 replicates on the built-in kernel estimates in R Wand and Jones (1994). We pick $a = 1.35$, which is roughly the 80-percentile of the data. Beyond this value (shown by the gray vertical line), the density function can be seen to be non-decreasing and convex. The signs of the estimates of the density and its derivative, however, are not as clear. A risk-averse user in this case would use $D = 2$, and set $J_2 = \{1\}$.

We calibrate the normal confidence intervals for $E_F[(X - a)^+_j]$. To account for simultaneous estimation, we apply a Bonferroni correction in constructing these intervals together with those for $F^{(j)}(a)$.

We apply Corollary 2 and Algorithm 1 to solve Program (33) for all possible combinations of $p \in \{90,99,99.9,99.99\}$, $J_1 \subseteq \{0,1,2,3,4\}$, $D \in \{0, \ldots, 5\}$, $J_2 \subseteq \{1,2,\min(3,D)\}$ if $D \geq 1$ and $J_2 = \emptyset$ if $D = 0$. For $D > 1$, we use numerical differentiation on the density estimates and apply the same bootstrap calibration procedure described before.

The results are displayed in Figure 2. Each point gives, for a given combination of the parameters $D$, $p$, and the sets $J_1$ and $J_2$, the relative error between the output of Program (33) and the true value of $P(X \geq q_p)$. For a given $p$, the smallest relative error decreases with $D$, as the intuition suggests. In addition, the smallest relative error across all $D$ values increases with $p$. This can be attributed to the fact that the non-tail data are less informative as we infer on quantities associated with farther part of the tail. The large circles show the output with no moment constraints (in particular, without $\gamma_{0,1} \leq F(a) \leq \overline{\gamma}_{0,1}$ discussed in Section 2), which can be shown to give extremely conservative bounds.

Table 1 shows, for each given $D$ and each of the four values of $p$, the sets $J_1^*$ and $J_2^*$ giving the smallest relative error obtained across all possible combinations of the sets $J_1$ and $J_2$ tested (as shown in Figure 3). The relative error is defined as the relative increase of the optimization output value over the truth. This is from one set of data (without replication),
so the value of relative error may vary. However, it can show some general pattern. In all four cases, the optimization outputs appear to capture the order of magnitude of the true underlying probability. The relative error in the case $D = 0$ seems to be significantly larger than using at least $D = 1$ (i.e., monotonicity of the tail density). The gain in relative error decreases as $D$ increases to 2 (i.e., convexity of the tail density). For $p = 90$- or 99-percentile, the relative error when using $D = 2$ and one or two moment constraint is kept at a decimal. For $p = 99.9$- or 99.99-percentile, the relative error is larger, but encouragingly, it is still within a single digit. The table also shows that increasing $D$ further does not result in dramatic improvement. This suggests that adding a monotonicity constraint on higher order derivatives without including a bound on the derivative itself is negligible.
| $D$ | $J_1^*$ | $J_2^*$ | Optimal Objective Value | Relative Error |
|-----|---------|---------|-------------------------|----------------|
| 0   | {0}     | {}      | 2.4e-01                 | 1.401          |
| 1   | {0, 1}  | {}      | 1.78e-01                | 0.777          |
| 2   | {0, 1}  | {}      | 1.64e-01                | 0.635          |
| 3   | {0, 1}  | {}      | 1.57e-01                | 0.573          |
| 4   | {0, 1}  | {}      | 1.54e-01                | 0.539          |
| 5   | {0, 1}  | {}      | 1.52e-01                | 0.516          |

(a) $p = 90$

| $D$ | $J_1^*$ | $J_2^*$ | Optimal Objective Value | Relative Error |
|-----|---------|---------|-------------------------|----------------|
| 0   | {1, 3}  | {}      | 4.82e-02                | 3.817          |
| 1   | {0, 3}  | {}      | 2.03e-02                | 1.032          |
| 2   | {0, 3}  | {}      | 1.66e-02                | 0.665          |
| 3   | {0, 3}  | {}      | 1.5e-02                 | 0.505          |
| 4   | {0, 3}  | {}      | 1.41e-02                | 0.414          |
| 5   | {0, 3}  | {}      | 1.36e-02                | 0.356          |

(b) $p = 99$

| $D$ | $J_1^*$ | $J_2^*$ | Optimal Objective Value | Relative Error |
|-----|---------|---------|-------------------------|----------------|
| 0   | {1, 3}  | {}      | 6.86e-03                | 5.856          |
| 1   | {1, 3}  | {}      | 2.89e-03                | 1.887          |
| 2   | {1, 3}  | {}      | 2.35e-03                | 1.349          |
| 3   | {1, 3}  | {1}     | 2.1e-03                 | 1.098          |
| 4   | {1, 3}  | {1}     | 1.95e-03                | 0.952          |
| 5   | {1, 3}  | {1}     | 1.86e-03                | 0.856          |

(c) $p = 99.9$

| $D$ | $J_1^*$ | $J_2^*$ | Optimal Objective Value | Relative Error |
|-----|---------|---------|-------------------------|----------------|
| 0   | {2, 3}  | {}      | 1.97e-03                | 18.688         |
| 1   | {2, 3}  | {}      | 8.26e-04                | 7.259          |
| 2   | {1, 3}  | {1}     | 6.66e-04                | 5.658          |
| 3   | {1, 3}  | {1}     | 5.86e-04                | 4.862          |
| 4   | {1, 3}  | {1}     | 5.4e-04                 | 4.397          |
| 5   | {1, 3}  | {1}     | 5.09e-04                | 4.092          |

(d) $p = 99.99$

Table 1  For each one of the five values of $D$ and for all four values of $p$, the sets $J_1^*$ and $J_2^*$ are the sets giving the smallest relative error obtained across all possible combinations of the sets $J_1$ and $J_2$ tested.
Figure 3  Relative error (defined as the relative increase of the optimization output value over the truth) for different \( p \) and choices of \( D \) and other constraints in the optimization. The solid lines represent the maximal relative error for a given value \( p \).

We conclude this section by pointing out some subtle numerical issues when implementing the algorithm, related to the choice of \( C \) in the compact-support moment problem formulation [24]. Note that \( C \) is not known in the specification and needs to be chosen through trial and error. In our implementation, for all cases choosing \( C \) in the tens work. However, one may encounter other examples in which \( C \) needs to be chosen much higher. For instance, if we consider using \( e^X \) instead of \( X \) in our current data set, we found that the proper \( C \) is in the range of thousands. This could cause numerical instability in R, but could potentially be well-implemented with more powerful optimization software.

5. Conclusion

We have investigated an optimization-based approach to bound expectation-type extremal performance measures. The approach utilizes constraints to encode information about the monotonicity-type behaviors of the tail and moments and aims to compute the worst-case value among all tail distributions subject to these constraints. We have developed two results, one on the transformation from monotonicity constraints to moment constraints by using elementary integration by parts and change of measures, and show that in general there can be multiple equivalent transformed formulations. We have also developed a method to transform an infinite-support moment problem into a compact-support moment problem, which avoids non-convergence issues when running techniques like generalized
linear programming due to escaping masses arising from the infinite support. A numerical example demonstrates the application of our approach and theoretical results.

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Appendix

EC.1. Technical Proofs for Section 3.1

Proof of Lemma 1: Let $D \in \mathbb{N}^*$ and $F \in \mathcal{P}^D(a, \infty)$. The proof focuses on the case when $a = 0$; this is without loss of generality since $F \in \mathcal{P}^D(a, \infty)$ if and only if $F(\cdot + a) \in \mathcal{P}^D(\mathbb{R}^+)$. 

Item 1: Since $F_{+}^{(D)}$ is monotone, it has a limit, say $l$, at $\infty$. If $l > 0$, the function $F^{(D-1)}$ is ultimately increasing, and there exists $K \in \mathbb{R}^+$ such that

$$F^{(D-1)}(x) - F^{(D-1)}(K) = \int_{K}^{x} F_{+}^{(D)}(u) du \to \infty \text{ as } x \to \infty \quad (EC.1)$$

Hence, $\lim_{x \to \infty} F^{(D-1)}(x) = \infty$. One can repeat this argument to show that $F^{(j)}$ is ultimately increasing and $\lim_{x \to \infty} F^{(j)}(x) = \infty$ for all $j \in \{0, D - 1\}$. This is a contradiction as $F$ is bounded. In the same way, we can also prove that the limit $l$ cannot be negative, for otherwise $F$ would be ultimately decreasing. We therefore conclude that $l = 0$.

Item 2: By definition, the function $F_{+}^{(D)}$ is either non-increasing or non-decreasing, and converges to 0 as shown in Item 1. As such, it never changes sign and $F^{(D-1)}$ is monotone as well. Proceeding by induction, we obtain

$$\mathcal{P}^D(\mathbb{R}^+) \subset \mathcal{P}^{D-1}(\mathbb{R}^+) \subset \ldots \subset \mathcal{P}^0(\mathbb{R}^+) = \mathcal{P}(\mathbb{R}^+)$$

Item 3: Building upon the proof of Item 1, we see that if $F_{+}^{(D)}$ is non-decreasing, it is non-positive and so $F^{(D-1)}$ is non-increasing and non-negative. Similarly, if $F_{+}^{(D)}$ is non-increasing (non-negative) then $F^{(D-1)}$ is non-decreasing (non-positive). Repeating this logic, the derivatives of odd order $j$ must have the same sign as $F_{+}^{(1)}$ which we know to be non-negative. \hfill \Box

Remark EC.1. It is straightforward to see that the proofs and hence the statements in Lemma 1 hold in the more general case where $F$ belongs to $\mathcal{Q}^D[a, \infty)$ defined in Section 3.1.

The proof of Theorem 2 requires the following three propositions.
Proposition EC.1. Let $D \in \mathbb{N}$, $g$ be a locally integrable function on $\mathbb{R}^+$ that is either bounded below or above, and $F \in \mathcal{Q}^D(\mathbb{R}^+)$. Then for any $j \in \{0, \ldots, D\}$, we have

$$
\int_0^\infty gdF = \int_0^\infty g^{(-j)}dP^{(j)} \tag{EC.2}
$$

where $P^{(j)}(x) = (-1)^j F_+^{(j)}(x)$ for all $x \in \mathbb{R}^+$.

Proof of Proposition EC.1. The proposition trivially holds for $D = 0$ so we focus on the case $D \geq 1$. First, we establish the validity of the statement when $g$ is non-negative and $D = 1$. We then generalize the result to unsigned functions that are either bounded above or below. The conclusion will hold for any $D > 1$ by recursing the argument.

Step 1: Because the function $g$ is locally integrable, its first order antiderivative $g^{(-1)}$ exists and is continuous. In addition, the right derivative $F_+^{(1)}$ is of bounded variation. The integral $\int_0^x g^{(-1)}dF_+^{(1)}$ therefore exists for all $x \geq 0$ and, with an integration by part, we obtain that for all $x \in \mathbb{R}^+$

$$
\int_0^x g^{(-1)}dF_+^{(1)} + \int_0^x F_+^{(1)}dg^{(-1)} = g^{(-1)}(x)F_+^{(1)}(x) = -g^{(-1)}(x)\int_x^\infty dF_+^{(1)} \tag{EC.3}
$$

where the last equality is a consequence of Lemma 1.1. Because $g$ is non-negative, the function $g^{(-1)}$ is non-decreasing and non-negative. Hence,

$$
0 \leq -g^{(-1)}(x)\int_x^\infty dF_+^{(1)} \leq -\int_x^\infty g^{(-1)}dF_+^{(1)}, \quad \text{for all } x \in \mathbb{R}^+ \tag{EC.4}
$$

In addition,

$$
\int_0^x F_+^{(1)}dg^{(-1)} = \int_0^x F_+^{(1)}(u)g(u)du = \int_0^x gdF \tag{EC.5}
$$

Combining (EC.3) and (EC.4), we have

$$
0 \leq \int_0^x g^{(-1)}dF_+^{(1)} + \int_0^x F_+^{(1)}dg^{(-1)} = -g^{(-1)}(x)\int_x^\infty dF_+^{(1)} \leq -\int_x^\infty g^{(-1)}dF_+^{(1)}, \quad \text{for all } x \in \mathbb{R}^+ \tag{EC.6}
$$

Subtracting $\int_0^x g^{(-1)}dF_+^{(1)}$ in (EC.6) gives

$$
-\int_0^x g^{(-1)}dF_+^{(1)} \leq \int_0^x F_+^{(1)}dg^{(-1)} \leq -\int_0^x g^{(-1)}dF_+^{(1)} - \int_x^\infty g^{(-1)}dF_+^{(1)}, \quad \text{for all } x \in \mathbb{R}^+ \tag{EC.7}
$$
Substituting (EC.5) into (EC.7), we have
\[- \int_0^x g^{(-1)} dF_+^{(1)} \leq \int_0^x g dF \leq - \int_0^\infty g^{(-1)} dF_+^{(1)}, \quad \text{for all } x \in \mathbb{R}^+ \quad (\text{EC.8})\]

Taking the limit on both sides of (EC.8), we obtain (EC.2).

**Step 2:** We now consider the case when \( g \) is an unsigned function bounded below by a constant \( m \in \mathbb{R} \). Inequality (EC.2) then applies for both the function \( g - m \) and the constant function 1. As a result, equalities \( \int_0^\infty (g - m) dF = \int_0^\infty (g^{(-1)} - mx) dP^{(1)} \) and \( \int_0^\infty dF = \int_0^\infty x dP^{(1)} \) hold. In fact, the last equality is bounded since \( F \in Q(\mathbb{R}^+) \), and
\[
\int_0^\infty g dF = \int_0^\infty (g - m) dF + m \int_0^\infty dF = \int_0^\infty (g^{(-1)} - mx) dP^{(1)} + m \int_0^\infty x dP^{(1)} = \int_0^\infty g^{(-1)} dP^{(1)} \quad (\text{EC.9})
\]

When \( g \) is an unsigned function bounded above by a constant \( M \in \mathbb{R} \). Inequality (EC.2) then applies for the function \( (M - g) \) and
\[- \int_0^\infty g dF = \int_0^\infty (M - g) - MD + \int_0^\infty g^{(-1)} dP^{(1)} = - \int_0^\infty g^{(-1)} dP^{(1)}. \]
This concludes our proof. \hfill \Box

**Proposition EC.2.** Let \( D \in \mathbb{N} \setminus \{0\} \), \( F \in Q^D(\mathbb{R}^+) \), and \( P(x) = (-1)^D F_+^{(D)}(x) \) for all \( x \in \mathbb{R}^+ \). Then,
1. \((-1)^{(j+1)} F_+^{(j)}(0) = \int_{[(D-j)!!]}^u dP(u) \) for all \( j \in \{1, \ldots, D\} \)
2. \( \lim_{x \to \infty} F(x) = \int \frac{u^D}{D!} dP(u) \)

By definition of the function \( P^{(j)} \) defined in Proposition EC.1, we have
\[
\lim_{x \to \infty} (-1)^j \left[ F_+^{(j)}(x) - F_+^{(j)}(0) \right] = \int_0^\infty dP^{(j)}(u)
\]
Applying Proposition EC.1 with \( g(u) \equiv 1 \) also gives
\[
\lim_{x \to \infty} (-1)^j \left[ F_+^{(j)}(x) - F_+^{(j)}(0) \right] = \int_0^\infty g^{(-(D-j))} dP^{(D-j)+j} = \int_0^\infty \frac{u^{D-j}}{(D-j)!} dP(u) \quad (\text{EC.10})
\]
The first item then follows from \( \lim_{x \to \infty} F_+^{(j)}(x) = 0 \) for all \( j \in \{1, \ldots, D\} \) by Lemma 1.1. The second item is a consequence of the continuity of \( F \) when \( D \geq 1 \) and the definition of \( Q^D(\mathbb{R}^+) \) that \( F(0) = 0 \).

**Proposition EC.3.** Let \( D \in \mathbb{N} \setminus \{0\} \). A function \( F \) is an element of \( Q^D(\mathbb{R}^+) \) if and only if there exists a function \( Q \) such that \( Q(x) \in Q(\mathbb{R}^+), \; Q(0) = 0, \; \int x^D dQ(x) < \infty, \) and

\[
D^j F(x) = \int (u^D - (u - x)^D I(u > x)) dQ(u) \quad \text{for all} \; x \in \mathbb{R} \tag{EC.11}
\]

In fact, \( Q(x) = (-1)^D \left[F^{(D)}(x) - F^{(D)}(0)\right] \) for all \( x \in \mathbb{R}^+ \).

We first show that any function \( F \in Q^D(\mathbb{R}^+) \) can be expressed in the form of \( (EC.11) \). Let \( h(u, x) \) be the function defined as \( D^j h(u, x) = u^D - (u - x)^D I(u > x) \). Then for all \((u, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \) and \( j \in \{0, \ldots, D\} \),

\[
\frac{\partial^j h}{\partial u^j}(u, x) = \frac{1}{(D - j)!} \left[u^{D-j} - (u - x)^{D-j} I(u > x)\right] \tag{EC.12}
\]

In particular, \( \frac{\partial^j h}{\partial u^j}(0, x) = 0 \) for all \( j \in \{0, \ldots, D - 1\} \) and \( \frac{\partial^D h}{\partial u^D}(u, x) = I(u \leq x) \). The function \( h(u, x) \) is therefore the \( D^{th} \) order anti-derivative with respect to \( u \) vanishing at \( 0 \) of the function \( I(u \leq x) \). Since \( F(x) = \int I(u \leq x) dF(u) \), an application of Proposition EC.1 shows that the distribution function \( F \) can be written as

\[
F(x) = \int h(u, x) dP(u)
\]

where \( P(x) = (-1)^D F_+^{(D)}(x) \) for all \( x \in \mathbb{R}^+ \). Hence, \( Q(x) = P(x) - P(0) \) and \( Q(0) = 0 \) trivially holds. From Lemma 1.3 and Remark EC.1 we also have \( Q(x) \in Q(\mathbb{R}^+) \). In addition, \( F(x) = \int h(u, x) dP(u) = \int h(u, x) dQ(u) \). Lastly, the integral \( \int u^D / D! dP(u) \) is bounded since it is the limit of the distribution function \( F \) by Proposition EC.2 and hence so is \( \int u^D dQ(u) \).

We now focus on the other direction of the statement, i.e. we consider the case when \( Q \) is a distribution function on \( \mathbb{R}^+ \) satisfying \( \int u^D dQ(u) < \infty, \; Q(0) = 0, \) and \( F \) is as defined in \( (EC.11) \). The function \( F \) is then absolutely continuous and non-decreasing. Moreover, we have \( (D - 1)! F_+^{(1)}(x) = \int (u - x)^{D-1} I(u > x) dQ(u) \) for all \( x \in \mathbb{R}^+ \) by an interchange of derivative and the integral, justified since \( u^D - (u - x)^D I(u > x) \) is \( Q \)-integrable and
$(u - x)^{D-1}I(u > x)$ is bounded by a $Q$-integrable function, which in turn is guaranteed since \( \int u^D dQ < \infty \) (e.g., Theorem 6.28 in Klenke (2013)).

Iteratively, we obtain \((-1)^{D+1} F_+^{(D)}(x) = \int I(u > x)dQ(u)\) for all \(x \in \mathbb{R}^+\). Hence, \((-1)^D \left[ F_+^{(D)}(x) - F_+^{(D)}(0) \right] = Q(x) - Q(0) = Q(x)\). As \(Q\) belongs to \(Q(\mathbb{R}^+)\), the function \(F_+^{(D)}\) is monotone. It remains to show that \(F\) is bounded to conclude that \(F \in Q^D(\mathbb{R}^+)\). Because \(F\) is non-decreasing, it is enough to prove that its limit is bounded. By definition, the \(D^{th}\) moment of \(Q\) is bounded. Therefore, using the monotone convergence theorem, we have

\[
\lim_{x \to \infty} D! F(x) = \int \lim_{x \to \infty} \left[ u^D - (u - x)^D I(u > x) \right] dQ(u) = \int u^D dQ(u) < \infty
\]

**Proof of Theorem 2**: The distribution function \(F\) is an element of \(Q^D[a, \infty)\) if and only if \(F(x + a)\) is an element of \(Q^D(\mathbb{R}^+)\). When \(D = 0\), the set \(J_2\) is empty by definition and program (5) can be reformulated as (14) by applying the change of variables \(J = J_1\), \(P = F \circ u_a\), \(H = h \circ u_a\), and \(G_j = g_{j_1} \circ u_a\) for all \(j \in J_1\), so that the conclusion holds. In the remainder of this proof, we focus on the case \(D \geq 1\).

With Assumption [1], Proposition [EC.1] allows us to reformulate the objective value and the first set of inequality constraints as follows:

\[
\int h(x) dF(x) = \int h(x + a) dF(x + a) = \int (h \circ u_a)^{(-D)} dP \tag{EC.13}
\]

\[
\int g_{j,1}(x) dF(x) = \int g_{j,1}(x + a) dF(x + a) = \int (g_j \circ u_a)^{(-D)} dP, \text{ for all } j \in J_1 \tag{EC.14}
\]

where \(P(x) = (-1)^D F_+^{(D)}(x + a)\) for all \(x \in \mathbb{R}^+\). Moreover, Proposition [EC.2] implies that

\[
(-1)^{j+1} F_+^{(j)}(a) = \int \frac{x^{D-j}}{(D-j)!} dP, \text{ for all } j \in J_2 \tag{EC.15}
\]

So (5) is the same as

\[
\begin{align*}
\sup_F & \int (h \circ u_a)^{(-D)} dQ \\
\text{subject to} & \int (g_j \circ u_a)^{(-D)} dQ \leq \gamma_{j,1} \quad \text{for all } j \in J_1 \\
& \gamma_{j,2} \leq \int x^{D-j}/(D-j)! dQ \leq \gamma_{j,2} \quad \text{for all } j \in J_2 \\
& Q(x) = (-1)^D \left[ F_+^{(D)}(x + a) - F_+^{(D)}(a) \right] \quad \text{for all } x \in \mathbb{R} \\
& F(x + a) \in Q^D(\mathbb{R}^+) \tag{EC.16}
\end{align*}
\]
By Proposition EC.3, \( F(\cdot + a) \in \mathcal{Q}^D(\mathbb{R}^+) \) if and only if there exists \( Q(\cdot) \in \mathcal{Q}(\mathbb{R}^+) \) such that \( Q(0) = 0, \int x^D dQ(x) \) is bounded, and

\[
D!F(x + a) = \int u^D - (u - x)^D I(u > x)dQ(u) \tag{EC.17}
\]

for all \( x \in \mathbb{R} \). Moreover, \( Q(\cdot) \) must satisfy \( Q(x) = (-1)^D \left[ F_+^D(x + a) - F_+^D(a) \right] \) for all \( x \in \mathbb{R} \). Therefore, we obtain that (EC.16) is the same as

\[
\begin{align*}
\sup_P & \quad \int (h \circ u_a)^{(-D)}dQ \\
\text{subject to} & \quad \int (g_j \circ u_a)^{(-D)}dQ \leq \gamma_{j,1} \quad \text{for all } j \in J_1 \\
& \quad \gamma_{j,2} \leq \int x^{D-j}/(D-j)!dQ(x) \leq \overline{\gamma}_{j,2} \quad \text{for all } j \in J_2 \\
& \quad \int x^D dQ < \infty \\
& \quad Q(0) = 0 \\
& \quad D!F(x + a) = \int u^D - (u - x)^D I(u > x)dQ(u) \\
& \quad Q \in \mathcal{Q}(\mathbb{R}^+) \tag{EC.18}
\end{align*}
\]

The last inequality constraint \( \int x^D dQ < \infty \) is redundant and can be dropped, since \( \bar{F}(a) = \int x^D / D!dQ(x) \leq \overline{\gamma}_{0,1} \) must be one of the constraint of Program (EC.18) by construction. Moreover, since \( (h \circ u_a)^{(-D)}(0) = (g_j \circ u_a)^{(-D)}(0) = 0 \), the constraint \( Q(0) = 0 \) impacts neither the feasible set nor the objective value and can also be dropped. Hence, (EC.18) is the same as

\[
\begin{align*}
Z^* = \sup_P & \quad \int (h \circ u_a)^{(-D)}dQ \\
\text{subject to} & \quad \int (g_j \circ u_a)^{(-D)}dQ \leq \gamma_{j,1} \quad \text{for all } j \in J_1 \\
& \quad \gamma_{j,2} \leq \int x^{D-j}/(D-j)!dQ \leq \overline{\gamma}_{j,2} \quad \text{for all } j \in J_2 \\
& \quad D!F(x + a) = \int u^D - (u - x)^D I(u > x)dQ(u) \\
& \quad Q \in \mathcal{Q}(\mathbb{R}^+) \tag{EC.19}
\end{align*}
\]

Since \( \int x^D dQ < \infty \), we can define a distribution function \( \tilde{Q} \in \mathcal{Q}(\mathbb{R}^+) \) absolutely continuous with respect to \( Q \) via \( d\tilde{Q} = x^{D-J}dQ \), i.e., the Radon-Nikodym derivative given by \( \frac{d\tilde{Q}}{dQ} = x^{D-J} \), where \( J \) can be taken as any integer in \( \{0, \ldots, D\} \). Converting the decision
variable from $Q$ to $\tilde{Q}$ in \textbf{(EC.19)} gives

$$Z^* = \sup_P \int H d\tilde{Q}$$

subject to

$$\int G_j d\tilde{Q} \leq \gamma_{j,1}$$

for all $j \in J_1$

$$\gamma_{j,2} \leq \int G_j d\tilde{Q} \leq \gamma_{j,2}$$

for all $j \in J_2$ \textbf{(EC.20)}

$$D!F(x + a) = \int \left[u^D - (u - x)^D I(u > x)\right] u^J d\tilde{Q}(u)$$

$$\tilde{Q} \in Q(\mathbb{R}^+)$$

The constraint defining the function $F$ does not affect \textbf{(EC.20)}. Therefore, \textbf{(5)} and \textbf{(15)} have the same optimal value, and if $\tilde{Q}^*$ is an optimal solution of \textbf{(15)}, then the function $F^*$ defined as

$$D!F^*(x + a) = \int u^J (1 - (1 - x/u)^D I(u > x)) d\tilde{Q}^*(u)$$ \textbf{(EC.21)}

is an optimal solution of \textbf{(5)}.

\textbf{□}

\textbf{EC.2. Technical Proofs for Section 3.2}

\textit{Proof of Theorem 4:} Let $x \in \mathbb{R}^+ \cup \{\infty\}$. By Assumption 2, Theorem 3 and Remark \textbf{EC.2} below, strong duality holds and we have

$$Z^* = \inf_y \sum_{j \in J} y_j \gamma_j$$

subject to

$$\sum_{j \in J} y_j G_j(u) \geq H(u)$$

for all $u \in \mathbb{R}^+$

$$y_j \geq 0$$

for all $j \in J$ \textbf{(EC.22)}

which is the dual formulation of \textbf{(14)}. Suppose that $J(x)$ is non-empty. Then, we consider the sequence $u_n$ in the definition of $J(x)$. For all $i \in J(x)$, $u_n \in \text{supp}(G_i)$ and \textbf{(EC.22)} satisfies the implicit constraint

$$\limsup_{u_n \to x} \left(\sum_{j \in J \setminus \{i\}} y_j \frac{G_j(u_n)}{|G_i(u_n)|}\right) \geq \limsup_{u \to x} \frac{H(u_n)}{|G_i(u_n)|}$$ \textbf{(EC.23)}

By the definition of $J(x)$, \textbf{(EC.23)} gives

$$0 \leq \limsup_{u_n \to x} \frac{H(u_n)}{|G_i(u_n)|} \leq \sum_{j \in J \setminus \{i\}} y_j \limsup_{u_n \to x} \frac{G_j(u_n)}{|G_i(u_n)|} - y_i \leq -y_i \leq 0$$ \textbf{(EC.24)}

As a consequence, the dual multipliers $y_i$’s must be 0 for all $i \in J(x)$. \textbf{(EC.22)} is then equivalent to
\[ Z^* = \inf_{y} \sum_{j \in \mathcal{J} \setminus \mathcal{J}(x)} y_j \gamma_j \]

subject to \[ \sum_{j \in \mathcal{J} \setminus \mathcal{J}(x)} y_j G_j(u) \geq H(u) \quad \text{for all } u \in \mathbb{R}^+ \] (EC.25)

\[ y_j \geq 0 \quad \text{for all } j \in \mathcal{J} \setminus \mathcal{J}(x) \]

Suppose \( \mathcal{J} = \mathcal{J}(x) \). When \( \sup \{ H(u) | u \in \mathbb{R}^+ \} > 0 \), (EC.25) becomes infeasible because of the constraint \( 0 \geq H(u) \) for all \( u \in \mathbb{R}^+ \). When \( \sup \{ H(u) | u \in \mathbb{R}^+ \} \leq 0 \), then \( Z^* = 0 \). The case \( \mathcal{J}(x) \subsetneq \mathcal{J} \) follows from strong duality again. \( \square \)

**Proof of Lemma 3:** Since each constraint in program (22) involves a lower bound and an upper bound, the set \( \mathcal{J}(x) \) defined in (20) becomes

\[ \mathcal{J}(x) = \left\{ i \in \mathcal{J} \mid \exists u_n \in \text{supp}(G_i) \text{ s.t. } \limsup_{u_n \to x} \frac{H(u_n)}{|G_i(u_n)|} \geq 0 \text{ and } \limsup_{u_n \to x} \frac{-G_i(u_n)}{|G_i(u_n)|} \leq 0 \text{ and } \limsup_{u_n \to x} \frac{G_i(u_n)}{|G_i(u_n)|} \leq 0 \quad \forall j \in \mathcal{J} \right\} \] (EC.26)

For a given function \( \tilde{G}_i \), the last two inequalities cannot hold at the same time for \( j = i \) since \( \limsup_{u_n \to x} \frac{\tilde{G}_i(u_n)}{|G_i(u_n)|} \) is either equal to 1 or \(-1\). Hence \( \mathcal{J}(x) = \emptyset \). \( \square \)

**Proof of Theorem 5:** Applying Theorem 4 if \( \mathcal{J} = \mathcal{J}(\infty) \), we fall into the trivial scenarios of the theorem yielding the first two items of Theorem 5. Otherwise, the constraints whose index fall in the set \( \mathcal{J}(\infty) \) can be dropped and Theorem 3 together with Remark EC.2 below gives

\[ Z^* = \inf_{y} \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j \gamma_j \]

subject to \[ \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j G_j(u) \geq H(u) \quad \text{for all } u \in \mathbb{R}^+ \] (EC.27)

\[ y_j \geq 0 \quad \text{for all } j \in \mathcal{J} \setminus \mathcal{J}(\infty) \]

Under Assumption 3 (EC.27) satisfies the implicit constraints

\[ \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j \limsup_{u \to \infty} \frac{G_j(u)}{G_M(u)} \geq \limsup_{u \to \infty} \left( \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j \frac{G_j(u)}{G_M(u)} \right) \geq \limsup_{u \to \infty} \frac{H(u)}{G_M(u)} = \lambda_M \] (EC.28)

When \( \lambda_M = \infty \), (EC.28) deems any solution with \( y_j \in \mathbb{R} \) infeasible, and hence \( Z^* = +\infty \). In the remainder of this proof, we only consider the case when \( \lambda_M \) is finite.
Since (14) is a feasible program, there exists a sequence of feasible solutions \( P^{(k)} \) such that \( \int HdP^{(k)} \to Z^* \), and because \( P^{(k)} \in \mathcal{Q}(\mathbb{R}^+) \), the integral \( \int dP^{(k)} \) is bounded for all \( k \in \mathbb{N} \). Hence, \( Z^* = \lim_{k \to \infty} Z_k \) where

\[
Z_k = \sup_P \int HdP \\
\text{subject to } \int dP = \nu^{(k)} \\
\int G_j dP \leq \gamma_j \text{ for all } j \in \mathcal{J} \setminus \mathcal{J}(\infty) \\
P \in \mathcal{Q}(\mathbb{R}^+) \tag{EC.29}
\]

and \( \nu^{(k)} = \int dP^{(k)} \). Based on Theorem EC.1 below, it is sufficient to investigate the sequences \( P^{(k)} \) with at most \( N \) point supports where \( N \) is the number of linearly independent functions in the set \( \{(G_j)_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)}, 1\} \), i.e. \( P^{(k)} \in \mathcal{Q}_N(\mathbb{R}^+) \). The sequence \( P^{(k)} \) can then be represented by \( N \) couples of point masses and point supports \((p_i^{(k)}, x_i^{(k)})\) where we assume without loss of generality that \( x_1^{(k)} \leq \ldots \leq x_N^{(k)} \). In particular, the sequence of

\[
\sum_{i=1}^N p_i^{(k)} \leq \sum_{i=1}^N p_i^{(k)} G_j \left(x_i^{(k)}\right) \leq \gamma_j \tag{EC.30}
\]

Next, we define the sequence \( s^{(k')} = \sum_{i \in I} p_i^{(k')} \left| G_M \left(x_i^{(k')}\right) \right| \) where \( I \) is the set containing the indexes of the support points which are unbounded for some subsequence, i.e.

\[
I = \left\{ i \in 1, \ldots, N \mid \lim_{k \to \infty} x_i^{(k)} = \infty \text{ for some subsequence indexed by } k \in \mathbb{N} \right\} \tag{EC.31}
\]

Moreover, we define \( k' \) as the index of the sequence associated with the smallest element in the set \( I \) if the latter is not empty and \( k' = k \) otherwise. As defined, the sequence \( s^{(k')} \) is bounded. To see this, note that by definition of the set \( \mathcal{J} \setminus \mathcal{J}(\infty) \), and under Assumptions 3 and 4 there exists some \( M \in \mathcal{J} \setminus \mathcal{J}(\infty) \) such that either \( \lambda_{j,M} > 0 \) for some \( j \in \mathcal{J} \setminus \mathcal{J}(\infty) \) or \( \lambda_M < 0 \). Furthermore, the quantity \( \lambda_{j,M} \) is finite by the same assumptions so that when \( \lambda_{j,M} > 0 \), we have for all \( \varepsilon_1 \in (0, \lambda_{j,M}) \) and \( k' \) large enough,

\[
\gamma_j \geq \sum_{i=1}^N p_i^{(k')} G_j \left(x_i^{(k')}\right) \geq \sum_{i \notin I} p_i^{(k')} G_j \left(x_i^{(k')}\right) + \sum_{i \in I} p_i^{(k')} \left| G_M \left(x_i^{(k')}\right) \right| \frac{G_j \left(x_i^{(k')}\right)}{\left| G_M \left(x_i^{(k')}\right) \right|}
\]
\[ \gamma_j \geq \sum_{i \notin \mathcal{I}} p_i^{(k')} G_j \left( x_i^{(k')} \right) + \lim_{k' \to \infty} \left( p_i^{(k')} G_j \left( x_i^{(k')} \right) \right) \]
\[ \geq \sum_{i \notin \mathcal{I}} p_i^{(k')} G_j \left( x_i^{(k')} \right) + (\lambda_{j,M} - \varepsilon_1) s^{(k')} \] (EC.32)

where the last inequality follows from the fact that

\[ \lim_{k' \to \infty} x_i^{(k')} = \infty, \quad \text{for all } i \in \mathcal{I} \] (EC.33)
\[ \limsup_{k' \to \infty} x_i^{(k')} < \infty \quad \text{for all } i \notin \mathcal{I} \] (EC.34)

Hence, \( \sum_{i \notin \mathcal{I}} p_i^{(k')} G_j \left( x_i^{(k')} \right) \) is finite and (EC.32) implies the boundedness of the sequence \( s^{(k')} \) when \( \lambda_{j,M} > 0 \) for some \( j \in \mathcal{J} \setminus \mathcal{J}(\infty) \). When \( \lambda_M < 0 \), we can show with a similar argument as in the derivation of (EC.32) that for all \( \varepsilon_2 \in (0, -\lambda_M) \) and \( k' \) large enough,

\[ \sum_{i=1}^{N} p_i^{(k')} H \left( x_i^{(k')} \right) \leq \sum_{i \notin \mathcal{I}} p_i^{(k')} H \left( x_i^{(k')} \right) + s^{(k')} \]
\[ \left\{ \begin{array}{ll}
(\lambda_M + \varepsilon_2) & \text{if } -\infty < \lambda_M \\
-\varepsilon_2 & \text{if } \lambda_M = -\infty
\end{array} \right. \] (EC.35)

The LHS in (EC.35) is bounded below since it converges to \( Z^* \) and (14) is consistent. In addition, the sum in the RHS of (EC.35) is finite by Assumption 6. The sequence \( s^{(k')} \) is therefore bounded when \( -\infty < \lambda_M < 0 \). Last but not least, \( s^{(k')} \) must be 0 for all \( k' \) large enough when \( \lambda_M = -\infty \); otherwise, we could choose \( \varepsilon_2 \) arbitrarily large and have the RHS tend to \(-\infty\) as \( \varepsilon_2 \) grows.

Consequently, we have shown that \( s^{(k')} \) is bounded whether \( \lambda_{j,M} > 0 \) or \( \lambda_M < 0 \), so there exists a subsequence \( k'' \) such that

\[ \left( p_i^{(k''}, x_i^{(k'')} \right) \to (p_i^*, x_i^*) \text{ for all } i \notin \mathcal{I} \]
\[ s^{(k'')} \to s^* \quad \text{where } s^* = 0 \text{ if } \lambda_M = -\infty \] (EC.36)

where \( p_i^*, x_i^*, \) and \( s^* \) are non-negative finite quantities, and for all \( j \in \mathcal{J} \setminus \mathcal{J}(\infty), \)

\[ \gamma_j \geq \lim_{k'' \to \infty} \sum_{i=1}^{N} p_i^{(k'')} G_j \left( x_i^{(k'')} \right) \]
\[ \geq \sum_{i \notin \mathcal{I}} \liminf_{k'' \to \infty} \left( p_i^{(k'')} G_j \left( x_i^{(k'')} \right) \right) + \lim_{k'' \to \infty} \left( \sum_{i \in \mathcal{I}} p_i^{(k'')} G_j \left( x_i^{(k'')} \right) \right) \]
\[ = \sum_{i \notin \mathcal{I}} p_i^* G_j \left( x_i^* \right) + \lambda_{j,M} s^* \] (EC.37)
where the last inequality is a consequence of $G_j$ being lower semi-continuous by Assumption 5. We have therefore shown that $(P^*, s^*)$, where $P^*$ is the distribution function with bounded point masses and support points given by $(p_i^*, x_i^*)_{i \notin I}$, is a feasible solution of Program (24). Using a similar argument as in the derivation of inequality (EC.37) and the fact that $H$ is upper semi-continuous in Assumption 6, we can show that $Z^* \leq \sum_{i \notin I} p_i^* H(x_i^*) + \lambda_M s^*$. As a result, $Z^* < \infty$ when $H$ is bounded on any compact subset of $\mathbb{R}^+$, and (24) returns an upper bound to $Z^*$ since $(P^*, s^*)$ is a feasible solution of (24).

We now prove that $Z^*$ is also an upper bound to (24). We do so by noting that the LHS in (EC.28) is equal to $\sum_{j \in J \setminus J(\infty)} y_j \lambda_j M$. Hence, the implicit constraint $\sum_{j \in J \setminus J(\infty)} y_j \lambda_j M \geq \lambda_M$ must hold for (EC.27) and we can add it to the constraint set of the latter. Based on Theorem 3 and Remark EC.2, strong duality also holds for (EC.27) when the constraint $\sum_{j \in J \setminus J(\infty)} y_j \lambda_j M \geq \lambda_M$ is added to the formulation. We then have

$$Z^* = \sup_{P,s} \int H dP + \lambda_M s$$

subject to $\int G_j dP + \lambda_j M s \leq \gamma_j$ for all $j \in J \setminus J(\infty)$

$$P \in Q(\mathbb{R}^+), s \geq 0$$

(EC.38)

The feasible region of (EC.38) includes that of (24), so (24) is bounded above by $Z^*$. Hence, (14) and (24) have the same optimal objective value. This concludes our proof. □

**Remark EC.2.** Under Assumption 2, strong duality continues to hold for (21) when $x = \infty$ and (EC.38), i.e. the optimal value of

$$\sup_{P} \int H dP$$

subject to $\int G_j dP \leq \gamma_j$ for all $j \in J \setminus J(\infty)$

$$P \in Q(\mathbb{R}^+)$$

(EC.39)

is equal to that of

$$\inf_{y} \sum_{j \in J \setminus J(\infty)} y_j \gamma_j$$

subject to $\sum_{j \in J \setminus J(\infty)} y_j G_j(u) \geq H(u)$ for all $u \in \mathbb{R}^+$

$$y_j \geq 0$$

for all $j \in J \setminus J(\infty)$
and the optimal value of

$$\sup_{P, s} \int H dP + \lambda_M s$$

subject to

$$\int G_j dP + \lambda_{j,M} s \leq \gamma_j \text{ for all } j \in \mathcal{J} \setminus \mathcal{J}(\infty)$$

$$P \in \mathcal{Q}(\mathbb{R}^+), s \geq 0$$

is equal to that of

$$Z^* = \inf_y \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j \gamma_j$$

subject to

$$\sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j G_j(u) \geq H(u) \text{ for all } u \in \mathbb{R}^+$$

$$\sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j \lambda_{j,M} \geq \lambda_M$$

$$y_j \geq 0 \text{ for all } j \in \mathcal{J} \setminus \mathcal{J}(\infty)$$

To see these, note that (EC.39) can be similarly written in the form (18) but with less inequalities than those in $\tilde{\mathcal{J}}$ and some equalities in $\tilde{\mathcal{J}}'$ becoming inequalities. The interior point conditions there can be verified to hold for these new reduced set of constraints. On the other hand, (EC.40) has the same form as (EC.39) except that we can view the decision variable (e.g., the distribution) as having support on $\mathbb{R}^+$ together with a point mass $s$ on one augmented point. The interior point conditions held for (EC.39) can be translated to this case by merely considering $s = 0$.

**Theorem EC.1.** Let $H : \mathbb{R}^+ \to \mathbb{R}$ and $G_j : \mathbb{R}^+ \to \mathbb{R}$ be measurable functions for all $j$ in a finite index set $\mathcal{J}$. Then programs (EC.41) and (EC.42) below have the same optimal objective value:

$$\sup_P \int H dP$$

subject to

$$\int dP \leq \nu$$

$$\int G_j dP \leq \gamma_j \text{ for all } j \in \mathcal{J}$$

$$P \in \mathcal{Q}(\mathbb{R}^+)$$

$$\sup_P \int H dP$$

subject to

$$\int dP \leq \nu$$

$$\int G_j dP \leq \gamma_j \text{ for all } j \in \mathcal{J}$$

$$P \in \mathcal{Q}_N(\mathbb{R}^+)$$

where $\nu \in \mathbb{R}^+$, $\gamma_j \in \mathbb{R}$ for all $j \in \mathcal{J}$, and $N$ is the number of linearly independent functions in the sequence $\{(G_j)_{j \in \mathcal{J}}, 1\}$.
We partition the feasible region of (EC.41) into two subregions, one with the additional constraint \( \int dP > 0 \), and another with \( \int dP = 0 \). We consider two programs, each one the same as (EC.41) but with the respective additional constraint. Clearly, the maximum of these two programs have the same optimal value as (EC.41). We show that each program is equivalent to (EC.42) with the corresponding additional constraint, and since the maximum of these equivalent programs has the same optimal value as (EC.42), we conclude the theorem.

In the case \( \int dP = 0 \), the equivalence trivially holds. In the alternate case \( \int dP > 0 \), the inequality \( \int dP \leq \nu \) becomes \( \int dP = \nu - s \) for some \( s \in [0, \nu) \). By applying a change of variable \( P(x) := P(x)/(\nu - s) \), the subprogram considered here can be reformulated as

\[
\sup_{s \in [0, \nu)} \sup_{P} (\nu - s) \int H dP \\
\text{subject to} \quad \int dP = 1 \\
\int G_j dP \leq \gamma_j/(\nu - s) \quad \text{for all } j \in J \\
P \in Q(R^+) \quad \text{(EC.43)}
\]

The feasible region of the inner program in (EC.43) is the set of probability measures defined on \( R^+ \). Since all probability measures on in \( R^+ \) (a Polish space) are regular, Theorem EC.2 applies to conclude that (EC.43) is equivalent to

\[
\sup_{s \in [0, \nu)} \sup_{P} (\nu - s) \int H dP \\
\text{subject to} \quad \int dP = 1 \\
\int G_j dP \leq \gamma_j/(\nu - s) \quad \text{for all } j \in J \\
P \in Q_N(R^+) \quad \text{(EC.44)}
\]

By changing back the variable, we see that (EC.44) is the same as (EC.42) with the additional constraint \( \int dP > 0 \). We therefore conclude our theorem. \( \square \)

**Theorem EC.2** (A particular case of Theorem 3.2 Winkler (1988)). Let \( \mathcal{X} \) be a Hausdorff space, \( \mathcal{F} \) be the Borel \( \sigma \)-field, \( P_r(\mathcal{X}) \) be the set of regular probability measures on \( \mathcal{X} \). In addition, let \( f_1, \ldots, f_n \) be measurable functions, \( c_1, \ldots, c_n \) are real values, and

\[
\mathcal{H} = \left\{ q \in P_r(\mathcal{X}) : f_i \text{ is } q\text{-integrable and } \int f_i dq \leq c_i, \ 1 \leq i \leq n \right\}
\]
In addition, let $g$ be a function on $\mathcal{X}$ integrable for every $q \in \mathcal{H}$ (possibly with integral values $\infty$ or $-\infty$). Then,

$$\sup \left\{ \int_{\mathcal{X}} g dq : q \in \mathcal{H} \right\} = \sup \left\{ \int_{\mathcal{X}} g dq : q \in \text{ex} \mathcal{H} \right\}$$

where $\text{ex} \mathcal{H}$ denotes the set of all extreme points of $\mathcal{H}$, i.e.

$$\text{ex} \mathcal{H} = \left\{ q \in \mathcal{H} : q = \sum_{i=1}^{N} t_i \cdot \delta(x_i), \ t_i > 0, \ \sum_{i=1}^{N} t_i = 1, \ x_i \in \mathcal{X}, \ 1 \leq N \leq n + 1, \text{ the vectors } (f_1(x_i), \ldots, f_n(x_i), 1), \ 1 \leq i \leq N, \text{ are linearly independent} \right\}$$

By Proposition 3.1 [Winkler (1988)], $G(q) = \int_{\mathcal{X}} g dq$ is a measure affine functional and Theorem 3.2 of [Winkler (1988)] holds. In addition, Examples 2.1(a) in [Winkler (1988)] mentions that the set $P$ in Theorem 2.1 of [Winkler (1988)] can be chosen to be the set of all regular probability measures. As such, the extreme points of $\mathcal{H}$ in Theorem 3.2 of [Winkler (1988)] are precisely the ones defined in Theorem 2.1(a) of [Winkler (1988)].

**Remark EC.3.** In the proof of Theorem 5, Assumption 7 is only used to ensure the boundedness of the sequence $P^{(k)}$. In fact, Theorem 5 would still hold provided that $\liminf_{k \to \infty} \int dP^{(k)} < \infty$. In this case, there would be a subsequence $k''$ such that $\int dP^{(k'')} < \infty$ and the rest of the proof would remain valid.

**Proof of Corollary 2.** Program (15) can be reformulated as

$$\sup_{P} \int H dP$$

subject to

$$\begin{align*}
\int G_{j,1} dP &\leq \gamma_{j,1} \quad \text{for all } j \in J_1 \\
n\int -G_{j,1} dP &\leq -\gamma_{j,1} \quad \text{for all } j \in J_1 \\
n\int G_{j,2} dP &\leq \gamma_{j,2} \quad \text{for all } j \in J_2 \\
n\int -G_{j,2} dP &\leq -\gamma_{j,2} \quad \text{for all } j \in J_2 \\
P &\in Q(\mathbb{R}^+) 
\end{align*}$$

(EC.45)

We verify all the assumptions needed to invoke Theorem 5. By definition, $G_{j,1}$ and $G_{j,2}$ are polynomials for all $j$, and by our choice of $M$, $\lim_{x \to \infty} G_{j,1}(x)/|G_{M,1}(x)| = \delta_{j,M}$ is well-defined and finite for all $j \in J_1$. The same holds for $\lim_{x \to \infty} G_{j,2}(x)/|G_{M,1}(x)|$ which is null for all $j \in J_2$. Thus Assumptions 3 and 4 hold. Since $\limsup_{x \to \infty} G_{M,1}(x)/|G_{M,1}(x)| = 1 > 0$,
we also have that the set $\mathcal{J}(\infty)$ is empty, so that cases 1 and 2 in Theorem 5 do not occur. In addition, Assumption 5 is trivially verified.

When $D \geq 1$, the function $H(x) = x^{1-D}(h \circ u_a)(-D)(x)$ is continuous for $x > 0$. From the generalized L'Hôpital’s rule, we can also show that $H$ is bounded on any compact subset of $\mathbb{R}^+$ since

$$
\liminf_{x \to 0} (h \circ u_a)(-J)(x) \leq \liminf_{x \to 0} H(x) \leq \limsup_{x \to 0} H(x) \leq \limsup_{x \to 0} (h \circ u_a)(-J)(x)
$$

(EC.46)

Assumption 6 therefore holds in this case since both ends of (EC.46) are bounded by the definition of $(h \circ u_a)(-J)$. In particular, they are equal to 0 when $J \geq 1$. When $D = 0$, Assumption 6 is also satisfied since we have assumed $h$ upper semi-continuous in this case.

Lastly, Assumption 7 is also satisfied since $G_{J,2} = 1/(D - J)!$ when $J$ is not empty and $G_{0,1} = 1/D!$ otherwise, which correspondingly can serve as the constraint function needed in Assumption 7.

From Theorem 5 item 3a, we have $Z^* = \infty$ if $\lambda_M = \limsup_{u \to \infty} H(u)/|G_{M,1}(u)| = \infty$. Otherwise, Theorem 5 item 3b concludes that program (EC.45) can be reformulated as

$$
\sup_{P,s} \int H dP + \lambda_M s
$$

subject to

$$
\int G_{j,1} dP + s\delta_{jM} \leq \tau_{j,1} \quad \text{for all } j \in J_1
$$

$$
\int -G_{j,1} dP - s\delta_{jM} \leq -\gamma_{j,1} \quad \text{for all } j \in J_1
$$

$$
\int G_{j,2} dP \leq \tau_{j,2} \quad \text{for all } j \in J_2
$$

$$
\int -G_{j,2} dP \leq -\gamma_{j,2} \quad \text{for all } j \in J_2
$$

$$
\int -G_{j,2} dP \leq -\gamma_{j,2} \quad \text{for all } j \in J_2
$$

$$
s = 0 \text{ if } \lambda_M = -\infty
$$

$$
s \geq 0
$$

$$
P \in Q_N(\mathbb{R}^+)
$$

which is equivalent to (26).

Proof of Corollary 3: We prove the corollary by showing that (14) admits, in either case, a sequence of feasible solutions $P^{(k)}$ such that $\int |G| dP^{(k)} \to \infty$ and $\int H dP^{(k)} \to Z^*$.

Item 1: In this case, (14) has a sequence of feasible solution $P^{(k)}$ such that $\int H dP^{(k)} \to Z^* = \infty$. If $\liminf_{u \to \infty} |G(u)/H(u)| = l$ where $l > 0$, then for all $\varepsilon \in (0, l)$ and $x$ large enough,

$$
\int_x^\infty |G| dP^{(k)} = \int_x^\infty \left| \frac{G}{H} \right| dP^{(k)} \geq \int_x^\infty \left| \frac{G}{H} \right| H dP^{(k)} \geq (l - \varepsilon) \int_x^\infty H dP^{(k)}
$$

(EC.48)
which converges to \( \infty \) as \( k \) grows, and so \( \int |G|dP^{(k)} \rightarrow \infty \).

**Item 2:** We start by proving that the following program is unbounded.

\[
\begin{align*}
\sup_P & \quad \int |G|dP \\
\text{subject to} & \quad \int G_j dP \leq \gamma_j \quad \text{for all } j \in \mathcal{J} \\
P & \in \mathcal{Q}(\mathbb{R}^+) \quad \text{(EC.49)}
\end{align*}
\]

We do so by applying Theorem 5 with the function \( H \) set to \( |G| \). If \( \mathcal{J} = \mathcal{J}(\infty) \), we are in the case of Theorem (5)(1) since \( G \) is not identically 0 and therefore \( \sup_{x \in \mathbb{R}^+} |G(x)| > 0 \). Otherwise, we are in Theorem 5(3a) since we have \( \limsup_{x \to \infty} |G(x)/G_M(x)| = \infty \). In either case, we obtain that (EC.49) is unbounded. So there exists a sequence of feasible solution \( P^{(k)} \) for (14) satisfying \( \int |G|dP^{(k)} \rightarrow \infty \). In addition, we have \( \limsup_{k \to \infty} \int dP^{(k)} < \infty \). To see this, note that for all \( j \in \mathcal{J} \)

\[
\inf_{u \in \mathbb{R}^+} G_j(u) \int dP^{(k)} \leq \int G_j dP^{(k)} \leq \gamma_j
\]

By assumption, there exists \( j \in \mathcal{J} \) such that \( \inf_{x \in \mathbb{R}^+} G_j(x) > 0 \). As a result,

\[
\limsup_{k \to \infty} \int_0^x |G|dP^{(k)} \leq \sup_{u \in [0,x]} |G(u)| \limsup_{k \to \infty} P^{(k)}(x) \leq \gamma_j \sup_{u \in [0,x]} |G(u)|/ \inf_{u \in \mathbb{R}^+} G_j(u) < \infty
\]

for all \( x \in \mathbb{R}^+ \). Consequently, \( \lim_{k \to \infty} \int_0^x |G|dP^{(k)} = \infty \) for all \( x \in \mathbb{R}^+ \). Furthermore, if \( \liminf_{u \to \infty} H(u)/|G(u)| = l > 0 \), then for all \( \varepsilon \in (0, l) \) and \( x \) large enough,

\[
\int HdP^{(k)} = \int_0^x HdP^{(k)} + \int_x^\infty HdP^{(k)} \\
\geq \inf_{u \in [0,x]} H(u)P^{(k)}(x) + \int_x^\infty \frac{H}{|G|} |G|dP^{(k)} \\
\geq \inf_{u \in [0,x]} H(u)P^{(k)}(x) + (l - \varepsilon) \int_x^\infty |G|dP^{(k)} \quad \text{(EC.52)}
\]

The first and second terms in the RHS are respectively finite and unbounded when \( k \) grows. So the LHS goes to \( \infty \) with \( k \).

**Item 3:** Using a similar argument as the proof of Item 2, we apply Theorem 5(1) and 5(3a) to show that

\[
\begin{align*}
\sup_P & \quad \int |G|dP \\
\text{subject to} & \quad \int G_j dP \leq \gamma_j \quad \text{for all } j \in \mathcal{J} \\
\int HdP & = Z^* \\
P & \in \mathcal{Q}(\mathbb{R}^+) \quad \text{(EC.53)}
\end{align*}
\]
is unbounded, where $Z^*$ is the optimal objective value of (14). This implies the existence of a sequence of distribution functions $P^{(k)}$ satisfying $\lim_{k \to \infty} \int |G| dP^{(k)} = \infty$, $\int G_j dP^{(k)} \leq \gamma_j$ for all $j \in J$, and $\int H dP^{(k)} \to Z^*$. Because (27) is bounded above by (14), this concludes that (27) and (14) have the same optimal objective value. \qed
Algorithm 2 Finding a feasible solution of \((24)\) when \(\mathcal{J} \setminus \mathcal{J}(\infty)\) is not empty

**Inputs:** Provide the parameters \(\gamma_j\) and the functions \(G_j\) for all \(j \in \mathcal{J} \setminus \mathcal{J}(\infty)\), and the function \(H\). Compute the quantities \(\lambda_{j,M}, j \in \mathcal{J} \setminus \mathcal{J}(\infty)\) and \(\lambda_M\). Also specify a big number \(C \in \mathbb{R}^+\).

**Initialization:**
- SET \(x_1\) to an arbitrary value of the set \([0, C]\)

**Procedure:** For each iteration \(k = 1, 2, \ldots\), given \((x_i)_{i=1}^k\):

1. (Master problem) Solve

\[
Z^k = \min_{s,r,p} \quad r \\
\text{subject to } -r + \sum_{i=1}^k G_j(x_i)p_i + \lambda_{j,M}s \leq \gamma_j, \forall j \in \mathcal{J} \setminus \mathcal{J}(\infty) \\
s \geq 0 \\
s = 0 \text{ if } \lambda_M = -\infty \\
r, p_i \geq 0 \quad \forall i = 1, \ldots, k
\]

Let \((p_i^k)_{i=1}^k, r^k, s^k\) be the optimal solution. Find the dual multipliers \((y_j^k)_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)}\) of dual multipliers satisfying

\[
\left(\sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j^k G_j(x_i)\right) p_i^k = 0, \quad \forall i = 1, \ldots, k \\
\left(\sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j^k \lambda_{j,M}\right) s^k = 0 \\
\left(-1 - \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j^k\right) r^k = 0 \\
y_j^k \geq 0 \quad \text{for all } j \in \mathcal{J} \setminus \mathcal{J}(\infty)
\]

2. (Subproblem) Find \(x_{k+1}\) that minimizes

\[
\rho^k(u) = \sum_{j \in \mathcal{J} \setminus \mathcal{J}(\infty)} y_j^k G_j(u), \quad \text{where } u \in [0, C]
\]

- SET \(\epsilon^k = \rho^k(x_{k+1})\)
- IF \(\epsilon^k \leq 0\) and \(r = 0\), STOP and RETURN \((x_i, p_i^k, s^k)_{i=1}^k\)
- IF \(\epsilon^k \leq 0\) and \(r > 0\), STOP; The problem is inconsistent.
Note that if we focus on program (14) instead of (24), we can similarly run Algorithm 2 but setting $\lambda_{j,M} = 0$ for all $j \in J \setminus J(\infty)$, to obtain a feasible solution. However, as we have discussed, (14) operates on an unbounded domain and may not bear an optimal solution to whom the algorithm can converge.