Linearization through symmetries for discrete equations

D Levi and C Scimiterna

Dipartimento di Matematica e Fisica dell’Università Roma Tre and Sezione INFN di Roma Tre, via della Vasca Navale 84, I-00146 Roma, Italy

E-mail: levi@roma3.infn.it

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Abstract

We show that one can devise through the symmetry approach a procedure to check the linearizability of a difference equation via a point or a discrete Cole–Hopf transformation. If the equation is linearizable, then the symmetry provides the linearizing transformation. At the end, we present a few examples of applications for equations defined on four lattice points.

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1. Introduction

In recent years, the use of partial difference equations (PDEs) has played an increasingly significant role in physics and mathematics. On the one hand, discrete systems are believed to be at the base of the basic laws of physics (see for example the recent literature on quantum gravity [25]) and on the other hand, with the increased use of computers, discretizations are playing an increasingly important role in solving numerically differential equations [7].

Calogero [5] introduced a heuristic distinction between nonlinear PDEs which are ‘C-integrable’ and ‘S-integrable’, namely equations that are linearizable by an appropriate change of variables (i.e. by an explicit redefinition of the dependent variable and maybe in some cases also the independent variables), and those equations that are integrable via the inverse scattering transform (IST) [4]. In the cited reference, Calogero used the asymptotic behavior to find equations belonging to these two classes of equations as it usually preserves integrability and linearizability. However, this approach, although very fruitful, is often very cumbersome and not always exhaustive (see for example the results of [8] for discrete equations).

A more intrinsic approach to integrability is based on the existence of symmetries. The well-known notion of higher symmetry is at the base of this approach together with the notion of IST. The mutual influence of these theories has led to the fundamental abstract concept of formal symmetry [23, 26], more basic than symmetry as it also provides higher conservation
laws, B"acklund transformations and Lax pair representation. In this sense, a formal symmetry is a universal object. However, the derivation of formal symmetries is difficult to apply in the case of difference equations [18]. Moreover, the distinction between C- and S-integrable equations is only at the level of the conservation laws as linearizable equations have no local conservation laws of arbitrary high order [26].

A well-established result in the framework of Lie theory for proving the linearizability of nonlinear PDEs is provided by Kumei and Bluman [11] (for a recent extended review see [1]) based on the analysis of the symmetry properties of linear PDEs. Following the analogy of the continuous case, we will formulate similar theorems for linearizable partial difference equations (P\Delta Es) by which we recover part of the results obtained by assuming the existence of linearizing transformations [13, 14, 22]. Partial results in this direction for the case of autonomous difference equations defined on a fixed lattice have been obtained by Quispel and collaborators [3, 20, 21].

In section 2, we review for the sake of completeness and clarity the results presented by Bluman et al in [1] for the linearization of nonlinear PDEs both by point and nonlocal transformations and consider examples of their applications. Section 3 is devoted to the introduction of P\Delta Es, their symmetry analysis and the discretization of the theorems presented in [1]. A series of examples is presented in section 4, while in section 5 we summarize the results and present conclusions and outlooks. In the appendix, a theorem complements the results of section 3 by showing that equations possessing an infinite-dimensional symmetry algebra depending on an arbitrary function are linear and the arbitrary function satisfies a linear homogeneous equation.

2. Linearization of PDEs through symmetries

In [11], Bluman and Kumei introduce a series of theorems dealing with the conditions for a nonlinear PDE to be transformable into a linear one by contact transformations. Here, in the following, we will limit ourselves to the case of just point transformations as these are the relevant ones in the discrete case [15, 16]. In more recent works, Bluman and Kumei [2] extended the consideration to the case where we have non-invertible transformations between a nonlinear and a linear PDE.

The basic observation is that a linear PDE,

\[
L v(y) = \mathcal{F}(y),
\]

where \( L \) is a \( \psi \)-independent but possibly \( y \)-dependent linear operator and \( \mathcal{F}(y) \) is the inhomogeneous term, has one point symmetry of infinitesimal symmetry generator

\[
\hat{X} = w(y) \frac{\partial}{\partial v}
\]

depending on a function \( w \) which satisfies the homogeneous linear equation

\[
L w(y) = 0,
\]

as any solution of (1) is the sum of a particular solution plus the general solution of the associated homogeneous equation. As the existence of an infinitesimal generator of the form (2) is preserved when we transform a linear equation into a nonlinear one by an invertible point transformation, following [1, section 2.4], we can state the following theorem for the existence of an invertible linearization mapping of a nonlinear PDE.

**Theorem 1.** A nonlinear PDE

\[
\mathcal{E}_n(x, u, u_x, \ldots u_{n\alpha}) = 0
\]
of order n for a scalar function u of an r-dimensional (r ≥ 2) vector x is linearizable by an
invertible point transformation to a linear equation (1) for v only if it possesses a symmetry
generator
\[ \hat{X} = \sum_{i=1}^{r} \xi_i(x, u) \partial_{x_i} + \phi(x, u) \partial_u, \quad \xi_i(x, u) = \alpha_i(x, u)w(y), \]
where \( \omega \) and \( \alpha_i \) are given functions of their arguments and \( w(y) \) is an arbitrary solution of (3).

Following [11], we can state the sufficient conditions for the existence of an invertible
linearization mapping of a nonlinear PDE. The following theorem defines the point
transformation.

**Theorem 2.** If a symmetry generator for the nonlinear PDE (4) as specified in theorem 1 exists
and if there exist r functionally independent solutions \( \Phi_i(x, u), i = 1, \ldots, r \), of the linear,
homogeneous, first-order PDE for a scalar function \( \Phi_1(x, u) \)
\[ \sum_{i=1}^{r} \alpha_i(x, u) \Phi_i(x, u)_x + \omega(x, u) \Phi(x, u)_u = 0, \]
and a particular solution \( \Psi(x, u) \) of the linear, inhomogeneous, first-order PDE for a scalar
function \( \Psi_1(x, u) \)
\[ \sum_{i=1}^{r} \alpha_i(x, u) \Psi_i(x, u)_x + \omega(x, u) \Psi(x, u)_u = 1, \]
then the invertible point transformation which transforms (4) to the linear PDE (1) is given by
\[ y_i = \Phi_i(x, u), \quad i = 1, \ldots, r, \]
\[ v = \Psi(x, u). \]

If a given linearizable nonlinear PDE does not have local symmetries of the form (5), i.e.
its local symmetries do not satisfy the criteria of theorem 1, then it could still happen, as shown
in [1, section 4.3], that a nonlocally related system has an infinite set of local symmetries that
yields an invertible mapping of the nonlocally related system to some linear system of PDEs.
Consequently, the invertible mapping of the nonlocally related system to a linear system will
provide a nonlocal (non-invertible) mapping of the given nonlinear PDE to a linear PDE. This
non-invertible transformation will be a kind of Cole–Hopf transformation [6, 10]. In this case,
however, we have to generalize theorem 2 to take into account the fact that we are dealing
with a system of equations.

**Theorem 3.** Let us consider a system of nonlinear PDEs
\[ \mathcal{E}_n^{(1)}(x, u, v, u_x, v_x, \ldots, u_{nx}, v_{nx}) = 0, \quad \mathcal{E}_n^{(2)}(x, u, v, u_x, v_x, \ldots, u_{nx}, v_{nx}) = 0 \]
of order n for two scalar functions u and v of an r-dimensional (r ≥ 2) vector x which
possesses a symmetry generator
\[ \hat{X} = \sum_{i=1}^{r} \xi_i(x, u, v) \partial_{x_i} + \phi(x, u, v) \partial_v + \psi(x, u, v) \partial_u, \quad \xi_i(x, u, v) = \sum_{j=1}^{2} \alpha_{ij}(x, u, v)w^{(j)}(y), \]
\[ \phi(x, u, v) = \sum_{j=1}^{2} \beta_{ij}(x, u, v)w^{(j)}(y), \quad \psi(x, u, v) = \sum_{j=1}^{2} \gamma_{ij}(x, u, v)w^{(j)}(y), \]
with $\alpha_i$, $\beta_j$ and $\gamma_j$ the given functions of their arguments and the function $w = (w^{(1)}(y), w^{(2)}(y))$ satisfying the linear homogeneous equations
\begin{equation}
M(y)w(y) = 0,
\end{equation}
with $y$ an $r$-dimensional vector depending on $u$, $v$ and the vector $x$ and $M$ a $2 \times 2$ matrix linear operator.

The invertible transformation
\begin{equation}
\begin{align*}
z^{(1)}(y) &= F^{(1)}(x, u, v), \\
z^{(2)}(y) &= F^{(2)}(x, u, v), \\
y &= G(x, u, v),
\end{align*}
\end{equation}
which transforms (10) to the system of linear PDEs $M(y)z(y) = 0$, (12) with $y$ an $r$-dimensional vector depending on $u$, $v$ and the vector $x$ and $M$ a $2 \times 2$ matrix linear operator.

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The conditions of theorem 1 are satisfied with $\omega = e^{-u}$ and $\alpha_i = 0$. We can apply theorem 2 and we obtain
\begin{equation}
\begin{align*}
\Phi_1 &= x, \\
\Psi_1 &= u, \\
\end{align*}
\end{equation}
from (6), (7), $\Phi_1 = x$ and $\Phi_2 = t$ as from (6) $\Phi_0 = 0$, while from (7), $\Psi(x, u)$ satisfies the equation $\Psi(x, u) = e^x$ i.e.
\begin{equation}
\begin{align*}
u &= \log_e(w).
\end{align*}
\end{equation}
Equation (18) is the linearizing transformation for the potential Burgers equation (16).

2.2. A nonlinear PDE linearizable by a non-invertible transformation

The standard example in this class is the Burgers equation
\begin{equation}
\begin{align*}
u_t &= \nu_{xx} + uu_x = \left[u_x - \frac{1}{2}u^2\right]_x,
\end{align*}
\end{equation}
\begin{equation}
\begin{align*}
u_t &= \nu_{xx} + uu_x = \left[u_x - \frac{1}{2}u^2\right]_x,
\end{align*}
\end{equation}
linearizable by a Cole–Hopf transformation. As (19) has no infinite-dimensional symmetry algebra but can be written as a conservation law, we can introduce a potential function \( v(x, t) \) and (19) can be written as the system

\[
    \begin{align*}
        v_x &= 2u, \\
        v_t &= 2ux - u^2.
    \end{align*}
\]

Applying theorem 3, we can find an infinite-dimensional symmetry of the form of (11). In fact, solving the determining equations, apart from terms corresponding to a finite-dimensional algebra, we obtain an infinite-dimensional dilation symmetry given by

\[
    \psi = 4\omega(x, t) e^{2\tau}, \quad \phi = \frac{1}{2} \psi_x + u \psi_t = e^{2\tau}[2\omega_x + \omega u],
\]

where \( \omega(x, t) \) satisfies the linear heat equation \( \omega_t = \omega_{xx} = 0 \).

The linearizing transformation can be obtained from theorem 3. Let us define \( w^{(1)}(y) = \omega(x, t), \) \( w^{(2)}(y) = \omega_4(x, t) \) and take \( G_1 = x \) and \( G_2 = t \) as functionally independent solutions of (14). As \( \alpha_1 = 0 \) and \( \gamma_1 = 4e^{2\tau}, \gamma_2 = 0, \beta_1 = u e^{2\tau} \) and \( \beta_2 = 2 e^{2\tau} \), we obtain as a particular solution of (15)

\[
    F^{(1)} = -e^{-\tau}, \quad F^{(2)} = \frac{1}{2} u e^{-\tau}.
\]

Equation (13) implies \( \xi^{(1)} = -e^{-\tau} \) and \( \xi^{(1)} = \frac{1}{2} u e^{-\tau} \) and from it we obtain as a linearizing transformation the Cole–Hopf transformation

\[
    u = -\frac{\xi^{(1)}}{\xi^{(2)}}.
\]

3. P\AE s and their linearization

We consider here the problem of the linearization of a P\AE.

It is well known that difference equations can be obtained on a given grid \( x_{n,m}, t_{n,m}, \) starting from physical, chemical or biological problems dealing with lattice systems, or as the discretization of differential equations with symmetries which we want to preserve. In this second case, we can use the freedom of the grid to obtain a difference scheme which preserves part or all of the symmetries of the continuous equation.

In this first approach to the problem of linearization of the difference equation, we will consider the case where the grid is preassigned and assumed to be fixed, with constant lattice spacing. Moreover, for simplicity we will consider autonomous equations defined on a two-dimensional grid so that there is no privileged position and we can write the dependent variables just in terms of the shifts with respect to the reference point \( u_{0,0} = u_{0,0} \) on the lattice.

A P\AE of order \( N \cdot N' \) for a function \( u_{n,m} \) will be a relation between \( N \cdot N' \) points in the two-dimensional grid, i.e.

\[
    \mathcal{E}_{N \cdot N'}(u_{0,0}, u_{1,0}, \ldots, u_{N,0}, u_{0,1}, \ldots, u_{N,1}, \ldots, u_{N,N'}) = 0.
\]

A continuous symmetry for equations of the form (23), where the lattice is fixed, i.e. the two variables \( x_{n,m} \) and \( t_{n,m} \) are completely specified as \( x_{n,m} = h_n + x_0 \) and \( t_{n,m} = h_m + t_0 \) with \( h_n, h_m, x_0 \) and \( t_0 \) the given constants, is given just by dilations

\[
    \mathcal{X}_{n,m} = \chi_{n,m}(u_{n,m}) \partial_{u_{n,m}}.
\]

It is easy to show that a linear P\AE of order \( N \cdot N' \) for a function \( v_{n,m} \),

\[
    \mathcal{F}_{N \cdot N'} = b(n, m) + \sum_{(i,j)=(0,0)}^{(N,N')} a_{i,j}(n, m) v_{n+i,m+j} = 0,
\]
has a symmetry
\[ \hat{X}_{n,m} = \phi_{n,m} \partial_{v_{n,m}}, \]  
where \( \phi_{n,m} \) is a solution of the homogeneous part of (25). It is not at all obvious, however, that an equation (23) having a symmetry (26) is linear and that the function \( \phi_{n,m} \) must satisfy a homogeneous linear equation. We leave the proof of this proposition to the appendix. The symmetry (26) is due to the superposition principle for linear equations. If the nonlinear equation (23) is linearizable by an invertible point transformation, then the symmetry (26) must be preserved. This is the content of theorem 1, which we presented in the previous section in the case of PDEs and this must still be valid here. So we can state the following theorem.

**Theorem 4.** A PDE (23) is linearizable by an invertible point transformation only if it has a point symmetry of the form
\[ \hat{X}_{n,m} = \alpha_{n,m}(u_{n,m}) \phi_{n,m} \partial_{v_{n,m}}, \]  
where the function \( \phi_{n,m} \) satisfies a linear PDE of the form (25) with \( b(n,m) = 0 \).

The proofs of theorem 4 and of the following theorem 5 are the same as those presented by Bluman and Kumei [2] in the continuous case and so we will not repeat them here.

As in the case of PDEs, we can present the following theorem which provides the point symmetry of the form
\[ \phi_{n,m} \partial_{v_{n,m}}. \]  
As in the continuous case, if (23) has no symmetries of the form considered in theorem 4 we can introduce some potential variables. On the lattice, there are infinitely many ways to introduce a potential variable as there are infinitely many ways to define a first derivative. Thus, it seems to be advisable to check the equation with a linearizability criterion such as the algebraic entropy [24] before looking for potential variables.

The simplest way to introduce a potential symmetry is by writing the difference equation (23) as a system
\[ v_{n+1,m} = E^{(1)}_{n,m}(u_{n,m}, \ldots), \quad v_{n,m+1} = E^{(2)}_{n,m}(u_{n,m}, \ldots), \]  
in such a way,
\[ E_{N,N'}(u_{0,0}, u_{1,0}, \ldots, u_{N,0}, 0, u_{0,1}, \ldots, u_{N,1}, \ldots, u_{N,N'}) = E^{(1)}_{n,m+1} - E^{(2)}_{n+1,m}. \]  
However, it is easy to show in full generality that the symmetries for (30) and (23) are the same.

We can introduce potential symmetries by the following system:
\[ v_{n+1,m} - v_{n,m} = E^{(1)}_{n,m}(u_{n,m}, \ldots), \quad v_{n,m+1} - v_{n,m} = E^{(2)}_{n,m}(u_{n,m}, \ldots), \]  
in such a way,
\[ E_{N,N'}(u_{0,0}, u_{1,0}, \ldots, u_{N,0}, 0, u_{0,1}, \ldots, u_{N,1}, \ldots, u_{N,N'}) = [E^{(1)}_{n,m+1} - E^{(1)}_{n,m}] - [E^{(2)}_{n+1,m} - E^{(2)}_{n,m}]. \]
i.e. the nonlinear difference equation is written as a discrete conservation law. As (32) is a system, in order to construct the symmetries we have to generalize the linearization theorem as we did in the continuous case.

**Theorem 6.** Let us consider a system of nonlinear PΔEs

\[
F_{n,m}^{(1)}(u_{n,m}, \ldots, v_{n,m}, \ldots) = 0, \quad F_{n,m}^{(2)}(u_{n,m}, \ldots, v_{n,m}, \ldots) = 0
\]  

of order \( N \cdot N' \) for two scalar functions \( u_{n,m} \) and \( v_{n,m} \) of the two indices \( n \) and \( m \) which possess a symmetry generator

\[
\tilde{X}_{n,m} = \phi_{n,m}(u_{n,m}, v_{n,m})\partial_{u_{n,m}} + \psi_{n,m}(u_{n,m}, v_{n,m})\partial_{v_{n,m}},
\]

\[
\phi_{n,m}(u_{n,m}, v_{n,m}) = \sum_{j=1}^{2} \beta^{(j)}(u_{n,m}, v_{n,m}) w_{n,m}^{(j)},
\]

\[
\psi_{n,m}(u_{n,m}, v_{n,m}) = \sum_{j=1}^{2} \gamma^{(j)}(u_{n,m}, v_{n,m}) w_{n,m}^{(j)},
\]

with \( \beta^{(j)} \) and \( \gamma^{(j)} \) the given functions of their arguments and the function \( w = (w_{n,m}^{(1)}, w_{n,m}^{(2)}) \) satisfying the linear difference equations

\[
L_{n,m} w_{n,m} = 0.
\]  

The invertible transformation

\[
z_{n,m}^{(1)} = K_{n,m}^{(1)}(u_{n,m}, v_{n,m}), \quad z_{n,m}^{(2)} = K_{n,m}^{(2)}(u_{n,m}, v_{n,m}),
\]

which transforms (34) to the system of linear PΔEs \( L_{n,m} z_{n,m} = g_{n,m} \) for \( z_{n,m} = (z_{n,m}^{(1)}, z_{n,m}^{(2)}) \), is given by the solution of the linear inhomogeneous first-order system of PDEs for the function \( K = (K_{n,m}^{(1)}(u_{n,m}, v_{n,m}), K_{n,m}^{(2)}(u_{n,m}, v_{n,m})) \)

\[
\beta_{n,m}^{(k)}(u_{n,m}, v_{n,m}) K_{n,m}^{(j)} + \gamma_{n,m}^{(k)}(u_{n,m}, v_{n,m}) K_{n,m}^{(j)} = \delta_{k}^{j},
\]

where \( \delta_{k}^{j} \) is the standard Kronecker symbol.

4. Examples

Here, we present at first one simple example of the linearizable PΔE belonging to each of the cases considered in the previous section, to verify the applicability of theorems 4–6. For concreteness and for comparison with the previous literature [13], we limit ourselves to the case where the nonlinear difference equation involves at most four lattice points. Then, we apply theorems 4–6 to the classification of linearizable PΔEs on four lattice points.

4.1. Dispersive PΔE linearizable by a point transformation

Let us consider the class of real dispersive multilinear equations on the square of type \( Q_{+} \)

\[
Q_{+} \triangleq a_{1}(u_{n,m} + u_{n+1,m+1}) + a_{2}(u_{n+1,m} + u_{n,m+1})
\]

\[
+ (\alpha_{1} - \alpha_{2}) u_{n,m} u_{n+1,m+1} + (\alpha_{1} + \alpha_{2}) u_{n,m+1} u_{n+1,m+1}
\]

\[
+ (\beta_{1} - \beta_{2}) u_{n,m} u_{n,m+1} + (\beta_{1} + \beta_{2}) u_{n+1,m} u_{n+1,m+1}
\]

\[
+ \gamma_{1} u_{n,m} u_{n+1,m+1} + \gamma_{2} u_{n+1,m} u_{n,m+1}
\]

\[
+ (\xi_{1} - \xi_{2}) u_{n,m} u_{n+1,m} u_{n,m+1} + (\xi_{1} + \xi_{2}) u_{n,m} u_{n+1,m} u_{n+1,m+1}
\]

\[
+ (\xi_{2} - \xi_{1}) u_{n+1,m} u_{n+1,m+1} u_{n,m+1} + (\xi_{2} + \xi_{1}) u_{n,m} u_{n,m+1} u_{n+1,m+1}
\]

\[
+ \xi u_{n,m} u_{n+1,m} u_{n,m+1} u_{n+1,m+1} = 0.
\]  

[36]
Inside this family, the equation defined by
\[\alpha_1 = \beta_1 = \frac{(a_1 + a_2)\gamma_1}{2a_1}, \quad \alpha_2 = \beta_2 = 0, \quad \gamma_2 = \frac{a_2\gamma_1}{a_1},\]
\[\xi_1 = \xi_2 = \frac{3(a_1 + a_2)\gamma_1^2}{8a_1^2}, \quad \xi_3 = \xi_4 = \frac{(a_1 - a_2)\gamma_1^2}{8a_1^2}, \quad \zeta = \frac{(a_1 + a_2)\gamma_3^3}{4a_1^3}\]
admits the infinitesimal generator of point symmetries
\[\tilde{\mathcal{X}} = 2a_1\phi_{n,m}\left(1 + \frac{\gamma_1}{2\alpha_1}\right)^2 \partial_{u,n,m},\]
where \(\phi_{n,m}\) satisfies the homogeneous linear equation \(\phi_{n,m} + \phi_{n+1,m+1} + \frac{\gamma_1}{a_1}\left(\phi_{n+1,m} + \phi_{n,m+1}\right) = 0\). Introducing this result in theorem 5 with \(\alpha(u_{n,m}) = 2a_1\left(1 + \frac{\gamma_1}{2\alpha_1\gamma_1}\right)^2\), we obtain that the transformation
\[v_{n,m} = -\frac{1}{\gamma_1\left(1 + \frac{\gamma_1}{2\alpha_1}\right)} + \kappa,\]
with \(\kappa\) an arbitrary integration constant, linearizes our equation to
\[v_{n,m} + v_{n+1,m+1} + \frac{a_2}{a_1}\left(v_{n+1,m} + v_{n,m+1}\right) + \frac{2(a_1 + a_2)(1 - \kappa\gamma_1)}{a_1\gamma_1} = 0,\]
which, choosing \(\kappa = 1/\gamma_1\), is homogeneous.

4.2. Nonlinear PDE linearizable by a noninvertible transformation
As an example in this class, let us consider the discrete Burgers equation
\[u_{n+1,m}[p + u_{n+1,m+1} - u_{n,m}[p + u_{n+1,m}] = 0.\]
Equation (44) has been presented by Levi et al [12] as a Bäcklund transformation of the Burgers hierarchy of differential–difference equations. One can show that (44) has no infinite-dimensional symmetries and thus cannot be linearized by point transformations. However, we can introduce a potential function \(v_{n,m}\) and rewrite (44) as a system for two coupled equations
\[v_{n,m+1} = u_{n,m}v_{n,m}, \quad v_{n+1,m} = \frac{1}{p + u_{n+1,m}} v_{n,m}.\]
Applying theorem 6, we can find for the system (45) an infinite-dimensional symmetry of the form (35) by solving the corresponding symmetry determining equations
\[\psi_{n,m+1} - \phi_{n,m}v_{n,m} - u_{n,m}\psi_{n,m} = 0, \quad \psi_{n+1,m} + \frac{\phi_{n+1,m}v_{n,m}}{(p + u_{n+1,m})^2} - \frac{\psi_{n,m}}{p + u_{n+1,m}} = 0.\]
Choosing \(u_{n,m} = v_{n,m}\) and \(u_{n+1,m} = w_{n,m+1}\), where \(w_{n,m}\) satisfies the linear PDE
\[w_{n+1,m+1} = w_{n,m} - pw_{n+1,m},\]
the solution of the symmetry determining equations is
\[\rho^{(1)} = -\frac{u_{n,m}}{v_{n,m}}, \quad \rho^{(2)} = \frac{1}{v_{n,m}}, \quad \gamma^{(1)} = 1, \quad \gamma^{(2)} = 0.\]
Using (47), (38) reads
\[-\frac{u_{n,m}}{v_{n,m}} K_{n,m,s,n}^{(1)} + K_{n,m,s,n}^{(1)} = 1, \quad \frac{1}{v_{n,m}} K_{n,m,s,n}^{(1)} = 0,\]
\[-\frac{u_{n,m}}{v_{n,m}} K_{n,m,s,n}^{(2)} + K_{n,m,s,n}^{(2)} = 0, \quad \frac{1}{v_{n,m}} K_{n,m,s,n}^{(2)} = 1,\]
whose solution is \(K_{n,m}^{(1)} = v_{n,m}\) and \(K_{n,m}^{(2)} = u_{n,m}v_{n,m}\). Equation (37) gives
\[\zeta_{n,m}^{(1)} = v_{n,m}, \quad \zeta_{n,m}^{(2)} = u_{n,m}v_{n,m},\]
from which we get the linearizing Cole–Hopf transformation \(u_{n,m} = \frac{\zeta_{n,m}^{(1)}}{2\zeta_{n,m}^{(2)}}\).
4.3. Classification of PΔEs on a square lattice linearizable by point transformations

We consider a general autonomous equation defined on a square lattice:

\[ \mathcal{F}(u_{0,0}, u_{0,1}, u_{1,0}, u_{1,1}) = 0, \]  
(50)

where, for convenience, we just write down the shift with respect to the reference point of indices \( n \) and \( m \). If we assume that \( u_{1,1} \) is present in (50), then we can rewrite the equation as

\[ u_{1,1} = F(u_{0,0}, u_{0,1}, u_{1,0}). \]  
(51)

Following theorem 4, we look for an infinitesimal symmetry generator of the form

\[ \tilde{X}_{0,0} = \alpha_{0,0}(u_{0,0})\phi_{0,0}(\partial_{u_{0,0}}), \]  
(52)

where the function \( \phi \) solves a linear homogeneous equation, i.e.

\[ \tilde{\mathcal{E}}\phi_{0,0} = 0, \quad \tilde{\mathcal{E}} = a + bT_1 + cT_2 + dT_1T_2, \]  
(53)

with \( T_1 \) and \( T_2 \) operators such that \( T_1\phi_{0,0} = \phi_{1,0} \) and \( T_2\phi_{0,0} = \phi_{0,1} \). If \( d \neq 0 \), then we can write (53) as

\[ \phi_{1,1} = -\frac{1}{d}[a\phi_{0,0} + b\phi_{1,0} + c\phi_{0,1}]. \]  
(54)

In this setting, \( u_{0,0}, u_{0,1}, u_{1,0}, \) and \( \phi_{0,1} \), with \( i, j = 0, 1 \) are independent variables. If (52) is a generator of the symmetries of (51), then we must have

\[ \tilde{X}\mathcal{F}|_{u_{1,1}=F} = 0 \iff F_{\alpha_{0,0}}\phi_{0,0}(u_{0,0}) + F_{\alpha_{1,0}}\phi_{1,0}(u_{1,0}) + F_{\alpha_{0,1}}\phi_{0,1}(u_{0,1}) = 0. \]  
(55)

As \( \phi_{0,0}, \phi_{1,0} \) and \( \phi_{0,1} \) are independent variables, we obtain from (55) three equations relating the function \( \alpha \), intrinsic of the symmetry, with the function \( F \), intrinsic of the nonlinear equation:

\[ a\alpha_{1,1}(F) + dF_{\alpha_{0,0}}(u_{0,0}) = 0, \quad b\alpha_{1,1}(F) + dF_{\alpha_{1,0}}(u_{1,0}) = 0, \quad c\alpha_{1,1}(F) + dF_{\alpha_{0,1}}(u_{0,1}) = 0. \]  
(56)

As in (56), up to a constant, the first term is the same for all three equations; we can rewrite them as a system of PDEs for the function \( F \) depending on \( \alpha \)

\[ \frac{1}{a}F_{\alpha_{0,0}}(u_{0,0}) = \frac{1}{b}F_{\alpha_{1,0}}(u_{1,0}) = \frac{1}{c}F_{\alpha_{0,1}}(u_{0,1}). \]  
(57)

which can be solved on the characteristic, giving \( F \) as a function of the symmetry variable:

\[ \xi = ag(u_{0,0}) + bg(u_{1,0}) + cg(u_{0,1}), \quad \alpha(x) = \frac{1}{g}(\xi). \]  
(58)

Introducing this result in theorem 5, we obtain \( \psi(u_{0,0}) = \int_{a(x)}^{ag(x)} g(\xi) d\xi = g(u_{0,0}) + \kappa, \) with \( \kappa \) being an arbitrary integration constant. Then, (56) gives that any linearizable nonlinear PΔE on a four-point lattice must be written as

\[ dF_{\xi} + \alpha(F(\xi)) = 0 \rightarrow F = g^{-1}\left(\frac{\xi - \xi_0}{d}\right), \]  
(59)

where by \( g^{-1}(x) \) we mean the inverse of the function \( g(x) \) given in (58).

Let us note that from (57) we can obtain the six necessary linearizability conditions we introduced in [14] to classify linearizable, multilinear equations on the four lattice points, that is,

\[ A(x, u_{0,1}) = \frac{F_{\alpha_{0,0}}}{F_{\alpha_{1,0}}} \bigg|_{u_{0,0}=u_{0,1}=x} = \frac{a}{b}, \]  
\( \forall x, u_{0,1}. \)  
(60a)
are constants expressed in term of \( a \).

In this case is a fraction of a second-order polynomial over a third-order polynomial. The only \( g \) is \( \phi \) and its inverse. As a trivial example, we can choose \( \psi \), \( \phi \) and \( \psi \) as the linear fractional function

So, linearizable equations on four lattice points are characterized by a function \( g(x) \) and its inverse. As a trivial example, we can choose \( g(x) = e^x \) and we obtain that the nonlinear equation \( u_{1,1} = \log(\alpha u_{1,0} + \beta e^{u_{1,0}} + \gamma e^{u_{0,1}} + k) \) linearizes to \( \psi_{1,1} = \alpha \psi_{0,0} + \beta \psi_{1,0} + \gamma \psi_{0,1} \). The corresponding function \( \alpha \) is \( \alpha(x) = e^{-x} \) and the linearizing transformation is \( \psi_{0,0} = e^{u_{0,0}} + \kappa \).

In section 4.1, we have shown that there exists a real, multilinear, dispersive equation on the square lattice belonging to the \( Q^+ \) class (39, 40) which is linearizable. It is interesting to find the corresponding function \( g(x) \) in terms of which we can linearize it. The function \( F \) in this case is a fraction of a second-order polynomial over a third-order polynomial. The only function \( g \) which provides this structure is \( g(x) = \frac{1}{1+\xi} \) which gives \( F = \frac{1}{1+\xi} \frac{d}{d-u} + \xi \).

For the sake of simplicity, we set in (32) \( \phi^{(2)}_{n,m} = u_{n,m} \). If we want equation (23) to be on the square, then we have to choose \( \phi^{(1)}_{n,m} = u_{n,m} (\delta_{n,m}, u_{n+1,m}) \). The application of the prolongation of the infinitesimal generator (35) to the second equation in (32) gives [17]

Then, the prolongation of the infinitesimal generator (35) applied to the first equation in (32) gives

where

To comply with theorem 6, we look for an infinitesimal coefficient of the infinitesimal generator (35) of the form

\[
\psi_{0,0} = w_{0,0}^{(1)} \frac{\partial}{\partial v_{0,0}} (v_{0,0}) + w_{0,0}^{(2)} \frac{\partial}{\partial u_{0,0}} (v_{0,0}).
\]
where the functions \( w_{n,m}^{(1)} \) and \( w_{n,m}^{(2)} \) satisfy a linear partial difference equation on the square lattice:

\[
\begin{align*}
w_{0,0}^{(1)} &= a_{0,0}^{(1)} w_{0,1}^{(1)} + a_{0,0}^{(2)} w_{1,0}^{(1)} + a_{0,0}^{(3)} w_{1,1}^{(1)}, \\
w_{0,0}^{(2)} &= b_{0,0}^{(1)} w_{0,1}^{(2)} + b_{0,0}^{(2)} w_{1,0}^{(2)} + b_{0,0}^{(3)} w_{1,1}^{(2)},
\end{align*}
\]

Introducing (64) and (65) into (62) and taking into account that we can always choose \( w_{0,1}^{(1)}, w_{1,0}^{(1)}, w_{0,1}^{(2)}, w_{1,0}^{(2)} \) and \( w_{1,1}^{(1)} \) as independent variables, we obtain the following system of coupled equations for \( \gamma^{(1)} \):

\[
\begin{align*}
\gamma_{1,0}^{(1)}(v_0 + g_0, 0) &\left(1 + \frac{\partial g_0}{\partial u_1}ight) - a_{0,0}^{(2)} \gamma_{0,0}^{(1)}(v_0, 0) \left(1 + \frac{\partial g_0}{\partial u_0}ight) = 0, \\
\gamma_{0,1}^{(1)}(v_0 + u_0 + g_0, 0) &\left(1 + \frac{\partial g_0}{\partial u_0}ight) = 0, \\
\gamma_{1,1}^{(1)}(v_0 + u_0 + g_0, 0) &\left(1 + \frac{\partial g_0}{\partial u_0}ight) = 0,
\end{align*}
\]

and similar ones for the function \( \gamma_{n,m}^{(2)}(v_0, 0) \). Adding (66) multiplied by \( a_{0,0}^{(1)} \) to (68) multiplied by \( a_{0,0}^{(1)} \), we obtain

\[
\frac{a_{0,0}^{(3)}}{a_{0,0}^{(1)}} \left(1 + \frac{\partial g_0}{\partial u_0}ight) = a_{0,0}^{(2)} \frac{\partial g_0}{\partial u_0}.
\]

In (70) appears the function \( g_{1,0} = g_{1,0}(u_1, u_2, 0) \), and if \( \frac{\partial g_{1,0}}{\partial u_0^2} \neq 0 \), we obtain a linear differential equation for \( g_0 \), whose solution is \( g_0 = g_{0,0}(u_{0,0}) + g_{0,1}(u_{0,0})u_1 \). Introducing this solution in (70), we obtain \( g_0 = g_{0,0} + g_{0,1}(u_{0,0})u_1 + g_{0,2}(u_{0,0}), i.e. a linear equation. By choosing, in place of (65), the most general linear coupled system of difference equations on the square lattice for \( w^{(1)} \) and \( w^{(2)} \), we would obtain the same result. So the introduced potential equation (32) does not provide linearizable discrete equations.

As we saw in section 4.2 for a difference–version of the Burgers equation, a different possibility is given by rewriting (32) as

\[
\begin{align*}
v_{n+1,m}/v_{n,m} &= \mathcal{E}_{n,m}^{(1)}(u_{n,m}, \ldots), \\
v_{n,m+1}/v_{n,m} &= \mathcal{E}_{n,m}^{(2)}(u_{n,m}, \ldots).
\end{align*}
\]

Equations (32) and (71) are transformable one into the other by defining \( v_{n,m} = \log(w_{n,m}) \) and redefining appropriately the functions \( \mathcal{E}_{n,m}^{(1)} \) and \( \mathcal{E}_{n,m}^{(2)} \). However, in doing so, if \( w_{n,m} \) satisfies a linear equation, then this will not be the case for \( v_{n,m} \). So the fact that the ansatz (32) does not give rise to linearizable equations is not in contradiction to the fact that (44) is linearizable.

The compatibility of (71) implies

\[
\mathcal{E}_{n+1,m}^{(1)}(u_{n+1,m}, \ldots) \mathcal{E}_{n,m}^{(2)}(u_{n,m}, \ldots) = \mathcal{E}_{n+1,m}^{(2)}(u_{n+1,m}, \ldots) \mathcal{E}_{n,m}^{(1)}(u_{n,m}, \ldots).
\]

If (72) is constrained to be an equation on the square lattice, then we must have \( \mathcal{E}_{n,m}^{(1)}(u_{n,m}, \ldots) = \mathcal{E}_{n,m}^{(1)}(u_{n,m}, u_{n+1,m}) \) and \( \mathcal{E}_{n,m}^{(2)}(u_{n,m}, \ldots) = \mathcal{E}_{n,m}^{(2)}(u_{n,m}, u_{n,m+1}) \). Let us take for simplicity \( \mathcal{E}_{n,m}^{(2)}(u_{n,m}, u_{n,m+1}) = \mathcal{E}_{n,m}^{(2)}(u_{n,m}, u_{n,m+1}) = u_{n,m} \).
Let us look for the symmetries of (71). Applying the infinitesimal generator (35) to the right-hand equation in (71), we obtain

$$\psi_{0,0} = \psi_{0,0}(v_{0,0}), \quad \phi_{0,0} = \frac{\psi_{0,1}(v_{0,1}) - u_{0,0}\psi_{0,0}(v_{0,0})}{v_{0,0}}.$$  \hspace{1cm} (73)

Then, the determining equation associated with the left-hand equation in (71) is given by

$$\psi_{1,0}(v_{1,0}) = \left[ \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{0,0}} \phi_{0,0} + \frac{\partial \xi_{0,1}^{(1)}}{\partial u_{1,0}} \right] v_{0,0} + \xi_{0,0}^{(1)} \psi_{0,0}(v_{0,0}),$$  \hspace{1cm} (74)

where the functions $\phi_{i,j}$ are expressed in term of the functions $\psi_{i,j}$ through (73).

As we look for linearizable equations, from theorem 6 it follows that we must have

$$\psi_{0,0}(v_{0,0}) = \sum_{j=1}^{2} w_{0,j} \gamma_{0,j}^{(j)}(v_{0,0}),$$  \hspace{1cm} (75)

where the discrete functions $w_{0,j}$ satisfy a linear difference equation on the square. We can assume that the coefficient of $w_{0,j}^{(i)}$ is always different from zero so that we have

$$w_{1,1}^{(1)} = a_{0,0} w_{0,0}^{(1)} + b_{0,1} w_{0,1}^{(1)} + b_{1,0} w_{1,0}^{(1)} + a_{0,0} w_{0,0}^{(2)} + f_{0,0} w_{0,0}^{(2)} + \epsilon_{0,0} w_{1,0}^{(2)}.$$  \hspace{1cm} (76)

In such a case, the variables $w_{0,0}^{(1)}$, $w_{0,1}^{(1)}$ and $w_{0,j}$, $j = 1, 2$, are independent and (74) splits into three couples of equations relating the functions $\gamma_{0,0}^{(j)}(v_{0,0})$, $j = 1, 2$, with the function $\xi_{0,0}^{(1)}$

$$\gamma_{1,0}^{(1)} \gamma_{0,0}^{(1)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{0,0}} [b_{1,1} \gamma_{1,1}^{(1)} + b_{1,0} \gamma_{1,1}^{(2)} - u_{1,0} \gamma_{1,1}^{(1)}],$$  \hspace{1cm} (77)

$$\gamma_{1,0}^{(2)} \gamma_{0,0}^{(1)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{1,0}} [c_{1,1} \gamma_{1,1}^{(1)} + c_{1,0} \gamma_{1,1}^{(2)} - u_{1,0} \gamma_{1,1}^{(2)}],$$  \hspace{1cm} (78)

$$\frac{\partial \xi_{0,0}^{(1)}}{\partial u_{0,0}} \gamma_{0,0}^{(1)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{1,0}} \gamma_{0,0}^{(1)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{1,0}} \gamma_{0,0}^{(1)} = 0,$$  \hspace{1cm} (79)

$$\frac{\partial \xi_{0,0}^{(1)}}{\partial u_{0,0}} \gamma_{0,0}^{(2)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{1,0}} \gamma_{0,0}^{(2)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{1,0}} \gamma_{0,0}^{(2)} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{1,0}} \gamma_{0,0}^{(2)} = 0,$$  \hspace{1cm} (80)

where $v_{0,1} = u_{0,0} v_{0,0}$, $v_{1,0} = \frac{\partial \xi_{0,0}^{(1)}}{\partial u_{0,0}}$, $v_{1,1} = u_{1,0} \xi_{0,0}^{(1)}$ and $v_{1,0}$ due to (71). As $\gamma_{0,0}^{(1)}$ and $\gamma_{0,0}^{(2)}$ cannot be simultaneously zero, we have that $\frac{\partial \xi_{0,0}^{(1)}}{\partial u_{0,0}} \neq 0$. For the same reason, in all generality, we can take $\gamma_{0,0}^{(1)} \neq 0$.

As $\xi_{0,0}^{(1)}$ is a function of $u_{0,0}$ and $\gamma_{0,0}^{(j)}$ is a function of $v_{0,0}$, we obtain from (77) and (78)

$$\gamma_{1,0}^{(1)} \gamma_{0,0}^{(1)} = b_{0,0} \gamma_{0,0}^{(1)} + b_{0,1} \gamma_{0,0}^{(2)}, \quad \gamma_{1,0}^{(2)} \gamma_{0,0}^{(2)} = c_{0,0} \gamma_{0,0}^{(2)} + c_{0,1} \gamma_{0,0}^{(1)}.$$  \hspace{1cm} (83)
From (79) and (80), we obtain
\[
\frac{\partial \varepsilon^{(1)}_0}{\partial u_{0,0}} = \kappa^{(1)}_{0,0},
\]
\[
-\gamma^{(1)}_{0,0,0} = a^{(1)}_{0,0} \gamma^{(1)}_{1,1} + a^{(2)}_{0,0} \gamma^{(2)}_{1,1}.
\]
while from (81) and (82), we obtain
\[
\frac{\partial \varepsilon^{(2)}_0}{\partial u_{0,0}} = \kappa^{(2)}_{0,0},
\]
\[
-\gamma^{(2)}_{0,0,0} = a^{(1)}_{0,0} \gamma^{(1)}_{1,1} + a^{(2)}_{0,0} \gamma^{(2)}_{1,1}.
\]
When \(\kappa^{(0)}_{0,0} \neq 0\), differentiating the formulas (83) with respect to \(v_{1,0}\), we have that \(\gamma^{(j)}_{0,0}\), \(j = 1, 2\) have to depend only on the indices but not on the field \(v_{0,0}\). Hence \(\gamma^{(j)}_{0,0}(v_{0,0}) = a^{(j)}_{0,0}\), \(j = 1, 2, a^{(j)}_{0,0} \neq 0\). The same happens if \(\kappa^{(0)}_{0,0} = 0\) and \(\kappa^{(2)}_{0,0} \neq 0\), as can be seen differentiating the formulas (85) with respect to \(v_{0,1}\). Solving the equations (83), (84) and (85) for \(\varepsilon^{(1)}_{0,0}\), we obtain
\[
\varepsilon^{(1)}_{0,0} = \frac{\kappa^{(2)}_{0,0} a^{(0)}_{0,0} + \kappa^{(1)}_{0,0}}{\kappa^{(0)}_{0,0} - u_{1,0}}.
\]
The resulting class of linearizable PΔEs (86) is an extension of the Burgers equations (44) [27]:
\[
\left(\kappa^{(0)}_{0,0} - u_{1,0}\right) \left(\kappa^{(2)}_{0,0} u_{0,0} + \kappa^{(1)}_{0,1}\right) u_{0,0} - \left(\kappa^{(0)}_{0,1} - u_{1,1}\right) \left(\kappa^{(2)}_{0,0} u_{0,0} + \kappa^{(1)}_{0,0}\right) u_{1,0} = 0,
\]
which reduces to it when \(\kappa^{(0)}_{0,0} = -p \neq 0, \kappa^{(1)}_{0,0} = -1\) and \(\kappa^{(2)}_{0,0} = 0\). Other two autonomous Burgers equations are obtained taking \(\kappa^{(0)}_{0,0} \neq 0, \kappa^{(1)}_{0,0} = 0\) and \(\kappa^{(2)}_{0,0} \neq 0\) or \(\kappa^{(0)}_{0,0} = 0, \kappa^{(1)}_{0,0} = 1\) and \(\kappa^{(2)}_{0,0} \neq 0\). In the autonomous case \(\kappa^{(1)}_{0,0}\) in all generality can be taken to be either 0 or 1, however, if we want (87) to be defined on the square, then \(\kappa^{(j)}_{0,0} \neq 0, j = 0, 1, 2\). Moreover, in the autonomous case, if \(\kappa^{(2)}_{0,0} \neq 0, \kappa^{(1)}_{0,0} = 1\) and \(\kappa^{(0)}_{0,0} = 0\), by the transformation \(u_{0,0} = \kappa^{(0)}_{0,0} u + \kappa^{(2)}_{0,0} u + \kappa^{(1)}_{0,0} u\), where \((\alpha, \beta, \gamma, \delta)\), \(j = 1, 2\) are arbitrary parameters, \(u_{0,0}\) will satisfy the Hietarinta equation [9]
\[
\begin{align*}
\alpha_{0,0} + \alpha_{1,1} + \alpha_{2,0} & = \alpha_{1,0} + \alpha_{0,1} + \alpha_{1,0}, \\
\beta_{0,0} + \beta_{1,1} + \beta_{2,0} & = \beta_{1,0} + \beta_{2,0} + \beta_{0,1}, \\
\gamma_{0,0} + \gamma_{1,1} + \gamma_{2,0} & = \gamma_{1,0} + \gamma_{2,0} + \gamma_{0,1}, \\
\delta_{0,0} + \delta_{1,1} + \delta_{2,0} & = \delta_{1,0} + \delta_{2,0} + \delta_{0,1},
\end{align*}
\]
with \(\kappa^{(0)}_{0,0} = 0\). Once the arbitrary parameters \(\kappa^{(j)}_{0,0} \neq 0, j = 0, 1, 2\) have been chosen, it remains to find a particular solution, if possible, of (83), (84) and (85) with \(\gamma^{(j)}_{0,0}(v_{0,0}) = a^{(j)}_{0,0}\), \(j = 1, 2\). Choosing \(a^{(1)}_{0,0} = 0\), we have that \(d^{(1)} = e^{(1)} = f^{(1)} = 0\) and we get the following system of non-autonomous difference equations for \(a^{(1)}_{0,0}\)
\[
\begin{align*}
a^{(1)}_{0,0} \alpha^{(1)}_{1,1} & = -\kappa^{(1)}_{0,0} \alpha^{(1)}_{0,0}, & b^{(1)}_{0,0} \alpha^{(1)}_{1,1} & = \kappa^{(1)}_{0,0} \alpha^{(1)}_{0,0}, & c^{(1)}_{0,0} \alpha^{(1)}_{1,1} & = -\kappa^{(2)}_{0,0} \alpha^{(1)}_{0,0}, \\
a^{(1)}_{0,0} \beta^{(1)}_{1,1} & = -\kappa^{(1)}_{0,0} \beta^{(1)}_{0,0}, & b^{(1)}_{0,0} \beta^{(1)}_{1,1} & = \kappa^{(1)}_{0,0} \beta^{(1)}_{0,0}, & c^{(1)}_{0,0} \beta^{(1)}_{1,1} & = -\kappa^{(2)}_{0,0} \beta^{(1)}_{0,0}, \\
a^{(1)}_{0,0} \gamma^{(1)}_{1,1} & = -\kappa^{(1)}_{0,0} \gamma^{(1)}_{0,0}, & b^{(1)}_{0,0} \gamma^{(1)}_{1,1} & = \kappa^{(1)}_{0,0} \gamma^{(1)}_{0,0}, & c^{(1)}_{0,0} \gamma^{(1)}_{1,1} & = -\kappa^{(2)}_{0,0} \gamma^{(1)}_{0,0},
\end{align*}
\]
from which we can see that \(a^{(1)}_{0,0} \neq 0, b^{(1)}_{0,0} \neq 0\) and \(c^{(1)}_{0,0} \neq 0\). This system is compatible only if
\[
\begin{align*}
a^{(1)}_{0,0} & = \kappa^{(1)}_{0,0}, & b^{(1)}_{0,0} & = \kappa^{(1)}_{0,0} \kappa^{(0)}_{0,0}, & c^{(1)}_{0,0} & = -\kappa^{(2)}_{0,0}, \\
b^{(1)}_{0,0} & = \kappa^{(1)}_{0,0} \kappa^{(0)}_{0,0}, & c^{(1)}_{0,0} & = -\kappa^{(2)}_{0,0},
\end{align*}
\]
which can be solved taking \(a^{(1)}_{0,0} = -\kappa^{(1)}_{0,0}, b^{(1)}_{0,0} = \kappa^{(0)}_{0,0}\) and \(c^{(1)}_{0,0} = -\kappa^{(2)}_{0,0}\). This choice implies that \(a^{(1)}_{0,0}\) is just a constant, which in all generality can be taken to be equal to 1. From (73) it
follows our system (71) and admits the infinitesimal generator
\[ \dot{X}_{n,m} = \frac{w_{0,1}^{(1)} - u_{0,0} w_{0,0}^{(1)}}{v_{0,0}} \partial_{u_{0,0}} + w_{0,0}^{(1)} \partial_{v_{0,0}}, \]
where the function \( w_{0,0}^{(1)} \) is an arbitrary solution of the difference equation
\[ w_{1,1}^{(1)} = -\kappa_{0,0}^{(1)} w_{0,0}^{(1)} + \kappa_{0,0}^{(1)} w_{1,0}^{(1)} - \kappa_{0,0}^{(2)} w_{0,1}^{(1)}, \tag{89} \]
from which it follows that \( w_{0,0}^{(2)} = w_{0,1}^{(1)} \), \( f_{0,0}^{(1)} (u_{0,0}, v_{0,0}) = -\frac{u_{0,0}}{v_{0,0}} \) and \( g_{0,0}^{(2)} (u_{0,0}, v_{0,0}) = \frac{1}{v_{0,0}} \).

Finally from theorem (6) we get \( \tau_{0,0}^{(1)} = v_{0,0} \), \( \tau_{0,0}^{(2)} = u_{0,0} v_{0,0} \) so that (87) is linearized to (89) for \( z_{0,0}^{(1)} \) by the discrete Cole–Hopf transformation \( u_{0,0} = \frac{\tau_{0,0}^{(1)}}{\tau_{0,0}^{(2)}} \).

The remaining case \( \kappa_{0,0}^{(0)} = 0, \kappa_{0,0}^{(2)} = 0 \), as can be seen solving (83), (84) and (85), implies \( \kappa_{0,0}^{(1)} \neq 0 \) and the linear equation \( u_{n+1,m+1} = u_{n,m} \), equivalent to (44) with \( p = 0 \), which is solved by \( u_{n,m} = f(n - m) + g((-1)^{n+m}) \), where \( f \) and \( g \) are arbitrary functions of their arguments.

5. Conclusions

In this paper, we studied from the point of view of the symmetries, the linearization through invertible or non-invertible transformations of nonlinear partial difference equations defined on a fixed non-transformable lattice. We find the linearizability conditions in strict analogy to the continuous case and apply them to the classification of linearizable P\( \Delta \)Es on a lattice of four points in the plane. The results we obtain are compatible with the results obtained previously by requiring the existence of a linearizing transformation; however, the classification turns out to be easier. As a change from [3, 20, 21], in this paper we consider the general case of nonlinear P\( \Delta \)Es linearizable by point transformations and discrete Cole–Hopf transformations, not just the case of autonomous nonlinear P\( \Delta \)Es linearizable by point transformations.

Work is in progress to extend these results to the case of linearizable equations belonging to more general plane lattices and nonlinear partial difference equations defined on transformable lattices.

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Appendix

In this appendix, we state and prove a theorem on the necessary and sufficient conditions for a partial difference equation to be linear. For the sake of simplicity of the presentation,

\[ \dot{X}_{n,m} = \frac{w_{0,1}^{(1)} - f_{0,0}(u_{0,0}) w_{0,0}^{(1)}}{f_{0,0}(u_{0,0}) v_{0,0}} \partial_{u_{0,0}} + w_{0,0}^{(1)} \partial_{v_{0,0}}. \]

1 These results can be immediately generalized to the case where \( \hat{c}_{0,0}^{(2)} (u_{0,0}, u_{0,1}) = f_{0,0} (u_{0,0}) \), with \( f_{0,0} \) an arbitrary function. In this case the infinitesimal generator takes the form
we limit our considerations to the case where the equation is defined on four points (see figure A1), i.e.
\[ \mathcal{E}(n, m, u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0. \]  
(A.1)

**Theorem 7.** The necessary and sufficient condition for a discrete equation (A.1) to have a symmetry with an infinitesimal generator \( \hat{X} = \phi_{n,m} \frac{\partial}{\partial u_{n,m}} \) is that it is linear. The function \( \phi_{n,m} \) must satisfy the following linear homogeneous discrete equation:
\[ F(\phi_{n,m}, \phi_{n+1,m}, \phi_{n+1,m+1}) = \phi_{n+1,m+1} - a_{n,m} \phi_{n,m} - b_{n,m} \phi_{n,m+1} - c_{n,m} \phi_{n+1,m} = 0. \]  
(A.2)

**Proof.** It is almost immediate to prove that a linear partial difference equation defined on four lattice points (A.1) has a symmetry (26), where \( \phi_{n,m} \) satisfies a homogeneous linear equation.

Not so easy is the proof that an equation which has such a symmetry (26) must be linear. In the full generality, (A.2) can be rewritten as
\[ \phi_{n+1,m+1} = F(n, m, \phi_{n,m}, \phi_{n+1,m}, \phi_{n,m+1}), \]  
(A.4)

and, by assumption, (A.1) does not depend on \( \phi_{n,m} \) or (A.4) on \( u_{n,m} \).

Let us prolong the symmetry generator (26) to all points contained in (A.1):
\[ \text{pr} \hat{X} = \phi_{n,m} \frac{\partial}{\partial u_{n,m}} + \phi_{n+1,m} \frac{\partial}{\partial u_{n+1,m}} + \phi_{n,m+1} \frac{\partial}{\partial u_{n,m+1}} + \phi_{n+1,m+1} \frac{\partial}{\partial u_{n+1,m+1}}. \]  
(A.5)

The most generic equation (A.1) having the symmetry (A.5) will be written in terms of its invariants
\[ \mathcal{E}(n, m, K_1, K_2, K_3) = 0, \]  
(A.6)

with
\[ K_1 = \frac{u_{n+1,m+1}}{\phi_{n,m+1}} - \frac{u_{n,m}}{\phi_{n,m}}, \quad K_2 = \frac{u_{n+1,m}}{\phi_{n+1,m}} - \frac{u_{n,m}}{\phi_{n,m}}, \quad K_3 = \frac{u_{n,m+1}}{\phi_{n,m+1}} - \frac{u_{n+1,m}}{\phi_{n+1,m}}. \]  
(A.7)
As (A.6) depends on $n, m$, we can with no loss of generality replace the invariants (A.7) in (A.6) by the functions

$$\tilde{K}_1 = u_{n,m+1} - u_{n,m} \frac{\phi_{n,m+1}}{\phi_{n,m}}, \quad \tilde{K}_2 = u_{n+1,m} - u_{n,m} \frac{\phi_{n+1,m}}{\phi_{n,m}}, \quad \tilde{K}_3 = u_{n+1,m+1} - u_{n,m} \frac{F(n, m, \phi_{n,m}, \phi_{n+1,m}, \phi_{n,m+1})}{\phi_{n,m}}.$$  (A.8)

Invariance of (A.6) then requires

$$\frac{\partial \mathcal{E}}{\partial \phi_{n,m}} = \frac{\partial \mathcal{E}}{\partial \phi_{n+1,m}} = \frac{\partial \mathcal{E}}{\partial \phi_{n,m+1}} = 0,$$  i.e.

$$\frac{\partial \mathcal{E}}{\partial \tilde{K}_1} \left( u_{n,m} \frac{\phi_{n,m+1}}{\phi_{n,m}^2} + \frac{\partial \mathcal{E}}{\partial \tilde{K}_2} \left( u_{n,m} \frac{\phi_{n+1,m}}{\phi_{n,m}^2} \right) + \frac{\partial \mathcal{E}}{\partial \tilde{K}_3} \left( F \frac{\phi_{n,m+1}}{\phi_{n,m}} \right) u_{n,m} = 0 \right),$$  (A.9)

$$\frac{\partial \mathcal{E}}{\partial \tilde{K}_2} \left( u_{n,m} \frac{\phi_{n+1,m}}{\phi_{n,m}^2} \right) u_{n,m} = 0,$$  (A.10)

$$\frac{\partial \mathcal{E}}{\partial \tilde{K}_3} \left( u_{n,m} \frac{\phi_{n,m+1}}{\phi_{n,m}^2} \right) u_{n,m} = 0.$$  (A.11)

As $\frac{\partial \mathcal{E}}{\partial \tilde{K}_i} \neq 0$ and (A.9)–(A.11) are a homogeneous system of algebraic equations, the determinant of the coefficients must be zero. Consequently, the function $F$ must satisfy the first-order linear partial differential equation

$$F - \phi_{n,m} F \phi_{n,m} - \phi_{n,m+1} F \phi_{n+1,m} - \phi_{n+1,m} F \phi_{n,m+1} = 0,$$  (A.12)

i.e. $F$ is given by

$$F = \phi_{n,m} f(\xi, \tau), \quad \xi = \frac{\phi_{n+1,m}}{\phi_{n,m}}, \quad \tau = \frac{\phi_{n,m+1}}{\phi_{n,m}},$$  (A.13)

where $f(\xi, \tau)$ is an arbitrary function of its arguments.

For $F$ given by (A.13), the system (A.9)–(A.11) reduces to the following two equations for $\frac{\partial \mathcal{E}}{\partial \tilde{K}_i}$,

$$\frac{\partial \mathcal{E}}{\partial \tilde{K}_1} f_\xi + f_\tau \frac{\partial \mathcal{E}}{\partial \tilde{K}_3} = 0, \quad \frac{\partial \mathcal{E}}{\partial \tilde{K}_2} + f_\tau \frac{\partial \mathcal{E}}{\partial \tilde{K}_3} = 0,$$  (A.14)

whose solution is obtained by solving (A.14) on the characteristics

$$\mathcal{E} = \mathcal{E}(n, m, L),$$

$$L = u_{n+1,m+1} - f u_{n,m} - f_\xi \left( u_{n,m+1} - u_{n,m} \frac{\phi_{n,m+1}}{\phi_{n,m}} \right) - f_\tau \left( u_{n+1,m} - u_{n,m} \frac{\phi_{n+1,m}}{\phi_{n,m}} \right).$$  (A.15)

Requiring that $\mathcal{E}$ be independent of $\phi_{n,m}, \phi_{n+1,m}$ and $\phi_{n,m+1}$, we obtain $f_\xi \xi = f_\tau \tau = f_\xi \tau = 0$, i.e.

$$f = a_{n,m} \xi + c_{n,m} \tau, \quad F = a_{n,m} \phi_{n,m} + b_{n,m} \phi_{n,m+1} + c_{n,m} \phi_{n+1,m},$$  (A.16)

$$L = a_{n,m+1} - a_{n,m} u_{n,m} - b_{n,m} u_{n,m+1} - c_{n,m} u_{n+1,m}.$$  (A.17)

$\mathcal{E} = 0$ is a (non-autonomous, maybe transcendental) equation for $L$ which, when solved, gives $L = d_{n,m}$, where $d_{n,m}$ stands for the set of the zeros of the equation (in addition to $n$ and $m$ possibly dependent on a set of parameters). In conclusion, $u_{n,m}$ must satisfy the linear equation

$$u_{n+1,m+1} - a_{n,m} u_{n,m} - b_{n,m} u_{n,m+1} - c_{n,m} u_{n+1,m} - d_{n,m} = 0.$$  (A.18)
Remark 1. The proof of theorem 7 does not depend on the position of the four lattice points considered in (A.1). The same result is also valid if the four points are put on the triangle shown in figure A2, i.e.

\[ \mathcal{E}(n, m, u_{n-1, m}, u_{n, m}, u_{n+1, m}, u_{n, m+1}) = 0. \] (A.19)

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