Asymptotic Composite Estimation

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Abstract

Composition methodologies in the current literature are mainly to promote estimation efficiency via direct composition, either, of initial estimators or of objective functions. In this paper, composite estimation is investigated for both estimation efficiency and bias reduction. To this end, a novel method is proposed by utilizing a regression relationship between initial estimators and values of model-independent parameter in an asymptotic sense. The resulting estimators could have smaller limiting variances than those of initial estimators, and for nonparametric regression estimation, could also have faster convergence rate than the classical optimal rate that the corresponding initial estimators can achieve. The simulations are carried out to examine its performance in finite sample situations.

Key words: Asymptotic representation, model-independent parameter, asymptotic composite regression, composite quantile regression.

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1. INTRODUCTION

Composition methodologies in statistics have received much attention in the literature. The earlier work may ascend to jackknife (Quenouille 1949, Quenouille 1956, Gray and Schucany 1972, Tuky 1958), a special composition approach that combines leave-one-out versions (or leave-many-out versions) of a traditional estimator (e.g., the least squares estimator) to construct an improved estimator. For a comprehensive review see Miller (1974). Recently the notion of composition has been further developed to several settings mainly for enhancing estimation efficiency. Zou and Yuan (2008) proposed a composite quantile linear regression via directly combining objective functions, by which the estimation efficiency is improved. Kai, Li and Zou (2010) extended it to construct efficiency-improved nonparametric regression estimation through directly combining the initial estimators. For the further developments of this methodology in semiparametric settings, see Kai, Li and Zou (2011). Composite models such as model averaging are obtained in spirit from the composition idea. By averaging the selected models beforehand, a refined model can be obtained; see for example Wang, Zhang and Zou (2010), Hansen (2007) and Hoeting et al. (1999).

From all the aforementioned works, although they respectively treat their related models for composite estimation construction, we note that, to construct a composite estimator, a model-independent parameter plays a crucial role. This parameter is not the one of interest for us to estimate, but with different values, several initial estimators for the parameter of interest can be defined, and then a composite estimator can be constructed. This is the common feature in all composite methodologies in the literature. The examples of model-independent parameter are the size of blocks in composite likelihood, the quantile in quantile regression estimation, and the bandwidth in kernel estimation for nonparametric regression.

It is worthwhile to note the following issues that are of interest to answer. Most
of the current composition methodologies in the literature have been developed from case to case. It is of interest to develop a generic framework for composition methodology. To this end, the key is to establish a generic relationship between estimation and model-independent parameter such that it can be used as a basis for composition estimation construction. Two of the popularly used approaches in the literature have the potential. First is the use of composite objective function. An example is Zou and Yuan (2008) who proposed composite quantile regression (CQR) with improved estimation efficiency in parametric setup. But Sun, Gai and Lin (2013) showed that for nonparametric quantile regression, the weights in composite objective function asymptotically play no role in enhancing estimation efficiency. The other is to directly combine initial estimators to form a composite estimator. This method usually cannot however work on bias reduction when initial estimators are biased such as nonparametric regression estimation. It is worth pointing out that bias reduction is another important issue as most of existing methods can only provide biased estimations.

In contrast, we find that the asymptotic representations of several estimations can offer us a way to establish a general framework: the asymptotic composite regression (ACR). This method has the following desirable features.

1. (*Generality*) The generic framework allows that, as long as an estimator has an asymptotically linear representation with a model-independent parameter, a composite estimator can then be constructed by a regression combination of several initial estimators according to different values of this parameter.

2. (*Variance reduction*) By selecting proper weights, the ACR is shown to be asymptotically more efficient than those obtained by existing composite methods such as the composite maximum likelihood and the composite least squares.

3. (*Bias reduction*) This is particularly useful for bias estimation that is usually
the case in the literature. This advantage of the ACR could result in faster convergence rate of biased estimation. For example, under the same regularity conditions, the corresponding ACR of the biased Nadaraya-Watson estimator of nonparametric regression can have faster convergence rate than the classical optimal one. It is worthwhile to point out that although the ACR estimator seems still to have a kernel estimation type, the above rate-accelerated property is acquired by composition, rather than by a delicately chosen kernel function. Thus, the Nadaraya-Watson estimator cannot possess this property. Further, the composition may be readily applied to other nonparametric smoothing estimations.

The rest of the paper is organized as follows. In Section 2 we review the asymptotic representation of parametric estimation and further examine three examples to motivate a general framework of relationship between estimator and model-independent parameter. In Section 3, the ACR is defined and the relevant parametric and nonparametric estimations are obtained. In Section 4, the accelerated convergence and efficiency of the new estimators are investigated, and the applications for the three important models are presented. Simulation studies are given in Section 5 and the proofs of the theorems are postponed to the Appendix.

2. MOTIVATING EXAMPLES AND ASYMPTOTIC REPRESENTATION

To motivate the methodology development, we first review asymptotic representations of parametric and nonparametric estimations in several settings. Let $F$ be the true distribution function of a random variable $X$ and $F_n$ be the empirical distribution function based on i.i.d observations $X_1, \ldots, X_n$ from $X$. Consider functional estimators of a parameter $\theta = T(F)$ of the form $\hat{\theta} = T(F_n)$ for some smooth functional $T$
having the influence function

\[ I(x) = \lim_{\varepsilon \to 0} [T((1 - \varepsilon)F + \varepsilon\delta_x) - T(F)]/\varepsilon, \]

where \( \delta_x \) is the unit point mass at \( x \). Under some regularity conditions (see, e.g., Shao, 1991), we have the following asymptotic representation:

\[ \hat{\theta} - \theta = \frac{1}{n} \sum_{i=1}^{n} I(X_i) + \epsilon_n, \]

where \( \epsilon_n = O_p(1/n) \) with a mean of order \( O(1/n) \) and a variance of order \( O(1/n^2) \). Particularly, for the maximum likelihood estimator, \( \theta = T(F) \) is defined as the solution of the equation \( \int (\partial/\partial \theta) \log f_\theta(X) dF(x) = 0 \) and so \( I(x) = J^{-1}(\partial/\partial \theta) \log f_\theta(X) \), where \( J = -E[(\partial^2/\partial \theta \partial \theta') \log f_\theta(X)] \) and \( f_\theta(x) \) is the density function of \( X \).

In the above asymptotic representation, \( \frac{1}{n} \sum_{i=1}^{n} I(X_i) \) is the leading term and determines the asymptotic property of the estimator \( \hat{\theta} \). In some situations, this term could depend on another parameter. More precisely, the above asymptotic representation often has the following form:

\[ \hat{\theta}_\tau - \theta = \frac{1}{n} \sum_{i=1}^{n} I(X_i, \tau) + \epsilon_n(\tau), \tag{2.1} \]

for some parameter \( \tau \). In the asymptotic representation (2.1), the model parameter of interest \( \theta \) (or the model itself) is unrelated to the additional parameter \( \tau \) whereas the asymptotic representation (or the estimator) depends on it. Thus in this paper we call \( \tau \) the model-independent parameter. For illustration, we examine the following motivating examples.

**Example 1 (Linear quantile regression)**. The conditional 100\( \tau \)% quantile of \( Y|X \) is

\[ \beta^T X + b_\tau, \]

where \( b_\tau \) is the 100\( \tau \)% quantile of \( Y - \beta^T X \). Without loss of generality, assume that \( E(Y - \beta^T X - b_\tau|X) = 0 \). The quantile regression estimator of \( (b_\tau, \beta^T)^T \) can be
obtained as
\[
\left( \hat{b}_\tau, \hat{\beta}_\tau \right) = \arg \min_{b, \beta} \sum_{i=1}^{n} \rho_\tau(Y_i - b - \beta^T X_i),
\]
where \( \rho_\tau(t) = \tau t_+ + (1-\tau) t_- \) is the so-called check function with + and − standing for positive and negative parts, respectively. Denote \( F_i(y) = F(y|X_i) = P(Y_i < y|X_i) \) and suppose that \( F_i(y), i = 1, \cdots, m \), are i.i.d. with a common density function \( f(y) > 0 \) for all \( y \). Under some regularity conditions (see, e.g., Bahadur 1966; Kiefer 1967; Koenker 2005), we have the following Bahadur representation:
\[
\hat{\beta}_\tau - \beta = \xi(\tau, \beta) \varphi_n + \epsilon_n(\tau),
\]
where \( \varphi_n = n^{-1/2}, \epsilon_n(\tau) = O_p(n^{-3/4}), \)
\[
\xi(\tau, \beta) = f^{-1}(Q(\tau)) D^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i(\tau - I(Y_i \leq b_\tau + \beta^T X_i)),
\]
\[
D = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T,
\]
and \( Q(\tau) = F^{-1}(\tau|X) \), the \( \tau \)th quantile of \( Y \). Here \( \xi(\tau, \beta) \) is of order \( O_p(1) \), and in the next section we will show that under a mild condition \( \xi(\tau, \beta) \) can be estimated.

We can see that the regression coefficient \( \beta \) is independent of \( \tau \), but the asymptotic representation of the estimator \( \hat{\beta}_\tau \) depends on \( \tau \). The argument can be applied to nonlinear parametric models. □

**Example 2 (Nonparametric regression).** Consider the following nonparametric regression:
\[
Y = r(X) + e,
\]
where \( r(x) \) is a smooth nonparametric regression function for \( x \in [0, 1] \), the error term satisfies \( E(e|X) = 0 \) and \( \text{Var}(e|X) = \sigma^2 \). We now give two asymptotic representations for the kernel estimator of \( r(x) \) with \( x \in (0, 1) \). As is known, \( x \in (0, 1) \) is not a necessary constraint, we use it only for simplicity of presentation. It is well known
that under certain regularity conditions with second order continuous and bounded derivatives, a commonly used kernel estimator \( \hat{r}_\tau(x) \) (e.g., Nadaraya-Watson estimator, we write it as the N-W estimator throughout the rest of the paper) of the regression function \( r(x) \) has the mean value:

\[
E(\hat{r}_\tau(x)) = r(x) + \frac{1}{2} \left\{ r''(x) + 2 \frac{r'(x)f'_X(x)}{f_X(x)} \right\} \mu_2(K)h^2 + O(h^4), \ x \in (0, 1),
\]

where \( f_X(x) \) is the density function of \( X \), \( \mu_2(K) = \int u^2 K(u)du \), \( K(x) \) is a kernel function and \( h \) is a bandwidth satisfying \( h = \tau n^{-\eta} \) for constants \( \tau > 0 \) and \( 0 < \eta < 1 \).

Then we have the following asymptotic representation

\[
\hat{r}_\tau(x) - r(x) = \xi(\tau) \varphi_{1n} + \epsilon_n(\tau), \ x \in (0, 1),
\]

where \( \xi(\tau) = \tau^2, \varphi_{1n} = \frac{1}{2} \left\{ r''(x) + 2 \frac{r'(x)f'_X(x)}{f_X(x)} \right\} \mu_2(K)n^{-2\eta} \) and \( \epsilon_n = \hat{r}_\tau(x) - E(\hat{r}_\tau(x)) + O(n^{-4\eta}) \). Here \( \epsilon_n \) has mean of order \( O(n^{-4\eta}) \) and variance of order \( n^{-(1-\eta)} \) and therefore is of order \( o_p(n^{-2\eta}) \) provided that \( 0 < \eta < 1/5 \).

Also we can use the Bahadur representation (see, e.g., Bhattacharya and Gangopadhyay 1990; Chaudhuri 1991; Hong 2003) to construct a relationship between the estimator and the model-independent parameter. Under regularity conditions (including the condition in Theorem 3.4(2) given in Section 3), the N-W estimator \( \hat{r}_\tau(x) \) has following Bahadur representation:

\[
\hat{r}_\tau(x) - r(x) = \xi(\tau, r) \varphi_{2n} + \epsilon_n(\tau), \ x \in (0, 1),
\]

where \( \epsilon_n(\tau) \) is of order \( O_p(n^{-3(1-\eta)/4}) \), \( \varphi_{2n} = n^{-(1-\eta)/2} \),

\[
\xi(\tau, r) = n^{-(1+\eta)/2}v^{-1}_\tau(x) \sum_{i=1}^{n} K_\tau(X_i - x)(Y_i - r(x)), \ x \in (0, 1),
\]

\[
v_\tau(x) = \int K(u)f_X(x + hu)du \quad \text{and} \quad K_\tau(x) = h^{-1}K(x/h) \quad \text{with} \quad h = \tau n^{-\eta}. \]

Here \( \xi(\tau, r) \) is of order \( O_p(1) \) and obviously can be estimated.
The two representations above show that the asymptotic representations for non-parametric regression are also related to a model-independent parameter $\tau$ (or $h$).

**Example 3 (Blockwise likelihood).** Blockwise composite likelihood (see, e.g., Varin, Reid and Firth 2011) is usually used for models with dependent data. In this example, we consider the blockwise empirical likelihood. Let $Y_1, \ldots, Y_n$ be dependent observations from an unknown $d$-variate distribution $f(y; \theta)$, where the parameter vector $\theta \in \Theta \subset \mathbb{R}^p$. The information about $\theta$ and $f(y; \theta)$ is available in the form of an unbiased estimating function $u(y; \theta)$, i.e. $E(u(Y; \theta^0)) = 0$, where $\theta^0$ is the true value of $\theta$ and $u(y; \theta)$ is a given function vector: $\mathbb{R}^d \times \Theta \to \mathbb{R}^r$ with $r \geq p$. Let $M$ and $L_\tau$ be integers satisfying $M = \lfloor n^{1-c} \rfloor$ and $L_\tau = \lfloor \tau n^{1-c} \rfloor$ for some constants $0 < c \leq 1$ and $0 < \tau \leq 1$, where $\lfloor x \rfloor$ stands for the integer part of $x$. Denote $B_i = (Y_{(i-1)L_\tau+1}, \ldots, Y_{(i-1)L_\tau+M})^\tau$, $i = 1, \ldots, Q_\tau$, where $Q_\tau = \lfloor (n-M)/L_\tau \rfloor + 1$. It can be verified that $Q_\tau = O(n^c)$.

We can see that $B_i$ are blocks of observations, $M$ is the window-width, and $L_\tau$ is the separation between the block start points. The observation blocks $B_i$ are used to construct the following estimating function:

$$U_i(\theta, \tau) = \frac{1}{M} \sum_{k=1}^{M} u(Y_{(i-1)L_\tau+k}; \theta).$$

Then, the blockwise empirical Euclidean log-likelihood ratio for dependent data is defined as

$$l_\tau(\theta) = \sup \left\{ \frac{1}{2} \sum_{i=1}^{Q_\tau} (Q_\tau p_i - 1)^2 \left| \sum_{i=1}^{Q_\tau} p_i = 1, p_i \geq 0, \sum_{i=1}^{Q_\tau} p_i U_i(\theta, \tau) = 0 \right. \right\},$$

and the empirical Euclidean likelihood estimator of $\theta$ is defined as

$$\hat{\theta}_\tau = \sup_{\theta \in \Theta} l_\tau(\theta).$$

Here we only consider the case of $p = r = 1$. It follows from the asymptotic representation given in the proof of Theorem 2 of Lin and Zhang (2001) that under certain
regularity conditions, the following asymptotic representation holds:

\[
\hat{\theta}_\tau - \theta = \xi(\tau, \theta) \varphi_n + o_p\left(\frac{1}{\sqrt{n}}\right),
\]

where

\[
\xi(\tau, \theta) = \sqrt{n} \bar{U}(\theta, \tau), \quad \varphi_n = \frac{1}{\sqrt{n} \Delta(\theta)},
\]

\[
\bar{U}(\theta, \tau) = \frac{1}{Q^\tau} \sum_{i=1}^{Q^\tau} U_i(\theta, \tau) \quad \text{and} \quad \Delta(\theta) = E(u'(Y; \theta)) \quad \text{with} \quad u'(y; \theta) \quad \text{being the derivative of} \quad u(y; \theta) \quad \text{with respect to} \quad \theta.
\]

Here \(\xi(\tau, \theta)\) is of order \(O_p(1)\) and the model parameter \(\theta\) is also free of \(\tau\) but the asymptotic representation given above depends on it. For this estimator, \(c\) could also be regarded as a model-independent parameter. But for simplicity, we do not take this case into account.

A common feature of all the asymptotic representations in Examples 1-3 is the formulation of (2.1). Also we can easily find other examples to have the common feature of this formulation. We list a few here: the relationship between penalty based estimators (e.g., the LASSO estimator) and penalty parameter; between B-spline estimator and the number of knots; between wavelet estimator and the bandwidth. Thus, this generic method may readily be extended to handle other estimations with both bias and variance reduction.

3. ASYMPTOTIC COMPOSITE ESTIMATION

3.1. A Regression Modeling via Asymptotic Representation

The asymptotic representations in (2.1) and Examples 1-3 reveal the relationship between model parameter of interest and model-independent parameter. Thus it offers us an useful way to construct new composite estimation in a general framework.

Note that we can define estimators according to different values \(\tau_k, k = 1, \cdots, m\), of the model-independent parameter \(\tau\). For example, for quantile regression estimation, \(\tau_k, k = 1, \cdots, m\), are different quantile positions; for nonparametric regression, \(\tau_k\)
are determined by different $V_k$-fold cross-validations, $k = 1, \cdots, m$. Let $\hat{\theta}_{r_k}$ be the corresponding estimators. We then regress $\hat{\theta}_{r_k}$ on $\tau_k$ to construct a new estimator of $\theta$ as the intercept of the following regression model (or $\theta$ regression model):

$$\hat{\theta}_{r_k} = \theta + g(\tau_k) + \epsilon_n(\tau_k), \ k = 1, \cdots, m. \quad (3.1)$$

Here $g(\cdot)$ is an unknown function. For the sake of identifiability, based on Examples 1-3, we assume (3.1) has the following framework:

$$\hat{\theta}_{r_k} = \theta + \xi(\tau_k, \theta) \varphi_n + \epsilon_n(\tau_k), \ k = 1, \cdots, m, \quad (3.2)$$

where $\xi(\tau, \theta)$ is a known function of $(\tau, \theta)$ and is of order $O_p(1)$, and $\varphi_n$ may be an unknown function with respect to $\theta$, but is independent of $\tau$. We further assume the following condition:

$(C1) \varphi_n = O_p(n^{-\delta_1})$ and $\epsilon_n(\tau_k) = O_p(n^{-\delta_2})$, where $\delta_1$ and $\delta_2$ are positive constants satisfying $\delta_1 < \delta_2$.

The above condition is not restrictive and several estimators, say, those in Examples 1-3, satisfy it. The condition determines the convergence rate of every term on the right-hand side of (3.2) and thus $\epsilon_n(\tau_k)$ could be regarded as the error term. We call model (3.2) (or (3.1)) the asymptotic composite regression (ACR) because it is established by using the asymptotic representation of estimator and a regression idea with the estimator as the response variable $\hat{\theta}_r$, and the model-independent parameter as the covariate $\tau$. Thus a composite estimator of $\theta$ is just the estimator of the intercept on the right-hand side of (3.2).

### 3.2. Estimation

Because $\xi(\tau, \theta)$ may be related to $\theta$, we first construct an initial estimator $\hat{\theta}$ to replace it. Denote $\hat{\xi}(\tau) = \xi(\tau, \hat{\theta})$. Here the initial estimator $\hat{\theta}$ may depends on $\tau$. We consider two different cases separately.
(1) We first consider the case of \( \varphi_n \) being an unknown function. Because \( \delta_2 > \delta_1 \), \( \xi(\tau_k, \theta) \varphi_n \) is the leading term of equation (3.2). In this case we ignore \( \epsilon_n(\tau_k) \) and then construct a composite estimator \( \tilde{\theta} \) of \( \theta \) as the first component of the following minimizers:

\[
\left( \tilde{\theta} \right) = \arg\min_{\theta, \varphi_n} \frac{1}{m} \sum_{k=1}^{m} w_k (\hat{\theta}_{\tau_k} - \theta - \hat{\xi}(\tau_k) \varphi_n)^2, \tag{3.3}
\]

where \( w_k, k = 1, \ldots, m, \) are weights satisfying \( \sum_{k=1}^{m} w_k = 1 \). The estimator can be expressed as

\[
\tilde{\theta} = \sum_{k=1}^{m} w_k \hat{\theta}_{\tau_k} - \tilde{\varphi}_n \tilde{\xi}, \tag{3.4}
\]

where \( \tilde{\xi} = \sum_{k=1}^{m} w_k \hat{\xi}(\tau_k) \) and

\[
\tilde{\varphi}_n = \frac{\sum_{k=1}^{m} w_k \hat{\theta}_{\tau_k} \left( \hat{\xi}(\tau_k) - \tilde{\xi} \right)}{\sum_{k=1}^{m} w_k \left( \hat{\xi}(\tau_k) - \tilde{\xi} \right)^2}.
\]

(2) If \( \varphi_n \) is given, \( \theta \) can be simply estimated as

\[
\tilde{\theta} = \sum_{k=1}^{m} w_k \left( \hat{\theta}_{\tau_k} - \hat{\xi}(\tau_k) \varphi_n \right). \tag{3.5}
\]

We call \( \tilde{\theta} \) defined in (3.4) and (3.5) the ACR estimator. The theoretical properties for the estimators will be given in Section 4.

3.3. Estimators for the Three Examples

Now we construct the corresponding composite estimators for the three examples mentioned in Section 2.

(a) Asymptotic composite quantile regression estimation. For the quantile regression estimation given in Example 1, suppose that the conditional density function \( f_e(\cdot | X) \) of the error \( e \) is given. We choose the initial estimators \( \hat{b}_r \) and \( \hat{\beta}_r \) respectively
of \( b_\tau \) and \( \beta \) as the quantile regression estimators defined in Example 1. According to (3.5), the ACR estimator has the form:

\[
\hat{\beta} = \frac{1}{\hat{f}(Q(\tau))n} \hat{D}_n^{-1} \sum_{i=1}^n X_i(\tau - I(Y_i \leq \hat{b}_\tau + \hat{\beta}_\tau X_i))
\]

where \( \hat{f}(Q(\tau)) = f(\hat{b}_\tau | X), \hat{D}_n = \frac{1}{n} \sum_{i=1}^n X_iX_i^\tau \) and \( \tau_k, k = 1, \ldots, m \), are different quantile positions.

(b) **Asymptotic composite nonparametric regression estimation.** Here we only use the N-W estimator as the initial estimator, which is defined as

\[
\hat{r}_\tau(x) = \frac{\sum_{i=1}^n Y_i K_\tau(X_i - x)}{\sum_{i=1}^n K_\tau(X_i - x)}, \quad x \in (0, 1).
\]

The asymptotic representations in Example 2 and the estimators (3.4) and (3.5) result in that two ACR estimators \( \hat{r}_i(x) \) of the regression function \( r(x) \) for \( x \in (0, 1) \) can be defined as

\[
\hat{r}_1(x) = \sum_{k=1}^m w_k \hat{r}_{\tau_k}(x) - \hat{\varphi}_{1n} \tau^2 \quad \text{and}
\]

\[
\hat{r}_2(x) = \sum_{k=1}^m w_k \left( \hat{r}_{\tau_k}(x) - n^{-\eta/2} \hat{\xi}(\tau_k) \right), \quad x \in (0, 1),
\]

where \( \tau_k \) are about the bandwidths \( h_k = \tau_k n^{-\eta}, k = 1, \ldots, m \),

\[
\hat{\varphi}_{1n} = \frac{\sum_{k=1}^m w_k \tau_k^2}{\sum_{k=1}^m \sum_{k=1}^m w_k \tau_k^2}, \quad \tau^2 = \sum_{k=1}^m w_k \tau_k^2,
\]

\[
\hat{\xi}(\tau) = n^{-1} v^{-1}(x) \sum_{i=1}^n K_\tau(X_i - x)(Y_i - \hat{r}_\tau(x)).
\]

In practical use, we may determine \( h_k = \tau_k n^{-\eta} \) by different \( V_k \)-fold cross-validations.

(c) **(Blockwise empirical likelihood estimation).** Consider blockwise empirical likelihood in Example 3. The blockwise empirical Euclidean log-likelihood ratio has the
following closed representation:
\[ l_\tau (\theta) = -\frac{Q_\tau}{2} \bar{U}(\theta, \tau) S^{-1}(\theta, \tau) \bar{U}(\theta, \tau), \]

where \( S(\theta, \tau) = \frac{1}{Q_\tau} \sum_{i=1}^{Q_\tau} (U_i(\theta, \tau) - \bar{U}(\theta, \tau))(U_i(\theta, \tau) - \bar{U}(\theta, \tau))^T \); see, e.g., Lin and Zhang (2001). Given \( \tau = \tau_k \), denote by \( \hat{\theta}_{\tau_k} \) the initial estimators of \( \theta \) by maximizing the above likelihood function. For simplicity, we here only consider the case with \( p = r = 1 \), i.e., both the parameter \( \theta \) and the unbiased estimating function \( u(y; \theta) \) are scalar. It follows from (3.4) that the composite estimator can be expressed as

\[ \tilde{\theta} = \sum_{k=1}^{m} w_k \hat{\theta}_{\tau_k} - \hat{\varphi}_n \tilde{\xi}, \quad (3.8) \]

where \( \tilde{\xi} = \sum_{k=1}^{m} w_k \hat{\xi}(\tau_k), \hat{\xi}(\tau_k) = \sqrt{n} \bar{U}(\hat{\theta}_{\tau_k}, \tau_k) \) and

\[ \hat{\varphi}_n = \frac{\sum_{k=1}^{m} w_k \hat{\theta}_{\tau_k} \left( \hat{\xi}(\tau_k) - \bar{\xi} \right)}{\sum_{k=1}^{m} w_k \left( \hat{\xi}(\tau_k) - \bar{\xi} \right)^2}. \]

4. THEORETICAL PROPERTIES AND OPTIMAL WEIGHTS

It is known that if (3.2) were a true linear regression model, the least squares estimator would be unbiased with minimum variance in certain sense. However, this model is only a linear model in form whose error term has a bias of order \( O(n^{-\delta_2}) \) in probability and the main part tends to zero at a certain convergence rate.

In this section we suppose \( \theta \) is a scalar parameter for simplicity. When \( \xi(\tau) \) is free of \( \theta \), we define the regenerated weights by

\[ \tilde{w}_k = w_k - \frac{w_k (\hat{\xi}(\tau_k) - \bar{\xi})}{\sum_{k=1}^{m} w_k (\hat{\xi}(\tau_k) - \bar{\xi})^2}, \]

which are free of the initial estimators and still satisfy \( \sum_{k=1}^{m} \tilde{w}_k = 1 \). We have the following theorem.
Theorem 4.1. When $\xi(\tau)$ is unrelated to $\theta$, then the ACR estimator $\tilde{\theta}$ satisfies

$$\tilde{\theta} - \theta = \sum_{k=1}^{m} \tilde{w}_k \epsilon_n(\tau_k),$$

where $\epsilon_n(\tau_k)$ are the error terms of the asymptotic representation defined in (3.2).

Remark 4.1. Interestingly, from the representation in the theorem and (C1) we can see that when $\xi(\tau)$ is free of $\theta$, the convergence rate of the ACR estimator is faster than those of the initial estimators. The first estimator in (3.7) has this property; see the details in Theorem 4.4 below. In other words, the ACR estimator can be super-efficient in certain scenarios. Remark 4.2 (a) given below will further verify this point of view. Theorem 4.1 also implies that the ACR estimator is bias-reduced. In the following, we give a result showing that the ACR method can reduce the variance in finite sample cases.

Define regenerated weights as

$$\tilde{w}_k = w_k - \bar{\xi} \frac{w_k(\xi(\tau_k) - \bar{\xi})}{\sum_{k=1}^{m} w_k(\xi(\tau_k) - \bar{\xi})^2}, \ k = 1, \cdots, m.$$  

Let $w = (w_1, \cdots, w_m)^T$, $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_m)^T$ and $1$ be a $m$-dimensional column vector with all components 1.

Theorem 4.2. When $\xi(\tau)$ is free of $\theta$, the variance of the ACR estimator $\tilde{\theta}$ can be expressed as

$$\text{Var}(\tilde{\theta}) = \bar{w}^T \Sigma_{\theta} \bar{w}.$$  

Particularly, when the original weight vector $w$ are chosen by the following equation:

$$\bar{w} = (1^T \Sigma_{\theta}^{-1} 1)^{-1} \Sigma_{\theta}^{-1} \bar{1},$$  

then,

$$\text{Var}(\hat{\theta}) \leq \text{Cov}(\hat{\theta}_{\tau_k}) \text{ for } k = 1, \cdots, m.$$  

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For the theorem, we have the following remark:

**Remark 4.2.**

(a) Denote \( \mathbf{w}^{*} = (w_{1}^{*}, \ldots, w_{m}^{*})^{T} = (1^{T}\Sigma_{\hat{\theta}}^{-1}\mathbf{1})^{-1}\Sigma_{\hat{\theta}}^{-1}\mathbf{1} \), which is the optimal weight vector that minimizes \( Var(\tilde{\theta}) = \tilde{\mathbf{w}}^{T}\Sigma_{\hat{\theta}}\tilde{\mathbf{w}} \) subject to \( \sum_{k=1}^{m} \tilde{w}_{k} = 1 \). Then, the theorem shows that we should choose the original weights \( w_{k} \) by equations:

\[
\frac{w_{k} - \bar{\xi}}{\sum_{k=1}^{m} w_{k}(\xi(\tau_{k}) - \bar{\xi})^{2}} = w_{k}^{*}, \quad k = 1, \ldots, m.
\]

Denote by \( \mathbf{w}_{k}^{0}, k = 1, \ldots, m \), the solutions of the above equations. Note that for arbitrary weights \( \mathbf{w}_{k} \), \( \sum_{k=1}^{m} w_{k}(\lambda_{k} - \bar{\lambda}) = 0 \), and the optimal weights \( w_{k}^{*}, k = 1, \ldots, m \), satisfy \( \sum_{k=1}^{m} w_{k}^{*} = 1 \). Thus, for the solutions of the above equations, the regularization condition still holds, formally,

\[
\sum_{k=1}^{m} \mathbf{w}_{k}^{0} = 1,
\]

which is a key condition for bias correction. However, the above \( m \) equations are nonlinear, we cannot get closed forms of the solutions of \( m \) unknown weights \( w_{k}, k = 1, \ldots, m \). Thus numerical methods are required.

(b) The above two theorems ensures that the ACR can simultaneously reduce bias and variance in certain cases. This sheds the insights on the potential merits of the ACR. Of course, the key condition is that \( \xi(\tau) \) is unrelated to \( \theta \). The first estimator in (3.7) satisfies this condition. However, this condition is unsatisfied sometimes. For instance, the estimator in (3.6), the second estimator in (3.7) and the estimator in (3.8) do not satisfy this condition. In this situation, only their asymptotic properties can be obtained. In the following, we will investigate the asymptotic properties for the important estimators (3.6)-(3.8) regardless of whether this condition is satisfied or not.
Consider the composite quantile regression estimator (3.6). In addition to those given in Example 1, we need the following conditions:

(C2) \( \max_{1 \leq i \leq n} \|X_i\| \leq cn^\nu \) for some constants \( c > 0 \) and \( 0 \leq \nu < 1/2 \).

(C3) The conditional density function \( f_e(u|x) \) of the error \( e \) is continuously differentiable.

The following theorem states the asymptotic normality of the estimator.

**Theorem 4.3.** For the linear regression model in Example 1, when (C2) and (C3) hold, the ACR estimator (3.6) has the following asymptotic representation:

\[
\tilde{\beta} - \beta = D^{-1} \frac{1}{n} \sum_{i=1}^{n} X_i \sum_{k=1}^{m} w_k f^{-1}(Q(\tau_k))(\tau_k - I(Y_i \leq b_{\tau_k} + \beta^T X_i)) + O_p(n^{-3/4}).
\]

Consequently,

\[
\sqrt{n}(\tilde{\beta} - \beta) \xrightarrow{D} N(0, w^T A_0 w D^{-1}),
\]

where \( w = (w_1, \ldots, w_m)^T \) and

\[
A_0 = \left( \frac{\min(\tau_k, \tau_{k'}) (1 - \max(\tau_k, \tau_{k'}))}{f(Q(\tau_k)) f(Q(\tau_{k'}))} \right)_{k,k'=1}^m.
\]

**Remark 4.3.** Particularly, when \( w_k = f(Q(\tau_k)) / \sum_{k=1}^{m} f(Q(\tau_k)) \), the limiting covariance of the ACR estimator (3.6) is the same as that of the composite estimator proposed by Zou and Yuan (2008). Furthermore, we can obtain the optimal weights in the following way. By minimizing the limiting variance given in Theorem 4.3, we see that the optimal weight vector has the form

\[
w^* = \min_{\|w\|_1 = 1} w^T A_0 w.
\]
By Lagrange multipliers, we see that the optimal weight vector has the following closed representation:

\[ w^* = \left(1^T A_0^{-1} \right)^{-1} A_0^{-1} 1, \]

and, as a result, the optimal limiting covariance of \( \sqrt{n}(\tilde{\beta} - \beta) \) is

\[ w^{*T} A_0 w^* D^{-1} = \left(1^T A_0^{-1} 1 \right)^{-1} D^{-1}. \]

With this optimal weight, the resulting estimator is more efficient than the composite estimator of Zou and Yuan (2008), but is the same as in Koenker (1984). In (3.6), when \( f(Q(\tau_k)) \) are estimated consistently, we can get a consistent estimator of the optimal weight vector \( w^* \). For a different problem (Fan and Wang 2011), the choice of the optimal weights was discussed, but, the theoretical justification in the scenario under study was not explored before.

We now investigate the asymptotic property of the estimators in (3.7) for the nonparametric regression model defined in Example 2. We consider the following two regularity conditions respectively:

(C4) Kernel function \( K(u) \) is symmetric with respect to \( u = 0 \), and satisfies \( \int K(u)du = 1, \int u^2K(u)du < \infty \) and \( \int u^2K^2(u)du < \infty \). Regression function \( r(x) \) defined in Example 2 and density function \( f_X(x) \) of \( X \) have the second-order continuous and bounded derivatives and \( f_X(x) > 0 \) for all \( x \).

(C5) Kernel function \( K(u) \) is symmetric with respect to \( u = 0 \), and satisfies \( \int K(u)du = 1, \int u^4K(u)du < \infty \) and \( \int u^2K^2(u)du < \infty \). Functions \( r(x) \) and \( f_X(x) \) have the fourth-order continuous and bounded derivatives and \( f_X(x) > 0 \) for all \( x \).

It is well known that for the N-W estimator, the convergence rate is related to two factors: bandwidth selection and smoothness of the regression function. Generally speaking, the more smooth the regression function is, the faster the rate can achieve.
when larger bandwidth and higher order kernel function are used. We note that condition (C4) is the typical condition for the N-W estimator when only second order derivatives are assumed for the smoothness of the regression function. However, condition (C5) is of interest. It assumes the smoothness with the fourth order derivatives, but does not require the higher order kernel. For the N-W estimator, its convergence rate cannot be accelerated, whereas the ACR estimator can. The following theorem states these.

Denote
\[ s_k(w) = 1 - \frac{\tau^2 (\tau_k^2 - \tau^2)}{\sum_{k=1}^{m} w_k (\tau_k^2 - \tau^2)^2}, \quad g_k = w_k - \frac{\tau^2 w_k (\tau_k^2 - \tau^2)}{\sum_{k=1}^{m} w_k (\tau_k^2 - \tau^2)^2} \]

and
\[ A_1(w) = \frac{\left( s_k(w) s_j(w) \right)}{\tau_k \tau_j} \left( \int K\left( \frac{u}{\tau_k} \right) K\left( \frac{u}{\tau_j} \right) du \right)_{k,j=1}^m, \]
\[ A_2 = \left( \frac{1}{\tau_k \tau_j} \int K\left( \frac{u}{\tau_k} \right) K\left( \frac{u}{\tau_j} \right) du \right)_{k,j=1}^m. \]

We have the following results.

**Theorem 4.4.** Suppose \( h_k = \tau_k n^{-\eta}, k = 1, \ldots, m, 0 < \eta < 1. \)

1. Under Condition (C4) or (C5), there is an \( c_n(x) = o(n^{-2\eta}) \) or \( c_n(x) = n^{-4\eta} g(x) \sum_{k=1}^{m} g_k \tau_k^4 \)

accordingly, the ACR estimator \( \hat{r}_1(x) \) in (3.7) achieves the following asymptotic normality:

\[ \sqrt{n^{1-\eta}} \left( \hat{r}_1(x) - r(x) - c_n(x) \right) \overset{D}{\to} N\left( 0, \sigma^2 \frac{A_1(w) w^T}{f_X(x)} \right), \quad x \in (0, 1). \]

2. For \( \hat{r}_2(x) \) in (3.7), under Condition (C4), if \( 1/5 \leq \eta < 1 \), then

\[ \sqrt{n^{1-\eta}} \left( \hat{r}_2(x) - r(x) - n^{-2\eta} d(x) \sum_{k=1}^{m} w_k \tau_k^2 \right) \overset{D}{\to} N\left( 0, \sigma^2 \frac{A_2 w^T}{f_X(x)} \right), \quad x \in (0, 1), \]

where \( d(x) \) is a given function.

**Remark 4.4.** We have the further remarks on the ACR estimators beyond the above comments on conditions (C4) and (C5).
(a) *Convergence acceleration.* The above result about \( \tilde{r}_1(x) \) shows the importance of bias reduction. As is known, the kernel estimation is biased, which is the case for all nonparametric smoothers in the literature. To achieve an optimal convergence rate and the asymptotic normality, bandwidth selection must balance between bias and variance terms. Under Condition (C4), the bias \( c_n(x) \) of the N-W estimator has the classical optimal rate \( O(n^{-2y}) \) and is impossible to be improved through selecting a kernel function because the smoothness assumption is only up to the second order derivative. In contrast, the bias of \( \tilde{r}_1(x) \) is \( o(n^{-2y}) \). This rate-accelerated bias can then play a very important role for us to get a convergence rate faster than the classical optimal rate. That is, when the bandwidth is selected to be \( h = O(n^{-1/5}) \), the typical optimal convergence rate of the N-W estimator is \( O(n^{-2/5}) \), whereas the ACR estimator \( \tilde{r}_1(x) \) has the rate of \( o(n^{-2/5}) \). Under Condition (C5), when the optimal bandwidth \( h = O(n^{-1/9}) \) is used, \( \tilde{r}_1(x) \) behaves like the N-W estimator constructed by higher order kernel; both estimators have the same convergence rate of order \( O(n^{-4/9}) \). It is worth pointing out that, without use of higher kernel function, the N-W estimator is not possible to have this rate. Thus, under the same conditions on the smoothness of the regression function and kernel function, the estimator \( \tilde{r}_1(x) \) has a faster convergence rate than the classical N-W estimator does. Also it will be shown later that, by the optimal weight, the limiting variance of the ACR estimator \( \tilde{r}_1(x) \) can be smaller than that of the N-W estimator.

For \( \tilde{r}_2(x) \), the convergence rate cannot be faster. However, we will verify the estimation efficiency can be promoted as well.

(b) *Weight selection and estimation efficiency.* Invoking the same argument as in Remark 4.3, we have that the optimal weight vector for the second estimator \( \tilde{r}_2(x) \) can be expressed as

\[
\mathbf{w}_2^* = \left( \mathbf{1}^T A_2^{-1} \mathbf{1} \right)^{-1} A_2^{-1} \mathbf{1}.
\]
However, $A_1(\mathbf{w})$ for the first estimator $\tilde{r}_1(x)$ depends on the weight vector $\mathbf{w}$ as well. Thus, $\tilde{r}_1(x)$ has no a closed form for the corresponding optimal weight vector. To handle the problem, we approximate $A_1(\mathbf{w})$ by

$$A_1 = \left( \frac{s_k s_j}{\tau_k \tau_j} \int K\left( \frac{u}{\tau_k} \right) K\left( \frac{u}{\tau_j} \right) du \right)_{k,j=1}^m,$$

where $s_k = 1 - \frac{\tau^2_k (r^2_k - r^2)}{\sum (r^2_k - r^2)^2}$ is free of the weights vector $\mathbf{w}$. A “sub-optimal” weight vector for $\tilde{r}_1(x)$ with the above $A_1$ is then

$$\mathbf{w}^*_1 = (1^T A_1^{-1} \mathbf{1})^{-1} A_1^{-1} \mathbf{1}.$$

With the weights $\mathbf{w}^*_1$ and $\mathbf{w}^*_2$, $\sqrt{n^{1-\eta}(\tilde{r}_1(x) - r(x))}$ and $\sqrt{n^{1-\eta}(\tilde{r}_2(x) - r(x))}$ have the limiting variances as

$$(1^T A_1^{-1} \mathbf{1})^{-1} \frac{\sigma^2}{f_X(x)} \text{ and } (1^T A_2^{-1} \mathbf{1})^{-1} \frac{\sigma^2}{f_X(x)},$$

respectively. The two limiting variances may be smaller than those of the common kernel estimators. For example, when kernel function is chosen as $K(u) = e^{-u^2}/\sqrt{2\pi}$, then

$$A_1 = \left( \frac{s_k s_j}{(2\pi)^{1/2} \sqrt{\tau^2_k + \tau^2_j}} \right)_{k,j=1}^m,$$

$$A_2 = \left( \frac{1}{(2\pi)^{1/2} \sqrt{\tau^2_k + \tau^2_j}} \right)_{k,j=1}^m.$$

It is known that when the kernel function is chosen as the above, the limiting variance of the N-W estimator is $\frac{\sigma^2}{2\sqrt{\pi f_X(x)}}$, which is just a special case of the variances in (4.1) with $m = 1$ and $\tau_1 = 1$. Thus, when $\min\{\tau_k; k = 1, \ldots, m\} < 1 < \max\{\tau_k; k = 1, \ldots, m\}$ and the above weights are used, the limiting variances of the ACR estimators are smaller than that of the N-W estimator.

(c) Kernel selection. As commented above, the ACR estimators can have either faster rate or smaller limiting variance. From the proof, we can see that the estimators are still the kernel estimation types. A natural concern is whether
the classical N-W estimator could also enjoy this rate-acceleration property through a delicate selection of kernel function. However, when looking into the detail of the proof, we can see that for a single N-W estimator, it is not possible to find such a kernel function, while it does be due to the composition. Therefore, this does show the advantage of the ACR.

We now deal with the asymptotic property of the composite empirical likelihood estimator defined in (3.8). Assume the following condition:

\( (C6) \quad \sqrt{n}(\bar{U}(\theta^0, \tau_1), \ldots, \bar{U}(\theta^0, \tau_m))^T \overset{D}{\longrightarrow} N(0, A_3(\theta^0)) \)

where \( A_3(\theta^0) \) is a positive definite matrix.

Clearly, this condition is mild for some common types of dependent data because \( \bar{U}(\theta^0, \tau) \) is actually an average of some functions with zero mean; see, e.g., Dimitris and Joseph (1992), and Lin and Zhang (2001).

**Theorem 4.5** Under Condition (C6), the composite blockwise empirical likelihood estimator (3.8) satisfies

\( \sqrt{n}(\tilde{\theta} - \theta^0) \overset{D}{\longrightarrow} N(0, \Delta^{-2}(\theta^0)w^T A_3(\theta^0)w) \).

**Remark 4.5.** Invoking the same arguments as used in Remarks 4.3 and 4.4, the optimal weight vector \( w^* \) can be designed, and with the optimal weight vector, the ACR estimator is more efficient than the initial estimators; the details are omitted here.

In short, all the theorems above reveal that the ACR method can improve original estimators in the sense of either faster convergence rate or better estimation efficiency. Moreover, by summarizing all the theorems, we can get the following general conclusions: (1) When \( \xi(\tau) \) is free of the model parameter \( \theta \), the ARC estimator may be
bias-reduced when the original estimator is biased, and consequently the convergence rate could be faster than the classical one; (2) If \( \xi(\tau) \) depends on \( \theta \), the estimation efficiency of the ARC estimator can be improved by choosing proper weights. These two general conclusions can be proved theoretically. But complex conditions and expressions are required; the details are thus omitted here.

5. SIMULATION STUDIES

In this section we examine the finite sample behaviors of the newly proposed estimators by simulation studies. Mean squared error (for parametric model) and mean integrated squared error (for nonparametric model) are used to evaluate the efficiency of involved estimators. We also report the simulation results for estimation bias because the initial idea of our method is to reduce estimation bias.

*Experiment 1.* Consider the linear regression in Example 1. Let \( \hat{\beta}_r \) be the common quantile regression estimator defined in Example 1 and \( \tilde{\beta} \) be the ACE defined by (3.6). Here we also consider the composite quantile regression (CQR) estimator \( \hat{\beta} \) proposed by ?, which is constructed by minimizing composite objective function as

\[
(\hat{\beta}^\tau, \hat{b}_{r_1}, \cdots, \hat{b}_{r_m})^T = \arg \min_{\beta, b_{r_1}, \cdots, b_{r_m}} \sum_{i=1}^{n} \sum_{k=1}^{m} \rho_{\tau_k}(Y_i - b_{r_k} - \beta^\tau X_i).
\] (5.1)

The samples respectively with size 100, 200 and 400 are generated from the model

\[ Y = X^\tau \beta + \epsilon, \]

where \( \beta = (3, 2, 1, -1, -2)^T \), the predictors \( X = (X_1, X_2, \cdots, X_5)^T \) follow a multivariate normal distribution \( N(0, \Sigma) \) with \( (\Sigma)_{i,j} = 0.5^{|i-j|} \) for \( 1 \leq i, j \leq 5 \), and the error term \( \epsilon \sim Gamma(1) \). We choose \( \tau = 0.5 \) to construct the common quantile regression (QR) estimator \( \hat{\beta}_r \) and select \( \tau_k = \frac{k}{10} \) for \( k = 1, 2, \cdots, 9 \) to construct both the CQR estimator \( \hat{\beta} \) and the ACE \( \tilde{\beta} \). Empirical bias and mean squared error (MSE) of the three estimators over 200 replications are reported in Table 1. In this setting
Table 1: Simulation results in Experiment 1

| n   | ACE Bias | CQR Bias | QR Bias | Bias | MSE | Bias | CQR Bias | QR Bias | Bias | MSE | Bias | CQR Bias | QR Bias | Bias | MSE |
|-----|----------|----------|---------|-------|-----|-------|----------|---------|-------|-----|-------|----------|---------|-------|-----|
| 100 | 0.0002   | 0.0045   | 0.0081  | 0.0007 | 0.0074 | -0.0009 | -0.0049 | -0.0033 | -0.0071 | -0.0004 | 0.00035 | 0.0002   | -0.0007 | -0.0063 | -0.0014 | 0.0035 |
|     | 0.0006   | -0.0024  | -0.0066 | -0.0037 | 0.0116 | 0.0013  | 0.0036   | 0.0072  | -0.0013 | 0.0006  | 0.0039   | 0.0005  | 0.0031  | 0.0016  | 0.0035 |
|     | 0.0111   | -0.0030  | -0.0057 | 0.0016  | 0.0150 | 0.0011  | 0.0036   | 0.0073  | 0.0058  | 0.0005  | 0.0035  | 0.0006  | 0.0037  | 0.0016  | 0.0035 |
| 200 | -0.0036  | -0.0019  | 0.0026  | 0.0002  | 0.0172 | -0.0024 | -0.0060  | -0.0085 | -0.0053 | 0.0002  | -0.0043 | 0.0002  | -0.0007 | -0.0029 | -0.0017 | 0.0003 |
|     | 0.0002   | 0.0003   | 0.0004  | 0.0034  | 0.0172 | 0.0009  | 0.0024   | 0.0044  | 0.0057  | 0.0004  | 0.0029  | 0.0012  | 0.0011  | 0.0026  | 0.0004 |
| 400 | -0.0007  | -0.0007  | 0.0020  | -0.0007 | 0.0077 | -0.0009 | -0.0060  | -0.0085 | -0.0013 | -0.0007 | 0.00035 | 0.0002  | -0.0007 | -0.0029 | -0.0017 | 0.0003 |
|     | 0.0006   | 0.0006   | 0.0005  | 0.0005  | 0.0004 | 0.0004  | 0.0005   | 0.0016  | 0.0058  | 0.0005  | 0.0004  | 0.0005  | 0.0006  | 0.0005  | 0.0005 |
|     | 0.0005   | 0.0005   | 0.0005  | 0.0005  | 0.0004 | 0.0004  | 0.0005   | 0.0016  | 0.0058  | 0.0005  | 0.0004  | 0.0005  | 0.0006  | 0.0005  | 0.0005 |
|     | 0.0005   | 0.0005   | 0.0005  | 0.0005  | 0.0004 | 0.0004  | 0.0005   | 0.0016  | 0.0058  | 0.0005  | 0.0004  | 0.0005  | 0.0006  | 0.0005  | 0.0005 |

The ACE $\hat{\beta}$ is clearly the best one in both bias and variance reduction, and the QR estimator $\hat{\beta}_r$ is reasonably inferior to the other two competitors.

Experiment 2. For the nonparametric regression

$$Y_i = r(X_i) + \epsilon_i, i = 1, \cdots, n,$$

the common local constant (LC) estimator (kernel estimator) is defined as

$$\hat{r}_h(x) = \frac{\sum_{i=1}^{n} Y_i K\left(\frac{X_i - x}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)}.$$  (5.2)

As a comparison, we here consider a composite objective function method, which is defined by following way: for $h_k = \tau_k n^{-\eta}, k = 1, \cdots, m$, define a composite local
constant (CLC) estimator as
\[ \hat{r}(x) = \arg \min_a \sum_{i=1}^{n} \sum_{k=1}^{m} (Y_i - a)^2 K\left( \frac{X_i - x}{h_k} \right). \]

This estimator has the following closed representation:
\[ \hat{r}(x) = \frac{\sum_{i=1}^{n} \sum_{k=1}^{m} Y_i K\left( \frac{X_i - x}{h_k} \right)}{\sum_{i=1}^{n} \sum_{k=1}^{m} K\left( \frac{X_i - x}{h_k} \right)}. \] (5.3)

Thus such an estimator can be regarded as an indirect composition of the LC estimators (5.2) with different bandwidths. Now we compare the ACE estimators defined by (3.7) with the LC estimator and the CLC estimator mentioned above via simulation studies. Consider the regression function \( r(X) = \sin(2\pi X) \), \( X \sim U(0, 1) \), \( \epsilon \sim N(0, 0.5^2) \) with the sample size \( n = 100, 200 \) and 400 respectively. In this experiment, the Epanechnikov kernel \( K(u) = 0.75(1 - u^2)1_{|u| \leq 1} \) is employed, and for simplicity the equal weights are used in the ACE. In the local constant estimation procedure, the bandwidth \( h \) is chosen by two-fold cross-validation. Then \( \tau_k \)'s are chosen so that \( h_k \)'s are around \( h \). Simulation results are tabulated in Table 2, in which MISE denotes the empirical mean integrated squared errors through 200 replications.

By comparing MISEs of the three estimators, we see that the ACE behaves the best among the three estimators even the optimal weights are not employed. Meanwhile, we notice that the CLC estimator \( \hat{r}(x) \) given in (5.3) is the worst one. The above two findings indicate that the ACE is an efficient composite estimator, whereas the competitor such as the CLC through a composite objective function is not always efficient.

**Experiment 3.** Consider the following linear regression model:
\[ Y_i = \theta X_i + \varepsilon_i, \quad i = 1, \ldots, n, \]
Table 2: MISE for nonparametric estimators in Experiment 2

|        | n=100   | n=200   | n=400   |
|--------|---------|---------|---------|
| LC     | 0.0228  | 0.0136  | 0.0075  |
| CLC    | 0.0262  | 0.0176  | 0.0121  |
| ACE    | 0.0206  | 0.0121  | 0.0068  |

where $\theta$ is a scalar parameter. In this model the errors $\varepsilon_i, i = 1, \cdots, n$, are dependent, satisfying

$$\varepsilon_i = a\varepsilon_{i-1} + \varepsilon_i, i = 2, \cdots, n, \varepsilon_1 = \varepsilon_1,$$

where $0 < |a| < 1$ and $\varepsilon_i, i = 1, \cdots, n$, are independent and identically distributed from $N(0, 1)$. In this case, an unbiased estimating function is chosen to be $u(\theta) = X_i(Y_i - \theta X_i)$. Thus the corresponding blockwise estimating function can be expressed as

$$U_i(\theta) = \frac{1}{M} \sum_{k=1}^{M} X_{(i-1)L_r+k}(Y_{(i-1)L_r+k} - \theta X_{(i-1)L_r+k}).$$

We first consider method 1: the blockwise empirical Euclidean likelihood defined in Subsection 3.3, by which the blockwise empirical Euclidean log-likelihood ratio has the following closed representation:

$$l_\tau(\theta) = -\frac{Q_\tau}{2S} \bar{U}^2(\theta),$$

where $S(\theta) = \frac{1}{Q_\tau} \sum_{i=1}^{Q_\tau} (U_i(\theta) - \bar{U}(\theta))^2$ and $\bar{U}(\theta) = \frac{1}{Q_\tau} \sum_{i=1}^{Q_\tau} U_i(\theta)$. In the simulation, $\theta$ is chosen to be $\theta = 5$, the window-width is $M = \lfloor n^{1/3} \rfloor$ (i.e., $c = 1/3$), $\tau$ is determined by the CV and the different values for the ACE are taken around $\tau$. We now compare the ACE defined by (3.8) and the blockwise empirical likelihood estimation (BELE) obtained by minimizing $l_\tau(\theta)$ in (5.4). Simulation results tabulated in Table 3 are obtained through 200 replications. We can see that the ACE behaves slightly better than the BELE does in the sense that the bias and MSE of the ACE are slightly but uniformly smaller than those of the BELE.
To further examine the behaviour of our method, now we consider method 2, an approximate method, as follows. It is known that $S$ in (5.4) is a consistent estimator of the variance of the error. If $S$ is ignored, the blockwise empirical Euclidean log-likelihood ratio has the following approximate representation:

$$l_\tau(\theta) \propto -\frac{Q_\tau}{2} \mathcal{L}^2(\theta).$$

In the simulation, $\theta$ is chosen to be $\theta = 2.5$, the window-width is $M = \lfloor n^{1/2} \rfloor$ (i.e., $c = 1/2$), the other conditions are designed as in method 1. We now compare the ACE defined by (3.8) and the approximate BELE obtained by minimizing $l_\tau(\theta)$ in (5.5). The following Table 4 reports the simulation results about bias and MSE for different combinations of $X_i \sim N(0, 1)$, $X_i \sim N(0.3, 1)$, $a = 0.1, 0.3, -0.3$ and $n = 100, 200, 300$, respectively. By comparing Table 3 and Table 4, we see that when $S$ is removed from the likelihood ratio, the approximate BELE runs into problems but the ACE still works well. More precisely, we have the following findings: (1) When $X_i \sim N(0.3, 1)$, the behavior of the BELE is relatively stable. The ACE works better than the BELE in the sense that both the bias and the MSE of the new estimator are smaller than those obtained by the BELE; (2) When $X_i \sim N(0, 1)$, the MSE of the BELE is quite large showing that the BELE is very unstable. In contrast, the ACE still works very well with much smaller bias and the MSE.

These simulation results and the definition in (3.8) show that when the algorithm for obtaining the initial estimators $\hat{\theta}_n$ is not stable, the ACE can efficiently improve the performance. Thus the ACE is an efficient composite method specially for the case when the original estimator is unstable.
Table 3: Simulation results of method 1 in Experiment 3

| $X$ | $n$  | $a$  | BELE       |          | ACE       |          |
|-----|------|------|------------|----------|-----------|----------|
|     |      |      | Bias       | MSE      | Bias      | MSE      |
| $N(0,1)$ | 100  | 0.1  | −0.0081    | 0.0121   | 0.0055    | 0.0111   |
|      | 0.5  |      | 0.0017     | 0.0145   | −0.0013   | 0.0133   |
|      | 0.9  |      | −0.0018    | 0.0696   | −0.0006   | 0.0661   |
|      | 200  | 0.1  | −0.0110    | 0.0048   | −0.0106   | 0.0046   |
|      | 0.5  |      | −0.0070    | 0.0072   | −0.0051   | 0.0070   |
|      | 0.9  |      | −0.0055    | 0.0331   | −0.0053   | 0.0317   |
|      | 400  | 0.1  | 0.0057     | 0.0033   | −0.0054   | 0.0031   |
|      | 0.5  |      | −0.0043    | 0.0034   | −0.0038   | 0.0032   |
|      | 0.9  |      | 0.0034     | 0.0136   | 0.0015    | 0.0130   |

Table 4: Simulation results of method 2 in Experiment 3

| $X$ | $n$  | $a$  | BELE       |          | ACE       |          |
|-----|------|------|------------|----------|-----------|----------|
|     |      |      | Bias       | MSE      | Bias      | MSE      |
| $N(0.3,1)$ | 100  | 0.1  | 0.0222     | 0.4264   | 0.0020    | 0.0204   |
|      | 0.3  |      | −0.0408    | 0.2997   | −0.0052   | 0.0244   |
|      | −0.3 |      | −0.0644    | 2.9066   | −0.0088   | 0.0279   |
|      | 200  | 0.1  | −0.0089    | 0.0398   | 0.0076    | 0.0090   |
|      | 0.3  |      | 0.0108     | 0.1749   | 0.0075    | 0.0127   |
|      | −0.3 |      | −0.0117    | 0.0272   | 0.0050    | 0.0101   |
|      | 300  | 0.1  | 0.0028     | 0.0209   | 0.0020    | 0.0060   |
|      | 0.3  |      | 0.0057     | 0.0310   | 0.0026    | 0.0080   |
|      | −0.3 |      | 0.0041     | 0.0159   | 0.0020    | 0.0064   |
| $N(0,1)$ | 100  | 0.1  | −0.3241    | 17.9075  | −0.0110   | 0.0253   |
|      | 0.3  |      | 0.0449     | 17.3460  | 0.0037    | 0.0271   |
|      | −0.3 |      | −0.2882    | 16.1170  | −0.0012   | 0.0235   |
|      | 200  | 0.1  | 0.1270     | 30.6297  | 0.0153    | 0.0124   |
|      | 0.3  |      | 0.1569     | 104.3950 | −1.9725e−5| 0.0142   |
|      | −0.3 |      | 0.2147     | 10.3675  | −0.0021   | 0.0120   |
|      | 300  | 0.1  | 0.3470     | 28.7330  | 0.0047    | 0.0065   |
|      | 0.3  |      | 0.1275     | 11.3672  | −0.0030   | 0.0063   |
|      | −0.3 |      | 0.0563     | 13.9492  | −0.0061   | 0.0057   |
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