HOCHSCHILD COHOMOLOGY AND ATIYAH CLASSES

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Abstract. In this paper we prove that on a smooth algebraic variety the HKR-morphism twisted by the square root of the Todd genus gives an isomorphism between the sheaf of poly-vector fields and the sheaf of poly-differential operators, both considered as derived Gerstenhaber algebras. In particular we obtain an isomorphism between Hochschild cohomology and the cohomology of poly-vector fields which is compatible with the Lie bracket and the cupproduct. The latter compatibility is an unpublished result by Kontsevich.

Our proof is set in the framework of Lie algebroids and so applies without modification in much more general settings as well.

1. Introduction

In the body of the paper we will work over a ringed site over a field of characteristic zero \(k\). Thus our results are for example applicable to stacks. However in this introduction we will state our results for a commutatively ringed space \((X, \mathcal{O}_X)\).

By definition a Lie algebroid on \(X\) is a sheaf of Lie algebras \(\mathcal{L}\) which is an \(\mathcal{O}_X\)-module and is equipped with an action \(\mathcal{L} \times \mathcal{O}_X \to \mathcal{O}_X\) with properties mimicking those of the tangent bundle (see §4.2 for a more precise definition). Throughout \(\mathcal{L}\) will be a locally free Lie algebroid over \((X, \mathcal{O}_X)\) of constant rank \(d\).

The advantage of the Lie algebroid framework is that it allows one to treat the algebraic/complex analytic and \(C^\infty\)-case in a uniform way.

Example 1.1. The following are examples of (locally free) Lie algebroids.

1. The sheaf of vector fields on a \(C^\infty\)-manifold.
2. The sheaf of holomorphic vector fields on a complex analytic variety.

1991 Mathematics Subject Classification. Primary 14F99, 14D99.
Key words and phrases. Deformation quantization, Atiyah classes.
The second author is a director of research at the FWO.
The results of this paper were partially obtained while the second author was visiting the Université Claude Bernard at Lyon. He hereby thanks the latter for its kind hospitality.
The sheaf of \( \mathcal{L} \)-poly-vector fields on a smooth algebraic variety.

(4) \( \mathcal{O}_X \otimes \mathfrak{g} \) where \( \mathfrak{g} \) is the Lie algebra of an algebraic group acting on a smooth algebraic variety \( X \).

**Example 1.2.** Assume that \( X \) is an affine integral singular algebraic variety. Then \( \mathcal{T}_X \) is not locally free. However it is always possible to construct a locally free sub Lie algebroid \( \mathcal{L} \subset \mathcal{T}_X \). So our setting applies to some extent to the singular case as well.

The Atiyah class \( A(\mathcal{L}) \) of \( \mathcal{L} \) is the element of \( \text{Ext}^1_X(\mathcal{L}, \mathcal{L}^* \otimes \mathcal{L}) \) which is the obstruction against the existence of an \( \mathcal{L} \)-connection on \( \mathcal{L} \). The \( i \)’th \((i > 0)\) scalar Atiyah class \( a_i(\mathcal{L}) \) is defined as

\[
\text{Alt} \text{Tr}(A(\mathcal{L}))^i \in H^i(X, (\wedge^i \mathcal{L})^*)
\]

In the \( C^\infty \) or affine case we have \( a_i(\mathcal{L}) = 0 \) as the cohomology groups \( H^i(X, (\wedge^i \mathcal{L})^*) \) vanish. If \( X \) is a Kahler manifold and \( \mathcal{L} \) is the sheaf of holomorphic vector fields then \( a_i(\mathcal{T}_X) \) coincides with the \( i \)’th Chern class of \( \mathcal{T}_X \) (see e.g. [18, (1.4)]).

The Todd class of \( \mathcal{L} \) is defined as

\[
td(\mathcal{L}) = \det(q(A(\mathcal{L})))
\]

where

\[
q(x) = \frac{x}{1 - e^{-x}}
\]

One sees without difficulty that \( \text{td}(\mathcal{L}) \) can be expanded formally in terms of \( a_i(\mathcal{L}) \).

The sheaf of \( \mathcal{L} \)-poly-vector fields on \( X \) is defined as \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) = \bigoplus_i \wedge^i \mathcal{L} \). This agrees with the standard definitions in case \( \mathcal{L} \) is one of the variants of the tangent bundle described in the examples above. It is easy to prove that \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \) is a sheaf of Gerstenhaber algebras on \( X \).

In the case that \( X \) is a \( C^\infty \)-manifold Kontsevich introduced the sheaf of so-called poly-differential operators on \( X \). This is basically a localized version of the Hochschild complex. It is straightforward to construct a Lie algebroid generalization \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \) of this concept as well (see [3] or §4.2.2). Like \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \), \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \) is equipped with a Lie bracket and an associative cupproduct but these operations satisfy the Gerstenhaber axioms only up to globally defined homotopies (see e.g. [15]).

The so-called Hochschild-Kostand-Rosenberg map is a quasi-isomorphism between \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \) and \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \) [3, 39]. This paper is concerned with the failure of the HKR-map to be compatible with the Lie brackets and cupproducts on \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \) and \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \).

Let \( D(X) \) be the derived category of sheaves of \( k \)-vector spaces. This category is equipped with a symmetric monoidal structure given by the derived tensor product. As indicated above \( D^\mathcal{L}_{\text{poly}}(X) \) is a Gerstenhaber algebra in \( D(X) \). We have the following result

**Theorem 1.3.** (see §10) The map in \( D(X) \)

\[
T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X) \xrightarrow{\text{HKR} \circ (\text{td}(\mathcal{L}))^{1/2}} D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X)
\]

is an isomorphism of Gerstenhaber algebras in \( D(X) \).

Applying the hypercohomology functor \( H^*(X, -) \) we get

**Corollary 1.4.** The map

\[
\bigoplus_{i,j} H^i(X, \wedge^i \mathcal{L}) \xrightarrow{\text{HKR} \circ (\text{td}(\mathcal{L}))^{1/2}} H^*(X, D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X))
\]

is an isomorphism of Gerstenhaber algebras.

\(^1\)It is not entirely trivial to make the Hochschild complex into a (pre)sheaf as the assignment \( U \mapsto C^*(\Gamma(U, \mathcal{O}_U)) \) is not compatible with restriction.
Let us restrict to the setting where $X$ is a smooth algebraic variety and $L = \mathcal{T}_X$. In that case it follows from the proof of [34, Thm 3.1(1)] together with [22, Thm 7.5.1] that the righthand side of (1.2) can be viewed as the Hochschild cohomology $\text{HH}^*(X)$ of $X$ (in the sense that it controls for example the deformation theory of $\text{Mod}(\mathcal{O}_X)$). So we may rephrase Corollary 1.4 as

**Corollary 1.5.** There is an isomorphism of Gerstenhaber algebras

$$\bigoplus_{i,j} H^j(X, \wedge^i \mathcal{T}_X) \xrightarrow{\text{HKR} \circ (\text{td}(L)^{1/2} \wedge -)} \text{HH}^*(X)$$

A version of this result which refers only to the cupproduct was proved by Kontsevich (see [7, Thm 5.1]). For the cupproduct one can use Swan’s definition of Hochschild cohomology [30]

$$\text{HH}^*(X) = \text{Ext}^*_{\mathcal{O}_X \times \mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

as Yekutieli [38, 39] shows that there is an isomorphism

$$\mathbb{H}^*(X, D_{\text{poly}}(\mathcal{O}_X)) \to \text{Ext}^*_{\mathcal{O}_X \times \mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

which is compatible with the cupproduct on the left and the Yoneda product on the right.

**Remark 1.6.** The algebra isomorphism (1.3) is part of a more general conjecture by Caldararu [7] which involves also the Hochschild homology of $X$. Other parts of this conjecture were proved by Markarian and Ramadoss [24, 27].

**Remark 1.7.** The cupproduct on $T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X)$ and $D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X)$ is $\mathcal{O}_X$-linear and hence these objects can also be considered as algebras in $D(\text{Mod}(\mathcal{O}_X))$. Likewise the map (1.1) can be viewed as an isomorphism in $D(\text{Mod}(X))$.

The cupproduct on $T^\mathcal{L}_{\text{poly}}(\mathcal{O}_X)$ is commutative and the cupproduct on $D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X)$ is commutative up to a homotopy given by the bullet product [15]. However the latter is not $\mathcal{O}_X$-linear. Hence $D^\mathcal{L}_{\text{poly}}(\mathcal{O}_X)$ is not commutative in $D(\text{Mod}(\mathcal{O}_X))$ and thus Theorem 1.3 does not hold in $D(\text{Mod}(\mathcal{O}_X))$ even if we consider only the cupproduct.

This situation is reminiscent of the Duflo isomorphism $S\mathfrak{g} \to U\mathfrak{g}$ which only becomes an algebra isomorphism after taking invariants. The analogue of taking invariants in our setting is taking global sections.

We thank Andrei Caldararu for bringing this point to our attention.

If we look only at the Lie algebra structure we actually prove a result which is somewhat stronger than Theorem 1.3. Let $\text{HoLieAlg}(X)$ be the category of sheaves of DG-Lie algebras with quasi-isomorphisms inverted.

**Theorem 1.8.** (see §7.4) The isomorphism (1.1) between $T^\mathcal{L}_{\text{poly}}(X)$ and $D^\mathcal{L}_{\text{poly}}(X)$ is obtained from an isomorphism in $\text{HoLieAlg}(X)$.

This theorem can be considered as a generalization of global formality results in [4, 11, 12, 20, 34, 40]. Global formality on the sheaf level is important for deformation theory. See for example [5, 19, 21, 34, 40].

The isomorphism in $\text{HoLieAlg}(X)$ is obtained by globalizing a local formality isomorphism [20, 31]. If we take Kontsevich’s local formality isomorphism then we obtain compatibility with cupproduct in Theorem 1.3 from the compatibility of the local formality isomorphism with tangent cohomology [20, 23, 25]. Kontsevich’s local formality isomorphism is only defined when the ground field contains $\mathbb{R}$ but this is not a problem since we show that it is sufficient to prove Theorem 1.3 over a suitable extension of the base field.

An alternative approach to Theorem 1.3 could be to work directly in the setting of $G_\infty$-algebras. Unfortunately it is unknown if Kontsevich’s $L_\infty$-morphism can be lifted to a
$G_\infty$-morphism. In [6] we will use Tamarkin’s local $G_\infty$-formality isomorphism to construct a $G_\infty$-quasi-isomorphism between $T^\mathcal{L}_{\text{poly}}(X)$ and $D^\mathcal{L}_{\text{poly}}(X)$. In the case that $\mathcal{L}$ is a tangent bundle this was proved recently in [14] using very different methods. Like in [14] we are unfortunately not able to write down the resulting isomorphism on hypercohomology. Thus in this way we obtain a result which is less precise than Corollary 1.5. This is why we have decided to publish the current paper separately.

2. Acknowledgement

This paper is hugely in debt to Kontsevich’s fundamental work on formality. In particular without the many deep results and insights contained in [20] this paper could not have been written.

Our proof of the global formality result Theorem 1.8 follows the general outline of [34] which in turn was heavily inspired by [40]. We use in an essential way an algebraic version of formal geometry. Algebraic versions of formal geometry were introduced independently and around the same time by Bezrukavikov and Kaledin in [2] and Yekutieli in [40]. The language we use is closer to [40]. As a result various technical statements can be traced back in some form to [40].

We wish to thank Andrei Caldararu, Vasily Dolgushev, Charles Torossian and Amnon Yekutieli for useful conversations and comments.

3. Notations and conventions

Throughout this paper $k$ is a field of characteristic zero. Unadorned tensor products are over $k$.

Many of the objects we use are equipped with some kind of topology, but if an object is introduced without a specified topology we assume that it is equipped with the discrete topology.

If an object carries a natural grading then all constructions associated to it are implicitly performed in the graded context. This applies in particular to completions.

Since all our constructions are natural in the sense that they do not depend on any choices we work mostly with rings and modules instead of with sheaves since this often simplifies the notations. However we freely sheafify such construction if needed.

On a double (or higher) complex we use the Koszul sign convention with respect to total degree.

4. Preliminaries

4.1. Categories of vector spaces. Below we will work with various enhanced symmetric monoidal categories of $k$-vector spaces. Which category we work in will usually be clear from the context but in order to be precise we list here the various possibilities.

4.1.1. Complete topological vector spaces. For us a complete topological vector space $V$ will be a topological vector space whose topology is generated by a separated, exhaustive descending filtration $V = F_0V \supset F_1V \supset \cdots$. This filtration is however not considered as part of the structure.

The completed tensor product

$$V \hat{\otimes} W = \text{proj lim}_p (V \otimes W/(F_p V \otimes W + V \otimes F_p W))$$

makes the category of complete topological vector spaces into a symmetric monoidal category.
4.1.2. Filtered complete topological vector spaces. A filtered complete topological vector space is by definition a topological vector space $V$, equipped with an ascending separated, exhaustive filtration $F^m V$ (which is considered part of the structure) such that each $F^m V$ is a complete topological vector space and the inclusion maps $F^m V \hookrightarrow F^{m+1} V$ are continuous.

The (completed) tensor product of two filtered complete topological vector space $V$ and $W$ is defined by

$$F^m (V \otimes W) = \sum_{p+q=m} F^p V \otimes F^q W$$

(where the summation sign refers of course to convergent sums).

4.1.3. Graded filtered complete topological vector spaces. Graded filtered complete topological vector spaces are the most general objects we will encounter below. These are simply graded objects over the linear category of filtered complete topological vector spaces. The DG-vector spaces are the most general objects we will encounter below. These are simply graded

Graded filtered complete topological spaces.

(4.2.1) Lie algebroids. Below $R$ is a commutative $k$-algebra and $L$ is a Lie algebroid over $R$ which is free of rank $d$. Namely, $L$ is a Lie $k$-algebra equipped with an $R$-module structure and a Lie algebra map $\rho : L \to \text{Der}(R)$ such that $[l_1, rl_2] = r[l_1, l_2] + \rho(l_1)(r)l_2$ for $l_1, l_2 \in L$ and $r \in R$. $\rho$ is called the anchor map and we usually suppress it from the notations writing $l(r)$ instead of $\rho(l)(r) (l \in L, r \in R)$. In particular, $R \oplus L$ becomes a Lie algebra with bracket given by $[(r, l), (r', l')] = (l(r') - l'(r), [l, l'])$.

Associated to $L$ there are various constructions which are analogous to constructions occurring for enveloping algebras and rings of differential operators. In the next few paragraphs we fix some notations for them and recall the properties we need. For more information the reader is referred to [3, 4, 26, 37].

4.2.1 The enveloping algebra of a Lie algebroid. Let $UL$ be the enveloping algebra associated to $L$. It is the quotient of the enveloping algebra associated to the Lie algebra $R \oplus L$ by the following relations: $r \otimes l = rl (r \in R, l \in R \oplus L)$. If we want to emphasize $R$ then we write $URL$. $UL$ has a canonical filtration obtained by respectively assigning length 0 and 1 to elements of $R$ and $L$. We equip $UL$ with the left $R$-module structure given by the natural embedding $R \to UL$ and we view $UL$ as an $R$-bimodule with the same left and right structure. For this bimodule structure $UL$ is a cocommutative $R$-coring in the sense that there is a natural cocommutative coassociative comultiplication $\Delta : UL \to UL \otimes_R UL$ and counit $\epsilon : UL \to R$. Assuming the Sweedler convention the comultiplication is defined by

$$\Delta(f) = f \otimes 1 \quad \text{for } f \in R$$

$$\Delta(l) = l \otimes 1 + 1 \otimes l \quad \text{for } l \in L$$

$$\Delta(DE) = D(1)E(1) \otimes D(2)E(2) \quad \text{for } D, E \in UL$$

Note that it requires some verification to show that this is well defined. To do this note that $UL \otimes_R UL$ is a right $UL \otimes UL$-module in the obvious way. One proves inductively on the length of $D$, expressed as a product of elements of $L$, that in $UL \otimes_R UL$ one has

$$(D(1) \otimes D(2))(f \otimes 1 - 1 \otimes f) = 0$$

for $f$ in $R$. It follows immediately that if $E' \otimes E'' \in UL \otimes_R UL$ then

$$(D(1) \otimes D(2)) \cdot (E' \otimes E'') \overset{\text{def}}{=} D(1)E' \otimes D(2)E''$$

is well defined.

There is a unique way to extend the anchor map into an algebra morphism $\rho : U(L) \to \text{End}(R)$. As before we write $D(r)$ instead of $\rho(D)(r) (D \in UL, r \in R)$, and then the counit on $UL$ is given by

$$\epsilon(D) = D(1).$$
$U L$ is a so-called “Hopf algebroid with anchor” [37]. As we are in the cocommutative case this is expressed by the property

$$D_{(1)}(f)D_{(2)} = Df \quad (f \in R, D \in U L).$$

4.2.2. $L$-poly-vector fields and $L$-poly-differential operators. $T^L_{\text{poly}}(R)$ is the Lie algebra of $L$-poly-vector fields [3]. I.e. it is the graded vector space $\wedge_R(L)[1]$ equipped with the graded Lie bracket obtained by extending the Lie bracket on $L$. We equip $T^L_{\text{poly}}(R)$ with the standard cupproduct (which is of degree one with our shifted grading). In this way $T^L_{\text{poly}}(R)$ becomes a (shifted) Gerstenhaber algebra.

$D^L_{\text{poly}}(R)$ is the DG-Lie algebra of $L$-poly-differential operators [3]. I.e. it is the graded vector space $\text{poly}(L)[1]$ equipped with the natural structure of a DG-Lie algebra [3]. The Lie bracket on $D^L_{\text{poly}}(R)$ given by $[D_1, D_2]_G = D_1 \bullet D_2 - (-1)^{|D_1||D_2|} D_2 \bullet D_1$, where

$$D_1 \bullet D_2 = \sum_{i=0}^{[D_2]} (-1)^i \delta_{D_1} \delta_{D_2} \cdot \Delta^{[D_2]} \otimes \text{id} \otimes \text{id}^{[D_1]-i}(D_1) \cdot (1 \otimes \Omega \otimes D_2 \otimes 1 \otimes D_1^{-i}).$$

Let $m = 1 \otimes 1 \in D^L_{\text{poly}}(R)_1$. Then the differential $d$ on $D^L_{\text{poly}}(R)$ is given by

$$d(m) = [m, -].$$

For the cupproduct we use the sign-modification by Gerstenhaber-Voronov [15]. This sign-modification is necessary to make the cohomology of $D^L_{\text{poly}}(R)$ into a (shifted) Gerstenhaber algebra. We put

$$D_1 \cup D_2 = (-1)^{|D_1||D_2|} D_1 \otimes D_2.$$

There is a HKR-theorem relating $T^L_{\text{poly}}(R)$ and $D^L_{\text{poly}}(R)$ [3]. Namely the map

$$\mu : l_1 \wedge \cdots \wedge l_n \mapsto (-1)^{n(n-1)/2} \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(n)}$$

defines a quasi-isomorphism between $(T^L_{\text{poly}}(R), 0)$ and $(D^L_{\text{poly}}(R), d)$ which induces an isomorphism of shifted Gerstenhaber algebras on cohomology. Note that for this last fact to be true one needs the unconventional sign in (4.1).

4.2.3. Algebraic functoriality. The formation of $U L$, $T^L_{\text{poly}}(R)$ and $D^L_{\text{poly}}(R)$ depends functorially on $L$ in a suitable sense.

**Definition 4.2.1.** An algebraic morphism of Lie algebroids

$$(R, L) \rightarrow (T, M)$$

is a pair $(\varrho, \ell)$ of an algebra morphism $\varrho : R \rightarrow T$ and a Lie algebra morphism $\ell : L \rightarrow M$ such that for any $r \in R$ and any $l \in L$,

$$\varrho(\ell(r)) = \ell(\varrho(r)) \quad \text{and} \quad \ell(rl) = \varrho(r)\ell(l).$$

For any algebraic morphism $(R, L) \rightarrow (T, M)$ there are obvious associated maps

$$U_{R L} \rightarrow U_{T M},$$

$$T^L_{\text{poly}}(R) \rightarrow T^M_{\text{poly}}(T),$$

$$D^L_{\text{poly}}(R) \rightarrow D^M_{\text{poly}}(T),$$

that are compatible with all algebraic structures.
4.2.4. Pairings, the De Rham complex and L-connections. Put \( L^* = \text{Hom}_R(L, R) \). We identify \( \wedge_R^* L \) with the \( R \)-dual of \( \wedge_R^0 L \) via the pairing
\[
(\sigma_1 \wedge \cdots \wedge \sigma_n, l_1 \wedge \cdots \wedge l_n) = \det \sigma_i(l_j).
\]
If \( \tau \in L^* \) then we denote contraction by \( \tau \) acting on \( \wedge_R^0 L \) by \( \tau \wedge - \). I.e.
\[
\tau \wedge (l_1 \wedge \cdots \wedge l_n) = \sum_i (-1)^{i-1} \tau(l_i)(l_1 \wedge \cdots \wedge \hat{l}_i \wedge \cdots \wedge l_n)
\]
We make \( \wedge_R L \) into a \( \wedge_R L^* \)-module by extending the \( - \wedge - \)-action. I.e.
\[
(\tau_1 \wedge \cdots \wedge \tau_m) \wedge (l_1 \wedge \cdots \wedge l_n) = \tau_1 \wedge (\cdots \wedge (\tau_m \wedge (l_1 \wedge \cdots \wedge l_n)) \cdots)
\]
An easy verification shows
\[
(\sigma_1 \wedge \cdots \wedge \sigma_n, l_1 \wedge \cdots \wedge l_n) = (\sigma_{m+1} \wedge \cdots \wedge \sigma_n, (\sigma_m \wedge \cdots \wedge \sigma_1) \wedge (l_1 \wedge \cdots \wedge l_n))
\]
The Lie algebroid analogue for the De Rham complex is a DG-algebra which as graded algebra is equal to \( \wedge_R^* L \). With the identification (4.2) the differential on \( \wedge_R L^* \) is given by the usual formula for differential forms [35, Prop 2.25(f)]:
\[
d\omega(l_0, \ldots, l_n) = \sum_{i=0}^n (-1)^i l_i(\omega(l_0, \ldots, \hat{l}_i, \ldots, l_n)) + \sum_{i<j} (-1)^{i+j} \omega([l_i, l_j], l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_n).
\]
In other words the anchor map \( \rho : L \to \text{Der}_k(R) = \text{Hom}_R(\Omega^1_R, R) \) dualizes to a morphism of DG-algebras
\[
\rho^* : \Omega^* R \to \wedge_R^* L.
\]
From (4.4) we deduce in particular \( (df)(l) = l(f) \) for \( f \in R, l \in L \), and if \( (l_i)_k \) is an \( R \)-basis of \( L \) then
\[
d(l^*_k)(l_i, l_j) = -l^*_k([l_i, l_j]).
\]
In other words \( (\wedge_R L^*, d) \) completely encodes the Lie-algebroid structure of \( L \).

Remark 4.2.2. In the literature a morphism between Lie algebroids \( (R, L) \to (T, M) \) is usually defined as a morphism of DG-algebras \( \eta : \wedge_T M^* \to \wedge_R^* L \). See e.g. [8]. One could call such morphisms “geometric” to differentiate them from the algebraic ones we use. We have already encountered one geometric morphism, namely (4.5).

If \( M \) is an \( R \)-module then a \( L \)-connection on \( M \) is a map \( L \otimes M \to M : l \otimes m \mapsto \nabla_l(m) \) with the following properties: for \( l, l_1, l_2 \in L, m \in M, f \in R \) we have
\[
\nabla_l(fm) = f(l(m)) + f \nabla_l(m),
\]
\[
\nabla_{l_1}(m) = f \nabla_{l_1}(m).
\]
The connection is flat if in addition we have
\[
[\nabla_{l_1}, \nabla_{l_2}] = \nabla_{[l_1, l_2]}.
\]
In that case \( M \) automatically becomes a left \( UL \)-module. Moreover, if \( (l_i)_k \) is a basis of \( L \) then we put a left \( \wedge_R L^* \)-DG-module structure on \( \wedge_R L^* \otimes_R M \) by defining the differential as
\[
\nabla(\omega \otimes m) = d\omega \otimes m + \sum_i l^*_i \omega \otimes \nabla_{l_i}(m).
\]
Recall that if \( C \) is a commutative \( R \)-DG-algebra then a flat connection on \( M \) is a derivation of square zero on \( C \otimes_R M \) of degree one which makes \( C \otimes_R M \) into a DG-C-module\(^2\). Thus \( \nabla \) is a flat \( \wedge_R L^* \)-connection on \( M \).

\(^2\)We recall that if \( M \) is in a category of complete topological vector spaces then one has to write \( C \hat{\otimes} M \) instead.
4.2.5. L-jets. Let \((UL)_{\leq n}\) be the elements of degree \(\leq n\) for the canonical filtration on \(UL\) introduced in \S 4.2.1. The \(L\)-jets are defined as
\[ J^nL = \text{Hom}_R((UL)_{\leq n}, R) \]
(this is unambiguous, as the left and right \(R\)-module structures on \(UL\) are the same, see \S 4.2.1). We also put
\[ JL = \text{proj lim}_n J^nL. \]
We now formulate some properties of \(JL\). Most of these properties hold for \(J^nL\) as well. \(JL\) has a natural commutative algebra structure obtained from the comultiplication on \(UL\). Thus for \(\phi_1, \phi_2 \in JL, D \in UL\) we have
\[ (\phi_1 \phi_2)(D) = \phi_1(D(1)) \phi_2(D(2)), \]
and the unit in \(JL\) is given by the counit on \(UL\). It is well-known that \(JL\) has a lot of extra structure which we now elucidate. First of all there are two distinct monomorphisms of \(k\)-algebras
\[ \alpha_1 : R \rightarrow JL : r \mapsto (D \mapsto r \epsilon(D)), \]
\[ \alpha_2 : R \rightarrow JL : r \mapsto (D \mapsto D(r)). \]
It will be convenient to write \(R_1 = \alpha_1(R)\) and to view \(JL\) as an \(R_1 - R_2\)-bimodule.

Define \(\epsilon : JL \rightarrow R\) by \(\epsilon(\phi) = \phi(1)\) and put \(J^cL = \ker \epsilon\). It is easy to see that \(\epsilon \circ \alpha_1 = \epsilon \circ \alpha_2 = \text{id}_R\). We conclude that
\[ JL = R_1 \oplus J^cL = R_2 \oplus J^cL \]
The filtration on \(JL\) induced by (4.7) coincides with the \(J^cL\)-adic filtration. If we filter \(JL\) with the \(J^cL\)-adic filtration then we obtain
\[ \text{gr} JL = S^*_R L^* \]
and the \(R_1\) and \(R_2\)-action on the r.h.s. of this equation coincide (here and below the letter \(S\) stands for “symmetric algebra”).

There are also two different commuting actions by derivations of \(L\) on \(JL\). Let \(l \in L, \phi \in JL, D \in UL\).
\[ 1\nabla_l(\phi)(D) = l(\phi(D)) - \phi(lD) \]
\[ 2\nabla_l(\phi)(D) = \phi(lD) \]
Again it will be convenient to write \(L_l\) for \(L\) acting by \(l\nabla\). Then \(l\nabla\) defines a flat \(L_l\)-connection on \(JL\), considered as an \(R_l\)-module. Thus \(JL\) becomes a \(UL_1 - UL_2\)-bimodule (with both \(UL_1\) and \(UL_2\) acting on the left). For some of the verifications below we note that the \(UL_2\) action on \(JL\) takes the simple form
\[ (D \cdot \phi)(E) = \phi(ED) \]
(for \(D, E \in UL_2, \phi \in JL\)).

The induced actions on \(\text{gr} JL = S^*_R L^*\) of \(l \in L\), considered as an element of \(L_1\) and \(L_2\), are given by the contractions \(i_{-1}\) and \(i_1\).

**Example 4.2.3.** In case \(R\) is the coordinate ring of a smooth affine algebraic variety and \(L = \text{Der}_k(R)\) then we may identify \(JL\) with the completion of \(R \otimes R\) at the kernel of the multiplication map \(R \otimes R \rightarrow R\). The two actions of \(R\) on \(JL\) are respectively \(R \otimes 1\) and \(1 \otimes R\).

Similarly a derivation on \(R\) can be extended to \(R \otimes R\) in two ways by letting it act respectively on the first and second factor. Since derivations are continuous they act on adic completions and hence in particular on \(JL\). This provides the two actions of \(L\) on \(JL\).
As $\nabla_1$ acts by derivation on $JL$ it is easy to see that the resulting $\wedge R_1 L^*_1$-DG-module $(\wedge R_1 L^*_1 \otimes R_1 JL, \nabla_1)$ (see (4.6)) is actually a commutative $\wedge R_1 L^*_1$-DG-algebra. The following result is well-known.

**Proposition 4.2.4.** The inclusion $\alpha_2 : R \hookrightarrow JL$ defines a quasi-isomorphism

$$R \longrightarrow \wedge R_1 L^*_1 \otimes R_1 JL ; r \longmapsto 1 \otimes \alpha_2(r).$$

**Proof.** It is easy to see that if $r \in R_2$ then $\nabla_1(1 \otimes r) = 0$. To prove that we obtain a quasi-isomorphism we filter $JL$ by the $J^c L$-adic filtration. We obtain the following associated graded complex

$$0 \to R \to S_R L^* \to L^* \otimes_R S_R L^* \to \cdots \to \wedge^d_R L^* \otimes S_R L^* \to 0$$

(4.10)

where the differential is given by $- \sum_j l^*_j \otimes i_{l_j}$ for a basis $(l_j)_j$ of $L$. It is easy to see that (4.10) is exact. \qed

### 4.3. Relative poly-vector fields, poly-differential operators and forms.

#### 4.3.1. Definitions.

We need relative poly-differential operators and poly-vector fields. So assume that $A \to B$ is a morphism of commutative DG-$k$-algebras. Then

$$T_{poly, A}(B) = \bigoplus_n T^n_{poly, A}(B)$$

$$D_{poly, A}(B) = \bigoplus_n D^n_{poly, A}(B)$$

where $T^n_{poly, A}(B) = \bigwedge^{n+1}_B \text{Der}_A(B)$. Similarly $D^n_{poly, A}(B)$ is the set of maps with $n + 1$ arguments $B \otimes_A \cdots \otimes_A B \to B$ which are differential operators when we equip $B$ with the diagonal $B \otimes_A \cdots \otimes_A B$-algebra structure.

If we consider $A$ and $B$ just as algebras then $T_{poly, A}(B)$ and $D_{poly, A}(B)$ are DG-Lie algebras in the usual way. The differential on $T_{poly, A}(B)$ is trivial and the differential on $D_{poly, A}(B)$ is the restriction of the Hochschild differential. We denote it by $d_{Hoch}$. The differential $d_B$ on $B$ induces a differential $[d_B, -]$ on $T_{poly, A}(B)$, $D_{poly, A}(B)$ which commutes with $d_{Hoch}$ on the latter. The total differentials on $T^n_{poly, A}(B)$ and $D^n_{poly, A}(B)$ are respectively $[d_B, -]$ and $[d_B, -] + d_{Hoch}$.

Recall that one can also consider the DG-algebra $\Omega_{B/A}$ of relative differentials. The differential is $d_B + d_{DR}$. There is a contraction map between relative one-differentials and relative vector fields: it is defined as the $B$-linear map

$$\Omega^1_{B/A} \otimes \text{Der}_A(B) \to B : fdg \otimes \xi \mapsto f \xi(g).$$

#### 4.3.2. Relation with $L$-jets.

Let us begin by the observation that

$$L_2 \to \text{Der}_{R_1}(JL) : l \mapsto (\theta \mapsto \nabla_1(\theta))$$

is a Lie algebra morphism.\footnote{Here and in the rest of the paper the symbol “Ω” is used in the sense of continuous differentials (since we usually deal with complete topological vector spaces). See [34, §3.4] for a more detailed discussion of continuous differentials in a slightly restricted case. There are no surprises.}

**Lemma 4.3.1.** (4.11) yields a well-defined isomorphism of $JL$-modules

$$JL \otimes_{R_1} L_2 \to \text{Der}_{R_1}(JL) : \phi \otimes l \mapsto (\theta \mapsto \phi \cdot \nabla_1(\theta)).$$

**Proof.** It suffices to check that this is the case for the associated graded modules for the $J^c L$-adic filtration, which is easy. \qed

\footnote{Together with $R_2 \to JL$ this actually is an algebraic morphism of Lie algebroids.}
Let $D_{R_1}(JL)$ be the ring of differential operators of $JL$ relative to $R_1$, considered as a $R_2$-module. Since the $L_2$-action on $JL$ commutes with the $R_1$-action we obtain a ring homomorphism

\[(4.13) \quad UL_2 \to D_{R_1}(JL): D \mapsto (\theta \mapsto D(\theta)).\]

It is easy to check that together with $R_2 \to JL$ this gives a Hopf algebroid homomorphism $(R_2, UL_2) \to (JL, D_{R_1}(JL))$.

**Lemma 4.3.2.** (4.13) yields a well-defined isomorphism of $JL$-modules.

\[(4.14) \quad JL \otimes_{R_2} UL_2 \to D_{R_1}(JL): \phi \otimes D \mapsto (\theta \mapsto \phi D(\theta)).\]

**Proof.** It is easily verified that this map is well-defined. To prove that it is an isomorphism we extend the natural filtration on $UL_2$ to a filtration on the r.h.s. of (4.14) and we filter the r.h.s. by order of differential operators. We then obtain a map

\[JL \otimes_{R_2} SL_2 = SJL(JL \otimes_{R_2} L_2) \to SJL(\text{Der}_{R_1}(JL))\]

which is induced from the natural map $JL \otimes_{R_2} L_2 \to \text{Der}_{R_1}(JL)$. This map is an isomorphism by Lemma 4.3.1. $\square$

Using again that the $L_2$-action on $JL$ commutes with the $R_1$-action we obtain natural DG-Lie algebra morphisms

\[(4.15) \quad T_{\text{poly}}^{L_2}(R_2) \to T_{\text{poly}, R_1}(JL),\]

\[D_{\text{poly}}^{L_2}(R_2) \to D_{\text{poly}, R_1}(JL).\]

We obtain the following (see [40, Lemma 5.1(22)]):

**Lemma 4.3.3.** The maps (4.15) induce well-defined isomorphisms of $JL$-modules.

\[JL \otimes_{R_2} T_{\text{poly}}^{L_2}(R_2) \to T_{\text{poly}, R_1}(JL)\]

\[JL \otimes_{R_2} D_{\text{poly}}^{L_2}(R_2) \to D_{\text{poly}, R_1}(JL).\]

**Proof.** It easy to check that these maps are well-defined. As an example we prove that the second map is an isomorphism. We have isomorphisms of vector spaces

\[D_{\text{poly}, R_1}(JL) = T_{JL}(D_{R_1}(JL))[1] = T_{JL}(JL \otimes_{R_2} UL_2)[1] = JL \otimes_{R_2} T_{R_2}(UL_2)[1] = JL \otimes_{R_2} D_{\text{poly}}^{L_2}(R_2).\]

In the first line we have used Lemma 4.3.2. One now easily shows that the resulting isomorphism $JL \otimes_{R_2} D_{\text{poly}}^{L_2}(R_2) \cong D_{\text{poly}, R_1}(JL)$ is indeed the morphism given in the statement of the lemma. $\square$

We also have:

**Lemma 4.3.4.** Let $C$ be a commutative $R_1$-DG-algebra. The canonical maps

\[(4.16) \quad \begin{cases} (\hat{C} \otimes_{R_1} JL) \otimes_{R_2} T_{\text{poly}}^{L_2}(R_2) \to T_{\text{poly}, C}(\hat{C} \otimes_{R_1} JL) \\ (\hat{C} \otimes_{R_1} JL) \otimes_{R_2} D_{\text{poly}}^{L_2}(R_2) \to D_{\text{poly}, C}(\hat{C} \otimes_{R_1} JL) \end{cases}\]

obtained by linearly extending the canonical maps

\[T_{\text{poly}}^{L_2}(R_2) \to T_{\text{poly}, C}(\hat{C} \otimes_{R_1} JL)\]

\[D_{\text{poly}}^{L_2}(R_2) \to D_{\text{poly}, C}(\hat{C} \otimes_{R_1} JL)\]

are well-defined isomorphisms. If $JL$ carries a flat $C$-connection $\nabla$ which is compatible with the $R_2$-action on $\hat{C} \otimes_{R_1} JL$ then $\nabla \otimes \text{id}$ on the left of (4.16) corresponds to $[\nabla, -]$ on the right.
Proof. We restrict ourselves to the case of poly-differential operators. The case of poly-vector fields is similar. Using the fact that $JL$ is formally smooth and formally of finite type over $R$ we easily deduce that the canonical map

$$C \otimes_{R_{1}} D_{\text{poly},R_{1}}(JL) \to D_{\text{poly},C}(C \otimes_{R_{1}} JL)$$

is an isomorphism. Combining this with Lemma 4.3.4 yields the required isomorphism. It is easily seen that this isomorphism yields the asserted compatibility for flat $C$-connections.

4.3.3. Differentials and $L$-jets. Let us introduce a $JL$-linear map

$$(4.17) \quad \Omega_{JL/R_{1}}^{1} \to JL \otimes_{R_{2}} L_{2}^{*} : \phi d\theta \mapsto \phi \otimes \tilde{\theta}$$

with $\tilde{\theta}(l) \defeq \tilde{\nabla}_{l}(\theta)$ for any $l \in L_{2}$. If we respectively denote (4.17) and (4.11) by $u$ and $v$ then by definition we have

$$u(\xi)(l) = \xi(v(l)) \quad (\forall \xi \in \Omega_{JL/R_{1}}^{1}) .$$

It then follows from taking the $R$-dual of (4.12) that (4.17) is an isomorphism, and that the restriction $L_{2} \to \Omega_{JL/R_{1}}^{1}$ to $L_{2}^{*}$ of its inverse fits into the commutative diagram

$$(4.18) \quad L_{2} \otimes L_{2} \xrightarrow{\text{contraction}} R_{2}$$

The next result follows by applying $\wedge(-)$ to the inverse of (4.17) and is parallel to lemma 4.3.1:

Lemma 4.3.5. The maps $R_{2} \to JL$ and $L_{2}^{*} \to \Omega_{JL/R_{1}}^{1}$ induce a DG-algebra morphism

$$\wedge_{R_{2}} L_{2}^{*} \to \Omega_{JL/R_{1}}^{1}$$

that extends to an isomorphism $JL \otimes_{R_{2}} \wedge_{R_{2}} L_{2}^{*} \to \Omega_{JL/R_{1}}^{1}$ of $JL$-modules. This isomorphism is compatible with differentials if we put on $JL \otimes_{R_{2}} \wedge_{R_{2}} L_{2}^{*}$ the canonical differential obtained from the $L_{2}$-connection on $JL$.

Proof. We may view (4.17) as an isomorphism between the Lie algebroids over $JL$ given by $JL \otimes_{R_{2}} L_{2}$ and $\text{Der}_{R_{1}}(JL)$ where the first Lie algebroid structure is deduced from the $L_{2}$-connection on $JL$.

As a result we obtain an isomorphism between the corresponding DG-algebras:

$$JL \otimes_{R_{2}} \wedge_{R_{2}} L_{2}^{*} = \bigwedge_{JL}(JL \otimes_{R_{2}} L_{2}^{*}) \cong \bigwedge_{JL} \text{Der}_{R_{1}}(JL)^{*} = \Omega_{JL/R_{1}}^{1}$$

The statement of the lemma follows easily from this. \hfill \Box

We also have the following analogue of Lemma 4.3.4:

Lemma 4.3.6. Let $C$ be a commutative $R_{1}$-DG-algebra. The canonical map

$$(4.19) \quad (C \otimes_{R_{1}} JL) \otimes_{R_{2}} (\wedge_{R_{2}} L_{2}^{*}) \to \Omega_{C \otimes_{R_{1}} JL/C}$$

is an isomorphism. If $JL$ carries a flat $C$-connection $\nabla$ which is compatible with the $R_{2}$-action on $C \otimes_{R_{1}} JL$ then $\nabla \otimes \text{id}$ on the left corresponds to the differential $d\nabla$ on the right.

Here $d\nabla$ is characterized by the properties that it coincides with $\nabla$ on $C \otimes_{R_{1}} JL = \Omega_{C \otimes_{R_{1}} JL/C}^{0}$ and that it commutes with the De Rham differential.

5. Coordinate spaces

We keep the notations of the previous section.
5.1. The coordinate space of a Lie algebroid. The following definition is inspired by [40]. We define \( R^\text{coord,L} \) as the commutative \( R_1 \)-algebra which trivializes \( JL \), i.e. which is universal for the property that there is an isomorphism of \( R^\text{coord,L} \)-algebras

\[
t : R^\text{coord,L} \otimes_{R_1} JL \cong R^\text{coord,L}[[t_1, \ldots, t_d]]
\]

such that \( R^\text{coord,L} \otimes_{R_1} JL \) is mapped to \( \{t_1, \ldots, t_d\} \).

For use below we give an explicit description of \( R^\text{coord,L} \). Since we have assumed \( L \) to be free of rank \( d \) we may assume that

\[
JL \cong R_1[[x_1, \ldots, x_d]]
\]

where \((x_i)_i\) is mapped to a basis of \( J^eL/\langle J^eL \rangle^2 \cong L^* \). Let \( T \) be the polynomial ring over \( R_1 \) in the variables \( y_{i,a_1 \cdots a_d} \) where \( i = 1, \ldots, d \), \( a_j \in \mathbb{N} \) and \( (a_1, \ldots, a_d) \neq (0, \ldots, 0) \). Then \( R^\text{coord,L} \) is the localization of \( T \) at \( det(y_{i,c_j}) \) where \( c_j = (0, \ldots, 1, \ldots, 0) \) with the 1 occurring in the \( j \)'th place and \( t \) is given by

\[
t(x_i) = \sum_{i,a_1 \cdots a_d} y_{i,a_1 \cdots a_d} t_1^{a_1} \cdots t_d^{a_d}
\]

Since \( R^\text{coord,L} \) is universal any \( k \)-linear automorphism \( \alpha \) of \( k[[t_1, \ldots, t_d]] \) yields a corresponding unique \( R_1 \)-linear automorphism \( \tilde{\alpha} \) of \( R^\text{coord,L} \) such that the combined automorphism \((\tilde{\alpha}, \alpha)\) of \( R^\text{coord,L}[[t_1, \ldots, t_d]] \) leaves the image under \( t \) of \( JL \) pointwise invariant. Since \( \text{GL}_d(k) \) acts on \( k[[t_1, \ldots, t_d]] \) we obtain a corresponding \( R_1 \)-linear action of \( \text{GL}_d(k) \) on \( R^\text{coord,L} \).

In fact if we write

\[
R^\text{coord,L} \cong R_1 \otimes S^\text{coord}
\]

with \( S^\text{coord} = k[[y_{i,a_1 \cdots a_d}]]_{\text{det}(y_{i,c_j})} \) then the \( \text{GL}_d(k) \)-action on \( R^\text{coord,L} \) is obtained from a \( \text{GL}_d(k) \)-action on \( S^\text{coord} \) which preserves the finite dimensional vector spaces with basis \( \{y_{i,a_1 \cdots a_d} | \sum_i a_i = s\} \). Needless to say that the decomposition (5.3) depends on the choice of the generators \((x_i)_i\) of \( J^eL \).

It follows in particular that the \( \text{GL}_d(k) \)-action is rational. Hence we may consider the derived action of \( \text{gl}_d(k) \) on \( R^\text{coord,L} \). The action of \( \text{GL}_d(k) \) on \( k[[t_1, \ldots, t_d]] \) also yields a derived action of \( \text{gl}_d(k) \). The two actions of \( \text{gl}_d(k) \) satisfy the following compatibility.

\[
\text{Lemma 5.1.1.} \quad \text{For } v \in \text{gl}_d(k) \text{ let } L_v \text{ be the action of } v \text{ on } R^\text{coord,L}[[t_1, \ldots, t_d]] \text{ obtained by linearly extending the action of } v \text{ on } k[[t_1, \ldots, t_d]].
\]

\[
\text{Let } L_v \text{ be the action of } v \text{ on } R^\text{coord,L}[[t_1, \ldots, t_d]] \text{ obtained by linearly extending the action of } v \text{ on } R^\text{coord,L}. \text{ Then } L_v + L_v \text{ is zero on } t(JL).
\]

\[
\text{Proof.} \quad \text{This is the derived version of the fact that } \text{GL}_d(k) \text{ leaves } t(JL) \text{ pointwise fixed.} \quad \square
\]

5.2. On some DG-algebras associated to coordinate spaces.

5.2.1. The DG-algebra \( C^\text{coord,L} \). Put \( C^\text{coord,L} = \Omega_{R^\text{coord,L}} \otimes_{\Omega_{R_1}} \wedge R_1 L_1^* \). Using the DG-algebra structures on \( \wedge R_1 L_1^* \) we see that \( C^\text{coord,L} \) is naturally a commutative \( \wedge R_1 L_1^* \)-DG-algebra.

As we have not put any restrictions on \( R \), the DG-algebras \( \Omega_{R_1} \) and \( \Omega_{R^\text{coord,L}} \) may be enormous objects. However only their “difference” matters and this is controlled by (5.3).

In fact from (5.3) we obtain a coordinate dependent isomorphism of DG-algebras

\[
C^\text{coord,L} = \Omega_{S^\text{coord}} \otimes \wedge R_1 L_1^*
\]

We will now form the completed tensor product over \( R^\text{coord,L} \) of the domain and codomain of the map (5.1) with \( C^\text{coord,L} \) ignoring the differentials. For the domain we get

\[
C^\text{coord,L} \otimes_{R^\text{coord,L}} R^\text{coord,L} \otimes_{R_1} JL = C^\text{coord,L} \otimes_{R_1} JL = \Omega_{R^\text{coord,L}} \otimes_{\Omega_{R_1}} (\wedge R_1 L_1^* \otimes R_1 JL)
\]

For the codomain we get

\[
C^\text{coord,L} \otimes_{R^\text{coord,L}} R^\text{coord,L}[[t_1, \ldots, t_d]] = C^\text{coord,L}[[t_1, \ldots, t_d]]
\]
So we obtain an isomorphism of graded algebras
\[ \tilde{t} : C^{\text{coord},L} \hat{\otimes} R_1 \rightarrow C^{\text{coord},L}[[t_1, \ldots, t_d]] \]
Both domain and codomain of \( \tilde{t} \) carry a natural differential. The differential on \( C^{\text{coord},L} \hat{\otimes} R_1 \rightarrow C^{\text{coord},L}[[t_1, \ldots, t_d]] \) is obtained by combining the ordinary differential on \( \Omega^{\text{coord},L} \) and the differential \( ^1 \nabla \) on \( \wedge R_1 L_1^* \hat{\otimes} R_1 \rightarrow C^{\text{coord},L}[[t_1, \ldots, t_d]] \) using the right hand side of (5.4). The differential on \( C^{\text{coord},L}[[t_1, \ldots, t_d]] \) is obtained from extending the differential on \( C^{\text{coord},L} \). Let us denote the resulting differentials by \( ^1 \nabla^\text{coord} \) and \( d \) respectively.

The constructed differentials transform the obvious morphisms of graded algebras
\[ C^{\text{coord},L} \rightarrow \Omega^{\text{coord},L} \hat{\otimes} R_1 \rightarrow C^{\text{coord},L}[[t_1, \ldots, t_d]] \]
into morphisms of DG-algebras.
By the middle equality in (5.4) \( ^1 \nabla^\text{coord} \) may be viewed as a flat \( C^{\text{coord},L} \)-connection on \( JL \). The map
\[ (\tilde{t} \circ ^1 \nabla^\text{coord} \circ \tilde{t}^{-1} - d) : C^{\text{coord},L}[[t_1, \ldots, t_d]] \rightarrow C^{\text{coord},L}[[t_1, \ldots, t_d]] \]
is now a \( C^{\text{coord},L} \)-linear derivation.

From [34, §6.4] we obtain the existence of elements \( \omega^i \in C^{\text{coord},L}[[t_1, \ldots, t_d]]_1 \) such that for
\[ \omega = \sum_i \omega^i \partial/\partial t_i \in C^{\text{coord},L} \hat{\otimes} \text{Der}_k([t_1, \ldots, t_d]) \]
we have \( \tilde{t} \circ ^1 \nabla^\text{coord} \circ \tilde{t}^{-1} = d + \omega \) and furthermore \( \omega \) satisfies the Maurer-Cartan equation
\[ d\omega + \frac{1}{2}[\omega, \omega] = 0 \]
in \( C^{\text{coord},L} \hat{\otimes} \text{Der}_k([t_1, \ldots, t_d]) \).

5.2.2. The \( \mathfrak{gl}_d(k) \)-action on \( C^{\text{coord},L} \) and the Maurer-Cartan form. We need the following result.

**Lemma 5.2.1.** For \( v \in \mathfrak{gl}_d(k) \) let \( i_v \) be the derivation on \( C^{\text{coord},L} = \Omega^{\text{coord},L} \hat{\otimes} R_1 \wedge R_1 L_1^* \)
obtained by linearly extending the contraction of \( \Omega^{\text{coord},L} \) with the action as \( R_1 \)-derivation of \( v \) on \( \Omega^{\text{coord},L} \) (cf. §5.1). Extend \( i_v \) to a map of degree \(-1\) from \( C^{\text{coord},L} \hat{\otimes} \text{Der}_k([t_1, \ldots, t_d]) \) to itself. Then we have
\[ i_v \omega = 1 \otimes v \]
where both sides are considered as elements of \( R^{\text{coord},L} \hat{\otimes} \text{Der}_k([t_1, \ldots, t_d]) \).

**Proof.** We may prove (5.7) by evaluation on an arbitrary element \( g \in R^{\text{coord},L}[[t_1, \ldots, t_d]] \).
Since \( \omega = \sum \omega_i(\partial/\partial t_i) \) we have \( i_v(\omega) = \sum_i i_v(\omega_i)(\partial/\partial t_i) \) and hence \( (i_v(\omega))(g) = i_v(\omega(g)) \).
Thus we need to show
\[ i_v \circ \omega = L_v \]
as operators on \( R^{\text{coord},L}[[t_1, \ldots, t_d]] \) (as in Lemma 5.1.1 \( L_v \) is the extension of the \( v \)-action on \( k[[t_1, \ldots, t_d]] \)).

It is clear that \( R^{\text{coord},L}[[t_1, \ldots, t_d]] \) is topologically generated by \( R^{\text{coord},L} \) and \( t(JL) \). Since the operators occurring are \( R^{\text{coord},L} \)-linear it is sufficient to prove the identity
\[ i_v \circ \omega \circ t = L_v \circ t \]
as operators on \( JL \). We may rewrite the l.h.s. of (5.9) as follows
\[ i_v \circ \omega \circ t = i_v \circ (d + \omega) \circ t = i_v \circ d \circ t = i_v \circ \tilde{t} \circ ^1 \nabla^\text{coord} - L_v \circ t = L_v \circ t. \]
In the second equality we have used the Cartan relation $L_\theta = d \circ i_\theta + i_\theta \circ d$ and the fact that the term $d \circ i_\theta$ acts as zero on $R^{\text{coord}, L}[t_1, \ldots, t_d]$ (for degree reasons). We have also used (5.5).

In the third equality we have used the fact that $JL$ is mapped to $\wedge_R L_1^* \otimes_R JL$ under $\nabla^{\text{coord}}$ and the image of $\wedge_R L_1^* \otimes_R JL$ in $C^{\text{coord}, L} \otimes \text{Der}_R(k[[t_1, \ldots, t_d]])$ under $t$ lies in the part generated by $R^{\text{coord}, L}$ and $\text{Der}_R(k[[t_1, \ldots, t_d]])$ and on this part $i_\theta$ is zero. □

5.3. The affine coordinate space of a Lie algebroid. We put $R^{\text{aff}, L} = (R^{\text{coord}, L})^{\text{Gl}_R(k)}$.

We easily verify from (5.3) that $R^{\text{aff}, L}$ is of the form $R^1 \otimes S^{\text{aff}}$ with $S^{\text{aff}} = (S^{\text{coord}})^{\text{Gl}_R(k)}$ and furthermore $S = k((z_i))$ for a set of (infinitely many) variables $(z_i)$.

Below we will also use the DG-algebra $C^{\text{aff}, L} = \Omega_{R^{\text{aff}, L}} \otimes \wedge_R L_1^*$. Exactly as for $C^{\text{coord}, L}$ one produces a flat $C^{\text{aff}, R}$-connection on $JL$ which we denote by $\nabla^{\text{aff}}$. Furthermore we have a coordinate dependent isomorphism of DG-algebras (5.11)

\[ C^{\text{aff}, L} = \Omega_{S^{\text{aff}}} \otimes \wedge_R L_1^*. \]

**Lemma 5.3.1.** For any free $R_2$-module $M$ we have that

\[ M \to (C^{\text{aff}, L} \otimes_R JL, \nabla^{\text{aff}}) \overset{\hat{\circ}}{\otimes}_{R_2} M \]

is a quasi-isomorphism.

**Proof.** Using the decomposition (5.11) we need to show that

\[ M \to \Omega_{S^{\text{aff}}} \otimes \wedge_R L_1^* \otimes_R JL \overset{\hat{\circ}}{\otimes}_{R_2} M \]

is a quasi-isomorphism. We filter the r.h.s. of (5.12) with the $J^c L$-adic filtration. This means we have to show that

\[ M \to \Omega_{S^{\text{aff}}} \otimes \wedge_R L^* \otimes_R SL^* \otimes_R M \]

is a quasi-isomorphism.

Using the proof of Proposition 4.2.4 we see that we may replace $\wedge_R L^* \otimes_R SL^*$ by $R$. Furthermore since $S^{\text{aff}}$ is a polynomial ring we also find that $\Omega_{S^{\text{aff}}}$ is quasi-isomorphic to $k$. Thus we are done. □

5.4. The universal property of the affine coordinate space. The affine coordinate space has a universal property, similar to (5.1).

**Proposition 5.4.1.** There is a filtered isomorphism of $R$-algebras (5.13)

\[ R^{\text{aff}, L} \otimes_R JL \cong R^{\text{aff}, L} \otimes_R SL^* \]

which induces the canonical isomorphism

\[ R^{\text{aff}, L} \otimes_R \text{gr}(JL) \cong R^{\text{aff}, L} \otimes_R SL^* \]

obtained by extending (4.9). Moreover $R^{\text{aff}, L}$ is universal for the existence of such an isomorphism.

**Proof.** We start with the isomorphism (5.14)

\[ R^{\text{coord}, L} \otimes_R JL \cong R^{\text{coord}, L}[t_1, \ldots, t_d] \]

It follows from the discussion after (5.2) that this isomorphism if $\text{Gl}_R$-equivariant is equipped with a $\text{Gl}_R$-action which is a combination of the linear action on the $(t_i)$’s and the extension of the $\text{Gl}_R$-action on $R^{\text{coord}, L}$. In particular this $\text{Gl}_R$-action preserves the $(t_i)$-grading.

Put

\[ \tilde{L}^* = R^{\text{coord}, L}t_1 + \cdots + R^{\text{coord}, L}t_d \]

considered as a $\text{Gl}_R$-module. Then (tautologically) we have a $\text{Gl}_R$-equivariant isomorphism

\[ R^{\text{coord}, L}[t_1, \ldots, t_d] \cong S_{R^{\text{coord}, L}(\tilde{L}^*)} \]
Combining this with (5.14) we get a \( \text{Gl}_d \)-equivariant isomorphism
\[
R^{\text{coord},L} \otimes_R J L \cong S_{R^{\text{coord},L}}(L^*)^* \]
and looking at degree one of the associated graded rings:
\[
R^{\text{coord},L} \otimes_R L^* \cong \hat{L}^* \]
Thus
\[
(5.15) \quad R^{\text{coord},L} \otimes_R J L \cong S_{R^{\text{coord},L}}(R^{\text{coord},L} \otimes_R L^*)^* \cong R^{\text{coord},L} \otimes \hat{S}L^* \]
It now suffices to take \( \text{Gl}_d \)-invariants to get the isomorphism (5.13).

We will only sketch the proof of universality since we will not need it in the sequel. Assume that we have an isomorphism
\[
W \otimes_R J L \cong W \otimes_R \hat{S}L^* \]
such that
\[
W \otimes_R \text{gr}(J L) \cong W \otimes_R SL^* \]
We need to construct a corresponding morphism \( \text{Raff},L \rightarrow W \).

We let \( \hat{W} \) be the commutative \( W \)-algebra which is universal for the existence of an isomorphism
\[
\hat{W} \otimes_W (W \otimes_R L) \cong \hat{W} t_1 + \cdots + \hat{W} t_d \]
Thus \( \text{Spec} \hat{W} / \text{Spec} W \) is a \( \text{Gl}_d \)-torsor and in particular \( \hat{W}^{\text{Gl}_d} \cong W \). We then have
\[
\hat{W} \otimes_R J L = \hat{W} \otimes_W (W \otimes_R J L) = \hat{W} \otimes_W (W \otimes_R \hat{S}L^*) = \hat{W} \otimes_R \hat{S}L^* = S_{\hat{W}}(\hat{W} t_1 + \cdots + \hat{W} t_d)^* \]
Hence there exists a corresponding morphism \( R^{\text{coord},L} \rightarrow \hat{W} \). Taking \( \text{Gl}_d \)-invariants yields the requested morphism \( \text{Raff},L \rightarrow W \). One easily checks that this morphism satisfies the appropriate uniqueness properties. \( \square \)

6. \( \infty \)-\textit{ALGEBRAS}

In this section we recall some properties of \( \infty \)-algebras and we fix some notations.

6.1. \( \infty \)-\textit{ALGEBRAS AND MORPHISMS}. An \( \infty \)-\textit{structure} on a vector space \( \mathfrak{g} \) is a coderivation \( Q \) of degree one on \( S(\mathfrak{g}[1]) \) which has square zero. Such a coderivation is fully determined its “Taylor coefficients” which are the coefficients
\[
Q_i : S^i(\mathfrak{g}[1]) \xrightarrow{\text{inclusion}} S(\mathfrak{g}[1]) \xrightarrow{Q} S(\mathfrak{g}[1]) \xrightarrow{\text{projection}} \mathfrak{g}[1] \]
If for \( a, b \in \mathfrak{g} \) one puts
\[
(6.1) \quad da = -Q_1(a) \quad \text{and} \quad [a, b] = (-1)^{|a|}Q_2(a, b) .
\]
then \( d^2 = 0 \) and \( d \) is a derivation of degree one of \( \mathfrak{g} \) with respect to the binary operation of degree zero \([-, -]\). If \( \partial^i Q = 0 \) for \( i > 2 \) then \( \mathfrak{g} \) is a DG-Lie algebra. Conversely any DG-Lie algebra can be made into an \( \infty \)-algebra by defining \( Q_1, Q_2 \) according to (6.1) and by putting \( Q_i = 0 \) for \( i > 2 \).

A morphism of \( \infty \)-algebras \( \mathfrak{g} \rightarrow \mathfrak{h} \), or \( \infty \)-\textit{morphism} is by definition an augmented coalgebra map of degree zero \( S(\mathfrak{g}[1]) \rightarrow S(\mathfrak{h}[1]) \) commuting with \( Q \). A morphism of \( \infty \)-algebras is again determined by its Taylor coefficients
\[
\psi_1 : S^i(\mathfrak{g}[1]) \xrightarrow{\text{inclusion}} S(\mathfrak{g}[1]) \xrightarrow{\psi} S(\mathfrak{h}[1]) \xrightarrow{\text{projection}} \mathfrak{h}[1] \]
One has \( d\psi_1 = \psi_1 d \) and hence \( \psi_1 \) defines a morphism of complexes.

The above notions make sense in any symmetric monoidal category. We will use them in the case of filtered complete linear topological vector spaces. Of course in that case the symmetric products have to be replaced by completed symmetric products.

6.2. **Twisting.** Assume that \( \psi : g \to h \) is a \( L_\infty \)-morphism between \( L_\infty \)-algebras. We assume in addition that we are in a complete filtered setting (in the category of graded vector spaces). I.e. there are ascending filtrations \( (F_n g)_n \) \((F_n h)_n\), on \( g, h \) and furthermore \( g, h \) are graded complete for the topologies induced by these filtrations. We assume in addition that \( \psi \) is compatible with the filtrations (i.e. it is a filtered map for the induced filtrations on \( S(g[1]) \) and \( S(h[1]) \)).

Let \( \omega \in F_{-1}g_1 \) be a solution of the \( L_\infty \)-Maurer-Cartan equation in \( g \).

\[
(6.2) \quad \sum_{i \geq 1} \frac{1}{i!} Q_i(\omega^i) = 0.
\]

Define \( Q_\omega, \psi_\omega \) and \( \omega' \) by

\[
(6.3) \quad Q_{\omega,i}(\gamma) = \sum_{j \geq 0} \frac{1}{j!} Q_{i+j}(\omega^j \gamma) \quad \text{(for} \ i > 0 \text{)}
\]

\[
(6.4) \quad \psi_{\omega,i}(\gamma) = \sum_{j \geq 0} \frac{1}{j!} \psi_{i+j}(\omega^j \gamma) \quad \text{(for} \ i > 0 \text{)}
\]

\[
(6.5) \quad \omega' = \sum_{j \geq 1} \frac{1}{j!} \psi_j(\omega^j)
\]

for \( \gamma \in S^i(g[1]) \). Then by [41, Thm 3.21,3.27] \( \omega' \in F_{-1}h_1 \) is a solution of the Maurer-Cartan equation in \( h \) and furthermore \( g, h \), when equipped with \( Q_\omega, Q_{\omega'} \) are again \( L_\infty \)-algebras. If we denote these by \( g_\omega \) and \( h_{\omega'} \) then \( \psi_\omega \) defines a filtered \( L_\infty \)-map \( g_\omega \to h_{\omega'} \).

**Remark 6.2.1.** Formulas similar to (6.3-6.5) also appear at other places in the literature e.g. [13, eq. (60)(61)] and [32, eq. (4.4)]. They are implicit in the language of formal \( Q \)-manifolds employed by Kontsevich in [20].

If \( g \) is a DG-Lie algebra then the \( L_\infty \)-Maurer-Cartan equation translates into the usual Maurer-Cartan equation

\[
(6.6) \quad d\omega + \frac{1}{2} [\omega, \omega] = 0
\]

and we obtain \( Q_{\omega,1}(\gamma) = Q_1(\gamma) + Q_2(\omega \gamma), \ Q_{\omega,2}(\gamma) = Q_2(\gamma) \) and \( Q_{\omega,i}(\gamma) = 0 \) for \( i \geq 3 \).

Translated into differentials and Lie brackets we get

\[
(6.7) \quad d\omega = d + [\omega, -] \quad \text{and} \quad [-, -]_\omega = [-, -]_\omega.
\]

6.3. **Descent for \( L_\infty \)-morphisms.** Assume that \( g \) is an algebra over a DG-operad \( O \) with underlying graded operad \( \bar{O} \) and consider a set of \( \bar{O} \)-derivations of degree -1 \( (i_v)_v \in S \) on \( g \). Put \( L_v = di_v + i_v d \). This is a derivation of \( g \) of degree zero which commutes with \( d \).

Put

\[
(6.8) \quad g^S = \{ w \in g \mid \forall v \in S : i_v w = L_v w = 0 \}
\]

It is easy to see that \( g^S \) is an algebra over \( O \) as well. Informally we will call such a set of derivations \( (i_v)_v \in S \) an \( S \)-action.

**Remark 6.3.1.** By definition the notion of an \( S \)-action only depends on the graded structure of \( g \). However the construction of \( g^S \) also depends on the differential.

The following is a slightly strengthened version of [34, Prop. 7.6.3].
Proposition 6.3.2. Assume that \( \psi \) is an \( L_\infty \)-morphism \( g \to h \) between \( L_\infty \)-algebras equipped with an \( S \)-action as above. Assume that \( \psi \) commutes with the \( S \)-action in the sense that for all \( v \in S \),
\[
i_v \psi_n(w_1, \ldots, w_n) = \sum (-1)^{|w_1| + \cdots + |w_{l-1}| - (l-1)} \psi_n(w_1, \ldots, i_v(w_l), \ldots, w_n)
\]
Then \( \psi \) descends to an \( L_\infty \)-morphism \( \psi^S : g^S \to h^S \).

6.4. Compatibility with twisting. Assume that \( g, h \) are topological DG-Lie algebras and \( \psi \) is an \( L_\infty \)-morphism \( g \to h \). As in §6.2 we assume that \( g, h \) are complete filtered and \( \psi \) is compatible with the filtration. Our aim is to understand the behavior of \( S \)-actions under twisting.

Assume that \( g \) and \( h \) are equipped with a \( S \)-action and assume that \( \psi \) commutes with this action (as in Proposition 6.3.2). Let \( \omega \in F_{-1}g_1 \) be a solution to the Maurer-Cartan equation. Since twisting does not change the Lie bracket (see (6.7)), \( S \) acts on \( g_\omega \) and \( h_\omega \) as well. The following is [34, Prop. 7.7.1].

Proposition 6.4.1. Assume that for \( i \geq 2 \) and all \( v \in S, \gamma \in S^{i-1}(g[1]) \) we have
\[
(6.9) \quad \psi_i(i_v \omega \cdot \gamma) = 0
\]
Then \( \psi_\omega \) commutes with the \( S \)-action on \( g_\omega \) and \( h_\omega \).

7. Formality for Lie algebroids

In this section we prove Theorem 1.8. We first prove a more precise result in the ring case. To do so we use the existence of the desired \( L_\infty \)-quasi-isomorphism in the local case, and extend it to the ring case with the help of coordinate spaces constructed in the previous section. We end the proof by sheafifying the ring case, using appropriate functorial properties.

7.1. The formality in the ring case and its functorial properties.

Theorem 7.1. Let \( R \) be a \( k \)-algebra. Assume that \( L \) is a Lie-algebroid over \( R \) which is free of rank \( d \). There exists a canonical DG-Lie algebra \( T^L \) together with \( L_\infty \)-quasi-isomorphisms
\[
(7.1) \quad T^L_{\text{poly}}(R) \longrightarrow T^L \longrightarrow D^L_{\text{poly}}(R)
\]
such that the induced map
\[
(7.2) \quad T^L_{\text{poly}}(R) \longrightarrow H^*(D^L_{\text{poly}}(R))
\]
is given by the HKR-formula (4.1).

The DG-Lie algebra \( T^L \) and the quasi-isomorphisms in (7.1) are functorial in the following sense: assume that \( \phi : (R, L) \to (T, M) \) is an algebraic morphism of Lie algebroids which induces an isomorphism
\[
(7.3) \quad T \otimes_R L \cong M
\]
then there is an associated commutative diagram
\[
(7.4) \quad T^L_{\text{poly}}(R) \longrightarrow T^L \longrightarrow D^L_{\text{poly}}(R)
\]
\[
(7.5) \quad T^M_{\text{poly}}(T) \longrightarrow T^M \longrightarrow D^M_{\text{poly}}(T)
\]
and \( T^\phi = T^\phi \circ T^\phi \).

The proof of this theorem will take the greater part of the next two subsections.
7.2. The local formality quasi-isomorphism. Let $F = k[[t_1, \ldots, t_d]]$. Kontsevich (over the reals) and Tamarkin (over the rationals) construct an $L_\infty$-quasi-isomorphism [20, 31]
(7.4)\[U : T_{\text{poly}}(F) \to D_{\text{poly}}(F)\]
where $U_t$ is given by the HKR formula\(^5\)
(7.5)\[U_t(\partial_i \wedge \cdots \wedge \partial_i) = (-1)^{(p-1)/2} \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^{\sigma} \partial_{\sigma(i)} \wedge \cdots \wedge \partial_{\sigma(p)}\]
with $\partial_i = \partial / \partial t_i$. This quasi-isomorphism has two supplementary properties which are crucial for its extension to the global case.
(P4) $U_q(\gamma_1 \cdots \gamma_q) = 0$ for $q \geq 2$ and $\gamma_1, \ldots, \gamma_q \in T^{\text{poly}, 1}(F)$,\(^6\)
(P5) $U_q(\gamma \alpha) = 0$ for $q \geq 2$ and $\gamma \in \mathfrak{gl}_q(k) \subset T^{\text{poly}, 1}(F)$.
For Tamarkin’s quasi-isomorphism the fact that properties (P4) and (P5) hold has been proved in [16].

7.3. Proof of Theorem 1.8 in the ring case.

7.3.1. Resolutions. In this section we construct resolutions of $T^L_{\text{poly}}(R)$ and $D^L_{\text{poly}}(R)$. These are jet analogues of the Dolgushev-Fedosov resolutions in [3, Section 2].

Since the action of $UL_2$ on $C^{\text{aff}, L} \otimes R$, $JL$ commutes with $\nabla^{\text{aff}}$ we obtain morphisms of DG-Lie algebras:
(7.6)
\[
\begin{align*}
T^L_{\text{poly}}(R_2) & \to T_{\text{poly}, C^{\text{aff}, L}}(C^{\text{aff}, L} \otimes R, JL, 1^{\text{aff}}) \\
D^L_{\text{poly}}(R_2) & \to D_{\text{poly}, C^{\text{aff}, L}}(C^{\text{aff}, L} \otimes R, JL, 1^{\text{aff}})
\end{align*}
\]

Proposition 7.3.1. The morphisms in (7.6) are quasi-isomorphisms.

Proof. This follows from lemmas 4.3.4 and 5.3.1. \qed

7.3.2. The formality map on coordinate spaces. The local $L_\infty$-quasi-isomorphism
\[U : T_{\text{poly}}(F) \to D_{\text{poly}}(F)\]
extends linearly to an $L_\infty$-quasi-isomorphism
\[\tilde{U} : C^{\text{coord}, L} \otimes T_{\text{poly}}(F) \to C^{\text{coord}, L} \otimes D_{\text{poly}}(F)\]
One easily verifies that the canonical maps
(7.7)
\[
\begin{align*}
C^{\text{coord}, L} \otimes T_{\text{poly}}(F) & \to T_{\text{poly}, C^{\text{coord}, L}}(C^{\text{coord}, L} \otimes F) \\
C^{\text{coord}, L} \otimes D_{\text{poly}}(F) & \to D_{\text{poly}, C^{\text{coord}, L}}(C^{\text{coord}, L} \otimes F)
\end{align*}
\]
are isomorphisms of DG-Lie algebras. Thus we obtain a corresponding $L_\infty$-quasi-isomorphism
\[\tilde{U} : T_{\text{poly}, C^{\text{coord}, L}}(C^{\text{coord}, L} \otimes F) \to D_{\text{poly}, C^{\text{coord}, L}}(C^{\text{coord}, L} \otimes F)\,.
\]
In Section 5 (§ 5.2.1) we have constructed an isomorphism of $C^{\text{coord}, L}$-DG-algebras
\[\tilde{t} : (C^{\text{coord}, L} \otimes R, JL, 1^{\text{coord}}) \to (C^{\text{coord}, L} \otimes F, d + \omega)\,.
\]
Therefore we obtain an isomorphism of Lie algebras
(7.8)\[\tilde{t}^{-1} \circ \circ \tilde{t} : D_{\text{poly}, C^{\text{coord}, L}}(C^{\text{coord}, L} \otimes R, JL) \to D_{\text{poly}, C^{\text{coord}, L}}(C^{\text{coord}, L} \otimes F)\,.
\]
\(^5\)The sign $(-1)^{(p-1)/2}$ is not present in Kontsevich’s setting. In this paper we slightly modify Kontsevich’s quasi-isomorphism. See §9.
\(^6\)For degree reasons, this is always true if $q > 2$. 

The Hochschild differential on the left is sent to the Hochschild differential on the right. The differential \( [t, \omega] \) on the left is sent to \([d + \omega, -] \) on the right. Then it follows using (6.7) from \$6.2\) that \( t \) defines an isomorphism of DG-Lie algebras

\[
(7.9) \quad D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \cong D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F, d) \omega.
\]

Similarly we have an isomorphism of DG-Lie algebras

\[
(7.10) \quad T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \cong T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F, d) \omega.
\]

We now use the grading on \( T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \) and \( D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \) obtained from the \( C^{\text{coord}, L} \)-grading on \( C^{\text{coord}, L} \otimes F \) as a filtration. Thus

\[
F_{-n}(T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F)) = T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F)_{\geq n}
\]

and similarly for \( D_{\text{poly}} \). Since these filtrations are finite in each degree for the total gradings on \( T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \) and \( D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \) these DG-Lie algebras are graded complete. It is also clear that \( \tilde{U} \) is compatible with \( F \). Finally since \( \omega \in T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \) it follows that \( \omega \in F_{-1}(T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F)). \) Thus the twisting formalism exhibited in \$6.2\) applies and we obtain an \( L_{\infty} \)-morphism

\[
(7.11) \quad \tilde{U}_\omega : T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \to D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes F) \omega
\]

since by (P4) (using the notation of (6.5)) one has

\[
\omega' = \sum_{j \geq 1} \frac{1}{j!} \tilde{U}_j(\omega^j) = \tilde{U}_1(\omega) = \omega.
\]

Hence using (7.9) and (7.10) we have an \( L_{\infty} \)-morphism

\[
(7.12) \quad \gamma^{\text{coord}} : T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{coord}} 
\]

7.3.3. The formality map on affine coordinate spaces. First remark that \( \tilde{U}_L \) descends under the \( \mathfrak{g}_d(k) \)-action. Namely, given the facts that \( \tilde{U} \) clearly commutes with the \( \mathfrak{g}_d(k) \)-action (in the sense of Proposition 6.3.2) and that, using (5.7) and (P5), \( \tilde{U}_\gamma(\bar{i}_d(\omega) \cdot \gamma) = 0 \) for any \( \gamma \in \mathfrak{g}_d(k) \) and \( i \geq 2 \); we can apply the criteria given by Propositions 6.3.2 and 6.4.1 and obtain an \( L_{\infty} \)-morphism

\[
(7.13) \quad \gamma^{\text{aff}} : T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \to D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}}.
\]

Here the notation \( (-) \cdot \bar{i}_d \) is used in the sense of (6.8) and \( \mathfrak{g}_d(k) \) acts by the derivation of the \( \mathfrak{g}_d(k) \)-action on the factor \( \Omega_{\text{poly}, L} \otimes R \otimes R, \wedge R, L_1 \). There are morphisms of DG-Lie algebras

\[
(7.14) \quad T_{\text{poly}, \text{coord}, L}(C^{\text{aff}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \to T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}}
\]

\[
D_{\text{poly}, \text{coord}, L}(C^{\text{aff}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \to D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}}
\]

obtained by extending \( C^{\text{aff}, L} \)-linear poly-vector fields and poly-differential operators to \( C^{\text{coord}, L} \)-linear ones. We claim that these maps yield isomorphisms of DG-Lie algebras

\[
(7.15) \quad T_{\text{poly}, \text{coord}, L}(C^{\text{aff}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \to T_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \bar{i}_d \to D_{\text{poly}, \text{coord}, L}(C^{\text{aff}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \bar{i}_d \to D_{\text{poly}, \text{coord}, L}(C^{\text{coord}, L} \otimes R, JL, 1) \otimes \gamma^{\text{aff}} \bar{i}_d.
\]

Using Lemma 4.3.4 and using the fact that \( T^{L_2}(R_2) \) and \( D^{L_2}(R_2) \) are free \( R_2 \)-modules and that \( JL \) is a topologically free \( R_1 \)-module it is sufficient to prove that

\[
(7.16) \quad (\Omega_{\text{poly}, L} \otimes R, \wedge R_1) \otimes \gamma^{\text{aff}} = \Omega_{\text{aff}, L} 
\]

Using the notations of Section 5 the isomorphism (7.16) follows from

\[
\Omega^{\gamma^{\text{aff}}(k)} \cong \Omega^{\gamma^{\text{aff}}(k)}
\]

This follows easily from the fact that \( \mathfrak{g}_d(k) \) acts freely on \( \text{Spec} S^{\text{coord}} \).
Therefore (7.13) now yields an $L_{\infty}$-morphism 

$$\psi^\aff : T_{\poly, C^{\aff, L}}(C^{\aff, L} \otimes_R 1, \mathcal{J}L, 1^\aff \mathcal{V}) \longrightarrow D_{\poly, C^{\aff, L}}(C^{\aff, L} \otimes_R 1, \mathcal{J}L, 1^\aff \mathcal{V}) .$$

7.3.4. End of the proof. We have constructed $L_{\infty}$-morphisms

$$T_{\poly}^L (R_2) \xrightarrow{\approx} T_{\poly, C^{\aff, L}}(C^{\aff, L} \otimes_R 1, \mathcal{J}L) \xrightarrow{\psi^\aff} D_{\poly, C^{\aff, L}}(C^{\aff, L} \otimes_R 1, \mathcal{J}L) .$$

such that the horizontal maps are quasi-isomorphisms.

We put $t^L = D_{\poly, C^{\aff, L}}(C^{\aff, L} \otimes_R 1, \mathcal{J}L)$. Then the lower horizontal map in (7.17) yields the rightmost quasi-isomorphism in (7.1).

We will prove below that the composition

$$T_{\poly}^L (R_2) \rightarrow H^*(t^L) \xrightarrow{\approx-1} H^*(D_{\poly}^L (R_2))$$

coincides with the HKR-isomorphism. It follows in particular that the diagonal map in (7.17)

$$T_{\poly}^L (R_2) \rightarrow t^L$$

is an $L_{\infty}$-quasi-isomorphism as well. This is the leftmost quasi-isomorphism in (7.1). We leave to the reader the tedious but straightforward verification of the functoriality of $t^L$.

To prove that the map on cohomology is given by the HKR-map we regard the complexes $C^{\aff, L}$ and $C^{\coord, L}$ is horizontal. We write the coordinates for the double grading as couples $(p, q)$ where $p$ is the column index.

According to (6.4) $\mathcal{U}^{\omega, 1}$ is given by

$$\hat{\mathcal{U}}^{\omega, 1}(\gamma) = \sum_{j \geq 0} \frac{1}{j!} \hat{U}_{j+1}(\omega^j \gamma) .$$

Now $\hat{U}_{j+1}$ is homogeneous for the column grading and of degree $1 - (j + 1)$ for the Hochschild grading (the row grading), thus it has bidegree $(0, -j)$. Since $\omega$ lives in $C_1^{\coord, L} \otimes T_{\poly}^0(F)$ it has bidegree $(1, 0)$, and hence $\hat{U}_{j+1}(\omega^j \gamma)$ has bidegree $(j, -j)$.

Let $\hat{U}_{\omega, 1}$ be the component of $\hat{\mathcal{U}}^{\omega, 1}$ indexed by $j$ in (7.18).

**Lemma 7.3.2.** We have the following commutative diagram

$$T_{\poly}^L (R_2) \longrightarrow C^{\coord, L} \otimes T_{\poly}(F) \xrightarrow{\mu} C^{\coord, L} \otimes D_{\poly}(F)$$

where the horizontal arrows are inclusions obtained from the action by derivations of $L_2$ on $C^{\coord, L} \otimes_R 1, \mathcal{J}L \cong C^{\coord, L} \otimes F$ (see (5.1)) and $\mu$ is the HKR-map (4.1).

**Proof.** This is almost a tautology. Let $l_1, \ldots, l_n \in L$ and denote by $\delta_i$ the derivation on $C^{\coord, L} \otimes F$ corresponding to $l_i$. Then

$$\hat{U}_{\omega, 1}(\delta_1 \wedge \cdots \wedge \delta_n) = \hat{U}_1(\delta_1 \wedge \cdots \wedge \delta_n) = (-1)^{n(n-1)/2} \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \delta_{\sigma(1)} \otimes \cdots \otimes \delta_{\sigma(n)} .$$

This implies the commutativity of (7.19). \qed
Since the maps in (7.14) are inclusions $V_{1}^{\text{aff}}$ has the same grading properties as $\hat{U}_{d,1}$. In particular it maps $T_{\text{poly},C^{\text{aff}},L}(C^{\text{aff}},L \otimes_{R_{1}} JL)_{p,q}$ to $\oplus_{j}D_{\text{poly},C^{\text{aff}},L}(C^{\text{aff}},L \otimes_{R_{1}} JL)_{p+j,q-j}$. Let $V_{1}^{j}$ be the component of $V_{j}$ indexed by $j$ in this decomposition. Thus we obtain a commutative diagram

\[
\begin{array}{ccc}
T_{\text{poly}}^{L_{2}}(R_{2}) & \longrightarrow & T_{\text{poly},C^{\text{aff}},L}(C^{\text{aff}},L \otimes_{R_{1}} JL) \\
\downarrow \phi & & \downarrow V_{1}^{\text{aff},0} \\
D_{\text{poly}}^{L_{2}}(R_{2}) & \longrightarrow & D_{\text{poly},C^{\text{aff}},L}(C^{\text{aff}},L \otimes_{R_{1}} JL)
\end{array}
\]

The following lemma ends the proof of the theorem (see [40, Thm. 7.1]).

**Lemma 7.3.3.** $V_{1}^{\text{aff},0}$ and $V_{1}^{\text{aff}}$ induce the same maps on cohomology.

**Proof.** We filter $T_{\text{poly},C^{\text{aff}},L}(C^{\text{aff}},L \otimes_{R_{1}} JL)$ and $D_{\text{poly},C^{\text{aff}},L}(C^{\text{aff}},L \otimes_{R_{1}} JL)$ according to the column index.

The $E_{1}$ term of the resulting spectral sequences consists of the cohomology of the columns. Using (4.16) we have to compute the cohomology of $(C^{\text{aff}},L \otimes_{R_{1}} JL) \otimes_{R_{2}} T_{\text{poly}}^{L_{2}}(R_{2})$ and $(C^{\text{aff}},L \otimes_{R_{1}} JL) \otimes_{R_{2}} D_{\text{poly}}^{L_{2}}(R_{2})$ for the second factor. We obtain $(C^{\text{aff}},L \otimes_{R_{1}} JL) \otimes_{R_{2}} T_{\text{poly}}^{L_{2}}(R_{2})$ and $(C^{\text{aff}},L \otimes_{R_{1}} JL) \otimes_{R_{2}} H(D_{\text{poly}}^{L_{2}}(R_{2}))$ (the latter because $D_{\text{poly}}^{L_{2}}(R_{2})$ is a complex consisting of filtered projective $R_{2}$-modules with filtered projective cohomology).

Using Lemma 5.3.1 we obtain that the $E_{2}$ terms are given by $T_{\text{poly}}^{L_{2}}(R_{2})$ and $H(D_{\text{poly}}^{L_{2}}(R_{2}))$. It is now clear that $V_{1}^{\text{aff},0}$ and $V_{1}^{\text{aff}}$ induce indeed the same map on cohomology. \qed

### 7.4. Proof of Theorem 1.8 in the sheaf case

As indicated in the introduction we can prove a result which slightly more general than Theorem 1.8. We work over a ringed site $(C, O)$ and $\mathcal{L}$ is a Lie algebroid locally free of rank $d$ on $(C, O)$. The DG-Lie algebras $T_{\text{poly}}^{\mathcal{L}}(O)$, $D_{\text{poly}}^{\mathcal{L}}(O)$ are obtained by sheafifying the presheaves

\[
\begin{align*}
U & \mapsto T_{\text{poly}}^{\mathcal{L}(U)}(O(U)) \\
U & \mapsto D_{\text{poly}}^{\mathcal{L}(U)}(O(U))
\end{align*}
\]

for $U \in \text{Ob}(C)$.

**Theorem 7.4.1.** There is an isomorphism between $T_{\text{poly}}^{\mathcal{L}}(O)$ and $D_{\text{poly}}^{\mathcal{L}}(O)$ in $\text{HoLieAlg}(O)$, the homotopy category of sheaves of DG-Lie algebras, which induces the HKR-isomorphism on cohomology.

**Proof.** We replace $\mathcal{C}$ with the full subcategory consisting of $U \in \mathcal{C}$ such that there is an isomorphism $\mathcal{L} \mid U \cong (O \mid U)^d$ (this does not change the category of sheaves).

If $p : U \rightarrow V$ is now a map in $\mathcal{C}$ then since $\mathcal{L}(V) \cong (O(V))^d$, $\mathcal{L}(U) \cong (O(U))^d$ we have that the restriction morphism

\[
p^* : (O(V), \mathcal{L}(V)) \rightarrow (O(U), \mathcal{L}(U))
\]

satisfies the condition (7.2), i.e. $O(U) \otimes_{O(V)} \mathcal{L}(V) \cong \mathcal{L}(U)$.

Put $pT_{\text{poly}}^{\mathcal{L}}(U) = t^{\mathcal{L}(U)}$ where $t^{\mathcal{L}(U)}$ is as in Theorem 7.1. Then $pT_{\text{poly}}^{\mathcal{L}}$ is a presheaf of DG-Lie algebras. Let $pT_{\text{poly}}^{\mathcal{L}}(O)$ and $pD_{\text{poly}}^{\mathcal{L}}(O)$ be respectively the presheaves of DG-Lie algebras of $\mathcal{L}$-poly-vector fields and $\mathcal{L}$-poly-differential operators.

From the commutative diagram (7.3) we now deduce the existence of $L_{\infty}$-quasi-isomorphisms of presheaves

\[
(7.21) \quad pT_{\text{poly}}^{\mathcal{L}}(O) \rightarrow pT_{\text{poly}}^{\mathcal{L}} \leftarrow pD_{\text{poly}}^{\mathcal{L}}(O)
\]

Let $t^{\mathcal{L}}$ be the sheafification of $pT_{\text{poly}}^{\mathcal{L}}$. Sheafifying (7.21) finishes the proof. \qed
8. Atiyah classes and jet bundles

In this section we relate Atiyah classes to jet bundles. That this is possible is well known (see e.g. [18, §4]) although we could not find the exact result we need (Proposition 8.4.2 below) in the literature.

8.1. Reminder. We define $(\mathcal{C}, \mathcal{O}, \mathcal{L})$ as in §7.4. Let $\mathcal{E}$ be an arbitrary $\mathcal{O}$-module. The Atiyah class $A_{\mathcal{L}}(\mathcal{E}) \in \text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^∗ \otimes \mathcal{E})$ is the obstruction against the existence of an $\mathcal{L}$-connection (not necessarily flat) on $\mathcal{E}$.

Let us briefly recall how $A_{\mathcal{L}}(\mathcal{E})$ is constructed. By (4.8) we have $J^1\mathcal{L} = \mathcal{O}_1 \oplus \mathcal{L}^∗ = \mathcal{O}_2 \oplus \mathcal{L}^∗$ as $\mathcal{O}_1$ and $\mathcal{O}_2$-algebras.

We consider the short exact sequence of $\mathcal{O}_1$-modules

\[(8.1) \quad 0 \to \mathcal{L}^∗ \otimes_{\mathcal{O}} \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E} \to \mathcal{E} \to 0\]

The class of this sequence in $\text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^∗ \otimes \mathcal{E})$ is $A_{\mathcal{L}}(\mathcal{E})$. To see that this is the obstruction against the existence of a connection let $\beta_0 : \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E}$ be the canonical splitting (as sheaves of abelian groups) of (8.1) obtained from the decomposition $J^1\mathcal{L} = \mathcal{O}_2 \oplus \mathcal{L}$. Then any splitting $\beta : \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E}$ as $\mathcal{O}_1$ modules yields a connection $\nabla : \mathcal{E} \to \mathcal{L}^∗ \otimes_{\mathcal{O}} \mathcal{E}$ given by $\beta - \beta_0$. It is easy to see that this construction is reversible.

Taking powers and symmetrizing we obtain an element $a(\mathcal{E}) \in \text{Ext}^n_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^n \mathcal{L}^* \otimes \mathcal{E})$. The $n$'th (scalar) Atiyah class $\alpha_n(\mathcal{E}) \in H^n(\mathcal{C}, \wedge^n \mathcal{L}^*)$ of $\mathcal{E}$ is the trace of $a(\mathcal{E})^n$.

8.2. Atiyah classes: algebraic background. We need some functoriality properties of the Atiyah class. To deduce these cleanly we work in a somewhat more abstract setting and introduce some ad hoc terminology.

Let $\text{Sh}^{bi}(\mathcal{C})$ be the category of sheaves of abelian groups on $\mathcal{C}$ graded by $\mathbb{Z}^2$. If $\mathcal{F} \in \text{Sh}^{bi}(\mathcal{C})$ and $f$ is a section of $\mathcal{F}_{i,j}$ then $|f| = i + j$ is the total degree of $f$. As always apply the Koszul sign convention with respect to total degree.

The category $\text{Sh}^{bi}(\mathcal{C})$ is equipped with two obvious shift functors each of total degree one

\[\mathcal{F}[1]_{i,j} = \mathcal{F}_{i+1,j}\]
\[\mathcal{F}(1)_{i,j} = \mathcal{F}_{i,j+1}\]

Definition 8.2.1. (1) A bigraded $\text{DG}$-algebra on $\mathcal{C}$ is a bigraded sheaf of algebras $\mathcal{A}$ on $\mathcal{C}$ equipped with a derivation $d_{\mathcal{A}}$ of degree $(1,0)$ such that $d_{\mathcal{A}}^2 = 0$.

(2) A $\text{dDG}$-algebra $\mathcal{A}$ on $\mathcal{C}$ is a bigraded sheaf of DG-algebras on $\mathcal{C}$ equipped with an additional derivation $d_{\mathcal{A}}$ of degree $(0,1)$ such that $d_{\mathcal{A}}d_{\mathcal{A}} + d_{\mathcal{A}}d_{\mathcal{A}} = 0$.

(3) Assume that $\mathcal{A}$ is a bigraded DG-algebra. A $\text{DG}$-$\mathcal{A}$ module is a bigraded sheaf of $\mathcal{A}$-modules $\mathcal{M}$ equipped with an additive map $d_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ of degree $(1,0)$ such that $d_{\mathcal{M}}^2 = 0$ and such that

\[d_{\mathcal{M}}(am) = d_{\mathcal{A}}(a)m + (-1)^{|a|}a d_{\mathcal{M}}(m)\]

for $a, m$ homogeneous sections of $\mathcal{A}$ and $\mathcal{M}$. We denote the category of $\text{DG}$-modules over $\mathcal{A}$ by $\text{DGMod}(\mathcal{A})$.

(4) Assume that $\mathcal{M}$ is DG-module over a dDG-algebra $\mathcal{A}$. Then a connection on $\mathcal{M}$ is an additive map $d_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ of degree $(0,1)$ such that

\[d_{\mathcal{M}}(am) = d_{\mathcal{A}}(a)m + (-1)^{|a|}a d_{\mathcal{M}}(m)\]

(5) The functors $?^[1]$ and $?^(1)$ change the signs of both $d_{\mathcal{M}}$ and $\bar{d}_{\mathcal{M}}$, when applicable.

(6) Assume that $\mathcal{M}$ is a DG-$\mathcal{A}$-module over a dDG-algebra $\mathcal{A}$, equipped with a connection. Then the curvature of $\mathcal{M}$ is defined as $R_{\mathcal{M}} = - (d_{\mathcal{M}}d_{\mathcal{M}} + d_{\mathcal{M}}d_{\mathcal{M}})$. This is a map $R_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}(1)[1]$ in $\text{DGMod}(\mathcal{A})$.

(7) The derived category of $\text{DGMod}(\mathcal{A})$, equipped with the shift functor $?^[1]$, is denoted by $D(\mathcal{A})$. 

}
Example 8.2.2. Let $A \to B$ be a morphism of sheaves of commutative DG-algebras. Then $\Omega_{B/A}$ is a dDG-algebra. The bigrading comes from the internal (coming from $B$) and external (exterior) degrees. The degree $(0, 1)$ derivation $d$ is the De Rham differential and the degree $(0, 1)$ derivation $\bar{d}$ is characterised by the property that it commutes with $d$ and that it coincides with $d_B$ on $B = \Omega^0_{B/A}$.

Assume that $A$ is a dDG-algebra. We define a bigraded DG-algebra.

$$J^1 A = A \oplus A \epsilon$$

where $\epsilon$ satisfies $\bar{d}_A(\epsilon) = 0 = \epsilon^2$, has degree $(0, -1)$ and $a \epsilon = (-1)^{|a|} a \epsilon$

We have two algebra morphisms commuting with $\bar{d}_A$:

$$i_1 : A \to J^1 A : a \mapsto a$$

$$i_2 : A \to J^1 A : a \mapsto a + \epsilon d_A(a)$$

We view $J^1 A$ as a DG-$A$-bimodule via $i_1$, $i_2$.

We get an associated exact sequence of $A$-$A$ bimodules

$$0 \to A \epsilon \to J^1 A \to A \to 0$$

(8.2)

Let $M \in \text{DGMod}(A)$. Tensoring (8.2) on the right by $M$ we obtain an exact sequence in $\text{DGMod}(A)$

$$0 \to M(1) \to J^1 A \otimes_A M \to M \to 0$$

(8.3)

with

$$M(1) \to J^1 A \otimes_A M : n \mapsto \epsilon \otimes n$$

$$J^1 A \otimes_A M \to M : (a + b \epsilon) \otimes m \mapsto am$$

Definition 8.2.3. Let $M \in \text{DGMod}(A)$. The Atiyah class $A(M)$ of $M$ is the element of $\text{Hom}^1_{\text{D}(A)}(M, M(1))$ representing the exact sequence (8.3).

Lemma 8.2.4. If $M$ has a connection then $A(M) = R_M$. In other words $A(M)$ is represented by an actual map

$$R_M : M \to M(1)[1]$$

of bigraded $A$-modules.

Proof. If $M$ has a connection $d_M$ then the map

$$\beta : M \to J^1 A \otimes_A M : m \mapsto 1 \otimes m - \epsilon \otimes d_M(m)$$

defines a right splitting of (8.3) as graded $A$-modules. The corresponding left splitting is

$$\alpha : J^1 A \otimes_A M \to M(1) : 1 \otimes m + \epsilon \otimes n \mapsto d_M(m) + n$$

Since (8.3) is split its corresponding class in $\text{Hom}^1_{\text{D}(A)}(M, M(1))$ is given by $-\alpha d_M \beta$. One computes that this is equal to $R_M$. \hfill \Box

Lemma 8.2.5. The Atiyah class is functorial in the following sense. Assume that we have a morphism of dDG-algebras $\theta : A \to B$ and DG-modules $M$, $N$ over $A$ and $B$ as well as an additive map $\psi : M \to N$ of degree zero which is compatible with the differentials and $\theta$

\footnote{To see this one should think of a degreewise split exact sequence as a shift to the left of a standard triangle constructed from a mapping cone. See [17, I§2].}
in the sense that $\psi \circ d_M = d_N \circ \phi$ and $\psi(am) = \theta(a)\psi(m)$. Then the following diagram is commutative in $D(A)$

\[
\begin{array}{ccc}
M & \xrightarrow{A(M)} & M[1][1] \\
\downarrow \psi & & \downarrow \psi \\
A\mathcal{N} & \xrightarrow{A(\mathcal{N})} & A\mathcal{N}[1][1]
\end{array}
\]

**Proof.** This follows from the functoriality of the exact sequence (8.3).

### 8.3. Scalar Atiyah classes

Let $A$ be a dDG-algebra on $C$ and let $\mathcal{M} \in \text{DGMod}(A)$. We assume in addition that $\mathcal{M}$ is locally free of constant rank $e$ over $C$. I.e. the topology on $C$ has a basis $B$ such that for $U \in B$ we have that $\mathcal{M}_U \cong A_U^{d\times e}$ as bigraded $A$-modules. We may now view $A(M)^n$ as an element of $\text{Hom}^{\text{B}}_{D(A)}(\mathcal{M}, \mathcal{M}(n))$, or since $\mathcal{M}$ is locally free, as an element of

$$H^n(C, \text{End}_A(\mathcal{M})^*, n)$$

where $H^n$ denotes hypercohomology and $\text{End}_A(\mathcal{M})^* \otimes A^n$ is equipped with the differential $[d_M, -]$.

It is easy to check locally that the trace map

$$\text{Tr} : \text{End}_A(\mathcal{M}) \rightarrow A$$

is in $\text{DGMod}(A)$. Thus we obtain a map on hypercohomology

$$\text{Tr} : H^n(C, \text{End}_A(\mathcal{M})^* \otimes A^n) \rightarrow H^n(C, A^*)$$

We call

$$a_n(M) = \text{Tr}(a(M)^n) \in H^n(C, A^*)$$

the $n$'th (scalar) Atiyah class of $M$.

**Lemma 8.3.1.** Assume that we have a morphism $\theta : A \rightarrow B$ of dDG-algebras and assume that $M \in \text{DGMod}(A)$ is locally free of rank $e$. Put $N = B \otimes_A M$. Then $N \in \text{DGMod}(B)$ is locally free of rank $e$. We have

$$a_n(N) = H^n(\theta)(a_n(M))$$

where $H^n(\theta)$ is the natural map

$$H^n(\theta) : H^n(C, A^*) \rightarrow H^n(C, B^*)$$

**Proof.** The commutative diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{A(M)} & \mathcal{M}[n](n) \\
\downarrow & & \downarrow \\
\mathcal{N} & \xrightarrow{A(N)} & \mathcal{N}[n](n)
\end{array}
\]

obtained from Lemma 8.3.1 may be translated into saying that $A(N)^n$ is the image of $A(M)^n$ under the induced map

$$H^n(C, \text{End}_A(\mathcal{M})^* \otimes A^n) \xrightarrow{B \otimes_A -} H^n(C, \text{End}_B(N)^* \otimes A^n)$$

One verifies locally that there is a commutative diagram of DG-modules

\[
\begin{array}{ccc}
\mathcal{E}_{\text{nd}_A(M)} & \xrightarrow{B \otimes_A -} & \mathcal{E}_{\text{nd}_B(N)} \\
\downarrow \text{Tr} & & \downarrow \text{Tr} \\
A & \xrightarrow{-} & B
\end{array}
\]

This finishes the proof. □
Example 8.3.2. We explain how the Atiyah class constructed in §8.1 fits into this framework. We define \( \mathcal{A} \) as the De Rham complex \( (\wedge L_1^*, d) \) and put it in degrees \((0,\ast)\). We define \( \mathcal{M} = \mathcal{A} \otimes \mathcal{O} \mathcal{E} \).

The Atiyah class \( A_\mathcal{L}(\mathcal{E}) \overset{\text{def}}{=} A(\mathcal{M}) \) now becomes an element of

\[
\text{Ext}^1_{\text{gr}(\mathcal{A})}(\mathcal{M}, \mathcal{M}(1)) = \text{Ext}^1_{\text{gr}(\mathcal{A})}(\mathcal{A} \otimes \mathcal{O} \mathcal{E}, (\mathcal{A} \otimes \mathcal{O} \mathcal{E})(1)) \\
= \text{Ext}^1_{\text{gr}(\mathcal{O})}(\mathcal{E}, (\mathcal{A} \otimes \mathcal{O} \mathcal{E})(1)) \\
= \text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes \mathcal{O} \mathcal{E})
\]

It is easy to see that \( A_\mathcal{L}(\mathcal{E}) \in \text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes \mathcal{O} \mathcal{E}) \) represents the part of degree zero of \((8.3)\). This is

\[
0 \to \mathcal{L}^* \otimes \mathcal{O} \mathcal{E} \to J^1 \mathcal{L} \otimes \mathcal{O} \mathcal{E} \to \mathcal{E} \to 0
\]

Hence \( A_\mathcal{L}(\mathcal{E}) \) coincides with our previous definition. It is easy to deduce from this that we also get the same \( a_{n,\mathcal{L}}(\mathcal{E}) \).

8.4. Atiyah classes from jet bundles. We assume we are in the setting from §8.1. As outlined in the previous section we will work with bigraded sheaves.

We first consider the \( L_2\)-De Rham complex \( \wedge L_2^* \) as a dDG-algebra concentrated in degrees \((0,\ast)\) with \( \mathcal{d} = 0 \). We then let \( C \) be a commutative DG-algebra such that \( J\mathcal{L} \) is equipped with a flat \( C \)-connection \( \nabla \). Thus \( (C \wedge \mathcal{O}_1, J\mathcal{L}, \nabla) \) becomes a DG-algebra (actually a DG-C-algebra). From Lemma 8.3.6 we obtain a morphism of dDG-algebras

\[
\theta : (\wedge L_2^*, \mathcal{d} = 0) \to \Omega_{C \wedge \mathcal{O}_1, J\mathcal{L}/C}
\]

If we set \( \mathcal{M} = (\wedge L_2^* \otimes \mathcal{O}_2, \mathcal{d} = 0) \in \text{DGMod}(\wedge L_2^*, \mathcal{d} = 0) \) then we obtain from Lemma 8.3.1 and Example 8.3.2

\[
(8.4) \quad \mathbb{H}(\theta)(a_n(\mathcal{L})) = \mathbb{H}(\theta)(a_n(\mathcal{M})) = a_n(\Omega_{C \wedge \mathcal{O}_1, J\mathcal{L}/C} \wedge \mathcal{L}^2) \overset{\text{def}}{=} a_n(\Omega_{C \wedge \mathcal{O}_1, J\mathcal{L}/C} \mathcal{O} \mathcal{E}) \cdot \hat{\mathcal{N}}.
\]

Finally recall that \( (C \wedge \mathcal{O}_1, J\mathcal{L}) \wedge \mathcal{O}_2 \mathcal{L}_2 \overset{\text{def}}{=} \mathcal{D} \mathcal{E} \mathcal{C}(C \wedge \mathcal{O}_1, J\mathcal{L}) \), therefore

\[
\hat{\mathcal{N}} \overset{\text{def}}{=} (\Omega_{C \wedge \mathcal{O}_1, J\mathcal{L}/C}) \wedge \mathcal{O} \mathcal{L} \mathcal{L} \mathcal{E} \mathcal{C}(C \wedge \mathcal{O}_1, J\mathcal{L}) \, \hat{\mathcal{N}}.
\]

The fact that \( \mathbb{H}(\theta)(a_n(\mathcal{L})) = a_n(\hat{\mathcal{N}}) \) provides a mean to compute \( a_n(\mathcal{L}) \) if we can compute \( a_n(\hat{\mathcal{N}}) \). The latter can be accomplished if we can put a connection on \( \hat{\mathcal{N}} \) (see Lemma 8.2.4)

Lemma 8.4.1. Assume that there is an isomorphism of graded \( C \)-algebras

\[
(8.5) \quad \pi : C \wedge \mathcal{O}_1, J\mathcal{L} \to C \wedge \mathcal{O} \hat{\mathcal{S}} \mathcal{L}^*
\]

which induces the identity map \( C \otimes \mathcal{O} \mathcal{gr} J\mathcal{L} \overset{\text{def}}{=} C \otimes \mathcal{O} \mathcal{S} \mathcal{L}^* \). Then \( \hat{\mathcal{N}} \), as introduced above, has a connection.

Proof. We use the isomorphism \((8.5)\) to transport the differential \( \nabla \) (defined on \( B \overset{\text{def}}{=} C \wedge \mathcal{O} J\mathcal{L} \)) to a differential \( \hat{\nabla} \) on \( \hat{B} \overset{\text{def}}{=} C \wedge \mathcal{O} \hat{\mathcal{S}} \mathcal{L}^* \). Note that this differential does not have a simple expression. As for \( \nabla \), we extend \( \hat{\nabla} \) to a unique differential on \( \Omega_{B/C} \overset{\text{def}}{=} C \otimes \Omega \hat{\mathcal{S}} \mathcal{L}^*/\mathcal{O} \) in such a way that it commutes with the De Rham differential \( \hat{\mathcal{d}}_{\text{DR}} \).

We now put

\[
\hat{\mathcal{N}}_0 = \mathcal{D} \mathcal{E} \mathcal{C}(\hat{B}) \overset{\text{def}}{=} \hat{B} \wedge \mathcal{O} \mathcal{L} \quad \text{and} \quad \hat{\mathcal{N}} = \Omega_{B/C} \hat{B} \hat{\mathcal{N}}_0 = \Omega_{B/C} \wedge \mathcal{O} \mathcal{L}.
\]

The isomorphism \((8.5)\) between \( B \) and \( \hat{B} \) yields isomorphims between \( \Omega_{B/C} \mathcal{L} \) and \( \Omega_{B/C} \hat{\mathcal{L}} \), between \( \mathcal{N}_0 \) and \( \hat{\mathcal{N}}_0 \) and between \( \mathcal{N} \) and \( \hat{\mathcal{N}} \). We can now define a connection on \( \hat{\mathcal{N}} = \Omega_{B/C} \wedge \mathcal{O} \mathcal{L} \) by putting

\[
d_{\hat{\mathcal{N}}}(b \otimes l) = \hat{\mathcal{d}}_{\text{DR}}(b) \otimes l
\]
It is easy to see that this is well-defined. Transporting accross the isomorphism $N \cong \hat{N}$ yields a connection $d_N$ on $N$.

The DG-algebras $C^{\text{coord} \cdot \mathcal{L}}$ and $C^{\text{aff} \cdot \mathcal{L}}$ are equipped with a canonical map $\wedge \mathcal{L}^* \to C^{\text{coord} \cdot \mathcal{L}}$ and $\wedge \mathcal{L}^* \to C^{\text{aff} \cdot \mathcal{L}}$ as follows from the definitions in §5.2.1 and §5.3.

In addition condition (8.5) applies with $C = C^{\text{aff} \cdot \mathcal{L}}$ and $C = C^{\text{coord} \cdot \mathcal{L}}$, see (5.13). We have natural morphisms

$$\wedge \mathcal{L}_2 \overset{\delta}{\rightarrow} C^{\text{aff} \cdot \mathcal{L}} \hat{\otimes} \Omega_{J \mathcal{L}/\mathcal{O}_1} \overset{\psi}{\rightarrow} C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \Omega_{J \mathcal{L}/\mathcal{O}_1} \overset{h_a}{\cong} C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \Omega_{\mathcal{F}/k}$$

Below we decorate notations referring to $C^{\text{aff} \cdot \mathcal{L}}$ and $C^{\text{coord} \cdot \mathcal{L}}$ by superscripts “aff” and “coord” respectively. For example we define $N^{\text{aff}}$ and $N^{\text{coord}}$ like $N$ in the above discussion where we replace $C$ by $C^{\text{aff} \cdot \mathcal{L}}$ and $C^{\text{coord} \cdot \mathcal{L}}$.

**Proposition 8.4.2.** Write the Maurer-Cartan form (see (5.6)) as

$$\omega = \sum_{i, \alpha} \eta^\alpha_i \omega^i_\alpha \partial_i$$

with $\eta^\alpha_i \in (C^{\text{coord} \cdot \mathcal{L}})_1$, $\omega^i_\alpha \in F$. Then we have as elements of complexes

$$(\mu \psi)(a_n(N^{\text{aff}})) = \text{Tr}(\Xi^n)$$

where $\Xi$ is the matrix with entries

$$\sum_{\alpha} \eta^\alpha_i d_F(\partial_j \omega^i_\alpha)$$

Furthermore as cohomology classes we have

$$\mathbb{H}(\theta)(a_n(\mathcal{L})) = a_n(N^{\text{aff}})$$

**Proof.** The identity (8.6) is (8.4).

We can use the canonical connection on $N^{\text{aff}}$ and $N^{\text{coord}}$ exhibited in the proof of Lemma 8.4.1 to compute $a_n(N^{\text{aff}})$ and $a_n(N^{\text{coord}})$ (using Lemma 8.2.4). Since these connections are compatible we get

$$a_n(N^{\text{coord}}) = \psi(a_n(N^{\text{aff}}))$$

as elements of complexes.

We now compute $a_n(N^{\text{coord}})$ explicitly. We have identifications (e.g. (5.14) and (5.15))

$$C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{O}_1, \mathcal{L} \cong C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{O} \hat{S} \mathcal{L}_1 \cong C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} F$$

Using these identifications we have

$$N^{\text{coord}} = C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{O} \hat{\Omega}_{\hat{S} \mathcal{L}^*/\mathcal{O}} \hat{\otimes} \mathcal{O} \mathcal{L} = C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{O} \hat{\Omega}_F \hat{\otimes} \sum_i k \partial_i$$

where $\partial_i = \partial / \partial t_i$. We have

$$\partial_1, \ldots, \partial_n \in C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{L} \subset C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{O} \hat{\Omega}_{\hat{S} \mathcal{L}^*/\mathcal{O}} \hat{\otimes} \mathcal{L}$$

and since $d_{N^{\text{coord}}}$ is zero on $C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{L}$ we deduce

$$d_{N^{\text{coord}}}(\partial_i) = 0$$

For further computation we use the identification

$$N^{\text{coord}} = C^{\text{coord} \cdot \mathcal{L}} \hat{\otimes} \mathcal{O} \hat{\Omega}_F \hat{\otimes} \sum_i k \partial_i$$

where $d_{N^{\text{coord}}}$ acts as

$$d_{N^{\text{coord}}}(c \hat{\otimes} \omega \hat{\otimes} \partial_i) = (-1)^{|\omega|} c \hat{\otimes} d_F \omega \hat{\otimes} \partial_i$$
Remember from §5.2.1 that the differential \( \partial^\text{coord} = 1 + \sum_i \eta_i \omega_i^\text{a} \partial_i \) is given by
\[
d_{\text{coord},F} \otimes 1 + \sum_{i,\alpha} \eta_i \omega_i^\alpha \partial_i
\]
where we think of \( \partial_i \) as a Lie derivative. We compute
\[
\partial^\text{coord} (\partial_j) = [d_{\text{coord},F} \otimes 1 + \sum_{i,\alpha} \eta_i \omega_i^\alpha \partial_i, \partial_j]
= \sum_{i,\alpha} (\eta_i \partial_j \omega_i^\alpha) \partial_i
\]
and hence
\[
R^\text{coord} (\partial_j) = -d_{\text{coord},F} \partial^\text{coord} + d_{\text{coord},F} \partial^\text{coord})(\partial_j)
= -d_{\text{coord},F} (\sum_{i,\alpha} \eta_i \partial_j \omega_i^\alpha) \partial_i
= \sum_{i,\alpha} \eta_i d_F (\partial_j \omega_i^\alpha) \partial_i
\]
Thus \( \mu(a_n(N^\text{coord})) = \text{Tr}(\Xi^n) \) where \( \Xi \) is as in the statement of the proposition. This finishes the proof. \( \square \)

9. The Kontsevich local formality quasi-isomorphism

9.1. The \( L_\infty \)-morphism. In this section we assume that \( k \) contains the reals and we describe the exact form of the Kontsevich local formality morphism.

As above let \( F = k[[t_1, \ldots, t_d]] \) and \( T^\text{poly}(F), D^\text{poly}(F) \) are respectively the Lie algebras of poly-vector fields and poly-differential operators over \( F \). We equip \( T^\text{poly}(F) \) and \( D^\text{poly}(F) \) with the shifted Gerstenhaber structures introduced in §4.2.2. For \( \gamma \in T^\text{poly}_n(F) \) we put
\[
\gamma^{t_1, \ldots, t_{n+1}} = (dt_{i_1} \wedge \cdots \wedge dt_{i_{n+1}})
\]
where \( (\cdot, \cdot) \) is the pairing introduced in (4.2).

The Kontsevich local formality isomorphism \( U : T^\text{poly}(F) \to D^\text{poly}(F) \) is defined as follows.
We put
\[
U_n = \sum_{m \geq 0} \sum_{F \in G_{n,m}} W_F U_F
\]
where the \( W_F \) are some coefficients to be defined below and where \( G_{n,m} \) is a set of directed graphs \( \Gamma \) described as follows

1. There are \( n \) vertices of the “first type” labeled by \( 1, \ldots, n \).
2. There are \( m \) vertices of the “second type” labeled by \( 1, \ldots, m \).
3. The vertices of the second type have no outgoing arrow.
4. There are no loops and double arrows.
5. There are \( 2n + m - 2 \) edges.
6. All edges carry a distinct label.

For use below we also introduce \( G_{n,m,\epsilon} \) which is defined in the same way except that the number of edges of the graphs should be equal to \( 2n + m - 2 - \epsilon \). The number of edges in a graph is denoted by \( |\Gamma|\).

For a vertex \( v \) of \( \Gamma \) we denote the incoming and outgoing edges of \( v \) by \( \text{In}(v) \) and \( \text{Out}(v) \) respectively. Let \( \Gamma_i \) be the vertices of the \( i \)th kind for \( i = 1,2 \). Let \( \gamma_i \in T^\text{poly}(F) \) and put \( k_i = |\gamma_i| \). By definition \( U_{\Gamma}(\gamma_1 \cdots \gamma_n) \) is zero unless \( |\text{Out}(i)| = k_i + 1 \). In that case
\[
U_{\Gamma}(\gamma_1 \cdots \gamma_n)(f_1 \cdots f_m) = \prod_{v \in \Gamma_1} \partial_{r_1} \cdots \partial_{r_d} \gamma_{a_1}^{s_1} \cdots \gamma_{a_{k+1}}^{s_{k+1}} \prod_{v \in \Gamma_2} \partial_{l_1} \cdots \partial_{l_d} f_v
\]
where we assume that the ordering on the labels $s_1 \cdots s_d$ is such that $s_1 < \cdots < s_{k_v+1}$.

The coefficients $W_G$ are defined as integrals over configuration spaces. Let $\mathcal{H}$ be the upper half plane and let $\mathbb{R}$ be its horizontal boundary. The group

$$G^{(1)} = \{ z \mapsto az + b \mid a, b \in \mathbb{R}, a > 0 \}$$

acts on $\mathcal{H} \cup \mathbb{R}$. $\mathcal{C}_{n,m}^+$ is the quotient $\text{Conf}_{n,m}^+/G^{(1)}$ where $\text{Conf}_{n,m}^+$ is the space of configurations of $n$ distinct points $p_1, \ldots, p_n$ in $\mathcal{H}$ and $m$ distinct points $q_1, \ldots, q_m$ in $\mathbb{R}$ such that $q_1 < \cdots < q_m$. The manifold $\mathcal{C}_{n,m}^+$ will be oriented as follows (see [1]). One puts $p_1$ in a fixed position and uses the coordinates of the other points to identify $\mathcal{C}_{n,m}^+$ with an open subset of the affine space $\mathbb{A} = \mathbb{C}^{n-1} \times \mathbb{R}^m$. One then transfers the standard orientation on $\mathbb{A}$ to $\mathcal{C}_{n,m}^+$.

For $\Gamma \in G_{n,m,\epsilon}$ put

$$\kappa_\Gamma = \bigwedge_{e \in \{ \text{edges of } \Gamma \}} d\phi_e$$

where the (multi-valued) function $\phi_e$ on $\mathcal{C}_{n,m}^+$ is defined as $\tilde{\phi}_e/2\pi$ where $\tilde{\phi}_e$ is computed as in the following image.

Thus if $e$ is an edge in $\Gamma$ from $p$ to $q$ then we embed $e$ as a line in the hyperbolic plane $\mathcal{H}$ and we measure the angle $\phi_e$ in the counter clockwise direction as indicated in the drawing.

The ordering of the edges in the product $\bigwedge_e d\phi_e$ is first according to the ordering of the starting vertices in the set $\Gamma_1$ (the ordering is by label) and then according to an arbitrarily chosen but fixed ordering on outgoing edges.

Now we put\footnote{This definition differs by a sign from Kontsevich’s definition. Kontsevich’s definition necessitates an unpleasant sign change in the definition of the Lie bracket on $T_{\text{poly}}(F)$ (see [1]). Moreover this sign change destroys the Gerstenhaber property of $T_{\text{poly}}(F)$. A tedious computation shows that with our definition no sign changes for the Lie bracket are necessary.}

(9.1) \[ W_G = (-1)^{|\Gamma|(|\Gamma|-1)/2} \int_{\mathcal{C}_{n,m}^+} \kappa_\Gamma \]

One easily sees that the product $W_G U_\Gamma$ is independent of the chosen ordering on outgoing edges.

Assume $n = 1$. In that case $G_{1,m}$ contains only one graph $\Gamma_0$ and $\kappa_{\Gamma_0} = (-1)^{m(m-1)/2}/m!$. Furthermore

$$U_{\Gamma_0}(\gamma)(f_1, \ldots, f_m) = \gamma^{1 \cdots m} \partial_{i_1} f_1 \cdots \partial_{i_m} f_m$$

$$= \langle df_1 \cdots df_m, \gamma \rangle$$

where $\langle -,- \rangle$ is as in (4.2). Hence

(9.2) \[ U_1(\gamma)(f_1, \ldots, f_m) = (-1)^{m(m-1)/2} \frac{1}{m!} \langle df_1 \cdots df_m, \gamma \rangle \]

It is easy to see that $U_1$ coincides with the HKR map $T_{\text{poly}}(F) \rightarrow D_{\text{poly}}(F)$ as defined by (7.5).
9.2. **Compatibility with cupproduct.** Let \( C \) be a commutative DG-\( k \)-algebra (equipped with some topology). Then we may extend \( U \) to an \( L_{\infty} \)-morphism

\[
\tilde{U} : C \otimes T_{\text{poly}}(F) \to C \otimes D_{\text{poly}}(F)
\]

Assume now that we have a solution \( \omega = \sum_\alpha \eta_\alpha \omega_\alpha \) to the Maurer-Cartan equation in \( (C^1 \otimes T^0_{\text{poly}}(F)) \). By property (P4) we know that \( U_1(\omega) \) is a solution to the Maurer Cartan equation in \( C^1 \otimes D^0_{\text{poly}}(F) \). It will be convenient to denote \( U_1(\omega) \) also by \( \omega \).

Under suitable convergence hypotheses we know that there exists a twisted \( L_{\infty} \)-morphism (see §7.2)

\[
\tilde{U}_\omega : (C \otimes T_{\text{poly}}(F))_\omega \to (C \otimes D_{\text{poly}}(F))_\omega
\]

Kontsevich sketches a proof that \( U_\omega \) commutes with cup product up to homotopy.\(^9\) A more detailed proof in the slightly restricted case that \( \omega \in T^1_{\text{poly}}(F) \) (i.e. a Poisson bracket) was given in [23]. In [25] it is even shown that \( \tilde{U}_\omega \) can be extended to an \( A_{\infty} \)-morphism (this is again in the case \( \omega \in T^1_{\text{poly}}(F) \)).

For the benefit of the reader we will state a result below which will be sufficient for the sequel. It can be obtained by copying the proof of [23], taking into account our modified sign conventions.

It is well known that \( C^+_{n,m} \) can be canonically compactified as a manifold with corners \( \bar{C}^+_{n,m} \) [20]. Let \( \bar{C}_{2,0} = \bar{C}^+_{2,0} \) (the “+” is superfluous) be the “Eye” as in the following figure:

![Eye diagram](image)

The upper outer boundary is where \( p_1 \) approaches the real line, the lower outer boundary is where \( p_2 \) approaches the real line. The inner boundary is where \( p_1 \) and \( p_2 \) approach each other, away from the real line.

The right corner is the locus where \( p_1, p_2 \) both approach the real line with \( p_1 \) to the left of \( p_2 \). Following Kontsevich [20] we have indicated a path \( \xi : [0,1] \to \bar{C}^+_{2,0} \) from a point on the inner boundary (labeled “0”) to the right corner (labeled “1”).

Next we consider the map

\[
F : C^+_{n,m} \to \bar{C}_{2,0}
\]

which is given by projection onto the first two points. One may show that this map can be extended to a map

\[
\bar{F} : C^+_{n,m} \to \bar{C}_{2,0}
\]

and we put \( Z_{n,m} = \bar{F}^{-1}(0,1] \). We orient \( Z \) by the normal \( dn \) (as indicated in the above figure).

For \( \Gamma \in G_{n,m,1} \) we put

\[
\bar{W}_\Gamma = (-1)^{||\Gamma||-1/2} \int_{Z_{n,m}} \kappa_{\Gamma}
\]

After a tedious computation, mimicking [23], we obtain the following.

\(^9\)Kontsevich’s proves this in fact for general solutions of the Maurer-Cartan equation in \( C \otimes T_{\text{poly}}(F) \).
\textbf{Proposition 9.1.} For $\alpha, \beta \in T_{\text{poly}}(F)$ we have as maps from $(C \otimes T_{\text{poly}}(F))_\omega$ to $(C \otimes D_{\text{poly}}(F))_\omega$
\[ \tilde{U}_{\omega, 1}(\alpha) \cup \tilde{U}_{\omega, 1}(\beta) = \tilde{U}_{\omega, 1}(\alpha \cup \beta) + d(H(\alpha, \beta)) - H(d\alpha, \beta) - (-1)^{\alpha + 1}H(\alpha, d\beta) = 0 \]
where
\[ H(\alpha, \beta) = \sum_{n,m \geq 0, \Gamma \in G_{n,m,1}} (-1)^{m-1} \frac{1}{(n-2)!} \tilde{W}_\Gamma \mathcal{U}_\Gamma(\alpha \beta \omega^{n-2}) \]
(the operators $\mathcal{U}_\Gamma$ are extended multilinearly to $C \otimes T_{\text{poly}}(F)$). In particular $\tilde{U}_{\omega, 1}$ commutes with cupproduct, up to a natural homotopy.

10. Proof of Theorem 1.3

In this section we will initially assume that $k$ contains the reals and we let the local formality morphism $\mathcal{U}$ in (7.4) be the one defined by Kontsevich (as in §9).

10.1. The local case. Combining the $L_\infty$-morphisms (7.11)(7.9)(7.10)(7.12)(7.15)(7.13) we obtain a commutative diagram
\[ \begin{array}{ccc}
T_{\text{poly}}^2(R_2) & \longrightarrow & T_{\text{poly, aff}, L}(C_{\text{aff}, L} \otimes_{R_1} JL) \\
\gamma_{\text{aff}} & \downarrow & \gamma_{\text{coord}} \\
D_{\text{poly}}^2(R_2) & \longrightarrow & D_{\text{poly, aff}, L}(C_{\text{aff}, L} \otimes_{R_1} JL) \\
\end{array} \]
where the horizontal maps are strict morphisms (i.e. the only the first Taylor coefficient is non-zero).

Our aim is to sheafify diagram (10.1) and to look at the result in the derived category of $\mathcal{O}$-modules. To determine the result it is sufficient to understand the $(-1)_1$ part of (10.1).

I.e.
\[ \begin{array}{ccc}
T_{\text{poly}}^2(R_2) & \longrightarrow & T_{\text{poly, coord}, L}(C_{\text{coord}, L} \otimes_{R_1} JL) \\
\gamma_{\text{aff}} & \downarrow & \gamma_{\text{coord}} \\
D_{\text{poly}}^2(R_2) & \longrightarrow & D_{\text{poly, coord}, L}(C_{\text{coord}, L} \otimes_{R_1} JL) \\
\end{array} \]

\textbf{Lemma 10.1.1.} The map $\gamma_{\text{aff}} : T_{\text{poly, aff}, L}(C_{\text{aff}, L} \otimes_{R_1} JL) \rightarrow D_{\text{poly, aff}, L}(C_{\text{aff}, L} \otimes_{R_1} JL)$ commutes with the Lie bracket and the cupproduct up to homotopies which are functorial for algebraic Lie algebroid morphisms which satisfy (7.2).

\textbf{Proof.} For the Lie bracket this is clear since $\gamma_{\text{aff}}$ is obtained from a $L_\infty$-morphism.

For the cupproduct we need to show that the homotopy $H$ defined by (9.3) descends to a map $T_{\text{poly, aff}, L}(C_{\text{aff}, L} \otimes_{R_1} JL)^2 \rightarrow D_{\text{poly, aff}, L}(C_{\text{aff}, L} \otimes_{R_1} JL)$. This is a computation similar to the proof of Proposition 6.4.1 combined with (5.7). We need the following version of (P5).

- $\tilde{W}_\Gamma \mathcal{U}_\Gamma(\gamma_\Omega) = 0$ for $q \geq 3$ (q being the number edges of the “first type” in $\Gamma$) and $\gamma \in \mathfrak{g}l_q(k) \subset T_{\text{poly}, 1}(F)$.

This is proved in exactly the same way as (P5). See [20, §7.3.3.1]. ☐

We will now evaluate the formula for $\tilde{U}_{\omega, 1}(\gamma)$ where we assume $\gamma \in T_{\text{poly}}(F)$.
\[ \tilde{U}_{\omega, 1}(\gamma) = \sum_{j \geq 0} \frac{1}{j!} \tilde{U}_{j+1}(\omega^j \gamma) \]
We may write
\[ \omega = \sum_\alpha \eta_\alpha \omega_\alpha \]
with \( \eta_\alpha \in C^1_{\text{coord}} \) and \( \omega_\alpha \in T^0_{\text{poly}}(F) \). Below we suppress the summation sign over \( \alpha \). Thus
\[ \tilde{U}_{j+1}(\omega^j \gamma) = \eta_{\alpha_j} \cdots \eta_{\alpha_1} U_{j+1}(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma) \]

To understand the (absence of) signs in this formula we note that we consider \( \tilde{U} \) as a degree zero map \( S_{j+1}((C^\text{coord}, L \otimes T_{\text{poly}}(F))[1]) \to (C^\text{coord}, L \otimes D_{\text{poly}}(F))[1] \). Thus \( \omega \) has even degree when appearing as argument to \( \tilde{U} \). However the \( \omega_\alpha \) as argument to \( \tilde{U} \) have odd degree. By contrast the degree of the \( \eta_\alpha \) is unchanged.

We need to enumerate the graphs contributing to the evaluation of \( U_{j+1}(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma) \).

We need to consider the following type of graphs.

1. There are \( j \) vertices labeled by \( \omega_{\alpha_1}, \ldots, \omega_{\alpha_j} \). These have 1 outgoing arrow.
2. There is 1 vertex labeled \( \gamma \) which has \( p + 1 \) outgoing arrows.
3. There are \( m = p - j + 1 \) vertices labeled by elements \( f_1, \ldots, f_m \in F \).

The edges leaving \( \gamma \) are ordered by their ending vertex where we extend the implied ordering of vertices of the first kind to all vertices via \( \omega_{\alpha_1} < \cdots < \omega_{\alpha_j} < \gamma < f_1 < \cdots < f_m \).

Since there are no loops and double edges we find that \( \gamma \) is connected through an outgoing arrow with all other vertices. It remains to allocate the \( j \) arrows emanating from the vertices labeled \( \omega_\alpha \).

Recall that by [Ko, §7.3.1.1 §7.3.3.1] that \( W_{\Gamma} \) is zero if \( \Gamma \) contains one of the following subgraphs.

\[ \begin{array}{c}
\text{(10.2)} \\
 p \\
 q \\
 r
\end{array} \]

or

\[ \begin{array}{c}
\text{(10.3)} \\
 p \\
 q \\
 r
\end{array} \]

or

\[ \begin{array}{c}
\text{(10.4)} \\
 q \\
 p
\end{array} \]

where \( q \) has no additional incoming or outgoing vertices.

If there is an \( \omega_\alpha \) which does not have an incoming arrow from another \( \omega_\alpha \) then we are in one of the situations (10.2)(10.3) or (10.4) and hence \( W_{\Gamma} = 0 \). The remaining graphs are of
the form

\( \omega_1 \alpha_1 \sigma \)

(10.5)

where \( \sigma \) is a permutation of \( \{1, \ldots, j\} \).

Let us define \( \Sigma_k \) as the following “opposite wheel”

\[
\text{and put } W_k = W_{\Sigma_k}. \text{ To fix the sign we order the vertices according to their labels and the outgoing edges of the central vertex according to their ending vertex.}
\]

Now we compute \( W_\Gamma \) and \( U_\Gamma \) for a graph as in (10.5). First we consider \( W_\Gamma \). Assume that there are \( s \) wheels of size \( l_1, \ldots, l_s \) respectively.

Write \( g_i \) for the edge emanating in \( \omega_1 \) for \( i = 1, \ldots, j \) and \( e_i \) for the edge connecting \( \gamma \) to \( \omega \) for \( i = 1, \ldots, j \). Finally write \( h_i \) for the edge connecting \( \gamma \) to \( f_i \) for \( i = 1, \ldots, m \). Then

\[
W_\Gamma = (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j} d\phi_{e_1} \cdots d\phi_{e_j} d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}
\]

To evaluate the integral we may put \( \gamma \) in \( i \in H \). This reduces the symmetry group \( G^{(1)} \) to the identity. We may clearly choose the \( \phi_{h_i} \) freely apart from the fact that \( \phi_{h_1} < \cdots < \phi_{h_{p+1-j}} \).

Thus we get

\[
W_\Gamma = (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j} d\phi_{e_1} \cdots d\phi_{e_j} d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}
\]

\[
= (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j} d\phi_{e_1} \cdots d\phi_{e_j} d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}
\]

\[
= (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j} d\phi_{e_1} \cdots d\phi_{e_j} d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}
\]

\[
= (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j} d\phi_{e_1} \cdots d\phi_{e_j} d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}
\]

\[
= (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j} d\phi_{e_1} \cdots d\phi_{e_j} d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}
\]

where in the last line we have used the identities \( (2l(2l-1))/2 \equiv l \mod 2 \) and \( \sum_i l_i = j \).

Now we compute \( U_\Gamma(\omega_1 \cdots \omega_j \gamma)(f_1, \ldots, f_m) \). We are short of symbols so we use the same symbol for an edge and for its corresponding index. Of course we use the same ordering of

\[
\text{...}
\]
the edges as above. We find

\[
U_T(\omega_{i_1} \cdots \omega_{i_m}) (f_1, \ldots, f_m) =
\]
\[
(\partial_{e(x)}(\partial_{e(y)}(\partial_{e(z)}(\partial_{e(w)}(\omega_{i_1} \cdots \omega_{i_m})) \cdot (\partial_{h_1}(f_1 \cdots \partial_{h_m}(f_m)))) \gamma e_1 \cdots e_h_1 \cdots e_h_m)
\]

We need a more concise way of writing this. Let \( \Xi_\alpha \) be the matrix of 1-forms \( d(\partial_{e(\omega)}(\omega)) = \partial_{e(\omega)}(\omega) d(\omega) \). Then \( U_T(\omega_{i_1} \cdots \omega_{i_m}) (f_1, \ldots, f_m) \) is equal to

\[
(-1)^\sigma \langle \text{Tr}(\Xi_{\alpha(1)} \cdots \Xi_{\alpha(s(t))}) \rangle \cdot \text{Tr}(\Xi_{\alpha(t_1+\cdots+t_{s-1}+1)} \cdots \Xi_{\alpha(t_1+\cdots+t_{s-1}+1)}) \rangle \cdot \text{Tr}(\Xi_{\alpha(t_1)} \cdots \Xi_{\alpha(t_s)}) \wedge \gamma)
\]

An easy computation yields

\[
U_T(\omega_{i_1} \cdots \omega_{i_m}) =
\]
\[
(-1)^\sigma (-1)^{m(m-1)/2} m! \text{HKR}(\text{Tr}(\Xi_{i_1} \cdots \Xi_{i_{s-1}+1})) \cdots \text{Tr}(\Xi_{i_{s-1}+1} \cdots \Xi_{i_1})) \wedge \gamma)
\]

Put

\[
\tilde{U}_T(\omega_{i_1} \cdots \omega_{i_m})(f_1, \ldots, f_n) = \eta_{i_1} \cdots \eta_{i_m} U_T(\omega_{i_1} \cdots \omega_{i_m})(f_1, \ldots, f_m)
\]

An easy computation yields

\[
\tilde{U}_T(\omega_{i_1} \cdots \omega_{i_m})(f_1, \ldots, f_n) = \eta_{i_1} \cdots \eta_{i_m} \text{HKR}(\text{Tr}(\Xi_{i_1} \cdots \Xi_{i_{s-1}+1})) \cdots \text{Tr}(\Xi_{i_{s-1}+1} \cdots \Xi_{i_1})) \wedge \gamma)
\]

where we have extended \(- \wedge -\) and HKR(\(-\)) to operations over \( C^{\text{coord.}} \) and where \( \Xi \) is the matrix \( \eta_{i_1} \eta_{i_2} \cdots \eta_{i_n} \) of elements of \( C^{\text{coord.}} \otimes \Omega^1_k \). The entries of \( \Xi \) have even total degree so the traces \( \text{Tr}(\Xi) \) commute.

Now note the following simple identities

\[
\frac{j(j-1)}{2} = \frac{\left(\sum_i l_i\right)^2 - \left(\sum_i l_i\right)}{2} = \sum_i l_i l_j + \sum_i \frac{l_i(l_i-1)}{2}
\]

\[
\frac{(m+2j)(m+2j-1)}{2} = \frac{m(m-1)}{2} + j \mod 2
\]

Collecting all signs we deduce

\[
W_T \tilde{U}_T(\omega_{i_1} \cdots \omega_{i_m})(f_1, \ldots, f_n) = \eta_{i_1} \cdots \eta_{i_m} \text{HKR}(\text{Tr}(\Xi_{i_1} \cdots \Xi_{i_{s-1}+1})) \cdots \text{Tr}(\Xi_{i_{s-1}+1} \cdots \Xi_{i_1})) \wedge \gamma)
\]

Putting temporarily \( X_f = (-1)^{(m-1)/2} W_f \text{Tr}(\Xi) \)

we find

\[
W_T \tilde{U}_T(\omega_{i_1} \cdots \omega_{i_m})(f_1, \ldots, f_n) = \text{HKR}(X_{i_1} \cdots X_{i_m} \wedge \gamma))
\]

Now we have to enumerate the number of possible graphs \( \Gamma \). Ordering the size of the wheels in increasing order we get a partition \( \tau = (1 \cdots 1 \cdots r \cdots r) \) where \( r \) occurs \( \tau \) times. The number of distinct graphs corresponding to such a partition is

\[
\frac{j!}{\tau_1! \cdots \tau_r! (\tau_1 \cdots \tau_r)}
\]

Thus we find that

\[
\sum_{j \geq 0} \frac{1}{j!} \tilde{U}_{j+1}(\omega_{i_1} \cdots \omega_{i_m})(f_1, \ldots, f_n) = \sum_{\tau_1, \cdots, \tau_r} \frac{1}{\tau_1! \cdots \tau_r! (\tau_1 \cdots \tau_r)} \text{HKR}(X_{i_1}^{\tau_1} X_{i_2}^{\tau_2} \cdots X_{i_r}^{\tau_r}) \wedge \gamma)
\]

Formally we have

\[
e^{X_{i_1} \cdots X_{i_r}} = \sum_{\tau} \frac{1}{\tau!} \text{HKR}(X_{i_1}^{\tau_1} X_{i_2}^{\tau_2} \cdots X_{i_r}^{\tau_r})
\]
so that we find
\[ \sum_{j \geq 0} \frac{1}{j!} \tilde{U}_{j+1}(\omega^j \gamma) = \text{HKR}(e^{X_1+X_2/2+\cdots} \wedge \gamma) \]

So if we put
\[ \Theta = \sum_l (-1)^{(l-1)/2} \frac{1}{l} W_l \Xi^l \]
then
\[ \sum_{j \geq 0} \frac{1}{j!} \tilde{U}_{j+1}(\omega^j \gamma) = \text{HKR}(e^{\text{Tr}(\Theta)} \wedge \gamma) \]
\[ = \text{HKR}(\det(e^{\Theta}) \wedge -) \]

This formula was proved under the assumption that \( \gamma \in T_{\text{poly}}(F) \). However by linear extension it follows that it remains true if \( \gamma \in C_{\text{coord}}^L \hat{\otimes} T_{\text{poly}}(F) \). Thus our final formula is
\[ (10.6) \quad \tilde{U}_{\omega,1} = \text{HKR}(\det(e^{\Theta}) \wedge -) \]

We now analyze the series \( \Theta \) more detail. We need to know the value of \( W_l \). As explained to us by Torossian this can be obtained from the work of Cattaneo and Felder on the quantization of coisotropic submanifolds \([9, 10]\). See \([36, \text{Thm 18}]\). As an alternative one can use a tedious but elementary computation using Stokes theorem \([33, (1.1)]\). The result is the following.

**Lemma 10.1.2.** We have
\[ (10.7) \quad W_l = -(-1)^{(l+1)/2} \beta_l \]
where \( \beta_n \) is the \( n \)'th modified Bernoulli number \( \beta_n \) which is defined by
\[ \sum_l \beta_l x^l = \frac{1}{2} \log \frac{e^x/2 - e^{-x/2}}{x} \]

Let us finally also mention that a result similar to (10.7) was announced by Shoikhet in \([29, \S 2.3.1]\). It can presumably be obtained from the methods in \([28]\).

Substituting we find
\[ \Theta = -\sum_l (-1)^l \beta_l \Xi^l \]
\[ = -\frac{1}{2} \log \frac{e^{\Xi/2} - e^{-\Xi/2}}{\Xi} \]
and hence
\[ e^\Theta = \sqrt{\frac{\Xi}{e^{\Xi/2} - e^{-\Xi/2}}} \]

It will be convenient for a module \( N \) with a connection to introduce the modified Todd class as follows
\[ \tilde{\text{td}}(N) = \det(\tilde{q}(A(N))) \]
where
\[ \tilde{q}(x) = \frac{x}{e^{x/2} - e^{-x/2}} \]
It follows from Proposition 8.4.2 that the following diagram is commutative.

\[
\begin{array}{ccc}
T_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes R_1, JL) & \longrightarrow & C^{\text{coord},L} \otimes T_{\text{poly}}(F) \\
\tilde{\eta}_{L}(N^{\text{aff},L})^{\frac{1}{2}} \wedge - & \downarrow & \text{det } e^{\eta} \wedge - \\
T_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes R_1, JL) & \longrightarrow & C^{\text{coord},L} \otimes T_{\text{poly}}(F)
\end{array}
\]

Thus we get

\[\eta^{\text{aff}}_{\text{I}} = \text{HKR} \circ (\tilde{\eta}(N^{\text{aff}})^{1/2} \wedge -)\]

### 10.2. The global case

Now we globalize things. Let \((C, O, L)\) be as in §7.4. Then it follows from Proposition 8.4.2 (specifically (8.6)) that the following diagram is commutative in \(D(O)\).

\[
\begin{array}{ccc}
T_{\text{poly}}(O) & \longrightarrow & T_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes O_1, JL) \\
\tilde{\eta}_{L}(1/2) \wedge - & \downarrow & \text{det } e^{\eta} \wedge - \\
T_{\text{poly}}(O) & \longrightarrow & T_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes O_1, JL)
\end{array}
\]

so that we get a commutative diagram in \(D(O)\).

\[
\begin{array}{ccc}
T^L_{\text{poly}}(O) & \longrightarrow & T_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes O_1, JL) \\
\text{HKR}(\tilde{\eta}(L)^{1/2} \wedge -) & \downarrow & \eta^{\text{aff}}_{\text{I}} \\
D^L_{\text{poly}}(O) & \longrightarrow & D_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes O_1, JL)
\end{array}
\]

where \(\tilde{\eta}(L)^{1/2}\) is as defined in the introduction. Since the horizontal isomorphisms as well as \(\eta^{\text{aff}}_{\text{I}}\) are Gerstenhaber algebra morphisms (Lemma 10.1.1) the same holds for \(\text{HKR}(\tilde{\eta}(L)^{1/2} \wedge -)\) as well. This finishes the proof of Theorem 1.3 in the case \(k\) contains the reals and the Todd class is replaced by the modified Todd class.

### 10.3. Proof for the ordinary Todd class

We have

\[
\tilde{\eta}(L) = \text{td}(L) \det(e^{-A(L)/2}) = \text{td}(L) e^{-\text{Tr}(A(L))/2} = \text{td}(L) e^{-\alpha_1(L)/2}
\]

In other words it is sufficient to prove that \(e^{-\alpha_1(L)/4} \wedge -\) defines an automorphism of \(T^L_{\text{poly}}(O)\) as a Gerstenhaber algebra in \(D(X)\).

We may as well prove that \(e^{-\text{Tr}(\Xi)/4} \wedge -\) is compatible with the Lie bracket and the cupproduct on \(C^{\text{coord}} \otimes T_{\text{poly}}(F)\) or equivalently that \(\text{Tr}(\Xi) \wedge -\) is a derivation for these operations. We have \(\text{Tr}(\Xi) = \sum_{i,\alpha} \eta_\alpha d(\partial_i \omega^\alpha_\alpha)\). Put \(b_\alpha = \sum \partial_i \omega^\alpha_\alpha\). Since everything is \(C^{\text{coord,L}}\) linear it is sufficient to prove that \(db_\alpha \wedge -\) is a derivation for the Lie algebra and cupproduct on \(T_{\text{poly}}(F)\).

Since this fact is clear for the cupproduct we only look at the Lie bracket. For \(D, E \in T^0_{\text{poly}}(F)\) we have

\[
[db_\alpha, [D, E]] = [D, E]([b_\alpha]) = D(E(b_\alpha) - E(D(b_\alpha))) = D(db_\alpha \wedge E) - E(db_\alpha \wedge D) = [D, db_\alpha] + [db_\alpha, D, E]
\]

finishing the proof.
10.4. **Arbitrary base fields.** Now we let $k$ be arbitrary (of characteristic zero) and we choose an embedding $k \subset \mathbb{C}$. It follows from the formulas (10.6) and (10.8) that $U_{\omega,1}$, while initially defined over $\mathbb{C}$ ($\mathbb{R}$ in fact), actually descends to $k$. We will denote the descended morphism by $u$.

Now note the following.

**Proposition 10.4.1.**

1. There exist $w_r \in k$ for $\Gamma \in G_{m,n}$, which are zero when $W_T$ is zero, such that for

$$l(\alpha, \beta) = \sum_{n,m \geq 0, \Gamma \in G_{n,m}} w_r(U_{\alpha \beta \omega^{n-2}})$$

we have

$$[u(\alpha), u(\beta)] - u([\alpha, \beta]) + d(l(\alpha, \beta)) - l(da, \beta) - (-1)^{|a|}l(\alpha, d\beta) = 0 \quad (10.10)$$

2. There exist $\tilde{w}_r \in k$ for $\Gamma \in G_{m,n,1}$, which are zero when $\tilde{W}_T$ is zero, such that for

$$h(\alpha, \beta) = \sum_{n,m \geq 0, \Gamma \in G_{n,m,1}} w_r(U_{\alpha \beta \omega^{n-2}})$$

we have

$$u(\alpha) \cup u(\beta) - u(\alpha \cup \beta) + d(h(\alpha, \beta)) - h(da, \beta) - (-1)^{|a|}h(\alpha, d\beta) = 0 \quad (10.11)$$

**Proof.** The equations (10.10)(10.11) are linear in $w_r, \tilde{w}_r$. By the fact that $U_{\omega}$ is a $L_{\omega}$-morphism and Proposition 9.1 there is a solution over $\mathbb{C}$. Hence there is a solution over $k$ (for example obtained by applying an arbitrary projection $\mathbb{C} \to k$). \qed

We can now proceed as in §10.1,10.2 (using an analogue of Lemma 10.1.1) to construct a commutative diagram

$$\begin{array}{ccc}
T_{poly}^{\mathbb{R}}(\mathcal{O}_2) & \xrightarrow{\text{eff}} & T_{poly,coord}^{\mathbb{R}}(C^{\mathbb{R}} \otimes \mathcal{O}_1 J \mathcal{L}) \\
\downarrow \text{eff} & & \downarrow \text{eff} \\
D_{poly}^{\mathbb{R}}(\mathcal{O}_2) & \xrightarrow{\text{eff}} & D_{poly,coord}^{\mathbb{R}}(C^{\mathbb{R}} \otimes \mathcal{O}_1 J \mathcal{L})
\end{array}$$

where $\text{eff}$ commutes both with the Lie bracket and the cupproduct up to a global homotopy. Using the fact that the formula (10.6) continues to hold

$$u = \text{HKR} \circ (\det(\Theta) \wedge -)$$

we can now continue as in §10.2 to finish the proof of Theorem 1.3.

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