The “non triviality” of a $\Phi^4_4$ model, III
the “Osterwalder-Schrader Positivity”

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Abstract

The present paper, III, is the third part of a series of papers, under the global title “the non triviality of a $\Phi^4_4$ model”. Parts I and II have been previously completed. In them thanks to the properties we dubbed “splitting -tree structure”, and “alternating signs”, which characterize our connected Green’s functions, we have constructed a unique non trivial solution to a $\Phi^4_4$ nonlinear renormalized system of equations of motion in Euclidean space.

In the present work, we show how, by application of these properties, the solution of our $\Phi^4_4$ model verifies the Osterwalder-Schrader Positivity requirement. This result complements those obtained in I and II where, apart from the Positivity, the Axiomatic Quantum Field theory properties have been established.

The O.S. Positivity is verified under a condition on the physical coupling constant relatively weaker than the one imposed in order to obtain the convergence of the $\Phi^4_4$ mapping to the unique non trivial solution.
## Contents

1 **Introduction**
   1 The verification by the $\Phi^4_4$ solution of the Relativistic and Euclidean Quantum Field Theory Axioms ...................... 1
   2 The equivalence between Q.F.T. Axioms in Euclidean space and the corresponding in Relativistic Minkowski Space .............. 2
      1 The Chart of Osterwalder-Schrader ......................... 2
      2 Plan of the paper .......................................... 3

2 **The O.S.P. conditions in momentum space**
   1 In $x$- space .................................................. 3
   2 The O.S.P.n conditions in q-space of non connected Green’s functions $\tau^{n+1}$ .................................................. 4
   3 The (O.S.P.n) conditions in q-space for the connected $H^{n+1}$ Green’s functions) .................................................. 5

3 **Verification of the (O.S.P.n) conditions by the $\Phi^4_4$ solution**
   1 The auxiliary Lemmas .......................................... 8
   2 The Main result .............................................. 10

4 **APPENDICES**
   1 The first examples ........................................... 15
   2 The proof of Theorem 3.1 ..................................... 18
   3 Proof of Lemma 3.2 ........................................... 22
   4 Reminders ..................................................... 25
1 Introduction

1 The verification by the $\Phi^4_4$ solution of the Relativistic and Euclidean Quantum Field Theory Axioms

The subject of verification of the axioms of Q.F.T in Minkowski [1] [2], [3] or Euclidean [4] [5] space by various physical interaction models between elementary particles has a long history.

The literature concerning the construction of quantum field theory models with Euclidean Green’s functions characterized by analogous features with those verified by the Wightman functions, began already in the 50’s.

In [6] J. Schwinger, presented the four-dimensional Euclidean formulation of Quantum Field Theory during the Annual International Conference on High Energy Physics at CERN, and published in the Proceedings of the National Academy of Sciences in 1958.

Later K. Symanzik in [7] proposed models in the Euclidean Quantum field theory context.

The equivalence between the axioms of Q.F.T. in Minkowski and Euclidean space appears in its most rigorous form during the years 1973-75 with the works of K. Osterwalder - R. Schrader [4] [5] [10], V. Glimm, E. Nelson [9], and J. Frölich in [12].

By the end of the seventies and until recently many papers did appear concerning the particular Axiomatic Q.F.T. property of Positivity under the name “O.S. Positivity” and more frequently of “Reflection Positivity” (cf. J. Glimm and A. Jaffe, in [11] and J. Frölich in [12]). In particular we would like to refer the reader to the opening talk of A. Jaffe [13] “Reflection Positivity Then and Now” at the conference dedicated to the memory of R. Schrader on November 20, 2017 held at the Mathematical Research Institute, Oberwolfach, Germany. The author not only expresses his enthusiasm for the discovery of the “O.S. reconstruction theorem” but he points out how the principle of “reflection positivity” plays a crucial role in many domains of mathematical physics (cf. [14]).

In the present work, the verification of the Osterwalder-Schrader axioms by our $\Phi^4_4$ solution completes our program of I [17] and II [18] towards the construction of an Axiomatic Q.F.T. [3] model.

Briefly, in [20] starting from the equation of motion and inspired by Zimmernann’s work [21], we introduced the “Renormalized Normal product” and established an equivalent infinite dynamical system of equations of motion in “four dimensions” for the Green’s functions (the “vacuum expectation values” of the theory) which has the form reminded in Appendix 4.4.

Now, we complete the results that we established previously partially in ref. [15] for the renormalized equations of motion, recently in [16] and more precisely in [17], [18] for the solution of these $\Phi^4_4$ equations of motion.

As a matter of fact, the linear Axiomatic Q.F.T. properties together with the distribution property, Euclidean covariance and symmetry together with the lin-
ear Axiomatic Q.F.T. analyticity properties (in complex Minkowski space) related to the locality, spectrum and uniqueness of the vacuum (associated to the cluster property) have been established for our $\Phi^4$ model.

These results ensured in some sense the coherence of our scheme but not completely. As far as the positivity property is concerned, the complete results in four (and automatically in all smaller) dimensions constitute the purpose of the present paper.

More precisely we prove that under the “weak condition” ($\Lambda < 1/6$) imposed on the physical coupling constant the infinite sequence of Green’s functions whose connected part is the solution of the $\Phi^4$ equations of motion in Euclidean momentum space in [17] and [18], verifies the set of Osterwalder-Schrader Positivity Axioms (O.S.P) [5].

In this way we ensure that the infinite sequence-solution is no longer formal but in view of the reconstruction theorem (cf.[2], and [5]), it is a well defined infinite sequence of Green’s functions equivalent to a nontrivial Wightman Q.F.T.

Remark 1.1 We point out that by saying “weak” condition ($\Lambda < 1/6$) imposed on the coupling constant, we simply mean that it is relatively less restrictive than the conditions we imposed in order to obtain the local contractivity ($\Lambda < 0.04$) and the corresponding to the stability of $\Phi^4_1$-iteration ($\Lambda < 0.05$) for the construction of the $\Phi^4_1$ non trivial solution obtained in [17] and [18].

2 The equivalence between Q.F.T. Axioms in Euclidean space and the corresponding in Relativistic Minkowski Space

1 The Chart of Osterwalder-Schrader

Let us remind that the main theorem proved in [4] or [5] is represented by the following chart of equivalences which connects the Euclidean Axioms of Osterwalder-Schrader and the Relativistic Wightman Axioms [2]

| EUCLIDEAN         | RELATIVISTIC                  |
|-------------------|-------------------------------|
| $(Euc.1)$          | $(Rel.1)$                      |
| $(c.f.[4])$        | $(c.f.[5])$                    |
| $Euc.1$            | $Rel.1$                        |
| $\equiv$           | $\equiv$                       |
| $\left(\begin{array}{c}
\text{Temperedness} \\
\text{Covariance} \\
\text{Positivity}
\end{array}\right)$ | $\left(\begin{array}{c}
\text{Temperedness} \\
\text{Covariance} \\
\text{Positivity} \\
\text{Spectrum}
\end{array}\right)$ |
| $Euc.1$            | $Rel.1$                        |
| $\equiv$           | $\equiv$                       |
| $\left(\begin{array}{c}
\text{Symmetry}
\end{array}\right)$ | $\left(\begin{array}{c}
\text{Locality}
\end{array}\right)$ |
| $Euc.1$            | $Rel.1$                        |
| $\equiv$           | $\equiv$                       |
| $\left(\begin{array}{c}
\text{Cluster}
\end{array}\right)$ | $\left(\begin{array}{c}
\text{Cluster}
\end{array}\right)$ |

(1.1)
2 Plan of the paper

In the next section we recall the definition of the O.S.P. conditions in x-Euclidean space, and present the analogous expression in terms of the non connected Green’s functions (time order product’s expectation values in our formalism). We then express them in terms of connected components.

By application of the Fourier transform together with the symmetry properties and Euclidean invariance, we reformulate the positivity in terms of the so called O.S.P.n conditions in the Euclidean four momentum space in terms of our Green’s functions sequences, namely truncated (connected) completely amputated with respect to the free propagators Green’s functions.

We complete this section by an auxiliary lemma which represents the starting point of the recursion used in the proof of the main theorem presented in section 3.

In the third section we establish the O.S.P.n conditions in momentum space for the non connected and connected part contributions. We first present two auxiliary lemmas and then the theorem 3.1 which yields as corollary the main result theorem 3.2.

In the appendices we give the detailed proofs of our statements together with some necessary reminders from [17][18].

The basic tools of the proof are again the "alternating signs” and the "splitting" or factorization properties of the Green’s functions in terms of “tree type” functions established previously in all dimensions $r$ with $0 \leq r \leq 4$ and at every value of the external momenta.

As a matter of fact the signs and the “tree type splitting” (or factorization) properties of the connected Green’s functions provide the possibility to obtain another decomposition of the non connected (non truncated) Green’s function $\tau^{n+1}$ in terms of its connected parts. This decomposition is different but equivalent to the “classical” one of definition 2.3 (cf. equation 2.8 reminded later in section 3), and we present it by Lemma 3.2. As a matter of fact it results from the successive application of the “tree type” decomposition $\frac{G^{n+1}}{(-6A)}$.

2 The O.S.P. conditions in momentum space

1 In x-space

In [4][5] the following conditions have been established by Osterwalder - Schrader in the Euclidean x-space.

$$\sum_{M,N} G_{(M+N)}(\Theta g_M \times g_N) \geq 0 \quad (2.2)$$

Where $G$ means the Schwinger functions [24] (distributions) in Euclidean x-space and it corresponds to the Wightman distributions in Minkowski space.
\( g_M \) belongs to the space of test functions \( S(\mathbb{R}^4) \), \( (g_M^*) \) means complex conjugate of \( g_M \) and

\[
(\Theta g)_M(x_1, \ldots, x_M) = g_M(\partial x_1 \ldots, \partial x_M).
\]

where for every vector \( x = \{x^0, \vec{x}\} \in \mathbb{R}^4 \), \( \partial x = \{-x^0, \vec{x}\} \).

In all that follows we denote by \( \tau^{n+1} \) the Fourier transform (in the sense of distributions) in \( q \)-space of the tempered distribution \( \mathcal{G} \). The connected (completely amputated with respect to the free propagators) parts of \( \tau^{n+1} \), correspond (following our prescriptions) to the \( H^{n+1} \) Green’s functions solutions of the equations that we introduced and studied in [22], [16], [17] and [18].

2 The O.S.P.n conditions in \( q \)-space of non connected Green’s functions \( \tau^{n+1} \)

By application of the isomorphisms of Fourier transform and its inverse on the product space of test functions \( S(\mathbb{R}^4) \times S(\mathbb{R}^4) \),

\[
\mathcal{F} : S(\mathbb{R}^4) \times S(\mathbb{R}^4) \to S(\mathbb{R}^4) \times S(\mathbb{R}^4)
\]

we directly obtain the corresponding positivity conditions for the non truncated Green’s functions (or time order product) in \( q \)-space (momentum space).

\[
\forall \ n = 2k + 1, \ k \in \mathbb{N} \\
\sum_{1 \leq M \leq n, 1 \leq N \leq n} \int_{q(M+N) = q(n+1)} \tau^{n+1}(q(n+1))\delta(Q_{n+1})\hat{f}(M)(q(M))\hat{f}(N)(q(N))dq(M)dq(N) \geq 0
\]

\[
q_{(M+N)} = q_{(n+1)} = q_{(M)} \cup q_{(N)}; \ q_{(M)} \cap q_{(N)} = \{q_M\} \subset q_{(M)}, \ q_M = q_i \in q_{(N)}
\]

and

\[
Q_{n+1} = \sum_{i=1}^{n+1} q_i
\]

(2.4)

Remarks 2.1

1. Here \( \hat{f} \) means the Fourier transform of an arbitrary test function \( f \in S(\mathbb{R}^4)_x \).

2. Following [5] or equivalently [2] when \( n \to \infty \) the conditions 2.2 in \( x \)-space (or 2.3 in \( q \)-space) ensure the positivity of the norm of every infinite dimensional vector of test functions \( \{\hat{f}_n\}_{n \in \mathbb{N}} \), associated with the hermitian form (scalar product) given in terms of the tempered distribution \( \tau \).

3. Note that the Euclidean-translation invariance in \( x \)–space leads to the total energy momentum conservation which is expressed by the “\( \delta - function \)” \( \delta(Q_{n+1}) \) appearing in the above formula.
4. As we noticed before, the above form of the O.S.P. conditions are not suitable to be studied by our method because the characteristic bounds, signs, splitting, and tree structure properties of the $\Phi_b^4$ solution established in [17]-[18] and recalled in the Appendix 4.4 are expressed in terms of the truncated or connected and completely amputated with respect to the free propagators Green’s functions $H^{n+1}$.

Therefore, taking into account the decomposition formula in connected parts of every inverse Fourier transform of the $\tau^{n+1}$ function in x-space and then by application of:

a. the isomorphisms of Fourier transform and its inverse on the product space of test functions $S(\mathbb{R}^4M) \times S(\mathbb{R}^4N)$,

b. the symmetry and Euclidean invariance of every connected Green’s function in x-space,

we obtain in a more appropriate expression of the O.S.P. conditions in Euclidean momentum $q$-space. We also notice that we shall use the notation (O.S.P.n) for reference either to the above set of inequalities 2.4 (non connected expression) or to the following (connected expressions) 2.8 or 2.9.

5. Moreover, the fact that every connected part Green’s function $H^{n+1}$ is a uniquely defined tempered distribution in the space $S'(\mathbb{R}^{4n})$ as solution of the equations of motion and continuous with respect to each one of its arguments, the other being constant, we are allowed to apply the Schwartz-Nuclear Theorem [19] and target all the proofs which follow to test functions which belong to the dense subset (of $S(\mathbb{R}^4M) \times S(\mathbb{R}^4N)$) of all linear combinations of the tensor product functions, namely:

**Definition 2.1** (Factorization of the test functions)

\[
\tilde{f}_{(N)} \in S(\mathbb{R}^4) \times S(\mathbb{R}^4) \ldots \ldots S(\mathbb{R}^4) \\
\tilde{f}_{(N)}(q_{(N)}) = \prod_{1 \leq l \leq N} f_{(1)}^{(l)}(q_l) \tag{2.5}
\]

Notice that in the following for simplicity we often omit the subscript 1 from $f_{(1)}^{(l)}$ and write $f_{(N)}$ instead of $\tilde{f}_{(N)}$.

3 The (O.S.P.n) conditions in q-space for the connected $H^{n+1}$ Green’s functions

**Definition 2.2** $\forall n = 2j + 1, \ j \in \mathbb{N}$ we consider the set of odd positive integers indices:

\[
(n) = \{1, 3, 5 \ldots , n\} \tag{2.6}
\]
We introduce the set $\pi(n)$ of all partitions of $(n)$ as follows:

A sequence $J$ of non empty disjoint subsets of $(n)$ belongs to $\pi(n)$, if:

$$J = (J_1, J_2, \ldots, J_k) \quad k \leq n$$

and

$$\forall i \in (1, 2, \ldots, k), \quad j \in (1, 2, \ldots, k), \text{ with } i \neq j \quad J_i \cap J_j = \emptyset, \quad \bigcup_{1 \leq i \leq k} J_i = (n).$$

Moreover $\text{Card} J_l = j_l$, where $j_l, \ l \in (1, 2, \ldots, k)$, are odd integers such that,

$$j_1 \geq j_2 \geq \ldots j_{k-1} \geq j_k, \quad \text{and } \sum_{i=1}^{k} j_i = n$$

In the particular case $k = 3$ we often use the notation $\pi(n)(3)$ for the set of partitions $I = (I_1, I_2, I_3)$ with $\text{Card} I_l = i_l$ where $i_l, \ l \in (1, 2, 3)$ are odd integers such that: $i_1 \geq i_2 \geq i_3$ and $\sum_{l=1}^{3} i_l = n$

Definition 2.3 (Connected parts’ form of the (O.S.P.n) conditions and matrix representations)

We first consider the standard decomposition of the non connected time order product in terms of the connected parts (and resp. connected completely amputated with respect to the free propagators):

$$\tau^{n+1}(q_{(n+1)}) = \sum_{J \in \pi(n)} C_{(j_1, \ldots, j_k)} \prod_{1 \leq l \leq k} \eta^{j_l+1}(q_{j_l+1}) \delta(Q_{j_l+1})$$

(or respectively)

$$\tau^{n+1}(q_{(n+1)}) = \sum_{J \in \pi(n)} C_{(j_1, \ldots, j_k)} \prod_{1 \leq l \leq k} H^{j_l+1}(q_{j_l+1}) \prod_{1 \leq r \leq j_l} \Delta F(q_r) \delta(Q_{j_l+1})$$

here: $C_{(j_1, \ldots, j_k)} = \frac{n!}{j_1! \ldots j_{k-1}! j_k!}$

(2.8)

So equivalently with (2.4) we have to ensure that, for every $n = 2r + 1, \ r \in \mathbb{N}$ and $\forall (q, \Lambda) \in \mathcal{E}^{kn} \times \mathbb{R}^+$,

$$\sum_{1 \leq N \leq n \leq M \leq n+1} \sum_{J \in \pi(n)} \int_{f(M)} \prod_{1 \leq l \leq k} H^{j_l+1} \prod_{1 \leq r \leq j_l} \Delta F(q_r) f(N(q)) dq \geq 0$$

(2.9)

Here $f(M)$ (resp. $f(N)$) are the factorized test functions defined on the corresponding cartesian products of euclidean momentum spaces as introduced before by (2.5)

Moreover the “$\delta$ – function” $\delta(Q_{j_l+1})$ (which appears in (2.8) for every connected part and expresses the total energy momentum conservation, resulting from the Euclidean-translation invariance in $x$-space) has disappeared in (2.9) after the integration (Fubini) with respect to every “last” momentum variable:

$$q_{j_l+1} = - \sum_{i=1}^{l} q_{j_i}$$
and \( dq(n) \) is an abbreviated notation for the Euclidean measure:

\[
dq(n) = \prod_{1 \leq m \leq n} dq_m
\]

in the space of \( n \) independent momentum variables. Finally we notice that often we simplify the notation of the arguments for the set of \( n \) independent moments and right \( q \) instead of \( q(n) \).

Notice that every term of the sum in (2.4) (resp. of double sum 2.9) is a hermitean form that can be represented as an element of a matrix representation as in the examples of 4.25 (given in Appendix 4.1).

For practical raisons we shall often use a three parts decomposition of \( \tau^{n+1} \):

\[
\tau^{n+1} = T_1^n + T_2^n + T_3^n
\]

(2.10)

where:

\[
T_1^n = H^{n+1} \prod_{l=1}^{n} \Delta F(q_l)
\]

\[
T_2^n = \sum_{I \in \mathcal{W}_n(3)} C(I) \prod_{l=1,2,3}^{n} H^{i_l+1} \Delta F(q_{i_l})
\]

(2.11)

with:

\[
C(I) = \frac{n!}{i_1!i_2!i_3!}
\]

\[
T_3^n = \sum_{J \in \mathcal{W}_n} C(j_1, \ldots, j_k) \prod_{1 \leq l \leq k} H^{j_l+1} \Delta F(q_{j_l})
\]

In the following proofs we rename the above decomposition (together with the equivalent one previously given by the formula 2.8) as the “classical connected parts decomposition”.

Moreover, for every term in 2.8 and 2.11 (resp. for every partition \( J \in \mathcal{W}_n \) i.e. every term in the sum \( \sum_{1 \leq M \leq n, 1 \leq N \leq n} \sum_{M+N \leq n+1} \langle f(M), \tau^{n+1} f(N) \rangle \)) we also simplify the notation and write:

\[
\langle f(M), \tau^{n+1} f(N) \rangle \quad \text{(and respectively: } \langle f(M), \prod_{1 \leq l \leq k} H^{j_l+1} f(N) \rangle \text{)}
\]

so the \( (O.S.P.n) \) conditions 2.4 can be written as follows:

\[
\sum_{1 \leq M \leq n, 1 \leq N \leq n} \langle f(M), \tau^{n+1} f(N) \rangle \geq 0
\]

(2.12)

(And respectively the corresponding connected form of 2.9 \( (O.S.P.n) \) conditions:

\[
\sum_{1 \leq M \leq n, 1 \leq N \leq n} \sum_{J \in \mathcal{W}_n} \langle f(M), \prod_{1 \leq l \leq k} H^{j_l+1} f(N) \rangle \geq 0
\]

or in terms of the three parts decomposition 2.11:

\[
\sum_{1 \leq M \leq n, 1 \leq N \leq n} \langle f(M), (T_1^n + T_2^n + T_3^n) f(N) \rangle \geq 0
\]

(2.13)
Remark 2.1 As one can see on the examples 4.25, the positivity \((O.S.P.n)\) conditions for every fixed \(n\) during the recursive procedure of our proof will be given in terms of the sums of only the left upper triangular matrix elements corresponding to \(M + N \leq n + 1\).

For example let us write the corresponding conditions to be ensured for \(n \leq 5\):

For \(n = 1\):
\[
\langle f(1), \tau^2 f(1) \rangle \geq 0
\]

For \(n = 3\):
\[
\langle f(1), \tau^2 f(1) \rangle + 2 \Re \langle f(1), \tau^4 f(3) \rangle + \langle f(2), \tau^4 f(2) \rangle \geq 0
\]
(2.14)

For \(n = 5\):
\[
\langle f(1), \tau^2 f(1) \rangle + 2 \Re \langle f(1), \tau^4 f(3) \rangle + \langle f(2), \tau^4 f(2) \rangle + 2 \Re \langle f(1), \tau^6 f(5) \rangle + 2 \Re \langle f(2), \tau^6 f(4) \rangle + 2 \Re \langle f(3), \tau^6 f(3) \rangle \geq 0
\]
(2.15)

In Appendix 4.1 we show the following:

Lemma 2.1 The \((O.S.P.n)\) conditions for \(n \leq 5\) are verified under the “weak” condition \(\Lambda < 1/6\). (cf. remark 1.1)

In the next section and by using the results of the previous lemma 2.1 as starting point we establish recurrently the O.S.P. conditions for every \(n\) under the same condition on the coupling constant: \(\Lambda < 1/6\).

3 Verification of the \((O.S.P.n)\) conditions by the \(\Phi^4_4\) solution

1 The auxiliary Lemmas

Before the main result given by the theorem 3.1, we present the following two useful auxiliary statements. The first one presents the “complete splitting-factorization” properties verified by the bounds \(H^{n+1}_{\min}\) in terms of the \(H^2\)-point functions. Moreover an evident bound is established for all \(n \geq 5\) by using the reminders of proposition 4.1 and definition 4.3. The proof is directly obtained recurrently by using the definitions 4.95, 4.96, 4.100.

The second Lemma relates the non connected Green’s function: \(\tau^{n+1}\) with all the “preceding” non connected i.e. \(\tau^{i+1}, (i = 1, 3, \ldots, n - 2)\) and it constitutes the pivot of the recurrent proof of the theorem. The proof is given in Appendix 4.3.

Lemma 3.1 The complete splitting
\(
\forall n \geq 7\) the following “complete splitting” properties are verified by the bounds \(H^{n+1}_{\min}\) in terms of the \(H^2\)-point functions.
\( |H_{\text{min}}^{n+1}| = \prod_{m=3}^{n} \delta_{m,\text{min}} \tilde{T}_m \prod_{l=1}^{n} H^2(q_l) \Delta_F(q_l) \) \hspace{1cm} (3.16)

(For the number \( \tilde{T}_n \) cf. remark 4.2)

\( \forall n \geq 5 \quad \delta_{n,\text{max}} < 3\Lambda n(n-1) \) \hspace{1cm} (3.17)

**Lemma 3.2** We suppose that the following properties are valid \( \forall \bar{n} \leq n - 2 \), then:

1. If \( H^{n+1} > 0 \),
   a) \( (\tau^{n+1} - T_1^n) = \sum_{I \in \omega_n(3)} C(I) \prod_{l=1,2,3} \tau^{i_l+1} \) \hspace{1cm} (3.18)
   here: \( C(I) = \frac{n!}{i_1!i_2!i_3!} \)
   b) \( \forall (q, \Lambda) \in \mathcal{E}_{4n} \times ]0, 1/6[, \quad \tau^{n+1} - T_1^n \geq 0 \) \hspace{1cm} (3.19)

2. If \( H^{n+1} < 0 \),
   a) \( \frac{|C_{n+1}|}{6\Lambda} \geq \frac{n(n-1)\tilde{T}_n}{2} \prod_{m=3}^{n-2} \delta_{m,\text{min}} \tilde{T}_m \prod_{l=1}^{n} H^2(q_l) \Delta_F(q_l)^2 \)
   \( T_1^n + T_2^n \geq \frac{n(n-1)}{2} \tilde{T}_n H_{\text{min}}^{n-1} \prod_{l=1}^{n} H^2(q_l) \Delta_F(q_l)^2 \left\{ 1 - \frac{2\delta_{\text{max}}}{n(n-1)} \right\} \)
   and \( \forall (q, \Lambda) \in \mathcal{E}_{4n} \times ]0, 1/6[, \quad T_1^n + T_2^n \geq 0 \) \hspace{1cm} (3.20)
   b) \( T_3^n = \sum_{I \in \omega_n(3)} C(I) \prod_{l=1,2,3} \tilde{\tau}^{i_l+1}(q_l) \)
   here: \( \tilde{\tau}^{i_l+1} = \tau^{i_l+1} - T_1^{i_l+1} \) if \( l = 1 \)
   \( \tilde{\tau}^{i_l+1} = \tau^{i_l+1} \) if \( l = 2, 3 \) \hspace{1cm} (3.21)
   and \( \forall (q, \Lambda) \in \mathcal{E}_{4n} \times ]0, 1/6[, \quad T_3^n \geq 0 \)
2 The Main result

Theorem 3.1 For every \( n = 2k + 1, k \in \mathbb{N} \) and for all integers \( M, N \) with \( 1 \leq M \leq n, 1 \leq N \leq n; M + N \leq n + 1 \) the following lower positive bounds are verified by the non connected Green’s functions.

1. \( \quad \text{if} \quad H^{n+1} > 0 \)
\( \Rightarrow \quad \forall (q, \Lambda) \in \mathcal{E}^{4n} \times ]0, 1/6[ \)
\[ \Re\langle f_{(M)}, \tau^{n+1} f_{(N)} \rangle \geq h(n, \Lambda) \| f \|^{2} (G_{1})^{n-1} \geq 0 \]
where
\[ \| f \|^{2} = \int |f^{(M)}(q)|^{2} (H^{2} \Delta_{F}^{2})(q_{M})dq_{M} \]
\[ (G_{1})^{n-1} = (-1)^{(n-1)} \left\{ \sup_{(i)} \int |f^{(i)}(q_{i})| (H^{2} \Delta_{F}^{2})(q_{i})dq_{i} \right\}^{n-1} \]
and
\[ h(n, \Lambda) = \frac{(n-2)(n-3)}{2} \prod_{m=3}^{n} \delta_{m, \text{min}} \bar{T}_{m} \times \left\{ 1 - \frac{2\delta_{n-2, \text{max}}}{(n-2)(n-3)} \right\} \geq 0 \]
(3.22)

2. \( \quad \text{if} \quad H^{n+1} < 0 \)
\( \Rightarrow \quad \forall (q, \Lambda) \in \mathcal{E}^{4n} \times ]0, 1/6[ \)
\[ \Re\langle f_{(M)}, \tau^{n+1} f_{(N)} \rangle \geq \hat{h}(n, \Lambda) \| f \|^{2} (G_{1})^{n-1} \]
where \( \| f \|^{2} \) and \( (G_{1})^{n-1} \) are the same as before.
and :
\[ \hat{h}(n, \Lambda) = \frac{n(n-1)\bar{T}_{n}}{2} \left\{ 1 - \frac{2\delta_{\text{max}}}{n(n-1)} \right\} \prod_{m=3}^{n-2} \delta_{m, \text{min}} \bar{T}_{m} \geq 0 \]
(3.23)

The proof of Theorem 3.1 is presented in Appendix 4.2

Finally, as a corollary we directly obtain our main result:

Theorem 3.2 For every \( n = 2k + 1, k \in \mathbb{N} \) the \((O.S.P.n)\) conditions 2.12 are verified under the following “weak condition” imposed on the physical coupling constant:
\[ \Lambda < \frac{1}{6} \]
(3.24)
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4 APPENDICES

1 The first examples

APPENDIX 4.1

1. The matrix representations of $\langle f(M), \tau^4 f(N) \rangle$ and $\langle f(M), \tau^6 f(N) \rangle$. As we noticed before (cf. remark 2.1) the matrix elements denoted by the symbol (*) are not taken into account in formulas 2.12 and 2.13 because they are such that $M + N > n + 1$.

$$P_3 = \begin{pmatrix}
(1,1) & 0 & (1,3) \\
0 & (2,2) & 0 \\
(3,1) & 0 & (*)
\end{pmatrix}$$

$$P_5 = \begin{pmatrix}
(1,1) & 0 & (1,3) & 0 & (1,5) \\
0 & (2,2) & 0 & (2,4) & 0 \\
(3,1) & 0 & (3,3) & 0 & (*) \\
0 & (4,2) & 0 & (*) & 0 \\
(5,1) & 0 & (*) & 0 & (*)
\end{pmatrix}$$ (4.25)

2. Proof of Lemma 2.1 (The proof for $n \leq 5$)

a) For $n = 1$ by using the positivity of $H^2$ we have trivially:

$$\langle f(1), \tau^2 f(1) \rangle = \langle f(1) H^2 f(1) \rangle = \int |f^{(1)}(q_1)|^2 H^2(q_1)|\Delta_F(q_1)|^2 dq_1 \geq 0 \quad (4.26)$$

b) For $n = 3$ we estimate every connected contribution, of

$$2\Re\langle f(1), \tau^4 f(3) \rangle \text{ and } \langle f(2), \tau^4 f(2) \rangle \quad (4.27)$$

By using the decomposition of $\tau^4$ into its connected parts:

$$H^4 \prod_{l=1}^3 \Delta_F(q_l) \text{ and } \prod_{l=1}^3 (H^2 \prod_1^3 \Delta_F^2(q_l))$$

we have

$$\Re\langle f(1), \tau^4 f_3 \rangle = \Re\langle f(1), H^4 \prod_{l=1}^3 \Delta_F f_3 \rangle + \Re\langle f(1), \prod_{l=1}^3 H^2_l \Delta_F^2 f_3 \rangle \quad (4.28)$$

We consider the first term of the r.h.s. of 4.28. By application of splitting and sign properties of $H^4$ (cf. definition 4.2) together with the factorized test functions (in view of the nuclear theorem as explained in
definition 2.1) and by Fubini’s theorem we write:

\[ \Re \langle f(1), H^4 f(3) \rangle = -\Re \int \bar{f}(1)(q_1) \delta_3(q(3)) \prod_{l=1}^{3} (H^2 \Delta F^2_l)(q_l) f^{(l)}(q_l) dq_l \]

\[ \geq -|\Re \int \prod_{l=2}^{3} (f^{(l)} H^2 \Delta F^2_l)(q_l) dq_l \left\{ \int |f^{(1)}(q_1)|^2 \delta_3(q) (H^2 \Delta F^2)(q_1) dq_1 \right\}| \]

The integral with respect to \( q_1 \) is positive and taking into account definition 4.3 for the upper bound of the splitting function:

\[ \forall (q, \Lambda) \in E^{12} \times [0, 0.04] \quad \delta_3(q, \Lambda) \leq \delta_{3, \text{max}} < 6\Lambda \]

we finally obtain:

\[ \Re \langle f(1), H^4 f(3) \rangle \geq -6\Lambda \|f^{(1)}\|^2 (G_1)^2 \]

where we used the following notations:

\[ \|f\|^2 = \int |f^{(1)}(q_1)|^2 (H^2 \Delta F^2_l)(q_1) dq_1 \]

and,

\[ (G_1)^2 = (-1)^2 \left\{ \sup_{(l)} \int |f^{(l)}(q_l)| (H^2 \Delta F^2_l)(q_l) dq_l \right\}^2 \]

In an analogous way, and by using the positive sign of \( H^2 \) point function the second term of the r.h.s. of 4.28 yields:

\[ \Re \langle f(1), \prod_{l=1}^{3} H^2 \Delta F^2_l, f(3) \rangle \]

\[ \geq \Re \int |f^{(1)}(q_1)|^2 H^2 \Delta F^2_{l}(q_l) dq_1 (-1)^2 \left\{ \sup_{(l)} \int |f^{(l)}(q_l)| (H^2 \Delta F^2_l)(q_l) dq_l \right\}^2 \]

\[ \geq \|f\|^2 (G_1)^2 \geq 0 \]

Now, by inserting the results 4.31 and 4.34 in 4.28 we obtain:

\[ \Re \langle f(1), \tau^4 f(3) \rangle \geq (1 - 6\Lambda) \|f\|^2 (G_1)^2 \geq 0 \]

under the condition \( \Lambda < \frac{\Lambda}{6} \)

By using analogous arguments to the ones we previously presented, a similar result is obtained for the contribution of the term \( \langle f(2), \tau^4 f(2) \rangle \geq 0 \). Precisely:

\[ \langle f(2), \tau^4 f(2) \rangle \geq (1 - 6\Lambda) \|f\|^2 (G_1)^2 \geq 0 \]
under the condition $\Lambda < \frac{1}{6}$.

Finally:

$$\sum_{1 \leq M \leq 3, 1 \leq N \leq 3, M+N \leq 4} \langle f_N, \tau^4 f_{M} \rangle \geq 0$$

(4.37)

under the condition $\Lambda < \frac{1}{6}$.

c) For $n = 5$

Following [2.15] the supplementary condition to ensure is the positivity of the sum:

$$2 \Re \langle f(1), \tau^6 f(5) \rangle + 2 \Re \langle f(2), \tau^6 f(4) \rangle + \langle f(3), \tau^6 f(3) \rangle \geq 0$$

(4.38)

with:

$$\tau^6 = H^6 \prod_{l=1}^{5} \Delta_F(q_l)$$

$$+ \frac{5!}{3!2} H^4 \prod_{l=1}^{3} \Delta_F(q_l) \prod_{l=4}^{5} (H^2 \Delta^2_F(q_l)) (q_l)$$

$$+ \frac{5!}{3!2} \prod_{l=1}^{3} (H^2 \Delta^2_F(q_l)) (q_l) \prod_{l=4}^{5} (H^2 \Delta^2_F(q_l)) (q_l)$$

(4.39)

The procedure being similar for each one of the terms in [4.38] we give the proof only for $2 \Re \langle f(1), \tau^6 f(5) \rangle,$ and for each term of the connected parts decomposition [4.39]

- By using the splitting and sign properties of $H^6$ and $H^4$ (cf. def.4.2c) we have:

$$H^6 = \delta_5(q(5)) \delta_3(q(3)) \prod_{l=1}^{5} (H^2 \Delta^2_F(q_l))$$

(4.40)

Then, as before we apply the factorization of the test functions and Fubini’s theorem on the first term connected part contribution in [4.39] and write as follows:

$$\Re \langle f(1), H^6 \prod \Delta_F f(5) \rangle = \Re \int \prod_{l=2}^{5} f^{(1)}(q_l)$$

$$\left\{ \int |f^{(1)}(q_1)|^2 \delta_5(q(5)) \delta_3(q(3)) (H^2 \Delta^2_F(q_1)) dq_1 \right\} (H^2 \Delta^2_F(q_l)) dq_l$$

(4.41)

The integral with respect to $q_1$ being positive and taking into account definition [4.3] for the lower bounds of the (solution in [17]) splitting functions, precisely:

$$\forall (q, \Lambda) \in \mathcal{E}^{12} \times [0, 0.04] \quad \delta_3(q, \Lambda) \geq \delta_{3, \text{min}}(\Lambda)$$

and

$$\forall (q, \Lambda) \in \mathcal{E}^{20} \times [0, 0.04] \quad \delta_5(q, \Lambda) \geq \delta_{5, \text{min}}(\Lambda)$$

(4.42)

the factorization of the test functions and Fubini theorem, we finally obtain:

$$\Re \langle f(1), H^6 \prod \Delta_F f(5) \rangle \geq \delta_{5, \text{min}} \delta_{3, \text{min}}(\Lambda) \| f \|^2 (G_1)^4 \geq 0$$

(4.43)
without any supplementary condition on $\Lambda$.

We also notice that the real positive numbers $\|f\|^2$ (and resp. $(G_1)^4$) are defined by analogy with $\|$ and $(G_1)\%$ respectively.

- For the second and third terms we proceed by analogy. We take the sum of them and by using the sign-splitting property of $H^4$ and the corresponding results of $4.31$ and $4.34$ we write:

$$\frac{5!}{3!^2} \Re \langle f(1), \tau^4 f(3) \rangle \geq \frac{5!}{3!^2} (1 - 6\Lambda) \|f\|^2 (G_1)^4 \geq 0 \quad \text{under the condition} \quad 0 < \Lambda < \frac{1}{6}$$

Finally

$$2 \Re \langle f(1), \tau^6 f(5) \rangle \geq \left\{ \delta_{\text{min}} \delta_{\text{min}} (\Lambda) + \frac{5!}{3!^2} (1 - 6\Lambda) \right\} \|f\|^2 (G_1)^4 \geq 0$$

under the condition $0 < \Lambda < \frac{1}{6}$

(4.44)

Conclusion

From the results of a) b) c) we finally obtain:

$$\sum_{1 \leq M \leq 5} \sum_{M \leq N \leq n} \langle f(N), \tau^6 f(M) \rangle \geq 0$$

under the condition $\Lambda < \frac{1}{6}$

(4.45)

2 The proof of Theorem 3.1

APPENDIX 4.2

We suppose that the statement holds $\forall \bar{n} \leq n - 2$

1. a) Let $H^{n+1} > 0$ (or $T_1^n > 0$).

For an arbitrary couple $(M, N)$ with $M \leq n, N \leq n$ and $N + M = n + 1$ we consider the corresponding first term in the sum $\sum_{\omega_n}$ of equation 2.8 (that means when $k = 1$). We suppose that the momentum variables are ordered and the test functions factorized then by Fubini’s
theorem we write:
\[
\mathcal{R} \left\langle f(M) H_{n+1} \prod_{M=1}^{N} \Delta_F f(N) \right\rangle = \mathcal{R} \int \left\{ I_{nt}(q(n-1)) \right\} \left( \prod_{i=1}^{N} f^{(i)}(q_i) \Delta_F(q_i) dq_i \right) \prod_{l=2}^{N} f^{(l)}(q_l) \Delta_F(q_l) dq_l
\]
Here:
\[
I_{nt}(q(n-1)) = \int |f^{(M)}(q_M)|^2 H^{(n+1)}(q_{(n)}) \Delta_F(q_M) dq_M
\]
(4.46)
The positivity of the integrand of \( I_{nt}(q(n-1)) \) allows us to take the lower bound of \( H_{n+1} \) by using lemma 3.1 (of complete splitting) and by following an analogous procedure as the one of \( H_6 \) we obtain:
\[
\mathcal{R} \left\langle f(M) H_{n+1} \prod_{M=1}^{N} \Delta_F f(N) \right\rangle \geq \prod_{m=3}^{n} \delta_{m,min} \mathcal{T}_m \| f \|^2 \prod_{i=1}^{N} (-1)^{(i)} \sup_i \int |f^{(i)}(q_i)| H^{(2)}(q_i) \Delta_F(q_i) dq_i \prod_{l=2}^{N} (-1)^{(l)} |f^{(l)}(q_l)| H^{(2)}(q_l) \Delta_F(q_l) dq_l
\]
\[
\geq \prod_{m=3}^{n} \delta_{m,min} \mathcal{T}_m \| f \|^2 (-1)^{(n-1)} \left\{ \sup_i \int |f^{(i)}(q_i)| (H_{n+1} \Delta_F)(q_i) dq_i \right\}^{n-1}
\]
\[
\geq \prod_{m=3}^{n} \delta_{m,min} \mathcal{T}_m \| f \|^2 (G_1)^{n-1} \geq 0
\]
(4.47)
So we obtain the positivity without any supplementary condition on the coupling constant than \( 0 < \Lambda < \frac{1}{6} \) which is required by the recurrence hypothesis. We notice that \( \| f_1 \|^2 \) (and resp. \( (G_1)^{(n-1)} \)) are always defined by analogy with 4.32 (and 4.33 respectively).

\[\blacksquare\]

b) Taking into account the notations 2.11 we have to show that:
\[
\forall (M, N) \text{ such that } 1 \leq M \leq \frac{n+1}{2} \leq N \leq n,
\forall (q, \Lambda) \in \mathcal{E}^{4n} \times [0, 1/6], \sum_{1 \leq M \leq n, 1 \leq N \leq n, N + M = n+1} \langle f(M), (T_2^n + T_3^n) f(N) \rangle \geq 0
\]
(4.48)
or equivalently by application of Lemma 3.2 1 a) equation 3.18 show that:
\[
\forall (q, \Lambda) \in \mathcal{E}^{4n} \times [0, 1/6], \mathcal{R} \left( f(M) \sum_{I \in \omega_n(3)} C(I) \prod_{l=1}^{3} e^{-i \tau l+1} f(N_l) \right) \geq 0
\]
(4.49)
We always suppose that the moments are ordered and the test functions factorized, so by Fubini’s theorem we write:

\[
\Re \langle f(M) \sum_{I \in \mathcal{N}(3)} C(I) \prod_{l=1}^{3} \tau_{i_l+1} f(N) \rangle = \Re \int \left\{ \text{Int} \left( q(n-1) \right) \right\} \prod_{i=1}^{M-1} f^{(i)}(q_i) dq_i \prod_{l=2}^{N} f^{(i)}(q_l) dq_l
\]

(4.50)

Where:

\[
\text{Int} \left( q(n-1) \right) = \int |f^{(M)}(qM)|^2 \sum_{I \in \mathcal{N}(3)} C(I) \prod_{i=1}^{l=2,3} \tau_{i_{i_l}+1} (q(i_l)) dqM
\]

(4.51)

Notice that following Lemma [3.2] 1 b) (and the recurrence hypothesis) each one of the terms and consequently the sum itself are positive. So a lower bound of the sum could be the “first” term-contribution \( \hat{T} = (n - 2, 1, 1) \) to obtain:

\[
\text{Int} \left( q(n-1) \right) \geq C_{T=_{(n-2,1,1)}} \int |f^{(M)}(qM)|^2 \tau_{n-1} (q(n-2)) dqM \times \prod_{l=n-1}^{n} H^2(q_l) \Delta_F^2(q_l)
\]

(4.52)

Now by [2.11] we have:

\[
\tau^{(n-1)} = T_1^{(n-2)} + T_2^{(n-2)} + T_3^{(n-2)}
\]

(4.53)

But, \( H^{n-1} < 0 \) so by the recurrence hypothesis of lemma [3.2] 2.a and b) (precisely on \( \delta_{n-2} \)), and then by application of Lemma [3.1] on \( H^{(n-3)} > 0 \), we obtain:

\[
\forall (q, \Lambda) \in \mathcal{E}^{4n} \times ]0, 1/6[ \\
T_1^{(n-2)} + T_2^{(n-2)} \geq \frac{(n-2)(n-3)}{2} \prod_{m=3}^{n-4} \delta_{m, min} T_m \\
\times \prod_{i=1}^{n-2} H^{2}(q_i)[\Delta_F(q_i)]^2 \left\{ 1 - \frac{2\delta_{n-2, max}}{(n-2)(n-3)} \right\} \geq 0
\]

(4.54)

and \( T_3^{(n-2)} \geq 0 \)

So, by taking into account [4.54] inside [4.53] a lower bound of \( \text{Int}(q_{(n-1)}) \) is obtained that we insert in equation [4.51]. Finally, we add the proof of
a) \[4.47\] for \(T_1^n\), and the result is as follows:

\[
\forall \text{ positive integers } N, M, \text{ such that } 1 \leq M \leq n, 1 \leq N \leq n \text{ with: } N + M = n + 1 \text{ and } \forall (q, \Lambda) \in E^{4n} \times [0, 1/6[,
\]

\[
\Re(f(M), (T_1^n + T_2^n + T_3^n)f(N)) \geq h(n, \Lambda) \| f \|^2 (G_1)^{n-1} \geq 0
\]

where:

\[
h(n, \Lambda) = \frac{(n-2)(n-3)}{2} \prod_{m=3}^{n} \delta_{m,min} \bar{T}_m \times \left\{ 1 - \frac{2\delta_{n-2,max}}{(n-2)(n-3)} \right\} \geq 0
\]

(4.55)

This completes the proof of the theorem \[3.1\] for \(H^{n+1} > 0\).

\[\blacksquare\]

2. Case \(H^{n+1} < 0\).

\[\text{a) We first have to show the positivity of:} \]

\[
2\Re(f(M), (T_1^n + T_2^n)f(N))
\]

We suppose always that the moments are ordered and the test functions factorized then by Fubini’s theorem we write:

\[
\Re(f(M)(T_1^n + T_2^n)f(N)) = \Re \left\{ I_{nt}(q(n-1)) \right\} \prod_{i=1}^{M-1} f^{(i)}(q_i) dq_i \prod_{l=2}^{N} f^{(l)}(q_l) dq_l
\]

(4.56)

Here:

\[
I_{nt}(q(n-1)) = \int |f^{(M)}(q_M)|^2 (T_1^n + T_2^n)(q(n)) dq_M
\]

By using Lemma \[3.2\] 2.a) we apply the positive lower bound (cf. \[3.20\]) of \(T_1^n + T_2^n\), and then by application of Lemma \[3.1\] on \(H^{n-1}\), we finally obtain:

\[
\Re(f(M)(T_1^n + T_2^n)f(N)) \geq \| f \|^2 (-1)^{(n-1)} \left\{ \sup_{(i)} \int |f^{(i)}(q_i)| (H^2 \Delta F)(q_i) dq_i \right\}^{n-1} \times \frac{n(n-1)\bar{T}_n}{2} \left\{ 1 - \frac{2\delta_{nmax}}{n(n-1)} \right\} \prod_{m=3}^{n-2} \delta_{m,min} \bar{T}_m \geq 0
\]

(4.57)

under the condition \(0 < \Lambda < \frac{1}{6}\).

\[\blacksquare\]

\[\text{b) The third term } T_3^n \text{ in } 2.13 \text{ of the remaining connected parts is exactly the sum appearing on the r.h.s. of } 3.21. \]

By the recurrence hypothesis every
term of this sum (and evidently the sum itself) is positive (Lemma 3.2 b) (for \( \Lambda < 1/6 \)) so we can take a lower bound of this sum by the “first term” \( \tilde{\tau}^{n-1} = \tau^{n-1} - T_1^{n-1} \) and the recurrence hypothesis of the positive \( H^{n-1} \) by proceeding as in the previous proof of \( H^{n+1} > 0 \). The remaining procedure is then analogous to that of \( T_2^n + T_3^n \) (cf. results: 4.47 4.48).

\[ (f(M) \sum_{I \in \mathcal{I}_n(3)} C(I) \prod_{l=1}^{3} \tilde{\tau}_{i_l+1} f_N) \geq (f(M), (T_2^{n-2} + T_3^{n-2}) f_N) \geq 0 \quad \forall \Lambda \in ]0, \frac{1}{6} [ \]

3 Proof of Lemma 3.2

APPENDIX 4.3 We suppose that the Lemmas 3.1, 3.2 are verified \( \forall \bar{n} \leq n - 2 \). Then,

1. if \( H^{n+1} > 0 \), we show that:

\[ (\tau^{n+1} - T_1^{n}) = \sum_{I \in \mathcal{I}_n(3)} C(I) \prod_{l=1,2,3} \tau_{i_l+1} \]

with:
\[ C(I) = \frac{n!}{i_1!2!i_3!} \]

or equivalently, following formulas 2.11

\[ T_2^n + T_3^n = \sum_{I \in \mathcal{I}_n(3)} C(I) \prod_{l=1,2,3} \tau_{i_l+1} \]

In other words we have to reformulate the “classical” decomposition of every non connected Green’s function \( \tau^{n+1} \) of definition 2.8 to a “tree” type (cf. 4.76) recursive expression in terms of the preceding non connected \( \tau^{i+1} \)’s \( \forall i \leq n - 2 \).

Remark 4.1 We recall that in the standard definition of 4.76 (cf. 22 c)

\( \forall I \in \mathcal{I}_n(3) \) with \( I = (I_1, I_2, I_3) \), \( \text{Card} I_l = i_l \) where \( i_l, l \in \{1, 2, 3\} \)

are odd integers such that: \( i_1 \geq i_2 \geq i_3 \), and \( \sum_{l=1}^{3} i_l = n \).

The starting point is the tree structure of \( T_2^n \) namely:

\[ T_2^n = \sum_{I \in \mathcal{I}_n(3)} C(I) \prod_{l=1,2,3} H^{i_l+1} \Delta_F(q_l) = \frac{C^{n+1}}{-6\Lambda} \]
This expression translates the first step decomposition of \( \tau^{n+1} \) as a sum of three factors’ products (or “tree type in smaller non connected parts”).

Before giving the details of the proof let us introduce some useful notations in order to display the relationship between the corresponding terms of \( T_2^n \) and \( T_3^n \):

\( i) \) Notation for every term of \( T_2^n \):

\[
T_{3,3}^{(i_1,1)}(q(n)) = C(I) \prod_{l=1,2,3} H_{i_l+1}^{i_l+1} \Delta F(q_{i_l}) \quad (4.62)
\]

In other words we shall often describe \( T_2^n \) as follows:

\[
T_2^n = \sum_{I \in \omega_n(3)} C(I) T_{3,3}^{(i_1,1)} \quad (4.63)
\]

Example: If \( i_1 = n - 2 \), then

\[
T_{3,3}^{(n-2,1)}(q(n)) = \frac{n!}{(n-2)!2} H^{n-1}(q(n-2)) \prod_{l=n-1}^{n} H^2(q_l) \Delta F(q_l)^2 \quad (4.64)
\]

\( ii) \) \( \forall \) fixed \( k = 3, 5, 7, \ldots i_1, i_1 + 2, \) we use the following notation of the \((k-2)\)th order step of the classical type development of \( H_{i_1+1} \) (or of the decomposition (2.11) of \( \tau_{i_1+1} \)) in terms of sums of “tree type” products. It corresponds to the \( k\)th order “new tree type decomposition of \( \tau^{n+1} \):

\[
T_{3,k}^{(i_1,k-2)} = \frac{n!}{i_1!i_2!i_3!} C(j_1,\ldots,j_k) \prod_{1 \leq l \leq k} H_{j_l+1}^{(j_l+1)} \quad (4.65)
\]

here: \( C(j_1,\ldots,j_k) = \frac{i_1!}{j_1!\ldots j_{k-1}! j_k!} \)

Then, starting from the first term-triplet of the sum (4.61) precisely: \( T_{3,3}^{(n-2,1)} \) (given previously by (4.64) we proceed in a decreasing order of \( i_1 \)’s and repeat for every triplet in formula of (4.61) the same recurrent procedure which follows:

By keeping invariant the product \( H^{i_2+1} H^{i_3+1} \) we obtain the contribution of the triplet to the step \( k = 5 \) (reminder: \( k=5 \) factors) of the classical expansion of \( \tau^{n+1} \), by writing the tree type expansion \( T_{2}^{(1)} \), of \( H^{i_1+1} \), so:

\[
T_{3,5}^{(i_1,3)} = \frac{n! \prod_{l=2,3} H_{i_l+1}^{i_l+1}}{i_1!i_2!i_3!} \sum_{I \in \omega_{i_1}(3)} \frac{i_1!}{i_1^{(1)}i_2^{(1)}i_3^{(1)}} \prod_{l=1,2,3} H_{i_l}^{(i_l+1)} \Delta F(q_{i_l}) \quad (4.66)
\]
For the following step \( k = 7 \) we apply by decreasing order of every triplet the same tree type expansion of every first factor \( H_{i_1}^{l_1+1} \), and so on . . . In other words we develop successively the \( H_{i_1}^{l_1+1} \) (the first factor) step by step in “tree type” \( T_1^{l_2} \) or \( C_{i_1+1}^{l_1+1} / (-6\Lambda) \) sums of products of “smaller” and “smaller” connected Green’s functions. Every new step \( k \) is obtained from the step \( k - 2 \) by developing the first \( H_{i_1}^{l_1+1} \) by decreasing order of the triplets of the previous \( T_1^{l_2} \). So, we obtain in a precise order the terms: \( T_3^{n,(i_1,k-2)} H_{i_2}^{l_2+1} H_{i_3}^{l_3+1} \) from \( 5 \leq k \) up to \( k \leq i_1 + 2 \) (cf. definitions [2.11]).

The sum, of all the intermediate steps at this level yields:

\[
\left( \sum_{k=3,5,\ldots,i_1+2} T_3^{n,(i_1,k-2)} \right) H_{i_2}^{l_2+1} H_{i_3}^{l_3+1} = \tau_{i_1+1}^{l_1+1} H_{i_2}^{l_2+1} H_{i_3}^{l_3+1} \quad (4.67)
\]

By keeping unchanged \( \tau_{i_1+1}^{l_1+1} H_{i_3}^{l_3+1} \) we continue in an analogous way the reconstruction of \( \tau_{i_2+1}^{l_2+1} \) by using the “new type in smaller connected parts decomposition” successively of \( H_{i_2}^{l_2+1} \) in terms of sums of tree type products. The result is equal to \( \tau_{i_1+1}^{l_1+1} \tau_{i_2+1}^{l_2+1} H_{i_3}^{l_3+1} \). The proof of every triplet reconstruction is completed by the synthesis in an analogous way of \( \tau_{i_3+1}^{l_3+1} \).

\[ \blacksquare \]

b) We take into account the proven property [4.59] and the recurrence hypothesis of both lemmas for every \( i_1 \leq n - 2 \) then the proof of positivity under the condition \( \Lambda \leq 1/6 \) is automatically obtained:

\[
\forall (q, \Lambda) \in \mathcal{E}^{4n} \times [0, 1/6],\quad \tau^{n+1} - T_1^n \geq 0 \quad (4.68)
\]

\[ \blacksquare \]

2. Let \( H^{n+1} < 0 \).

a) We first show the properties [3.20]

The lower bound of the tree global term \( \frac{|C_{3,1}^{n+1}|}{6\Lambda} \). We consider the definition [2.11] of \( T_1^n \) and \( T_2^n \). By using on one hand the definitions [4.2] [4.84] of the splitting of \( H^{n+1} \) and on the other hand the definition [4.76] of the tree term of the mapping \( C_{n+1} \), we associate with every connected contribution of the tree \( C_{n+1} \), the corresponding connected contribution \( \prod_{1 \leq l \leq k} H_{i_l}^{l_l+1} \) in the sum

\[
\sum_{l \in \omega_n(3)} C_{(l)} \prod_{1 \leq l \leq k} H_{i_l}^{l_l+1}
\]

and obtain:

\[
T_1^n + T_2^n = H^{n+1} \prod_{l=1}^n (\Delta F(q_l)) + \sum_{l \in \omega_n(3)} C_{(l)} \prod_{l=1,2,3} H_{i_l}^{l_l+1} \Delta F(q_{i_l})
= \frac{|C_{n+1}^{n+1}|}{6\Lambda} \left\{ 1 - 2 \frac{\delta_n(q, \Lambda)}{n(n-1)} \right\} \quad (4.69)
\]
In order to obtain the positivity property of (3.20) we take the upper bound of the splitting function (cf. 4.84 in the reminders):

\[ \delta_{n,\text{min}}(\Lambda) \leq \delta_n(\tilde{q}, \Lambda) \leq \delta_{n,\text{max}}(\Lambda) \tag{4.70} \]

then, by application of the bound lemma (3.1 ii) we verify:

\[ T_1^n + T_2^n \geq 0 \]

under the condition \( \Lambda < 1/6 \) \tag{4.71}

b) The “tree decomposition” of the term \( T_3^n \) (cf.2.11 in 3.21) is established by the same procedure developed before in the case of the corresponding \( T_3^n \) of \( H^{n+1} > 0 \), with a small difference: the analogous triplets do not contain their first factor:

\[ T_{1i}^n = H^{n+1} \prod_{j=1}^{i_1} \Delta F(q_j) \tag{4.72} \]

because it has been taken into account in the term \( T_2^n \).

Moreover, by using the same arguments we presented before for \( H^{n+1} > 0 \) this term is positive (recurrently).

4 Reminders

APPENDIX 4.4

1. The equations of motion established in [20]

Definition 4.1

\[
H^2(q, \Lambda) = -\frac{\Lambda}{\gamma + \rho} \left\{ \left[ N_3^{(3)} H^4 \right] - \Lambda \alpha H^2(q, \Lambda) \Delta F(q) \right\} + \frac{(q^2 + m^2)\gamma}{\gamma + \rho} \tag{4.73}
\]

(Here \( m > 0 \) and \( \Lambda > 0 \) are the physical mass and coupling constant of the interaction model, and \( \alpha, \beta, \gamma \), are physically well defined quantities associated to this model, the so called renormalization constants). Moreover,

\( \forall n \geq 3, \ (q, \Lambda) \in \mathcal{E}^4n \times \mathbb{R}^+ \)

\[ H^{n+1}(q, \Lambda) = \frac{1}{\gamma + \rho} \left\{ \left[ A^{n+1} + B^{n+1} + C^{n+1} \right](q, \Lambda) + \Lambda \alpha H^{n+1}(q, \Lambda) \Delta F(q) \right\} \tag{4.74} \]

with:

\[
A^{n+1}(q, \Lambda) = -\Lambda [N_3^{(n+2)} H^{n+3}](q, \Lambda); \\
B^{n+1}(q, \Lambda) = -3\Lambda \sum_{n_2(n_j)} [N_2^{(j_2)} H^{j_2+2} N_1^{(j_1)} H^{j_1+1}](q, \Lambda) \tag{4.75}
\]
\[ C^{n+1}(q, \Lambda) = -6\Lambda \sum_{I \in \mathbb{Z}_n(3)} \prod_{l=1,2,3} [N_1^{(l)}|H^{n+1}](q_l, \Lambda) \quad (4.76) \]

Here the notations:
\[ [N_3^{(n+2)} H^{n+3}], \ [N_2^{(j_2)} H^{j_2+2} N_1^{(j_1)} H^{j_1+1}] \text{ and } \prod_{l=1,2,3} [N_1^{(l)} H^{l+1}] \quad (4.77) \]

represent the \( \Phi_4^4 \) operations which have been introduced in the “Renormalized G-Convolution Product” (R.G.C.P) context of the references \([25]\ [26]\).

Briefly, the two loop \( \Phi_4^4 \) operation is defined by:
\[ [N_3^{(n+2)} H^{n+3}] = \int R_G^{(3)} \, H^{n+3} \prod_{i=1,2,3} \Delta_F(l_i) \, d^4k_1 d^3k_2 \quad (4.78) \]

with \( R_G^{(3)} \) the corresponding renormalization operator for the two loop graph.

The analogous expression for the one loop \( \Phi_4^4 \) operation is the following:
\[ [N_2^{(j_2)} H^{j_2+2} N_1^{(j_1)} H^{j_1+1}] = (H^{j_1+1} \Delta_F)(q_{j_1}) \int R_G^{(2)} H^{j_2+2} \prod_{i=1}^2 \Delta_F(l_i) d^4k \quad (4.79) \]

The method is based on the proof of the existence and uniqueness of the solution of the corresponding infinite system of dynamical equations of motion verified by the sequence of the Schwinger functions, i.e the connected, completely amputated with respect to the free propagator Green’s functions:
\[ H = \{ H^{n+1} \}_{n=2k+1, k \in \mathbb{N}} \quad (4.80) \]

in the Euclidean \( r \)-dimensional momentum space, \( \mathbb{E}^{rn} \) (where \( 0 \leq r \leq 4 \)).

2. The subset \( \Phi_R \subset \mathcal{B}_R \) \([17]\)

**Definition 4.2** We say that a sequence \( H \in \mathcal{B}_R \) belongs to the subset \( \Phi_R \), if the following properties are verified:

(a) \( \forall (q, \Lambda) \in \mathbb{E}^4 \times ]0, 0.04[ \)
\[ H^2(q, \Lambda) = (q^2 + m^2)(1 + \delta_1(q, \Lambda) \Delta_F) \]
with \( \delta_1(q, \Lambda) \Delta_F(q)|_{q^2+m^2=0} = 0 \ or \ H^2 \Delta_F(q)|_{q^2+m^2=0} = 1 \)
and \( H_{\min}^2(q) \leq H^2(q, \Lambda) \leq H_{\max}^2(q, \Lambda) \)

with \( H_{\max}^2(q, \Lambda) = \gamma_{\max}((q^2 + m^2) + 6\Lambda^2(q^2 + m^2) \frac{q^2}{m^2}); \)
\[ H_{\min}^2(q) = q^2 + m^2 \quad (4.81) \]
(b) For every \( n = 2k + 1, k \in \mathbb{N}^* \) the function \( H^{n+1} \), belongs to the class \( A_{4n}^{(\alpha_n, \beta_n)} \) of Weinberg functions such that \( \forall S \subset E^{4n} \) the corresponding asymptotic indicatrices are given by:

\[
\alpha_n(S) = \begin{cases} 
-(n - 3) & \text{if } S \not\subset \text{Ker } \lambda_n \\
0 & \text{if } S \subset \text{Ker } \lambda_n \\
\beta_n(S) = n \beta_1 & \forall S \subset E^{4n} \\
(\text{with } \beta_1 \in \mathbb{N} \text{ arbitrarily large})
\end{cases} \tag{4.82}
\]

(c) There is an increasing and bounded (with respect to \( n \)) associated positive sequence: \( \{\delta_n(q, \Lambda)\}_{n=2k+1, k \in \mathbb{N}^*} \) of splitting functions \( \in D \) which belong to the class \( A_{(0,0)}^n \) of Weinberg functions for every \( n \geq 3 \) such that \( H \) is a tree type sequence. More precisely:

i) \( \forall (q, \Lambda) \in E^{12} \times [0, 0.04] \)

\[
H^4(q\Lambda) = -\delta_3(q, \Lambda) \prod_{l=1,2,3} H^2(q_l, \Lambda) \Delta F(q)
\]

with \( \delta_3(q, \Lambda) \sim \Lambda^{-1} \)

For every finite fixed \( \bar{q} \in E^{12} \)

\[
\lim_{\Lambda \to 0} \frac{\delta_3(\bar{q}, \Lambda)}{\Lambda} = 6
\]

and \( \forall \Lambda \in [0, 0.04] \)

\( \delta_{3,\text{min}}(\Lambda) \leq \delta_3(q, \Lambda) \leq \delta_{3,\text{max}}(\Lambda) \) \tag{4.83}

ii) For every \( n = 2k + 1, k \geq 2 \) and \( \forall (q, \Lambda) \in E^{12} \times [0, 0.04] \):

\[
H^{n+1}(q, \Lambda) = \frac{\delta_n(q, \Lambda) C^{n+1}(q, \Lambda)}{3 \Lambda n(n - 1)}
\]

with \( \delta_n(q, \Lambda) \sim \Lambda^{-1} \)

For every finite fixed \( \bar{q} \in E^{4n} \)

\[
\lim_{\Lambda \to 0} \frac{\delta_n(\bar{q}, \Lambda)}{\Lambda} \sim 3n(n - 1)
\]

and \( \forall \Lambda \in [0, 0.04] \)

\( \delta_{n,\text{min}}(\Lambda) \leq \delta_n(q, \Lambda) \leq \delta_{n,\text{max}}(\Lambda) \) \tag{4.84}

Here \( \{\delta_{n,\text{min}}\}, (\text{but not } \{\delta_{n,\text{max}}\}) \) are the splitting sequences lower bounds of the solution of the zero dimensional problem (cf. definition 4.3 of the reminders).

iii) Moreover there is a finite number \( \delta_\infty \in \mathbb{R}^+ \) a uniform bound independent of \( H \) such that:

\[
\lim_{n \to \infty} \delta_n(q, \Lambda) \leq \delta_\infty \quad \forall \Lambda \in [0, 0.04] \tag{4.85}
\]

(d) The renormalization functions \( \alpha, \rho \) and \( \gamma \), appearing in the definition of \( M \) are well defined real analytic functions of \( q^2 \) and \( \Lambda \), and yield
at the limits \((q^2 + m^2) = 0\) and \(q = 0\) the physical conditions of renormalization required by the two-point and four-point functions:

\[
\alpha(q, \Lambda) = [N_3^{(3)} H^4(q, \Lambda)] \quad \text{and} \quad \tilde{\alpha}(\Lambda) = [N_3^{(3)} H^4(q, \Lambda)]_{|q^2 + m^2 = 0}
\]

with: \(a_{\text{min}}(\Lambda) \leq \tilde{\alpha}(\Lambda) \leq a_{\text{max}}(\Lambda)\)

\[
\rho(q, \Lambda) = \left[ \frac{\partial}{\partial q^2} [N_3^{(3)} H^4(q, \Lambda)] \right], \quad \text{and} \quad \tilde{\rho}(\Lambda) = \rho(q, \Lambda)_{|q^2 + m^2 = 0}
\]

with: \(\rho_{\text{min}}(\Lambda) \leq \tilde{\rho}(\Lambda) \leq \rho_{\text{max}}(\Lambda)\)

\[
\gamma(q, \Lambda) = \left[ -6\Lambda \prod_{l=1,2,3} H^2(q_{\ell}) \Delta F(q_{\ell}) \right] \frac{H^4(q)}{H^4(q)}
\]

and \(\tilde{\gamma}(\Lambda) = \gamma(q, \Lambda)_{|q = 0}\) with \(\gamma_{\text{min}}(\Lambda) \leq \tilde{\gamma}(\Lambda) \leq \gamma_{\text{max}}(\Lambda)\)

3. **Definition 4.3** The upper and lower bounds of the splitting sequences and of the renormalization parameters (cf. [17]):

\[
\forall \Lambda \in [0, 0.04] \quad \delta_{3,\text{max}}(\Lambda) = \frac{6\Lambda}{1 + \rho_0 + \Lambda |a_0| + 6d_0}; \quad \delta_{3,\text{min}}(\Lambda) = \frac{6\Lambda}{1 + 9\Lambda (1 + 6\Lambda^2)}
\]

and \(\forall n \geq 5\)

\[
\delta_{n,\text{max}}(\Lambda) = \frac{3\Lambda n (n - 1)}{1 + \rho_0 + \Lambda |a_0| + n(n - 1)d_0}
\]

with:

\[
a_0 = -\delta_{3,\text{min}} [N_3^{(3)} q^2 + m^2 = 0]; \quad \rho_0 = \Lambda \delta_{3,\text{min}} \left[ \frac{\partial}{\partial q^2} [N_3^{(3)}]_{q^2 + m^2 = 0} \right]
\]

and

\[
\delta_{n,\text{min}}(\Lambda) = \frac{3\Lambda n(n - 1)}{\gamma_{\text{max}} + \rho_{\text{max}} + \Lambda |a_{\text{max}}| + 3\Lambda n(n-1)}
\]

with

\[
\gamma_{\text{max}} = 1 + 9\Lambda (1 + 6\Lambda^2), \quad \gamma_{\text{min}} = 1, \quad \rho_{\text{max}} = 6\Lambda^2 \left[ \frac{\partial}{\partial q^2} [N_3^{(3)}]_{q^2 + m^2 = 0} \right] \quad \text{and} \quad |a_{\text{max}}| = 6\Lambda [N_3^{(3)} q^2 + m^2 = 0]
\]

4. **The signs and bounds**

The following properties have been established in [17][18] at every order of the \(\Phi_4^4\)-iteration consequently, the sequence \(\{H\}\) solution of the contractive mapping \(\mathcal{M}^*\) also verifies the following:

**Proposition 4.1** \(\forall \Lambda \in [0, 0.04]\)
i) \( \forall q \in \mathcal{E}^4(q) \)
\[
H^2(q, \Lambda) > 0, \quad H^2_{\text{min}}(q, \Lambda) \leq H^2(q, \Lambda) \leq H^2_{\text{max}}(q, \Lambda)
\]
with \( H^2_{\text{min}}(q, \Lambda) = q^2 + m^2 \)
and
\[
H^2_{\text{max}}(q, \Lambda) = \gamma_{\text{max}}[(q^2 + m^2) + 6\Lambda^2(q^2 + m^2)^{\pi^2/54}] \] (4.90)

ii) The global term ("\( \Phi^4 \) operation")
\[
C^{n+1}(q, \Lambda) = -6\Lambda \sum_{\omega_n(I)} \prod_{l=1,2,3} [\mathcal{A}^{(i)}_1 H^{i+1}](q_i, \Lambda) \] (4.91)

verifies the following properties:

a. The "good sign" property:
\[
\forall n = 2k + 1 \quad (k \geq 1) \quad C^{n+1} = (-1)^{\frac{n-1}{2}} |C^{n+1}| \] (4.92)

b. It is a R.P. C. (cf. [17]) consequently it verifies Euclidean invariance and linear axiomatic quantum field theory properties.

c. For every \( n = 2k + 1, k \geq 1 \) the function \( C^{n+1}(q, \Lambda) \), belongs to the class \( A^{(\alpha_n \beta_n)}_{\mathcal{E}(q)} \) of Weinberg functions such that \( \forall S \subset \mathcal{E}(q) \) the corresponding asymptotic indicatrices are given by:
\[
\alpha_n(S) = \begin{cases} 
-(n - 3), & \text{if } S \not\subset \text{Ker } \lambda_n \\
0, & \text{if } S \subset \text{Ker } \lambda_n
\end{cases} \] (4.93)
\[
\beta_n = \beta(1) n \quad \forall S \subset \mathcal{E}(q) \] (4.94)

d. For every \( n = 2k + 1, k \geq 1 \)
\[
|C^{n+1}_{\text{min}}(q, \Lambda)| \leq |C^{n+1}(q, \Lambda)| \leq |C^{n+1}_{\text{max}}(q, \Lambda)|
\]
with:
\[
|C^{n+1}_{\text{min}}(q, \Lambda)| = 3\Lambda (n - 1) T_n |H^{n-1}(q_{(n-2)})| \prod_{l=2,3} (H^2 \Delta_F)(q_l)
\]
\[
|C^{n+1}_{\text{max}}(q, \Lambda)| = 3\Lambda (n - 1) \tilde{T}_n |H^{n-1}(q_{(n-2)})| \prod_{l=2,3} (H^2 \Delta_F)(q_l)
\] (4.95)

Remark 4.2
Notice that in the last formula we take into account the result of ref. [22] c] on the number \( T_n \) (and \( \tilde{T}_n \)) of different partitions inside the tree terms. Precisely:

for \( n = 3, n = 5 \) \( T_n = 1 \)

and,
\[
\forall n \geq 7 \quad T_n = \left\lfloor \frac{(n-3)^2}{48} \right\rfloor + \frac{(n-3)}{3} + 1
\] (4.96)
(\text{where } \left\lfloor . \right\rfloor \text{ means integer part})

and
\[
\tilde{T}_n = \left\lfloor \frac{(n-3)^2}{48} \right\rfloor + \frac{(n-3)}{3}
\]

29
iii) \[ \forall n = 2k + 1 \ (k \geq 1) \quad H^{n+1} = (-1) \frac{n-1}{2} |H^{n+1}| \] (4.97)

iv) \[ \forall n = 2k + 1 \ (k \geq 1) \quad |H^{n+1}_{\min}| \leq |H^{n+1}| \leq |H^{n+1}_{\max}| \] (4.98)

Here \( H^{n+1}_{\max} \) is defined as follows:

\[ |H^4_{\max}| = \delta_{3,\max} \prod_{l=1,2,3} H^2(q_l, \Lambda) \Delta F(q_l) \] (4.99)

Then recurrently \( \forall n = 2k + 1 - k \geq 2 \) and by using the preceding definitions of \( |C^+_{\max}(q, \Lambda)| \) and \( |C^+_{\min}(q, \Lambda)| \), we obtain the bounds:

\[ |H^{n+1}_{\max}(q_{(n)})| = \delta_{n,\max} T_n \Delta F(\sum_{i=1}^{n-2} q_i) |H^{n-1}_{\max}(q_{(n-2)})| \prod_{l=2,3} H^2(q_l) \Delta F(q_l) \]

\[ |H^4_{\min}(q_{(3)})| = \delta_{3,\min} \prod_{l=1,2,3} H^2(q_l, \Lambda) \Delta F(q_l) \]

\[ |H^{n+1}_{\min}(q_{(n)})| = \delta_{n,\min} \tilde{T}_n \Delta F(\sum_{i=1}^{n-2} q_i) |H^{n-1}_{\min}(q_{(n-2)})| \prod_{l=2,3} H^2(q_l, \Lambda) \Delta F(q_l) \] (4.100)