STATIONARY QUANTUM BGK MODEL FOR BOSONS AND FERMIONS IN A BOUNDED INTERVAL

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Abstract. In this paper, we consider the existence problem for a stationary relaxational models of the quantum Boltzmann equation. More precisely, we establish the existence of mild solution to the fermionic or bosonic quantum BGK model in a slab with inflow boundary data. Unlike the classical case, it is necessary to verify that the quantum local equilibrium state is well-defined, and the transition from the non-condensed state to the condensated state (Bosons), or from the non-saturated state to the saturated state (Fermions) does not arise in our solution space.

1. Introduction

The stationary quantum BGK model \[27, 30, 37, 39, 40, 42, 48, 49, 51\] in a bounded interval reads

\[
p_i \frac{\partial f}{\partial x} = \frac{N}{\tau} (\mathcal{K}(f) - f),
\]

subject to boundary conditions:

\[
f(0, p) = f_L(p) \quad \text{for} \quad p_1 > 0, \quad f(1, p) = f_R(p) \quad \text{for} \quad p_1 < 0.
\]

The momentum distribution function \(f(x, p)\) depends on the position \(x \in [0, 1]\) and the momentum \(p \in \mathbb{R}^3\). The Knudsen number \(\tau > 0\) measures how rarefied the gas system is, and is defined by the ratio between the characteristic length and mean free path. Throughout this paper, \(\mathcal{K}\) denotes the local equilibrium of the system. For bosonic case, it represents the Bose-Einstein distribution without condensation, and in the fermionic case, it represents the non-saturated Fermi-Dirac distribution, which will be defined below. To present the exact form of \(\mathcal{K}\), we first define the macroscopic mass, momentum and energy:

\[
N_f(x) = \int_{\mathbb{R}^3} f(x, p) dp, \quad P_f(x) = \int_{\mathbb{R}^3} f(x, p) p dp, \quad E_f(x) = \int_{\mathbb{R}^3} f(x, p) |p|^2 dp.
\]

We then introduce the equilibrium parameter \(a\) and \(c\) defined by ( + sign is for fermion and − sign is for boson, see \([7, 39]\)):

\[
\frac{N_f(x)}{(E_f(x) - |P_f(x)|^2/N_f(x))^2} = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{1 + c(x)} \pm 1} dp}{\left(\int_{\mathbb{R}^3} e^{1 + c(x)} \pm 1 \right)^2}.
\]

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and
\begin{equation}
(a(x) = \left(\int_{\mathbb{R}^3} \frac{1}{e^{\beta_k(x)|p|^2 + c(x) + 1} dp}\right)^{\frac{2}{3}} N(x)^{-\frac{2}{3}}.
\end{equation}

Note that $c$ is determined implicitly. For the later convenience, we define
\begin{equation}
\beta_k(c) = \frac{\int_{\mathbb{R}^3} \frac{1}{e^{\beta_k(x)|p|^2 + c(x) + 1} dp}}{\left(\int_{\mathbb{R}^3} \frac{1}{e^{\beta_k(x)|p|^2 + c(x) + 1} dp}\right)^{\frac{2}{3}}},
\end{equation}

The relations (1.4) and (1.5) arise from the requirement that $F, B$ must share the same mass, momentum and energy with $f$ (See [7]). Now we are ready to define the local quantum equilibriums. [7, 15, 32, 39, 52]

- **Bose-Einstein distribution:** The local equilibrium for bosons is defined as follows:
\begin{equation}
\begin{cases}
B_1(f) = \left(e^{a(x)||p - \frac{\beta_k(x)}{N(x)}|2 + c(x) + 1}\right)^{-1}, & \text{if } \frac{N}{(E - \frac{P^2}{N})^{\frac{3}{2}}} \leq \beta_B(0), \\
B_2(f) = \left(e^{a(x)||p - \frac{\beta_k(x)}{N(x)}|2 - 1}\right)^{-1} + k(x)\delta_{p = \frac{\beta_k(x)}{N(x)}}, & \text{otherwise},
\end{cases}
\end{equation}

where
\begin{equation}
k(x) = N(x) - \beta_B(0) \left(E(x) - \frac{P(x)^2}{N(x)}\right)^{\frac{3}{2}}.
\end{equation}

The dirac delta function corresponds to Bose-Einstein condensation. $B_1$ corresponds to the non-condensation case, while $B_2$ is referred as the condensation case.

- **Fermi-Dirac distribution:** The local equilibrium for fermions is defined as follows:
\begin{equation}
\begin{cases}
F_1(f) = \left(e^{a(x)||p - \frac{\beta_k(x)}{N(x)}|2 + c(x) + 1}\right)^{-1}, & \text{if } \frac{N}{(E - \frac{P^2}{N})^{\frac{3}{2}}} < \beta(-\infty), \\
F_2(f) = \chi_{|p - \frac{\beta_k(x)}{N(x)}| \leq \sqrt{\frac{2N(x)}}{\beta_k(x)}} \chi, & \text{otherwise},
\end{cases}
\end{equation}

where $\chi_A$ denotes the characteristic function on $A$, and the second case of $F_2$ is called the saturated Fermi-Dirac distribution.

Throughout this paper, we will use $B(f)$ to denote the Bose-Enstein distribution without condensation $B_1(f)$, while $F(f)$ is used to denote the non-saturated Fermi-Dirac distribution $F_1(f)$. Also, $K(f)$ denotes either $B(f)$ or $F(f)$.

1.1. **Brief history.** The slab problem corresponds to the situation where there is a gas flow between two parallel gas-emitting plates of infinite size. This arise often in science and engineering, and attracted the interest of many researchers. In the case of the Boltzmann equation, the first mathematical study can be traced back to [2], where the existence of a measure valued solution were investigated. In the framework of weak solutions, Arkeryd and Nouri considered the existence of $L^1$ solution for the inflow boundary conditions in [4] and for the diffusive reflection conditions in [1]. These results were extended to gas mixture problem by Brull [12,13]. Gomeshi studied the existence of unique mild solutions under the condition that the Knudsen number is sufficiently large in [24]. For the related 3d problem near equilibrium, see [19, 20].
In the case of BGK type model, Ukai studied stationary Boltzmann BGK model in slab for fixed large boundary data in \[47\] using a Schauder type fixed point theorem. Nouri \[39\] established the existence of weak solutions for the stationary quantum BGK model with a discretized condensation term in a slab. Bang and Yun obtained the existence and uniqueness of mild solutions for the ES-BGK model under the assumption that gas is sufficiently rarefied and inflow datas are not concentrated on \(p = 0\) in \[8\].

The mathematical research for the quantum relaxation model has just started, and the literature remains extremely limited. The first mathematical study was carried out by Nouri as mentioned above. In \[9, 10\], Braukhoff obtained analytic solutions of quantum BGK type model arising in the study of ultracold fermionic clouds. The global existence and asymptotic behavior of fermionic quantum BGK model near a global Fermi-Dirac distribution were studied by the authors in \[7\]. Presently, authors are not aware of any further analytical results on the quantum BGK models. We refer to \[22, 26, 28, 29, 37, 43, 48, 50\] for numerical studies on the quantum BGK model.

Quantum Boltzmann equation, on the other hand, has seen more progress. We refer to \[11, 16, 17, 18, 31, 32, 33, 35, 34, 38, 44\] for homogeneous problem, and \[1, 3, 5, 14\] for inhomogeneous problems.

1.2. Notations. We define notations and norms that are frequently used throughout this paper.

- Throughout this paper, we fix \(B(f) = B_1(f)\) and \(F(f) = F_1(f)\). Also, \(K(f)\) denotes either \(B(f)\) or \(F(f)\).
- Every constants \(C\) are defined generically. We also use \(C_{a,b,\cdots}\) when it is necessary to explicitly show the dependence on \(a, b, \cdots\). Especially, we denote \(C_{l,u}\) when the constant depends only on the constants defined in \[2.2\] and \(k\).
- When there’s no risk of confusion, we suppress the dependence of the macroscopic fields on \(f\), and denote \(N, P, E\) instead of \(N_f, P_f\) and \(E_f\).
- We define our weighted \(L^1\) norm and weighted \(L^\infty\) as follows:
  \[
  \sup_x ||f||_{L^1} = \sup_x \left\{ \int_{\mathbb{R}^3} |f(x,p)| (1 + |p|^2) dp \right\},
  \]
  \[
  ||f||_{L^\infty} = \sup_{x,p} |f(x,p)| (1 + |p|^2).
  \]

- We use the following notation (See Ch 3):
  \[
  \beta_B^{-1} = (\beta_B|_{[0,\infty)})^{-1}, \quad \beta_F^{-1} = (\beta_F|_{(-\infty,0]})^{-1}.
  \]

This paper is organized as follows. In Section 2 we present the main result and give an example of boundary data satisfying the assumption of main theorem. Section 3 is devoted to the fixed point setup of the problem. We define the solution space and prove that the equilibrium is well defined in this space. Some useful estimates are also introduced in this section. In Section 4, we establish that the solution operator maps the solution space into itself. We prove the main theorem in the final Section 5 by showing that the solution operator is a contraction mapping.

2. Main result

In this section we present our main results. For brevity we denote
\[
(2.1) \quad f_{LR}(p) = f_L(p)1_{p_1>0} + f_R(p)1_{p_1<0},
\]
and define the following quantities:

\[ a_u = 2 \int_{\mathbb{R}^3} f_{LR} dp, \quad a_l = \int_{\mathbb{R}^3} e^{-\frac{a}{|p|^{1/2}}} f_{LR} dp, \quad a_s = \int_{\mathbb{R}^3} \frac{1}{|p|^1} f_{LR} dp, \]

\[ c_u = 2 \int_{\mathbb{R}^3} f_{LR} |p|^2 dp, \quad c_l = \int_{\mathbb{R}^3} e^{-\frac{a}{|p|^{1/2}}} f_{LR} |p|^2 dp, \quad c_s = \int_{\mathbb{R}^3} \frac{1}{|p|^1} f_{LR} |p|^2 dp, \]

and

\[ k = \left( \int_{p_1 > 0} e^{-\frac{a}{|p|^{1/2}}} f_L(p) |p_1| dp \right) \left( \int_{p_1 < 0} e^{-\frac{a}{|p|^{1/2}}} f_R(p) |p_1| dp \right). \]

**Definition 2.1.** We say that \( f \in L^1_2([0,1] \times \mathbb{R}^3_p) \) is a mild solution of (1.1) if \( f \) satisfies

\[
f(x,p) = e^{-\frac{a}{|p|^{1/2}}} f_L(p) \left( \int_0^x e^{-\frac{b}{|p|^{1/2}}} f_L(y) dy \right) f_L(p) + \frac{1}{\tau |p_1|} \int_0^x e^{-\frac{a}{|p|^{1/2}}} f_L(y) dy f_L(p) \]

and

\[
f(x,p) = e^{-\frac{a}{|p|^{1/2}}} f_R(p) \left( \int_x^1 e^{-\frac{b}{|p|^{1/2}}} f_R(y) dy \right) f_R(p) + \frac{1}{\tau |p_1|} \int_x^1 e^{-\frac{a}{|p|^{1/2}}} f_R(y) dy f_R(p) \]

for \( p_1 > 0 \), \( p_1 < 0 \), respectively.

Now we state our main results.

**Theorem 2.2.** Assume \( f_L \) and \( f_R \) satisfy the following conditions:

1. **Boundary data are non-negative:**
   \[ f_{LR} \geq 0, \]

2. **Boundary data satisfy the following integrability conditions:**
   \[ f_{LR}, \quad \frac{1}{|p_1|} f_{LR} \in L^1_2, \]

3. **Contributions of the inflow from the boundary in \( p_2 \) and \( p_3 \) directions are negligible:**
   \[ \int_{\mathbb{R}^2} f_{LP_1 dp_2 dp_3} = \int_{\mathbb{R}^2} f_{RP_1 dp_2 dp_3} = 0. \quad (i = 2, 3) \]

We assume further that

\[ \frac{a u_0}{k^{1/2}} < \beta_B(0) \quad \text{(Boson),} \quad \frac{a u_0}{k^{1/2}} < \beta_F(-\ln 3) \quad \text{(Fermion).} \]

Then for sufficiently large \( \tau \), there exists a unique non-negative mild solution \( f \) of (1.1) satisfying

\[ a_l \leq N(x) \leq a_u, \quad c_l \leq E(x) \leq c_u, \]

and

\[ E(x) N(x) - |P(x)|^2 \geq k. \]

**Remark 2.3.** (1) The meaning of assumption (2.4) will be considered in Chapter 3. (2) Note that in (2.4), the fermion case is restricted to \( \beta_F(-\ln 3) \). This is because we don’t know yet whether \( \beta_F(c) \) for fermion is a strictly monotone decreasing function in the whole range, even though the numerics indicate in that way. This is left as a future project. (3) Extending
this result to include the condensed state (Boson) and the saturated state (Fermion) will be interesting, and is left for the future.

Before we move on to the proof of the theorem, we present a simple example of boundary data which satisfies the assumption of Theorem 2.2 (1), (2), (3) and (2.4) for bosons. Example for fermionic particles can be constructed similarly. We define

\[ f_L(p) = C_L r_1 \leq p_r \leq r_2 e^{-\frac{|p|^2}{2} - \frac{|p|^2}{2}}, \quad f_R(p) = C_R r_2 \leq p_r \leq -r_1 e^{-\frac{|p|^2}{2} - \frac{|p|^2}{2}}, \]

for some \( C_L, C_R > 0 \) and \( r_1, r_2 > 0 \) to be determined soon. Since it can be readily checked that they satisfy the conditions (1), (2), (3) of Theorem 2.2, we check the condition (2.4) only. We first compute \( a_u \) as

\[
a_u = 2 \int_{\mathbb{R}^3} f_L dp \]

\[
= 2 \left( C_L \int_{r_1}^{r_2} dp_1 \left( \int_{-\infty}^{\infty} e^{-\frac{|p|^2}{2}} dp_2 \right)^2 + C_R \int_{-r_2}^{r_1} dp_1 \left( \int_{-\infty}^{\infty} e^{-\frac{|p|^2}{2}} dp_2 \right)^2 \right) \]

\[
= 4\pi (C_L + C_R)(r_2 - r_1). \]

We then compute

\[
\int_{p_1 > 0} e^{-\frac{4\pi}{r_1} f_L(p)} dp_1 \geq \int_{\mathbb{R}^3} f_L(p) dp \]

\[
= C_L e^{-\frac{4\pi(C_L + C_R)(r_2 - r_1)}{r_1}} \int_{r_1}^{r_2} dp_1 \left( \int_{-\infty}^{\infty} e^{-\frac{|p|^2}{2}} dp_2 \right)^2 \]

\[
= \pi C_L e^{-\frac{4\pi(C_L + C_R)(r_2 - r_1)}{r_1}} (r_2 - r_1), \]

and, similarly,

\[
\int_{p_1 < 0} e^{-\frac{4\pi}{r_1} f_R(p)} dp_1 \geq \pi C_R e^{-\frac{4\pi(C_L + C_R)(r_2 - r_1)}{r_1}} (r_2 - r_1), \]

to get

\[
k = \left( \int_{p_1 > 0} e^{-\frac{4\pi}{r_1} f_L(p)} dp_1 \right) \left( \int_{p_1 < 0} e^{-\frac{4\pi}{r_1} f_R(p)} dp_1 \right) \]

\[
= \pi^2 C_L C_R e^{-\frac{8\pi(C_L + C_R)(r_2 - r_1)}{r_1}} (r_2 - r_1)^2. \]

Hence we derive

\[
\frac{a_u}{k^2} = \frac{(4\pi(C_L + C_R)(r_2 - r_1))^{\frac{\delta}{2}}}{\left( \pi^2 C_L C_R e^{-\frac{8\pi(C_L + C_R)(r_2 - r_1)}{r_1}} (r_2 - r_1)^2 \right)^{\frac{\delta}{2}}} \]

\[
= 4^{\frac{\delta}{2}} \pi^{\frac{\delta}{2}} \frac{(C_L + C_R)^{\frac{\delta}{2}} (r_2 - r_1)^{\frac{\delta}{2}}}{(C_L C_R)^{\frac{\delta}{2}} (r_2 + r_1)^{\frac{\delta}{2}}} \frac{28\pi(C_L + C_R)(r_2 - r_1)}{r_1}. \]

This shows that a proper choice of \( C_L, C_R \) and \( r_1, r_2 \) gives the desired condition.
3. Fixed point set-up

We define the solution space by
\[ \Lambda = \{ f \in L^1_2([0,1] \times \mathbb{R}^3) \mid f \text{ satisfies } (A,B,C) \} , \]
endowed with the metric \( d(f,g) = \sup_{x \in [0,1]} ||f - g||_{L^2} \).

- (A) \( f \) is non-negative:
  \[ f(x,p) \geq 0 \quad \text{for } x, p \in [0,1] \times \mathbb{R}^3. \]

- (B) Mass and energy satisfy
  \[ a_t \leq \int_{\mathbb{R}^3} f(x,p)dp \leq a_u, \quad c_t \leq \int_{\mathbb{R}^3} f(x,p)||p||^2dp \leq c_u. \]

- (C) \( f \) satisfies
  \[ \left( \int_{\mathbb{R}^3} f(x,p)dp \right) \left( \int_{\mathbb{R}^3} f(x,p)||p||^2dp \right) - \left| \int_{\mathbb{R}^3} f(x,p)pdp \right|^2 \geq k. \]

### 3.1. Determination of \( a, b \) and \( c \)

We first verify that for any distribution function \( f \) that lies in \( \Lambda \), the nonlinear relations (1.4) and (1.5) admit a unique set of solutions \( a \) and \( c \), so that the local equilibrium \( K(f) \) is well defined. It is clear that \( a \) is uniquely determined by (1.5) once the unique existence of \( c \) is determined from (1.4). Note that, in view of the definition of (1.6), the nonlinear relation (1.4) is rewritten by
\[ \beta F(c) = \frac{N(x)}{\left( E(x) - \frac{|P(x)|^2}{N(x)} \right)^{\frac{2}{3}}} \]

(3.1)

Therefore, it is sufficient to show that \( \beta F \) is a monotone function, and r.h.s of (3.1) lies in the range of \( \beta F \). For this we recall the following lemma:

**Lemma 3.1.** [7, 32] The function \( \beta_B \) and \( \beta_F \) defined in (1.6) satisfy the following properties.

1. \( \beta_B \) is strictly decreasing on \([0, \infty)\) and its range is \((0, \beta(0))\).
2. \( \beta_F \) is strictly decreasing on \((-\ln 3, \infty)\) and its range is \((0, \beta(-\ln 3))\).

**Proof.** Proof for (1) can be founded in [32], and the proof for (2) can be founded in [7]. \(\square\)

**Lemma 3.2.** Assume \( f \in \Lambda \). Then \( a \) and \( c \) are uniquely determined from (1.4) and (1.5), and \( K(f) \) is well-defined. Moreover, \( K(f) \) is not condensed (Bosonic case) nor saturated (Fermionic case). That is, no transition from \( B_1(f) \) to \( B_2(f) \), or \( F_1(f) \) to \( F_2(f) \) occurs.

**Proof.** (Boson): We note from (2.4), (2.5) and (2.6) that
\[ 0 < \frac{N}{\left( E - |P|^2/N \right)^{\frac{2}{3}}} \leq \frac{N^{\frac{2}{3}}}{(EN - |P|^2)^{\frac{2}{3}}} \leq \frac{a_u^{\frac{2}{3}}}{k^{\frac{2}{3}}} < \beta_B(0). \]

Therefore, in view of Lemma 3.1 the interval \( (0, N^{\frac{2}{3}}/(EN - |P|^2)^{\frac{2}{3}}) \) lies in the range of \( \beta_B \), and we can fix a unique \( c \) satisfying (3.1) by the monotonicity of \( \beta_B \) obtained in Lemma 3.1 which in turn leads to the determination of \( a \) by (1.5). Note also from (1.7) that this guarantees that the condensation does not arise if \( f \in \Lambda \). In conclusion, \( B(f) \) is well-defined.
for $f \in \Lambda$.  

- (Fermion): Similarly, combining second condition of (2.4) with (2.5) and (2.6) yields

$$\frac{N}{(E - \frac{|P|^2}{N})^{3/5}} = \frac{N^*}{(EN - |P|^2)^{3/5}} \leq \frac{a_u^{\frac{8}{5}}}{k^{\frac{8}{5}}} < \beta_f (-\ln 3),$$

for fermion case. Therefore, by the exactly same argument, we can conclude that $a$ and $c$ are uniquely determined for $f \in \Lambda$, and the transition from the non-saturated state $F_1(f)$ to the saturated state $F_2(f)$ does not happen.

In view of this consideration, we can uniquely determine $c$ satisfying (1.3). For brevity, we slightly abuse the notation to denote as

$$\beta_B^{-1} = (\beta_B |_{0, \infty})^{-1}, \quad \beta_f^{-1} = (\beta_f |_{-\ln 3, \infty})^{-1},$$

and $\beta_K^{-1}$ will denote

$$\beta_K^{-1} = \beta_B^{-1} \quad \text{(Boson)} \quad \text{and} \quad \beta_K^{-1} = \beta_f^{-1} \quad \text{(Fermion)}.$$

We first consider the range of $a$ and $c$ when they are constructed from an element of $\Lambda$.

**Lemma 3.3.** Let $f \in \Lambda$, and the boundary data $f_{LR}$ satisfy (2.4). Define $a_*, a^*, c_*, c^*$ by

$$c_\ast = \beta_K^{-1} \left( \frac{a_u^{\frac{8}{5}}}{k^{\frac{8}{5}}} \right), \quad c^\ast = \beta_K^{-1} \left( \frac{a_l^{\frac{8}{5}}}{(a_u c_u)^{\frac{8}{5}}} \right),$$

and

$$a_\ast = \left( \int_{\mathbb{R}^3} \frac{1}{e^{p^2 + c_-} + 1} dp \right)^{\frac{3}{5}} a_u^{-\frac{2}{5}}, \quad a^\ast = \left( \int_{\mathbb{R}^3} \frac{1}{e^{p^2 + c_-} + 1} dp \right)^{\frac{3}{5}} a_l^{-\frac{2}{5}}.$$

Then, the equilibrium parameter $a$ and $c$ satisfy

$$0 \leq a_* \leq a \leq a^*.$$  

and

$$-\ln 3 \leq c_* \leq c \leq c^* \quad \text{(Fermion)} \quad \text{and} \quad 0 \leq c_* \leq c \leq c^* \quad \text{(Boson)}.$$  

In the case of fermion, we note that $-\ln 3 \leq c_\ast$.

**Proof.** (1) **Estimates for $c$:** From (1.4) and (1.6), we have

$$\beta_K(c) = \frac{N}{(E - \frac{|P|^2}{N})^{3/5}} = \frac{N^*}{(EN - |P|^2)^{3/5}}.$$  

Since $f \in \Lambda$ we have $a_l \leq N \leq a_u$, $E \leq c_u$, and $EN - |P|^2 \geq k$, so that

$$\frac{a_l^{\frac{8}{5}}}{(a_u c_u)^{\frac{8}{5}}} \leq \beta_K(c) \leq \frac{a_u^{\frac{8}{5}}}{k^{\frac{8}{5}}}.$$  

Now, since Lemma 3.1 implies that $\beta_K^{-1}$ is strictly decreasing, and the closed interval $[a_l^{\frac{8}{5}}/(a_u c_u)^{3/5}, a_u^{\frac{8}{5}}/k^{3/5}]$ lies in the range of $\beta_K(c)$, we have

$$0 \leq \beta_K^{-1} \left( \frac{a_u^{\frac{8}{5}}}{k^{\frac{8}{5}}} \right) \leq c \leq \beta_K^{-1} \left( \frac{a_l^{\frac{8}{5}}}{(a_u c_u)^{\frac{8}{5}}} \right).$$
to get the desired estimates for \( c \).

(2) **Estimates for \( a \):** We recall (1.3). Then from \( a_l \leq N \leq a_u \) and estimates of \( c \) established above, we find

\[
\left( \int_{\mathbb{R}^3} \frac{1}{|p|^2 + c^* + 1} dp \right)^\frac{2}{3} a_u^{-\frac{2}{3}} \leq a \leq \left( \int_{\mathbb{R}^3} \frac{1}{|p|^2 + c^* + 1} dp \right)^\frac{2}{3} a_l^{-\frac{2}{3}}.
\]

For boson case, \( c_s \geq 0 \) implies the positivity of \( a_s \). For fermion case, positivity of \( a_s \) is trivial. This completes the proof. \( \square \)

3.2. **Solution operator.** By Lemma 3.2, the following solution operator \( \Phi \) is well-defined on \( \Lambda \):

**Definition 3.4.** We define our solution operator \( \Phi \) as

\[
\Phi(f) = \Phi^+(f)1_{p_1>0} + \Phi^-(f)1_{p_1<0},
\]

where

\[
\Phi^+(f)(x,p) = e^{-\frac{|f(y)|}{2\tau_{p_1}}} \int_x^y e^{rac{|f(z)|}{2\tau_{p_1}}} N_f(y)dy f_L(p)
\]

\[
+ \frac{1}{\tau|p_1|} \int_x^y e^{-\frac{|f(y)|}{2\tau_{p_1}}} \int_x^y N_f(y) dz N_f(y)K(f)dy
\]

\text{if } p_1 > 0,
\]

and

\[
\Phi^-(f)(x,p) = e^{-\frac{|f(y)|}{2\tau_{p_1}}} \int_x^y e^{rac{|f(z)|}{2\tau_{p_1}}} N_f(y)dy f_R(p)
\]

\[
+ \frac{1}{\tau|p_1|} \int_x^y e^{-\frac{|f(y)|}{2\tau_{p_1}}} \int_x^y N_f(y) dz N_f(y)K(f)dy
\]

\text{if } p_1 < 0.
\]

In the remaining sections, we show that \( \Phi \) has a unique fixed point in \( \Lambda \) if \( \tau \) is sufficiently large. We first prove several estimates on the quantum local equilibrium.

**Lemma 3.5.** Let \( f \in \Lambda \), then there exists a constant \( C_{l,u} \) depending only on the quantities in (2.2) and \( k \) such that

\[
K(f)(1 + |p|^2) \leq C_{l,u} e^{-\frac{k}{|p|^2}}.
\]

**Proof.** We only consider \( B(f)|p|^2 \). By an explicit computation, we have

\[
B(f)|p|^2 = \frac{|p|^2}{e^{a|\frac{p-N}{N}|^2 + c - 1}}
\]

\[
\leq \frac{|p|^2}{\left( e^{\frac{a}{N}|p-N|^2 + \frac{c}{a}} - 1 \right) \left( e^{\frac{a}{N}|p-N|^2 + \frac{c}{a}} + 1 \right)}
\]

\[
\leq \frac{2|p-N|^2 + 2\frac{p^2}{N}}{e^{\frac{a}{N}|p-N|^2 + \frac{c}{a}} - 1} \frac{1}{e^{\frac{a}{N}|p-N|^2 + \frac{c}{a}} + 1}
\]

In last line, we used \( a^2 \leq 2|a - b|^2 + 2b^2 \). Then, we observe

\[
|P| \leq \frac{a_u + c_u}{a_l},
\]

which follows from \( |P| \leq a_u + c_u \), and use the boundedness of \( e^{\frac{a^2}{N}+\frac{1}{a^2}} \) to get

\[
B(f)|p|^2 \leq \frac{C_{a_u,c_u}}{e^{\frac{a^2}{N}+\frac{1}{a^2}} + 1}.
\]
Now, since $|a - b|^2 \geq a^2/2 - b^2$, we have

$$B(f)|p|^2 \leq \frac{C_{l,u}}{e^{\frac{|x|^2}{2p|p_1|}}} \leq \frac{C_{l,u}}{e^{-\frac{|x|^2}{2p|p_1|} + 1}} = \frac{C_{l,u}}{e^{-\frac{|x|^2}{2p|p_1|} + \frac{1}{2}}}.$$  

We then use (5.5) again to get the desired result:

$$B(f)|p|^2 \leq C_{l,u}e^{-\frac{|x|^2}{2p|p_1|}}e^{-\frac{|p|^2}{2}} \leq C_{l,u}e^{-\frac{|p|^2}{2}}.$$

The following decay estimates are crucially used throughout the paper. The proof can be found in [8]. We provide detailed proof for reader’s convenience.

**Lemma 3.6.** We have

$$\int_0^x \int_0^\infty \frac{1}{\tau|p_1|} e^{-\frac{a(x-y)}{|p_1|}} e^{-C_{l,u}|p|^2} dp_1 dy \leq C_{l,u} \left( \ln \frac{\tau + 1}{\tau} \right).$$

**Proof.** We divide the integral domain of $p_1$ into three parts:

$$A = \left\{ \int_0^x \int_0^{p_1} + \int_0^x \int_{p_1}^{\tau} + \int_0^x \int_{\tau}^{p_1} \right\} \frac{1}{\tau|p_1|} e^{-\frac{a(x-y)}{|p_1|}} e^{-C_{l,u}|p|^2} dp_1 dy$$

$$\equiv I + II + III.$$

Integrating in $y$ first, we get

$$I = \int_0^{p_1} \int_0^{x} \frac{1}{\tau|p_1|} e^{-\frac{a(x-y)}{|p_1|}} e^{-C_{l,u}|p|^2} dy dp_1$$

$$= \frac{1}{a_1} \int_0^{p_1} \left\{ 1 - e^{-\frac{a_1 x}{|p_1|}} \right\} e^{-C_{l,u}|p|^2} dp_1$$

$$\leq \frac{1}{a_1} \tau.$$

We start similarly for $II$:

$$II = \int_{\tau}^{p_1} \int_0^{x} \frac{1}{\tau|p_1|} e^{-\frac{a(x-y)}{|p_1|}} e^{-C_{l,u}|p|^2} dy dp_1$$

$$\leq \frac{1}{a_1} \int_{\tau}^{p_1} \left\{ 1 - e^{-\frac{a_1 x}{|p_1|}} \right\} e^{-C_{l,u}|p|^2} dp_1.$$

Then we expand $e^{-\frac{a_1 x}{|p_1|}}$ in Taylor expansion to obtain

$$II \leq \frac{1}{a_1} \int_{\tau}^{p_1} \left\{ \left( \frac{a_1}{\tau|p_1|} \right)^2 + \frac{1}{3!} \left( \frac{a_1}{\tau|p_1|} \right)^3 + \cdots \right\} dp_1$$

$$\leq \frac{1}{a_1} \int_{\tau}^{p_1} \left( \frac{a_1}{\tau|p_1|} \right)^2 dp_1 + \frac{1}{a_1} \int_{\tau}^{p_1} \left( \frac{a_1}{\tau|p_1|} \right)^3 dp_1 + \cdots.$$

Then, since

$$\int_{\tau}^{p_1} \left( \frac{1}{p_1} \right)^n dp = \left[ \frac{1}{n - 1} \right] \left( \frac{1}{p_1} \right)^n = \frac{1}{n - 1} \left( \frac{p_1^{n-2} - 1}{p_1^n - 1} \right).$$
We can bound $II$ by
\[
\frac{1}{\tau} \ln \tau^2 + \frac{1}{2!} \frac{a_1}{\tau} \tau^2 - 1 + \frac{1}{2} \frac{a_2}{\tau^2} \tau^4 - 1 + \frac{1}{3} \frac{a_3}{\tau^3} \tau^6 - 1 + \cdots
\]
\[
\leq \frac{1}{\tau} \ln \tau^2 + \frac{1}{2!} \frac{a_1}{\tau} + \frac{1}{3!} \frac{a_2}{\tau} + \frac{1}{4!} \frac{a_3}{\tau} + \cdots
\]
\[
= \frac{1}{\tau} \ln \tau^2 + \frac{a_1}{a_1} \frac{1}{\tau},
\]
where we used $(\tau^n - 1)/\tau^n \leq 1$ in second line. Finally, by using $e^{-a|x-y|/(\tau|p|)} < 1$, we estimate $III$ as
\[
III = \int_{p_1 > \tau} \left\{ \int_0^\tau \frac{1}{\tau|p_1|} e^{-a(x-y)/|p_1|} dy \} e^{-C_{l,u} p_1^2} dp_1 \leq \frac{1}{\tau^2} \int_\mathbb{R} e^{-C_{l,u} p_1^2} dp_1 \leq C_{l,u} \frac{1}{\tau^2}.
\]
Combining the above estimates gives the desired results for sufficiently large $\tau$:
\[
I + II + III \leq C_{l,u} \left\{ \frac{1}{\tau} + \frac{1}{\ln \tau^2 + \frac{1}{\tau}} \right\} \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right).
\]

4. Φ MAPS Λ INTO Λ

The main result of this section is stated in the following proposition.

**Proposition 4.1.** Let $f_{LR}$ satisfies the assumptions in Theorem 2.2. Then, there exists $\tau_0$ such that if $\tau \geq \tau_0$, then the solution operator $\Phi$ maps $\Lambda$ into $\Lambda$.

**Proof.** The proof is given in the following Lemma 4.1, 4.2, 4.3 and 4.5. □

**Lemma 4.1.** Let $f \in \Lambda$. Assume $f_{LR}$ satisfies all the assumptions of the Theorem 2.2. Then $\Phi(f)$ satisfies the following estimates:
\[
\Phi(f)(x,p) \geq 0.
\]

**Proof.** Thanks to Lemma 3.3, the local equilibrium is strictly positive:
\[
\mathcal{K}(f) = \frac{1}{e^{a|x-p_e|^2 + c} + 1} \geq \frac{1}{e^{a|x-p_e|^2 + c} + 1} > \frac{1}{e^{c} + 1} > 0.
\]

Therefore, we have from (3.3) and (3.3) that
\[
\Phi^+(f)(x,p) \geq e^{-\frac{a}{\tau|p_1|} \int_\tau^\tau N_{(y)}(u)dy} f_L(p) \geq 0, \quad \text{if} \quad p_1 > 0,
\]
\[
\Phi^-(f)(x,p) \geq e^{-\frac{a}{\tau|p_1|} \int_\tau^\tau N_{(y)}(u)dy} f_R(p) \geq 0, \quad \text{if} \quad p_1 < 0,
\]
which gives desired result. □

**Lemma 4.2.** Let $f \in \Lambda$. Assume $f_{LR}$ satisfies all the assumptions of the Theorem 2.2, then $\Phi(f)$ also satisfies the following inequality.
\[
\int_{\mathbb{R}^3} \Phi(f) dp \geq a_1, \quad \int_{\mathbb{R}^3} \Phi(f)|p|^2 dp \geq c_1.
\]
We then integrate with respect to $|p|^2 dp$ to get the desired results:

$$\int_{\mathbb{R}^3} \Phi(f)|p|^2 dp \geq \int_{\mathbb{R}^3} e^{-\frac{a_u}{|p|^2}} f_{LR}|p|^2 dp = c_1.$$

**Lemma 4.3.** Let $f \in \Lambda$. Assume $f_{LR}$ satisfies all the assumptions of the Theorem. Then $\Phi(f)$ satisfies the following estimates:

$$\int_{\mathbb{R}^3} \Phi(f) dp \leq a_u, \quad \int_{\mathbb{R}^3} \Phi(f)|p|^2 dp \leq c_u,$$

for sufficiently large $\tau$.

**Proof.** We only consider the second inequality. We integrate with respect to $|p|^2 dp$ to get

$$\int_{\mathbb{R}^3} \Phi(f)|p|^2 dp = \int_{\mathbb{R}^3} e^{-\frac{a_u}{|p|^2}} f_{LR}(p)|p|^2 dp$$

(4.3)

We then recall Lemma 3.2 and use $a_l \leq N_f \leq a_u$ to bound the second term as

$$\int_{p_1 > 0} \frac{1}{|p_1|} \int_{0}^{x} e^{-\frac{a_u}{|p_1|^2}} f_{LR}(p)|p|^2 dp dy dp$$

$$\leq C_{l,u} \int_{0}^{x} \frac{1}{|p_1|} e^{-\frac{a_u}{|p_1|^2}} f_{LR}(p)|p|^2 dp dy dp$$

Therefore, we have from Lemma 3.4 that

$$\int_{p_1 > 0} \Phi^+(f)|p|^2 dp \leq \int_{p_1 > 0} f_L(p)|p|^2 dp + C_{l,u} \left( \ln \frac{\tau + 1}{\tau} \right).$$

Similarly, we can derive

$$\int_{p_1 < 0} \Phi^-(f)|p|^2 dp \leq \int_{p_1 < 0} f_R(p)|p|^2 dp + C_{l,u} \left( \ln \frac{\tau + 1}{\tau} \right).$$
so that

\[ \int_{\mathbb{R}^3} \Phi(f)|p|^2 dp \leq \frac{1}{2} c_u + C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right), \]

which gives the desired result for sufficiently large \( \tau \).

\[ \square \]

**Lemma 4.4.** Let \( f \in \Lambda \). Assume \( f_{LR} \) satisfies all the assumptions of the Theorem. Then, for sufficiently large \( \tau \), we have

\[ \left| \int_{\mathbb{R}^3} \Phi(f)p_i dp \right| \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right), \]

for \( i = 2, 3 \).

**Proof.** We only consider the case \( i = 2 \). For this, we integrate \( \Phi \) with respect to \( p_2 dp_3 dp_3 \): \( \Phi(f)(x,p)p_2 dp_3 dp_3 \)

\begin{align*}
\int_{\mathbb{R}^2} \Phi^+(f)(x,p)p_2 dp_3 dp_3 &= \int_{\mathbb{R}^2} e^{-\frac{\pi}{|p_1|} \int_0^x N_f(y)dy} f_L(p)p_2 dp_3 dp_3 \\
&+ \int_{\mathbb{R}^2} \frac{1}{\tau|p_1|} \int_0^x e^{-\frac{\pi}{|p_1|} \int_0^x N_f(z)dz} N_f(y) \Phi(f) p_2 dp_3 dp_3.
\end{align*}

We note that the first term in r.h.s vanishes due to the assumption (3) of Theorem 2.2

\[ \int_{\mathbb{R}^2} e^{-\frac{\pi}{|p_1|} \int_0^x N_f(y)dy} f_L(p)p_2 dp_3 dp_3 = 0. \]

For the second term, we use \( a_1 \leq N_f \leq a_u \) and employ Lemma 3.5 to derive

\[ \left| \int_{\mathbb{R}^2} \Phi^+(f)(x,p)p_2 dp_3 dp_3 \right| \leq C_{l,u} \frac{1}{|p_1|} \int_0^x e^{-\frac{\pi}{|p_1|} \int_0^x N_f(z)dz} N_f(y) e^{-C_{l,u}|p_1|^2} dy \\
\leq C_{l,u} \frac{a_u}{\tau|p_1|} \int_0^x e^{-\frac{a_u(x-y)}{|p_1|}} e^{-C_{l,u}|p_1|^2} dy. \]

Now we integrate with respect to \( dp_1 \) on \( p_1 > 0 \) to obtain

\[ \left| \int_{p_1 > 0} \Phi^+(f)(x,p)p_2 dp \right| \leq \int_{p_1 > 0} \left| \int_{\mathbb{R}^2} \Phi^+(f)(x,p)p_2 dp_3 dp_3 \right| dp_1 \\
\leq C_{l,u} \int_{p_1 > 0} \frac{1}{\tau|p_1|} \int_0^x e^{-\frac{a_u(x-y)}{|p_1|}} e^{-C_{l,u}|p_1|^2} dy dp_1. \]

Therefore, we have from Lemma 3.5 that

\[ \left| \int_{p_1 > 0} \Phi^+(f)(x,p)p_2 dp \right| \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right). \]

Similarly, we have

\[ \left| \int_{p_1 < 0} \Phi^+(f)(x,p)p_2 dp \right| \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right), \]

which gives the desired result.

\[ \square \]

**Lemma 4.5.** Let \( f \in \Lambda \). Assume \( f_{LR} \) satisfies all the assumptions of the Theorem. Then, for sufficiently large \( \tau \), we have

\[ \left( \int_{\mathbb{R}^3} \Phi(f) dp \right) \left( \int_{\mathbb{R}^3} \Phi(f)|p|^2 dp \right) - \int_{\mathbb{R}^3} \Phi(f) dp \leq C_{l,u} \left( \frac{\ln \tau + 1}{\tau} \right)^2 \geq k. \]
Proof. We have from Cauchy-Schwarz inequality that
\[
\int_{\mathbb{R}^3} \Phi(f)dp \int_{\mathbb{R}^3} \Phi(f)|p|^2 dp - \int_{\mathbb{R}^3} \Phi(f)dp \leq \left( \int_{\mathbb{R}^3} |p|\Phi(f)dp \right)^2 - \left| \int_{\mathbb{R}^3} p\Phi(f)dp \right|^2
\]
(4.4)
\[
\geq \left( \int_{\mathbb{R}^3} |p_1|\Phi(f)dp \right)^2 - \left| \int_{\mathbb{R}^3} p_1\Phi(f)dp \right|^2
\]
\[
= \left( \int_{\mathbb{R}^3} |p_1|\Phi(f)dp \right)^2 - \left| \int_{\mathbb{R}^3} p_1\Phi(f)dp \right|^2 - R,
\]
where
\[
R = M_2^2 + M_3^2 + 2M_1M_2 + 2M_2M_3 + 2M_1M_3,
\]
and
\[
M_i = \int_{\mathbb{R}^3} p_i\Phi(f)dp \quad \text{for} \quad i = 1, 2, 3.
\]
From Lemma 4.3 we can bound \( M_1 \) as
\[
M_1 \leq \left| \int_{\mathbb{R}^3} (1 + |p|^2)\Phi(f)dp \right| \leq a_u + c_u,
\]
and \( M_2, M_3 \) decay as Lemma 4.4.
\[
M_2, M_3 \leq C_{l,u}\left( \ln \frac{\tau + 1}{\tau} \right).
\]
Therefore,
\[
R \leq C_{l,u}\left( \ln \frac{\tau + 1}{\tau} \right).
\]
On the other hand, we use \( a^2 - b^2 = (a + b)(a - b) \) to get
\[
\left( \int_{\mathbb{R}^3} |p_1|\Phi(f)dp \right)^2 - \left| \int_{\mathbb{R}^3} p_1\Phi(f)dp \right|^2
\]
(4.6)
\[
\geq \left\{ \int_{\mathbb{R}^3} (|p_1| + p_1)\Phi(f)dp \right\} \left\{ \int_{\mathbb{R}^3} (|p_1| - p_1)\Phi(f)dp \right\}
\]
\[
= 4 \int_{p_1 > 0} p_1\Phi(f)dp \int_{p_1 < 0} |p_1|\Phi(f)dp,
\]
and observe that from the definition of \( \Phi \), and property (B) of \( \Lambda \): \( N_f \leq a_u \) that
\[
\int_{p_1 > 0} p_1\Phi(f)dp \geq \int_{p_1 > 0} p_1 e^{-\frac{1}{\tau p_1}|f_{L}(y)|^2} f_{L}(p)dp \geq \int_{p_1 > 0} p_1 e^{-\frac{a_u}{\tau p_1}} f_{L}(p)dp
\]
and
\[
\int_{p_1 < 0} |p_1|\Phi(f)dp \geq \int_{p_1 < 0} |p_1| e^{-\frac{1}{\tau p_1}|f_{L}(y)|^2} f_{R}(p)dp \geq \int_{p_1 < 0} |p_1| e^{-\frac{a_u}{\tau p_1}} f_{R}(p)dp.
\]
We insert these lower bounds into (4.6) and recall the definition of \( k \) in (2.3) to obtain
\[
\left( \int_{\mathbb{R}^3} |p_1|\Phi(f)dp \right)^2 - \left| \int_{\mathbb{R}^3} p_1\Phi(f)dp \right|^2 \geq 4k.
\]
(4.7)
From (4.4), (4.5) and (4.7), we have
\[
\left( \int_{\mathbb{R}^3} \Phi(f) dp \right) \left( \int_{\mathbb{R}^3} \Phi(f) |p|^2 dp \right) - \left| \int_{\mathbb{R}^3} \Phi(f) p dp \right|^2 \geq 4k - C_{t,u} \left( \frac{\ln \tau + 1}{\tau} \right),
\]
which, for sufficiently large \( \tau \), gives the desired result. \( \square \)

5. CONTINUITY OF QUANTUM EQUILIBRIUM \( \mathcal{K} \)

In this section, we establish the continuity property of the quantum equilibrium \( \mathcal{K} \), which is crucially used to show the contractiveness of \( \Phi \) in Section 5.

5.1. Transitional quantum local equilibrium \( \mathcal{K}(\theta) \). In this subsection, we define a transitional quantum local equilibrium. We start with the convexity of our solution space.

Lemma 5.1. Let \( f, g \in \Lambda \), Then the linear combination \((1 - \theta)f + \theta g \) lies in \( \Lambda \) for \( \theta \in [0, 1] \).

Proof. Since the conditions \((A)\) and \((B)\) of \( \Lambda \) are trivially satisfied, we only consider \((C)\). For this, we define a functional \( G \) by
\[
G(f) = \left( \int_{\mathbb{R}^3} f(x, p) dp \right) \left( \int_{\mathbb{R}^3} f(x, p) |p|^2 dp \right) - \left( \int_{\mathbb{R}^3} f(x, p) p dp \right)^2,
\]
and a matrix \( M \) by
\[
M(f) = \begin{pmatrix} N_f & P_f \\ P_f & E_f \end{pmatrix},
\]
for \( f \in \Lambda \). We note that
\[
G(f) = \det M(f).
\]
Then, by Brum-Minkowski inequality, we have for \( f, g \in \Lambda \)
\[
G(\theta f + (1 - \theta)g) = \det M(\theta f + (1 - \theta)g)
\geq \{ \det M(f) \}^\theta \{ \det M(g) \}^{1-\theta}
\geq \{ G(f) \}^\theta \{ G(g) \}^{1-\theta}
\geq k^\theta k^{1-\theta}
= k.
\]
Therefore, \( \theta f + (1 - \theta)g \in \Lambda \). \( \square \)

We now define the transitional macroscopic fields constructed from the linear combination \( \theta f + (1 - \theta)g \) as
\[
(N_\theta, P_\theta, E_\theta) = (1 - \theta)(N_f, P_f, E_f) + \theta(N_g, P_g, E_g),
\]
for \( \theta \in [0, 1] \). Now, since we have shown in Lemma 5.1 that \( \theta f + (1 - \theta)g \in \Lambda \), the existence of the unique quantum equilibrium \( \mathcal{K}(\theta) \):
\[
\mathcal{K}(\theta) = \frac{1}{e^{a_\theta(x)p^2} + e^{a_\theta(x)(1-p)^2} + e^{a_\theta(x)}} \pm 1
\]
which shares the same mass, momentum and energy with \( \theta f + (1 - \theta)g \): 
\[
\int_{\mathbb{R}^3} \mathcal{K}(\theta) dp = N_\theta, \quad \int_{\mathbb{R}^3} \mathcal{K}(\theta) dp = P_\theta, \quad \int_{\mathbb{R}^3} \mathcal{K}(\theta) dp = E_\theta
\]
is guaranteed by Lemma 3.2. We also recall from Lemma 3.3 that $a_\theta$ and $c_\theta$ are determined by

\begin{equation}
(5.1) \quad c_\theta = \beta_K^{-1} \left( \frac{N_\theta}{(E_\theta - \frac{P^2}{2N_\theta})^{\frac{3}{2}}} \right), \quad a_\theta = \left( \frac{\int_{\mathbb{R}^3} \frac{1}{e^{\frac{1}{2} + 1 + a \frac{p^2}{2N_\theta}}} dp}{N_\theta} \right)^{\frac{3}{2}},
\end{equation}

and satisfy

\[ a_* \leq a_\theta \leq a^*, \quad c_* \leq c_\theta \leq c^*. \]

for some positive constants $a_*$, $a^*$, $c^*$ and $c_*$.

5.2. Derivatives of $F(\theta)$. We now derive derivative estimates of $a_\theta$ and $c_\theta$, which will be needed later in the proof of the continuity estimate of $K(\theta)$. We first need the following estimate of $\beta_K$.

Lemma 5.2. Let $f, g \in \Lambda$, then $\beta_K$ defined in (1.6) satisfies

\[ \left| \frac{1}{\beta'_K(c\theta)} \right| < C_{l,u}, \]

where $C_{l,u}$ depends on constants of (5.2) and $k$.

Proof. By definition given in (1.6), $\beta_K$ is an infinitely differentiable function. On the other hand, Lemma 5.1 implies that $\beta'_K(c) < 0$. Therefore, we see from Lemma 3.3 that $\beta'_K(c)$ is a strictly negative continuous function defined on a closed interval $[c_*, c^*]$. Hence, there exists positive $C$ such that $|\beta'_K(c)| \geq C$, which gives the desired result. \qed

Lemma 5.3. We have

\[ \left| \left( \frac{\partial c_\theta}{\partial N_\theta}, \frac{\partial c_\theta}{\partial P_\theta}, \frac{\partial c_\theta}{\partial E_\theta} \right) \right| \leq C_{l,u}. \]

Proof. Recall that $c_\theta$ is function of $N_\theta$, $P_\theta$ and $E_\theta$:

\[ c_\theta = \beta_K^{-1} \left\{ \frac{N_\theta}{(E_\theta - \frac{P^2}{2N_\theta})^{\frac{3}{2}}} \right\}. \]

(1) By an explicit computation, we get

\[ \left| \frac{\partial c_\theta}{\partial N_\theta} \right| = \frac{1}{|\beta'_K(c\theta)|} \left| \frac{N_\theta}{\left( E_\theta - \frac{P^2}{2N_\theta} \right)^{\frac{3}{2}}} \right| = \frac{1}{|\beta'_K(c\theta)|} \left| \frac{E_\theta - \frac{P^2}{2N_\theta}}{\left( E_\theta - \frac{P^2}{2N_\theta} \right)^{\frac{3}{2}}} \right|. \]

We then use $N_\theta \leq a_u$, $E_\theta \leq c_u$ and $N_\theta E_\theta - P^2_\theta \geq k$ together with Lemma 5.2 to obtain

\[ \left| \frac{\partial c_\theta}{\partial N_\theta} \right| \leq \frac{1}{|\beta'_K(c\theta)|} \frac{a_u c_u}{k^{\frac{3}{2}}} \leq C_{l,u}. \]

(2) Similarly, we compute

\[ \left| \frac{\partial c_\theta}{\partial P_\theta} \right| = \frac{1}{|\beta'_K(c\theta)|} \left| \frac{\partial}{\partial P_\theta} \left( \frac{N_\theta}{\left( E_\theta - \frac{P^2}{2N_\theta} \right)^{\frac{3}{2}}} \right) \right| = \frac{1}{|\beta'_K(c\theta)|} \left| \frac{0}{E_\theta - \frac{P^2}{2N_\theta}} \right|^\frac{3}{2}. \]
Since $|P_0| \leq a_u + c_u$, we have
\[
\left| \frac{\partial c_0}{\partial P_0} \right| \leq \frac{1}{\theta^{\prime}(c_0)} \frac{\theta(a_u + c_u) a_u^{\frac{2}{3}}}{k^{\frac{1}{3}}} \leq C_{l,u}.
\]

(3) In an almost identical manner, we compute
\[
\left| \frac{\partial c_0}{\partial E_0} \right| = \left| \frac{1}{\theta^{\prime}(c_0)} \right| \frac{\partial}{\partial E_0} \left( \frac{N_0}{(E_0 - \frac{L^2}{N_0})^{\frac{3}{2}}} \right) = \left| \frac{1}{\theta^{\prime}(c_0)} \right| \frac{\frac{3}{2} a_u A}{k^{\frac{1}{3}}} \leq C_{l,u}.
\]

\[\square\]

Lemma 5.4. We have
\[
\left| \left( \frac{\partial a_0}{\partial N_0}, \frac{\partial a_0}{\partial P_0}, \frac{\partial a_0}{\partial E_0} \right) \right| \leq C_{l,u}.
\]

Proof. (1) We recall (5.1) and compute
\[
\frac{\partial a_0}{\partial N_0} = \frac{2}{3} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right)^{-\frac{2}{3}} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right) = \frac{2}{3} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right)^{-\frac{2}{3}} \left( \frac{N_0 \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp - \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0^2} \right).
\]

It then follows directly from from $a_t \leq N_0 \leq a_u$, Lemma 5.3 and Lemma 5.3 that
\[
\frac{\partial a_0}{\partial N_0} \leq C \left( \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp \right)^{-\frac{2}{3}} \left( C_{l,u} a_u \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp + \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp \right) \leq C_{l,u}.
\]

(2) In a similar manner, we have
\[
\left( \frac{\partial a_0}{\partial P_0} \right)_i = \frac{2}{3} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right)^{-\frac{2}{3}} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right) = \frac{2}{3} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right)^{-\frac{2}{3}} \left( \frac{\int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp}{N_0} \right) \leq C \left( \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp \right)^{-\frac{2}{3}} \left( C_{l,u} a_u \int_{R^3} \frac{1}{e^{\theta a_u a_0} \pm 1} \, dp \right) \leq C_{l,u}.
\]

(3) Replacing $\frac{\partial}{\partial N_0}$ by $\frac{\partial}{\partial E_0}$ in (2), we get the same result for $\frac{\partial a_0}{\partial E_0}$. \[\square\]
5.3. Continuity of $\mathcal{K}$. We now prove the main result of this section:

**Proposition 5.1.** Let $f, g \in \Lambda$. Then the quantum equilibrium $\mathcal{K}$ satisfies the following property:

$$|\mathcal{K}(f) - \mathcal{K}(g)| \leq C_{l,u} \sup_x ||f - g||_{L^1} e^{-C_{l,u}||p||^2}.$$  

**Proof.** We apply Taylor's theorem around $\theta = 0$ to have

$$\mathcal{K}(1) - \mathcal{K}(0) = \int_0^1 \mathcal{K}'(\theta) d\theta,$$

so that

$$\mathcal{K}(f) - \mathcal{K}(g) = (N_g - N_f) \int_0^1 \frac{\partial \mathcal{K}(\theta)}{\partial N_g} d\theta$$

$$+ (P_g - P_f) \int_0^1 \frac{\partial \mathcal{K}(\theta)}{\partial P_g} d\theta$$

$$+ (E_g - E_f) \int_0^1 \frac{\partial \mathcal{K}(\theta)}{\partial E_g} d\theta.$$  

To estimate the first integral, we compute

$$\frac{\partial \mathcal{K}(\theta)}{\partial N_g} = \frac{-\left\{ \frac{\partial a_e}{\partial N_g} |p - \frac{P_0}{N_g}|^2 + a_0 \frac{2P_0}{N_g} (p - \frac{P_0}{N_g}) + \frac{\partial c_0}{\partial N_g} \right\} e^{a_e|p - \frac{P_0}{N_g}|^2 + c_0}}{(e^{a_e|p - \frac{P_0}{N_g}|^2 + c_0} \pm 1)^2}.$$  

From Lemma 3.5, we observe $\mathcal{K}(f) \leq C_{l,u}$ to obtain

$$\frac{\partial \mathcal{K}(\theta)}{\partial N_g} \leq C_{l,u} \left( |p - \frac{P_0}{N_g}|^2 + |p - \frac{P_0}{N_g}| + 1 \right) e^{a_e|p - \frac{P_0}{N_g}|^2 + c_0} \pm 1.$$  

With these computations and Lemma 5.3, Lemma 5.3 and Lemma 5.4, we get

$$\frac{\partial \mathcal{K}(\theta)}{\partial N_g} \leq C_{l,u} \left( |p|^2 + 1 \right) \mathcal{K}(\theta).$$  

Since $|P| \leq a_u + c_u$ and $N_g \geq a_t$, we find

$$\left| \frac{\partial \mathcal{K}(\theta)}{\partial N_g} \right| \leq C_{l,u} \left( |p|^2 + 1 \right) \mathcal{K}(\theta),$$  

which, thanks to Lemma 3.5 gives

$$\left| \frac{\partial \mathcal{K}(\theta)}{\partial N_g} \right| \leq C_{l,u} e^{-C_{l,u}||p||^2}.$$  

Similarly, we have ($i = 1, 2, 3$)

$$\left| \frac{\partial \mathcal{K}(\theta)}{\partial P_g} \right| \leq C_{l,u} \left( |p|^2 + 1 \right) e^{a_e|p - \frac{P_0}{N_g}|^2 + c_0} \pm 1$$

$$\leq C_{l,u} \left( |p|^2 + 1 \right) e^{-C_{l,u}||p||^2}.$$
and
\[
\left| \frac{\partial K(g)}{\partial E_\theta} \right| = -\left\{ \frac{\partial a_\theta}{\partial E_\theta} |p - P_\theta|_N^2 + \frac{\partial c_\theta}{\partial E_\theta} \right\} e^{a_\theta |p - P_\theta|_N^2 + c_\theta} (e^{a_\theta |p - P_\theta|_N^2 + c_\theta} \pm 1)^2
\leq C_{L,u} (|p|^2 + 1) \frac{1}{e^{a_\theta |p - P_\theta|_N^2 + c_\theta} \pm 1}
\leq C_{L,u} e^{-C_{L,u}|p|^2}.
\]

Substituting these estimates into \((5.2)\) yields the desired result:
\[
|K(f) - K(g)|
\leq \left( \left| \int_{\mathbb{R}^3} (f - g)dp \right| + \left| \int_{\mathbb{R}^3} (f - g)pdp \right| + \left| \int_{\mathbb{R}^3} (f - g)|p|^2dp \right| \right) C_{L,u} e^{-C_{L,u}|p|^2}
\leq C_{L,u} \sup_x ||f - g||_{L^1_2} e^{-C_{L,u}|p|^2}.
\]

\[\square\]

6. \(\Phi\) IS CONTRACTIVE IN \(\Lambda\)

It remains to show that \(\Phi\) is a contraction mapping in \(\Lambda\) for sufficiently large \(\tau\).

**Proposition 6.1.** Let \(f, g \in \Lambda\) and \(f_{L^2}\) satisfies all the assumptions of the Theorem, then, for sufficiently large \(\tau\), \(\Phi\) satisfies
\[
\sup_{x \in [0,1]} ||\Phi(f) - \Phi(g)||_{L^2_1} \leq \alpha \sup_{x \in [0,1]} ||f - g||_{L^2_1},
\]
for some constant \(0 < \alpha < 1\).

**Proof.** We only estimate \(\Phi^+\). Let
\[
\Phi^+(f) = I(f) + II(f, f, f),
\]
where
\[
I(f) = e^{-\frac{1}{\tau|p_1|} \int_0^\tau N_f(y)dy} f_{L^1}(p),
\]
and
\[
II(f, g, h) = \frac{1}{\tau|p_1|} \int_0^\tau e^{-\frac{1}{\tau|p_1|} \int_0^\tau N_f(z)dz} N_g(y)K(h)dy.
\]

• **Estimates for \(I(f) - I(g)\):** Consider
\[
I(f) - I(g) = \left\{ e^{-\frac{1}{\tau|p_1|} \int_0^\tau N_f(y)dy} - e^{-\frac{1}{\tau|p_1|} \int_0^\tau N_g(y)dy} \right\} f_{L^1}(p),
\]
which, by mean value theorem, can be rewritten as
\[
e^{-\frac{1}{\tau|p_1|} \int_0^\tau N_f(y)dy} - e^{-\frac{1}{\tau|p_1|} \int_0^\tau N_g(y)dy}
= -\frac{1}{\tau|p_1|} e^{-\frac{1}{\tau|p_1|} \int_0^\tau (1-\mu)N_f(y)+\mu N_g(y)dy} \int_0^\tau N_f(y) - N_g(y)dy,
\]
for some $0 < \mu < 1$. Since we have $N_f, N_g \geq a_l$, we see that

$$|I(f) - I(g)| \leq \frac{1}{\tau|p_1|} e^{-\frac{|s|}{\tau|p_1|}} \int_{0}^{x} (1-\theta)N_f(y) + \delta N_g(y)dy \int_{0}^{x} |N_f(y) - N_g(y)|dyf_L(p)$$

$$\leq \frac{1}{\tau|p_1|} e^{-\frac{|s|}{\tau|p_1|}} \sup_{x \in [0,1]} ||f - g||_{L^2} f_L(p),$$

where we used

$$|N_f(y) - N_g(y)| \leq \sup_{x \in [0,1]} ||f - g||_{L^2}.$$ 

Now we integrate each term with respect to $(1 + |p|^2)dp$ on $p_1 > 0$:

$$\int_{p_1 > 0} |I(f) - I(g)|(1 + |p|^2)dp \leq \int_{p_1 > 0} \frac{1}{\tau|p_1|} e^{-\frac{|s|}{\tau|p_1|}} f_L(p)(1 + |p|^2)dp \sup_{x \in [0,1]} ||f - g||_{L^2}$$

$$\leq \frac{1}{\tau} \int_{p_1 > 0} \frac{1}{|p_1|} f_L(p)(1 + |p|^2)dp \sup_{x \in [0,1]} ||f - g||_{L^2}$$

$$\leq \frac{1}{\tau} (a_s + c_s) \sup_{x \in [0,1]} ||f - g||_{L^2},$$

to get the desired result.

- **Estimates for $II(f) - II(g)$:** We split it as

$$II(f, f, f) - II(g, g, g) = \{II(f, f, f) - II(g, f, f)\} + \{II(g, f, f) - II(g, g, f)\}$$

$$+ \{II(g, g, f) - II(g, g, g)\}$$

= $I_1 + I_2 + I_3$.

(i) Estimate of $I_1$: In a similar manner as in (6.1), we get

$$e^{-\frac{|s|}{\tau|p_1|}} \int_{0}^{x} f_N(z)dz - e^{-\frac{|s|}{\tau|p_1|}} \int_{0}^{x} f_N(z)dz \leq \sup_{x \in [0,1]} ||f - g||_{L^2}$$

$$\leq \frac{C}{a_t} e^{-\frac{|s|}{2\tau|p_1|}} \sup_{x \in [0,1]} ||f - g||_{L^2}.$$ 

In last line, we used $xe^{-x} \leq Ce^{-\frac{x}{2}}$. From this, we see that

$$\int_{p_1 > 0} |II_1|(1 + |p|^2)dp$$

$$\leq \frac{1}{\tau|p_1| \int_{0}^{x} e^{-\frac{|s|}{\tau|p_1|}} f_N(z)dz - e^{-\frac{|s|}{\tau|p_1|}} f_N(z)dz} |N_f(y)K(f)(1 + |p|^2)dydp$$

$$\leq \frac{Ca_u}{a_t} \int_{p_1 > 0} \frac{1}{|p_1|} \int_{0}^{x} e^{-\frac{|s|}{2\tau|p_1|}} \sup_{x \in [0,1]} ||f - g||_{L^2}.$$ 

We then apply Lemma 3.5 to obtain

$$\int_{p_1 > 0} |II_1|(1 + |p|^2)dp \leq C_{t,u} \int_{p_1 > 0} \frac{1}{|p_1|} \int_{0}^{x} e^{-\frac{|s|}{2\tau|p_1|}} e^{-C_{t,u}|p|^2} dydp \sup_{x \in [0,1]} ||f - g||_{L^2}$$

and Lemma 3.6 to obtain

$$\int_{p_1 > 0} |II_1|(1 + |p|^2)dp \leq C_{t,u} \left(\frac{\ln \tau + 1}{\tau}\right) \sup_{x \in [0,1]} ||f - g||_{L^2}.$$
We then apply the continuity property of $K$ to obtain the desired contractive estimate for $\Phi$ when $\tau > 0$.

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