Bosonic continuum theory of one-dimensional lattice anyons

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Anyons with arbitrary exchange phases exist on 1D lattices in ultracold gases. Yet, known continuum theories in 1D do not match. We derive the continuum limit of 1D lattice anyons via interacting bosons. The theory maintains the exchange phase periodicity fully analogous to 2D anyons. This provides a mapping between experiments, lattice anyons, and continuum theories, including Kundu anyons with a natural regularization as a special case. We numerically estimate the Luttinger parameter as a function of the exchange angle to characterize long-range signatures of the theory.

Acquiring “any phase” when spatially exchanged anyons break the dichotomic classification of quantum particles into bosons and fermions. Anyons have inspired theoretical and experimental physicists for decades [1–11]. Recently, scientific and technological interest increased again because of possible applications of non-abelian anyons in topological quantum computing [12, 13] and the apparent detection of Majorana zero modes [14–17]. Even though anyons are often associated to be exclusively two-dimensional, they have also been discussed in a wide range of one-dimensional (1D) systems that differ significantly from their two-dimensional continuous counterparts, this discrete construction side-steps the most interesting aspects of anyon physics: The continuous exchange of two particles. To our knowledge, the relation between 1D Leinaas-Myrheim particles, the Kundu model, and the experimentally accessible lattice anyons has up to now been an open problem. In particular, no corresponding bosonic continuum Hamiltonian exists that is 2π-periodic in the anyonic exchange angle.

In this paper, we provide the continuum theory of the anyon Hubbard model. We thereby obtain an explicit mapping between experiments on one-dimensional bosonic lattices, lattice theories of anyons, and general theories in the continuum, including Kundu anyons as a special case. Deriving a naive long wave-length limit is insufficient [31]. Instead, it is necessary to use the bosonic form of the Hamiltonian and to consider all orders of the phase angle. This leads to statistically induced current-density as well as two- and three-particle interactions, but in turn results in the 2π-periodicity in the anyonic phase angle even in 1D. We emphasize that the bosonic form of the Hamiltonian is experimentally directly accessible, in stark contrast to the purely anyonic description, which encodes the topological character of the exchange implicitly in the creation algebra or the boundary conditions of the wave functions. Additionally, the bosonic form facilitates theoretical calculations, since the anyonic exchange algebra is not preserved under unitary transformations. We furthermore provide numerical results using the density matrix renormalization group (DMRG) algo-

\[ \hat{a}_i \hat{a}_j^\dagger - e^{-i\theta \text{sgn}(i-j)} \hat{a}_j^\dagger \hat{a}_i = \delta_{i,j}, \]
rithm to illustrate how the continuum limit is approached and discuss the implications for the effective low-energy Tomonaga-Luttinger liquid theory.

Our starting point is the anyon Hubbard model on a 1D lattice with \( L \) sites:

\[
H = -J \sum_{j=1}^{L-1} \left( \hat{a}^\dagger_j \hat{a}_{j+1} + \text{h.c.} \right) + U / 2 \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1),
\]

(2)

where the anyonic operators obey the algebra in Eq. (11) and \( \hat{n}_j = \hat{a}^\dagger_j \hat{a}_j \) is the particle number operator. Bosons are described by this model at \( \theta = 0 \). For \( \theta = \pi \) the on-site quantum brackets Eq. (11) remain bosonic, so that ordinary fermions are not included but so-called pseudofermions instead. The anyon Hubbard model breaks spatial inversion symmetry and time reversal symmetry but obeys a generalized inversion symmetry [54]. By the Jordan-Wigner transformation [30]

\[
\hat{a}_j = e^{i \theta} \sum_{k<j} \hat{n}_k \hat{b}_j,
\]

(3)

the relations in Eq. (11) can be exactly represented by bosonic operators \( \hat{b}_j \) using the Hamiltonian

\[
H = -J \sum_{j=1}^{L-1} \hat{b}^\dagger_j e^{i \theta \hat{n}_j} \hat{b}_{j+1} + \text{h.c.} + U / 2 \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1).
\]

(4)

Here, the hopping depends on the occupation number in form of a Peierls factor \( e^{i \theta \hat{n}_j} \), which plays a central role for experimental realizations [30]. Interestingly, the anyon Hubbard interaction \( \hat{n}_j (\hat{n}_j - 1) \) is independent of the anyonic phase because \( \hat{n}_j = \hat{a}^\dagger_j \hat{a}_j = \hat{b}^\dagger_j \hat{b}_j \). In the continuum limit it leads to a simple two-body interaction term [54, 56], which can be added to the effective interaction terms arising from the anyonic exchange symmetry in the following.

The continuum limit of the kinetic part is defined by taking the lattice spacing to zero

\[
\hat{H}_{\text{cont.}} = \lim_{L \to \infty, a \to 0} -J \sum_{j=1}^{L-1} \hat{b}^\dagger_j e^{i \theta \hat{n}_j} \hat{b}_{j+1} + \text{h.c.}.
\]

(5)

Following the procedure of Ref. [53], the bosonic field operator in the continuum is

\[
\hat{\Psi}_B(x) = \lim_{a \to 0} \hat{b}_j / \sqrt{a},
\]

(6)

which results in the bosonic commutator

\[
\left[ \hat{\Psi}_B(\tilde{x}), \hat{\Psi}_B(x) \right] = \lim_{a \to 0} \frac{\delta_{\tilde{x},x}}{a} \equiv \delta(x - \tilde{x}).
\]

(7)

Because of the delta-function, it is important that all expressions are normal ordered before taking the continuum limit in order to avoid divergences. Furthermore, we expand the bosonic operator as

\[
\hat{\Psi}_B(x + a) \approx \hat{\Psi}_B(x) + a \partial_x \hat{\Psi}_B(x) + \frac{a^2}{2} \partial_x^2 \hat{\Psi}_B(x).
\]

(8)

In order to keep the full dependence on the anyonic phase angle \( \theta \) it is crucial to express the Peierls factor in normal ordered form, which is possible to all orders by [57]

\[
e^{i \theta \hat{n}_j} = \sum_{q=0}^{\infty} \frac{(i \theta)^q}{q!} \hat{n}_j^q = \sum_{q=0}^{\infty} \frac{(i \theta)^q}{q!} \sum_{m=0}^{q} S(q, m) (\hat{b}_j^\dagger)^m (\hat{b}_j)^m,
\]

(9)

where \( S(q, m) \) are the Stirling numbers of second kind [55]. Going to the continuum limit according to Eq. (9) and Eq. (5), we observe that each operator \( \hat{b}_j \) carries a factor of \( \sqrt{a} \) and each derivative a factor of \( a \), which yields an overall scale \( J a^2 = \hbar^2 / 2m_{\text{eff}} \) that is set to unity in the following. By neglecting higher powers in \( a \) and resumming the Peierls factor in Eq. (9), we finally obtain the following Hamiltonian density in the continuum limit

\[
\hat{H}_{\text{cont.}} = \hat{H}_{\text{kin}} + \hat{H}_{\text{int}} + \hat{H}_3,
\]

(10)

with

\[
\hat{H}_{\text{kin}} = \partial_x \hat{\Psi}_B^\dagger(x) \partial_x \hat{\Psi}_B(x),
\]

\[
\hat{H}_{\text{int}} = V_2(\theta) \hat{\rho}_B^2(x) + V_3(\vartheta) \hat{\rho}_B(x),
\]

\[
\hat{H}_3 = V_3(\theta) \hat{\rho}_B(x) \hat{J}_B(x),
\]

(11)

where \( \hat{J}_B(x) = -i \left( \hat{\Psi}_B^\dagger(x) \partial_x \hat{\Psi}_B(x) - \text{h.c.} \right) \) is the current operator and \( \hat{\rho}_B(x) = \hat{\Psi}_B(x) \hat{\Psi}_B(x) \) the density operator. We have furthermore dropped trivial contributions proportional to the density, since they are constant shifts of energy for a given particle number. The coupling constants of the theory are

\[
V_2(\theta) = \text{Im} \left[ \hat{V}_2(\theta) \right] = -\sin(\theta)
\]

(12)

\[
V_2(\vartheta) = - \frac{2 \text{Re} \left[ \hat{V}_2(\theta) \right]}{a} = 2 \frac{[1 - \cos(\theta)]}{a},
\]

(13)

\[
V_3(\theta) = - \text{Re} \left[ \hat{V}_2(\theta)^2 \right] = - \left[ 1 - 2 \cos(\theta) + \cos(2\theta) \right],
\]

(14)

with \( \hat{V}_2(\theta) = 1 - e^{i \theta} \). The two-body interaction \( V_2(\theta) \) diverges when \( a \to 0 \) if \( \theta \neq 0 \), which is in fact also the case for the ordinary Bose-Hubbard interaction \( U \) [54]. In the Hubbard model this can be resolved by rescaling \( U \) with the overall density, leading to a renormalized theory which recovers the so-called Tonks-Girardeau limit with infinitely strong interactions for low densities [60, 61]. However, in the anyonic theory, the angle \( \theta \) cannot simply be rescaled or renormalized without changing the anyonic phase angle, which would violate the topological character. As we will see below, the dependence on the lattice spacing \( a \) will allow us to define regularized coupling constants, which are fixed by experimental parameters. Moreover, we recognize a current-density interaction potential \( V_J \) and a three-body interaction \( V_3 \), so that the
full theory cannot be derived by only considering two-particle scattering processes \[31\].

We furthermore observe that the resulting bosonic model in Eq. (10) has the same structure as the integrable Kundu model \[29\] albeit with different coupling constants. The Kundu model is described by

\[
V_j^{\text{Kundu}}(\theta) = -\theta
\]
\[
V_2^{\text{Kundu}}(\theta) = \theta^2 \int dx \delta^2(x) + c
\]
\[
V_3^{\text{Kundu}}(\theta) = \theta^2,
\]

where the parameter \(c\) is related to the Hubbard interaction \(U = ac\), which describes a possible additional interaction of the equivalent anyonic form of the Kundu model \[29\]

\[
\hat{H}_K = \int_{-\infty}^{\infty} dx \ \partial_x \hat{\Psi}_A(x) \partial_x \hat{\Psi}_A(x) + \int_{-\infty}^{\infty} dx \ c \hat{\Psi}_A(x) \hat{\Psi}_A(x) \hat{\Psi}_A(x) \hat{\Psi}_A(x),
\]

Here, the anyonic operators \(\hat{\Psi}_A\) are related to the bosonic ones \(\hat{\Psi}_B\) by the continuum version of the Jordan-Wigner transformation \[29\]

\[
\hat{\Psi}_A(x) = \hat{\Psi}_B(x) e^{i\theta \int_{-\infty}^{x} n(y)dy}.
\]

This relation naively looks like a straight-forward continuum limit of Eq. (6), but such an approximation does not capture the full topological character since the crucial symmetry \(\theta \to \theta + 2\pi\) is lost.

By comparing the coupling constants, we see that the Kundu model corresponds to the special case of small anyon angles \(\theta\) in the continuum model \(\hat{H}_{\text{cont.}}\) in Eq. (10). Moreover, the singular behavior of a double delta function is replaced by the \(1/a\) divergence of \(V_2\) in Eq. (13).

Therefore, our derived continuum model is more general and introduces a well-defined limiting procedure how the original Kundu model must be interpreted. Namely, we find that one has to extrapolate the experimental results when lowering the lattice spacing \(a\) towards zero while keeping \(L\) finite. Obviously, changing \(a\) directly is difficult in an optical lattice, but it is possible to reach this limit otherwise by noting that the density \(\rho_0 = N/L\) remains finite in the continuum limit. For a given density \(\rho_0\), the continuum limit can then be effectively achieved by extrapolating \(a = N/L\rho_0 \to 0\) by systematically lowering the number of particles per site \(N/L\).

We illustrate this procedure by means of a numerical experiment using the DMRG algorithm \[62\], which is a powerful tool to analyze the properties of the proposed continuum theory in Eq. (10). We simulate non-interacting anyons by bosons with an occupation-dependent hopping in Eq. (5) using up to 500 DMRG states in finite systems with fixed boundary conditions at the edges. For the values of \(\theta\) and the relatively low densities used for the simulations below, we find that the numerical restriction to maximal two bosons per site gives good results. It is well-established that the local density can be considered as a convenient observable to analyze the interaction strength in 1D \[32, 63, 65\], since characteristic density oscillations develop near the edges due to collective modes, which ultimately are related to Friedel oscillations in the fermion limit \[68\]. An interacting bosonic system gradually develops density oscillations near edges \[50\] that grow with increasing interactions and with decreasing densities analogous to the Fermi-Hubbard model \[67\].

In Fig. (4), we see that the corresponding density oscillations in a non-interacting anyon gas grow with \(\theta\), which plays the role of an effective interaction as expected. Using \(N = 5\) and \(L = 50\), the densities gradually build up with increasing values of \(\theta\). Keeping \(\theta = 0.5\pi\) fixed in Fig. (4), we observe how the characteristic oscillations build up with lower densities analogously to the behavior in the Hubbard model \[67\]. We see that choosing a small but finite density \(N/L\) creates a natural cutoff similar to choosing a finite \(a\) in the bosonization procedure to renormalize diverging terms in impurity problems \[69\].

In order to understand the oscillations on a more quantitative level, it is often possible to derive the Tomonaga-Luttinger liquid theory for the low-energy excitation of interacting bosons \[70, 72\]. However, in this case, a microscopic derivation of such a model is involved, since it would require to keep all orders of particle-interactions in Eq. (9). Even for known models where a derivation is possible, the Tomonaga-Luttinger liquid theory depends on nonuniversal constants that can only accurately be determined by comparison with exact results \[53\].

We therefore turn to numerical methods to determine the properties of the Luttinger liquid theory for our proposed model, most of which surprisingly can be inferred directly by using the local density data in Fig. (4). It is known that the long-range decay of the oscillations away from the edge is governed by a powerlaw \[63, 64\], where the exponent, for spinless models, is the famous Luttinger parameter \(K\) \[65\] that characterizes the two-particle interaction strength. In particular, the oscillations in the local density of a finite-size Tomonaga-Luttinger liquid follow the analytic expression \[67\]

\[
n_j/a \approx \rho + A \cos[2\pi(ja - \ell/2)] \left(\ell \sin \frac{\pi ja}{\ell}\right)^{-K},
\]

where \(\rho\) is the average density near the middle of the chain \(\ell/2 \equiv (L + 1)a/2\). As shown in Fig. (4) this expression describes the local density very well for all \(\theta > 0\) and results in a non-trivial exponent \(K\) which approaches unity for small densities and \(\theta \to \pi\). The fits are spatially limited by a cut-off distance from the edges, which indi-
FIG. 1. DMRG results for the local particle density \( n_j \).
(a) For a chain of \( L = 50 \) sites with 5 anyons with different \( \theta \). (b) For \( \theta = 0 \) and \( \pi \) and a chain length \( L = 100 \) the Friedel oscillations increase with decreasing particle number \( N \). Solid lines are fits to power-law-decaying oscillations in Eq. (20), from which the Luttinger parameters \( K \) are extracted (insets).

cates the range of validity of the Tomonaga-Luttinger liquid theory. Notably, the cut-off distance increases with the Luttinger parameter \( K \). It hence becomes increasingly difficult to describe weaker oscillations, but the data is still consistent with the expected behavior of free bosons \( K \to \infty \) for \( \theta \to 0 \), see inset of Fig. 1a. In the opposite limit of \( \theta = \pi \), the solid line in Fig. 1b is given by the analytic expression for free fermions \( 47 \) over the entire system, which means that the pseudo-fermions are well described by ordinary fermions with \( K = 1 \) in this case \( 31,32 \). The Tomonaga-Luttinger parameters in the insets of Fig. 1 determine relevant long-range correlations \( 40 \) including energy-dependent quantities like the local density of states \( 23 \). The numerical data therefore not only confirms the stability of a Tomonaga-Luttinger liquid ground state but also predicts the dominant correlations of the continuum anyon model quantitatively as a function of \( \theta \) and \( \rho_0 \). In experimental setups, the confinement is commonly given by a harmonic trap, which also gives rise to density oscillations \( 60 \), described by corresponding fit functions \( 45 \). Thus, experimental measurements of the local densities \( 23 \) in optical traps can be used to extract these characteristic exponents.

In conclusion, 1D anyons on a lattice inspired our development of a continuum theory of 1D anyons in terms of interacting bosons, which keeps the topological character of abelian anyonic exchange in contrast to previously discussed Hamiltonians in 1D. The representation of anyons as normal ordered bosons with modified hopping is crucial when taking the continuum limit. We take all orders of the anyonic phase into account, resulting in a Hamiltonian that includes current-density as well as two- and three-particle bosonic interactions with \( 2\pi \)-periodic coefficients in the anyonic phase angle. To our knowledge, a description of 1D continuum anyons that captures this topological hallmark was an open problem and the theory extends the two-dimensional concept of a \( 2\pi \)-periodic anyonic exchange phase to 1D. The known Kundu model is contained in the limit of a vanishing statistical angle. Our work therefore uncovers a unifying, physically motivated continuum theory of one-dimensional abelian anyons that derives from the original idea of an anyonic exchange phase in 2D. We thereby connect recent experiments in ultracold atomic gases to the seminal considerations of Leinaas and Myrheim as well as Wilczek, which inspired the name ‘anyon’ \( 1 \). We furthermore determine the characteristic Luttinger parameter from the decay of density oscillations.

T.P. and M.T. acknowledge support by the Cluster of Excellence “CUI: Advanced Imaging of Matter” of the Deutsche Forschungsgemeinschaft (DFG) – EXC 2056 – project ID 390715994. M.B., K.J., A.P., and S.E. were funded by the DFG – project-id 27762539 – TRR 185 via the SFB/Transregio 185: “OSCAR – Open System Control of Atomic and Photonic Matter”.

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