Simple Extensions of Polytopes

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Abstract

We introduce the simple extension complexity of a polytope \( P \) as the smallest number of facets of any simple (i.e., non-degenerate in the sense of linear programming) polytope which can be projected onto \( P \). We devise a combinatorial method to establish lower bounds on the simple extension complexity and show for several polytopes that they have large simple extension complexities. These examples include both the spanning tree and the perfect matching polytopes of complete graphs, uncapacitated flow polytopes for non-trivially decomposable directed acyclic graphs, and random 0/1-polytopes with vertex numbers within a certain range. On our way to obtain the result on perfect matching polytopes we improve on a result of Padberg and Rao’s on the adjacency structures of those polytopes.

1 Introduction

In combinatorial optimization, linear programming formulations are a standard tool to gain structural insight, derive algorithms and to analyze computational complexity. With respect to both structural and algorithmic aspects of linear optimization over a polytope \( P \) can be replaced be linear optimization over any (usually higher dimensional) polytope \( Q \) of which \( P \) can be obtained as the image under a linear map (which we refer to as a projection). Such a polytope \( Q \) (along with a suitable projection) is called an extension of \( P \).

Defining the size of a polytope as its number of facets, the smallest size of any extension of the polytope \( P \) is known as the extension complexity \( xc(P) \) of \( P \). It has turned out in the past that for several important polytopes related to combinatorial optimization problems the extension complexity is bounded polynomially in the dimension. One of the most prominent examples is the spanning tree polytope of the complete graph \( K_n \) on \( n \) nodes, which has extension complexity \( \mathcal{O}(n^3) \) [10].

After Rothvoß [14] showed that there are 0/1-polytopes whose extension complexities cannot be bounded polynomially in their dimensions, only very recently Fiorini et al. [5] could prove that the extension complexities of some concrete and important examples of polytopes like traveling salesman polytopes cannot be bounded polynomially. Similar results have then also been deduced for several other polytopes associated with NP-hard optimization problems, e.g., by Avis and Tiwary [1] and Pokutta and van Vyve [13]. Very recently, Rothvoß [15] showed that also the perfect matching polytope of the complete
Graph (with an even number of nodes) has exponential extension complexity, thus exhibiting the first polytope with this property that is associated with a polynomial time solvable optimization problem.

The first fundamental research with respect to understanding extension complexities was Yannakakis’ seminal paper [17] of 1991. Observing that many of the nice and small extensions that are known (e.g., the polynomial size extension of the spanning tree polytope of $K_n$ mentioned above) have the nice property of being symmetric in a certain sense, he derived lower bounds on extensions with that special property. In particular, he already proved that both perfect matching polytopes as well as traveling salesman polytopes do not have polynomial size symmetric extensions.

It turned out that requiring symmetry in principle actually can make a huge difference for the minimal sizes of extensions (though nowadays we know that this is not really true for traveling salesman and perfect matching polytopes). For instance, Kaibel, Theis, and Pashkovich [9] showed that the polytope associated with the matchings of size $\lfloor \log n \rfloor$ in $K_n$ has polynomially bounded extension complexity although it does not admit symmetric extensions of polynomial size. Another example is provided by the permutahedron which has extension complexity $\Theta(n \log n)$ [8], while every symmetric extension of it has size $\Omega(n^2)$ [12].

These examples show that imposing the restriction of symmetry may severely influence the smallest possible sizes of extensions. In this paper, we investigate another type of restrictions on extensions, namely the one arising from requiring the extension to be a non-degenerate polytope. A $d$-dimensional polytope is called simple if every vertex is contained in exactly $d$ facets. We denote by $sxc(P)$ the simple extension complexity, i.e., the smallest size of any simple extension of the polytope $P$.

Simplicity is both a property that is interesting from practical (primal non-degeneracy of linear programs) as well as from theoretical (large parts of the combinatorial/extremal theory of polytopes deal with simple polytopes) point of views. And similarly to the restriction to symmetric extensions, there are also nice examples of simple extensions of certain polytopes relevant in optimization. For instance, generalizing the well-known fact that the permutahedron is a zonotope, Wolsey showed in the late 80’s (personal communication) that, for arbitrary processing times, the completion time polytope for $n$ jobs is a projection of an $O(n^2)$-dimensional cube. The main results of this paper show, however, that for several polytopes relevant in optimization (among them both perfect matching polytopes and spanning tree polytopes) insisting on simplicity causes exponential sizes of the extensions. More precisely, we establish that for the following polytopes the simple extension complexity equals their number of vertices (note that the number of vertices of $P$ is a trivial upper bound for $sxc(P)$, realized by the extension obtained from writing $P$ as the convex hull of its vertices):

- Perfect matching polytopes of complete graphs (Theorem 6.1)
- Uncapacitated flow polytopes of non-decomposable acyclic networks (Theorem 5.1)
- (Certain) random $0/1$-polytopes (Theorem 2.8)
- Hypersimplices (Theorem 3.1)
Furthermore, we prove that

- the spanning tree polytope of the complete graph with \( n \) nodes has simple extension complexity at least \( \Omega(2^{n-o(n)}) \) (Theorem 4.2).

Let us make a brief digression on the potential relevance of simple extensions with respect to questions related to the diameter of a polytope, i.e., the maximal distance (minimum number of edges on a path) between any pair of vertices in the graph of the polytope. We denote by \( \Delta(d, m) \) the maximal diameter of any \( d \)-dimensional polytope with \( m \) facets. It is well-known that \( \Delta(d, m) \) is attained by simple polytopes. A necessary condition for a polynomial time variant of the simplex-algorithm to exist is that \( \Delta(d, m) \) is bounded by a polynomial in \( d \) and \( m \) (thus by a polynomial in \( m \)). In fact, in 1957 Hirsch even conjectured (see [3]) that \( \Delta(d, m) \leq m - d \) holds, which has only rather recently been disproved by Santos [16]. However, still it is even unknown whether \( \Delta(d, m) \leq 2m \) holds true, and the question, whether \( \Delta(d, m) \) is bounded polynomially (i.e., whether the polynomial Hirsch-conjecture is true) is a major open problem in Discrete Geometry.

In view of the fact that linear optimization over a polytope can be performed by linear optimization over any of its extensions, a reasonable relaxed version of that question might be to ask whether every \( d \)-dimensional polytope \( P \) with \( m \) facets admits an extension whose size and diameter both are bounded polynomially in \( m \). Stating the relaxed question in this naive way, the answer clearly is positive, as one may construct an extension by forming a pyramid over \( P \) (after embedding \( P \) into \( \mathbb{R}^{\dim(P)+1} \)), which has diameter two. However, in some accordance with the way the simplex algorithm works by pivoting between bases rather than only by proceeding along edges, it seems to make sense to require the extension to be simple (which a pyramid, of course, in general is not). But still, this is not yet a useful variation, since our result on flow polytopes mentioned above shows that there are polytopes that even do not admit a polynomial size simple extension at all. Therefore, we propose to investigate the following question, whose positive answer would be implied by establishing the polynomial Hirsch-conjecture (as every polytope is an extension of itself).

**Question 1.1.** Does there exist a polynomial \( q \) such that every simple polytope \( P \) with \( m \) facets has a simple extension \( Q \) with at most \( q(m) \) many facets and diameter at most \( q(m) \)?

The paper is structured as follows: We first devise some techniques to bound the simple extension complexity of a polytope from below (Section 2). Then we deduce our results on hypersimplices (Section 3), spanning tree polytopes (Section 4), flow polytopes (Section 5), and perfect matching polytopes (Section 6). The core of the latter part is a strengthening of a result of Padberg and Rao [11] on adjacencies in the perfect matching polytope (Theorem 6.6) which may be of independent interest.

Let us end this introduction by remarking that the concept of simplicial extensions is not interesting, since it is rather easy to see that for every polytope \( P \) the smallest possible simplicial (i.e., all facets are simplices) polytope that is an extension of \( P \) is the simplex with as many vertices as \( P \) has.
2 Bounding Techniques

Let $P \subseteq \mathbb{R}^n$ be a polytope with $N$ vertices. The faces of $P$ form a graded lattice $\mathcal{L}(P)$, ordered by inclusion (see [18]).

Clearly, $P$ is the set of all convex combinations of its vertices, immediately providing an extended formulation of size $N$:

$$P = \text{proj}_x \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^V : x = \sum_{v \in V} y_v v, \sum_{v \in V} y_v = 1 \right\}$$

Here, $\text{proj}_x (\cdot)$ denotes the projection onto the space of $x$-variables and $V$ is the set of vertices of $P$. Note that this trivial extension is simple since the extension is an $(N-1)$-simplex.

An easy observation for extensions $P = \pi(Q)$ is that the assignment $F \mapsto \pi^{-1}(F) \cap Q$ defines a map $j$ which embeds $\mathcal{L}(P)$ into $\mathcal{L}(Q)$, i.e., it is one-to-one and preserves inclusion in both directions (see [4]). Note that this embedding furthermore satisfies $j(F \cap F') = j(F) \cap j(F')$ for all faces $F, F'$ of $P$ (where the nontrivial inclusion $j(F) \cap j(F') \subseteq j(F \cap F')$ follows from $\pi(j(F) \cap j(F')) \subseteq \pi(j(F)) \cap \pi(j(F')) = F \cap F'$). We use the shorthand notation $j(v) := j(\{v\})$ for vertices $v$ of $P$.

We consider the face-vertex non-incidence graph $G_N(P)$ which is a bipartite graph having the faces and the vertices of $P$ as the node set and edges $\{F, v\}$ for all $v \notin F$. Every facet $f$ of an extension induces two node sets of this graph in the following way:

$$\mathcal{F}(\hat{f}) := \{ F \text{ face of } P : j(F) \subseteq \hat{f} \}$$
$$\mathcal{V}(\hat{f}) := \{ v \text{ vertex of } P : j(v) \notin \hat{f} \}$$

We call $\mathcal{F}(\hat{f})$ and $\mathcal{V}(\hat{f})$ the set of faces (resp. vertices) induced by the facet $\hat{f}$ (with respect to the extension $P = \pi(Q)$). Typically, the extension and the facet $\hat{f}$ are fixed and we just write $\mathcal{F}$ (resp. $\mathcal{V}$). It may happen that $\mathcal{V}(\hat{f})$ is equal to the whole vertex set, e.g. if $\hat{f}$ projects into the relative interior of $P$. If $\mathcal{V}(\hat{f})$ is a proper subset of the vertex set we call facet $\hat{f}$ proper w.r.t. the projection.

For each facet $\hat{f}$ of an extension of $P$ the face and vertex sets together induce a biclique (i.e., complete bipartite subgraph) in $G_N(P)$. It follows from Yannakakis [17] that every edge in $G_N(P)$ is covered by at least one of those induced bicliques. We provide a brief combinatorial argument for this (in particular showing that we can restrict to proper facets) in the proof of the following proposition.

**Proposition 2.1.** Let $P = \pi(Q)$ be an extension.

Then the subgraph of $G_N(P)$ induced by $\mathcal{F}(\hat{f}) \cup \mathcal{V}(\hat{f})$ is a biclique for every facet $\hat{f}$ of $Q$. Furthermore, every edge $\{F, v\}$ of $G_N(P)$ is covered by at least one of the bicliques induced by a proper facet.

**Proof.** Let $\hat{f}$ be one of the facets and assume that an edge $\{F, v\}$ with $F \in \mathcal{F}(\hat{f})$ and $v \in \mathcal{V}(\hat{f})$ is not present in $G_N(P)$, i.e., $v \notin F$. From $v \in F$ we obtain $j(v) \subseteq j(F) \subseteq \hat{f}$, a contradiction to $v \notin \mathcal{V}(\hat{f})$. 


To prove the second statement, let \( \{ F, v \} \) be any edge of \( G_N(P) \), i.e., \( v \notin F \). Observe that the preimages \( G := j(F) \) and \( g := j(v) \) are also not incident since \( j \) is a lattice embedding. As \( G \) is the intersection of all facets of \( Q \) it is contained in (the face-lattice of a polytope is coatomic), there must be at least one facet \( \hat{f} \) containing \( G \) but not \( g \) (since otherwise \( g \) would be contained in \( G \)), yielding \( F \in \mathcal{F}(\hat{f}) \) and \( v \in \mathcal{V}(\hat{f}) \).

If \( F \neq \emptyset \), any vertex \( w \in F \) satisfies \( j(w) \subseteq G \subseteq \hat{f} \) and hence \( \hat{f} \) is a proper facet. If \( F = \emptyset \), let \( v \) be any vertex of \( P \) distinct from \( v \). The preimages \( j(v) \) and \( j(w) \) clearly satisfy \( j(v) \notin j(w) \). Again, since the face-lattice of \( Q \) is coatomic, there exists a facet \( \hat{f} \) with \( j(w) \subseteq \hat{f} \) but \( j(v) \notin \hat{f} \). Hence, \( \hat{f} \) is a proper facet and (since \( \emptyset = F \subseteq \hat{f} \)) \( F \in \mathcal{F}(\hat{f}) \) and \( v \in \mathcal{V}(\hat{f}) \) holds.

Before moving on to simple extensions we mention two useful properties of the induced sets. Both can be easily verified by examining the definitions of \( \mathcal{F} \) and \( \mathcal{V} \). See Figure 1 for an illustration.

**Lemma 2.2.** Let \( \mathcal{F} \) and \( \mathcal{V} \) be the face and vertex sets induced by a facet of an extension of \( P \), respectively.

Then \( \mathcal{F} \) is closed under taking subfaces and \( \mathcal{V} = \{ v \text{ vertex of } P : v \notin \bigcup \mathcal{F} \} \).

![Figure 1: The Sets \( \mathcal{F} \) and \( \mathcal{V} \) in the Face Lattice.](image)

For the remainder of this section we assume that the extension polytope \( Q \) is a simple polytope and that \( \mathcal{F} \) and \( \mathcal{V} \) are face and vertex sets induced by a facet of \( Q \).

**Theorem 2.3.** Let \( \mathcal{F} \) and \( \mathcal{V} \) be the face and vertex sets induced by a facet of a simple extension of \( P \), respectively. Then

...
(a) all pairs \( (F,F') \) of faces of \( P \) with \( F \cap F' \neq \emptyset \) and \( F,F' \notin \mathcal{F} \) satisfy \( F \cap F' \notin \mathcal{F} \),
(b) the (inclusion-wise) maximal elements in \( \mathcal{F} \) are facets of \( P \),
(c) and every vertex \( v \notin V \) is contained in some facet \( F \) of \( P \) with \( F \in \mathcal{F} \).

Proof. Let \( \hat{F} \) be the facet of \( Q \) inducing \( \mathcal{F} \) and \( V \) and \( F,F' \) two faces of \( P \) with non-empty intersection. Since \( F \cap F' \neq \emptyset \), we have \( j(F \cap F') \neq \emptyset \), thus the interval in \( \mathcal{L}(Q) \) between \( j(F \cap F') \) and \( Q \) is a Boolean lattice (because \( Q \) is simple). Suppose \( F \cap F' \in \mathcal{F}(\hat{F}) \). Then \( \hat{F} \) is contained in that interval and it is a coatom, hence it contains at least one of \( j(F) \) and \( j(F') \) due to \( j(F) \cap j(F') = j(F \cap F') \). But this implies \( j(F) \in \mathcal{F} \) or \( j(F') \in \mathcal{F} \), proving (a).

For (b), let \( F \) be an inclusion-wise maximal face in \( \mathcal{F} \) but not a facet of \( P \). Then \( F \) is the intersection of two faces \( F_1 \) and \( F_2 \) of \( P \) properly containing \( F \). Due to the maximality of \( F \), \( F_1, F_2 \notin \mathcal{F} \) but \( F_1 \cap F_2 \in \mathcal{F} \), contradicting (a).

Statement (c) follows directly from (b) and Lemma 2.2. \( \square \)

In order to use the Theorem 2.3 for deriving lower bounds on the sizes of simple extensions of a polytope \( P \), one needs to have good knowledge of parts of the face lattice of \( P \). The part one usually knows most about is formed by the vertices and edges of \( P \). Therefore, we specialize Theorem 2.3 to these faces for later use.

Let \( G = (V,E) \) be a graph and denote by \( \delta(W) \subseteq E \) the cut-set of a node-set \( W \). Define the common neighbor operator \( \Lambda(\cdot) \) by

\[
\Lambda(W) := W \cup \{v \in V : \exists \{u,v\}, \{v,w\} \in \delta(W) : u \neq w\} .
\]  

(2)

A set \( W \subseteq V \) is then a (proper) common neighbor closed (for short \( \Lambda \)-closed) set if \( \Lambda(W) = W \) (and \( W \neq V \)) holds. We call sets \( W \) with a minimum node distance of at least 3 (i.e., the distance-2-neighborhood of a node \( w \in W \) does not contain another node \( w' \in W \) isolated. Isolated node sets are clearly \( \Lambda \)-closed. Note that singleton sets are isolated and hence proper \( \Lambda \)-closed. In particular, the vertex sets induced by the facets of the trivial extension (see beginning of Section 2) are the singleton sets.

Using this notion, we obtain the following corollary of Theorem 2.3.

**Corollary 2.4.** The vertex set \( V \) induced by a proper facet of a simple extension of \( P \) is a proper \( \Lambda \)-closed set.

Proof. Theorem 2.3 implies that for every \( \{u,v\}, \{v,w\} \) of (distinct) adjacent edges of \( P \), we have

\[
\{u,v\}, \{v,w\} \notin \mathcal{F} \implies \{v\} \notin \mathcal{F} .
\]

Due to Lemma 2.2, \( V = \{v \text{ vertex of } P : v \notin \mathcal{F} \} \), where \( \mathcal{F} \) is the face set induced by the same facet. Hence, \( v \notin V \) implies \( \{u,v\} \in \mathcal{F} \) or \( \{v,w\} \in \mathcal{F} \), thus \( u \notin V \) or \( w \notin V \) and we conclude that \( V \) is \( \Lambda \)-closed.

Furthermore, \( V \) is not equal to the whole vertex set of \( P \) since the given facet is proper. \( \square \)

We just proved that every biclique \( \mathcal{F} \cup V \) induced by a (proper) facet from a simple extension must satisfy certain properties. The next example shows that these properties are not sufficient for an extension polytope to be simple.
Example 2.5. Define $m_1, \ldots, m_7 \in \mathbb{R}^3$ to be the columns of the matrix

$$M = \begin{pmatrix} 1 & 5 & 1 & 0 & -1 & -5 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -4 & 0 & 1 & 0 & -4 & 0 \end{pmatrix},$$

and let $Q := \text{conv} \{m_1, \ldots, m_7\} \subseteq \mathbb{R}^3$ be their convex hull. The vertex $m_4$ has 4 neighbors, that is, $Q$ is not simple. Let $P$ be the projection of $Q$ onto the first two coordinates. Observe that $P$ is a 6-gon and that the only relevant types of faces $F, F'$ are adjacent edges of $P$. It is quickly verified that all induced face and vertex sets satisfy Theorem 2.3 and Corollary 2.4, respectively.

Note that this example only shows that we cannot decide from the biclique covering whether the extension is simple. It may still be true that for such biclique coverings there always exists a simple extension.

The polytope $Q$ from the example can be used to show that Corollary 2.4 is indeed a specialization of Theorem 2.3 (a). To see this, consider the set $\mathcal{F}$ of faces consisting of $\text{conv} \{m_1, m_4, m_5\}$, $\text{conv} \{m_3, m_4, m_7\}$ and all their subfaces. Lemma 2.2 implies $V = \{m_2, m_6\}$ which is proper $\Lambda$-closed. But $\mathcal{F}$ does not satisfy Theorem 2.3 (a) for the choice $F := \text{conv} \{m_1, m_2, m_3, m_4\} \notin \mathcal{F}$, $F' := \text{conv} \{m_4, m_5, m_6, m_7\} \notin \mathcal{F}$ since $F \cap F' = \{m_4\} \in \mathcal{F}$.

Nevertheless we can obtain useful lower bounds from Theorem 2.3 and Corollary 2.4.

Corollary 2.6. The node set of a polytope $P$ can be covered by $\text{sxc}(P)$ many proper $\Lambda$-closed sets.

Lemma 2.7. Let $P$ be a polytope and $G$ its graph. If all proper $\Lambda$-closed sets in $G$ are isolated then the simple extension complexity of $P$ is greater than the maximum size of the neighborhood of any node of $G$.

Proof. Let $w$ be a node maximizing the size of the neighborhood and let $W$ be the neighborhood of $w$. Since no isolated set can contain more than one node from $W \cup \{w\}$, Corollary 2.6 implies the claim. \qed
Using knowledge about random 0/1 polytopes, we can easily establish the following result.

**Theorem 2.8.** There is a constant $\sigma > 0$ such that a random $d$-dimensional 0/1-polytope $P$ with at most $2^d$ vertices asymptotically almost surely has a simple extension complexity equal to its number of vertices.

**Proof.** It is one of the main results in the thesis [7] that there is such a $\sigma$ ensuring that a random $d$-dimensional 0/1-polytope $P$ with at most $2^d$ vertices asymptotically almost surely has every pair of vertices adjacent. Since in this situation the only proper $\Lambda$-closed sets are the singletons, Corollary 2.6 yields the claim. \qed

## 3 $k$-Hypersimplex

Let $\Delta(k)$ denote the $k$-hypersimplex in $\mathbb{R}^n$, i.e., the 0/1-cube intersected with the hyperplane $\langle 1_n, x \rangle = k$. Observe that $\Delta(k)$ is almost simple in the sense that its dimension is $n - 1$, but every vertex lies in exactly $n$ facets. With this in mind, the following result may seem somewhat surprising.

**Theorem 3.1.** Let $1 \leq k \leq n - 1$. The simple extension complexity of $\Delta(k) \subseteq \mathbb{R}^n$ is equal to its number of vertices $\binom{n}{k}$.

**Proof.** The case of $k = 1$ or $k = n - 1$ is clear since then $\Delta(k)$ is an $(n - 1)$-dimensional simplex.

Let $2 \leq k \leq n - 2$ and $\mathcal{F}$ and $\mathcal{V}$ be face and vertex sets induced by a proper facet of a simple extension of $\Delta(k)$.

Since every vertex $v$ of $\Delta(k)$ has $v_i = 0$ or $v_i = 1$, at most one of the facets $x_i \geq 0$ or $x_i \leq 1$ can be in $\mathcal{F}$ for every $i \in [n]$ (otherwise $\mathcal{V}$ would be empty). We can partition $[n]$ into $L \cup U \cup R$ such that $L$ (resp. $U$) contains those indices $i \in [n]$ such that the facet corresponding to $x_i \geq 0$ (resp. $x_i \leq 1$) is in $\mathcal{F}$ and $R$ contains the remaining indices. Lemma 2.2 yields

$$\mathcal{V} = \{ v \text{ vertex of } \Delta(k) : v_L = 1, \ v_U = \emptyset \} \ .$$

We now prove that a node set $\mathcal{V}$ of this form is proper $\Lambda$-closed only if $|\mathcal{V}| = 1$. Then, Corollary 2.6 yields the claim.

Indeed, if we have $|\mathcal{V}| > 1$, then there exist vertices $u, w \in \mathcal{V}$ and indices $i, j \in R$ such that $u_i = w_j = 1$, $u_j = w_i = 0$, and $u_l = w_l$ for all $l \notin \{i, j\}$ (see Figure 3). Choose any $s \in L \cup U$ and observe that, since $u, w \notin \mathcal{V}$, $u_s = w_s = 1$ if $s \in L$ and $u_s = w_s = 0$ if $s \in U$. The following vertex is easily checked to be adjacent to $u$ and $w$ (min and max must be read component-wise):

$$v := \begin{cases} 
\max(u, w) - \alpha^s & \text{if } s \in L \\
\min(u, w) + \alpha^s & \text{if } s \in U 
\end{cases}$$

As $v_s = 0$ if $s \in L$ and $v_s = 1$ if $s \in U$, $v \notin \mathcal{V}$. This contradicts the fact that $\mathcal{V}$ is $\Lambda$-closed. \qed
| \( L \) | \( U \) | \( R \) |
|---|---|---|
| 1, 1, ..., 1 | 0, 0, ..., 0 | 0, 0, ..., 1 |
| \( s \) | \( v \notin \mathcal{V} \) |

Figure 3: Vertices of \( \Delta(k) \) in \( \mathcal{V} \) for a Biclique.

4 Spanning Tree Polytope

In this section we bound the simple extension complexity of the spanning tree polytope \( P_{\text{spt}}(K_n) \) of the complete graph \( K_n \) with \( n \) nodes.

**Lemma 4.1.** All proper \( \Lambda \)-closed sets in the graph of \( P_{\text{spt}}(K_n) \) are isolated.

**Proof.** Two vertices of \( P_{\text{spt}}(K_n) \) are adjacent if and only if the symmetric difference of the corresponding spanning trees consists of exactly two edges. Throughout the proof, we will identify vertices with the corresponding spanning trees.

Suppose \( V \) is a proper \( \Lambda \)-closed set that is not isolated. Then there are spanning trees \( T_1, T_2 \in V \) and \( T_3 \notin V \), such that \( T_1 \) is adjacent to both \( T_2 \) and \( T_3 \), but \( T_2 \) and \( T_3 \) are not adjacent.

\[ T_1 \cap T_2 \] is a forest with exactly two components having vertex sets \( X \) and \( Y \). Let \( e \in T_1 \) and \( f \in T_2 \) be the edges in \( T_1 \cup T_2 \) connecting \( X \) and \( Y \), \( \{g\} = T_1 \setminus T_3 \), and \( \{h\} = T_3 \setminus T_1 \). We have \( g \neq e \), since \( T_2 \setminus T_3 \subseteq T_1 \cup \{e\} \setminus T_3 \subseteq \{g,e\} \) cannot have cardinality one, because \( T_2 \) and \( T_3 \) are not adjacent.
Therefore, let w.l.o.g. \( g \) be an edge in \( T_1[X] \) and let \( X' \) and \( X'' \) be the components of \( T_1 \setminus \{e\} \) it connects such that \( X' \cap e = \emptyset \). Define \( F := T_1 \cap T_2 \cap T_3 \) and observe \( T_1 = F \cup \{e, g\}, T_2 = F \cup \{f, g\}, \) and \( T_3 = F \cup \{e, h\} \). There are two possible cases for \( h \):

**Case 1:** \( h \) connects \( Y \) with \( X' \) or \( X'' \).

Let \( T' := F \cup \{g, h\} \) and observe that \( T' \) is a spanning tree since \( g \) connects \( X' \) with \( X'' \) and \( h \) connects one of both with \( Y \). Obviously, \( T' \) is adjacent to \( T_1, T_2, \) and \( T_3 \). Since \( T' \) is adjacent to \( T_1 \) and \( T_2, T' \in \Lambda(\mathcal{V}) = \mathcal{V} \). Since \( T_3 \) is adjacent to \( T_1, T' \in \mathcal{V} \), this in turn implies the contradiction \( T_3 \in \mathcal{V} \).

**Case 2:** \( h \) connects \( X' \) with \( X'' \).

Let \( j \) be any edge connecting \( X' \) with \( Y \) (recall that we dealing with a complete graph) and let \( T' := F \cup \{g, j\} \) which is a spanning tree adjacent to \( T_1 \) and \( T_2 \) and hence \( T' \in \Lambda(\mathcal{W}) = \mathcal{W} \). Clearly, \( T'' := F \cup \{e, j\} \) is a spanning tree adjacent to \( T_1 \) and \( T'' \) hence \( T'' \in \mathcal{V} \). Finally, let \( T''' := F \cup \{h, j\} \) be a third spanning tree adjacent to \( T' \) and \( T'' \). Again, we have \( T''' \in \mathcal{V} \) due to \( \Lambda(\mathcal{W}) = \mathcal{W} \).

Since \( T_3 \) is adjacent to \( T_1 \) and \( T''' \), exploiting \( \Lambda(\mathcal{V}) = \mathcal{V} \) once more yields the contradiction \( T_3 \in \mathcal{V} \).

Using this result we immediately get a lower bound of \( \Omega\left(n^3\right) \) for the simple extension complexity of \( P_{spt}(K_n) \) since the maximum degree of its graph is of that order. However, we can prove a much stronger result.

**Theorem 4.2.** The simple extension complexity of the spanning tree polytope of \( K_n \) is in \( \Omega(2^{\sqrt{n-o(n)}}) \).

**Proof.** Assume \( n \geq 5 \) and let \( s, t \) be any two distinct nodes of \( K_n \). Consider certain subsets on the other nodes

\[ W := \{W \subseteq V \setminus \{s, t\} : |W| = \lfloor n/2 \rfloor\} . \]
Let \( k := \lceil n/2 \rceil \), fix some ordering of the nodes \( w_1, w_1, \ldots, w_k \in W \) for each \( W \in \mathcal{W} \) and define a specific tree \( T(W) \)

\[
T(W) := \{\{s, w_1\}, \{w_k, t\}\} \\
\cup \{\{w_i, w_{i+1}\} : i \in [k - 1]\} \\
\cup \{\{t, v\} : v \notin (W \cup \{s, t\})\}
\]

as depicted in Figure 5. We will now prove that for each simple extension of \( P_{sp}(K_n) \) every such \( T(W) \) must be in a different induced vertex set.

Let \( W \in \mathcal{W} \) be some set \( W \) with tree \( T(W) \). Let \( \mathcal{F} \) and \( \mathcal{V} \) be the face and vertex sets, respectively, induced by a proper facet of a simple extension such that \( T(W) \) is in \( \mathcal{V} \). Construct an adjacent tree \( T' \) as follows.

Choose some vertex \( y \in W \) and let \( x-y-z \) be a subpath of the \( s-t \)-path in \( T(W) \) in that order. Note that \( \{x, y, z\} \subseteq W \cup \{s, t\} \). Denote by \( a, b, c \) the edges \( \{x, y\} \), \( \{x, z\} \), and \( \{y, z\} \), respectively.

Let \( T'' = T(W) \setminus \{a\} \cup \{b\} \). Because \( T'' \) is adjacent to \( T(W) \), \( T'' \notin \mathcal{V} \) by Lemma 4.1. Hence, due to Lemma 2.2, there must be a facet \( F \in \mathcal{F} \) defined by \( x(E[U]) \leq |U| - 1 \) (with \( |U| \geq 2 \)) which contains \( T'' \) but not \( T(W) \). Hence, we have \( |T(W)[U]| < |U| - 1 \) and \( |T''[U]| = |U| - 1 \). This implies \( |T(W) \cap \delta(U)| \geq 2 \) and \( |T'' \cap \delta(U)| = 1 \). Obviously, \( a \in \delta(U) \) and \( b \notin \delta(U) \).

Then \( x, z \in U \) if and only if \( y \notin U \) because \( a \notin \delta(U) \) and \( b \notin \delta(U) \). Hence, \( e \in \delta(U) \), i.e., \( T \cap U = \{e\} \). Due to \( |U| \geq 2 \), this implies \( U = V \setminus \{y\} \).

As this can be argued for any \( y \in W \), we have that the facets defined by \( V \setminus \{y\} \) are in \( \mathcal{F} \) for all \( y \in W \). Hence, \( \mathcal{V} \) contains only trees \( T \) for which \( |T \cap \delta(V \setminus \{y\})| = |T \cap \delta(\{y\})| \geq 2 \), i.e., no leaf of \( T \) is in \( W \).

This shows that for distinct sets \( W, W' \in \mathcal{W} \), any vertex set \( \mathcal{V} \) induced by a proper facet of a simple extension that contains \( T(W) \) containing \( T(W) \) does not contain \( T(W') \) because any vertex \( v \in W \setminus W' \) is a leaf of \( T(W') \). Hence, the number of simple bicliques is at least

\[
|\mathcal{W}| = \binom{n-2}{\lceil n/2 \rceil} \in \Omega(2^{n-o(n)})
\]

\(\blacksquare\)
5 Flow Polytopes for Acyclic Networks

Many extended formulations model the solutions to the original formulation via a path in a specifically constructed directed acyclic graph. The size of the construction then equals the number of arcs in that graph since the paths from two fixed nodes \( s \) and \( t \) arise as vertices of the corresponding flow polytope whose facets correspond to nonnegativity constraints on arcs. Such a network formulation can be easily decomposed into two independent formulations if a node \( v \) exists such that every \( s \rightarrow t \)-path traverses \( v \). We are now interested in the simple extension complexities of flow polytopes of \( s \rightarrow t \)-networks that cannot be decomposed in such a trivial way.

Let \( D = (V, A) \) be a directed acyclic graph with fixed source \( s \in V \) and sink \( t \in V \). By \( \mathcal{P}_{s,t}(D) \) we denote the arc-sets of \( s \rightarrow t \)-paths in \( D \). For some path \( P \in \mathcal{P}_{s,t}(D) \) and nodes \( u, v \in V(P) \), we denote by \( P_{(u,v)} \) the subpath of \( P \) going from \( u \) to \( v \).

We consider the flow polytope \( P_{s\rightarrow t\text{-flow}}(D) \), i.e., the set of all \( s \rightarrow t \)-flows in \( D \) of value one. The facets of \( P_{s\rightarrow t\text{-flow}}(D) \) correspond to the nonnegativity constraints \( y_a \geq 0 \) for some \( a \in A \). Clearly, the vertices correspond to \( \mathcal{P}_{s,t}(D) \). A path \( P \in \mathcal{P}_{s,t}(D) \) is non-incident to a facet \( y_a \geq 0 \) if and only if \( a \in P \). Two paths \( P, P' \in \mathcal{P}_{s,t}(D) \) are adjacent vertices of the polytope if and only if their symmetric difference consists of two paths from \( x \) to \( y \) (\( x, y \in V, x \neq y \)) without common inner nodes (see [6]). Our main result in this section is the following:

**Theorem 5.1.** Let \( D = (V, A) \) be a directed acyclic graph with source \( s \in V \) and sink \( t \in V \) such that for every node \( v \in V \setminus \{s, t\} \) there exists an \( s \rightarrow t \)-path in \( D \) which does not traverse \( v \).

Then the simple extension complexity of \( P_{s\rightarrow t\text{-flow}}(D) \subseteq \mathbb{R}_{+}^{A} \) is equal to the number of distinct \( s \rightarrow t \)-paths \( |\mathcal{P}_{s,t}(D)| \).

**Proof.** Let \( \mathcal{F} \) and \( \mathcal{V} \) be the face and vertex sets induced by a proper facet of a simple extension of \( P_{s\rightarrow t\text{-flow}}(D) \), respectively. Assume for the sake of contradiction \( |\mathcal{V}| \geq 2 \). By Theorem 2.3 (b), the (inclusion-wise) maximal faces in \( \mathcal{F} \) are facets. Let \( \emptyset \neq B' \subseteq A \) be the arc set corresponding to these facets. By Lemma 2.2, \( \mathcal{V} \) is the set of (characteristic vectors of) paths \( P \in \mathcal{P}_{s,t}(D) \) satisfying \( P \supseteq B' \). Let \( B \subseteq A \) be the set of arcs common to all such paths and note that \( B 
subseteq B' \).

By construction, for any path \( P \in \mathcal{V} \) and any arc \( a \in P \setminus B \), there is an alternative path \( P' \in \mathcal{V} \) with \( a \notin P' \).

Let us fix one of the paths \( P \in \mathcal{V} \). Let, without loss of generality, \( (x', x) \in B \) be such that the arc of \( P \) leaving \( x \) (exists and) is not in \( B \). If such an arc does not exist, since \( B \supseteq P \), there must be an arc \( (x, x') \in B \) such that the arc of \( P \) entering \( x \) is not in \( B \). In this case, revert the directions of all arcs in \( D \) and exchange the roles of \( s \) and \( t \) and apply subsequent arguments to the new network. Let \( y \) be the first node on \( P|_{(x,t)} \) different from \( x \) and incident to some arc in \( B \) or, if no such \( y \) exists, let \( y := t \). Paths in \( \mathcal{V} \) must leave \( x \) and enter \( y \) but may differ inbetween. The set of traversed nodes is defined as

\[
S := \{ v \in V \setminus \{x, y\} : \exists x \rightarrow v \rightarrow y \text{-path in } D \}.
\]

By construction, \( x \notin \{s, t\} \) and by the assumptions of the Theorem there exists a path \( P' \in \mathcal{P}_{s,t}(D) \) which does not traverse \( x \). Let \( s' \) be the last node on \( P|_{(s,x)} \)

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that is traversed by $P'$. Analogously, let $t'$ be the first node of $V(P|_{(s,t)}) \cup S$ that is traversed by $P'$. Note that $t' \neq x$ since $t'$ is traversed by $P'$ but $x$ is not.

We now distinguish two cases for which we show that $V$ is not $\Lambda$-closed yielding a contradiction to Corollary 2.4:

![Figure 6: Construction for Case 1 in the Proof of Theorem 5.1.](image)

**Case 1: $t' \in S$.**

By definition of $S$ there must be an $x$-$t'$-$y$-path $W$. Let $(z, t') \in W$ be the arc of $W$ entering $t'$. By definition of $y$, we conclude that $(z, t') \notin B$. Hence, there is an alternative $x$-$y$-path $W' \neq W$ which does not use $(z, t')$. We choose $W'$ such that it uses as many arcs of $W|_{(t',y)}$ as possible. Construct the following three paths (see Figure 6):

$P_1 := P|_{(s,x)} \cup W \cup P|_{(y,t)}$
$P_2 := P|_{(s,x)} \cup W' \cup P|_{(y,t)}$
$P_3 := P|_{(s,s')} \cup P'|_{(s',t')} \cup W|_{(t',y)} \cup P|_{(y,t)}$

By construction $P_1, P_2 \in V$ but $P_3 \notin V$. $P_1$ and $P_3$ are adjacent in $P_{s-t}$-$flow(D)$ since they only differ in the disjoint paths from $s'$ to $t'$. Analogously, $P_2$ and $P_3$ are adjacent and thus, contradicting the fact that $V$ is $\Lambda$-closed.

![Figure 7: Construction for Case 2 in the Proof of Theorem 5.1.](image)
Case 2: $t' \notin S$.

Let $W := P((s,x))$ and let $W'$ be a different $x$-$y$-path which must exist by definition of $y$. Construct the following three paths (see Figure 7):

$$
P_1 := P = P((s,x)) \cup W \cup P((y,t))
$$
$$
P_2 := P((s,x)) \cup W' \cup P((y,t))
$$
$$
P_3 := P((s,s')) \cup P'((s',t')) \cup P((t,t'))
$$

By construction $P_1, P_2 \in \mathcal{V}$ but $P_3 \notin \mathcal{V}$ since it does not use $(x',x) \in B$. $P_1$ and $P_3$ as well as $P_2$ and $P_3$ are adjacent in $P_{s,t}$-flow ($D$) since they only differ in the disjoint paths from $s'$ to $t'$. Again, this contradicts the fact that $\mathcal{V}$ is $\Lambda$-closed.

6 Perfect Matching Polytope

The matching polytope and the perfect matching polytope of a graph $G = (V,E)$ are defined as

$$
P_{\text{match}}(G) := \text{conv} \{ \chi(M) : M \text{ matching in } G \}
$$
$$
P_{\text{match}}^{\text{perf}}(G) := \text{conv} \{ \chi(M) : M \text{ perfect matching in } G \}
$$

where $\chi(M) \in \{0,1\}^E$ is the characteristic vector of the set $M \subseteq E$, i.e., $\chi(M)_e = 1$ if and only if $e \in M$. We mainly consider the (perfect) matching polytope of the complete graph with $2n$ nodes $P_{\text{match}}^{\text{perf}}(K_{2n})$. Our main theorem here reads as follows:

**Theorem 6.1.** The simple extension complexity of the perfect matching polytope of $K_{2n}$ is equal to its number of vertices $\frac{(2n)!}{n!2^n}$.

We first give the high-level proof which uses a structural result presented afterwards.

**Proof.** The proof is based on Theorem 6.6. It states that for any three perfect matchings $M_1$, $M_2$, $M_3$ in $K_{2n}$, where $M_1$ and $M_2$ are adjacent (i.e., the corresponding vertices are adjacent), $M_3$ is adjacent to both $M_1$ and $M_2$ or there exists a fourth matching $M'$ adjacent to all three matchings.

Let $P = P_{\text{match}}^{\text{perf}}(K_{2n})$ and suppose that $\mathcal{V}$ is a proper $\Lambda$-closed set with $|\mathcal{V}| \geq 2$. Since the polytope’s graph is connected there exists a matching $M_1 \notin \mathcal{V}$ adjacent to some matching $M_2 \in \mathcal{V}$. Let $M_3 \in \mathcal{V} \setminus \{M_2\}$. As $\mathcal{V}$ is $\Lambda$-closed and $M_3 \notin \mathcal{V}$, $\{M_1, M_2, M_3\}$ cannot be a triangle. Hence, by Theorem 6.6 mentioned above, there exists a common neighbor matching $M'$. Since $M'$ is adjacent to $M_2$ and $M_3$, we conclude $M' \in \mathcal{V}$. But now $M_1 \notin \mathcal{V}$ is adjacent to the two matchings $M_2$ and $M'$ from $\mathcal{V}$ contradicting the fact that $\mathcal{V}$ is $\Lambda$-closed.

Hence all proper $\Lambda$-closed sets are singletons which implies the claim due to Corollary 2.6.

Since $P_{\text{match}}^{\text{perf}}(K_{2n})$ is a face of $P_{\text{match}}(K_{2n})$ and simple extensions of polytopes induce simple extensions of their faces we obtain the following corollary for the latter polytope.

**Corollary 6.2.** The simple extension complexity of the matching polytope of $K_{2n}$ is at least $\frac{(2n)!}{n!2^n}$.
6.1 Adjacency Result for the Perfect Matching Polytope

We now turn to the mentioned result on the adjacency structure of the perfect matching polytope of $K_{2n}$. It is a generalization of the diameter result of Padberg and Rao in [11].

Clearly, the symmetric difference $M \Delta M'$ of two perfect matchings is always a disjoint union of alternating cycles. Chvátal [2] showed that (the vertices corresponding to) two perfect matchings $M$ and $M'$ are adjacent if and only if $M \Delta M'$ forms a single alternating cycle. We start with an easy construction and modify the resulting matching later.

![Figure 8: Lemma 6.3 for a 10-cycle and a 12-cycle.](image)

**Lemma 6.3.** For any adjacent perfect matchings $M_1$, $M_2$ there exists a perfect matching $M'$ adjacent to $M_1$ and $M_2$ that satisfies

$$|M_1 \Delta M_2| = |M_1 \Delta M'| = |M_2 \Delta M'|.$$

**Proof.** Let $v_0, v_1, \ldots, v_{2l-1}, v_{2l} = v_0$ be the set of ordered nodes of the cycle $M_1 \Delta M_2$. If $l$ is odd, $M' := \{\{v_i, v_{i+3}\} : i = 0, 2, 4, 6, \ldots, 2l - 2\}$ induces $M_i \cdot M'$-cycles visiting the nodes in the following order:

- $M_1 \Delta M'$: $v_0, v_3, v_2, v_5, v_4, v_7, v_6, \ldots, v_{2l-1}, v_{2l-2}, v_1, v_0$
- $M_2 \Delta M'$: $v_0, v_3, v_4, v_7, v_8, \ldots, v_{2l-3}, v_{2l-2}, v_{2l-4}, v_{2l-1}, v_0$

If $l$ is even (identifying $v_{2l+1} = v_1$),

$$M' := \{\{v_i, v_{i+3}\} : i = 4, 6, \ldots, 2l - 2\} \cup \{\{v_0, v_2\}, \{v_3, v_5\}\}$$

induces $M_i \cdot M'$-cycles visiting the nodes in the following order:

- $M_1 \Delta M'$: $v_0, v_2, v_3, v_5, v_4, v_7, v_6, \ldots, v_{2l-1}, v_{2l-2}, v_1, v_0$
- $M_2 \Delta M'$: $v_0, v_2, v_1, v_{2l-2}, v_{2l-3}, \ldots, v_6, v_5, v_3, v_4, v_7, v_8, \ldots, v_{2l-4}, v_{2l-1}, v_0$
Figure 8 shows examples for both cases. It is easy to see that the cycles have the correct size. In order to produce a perfect matching on all nodes we simply add $M_1 \cap M_2$ to $M'$.

![Figure 9: Lemma 6.4 with 3 outer cycles.](image)

Suppose there is a third perfect matching $M_3$ and we want to make $M'$ adjacent to this matching as well. The idea is to iteratively connect $M_i$-$M'$-cycles until only one cycle is left. The next lemma states that this is easy if at only one of the cycles in question uses any of the nodes $v_0, v_1, \ldots, v_{2l-1}$.

**Lemma 6.4.** Let $M_1$ and $M_2$ be two adjacent perfect matchings and $M_3$ a third perfect matching. Define $V^*$ to be the node set of $M_1 \Delta M_2$. Then there exists a perfect matching $M'$ adjacent to $M_1$ and $M_2$ which satisfies:

(i) $|M_1 \Delta M'| = |M_2 \Delta M'| \geq |M_1 \Delta M_2|

(ii) Every cycle $C \subseteq M_3 \Delta M'$ has $V(C) \cap V^* \neq \emptyset$

**Proof.** Let $\overline{M}$ be the perfect matching adjacent to $M_1$ and $M_2$ constructed in Lemma 6.3. We now enlarge the $M_i$-$\overline{M}$-cycles ($i = 1, 2$) in order to remove $M_3$-$\overline{M}$-cycles outside of $V^*$.

Let $\{u_0, v_0\}$ be an $\overline{M}$-edge with $u_0, v_0 \in V^*$. Let $C_1, C_2, \ldots, C_s$ be all $M_3$-$\overline{M}$-cycles with $V(C_i) \cap V^* = \emptyset$ and let, for $i = 1, 2, \ldots, s$, $\{u_i, v_i\} \in C_i \cap \overline{M}$ be any $\overline{M}$-edge of $C_i$. Define $M'$ to be

$$M' := (\overline{M} \setminus \{(u_i, v_i) : i = 0, 1, \ldots, s\}) \cup \{(u_i, v_{i+1}) : i = 0, 1, \ldots, s\} \quad \text{(4)}$$

where $v_{s+1} = v_0$ (see Figure 9).

We now verify that $M'$ is adjacent to $M_i$ ($i = 1, 2$). Since the cycles $C_1, \ldots, C_s$ do not touch $V^*$, $\overline{M}$ and $M_i$ coincide outside $V^*$. Hence, the modification
replaces the $M$-edge $\{u_0, v_0\}$ by an alternating $M$-path from $u_0$ to $v_0$ which visits exactly 2 nodes of each $C_i$. This also proves Property (i) as it was satisfied by $M$ already.

Since $M'$ coincides with $M_2$ outside $V^* \cup V(C_1) \cup \ldots \cup V(C_s)$, and since the alternating $M_i$-$M'$-cycle containing $u_0$ and $v_0$ contains the paths $C_i \cap \{u_0, v_0\}$ for all $i \in [s]$, Property (ii) is satisfied as well. 

Remark 6.5. Equality in Property (i) of Lemma 6.4 holds if and only if $s = 0$.

We are now ready to establish our main combinatorial result on matchings, a generalization of the diameter result of Padberg and Rao [11]. For this we continue connecting $M_i$-$M'$-cycles although this requires more work than in Lemma 6.4 since the cycles touch $V^*$ now.

Theorem 6.6. Let $M_1$ and $M_2$ be two adjacent perfect matchings and $M_3$ a third perfect matching. Then the three matchings are pairwise adjacent or there exists a perfect matching $M'$ adjacent to all three.

Proof. We start with the perfect matching $M'$ constructed in Lemma 6.4. Let $V^*$ be the node set of $M_1 \Delta M_2$. We first remove the nodes matched by $M_1 \cap M_2 \cap M'$ since $M'$ coincides with all three other matchings and this will never change. Furthermore, we will consider edges in $M' \cap M_3$ as degenerate $M'$-$M_3$-cycles (of length 2). Note that by the construction in Lemma 6.4, the only degenerate $M'$-$M_3$-cycles are contained in $V^*$.

We denote by $c(M_3, M')$ the number of (maybe degenerate) $M_3$-$M'$-cycles, i.e., the number of connected components of $M_3 \Delta M'$ plus $|M_3 \cap M'|$. Note that all such cycles touch $V^*$ by Lemma 6.4 and since we removed matched by $M_1 \cap M_2 \cap M'$.

We fix some arbitrary node $\bar{v} \in V^*$. Since $M' \cap (M_1 \Delta M_2) = \emptyset$ initially, the following invariant trivially holds, where $\bar{C}$ is the component in $M_3 \Delta M'$ containing node $\bar{v}$:

$$M' \cap (M_1 \Delta M_2) \subseteq \bar{C}. \quad (5)$$

The following algorithm establishes the existence of matching $M'$:

1. Let $\bar{C}$ be the $M_3$-$M'$-cycle going through $\bar{v}$, which may be degenerate.
2. Let $C'$ be another $M_3$-$M'$-cycle (maybe degenerate) connected to $\bar{C}$ by some edge $e \in M_j \setminus M_k$ (for some $j = 1, 2$ and $k = 3 - j$).
3. If $|M_i \Delta M'| = 4$, first exchange $j$ and $k$, and then choose some new edge $e \in M_j \setminus M_k$.
4. Let this edge be $e = \{u_1, u_2\}$ with $u_1 \in V(\bar{C})$ and $u_2 \in V(C')$. Let $f_1 = \{u_1, v_1\}, f_2 = \{u_2, v_2\} \in M'$ be the edges matching $u_1$ and $u_2$.
5. If $f_1 \notin M_k$ and $u_2$ comes before $v_2$ on a walk on $M_k \Delta M'$ starting in $u_1$ with edge $f_1$, go to Step 6, and otherwise go to Step 7.
6. Let $M' := (M' \setminus \{f_1, f_2\}) \cup \{\{u_1, u_2\}, \{v_1, v_2\}\}$ and go to Step 8.
7. Let $M' := (M' \setminus \{f_1, f_2\}) \cup \{\{u_1, v_2\}, \{u_2, v_1\}\}$ and go to Step 8.
8. If $c(M_3, M') \geq 2$, go to Step 1.
When the procedure terminates, $M_3 \Delta M'$ is one cycle, i.e., $M_3$ and $M'$ are adjacent. It remains to prove that it is well-defined (existence of certain objects used in Steps 2, 3, 5) and that $M'$ is always adjacent to $M_1$ and $M_2$ unless $M'$ becomes equal to $M_1$ or $M_2$ (in which case, $M_1$, $M_2$ and $M_3$ must be pairwise adjacent).

Step 2 is well-defined since $M_j \Delta M_k = M_1 \Delta M_2$ is a single cycle visiting all nodes in $V'$. Since all $M_3$-$M'$-cycles contain nodes in $V'$, such an edge exists. The existence of the alternative edge $e$ in Step 3 is proved later.
Because \( e \in M_1 \setminus M_2 \) and \( e \notin V(\hat{C}) \), by Invariant (5) edge \( e \) is not contained in \( M' \), in particular we have \( f_1, f_2 \neq e \) and \( u_1, u_2, v_1, v_2 \) are pairwise distinct nodes. Again by Invariant (5), \( f_2 \notin M_k \), because \( f_2 \notin M_j \) (\( e \in M_j \) and \( e \cap f_2 \neq \emptyset \)) and \( f_2 \notin V(\hat{C}) \).

If \( f_1 \notin M_k \), we must prove that the walk in Step 5 exists. Since \( f_1, f_2 \notin M_k \), \( f_1, f_2 \in M_k \Delta M' \). We can assume that so far \( M' \) and \( M_k \) are adjacent, that is, \( M_k \Delta M' \) is a single cycle. Hence, the walk will visit the nodes \( u_2 \) and \( v_2 \).

![Figure 12: Case with \(|M_1 \Delta M_2| = 4\).](image)

We now see how the modified matching \( M' \) relates to the other matchings. In Steps 6 and 7, the new matching \( M' \) connects two \( M_j-M' \)-cycles \( \hat{C} \) and \( C' \), decreasing \( c(M_j, M') \) by 1. For the cycle \( M_k \Delta M' \), only the direction of the walk of Step 5 is reversed. Hence, \( M_k \) and \( M' \) are still adjacent and \(|M_k \Delta M'| \) remains unchanged.

In Step 6 (see Figure 10), the new \( M_j-M' \)-cycle visits the edge \( \{v_1, v_2\} \) instead of the path \( v_1-u_1-u_2-v_2 \). \( M_j \) and \( M' \) are still adjacent or they are equal. In the former case, \(|M_j \Delta M'| \) was decreased by 2. This is the only modification where an edge of \( M_j \Delta M_2 \) can enter \( M' \). But since this edge is in \( \hat{C} \) afterwards, Invariant 5 remains true.

In Step 7 (see Figure 11), the new \( M_j-M' \)-cycle did not change. In fact, only the direction of edge \( e \) is reversed. Hence, \( M_j \) and \( M' \) are still adjacent and \(|M_j \Delta M'| \) remains unchanged.

To summarize, the steps decrease \( c(M_5, M') \) by 1 in every iteration and leave the adjacency of \( M_i \) and \( M' \) intact, except that \( M' \) can become equal to \( M_j \) if \(|M' \Delta M_j| \) was equal to 4 before.

Step 3 tries to avoid making \( M' \) equal to \( M_j \) which may happen if \(|M_j \Delta M'| = 4 \). In this case, since \( c(M_3, M') \leq \frac{1}{2}|V'| \) was satisfied at the beginning and \( c(M_3, M') \) was decreased by 1 in every iteration (and we are not done, i.e., \( c(M_3, M') \geq 2 \)), the number of iterations is at most \( \frac{1}{2}|V'| - 2 \). In every iteration, \(|M_j \Delta M'| \) is decreased by at most 2 and at the beginning it was at least \(|V'| \). Hence, if \(|M_j \Delta M'| = 4 \), there must have been exactly \( \frac{1}{2}|V'| - 2 \) iterations and in
each of them $j$ was the same and Step 6 was chosen. In particular, $|M_k \Delta M'|$ was never decreased, that is, $V(M_k \Delta M') \supseteq V^*$. Hence, there is an edge $e \in M_k$ connecting the only two remaining cycles $\bar{C}$ and $C'$. This establishes that Step 3 is well-defined.

Step 3 tries to avoid the situation in which $|M_j \Delta M'| = 4$ and Step 6 is applied. But if also $|M_k \Delta M'| = 4$ holds, Step 6 would make $M'$ equal to $M_k$ (since $j$ and $k$ are exchanged before this step). According to the last paragraph, this can only happen if $|V^*| = 4$ holds and $|M_k \Delta M'|$ was equal to $|V^*|$ at the beginning. The last statement implies $s = 0$ by Remark 6.5. It is now easy to verify that then $M_1$, $M_2$, and $M_3$ must be pairwise adjacent (see Figure 12).

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