Boundary deconfined quantum criticality at transitions between symmetry-protected topological chains

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Decades of research have revealed a deep understanding of topological quantum matter with protected edge modes. We report that even richer physics emerges when tuning between two topological phases of matter whose respective edge modes are incompatible. The frustration at the edge leads to novel boundary physics, such as symmetry-breaking phases with exotic non-Landau transitions—even when the edge is zero-dimensional. As a minimal case study we consider spin chains with $Z_3 \times Z_3$ symmetry, exhibiting two nontrivial symmetry-protected topological (SPT) phases. At the bulk 1+1D critical transition between these SPT phases, we find two stable 0+1D boundary phases, each spontaneously breaking one of the $Z_3$ symmetries. Furthermore, we find that a single boundary parameter tunes a non-Landau boundary critical transition between these two phases. This constitutes a 0+1D version of an exotic phenomenon driven by charged vortex condensation known as deconfined quantum criticality. This work highlights the rich unexplored physics of criticality between nontrivial topological phases and provides insights into the burgeoning field of gapless topological phases.

Symmetry-protected edge modes in phases of matter are well-understood when there is a finite energy gap to creating excitations in the bulk [1–3]. For instance, in 1D systems [4–8] this leads to topologically protected ground state degeneracies which are exponentially localized near the endpoints [9, 10]. However, edge modes at phase transitions and criticality [11–42] remain a fertile area of study. Although such edge modes delocalize and disappear at phase transitions to the trivial phase [Fig. 1(a)] [43–49], it has been realized that transitions to other phases—such as spontaneous symmetry-breaking phases [Fig. 1(b)] [28, 37, 50–53]—can leave part of the edge mode intact.

This raises a question which is fundamental for understanding the interplay between topology and criticality: What is the fate of edge modes when the system transitions from one topological phase to another nontrivial topological phase? To what extent do edge modes survive at the critical point? While previous work has studied this question in the non-interacting fermion case [33, 37, 54], here we explore a more generic and richer framework. Importantly, in our work, the edge modes of one phase are incompatible with those of the other phase due to differences in how they realize the symmetry action. The resulting frustration gives rise to fascinating boundary effects when both types of edge modes are forced to coexist and compete at criticality.

As a minimal example of this scenario, we study a transition between two $Z_3 \times Z_3$-symmetric spin chain Hamiltonians [55], each phase hosting protected edge modes transforming under distinct projective representations [4]. These two gapped phases are simple examples of the more general phenomenon of symmetry-protected topological (SPT) phases. We find that edge degeneracy typically persists at the critical point in two possible ways; more precisely, there are two conformal boundary conditions each spontaneously breaking one of the two $Z_3$ symmetries [Fig. 1(c)]. These can be thought of as spontaneous symmetry-breaking phases in zero spatial dimensions. Moreover, we find a direct continuous boundary transition between these two [Fig. 1(d)], where one sym-

FIG. 1. SPT edge modes, criticality, and boundary DQCP. Panels highlight what happens to SPT edge modes when tuning to quantum criticality. Based on end-to-end long-range order of boundary order parameters: (a) tuning to trivial phase destroys edge modes and (b) edge modes can persist upon tuning toward a nontrivial phase, like a symmetry-breaking phase [28, 37, 53]. (c) In this work, we show a richer phenomenology at a transition between two distinct SPT phases protected by $Z_3 \times Z_3$ symmetry [Eq. (5)]; there are distinct symmetry-breaking boundary conditions at criticality. (d) Moreover, there is a direct continuous transition (“DQCP”) between these two by tuning a boundary parameter.
metry breaks exactly when another is restored. This is a stark violation of the conventional Landau paradigm of phase transitions which posits that symmetry subgroups only break one at a time. In fact, it is a 0+1D manifestation of a deconfined quantum critical point (DQCP), an exotic phenomenon originally proposed for 2+1D [56–76] and recently explored in 1+1D [77–84]. Indeed, we discuss how even the mechanism is quite similar to that in higher dimensions, namely, condensing defects for one symmetry-breaking order gives rise to long-range order for the other [58].

Moreover, we show that the bulk critical point itself has a nontrivial topological invariant—making it an instance of gapless SPT or symmetry-enriched criticality [37]. The conventional lore for topologically nontrivial SPT phases, shown rigorously in the gapped case, is that edge modes are guaranteed by a bulk-boundary correspondence. However, at the boundary critical point, we report here, edge modes disappear. This shows that the notion of bulk-boundary correspondence is more subtle for gapless SPT phases, opening up exciting future research directions.

\[ Z_3 \times Z_3 \text{ cluster SPT chains.} \]

Define shift and clock matrices:

\[
X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\]

where \( \omega = e^{2\pi i/3} \). We consider quantum chains respecting the \( Z_3 \times Z_3 \) symmetry generated on even and odd sublattices:

\[
U^e = \prod_j X_{2j} \quad \text{and} \quad U^o = \prod_j X_{2j+1}.
\]

Following Ref. [55], we define “cluster Hamiltonians” [51] for two distinct non-trivial \( Z_3 \times Z_3 \) SPT phases

\[
H_\omega = -\sum_j (Z_{2j-1}X_{2j}Z_{2j+1}^\dagger + Z_{2j}X_{2j+1}^\dagger + X_{2j+2}^\dagger + \text{h.c.})
\]

\[
H_{\bar{\omega}} = -\sum_j (Z_{2j-1}X_{2j}^\dagger Z_{2j+1} + Z_{2j}X_{2j+1}Z_{2j+2}^\dagger + \text{h.c.}).
\]

The effective low-energy action of \( U^e \) and \( U^o \) on each boundary is such that they commute only up to a projective phase \( \omega \) or \( \bar{\omega} \), leading to a threefold degenerate ground state space (per edge) [55].

The edge projective symmetry action is detectable via bulk string order parameters. That is, among operators of the form \( \cdots X_{2j-6}X_{2j-4}X_{2j-2}O_{2j} \) (a \( Z_3 \)-string operator), only those with \( Z_3 \)-charged \( O_{2j} \) have long-range order (LRO) [85], and vice versa for the other symmetry. For instance, \( H_\omega \) has LRO in an \( \omega \)-charged \( Z_3 \)-string operator

\[
\lim_{|k-j| \to \infty} \langle Z_{2j-1}X_{2j}X_{2j+1} \cdots X_{2k}Z_{2k+1}^\dagger \rangle = 1,
\]

while \( H_{\bar{\omega}} \) has an \( \bar{\omega} \)-charged \( Z_3 \)-string operator. While the left hand side of Eq. (4) is unity only for the fixed-point Hamiltonian \( H_\omega \), it remains nonzero throughout the SPT phase [85].

**Numerical method.**—We confirm our CFT analysis using density matrix renormalization group (DMRG) simulations [86, 87] on finite chains of lengths \( 25 \lesssim L \lesssim 125 \). At each length, we considered the limit of bond dimension \( \chi \to \infty \), with simulations run up to \( \chi = 170 \) found to sufficiently guarantee convergence for ground state end-to-end correlators and excited state energy levels. For technical efficiency, instead of implementing the cluster Hamiltonian directly, we simulated a unitarily-equivalent three-state Potts chain as described in Ref. [88].

**Criticality and boundary symmetry breaking.**—We study a linear interpolation between the two non-trivial cluster Hamiltonians (3):

\[
H(s, b) = (1 + s)H_\omega + (1 - s)H_{\bar{\omega}} - b(X_1 + X_{2N+1} + \text{h.c.}).
\]

Since we are interested in edge behavior, we have open boundary conditions with \( j \in [1, 2N + 1] \) and boundary tuning parameter \( b \) to explore generic boundary behavior.

This model exhibits a direct transition at the midpoint \( s = 0 \). In fact, a local unitary (the SPT entangler [89]) maps \( H_1 \mapsto H_\omega \mapsto H_{\bar{\omega}} \mapsto H_1 \), where \( H_1 = -\sum_j X_j + \text{h.c.} \) is a trivial phase. So the bulk critical point can be mapped to one between the trivial and SPT phase, \( H_1 + H_{\bar{\omega}} \), which in Ref. [44] was found to be be described by a certain orbifold of two copies of the three-state Potts conformal field theory (Potts\( ^2 \) CFT). However, these entangler transformations do not apply for open boundary conditions, and we will find \( H_\omega + H_{\bar{\omega}} \) has much richer boundary criticality than \( H_1 \) or \( H_{\bar{\omega}} \); we will also discuss how a bulk symmetry-protected topological invariant detects this difference. We note that this \( s = 0 \) critical point belongs to a one-parameter family of theories stabilized by \( Z_3 \times Z_3 \), translation, and charge conjugation (see Refs. [83, 88, 90, 91] therein for details on adjacent bulk phases).

Unlike in gapped SPT phases, string operators [Eq. (4)] no longer have LRO at criticality. Instead they decay algebraically with universal exponents distinguishing \( H_\omega + H_{\bar{\omega}} \) and \( H_1 \). For example, considering charges of the ‘lightest’ string operators, i.e., those with the smallest such exponents, \( H_\omega + H_{\bar{\omega}} \) has two degenerate \( U^e \)-string operators with \( U^o \) charges \( \{\omega, \bar{\omega}\} \) [e.g., the lattice string operator Eq. (4)], while \( H_1 + H_\omega \) has charges \( \{1, \omega\} \); these correspond to string operators with LRO in the nearby symmetric phases. This bulk topological invariant proves that these two CFTs cannot be
connected by a $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric path without passing through a multi-critical point or tuning off criticality.

In gapped SPTs, LRO of charged strings (4) directly imply edge modes. Analogously, we might expect a similar bulk-boundary correspondence can distinguish the “trivial” transition $H_1 + H_o$ from the “topological” one $H_o + H_\omega$. To explore this, we turn to a more concrete analysis of Eq. (5), using analytic and numerical methods inspired by Ref. 37.

In the fine-tuned case $b = 0$, zero mode operators $Z_1$ and $Z_{2N+1}$ commute with $H$ [Eq. (5)]. Their $\mathbb{Z}_2$ charge implies a threefold degenerate spectrum. Morally, this implies a threefold degenerate spectrum. Morally, $Z_1$ and $Z_{2N+1}$ are order parameters for a spontaneous symmetry-breaking (SSB) boundary and are indeed LRO in time. They are also phase-locked across the critical bulk: $\langle Z_1 \rangle = \langle Z_{2N+1} \rangle$ in all three ground states (i.e., unlike in gapped SPTs, degeneracies are not independent for both edges). We call this boundary phase “o-SSB.” Note that bulk gaplessness is what ensures a well-defined boundary SSB; in contrast, in gapped SPT phases end-to-end LRO requires a certain basis of degenerate ground states. Indeed, gapped SPT edge modes, being genuine zero-dimensional systems, have no robust notion of “phase of matter.”

Adding nonzero $b$ splits degeneracy for finite systems, similar to the exponentially small finite-size splitting of gapped SPTs [10, 92]. At criticality ($s = 0$), edge modes split algebraically $\sim 1/N^\alpha$ with boundary-condition-dependent exponent $\alpha$. Crucially $\alpha > 1$, such that degeneracy is relative to bulk finite-size splitting $\sim 1/N$. We numerically confirm this faster-than-$1/N$ splitting, and hence boundary stability, in Fig. 2(a). Later we derive a mapping to the Potts model, implying the universal exponent $\alpha = 5/3$.

The other easy limit, $b \to \infty$, projects $X_1 = X_{2N+1} = 1$, i.e. throws out sites 1 and $2N + 1$ and operators acting on them. One has the same model as when $b = 0$, but with $j \in [2, 2N]$. The story above repeats, except now $U^c$ is spontaneously broken at the boundary, not $U^o$. This is a distinct boundary phase (“e-SSB”) from $b = 0$, raising the question of what boundary transition occurs as we tune the boundary coupling.

We note this perturbatively stable boundary symmetry breaking requires the exotic topological nature of the model $H_o + H_\omega$. For comparison, the trivial gapless theory $H_1 + H_o$ has a generically nondegenerate conformal spectrum and no boundary symmetry breaking except at some unstable fine-tuned boundary points. The difference lies in the so-called boundary disorder operators, i.e., operators toggling between superselection sectors of 0+1D SSB ground states. For $H_1 + H_o$, perturbing a fine-tuned degenerate edge with infinitesimal $X_1$ will disorder the 0+1D SSB and flow to a unique symmetry-preserving edge. In contrast, $H_o + H_\omega$ has no RG-relevant symmetry-allowed boundary disorder operator with which we can perturb the edge. This intuitively matches the bulk topological invariants and also follows from CFT (explained in Refs [88, 93]). The relevant perturbation, shown in Table I, carries nontrivial charge under the unbroken symmetry and thus cannot be generated under RG!

Finally we remark that the boundary order parameters and disorder operators for the gapless regime match localized projective symmetry generators from the adjacent gapped SPT phases. Although the gapped SPT phase is agnostic with respect to automorphisms of $\mathbb{Z}_3 \times \mathbb{Z}_3$, the gapless theory selects a specific choice of projective symmetry generators to play the role of boundary order parameter or disorder operator. This physically corresponds to the fact that the boundary of the gapless phase has genuine 0+1D SSB, in contrast to the edge of a gapped SPT phase.

**DQCP in zero dimensions.**—To recap, for $b = 0$, $H_o + H_\omega$ with open boundaries spontaneously breaks the odd-sublattice $\mathbb{Z}_3$ symmetry, while for $b \to \infty$ it breaks the even one. It turns out these two phases persist for all $b$, except at $b = 1$, where there is a direct transition. This boundary transition is continuous, and both symmetries are unbroken there. Indeed, for $b = 1$, we find no ground state degeneracy (Fig. 2), contrary to a naive expectation from the bulk topological invariant.

Tuning left and right boundary couplings simultaneously (see Eq. (5)) lets us use order parameters’ end-to-end correlations to detect the transition, which occurs independently on both edges. In particular, $\langle Z_1 Z_{2N+1}^\dagger \rangle$ is nonzero in the $\omega$-SSB boundary phase ($0 \leq b < 1$) and zero in the $\epsilon$-SSB boundary phase ($b > 1$), and vice versa.

![FIG. 2. Edge modes and boundary dissolution at SPT criticality.](image-url)
TABLE I. The most relevant disorder operators of the odd symmetry-breaking boundary \((b < 1)\) are order parameters of the even symmetry-breaking boundary \((b > 1)\) and vice versa. Here we show left boundary lattice expressions\(^2\). Restoring one symmetry requires condensing said disorder operator, thereby spontaneously breaking the other symmetry; this is the mechanism leading to the \(0+1\)D ‘non-Landau’ DQCP.

| Order operator | Disorder operator |
|----------------|-------------------|
| \(\sigma\)-SSB  | \(Z_1\) \(X_1Z_2^{(1)}\) | \(\sigma\)-SSB  | \(Z_1\) \((\text{or } X_2Z_3^{(1)})\) |

\(\alpha\) and \(\beta\) are the anomalous symmetries of the Potts chain. The square root gives the boundary vacuum expectation value \((\text{vev})\). Using DMRG, we obtain Fig. 1(d) and clearly see the direct continuous transition at \(b = 1\). Later, we analytically show both vevs vanish at \(b = 1\) with unbroken symmetry and no ground state degeneracy.

This continuous SSB-to-SSB transition resembles deconfined quantum criticality points (DQCP) in higher dimensions. A key feature of DQCP is that the “vortex” in one ordered phase is charged under the symmetry broken in the other. Thus they cannot simultaneously condense, leading to a Landau-forbidden transition. The same mechanism prevails here, with the role of vortices played by relevant boundary disorder operators of Table I. Another salient DQCP feature is an emergent symmetry exchanging nearby SSB phases, which we will show indeed occurs at \(b = 1\). Despite these similarities, an anomalous symmetry is arguably missing. Indeed, a \textit{bona fide} zero-dimensional anomaly is usually understood to be a projective representation; since this implies degeneracy, it cannot be present at \(b = 1\). Thus, following Ref. \(83\), we use the term DQCP in a slightly broader context, namely, a non-Landau transition between distinct SSB phases stabilized by condensing charged defect operators.

We numerically verify symmetry restoration and nondegeneracy at \(b = 1\) by computing a finite chain’s spectrum, Fig. 2(b). Remarkably, this spectrum coincides with the known analytic result for a single Potts chain with periodic\(\) (twisted) boundary conditions \(94\). This is no coincidence, as we now demonstrate.

\textbf{Mapping to single Potts chain.}—Remarkably, the open chain in Eq. (5) is unitarily equivalent to a single three-state Potts chain on a ring with some defects depending on \(b\). The mapping is summarized in Fig. 3 (further details are provided in the Supplemental Material \(88\)). The Z_3 physical symmetry is the global Z_3 symmetry of the Potts ring, while the eigenvalues of the \(\text{Z}_3\) generator label the \(\text{Z}_3\) twisted boundary conditions of the Potts ring. The result is that \(b\) tunes the strength of a single exchange term on opposite sides of the ring. The DQCP at \(b = 1\) corresponds to translation symmetry, where the spectrum matches that of the Potts chain on a ring, which is nondegenerate.

With this mapping, dominant boundary operators are identified through the Potts defect conformal field theory \(95-100\) confirming our claims in Fig. 1. For example, at \(b < 1\), the dominant symmetry-allowed boundary perturbation corresponds to an irrelevant \(\psi^A\psi^{B\dagger}\) CFT operator of dimension \(4/3\) coupling the two chains’ endpoints (Fig. 3), while the \(\text{Z}_3\) charged disorder operator corresponds to \(\psi^A\) of dimension \(2/3\) leading to the \(|s|^{5/9}\) scaling of Fig. 1(c). Similary, at the DQCP, scalings of Figs. 1(c) and 1(d) arise from the 2/15 dimensional \(\text{Z}_3\) charged boundary operators \(\sigma\bar{\sigma} (\mu\bar{\mu})\) and the 4/5 dimensional the symmetric boundary perturbation \(c\).
“delocalized” QCP.

**Outlook.**—We studied a minimal example of competition between two inequivalent types of topologically-protected edge mode with $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. We found that as a result of this competition, there are effectively fewer edge modes at criticality, and they organize themselves into one of two distinct boundary-symmetry-breaking phases breaking only a three-fold subgroup. Most strikingly, there is an unconventional direct boundary transition between these two symmetry-breaking regimes. At this boundary transition, edge modes disappear and emergent features of a deconfined quantum critical point appear. These results were obtained using conformal field theory and tensor network simulations on a critical one-dimensional open-chain lattice model on an open chain with a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry.

SPT transitions and edge modes of gapless systems merit further study. Our results encourage exploring other direct transitions between nontrivial SPT phases, where, as we have exemplified, novel boundary physics is expected. Examples include boundaries of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^T$ SPT-SPT' transition in Eq.(28) of Ref. 37 and the $c = 2$ multicritical point where all three $\mathbb{Z}_3 \times \mathbb{Z}_3$ SPT phases meet. Another major open question regards bulk-boundary correspondence for gapless SPT phases. Remarkably we have found that even with a nontrivial bulk topological invariant, boundary edge modes can disappear in a boundary DQCP. It remains unknown how general this phenomenon is. Insights might also be gained by understanding boundary conditions as RG flows to critical points [101–103]. Another open question is to explore higher-dimensional analogs, such as transitions between nontrivial $2+1D$ $\mathbb{Z}_n$ SPTs.

Finally, it would be exciting to explore these phenomena in experiment. Intriguingly, $\mathbb{Z}_3 \times \mathbb{Z}_3$ SPT phases have been predicted in optical lattices of cold alkaline-earth atoms [104, 105]. While numerical simulations found a direct first-order transition between the two nontrivial SPT phases, our work suggests a broader phase diagram can have a direct continuous transition, where one would observe $0+1D$ boundary DQCP. To facilitate such experimental explorations, one can map the three-body cluster Hamiltonian to a two-body interacting system [106], similar to what has been done for the $\mathbb{Z}_2$ case [107].

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[1] Zheng-Cheng Gu and Xiao-Gang Wen, “Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order,” Phys. Rev. B 80, 155131 (2009).
[2] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen, “Symmetry protected topological orders and the group cohomology of their symmetry group,” Phys.Rev. B87, 155114 (2013).
[3] T. Senthil, “Symmetry-protected topological phases of quantum matter,” Annual Review of Condensed Matter Physics 6, 299–324 (2015).
[4] Frank Pollmann, Erez Berg, Ari M. Turner, and Masaki Oshikawa, “Entanglement spectrum of a topological phase in one dimension,” Physical Review B 81 (2010), 10.1103/PhysRevB.81.064439, arXiv: 0910.1811.
[5] Ari M. Turner, Frank Pollmann, and Erez Berg, “Topological phases of one-dimensional fermions: An entanglement point of view,” Phys. Rev. B 83, 075102 (2011).
[6] Lukasz Fidkowski and Alexei Kitaev, “Topological phases of fermions in one dimension,” Physical Review B 83 (2011), 10.1103/physrevb.83.075103.
[7] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, “Complete classification of 1d gapped quantum phases in interacting spin systems,” Physical Review B 84 (2011), 10.1103/PhysRevB.84.235128, arXiv: 1103.3323.
[8] Schuch, N. and Pérez-Garcia, D. and Cirac, J. I., “Classifying quantum phases using matrix product states and projected entangled pair states,” Phys. Rev. B 84, 165139 (2011).
[9] Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki, “Valence bond ground states in isotropic quantum antiferromagnets,” Comm. Math. Phys. 115, 477–528 (1988).
[10] T Kennedy, “Exact diagonalisations of open spin-1 chains,” Journal of Physics: Condensed Matter 2, 5737–5745 (1990).
[11] J. P. Kestner, Bin Wang, Jay D. Sau, and S. Das Sarma, “Prediction of a gapless topological Haldane liquid phase in a one-dimensional cold polar molecular lattice,” Phys. Rev. B 83, 174409 (2011).
[12] Meng Cheng and Hong-Hao Tu, “Majorana edge states in interacting two-chain ladders of fermions,” Phys. Rev. B 84, 094503 (2011).
[13] Łukasz Fidkowski, Roman M. Lutchyn, Chetan Nayak, and Matthew P. A. Fisher, “Majorana zero modes in one-dimensional quantum wires without long-ranged superconducting order,” Phys. Rev. B 84, 195436 (2011).
[14] Jay D. Sau, B. I. Halperin, K. Flensberg, and S. Das Sarma, “Number conserving theory for topologically protected degeneracy in one-dimensional fermions,” Phys. Rev. B 84, 144509 (2011).
[15] J. Ruhman, E. G. Dalla Torre, S. D. Huber, and E. Alt-
man, “Nonlocal order in elongated dipolar gases,” Phys. Rev. B 85, 125121 (2012).

[16] Tarun Grover and Ashish Vishwanath, “Quantum Criticality in Topological Insulators and Superconductors: Emergence of Strongly Coupled Majoranas and Supersymmetry,” arXiv e-prints, arXiv:1206.1332 (2012), arXiv:1206.1332 [cond-mat.str-el].

[17] Christina V. Kraus, Marcello Dalmonte, Mikhail A. Baranov, Andreas M. Läuchli, and P. Zoller, “Majorana Edge States in Atomic Wires Coupled by Pair Hopping,” Phys. Rev. Lett. 111, 173004 (2013).

[18] Gerardo Ortiz, Jorge Dukelsky, Emilio Cobanera, Carlos Esebbag, and Carlo Beenakker, “Many-Body Characterization of Particle-Conserving Topological Superfluids,” Phys. Rev. Lett. 113, 267002 (2014).

[19] Anna Keselman and Erez Berg, “Gapless symmetry-protected topological phase of fermions in one dimension,” Phys. Rev. B 91, 235309 (2015).

[20] Jonathan Ruhman, Erez Berg, and Ehud Altman, “Topological States in a One-Dimensional Fermi Gas with Attractive Interaction,” Phys. Rev. Lett. 114, 100401 (2015).

[21] Nikolaos Kainaris and Sam T. Carr, “Emergent topological properties in interacting one-dimensional systems with spin-orbit coupling,” Phys. Rev. B 92, 035139 (2015).

[22] Fernando Iemini, Leonardo Mazza, Davide Rossini, Rosario Fazio, and Sebastian Diehl, “Localized Majorana-Like Modes in a Number-Conserving Setting: An Exactly Solvable Model,” Phys. Rev. Lett. 115, 156402 (2015).

[23] Nicolai Lang and Hans Peter Büchler, “Topological states in a microscopic model of interacting fermions,” Phys. Rev. B 92, 041118(R) (2015).

[24] Gerardo Ortiz and Emilio Cobanera, “What is a particle-conserving Topological Superfluid? The fate of Majorana modes beyond mean-field theory,” Annals of Physics 372, 357 – 374 (2016).

[25] Arianna Montorsi, Fabrizio Dolcini, Rita C. Iotti, and Fausto Rossi, “Symmetry-protected topological phases of one-dimensional interacting fermions with spin-charge separation,” Phys. Rev. B 95, 245108 (2017).

[26] Zhiyuan Wang, Youjiang Xu, Han Pu, and Kaden R. A. Hazzard, “Number-conserving interacting fermion models with exact topological superconducting ground states,” Phys. Rev. B 96, 115110 (2017).

[27] Jonathan Ruhman and Ehud Altman, “Topological degeneracy and pairing in a one-dimensional gas of spinless fermions,” Phys. Rev. B 96, 085133 (2017).

[28] Thomas Scaffidi, Daniel E. Parker, and Romain Vasseur, “Gapless Symmetry-Protected Topological Order,” Phys. Rev. X 7, 041048 (2017).

[29] K. Guther, N. Lang, and H. P. Büchler, “Ising anyonic topological phase of interacting fermions in one dimension,” Phys. Rev. B 96, 121109 (2017).

[30] Nikolaos Kainaris, Raul A. Santos, D. B. Gutman, and Sam T. Carr, “Interaction induced topological protection in one-dimensional conductors,” Fortschrritte der Physik 65, 1600054 (2017).

[31] Hong-Chen Jiang, Zi-Xiang Li, Alexander Seidel, and Dung-Hai Lee, “Symmetry protected topological Luttinger liquids and the phase transition between them,” Science Bulletin 63, 753 – 758 (2018).

[32] Rui-Xing Zhang and Chao-Xing Liu, “Crystalline Symmetry-Protected Majorana Mode in Number-Conserving Dirac Semimetal Nanowires,” Phys. Rev. Lett. 120, 156802 (2018).

[33] Ruben Verresen, Nick G. Jones, and Frank Pollmann, “Topology and Edge Modes in Quantum Critical Chains,” Phys. Rev. Lett. 120, 057001 (2018).

[34] Daniel E. Parker, Thomas Scaffidi, and Romain Vasseur, “Topological Luttinger liquids from decorated domain walls,” Phys. Rev. B 97, 165114 (2018).

[35] Anna Keselman, Erez Berg, and Patrick Azaria, “From one-dimensional charge conserving superconductors to the gapless Haldane phase,” Phys. Rev. B 98, 214501 (2018).

[36] Chun Chen, Wei Yan, C. S. Ting, Yan Chen, and F. J. Burnell, “Flux-stabilized majorana zero modes in coupled one-dimensional fermi wires,” Phys. Rev. B 98, 161106 (2018).

[37] Ruben Verresen, Ryan Thorngren, Nick G. Jones, and Frank Pollmann, “Gapless topological phases and symmetry-enriched quantum criticality,” Phys. Rev. X 11, 041059 (2021).

[38] Carlos M. Duque, Hong-Ye Hu, Yi-Zhuang You, Vedika Khemani, Ruben Verresen, and Romain Vasseur, “Topological and symmetry-enriched random quantum critical points,” Phys. Rev. B 103, L100207 (2021).

[39] Oleksandr Balabanov, Daniel Erkensten, and Henrik Johannesson, “Topology of critical chiral phases: Multiband insulators and superconductors,” Phys. Rev. Research 3, 043048 (2021).

[40] Shun-Chiao Chang and Pavan Hosur, “Absence of Friedel oscillations in the entanglement entropy profile of one-dimensional intrinsically gapless topological phases,” (2022), arXiv:2201.07260 [cond-mat.str-el].

[41] Joana Fraxanet, Daniel González-Cuadrá, Tilman Pfau, Maciej Lewenstein, Tim Langen, and Luca Barbiero, “Topological quantum critical points in the extended bose-hubbard model,” Phys. Rev. Lett. 128, 043402 (2022).

[42] Oleksandr Balabanov, Carlos Ortega-Taberner, and Maria Hermanns, “Quantization of topological indices in critical chains at low temperatures,” Phys. Rev. B 106, 045116 (2022).

[43] Lokman Tsui, Hong-Chen Jiang, Yuan-Ming Lu, and Dung-Hai Lee, “Quantum phase transitions between a class of symmetry protected topological states,” Nuclear Physics B 896, 330 – 359 (2015), arXiv:1503.06794 [cond-mat.str-el].

[44] Lokman Tsui, Yen-Ta Huang, Hong-Chen Jiang, and Dung-Hai Lee, “The phase transitions between $Z_n \times Z_n$ bosonic topological phases in $1+1$D, and a constraint on the central charge for the critical points between bosonic symmetry protected topological phases,” Nuclear Physics B 919, 470 – 503 (2017).

[45] Nick Bultinck, “Uv perspective on mixed anomalies at critical points between bosonic symmetry-protected phases,” Phys. Rev. B 100, 165132 (2019).

[46] Maxime Dupont, Snir Gazit, and Thomas Scaffidi, “From trivial to topological paramagnets: The case of $Z_2$ and $Z_2^3$ symmetries in two dimensions,” Phys. Rev. B 103, 144437 (2021).

[47] Maxime Dupont, Snir Gazit, and Thomas Scaffidi, “Evidence for deconfined $u(1)$ gauge theory at the transition between toric code and double semion,” Phys. Rev. B 103, L140412 (2021).
[48] Nathanan Tantivasadakarn, Ryan Thorngren, Ashvin Vishwanath, and Ruben Verresen, “Building models of topological quantum criticality from pivot Hamiltonians,” SciPost Phys. 14, 013 (2023).

[49] Nathanan Tantivasadakarn, Ryan Thorngren, Ashvin Vishwanath, and Ruben Verresen, “Pivot Hamiltonians as generators of symmetry and entanglement,” SciPost Phys. 14, 012 (2023).

[50] Masuo Suzuki, “Relationship among Exactly Soluble Models of Critical Phenomena. I)*2D Ising Model, Dimer Problem and the Generalized XY-Model,” Progress of Theoretical Physics 46, 1337 (1971).

[51] Hans J. Briegel and Robert Raussendorf, “Persistent entanglement in arrays of interacting particles,” Phys. Rev. Lett. 86, 910–913 (2001).

[52] W. Son, L. Amico, R. Fazio, A. Hamma, S. Pascazio, and V. Vedral, “Quantum phase transition between cluster and antiferromagnetic states,” EPL (Europhysics Letters) 95, 50001 (2011).

[53] Daniel E. Parker, Romain Vasseur, and Thomas Scaffidi, “Topologically Protected Long Edge Coherence Times in Symmetry-Broken Phases,” Phys. Rev. Lett. 122, 240605 (2019).

[54] Ruben Verresen, “Topological and edge states survive quantum criticality between topological insulators,” (2020), arXiv:2003.05453 [cond-mat.str-el].

[55] Scott D. Geraedts and Olexei I. Motrunich, “Exact search of noether’s theorem at the deconfined quantum critical point,” Phys. Rev. B 98, 174421 (2018).

[56] Kun Chen, Yuan Huang, Youjin Deng, A. B. Kuklov, Adam Nahum, P. Serna, J. T. Chalker, M. Ortuño, and V. Vedral, “Quantum phase transition between an antiferromagnet and a valence-bond-solid transition,” Phys. Rev. Lett. 110, 185701 (2013).

[57] Adam Nahum, P. Serna, J. T. Chalker, M. Ortuño, and A. M. Somoza, “Emergent so(5) symmetry at the née to valence-bond-solid transition,” Phys. Rev. Lett. 115, 267203 (2015).

[58] Scott D. Geraedts and Olexei I. Motrunich, “Emergent so(5) symmetry at the columnar ordering transition in the classical cubic dimer model,” Phys. Rev. Lett. 122, 080601 (2019).

[59] Jun Takahashi and Anders W. Sandvik, “Valence-bond solids, vestigial order, and emergent so(5) symmetry in a two-dimensional quantum magnet,” Phys. Rev. Research 2, 033459 (2020).

[60] Chong Wang, Adam Nahum, Max A. Metlitski, Cenke Xu, and T. Senthil, “Deconfined quantum critical points: Symmetries and dualities,” Physical Review X 7 (2017), 10.1103/physrevx.7.031051.

[61] Rajesh Kocher, Stephen Powell, and Adam Nahum, “Emergent so(5) symmetry at the columnar ordering transition in the classical cubic dimer model,” Phys. Rev. Lett. 122, 080601 (2019).

[62] Zi-Xiang Li, Shao-Kai Jian, and Hong Yao, “Deconfined quantum criticality and emergent so(5) symmetry in fermionic systems,” (2019).

[63] Sven Ma, Yu-Zhuang You, and Zi Yang Meng, “Role of noether’s theorem at the deconfined quantum critical point,” Phys. Rev. Lett. 122, 175701 (2019).

[64] N. V. Prokof’ev, and B. V. Svistunov, “Deconfined criticality and emergent so(5) symmetry in fermionic systems,” SciPost Phys. 7, 013 (2023).

[65] Zhenjiu Wang, Michael P. Zaletel, Roger S. K. Mong, and Fakher F. Assaad, “Phases of the (2+1) dimensional Ising ferromagnet, and conserved current at a one-dimensional incarnation of noether’s theorem at the deconfined quantum critical point,” Physical Review B 100, 247202 (2019).

[66] Ashvin Vishwanath, Leon Balents, Subir Sachdev, and Matthew P. A. Fisher, “Deconfined Quantum Critical Points,” Science 303, 1490–1494 (2004), https://science.sciencemag.org/content/303/5663/1490.full.

[67] Nathanan Tantivasadakarn, Ryan Thorngren, Ashvin Vishwanath, and T. Senthil, “Quantum criticality beyond the landau-ginzburg-wilson paradigm,” Phys. Rev. B 70, 144407 (2004).

[68] Michael Levin and T. Senthil, “Deconfined quantum criticality and né el order via dimer disorder,” Physical Review B 70 (2004), 10.1103/physrevb.70.220403.

[69] Leon Balents, Lorenz Bartosch, Anton Burkov, Subir Sachdev, and Krishnendu Sengupta, “Putting competing orders in their place near the mott transition,” Phys. Rev. B 71, 144508 (2005).

[70] Ashvin Vishwanath, L. Balents, and T. Senthil, “Quantum criticality and deconfinement in phase transitions between valence bond solids,” Phys. Rev. B 69, 224416 (2004).

[71] Pouyan Ghaemi and T. Senthil, “Néel order, quantum spin liquids, and quantum criticality in two dimensions,” Phys. Rev. B 73, 054415 (2006).

[72] Anders W. Sandvik, “Evidence for deconfined quantum criticality in a two-dimensional heisenberg model with four-spin interactions,” Phys. Rev. Lett. 98, 227202 (2007).

[73] Tarun Grover and T. Senthil, “Quantum spin nematics, dimerization, and deconfined criticality in quasi-1d spin-one magnets,” Phys. Rev. Lett. 98, 247202 (2007).

[74] Roger G. Melko and Ribhu K. Kaul, “Scaling in the fan of an unconventional quantum critical point,” Phys. Rev. Lett. 100, 017203 (2008).

[75] Anders W. Sandvik, “Continuous quantum phase transition between an antiferromagnet and a valence-bond solid in two dimensions: Evidence for logarithmic corrections to scaling,” Phys. Rev. Lett. 104, 177201 (2010).

[76] Kun Chen, Yuan Huang, Youjin Deng, A. B. Kuklov, N. V. Prokof’ev, and B. V. Svistunov, “Deconfined criticality in the heisenberg model with ring-exchange interactions,” Phys. Rev. Lett. 110, 185701 (2013).

[77] Adam Nahum, P. Serna, J. T. Chalker, M. Ortuño, and A. M. Somoza, “Emergent so(5) symmetry at the née to valence-bond-solid transition,” Phys. Rev. Lett. 115, 267203 (2015).

[78] Gaoyong Sun, Bo-Bo Wei, and Su-Peng Kou, “Fidelity as a probe for a deconfined quantum critical point,”
Phys. Rev. B 100, 064427 (2019).

[82] Sibin Yang, Dao-Xin Yao, and Anders W. Sandvik, “Deconfined quantum criticality in spin-1/2 chains with long-range interactions,” (2020).

[83] Brenden Roberts, Shenghan Jiang, and Oleksi I. Motrunich, “One-dimensional model for deconfined criticality with $Z_3 \times Z_3$ symmetry,” Phys. Rev. B 103, 155143 (2021).

[84] Carolyn Zhang and Michael Levin, “Exactly solvable model for a deconfined quantum critical point in 1d,” Phys. Rev. Lett. 130, 026801 (2023).

[85] Frank Pollmann and Ari M. Turner, “Detection of symmetry-protected topological phases in one dimension,” Phys. Rev. B 86, 125441 (2012).

[86] Steven R. White, “Density matrix formulation for quantum renormalization groups,” Phys. Rev. Lett. 69, 2863–2866 (1992).

[87] Johannes Hauschild and Frank Pollmann, “Efficient numerical simulations with Tensor Networks: Tensor Network Python (TenPy),” SciPost Phys. Lect. Notes , 5 (2018).

[88] Luiz H. Santos, “Rokhsar-Kivelson models of bosonic symmetry-protected topological states,” Phys. Rev. B 91, 155150 (2015).

[89] John Cardy, “Bulk renormalization group flows and boundary states in conformal field theories,” SciPost Physics 3 (2017), 10.21468/scipostphys.3.2.011.

[90] Ruben Verresen, Roderich Moessner, and Frank Pollmann, “One-dimensional symmetry protected topological phases and their transitions,” Phys. Rev. B 96, 165124 (2017).

[91] Jonas A. Kjäll, Michael P. Zaletel, Roger S. K. Mong, Jens H. Bardarson, and Frank Pollmann, “Phase diagram of the anisotropic spin-2 XXZ model: Infinite-system density matrix renormalization group study,” Phys. Rev. B 87, 235106 (2013).
Supplemental Material

Mapping to single Potts chain

The nonlocal unitary map of Fig. 3 of the main text from the cluster to Potts model is made explicit in Table II.

| Potts | Cluster |
|-------|---------|
| $X_1 X_{1-j}$ | $X_2 < j < 2N$ |
| $Z_1^j Z_{1-j}^j$ | $Z_2 < j < 2N$ |
| $Z_{j=1} Z_{j=2}^2 Z_{j=1}$ | $X_{1-j} < j < 1 - 2N$ |
| $W \prod_{k=1}^N X_k^j X_{1-k}$ | $Z_{1-j} < j < 1 - 2N + 1$ |
| $Y^j Z_N Z_{N+1}^j$ | $X_{2N+1}$ |

TABLE II. Mapping from cluster to Potts: An exact equivalence between the cluster model on an open chain with odd-site endpoints and the Potts model. Here the cluster model is defined on all integer sites $1 \leq j \leq 2N + 1$. The Potts chain is defined on integer sites $1 \leq j \leq 2N$ with periodic boundary conditions $j \equiv j + 2N$; the Potts chain couples to an additional qutrit described by clock and shift matrices $Y$ and $W$ representing a gauged degree of freedom.

Boundary CFT and Defect CFT Analysis

Thanks to the above mapping, instead of directly identifying the boundary conditions of the gapless cluster model, we can indirectly characterize them using some well-known defect descriptions of the Potts model. The low energy description of the bond-strength defect $b$ in the Potts Hamiltonian takes three universal forms for different regimes of $b$, shown in Fig.1 and Table III. For each regime, we summarize the important CFT results.

$$\cdots - \tilde{Z}_{-2} \tilde{Z}_{-1} - \tilde{X}_{-1} - \tilde{Z}_{-1} \tilde{Z}_{0}^j - \tilde{X}_0 - b \tilde{Z}_0 \tilde{Z}_{1}^j - \tilde{X}_1 - \tilde{Z}_1 \tilde{Z}_{2}^j - \tilde{X}_2 - \tilde{Z}_2 \tilde{Z}_{3}^j - \tilde{X}_3 \cdots + \text{h.c.}$$

| $b$ | $[0, 1)$ | $1$ | $(1, \infty)$ |
|-----|---------|-----|-------------|
| RG Flow | $b \to 0$ | $b = 1$ | $b \to \infty$ |
| Potts Defect | (free, free) No defect $(A, A) + (B, B) + (C, C)$ |
| Cluster Boundary | o-SSB DQCP e-SSB |

TABLE III. Different regimes of boundary perturbation strength $b$

Boundary-SSB

The o-SSB and e-SSB conformal boundary conditions map to each other by the Potts model’s Kramers Wannier duality. Without loss of generality let us consider o-SSB. The underlying theory has three different conformal vacua which each return nonzero vev for $Z^a_0$-charged lattice order parameters. We can denote the local $\mathcal{W}_3^a$ primary boundary CFT operators as $\psi_A$ and $\psi_B$, with $Z^a_0$ charge $\omega$ and conformal dimension $2/3$, as well as all fusion products of these. The most relevant neutral boundary operator $\psi_A \psi_B$ has dimension $4/3 > 1$, explaining the perturbative stability of this boundary condition and the finite size gap scaling exponent $2(4/3) - 1 = 5/3$. Meanwhile, the relevant boundary operators $\psi_{A/B}$ are charged and play the role of boundary disorder operators.
The conformal defect that arises by changing the strength $b$ of one bond in the Potts model. (a) For $b \geq 0$ there are three different defects in the field theory. At $b = 1$ there is no defect (identity defect, shown in purple), which is unstable to local perturbation. Tuning $b$ below or above 1 triggers an RG flow to the stable defects (free, free) (shown in blue) or $\sum_{cyc} (A, A) \equiv (A, A) + (B, B) + (C, C)$ (shown in red) respectively. (b) Conformal defects for complex $b = |b|e^{i\theta}$ computed numerically (colored dots) from long range order of cluster order parameters. A topological twist defect relates $\theta$ and $\theta + 2\pi/3$.

### DQCP

The theory with the DQCP boundary on both sides has a unique conformal vacuum and has local boundary CFT operators in one-to-one correspondence with the Potts model’s local bulk operators and nonlocal $\mathbb{Z}_3$-twisted-sector operators. The most relevant neutral boundary operator $\epsilon$ has dimension $\Delta_\epsilon = 4/5$, leading to the perturbative instability of the DQCP boundary condition upon changing $b$ above or below 1. Furthermore, the $e$-SSB’s and $a$-SSB’s charged order parameters correspond to the DQCP’s charged operators $\sigma$ and $\mu$ respectively and are manifestly interchanged by the Potts Kramers-Wannier order-disorder duality. In agreement with the numerics of Fig. 1 (d) of the main text, as $b \rightarrow 1$ boundary order parameters vanish as $\propto (b - 1)^{\Delta_\epsilon/\Delta_\sigma} = (b - 1)^{2/3}$ from the $e$-SSB regime and $\propto (1 - b)^{\Delta_\epsilon/\Delta_\sigma} = (1 - b)^{2/3}$ from the $a$-SSB regime. From the gapped SPT phases as $s \rightarrow 0$, both boundary order parameters vanish as $\propto |s|^{\Delta_\epsilon/\Delta_\sigma} = |s|^{1/3}$ where $\Delta_\sigma = 4/5$ is the dimension of the bulk operator perturbing to the gapped SPT.

### Boundary State Analysis

Boundary conditions of the cluster model can be expressed in terms of boundary states living in the cluster model’s bulk Hilbert space [93]. Here we write down explicitly some of these boundary states. For convenience states are written using the notation of the Potts$^2$ model, keeping in mind the actual Hilbert space is its antidiagonal $\mathbb{Z}_3$ orbifold.

$$
|\text{o-ssb}\rangle = 3\sqrt{3}N^2 (|0\rangle - |3\rangle - \sqrt{\varphi}|2/5\rangle + \sqrt{\varphi}|7/5\rangle)_{A} (|0\rangle - |3\rangle - \sqrt{\varphi}|2/5\rangle + \sqrt{\varphi}|7/5\rangle)_{B}
$$

$$
|\text{e-ssb}\rangle = 3\sqrt{3}N^2 (|0\rangle + |3\rangle + \sqrt{\varphi}|2/5\rangle + \sqrt{\varphi}|7/5\rangle)_{A} (|0\rangle + |3\rangle + \sqrt{\varphi}|2/5\rangle + \sqrt{\varphi}|7/5\rangle)_{B}
$$

\begin{equation}
|\text{DQCP}\rangle = \sqrt{3} \sum_{h} \sum_{J} |h, J\rangle_{A} |h, J\rangle_{B}
\end{equation}

Here $\phi$ is the golden ratio and $N = \left(\frac{5 - \sqrt{5}}{30}\right)^{1/4}$. The notation $|h, J\rangle_{A/B}$ refers schematically to the $J$-th descendant of Potts bulk primary $h$, while $|\phi\rangle_{A/B}$ refers to the Potts model Ishibashi state. $|\text{DQCP}\rangle$ is a simple boundary condition, while $|\text{o/e-SSB}\rangle$ each decompose into a sum of three simple $\mathbb{Z}_3^{a/c}$-breaking boundary states. For example $|\text{e-ssb}\rangle$ is the sum of three distinct boundary states $|\text{e-a}\rangle$, where

\begin{equation}
|\text{e-a}\rangle \equiv \frac{1}{\sqrt{3}} \sum_{b=0}^{2} |\omega^{a+b}\rangle_{A} |\omega^{a-b}\rangle_{B}
\end{equation}

$$
|\omega^{a}\rangle \equiv N (|0\rangle + |3\rangle + \omega^{a}|\psi\rangle + \omega^{-a}|\psi^\dagger\rangle + \sqrt{\varphi} (|2/5\rangle + |7/5\rangle) + \omega^a|\sigma\rangle + \omega^{-a}|\sigma^\dagger\rangle))
$$
Adjacent Gapped Phases

Our gapless model $H_\omega + H_{\bar{\omega}}$ is not adjacent to a trivial gapped phase. Its nearby phases are the two gapped SPT phases, the two gapped SSB phases breaking the odd or even $\mathbb{Z}_3$ subgroups, and transitions between these. These can be identified by considering relevant bulk CFT operators [83, 90, 91]. Imposing translation and charge conjugation symmetries (in addition to $\mathbb{Z}_3 \times \mathbb{Z}_3$) stabilizes a one-parameter family of models containing $H_\omega + H_{\bar{\omega}}$, a first order transition between the gapped SSB phases, and a first order transition between the gapped SPT phases. Adding the bulk odd (even) SSB perturbation to a semi-infinite region of the infinite gapless $H_\omega + H_{\bar{\omega}}$ chain results in the o-SSB (e-SSB) boundary condition.