Monochromatic Clique Decompositions of Graphs

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Abstract: Let $G$ be a graph whose edges are colored with $k$ colors, and $\mathcal{H} = (H_1, \ldots, H_k)$ be a $k$-tuple of graphs. A monochromatic $\mathcal{H}$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a monochromatic copy of $H_i$ in color $i$, for some $1 \leq i \leq k$. Let $\phi_k(n, \mathcal{H})$ be the smallest number $\phi$, such that, for every order-$n$ graph and every $k$-edge-coloring, there is a monochromatic $\mathcal{H}$-decomposition with at most $\phi$ elements. Extending the previous results of Liu and Sousa [Monochromatic $K_r$-decompositions of graphs, J Graph Theory 76 (2014), 89–100], we solve this problem when each graph in $\mathcal{H}$ is a clique and $n \geq n_0(\mathcal{H})$ is sufficiently large. © 2015 The Authors Journal of Graph Theory Published by Wiley Periodicals, Inc. J. Graph Theory 80: 287–298, 2015

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1. INTRODUCTION

All graphs in this article are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [3].

Given two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a subgraph isomorphic to $H$. Let $\phi(G, H)$ be the smallest possible number of parts in an $H$-decomposition of $G$. It is easy to see that, if $H$ is nonempty, we have $\phi(G, H) = e(G) - \nu_H(G)(e(H) - 1)$, where $\nu_H(G)$ is the maximum number of pairwise edge-disjoint copies of $H$ that can be packed into $G$. Dor and Tarsi [4] showed that if $H$ has a component with at least three edges then it is NP-complete to determine if a graph $G$ admits a partition into copies of $H$. Thus, it is NP-hard to compute the function $\phi(G, H)$ for such $H$. Nonetheless, many exact results were proved about the extremal function

$$\phi(n, H) = \max\{\phi(G, H) \mid \nu(G) = n\},$$

which is the smallest number such that any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi(n, H)$ elements.

This function was first studied, in 1966, by Erdős et al. [6], who proved that $\phi(n, K_3) = t_2(n)$, where $K_3$ denotes the complete graph (clique) of order $3$, and $t_r(n)$ denotes the number of edges in the Turán graph $T_{r-1}(n)$, which is the unique $(r-1)$-partite graph on $n$ vertices that has the maximum number of edges. A decade later, Bollobás [2] proved that $\phi(n, K_r) = t_{r-1}(n)$, for all $n \geq r \geq 3$.

Recently, Pikhurko and Sousa [13] studied $\phi(n, H)$ for arbitrary graphs $H$. Their result is the following.

**Theorem 1.1** [13]. Let $H$ be any fixed graph of chromatic number $r \geq 3$. Then,

$$\phi(n, H) = t_{r-1}(n) + o(n^2).$$

Let $\text{ex}(n, H)$ denote the maximum number of edges in a graph on $n$ vertices not containing $H$ as a subgraph. The result of Turán [20] states that $T_{r-1}(n)$ is the unique extremal graph for $\text{ex}(n, K_r)$. The function $\text{ex}(n, H)$ is usually called the Turán function for $H$. Pikhurko and Sousa [13] also made the following conjecture.

**Conjecture 1.2** [13]. For any graph $H$ of chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$ for all $n \geq n_0$.

A graph $H$ is edge-critical if there exists an edge $e \in E(H)$ such that $\chi(H) > \chi(H - e)$, where $\chi(H)$ denotes the chromatic number of $H$. For $r \geq 4$, a clique-extension of order $r$ is a connected graph that consists of a $K_{r-1}$ plus another vertex, say $v$, adjacent to at most $r - 2$ vertices of $K_{r-1}$. Conjecture 1.2 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [18] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [17, 19]. Later, Özkahya and Person [12] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Their result is the following.

**Theorem 1.3** [12]. For any edge-critical graph $H$ with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\text{ex}(n, H)$ is the Turán graph $T_{r-1}(n)$.

Recently, as an extension of Özkahya and Person’s work (and as further evidence supporting Conjecture 1.2), Allen et al. [1] improved the error term obtained by Pikhurko
and Sousa in Theorem 1.1. In fact, they proved that the error term \( o(n^2) \) can be replaced by \( O(n^{2-a}) \) for some \( a > 0 \). Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.3 since the error term \( O(n^{2-a}) \) that they obtained vanishes for every edge-critical graph \( H \).

Motivated by the recent work about \( H \)-decompositions of graphs, a natural problem to consider is the Ramsey (or colored) version of this problem. More precisely, let \( G \) be a graph on \( n \) vertices whose edges are colored with \( k \) colors, for some \( k \geq 2 \) and let \( \mathcal{H} = (H_1, \ldots, H_k) \) be a \( k \)-tuple of fixed graphs, where repetition is allowed. A \textit{monochromatic} \( \mathcal{H} \)-decomposition of \( G \) is a partition of its edge set such that each part is either a single edge, or forms a monochromatic copy of \( H_i \) in color \( i \), for some \( 1 \leq i \leq k \). Let \( \phi_k(G, \mathcal{H}) \) be the smallest number, such that, for any \( k \)-edge-coloring of \( G \), there exists a monochromatic \( \mathcal{H} \)-decomposition of \( G \) with at most \( \phi_k(G, \mathcal{H}) \) elements. Our goal is to study the function

\[
\phi_k(n, \mathcal{H}) = \max\{\phi_k(G, \mathcal{H}) \mid v(G) = n\},
\]

which is the smallest number \( \phi \) such that, any \( k \)-edge-colored graph of order \( n \) admits a monochromatic \( \mathcal{H} \)-decomposition with at most \( \phi \) elements. In the case when \( H_i \cong H \) for every \( 1 \leq i \leq k \), we simply write \( \phi_k(G, H) = \phi_k(G, \mathcal{H}) \) and \( \phi_k(n, H) = \phi_k(n, \mathcal{H}) \).

The function \( \phi_k(n, K_r) \), for \( k \geq 2 \) and \( r \geq 3 \), has been studied by Liu and Sousa [11], who obtained results involving the Ramsey numbers and the Turán numbers. Recall that for \( k \geq 2 \) and integers \( r_1, \ldots, r_k \geq 3 \), the \textit{Ramsey number} for \( K_{r_1}, \ldots, K_{r_k} \), denoted by \( R(r_1, \ldots, r_k) \), is the smallest value of \( s \), such that, for every \( k \)-edge-coloring of \( K_s \), there exists a monochromatic \( K_{r_i} \) in color \( i \), for some \( 1 \leq i \leq k \). For the case when \( r_1 = \cdots = r_k = r \), for some \( r \geq 3 \), we simply write \( R_k(r) = R(r_1, \ldots, r_k) \). Since \( R(r_1, \ldots, r_k) \) does not change under any permutation of \( r_1, \ldots, r_k \), without loss of generality, we assume throughout that \( 3 \leq r_1 \leq \cdots \leq r_k \). The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite [15]. To this date, the values of \( R(3, r_2) \) have been determined exactly only for \( 3 \leq r_2 \leq 9 \), and these are shown in the following table [14].

| \( r_2 \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| \( R(3, r_2) \) | 6 | 9 | 14 | 18 | 23 | 28 | 36 |

The remaining Ramsey numbers that are known exactly are \( R(4, 4) = 18, R(4, 5) = 25, \) and \( R(3, 3, 3) = 17 \). The gap between the lower bound and the upper bound for other Ramsey numbers is generally quite large.

For the case \( R(3, 3) = 6 \), it is easy to see that the only 2-edge-coloring of \( K_5 \) not containing a monochromatic \( K_3 \) is the one where each color induces a cycle of length 5. From this 2-edge-coloring, observe that we may take a “blow-up” to obtain a 2-edge-coloring of the Turán graph \( T_5(n) \), and easily deduce that \( \phi_2(n, K_3) \geq t_5(n) \). See Figure 1.

This example was the motivation for Liu and Sousa [11] to study \( K_r \)-monochromatic decompositions of graphs, for \( r \geq 3 \) and \( k \geq 2 \). They have recently proved the following result.

\[ Journal\ of\ Graph\ Theory\ DOI\ 10.1002/jgt \]
Theorem 1.4 [11].

(a) $\phi_k(n, K_3) = t_{R_3} - 1(n) + o(n^2)$;
(b) $\phi_k(n, K_3) = t_{R_3} - 1(n)$ for $k = 2, 3$ and $n$ sufficiently large;
(c) $\phi_k(n, K_r) = t_{R_r} - 1(n)$, for $k \geq 2, r \geq 4$ and $n$ sufficiently large.

Moreover, the only graph attaining $\phi_k(n, K_r)$ in cases (b) and (c) is the Turán graph $T_{R_r} - 1(n)$.

They also made the following conjecture.

Conjecture 1.5 [11]. Let $k \geq 4$. Then $\phi_k(n, K_3) = t_{R_3} - 1(n)$ for $n \geq R_3(n)$.

Here, we will study an extension of the monochromatic $K_r$-decomposition problem when the clique $K_r$ is replaced by a fixed $k$-tuple of cliques $C = (K_{r_1}, \ldots, K_{r_k})$. Our main result, stated in Theorem 1.6, is clearly an extension of Theorem 1.4. Also, it verifies Conjecture 1.5 for sufficiently large $n$.

Theorem 1.6. Let $k \geq 2$, $3 \leq r_1 \leq \cdots \leq r_k$, and $R = R(r_1, \ldots, r_k)$. Let $C = (K_{r_1}, \ldots, K_{r_k})$. Then, there is an $n_0 = n_0(r_1, \ldots, r_k)$ such that, for all $n \geq n_0$, we have

$$\phi_k(n, C) = t_{R_r}(n).$$

Moreover, the only order-$n$ graph attaining $\phi_k(n, C)$ is the Turán graph $T_{R_r} - 1(n)$ (with a $k$-edge-coloring that does not contain a color-$i$ copy of $K_r$ for any $1 \leq i \leq k$).

The upper bound of Theorem 1.6 is proved in Section 2. The lower bound follows easily by the definition of the Ramsey number. Indeed, take a $k$-edge-coloring $f'$ of the complete graph $K_{R_r}$ without a monochromatic $K_r$ in color $i$, for all $1 \leq i \leq k$. Note that $f'$ exists by definition of the Ramsey number $R = R(r_1, \ldots, r_k)$. Let $u_1, \ldots, u_{R_r}$ be the vertices of the $K_{R_r}$. Now, consider the Turán graph $T_{R_r} - 1(n)$ with a $k$-edge-coloring $f$ that is a “blow-up” of $f'$. That is, if $T_{R_r} - 1(n)$ has partition classes $V_1, \ldots, V_{R_r}$, then for $v \in V_j$ and $w \in V_{\ell}$ with $j \neq \ell$, we define $f(vw) = f'(u_ju_{\ell})$. Then, $T_{R_r} - 1(n)$ with this $k$-edge-coloring has no monochromatic $K_r$ in color $i$, for every $1 \leq i \leq k$. Therefore, $\phi_k(n, C) \geq \phi_k(T_{R_r} - 1(n), C) = t_{R_r} - 1(n)$ and the lower bound in Theorem 1.6 follows.

In particular, when all the cliques in $C$ are equal, Theorem 1.6 completes the results obtained previously by Liu and Sousa in Theorem 1.4. In fact, we get the following direct corollary from Theorem 1.6.

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Corollary 1.7. Let $k \geq 2$, $r \geq 3$ and $n$ be sufficiently large. Then,

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

Moreover, the only order-$n$ graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R_k(r)-1}(n)$ (with a $k$-edge-coloring that does not contain a monochromatic copy of $K_r$).

2. PROOF OF THEOREM 1.6

In this section, we will prove the upper bound in Theorem 1.6. Before presenting the proof we need to introduce the tools. Throughout this section, let $k \geq 2$, $3 \leq r_1 \leq \ldots \leq r_k$ be an increasing sequence of integers, $R = R(r_1, \ldots, r_k)$ be the Ramsey number for $K_{r_1}, \ldots, K_{r_k}$, and $C = (K_{r_1}, \ldots, K_{r_k})$ be a fixed $k$-tuple of cliques.

We first recall the following stability theorem of Erdős and Simonovits [5, 16].

Theorem 2.1 (Stability Theorem [5,16]). Let $r \geq 3$, and $G$ be a graph on $n$ vertices with $e(G) \geq t_{r-1}(n) + o(n^2)$ and not containing $K_r$ as a subgraph. Then, there exists an $(r-1)$-partite graph $G'$ on $n$ vertices with partition classes $V_1, \ldots, V_{r-1}$, where $|V_i| = \frac{n}{r-1} + o(n)$ for $1 \leq i \leq r-1$, that can be obtained from $G$ by adding and subtracting $o(n^2)$ edges.

Next, we recall the following result of Győri [7, 8] about the existence of edge-disjoint copies of $K_r$ in graphs on $n$ vertices with more than $t_{r-1}(n)$ edges.

Theorem 2.2 [7,8]. For every $r \geq 3$ there is $C$ such that every graph $G$ with $n \geq C$ vertices and $e(G) = t_{r-1}(n) + m$ edges, where $m \leq \binom{r}{2} / C$, contains at least $m - Cm^2 / n^2$ edge-disjoint copies of $K_r$.

Now, we will consider coverings and packings of cliques in graphs. Let $r \geq 3$ and $G$ be a graph. Let $\mathcal{K}$ be the set of all $K_r$-subgraphs of $G$. A $K_r$-cover is a set of edges of $G$ meeting all elements in $\mathcal{K}$, that is, the removal of a $K_r$-cover results in a $K_r$-free graph. A $K_r$-packing in $G$ is a set of pairwise edge-disjoint copies of $K_r$. The $K_r$-covering number of $G$, denoted by $\tau_r(G)$, is the minimum size of a $K_r$-cover of $G$, and the $K_r$-packing number of $G$, denoted by $\nu_r(G)$, is the maximum size of a $K_r$-packing of $G$. Next, a fractional $K_r$-cover of $G$ is a function $f : E(G) \to \mathbb{R}_+$, such that $\sum_{e \in E(H)} f(e) \geq 1$ for every $H \in \mathcal{K}$, that is, for every copy of $K_r$ in $G$ the sum of the values of $f$ on its edges is at least 1. A fractional $K_r$-packing of $G$ is a function $p : \mathcal{K} \to \mathbb{R}_+$ such that $\sum_{H \in \mathcal{K}, e \in E(H)} p(H) \leq 1$ for every $e \in E(G)$, that is, the total weight of $K_r$’s that cover any edge is at most 1. Here, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. The fractional $K_r$-covering number of $G$, denoted by $\tau_r^*(G)$, is the minimum of $\sum_{e \in E(G)} f(e)$ over all fractional $K_r$-covers $f$, and the fractional $K_r$-packing number of $G$, denoted by $\nu^*_r(G)$, is the maximum of $\sum_{H \in \mathcal{K}} p(H)$ over all fractional $K_r$-packings $p$.

One can easily observe that

$$\nu_r(G) \leq \tau_r(G) \leq \left(\frac{r}{2}\right) \nu_r(G).$$

For $r = 3$, we have $\tau_3(G) \leq 3\nu_3(G)$. A long-standing conjecture of Tuza [21] from 1981 states that this inequality can be improved as follows.
Conjecture 2.3 [21]. For every graph $G$, we have $\tau_3(G) \leq 2\nu_3(G)$.

Conjecture 2.3 remains open although many partial results have been proved. By using the earlier results of Krivelevich [10], and Haxell and Rödl [9], Yuster [22] proved the following theorem which will be crucial to the proof of Theorem 1.6. In the case $r = 3$, it is an asymptotic solution of Tuza’s conjecture.

Theorem 2.4 [22]. Let $r \geq 3$ and $G$ be a graph on $n$ vertices. Then

$$\tau_r(G) \leq \left\lfloor \frac{r^2}{4} \right\rfloor \nu_r(G) + o(n^2). \quad (1)$$

We now prove the following lemma that states that a graph $G$ with $n$ vertices and at least $t_{R-1}(n) + \Omega(n^2)$ edges falls quite short of being optimal.

Lemma 2.5. For every $k \geq 2$ and $c_0 > 0$ there are $c_1 > 0$ and $n_0$ such that for every graph $G$ of order $n \geq n_0$ with at least $t_{R-1}(n) + c_0 n^2$ edges, we have $\phi_k(G, C) \leq t_{R-1}(n) - c_1 n^2$.

Proof. Suppose that the lemma is false, that is, there is $c_0 > 0$ such that for some increasing sequence of $n$ there is a graph $G$ on $n$ vertices with $e(G) \geq t_{R-1}(n) + c_0 n^2$ and $\phi_k(G, C) \geq t_{R-1}(n) + o(n^2)$. Fix a $k$-edge-coloring of $G$ and, for $1 \leq i \leq k$, let $G_i$ be the subgraph of $G$ on $n$ vertices that contains all edges with color $i$.

Let $m = e(G) - t_{R-1}(n)$, and let $s \in \{0, \ldots, k\}$ be the maximum such that

$$r_1 = \ldots = r_s = 3.$$

Let us very briefly recall the argument from [11] that shows $\phi_k(G, C) \leq t_{R-1}(n) + o(n^2)$, adopted to our purposes. If we remove a $K_{r_i}$-cover from $G_i$ for every $1 \leq i \leq k$, then we destroy all copies of $K_R$ in $G$. By Turán’s theorem, at most $t_{R-1}(n)$ edges remain. Thus,

$$\sum_{i=1}^k \tau_{r_i}(G_i) \geq m. \quad (2)$$

By Theorem 2.4, if we decompose $G$ into a maximum $K_{r_i}$-packing in each $G_i$ and the remaining edges, we obtain that

$$\phi_k(G, C) \leq e(G) - \sum_{i=1}^k \left( \binom{r_i}{2} - 1 \right) \nu_{r_i}(G_i) \leq t_{R-1}(n) + m - \sum_{i=1}^k \left( \binom{r_i}{2} - 1 \right) \tau_{r_i}(G_i) + o(n^2) \quad (3)$$

$$\leq t_{R-1}(n) + m - \sum_{i=1}^k \tau_{r_i}(G_i) - \frac{1}{4} \sum_{i=s+1}^k \tau_{r_i}(G_i) + o(n^2) \leq t_{R-1}(n) + o(n^2).$$

The third inequality holds since $(\binom{r}{2} - 1)/[r^2/4] \geq 5/4$ for $r \geq 4$ and is equal to 1 for $r = 3$. 

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Let us derive a contradiction from this by looking at the properties of our hypothetical counterexample $G$. First, all inequalities that we saw have to be equalities within an additive term $o(n^2)$. In particular, the slack in (2) is $o(n^2)$, that is,

$$\sum_{i=1}^{k} \tau_{r_i}(G_i) = m + o(n^2). \quad (4)$$

Also, $\sum_{i=s+1}^{k} \tau_{r_i}(G_i) = o(n^2)$. In particular, we have that $s \geq 1$. To simplify the later calculations, let us redefine $G$ by removing a maximum $K_{r_i}$-packing from $G_i$ for each $i \geq s + 1$. The new graph is still a counterexample to the lemma if we decrease $c_0$ slightly, since the number of edges removed is at most $\sum_{i=s+1}^{k} \binom{n}{2} \tau_{r_i}(G_i) = o(n^2)$.

Suppose that we remove, for each $i \leq s$, an arbitrary (not necessarily minimum) $K_3$-cover $F_i$ from $G_i$ such that

$$\sum_{i=1}^{s} |F_i| \leq m + o(n^2). \quad (5)$$

Let $G' \subseteq G$ be the obtained $K_{r_i}$-free graph. (Recall that we assumed that $G_i$ is $K_{r_i}$-free for all $i \geq s + 1$.) Let $G_i' \subseteq G_i$ be the color classes of $G'$. We know by (5) that $e(G') \geq t_{r_i-1}(n) + o(n^2)$. Since $G'$ is $K_{r_i}$-free, we conclude by the Stability Theorem (Theorem 2.1) that there is a partition $V(G) = V(G') = V_1 \cup \ldots \cup V_{R-1}$ such that

$$\forall i \in \{1, \ldots, R-1\}, \quad |V_i| = \frac{n}{R-1} + o(n) \quad \text{and} \quad |E(T) \setminus E(G')| = o(n^2), \quad (6)$$

where $T$ is the complete $(R-1)$-partite graph with parts $V_1, \ldots, V_{R-1}$.

Next, we essentially expand the proof of (1) for $r = 3$ and transform it into an algorithm that produces $K_3$-coverings $F_i$ of $G_i$, with $1 \leq i \leq s$, in such a way that (5) holds but (6) is impossible whatever $V_1, \ldots, V_{R-1}$ we take, giving the desired contradiction.

Let $H$ be an arbitrary graph of order $n$. By the LP duality, we have that

$$\tau^*_r(H) = v^*_r(H). \quad (7)$$

By the result of Haxell and Rödl [9] we have that

$$v^*_r(H) = v_r(H) + o(n^2). \quad (8)$$

Krivelevich [10] showed that

$$\tau_3(H) \leq 2\tau^*_3(H). \quad (9)$$

Thus, $\tau_3(H) \leq 2v_3(H) + o(n^2)$ giving (1) for $r = 3$.

The proof of Krivelevich [10] of (9) is based on the following result.

**Lemma 2.6.** Let $H$ be an arbitrary graph and $f : E(H) \to \mathbb{R}_+$ be a minimum fractional $K_3$-cover. Then $\tau_3(H) \leq \frac{3}{2} \tau^*_3(H)$ or there is $xy \in E(H)$ with $f(xy) = 0$ that belongs to at least one triangle of $H$.

**Proof.** If there is an edge $xy \in E(H)$ that does not belong to a triangle, then necessarily $f(xy) = 0$ and $xy$ does not belong to any optimal fractional or integer $K_3$-cover. We can remove $xy$ from $E(H)$ without changing the validity of the lemma. Thus, we can assume that every edge of $H$ belongs to a triangle.

Suppose that $f(xy) > 0$ for every edge $xy$ of $H$, for otherwise we are done. Take a maximum fractional $K_3$-packing $p$. Recall that it is a function that assigns a weight

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Let \( p(xyz) \in \mathbb{R}_+ \) to each triangle \( xyz \) of \( H \) such that for every edge \( xy \) the sum of weights over all \( K_3 \)'s of \( H \) containing \( xy \) is at most 1, that is,

\[
\sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) \leq 1, \tag{10}
\]

where \( \Gamma(v) \) denotes the set of neighbors of the vertex \( v \) in \( H \).

This is the dual LP to the minimum fractional \( K_3 \)-cover problem. By the complementary slackness condition (since \( f \) and \( p \) are optimal solutions), we have equality in (10) for every \( xy \in E(H) \). This and the LP duality imply that

\[
\tau^*_3(H) = \nu^*_3(H) = \frac{1}{3} \sum_{\text{triangle } xyz} p(xyz) = \frac{1}{3} e(H).
\]

On the other hand \( \tau_3(H) \leq \frac{1}{2} e(H) \): take a bipartite subgraph of \( H \) with at least half of the edges; then the remaining edges form a \( K_3 \)-cover. Putting the last two inequalities together, we obtain the required result. \( \blacksquare \)

Let \( 1 \leq i \leq s \). We now describe an algorithm for finding a \( K_3 \)-cover \( F_i \) in \( G_i \). Initially, let \( H = G_i \) and \( F_i = \emptyset \). Repeat the following.

Take a minimum fractional \( K_3 \)-cover \( f \) of \( H \). If the first alternative of Lemma 2.6 is true, pick a \( K_3 \)-cover of \( H \) of size at most \( \frac{1}{3} \tau^*_3(H) \), add it to \( F_i \) and stop. Otherwise, fix some edge \( xy \in E(H) \) returned by Lemma 2.6. Let \( F' \) consist of all pairs \( xz \) and \( yz \) over \( z \in \Gamma(x) \cap \Gamma(y) \). Add \( F' \) to \( F_i \) and remove \( F' \) from \( E(H) \). Repeat the whole step (with the new \( H \) and \( f \)).

Consider any moment during this algorithm, when we had \( f(xy) = 0 \) for some edge \( xy \) of \( H \). Since \( f \) is a fractional \( K_3 \)-cover, we have that \( f(xz) + f(yz) \geq 1 \) for every \( z \in \Gamma(x) \cap \Gamma(y) \). Thus, if \( H' \) is obtained from \( H \) by removing \( 2\ell \) such pairs, where \( \ell = |\Gamma(x) \cap \Gamma(y)| \), then \( \tau^*_3(H') \leq \tau^*_3(H) - \ell \) because \( f \) when restricted to \( E(H') \) is still a fractional cover (although not necessarily an optimal one). Clearly, \( |F_i| \) increases by \( 2\ell \) during this operation. Thus, indeed we obtain, at the end, a \( K_3 \)-cover \( F_i \) of \( G_i \) of size at most \( 2\tau^*_3(G_i) \).

Also, by (7) and (8) we have that

\[
\sum_{i=1}^{s} |F_i| \leq 2 \sum_{i=1}^{s} \nu_3(G_i) + o(n^2).
\]

Now, since all slacks in (3) are \( o(n^2) \), we conclude that

\[
\sum_{i=1}^{s} \nu_3(G_i) \leq \frac{m}{2} + o(n^2)
\]

and (5) holds. In fact, (5) is equality by (4).

Recall that \( G'_i \) is obtained from \( G_i \) by removing all edges of \( F_i \) and \( G' \) is the edge-disjoint union of the graphs \( G'_i \). Suppose that there exist \( V_1, \ldots, V_{R-1} \) satisfying (6). Let \( M = E(T) \setminus E(G') \) consist of missing edges. Thus, \( |M| = o(n^2) \).

Let

\[
X = \{ x \in V(T) \mid \deg_M(x) \geq c_2 n \},
\]

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where we define $c_2 = (4(R - 1))^{-1}$. Clearly,

$$|X| \leq 2|M|/c_2n = o(n).$$

Observe that, for every $1 \leq i \leq s$, if the first alternative of Lemma 2.6 holds at some point, then the remaining graph $H$ satisfies $\tau_3^s(H) = o(n^2)$. Indeed, otherwise by $\tau_3(G_i) \leq 2\tau_3^s(G_i) - \tau_3^s(H)/2 + o(n^2)$ we get a strictly smaller constant than $2$ in (9) and thus a gap of $\Omega(n^2)$ in (3), a contradiction. Therefore, all but $o(n^2)$ edges in $F_i$ come from some parent edge $xy$ that had $f$-weight $0$ at some point.

When our algorithm adds pairs $xz$ and $yz$ to $F_i$ with the same parent $xy$, then it adds the same number of pairs incident to $x$ as those incident to $y$. Let $P$ consist of pairs $xy$ that are disjoint from $X$ and were a parent edge during the run of the algorithm. Since the total number of pairs in $F_i$ incident to $x$ is at most $n|X| = o(n^2)$, there are $|F_i| - o(n^2)$ pairs in $F_i$ such that their parent is in $P$.

Let us show that $y_0$ and $y_1$ belong to different parts $V_j$ for every pair $y_0y_1 \in E(P)$. Suppose on the contrary that, say, $y_0, y_1 \in V_j$. For each $2 \leq j \leq R - 1$ pick an arbitrary $y_j \in V_j \setminus (\Gamma_M(y_0) \cup \Gamma_M(y_1))$. Since $y_0, y_1 \notin X$, the possible number of choices for $y_j$ is at least

$$\frac{n}{R - 1} - 2c_2n + o(n) = \frac{n}{R - 1} - 3c_2n.$$

Let

$$Y = \{y_0, \ldots, y_{R-1}\}.$$

By the above, we have at least $(\frac{n}{R - 1} - 3c_2n)^{R-2} = \Omega(n^{R-2})$ choices of $Y$. Note that by the definition, all edges between $\{y_0, y_1\}$ and the rest of $Y$ are present in $E(G')$. Thus, the number of sets $Y$ containing at least one edge of $M$ different from $y_0y_1$ is at most

$$|M| \times n^{R-4} = o(n^{R-2}).$$

This is $o(1)$ times the number of choices of $Y$. Thus, for almost every $Y, H = G'[Y]$ is a clique (except perhaps the pair $y_0y_1$). In particular, there is at least one such choice of $Y$; fix it. Let $i \in \{1, \ldots, k\}$ be arbitrary. Adding back the pair $y_0y_1$ colored $i$ to $H$ (if it is not there already), we obtain a $k$-edge-coloring of the complete graph $H$ of order $R$. By the definition of $R = R(r_1, \ldots, r_k)$, there must be a monochromatic triangle on $abc$ of color $h \leq s$. (Recall that we assumed at the beginning that $G_j$ is $K_{ij}$-free for each $j > s$.) But $abc$ has to contain an edge from the $K_3$-cover $F_i$, say $ab$. This edge $ab$ is not in $G'$ (it was removed from $G$). If $a, b$ lie in different parts $V_j$, then $ab \in M$, a contradiction to the choice of $Y$. The only possibility is that $ab = y_0y_1$. Then $h = i$. Since both $y_0c$ and $y_1c$ are in $G'_i$, they were never added to the $K_3$-cover $F_i$ by our algorithm. Therefore, $y_0y_1$ was never a parent, which is the desired contradiction.

Thus, every $xy \in P$ connects two different parts $V_j$. For every such parent $xy$, the number of its children in $M$ is at least half of all its children. Indeed, for every pair of children $xz$ and $yz$, at least one connects two different parts; this child necessarily belongs to $M$. Thus,

$$|F_i \cap M| \geq \frac{1}{2} |F_i| + o(n^2).$$
(Recall that parent edges that intersect $X$ produce at most $2n|X| = o(n^2)$ children.) Therefore,

$$|M| \geq \frac{1}{2} \sum_{i=1}^{s} |F_i| + o(n^2) \geq \frac{m}{2} + o(n^2) = \Omega(n^2),$$

contradicting (6). This contradiction proves Lemma 2.5. ■

We are now able to prove Theorem 1.6.

**Proof of the upper bound in Theorem 1.6.** Let $C$ be the constant returned by Theorem 2.2 for $r = R$. Let $n_0 = n_0(r_1, \ldots, r_k)$ be sufficiently large to satisfy all the inequalities we will encounter. Let $G$ be a $k$-edge-colored graph on $n \geq n_0$ vertices. We will show that $\phi_k(G, C) = t_{R-1}(n)$ with equality if and only if $G = T_{R-1}(n)$, and $G$ does not contain a monochromatic copy of $K_{r_i}$ in color $i$ for every $1 \leq i \leq k$.

Let $e(G) = t_{R-1}(n) + m$, where $m$ is an integer. If $m < 0$, we can decompose $G$ into single edges and there is nothing to prove.

Suppose $m = 0$. If $G$ contains a monochromatic copy of $K_{r_i}$ in color $i$ for some $1 \leq i \leq k$, then $G$ admits a monochromatic $C$-decomposition with at most $t_{R-1}(n) - \binom{r_i}{2} + 1 < t_{R-1}(n)$ parts and we are done. Otherwise, the definition of $R$ implies that $G$ does not contain a copy of $K_R$. Therefore, $G = T_{R-1}(n)$ by Turán’s theorem and $\phi_k(G, C) = t_{R-1}(n)$ as required.

Now suppose $m > 0$. We can also assume that $m < \binom{r_i}{2}/C$ for otherwise we are done: $\phi_k(G, C) < t_{R-1}(n)$ by Lemma 2.5. Thus, by Theorem 2.2, the graph $G$ contains at least $m - Cm^2/n^2 > \frac{m}{2}$ edge-disjoint copies of $K_R$. Since each $K_R$ contains a monochromatic copy of $K_{r_i}$ in the color-$i$ graph $G_i$, for some $1 \leq i \leq k$, we conclude that $\sum_{i=1}^{k} v_{r_i}(G_i) > \frac{m}{2}$, so that $\sum_{i=1}^{k} \left( \binom{r_i}{2} - 1 \right) v_{r_i}(G_i) \geq \sum_{i=1}^{k} 2v_{r_i}(G_i) > m$. We have

$$\phi_k(G, C) = e(G) - \sum_{i=1}^{k} \binom{r_i}{2} v_{r_i}(G_i) + \sum_{i=1}^{k} v_{r_i}(G_i) < t_{R-1}(n),$$

giving the required. ■

**Remark.** By analyzing the above argument, one can also derive the following stability property for every fixed family $C$ of cliques as $n \to \infty$: every graph $G$ on $n$ vertices with $\phi_k(G, C) = t_{R-1}(n) + o(n^2)$ is $o(n^2)$-close to the Turán graph $T_{R-1}(n)$ in the edit distance.

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