Fidelity of Gaussian Channels

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Abstract. A noisy Gaussian channel is defined as a channel in which an input field mode is subjected to random Gaussian displacements in phase space. We introduce the quantum fidelity of a Gaussian channel for pure and mixed input states, and we derive a universal scaling law of the fidelity for pure initial states. We also find the maximum fidelity of a Gaussian channel over all input states. Quantum cloning and continuous-variable teleportation are presented as physical examples of Gaussian channels to which the fidelity results can be applied.

1. Introduction

A principal aim of quantum information theory [1] is to determine the ultimate limits on capacity, entropy, or fidelity of quantum information transmitted in noisy quantum channels [2, 3, 4, 5, 6, 7]. Quantum channels with noise are examples of open quantum systems that interact with an environment. The environment produces classical and/or quantum fluctuations that generally degrade the input states. The channel is described by a linear map \( \rho \mapsto \Phi(\rho) \), which takes an input state \( \rho \) to an output state \( \Phi(\rho) \). The efficacy of the channel can be characterized by a quantum fidelity \( F(\rho, \gamma) \), which measures the quality of transmitted information as a function of the input state \( \rho \) and the noise parameter \( \gamma \) that describes the environment.

A special class of noisy quantum channels consists of bosonic channels with excess noise described by random Gaussian shifts in phase space [8, 9, 10, 11]. Such quantum Gaussian channels have attracted considerable attention in the framework of quantum information with continuous variables. Recent investigations devoted to bosonic Gaussian channels have addressed the following problems:

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lower bounds on the capacities [10], the question of multiplicativity of the maximal $p$-norm output purities [12], and the Holevo-Werner additivity of entropies at the output of the channels [13].

The purpose of this paper is to describe the quantum fidelity of Gaussian channels, using as a tool phase-space methods related to Wigner and Weyl functions. A general scaling law for quantum fidelity for pure input states is derived. We argue that this scaling law reflects the duality of the Wigner and Weyl phase-space distributions related to sub-Planck [14] and large scales of the quantum state $\rho$ (for an extended discussion of this duality, see Ref. [15]). We show that if the devices processing quantum information can be built from linear optical elements in an extended Hilbert space, the resulting Gaussian channels have important physical applications. We discuss quantum cloning and a continuous-variable teleportation protocol as examples of Gaussian noise channels.

The paper is structured as follows. In Sec. 2, the description of quantum channels in terms completely positive maps (CPM) is recalled. In Sec. 3, the Gaussian channel is defined in terms of random Gaussian shifts and is related to Wigner and Weyl functions in phase space. In Sec. 4, the quantum fidelity of a Gaussian channel is introduced for pure and mixed input states. In Sec. 5, a universal scaling law of the fidelity for pure initial states is derived, and in Sec. 6, we find the maximum fidelity of a Gaussian channel over all input states. Section 7 presents as examples explicit calculations of the fidelity for number states and squeezed states. Section 8 applies two different kinds of fidelities for mixed input states to a thermal input. In Sec. 9, we discuss quantum cloning and continuous-variable teleportation as physical examples of bosonic Gaussian channels. Some final thoughts are presented in Sec. 10.

2. Quantum channels and completely positive maps

Consider a quantum system described by a Hilbert space $\mathcal{H}$, with a given density operator $\rho$. A noisy quantum channel is a linear and trace-preserving map of the quantum state $\rho$, given by the Kraus decomposition

$$\Phi(\rho) = \sum_{i} K_{i}\rho K_{i}^{\dagger},$$

(1)

where the completeness condition $\sum_{i} K_{i}^{\dagger}K_{i} = I$ makes the map trace-preserving [16]. We call $\rho$ the input state and the transformed $\Phi(\rho)$ the output state.

The overall system-environment state is described by a density operator that evolves unitarily through the quantum channel. Most often, one is interested in the system alone, which is described by a reduced density operator obtained by tracing over the environment degrees of freedom. As a result of such reduction, a quantum channel with noise is characterized by a linear, trace-preserving, and
completely positive map. A completely positive map is defined in the following way: if the system undergoes the dynamics described by $\Phi$ and a reference system $R$ of arbitrary Hilbert-space dimension experiences no dynamics, a situation described by the overall superoperator $\Phi \otimes I_R$, where $I_R$ is the identity superoperator for the reference system, then an arbitrary (potentially entangled) joint state of the system and reference system is mapped to a positive output state; i.e., the superoperator $\Phi \otimes I_R$ maps positive operators to positive operators.

A completely positive map is required to describe reduced dynamics because it implies that such dynamics arises from a unitary evolution of the system and an environment,

$$\Phi(\rho) = \text{Tr}_E[U(\rho \otimes |e\rangle\langle e|)U^\dagger].$$

Here the environment degrees of freedom are denoted by $E$ and $|e\rangle$ is some initial state of the environment. The following statements about a linear and trace-preserving map $\Phi$ are equivalent: (i) $\Phi$ is completely positive, (ii) $\Phi$ has a Kraus decomposition as in Eq. (1), and (iii) $\Phi$ is the reduced dynamics for some system-environment unitary dynamics as in Eq. (2).

### 3. Bosonic Gaussian Channels

What we mean by a bosonic Gaussian channel in this paper is a completely positive map that results from zero-mean random Gaussian shifts in the phase space of a bosonic mode $|$S$. This CPM acts on input states $\rho$ in the following way:

$$\Phi(\rho) = \int d^2 \alpha \mathcal{G}(\alpha) D(\alpha) \rho D(\alpha).$$

Here $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator in the phase space of the bosonic mode, where the bosonic creation and annihilation operators obey the commutation relation $[a, a^\dagger] = 1$, and the classical-noise Gaussian distribution,

$$\mathcal{G}(\alpha) = \frac{2}{\pi \gamma} e^{-2|\alpha|^2/\gamma},$$

has zero mean and phase-space variance $\sigma^2 = \gamma/2$. The complex number $\alpha = (q + ip)/\sqrt{2}$ corresponds to the phase-space points of a single-mode harmonic oscillator described by position $q$ and momentum $p$. Notice that the Gaussian channel of Eq. (3) is a unital map; i.e., it leaves the identity operator unaffected. More importantly, this Gaussian channel preserves the expectation value of $a$; i.e., this kind of Gaussian channel cannot have any overall attenuation or amplification. The Gaussian channel (3) is in the Kraus form (1) if we make the identification $K_i \rightarrow \sqrt{\mathcal{G}(\alpha)} D(\alpha)$.

In Sec. (4) we show that such Gaussian channels can be implemented in the framework of linear optical elements in an extended Hilbert space. In such cases
the CPM (3) can be derived from the general relation (2). We will discuss quantum cloning and continuous-variable teleportation as physical examples of bosonic Gaussian channels resulting from a reduction of quantum systems that interact with an environment.

In our discussion of the Gaussian channel, we will analyze various phase-space overlaps. We use the following definitions of the phase-space Wigner and Weyl functions. The Wigner function of an arbitrary density operator $\rho$ is [17] given by

$$W_{\rho}(\alpha) = \frac{2}{\pi} \text{Tr} \left[ \rho D(\alpha)(-1)^{a^\dagger a} D(\alpha)^\dagger \right] = \int \frac{d^2 \beta}{\pi} e^{\alpha \beta^* - \alpha^* \beta} C_{\rho}(\beta),$$

(5)

where $C_{\rho}(\alpha) = \text{Tr}[\rho D(\alpha)]$ is the Weyl characteristic function and $\rho$ is an arbitrary density operator. We find it useful to rewrite the Gaussian CPM (3) in terms of the phase-space Wigner functions of the input and the output states:

$$W_{\Phi(\rho)}(\alpha) = \int d^2 \beta G(\alpha - \beta) W_{\rho}(\beta).$$

(6)

The counterpart of this relation in terms of the Weyl functions of the input and output states is

$$C_{\Phi(\rho)}(\alpha) = e^{-\gamma |\alpha|^2/2} C_{\rho}.$$  

(7)

4. Fidelity of a Gaussian channel

4.1. Fidelity for pure input states

If the input state to the Gaussian channel is a pure state $\rho = |\Psi\rangle\langle\Psi|$, the channel fidelity is given as a quantum overlap between the input and the output state. In this case the channel fidelity is

$$F(\Psi, \gamma) = \langle \Psi | \Phi(\rho) | \Psi \rangle.$$  

(8)

Using the phase-space Wigner functions of the input and output state, we can rewrite this fidelity as a phase-space overlap

$$F(\Psi, \gamma) = \pi \int d^2 \alpha W_{\Phi(\rho)}(\alpha) W_{\Psi}(\alpha) = \pi \int d^2 \alpha d^2 \beta G(\alpha - \beta) W_{\Phi}(\alpha) W_{\Psi}(\beta).$$  

(9)

Another useful form for this fidelity, this time in terms of the Weyl functions, comes from substituting the output state (3) directly into Eq. (8):

$$F(\Psi, \gamma) = \int d^2 \alpha G(\alpha) |C_{\Phi}(\alpha)|^2.$$  

(10)
4.2. Fidelity for mixed input states

An appropriate measure for assessing the fidelity of an mixed input state is the entanglement fidelity [18], which is defined in the following way. Imagine that the input mixed state $\rho$ is purified to a state $|\psi\rangle$ of the original mode and a reference mode $R$. An example of such a purification is

$$|\psi\rangle = \sqrt{\rho} \otimes I_R \sum_n |n\rangle \otimes |n\rangle = \sum_n \sqrt{\rho} |n\rangle \otimes |n\rangle.$$  \hspace{1cm} (11)

The input mode is now entangled with the reference mode, and the purified state belongs to an enlarged Hilbert space $H \otimes H_R$. We now suppose that the original mode is subjected to the Gaussian channel while the reference mode is left untouched. The resulting output state is

$$\Phi \otimes I_R(|\psi\rangle\langle\psi|) = \int d^2 \alpha \mathcal{G}(\alpha) D(\alpha) \otimes I_R |\psi\rangle\langle\psi| I_R \otimes D^\dagger(\alpha).$$  \hspace{1cm} (12)

The entanglement fidelity is now defined to be the fidelity of this joint output state with the purified input state:

$$F(\rho, \gamma) = \langle \psi | \Phi \otimes I_R(|\psi\rangle\langle\psi|) | \psi \rangle.$$  \hspace{1cm} (13)

The entanglement fidelity is a property of the system state $\rho$ even though the purification is not unique. The entanglement fidelity reduces to the fidelity of Eq. (8) in the case of pure input states.

It is now easy to see that the entanglement fidelity for a Gaussian channel has the form

$$F(\rho, \gamma) = \int d^2 \alpha \mathcal{G}(\alpha)|C_\rho(\alpha)|^2,$$  \hspace{1cm} (14)

which is the same as the corresponding form (10) of the pure-state fidelity. Notice, however, that the entanglement fidelity is not given by the mixed-state version of Eq. (9). A more thorough discussion of the entanglement fidelity for mixed input states is given in Ref. [15].

Another possible fidelity measure for a mixed-state input regards the mixed state as coming from a particular ensemble of states, $\{p_n, \Psi_n\}$, by which we mean that in each experimental run, one of the states $|\Psi_n\rangle$ is selected randomly with probability $p_n$. Thus, in each run, the Gaussian channel delivers with probability $p_n$ the initial state with fidelity $F(\Psi_n, \gamma)$. A statistical average leads to a mean ensemble fidelity given by

$$\bar{F}(\{p_n, \Psi_n\}, \gamma) = \sum_n p_n F(\Psi_n, \gamma) = \sum_n p_n \langle \Psi_n | \Phi(|\Psi_n\rangle\langle\Psi_n|) | \Psi_n \rangle.$$  \hspace{1cm} (15)

The mean ensemble fidelity depends on the particular ensemble used to make up an input density operator $\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$. The convexity of the entanglement fidelity means that the entanglement fidelity for a mixed input state $\rho$ is less than or equal to the mean ensemble fidelity for any ensemble corresponding to $\rho$. 
5. Scaling law of channel fidelity

If we assume that the input state is given by a pure state, the fidelity is given by the quantum overlap formula (8), which leads to the equivalent phase-space formulas (9) and (10). Here we use these phase-space formulas to show that the fidelity of the Gaussian channel obeys a universal scaling law.

Applying the Fourier transform relation (5) between the Wigner and Weyl functions to Eqs. (9) and (10), we obtain two new formulas for the fidelity. The result is four equivalent forms for the channel fidelity:

\[
F(\Psi, \gamma) = \frac{2}{\pi \gamma} \int d^2 \alpha e^{-2|\alpha|^2/\gamma} |C_\Psi(\alpha)|^2 \\
= \frac{2}{\gamma} \int d^2 \alpha d^2 \beta e^{-2|\alpha-\beta|^2/\gamma} W_\Psi(\alpha) W_\Psi(\beta) \\
= \frac{1}{\pi} \int d^2 \alpha e^{-\gamma|\alpha|^2/2} |C_\Psi(\alpha)|^2 \\
= \int d^2 \alpha d^2 \beta e^{-\gamma|\alpha-\beta|^2/2} W_\Psi(\alpha) W_\Psi(\beta).
\] (16)

The first two lines are rewrites of Eqs. (9) and (10). The third line comes from writing the fidelity (8) as an overlap of the input and output Weyl characteristic functions; it is thus also obtained by Fourier transforming the Wigner functions in the integrand of the second line. Similarly, the last line is obtained by Fourier transforming the Weyl functions in the integrand of the first line.

The first and third forms (and the second and fourth) show us that

\[
F(\Psi, \gamma) = \frac{2}{\gamma} F(\Psi, 4/\gamma).
\] (17)

This scaling law reflects a duality between Wigner functions and Weyl characteristic functions. This duality is related to sub-Planck structures of the Wigner functions [14]. For a given input state, the Wigner function has two important scales: a small scale \( l \) and a large scale \( L \). The small scale \( l \) characterizes the sub-Planck phase-space structures in the input state’s Wigner function. The large scale \( L \) characterizes the scale over which the Wigner function is nonnegligible. For pure states, these two scales are related by an uncertainty relation, \( L l \sim 1 \). Since the Weyl characteristic function is the Fourier transform of the Wigner function, these two scales appear inversely in the Weyl function.

The scaling law (17) displays this duality. If we look at the fidelity in the form of the second relation in Eq. (10), we see that the fidelity between the input state and the output state approaches unity if the dispersion of the Gaussian channel satisfies \( \sigma = \sqrt{\gamma/2} \leq l \) so that the integral approaches \( \text{Tr}(\rho^2) \), which is 1 for a pure state; in this case, all the small-scale phase-space structure of the Wigner
function is well transmitted through the noisy channel. The dual form of the fidelity, expressed in the last relation of Eq. (16), says that to get good fidelity, we need to have $1/\sigma = \sqrt{2/\gamma} \geq L$, so that the integral reduces to the square of the integral over the entire Wigner function. Putting these two results together gives the phase-space uncertainty relation, $Ll \sim 1$.

In terms of the dispersion, the scaling law has the form

$$F(\Psi, \sigma) = \frac{1}{\sigma^2} F(\Psi, 1/\sigma).$$

(18)

Additional discussion of the relation between fidelity and sub-Planck structure and of the phase-space uncertainty relation can be found in Ref. [15].

6. Maximum fidelity

From these considerations, we can derive the maximum channel fidelity that can be achieved by any initial pure state. We show that the maximum fidelity is given by the coherent-state fidelity for all values of $\gamma$. To demonstrate this, return to the expression for the fidelity given by the last formula in Eq. (16):

$$F(\Psi, \gamma) = \int d^2 \alpha d^2 \beta e^{-\gamma|\alpha - \beta|^2/2} W_\Psi(\alpha) W_\Psi(\beta).$$

(19)

The task can be restated as finding the pure state that maximizes this overlap.

Notice that this fidelity can be thought of as the average value of $e^{-\gamma|\alpha - \beta|^2/2}$ with respect to a pure product copy state, $|\Psi_A\rangle \otimes |\Psi_B\rangle$, of two modes, $A$ and $B$: the joint Wigner function of the two modes is $W_{AB}(\alpha, \beta) = W_\Psi(\alpha) W_\Psi(\beta)$. Introducing modes $C$ and $D$, with annihilation operators $c = (a + b)/\sqrt{2}$ and $d = (a - b)/\sqrt{2}$ and corresponding c-number variables $\chi = (\alpha + \beta)/\sqrt{2}$ and $\delta = (\alpha - \beta)/\sqrt{2}$, we can rewrite the fidelity as

$$F(\Psi, \gamma) = \int d^2 \delta e^{-\gamma|\delta|^2} W_D(\delta),$$

(20)

where $W_D(\delta) = \int d^2 \chi W_{AB}(\chi, \delta)$. What we see is that the fidelity is the expectation value of the mode-$D$ operator $A_\gamma$ whose symmetrically ordered associated function is $e^{-\gamma|\delta|^2}$. Letting $\gamma = (\bar{n} + 1/2)^{-1}$, we see that $A_\gamma$ is given by $\bar{n} + 1/2$ times the density operator for a thermal state of mode $D$ whose mean number of photons is $\bar{n} = \gamma^{-1}(1 - \gamma/2)$. Thus we can write the fidelity as

$$F(\Psi, \gamma) = \text{Tr}(A_\gamma \rho_D),$$

(21)
where

\[ A_\gamma = \frac{1}{1 + \frac{\gamma}{2}} \left( \frac{1 - \gamma/2}{1 + \gamma/2} \right)^{d_1 d_2} \]

\[ = \frac{1}{1 + \gamma/2} \left( \frac{1 - \gamma/2}{1 + \gamma/2} \right)^{(a^\dagger b^\dagger)(a-b)/2} . \]

(22)

Notice that \( A_{\gamma=0} = I_D \), confirming that the fidelity is 1 regardless of the input state for \( \gamma = 0 \). Generally we can bound the fidelity by the largest eigenvalue of \( A_\gamma \):

\[ \mathcal{F}(\Psi, \gamma) \leq \left( \text{max eigenvalue of } A_\gamma \right) = \frac{1}{1 + \gamma/2} . \]

(23)

The reason this is the largest eigenvalue is that the factor in large parentheses in the expression for \( A_\gamma \) has magnitude \( \leq 1 \), which means that the largest eigenvalue, corresponding to the vacuum state for mode \( D \), is \((1 + \gamma/2)^{-1}\). Since coherent states saturate the upper bound, we can write

\[ \mathcal{F}_{\text{max}}(\gamma) = \frac{1}{1 + \gamma/2} . \]

(24)

The bound on the expectation value of \( A_\gamma \) is useful in other applications than the noisy Gaussian channels considered here. For that purpose, note that the bound holds for all joint states \( \rho_{AB} \) of modes \( A \) and \( B \)—i.e., it holds for the expectation value \( \text{Tr}(A_\gamma \rho_{AB}) \)—not just for the pure product copy states that are relevant to the channel fidelity. A joint state achieves the expectation value bound if and only if mode \( D \) is in vacuum; i.e., the state \( \rho_{AB} \) is the state of the two output modes, \( a = (c + d)/\sqrt{2} \) and \( b = (c - d)/\sqrt{2} \), of a 50:50 beamsplitter that has vacuum incident on its mode-\( D \) input.

If we specialize to pure product input states, \( |\Psi_A \rangle \otimes |\Phi_B \rangle \), we can say much more about when the bound is achieved. Since mode \( D \) is in vacuum, we have

\[ 0 = d|\Psi_A \rangle \otimes |\Phi_B \rangle = \frac{1}{\sqrt{2}}(a - b)|\Psi_A \rangle \otimes |\Phi_B \rangle , \]

(25)

which implies that

\[ a|\Psi_A \rangle \otimes |\Phi_B \rangle = |\Psi_A \rangle \otimes b|\Phi_B \rangle . \]

(26)

This requires that \( a|\Psi_A \rangle = \langle \Phi_B |b|\Phi_B \rangle |\Psi_A \rangle = a|\Psi_A \rangle \) and \( b|\Phi_B \rangle = \langle \Psi_A |a|\Psi_A \rangle |\Phi_B \rangle = \alpha|\Phi_B \rangle \), i.e., that \( |\Psi_A \rangle \) and \( |\Phi_B \rangle \) are the same coherent state \( |\alpha \rangle \). Thus the maximum channel fidelity is achieved if and only if the input state is a coherent state.
7. Examples of pure states in a Gaussian channel

7.1. Number states

Let us take as an example the case of an input state that is a number state $|n\rangle$ of a harmonic oscillator. In this case the fidelity of transmission through a Gaussian channel is

$$F(|n\rangle, \gamma) = \frac{2}{\pi \gamma} \int d^2 \alpha e^{-2|\alpha|^2/\gamma} |\langle n|D(\alpha)|n\rangle|^2.$$  \hspace{1cm} (27)

Using the property

$$\langle n|D(\alpha)|n\rangle = e^{-|\alpha|^2/2} L_n(|\alpha|^2),$$  \hspace{1cm} (28)

where $L_n$ denotes the $n$th-order Laguerre polynomial, one can calculate an exact expression for the fidelity generating function for all number states:

$$F(\gamma, \lambda) = \sum_{n=0}^{\infty} \lambda^n F(|n\rangle, \gamma) = \frac{1}{\sqrt{(1 + \gamma/2)^2 - 2\lambda(1 + \gamma^2/4) + \lambda^2(1 - \gamma/2)^2}}.$$  \hspace{1cm} (29)

The resulting fidelity of a Gaussian channel with a number state at the input is

$$F(|n\rangle, \gamma) = \frac{(1 - \gamma/2)^n}{(1 + \gamma/2)^{n+1}} P_n\left(\frac{1 + \gamma^2/4}{1 - \gamma^2/4}\right),$$  \hspace{1cm} (30)

where $P_n(x)$ is a Legendre polynomial. For $\gamma = 1, 2$, this becomes

$$F(|n\rangle, 1) = \frac{2}{3^{n+1}} P_n(5/3), \quad F(|n\rangle, 2) = \frac{(2n)!}{2^{2n+1}(n!)^2}. \hspace{1cm} (31)$$

Using a series expansion of the generating function (29) or working directly with the expression (30), one can easily calculate the fidelities for the lowest number states:

$$F(|1\rangle, \gamma) = \frac{4 + \gamma^2}{(2 + \gamma)^3}, \quad F(|2\rangle, \gamma) = \frac{16 + 16\gamma^2 + \gamma^4}{(2 + \gamma)^5}. \hspace{1cm} (32)$$

Techniques similar to those used in this section can be used to calculate the channel fidelity for an input state that is an arbitrary superposition of number states, but in the absence of some general technique like the generating function (29), the calculation becomes increasingly tedious as higher number states are included in the superposition. As an example, for input state $|\Psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, the channel fidelity is

$$F(\Psi, \gamma) = \frac{1 + 3\gamma/4 + \gamma^2/4}{(1 + \gamma/2)^3}. \hspace{1cm} (33)$$
7.2. Squeezed state

A squeezed state of a one-dimensional harmonic oscillator is given by the following formula \[19\]:

\[
|\mu\rangle = (1 - |\mu|^{2})^{1/4}e^{-\mu a^\dagger/2}|0\rangle. \tag{34}
\]

Here

\[
|\mu| = \tanh r = \sqrt{\frac{n}{1 + \bar{n}}}, \tag{35}
\]

where \(r\) is the usual squeeze parameter and \(\bar{n}\) is the mean number of oscillator quanta. For \(\mu = 0, \bar{n} = 0\), and the squeezed state reduces to the ground state of the one-dimensional harmonic oscillator. The fidelity of a Gaussian channel with a squeezed state at the input is

\[
F(|\mu\rangle, \gamma) = \frac{1}{\sqrt{1 + (2\bar{n} + 1)\gamma + \gamma^2/4}}. \tag{36}
\]

Figure 1 depicts the fidelity of various states discussed in this section, as functions of the channel noise \(\gamma\).

8. Example of mixed-state fidelities for a Gaussian channel

Because of the many experimentally uncontrollable properties of the input states to a quantum channel, in reality we have to deal with sources described by a mixed state or, in some cases, by a particular statistical ensemble of incoming pure states. In Sec. 4.2 we introduced the entanglement fidelity and the mean ensemble fidelity as fidelity measures to characterize these situations. We note that since the mean ensemble fidelity \[15\] is an average of pure-state fidelities, it satisfies the scaling law \[17\] and the upper bound \[23\]. Moreover, since the entanglement fidelity is bounded above by the mean ensemble fidelity, we can write generally that

\[
F(\rho, \gamma) \leq \bar{F}(\{p_n, \Psi_n\}, \gamma) \leq \frac{1}{1 + \gamma/2} \tag{37}
\]

for any ensemble that corresponds to the input state \(\rho\).

In this section we illustrate the two mixed-state fidelities by considering an ensemble of number states, \(\{p_n, |n\rangle\}\), that are selected with the Bose-Einstein probabilities

\[
p_n = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}}. \tag{38}
\]

The corresponding density operator is the thermal density operator

\[
\rho = \sum_{n=0}^{\infty} p_n |n\rangle \langle n| = \frac{1}{1 + \bar{n}} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^{a^\dagger a}, \tag{39}
\]
where \( \bar{n} \) is the mean number of thermal quanta.

In this case we can apply the formula (29) to find the mean ensemble fidelity. A simple calculation gives

\[
\bar{F} = \frac{1}{\sqrt{1 + (2\bar{n} + 1)\gamma/2 - \bar{n}(1 + \bar{n})\gamma^2}} = \frac{1}{\sqrt{1 + (2\bar{n} + 1)\gamma + \gamma^2/4}},
\]

(40)

the same as the fidelity (36) for an input squeezed state with the same mean number of quanta. It is easy to verify that this ensemble fidelity satisfies the scaling law (17). The entanglement fidelity follows from inserting the Weyl function for a thermal state, \( C_\rho(\alpha) = e^{-|\alpha|^2/2}\gamma}, into Eq. (14), which gives

\[
F(\rho, \gamma) = \frac{1}{1 + (2\bar{n} + 1)\gamma/2}.
\]

(41)

It is trivial to see in this case that \( F(\rho, \gamma) \leq \bar{F} \), with equality holding if and only if \( \bar{n} = 0 \) or \( \gamma = 0 \). In Fig. 2 we have plotted, as functions of the channel noise \( \gamma \), the mean ensemble fidelity and the entanglement fidelity for an ensemble with Bose-Einstein statistics.
Fig. 2: Plots of the entanglement fidelity (lower curve) and the mean ensemble fidelity (middle curve) as functions of $\gamma$ for $\bar{n} = 1$. For reference, the upper curve gives the fidelity of a coherent state.

9. Physical examples of Gaussian channels

9.1. Cloning

As an initial example of a Gaussian channel, we now investigate a very simple experimental setup used for approximate cloning of the states of a field mode. The no-cloning theorem [20][21][22] shows that a universal and faithful cloning machine, which would clone an arbitrary input quantum state perfectly, is incompatible with quantum mechanics. It is possible, however, to find imperfect cloning machines that copy quantum states with some loss of quantum fidelity. The simplest device designed to clone quantum states is a 50:50 beam splitter that is preceded by an amplifier with amplitude gain of $\sqrt{2}$ to compensate for the reduction in signal amplitude at the beam splitter. This setup is depicted in Fig. 3.

The input (pure) state of mode $A$ is cloned into two outgoing clones in modes $C$ and $D$. The setup of this cloning device requires only linear optics and linear amplification. As a result, the annihilation operators for the outgoing modes of
Fig. 3: Cloning experimental setup with an amplifier of amplitude gain $\sqrt{2}$. BS denotes a 50:50 beam splitter. The annihilation operators of the input mode, the vacuum mode, and the modes that carry the clones are indicated.

the clones are $c = \frac{1}{\sqrt{2}}(\sqrt{2}a + a_\text{amp}^\dagger + v) = a + \frac{1}{\sqrt{2}}(a_\text{amp}^\dagger + v)$, $d = \frac{1}{\sqrt{2}}(\sqrt{2}a + a_\text{amp}^\dagger - v) = a + \frac{1}{\sqrt{2}}(a_\text{amp}^\dagger - v)$, (42)

where $a_\text{amp}^\dagger$ is a vacuum-noise creation operator that describes the noise introduced by the amplifier, and $v$ is the annihilation operator for the vacuum-mode $V$ that is incident on the unused port of the beam splitter. The whole dynamics of the cloning process, including amplification, is described by a unitary transformation in the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_\text{amp} \otimes \mathcal{H}_V$ of the input mode $A$, the amplifier-noise mode, and the vacuum mode $V$. As a result of this transformation, the reduced dynamics of such a cloning device corresponds to a Gaussian channel with $\gamma = 1$.

For all initial pure states, the cloning fidelity is given by

$$F_{\text{clone}}(\Psi) = F(\Psi, \gamma = 1) = 2F(\Psi, 4),$$

(43)

where the latter equality is a consequence of the scaling law [17]. The general fidelity bound [23] for a Gaussian channel gives us a bound on the cloning fidelity, $F_{\text{clone}} \leq F_{\text{max}}(\gamma = 1) = \frac{2}{3}$, with equality achieved only for cloning of coherent states.
9.2. Continuous-variable teleportation

As a second example, we consider a continuous-variable quantum teleportation protocol, which is effectively equivalent to a Gaussian channel [24]. Quantum teleportation is a process that can transfer an arbitrary quantum state from a system held by one party, usually called Alice, to a system held by a second party, usually called Bob [25, 26, 27]. The process requires a pair of systems, shared by Alice and Bob, in an entangled state—the entangled resource—and an amount of classical information transmitted from Alice to Bob.

The entangled resource is ideally a pure two-mode squeezed state [19] of two modes, \( A \) and \( B \), which have annihilation operators \( a \) and \( b \). Such states can be described by a Gaussian Wigner function, \( W_{AB}(\alpha, \beta) \), that is specified by the following nonvanishing second moments:

\[
\begin{align*}
    n + \frac{1}{2} &= \int d^2 \alpha \, d^2 \beta \, |\alpha|^2 W_{AB}(\alpha, \beta) = \int d^2 \alpha \, d^2 \beta |\beta|^2 W_{A,B}(\alpha, \beta), \\
    m &= \int d^2 \alpha \, d^2 \beta \, \alpha \beta W_{AB}(\alpha, \beta) = \int d^2 \alpha \, d^2 \beta \, \alpha^* \beta^* W_{AB}(\alpha, \beta). \quad (44)
\end{align*}
\]

The moments must satisfy \( n \geq 0 \) and \( \sqrt{n(n+1)} \geq |m| \) to ensure that the Wigner distribution corresponds to a valid quantum state. The state is pure if and only if \( \sqrt{n(n+1)} = |m| \), in which case it is a two-mode squeezed state. When the state is pure, the limit \( |m| \to \infty \) gives the original entangled state of Einstein, Podolsky, and Rosen [28], with \( m \) negative leading to a Wigner function proportional to \( \delta(q_A + q_B)\delta(p_A - p_B) \), which are the ideal correlations for the teleportation protocol we are considering, and \( m \) positive leading to a Wigner function proportional to \( \delta(q_A - q_B)\delta(p_A + p_B) \). The correlated state of Eq. (44) is separable (unentangled) if and only if \( n \geq |m| \) [29].

The state to be teleported is an input pure state \( \rho \) of a mode \( V \) in Alice’s possession, which has annihilation operator \( v \). The protocol consists of (i) Alice’s measuring the two (commuting) homodyne quadratures contained in the Hermitian real and imaginary parts of the operator \( v + a^\dagger \); (ii) Alice’s communicating to Bob the (complex) result \( \xi \) of this measurement; and (iii) Bob’s displacing the complex amplitude of his mode \( B \) by \( \xi \).

The efficacy of the protocol is quantified by the fidelity between the output state of mode \( B \) and the input state \( |\psi\rangle \), averaged over the possible measurement results. The teleportation protocol involves three modes in an extended Hilbert space \( \mathcal{H}_V \otimes \mathcal{H}_A \otimes \mathcal{H}_B \). Reduction of this protocol to the Hilbert space of the incoming mode reduces the teleportation protocol to a Gaussian channel with quantum fidelity given by Eq. (44), where the Gaussian noise distribution is calculated from the following relation:

\[
    G(\nu) = \int d^2 \alpha \, d^2 \beta \, \delta(\beta + \alpha^* - \nu) W_{AB}(\alpha, \beta). \quad (45)
\]
Simple calculation involving the Wigner function of the entangled resource leads to a Gaussian distribution for $G(\nu)$, with the noise parameter given by $\gamma = 2[1+2(n+m)]$. A separable resource has $n \geq |m|$ and, accordingly, $\gamma \geq 2$. For $\sqrt{n(n+1)} \geq m \geq n$, the correlated state is entangled, but with the wrong sort of correlations for the protocol we are considering, so $\gamma \geq 2$. For $\sqrt{n(n+1)} \geq -m \geq n$, the correlated state is entangled and $0 \leq \gamma \leq 2$. Perfect teleportation is achieved if $\gamma = 0$, which corresponds to $m = -\sqrt{n(n+1)} \rightarrow -\infty$ ($m + n = -1/2$), i.e., to a pure entangled state with perfect EPR correlations. For pure input states, we conclude that the maximum teleportation fidelity for a given entangled resource $\gamma$ is

$$F_{\text{max}}(\gamma) = \frac{1}{1 + \gamma/2} = \frac{1}{2(1 + n + m)}$$

and is achieved if and only if the input state is a coherent state.

We have used the fidelity bound [24] to get two other interesting results [24]. The first is that the maximum fidelity for teleporting a coherent state using the standard protocol, but with any separable state for modes $A$ and $B$ [not necessarily a state of the Gaussian form specified by the moments (44)], is $1/2$. The second result has to do with local hidden-variable models for continuous-variable teleportation. The teleportation of any Gaussian input state can be described within a local hidden-variable model, no matter what fidelity is achieved in the teleportation; the hidden-variable model is based on the classical phase-space variables of the Wigner distribution. For non-Gaussian pure input states, we have shown that the value $\gamma = 1$ plays a special role: each non-Gaussian pure input state $|\Psi\rangle$ has its own threshold fidelity, $F(\Psi, 1) < F_{\text{max}}(\gamma = 1) = 2/3$, below which its teleportation can be accommodated within an extended phase-space hidden-variable model and above which it cannot.

10. Conclusion

In this paper we follow Hall [8] in defining a Gaussian channel as one in which an input mode is subjected to random Gaussian displacements in phase space. Such channels arise naturally whenever field modes undergo linear optical transformations, linear amplification, and measurements of quadrature components, provided that the overall channel preserves the mean complex amplitude of the input mode. We introduce the quantum fidelity for both pure and mixed inputs and derive the maximum fidelity that can be achieved over all input states. This bound—and related ones that might come from using similar theoretical techniques—should prove useful in analyzing the performance of Gaussian channels.
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