Equilibrium Spin Distribution From Detailed Balance

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Abstract As the core ingredient for spin polarization, the equilibrium spin distribution function that eliminates the collision terms is derived from the detailed balance principle. The kinetic theory for interacting fermionic systems is applied to the Nambu–Jona-Lasinio model at quark level. Under the semi-classical expansion with respect to $\hbar$, the kinetic equations for the vector and axial-vector distribution functions are obtained with collision terms. For an initially unpolarized system, spin polarization can be generated at the first order of $\hbar$ from the coupling between the vector and axial-vector charges. Different from the classical transport theory, the collision terms in a quantum theory vanish only in global equilibrium with Killing condition.

1 Introduction

The single-particle distribution function is of fundamental significance in off-equilibrium kinetic theory as well as many-body physics in equilibrium state. It has long been well known that for a system consisting of identical particles in thermodynamic equilibrium, the average number of particles in a single-particle state is described by Boltzmann distribution for non-relativistic system, Bose-Einstein distribution for bosons, and Fermi-Dirac distribution for fermions. However, in general, when spin of fermions is an independent degree of freedom, the distribution for spin-1/2 particles needs to be extended to describe the thermodynamical equilibrium of spin degrees of freedom [1]. By analyzing the density matrix for spin-1/2 particles, it is found that the non-even population of the polarization states arises from a steady gradient of temperature, and is orthogonal to particle momentum [1]. The same equilibrium distribution is also derived in Refs. [2,3] by analyzing the free streaming spin transport equation. On the experimental side, the spin polarization effect in heavy ion collisions has attracted intense attention [4–7]. A large global angular momentum is produced in non-central heavy ion collisions and the spin of hadrons emitted is aligned with the direction of the global angular momentum [8–10]. The magnitude of the global polarization of $\Lambda$ baryons can be very well described by models based on relativistic hydrodynamics with thermodynamic equilibrium of spin degrees of freedom [11–16]. The distribution function of a system of spin-1/2 particles is thus not only of significant importance for theoretical interest, but also required to explain the experimental data. As a matter of fact, different forms of equilibrium distribution functions are proposed based on different arguments. The most optimal situation would be to derive an equilibrium form from the entropy production [17] or the collision terms for particles with spin. In a classical kinetic theory, as one of the basic requirements, the local equilibrium is defined by means of detailed balance, namely the vanishing of the collision kernel in the Boltzmann equation [18].

In this work, we investigate the detailed balance principle in a quantum kinetic theory. For spin transport at the leading order in $\hbar$, the vanishing of the collision terms together with the Killing condition lead to global equilibrium spin distribution function. Different from the classical kinetic theory where the collision terms are eliminated by the local equilibrium distribution, in a quantum kinetic theory, the collision terms vanish only in global equilibrium.

The spin related anomalous transport phenomena in heavy ion collisions, such as chiral magnetic effect [19,20] and chiral vortical effect [21], call for the spin related transport theory and hydrodynamic theory. The chiral kinetic theory [22–31] is developed to describe these anomalous transport of massless fermions and is further extended to the spin transport of massive fermions [32–36]. Recently, it is also extended from the free streaming scenario to including collisional effects [37–42]. The general framework of spin transport with collision terms is derived based on the Keldysh formalism [38]. This framework is then applied to the weakly coupled quark-gluon plasma at high temperature to compute
the spin-diffusion term for massive quarks up to the leading logarithmic order \[38\] and weakly coupled quantum electrodynamics plasma \[42\]. In this work, we investigate the collision terms in spin transport theory in the framework developed in Ref. \[38\]. In order to include fermionic 2-by-2 scattering, we consider the interaction among fermions by adopting the Nambu–Jona-Lasinio (NJL) model and calculate the collisional self-energy by taking semi-classical (\(\hbar\)) expansion and non-perturbative (1/\(N_c\)) expansion \[43\]. For massive fermions, spin is an independent degree of freedom, we take vector and axial-vector components of the Wigner function as independent degrees of freedom and derive their kinetic equations at classical level and the leading order in \(\hbar\). The vector and axial-vector components in equilibrium state are derived by requiring the detailed balance of the kinetic equations. Since our goal is to derive the spin distribution in \(\hbar\) order of \(\hbar\), we have dropped the real parts of the retarded and advanced self-energies and of the retarded propagators. The star product of two functions \(A(q, X)\) and \(B(q, X)\) is generated from the Wigner transformation and stands for the shorthand notation of the following calculations

\[
A \star B = AB + \frac{i\hbar}{2} [AB]_{PB} + O(\hbar^2),
\]

where the Poisson bracket is \([AB]_{PB} = (\partial_\mu A)(\partial_\mu B) - (\partial_\mu A)(\partial_\mu B)\). The commutators are \([F, G] = FG - GF\), \([F, G] = FG - GF\), \([F, G] = FGF + GF\), and \([F, G] = FGF + GF\) for arbitrary matrix-valued functions.

Different Dirac components of the Wigner function have different physical meanings. Performing the spin decomposition of the Wigner function, one obtains various components,

\[
S^< = S + i\gamma^5 \gamma^\mu + \partial_\mu \gamma^\mu + \frac{1}{2} \sigma^{\mu\nu} D_{\mu\nu},
\]

\[
S^> = \bar{S} + i\gamma^5 \gamma^\mu + \bar{\partial}_\mu \gamma^\mu + \bar{\sigma}^{\mu\nu} D_{\mu\nu},
\]

with \(\sigma^{\mu\nu} = i(\gamma^\mu, \gamma^\nu)/2\) and \(\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\). Similarly, the collisions terms in (2) are also decomposed by the Clifford algebra,

\[
C = [\Sigma^<, S^>], \quad [\Sigma^>, S^<],
\]

\[
= C_S + i\gamma^5 C_\rho + \gamma^\mu C_\rho \mu + \gamma^5 \gamma^\mu C_\mu + \frac{1}{2} \sigma^{\mu\nu} C_{T_{\mu\nu}},
\]

\[
D = [\Sigma^<, S^>], \quad [\Sigma^>, S^<],
\]

\[
= D_S + i\gamma^5 D_\rho + \gamma^\mu D_\rho \mu + \gamma^5 \gamma^\mu D_\mu + \frac{1}{2} \sigma^{\mu\nu} D_{T_{\mu\nu}}.
\]

Note that \(C\) and \(D\) contain both the loss and gain terms, they can be recognized as \(I^<_{\text{gain}} = [\Sigma^<, S^>]\), \(I^>_{\text{gain}} = [\Sigma^>, S^<]\), \(I^a_{\text{gain}} = [\Sigma^<, S^>]\) and \(I^a_{\text{gain}} = [\Sigma^>, S^<]\), with \(c\) and \(a\) denoting commutator and anti-commutator respectively. Since \(\Sigma\) and \(S\) are both \(4 \times 4\) matrices, their multiplication is non-commutative. The same spin decomposition for the self-energies is required to further derive the constraint and transport equa-
tions for the spin components,

\[ \Sigma^+ = \Sigma_S + i \Sigma_P \gamma^5 + \Sigma_{\mu\nu} \gamma^\mu \gamma^\nu + \frac{1}{2} \Sigma_{T\mu\nu} \sigma^{\mu\nu}, \]

\[ \Sigma^- = \Sigma_S + i \Sigma_P \gamma^5 + \Sigma_{\mu\nu} \gamma^\mu \gamma^\nu + \frac{1}{2} \Sigma_{T\mu\nu} \sigma^{\mu\nu}. \]

From the sum and difference of the Kadanoff–Baym equations (2) as well as the decomposition of the Wigner functions (4) and of the collision terms (5), one derives ten equations for the components,

\[ p_\mu \gamma^\mu - m S = \frac{i \hbar}{4} C_S, \]

\[ 2 m p - h \nabla_S A^\mu = - \frac{i \hbar}{2} C_P, \]

\[ 2 p_\gamma S - 2 m \nabla^\gamma S_{\mu\nu} = \frac{i \hbar}{2} C_{\mu\nu}, \]

\[ 2 \hbar \nabla_\mu \nabla_\nu S_{\mu\nu} = \frac{i \hbar}{2} C_{\mu\nu}, \]

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The classical components are on the mass shell \((p^2 - m^2) \gamma_{\mu} = 0\) and \((p^2 - m^2) A_{\mu} = 0\). Their transport equations are

\[ (p \cdot \nabla) \gamma_{\mu} = \frac{m}{2} D_{\mu} + \frac{i}{2} p^\nu C_{\mu\nu}^T, \]

\[ (p \cdot \nabla) A_{\mu} = \frac{m}{2} D_{\mu} - \frac{i}{2} p^\nu C_{\mu\nu}^T. \]

The four components of \(V_{\mu}\) and \(A_{\mu}\) are not all independent. The constraints \(p_\mu A_{\mu} = 0\) and \(p_\mu \gamma_{\mu} = 0\) indicate that \(V_{\mu}\) has three independent components and \(A_{\mu}\) has only one independent component. Considering the similar restrictions at \(O(h)\), there are the same number of independent components for \(A_{(1)}\) and \(V_{(1)}\). In order to keep the description covariant and symmetric, we derive in the following the transport equations for \(V_{\mu}\) and \(A_{\mu}\), but keep in mind that \(V_{\mu}\) and \(A_{\mu}\) has redundant components and that the system has 4 independent degrees of freedom in total: one is number density and the other three are spin density.

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\[ (p \cdot \nabla) A_{\mu} = \frac{m}{2} D_{\mu} - \frac{i}{2} p^\nu C_{\mu\nu}^T. \]
$p_{\mu}V_{\nu}^{(1)} = -\frac{1}{2m}\epsilon_{\mu\nu\lambda\beta}p^\alpha\Sigma^{(0)}_A A^{(0)\beta} + \frac{1}{2}\epsilon_{\mu\nu\lambda\beta}\nabla^\alpha A^{(0)\beta}$

$$-\frac{1}{2}\epsilon_{\mu\nu\lambda\beta}\Sigma^{(0)}_V A^{(0)\beta} - \frac{1}{2m}p_{\mu}\Sigma^{(0)\beta}_p A^{(0)\nu},$$

$$p_{\mu}A^{(1)\mu} = -\frac{1}{2m}p^\nu\Sigma^{(0)\nu}_p V^{(0)} - \frac{1}{2m}p^\nu\Sigma^{(0)\nu}_T p^{(0)\nu} A^{(0)\nu}. \quad (13)$$

Since the right hand side of the constraint and transport equations (14) and (13) contain only the $O(\hbar^0)$ components, $V^{(1)}_\mu$ contains only one independent component representing the first order correction to the number density, and $A^{(1)\mu}$ contains only three independent components representing the first order correction to the spin density. The on-shell relations become

$$(p^2 - m^2)\gamma^{(1)}_\mu = m^2 - \frac{1}{2}\Sigma^0\mu + \frac{m}{2}\Sigma T\nu V^\nu$$

$$+ \frac{1}{2}\epsilon_{\nu\lambda\beta\mu}\Sigma^{(0)\beta}_V A^{(0)\nu},$$

$$(p^2 - m^2)A^{(1)\mu} = -\frac{1}{2m}p^\nu\Sigma^0\nu V^\nu + \frac{1}{2}\Sigma^0\mu V^\mu$$

$$+ \epsilon_{\mu\beta\nu\lambda}p^\alpha\Sigma^{(0)\beta}_A A^{(0)\nu}. \quad (14)$$

with components of self-energy and Wigner function at order $O(\hbar^0)$. The transport equations for the first order components $V^{(1)}_\mu$ and $A^{(1)\mu}$ are

$$(p \cdot \nabla)\gamma^{(1)}_\mu = -m\Sigma^{(0)}_S \gamma^{(1)}_\mu - p^\nu\Sigma^{(0)\nu}_V V^\nu$$

$$+ p_{\mu}\Sigma^{(0)\nu}_A A^{(0)\nu} + \frac{p^\nu}{m}\epsilon_{\rho\sigma\nu\mu}p^\rho\Sigma^{(0)\nu}_T A^{(0)\sigma}$$

$$- \frac{m}{2}\epsilon_{\sigma\nu\lambda\beta}\Sigma^{(0)\lambda}_T A^{(0)\nu},$$

$$- m\Sigma^{(1)\nu}_S \gamma^{(0)}_\mu - p^\nu\Sigma^{(0)\nu}_V V^\nu + p_{\mu}\Sigma^{(1)\nu}_A A^{(0)\nu}$$

$$+ \frac{p^\nu}{m}\epsilon_{\rho\sigma\nu\mu}p^\rho\Sigma^{(1)\nu}_T A^{(0)\sigma},$$

$$- \frac{m}{2}\epsilon_{\sigma\nu\lambda\beta}\Sigma^{(1)\lambda}_T A^{(0)\nu},$$

$$- \frac{1}{2m}p^\nu\Sigma^{(0)\nu}_T (p^\nu A^{(0)\nu}) \text{P.B.}$$

$$+ \frac{m}{2}\Sigma^{(0)\nu}_T V^\nu \text{P.B.} - \frac{1}{2}\epsilon_{\nu\lambda\beta\mu}\Sigma^{(0)\beta}_V A^{(0)\nu},$$

$$- \frac{1}{2}\epsilon_{\nu\lambda\beta\mu}\Sigma^{(0)\beta}_V A^{(0)\nu} - \frac{1}{2m}p_{\mu}(\nabla^\nu \Sigma^{(0)\nu}_V) A^{(0)\nu},$$

$$+ \frac{1}{2m}p^\nu\Sigma^{(0)\nu}_V A^{(0)\nu} - \frac{1}{2m}p^\nu\Sigma^{(0)\nu}_T A^{(0)\nu},$$

$$- \frac{1}{2}\epsilon_{\nu\lambda\beta\mu}\Sigma^{(0)\beta}_V A^{(0)\nu} \quad (15)$$

and

$$(p \cdot \nabla)A^{(1)}_\mu = -m\Sigma^{(0)}_S A^{(1)}_\mu - p^\nu\Sigma^{(0)}_V A^{(1)}_\nu$$

$$- p^\nu\Sigma^{(1)}_A A^{(1)}_\nu - \frac{1}{2}\epsilon_{\nu\lambda\beta\mu}\Sigma^{(0)\beta}_T A^{(0)\nu} \quad (16)$$

The first two lines in equations (15) and (16) are dynamical effects which contain for instance the diffusion effect. These terms have the same structure as the collision terms in the classical limit (11). The last three lines in both transport equations are related to the derivatives of self-energies and distribution functions, which are inhomogeneous effects. As we will see in the following, these inhomogeneous effects produce spin polarization from the thermal vorticity.

### 3 Fermionic 2 by 2 scattering

In this paper, we focus on deriving the equilibrium distribution from the detailed balance principle. To this end, the interaction needs to be specified to obtain the explicit expression of the off-diagonal self-energies $\Sigma^<$ and $\Sigma^>$. Considering the fact that, while the process from non-equilibrium to equilibrium depends strongly on the interaction among particles of the system, the equilibrium distribution itself should be independent of the details of the interaction. Therefore, we adopt a NJL-type model with only scalar interaction and calculate the fermionic 2 by 2 scattering.

$$\mathcal{L} = \bar{\psi} (i\hbar \partial_\mu - m) \psi + G(\bar{\psi} \psi)^2. \quad (17)$$

In general a large part of the light fermion mass comes from the chiral condensate. To simplify the calculation, we work here in the chiral restored phase and consider only the current mass. Due to the nature of the strong coupling, we take two
expansions, one is in the inverse number of colors \(1/N_c\) and the other in \(h\). Directly translating from the diagrams [43] and performing the Wigner transformation, the self-energies to the leading order (LO) and next to the leading order (NL) of the \(1/N_c\) expansion can be explicitly expressed as

\[
\Sigma_{\text{LO}}(X, p) = G^2 \int dP \ S^c(X, p_1) \text{Tr} \left[ S^c(X, p_2) S^c(X, p_3) \right],
\]

\[
\Sigma_{\text{NL}}(X, p) = -G^2 \int dP \ S^c(X, p_1) S^\gamma(X, p_2) S^c(X, p_3)
\]

(18)

with \(\int dP = \int \frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi)^6} \delta(p - p_1 + p_2 - p_3)\) for the momentum integral. The lesser self-energy \(\Sigma_{\text{LO}}^c\) and \(\Sigma_{\text{NL}}^c\) can be obtained by taking the exchange \(S^c \leftrightarrow S^\gamma\) in (18). As clarified in [43], the self-energies \(\Sigma_{\text{LO}}^c\) and \(\Sigma_{\text{NL}}^c\) correspond to different scattering channels. Since the detailed balance requires that the gain term and the loss term cancel with each other in arbitrary collision channel, we consider, to simplify the calculation, only the collisional self-energy at leading order.

The spin decomposition of \(\Sigma_{\text{LO}}\) follows simply from that of \(S^c(X, p_1)\), since the factor \(\text{Tr}(S^c(X, p_2)S^c(X, p_3)) = S^2S^3 + p^2p^3 + \gamma_-^{\mu}V^{\mu\nu} - A^2_{\mu}A^3_{\nu} + \frac{1}{2}S^3_{\mu\nu}S_{\mu\nu}\) is a number. The self-energy \(\Sigma_{\text{LO}}\) can be decomposed as \(\Sigma_{\text{LO}} = G^2 \int dP \ \text{Tr}(S^c(X, p_2)S^c(X, p_3)) S^c(X, p_1)\). For instance, \(\Sigma_S\) corresponds to \(S\), \(\Sigma_{V_{\mu}}\) to \(V_{\mu}\), and \(\Sigma_{A_{\mu}}\) to \(A_{\mu}\).

3.1 Classical limit

Taking the spin components of the self-energy, one obtains the transport equations for \(V^{(0)}_{\mu}\) and \(A^{(0)}_{\mu}\) including collision terms. Considering the relation between \(S^{(0)}\) and \(V^{(0)}\) and the fact that \(V^{(0)}_{\mu}\) can be decomposed to \(V^{(0)}_{\mu} \propto \delta(p^2 - m^2) \mu \nu f_V\), it would be convenient to derive the collision terms in transport equations for the vector charge distribution \(f_V\) and the axial-vector charge distribution \(f_A\) from the following two equations,

\[
p \cdot \nabla S^{(0)} = -G^2 \int \left[ \left( 1 + \frac{p_2 \cdot p_3}{m^2} \right) (S^2S^3 - A^2_{\mu}A^3_{\mu}) + \frac{p_\mu p_\nu}{m^2} A^2_{\mu}A^3_{\nu} \right] \left( m + \frac{p_1 \cdot p_1}{m} \right) (S^2S^3).
\]

\[
p \cdot \nabla A^{(0)} = -G^2 \int \left[ \left( 1 + \frac{p_2 \cdot p_3}{m^2} \right) (S^2S^3) \right.
\]

(19)

\[
\left. - \frac{p_\mu p_\nu}{m^2} A^2_{\mu}A^3_{\nu} \right] \left[ \left( m + \frac{p_1 \cdot p_1}{m} \right) (S^2S^3) - \frac{p_\mu p_\nu}{m^2} A^2_{\mu}A^3_{\nu} \right].
\]

Note that all the components on the right hand side are at leading order of \(h\). Before moving on to analyzing the scattering channels, we first recall the classical free fermion solution of the Wigner function [35]. From the definition of the Wigner function as well as the contour green’s function, the classical Wigner function for a free fermion system is given by

\[
S^c(X, p) = \frac{\delta(p^2 - m^2)}{(2\pi)^3} \left\{ \begin{array}{ll}
\theta(p^0) \tilde{u}_s(p) u_r(p) f^r_{q}(X, p) & \text{if } s = + \text{ and } r = +, \\
\theta(-p^0) \tilde{u}_s(p) u_r(p) f^r_{q}(X, p) & \text{if } s = + \text{ and } r = -,
\end{array} \right.
\]

(21)

where \(s, r = \pm 1\) denote the spin up and down along the direction set by the unit vector \(n^{\pm\mu}(X, p) = \delta_{sr} - f^r_{q}(X, p)\) and \(f^r_{q}(X, -p) = \delta_{sr} - f^r_{q}(X, -p)\) can be obtained from the ensemble average of the creation and annihilation operators, and \(f^r_{q}\) is an element of a 2 \(\times 2\) Hermitian matrix which can be diagonalized to give \(f^r_{q}\). One can easily find that \(\tilde{f}^{sr}_{q}\) gives \(1 - f^r_{q}\) after diagonalization. The mean polarization vector is defined as \(n^{\pm\mu}(X, p) = \theta(p^0)n^{\pm\mu}(X, p) - \theta(-p^0)n^{-\mu}(X, -p)\) with

\[
n^{\pm\mu}(X, p) = \pm \left( \frac{n^+ \cdot p}{m}, n^+ + \frac{n^+ \cdot p}{m} \right),
\]

(22)

where \(n^\pm\) is the direction of the mean polarization for particles (+) and anti-particles (-) satisfying \(n^+ \cdot n^- = -1\), and the momentum \(p\) is measured in the Particle Rest Frame. The mean polarization is a unit time-like vector satisfying \(n^{(0)}_{\mu}(X, p)n^{(0)}_{\mu}(X, p) = -1\). The spin decomposition of \(S^c\) can be obtained by simply taking the exchange \(f_q \leftrightarrow \tilde{f}_q\) and \(f^r_{q} \leftrightarrow f^r_{q}\). Note that \(f^r_{q}\) is the particle distribution parallel (\(s = +\)) and anti-parallel (\(s = -\)) to the unit vector \(n^{\pm\mu}\). The vector charge distribution \(f_V\) and axial charge distribution \(f_A\) are combinations of \(f^+_q, f_V = f^+_q + f^-_q\) and \(f_A = f^+_q - f^-_q\). The magnitude of polarization can be defined through the positive quantity \(\zeta_{q/q}(X, p) = f_{q/q}(X, p)\text{e}(X, p)\) and \(f^r_{q} = f_{q/q}(1 + s\zeta_{q/q}(X, p))\). If \(\zeta_{q/q}(X, p) = 1\) corresponds to a pure state, while \(\zeta_{q/q}(X, p) < 1\) describes a mixed state [45]. It is worth noticing that, for outgoing particles there is \(\tilde{f}^r_q = 1 - f^r_q\) which leads to \(f_V = 1 - f_V\) and \(f_A = -f_A\). This can be understood from the fact that, the axial distribution function \(f_A\) comes from the off-diagonal
component of $f^{sr}$, and the relation $\tilde{f}^{sr} = \delta_{rs} - f^{sr}$ results in $\tilde{f}_A = -f_A$. With $f_V$ and $f_A$, the classical components can be rewritten as
\begin{equation}
\begin{split}
\mathcal{V}^{(0)}(X, p) &= \frac{2p_\mu}{(2\pi)^3 2E_p} \left\{ \delta(p^0 - E_p) f_{Vq}(X, p) \\
+ &\delta(p^0 + E_p) f_{\bar{V}i\bar{q}}(X, -p) \right\}, \\
\mathcal{A}^{(0)}_\mu(X, p) &= \frac{2m}{(2\pi)^3 2E_p} \left\{ \delta(p^0 - E_p) n^{+}_\mu(X, p) f_{Aq}(X, p) \\
- &\delta(p^0 + E_p) n^{-}_\mu(X, -p) f_{A\bar{q}}(X, -p) \right\}.
\end{split}
\end{equation}

The similar relations for $\tilde{\mathcal{V}}^{(0)}$ and $\tilde{\mathcal{A}}^{(0)}_\mu$ can be obtained by taking the exchange $f_{Vq}/f_{\bar{V}i\bar{q}} \leftrightarrow f_{\bar{V}i\bar{q}}/f_{Vq}$ and $f_{Aq}/f_{A\bar{q}} \leftrightarrow f_{A\bar{q}}/f_{Aq}$.

Let’s consider the particle sector of the transport equations for $\mathcal{V}^{(0)}_\mu$ and $\mathcal{A}^{(0)}_\mu$, namely the terms with the delta function $\delta(p^0 - E_p)$. While both particles and anti-particles exist in $S^<(p_2)$, $S^>(p_3)$ and $S^<(p_1)$, corresponding to different scattering processes, only three of the eight channels are allowed by the energy-momentum conservation. Each channel contains a product of four quark and anti-quark distribution functions $f_{Vq}$ and $f_{\bar{V}i\bar{q}}$. One may attribute a diagram to each of these processes in a loose sense, by assigning $f_{Vq}$ to an incoming quark, $f_{\bar{V}i\bar{q}}$ to an incoming anti-quark, $f_{V\bar{q}}$ to a outgoing quark, and $f_{\bar{V}i\bar{q}}$ to an outgoing anti-quark. The allowed channels correspond to the quark–quark scattering and quark–antiquark scattering. The other channels involving particle and antiparticle creation and annihilation can be categorized as off-shell processes. Together with the gain term, one can obtain the transport equation for the vector charge. In the following, when considering detailed balance, we only focus on the first channel, namely the quark–quark scattering.

In kinetic theory, the local equilibrium state is specified by the distribution functions that eliminate the collision kernel. This implies that the distribution functions must depend only on the linear combination of the conserved quantities, namely the particle number, energy and momentum, and angular momentum.

We first focus on the collision terms for $p \cdot \nabla S^{(0)}$ in equation (19). The collision terms include three parts: the term with only vector charge distribution $f_V$, the term with only axial charge distribution $f_A$, and the term with both $f_V$ and $f_A$. The term involving only $f_V$ is

\begin{equation}
m(m^2 + p_2 \cdot p_3)(m^2 + p \cdot p_1) \left( f^{p_0}_V f^{p_s}_V f^{p_2}_V f^{p_3}_V - f^{p_0}_V f^{p_s}_V f^{p_3}_V f^{p_2}_V \right),
\end{equation}

where we have neglected the subscript $q$ in $f_{Vq}$ and $f_{Aq}$. The detailed balance requires all the collision terms to vanish. This implies that the local equilibrium distribution $f_V$ is the Fermi-Dirac function. Requiring that $\mathcal{A}^{(0)}_\mu$ has the structure $\mathcal{A}^{(0)}_\mu = mn_{\mu} f_A$, one can easily show that the term containing only $A_{\mu}$ is

\begin{equation}
m[(p_2 \cdot n_3)(p_3 \cdot n_2)(p \cdot n_1)(p_1 \cdot n) \\
+ (m^2 + p_2 \cdot p_3)(m^2 + p \cdot p_1)(n_1 \cdot n_2)(n_2 \cdot n_3) \\
- (m^2 + p_2 \cdot p_3)(p \cdot n_1)(p_1 \cdot n)(n_3 \cdot n_2) \\
- (m^2 + p \cdot p_1)(n_1 \cdot n)(p_2 \cdot n_3)(p_3 \cdot n_2)] \\
\times (f^{p_0}_A f^{p_1}_A f^{p_2}_A f^{p_3}_A - f^{p_0}_A f^{p_1}_A f^{p_3}_A f^{p_2}_A).
\end{equation}

Considering $\tilde{f}_A = -f_A$, the detailed balance in local equilibrium state requires $\tilde{f}_A f^{p_1}_A f^{p_2}_A f^{p_3}_A - f^{p_0}_A f^{p_1}_A f^{p_3}_A f^{p_2}_A = 0$. This does not have any restriction on the equilibrium distribution function. Finally, the term involving mixture of $f_V$ and $f_A$ is

\begin{equation}
m(m^2 + p_2 \cdot p_3)((p \cdot n_1)(p_1 \cdot n) - (m^2 + p \cdot p_1)n_1 \cdot n) \times (f^{p_0}_V f^{p_1}_V f^{p_2}_V f^{p_3}_V - f^{p_0}_V f^{p_2}_V f^{p_1}_V f^{p_3}_V) \\
+ m(m^2 + p \cdot p_1)((p_2 \cdot n_3)(p_3 \cdot n_2) \\
- (m^2 + p_2 \cdot p_3)n_2 \cdot n_3) \times (f^{p_1}_V f^{p_2}_V f^{p_3}_V f^{p_0}_V - f^{p_1}_V f^{p_3}_V f^{p_2}_V f^{p_0}_V).
\end{equation}

A trivial solution of the detailed balance for this term is $f_A = 0$.

We then consider the collision terms for $p \cdot \nabla A^{(0)}_\mu$ in equation (19). The terms in the momentum integral can be simplified as

\begin{equation}
-m(m^2 + p_2 \cdot p_3)(m^2 + p_1 \cdot p)n_\mu f_A (f^{p_1}_V f^{p_2}_V f^{p_3}_V \\
+ f^{p_2}_V f^{p_3}_V f^{p_1}_V) \\
+ \left[ (m^2 + p_1 \cdot p)n_\mu - (p \cdot n_1)(p_\mu + p_1 \mu) \right] \\
\times m(m^2 + p_2 \cdot p_3) f_A (f^{p_0}_V f^{p_1}_V f^{p_2}_V + f^{p_3}_V f^{p_2}_V f^{p_1}_V) \\
- m(m^2 + p_1 \cdot p)n_\mu \\
\times \left[ (p_2 \cdot n_3)(p_3 \cdot n_2) - (m^2 + p_2 \cdot p_3)n_2 \cdot n_3 \right] f_A (f^{p_1}_V f^{p_2}_V f^{p_3}_V) \\
+ n_\mu \left[ (p_2 \cdot n_3)(p_3 \cdot n_2) - (m^2 + p_2 \cdot p_3)n_2 \cdot n_3 \right] f_A (f^{p_1}_V f^{p_2}_V f^{p_3}_V),
\end{equation}

where we have used the relations $\tilde{f}_V + f_V = 1$ and $\tilde{f}_A = -f_A$. Again the detailed balance is satisfied by the trivial equilibrium solution $f_A = 0$ at classical level. Since the couplings between spin and external background field, vorticity, and orbital angular momentum are all at first order, and the strong interaction under consideration does not introduce any preferable direction, there cannot be any physical mechanism that affects the rotational symmetry of the system as well as the spin at classical level. Therefore, if the system is not initially polarized, the spin distribution will keep zero during the evolution of the system.

Considering the relations between spin components of the Wigner function shown in equation (9), the above solution
of \( f_V \) and \( f_A \) in local equilibrium state leads to a nonzero scalar component \( S^{(0)} \neq 0 \) and a vanishing tensor component \( S^{(0)}_{\mu\nu} = 0 \). The pseudo-scalar component at order \( \mathcal{O}(h) \) is directly shown to be vanished by the constraint equations. Since the axial charge current appears at order \( \mathcal{O}(h) \), it is necessary to analyze the transport equations at the leading order in \( h \). In an initially unpolarized system, when considering the fermionic 2 by 2 collisions, the spin polarization can be produced as a quantum effect. The equilibrium spin polarization is then derived from the detailed balance of the collision terms in the transport equations at order \( \mathcal{O}(h) \).

3.2 Collision term at quantum level

The classical solution \( A^{(0)}_{\mu} = 0 \) in equilibrium state greatly simplifies the transport equations (15) and (16). With the NJL model one can further verify that the vanishing \( A^{(0)}_{\mu} \) leads to vanishing \( \Sigma^0_{\mu} \) and \( \Sigma^0_{\mu\nu} \). At the leading order in \( 1/N_c \) expansion, \( \Sigma^0_{\mu\nu} \) can also be shown to vanish. With these conditions, the vector component and axial-vector component are still on-shell at order \( \mathcal{O}(h) \). The transverse equations for \( \Sigma^0_{\mu} \) at order \( \mathcal{O}(h) \) are both at the order \( \mathcal{O}(h^0) \) and \( \mathcal{O}(h^0) \) as defined as \( S^{(1)}_{\nu} \). The components \( \Sigma^0_{\mu}, \Sigma^0_{\nu} \) and \( \Sigma^0_{\nu\mu} \) are all involved. At the leading order in \( 1/N_c \), taking \( \mathcal{S} \) and \( \mathcal{A}_{\mu} \) as independent components, and considering the relations between the spin components shown in (9), the components of the self-energy can be evaluated and are presented in Appendix B. With the known self-energy, the loss term on the right hand side of the transport equations (28) and (29) can be evaluated explicitly. For instance, the loss term in (28) is given by

\[
I^{(1)}_{V, \text{loss}} = G^2 p^\mu \int_{qk} \left[ \frac{m^2 + (k + q) \cdot k}{m^3} \left( \frac{m + p \cdot (p + q)}{m^2} \right) \right] \\
\times \left\{ S^{(0)}(k + q) \tilde{S}^{(0)}(k) \tilde{S}^{(0)}(p + q) S^{(1)}(p) + S^{(1)}(k + q) \tilde{S}^{(0)}(k) \tilde{S}^{(0)}(p + q) S^{(0)}(p) + S^{(0)}(k) \tilde{S}^{(1)}(k) \tilde{S}^{(0)}(p + q) S^{(0)}(p) + S^{(0)}(k) \tilde{S}^{(0)}(k) \tilde{S}^{(1)}(p + q) S^{(0)}(p) \right\}.
\]

The loss term on the right hand side of the transport equation (29) can be similarly expressed as

\[
I^{(1)}_{A, \text{loss}} = G^2 \int_{qk} \left\{ \left( 1 + \frac{(k + q) \cdot k}{m^2} \right) \left( \frac{m + p \cdot (p + q)}{m} \right) \right\} \\
\times \left\{ S^{(0)}(k + q) \tilde{S}^{(0)}(k) \tilde{S}^{(0)}(p + q) \tilde{A}^{(1)}_{\mu}(p) + \left( 1 + \frac{(k + q) \cdot k}{m^2} \right) \left( \frac{m + p \cdot (p + q)}{m} \right) S^{(0)}(k + q) \right\} \\
\times \left\{ \tilde{A}^{(1)}_{\mu}(p) + \frac{(k + q) \cdot k}{m^2} \right\} \tilde{S}^{(0)}(k) S^{(0)}(p) \tilde{A}^{(1)}_{\mu}(p) + \frac{(k + q) \cdot k}{m^2} \right\} \tilde{S}^{(0)}(k) S^{(0)}(p) \tilde{A}^{(1)}_{\mu}(p) \right\}.
\]

The substitution \( p_2 \rightarrow k + q \) and \( p_3 \rightarrow k \), and carrying out the integration over \( p_1 \) using the \( \delta \)-function, the zeroth- and first-order self-energy becomes

\[
\begin{align*}
\Sigma^{(0)}_{\nu}(p) &= G^2 \int_{qk} S^{(0)}(p + q) \text{Tr} \left\{ S^{<}(k + q) S^{>}(k) \right\}^{[0]}(p), \\
\Sigma^{(1)}_{\nu}(p) &= G^2 \int_{qk} S^{(1)}(p + q) \text{Tr} \left\{ S^{<}(k + q) S^{>}(k) \right\}^{[0]}(p) \\
&\quad + G^2 \int_{qk} S^{(0)}(p + q) \text{Tr} \left\{ S^{<}(k + q) S^{>}(k) \right\}^{(1)}(p) \\
&= \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} \left\{ S^{(0)}(k) \tilde{S}^{(0)}(k) \tilde{S}^{(0)}(p + q) \tilde{S}^{(0)}(p) \right\}.
\end{align*}
\]
The first order component \( A^{(1)}_\mu \) appears in the first three terms, and the last two terms contain only the classical scalar component \( S^{(0)} \) and its derivative. The appearance of the terms involving purely classical components indicates that the spin polarization \( A^{(1)}_\mu \) can be generated by collisions. Since such terms involve also spatial derivatives of \( S^{(0)} \), the spin polarization does not appear in homogeneous systems. When the system achieves local equilibrium, the detailed balance requires that the gain term and loss term cancel to each other. In the following, we show that the detailed balance in a quantum kinetic theory is fulfilled only in global equilibrium. In this sense the solution \( A^{(1)}_\mu \) that eliminates the collision term is

\[
A^{(1)}_\mu = \frac{1}{(2\pi)^3 E_p} \epsilon_{\mu\nu\rho\lambda} p^\nu \nabla^\sigma \beta^\lambda f'_\nu(p).
\]

\[\text{(36)}\]

For a system in non-equilibrium state, particles can have different momentum. When the system is in equilibrium state, however, all the particles are thermalized and have only one momentum scale (here the momentum \( p \)). Therefore, the second term in above solution which depends on two momenta \( p \) and \( q \) should vanish. This requires the Killing condition \( \nabla^\sigma \beta^\lambda + \nabla^\lambda \beta^\sigma = 0 \). This is to say that the detailed balance in a quantum kinetic theory is fulfilled only in global equilibrium. In this sense the solution \( A^{(1)}_\mu \) that eliminates the collision term is called as the global equilibrium distribution. The result here is consistent with the conclusion in previous studies on non-local collisions \[39\] and relaxation from spin chemical potential to thermal vorticity in global equilibrium \[46–49\]. The solution indicates that in an initially unpolarized system, non-zero spin polarization can be generated from the collision terms, especially the coupling between vector and axial-vector charges. Different from the classical transport theory where the collision terms can be eliminated by the local equilibrium distribution, for the spin transport at order \( O(h) \), the collision terms vanish only in global equilibrium.

In the above calculations we have used a simple NJL model with only scalar interaction. We believe that the equilibrium distribution should be independent of the interaction model, while the process from non-equilibrium to equilibrium depends strongly on the interaction itself. However, as far as we know, this is not yet strictly proven in quantum kinetic theory. We checked this problem in the NJL model and found that, 1) the above conclusions do not depend on the coupling constant \( G \), namely the strength of the interaction, and 2) with other interaction channels such as the pseudoscalar channel we obtain the same equilibrium distribution.

4 Angular momentum conservation

In the last section, we have verified that the distribution function in global equilibrium eliminates the collision terms, as
required by the detailed balance principle. For an initially un-polarized system, the spin can get polarized by collisions, indicating the conversion between orbital and spin angular momentum. In this section, we check the total angular momentum conservation of the system.

The energy-momentum tensor and spin tensor are related through the vector component and momentum conservation of the system.

Taking momentum integral over the last equation of (8) leads to the Wigner function through the vector component and

\[ T^{\mu\nu} = \int d^4 p \rho^{\mu\nu} p^\nu \]

\[ S^{\rho,\mu\nu} = -\frac{1}{2} \int d^4 p e^{\rho\mu\nu\lambda} A_\lambda, \]

and the total angular momentum contains the orbital part and spin part,

\[ M_{\rho,\mu\nu} = x_\mu T_{\rho\nu} - x_\nu T_{\rho\mu} + \hbar S_{\rho,\mu\nu}. \tag{38} \]

Taking momentum integral over the last equation of (8) leads to

\[ \partial^\rho M_{\rho,\mu\nu} = \frac{\hbar}{4} \int d^4 p \ D T_{\mu\nu}. \tag{39} \]

At classical level with \( \hbar = 0 \), there is clearly

\[ \partial^\rho M_{\rho,\mu\nu}^{(0)} = 0, \tag{40} \]

which means orbital angular momentum conservation. At first order in \( \hbar \), there is

\[ \partial^\rho M_{\rho,\mu\nu}^{(1)} = \frac{1}{4} \int d^4 p \ D T_{\mu\nu}^{(0)} \tag{41} \]

with the classical collision term

\[ D_{T_{\mu\nu}}^{(0)} = -2 \left[ \Sigma S_{\mu\nu} + \Sigma T_{\mu\nu} + S + \epsilon_{\mu\nu\alpha\beta} \left( \Sigma A^{\alpha\beta} - \Sigma A^{\alpha\beta} - \frac{1}{2} \Sigma A^{\alpha\beta} \right) \right]^{(0)}. \tag{42} \]

Considering the classical constraints \( P^{(0)}, A^{(0)}_\mu, S^{(0)}_{\mu\nu} = 0 \) and \( \Sigma^{(0)} T_{\mu\nu}^{(0)}, \Sigma^{(0)} A^{(0)} = 0 \), we have again the orbital angular momentum conservation,

\[ D_{T_{\mu\nu}}^{(0)} = 0, \tag{43} \]

\[ \partial^\rho M_{\rho,\mu\nu}^{(1)} = 0. \]

At second order in \( \hbar \), the transfer from orbital angular momentum to spin angular momentum starts. In this case, the collision term controlling the total angular momentum conservation becomes

\[ D_{T_{\mu\nu}}^{(1)} = -2 \left[ \Sigma S_{\mu\nu} + \Sigma T_{\mu\nu} + S + \epsilon_{\mu\nu\alpha\beta} \left( \Sigma A^{\alpha\beta} - \Sigma A^{\alpha\beta} - \frac{1}{2} \Sigma A^{\alpha\beta} \right) \right]^{(1)} + \left[ \Sigma V_{\mu}[V_{\nu}] \right]_{P.B.}. \tag{44} \]

Taking its momentum integration and considering the asymmetry under momentum exchange \( p \leftrightarrow p_1 \) and \( p_2 \leftrightarrow p_3 \), the two integrals of the above first-order components disappear,

\[ \frac{-2}{4} \int d^4 p \left[ \Sigma S_{\mu\nu} + \Sigma \epsilon_{\mu\nu\alpha\beta} A^{\alpha\beta} \right] \]

\[ = (\epsilon_{\mu\nu} \epsilon_{\alpha\beta}) \int d^4 p d^4 p_1 d^4 p_2 d^4 p_3 (m^2 + p_2 \cdot p_3) \]

\[ \times \delta(p - p_1 + p_2 - p_3) \delta(p^2 - m^2) \delta(p_1^2 - m^2) \]

\[ \times \delta(p_2^2 - m^2) \delta(p_2^2 - m^2) \times \left[ f(p_2) f(p_3) \Sigma S_{\mu\nu}(p) \right] \]

\[ \times \Sigma S_{\mu\nu}(p) - \bar{f}(p_2) \bar{f}(p_3) \Sigma S_{\mu\nu}(p) + f(p_2) \bar{f}(p_3) \]

\[ \times \Sigma S_{\mu\nu}(p) + f(p_2) \bar{f}(p_3) \Sigma S_{\mu\nu}(p) \]

\[ \times \left( \Sigma A^{\alpha\beta} \Sigma A^{\alpha\beta} \right) = 0. \]

\[ \frac{1}{4} \int d^4 p \left( \Sigma V_{\mu}[V_{\nu}] \right)_{P.B.} = 0. \tag{46} \]

Therefore, the total angular momentum is conserved at the second order in \( \hbar \),

\[ \partial^\rho M_{\rho,\mu\nu}^{(2)} = \frac{\hbar}{4} \int d^4 p \ D T_{\mu\nu}^{(1)} = 0. \tag{47} \]

\section{5 Conclusion}

Spin is a quantum effect and is normally neglected in a classical transport theory. In this work, we addressed the problem of spin polarization in the Wigner function formalism of quantum kinetic theory. While non-equilibrium distributions are related to the details of the interaction of the system, namely the collision terms, the corresponding equilibrium distributions are determined only by the detailed balance between the loss and gain terms, namely the disappearing of the total collision terms. We obtained the equilibrium spin distribution by requiring the detailed balance for the Kadanoff–Baym equations. To be specific, we take a NJL model as an example to calculate the collision terms in the constraint and transport equations at classical level and to the leading order in \( \hbar \). We found that, for an initially non-polarized system without external electromagnetic
fields, while the equilibrium spin distribution is trivial at
classical level, the quantum correction internally generated by
the inhomogeneous vorticity of the system leads to a non-trivial
spin distribution. Different from the classical transport theory
where the collision terms are eliminated by the local equilib-
rium distribution, for the spin transport in a quantum kinetic
theory, the collision terms vanish only in global equilibrium.

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Appendix A. Spin decomposition and semiclassical expansion

The collision terms in the Kadanoff–Baym equations (2),
[Σ⁺, S⁻]*, − [Σ⁺, S⁺], [Σ⁺⁺, S⁻], ⁴ S⁺⁺, ⁴ S⁺, ⁴ S⁺⁺, [Σ⁺⁺, S⁺⁻]*, are
4 × 4 matrices which should be decomposed with the Cliff-
ord algebra. To the lowest order of h the loss terms can be
decomposed as

\[ [\Sigma^+, \ S^-]^{(0)}_1 = +2i \left[ A^\mu P + S^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ = \frac{1}{4} \left[ \gamma^5 \right] \]  

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]

\[ + 2 \left[ \Sigma^\mu S + \Sigma^\mu T + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} T_{A^\mu S^{\mu\nu}} \right] \gamma^5 \gamma^\mu \]
The spin decomposition of the gain terms can be obtained similarly by taking the exchanges \( \Sigma^> \leftrightarrow \Sigma^< \) and \( S^< \leftrightarrow S^> \).

**Appendix B. Components of self-energy**

The transport equation for \( A_{\mu}^{(1)} \) contains \( \Sigma_{S}^{(0)}, \Sigma_{V}^{(0)} \) and \( \Sigma_{A}^{(1)}, \Sigma_{T}^{(1)} \). To the leading order of the \( 1/N_c \) expansion, taking \( \Sigma \) and \( A_{\mu} \) as independent components and considering the relations between the spin components (9), the components of the zeroth-order self-energy can be evaluated as

\[
\Sigma_{S}^{(0)}(p) = G^2 \int \frac{d^4k}{m^2} \frac{[m^2 + (k + q) \cdot k]}{m^2} \times S^{(0)}(k + q) S^{(0)}(k) S^{(0)}(p + q),
\]

\[
\Sigma_{V}^{(0)}(p) = G^2 \int \frac{d^4k}{m^2} \frac{[m^2 + (k + q) \cdot k]}{m^2} \times S^{(0)}(k + q) S^{(0)}(k) S^{(0)}(p + q).
\]

For the first order self-energy, it is related to the first-order components of the Wigner function. \( \Sigma_{A}^{(1)}(p) \) and \( \Sigma_{T}^{(1)}(p) \) can be obtained via simply replacing the product of three zeroth-order components in \( \Sigma_{S}^{(0)}(p) \) and \( \Sigma_{V}^{(0)}(p) \) by three same products but with one first-order component, and \( \Sigma_{A}^{(1)}(p) \) and \( \Sigma_{T}^{(1)}(p) \) can be evaluated as

\[
\Sigma_{A}^{(1)}(p) = G^2 \int \frac{d^4k}{m^2} \frac{[m^2 + (k + q) \cdot k]}{m^2} \times S^{(0)}(k + q) S^{(0)}(k) A_{\mu}^{(1)}(p + q),
\]

\[
\Sigma_{T}^{(1)}(p) = G^2 \int \frac{d^4k}{m^2} \frac{[m^2 + (k + q) \cdot k]}{m^2} \times \frac{(p + q)_{\nu}}{2m^2} S^{(0)}(k + q) S^{(0)}(k) \nabla_{\nu} S^{(0)}(p + q).
\]
