Abstract

Let $X$ be a complex projective curve which is smooth and irreducible of genus 2. The moduli space $\mathcal{M}_2$ of semistable symplectic vector bundles of rank 4 over $X$ is a variety of dimension 10. After assembling some results on vector bundles of rank 2 and odd degree over $X$, we construct a generically finite cover of $\mathcal{M}_2$ by a family of 5-dimensional projective spaces, and outline some applications.

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1 Introduction

Let $X$ be a complex projective smooth irreducible curve of genus $g$ with structure sheaf $\mathcal{O}_X$ and canonical bundle $K_X$. In this section, we review the notion of a symplectic vector bundle over $X$ and introduce the moduli space we will be studying.
Symplectic vector bundles

**Definition:** A symplectic vector bundle over $X$ is a pair $(W, \theta)$ where $W$ is a vector bundle over $X$ and $\theta$ is a bilinear nondegenerate antisymmetric form on $W \times W$ with values in a line bundle $L$.

For us, $L$ will always be the trivial bundle $O_X$. Such a bundle $W$ is necessarily of even rank. If there is no ambiguity, we write $W$ for the pair $(W, \theta)$. For example, if $W$ is simple then $\theta$ is unique up to nonzero scalar multiple.

A subbundle $E \subset W$ is isotropic if $\theta(E, E) = 0$. For any subbundle $E \subseteq W$, we have the short exact vector bundle sequence

$$0 \to E^\perp \to W \to E^* \to 0$$

where the surjection is the map $w \mapsto \theta(w, \cdot)|_E$, and

$$E^\perp = \{ w \in W : \theta(w, E) = 0 \}$$

is the orthogonal complement of $E$. Clearly $E$ is isotropic if and only if $E \subseteq E^\perp$; this shows that the rank of an isotropic subbundle is at most $\frac{1}{2} \text{rk}(W)$. An isotropic subbundle of maximal rank is called a Lagrangian subbundle; clearly $E$ is Lagrangian if and only if $E = E^\perp$.

Conversely, one would like to know when a short exact sequence

$$0 \to E \to W \to E^* \to 0$$

is induced by a symplectic form. We have

**Criterion 1** An extension $0 \to E \to W \to E^* \to 0$ has a symplectic structure with respect to which $E$ is isotropic if and only if $W$ is isomorphic as a vector bundle to an extension whose cohomology class belongs to $H^1(X, \text{Sym}^2 E)$.

**Proof**
Due to S. Ramanan; see [9], § 2 for a proof.

Moduli of vector bundles over $X$

We recall the notions of stability and semistability for vector bundles. The slope of a bundle $W \to X$ is the ratio $\text{deg}(W)/\text{rk}(W)$, denoted $\mu(W)$. Then $W$ is stable (respectively, semistable) if for all proper nonzero subbundles $V \subset W$ we have

$$\mu(V) < \mu(W) \quad (\text{respectively, } \mu(V) \leq \mu(W)).$$
Now let $r$ and $d$ be integers with $r \geq 1$. The moduli space of semistable vector bundles of rank $r$ and degree $d$ over $X$ is denoted $U_X(r,d)$. For any line bundle $L \to X$ of degree $d$, the closed subvariety of $U_X(r,d)$ of bundles with determinant $L$ is denoted $SU_X(r,L)$. References for these objects include Seshadri [25] and Le Potier [15]. The variety $U_X(1,d)$ is the $d$th Jacobian variety of $X$, and will be denoted $J^d_X$. See for example Birkenhake–Lange [5], Chap. 11, for details.

The main object of interest for us is the moduli space of semistable symplectic vector bundles of rank $2n$ over $X$, which we denote $M_n$. Since such a bundle always has trivial determinant, there is a forgetful map $M_n \to SU_X(2n,O_X)$. We have

**Theorem 2** $M_n$ is canonically isomorphic to the moduli space of semistable principal $Sp_n\mathbb{C}$-bundles over $X$, and the forgetful map $M_n \to SU_X(2n,O_X)$ is an injective morphism.

**Proof** (Sketch; see [8], Chap. 1 for details.)

For any smooth variety $Y$, there is an equivalence between the groupoids

$$
\{\text{principal Sp}_n\mathbb{C}\text{-bundles over } Y\} + \{\text{isomorphisms}\}
$$

and

$$
\{\text{symplectic vector bundles of rank } 2n \text{ over } Y\} + \{\text{vector bundle isomorphisms respecting the symplectic structures}\}.
$$

Results of Ramanan [20] and Ramanathan [23] on orthogonal bundles are easily adapted to the symplectic case to show that the notions of semistability and $S$-equivalence as principal $Sp_n\mathbb{C}$-bundle and vector bundle coincide under this equivalence. This shows that the map $M_n \to SU_X(2n,O_X)$ is an injective morphism. □

**Remark:** In particular, any two symplectic forms on a polystable vector bundle differ by a vector bundle automorphism.

Thm. 2 allows us to use results of Ramanathan [21], [22] (especially Thm. 5.9) and [23] on moduli of principal $G$-bundles to deduce information about $M_n$. We find that $M_n$ is a projective variety of dimension $n(2n+1)(g-1)$ which is irreducible and normal. Moreover, the locus of stable vector bundles in $M_n$ is dense.
Statement of the main theorem

Motivation: Consider firstly the rank 2 case. The following is well known, but we give a proof which illustrates Criterion 1.

**Proposition 3** A bundle $W \to X$ of rank 2 has an $O_X$-valued symplectic form if and only if it has trivial determinant.

**Proof**
If $W$ is symplectic of rank 2 then any line subbundle $L \subset W$ is clearly Lagrangian, so $W$ is an extension of $L^{-1}$ by $L$. Thus $W$ has trivial determinant.

Conversely, if $\det(W) = O_X$ then any line subbundle $L \subset W$ induces an exact sequence $0 \to L \to W \to L^{-1} \to 0$, with class in $H^1(X, \text{Hom}(L^{-1}, L))$. But since $L$ has rank 1, we have $\text{Hom}(L^{-1}, L) = \text{Sym}^2 L$, so $W$ is symplectic by Criterion 1. □

This gives rise to an identification between $\mathcal{M}_1$ and $SU_X(2, O_X)$. When $X$ is nonhyperelliptic of genus 3, Narasimhan and Ramanan [17] construct a generically finite cover of $\mathcal{M}_1$ (in this case isomorphic to the Coble quartic) by the union of the spaces $\mathbb{P}H^1(X, L^{-2})$ as $L$ ranges over $J^2_X$. This description is a useful tool: for example, it was used by Pauly in [19] to prove that the Coble quartic is self-dual.

In the present paper we give a similar description of $\mathcal{M}_2$ when $X$ has genus 2. The main tool is Criterion 1. We will consider a union of projective spaces of the form $\mathbb{P}H^1(X, \text{Sym}^2 E)$ as $E$ ranges over a certain family of vector bundles over $X$.

**Notation:** Let $V$ be a vector space and $v \in V$ a nonzero vector. Then we denote by $(v)$ the point defined by $v$ in the projective space $\mathbb{P}V$. We use similar notation for points of projectivised vector bundles.

The moduli space $\mathcal{U}_X(2, -1)$ of semistable vector bundles of rank 2 and degree $-1$ over $X$ is an irreducible variety of dimension $4g - 3 = 5$. Since $\text{gcd}(2, -1) = 1$, we have a diagram
where $E$ is a Poincaré bundle. By Riemann–Roch and semistability, the sheaf $R^1p_*(\text{Sym}^2E)$ is locally free of rank 6 over $U_X(2, -1)$. We denote $\mathbb{B}$ the associated $\mathbb{P}^5$-bundle over $U_X(2, -1)$. The fibre of $\mathbb{B}$ over a bundle $E$ is isomorphic to $\mathbb{P}H^1(X, \text{Sym}^2E)$; henceforth we denote this space $\mathbb{P}^5_E$.

By the results in Seshadri [25], Appendix II, there exists a vector bundle over $\mathbb{B} \times X$ whose restriction to $\{(\langle\delta\rangle, E)\} \times X$ is isomorphic as a vector bundle to the extension $0 \to E \to W \to E^* \to 0$ defined by $\delta$. Thus there exists a classifying map $\Phi: \mathbb{B} \to \mathcal{M}_2$ by the moduli property of $\mathcal{M}_2$. A priori, $\Phi$ is only a rational map. The main result of this paper is

**Theorem 4** The moduli map $\Phi$ is a surjective morphism which is generically finite of degree 24.

The strategy for the proof is as follows. Firstly, we show that $\Phi$ is defined everywhere, so it is in fact a morphism. Then we consider a general (in a sense to be made precise) symplectic extension $0 \to E \to W \to E^* \to 0$ and show that it admits only finitely many isotropic subbundles of rank 2 and degree $-1$. It is then easy to show that $\Phi$ is generically finite. By dimension count, irreducibility and projectivity of the spaces, it is surjective. Finally, we use results of Lange and Newstead [14] to calculate the degree of $\Phi$.

We conclude with some actual and future applications for this description of $\mathcal{M}_2$.

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## 2 Extensions and principal parts

In this brief section we state some results on extensions of vector bundles which will be used throughout the paper. Firstly, we recall some notions from Kempf [10]. The sheaf of sections of a vector bundle $(W, L, O_X, K_X)$ will be denoted by the corresponding script letter $(\mathcal{W}, \mathcal{L}, \mathcal{O}_X, \mathcal{K}_X)$. For any
vector bundle $V \to X$, there is a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{V} \to \text{Rat}(V) \to \text{Prin}(V) \to 0$$

where $\text{Rat}(V)$ is the sheaf of rational sections of $V$ and $\text{Prin}(V)$ that of principal parts with values in $V$. Taking global sections, we get

$$0 \to H^0(X,V) \to \text{Rat}(V) \to \text{Prin}(V) \to H^1(X,V) \to 0. \quad (1)$$

We write $\overline{s}$ for the principal part of $s \in \text{Rat}(V)$, and we denote the cohomology class of $q \in \text{Prin}(V)$ by $[q]$. Recall that an elementary transformation of a vector bundle $F \to X$ is the bundle corresponding to the kernel of a surjective map from $F$ to a torsion sheaf.

**Theorem 5** Let $E$ and $F$ be vector bundles over $X$ and suppose that there are no nonzero maps $F \to E$. Let $0 \to E \to W \to F \to 0$ be an extension of class $\delta(W) \in H^1(X,\text{Hom}(F,E))$.

(i) There is a bijection between

$$\left\{ \begin{array}{l}
\text{principal parts } p \in \text{Prin}(\text{Hom}(F,E)) \\
\text{such that } \delta(W) = [p]
\end{array} \right\}$$

and

$$\left\{ \begin{array}{l}
\text{elementary transformations of } F \\
\text{lifting to vector subbundles of } W
\end{array} \right\}$$

given by $p \leftrightarrow \text{Ker}(p : F \to \text{Prin}(E))$.

(ii) If $F = E^*$ and $\delta(W)$ is symmetric, so that by Crit. 1 we have a symplectic form on $W$, then the subbundle associated to $\text{Ker}(p)$ is isotropic if and only if $p$ is itself a symmetric principal part$^1$.

**Proof**

See [9], § 3.

3 Vector bundles of rank 2 and odd degree over a curve of genus 2

Here we assemble some technical results which will be needed later.

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$^1$This is a stronger requirement than that the class $[p]$ be symmetric. By (1), the class $[p]$ is symmetric if $\alpha p = \overline{\alpha}$ for some global rational section $\alpha$ of $\text{Hom}(E^*,E)$. 

6
A cover of another moduli space

The moduli space $U_X(2, 2k + 1)$ of semistable vector bundles of rank 2 and degree $2k + 1$ over $X$ is a variety of dimension 5 which is smooth and irreducible.

We give another description of $U_X(2, 2k + 1)$, which will in fact illustrate the strategy of the proof of our main result. In the same way as we arrived at the $\mathbb{P}^5$-bundle $\mathcal{B} \rightarrow U_X(2, -1)$, we construct a projective bundle $\mathcal{S} \rightarrow J^k_{X^2}$ whose fibre at $(M, L)$ is isomorphic to $\mathbb{P}H^1(X, \text{Hom}(M, L)) = \mathbb{P}^1$. This admits a moduli map $\Phi_\mathcal{S} : \mathcal{S} \rightarrow U_X(2, 2k + 1)$.

**Lemma 6** The map $\Phi_\mathcal{S}$ is a surjective morphism of degree 4. In particular, every $E \in U_X(2, 2k + 1)$ fits into 1, 2, 3 or (generically) 4 short exact sequences of the form

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$$

(2)

where $L$ and $M$ are line bundles of degree $k$ and $k + 1$ respectively.

**Proof**

Let $E$ belong to the image of $\Phi_\mathcal{S}$, so $E$ is a stable extension of type (2). Suppose $L' \subset E$ is a line subbundle of degree $k$. If $L' \neq L$ then $L'$ must be of the form $M(-x)$ for some point $x \in X$. Conversely, by Thm. 5 (i) a subsheaf $\mathcal{M}(-x) \subset \mathcal{M}$ lifts to a line subbundle of $E$ if and only if $\delta(E)$ can be represented by a principal part which has a simple pole at $x$ and is otherwise regular.

Now by analogy with Kempf–Schreyer [11], § 1, we note that the standard map

$$\phi_{|K_XML^{-1}|^*} : X \rightarrow |K_XML^{-1}|^* \cong \mathbb{P}H^1(X, \text{Hom}(M, L)) = \mathbb{P}^1$$

can be defined by sending a point $x \in X$ to the line spanned by the cohomology class of a principal part which has a simple pole at $x$ and is otherwise regular, when this class is nonzero. Since $\phi_{|K_XML^{-1}|^*}$ is surjective of degree at most 3 and $\delta(E)$ is nonzero, there are at most three, and generically three, points $x \in X$ such that $M(-x)$ is a subbundle of $E$.

Therefore $E$ occurs as an extension of type (2) for at most four pairs $(M, L)$. Moreover, two distinct classes in the pencil $|K_XML^{-1}|^*$ define non-isomorphic vector bundles: it is easy to check that there is at least one degree $k$ line subbundle which belongs to one but not to the other. Hence $\Phi_\mathcal{S}$ is quasifinite of degree 4. By dimension count, it is dominant. Now it is not hard to see that every nontrivial extension of type (2) is stable, so $\Phi_\mathcal{S}$ is a morphism. Since $\mathcal{S}$ and $U_X(2, 2k + 1)$ are projective varieties, the image is closed, so it is surjective. □
Note: Lange and Narasimhan prove in [13], Prop. 4.2, that a vector bundle of rank 2 and degree \(2k + 1\) over \(X\) has at most 4 subbundles of degree \(k\).

We will need

**Lemma 7** Let \(V\) and \(L\) be vector bundles of ranks \(n\) and 1 respectively, and let \(f: L \to V\) be a homomorphism. Then \(f\) factorises via a map \(L(x) \to V\) if and only if \(f\) is zero at \(x\).

**Proof**
Narasimhan–Ramanan [18], Lemma 5.3.

**Proposition 8** Let \(E \to X\) be a (semi)stable bundle of rank 2 and degree \(2k + 1\). Then \(h^0(X, \text{Hom}(L, E)) \leq 1\) for all \(L \in J^k_X\).

**Proof**
Let \(E\) be a vector bundle of rank 2 and degree \(2k + 1\) and suppose that \(h^0(X, \text{Hom}(L, E)) \geq 2\) for some \(L \in J^k_X\). Firstly, note that if any nonzero map \(L \to E\) has a zero then \(E\) would have a line subbundle of degree at least \(k + 1\) by Lemma 7, so would not be semistable. Thus we can suppose that \(E\) is an extension \(0 \to L \to E \to M \to 0\) for some \(M \in J^{k+1}_X\). By hypothesis, there also exists a copy of \(L\) in \(E\) whose projection to \(M\) is generically nonzero, so \(L = M(-x)\) for some \(x \in X\). Therefore, by Thm. 5 (i) the class \(\delta(W)\) can be represented by a principal part \(p\) with just one pole which is simple and at \(x\).

But since \(L = M(-x)\), the bundle \(M^{-1}L(x)\) is trivial, so has a global section which is nonzero at \(x\). This gives a global rational section of \(M^{-1}L\) with just a simple pole at \(x\). Hence \([p] = 0\) by (1), so \(E\) is a trivial extension. In particular, it is not semistable. \(\square\)

**Genericity of bundles in \(U_X(2, -1)\)**

In this section, we show that some conditions which we will need later are satisfied by a generic stable bundle of degree \(-1\) and rank 2 over \(X\). We will need

**Lemma 9** Every semistable vector bundle \(F\) of rank at most 3 and slope 1 over a curve of genus 2 satisfies \(h^0(X, M \otimes F) = 0\) for generic \(M \in J^0_X\).

**Proof**
This is a special case of Raynaud [24], Cor. 1.7.4.
Proposition 10 For generic $E \in \mathcal{U}_X(2, -1)$, the bundle $\text{Hom}(E, E^*)$ has no global sections.

Proof
We have $h^0(X, \text{Hom}(E, E^*)) = h^0(X, \det E^*) + h^0(X, \text{Sym}^2 E^*)$, so it suffices to show that the subsets of $\mathcal{U}_X(2, -1)$ where $h^0(X, \det E^*) > 0$ and where $h^0(X, \text{Sym}^2 E^*) > 0$ are each of codimension 1.

Firstly, $h^0(X, \det E^*) > 0$ if and only if $(\det E^*)^{-1}$ is effective. The set of such $E$ is the inverse image of $\text{Supp}(\Theta)$ under the map $\mathcal{U}_X(2, -1) \to J_X^1$ defined by $E \mapsto (\det E)^{-1}$. Clearly this map is surjective, so the inverse image of a divisor is a divisor.

For the rest: $\text{Sym}^2 E^*$ is of slope $1 = g - 1$ for all $E \in \mathcal{U}_X(2, -1)$, so we expect that if the set

$$\{ E \in \mathcal{U}_X(2, -1) : h^0(X, \text{Sym}^2 E^*) > 0 \}$$

is not equal to $\mathcal{U}_X(2, -1)$ then it is the support of a divisor. Since $\mathcal{U}_X(2, -1)$ is irreducible, then, it suffices to exhibit one $E$ such that $h^0(X, \text{Sym}^2 E^*) = 0$. Choose any $E \in \mathcal{U}_X(2, -1)$. If $h^0(X, \text{Sym}^2 E^*) = 0$ then we are done. If not, we note that $\text{Sym}^2 E$ is semistable for all $E \in \mathcal{U}_X(2, -1)$ by Le Potier [15], p. 161. Thus, by Lemma 9, there exists at least one $M \in J_X^0$ such that

$$h^0(X, M \otimes \text{Sym}^2 E^*) = 0.$$ 

Let $N$ be any square root of $M$. Since

$$M \otimes \text{Sym}^2 E^* = N^2 \otimes \text{Sym}^2 E^* \cong \text{Sym}^2(N \otimes E^*),$$

by construction the bundle $E' := N^{-1} \otimes E$, which is stable of degree $-1$ and rank 2, satisfies $h^0(X, \text{Sym}^2(E'^*)) = 0$. This completes the proof of the proposition. □

Recall that the Weierstrass points of $X$ are the 6 points fixed by the hyperelliptic involution $\iota: X \to X$.

Proposition 11 A general $E \in \mathcal{U}_X(2, -1)$ satisfies the following:

- $E^*$ has no degree 1 line bundle quotients $L$ such that $L^2 = O_X(2x)$.
- $\det(E^*)^2$ is of the form $O_X(a + b)$ for distinct $a, b \in X$.
- Furthermore, neither $a$ nor $b$ is a Weierstrass point.
Proof
We define some hypersurfaces in $J^1_X \times J^0_X$. Firstly, let $Y$ be the set
$$\{ L \in J^1_X : L^2 \text{ is of form } O_X(2x) \text{ for some } x \in X \},$$
the union of all translates of the theta divisor by line bundles of order 2 in $J^0_X$. We let $H_1 := Y \times J^0_X$.

Next, consider the curve $X_2 := \{ O_X(2x) : x \in X \}$ in $J^2_X$. Let $H_2$ be the inverse image in $J^1_X \times J^0_X$ of $X_2$ by the map $J^1_X \times J^0_X \to J^2_X$ given by $(L, M) \mapsto L^2M^2$.

Finally, for each Weierstrass point $w \in X$ we have the set
$$\{ O_X(w + x) : x \in X \} =: \Theta + w.$$ Write $H_3$ for the inverse image of the union of the six $\Theta + w$ by the map $(L, M) \mapsto L^2M^2$.

Let $H := H_1 \cup H_2 \cup H_3$. Now recall from Lemma 6 that we have a cover $\Phi_S : S \to U_X(2, 1)$ (here $k = 0$). If any of the above conditions were not satisfied by a general $E \in U_X(2, -1)$ then the restriction of $\Phi_S$ to the subvariety $S|_H \subset S$ would be dominant. But this is impossible since $\dim(S|_H) = 4$ but $\dim(U_X(2, 1)) = 5$. The proposition follows. □

A ruled surface in $\mathbb{P}^5$

In this section, we fix $E \in U_X(2, -1)$ and describe a map of the ruled surface $\mathbb{P}E$ into $\mathbb{P}^5_E$. We then study the extent to which this map fails to be an embedding for generic $E$.

We generalise slightly the aforementioned approach from § 1 of Kempf–Schreyer [11]. For any $x \in X$, we can form the exact sheaf sequence

$$0 \to \text{Sym}^2E \to (\text{Sym}^2E)(x) \to \frac{(\text{Sym}^2E)(x)}{\text{Sym}^2E} \to 0$$

whose cohomology sequence begins

$$0 \to H^0(X, (\text{Sym}^2E)(x)) \to (\text{Sym}^2E)(x)|_x \xrightarrow{\delta} H^1(X, \text{Sym}^2E) \to \ldots$$

The second term can be identified with the set of $\text{Sym}^2E$-valued principal parts which are regular except for possibly a simple pole at $x$. Moreover, there is a canonical isomorphism $\mathbb{P}(\text{Sym}^2E)(x)|_x \xrightarrow{\sim} \mathbb{P}\text{Sym}^2E|_x$.

For each $x \in X$, then, we define a map $\psi_x : \mathbb{P}E|_x \to \mathbb{P}H^1(X, \text{Sym}^2E)$ by the composition

$$\mathbb{P}E|_x \xrightarrow{\text{Segre}} \mathbb{P}\text{Sym}^2E|_x \xrightarrow{\sim} \mathbb{P}(\text{Sym}^2E)(x)|_x \xrightarrow{\mathbb{P}\delta} \mathbb{P}H^1(X, \text{Sym}^2E).$$ (3)
We define \( \psi: \mathbb{P}E \to \mathbb{P}H^1(X, \text{Sym}^2E) \) to be the product of the \( \psi_x \) over all \( x \in X \). We will need

**Proposition 12** For general \( E \in \mathcal{U}_X(2, -1) \), the bundle \( K_X \otimes \text{Sym}^2E^* \) is generated by global sections.

**Proof**

For any \( x \in X \), we have an exact cohomology sequence

\[
0 \to H^0(X, K_X(-x) \otimes \text{Sym}^2E^*) \to H^0(X, K_X \otimes \text{Sym}^2E^*) \to (K_X \otimes \text{Sym}^2E^*)|_x \to H^1(X, K_X(-x) \otimes \text{Sym}^2E^*) \to 0,
\]

whence \( K_X \otimes \text{Sym}^2E^* \) is generated by global sections if and only if

\[
h^1(X, K_X(-x) \otimes \text{Sym}^2E^*) = h^1(X, (\text{Sym}^2E^*)(\iota(x))) = 0
\]

for all \( x \in X \). By Serre duality, this is equal to

\[
h^0(X, K_X(-\iota(x)) \otimes \text{Sym}^2E) = h^0(X, (\text{Sym}^2E)(\iota(x))) = h^0(X, \text{Hom}(O_X(-x), \text{Sym}^2E)).
\]

Now let \( M \in J_X^0 \) be a bundle which is not of order 2, and \( N \in J_X^{-1} \) such that \( N^{-1}M^{-1} \) is not effective; clearly the general \( (N, M) \in J_X^{-1} \times J_X^0 \) satisfies these conditions. We will show that for all but at most two extensions

\[
0 \to N \to E \xrightarrow{\xi} M \to 0,
\]

there are no maps \( O_X(-x) \to \text{Sym}^2E \) for any \( x \in X \). Let \( E \) be such an extension, with class \( p_1 + p_2 + p_3 \in |K_XMN^{-1}| \). It is not hard to see that the map \( E \otimes E \to M \otimes E \) given by \( e \otimes f \mapsto \frac{1}{2}[(e\otimes f) + (f \otimes e)] \) induces another exact sequence

\[
0 \to N^2 \to \text{Sym}^2E \to M \otimes E \to 0.
\]

Since \( M \) is not of order 2, there is a unique pair of points \( q_1, q_2 \in X \) such that \( K_XM^2 = O_X(q_1 + q_2) \).

If there is a nonzero map \( O_X(-x) \to \text{Sym}^2E \) then \( O_X(-x) \) must be a subbundle of \( M \otimes E \), equivalently \( M^{-1}(-x) \) must be a subbundle of \( E \). But the degree \(-1\) subbundles of \( E \) are exactly \( N \) and the \( M(-p_i) \). If \( N = M^{-1}(-x) \) then \( N^{-1}M^{-1} \) is effective, contrary to hypothesis.

If \( M(-p_i) = M^{-1}(-x) \) for some \( i \) then \( M^2 = O_X(p_i - x) \), so we obtain \( K_XM^2 = O_X(p_i + \iota(x)) \). This means that the only extensions of \( M \) by \( N \)
such that $M^{-1}(-x)$ belongs to $E$ are those where $p_i = q_j$ for some $i$ and $j$. This can happen for at most two classes in $|K_XMN^{-1}|$.

In summary, for all but at most two extensions of $M$ by $N$, there are no maps $O_X(-x) \to \text{Sym}^2E$. The proposition follows. □

Now write $\pi$ for the projection $\mathbb{P}E \to X$ and let $\Upsilon \to \mathbb{P}E$ be the line bundle $\pi^*K_X \otimes \mathcal{O}_{\mathbb{P}E}(2)$.

**Proposition 13** There is a natural identification $\mathbb{P}^5 \cong |\Upsilon|^*$ under which $\psi$ coincides with the natural map $\phi_{|\Upsilon|}: \mathbb{P}E \dashrightarrow |\Upsilon|^*$. In particular, $\psi$ is algebraic. Moreover, for general $E$, it is defined everywhere.

**Proof** (Sketch; see [8], Prop. 4.11 for details)

Firstly, using the fact that $K_X$ and $\text{Sym}^2E^*$ are locally trivial on $X$, one sees that sections of $\Upsilon$ over $\mathbb{P}E$ can be interpreted naturally as sections of $K_X \otimes \text{Sym}^2E^*$ over $X$. Thus for any $u \in \mathbb{P}E$, a principal part $p$ defining $\psi(u)$ defines a linear form $[p]$ on $H^0(\mathbb{P}E, \Upsilon)$ by Serre duality. One shows that $\text{Ker}[p]$ contains the subspace $H^0(\mathbb{P}E, \Upsilon - u)$ of sections vanishing at $u$. Now by Prop. 12, for general $E$, the bundle $K_X \otimes \text{Sym}^2E^*$ is generated by global sections, so there is a section of $\Upsilon$ not vanishing at $u$. By this and the fact that no global rational section of $K_X$ has just one simple pole, we find that $[p]$ is nonzero. Hence $H^0(\mathbb{P}E, \Upsilon - u)$ is a hyperplane. It follows that $\psi$ is identified with $\phi_{|\Upsilon|}$ and is a morphism. □

We now determine the extent to which $\psi$ fails to be an embedding in general.

**Lemma 14** For general $E \in \mathcal{U}_X(2, -1)$, the map $\psi$ is an embedding except at a finite number of points.

**Proof**

Since $E$ is general, we can suppose that $\psi$ is a morphism by Props. 12 and 13. We distinguish three ways in which it can fail to be an embedding:

(i) $\psi(u) = \psi(v)$ for some $u, v \in \mathbb{P}E$ lying over distinct $x, y \in X$.

(ii) $\psi(u) = \psi(v)$ for distinct $u$ and $v$ in a fibre $\mathbb{P}E|_x$.

(iii) The differential of $\psi$ is not injective at a point $u$ in some $\mathbb{P}E|_x$.

Recall that we have the cohomology sequence

$$0 \to \text{Rat}(\text{Sym}^2E) \to \text{Prin}(\text{Sym}^2E) \xrightarrow{\delta} H^1(X, \text{Sym}^2E) \to 0 \quad (4)$$

which is exact since $h^0(X, \text{Sym}^2E) = 0$ by semistability.
Suppose that (i) occurs. Let \( \tilde{u} \) and \( \tilde{v} \) be lifts of \( u \) and \( v \) to \( E \) and suppose that \( p \) and \( q \) are Sym\(^2\)\(E\)-valued principal parts with simple poles along \( \tilde{u} \otimes \tilde{u} \) and \( \tilde{v} \otimes \tilde{v} \) respectively. Then

\[
\langle [p] \rangle = \psi(u) = \psi(v) = \langle [q] \rangle.
\]

After normalising if necessary, we can assume that the class \([p - q]\) is zero in \( H^1(X, \text{Sym}^2E) \). Then by exactness of (4), there exists a global rational section \( \alpha \) of \( \text{Sym}^2E \) with principal part \( p - q \). This is a global regular section of \((\text{Sym}^2E)(x + y)\) with \( \alpha(x) \) and \( \alpha(y) \) decomposable. The generic rank of the corresponding symmetric map \( \alpha : E^* \to E(x + y) \) may be 1 or 2. If it is 2 then \( \det(\alpha) \) is generically nonzero, so has

\[
\deg(E^*) - \deg(E(x + y)) = 2
\]

zeroes. It follows that \( \det(\alpha) \in H^0(X, \mathcal{O}_X(x + y)) \). But we also have

\[
\det(\alpha) \in H^0(X, \text{Hom}(\det E^*, (\det E)(2x + 2y))),
\]

so \( (\det E)^2 = \mathcal{O}_X(-x - y) \).

Now since \( E^* \) is of rank 2, there is an isomorphism \( E \cong E^* \otimes \det(E) \), which induces another isomorphism \( \text{Sym}^2E \cong \text{Sym}^2E^* \otimes \det(E)^2 \). Thus

\[
(\text{Sym}^2E)(x + y) \cong \text{Sym}^2E \otimes (\det E)^{-2} \cong \text{Sym}^2E^*,
\]

so \( \alpha \) defines a nonzero section of \( \text{Sym}^2E^* \). But by Prop. 10, for general \( E \) there are no such \( \alpha \).

Therefore we can suppose \( \alpha \) to be generically of rank 1. This means that \( \alpha \) factorises as

\[
\mathcal{E}^* \to \mathcal{L} \to \mathcal{E}(x + y)
\]

where \( \mathcal{L} \) is an invertible subsheaf of \( \mathcal{E}(x + y) \). By stability of \( E^* \), the degree of \( \mathcal{L} \) is at least 1, and by stability of \( E(x + y) \), it is at most 1. Therefore it is 1 and in fact we have a sequence of vector bundle maps \( E^* \to L \to E(x + y) \).

Since \( \alpha \) is symmetric, this is identified with

\[
E^* \to L^{-1}(x + y) \to E(x + y),
\]

so \( L^2 = \mathcal{O}_X(x + y) \). We see that \( L^{-1} \) is a subbundle of \( E \) and the two points of \( \mathbb{P}E \) that are contracted are those corresponding to the fibres \( L^{-1}|_x \) and \( L^{-1}|_y \).

In summary, for generic \( E \), the map \( \psi \) contracts two points of \( \mathbb{P}E \) only if they are the images of \( N|x \) and \( N|y \) for some degree \(-1\) subbundle \( N \) of \( E \) such that \( N^{-2} = \mathcal{O}_X(x + y) \). By Lemma 6, there are at most four such \( N \).
By Prop. 4, we can assume that $N^{-2} \neq K_X$ (otherwise the quotient $N^{-1}$ of $E^*$ would satisfy $(N^{-1})^2 = O_X(2w)$ for any Weierstrass point $w$), so there is a unique pair $x, y$ for each $N$. Moreover, by Prop. 8, there is only one independent map $N \to E$, so there is only one possibility for the points $N|_x$ and $N|_y$ in $\mathbb{P}E$.

Putting all this together, we have at most eight points of $\mathbb{P}E$ at which $\psi$ fails to be injective in this way.

Case (ii) is simpler. Proceeding as above, we get a symmetric map $\alpha: E^* \to E(x)$ which is of rank 2 at $x$. Since the bundles have the same rank and degree, it is an isomorphism. Twisting $\alpha^{-1}$ by $O_X(-x)$, we get a symmetric map $E \to E^*(-x)$. Composing with the generic inclusion $E^*(-x) \to E^*$, we obtain a nonzero global section of $\text{Sym}^2 E^*$ so, by Prop. 10, the bundle $E$ is not general.

Lastly, suppose (iii) happens. This means that the space of global sections of $\Upsilon$ which vanish to order 2 at $u$ is of dimension greater than expected, that is, at least 4. Let $e, f$ be a local basis for $E$ with $(e|_x) = u$, and consider also the dual local basis $e^*, f^*$ of $E^*$. Let $z$ be a local coordinate on $X$ centred at $x$. Near $u$, a section of $\Upsilon$ vanishing to order 2 at $u$, viewed as a section of $K_X \otimes \text{Sym}^2 E^*$, has the form

$$dz \otimes \left(c_0(f^* \otimes f^*) + b_1 z(e^* \otimes f^* + f^* \otimes e^*) + c_1 z(f^* \otimes f^*)ight)$$

$$+ \text{terms of higher order in } z$$

for some $c_0, b_1, c_1 \in \mathbb{C}$. Now clearly the contractions of such a section against the principal parts

$$\frac{e \otimes e}{z}, \quad \frac{e \otimes f + f \otimes e}{z} \quad \text{and} \quad \frac{e \otimes e}{z^2}$$

are all regular, so the cohomology classes of these principal parts belong to the space$^2$ $\text{Ker}(H^0(\mathbb{P}E, \Upsilon)^* \to H^0(\mathbb{P}E, \Upsilon - 2u)^*)$. By hypothesis, this has dimension at most 2, so, by exactness of (4), there is a relation

$$\lambda_1 \left[\frac{e \otimes e}{z}\right] + \lambda_2 \left[\frac{e \otimes f + f \otimes e}{z}\right] + \lambda_3 \left[\frac{e \otimes e}{z^2}\right] = 0$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, not all zero. This means that there is a global rational section of $\text{Sym}^2 E$ with principal part

$$\frac{\lambda_1 e \otimes e}{z} + \frac{\lambda_2 (e \otimes f + f \otimes e)}{z} + \frac{\lambda_3 e \otimes e}{z^2}.$$ 

$^2$The projectivisation of this space is the embedded tangent space to $\psi(\mathbb{P}E)$ at $\psi(u)$. 

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If \( \lambda_3 = 0 \) then in fact we are in situation (ii), so \( E \) is not general by Prop. 10. Assuming the contrary, there is a regular symmetric map \( \alpha: E^* \to E(2x) \) which has rank 1 at \( x \). Again, there are two possibilities: the generic rank could be either 1 or 2. If it is 1 then as before \( \alpha \) factorises as \( E^* \to L \to E(2x) \) where \( L \) is a degree 1 line bundle quotient of \( E^* \). Since \( \alpha \) is symmetric, we have as before \( L^2 = O_X(2x) \), so \( E \) is not general, this time by Prop. 11.

Thus we can suppose that \( \alpha \) has generic rank 2. Then we have the sheaf sequence

\[
0 \to E^* \xrightarrow{\alpha} E(2x) \to T \to 0
\]

where \( T \) is a torsion sheaf. Now the determinant of \( \alpha \) vanishes at two points (counted with multiplicity) because \( \deg(E(2x)) - \deg(E^*) = 2 \). One of these points is \( x \); call the other one \( y \) (they coincide if and only if \( \lambda_2 = 0 \)). So \( \det(\alpha) \in H^0(X, O_X(x + y)) \).

By genericity and Prop. 11, we have \( \det(E^*)^2 = O_X(a + b) \) for distinct \( a, b \in X \), neither one a Weierstrass point. Then \( (\det E)^2(4x) = O_X(4x - a - b) = O_X(3x) = O_X(y + a + b) \).

We show that there are only a finite number of solutions \( x, y \in X \) to this equation. Otherwise, it is not hard to see that for each \( x \in X \) we would have \( y(x) \in X \) such that

\[
O_X(3x) = O_X(y(x) + a + b)
\]

(\( y(x) \) is well defined if it exists because if \( y + a + b \) and \( y' + a + b \) are linearly equivalent then clearly \( y = y' \)). In particular, \( O_X(3a) = O_X(y(a) + a + b) \).

Then either \( b + y(a) = 2a \), contradicting the hypothesis \( b \neq a \), or \( a \) is a base point of \( O_X(3a) \), whence \( a \) is a Weierstrass point, again contradicting genericity. Thus there are only finitely many \( x \in X \) such that \( y \) and \( \alpha \) with these properties can exist.

We show that if \( y \) does exist, there is at most one independent \( \alpha \). For otherwise,

\[
\det: H^0(X, \text{Sym}^2 E(2x)) \to H^0(X, O_X(x + y))
\]

has positive-dimensional fibres, so there is a nonzero symmetric homomorphism \( \beta: E^* \to E(2x) \) with determinant everywhere zero, that is, of rank generically 1. But we saw above that, by genericity, this is impossible.

In summary, for general \( E \), the differential of \( \psi \) can fail to be injective at only finitely many points of \( \mathbb{P}E|_x \). This completes the proof of the lemma. \( \square \)
4 Proof of the main theorem

In this section we will prove Thm. 4. We begin by studying some special loci of $\mathbb{P}^5_E$.

Stability of bundles in $\mathbb{P}^5_E$

We investigate the loci of strictly semistable bundles in $\mathbb{P}^5_E$ and prove that the general extension represented in $\mathcal{B}$ is a stable vector bundle. We begin by stating a technical result:

**Proposition 15** Let $W \to X$ be any self-dual vector bundle. Then $W$ is stable (resp., semistable) if and only if it contains no destabilising (resp., desemistabilising) subbundles of rank at most $\frac{1}{2}\text{rk}(W)$.

**Proof**

This is straightforward to check; we remark that it is true whether $W$ has even or odd rank.

**Lemma 16** For any $E \in \mathcal{U}_X(2,-1)$, every nontrivial symplectic extension $W$ of $E^*$ by $E$ is semistable.

**Proof**

Since a symplectic vector bundle is self-dual, by Prop. 15 it is enough to show that $W$ contains no desemistabilising subbundles of rank at most 2.

Firstly, suppose $L \subset W$ is a line subbundle of degree at least 1. Since $E$ is stable, $h^0(X,\text{Hom}(L,E^*)) > 0$. But since $L$ and $E^*$ are stable and $\mu(L) \geq 1 > \mu(E^*) = \frac{1}{2}$, this is impossible.

Next, suppose $F \subset W$ is a subbundle of rank 2 and degree at least 1. Since we have just seen that $W$ contains no line subbundles of positive degree, $F$ must be stable. Firstly, if the composed map $F \to W \to E^*$ were zero then we would have a nonzero map $F \to E$, contradicting stability of $F$ and $E$. Therefore the composed map is nonzero. Since $F$ and $E^*$ are stable, it must be an isomorphism, so $W$ is a trivial extension. □

This lemma shows in particular that the classifying map $\Phi: \mathcal{B} \to \mathcal{M}_2$ is defined everywhere, so it is a morphism.

We quote a result of Narasimhan and Ramanan:
Lemma 17 Consider a diagram of vector bundles over $X$

\[
\begin{array}{cccc}
0 & \rightarrow & E & \rightarrow W & \rightarrow F & \rightarrow 0 \\
& & & \uparrow f & & \\
& & & V & & \\
\end{array}
\]

where the top row is exact. Then $f$ factorises via a map $V \rightarrow W$ if and only if the class $\delta(W)$ of the extension belongs to the kernel of the induced map $f^*: H^1(X, \text{Hom}(F, E)) \rightarrow H^1(X, \text{Hom}(V, E))$.

Proof
Narasimhan–Ramanan [18], Lemma 3.1.

Lemma 18 The generic symplectic bundle in $\mathbb{B}$ is a stable vector bundle.

Proof
We will prove this by determining the classes in a general $\mathbb{P}_E^5$ representing vector bundles which contain a subbundle of degree 0; this will also be used later. Again, by Prop. 15 it suffices to check for degree 0 subbundles of ranks 1 or 2.

Firstly, suppose $M \subset W$ is a line subbundle of degree 0. Since $E$ is stable, $h^0(X, \text{Hom}(M, E^*)) > 0$; then in fact $M$ is a subbundle of $E^*$ by Lemma 7 since $E^*$ is stable. By Lemma 6, there are at most 4 possibilities for $M \in J^0_X$.

Conversely, by Lemma 17, a map $j: M \hookrightarrow E^*$ lifts to $W$ if and only if $\delta(W)$ belongs to the kernel of the induced map $j^*: H^1(X, \text{Sym}^2 E) \rightarrow H^1(X, \text{Hom}(M, E))$.

We check that this map is surjective. By Serre duality, it is equivalent to check that the transposed map $H^0(X, K_X M \otimes E^*) \rightarrow H^0(X, K_X \otimes \text{Sym}^2 E^*)$, which by abuse of notation we also denote $j$, is injective. Now the induced map

\[
j: H^0(X, K_X M \otimes E^*) \rightarrow H^0(X, K_X \otimes E^* \otimes E^*)
\]

is injective, because the functor $K_X \otimes - \otimes E^*$ and the global section functor are left exact (the latter because these sheaves are locally free). Since $H^0(X, K_X \otimes E^* \otimes E^*)$ is the direct sum

\[
H^0(X, K_X \otimes \text{Sym}^2 E^*) \oplus H^0(X, K_X \otimes \bigwedge^2 E^*),
\]

we have to show that $\text{Im}(j) \cap H^0(X, K_X \otimes \bigwedge^2 E^*) = \{0\}$. Now $E$ is of rank 2, so the latter space is just $H^0(X, K_X \otimes (\det E)^{-1})$, and any nonzero section
therein has 3 zeroes (counted with multiplicity). But if a map $K_{X}^{-1} M^{-1} \to E^{\ast}$ vanished at 3 points, $E^{\ast}$ would contain a line subbundle of degree at least 1 by Lemma 7; this would contradict the stability of $E^{\ast}$.

Thus the restriction of $j^{\ast}$ to $H^{1}(X, \text{Sym}^{2} E)$ is surjective. Since $\text{Sym}^{2} E$ and $\text{Hom}(M, E)$ have no global sections, the kernel of $j^{\ast}$ is of dimension

$$-\chi(X, \text{Sym}^{2} E) + \chi(\text{Hom}(M, E)) = 6 - 3 = 3.$$ 

Hence there is a union of between 1 and 4 projective planes in $\mathbb{P}_{E}^{5}$ representing extensions which are destabilised by a line subbundle of degree 0.

Now we consider a destabilising subbundle $G \subset W$ of rank 2. We will use the map $\psi : \mathbb{P}E \to \mathbb{P}_{E}^{5}$ defined in section 3.

**Proposition 19** Let $E \in \mathcal{U}_{X}(2, -1)$ be general, and let $W$ be a nontrivial symplectic extension of $E^{\ast}$ by $E$. Then $W$ is destabilised by a subbundle of rank 2 and degree 0 if and only if $\langle \delta(W) \rangle$ belongs to $\psi(\mathbb{P}E)$.

**Proof**

Let $G \subset W$ be a subbundle of rank 2 and degree 0. Then the rank of $G \cap \mathcal{E}$ is 2, 1 or 0. It cannot be 0 because then the image of $G$ in $\mathcal{E}^{\ast}$ would be a torsion subsheaf of length 1, of which there are none. If $G \cap \mathcal{E}$ were an invertible subsheaf $\mathcal{L}$ then we would have a diagram

$$
\begin{array}{cccccc}
0 & \to & \mathcal{E} & \to & \mathcal{W} & \to & \mathcal{E}^{\ast} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{L} & \to & \mathcal{G} & \to & \mathcal{M} & \to & 0
\end{array}
$$

where $\mathcal{M}$ is an invertible subsheaf of $\mathcal{E}^{\ast}$. Since $E$ and $E^{\ast}$ are stable, we have $\deg(\mathcal{L}) \leq -1$ and $\deg(\mathcal{M}) \leq 0$. But then $\deg(G) \leq -1$, a contradiction.

Hence $G$ is an elementary transformation $0 \to \mathcal{G} \to \mathcal{E}^{\ast} \to \mathcal{C}_{x} \to 0$ for some point $x \in X$. Therefore, by Thm. 5 (i), the class $\delta(W)$ is of the form $[q]$ where $q \in \text{Prin}(\text{Hom}(E^{\ast}, E)) \simeq \text{Hom}(\mathcal{E}^{\ast}, \text{Prin}(E))$ has kernel $G$. Clearly $q$ is supported at $x$ with a simple pole along $f \otimes f' f$ for some nonzero $f, f' \in E|_{x}$.

Now since $W$ is symplectic, $t q - q = \overline{\alpha}$ for some global rational section $\alpha$ of $\text{Hom}(E^{\ast}, E)$. Since $\overline{\alpha}$ is clearly antisymmetric, $\frac{\alpha - t \overline{\alpha}}{2}$ defines a global rational section of $\det E$ with principal part $\overline{\alpha}$. Now if $\alpha$ is nonzero then it has a single simple pole at $x$, so is a global regular section of $(\det E)(x)$. But this is nonzero only if $\det E = O_{X}(-x)$, which it is not since $E$ is general. Therefore $f'$ is proportional to $f$ and $\langle \delta(W) \rangle = \psi(f)$.
Conversely, suppose that $\delta(W)$ lies over $\psi(f)$ for some nonzero $f \in E|_x$. By the definition of $\psi$, the class $\delta(W)$ can be represented by $q \in \text{Prin}(\text{Sym}^2 E)$ supported at one point $x \in X$ and with a simple pole along $f \otimes f$. By Theorem 5 (i), the kernel of $q$ lifts to a subbundle of $W$ which clearly has rank 2 and degree 0. □

In summary, we have shown that the locus of extensions in a general $\mathbb{P}_E^5$ containing a subbundle of degree 0 is of dimension 2. The complement of this locus in $\mathbb{P}_E^5$ consists of classes defining stable vector bundles. Lemma 18 follows. □

Maximal Lagrangian subbundles

Let $E \in U_X(2, -1)$ be general and consider a general stable symplectic extension $W$ of $E^*$ by $E$. We will show that $W$ has finitely many Lagrangian subbundles of degree $-1$. Suppose $F \subset W$ is such a subbundle. There are three possibilities for the rank of the sheaf $\mathcal{F} \cap \mathcal{E}$: these are 2, 1 and 0. If it is 2 then it is not hard to see that $F = E$.

**Proposition 20** A general extension in $\mathbb{P}_E^5$ contains no subbundles of rank 2 and degree $-1$ intersecting $E$ generically in rank 1.

**Proof** (idea from Lange–Newstead [14], Prop. 2.4)

Let $F \subset W$ be such a subbundle; if $W$ is stable then clearly $F$ is too. We write $\mathcal{L}$ for the subsheaf $\mathcal{F} \cap \mathcal{E}$, which is invertible because, say, $\mathcal{E}$ is locally free. By stability, $\text{deg}(\mathcal{L}) \leq -1$. The image $\mathcal{M}$ of $\mathcal{F}$ in $\mathcal{E}^*$ is coherent, hence invertible because $\mathcal{E}^*$ is locally free. Since $E^*$ is stable, $\text{deg}(\mathcal{M}) \leq 0$. But we also have

$$\text{deg}(\mathcal{M}) = -1 - \text{deg}(\mathcal{L}) \geq 0.$$  

Thus $\text{deg}(\mathcal{M}) = 0$ and $\text{deg}(\mathcal{L}) = -1$. By stability, these subsheaves must in fact correspond to line subbundles by Lemma 7. Therefore, by Lemma 6 there are at most four possibilities for each of the corresponding line subbundles of $E$ and $E^*$.

Conversely, let $L \subset E$ and $M \subset E^*$ be line subbundles of degree $-1$ and 0 respectively, and let $0 \to L \to F \to M \to 0$ be a nontrivial extension. We want to know when the composed map $g_F: F \to M \to E^*$ factorises via $W$. By Lemma 17, this is equivalent to $\delta(W)$ belonging to the kernel of the induced map

$$g^*_F: H^1(X, \text{Sym}^2 E) \to H^1(X, \text{Hom}(F, E)).$$  

Clearly, this factorises via $H^1(X, \text{Hom}(M, E)) = \mathbb{C}^3$. 

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Now if $F$ were isomorphic to $E$, we would have a nonzero bundle map $E \to E^*$, contradicting genericity. Since $F$ and $E$ are stable, this implies that $h^0(X, \text{Hom}(F, E)) = 0$. Therefore we have an exact cohomology sequence

$$0 \to H^0(X, \text{Hom}(L, E)) \to H^1(X, \text{Hom}(M, E)) \to H^1(X, \text{Hom}(F, E)) \to H^1(X, \text{Hom}(L, E)) \to 0.$$  

Since $h^0(X, \text{Hom}(L, E)) = 1$, the map

$$H^1(X, \text{Hom}(M, E)) \to H^1(X, \text{Hom}(F, E))$$

has rank 2. But we recall from the proof of Lemma 18 that the map $H^1(X, \text{Sym}^2 E) \to H^1(X, \text{Hom}(M, E))$ is surjective. Therefore the rank of $g_F^*$ is also 2, so its kernel is of dimension 4.

We now make a dimension count. There is a 0-dimensional choice for $L$ and $M$, a 1-dimensional choice for the extension $F$ and each $g_F^*$ has a 4-dimensional kernel, giving a $\mathbb{P}^3$ in $\mathbb{P}_E^5$. Hence the set of extensions in $\mathbb{P}_E^5$ admitting such an $F$ is of codimension at least 1 in $\mathbb{P}_E^5$. The proposition follows.  

The last possibility is that $\dim(F|_x \cap E|_x) = 0$ for all but finitely many $x \in X$. Thus $F$ is an elementary transformation

$$0 \to \mathcal{F} \to \mathcal{E}^* \to T \to 0$$

for some torsion sheaf $T$ of length 2. The following result is in the spirit of Lange–Narasimhan [13], Prop. 1.1. We recall that a 2-secant to a variety $Y \subset \mathbb{P}^n$ is the projective linear span of two distinct points of $Y$ or else a line tangent to some point of $\psi(\mathbb{P}E)$.

**Lemma 21** Let $W$ be a stable symplectic extension of $E^*$ by $E$. Then the number of degree $-1$ elementary transformations of $E^*$ which lift to isotropic subbundles of $W$ is bounded by the number of 2-secants to the surface $\psi(\mathbb{P}E) \subset \mathbb{P}_E^5$ which pass through $\langle \delta(W) \rangle$.

**Proof**
Consider an elementary transformation $0 \to \mathcal{F} \to \mathcal{E}^* \to T \to 0$ such that $F$ lifts to an isotropic subbundle of $W$ of degree $-1$. Since $E$ and $E^*$ are stable and $\mu(E^*) > \mu(E)$, there are no maps $E^* \to E$. Therefore, by Thm. 5 (ii) the sheaf $\mathcal{F}$ is the kernel of a symmetric $p \in \text{Prin}(\text{Sym}^2 E)$ such that $\delta(W) = [p]$. There are three possibilities for $T$, which is identified with the image of $p$: $\mathcal{E}^* \to \text{Prin}(E)$:
(i) $C_x \oplus C_y$ for distinct $x, y \in X$.

(ii) $C_x^{\oplus 2}$ for some $x \in X$.

(iii) $C_{2x}$ for some $x \in X$.

We treat each case in turn.

(i) Here $p$ is a sum of two principal parts $p_1$ and $p_2$ with simple poles along vectors $g \otimes g \in \text{Sym}^2 E|_x$ and $h \otimes h \in \text{Sym}^2 E|_y$ respectively. They are symmetric because $p$ is and decomposable because $p$ is not surjective to $E(x)$ or $E(y)$. We claim that $\psi\langle g \rangle \neq \psi\langle h \rangle$. For, otherwise there is a relation $[p_1] = \lambda [p_2]$ for some $\lambda \in \mathbb{C}^*$. But then $\delta(W)$ is proportional to $[p_1]$, say, so $\langle \delta(W) \rangle \in \psi(E)$. By Prop. 19, then, $W$ is not stable, contrary to hypothesis.

Thus $\langle \delta(W) \rangle$ lies on the secant spanned by the distinct points $\psi\langle g \rangle$ and $\psi\langle h \rangle$.

(ii) Here $p$ is supported at one point $x \in X$. The image of the map $p: E^* \to \text{Prin}(E)$ is isomorphic to $T$, so is of length 2 and has only simple poles. Since $E$ is of rank 2, the map $p$ is surjective to the subsheaf

$$\frac{E(x)}{E} \subset \text{Prin}(E).$$

Therefore $p$ has a simple pole along some indecomposable vector in $\text{Sym}^2 E$ and is otherwise regular. Such a vector can be written as $g \otimes g + h \otimes h$ for some linearly independent $g, h \in E|_x$ (for example, because the image of $\langle g \rangle \mapsto \langle g \otimes g \rangle$ is nondegenerate in $\mathbb{P} \text{Sym}^2 E|_x$). As above, if $\psi\langle g \rangle = \psi\langle h \rangle$ then $\langle \delta(W) \rangle$ belongs to the image of $\psi$, contradicting stability. Thus $\langle \delta(W) \rangle$ lies on the secant to $\psi\langle g \rangle$ and $\psi\langle h \rangle$.

(iii) Again, $p$ is symmetric and supported at one point $x \in X$, where this time it has a double pole. Since $p$ is not surjective as a map $E^* \to E(2x)|_x$, the double pole must be along a decomposable vector $g \otimes g \in \text{Sym}^2 E|_x$. The image of $p$ is a torsion sheaf of length 2, so it must be equal to the subsheaf of $E(2x)/E$ of principal parts with poles of order up to 2 in the direction of $g \subset E|_x$, and $p$ has poles in no direction other than $g \otimes g$.

Recall that $\Upsilon \to \mathbb{P}E$ is the line bundle $\pi^*K_X \otimes O_{\mathbb{P}E}(2)$. Let $e, f$ be a local basis for $E$ near $x$ such that $e|_x = g$. Examining the local expression from the proof of Lemma 14 for a section of $\Upsilon$ vanishing to order 2 at $\langle g \rangle =: u$,
we see that the contraction of such a section against a principal part of the form
\[ \frac{e \otimes e}{z^2} + \frac{\lambda e \otimes e}{z} \]
for any \( \lambda \in \mathbb{C} \), is regular. Thus \( \langle [p] \rangle \) belongs to the embedded tangent space
\[ \mathbb{P} \ker \left( H^0(\mathbb{P}E, \Upsilon)^* \to H^0(\mathbb{P}E, \Upsilon - 2\langle g \rangle)^* \right) \]
to \( \psi(\mathbb{P}E) \) at \( \psi\langle g \rangle \). In particular, \( \langle [p] \rangle \) lies on a 2-secant to \( \psi(\mathbb{P}E) \). This completes the proof of Lemma 21.

\[ \square \]

**Remark:** It is intriguing that in case (iii), in fact \( \langle \delta(W) \rangle \) belongs to a particular line in the embedded tangent space to \( \psi(\mathbb{P}E) \), namely that spanned by classes of principal parts with single and double poles along \( v \otimes v \). The exact relationship between secants to \( \psi(\mathbb{P}E) \) and subsheaves lifting to \( W \) is still under investigation.

We return to the proof of Theorem 4. By Lemma 14, there are at most a finite number of points where \( \psi \) fails to be an embedding. Therefore, the union of all 2-secants to \( \psi(\mathbb{P}E) \) is of dimension at most 5 and through a general point of \( \mathbb{P}_E^5 \) there pass a finite number of 2-secants to \( \psi(\mathbb{P}E) \).

Let \( W \) be a general stable symplectic extension of \( E^* \) by \( E \). By Prop. 20 and Lemma 21, the bundle \( W \) has finitely many mutually nonisomorphic isotropic subbundles of rank 2 and degree \(-1\). Hence \( W \) is represented in a finite number of fibres of \( \mathbb{B} \).

**Proposition 22** Let \( E \to X \) be a stable bundle of rank 2 and degree \(-1\) such that there are no nonzero maps \( E \to E^* \). Let \( W \) and \( W' \) be two extensions of \( E^* \) by \( E \). Then \( W \cong W' \) if and only if \( \delta(W') = \lambda \delta(W) \) for some \( \lambda \in \mathbb{C^*} \).

**Proof**
This is a special case of Narasimhan–Ramanan [18], Lemma 3.3.

Write \( U \) for the dense subset \( \{ E \in \mathcal{U}_X(2, -1) : h^0(X, \text{Hom}(E, E^*)) = 0 \} \) of \( \mathcal{U}_X(2, -1) \) and denote its complement \( \Delta \). We observe that a general bundle \( W \in \mathcal{M}_2 \) cannot contain an isotropic subbundle \( E \) belonging to \( \Delta \): otherwise, the restriction of \( \Phi \) to \( \mathbb{B}_{|\Delta} \) would be dominant, which is impossible since \( \dim(\mathbb{B}_{|\Delta}) = 9 \) and \( \dim(\mathcal{M}_2) = 10 \). Thus we can assume that the finitely many fibres of \( \mathbb{B} \) in which our general \( W \) is represented all lie over \( U \). By Prop. 22, it occurs only once in each of these fibres. Thus the fibre of \( \Phi \) over \( W \) is finite. By semicontinuity, \( \Phi \) is generically finite. Since \( \mathbb{B} \) and \( \mathcal{M}_2 \) are...
of the same dimension, $\Phi$ is dominant. Since it a morphism of projective varieties, its image is closed, so it is actually surjective.

**The degree of $\Phi$**

We use some results of Lange and Newstead to calculate the degree of $\Phi$.

**Theorem 23** Let $W$ be a general vector bundle of rank $n$ and degree $d$ over $X$. Suppose that $n \geq 4$ is even and $2d + 4 \equiv 0 \mod n$ with $\frac{2d+4}{n}$ odd. Then the number of subbundles of rank 2 and maximal degree, counted with multiplicity, is equal to $\frac{n^3}{15}(n^2 + 2)$.

**Proof**

Lange–Newstead [14], pp. 7–10.

We check that a general $W \in \mathcal{M}_2$ is general enough in the sense of Lange and Newstead. We follow the notation of [14]: for a subbundle $F \subseteq W$, we define

$$s(W, F) = -\text{rk}(W)\deg(F) + \text{rk}(F)\deg(W)$$

and

$$s_{n'}(W) := \min\{s(W, F) : F \subseteq W \text{ of rank } n'\}.$$ 

For us, $n' = 2$, $g = 2$ and $n = 4$. We check the generality conditions, which are listed on p. 6 of [14]:

(i) $s_{n'}(W) = n'(n - n')(g - 1)$.

(ii) $s_{n_1}(W) \geq n_1(n - n_1)(g - 1)$ for all $n_1 \in \{1, \ldots, n' - 1\}$.

(iii) $W$ has only finitely many maximal subbundles of rank $n'$.

For (i), note that $s_2(W) = \min\{-4\deg(F) : F \subseteq W \text{ of rank } 2\} = 4$ since $W$ is stable but contains a Lagrangian subbundle of degree $-1$. On the other hand, $n'(n - n')(g - 1) = 2(4 - 2)(2 - 1) = 4$.

In (ii), the only value of $n_1$ that we need to check is 1. We have

$$s_1(W) = \min\{s(W, L) : L \text{ a line subb. of } W\}$$

$$= \min\{-4\deg(L) : L \text{ a line subb. of } W\}$$

$$= 4$$

since $W$ is stable by Lemma 18. On the other hand, we have

$$n_1(n - n_1)(g - 1) = 3.$$
As for (iii): we saw in the preceding section that apart from $E$, all maximal rank 2 subbundles of a general $W$ lift from elementary transformations of $E^*$. By Lemma 21, there are at most finitely many isotropic subbundles of this type. It will suffice, therefore, to show that all subbundles of rank 2 and degree $-1$ of a general $W$ are isotropic.

Let $F \subset W$ be such a subbundle. The symplectic form on $W$ restricts to a global section of $\wedge^2 F^*$, that is, $(\det F)^{-1}$. To compute this line bundle, we note that $F$ fits into a short exact sequence

$$0 \to F \to E^* \xrightarrow{p} T \to 0$$

where $T$ is a torsion sheaf of length 2 and $p \in \text{Prin}(\text{Hom}(E^*, E))$ is such that $\delta(W) = [p]$. We have $\text{Supp}(T) = \text{Supp}(p) = \{x, y\}$ for points $x, y \in X$ which are not necessarily distinct. Thus

$$(\det F)^{-1} = (\det E)(x + y).$$

This has a nonzero section only if it is effective. In order for $W$ to contain a nonisotropic subbundle of rank 2 and degree $-1$, then, $\langle \delta(W) \rangle$ must lie on a line joining points in the fibres of $\psi(\mathbb{P}E)$ over $x$ and $y$ for some $x + y$ such that $\det(E)(x + y)$ is effective. For each such divisor $x + y$ the family of such classes $\langle \delta(W) \rangle$ in $\mathbb{P}^5_E$ has dimension at most

$$\dim(\mathbb{P}E|_x) + \dim(\mathbb{P}E|_y) + \dim(\mathbb{P}^1) = 3.$$

By genericity, $(\det E)^{-1}$ is not effective so $\det E$ is of the form $O_X(q - r - s)$ with $O_X(r + s) \neq K_X$. We determine the divisors $x + y$ such that the bundle $(\det E)(x + y)$ is effective, that is,

$$O_X(q - r - s + x + y) = O_X(t) \text{ for some } t \in X.$$

Equivalently, we seek all solutions $x, y, t \in X$ to the equation

$$O_X(q + x + y) = O_X(r + s + t).$$

Let $t \in X$. If $|r + s + t|$ has a base point then this must be $r$ or $s$ because $r + s \not\in |K_X|$, so in particular $q$ is not a base point. Therefore there is a unique divisor $x + y$ such that $q + x + y \sim r + s + t$. Letting $t$ vary, we find a one-dimensional family of pairs $x + y$ such that $\det(E)(x + y)$ is effective.

Furthermore, if we had started with a different representative $q' - r' - s'$ for $\det E$ then we would have obtained the same family of $x + y$. Explicitly,

$$O_X(q + x + y) = O_X(r + s + t) \iff q - r - s \sim t - x - y$$

$$\iff q' - r' - s' \sim t - x - y \text{ by hypothesis}$$

$$\iff O_X(q' + x + y) = O_X(r' + s' + t).$$

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Also, since for each \( t \in X \) there exists unique such \( x + y \), these are all the solutions.

Putting all this together, we find that the locus of \( \langle \delta(W) \rangle \) in \( \mathbb{P}_E^5 \) such that \( W \) can contain a nonisotropic subbundle of rank 2 and degree \(-1\) is of dimension at most 4. Therefore, for general \( \langle \delta(W) \rangle \in \mathbb{P}_E^5 \), all maximal subbundles lifting from elementary transformations of \( E^* \) are isotropic.

Thus \( W \) has finitely many maximal subbundles of rank 2. In summary, we find that a general symplectic bundle is indeed general in the sense of Lange and Newstead.

In our case, the number \( \frac{n^3}{12} (n^2 + 2) \) is 24. To verify that \( \deg \Phi = 24 \), we need to check that all the subbundles of a general \( W \in \mathcal{M}_2 \) are isotropic and distinct. We have just checked isotropy. For distinctness: let \( E \subset W \) be a maximal subbundle of rank 2 and suppose that \( h^0(X, \text{Hom}(E, W)) \geq 2 \). By generality, some copy of \( E \) in \( W \) is isotropic, so \( W/E \cong E^* \) and we get a diagram

\[
\begin{array}{c}
0 \rightarrow E \rightarrow W \rightarrow E^* \rightarrow 0 \\
\end{array}
\]

where the composed map is nonzero. But this yields a nonzero map \( E \rightarrow E^* \), contradicting genericity.

In summary, \( \Phi : \mathbb{B} \rightarrow \mathcal{M}_2 \) is generically finite of degree 24. This completes the proof of Theorem 4, our main result. \( \square \)

**Remark:** Some fibres of \( \Phi \) may be of positive dimension. (This is analogous, for example, to the fact that there exist vector bundles of rank 2 and degree 0 over a curve of genus 3 which admit infinitely many maximal line subbundles.) If, for example, \( \psi : \mathbb{P}E \rightarrow \mathbb{P}_E^5 \) failed to be an embedding over a locus of positive dimension in \( \mathbb{P}E \) then there might be points of \( \mathbb{P}_E^5 \) lying on infinitely many 2-secants to \( \psi(\mathbb{P}E) \). Then Lemma 21 might lead us to expect that that infinitely many degree \(-1\) elementary transformations of \( E \) would lift to the corresponding extensions of \( E^* \) by \( E \). This will be a subject of future study.

## 5 Applications and future work

We give one immediate application of this construction.
Corollary 24  The moduli space $\mathcal{M}_2$ is of Kodaira dimension $-\infty$.

Proof
Recall that we have a proper surjective map $\mathbb{B} \to \mathcal{U}_X(2, -1)$ whose fibres are projective spaces. Hence $\mathbb{B}$ is uniruled, so is of Kodaira dimension $-\infty$. Since by Thm. 4 there is a morphism $\mathbb{B} \to \mathcal{M}_2$ which is surjective and generically finite, the same is true for $\mathcal{M}_2$. □

Another application of Thm. 4 is given, in the last chapter of [8], to the study of theta divisors of symplectic vector bundles over a curve of genus 2. We sketch the result. Let $W$ be a semistable symplectic bundle of rank 4 and consider the set

$$S(W) = \{ L \in J^1_X : h^0(X, \text{Hom}(L, W)) > 0 \}. $$

For general $W$, this set is the support of an even $4\Theta$ divisor $D(W)$, and this defines a rational map $D : \mathcal{M}_2 -\to |4\Theta|_+ = \mathbb{P}^9$. The line bundle $D^*O(1) =: \Xi$ is called the determinant bundle on $\mathcal{M}_2$.

For certain $W$, however, $S(W)$ is the whole Jacobian; equivalently, $W$ is a base point of $|\Xi|$. If one wishes to find such bundles, by Thm. 4 it suffices to look for them in the spaces $\mathbb{P}^5_E$. In [8], Chap. 6, we find necessary conditions on $E \in \mathcal{U}_X(2, -1)$ for the existence of an extension $W$ in $\mathbb{P}^5_E$ such that $S(W) = J^1_X$. We hope that this will lead to a full description of the base locus of $|\Xi|$. See Beauville [2], [4] and Raynaud [24] for general results on this type of question.

It is natural to ask whether a similar description of $\mathcal{M}_2$ exists in higher genus. One calculates that for the dimensions to work out as in the genus 2 case, we should consider symplectic extensions $0 \to E \to W \to E^* \to 0$ where $E \to X$ is a vector bundle of rank 2 and degree $1 - g$. The $\mathbb{P}^5_E$ are replaced by projective spaces of dimension $6g - 7$. However, there is no Poincaré bundle over $\mathcal{U}_X(2, g - 1) \times X$ if $g - 1$ is not relatively prime to 2, so we are forced to restrict to the case of even genus.

If symplectic bundles in higher even genus are still general enough then, by Lange–Newstead [14], Prop. 2.4, the problem involves determining those elementary transformations $0 \to F \to \mathcal{E}^* \to T \to 0$, where $T$ is a torsion sheaf of length $2g - 2$, which lift to the above $W$. Now one expects that the $(2g - 2)$nd secant variety to a nondegenerate surface in $\mathbb{P}^{6g-7}$ with a finite number of singular points will be of dimension $6g - 7$, so Lemma 21 might lead us to conjecture that such an extension would indeed have only finitely many Lagrangian subbundles of degree $g - 1$. This will be studied in the future.
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