The Signature Problem for Embedded Space-times

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March 24, 2022

Abstract

The compatibility between general relativity and the property that space-times are embeddable manifolds is further examined. It is shown that the signature of the embedding space is uniquely determined provided the embedding space is real and its dimension is kept to the minimal. Signature changes produce complex embeddings which in turn may induce topological changes in the space-time. Space-time signature preserving symmetries identify the twisting vector as a real connection on space-time whose curvature is described by the Ricci’s equation in terms of the second fundamental form.

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1 Introduction

The four-dimensionality of space-time is deeply rooted in experimental facts. On such grounds there is no direct evidence to support the idea of a higher dimensional physical space, at least with today’s available level of high energy physics ($\approx 10^{19}$ GeV). Still, the hypothesis of higher dimensional of space-time $M_D$ appears to be consistent with the extra degrees of freedom required by some unified theories. The two best known examples of geometric unification are based on higher dimensions. Kaluza-Klein theory, mimics general relativity to the extent that it uses the same Einstein-Hilbert action with the metric as the dynamical variable. On the other hand, string theory appeals to the notion of a minimal manifold, using the Nambu-Goto action with the embedding coordinates as the dynamical variables.

In those high dimensional models, supposedly all dimensions were once, all accessible. However, at some later time, $D - 4$ of those dimensions become invisible so to speak, to any observer with "low energy" probes (anything below Planck’s energy: $10^{19}$GeV). Such "dimensional reduction" is usually explained by the ingenious spontaneous compactification mechanisms $M_D \to V_4 \times B_n$ where $V_4$ is the four dimensional space-time and $B_n$ is some compact internal space with distinct characteristics in each theory [1]. In these particular constructions, there are no differential-geometric constraints imposed on the way $V_4$ and $B_n$ are put together, resulting that the compact internal spaces play little role in the classical dynamics of $V_4$.

Another, perhaps more natural, way of introducing higher dimensions is to use the fact that all manifolds, including space-times are embeddable into some higher dimensional space $M_D$ [2], [3]. However, it is not clear that this mathematical property is compatible with the physics of the space-times. There are at least two basic problems which must be solved before we can make good use of the space-time embeddings. The first problem (the signature problem) refers to the existence of different embedding signatures for the same space-time. The other problem refers to the physical role, if any, of the extrinsic curvature.

The number of dimensions $D$ depends on the differentiable nature of the embedding functions. If we agree with Janet and Cartan that those functions are analytic, then the embedding space has dimension $D \leq d(d + 1)/2$ [4]. However, analytic functions may be too special as compared to differentiable functions to describe high energy physics. If we assume the more likely differentiable embedding, the number of dimensions becomes $D \leq d(d + 3)/2$ [5]. Of course, in most known situations we need less than those limits and it makes sense to adopt a principle of economy of dimensions: If a given space-time has been proven to be embeddable in $D$ dimensions, then we will not use more than $D$ dimensions. Furthermore, the fact that it is always possible to find an embedding in a flat space, we will assume that our embedding space is flat. However, we admit that physics may not act that way.
As in the Kaluza-Klein model, it is possible that the embedding space is dynamical with a curvature and topology that changes with time.

Consider a space-time $V_4$ with metric $g_{ij}$, solution of Einstein’s equations and its local isometric embedding in a flat $D$-dimensional manifold $\mathcal{M}_D$. That is, a 1:1 map

$$\mathcal{Y} : V_4 \rightarrow \mathcal{M}_D$$

such that

$$g_{ij} = \mathcal{Y}^\mu \mathcal{Y}^\nu \partial_\mu \partial_\nu, \quad \mathcal{N}_A^\mu \mathcal{Y}^\nu \partial_\mu \partial_\nu = 0, \quad \mathcal{N}_A^\mu \mathcal{N}_B^\nu \partial_\mu \partial_\nu = g_{AB} = \epsilon_A \delta_{AB} \quad (1)$$

where $x^i$ are coordinates in space-time and $\mathcal{N}_A$ are $D - 4$ vector fields orthogonal to the embedded space-time. Here $\epsilon_A = \pm 1$ and $\partial_\mu \partial_\nu$ denote the components of the metric of $\mathcal{M}_D$ in the embedding coordinates $\mathcal{Y}^\mu$. If we prefer, we may use Cartesian coordinates where the metric of $\mathcal{M}_D$ has components $\eta_{\mu \nu}$.

The embedding functions $\mathcal{Y}^\mu (x^i)$ can be obtained by integrating the Gauss and Weingarten equations:

$$\mathcal{Y}^\mu_{;ij} = g^{MN} b_{ijA} \mathcal{N}^\mu_N \quad (2)$$

$$\mathcal{N}_A^\mu_{;j} = -g^{mn} b_{jmA} \mathcal{Y}^\mu_{,n} + g^{MN} A_{jAM} \mathcal{N}^\mu_N \quad (3)$$

where $b_{ijA}$ are the components of the second fundamental form and $A_{iAB}$ are the components of the twisting vector. Since $\mathcal{M}_D$ is flat we may always choose $\mathcal{Y}^\mu$ as Cartesian coordinates we obtain explicitly

$$b_{ijA} = -\mathcal{Y}^\mu_{;i} \mathcal{N}^\nu_{A,j} \eta_{\mu \nu} = \mathcal{Y}^\mu_{,ij} \mathcal{N}_A^\nu \eta_{\mu \nu} \quad (4)$$

It follows that $b_{ijA}$ is symmetric in the first two indices. Likewise, the expression of the twisting vector is:

$$A_{iAB} = \mathcal{N}_A^\mu \mathcal{N}^\nu_{B,i} \eta_{\mu \nu} \quad (5)$$

so that $A_{iAB} = -A_{iBA}$. These two quantities, determine completely the extrinsic geometry of the space-time, giving a measure of the local shape of the space-time as compared to the tangent space. Obviously, if the embedding of the space-time is given by the embedding coordinates as for example in [7], then all we have to do is to calculate $\mathcal{N}_A^\mu$, $b_{ijA}$ and $A_{iAB}$ [8]. However, if we do so we learn very little over what we already know from the intrinsic geometry. The situation may be different if we assume that the embedding is not known but that it results from the space-time dynamics. Since in general $b_{ijA}$ and $A_{iAB}$ are independent of the metric, we may express this dynamics in terms of those variables instead of the embedding.

\[\text{Lower case Latin indices run from 1 to 4 and capital Latin indices run from 5 to D, where D is the smallest possible embedding dimension. All Greek indices run from 1 to D. The indicated antissymmetrization applies only to the indice of the same kind near the brackets.}\]
coordinates $Y^\mu$. The integrability conditions for (2) and (3) are the well known Gauss-Codazzi-Ricci equations for sub manifolds which may be written as

\begin{align}
R_{ijkl} &= 2g^{MN}b_{i[kM}b_{j]lN}, \\
b_{i[A;k]} &= g^{MN}A_{l[kM}b_{j]lN}, \\
A_{[jAB;k]} + g^{MN}A_{[jMA}A_{k]NB} &= -g^{mn}b_{m[jA}b_{k]nB}.
\end{align}

There are some specific procedures for integrating these equations, as for example in \cite{9},\cite{10}. In the next section we look at the signature problem and in section 3 we deal with the interpretation of the twisting vector as a gauge field.

2 The Signature Problem

Assuming that the space-time has Lorentz signature, then the embedding space has necessarily a pseudo Euclidean signature, possibly with several time-like directions, one of them necessarily lying on the tangent plane. It is possible to find different embedding signatures for the same space-time. If the extrinsic properties of the space-time are to be physically relevant, then such ambiguity is not acceptable. The mentioned principle of economy of dimensions has the following consequence:

**Theorem 1** IF $D$ is the smallest dimension in which we can isometrically embed a non-flat space-time $V_4$ in a real space $M_D$, then the signature of this space is unique.

Suppose that we have two embeddings of the same space-time $Y : V_4 \to M_D$ and $Y' : V_4 \to M'_D$ which differ only in signature: $p - q$ in the first case and $p' - q'$ in the second case. Since the tangent spaces to $V_4$ have the same Minkowski signature, without loss of generality they may be identified. That is, we may define a map $T : M_D \to M'_D$ such that its derivative $T^*$ restricted to the tangent space $TV_4$ is the identity: $T^*|_{TV_4} = T^*_1 = Id$. On the other hand, the restriction $T^*_2$ to the subspaces $V_4^\perp$ of $M_D$ orthogonal to the space-time, $T^*_2|_{TV_4^\perp}$ is a general linear transformation (figure 1). In terms of the embedding coordinates and normal vectors this is equivalent to

\begin{equation}
Y'_\mu,\iota = Y_{\mu,i}, \text{ and } \mathcal{N}'_B = T^A_B\mathcal{N}_A
\end{equation}

From (4), the second fundamental form transforms as $b'_{ijB} = T^A_Bb_{ijA}$ so that for the second embedding Gauss equation is

\begin{equation}
R_{ijkl} = 2g^{MN}b'_{i[kM}b'_{j]lN}.
\end{equation}
Fig. 1: Two embeddings of $V_4$

Comparing with Gauss’ equation in (6) we obtain

$$2 \sum_{A=5}^{D} g^{\mu AB} b_{i[kA]b[|j]B} = 0,$$

(9)

where we have denoted

$$g^{\mu AB} = g^{AB} - g^{MN} T^A_M T^B_N.$$  

(10)

It remains to see if (9) admits a non trivial solution $g^{\mu AB}$ of the form $\epsilon_A^\mu \delta_{AB}$, with $\epsilon^\mu_A = \pm 1$. We have the following possibilities

a) All $g^{\mu AB}$ coincide with $g^{AB}$:

$g^{\mu AB} = g^{AB} \ \forall \ A, B = 5, ..., D$. In this case we have $g^{MN} T^A_M T^B_N = 0$ which is not possible since the left hand side of equation (8) becomes identically zero, contradicting the hypothesis of a non flat $V_4$.

b) Only some values of $g^{\mu AB}$ coincide with $g^{AB}$.

For example, suppose that $g^{\mu AB} \neq g^{AB}$ for $A, B = 5, ..., D_1$ and $g^{\mu AB} = g^{AB}$ for $A, B = D_1 + 1, ..., D$, where $5 < D_1 < D$. From (9), it follows that

$$2 \sum_{A,B=5}^{D_1} g^{\mu AB} b_{i[kA]b[|j]B} + 2 \sum_{A,B=D_1+1}^{D} g^{AB} b_{i[kA]b[|j]B} = 0.$$

Therefore, replacing the last term in the Gauss equation of (6), we get

$$R_{ijkl} = 2 \sum_{A,B=5}^{D_1} g^{AB} b_{i[kA]b[|j]B} + 2 \sum_{A,B=D_1+1}^{D} g^{AB} b_{i[kA]b[|j]B} = 2 \sum_{A,B=5}^{D_1} (g^{AB} - g^{\mu AB}) b_{i[kA]b[|j]B}.$$

Since, the quadratic form $g^{AB} - g^{\mu AB}$ can always be diagonalized, we may write $g^{AB} - g^{\mu AB} = g^{\nu AB} = \epsilon_A^\nu \delta_{AB}$. Therefore the last equation corresponds to Gauss’
In our case, the complexification of $M$ is a complex structure. The resulting complex manifold $M$ time, except for a difference in topology: In Kasner \[7\]: $F_{\text{K}}$, and Fronsdal \[12\]: $F_{\text{F}}$.

\[ (11) \]

\[
\begin{cases}
+ : \mathcal{M}_D \times \mathcal{M}_D \to \mathcal{M}_D \times \mathcal{M}_D \text{ given by } (u, v) + (w, x) = (u + w, v + x) \\
* : \mathcal{M}_D \times \mathcal{M}_D \to \mathcal{M}_D \times \mathcal{M}_D \text{ given by } (u, v) * (w, t) = (uw - vt, vw + ux)
\end{cases}
\]

In our case, the complexification of $M$ induced by $T$ occurs only on the subspace of $M$ orthogonal to the space-time $V_4$ which remains real and preserves its light cone structure. The resulting complex manifold $\mathcal{M}_D / \mathcal{C}$, defines a "complex embedding" of a real space-time.

As a classic example of the signature change problem consider two well known embeddings of the Schwarzschild space-time in six dimensional pseudo Euclidean flat spaces:

Kasner \[4\]: $K : V_4 \to M_6$, $ds^2 = dY_1^2 + dY_2^2 - dY_3^2 - dY_4^2 - dY_5^2 - dY_6^2$.

Fronsdal \[12\]: $F : V_4 \to M_6'$, $-ds^2 = dY_1^2 - dY_2^2 - dY_3^2 - dY_4^2 - dY_5^2 - dY_6^2$, given by (here we assume mass units such that $2m = 1$):

\[
K \begin{cases}
\mathcal{Y}_1 = (1 - 1/r)^{1/2}\cos \theta \\
\mathcal{Y}_2 = (1 - 1/r)^{1/2}\sin \theta \\
\mathcal{Y}_3 = f(r), \ (df/dr)^2 = \frac{1 + 4\varphi^2}{4R^2(r-1)} \\
\mathcal{Y}_4 = r\sin \theta \sin \phi \\
\mathcal{Y}_5 = r\sin \theta \cos \phi \\
\mathcal{Y}_6 = r\cos \theta
\end{cases}
\quad \text{and} \quad F \begin{cases}
\mathcal{Y}'_1 = 2(1 - 1/r)^{1/2}\sinh(t/2) \\
\mathcal{Y}'_2 = 2(1 - 1/r)^{1/2}\cosh(t/2) \\
\mathcal{Y}'_3 = g(r), \ (dg/dr)^2 = \frac{(r^2 + r - 1)}{r^2} \\
\mathcal{Y}'_4 = r\sin \theta \sin \phi \\
\mathcal{Y}'_5 = r\sin \theta \cos \phi \\
\mathcal{Y}'_6 = r\cos \theta
\end{cases}
\]

In the first case we have two time-like dimensions while in the second case we have only one (we are using $-ds^2$ instead of $ds^2$. Both correspond to the same space-time, except for a difference in topology: In $K$, the space-time extends only to $r = 1$,
while in $F$ it extends to $r=0$. The second embedding corresponds in fact to Kruskal’s space-time, or the maximal analytic extension of the Schwarzschild space-time.

Notice that the embedding defined by Kasner is not causal. Any curve in the plane $(Y_1, Y_2)$ with a parameter range greater than $2\pi$ is closed in that embedding \cite{12}. Since we are required to perform genuine non-local experiments to apply the equivalence principle and to distinguish causal and non causal propagations, we cannot rely on the implicit function theorem alone to characterize an embedding properly. Unless he remains strictly local, an observer in the space-time would be able to detect if his space-time is embedded or not simply by observing a classical breaking of causality. In essence we are saying that the Kasner embedding cannot be used as a physical embedding. Nonetheless, the Schwarzschild space-time can be seen as a subset embedded in Kruskal’s space-time defined by an extension map $\Psi$ \cite{13}. That is, there is a third embedding of Schwarzschild’s space-time, given by composite map $F \circ \Psi$. This embedding is consistent with the one defined by Fronsdal and it has the appropriate signature.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{embedding_diagram}
\caption{Two embeddings of Schwarzschild space-time}
\end{figure}

The matrix representing $T$ is

\[(T^A_B) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.\]

Replacing in (11) with $g_{55} = 1$, $g_{66} = -1$ and $g'_{55} = 1$, $g'_{66} = 1$, we obtain

\[a^2 + b^2 = 1, \quad c^2 + d^2 = -1, \quad ac + bd = 0, \quad (ad - bc)^2 = -1\]

One possible solution is $a = d = 0$, $c = i$, $b = 1$, so that $T$ is indeed complex.

The change of signature of $\mathcal{M}_D$ may have some topological consequences. This can be seen by taking the embedding diagrams for Schwarzschild (Kasner) and Kruskal (Fronsdal) space-times: In figure 3, the circle in the left represents an open sphere $S^2$ which intersects the semi planes $\mathbb{R}^2_1$ and $\mathbb{R}^2_{11}$, excluding the plane $r = 2m = 1$. The corresponding topology is then $(\mathbb{R}^2 \cup \mathbb{R}^2_{11}) \times S^2$. On the other hand, in the right hand side the topology is $\mathbb{R}^2 \times S^2$ \cite{18}.

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The above result shows that by use of complex embeddings it is possible to preserve the space-time signature while altering only the signature of the embedding space. For example, we may use complex transformations to make $M_D$ truly Minkowskian (with just one time like dimension) and use it as a fixed background in a canonical quantization procedure while keeping intact the space-time signature. In this case the group of rotations of the normal vectors is $SO(D - 4)$ whose importance will be seen in the next section. This may be relevant for the recent debate on the need or not of changing the space-time signature to apply path integrals in quantum cosmology (see e.g. [14], [15], [17]), provided it could be made dynamical. That is, considering the embedding equations as part of the dynamical equations, together with Einstein’s equations. In this case, any 4-surface of discontinuity of the second fundamental form $b_{ij}$ may induce a classical change of signature of $M_D$ in a process analogous to that described in [13] and eventual topological changes [16]. To see how this dynamics takes place we need an easier way to interpret the fundamental equations (6) which control the extrinsic curvature of the space-time.

3 The Twisting Connection

In the following we consider the embeddings of a given space-time in a real space with the same signature $p + q$. Therefore the signature preserving symmetry is $T = SO(p - 3, q - 1)$.

The fundamental theorem of sub manifolds says that given the symmetric tensor $g_{ij}$, $D - 4$ tensors $b_{ijA}$ and $(D - 4)(D - 5)/2$ vectors $A_{iAB}$ satisfying (6) then there

\[ \mathcal{R}_{II}^{2} \quad \mathcal{R}_{i}^{2} \quad \Psi \quad \mathcal{M}_{D} (+ + - - - -) \quad \mathcal{M}'_{D} (+ - - - - -) \]

Fig.3: Topology change with signature change

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\(^{2}\)We consider as equivalent signatures which differ only by a factor $-1$, or by a mere relabelling of the embedding coordinates.
is a 4-dimensional sub manifold of a flat space $\mathcal{M}_D$ which has $g_{ij}$ as its metric, $b_{ijA}$ as its second fundamental form and $A_{iAB}$ as its twisting vector.

When we apply this theorem to a space-time, we have an embedded manifold which acts as the arena for low and high energy phenomena. We could well ask what prevents the space-time from “diluting” into the ambient space. That is, why it holds together as a four-dimensional sub manifold of $\mathcal{M}_D$? Put in another way, a high energy particle collision or a pair creation could in principle eject a particle from space-time into the ambient space. However at the current energy level this is not observed. It appears that at this level of energy the space-time submanifold is stable. In the following we show that the twisting vector $A_{iAB}$ may play a role in that dynamics.

Since we are not assuming any external forces acting on the space-time, we may take those curves as geodesics of $\mathcal{M}_D$. For each direction $N_A$ we may define the parameter $x^A$ so that the geodesic coordinates are $Z^\mu = Z^\mu(x^i, x^A)$:

$$\frac{\partial^2 Z^\gamma}{\partial x^A \partial x^A} + \Gamma^\gamma_{\alpha\beta} \frac{\partial Z^\alpha}{\partial x^A} \frac{\partial Z^\beta}{\partial x^A} = 0. \quad (12)$$

For simplicity but without loss of generality, let us take $Z^\mu$ as geodesic coordinate of $\mathcal{M}_D$ (actually, since $\mathcal{M}_D$ is flat we may take Cartesian coordinates). In this case $\Gamma^\gamma_{\alpha\beta} = 0$ and we may write $Z^\mu(x^i, x^A) = \mathcal{Y}^\mu(x^i) + x^A N_A^\mu$. The metric of $\mathcal{M}_D$ in the Gaussian coordinate system $(x^i, x^A)$ is

$$G'_{\alpha\beta} = Z^\mu_{\alpha} Z^\nu_{\beta} G_{\mu\nu} = \left( \tilde{g}_{ij} + x^A x^B g^{MN}_{ij} A_{iMA} A_{jNB} \ g_{iA} \ g_{AB} \right) \quad (13)$$

where we have denoted

$$\tilde{g}_{ij} = g_{ij} - 2x^A b_{ijA} + x^A x^B g^{mn} b_{imA} b_{jnB}, \text{ and } g_{iA} = x^M A_{iMA}.$$ 

Now let us consider a remarkable property of the twisting vector:

**Theorem 2** Under an infinitesimal pseudo rotation of the normal vectors $N$, the twisting vector transforms as:

$$A'_{iAB} = A_{iAB} - f^{EF}_{AB, MN} A_{iEF} \Theta^{MN} - \Theta_{AB,i} \quad (14)$$

where $f^{EF}_{AB, MN}$ denote the structure constants and $\Theta^{MN}$ denote the parameters of the group $SO(p-3,q-1)$.

From (13) we may express the torsion vector as $A_{iAB} = \partial G_{iA}/\partial x^A$. Therefore under an infinitesimal transformation of $SO(p-3,q-1)$:

$$x^i = x^i, \quad x^A = x^A + \xi^A.$$
Keeping only the linear terms in $\xi$, the infinitesimal transformation of $A_{iAB}$ is:

$$A'_{iAB} = \frac{\partial G'}{\partial x^B} = (\delta^M_A - \xi^M_A) \frac{\partial}{\partial x^M} \left( (\delta^\mu_i - \xi^\mu_i)(\delta^\nu_A - \xi^\nu_A) G_{\mu\nu} \right).$$

Since $\xi^k = 0$ and $\xi^A = \Theta^A_M(x^i)x^M$ we end up with

$$A'_{iAB} = A_{iAB} - 2g^{MN}A_{iM[q} \Theta_{NB]} - \Theta_{A[i}g_{M]} \Theta_{A]}^{M}.$$

The Lie algebra of $SO(p - 3, q - 1)$ with generators $L_{AB}$ is given by $[L_{MN}, L_{PQ}] = f_{AB}^{MNPQ}L_{AB}$, where

$$f_{AB}^{MNPQ} = 4\alpha \delta^N_A g^M[P \delta^Q_B] \text{ and } f_{ABMN}^{PQ} = 4\alpha \delta^P_A g^B[M \delta^Q_N],$$

where $\alpha$ is a some normalization constant. Therefore,

$$A'_{iAB} = A_{iAB} - 2A_{iEF} \frac{1}{4\alpha} f^{EF}_{ABMN} \Theta^{MN} - \Theta^{M} g_{MB} A_{iM}.$$

Hence, for $\alpha = 1/2$ we obtain (14), which is the same as the transformation of a gauge potential in Yang-Mills theory, with gauge group $SO(p - 3, q - 1)$.

Defining Lie-algebra valued "twisting" vector field

$$A_i = A_{iAB}L^{AB}.$$

The transformation (14) suggests that $A_i$ induce a gauge-like connection in $V_4$, the twisting connection, with the corresponding “gauge” covariant derivative operator by

$$D_i = \nabla_i + \beta A_i,$$

where $\beta$ is another constant to be appropriately chosen, not necessarily meaning a coupling constant. This covariant derivative acts on Lie algebra valued functions $f$ as $D_if = \nabla_i f + \beta[A_i, f]$. In particular, for scalar functions $f$, $D_if = \nabla_i f$, so that $D_ig_{jk} = 0$. Using the fact that $\nabla_i L^{AB} = 0$, we obtain the commutator

$$[D_i, D_j] = \beta(\nabla_i A_j - \nabla_j A_i + \beta[A_i, A_j]),$$

Next we consider the Clifford algebra associated with the metric $g^{AB}$ defined by

$$E^AE^B + E^BE^A = 2g^{AB}E^0$$

where $E^0$ is the identity element $E^AE^0 = E^0E^A = E^A$. This algebra is closely related with the isometry group of $g^{AB}$. In fact, if the Lie algebra of this group is generated by $L^{AB}$, then [14]

$$L^{AB} = \frac{1}{\gamma}[E^A, E^B],$$

where $\gamma$ is a normalization constant.
where again $\gamma$ is another scale constant to be chosen. The indices $A, B, \ldots$ are raised and lowered with $g_{AB}$ and $g^{AB}$ such that $E^A = g^{AB}E_B$. Therefore given the coefficients of the second fundamental form $b_{ijA}$, we may define the Clifford algebra valued tensors $b_{ij} = b_{ijA}E^A$.

**Theorem 3** If $F_{ij}$ is the curvature associated with the twisting connection, then Codazzi’s and Ricci’s equations are respectively equivalent to

$$D_i[b_{ij}] = 0 \quad (19)$$

$$F_{ij} = -2g^{mn}b_{m[i}b_{j]n} \quad (20)$$

In fact, since $D_i$ and $[D_i, D_j]$ are Lie-algebra valued functions, we may write $[D_i, D_j] = [D_i, D_j]_{AB}L^{AB}$, where we have denoted (from (17))

$$[D_i, D_j]_{AB} = \beta \left( \nabla_iA_{jAB} - \nabla_jA_{iAB} + \beta A_{iMN}A_{jPQ}f^{MNPQ}_{AB} \right). \quad (21)$$

From the definition of structure constants it follows that

$$f^{MNPQ}_{AB}L^{AB} = [L^{MN}, L^{PQ}]_{AB}L^{AB}$$

Therefore, (21) may be written as

$$[D_i, D_j]_{AB} = 2\beta(\nabla_iA_{jAB} + \beta g^{MN}A_{i[kA}A_{j]NB}) \quad (22)$$

Comparing the right hand side of this expression to the left hand side of Ricci’s equation in (3) we obtain with $\beta = -1$, $D_i = \nabla_i - A_i$ and

$$[D_i, D_j] = g^{mn}b_{m[i}b_{j]n} \frac{1}{\gamma}[E^A, E^B] = \frac{4}{\gamma}g^{mn}b_{m[i}b_{j]n} \quad (23)$$

To complete the demonstartion, introduce the notation

$$D^{N}_{kA} = \delta^N_A \nabla_k b_{j+iN} - \beta g^{MN}A_{kAM}.$$  

Then the second equation (6) can be written as

$$D^{N}_{kA}b_{j+iN} = 0, \quad (24)$$

On the other hand, using the definition of $D_i$, the gauge covariant derivative of $b_{ij}$ is given by

$$D_k b_{ij} = \nabla_k b_{ij} - [A_k, b_{ij}] \quad (25)$$

but, we can easily see that

$$[A_k, b_{ij}] = A_k b_{ij} - b_{ij}A_k = \frac{8}{\gamma}g^{AB}A_{kCB}b_{ijA}E^C.$$
Consequently,

\[ D_k b_{ji} = \left( \delta^M_A \nabla_k - \frac{8}{\gamma} g^{MN} A_{kAN} \right) b_{ijM} E^A \]

Comparing with (24) it follows that for \( \gamma = 8 \) we obtain Codazzi’s equation (19):

\[ D_{[k} b_{lj]} = D_{[k} b_{lj]} M E^A = 0 \]

Finally, the curvature associated with \( A_i \) is \( F_{ij} = [D_i, D_j] \), so that from (23) we obtain (20).

As we see, Gauss and Ricci’s equations are equivalent in the sense that the curvature tensors of the Levi-Civita and twisting connections in terms of the variable \( b_{ij} \), which acts as a source field subjected to Codazzi’s equation.

For completeness we may also write Gauss equation in the same algebraic form. This is easily accomplished using the definition of \( E^A \) in the first equation of (6), obtaining

\[ R_{ijkl} E^0 = b_{i[k} b_{lj]} - b_{j[k} b_{li]} \]

We conclude that the conditions for the embedding of a space-time may be compatible with with the physics of the space-time physics, provided the integrability conditions are included as part of the dynamics and with the adoption of the principle of economy of dimensions.

The hidden internal indices \( A, B, .. \) in the algebraic form of the equations (19), (20) and (2), merely reflect the degrees of freedom for the embedding which is defined up to a transformation of the normal vectors. As such they do not affect the number of independent equations. To understand the space-time as a four-dimensional submanifold and why it stays like that, depends on further understanding of \( b_{ij} \) and \( A_i \) as physical fields in addition to the metric (the gravitational field). This will be dealt with in a subsequent paper.

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