Post-Buckling Analysis of Curved Beams

Stretchability of the stretchable and flexible electronics involves the post-buckling behaviors of internal connectors that are designed into various shapes of curved beams. The linear displacement–curvature relation is often used in the existing post-buckling analyses. Koiter pointed out that the post-buckling analysis needs to account for curvature up to the fourth power of displacements. A systematic method is established for the accurate post-buckling analysis of curved beams in this paper. It is shown that the nonlinear terms in curvature should be retained, which is consistent with Koiter’s post-buckling theory. The stretchability and strain of the curved beams under different loads can be accurately obtained with this method. [DOI: 10.1115/1.4035534]

Keywords: post-buckling, stretchability, curved beam, curvature, finite deformation

1 Introduction

Curved beams have simple geometry and facilitative fabrication and are widely used in the fields of electronic, aerospace, and architecture. Recently, curved beams are treated as internal connectors of stretchable electronics due to their high stretchability with small strain [1–13]. There are many functional stretchable and flexible electronics based on different shapes of curved beams, such as epidermal health/wellness monitors [14–18], sensitive electronic skins [19–23], and spherical-shaped digital cameras [24–26]. The post-buckling behaviors, especially lateral and out-of-plane buckling, provide the flexibility and stretchability of the electronics [27], which are the essential properties of a robust technology of assembling various microstructures [28–30].

There are many researchers studying up on the stability of planar curved beams. Timoshenko and Gere [31] presented initial buckling behaviors of circular beams. The analytical models for post-buckling behaviors of the inextensible ring under uniform radial pressure were developed by Carrier [32] and Budiansky [33]. A systematic variational approach of space curved beams was developed by Liu and Lu and employed on buckling behavior and critical load of the serpentine structure [34]. Many results of the post-buckling behaviors of curved beams are obtained by the semi-analytical energy approach and finite-element method.
where the curvature
\[ j = \frac{\partial}{\partial s} \] of deformed curved beam, and the elongation of the deformed beam, 
\[ s = \int C_2 \] can be given as
\[ E_3 = \frac{\partial p}{\partial s} \] (2.1)

The curvature of the curved beam was defined by Love [40] as
\[ \frac{\partial E_i}{\partial s} = K \times E_i \quad (i = 1, 2, 3) \] (2.2)

where the curvature \( K = K_1 E_1 + K_2 E_2 + K_3 E_3 \), \( K_1 \) and \( K_2 \) are the curvatures along the axes in cross section, and \( K_3 \) denotes the twist along the tangential direction of the centroid line.

The centroid line of curved beam is deformed from \( P \) to \( \bar{P} = \bar{P} + U \), where \( U \) is the displacement of the beam. The local triad vectors of the deformed curved beam are \( e_i(i = 1, 2, 3) \), where \( e_1 \) and \( e_2 \) are the orthogonal unit vectors in the cross section of deformed curved beam, and \( e_3 \) is the unit vector along the tangential direction of the centroid line of deformed beam, which is defined as
\[ e_3 = \frac{\partial \bar{P}}{\partial s} \] (2.3)

where \( s \) is the arc-length coordinate of the deformed beam, which is the function of the arc-length of initial curved beam, \( S \). The elongation of the deformed beam, \( \lambda \), can be obtained from
\[ \lambda = \frac{ds}{dS} \] (2.4)

The curvature of the deformed curved beam also can be given as [40]
\[ \frac{\partial e_i}{\partial s} = \kappa \times e_i \quad (i = 1, 2, 3) \] (2.5)

where \( \kappa = \kappa_1 e_1 + \kappa_2 e_2 + \kappa_3 e_3 \), \( \kappa_1 \) and \( \kappa_2 \) are the components of curvature along sectional vectors \( e_1 \) and \( e_2 \), respectively, and \( \kappa_3 \) denotes the twist along the tangent direction, \( e_3 \). Here, the twist angle, \( \phi \), is defined with \( \kappa_3 \) as
\[ \phi = \int \kappa_3 ds \] (2.6)

where the axis of the twist angle changes with the location.

2.2 Equilibrium Equations and Constitutive Relation of Curved Beam. The internal force, \( t = t_i e_i \), and moment, \( m = m_i e_i \), of the curved beam satisfy the following equilibrium equations [41]:

\[ \frac{dt_i}{ds} + p_i = 0 \] (2.7)

\[ \frac{dm_i}{ds} + e_3 \times t_i + q_i = 0 \] (2.7)

where \( p = p_i e_i \), and \( q = q_i e_i \) are the distributed force and moment per unit length in the deformed beam, respectively. The derivative symbols have the relation, \((1/\lambda)(\partial(\lambda)/\partial s) = (\partial(\lambda)/\partial s)\). The internal force, \( t_i \), is conjugate with the elongation, \( \lambda \), and has the constitutive relation
\[ t_i = E A (\lambda - 1) \] (2.8)

where \( E \) and \( A \) are the elastic modulus and the cross section area of the beam, respectively.

The moment, \( m_i \), is not conjugate with the curvature, \( \kappa \), but it is conjugate with the Lagrangian curvature, \( \dot{\kappa} \), which is defined as
\[ \frac{dm_i}{\partial s} = \dot{\kappa} 	imes e_i \quad (i = 1, 2, 3) \] (2.9)

where the Lagrangian curvature, \( \dot{\kappa} \), has the relation with curvature, \( \kappa \), as \( \dot{\kappa} = \lambda \dot{\kappa} \).

The constitutive relation between Lagrangian curvatures (abbreviated hereafter simply to curvatures) and moments is
\[ m_1 = EI_1 (k_1 - K_1) \]
\[ m_2 = EI_2 (k_2 - K_2) \]
\[ m_3 = GI (k_3 - K_3) \] (2.10)

where \( G \) is the shear modulus of the beam, \( I_1 \) and \( I_2 \) are the sectional moment of inertia about the local coordinates, and \( J \) is the polar moment of inertia of the cross section of beam.

The components of the equilibrium equation expressed with the components of internal forces, moments, and the curvatures can be given as
\[ \frac{dt_i}{ds} + \epsilon_{ijk} \dot{k}_j t_k + \lambda p_i = 0 \quad (i = 1, 2, 3) \] (2.11)

\[ \frac{dm_i}{ds} + \epsilon_{ijk} \dot{k}_j m_k - \epsilon_{ijk} \dot{\lambda} t_j + \lambda q_i = 0 \quad (i = 1, 2, 3) \]
where \( \epsilon_{ijk} \) are the components of Eddington tensor.

3 The Deformation Variables in Terms of Displacement for Planar Curved Beam

The planar curved beam, which has only one nonzero component of initial curvature, \( K_i \), is widely used. We will focus on the planar curved beam in Secs. 3–7. The displacement, \( U \), can decompose in the local coordinates, \( E_i (i = 1, 2, 3) \), as \( U = U_1 E_1 + U_2 E_2 + U_3 E_3 \). The elongation defined by Eq. (2.4) can be expressed through the components of the displacement as

\[
\lambda = \left\{ 1 + \frac{2(dU_1)}{ds} + U_3 K_1 \right\}
+ \left[ \left( \frac{dU_1}{ds} \right)^2 + \left( \frac{dU_2}{ds} - U_3 K_1 \right)^2 + \left( \frac{dU_3}{ds} + U_3 K_1 \right)^2 \right]^{\frac{1}{2}}
\]

(3.1)

The relations between the local triads on the initial and deformed curved beams are given by the coefficients \( a_{ij} \) as

\[
e_i = a_{ij} E_j \quad (i = 1, 2, 3)
\] (3.2)

\( E_1 \) and \( E_2 \) are the tangent directions of the initial and deformed curved beams, respectively. By substituting Eqs. (2.1) and (2.3) into Eq. (3.2), the coefficients, \( a_{ij} (i = 1, 2, 3) \), can be derived as

\[
a_{31} = \frac{1}{\lambda} \frac{dU_1}{ds}, \quad a_{32} = \frac{1}{\lambda} \left( \frac{dU_2}{ds} - U_3 K_1 \right), \quad a_{33} = \frac{1}{\lambda} \left( 1 + \frac{dU_3}{ds} + U_3 K_1 \right)
\] (3.3)

The local coordinates, \( e_i (i = 1, 2, 3) \) and \( E_i (i = 1, 2, 3) \), are orthonormal (i.e., \( e_i \cdot e_j = \delta_{ij} \) and \( E_i \cdot E_j = \delta_{ij} \)), which lead the coefficients, \( a_{ij} \), to satisfy the following equations:

\[
a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1 \quad (i = 1, 2, 3)
\]

(3.4)

The relations between the coefficients, \( a_{ij} (i = 1, 2, j = 1, 2, 3) \), the displacements, \( U_i \), and the twist angle, \( \phi \), are complicated and cannot be explicitly expressed as Eq. (3.3). Substitution of Eqs. (3.1) and (3.3) into Eq. (3.4) and expansion of the coefficients, \( a_{ij} \), to the powers of generalized displacements lead to

\[
\{ a_{ij} \} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
+ \begin{bmatrix}
0 & \psi_1[i] & -\frac{dU_1}{ds} \\
-\psi_1[i] & 0 & -\frac{dU_2}{ds} + U_3 K_1 \\
\frac{dU_1}{ds} & \frac{dU_2}{ds} - U_3 K_1 & 0
\end{bmatrix}
+ \{ a_{ij}^{(2)} \} + \{ a_{ij}^{(3)} \} + \{ \cdots \} \quad (i, j = 1, 2, 3)
\] (3.5)

After substituting Eqs. (3.2) and (3.5) into Eq. (2.9), the curvatures can be expressed in terms of generalized displacements as

\[
\begin{bmatrix}
\hat{k}_1 \\
\hat{k}_2 \\
\hat{k}_3
\end{bmatrix} = \begin{bmatrix}
K_1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
\frac{d^2 U_1}{ds^2} + \frac{d(U_1 K_1)}{ds} \\
\frac{d^2 U_1}{ds^2} - \psi_1[i] K_1 \\
\frac{d\phi}{ds}
\end{bmatrix} + \{ \hat{a}_{ij}^{(2)} \} + \{ \hat{a}_{ij}^{(3)} \} + \{ \cdots \}
\]

(3.6)

where \( \hat{a}_{ij}^{(2)} \) and \( \hat{a}_{ij}^{(3)} \) are the second and third power of displacements, \( U_1 \), and twist angle, \( \phi \), which are given in Appendix A.

The nonlinear terms (i.e., \( \hat{a}_{ij}^{(2)} \) and \( \hat{a}_{ij}^{(3)} \)) are important for the analysis of the post-buckling behaviors of the curved beams based on the Koiter’s theory. As far as the authors are aware, the power expansion of deformation for Euler–Bernoulli curved beams (without assuming inextensibility of the beams) is a new result. The results obtained by Su et al. [41] for straight beams follow as a special case.

4 The Perturbation Solution for Post-Buckling of the Planar Curved Beam

The planar curved beam will be buckling in-plane or out-of-plane due to the planar loads (e.g., \( p_1 = 0 \)). The method of perturbation is used to solve differential equations (2.11) substituted with variables of the deformation and displacement, where a small ratio, \( a \), of the maximum deflection to the characteristic length (e.g., the beam length, \( L_S \), or the initial curvature radius of beam, \( R \)) is introduced. The generalized displacements from buckling, expanded to the powers of \( a \), can be written as

\[
\begin{align*}
U_1 &= a U_{1(1)} + a^2 U_{1(2)} + a^3 U_{1(3)} + O(a^4) L_S \\
U_2 &= a U_{2(1)} + a^2 U_{2(2)} + a^3 U_{2(3)} + O(a^4) L_S \\
U_3 &= a U_{3(1)} + a^2 U_{3(2)} + a^3 U_{3(3)} + O(a^4) L_S \\
\phi &= a \phi_{(1)} + a^2 \phi_{(2)} + a^3 \phi_{(3)} + O(a^4)
\end{align*}
\]

(4.1)

By substituting Eq. (4.1) into Eqs. (3.1) and (3.6), the elongation and curvatures can be also expanded to the powers of the small ratio, \( a \), as

\[
\lambda = 1 + a \lambda_{(1)} + a^2 \lambda_{(2)} + a^3 \lambda_{(3)} + O(a^4)
\]

(4.2)

\[
\hat{k}_i = \hat{k}_{i(0)} + a \hat{k}_{i(1)} + a^2 \hat{k}_{i(2)} + a^3 \hat{k}_{i(3)} + O(a^4) L_S^{-1} \quad (i = 1, 2, 3)
\]

(4.3)

where \( \hat{k}_{i(0)} = K_i \), \( \hat{k}_{i(1)} = 0 \), \( \lambda_{(2)} \) and \( \lambda_{(3)} \) are the functions of generalized displacements, and

\[
\hat{\lambda}_{(1)} = \frac{dU_{1(1)}}{ds} + U_{2(1)} K_1
\]

(4.4)

\[
\hat{\lambda}_{(2)} = \frac{d^2 U_{2(1)}}{ds^2} + \frac{d(U_{3(1)} K_1)}{ds}
\]

(4.5)

\[
\hat{\lambda}_{(3)} = \frac{d\phi_{(1)}}{ds}
\]

where \( \phi_{(1)} = \phi_{(1)} - \int K_1 (dU_{1(1)}/ds) ds \).

The coefficient, \( a_{ij} \), can be expanded to the powers of the small ratio, \( a \), as

\[
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\]
where \( \theta_{ij(k)} \) are the functions of generalized displacements

\[
\theta_{ij(1)} = \begin{cases} 
0 & -\frac{dU_{1(1)}}{dS} \\
\psi_{1(1)} & 0 \\
\frac{dU_{1(1)}}{dS} & -U_{1(1)}K_1 \\
\frac{dU_{2(1)}}{dS} & -U_{3(1)}K_1 \\
& 0 
\end{cases}
\]  

The expressions of \( \lambda_{(k)} \), \( \hat{\kappa}_{(k)} \), and \( \theta_{ij(k)} \) for \( k = 2 \) are given in Appendix B.

Substituting Eqs. (4.2) and (4.3) into constitutive relations (2.8) and (2.10) and considering the equilibrium equations, the internal forces and moments can be expanded to powers of the small ratio \( a \) as

\[
t_i = t_i(0) + a t_i(1) + a^2 t_i(2) + a^3 t_i(3) + O(a^4)EA \tag{4.8}
\]

\[
m_i = m_i(0) + a m_i(1) + a^2 m_i(2) + a^3 m_i(3) + O(a^4)EI L_0^{-1} \tag{4.9}
\]

where \( t_i(0) \) and \( m_i(0) \) are the forces and moments at the onset of buckling, and the relation between \( t_i(k) \), \( m_i(k) \), and \( \lambda_{(k)} \), \( \hat{\kappa}_{(k)} \) (\( k \geq 1 \)) can be written as

\[
t_i(k) = E A \lambda_{(k)} \tag{4.10}
\]

\[
m_i(k) = E I \hat{\kappa}_{(k)} \tag{4.10a}
\]

\[
m_i(k) = G J \hat{\kappa}_{(k)} \tag{4.10b}
\]

After substituting the forces (4.8) and moments (4.9), equilibrium equations (2.11) can be decomposed as

\[
\frac{dt_i(0)}{dS} + \epsilon_{ijk} \frac{d^2 t_j(0)}{dS^2} + p_i(0) = 0 \quad (i = 1, 2, 3) \tag{4.11}
\]

and

\[
\frac{dm_i(0)}{dS} + \frac{\lambda_{(1)}}{S_0} + \epsilon_{ijk} \frac{d^2 m_j(0)}{dS^2} + q_i(0) = 0 \quad (i = 1, 2, 3) \tag{4.12}
\]

where \( \lambda_{(0)} = 1 \), and the loads are expanded to the power series of \( a \) as

\[
p_i = p_i(0) + a p_i(1) + a^2 p_i(2) + a^3 p_i(3) + O(a^4)EA L_0^{-1} \tag{4.13}
\]

\[
q_i = q_i(0) + a q_i(1) + a^2 q_i(2) + a^3 q_i(3) + O(a^4)EI L_0^{-2}
\]

where \( p_i(0) \) and \( q_i(0) \) are the critical loads at bifurcation point.

The differential equations in this section, in which internal forces and moments are substituted with deformation variables and displacements, are solved in the following steps.

**Step 1:** Solve Eq. (4.12) for \( n = 1 \) with the corresponding boundary conditions to determine the buckling mode for the leading order, \( \lambda_{(1)} \), \( \kappa_{(1)} \), and \( U_{(1)} \), of elongations, curvatures, and displacements, and the critical loads at the onset of buckling, \( p_i(0) \) and \( q_i(0) \).

**Step 2:** Solve Eq. (4.12) for \( n = 2 \) with the corresponding boundary conditions to determine the second order \( \lambda_{(2)} \), \( \kappa_{(2)} \), and \( U_{(2)} \) of elongation, curvatures, and displacements, and the increment of loads, \( p_i(1) \) and \( q_i(1) \).

**Step 3:** Solve Eq. (4.12) for \( n = 3 \) with the corresponding boundary conditions to determine the buckling mode for the third order \( \lambda_{(3)} \) and \( \kappa_{(3)} \) of elongation and curvatures, and the increment of loads, \( p_i(2) \) and \( q_i(2) \).

### 5 In-Plane Post-Buckling Behavior of Elastic Ring

As illustrated in Fig. 2, the uniform distributed radial load, \( \bar{p}_2 \), which remains normal to the centroid line during the deformation, is applied on an elastic thin ring. The arc-length, \( S \), is clockwise counted from \( A \). The ring at \( A \) is simply supported, and the displacement along tangent direction at \( B \) is constrained, these boundary conditions can be written as

\[
U_{i(0)}|_{S=0} = 0 \quad (i = 1, 2, 3, \ldots) \tag{5.1}
\]

\[
U_{3(0)}|_{S=\pi R} = 0 \quad (n = 1, 2, 3, \ldots) \tag{5.2}
\]

where \( R = 1/K_1 \) is the initial curvature radius of ring. The displacements, deformations, and forces/moments are periodical due to the periodical deformation of the ring, i.e.,

\[
(\cdot)|_{S=\pi R} = (\cdot)|_{S=2\pi R} \quad (0 \leq S \leq 2 \pi R) \tag{5.3}
\]

where \( (\cdot) \) denotes \( \lambda \), \( \hat{\kappa}_{(n)} \), \( \hat{\kappa}_{(n)} \), \( \lambda_{(n)} \), \( t_i(0) \), \( m_i(0) \), \( U_i \), \( U_{ij(0)} \), \( \phi \), \( \psi_{ij(0)} \), and \( \theta_{ij(0)} \).

The width of the ring section, \( w \), is much larger than its thickness, \( t \), and the ring will be buckling in the initial plane of the curved beam under the load, \( \bar{p}_2 = -\bar{p}_2 \). The out-of-plane components of force, moment, and displacement are zero, i.e., \( t_i = 0 \), \( m_j = 0 \), \( \psi_{ij(0)} = 0 \) \( \lambda_{(1)} = 0 \), \( \kappa_{(1)} = 0 \), \( U_1 = 0 \), and \( \phi = 0 \). The internal force and moment at the onset of buckling, \( t_{2(0)} \), \( t_{3(0)} \), and \( m_1(0) \), satisfy Eq. (4.11) and have the relation with critical load, \( \bar{p}_2(0) \), as

\[
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\]

Fig. 2 Schematic illustration of elastic ring under uniform compression
\[ t_{3(0)} = -p_{2(0)}R, \quad t_{2(0)} = 0, \quad m_{1(0)} = 0 \quad (5.4) \]

After substituting constitutive relation (4.10) into equilibrium equations (4.12) for \( n = 1 \) and eliminating \( t_{2(1)} \) and \( \hat{k}_{1(1)} \), the differential equation for elongation, \( \hat{\lambda}_{1(1)} \), is derived as

\[
\frac{d^3 \hat{\lambda}_{1(1)}}{ds^3} + k_1^2 \frac{d\hat{\lambda}_{1(1)}}{ds} = 0 \quad (5.5)
\]

where \( k_1^2 = [1 + p_{2(0)} R/(EA) + p_{2(0)} R^3/(EI_1)]/R^2 \). Solution of Eq. (5.5) is

\[
\hat{\lambda}_{1(1)} = C_{12} \cos (k_1 s) + C_{11} \sin (k_1 s) + C_{10}
\]

where \( C_{10}, C_{11}, \) and \( C_{12} \) are the parameters to be determined. The critical load, \( p_{2(0)} \), can be determined by the periodic condition of elongation, \( \hat{\lambda}_{1(1)} \), as

\[
p_{2(0)} = \frac{3EI_1}{R^3} (1 + c_1)^{-1}
\]

where \( c_1 = Ei/(ER^2) \). \( \hat{k}_{1(1)} \) is also obtained by substituting Eqs. (4.10) and (5.6) into Eq. (4.12) as

\[
\hat{k}_{1(1)} = \frac{1}{3Rc_1} \left[ C_{10} + c_1 \left( \frac{p_{2(1)}}{CI_1} (1 + c_1) + 4C_{10} \right) \right]
\]

and

\[
-3 \left( C_{11} \sin \frac{2S}{R} + C_{12} \cos \frac{2S}{R} \right)
\]

The differential equations for \( U_{2(1)} \) and \( U_{3(1)} \) can be derived from Eqs. (4.4) and (4.5) as

\[
\frac{dU_{3(1)}}{ds} + \frac{U_{2(1)}}{R} = \hat{\lambda}_{1(1)}
\]

\[
\frac{d^2 U_{2(1)}}{ds^2} - \frac{1}{R} \frac{dU_{3(1)}}{ds} = -\hat{\lambda}_{1(1)}
\]

The solutions of \( U_{2(1)} \) and \( U_{3(1)} \) can be obtained by the boundary conditions (5.1) and (5.2) and the periodic condition (5.3) as

\[
U_{2(1)} = -R \cos \frac{S}{R} + R \cos \frac{2S}{R} \left( 1 + c_1 \right) \frac{p_{2(1)}}{EI_1} \left( 1 - \cos \frac{2S}{R} \right)
\]

\[
U_{3(1)} = R \sin \frac{S}{2} \left( 1 + 4c_1 \right) \frac{p_{2(1)}}{EI_1} \sin \frac{2S}{R}
\]

and the parameters, \( C_{10}, C_{11}, \) and \( C_{12} \), are also determined as

\[
C_{10} = \frac{p_{2(1)}}{EI_1} \left( 1 + c_1 \right) / 1 + 4c_1, \quad C_{11} = 0, \quad C_{12} = \left[ \frac{3c_1}{1 + c_1} + \frac{3c_1^2}{1 + 4c_1} \frac{p_{2(1)}}{EI_1} \right]
\]

where \( U_{2(1)} \) is assumed to be symmetrical about \( AB \), and max(\( U_{2(1)} \)) = 2R.

Substitution of constitutive relation (4.10) into equilibrium equations (4.12) for \( n = 2 \) and elimination of \( t_{2(2)} \) and \( \hat{k}_{1(2)} \) give the differential equation for elongation, \( \hat{\lambda}_{2(2)} \), as

\[
\frac{d^2 \hat{\lambda}_{2(2)}}{ds^3} + k_1^2 \frac{d\hat{\lambda}_{2(2)}}{ds} = F_{22}(S)
\]

The solution of Eq. (5.12) is

\[
\hat{\lambda}_{2(2)} = C_{22} \cos (k_2 s) + C_{21} \sin (k_2 s) + C_{20}
\]

\[
\int_0^S \sin (k_2 (S - \zeta)) F_{22}(\zeta) d\zeta / k_2
\]

where \( \hat{\lambda}_{2(2)} \) is also periodic, which determined the increment of load, \( p_{2(1)} = 0 \), and the parameters, \( C_{20}, C_{21}, \) and \( C_{22} \) will be determined with the boundary conditions and orthonormality condition [37,38].

The curvature, \( \hat{k}_{1(2)} \), is also obtained by substitution of Eqs. (4.10) and (5.13) into Eq. (4.12) as

\[
\hat{k}_{1(2)} = -\frac{3(23 + 68c_1)}{16R(1 + c_1)^2} + \frac{1 + c_1}{3R} \frac{p_{2(1)}}{EI_1} + \frac{1 + 4c_1}{3Rc_1} C_{20} - \frac{C_{21}}{c_1} \frac{2S}{R}
\]

\[
\frac{1}{4R} \left[ \frac{45}{(1 + c_1)^2} + \frac{4C_{22}}{c_1} \right] \cos \frac{2S}{R} + \frac{9(1 - 4c_1)}{16R(1 + c_1)^2} \cos \frac{4S}{R}
\]

The differential equations for \( U_{2(2)} \) and \( U_{3(2)} \) are given by substitution of Eqs. (5.13) and (5.14) into Eqs. (B1) and (B2) as

\[
\frac{d^2 U_{2(2)}}{ds^2} - \frac{1}{R} \frac{dU_{3(2)}}{ds} = F_{22}(S)
\]

\[
\frac{dU_{2(2)}}{ds} + \frac{U_{3(2)}}{R} = F_{23}(S)
\]

Fig. 3 The ratio of load to critical load, \( p_2/p_{20}\), versus the normalized displacement, \( U_{\text{max}}/(2R) \), during post-buckling, which is consistent with the results of Carrier's model.
where

\[ F_{22}(S) = -\kappa_{1(2)} + \frac{d}{dS} \left( \frac{dU_{1(4)}dU_{2(1)}}{dS} \right) + \frac{1}{2R} \frac{d^2 \left( U_{2(1)}^2 - U_{3(1)}^2 \right)}{dS^2} - \frac{1}{R} \frac{d \left( U_{2(1)}U_{1(4)} \right)}{dS} \]

\[ F_{23}(S) = \lambda_{(2)} - \frac{1}{2} \left( \frac{dU_{2(1)}}{dS} - \frac{1}{R} U_{3(1)} \right)^2 \]

and the underlined terms in \( F_{22}(S) \) come from the nonlinear part of curvature (B2).

The solutions of Eq. (5.15) with the boundary conditions (5.1) and (5.2) are

\[
U_{2(2)} = \frac{R}{80(1 + c_1)^2(1 + 4c_1)} \left\{ 180c_1(1 + 2c_1) - 3c_1(61 + 12c_1)\cos \frac{S}{R} \right. \\
\left. - 5 \left[ 9 + 16c_1(1 + c_1)^3 \frac{p_{2(2)}R^3}{EI_1} \right] \left( 1 - \cos \frac{S}{R} \right) + 3c_1(1 + 4c_1)\cos \frac{4S}{R} \right\} \\
+ \frac{3Rc_1}{5(1 + c_1)^2} \left( \cos \frac{S}{R} - \cos \frac{4S}{R} \right) \\
U_{3(2)} = -\frac{R}{320(1 + c_1)^3(1 + 4c_1)} \left\{ 4 \left( 45 - 3c_1(61 + 124c_1) + 80c_1(1 + c_1)^3 \frac{p_{2(2)}R^3}{EI_1} \right) \sin \frac{S}{R} \right. \\
\left. + (3 + 4c_1)(-15 + 76c_1)\sin \frac{4S}{R} \right\} \\
+ \frac{3Rc_1}{20(1 + c_1)^2} \left( -4 \sin \frac{S}{R} + \sin \frac{4S}{R} \right)
\]

where the underlined terms are derived from the nonlinear terms of the curvature, and \( U_{2(2)} \) is assumed to be symmetrical about \( AB \). The parameters, \( C_{20}, C_{21}, \) and \( C_{22} \), are also determined as

\[
C_{20} = -c_1(1 + c_1)\frac{p_{2(2)}R^3}{(1 + 4c_1)} + \frac{9c_1(23 + 68c_1)}{16(1 + c_1)^2(1 + 4c_1)} \]

\[
C_{21} = 0, \quad C_{22} = -\frac{45c_1}{4(1 + c_1)^3} \quad (5.17)
\]

Substitution of constitutive relation (4.10) into equilibrium equation (4.12) for \( n = 3 \) and elimination of \( t_{2(3)} \) and \( \kappa_{1(3)} \) give the differential equation for elongation, \( \lambda_{(3)} \), as

\[
\frac{d^3 \lambda_{(3)}}{dS^3} + k_1^2 \frac{d \lambda_{(3)}}{dS} = F_3(S)
\]

where

\[
F_3(S) = \frac{6c_1}{R^3(1 + c_1)(1 + 4c_1)} \left[ \frac{27(3 - 24c_1 - 16c_1^2)}{32(1 + c_1)^2} - (1 - c_1 + 4c_1^2)\frac{p_{2(2)}R^3}{EI_1} \right] \sin \frac{2S}{R} \\
+ \frac{1053c_1(-3 + 4c_1)}{16R^3(1 + c_1)^3} \sin \frac{6S}{R}
\]

The solution of Eq. (5.18) is

\[
\lambda_{(3)} = C_{32} \cos(k_1S) + C_{31} \sin(k_1S) + C_{30} + \int_0^S \frac{\cos(k_1(\zeta - \zeta'))F_3(\zeta') d\zeta'}{k_1} \quad (5.19)
\]

where \( C_{30}, C_{31}, \) and \( C_{32} \) are the parameters, and the increment of load, \( p_{2(2)} \), can be determined by the periodical condition of elongation, \( \lambda_{(3)} \),

\[
p_{2(2)} = \frac{(3 - 24c_1 - 16c_1^2)}{(1 + c_1)^2(1 - c_1 + 4c_1^2)} \frac{27EI_1}{32R^3} \quad (5.20)
\]

The thickness, \( t \), of the cross section of beam is much smaller than the radius, \( R \), which indicates \( c_1 \ll 1 \). The load, \( p_2 \), normalized by critical load, \( p_{2(0)} \), can be simplified as

\[
\frac{p_2}{p_{2(0)}} = 1 + \frac{a^2}{27} \left( \frac{U_{2(2)}}{2R} \right)^2 \approx 1 + \frac{27a^2}{32} = 1 + \frac{27}{32} \left( \frac{U_{2(2)}}{2R} \right)^2
\]

where \( a = U_{2(2)}/(2R) \) has been used, and it is the same with the result of Budiansky [33].

Figure 3 shows that the normalized load, \( p_2/p_{2(0)} \), increases with the normalized maximum displacement, \( U_{2(2)}/(2R) \), which is consistent with Carrier’s model [32]. It indicates that the elongation can be neglected due to the inextensibility of elastic ring in Carrier’s model.

6 Lateral Buckling of Circular Beam

The curved beam is widely used as interconnector, which is often freestanding and connects the sensors in the stretchable and
flexible electronics [2]. The lateral buckling of the freestanding interconnector will happen because the thickness of the beam cross section, $t$, is much larger than its width, $w$ [1,2,16]. The thickness direction is along the radial direction of the circular beam, which is consistent with it in Sec. 5. The displacement, $U_1$, and rotation, $\phi$, which are the odd powers of the small ratio, $a$, of the maximum deflection of $U_1$ to beam length, are the primary displacements, while the secondary displacements, $U_2$ and $U_3$, are the even powers of the small ratio, $a$ [41]. The curvatures, $\kappa_2$ and $\kappa_3$, are the odd powers of the small ratio, $a$, and the elongation, $\lambda$, and curvature, $\kappa_1$, are the even powers of the small ratio, $a$.

### 6.1 The Lateral Buckling of Circular Beam Under Bending Moment

As shown in Fig. 4, the bending moment, $M$, is applied on the circular beam at the ends. This beam with length, $L$, which subtends the angle, $\alpha$, is simply supported in the plane and the out-of-plane, and beam ends cannot rotate around the centroid of beam, but the right beam end can freely slide in the plane of the beam, i.e.,

$$
U_2|_{s=L} = U_3|_{s=L} = 0
$$

(6.1)

$$
(U_E E_2 + U_3 E_3)|_{s=L} \cdot \left( \cos \frac{\pi}{2} E_2 - \sin \frac{\pi}{2} E_3 \right)|_{s=L} = 0
$$

$$
t|_{s=L} \cdot \left( \sin \frac{\pi}{2} E_2 + \cos \frac{\pi}{2} E_3 \right)|_{s=L} = 0
$$

$$
m|_{s=0} = m|_{s=L} = M
$$

By substituting Eq. (3.2) into the above equations, the boundary conditions can be expanded with respect to the perturbation parameter, $a$, as

$$
U_2(s=0) = U_3(s=0) = 0
$$

(6.2)

$$
(U_2(s) \cos \frac{\pi}{2} - U_3(s) \sin \frac{\pi}{2})|_{s=L} = 0
$$

(6.3)

$$
\sum_{i=0}^{\infty} \left( t_i(a_{21}(i-1) \sin \frac{\pi}{2} + t_i(a_{31}(i-1) \cos \frac{\pi}{2}) \right)|_{s=L} = 0
$$

$$
m(s=0) = m(s=L) = M
$$

The internal forces and moments at the onset of buckling, $t_1(0)$, $m_0(0)$ $(i = 1, 2, 3)$, satisfy Eq. (4.11) and have the relation with critical load, $M_{(0)}$, as

$$
t_1(0) = t_2(0) = t_3(0) = 0, m_2(0) = m_3(0) = 0, m_1(0) = M_{(0)}
$$

(6.5)

Substitution of constitutive relation (4.10) into equilibrium equations (4.12) for $n = 1$ and elimination of $k_{3(1)}$ and $t_{1(1)}$ give the differential equation for $k_{2(1)}$ as

$$
\frac{d^2 k_{2(1)}}{ds^2} + \frac{k_{2(1)}}{R} = 0
$$

(6.6)

where $k_{2(1)} = |M_{(0)} R/(E I_2) - 1|/|M_{(0)} R c_2/(E I_2) - c_3|/(R^2 c_1)$, $c_2 = E I_2/(E A R^2)$, and $c_3 = G J/(E A R^2)$. The solution of Eq. (6.6) is

$$
k_{2(1)} = C_{41} \cos (k_2 S) + C_{40} \sin (k_2 S)
$$

(6.7)

where the parameters, $C_{40}$ and $C_{41}$, will be determined by the boundary conditions. The boundary conditions, $\bar{k}_{2(1)}|_{S=0} = \bar{k}_{2(1)}|_{S=L} = 0$, which are derived by substitution of constitutive relation (4.10) into the boundary conditions (6.4), determine the critical load as

$$
M_{(0)} = \frac{E I_2}{2 R} \left[ 1 + \frac{c_3}{c_2} - \frac{1}{2} \frac{E I_2}{c_2} \right] + \frac{R^2}{\pi^2}
$$

(6.8)

where the normalized angle, $\bar{\alpha} = \alpha/\pi$, and $k_2$ can be simplified to $1/(R \bar{\alpha})$. When $\bar{\alpha} = 1$, one of the two values of the critical load in Eq. (6.8) is zero, which corresponds to the freedom of a semicircular beam to rotate about the diameter connecting the two ends, the other value, $M_{(0)} = \pi E I_2 (1 + c_3/c_2)/L$, is the critical load for the semicircular beam. When the curved beam is shallow, i.e., $R \gg L$, the critical load, $M_{(0)}$, will approach to the critical load for the straight beam, $(\pi E I_2 L)/c_3/c_2$ [31].

The $k_{3(1)}$ can be obtained by the substitution of constitutive relation (4.10) and Eq. (6.7) into equilibrium equations (4.12) for $n = 1$ as

$$
k_{3(1)} = \frac{\bar{\alpha} c_2}{c_3} \left( \frac{M_{(0)} R}{E I_2} - 1 \right) \left( C_{40} \left( 1 - \cos \frac{S}{2 R} \right) + C_{41} \sin \frac{S}{2 R} \right) + C_{42}
$$

(6.9)

where the parameter, $C_{42}$, will be determined by the boundary condition.

The differential equations for $U_1(1)$ and $\phi_1(1)$ can be derived from Eq. (4.5) as

$$
\frac{d^2 U_1(1)}{ds^2} + \frac{1}{R^2} U_1(1) = k_1(1) + \frac{1}{R} \phi_1(1)
$$

(6.10)

The solutions of Eq. (6.10), which satisfy the boundary conditions (6.4), are

$$
\phi_1(1) = \frac{\pi c_2 \bar{\alpha} (\bar{\alpha}^2 - 1) \left( \frac{M_{(0)} R}{E I_2} - 1 \right)}{\bar{\alpha}^2 c_2 \left( \frac{M_{(0)} R}{E I_2} - 1 \right) - c_3} \sin \frac{S}{2 R}
$$

(6.11)

$$
U_1(1) = \frac{\pi \bar{\alpha} R \sin \frac{S}{2 R}}{\bar{\alpha}^2 c_2 \left( \frac{M_{(0)} R}{E I_2} - 1 \right) - c_3}
$$

(6.12)

where the maximum of $U_1(1)$ is $L$, and the parameters, $C_{40}$, $C_{41}$, and $C_{42}$, are

$$
C_{40} = \frac{\pi}{\bar{\alpha}^2 c_2 \left( \frac{M_{(0)} R}{E I_2} - 1 \right) - c_3}, \quad C_{41} = 0,
$$

$$
C_{42} = \frac{\pi c_2 \bar{\alpha} (\bar{\alpha}^2 - 1) \left( 1 - \frac{M_{(0)} R}{E I_2} \right)}{\bar{\alpha}^2 c_2 \left( 1 - \frac{M_{(0)} R}{E I_2} \right)}
$$

(6.13)

Substitution of constitutive relation (4.10) into equilibrium equations (4.12) for $n = 2$ and elimination of $\lambda_2(2)$ and $t_2(2)$ give the differential equation for leading terms of $k_1$ and $k_{1(2)}$, as
where $k_3 = 1/R^2$, and $F_{S1}(S) = (c_2 - c_3)/c_1 [(d^2/S)k_{S1}(S)] + (1/R^2)\kappa_{S1}(S)$. The solution of Eq. (6.13) is

$$\dot{\kappa}_{S1}(S) = C_{S2} \cos (k_3 S) + C_{S1} \sin (k_3 S) + C_{S0} + \int_0^S \frac{\sin (k_3 (S - \zeta)) F_{S1}(\zeta)}{k_3} \, d\zeta.$$  

(6.14)

The elongation, $\hat{\lambda}_{(2)}$, can be obtained by substitution of constitutive relation (4.10) and the above equation into equilibrium equations (4.12) for $n = 2$ as

$$\dot{\lambda}_{(2)} = -c_1 R \left( C_{S2} \cos \frac{S}{R} + C_{S1} \sin \frac{S}{R} \right) + \frac{\pi^2 c_3}{c_1} (c_2 - c_3) \left( \frac{S - \zeta}{R} \right)^2 \left( \frac{M_{00} R}{EI_2} - 1 \right)^2 \cos \frac{2S}{R}$$  

$$+ c_1 R^3 \int_0^S \cos \frac{S - \zeta}{R} F_{S1}(\zeta) d\zeta.$$  

(6.15)

The differential equations for the leading terms, $U_{2(2)}$ and $U_{3(2)}$, of displacements, $U_2$ and $U_3$, are derived by substitution of Eq. (6.14) into Eqs. (61) and (B2) as

$$\frac{d^2 U_{2(2)}}{dS^2} + \frac{U_{2(2)}}{R^2} = F_{S2}(S)$$  

(6.16)

$$\frac{dU_{3(2)}}{dS} = \dot{\lambda}_{(2)} - \frac{U_{2(2)}}{R} - \frac{1}{2} \left( \frac{dU_{1(1)}}{dS} \right)^2,$$

where

$$F_{S2}(S) = -\kappa_{S1}(S) + \frac{\lambda_{(2)}}{R} - \frac{(dU_{1(1)}/dS)^2}{(2R) + \psi_{11}^{(1)}(d^2U_{1(1)}/dS^2) - \left( \psi_{11}^{(1)} \right)^2 + (dU_{1(1)}/dS)^2} / (2R).$$

and the underlined terms are derived from the nonlinear terms of curvature.

The solutions of Eq. (6.16), which satisfy the boundary conditions (6.3), are

$$U_{2(2)}(S) = C_{S3} \cos \frac{S}{R} + C_{S4} \sin \frac{S}{R} + \int_0^S \frac{\sin (S - \zeta)}{R} F_{S2}(\zeta) d\zeta,$$

$$U_{3(2)}(S) = \int_0^S \left[ \frac{T_{2(2)}}{R} - \frac{U_{2(2)}}{R} - \frac{1}{2} \left( \frac{dU_{1(1)}}{dS} \right)^2 \right] d\zeta + C_{S5},$$

(6.17)

and the parameters, $C_{S0}$, $C_{S1}$, $C_{S2}$, $C_{S3}$, $C_{S4}$, and $C_{S5}$, are determined as

$$C_{S5} = C_{S3} = C_{S1} = 0$$

$$C_{S2} = \frac{\pi^2 c_3}{c_1} \left( \frac{c_2}{c_3} - 1 \right) \left( \frac{M_{00} R}{EI_2} - 1 \right)$$

$$- \frac{S^2 R}{2} \left[ \frac{c_3}{c_2} \left( \frac{M_{00} R}{EI_2} - 1 \right) \right]$$

$$C_{S4} = -\frac{\pi^2 R^2}{2} C_{S2} + \frac{\pi R c_2}{c_1} \frac{M_{00} R}{EI_2} + \frac{\pi^4 (S^2 - 1)}{8S} \left( \frac{M_{00} R}{EI_2} - 1 \right)^2$$

$$C_{S0} = -C_{S2} + \frac{c_2 R c_1}{M_{00} R}$$

(6.18)

Substitution of constitutive relation (4.10) into equilibrium equations (4.12) for $n = 3$ and elimination of $\kappa_{S3}$ and $t_{1(3)}$ give the differential equation for $\kappa_{S3}$ as

$$\frac{d^2 \kappa_{S3}}{dS^2} + k_3^2 \kappa_{S3} = F_0(S)$$

(6.19)

where

$$F_0(S) = \left( \left( 1 - c_1/c_2 \right) M_{00} R c_2 / (EI_2 c_3) - 1 \right) \kappa_{S1} \kappa_{S2} - R (c_1/c_2 - c_3/c_2) [d(\kappa_{S1} \kappa_{S2})/dS] / R.$$
The solution of Eq. (6.19) is

$$\dot{\kappa}_{2(3)} = C_{66} \cos(k_2 S) + C_{60} \sin(k_2 S) + \int_0^S \frac{\sin[k_2(S - \zeta)]F_\alpha(\zeta)}{k_2} \, d\zeta$$

(6.20)

where $C_{60}$ and $C_{66}$ are the parameters. $\dot{\kappa}_{2(3)}$ satisfies the boundary conditions $\dot{\kappa}_{2(3)}|_{S=0} = \dot{\kappa}_{2(3)}|_{S=L_4} = 0$, which determine the load increment

$$M_{(2)} = \frac{EI_2}{R} \left[ \pi^2(\bar{a}^2 - 1)^2 \left( \frac{c_2}{c_3} - 1 \right) \left[ 1 + 3 \frac{c_3}{c_2} - 4 \frac{M_{(0)}R}{EI_2} \right] \left( 1 - \frac{M_{(0)}R}{EI_2} \right) \right]$$

(6.21)

For the narrow rectangular section of beam, which thickness is much larger than its width, i.e., $t \gg w$, the stiffness ratios, $c_2$ and $c_3$ are much smaller than $c_1$, i.e., $c_1 \gg c_2 \sim c_3$. The load increment in Eq. (6.21) can be simplified to

$$M_{(2)} \approx \frac{EI_2}{R} \pi^2(\bar{a}^2 - 1)^2 \left( \frac{c_2}{c_3} - 1 \right) \left( 1 + 3 \frac{c_3}{c_2} - 4 \frac{M_{(0)}R}{EI_2} \right) \left( 1 - \frac{M_{(0)}R}{EI_2} \right)$$

(6.22)

The ratio of $c_1$ to $c_2$ is $2/(1 + \nu)$, which only depends on the Poisson’s ratio, Eq. (6.22) can be written as $M_{(2)} \approx (EI_2/R)f_1(\nu)$. The ratio of bending moment load, $M$, to the critical load, $M_{(0)}$, is

$$\frac{M}{M_{(0)}} = 1 + a^2 \frac{M_{(2)}}{M_{(0)}} \approx 1 + \left( \frac{U_{1max}}{L_0} \right)^2 \frac{\pi^2(\bar{a}^2 - 1)^2}{4 \frac{M_{(0)}R}{EI_2}} \left( 1 - \frac{M_{(0)}R}{EI_2} \right)$$

(6.23)

where the small ratio was defined as $a = U_{1max}/L_0$. The maximum principal strain in the beam can be obtained as

$$\varepsilon_{max} = (aw)_{max} \left[ \frac{1 - \nu}{4} \dot{\kappa}_{2(1)} + 1 \right] + \frac{1}{2} \sqrt{\frac{(1 - \nu)^2}{4} \kappa_{2(1)}^2 + \kappa_{3(1)}^2}$$

(6.24)

The shortening ratio of the distance, $\gamma_d$, between the beam ends due to the bending moment is defined as

$$\gamma_d = \frac{|U|_{S=L_4}}{2R \sin \frac{\pi a}{2}}$$

(6.25)

The ratio of the maximum principal strain in beam to $\sqrt{\gamma_d (w/R)}$ is

$$\gamma = \frac{\varepsilon_{max} R}{\sqrt{\gamma_d w}} = f_2(\bar{a}, \nu)$$

(6.26)

where the function, $f_2(\bar{a}, \nu)$, depends only on the shape and the Poisson’s ratio of beam. As shown in Fig. 5(a), the effect of Poisson’s ratio on the ratio, $\gamma$, can be neglected, especially for the curvature with the nonlinear terms. But the effect of the normalized angle, $\bar{a}$, on the ratio, $\gamma$, is significant as shown in Fig. 5(b), where the Poisson’s ratio is 0.42 for gold, which is the primary material of the interconnector in the stretchable and flexible electronics. The value gap between the ratio, $\gamma$, with and without the nonlinear terms in curvature would be larger than 100% for $\bar{a} > 5/6$. Figure 6 shows the normalized maximum principal strain, $\varepsilon_{max} R/w$, versus the shortening ratio of the ends distance, $\gamma_d$, for $\nu = 0.42$, which indicates that the nonlinear terms in curvatures should be considered for the strain of beam. There is 73% increase in the normalized maximum strain, $\varepsilon_{max} R/w$, without the nonlinear terms of curvature for $\bar{a} = 2/3$ and $\gamma_d = 0.3$ from the

![Fig. 4 Schematic illustration of boundary conditions of circular beam under bending moment load](http://appliedmechanics.asmedigitalcollection.asme.org/pdfaccess.ashx?url=/data/journals/jamcav/935918/ on 02/17/2017 Terms of Use: http://www.asme.org)
normalized maximum principal strain with the nonlinear terms, which indicates that the stretchability of circular beam is underestimated when the nonlinear terms are neglected. Figure 6 also shows that the relation between the normalized strain, $\varepsilon_{\text{max}}R/w$, and the shortening ratio, $\gamma_d$, depends on the normalized angle, $\tilde{\alpha}$, and the longer circular beams have the higher stretchability with the same critical normalized strain. This model can be used to analyze the stretchability of the serpentine bridge, which can be simplified as two semicircular beams [2]. The maximum strain in the bridge fabricated with gold can be obtained as $\varepsilon_{\text{max}} = 0.9365 (w/R)\sqrt{\text{pre}/(1 + \text{pre})}$ and $\varepsilon_{\text{max}} = 2.264(w/R)\sqrt{\text{pre}/(1 + \text{pre})}$ for the curvatures with and without the nonlinear terms, where the prestrain, $\text{pre}$, which is applied on the soft substrate, is related to the shortening ratio, $\gamma_d$, by $\text{pre} = \gamma_d/(1 - \gamma_d)$. For the design of the stretchability of the serpentine bridge, the nonlinear terms of the curvature should be considered in the analysis process of the post-buckling behavior of the curved beam.

The finite-element method (FEM) of the commercial software ABAQUS, where the shell element S4R is used since the width of the cross section, $w$, is much smaller than its thickness, $t$, is adopted to simulate the post-buckling behaviors of the curved beams under the bending moments. The distributions of the twist angle $\phi$ of the circular beams for the shortening ratios, $\gamma_d = 0.1, 0.2,$ and 0.3, are shown in Fig. 7, where the elastic modulus and Poisson’s ratio are 79.5 GPa and 0.42 for gold, and the length, $L$, thickness, $t$, width, $w$, and the normal angle, $\tilde{\alpha}$, of the circular beam are 900 mm, 30 mm, 1 mm, and 2/3, respectively. Figure 7 shows that the theoretical results are consistent with the FEM results.

### 6.2 The Lateral Buckling of Circular Beams Under Uniform Pressure

As shown in Fig. 8, the uniform pressure, $P_2 = -P_2$, which is along the thickness direction of cross section (i.e., $-e_2$) during the deformation, is applied on the circular beam, the lateral buckling will happen because the thickness of the beam section, $t$, is much larger than its width, $w$. The power orders of displacements are similar to those in Sec. 6.1. The beam with length, $L$, which subtends the angle, $\alpha$, is simply supported in the plane and out-of-plane, and the beam ends cannot rotate around the centroid of beam, but can freely slide toward the arch center in the plane, i.e.,

$$U_3|_{S=0} = U_3|_{S=L_S} = 0$$
$$m_1|_{S=0} = m_1|_{S=L_S} = 0$$
$$$(6.27)$$

$$m_1|_{S=0} = m_1|_{S=L_S} = 0$$

$$U_1|_{S=0} = U_1|_{S=L_S} = 0$$
$$e_2|_{S=0} = e_2|_{S=L_S} = 0$$

$$m_2|_{S=0} = m_2|_{S=L_S} = 0$$

The deformation of the model is symmetrical about the midline (i.e., the dotted–dashed line in Fig. 8) to avoid the rigid movement. By substituting Eq. (3.2) into the above boundary conditions, the expanded formulas with respect to the perturbation parameter, $a$, of the boundary conditions (6.27) and (6.28) are

$$U_3|_{S=0} = U_3|_{S=L_S} = 0$$
$$m_1|_{S=0} = m_1|_{S=L_S} = 0$$

$$\sum_{i=0}^{N} \left\{ U_i|^S_{S=0} \right\} = \sum_{i=0}^{N} \left\{ U_i|^S_{S=L_S} \right\} = 0$$

$$U_1|_{S=0} = U_1|_{S=L_S} = 0$$
$$a_2|_{S=0} = a_2|_{S=L_S} = 0$$

$$m_2|_{S=0} = m_2|_{S=L_S} = 0$$

The formulas and solving of the governing equations are similar to those in Sec. 6.1. Replacing $k_2, k_3, F_{S2}(S), F_{S2}(S)$, and $F_{S6}(S)$ in Sec. 6.1 with the following $k_2, k_3, F_{S4}(S), F_{S2}(S)$, and $F_{S6}(S)$, which are

---

Fig. 5 The ratio, $\gamma_d$ with and without the nonlinear terms in curvature: (a) the ratio, $\gamma_d$, versus the Poisson’s ratio $\nu$ for the normalized angle $\tilde{\alpha} = 1/4, 1/2$, and $2/3$. (b) The ratio, $\gamma_d$, versus the normalized angle $\tilde{\alpha}$ for gold (i.e., $\nu = 0.42$).

Fig. 6 The normalized maximum principal strain $\varepsilon_{\text{max}}R/w$ with and without the nonlinear terms in curvature versus the shortening ratio $\gamma_d$ for $\nu = 0.42$ and $\tilde{\alpha} = 1/4, 1/2$, and $2/3$
\[ k_2^2 = \frac{1}{R^2} \left( 1 + \frac{\bar{p}_{2(0)} R}{EI_2} \right) \quad k_3^2 = \frac{1}{R^2} \left( 1 + \frac{c_2}{c_1} \frac{\bar{p}_{2(0)} R}{EI_2} \right) \]  
\[ \tilde{F}_{31}(S) = \frac{1}{R^2 c_1} \left[ -2Rc_3 \frac{d \tilde{K}_{3(1)}}{dS} + R^2(3c_2 - 2c_3) \frac{d \tilde{K}_{2(1)}}{dS} + R^2(2c_2 - c_3) \frac{d^2 \tilde{K}_{2(1)}}{dS^2} \tilde{K}_{3(1)} \right] \]
\[ + c_2 \tilde{K}_{2(1)} \left( 1 + \frac{\bar{p}_{2(0)} R}{EI_2} \right) \left( \tilde{K}_{3(1)} - R \frac{d \tilde{K}_{2(1)}}{dS} + R^2(2c_2 - c_3) \frac{d^2 \tilde{K}_{2(1)}}{dS^2} \right) \]
\[ \tilde{F}_{52}(S) = -\tilde{K}_{3(2)} + \tilde{\lambda}_{3(2)} \frac{1}{R} \left( \frac{d \tilde{U}_{1(1)}}{dS} \right)^2 + \tilde{\psi}_{1(2)} ^2 R \left( \frac{d \tilde{U}_{1(1)}}{dS} \right)^2 - \frac{1}{2R} \left( \tilde{\psi}_{1(2)} + \frac{d \tilde{U}_{1(1)}}{dS} \right)^2 \]
\[ \tilde{F}_{6}(S) = \frac{1}{R^2 c_2} \left\{ \tilde{K}_{2(1)} \left[ R^2(c_2 - c_1) \tilde{K}_{3(1)}^2 + \tilde{\lambda}_{3(2)} \right] + R\tilde{K}_{1(2)} \left[ (c_1 - c_2) \tilde{K}_{2(1)} + R(c_3 - c_1) \frac{d \tilde{K}_{3(1)}}{dS} \right] \right\} \]
\[ - c_2 R \frac{d}{dS} \left( \tilde{\lambda}_{3(2)} \tilde{K}_{3(1)} \right) + R^2 c_2 \frac{d}{dS} \left( \tilde{\lambda}_{3(2)} \frac{d \tilde{K}_{2(1)}}{dS} \right) + R^2 \left[ (2c_1 - c_3) \tilde{K}_{3(1)} \frac{d \tilde{K}_{2(1)}}{dS} \right] \]

The critical load, \( \bar{p}_{2(0)} = EI_2/R^3(\tilde{a}^2 - 1) \), is obtained by the boundary conditions (6.30), which is consistent with the result of previous study [42]. When \( 1 \gg c_1 \gg c_2 \sim c_3 \), the load increment can be simplified to
\[ \bar{p}_{2(2)} = \frac{EI_2}{R^3} \left[ f_{31}(\nu, \tilde{a}) + f_{32}(\nu, \tilde{a}) \right] \]
(6.33)

where
\[ f_{31}(\nu, \tilde{a}) = \frac{\pi^2 (1 - \tilde{a}^2)^2 \left[ \tilde{a}^2 (17 - \nu)(1 + \nu) + 16(3 + 2\nu) - 2\tilde{a}^2 (22 + 33\nu + \nu^2) \right]}{8\tilde{a}^2 (\tilde{a}^2 - 4) \left[ 2 + \tilde{a}^2 (1 + \nu) \right]^2} \]
\[ f_{32}(\nu, \tilde{a}) = \frac{\pi^2 (1 - \tilde{a}^2)^2 \left[ 8 + 4\nu + \tilde{a}^2 (1 + \nu)^2 \right]}{16\tilde{a}^2 (\tilde{a}^2 - 4) \left[ 2 + \tilde{a}^2 (1 + \nu) \right]^2} \]
(6.34)

The function \( f_{32}(\nu, \tilde{a}) \) is derived from the nonlinear terms of curvature.

The ratio of the uniform pressure to the critical load is
\[ \frac{p_2}{\bar{p}_{2(0)}} = 1 + a^2 \frac{p_{2(2)}}{\bar{p}_{2(0)}} \approx 1 + \left( \frac{U_{\text{max}}}{L_0} \right)^2 \left[ f_{31}(\nu, \tilde{a}) + f_{32}(\nu, \tilde{a}) \right] \frac{1}{\tilde{a}^2 - 1} \]
(6.35)

![Fig. 7 The distributions of the twist angle of the circular beam for the different shortening ratios, \( \gamma_d = 0.1, 0.2, \) and 0.3](image1)

![Fig. 8 Schematic illustration of boundary conditions of circular beam under uniform pressure](image2)
Figure 9 shows the normalized load, $\frac{p_2}{p_{2,0}}$, versus the normalized out-of-plane displacement, $\frac{U_{\text{max}}}{L_5}$, which indicates that the nonlinear terms of curvature have significant impact on the analysis of the post-buckling behaviors of circular beam. The uniform pressure increases in the buckling process, but it decreases since the nonlinear terms of curvature is neglected.

7 Conclusions and Discussion

A systematic method is established for post-buckling analysis of curved beams in this paper. The deformation variables of the curved beam are up to the third power of generalized displacements due to the necessity of the fourth power of generalized displacements in the potential energy for the post-buckling analysis [37–39]. The currently prevailing post-buckling analyses, however, are accurate only to the second power of generalized displacements in the potential energy since some authors assume a linear displacement–curvature relation. Although the effect of the nonlinear terms of curvature on in-plane post-buckling behavior of planar beam is not so critical, it has significant impact on the lateral post-buckling. When the bending moment is applied on circular beam, the strain in beam is overvalued (~73%) due to the neglect of the nonlinear terms in curvature expression, which will lead to underestimate of the stretchability of circular beam. The difference of post-buckling behavior also stems from the neglect of the nonlinear terms for the lateral buckling under the uniform pressure. In summary, the nonlinear terms in curvature cannot be neglected in the lateral buckling behavior of the curved beam.

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Appendix A

The second and third powers of generalized displacements in coefficients and curvatures are

\[ a_{11}^{(2)} = -\frac{1}{2} \left( \psi_1^{[1]} \right)^2 + U_1^{[2]} \]

\[ a_{12}^{(2)} = -\frac{1}{2} U_1^{[2]} U_2^{[2]} + \frac{1}{2} U_1^{[2]} U_2^{[2]} + \psi_1^{[2]} \]

\[ a_{13}^{(2)} = U_2^{[2]} U_1^{[2]} - \psi_1^{[2]} U_2^{[2]} + \left( U_1^{[2]} U_2^{[2]} + \psi_1^{[2]} U_1^{[2]} \right) K_1 \]

\[ a_{21}^{(2)} = -\frac{1}{2} U_1^{[2]} U_2^{[2]} + \frac{1}{2} U_1^{[2]} U_3^{[2]} - \psi_2^{[2]} \]

\[ a_{22}^{(2)} = -\frac{1}{2} \left( \psi_2^{[1]} \right)^2 + (U_2^{[2]} - U_3^{[2]})^2 \]

\[ a_{23}^{(2)} = \psi_2^{[1]} U_1^{[2]} + (U_2^{[2]} + U_3^{[2]})(U_2^{[2]} - U_3^{[2]} K_1) \]

\[ a_{31}^{(2)} = -\left( U_1^{[2]} + U_2^{[2]} K_1 \right) U_1^{[2]} \]

\[ a_{32}^{(2)} = -(U_2^{[2]} + U_3^{[2]} K_1)(U_2^{[2]} - U_3^{[2]} K_1) \]

\[ a_{33}^{(2)} = -\frac{1}{2} U_2^{[2]} + (U_2^{[2]} - U_3^{[2]} K_1)^2 \]

\[ a_{11}^{(3)} = -\frac{1}{2} \psi_1^{[2]} U_1^{[2]} U_2^{[2]} - \psi_1^{[2]} U_2^{[2]} + U_1^{[2]} U_3^{[2]} + \left( U_1^{[2]} U_2^{[2]} + \frac{1}{2} \psi_1^{[2]} U_1^{[2]} U_3^{[2]} \right) K_1 \]

\[ a_{12}^{(3)} = \frac{1}{4} \psi_1^{[2]} \left( U_1^{[2]} - U_2^{[2]} \right) + U_1^{[2]} U_3^{[2]} + \psi_1^{[2]} \]

\[ + \left[ U_1^{[2]} U_2^{[2]} - U_1^{[2]} U_3^{[2]} + \frac{1}{2} \psi_1^{[2]} U_2^{[2]} U_3^{[2]} \right] K_1 + \left( -\frac{1}{4} \psi_1^{[2]} U_2^{[2]} - U_1^{[2]} U_2^{[2]} \right) K_1^2 \]

\[ a_{13}^{(3)} = \frac{1}{2} \psi_1^{[2]} U_2^{[2]} + \psi_1^{[2]} U_3^{[2]} - \psi_1^{[2]} U_2^{[2]} - U_1^{[2]} \left[ U_2^{[2]} + \frac{1}{2} \left( U_1^{[2]} + U_2^{[2]} \right) \right] \]

\[ + \left[ U_3^{[2]} - \psi_1^{[2]} (U_1^{[2]} U_3^{[2]} - U_2^{[2]} U_3^{[2]} - U_3^{[2]} U_3^{[2]} + 2 U_2^{[2]} U_3^{[2]} + 2 U_2^{[2]} U_3^{[2]} \right] K_1 \]

\[ + \left( -\psi_1^{[2]} U_2^{[2]} + \frac{1}{2} U_2^{[2]} U_2^{[2]} - U_1^{[2]} U_3^{[2]} \right) K_1 \]

\[ (A4) \]
\[ a_{21}^{(3)} = \frac{1}{4} \psi^{(i)} \left( U_{1}^2 - U_{2}^2 \right) + U_{1} U_{3} U_{4} - \psi^{(i)} + \left[ U_{1} U_{2} U_{3} - U_{1} U_{4} + \frac{1}{2} \psi^{(i)} U_{3} U_{4} \right] K_{1} + \left( -\frac{1}{4} \psi^{(i)} U_{3}^2 - U_{1} U_{2} U_{3} \right) K_{1}^2 \]

\[ a_{22}^{(3)} = \frac{1}{2} \psi^{(i)} U_{1} U_{2} - \psi^{(i)} \psi^{(i)} + U_{2} U_{3} + \left( U_{1} U_{2} - \frac{1}{2} \psi^{(i)} U_{3} U_{4} - 2 U_{1} U_{2} U_{3} \right) K_{1} + \left( U_{1} U_{2} - 2 U_{2} U_{3} U_{4} \right) K_{1}^2 + U_{2} U_{3} K_{1}^3 \]

\[ a_{23}^{(3)} = \frac{1}{2} \left( \psi^{(i)} \right)^2 U_{2} + \psi^{(i)} U_{1} U_{3} + \psi^{(i)} U_{1} U_{2} - U_{1} U_{3} - \frac{1}{2} \left( U_{2}^2 + U_{3}^2 \right) \]

\[ + \left( -\psi^{(i)} U_{1} U_{2} - \frac{1}{2} U_{1} \left( \psi^{(i)} \right)^2 + U_{1} U_{2} + U_{3} \left( U_{2}^2 - U_{3}^2 \right) - 2 U_{2} U_{3} U_{4} \right) K_{1} + \left( 2 U_{2} U_{3} U_{4} + \frac{3}{2} U_{2} U_{3} - U_{2} U_{3} \right) K_{1}^2 + \left( U_{2} U_{3} - \frac{1}{2} U_{2} \right) K_{1}^3 \]

\[ a_{31}^{(3)} = U_{1} \left[ U_{3}^2 \left( U_{1}^2 + U_{2}^2 \right) \right] + \left( U_{1} U_{2} - \frac{1}{2} \psi^{(i)} U_{3} U_{4} \right) K_{1} + \left( U_{1} U_{2} - 2 U_{2} U_{3} U_{4} \right) K_{1}^2 \]

\[ a_{32}^{(3)} = U_{2} \left[ U_{3} \left( U_{2}^2 + U_{1}^2 \right) \right] + \frac{1}{2} \psi^{(i)} U_{2} U_{3} + \left( U_{2} U_{3} - U_{1} U_{4} \right) K_{1} + \left( U_{2} U_{3} - 2 U_{1} U_{2} U_{3} \right) K_{1}^2 \]

\[ a_{33}^{(3)} = \psi^{(i)} U_{1} U_{2} + \psi^{(i)} U_{2} U_{3} + \left( U_{1} U_{2} - 2 U_{2} U_{3} U_{4} \right) K_{1} + \left( U_{1} U_{2} - 2 U_{2} U_{3} U_{4} \right) K_{1}^2 + U_{2} U_{3} K_{1}^3 \]

\[ \dot{\kappa}_{1}^{(3)} = \psi^{(i)} U_{1} U_{2} + \frac{1}{2} \left( \psi^{(i)} \right)^2 U_{2} - \psi^{(i)} \psi^{(i)} U_{3} U_{4} + U_{1} U_{2} U_{3} + \frac{1}{2} U_{1} U_{2} U_{3} - 2 U_{2}^2 U_{3} U_{4} \]

\[ + \left( U_{1} U_{2} U_{3} - U_{1} U_{4} \right) U_{2} + \left( U_{1} U_{2} - \frac{1}{2} \psi^{(i)} U_{3} U_{4} - 2 U_{1} U_{2} U_{3} \right) \right] K_{1} + \left( U_{1} U_{2} - 2 U_{2} U_{3} U_{4} \right) K_{1}^2 \]

\[ \dot{\kappa}_{2}^{(3)} = \psi^{(i)} U_{1} U_{2} - \frac{1}{2} \left( \psi^{(i)} \right)^2 U_{2} - \psi^{(i)} \psi^{(i)} U_{3} U_{4} + \psi^{(i)} \psi^{(i)} U_{3} U_{4} - \psi^{(i)} U_{3} U_{4} - U_{3}^2 \]

\[ + \frac{1}{2} U_{1} U_{2} U_{3} - \frac{1}{2} U_{1} U_{2} U_{3} U_{4} + \frac{1}{2} U_{1} U_{2} U_{3} \]
\[
\psi^{[2]} = \left[ \frac{1}{2} \left( U'_{1j} U''_{1j} - U''_{1j} U'_{1j} \right) - U'_{1j} U_{1j} K'_1 + \frac{1}{2} \left( U'_{1j} U_{3j} K'_1 \right)' + U'_{1j} U_{2j} K'_1 \right] \, \text{d}S
\]  
(A9)

\[
\psi^{[3]} = \left\{ \begin{array}{l}
\left\{ -\frac{1}{2} \psi^{[1]} \left( \psi^{[1]} \right)^2 + \frac{1}{4} \left[ -\psi^{[1]} \left( U''_{1j} + U''_{2j} \right) \right] + \left( U''_{1j} U''_{2j} - U''_{1j} U''_{2j} \right) \left( U''_{1j} \right)' \\
+ \left( \frac{1}{2} \psi^{[1]} U'_{1j} U_{3j} + U'_j U_{1j} K'_1 \right) \right\} K'_1 \, \text{d}S \\
\frac{1}{2} \left[ \psi^{[1]} U'_{1j} U_{3j} \right]' + \frac{1}{2} U'_j U''_{1j} \left( U''_{1j} + U''_{2j} \right) - U''_{1j} \left( U'_j U_{1j} - U'_{1j} U_{2j} \right) - U''_{1j} U_{2j} \left( U''_{1j} - U''_{2j} \right) \right \} K'_1 \\
+ \left( U'_{1j} U_{3j} \right)' + U'_j \left( \frac{1}{2} \psi^{[1]} U_{1j} + \frac{1}{4} \psi^{[1]} U_{1j} \right) \left( K'_1 \right)' - U'_j \left( U''_{1j} - U''_{2j} \right) \right \} K'_1 \\
\end{array} \right. 
\]  
(A10)

Appendix B

The expressions of \( \lambda_{(2)} \), \( \kappa_{(2)} \), and \( a_{(2)} \) are

\[
\dot{\lambda}_{(2)} = U'_{2j} + U_{2j} K'_1 + \frac{1}{2} \left[ U''_{1j} + \left( U''_{2j} - U''_{1j} \right) \right] 
\]  
(B1)

\[
\begin{aligned}
\dot{\kappa}_{(2)} &= \left\{ \begin{array}{l}
- U''_{2j} + \left( U_{2j} K'_1 \right)' + \psi^{[1]} \left[ U''_{1j} + \left( U''_{2j} - U''_{1j} \right) \right]' + \left( U''_{1j} U''_{2j} - U''_{1j} U''_{2j} \right) K'_1 \\
+ \left( \left( U''_{2j} - U''_{1j} \right) \right)' + \frac{1}{2} \left[ \left( \psi^{[1]} \right)^2 + U''_{1j} \right] \end{array} \right\} \right\} K'_1 - \left( U''_{2j} U_{2j} + \frac{1}{2} \psi^{[1]} U_{2j} K'_1 \right) \\
\dot{\kappa}_{(3)} &= \psi^{[2]} \\
\end{aligned}
\]  
(B2)

\[
\begin{aligned}
a_{11(2)} &= - \frac{1}{2} \left[ \left( \psi^{[1]} \right)^2 + U''_{1j} \right] \\
a_{12(2)} &= \psi^{[1]} - \frac{1}{2} U'_{1j} U''_{2j} + \frac{1}{2} U'_{1j} U_{2j} K'_1 + \psi^{[2]} \\
a_{13(2)} &= - U'_{1j} + U''_{1j} U_{2j} - \psi^{[1]} U''_{2j} + \left( U''_{1j} U_{2j} + \psi^{[1]} U_{2j} \right) K'_1 \\
a_{21(2)} &= - \psi^{[1]} - \frac{1}{2} U'_{1j} U''_{2j} + \frac{1}{2} U'_{1j} U_{2j} K'_1 - \psi^{[2]} \\
a_{22(2)} &= - \frac{1}{2} \left[ \left( \psi^{[1]} \right)^2 + U''_{1j} \right] \\
a_{23(2)} &= - U''_{1j} + U_{2j} K'_1 + \psi^{[1]} U''_{2j} + \left( U''_{1j} + U_{2j} \right) K'_1 \right) \\
\end{aligned}
\]  
(B3)

\[
\begin{aligned}
a_{31(2)} &= U'_{1j} - \left( U''_{2j} - U''_{1j} \right) \left( U''_{1j} \right)' \\
a_{32(2)} &= U''_{2j} - U_{2j} K'_1 - \left( U''_{2j} - U_{2j} \right) K'_1 \right) \\
a_{33(2)} &= - \frac{1}{2} \left[ U''_{2j} \right]' + \left( U''_{2j} - U_{2j} \right) K'_1 \right] } \\
\end{aligned}
\]  
(B4)

where \( \psi^{[2]} = \psi^{[2]} - \int K'_1 U''_{1j} \, \text{d}S \)

\[
\psi^{[2]} = \left[ \frac{1}{2} \left( U'_{1j} U''_{2j} - U''_{1j} U''_{2j} \right) - U'_{1j} U_{2j} K'_1 + \frac{1}{2} \left( U''_{1j} U_{2j} K'_1 \right)' + U''_{1j} U_{2j} K'_1 \right] \, \text{d}S 
\]
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