OPTIMAL RATE OF CONDENSATION FOR TRAPPED BOSONS IN
THE GROSS–PITAЕVSKII REGIME

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Abstract. We study the Bose–Einstein condensates of trapped Bose gases in the Gross–Pitaevskii regime. We show that the ground state energy and ground states of the many-body quantum system are correctly described by the Gross–Pitaevskii equation in the large particle number limit, and provide the optimal convergence rate. Our work extends the previous results of Lieb, Seiringer and Yngvason on the leading order convergence, and of Boccato, Brennecke, Cenatiempo and Schlein on the homogeneous gas. Our method relies on the idea of ‘completing the square’, inspired by recent works of Brietzke, Fournais and Solovej on the Lee–Huang–Yang formula, and a general estimate for Bogoliubov quadratic Hamiltonians on Fock space.

Keywords: Trapped Bose gases, Gross–Pitaevskii equation, Bose–Einstein condensation, Optimal bounds.

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1. Introduction

Since the first experimental realizations of Bose–Einstein condensation in cold atomic gases in 1995 [1, 14], the rigorous understanding of the condensation from basic laws of quantum physics has become a major problem in mathematical physics. In the present paper, we will investigate this issue for a system of trapped bosons and provide a quantitative justification of the condensation for the low-lying eigenstates.

We consider a system of \( N \) bosons in \( \mathbb{R}^3 \) described by the Hamiltonian

\[
H_N = \sum_{j=1}^{N} (-\Delta x_j + V_{\text{ext}}(x_j)) + \sum_{1 \leq j < k \leq N} N^2 V(N(x_j - x_k))
\]

on the bosonic space \( L^2(\mathbb{R}^3)^{\otimes N} \). Here the external potential, which satisfies \( V_{\text{ext}}(x) \to +\infty \) as \( |x| \to +\infty \), serves to trap the system. The interaction potential \( V \) is non-negative and sufficiently smooth. The Hamiltonian \( H_N \) with the core domain \( C^\infty_c(\mathbb{R}^3)^{\otimes N} \) is bounded from below and can be extended to be a self-adjoint operator by Friedrichs’ method.

Note that the range of the interaction is of order \( N^{-1} \), much smaller than the average distance of the particles (which is of order \( N^{-1/3} \) as the system essentially occupies a volume of order 1). Therefore, any particle interacts with very few others, namely the system is very dilute. On the other hand, the interaction potential is very strong in its range (the strength is of order \( N^2 \)), making the correlation of particles complicated. This so-called Gross–Pitaevskii regime is relevant to the physical setup in [1, 14], and its mathematical analysis is both interesting and difficult.

To the leading order, the macroscopic properties of the system are well captured by the famous Gross–Pitaevskii theory [25, 40]. In this theory, a quantum particle is effectively felt by the others as a hard sphere whose radius is the scattering length of the interaction potential. Recall that the scattering length \( a \) of the potential \( V \) is defined by the variational formula

\[
8\pi a = \inf \left\{ \int_{\mathbb{R}^3} 2|\nabla f|^2 + V|f|^2, \lim_{|x| \to \infty} f(x) = 1 \right\}.
\]

When \( V \) is sufficiently smooth, (2) has a minimizer \( 0 \leq f \leq 1 \) and it satisfies

\[
(-2\Delta + V)f = 0.
\]

The scattering length can then be recovered from the formula

\[
8\pi a = \int V f.
\]

By scaling, the scattering length of \( V_N = N^2 V(N\cdot) \) is \( aN^{-1} \). If we formally replace the interaction potential \( V_N(x - y) \) in \( H_N \) by the Dirac-delta interaction \( 8\pi aN^{-1} \delta_0(x - y) \), and insert the ansatz of full condensation \( \Psi_N = u^{\otimes N} \), then we obtain the Gross–Pitaevskii approximation for the ground state energy per particle

\[
\epsilon_{\text{GP}} = \inf_{\|u\|_{L^2(\mathbb{R}^3)} = 1} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_{\text{ext}}|u|^2 + 4\pi a|u|^4).
\]

It is well-known that the variational problem (3) has a minimizer \( \varphi_{\text{GP}} \geq 0 \). This minimizer is unique (up to a constant phase) and solves the Gross–Pitaevskii equation

\[
(-\Delta + V_{\text{ext}} + 8\pi a\varphi_{\text{GP}}^2 - \mu) \varphi_{\text{GP}} = 0, \quad \mu \in \mathbb{R}.
\]
Note that the expectation of $H_N$ against the uncorrelated wave function $\varphi^\otimes N_{GP}$ gives us a formula like (5) but with $4\pi a$ replaced by the larger value $(1/2) \int V$. Thus in the Gross–Pitaevskii regime, the particle correlation due to the two-body scattering process plays a crucial role.

The rigorous derivation of the Gross–Pitaevskii theory from the many-body Schrödinger theory is the subject of many important works in the last two decades; see [31, 28, 29, 37, 6, 7, 8] for low-lying eigenstates, [18, 17] for thermal equilibrium states, and [20, 19, 39, 4, 10] for dynamics. In regard to the ground state energy $E_N := \inf \text{Spec}(H_N) = \inf_{\Psi \in L^2(\mathbb{R}^3)^\otimes N, \|\Psi\|_{L^2} = 1} \langle \Psi, H_N \Psi \rangle$, Lieb, Seiringer and Yngvason [31] proved that

$$\lim_{N \to \infty} E_N = e_{GP}. \quad (7)$$

Later, Lieb and Seiringer [29] proved that the ground state $\Psi_N$ of $H_N$ exhibits the complete Bose–Einstein condensation, namely

$$\lim_{N \to \infty} \frac{\gamma^{(1)}_{\Psi_N}}{N} = |\varphi_{GP}\rangle \langle \varphi_{GP}| \quad (8)$$

in the trace norm. See also [37] for a simplified proof of these results. Here the one-body density matrix $\gamma^{(1)}_{\Psi_N}$ of $\Psi_N$ is an operator on $L^2(\mathbb{R}^3)$ with kernel

$$\gamma^{(1)}_{\Psi_N}(x, y) = N \int_{(\mathbb{R}^3)^{N-1}} \Psi_N(x, x_2, ..., x_N) \overline{\Psi_N(y, x_2, ..., x_N)} dx_2 ... dx_N.$$

In particular, $\gamma^{(1)}_{u \otimes N} = |u\rangle \langle u|$. Note that [3] may hold even if $\Psi_N$ and $\varphi^\otimes N_{GP}$ are not close in the usual norm topology.

A special case of trapped systems is the homogeneous gas, where $H_N$ acts on $L^2(\mathbb{T}^3)^\otimes N$ instead of $L^2(\mathbb{R}^3)^\otimes N$, with $\mathbb{T}^3$ the unit torus in three dimensions, and the external potential is ignored. For this translation-invariant system, it is easy to see that $e_{GP} = 4\pi a$, $\varphi_{GP} = 1$.

In this case, recently Boccato, Brennecke, Cenatiempo and Schlein [6, 7] proved that

$$E_N = N e_{GP} + O(1), \quad \langle \varphi_{GP}, \gamma_{\Psi_N} \varphi_{GP} \rangle = N + O(1), \quad (9)$$

improving upon the leading order convergence in [28]. These optimal bounds are crucial inputs for the analysis of the excitation spectrum of $H_N$ in [8]. Similar bounds were obtained earlier for the quantum dynamics in [4, 10].

**Main result.** In the present paper we aim at providing an alternative approach to the optimal condensation [9] and extending it to inhomogeneous trapped gases. Our main result is

**Theorem 1** (Optimal condensation for trapped Bose gases). Let $0 \leq V_{\text{ext}} \in C^1(\mathbb{R}^3)$ satisfy $|\nabla V_{\text{ext}}(x)|^2 \leq 2V_{\text{ext}}(x)^3 + C$ and $V_{\text{ext}}(x) \to \infty$ as $|x| \to \infty$. Let $0 \leq V \in L^1(\mathbb{R}^3)$ be radial with compact support such that its scattering length $a$ is small. Let $\varphi_{GP}$ be the Gross–Pitaevskii minimizer for $e_{GP}$ in [5]. Let $E_N$ be the ground state energy of the Hamiltonian $H_N$ in [1]. Then we have the energy bound

$$|E_N - Ne_{GP}| \leq C \quad (10)$$
and the operator bound

\[ H_N \geq N e_{\text{GP}} + C^{-1} \sum_{i=1}^{N} (1 - |\varphi_{\text{GP}}\rangle \langle \varphi_{\text{GP}}|)_{x_i} - C \quad \text{on} \quad L^2(\mathbb{R}^3)^{\otimes N}. \]  

(11)

Here \( C > 0 \) is constant independent of \( N \).

A direct consequence of Theorem 1 is

**Corollary 2.** For any wave function \( \Psi_N \) in \( L^2(\mathbb{R}^3)^{\otimes N} \), we have

\[ 0 \leq N - \langle \varphi_{\text{GP}}, \gamma_N^{(1)} \varphi_{\text{GP}} \rangle \leq C(\langle \Psi_N, H_N \Psi_N \rangle - E_N + 1). \]  

(12)

In particular, if \( \Psi_N \) is a ground state of \( H_N \), then

\[ N - \langle \varphi_{\text{GP}}, \gamma_N^{(1)} \varphi_{\text{GP}} \rangle \leq C. \]  

(13)

Proof strategy. Our proof contains two main steps.

- **First**, we factor out the particle correlation by simply ‘completing the square’. The idea is inspired by recent works of Brietzke–Fournais–Solovej [11] and Fournais–Solovej [22] on the Lee–Huang–Yang formula. This step allows us to bound \( H_N \) by a quadratic Hamiltonian on Fock space, up to an energy shift.
- **Second**, we estimate the ground state energy of the quadratic Hamiltonian. In the homogeneous case, this step can be done by ‘completing the square’ again, as realized already in 1947 by Bogoliubov [9]. In the inhomogeneous case, the analysis is significantly more complicated. We will derive a sharp lower bound for general quadratic Hamiltonians, and then apply it to the problem at hand.

Our method is different from that of [6, 7]. For the reader’s convenience, we will quickly present our proof in the homogeneous case. Then we explain further details in the inhomogeneous case.

We use the Fock space formalism, which is recalled in Section 2. Our convention of the Fourier transform is

\[ \hat{g}(p) = \int g(x)e^{-ip\cdot x}dx. \]
Homogeneous case. Let us focus on the main estimate \[11\]. Let \( P = |\varphi_{\text{GP}}\rangle \langle \varphi_{\text{GP}}| \) and let \( f_N(x) = f(Nx) \) where \( f \) is the scattering solution of \[9\]. Since \( V_N \geq 0 \) we have

\[
(1 - P \otimes P f_N)V_N(1 - f_N P \otimes P) \geq 0
\]  

(14)

where \( V_N \) and \( f_N \) are the multiplication operators by \( N^2 V(N(x - y)) \) and \( f(N(x - y)) \) on the two-particle space. Expanding (14) leads to the operator inequality

\[
H_N \geq \sum_{p \neq 0} \left( |p|^2 a_p^* a_p + \frac{1}{2} \int_{\mathbb{R}^3} V_N(p) a_p^* a_{-p}^* a_0 a_0 + \frac{1}{2} f_N V_N(p) a_0^* a_0 a_p a_{-p} \right)
+ \frac{1}{2} \left( \left( 2f_N - f_N^2 \right) V_N \right) a_0^* a_0 a_0 a_0.
\]  

(15)

Here \( a_p^* \) and \( a_p \) are the creation and annihilation operators of momentum \( p \in 2\pi \mathbb{Z}^3 \) on Fock space. Note that the form of the ‘square’ in (14) is slightly different from that of \([11, 22]\) as we factor out completely the ‘cubic contribution’ to make the analysis shorter (in \([11]\) the cubic terms are estimated further by the Cauchy–Schwarz inequality).

Next, recall that by an extension of Bogoliubov’s method \([32, \text{Theorem 6.3}]\), we have

\[
A(b_p^* b_p + b_{-p}^* b_{-p}) + B(b_p^* b_{-p} + b_p b_{-p}) \geq - (A - \sqrt{A^2 - B^2}) \left[ b_p b_p^* + b_{-p} b_{-p}^* \right]
\]

for all constants \( A \geq B \geq 0 \) and operators \( b_p, b_{-p} \) on Fock space. Taking

\[
b_p = N^{-1/2} a_0^* a_p, \quad b_p^* b_p \leq a_p^* a_p, \quad [b_p, b_p^*] \leq 1, \quad \forall 0 \neq p \in 2\pi \mathbb{Z}^3
\]

we find that

\[
\sum_{p \neq 0} \left( (|p|^2 - \mu) a_p^* a_p + \frac{1}{2} \int_{\mathbb{R}^3} V_N(p) a_p^* a_{-p}^* a_0 a_0 + \frac{1}{2} f_N V_N(p) a_0^* a_0 a_p a_{-p} \right)
\]

\[
\geq \frac{1}{2} \sum_{p \neq 0} \left( |p|^2 - \mu \right) (b_p b_p + b_{-p}^* b_{-p}) + N \int_{\mathbb{R}^3} V_N(p) b_p^* b_{-p}^* + \int_{\mathbb{R}^3} V_N(p) b_p b_{-p}
\]

\[
\geq \frac{1}{2} \sum_{p \neq 0} \left( |p|^2 - \mu - \sqrt{|p|^2 - \mu}^2 - |N \int_{\mathbb{R}^3} V_N(p) |^2 \right)
\]  

(16)

for any constant \( 0 < \mu < 4\pi^2 - 8\pi a \) (we used \( N \| \int_{\mathbb{R}^3} V_N(p) \|_{L^\infty} \leq 8\pi a \)). It is straightforward to see that the right side of (16) equals

\[
- \frac{N^2}{4} \sum_{p \neq 0} \frac{|\int_{\mathbb{R}^3} V_N(p)|^2}{4|p|^2} + O(1) = - \frac{N^2}{4} \int_{\mathbb{R}^3} \frac{|\int_{\mathbb{R}^3} V_N(p)|^2}{4|p|^2} \frac{dp}{(2\pi)^3} + O(1)
\]

\[
= - \frac{N^2}{2} \int_{\mathbb{R}^3} V_N f_N (1 - f_N) + O(1).
\]  

(17)

Here we have used Plancherel’s identity and the fact that \( \hat{V}_N \hat{f}_N(p) = 2|p|^2 (1 - f_N)(p) \) which follows from the scattering equation \[3\].
Finally, inserting \([16, 17]\) in \([15]\) we conclude that, for any \(\mu < 4\pi^2 - 8\pi a\),

\[
H_N \geq \mu N_+ - \frac{N^2}{2} \int V_N f_N (1 - f_N)
+ \frac{1}{2} (N - N_+) (N - N_+ - 1) \int (2f_N - f_N^2) V_N + O(1)
\]

\[
\geq (\mu - 16\pi a) N_+ + \frac{N^2}{2} \int f_N V_N + O(1) = (\mu - 16\pi a) N_+ + 4\pi a N + O(1)
\]

(18)

where \(N_+ := N - a_0^* a_0\). If \(a < \pi/6\), we can choose \(\mu\) such that \(16\pi a < \mu < 4\pi^2 - 8\pi a\) and conclude the proof of \([11]\).

**Inhomogeneous case.** Using \([14]\) we obtain a lower bound of the form

\[
H_N \geq N \int_{\mathbb{R}^3} (|\nabla \varphi_{GP}|^2 + V_{ext} |\varphi_{GP}|^2) + \frac{N^2}{2} \int_{\mathbb{R}^3} \left( \left( (2f_N - f_N^2) V_N \right) \star \varphi_{GP}^2 \right)
\]

\[
\quad + \operatorname{inf} \operatorname{Spec}(\mathbb{H}_{Bog}) + (\mu - \mu_1) N_+ + O(1)
\]

(19)

where

\[
N_+ = d\Gamma(Q), \quad \mu_1 = \int_{\mathbb{R}^3} (|\nabla \varphi_{GP}|^2 + V_{ext} |\varphi_{GP}|^2) + 32\pi a \| \varphi_{GP} \|^2_{L^\infty},
\]

and \(\mathbb{H}_{Bog}\) is an operator on the excited Fock space \(\mathcal{F}(QL^2(\mathbb{R}^3))\) defined by

\[
\mathbb{H}_{Bog} = d\Gamma(H) + \frac{1}{2} \int K(x, y)(a_x^* a_y^* + a_x a_y) \, dx \, dy,
\]

\[
H = Q(-\Delta + V_{ext} - \mu) Q, \quad Q = 1 - P
\]

(20)

and

\[
K(x, y) = (Q \otimes Q \tilde{K}(\cdot, \cdot))(x, y), \quad \tilde{K}(x, y) = \varphi_{GP}(x) \varphi_{GP}(y)(NV_N f_N)(x - y).
\]

Note that \([13]\) allows us to choose \(\mu > \mu_1\) such that \(H > \|K\|_{\text{op}}\), where \(K\) is the operator with kernel \(K(x, y)\). Therefore, in principle, the quadratic Hamiltonian \(\mathbb{H}_{Bog}\) can be diagonalized by a Bogoliubov transformation; see \([24, 3, 36, 16]\) for recent results. However, extracting an explicit lower bound is not straightforward. We will prove the following general lower bound, which is of independent interest,

\[
\mathbb{H}_{Bog} \geq \frac{1}{4} \operatorname{Tr} \left( H^{-1} K^2 \right) - C \| K \|_{\text{op}} \operatorname{Tr}(H^{-2} K^2)
\]

(21)

(see Theorem \([5]\)). The simpler general lower bound

\[
\mathbb{H}_{Bog} \geq \frac{1}{2} \operatorname{Tr} \left( H^{-1} K^2 \right)
\]

(22)

is well-known; see \([12\) Theorem 5.4], \([36\) Theorem 2] and \([16\) Theorem 3.23]. The significance of \([12]\) is that we get the optimal constant \((-1/4)\) for the main term which is crucial for our application.

It remains to evaluate the right side of \([21]\) for \(H\) and \(K\) in \([20]\). For a heuristic calculation, let us replace \(H\) and \(K\) by \(-\Delta\) and \(\tilde{K}\), respectively. We can write

\[
\operatorname{Tr} \left( (-\Delta)^{-1} \tilde{K}^2 \right) = N^2 \operatorname{Tr} \left( \varphi_{GP}(x) (\int_N V_N (p) ) \varphi_{GP}(x) p^{-2} \varphi_{GP}(x) (\int_N V_N (p) ) \varphi_{GP}(x) \right).
\]
where $\varphi_{\text{GP}}(x)$ and $v(p)$ are multiplication operators in the position and momentum spaces. If we could commute $\varphi_{\text{GP}}(x)$ and $p^{-2}$, then the above trace would become

$$N^2 \text{Tr} \left( \varphi_{\text{GP}}(x)(\tilde{f}_N V)(p)\varphi_{\text{GP}}^2(x)|p|^{-2}(\tilde{f}_N V)(p)\varphi_{\text{GP}}(x) \right)$$

$$= 2N^2 \text{Tr} \left( \varphi_{\text{GP}}^2(x)(\tilde{f}_N V)(p)\varphi_{\text{GP}}^2(x)(1 - \tilde{f}_N)(p) \right)$$

$$= 2N^2 \int \varphi_{\text{GP}}^2(x)(V_N f_N(1 - f_N))(x - y)\varphi_{\text{GP}}^2(y) dx dy.$$

Here we have used $\tilde{V}_N f_N(p) = 2|p|^2(1 - \tilde{f}_N)(p)$ thanks to the scattering equation and the equality between the Hilbert–Schmidt norm of operators and the $L^2$-norm of operator kernels. This heuristic calculation can be made rigorous by using the Kato–Seiler–Simon inequality [42, Theorem 4.1] to control several commutators. We can also bound $\text{Tr}(H^{-2}K^2)$ by $O(1)$ in the same way.

In summary, we obtain from (19) and (21) that

$$H_N \geq N \int_{\mathbb{R}^3} |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}}|\varphi_{\text{GP}}|^2 + \frac{N^2}{2} \int_{\mathbb{R}^3} \left( (2f_N - f_N^2) V_N \ast \varphi_{\text{GP}}^2 \right)$$

$$+ \frac{N^2}{2} \int (V_N f_N(1 - f_N) \ast \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 + (\mu - \mu_1)N_+ + O(1)$$

$$= N \int_{\mathbb{R}^3} |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}}|\varphi_{\text{GP}}|^2 + \frac{N^2}{2} \int (V_N f_N \ast \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 + (\mu - \mu_1)N_+ + O(1)$$

$$= N \int_{\mathbb{R}^3} |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}}|\varphi_{\text{GP}}|^2 + 4\pi \int \varphi_{\text{GP}}^4 + (\mu - \mu_1)N_+ + O(1).$$

Here we have used $NV_N f_N \approx 8\pi a\delta_0$ in the last estimate. Thus (11) holds true.

The energy upper bound $E_N \leq N\epsilon_{\text{GP}} + O(1)$ is a separate issue, which is conceptually easier. It is known that in the Fock space setting, a good trial state is of the form

$$W(\sqrt{N}\varphi_{\text{GP}})\Gamma W(\sqrt{N}\varphi_{\text{GP}})^*$$

where $W(g) = e^{a(g) - a^*(g)}$ is the Weyl operator and $\Gamma'$ is an appropriate quasi-free state; see Benedikter–Porta–Schlein [5, Appendix A] (similar constructions in the homogeneous case can be found in [21, 38]). This construction can be adapted to the $N$-particle Hilbert space, using the unitary operator $U_N$ introduced by Lewin–Nam–Serfaty–Solovej [26] instead of the Weyl operator and a modified version of $\Gamma'$.

**Organization of the paper.** In Section 2 we recall some standard facts on the scattering length, Gross–Pitaevskii theory, Fock space formalism and quasi-free states. Then we prove the operator lower bound (11) in Section 3 and the energy upper bound in Section 4.

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2. Preliminaries

2.1. Scattering length. We recall some basic properties of the scattering length from [30, Appendix C]. Under our assumptions on the potential $V$, the scattering problem (2) has a unique minimizer $f$. The minimizer is radially symmetric, $0 \leq f \leq 1$ and

$$(-2\Delta + V(x))f(x) = 0, \quad 8\pi a = \int V f.$$  \hfill (23)

Moreover, the function $\omega = 1 - f$ vanishes at infinity, more precisely

$$0 \leq \omega(x) \leq \frac{C}{|x| + 1}, \quad \forall x \in \mathbb{R}^3.$$  \hfill (24)

By scaling, the function $f_N(x) = f(Nx)$ solves the scattering problem for $V_N(x) = N^2V(Nx)$, namely

$$(-2\Delta + V_N(x))f_N(x) = 0, \quad \frac{8\pi a}{N} = \int V_N f_N.$$  \hfill (25)

Thus the function $\omega_N = 1 - f_N$ vanishes at infinity,

$$0 \leq w_N(x) \leq \frac{C}{|Nx| + 1},$$  \hfill (26)

and satisfies

$$-2\Delta \omega_N = V_N f_N \quad \text{in } \mathbb{R}^3.$$  \hfill (27)

2.2. Gross–Pitaevskii theory. Under our assumption on the external potential $V_{ext}$, the minimization problem [5] has a minimizer $\varphi_{GP} \geq 0$ under the constraint

$$\varphi_{GP} \in H^1(\mathbb{R}^3), \quad \int |\varphi_{GP}|^2 = 1, \quad \int V_{ext} |\varphi_{GP}|^2 < \infty.$$  \hfill (28)

Moreover, the minimizer is unique (up to a constant phase) and satisfies the Euler–Lagrange equation (6); see [31, Appendix A] for details.

From (6) and the fact $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, we find that $(-\Delta + V_{ext})\varphi_{GP} \in L^2(\mathbb{R}^3)$. Under the extra condition $|\nabla V_{ext}|^2 \leq 2V_{ext}^3 + C$ we can show that $\Delta \varphi_{GP} \in L^2(\mathbb{R}^3)$ as follows. Replacing $V_{ext}$ by $V_{ext} + 1$ if necessary, we can assume that $V_{ext} \geq 1$. By the IMS formula

$$g^2(x)(-\Delta) + (-\Delta)g^2(x) = 2g(x)(-\Delta)g(x) - 2|\nabla g(x)|^2, \quad \forall 0 \leq g \in H^1,$$  \hfill (29)

we can write

$$(-\Delta + V_{ext})^2 = \Delta^2 + V_{ext}^2 + V_{ext}(-\Delta) + (-\Delta)V_{ext}$$

$$= \Delta^2 + V_{ext}^2 + 2\sqrt{V_{ext}}(-\Delta)\sqrt{V_{ext}} - 2|\nabla \sqrt{V_{ext}}|^2.$$  \hfill (30)

The condition $|\nabla V_{ext}|^2 \leq 2V_{ext}^3 + C$ ensures that

$$2|\nabla \sqrt{V_{ext}}|^2 = \frac{|\nabla V_{ext}|^2}{2V_{ext}} \leq V_{ext}^2 + C.$$  \hfill (31)

Therefore, we conclude that

$$(-\Delta + V_{ext})^2 \geq \Delta^2 - C \quad \text{on } L^2(\mathbb{R}^3).$$  \hfill (32)

Consequently, from $(-\Delta + V_{ext})\varphi_{GP} \in L^2(\mathbb{R}^3)$ we deduce that $\Delta \varphi_{GP} \in L^2(\mathbb{R}^3)$. In summary, we have $\varphi_{GP} \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. 
We may also define the operator-valued distributions \( a \) for the operator domain \( D(-\Delta + V_{\text{ext}}) \) is a subspace of \( H^2(\mathbb{R}^3) \). In general, if \( 0 \leq V_{\text{ext}} \in L^2_{\text{loc}}(\mathbb{R}^d) \), then \(-\Delta + V_{\text{ext}}\) is essentially self-adjoint with core domain \( C_c^\infty(\mathbb{R}^d) \) by Kato’s theorem [11] Theorem X.28, but \( D(-\Delta + V_{\text{ext}}) \) may be different from \( D(-\Delta) \cap D(V_{\text{ext}}) \) [11] Theorem X.32. See [28] [10] for further discussions in this direction.

2.3. Fock space formalism. Let \( \mathcal{F} = L^2(\mathbb{R}^d) \) (or a closed subspace of \( L^2(\mathbb{R}^d) \)) be the Hilbert space of one particle. The bosonic Fock space is defined by

\[
\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n
\]

where the number of particles can vary. For any \( g \in \mathcal{H} \), we can define the creation and annihilation operators \( a^* \) and \( a \) on Fock space by

\[
(a^*(g) \Psi)(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} g(x_j) \Psi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}),
\]

\[
(a(g) \Psi)(x_1, \ldots, x_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} g(x_n) \Psi(x_1, \ldots, x_n) dx_n, \quad \forall \Psi \in \mathcal{H}^n, \quad \forall n.
\]

These operators satisfy the canonical commutation relations

\[
[a(g_1), a(g_2)] = [a^*(g_1), a^*(g_2)] = 0, \quad [a(g_1), a^*(g_2)] = \langle g_1, g_2 \rangle, \quad \forall g_1, g_2 \in \mathcal{H}.
\]

We may also define the operator-valued distributions \( a^*_x \) and \( a_x \), with \( x \in \mathbb{R}^d \), by

\[
a^*_x = \sum_{n=1}^{\infty} f_n(x) a^*(f_n), \quad a_x = \sum_{n=1}^{\infty} f_n(x) a(f_n)
\]

where \( \{f_n\}_{n=1}^{\infty} \) is an orthonormal basis of \( \mathcal{H} \) (the definition is independent of the choice of the basis). Equivalently, we have

\[
a^*(g) = \int_{\mathbb{R}^d} g(x) a^*_x dx, \quad a(g) = \int_{\mathbb{R}^d} g(x) a_x dx, \quad \forall g \in \mathcal{H}.
\]

The canonical commutation relations can be rewritten as

\[
[a^*_x, a_y] = [a_x, a^*_y] = 0, \quad [a_x, a^*_y] = \delta(x - y), \quad \forall x, y \in \mathbb{R}^d.
\]

These creation and annihilation operators can be used to express several important observables. For example, the particle number operator can be written as

\[
\mathcal{N} := \bigoplus_{n=0}^{\infty} n 1_{\mathcal{H}^\otimes n} = \sum_n a^*(u_n) a(u_n) = \int_{\mathbb{R}^d} a^*_x a_x dx.
\]

Here \( \{u_n\} \) is any orthonormal basis for \( \mathcal{H} \). More generally, for any one-body self-adjoint operator \( A \) we have

\[
d\Gamma(A) := \bigoplus_{n=0}^{\infty} \left( \sum_{i=1}^{n} A_{x_i} \right) = \sum_{m,n} \langle u_m, A u_n \rangle a_m^* a_n = \int_{\mathbb{R}^d} a^*_x A_x a_x dx.
\]
For $H_N$ in (1), we can write

$$H_N = d\Gamma(-\Delta + V_{\text{ext}}) + \frac{1}{2} \sum_{m,n,p,q} \langle u_m \otimes u_n, V_N u_p \otimes u_q \rangle a^*(u_m)a(u_n)a(u_p)a(u_q)$$

$$= d\Gamma(-\Delta + V_{\text{ext}}) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_N(x-y)a^*_x a_y a_x a_y \, dx \, dy.$$  

(29)

2.4. **Quasi-free states.** Let $\Gamma$ be a (mixed) state on Fock space with finite particle number expectation, namely $\langle N \rangle_{\Gamma} = \text{Tr}(N \Gamma) < \infty$. We call $\Gamma$ a quasi-free state if it satisfies Wick’s Theorem:

$$\langle a^\#(f_1) a^\#(f_2) \cdots a^\#(f_{2n}) \rangle_{\Gamma} = \sum_{\sigma} \prod_{j=1}^{n} \langle a^\#(f_{\sigma(2j-1)}) a^\#(f_{\sigma(2j)}) \rangle_{\Gamma},$$

$$\langle (a^\#(f_1) a^\#(f_2) \cdots a^\#(f_{2n-1})) \rangle_{\Gamma} = 0, \ \forall f_1, \ldots, f_n \in \mathcal{H}, \forall n \in \mathbb{N}.$$  

Here $a^\#$ is either the creation or annihilation operator and the sum is taken over all permutations $\sigma$ satisfying $\sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\}$ for all $j$.

By the definition any quasi-free state is determined uniquely by its one-body density matrices $(\gamma_{\Gamma}, \alpha_{\Gamma})$, where $\gamma_{\Gamma} : \mathcal{H} \to \mathcal{H}$ and $\alpha_{\Gamma} : \mathcal{H} \to \mathcal{H}^* \equiv \overline{\mathcal{H}}$ defined by

$$\langle g_1, \gamma_{\Gamma} g_2 \rangle = \langle a^\star(g_2)a(g_1) \rangle_{\Gamma}, \quad \langle \overline{\alpha_{\Gamma} g_2}, \alpha_{\Gamma} g_1 \rangle = \langle a^\star(g_2)a^\star(g_1) \rangle_{\Gamma}, \ \forall g_1, g_2 \in \mathcal{H}.$$  

It is well-known (see e.g. [34, Theorem 3.2]) that any given operators $(\gamma, \alpha)$, with $\gamma : \mathcal{H} \to \mathcal{H}$ and $\alpha : \mathcal{H} \to \mathcal{H}^* \equiv \overline{\mathcal{H}}$, are the one-body density matrices of a (mixed) quasi-free state with finite particle number expectation if and only if

$$\gamma \geq 0, \quad \text{Tr} \gamma < \infty, \quad \overline{\alpha} = \alpha^\star, \quad \begin{pmatrix} \gamma & \alpha^\star \\ \alpha & 1 + \overline{\gamma} \end{pmatrix} \geq 0 \quad \text{on} \quad \mathcal{H} \oplus \mathcal{H}^*.$$  

(30)

Here we write $\overline{A} = JAJ$ for short, with $J$ the complex conjugation, namely $\overline{Ag} = (A\overline{g})$.

The reader may think of the quasi-free states as “Gaussian quantum states”. In particular, the contribution of sectors with high particle numbers decays very fast. In fact, if $\Gamma$ is a quasi-free state, then

$$\langle N^\ell \rangle_{\Gamma} \leq C_\ell (1 + \langle N \rangle_{\Gamma})^\ell, \quad \forall \ell \geq 1.$$  

(31)

Here the constant $C_\ell$ depends only on $\ell$ (see [33, Lemma 5]).

3. **Lower bound**

In this section, we will prove the operator lower bound (11).

**Lemma 4 (Lower bound).** Let $V_{\text{ext}}$ and $V$ be as in Theorem 1, where the scattering length $a$ of $V$ is small so that (13) holds true. Then

$$H_N \geq Ne_{\text{GP}} + C^{-1} \sum_{i=1}^{N} Q_{\xi_i} - C \quad \text{on} \quad L^2(\mathbb{R}^3)^{\otimes N}$$

with $Q = 1 - |\varphi_{\text{GP}}\rangle \langle \varphi_{\text{GP}}|$. The constant $C > 0$ is independent of $N$. 


Lemma 5. Let $\mu := \int_{\mathbb{R}^3} (|\nabla \varphi_{GP}|^2 + V_{ext} |\varphi_{GP}|^2) + 32\pi a \|\varphi_{GP}\|_{L^\infty}^2$,  
(32)

$$
\mu_2 := \inf_{\|u\|_{L^2=1}} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_{ext}|u|^2) - 8\pi a \|\varphi_{GP}\|_{L^\infty}^2.
$$
(33)

Note that $\mu_1 < \mu_2$ thanks to (13). Our starting point is

Lemma 5. Let $\mu_1 < \mu < \mu_2$. Then under the notations in (20) we have

$$
H_N \geq N \int_{\mathbb{R}^3} (|\nabla \varphi_{GP}|^2 + V_{ext} |\varphi_{GP}|^2) + \frac{N^2}{2} \int_{\mathbb{R}^3} (((2f_N - f_N^2)V_N) * \varphi_{GP}^2) + (\mu - \mu_1)N_+ + \inf \text{Spec}(\mathbb{H}_{Bog}) - C.
$$
(34)

Proof. We write $\varphi = \varphi_{GP}$ for short. Denote $P = |\varphi\rangle \langle \varphi| = \mathbb{1} - Q$. Let $f_N$ be the scattering solution of $V_N(x) = N^2 V(Nx)$ as in (24). Expanding the operator inequality (12), we obtain

$$
V_N \geq P \otimes P (2f_N - f_N^2) V_N P \otimes P + (P \otimes P f_N V_N Q \otimes Q + P \otimes P f_N V_N Q \otimes P + h.c.)
$$
(35)

Let $\{\varphi_n\}_{n=0}^{\infty}$ be an orthonormal basis for $L^2(\mathbb{R}^3)$ with $\varphi_0 = \varphi$ and denote $a_n := a(\varphi_n)$. From (35) we have the operator inequality in $L^2(\mathbb{R}^3) \otimes N$:

$$
H_N \geq H_0 + H_1 + H_2
$$
(36)

where

$$
H_0 = \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + V_{ext} \varphi^2) + \frac{1}{2} \int_{\mathbb{R}^3} (((2f_N - f_N^2)V_N) * \varphi^2)\varphi^2 a_0^* a_0 a_0 a_0,
$$

$$
H_1 = a^*(Q(-\Delta + V_{ext}) \varphi) a_0 + a^*(Q((f_N V_N) * \varphi^2) \varphi) a_0 a_0 a_0 + h.c.,
$$

$$
H_2 = \frac{1}{2} \sum_{m,n \geq 1} (\langle \varphi_m, (-\Delta + V_{ext}) \varphi_n \rangle a_m^* a_n + N^{-1} \langle \varphi_m \otimes \varphi_n, K \rangle a_m^* a_n a_0 a_0 + h.c.)
$$

Analysis of $H_0$. Using (18) again we have

$$
H_0 = \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + V_{ext} \varphi^2)(N - N_+)
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^3} (((2f_N - f_N^2)V_N) * \varphi^2)\varphi^2(N - N_+)(N - N_+ - 1)
$$

$$
\geq N \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + V_{ext} \varphi^2) + \frac{N^2}{2} \int_{\mathbb{R}^3} (((2f_N - f_N^2)V_N) * \varphi^2)\varphi^2
$$

$$
- \left( \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + V_{ext} \varphi^2) + N \int_{\mathbb{R}^3} (((2f_N - f_N^2)V_N) * \varphi^2)\varphi^2 \right) N_+.
$$

Then using

$$
0 \leq \int_{\mathbb{R}^3} N(((2f_N - f_N^2)V_N) * \varphi^2)\varphi^2 \leq 2N \|f_N V_N\|_{L^1} \|\varphi\|_{L^4}^4 \leq 16\pi a \|\varphi\|_{L^\infty}^2
$$
and the definition of $\mu_1$ in (32), we obtain
\[
\mathcal{H}_0 \geq N \int_{\mathbb{R}^3} \left( |\nabla \varphi|^2 + V_{\text{ext}} \varphi^2 \right) + \frac{N^2}{2} \int_{\mathbb{R}^3} \left( \left( (2f_N - f_N^2) V_N \right) * \varphi^2 \right) \varphi^2 \\
+ (16\pi a \|\varphi\|_{L^\infty}^2 - \mu_1) N_+ - C.
\] (37)

**Analysis of $\mathcal{H}_1$.** We have
\[
\mathcal{H}_1 = a^* (Q(-\Delta + V_{\text{ext}}) \varphi) a_0 + a^* (Q((f_N V_N) * \varphi^2) \varphi) (N - N_+) a_0 + \text{h.c.}
\]
\[
= a^* (Q(-\Delta + V_{\text{ext}} + (f_N V_N) * \varphi^2) \varphi) a_0 - a^* (Q((f_N V_N) * \varphi^2) \varphi) (N_+ a_0 + \text{h.c.})
\]
\[
= a^* (Q((f_N V_N) * \varphi^2 - 8\pi a \varphi^2) \varphi) a_0 - a^* (Q((f_N V_N) * \varphi^2) \varphi) N_+ a_0 + \text{h.c.} \quad \text{(38)}
\]
Here in the last equality we have used the Gross–Pitaevskii equation (6). For the first term on the right side of (38), we use
\[
\| g \|_{L^2} \leq \| (f_N V_N) * \varphi^2 - 8\pi a \varphi^2 \|_{L^\infty}
\]
we have
\[
\| g \|_{L^2} \leq \| (f_N V_N) * \varphi^2 - 8\pi a \varphi^2 \|_{L^\infty} \\
\leq \| N (f_N V_N * \varphi^2) \|_{L^1} - \| \int_{\mathbb{R}^3} |\tilde{fV}(p/N) - \tilde{fV}(0)||\varphi^2| dp \\
\leq \frac{1}{N} \| x |fV| L^1 \| \varphi^2 \|_{H^{1/2}} \leq C.
\]
Therefore, by the Cauchy–Schwarz inequality
\[
\pm (a^* (g) a_0 + a^*_0 a (g)) \leq N a^* (a (g)) + N^{-1} a^*_0 a_0 \leq N \| g \|_{L^2} N_+ + 1 \leq C.
\]
For the second term on the right side of (38), we use
\[
\| Q((f_N V_N) * \varphi^2) \varphi \|_{L^2} \leq \| (f_N V_N) * \varphi^2 \|_{L^\infty} \leq \| f_N V_N \|_{L^1} \| \varphi^2 \|_{L^\infty} \leq \frac{8\pi a \|\varphi\|_{L^\infty}^2}{N},
\]
and the Cauchy–Schwarz inequality
\[
\pm (a^* (Q((f_N V_N) * \varphi^2) \varphi) ) N_+ a_0 + \text{h.c.}
\]
\[
\leq \varepsilon^{-1} a^* (Q((f_N V_N) * \varphi^2) \varphi) a (Q((f_N V_N) * \varphi^2) \varphi) + \varepsilon a^*_0 N_+^2 a_0
\]
\[
\leq \varepsilon^{-1} \| Q((f_N V_N) * \varphi^2) \varphi \|_{L^2}^2 N_+ + \varepsilon N N_+^2
\]
\[
\leq \varepsilon^{-1} \left( \frac{8\pi a \|\varphi\|_{L^\infty}^2}{N} \right)^2 N_+ + \varepsilon N N_+^2. \quad \text{(39)}
\]
Optimizing over $\varepsilon > 0$ we can replace the right side of (39) by $16\pi a \|\varphi\|_{L^\infty}^2 N_+$. Thus
\[
\mathcal{H}_1 \geq -16\pi a \|\varphi\|_{L^\infty}^2 N_+ - C. \quad \text{(40)}
\]

**Analysis of $\mathcal{H}_2$.** We will prove that
\[
\mathcal{H}_2 - \mu N_+ \geq \inf \text{Spec}(\mathbb{H}_{\text{Bog}}).
\] (41)
The main difficulty in (41) is to remove the factors $N^{-1} a_0^* a_0$ and $N^{-1} a_0 a_0$ in $\mathcal{H}_2$.
Recall the operators $H, K$ defined in (20). For any (mixed) state $\Gamma$ on $L^2(\mathbb{R}^3)^{\otimes N}$, we can write
\[
\langle \mathcal{H}_2 - \mu N_+ \rangle_{\Gamma} = \text{Tr} (H \gamma) + \Re \text{Tr} (K \alpha).
\]
where $\mathcal{H} : \mathcal{H}_+^* \to \mathcal{H}_+$ is the operator with kernel $K(x, y)$ and $\gamma : \mathcal{H}_+ \to \mathcal{H}_+$, $\alpha : \mathcal{H}_+ \to \mathcal{H}_+$ are operators defined by

$$\langle g_1, \gamma g_2 \rangle = \langle a^*(g_2)a(g_1) \rangle_{\mathcal{F}}, \quad (\overline{g_1}, \alpha g_2) = N^{-1} \langle a^*(g_2)a^*(g_1)a_0a_0 \rangle_{\mathcal{F}}, \quad \forall g_1, g_2 \in \mathcal{H}_+.$$  

Then we have $\gamma \geq 0$, $\text{Tr} \gamma = \langle N_+ \rangle < \infty$ and $\alpha^* = \overline{\gamma}$. Moreover, for all $g_1, g_2 \in \mathcal{H}_+$, by the Cauchy–Schwarz inequality, we have

$$\pm (a^*(g_1)a^*(g_2)a_0a_0 + \text{h.c.}) \leq a^*(g_1)a_0(a^*(g_1)a_0)^* + (a^*(g_2)a_0)^*a^*(g_2)a_0$$

$$= a^*(g_1)a_0((N - N_+) + a_0a_0)^*(N - N_+)$$

$$\leq \text{Na}^*(g_1)a(g_1) + \text{N}a(g_2)a^*(g_2).$$

Here we have used $a^*(g_1)a(g_1)(N_+ - 1) \geq 0$. Consequently, for all $g_1, g_2 \in \mathcal{H}_+$,

$$\langle \frac{g_1}{\sqrt{g_2}}, \gamma \frac{g_1}{\sqrt{g_2}} \rangle_{\mathcal{F}_+} = \langle g_1, \gamma g_1 \rangle + \langle g_2, (1 + \gamma)g_2 \rangle + (\overline{g_2}, \alpha g_1) + \langle g_1, \alpha^* \overline{g_2} \rangle$$

$$= \langle a^*(g_1)a(g_1) + a_0a_0, a^*(g_2) + N^{-1}a^*(g_2)a^*(g_1)a_0a_0 + N^{-1}a_0^*a_0a(g_2)a(g_1) \rangle_{\mathcal{F}} \geq 0.$$  

Thus $(\gamma, \alpha)$ satisfies the conditions in (30). Hence, there exists a mixed quasi-free state \( \Gamma' \) on Fock space $\mathcal{F}^\times(\mathcal{H}_+)$ such that $(\gamma, \alpha)$ are its one-body density matrices. Therefore,

$$\langle \mathcal{H}_2 - \mu N_+ \rangle_{\Gamma'} = \text{Tr} (H \gamma) + \text{Tr} (K \alpha) = \text{Tr} (H_{\text{Bog}} \Gamma') \geq \text{inf Spec}(H_{\text{Bog}}).$$

Thus (11) holds true.

Conclusion. Inserting (37), (40) and (11) in (39) we obtain (43).

3.2. A general bound for quadratic Hamiltonians. Now we prove a general lower bound on quadratic Hamiltonians on Fock space, which is of independent interest.

Theorem 6 (Lower bound for quadratic Hamiltonians). Let \( \mathcal{F} \) be a closed subspace of $L^2(\mathbb{R}^d)$. Let $K$ be a self-adjoint bounded operator on $\mathcal{F}$ with real-valued symmetric kernel $K(x, y) = K(y, x)$. Let $H > 0$ be a self-adjoint operator on $\mathcal{F}$ such that $H$ has compact resolvent with real-valued eigenfunctions, $H^{-1/2}K$ is a Hilbert-Schmidt operator, and $H \geq (1 + \varepsilon)||K||_{\text{op}}$ for a constant $\varepsilon > 0$. Then

$$\text{d} \Gamma(H) + \frac{1}{2} \iint K(x, y)(a_x^*a_y^* + a_xa_y) dx dy \geq -\frac{1}{4} \text{Tr} (H^{-1}K^2) - C_\varepsilon ||K||_{\text{op}} \text{Tr}(H^{-2}K^2)$$

(42)

on the Fock space $\mathcal{F}(\mathcal{F})$. Here the constant $C_\varepsilon > 0$ depends only on $\varepsilon$.

Remark 7. The constant $-1/4$ is optimal. In fact, if $H$ and $K$ commute, then the ground state energy of the quadratic Hamiltonian is

$$\frac{1}{2} \text{Tr} \left( \sqrt{H^2 - K^2} - H \right) = -\frac{1}{2} \text{Tr} \left( \frac{K^2}{H + \sqrt{H^2 - K^2}} \right)$$

which is close to $-\langle 1/4 \rangle \text{Tr}(H^{-1}K^2)$ when $H$ is significantly bigger than $K$.

Proof of Theorem 6. First, let us assume that $K$ is trace class. Then following the analysis of Grech–Seiringer [24] Section 4], we see that the ground state energy of the quadratic Hamiltonian in (42) is $\frac{1}{2} \text{Tr}(E - H)$ where

$$E := (D^{1/2}(D + 2K))^{1/2}, \quad D := H - K \geq 0.$$  

Using the formula

$$x = \frac{2}{\pi} \int_0^\infty \frac{x^2}{x^2 + t^2} dt, \quad \forall x \geq 0,$$
and the resolvent identity, we can rewrite

\[
E - D = \frac{2}{\pi} \int_0^\infty \left( \frac{1}{D^2 + t^2} - \frac{1}{E^2 + t^2} \right) t^2 dt = \frac{2}{\pi} \int_0^\infty \frac{1}{D^2 + t^2} D^{1/2} (2K) D^{1/2} \frac{1}{E^2 + t^2} t^2 dt
\]

\[
= \frac{2}{\pi} \int_0^\infty \frac{1}{D^2 + t^2} D^{1/2} (2K) D^{1/2} \frac{1}{D^2 + t^2} t^2 dt
\]

\[
- \frac{2}{\pi} \int_0^\infty \left( \frac{1}{D^2 + t^2} D^{1/2} (2K) D^{1/2} \right)^2 \frac{1}{D^2 + t^2} t^2 dt
\]

\[
+ \frac{2}{\pi} \int_0^\infty \left( \frac{1}{D^2 + t^2} D^{1/2} (2K) D^{1/2} \right)^3 \frac{1}{E^2 + t^2} t^2 dt
\]

\[=: (I) + (II) + (III). \tag{43} \]

Dealing with (I). Using the cyclicity of the trace and

\[
\frac{2}{\pi} \int_0^\infty \frac{xt^2}{(x^2 + t^2)^2} dt = \frac{1}{2}, \quad \forall x > 0,
\]

we have

\[
\text{Tr}(I) = \frac{2}{\pi} \text{Tr} \int_0^\infty \frac{Dt^2}{(D^2 + t^2)^2} (2K) dt = \text{Tr}(K). \tag{44}
\]

Dealing with (II). Note that \( D = H - K > 0 \) has compact resolvent (since \( H \) has compact resolvent and \( K \) is compact). Therefore, we can write

\[
D = \sum_j D_j |\varphi_j\rangle \langle \varphi_j|
\]

with positive eigenvalues \( (D_j) \) and an orthonormal basis of eigenvectors \( (\varphi_j) \). Therefore,

\[
\text{Tr} (II) = -\frac{8}{\pi} \text{Tr} \int_0^\infty \frac{D}{(D^2 + t^2)^2} K \frac{D}{D^2 + t^2} K t^2 dt
\]

\[
= -\frac{8}{\pi} \text{Tr} \int_0^\infty \sum_{i,j} \frac{D_j}{(D_j^2 + t^2)^2} |\varphi_j\rangle \langle \varphi_j| K \frac{D_j}{D_j^2 + t^2} |\varphi_i\rangle \langle \varphi_i| K t^2 dt
\]

\[
= -\frac{8}{\pi} \sum_{i,j} |\langle \varphi_i, K \varphi_j \rangle|^2 \int_0^\infty \frac{D_j D_j}{(D_j^2 + t^2)^2 (D_i^2 + t^2)} t^2 dt.
\]

Using

\[
\frac{8}{\pi} \int_0^\infty \frac{xy}{(x^2 + t^2)^2 (y^2 + t^2)} t^2 dt = \frac{2y}{(x+y)^2} \leq \frac{1}{2x}, \quad \forall x, y > 0
\]

we find that

\[
\text{Tr} (II) \geq -\frac{1}{2} \sum_{i,j} |\langle \varphi_i, K \varphi_j \rangle|^2 D_j^{-1} = -\frac{1}{2} \text{Tr} (KD^{-1}K). \tag{45}
\]

Dealing with (III). By Hölder’s inequality for Schatten norm \[42, \text{Theorem 2.8}],

\[
\text{Tr}(III) = \frac{16}{\pi} \left| \int_0^\infty \text{Tr} \left( \left( \frac{D}{D^2 + t^2} K \right)^3 \frac{t^2}{E^2 + t^2} D^{-1/2} \right) dt \right|
\]

\[
\leq \frac{16}{\pi} \int_0^\infty \left\| \frac{D}{D^2 + t^2} K \right\|_\text{op}^3 \left\| \frac{t^2}{E^2 + t^2} D^{-1/2} \right\|_\text{op} dt.
\]
Then by the Araki–Lieb–Thirring inequality [33,2],
\[
\left\| \frac{D}{D^2 + t^2} K \right\|_{\mathfrak{S}^3}^3 = \text{Tr} \left( \left( K \left( \frac{D}{D^2 + t^2} \right)^2 K \right)^{3/2} \right) = \text{Tr} \left( \left( \frac{D}{D^2 + t^2} K^2 \frac{D}{D^2 + t^2} \right)^{3/2} \right) \\
\leq \text{Tr} \left( \left( \frac{D}{D^2 + t^2} \right)^{3/2} |K|^3 \left( \frac{D}{D^2 + t^2} \right)^{3/2} \right) = \text{Tr} \left( \frac{D^3}{(D^2 + t^2)^3} |K|^3 \right).
\]

Here we have used the fact that $|A| = \sqrt{A^*A}$ and $\sqrt{AA^*}$ have the same non-zero eigenvalues (with multiplicity). On the other hand, using $H \geq (1 + \varepsilon)\|K\|_{\text{op}}$ we find that
\[
D + 2K = H + K \geq C_\varepsilon^{-1}(H - K) = C_\varepsilon^{-1}D
\]
for any large constant $C_\varepsilon$ satisfying $(C_\varepsilon + 1)/(C_\varepsilon - 1) \leq 1 + \varepsilon$. Hence,
\[
E^2 = D^{1/2}(D + 2K)D^{1/2} \geq C_\varepsilon^{-1}D^2.
\]

Reversely, we also have $D + 2K \leq C_\varepsilon D$, and hence
\[
E^2 = D^{1/2}(D + 2K)D^{1/2} \leq C_\varepsilon D^2.
\]

Since the mapping $0 \leq A \mapsto \sqrt{A}$ is operator monotone, we deduce that $D^{1/2}E^{-1/2}$ and $E^{1/2}D^{-1/2}$ are bounded operators. Therefore,
\[
\left\| D^{1/2} \frac{i^2}{E^2 + t^2} D^{-1/2} \right\|_{\text{op}} = \left\| D^{1/2} E^{-1/2} \frac{i^2}{E^2 + t^2} E^{1/2} D^{-1/2} \right\|_{\text{op}} \\
\leq \left\| D^{1/2} E^{-1/2} \right\|_{\text{op}} \left\| \frac{i^2}{E^2 + t^2} \right\|_{\text{op}} \left\| E^{1/2} D^{-1/2} \right\|_{\text{op}} \leq C_\varepsilon.
\]

We conclude that
\[
|\text{Tr}(\text{III})| \leq C_\varepsilon \int_0^\infty \text{Tr} \left( \frac{D^3}{(D^2 + t^2)^3} |K|^3 \right) dt \leq C_\varepsilon \text{Tr}(D^{-2} |K|^3). \quad (46)
\]

In the last estimate we have used the identity
\[
x^{-2} = \frac{16}{3\pi} \int_0^\infty \frac{x^3}{(x^2 + t^2)^3} dt, \quad \forall x > 0.
\]

**Conclusion in the trace class case.** Inserting [41], [45], [46] in [43] we find that
\[
\text{Tr}(E - H) = \text{Tr}(E - D) - \text{Tr}(K) \geq -\frac{1}{2} \text{Tr}(D^{-1} K^2) - C_\varepsilon \text{Tr}(D^{-2} |K|^3).
\]

Let us replace $D = H - K$ by $H$ on the right side. Using $H \geq (1 + \varepsilon)\|K\|_{\text{op}}$ and the Cauchy–Schwarz inequality we have
\[
D^2 = (H - K)^2 = H^2 + K^2 - HK - KH \geq (1 - \eta)H^2 - (\eta^{-1} - 1)K^2 \\
\geq (1 - \eta)H^2 - (\eta^{-1} - 1)(1 + \varepsilon)^{-2}H^2 \geq (C_\varepsilon)^{-1}H^2.
\]

Here the constant $0 < \eta < 1$ is chosen sufficiently close to 1 (depending on $\varepsilon$). Therefore,
\[
\text{Tr}(D^{-2} |K|^3) \leq C_\varepsilon \text{Tr}(H^{-2} |K|^3) \leq C_\varepsilon \|K\|_{\text{op}} \text{Tr}(H^{-2} K^2).
\]

By the resolvent identity and Hölder’s inequality for Schatten norm [42, Theorem 2.8],
\[
|\text{Tr} \left( (D^{-1} - H^{-1}) K^2 \right) | = |\text{Tr} \left( D^{-1} K H^{-1} K^2 \right) | \\
\leq \|D^{-1} H\|_{\text{op}} \|H^{-1} K\|_{\mathfrak{S}^2} \|K\|_{\text{op}} \leq C_\varepsilon \|K\|_{\text{op}} \text{Tr}(H^{-2} K^2)
\]

Recall that from \((36, \text{Theorem } 2)\), the bound \((22)\) holds true:
\[
\text{Tr}(E - H) \geq -\frac{1}{2} \text{Tr} \left( D^{-1} K^2 \right) - C \varepsilon \text{Tr} \left( D^{-2} |K|^2 \right) \\
\geq -\frac{1}{2} \text{Tr} \left( H^{-1} K^2 \right) - C \varepsilon \|K\|_{\text{op}} \text{Tr} \left( H^{-2} K^2 \right).
\]

**Removing the trace class condition.** Finally, let us remove the trace class condition on \(K\). Recall that \(H\) has compact resolvent. For every \(n \in \mathbb{N}\) we introduce the spectral projection
\[
P_n = \mathbbm{1}(H \leq n), \quad Q_n = \mathbbm{1}(H > n)
\]
and decompose
\[
K = K^{(1)}_n + K^{(2)}_n, \quad K^{(1)}_n = \frac{1}{2} (P_n K + KP_n), \quad K^{(2)}_n = \frac{1}{2} (Q_n K + KQ_n).
\]

Note that \(K^{(1)}_n\) is a self-adjoint finite-rank operator because \(P_n\) is finite-rank. Moreover, \(K^{(1)}_n\) has a real-valued symmetric kernel because \(K\) has the same property and \(H\) has real-valued eigenvalues. By the triangle inequality,
\[
\|K^{(1)}_n\|_{\text{op}} \leq \frac{1}{2} \|P_n K\|_{\text{op}} + \|KP_n\|_{\text{op}} \leq \|K\|_{\text{op}},
\]
\[
\|H^{-s} K^{(1)}_n\|_{\overline{\varepsilon}_2} \leq \frac{1}{2} \|H^{-s} P_n K\|_{\overline{\varepsilon}_2} + \|H^{-s} K P_n\|_{\overline{\varepsilon}_2} \leq \|H^{-s} K\|_{\overline{\varepsilon}_2}, \quad \forall s \geq 1/2.
\]

Take \(\eta \in (0, \varepsilon/2)\). Using \(H \geq (1 + \varepsilon)\|K\|_{\text{op}}\) we find that
\[
(1 - \eta) \geq \left(1 - \frac{\varepsilon}{2}\right) (1 + \varepsilon) \|K\|_{\text{op}} \geq \left(1 + \frac{\varepsilon(1 - \varepsilon)}{2}\right) \|K^{(1)}_n\|_{\text{op}}.
\]

Applying \((12)\) in the trace class case with \((H, K)\) replaced by \((1 - \eta)H, K^{(1)}_n\) we get
\[
(1 - \eta) \text{d} \Gamma(H) + \frac{1}{2} \iint K^{(n)}_1(x,y)(a_x^* a_y + a_x a_y) dxdy \\
\geq -\frac{1}{4(1 - \eta)} \|H^{-1/2} K^{(1)}_n\|_{\overline{\varepsilon}_2}^2 - C \varepsilon \|K^{(1)}_n\|_{\text{op}} \|H^{-1} K^{(1)}_n\|_{\overline{\varepsilon}_2}^2 \\
\geq -\frac{1}{4(1 - \eta)} \|H^{-1/2} K\|_{\overline{\varepsilon}_2}^2 - C \varepsilon \|K\|_{\text{op}} \|H^{-1} K\|_{\overline{\varepsilon}_2}^2, \quad \forall n \geq 1.
\]

Next, note that \(K^{(2)}_n\) is also self-adjoint and has a real-valued symmetric kernel. Moreover, \(Q_n \to 0\) strongly as \(n \to \infty\) (namely \(|Q_n u| \to 0\) for all \(u \in \mathcal{H}\)) since \(H\) has compact resolvent. Since \(H^{-1/2}K\) is Hilbert-Schmidt, we deduce that
\[
\|H^{-1/2} K^{(2)}_n\|_{\overline{\varepsilon}_2} \leq \frac{1}{2} \|H^{-1/2} Q_n K\|_{\overline{\varepsilon}_2} + \|H^{-1/2} K Q_n\|_{\overline{\varepsilon}_2} \to 0 \quad \text{as } n \to \infty.
\]

Recall that from \((36, \text{Theorem } 2)\), the bound \((22)\) holds true if \(\|H^{-1/2} K H^{-1/2}\|_{\text{op}} < 1\) and \(\|H^{-1/2} K\|_{\overline{\varepsilon}_2} < \infty\). Using \((22)\) with \((H, K)\) replaced by \((\eta H, K^{(2)}_n)\), we find that
\[
\eta \text{d} \Gamma(H) + \frac{1}{2} \iint K^{(2)}_n(x,y)(a_x^* a_y + a_x a_y) dxdy \geq -\frac{1}{2\eta} \|H^{-1/2} K^{(2)}_n\|_{\overline{\varepsilon}_2} \to 0
\]
as \(n \to \infty\). Putting \((17)\) and \((48)\) together, we find that
\[
\text{d} \Gamma(H) + \frac{1}{2} \iint K(x,y)(a_x^* a_y + a_x a_y) dxdy \geq -\frac{1}{4(1 - \eta)} \|H^{-1/2} K\|_{\overline{\varepsilon}_2}^2 - C \varepsilon \|K\|_{\text{op}} \|H^{-1} K\|_{\overline{\varepsilon}_2}^2
\]
Taking \(\eta \to 0\) we obtain \((12)\). This completes the proof of Theorem \(6\).
\(\square\)
3.3. Explicit lower bound for $\mathbb{H}_{\text{Bog}}$. Now we apply Theorem 6 to compute an explicit lower bound for the quadratic Hamiltonian $\mathbb{H}_{\text{Bog}}$ in Lemma 5.

Lemma 8 (Lower bound for $\mathbb{H}_{\text{Bog}}$). For $\mathbb{H}_{\text{Bog}}$ in Lemma 5 we have

$$\inf \text{Spec}(\mathbb{H}_{\text{Bog}}) \geq -\frac{N^2}{2} \int_{\mathbb{R}^3} (V_N f_N(1 - f_N) \ast \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 - C.$$  \hfill (49)

Proof. We will write $\varphi = \varphi_{\text{GP}}$ for short. Recall the notations $H, K$ in (20).

**Lower bound by Theorem 6.** Since $\varphi \geq 0$, the kernel $K(x, y)$ is symmetric and real-valued. Thus the operator $H$ is symmetric. It is bounded with $\|K\|_{\text{op}} \leq 8\pi a \|\varphi\|_{L^\infty}$ because for all $g_1, g_2 \in \mathcal{S}_+$ we have

$$\langle g_1, Kg_2 \rangle = \int \frac{g_1(x) \varphi(x)(Nf_N V_N(x - y)) \varphi(y)g_2(y)dx}{\|g_1\|_{L^2} \|g_2\|_{L^2} \|Nf_N V_N\|_{L^1}} \leq 8\pi a \|\varphi\|_{L^\infty}^2 \|g_1\|_{L^2} \|g_2\|_{L^2}.$$  \hfill (50)

On the other hand, since $\mu_2 > \mu$ we have

$$H = Q(-\Delta + V_{\text{ext}} - \mu)Q \geq \mu_2 - \mu + 8\pi a \|\varphi\|_{L^\infty}^2 \geq (1 + \varepsilon) \|K\|_{\text{op}}$$

for a small constant $\varepsilon > 0$ independent of $N$. Moreover, $H$ has compact resolvent since $V_{\text{ext}}(x) \to \infty$ as $|x| \to \infty$. Thus we can apply Theorem 6 and obtain

$$\inf \text{Spec}(\mathbb{H}_{\text{Bog}}) \geq -\frac{1}{4} \text{Tr}_{\mathcal{S}_+} (H^{-1}K^2) - C \text{Tr}_{\mathcal{S}_+} (H^{-2}K^2).$$  \hfill (51)

Replacing $H$ and $K$ by $1 - \Delta$ and $\tilde{K}$. We can interpret $H = Q(-\Delta + V_{\text{ext}} - \mu)Q$ as an operator on $L^2(\mathbb{R}^3)$. Then using $V_{\text{ext}} \geq 0$ and $Q = 1 - |\varphi\rangle \langle \varphi|$ we have

$$H \geq Q(-\Delta)Q - \mu = -\Delta + |\varphi\rangle \langle \Delta \varphi| + |\Delta \varphi\rangle \langle \varphi| + \|\nabla \varphi\|_{L^2(\mathbb{R}^3)}^2 |\varphi\rangle \langle \varphi| - \mu \geq -\Delta - 2 \|\Delta \varphi\|_{L^2} - \mu$$

on $L^2(\mathbb{R}^3)$. \hfill (52)

Moreover, by (28) and the Cauchy-Schwarz inequality,

$$H^2 \geq Q(-\Delta + V_{\text{ext}} - \mu)^2 Q - \|Q(-\Delta + V_{\text{ext}} - \mu)\varphi\|_{L^2(\mathbb{R}^3)}^2 \geq \frac{1}{2} Q(-\Delta)^2 Q - C = \frac{1}{2} \|\varphi\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|\Delta \varphi\|_{L^2}^2 - C \geq \frac{1}{4} \|\Delta \varphi\|_{L^2}^2 - C$$

on $L^2(\mathbb{R}^3)$. \hfill (53)

Since $H_{\mathcal{S}_+}$ is strictly positive, we also have, for any large constant $C_0 > 0$,

$$C_0^2 H^2 \geq H^2 + C_0$$

on $\mathcal{S}_+$. \hfill (54)

Recall that the mapping $t \mapsto t^{-1}$ is operator monotone for $t > 0$. Moreover, if $A$ is a self-adjoint positive operator on $L^2(\mathbb{R}^3)$ that commutes with $Q$, then

$$Q(AQ)^{-1}_{\mathcal{S}_+} Q = QA^{-1} Q$$

on $L^2(\mathbb{R}^3)$. \hfill (55)

by Spectral Theorem. Therefore,

$$QH_{\mathcal{S}_+} H^2 \leq C_0^2 Q(H^2 + C_0)^{-1} Q \leq CQ(H^2 + C_0)^{-1} Q \leq CQ(1 - \Delta)^{-2} Q$$

on $L^2(\mathbb{R}^3)$. \hfill (56)
Similarly,
\[
QH^{-1}_{\mathcal{O}+} Q = (Q(H + C))^{-1}_{\mathcal{O}+} Q + CQH(H + C)_{\mathcal{O}+}^{-1} Q
\]
\[
\leq Q(H + C)^{-1} Q + CQH^2_{\mathcal{O}+} Q
\]
\[
\leq Q(1 - \Delta)^{-1} Q + CQ(1 - \Delta)^{-2} Q \quad \text{on } L^2(\mathbb{R}^3).
\]  

Next we replace $K$ by $\tilde{K}$. Following [51], we have $\|K\|_{\text{op}} \leq C$ where $K$ is the operator on $L^2(\mathbb{R}^3)$ with kernel $K(x,y)$. Using $K^2 \leq Q\tilde{K}^2 Q$ on $\mathcal{O}^3$ and [56] we can estimate
\[
\text{Tr}_{\mathcal{O}+} (H^{-2}K^2) \leq \text{Tr}_{\mathcal{O}+} \left( H^{-2}Q\tilde{K}^2 Q \right) = \text{Tr}_{L^2(\mathbb{R}^3)} \left( QH^{-2} Q\tilde{K}^2 \right)
\]
\[
\leq C\text{Tr} \left( Q(1 - \Delta)^{-2} Q\tilde{K}^2 \right) = C\text{Tr} \left( (1 - \Delta)^{-2} Q\tilde{K}^2 Q \right)
\]
\[
\leq C\text{Tr} \left( (1 - \Delta)^{-2} \tilde{K}^2 \right) + C.
\]  

Similarly, from [52] we deduce that
\[
\text{Tr}_{\mathcal{O}+} (H^{-1}K^2) \leq \text{Tr}_{\mathcal{O}+} \left( QH^{-1} Q\tilde{K}^2 \right) \leq \text{Tr} \left( (Q(1 - \Delta)^{-1} Q + CQ(1 - \Delta)^{-2} Q) \tilde{K}^2 \right)
\]
\[
\leq \text{Tr} \left( (1 - \Delta)^{-1} \tilde{K}^2 \right) + C\text{Tr} \left( (1 - \Delta)^{-2} \tilde{K}^2 \right) + C.
\]  

Thus [51] reduces to
\[
\inf \text{Spec}(\mathcal{H}_{\text{Bog}}) \geq -\frac{1}{4} \text{Tr} \left( (1 - \Delta)^{-1} \tilde{K}^2 \right) - C\text{Tr} \left( (1 - \Delta)^{-2} \tilde{K}^2 \right) - C.
\]  

**Evaluation of traces in [60].** Note that the operator $\tilde{K}$ with kernel $\varphi(x)N\tilde{f}_N V_N (x - y)\varphi(y)$ can be written as
\[
\tilde{K} = \varphi(x)N\tilde{f}_N V_N (p)\varphi(x) \quad \text{on } L^2(\mathbb{R}^3)
\]  

where $\varphi(x)$ and $v(p)$ are the multiplication operators on the position and momentum spaces (the derivation of [61] uses $\hat{u} \ast \hat{v} = \hat{u}\hat{v}$). Recall the Kato–Seiler–Simon inequality on Schatten norms [42, Theorem 4.1]:
\[
\|u(x)v(p)\|_{L^r} \leq C_{d,r} \|u\|_{L^r(\mathbb{R}^d)} \|v\|_{L^r(\mathbb{R}^d)}, \quad 2 \leq r < \infty.
\]  

Consequently,
\[
\text{Tr} \left( (1 - \Delta)^{-2} \tilde{K}^2 \right) = \text{Tr} \left( \varphi(x)N(\tilde{f}_N V_N)(p)\varphi(x) \frac{1}{(1 + p^2)^2} \varphi(x)N(\tilde{f}_N V_N)(p)\varphi(x) \right)
\]
\[
\leq \|\varphi\|^2_{L^\infty(\mathbb{R}^3)} \|N\tilde{f}_N V_N\|^2_{L^\infty(\mathbb{R}^3)} \|\varphi(x)(p^2 + 1)^{-1}\|^2_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \|\varphi\|^2_{L^\infty(\mathbb{R}^3)} \|N\tilde{f}_N V_N\|^2_{L^\infty(\mathbb{R}^3)} \|\varphi\|^2_{L^2(\mathbb{R}^3)} \|\varphi(x)(p^2 + 1)^{-1}\|^2_{L^2(\mathbb{R}^3)} \leq C.
\]  

Here we have used Hölder’s inequality for Schatten spaces and $\|N\tilde{f}_N V_N\|_{L^\infty} \leq 8\pi a$.

Next, consider
\[
\text{Tr} \left( (1 - \Delta)^{-1} \tilde{K}^2 \right) = \text{Tr} \left( \varphi(x)N(\tilde{f}_N V_N)(p)\varphi(x) \frac{1}{1 + p^2} \varphi(x)N(\tilde{f}_N V_N)(p)\varphi(x) \right).
\]
Let us decompose

\[ 2\varphi(x)(1 + p^2)^{-1}\varphi(x) - \varphi^2(x)(1 + p^2)^{-1} - (1 + p^2)^{-1}\varphi^2(x) \]

\[ = - [\varphi(x), [\varphi(x), (1 + p^2)^{-1}]] - [\varphi(x), (1 + p^2)^{-1}][\varphi(x), p^2](1 + p^2)^{-1} \]

\[ = [\varphi(x), (1 + p^2)^{-1}] [\varphi(x), p^2](1 + p^2)^{-1} + (1 + p^2)^{-1} [\varphi(x), [\varphi(x), p^2]] (1 + p^2)^{-1} \]

\[ = - 2(1 + p^2)^{-1}[\varphi(x), p^2](1 + p^2)^{-1}[\varphi(x), p^2](1 + p^2)^{-1} \]

\[ - 2(1 + p^2)^{-1}\nabla\varphi(x)|^2(1 + p^2)^{-1}, \]

where we have used

\[ [\varphi(x), (p^2 + 1)^{-1}] = - (p^2 + 1)^{-1}[\varphi(x), p^2](p^2 + 1)^{-1} \]

and the IMS formula \[27\] for \([\varphi(x), [\varphi(x), p^2]]\). This gives

\[
\varphi(x)N\overline{f_N V_N(p)}\varphi(x) \frac{1}{1 + p^2} - \varphi(x)N\overline{f_N V_N(p)}\varphi(x) = \frac{1}{2} \varphi(x) \overline{f_N V_N(p)} \left( \frac{\varphi^2(x)}{1 + p^2} + \frac{1}{1 + p^2} \varphi^2(x) \right) \frac{N\overline{f_N V_N(p)}}{1 + p^2} \varphi(x) - \varphi(x)N\overline{f_N V_N(p)} \frac{1}{1 + p^2} [\varphi(x), p^2] \frac{1}{1 + p^2} N\overline{f_N V_N(p)} \varphi(x) - \varphi(x)N\overline{f_N V_N(p)} \frac{1}{1 + p^2} \nabla\varphi(x)|^2 \frac{1}{1 + p^2} N\overline{f_N V_N(p)} \varphi(x) =: (I) + (II) + (III). \tag{64}
\]

**Dealing with (I).** For the main term (I), we write

\[ \text{Tr}(I) = \Re \text{Tr} \left( \frac{\varphi^2(x)N\overline{f_N V_N(p)}\varphi^2(x)N\overline{f_N V_N(p)}}{1 + p^2} \right) \]

\[ = \Re \text{Tr} \left( \frac{\varphi^2(x)N\overline{f_N V_N(p)}\varphi^2(x)N\overline{f_N V_N(p)}}{p^2} \right) - \Re \text{Tr} \left( \frac{N\overline{f_N V_N(p)}\varphi^2(x)N\overline{f_N V_N(p)}}{p^2(1 + p^2)} \varphi^2(x) \right). \]

The first term can be computed exactly using the scattering equation \[26\]

\[ \Re \text{Tr} \left( \frac{\varphi^2(x)N\overline{f_N V_N(p)}\varphi^2(x)}{p^2} \right) \]

\[ = 2\Re \text{Tr} \left( \frac{\varphi^2(x)N\overline{f_N V_N(p)}\varphi^2(x)N(1 - f_N(p))}{p^2} \right) = 2N^2 \int_{\mathbb{R}^3} (V_Nf_N(1 - f_N) + \varphi^2) \varphi^2. \]

Here we have used the following identity

\[ \text{Tr} \left( \frac{\varphi_1(x)\overline{g_1(p)}\varphi_2(x)\overline{g_2(p)}}{p^2} \right) = \langle \overline{g_1(p)}\varphi_1(x), \varphi_2(x)\overline{g_2(p)} \rangle_{L^2} \]

\[ = \langle \langle \overline{g_1(p)}\varphi_1(x) \rangle(\cdot, \cdot), \langle \varphi_2(x)\overline{g_2(p)} \rangle(\cdot, \cdot) \rangle_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \]

\[ = \iint g_1(y - z)\overline{\varphi_1(z)}\varphi_2(y)g_2(y - z)dydz. \tag{65} \]
This is based on the equality between the Hilbert–Schmidt norm of operators and the $L^2$-norm of operator kernels (the kernel of operator $\tilde{g}_1(x)$ is $g_1(y - z)\varphi(z)$, similarly to [61]). Here in our case $\varphi(x) \geq 0$ and $f_NV(x)$ is real-valued since $V_Nf_N$ is radial.

The second term can be estimated by Hölder’s and the Kato–Seiler–Simon inequalities:

$$\left| \text{Tr} \left( N\widehat{f_NV_N}(p)\varphi^2(x) \frac{N\widehat{f_NV_N}(p)}{p^2(1 + p^2)} \varphi^2(x) \right) \right| \leq \left\| N\widehat{f_NV_N} \right\|_{L^\infty}^2 \left\| \varphi(x) \right\|_{L^\infty}^2 \left\| \varphi^2 \right\|_{L^2}^2 \left\| \frac{1}{p^2(1 + p^2)} \right\|_{L^2}^2 \leq C \left\| N\widehat{f_NV_N} \right\|_{L^\infty}^2 \left\| \varphi \right\|_{L^4}^4 \left\| \frac{1}{p^2(1 + p^2)} \right\|_{L^2} \leq C.$$ 

Here we used again $\left\| N\widehat{f_NV_N} \right\|_{L^\infty} \leq 8\pi a$. Thus

$$\text{Tr}(I) = 2N^2 \int_{\mathbb{R}^3} (V_Nf_N(1 - f_N) * \varphi^2) \varphi^2 + O(1).$$

**Dealing with (II).** By expanding further

$$[\varphi(x), p^2] = (\Delta \varphi)(x) + 2(\nabla \varphi(x)) \cdot \nabla$$

and using the triangle inequality we have

$$|\text{Tr}(II)| = \left| \text{Tr} \left( \varphi(x) \frac{N\widehat{f_NV_N}(p)}{1 + p^2} \varphi(x), p^2 \right) \frac{1}{1 + p^2}[\varphi(x), p^2] \frac{1}{1 + p^2} N\widehat{f_NV_N}(p) \varphi(x) \right|$$

$$\leq \left| \text{Tr} \left( \varphi(x) \frac{N\widehat{f_NV_N}(p)}{1 + p^2} (\Delta \varphi(x)) \frac{1}{1 + p^2} (\Delta \varphi(x)) \frac{1}{1 + p^2} N\widehat{f_NV_N}(p) \varphi(x) \right) \right|$$

$$+ 4 \left| \text{Tr} \left( \varphi(x) \frac{N\widehat{f_NV_N}(p)}{1 + p^2} (\nabla \varphi(x)) \frac{1}{1 + p^2} (\nabla \varphi(x)) \frac{1}{1 + p^2} N\widehat{f_NV_N}(p) \varphi(x) \right) \right|$$

$$+ 4 \left| \text{Tr} \left( \varphi(x) \frac{N\widehat{f_NV_N}(p)}{1 + p^2} (\nabla \varphi(x)) \frac{1}{1 + p^2} (\nabla \varphi(x)) \frac{1}{1 + p^2} N\widehat{f_NV_N}(p) \varphi(x) \right) \right|.$$ 

Then by Hölder’s and the Kato–Seiler–Simon inequalities,

$$|\text{Tr}(II)| \leq \left\| \varphi \right\|_{L^\infty}^2 \left\| N\widehat{f_NV_N} \right\|_{L^\infty}^2 \left\| (\Delta \varphi(x)) \frac{1}{1 + p^2} \right\|_{L^\infty}^2$$

$$+ 4 \left\| \varphi \right\|_{L^\infty}^2 \left\| N\widehat{f_NV_N} \right\|_{L^\infty}^2 \left\| (\Delta \varphi(x)) \right\|_{L^\infty}^2 \left\| \nabla \varphi(x) \right\|_{L^2} \left\| \frac{1}{1 + p^2} \right\|_{L^\infty}$$

$$+ 4 \left\| \varphi \right\|_{L^\infty}^2 \left\| N\widehat{f_NV_N} \right\|_{L^\infty}^2 \left\| \nabla \varphi(x) \right\|_{L^8} \left\| \frac{1}{1 + p^2} \right\|_{L^\infty} \times$$

$$\left\| \nabla \varphi(x) \right\|_{L^4} \left\| \frac{1}{1 + p^2} \right\|_{L^8} \left\| N\widehat{f_NV_N} \right\|_{L^\infty} \left\| \frac{1}{1 + p^2} \varphi(x) \right\|_{L^4} \leq C \left\| \varphi \right\|_{L^\infty}^2 \left\| N\widehat{f_NV_N} \right\|_{L^\infty}^2 \left\| \Delta \varphi \right\|_{L^2}^2 \left\| \frac{1}{1 + p^2} \right\|_{L^2}^2.$$
Dealing with (III). This term is negative and can be ignored for an upper bound. Nevertheless, we can bound it by Hölder’s and the Kato–Seiler–Simon inequalities,

\[ |\text{Tr} (\Pi)| = |\text{Tr} \left( \varphi(x)Nf_{\hat{N}} \nabla \varphi(p) \left( \frac{1}{1 + p^2} \right) \nabla \varphi(x) \right) | \leq \| \nabla \varphi \|^2_{L_2} \| Nf_{\hat{N}} \|^2_{L_2} \| \nabla \varphi \|^2_{L^4} \left( \frac{1}{1 + p^2} \right) \leq C. \]

In summary, we deduce from (64) that

\[ \text{Tr}((1 - \Delta)^{-1} K^2) = \text{Tr}\left( \varphi(x)Nf_{\hat{N}} \nabla \varphi(p) \left( \frac{1}{1 + p^2} \right) \nabla \varphi(x) \right) \]

\[ = 2N^2 \int_{\mathbb{R}^3} \left( V_N f_N (1 - f_N) * \varphi^2 \right) \varphi^2 + O(1). \] (66)

Inserting (65) and (66) in (60) we obtain the desired estimate (49):

\[ \inf \text{Spec} (\mathcal{H}_{\text{Bog}}) \geq -\frac{1}{2} N^2 \int_{\mathbb{R}^3} \left( V_N f_N (1 - f_N) * \varphi^2 \right) \varphi^2 - C. \]

\[ \square \]

3.4. Conclusion of lower bound.

Proof of Lemma 4. From Lemma 3 and Lemma 8 we have

\[ H_N \geq N \int_{\mathbb{R}^3} \left( |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}} |\varphi_{\text{GP}}|^2 \right) + \frac{N^2}{2} \int_{\mathbb{R}^3} \left( (((2f_N - f_N^2)V_N) * \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 \right) \]

\[ + (\mu - \mu_1) N_+ + \inf \text{Spec}(\mathcal{H}_{\text{Bog}}) - C \]

\[ \geq N \int_{\mathbb{R}^3} \left( |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}} |\varphi_{\text{GP}}|^2 \right) + \frac{N^2}{2} \int_{\mathbb{R}^3} \left( (((2f_N - f_N^2)V_N) * \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 \right) \]

\[ + (\mu - \mu_1) N_+ - \frac{1}{2} N^2 \int_{\mathbb{R}^3} (V_N f_N (1 - f_N) * \varphi^2) \varphi^2 - C \]

\[ = N \int_{\mathbb{R}^3} \left( |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}} |\varphi_{\text{GP}}|^2 \right) + \frac{N^2}{2} \int_{\mathbb{R}^3} \left( ((f_N V_N) * \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 \right) \]

\[ + (\mu - \mu_1) N_+ - C. \]

It remains to show that

\[ \frac{N^2}{2} \int_{\mathbb{R}^3} \left( ((f_N V_N) * \varphi_{\text{GP}}^2) \varphi_{\text{GP}}^2 \right) = N 4\pi a \int_{\mathbb{R}^3} |\varphi_{\text{GP}}|^4 + O(1). \] (67)
In fact, we have

\[
\left| N \int \left( (f_N V_N) \phi_{GP}^2 - 8\pi a \int |\phi_{GP}|^4 \right) \right|
\]

\[
= \left| \int N f_N V_N(k) |\phi_{GP}(k)|^2 dk - \sqrt{V}(0) \int |\phi_{GP}(k)|^2 dk \right|
\]

\[
= \left| \int \left( \sqrt{V}(k/N) - \sqrt{V}(0) \right) |\phi_{GP}(k)|^2 dk \right|
\]

\[
\leq \left\| \nabla_k \sqrt{V} \right\|_{L^\infty} \int |k/N| |\phi_{GP}(k)|^2 dk \leq CN^{-1} \|x|fV\|_{L^1} \|\phi_{GP}\|_{H^{1/2}} \leq CN^{-1}.
\]

Thus (65) holds true. Hence we find that

\[
H_N \geq N \int_{\mathbb{R}^3} \left( |\nabla \phi_{GP}|^2 + V_{ext} |\phi_{GP}|^2 \right) + 4\pi a N \int_{\mathbb{R}^3} |\phi_{GP}|^4
\]

\[
+ (\mu - \mu_1) N^+ - C = N e_{GP} + (\mu - \mu_1) N^+ - C.
\]

Since \( \mu > \mu_1 \) and \( N^+ = \sum_{i=1}^N Q_{x_i} \), the proof of Lemma 4 is complete. \( \square \)

4. Upper bound

In this section prove the missing energy upper bound.

4.1. Construction of the trial state. Let us explain the construction of the trial state. In the Fock space setting, it is known \[5, Appendix A\] that we can reach the energy \( Ne_{GP} + O(1) \) using trial states of the form

\[
W(\sqrt{N} \phi_{GP}) \Gamma W(\sqrt{N} \phi_{GP})^*.
\]

where \( W(g) = e^{a(g) - a^*(g)} \) is the Weyl operator and \( \Gamma' \) is an appropriate quasi-free state. In the following, we will adapt this construction to the \( N \)-particle Hilbert space. We will use the unitary operator \( U_N \) introduced in \[26\] instead of the Weyl operator and modify the quasi-free state slightly. Denote

\[
Q = 1 - |\phi_{GP} \rangle \langle \phi_{GP}|, \quad \mathcal{H}^+_N = Q L^2(\mathbb{R}^3).
\]

As explained in \[26\], any function \( \Psi_N \in L^2(\mathbb{R}^3)^{\otimes N} \) admits a unique decomposition

\[
\Psi_N = \phi_{\otimes N} \xi_0 + \phi_{\otimes N-1} \otimes \xi_1 + \phi_{\otimes N-2} \otimes \otimes \xi_2 + \ldots + \phi_{\otimes N-k} \otimes \otimes \xi_k \]

with \( \xi_k \in H^+_{\otimes k} \) (with the convention that \( \xi_0 \in \mathbb{C} \)). This defines a unitary map \( U_N \) from \( L^2(\mathbb{R}^3)^{\otimes N} \) to \( F^{\leq N}(\mathcal{H}_N^+) \), the truncated Fock space with particle number \( N \leq N^+ \), by

\[
U_N \left( \sum_k \phi_{\otimes N-k} \otimes \otimes \xi_k \right) = \bigoplus_{k=0}^N \xi_k.
\]

Next, let \( k \) be the Hilbert–Schmidt operator on \( L^2(\mathbb{R}^3) \) with kernel

\[
k(x,y) = \phi_{GP}(x) N(1 - f_N(x-y)) \phi_{GP}(y),
\]

with \( f_N \) the scattering solution in \[24\]. Define \( \gamma : \mathcal{H}_N^+ \to \mathcal{H}_N^+ \), \( \alpha : \mathcal{H}_N^+ \to \mathcal{H}_N^+ \equiv \mathcal{H}_N^+ \) by

\[
\gamma = Q k^2 Q, \quad \alpha = \overline{Q} k Q.
\]
Then we have \( \gamma \geq 0, \operatorname{Tr} \gamma \leq \operatorname{Tr} k^2 < \infty, \alpha^* = Qk\bar{\gamma} = \bar{\pi} \) and for all \( g_1, g_2 \in \mathcal{F}_+ \)
\[
\left\langle \left( g_1, 2 \right), \left( g_2, 2 \right) \right\rangle_{\mathcal{F}_+} = \langle g_1, k^2 g_1 \rangle + \langle g_2, (1 + k^2) g_2 \rangle + 2R(\bar{\gamma} k g_1) \geq 0
\]
by the Cauchy–Schwarz inequality. Thus \( (\gamma, \alpha) \) satisfies (69). Hence, there exists a unique (mixed) quasi-free state \( \Gamma \) on the excited Fock space \( \mathcal{F}(\mathcal{F}_+) \) such that \( (\gamma, \alpha) \) are its one-body density matrices, namely
\[
\langle g_1, k^2 g_2 \rangle = \langle a^*(g_2)a(g_1) \rangle_{\Gamma}, \quad \langle \bar{\gamma} k g_2 \rangle = \langle a^*(g_2)a^*(g_1) \rangle_{\Gamma}, \quad \forall g_1, g_2 \in \mathcal{F}_+.
\]

In this section we will prove

**Lemma 9** (Upper bound). Let \( V_{\text{ext}}, V \) as in Theorem 1 (but without the technical condition (13)). Let \( U_N = U_N(\varphi_{\text{GP}}) \) be as in (68) and let \( \Gamma \) be as in (69). Let \( \mathbb{1}_{\leq N} = \mathbb{1}(N \leq N) \) be the truncation on the particle number operator. Then
\[
\Gamma_N := U_N^* \mathbb{1}_{\leq N} \Gamma \mathbb{1}_{\leq N} U_N
\]
is a non-negative operator on \( L^2(\mathbb{R}^3)^{\otimes \leq N} \) with \( |1 - \operatorname{Tr} \Gamma_N| \leq C_s N^{-s} \) for any \( s \geq 1 \), and
\[
\operatorname{Tr}(H_N \Gamma_N) \leq N\varepsilon_{\text{GP}} + C
\]
with the Hamiltonian \( H_N \) in (11). Consequently,
\[
E_N \leq \frac{\operatorname{Tr}(H_N \Gamma_N)}{\operatorname{Tr} \Gamma_N} \leq N\varepsilon_{\text{GP}} + C.
\]

**Remark 10.** In the Fock space setting in [5, Appendix A], the quasi-free state \( \Gamma' \) is constructed using an explicit Bogoliubov transformation of the form
\[
T_0 = \exp \left( \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} k(x, y) (a_k^* a_y^* - a_x a_y) \mathrm{d}x \mathrm{d}y \right).
\]
Its action on creation and annihilation operators is given for any \( g \in L^2(\mathbb{R}^3) \) by
\[
T_0^* a^*(g) T_0 = a^*(\text{ch}(k) g) + a(\text{sh}(k) g),
\]
where
\[
\text{ch}(k) = \sum_{n \geq 0} \frac{(k \bar{k})^n}{(2n)!} \quad \text{and} \quad \text{sh}(k) = \sum_{n \geq 0} \frac{(k \bar{k})^n k}{(2n + 1)!}.
\]
The one-body density matrices of \( \Gamma' \) can be computed in terms of \( \text{ch}(k) \) and \( \text{sh}(k) \). Our construction of the quasi-free state \( \Gamma \) is slightly different as its one-body density matrices are given exactly in terms of \( k \). This makes the energy computation easier.

We divide the proof of Lemma 9 into several steps.

### 4.2. Operator bound on Fock space.
First, we analyse the action of the unitary transformation \( U_N \).

**Lemma 11** (Operator bound on Fock space). We have the operator inequality
\[
\mathbb{1}_{\leq N} U_N H_N U_N^* \mathbb{1}_{\leq N} \leq \mathbb{1}_{\mathcal{F}_+}(\mathcal{G}_N + C(N + 1)^6) \mathbb{1}_{\mathcal{F}_+} \quad \text{on} \quad \mathcal{F}((\mathcal{F}_+),
\]

(70)
where \( \mathcal{G}_N \) is the following operator on the full Fock space \( \mathcal{F}(L^2(\mathbb{R}^3)) \):

\[
\mathcal{G}_N = N \int \left( |\nabla \varphi_{\text{GP}}|^2 + V_{\text{ext}}|\varphi_{\text{GP}}|^2 \right) + \frac{N^2}{2} \int (V_N * \varphi_{\text{GP}}^2) + \sqrt{N} \left( a \left( (-\Delta + V_{\text{ext}} + NV_N * \varphi_{\text{GP}}^2) \varphi_{\text{GP}} \right) + \text{h.c.} \right) + d\Gamma(-\Delta + V_{\text{ext}}) + \frac{N}{2} \int V_N(x - y) \varphi_{\text{GP}}(x) \varphi_{\text{GP}}(y) (a_x^* a_y + \text{h.c.}) \, dx \, dy \\
+ \sqrt{N} \int V_N(x - y) \varphi_{\text{GP}}(x) (a_y^* a_x a_y + \text{h.c.}) \, dx \, dy \\
+ \frac{1 + CN^{-1}}{2} \int V_N(x - y) a_x^* a_y^* a_x a_y \, dx \, dy.
\]

**Proof.** Let us write \( \varphi = \varphi_{\text{GP}} \) for short. After a straightforward computation as in [26, Section 4] (see also [27, Appendix B]) using (29) and the rules

\[
U_N a^*(\varphi) a(g_1) U_N^* = \sqrt{N - N_+} a(g_1), \quad U_N a^*(g_1) a(g_2) U_N^* = a^*(g_1) a(g_2), \quad \forall g_1, g_2 \in \mathfrak{H}_+
\]

we obtain

\[
\mathbb{1} \lesssim N U_N H_N U_N^* \mathbb{1} \lesssim N = \mathbb{1}_{\mathcal{F}_+} \mathbb{1} \lesssim N \mathcal{G}_N \mathbb{1} \lesssim N \mathcal{G}_N \mathbb{1}_{\mathcal{F}_+} \quad \text{on} \quad \mathcal{F}_+,
\]

where \( \mathcal{G}_N \) is the following operator on the truncated Fock space \( \mathbb{1} \lesssim N \mathcal{F}(L^2(\mathbb{R}^3)) \):

\[
\mathcal{G}_N = \left( (N - N_+) \int (|\nabla \varphi|^2 + V_{\text{ext}}|\varphi|^2) + \frac{1}{2} (N - N_+) (N - N_+ - 1) \int (V_N * \varphi^2) \varphi^2 \right) \\
+ \left( \sqrt{N - N_+} a((-\Delta + V_{\text{ext}}) \varphi + \text{h.c.}) \right) \\
+ \left( (N - N_+ - 1) \sqrt{N - N_+} \left( (V_N * \varphi^2) \varphi + \text{h.c.} \right) \right) \\
+ (d\Gamma(-\Delta + V_{\text{ext}}) + (N - N_+) d\Gamma(V_N * \varphi^2 + N^{-1} K)) \\
+ \left( \frac{1}{2} \left\| K(x, y) a_x^* a_y^* \right\| dx \, dy \right) \sqrt{(N - N_+) (N - N_+ - 1)} \, + \text{h.c.} \right) \\
+ \left( \sqrt{N - N_+} \int V_N(x - y) \varphi(x) a_y^* a_{x'} a_{y'} \, dx \, dy \right) + \text{h.c.} \\
+ \frac{1}{2} \int V_N(x - y) a_x^* a_y^* a_x a_y \, dx \, dy
\]

\[
=: (I) + (II) + (III) + (IV) + (V) + (VI) + (VII)
\]

where \( K \) is the operator on \( L^2(\mathbb{R}^3) \) with kernel \( K(x, y) = \varphi(x) NV_N(x - y) \varphi(y) \). Here unlike the presentation in [26, 27], we do not put the projection \( Q \) in the expression of \( \mathcal{G}_N \) because we have introduced the projection \( \mathbb{1}_{\mathcal{F}_+} \) in (71).

Now let us simplify further \( \mathcal{G}_N \), which is a proper operator on \( \mathbb{1} \lesssim N \mathcal{F} \).

**Analysis of (I).** Using \( |N - N_+| \leq N \), we have

\[
(I) \leq N \int \left( |\nabla \varphi|^2 + V_{\text{ext}}|\varphi|^2 \right) + \frac{N^2}{2} \int (V_N * \varphi^2) \varphi^2.
\]
Analysis of (II). Let us replace $\sqrt{N - N_+}$ by $\sqrt{N}$. By the Cauchy–Schwarz inequality we have
\[
\pm \left( \left( \sqrt{N - N_+} - \sqrt{N} \right) a((-\Delta + V_{\text{ext}})\varphi) + \text{h.c.} \right) \\
\leq N \left( \sqrt{N - N_+} - \sqrt{N} \right)^2 + N^{-1} a^*((-\Delta + V_{\text{ext}})\varphi)a((-\Delta + V_{\text{ext}})\varphi) \\
\leq N \left( \frac{N_+}{\sqrt{N - N_+} + \sqrt{N}} \right)^2 + N^{-1} \|(-\Delta + V_{\text{ext}})\varphi\|_{L^2}^2 \leq C(N^2 + 1). \tag{74}
\]
Here we have used $a^*(g)a(g) \leq N\|g\|_{L^2}^2$ and the fact that $(-\Delta + V_{\text{ext}})\varphi \in L^2$.

Analysis of (III). We can replace $(N - N_+)\sqrt{N - N_+}$ by $N\sqrt{N}$ as
\[
\pm \left( \left( (N - N_+) - N\sqrt{N} \right) a ((V_N \ast \varphi^2) \varphi) + \text{h.c.} \right) \\
\leq N^{-1} \left( (N - N_+) - N\sqrt{N} \right)^2 + Na^* ((V_N \ast \varphi^2) \varphi)a ((V_N \ast \varphi^2) \varphi) \\
\leq C(N_+^2 + 1) + N\| (V_N \ast \varphi^2) \varphi\|_{L^2}^2 \leq C(N^2 + 1). \tag{75}
\]
Here we have used
\[
\| (V_N \ast \varphi^2) \varphi\|_{L^2} \leq \|V_N \ast \varphi^2\|_{L^\infty} \|\varphi\|_{L^2} \leq \|V_N\|_{L^1} \|\varphi^2\|_{L^\infty} \|\varphi\|_{L^2} \leq CN^{-1}.
\]

Analysis of (IV). Similarly to (54) we have $\|K\|_{\text{op}} \leq C$. Combining with the uniform bound $\|V_N \ast \varphi^2\|_{L^\infty} \leq CN^{-1}$ used above, we have
\[
\pm (N - N_+)d\Gamma(V_N \ast \varphi^2 + N^{-1}K) \leq CN. \tag{76}
\]

Analysis of (V). We can replace $N^{-1} \sqrt{(N - N_+)(N - N_+ - 1)}$ by 1 as
\[
\pm \left( \int K(x,y)a_x^*a_y^*\left( \sqrt{(N - N_+)(N - N_+ - 1)} \right) - 1 \right) \, dx \, dy + \text{h.c.} \\
\leq \int \left( |K(x,y)|N^2 \left( \frac{\sqrt{(N - N_+)(N - N_+ - 1)}}{N} - 1 \right)^2 + \frac{|K(x,y)|}{N^2} a_x^*a_y^*a_xa_y \right) \, dx \, dy \\
\leq C \int (NV_N(x - y) (\varphi^2(x) + \varphi^2(y)) N_+^2 + N^{-1} \|\varphi\|_{L^\infty}^2 V_N(x - y) a_x^*a_y^*a_xa_y) \, dx \, dy \\
\leq CN_+^2 + CN^{-1} \int V_N(x - y) a_x^*a_y^*a_xa_y dx \, dy. \tag{77}
\]

Analysis of (VI). We can replace $\sqrt{N - N_+}$ by $\sqrt{N}$ as
\[
\pm \left( \left( \sqrt{N - N_+} - \sqrt{N} \right) \int V_N(x - y)(\varphi(x)a_y^*a_xa_y + \text{h.c.}) \, dx \, dy \right) \\
\leq N \|\varphi\|_{L^\infty}^2 \int |V_N(x - y)|a_y^* \left( \sqrt{N - \sqrt{N - N_+}} \right)^2 a_y \, dx \, dy \\
\quad + N^{-1} \int V_N(x - y)a_x^*a_y^*a_xa_y \, dx \, dy \\
\leq CN^2 + N^{-1} \int V_N(x - y) a_x^*a_y^*a_xa_y dx \, dy. \tag{78}
\]
Here we have used \((\sqrt{N} - \sqrt{N-1})^2 \leq N^{-1}N_+ \leq N_+\) and
\[
\int a_y^*N_+a_ydy = \int a_y^*a_y(N_+ + 1)dy = N(N_+ + 1) \leq 2N^2.
\]

**Conclusion.** Inserting (73)–(78) in (72), we deduce from (71) that
\[
\mathbb{1}^{\leq N} U_N H_N U_N^\dagger \mathbb{1}^{\leq N} \leq \mathbb{1}_{F_+} \mathbb{1}^{\leq N} (G_N + C(N + 1)^2) \mathbb{1}^{\leq N} \mathbb{1}_{F_+} \text{ on } F_+.
\] (79)

Now we remove the cut-off \(\mathbb{1}^{\leq N}\) on the right side of (79). For all terms which are positive and commute with \(N\), the cut-off \(\mathbb{1}^{\leq N}\) can be removed for an upper bound. It remains to consider the operator
\[
F := \sqrt{N} \left( a \left( (-\Delta + V_{\text{ext}} + NV_N \ast \varphi^2 ) \varphi \right) + \text{h.c.} \right) + \frac{1}{2} \int \int K(x,y) (a_x^*a_y^* + \text{h.c.}) dx \, dy
\]
\[
+ \sqrt{N} \int V_N(x-y) \varphi(x)(a_y^*a_xa_y + \text{h.c.}) dx \, dy
\]
on \(F\). By the Cauchy–Schwarz inequality we can bound
\[
\pm F \leq N + N \| (-\Delta + V_{\text{ext}} + NV_N \ast \varphi^2 ) \varphi \|^2_{L^2} + \int \int (|K(x,y)|^2 + a_x^*a_xa_y^*a_y) \, dx \, dy
\]
\[
+ \int (N|V_N(x-y)|^2)^2 |\varphi(x)|^2 a_y^*a_y + a_x^*a_xa_y^*a_y \, dx \, dy
\]
\[
\leq C(N^3 + N^2).
\] (80)

Denote
\[
F_1 := F + C_0(N^3 + N^2) \geq 0 \quad \text{and} \quad \mathbb{1}^{> N} = \mathbb{1} - \mathbb{1}^{\leq N}.
\]
By the Cauchy-Schwarz inequality and (80) we can bound
\[
\mathbb{1}^{\leq N} F^{\leq N} - F = -\mathbb{1}^{\leq N} F_1^{> N} - \mathbb{1}^{> N} F_1^{\leq N} - \mathbb{1}^{> N} F_1^{> N}
\]
\[
\leq N^{-3} \left( \mathbb{1}^{\leq N} F_1^{\leq N} \right) + N^3 \left( \mathbb{1}^{> N} F_1^{> N} \right) - \mathbb{1}^{> N} F_1^{> N}
\]
\[
\leq CN^{-3} (N^3 + N^2) \mathbb{1}^{\leq N} + CN^3 (N^3 + N^2) \mathbb{1}^{> N} \leq C(N + 1)^6.
\]
Thus in conclusion, we have
\[
\mathbb{1}^{\leq N} G_N \mathbb{1}^{\leq N} \leq G_N + C(N + 1)^6.
\]
Inserting this in (79) we conclude the proof of Lemma 11.

4.3. **Conclusion of upper bound.**

**Proof of Lemma 11** Now consider the mixed state \(\Gamma_N = U_N^\dagger \mathbb{1}^{\leq N} \Gamma \mathbb{1}^{\leq N} U_N\). Again we will write \(\varphi = \varphi_{\text{GP}}\) for short.

**Trace normalization.** Since
\[
\mathbb{1}^{> N} := \mathbb{1} - \mathbb{1}^{\leq N} \leq N^s N^{-s}, \quad \forall s \geq 1
\] (81)
we have
\[
0 \leq 1 - \text{Tr} \Gamma_N = 1 - \text{Tr} \left( \mathbb{1}^{\leq N} \Gamma \right) = \text{Tr} \left( \mathbb{1}^{> N} \Gamma \right)
\]
\[
\leq N^{-s} \text{Tr} \left( N^s \Gamma \right) \leq N^{-s} C_s(1 + \text{Tr} (N\Gamma))^s \leq C_s N^{-s}, \quad \forall s \geq 1.
\] (82)
Here we have used (31) and \(\text{Tr}(N\Gamma) = \text{Tr} \gamma \leq \text{Tr} k^2 \leq C\).
Energy expectation. Thanks to Lemma 11 we have
\[ \text{Tr}(H_N \Gamma_N) = \text{Tr}\left(1^{\otimes N} U_N H_N U_N^{\dagger} 1^{\otimes N} \Gamma\right) \leq \text{Tr}\left((G_N + C(\mathcal{N} + 1)^6) \Gamma\right). \] (83)

Using (31) again we have \(\text{Tr}((\mathcal{N} + 1)^6 \Gamma) \leq C\). Moreover, since \(\Gamma\) is a quasi-free state on \(\mathcal{F}_+\) with the one-body density matrices \((\gamma, \alpha)\), by Wick Theorem we have
\[
\text{Tr}(G_N \Gamma) = N \int \left(\left|\nabla \varphi\right|^2 + V_{\text{ext}} \varphi^2\right) + \frac{N^2}{2} \int \left(V_N \ast \varphi^2\right) \varphi^2
\]
\[
+ \text{Tr}((-\Delta + V_{\text{ext}})\gamma) + \Re \int \int V_N(x-y) \varphi(x) \varphi(y) \alpha(x,y) dx dy
\]
\[
+ \frac{1 + CN^{-1}}{2} \int \int V_N(x-y) (\gamma(x,x)\gamma(y,y) + |\gamma(x,y)|^2 + |\alpha(x,y)|^2) dx dy.
\] (84)

It remains to evaluate the right side of (84) term by term.

Kinetic energy. Using \(\gamma = Qk^2Q = (1-P)k^2(1-P)\) we can decompose
\[ \text{Tr}(-\Delta \gamma) = \text{Tr}(-\Delta k^2) + 2\Re \text{Tr}(\Delta P k^2) + \text{Tr}(-\Delta P^2 k^2). \]

We have
\[ \text{Tr}(-\Delta P k^2) = \langle \varphi, -\Delta \varphi \rangle \langle \varphi, k^2 \varphi \rangle \leq C, \]
\[ |\text{Tr}(\Delta P k^2)| = |\langle \varphi, k^2 (\Delta \varphi) \rangle| \leq \|\varphi\|_{L^2} \|\Delta \varphi\|_{L^2} \|k^2\|_{\text{op}} \leq C. \]

Now consider the main term \(\text{Tr}(-\Delta k^2)\). Similarly to (61), we write the operator \(k\) as
\[ k = \varphi(x) N \omega_N(p) \varphi(x) \text{ on } L^2(\mathbb{R}^3) \]
where \(\omega_N = 1 - f_N\) and \(\varphi(x), \omega_N(p)\) are multiplication operators on the position and momentum spaces. By the IMS formula (27) we can decompose
\[
\text{Tr}(-\Delta k^2) = N^2 \text{Tr} \left( \varphi(x) p^2 \varphi(x) \omega_N(p) \varphi(x) \right) \omega_N(p) \right)
\]
\[
= \frac{N^2}{2} \text{Tr} \left( \left(\varphi^2(x) p^2 + p^2 \varphi^2(x)\right) \omega_N(p) \varphi^2(x) \omega_N(p)\right)
\]
\[
+ N^2 \text{Tr} \left( |\nabla \varphi(x)|^2 \omega_N(p) \varphi^2(x) \omega_N(p)\right).
\]
The first term can be computed exactly using the scattering equation (26) and (63):
\[
\frac{N^2}{2} \text{Tr} \left( \left(\varphi^2(x) p^2 + p^2 \varphi^2(x)\right) \omega_N(p) \varphi^2(x) \omega_N(p)\right) = \frac{N^2}{2} \int N \left((V_N f_N \omega_N) \ast \varphi^2\right) \varphi^2.
\]
The second term can be bounded using (63)
\[
N^2 \text{Tr} \left( |\nabla \varphi(x)|^2 \omega_N(p) \varphi^2(x) \omega_N(p)\right) = N^2 \int \left(\omega_N^2 \ast \varphi^2\right) |\nabla \varphi|^2 \leq C \|\nabla \varphi\|_{L^2}^4.
\]
Here we have used
\[
N^2 \|\omega_N^2 \ast \varphi^2\|_{L^\infty} \leq C \|x|^{-2} \ast \varphi^2\|_{L^\infty} \leq C \|\nabla \varphi\|_{L^2}^2
\] (85)
by (29) and Hardy’s inequality \(|x|^{-2} \leq 4(-\Delta)\). Thus
\[
\text{Tr}(-\Delta \gamma) = \frac{N^2}{2} \int N \left((V_N f_N (1 - f_N)) \ast \varphi^2\right) \varphi^2 + O(1). \] (86)
**External potential energy.** Using (85) again, we have

\[ |(k^2)(x, y)| = \left| \int dz k(x, z) k(z, y) \right| = N^2 \int dz \varphi(z) \omega_N(x - z) \varphi^2(z) \omega_N(z - y) \varphi(y) \]

\[ \leq \varphi(x) \varphi(y) \frac{N^2}{2} \int dz \left( \omega_N^2(x - z) + \omega_N^2(z - y) \right) \varphi^2(z) \leq C \varphi(x) \varphi(y). \]

Hence,

\[ |\gamma(x, y)| = \left| \left[ (1 - P)k^2(1 - P) \right] (x, y) \right| \]

\[ = \left| (k^2)(x, y) - \varphi(x) \int dz \varphi(z) (k^2)(z, y) - \varphi(x) \int dz \varphi(z) (k^2)(x, z) \right. \]

\[ + \varphi(x) \varphi(y) \int dx dy \varphi(x) (k^2)(x, y) \right| \]

\[ \leq C \varphi(x) \varphi(y). \]  

(87)

Consequently,

\[ \text{Tr}(V_{\text{ext}} \gamma) = \int V_{\text{ext}}(x) \gamma(x, x) dx \leq C \int V_{\text{ext}} \varphi^2 \leq C. \]  

(88)

**Bogoliubov pairing energy.** Since \( \alpha(x, y) = (Q \otimes Qk)(x, y) \), by decomposing \( Q = 1 - P \) we have

\[ |\alpha(x, y) - k(x, y)| = \left| -\varphi(x) \left( \varphi^2 \ast N \omega_N \right)(y) \varphi(y) - \varphi(x) \left( \varphi^2 \ast N \omega_N \right)(x) \varphi(y) \right. \]

\[ + \varphi(x) \varphi(y) \int \left( N \omega_N \ast \varphi^2 \right) \varphi^2 \right| \]

\[ \leq C \varphi(x) \varphi(y). \]  

(89)

Here we have used

\[ N \left\| \omega_N \ast \varphi^2 \right\|_{L^\infty} \leq C \left\| x^{-1} \ast \varphi^2 \right\|_{L^\infty} \leq C \]

which is similar to (85). Thus

\[ \Re N \int V_N(x - y) \varphi(x) \varphi(y) \alpha(x, y) dx dy \]

\[ \leq \Re N \int V_N(x - y) \varphi(x) \varphi(y) \left( k(x, y) + C \varphi(x) \varphi(y) \right) dx dy \]

\[ = -N^2 \int (V_N \omega_N)(x - y) \varphi^2(x) \varphi^2(y) dx dy + CN \int V_N(x - y) \varphi^2(x) \varphi^2(y) dx dy \]

\[ = -N^2 \int \left( (V_N - f_N) \ast \varphi^2 \right) \varphi^2 + O(1). \]  

(90)

In the last estimate we have used

\[ N \int dx dy V_N(x - y) \varphi^2(x) \varphi^2(y) \leq C \left\| \varphi \right\|_{L^\infty}^2 \left\| \varphi \right\|_{L^2}^2 N \left\| V_N \right\|_{L^1} \leq C. \]  

(91)

**Interaction energy.** Using (77) and (91) we have

\[ \int dx dy V_N(x - y) \left( \gamma(x, x) \gamma(y, y) + |\gamma(x, y)|^2 \right) \leq CN^{-1}. \]  

(92)

Moreover, by (85) and the Cauchy–Schwarz inequality,

\[ |\alpha(x, y)|^2 \leq (1 + N^{-1})|k(x, y)|^2 + CN|\varphi(x)|^2|\varphi(y)|^2. \]
Therefore,
\[ \int V_N(x, y)|\alpha(x, y)|^2dxdy \leq N^2 \int ((V_N(1 - f_N))^2) \varphi^2 + C. \] (93)

**Conclusion.** Inserting (86), (88), (90), (92), (93) in (83)-(84) we find that
\[ \text{Tr}(H_N \Gamma_N) \leq N \int (|\nabla \varphi|^2 + V_{\text{ext}} \varphi^2) + N^2 \int (V_N * \varphi^2) \varphi^2 
\quad + \frac{N^2}{2} \int ((V_N f_N(1 - f_N))^2) \varphi^2 - N^2 \int ((V_N(1 - f_N))^2) \varphi^2 
\quad + \frac{N^2}{2} \int ((V_N(1 - f_N))^2) \varphi^2 + C 
\quad = N \int (|\nabla \varphi|^2 + V_{\text{ext}} \varphi^2) + N^2 \int ((V_N f_N)^2) \varphi^2 + C. \]

Finally, by Young’s inequality we have
\[ \frac{N^2}{2} \int (V_N f_N * \varphi^2) \varphi^2 \leq \frac{N^2}{2} \|(V_N f_N) * \varphi^2\|_{L^2} \|\varphi^2\|_{L^2} \leq \frac{N^2}{2} \|V_N f_N\|_{L^1} \|\varphi^2\|^2_{L^2} = 4\pi aN \int \varphi^4. \]

Thus
\[ \text{Tr}(H_N \Gamma_N) \leq N \int (|\nabla \varphi|^2 + V_{\text{ext}} \varphi^2) + 4\pi aN \int \varphi^4 + C = N\epsilon_{\text{GP}} + C. \]

Finally, by the variational principle
\[ E_N = \inf \text{Spec}(H_N) \leq \frac{\text{Tr}(H_N \Gamma_N)}{\text{Tr}\Gamma_N} \leq \frac{N\epsilon_{\text{GP}} + C}{1 + CN^{-1}} \leq N\epsilon_{\text{GP}} + O(1). \]

This ends the proof of Lemma 9. \[\square\]

The proof of Theorem 1 is complete.

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