Off-shell gluon amplitudes in QCD

Naser Ahmadiniaz

Center for Relativistic Laser Science,
Institute for Basic Science (IBS), Gwangju 61005, Korea and
Department of Physics, Kunsan National University, Kunsan 54150, Korea

E-mail: ahmadiniaz@ibs.re.kr

To the memory of a good friend and colleague Victor Villanueva.

Abstract. In this contribution we discuss off-shell gluon amplitudes from worldline formalism perspective. We use the Bern-Kosower master formula and two IBP methods to extract the form factor of the off-shell three- and four-gluon vertices. For the four-gluon case we also discuss its representation in $\mathcal{N}=4$ SYM.

1. Introduction

In theoretical physics quantum chromodynamics (QCD) is a theory of strong interaction, which is a non-abelian theory that describes interaction between QCD ingredients as quarks and gluons. It has the $SU(3)$ symmetry group which is free in the ultraviolet (UV) and confining in the infrared (IR) regions. Comparing the IR behavior of QCD with quantum electrodynamics (QED), the former has two complications: first its field quanta (gluons) carry charges hence are self-coupled; and second, they couple to massless particles. In this paper we discuss the off-shell vertex functions in QCD, there are several reasons to study these objects.

During the last decade the IR behavior of the Yang-Mills Green functions has been studied extensively. The basic object of interest in QCD is the one-particle irreducible (‘1PI’) off-shell $N$-gluon Green’s function. It contains important information on the IR regime of QCD (see, e.g., [1]) and also important ingredients for the matching perturbative information with lattice data (see, e.g., [2]). They are essential ingredients for the Schwinger-Dyson equations (SDEs) and for the full exploitation of the renormalization group. So, for these off-shell vertices one needs new methods because the ordinary one based on the analysis of the Ward identity is too complicated especially beyond the three-gluon vertex. In the context of SDEs, because of the lack of the four-gluon vertex structure (form factors) only two and three-point amplitudes were used with their full loop correction structure (explicit computation for the $N = 4$ is absent and it is stuck at the $N = 2$ and $N = 3$ level), so a need for the inclusion of the four-gluon vertex is already felt [3]. On the other hand, there is the calculation of on-shell matrix elements which have seen substantial progress during the last decade, particularly for massless and/or SUSY cases. Some new techniques have emerged, such as unitarity-based methods [4, 5], twistors [6], BCFW recursion [7, 8], and Grassmannians [9, 10] (recent reviews of these ideas can be found in [11] and [12]). So, it is obvious that the off-shell $N$-gluon amplitudes (see Fig. 1) are essential...
for studying the low energy regime of QCD. In the rest of this section we make a short review on what is known about the $N$-gluon vertices.

In 1980 Ball and Chiu [13] studied the off-shell gluon amplitudes for the gluon loop in Feynman gauge. They derived a form factor (“Ball-Chiu”) decomposition of the three-gluon vertex by analyzing the Ward identities, their form factor is valid to all loop orders. Later in 1989, Cornwall and Papavassiliou used the pinch techniques to construct a “gauge invariant three-gluon vertex” in [14] (see [15] for a review of this technique). In 1992 Freedman et al. studied the conformal properties of this vertex [16]. Papavassiliou in 1993 studied the four-gluon vertex, he obtained the Ward identity for this object [17]. The gluon loop contribution to the one-loop three-gluon vertex in arbitrary covariant gauge was calculated by Davydychev, Osland and Tarasov [18] in 1996. Later, Davydychev, Osland and Saks [19] calculated the generalized massive case for the fermion loop. The extension of the one-loop three-gluon vertex to various dimension was studied by Binger and Brodsky in 2006 [20] using the background field method [21, 22]. Besides the gluon and fermion loop cases, they also include the scalar loop, as indeed for SUSY extension of QCD, and they derived various sum rules relevant to the SUSY case. They also verified the equivalence of the gluon loop case obtained by the background field method (with quantum Feynman gauge) with the pinch technique which was suggested in [23, 24]. Two loop correction to the three-gluon vertex has been obtained for some very special momentum configurations, see [25, 26, 27].

![Figure 1. $N$-gluon amplitudes with an arbitrary number of gluons.](image)

In this contribution we extend the string-inspired approach to perturbative QCD, originally developed in the on-shell context by Bern and Kosower [28, 29, 30] to the off-shell case, explicitly for $N = 3$ and $N = 4$, to show that it is extremely promising as a tool for the derivation of form factor decomposition of the $N$-gluon amplitudes, see also [31].

2. Ball-Chiu form factor decomposition for the three-gluon vertex

From QCD Lagrangian $L_g = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$ there are three- and four-gluon self interactions.

Since gluons carry color charges, their self-interactions are the main differences between QCD and QED. In fact these self-interactions are the source of many unique features of QCD, such as asymptotic freedom, chiral symmetry breaking and color confinement. The three-gluon coupling vertex can be written as

$$-igf^{a_1a_2a_3}[g_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + \text{cycl}],$$

where $k_i$'s are the momenta of the gluons and $f^{a_1a_2a_3}$ is the gauge group structure constant and “cycl” stands for cyclic permutation of the three gluons. The one-loop correction 1PI to
the three-gluon vertex, (see Fig. 2 for the scalar loop particle) had been obtained by Ball and Chiu more than three decades ago by analyzing the Ward identity [13]. This object can be decomposed in the following form

$$\Gamma_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = A(k_1^2, k_2^2, k_3^2)g_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + B(k_1^2, k_2^2, k_3^2)g_{\mu_1\mu_2}(k_1 + k_2)_{\mu_3} + C(k_1^2, k_2^2, k_3^2)[k_{\mu_1}k_{\mu_2} - k_1 \cdot k_2]_{\mu_3}(k_1 - k_2)_{\mu_3} + \frac{1}{3}S(k_1^2, k_2^2, k_3^2)(k_{\mu_1}k_{\mu_2}k_{\mu_3} + k_{1\mu_2}k_{2\mu_3}k_{3\mu_1}) + F(k_1^2, k_2^2, k_3^2)[k_{\mu_1\mu_2}(k_1 \cdot k_3)_{\mu_3}k_2 - k_1 \cdot k_2g_{\mu_1\mu_2}][k_{\mu_3}k_1 \cdot k_3 - k_{1\mu_3}k_2 \cdot k_3] + H(k_1^2, k_2^2, k_3^2)(-g_{\mu_1\mu_2}[k_{1\mu_3}k_2 \cdot k_3 - k_{2\mu_3}k_1 \cdot k_3] + \frac{1}{3}(k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} - k_{1\mu_2}k_{2\mu_3}k_{3\mu_1})) + \text{cyclic permutations of } (k_1, \mu_1), (k_2, \mu_2), (k_3, \mu_3).$$  

(2)

The form factors $A, B, C, F, H, S$ have the following properties:

- $A, C$ and $F$ are symmetric, $B$ is antisymmetric, $H$ is totally symmetric and $S$ is totally antisymmetric function with respect to interchange of any pair of arguments.
- $F$ and $H$ are manifestly transversal.
- $S$ which is totally antisymmetric remains zero for any order of one-loop correction. $S$ turns out to be zero at one-loop, and there are arguments that it vanishes even at any order in perturbation theory [38].

This is a universal form factor decomposition and it is valid for any kind of particle in the loop (scalar, spinor or gluon) and for higher loop corrections the only change will be the form of the coefficient functions. At tree level, from Eq. (1), the only non-vanishing function is $A = 1$.

3. The string-inspired formalism

The starting point for our calculation is the following master formula which was obtained by Bern and Kosower in their analysis for the infinite string limit of string amplitude, see [28, 29, 30]

$$\Gamma^{a_1 \ldots a_N}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] = (-ig)^N\text{tr}(T^{a_1} \ldots T^{a_N}) \int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T} \times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{N-2}} d\tau_{N-1} \times \exp\left\{ \sum_{i,j=1}^N \left[ \frac{1}{2}G_{Bij}k_i \cdot k_j - i\tilde{G}_{Bij}\varepsilon_i \cdot k_j + \frac{1}{2}\tilde{G}_{Bij}\varepsilon_i \cdot \varepsilon_j \right]\right\}_{\text{lin}(\varepsilon_1 \ldots \varepsilon_N)}.$$  

(3)
For the scalar loop particle, this master formula is a $D$-dimensional parameter integral representation of the color-ordered $1PI$ $N$-gluon amplitude. Here $T$ is the total proper time of the loop particle, $m$ its mass, $\tau_i$ fixes the location of the $i$th gluon and $\varepsilon_i$ is its polarization vector. $T^a$ is the generator of the gauge group, $\int Dx$ is an integral over closed trajectories in Minkovski space-time with periodicity $T$. Translational invariance in proper-time has been used to set $\tau_N = 0$. The “bosonic” worldline Green’s function is denoted by $G_{Bij} \equiv G_B(\tau_i, \tau_j)$ defined by

$$G_{Bij} = |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T},$$  \hspace{1cm} (4)$$

we also need its first and second derivatives which are given by

$$\dot{G}_{Bij} = \text{sign}(\tau_i - \tau_j) - \frac{2(\tau_i - \tau_j)}{T},$$
$$\ddot{G}_{Bij} = 2\delta(\tau_i - \tau_j) - \frac{2}{T}. $$ \hspace{1cm} (5)$$

The master formula in the Bern-Kosower formalism is a generating functional for the full on-shell gluon amplitude in any $N$-order for the scalar, spinor and gluon loop. After obtaining the scalar loop correction to the problem in hand one can follow the replacement rules introduced by Bern and Kosower [28, 29, 30] to find the spinor- and gluon-loop contribution to the amplitude. These replacement rules can be summarized as follows:

First step: for fixed $N$, expand the generating exponential and take only the terms linear in all polarization vectors which leads to a polynomial ($P_N$) in terms of the Green’s function and its first and second derivatives. Then, by using some suitable integration-by-parts (IBPs) we remove all second derivatives $\ddot{G}_{Bij}$. These sets of IBPs transform $P_N$ to a new polynomial ($Q_N$) which is written in terms of $\dot{G}_{Bij}$’s only.

Final step: apply two following types of pattern-matching rules:

- Although the master formula generates only the $1PI$ contribution to the scattering matrix, one can obtain the missing reducible part (from field theory point of view) by means of the “tree replacement rules”.
- The spinor- and gluon-loop contributions are obtained by applying the “loop replacement rules” to the one from the scalar-loop.

Bern, Dixon and Kosower applied this string-inspired method to recalculate the one-loop correction to five gluon amplitude (on-shell) for the first time in [32].

4. The worldline path integral approach
In 1992, Strassler [33] used worldline representation of the gluinoic effective action and rederived the Bern-Kosower master formula and replacement rules. This path integral for the scalar loop is written as

$$\Gamma[A] = \text{tr} \int_0^\infty dT \frac{e^{-m^2T}}{T} \int Dx(\tau) P e^{-\int_0^T d\tau \left[ \frac{1}{2} \dot{x}^2 + ig \dot{x} \cdot A(x(\tau)) \right]},$$ \hspace{1cm} (6)$$

where $A_{\mu} = A_{\mu}^a T^a$ and $P$ denotes path ordering. The Bern-Kosower version of this master formula was derived for on-shell case to calculate the scattering amplitude but Strassler’s version as well as the replacement rules hold off-shell. In [34] Strassler used some IBPs to remove all
\[ f^{\mu\nu}_i \equiv k^i_\mu \varepsilon^\nu_i - \varepsilon^\mu_i k^\nu_i, \]  
(7)

in the bulk, and color commutators \([T^{a_i}, T^{a_j}]\) as boundary terms. Putting them together, they produce full nonabelian field strength tensors

\[ F^{\mu\nu}_a = F^a_{\mu\nu}(\partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu}) + ig[A^b_{\mu} T^b, A^c_{\nu} T^c], \]
(8)

in the low-energy effective action. Thus by means of the IBP, interestingly gauge invariant tensor structures emerge but at the integrand level. The IBP contains removing all \( \hat{G}_{Bij} \)'s by adding some total derivatives but there is not a unique way to do that, obviously it has to be done in a way to preserve the Bose symmetry between the gluons. This ambiguity begins at the four-point level (for the two- and three-point cases there is no ambiguity), Strassler started to investigate this ambiguity in [34], but did not present a general algorithm that would preserve the full permutation symmetry. Such an algorithm was found later by Schubert [35, 36]. It leads to a representation of the one-loop \( N \)-gluon amplitudes that is manifestly covariant in the sense explained above, but is still not optimized from the point of view of the SDEs, since it does not lead to a clean separation of the vertices into transversal and longitudinal parts at the integrand level. As one of the main results of our collaboration with Victor, in [37] we solved this problem by introducing a more general class of total derivative terms that allow one to absorb all non-transversality into the boundary terms of the IBP procedure. The price to pay is that the arising integrands now in general involve spurious non-localities. Thus in [37] two algorithms emerged that both can be applied for an arbitrary number of gluons, but they are, in some sense, complementary:

- To obtain a term-by-term match with the low energy effective action (which is essential to get the correct structures) we use only local total derivatives to remove the \( \hat{G}_{Bij} \)'s. This leads to what we call the “Q-representation”.

- To get the transversality of the bulk terms and Ball-Chiu form factors both local and nonlocal total derivatives are needed. This leads to what we call the “S-representation”.

In [38], we applied both algorithms to the three-point case and showed that, in particular, the second algorithm generates the Ball-Chiu tensor decomposition, for all possible loop particles. The explicit form factors of the four-gluon vertex is not known yet because of its rich tensorial structure, then using conventional methods and analyzing the Ward identity to find them is a cumbersome task. Very recently [39, 40], we carried out the same program which is the generalization of the Ball-Chiu decomposition for the four-gluon vertex. In this contribution we briefly present our calculations for \( N = 3 \) and for \( N = 4 \), the latter has a very compact representation in \( N = 4 \) SYM theory which will be discussed shortly. The final form factors of the four-gluon vertex will appear soon [39].

5. The Q-representation of the three-gluon vertex

We consider the two- and three-point levels to find their Q-representation. For example for the two point case we have the following \( P \)-polynomial

\[ P_2 = \hat{G}_{B12} \varepsilon_1 \cdot k_2 \hat{G}_{B21} \varepsilon_2 \cdot k_1 - \hat{G}_{B12} \varepsilon_1 \cdot \varepsilon_2, \]
(9)

by adding a total derivative as

\[ -\partial_2 \left( \hat{G}_{B12} \varepsilon_1 \cdot \varepsilon_2 e^{(i)} \right), \]
\( P_2 \) can be transformed to \( Q_2 \)
\[
P_2 \rightarrow Q_2 = \hat{G}_{B12}\hat{G}_{B21}(\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_1 - \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot k_2) = \hat{G}_{B12}\hat{G}_{B21}\text{tr}(f_1 f_2). \tag{10}
\]

For the three point case in a similar way by adding
\[
-\partial_2\left(\hat{G}_{B12}\varepsilon_1 \cdot \varepsilon_2\hat{G}_{B3i}\varepsilon_3 \cdot \varepsilon_1 e^{(i)}\right),
\]
(and other two similar total derivatives respect to variables 2 and 3) to
\[
P_3 = \hat{G}_{B11}\varepsilon_1 \cdot k_1\hat{G}_{B2j}\varepsilon_2 \cdot k_j\hat{G}_{B3k}\varepsilon_3 \cdot k_k - \left[\hat{G}_{B12}\varepsilon_1 \cdot \varepsilon_2\hat{G}_{B3i}\varepsilon_3 \cdot k_i + 2 \text{ perm}\right],
\tag{11}
\]
we obtain a new polynomial \( Q_3 \) which is
\[
P_3 \rightarrow Q_3 = Q_3^3 + Q_3^2,
\tag{12}
\]
with
\[
Q_3^3 = \hat{G}_{B12}\hat{G}_{B23}\hat{G}_{B31}\text{tr}(f_1 f_2 f_3),
Q_3^2 = \hat{G}_{B12}\hat{G}_{B21}\text{tr}(f_1 f_2)\hat{G}_{B3i}\varepsilon_3 \cdot k_i + 2 \text{ perm} \quad i = 1, 2, 3,
\tag{13}
\]
where the upper indices refer to the “cycle content”; e.g. \( Q_3^2 \) contains a factor \( \hat{G}_{B12}\hat{G}_{B23}\hat{G}_{B31} \) whose indices form a closed cycle involving three points, called “three-cycle”. Note that \( G_{Bii} = 0 \).

Finally the \( Q \)-representation of the three-gluon vertex can be represented as \([38]\)
\[
\Gamma = \frac{g^3}{(4\pi)^2} \text{tr}(T^{a_1}[T^{a_2}, T^{a_3}]) (\Gamma^3 + \Gamma^2 + \Gamma^{bt}),
\tag{14}
\]
where
\[
\Gamma^3 = -\int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \int_0^T d\tau_1 \int_0^\tau_1 d\tau_2 Q_3^3 \exp \left\{ \sum_{i,j=1}^3 \frac{1}{2} G_{Bij} k_i \cdot k_j \right\},
\]
\[
\Gamma^2 = \Gamma^3(Q_3^2 \rightarrow Q_3^3),
\]
\[
\Gamma^{bt} = \int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \int_0^T d\tau_1 \hat{G}_{B12}\hat{G}_{B21} \left[ \varepsilon_3 \cdot f_1 \cdot \varepsilon_2 e^{G_{B1i}k_1 \cdot (k_2+k_3)} + \text{cycl} \right].
\tag{15}
\]

The \( \Gamma^{bt} \) contains the boundary term contributions. We performed the calculation for the scalar loop, to obtain the spinor loop case we apply the “Bern-Kosower replacement rules” by the following “loop replacement rules”
\[
\hat{G}_{B_{1i1}}\hat{G}_{B_{2i2}}\cdots\hat{G}_{B_{ni1}} \rightarrow \hat{G}_{B_{1i1}}\hat{G}_{B_{2i2}}\cdots\hat{G}_{B_{ni1}} - G_{F_{i1}i}G_{F_{i2}i}G_{F_{i3}i} \cdots G_{F_{ni1}}, \tag{16}
\]
where \( G_{F_{ij}} = \text{sign}(\tau_i - \tau_j) \) is the “fermionic” worldline Green’s function. These rules for the three-gluon vertex are simply
\[
\hat{G}_{B_{ij}}\hat{G}_{B_{ji}} \rightarrow \hat{G}_{B_{ij}}\hat{G}_{B_{ji}} - G_{F_{ij}}G_{F_{ji}},
\hat{G}_{B_{12}}\hat{G}_{B_{23}}\hat{G}_{B_{31}} \rightarrow \hat{G}_{B_{12}}\hat{G}_{B_{23}}\hat{G}_{B_{31}} - G_{F12}G_{F23}G_{F31}. \tag{17}
\]
For the gluon loop there are similar replacement rules but with different coefficient for the fermionic Green’s function which depends on the number of external legs, for three external gluons they read

$$\dot{G}_{Bij}\dot{G}_{Bji} \rightarrow \dot{G}_{Bij}\dot{G}_{Bji} - 4G_{Fij}G_{Fji},$$
$$\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} \rightarrow \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} - 4G_{F12}G_{F23}G_{F31}. $$

(18)

Note that, the gluon vertex obtained in this way corresponds to the background field method with quantum Feynman gauge [33, 41].

5.1. Comparison with the effective action

The general form of the low-energy expansion of the one-loop QCD effective action induced by a loop particle of mass $m$ (see [42, 43])

$$\Gamma[F] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \int dx_0 \sum_{n=2}^{\infty} \frac{(-T)^n}{n!} O_n[F],$$

(19)

where $O_n[F]$ is a Lorentz and gauge invariant expression of mass dimension $2n$. To the lowest orders and for the scalar loop case one needs

$$O_2 = c_2 g^2 F_{\mu\nu} F_{\mu\nu},$$
$$O_3 = c_3^3 i g^3 F_{\kappa\lambda} F_{\mu\nu} F_{\mu\kappa\lambda} + c_3^2 g^2 D_{\lambda} F_{\mu\nu} D^{\lambda} F_{\mu\nu},$$

(20)

where only the coefficients $c_2, c_3^3$ depend on the spin of the loop particle, for the scalar case they read

$$c_2 = -\frac{1}{6}, \quad c_3^3 = -\frac{2}{15}, \quad c_3^2 = -\frac{1}{20}. $$

(21)

Now, we can compare our final results with the low-energy effective action which leads to the following correspondences

$$\Gamma^3 \leftrightarrow F^\kappa_\lambda F^\mu_\lambda F^\kappa_\mu = f^\kappa_\lambda f^\mu_\lambda f^\kappa_\mu + \text{higher point terms},$$
$$\Gamma^2 \leftrightarrow (\partial + igA) F(\partial + igA) F,$$
$$\Gamma^{bt} \leftrightarrow (f + ig[A,A]) (f + ig[A,A]).$$

(22)

6. The S-representation of the three-gluon vertex

For S-representation, consider the first term of $Q_3^2$, choose a momentum vector $r_3$ such that $r_3 \cdot k_3 \neq 0$ and add the following non-local total derivative

$$-\frac{r_3 \cdot \varepsilon_3}{r_3 \cdot k_3} \text{tr}(f_1 f_2) \partial_3 \left(\dot{G}_{B12}\dot{G}_{B21} e^{(3)}\right).$$

(23)

In addition to this term we need two similar total derivatives for the second and third terms of $Q_3^2$ to transform $Q_3$ to a new form as

$$Q_3 \rightarrow S_3 = S_3^3 + S_3^2,$$

(24)
with
\[ S_3^3 = Q_3^3 = \frac{1}{2} \mathcal{G}_{B12} \mathcal{G}_{B23} \mathcal{G}_{B31} \text{tr} (f_1 f_2 f_3), \]
\[ S_3^2 = \frac{1}{2} \mathcal{G}_{B12} \mathcal{G}_{B21} \text{tr} (f_1 f_2) \mathcal{G}_{B3k} \frac{r_3 \cdot f_3 \cdot k_k}{r_3 \cdot k_3} + 2 \text{ perm}. \]

(25)

There is a difference between \( Q_3^2 \) and \( S_3^2 \), in the latter all polarization vectors \( \varepsilon_i \) have been absorbed into tensors \( f_i \). In the S-representation scheme we make all the bulk terms manifestly transversal at the integrand level. Finally the S-representation of the three-gluon vertex is written as
\[ \tilde{\Gamma} = \frac{g^3}{(4\pi)^2} \text{tr}(T^{a_1} [T^{a_2}, T^{a_3}]) \tilde{\Gamma}^3 + \tilde{\Gamma}^2 + \tilde{\Gamma}^{bt}, \]

(26)

with
\[ \tilde{\Gamma}^3 = -\int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 S_3^3 \exp \left\{ \sum_{i,j=1}^3 \frac{1}{2} \mathcal{G}_{Bij} k_i \cdot k_j \right\}, \]
\[ \tilde{\Gamma}^2 = \tilde{\Gamma}^3 (S_3^3 \rightarrow S_3^2), \]
\[ \tilde{\Gamma}^{bt} = \int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \int_0^T d\tau_1 \mathcal{G}_{B12} \mathcal{G}_{B21} \left\{ [\varepsilon_3 \cdot f_1 \cdot \varepsilon_2 - \frac{1}{2} \text{tr}(f_1 f_2) \rho_3 + \frac{1}{2} \text{tr}(f_3 f_1) \rho_2] \right. \]
\[ \times \mathcal{E}^{G_{B2k1}(k_2+k_3)} + \text{cycl.,} \right\}, \]

(27)

where \( \rho_i := \frac{r_i \cdot \varepsilon_i}{r_i \cdot k_i} \). By the following cyclic choice
\[ r_1 = k_2 - k_3, \quad r_2 = k_3 - k_1, \quad r_3 = k_1 - k_2, \]
we get a term-by-term match with the Ball-Chiu decomposition:
\[
\begin{align*}
H(k_1^2, k_2^2, k_3^2) &= C(r) \frac{d_0 g^2}{(4\pi)^{D/2}} \Gamma \left( 3 - \frac{D}{2} \right) I^{D}_{3,B}(k_1^2, k_2^2, k_3^2), \\
A(k_1^2, k_2^2, k_3^2) &= C(r) \frac{d_0 g^2}{2 (4\pi)^{D/2}} \Gamma \left( 2 - \frac{D}{2} \right) \left[ I^{D}_{3,B}(k_1^2) + I^{D}_{3,B}(k_2^2) \right], \\
B(k_1^2, k_2^2, k_3^2) &= C(r) \frac{d_0 g^2}{2 (4\pi)^{D/2}} \Gamma \left( 2 - \frac{D}{2} \right) \left[ I^{D}_{3,B}(k_1^2) - I^{D}_{3,B}(k_2^2) \right], \\
F(k_1^2, k_2^2, k_3^2) &= C(r) \frac{d_0 g^2}{(4\pi)^{D/2}} \Gamma \left( 3 - \frac{D}{2} \right) \frac{I^{D}_{2,B}(k_1^2, k_2^2, k_3^2) - I^{D}_{2,B}(k_2^2, k_1^2, k_3^2)}{k_1^2 - k_2^2}, \\
C(k_1^2, k_2^2, k_3^2) &= C(r) \frac{d_0 g^2}{(4\pi)^{D/2}} \Gamma \left( 2 - \frac{D}{2} \right) \frac{I^{D}_{3,B}(k_2^2) - I^{D}_{3,B}(k_1^2)}{k_1^2 - k_2^2}, \\
S(k_1^2, k_2^2, k_3^2) &= 0,
\end{align*}
\]

(28)

where \( d_0 = 1 \) for the scalar loop and we have used \( \text{tr}(T^{a_1} [T^{a_2}, T^{a_3}]) = i C(r) \ f^{a_1 a_2 a_3} \) for \( SU(N) \) one has \( C(N) = 1 \) for the fundamental and \( C(G) = N \) for the adjoint representation.
The form factors in (28) are for the scalar loop, but due to the loop replacement rules (17) and (18), the spinor and gluon loop cases can be obtained and they are different in the coefficient functions on the right hand side. We find the following Feynman-Schwinger parameter integral representations for the scalar case:

\[
I^{D}_{3,B}(k_1^2, k_2^2, k_3^2) = \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(1 - \alpha_1 - \alpha_2 - \alpha_3)(1 - 2\alpha_1)(1 - 2\alpha_2)(1 - 2\alpha_3)}{(m^2 + \alpha_1 \alpha_2 k_1^2 + \alpha_2 \alpha_3 k_2^2 + \alpha_1 \alpha_3 k_3^2)^{3 - \frac{D}{2}}},
\]

\[
I^{D}_{2,B}(k_1^2, k_2^2, k_3^2) = \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(1 - \alpha_1 - \alpha_2 - \alpha_3)(1 - 2\alpha_2)^2(1 - 2\alpha_1)}{(m^2 + \alpha_1 \alpha_2 k_1^2 + \alpha_2 \alpha_3 k_2^2 + \alpha_1 \alpha_3 k_3^2)^{3 - \frac{D}{2}}},
\]

\[
I^{D}_{M,B}(p^2) = \int_0^1 d\alpha \frac{(1 - 2\alpha)^2}{(m^2 + \alpha(1 - \alpha)p^2)^{3 - \frac{D}{2}}}.\]

(29)

7. The four-gluon vertex

\(P\)-polynomial for the four-gluon vertex is more complicated than the two- and three-point cases. After expanding the Bern-Kosower master formula and keeping the terms linear in each polarization vector one gets

\[
\Gamma^{a_1a_2a_3a_4} = g^4 \text{tr}(T^{a_1} \ldots T^{a_4}) \int_0^\infty dT (4\pi T)^{-D/2} e^{-m^2 T} \times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 P_4 \exp \left\{ \sum_{i,j=1}^{4} \frac{1}{2} G_{Bij} k_i \cdot k_j \right\},
\]

(30)

with the following polynomial

\[
P_4 = \begin{array}{c}
\hat{G}_{B12}\varepsilon_1 \cdot \varepsilon_2 \hat{G}_{B34}\varepsilon_3 \cdot \varepsilon_4 + \hat{G}_{B13}\varepsilon_1 \cdot \varepsilon_3 \hat{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 + \hat{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \hat{G}_{B23}\varepsilon_2 \cdot \varepsilon_3 \\
- \hat{G}_{B11}\varepsilon_1 \cdot k_i \hat{G}_{B2j} k_j \cdot \hat{G}_{B33}\varepsilon_3 \cdot \varepsilon_4 - \hat{G}_{B14}\varepsilon_1 \cdot \varepsilon_4 \hat{G}_{B22} k_j \cdot \hat{G}_{B3j} k_j \cdot \hat{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \\
- \hat{G}_{B11}\varepsilon_1 \cdot k_i \hat{G}_{B2j} k_j \cdot \hat{G}_{B33}\varepsilon_3 \cdot \varepsilon_4 - \hat{G}_{B22}\varepsilon_2 \cdot \varepsilon_4 \hat{G}_{B33}\varepsilon_3 \cdot \varepsilon_4 - \hat{G}_{B2j} k_j \cdot \hat{G}_{B3j} k_j \cdot \hat{G}_{B24}\varepsilon_2 \cdot \varepsilon_4 \\
+ \hat{G}_{B11}\varepsilon_1 \cdot k_i \hat{G}_{B2j} k_j \cdot \hat{G}_{33} k_k \cdot \hat{G}_{B4i} \varepsilon_4 \cdot k_k
\end{array}.
\]

(31)

There are some dummy indices in \(P_4\) which are to be summed. Q-representation for the scalar loop case after adding some total derivatives is obtained after replacing \(P_4\) by \(Q_4\) in Eq. (30) with

\[
Q_4 = Q_4^1 + Q_4^2 + Q_4^3 - Q_4^{22},
\]

\[
Q_4^1 = \hat{G}(234) + \hat{G}(1243) + \hat{G}(1324),
\]

\[
Q_4^2 = \hat{G}(123)T(4) + \hat{G}(234)T(1) + \hat{G}(341)T(2) + \hat{G}(412)T(3),
\]

\[
Q_4^3 = \hat{G}(12)T(34) + \hat{G}(13)T(24) + \hat{G}(14)T(23) + \hat{G}(23)T(14)
+ \hat{G}(24)T(13) + \hat{G}(34)T(12),
\]

\[
Q_4^{22} = \hat{G}(12)\hat{G}(34) + \hat{G}(13)\hat{G}(24) + \hat{G}(14)\hat{G}(23),
\]

(32)
with the following compact notation
\[
\hat{G}(i_1 i_2 \cdots i_n) := \hat{G}_{B_{i_1 i_2}} \hat{G}_{B_{i_2 i_3}} \cdots \hat{G}_{B_{i_{n-1} i_n}} \left( \frac{1}{2} \right)^{\delta_{n,2}} \text{tr}(f_{i_1} f_{i_2} \cdots f_{i_n}),
\]
\[
T(i) := \sum_r \hat{G}_{Bir} \varepsilon_i \cdot k_r,
\]
\[
T(ij) := \sum_{r,s} \left\{ \hat{G}_{Bir} \varepsilon_i \cdot k_r \hat{G}_{Bjs} \varepsilon_j \cdot k_s + \frac{1}{2} \hat{G}_{Bir} \varepsilon_i \cdot k_r \left[ \hat{G}_{Bjr} k_i \cdot k_r - \hat{G}_{Bjr} k_j \cdot k_r \right] \right\}.
\]
\[
(33)
\]
\(T(i)\) and \(T(ij)\) are called one- and two-tail respectively, see [37] for more details. As it is obvious from the structure of \(P_4\) removing all \(\hat{G}_{Bij}\)'s is much more involved than the lower points. IBP for the four-gluon vertex leads to two type of boundary terms: single boundary terms which involve structure of three-point integrals and double boundary terms with two-point integrals. The following rules emerge:

- Each single boundary term, say for the limit \(3 \to 4\), matches some bulk term in the Q-representation of the three-gluon vertex, with momenta \((k_1, k_2, k_3 + k_4)\), and \(f_3 = k_3 \otimes \varepsilon_3 - \varepsilon_3 \otimes k_3\) replaced by \(\varepsilon_3 \otimes \varepsilon_4 - \varepsilon_4 \otimes \varepsilon_3\).
- Each double boundary term, say for the limit \(1 \to 2, 3 \to 4\), matches the bulk term in the Q-representation of the two-point function, with momenta \((k_1 + k_2, k_3 + k_4)\), and the double replacement

\[
\begin{align*}
&f_1 = k_1 \otimes \varepsilon_1 - \varepsilon_1 \otimes k_1 \to \varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1, \\
&f_2 = k_2 \otimes \varepsilon_2 - \varepsilon_2 \otimes k_2 \to \varepsilon_3 \otimes \varepsilon_4 - \varepsilon_4 \otimes \varepsilon_3.
\end{align*}
\]
\[
(34)
\]

These boundary terms actually combine to some terms from the bulk contribution to complete a \(f_i\) to a full nonabelian field strength tensor \((F_i)\), i.e:
\[
f_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu].
\]
\[
(35)
\]

The Q-representation of the one-loop four-gluon vertex leads to 19 different structures, which five of them are just the two- and three-point form factors showing up at the four-point level as boundary terms and only 14 true four-point tensors appear, see [39, 40]. The S-representation of the four-gluon vertex will be discussed elsewhere [39] but as was mentioned earlier for the three-gluon case for this representation the bulk terms are written completely in terms of the field strength tensor \(f_i\), so that all non-transversality has been pushed into the boundary terms. To obtain the Ball-Chiu type form factor decomposition one needs to introduce the four vector \(r_i\) to satisfy the condition \(r_i \cdot k_i \neq 0\) and after fixing them the form factors are obtained. To compare with the low energy effective action besides \(O_2\) and \(O_3\) one needs \(O_4\) as well which is written as [43]
\[
O_4 = -\frac{2}{35} g^4 F_{\kappa \lambda} F_{\mu \nu} F_{\mu \nu} F_{\kappa \lambda} + \frac{4}{35} g^4 F_{\kappa \lambda} F_{\mu \nu} F_{\mu \nu} F_{\nu \mu} - \frac{1}{21} g^4 F_{\kappa \lambda} F_{\mu \nu} F_{\mu \nu} F_{\nu \mu} - \frac{8}{105} i g^2 F_{\kappa \lambda} D_\lambda F_{\mu \nu} D_\nu F_{\nu \mu} - \frac{6}{35} i g^3 F_{\kappa \lambda} D_\mu F_{\mu \nu} D_\nu F_{\nu \mu} + \frac{11}{420} g^4 F_{\kappa \lambda} F_{\mu \nu} F_{\lambda \mu} F_{\nu \mu} + \frac{1}{70} g^2 D_\kappa D_\lambda F_{\mu \nu} D_\kappa D_\lambda F_{\nu \mu}.
\]
\[
(36)
\]
As a check, we have matched the low-energy limit of our final results for the four-gluon vertex and found complete agreement [39].
8. Off-shell one-loop four-gluon vertex in $\mathcal{N}=4$ SYM

In recent years, tremendous progress has been made towards a more complete understanding of the scattering amplitudes in $\mathcal{N}=4$ SYM theory. $\mathcal{N}=4$ SYM is special in a sense that it has more symmetries than other gauge theories. It is understood that the theory is UV finite in perturbation theory. Since in dimensional regularization the bubble diagrams are the source of UV divergences one may think that, the one-loop four-gluon amplitude in $\mathcal{N}=4$ SYM is built out of the triangle and box diagrams but not the bubbles. But it turns out that in one-loop $\mathcal{N}=4$ SYM theory the triangle and bubbles vanish [44, 45]. For the two-point case it is easy to see that, in the massless case, the only way it can be finite is that it vanishes and for the three-point case, some of the diagrams are zero since they are proportional to the mass then in the massless limit they vanish, and the rest of the diagrams cancel each other, so three-point function in $\mathcal{N}=4$ SYM vanishes, see [46] for more details. Four-gluon vertex in $\mathcal{N}=4$ SYM theory is the first non-vanishing vertex and it turned to be extremely simple: all boundary terms cancel out since they would covariantize the nonexisting lower point amplitudes and the bulk term factors as

$$\Gamma^{a_1a_2a_3a_4} = 4g^4 \text{tr} (T^{a_1}T^{a_2}T^{a_3}T^{a_4})F^4_{ss} B(1234) + \text{non-cyclic permutations.}$$  

$$B(1234)$$ represents the off-shell scalar box integral with momenta $k_1, \ldots, k_4$, and

$$F^4_{ss} = \text{tr} (f_1f_2f_3f_4) + \text{tr} (f_1f_2f_4f_3) + \text{tr} (f_1f_3f_2f_4) - \frac{1}{4} \text{tr} (f_1f_2)\text{tr} (f_3f_4) - \frac{1}{4} \text{tr} (f_1f_3)\text{tr} (f_2f_4) - \frac{1}{4} \text{tr} (f_1f_4)\text{tr} (f_2f_3),$$  

summarizes the whole Lorentz structure contained in this vertex. $F^4_{ss}$ is well known for string theorists, it appears in the low energy effective action of the open superstring; see [47] for example. But from quantum field theory point of view the structure of the off-shell four-gluon vertex in $\mathcal{N}=4$ SYM is new.

9. Summary and Outlook

We have presented two IBP algorithms (Q and S representations) which generate the form factor decomposition of the $N$-gluon off-shell amplitude compatible with Bose symmetry and gauge invariance using string-inspired formalism. Applying these two IBP algorithms we have calculated in a unifying way the one-loop correction to the three- and four-gluon vertices for the scalar, spinor and gluon loop cases obtained directly from the Bern-Kosower master formula. The parameter integrals appearing in the form factors are obtained directly from the Bern-Kosower master formula. The Feynman-Schwinger parameter representation of these integrals are suitable for numerical purposes. In particular, the four-point case leads to a natural generalization of the Ball-Chiu form factor decomposition which leads to 19 tensor structures. It is distinguished by the fact that all true four-point terms are manifestly transversal, so that all longitudinal components are pushed to the boundary terms which represent the lower-point (two- and three-point) integrals. Four-point case has an extremely compact form in $\mathcal{N}=4$ SYM theory which we have presented in the last section. The main advantage of string-inspired formalism is the fact that it is possible to generate the $N$-gluon vertex without going through the usual tedious analysis of the nonabelian Ward identity.

10. Acknowledgements

I would like to thank C. Schubert for his reading of the manuscript and helpful comments and discussions. This work was supported by IBS (Institute for Basic Science) under grant IBS-R012-D1.
References

[1] M. Pelaez, M. Tissier and N. Wschebor, Phys. Rev. D 88 (2013) 125003, arXiv: 1310.2594 [hep-th].
[2] R. Alkofer, M. Q. Huber and K. Schwenzer, Eur. Phys. J. C 62 (2009) 761, arXiv:0812.4045 [hep-ph].
[3] C. Kellermann and C. Fischer, Phys. Rev. D 78 (2008) 025015, arXiv: 0801.2697 [hep-th];
   V. Mader and R. Alkofer, PoS ConfinementX (2012) 063, arXiv:1301.7498 [hep-th].
[4] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 425 (1994) 217, hep-ph/9403226.
[5] Z. Bern and Y.-t. Huang, J. Phys. A 44 (2011) 454003, arXiv:1103.1869 [hep-th].
[6] E. Witten, Comm. Math. Phys. 252 (2004) 189, hep-th/0312171.
[7] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 715 (2005), 499, hep-th/0412308.
[8] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94 (2005) 181602, hep-th/0501052.
[9] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP 1003 (2010) 020, arXiv:0907.5418 [hep-th].
[10] L. J. Mason and D. Skinner, JHEP 0911 (2009) 045, arXiv:0909.0250 [hep-th].
[11] H. Elvang and Y.-t. Huang, arXiv:1308.1697 [hep-th].
[12] L. J. Dixon, SLAC-PUB-15775, arXiv:1310.5353 [hep-ph].
[13] J. S. Ball and T. W. Chiu, Phys. Rev. D 22 (1980) 2250, Erratum ibid 23 (1981) 3085.
[14] J. M. Cornwall and J. Papavassiliou, Phys. Rev. D 40 (1989) 3474.
[15] D. Binosi and J. Papavassiliou, Phys. Rept. 479 (2009) 1, arXiv:0909.2536 [hep-ph].
[16] D. Z. Freedman, G. Grignani, K. Johnson and N. Rius, Ann. Phys. (N. Y.) 218 (1992) 75.
[17] J. Papavassiliou, Phys. Rev. D 47 (1993) 4728.
[18] A. I. Davydychev, P. Osland and O. V. Tarasov, Phys. Rev. D 54(1996) 4087, hep-ph/9605348; Erratum-ibido 59 (1999) 10991.
[19] A. I. Davydychev, P. Osland and L. Saks, JHEP 0108 (2001) 050, hep-ph/0105072.
[20] M. Binger and S. J. Brodsky, Phys. Rev. D 74 (2006) 054016, hep-ph/0602199.
[21] L. F. Abbott, Nucl. Phys. B 185 (1981) 189.
[22] L. F. Abbott, M. T. Grisaru and R. K. Schaefer, Nucl. Phys. B 229 (1983) 372.
[23] A. Denner, G. Weiglein and S. Dittmaier, Phys. Lett. B 333 (1994) 420.
[24] J. Papavassiliou, Phys. Rev. D 51 (1995) 856.
[25] A. I. Davydychev, P. Osland and O. V. Tarasov, Phys. Rev. D 58 (1998) 036007, hep-ph/9801380.
[26] A. I. Davydychev and P. Osland, Phys. Rev. D 59 (1999) 014006, hep-ph/9906522.
[27] J. A. Gracey, Phys. Rev. D 84 (2011) 085011, arXiv:1108.4806 [hep-ph].
[28] Z. Bern and D. A. Kosower, Phys. Rev. Lett. 66 (1991) 1669.
[29] Z. Bern and D. A. Kosower, Nucl. Phys. B 362 (1991) 389.
[30] Z. Bern and D. A. Kosower, Nucl. Phys. B 379 (1992) 451.
[31] J. Cornwall, arXiv:1311.1827 [hep-ph], to appear in the proceedings of this workshop.
[32] Z. Bern, L. Dixon and D. A. Kosower, Phys. Rev. Lett. 70 (1993) 2677.
[33] M. J. Strassler, Nucl. Phys. B 385 (1992) 145, hep-ph/9205205.
[34] M. J. Strassler, “Field theory without Feynman diagrams: a demonstration using actions induced by heavy particles”, SLAC-PUB-5978 (1992) (unpublished).
[35] C. Schubert, Eur. Phys. J. C 5 (1998) 693, hep-th/9710067.
[36] C. Schubert, Phys. Rept. 355 (2002) 73, arXiv:hep-th/0101036.
[37] N. Ahmadiniaz, C. Schubert and V.M. Villanueva, JHEP 1301 (2013) 312, arXiv:1211.1821 [hep-th].
[38] N. Ahmadiniaz and C. Schubert, Nucl. Phys. B 869 (2013) 417, arXiv:1210.2331 [hep-ph].
[39] N. Ahmadiniaz and C. Schubert, “String-inspired form factor decompositions of the four-gluon vertex”, in preparation.
[40] N. Ahmadiniaz and C. Schubert, International Journal of Modern Physics E 25 (2016) 1642004.
[41] M. Reuter, M. G. Schmidt and C. Schubert, Ann. Phys. (N.Y.) 259 (1997) 313, hep-th/9610191.
[42] A. van de Ven, Nucl. Phys. B 250 (1985) 593.
[43] D. Fliegner, P. Haberl, M.G. Schmidt and C. Schubert, Ann. Phys. (N.Y.) 264 (1998) 51, hep-th/9707189.
[44] O. Piguet and A. Rouet, Nucl. Phys. B 108 (1976) 265.
[45] D. Storey, Phys. Lett. B 105 (1981) 171.
[46] Stefano Kovacs, “N = 4 supersymmetric Yang-Mills theory and the AdS/SCFT correspondence”, PhD Thesis (1998) (Università di Roma, Tor Vergata), arXiv: hep-th/9908171.
[47] J. Broedel and L. J. Dixon, JHEP 1210 (2012) 091, arXiv:1208.0876 [hep-th].