ACYCLIC ORIENTATIONS AND THE CHROMATIC POLYNOMIAL OF SIGNED GRAPHS

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ABSTRACT. We present a new correspondence between acyclic orientations and coloring of a signed graph (symmetric graph). Goodall et al. introduced a bivariate chromatic polynomial \( \chi_G(k, l) \) that counts the number of signed colorings using colors \( 0, \pm 1, \ldots, \pm k \) along with \( l - 1 \) symmetric colors \( 0_1, \ldots, 0_{l-1} \). We show that the evaluation of the bivariate chromatic polynomial \( |\chi_G(-1, 2)| \) is equal to the number of acyclic orientations of the signed graph modulo the equivalence relation generated by swapping sources and sinks. We present three proofs of this fact, a proof using toric hyperplane arrangements, a proof using deletion-contraction, and a direct proof.

1. Introduction

In 1973, Richard Stanley [Sta73] found a simple yet astonishing connection between the chromatic polynomial and acyclic orientations of a graph. Let \( \chi_G \) be the chromatic polynomial of \( G \). He showed that \( \chi_G(-1) \) is, up to a sign, equal to the number of acyclic orientations of \( G \).

Later on, Greene and Zaslavsky [GZ83] found a simple yet different connection between these two objects. Namely, they showed that the linear coefficient of \( \chi_G \) is, up to a sign, equal to the number of acyclic orientations of \( G \) with a unique fixed sink. Furthermore, this number equals an evaluation of the Tutte polynomial \( T_G(1, 0) \). This number also counts equivalence classes of acyclic orientations under the flip move. Specifically, two acyclic orientations are flip equivalent to each other if we can convert a sink (or source) to a source (or sink) by reversing all edges adjacent to that vertex. This flip move was introduced by Pretzel [Pre86], and further studied by many past research [Pro02; DMR16].

In this work, we generalize these results to symmetric graphs. Symmetric graphs are a variation of signed graphs introduced by Harary [Har53] and further studied by Zaslavsky [Zas82a; Zas82b; Zas82c]. In the literature, they are also called double covering graphs in [Zas82c] or skew-symmetric graphs in [Bab06; Tut67]. Simply speaking, symmetric graphs are graphs with an involution \( \iota \) such that \( uv \) is an edge if and only if \( \iota(u)\iota(v) \) is an edge. Notions such as the chromatic polynomial and acyclic orientations are all generalizable to symmetric graphs (details in Section 2). Our main result is that, the number of equivalence classes of acyclic orientations modulo certain flip moves, is equal to a specialization of the bivariate chromatic polynomial of symmetric graph introduced by [Goo+21].

There are several reasons why this result was not previously discovered. First, this number is not a simple evaluation of the Tutte polynomial of the graphical matroid of the symmetric graph (also known as the frame matroid in [Zas82c]). In fact, it is
an evaluation only if we consider the trivariate Tutte polynomial introduced recently by [Goo+21]. Secondly, while the bivariate chromatic polynomial satisfies a simple deletion-contraction recurrence, there is no such obvious recurrence for the number of acyclic orientations modulo flip moves, which is different from the case of ordinary graphs. Therefore, a simple induction proof does not exist.

In fact, both colorings and acyclic orientations are deeply connected to the geometry of toric hyperplane arrangements, which were studied recently in [NPS02; ERS09; DS19]. We will first introduce the basics of toric hyperplane arrangements in Section 3, and then in Section 4 we define the graphical toric hyperplane arrangement of $G$ introduced by [GZ83]. Analogous to the results in [DMR16], we show that there is a bijection between acyclic orientations of $G$ modulo flip moves and connected chambers in the graphical toric hyperplane arrangement of $G$. From there, we prove our main results in two ways: a finite field method from [ERS09] in Section 4, and a deletion-restriction proof in Section 5. Finally, we will include a direct proof of the result without the use of geometry in Section 6. The proof uses the reciprocity idea of Stanley’s [Sta73], and a direct induction.

2. Preliminaries: Symmetric Graphs

2.1. Signed Graphs and Symmetric Graphs. A signed graph $\Sigma = (\Gamma, \sigma)$ is a pair of an unsigned graph $\Gamma = (V, E)$ equipped with a function $\sigma$ which labels each edge (except an half edge) in $\Gamma$ positive or negative. The underlying unsigned graph $\Gamma$ may have not only links and loops but also half edges (which has only one endpoint).

Although signed graph appears more often in literature, there is an alternative form of a signed graph which we will use below.

**Definition 2.1.** A symmetric graph is a graph $G = (V, E)$ on vertices

$$V = \{v_{-n}, v_{-(n-1)}, \ldots, v_0, v_1, \ldots, v_n\}$$

such that if $(v_i, v_j) \in E$, then $(v_{-i}, v_{-j}) \in E$.

There is a natural correspondence between signed graphs $\Sigma$ on $n$ vertices and symmetric graphs $G$ on $2n + 1$ vertices. The correspondence is summarized in Table 1.

| Signed Graph $\Sigma$ | Symmetric Graph $G$ |
|----------------------|-------------------|
| Vertex $u_i$         | Vertex pair $\{v_i, v_{-i}\}$ |
| Positive Link $u_iu_j$ | Edges $v_iv_j$ and $v_{-i}v_{-j}$ |
| Negative Link $u_iu_j$ | Edges $v_iv_{-j}$ and $v_{-i}v_j$ |
| Positive Loop $u_i$  | Loops $v_i$ and $v_{-i}$ |
| Negative Loop $u_i$  | Edge $v_iv_{-i}$ |
| Half edge $u_i$      | Edges $v_0v_i$ and $v_0v_{-i}$ |

**Example 2.2.** Figure 1 shows a signed graph and the corresponding symmetric graph. The signed graph on the left contains a positive link, a negative link, and a half edge.
Definition 2.3. A symmetric graph is simple if it is simple as a graph. It is weakly connected if vertex $v_i$ is connected to $v_{-i}$ for all $i \neq 0$.

Without explicitly saying, we will assume all symmetric graphs are simple in the remaining paper. That is, it contains neither loops nor multiple edges.

2.2. Colorings of symmetric graphs. In [Goo+21], Goodall et al. introduced a bivariate $(k,l)$-coloring of a symmetric graph $G$ using colors $\{0, \pm 1, \ldots, \pm k\}$ along with an extra set of colors $\{0_1, \ldots, 0_{l-1}\}$ where we rule that $0_i = -0_i$.

Definition 2.4. Let $k, l \geq 1$ be integers. A proper $(k,l)$-coloring of a symmetric graph $G$ is an assignment $f : V \to \{0, \pm 1, \ldots, \pm k\} \cup \{0_1, \ldots, 0_{l-1}\}$ such that

- $f(v_0) = 0$;
- $f(v_i) = -f(v_{-i})$ for any $i \neq 0$;
- $f(v_i) \neq f(v_j)$ for any edge $v_iv_j \in E$.

Goodall et al. [Goo+21, Theorem 6.5] showed that the number of proper $(k,l)$-colorings of $G$ is also a polynomial in $k$ and $l$, called the bivariate chromatic polynomial $\chi_G(k,l)$. When $l = 1$, a proper $(k,1)$-coloring is also called a proper $k$-coloring, which was first introduced by Zaslavsky in [Zas82b].

Remark 2.5. It is an interesting question to find combinatorial interpretations for integer evaluations of the bivariate chromatic polynomial $\chi_G(k,l)$ when $k \leq 0$. The case $l = 1$ is well-studied, and later Theorem 2.10 will give an example for $l = 2$. This question is still open even for the case $l = 0$, which is called proper non-zero $k$-colorings by Zaslavsky in [Zas82b].

Example 2.6. We calculate the bivariate chromatic polynomial $\chi_G(k,l)$ of the symmetric graph $G$ in Figure 1b. We consider two cases based on the coloring of $v_2$.

- If $f(v_2) = 0_i$ for some $i$, then $v_i$ can be colored any color other than $0_i$. There are $2k + l - 1$ choices for $f(v_i)$ and $l - 1$ choices for $i$.
- If $f(v_2) = j$ for some $r \in \{\pm 1, \ldots, \pm k\}$, then $v_1$ can be colored any color other than $\pm j$. There are $2k + l - 2$ choices for $f(v_1)$ and $2k$ choices for $j$.

Combining the two cases,

\[ \chi_G(k,l) = (2k + l - 1)(l - 1) + (2k + l - 2)2k. \]
2.3. Acyclic orientations of symmetric graphs.

**Definition 2.7.** A symmetric acyclic orientations of symmetric graph $G$ is an acyclic orientation of $G$ such that if $v_i \rightarrow v_j$ then $v_{-j} \rightarrow v_{-i}$.

The acyclic orientation of a symmetric graph (or signed graph) was studied by Zaslavsky in [Zas82b; Zas91]. Analogous to Richard Stanley’s celebrated result [Sta73] where the number of acyclic orientations of an ordinary graph is equal to its chromatic polynomial evaluated at $-1$, Greene and Zaslavsky in [GZ83] proved the following result.

**Theorem 2.8 ([GZ83, Theorem 9.1]).** For a symmetric graph $G$ with $2n+1$ vertices, the number of symmetric acyclic orientations of $G$ is equal to $(-1)^n \chi_G(-1,1)$.

**Definition 2.9.** In a symmetric acyclic orientation $\omega$ of symmetric graph $G$, if $v_i (i \neq 0)$ is a source (sink) vertex of $\omega$, then $v_{-i}$ is a sink (source). If $v_i v_{-i}$ is not an edge, then we call $(v_i, v_{-i})$ a source-sink pair. Reversing the direction of all edges adjacent to the source-sink pair $(v_i, v_{-i})$, we get another symmetric acyclic orientation $\omega'$ of $G$, and we say $\omega$ and $\omega'$ differ by a flip, denoted $\omega \sim \omega'$. The transitive closure of the flip operation generates an equivalence relation on the set of all symmetric acyclic orientation $\text{Acyc}(G)$ denoted by $\sim$.

The following is our main theorem.

**Theorem 2.10.** For a weakly connected symmetric graph $G$ with $2n+1$ vertices, we have

$$|\text{Acyc}(G)/\sim| = (-1)^n \chi_G(-1,2).$$

**Example 2.11.** We consider the symmetric graph $G$ in Figure 1b. On the one hand, its bivariate chromatic polynomial is computed in Equation (1) and we can evaluate it

$$\chi_G(-1,2) = 3.$$

On the other hand, $G$ has exactly 3 equivalence classes of symmetric acyclic orientations as shown in Figure 2.

We will present three proofs for Theorem 2.10 in the following sections. The first two proofs relate both numbers to the number of chambers in a graphical toric hyperplane arrangement corresponding to $G$, while the last proof uses a direct induction.

3. Preliminaries: Toric Hyperplane Arrangements

**Definition 3.1.** Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be the union of (rational) hyperplanes in $\mathbb{R}^n$ where in $H_i : \vec{h}_i \cdot \vec{x} = b_i$, all coordinates $\vec{h}_i$ and $b_i$ are rational. The toric hyperplane arrangement $\mathcal{H}_{\text{tor}}$ is the image of $\mathcal{H}$ under the projection map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$. Its affine hyperplane arrangement is the preimage $\mathcal{H}_{\text{aff}} = \pi^{-1}(\mathcal{H}_{\text{tor}})$.

In simple terms, toric hyperplane arrangements are projections of hyperplane arrangements onto a tori. It have been studied extensively by many previous research, for example [NPS02; ERS09; Moc12; DMR16]. In this section, we will state some basic facts about toric hyperplane arrangements without proof. Most of the results can be found in [ERS09].
Figure 2. Equivalence classes of symmetric acyclic orientations of $G$ in Figure 1b

Definition 3.2. Let $V$ be a $k$-dimensional (rational) affine subspace of $\mathbb{R}^n$, where $V = \{ \vec{x} : A\vec{x} = \vec{b} \}$ for some matrix $A$ and vector $\vec{b}$ with rational entries. Then the image $\pi(V)$ is called a $k$-dimensional toric subspace of $\mathbb{R}^n/\mathbb{Z}^n$. An $(n-1)$-dimensional toric subspace is also called a toric hyperplane.

Lemma 3.3 ([ERS09, Lemma 3.1]). A $k$-dimensional toric subspace is homeomorphic to the $k$-dimensional torus $\mathbb{R}^k/\mathbb{Z}^k$. Moreover, the intersection of toric subspaces $V \cap W$ is a disjoint union of toric subspaces $V \cap W = \bigcup_{i=1}^{r} U_i$ for some integer $r$.

Definition 3.4. For a toric hyperplane arrangement $\mathcal{H}_{\text{tor}} = \{H_1, \ldots, H_m\}$, define the intersection lattice $\mathcal{L}(\mathcal{H})$ as the set of all connected components (toric subspaces) of all possible intersections of toric hyperplanes $H_{i_1} \cap \cdots \cap H_{i_r}$, ordered by inverse inclusion. It has a unique minimal element $\hat{0}$ corresponding to the intersection of empty set, which is the entire space $\mathbb{R}^n/\mathbb{Z}^n$ itself. The toric characteristic polynomial $\chi_{\mathcal{H}}(q)$ is the characteristic polynomial of poset $\mathcal{L}(\mathcal{H})$

$$\chi_{\mathcal{H}}(q) = \sum_{x \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, x) \cdot q^{\dim(x)}.$$  

Example 3.5. Figure 3 shows an example of a toric hyperplane arrangement with hyperplanes $x_1 = 2x_2$, $2x_1 = x_2$ and $x_1 = x_2$. The corresponding intersection lattice $\mathcal{L}(\mathcal{H})$ has toric characteristic polynomial $\chi_{\mathcal{H}}(q) = q^2 - 3q + 4$. 

(A) Class 1  (B) Class 2  (C) Class 3
For a toric hyperplane arrangement $H_{tor}$, the connected components of $\mathbb{R}^n / \mathbb{Z}^n - H_{tor}$ are called \textit{toric chambers} of $H_{tor}$. Similarly, for its affine hyperplane arrangement $H_{aff}$, the connected components of $\mathbb{R}^n - H_{aff}$ are called \textit{affine chambers} of $H_{aff}$. The set of affine chambers is denoted as $\text{Cham}(H_{aff})$.

Recall that a hyperplane arrangement is \textit{essential} if the normal vectors of the hyperplanes span the entire space $\mathbb{R}^n$. When the toric arrangement $H_{tor}$ is essential, we can determine the number of toric chambers of $H_{tor}$ using the toric characteristic polynomial.

\textbf{Theorem 3.6 (ERS09, Theorem 3.6).} If $\mathcal{H}$ is essential, then

$$|\text{Cham}(H_{tor})| = (-1)^n \chi_{\mathcal{H}}(0).$$

\section{4. A FIRST PROOF: GRAPHICAL TORIC HYPERPLANE ARRANGEMENTS}

In order to prove the main theorem, we will show that both sides of Theorem 2.10 are equal to the number of chambers in a graphical toric hyperplane arrangement associated to $G$, which is defined below.

\textbf{Definition 4.1.} Given a symmetric graph $G = (V, E)$ with $2n + 1$ vertices, define its \textit{graphical toric hyperplane arrangement} $B_{tor}(G)$ by the union of the following hyperplanes in $\mathbb{R}^n / \mathbb{Z}^n$.

$$B_{tor}(G) = \bigcup_{0<i<j\leq n} \{x_i - x_j = 0\} \cup \bigcup_{0<i<j\leq n} \{x_i + x_j = 0\} \cup \bigcup_{0<i\leq n} \{x_i = 0\} \cup \bigcup_{0<i\leq n} \{2x_i = 1\}.$$

Define $B_{aff}(G) := \pi^{-1}(B_{tor}(G))$ as the corresponding \textit{graphical affine hyperplane arrangement}.

\textbf{Example 4.2.} For the symmetric graph $G$ in Figure 1b, Figure 4 shows the affine hyperplane arrangement $B_{aff}(G)$ and toric hyperplane arrangement $B_{tor}(G)$ corresponding to $G$, with their chambers colored in different colors.
Lemma 4.3. The toric arrangement $B_{\text{tor}}(G)$ is essential if and only if $G$ is weakly connected.

Proof. If $G$ is not weakly connected, there exists index $i > 0$ such that $v_i$ and $v_{-i}$ are not connected. Consider the connected component $S$ containing $v_i$. There are two facts about $S$.

- $v_0 \notin S$. Otherwise by symmetry of $G$, there is a path $v_i \rightarrow v_0 \rightarrow v_{-i}$;
- If $v_j \in S$, then $v_{-j} \notin S$. This is because $v_i$ and $v_j$ are connected implies $v_{-j}$ and $v_{-i}$ are connected.

Denote the connected component $S = \{v_i, v_{i_1}, \ldots, v_{i_p}\} \cup \{v_{-j_1}, \ldots, v_{-j_q}\}$ with distinct indices $i_1, \ldots, i_p, j_1, \ldots, j_q > 0$. Then all normal vectors in $B_{\text{tor}}(G)$ are perpendicular to the vector

$$\vec{e}_S = \vec{e}_i + \vec{e}_{i_1} + \cdots + \vec{e}_{i_p} - \vec{e}_{j_1} - \cdots - \vec{e}_{j_q},$$

where $\vec{e}_i$ is the vector with a 1 in the $i$th coordinate and 0’s elsewhere.

If $B_{\text{tor}}(G)$ is not essential, then there exists a non-zero vector $\vec{w} = (w_1, \ldots, w_n)$ which is perpendicular to all normal vectors. Consider a non-zero entry $w_i \neq 0$. We claim that $v_i$ and $v_{-i}$ are not connected in $G$. Otherwise there is a path from $v_i$ to $v_{-i}$. If we sum up the normal vectors associated to each edge along the path, we get $2\vec{e}_i$. Since $\vec{w}$ is perpendicular to all normal vectors, we have $w_i = \vec{w} \cdot \vec{e}_i = 0$, which is a contradiction. \qed

4.1. Chambers of $B_{\text{tor}}(G)$ and Acyclic Orientations of $G$. A point $x$ in $\mathbb{R}^n/\mathbb{Z}^n$ does not have a well-defined coordinate $(x_1, \ldots, x_n)$. However, if we translate every coordinate $x_i$ by an integer we can assume each $x_i$ lies in the interval $(-0.5, 0.5]$, and this coordinate is well-defined. For each point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n/\mathbb{Z}^n - B_{\text{tor}}(G)$, we denote $x_0 = 0$ and $x_{-i} = -x_i$ for any $1 \leq i \leq n$. Now we define an acyclic orientation $\omega(x)$ of $G$ as follows: for any $v_i, v_j \in E(G)$ ($i, j \in [-n, n]$), since $x_i \neq x_j$, we have either

- $x_i < x_j$, then we direct $v_j \rightarrow v_i$, or
- $x_i > x_j$, then we direct $v_i \rightarrow v_j$. 

![Figure 4](image-url)
Notice that this orientation $\omega(x)$ is indeed symmetric since $x_i < x_j \iff x_{-i} > x_{-j}$. Denote this map $x \mapsto \omega(x)$ as $\beta_G : (\mathbb{R}^n / \mathbb{Z}^n - \mathcal{B}_{\text{tor}}(G)) \to \text{Acyc}(G)$.

**Theorem 4.4.** The map $\beta_G$ induces an isomorphism $\bar{\beta}_G$ from $\text{Cham}(\mathcal{B}_{\text{tor}}(G))$ to $\text{Acyc}(G)/ \sim$ as follows:

\[
\begin{array}{ccc}
\mathbb{R}^n / \mathbb{Z}^n - \mathcal{B}_{\text{tor}}(G) & \xrightarrow{\beta_G} & \text{Acyc}(G) \\
\downarrow & & \downarrow \\
\text{Cham}(\mathcal{B}_{\text{tor}}(G)) & \xrightarrow{\bar{\beta}_G} & \text{Acyc}(G)/ \sim
\end{array}
\]

In other words, two points $x, y$ lie in the same chamber in $\text{Cham}(\mathcal{B}_{\text{tor}}(G))$ if and only if $\beta_G(x) \sim \beta_G(y)$.

**Proof.** **Well-definition:** We need to show that if two points $x, y$ lie in the same chamber $c$, then $\beta_G(x) \sim \beta_G(y)$. There exists a path $\gamma$ in $c$ between $x$ and $y$. Since $c$ is open, we can assume $\gamma$ takes steps in coordinate directions only, and therefore reduce to the case where $x$ and $y$ differ in only one coordinate $x_i \neq y_i$. Since $\beta_G(x)$ only changes when passing through the boundary line $x_i = 0.5$, one may assume $x_i = 0.5 - \varepsilon$ and $y_i = 0.5 + \varepsilon$ for some arbitrarily small $\varepsilon > 0$. In such case $\beta_G(x)$ and $\beta_G(y)$ differs by flipping the source-sink pair $(v_i, v_{-i})$, so $\beta_G(x) \sim \beta_G(y)$.

**Surjectivity:** We only need to show that $\beta_G$ is surjective. For any symmetric acyclic orientation $\omega$ of $G$, we can find a symmetric linear extension $\omega_{-n} < \cdots < \omega_{-1} < \omega_0 < \omega_1 < \cdots < \omega_n$ of all vertices $V(G) = [-n, n]$ that is compatible with $\omega$, and that $\omega_i + \omega_{-i} = 0$ for all $i = 0, 1, \ldots, n$. Then we choose real numbers $-0.5 < x_{\omega_{-n}} < \cdots < x_{\omega_{-1}} < x_0 = 0 < x_{\omega_1} < \cdots < x_{\omega_n} < 0.5$ and let $x = (x_1, \ldots, x_n)$, so that $\beta_G(x) = \omega$.

**Injectivity:** We need to show that for any two points $x, y$, if $\beta_G(x) = \beta_G(y)$ or $\beta_G(x)$ and $\beta_G(y)$ differ by a flip, then $x, y$ lie in the same chamber. If $\beta_G(x) = \beta_G(y)$, then the segment connecting $x$ and $y$ will not cross any hyperplanes. Therefore $x$ and $y$ lie in the same chamber. If $\beta_G(x)$ and $\beta_G(y)$ differ by a source-sink flip at $(v_i, v_{-i})$, assume $v_i$ is the source and $v_{-i}$ is the sink. Then the segment connecting $x$ and $y + \vec{e}_i = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ will not cross any hyperplanes. Therefore $x$ and $y$ lie in the same chamber. \(\Box\)

### 4.2. Chambers of $\mathcal{B}_{\text{tor}}(G)$ and the Bivariate Chromatic Polynomial of $G$.

According to Theorem 3.6, the number of chambers $|\text{Cham}(\mathcal{B}_{\text{tor}}(G))|$ is related to the toric characteristic polynomial $\chi_{\mathcal{B}_{\text{tor}}(G)}$. The following lemma in [ERS09] implies a relationship between $\chi_{\mathcal{B}_{\text{tor}}(G)}$ and the bivariate chromatic polynomial $\chi_G$.

**Lemma 4.5** ([ERS09, Theorem 3.7]). Given a toric hyperplane arrangement $\mathcal{H}_{\text{tor}}$, there exists infinitely many positive integers $q$ such that the toric characteristic polynomial evaluated at $q$ is given by the number of lattice points in $(\frac{1}{q} \mathbb{Z})^n / \mathbb{Z}^n$ that do not lie on
any of the toric hyperplanes $H_i$, that is,
\[ \chi_H(q) = \left| \left( \frac{1}{q} \mathbb{Z} \right)^n / \mathbb{Z}^n - H_{\text{tor}} \right|. \]

**Proposition 4.6.** For a symmetric graph $G$, the relationship between the bivariate chromatic polynomial $\chi_G(k, l)$ and the characteristic polynomial of $\mathcal{L}(\mathcal{B}_{\text{tor}}(G))$ is the following:

\[ \chi_{\mathcal{B}_{\text{tor}}(G)}(q) = \chi_G \left( \frac{q - 2}{2}, 2 \right). \]

**Proof.** Assume $q = 2k + 2$ is even. Given any point in the lattice $\left( \frac{1}{q} \mathbb{Z} \right)^n / \mathbb{Z}^n$ that do not lie on $\mathcal{B}_{\text{tor}}(G)$

\[ x = \left( \frac{x_1}{q}, \ldots, \frac{x_n}{q} \right) \in \left( \frac{1}{q} \mathbb{Z} \right)^n / \mathbb{Z}^n - \mathcal{B}_{\text{tor}}(G) \]

where $x_1, \ldots, x_n$ are integers in $[-k, k + 1]$, we can define a unique $(k, 2)$-coloring $f$ of $G$ with colors $\{0, \pm 1, \ldots, \pm k\} \cup \{0'\}$ given by

\[ f(v_i) = \begin{cases} 
0' & \text{if } x_i = k + 1, \\
x_i & \text{otherwise.}
\end{cases} \]

It is not hard to show that this coloring is indeed proper, and all proper colorings come from this way. Therefore, the number of such lattice points $x$ is equal to $\chi_G(k, 2)$. On the other hand, by Lemma 4.5 the same number is equal to $\chi_{\mathcal{B}_{\text{tor}}(G)}(q)$ for infinitely many $q$. This concludes our proof. \qed

We are now in position to prove the main theorem.

**Theorem 2.10.** For a weakly connected symmetric graph $G$ with $2n + 1$ vertices, we have

\[ |\text{Acyc}(G)/\sim| = (-1)^n \chi_G(-1, 2). \]

**Proof.** By Theorem 4.4, Theorem 3.6 and Proposition 4.6, we have

\[ |\text{Acyc}(G)/\sim| = |\text{Cham } \mathcal{B}_{\text{tor}}(G)| = (-1)^n \chi_{\mathcal{B}_{\text{tor}}(G)}(0) = (-1)^n \chi_G(-1, 2). \]

\qed

**Remark 4.7.** Novik et al. [NPS02] proved that the number of chambers in a toric arrangement $|\text{Cham}(\mathcal{H}_{\text{tor}})|$ is equal to the evaluation of the Tutte polynomial $T_\mathcal{H}(1, 0)$ if $\mathcal{H}$ is a unimodular hyperplane arrangement (if the principal normal vectors of $\mathcal{H}$ form a unimodular matrix). It turns out that $\mathcal{B}(G)$ is one of the simplest non-unimodular hyperplane arrangement, and this work might shed light on how to generalize the results in [NPS02] to the general non-unimodular case. In [DM13], D’Adderio and Moci proposed using arithmetic Tutte polynomials to calculate the number of chambers in a toric arrangement, but their method only works when $\mathcal{H}$ is central, and cannot deal with hyperplanes such as $2x_i = 1$ in our case. So far, it is unknown how are the arithmetic Tutte polynomials related to $\chi_G(k, l)$ in this paper.
5. A second proof: Deletion-Contraction

In this section, we present an alternative proof to the fact that the number of chambers in $B_{tor}(G)$ equals $(-1)^n \chi_G(-1, 2)$ using deletion-contraction recurrences.

**Definition 5.1.** For a toric hyperplane arrangement $\mathcal{H}$ of dimension $n$ and a toric hyperplane $H \in \mathcal{H}$, the intersection $\mathcal{H} \cap H$ is also a toric hyperplane arrangement of dimension $n - 1$. We call $\mathcal{H} \cap H$ a restriction. If we take away the hyperplane $H$, the remaining hyperplane arrangement $\mathcal{H} - H$ is called a deletion.

**Theorem 5.2** (Deletion-restriction of toric hyperplane arrangements). Given an essential toric hyperplane arrangement $\mathcal{H}$ and a hyperplane $H \in \mathcal{H}$,

$$|\text{Cham } \mathcal{H}| = \begin{cases} |\text{Cham } (\mathcal{H} \cap H)| + |\text{Cham } (\mathcal{H} - H)| & \text{if } \mathcal{H} - H \text{ is essential,} \\ |\text{Cham } (\mathcal{H} \cap H)| & \text{otherwise.} \end{cases}$$

**Proof.** Proof analogous to the proof of Theorem 3.6 (see [ERS09, Theorem 3.6]). □

**Definition 5.3.** Given a symmetric graph $G = (V, E)$, an edge $e = v_i v_j \in E$ is contractible if $i, j \neq 0$ and $j \neq -i$. Given a contractible edge $e = v_i v_j$, denote the edge $-e = v_{-i} v_{-j} \in E$. The deletion $G - e$ is the graph obtained by deleting edges $e$ and $-e$ from $G$. The contraction $G/e$ is the graph obtained by contracting edges $e$ and $-e$ in $G$.

**Theorem 5.4** (Deletion-contraction of colorings). Given a symmetric graph $G$ and a contractible edge $e$, we have

$$\chi_G(k, l) = \chi_{G-e}(k, l) - \chi_{G/e}(k, l).$$

**Proof.** Assume $e = v_i v_j$. We only need to prove that the equality holds for any integers $k, l \geq 1$. In such case $\chi_{G-e}(k, l)$ counts the number of proper $(k, l)$-colorings of $G - e$. For each of these colorings, we can split them into two categories based whether $v_i$ and $v_j$ have the same color. If $v_i$ and $v_j$ have different colors, the number of such colorings is counted by $\chi_G(k, l)$; if $v_i$ and $v_j$ have the same color, we can contract $e = v_i v_j$ and the number of such colorings is counted by $\chi_{G/e}(k, l)$. Therefore, $\chi_{G-e}(k, l) = \chi_{G}(k, l) + \chi_{G/e}(k, l)$. □

Finally, Theorem 2.10 is a direct corollary of the following theorem.

**Theorem 5.5.** For any symmetric graph $G$ with $2n + 1$ vertices, we have

$$(-1)^n \chi_G(-1, 2) = \begin{cases} |\text{Cham } (B_{tor}(G))| & \text{if } G \text{ is weakly connected,} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Use induction. In the base case, $G$ has the form of the following graph in Figure 5. In such case, the left hand side is $\chi_G(k, l) = (2k + l)^a \cdot (2k + l - 1)^b \cdot (2k)^c$. The right hand side when $a = 0$ is the toric hyperplane arrangement of toric hyperplanes \( \{x_j = 0\} \cup \{x_k = 0\} \cup \{x_i = 1/2\} \) in the tori. It cuts the tori into $2^c$ chambers. Easy to check that the number matches. The induction step follows from Theorem 5.2 and Theorem 5.4. □
6. A third proof: Direct Induction

In this section, we will present a direct proof of Theorem 2.10 without using the geometric interpretations. We will start with the following theorem that evaluates \((-1)^n \chi_G(-1, 2)\).

Definition 6.1. In a symmetric graph \(G\) with \(2n + 1\) vertices, a set \(I \subseteq \{1, 2, \ldots, n\}\) is an independent set if the vertices \(V(I) = \{v_i : i \in I\} \cup \{-i \in I\}\) is an independent set in \(G\). Let \(G - I\) denote the symmetric graph obtained by deleting all vertices in \(V(I)\) from \(G\).

Theorem 6.2. Given a symmetric graph \(G\) with \(2n + 1\) vertices, we have

\[
(-1)^n \chi_G(-1, 2) = \sum_{I \subseteq [n] \text{ indep.}} (-1)^{|I|} |\text{Acyc}(G - I)|.
\]

Proof. Given a proper \((k, 2)\)-coloring of \(G\) using colors \(\{0, \pm 1, \ldots, \pm k\} \cup \{0'\}\), let \(I\) be the independent set of vertices colored by \(0'\). Then the number \(\chi_G(k, 2)\) is equal to the number of tuples \((I, f)\), where \(I\) is an independent set of \(G\) and \(f\) is a proper \(k\)-coloring of the subgraph \(G - I\). This implies that

\[
\chi_G(k, 2) = \sum_{I \subseteq [n] \text{ indep.}} \chi_{G-I}(k, 1).
\]

The theorem then follows from letting \(k = -1\) and applying Theorem 2.8, which says that \(|\text{Acyc}(G)| = (-1)^n \chi_G(-1, 1)\).

Example 6.3. We show an example of Theorem 6.2 using the graph \(G\) in Figure 1b. The independent sets of \(G\) are \(I = \emptyset, \{1\}\) and \(\{2\}\). Therefore, Theorem 6.2 tells us that the evaluation of the bivariate chromatic polynomial \((-1)^n \chi_G(-1, 2)\) is equal to the alternating sum in Figure 6. The number of acyclic orientations of the three subgraphs on the right hand side in Figure 6 are 6, 2 and 1 respectively (see Figure 2 for all 6 orientations of the first subgraph), and the theorem holds since \(3 = 6 - 2 - 1\).

In order to prove the main theorem, what is left is to show that the right hand side of Theorem 6.2 satisfies

\[
(2) \quad |\text{Acyc}(G)/\sim| = \sum_{I \subseteq [n] \text{ indep.}} (-1)^{|I|} |\text{Acyc}(G - I)|.
\]

The strategy is to use induction. We would like to split each term \((-1)^{|I|} |\text{Acyc}(G - I)|\) in the alternating sum into two categories based on whether \(i \in I\) or \(i \notin I\) for some fixed
index $i \in [n]$. If $i \in I$, then the sum reduces to the case of $G - \{i\}$. If $i \notin I$, the sum reduces to a version of $G$ where $i$ is “frozen”. Namely, $i$ is not allowed to be picked out into the independent set $I$. This motivates the following definition.

**Definition 6.4.** A frozen symmetric graph is a pair $\Gamma = (G, S)$ of symmetric graph $G$ and a set $S \subseteq [n]$. Vertices in $V(S) = \{v_i : i \in S\} \cup \{v_{-i} : i \in S\}$ are called frozen vertices; other vertices are called free vertices. Two acyclic orientations $\omega$ and $\omega'$ of $G$ differ by an $S$-flip if one can reverse all arrows adjacent to a free source-sink pair $(v_i, v_{-i})$ where $i \notin S$ in $\omega$ to obtain $\omega'$. In other words, we are not allowed to flip the frozen vertices. The transitive closure of the $S$-flip operation generates an equivalence relation on the set of all symmetric acyclic orientation $\text{Acyc}(G)$ denoted by $\sim_S$.

Also, we say $\Gamma = (G, S)$ is weakly connected if vertex $v_i$ is connected to $v_{-i}$ in $G$, or connected to a frozen vertex, for all $i \neq 0$. This definition is compatible with the definition of weakly-connectedness for regular symmetric graphs in Definition 2.3 (when $S = \emptyset$). We will assume the frozen symmetric graph $\Gamma$ is weakly-connected in the rest of the paper. The following theorem is a generalization of Equation (2).

**Theorem 6.5.** Given a frozen symmetric graph $\Gamma = (G, S)$, we have

$$|\text{Acyc}(G)/\sim_s| = \sum_{I \subseteq [n] \text{ indep.}} (-1)^{|I|} |\text{Acyc}(G - I)|.$$ 

The proof of Theorem 6.5 relies on the following definitions and a key lemma.

**Definition 6.6.** Given a frozen symmetric graph $\Gamma = (G, S)$ and index $i \notin S$ where $v_i, v_{-i}$ are not adjacent, denote the graph $\Gamma_i := (G, S \cup \{i\})$ called the freezing of $G$ at $i$. Let $G - i$ be the subgraph of $G$ obtained by deleting vertices $v_i$ and $v_{-i}$, and let $N(i) \subseteq [n]$ denote the set of indices $j$ where $v_j$ is a neighbour of $v_i$ or $v_{-i}$. Denote the graph $\Gamma - i := (G - i, S \cup N(i))$ called the deletion of $G$ at $i$.

**Remark 6.7.** The freezing $\Gamma_i$ is obtained by simply “freezing” the vertices $v_i$ and $v_{-i}$ in $\Gamma$. To obtain the deletion $\Gamma - i$, one deletes vertices $v_i$ and $v_{-i}$, and then “freezes” all neighbours of them.

**Example 6.8.** Figure 7 shows an example of freezing and deletion of a frozen symmetric graph $\Gamma = (G, \{2, 4, 5\})$ at index 4. Frozen vertices are labeled by squares.

Now we present the key lemma that is crucial to the proof of Theorem 6.5.
Lemma 6.9 (Freezing-Deletion of acyclic orientations). Given a frozen symmetric graph \( \Gamma = (G, S) \) and \( i \notin S \) where \( v_i, v_{-i} \) are not adjacent, denote \( \Gamma_i = (G, S \cup \{i\}) \) and \( \Gamma - i = (G - i, S \cup N(i)) \). We have

\[
|\text{Acyc}(G)/\sim_S| = |\text{Acyc}(G)/\sim_{S \cup \{i\}}| - |\text{Acyc}(G - i)/\sim_{S \cup N(i)}|.
\]

We will postpone the proof for Lemma 6.9 to the last subsection. Before that, we will show how Lemma 6.9 implies Theorem 6.5, and thus the main result.

Proof of Theorem 6.5. We prove by induction on \( (n, |S|) \) where the second term \( |S| \) goes in reverse order. In the base case, \( S = \{i \in [n]: v_i \) and \( v_{-i} \) are not adjacent \}, and both sides are equal to \( |\text{Acyc}(G)| \). For the induction step, pick a random index \( i \notin S \) and \( v_i, v_{-i} \) not adjacent. We split the alternating sum on the right hand side into two cases: \( i \in I \) or \( i \notin I \).

\[
\sum_{\substack{I \subseteq [n] \text{ indep.} \\text{I \cap S = \emptyset} \\text{i \notin I}}} (-1)^{|I|} |\text{Acyc}(G - I)| + \sum_{\substack{I \subseteq [n] \text{ indep.} \\text{I \cap S = \emptyset} \\text{i \in I}}} (-1)^{|I|} |\text{Acyc}(G - I)|
\]

\[
= \sum_{\substack{I \subseteq [n] \text{ indep.} \\text{I \cap (S \cup \{i\}) = \emptyset}}} (-1)^{|I|} |\text{Acyc}(G - I)| - \sum_{\substack{I' \subseteq [n] - i \text{ indep.} \\text{I' \cap S = \emptyset} \\text{I' \cap N(i) = \emptyset}}} (-1)^{|I'|} |\text{Acyc}(G - i - I')|.
\]

When \( i \notin I \), it reduces to the case of \( \Gamma_i \), and when \( i \in I \), it reduces to the case \( \Gamma - i \). The induction step then follows from Lemma 6.9. \( \square \)

Specifically, when \( S = \emptyset \) we get the following corollary.

Corollary 6.10. Given a symmetric graph \( G \), we have

\[
|\text{Acyc}(G)/\sim| = \sum_{I \subseteq [n] \text{ indep.}} (-1)^{|I|} |\text{Acyc}(G - I)|.
\]

Theorem 2.10 then follows from Theorem 6.2 and Corollary 6.10.

6.1. Proof of Lemma 6.9. For a frozen symmetric graph \( \Gamma = (G, S) \), consider a graph \( \mathcal{G}(\Gamma) \) whose vertex set is the set of all possible acyclic orientations of \( G \), and there is an edge between two orientations \( \omega \) and \( \omega' \) of \( G \) if they differ by an \( S \)-flip. Then the number \( |\text{Acyc}(G)/\sim_S| \) is the number of connected components of \( \mathcal{G}(\Gamma) \).
Now for \( \Gamma = (G, S) \) and \( j \not\in S \), \( \mathcal{G}(\Gamma_j) \) has the same vertex set as \( \mathcal{G}(\Gamma) \), so we can view \( \mathcal{G}(\Gamma_j) \) as an induced subgraph of \( \mathcal{G}(\Gamma) \) with the same vertex set and fewer edges. For \( \mathcal{G}(\Gamma - j) \), we can add \( v_j \) as a source and \( v_{-j} \) as a sink to each vertex of \( \mathcal{G}(\Gamma - j) \). Then we can identify \( \mathcal{G}(\Gamma - j) \) as a induced subgraph of \( \mathcal{G}(\Gamma) \) in this case as well, but with both fewer vertices and fewer edges. Denote \( \# \text{Comp}(\mathcal{G}) \text{ as the number of connected components in a graph } \mathcal{G} \). Our goal is to show the following

\[
\# \text{Comp}(\mathcal{G}(\Gamma)) = \# \text{Comp}(\mathcal{G}(\Gamma_j)) - \# \text{Comp}(\mathcal{G}(\Gamma - j)).
\]

Now since \( \mathcal{G}(\Gamma_j) \) and \( \mathcal{G}(\Gamma - j) \) are both induced subgraphs of \( \mathcal{G}(\Gamma) \), we can restrict our attention to a single connected component \( C \subseteq \mathcal{G}(\Gamma) \). We only need to show that

\[
(3) \quad \# \text{Comp}(\mathcal{G}(\Gamma_j) \cap C) = \# \text{Comp}(\mathcal{G}(\Gamma - j) \cap C) + 1.
\]

We would like to define some “weight function” for each orientation \( \omega \in \mathcal{C} \), such that it satisfies some nice properties.

**Proposition 6.11.** There exists a family of height functions \( \{ h_\omega : V(G) \to \mathbb{R} \}_{\omega \in \mathcal{C}} \) that satisfies the following properties:

1. (Frozen zeros) For any frozen vertex \( v \in V(S) \), \( h_\omega(v) = 0 \).
2. (Symmetric) \( h_\omega(v_i) + h_\omega(v_{-i}) = 0 \) for any \( i \in [n] \).
3. (Compatible with orientation) If \( u \to v \) in orientation \( \omega \), then \( 0 \leq h_\omega(u) - h_\omega(v) < 1 \), and \( h_\omega(u) = h_\omega(v) \) if they are both zeros.
4. (Compatible with flips) For any adjacent \( \omega, \omega' \in \mathcal{C} \) that differ by flipping \((v_i, v_{-i})\) from a source/sink pair to a sink/source pair in \( \omega \), we have

\[
h_\omega(v) - h_{\omega'}(v) = \begin{cases} 
-1, & \text{if } v = v_i, \\
1, & \text{if } v = v_{-i}, \\
0, & \text{otherwise}.
\end{cases}
\]

5. (Integral) For any \( \omega, \omega' \in \mathcal{C} \) and vertex \( v \), \( h_\omega(v) - h_{\omega'}(v) \) is an integer.
6. (Uniqueness) If \( h_\omega = h_{\omega'} \), then \( \omega = \omega' \).

**Proof.** Define \( C \subseteq G \) to be a closed cycle of \( \Gamma = (G, S) \) if \( C \) is a (directed) loop or \( C \) is a (directed) path that starts and ends at two frozen vertices. For any closed cycle \( C \subseteq G \) and an orientation \( \omega \) of \( G \), define the circulation of \( \omega \) around \( C \) as \( c_\omega(C) = |C_\omega^+| - |C_\omega^-| \), where \( C_\omega^+ \) is the set of forward edges in \( C \) and \( C_\omega^- \) is the set of backward edges in \( C \). Note that flipping at a source-sink pair will not change the circulation of any closed cycle. Therefore, \( c = c_\omega \) is fixed across all \( \omega \in \mathcal{C} \).

To proceed with the proof, we need an extra ingredient: a real-valued function \( f_\omega \) on the set of directed edges of \( G \) (in both directions, denoted as \( E(G) \)). To define \( f_\omega \), we first define \( f_\omega : \tilde{E}(G) \to \mathbb{R} \) for any \( \omega \in \mathcal{C} \) as follows:

\[
f_\omega(\tilde{e}) = \begin{cases} 
1, & \text{if } \tilde{e} \in \omega, \\
0, & \text{if } \tilde{e} \not\in \omega.
\end{cases}
\]

It is not hard to check that \( f_\omega \) satisfies the following four properties:

1. \( 0 \leq f_\omega(\tilde{e}) \leq 1 \);
2. \( f_\omega(\tilde{e}) + f_\omega(-\tilde{e}) = 1 \);
Since we flipped Equation (4), we only need to check that the definition of \( \vec{e} \) determined. In the second case, \( h(\omega, \omega) \) is equal to \( \sum_{v \in C} f_\omega(\vec{e}) = \frac{1}{2}(|C| + c(C)) \).

Now we define \( f_C(\vec{e}) \) to be the average of all \( f_\omega(\vec{e}) \) ranging over all \( \omega \in C \). Therefore \( f_C \) satisfies the four properties above as well. Now we claim that there is a unique function \( h_\omega : V(G) \to \mathbb{R} \) for any \( \omega \in C \) such that it satisfies properties (1), (2), and for any \( \vec{e} = u \to v \),

\[
h_\omega(u) - h_\omega(v) = \begin{cases} 1 - f_C(\vec{e}), & \text{if } \vec{e} \in \omega, \\ -f_C(\vec{e}), & \text{if } \vec{e} \notin \omega. \end{cases}
\]

Uniqueness: Since \( \Gamma \) is simply connected, there exists a directed path \( P = e_1, \ldots, e_k \) either between \( v_i \) and \( v_{-i} \), or between \( v_i \) and a frozen vertex \( u \). In the first case, \( h_\omega(v_i) - h_\omega(v_{-i}) = t - f_C(e_1) - \cdots - f_C(e_k) \) where \( t \) is the number of \( e_i \) such that \( e_i \in \omega \). Combine with the fact that \( h_\omega(v_i) + h_\omega(v_{-i}) = 0 \), we know that \( h_\omega(v_i) \) and \( h_\omega(v_{-i}) \) are uniquely determined. In the second case, \( h_\omega(v_i) = h_\omega(v_i) - h_\omega(u) = t - f_C(e_1) - \cdots - f_C(e_k) \) for the same \( t \), so again \( h_\omega(v_i) \) is uniquely determined.

Existence: We only need to check that the definition of \( h_\omega \) is self-consistent on every closed cycle \( C \). Let \( e_1, \ldots, e_k \) be edges on the closed cycle, then we only need to show that \( 0 = t - f_C(e_1) - \cdots - f_C(e_k) \), where \( t \) is the number of forward edges on \( C \), which is also equal to \( \sum_{i=1}^{k} f_C(e_i) \) by property (d) of \( f_C \).

Finally, we show that \( h_\omega \) satisfies properties (3), (4) and (5). For (3), Equation (4) guarantees that \( 0 \leq h_\omega(u) - h_\omega(v) \leq 1 \) for \( \vec{e} = u \to v \in \omega \). If \( h_\omega(u) - h_\omega(v) = 1 \), then \( f_C(e) = 0 \), which implies that \( \vec{e} \notin \omega \) for all \( \omega' \in C \), which contradicts with the case \( \omega' = \omega \) itself. If \( h_\omega(u) - h_\omega(v) = 0 \), then \( f_C(e) = 1 \), which implies that \( \vec{e} \in \omega \) for all \( \omega' \in C \). In other words, the direction of the edge \( \vec{e} \) has never changed in all orientations in \( C \), so neither \( u \) nor \( v \) has ever been flipped in \( C \), otherwise it will change the direction of \( \vec{e} \). As a result, we can assume \( u \) and \( v \) are frozen vertices from the start, so \( f_\omega(u) = f_\omega(v) = 0 \).

For (4), for any vertex \( v_j \neq v_i, v_{-i} \), since we never flipped \( v_j \) or \( v_{-j} \), the path \( P \) between \( v_j \) and \( v_{-j} \) (or a frozen vertex \( u \)) has the same circulation in \( \omega \) and \( \omega' \) (see the Uniqueness section). As a result, \( h_\omega(v_j) = h_\omega(v_j) \). Now assume \( v_k \) is adjacent to \( v_i \), and by Equation (4), since we flipped \( v_i \), the evaluation of \( h_\omega(v_i) \) changes by 1. The same for \( h_\omega(v_{-i}) \). (5) is obvious from (4).

For (6), assume \( h_\omega = h_\omega' \). Property (3) tells us that the orientation of most edges \( e = (u, v) \) are known by comparing \( h_\omega(u) \) and \( h_\omega(v) \), unless \( h_\omega(u) = h_\omega(v) = 0 \). In that case, the direction of the edge \( \vec{e} \) has never changed in all orientations in \( C \), which implies that \( \omega = \omega' \).

\[\Box\]

**Proposition 6.12.** For any \( \omega, \omega' \in C \), there exists a path \( \omega = \omega_0, \omega_1, \ldots, \omega_n = \omega' \) in \( C \) such that only the vertices in the set \( \{v : h_\omega(v) \neq h_\omega'(v)\} \) are flipped.

*Proof.* We induct on the value \( \Delta(\omega, \omega') := \sum_{v \in V(G)} |h_\omega(v) - h_\omega'(v)| \). This value is an integer because of property (5). When \( \Delta(\omega, \omega') = 0 \) we have \( h_\omega = h_\omega' \) so \( \omega = \omega' \). Assume \( \Delta(\omega, \omega') > 0 \). Assume \( v \) is the vertex satisfying \( h_\omega(v) \neq h_\omega'(v) \) and maximized \( \max\{h_\omega(v), h_\omega'(v)\} \). WLOG assume \( h_\omega(v) > h_\omega'(v) \). Obviously \( h_\omega(v) > 0 \) (otherwise replace \( v = v_i \) with \( v_{-i} \)). We claim that \( v \) is a source in \( \omega \). Otherwise there exists some
Proposition 6.12, we can find a path from $\omega$. As a result $h_\omega(u) > h_\omega(v)$. By maximality, we must have $h_\omega(u) = h_\omega'(u)$, but then $h_\omega'(u) - h_\omega'(v) > h_\omega(v) - h_\omega'(v) \geq 1$ by (5), which contradicts with (3).

Now we can flip $v$ in $\omega$ to obtain $\omega''$, and the value of $\Delta(\omega'', \omega')$ decreases. By induction hypothesis, there is a path between $\omega'', \omega'$, which concludes the proof. \qed

Corollary 6.13. For any $\omega, \omega' \in \mathcal{G}(\Gamma - j) \cap \mathcal{C}$, they belong to the same connected component in $\text{Comp}(\mathcal{G}(\Gamma) \cap \mathcal{C})$ if and only if $h_\omega(v_j) = h_\omega'(v_j)$.

Proof. Because of Proposition 6.12, we can find a path from $\omega$ to $\omega'$ that avoids flipping $v_j$, so the path lives in $\mathcal{G}(\Gamma_j) \cap \mathcal{C}$. \qed

Corollary 6.14. For any $\omega, \omega' \in \mathcal{G}(\Gamma - j) \cap \mathcal{C}$, they belong to the same connected component in $\text{Comp}(\mathcal{G}(\Gamma - j) \cap \mathcal{C})$ if and only if $h_\omega(v_j) = h_\omega'(v_j)$.

Proof. $h_\omega(v_j) = h_\omega'(v_j)$ implies that $h_\omega(u) = h_\omega'(u)$ for every $u \in N(j)$. Because of Proposition 6.12, we can find a path from $\omega$ to $\omega'$ that avoids flipping $v_j$ and all vertices in $N(j)$, so the path lives in $\mathcal{G}(\Gamma - j) \cap \mathcal{C}$. \qed

Proof of Lemma 6.9. Assume $h_\omega(v_j)$ can hold $d$ different values $N_1 < N_2 < \cdots < N_d$ over all $\omega \in \mathcal{C}$. Then by Corollary 6.13 the number of connected components $\#\text{Comp}(\mathcal{G}(\Gamma_j) \cap \mathcal{C}) = d$. Now we claim that $h_\omega(v_j)$ can hold $d - 1$ different values $N_2, \ldots, N_d$ over all $\omega \in \mathcal{G}(\Gamma - j) \cap \mathcal{C}$. First, $h_\omega(v_j) \neq N_1$. Otherwise, we can flip $(v_j, v_{j-1})$ to get $\omega'$, but $h_\omega'(v_j) = N_1 - 1 < N_1$, which contradicts the fact that $N_1$ is the minimal possible value. On the other hand, for any $2 \leq i \leq d$, there exists a path $\omega_0, \ldots, \omega_n \in \mathcal{C}$ such that $h_{\omega_0}(v_j) = N_i$ and $h_{\omega_n}(v_j) = N_1$. Find the last orientation $\omega_k$ in the path such that $h_{\omega_k}(v_j) = N_i$. Then $v_j$ must be a source in $\omega_k$, which implies that $\omega_k \in \mathcal{G}(\Gamma - j) \cap \mathcal{C}$. This concludes the proof of the claim. By Corollary 6.14, we have $\#\text{Comp}(\mathcal{G}(\Gamma_j) \cap \mathcal{C}) = d - 1$. Therefore, Equation (3) holds. \qed

Remark 6.15. The height function $h_\omega$ in Proposition 6.11 is inspired by James Propp’s construction of height functions for ordinary type A graphs in [Pro02]. In Propp’s paper, he used the height function to show that there is a poset structure on each equivalence class of $\text{Acyc}(\mathcal{G})/\sim$, which is defined by $\omega \succ \omega'$ if we flip source $i(\neq u)$ to sink $i$ ($u$ is a given fixed vertex). Moreover, the poset is a lattice. As a result, each of the lattice has a unique maximum element: acyclic orientation of $\mathcal{G}$ with a unique sink $u$. This maximum element gives us a representative object for each equivalence class in $\text{Acyc}(\mathcal{G})/\sim$ in the type A case, which helps to set up concrete bijections. However, all the methods we just described do not work for type B symmetric graphs, because there is not an obvious poset structure, and even if we force a poset structure, the poset will not be a lattice and will not have a unique representative object. Therefore, it will be interesting to know if one can find a well-defined representative object through some other methods for each equivalence class in $\text{Acyc}(\mathcal{G})/\sim$ in the type B case, in order to have a bijective proof of Lemma 6.9.

Remark 6.16. Another open question raised by Alex Postnikov is, whether we can find a Lie theoretic analog of Theorem 2.10. In other words, can we find a way to prove Theorem 2.10 for both ordinary graphs and symmetric graphs simultaneously, while potentially generalizing to other Lie types.
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