Continuum models of discrete heterogeneous structures and saturated porous media: constitutive relations and invariance of internal interactions

G L Brovko, A G Grishayev and O A Ivanova
Theory of Elasticity Dept, Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Leninskiye Gory, Moscow, 119992, RUSSIA
E-mail: glb@mech.math.msu.su, glb@dataforce.net

Abstract. Approaches to mechanical (constructive) modeling of nonhomogeneous media with complicated structure are presented. The constitutive and motion equations of special type discrete systems are obtained as average equations demonstrating the Cosserat type properties of the systems. The equations for saturated porous media are proposed with special attention to the different types of inner interactions. The invariance properties of these interactions and their quantitative contributions are analyzed using the principle of material objectivity the methods of measurement theory. The invariances of compound (force, moment) internal interactions are studied.

1. Introduction

Classic continuum mechanics was initiated by European scientists during the 18th-19th centuries and achieved its self-dependent theoretical status in the 20th century with the appearance of the general theory of constitutive relations. Present-day views in classic continuum mechanics, mechanics of solids and the theory of elasticity are reflected in modern courses and investigations based on general positions of contemporary scientific literature have achieved classic status as fundamental problems (see, for example, [1–14]) and in special studies (see [15–25]).

In the 19th century, attempts were made to describe material structure from beyond the classical perspective. The first well-composed theoretical realization was given in the famous work by the Cosserat brothers [26] and intensively developed in different theories of moment type during 1950–60s and later [27–42]. The development of moment theories of Cosserat type media including pseudo-continua concentrated mainly on mathematical questions and was accompanied, as a rule, only by indirect experiments and did not touch upon real examples and concrete applications.

At the same time interest rose in inhomogeneous structures (composites, mixtures, conglomerates) with interacting phases. Since then different theories for such structures including saturated porous media mainly based on the concept of interpenetrative continua were developed, concrete static and dynamic problems were solved [43–56], and special attention was paid to the important problem of invariance of interaction forces [55, 56]. A variety of natural and artificial porous materials, different regimes of gas and fluid flow through filters, ground layers, perforated and lattice structures demonstrate the importance of such investigations from theoretical and practical points of view.
In the present paper, the method of mechanical (constructive) modeling proposed in [57] is used for building mathematical continuum models of discrete and heterogeneous structures. The method consists of detailed examination of mechanical properties of a representative element of the structure and derivation of the appropriate continuum relations characterizing, with sufficient precision, constitutive properties and motion laws of the element. In application to discrete constructions of special type and saturated porous media, the method demonstrates the realization of the interpenetrative continua concept. The results obtained on the base of approaches developed in [51, 57–59] are presented for Cosserat type structures [60–62] and saturated porous media [63–66] together with some new results.

1.1. Models of Cosserat type

The usage of the method for one- and two-dimensional special type discrete constructions leads to the averaged continuum forms of constitutive and motion equations which coincide with the corresponding equations of a Cosserat point. All Cosserat continuum kinematic and dynamic characteristics get their direct clear mechanical sense in terms of the initial discrete models. The cases of uncoupled (disconnected) and momentless (without moment stresses) free Cosserat models as well as constrained pseudo-continuum models are distinguished. The features of motions and equilibrium forms of the obtained models are examined. In particular, the simple examples of one-dimensional continuum models with elastic properties demonstrate the presence of exactly two forms and two respective frequencies in each mode of free vibration; the case of forced vibrations (e.g. under steady air- or fluid-flow) shows the possibility of disturbing the stability of vibrations and transition to the regime of divergence. The models with elastic – perfect plastic properties possess an infinite number of equilibrium forms (in a certain range).

Thus the building of Cosserat continuum models from appropriate discrete structures through the indicated method of mechanical (constructive) modeling, on the one hand, permits to immediate clarification of the mechanical sense of all Cosserat characteristics and, on the other hand, it demonstrates the principal availability for real existence (naturally or artificially) of Cosserat type continua and drops a hint at possible ways to technologically manufacture them.

1.2. Modeling of saturated porous media

Using the method of mechanical modeling, the new model of saturated porous media was built. In contrast to Biot’s one it is able to circumscribe deformations of a skeleton at arbitrary strains (in the Lagrangean description) and arbitrary motions of the fluid (gas) through the skeleton (in the Eulerian description). The model is characterized by specific volume values (regulated with continuity equations) of solid (skeleton) and liquid (fluid) components, by mixed type of constitutive equations of the components and by specific force (and moment) interactions between the components. The new model clarifies the roles played by Biot’s constants by restricting it to small motions.

Also the new model provides a new list of internal interactive forces (between skeleton and fluid) including the static one and three types of dynamic forces: the front-face force, the viscous drag force (of Darcy’s type) and inertial resistance force (of Biot’s type).

1.3. Analysis of interactive forces in porous media

Investigation of the interactive forces using the material objectivity principle leads to general reduced forms. The inertial resistance force in the new model is more specialized than the corresponding force in Biot’s model (the Biot’s force does not satisfy the objectivity principle) and coincides with Biot’s model only for one-dimensional dynamics (e.g. longitudinal wave propagation).

Additional examination of interactive forces with the methods of measurement theory gives more detailed information disclosing the quantitative features of the dependence of forces on
the kinematic characteristics of mutual motion of components. It permits estimation of the comparative effects of different types of forces in different flow regimes (the corresponding table is presented).

1.4. Study of compound internal interactions

Analogously to the previous, the invariance properties of composite (force and moment) internal interactions are studied by means of the objectivity principle in more complicated cases when forces and moments depend on relative displacements (speed, acceleration) and mutual rotations (vortices). The reduced forms are obtained for the cases of solid-solid and solid-fluid aggregates. The results for saturated porous media show that the fluid-flow may act upon the skeleton not only with front-face force, but with shifting (“lifting”) force as well as with overturning and rotating (“screw”) moments.

2. Method of mechanical (constructive) modeling in building Cosserat point models

The method consists of the following steps:

a) detailed analysis of the structure of a medium with complicated properties and elaboration of the appropriate discrete model,

b) derivation of motion and constitutive equations for the discrete model,

c) obtaining the continuum forms of equations by an averaging procedure.

In practice, step a) is the most creative, step b) is mainly technical and not so hard in its realization, while step c) usually based on certain assumptions and an averaging procedure of the appropriate type is the most difficult to explore (and may lead to different results depending on the assumptions).

We shall illustrate the scheme using the example of one-dimensional Cosserat models concentrating attention on one model of a supplied beam.

2.1. Model of a supplied Cosserat beam

2.1.1. Discrete construction. Let us consider plane-parallel bending-tension motions of a homogeneous beam with constant section-area supplied by identical rigid massive inclusions (in the form of wheels) periodically placed along the longitudinal line of the beam on aces (spindles) perpendicular to the bending plane of the beam. Let the inclusions be capable of rotation around their spindles and let them be connected with the nearby carrying elements of the beam by elastic joints as well as connected with their nearest neighbours by elastic belt drives through transmissive weightless pulleys.

The initial (undeformed) and deformed configurations of the supplied beam in its motion (in-plane bending-tension of the supporting beam and rotations of rigid massive inclusions) are schematically shown on Fig 1. We shall use the designation $\xi$ for the $x$ coordinate of the longitudinal line of the supporting beam in the undeformed configuration: $\xi$ is a Lagrangean (“material”) coordinate.

A cell of the construction is shown in Fig 1. Its image in a deformed configuration of the supplied beam is shown in Fig 2 together with forces and moments acting on its elements.

2.1.2. Equations of the discrete model. The kinematic characteristics of the cell are the following: $\mathbf{r}$ — radius-vector of the center-point of the cell, $\mathbf{r}^+$ and $\mathbf{r}^-$ — radius-vectors of the right and the left edges of the beam-element of the cell, $\lambda^+$ and $\lambda^-$ — elongation factors of the right and the left beam-elements of the cell, $\varphi_{\text{beam}}$ — absolute (with respect to $x$ axis) rotation angle of the beam-element tangent at the center-point, $\varphi_{\text{beam}}^+/2$ and $\varphi_{\text{beam}}^-/2$ — angles between the center-point tangent and the tangents at the edges, $\varphi_{\text{incl}}$ — absolute rotation angle of the
Figure 1. Supplied beam in pre-deformed (a) and deformed (b) configurations. Massive inclusions are shown as dark circles, transmissive weightless pulleys as transparent circles, belt drive as a dashed line. A cell of the construction is picked out with a chain line rectangle.

Figure 2. Deformed cell: external actions and internal moments.

massive inclusion, $\varphi_{incl}^+/2$ and $\varphi_{incl}^-/2$ — rotation angles of the right and the left transmissive weightless pulleys relative to the massive inclusion rotation.

External (relative to the cell) forces and moments are formed by the ones external to the whole system and internal forces and moments of the system playing the external role relative to the cell. The first type external actions are: $\mathbf{b}$ — integrated vector of external forces acting on the cell, $m_{beam}$ — external moment acting on the beam-element of the cell, $m_{incl}$ — external moment acting on massive inclusion. The second type external actions are: $\mathbf{F}^+$ and $\mathbf{F}^−$ — vectors of forces acting on the right and the left edges of the cell-beam from the remained parts of the construction, $M_{beam}^+$ and $M_{beam}^−$ — moments acting on the edges of the cell-beam from the remained parts of the construction, $M_{incl}^+$ and $M_{incl}^−$ — moments acting on the right and
the left transmissive weightless pulleys from the inclusions of the neighbour cells.

Inner interactions in the cell are: \( P^+ \) and \( P^- \) — tension resistance forces of the right and the left parts of the cell-beam, \( M^\text{bend}_\text{incl} \) and \( M^\text{bend}_\text{int} \) — bending resistance moments of the right and the left parts of the cell-beam, \( M^\text{int}_\text{incl} \) and \( M^\text{int}_\text{int} \) — moments acting from the right and the left transmissive weightless pulleys on the massive inclusion of the cell, \( M^*\text{incl} \rightarrow \text{beam} \) — moment acting from the massive inclusion on the cell-beam.

Then designating the mass of the cell by \( m_\text{cell} \) and the moment of inertia of the massive inclusion by \( J^\text{incl} \), taking into account inertial terms of the translation motion of the cell and the rotation motion of the inclusion and neglecting the inertial terms of tension and rotation of the beam-elements and belt drives, we obtain the expression of the virtual work of the cell:

\[
\delta A = b\delta r + F^+\delta r^+ - F^-\delta r^- +
M^\text{beam}_\text{incl}\delta (\varphi^\text{beam} + \Delta\varphi^\text{beam}_\text{incl}) - M^\text{beam}_\text{int}\delta (\varphi^\text{beam} - \Delta\varphi^\text{beam}_\text{incl}) +
+m^\text{beam}\delta\varphi^\text{beam} + m^\text{incl}\delta\varphi^\text{incl} - P^+\frac{a}{2}\delta\lambda^+ - P^-\frac{a}{2}\delta\lambda^- -
-M^\text{bend}_\text{incl}\delta\varphi^\text{beam} - M^\text{bend}_\text{int}\delta\varphi^\text{beam} - M^*\text{incl} \rightarrow \text{beam}\delta\varphi^\text{incl} +
+ (M^\text{incl}_\text{incl} - M^\text{incl}_\text{int}) \frac{\delta}{\delta \varphi} (\varphi^\text{incl} + \Delta\varphi^\text{incl}_\text{incl}) -
\frac{\delta}{\delta \varphi} (\varphi^\text{incl} - \Delta\varphi^\text{incl}_\text{incl}) +
+ (M^\text{int}_\text{incl} - M^\text{int}_\text{int}) \delta\varphi^\text{incl} - m_\text{cell}\ddot r\delta r - J^\text{incl}\ddot\varphi^\text{incl}\delta\varphi^\text{incl}.
\]

Using the principle of a virtual work and the notation for specific (linear) external actions

\[
\hat f := \frac{b}{a}, \quad \bar m := \frac{m^\text{beam}}{a}, \quad \bar m^\text{incl} := \frac{m^\text{incl}}{a},
\]

for specific moment interaction, mass density and moment of inertia of inclusion

\[
\bar M^\text{incl} \rightarrow \text{beam} := \frac{M^*\text{incl} \rightarrow \text{beam}}{a}, \quad \bar \rho := \frac{m_\text{cell}}{a}, \quad \bar J := \frac{J^\text{incl}}{a}
\]

as well as the notation

\[
\bar A := \frac{A^+ + A^-}{2}, \quad \frac{\delta A}{\delta \xi} := \frac{A^+ - A^-}{a}
\]

for average values and for difference quotients of quantities \( A^+ \) and \( A^- \) (and \( T^+ := F^+ \cdot e, T^- := F^- \cdot e, Q^+ := F^+ \cdot n, Q^- := F^- \cdot n \)), we obtain the equations for the discrete model.

These equations include the system of dynamic equilibrium equations

\[
\frac{\delta F}{\delta \xi} + \hat f - \bar \rho \hat r = 0,
\]

\[
\frac{\delta M^\text{beam}}{\delta \xi} + \bar Q \lambda + \bar m + \bar M^\text{incl} \rightarrow \text{beam} = 0,
\]

\[
\frac{\delta M^\text{incl}}{\delta \xi} + \bar m^\text{incl} - \bar M^\text{incl} \rightarrow \text{beam} - \bar J \ddot \varphi^\text{incl} = 0
\]

and the equations of a "smooth coincidence" between external and internal (determined by constitutive equations) force parameters

\[
\bar T = \bar \dot P, \quad \bar M^\text{beam} = \bar M^\text{bend}, \quad \bar M^\text{incl} = \bar M^\text{int},
\]

The theorem of kinetic energy

\[
\hat W(e) = \hat K + \hat W(i)
\]
leads (while eqns (2.5), (2.6) hold) to expressions of specific kinetic energy
\[ \ddot{K} = \frac{1}{2} \ddot{\rho} |\dot{r}|^2 + \frac{1}{2} \ddot{J} \dot{\varphi}_{\text{incl}}^2 \] (2.8)
and specific power of internal forces
\[ \ddot{W}_{(i)} = \dot{P} \cdot \dot{\lambda} + \dot{M}_{\text{bend}} \cdot \left( \frac{\partial \varphi_{\text{beam}}}{\partial \xi} \right) + \dot{M}_{\text{int}} \cdot \left( \frac{\partial \varphi_{\text{incl}}}{\partial \xi} \right) + \dot{M}_{\text{incl} \rightarrow \text{beam}} \cdot (\varphi_{\text{incl}} - \varphi_{\text{beam}}). \] (2.9)
Eqn (2.9) marks out the pairs of conjugate internal force and kinematic parameters
\[ \dot{P} \sim \lambda, \quad \dot{M}_{\text{bend}} \sim \frac{\partial \varphi_{\text{beam}}}{\partial \xi}, \quad \dot{M}_{\text{int}} \sim \frac{\partial \varphi_{\text{incl}}}{\partial \xi}, \quad \dot{M}_{\text{incl} \rightarrow \text{beam}} \sim (\varphi_{\text{incl}} - \varphi_{\text{beam}}) \] (2.10)
and shows that the constitutive equations of the cell should be represented \([1, 3, 10]\) by the
dependence of the set \(S = (\dot{P}, \dot{M}_{\text{bend}}, \dot{M}_{\text{int}}, \dot{M}_{\text{incl} \rightarrow \text{beam}})\) of internal generalized forces on the
retrospective history of the set \(E = \left( \lambda, \frac{\partial \varphi_{\text{beam}}}{\partial \xi}, \frac{\partial \varphi_{\text{incl}}}{\partial \xi}, (\varphi_{\text{incl}} - \varphi_{\text{beam}}) \right)\) of internal kinematic
parameters:
\[ S(t) = \mathcal{F} \left[ E'(s) \right]_{s \geq 0}, \] (2.11)
where \(E'(s) := E(t - s)\) is the pre-history of the process \(E(\tau)\) with \(\tau \leq t\).
The analysis of objectivity types \([58, 66]\) of the parameters in (2.10) shows that the principle
of material frame-indifference \([1, 3]\) does not restrict the relation (2.11), and, in accordance with
the principle of macroscopic determinability \([10]\), the relation (2.11) is the general reduced form
of the constitutive relation of the cell.
The simplest variant of constitutive equations (2.11)
\[ \dot{P} = C_{\text{tens}}(\lambda - 1), \quad \dot{M}_{\text{bend}} = C_{\text{bend}} \frac{\partial \varphi_{\text{beam}}}{\partial \xi}, \quad \dot{M}_{\text{int}} = C_{\text{incl}} \frac{\partial \varphi_{\text{incl}}}{\partial \xi}, \quad \dot{M}_{\text{incl} \rightarrow \text{beam}} = C_{\text{incl} \rightarrow \text{beam}} \left( \varphi_{\text{incl}} - \varphi_{\text{beam}} \right) \] (2.12)
describes elastic properties of the construction: \(C_{\text{tens}}, C_{\text{bend}}, C_{\text{incl}}, C_{\text{incl} \rightarrow \text{beam}}\) are the constants
of elastic resistance to tension, bending, mutual rotations of inclusions, and rotations of
inclusions relative to the beam.
In particular, at small extension of the beam when \(\lambda - 1 =: \varepsilon \ll 1\) and thus \(\frac{\partial \varphi_{\text{beam}}}{\partial \xi} = \kappa(1 + \varepsilon) \equiv \kappa\) (\(\kappa\) is the curvature of the supporting beam line), the second of the relations (2.12)
turns into the classic form \([15, 20, 22, 67]\): \(\dot{M}_{\text{bend}} = C_{\text{bend}} \kappa\).

### 2.1.3. Continuum equations.
In the long-wave approximation, the averaging procedure as \(a \rightarrow 0\) applied to (2.5), (2.6) leads to the system
\[ \frac{\partial \mathbf{F}}{\partial \xi} + \mathbf{f} - \rho \ddot{\mathbf{r}} = 0, \]
\[ \frac{\partial M_{\text{beam}}}{\partial \xi} + Q \lambda + m + M_{\text{incl} \rightarrow \text{beam}} = 0, \]
\[ \frac{\partial M_{\text{int}}}{\partial \xi} + m_{\text{in}} - M_{\text{incl} \rightarrow \text{beam}} - J \dot{\varphi}_{\text{incl}} = 0 \] (2.13)

\[ T \equiv P, \quad M_{\text{beam}} \equiv M_{\text{bend}}, \quad M_{\text{incl}} \equiv M_{\text{int}} \] (2.14)
(with analogous designations for quantities evaluated in the limit).
The power of internal forces (2.9) takes the form
\[
W_i = P \cdot \dot{\lambda} + M_{\text{bend}} \cdot \left( \frac{\partial \varphi_{\text{beam}}}{\partial \xi} \right) + M_{\text{int}} \cdot \left( \frac{\partial \varphi_{\text{incl}}}{\partial \xi} \right) + M_{\text{incl} \rightarrow \text{beam}} \cdot (\varphi_{\text{incl}} - \varphi_{\text{beam}}),
\]
and the constitutive equations (2.12) are
\[
P = C_{\text{tens}}(\lambda - 1), \quad M_{\text{bend}} = C_{\text{bend}} \frac{\partial \varphi_{\text{beam}}}{\partial \xi}, \quad M_{\text{int}} = C_{\text{incl}} \frac{\partial \varphi_{\text{incl}}}{\partial \xi},
\]
\[
M_{\text{incl} \rightarrow \text{beam}} = C_{\text{incl} \rightarrow \text{beam}} (\varphi_{\text{incl}} - \varphi_{\text{beam}}).
\]

Eqs (2.13) express the conditions of the dynamic equilibrium of a one-dimensional Cosserat continuum (in this plane case).

In the general case, the substitution of the constitutive equations (2.16) into (2.13) (with (2.14)) leads to the system of one vector and two scalar equations for unknown functions \( r, \varphi_{\text{incl}}, Q \) (\( \lambda \) and \( \varphi_{\text{beam}} \) are expressed through \( r \)).

2.1.4. Particular and special models. The particular cases are the following.

Disconnected, or uncoupled model. In the case of absence of interactions between the beam and inclusions, i.e. when \( M_{\text{incl} \rightarrow \text{beam}} = 0 \), the first two equations (2.13) describe the motion of the beam (weighted by inclusions) and the third one independently describes dynamical rotation of the inclusions. The first two equations of the disconnected model coincide with known equations of the classic theory of beams [15,20,22,67].

Momentless model. Here \( M_{\text{incl}} \equiv M_{\text{int}} \equiv 0 \) (for \( C_{\text{incl}} = 0 \)). The “belt connections” vanish, but \( M_{\text{incl} \rightarrow \text{beam}} \) and \( \varphi_{\text{incl}} \) must not be equal to 0.

Pseudo-continuum (constrained model). The inner kinematic constrain \( \varphi_{\text{incl}} \equiv \varphi_{\text{beam}} \) eliminates \( \varphi_{\text{incl}} \) from the set of decision variables, and the reaction (“supporting force”) \( M_{\text{incl} \rightarrow \text{beam}} \) becomes undetermined by constitutive equations (it is to be determined using the whole system of equations).

2.2. Small motions of a supplied beam

2.2.1. Linearization of equations. Let displacements \( u \) and \( w \), specific elongation \( \varepsilon = \lambda - 1 \) and the angles of the beam elements \( \varphi_{\text{beam}} \) be small and the approximate equalities hold:
\[
\xi \cong x, \quad \frac{\partial}{\partial \xi} \cong \frac{\partial}{\partial x}, \quad \lambda \cong 1, \quad \varphi_{\text{beam}} \cong \frac{\partial w}{\partial x}, \quad \kappa \cong \frac{\partial \varphi_{\text{beam}}}{\partial x} \cong \frac{\partial^2 w}{\partial x^2}.
\]

Let the value ranges of \( \varepsilon, \kappa, \frac{\partial \varphi_{\text{incl}}}{\partial x}, \varphi_{\text{incl}} - \varphi_{\text{beam}} \) be small enough to permit linear constitutive equations of the elastic type (2.16) (see (2.17)) as follows (in classic notation):
\[
P = E S_{\text{cross}} \varepsilon, \quad M_{\text{bend}} = E J_{\text{cross}} \kappa, \quad M_{\text{int}} = C \frac{\partial \varphi_{\text{incl}}}{\partial \xi},
\]
\[
M_{\text{incl} \rightarrow \text{beam}} = K (\varphi_{\text{incl}} - \varphi_{\text{beam}}),
\]
where \( E \) is the Young module of the beam material, \( S_{\text{cross}} \) is the area, \( J_{\text{cross}} \) is the moment of inertia of the cross-section of the beam, \( C \) is the stiffness coefficient of the belt drive and \( K \) is the elasticity constant of hinging (between inclusions and the beam).

Then the substitution of (2.18) in (2.13) leads to the equations
\[
E S_{\text{cross}} \frac{\partial^2 u}{\partial x^2} + f_x = \rho \frac{\partial^2 u}{\partial t^2},
\]
\[ \frac{\partial Q}{\partial x} + f_y = \rho \frac{\partial^2 w}{\partial t^2}, \]  
(2.20)

\[ E J_{\text{cross}} \frac{\partial^3 w}{\partial x^3} + m + Q + K \left( \varphi_{\text{incl}} - \frac{\partial w}{\partial x} \right) = 0, \]  
(2.21)

\[ C \frac{\partial^2 \varphi_{\text{incl}}}{\partial x^2} - K \left( \varphi_{\text{incl}} - \frac{\partial w}{\partial x} \right) + m_{\text{in}} = J \frac{\partial^2 \varphi_{\text{incl}}}{\partial t^2}. \]  
(2.22)

Here the external forces \( f_x, f_y \) and external moments \( m, m_{\text{in}} \) are considered as known. Hence the four equations (2.19)-(2.22) are a system for four unknown functions \( u, w, \varphi_{\text{incl}}, Q \) with arguments \( x \) and \( t \).

Eqn (2.19) is obviously independent of (2.20)-(2.22): it describes the longitudinal motion (longitudinal waves with expansion velocity
\[ c = \sqrt{\frac{S_{\text{cross}} E}{\rho}}. \]

Therefore, let us consider only (2.20)-(2.22).

Eliminating \( Q \) from (2.20)-(2.22) gives a system of two partial differential equations for two functions \( w \) and \( \varphi_{\text{incl}} \):

\[ -E J_{\text{cross}} \frac{\partial^4 w}{\partial x^4} + K \frac{\partial^2 w}{\partial x^2} - K \frac{\partial \varphi_{\text{incl}}}{\partial x} - \frac{\partial m}{\partial x} + f_y = \rho \frac{\partial^2 w}{\partial t^2}, \]

\[ C \frac{\partial^2 \varphi_{\text{incl}}}{\partial x^2} - K \varphi_{\text{incl}} + K \frac{\partial w}{\partial x} + m_{\text{in}} = J \frac{\partial^2 \varphi_{\text{incl}}}{\partial t^2}. \]  
(2.23)

For the disconnected model \((K = 0)\) the system (2.23) simplifies because the equations decouple. The first one is the equation for beam deflection \( w \) and agrees with the well-known Zhuravskiy equations [20] of the classic theory (taking into account the mass of inclusions). The second equation describes the rotations of the array of massive inclusions under the external linear-specific moment \( m_{\text{in}} \), and in the case of \( C = 0 \) (momentless model) the disk rotations are independent of one another.

2.2.2. Free vibrations. Excluding the consideration of longitudinal movement (eqns (2.23)) let us turn to the natural free linear vibrations (bending and rotations) of the system in the general case (when \( K \neq 0 \) and \( C \neq 0 \)) in the absence of external forces and moments: \( f_y = 0, m = 0 \) and \( m_{\text{in}} = 0 \).

Let the boundary conditions be as follows (pinning of the edges \( x = 0 \) and \( x = l \), the ends of the beam and end inclusions are free from moment actions):

\[ w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial \varphi_{\text{incl}}}{\partial x} = 0. \]  
(2.24)

Finding the solution to the problem (2.23), (2.24) in the form

\[ w = C_w(x) e^{i\omega t}, \quad \varphi_{\text{incl}} = C_\varphi(x) e^{i\omega t}, \]  
(2.25)

where \( C_w(x) \) and \( C_\varphi(x) \) are the amplitude functions, and \( \omega \) is the angular (radian) frequency, one obtains the system of ordinary differential equations:

\[ E J_{\text{cross}} \frac{d^4 C_w(x)}{dx^4} - K \frac{d^2 C_w(x)}{dx^2} + K \frac{d^4 C_w(x)}{dx^4} = \rho \omega^2 C_w(x), \]

\[ -C \frac{d^2 C_\varphi(x)}{dx^2} + K C_\varphi(x) - K \frac{d^4 C_w(x)}{dx^4} = J \omega^2 C_\varphi(x). \]  
(2.26)

Consider amplitude functions (satisfying the boundary conditions (2.24)) of the form

\[ C_w(x) = A \sin px, \quad C_\varphi(x) = B \cos qx + D, \]  
(2.27)

1 Remind that \( \rho \) denotes the linear mass density of the supplied beam.
where \( A, B \) and \( D \) are arbitrary constants, while \( p \) and \( q \) are expressed by equalities

\[
p = \frac{\pi k}{l}, \quad q = \frac{\pi m}{l} \quad (k, m \in \mathbb{N}),
\]

Eqns (2.26) yield

\[
E J_{\text{cross}} A p^4 \sin px + K A p^2 \sin px - K B q \sin qx = \rho \omega^2 A \sin px, \\
C B q^2 \cos qx + K B \cos qx + K D - K A p \cos px = J \omega^2 B \cos qx + J \omega^2 D.
\]

Eqn (2.29) hold identically if \( p = q \) and \( D = 0 \) and (2.29) turns into the system of two linear algebraic equations for constants \( A \) and \( B \):

\[
(-E J_{\text{cross}} p^4 - K p^2 + \rho \omega^2) A + K p B = 0, \\
K p A + (-C p^2 - K + J \omega^2) B = 0.
\]

(2.30)

The system (2.30) has a non-trivial solution if and only if its determinant is equal to zero:

\[
\det \begin{pmatrix}
-E J_{\text{cross}} p^4 - K p^2 + \rho \omega^2 & K p \\
K p & -C p^2 - K + J \omega^2
\end{pmatrix} = 0,
\]

that is

\[
\rho J \omega^4 - (E J_{\text{cross}} p^4 + (\rho C + K J) p^2 + K \rho) \omega^2 + \rho J (\rho C + K J) p^2 + \rho^2 K^2 = 0.
\]

(2.31)

The eqn (2.31) is biquadratic in the angular frequency \( \omega \). Considering it as a quadratic equation for \( \omega^2 \) we have the expression of its discriminant:

\[
D(p) = (E J_{\text{cross}} p^4)^2 - 2 E J_{\text{cross}} (\rho C - K J) p^2 + \left[ (\rho C - K J)^2 - 2 \rho K J E J_{\text{cross}} \right] p^4 + 2 \rho K (\rho C + K J) p^2 + \rho^2 K^2.
\]

(2.32)

For the discriminant to be non-negative\(^2\) both roots of this quadratic equation must be real and positive:

\[
\omega_{1,2}^2(p) = \frac{E J_{\text{cross}} p^4 + (\rho C + K J) p^2 + K \rho \pm \sqrt{D(p)}}{2 \rho J}
\]

(2.33)

hence, the solutions to the equation (2.31) are the real nonzero numbers \( \pm \omega_1(p), \pm \omega_2(p) \) (let us agree that \( \omega_1(p) > 0, \omega_2(p) > 0 \) and \( \omega_1(p) \leq \omega_2(p) \)).

If \( K \neq 0 \) (general case), then, for each \( p \) of the form (2.28) and each corresponding (by (2.33)) one of two obtained values of \( \omega \), the non-trivial solution of the system (2.30) for \( A \) and \( B \) is

\[
A = a(p, \omega) \cdot \Phi, \quad B = \Phi,
\]

(2.34)

where \( \Phi \) is an arbitrary (nonzero) constant, and

\[
a(p, \omega) = \frac{K p}{E J_{\text{cross}} p^4 + K p^2 - \rho \omega^2} = \frac{C p^2 + K - J \omega^2}{K p}.
\]

(2.35)

\(^2\) A negative discriminant leads to imaginary values of \( \omega \), so the solution (2.25) exponentially decreases or increases in time. Such behavior has no place in conservative (linearly elastic) systems.
where the signs + or − as well as the integer \( j \) match in both equalities, \( \Phi_j \) is an arbitrary nonzero constant, \( p \) is any value of (2.28), \( \omega_j = \omega_j(p) \) and \( a_j = a(p, \omega_j) \) are determined by the formulae (2.33) and (2.35).

Due to the linearity of the problem (2.23), (2.24) each linear combination of pairs of functions (2.36) with constant coefficients is a solution too. Every possible real-valued combination is a pair of sums (with natural \( k \)):

\[
\begin{align*}
    w &= \sum_k \sin \frac{\pi k x}{l} \left[ a_1^{(k)} \Phi_1^{(k)} \sin (\omega_1^{(k)} t + \varphi_1^{(k)}) + a_2^{(k)} \Phi_2^{(k)} \sin (\omega_2^{(k)} t + \varphi_2^{(k)}) \right], \\
    \varphi_{\text{incl}} &= \sum_k \cos \frac{\pi k x}{l} \left[ \Phi_1^{(k)} \sin (\omega_1^{(k)} t + \varphi_1^{(k)}) + \Phi_2^{(k)} \sin (\omega_2^{(k)} t + \varphi_2^{(k)}) \right],
\end{align*}
\]

(2.37)

where \( \Phi_j^{(k)}, \varphi_j^{(k)} \) are arbitrary constants \((j = 1, 2)\), and, using (2.33), (2.35),

\[
\omega_j^{(k)} = \omega_j \left( \frac{\pi k}{l} \right), \quad a_j^{(k)} = a \left( \frac{\pi k}{l}, \omega_j^{(k)} \right) \quad (j = 1, 2).
\]

(2.38)

The solutions (2.37) are either finite sums or infinite series for which uniform convergence is provided by the convergence of the majorant series

\[
\sum_k |\Phi_j^{(k)}|, \quad \sum_k |a_j^{(k)} \Phi_j^{(k)}| \quad (j = 1, 2).
\]

The pair of functions represents the general form for the solution of the problem (2.23), (2.24) (under the stated assumptions).

The essential feature of the solution is that, for each oscillation mode determined by a natural \( k \), there exist exactly two values of frequency and two forms of oscillations.

In the case of the disconnected model \((K = 0)\) the analysis is simpler and the solution corresponds to independent oscillations of the beam and the system of inclusions.

2.2.3. Example of discrete construction. Let us consider the “antenna type” frame construction (Fig 3) of length 1 m made from steel rods of square section with side 5 mm; cross rods of 250 mm length (condition rigidly, rigidly joint to the supporting rod) play the role of massive inclusions, their tips are connected by rubber thread of diameter 0.7 mm (belt drive).

![Figure 3](image_url)

Figure 3. Discrete construction (antenna type).

The constants of this model are:

\[
\begin{align*}
    E &= 2 \cdot 10^{11} \frac{N}{m^2}, \quad \rho = 0.61 \frac{kg}{m}, \quad J = 7.8 \cdot 10^{-3} \, kg \cdot m, \\
    J_{\text{cross}} &= 5.2 \cdot 10^{-11} \, m^4, \quad K = 250 \, N, \quad C = 25 \, N \cdot m^2.
\end{align*}
\]
Calculations of the discriminant $D$, frequencies $\omega_1^{(k)}$, $\omega_2^{(k)}$ and multipliers $a_1^{(k)}$, $a_2^{(k)}$ vs $k$ are presented in Fig 4, Fig 5 and Fig 6.

For this model, all the values of the discriminant (2.32) calculated for $k = 1, 2, \ldots, 100$ (in view of (2.28)) are positive, and the discriminant increases on the interval $4 \leq k \leq 100$ what could be seen on Fig 4; values of the discriminant on the interval $1 \leq k \leq 6$ are presented in more detail in Fig 4 (b). Values (2.38) of the both frequencies $\omega_1^{(k)}$ and $\omega_2^{(k)}$ increase on the interval $1 \leq k \leq 100$ (see Fig 5). Values (2.38) of the coefficients $a_1^{(k)}$ and $a_2^{(k)}$ vs $k$ (expressing the quotients $A/B$ of amplitude coefficients according to (2.34)) are shown in Fig 6. For lower frequencies $\omega_1^{(k)}$, corresponding values of $a_1^{(k)}$ are all positive and rapidly tend to zero with $k$ increasing. Values of $a_2^{(k)}$ correspond to higher frequencies $\omega_2^{(k)}$, are all negative and their absolute values are unbounded in $k$.

For example, for the mode with $k = 1$, there are exactly two oscillation forms:

1) with angular (radian) frequency $\omega_1^{(1)} = 59.6$ sec$^{-1}$ (oscillation frequency $\nu_1^{(1)} = 9.49$ Hz) and coefficient $a_1^{(1)} = 0.5972$ m and
2) with angular frequency $\omega_2^{(1)} = 256.6 \text{ sec}^{-1}$ (oscillation frequency $\nu_2^{(1)} = 40.86 \text{ Hz}$) and coefficient $a_2^{(1)} = -0.0214 \text{ m}$.

Figure 6. Graphs of $a_1^{(k)}$ and $a_2^{(k)}$ (measured in meters) in the ranges (a) $1 \leq k \leq 100$ and (b) $1 \leq k \leq 5$ ($a_1^{(k)}$ — circles, $a_2^{(k)}$ — daggers).

Figure 7. The first oscillation mode: $k = 1$. Configurations of the supporting rod (a) and rotation angles of supporting rod (b) at maximal deflections in the first (circles) and in the second (daggers) oscillation forms (deflections (a) are measured in meters). The solid line on (b) shows the rotation angles of the rods-inclusions (the same in both oscillation forms) corresponding to maximal deflections (a).

Fig 7 (a) demonstrates the configurations of the supporting rod corresponding to maximal deflections in the first and in the second oscillation forms of the first mode ($k = 1$). Fig 7 (b) shows graphs of rotation angles (maximal values) of rods-inclusions and elements of the supporting rod.

Fig 7 (b) shows (for the first oscillation mode $k = 1$) that the first oscillation form (low frequency) corresponds to “concomitant” (in one and the same direction) rotations of inclusions and the elements of the supporting rod (Fig 8 (a)), while the second oscillation form (higher frequency) corresponds to “counter” (in opposite directions) rotations (Fig 8 (b)).

In the case of “counter” rotations characterized by higher frequencies, the entire system has stronger stiffness than during “concomitant” motions (with lower frequencies). Calculations for $k = 1$ show (as can be seen in Fig 7 as well as in the graphs in Fig 6 at $k = 1$) that if maximal
rotations of the end inclusions are equal to $5^\circ$ then the maximal value of deflection (in the middle of supporting rod) reaches 5 cm in the first form and only 2 mm (with the opposite sign) in the second form.

At that, as Fig 6 shows for other modes (with increasing $k$), low frequency motions in the first form ("concomitant" motions) are characterized by the inclusion rotations prevailing over the deflections, while high frequency motions in the second form ("counter" rotations) dominate over the inclusion rotations.

2.2.4. Vibrations in a fluid-flow (gas-flow). The same model of the supplied beam was considered in view of eqns (2.23) in the case when $f_y = 0$ and $m = 0$, but

$$m_{in} = g\varphi_{incl}$$  \hspace{1cm} (2.39)

with the coefficient $g$. At $g > 0$ the moment $m_{in}$ tends to increase the angular deviation of inclusions, and at $g < 0$ it tends to decrease the angular deviation.

The condition (2.39) could be interpreted as the influence of a homogeneous gas- or fluid-flow with constant velocity $v$ and mass density $\rho^*$ impinging on inclusions equipped with winglets turned towards the flow ($g > 0$) or along the flow ($g < 0$). At small angular deviations $\varphi_{incl}$ of inclusions the quantity $g$ may be set to a constant:

$$g = c\rho^*v^2,$$  \hspace{1cm} (2.40)

where $c$ is an aerodynamic constant with the same sign as $g$.

The same approach as in the case of free vibrations leads us to a new expression for the discriminant

$$D(p) = (JEJ_{\text{cross}})^2 p^8 + 2JEJ_{\text{cross}} (KJ - \rho C) p^6 +$$

$$+ \left[ (\rho C - KJ)^2 - 2\rho J E J_{\text{cross}} (K - g) \right] p^4 +$$

$$+ [2\rho (K - g)(\rho C + KJ) + 4\rho JKg] p^2 + \rho^2 (K - g)^2$$  \hspace{1cm} (2.41)

and expressions for frequencies

$$\omega_1^2 = \frac{JEJ_{\text{cross}} p^4 + (\rho C + KJ) p^2 + \rho (K-g) - \sqrt{D(p)}}{2\rho J},$$

$$\omega_2^2 = \frac{JEJ_{\text{cross}} p^4 + (\rho C + KJ) p^2 + \rho (K-g) + \sqrt{D(p)}}{2\rho J}$$  \hspace{1cm} (2.42)

(the case $g = 0$ coincides with free vibrations).

The calculation of the similar "antenna type" construction in the gas-flow shows the dependence of squared frequencies on $g$ and $k$ (Fig 9, Fig 10).

In all cases the negative value of a squared frequency leads to the phenomenon of "divergence": exponential one-way deviation of the system (instead of oscillation).
2.3. Other Cosserat models

2.3.1. Supplied beam with plastic properties. Similar one-dimensional Cosserat models with elastic — perfect plastic properties are able to have an infinite number of equilibrium forms (in a certain range).

2.3.2. Models of higher dimensions. The method of mechanical (constructive) modeling was applied to similar 2-dimensional systems.

Motion equations (in Euler description) of corresponding 2-dimensional Cosserat continuum model have appeared in the form:

\[\text{div} \, S + \rho f - \rho \ddot{u} = 0,\quad \text{div} \, M + 2 \text{coax} \, S + \rho g - \rho j \cdot \dot{\omega} = 0,\]

where \(u\) and \(\omega\) are the displacement and rotation vectors (in-plane movement), \(S\) and \(M\) are stress and moment-stress tensors, \(\text{coax} \, S\) is the coaxial vector to \(S\), \(f\) and \(g\) are the external force and external moment, \(\rho\) is the mass density, \(j\) is the moment of inertia (2nd rank tensor).

The equivalent equations may be obtained in the Lagrangian description and three-dimensional models are under construction.
2.4. Closing remark
Thus, building Cosserat continuum models from appropriate discrete structures through the indicated method of mechanical (constructive) modeling,

• on the one hand, immediately clarifies the mechanical sense of all Cosserat characteristics,

• on the other hand, demonstrates the principal availability for real existence (naturally or artificially) of Cosserat type continua and drops a hint at possible ways for their technological manufacture.

3. Saturated porous media
Using the method of mechanical modeling, a new model of saturated porous media was built in the framework of the hypothesis of interpenetrative continua. In contrast to Biot’s theory it is able to circumscribe deformations of a skeleton at arbitrary strains (in the Lagrangean description) and arbitrary motions of the fluid (gas) through the skeleton (in the Eulerian description). The model is characterized by specific volume values (regulated with continuity equations) of solid (skeleton) and liquid (fluid) components, by mixed type of constitutive equations of the components and by specific force (and moment) interactions between the components.

3.1. Equations of the model
3.1.1. General form of equations. In a simple case of two-phase saturated porous media (with an incompressible material of skeleton and compressible or incompressible fluid), at small skeleton deformations the system of equations is

— motion equations of solid and fluid components

\[
\text{div}\sigma_s + g_s + f_s = \rho_s \frac{\partial^2 u_s}{\partial t^2},
\]

(3.1)

\[
\text{div}\sigma_f + g_f + f_f = \rho_f \frac{d v_f}{d t}.
\]

(3.2)

— mass conservation equations for fluid and solid components

\[
\frac{\partial}{\partial t} \rho_f + \text{div} (\rho_f v_f) = 0,
\]

\[
m = m_0 + (1 - m_0) \text{div} u_s,
\]

(3.3)

— constitutive equations for the pair of components

\[
\sigma_s = -(1 - m)p_p I + \lambda_0 \text{div} u_s I + 2\mu_s \text{sym} \nabla u_s,
\]

(3.4)

\[
\sigma_f = -mp_p I + \lambda_0 \text{div} v_f I + 2\mu_f \text{sym} \nabla v_f,
\]

(3.5)

— expressions of effective mass densities \(\rho_s\) of solid and \(\rho_f\) of fluid phases with the true densities \(\rho_{s\text{m}}\) and \(\rho_{f\text{m}}\) of their materials

\[
\rho_s = (1 - m)\rho_{s\text{m}},
\]

(3.6)

\[
\rho_f = m\rho_{f\text{m}},
\]

(3.7)

— and the state equation of the fluid expressing the dependence between true pore pressure \(p_p\) and true mass density (for compressible fluid)

\[
p_p = \varphi_f(\rho_{f\text{m}})
\]

(3.8)
or the incompressibility condition (for an incompressible fluid)
\[ \rho_{\text{fin}} = \rho_{\text{fin}0} \equiv \text{const}. \quad (3.9) \]

Here the known quantities are external body-forces (volume densities) \( g_s, g_f \), initial porosity \( m_0 \), effective characteristics of the skeleton elasticity \( \lambda_s, \mu_s \) and fluid viscosity \( \lambda_f, \mu_f \) and the constant density \( \rho_{\text{sm}} \) of the solid material.

The unknown functions are the effective stress tensors of the skeleton \( \sigma_s \) and the fluid phase \( \sigma_f \), the displacement vector \( u_s \) of the solid phase and the velocity vector \( v_f \) of the fluid phase, porosity \( m \), effective mass densities \( \rho_s, \rho_f \) and fluid viscosity \( \rho_{\text{fin}} \) of the fluid, true porous pressure \( p_p \) and true fluid mass density \( \rho_{\text{fm}} \).

Hence, the equations (3.1)-(3.9) (for compressible fluid) form a system of 23 equations for 23 unknown “scalar” functions. Initial and boundary conditions must be included in the formulation of the value boundary problem.

3.1.2. Equations in terms of displacement and velocity vectors. For the case of constant \( \lambda_s, \mu_s, \lambda_f, \mu_f \), substituting the stress tensors from the constitutive equations (3.4), (3.5) in the motion equations (3.1), (3.2) and eliminating the effective densities \( \rho_s, \rho_f \) using (3.6), (3.7), the system (3.1)-(3.9) can be rewritten in the form of Lame type (for a compressible fluid):
\[
\begin{align*}
-\nabla \left( (1 - m)p_p \right) + (\lambda_s + \mu_s) \nabla \text{div} u_s + \\
(1 - m) \rho_{\text{sm}} g_s + f_s &= (1 - m) \rho_{\text{sm}} \frac{\partial^2 u_s}{\partial t^2}, \\
-\nabla \left( m \rho_{\text{fin}} \right) + (\lambda_f + \mu_f) \nabla \text{div} v_f + \\
m \rho_{\text{fin}} g_f - f_f &= m \rho_{\text{fin}} \frac{dv_f}{dt}, \\
m &= m_0 + (1 - m_0) \text{div} u_s, \\
\frac{\partial (m \rho_{\text{fin}})}{\partial t} + \text{div}(m \rho_{\text{fin}} v_f) &= 0, \\
p_p &= \varphi_f (\rho_{\text{fin}}).
\end{align*}
\]

For an incompressible fluid the last equation of (3.10) should be replaced by the incompressibility condition (3.9) (with \( \rho_{\text{fin}0} \) given as data).

The system (3.10) consists of 2 vector and 3 scalar equations for 2 unknown vector functions \( u_s, v_f \) and 3 unknown scalar functions \( m, p_p, \rho_{\text{fin}} \) (9 “scalar” equations for 9 unknown “scalar” functions).

The system (3.10) is nonlinear. It does not belong to a classic type but it coincides with classic equations in special simple cases.

Remark. In both general and Lame type systems the vector \( f_s \) is the interactive (volume-density) body-force from the fluid on the skeleton and vector \( f_f \) is the reaction force: for a two-phase system \( f_s = -f_f \).

Interactive forces should be determined by constitutive relations.

3.2. Interactive forces

The detailed analysis of interactive forces permits representation of the interactive force \( f \) (for example, \( f \equiv f_s \)) as the sum of static and dynamic components
\[ f = f_{\text{stat}} + f_{\text{dyn}}, \]
where
\[ f_{\text{stat}} = -p_p \text{grad} m. \]

The dynamic component \( f_{\text{dyn}} \) should be analyzed using the material objectivity principle and measurement theory methods.
3.2.1. Frame-invariancy of dynamic interactive forces. Abbreviate \( f_{\text{dyn}} \) as \( f \).

Suppose that \( f \) is a function of velocities and accelerations of fluid and skeleton phases

\[
f = p(v_f, v_s, w_f, w_s)
\]

(3.13)

Using the material objectivity principle we obtain the following result.

**Theorem 1.** The general reduced form of a dynamic interactive force (3.13) is expressed by the equation

\[
f = p(v_r, \dot{v}_r)v_r^0,
\]

(3.14)

where \( v_r \) is the magnitude of the relative velocity \( v_r = v_f - v_s \), \( v_r^0 = v_r/v_r \) is the unit vector along the relative velocity, \( \dot{v}_r = w_r \cdot v_r^0 \) is the derivative of \( v_r \) by time (equal to the projection of the acceleration vector along the relative velocity direction) and \( p \) is a material scalar function.

**Remark.** The problem of invariance properties of inertial interactive forces was formulated and studied in [55, 56]. In [55] a simple direct approach to the problem was formulated and partially studied. The approach of [56] is more complicated, it is addressed to the more general case with and gives the solution of the problem in terms of an elegantly introduced class of objective relative accelerations specified by a coefficient named as constitutive (subject to be determined). Here, in theorem 1 we get the complete solution of the simple formulation of the problem [55] with the result immediately in terms of usual relative accelerations (and velocities) without any additional coefficients.

3.2.2. Analysis by the methods of measurement theory. Let the magnitude \( f \equiv |f| \) of a (dynamic) interactive force \( f \) be a function

\[
f = p(\rho, v, d, m, \mu, w),
\]

(3.15)

where \( \rho \equiv \rho_f \) is the fluid density, \( v \equiv v_r \) is the magnitude of the relative velocity, \( d \) is the typical size of the porous structure, \( m \) is the porosity, \( \mu \) is the coefficient of viscosity of the fluid and \( w \equiv \dot{v}_r \) is a collinear relative acceleration.

Owing to the \( \pi \)-theorem [9], we obtain

\[
p(\rho, v, d, m, \mu, w) \equiv \frac{\rho v^2 d}{\pi} \varphi \left( m, \frac{1}{\text{Re}}, \frac{1}{\text{B}} \right),
\]

(3.16)

where \( \text{Re} = \frac{\rho v d}{\mu} \equiv \frac{\omega d}{v} \) is the Reynolds number (\( \nu \) is the kinematic viscosity), \( B = \frac{\omega^2 d}{\nu} \) is a new dimensionless parameter.

Suppose \( \varphi \) from (3.16) is a linear function of its two last arguments,

\[
\varphi \left( m, \frac{1}{\text{Re}}, \frac{1}{\text{B}} \right) \equiv \varphi_F(m) + \varphi_D(m) \frac{1}{\text{Re}} + \varphi_B(m) \frac{1}{\text{B}},
\]

with dimensionless coefficients \( \varphi_F(m), \varphi_D(m), \varphi_B(m) \) depending on porosity \( m \).

Thus, the magnitude of the dynamic interactive force is

\[
f = f_F + f_D + f_B,
\]

(3.17)

where

\[
\begin{align*}
f_F & \equiv \frac{\rho v^2}{\pi} \varphi_F(m) \equiv \rho \varphi_F(m) \frac{v^2}{d}, \\
f_D & \equiv \frac{\rho v^2}{\pi} \varphi_D(m) \frac{1}{\text{Re}} \equiv \rho \varphi_D(m) \frac{v^2}{\nu}, \\
f_B & \equiv \frac{\rho v^2}{\pi} \varphi_B(m) \frac{1}{\text{B}} \equiv \rho \varphi_B(m) w.
\end{align*}
\]

(3.18)

The items \( f_F, f_D, f_B \) may be interpreted respectively as dynamic forces of
• **frontal resistance** (dynamic velocity pressure),
• **viscous resistance** (Darcy law),
• **inertial resistance** (of Biot added mass type).

As a result, for arbitrary motions, the vector of dynamic interactive force has the form

\[ \mathbf{f} = \mathbf{f}_F + \mathbf{f}_D + \mathbf{f}_B \equiv \left( c_0 (\rho_f, m, d) v_r + d_0 (m, d, \mu_f) + b_0 (\rho_f, m) \frac{d}{dt} \right) \mathbf{v}_r, \]  

(3.19)

where the coefficients

\[ c_0 (\rho_f, m, d) = \varphi_F (m) \frac{\rho_f}{d}, \quad d_0 (m, d, \mu_f) = \frac{\varphi_D (m) \mu_f}{d^2}, \quad b_0 (\rho_f, m) = \varphi_B (m) \rho_f \]

could be considered in specific cases as constants.

3.2.3. **Estimation of contributions of dynamic forces.** Comparison of the items in (3.17) (assuming the same order of the values \( \varphi_F (m) \), \( \varphi_D (m) \) and \( \varphi_B (m) \)) leads to a family of approximations to the dynamic force \( \mathbf{f} \) in different dynamical regimes (see table 1).

| Table 1. Comparative contributions of the dynamic forces. |
|-----------------------------------------------------------|
| Re \( \ll 1 \) | Re \( \sim 1 \) | Re \( \gg 1 \) |
|----------------|----------------|----------------|
| B \( \ll 1 \) | 0 + \( f_D + f_B \) | 0 + 0 + \( f_B \) |
| B \( \sim 1 \) | 0 + \( f_D + 0 \) | \( f_F + f_D + f_B \) | \( f_F + 0 + f_B \) |
| B \( \gg 1 \) | 0 + \( f_D + 0 \) | \( f_F + f_D + 0 \) | \( f_F + 0 + 0 \) |

3.3. **Conclusion**

The nonlinearity of the equations ((3.1)-(3.10)), the presence of static ((3.11), (3.12)) and special forms for dynamic ((3.14), (3.19)) interaction forces show the essential difference of the proposed model from the classic ones. The expression of inertial resistance force ((3.18)-(3.19)) demonstrates the principal distinctions from Biot added mass force in the general case (within the domain of applicability of Biot’s model); it coincides with Biot’s force in the case of collinearity of relative velocity and relative acceleration vectors (one-dimensional motions) and clarifies, in this case, the sense of added mass density of Biot’s model as an inertial resistance characteristic.

Table 1 may be used to simplify the interactive force (indicated in the table) for different practical calculations.

4. **Compound internal interactions**

As a generalization of the previous considerations, let us consider compound interactions consisting of body forces \( \mathbf{f} \) and moments \( \mathbf{M} \):

\[ T = (\mathbf{f}, \mathbf{M}). \]
4.1. Interactions in solid two-phase media

Let the two-phase medium be formed from elastic components (denoted by indices (1) and (2)). Let \( T \) be a function

\[
T = \varphi \left( u_r, A^{(1)}, A^{(2)} \right),
\]

where \( u_r \) is the vector of mutual displacement of phases, \( A^{(1)} \) and \( A^{(2)} \) are the deformation gradients (affinors) of each phase (relatively to a reference configuration).

**Theorem 2.** The general reduced form of compound internal interactions (4.1) in solid two-phase media is as follows

\[
f = Q^{(1)} \cdot \varphi_t \left( Q^{(1)T} \cdot u_r, X^{(1)}, X^{(2)}, Q^{(1)T} \cdot Q^{(2)} \right),
\]

\[
M = Q^{(1)} \cdot \Phi_M \left( Q^{(1)T} \cdot u_r, X^{(1)}, X^{(2)}, Q^{(1)T} \cdot Q^{(2)} \cdot Q^{(1)T} \right),
\]

where \( Q^{(i)}, X^{(i)} \) are the orthogonal and symmetric parts of the polar decomposition of affinor \( A^{(i)} \), namely, \( A^{(i)} = Q^{(i)} \cdot X^{(i)} \) \((i = 1, 2)\), and \( \varphi_t, \Phi_M \) are material functions.

In the case of small strains the relations (4.2) are reduced to the form

\[
f = \psi_f \left( u_r, \varepsilon^{(1)}, \varepsilon^{(2)}, \omega^{(2)} - \omega^{(1)} \right),
\]

\[
M = \Psi_M \left( u_r, \varepsilon^{(1)}, \varepsilon^{(2)}, \omega^{(2)} - \omega^{(1)} \right),
\]

where \( \varepsilon^{(i)}, \omega^{(i)} \) are the linear strain and rotation tensors.

4.2. Compound interactions in two-phase porous saturated media

Consider a saturated porous medium with elastic skeleton ((s) — skeleton, (f) — fluid). Let

\[
T = \varphi \left( v_r, w_r, D^{(f)}, D^{(s)}, A^{(s)} \right),
\]

where \( v_r \) and \( w_r \) are the relative velocity and acceleration, \( D^{(f)} \) and \( D^{(s)} \) are the stretching tensors of the fluid and the skeleton and \( A^{(s)} \) is the deformation gradient of the skeleton.

The principle of material objectivity is satisfied if the constitutive relation (4.4) has the form

\[
T = Q^{(s)} \cdot \varphi \left( Q^{(s)T} \cdot v_r, w_r \cdot v_r, Q^{(s)T} \cdot \Omega_r \cdot Q^{(s)}, Q^{(s)T} \cdot V^{(f)}, Q^{(s)}, \varepsilon^{(s)}, \varepsilon^{(s)} \right),
\]

where \( Q^{(s)} \) is the orthogonal part of polar decomposition of the affinor \( A^{(s)} \), \( \Omega_r = \Omega^{(f)} - \Omega^{(s)} \) is the relative spin tensor, \( V^{(f)} \) is the strain rate tensor of the fluid phase, \( \varepsilon^{(s)} \) is the Green strain tensor of the skeleton, and the notation \( Q \cdot T = Q \cdot (f, M) := (Q \cdot f, Q \cdot M \cdot Q^{T}) \) is used.

Using the methods of [68] one can see that the simplified version of (4.4) in the form

\[
T = \varphi \left( v_r, w_r \cdot v_r, \Omega_r \right)
\]

satisfies (5.5) if interactive actions have the form

\[
f = a v_r + b \Omega_r \cdot v_r + c \Omega_r^2 \cdot v_r + d \varepsilon : \Omega_r,
\]

\[
M = A \Omega_r + B \text{skw}(v_r \otimes \Omega_r \cdot v_r) + C \text{skw}(v_r \otimes \Omega_r^2 \cdot v_r) + D \varepsilon \cdot v_r,
\]
where $a$, $b$, $c$, $d$, $A$, $B$, $C$, $D$ are scalar functions of the scalar product $\mathbf{w}_r \cdot \mathbf{v}_r$ and mutual invariants of $\mathbf{v}_r$ and $\Omega_r$, and $\epsilon$ is the Levi–Civita tensor.

The relations (4.7) and (4.8) mean, in particular, the possibility for the appearance of front resistance force, lifting (shifting) force as well as overturning and rotating ("screw") moments exerted by the fluid on the skeleton.

This work supported by the Russian Foundation for Basic Research (project 06-01-00565 and project 06-01-10832).

References

[1] Noll W 1958 A mathematical theory of the mechanical behavior of continuous media Arch. Rat. Mech. Anal. 2 197-226

[2] Truesdell C A and Noll W 1965 The Nonlinear Field Theories of Mechanics (Encyclopedia of Physics vol III/3) (Springer-Verlag) (Second Edition, 1992. Third Edition, 2004)

[3] Truesdell C A 1972 A First Course in Rational Continuum Mechanics (Baltimore, Maryland: The Johns Hopkins University)

[4] Sedov L I 1962 Introduction to Continuum Mechanics (Moscow: Fizmatgiz)

[5] Mandel J 1966 Cours de Mécanique des milieux continus (vol I, II) (Paris: Gautier-Villars) (Reprint 1994)

[6] Eringen A C 1967 Mechanics of Continua (New York: Wiley)

[7] Sedov L I 1973 Continuum Mechanics (Moscow: Nauka)

[8] Germain P 1973 Course de Meccanique des Milieux Continus. Théorie Générale (Paris: Masson et C)

[9] Sedov L I 1977 Methods of Similarity and Measurements in Mechanics (Moscow: Nauka)

[10] Ilyushin A A 1990 Continuum Mechanics (Moscow: Moscow University Press)

[11] Mueller I 1984 Thermodynamics (Boston: Pitman)

[12] Silhavy M 1997 The Mechanics and Thermodynamics of Continuous Media (Berlin: Springer)

[13] Wilmanski K 1998 Thermomechanik von Kontinuen (Berlin: Springer)

[14] Liu I-Shih 2002 Continuum Mechanics (Berlin–Heidelberg–New York: Springer-Verlag)

[15] Love A E H 1927 A Treatise on the Mathematical Theory of Elasticity (Cambridge: University Press)

[16] Novozhilov V V 1948 Foundations of Nonlinear Theory of Elasticity (Moscow–Leningrad: Gostekhizdat)

[17] Ilyushin A A 1948 Plasticity. Elastic-Plastic Deformations (Moscow–Leningrad: Gostekhizdat)

[18] Ilyushin A A 1963 Plasticity. Foundations of the General Mathematical Theory (Moscow: Academy of Sciences of the USSR)

[19] Hill R 1983 The Mathematical Theory of Plasticity (Oxford)

[20] Ilyushin A A and Lenski V S 1959 Strength of Materials (Moscow: Fizmatgiz)

[21] Bell J F 1973 Mechanics of Solids I (Experimental Foundations of Mechanics of Solids) vol VI/a/1 ed C Truesdell (Berlin–Heidelberg–New York: Springer-Verlag)

[22] Novacki W 1970 Theory of Elasticity (Warsaw: Państwowe Wydawnictwo Naukowe)

[23] Lourie A I 1980 Nonlinear Theory of Elasticity (Moscow: Nauka)

[24] Ciarell P G 1988 Mathematical Elasticity vol 1 Three-Dimensional Elasticity (Studies in Mathematics and its Applications vol 20 ed J-L Lions et all) (Amsterdam–New York–Oxford–Tokio: North-Holland)

[25] Antman S S 2005 Nonlinear Problems of Elasticity (N.Y.: Springer)

[26] Truesdell C and Toupin R A 1960 The Classical Field Theories (Handbuch der Physik vol 3/1) (Berlin: Springer) 226-793

[27] Aseev S E and Kuvshinskii E V 1960 Basic equations of the theory of elasticity for media with rotatory interactions of particles Fizika tverdogo tela (Physics of Solids) 2 No 7 1399-409

[28] Toupin R A 1962 Elastic materials with couple-stresses Arch. Rat. Mech. Anal. 11 No 5 385-414

[29] Mindlin R D and Tiersten H F 1962 Effects of couple-stresses in linear elasticity Arch. Rat. Mech. Anal. 11 No 5 415-48

[30] Palmov V A 1964 Basic equations of the theory of non-symmetric elasticity Prikladnaya Matematika i Mekhanika (Applied Mathematics and Mechanics) 28 No 3 401-08

[31] Green A E 1965 Micro-materials and multipolar continuum mechanics Int. J. Eng. Sci. 3 No 5 533-37

[32] Lomakin V A 1970 Statistical Problems in Mechanics of Solids (Moscow: Nauka)

[33] Ilyushin A A and Lomakin V A 1971 Moment theories in mechanics of solids Strength and Plasticity (Moscow: Nauka) pp 54-60

[34] Kunin I A 1975 Theory of elastic media with microstructure (Moscow: Nauka)

[35] Green A E and Naghdi P M 1991 A thermomechanical theory of a Cosserat point with application to composite materials Q. J. Mech. Appl. Math. 44 335-55
[67] Popov E P 1986 Theory and Calculations for Flexible Elastic Rods (Moscow: Nauka)
[68] Spencer A J M 1971 Theory of Invariants (Continuum Physics vol I part III ed A C Eringen) (New York–London: Academic) pp 239-353