On a variant of Hardy inequality between weighted Orlicz spaces

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Abstract

Let $M$ be an $N$-function satisfying the $\Delta_2$—condition, let $\omega, \varphi$ be two other functions, $\omega \geq 0$. We study Hardy-type inequalities

$$\int_{\mathbb{R}^+} M(\omega(x)|u(x)|)\exp(-\varphi(x))dx \leq C \int_{\mathbb{R}^+} M(|u'(x)|)\exp(-\varphi(x))dx,$$

where $u$ belongs to some dilation invariant set $\mathcal{R}$ contained in the space of locally absolutely continuous functions. We give sufficient conditions the triple $(\omega, \varphi, M)$ must satisfy in order to have such inequalities valid for $u$ from a given set $\mathcal{R}$. The set $\mathcal{R}$ can be smaller than the set of Hardy transforms. Bounds for constants, retrieving classical Hardy inequalities with best constants, are also given.

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1 Introduction

General framework and classical approach. Hardy type inequalities have been the subject of intensive research, going back to Hardy, who in the early 1920’s ([25, 26]) obtained inequalities of the form

$$\int_{\mathbb{R}^+} |u(t)|^p t^{\alpha-p} dt \leq C \int_{\mathbb{R}^+} |u'(t)|^p t^{\alpha} dt. \quad (1.1)$$

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Over the decades to come, this inequality was generalized to
\[
\left( \int_{\mathbb{R}^+} \left( \int_0^t |f(\tau)|^q d\mu(t) \right)^\frac{1}{q} \right)^\frac{1}{p} \leq C \left( \int_{\mathbb{R}^+} |f(t)|^p d\nu(t) \right)^\frac{1}{p},
\]
with two nonnegative Radon measures \(\mu, \nu\), and further to inequalities in the Orlicz-space setting:
\[
Q^{-1} \left( \int_{\mathbb{R}^+} Q(\theta(x)|Tf(x)|)w(x)dx \right) \leq P^{-1} \left( \int_{\mathbb{R}^+} P(C\rho(x)|f(x)|)v(x)dx \right).
\]

Inequalities in \(L^p\) were studied by Muckenhoupt [46], Mazya and Rozin [44], Bradley [9], Kokilashvili [33], Sinnamon [53], Sawyer [51], Bloom and Kerman [6], Stepanow [54], and many others (we refer to the monographs [38, 36, 37, 34, 45] and references therein). As to the Orlicz-space result (1.2), several authors contributed to the complete characterization of admissible weights \(\theta, w, \rho, v\) and nondecreasing functions \(P, Q\), which allow for (1.2) with \(Tf(x) = \int_0^x K(x, y)f(y)dy\) being the generalized Hardy operator with kernel \(K\). To name just a few, we refer to the papers of Bloom-Kerman [5, 6], Lai [40, 41, 42, 43], Heinig-Maligranda [28], Bloom-Kerman [6], Heinig-Lai [27] and their references. For a more detailed account of such results, we refer to Sections 3.1, 3.2.

Hardy-type inequalities are widely applicable in the PDE theory and in functional analysis. For example, one can derive various Sobolev embedding theorems in the \(L^p\) setting, which can then be used to prove the existence of solutions of the Cauchy problem in the elliptic and parabolic PDEs (see e.g. [7, 22, 36, 15, 44, 55, 39]), to study of the asymptotic behaviour of solutions ([2, 57]), as well as their stability [10, 11]. They are present in the probability theory (see e.g. [17, 50, 19]). Hardy inequalities are also of separate interest (see e.g. [37, 38, 56]). For latest results, see the very recent papers [7, 13, 23, 47] and their references.

The investigation of weighted or nonweighted Orlicz-Sobolev spaces defined by an \(N\)-function different from \(\lambda^p\) is suggested by physical models (see e.g. [1, 3, 14, 48, 15]). Therefore it is worthwhile to examine Hardy-type inequalities in general Orlicz spaces as well.

One of the central problems evolving around Hardy inequalities can be expressed as follows. Consider sets of Hardy transforms \(\mathcal{H}\) or conjugate Hardy transforms \(\mathcal{H}^*\):
\[ H = \{ u(t) = \int_0^t f(s)ds, \int_0^a |f(\tau)|d\tau < \infty \text{ for every } a > 0 \} = \{ u \in W^{1,1}_\text{loc}(\mathbb{R}_+) : \lim_{r \to 0} u(r) = 0 \}; \]
\[ H^* = \{ u(t) = \int_t^\infty f(s)ds, \int_a^\infty |f(\tau)|d\tau < \infty \text{ for every } a > 0 \} = \{ u \in W^{1,1}_\text{loc}(\mathbb{R}_+) : \lim_{r \to \infty} u(r) = 0 \}. \quad (1.3) \]

and let \( Tf \) be either the Hardy transform of \( f \), or the conjugate Hardy transform of \( f \). Then one deals with the following problem.

**Problem 1 (classical).** Given two \( N \)-functions \( Q, P \), describe all possible weights \( (\theta, \omega, \rho, v) \) for which inequality (1.2) follows with some constant \( C \) independent of \( f \) (with \( T \) the Hardy transform, or the conjugate Hardy transform).

This problem has been solved completely for the Hardy transform by Bloom and Kerman in [6] for modular functions \( P \) and \( Q \) such that \( Q \) dominates \( P \) in some special sense (for the details see Section 3.2). Further generalization of (3.3), without the domination restriction can be found in Lai's paper [43]. Therefore Problem 1 can be considered as a closed and classical one.

**Another approach and its motivation.** We are concerned with another problem, which is expressed as follows.

**Problem 2 (general).** Given two \( N \)-functions \( Q, P \) and weights \( (\theta, \omega, \rho, v) \), find a possibly big dilation invariant set \( \mathcal{R} \) contained is the set of locally absolutely continuous functions for which the inequality

\[ Q^{-1}\left( \int_{\mathbb{R}_+} Q(\theta(x)|u(x)|)w(x)dx \right) \leq P^{-1}\left( \int_{\mathbb{R}_+} P(C\rho(x)|u'(x)|)v(x)dx \right), \quad (1.4) \]

would hold with some constant \( C \) independent of \( u \in \mathcal{R} \).

Let us make several comments here.

It can happen that the given weights \( (\theta, \omega, \rho, v) \) obey the known requirements for the validity of (1.2), say for \( Tf \) being the Hardy transform of \( f \), as described in Problem 1. In such a case the set \( \mathcal{R} \) contains the full set of Hardy transforms \( H \). On the other hand, if this requirement is not fulfilled, we cannot expect (1.4) to hold for every \( u \in \mathcal{H} \). In such a case \( \mathcal{R} \cap \mathcal{H} \) will be a proper subset of \( \mathcal{H} \). The solution to Problem 2 would therefore lead to...
Hardy-type inequalities within a (possibly) narrower class of functions than this required so far.

Since the set of Hardy transforms (resp. conjugate Hardy transforms) is dilation invariant, we want that our set $\mathcal{R}$ be dilation invariant as well. This means by definition that if $u \in \mathcal{R}$ then for every $\lambda > 0$ the function $u_\lambda(x) := u(\lambda x)$ also belongs to $\mathcal{R}$.

If we substitute $u = Tf$ in (1.4), where $Tf$ is either Hardy transform or conjugate Hardy transform, then $u' = f$, so that (1.4) reduces to the special case of (1.2).

**The reduced problem and partial answers.** In this paper we deal with the special variant of Problem 2, which reads as follows.

**Problem 3 (reduced).** Given an $N$–function $M$ satisfying the $\Delta_2$–condition, and a pair of functions $(\omega, \varphi)$ where $\omega \geq 0$, describe the possibly big dilation invariant set $\mathcal{R}$ contained is the set of locally absolutely continuous functions for which inequality

$$
\int_{\mathbb{R}^+} M(\omega(x)|u(x)|)\exp(-\varphi(x))dx \leq C \int_{\mathbb{R}^+} M(|u'(x)|)\exp(-\varphi(x))dx, \quad (1.5)
$$

follows with some constant $C$ independent on $u \in \mathcal{R}$.

Variants of (1.5) with $M(\lambda) = \lambda^p$ and $\mathcal{R}$ determined by the constraints concerning $M, \omega, \phi$ have been studied e.g. in the papers [4, 16, 20, 21], see also their references. To the best of our knowledge their extension to Orlicz setting was not considered so far.

Our main result formulated in Theorem 2.1 states that if the triple of functions $(\omega, \varphi, M)$ satisfies certain simple compatibility conditions, then we can indicate a dilation invariant set $\mathcal{R}$ such that (1.5) holds for every $u \in \mathcal{R}$. Moreover, we give some bounds on the constant $C$, which can be expressed in terms of the Simonenko lower and upper index (see [52] and [24], [18] for interesting related results).

The decision whether $u \in \mathcal{R}$ is based on its behavior near zero and near infinity, which is very natural in problems arising from PDE’s: when analyzing a particular equation one can often say that its solution (i.e. our function $u$) has some ‘good’ properties near the boundary, expressed in terms of its rate of decay near the boundary.

As an illustration we derive the classical Hardy inequalities (1.1) with best constants (see Section 3.1.1 for discussion).

We also obtain sufficient conditions for (1.5) to hold for every $u \in \mathcal{H}$.
(see Proposition 5.2 in Section 5.2.1). These conditions can be easily implemented in practice, and since the verification of the classical Bloom-Kerman conditions (5.6) from [6] seems rather hard, by using our approach one can avoid the verification of Bloom-Kerman conditions and quickly deduce that (1.5) is satisfied for every \( u \in \mathcal{H} \).

Perhaps it is even more interesting to deal with the case when Bloom-Kerman conditions (5.6) are not satisfied, so that inequality (1.5) is not valid for every \( u \in \mathcal{H} \). Then we find a set \( \mathcal{R} \) such that \( \mathcal{R} \cap \mathcal{H} \) is smaller than \( \mathcal{H} \) and (1.5) holds for every its element (see Section 5.2.2).

As a particular type of inequalities alike (1.5), we analyze those with \( \omega = |\varphi'| \) (see Section 3.3), and in the class of admissible \( \varphi' \)'s we obtain the inequality

\[
\int_{\mathbb{R}^+} M(|\varphi'(x)|u(x)|\exp(-\varphi(x))dx \leq C \int_{\mathbb{R}^+} M(|u'(x)|)\exp(-\varphi(x))dx.
\]

For \( M(\lambda) = \lambda^p \) and \( \psi(x) = \exp\{-\frac{\varphi(x)}{p}\} \) we get

\[
\int_{\mathbb{R}^+} (|\psi'(x)u(x)|)^p dx \leq C \int_{\mathbb{R}^+} (|u'(x)\psi(x)|)^p dx,
\]

which is nothing but a particular case of Caccioppoli inequality on \( \mathbb{R}^+ \) (see e.g. [12, 29]). Caccioppoli inequalities are commonly used in the regularity theory, and so we believe that our variant (1.6) can be used in the regularity theory as well.

2 Preliminaries and statements of main results

2.1 Preliminaries

Orlicz spaces

Let us recall some preliminary facts about Orlicz spaces, referring e.g. to [49] for details. Here we deal with Orlicz spaces of functions defined on \( \mathbb{R}^+ \).

Suppose that \( \mu \) is a positive Radon measure on \( \mathbb{R}^+ \) and let \( M : [0, \infty) \to [0, \infty) \) be an \( N \)-function, i.e. a continuous convex function satisfying \( \lim_{\lambda \to 0} \frac{M(\lambda)}{\lambda} = 0 \) and \( \lim_{\lambda \to \infty} \frac{M(\lambda)}{\lambda} = \infty \).
The weighted Orlicz space $L^M_\mu$ we deal with is by definition the space

$$L^M_\mu \overset{def}{=} \{ f : \mathbb{R}_+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}_+} M\left(\frac{|f(x)|}{K}\right) d\mu(x) \leq 1 \text{ for some } K > 0\},$$

equipped with the Luxemburg norm

$$\|f\|_{L^M_\mu} = \inf\{ K > 0 : \int_{\mathbb{R}_+} M\left(\frac{|f(x)|}{K}\right) d\mu(x) \leq 1 \}.$$  

This norm is complete and turns $L^M_\mu$ into a Banach space. When $\mu$ is the Lebesgue measure, it is dropped from the notation. For $M(\lambda) = \lambda^p$ with $p > 1$, the space $L^M_\mu$ coincides with the usual $L^p_\mu$ space (defined on $\mathbb{R}_+$).

The symbol $M^*$ denotes the complementary function of an $N-$function $M$, i.e. its Legendre transform: for $y, \geq 0$, $M^*(y) = \sup_{x>0}[xy - M(x)]$. It is again an $N-$function and from its definition we get the Young inequality:

$$xy \leq M(x) + M^*(y), \text{ for } x, y \geq 0.$$  

$M$ is said to fulfill the $\Delta_2-$condition if and only if, for some constant $c > 0$ and every $\lambda > 0$, we have

$$M(2\lambda) \leq cM(\lambda). \quad (2.1)$$

In the class of differentiable convex functions the $\Delta_2-$condition is equivalent to:

$$\lambda M'(\lambda) \leq \tilde{c}M(\lambda),$$

satisfied for every $\lambda > 0$, with the constant $\tilde{c}$ being independent of $\lambda$ (see e.g. [35], Theorem 4.1).

We will need the following property of modulars: (see [35], formula (9.21))

$$\int_{\mathbb{R}_+} M\left(\frac{f(x)}{\|f\|_{L^M_\mu}}\right) d\mu(x) \leq 1. \quad (2.2)$$

When $M$ satisfies the $\Delta_2-$condition, then (2.2) becomes an equality.

The function $M_1$ is said to dominate $M_2$ if there exist two positive constants $K_1, K_2$ s.t. $M_2(\lambda) \leq K_1M_1(K_2\lambda)$ for every $\lambda > 0$. In such case we have

$$\| \cdot \|_{L^{M_2}_\mu} \leq K \| \cdot \|_{L^{M_1}_\mu}, \text{ with } K = K_2(K_1 + 1). \quad (2.3)$$

Functions $M_1$ and $M_2$ are called equivalent when $M_2$ dominates $M_1$ and $M_1$ dominates $M_2$. In particular equivalent N-functions give raise to equivalent Luxemburg norms.
On the set $\mathcal{L}_\mu^M = \{u \text{ measurable: } \int M(|u|)d\mu < \infty \}$, one introduces the so-called dual norm:

$$
\|u\|_{L^M_\mu} = \sup \left\{ \int_{\mathbb{R}_+} u(x)v(x) \, d\mu(x) : v \in L^M_\mu, \int_{\mathbb{R}_+} M^*(|v(x)|) \, d\mu(x) \leq 1 \right\}.
$$

The advantage of this norm is the Hölder-type inequality:

$$
\text{for } f \in L^M_\mu, \ g \in L^{M^*}_\mu, \ \int_{\mathbb{R}_+} f \cdot g \, d\mu \leq \|f\|_{L^M_\mu} \|g\|_{L^{M^*}_\mu}, \quad (2.5)
$$

When $M$ satisfies the $\Delta_2$—condition, then $\mathcal{L}_\mu^M = L^M_\mu$, and in general, the Orlicz space $L^M_\mu$ is the completion of $\mathcal{L}_\mu^M$ in the dual norm. The Luxemburg norm and the dual norm are equivalent:

$$
\|u\|_{L^M_\mu} \leq \|u\|_{L^M_\mu} \leq 2\|u\|_{L^M_\mu}.
$$

**Assumptions.**

Throughout the paper we assume:

(M) $M: [0, \infty) \to [0, \infty)$ is a differentiable $N$—function, i.e. $M$ is convex, $M(0) = M'_+(0) = 0$, $M(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$, and moreover $M$ satisfies the condition:

$$
d_M \frac{M(\lambda)}{\lambda} \leq M'(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \quad \text{for every } \lambda > 0, \quad (2.6)
$$

where $D_M \geq d_M \geq 1$.

(\(\mu\)) $\mu$ is a Radon measure on $\mathbb{R}_+$, absolutely continuous with respect to the Lebesgue measure and $\mu(dr) = e^{-\varphi(r)}$, $\varphi \in C^2(\mathbb{R}_+)$, $\varphi'$ does never vanish,

(\(\omega\)) $\omega: (0, \infty) \to [0, \infty)$ is a $C^1$—function.

**Remark 2.1** The latter inequality in (2.6) implies that $M$ satisfies $\Delta_2$—condition (see (2.1)). The condition $d_M > 1$ is equivalent to the fact that also $M^*$ satisfies the $\Delta_2$—condition (see e.g. [35], Theorem 4.3 or [30], Proposition 4.1). Moreover, for any $N$—function $M$, the left-hand side in (2.6) holds with $d_M = 1$.

If $d_M$ and $D_M$ are the best possible constants in (2.6) they obey the definition of Simonenko lower and upper index of $M$ and are related to Boyd indices of $L^M(\mathbb{R}^n, \mu)$ (see [8], [52] for definitions, [18], [24], [58] for discussion on those and other indices of Orlicz spaces).
Conditions

Before we proceed, we need to introduce the following quantities. We set

\[ \Omega := \{ r \in \mathbb{R}_+: \omega(r)u(r) \neq 0 \} \quad (2.7) \]

\[ F := \{ r \in \mathbb{R}_+: \omega(r) \neq 0, \omega'(r)\varphi'(r) > 0 \} \]

\[ G := \{ r \in \mathbb{R}_+: \omega(r) \neq 0, \omega'(r)\varphi'(r) < 0 \} \]

Then we define:

\[ b_1(r, \omega, \varphi, M) = \left( 1 + \frac{\varphi''(r)}{(\varphi'(r))^2} \right) \frac{\omega'(r)}{\omega(r)\varphi'(r)} \left[ d_M \chi_G(r) + D_M \chi_F(r) \right] ; \]

\[ b_1 = b_1(\omega, \varphi, M) := \inf \{ b_1(r, \omega, \varphi, M) : r \in \mathbb{R}_+ \} ; \quad (2.8) \]

\[ b_2(r, \omega, \varphi, M) = \left( -1 - \frac{\varphi''(r)}{(\varphi'(r))^2} + \frac{\omega'(r)}{\omega(r)\varphi'(r)} \right) \left[ d_M \chi_F(r) + D_M \chi_G(r) \right] ; \]

\[ b_2 = b_2(\omega, \varphi, M) := \inf \{ b_2(r, \omega, \varphi, M) : r \in \mathbb{R}_+ \} ; \]

\[ L = L(\omega, \varphi) := \sup \left\{ \frac{\omega(r)}{|\varphi'(r)|} : r \in (0, \infty), \varphi'(r) \neq 0 \right\} . \quad (2.9) \]

We use the convention \( \sup \emptyset = -\infty, \inf \emptyset = +\infty, c/\infty = 0, f \chi_A \) is the function \( f \) extended by 0 outside \( A \).

2.2 Main results

Our area of interest will be those triples \((M, \varphi, \omega)\) for which either

(B1) \( b_1 > 0, L < \infty \), or

(B2) \( b_2 > 0, L < \infty \).

We will deal with the following function:

\[ h^u(r) = h^{u(\omega, \varphi, M)}(r) = \frac{1}{\varphi'(r)} M(\omega(r)|u(r)|) , \quad (2.10) \]

which is well defined since \( \varphi'(r) \) is never zero.

Let us introduce the following two classes of functions:

\[ \mathcal{R}^+_\omega := \{ u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+) : \exists s_n \to 0, R_n \to -\infty : \lim_{n \to \infty} (h^u(R_n) e^{-\varphi(R_n)} - h^u(s_n) e^{-\varphi(s_n)}) \geq 0 \} ; \quad (2.11) \]

\[ \mathcal{R}^-\omega := \{ u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+) : \exists s_n \to 0, R_n \to -\infty : \lim_{n \to \infty} (h^u(R_n) e^{-\varphi(R_n)} - h^u(s_n) e^{-\varphi(s_n)}) \leq 0 \} ; \quad (2.12) \]
not precluding the limits from being infinite. For simplicity we will usually omit \((\omega, \varphi, M)\) from the notation.

Note that both sets \(R^+\) and \(R^-\) contain the set of compactly supported \(W^{1,1}\) functions and that they sum up to the whole set \(W^{1,1}_{loc}(\mathbb{R}^+).\) Moreover, they are dilation invariant, i. e. for every \(\lambda > 0\) and \(u \in R\) (where \(R\) is either \(R^+\) or \(R^-\)) we have \(u_{\lambda}(x) := u(\lambda x) \in R.\)

Our main result reads as follows.

\[ \text{Theorem 2.1} \quad \text{Suppose that } M, \varphi, \omega \text{ satisfying } (M), (\mu), (\omega) \text{ are such that Condition } (B1) \text{ (respectively } (B2) \text{) holds true. Then} \]

\[ \int_{\mathbb{R}^+} M(\omega(r)) |u(r)| \mu(dr) \leq C \int_{\mathbb{R}^+} M(|u'(r)|) \mu(dr) \quad (2.13) \]

holds for every \(u \in R^+_M(\omega, \varphi, M)\) (resp. for every \(u \in R^-_M(\omega, \varphi, M)\)), where \(C = c(\frac{LD^2}{b_2\mu})\) (respectively \(C = c(\frac{LD^2}{b_2\mu})\)), \(c(x) = \max(x^{d_M}, x^{D_M})\).

As a direct consequence we also obtain the following theorem.

\[ \text{Theorem 2.2} \quad \text{Suppose that } M, \varphi, \omega \text{ satisfying } (M), (\mu), (\omega) \text{ are such that Condition } (B1) \text{ (respectively } (B2) \text{) holds true. Then the inequality} \]

\[ \|\omega u\|_{L^\mu_M} \leq \tilde{C} \|u'\|_{L^\mu_M} \]

holds for every \(u \in R^+_M(\omega, \varphi, M)\) (resp. for every \(u \in R^-_M(\omega, \varphi, M)\)), where \(\tilde{C} = c(\frac{LD^2}{b_2\mu}) + 1\) (respectively \(\tilde{C} = c(\frac{LD^2}{b_2\mu}) + 1\)), \(c(x) = \max(x^{d_M}, x^{D_M})\).

3 \hspace{1em} \text{Particular cases}

The main goal of this section is to illustrate Theorem 2.1 in various contexts. At first we discuss the case when \(M(\lambda) = \lambda^p\) (Subsection 3.1). Then we turn our attention to a general \(M\) falling within our scope (Subsection 3.2). Finally, in Subsection 3.3, we restrict ourselves to the special choice of weights \(\omega = |\varphi'|\).

3.1 \hspace{1em} \text{Inequalities in the } L^p \text{ setting}

When \(M(\lambda) = \lambda^p\) with \(p > 1\), our conditions get simpler. In particular, \(d_M = D_M = p\), and since \(\varphi'\) is assumed to be nonzero everywhere, we have:

\[ b_1(r, \varphi, \omega) = 1 + \frac{\varphi''(r)}{(\varphi'(r))^2} - p \frac{\omega'(r)}{\omega(r)\varphi'(r)} \chi_{\{\omega(r) \neq 0\}} = -b_2(r, \varphi, \omega). \]
In particular, our Theorem yields results when $L < \infty$, and either $\inf_{r>0} b_1(r, \varphi, \omega) > 0$, or $\sup_{r>0} b_1(r, \varphi, \omega) < 0$.

### 3.1.1 Classical Hardy inequalities

As the first example illustrating our methods, we show that we can get the classical Hardy inequality. This inequality reads as follows (see e.g. [26], Theorem 330 for the classical source, Theorem 5.2 in [36], or [38] for the statement, historical framework and discussion).

**Theorem 3.1** Let $1 < p < \infty$, $\alpha \neq p - 1$. Suppose that $u = u(t)$ is an absolutely continuous function in $(0, \infty)$ such that $\int_0^\infty |u(t)|^p t^\alpha dt < \infty$, and let

$$
    u^+(0) := \lim_{t \to 0} u(t) = 0 \quad \text{for} \quad \alpha < p - 1, \\
    u(\infty) := \lim_{t \to \infty} u(t) = 0 \quad \text{for} \quad \alpha > p - 1.
$$

Then the following inequality holds:

$$
    \int_0^\infty |u(t)|^p t^{\alpha - p} dt \leq C \int_0^\infty |u'(t)|^p t^\alpha dt,
$$

where $C = \left( \frac{p}{|\alpha - p + 1|} \right)^p$.

We consider the case $\alpha \neq 0$. Let us explain how this theorem follows from our results.

Setting

$$
    M(r) = r^p, \quad \mu(dr) = r^\alpha dr = \exp(\alpha \ln r), \quad \omega(r) = \frac{1}{r},
$$

we have

$$
    \varphi(r) = -\alpha \ln r, \quad \varphi'(r) = \frac{-\alpha}{r}, \quad \varphi''(r) = \frac{\alpha}{r^2},
$$

$$
    \omega'(r) = -\frac{1}{r^2}, \quad \omega'(r) \varphi'(r) = \frac{\alpha}{r^2} > 0, \quad d_M = D_M = p.
$$

By a direct check we see that:

$$
    b_1 = \frac{\alpha - (p - 1)}{\alpha}, \quad b_2 = \frac{(p - 1) - \alpha}{\alpha}, \quad L = \frac{1}{|\alpha|}.
$$

Therefore for $\alpha > p - 1$ and for $\alpha < 0$ we have $b_1 > 0$, while for $0 < \alpha < p - 1$ we have $b_2 > 0$. In either case $\varphi'$ does never vanish. The constant $C$ in (2.13) is equal to $\left( \frac{p}{|\alpha - (p - 1)|} \right)^p$, which coincides with the classical statement.
The only thing that remains to be checked is that any function \( u \) as in the statement of Theorem 3.1 for which the right hand side in (3.1) is finite belongs to \( \mathcal{R}^+ \) when \( \alpha > p - 1 \) or \( \alpha < 0 \), and to \( \mathcal{R}^- \) when \( 0 < \alpha < (p - 1) \).

We will use standard arguments (see e.g. [36], proof of Theorem 5.2).

First suppose that \( \alpha > p - 1 \), and let \( u \) be as in the assumptions of Theorem 3.1. Then, for any \( t > 0 \) we define \( U(t) := \int_{t}^{\infty} |u(\tau)|d\tau \). One has:

\[
U(t) = \int_{t}^{\infty} |u'(\tau)|d\tau \leq \int_{t}^{\infty} |u'(\tau)| \frac{\alpha}{\tau^{\frac{\alpha}{p} - \frac{1}{p}}}d\tau \\
\leq \left( \int_{t}^{\infty} |u'(\tau)|^{p} \tau^{\alpha}d\tau \right)^{\frac{1}{p}} \left( \int_{t}^{\infty} \tau^{-\frac{\alpha}{p} - 1}d\tau \right)^{\frac{p-1}{p}} \\
= \left( \frac{p - 1}{\alpha - (p - 1)} \right)^{\frac{p-1}{p}} \left( \int_{t}^{\infty} |u'(\tau)|^{p} \tau^{\alpha}d\tau \right)^{\frac{1}{p}} t^{\frac{1}{p}(\alpha - (p - 1))} < \infty.
\]

From this chain of inequalities and the condition \( \int_{0}^{\infty} |u'(\tau)|^{p} \tau^{\alpha}d\tau < \infty \) we infer not only that \( u \in \mathcal{W}_{1,1,1}^{1}(0, \infty) \) and that \( U \) is well defined, but also that \( \lim_{R \to \infty} U(R)^p R^{\alpha - (p - 1)} = 0 \). Taking into account that \( \lim_{t \to \infty} u(t) = 0 \), we have

\[
|u(t)| = \left| \int_{t}^{\infty} u'(\tau)d\tau \right| \leq \int_{t}^{\infty} |u'(\tau)|d\tau = U(t),
\]

and therefore \( \lim_{R \to \infty} |u(R)|^p R^{\alpha - (p - 1)} = 0 \) as well.

Recall now the formulas defining the class \( \mathcal{R}^+ \). The function \( h(r)e^{-\varphi(r)} \) appearing there is in present situation equal to \(-\frac{1}{\alpha} |u(r)|^{p} r^{\alpha - (p - 1)}\), vanishing for \( r \)'s tending to infinity. Therefore \( u \in \mathcal{R}^+ \).

When \( \alpha < p - 1 \), then we proceed similarly, but now we take \( U(t) := \int_{0}^{t} |u'(\tau)|d\tau \). Again, \( U \) is well defined and \( \lim_{r \to 0} U(r)^p r^{\alpha - (p - 1)} = 0 \), and since now \( |u(r)| \leq U(r) \) as well, this permits to assert that \( u \in \mathcal{R}^- \). We are done.

### 3.1.2 General approach within \( L^p \)-spaces

The following result is considered classical now (see e.g. [44], Theorem 1 of Section 1.3.1).

**Theorem 3.2** Let \( \mu, \nu \) be the nonnegative Borel measures on \((0, \infty)\), let \( \nu^* \) be the absolutely continuous part of \( \nu \) and \( 1 \leq p \leq q \leq \infty \). Then inequality

\[
\left( \int_{\mathbb{R}^+} \left( \int_{0}^{t} f(\tau)d\tau \right)^{\frac{q}{p}} d\mu(t) \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^+} |f(t)|^{p} d\nu(t) \right)^{\frac{1}{p}} \tag{3.2}
\]
holds for an arbitrary locally integrable function \( f \) if and only if

\[
B := \sup (\mu[r, \infty])^{\frac{1}{q}} \left( \int_0^r \left( \frac{d\nu^*}{d\tau} \right)^{-\frac{1}{p-1}} d\tau \right)^{\frac{p-1}{p}} < \infty.
\]

The case \( p = q \geq 1 \) is due to Muckenhoupt [46]. Extensions for general coefficients \( p, q \) were proven by Mazya and Rozin ([44], Theorem 1 of Section 1.3.1), Bradley [9] and Kokilashvili [33]. Some other generalizations (admitting also \( p, q \) below 1) were obtained by Sinnamon [53], Sawyer [51], Bloom and Kerman [6], Stepanov [54] and others.

Observe that inequality (2.13) corresponding to \( \omega(r) = r \) is a particular case of (3.2) when one takes \( \nu(dr) = e^{-\varphi(r)}dr \), \( p = q \), \( \mu(dr) = r^p e^{-\varphi(r)}dr \), but only for the representant \( u(t) = \int_t^0 u'(\tau)d\tau \). In particular \( u^+(0) = 0 \), which we didn’t require (Theorem 3.1 shows that in general the condition \( u^+(0) = 0 \) may not hold). In this case (3.2) reads

\[
\int_{\mathbb{R}^+} |u(\tau)\omega(\tau)|^p d\nu(\tau) \leq C \int_{\mathbb{R}^+} |u'(\tau)|^p d\nu(\tau),
\]

and \( u(t) = \int_0^t u'(s)ds \), \( u \) is an absolutely continuous function. Condition \( B < \infty \) is equivalent to

\[
\sup_{r>0} \left( \int_r^\infty x^p \exp(-\varphi(x)) dx \right) \left( \int_0^r \exp\left(\frac{\varphi(x)}{p-1}\right) dx \right)^{p-1} < \infty.
\]

It is of different nature than our conditions (B1) and (B2) and usually not easy to handle. But since \( B < \infty \) is equivalent to the inequality (3.2) holding for all \( u \) in the set of Hardy transforms \( \mathcal{H} \) (see (1.3)), our assumptions can serve as a tool towards verifying \( B < \infty \).

We may as well deal with the set of conjugate Hardy transforms \( u = -\int_t^\infty f(\tau)d\tau \in \mathcal{H}^\ast \) (see (1.3)) instead of \( u = \int_0^t f(\tau)d\tau \in \mathcal{H} \) as illustrated in Theorem 3.1, the case \( \alpha > p - 1 \).

It may happen that inequalities (3.2) do not hold in general on all set of Hardy transforms \( u = \int_0^t f(\tau)d\tau \in \mathcal{H} \), but they hold on a set essentially smaller than \( \mathcal{H} \), which we will indicate in the sequel. These are the sets \( \mathcal{R}^+, \mathcal{R}^- \) defined by (2.11) and (2.12).
3.2 Results in Orlicz spaces

The series of papers of Maz'ya, Bloom-Kerman and Lai [44, 6, 40, 41, 42, 43, 27] is concerned with inequalities in the form

\[ Q^{-1} \left( \int_{\mathbb{R}^+} Q(\omega(x)Tf(x))r(x)dx \right) \leq P^{-1} \left( \int_{\mathbb{R}^+} P(C\rho(x)f(x))v(x)dx \right), \] (3.3)

where \( Tf \) is the Hardy-type operator

\[ Tf(x) = \int_0^x K(x, y)f(y)dy, \quad x > 0, \]

with a suitable kernel \( K \). The case \( K = 1 \), corresponding to the classical Hardy operator, is included there too. \( P \) and \( Q \) are assumed to be certain nondecreasing functions on \( \mathbb{R}^+ \) satisfying

\[ \lim_{t \to 0^+} P(t) = \lim_{t \to 0^+} Q(t) = 0, \quad \lim_{t \to \infty} P(t) = \lim_{t \to \infty} Q(t) = \infty. \]

Bloom and Kerman proved in [6] that within the class of modular functions \( P \) and \( Q \) satisfying the following domination property:

there exists a constant \( \eta > 0 \) for which \( \sum QP^{-1}(a_j) \leq QP^{-1}(\eta \sum a_j) \),

whenever \( \{a_j\} \) is an arbitrary nonnegative sequence,

(3.3) is then equivalent to the conditions

\[ \int_0^y P^* \left( \frac{G(\epsilon, y)K(y, x)}{B\epsilon v(x)\rho(x)} \right) v(x)dx \leq G(\epsilon, y) < \infty \quad \text{and} \]
\[ \int_0^y P^* \left( \frac{H(\epsilon, y)}{B\epsilon v(x)\rho(x)} \right) v(x)dx \leq H(\epsilon, y) < \infty, \]

holding for all \( y > 0 \) and \( \epsilon > 0 \), where \( P^* \) is the Legendre transform of \( P \),

\[ G(\epsilon, y) = PQ^{-1} \left( \int_y^\infty Q(\epsilon \omega(x))r(x)dx \right), \]
\[ H(\epsilon, y) = PQ^{-1} \left( \int_y^\infty Q(\epsilon \omega(x)K(y, x))r(x)dx \right), \]

\( B > 0 \) is a constant.

In our particular case: \( P = Q(= M) \) with \( M \) satisfying the \( \Delta_2 \)-condition, \( r = v = \exp(-\varphi(x)) \), \( K \equiv 1 \), \( \rho \equiv 1 \) inequality (3.3) reduces to

\[ \int_{\mathbb{R}^+} M(\omega(x)Tf(x))\exp(-\varphi(x))dx \leq C \left( \int_{\mathbb{R}^+} M(f(x))\exp(-\varphi(x))dx \right), \] (3.5)
which is the type of inequality we are dealing with.

In this case conditions (3.4) are simpler and become
\[ \int_0^y M^*(\frac{G(\epsilon,y)}{\epsilon\exp(-\varphi(x))}) \exp(-\varphi(x))dx \leq G(\epsilon,y) < \infty, \]
holding for all \( y > 0 \) and \( \epsilon > 0 \), where
\[ G(\epsilon,y) = \int_y^\infty M(\epsilon\omega(x))\exp(-\varphi(x))dx, \]
\( B > 0 \) is a constant. Further generalization of (3.3), without the restriction \( P << Q \), can be found in Lai’s paper [43].

Condition (3.6) as well as Lai’s condition are not implemented easily: in practice, for given \( \omega, \varphi, M \), it is usually hard to see whether (3.6) holds or not. Conditions (B1) and (B2) are much simpler. When they are satisfied, and when we know that the Hardy operator \( Tf(x) \) (or the dual Hardy operator \( T^*f(x) = \int_x^\infty f(\tau)d\tau \), see [43], the last remark on page 671) belongs to the set \( \mathcal{R}^- \) or \( \mathcal{R}^+ \), then inequality (3.5) is just the statement of Theorem 2.1.

### 3.3 Special choice of weights. The case of \( \omega = |\varphi'| \)

Another case that substantially simplifies the approach is that of \( \omega = |\varphi'| \) (in fact this was used in the proof of the classical Hardy inequality). Since we require \( \varphi \) to be \( C^1 \) with nonzero derivative, \( \varphi' \) is either always positive, or always negative.

This time around, we have:

\[
\begin{align*}
  b_1(r,|\varphi'|,\varphi,M) &= \begin{cases} 
    1 + (1 - d_M) \frac{\varphi''(r)}{(\varphi'(r))^2} & \text{if } \varphi''(r) \leq 0, \\
    1 + (1 - D_M) \frac{\varphi''(r)}{(\varphi'(r))^2} & \text{if } \varphi''(r) \geq 0,
  \end{cases} \\
  b_2(r,|\varphi'|,\varphi,M) &= \begin{cases} 
    -1 + (D_M - 1) \frac{\varphi''(r)}{(\varphi'(r))^2} & \text{if } \varphi''(r) \leq 0, \\
    -1 + (d_M - 1) \frac{\varphi''(r)}{(\varphi'(r))^2} & \text{if } \varphi''(r) \geq 0,
  \end{cases}
\end{align*}
\]

\[ L = 1. \]

In particular (as \( D_M \geq d_M \geq 1 \)) we get:

\[ b_1 > 0 \quad \text{if and only if } \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} < \frac{1}{D_M - 1} =: \tilde{D}_M; \]

\[ b_2 > 0 \quad \text{if and only if } \inf_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} > \frac{1}{d_M - 1} =: \tilde{d}_M. \]

This leads to the following conclusion, which is of separate interest.
Corollary 3.1 Assume that conditions (M), (µ) are satisfied, and 

\[ \left( \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} < \tilde{D}_M \right) \text{ or } \left( \inf_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} > \tilde{d}_M \right). \]

Then the inequalities

\[ \int_{\mathbb{R}_+^*} M(|\varphi'(r)||u(r)|) \exp(-\varphi(r)) dr \leq C \int_{\mathbb{R}_+^*} M(|u'(r)|) \exp(-\varphi(r)) dr; \]

\[ \|\varphi'u\|_{L^M} \leq \tilde{C} \|u'\|_{L^M} \]

hold for every \( u \in \mathcal{R}^+_{(|\varphi'|, \varphi, M)} \) with \( C = c(D_{2M}^2), \tilde{C} = c(D_{2M}^2) + 1 \) in the first case, and for every \( u \in \mathcal{R}^-_{(|\varphi'|, \varphi, M)} \) with \( C = c(D_{2M}^2), \tilde{C} = c(D_{2M}^2) + 1 \) in the other case, \( c(r) = \max(r^dM, r^{-dM}) \).

**Example 3.1 (classical inequalities)** To illustrate this corollary we consider again

\[ M(\lambda) = \lambda^p, \quad p > 1, \quad \varphi(\lambda) = -\alpha \ln r \]

as in Section 3.1.1. In this case we have

\[ d_M = D_M = p, \quad \frac{\varphi''}{(\varphi')^2} \equiv \frac{1}{\alpha}. \]

We have

\[ \frac{1}{\alpha} \equiv \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} < \frac{1}{p - 1} = \tilde{D}_M \quad \text{for } \alpha \in (-\infty, 0) \cup (p - 1, \infty), \]

\[ \frac{1}{\alpha} \equiv \inf_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} > \frac{1}{p - 1} = \tilde{d}_M \quad \text{for } \alpha \in (0, p - 1). \]

Therefore classical Hardy inequality follows from Corollary 3.1 as well.

**Example 3.2 (Hardy inequalities with logarithmic-type weights)**

As another illustration we show what kind of inequality can be obtained for measures \( \mu(dr) = r^\alpha(\ln(1 + r))^\beta \, dr \), with \( \alpha > 0, \beta > 0 \).

In this case we have

\[ \varphi(r) =: \varphi_{\alpha, \beta}(r) = -\alpha \ln r - \beta \ln \ln(1 + r), \]

\[ \varphi'(r) = -\frac{\alpha}{r} - \frac{\beta}{\ln(1 + r)} \frac{1}{1 + r}, \]

\[ \varphi''(r) = \frac{\alpha}{r^2} + \frac{\beta}{(1 + r)^2} \frac{1}{\ln(1 + r)} \left( 1 + \frac{1}{\ln(1 + r)} \right). \]
Choose $\omega(r) = |\varphi'(r)|$, then
\[ b_1(r) = 1 + (1 - D_M) \frac{\varphi''(r)}{(\varphi'(r))^2}, \]
and so $b_1 = 1 - (D_M - 1) \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2}$. As the derivative $\varphi'(r)$ is of order $\frac{1}{r}$ and $\varphi''(r)$ is of order $\frac{1}{r^2}$ on $\mathbb{R}^+$, the supremum involved is finite. Denote
\[ s_{\alpha,\beta} = \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2}, \]
then
\[ b_1 > 0 \iff D_M < 1 + \frac{1}{s_{\alpha,\beta}}. \]

As $\frac{1}{r} \sim \omega(r)$, we arrive at the following.

**Theorem 3.3** Suppose $\alpha, \beta > 0$ and let $M$ be such an $N$–function satisfying (M) that $D_M < 1 + \frac{1}{s_{\alpha,\beta}}$, where $s_{\alpha,\beta}$ is given by (3.7). Then there exists a constant $C > 0$ such that the inequality
\[ \int_{\mathbb{R}^+} M \left( \frac{|u(r)|}{r} r^\alpha (\ln(1 + r))\beta \right) dr \leq C \int_{\mathbb{R}^+} M(|u'(r)|) r^\alpha (\ln(1 + r))\beta dr \]
holds for all $u \in \mathcal{R}^+(\varphi', \varphi, M) \supset C_0^1(\mathbb{R}^+)$. Similar analysis can be performed for negative $\alpha$ or $\beta$.

Practical information how to easily verify the assumption whether $u \in \mathcal{R}^+$ or $u \in \mathcal{R}^-$ will be provided in Subsection 5.1.

## 4 Proofs of Theorems 2.1 and 2.2

Before we pass to the actual proofs, let us formulate two easy lemmas concerning Young functions. Although these lemmas may be known to the specialists (see e.g. [24], [52] and [18] for related results), for readers’ convenience we submit the proofs.

**Lemma 4.1** Suppose that $M$ is a differentiable $N$–function. Then we have.
Suppose that there exists a constant $D_M \geq 1$ such that
\[ M'(r) \leq D_M \frac{M(r)}{r}, \quad \text{for every } r > 0. \quad (4.1) \]
Then for all $r > 0$, $\lambda \geq 1$
\[ M(\lambda r) \leq \lambda^{D_M} M(r). \]

(ii) Suppose that there exists a constant $d_M \geq 1$ such that
\[ d_M \frac{M(r)}{r} \leq M'(r) \quad \text{for every } r > 0. \]
Then for all $r > 0$, $\lambda \leq 1$
\[ M(\lambda r) \leq \lambda^{d_M} M(r). \]

(iii) Suppose that there exist constants $1 \leq d_M \leq D_M$ such that
\[ d_M \frac{M(r)}{r} \leq M'(r) \leq D_M \frac{M(r)}{r} \quad \text{for every } r > 0. \]
Then for all $r > 0$, $\lambda > 0$
\[ M(\lambda r) \leq \max(\lambda^{d_M}, \lambda^{D_M}) M(r) := c(\lambda) M(r). \quad (4.2) \]

We recall Remark 2.1 about interpretation of constants $d_M, D_M$.

**Proof.** We only prove part (i). Part (ii) is proven analogously, while part (iii) is their direct consequence.

From (4.1) we get $\frac{M'(r)}{M(r)} \leq \frac{D_M}{r}$, and further, for any $r > 0, \lambda > 1$
\[ \int_r^{\lambda r} \frac{M'(t)}{M(t)} \, dt \leq \int_r^{\lambda r} \frac{D_M}{t} \, dt, \]
which after integrating gives $[\ln M(t)]_r^{\lambda r} \leq [\ln t^{D_M}]_r^{\lambda r}$, and further $M(\lambda r) \leq \lambda^{D_M} M(r)$. \hfill \Box

**Lemma 4.2** Suppose that $M$ is a differentiable $N$–function, and let $1 \leq d_M \leq D_M$ be two constants such that
\[ d_M \frac{M(r)}{r} \leq M'(r) \leq D_M \frac{M(r)}{r}, \quad \text{for every } r > 0. \quad (4.3) \]
Then for every $r, s > 0$ the following estimate holds true:
\[ \frac{M(r)}{r} s \leq \frac{D_M - 1}{d_M} M(r) + \frac{1}{d_M} M(s). \quad (4.4) \]
Proof. Using the Young inequality \( rs \leq M^*(r) + M(s) \) together with (4.3) we have:

\[
\frac{M(r)}{r} s \leq \frac{1}{d_M} M'(r)s \leq \frac{M^*(M'(r))}{d_M} + \frac{M(s)}{d_M}.
\]

From the very definition of the conjugate function \( M^* \) we have

\[
M(r) = rM'(r) - M^*(M'(r)),
\]

and so

\[
M^*(M'(r)) \leq D_M M(r) - M(r) = (D_M - 1) M(r).
\]

Inserting this into (4.5) we get (4.4). □

Proof of Theorem 2.1. Suppose \( u \in \mathcal{R}^+_{(\omega,\varphi,M)} \) (resp. \( u \in \mathcal{R}^-_{(\omega,\varphi,M)} \)). Choose \( s_n \to 0, R_n \to \infty \) as in (2.11) (resp. (2.12)). To abbreviate, we denote

\[
J = \int_0^\infty M(\omega(r)|u(r)|) e^{-\varphi(r)} dr, \quad J_n = \int_{s_n}^{R_n} M(\omega(r)|u(r)|) e^{-\varphi(r)} dr,
\]

\[
H = \int_0^\infty M(|u'(r)|) e^{-\varphi(r)} dr, \quad H_n = \int_{s_n}^{R_n} M(|u'(r)|) e^{-\varphi(r)} dr,
\]

Let \( h^u \) be given by (2.10). Under our assumptions, it is well defined for every \( r > 0 \). Since \( u \) is \( W^{1,1}_{\text{loc}} \) and \( M \) is locally Lipschitz, we infer that \( h^u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+) \),

\[
(h^u)'(r) = \frac{d}{dr} \left( \frac{1}{\varphi'(r)} \right) M(\omega(r)|u(r)|) + \frac{1}{\varphi'(r)} M'(\omega(r)|u(r)|) (\omega'(r)|u(r)| + \omega(r)u'(r) \text{sgn} u(r)),
\]

(4.6)
in the sense of distributions and almost everywhere, and \( h \) is absolutely continuous on each interval \([s, R] \subseteq (0, \infty)\) (see e.g. [44], Theorems 1 and 2, Sec. 1.1.3). Moreover, for every \( R, s \) such that \( 0 < s < R < \infty \)

\[
\int_s^R (h^u)'(r) e^{-\varphi(r)} dr = h^u(r)e^{-\varphi(r)}|_s^R + \int_s^R M(\omega(r)|u(r)|) e^{-\varphi(r)} dr = \theta(R, s) + \int_s^R M(\omega(r)|u(r)|) e^{-\varphi(r)} dr,
\]

and so we have

\[
J_n = \int_{s_n}^{R_n} (h^u)'(r) e^{-\varphi(r)} dr - \theta_n,
\]

(4.7)
where \( \theta_n = \theta(R_n, s_n) \to \alpha \in [0, \infty] \) (resp. \( [-\infty, 0] \)).

Inserting (4.6) inside (4.7) and using (2.7) yields, after some rearrangement,

\[
J_n = \int_{s_n}^{R_n} \left( -\frac{\varphi''(r)}{\varphi'(r)^2} \right) M(\omega(r)|u(r)|)|e^{-\varphi(r)}dr \\
- \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|)|\omega'(r)||u(r)|\chi_G(r)e^{-\varphi(r)}dr \\
+ \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|) (|\omega'(r)||u(r)|\chi_F(r) + \omega(r)u'(r)\text{sgn} u(r)) e^{-\varphi(r)}dr - \theta_n
= I_n - II_n + III_n - \theta_n. \tag{4.8}
\]

From now on, the proofs for the two cases: (B1), with \( u \in \mathcal{R}^+ \), and (B2), with \( u \in \mathcal{R}^- \), differ slightly.

**Case 1.** Assumption (B1) is satisfied, \( u \in \mathcal{R}^+ \).

As \( M \) satisfies \(-M'(\lambda) \leq -d_M \frac{M(\lambda)}{\lambda} \), one has

\[
-II_n \leq - \int_{s_n}^{R_n} d_M M(\omega(r)|u(r)|) \frac{|\omega'(r)|}{\omega(r)|\varphi'(r)|} \chi_G(r)e^{-\varphi(r)}dr \\
= \int_{s_n}^{R_n} d_M M(\omega(r)|u(r)|) \frac{\omega'(r)}{\omega(r)|\varphi'(r)|} \chi_G(r)e^{-\varphi(r)}dr =: IV_n
\tag{4.9}
\]

As to the estimate of \( III_n \), first estimate \( u'(r)\text{sgn} u(r) \) by \( |u'(r)| \), and then use the inequality \( M'(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \). For every \( r \in \Omega \) (see (2.7)) we have:

\[
M'(\omega(r)|u(r)|) (|\omega'(r)||u(r)|\chi_F(r) + \omega(r)u'(r)\text{sgn} u'(r)) \\
\leq D_M \left[ M(\omega(r)|u(r)|) \frac{|\omega'(r)|}{\omega(r)} \chi_F(r) + M(\omega(r)|u(r)|) \frac{|u'(r)|}{|u(r)|} \right] \\
=: D_M [A_1(r) + A_2(r)]. \tag{4.10}
\]

This implies

\[
III_n \leq D_M \int_{\Omega \cap [s_n, R_n]} \frac{1}{|\varphi'(r)|} [A_1(r) + A_2(r)]e^{-\varphi(r)}dr =: V_n + VI_n. \tag{4.11}
\]

where

\[
V_n = \int_{\Omega \cap [s_n, R_n]} D_M \frac{|\omega'(r)|}{\omega(r)|\varphi'(r)|} \chi_F(r) M(\omega(r)|u(r)|) e^{-\varphi(r)}dr
\]
Consequently, since \( \delta > 0 \),

\[
\text{VI}_n = \int_{\Omega \cap [s_n, R_n]} D_M \frac{|u'(r)|}{|u(r)||\varphi'(r)|} M(\omega(r)|u(r)|) e^{-\varphi(r)} dr.
\]

Definition of the constant \( b_1 \), (2.8), yields

\[
b_1 J_n \leq J_n - I_n - IV_n - V_n,
\]

and the series of estimates (4.8), (4.9), (4.11) gives

\[
J_n = I_n - II_n + III_n - \theta_n \leq I_n + IV_n + V_n + VI_n - \theta_n.
\]

Combining the two we get

\[
b_1 J_n \leq VI_n - \theta_n.
\]

Consequently, since \( b_1 \) is assumed to be positive,

\[
J_n \leq \frac{D_M}{b_1} \int_{\Omega \cap [s_n, R_n]} \frac{A_2(r)}{|\varphi'(r)|} e^{-\varphi(r)} dr - \frac{\theta_n}{b_1}. \tag{4.12}
\]

Now we use the estimates from Lemmas 4.1 and 4.2. For any \( 0 < \delta \leq 1 \) (or, any \( \delta > 0 \) when \( M(\lambda) = \lambda^p \)) and \( r \in \Omega \)

\[
A_2(r) = \frac{\varphi'(r)}{\omega(r)} M(\omega(r)|u(r)|) |u'(r)| \leq \delta \omega(r) \left[ \frac{D_M - 1}{d_M} M(\omega(r)|u(r)|) + \frac{1}{d_M} M\left(\frac{|u'(r)|}{\delta}\right) \right] \leq \delta \omega(r) \left[ \frac{D_M - 1}{d_M} M(\omega(r)|u(r)|) + \frac{1}{d_M} c\left(\frac{1}{\delta}\right) M(|u'(r)|) \right]
\]

(Lemma 4.1 was used in the very last line).

Using now this estimate on \( A_2(r) \), we obtain from (4.12)

\[
J_n \leq \frac{D_M \delta}{b_1} \left( \frac{D_M - 1}{d_M} \int_{s_n}^{R_n} \frac{\omega(r)}{|\varphi'(r)|} M(\omega(r)|u(r)|) e^{-\varphi(r)} dr + \frac{1}{d_M} c\left(\frac{1}{\delta}\right) \int_{s_n}^{R_n} \frac{\omega(r)}{|\varphi'(r)|} M(|u'(r)|) e^{-\varphi(r)} dr \right) - \frac{\theta_n}{b_1} \leq \frac{D_M L \delta}{b_1} \left( \frac{D_M - 1}{d_M} J_n + \frac{1}{d_M} c\left(\frac{1}{\delta}\right) H_n \right) - \frac{\theta_n}{b_1},
\]

or, after rearranging:

\[
\frac{\theta_n}{b_1} + \left( 1 - \frac{D_M L \delta}{b_1 d_M} (D_M - 1) \right) J_n \leq \frac{D_M L \delta}{b_1 d_M} c\left(\frac{1}{\delta}\right) H_n.
\]
Choose now $\delta_0 = \frac{b_d d_M}{LD_M}$, obtaining
\[
\frac{D_M \theta_n}{b_1} + J_n \leq c \left( \frac{1}{\delta_0} \right) H_n.
\]

Only now we let $n \to \infty$. As all limits: $\lim_{n \to \infty} J_n = J$, $\lim_{n \to \infty} H_n = H$, $\lim_{n \to \infty} \theta_n$ are well defined and nonnegative (finite or not), this implies
\[
J \leq c \left( \frac{LD_M^2}{b_1 d_M} \right) H
\]
and finishes the proof.

**Case 2.** We now prove the statement under the assumption (B2), for $u \in \mathcal{R}^-$. We start with an expression similar to (4.8), but the integrals are rearranged somewhat differently.

This time around we write
\[
J_n = \int_{s_n}^{R_n} \left( -\frac{\varphi''(r)}{(\varphi'(r))^2} \right) M(\omega(r)|u(r)|)e^{-\varphi(r)} \, dr
+ \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|)|\omega'(r)||u(r)|\chi_F e^{-\varphi(r)} \, dr
- \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|) (|\omega'(r)||u(r)|\chi_G + \omega(r)u'(r)\text{sgn } u(r)) e^{-\varphi(r)} \, dr - \theta_n
= I_n + II'_n - III'_n - \theta_n. \tag{4.13}
\]

Similarly as before, since $M'(\lambda) \geq d_M \frac{M(\lambda)}{\lambda}$, we get
\[
II'_n \geq \int_{s_n}^{R_n} d_M M(\omega(r)|u(r)|) \frac{\omega'(r)}{\omega'(r)} \chi_F(r)e^{-\varphi(r)} \, dr =: IV'_n
\]
(one can omit the absolute values because $r \in F$). Further, since $-M'(\lambda) \geq -d_M \frac{M(\lambda)}{\lambda}$, for every $r \in \Omega$ we have:
\[
-\frac{\omega'(r)}{\omega'(r)} (|\omega'(r)||u(r)|\chi_G(r) + \omega(r)|u(r)|) \\
\geq -d_M \left[ M(\omega(r)|u(r)|) \frac{\omega'(r)}{\omega'(r)} \chi_G(r) + M(\omega(r)|u(r)|) \frac{|u'(r)|}{|u(r)|} \right] \\
= -d_M [B(r) + A_2(r)],
\]
where $A_2(r)$ is the same as in (4.10). This last estimate combined with (4.13) imply
\[
-III'_n \geq -d_M \int_{\Omega \cap [s_n, R_n]} \frac{1}{|\varphi'(r)|} [B(r) + A_2(r)] e^{-\varphi(r)} \, dr := V'_n + VI'_n.
\]
On the other hand,

\[ b_2 J_n \leq I_n + IV_n' + V_n' - J_n \text{ and } J_n \geq I_n + II_n' - III_n' - \theta_n \geq I_n + IV_n' + V_n' + VI_n' - \theta_n. \]

We get \( b_2 J_n \leq -VI_n' + \theta_n \), which leads to

\[ J_n \leq D_M \frac{b_2}{b_2} \int_{\Omega \cap [s_n, R_n]} A_2(r) e^{-\varphi(r)} dr + \frac{\theta_n}{b_2}. \]

Now the proof follows along the same lines as the proof in the first case, starting from (4.12) up to its end, with \( b_2 \) replacing \( b_1 \) and \( \theta_n \) replacing \(-\theta_n\). We are done. \( \square \)

**Remark 4.1** Observe that for \( M(\lambda) = \lambda^p \) we have \( d_M = D_M = p \), \( c(x) = x^p \). Therefore under the assumptions of Theorem 2.1 the constant \( C \) equals to either \( (L_p b_1)^p \) if \( b_1 > 0 \) or \( (L_p b_2)^p \) if \( b_2 > 0 \).

**Proof of Theorem 2.2:** Without loss of generality we can assume that \( A = \| u' \|_{L_M^p} \) is finite. Let us substitute \( u_A := \frac{u}{\lambda} \) in (2.13). Then we get

\[ \int_{\mathbb{R}^+} \frac{M(C)}{C} \left( \frac{\omega |u|}{A} \right) \exp(-\varphi') \leq 1, \]

which implies \( \| \omega u \|_{L_M^p} \leq A. \) As \( N \)-functions \( \frac{M}{C} \) and \( M \) are equivalent, we get by (2.3) that \( \| \omega \mu \|_{L_M^p} \leq (C + 1) \| \omega u \|_{L_M^p} \leq (C + 1) A = (C + 1) \| u' \|_{L_M^p}. \) Therefore the result follows. \( \square \)

**Remark 4.2** One may compare our results with those recently proven in [7]. Namely, in Theorem 3.1 p. 416 the authors obtain the following inequality for the Gaussian measure, for \( M(\lambda) = \lambda^p \) :

\[ \int_{\mathbb{R}^+} |\omega u|^p \exp(-\frac{x^2}{2}) dx \leq \left( \frac{p}{p-1} \right)^p \int_{\mathbb{R}^+} |u'|^p \exp(-\frac{x^2}{2}) dx, \]

where \( \omega(x) = \frac{\exp(x^2/2(p-1))}{\int_0^\infty \exp(x^2/2(p-1)) dx} \), holding for every \( u \in W_0^{1,p}(\mathbb{R}^+, d\mu) \) (i.e. the completion of \( C_c^\infty(\mathbb{R}^+) \) in the weighted Sobolev space \( W_0^{1,p}(\mathbb{R}^+, d\mu) \), where \( \mu(x) = \exp(-x^2/2) dx \) is the Gaussian measure). In this case we have \( \varphi(x) = x^2/2 \) and the quantity \( \omega(x)/\varphi'(x) \) is not bounded as required by our Theorem 3.1. Instead, the weight \( \omega \) obeys a different requirement, which is the ODE: \( x\omega - (p-1)\omega' = (p-1)\omega^2. \)
5 Analysis of sets $\mathcal{R}$

5.1 Verification of the condition $u \in \mathcal{R}$

The analysis provided in this subsection is two-fold. At first we show an easy practical method to verify whether $u \in \mathcal{R}$ (here $\mathcal{R}$ equals to $\mathcal{R}^+$ or $\mathcal{R}^-$). Further analysis is devoted to the discussion when the condition

$$\int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty. \quad (5.1)$$

together with $u \in \mathcal{H}$ (respectively $u \in \mathcal{H}^*$) implies that $u \in \mathcal{R}$.

We start with the following result.

Proposition 5.1 Suppose $M, \varphi$ satisfy conditions $(M)$, $(\mu)$ $(\omega)$, $\varphi \in C^2(\mathbb{R}_+)$, $\varphi'$ does never vanish and $L < \infty$ (see 2.9). Then the following statements hold true.

(i) Assume that $\varphi'(R) \to 0$ as $R \to \infty$, and $u \in W^{1,1}_{loc}(\mathbb{R}_+)$ be such that $u(R)$ together with $u(R)e^{-\varphi(R)}$ are bounded next to $\infty$. Then for $\varphi' < 0$ we have $u \in \mathcal{R}^+_{(\omega, \varphi, M)}$, while for $\varphi' > 0$ we have $u \in \mathcal{R}^-_{(\omega, \varphi, M)}$.

(ii) Assume that $\varphi'(r) \to 0$ as $r \to 0$, and and $u \in W^{1,1}_{loc}(\mathbb{R}_+)$ be such that $u(r)$ together with $u(r)e^{-\varphi(r)}$ are bounded next to $0$. Then for $\varphi' > 0$ we have $u \in \mathcal{R}^+_{(\omega, \varphi, M)}$, while for $\varphi' < 0$ we have $u \in \mathcal{R}^-_{(\omega, \varphi, M)}$.

Proof. i): We have ($h^u$ was defined by (2.10)):

$$|h^u(R)| \exp(-\varphi(R)) = \frac{M(\omega(R)|u(R)|)}{|\varphi'(R)|} \exp(-\varphi(R))$$
$$= \frac{M(|\varphi'(R)||u(R)|\omega(R)|\varphi'(R)|)}{|\varphi'(R)|} \exp(-\varphi(R))$$
$$\leq c(L) \frac{M(|\varphi'(R)||u(R)|)}{|\varphi'(R)|} \exp(-\varphi(R)) \chi_{\{u(R)\neq 0\}},$$

where $c(\cdot)$ is defined in (4.2). Property $\frac{M(r)}{r} \to 0$ as $r \to 0$ and our assumptions imply

$$\frac{M(|\varphi'(R)||u(R)|)}{|\varphi'(R)||u(R)|} \chi_{\{u(R)\neq 0\}} \to 0 \text{ as } R \to \infty$$

and so $c(L)|u(R)| \exp(-\varphi(R))$ is bounded next to $\infty$. Therefore $h^u(R)|\exp(-\varphi(R)) \to 0$ as $R \to \infty$ and the statement now follows from definition of $\mathcal{R}^+$ and $\mathcal{R}^-$. 

ii): Similarly as in the proof of part i), we check that $|h^u(r)\exp(-\varphi(r))| \to 0$ as $r \to 0$. □

In the remaining part of this subsection we examine the property (5.1).

To proceed, let us set some additional notation.

First, recall $c(\cdot)$ from (4.2), and then define for $r > 0$

$$f_\varphi(r) = e^{-1}(e^{-\varphi(r)}),$$

$$(5.2)$$

$$A_\varphi(r) = \frac{1}{f_\varphi} \|L(M,r)(0,r)\|$$

$$B_\varphi(r) = \frac{1}{f_\varphi} \|L(M,r,\infty)\|.$$  

The norms considered here are the dual norms defined by (2.4).

We will distinguish two naturally appearing cases: the first one when $A_\varphi$ is well defined, and the other when $B_\varphi$ is well defined.

CASE 1. $A_\varphi(r)$ well defined for small $r$'s.

We are now to analyze the case when $u \in H$ which satisfies (5.1) belongs to $R$.

We start with a following lemma.

**Lemma 5.1** Assume that

1. $M, \varphi, \omega$ satisfy (M), (\varphi), (\omega),

2. $A_\varphi$ is well defined for small $r$'s and the function

$$K(r) = \frac{M(\omega(r)A_\varphi(r))}{|\varphi'(r)|} e^{-\varphi(r)}$$

is bounded next to 0.

Then for every $u \in W_{loc}^{1,1}(\mathbb{R}_+)$ such that $\int_0^\infty M(|u'(r)|)e^{-\varphi(r)} \, dr < \infty$ and $\lim_{r \to 0} u(r) = 0$ the function $h^u$ defined by (2.10) satisfies

$$\lim_{r \to 0} h^u(r)e^{-\varphi(r)} = 0.$$  

$$(5.3)$$

**Proof.** Set

$$U(r) = \int_0^r |u'(\rho)| \, d\rho.$$  

From inequality (2.5) we have

$$\int_0^r |u'(\rho)| \, d\rho = \int_0^r |u'(\rho)f_\varphi(\rho)| \cdot \frac{1}{|f_\varphi(\rho)|} \, d\rho \leq \|u' \cdot f_\varphi\|_{L(M)(0,r)} \cdot A_\varphi(r).$$  

$$(5.4)$$
We have
\[ \| u' f \|_{L^M(0,r)} \leq \| u' \|_{L^M(0,r)}. \] (5.5)

Indeed, if \( A = \| u' f \|_{L^M(0,r)} \) then we get from the definition of \( f \varphi \) (5.2) and the property (4.2):
\[
1 = \int_{(0,r)} M \left( \frac{|u'|}{A} \right) dx \leq \int_{(0,r)} c(\varphi) M \left( \frac{|u'|}{A} \right) dx = \int_{(0,r)} M \left( \frac{|u'|}{A} \right) \exp(-\varphi) dx.
\]

Therefore \( A \leq \| u' \|_{L^M(0,r)} \), which gives (5.5).

As dual and Luxemburg norms are equivalent, we get
\[
\| u' f \|_{L^M(0,r)} \leq A \| u' \|_{L^M(0,r)} \leq A \| u' \|_{L^M(0,\infty)} < \infty,
\]
where \( A \) is some universal constant. Therefore \( U \) is well-defined.

From the assumption \( \lim_{r \to 0} u(r) = 0 \) we get \( |u(r)| \leq U(r) \), and the estimate (5.4) holds true for \( u \) instead of \( U \) as well.

Now,
\[
|h^u(r)e^{-\varphi(r)}| = \frac{M(\omega(r)|u(r)|)}{|\varphi'(r)|} e^{-\varphi(r)} \leq \frac{M(\omega(r)|u'\|_{L^M(0,r)} A \varphi(r))}{|\varphi'(r)|} e^{-\varphi(r)} \leq \frac{M(\omega(r)A \varphi(r))}{|\varphi'(r)|} e^{-\varphi(r)} c(||u'\|_{L^M(0,r)})) = K(r)c(||u'\|_{L^M(0,r)}).
\]

But since \( c(x) \to 0 \) when \( x \to 0 \), and \( ||u'\|_{L^M(0,r)} \to 0 \) when \( r \to 0 \), the assertion (5.3) follows from the boundedness of \( K(r) \) for small \( r \)’s. \( \square \)

As a corollary, we obtain straight from the definition of \( R^+ \), \( R^- \):

**Corollary 5.1** Suppose that the assumptions 1, 2 of Lemma 5.1 are satisfied. Then we have.

i) When \( \varphi' > 0 \), then
\[
\{ u \in H, \int_{\mathbb{R}^+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \} \subset R^+;
\]

ii) When \( \varphi' < 0 \), then
\[
\{ u \in H, \int_{\mathbb{R}^+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \} \subset R^-.
\]

To illustrate the statements above we now discuss the following example.
Example 5.1 Let us consider the case \( \varphi(r) = -\alpha \ln r, \alpha < p - 1, M(\lambda) = \lambda^p \) and \( \omega(r) = \frac{1}{r} \) as in Theorem 3.1. In such a case we have \( M^*(\lambda) = c_p \lambda^{\frac{p}{p-1}} \) and, in this range of \( \alpha \)'s,

\[ f_\varphi(r) = r^{\frac{\alpha}{p}}, \quad M^*(f_\varphi(r)) = c_p r^{\frac{\alpha}{p-1}}, \quad A_\varphi(r) = a_p r^{\frac{p-1}{p}}. \]

Then \( K(r) \) is just a constant. Moreover, every function \( u \in H \) which satisfies

\[ \int_{0}^{\infty} |u(r)|^p x^\alpha dx < \infty \]

belongs to \( \mathcal{R}^-_{(\frac{1}{2}, x^\alpha, \lambda^p)} \) for \( \alpha < 0 \), and to \( \mathcal{R}^+_{(\frac{1}{2}, x^\alpha, \lambda^p)} \) when \( \alpha > 0 \).

**CASE 2.** \( B_\varphi(R) \) well defined for large \( R \)'s.

In this case we have the following dual statements dealing with the property that \( u \in H^* \) which satisfies (5.1) belongs to \( \mathcal{R} \).

**Lemma 5.2** Assume that

1. \( M, \varphi, \omega \) satisfy (M), (\( \varphi \)), (\( \omega \)),
2. \( B_\varphi \) is well defined and the function

\[ L(R) = \frac{M(\omega(R)B_\varphi(R))}{|\varphi'(R)|} e^{-\varphi(R)}, \]

is bounded next to infinity.

Then for every \( u \in H^* \) is such that \( \int_{0}^{\infty} M(|u'(r)|) e^{-\varphi(r)} dr < \infty \) the function \( h^u \) defined by (2.10) satisfies

\[ \lim_{R \to \infty} h^u(R) e^{-\varphi(R)} = 0. \]

**Proof.** It is almost identical with that of Lemma 5.1. We must replace now \( U(r) = \int_{0}^{R} |u'(\rho)| d\rho \) with \( U^*(R) = \int_{R}^{\infty} |u'(\rho)| d\rho \) and then proceed as before.

\( \Box \)

As a counterpart of Corollary 5.1 we assert the following.

**Corollary 5.2** Suppose that the assumptions of Lemma 5.2 are satisfied. Then we have.

i) When \( \varphi' > 0 \), then

\[ \{ u \in H^*, \quad \int_{\mathbb{R}^+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \} \subset \mathcal{R}^-; \]

ii) When \( \varphi' < 0 \), then

\[ \{ u \in H, \quad \int_{\mathbb{R}^+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \} \subset \mathcal{R}^+. \]
This result is illustrated by following example.

**Example 5.2** We again consider the case $\varphi(r) = -\alpha \ln r$, $M(\lambda) = \lambda^p$ and $\omega(r) = \frac{1}{r}$ as in Theorem 3.1, but now $\alpha > p - 1$. In such a case we have $B_{\varphi}(r) = b_p r^{-\frac{\alpha + p - 1}{p}}$ and $L(r)$ is just a constant. Therefore every function $u \in \mathcal{H}^*$ which satisfies $\int_0^\infty |u'(x)|^{p \alpha} dx < \infty$ belongs to $\mathcal{R}^+_{(\frac{1}{p}, p, \lambda)}$.

**Remark 5.1** Suppose that the assumptions of Theorem 2.1 are satisfied and let us put $\mathcal{R} = \mathcal{R}^+$ in case of (B1) and $\mathcal{R} = \mathcal{R}^-$ in case of (B2). One could ask whether the spaces

$$\{u \in \mathcal{R} : \int_{\mathbb{R}^+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty\},$$

playing the crucial role in the inequality (2.13) can possibly be nonlinear. We do not know the answer to this question.

### 5.2 Relation to Bloom-Kerman results

We are now to compare our results with that of Bloom and Kerman from [6].

**5.2.1 When our conditions imply Bloom and Kerman ones**

Here we will show an example where our assumptions yield inequality (3.5) for all functions $Tf = \int_0^t f(\tau) d\tau \in \mathcal{H}$, so inherently the Bloom-Kerman condition is satisfied.

Corollary 5.1 results in the following proposition.

**Proposition 5.2** Assume that $(M, \varphi, \omega)$ satisfy the assumptions 1,2 in Lemma 5.1 and either $(\varphi' > 0, b_1 > 0, L < \infty)$ or $(\varphi' < 0, b_2 > 0, L < \infty)$. Then we have.

(i) There exists a constant $C > 0$ such that inequality

$$\int_0^\infty M(\omega(x)|u(x)|) \exp(-\varphi(x)) dx \leq C \int_0^\infty M(|u'(x)|) \exp(-\varphi(x)) dx$$

holds for every $u \in \mathcal{H}$. 

(ii) The triple \((\omega, \varphi, M)\) satisfies the Bloom-Kerman condition:
\[
\int_0^y M^* \left( \frac{G(\epsilon, y)}{B \exp(-\varphi(x))} \right) \exp(-\varphi(x)) \, dx \leq G(\epsilon, y) < \infty, \tag{5.6}
\]
holding for all \(y > 0\) and \(\epsilon > 0\),
\[
G(\epsilon, y) = \int_y^\infty M(\epsilon \omega(x)) \exp(-\varphi(x)) \, dx,
\]

\(B > 0\) is a constant.

Proof. i): Statement i) is just a combination of Corollary 5.1 and Theorem 2.1.

ii): Let \(u(x) = \int_0^x f(\tau) \, d\tau = (Tf)(x) \in \mathcal{H}\) be the Hardy transform of \(f\). Then part i) implies
\[
\int_0^\infty M(\omega(x)Tf(x)) \exp(-\varphi(x)) \, dx \leq C \int_0^\infty M(f(x)) \exp(-\varphi(x)) \, dx,
\]
which is proven to be equivalent to the condition (5.6) (see Theorem 1.7 in [6] and our comments in Subsection 3.2). \(\Box\)

5.2.2 When Bloom and Kerman conditions are not satisfied

It may happen that our conditions are satisfied and the Bloom-Kerman conditions are not. In such a case inequalities (2.13) cannot hold for every Hardy transform \(u \in \mathcal{H}\) (see (1.3)) but they hold on proper subsets in the set of Hardy transforms. This is illustrated by the following result.

**Proposition 5.3** There exists a triple \((\omega, \varphi, M)\) such that conditions (M), (\(\mu\)), (\(\omega\)) are satisfied and

i) \((\omega, \varphi, M)\) satisfies (B1), in particular inequality
\[
\int_{\mathbb{R}_+} M(\omega(r)) \, |u(r)| \, \mu(dr) \leq C \int_{\mathbb{R}_+} M(|u'(r)|) \, \mu(dr) \tag{5.7}
\]
is satisfied whenever \(u \in \mathcal{R}^+_{(\omega, \varphi, M)}\).

ii) \((\omega, \varphi, M)\) does not satisfy the Bloom-Kerman condition (5.6).

iii) The set
\[
\mathcal{R}^{(0,+)}_{(\omega, \varphi, M)} := \mathcal{R}^+_{(\omega, \varphi, M)} \cap \mathcal{H} \subseteq \mathcal{H}
\]
is a proper subset of \(\mathcal{H}\). Moreover, inequality (5.7) is not satisfied for every \(u \in \mathcal{H}\), with the constant independent of \(u\).
Proof. Let \( p > 1 \) and
\[
M(\lambda) = \lambda^p, \quad \phi(x) = -\frac{1}{2}x^2, \quad \omega(x) = x.
\]
In particular conditions (M), (\( \mu \)), (\( \omega \)) are satisfied, \( b_1 = 1 + \frac{1}{2x}(p - 1) > 0 \), \( L = 1 \) and the condition (B1) is also satisfied. Therefore the result i) follows by Theorem 2.1.

ii): The Bloom-Kerman condition does not hold: one has just
\[
G(\epsilon, y) = \int_y^\infty (\epsilon x)^p e^{\frac{1}{2}x^2} dx = \infty,
\]
and so (5.6) is violated.

iii): The Laplace function
\[
u(r) = \int_0^r \exp(-\tau^2)d\tau,
\]
belongs to \( \mathcal{H} \setminus \mathcal{R}^{0,+}_{(\omega, \phi, M)} \) and does not satisfy (5.7).

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