A variational approach to the low energy properties of even-legged d-dimensional quantum spin systems

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Abstract

We develop a variational formalism in order to study the structure of low energy spectra of frustrated quantum spin systems. It is first applied to trial wavefunctions of ladders with one spin 1/2 on each site. We determine energy minima of these states. The variational ground state shows a finite energy gap with respect to the energies of states which span the Hilbert space and are orthogonal to it. This is the case for any size of the system. Under some justifiable approximations the argument can be extended to even-legged ladder systems in $2d$ and higher dimensional spaces. The Hamiltonian can contain spin-spin coupling interactions of any range. For specific values of the coupling strengths level degeneracies can occur.

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1 Introduction.

The low energy spectra of magnetic materials can be described by quantum spin models characterized by their geometric structure, the form of the interactions between the spins located on sites, the strength and the range of the couplings which fix the intensity of the interactions. The knowledge of the spectral properties of these systems is considered to be of prime importance for the understanding of specific phenomena such as superconductivity at high $T_c$. Investigations on this subject which go on for many years have concentrated in the recent past on specific structures, in particular the presence of ladders and stripes in 2d superconducting material, see f.i. [1, 2, 3, 4].

It has been observed that systems which expectedly behave like 2-leg ladders show a gap between the ground state and the first excited state [5] whereas those which behave like 3-leg ladders possess a continuous spectrum [6]. Theoretical investigations have been developed in order to study the properties of the ground state and low energy states with a particular interest for the existence or absence of such a gap. In 1d systems Lieb, Schultz and Mattis (LSM) [7] established that spin-1/2 systems which are invariant under translations and rotations either show no finite energy gap between the ground state and the first excited state, or the ground state is degenerate and spontaneously breaks the translational invariance creating a dimerized structure. In the first case the energy interval between the ground state and the first excited state $\Delta E$ goes to zero as $1/L$ when the length $L$ of the chain tends to $\infty$. A large amount of work on spectral properties of quantum spin systems followed and can be found in the litterature from the late 60s to the late 80s, see f. i. [8, 9, 10, 11, 12].

An important step was performed by Haldane who conjectured that the spectra of Heisenberg antiferromagnetic chains are gapless if the spins are half-integers and show a gap if the spins are integers [13, 14]. Later Shelton et al. [15] found that weakly coupled isotropic spin-1/2 antiferromagnetic Heisenberg chains always show a spectral gap. In an application of the nonlinear sigma-model Sierra [16] tried to extend the Haldane conjecture to ladders. He showed that in the framework of this model spin-1/2 even ladders exhibit a gapped spectrum whereas odd ladders show a gapless spectrum. Further tests of Haldane’s conjecture have been made, see f.i. [17, 18] and refs. therein. Oshikawa extended the LSM theorem to quantum spin systems.
with a conserved particle number on a periodic lattice in arbitrary dimensions and showed that an excitation gap opens when the number of particles per unit cell of the ground state is an integer \[19\]. The theorem was also found to be verified in chiral liquid spin systems in higher dimensions \[20\]. There it was however observed that the conclusions drawn in ref. \[19\] may not be universally verified. More recently Hastings \[21\] showed that \(\Delta E\) goes to zero like \(\ln L/L\) for spin-1/2 systems in higher dimensions when the Hamiltonian of the system is \(SU(2)\) invariant and sites are occupied by an odd number of spin-1/2 particles.

Further information has been obtained by means of specific models, in particular quantum spin systems, ladders and 2d networks, see in particular \[22, 23, 24, 26, 27\]. These studies deal with specific systems using different formal tools. They show non trivial low energy spectral properties and very rich phase diagrams. However the low energy properties of 2d and higher quantum spin systems remain an open question in a large number of cases.

In the present work we propose a further investigation concerning the existence of a finite energy gap \(\Delta E\) between the ground state and the low energy states of frustrated quantum spin systems. More specifically we consider spin 1/2 ladders and extend the analysis to even-legged 2d systems, as well as systems with an even number of chains in an arbitrary number of space dimensions. We show that the results are valid for any range of the spin-spin interaction.

The investigation relies on a variational approach. We consider first the case of spin-1/2 ladders. The central concept which governs the behaviour of the system is SO(4)-invariance of the Hamiltonian. We introduce a trial wavefunction in order to determine the energy extremum of the system and analyze the nature of this extremum. If this state corresponds to a minimum in energy we show that the states orthogonal to it are characterized by an energy gap with respect to this state except for specific values of the coupling constants which enter the Hamiltonian leading to degeneracies. We then show how the results can be extended to 2d and higher dimensional spaces for systems with an even number of legs. Finally we comment and discuss the outcome of the present investigations and draw conclusions.
2 The model.

We consider spin-1/2 ladders [28, 29] described by Hamiltonians of the following type

$$H^{(s,s)} = J_t \sum_{i=1}^{N} s_{i1} s_{i2} + J_l \sum_{<ij>} s_{i1} s_{j1} + J_l \sum_{<ij>} s_{i2} s_{j2} + J_{1c} \sum_{<ij>} s_{i1} s_{j2} \quad (1)$$

$$+ J_{2c} \sum_{<ij>} s_{i2} s_{j1}$$

where the indices 1 or 2 label the spin 1/2 vector operators $s_{ik}$ acting on the sites $i$ on both ends of a rung, in the second and third term $i$ and $j$ label nearest neighbours along the legs of the ladder. The fourth and fifth term correspond to diagonal interactions between nearest sites located on different legs. $N$ is the number of sites on a ladder and the coupling strengths $J_t, J_l, J_{1c}, J_{2c}$ are positive. In the sequel we fix $J_{1c} = J_{2c} = J_c$. The ladder is represented in Fig. 1.

The Hamiltonian (1) possesses $SO(4)$ symmetry by construction. By means of a spin rotation [38]

$$s_{i1} = \frac{1}{2} (S_i + R_i) \quad (2)$$

$$s_{i2} = \frac{1}{2} (S_i - R_i) \quad (3)$$

it can be expressed in the form

$$H^{(S,R)} = \frac{J_t}{4} \sum_{i=1}^{N} (S_i^2 - R_i^2) + J_l \sum_{<ij>} S_i S_j + J_2 \sum_{<ij>} R_i R_j \quad (4)$$

where the components $S_i^{(+)}$, $S_i^{(-)}$, $S_i^{(z)}$ and $R_i^{(+)}$, $R_i^{(-)}$, $R_i^{(z)}$ of the vector operators $S_i$ and $R_i$ are the $SO(4)$ group generators which can be written as

$$S_i^{(+)} = \sqrt{2} (X_i^{(11)(10)} + X_i^{(10)(1-1)}) = S_i^{(-)*}$$

$$S_i^{(z)} = X_i^{(11)(11)} - X_i^{(1-1)(1-1)}$$
Figure 1: Top: the original spin ladder. The coupling strengths are indicated as given in the text. Bottom: The ladder in the SO(4) representation. See the text.

\[ R_i^{(+)} = \sqrt{2}(X_i^{(11)(00)} + X_i^{(00)(1-1)}) = R_i^{(-)*} \]

\[ R_i^{(z)} = -(X_i^{(10)(00)} + X_i^{(00)(10)}) \]

where

\[ X_i^{(S'M'S_S)} = |SM_s\rangle_{i; i} \langle S'M_S'| \]

and the states \(|SM_S\rangle\) are generated by the coupling of spin 1/2 states

\[ |SM_s\rangle_i = \sum_{m_1, m_2} \langle 1/2 \ m_1 \ 1/2 \ m_2 | 1 \ 0 \ |1/2 \ m_1 \rangle_{i} |1/2 \ m_2 \rangle_i \]

to two particle singlet \((S = 0, M_S = 0)\) and triplet \((S = 1, M_S = 0, +1, -1)\) states.

In the new representation the coupling strengths are \(J_1 = (J_l + J_c)/2\) and \(J_2 = (J_l - J_c)/2\). \(J_1\) is always positive, the sign of \(J_2\) depends on the relative strength of the couplings between spins on sites located on the same leg \((J_l)\) and different legs \((J_c)\). The SO(4) symmetry is shown to be of particular interest for the characterization of the spectral properties of even-legged spin systems. This point appears clearly below.
3 Variational ansatz.

3.1 N-body wavefunction.

Variational techniques have been applied for a long time in order to investigate the properties of quantum spin systems, see f. i. refs. [11, 30, 31, 32, 33, 34, 35, 36, 37]. In analogy with the phenomenological BCS approach and the approach of refs. [39, 40, 24, 25] we introduce a trial wavefunction which takes a product form of two spin-1/2 states

$$|\Phi_0^{(N)}\rangle = \prod_{i=1}^{N} |\Psi_i^{(0)}\rangle \, .$$

Here $|\Psi_i^{(0)}\rangle$ is the wavefunction at the location $i$ which is chosen as a linear combination of singlet and triplet states $|SM\rangle_i$

$$|\Psi_i^{(0)}\rangle = \alpha_i |00\rangle_i + \beta_i |11\rangle_i + \beta_{01} |10\rangle_i + \beta_{11} |1+1\rangle_i \, .$$

with the normalization

$$|\alpha_i|^2 + |\beta_i|^2 + |\beta_{01}|^2 + |\beta_{11}|^2 = 1 \, .$$

The time-reversal symmetry of the Hamiltonian allows to introduce $|\beta_T|^2 = |\beta_{11}|^2$. A further constraint is imposed such that $\beta_{i1} = \beta_{iT} = \kappa \beta_{i0}$ where $\kappa$ is fixed as the ratio

$$\kappa = \frac{\langle 1/2 \ 1/2 \ 1/2 \ 1/2 |1 1 \rangle}{\langle 1/2 \ 1/2 \ 1/2 \ -1/2 |1 0 \rangle} = \sqrt{2} \, .$$

where the Clebsh-Gordan coefficients weigh the coupling of the two electron state $|1/2m_1; 1/2m_2\rangle_i$ to the states $|11\rangle_i$ and $|10\rangle_i$ respectively. Each site is occupied by a single spin. The constraint takes care of the relative weight of the $|1 1\rangle$ and the $|1 0\rangle$ two-spin states. Defining $\gamma_i = (1 + 2\kappa^2)^{1/2} \beta_{i0}$ leads to the normalization relation

$$|\alpha_i|^2 + |\gamma_i|^2 = 1 \, .$$
The imposed constraint fixes $\beta_i = \kappa \gamma_i / (1 + 2 \kappa^2)^{1/2} = \beta_i \tau$. The $\{\alpha_i\}$ and $\{\gamma_i\}$ are the variational quantities.

Using the expressions of the operators $S_i$ and $R_i$ defined above the energy $E^0_N = \langle \Phi_0^{(N)} | H^{(S,R)} | \Phi_0^{(N)} \rangle$ associate to the trial wavefunction $| \Phi_0^{(N)} \rangle$ is obtained as

$$E^0_N = 2E_T \sum_{i=1}^N |\gamma_i|^2 + E_S \sum_{i=1}^N (3|\alpha_i|^2 + |\gamma_i|^2) + E_{int1} + E_{int2} . \quad (10)$$

with $E_T = J_t/4$, $E_S = - J_t/4$,

$$E_{int1} = \frac{8 \kappa^2 J_1}{(1 + 2 \kappa^2)^2} \sum_{<ij>} |\gamma_i|^2 |\gamma_j|^2 . \quad (11)$$

and

$$E_{int2} = \frac{4 J_2}{(1 + 2 \kappa^2)} \sum_{<ij>} (Re(\alpha_i \gamma_i^*) Re(\alpha_j \gamma_j^*)) \quad (12)$$

$$+ \frac{8 \kappa^2 J_2}{(1 + 2 \kappa^2)} \sum_{<ij>} (Im(\alpha_i^* \gamma_i) Im(\alpha_j \gamma_j^*)) \quad (12)$$

### 3.2 Extrema of the energy

In order to fix the energy spectrum of the system the complex amplitudes $\alpha_i$ and $\gamma_i$ are varied under the normalization constraint Eq. (9)

$$\left( \frac{d}{d \alpha_i^*} + \frac{d \gamma_i}{d \alpha_i} \right) E^0_N = 0 . \quad (13)$$

We introduce the parametrization

$$\alpha_i = exp(i \phi_{1i}) sin(\theta_i/2) . \quad (14)$$

and

$$\gamma_i = exp(i \phi_{2i}) cos(\theta_i/2) . \quad (15)$$
Applying Eq. (13) to $E_N^0$ given by Eq. (10) and using Eq. (14) and Eq. (15) leads to

$$
\epsilon_i \exp(i(\phi_{1i} - \phi_{2i}) \sin(\theta_i) + (\cos^2(\theta_i/2) - \sin^2(\theta_i/2))\Delta_i = 0 .
$$

(16)

where

$$
\epsilon_i = (E_S - E_T) - \frac{4\kappa^2 J_1}{(1 + 2\kappa^2)^2} \sum_{<j>} |\gamma_j|^2 .
$$

(17)

The energy $\epsilon_i$ is a negative quantity, $\Delta_i$ is complex and given by

$$
\Delta_i = \frac{J_2}{1 + 2\kappa^2} \sum_{<j>} ((1 - 2\kappa^2)\alpha_j^* \gamma_j + (1 + 2\kappa^2)\alpha_j \gamma_j^*) .
$$

(18)

The symbol $< j >$ indicates the nearest neighbour of $i$, in practice $j = i + 1$. $\Delta_i$ can be written as

$$
\Delta_i = |\Delta_i| \exp(i(\phi_{1i} - \phi_{2i})) .
$$

(19)

The solution of Eq. (16) takes the form

$$
\epsilon_i \sin\theta_i + |\Delta_i| \cos\theta_i = 0
$$

(20)

with

$$
\sin\theta_i = -|\Delta_i| / E_i
$$

(21)

and

$$
\cos\theta_i = \epsilon_i / E_i .
$$

(22)

Here $E_i^2 = \epsilon_i^2 + |\Delta_i|^2$ and

$$
|\gamma_i|^2 = \frac{1}{2}(1 + \frac{\epsilon_i}{E_i}) .
$$

(23)

$$
|\alpha_i|^2 = \frac{1}{2}(1 - \frac{\epsilon_i}{E_i}) .
$$

(24)

$$
\Delta_i = \Delta_i^r + i \Delta_i^i .
$$

(25)

$$
\Delta_i^r = -\frac{J_2}{(1 + 2\kappa^2)} \sum_{<j>} \frac{\Delta_j^r}{E_j} .
$$

(26)

$$
\Delta_i^i = -\frac{2\kappa^2 J_2}{(1 + 2\kappa^2)} \sum_{<j>} \frac{\Delta_j^i}{E_j} .
$$

(27)
The energy $E_i$ can in principle be positive or negative. The expression of $\epsilon_i$ in Eq. (17) shows that the variational wavefunction $|\Phi_0^{(N)}\rangle$ postulated in Eq. (5) entangles nearest neighbouring site state wavefunctions.

The Hamiltonian of the spin ladder is fixed by the three coupling constants $J_t$, $J_1$ and $J_2$. A close inspection of the expressions of $\epsilon_i$ (Eq. (17)), $\Delta_i$ (Eq. (18)) shows that the amplitude $\alpha_i$ and hence $\gamma_i$ of each site can be determined self-consistently if the system is closed in such a way that site $N + 1$ is identified with site 1.

### 3.3 Nature of the extrema.

The nature of the extremum is given by the sign of the second derivative $d^2E_N^0$ of the energy $E_N^0$ with respect to $\alpha_i$ under the constraint fixed by Eq. (9) and the definition of $\gamma_i$. The explicit calculation leads to

$$d^2E_N^0 = \frac{J_t}{4}(-3 + \frac{|\alpha_i|^2}{|\gamma_i|^2}) + \frac{|\alpha_i|^2}{|\gamma_i|^2} \frac{8\kappa^2J_1}{(1 + 2\kappa^2)^2} \sum_{<j>} |\gamma_j|^2.$$ (28)

The sign of $d^2E_N^0$ depends on the relative strengths of $J_t$, $J_1$ which are positive quantities and the ratio of the amplitudes $|\alpha_i|^2$ and $|\gamma_i|^2$. In fact

$$\frac{|\alpha_i|^2}{|\gamma_i|^2} = \frac{E_i - \epsilon_i}{E_i + \epsilon_i}.$$ (29)

which is larger or smaller than 1 depending on the sign of $E_i$.

$d^2E_N^0$ is positive if $J_t$ is small enough so that the first term in Eq. (28) gets smaller than the second term which is strictly positive.

More precisely, for fixed $J_t$ and $J_1$ $d^2E_N^0$ is positive if $|\alpha_i|^2/|\gamma_i|^2$ gets larger than a minimum value,

$$|\alpha_i|^2/|\gamma_i|^2 \geq \frac{3}{1 + \frac{4}{J_t}F(\kappa, J_1)}.$$ (30)
where

\[ F(\kappa, J_1) = \frac{8\kappa^2 J_1}{(1 + 2\kappa^2)^2} \sum_{<j>} |\gamma_j|^2 \]

This may be realized if \( J_1 \) is large and (or) \(|\Delta_i|\) is small enough which may be the case for small \(|J_2| = |(J_I - J_c)|/2\).

4 Energy of the variational state. Nature of the spectrum.

4.1 Unprojected wavefunction.

Using the explicit expressions of the amplitudes \( \{\alpha_i\} \) and \( \{\gamma_i\} \) leads to the expression of the variational energy

\[ E_N^0 = (E_T + 2E_S)N + (E_T - E_S)\sum_i \frac{\epsilon_i}{E_i} \]

\[ + \frac{2\kappa^2 J_1}{(1 + 2\kappa^2)^2} \sum_i (1 + \frac{\epsilon_i}{E_i}) \sum_{<j>} (1 + \frac{\epsilon_j}{E_j}) \]

\[ + \frac{J_2}{(1 + 2\kappa^2)} \sum_i \frac{\Delta_i}{E_i} \sum_{<j>} \frac{\Delta_j}{E_j} - \frac{2J_2\kappa^2}{(1 + 2\kappa^2)} \sum_i \frac{\Delta_i}{E_i} \sum_{<j>} \frac{\Delta_j}{E_j} \]

We want to analyze the properties of the energy spectrum, in particular to see whether it is discrete or continuous.

Since the Hamiltonian of the system is real symmetric the space of states is spanned by a complete set of states which are orthogonal to each other.

\[ |\Phi_k^{(N)}\rangle = \prod_{i=1}^N |\psi_i^{(k)}\rangle \]

Orthogonality is realized if \( |\Phi_k^{(N)}\rangle \) differs from \( |\Phi_0^{(N)}\rangle \) by at least one state.
\[|\Psi_i^{(k)}\rangle = \alpha_i^{(k)}|00\rangle_i + \gamma_i^{(k)}|10\rangle_i. \]  
(32)

orthogonal to
\[|\Psi_i^{(0)}\rangle = \alpha_i^{(0)}|00\rangle_i + \gamma_i^{(0)}|10\rangle_i. \]  
(33)

where \(\alpha_i^{(0)}\) and \(\gamma_i^{(0)}\) stands for the former notation \(\alpha_i\) and \(\gamma_i\) of section 3.

If \(\alpha_i^{(k)} = -\gamma_i^{(0)*}\) and \(\gamma_i^{(k)} = \alpha_i^{(0)*}\) the wavefunction in which \(|\Psi_i^{(0)}\rangle\) is replaced by \(|\Psi_i^{(k)}\rangle\) is then orthogonal to it by construction. Using \(|\Psi_i^{(k)}\rangle\) the energy \(\epsilon_{i-1}^{(k)}\) corresponding to the location of site \((i-1)\) of the state \(|\Phi_N^{(N)}\rangle\) orthogonal to the corresponding state \(|\Psi_i^{(0)}\rangle\) in \(|\Phi_0^{(N)}\rangle\) is then given by \((\sum_{<j>}\) corresponds in fact to the site \(i\), see below Eq. 18

\[\epsilon_{i-1}^{(k)} = (E_S - E_T) - \frac{4\kappa^2 J_1}{(1 + 2\kappa^2)^2} |\gamma_i^{(k)}|^2. \]  
(34)

where \(|\gamma_i^{(k)}|^2 = |\alpha_i^{(0)}|^2\). Eq. (34) shows that in general
\[\epsilon_{i-1}^{(k)} \neq \epsilon_{i-1}^{(0)}. \]  
(35)

Using Eq. 18 it is easy to see that
\[\Delta_{i-1}^{(k)} = -\Delta_{i-1}^{(0)}. \]  
(36)

and hence
\[E_{i-1}^{(k)} \neq E_{i-1}^{(0)}. \]  
(37)

Consequently the states \(|\Psi_i^{(0)}\rangle\) and \(|\Psi_i^{(k)}\rangle\) are non degenerate states since

\[E_N^{k} - E_N^{0} = (E_T - E_S)(\frac{\epsilon_i^{(k)}}{E_i^{(k)}} - \frac{\epsilon_i^{(0)}}{E_i^{(0)}}) \]
\[+ \frac{2\kappa^2 J_1}{(1 + 2\kappa^2)^2} (1 + \frac{\epsilon_i}{E_i})(\frac{\epsilon_i^{(k)}}{E_i^{(k)}} - \frac{\epsilon_i^{(0)}}{E_i^{(0)}}) \]
\[+ \frac{J_2}{(1 + 2\kappa^2)} \Delta_i^{(k)} \frac{\Delta_i^{(k)}}{E_i^{(k)}} - \frac{\Delta_i^{(0)}}{E_i^{(0)}} \]
\[- \frac{2\kappa^2 J_2}{(1 + 2\kappa^2)} \frac{2}{E_i} (\frac{\Delta_i^{(k)}}{E_{i-1}^{(k)}} - \frac{\Delta_i^{(0)}}{E_{i-1}^{(0)}}) \]
is different from 0 whatever the number of sites except if \( |\Delta_{i-1}^{(k)}| = |\Delta_{i-1}^{(0)}| = 0 \) which may happen if \( J_2 = 0 \) i.e. \( J_l = J_c \).

The physical states which are orthogonal to the variational state are generally linear combinations of the orthogonal states \( |\Phi_k^{(N)}\rangle \). Since these states are non-degenerate with the variational ground state this property will be shared by the physical states in the general case, i.e. there is a finite gap between the variational state and the other physical states. As already mentioned above degeneracy in energy may occur in special cases, f.i. if \( J_2 = 0 \).

### 4.2 Projected wavefunction.

Since the Hamiltonian \( H^{(S,R)} \) commutes with the total spin projection \( M_{\text{tot}} \) the actual wavefunction has to be projected on a state of fixed total spin projection

\[
\mathcal{P}|\Phi_0^{(N)}\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{i\phi(M_{\text{tot}}-\sum_i S_z^i)} \prod_{i=1}^{N} \sum_{k=1}^{4} \eta^i_k |S_k M_k\rangle_i .
\]

(39)

where \( \eta^1_i = \alpha_i, \eta^2_i = \beta_i, \eta^3_i = \beta_i, \eta^4_i = \beta_i \).

The energy corresponding to the projected state \( \mathcal{P}|\Phi_0^{(N)}\rangle \) is then obtained from the hermitic matrix element

\[
E_P^0 = \frac{1}{2} \left[ \langle \Phi_0^{(N)} | H^{(S,R)} \mathcal{P} | \Phi_0^{(N)} \rangle + \langle \Phi_0^{(N)} | \mathcal{P} H^{(S,R)} | \Phi_0^{(N)} \rangle \right] .
\]

(40)

In the case where \( M_{\text{tot}} = 0 \) one gets

\[
\langle \Phi_0^{(N)} | H^{(S,R)} \mathcal{P} | \Phi_0^{(N)} \rangle = \sum_{p=0}^{\infty} \frac{(-1)^p \pi^{2p}}{(2p)!} \sum_{l_1, \ldots, l_n} c^l_{l_1, \ldots, l_n} \mathcal{C}^0_N(l_1, \ldots, l_n) .
\]

(41)

where \( c^l_{l_1, \ldots, l_n} \) is the combinatorial coefficient associated with the decomposition of \( \sum_i (S_i^z)^l_i, l = 2p = \sum_i l_i \) and
\[ E_N^0(l_1, \ldots, l_n) = \langle \Phi_0^{(N)} | H^{(S,R)} \prod_{i=1}^N \sum_{k=1}^4 \eta^i_k (M_k)^i | S_k M_k \rangle_i . \] (42)

where \{M_k\} are the projections of the states \( k \) on the sites \( i \).

Using the explicit expressions of the \{\alpha_i\}, \{\gamma_i\} and \{\Delta_i\} (eqs. (23) to (27)) \( E_N^0(l_1, \ldots, l_n) \) can be put in the form

\[ E_N^0(l_1, \ldots, l_n) = \frac{(2E_T + E_S)}{\lambda^2} \sum_i [\delta_{l_i0} + \kappa^2(1 + (-1)^{l_i})] |\gamma_i|^2 \]

\[ + 3E_S \sum_i \delta_{l_i0} |\alpha_i|^2 + \frac{\kappa^2 J_1}{\lambda^4} \sum_{<ij>} A(i,j) |\gamma_i|^2 |\gamma_j|^2 \]

\[ + \frac{J_2}{\lambda^2} \sum_{<ij>} [C(i,j) \alpha_i^* \alpha_j^* \gamma_i \gamma_j - \delta_{l_i0} \delta_{l_j0} \alpha_i \alpha_j \gamma_i^* \gamma_j^* + B(i,j) \alpha_i \alpha_j \gamma_i \gamma_j + B(j,i) \alpha_i^* \alpha_j^* \gamma_i^* \gamma_j^*] \]

where \( \lambda = (1 + 2\kappa^2)^{1/2} \) and

\[ A(i,j) = \kappa^2[1 + (-1)^{l_i+l_j} + (-1)^{l_i+1} + (-1)^{l_j+1}] \]

\[ + 2\delta_{l_i0} \delta_{l_j0} + \delta_{l_i0} (1 + (-1)^{l_i} + 2(-1)^{l_j}) \] (44)

\[ B(i,j) = \delta_{l_j0} [1 + (-1)^{l_i} + \delta_{l_i0}] . \] (45)

and

\[ C(i,j) = \delta_{l_i0} \delta_{l_j0} + (-1)^{l_i+1} + (-1)^{l_j+1} . \] (46)

One can then construct projected states \{\mathcal{P} | \Phi_k^{(N)} \rangle \} which are orthogonal to \( \mathcal{P} | \Phi_k^{(N)} \rangle \) in the same way as in section 4a) above and find out that the energy difference \( \Delta \mathcal{E} = \mathcal{E}_k^N - \mathcal{E}_0^N \) and the corresponding \( \Delta E_P = E_P^k - E_P^0 \) are different from zero except maybe if \( J_2 = 0 \).
5 Generalization to even-legged spin 1/2 systems in 2d with any type (short or long) range interaction.

It is straightforward to see that the present considerations can be applied to spin ladders which do not only contain interactions between neighbouring sites, but also interactions at any distance from each other. Such contributions introduce a quantitative but no qualitative change in the gap equations Eqs. (25), (26) and (27).

More generally, the results can be extended to 2d frustrated systems with an even number of ladder legs showing the same type of couplings between sites as those introduced on the ladder. They are obtained by adding explicit couplings between sites located on the neighbouring legs which belong to neighbouring ladders. If \( L = M \times N \) where \( M \) is the number of parallel ladders and \( N \) the length of the ladders, the Hamiltonian in the spin rotated representation (Eqs. (2) and (3)) can be put in the form

\[
H^{(S,R)} = \frac{J_t}{4} \sum_{i=1}^{L} (S_i^2 - R_i^2) + J_1 \sum_{<ij>} S_i S_j + J_2 \sum_{<ij>} R_i R_j + H^{(int)} . 
\]

(47)

with \( H^{(int)} = H^{(int1)} + H^{(int2)} \) and

\[
H^{(int1)} = \frac{J_{int}^r}{4} \sum_{i=1}^{L-N} (S_i S_{i+N} + S_i R_{i+N} - R_i S_{i+N} - R_i R_{i+N}) .
\]

(48)

\[
H^{(int2)} = \frac{J_{int}^d}{4} \left[ \sum_{i=1}^{L-N-1} (S_i S_{i+1+N} + S_i R_{i+N+1} - R_i S_{i+1+N+1} - R_i R_{i+1+N+1}) + \sum_{i=2}^{L-N} (S_{i-1} S_{i+N} + S_{i-1} R_{i+N} - R_{i-1} S_{i+N} - R_{i-1} R_{i+N}) \right]
\]

(49)

\( J_{int}^r \) is a coupling strength between sites in neighbouring chains belonging to neighbouring ladders along rungs and \( J_{int}^d \) acts between the same elements
Figure 2: The even-legged 2d spin system. The coupling strengths between successive ladders are indicated by $J_{\text{int}}^d$ and $J_{\text{int}}^r$ as written in the text.

along a diagonal linking neighbouring site (see Fig. 2).

The Hamiltonian Eq. (47) to Eq. (49) describes a unique chain which is constructed in such a way that the end of each chain $m \ (m = 1, \ldots, M)$ corresponding to the Hamiltonian $H^{(S,R)}$ Eq. (4) is connected to the beginning of the next one, see Fig. (3).

This introduces spurious terms $S_{kN}S_{(k+1)N}, \ R_{kN}R_{(k+1)N}, \ R_{kNS_{(k+1)N}}, \ S_{kNR_{(k+1)N}}, \ k = 1, \ldots, M - 1$ which couple the border sites of the consecutive spin chains. The contributions of their matrix elements should be subtracted from the expression of the energy after the variation. The number of terms of this type is however small compared to the total number of states if $M$ and $N$ are large ("surface" contributions). Neglecting them in the limit $L = M \times N \rightarrow \infty$ one can go through the calculations as before, obtain the same type of equations and draw the same type of conclusions as before.
More explicitly the expression of the energies is given by

$$\epsilon_i = (E_S - E_T) - \frac{4\kappa^2 J_1}{(1 + 2\kappa^2)^2} \sum_{<j>} |\gamma_j|^2$$

$$- \frac{\kappa^2 J'_{int}}{(1 + 2\kappa^2)^2} \sum_{(j)} |\gamma_j|^2 - \frac{2\kappa^2 J'_{int}}{(1 + 2\kappa^2)^2} \sum_{[j]} |\gamma_j|^2$$

and the complex gap $\Delta_i$ which is now the sum of three terms

$$\Delta_i = \Delta_{1i} + \Delta_{2i} + \Delta_{3i}.$$  

where

$$\Delta_{1i} = -\frac{J_2}{(1 + 2\kappa^2)} \sum_{<j>} \left((1 - 2\kappa^2)\alpha_j^* \gamma_j + (1 + 2\kappa^2)\alpha_j \gamma_j^*\right).$$

$$\Delta_{2i} = -\frac{J'_{int}}{4(1 + 2\kappa^2)} \sum_{(j)} \left((1 - 2\kappa^2)\alpha_j^* \gamma_j + (1 + 2\kappa^2)\alpha_j \gamma_j^*\right).$$

$$\Delta_{3i} = -\frac{J'_{int}}{2(1 + 2\kappa^2)} \sum_{[j]} \left((1 - 2\kappa^2)\alpha_j^* \gamma_j + (1 + 2\kappa^2)\alpha_j \gamma_j^*\right).$$

$< j > (= i + 1)$ stands for the site which is the right-nearest neighbour along a chain, $(j)$ corresponds to the nearest-neighbour of a site $i$ along a chain.
rung between chains belonging to nearest-neighbour ladders and \([j]\) to the cross-diagonal nearest-neighbour of site \(i\) located on the chain of the nearest-neighbour ladder.

The expression of the variational energy is formally the same as the one of section 4 except for the additional sum of terms corresponding to the couplings \(J'_{\text{int}}\) and \(J^d_{\text{int}}\). It is given by

\[
d^2 E_N = \frac{J_l}{4} (-3 + \frac{|\alpha_i|^2}{|\gamma_i|^2}) + \frac{|\alpha_i|^2}{|\gamma_i|^2 (1 + 2\kappa^2)^2} [4J_l \sum_{<j>} |\gamma_j|^2 + J^r_{\text{int}} \sum_{(j)} |\gamma_j|^2 + 2J^d_{\text{int}} \sum_{[j]} |\gamma_j|^2]
\]

The nature of the extremum (maximum or minimum) depends on \(J_l\), \(J_1\) and the additional coupling strengths \(J^r_{\text{int}}\) and \(J^d_{\text{int}}\). \(d^2 E_N\) is positive if \(|J_2| = |J_l - J_i|/2\), \(J^r_{\text{int}}\) and \(J^d_{\text{int}}\) are such that

\[
|\alpha_i|^2/|\gamma_i|^2 \geq \frac{3}{1 + \frac{4}{|J_l|} G(\kappa, J_1, J^r_{\text{int}}, J^d_{\text{int}})}
\]

where

\[
G(\kappa, J_1, J^r_{\text{int}}, J^d_{\text{int}}) = \frac{2\kappa^2}{(1 + 2\kappa^2)^2} [4J_1 \sum_{<j>} |\gamma_j|^2 + J^r_{\text{int}} \sum_{(j)} |\gamma_j|^2 + 2J^d_{\text{int}} \sum_{[j]} |\gamma_j|^2]
\]

Hence \(d^2 E_N\) is positive if \(J_l\), \(J^r_{\text{int}}\), \(J^d_{\text{int}}\) get large and \(J_l\) and (or) the \(\{\Delta_i\}\)s remain relatively small.

The discussion about the structure of the spectrum can be taken from section 4. In general this spectrum will show a gap between the ground state and excited states, except for specific values of specific coupling strengths. In fact degeneracy of states appears when the gaps \(\Delta_{ni}\) \((n = 1, 2, 3)\) disappear, which happens when \(J_2\), \(J^r_{\text{int}}\) and \(J^d_{\text{int}}\) are equal to zero.

The present results can evidently be extended to the case where one works with projected wavefunctions along the same lines as it has been shown in section 4.
6 Remarks, discussion.

The present derivations are subject to comments:

- The spin Hamiltonians $H$ which were introduced commute with also with the total spin $S_{\text{tot}}$. Hence the corresponding quantum numbers are conserved and characterize the states of the physical spectrum $|\Phi_0^{(N)}(S_{\text{tot}}, M_{\text{tot}})\rangle$. The trial wavefunction $|\Phi_0^{(N)}\rangle$ should be projected on a state with fixed $S_{\text{tot}}$. This changes quantitatively the energy $E_0^N$ and $E_0^P$. It should however not qualitatively affect the calculated energy differences and the existence of a gap between the variational ground state and other states of the infinite system.

- In practice $\beta_{i1} = \beta_{\pi}$ may be treated as independent variational parameters if the constraint introduced in section 3a) Eq. (3) is not taken into account. The corresponding component may be kept in the definition of $|\Psi_i^{(0)}\rangle$, Eq. (6).

- It is easy to see that the present formalism can be extended to any even-legged system of spins in any space dimension $D = nd$ with the proviso that, as in the $2d$ case, the contribution of terms coming from the border sites of the finite systems and which should not be present can be neglected when applying the variational procedure. This is realized if the system gets large. Since the algebraic expressions are rather cumbersome we do not write them out explicitly.

- The "linking procedure" used in Section 5 in order to generate a continuous linear system leads a posteriori to a symmetry constraint of $2d$ and systems of higher dimensionality which by construction possess the $SO(4)$ symmetry introduced in section 2. This is a constraint which may limit the range of the variational procedure in the sense that other symmetries could take over at different space dimensionalities.
7 Depeneracies.

7.1 Behaviour of $\epsilon_i$ and $\Delta_i$ at $J_2 = 0$.

As observed above in sections 4 and 5 the variational state gets degenerate with states orthogonal to it when the gaps $\{\Delta_i\}$ vanish. This is the case when $J_2 = 0$ (section 4), $J_2 = J^d_{\text{int}} = J^{nd}_{\text{int}} = 0$ (section 5). It is of interest to analyze more precisely the property of this point. In the sequel we concentrate on the case of a ladder system. The generalization to $2d$ and higher dimensions is straightforward.

It is easy to determine the behaviour of $\Delta_i$ and $\epsilon_i$ in the neighbourhood of $J_2 = 0$. The derivative of each gap with respect to $J_2$ reads

$$
\frac{d\Delta_i}{dJ_2} = \frac{1}{1 + 2\kappa^2 \frac{\Delta_i+1(J_2)}{E_{i+1}(J_2)}} - \frac{J_2}{1 + 2\kappa^2 \frac{\Delta'_i(J_2)}{E_{i+1}(J_2)}}
$$

(56)

where $\Delta'_i(J_2) = \frac{d\Delta_i(J_2)}{dJ_2}$ and $\epsilon'_i(J_2) = \frac{d\epsilon_i(J_2)}{dJ_2}$. $\Delta'_i(J_2) = 0$ for $J_2 = 0$ and is a continuous function of $J_2$ in the neighbourhood of this point.

Similarly $\epsilon'_i(J_2)$ is a continuous function of $J_2$ in the neighbourhood of this point.

7.2 Eigenstates at $J_2 = 0$.

We show that the states $\{|\Phi^{(N)}_k\rangle\}$, $k = 0$ and $k \neq 0$ are eigenstates of the Hamiltonian $H^{(S,R)}$ defined in Eq.(4) when $J_2 = 0$.

Indeed the part of the Hamiltonian proportional to $J_i$ is diagonal in the basis of states $\{|\Phi^{(N)}_k\rangle\}$, the third term does not contribute and the second term is a sum of terms of the form

$$
\langle \Phi^{(N)}_k | S_i S_j | \Phi^{(N)}_{k'} \rangle = \frac{8\kappa^2}{\lambda^4} \gamma_i^{(k)} \gamma_j^{(k')} \gamma_i^{(k)} \gamma_j^{(k')}
$$

(57)

Using the expression of $\gamma_i$ in Eq. (23) and the negative root solution of $E_i$, $E_i = -(\epsilon_i^2 + |\Delta_i|^2)^{1/2}$ leads to $\langle \Phi^{(N)}_k | S_i S_j | \Phi^{(N)}_{k'} \rangle = 0$ for any $k$ and $k'$, $i$.
and $j$. Hence the states $\{|\Phi_k^{(N)}\rangle\}$ are eigenstates of $H^{(S,R)}$ orthogonal to each other.

### 7.3 Behaviour of the energy interval between the states in the neighbourhood of $J_2 = 0$.

The energy intervals between the physical states which are linear combinations of states $\{|\Phi_k^{(N)}\rangle\}$ vary continuously with $J_2$ in the neighbourhood of $J_2 = 0$.

Using the expression given in Eq. (38) for a state $|\Phi_k^{(N)}\rangle$ which differs from the state $|\Phi_k^{(N)}\rangle$ by $n$ consecutive site wavefunctions $|\Psi_l^{(k)}\rangle$, (f.l. $l = N - n + 1, ..., N$) the energy differences between these states are given by

$$
\Delta E_N^{(kk')} = E_N^{(k)} - E_N^{(k')} = (E_T - E_S) \sum_l (\epsilon_l^{(k)} - \epsilon_l^{(k')}) + \frac{2\kappa^2 J_1}{(1 + 2\kappa^2)^2} \sum_l [(1 + \epsilon_{N-l})(\epsilon_l^{(k)} - \epsilon_l^{(k')}) + (\epsilon_{l+1}^{(k)} + \epsilon_{l+1}^{(k')}) + \epsilon_{l+1}^{(k)} - \epsilon_{l+1}^{(k')}] + \frac{J_2}{(1 + 2\kappa^2)} [\delta_l^{r(N-n+1)} - \delta_l^{r(N-n+1)} + \sum_l \delta_l^{r(k)}(\delta_{l+1}^{r(k)} - \delta_{l+1}^{r(k')})] - \frac{2\kappa^2 J_2}{(1 + 2\kappa^2)} [\delta_l^{i(N-n+1)} - \delta_l^{i(N-n+1)} + \sum_l \delta_l^{i(k)}(\delta_{l+1}^{i(k)} - \delta_{l+1}^{i(k')})]
$$

where $\epsilon_l^{(k)} = E_l^{(k)} / E^{(k)}$, $\delta_l^{r(k)} = \Delta_l^{r(k)} / E^{(k)}$ and $\delta_l^{i(k)} = \Delta_l^{i(k)} / E^{(k)}$ depend implicitly on $J_2$.

$\Delta E_N^{(kk')} \rightarrow 0$ continuously to zero for any $\{k, k'\}$ when $J_2 \rightarrow 0$ which shows that the degeneracy of states for this particular value of the coupling $J_2$. The same is true for the physical states which go continuously over to the states $\{|\Phi_k^{(N)}\rangle\}$ when $J_2 \rightarrow 0$. This is also the case when the states $|\Psi_l^{(k)}\rangle$ which differ in different states $|\Phi_k^{(N)}\rangle$ are non consecutive as considered above.
Close to \( J_2 = 0 \) \( E_N^{(k)} \) and hence \( \Delta E_N^{(kk')} \) behave like \( cnst \ast J_2^2 \) to lowest order in \( J_2 \) and \( E_N^{(k)} (J_2) \simeq E_N^{(k)} (-J_2) \)

Finally the same reasoning is valid in the projected framework dealing with \( \mathcal{P}|\Phi_k^{(N)} \rangle \) and \( \mathcal{P}|\Phi_{k'}^{(N)} \rangle \).

8 Generalization to a correlated variational wavefunction.

8.1 Correlated wavefunction.

The variational wavefunction which was postulated in Eq. (5) has a mean-field structure which includes implicit correlations between site states through Eq. (20) to Eq. (27) but no explicit correlations. In order to implement such correlations we go over to a pseudo-fermion representation of the operators \( S_i^{(+)} \), \( S_i^{(-)} \), \( S_i^{(z)} \) and \( R_i^{(+)} \), \( R_i^{(-)} \), \( R_i^{(z)} \).

\[
S_i^{(+)} = b_i^+ d_i + d_i^+ c_i
\]

\[
S_i^{(-)} = c_i^+ d_i + d_i^+ b_i
\]

\[
S_i^{(z)} = b_i^+ b_i + c_i^+ c_i + d_i^+ d_i
\]

\[
R_i^{(+)} = b_i^+ a_i + a_i^+ c_i
\]

\[
R_i^{(-)} = a_i^+ b_i + c_i^+ a_i
\]

\[
R_i^{(z)} = c_i^+ a_i + a_i^+ d_i
\]

where \( a_i^+, b_i^+, c_i^+, d_i^+ \) are anticommuting fermion operators which generate singlet and triplet states
\[ a_i^+ |0\rangle_i = |S = 0, M = 0\rangle_i \]
\[ b_i^+ |0\rangle_i = |S = 1, M = 1\rangle_i \]
\[ c_i^+ |0\rangle_i = |S = 1, M = -1\rangle_i \]
\[ d_i^+ |0\rangle_i = |S = 1, M = 0\rangle_i \]

and \(|0\rangle_i\) is the particle vacuum on site \(i\).

We introduce the variational trial wavefunction

\[ |\Upsilon_0^{(N)}\rangle = e^F |0\rangle. \]  \(\text{(59)}\)

where

\[ F = f_1 \sum_{i_1} \Omega_{i_1}^+ + f_2 \sum_{i_2} \sum_{i_1} \Omega_{i_1}^+ \Omega_{i_2}^+ + f_3 \sum_{i_3} \sum_{i_2} \sum_{i_1} \Omega_{i_1}^+ \Omega_{i_2}^+ \Omega_{i_3}^+ + \ldots \]  \(\text{(60)}\)

\(N\) is the total number of sites in the \(SO(4)\) representation, \(|0\rangle\) the particle vacuum and

\[ \Omega_{i_1}^+ = \alpha_{i_1} a_{i_1}^+ + \beta_{i_1} b_{i_1}^+ + \gamma_{i_1} c_{i_1}^+ + \delta_{i_1} d_{i_1}^+ \]
\[ [\Omega_{i_1}^+, \Omega_{i_m}^+] = 0 \]

Developing the exponential in Eq. (59) leads to

\[ |\Upsilon_0^{(N)}\rangle = [1 + F + \frac{1}{2!}F^2 + \frac{1}{3!}F^3 + \ldots]|0\rangle. \]  \(\text{(61)}\)

Since the creation operators \(\{\Omega_{i_1}^+\}\) are fermions and since the system should be occupied by two particles by site \(i\) (1 particle on each site of a rung \(i\)) the sum of terms in Eq. (60) and Eq. (61) is restricted to a finite
number of terms which all contain exactly $N$ distinct operators $\{\Omega^+_i\}$. This sum is written in a fixed order so that $i_1 < i_2 < i_3 < ... < i_N$. The state $|Y_0^{(N)}\rangle$ can be written as

$$|Y_0^{(N)}\rangle = \sum_{p=1}^{N} \frac{1}{p!} \sum B_N(p; n_1, n_2, ..., n_N) f^{(1)}_1 f^{(2)}_2 ... f^{(N)}_N \Omega^+_1 ... \Omega^+_N |0\rangle. \tag{62}$$

where the sum contains only the terms which generate $N$ operators $\{\Omega^+_i\}$ and

$$B_N(p; n_1, n_2, ..., n_N) = (-1)^p \frac{N!}{n_1!...n_p!}$$

with the constraint $n_1 + 2n_2 + ... + Nn_N = N$ and $P$ the number of permutations such that the operators $\{\Omega^+_i\}$ appear in increasing order with respect to their index, $i_1 < i_2 < i_3 < ... < i_N$.

In a realistic description the correlation functions $f_l$ could depend on the relative distance between the particles on the different sites, $f_l = f_l(i_1, ..., i_l)$. We do not take this fact into account. It is sensible to believe that this point is not crucial for the aim which we fix here, i.e. to show the qualitative behaviour of the system as already claimed before. It makes also physically sense to consider that the coefficients $f_l$, $(l \geq 2)$ which correlate $l$ particles are a product of two-body terms $f_l = C(l, 2)f_{(2)}$ where $C(l, 2) = l!/(2!(l - 2)!)$.

Under these conditions

$$|Y_0^{(N)}\rangle = \sum_{p=1}^{N} \frac{1}{p!} \sum B_N(p; n_1, ..., n_N) f^{(n_1)}_1 f^{(n_2, ..., n_N)}_2 \prod_{l=2}^{N}[C(l, 2)]^{n_l} |\Phi_0^{(N)}\rangle. \tag{63}$$

where $n_{(2, N)} = n_2 + ... + n_N$ with the constraint $n_1 + 2n_2 + ... + Nn_N = N$.

8.2 Variational wavefunction.

The structure of the correlated wavefunction $|Y_0^{(N)}\rangle$ is a product of a term which contains the correlations and the mean-field wavefunction $|\Phi_0^{(N)}\rangle$. In its
whole generality the search of extrema of the energy \( E_N^{(c)} = \langle \Upsilon_0^{(N)} | H^{(S,R)} | \Upsilon_0^{(N)} \rangle \) is in principle given by varying the coefficients \( \{ \alpha_i \}, \{ \beta_i \}, \{ \gamma_i \}, \{ \delta_i \} \) and \( f_{(1)}, f_{(2)} \) under the normalization constraint

\[
\langle \Upsilon_0^{(N)} | \Upsilon_0^{(N)} \rangle = 1.
\] (64)

One may consider a two-step procedure suggested by the structure of the wavefunction \( | \Upsilon_0^{(N)} \rangle \) in Eq. (63). A minimization on the wavefunction \( | \Phi_0^{(N)} \rangle \) is done first on \( E_N^{(0)} = \langle \Phi_0^{(N)} | H^{(S,R)} | \Phi_0^{(N)} \rangle \) as before in section 4. Then Eq. (64) is verified if

\[
C = \sum_{p,p'} \frac{1}{p!p'} \sum_{n_i,n'_i} \prod_{l=2}^{N} [C(l,2)]^{2n_l} B_N(p; n_1, \ldots, n_N) B_N(p'; n'_1, \ldots, n'_N) \] (65)

\[
f_{(1)}^{n_1+n'_1} f_{(2)}^{n_2+n'_2} = 1
\]

where the sums over \( \{ n_i \} \) are such that \( n_1 + 2n_2 + \ldots + Nn_N = N \) and the same for \( \{ n'_i \} \).

\( C \) should then be fixed by looking for values of \( f_{(1)} \) and \( f_{(2)} \) such that the normalization constraint given by Eq. (64) is realized and \( E_N^{(c)} \) is minimized.

By construction it appears that all the reasoning concerning the nature of the spectrum which was made in sections 4 and 5 remains then valid as long as solutions exist. Hence the same conclusions as those drawn there can be taken up in the present case.

If such a solution can be found it ensures the existence of a gap for any space dimension.

9 Summary and conclusions.

We introduced a variational approach in order to describe the low-lying energy states of frustrated quantum spin systems. We started with spin-1/2
ladders and extended the approach to even-legged systems in any space dimension $d$.

In a first step the postulated trial wavefunction was written as a superposition of spin singlet and triplet states in the framework of $SO(4)$ symmetry. The variational ansatz limited the wavefunction to a mean-field product of site states formally similar to the ansatz used in high-$T_c$ theory [41]. The local amplitudes of the site state components were correlated through nearest neighbour contributions. We looked for energy extrema and analyzed the conditions under which the variational procedure leads to energy minima. The realization of this property depends on the parameter subspace in which the coupling strengths of the spin-spin interactions are located. If the energy extrema do not correspond to minima the variational ansatz used for the trial wavefunction does not describe the structure of the physical ground state. In these cases the structure of this state must be different from the postulated ansatz.

In the cases where the trial wavefunction describes a state of minimum energy the present analysis shows that spectra of spin-1/2 quantum spin systems with two legs are gapped. The approach can in principle be generalized to spin-spin interactions of any range and any type of frustration if the postulated variational ground state wavefunction leads to an energy minimum. For specific values of the coupling constants the gap may vanish leading to degeneracies in energy.

The extension of the formalism to even-legged systems of dimension $d \geq 2$ was developed in section 5. The pertinence of such an extension which has been discussed in section 6 is warranted as long as the $SO(4)$ symmetry which characterizes even-legged ladders remains a symmetry which governs the system at higher space dimensions.

In a further step we suggested an extension of the variational trial wavefunction to include explicit correlations beyond the mean-field approximation. We introduced simplifications to the most general form of this wavefunction which may have quantitative but not qualitative effects on the conclusions. It is seen that the corresponding spectra may again show the characteristic gap obtained with the mean-field variational ansatz.
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