Oscillations and instabilities of fast rotating neutron stars

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Abstract. We present our results of our study of the oscillations and instabilities of rapidly rotating relativistic stars in the Cowling approximation. This is the first study of its kind and provides information on the effect of fast rotation on the oscillation spectra while it offers the possibility in studying the complete problem by including spacetime perturbations. Our study includes both axisymmetric and non-axisymmetric perturbations and provides limits for the onset of the secular bar mode rotational instability. We also present approximate formulae for the dependence of the oscillation spectrum from rotation. The results suggest that it is possible to extract the relativistic star’s parameters from the observed gravitational wave spectrum.

1. Introduction

During their evolution relativistic stars may undergo oscillations which can become unstable under certain conditions. Rotation strongly affects these oscillations and perturbed stars can become unstable if they rotate faster than some critical velocities. During these oscillatory phases of their lives compact stars emit copious amounts of gravitational waves which together with viscosity tend to suppress the oscillations. The oscillations are divided into different families according to the restoring force [1, 2]. If pressure is the main restoring force then these modes are called (pressure) p-modes, buoyancy is the restoring force of another class of oscillation mode, the g-modes while Coriolis force is the restoring force for the inertial modes. Spacetime induces another family of oscillations which couple only weakly to the fluid, these are the so-called w-modes [3]. There are more families of modes if one assumes the presence of crust [4, 5, 1, 6] or magnetic fields [7]. For a complete description of the relativistic star perturbation theory one may refer to [2, 8, 9].

During the last two decades, these studies became even more important due to the relations of the oscillations and instabilities to the emission of gravitational waves and the possibility of getting information about the stellar parameters (mass, radius, equation of state) by the proper analysis of the oscillation spectrum [10, 11, 12, 13, 14, 15, 16]. Still, all these studies were mainly dealing with non-rotating stars, because the combination of rotation and general relativity made both the analytic and numerical studies extremely involved. This led to certain approximations in studying rotating stars in GR. The most obvious of them include the so called “slow-rotation” and the “Cowling” approximation. Actually, both approximations were known and have been used extensively in Newtonian theory of stellar oscillations [17]. Here, we use the Cowling
approximation, where one typically neglects the perturbations of the Newtonian potential or the spacetime in the case of GR. This is a quite good approximation, both qualitatively and quantitatively, for the higher order $p$-modes, for the $g$-modes and the inertial modes while it is only qualitatively good for the $f$-modes, see for example [18].

This article is a short description of the work presented in [19] and [20].

2. The Perturbation Equations

The normal mode analysis of rotating neutron stars involves the solution of the full nonlinear set of Einstein’s equations of General Relativity together with the equations of motion for the fluid (we set $G = c = 1$ here)

\begin{align}
G_{\mu\nu} &= 8\pi T_{\mu\nu}, \\
\nabla^\mu T_{\mu\nu} &= 0,
\end{align}

where $G_{\mu\nu}$ is the Einstein-tensor and $T_{\mu\nu}$ is the energy-momentum–tensor. For a first step in solving this complicated problem, we will introduce some useful approximations.

First of all, we linearize equations (1) and (2), which means that we constrain our study to small perturbations around the equilibrium. Secondly, we will work in the so-called Cowling-approximation, which means that we will neglect all metric perturbations. This simplifies significantly the equations one has to solve since the space-time is considered as “frozen”; and we only have to solve the linearized version of (2). Under this assumption equation (2) will be written as

\begin{equation}
\nabla^\mu (\delta T_{\mu\nu}) = g_{\mu\kappa} (\partial_\mu \delta T_{\kappa\nu} - \Gamma^\lambda_{\kappa\mu} \delta T_{\lambda\nu} - \Gamma^\lambda_{\mu\nu} \delta T_{\kappa\lambda}) = 0,
\end{equation}

where $g_{\mu\nu}$ is the metric describing neutron star’s spacetime, $\Gamma^\lambda_{\kappa\mu}$ are the Christoffel symbols and $\delta T_{\mu\nu}$ is the Eulerian perturbation of the energy-momentum–tensor. We assume that the matter has no viscosity or shear stresses, i.e. that it can be described by a perfect fluid. Thus $\delta T_{\mu\nu}$ has the form

\begin{equation}
\delta T_{\mu\nu} = (\epsilon + p)(u_\mu \delta u_\nu + u_\nu \delta u_\mu) + (\delta p + \delta \epsilon) u_\mu u_\nu + \delta p g_{\mu\nu}.
\end{equation}

Here $\epsilon$ is the energy-density, $p$ is the pressure, $u_\mu$ is the 4-velocity and $\delta u_\mu$ are its perturbation. Energy density and pressure are not independent quantities but are connected via an equation of state (EoS) which we assume to be polytropic, i.e.

\begin{equation}
p = K \tilde{\rho}^{1+1/N} \text{ where } \epsilon = \tilde{\rho} + Np.
\end{equation}

Here $\tilde{\rho}$ is the rest-mass density, $K$ the polytropic constant, $N$ the polytropic exponent and $\Gamma = 1+1/N$ the polytropic index. For barotropic oscillations, both the unperturbed background and the perturbations are described by the same equation of state. In this case the pressure variation can be replaced by the corresponding energy-density variation via $\delta p = c_s^2 \delta \epsilon$, where $c_s$ is the speed of sound

\begin{equation}
c_s^2 = \frac{\partial p}{\partial \epsilon}.
\end{equation}

For polytropic EoS of the form (5) it is given by

\begin{equation}
c_s^2 = \frac{\Gamma p}{\epsilon + p}.
\end{equation}

The background model is a compact relativistic star that rotates uniformly up to its Kepler-limit, i.e. the point were it is torn apart by centrifugal forces. In this work we will adopt the metric
in a comoving frame of reference as it is described in [21]. In Lewis-Papapetrou coordinates 
\((\rho, \zeta, \varphi, t)\) the metric reads

\[
    ds^2 = e^{-2U} \left[ e^{2k} (d\rho^2 + d\zeta^2) + W^2 d\varphi^2 \right] - e^{2U} (dt + a d\varphi)^2,
\]

where \(U, k, W\) and \(a\) are depending on \(\rho\) and \(\zeta\). In general, the \((\varphi, t)\)-metric component is proportional to the function \(a\) and vanishes in the absence of rotation.

In contrast to nonrotating or slowly rotating objects, fast rotating neutron stars are deformed due to centrifugal forces. This means that it is no longer possible to decompose the angular part of the perturbation quantities in spherical harmonics as it was usually done for the non-rotating case (see [22]) or in the slow-rotation-approximation (for example [23] and [24]). Instead we can only separate the azimuthal dependence and the perturbation functions that we use will be written as

\[
    (\epsilon + p) W e^U \delta u_\rho = f_1(\rho, \zeta, t) e^{im\varphi},
\]

\[
    (\epsilon + p) W e^U \delta u_\zeta = f_2(\rho, \zeta, t) e^{im\varphi},
\]

\[
    (\epsilon + p) \delta u_\varphi = f_3(\rho, \zeta, t) e^{im\varphi},
\]

\[
    e_s^2 e^U \delta \epsilon = H(\rho, \zeta, t) e^{im\varphi},
\]

and we end up with the following set of four equations:

\[
    \frac{\partial f_1}{\partial t} = -W e^U \frac{\partial H}{\partial \rho} - e^{5U} \frac{\partial a}{\partial \rho} f_3 - \frac{W \partial U}{c_s^2} \frac{\partial}{\partial \rho} e^U H,
\]

\[
    \frac{\partial f_2}{\partial t} = -W e^U \frac{\partial H}{\partial \zeta} - e^{5U} \frac{\partial a}{\partial \zeta} f_3 - \frac{W \partial U}{c_s^2} \frac{\partial}{\partial \zeta} e^U H,
\]

\[
    \frac{\partial f_3}{\partial t} = \frac{im}{F} \left( a c_s^2 e^{4U} f_3 + W^2 H \right) + \frac{W a c_s^2 e^{3U-2k}}{F} \left( \frac{\partial f_1}{\partial \rho} + \frac{\partial f_2}{\partial \zeta} \right),
\]

\[
    \frac{\partial H}{\partial t} = \frac{im}{F} \left( a c_s^2 e^{4U} H + c_s^2 e^{4U} f_3 \right) + \frac{W a c_s^2 e^{3U-2k}}{F} \left( \frac{\partial f_1}{\partial \rho} + \frac{\partial f_2}{\partial \zeta} \right),
\]

where

\[
    F := a^2 c_s^2 e^{4U} - W^2.
\]

Keep in mind that since a comoving frame of reference is used here, there is no explicit dependence on the angular velocity \(\Omega\) in this set of equations.

Together with the correct boundary conditions and initial data, this system forms an appropriate description for linear perturbations on rotating compact objects; see [19] for more details.

3. Numerical Implementation

The utilization of a comoving frame of reference helped us to keep the formulation of the time evolution equations rather simple. The next step for a successful numerical treatment of the problem is the use of surface-fitted coordinates. Since the shape of a rapidly rotating equilibrium configuration is an oblate spheroid, the standard choice of spherical coordinates for covering the
physical domain of interest usually leads to problems at the surface of the compact object. In this case the surface now lies somewhere in between different $r = \text{const}$ slices, so one has to find the correct way of implementing the boundary conditions there. With surface-fitted coordinates on the other hand there is not such a problem; for all rotation rates, the coordinate system keeps attached to the surface. The following Figure 1 depicts this situation for a fast rotating star.

![Figure 1](image)

**Figure 1.** *Left panel:* A cut through a rapidly rotating neutron star and how it is covered by the coordinate system. $\sigma = \text{const.}$ lines are shown dotted while $\tau = \text{const.}$ lines are drawn solid *Right panel:* The transformation on a rectangular domain and the physical meaning of the boundaries.

The other ingredients are standard in numerical relativity; we use second order finite-difference stencils for spatial discretization in both dimensions and an Iterated Crank-Nicholson scheme with swapped weights for time-evolution, see [25, 26] for more details.

As it turned out, the numerical evolution of the equations was unstable i.e. after the first few time steps, high frequency oscillations of exponentially growing modes developed near the center of the star (i.e. at $\sigma = 0$). This is most likely due to the presence of a coordinate singularity at the origin but also because some of the coefficients on the right-hand-sides of our evolution system get nearly singular close to the center. It should be noted that the occurrence of singular terms in general doesn’t necessarily lead to a failure of the numerical scheme.

The solution to this problem was the utilization of additional viscous terms in the evolution equations. Since they do not represent any physical effect in the initial setup of the problem, they are commonly referred to as **artificial viscosity**. Here, to each of the four equations we added a Kreiss-Oliger like term of the form (see [27] for details)

$$
\mathcal{V}(f) = \alpha \left( D_\sigma^+ D_\tau^- + D_\sigma^- D_\tau^+ \right) f,
$$

(12)

where $f$ is a perturbation variable, $\alpha = \text{const.}$ is the dissipation coefficient, $(\sigma, \tau)$ are the spatial coordinates and $D^+, D^-$ are the standard forward- and backward-difference operators. By using dissipation coefficients $\alpha_i$, $i = 1 \ldots 4$ ranging from $10^{-3} - 10^{-4}$ it become possible to stabilize the time-evolution code against exponentially growing instabilities.

After a successful simulation, the real part of the complex solution is taken and a discrete Fourier-transformation at several points inside the star is performed on these data to extract the
oscillation frequencies. If $N$ is the number of points in this time series and $\Delta t$ is the temporal resolution, the corresponding frequency resolution and the Nyquist-frequency are given by

$$\Delta f = \frac{1}{N \Delta t} \quad \text{and} \quad f_c = \frac{1}{2 \Delta t}. \quad (13)$$

Since an explicit numerical scheme has been used for time-evolution, there are certain restrictions on the absolute value of $\Delta t$. The timestep cannot be arbitrarily large and depends on the spatial resolution of our grid (CFL-criterion). For most of the simulations, a timestep of the order of $\Delta t \approx 10^{-6}$ sec and $N \approx 10^4$ was used. The total evolution time then is $t_{max} \approx 30 - 40$ ms with $\Delta f \approx 15 - 30$ Hz and $f_c \approx 8 - 12$ kHz.

4. Results

We study first axisymmetric perturbations mainly to test the code against published results. Later we will proceed by studying non-axisymmetric perturbations for several polytropic equations of state.

4.1. Axisymmetric Perturbations

Following the procedure for the non-linear problem presented in [28], a density perturbation of the form

$$\delta \rho = A \rho_c \sin \left( \frac{\pi r}{r_s(\theta)} \right), \quad (14)$$

is used as initial data. Here $A$ is the perturbation amplitude, $\rho_c$ is the central density, $(r, \theta)$ denote spherical coordinates and $r_s(\theta)$ is the coordinate radius of the star. The next Figure 2 shows the variation of certain mode frequencies with rotation rate and compares it to the values found in [28]; for more details see [19].

![Figure 2. Left panel: The fundamental ($l = 2, m = 0$) frequency is plotted as function of the rotation rate Right panel: The same plot for the first overtone. For comparison, the results by [28] are shown as well](image)

The agreement between these two different methods is extremely good; the relative difference of the various results is lower than 3%.
Additionally, it is also possible to extract the shape of the eigenfunction which belongs to a certain oscillation. In fact this is how one usually identifies frequency peaks in the power spectrum. In Figure 3 we show the eigenfunctions for the $2f$-mode and its first overtone in the nonrotating limit. Keep in mind, that in the left panel the phase of the eigenfunction is displayed correctly while in the right panel the absolute value of the eigenfunction is depicted, i.e. the phase is not adjusted there. Again, more examples can be found in [19, 20].

![Figure 3](image)

**Figure 3.** *Left panel:* Eigenfunction of the $2f$-mode with corrected phase *Right panel:* The shape of the $2p_1$-eigenfunction; the node in radial direction is clearly visible

Low-frequency modes pose a special challenge to the computer program. In order to accurately determine these frequencies one needs long evolution times, see equation (13), and therefore the code has to be long-term stable as well. Once rotation sets in, a new class of modes restored by the Coriolis force appears with frequencies $\omega \sim \Omega$. We again tested our implementation against other published results for these so-called inertial modes. Figure 4 shows a comparison between results obtained by this linear code in the Cowling approximation and a nonlinear code using the CFC-approach, see [29].

![Figure 4](image)

**Figure 4.** Change of the $i_1$, $i_2$ and $i_{-2}$ inertial mode frequency with rotation rate. Note, that these modes are degenerate in the nonrotating limit

As in the case for pressure driven modes discussed above, the agreement between these two methods is very good. This gives us enough confidence to trust our handling of the artificial viscosity.
4.2. Nonaxisymmetric Perturbations

We will now turn to the study of non-axisymmetric oscillations on rotating compact objects with emphasis on the \( m = \pm 2 \) perturbations. In the nonaxisymmetric case, the frequencies of co- and counter-rotating modes split in the presence of rotation and there is a critical rotation rate for which these frequencies vanish for an inertial observer. This signals the onset of a rotational instability, the so-called CFS-instability. For higher rotation rates this oscillation will go unstable due to the emission of gravitational waves.

We study the onset of the f-mode instability for several polytropic equations of state. All results presented so far make use of \( \Gamma = 2 \), \( K = 100 \) and a fixed central rest-mass density \( \rho_c = 1.28 \times 10^{-3} \) in units where \( G = c = M_\odot = 1 \). In the nonrotating case, this leads to a stellar model with a gravitational mass of \( M = 1.4 \, M_\odot \) and a circumferential radius of \( R = 14.15 \text{ km} \). In addition to this polytropic model we have used two realistic equations of state for which polytropic fits are available. More specifically we take \( \Gamma = 2.46 \) and \( K = 0.00936 \) for EoS A and \( \Gamma = 2.34 \) and \( K = 0.0195 \) for EoS II; these values are given in geometric units and with [km] as length scale, see also [16]. The two new configurations that were chosen are very close to their maximum mass limit; in particular the background model for EoS A has a mass of \( M = 1.61 \, M_\odot \) and a radius of \( R = 9.51 \text{ km} \) while the EoS II model has a mass of \( M = 1.91 \, M_\odot \) and a radius of \( R = 11.68 \text{ km} \).

It is well known that the f-mode frequencies in the nonrotating case scale with the mean density of the background star. We therefore expect the highest frequency for the EoS A model followed by EoS II. The next Figure 5 shows our results for the different equations of state used here and confirms the prediction about the relative amplitude of the fundamental quadrupolar mode frequency.

As one can see from Figure 5, the CFS-instability is actually reached for all three different background models although the precise rotation rate where the modes become unstable varies from one EoS to another. The model labeled BU gets unstable just at its Kepler limit while the other two models only need 85 – 90% of their maximum angular velocity to enter the instability.

Despite the obvious differences in the variation of the f-mode frequencies for the distinct background models, there is a nice common feature that becomes apparent in a comoving coordinate system. If we normalize the oscillation frequencies with its value in the nonrotating limit and similarly the rotation frequency with the Kepler-limit, we arrive at Figure 6. For a large range of rotation frequencies, the three different frequency distributions for co- and counter-rotating modes can be fitted by a single 2nd-order polynomial which is drawn with green dots in Figure 6. If we write for all rotation parameters

\[
\frac{f}{f_0} = 1.0 + C^{(1)}_{lm} \left( \frac{\nu}{\nu_K} \right) + C^{(2)}_{lm} \left( \frac{\nu}{\nu_K} \right)^2,
\]

and make a least-square fit through the data points, one arrives at \( C^{(1)}_{22} = -0.25 \pm 0.02 \), \( C^{(2)}_{22} = -0.25 \pm 0.02 \) for \( m = 2 \) and \( C^{(1)}_{2-2} = 0.48 \pm 0.03 \), \( C^{(2)}_{2-2} = -0.55 \pm 0.04 \) in the \( m = -2 \) case.

5. Conclusions

In this work we presented a study of the oscillation properties of fast rotating relativistic stars for both axisymmetric and non-axisymmetric perturbations. The study was based on the Cowling approximation using a 2D version of the perturbation equations. The results for axisymmetric perturbations are in excellent agreement with earlier ones while the non-axisymmetric results are the first of their kind in the literature. We demonstrated the neutral points for the onset of the CFS instability and suggested possible normalizations which can be used in order to extract the parameters of the rotating star.
The method presented here (evolution of the 2D linearized equations) can be extended by including the perturbations of the spacetime. This will offer the possibility in testing the earlier results [30] for the onset of the secular instability in fast rotating stars. Finally, the effect of differential rotation on the spectra can be studied both for testing earlier, mainly Newtonian, results [31, 32] but more importantly in finding the correct dependence of the frequencies on rotation as well as the neutral points for the onset of the secular instabilities. This is actually an important step since newly born neutron stars are expected to rotate differentially at least during the stages that they will be in an oscillatory phase or the even when they are secularly unstable.

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Figure 6. Same as Figure 5 but normalized and in the comoving frame.

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