HOMOTOPY TYPE AND HOMOLOGY VERSUS VOLUME FOR ARITHMETIC LOCALLY SYMMETRIC SPACES

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Abstract. We study locally symmetric spaces associated with arithmetic lattices in semisimple Lie groups. We prove the following results about their topology: the minimal number of tetrahedra needed for a triangulation is at most linear in the volume and the Betti numbers are sub-linear in the volume except possibly in middle degree. The proof of these results uses the geometry of these spaces, namely the study of their thin parts. In this regard we prove that these spaces converge in the Benjamini–Schramm sense to their universal covers and give an explicit bound for the volume of the thin part for trace fields of large degree. The main technical ingredients for our proofs are new estimates on orbital integrals, a counting result for elements of small displacement, and a refined version of the Margulis lemma for arithmetic locally symmetric spaces.

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1. Introduction

This work deals with arithmetic locally symmetric spaces which are the quotients of symmetric spaces of noncompact type by arithmetic lattices, the prototypical example being $\text{SL}_d(\mathbb{Z})\backslash\text{SL}_d(\mathbb{R})/\text{SO}(d)$ (we recall the general definition in 2.5 below). These spaces are manifolds (or more generally orbifolds) of non-positive curvature and we are interested in bounding various topological invariants purely in terms of their volume. More
precisely we prove the two following results. Let $G$ be a semisimple Lie group with the symmetric space $X$ and let $M = \Gamma \backslash X$ be a finite volume locally symmetric space.

- If $\Gamma$ is arithmetic and $M$ is a manifold then $M$ is homotopy equivalent to a simplicial complex with $O_X(\text{vol}(M))$ vertices and degree $O_X(1)$. This confirms a conjecture of Gelander [35].
- For $\Gamma$ congruence arithmetic, $b_k(M) \sim \beta_k^2(G) \text{vol}(M)$, where $\beta_k^2(G)$ is the $k$th $L^2$-Betti number of $G$. In particular, the Betti numbers asymptotically vanish outside of the middle dimension.

The first result was conjectured by Gelander [35] in the more general setting of finite volume locally-$X$ spaces (not necessarily arithmetic), except when $X = H^3$. By super-rigidity results of Margulis, Corlette, Gromov-Schoen, we have a full proof of this conjecture except in the case where $X$ is either hyperbolic $n$-space when $n \geq 4$ or complex hyperbolic space. It is known that there are non-arithmetic lattices in all groups $\text{SO}(n,1) = \text{Isom}(\mathbb{H}^n)$ due to a construction by Gromov–Piatetski-Shapiro [38], and in $\text{SU}(n,1)$ for $n = 2, 3$ by constructions of Deligne–Mostow [26] (the existence of non-arithmetic lattices in $\text{SU}(n,1)$ for $n \geq 4$ being a major open question). Following [37], we say that a family of orbifolds $F$ has uniform homotopy complexity if there are constants $A, B$ such that every $M \in F$ is homotopy equivalent to some a simplicial complex $\mathcal{N}$ with at most $A \text{vol}(M)$ vertices and all vertices are bounded by $B$. Our first result can be restated as follows: for any symmetric space $X$ of noncompact type without Euclidean factors, the family of arithmetic locally–$X$–manifolds has uniform homotopy complexity.

The second result is a refinement of a theorem of Gromov who proved that $b_k(M) = O(\text{vol}(M))$ for negatively curved manifolds (and also non-positively curved manifolds, under the necessary assumption that the manifold and its Riemannian metric is analytic), which was written up in the book by Ballmann, Gromov and Schroeder [10]. In the context of compact locally symmetric spaces, using Matsushima’s formula it can be re-interpreted as a problem of limit multiplicities for cohomological automorphic representations (see 1.4 below for explanation on these notions). The latter problem was dealt with for higher-rank groups with property (T) in [2] Corollary 1.6 via the notion of Benjamini–Schrömm convergence and we use the same approach here: we prove that sequences of congruence arithmetic lattices always converge in this sense to the symmetric space (Theorem 1.1 below), and the result on Betti numbers for compact $M$ follows by usual arguments. For locally symmetric spaces of finite volume we can use a convergence result for Betti numbers of Abért–Bergeron–Biringer–Gelander [1] but the more general limit multiplicities problem remains open (see 1.4.1 below for some discussion). Note that for lattices in rank 1 groups, the assumption that they are not merely arithmetic is necessary for the conclusion about Betti numbers to hold, see for example [2] Example 1.7.

A large part of our work here is thus concerned with the study of Benjamini-Schramm convergence. This notion was introduced in a discrete setting by Abért–Glasner–Virág in [4], and it for locally symmetric spaces it was introduced in [2]. In that last paper Abért et al. prove that if $G$ is a higher rank semisimple Lie group with property (T) then any sequence of irreducible pairwise non-isometric finite volume locally symmetric spaces Benjamini-Schramm converges to $X$, and so does any sequence of quotients by congruence subgroups in a fixed lattice. Later on, convergence was proven for some sequences of noncommensurable arithmetic lattices in some rank 1 or products of rank
1 groups by the last author \[64\], Matz \[53\] and for sequences of congruence subgroups in higher-rank products (where property (T) does not necessarily hold) by Levit \[47\].

The present work is in part a follow-up to the work of the first author \[33\] where convergence was proven to hold for all torsion-free lattices in \( \text{PGL}_2(\mathbb{C}) \) and \( \text{PGL}_2(\mathbb{R}) \), and subsequent work with the last author \[34\] dealing with the presence of torsion elements. Using the same approach as in these papers we show that Benjamini–Schramm convergence holds for sequences of lattices where the degree of the trace field is bounded.

For the remaining cases we rely on another argument using a refinement of the Margulis lemma for arithmetic manifolds, which is a corollary of the “height gap theorem” of Breuillard \[22\]. Following this latter approach we are also able to get powerful estimates on the volume of the thin part (Theorem \[C\] below) which are strong enough to imply the Gelander conjecture. We can also use them to give quantitative upper bounds for Betti numbers \( b_k(M) \) in terms of the degree of the trace field of \( M \) (Theorem \[F\] below).

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1.1. Estimates on the volume of the thin part and the Gelander conjecture.

Our first result is the following, which was conjectured to hold in \[35\]; we call an arithmetic locally-\( X \)-manifold any quotient \( \Gamma \backslash X \) where \( \Gamma \) is a torsion-free arithmetic lattice in \( G \).

**Theorem A.** There are constants \( A, B \) dependent only on \( X \), such that every arithmetic locally-\( X \)-manifold \( M \) is homotopy equivalent to a simplicial complex \( N \) with at most \( A \text{vol}(M) \) simplices, where every vertex is incident to at most \( B \) simplices.

A standard consequence of this result is the following estimate on torsion homology (similar estimates for Betti numbers are well-known in the setting of locally symmetric space by the work of Ballmann–Gromov–Schroeder \[9\]). See e.g. \[8\] Section 5.2] for the argument deducing the bounds on homology from the existence of a triangulation with bounded degree; we note that in the case where \( X \) is of rank 1 a more general result is given in \[8\] Theorem 1.2]; the statement there eschews the arithmeticity assumption (in fact it is valid for all negatively curved manifolds), and is deduced from a weaker result regarding homotopy type, see Theorem 1.4 there.

**Theorem B.** There is a constant \( C \) depending only on \( X \) such that for any arithmetic locally-\( X \)-manifold \( M \) of finite volume and \( 0 \leq i \leq \dim(X) \) we have

\[
\log |H_i(M,\mathbb{Z})_{\text{tors}}| \leq C \text{vol}(M)
\]

where for an abelian group \( A \) we denote by \( A_{\text{tors}} \) its torsion subgroup.

We note that while we expect these two results to extend to \( X \)-orbifolds the arguments we supply are not sufficient for this extension, though they should provide a good starting point.

To construct the simplicial complex in Theorem \[A\] we use the same technique as in \[35\], namely taking the nerve of a good cover by balls; the proof consists in estimating the number of centers we need for such a cover in terms of the volume of \( M = \Gamma \backslash X \). This set of centers is going to be a sufficiently dense and sufficiently separated subset of points in \( M \), the difficulty is that the density and separation must depend on the local injectivity radius. However a straightforward argument shows that for a compact manifold we can always use a set with \( O \left( \text{vol}(M) + \int_{M_{\leq \varepsilon} \text{InjRad}_M(x)}^{-\dim(M)} dx \right) \) points (where \( \varepsilon \) is any
positive number). To guarantee that this is $O(\text{vol} \ M)$ we need estimates for the volume of the thin part $M_{\leq \varepsilon}$ of an arithmetic locally symmetric space in terms of the degree of its field of definition. This is given by the following result.

**Theorem C.** Let $G$ be a semisimple Lie group with associated symmetric space $X$. There are positive constants $c = c_X$, $\eta = \eta_X$ such that any arithmetic lattice $\Gamma \subset G$ with trace field $k$ satisfies

$$\text{vol}((\Gamma \setminus X)_{\leq \eta[k:Q]}) \leq \text{vol}(\Gamma \setminus X)e^{-c[k:Q]}.$$ 

See section 6 for the details of the argument. Note that it should also be possible to construct a complex with the same number of vertices and bounded degree which is homeomorphic to the manifold, using Delaunay triangulations (see 6.1). It follows from positivity of the simplicial volume [43] that for triangulations one cannot do better than $O(\text{vol})$ simplices. On the other hand it is not clear at present if this is also the case for simplicial complexes which are merely homotopy equivalent to locally symmetric spaces.

We do note that using $L^2$-invariants one can prove that given $B$, a simplicial complex of degree at most $B$ which is homotopy equivalent to a real or complex hyperbolic manifold $M$ of dimension $d$ must have at least $c(d, B) \text{vol}(M)$ simplices; see 6.2 below for some details.

It is possible to effectively compute such simplicial complexes. Work in progress by M. Lipnowski and A. Page [48] proposes an algorithm that does this in polynomial time in the volume, for general compact locally symmetric spaces, with a lower bound on the injectivity radius in the input.

1.1.1. Sketch of proof of Theorem C. The proof of Theorem C uses an enhanced version of the Margulis lemma which we call the “arithmetic Margulis lemma”, which follows from the work of Breuillard [22].

**Theorem 3.1.** Let $G$ be a real semi-simple Lie group. There exists a constant $\varepsilon_G > 0$ with the following property. Let $\Gamma \subset G$ be an arithmetic lattice with trace field $k$. Let $x \in X$. Then, the subgroup generated by the set

$$\{ \gamma \in \Gamma \mid d(x, \gamma x) \leq \varepsilon_G[k:Q] \}$$

is virtually nilpotent.

Roughly speaking, this theorem allows us to work “as if” the $2\eta[k:Q]$-thin part of $\Gamma \setminus X$ was decomposed into a disjoint collars of diameter $2\eta[k:Q]$, all embedded in $\Gamma \setminus X$, and the collars associated to distinct maximal tori were disjoint. In reality they can overlap, but any point is covered by at most $|k:Q|O(1)$ collars. This will not matter when we prove exponential bounds; see Propositions 1.2 1.3 for precise statements. In this ideal situation it suffices to prove that the growth of the volume of an embedded collar is exponential in the diameter, i.e. there exists $\delta > 0$ such that a collar of diameter $R + T$ is larger that $e^{\delta T}$ times the volume of a tube of diameter $R$. Then, since the total volume of all collars of diameter $2\eta[k:Q]$ is both smaller than $\text{vol}(\Gamma \setminus X)$ and larger than

1. If $R > 0$, we denote the $R$-thin part by $M_{\leq R}$, which is the set of points where the injectivity radius is smaller than $R$, see [24.2] below.

2. All this work would be rendered unnecessary if a positive answer to Lehmer’s problem, asking whether there is a universal lower bound for the Mahler measure for all algebraic integers. It would imply a uniform lower bound for the injectivity radius on all the arithmetic quotients $\Gamma \setminus X$. However, Lehmer’s problem has been an outstanding question in number theory for some decades, see [69] for a survey.
$e^{\delta n[k:Q]}$ times the volume of the tubes of diameter $\eta[k:Q]$ we get the result. The claim on the volumes of the tubes is a statement that is completely independent of the lattice. We deduce it from a result comparing the orbital integrals of functions of the form $1_{B(R)}$ when $R$ grows, where the comparison factor depends only on the group (Theorem 4.4); this result might be of independent interest and is stated as follows (see 2.6 below for the definition of the orbital integrals $O(\gamma,f)$).

Theorem 4.4. Let $G$ be a semisimple Lie group of finite center and let $\Gamma$ be a uniform lattice in $G$. There exists $C,\delta > 0$ (depending only on $G$) such that for every $R_1, R_2 \geq 0$ and every non-central $\gamma \in \Gamma$:

$$O(\gamma, 1_{B(R_1)}) < Ce^{-\delta R_2} O(\gamma, 1_{B(R_1+R_2)}).$$

Finally we note that Theorem C is optimal regarding the dependency of the thin part on the degree of the trace field. Indeed, as was indicated to us by Misha Belolipetsky, examples for this are given by the construction in [13, Section 6]. There, for any semisimple $G$, a sequence of number fields $k_n$ and arithmetic lattices $\Lambda_n$ in $G$ are constructed so that $[k_n : Q]$ goes to infinity, $\Lambda_n$ has trace field $k_n$ and $\text{vol}(\Lambda_n \setminus G) \leq c [k_n : Q]$ where $c_1$ is a constant depending only on $G$. For these lattices Theorem C thus shows that $\text{vol}(\Lambda_n \setminus X) \leq c [k_n : Q]$ where

$$\text{vol}(\Lambda_n \setminus X) \leq e^{-\delta \log \text{vol}(\Lambda_n \setminus X)} \leq (\text{vol}(\Lambda_n \setminus X))^{1-\alpha}$$

for some $\alpha > 0$ depending on $G$, and this is optimal (up to the determination of the best possible constants $\eta, \alpha$) since the volume of balls in $X$ grows exponentially.

1.2. Benjamini–Schramm convergence: non-effective results. Consider the following condition for a sequence of locally symmetric spaces $M_n$ of finite volume.

$$\forall R > 0 \lim_{n \to +\infty} \frac{\text{vol}((M_n)_{\leq R})}{\text{vol} M_n} = 0$$

In [2] it was called the “Benjamini–Schramm convergence to $X$”; see 8.1 below for more details on this notion of convergence. Theorem C establishes that this holds for a sequence with trace field of unbounded degree, with quantitative estimates on the rate of convergence. In general we prove the following statement.

Theorem D. Let $G$ be a noncompact semisimple Lie group, let $X$ be its symmetric space and let $\Gamma_n$ be sequence of pairwise non-conjugate congruence arithmetic lattices. Then, (1.1) holds for the sequence of locally symmetric spaces $\Gamma_n \setminus X$.

We note that this theorem remains valid when taking a sequence of arithmetic lattices which are not necessarily congruence but which contains no infinite subsequence of pairwise commensurable lattices (or even no infinite sequence of lattices contained in the same maximal arithmetic lattice), and that this apparently more general statement follows formally from the theorem stated above since maximal arithmetic lattices are congruence by Proposition 10.1; in fact our proof of Theorem D establishes this directly, but we chose the statement above for its simplicity.

1.2.1. Proof of Theorem D. We prove Theorem D by first distinguishing between the cases when the arithmetic lattices are defined using number fields of bounded or unbounded degree over $Q$ (see 2.5 below for the definition of the trace field). Namely, we will prove the two following theorems.

Theorem 8.4. Let $G$ be a noncompact semisimple Lie group and let $X$ be its symmetric space. Let $\Gamma_n$ in $G$ be a sequence of congruence arithmetic lattices in $G$ and assume that
the degree over \(\mathbb{Q}\) of the trace field of \(\Gamma_n\) goes to infinity with \(n\). Then (1.1) holds for the sequence of locally symmetric spaces \(\Gamma_n\backslash X\).

**Theorem 10.4.** Let \(G\) be a noncompact semisimple Lie group and let \(X\) be its symmetric space. Let \(\Gamma_n\) in \(G\) be a sequence of different maximal arithmetic lattices in \(G\) and assume that there exists \(d \in \mathbb{N}\) such that the trace field of each \(\Gamma_n\) is an extension of degree \(d\) of \(\mathbb{Q}\). Then (1.1) holds for the sequence of locally symmetric spaces \(\Gamma_n\backslash X\).

**Proof of Theorem D.** Let \(\Gamma_n\) be a sequence of congruence arithmetic lattices in \(G\). Passing to a sub-sequence we can assume that

\[
\lim_{n \to \infty} \frac{\text{vol}(\Gamma_n \backslash X)_{\leq R}}{\text{vol}(\Gamma_n \backslash X)} = \ell,
\]

for some \(\ell \geq 0\). Theorem (1) will follow once we show that any such limit \(\ell\) is zero. Let \(k_n\) be the trace field of \(\Gamma_n\) and let \(d_n = [k_n : \mathbb{Q}]\). If \(d_n\) are unbounded, then \(\ell = 0\) by Theorem 8.4. If \(d_n\) are bounded, choose a sequence of maximal arithmetic lattices \(\Delta_n\), such that \(\Gamma_n \subset \Delta_n\) for every \(n\). If the set of \(\Delta_n\) is finite then \(\ell = 0\) by [2, Theorem 1.12]. Finally, if the sequence \(\Delta_n\) is infinite, we can apply Theorem 10.4 to show

\[
\ell = \lim_{n \to \infty} \frac{\text{vol}(\Delta_n \backslash X)_{\leq R}}{\text{vol}(\Delta_n \backslash X)} = 0.
\]

\[
\square
\]

1.2.2. **Outline of the proofs of Theorems 10.4 and 8.4.** The proofs we give for these two statements are completely different from each other. We mention that through a very careful estimate on the geometric side of Selberg’s trace formula it should be possible, though quite difficult at some points, to give a uniform proof of both Theorem 10.4 and 8.4. Such approach was carried out in [33] for \(SL_2(\mathbb{R})\) and \(SL_2(\mathbb{C})\). While working on the previous versions of this paper we were able to do that for most classical types. The new strategy adopted in this paper results in a shorter and more elegant argument.

Theorem 8.4 can be deduced directly from Theorem C, which is an effective upper bound on the volume of the thin part. We also give a very short proof of Theorem 8.4 that by-passes Theorem C using invariant random subgroups and the arithmetic Margulis lemma (see Section 8.3).

In the case where the degree of the trace field is bounded we use the same approach as in [33, 34], combining the notion of Benjamini–Schramm convergence of locally symmetric spaces introduced in [2] with an estimate on the geometric side of Selberg’s trace formula [7]. Following [33], we can estimate the volume of the thin part by the trace of a certain convolution operator. Recall that \(X = G/K\), where \(K\) is a maximal compact subgroup of \(G\). There is a unique bi-\(K\)-invariant semi-metric on \(G\) such the image of an \(R\)-ball in \(G\) is precisely an \(R\)-ball in \(X\). Let \(1_R : G \to \mathbb{R}\) be the characteristic function of the ball of radius \(R\) in \(G\). A point \(\Gamma gK \in \Gamma \backslash X\) is in the \(R\)-thin part if and only if \(\gamma gK\) is in the \(R\)-ball around \(gK\), for some non-central \(\gamma \in \Gamma\). Therefore \(\Gamma gK \in \Gamma \backslash X \in (\Gamma \backslash X)_{\leq R}\) if and only if \(\sum_{\gamma \in \Gamma \backslash Z(\Gamma)} 1_R(g^{-1}\gamma g) > 0\). Benjamini-Schramm convergence will follow once we show that the integral

\[
\frac{1}{\text{vol}(\Gamma_n \backslash G)} \int_{G/\Gamma_n} \sum_{\gamma \in \Gamma_n \backslash Z(\Gamma)} 1_R(g^{-1}\gamma g) dg
\]
converges to zero as $n \to \infty$. After some algebraic manipulation, this integral is equal to

$$\frac{1}{\text{vol}(\Gamma_n \backslash G)} \sum_{[\gamma]_{\Gamma_n} \subset W} \text{vol}((\Gamma_n)_{\gamma} \backslash G_{\gamma}) \mathcal{O}(\gamma, 1_R),$$

where $[\gamma]_{\Gamma_n}$ is the conjugacy class of $\gamma$ in $\Gamma_n$, $\mathcal{O}(\gamma, 1_R)$ is the orbital integral

$$\mathcal{O}(\gamma, f) = \int_{G_{\gamma} \backslash G} 1_R(x\gamma x^{-1}) \, dx,$$

and $W$ is the set of non-central elements of $\Gamma_n$.

One can recognize that the above sum is the non-central contribution in the geometric side of the celebrated Selberg trace formula [7]. The conjugacy classes of elements in $\Gamma_n$ are difficult to parametrize, so instead we use the adelic trace formula as in [33] (see 10.5). The adelic trace formula replaces the sum over the conjugacy classes in $\Gamma_n$ by a sum over the rational conjugacy classes in certain semi-simple algebraic group defined over the trace field of $\Gamma_n$. The rational conjugacy classes are much easier to parametrize and the adelic orbital integrals are in a sense better behaved. Giving bounds for this summation of integrals in the adelic setting is the main contribution from this part of our work and the technical parts take up various sections. The main ingredients are known estimates for the volumes of adelic quotients of tori (mostly from [73]), strong bounds for local and global orbital integrals (given in Section 11) and the estimates on the number of rational conjugacy classes elements of small Weil height (given in Section 12). These bounds are quite delicate and difficult to obtain for conjugacy classes with large centralizers. Using a reduction argument adapted from [34] (see Theorem 8.3 and Lemma 10.8 for precise statements) we can restrict $W$ to be the set of highly regular conjugacy classes and still get the conclusion about the Benjamin-Schramm convergence. This part of the argument uses the Zariski density of invariant random subgroups proved in [2].

Because of the reduction step, the argument does not give an effective bound on the volume of the $R$-thin part of the $\Gamma_n \backslash X$, since we completely sidestep estimating the orbital integrals and covolumes of centralisers for irregular semisimple elements.

1.3. Betti numbers. The assumption that a sequence of locally symmetric space Benjamin-Schramm converges to $X$ (i.e. that it satisfies (1.1)) is natural in questions about the asymptotic growth of the topological invariants of $\Gamma_n \backslash X$. For example, by [11], we understand the growth of Betti numbers $b_i(\Gamma_n \backslash X)$ for such sequences in terms of the $L^2$-Betti numbers $\beta_i^{(2)}(X)$ of $X$. As an immediate corollary of [11] Corollary 1.4] and our theorem [10.4] we get the following result.

**Theorem E.** Let $\Gamma_n$ be a sequence of pairwise distinct, torsion-free irreducible congruence arithmetic lattices or pairwise non-commensurable irreducible arithmetic lattices in a semisimple Lie group $G$. Then

$$\lim_{n \to \infty} \frac{b_i(\Gamma_n \backslash X)}{\text{vol}(\Gamma_n \backslash X)} = \begin{cases} \beta_i^{(2)}(X) & \text{if } i = \dim(X)/2; \\ 0 & \text{otherwise}. \end{cases}$$

---

3For example, classifying conjugacy classes in $\text{SL}_n(\mathbb{Z})$ is more difficult than classifying conjugacy classes in $\text{SL}_n(\mathbb{Q})$
When all $\Gamma_n$ are uniform we do not need the general result of [11] to prove this as we explain in 1.4 just below. In the case where the degree of trace fields goes to infinity we can use the following stronger result which we prove using our explicit estimates from Theorem C (see 7, and we also give some explanations in 1.4).

**Theorem F.** Let $i \in 0, 1, \ldots, \dim X$. Then

$$b_i(\Gamma \setminus X) = \text{vol}(\Gamma \setminus X) \cdot \left( \beta_i^{(2)}(X) + O([k : \mathbb{Q}]^{-1}) \right)$$

for all uniform arithmetic lattices $\Gamma \subset G$.  

It is worth pointing out that this result is interesting only when $[k : \mathbb{Q}] \to \infty$, which excludes non-uniform lattices. The nonzero $L^2$-Betti numbers appear only for $i = \dim X / 2$ and $\beta_{\dim X / 2}^{(2)}(X)$ can be computed explicitly: it is equal to $\chi(X^d) / \text{vol}(X^d)$ where $X^d$ is the “compact dual” of $G$ endowed with the Haar measure compatible with that of $G$. A list of all simple Lie groups $G$ (up to isogeny) of non-compact type for which $\beta_{\dim X / 2}^{(2)}(G) \neq 0$ is as follows:

- all unitary groups $\text{SU}(n, m)$;
- all orthogonal groups $\text{SO}(n, m)$ with $nm$ even and the groups $\text{SO}^*(2n)$;
- all symplectic groups $\text{Sp}(n)$ and all groups $\text{Sp}(p, q)$ (isometries of quaternionic hermitian forms);
- some exceptional groups (at least one in each absolute type).

This is well-known, and can be recovered as follows: by [56, Theorem 1.1], a group is on the list if and only if it has a compact maximal torus. In terms of the classification by Vogan diagrams [42, VI.8] this means that the action of the Cartan involution on the Dynkin diagram is trivial. The list of such diagrams is given in this book, in Figure 6.1, p. 414 and Figure 6.2, p. 416 for classical and exceptional types respectively.

1.4. Limit multiplicities. Using Matsushima’s formula we can deduce both theorems above from the results on limit multiplicities. To explain this in detail we need to introduce some notations. If $\pi$ is a unitary representation of $G$, acting on a Hilbert space $\mathcal{H}_\pi$, and $\Gamma$ a lattice $G$, then $m(\pi, \Gamma)$ is the multiplicity of $\pi$ in $L^2(\Gamma \setminus G)$, that is

$$m(\pi, \Gamma) = \dim \text{Hom}(\pi, L^2(\Gamma \setminus G)).$$

If $\Gamma$ is uniform then the multiplicity $m(\pi, \Gamma)$ is finite, and nonzero for only countably many $\pi$. Moreover $L^2(\Gamma \setminus G) \cong \bigoplus_{\pi} \mathcal{H}_\pi^{n(\pi, \Gamma)}$ with the sum running over all isomorphism classes of unitary representations of $G$.

The problem of limit multiplicities in its most basic form asks whether for a sequence of (pairwise non-conjugated) lattices $\Gamma_n$ in $G$ we have

$$m(\pi, \Gamma_n) \to \infty, \quad d(\pi)$$

for all $\pi$, where $d(\pi)$ is the “formal degree” of $\pi$, which is nonzero unless $\pi$ is a discrete series [11]. On the other hand Matsushima’s formula [52] states the for every $i$ there is a finite set $C_i$ of unitary representations of $G$, called the cohomological representations of degree $i$, such that for any uniform lattice $\Gamma \subset G$

$$b_i(\Gamma \setminus X) = \sum_{\pi \in C_i} n(\pi, i) \cdot m(\pi, \Gamma)$$

(1.3)
where \( n(\pi, i) = \dim \text{Hom}_K(\bigwedge^i p, \pi) \) and \( p \) the representation of \( K \) on the tangent space of \( X \) at the identity coset. It is an integer depending only on \( \pi \) and the degree \( i \).

It is known that cohomological representations can be discrete series only in degree \( i = \dim(X)/2 \), so (1.3) and (1.4) then imply the convergence in Theorem E (with \( c(G) \) the sum of formal degrees of cohomological discrete series).

For a sequence of uniform lattices with bounded degree trace field, which are always uniformly discrete, the arguments in [2, 6.10] (see also [25]) imply (1.3). Using similar arguments, we give more precise estimates on the multiplicities using Theorem C resulting in Theorem F, which we do in Section 7.

On the other hand in [2, Theorem 1.2] a stronger result is claimed, namely that for any open regular bounded Borel subset \( S \) of the unitary dual of \( G \) we have the limit

\[
\nu_{\Gamma_n}(S) := \frac{1}{\text{vol}(\Gamma_n \backslash G)} \sum_{\pi \in S} m(\pi, \Gamma_n) \xrightarrow{n \to +\infty} \nu^G(S)
\]

for any uniformly discrete sequence \( \Gamma_n \) such that \( \Gamma_n \backslash X \) is Benjamini–Schramm convergent to \( X \), where \( \nu^G \) is the Plancherel measure of \( G \).

The proof of (1.5) rests on the two following claims:

1. The sequence of measures \( \nu_{\Gamma_n} \) on the unitary dual converges weakly to the Plancherel measure;
2. The Plancherel measure on the unitary dual is characterised by its values on Fourier transforms of smooth, compactly supported functions on \( G \) (see [2, Proposition 6.4]).

The first claim is holds in our setting, as a formal consequence of Benjamini–Schramm convergence in the bounded-degree case and (with more work) as a consequence of Theorem C in the other cases. The second claim is “Sauvageot’s density principle” from [66] and it is in this last reference that a gap has been found [55, footnote 7 in section 24.2]. Until this is fixed we cannot claim that (1.5) holds for all sequences of congruence lattices.

1.4.1. Limit multiplicities for non-cocompact lattices. For non-uniform lattices it is much harder to prove limit multiplicities for Benjamini–Schramm convergent sequences because the Arthur trace formula, which extends the Selberg trace formula to this setting, includes terms which are not directly related to the geometry. For congruence subgroups in a fixed arithmetic lattice this was dealt with by Finis–Lapid in [31, 32] and for some lattices in groups of type \( A_1 \) this was dealt with by Matz [53].

1.5. Further questions. To finish this introduction we state some open problems regarding locally symmetric spaces.

1.5.1. Optimal bounds for the volume of the thin part. The best possible statement in the vein of Theorems D and E would be the following.

**Conjecture 1.1.** Let \( G \) be a semisimple Lie group. There exists \( \alpha, \beta \) such that for any congruence arithmetic lattice \( \Gamma \) in \( G \) and \( M = \Gamma \backslash X \) the following holds

\[
\text{vol} \left(M_{\leq 1/\beta \log \text{vol}(M)} \right) \leq \text{vol}(M)^{1-\alpha}.
\]

This is proven in [33] for \( G = \text{PGL}_2(\mathbb{C}), \text{PGL}_2(\mathbb{R}) \). For congruence subgroups in a fixed arithmetic lattice a similar result is [2, Theorem 1.12]. Our quantitative results Theorem 10.7 and Theorem C also go in this direction, though they are much weaker (in different ways each) than what would be needed to address this conjecture.
1.5.2. **Non-congruence locally symmetric spaces.** The conclusion of the Gelander’s conjecture is true for hyperbolic surfaces and 2-orbifolds by the Gauss-Bonnet theorem. It is known to be false in general for hyperbolic 3–manifolds because of hyperbolic Dehn surgery, see [8, 1.4]. On the other hand it might very well be true for all other locally symmetric spaces, but our lack of understanding of the global structure on one hand, and the relative scarceness of examples on the other, for hyperbolic manifolds in dimensions $\geq 4$ seems at present to be a major obstacle to solving this one way or the other.

As mentioned, in contrast to what happens for higher-rank spaces\(^4\), for real and complex hyperbolic manifolds it is known that not all sequences BS-converge to hyperbolic space\(^5\). The following question of Shmuel Weinberger asks whether this is still true “generically”.

**Question 1.2.** Let $G$ be $\text{PO}(d,1)$, $d \geq 4$ or $\text{PU}(d,1)$, $d \geq 2$. Order the set of conjugacy classes of lattices in $G$ by covolume, so we get a sequence $\Gamma_1, \Gamma_2, \ldots$. Does

$$\frac{1}{n} \cdot \left\{ 1 \leq i \leq n : \frac{\text{vol}(\Gamma_i \backslash X) \leq R}{\text{vol}(\Gamma_1 \backslash X)} > \varepsilon \right\} \xrightarrow{n \to +\infty} 0$$

hold for all $\varepsilon > 0$ and $R > 0$?

For hyperbolic surfaces or 3–manifolds this question does not make sense because the ordering does not exist because of the failure of Wang’s finiteness theorem. For surfaces one can formulate similar questions using random models and they have a positive answer for discrete models (see e.g. [63, Appendix B]) or continuous ones (see [54]). In these dimensions it also makes sense to restrict to arithmetic manifolds for which Wang finiteness holds [18]; for surfaces it seems to be very likely that the answer would then be positive, see [50].

1.5.3. **Homotopy type of arithmetic orbifolds.** We expect that with a better understanding of the conical singularities\(^6\) in locally symmetric spaces, our proof of Gelander’s conjecture for manifolds could be extended to orbifolds. In dimension 3 results in this direction are given in [34], and the techniques of [31] might also be relevant.

1.6. **Outline of the paper.** Section 2 introduces the main objects we are interested in, the arithmetic lattices and their locally symmetric spaces, including the necessary background on Lie and algebraic groups. There we recall various standard results useful for our purposes which are not easy to find in the literature.

Section 3 gives the statement and proof of the Arithmetic Margulis lemma. In Section 4 we explain how to prove Theorem C using the arithmetic Margulis lemma and the comparison between orbital integrals which we establish in the next section 5. We deduce Theorems A and F in Sections 6 and 7 respectively.

Sections 8, 9 are preliminaries for the rest of the paper and many results therein are well known. Section 8 introduces Benjamini–Schramm convergence and gives a criterion for convergence, which we use right away to give a very short proof of Theorem 8.4. In Section 9 we recall various definitions and facts about the Galois cohomology of semisimple algebraic groups and algebraic tori. Sections 10 through 12 contain the proof of Theorem 10.4; the first explains the proof using standard results on the arithmetic of

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\(^4\)Conditionally on the congruence subgroup property in some cases.

\(^5\)For quaternionic and octonionic manifolds nothing is known one way or the other.

\(^6\)conical singularities correspond to torsion conjugacy classes in the lattice $\Gamma$. 

algebraic groups together assuming certain estimates on orbital integrals the number of elements of small height. These estimates are proven in the sections [3] and [2].

2. Notation, preliminaries

2.1. Number fields. Throughout this paper we will use \( k \) to denote a number field and \( \overline{k} \) an algebraic closure. The ring of integers of \( k \) will be denoted by \( \mathfrak{o}_k \). We will write \( \Delta_k \) for the field discriminant of \( k \). If \( L/k \) is a finite extension, we write \( \Delta_{L/k} \) for the relative discriminant, which is the ideal generated by the discriminants of all bases \( a_1, \ldots, a_{[L:k]} \in \mathfrak{o}_L \), of \( L \) over \( k \).

We will use \( V \) to denote the set of equivalence classes of valuations on \( k \). For each \( v \in V \) we will use the same letter to denote the usual normalisation\(^7\) of the equivalence class and \( | \cdot |_v \), for the associated norm on \( k \). The completion of \( k \) at \( v \) will be denoted by \( k_v \). We will denote the set of Archimedean valuations by \( V_\infty \) and non-Archimedean ones by \( V_f \). If \( v \in V_f \), then \( \mathfrak{o}_{k_v} \) will be the ring of integers in \( k_v \). For non-Archimedean valuations we will use the same letter \( p \) for a prime ideal of \( \mathfrak{o}_k \) and the corresponding \( p \)-adic valuation. If \( \tau \in \text{Hom}(k, \mathbb{C}) \) we will write \( | \cdot |_\tau \) for the valuation induced by the absolute value on \( \mathbb{C} \).

If \( a \in k \) we denote by \( N_{k/\mathbb{Q}}(a) \) its absolute norm which is defined by:

\[
N_{k/\mathbb{Q}}(a) = \prod_{v \in V_\infty} |a|_v = \prod_{v \in V_f} |a|_v^{-1}.
\]

and for an ideal \( I \), \( N_{k/\mathbb{Q}}(I) = |\mathfrak{o}_k/I| \). The two definitions coincide when \( I = (a) \) is principal. The Weil height of \( a \in k \) is defined to be

\[
h(a) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in V} \tau_v \log \max(1, |a|_v);
\]

where \( \tau_v = 2 \) if \( k_v = \mathbb{C} \) and \( v = 1 \) otherwise. Note that if \( a \in \mathfrak{o}_k \), then the sum can be restricted to \( v \in V_\infty \). This definition does not depend on the choice of the field \( k \) containing \( a \). The (logarithmic) Mahler measure is defined as

\[
m(a) = [\mathbb{Q}(a) : \mathbb{Q}]h(a).
\]

The adèle ring of \( k \) will be denoted by \( \mathbb{A} \), the infinite adèles \( \prod_{v \in V_\infty} k \) will be denoted by \( k_\infty \) and the finite ones \( \prod_{v \in V_f} k_v \) by \( \mathbb{A}_f \).

2.2. Algebraic groups and semi-simple elements. Throughout this paper we will use \( G \) to denote a semisimple algebraic group defined over \( k \); usually we will make this precise and add further hypotheses at every use of the notation. The adjoint representation of \( G \) on its \( k \)-Lie algebra will be denoted by \( \text{Ad} \). The group \( G(\mathbb{A}) \) and \( G(\mathbb{A}_f) \) will denote respectively the groups \( \mathbb{A} \) and \( \mathbb{A}_f \) points of \( G \) (see [38, pp. 248–249]).

Let \( \gamma \in G(k) \). We write \( G_\gamma \) for the centralizer of \( \gamma \), it will always be an algebraic subgroup of \( G \) defined over \( k \). If \( T \) is a torus in \( G \) we will denote by \( X^*(T) \) its group of characters defined over the algebraic closure \( \overline{k} \). It is a free abelian group of rank \( \dim(T) \) equipped with an action of the absolute Galois group of \( k \) [31].

We will write \( Z(G), Z(\mathbb{A}), Z(\mathbb{A}_f) \) for the centre of an algebraic group \( G \), a Lie group \( G \) or a discrete subgroup \( \Lambda \subset G \).

We recall that a semisimple element \( \gamma \in G(k) \) is said to be regular if the connected component of \( G_\gamma \) is a torus. This is equivalent to the following. Let \( T \leq G \) be a

\(^7\)absolute value in Archimedean places and \( |\varphi|_p = |\mathfrak{o}_k/p| \) where \( \varphi \in k_p \) is a uniformizer.
maximal torus with $\gamma \in T(k)$, and let $\Phi = \Phi(G, T) \subset X^*(T)$ be the root system of $(G, T)$ [58, 2.1.10]. Then, $\gamma$ is regular if and only if $\lambda(\gamma) \neq 1$ for all $\lambda \in \Phi$. An element $\gamma \in G(k)$ is said to be strongly regular if it is regular and $\lambda(\gamma) \neq \lambda'(\gamma)$ for all $\lambda \neq \lambda' \in \Phi$. The notion of strongly regular elements is not standard. We motivate its introduction by the following property.

**Proposition 2.2.** Let $\gamma$ be a strongly regular element of $G(k)$; then $G_\gamma$ is a maximal torus of $G$ which is split by the field $k[\lambda(\gamma) \mid \lambda \in \Phi]$, where $\Phi = \Phi(G, G_\gamma)$ is the root system of $G$ relative to the torus $G_\gamma$.

**Proof.** Since $\gamma$ is regular, the connected component $T$ of $G_\gamma$ is a maximal torus in $G$ ([70, 2.11]). We must prove that $G_\gamma = T$, i.e., that $G_\gamma$ is connected. We have that $G_\gamma \subset J$ where $J$ is the normaliser of $T$ in $G$, so that $W = J(k)/T(k)$ is the Weyl group of $G$. It suffices to prove that $\gamma^w \neq \gamma$ for all $w \in W, w \neq 1$. The Weyl group $W$ acts faithfully on $\Phi$ and for any $\lambda \in \Phi$ we have $\lambda(\gamma^w) = \lambda(w(\gamma))$. Choosing a $\lambda$ with $\lambda^w \neq \lambda$ we get that $\lambda(\gamma^w) \neq \lambda(\gamma)$ by strong regularity of $\gamma$. Hence, $\gamma^w \neq \gamma$. This proves that $G_\gamma$ is a torus.

Now we prove that $L = k[\lambda(\gamma) \mid \lambda \in \Phi]$ splits $T$. This amounts to showing that $Gal(L)$ acts trivially on $X^*(T)$, or on $\Phi$ [57]. Let $\sigma \in Gal(L)$ and let $\lambda \in \Phi$; we have $\lambda^\sigma(\gamma) = \lambda(\gamma^\sigma) = \lambda(\gamma)$. For all $\lambda' \in \Phi \setminus \{\lambda\}$ we have $\lambda'(\gamma) \neq \lambda(\gamma)$. It follows that $\lambda^\sigma = \lambda^\sigma$ must hold. Hence, $\sigma$ acts trivially on $\Phi$. \qed

Let $\gamma \in G(k)$ be a semisimple element. Choose a maximal torus $T$ containing it and let $\Phi = \Phi(G, T)$ be the associated root system. We define the (logarithmic) Mahler measure $m(\gamma)$ by:

$$m(\gamma) = \sum_{\lambda \in \Phi} [k : Q] h(\lambda(\gamma)).$$

where $h$ is the Weil height, defined by (2.1). The Weyl discriminant of $\gamma$ is

$$\Delta(\gamma) = \prod_{\lambda \in \Phi} (1 - \lambda(\gamma));$$

if all $\lambda(\gamma) \in \sigma_k$ then clearly $m(\Delta(\gamma)) \ll m(\gamma)$.

We say that an element $\gamma \in G(k)$ has integral traces if $tr \text{Ad}(\gamma^m) \in \sigma_k$, for every $m \in \mathbb{N}$.

**Proposition 2.2.** For any $d \in \mathbb{N}$ there exists an increasing function $f_d : [0, +\infty[ \to [0, +\infty[$ such that the following holds. Let $k$ be a number field with $[k : Q] \leq d$ and $G$ a semisimple $k$-group. For any strongly regular $\gamma \in G(k)$ which has integral traces in the adjoint representation, if $L$ is the minimal Galois splitting field for the centraliser $G_\gamma$, and $m(\gamma) \leq R$ then

$$N_{k/Q}(\Delta_{L/k}) \leq f_d(R).$$

**Proof.** Let $m = [L : k]$. By Proposition 2.1 the field $L$ is contained in $k[\lambda(\gamma) \mid \lambda \in \Phi]$. The degree $k[\lambda(\gamma) \mid \lambda \in \Phi] : k$ is at most the degree of the characteristic polynomial of $\text{Ad}(\gamma)$, so $m$ is bounded in terms of $G$. Since $\gamma$ has integral traces, $\lambda(\gamma)$ are algebraic integers. Let

$$\Sigma := \{\lambda(\gamma^\ell) \mid \lambda \in \Phi, \ell = 0, 1, \ldots, m - 1\} \subset \sigma_L.$$

There exist $\theta_1, \ldots, \theta_m \in \Sigma$ which form a basis of $L$ over $k$. By definition of the relative discriminant [41], $\Delta_{L/k}$ belongs to the principal ideal $I$ of $\sigma_k$ generated by $\det((\theta_i^{\ell_j})_{1 \leq i, j \leq m})$ where $Gal(L/k) = \{\sigma_1, \ldots, \sigma_m\}$. In particular, $N_{k/Q}(\Delta_{L/k}) \leq N_{k/Q}(I)$. 

Let \( \tau_1, \ldots, \tau_{[k:Q]} \) be the set of all embeddings \( k \to \mathbb{C} \). We extend each \( \tau_\ell \) to \( L \) in an arbitrary way. We have

\[
N_{k/Q}(\Delta_{L/k}) = \prod_{i=1}^{[k:Q]} |\det(\theta_i^{\tau_i})|_{1 \leq i,j \leq m} = \prod_{\ell=1}^{[k:Q]} |\det((\theta_i^{\tau_\ell})_{1 \leq i,j \leq m})|.
\]

Since \( m(\gamma) \) is bounded by \( R \), each absolute value \( |\lambda(\gamma)^{\tau_i}| \) is bounded by \( e^m R \) and consequently \( |\theta_i^{\tau_i}| \leq e^{m^2 R} \). Therefore, the absolute value of the product is bounded by a function depending only on \( R \) and \( d \). \( \square \)

A \( k \)-torus \( T \) is said to be \( k \)-anisotropic if \( X^\ast(T)^{\text{Gal}(k)} = 0 \). In other words, it has no nontrivial character defined over \( k \). This is equivalent to the quotient \( T(k) \backslash T(\mathbb{A}) \) being compact [58, Theorem 4.11]. A reductive \( k \)-group \( G \) is said to be \( k \)-anisotropic if all its \( k \)-tori are anisotropic.

2.3. Normalisation of measure.

2.3.1. Lie groups. We use the same normalisation of measure for reductive Lie groups as in [59] and [73]. Let \( G \) be a real reductive group. The space of right \( G \)-invariant real differential \((\dim G)\)-forms is a one dimensional vectorial space. Any non-zero such form will \( \omega \) gives rise to a right \( G \)-invariant measure \( \mu_\omega \) on \( G \) via the procedure described in [58, p. 167).

If \( G \) is a semisimple real Lie group, there exists a unique compact semisimple real Lie group \( \tilde{G} \), such that the complexifications \( G_\mathbb{C} \) and \( \tilde{G}_\mathbb{C} \) are isomorphic. Let \( \tilde{\omega} \) be the unique, up to sign, real differential form on \( \tilde{G} \), such that \( \mu_\omega \) is the Haar probability measure. We fix a (complex) isomorphism \( \iota : G_\mathbb{C} \to \tilde{G}_\mathbb{C} \) and define \( \omega := \iota^\ast(\tilde{\omega}) \). The form \( \omega \) turns out to be \( G \)-invariant, defined over \( \mathbb{R} \) and independent of the choice of \( \iota \).

We fix our choice of Haar measure on \( G \) to be \( \mu_\omega \).

If \( T \) is a \( \mathbb{R} \)-torus, the connected component of \( T(\mathbb{R}) \) is isomorphic to a product \( \text{SO}(2)^a \times (\mathbb{R}^\times)^b \). We normalise the Haar measure by taking the probability Haar measure on the \( \text{SO}(2) \)-factors and \( dt/|t| \) on the \( \mathbb{R}^\times \) factors.

If \( T \) is a torus over a non-Archimedean local field \( k_v \), then we normalise the Haar measure on \( T(k_v) \) so that the unique maximal compact subgroup has measure 1. If \( G \) is a semisimple group over \( k_v \), then we will normalise the Haar measure on \( G(k_v) \) by choosing a compact-open subgroup of mass 1 at each instance.

2.3.2. Homogeneous spaces. We recall how the quotient measures on homogeneous spaces are defined: if \( G \) is a locally compact group and \( H \) a closed unimodular subgroup, then for any choices of two of Haar measures \( dg \) on \( G \), \( dh \) on \( H \) and a \( G \)-invariant measure \( dx \) on \( G/H \) the third is uniquely determined so that the equality

\[
\int_G f dg = \int_{G/H} \int_H f(xh)dhdx
\]

holds for any compactly supported function \( f \) on \( G \) (note that \( \int_H f(xh)dh \) is a right-\( H \)-invariant function with compact support in the quotient).

2.4. Symmetric spaces. If \( G \) is a semisimple Lie group then the associated Riemannian symmetric space is \( X = G/K \) where \( K \) is a maximal compact subgroup, endowed with a \( G \)-invariant Riemannian metric. We use \( d_X \) to denote the distance function on \( X \). To normalize this metric we use the measure normalizations described above: if \( G \)
is (implicitly) endowed with a Haar measure we choose the Riemannian metric on $X$ so that its volume form coincides with the quotient measure where $\text{vol}(K) = 1$.

2.4.1. Semisimple isometries. If $g \in G$ is semisimple and does not generate a bounded subgroup then we will call it noncompact. Such an element admits a unique totally geodesic subspace $\min(g) \subset X$ on which it acts with the minimal translation length. The distance $d_X(x,gx)$ is constant for $x \in \min(g)$; we call it the minimal translation length and denote by $\ell(g)$. The following proposition is known to experts, though it is often only implicit in the literature.

**Proposition 2.3.** Let $X$ be the symmetric space of a semisimple Lie group $G$. There are constants $a_X, A_X > 0$ such that for any number field $k$, any semisimple $k$-group $G$ such that the symmetric space of $G(k_\infty)$ is isometric to $X$ (i.e. $G(k_\infty)$ is isogeneous to $G$ times a compact group) and any semisimple $\gamma \in G(k)$ which is non-compact in $G(k_\infty)$ and has integral traces we have

\begin{equation}
(2.4) \quad a_X m(\gamma) \leq \ell(\gamma) \leq A_X m(\gamma).
\end{equation}

**Proof.** By the hypothesis on $G(k_\infty)$, $\dim G$ is bounded by the dimension of $G$. So there exists an $n$ such that any such $G$ admits a faithful $k$-rational representation

$$\rho: G \to \text{SL}_n.$$ 

For any semisimple $\gamma \in G(k)$ we have that $\rho(\gamma)$ is semisimple and $m(\gamma) \asymp m(\rho(\gamma))$ with constants depending only on $n$ (since there are only finitely possible choices of $G$ and $\rho$ up to isomorphisms over $\bar{\mathbb{Q}}$).

For any $v \in V_\infty$ let $\rho_v: G(k_v) \to \text{SL}_n(k_v)$ be the map induced by $\rho$. Let $V_{nc}$ be the set of $v \in V_\infty$ such that $G(k_v)$ is non-compact. Then, $X$ embeds isometrically (up to a rescaling which depends only on $X$) in the symmetric space $Y = \prod_{v \in V_{nc}} Y_v$ where $Y_v$ is the symmetric space of $\text{SL}_n(\mathbb{R})$ or $\text{SL}_2(\mathbb{R})$ according to whether $v$ is real or complex. The embedding is induced by $\prod_{v \in V_{nc}} \rho_v$. We denote by $\ell_v(\gamma)$ the translation length of $\rho \otimes_v \mathbb{R}(\gamma)$ on $Y_v$. The translation length on $Y$ is asymptotic to $\sum_{v \in V_{nc}} \ell_v(\gamma)$, with constants depending only on the cardinality $|V_{nc}|$, which is bounded by $\dim(X)$.

We now estimate each $\ell_v(\gamma)$ using results in [45]. Note that the definition (2.1) in [45] for the Mahler measure is unconventional as they do not restrict to polynomials with $\mathbb{Z}$ coefficients; in our context, it corresponds to fixing $v$ and looking only at $\sum_{\lambda \in \Phi_v} \log \max(1, |\lambda(\rho(\gamma))|_v)$ (where $\Phi_v$ is the root system for $\text{SL}_n$). Hence, [45 Theorem 2.3] implies that $\ell_v \asymp \sum_{\lambda \in \Phi_v} \log \max(1, |\lambda(\rho(\gamma))|_v)$ with implicit constants depending only on $n$. It follows that

$$\sum_{v \in V_{nc}} \ell_v(\gamma) \asymp \sum_{v \in V_{nc}} \sum_{\lambda \in \Phi_v} \log \max(1, |\lambda(\rho(\gamma))|_v) = m(\rho(\gamma)).$$

By the first paragraph $m(\rho(\gamma))$ is comparable to $m(\gamma)$ and by the second $\sum_{v \in V_{nc}} \ell_v(\gamma)$ is comparable to $\ell(\gamma)$. The proof is finished. \qed

A stronger condition than being noncompact is $\mathbb{R}$-regularity. An element $g \in G$ is said to be $\mathbb{R}$-regular if $\text{Ad}(g)$ has the maximal number of eigenvalues of absolute value $\neq 1$. These elements can also be characterised by the following result, which is [12 Lemma 1.5].

**Lemma 2.4.** Let $\gamma$ be a semi-simple regular element of $G$. If $\gamma$ is $\mathbb{R}$-regular then its centralizer $G_\gamma$ contains a maximal $\mathbb{R}$-split torus of $G$. 

2.4.2. Injectivity radius and the thin part. Let \( d(g, h) := d_X(gK, hK) \). It is a bi-\( K \)-invariant, left \( G \)-invariant semi-metric on \( G \). Let

\[
B(R) = \{ g \in G : d_X(K, gK) \leq R \} = \{ g \in G : d(g, 1) \leq R \}.
\]

One can think of \( B(R) \) as a ball in \( G \), although it only corresponds to the semi-metric \( d \) on \( G \). Let \( \Lambda \subset G \) be a discrete subgroup of \( G \) and let \( x = \Lambda gK \in \Lambda \setminus X \) be a point in the locally symmetric space \( M = \Lambda \setminus X \). The injectivity radius of \( M \) at \( x \) is defined as

\[
\text{InjRad}_{\Lambda \setminus \Gamma}(x) = \sup \left\{ R \geq 0 : d_X(gK, \gamma gK) \geq \frac{R}{2} \text{ for every } \gamma \in \Lambda \setminus Z(\Lambda) \right\}
\]

\[
= \sup \{ R \geq 0 : \Lambda^g \cap B(2R) = Z(\Lambda) \}.
\]

If \( M \) is a manifold (that is, if \( \Lambda \) is torsion-free) then this is the same as the usual definition of injectivity radius. In general, this definition also takes into account the singular locus of the orbifold \( M \). We also recall that the global injectivity radius is defined by \( \text{InjRad}_M := \inf_{x \in M} \text{InjRad}_M(x) \).

For \( R > 0 \) the \( R \)-thin part of the locally symmetric space \( M \) is the set

\[
M_{\geq R} = \{ x \in M | \text{InjRad}_M(x) \leq R \}.
\]

The \( R \)-thick part \( (\Lambda \setminus X)_{\geq R} \) can be then defined as the closure of the complement of the \( R \)-thin part.

2.5. Arithmetic lattices. Let \( G \) be a semisimple Lie group. A lattice \( \Gamma \subset G \) is called arithmetic if there exists a \( \mathbb{Q} \)-group \( H \) and a surjective morphism \( \pi : H(\mathbb{R}) \to G \) with compact kernel such that \( \Gamma \) and \( \pi(\mathbb{H}(\mathbb{Z})) \) are commensurable (i.e. have a common finite index subgroup). An arithmetic locally symmetric space is the quotient of a symmetric space \( X = G/K \) by an arithmetic lattice in \( G \).

Alternatively, arithmetic lattices can be given a more constructive definition using the language of adèles. This point of view is particularly well adapted to writing down trace formulas so that is what we are going to use. Let \( k \) be a number field with adèle ring \( \mathbb{A} \) and let \( G \) be an adjoint semi-simple linear algebraic group over \( k \). We will say that the couple \( (G, k) \) is admissible (for \( G \)) if \( G(k \otimes \mathbb{R}) \simeq G \times K \) with \( K \) compact. Let \( U \) be an open compact subgroup of \( G(\mathbb{A}_f) \). Then

\[
\Gamma_U := G(k) \cap (G(k_\infty) \times U)
\]

is a lattice in \( (G(k_\infty) \times U) \) (see [55, Chapter 5]). By abuse of notation we will also write \( \Gamma_U \) for the projection of \( \Gamma_U \) to \( G \), which is still a lattice since \( U \) is compact.

In general, an arithmetic lattice can be defined as any lattice which is commensurable to some \( \Gamma_U \), and a congruence arithmetic lattice is one that contains some \( \Gamma_U \). We note that the Borel density theorem and the commensurability invariance of the trace field (see below) imply that the commensurability class of \( \Gamma_U \) is uniquely determined by the pair \((G, k)\) and vice-versa. We recall that the lattice \( \Gamma_U \) is uniform if and only if the \( k \)-group \( G \) is \( k \)-anisotropic [55, Theorem 4.12].

The (adjoint) trace field of an arithmetic lattice \( \Gamma \) is defined as the field \( k \) where \( (G, k) \) is the unique admissible pair for which \( \Gamma_U \) is commensurable to \( \Gamma \). Alternatively, the trace field is the field generated by the traces of elements of \( \Gamma \) in the adjoint representation:

\[
k = \mathbb{Q}(\text{tr Ad}(\gamma), \gamma \in \Gamma)
\]

That both definitions coincide is not obvious and follows from theorems of Vinberg. More precisely, let \( \Gamma' \) be an arithmetic lattice in \( G(k) \) which is commensurable to \( \Gamma \). Then
we have obviously that $\text{Ad}(\Gamma') \subset \text{Ad}(G(k))$ so $Q(\text{tr Ad}(\Gamma')) \subset k$; on the other hand we have $Q(\text{tr}(\Gamma')) = Q(\text{tr}(\Gamma'))$ as follows from [74, Theorem 3]. For the reverse inclusion $k \subset Q(\text{tr Ad}(\gamma), \gamma \in \Gamma)$ we use Vinberg’s theorem: let $l$ be the trace field, by [74, Theorem 1] $\text{Ad}(\Gamma)$ is contained in $\text{Ad}(G(l_\infty))$. If $l \subset k$, then the image of $G(l_\infty)$ in $G$ is a proper subgroup of $G$ and this contradicts the fact that $\Gamma$ is Zariski-dense in the Zariski topology on $G(k_\infty)$.

The following proposition, the proof of which also uses Vinberg’s results, will also be of use later; in particular it implies that if $\Gamma$ is an arithmetic lattice then we can apply (2.4) to any semisimple $\gamma \in \Gamma$.

**Proposition 2.5.** Let $\Gamma \subset G$ be an arithmetic lattice and let $\gamma \in \Gamma$ be a semisimple element. Let $T$ be a maximal torus containing $\gamma$. Let $\Phi$ be the roots associated to $T$ (over $\mathbb{C}$). Then $\lambda(\gamma) \in \mathbb{Z}$ for all $\lambda \in \Phi$.

**Proof.** By definition of an arithmetic lattice, there exists a finite-index subgroup $\Gamma_1 \subset \Gamma$ and a $\mathfrak{o}_k$-lattice $L$ in the $k$-Lie algebra of $G$ such that $\text{Ad}(\Gamma_1)L \subset L$. It follows that the ring generated by the traces $\text{tr}(\text{Ad}(\gamma)), \gamma \in \Gamma_1$ is contained in $\mathfrak{o}_k$. By [74, Theorem 3] so is the ring generated by the $\text{tr}(\text{Ad}(\gamma)), \gamma \in \Gamma$. In particular all eigenvalues of elements in $\text{Ad}(\Gamma)$ are algebraic integers, which implies that the $\lambda(\gamma)$ must be an algebraic integer as well. \hfill \square

We use this result to give a lower bound on the systole of an arithmetic locally symmetric space using a lower bound on Mahler measure due to E. Dobrowolski. Similar estimates have been obtained earlier in [14] and [37].

**Proposition 2.6.** There exists a constant $c$ depending only on $G$ such that if $\Gamma$ is an arithmetic lattice in $G$ with adjoint trace field $k$ and $d = [k : \mathbb{Q}] \geq 2$, then for any semisimple non-compact $\gamma \in \Gamma$ we have

$$\ell(\gamma) \geq \frac{c}{(\log d)^3}.$$  

**Proof.** The proof of the previous proposition shows that the minimal polynomial of $\text{Ad}(\gamma)$ belongs to $\mathfrak{o}_k$, so its roots $\lambda(\gamma)$ are algebraic integers of degree at most $d \cdot \dim(G)$. By [27, Theorem 1] it follows that $m(\lambda(\gamma)) \geq \frac{1}{2(\log (d) + \log \dim(G))}$ for all $\lambda$ such that $\lambda(\gamma)$ is not a root of unity. Since $\gamma$ is non-compact there exists at least one such $\lambda$ and it follows that $m(\gamma) \geq \frac{1}{2(\log (d) + \log \dim(G))} \gg \frac{1}{\log (d)^3}$, and the result follows by (2.4). \hfill \square

2.6. Orbital integrals. The following proposition should be well-known.

**Proposition 2.7.** Let $\gamma$ be a semisimple element in a semisimple Lie group $G$. Then the map from $G/G_\gamma$ to $G$ defined by $g \mapsto g\gamma g^{-1}$ is proper.

It follows immediately from this proposition that for any continuous compactly supported function $f$ the orbital integral $\mathcal{O}(\gamma, f)$ defined by

$$\mathcal{O}(\gamma, f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx$$

is convergent.
Below we give a complete proof of Proposition 2.7 in the case where \( \gamma \) is regular non-compact, which uses several results that will be of use later, when we study further the orbital integral themselves both in Archimedean and non-Archimedean contexts.

2.6.1. **Commutators.** In this section \( k_v \) is a local field, possibly non-Archimedean\(^8\) and \( G \) is a \( k_v \)-group.

**Lemma 2.8.** Let \( T \) a maximal split torus in \( G \) and \( \gamma \in T(k_v) \), \( \Phi = \Phi_k(G,T) \) the associated relative root system. Let \( N \) be a \( k_v \)-unipotent subgroup normalised by \( T \) and \( \Phi^+_{N} \subset \Phi \) the subset of roots in \( N \). We denote by \( t \mapsto n_\lambda(t) \) an isomorphism from \( k_v \) to the root subspace associated with \( \lambda \).

We fix a linear order on \( \Phi^+_{N} \) so that any root cannot be decomposed as a sum of larger roots. There are constants \( c^\lambda_n \in k_v \) (depending only on the choice of isomorphisms \( n_\lambda \)) and \( \varepsilon_{\lambda,\nu} \in \{0,1\} \) (depending only on the chosen order) for \( \nu \subset \Phi^+_{N}, \nu \in \nu, \lambda \in \Phi^+_{N} \) such that

\[
\gamma n^{-1} \gamma^{-1} n = \prod_{\lambda \in \Phi^+_{N}} n_\lambda \left( 1 - \lambda(\gamma) \right) t_\lambda + \sum_{\nu \subset \Phi^+_{N}, \sum_{\nu} = \lambda} c^\lambda_{\nu} \prod_{\nu \in \nu} \nu(\gamma)^{\varepsilon_{\lambda,\nu}} t_\nu,
\]

where \( n = \prod_{\lambda \in \Phi^+_{N}} n_\lambda(t_\lambda) \).

**Proof.** Let \( \lambda_1, \ldots, \lambda_m \) be the chosen linear ordering. We have

\[
\gamma n^{-1} \gamma^{-1} n = \prod_{i=m}^{1} n_{\lambda_i}(-\lambda_i(\gamma)t_i) \prod_{i=1}^{m} n_{\lambda_i}(t_i).
\]

For each \( i = 1, \ldots, m \) we have \( \gamma n_{\lambda_i}(t) \gamma^{-1} = n_{\lambda_i}(\lambda_i(\gamma)t) \). We use this relation to move all the components of the first product one by one starting with \( i = m \) to their rightful place in the second product. We arrive at the final formula, where constants depend only on the commutators \( [n_{\lambda_i}(1), n_{\lambda_j}(1)] \). \( \square \)

2.6.2. **Commutators at Archimedean places.** Here \( k_v = \mathbb{R} \) or \( \mathbb{C} \). If \( H, L \) are Lie groups with Lie algebras \( h, l \) then their tangent bundles admit a canonical isomorphism to \( H \times h, L \times l \) respectively and so the Jacobian of a differentiable map \( H \to L \) is just a map \( H \to \text{Hom}(h,l) \).

**Lemma 2.9.** In the situation of Lemma 2.8 assume in addition that \( \gamma \) acts non-trivially on every root subspace \( N_\lambda, \lambda \in \Phi^+_{N} \). Then the map

\[
n \mapsto \gamma n^{-1} \gamma^{-1} n
\]

is a diffeomorphism of \( N \) onto itself and its Jacobian determinant is constant.

**Proof.** Since \( N(k_v) \) is simply connected, it suffices to prove the second assertion. It follows from Lemma 2.8 that the Jacobian matrix is upper triangular in the basis of \( n \) coming from the chosen order on roots. Moreover the diagonal terms are constantly equal to \( 1 - \lambda(\gamma), \lambda \in \Phi^+_{N} \) and the additional hypothesis on \( \gamma \) ensures that these are all nonzero. \( \square \)

\(^8\)The computation works over any field but we will use it only for these cases.
2.6.3. Proof of Proposition \[2.7\]. Let $M := G_\gamma$, consider first the case where $M$ is a non-compact torus. In this case, $M$ is contained in a proper parabolic subgroup of $G$ so one can find a unipotent subgroup $N$ normalised by $M$ and maximal for this property. Then $MN$ is a parabolic subgroup of $G$ and we can choose a maximal compact subgroup $K$ of $G$ such that the Iwasawa decomposition $G = MNK$ holds (see \[12\] Proposition 7.31). Since $K$ is compact, it suffices to prove that the map $N \to G$, $n \mapsto n^{-1}\gamma^{-1}n$ is proper. By Lemma \[2.9\] it is a diffeomorphism onto the closed subset $\gamma^{-1}N$ of $G$, so we are finished.

In general $M$ might not be contained in a proper parabolic. However, if $M$ is non-compact then it contains a non-compact torus and we can always find a proper parabolic $P$ such that $P \cap M$ is cocompact in $M$ and a Levi component of $P$ is contained in $M$.

Then considering the sequence of maps $K \times P \to K \times P/(P \cap M) \to G/M$ we need to prove that the map $N/(N \cap M) \to G$, $n \mapsto n\gamma n^{-1}$, is proper, where $N$ is the unipotent radical of $P$. This can be achieved by the same argument as above, using an appropriate modification of Lemma \[2.9\].

Finally, if $M$ is compact then $\gamma$ is compact and does not commute with any semi-simple non-compact element of $G$ and it is possible to prove the proposition using the $KAK$-decomposition.

3. An arithmetic Margulis lemma

**Theorem 3.1.** Let $G$ be a real semi-simple Lie group. There exists a constant $\varepsilon_G > 0$ with the following property. Let $\Gamma \subseteq G$ be an arithmetic lattice with trace field $k$. Let $x \in X$. Then, the subgroup generated by the set

$$\{ \gamma \in \Gamma \mid d(x, \gamma x) \leq \varepsilon_G[k : \mathbb{Q}]\}$$

is virtually nilpotent.

**Remark 3.2.** If $\Gamma$ is uniform then our proof shows that the subgroup generated by $\{ \gamma \in \Gamma \mid d(x, \gamma x) \leq \varepsilon_G[k : \mathbb{Q}]\}$ is even virtually abelian.

We will extract \[3.1\] from the height gap theorem of \[22\], more precisely, using \[22\] Cor 1.7.

**Theorem 3.3 (\[22\] Cor 1.7).** There are constants $N_1 = N_1(d) \in \mathbb{N}, \varepsilon_1 = \varepsilon_1(d) > 0$ such that if $F$ is any finite subset of $\text{GL}_d(\mathbb{Q})$ containing 1 and generating a non-virtually solvable subgroup, then we may find $a \in F^{N_1}$ and an eigenvalue $\lambda$ of $a$ such that $h(\lambda) > \varepsilon$.

**Proof of Theorem 3.3**. Using the adjoint representation we embed $G(k) \hookrightarrow \text{GL}_{\dim G}(k)$. Let $F = \{ \gamma \in \Gamma \mid d(x, \gamma x) \leq \varepsilon_G[k : \mathbb{Q}]\}$. We will first prove that the subgroup generated by $F$ is virtually solvable and then upgrade it to a virtually nilpotent subgroup. The precise value of $\varepsilon_G$ will be fixed during the proof, it will depend only on $G, \varepsilon_0$ and $N_1$.

Assume for the sake of contradiction that $F$ is not virtually solvable. Using \[22\] Cor 1.7, we find an element $\gamma \in F^{N_1}$ with an eigenvalue $\lambda$ such that $h(\lambda) \geq \varepsilon_1$. From the definition of $m(\gamma)$, we deduce immediately that

$$m(\gamma) \geq [k : \mathbb{Q}]h(\lambda) \geq [k : \mathbb{Q}]\varepsilon_1.$$

By Proposition \[2.3\] there is a constant $a_X$ such that

$$\ell(\gamma) \geq a_X[k : \mathbb{Q}]\varepsilon_1.$$

Since $\gamma \in F^{N_1}$ we must have $d(x, \gamma x) \leq N_1\varepsilon_G$. Setting $\varepsilon_G < \frac{\varepsilon_1}{a_XN_1}$ we get a contradiction. This proves that for $\varepsilon_G$ small enough, the group generated by $F$ is virtually solvable.
It remains to show that by choosing \( \varepsilon_G \) small enough, the group \( \langle F \rangle \) is in fact virtually nilpotent. Let \( \varepsilon_1 \) be the ordinary Margulis constant for \( X \) and let \( d_0 \) be the maximal degree of the trace field of a non-uniform lattice in \( G \). By choosing \( \varepsilon_G \leq \varepsilon_1 / d_0 \) we can guarantee that Theorem 3.1 holds for all non-uniform arithmetic lattices in \( G \). We can now restrict to the uniform case. Recall that an arithmetic lattice \( \Gamma \) with a trace field \( k \) is uniform if and only if the group \( G \) used to construct it is \( k \)-anisotropic [58, Thm 5.5].

Let \( H \subset G \) be the connected component of the Zariski closure of \( \langle F \rangle \). By Lemma 3.1, \( H \) must be a abelian Lie group. We deduce that \( \langle F \rangle \) is virtually abelian. This concludes the proof of the theorem in the uniform case.

\[ \Box \]

**Lemma 3.4.** Let \( H \) be a connected solvable subgroup of a linear semi-simple group \( G \). Suppose that both \( H, G \) are defined over \( k \) and that \( G \) is anisotropic over \( k \). Then \( H \) is abelian.

**Proof.** Recall that a semi-simple algebraic group \( G \) over \( k \) is anisotropic if one of the following equivalent conditions is satisfied:

1. There are no non-trivial \( k \)-rational maps \( G_m \to G \)
2. There are no rational unipotent subgroups of \( G \).

Let \( N \) be the unipotent radical of \( H \). Then by [19, 10.6] the derived subgroup \([H,H] \) is contained in \( N \). However, \( N \) is a \( k \)-rational unipotent subgroup of \( G \) so it must be trivial. We conclude that \( H \) is abelian.

\[ \Box \]

4. **Thin part in unbounded degree**

4.1. **Effective estimates on the volume of the thin part.** Let \( \Gamma \) be a lattice in a semisimple Lie group \( G \) and let \( R > 0 \). For any \( \gamma \in \Gamma, \gamma \neq \text{Id} \), we define the following subset of \( G \):

\[ S^R_\gamma := \{g \in G \text{ such that } d(g^{-1} \gamma g, 1) < R\}, \]

We note that if \( \Gamma \) is uniform the volume of \( \Gamma_\gamma \setminus S^R_\gamma \) is always finite. We use the arithmetic Margulis lemma from the previous section to prove the following fact.

**Proposition 4.1.** There exists \( \varepsilon, d_0, m > 0 \) only depending on \( G \) such that if \( \Gamma \) is a lattice with trace field \( k \) with degree \( d := [k : \mathbb{Q}] \geq d_0 \), then every \( g \) in \( G \) lies in at most \( d^m \) sets of the form \( S^d_\gamma \) where \( \gamma \in \Gamma \).

**Proof.** Let \( \varepsilon \) be the constant \( \varepsilon_G \) given by the Theorem 3.1. Let \( S = \{\gamma_1, ..., \gamma_n\} \) be all the distinct elements in \( \Gamma \) such that \( g \in S^d_\gamma \) for all \( i = 1, 2, ..., n \), that is \( d_G(g^{-1} \gamma_i g, 1) < \varepsilon d \) for every \( i \). We will assume that \( d \) is large enough so that \( \Gamma \) is uniform.

Define \( \Gamma_1 \) to be the subgroup generated by \( g^{-1} S g \). By Theorem 3.1 the subgroup \( \Gamma_1 \) is virtually solvable. As \( \Gamma \) is uniform, we have that \( \Gamma_1 \) is virtually abelian (Lemma 3.4). Moreover, by Lemma 8.5 there exists \( C_1 > 0 \) only depending on \( G \) and an abelian subgroup \( \Gamma_2 \) of \( \Gamma_1 \) such that \( [\Gamma_1 : \Gamma_2] < C_1 \). We can suppose \( \Gamma_2 = \Gamma_3 \times \Gamma_4 \), where \( \Gamma_3 \) consists of noncompact elements and \( \Gamma_4 \) consists of torsion elements.

Let \( \gamma \) be a torsion element in \( \Gamma_4 \). Then its characteristic polynomial in the adjoint representation is a polynomial with coefficients in \( \mathfrak{o}_k \) and degree at most \( C_3 \) where \( C_3 \) is a constant depending only on \( G \). It follows that the degree of the minimal polynomial over \( \mathbb{Q} \) of an eigenvalue of \( \text{Ad} \gamma \) is at most \( C_3 d \) (recall that \( d = [k : \mathbb{Q}] \)). Since the Euler totient function grows at least like square root, it follows that the eigenvalues of \( \text{Ad} \gamma \) are all roots of unity of order at most \( C_4 d^2 \), and that the order of \( \gamma \) itself is at most...
\[ C_0 d^2. \] As \( \Gamma_4 \) is a finitely generated abelian group of rank at most \( \text{rk}(G) \), we conclude that the order of \( \Gamma_4 \) is at most \( d^{m_1} \) for all \( d \) sufficiently large and some fixed integer \( m_1 \).

Let \( x_0 \) be the identity coset in \( X = G/K \). Then for any \( g \in G \) we have \( d_G(1,g) = d_X(gx_0, x_0) \) (recall that \( d_G \) is the semi-distance on \( G \) defined by this). So \( |\Gamma_2 \cap B_G(1, \varepsilon d)| \) is equal to the number of points in the orbit of \( x_0 \) at distance at most \( \varepsilon d \) from \( x_0 \). Call this set \( \Sigma \).

Let \( T \) be the a maximal torus of \( G \) containing \( \Gamma_2 \). It stabilizes a flat subspace \( E \subset X \), let \( p_E : X \to E \) be the closest point projection. Since \( X \) is non-positively curved, the projection \( p_E \) does not increase distances. Consider the restriction \( p_E : \Sigma \to E \). This map is at most \( |\Gamma_4| \)-to-one. On the other hand, for \( \gamma \in \Gamma_3 \), it follows from Proposition 4.1 that we have \( \ell(\gamma) \geq \frac{c_1}{\log(d)} \) for some positive constant \( c_1 > 0 \). So the image of \( \Sigma \) in \( E \) is contained in an Euclidean lattice with minimal vector of length at least \( \frac{c_1}{\log(d)} \).

Therefore, by a packing argument we get that
\[
|\Gamma_2 \cap B_G(1, \varepsilon d)| \leq \frac{\text{Vol}(B_E(\varepsilon d/2 + 1))}{\text{Vol}(B_E(c_1(\log d)^{-3}))}|\Gamma_4|
\]
where \( B_E(r) \) is the ball of radius \( r \) in \( E \) and the quantity in the right hand side is bounded by \( d^m \) with \( m := m_1 + \dim(E) + 1 \) and \( d \) large.

Observe that for \( \gamma_1, \gamma_2 \in \Gamma \) and \( R > 0 \) we have \( \gamma_2 S^R_{\gamma_1} = S^R_{\gamma_2 \gamma_1^{-1} \gamma_2} \). Hence, the set
\[
S^R_{\gamma} := \bigcup_{\gamma \in \Gamma} S^R_{\gamma^{-1} \gamma \gamma}
\]
is right \( \Gamma \)-invariant.

**Proposition 4.2.** Let \( \varepsilon, m, d_0 > 0 \) be as in Proposition 4.1 and let \( \Gamma \) be an arithmetic lattice in \( G \) with trace field of degree \( d \geq d_0 \). For any \( \gamma \in \Gamma \) and \( R < \varepsilon d \) we have:
\[
\text{vol}(\Gamma \backslash S^R_{\gamma}) \geq d^{-m} \text{vol}(\Gamma \backslash S^R_{\gamma})
\]

**Proof.** We need to show that the obvious projection \( p : \Gamma \backslash S^R_{\gamma} \to \Gamma \backslash S^R_{\gamma} \) is at most \( d^4 \) to one. Suppose that pairwise distinct points \( \Gamma, g_1, \Gamma, g_2, \ldots, \Gamma, g_k \in \Gamma \backslash S^R_{\gamma} \) project to the same point in \( \Gamma \backslash S^R_{\gamma} \). Then, there exist different cosets \( \gamma_1, \gamma_2, \ldots, \gamma_n \) such that \( \gamma_i g_i \) are pairwise different conjugacy classes of \( \Gamma \) the intersection \( \cap_{i=1}^n \Gamma \backslash S^R_{\gamma_i} \) is trivial.

In a similar way we can prove the following:

**Proposition 4.3.** Let \( \varepsilon, m, d_0 > 0 \) be as in Proposition 4.1, let \( d \geq d_0 \), \( R < \varepsilon d \) and let \( \Gamma \) be an arithmetic lattice in \( G \) with trace field of degree \( d \). Then, for any \( n > d^m \), if \( \{\gamma_1\}, \{\gamma_2\}, \ldots, \{\gamma_n\} \) are pairwise different conjugacy classes of \( \Gamma \) the intersection \( \cap_{i=1}^n \Gamma \backslash S^R_{\gamma_i} \) is trivial.

Before giving the proof of our estimates on the volume of the thin part we state a bound on orbital integrals which will be essential in the argument.

**Theorem 4.4.** Let \( G \) be a semisimple Lie group with finite center and without compact factors. There exist constants \( C, \delta > 0 \) depending only on \( G \), such that for every \( R_1, R_2 \geq 0 \) and every non-central semi-simple element \( \gamma \in G \) we have:
\[
\mathcal{O}(\gamma, 1_{B(R_1)}) \leq C e^{-\delta R_2} \mathcal{O}(\gamma, 1_{B(R_1+R_2)}).
\]

The proof of the above estimate is postponed to Section 5 below.
Theorem 4.5. Let $G$ be a semisimple Lie group with the associated symmetric space $X$. There are constants $C_1, C_2, \delta, \varepsilon, m > 0$ depending only on $G$ with the following property. Let $\Gamma \subset G$ be an arithmetic lattice with trace field $k$ and let $d = [k : \mathbb{Q}]$. For any $R_1, R_2 \geq 0$ with $R_1 + R_2 \leq \varepsilon d$ we have:

$$\text{vol}((\Gamma \setminus \Gamma) \leq R_1) \leq C_2 e^{-\delta R_2} [k : \mathbb{Q}]^{2m} \text{vol}((\Gamma \setminus \Gamma) \leq R_1 + R_2).$$

Proof. Suppose that $\Gamma g K \in (\Gamma \setminus \Gamma) \leq R_1$. By the definition of the thin part (Section 2.4.2), there exists a $\gamma \in \Gamma \setminus \Gamma$ such that $d_X(g K, \gamma g K) \leq R_1$. Therefore, $g \in S^R_{\gamma}$. The natural projection map $\Gamma \setminus \Gamma \to \Gamma \setminus \Gamma$ is volume preserving, so it follows that

$$\text{vol}((\Gamma \setminus \Gamma) \leq R_1) \leq \sum_{[\gamma] \in \Gamma \setminus \Gamma} \text{vol}(\Gamma \setminus \Gamma S^{R_1}_{\gamma}) \leq \sum_{[\gamma] \in \Gamma \setminus \Gamma} \text{vol}(\Gamma \setminus \Gamma S^{R_1}_{\gamma}).$$

By the definition of $S^R_{\gamma}$

$$\text{vol}(\Gamma \setminus \Gamma S^{R_1}_{\gamma}) = \int_{\Gamma \setminus \Gamma} 1_{B(R_1)}(g^{-1} \gamma g) dg = \text{vol}(\Gamma \setminus \Gamma O(\gamma, 1_{B(R_1)})).$$

We get

$$\text{vol}((\Gamma \setminus \Gamma) \leq R_1) \leq \sum_{[\gamma] \in \Gamma \setminus \Gamma} \text{vol}(\Gamma \setminus \Gamma O(\gamma, 1_{B(R_1)}))$$

$$\leq C_2 e^{-\delta R_2} \sum_{[\gamma] \in \Gamma \setminus \Gamma} \text{vol}(\Gamma \setminus \Gamma O(\gamma, 1_{B(R_1 + R_2)}))$$

$$\leq C_2 e^{-\delta R_2} d^{m} \sum_{[\gamma] \in \Gamma \setminus \Gamma} \text{vol}(\Gamma \setminus \Gamma S^{R_1 + R_2}_{\gamma})$$

where the last line follows from Proposition 4.2. By Proposition 4.3 each point of $\Gamma \setminus \Gamma$ is covered by at most $d^{m}$ distinct sets $\Gamma \setminus \Gamma S^{R_1 + R_2}_{\gamma}/K$. We also know that each point of $\Gamma \setminus \Gamma$ covered by $\Gamma \setminus \Gamma S^{R_1 + R_2}_{\gamma}/K$ is in the $(R_1 + R_2)$-thin part. Hence,

$$\sum_{[\gamma] \in \Gamma \setminus \Gamma} \text{vol}(\Gamma \setminus \Gamma S^{R_1 + R_2}_{\gamma}) \leq d^{m} \text{vol}((\Gamma \setminus \Gamma)_{R_1 + R_2}).$$

Combining the inequalities together we obtain

$$\text{vol}((\Gamma \setminus \Gamma) \leq R_1) \leq C_2 e^{-\delta R_2} d^{2m} \text{vol}((\Gamma \setminus \Gamma)_{R_1 + R_2}).$$

We can now deduce Theorem C as an immediate corollary.

Proof of Theorem C. Let $\eta := \varepsilon/2$, $R_1 = \eta [k : \mathbb{Q}]$ and $R_2 = \varepsilon [k : \mathbb{Q}]/2$. We have vol((\Gamma \setminus \Gamma) \leq R_1 + R_2) \leq \text{vol}(\Gamma \setminus \Gamma)$ so Theorem 4.5 yields the conclusion of Theorem C.

5. COMPARISON BETWEEN ORBITAL INTEGRALS

Let $f \in C_c(G)$ be a compactly supported function. The orbital integral was defined as

$$O(\gamma, f) = \int_{G \setminus \Gamma} f(g^{-1} \gamma g) dg.$$

It is finite, by Proposition 2.7 In this section we prove Theorem 4.4.
By Lemma 2.7, \( \psi_d \) is non-decreasing in \( R \). It follows that such that

\[
\int_G \varphi(g) dg = 1 \quad \text{and the following property holds. For every proper closed reductive subgroup } H \subset G \text{ and every } f \in L^2(H \backslash G) \text{ we have }
\]

\[
\|f \ast \varphi\|_2 \leq e^{-\delta} \|f\|_2.
\]

**Proof.** \( G \) decomposes as a product of simple groups. It is clear that the lemma will follow once we prove it for each factor separately. We can therefore assume that \( G \) is a simple non-compact Lie group. Choose any function \( \varphi \) satisfying the assumption of the lemma. Let \( M \) be the operator of the right convolution by \( \varphi \). We need to show that \( \|M\|_{L^2(H \backslash G)} \leq e^{-\delta} \), i.e. that \( \|M\|_{L^2(H \backslash G)} \) is uniformly bounded away from 1. Suppose \( G \) has Kazhdan’s property (T) and consider the unitary representation \( \pi = \bigoplus_H L^2(H \backslash G) \), where \( H \) runs over the conjugacy classes of proper reductive subgroups of \( G \). This representation has no fixed vectors, so by property (T) it has no almost invariant vectors. Since the support of \( \varphi \) generates \( G \), we deduce that \( \|M\|_{\pi} = \sup_{\|v\| = 1} \langle v, \pi(\varphi)v \rangle < 1 \).

It remains to treat the cases of non-compact simple Lie groups without property (T). These are, up to isogeny, \( \text{SO}(n,1), \text{SU}(n,1), n \geq 2 \) (see [11, 3.5.4]). The compact factors of \( H \) do not affect the norm of the convolution by \( \varphi \), so we can assume that the pair \((G,H)\) is (i) \( (\text{SO}(n,1),\text{SO}(m,1)), 0 < m < n \), (ii) \( (\text{SU}(n,1),\text{SO}(m,1)), 0 < m \leq n \) or (iii) \( (\text{SU}(n,1),\text{SU}(m,1)) \), with \( 0 < m < n \). Let \( \Lambda \) be a uniform lattice in \( H \). Since \( L^2(H \backslash G) \) is a sub-representation of \( L^2(\Lambda \backslash G) \), it is enough to show that \( M \) has a uniform spectral gap on \( L^2(\Lambda \backslash G) \). The function \( \varphi \) is \( K \)-invariant so this will follow from the uniform spectral gap for the Laplace operator acting on \( L^2(\Lambda \backslash X) \). Let \( \lambda_0(\Lambda \backslash X) \) be the lowest eigenvalue of the Laplace operator acting on \( L^2(\Lambda \backslash X) \). Let \( \delta(\Lambda) \) be the critical exponent of \( \Lambda \) in \( G \); it is equal to \( m, m, 2m \) in cases (i),(ii),(iii) respectively. By [28, 29, 30] we have

\[
\lambda_0(\Lambda \backslash X) = \begin{cases} 
\frac{1}{2}(\dim \partial X)^2 & \text{if } 0 \leq \delta(\Lambda) \leq \frac{1}{2} \dim \partial X, \\
\delta(\Lambda)(\dim \partial X - \delta(\Lambda)) & \text{if } \frac{1}{2} \dim \partial X < \delta(\Lambda) \leq \dim \partial X.
\end{cases}
\]

It follows that \( \lambda_0(\Lambda \backslash X) \) is bounded away from 0 uniformly in \( m \). The lemma is proved.

We are ready to prove Theorem 4.1.

**Proof Theorem 4.1.** It is enough to prove that

\[
O(\gamma, 1_{B(R)}) \leq e^{-2\delta} O(\gamma, 1_{B(R+2)}),
\]

for every \( R \geq 0 \). The theorem will follow by induction and the fact that \( O(\gamma, 1_{B(R)}) \) is non-decreasing in \( R \). Consider the function \( \psi_R : G_\gamma \backslash G \to \mathbb{R}_{\geq 0} \), given by \( \psi_R(g) = 1_{B(R)}(g^{-1}\gamma g) \). Alternatively, we can describe \( \psi_R \) as the characteristic function of the set

\[
\{ g \in G_\gamma \backslash G \mid \text{d}(g, \gamma g) \leq R \},
\]

where \( \text{d} : G \times G \to \mathbb{R}_{\geq 0} \) is the right invariant semi-metric introduced in section 2.4.2. By Lemma 2.7, \( \psi_R \) is compactly supported and we have

\[
O(\gamma, 1_{B(R)}) = \|\psi_R\|_2^2.
\]
Observe that for any \( g \in G \backslash G \) such that \( d(g, \gamma g) \leq R \) and any \( h \in B(1) \) we have
\[
d(gh, \gamma gh) \leq d(g, gh) + d(\gamma g, g) + d(\gamma gh, \gamma g) \leq R + 2.
\]
Let \( \varphi \) be the function from the Lemma 5.1. Since \( \text{supp} \varphi \subset B(1) \), the above inequality implies that \( \text{supp} (\psi_R \ast \varphi) \subset \text{supp} (\psi_{R+2}) \). In particular, \( \psi_R \ast \varphi(g) = \psi_R \ast \varphi(g) \psi_{R+2}(g) \) for every \( g \in G \backslash G \). By the Cauchy-Schwartz inequality
\[
\| \psi_R \ast \varphi \|_2 \| \psi_{R+2} \|_2 \geq \left( \int_{G \backslash G} \psi_R \ast \varphi(g) dg \right)^2.
\]
\[
\| \psi_R \ast \varphi \|_2^2 \Omega(\gamma, 1_B(R+2)) \geq \left( \int_{G \backslash G} \psi_R(g) dg \right)^2.
\]
By Lemma 5.1, \( \| \psi_R \ast \varphi \|_2 \geq e^{-2\delta} \Omega(\gamma, 1_B(R+2)) \). We infer that
\[
e^{-2\delta} \Omega(\gamma, 1_B(R+2)) \geq \Omega(\gamma, 1_B(R)).
\]
This establishes \( \Omega \). □

6. GELANDER’S CONJECTURE

Suppose \( M \) is a manifold locally isometric to \( X \) (i.e. \( M = \Gamma \backslash X \)). For any \( x \in M \) we denote
\[
\varepsilon_x = \min(1, \text{InjRad}_M(x)).
\]
In this section we will prove the following result.

Proposition 6.1. Let \( X \) be a symmetric space of non-compact type without Euclidean factors. There exists \( A, B \) depending only on \( X \) such that any compact \( X \)-manifold is homotopy equivalent to a simplicial complex with at most
\[
A \int_M \varepsilon_x^{-\dim(X)} dx
\]
vertices and where every vertex belongs to at most \( B \) simplices.

The Gelander’s conjecture follows immediately from this and Theorem C.

Proof of Theorem A. Indeed, let \( M = \Gamma \backslash X \) with \( \Gamma \) arithmetic. Then by the proposition we have a homotopy equivalence between \( M \) and a simplicial complex with at most \( N \) vertices where
\[
N \ll X \int_M \varepsilon_x^{-\dim(X)} dx
\]
\[
\leq \text{InjRad}(M)^{-\dim(X)} \text{vol}(M_{\leq 1}) + \text{vol}(M)
\]
\[
\ll X \text{vol}(M) \left( e^{-c[k : \mathbb{Q}]} \log[k : \mathbb{Q}]^{-3\dim(X)} + 1 \right)
\]
where the last line is deduced using Dobrowolski’s bound (Proposition 2.6) and Theorem C. The term \( e^{-c[k : \mathbb{Q}]} \log[k : \mathbb{Q}]^{-3\dim(X)} \) is bounded independently of \( [k : \mathbb{Q}] \) so this proves that \( |S| = O(\text{vol}(M)) \) and finishes the proof of Theorem A. □
Before proving Proposition [6.1] we need a few preliminaries. We define, for \( r \in [1, +\infty[ \):

\[
C_X(r) = \sup_{1/2 > t > 0} \frac{\text{vol}_M(B_X(x_0, r t))}{\text{vol}_M(B_X(x_0, t))}
\]

where \( x_0 \) is an arbitrary point in \( X \). We say that a subset in a metric space is \( \eta \)-separated if all distances between distinct points are > \( \eta \). The constant \( C_X(r) \) will appear through the following packing lemma which we will use repeatedly in our arguments below.

**Lemma 6.2.** Let \( M \) be an \( X \)-manifold, \( 0 < \eta < \delta < 1/2 \) and \( P \) an \( \eta \)-separated subset of \( B_M(x, \delta) \). Then, \( |P| < C_X(\delta/\eta) \).

The proof is omitted. The main ingredient for the proof of Proposition [6.1] is the following lemma.

**Lemma 6.3.** Let \( M \) be an \( X \)-manifold. There exists a subset \( S \subset M \) such that:

1. For every \( x \in M \) there is a \( p \in S \) such that \( x \in B_M(p, \varepsilon_p/6) \);
2. If \( p, q \in S \) and \( p \neq q \) then \( d(p, q) > \max(\varepsilon_p, \varepsilon_q)/36 \).

Moreover such a set \( |S| \) satisfies

\[
|S| \lesssim_X \int_M \varepsilon_x^{-\dim(X)} dx.
\]

**Proof.** Such a set is given by any maximal subset which satisfies condition (2) so we have only to prove (6.1). For \( p \in S \) we define

\[
F_p(x) = \begin{cases} \text{vol}_M(B_M(p, \varepsilon_p/2))^{-1} & \text{if } x \in B_M(p, \varepsilon_p/4) \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
H_p(x) = \begin{cases} \text{vol}_M(B_M(x, \varepsilon_x/2))^{-1} & \text{if } p \in B_M(x, \varepsilon_x/2) \\ 0 & \text{otherwise} \end{cases}
\]

Now if \( x \in B_M(p, \varepsilon_p/4) \) then \( \varepsilon_x > \varepsilon_p/2 \) (since the ball of radius \( \varepsilon_p/2 \) around \( x \) is contained in the ball of radius \( \varepsilon_p \) around \( p \) and the latter is embedded). This implies that

\[
F_p(x) \leq H_p(x) \cdot \frac{\text{vol}_M(B_M(p, \varepsilon_p))}{\text{vol}_M(B_M(p, \varepsilon_p/2))} \leq C_X(2)H_p(x)
\]

for all \( p, x \).

On the other hand, if \( p \in B_M(x, \varepsilon_x/2) \), then \( \varepsilon_p \geq \varepsilon_x/2 \), so that the subset \( \{ q \in S : H_q(x) \neq 0 \} \) is \( \varepsilon_x/72 \)-separated by (2). By Lemma 6.2 implies the inequality

\[
|\{ q \in S : H_q(x) \neq 0 \}| \leq C_X(36).
\]

So, we get that

\[
\sum_{p \in S} F_p(x) \leq C_X \sum_{p \in S} H_p(x) \leq C_X(2)C_X(36) \text{vol}_M(B_M(x, \varepsilon_x/2))^{-1}
\]

for all \( x \in M \). Now we have \( \int_M F_p(x) dx = 1 \), so we get

\[
|S| = \int_M \sum_{p \in S} F_p(x) dx \leq C_X(2)C_X(36) \int_M \text{vol}_M(B_M(x, \varepsilon_x/2))^{-1} dx.
\]

On the other hand \( \text{vol}_M(B_M(x, \varepsilon_x/2)) \asymp_X \varepsilon_x^{-\dim(X)} \) since \( \varepsilon_x \leq 1 \), and (6.1) follows. \( \square \)
Proof of Proposition 6.4. We construct a simplicial complex by the same arguments as in \cite{35, 33, 10.1}. A good cover of a topological space $Z$ is a collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of open subsets of $Z$ such that $Z = \bigcup_{i=1}^{N} U_i$ and all the nonempty intersections between the $U_i$ are contractible. If $\mathcal{U}$ is any collection of subsets of $Z$, its nerve $N(\mathcal{U})$ is the simplicial complex with vertex set $\mathcal{U}$ and $U_0, \ldots, U_m$ is an $m$-simplex whenever $U_0 \cap \cdots \cap U_m$ is nonempty. The following proposition is \cite{21, Theorem 13.4}.

**Proposition 6.4.** If $\mathcal{U}$ is a good cover of $Z$, then $N(\mathcal{U})$ is homotopy equivalent to $Z$.

Let $M$ be a $X$-manifold. Pick $S \subset M$ as in Lemma 6.3. It follows from (1) that the balls $B_M(p, \varepsilon_p/6), p \in S$ are a cover of $M$, and each is contractible since $\varepsilon_p/6 < \text{InjRad}_M(p)$. Moreover, if $p, q \in S$ then $B_M(p, \varepsilon_p/6) \cap B_M(q, \varepsilon_q/6)$ is connected: assume otherwise and that $\varepsilon_q \leq \varepsilon_p$. Choose a lift $\tilde{q}$ of $q$ to $X$. Then $B_X(\tilde{q}, \varepsilon_q/6)$ intersects two distinct lifts of $B_M(p, \varepsilon_p/6)$, with centers $\tilde{p}_1, \tilde{p}_2$. We have

$$d_X(\tilde{p}_1, \tilde{p}_2) \leq \frac{\varepsilon_p}{6} + 2 \frac{\varepsilon_q}{6} + \frac{\varepsilon_p}{6} \leq \frac{2\varepsilon_p}{3},$$

which contradicts the fact that $\varepsilon_p \leq \text{InjRad}_M(p)$.

It follows that all the intersections between the balls $B(p, \varepsilon_p/6), p \in S$ are isometric to intersections of balls of $X$, which are contractible. We conclude that $B_M(p, \varepsilon_p/6), p \in S$ is a good cover of $M$, so by Proposition 6.4 $M$ is homotopy equivalent to a simplicial complex with at most $|S|$ vertices and by (6.1) this implies the first part of the proposition.

It remains to bound the number of simplices incident to any vertex. To do this it suffices to bound the degree of a vertex $p$ in the 1-skeleton of the nerve of the cover $B_M(p, \varepsilon_p/6), p \in S$. This follows from the same argument as above: if $q \in B_M(p, \varepsilon_p/6)$ then $\varepsilon_q > 5\varepsilon_p/6$, so all points in $S \cap B_M(p, \varepsilon_p/6)$ are $5\varepsilon_p/216$-separated and applying Lemma 6.2 it follows that $B_M(p, \varepsilon_p/6)$ contains at most $C_M(36/5)$ points of $S$. \hfill \Box

### 6.1. Triangulations

Though it is unnecessary for most applications, it is natural to ask whether it is possible to replace “homotopy equivalent” by “homeomorphic” in the statement of Theorem A or Proposition 6.1. If $M$ has constant curvature (i.e. if $M$ is hyperbolic) then a set of points such as the one used in the argument explained above can also be used to construct a “Delaunay triangulation” of $M$ under a mild genericity assumption which we can always arrange for the sets constructed for the proof of the theorem. In variable curvature it is much more complicated to construct Delaunay triangulations but this was achieved in \cite{16}; in this reference the authors describe an algorithm that, starting from a set of points such as that used for the proof of Gelander’s conjecture, outputs a modified set of points from which a Delaunay triangulation can be constructed by patching local Euclidean triangulations together, and then show that it is homeomorphic to the manifold by using center of mass maps from the simplices to the manifold. Unfortunately in this reference the authors use nets with constant separation, which cannot be applied to our situation since the delicate case is exactly when the injectivity radius might go to 0. Their construction should work in general as proven in work in progress \cite{17}.

### 6.2. A few words on lower bounds for the minimal number of simplices

In general it is not known whether the upper bound in Theorem A is sharp. We can mention a few results (which are essentially immediate consequences of well-known facts) going in this direction. If we consider only compact $M$ (but any $X$) then there is a $c > 0$
depending only on $X$ such that any simplicial complex which is homeomorphic to $M$ (rather than just homotopy equivalent) has at least $c \text{vol}(M)$ maximal simplices. This follows almost immediately from the proportionality principle for the simplicial volume and its positivity for locally symmetric spaces which are proven in [43].

For certain $X$ we can prove lower bounds for complexes homotopy equivalent to locally-$X$ manifolds using $L^2$-invariants. Let $\delta$ be the fundamental rank of the isometry group $G = \text{Isom}(X)$. It can be computed as the difference between the absolute rank of $G$ and the rank of a maximally compact torus, see [42, p. 386]). Then if $\delta = 0$ or $\delta = 1$, for any $B > 0$ there exists a constant $c > 0$ such that for any locally-$X$ manifold $M$ any simplicial complex which is homotopy equivalent to $M$ and of degree $B$ has at least $c \text{vol}(M)$ vertices. This holds for all symmetric spaces associated with orthogonal groups $\text{SO}(p,q)$ (which have $\delta = 0$ or 1 depending on whether $pq$ is even or odd) and unitary groups $\text{SU}(n,m)$ (which have $\delta = 0$), in particular for all real and complex finite-volume hyperbolic manifolds.

Let us sketch the argument for the proof. The hypothesis on $G$ implies that either the middle $L^2$-Betti number $\beta_d^{(2)}(G)$ of $X$ or its $L^2$-torsion $t^{(2)}(G)$ is nonzero (see [56, Theorem 1.1]). Then, for any locally-$X$ manifold $M$ of finite volume and any finite simplicial complex $Z$ which is homotopy equivalent to $M$ the corresponding combinatorial invariant $b_d^{(2)}(Z, \pi_1(Z))$ or $\tau^{(2)}(Z, \pi_1(Z))$ is equal to respectively $\text{vol}(M)\beta_d^{(2)}(G)$ or $\text{vol}(M)t^{(2)}(G)$. In the first case it follows that the Euler characteristic $\chi(Z) = (-1)^d/2 \text{vol}(M)\beta_d^{(2)}(G)$, so $Z$ must have at least $2 \text{vol}(M)\beta_d^{(2)}(G)/(d+1)$ $k$-simplices for some $0 \leq k \leq d$ of the same parity as $d/2$, and since we assumed its degree is bounded the same is true of the number of vertices. In the second case, it must hold that the logarithm Fuglede–Kadison determinant of at least one of the combinatorial Laplacians of the $L^2$-chain complex of the universal cover of $Z$ must be at least $c_0 \text{vol}(M)$ (this follows from the formula for the $L^2$-torsion in terms of these determinants [49, Definition 3.29], and the fact that they are always $\geq 1$, [49, Theorem 13.3(2)]). On the other hand the classical proof of the Hadamard inequality can also be applied to this setting since the universal cover of $Z$ is $L^2$-acyclic so its Laplacians have no kernel, and using the hypothesis on the degree this gives the conclusion in this case as well.

In the other direction we mention that in [3] the first author together with M. Abért, N. Bergeron and D. Gaboriau construct CW-complexes which are homotopy equivalent to congruence covers of certain locally symmetric spaces and have relatively few cells of low dimensions compared to the degree.

7. Betti numbers

7.1. Limit multiplicities and Betti numbers. By Matsushima’s formula, there exists a finite number of irreducible representations that control the size of the Betti numbers of $\Gamma \backslash X$. More precisely, for every $i = 1, 2, \ldots, \dim X$, there exists a finite collection of unitary irreducible representations $\{\pi_j^i\}$ such that the $i$-th Betti number

$$b_i(\Gamma \backslash X) = \dim(H^i(\Gamma \backslash X)) = \sum_j n(\pi_j^i, i)m(\pi_j^i, \Gamma)$$

where $m(\pi_j^i, \Gamma)$ is the multiplicity of $\pi_j^i$ in $L^2(G/\Gamma)$. In addition, if $i \neq 1/2 \dim(X)$ then none of the $\pi_j^i$ is a discrete series. Consequently, Theorem [4] follows from the following limit multiplicities result.
Theorem 7.1. Let $G$ be a semi-simple real Lie group and let $\pi$ be an irreducible unitary representation of $G$ which is not discrete series. Then, for any co-compact arithmetic lattice $\Gamma \subset G$ with the trace field $k$ we have

$$m(\pi, \Gamma) \leq \pi \left\{ \begin{array}{ll}
\frac{\text{vol}(\Gamma \setminus G) [k : \mathbb{Q}]^{-1}}{\text{vol}(\Gamma \setminus G) e^{-c_\pi [k : \mathbb{Q}]}} & \text{if } \pi \text{ is tempered}, \\
\text{vol}(\Gamma \setminus G) & \text{if } \pi \text{ in non-tempered},
\end{array} \right.$$ 

where $c_\pi > 0$ depends only on $\pi$.

In the rest of this section we prove this theorem. The proof is classical and follows the lines of [33, 10.2] and [2, 6.10-6.21], using arguments originating with D. Kazhdan and D. L. DeGeorge and N. R. Wallach [24].

7.2. Estimates for matrix coefficients. To apply the usual methods we need some well-known estimates for the growth of the $L^2$-norms of matrix coefficients for non-discrete series representations restricted to a ball in $G$. We distinguish further between tempered and non-tempered representations; in the first case we will prove that.

Proposition 7.2. Let $\pi \in \widehat{G}$ be a tempered representation which is not a discrete series. Let $v \in H_\pi$ be a $K$-finite vector, and $\phi(g) = \langle \pi(g)v, v \rangle$ the associated matrix coefficient. Then as $R \to +\infty$ we have:

$$\int_{B(R)} |\phi(g)|^2 \, dg \gg_{\pi} R^{2d+r}$$

This inequality will follow from the asymptotic equivalent for matrix coefficients due to Harish-Chandra (see [7, 3] below) and integration on $KAK$ decomposition of $G$. Using the same arguments we can also prove the following sharper result for non-tempered representations.

Proposition 7.3. Let $\pi \in \widehat{G}$ be a non-tempered representation. Let $v \in H_\pi$ be a $K$-finite vector, and $\phi(g) = \langle \pi(g)v, v \rangle$ the associated matrix coefficient. Then there exists $a_\pi > 0$ such that as $R \to +\infty$ we have:

$$\int_{B(R)} |\phi(g)|^2 \, dg \gg_{\pi} e^{a_\pi R}.$$
Proposition 7.4. Let \( f \) be a smooth \( C^\infty \) function in \( T \), then for any \( H_0 \in a \), we have
\[
\lim_{R \to \infty} \frac{1}{\text{vol}(B_{H_0}(R))} \int_{B_{H_0}(R)} f(q(H)) \, dH \to \int_T f
\]
Moreover the convergence is uniform over \( H_0 \in a \).

Let \( H_0 \) be a nonzero vector in the interior of \( a^+ \) such that \( P(H_0) \neq 0 \). Let \( \delta_1, \delta_2 > 0 \) be small enough, so that the closure of \( B_{H_0}(\delta_1) \) is contained in the interior of \( a^+ \) and for all \( H \in B_{H_0}(\delta_1) \) we have \( |P(H)| \geq \delta_2 \).

We have
\[
P(tH) = t^d \sum_{j=1}^m e^{i(tH,\alpha_j)} p_j(H).
\]
Let \( x_0 = q(H_0) = (\langle H_0, \alpha_1 \rangle, \ldots, \langle H_0, \alpha_m \rangle) \) be the projection of \( H_0 \) in \( T \subset \mathbb{T}^m \) and choose \( B_{x_0} \) a ball around \( x_0 \in T \) small enough so we can guarantee by (7.4) that for all \( H \in B_{H_0}(\delta_1) \) and all \( t > 0 \) such that \( q(th) \in B_{x_0} \) we have \( |P(th)| \geq \frac{1}{2} t^d \delta \).

Therefore for all \( H \in B_{H_0}(\delta_1) \) and all \( t > 0 \)
\[
\int_{B(R)} |\phi(g)|^2 \, dg \gg \int_K \int_K \int_{a^+} e^{\rho(H)<R} |\phi(k_1 \exp(H)k_2)|^2 \, dHdk_1dk_2.
\]
We choose \( \delta_2 \) small enough such that
\[
B_{R,H_0} := (\delta_2 R)B_{H_0}(\delta_2) \subseteq \{ H \in a^+ | \rho(H) < R \}
\]
and therefore
\[
\int_{a^+} e^{\rho(H)}|\phi(k_1 \exp(H)k_2)|^2 \, dH \gg \int_{B_{R,H_0}} e^{2\rho(H)}|\phi(k_1 \exp(H)k_2)|^2 \, dH
\]
Observe that \( B_{R,H_0} \) is a ball of radius \( O(R) \) which is contained in the cone \( \{ th \mid t > 0, H \in B_{H_0}(\delta_2) \} \) and therefore the estimate (7.5) together with proposition (7.4) (by taking \( f \) to be a smooth approximation of the characteristic function of \( B_{x_0} \)) give us:
\[
\frac{1}{\text{vol}(B_{R,H_0})} \int_{B_{R,H_0} \cap a^{-1}(B_{x_0})} e^{2\rho(H)}|\phi(k_1 \exp(H)k_2)|^2 \, dH \gg R^{2d} \text{vol}(B_{x_0})
\]
which easily implies (7.1).
7.2.2. Proof of Proposition 7.3. The argument is the same, except that as \( \pi \) is not tempered one of the \( \lambda_i \) appearing in (7.3) must satisfy \( \text{Re}(\lambda_i) > \rho \), by [41, Theorem 8.53]. By using the same argument as for (7.5) we get that

\[
|\phi(k_1 \exp(tH)k_2)| \gg e^{-(\rho + \varepsilon)(tH)t^d} \tag{7.6}
\]

where \( \varepsilon \in a^*_R \) is positive on \( a^+ \). Using the same integration scheme, we get that

\[
\frac{1}{\text{vol}(B_{R,H_0})} \int_{B_{R,H_0} \cap (B_{x_0})} e^{2\rho(H)}|\phi(k_1 \exp(H)k_2)|^2 \, dH \gg e^{-aR} \text{vol}(B_{x_0})
\]

for some \( a > 0 \) (depending on \( \pi \) via \( \varepsilon \)) so that (7.2) follows.

7.3. Proof of limit multiplicities. The proof of Theorem 7.1 follows well-known lines as well. By an argument of G. Savin [67] (see also [2, Lemma 6.15]), there exists a subspace \( W \subset L^2(\Gamma \setminus G) \) such that if

\[
\beta(x) = \sup_{f \in W, \|f\|=1} |f(x)|^2
\]

then

\[
m(\pi, \Gamma) = \int_{\Gamma \setminus G} \beta(x) \, dx. \tag{7.7}
\]

On the other hand there is the following estimate for \( \beta(x) \) (see [2, (6.21.1)]: there exists a vector \( v \in H_\pi \) (from which \( W \) is defined) such that for any \( r > 0 \) we have for any \( x \in \Gamma \setminus G \) that

\[
\beta(x) \leq \frac{1}{\|\phi_r\|^2} N_\Gamma(x, 2r) \tag{7.8}
\]

where

\[
\phi_r(g) = 1_{B(r)}(g)\langle \pi(g)v, v \rangle_{H_\pi}
\]

and

\[
N_\Gamma(x, \ell) = |\{ \gamma \in \Gamma : d(x, \gamma x) < \ell \}|.
\]

By Proposition 4.1 there exists \( m \in \mathbb{N} \) and \( \varepsilon > 0 \) such that

\[
N_\Gamma(x, 2r) \leq [k : \mathbb{Q}]^m
\]

for any \( r \leq \frac{\varepsilon}{2} [k : \mathbb{Q}] \). By the argument in the proof of [2, Lemma 6.20], it follows that for such \( r \) we have

\[
\int_{\Gamma \setminus G} N_\Gamma(x, 2r) \, dx \leq [k : \mathbb{Q}]^m \text{vol}((\Gamma \setminus G)_{\leq 2r}) + \text{vol}(\Gamma \setminus G)
\]

and by Theorem 4.1 this is \( (1 + o(1)) \text{vol}(\Gamma \setminus G) \) as \( [k : \mathbb{Q}] \to +\infty \). Using (7.7) and (7.8) with \( r = \frac{\varepsilon}{2} [k : \mathbb{Q}] \) we get

\[
m(\pi, \Gamma) \ll \frac{1}{\|\phi_r\|^2} \text{vol}(\Gamma \setminus G)
\]

and now Theorem 7.1 follows from Proposition 7.2 (when \( \pi \) is tempered) and Proposition 7.3 (when it is not).
8. Benjamini–Schramm convergence

In this section $G$ is an adjoint semisimple Lie group on which we fix a Haar measure (our statements will be independent of this choice). In the questions related the Benjamini-Schramm convergence we care only about the image of $G$ in the group of isometries of $X$, so passing to the adjoint group does not lose any information.

8.1. Benjamini–Schramm convergence.

8.1.1. Benjamini-Schramm convergence to the universal cover. The Benjamini-Schramm topology is a topology defined on a subset of locally symmetric spaces, including all finite volume locally symmetric spaces and the space $X$ itself. In this paper we shall care only about the convergence of a sequence of finite volume spaces $(\Gamma_n \backslash X)_{n \in \mathbb{N}}$ to $X$. We say the $(\Gamma_n \backslash X)_{n \in \mathbb{N}}$ Benjamini-Schramm converges to $X$ if for every $R > 0$ we have

$$\lim_{n \to \infty} \frac{\text{vol}(\langle \Gamma_n \backslash X \rangle \leq R)}{\text{vol}(\Gamma_n \backslash X)} = 0.$$ 

In more intuitive terms this means that the injectivity radius around a typical point of $\Gamma_n \backslash X$ gets very large as $n$ goes to infinity.

8.1.2. The space of subgroups and invariant random subgroups. Let $\text{Sub}(G)$ be the space of closed subgroups of $G$ equipped with the topology of Hausdorff convergence on compact sets. It is a compact $G$-space, often called the Chabauty space. The group $G$ acts on $\text{Sub}(G)$ by conjugation. By a slight abuse of notation, we can also speak about the injectivity radius of a subgroup of $G$:

$$\text{inj}(\Gamma) := \inf \{ d(\gamma K, K) \mid \gamma \in \Gamma \setminus Z(\Gamma) \}.$$ 

Note that the injectivity radius of a non-discrete group is always 0 but a discrete group might also have injectivity radius 0, if it intersects $K$ non-trivially. An invariant random subgroup of $G$ is a $G$-invariant probability measure on $\text{Sub}(G)$. To every lattice $\Gamma \subset G$ we associate an invariant random subgroup $\mu_\Gamma$ supported on the conjugacy class of $\Gamma$ as follows :

$$\mu_\Gamma := \frac{1}{\text{vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \delta_\gamma d\gamma.$$ 

We can use the weak-* topology on the space of probability measures on $\text{Sub}(G)$ to give an equivalent characterization of Benjamini-Schramm convergence as follows (see [2, Proposition 3.6]).

**Lemma 8.1.** Assume that $G$ is an adjoint group. A sequence of locally symmetric spaces $\Gamma_n \backslash X$ Benjamini-Schramm converges to $X$ if and only if the sequence of measures $\mu_{\Gamma_n}$ converges to $\delta_{\{1\}}$ in weak-* topology.

Using this lemma, the question of whether a sequence of locally symmetric spaces Benjamini-Schramm converges to $X$ can be often answered by studying the possible limits of $\mu_{\Gamma_n}$. One result which often makes that possible is the Zariski density for invariant random subgroups; see the remark following [2, Theorem 2.9] and [36, Theorem 1.9].

**Theorem 8.2.** Let $G$ be an adjoint semisimple Lie group without compact factors. Let $\mu$ be an ergodic invariant random subgroup of $G$. Then, there exists a normal subgroup $G_1$ of $G$ such that $\mu(\text{Sub}(G_1)) = 1$ and $\mu$-almost all subgroups are Zariski dense in $G_1$. 

In particular, if $G$ is simple and $\mu$ is an ergodic IRS, then either $\mu$ is trivial or it is Zariski dense almost surely.

8.2. Main criterion for convergence. Let $f$ be a smooth compactly supported function on $G$ (that is, $f \in C_c^\infty(G)$). Let $W$ be an open subset in $G$ which contains only semisimple elements and is invariant under conjugation. We define, for a lattice $\Gamma$

$$
\text{tr } R_W^\Gamma f := \sum_{[\gamma] \in W} \text{vol}(\Gamma \gamma \backslash G_\gamma) O(\gamma, f),
$$

where the sum is over all conjugacy classes $[\gamma] \in W$. Let us address briefly issues of convergence in (8.1). First we need that for all $\gamma \in W$ the covolume $\text{vol}(\Gamma \gamma \backslash G_\gamma)$ is finite, equivalently that $\Gamma \gamma \backslash G_\gamma$ is compact. If $\Gamma$ is uniform, this is always the case. On the other hand, if it is non-uniform, then we must ensure that the covolume is finite of all classes in $W$. The orbital integrals $O(\gamma, f)$ are always finite (see Proposition 2.7). Finally, since the support of $f$ intersects only finitely many conjugacy classes of $\Gamma$ there is no issue with the convergence of the sum.

We say that the subset $W$ is sufficiently dense in $G$, if for every discrete Zariski-dense subgroup $\Lambda$ in $G$, we have $\Lambda \cap W \neq \{1\}$.

Theorem 8.3. Let $G$ be a semisimple adjoint Lie group. Let $W$ be a sufficiently dense subset of $G$. Let $(\Gamma_n)$ be a sequence of lattices in $G$. Assume either that they are all uniform or that $(\Gamma_n) \gamma \backslash G_\gamma$ is compact for all $n$ and $\gamma \in \Gamma_n \cap W$.

If $G$ is simple and the following condition holds:

$$
\forall f \in C_c^\infty(G) \lim_{n \to +\infty} \left( \frac{1}{\text{vol}(\Gamma_n \backslash X)} \text{tr } R_W^\Gamma f \right) = f(1_G),
$$

then the sequence of locally symmetric spaces $\Gamma_n \backslash X$ is Benjamini–Schramm convergent to $X$.

If $G$ is not necessarily simple then the same holds assuming that for every non-trivial proper normal subgroup $H \leq G$ there exists a subset $F$ of $\text{Sub}_H$ such that $\Lambda \notin F$ for all Zariski-dense subgroups $\Lambda \subset H$ and a neighbourhood $\mathcal{U}_H$ of $\{\Lambda \in \text{Sub}_H : \Lambda \subset F\}$ in $\text{Sub}_G$ such that $g \Gamma_n g^{-1} \notin \mathcal{U}_H$ for all $n$ and $g \in G$.

8.2.1. Proof when $G$ is simple. Let $\mu_n$ be the invariant random subgroup of $G$ supported on the conjugacy class of $\Gamma_n$. We want to prove that any weak limit $\mu$ of a subsequence of $(\mu_n)$ is equal to the trivial IRS $\delta_{\{1\}}$. Since $G$ is simple, by Theorem 8.2 it suffices to prove that $\mu$ is supported on non-Zariski-dense subgroups to deduce that it must be trivial.

To this end we choose a covering $W = \bigcup_{C \in \mathcal{C}} C$ of $W$ where $\mathcal{C}$ is countable and every $C \in \mathcal{C}$ is compact. We can do this since $\text{Sub}_G$ is metrizable [23, Proposition 2]. Let $Q_C = \{\Lambda : \Lambda \cap C \neq \emptyset\}$. This is a Chabauty-closed subset of $\text{Sub}_G$. If $\nu$ is a nontrivial IRS, then it is Zariski dense with positive probability so there exists $C \in \mathcal{C}$ such that $\nu(Q_C) > 0$. We want to prove the opposite for $\mu$, which amounts to the following: for every $C$, there exists a non-negative Borel function $F$ on $\text{Sub}_G$ which is positive on $Q_C$ and such that $\int_{\text{Sub}_G} F(\Lambda) d\mu(\Lambda) = 0$.

Let us fix $C \in \mathcal{C}$ and prove this. There exists an open relatively compact subset $V$ with $C \subset V$ and $\overline{V} \subset W$. Choose any $f \in C^\infty(G)$ such that $f > 0$ on $C$ and $f = 0$ on
\[ F(\Lambda) = \begin{cases} \sum_{\lambda \in \Lambda} f(\lambda) & \text{if } \Lambda \text{ is discrete,} \\ 1 & \text{if } \Lambda \text{ is not discrete and intersects } C, \\ 0 & \text{otherwise.} \end{cases} \]

\( F \) is lower semi-continuous on \( \text{Sub}_G \), non-negative and positive on \( Q_C \). On the other hand, we have:

\[
\int_{\text{Sub}_G} F(\Lambda) \, d\mu_n(\Lambda) = \frac{1}{\text{vol}(\Gamma_n \backslash G)} \int_{G/\Gamma_n} \sum_{\gamma \in g\Gamma_n g^{-1}} f(\gamma) \, dg
\]

\[
= \frac{1}{\text{vol}(\Gamma_n \backslash G)} \sum_{[\gamma] \in C} \text{vol}(\Gamma_n \backslash \gamma(G)\gamma) \mathcal{O}_\gamma(f).
\]

By the so-called “Portemanteau theorem” [40, Theorem 13.16] the limit inferior of the left-hand side is larger or equal to \( \int_{\text{Sub}_G} F(\Lambda) \, d\mu(\Lambda) \). Finally, the hypothesis (8.2) implies that the right-hand side converges to 0. It follows that

\[
\int_{\text{Sub}_G} F(\Lambda) \, d\mu(\Lambda) = 0
\]

which finishes the proof.

8.2.2. *Proof in general.* Let \( \mu_n \) be the invariant random subgroup supported on the \( G \)-orbit of \( \Gamma_n \) and assume that \( \mu \) is not supported on the trivial subgroup. Then by Theorem 8.2 there is a morphism \( \pi : G \to H \) with kernel \( \ker(\pi) \neq \{1\} \) and

\[ \mu(\{\Lambda \in \text{Sub}_{\ker(\pi)} : \Lambda \text{ is Zariski-dense in } \ker(\pi)\}) > 0. \]

If \( \pi \) is trivial (that is \( \ker(\pi) = G \)) then the same arguments as in the proof in the simple case apply to give a contradiction. On the other hand, if \( \ker(\pi) \) is a proper subgroup of \( G \) then there exists a Zariski-dense subgroup \( \Lambda \subset \ker(\pi) \) such that \( \mu(C) > 0 \) for every Chabauty neighbourhood \( C \) of \( \Lambda \). Since \( \Lambda \not\in F \), we may choose \( C \) such that \( C \subset U_H \), and the hypothesis that \( g\Gamma_n g^{-1} \not\subset U_H \) for all \( n \) and \( g \in G \) then gives that \( \mu_n(C) = 0 \) for all \( H \). A standard argument then shows that this contradicts the weak convergence of \( \mu_n \) to \( \mu \).

8.3. *Short proof of a non-effective result for unbounded degree.* This short subsection is dedicated to the proof of the following result, which is the “second half” of Theorem D. It is strictly weaker than Theorem C but since the argument is very short we feel it deserves to be included.

**Theorem 8.4.** Let \( G \) be a noncompact semisimple Lie group and let \( X \) be its symmetric space. Let \( \Gamma_n \) in \( G \) be a sequence of arithmetic lattices in \( G \) and assume that the degree of the trace field of \( \Gamma_n \) over \( \mathbb{Q} \) goes to infinity with \( n \). Then, (1.1) holds for the sequence of locally symmetric spaces \( \Gamma_n \backslash X \).

**Proof.** We want to prove that (1.1) holds for the locally symmetric spaces \( \Gamma_n \backslash X \). According to 8.1 this is equivalent to proving that the invariant random subgroups \( \mu_{\Gamma_n} \) converge in weak topology to the Dirac mass \( \delta_{\{1\}} \) supported on the trivial subgroup. Since the space of invariant random subgroups on \( G \) is compact it suffices to prove that any accumulation point of a sequence \( (\mu_{\Gamma_n}) \) is equal to \( \delta_{\{1\}} \). The idea of the proof is quite simple. We show that Theorem 3.1 and Remark 3.2 force any limit to be supported
on the set of virtually abelian subgroups of $G$. Then we use Zariski density for IRS’ses to deduce that such a limit must be concentrated on the trivial subgroup.

Suppose to the contrary that $\mu_{\Gamma_n}$ accumulates at a non-trivial IRS $\mu$. This means that there exists a closed subgroup $\Lambda \subset G$, $\Lambda \neq \{1\}$ in the support of $\mu$, so that $\mu(V) > 0$ for any Chabauty-neighbourhood $V$ of $\Lambda$. It follows that $V$ intersects the conjugacy class of $\Gamma_n$ for infinitely many $n$. In particular, $\Lambda$ is a limit of a sequence $g_n \Gamma_b g_n^{-1}$ in the Chabauty topology. We will deduce from this and the Arithmetic Margulis Lemma (Theorem 3.1 and the following remark), that $\Lambda$ must be virtually abelian. This contradicts the Borel density for IRS (Theorem 8.2) and suffices to establish the theorem.

For this we need the following lemma, which will also be useful later.

**Lemma 8.5.** For any $m$ there exists an $A$ with the following property. Let $\Delta$ be a finitely generated subgroup of $\text{GL}_m(\mathbb{C})$ which contains an abelian subgroup of finite index composed only of semisimple elements. Then, $\Delta$ has an abelian subgroup of index at most $A$.

**Proof.** Let $H_1$ be a semi-simple normal abelian torsion-free finite-index subgroup of $\Delta$. The group $H_1$ can be conjugated into the group $D$ of diagonal matrices, so we may assume that $H_1 = \Delta \cap D$. In particular, $\Delta$ normalises the connected component of the Zariski closure $T$ of $H_1$, which is a torus. It follows that $\Delta / H_1$ embeds into a product of groups of the form $S_{i_j} \wr \text{PGL}_{i_j}(\mathbb{C})$ with $\sum_i i_j = m$. Indeed, the quotient of the normaliser of $T$ by $T$ is a direct product of groups of this form. In particular, $\Delta / H_1$ is linear group of degree at most $m^2$ (since $\text{PGL}_{i_j}$ has a linear representation of degree at most $i_j^2$). By Jordan’s theorem, it follows that there exists a subgroup $H_1 \subset H_2 \subset \Delta$ such that $H_2 / H_1$ is abelian and $|\Delta / H_2|$ is bounded by a constant $A_1$ depending only on $m$. Let $H_3$ be the subgroup of $\Delta$ acting trivially by conjugation on $H_1$. The quotient $\Delta / H_3$ acts freely on $H_1$, so it embeds into the automorphism group of the character module $X^*(T)$. The rank of $T$ is bounded by $m$, so $\Delta / H_3$ is isomorphic to a finite subgroup of $\text{GL}_m(\mathbb{Z})$. Therefore, $|\Delta / H_3|$ is bounded by a constant $A_2$ depending only on $m$. Now the group $H_4 := H_2 \cap H_3$ contains $H_1$ and has index at most $A_1 A_2$ in $\Delta$. By construction $H_1 \rightarrow H_4 \rightarrow H_4 / H_1$ is a central extension of an abelian torsion free group by a torsion abelian group. We will prove that it is abelian, which finishes the proof of our lemma. Our argument relies on the following (probably well-known) fact: any central extension of a torsion-free abelian group by a torsion abelian group is abelian.

We give a short proof of this last claim: let $A \rightarrow B \rightarrow C$ be such an extension and choose any $c_1, c_2 \in C$ and lifts $b_1, b_2$ to $B$. We need only prove that $b_1$ and $b_2$ commute with each other. To do so, we note that the commutator $[b_1, b_2] = b_1 b_2 b_1^{-1} b_2^{-1}$ belongs to $A$ and depends only on $c_1, c_2$ and not on the choice of lifts. Moreover if $c_3 \in C$ and $b_3$ is a lift to $B$ we have:

$$[b_1, b_2 b_3] = b_1 b_2 b_3 b_1^{-1} b_2^{-1} b_3^{-1}$$

$$= b_1 b_2 b_3 b_1^{-1} [b_1, b_3] b_2^{-1}$$

$$= [b_1, b_3] [b_1, b_2]$$

where the last line follows since $[b_1, b_3] \in A$ which is central in $B$. So the map $(b_1, b_2) \mapsto [b_1, b_2]$ is a $\mathbb{Z}$-bilinear map from $C$ to $A$, which has to be trivial since $C$ is torsion and $A$ is torsion-free. This means that $B$ is abelian. □

We go back to proving that $\Lambda$ must be virtually abelian. To simplify notation assume that the sequence $\Gamma_n$ actually converges to $\Lambda$ and that all $\Gamma_n$ are uniform. Almost all
of them are uniform anyways, since any lattice with the trace field of degree $> \dim(G)$ must be uniform. Let $\varepsilon_G$ be the constant given by Theorem 3.1 and recall that $k_{\Gamma_n}$ is the trace field of $\Gamma_n$. Fix $R > 0$. Since $[k_{\Gamma_n} : \mathbb{Q}] \to +\infty$, we can assume that $[k_{\Gamma_n} : \mathbb{Q}] \cdot \varepsilon_G > R + 10^{-2}$ for all $n$. Let

$$\Lambda_{n,R} = \langle \Gamma_n \cap B(R + 10^{-2}) \rangle, \Lambda_R = \langle \Lambda \cap B(R) \rangle.$$ 

It follows from Theorem 3.1, remark 3.2 and Lemma 8.5 that there exists $A > 0$ and a sequence $H_{n,R} \subset \Lambda_{n,R}$ of abelian subgroups such that $[\Lambda_{n,R} : H_{n,R}] \leq A$ for all $n$. The sequence $H_{n,R}$ converges to an abelian subgroup $H_R \subset \Lambda$; moreover (since any element of $\Lambda \cap B(R)$ is a limit of a sequence $\gamma_n \in \Gamma_n \cap B(R + 10^{-2})$) we see that $\Lambda_R \cap H_R$ has index at most $A$ in $\Lambda_R$.

In the previous paragraph we proved that for every $R > 0$, the set $\mathcal{A}_R$ of abelian subgroups contained in $\Lambda_R$ with index at most $A$ is nonempty. Each set $\mathcal{A}_R$ is finite since $\Lambda_R$ is finitely generated and whenever $R < R'$ and $H \in \mathcal{A}_R$ we have $H \cap \Lambda_R \in \mathcal{A}_{R'}$, so we can pick a $H_1 \in \mathcal{A}_1$ such that for infinitely many $N \in \mathbb{N}$ there exists a $H_N \in \mathcal{A}_N$ with $H_1 \subset H_N$. We can then pick an integer $N_2 > 1$ and $H_{N_2} \in \mathcal{A}_{N_2}$ such that $H_1 \subset H_{N_2}$ and for infinitely many $N \in \mathbb{N}$ there exists a $H_N \in \mathcal{A}_N$ with $H_{N_2} \subset H_N$. Iterating this, we construct an increasing sequence $N_i \in \mathbb{N}$ and $H_{N_i} \in \mathcal{A}_{N_i}$ which satisfy $H_{N_i} \subset H_{N_{i+1}}$. It follows that $H = \bigcup_{i \in \mathbb{N}} H_{N_i}$ is an abelian subgroup of $\Lambda = \bigcup_{i \in \mathbb{N}} \Lambda_{N_i}$ of index at most $A$. □

9. Galois cohomology

In the remainder of the paper we will make intensive use of Galois cohomology of number fields and local fields. We regroup here various facts (some standard and some new) that we will use repeatedly in the later sections.

9.1. Notation. We let $\text{Gal}(k)$ be the absolute Galois group of $\overline{k}$ over $k$. If $L/k$ is a finite normal extension then $\text{Gal}(L/k)$ denotes the corresponding finite Galois group.

Non-abelian cohomology. If $H$ is a $k$-group then $H(\overline{k})$ (respectively $H(L)$) is a $\text{Gal}(k)$-group (respectively $\text{Gal}(L/k)$-group) and we denote by $H^i(k, H)$ (respectively $H^i(L/k, H)$) the resulting non-abelian cohomology sets (for $i = 0, 1$) which are defined in [68, pp. 45–46]. In the case where $H$ is abelian, the higher cohomology groups are defined as well. If $H$ is non-abelian then $H^0(k, H) = H(k) = H^0(L/k, H)$ and $H^1(k, H)$ is the quotient set of

$$Z^1(k, H) = \{c : \text{Gal}(k) \to H(\overline{k}), c(\sigma \tau) = c(\sigma) \cdot c(\tau)^c\}$$

by the equivalence relation $b \sim c$ if $c(\sigma) = v^{-1}b(\sigma)v^c$ for some $v \in H(\overline{k})$. The definition for $H^1(L/k, H)$ is the same.

These sets are not groups unless $H$ is abelian, however they have a distinguished point which is the class of the trivial morphism from $\text{Gal}(k)$ to $H(\overline{k})$.

9.1.1. Exact sequences. If $H$ is a $k$-subgroup of a $k$-group $G$, then $V = G/H$ is a $k$-variety. We can define $H^0(k, V) := V(k)$ and there is a long exact sequence

$$1 \to H(k) \to G(k) \to V(k) \to H^1(k, H) \to H^1(k, G)$$

(9.1)

(here exact means that the preimage of the distinguished point is the image of the preceding map); see [68, Proposition 36].
If $H$ is normal in $G$, then $G/H$ is a $k$-group and the sequence extends to a further term $H^1(k, G/H)$; if $H$ is central in $G$, then it extends to $H^2(k, H)$ (see loc. cit., Propositions 38, 43).

**Shafarevich–Tate groups.** The Shafarevich–Tate group $\mathrm{III}^1(H)$ is the kernel of the map

$$H^1(k, H) \to \prod_{v \in V} H^1(k_v, H).$$

It is always a finite group (see [58, p. 284]).

### 9.2. Galois cohomology of tori over local fields.

#### 9.2.1. Nakayama–Tate. If $T$ is a torus, then its character group $X^*(T)$ is a Gal$(k)$-module. If $L$ is a splitting field of $T$, then the action of Gal$(k)$ factors through Gal$(L/k)$, so $X^*(T)$ is also a Gal$(L/k)$-module. The resulting cohomology groups are the same for both Galois groups, which follows from the inflation-restriction exact sequence. We have the following corollary of the local Nakayama–Tate theorem, see [58, Theorem 6.2].

**Theorem 9.1.** Let $T$ be an algebraic torus defined over a local field $k_v$ and split over a Galois extension $L_v/k_v$. Then there is an isomorphism $H^1(L_v/k_v, T) \simeq H^1(L_v/k_v, X^*(T))$.

#### 9.2.2. Bounds on Shafarevich–Tate groups.

**Lemma 9.2.** Let $T$ be an algebraic torus of dimension $r$ defined over $k$ and split over a Galois extension $L/k$. Then

$$|\mathrm{III}^1(T)| \leq |H^2(L/k, X^*(T))| \leq [L : k]^{r[L:k]^2}.$$

If $T$ is a maximal torus of a semisimple algebraic group $G$ then we can find $L$ with $[L : k] \leq (\dim G - \rk G)!$. In particular we have then that $|\mathrm{III}^1(T)| = O_{\dim G}(1)$.

**Proof.** By [58, Proof of Prop. 6.9 p. 306] there is a surjective map $H^2(L/k, X^*(T)) \twoheadrightarrow \mathrm{III}^1(T)$, which explains the first inequality. The second inequality is a standard bound on the size of cohomology groups of finite groups.

Now we prove the second statement. If $T$ is a maximal torus in a semisimple algebraic group $G$ we have an action on the root system $\pi : \Gal(k) \to \Aut(\Phi(G, T))$. Let us write $\rho : \Gal(k) \to \GL(X^*(T))$ for the action of the Galois group on the character module of $T$. The roots in $\Phi(G, T)$ span a subgroup of $X^*(T)$ of full rank so $\ker \pi = \ker \rho$. In particular $[\Gal(k) : \ker \rho] \leq |\Aut(\Phi)| \leq (\dim G - \rk G)!$ so the field $k^{\ker \rho}$, which is the minimal splitting field of $T$, is an extension of $k$ of degree at most $(\dim G - \rk G)!$. □

#### 9.2.3. Bounds on $H^1$.

**Lemma 9.3.** Let $k_v$ be a local field and let $T$ be a $k_v$-torus. Then $|H^1(k_v, T)|$ is bounded by a constant depending only on $\dim(T)$.

**Proof.** Let $L_v$ be the minimal splitting field of $T$. The Galois group $\Gal(L_v/k_v)$ acts on the character module $X^*(T)$ by automorphisms. Write $J$ for the image of $\Gal(L_v/k_v)$ in $\GL(X^*(T))$. Since $\GL_d(\Z)$ has only finitely many finite subgroups, there is a bound on $|J|$ depending only on $\dim(X^*(T) \otimes \Q) = \dim(T)$. By Theorem 9.1 we have $H^1(L_v/k_v, T) \simeq H^1(L_v/k_v, X^*(T))$. On the other hand we have

$$|H^1(J, X^*(T))| \leq |J|^{\rk G} \leq A^{\rk G},$$

which finishes the proof. □
9.3. Compact cohomology classes in tori. If $T$ is defined over a local field $k_v$ and $L_v/k_v$ is a finite extension, we write $T(L_v)^b$ for the unique maximal compact subgroup of $T(L_v)$. We introduce the notion of compact cohomology classes which will be crucial in the proof of Proposition

Definition 9.4. Let $T$ be an algebraic torus defined over $k_v$, let $L_v$ be the minimal Galois extension of $k_v$ splitting $T$. We say that a class $\alpha \in H^1(k_v, T)$ is compact if it has a representative cocycle $c \in Z^1(L_v/k_v, T(L_v)^b)$.

Proposition 9.5. Let $T$ be an algebraic torus defined over $k_v$ split by an unramified extension of $k_v$. Then every compact cohomology class in $H^1(k_v, T)$ is trivial.

Proof. We compute all the Galois cohomology groups using the minimal splitting extension $L_v/k_v$ which is unramified by hypothesis. Let $M$ be the multiplicative group over $k_v$, then there is a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : H^1(L_v/k_v, T) \times H^1(L_v/k_v, X^*(T)) \to H^2(L_v/k_v, M)$$

induced by the non-degenerate pairing $T \times X^*(T) \to M$ (the non-degeneracy of $\langle \cdot, \cdot \rangle$ is established in [58, p. 303]). A character $\chi \in X^*(T)$ takes integer values on $T(L_v)^b$, so if $\alpha \in H^1(L_v/k_v, T)$ is compact, then the linear form $\langle \alpha, \cdot \rangle$ on $H^1(L_v/k_v, X^*(T))$ takes values in the image of $H^2(L_v/k_v, o_{L_v}^\times)$. Since $L_v$ is unramified, the latter group is trivial by the Lemma 9.4 below. We get that $\langle \alpha, \cdot \rangle$ is trivial. Hence, $\alpha = 0$, by the non-degeneracy of the pairing. □

Lemma 9.6. The group $H^2(L_v/k_v, o_{L_v}^\times)$ vanishes for every unramified Galois extension $L_v/k_v$.

Proof. Let $k^u_v$ be the unramified closure of $k_v$. Using the inflation-restriction long exact sequence (also known as Hochschild-Serre sequence) we get

$$H^1(k^u_v/L_v, o_{k^u_v}^\times)_{\text{Gal}(L_v/k_v)} \to H^2(L_v/k_v, o_{L_v}^\times) \to H^2(k^u_v/k_v, o_{k^u_v}^\times).$$

The first term vanishes by [58, Theorem 6.8] so the map $H^2(L_v/k_v, o_{L_v}^\times) \to H^2(k^u_v/k_v, o_{k^u_v}^\times)$ is injective. To finish the proof it is enough to show that $H^2(k^u_v/k_v, o_{k^u_v}^\times) = 0$. This follows from the argument in the proof of [58, p. 299, Proposition 6.7], applied to the multiplicative torus $G_{m}$. □

10. Benjamini–Schramm convergence in bounded degree

In this section $G$ is a semisimple, simply connected Lie group. The reason to change the setting from the adjoint groups to simply connected ones is the Prasad volume formula, where many factors have been explicitly worked out only in the simply connected case.

10.1. Finer description of arithmetic lattices.

10.1.1. Bruhat–Tits buildings and parahoric subgroups. We will describe some results of Bruhat–Tits theory that will be useful later. Let $G$ be a simply connected, semisimple $k$-group and $v \in V_f$. To the $v$-adic group $G(k_v)$ there is an associated Euclidean building (see [72, 2.1]) which we will denote by $X = X(G, k_v)$.

If $T$ is a maximal $k_v$-split torus of $G$, there is a unique apartment $A \subset X$ which is preserved by $T(k_v)$. If $C$ is a chamber (i.e. a maximal simplex) of $A$, the associated Iwahori subgroup $I$ is the stabiliser of $C$ in $G(k_v)$. Facets of $C$ are in 1-1 correspondence
with the subsets of a basis of affine roots for $G, T$; the stabiliser of a facet (i.e. any compact subgroup containing the Iwahori subgroup) is called a parahoric subgroup (see [59, 0.5]).

Let $\Phi_a(G, \tilde{T})$ be the affine root system over $k_v$ and let $\Delta_{a,v}$ the associated Dynkin diagram. The vertices of the diagram correspond to the top dimensional facets of the boundary of $C$. For $\Theta_v \subset \Delta_{a,v}$ we denote by $C_{\Theta_v}$ the associated facet of $C$, which is a simplex in $X$. By the previous paragraph, any parahoric subgroup $U_v$ of $G(k_v)$ is conjugated to the stabiliser of a unique simplex $C_{\Theta_v}$, and we call $\Theta_v$ the type of $U_v$.

The group $U_v$ of type $\Theta_v$ is said to be special if $\Delta_a \setminus \Theta_v$ is isomorphic to the Dynkin diagram of $G$ over $k_v$. It is hyperspecial if and only if this remains true over any unramified finite extension of $k_v$ (see [59, 0.5, 0.6]).

10.1.2. Principal and maximal arithmetic lattices. A lattice of $\Gamma$ of $G$ is said to be maximal if it is maximal with respect to inclusion among lattices.

Let $k$ be a number field and $G$ a $k$-group such that there is a set of places $S_\infty \subset V_\infty$ with $\prod_{v \in S_\infty} G(k_v) \cong G$ and $G(k_v)$ is compact for all $v \in V_\infty \setminus S_\infty$. Note that then $G$ must be semisimple and simply connected. By 2.5 any congruence subgroup $\Gamma_U \subset G(k)$ is an arithmetic lattice in $G$.

If $U_v, v \in V_f$ are parahoric subgroups such that $U_v$ is hyperspecial for almost all $v$, and $U = \prod_{v \in V_f} U_v$ then the lattice $\Gamma_U$ is called a principal arithmetic subgroup of $G$. The following proposition is contained in [20, Proposition 1.4].

Proposition 10.1. A maximal arithmetic lattice in $G$ is the normalizer in $G$ of some principal arithmetic lattice.

10.1.3. Classification. The conjugacy class in $G$ of the principal arithmetic lattice $\Gamma_U$ is determined by the number field $k$, the isomorphism class of the $k$-group $G$, and the local groups $U_v, v \in V_f$ (equivalently the types $\Theta_v$). Regarding the group $G$, we will only be interested in the following data:

(1) The minimal field extension $\ell/k$ such that $G/\ell$ is an inner form of its split form, except when $G$ is of type $^6D_4$. In the latter case $\ell$ is defined as a subfield of this field with $\ell/k$ of degree 3 (see [59, 0.2]). In any case $\ell$ is uniquely determined up to Galois conjugacy.

(2) The finite set of places $v \in V_f$ such that $G$ is not quasi-split over $k_v$.

Regarding the compact-open subgroup $U$ we will be using the following information:

(3) The finite set of places $v \in V_f$ such that $U_v$ is not special;

(4) and those where $G$ splits over an unramified extension of $k_v$ but $U_v$ is not hyperspecial.

We will denote by $S_U$ the set of all "bad places" $v \in V_f$ which fall in one of the sets in 4 or 2.

10.2. Volume formula.

10.2.1. Strong approximation. Since $G$ is noncompact and $G$ is simply connected, the latter satisfies the strong approximation property with respect to infinite places [58, Theorem 7.12]. That is, $G(k)$ is dense in $G(k_f)$. Let $dg_\infty$ be the standard Haar measure

---

9Note that while it does not completely characterise the $k$-isomorphism class of $G$ it does so up to finite ambiguity.
on $G(k_\infty)$. Let $dg_f$ the Haar measure on $G(k_f)$ normalised so that $\text{vol}_{dg_f}(U) = 1$. Then, we have an isomorphism

$$
\Psi: (G(k)\backslash G(k)/U, dg_\infty \otimes dg_f) \to (\Gamma_U \backslash G, dg_\infty)
$$

of measured spaces, where we define quotient measures by taking counting measure on the discrete subgroups $G(k)$, $\Gamma_U$ and Haar probability measure on the compact subgroup $U$. The map $\Psi$ is simply induced by the first projection $G(k_f) = G(k_\infty) \times G(k_f) \rightarrow G(k_\infty)$.

10.2.2. Prasad’s volume formula. We will describe here part of the formula obtained by Gopal Prasad [59] for the covolume $\text{vol}(\Gamma_U \backslash G)$ of a principal arithmetic lattice in $G$ with respect to the standard Haar measure on $G$ (normalized as in section 2.3). The following proposition is a consequence of his results, tailored to our needs. In the sequel we use $C_G$ to denote a constant depending only on $G$.

**Proposition 10.2.** Let $k$ be a number field, let $G/k$ be a (simply connected) $k$-group such that $G(k_\infty)$ is isomorphic to $G \times$ a compact factor, and $U$ a compact-open subgroup which is a product of parahoric groups, so that $\Gamma_U$ is a principal arithmetic lattice. Then

$$
\text{vol}(\Gamma_U \backslash G) \geq C_G |S_U|^\frac{\dim(G)}{2} \cdot N_{k/\mathbb{Q}}(\Delta_\ell/k) \cdot \prod_{v \in S_U} q_v
$$

where $S_U, \ell$ are defined as in [10.3] above and $q_v$ is the cardinality of the residue field of $k_v$.

**Proof.** This follows immediately from the exact formula in [59] Theorem 3.7] together with the following further facts from this reference: in the infinite product $\mathcal{E}$ all factors are $>1$ and they are $>q_v$ at places in $S_U$ (Proposition 2.10(iv) in loc. cit.), and the exponent $s(G)$ of $N_{k/\mathbb{Q}}(\Delta_\ell/k)^{1/2}$ is at least 2 for all types (this can be checked case-by-case using the formulas given in loc. cit., 0.4). \qed

10.3. Index of maximal lattices. To estimate the covolumes of maximal arithmetic lattices, we will use the following result. It follows from standard arguments from [51] and [20], as in [12, Section 4] which deals the case of even orthogonal groups. We essentially follow the argument of Belolipetsky–Emery in [12, Section 4], cutting short where we do not need their degree of precision.

**Proposition 10.3.** Let $\Gamma_U$ be a principal arithmetic lattice in $G$ and $\Gamma = N_G(\Gamma_U)$ its normalizer. Let $\varepsilon > 0$. Then

$$
[\Gamma : \Gamma_U] \ll_\varepsilon C_G^{|S_U|} N_{k/\mathbb{Q}}(\Delta_\ell/k)^{\frac{3}{2} + \varepsilon} \Delta_\ell^{\frac{3}{2} + \varepsilon}.
$$

**Proof.** Let $h_\ell$ be the class number of $\ell$. We first prove that

$$
[\Gamma : \Gamma_U] \leq \begin{cases} 
C_G^{|S_U|} h_\ell & \text{if } G \text{ is not of type } ^1D_{2m}, \\
C_G^{|S_U|} h_\ell^2 & \text{otherwise}.
\end{cases}
$$

Let $\overline{G}$ be the adjoint group of $G$ and let $\pi$ be the morphism from $G$ to $\overline{G}$. We have an exact sequence

$$
1 \rightarrow \mathbb{Z}_G \rightarrow G \rightarrow \overline{G} \rightarrow 1,
$$

which gives rise to the following exact sequence in Galois cohomology:

$$
G(k) \xrightarrow{\pi} \overline{G}(k) \xrightarrow{\Delta} H^1(k, \mathbb{Z}_G).
$$

(10.1)
As $\Gamma_U \subset G(k)$ and the image of $\Gamma$ is contained in $\overline{G}(k)$, the map $\delta$ induces a map 
$\Gamma/\Gamma_U \to H^1(k, Z_G)$. By [51] Proposition 2.6(i), the following sequence is exact:

$$
1 \to Z_G(\overline{k})/Z_G(k) \to \Gamma/\Gamma_U \delta \to H^1(k, Z_G).
$$

(10.2)

It remains to identify the image of $\Gamma/\Gamma_U$ in the (infinite) group $H^1(k, Z_G)$. For this, note that we can lift the morphism $\overline{T}(k) \to \prod_{v \in V_f} Aut(\Delta_{a,v})$ (induced by the conjugation action of $\overline{G}(k)$ on parahorics) to a map $\xi : H^1(k, Z_G) \to \prod_{v \in V_f} Aut(\Delta_{a,v})$ using the cohomology exact sequence (11.1) since $G(k)$ acts trivially on the local Dynkin diagrams $\Delta_{a,v}$. Then, $\delta(\Gamma/\Gamma_U)$ acts trivially at every finite place outside of $S_U$ (see [12, 4.2]) and preserves the nontrivial types $\Theta_v$ at places in $S_U$. Together with (10.2) this implies that

$$
|\Gamma/\Gamma_U| \leq |Z_G(\overline{k})| \cdot \prod_{v \in S_U} |Aut(\Delta_{a,v})| \cdot |\{c \in H^1(k, Z_G) : \forall v \in V_f \setminus S_U, \xi(c)_v = Id\}|
$$

(10.3)

since $|Aut(\Delta)| \leq C_G$ for any Dynkin diagram and a constant $C_G$ depending on $G$ and $H_{\xi,S_U} := \{c \in H^1(k, Z_G) : \forall v \in V_f \setminus S_U, \xi(c)_v = Id\}$

We estimate the size of this set following [12]. We first need a proper description of $H^1(k, Z_G)$. According to the table on [58, p. 332], the center $Z_G$ is of the form $\mu_b \oplus \text{res}^{(1)}_{\ell/k}(\mu_b)$, where $\ell$ is the field defined in [1] above, $b$ is a positive integer depending only on $G$ and $\mu_b$ is the algebraic group defining $b$-root of unity. The only exception is in the case where $G$ is inner of type $D_{2m}$. Then $Z_G = \mu_2 \times \mu_2$. We will now restrict to the former two cases. There is always a map from $H^1(k, Z_G)$ to $\ell^x/(\ell^x)^b$. It is an isomorphism in case $\ell = k$ and otherwise comes from a long exact sequence in Galois cohomology. The size of its kernel is bounded by $b$, see (17) in [12]. Let

$$
\ell^x_{b,S_U} = \{x \in \ell : \forall v \in V_f \setminus S_U, v(x) = 0 \pmod{b}\}
$$

$$
\ell^x_b = \{x \in \ell : \forall v \in V_f, v(x) = 0 \pmod{b}\}
$$

Then $\ell^x_{b,S_U} \subset b \subset (\ell^x)^b$ and by [12, 4.9] (see also [20, Section 2]) we have that $H_{\xi,S_U} \subset \ell^x_{b,S_U}/(\ell^x)^b$. Hence

$$
|H_{\xi,S_U}| \leq |\ell^x_{b,S_U}/(\ell^x)^b|.
$$

By the weak approximation property for the multiplicative group, $\ell^x_{b,S_U}/(\ell^x)^b \cong (\mathbb{Z}/b\mathbb{Z})^{S_U}$. We have

$$
|\ell^x_{b,S_U}/(\ell^x)^b| = b^{S_U} \cdot |\ell^x_b/(\ell^x)^b|
$$

and [20, Proposition 0.12)] gives the estimate

$$
|\ell^x_b/(\ell^x)^b| \leq b^{[\ell:Q]} h_{\ell}
$$

so that

$$
|H_{\xi,S_U}| \leq b \cdot b^{S_U} |b^{[\ell:Q]} h_{\ell}|
$$

which (as $[\ell:Q]$ is assumed to be bounded) together with (10.3) finishes the proof when $G$ is not of type $^1D_{2m}$.

It remains to deal with $G$ of type $^1D_{2m}$. In this case $\ell = k$ and $Z_G = \mu_2 \times \mu_2$. The argument above yields $|\Gamma/\Gamma_U| \leq C_G^{S_U} \cdot b \cdot b^{S_U} |b^{[\ell:Q]} h_{\ell}|^2$. 


We can now finish the proof of the proposition. By the Brauer–Siegel Theorem, $h_\ell \ll_{\varepsilon} \Delta_\ell^{1/2+\varepsilon}$, so

$$h_\ell \ll_{\varepsilon} \Delta_\ell^{\frac{1}{2}+\varepsilon} = N_{k/Q}(\Delta_{\ell/k})^{\frac{1}{2}+\varepsilon} \Delta_{k}^{\frac{1}{2}+\varepsilon} \leq N_{k/Q}(\Delta_{\ell/k})^{\frac{1}{2}+\varepsilon} \Delta_{k}^{\frac{1}{2}+\varepsilon},$$

because $[\ell : k] \leq 3$. In the case of type $1D_{2m}$ we use Brauer-Siegel estimate to get $h_k^2 \ll_{\varepsilon} \Delta_k^{1+\varepsilon}$. Proposition follows.

Note that the image of $\ell^X_\ell$ in $h_\ell$ could be much smaller than the whole class group. This is where, for the fields of large discriminant, the main loss occurs. Conjecturally, the image is of size $\ll \Delta_{\ell}^\varepsilon$ for any $\varepsilon > 0$ but even getting an exponent $< 1/2$ seems hard and is not known in general (see [15] for more context and a result for $b = 2$).

10.4. Convergence. In this subsection we prove the following result, admitting estimates which we will prove in the next two sections.

**Theorem 10.4.** Let $G$ be a noncompact semisimple Lie group and let $X$ be its symmetric space. Let $\Gamma_n$ in $G$ be a sequence of different maximal arithmetic lattices in $G$ and assume that there exists $d \in \mathbb{N}$ such that the trace field of $\Gamma_n$ is an extension of degree $d$ of $\mathbb{Q}$. Then ([11]) holds for the sequence of locally symmetric spaces $\Gamma_n \backslash X$.

10.4.1. Adelic trace formula. For a function $f$ on $G$ and a principal arithmetic lattice $\Gamma_U$, we put $f_\gamma = f \otimes \mathbf{1}_{U_\infty} \otimes \mathbf{1}_U$, where $U_\infty$ is the product of $G(k_v)$ for $v \in S_\infty$ with $G(k_v)$ compact. For a rational conjugacy class $[\gamma]_{G(k)}$, we write $[\gamma]_{G(k)} \subset W$ if the Archimedean part satisfies $[\gamma]_{G(k_\infty)} \subset W \times U_\infty \subset G(k_\infty)$.

**Proposition 10.5.** Let $f, W$ be as in Theorem 10.4. We have

$$\text{tr} R^W_{\Gamma_U} f = \sum_{[\gamma]_{G(k)} \subset W} \text{vol}(G_\gamma(k)/G_{\gamma}(\mathbb{A})) \int_{G(\mathbb{A})/G_{\gamma}(\mathbb{A})} f_\gamma(x\gamma x^{-1}) dx \tag{10.4}$$

**Proof.** Let $\Psi$ be the isomorphism from $G(k) \backslash G(\mathbb{A})/U$ to $\Gamma_U \backslash G$ described in [10.2]. Let $F$ (resp. $F_\gamma$) be the $\Gamma_U$-invariant (resp. $G(\mathbb{A})$-invariant) function defined by $F(x) = \sum_{[\gamma]_{G(\mathbb{A})} \subset W} F(x\gamma x^{-1})$ (resp. $F_\gamma(x) = \sum_{[\gamma]_{G(\mathbb{A})} \subset W} f_\gamma(x\gamma x^{-1})$). Then we have that $F_\gamma = F \circ \Psi$ and since local measure are normalized so that $U_v$ has volume 1 it follows that $\int_{\Gamma \backslash G} F(x) dx = \int_{G(k) \backslash G(\mathbb{A})} F_\gamma(x) dx$ (see also the proof of Lemma 4.3 in [33]). On the other hand the usual unfolding trick (see for instance [7]) shows that $\int_{\Gamma \backslash G} F(x) dx$ (resp. $\int_{G(k) \backslash G(\mathbb{A})} F_\gamma(x) dx$) is equal to the term on the left-hand (resp right-hand) side of the equality (10.4), which proves the latter.

10.4.2. The case of principal arithmetic lattices.

**Proposition 10.6.** Let $W$ be a subset of the set of strongly regular, $\mathbb{R}$-regular semisimple elements in $G$. Let $\Gamma_U$ be a principal arithmetic lattice and let $k, S_U$ be defined as in 10.1.3. Then, for any $f \in C^\infty_c(G)$ we have

$$\text{tr} R^W_{\Gamma_U} f \ll f_d \Delta_k^{\dim(G)\frac{\dim(G)}{2} - \delta}$$

for any $\delta < \frac{\dim(G) - \dim(T)}{2}$, where $T$ is a maximal $k$-torus in $G$ and $d = \lfloor k : \mathbb{Q} \rfloor$.

In view of Proposition 10.5 this proposition is an immediate consequence of the following theorem (we will make this deduction at the end of this section). The theorem itself is the main technical contribution in this part of our work and the proof depends on estimates that we will prove in the next two sections.
Theorem 10.7. Let $G$ be an adjoint Lie group and $f$ a continuous, compactly supported function on $G$. Let $W$ be a subset of the set of strongly regular, $\mathbb{R}$-regular semisimple elements in $G$. For fixed $d \in \mathbb{N}$ and all number fields with $[k : \mathbb{Q}] = d$ and $G(k_\infty) \simeq G$ we have
\[
\sum_{[\gamma] \subset W} \text{vol}(G_\gamma(k) \backslash G_\gamma(k)) \cdot \int_{G_\gamma(k) \backslash G(k)} f_\gamma(x^{-1} \gamma x) dx \leq_{f,d} \Delta_k^{\dim(G) - \delta}
\]
for any $\delta < \frac{\dim(G) - \dim(T)}{2}$, where $T$ is a maximal $k$-torus in $G$.

Proof. Let
\[
\mathcal{O}(\gamma, f_\gamma) = \int_{G_\gamma(k) \backslash G(k)} f(x^{-1} \gamma x) dx
\]
We explain here how to estimate the sum
\[
\sum_{[\gamma] \subset W} \text{vol}(G_\gamma(k) \backslash G_\gamma(k)) \cdot \mathcal{O}(\gamma, f_\gamma),
\]
leaving most technical details to the next two sections. The support of $f_\gamma$ is compact in $G(k)$ so this is a finite sum. In fact, if we assume that $f$ has support in a ball $B_G(1, R)$, for some radius $R$, the we can restrict the sum to the conjugacy classes $[\gamma] \subset G(k)$ which satisfy $m(\gamma) \leq R$ and $\gamma \in U$.

For such $\gamma$, we prove in Theorem 11.1 that for the Haar measure normalised so that $\text{vol}(U) = 1$ we have
\[
(10.5) \quad \mathcal{O}(\gamma, f_\gamma) \ll_{f,[k:\mathbb{Q}]} 1.
\]

To estimate the co-volumes of the centralizers we follow the proof of [33, Lemma 6.4], where it is deduced from results of Ullmo and Yafaev [73]. For any $k$-torus $T$ we have
\[
\text{vol}(T(k) \backslash T(k)) \ll |\Delta_k|^\frac{r}{2} \cdot N_{k/\mathbb{Q}}(\Delta_{L/k})^{\frac{1}{2}} \cdot L(1, \chi_T)
\]
where $r = \dim(T)$ and $L(\cdot, \chi_T)$ is the Artin $L$-function associated with the character $\chi_T$ of the representation of $\text{Gal}(k)$ on $X^*(T)$. The $L$-function is holomorphic at 1 since under our assumptions the torus $T$ is anisotropic. Under our assumptions on $\gamma$, it follows from Proposition 2.2 that $N_{k/\mathbb{Q}}(\Delta_{L/k}) = O(1)$. For the term $L(1, \chi_T)$, by [73, Proposition 2.1] we have that for any $\varepsilon > 0$
\[
L(1, \chi_T) \ll_{\varepsilon, [L: \mathbb{Q}]} |\Delta_L|^\varepsilon
\]
and the right-hand side is equal to $N_{k/\mathbb{Q}}(\Delta_{L/k})^\varepsilon |\Delta_k|^{1/2 - [L:k]}$. Consequently
\[
(10.6) \quad \text{vol}(T(k) \backslash T(k)) \ll_{f,[k:\mathbb{Q}],\varepsilon} |\Delta_k|^{\varepsilon + \frac{r}{2}}
\]
for any $\varepsilon > 0$.

Let $\mathcal{C}(R, U)$ be the set of conjugacy classes $[\gamma] \subset G(k)$ which satisfy $m(\gamma) \leq R$ and $\gamma \in U$. It follows from (10.5), (10.6) that
\[
\sum_{[\gamma] \subset W} \text{vol}(G_\gamma(k) \backslash G_\gamma(k)) \cdot \mathcal{O}(\gamma, f_\gamma) \ll_{f,[k:\mathbb{Q}],\varepsilon} |\mathcal{C}(R, U)| \cdot |\Delta_k|^{\varepsilon + \frac{r}{2}}.
\]
In Theorem 12.1 we prove that $|\mathcal{C}(R, U)| \ll_{R,[k:\mathbb{Q}]} 1$. Writing $\delta = \frac{\dim(G) - r - \varepsilon}{2}$ we get that
\[
\sum_{[\gamma] \subset W} \text{vol}(G_\gamma(k) \backslash G_\gamma(k)) \cdot \mathcal{O}(\gamma, f_\gamma) \ll_{f,[k:\mathbb{Q}],\varepsilon} |\Delta_k|^\frac{\dim(G)}{2} - \delta.
\]
Since $\varepsilon > 0$ can be chose arbitrarily small, $\delta$ can be chosen arbitrarily close to $\frac{\dim(G) - r}{2}$. The proof is complete.

10.4.3. Preliminary lemmas. Various estimates on the volumes of centralizers or the orbital integrals become easier if we restrict to conjugacy classes with additional regularity properties. This is why we introduced the set $W$. To deal with maximal lattices, we will need the notion of strong $m$–regularity ($m \in \mathbb{N}$), which we will not use anywhere else in the paper. If $G$ is a semisimple group, $T$ a maximal torus in $G$ an element $g \in T$ is strongly $m$–regular if $\xi(g) \neq 1$ for all non-trivial characters $\xi \in X^*(T)$ such that $\xi$ is a product of at most $2m$ roots. We note that strongly $m$–regular elements for any $m$ are a Zariski-dense subset.

**Lemma 10.8.** Let $G$ be a semisimple Lie group and $m \in \mathbb{N}$. The set of strongly $m$–regular, $\mathbb{R}$–regular elements is sufficiently dense in $G$.

**Proof.** In [60] Prasad gives a proof that any Zariski-dense subgroup of a semisimple Lie group contains an $\mathbb{R}$–regular element. His argument actually implies the stronger statement above, as we now explain. At the beginning of the proof Prasad introduces a numerical invariant $m(\gamma)$ for $\gamma \in G$, which equals 1 if and only if $\gamma$ is $\mathbb{R}$–regular. In Lemma B he proves that for any Zariski-dense subgroup $\Lambda$ in $G$ the minimum $\min_{\gamma \in \Lambda} m(\gamma)$ is reached on the subset of regular elements; the rest of the proof establishes that $\min_{\gamma \in \Lambda} m(\gamma) = 1$. On the other hand, the proof of Lemma B immediately implies that the minimum is reached on the intersection of $\Lambda$ with any Zariski-open subset of $G$. In particular, it works for strongly $m$–regular elements.

**Lemma 10.9.** Let $L, H$ be two adjoint semisimple groups and $G = L \times H$. There exists a neighbourhood $V$ of $H$ in $G \setminus \{1\}$ which does not contain any irreducible arithmetic lattice of $G$.

**Proof.** Let $\| \cdot \|_H, \| \cdot \|_L$ be some norms on $H$ and $L$. Let $\pi_L, \pi_H$ be the projections from $G$ to $L, H$ respectively. We define the following open subsets of $\text{Sub}_G$:

$$V_{R, \varepsilon} = \{ \Lambda \in \text{Sub}_G : \exists g \in \Lambda \setminus \{1\} : \|\pi_L(g) - 1\|_L < \varepsilon, \|\pi_H(g)\| < R \}.$$ 

For any $R_\varepsilon > 0$ such that $\lim_{\varepsilon \to 0} R_\varepsilon = +\infty$ we have that $\bigcup_{R > 0} V_{R, \varepsilon}$ is a neighbourhood in $\text{Sub}_G \setminus \{ \{1\} \}$. We want to prove that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for every irreducible arithmetic lattice $\Gamma$ in $G$ the following is true. If $\gamma \in \Gamma \setminus \{1\}$ satisfies $\|\pi_L(g) - 1\|_L < \varepsilon$, then $\|\pi_H(g)\| \geq R_\varepsilon$, and $\lim_{\varepsilon \to 0} R_\varepsilon = +\infty$. By construction, the set $V := \bigcup_{R > 0} V_{R, \varepsilon}$ will not contain any irreducible arithmetic lattice of $G$.

Since $\Gamma$ is irreducible arithmetic and $G$ is adjoint, there exists a number field $k$, a $k$–group $G$ and a partition $V_\infty = S_1 \cup S_2 \cup S_3$ such that $H = G(k_{S_1}), L = G(k_{S_2}), G(k_{S_3})$ is compact and $\Gamma$ is an arithmetic subgroup of $G(k)$. In particular, the eigenvalues $\lambda_1, \ldots, \lambda_m$ of $\text{Ad}(\gamma)$ (where $\text{Ad}$ is the adjoint representation of $G$ on its $k$–Lie algebra) are algebraic integers. For each $i$

$$\prod_{v \in S_2} |\lambda_i - 1|_v \ll \varepsilon$$

(with constants depending only on the choice of norm on $L$) and

$$\prod_{v \in S_3} |\lambda_i - 1|_v < 2^{|S_3|} < 2^d.$$
On the other hand, since $\lambda_i$ are algebraic integers and they are not all equal to 1, there must be an $1 \leq i \leq m$ such that

$$\prod_{v \in V_{\infty}} |\lambda_i - 1| \geq 1.$$ 

It follows that

$$\prod_{v \in S_1} |\lambda_i - 1| \gg 2^{-d}\varepsilon^{-1}.$$ 

This finishes the proof, since $\|\pi_H(\gamma) - 1\| \gg \prod_{v \in S_1} |\lambda_i - 1|$ (with a constant depending only on the choice of norm on $H$) so can choose $R_\varepsilon \gg \varepsilon^{-1}$. □

10.4.4. The proof of Theorem 10.4 for uniform lattices. It follows from Lemmas 10.8, 10.9 and Theorem 8.3 that we need only to prove that (8.2) with $W$ being set of $R$-regular, strongly $m$-regular elements of $G$, for $m$ depending only on $G, d$, holds for any sequence $\Gamma_n$ of maximal arithmetic lattices whose trace fields have bounded degree. We will prove that

$$\text{tr} R_\Gamma^W 1_{B(R)} = o(\text{vol}(\Gamma \backslash G))$$

for a uniform maximal arithmetic lattice $\Gamma$ in $G$ with degree of trace field $[k_\Gamma : \mathbb{Q}] = d$, where $W$ is the set of strongly $m$-regular, $R$-regular elements. This implies (8.2) since we can approximate any compactly supported function by sums of translates of functions of the form $1_{B(R)}$ and in view of [42, Theorem 8.36] it implies directly the geometric statement (1.1)).

Let $\Gamma$ be such a lattice. By the description in Proposition 10.1 there exists a principal arithmetic lattice $\Gamma_U$ in $G$, such that $\Gamma = N_G(\Gamma_U)$. We recall Prasad’s volume formula and the index estimates we discussed above:

$$\text{vol}(\Gamma_U \backslash G) \geq C_G \Delta_k^\frac{\dim(G)}{2} \cdot N_{\mathbb{Q}/k}(\Delta_k) \cdot \prod_{v \in S_U} q_v,$$

(10.8)

$$[\Gamma : \Gamma_U] \leq C_G^{\left|S_U\right|} N_{\mathbb{Q}/k}(\Delta_k) \Delta_k^{\frac{3}{2}}$$

(10.9)

where $C_G$ is a constant depending only on $G$ (we take the largest between the two statements). We will finally need the following facts.

**Lemma 10.10.** Both

1. the exponent of the finite group $\Gamma / \Gamma_U$;
2. the number of $h \in G$ such that $h^m = g$ for a regular $g \in G$

are bounded by a constant depending only on $G$, the degree $[k : \mathbb{Q}]$ and $m$. We will denote upper bounds by $C_G$ since we assume that $[k : \mathbb{Q}]$ is bounded and that $m$ depends only on $G$.

**Proof.** By [51] Proposition 2.6(i)] there is an exact sequence

$$0 \to \mathbb{Z}_{\mathbb{G}(k)} \to \mathbb{Z}_{\mathbb{G}(k)} \to \Gamma_n / \Gamma_{U_n} \to H^1(k, \mathbb{Z}_{\mathbb{G}}).$$

The group $H^1(k, \mathbb{Z}_{\mathbb{G}})$ has finite exponent depending only on the absolute type of $G$ so the same must hold for $\Gamma / \Gamma_U$. This proves the first point. Also, for a regular element, its $m$–th roots belong to a torus and hence their number is bounded by a constant depending on $\dim(G)$ and the number of roots of unity in a finite extension of $k$ of degree at most the rank of $G$. This proves the second point. □
Note that \( m \)-th power of a strongly \( m \)-regular element is strongly regular. It follows from the lemma above that the map \( \gamma \mapsto \gamma^m \) from the set of semisimple, \( m \)-regular elements of \( \Gamma \) to strongly regular elements in \( \Gamma_U \) is at most \( C_G \)-to-one. On the other hand, it follows from the triangle inequality, that for any isometry \( g \in G \) we have
\[
1_{B(R)}(x^{-1}g^{-1}x) \leq 1_{B(mR)}(x^{-1}g^m x).
\]
For any \( g \in \Gamma \cap W \), the power \( \gamma^m \) is regular, so \( G_{\gamma^m} = G_{\gamma} \). We conclude that
\[
\text{vol}(\Gamma \setminus G_{\gamma}) = \text{vol}(\Gamma_{\gamma^m} \setminus G_{\gamma^m}) \leq \text{vol}((\Gamma_U)_{\gamma^m} \setminus G_{\gamma^m}).
\]
Putting this together we get
\[
\text{tr} R_s^{W} 1_{B(R)} \leq C_G \cdot \text{tr} R_s^{W'} 1_{B(mR)},
\]
where \( W' \) is the set of strongly regular, \( \mathbb{R} \)-regular elements of \( G \). By Proposition \ref{prop:stronglyRegularElements}, it follows that for any \( \varepsilon > 0 \) we have
\[
\text{tr} R_s^{W} 1_{B(R)} \ll_{G,R,\varepsilon,d} \Delta_k^{\dim G - (\dim G - \dim T) - \varepsilon} \cdot \frac{\text{vol}(\Gamma_U \setminus G)}{\Delta_k^{\dim G - \dim T} \cdot N_{k/q}(\Delta_{k/\ell}) \cdot \prod_{v \in S_U} q_v}
\]
Using \ref{prop:volGroup} we get
\[
\text{tr} R_s^{W} 1_{B(R)} \ll_{G,R,\varepsilon,d} \frac{\Delta_k^{\dim G - (\dim G - \dim T) - \varepsilon} \cdot N_{k/q}(\Delta_{k/\ell}) \cdot \prod_{v \in S_U} q_v}{\Delta_k^{\dim G - \dim T} \cdot \prod_{v \in S_U} q_v} \cdot \text{vol}(\Gamma \setminus G).
\]
Now using \ref{prop:volumeLowerBound} and the fact that \( C^{|S|} \ll \prod_{v \in S} q_v^{1/2} \) for any \( C > 1 \), we deduce that
\[
\text{tr} R_s^{W} 1_{B(R)} \ll_{G,R,\varepsilon,d} \frac{1}{\Delta_k^{\dim G - (\dim G - \dim T) - \varepsilon} \cdot N_{k/q}(\Delta_{k/\ell})^{1/2} \cdot \prod_{v \in S_U} q_v^{1/2} \cdot \text{vol}(\Gamma \setminus G)}.
\]
When \( G \) is not of type \( A_1 \) we have \( \dim G - \dim T \geq 4 \) and we see that \ref{prop:volumeUpperBound} holds, since for any \( C \) the denominator \( \Delta_k^{\dim G - (\dim G - \dim T) - \varepsilon} \cdot N_{k/q}(\Delta_{k/\ell})^{1/2} \cdot \prod_{v \in S_U} q_v^{1/2} \) is \( \leq C \) only for finitely many choices of \( k, \ell \) and \( S_U \). The remaining case where \( G \) is of type \( A_1 \) is dealt with in \ref{prop:volumeUpperBound}; we could also deduce it from our arguments here, going back to the proof of \ref{prop:volumeLowerBound} and observing that we can get the better bound \( \left[ \Gamma : \Gamma_U \right] \leq C^{|S_U|} \Delta_k^{1/2} \) for type \( A_1 \), since \( G \) is always inner so \( \ell = k \).

10.5. **The case of non-uniform lattices.** The proof of Theorem \ref{thm:main} when all \( \Gamma_n \) are non-uniform is exactly the same as in the case where they are uniform except that we need in addition to choose the conjugacy-invariant, sufficiently dense set \( W \) such that for any choice of a lattice \( \Gamma \) in \( G \) and any \( \gamma \in \Gamma \cap W \) the quotient \( \Gamma \gamma \setminus G_{\gamma} \) is compact. The following lemma shows that for this we can use the set of \( m \)-regular elements.

**Lemma 10.11.** Let \( G \) be a semisimple Lie group. There exists \( m = m(G) \) such that every non-uniform, irreducible arithmetic lattice \( \Gamma \) and every strongly \( m \)-regular element \( \gamma \in \Gamma \) the group \( \Gamma \gamma \) is a lattice in \( G_{\gamma} \).

**Proof.** Let \( G \) be an algebraic group over a number field \( k \) such that we have a surjective map \( G(k_{\infty}) \to G \) and \( \Gamma \subset G(k) \). The lattice is non-uniform and irreducible so \( G \) is not \( k \)-anisotropic. It follows that \( G \simeq G(k_{\infty}) \), because the latter has no compact factors. Let \( \gamma \) be a strongly regular element of \( G(k) \). The group \( \Gamma \gamma \) is a lattice in \( G_{\gamma} \) if and only if \( G_{\gamma} \) is \( k \)-anisotropic. Write \( T := G_{\gamma} \), it is an algebraic torus, by Proposition \ref{prop:tori}}
Torus $T$ is anisotropic if and only if it has no rational characters. The latter means that the action of $\text{Gal}(\mathbb{F}/k)$ on $X^*(T)$ has no non-trivial fixed points. Write $T = T(k_{\infty})$ and let $X^*(T)$ be the module of complex characters of $T$. Then
\[
G \cong \prod_{\nu} G(k_{\nu}), \quad T \cong \prod_{\nu} T(k_{\nu}) \quad \text{and} \quad X^*(T) \cong \bigoplus_{\nu} X^*(T),
\]
where $\nu$ runs over the archimedean places of $k$.

**Claim 1.** There exist $m_0 = m_0(G)$ such that if $X^*(T)_{\text{Gal}(\mathbb{F}/k)} \neq 0$, then there is a rational character $\xi \neq 0$ with $\|\xi\| \leq m_0$, where $\|\xi\|$ denotes the minimal number of roots (with multiplicites) needed to write down $\xi$ as a product. Indeed, the distance $\min\{\|\xi\| : \xi \in X^*(T)_{\text{Gal}(k)}, \xi \neq 0\}$ depends only on the image of $\text{Gal}(k)$ in $\text{GL}(X^*(T))$. Torus $T$ is rational, so $\text{Gal}(k)$ preserves the root system of $\Phi(G,T)$. This leaves only finitely many possibilities for the image and their number depends only on the root system of $G$. Root system of $G$ is determined by $G$ alone, which proves the claim.

**Claim 2.** Let $\xi$ be a rational character of $T$ with $\|\xi\| \leq m_0$. Write $\xi_{\nu}$ for the extension to $T(k_{\nu})$. Then the character
\[
\lambda := \prod_{\nu \text{ real}} \xi_{\nu} \prod_{\nu \text{ complex}} (\xi_{\nu}^{*} \xi_{\nu})
\]
satisfies $\|\lambda\| \leq [k : \mathbb{Q}]m_0$ and $\lambda(\gamma) = \pm 1$. The first assertion holds because $\|\cdot\|$ is subadditive. For the second, observe that $\lambda(\gamma) = N_{k/\mathbb{Q}}(\xi(\gamma))$ and that $\xi(\gamma) \in \mathcal{O}_k^\times$ because $\gamma \in \Gamma$.

Suppose $\gamma$ is such that $G_{\gamma}$ is not anisotropic. By combining the claims we construct a nontrivial character $\lambda^2$ with $\|\lambda^2\| \leq 2[k : \mathbb{Q}]m_0$ such that $\lambda^2(\gamma) = 1$. Non-uniform lattices have trace field of degree at most $\dim G$, so the lemma holds with $m = 2m_0 \dim G$. \hfill \Box

### 11. Local and Global Estimates for Orbital Integrals

This section is devoted to the proof of the following theorem. We note that similar results are proven for instance in [39, Proposition 3.13] but they are not sufficient for our purposes (we need to have no multiplicative constant in our local inequalities).

**Theorem 11.1.** Let $k$ be a number field, $G$ a simply connected semisimple $k$-group, $U$ a compact-open subgroup in $G(k_f)$ which is a product of parahoric subgroups. Fix the normalisation of Haar measure on $G(k_f)$ so that $\text{vol}(U) = 1$. Let $f \in C_c^\infty(G(k_{\infty}))$ and let $f_{\kappa} = f \otimes 1_U$. There exists $C_o$, depending on $R, [k : \mathbb{Q}]$, such that for any strongly regular, $\mathbb{R}$-regular element $\gamma \in G(k) \cap U$ with $m(\gamma) \leq R$ we have
\[
\mathcal{O}(\gamma, f_{\kappa}) \leq C_o \|f\|_{\infty}.
\]

**Proof.** We prove Theorem 11.1 assuming the local estimates given later. By assumption, we have $U = \prod_{\nu \in \mathcal{V}_f} U_{\nu}$, where $U_{\nu}$ is a parahoric subgroup of $G(k_{\nu})$. It follows that
\[
\mathcal{O}(\gamma, f_{\kappa}) = \mathcal{O}(\gamma, f_{\infty}) \cdot \prod_{\nu \in \mathcal{V}_f} \mathcal{O}(\gamma, 1_{U_{\nu}}).
\]
We can apply Proposition 11.2 and Proposition 11.5 together with Lemma 11.6 to estimate the right-hand side and get
\[
\mathcal{O}(\gamma, f_{\kappa}) \leq C_R \|f\|_{\infty} \cdot N_{k/\mathbb{Q}}(\Delta(\gamma))^a,
\]
\footnote{This hypothesis is unnecessary here.}
for some \( a > 0 \). The hypothesis that \( m(\gamma) \leq R \) and the fact that \( \Delta(\gamma) \) is an algebraic integer give a uniform bound on \( N_{k/Q}(\Delta(\gamma)) \), dependent only on \([k : Q]\) and \( R \). This finishes the proof. \( \square \)

11.1. Nonarchimedean orbital integrals. For this subsection we let \( v \in V_f \), so \( k_v \) is a local non-archimedean field. We denote the cardinality of the residue field of \( k_v \) by \( q \). We fix a simply connected semisimple \( k_v \)-group \( G \).

Let \( U_v \) be a maximal compact subgroup of \( G(k_v) \). We fix the Haar measure on \( G(k_v) \) so that \( \text{vol}(U_v) = 1 \). Let \( \gamma \in U_v \) be a semisimple regular element. Let \( G_\gamma \) be the centraliser of \( \gamma \) in \( G \). It is an algebraic subgroup whose identity component is a torus. Our goal is to estimate the following integral orbital:

\[
O(\gamma, 1_{U_v}) = \int_{G_v(k_v) \setminus G(k_v)} 1_{U_v}(x) dx.
\]

Namely, we will prove the following result.

**Proposition 11.2.** There exists a constant \( a > 0 \), which depends only on the absolute type of \( G \), such that for any \( \gamma \in G(k_v) \) as above we have:

\[
O(\gamma, 1_{U_v}) \leq |\Delta(\gamma)|_v^{-a}.
\]

**Proof.** The main ingredient in the proof is the following relation between the orbital integrals and Bruhat-Tits buildings. Recall that \( X = X(G, k_v) \) is the Bruhat–Tits building associated with the simply-connected group \( \tilde{G} \) over \( k_v \) (see \[10.1.1\]). We will prove that there exists \( b \) depending only on absolute type of \( G \) such that

\[
(11.1) \quad O(\gamma, 1_{U_v}) \leq |B_X(x_0, R)|, \quad R = b \cdot v(\Delta(\gamma))
\]

where \( B_X(x_0, R) \) is the set of vertices in the ball of radius \( R \) around \( x_0 \) in the building \( X \). By the estimates in \[14\ Section 3.3\], there exists constants \( c > 0 \) depending only on the absolute type of \( G \) such that \( |B_X(x_0, R)| \leq q^{-cR} \) for \( R \geq 1 \). Together with (11.1) this implies that

\[
O(\gamma, 1_{U_v}) \leq q^{-c b \cdot v(\Delta(\gamma))}
\]

and this is equal to \( |\Delta(\gamma)|_v^{-a} \) where \( a \) depends only on the absolute type of \( G \). \( \square \)

The rest of this subsection is dedicated to the proof of (11.1); we deal first with the case where \( G_\gamma \) is split over \( k_v \) and then deduce the general case by using embedding arguments.

11.1.1. Recall that we assume that the Haar measures on \( G(k_v), G_\gamma(k_v) \) are normalised so that the maximal compact subgroups \( U_v, G_\gamma(k_v) \) have measure 1. In the proofs below we will use the shorthand \( O(\gamma, 1_{U_v}) = O_\gamma \). We start with giving a more geometric formula for \( O_\gamma \). We assume that a chamber \( C \) of \( X \) has been fixed. Recall that the types of facets in \( X \) are indexed by the facets of \( C \). For a local type \( \Theta \), we denote by \( X_\Theta \) the set of translates of the facet \( C_\Theta \) under \( G(k_v) \).

**Lemma 11.3.** Let \( U_v \) be of type \( \Theta \) and let \((X_\Theta)^{\gamma}\) the chambers of type \( \Theta \) in \( X \) which are fixed by \( \gamma \). Then

\[
O(\gamma, 1_{U_v}) = \sum_{x \in G_\gamma(k_v) \setminus (X_\Theta)^{\gamma}} |G_\gamma(k_v)^b \cdot x|.
\]
Proof. Let $\alpha$ be a function on $G(k_v)$ such that $\int_{G_\gamma(k_v)} \alpha(tx)dt = 1$ for any $x \in G(k_v)$. It follows that
\[
\int_{G_\gamma(k_v) \times U_v} \alpha(x)dx = [G_\gamma(k_v) : G_\gamma(k_v) \cap xU_vx^{-1}]
= [G_\gamma(k_v)^b : G_\gamma(k_v)^b \cap xU_vx^{-1}].
\]
Now $1_{U_v}(x^{-1}\gamma x)$ is nonzero if and only if $\gamma$ fixes the corresponding coset $xU_v$. So
\[
\int_{G(k_v)} \alpha(x)1_{U_v}(x^{-1}\gamma x)dx = \sum_{xU_v \in G(k_v)/U_v} \int_{xU_v} \alpha(x)dy
= \sum_{G_\gamma(k_v) \times U_v \in G_\gamma(k_v) \times G(k_v)/U_v} [G_\gamma(k_v)^b : G_\gamma(k_v)^b \cap xU_vx^{-1}]
\]
which finishes the proof by identifying $G(k_v)/U_v$ with $X_{\Theta_v}$ and $[G_\gamma(k_v)^b : G_\gamma(k_v)^b \cap xU_vx^{-1}]$ with $|G_\gamma(k_v)^b \cdot x|$.

11.1.2. Proof of (11.1) in the split case. We assume here that the maximal torus $G_\gamma$ is split over $k_v$, so there is a unique apartment $A_\gamma$ of $X$ stabilised by $G_\gamma(k_v)$.

Lemma 11.4. Under the assumptions above there exists $\ell'$ depending only on the absolute type of $G$ such that any simplex of $X$ fixed by $\gamma$ lies at most at distance $\ell' \cdot v(\Delta(\gamma))$ from $A_\gamma$.

Proof. Let $C_\Theta$ be the simplex of $X$ fixed by $U_v$. We choose a chamber $C$ of $A_\gamma$ containing $C_\Theta$, so there exists a special vertex $x_0 \in C$ fixed by $\gamma$. Let $I \subseteq U_v$ be the Iwahori subgroup fixing $C$ pointwise. Let $\Phi$ be the (linear) root system of the pair $(G, G_\gamma)$. The chamber $C$, together with the vertex $x_0$, determine a set $\Phi^+$ of positive roots. Let $N$ be the associated maximal unipotent subgroup of $G$. Then according to [72, 3.3.2] we have the Iwasawa decomposition
\[
G(k_v) = G_\gamma(k_v) \cdot N(k_v) \cdot I.
\]
Let $C'_\Theta = gC_\Theta$ and write $g = ank$ the Iwasawa decomposition of $g$. If $\gamma$ preserves $C'_\Theta$ then we have $\gamma an \in aU_v$, and as $a \in G_\gamma(k_v)$ it follows that $n^{-1}\gamma n \in U_v$ and finally that $\gamma^{-1}n^{-1}\gamma \cdot n \in U_v$.

For $\lambda \in \Phi$ we denote by $N_\lambda$ the 1-parameter unipotent subgroup of $G$ and associated to $\lambda$. We will also use the notation set up in [72]: if $\lambda \in \Phi$ and $k \in \mathbb{Z}$ we let $\lambda + k$ be the affine function on $A_\gamma$ with vector part $\lambda$ and such that $(\lambda + k)(x_0) = k$. For such a function $f = \lambda + k$ let $X_f$ be the subgroup of $N_\lambda(k_v)$ fixing the half-apartment $f^{-1}([0, +\infty[)$. This determines an isomorphism $n_\lambda$ from the additive group $k_v$ to $N_\lambda$ such that $n_\lambda(0_{k_v}) = X_{\lambda}$.

We can describe the chamber $C$ as follows:
\[
C_\Theta = \bigcap_{\lambda \in \Phi^+ \setminus \Theta} \left( (\lambda^{-1}([0, +\infty[) \cap (1 - \lambda)^{-1}([0, +\infty[)) \cap \bigcap_{\lambda \in \Theta} \lambda^{-1}([0, +\infty[) \right)
\]
and it follows from [72, 3.1.1] that the product map defines a bijection
\[
U_v = \left( \prod_{\lambda \in \Phi^+} X_\lambda \right) \times G_\gamma(k_v)^b \times \left( \prod_{\lambda \in \Phi^+ \setminus \Theta} X_{1-\lambda} \times \prod_{\lambda \in \Theta} X_{-\lambda} \right).
\]
So \( \gamma^{-1}n\gamma \cdot n \in U_v \) is equivalent to it belonging to \( \prod_{\lambda \in \Phi^+} X_\lambda \). We let \( \Phi^+ = \{ \lambda_1, \ldots, \lambda_k \} \) and write

\[
n = n_{\lambda_k}(t_k) \cdots n_{\lambda_1}(t_1).
\]

Let \( b^j_{i,s} \) be the structure constants of a Chevalley basis of the Lie algebra of \( G \), determined by \([u_j, u_s] = b^j_{i,s} u_i\) if \( u_i \) are the basis vectors corresponding to the roots \( \lambda_i \). They are integers depending only on the absolute type of \( G \). We want to deduce from this the lower bounds for \( v \) which implies that \( \lambda_i \) are the basis vectors corresponding to the roots \( \lambda_i \). They are integers depending only on the absolute type of \( G \), see [71, Theorem 1 on p. 6]. It follows from Lemma 2.8 that (for concision we use the notation \( a_i = \lambda_i(\gamma) \)):

\[
(11.2) \quad n^{-1} \gamma n \gamma^{-1} = \prod_{i=1}^{k} n_{\lambda_i} \left( (a_i - 1)t_i + \sum_{\lambda_1 + \cdots + \lambda_m = \lambda_i} b^j_{i,j_1,\ldots,j_m} \prod_{l=1}^{m} t_{j_l} \right),
\]

where \( b^j_{i,j_1,\ldots,j_m} \) is a nonzero product of the \( b^i_{j,s} \) (depending on the chosen order on roots). We have arrived at the following reformulation:

\[
n^{-1} \gamma n \gamma^{-1} \in I \iff \forall i = 1, \ldots, k : \left( (a_i - 1)t_i + \sum_{\lambda_1 + \cdots + \lambda_m = \lambda_i} b^j_{i,j_1,\ldots,j_m} \prod_{l=1}^{m} t_{j_l} \right) \in \mathfrak{o}_{k_v}.
\]

We want to deduce from this the lower bounds for \( v(t_i) \), \( 1 \leq i \leq k \). Note that if \( N \) were abelian, this would be immediate, as all \( b^j_{i,s} \) would vanish and \( (a_i - 1)t_i \in \mathfrak{o}_{k_v} \) gives \( v(t_i) \geq -v(a_i - 1) \).

**Type A\(_2\)** We give first a simple example to illustrate the general principle, and point to a possible sharpening of our estimates. If \( G = \text{SL}_3 \) we can choose the \( \lambda_i, n_i \) so that all \( b^j_{i,s} = 0 \) for \( i = 1, 2 \) and \( b^3_{1,2} = 1 \). We get that

\[
v(t_i) \geq -v(a_i - 1), \quad i = 1, 2
\]

and

\[
v(t_3) \geq -v(a_3 - 1) \quad \text{or} \quad v((a_3 - 1)t_3) = v(a_2 t_1 t_2) \geq v(t_1)
\]

(for the last inequality we use \( v(a_2) \geq v(a_2 - 1) \geq -v(t_2) \)) so in the worst case we get that

\[
v(t_1) \geq -v(a_1 - 1), \quad v(t_2) \geq -v(a_2 - 1), \quad v(t_3) \geq -v(a_3 - 1) - v(a_1 - 1)
\]

which implies that \( v(t_i) \geq -v(\Delta(\gamma)) \) for all \( i \), and also \( \sum_i v(t_i) \geq -2v(\Delta(\gamma)) \).

**General case** Ordering the \( \lambda_i \)'s (as in Lemma 2.8) so that the \( l \)-th term in the lower central series of \( N \) is spanned by \( N_{\lambda_k}, \ldots, N_{\lambda_{k+1}} \) (with \( 1 = k_1 \leq k_2 \leq \cdots \leq k_m \)), we see that all \( a^i_{j,s} = 0 \) for \( 1 \leq i < k_1 \), so that \( v(t_i(a_i - 1)) \geq 0 \), hence \( v(t_i) \geq -v(a_i - 1) \) for those \( i \).

Now we proceed by induction on \( l \) to estimate \( v(t_i) \) for \( k_1 \leq i \leq k_{l+1} - 1 \). Assume that we know that for \( 1 \leq i \leq k_l - 1 \) we have \( v(t_i) \geq -\sum_{1 \leq j \leq k_l-1} b_j v(a_j - 1) \) for some nonnegative integers \( b_j \leq 2^{l-1} \). Now (11.2) implies that either \( v((a_i - 1)t_i) \geq 0 \), or

\[
v((a_i - 1)t_i) \geq -v \left( \sum_{\lambda_1 + \cdots + \lambda_m = \lambda_i} b^j_{i,j_1,\ldots,j_m} \prod_{s=1}^{m} a^s_{j_1,\ldots,j_s} t_{j_s} \right).
\]
In the first case we get that \( v(t_i) \geq -v(a_i - 1) \) and we are done. In the second case we get that
\[
v(t_i) \geq -v(a_i - 1) + \min_{\lambda_j + \cdots + \lambda_{jm} = \lambda_i} v \left( \prod_{s=1}^{m} a_{i_j} \cdots a_{i_k} t_{js} \right)
\]
and by the recursion hypothesis it follows that:
\[
v(t_i) \geq -v(a_i - 1) + \sum_{s=1}^{m} v(t_{js})\]
we get that:
\[
v(t_i) \geq -v(a_i - 1) - 2 \sum_{1 \leq j \leq k-i-1} b_{j} v(a_j - 1) \geq \sum_{1 \leq j \leq k-i-1} b'_{j} v(a_j - 1)
\]
with \( b'_j \leq 2^j \).

In conclusion: from the condition that \( n^{-1} \cdot \gamma n^{-1} \in I \), equivalent to \( (a_i - 1)t_i + \sum_{1 \leq j \leq k} a_{i_j} t_{ij} \in \Phi_k \), for all \( 1 \leq i \leq k \) we have arrived at the following set of inequalities:
\[
v(t_i) \geq -2^{m-1} \sum_j v(a_j - 1)
\]
Since
\[
-v(\Delta(\gamma)) = -v \left( \prod_{\lambda \in \Phi} (1 - \lambda(\gamma)) \right) \leq -\sum_{\lambda \in \Phi^+} v(1 - \lambda(\gamma)),
\]
the inequality following from \( -v(1 + \lambda(\gamma)) \geq -v(\lambda(\gamma)) \geq 0 \) for \( \lambda \in \Phi^+ \) we get that
(11.3)
\[
v(t_i) \geq -c \cdot v(\Delta(\gamma)), \quad 1 \leq i \leq k
\]
where \( c = 2^m \) depends only on the absolute type of \( G \).

We can finally conclude. Let \( x_1 \) be the vertex of \( A_\gamma \) at the tip of the intersection of the half-apartments \( \lambda_i^{-1} \{ -v(\Delta(\gamma)) \} \). We have just proven that if \( \gamma \) fixes \( gI \), with \( g = ank \), then \( n \) fixes \( x_1 \). In particular we get that
\[
d(A_\gamma, gC) = d(A_\gamma, nC) \leq d(x_1, nC) = d(x_1, C) \leq d(x_1, x_0) = c' \cdot v(\Delta(\gamma)).
\]
where \( c' \) depends only on \( c \) and the geometry of the apartment, so only on the absolute type of \( G \). As a fixed vertex belongs to a fixed chamber at the same distance from \( A_\gamma \), we get the conclusion. \hfill \Box

**Conclusion.** Let
\[
R := \sup_{x \in Fix_{X^\Theta}(\gamma)} d(x, A_\gamma) \leq c \cdot v(\Delta(\gamma)),
\]
the inequality following from by Lemma [11.4].

The group \( G_\gamma(k_v) \) acts transitively on \( A_\gamma^\Theta := A_\gamma \cap X^\Theta \) (this follows from Iwasawa decomposition). Moreover every vertex of \( A_\gamma \) is at distance at most 1 from a vertex of \( A_\gamma^\Theta \) (because every chamber contains a vertex of every type)It follows (as \( 2R \geq R + 1 \) for \( R \geq 1 \) that there exists a set \( \{ y_1, \ldots, y_n \} \) of representatives for the orbits of \( G_\gamma(k_v) \) on \( Fix_{X^\Theta}(\gamma) \) such that \( y_i \in B_X(x_0, 2R) \) for all \( i \). The subgroup \( G_\gamma(k_v)^b \) preserves this ball and the orbits \( G_\gamma(k_v)^b \cdot y_i \) are pairwise disjoint, so it follows that:
\[
\sum_{x \in G_\gamma(k_v) \setminus Fix_{X^\Theta}(\gamma)} |G_\gamma(k_v)^b x| \leq B_X(x_0, 2R).
\]
This finishes the proof of (11.1) in this case.

11.1.3. Non-split case. Let $L_v/k_v$ be a finite Galois extension splitting $G_\gamma$. Then there is an inclusion of buildings $X \subset X(\overline{G}, L_v) = X_{L_v}$ such that the image of $X$ is a totally geodesic subspace. The Galois group $\text{Gal}(L_v/k_v)$ acts by isometries on $X_{L_v}$.

**Tame case.** We suppose here that $L_v/k_v$ is unramified or tamely ramified. In these cases we have $X = X_{L_v}^{\text{Gal}(L_v/k_v)}$ by [22, 2.6.1], [65, Proposition 5.1.1] (see also [61]). The Galois group preserves $A_\gamma, L_v$, where it acts by affine isometries, so $A_\gamma, L_v \cap X = (A_\gamma, L_v)^{\text{Gal}(L_v/k_v)}$ is an affine subspace. The linear part of the Galois action on $A_\gamma, L_v$ is given by the action on the co-root space $X_1(G_\gamma)$. It follows that the dimension of its fixed points equals the $k_v$-rank of $G_\gamma$, which equals the $k_v$-rank of $G_\gamma$.

We define $A_\gamma := A_\gamma, L_v \cap X$, which by the discussion above is an affine subset of dimension $r$ in an apartment of $X$. It follows from Lemma 11.4 that $\text{Fix}_{X_\Theta}(\gamma)$ is contained in the $c' \cdot v(\Delta(\gamma))$-neighbourhood of $A_\gamma$, with respect to the metric induced on $X$ by that of $X_{L_v}$. This metric might not be the intrinsic metric on $X$, but it can only be multiplied by a factor depending only the Galois action on $X_1(G_\gamma)$, which is determined up to finite ambiguity by the absolute type of $G$. In particular the statement of the lemma remains valid in this case.

We let $R = \sup(d(x, A_\gamma) : x \in \text{Fix}_{X_\Theta}(\gamma))$, by the preceding comments we have that $R \leq c' \cdot v(\Delta(\gamma))$. In case $R = 0$ (that is $v(\Delta(\gamma)) = 0$) the set $X_\Theta \cap A_\gamma$ is nonempty, and the action of $G_{\gamma}(k_v)$ on it is transitive. It follows that $O_\gamma = 1$.

In case $R \geq 1$ the set $X_\Theta \cap A_\gamma$ might be empty. We fix representatives $x_1, \ldots, x_s$ of the $G_{\gamma}(k_v)$-orbits on $A_\gamma$, we may assume that $x_i \in B_X(x_1, 1)$ for $1 \leq i \leq s$. Since $\text{Fix}_{X_\Theta}(\gamma)$ is contained in the $R$-neighbourhood of $A_\gamma$ we get that

$$O_\gamma \leq |B_X(x_1, R + 1)| \leq |B_X(x_1, 2R)|.$$ 

This finishes the proof in case $G_\gamma$ is split by a unramified or tamely ramified extension of $k_v$.

**Wild ramification.** Assume now that $L_v/k_v$ is wildly ramified. This case is similar to the preceding but the set $A_\gamma, L_v \cap X$ itself might be empty. We note that we must have $v(\Delta(\gamma)) \geq 1$. We fix a vertex $x_0 \in \text{Fix}_{X_\Theta}(\gamma)$ which is as close as possible (in $X_{L_v}$) to $A_\gamma, L_v$. Then $\text{Fix}_{X_\Theta}(\gamma)$ is contained in the $(R - w)$-neighbourhood of $G_{\gamma}(k_v) \cdot x_0$ where $w = d(x_0, A_\gamma)$. We can then apply the exact same argument as above, which finishes the proof of (11.1).

11.2. Archimedean orbital integrals. In this section $k_v = \mathbb{R}$ or $k_v = \mathbb{C}$, as above $G$ is a semisimple $k_v$-group. $f$ is a function with compact support (say in the ball $B_{G(k_v)}(1, R)$ of radius $R$ around the identity, in a fixed Euclidean norm on $G(k_v)$). We want to estimate the orbital integrals

$$O(\gamma, f) = \int_{G_{\gamma}(k_v) \backslash G(k_v)} f(x^{-1}\gamma x)dx$$

for sufficiently regular $\gamma$. We will prove the following result; for a regular element $g \in G(k_v)$ and $\Phi_\mathbb{R}$ the root system of $G(k_v)$ relative to $G_{\gamma}(k_v)$ we denote

$$\Delta^+(\gamma) = \prod_{\lambda \in \Phi_\mathbb{R}} (1 - \lambda(\gamma)).$$
**Proposition 11.5.** There exists $a_+ \geq a_- > 0$ and $C > 0$, depending only on $G$, on $d_G(1, \gamma)$ and on $R$, such that for any $\mathbb{R}$-regular element $\gamma \in G(k_v)$ (with centraliser $G_\gamma$, in $G$) we have

$$O(\gamma, f) \leq C \cdot \|f\|_\infty \cdot |\Delta^+(\gamma)|.$$  

**Proof.** Let $G = G(k_v)$, $T = G_\gamma(k_v)$ and let $P$ be the minimal parabolic subgroup of $G$ containing the connected component of $T$. Let $A$ be a maximal $\mathbb{R}$-split sub-torus of $T$. Then $P$ has a Langlands decomposition $P = MAN$, with $N$ unipotent and $M$ semisimple. Since we assume $\gamma$ to be $\mathbb{R}$-regular, $A$ is a maximal $\mathbb{R}$-split torus of $G$ by Lemma 2.4 and $M$ is in fact compact.

Let $K$ be a maximal compact subgroup of $G$ containing $M$, such that $A$ is fixed by the Cartan involution of $G$ associated with $K$. Then we have the Iwasawa decomposition $G = KAN$ (see [42, Proposition 7.31]).

We have

$$O(\gamma, f) = \int_{A \backslash G/K} \int_K f(u^{-1}x^{-1}\gamma xu)du dx$$

$$= \int_{A \backslash G/K} \tilde{f}(x^{-1}\gamma x)dx$$

where $\tilde{f}(y) = \int_K f(u^{-1}gu)du$. As $\tilde{f}$ satisfies $\|\tilde{f}\|_\infty \leq \|f\|_\infty$ and its support is contained in $B(1, R + d)$ where $d$ is the diameter of $K$, we will assume in the sequel that $f = \tilde{f}$.

Let $\Phi^+_R$ be the positive roots associated with $N$ in the relative root system of $(G, A)$. We obtain

$$O(\gamma, f) = \int_N f(n^{-1}\gamma n)dn = \int_N f(\gamma \cdot y(n))dn$$

where $y(n) = \gamma^{-1}n^{-1}\gamma \cdot n$. By the proof of Lemma 2.9 we see that $y$ is a diffeomorphism of $N$ and its Jacobian equals $\prod_{\lambda \in \Phi^+_R}(1 - \lambda(\gamma)) = \Delta^+(\gamma)$. The Haar measure on $N$ is given by $\prod_{\lambda \in \Phi^+} dn_\lambda$ and it follows that

$$O(\gamma, f) = \Delta^+(\gamma) \int_N f(\gamma y)dy$$

We can conclude by putting $C = \int_N f(\gamma y)dy/\|f_\infty\| \leq \text{vol } B_N(1, R + d_G(1, \gamma)).$  

**11.2.1. All infinite places.**

**Lemma 11.6.** Let $G/k$ be a $k$-semisimple group. There exists a constant $C$ depending on the absolute type of $G$, on $R$ and on $[k : \mathbb{Q}]$ such that for any strongly regular, $\mathbb{R}$-regular element $\gamma \in G(k)$ with $m(\gamma) \leq R$ we have:

$$\prod_{v \in \mathcal{V}_\infty} |\Delta^+(\gamma)|_v \leq C \prod_{v \in \mathcal{V}_\infty} |\Delta(\gamma)|_v.$$  

**Proof.** We fix an absolute root system $\Phi$ for $(G, G_\gamma)$. For each archimedean place $v \in \mathcal{V}_\infty$ we get a relative root system $\Phi_R$ (which is made up of the restrictions of elements of $\Phi$ to the split part of $G_\gamma(k_v)$). It follows from [6] Propositions 2.2-4 that we can subdivide $\Phi$ into three subsets $\Phi_1, \Phi_2, \Phi_3$ such that:

- $\Phi_1 = \Phi_R \cap \Phi$ ;
- if $\lambda \in \Phi_2$ then $\overline{\lambda} \in \Phi$ and $(\lambda \cdot \overline{\lambda})^{1/2} \in \Phi_R$ ; we choose a set $\Phi_2' \subset \Phi_2$ such that $\Phi_2'$
- is the disjoint union of $\Phi_2'$ and its complex conjugate.
• if $\lambda \in \Phi_2$ then $\lambda$ takes imaginary values on the (real) Lie algebra of the $\mathbb{R}$-split part of $G_\gamma$.

We have
\begin{equation}
\left|\frac{\Delta^+(\gamma)}{\Delta(\gamma)}\right|_v = \prod_{\lambda \in \Phi_2} \frac{1 - |\chi(\gamma)|_v}{|(1 - \lambda(\gamma))(1 - \lambda(\gamma))|} \prod_{\lambda \in \Phi_3} |1 - \lambda(\gamma)|_v^{-1} \ll \prod_{\lambda \in \Phi_2 \cup \Phi_3} |1 - \lambda(\gamma)|_v^{-1}
\end{equation}
where the estimate follows from the facts that
\[
\frac{1 - |z|_v}{|(1 - z)(1 - \overline{z})|} \ll \frac{1}{|1 - z| + |1 + z|}
\]
and if $\lambda \in \Phi_2$ then $-\lambda \in \Phi_2$ as well.

Now we observe that for each $\lambda \in \Phi$, $1 - \lambda(\gamma)$ is a nonzero algebraic integer so that $\prod_{v \in V(k)} |1 - \lambda(\gamma)|_v \geq 1$. On the other hand, since we are assuming that $m(\gamma) \ll R$ there exists some $M$ depending only on $R$ and $X$ such that $|1 - \lambda(\gamma)|_v \leq M$ for all $v$. It follows that $|1 - \lambda(\gamma)|_v \geq M^{-[k:Q]}$, so the product in the rightmost term of (11.4) is bounded, which establishes the lemma. \hfill \Box

12. Estimate for the number of elements of small height

We fix a semisimple Lie group $G$. The main result of this section is the following estimate.

**Theorem 12.1.** For any $d \in \mathbb{N}$, $R > 0$ there exists a constant $C$ depending on $R, d$ and $G$ such that the following holds. For any number field $k$ with $[k : \mathbb{Q}] \leq d$, semisimple $k$-group $G$ with $G(k_\infty)$ isogenous to $G$ times a compact group, and any maximal compact subgroup $U = \prod_{v \in V_f} U_v$ of $G(\mathbb{A}_f)$, the number of conjugacy classes of strongly regular elements $\gamma \in G(k)$ such that $m(\gamma) \leq R$ and whose projection in $G(\mathbb{A}_f)$ is contained in $U$ is bounded by $C$.

We will use the following nomenclature:

• A conjugacy class in $G(k)$ will be called rational;

• A conjugacy class in $G(F)$ will be called geometric.

12.1. Geometric conjugacy classes.

**Lemma 12.2.** There are at most finitely many geometric conjugacy classes of semisimple elements in $G(k) \cap U$ which contain a representative of Mahler measure at most $R$.

**Proof.** Let $T$ be a maximal torus in $G$. Every semisimple element of $G(F)$ is conjugated into $T(F)$, since all maximal tori are conjugated over $F$ and every semisimple element is contained in some maximal torus.

Let $\Phi = \Phi(G, T)$ be the root system. For any $\gamma \in G(k) \cap U$ the polynomial
\[
F_\gamma(t) = \prod_{\lambda \in \Phi} (t - \lambda(\gamma))
\]
characterises the $G(F)$-conjugacy class of $\gamma$ up to finite ambiguity. It belongs to $O_k[t]$ and its Mahler measure is equal to that of $\gamma$. If $d = [k : \mathbb{Q}]$ we thus have that $\prod_{\sigma, k \to \mathbb{C}} F_\gamma^\sigma$ is a monic polynomial in $\mathbb{Z}[t]$ with Mahler measure at most $dR$ and degree equal to $d(\dim(G) - \text{rk}(G))$. By the Northcott property, there are at most finitely many such polynomials (see for example [41, proof of Lemma 3.1]) and this finishes the proof of the lemma. \hfill \Box
12.2. Rational conjugacy classes in a geometric conjugacy class. Let $\gamma$ be an element of $G(k)$ and let $U = \prod_{v \in V_f} U_v$ be a maximal compact subgroup of $G(\mathbb{A}_f)$. Write

$$S(\gamma) = \{[\gamma']_{G(k)} : \gamma' \in [\gamma]_{G(\overline{k})} \cap G(k)\};$$

$$S_U(\gamma) = \{[\gamma']_{G(k)} : \gamma' \in [\gamma]_{G(\overline{k})} \cap G(k), \gamma' \in U\}.$$

We have the following realisation of conjugacy classes as classes in Galois cohomology.

**Lemma 12.3.** There is a bijection $S(\gamma) \rightarrow \ker[H^1(k, G_{\gamma}) \rightarrow H^1(k, G)]$. Explicitly, if $g \in G(\overline{k})$ is such that $[g^{-1}\gamma g]_{G(k)} \in S(\gamma)$ then we send it to the cohomology class $[\sigma \mapsto g^\sigma g^{-1}] \in H^1(k, G_{\gamma})$.

**Proof.** The set of $k$-points in the $[\gamma]_{G(\overline{k})}$ is naturally identified with $G_{\gamma}\backslash G(k)$. The group $G(k)$ acts on $G_{\gamma}\backslash G(k)$ on the right and the set $S(\gamma)$ is identified with the orbits of $G(k)$. By exactness of the sequence (9.1) for $H = G_{\gamma}$, we get a bijection

$$S(\gamma) \simeq \ker[H^1(k, G_{\gamma}) \rightarrow H^1(k, G)].$$

Explicitly, if $g \in G(\overline{k})$ is such that $[g^{-1}\gamma g]_{G(k)} \in C(\gamma, k)$ then it is mapped to the cohomology class $[\sigma \mapsto g^\sigma g^{-1}] \in H^1(k, G_{\gamma})$. \hfill $\square$

In order to pinpoint the image of the subset $S_U(\gamma)$ via the bijection from Lemma 12.3 we will need to use a local-global principle for the Galois cohomology.

12.2.1. Local-global principle. Let us define the local counterparts of the sets $S(\gamma)$ and $S_U(\gamma)$. If $v \in V_\infty \cup V_f$ let

$$S_v(\gamma) = \{[\gamma']_{G(k_v)} : \gamma' \in [\gamma]_{G(\overline{k_v})} \cap G(k_v)\}.$$

In addition, we define

$$S_{U_v}(\gamma) = \{[\gamma']_{G(k_v)} : \gamma' \in [\gamma]_{G(\overline{k_v})} \cap G(k_v), \gamma' \in U_v\}.$$

We have a commutative diagram

\[
\begin{array}{ccc}
\prod_v H^1(k_v, G_{\gamma}) & \rightarrow & H^1(k, G_{\gamma}) \\
\phi & & \phi \\
\Pi_v S_{U_v}(\gamma) & \rightarrow & \Pi_v S_v(\gamma) \\
\end{array}
\]

**Lemma 12.4.** Let $\gamma$ be an $m$-regular element. Then $|S_U(\gamma)| \leq |\prod_v H^1(k_v, G_{\gamma})| \prod_v |S_{U_v}(\gamma)|$.

**Proof.** By Proposition 2.1 the group $G_{\gamma}$ is an algebraic torus so the map $\phi : H^1(k, G_{\gamma}) \rightarrow \prod_v H^1(k_v, G_{\gamma})$ is a group homomorphism with kernel $\prod_v H^1(k_v, G_{\gamma})$. In particular it is $|\prod_v H^1(k_v, G_{\gamma})|$-to-one so

\[
|S_U(\gamma)| \leq \left|\phi^{-1}\left(\prod_v S_{U_v}(\gamma)\right)\right| = |\prod_v H^1(k_v, G_{\gamma})| \prod_v |S_{U_v}(\gamma)|.
\]

$\square$
Using Lemma 12.4 we reduce the estimate on $|S_{U_v}(\gamma)|$ to the estimates on $|S_{U_v}(\gamma)|$ for $\nu \in V_k$ and an upper bound on $|\text{III}^1(G_v)|$. The latter will be provided by Lemma 9.2.

12.2.2. Conjugacy classes at the archimedean places. Let $v \in V_\infty$ and let $\gamma \in G(k_v)$ be an element whose centraliser is a torus. By Lemma 12.3 and the Nakayama–Tate theorem we have that $|S_{U_v}(\gamma)| \leq |H^1(\mathbb{C}/k_v, X^*(G_v))|$. But since $\dim(G_v)$ is bounded independently of $k, v, \gamma$ we have that $|H^1(\mathbb{C}/k_v, X^*(G_v))| < 2^{\dim(G_v)} = O(1)$. In conclusion, since there are at most $d$ Archimedean places we have

\[
(12.3) \prod_{v \in V_\infty} |S_{k_v}(\gamma)| = O(1).
\]

We remark that this statement is in fact independent of our hypothesis that $[k : \mathbb{Q}]$ remains bounded, since it can be proven that for Archimedean $v$ we have $|S_{k_v}(\gamma)| = 1$ as long as $G(k_v)$ is compact.

12.2.3. Conjugacy classes at the non-archimedean places.

**Proposition 12.5.** Let $v \in V_f$ and let $\gamma$ be a regular semi-simple element of $G(k_v)$ such that $|\Delta(\gamma)|_p = 1$ and $G_{\gamma}$ is split by an unramified extension of $k_v$. Then for any open compact subgroup $U_v$ of $G(k_v)$ we have $|S_{U_v}(\gamma)| = 1$.

**Proof.** The first step is the following lemma.

**Lemma 12.6.** Let $\gamma \in G(k_v)$ such that $|\Delta(\gamma)|_v = 1$ and $G_{\gamma}$ is split by a tamely ramified extension of $k_v$. Let $U_v$ be a maximal compact subgroup of $G(k_v)$. The image of $S_{U_v}(\gamma)$ in $\ker[H^1(k_v, G_{\gamma}) \to H^1(k_v, G)]$ consists only of compact classes.

**Proof.** Write $T = G_{\gamma}$. Let $L_v$ be a tamely ramified extension of $k_v$ splitting $T$. Let $[\gamma']|_{G(k_v)}$ be a class in $S_{U_v}(\gamma)$ with $\gamma' \in U_v$. Since $L_v$ splits $T$, we have $H^1(L_v, T) = \{1\}$ by Hilbert’s theorem 90 and so $\gamma$ and $\gamma'$ are conjugate in $G(L_v)$ according to Lemma 12.3.

Choose $g \in G(L_v)$ such that $\gamma' = g^{-1}\gamma g$. We need to show that the the image of $[\gamma']|_{G(k_v)}$ is a compact cohomology class. That amounts to showing that there exists $h \in G(k_v)$ such that $h \in T(L_v)g$ (so the cocycle associated with $h$ is cohomologous to that of $g$) and for all $\sigma \in \text{Gal}(L_v/k_v)$ we have $h^\sigma h^{-1} \in T(L_v)^b$ (so the cocycle is compact).

Let $X_{L_v}, X_{k_v}$ be the Bruhat–Tits buildings of $G(L_v), G(k_v)$ respectively. Then, as the extension $L_v/k_v$ is tamely ramified, it follows from 61 that we have an inclusion of simplicial complexes $X_{k_v} \hookrightarrow X_{L_v}$ and $X_{k_v} = X^{G(L_v/k_v)}_{L_v}$.

Let $\tau$ be the cell of $X_{k_v}$ stabilised by $U_v$ and $\bar{U}_v$ the stabiliser of $\tau$ in $G(L_v)$; this is a compact subgroup of $G(L_v)$. Since $\tau \in X^\text{Gal}(L_v/k_v)_{L_v}$, the group $\bar{U}$ is stable under $\text{Gal}(L_v/k_v)$. We have $\tau = \gamma' \tau$, so $\gamma g \tau = g \tau$. By Lemma 12.7 it follows that $g \in T(L_v)\bar{U}_v$. We write $g = th$ with $h \in \bar{U}_v, t \in T(L_v)$. Then, $h \in T(L_v)g$ and $h^\sigma h^{-1} \in \bar{U}_v$ so that $h^\sigma h^{-1} \in T(L_v)^b$. This concludes the proof. \qedhere

**Lemma 12.7.** If $\gamma \in G(L_v)$ is regular, then there is a unique apartment $A \subset X_{L_v}$ stabilised by $G_{\gamma}$. If in addition $\gamma$ belongs to a compact subgroup and $|\Delta(\gamma)|_v = 1$, then we have that $A = (X_{L_v})^\gamma$.

Moreover if $g \in G(L_v)$ and $\tau$ is a cell in $A$ such that $h \tau \in A$ as well then $h \in G_{\gamma}G(L_v)\text{Stab}_{G(L_v)}\tau$.
Proof. This follows immediately from Lemma 11.4.

Now we can finish the proof of the proposition. By Lemma 12.6 the image of $S_{U_v}(\gamma)$ consists only of compact cocycles in $H^1(k_v, G_\gamma)$. The centralizer $G_\gamma$ is split by an unramified extension of $k_v$, so by Proposition 9.5 the only compact cohomology class in $H^1(k_v, G_\gamma)$ is the trivial class. Hence $|S_{U_v}(\gamma)| = 1$. The proposition is proved.

We will also need a bound to deal with the finite number of places where the hypotheses of Proposition 12.5 are not satisfied. These bounds are deduced immediately from Lemmas 12.3 and 9.3. We record them in the following lemma.

Lemma 12.8. Let $v \in V_f$ and let $\gamma \in U_v$ be a regular semisimple element. We have $|S_{U_v}(\gamma)| \leq C$ where $C$ depends only on the absolute type of $G$.

12.3. Global conjugacy classes. We conclude here the proof of Theorem 12.1. Recall that $G$ is a $k$-group with $G(k_\infty)$ isomorphic to a fixed Lie group and $\gamma \in G(k)$ is strongly regular, satisfies $m(\gamma) \leq R$ and generates a compact subgroup at all non-archimedian places.

Define the following sets of finite places:

$S_1 = \{v \in V_f : |\Delta(\gamma)|_v \neq 1\};
S_2 = \{v \in V_f : G_\gamma \text{ is not split by an unramified } L_v/k_v\}.$

By Lemma 12.4, Lemma 12.3, Proposition 12.5 and Lemma 12.8, we have that $|S_{U_v}(\gamma)| \ll C^{|S_1|+|S_2|}$ for some $C$ depending only on $G$. Therefore, we must prove that $|S_1|, |S_2| = O_{R,G}(1)$.

For $S_1$ this is easy: we have $\Delta(\gamma) \in o_k$, so

$$2^{|S_1|} \leq \prod_{v \in S_1} q_v \leq N_k/q^\Delta(\gamma) \leq e^R.$$ 

In particular, $|S_1| \leq R/\log(2)$.

For $S_2$ we note that if $L$ is the minimal Galois extension of $k$ splitting $G_\gamma$, then

$$\prod_{v \in S_2} p_v = \Delta_{L/k}.$$

By Proposition 2.2 this is bounded by a constant depending only on $R, d$ and the absolute type of $G$.

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