ON A CONJECTURED REVERSE FABER-KRAHN INEQUALITY FOR A STEKLOV–TYPE LAPLACIAN EIGENVALUE

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Abstract. For a given bounded Lipschitz set $\Omega$, we consider a Steklov–type eigenvalue problem for the Laplacian operator whose solutions provide extremal functions for the compact embedding $H^1(\Omega) \hookrightarrow L^2(\partial \Omega)$. We prove that a conjectured reverse Faber–Krahn inequality holds true at least in the class of Lipschitz sets which are “close” to a ball in a Hausdorff metric sense. The result implies that among sets of prescribed measure, balls are local minimizers of the embedding constant.

1. Introduction

For any given open bounded Lipschitz set $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) the compact trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial \Omega)$ allows us to define the positive quantity

$$
\lambda(\Omega) = \min_{w \in H^1(\Omega), w \neq 0} \frac{\int_\Omega |Dw|^2 \, dx + \int_\Omega w^2 \, dx}{\int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1}},
$$

so that any extremal function $u$ (a function achieving the minimum in (1)) is the solution to a Steklov–type eigenvalue problem

$$
\begin{cases}
-\Delta u + u = 0 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = \lambda(\Omega) \, u & \text{on } \partial \Omega,
\end{cases}
$$

where $\nu$ is the outer unit normal to $\partial \Omega$. Problem (2) has been widely investigated for instance in [19, 23, 24, 25, 36, 40]. The eigenvalue $\lambda(\Omega)$ is the reversed squared norm of the trace embedding operator $T_\Omega : H^1(\Omega) \to L^2(\partial \Omega)$, and very often such a norm is also called “sharp embedding constant” drawing the attention to the fact that it is the smallest possible constant $C$ for which the Sobolev–Poincaré trace inequality

$$
\|w\|_{L^2(\partial \Omega)} \leq C \|w\|_{H^1(\Omega)}
$$

holds true for all $w \in H^1(\Omega)$.

Motivated by the study of the optimal embedding constants, and following the intuition that among sets of given volume $\lambda(\cdot)$ might be maximal on balls, J.D. Rossi proved (see [40]) that the ball is indeed a stationary set for the shape functional $\lambda(\cdot)$

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under smooth volume preserving perturbation of the domain. More recently A. Henrot\textsuperscript{1}, in analogy to the celebrated Brock and Weinstock inequality [10, 46] for the Steklov eigenvalue, proposed the following conjecture

**Conjecture 1.** For any given open bounded Lipschitz set $\Omega$, then

$$\lambda(\Omega) \leq \lambda(\Omega^2),$$

and therefore, among open bounded Lipschitz sets of given measure, the ball achieves the worst (least) embedding constant in (3).

As usual by $\Omega^2$ we denote the ball centered in the origin and having the same Lebesgue measure ($\mathcal{L}^n$) as $\Omega$.

Our main result is that balls are indeed local maximizers of $\lambda$ in the $L^\infty$ topology. We prove that:

**Theorem 1.1** (Main Theorem). Let $\Omega \subset \mathbb{R}^n$ be an open bounded Lipschitz set, then

$$\lambda(\Omega) \leq \lambda(\Omega^2),$$

provided up to translations

$$\{x \in \mathbb{R}^n : \text{dist}(x, \mathbb{R}^n \setminus \Omega^2) > \delta\} \subset \Omega \subset \{x \in \mathbb{R}^n : \text{dist}(x, \Omega^2) < \delta\},$$

for some positive constant $\delta$ that depends on $\mathcal{L}^n(\Omega)$ and $n$ only. Moreover equality holds in (4), under the constraint (5), if and only if $\Omega$ is a ball.

**Remark 1.** It is important to warn the reader that since $\lambda$ is not scaling invariant then $\delta$, in Theorem 1.1, has a genuine dependence on $\mathcal{L}^n(\Omega)$.

Isoperimetric inequalities for eigenvalues of elliptic operators (Laplacian above all) is an active field of research [30, 32]. Without presuming to give an exhaustive picture on the state-of-the-art, we consider just four remarkable examples (see for instance [5, 7, 10, 12, 16, 17, 18, 22, 30, 31, 32, 33, 41, 45, 46] for more details):

**(Faber–Krahn)**

$$\lambda_D(\Omega) = \min_{\substack{w \in H^1_0(\Omega) \setminus \{0\}}} \frac{\int_{\Omega} |Dw|^2 \, dx}{\int_{\Omega} w^2 \, dx}, \quad \lambda_D(\Omega) \geq \lambda_D(\Omega^2)$$

**(Szegö–Weinberger)**

$$\lambda_N(\Omega) = \min_{\substack{w \in H^1(\Omega) \setminus \{0\}, \int_{\Omega} w = 0}} \frac{\int_{\Omega} |Dw|^2 \, dx}{\int_{\Omega} w^2 \, dx}, \quad \lambda_N(\Omega) \leq \lambda_N(\Omega^2)$$

**(Brock–Weinstock)**

$$\lambda_S(\Omega) = \min_{\substack{w \in H^1(\Omega) \setminus \{0\}}} \inf_{c} \frac{\int_{\Omega} |Dw|^2 \, dx}{\int_{\partial \Omega} (w - c)^2 \, d\mathcal{H}^{n-1}}, \quad \lambda_S(\Omega) \leq \lambda_S(\Omega^2)$$

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(Bossel–Daners)

\[ \lambda_R(\Omega) = \min_{\substack{w \in H^1(\Omega) \\ w \neq 0}} \frac{\int_{\Omega} |Dw|^2 \, dx + \alpha \int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1}}{\int_{\Omega} w^2 \, dx}, \quad \lambda_R(\Omega) \geq \lambda_R(\Omega^\sharp) \text{ if } \alpha \geq 0. \]

Each of the previous minimization provides the least positive eigenvalue of the Laplacian for a specific boundary condition. The subscript \( D, N, S, R \) stand for Dirichlet, Neumann, Steklov, Robin, boundary conditions. Each of these eigenvalues is related to some embedding constant for an inequality of Sobolev–Poincaré type. In all the cases, among sets of given measure, balls are extremals. For more general optimal inequalities of Sobolev–Poincaré type we refer for instance to [37] and, among the others, also to some recent results in [4, 13, 14, 20, 21, 26, 35, 44].

Interestingly enough, very little is known about the Bossel–Daners inequality for negative \( \alpha \) (see [17]). On the other hand, by trivial scaling arguments and monotonicity of \( \lambda_R(\cdot) \) with respect to \( \alpha \), we will show that Conjecture 1 is equivalent to the following one.

**Conjecture 2.** For any given open bounded Lipschitz set \( \Omega \) and \( \alpha < 0 \), then

\[ \lambda_R(\Omega) \leq \lambda_R(\Omega^\sharp). \]

Actually, Conjecture 2 has been addressed in 1977 by Bareket in [6], where she also provided a partial answer in two dimensions: for a given smooth simply connected set which is “nearly circular” there exists \( \alpha < 0 \), with \( |\alpha| \) small enough, such that (6) holds true. To our knowledge, Conjecture 2 is still open (see, for instance, [11]). Furthermore, very recently it has been remarked that so far it is still unknown whether balls have any kind of local maximizing property (for \( \lambda_R(\cdot) \) with \( \alpha < 0 \))^2. From this point of view Theorem 1.1 provides a positive answer to the last question, that can be summarized in the following statement.

**Corollary 1.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded Lipschitz set, then

\[ \lambda_R(\Omega) \leq \lambda_R(\Omega^\sharp), \]

for a given \( \alpha < 0 \), provided up to translations

\[ \{ x \in \mathbb{R}^n : \text{dist}(x, \mathbb{R}^n \setminus \Omega^\sharp) > \delta \} \subset \Omega \subset \{ x \in \mathbb{R}^n : \text{dist}(x, \Omega^\sharp) < \delta \}, \]

for some positive constant \( \delta \) that depends on \( L^n(\Omega) \), \( n \) and \( \alpha \) only. Moreover equality holds in (7), under the constraint (8), if and only if \( \Omega \) is a ball.

Now, concerning the proof of our results, the maximization of the eigenvalue \( \lambda \) among Lipschitz sets is performed introducing a weighted isoperimetric problem which involves modified Bessel functions. It could be of some interest to mention that inequality \( \lambda_D(\Omega) \geq \lambda_D(\Omega^\sharp) \) was conjectured by Lord Rayleigh in 1877 and, as described in [18],

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2This open question was raised by the Working Group on “Low eigenvalues of Laplace operator”, during the Workshop on “New trends in shape optimization” held at De Giorgi Center, Pisa, July 2012. http://www.lama.univ-savoie.fr/ANR-GAOS/CRM%20Pisa/index.html
he provided a proof in the case of nearly circular sets (in the plane) using perturbation series involving Bessel functions.

We start by considering the function
\[ z(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|), \quad x \in \mathbb{R}^n, \]
where \( J_{\nu} \) denotes the modified Bessel function of order \( \nu \) (see for instance [1]). When \( \Omega = \Omega^\sharp \), \( z \) is the extremal function in (1), unique up to a multiplicative factor ([36]). Actually \( z \) is an analytic function in the whole \( \mathbb{R}^n \) where it solves
\[ -\Delta z + z = 0, \]
and it can be used as a test function in (1) even when \( \Omega \neq \Omega^\sharp \). For any given bounded Lipschitz set \( \Omega \) we define the notions of weighted volume and weighted perimeter by
\[ V(\Omega) = \int_\Omega (|Dz|^2 + z^2) \, dx, \]
and
\[ P(\Omega) = \int_{\partial \Omega} z^2 \, dH^{n-1}. \]

Then, by testing the right hand side of (1) with \( z \), we get
\[ \lambda(\Omega) \leq \frac{V(\Omega)}{P(\Omega)}, \]
with equality at least in the case \( \Omega = \Omega^\sharp \).

This naturally suggests a way to prove Conjecture 1 by looking for the inequality
\[ (9) \quad \frac{V(\Omega)}{P(\Omega)} \leq \frac{V(\Omega^\sharp)}{P(\Omega^\sharp)}. \]
Unfortunately in general (9) is false. For instance, let us consider \( x_0 \in \mathbb{R}^n \setminus \{0\} \). For what we have said before the function \( z(x - x_0) \) is the unique extremal in (1) when \( \Omega = B_\rho(x_0) \) is the ball of radius \( \rho \) centered at \( x_0 \), and in this case we have
\[ \frac{V(B_\rho(x_0))}{P(B_\rho(x_0))} > \lambda(B_\rho(x_0)) = \lambda(B_\rho(0)) = \frac{V(B_\rho(0))}{P(B_\rho(0))}. \]

We have used that \( \lambda(\cdot) \) is invariant under translations, and the resulting inequality emphasizes that the same is not true for the ratio \( \frac{V(\cdot)}{P(\cdot)} \). As a consequence, if there is any hope to prove (9), then it is crucial to carefully choose a suitable reference system for each \( \Omega \).

The paper is divided into two parts.

**Part 1.** In Section 2 we give a proof of Conjecture 1, by way of (9), for nearly spherical sets. To formulate the result we first need some definition.

**Definition 1.2 (\( \mathcal{N}(n, \varepsilon) \) functions).** For given \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) we denote by \( \mathcal{N}(n, \varepsilon) \) the set of functions \( v \in W^{1,\infty}(\mathbb{S}^{n-1}) \) such that
1.) \( \|v\|_{W^{1,\infty}} \leq \varepsilon \)

2.) \( \frac{1}{n} \int_{\mathbb{S}^{n-1}} (1 + v(\xi))^n \, d\sigma_\xi = \omega_n \) (Volume constraint)

3.) \( \int_{\mathbb{S}^{n-1}} (1 + v(\xi))^{n+1} \, d\sigma_\xi = 0 \) (Barycenter constraint).

Here we denote by \( \sigma_\xi \) the surface area measure on \( \mathbb{S}^{n-1} \) and we denote as usual by \( \omega_n \) the volume of the unit ball in \( \mathbb{R}^n \).

Then we need the notion of nearly spherical sets (see also [28]).

**Definition 1.3** \(((n, \omega, \varepsilon)\text{-}NS \text{ sets})\). Let \( \omega > 0 \) and \( 0 \leq \varepsilon \leq 1 \). We say that an open set \( \Omega \subset \mathbb{R}^n \) is \((n, \omega, \varepsilon)\text{-}NS \) (that is Nearly Spherical) if there exists \( v \in \mathcal{N}(n, \varepsilon) \) such that, possibly up to a translation, the boundary of \( \Omega \) in polar coordinates \( (r, \xi) \in [0, \infty) \times \mathbb{S}^{n-1} \), can be represented as

\[ r(\xi) = \rho(1 + v(\xi)). \]

Here \( \rho = (\omega/\omega_n)^{1/n} \) is the radius of the ball having same measure as \( \Omega \).

It is clear now why we labeled 2.) and 3.) by Volume constraint and Barycenter constraint respectively. In fact the Volume constraint condition implies that when \( \Omega \) is \((n, \omega, \varepsilon)\text{-}NS \) than \( \mathcal{L}^n(\Omega) = \omega \), while the Barycenter constraint condition establishes that in the reference frame where the boundary of \( \Omega \) is represented as

\[ r(\xi) = \rho(1 + v(\xi)) \]

then the barycenter of \( \Omega \) is placed in the origin.

The main statement of the section is the following.

**Theorem 1.4.** In any dimension \( n \) and for every positive constant \( \omega \) there exist two constants \( \varepsilon \) and \( K \) depending just on \( n \) and \( \omega \) such that, if \( \Omega \) is any \((n, \omega, \varepsilon)\text{-}NS \) set and \( v(\xi) \in \mathcal{N}(n, \varepsilon) \) provides the polar representation of its boundary, then

\[
\lambda(\Omega^\sharp) = \frac{V(\Omega^\sharp)}{P(\Omega^\sharp)} \geq \frac{V(\Omega)}{P(\Omega)} \left( 1 + K(n, \omega) \int_{\mathbb{S}^{n-1}} v^2(\xi) \, d\sigma_\xi \right) \\
\geq \lambda(\Omega) \left( 1 + K(n, \omega) \int_{\mathbb{S}^{n-1}} v^2(\xi) \, d\sigma_\xi \right).
\]

We explicitly observe that the above theorem provides an estimate on how close \( \Omega \) is to a ball in terms of the eigenvalue gap. Recent results for the first Dirichlet eigenvalue in this direction are contained, for example, in [8, 9, 29, 39].

**Part 2.** In Section 3 we prove Theorem 1.1. Our ultimate goal is to prove that for any given \( \omega > 0 \) there exists \( \delta > 0 \) such that the ball centered in the origin is the unique minimizer of

\[
P(\Omega) - \frac{P(\Omega^\sharp)}{V(\Omega^\sharp)} V(\Omega)
\]

among all finite perimeter sets \( \Omega \), having Lebesgue measure \( \mathcal{L}^n(\Omega) = \omega \), barycenter \( X(\Omega) \) in the origin, and such that \( B_{\rho-\delta} \subset \Omega \subset B_{\rho+\delta} \). Obviously this approach first require a natural generalization of notion of \( V \) and \( P \) for finite perimeter sets. We
end up then with a subtle weighted constrained isoperimetric problem which can not be elementarily tackled. However we replace such a constrained minimization by a penalized one, following an original powerful idea which has been introduced in [15] to prove a stability result for the classical isoperimetric inequality. Since then, such a technique has been further developed and refined in several different contexts, but among the others our construction is reminiscent of the one worked out in [2].

The core of the strategy is to show that the previous minimization is indeed equivalent to minimizing the functional

$$P(\Omega) - \frac{P(\Omega^\sharp)}{V(\Omega^\sharp)} V(\Omega) + A_1 |X(\Omega)| + A_2 |\mathcal{L}^n(\Omega) - \omega| + A_3 (V(\Omega \setminus B_{\rho+\delta}) + V(B_{\rho-\delta}\setminus\Omega))$$

for some choice of the positive constants $A_1$, $A_2$ and $A_3$.

For such a functional it is easy to show that for any given $\omega > 0$ and $\delta > 0$, every minimizer is a so called almost-minimizer for the Euclidean perimeter. We refer the interested reader to [3, 42, 43], as well as to [34] and the references therein. For our purposes it is enough to recall that a finite perimeter set $\Omega$ is an almost-minimizer for the Euclidean perimeter $\text{Per}(\Omega)$ if there exist two positive constants $K$ and $r_0$, such that

$$\text{Per}(\Omega) \leq \text{Per}(\tilde{\Omega}) + K r^n,$$

whenever $\Omega \Delta \tilde{\Omega} \subset B_r(x_0)$ and $0 < r < r_0$. An almost-minimizer for the perimeter $\Omega$ has reduced boundary which is a $C^{1,1/2}$ hypersurface. Moreover if $\Omega_h$ is any sequence of almost-minimizers with uniform constants $K$ and $r_0$, converging to a ball in $L^1$ as $h \to \infty$, then $\Omega_h$ is nearly spherical in the sense of Definition 1.3 for sufficiently large $h$. Furthermore if $v_h(\xi) \in \mathcal{N}$ is the corresponding function which, according to Definition 1.3, provides the polar representation of $\partial \Omega$, then $v_h \to 0$ in $C^{1,\alpha}(\mathbb{S}^{n-1})$ for every $\alpha \in (0,1/2)$, as $h \to \infty$.

Combining all these results, we have that for any given positive $\omega$, if $\delta > 0$ is small enough, every minimizer of the constrained functional (10) is a nearly spherical set. Therefore in view of Theorem 1.4, possibly choosing a smaller $\delta$, the minimizer is also unique and it is a ball.

2. Proof of Theorem 1.4

Throughout this section, whenever we consider a nearly spherical set, we will choose a reference frame such that the barycenter is set in the origin. Since $\lambda(\cdot)$ is invariant under translations our choice bears by no means a loss of generality.

For the sake of simplicity we split the proof in several steps. Throughout the paper, $B_r = B_r(0)$, with $r > 0$, and, given $\omega > 0$, we denote $\rho = (\omega/\omega_n)^{1/n}$, so that $B_{\rho}$ belongs to $(n,\omega,\varepsilon)-\text{NS}$.

We set

$$h_{\rho}(t) = \left((t\rho)^{1-\frac{2}{n}} I_{\mathbb{R}^{n-1}}(t\rho)\right)^2$$

and

$$f_{\rho}(t) = \frac{h_{\rho}'(t)}{2\rho} = (t\rho)^{2-n} I_{\mathbb{R}^{n-1}}(t\rho) I_{\mathbb{R}^{n}}(t\rho),$$

for some choice of the positive constants $A_1$, $A_2$ and $A_3$.
where we have used the first one of the following derivation rules for Bessel functions
\[ I'_\nu(s) = \frac{\nu}{s} I_{\nu}(s) + I_{\nu+1}(s) \quad s \in \mathbb{R}, \]
\[ I'_{\nu+1}(s) = I_{\nu}(s) - \frac{\nu + 1}{s} I_{\nu+1}(s) \quad s \in \mathbb{R}. \]

Before entering into the details we recall that (see for instance [1]) the functions \( s^{1-n/2}I_{\frac{n}{2}-1}(s) \) and \( s^{1-n/2}I_{\frac{n}{2}}(s) \) are increasing and analytic in \((0, \infty)\) and for all \( \alpha \in \mathbb{R} \) and \( n \in \mathbb{N} \) we have
\[ \lim_{s \to 0} s^{1-n/2}I_{\frac{n}{2}-1}(s) = \frac{1}{2\pi \Gamma(\frac{n}{2})}, \]
\[ \lim_{s \to +\infty} s^{\alpha}I_{\frac{n}{2}-1}(s) = +\infty. \]

Our starting point is the following estimate.

**Lemma 2.1.** If for some \( n \in \mathbb{N}, \omega > 0 \) and \( 0 < \varepsilon < 1 \) a set \( \Omega \) belongs to \((n, \omega, \varepsilon)\)-NS and \( v \in \mathcal{N}(n, \varepsilon) \) is the function that describes its boundary then
\[ \lambda(\Omega) \leq \frac{V(\Omega)}{P(\Omega)} = \frac{\int_{S^{n-1}} f_\rho(1 + v(\xi)) (1 + v(\xi))^{n-1} d\sigma_\xi}{\int_{S^{n-1}} h_\rho(1 + v(\xi)) (1 + v(\xi))^{n-1} \sqrt{1 + \frac{|Dv(\xi)|^2}{(1 + v(\xi))^2}} d\sigma_\xi}. \]
Moreover if \( \Omega = \Omega^\sharp \equiv B_\rho \) then equality holds in (18) and \( \lambda(\Omega^\sharp) = \frac{f_\rho(1)}{h_\rho(1)}. \)

**Proof.** The statement follows at once from the variational formulation of \( \lambda(\Omega) \) as
\[ \lambda(\Omega) = \min_{w \in H^1(\Omega)} \frac{\int_{\Omega} (|Dw|^2 + w^2) \, dx}{\int_{\partial\Omega} w^2 \, dH^{n-1}}, \]
and the definition of \( P(\Omega) \) and \( V(\Omega) \) given in the introduction.

Indeed, if \( z(x) = |x|^{1-\frac{n}{2}}I_{\frac{n}{2}-1}(|x|) \), one has
\[-\Delta z(x) + z(x) = 0 \quad \forall x \in \mathbb{R}^n \]
and
\[ \lambda(\Omega) \leq \frac{V(\Omega)}{P(\Omega)} = \int_{\Omega} (|Dz|^2 + z^2) \, dx \int_{\partial\Omega} \frac{\partial z}{\partial \nu} z \, dH^{n-1} \int_{\partial\Omega} z^2 \, dH^{n-1}, \]
where \( \nu \) is the unit outer normal to \( \partial\Omega. \)

Then, to go from (20) to (18) we just use the explicit representation of \( u \) in terms of modified Bessel functions together with the explicit representation of the boundary of \( \Omega \) in terms of \( v(\xi) \) according to Definition 1.3.
Finally, we observe that $z(x) = |x|^{1 - \frac{n}{2}} I_{\mathbb{R}^n_+}(|x|)$ is precisely the first eigenfunction of (2) when $\Omega$ is any ball centered in the origin. Indeed, $z(x)$ achieves the minimum on the right hand side of (19) anytime $\Omega = \Omega^\sharp$. □

The following Lemma, whose proof is included for completeness (see also [28]), holds.

**Lemma 2.2.** Let $n \in \mathbb{N}$. There exists a positive constant $C$ which depends on $n$ only, such that for any given $0 < \varepsilon < 1$ and for all $v \in \mathcal{N}(n, \varepsilon)$ then

$$(E1) \quad |(1 + v)^{n-1} - (1 + (n - 1)v + (n - 1)(n - 2)\frac{v^2}{2})| \leq C\varepsilon v^2 \quad \text{on } S^{n-1}.$$ 

$$(E2) \quad 1 + \frac{|Dv|^2}{2} - \sqrt{1 + \frac{|Dv|^2}{(1 + v)^2}} \leq C\varepsilon (v^2 + |Dv|^2) \quad \text{on } S^{n-1}.$$ 

Moreover

$$(21) \quad \left| \int_{S^{n-1}} v(\xi) d\sigma_\xi + \frac{n-1}{2} \int_{S^{n-1}} v^2(\xi) d\sigma_\xi \right| \leq C\varepsilon \|v\|^2_{L^2}.$$ 

**Proof.** Although the same constant $C$ appears in (E1), (E2) and (21), throughout this proof, for the sake of simplicity, the constant $C$ is meant to be any constant that may be determined in terms of $n$ alone.

That said, inequality (E1) follows immediately from

$$|1 + v)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} v^k,$$

using the fact that $|v| \leq \varepsilon$.

On the other hand, to prove (E2) we just use the trivial inequality $\sqrt{1 + s} \geq 1 + \frac{s}{2} - \frac{s^2}{4}$ which holds true for all nonnegative $s$, to get

$$\sqrt{1 + \frac{|Dv|^2}{(1 + v)^2}} \geq 1 + \frac{|Dv|^2}{2(1 + v^2)} - \frac{|Dv|^4}{4(1 + v^2)^2} \geq 1 + \frac{|Dv|^2}{2(1 + \varepsilon^2)} - \frac{|Dv|^4}{4} \geq 1 + \frac{|Dv|^2}{2} - C\varepsilon (v^2 + |Dv|^2).$$

Finally, since $v \in \mathcal{N}(n, \varepsilon)$ we know that

$$\frac{1}{n} \int_{S^{n-1}} (1 + v(\xi))^n d\sigma_\xi = \omega_n,$$

and integrating over $S^{n-1}$ the identity

$$(1 + v)^n = \sum_{k=0}^n \binom{n}{k} v^k,$$

we easily get also inequality (21). □

A trivial consequence of the analyticity of $f_\rho(\cdot)$ and $h_\rho(\cdot)$ is the following Lemma.
Lemma 2.3. Let \( n \in \mathbb{N} \) and \( \omega > 0 \), and let \( f \) and \( h \) be the functions defined in (12) and (13). There exists a positive constant \( K \) depending on \( n \) and \( \omega \) alone such that, for any given \( 0 < \varepsilon < 1 \) and for all \( v \in \mathcal{N}(n, \varepsilon) \) then

\[
\begin{align*}
(T1) \ |h_p(1 + v) - h_p(1) - h'_p(1)v - h''_p(1)\frac{v^2}{2}| & \leq K\varepsilon v^2 \quad \text{on } \mathbb{S}^{n-1} \\
(T2) \ |f_p(1 + v) - f_p(1) - f'_p(1)v - f''_p(1)\frac{v^2}{2}| & \leq K\varepsilon v^2 \quad \text{on } \mathbb{S}^{n-1}
\end{align*}
\]

Lemma 2.4 (Poincaré Inequality [28]). Let \( n \in \mathbb{N} \), then there exists a suitable positive constant \( C \) such that for any given \( 0 < \varepsilon < 1 \) and for all \( v \in \mathcal{N}(n, \varepsilon) \) then

\[
\|Dv\|^2_{L^2} \geq 2n(1 - C\varepsilon)\|v\|^2_{L^2}.
\]

Remark 2. We also notice that for any given function \( v(\xi) \in H^1(\mathbb{S}^{n-1}) \) a well known Poincaré Inequality holds

\[
\|Dv\|^2_{L^2} \geq (n - 1)\|v - \bar{v}\|^2_{L^2},
\]

where \( \bar{v} \) is the average of \( v \) over \( \mathbb{S}^{n-1} \).

Unluckily inequality (23) does not satisfy our needs since the constant \( n - 1 \) is not large enough to go further in the proof of Proposition 2.5. Indeed we will take advantage of the fact that we are working with functions belonging to \( \mathcal{N}(n, \varepsilon) \) which is a proper subset of \( H^1(\mathbb{S}^{n-1}) \), and therefore we are able to reach the better (larger) constant in (22). Indeed this Lemma means that we can get as close to the embedding constant \( 2n \) as we wish, provided \( \varepsilon \) is chosen small enough.

Proof of Lemma 2.4. As before, for the sake of simplicity, throughout the proof the constant \( C \) is meant to be any constant that may be determined in terms of \( n \) alone.

The proof is included for completeness (with a slightly different notation can be found also in [28]).

We recall (see for instance [38]) that \( v(\xi) \in L^2(\mathbb{S}^{n-1}) \) admits a Fourier expansion, in the sense that there exists a family of spherical harmonics \( \{Y_k(\xi)\}_{k \in \mathbb{N}} \) which satisfy for all \( k \geq 0 \)

\[-\Delta Y_k = k(k + n - 2)Y_k \quad \text{and} \quad \|Y_k\|_{L^2} = 1,
\]

such that

\[
v(\xi) = \sum_{k=0}^{+\infty} a_k Y_k(\xi) \quad \xi \in \mathbb{S}^{n-1}
\]

and the coefficients \( a_k \) are the projections of \( v \) onto the subspaces spanned by \( Y_k \):

\[
a_k = \int_{\mathbb{S}^{n-1}} v(\xi)Y_k(\xi)\,d\sigma_\xi,
\]

so that

\[
\|v\|^2_{L^2} = \sum_{k=0}^{+\infty} a_k^2.
\]

Notice that \( Y_0 = (n\omega_n)^{-1/2} \), therefore (21) implies

\[
|a_0| = (n\omega_n)^{-1/2} \left| \int_{\mathbb{S}^{n-1}} v(\xi)\,d\sigma_\xi \right| \leq (n\omega_n)^{-1/2} \left| \int_{\mathbb{S}^{n-1}} v^2(\xi)\,d\sigma_\xi \right| \left( \frac{n-1}{2} + C\varepsilon \right).
\]
Up to renaming the constant $C$, we get

$$|a_0| \leq C\varepsilon\|v\|_{L^2}.$$ 

Now, multiplying the identity

$$(1 + v)^{n+1} = \sum_{k=0}^{n+1} \binom{n}{k} v^k,$$ 

by $Y_1$ and integrating we get

$$|a_1| = \left| \int_{S^{n-1}} Y_1(\xi)(1 + v(\xi))^{n+1} \, d\sigma_\xi \right| \geq C\varepsilon\|v\|_{L^2}.$$ 

(24) Here we used Hölder’s inequality and the fact that the integral of $Y_1$ over $S^{n-1}$ is zero. Since in our notation the function $Y_1(\xi) = \xi \cdot \tau$ for some suitable vector $\tau$, then the Barycenter constraint 3.) implies that the lefthand side of (24) vanishes and

Finally we can conclude the proof observing on one hand that

$$\|v\|_{L^2}^2 = \sum_{k=0}^\infty a_k^2 = a_0^2 + a_1^2 + \sum_{k=2}^\infty a_k^2 \leq C\varepsilon\|v\|_{L^2}^2 + \sum_{k=2}^\infty a_k^2$$ 

and on the other hand

$$\|Dv\|_{L^2}^2 = \sum_{k=1}^\infty k(k + n - 2)a_k^2 \geq \sum_{k=2}^\infty k(k + n - 2)a_k^2 \geq 2n \sum_{k=2}^\infty a_k^2.$$ 

\[\Box\]

**Proposition 2.5.** For any given $n \in \mathbb{N}$ and $\omega > 0$, there exist two positive constants $K > 0$ and $0 < \varepsilon_0 < 1$ depending on $n$ and $\omega$ only, such that for all $0 < \varepsilon < \varepsilon_0$ and $\Omega \in (n, \omega, \varepsilon)$–NS, then

$$\frac{V(\Omega^\sharp)P(\Omega) - P(\Omega^\sharp)V(\Omega)}{n\omega_n} =$$

$$f_\rho(1) \int_{S^{n-1}} h_\rho(1 + v(\xi)) (1 + v)^{n-1} \sqrt{1 + \frac{|Dv(\xi)|^2}{(1 + v(\xi))^2}} \, d\sigma_\xi -$$

$$h_\rho(1) \int_{S^{n-1}} f_\rho(1 + v(\xi)) (1 + v(\xi))^{n-1} \, d\sigma_\xi \geq K \int_{S^{n-1}} v^2 \, d\sigma_\xi$$

whenever $v \in \mathcal{N}(n, \varepsilon)$ provides the polar representation of $\partial\Omega$ and $h_\rho$ and $f_\rho$ are the functions defined in (12) and (13).

**Proof of Proposition 2.5.** In what follows $K_1$ and $K_2$ are meant to be constants that may be determined in terms of $n$ and $\omega$ alone.
By using Lemmata 2.2, 2.3 and 2.4 we have

\[ f_\rho(1) \int_{S^{n-1}} h_\rho(1 + v) (1 + v)^{n-1} \sqrt{1 + \frac{|Dv|^2}{(1 + v)^2}} \, d\sigma - h_\rho(1) \int_{S^{n-1}} f_\rho(1 + v) (1 + v)^{n-1} \, d\sigma \]

\[ \geq \int_{S^{n-1}} v \left[ f_\rho(1) h_\rho'(1) - f_\rho'(1) h_\rho(1) \right] \, d\sigma \]

\[ + \int_{S^{n-1}} \frac{v^2}{2} \left[ f_\rho(1) h_\rho''(1) - f_\rho'(1) h_\rho(1) + 2(n - 1) \left( f_\rho(1) h_\rho'(1) - f_\rho'(1) h_\rho(1) \right) \right] \, d\sigma \]

\[ + \int_{S^{n-1}} h_\rho(1) f_\rho(1) \frac{|Dv|^2}{2} \, d\sigma - \varepsilon K_1 \| Dv \|_{L^2}^2 \]

\[ \geq \frac{1}{4n} \left[ (n - 1) \left( f_\rho(1) h_\rho''(1) - f_\rho'(1) h_\rho(1) \right) + (f_\rho(1) h_\rho''(1) - f_\rho'(1) h_\rho(1)) \right] \]

\[ + 2 n h_\rho(1) f_\rho(1) - \varepsilon K_2 \] \[ \| Dv \|_{L^2}^2 \]

provided \( \varepsilon \leq \varepsilon_0 \) and \( \varepsilon_0 \) is small enough.

Since \( K_2 \) does not depend on \( \varepsilon \) then \( \varepsilon K_2 \) can be chosen arbitrarily small provided \( \varepsilon_0 \) is small enough and the proof is complete if we just prove that

\[ (n - 1) \left( f_\rho(1) h_\rho''(1) - f_\rho'(1) h_\rho(1) \right) + (f_\rho(1) h_\rho''(1) - f_\rho'(1) h_\rho(1)) \]

\[ + 2 n h_\rho(1) f_\rho(1) > 0 \]

for all \( \rho > 0 \) (i.e. for all \( \omega > 0 \)).

Let us define \( \nu = \frac{n}{2} - 1 \) and for convenience let us rewrite inequality (25) as:

\[ \left[ \left( f_\rho(t) \frac{d}{dt} h_\rho(t) - h_\rho(t) \frac{d}{dt} f_\rho(t) \right) \right]' + 4(\nu + 1)t^{(2\nu - 1)} h_\rho(t) f_\rho(t) > 0 \]

in \( t = 1 \) and for all \( \rho > 0 \).

We multiply the left hand side of (26) by \( \rho^{2\nu - 1} \), and after the change of variables \( s = \rho t \) the inequality becomes

\[ \frac{d}{ds} \left[ s^{(2\nu+1)} \left( f_\rho \left( \frac{s}{\rho} \right) \frac{d}{ds} h_\rho \left( \frac{s}{\rho} \right) - h_\rho \left( \frac{s}{\rho} \right) \frac{d}{ds} f_\rho \left( \frac{s}{\rho} \right) \right) \right] \]

\[ + 4(\nu + 1)s^{(2\nu - 1)} h_\rho \left( \frac{s}{\rho} \right) f_\rho \left( \frac{s}{\rho} \right) > 0 \]

to be proven in \( s = \rho \) and for all \( \rho > 0 \).

According to (12) and (13) then

\[ h_\rho \left( \frac{s}{\rho} \right) = s^{-2\nu} T_\nu^2(s) \]

and

\[ f_\rho \left( \frac{s}{\rho} \right) = s^{-2\nu} T_\nu(s) T_{\nu+1}(s). \]

In the following we will repeatedly use (14) and (15), according to which
\[
\frac{d}{ds} \left( s^{-\nu} \mathcal{I}_{\nu} \right) = s^{-\nu} \mathcal{I}_{\nu+1}
\]
and
\[
\frac{d}{ds} \left( s^{\nu+1} \mathcal{I}_{\nu+1} \right) = s^{\nu+1} \mathcal{I}_{\nu}.
\]

Then we can easily determine that
\[
s^{(2\nu+1)} \left( f_{\rho} \left( \frac{s}{\rho} \right) \frac{d}{ds} h_{\rho} \left( \frac{s}{\rho} \right) - h_{\rho} \left( \frac{s}{\rho} \right) \frac{d}{ds} f_{\rho} \left( \frac{s}{\rho} \right) \right) = s^{2\nu} \mathcal{I}_{\nu+1}^2(s) + (2\nu+1) \mathcal{I}_{\nu}^2(s) \mathcal{I}_{\nu+1}(s) - s \mathcal{I}_{\nu}^4(s).
\]

Then, differentiating again by \( s \) and summing \( 4(\nu+1)s^{(2\nu-1)}h_{\rho} \left( \frac{s}{\rho} \right) f_{\rho} \left( \frac{s}{\rho} \right) \) we have
\[
\frac{d}{ds} \left[ s^{(2\nu+1)} \left( f_{\rho} \frac{d}{ds} h_{\rho} - h_{\rho} \frac{d}{ds} f_{\rho} \right) \right] + 4(\nu+1)s^{(2\nu-1)}h_{\rho}f_{\rho}
\]
\[
= s^{-(2\nu+1)} \left[ 2s^2 \mathcal{I}_{\nu}(s) \mathcal{I}_{\nu+1}^3(s) + 2(2\nu+1) s \mathcal{I}_{\nu}^3(s) \mathcal{I}_{\nu+1}^2(s) + ((2\nu+3) - 2s^2) \mathcal{I}_{\nu}^5(s) \mathcal{I}_{\nu+1}(s) \right].
\]

Finally (25) holds true for all \( \rho > 0 \) and \( n \geq 2 \) provided the function
\[
H_n(s) = 2s^2 \mathcal{I}_{\nu+1}^2(s) + 2(2\nu+1) s \mathcal{I}_{\nu}(s) \mathcal{I}_{\nu+1}(s) + (-2s^2 + 2\nu + 3) \mathcal{I}_{\nu}^2(s) > 0
\]
for all \( s > 0 \) and \( n \geq 2 \).

We can prove such an inequality for instance by differentiating with respect to \( s \)
\[
\frac{d}{ds} H_n(s) = \frac{1}{s} \left( H_n + 4s \mathcal{I}_{\nu+1} \mathcal{I}_{\nu+1} + (4\nu^2 + 4\nu - 1) \mathcal{I}_{\nu}^2 \right).
\]

Observing that \( (4\nu^2 + 4\nu - 1) \geq 0 \) if \( n \geq 3 \) and that \( \mathcal{I}_{\nu}(s) \) and \( \mathcal{I}_{\nu+1}(s) \) are positive for all \( s > 0 \), we deduce \( sH_n'(s) \geq H_n(s) \) provided \( n \geq 3 \). This immediately implies \( H_n(s) > 0 \) for all \( s > 0 \) and \( n \geq 3 \).

For \( n = 2 \) we set \( G(s) = s \frac{d}{ds} H_2(s) \), and differentiating once again
\[
\frac{d}{ds} G(s) = 6s \mathcal{I}_{\nu+1}^2 + 2s \mathcal{I}_{\nu}^2 > 0 \quad \forall s > 0,
\]
we get \( H_2 > 0 \) for all \( s > 0 \).

We can now conclude the proof of Theorem 1.4. For given \( n \in \mathbb{N} \) and \( \omega > 0 \) let \( 0 < \varepsilon_0 < 1 \) and \( K \) be the constant given by Proposition 2.5. Let \( 0 < \varepsilon < \varepsilon_0 \), and let \( \Omega \) be any set in \( (n, \omega, \varepsilon)\text{–NS} \). As usual let \( v \in \mathcal{N}(n, \varepsilon) \) be the function providing the polar representation of \( \partial \Omega \). We then use Lemma 2.1, Proposition 2.5, the monotonicity of \( h \).
and $f$, and the boundedness of the function $v$ to get

$$\lambda(\Omega^2) = \frac{V(\Omega^2)}{P(\Omega^2)} \geq \frac{V(\Omega)}{P(\Omega)} + \frac{n\omega_n K \int_{S^{n-1}} v^2 \, d\sigma_\xi}{P(\Omega^2) P(\Omega)}$$

$$= \frac{V(\Omega)}{P(\Omega)} \left( 1 + \frac{n\omega_n K \int_{S^{n-1}} v^2 \, d\sigma_\xi}{P(\Omega^2) V(\Omega)} \right)$$

$$= \frac{V(\Omega)}{P(\Omega)} \left( 1 + \frac{K \int_{S^{n-1}} v^2 \, d\sigma_\xi}{h_\rho(1) \int_{S^{n-1}} f_\rho(1 + v(\xi)) \, d\sigma_\xi} \right)$$

$$\geq \lambda(\Omega) \left( 1 + \frac{K \int_{S^{n-1}} v^2 \, d\sigma_\xi}{n\omega_n 2^{n-1} h_\rho(1) f_\rho(2)} \right),$$

which is exactly the inequality in the statement on Theorem 1.4 provided $K$ is renamed.

3. Proof of Theorem 1.1

First we use the same arguments employed in the proof of Lemma 2.1 and in particular the estimate given in (20) to observe that if $\Omega$ is any bounded Lipschitz subset of $\mathbb{R}^n$ then

$$\lambda(\Omega) \leq \frac{V(\Omega)}{P(\Omega)},$$

where $V(\Omega)$ and $P(\Omega)$ are the weighted volume and perimeter introduced in Section 1. Namely, using (14), we have

$$(27) \quad V(\Omega) = \int_\Omega |x|^{2-n} \left( I_{\mathbb{R}_2}^2(|x|) + I_{\mathbb{R}^n_2}^2(|x|) \right) \, dx,$$

and

$$(28) \quad P(\Omega) = \int_{\partial \Omega} |x|^{2-n} I_{\mathbb{R}_2^n}^2(|x|) \, d\mathcal{H}^{n-1}.$$

We observe that the notion of weighted volume and weighted perimeter in (27) and (28) can be extended to the whole class of sets of finite perimeter, provided topological boundary in (28) is replaced by the reduced boundary $\partial^* \Omega$ (see [34]).
For any measurable set $E$ we shall denote by $X(E)$ its barycenter and, for any given $\omega > 0$, we put
\[ \gamma_\omega = \frac{P(B_\rho)}{V(B_\rho)}, \]
the reciprocal of the eigenvalue $\lambda(B_\rho)$, remembering that $\rho = (\omega/\omega_n)^{1/n}$. Theorem 1.1 is then a consequence of the following result.

**Theorem 3.1.** For any given $n \in \mathbb{N}$ and $\omega > 0$ there exists a positive constant $\delta$ such that, in the class of finite perimeter sets, the ball $B_\rho$ is the unique minimizer of
\begin{equation}
\min_{\Omega} \{ J_0(\Omega) : \mathcal{L}^n(\Omega) = \omega, B_\rho - \delta \subset \Omega \subset B_\rho + \delta, X(\Omega) = 0 \},
\end{equation}
where
\begin{equation}
J_0(\Omega) = P(\Omega) - \gamma_\omega V(\Omega).
\end{equation}

According to what we observed in the introduction concerning the choice of the reference system, without the barycenter condition $X(\Omega) = 0$ the result is false. In principle this mandates the additional hypothesis in the statement of Theorem 1.1

\[ X(\Omega) = 0. \]

However, by trivial arguments one can easily prove that, if Theorem 1.1 is true under such an additional constraint then, possibly after taking a smaller positive $\delta$, it holds true even without it.

The proof of Theorem 3.1 is done in two steps: Proposition 3.2 and Proposition 3.3. We start by showing that the minimizers for (29) are actually the minimizers for an unconstrained minimum problem involving a suitable penalized functional, namely, we have:

**Proposition 3.2.** For every $n \in \mathbb{N}$ and $\omega > 0$ there exist some positive constants $\delta_0$, $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, such that, for every $0 < \delta < \delta_0$, the unconstrained functional
\begin{equation}
J(\Omega) = J_0(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 |\mathcal{L}^n(\Omega) - \omega| + \Lambda_3 (V(\Omega \setminus B_\rho + \delta) + V(B_\rho - \delta \setminus \Omega)),
\end{equation}
admits minimizers in the class of the sets with finite measure and perimeter, and $\Omega$ is a minimizer for (29) if and only if $\Omega$ is a minimizer for (31).

**Proof.** We proceed by steps.

**Claim 1.** There exist two positive constants $\Lambda_1$ and $\delta_0$, which depends on $\omega$ and $n$ only, such that, for all $0 < \delta < \delta_0$, there exists minimizers for the problem
\begin{equation}
\min_{\Omega} \{ J_0(\Omega) + \Lambda_1 |X(\Omega)| : \mathcal{L}^n(\Omega) = \omega, B_\rho - \delta \subset \Omega \subset B_\rho + \delta \}
\end{equation}
in the class of finite perimeter sets, and every minimizer is also a minimizer for (29).

Since the existence of minimizers for (32) is trivial, it is enough to show that there exist positive constants $\delta_0$ and $\Lambda_1$ depending on $n$ and $\omega$ only, such that whenever for some $a > 0$ and $\delta < \delta_0$ a set $\Omega$ satisfies
\begin{enumerate}[(a)]
\item $\mathcal{L}^n(\Omega) = \omega$, 
\item $B_\rho - \delta \subset \Omega \subset B_\rho + \delta$
\end{enumerate}

...
(c) $X(\Omega) = a e_1$, with $e_1 = (1,0,\ldots,0)$
then it is possible to find a set $\Omega$ satisfying the constraint (a) and (b) and moreover
$J_0(\Omega) + 2 \Lambda_1 |X(\Omega)| < J_0(\Omega) + 2 \Lambda_1 |X(\Omega)|$.

Firstly we show that there exists $\delta_0$ small enough, such that for any $\Omega$ satisfying
(a), (b), and (c) for some $\delta < \delta_0$, then certainly
\[
\mathcal{L}^n \{ x \in (\Omega \setminus B_{\rho-\delta}); x \cdot e_1 > 6\delta_0 \} > 0, \\
\mathcal{L}^n \{ x \in (B_{\rho+\delta} \setminus \Omega); x \cdot e_1 < -6\delta_0 \} > 0.
\]
Indeed we have
- $\mathcal{L}^n (B_{\rho+\delta} \setminus B_{\rho-\delta}) = 2n\omega_n \rho^{n-1}\delta + O(\delta^2)$
- $\mathcal{L}^n (\Omega \setminus B_{\rho-\delta}) = n\omega_n \rho^{n-1}\delta + O(\delta^2)$
- $\mathcal{L}^n \{ x \in (B_{\rho+\delta} \setminus B_{\rho-\delta}); |x \cdot e_1| < 6\delta_0 \} = 2(n-1)\omega_{n-1} \rho^{n-2}(\delta + O(\delta^2))(6\delta_0 + O(\delta^2))$

Therefore for $\delta_0$ small enough
\[
\frac{9}{10} n\omega_n \rho^{n-1}\delta \leq \mathcal{L}^n (\Omega \setminus B_{\rho-\delta}) \leq \frac{11}{10} n\omega_n \rho^{n-1}\delta \\
\frac{9}{10} n\omega_n \rho^{n-1}\delta \leq \frac{1}{2} \mathcal{L}^n (B_{\rho+\delta} \setminus B_{\rho-\delta}) \leq \frac{11}{10} n\omega_n \rho^{n-1}\delta \\
\mathcal{L}^n \{ x \in (B_{\rho+\delta} \setminus B_{\rho-\delta}); |x \cdot e_1| < 6\delta_0 \} \leq \frac{3}{10} n\omega_n \rho^{n-1}\delta.
\]
As a consequence assuming
\[
\mathcal{L}^n \{ x \in (\Omega \setminus B_{\rho-\delta}); x \cdot e_1 > 6\delta_0 \} = 0,
\]
then
\[
\mathcal{L}^n \{ x \in (\Omega \setminus B_{\rho-\delta}); x \cdot e_1 < -6\delta_0 \} \geq \frac{6}{10} n\omega_n \rho^{n-1}\delta \geq \frac{1}{2} \mathcal{L}^n (\Omega \setminus B_{\rho-\delta}),
\]
in contradiction with (c).

Similarly if we assume
\[
\mathcal{L}^n \{ x \in (B_{\rho+\delta} \setminus \Omega); x \cdot e_1 < -6\delta_0 \} = 0
\]
then
\[
\mathcal{L}^n \{ x \in (\Omega \setminus B_{\rho-\delta}); x \cdot e_1 < -6\delta_0 \} \geq \frac{6}{10} n\omega_n \rho^{n-1}\delta \geq \frac{1}{2} \mathcal{L}^n (\Omega \setminus B_{\rho-\delta})
\]
again contradicting (c).

From now on we consider $\delta_0$ fixed in term of $n$, $\omega$, and in what follows we name $C_i$, $i = 1,2,\ldots$, generic constants which depends on $n$, $\omega$ alone.

Let us denote by $E_0$ the union of all balls of radii $3\delta_0$ not intersecting $B_{\rho-\delta} \cap \{ x \in \mathbb{R}^n; x \cdot e_1 \leq 3\delta_0 \}$, and by $F_0$ the union of all balls of radii $3\delta_0$ included in $B_{\rho+\delta} \cap \{ x \in \mathbb{R}^n; x \cdot e_1 < 0 \}$.

Then we define $E_t = \{ x \in E_0; \text{dist}(x, \partial E_0) > t, t > 0 \}$ and $F_t = \{ x \in F_0; \text{dist}(x, \partial F_0) > t, t > 0 \}$, the level sets of the distance functions from the boundary of $E_0$ and $F_0$ respectively. Clearly, regardless the choice of $\Omega$ satisfying (a), (b), (c) we have
\[
\mathcal{L}^n (E_0 \cap \Omega) > 0 \\
\mathcal{L}^n (F_0 \setminus \Omega) > 0 \\
\mathcal{L}^n (E_{2\delta} \cap \Omega) = 0 \\
\mathcal{L}^n (F_{2\delta} \setminus \Omega) = 0
We will prove that for any given $\Omega$ satisfying (a), (b), and (c) there exists a positive constant $\sigma_0$ such that for all $0 < \sigma < \sigma_0$ we can find $\bar{t}, \tilde{t} \in [0, 2\delta]$ depending on $\sigma$ such that the following properties hold:

(i) $L^n(\Omega \cap E_t) = L^n(F_{\bar{t}} \setminus \Omega) = \sigma$
(ii) $L^n(E_t \cap B_{\rho-\delta}) = L^n(F_{\tilde{t}} \setminus B_{\rho+\delta}) = 0$
(iii) $P(\Omega \setminus E_t) < P(\Omega) + C_1 \sigma$
(iv) $P(\Omega \cup F_{\tilde{t}}) < P(\Omega) + C_1 \sigma$

Properties (i)–(ii) follow at once by the definition of $E_t$ and $F_{\tilde{t}}$.

Concerning (iii) we observe that, for $t \in (0, 2\delta)$, $E_t$ and $F_{\tilde{t}}$ are two families of $C^{1,1}$ sets with mean curvature bounded in terms of $\delta_0$ and $\omega$ and $n$ alone. Moreover, if for $x \in E_0$ such that $\text{dist}(x, \partial E_0) \leq 2\delta_0$ we set $T_E(x) = \nabla \text{dist}(x, \partial E_0)$, then for all $x \in \partial E_t$ the vector $T_E(x)$ is the inner unit normal to $\partial E_t$ at $x$. Since for $\text{dist}(x, \partial E_0) \leq 2\delta_0$ we have

$$\left| \text{div}(|x|^{-n}\mathcal{I}_{\frac{2}{2} - 1}(|x|)T_E(x)) \right| \leq \left| \nabla(|x|^{-n}\mathcal{I}_{\frac{2}{2} - 1}(|x|)) \right| + |x|^{-n}\mathcal{I}_{\frac{2}{2} - 1}(|x|) \right| \text{div} T_E \leq C_1$$

we have

$$|P(\Omega \setminus E_t) - P(\Omega)| \leq \int_{\Omega \setminus E_t} \text{div} \left( |x|^{-n}\mathcal{I}_{\frac{2}{2} - 1}(|x|)T_E(x) \right) dx \leq C_1 L^n(\Omega \cap E_t).$$

We can argue in the same way to deduce (iv) by using $F_0$ in place of $E_0$ and by defining the vector field $T_F(x) = \nabla \text{dist}(x, \partial F_0)$ for $\text{dist}(x, \partial F_0) \leq 2\delta_0$. We have

$$|P(\Omega \cup F_{\tilde{t}}) - P(\Omega)| \leq \int_{F_{\tilde{t}} \setminus \Omega} \text{div} \left( |x|^{-n}\mathcal{I}_{\frac{2}{2} - 1}(|x|)T_F(x) \right) dx \leq C_1 L^n(F_{\tilde{t}} \setminus \Omega).$$

Finally we observe that $V(\Omega \cup F_{\tilde{t}}) > V(\Omega)$, and using (16)–(17), then we also have $V(\Omega \setminus E_t) \geq V(\Omega) - C_2 \sigma$.

Once we have constructed the sets $E_t$ and $F_{\tilde{t}}$ the proof of the claim follows at once from (i)–(iv). Indeed let

$$\bar{\Omega} = (F_{\tilde{t}} \cup \Omega) \setminus E_t,$$

we have by construction

$$|X(\bar{\Omega})| \leq |X(\Omega)| - C_3 \sigma + O(\sigma^2) \leq |X(\Omega)| - C_4 \sigma$$

provided $\sigma$ is smaller than a positive quantity $\sigma_0$ which may depend on $\Omega$. Eventually we have

$$J_0(\bar{\Omega}) + \Lambda_1 |X(\bar{\Omega})| = P(\bar{\Omega}) - \gamma_\omega V(\bar{\Omega}) + \Lambda_1 |X(\bar{\Omega})| \leq P(\Omega) - \gamma_\omega V(\Omega) + \Lambda_1 |X(\Omega)| + (2C_1 + C_2 \gamma_\omega - C_4 \Lambda_1) \sigma < J_0(\Omega) + \Lambda_1 |X(\Omega)|$$

provided $\Lambda_1$ is large enough. Observe that $\Lambda_1$ can be explicitly computed in terms of $n$, $\omega$ and $\delta_0$ alone.
Claim 2. Let $\delta_0$ and $\Lambda_1$ be the constant given in Claim 1, then there exists a positive constant $\Lambda_2$, which depends on $\omega$ and $n$ only, such that, for every $0 < \delta < \delta_0$, the problem

$$
\min_{\Omega} \{ J_0(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 |\mathcal{L}^n(\Omega) - \omega| : B_{\rho-\delta} \subset \Omega \subset B_{\rho+\delta} \}
$$

admits minimizers in the class of finite perimeter sets, and every minimizer is also a minimizer for (32).

As in Claim 1 existence of minimizers is trivial and arguing as before, let us assume that for some $\delta \in (0, \delta_0)$, a set $\Omega$ satisfies

$$
B_{\rho-\delta} \subset \Omega \subset B_{\rho+\delta}
$$

and

$$
\mathcal{L}^n(\Omega) > \omega.
$$

In this case there exists $\tilde{\rho} \in (\rho - \delta, \rho + \delta)$ such that $\mathcal{L}^n(\Omega \cap B_{\tilde{\rho}}) = \omega$. If we set

$$
\tilde{\Omega} = \Omega \cap B_{\tilde{\rho}}
$$

then

$$
|P(\tilde{\Omega}) - P(\Omega)| \leq \int_{\Omega \setminus \tilde{\Omega}} \text{div} \left( |x|^{2-n} \mathcal{I}_{\frac{2}{2-n}}^\mathcal{I} \frac{x}{|x|} \right) dx \leq C_5 \mathcal{L}^n(\Omega \setminus \tilde{\Omega}).
$$

As in the previous claim by $C_i$ we denote positive constants depending just on $\delta_0$, $n$, and $\omega$.

Using (16) we deduce that

$$
V(\tilde{\Omega}) \geq V(\Omega) - C_6 \mathcal{L}^n(\Omega \setminus \tilde{\Omega}).
$$

Finally we observe that

$$
|X(\tilde{\Omega})| \leq |X(\Omega)| + (\rho + \delta_0) \mathcal{L}^n(\Omega \setminus \tilde{\Omega}),
$$

and therefore

$$
J_0(\tilde{\Omega}) + \Lambda_1 |X(\tilde{\Omega})| + \Lambda_2 |\mathcal{L}^n(\tilde{\Omega}) - \omega| = P(\tilde{\Omega}) - \gamma_\omega V(\tilde{\Omega}) + \Lambda_1 |X(\tilde{\Omega})|
\leq P(\Omega) - \gamma_\omega V(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 |\mathcal{L}^n(\Omega) - \omega|
+(C_5 + C_6 \gamma_\omega + (\rho + \delta_0) \Lambda_1 - \Lambda_2) \mathcal{L}^n(\Omega \setminus \tilde{\Omega})
< J_0(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 |\mathcal{L}^n(\Omega) - \omega|
$$

provided $\Lambda_2$ is large enough. Observe that $\Lambda_2$ can be explicitly computed in terms of $n$, $\omega$, $\delta_0$ and $\Lambda_1$ alone.

If now we assume that for some $\delta \in (0, \delta_0)$ a set $\Omega$ satisfies

$$
B_{\rho-\delta} \subset \Omega \subset B_{\rho+\delta}
$$

and

$$
\mathcal{L}^n(\Omega) < \omega,
$$

the previous arguments work as well provided $\tilde{\Omega} = \Omega \cup B_{\tilde{\rho}}$ for some $\tilde{\rho} \in (\rho - \delta, \rho + \delta)$ such that $\mathcal{L}^n(\tilde{\Omega}) = \omega$. 
Claim 3. Let $\delta_0$, $\Lambda_1$ and $\Lambda_2$ be the constants given in Claim 1 and Claim 2, then there exist a positive constant $\Lambda_3$, which depends on $\omega$ and $n$ only, such that, for every $0 < \delta < \delta_0$, the problem

\[
\min_{\Omega} \{ J_0(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 \mathcal{L}^n(\Omega) - \omega | + \Lambda_3 (V(\Omega \setminus B_{\rho+\delta}) + V(B_{\rho-\delta} \setminus \Omega)) \}
\]

admits minimizers in the class of the sets with finite measure and perimeter, and every minimizer is a minimizer for (33).

In the same spirit of the previous claims let us consider a set $\Omega$ which for some $\delta \in (0, \delta_0)$ satisfies

\[ V(\Omega \setminus B_{\rho+\delta}) + V(B_{\rho-\delta} \setminus \Omega) > 0. \]

We set

\[ \tilde{\Omega} = (\Omega \cap B_{\rho+\delta}) \cup B_{\rho-\delta}. \]

If $s(t)$ is any smooth function such that

- $0 \leq s(t) \leq 1$ for all $t \geq 0$
- $s(t) = 1$ for $|t - 1| \leq \frac{1}{4}$
- $s(t) = 0$ for $|t - 1| \geq \frac{1}{2}$
- $\|s\|_{C^1}$ is bounded by a constant $K$

then

\[
|P(\tilde{\Omega}) - P(\Omega)| \leq \left| \int_{\Omega \setminus B_{\rho+\delta}} \text{div} \left( s \left( \frac{|x|}{\rho} \right) |x|^{2-n} I_{\frac{2}{2}-1}(|x|) \frac{x}{|x|} \right) dx \right|
\]

\[ + \left| \int_{B_{\rho-\delta} \setminus \Omega} \text{div} \left( s \left( \frac{|x|}{\rho} \right) |x|^{2-n} I_{\frac{2}{2}-1}(|x|) \frac{x}{|x|} \right) dx \right| \]

\[ \leq C_7 \left( \mathcal{L}^n(B_{\rho-\delta} \setminus \Omega) + \mathcal{L}^n(\Omega \setminus B_{\rho+\delta}) \right). \]

Here we have used that $s(|x|/\rho) = 1$ on $B_{\rho+\delta}$ and $B_{\rho-\delta}$ which is true provided $\delta_0 < \rho/4$.

An assumption that we can always effort without loss of generality.

Observe also that

\[ \mathcal{L}^n(B_{\rho-\delta} \setminus \Omega) + \mathcal{L}^n(\Omega \setminus B_{\rho+\delta}) \leq C_8 (V(B_{\rho-\delta} \setminus \Omega) + V(\Omega \setminus B_{\rho+\delta})), \]

and using (16) and (17)

\[
|X(\tilde{\Omega})| - |X(\Omega)| \leq \rho \mathcal{L}^n(B_{\rho-\delta} \setminus \Omega) + \int_{\Omega \setminus B_{\rho+\delta}} |x| dx
\]

\[ \leq \rho \mathcal{L}^n(\Omega \setminus B_{\rho+\delta}) + C_9 \left| \int_{\Omega \setminus B_{\rho+\delta}} |x|^{2-n} I_{\frac{2}{2}-1}(|x|) dx \right| \]

\[ \leq C_{10} (V(B_{\rho-\delta} \setminus \Omega) + V(\Omega \setminus B_{\rho+\delta})). \]
Now we have
\[
J_0(\bar{\Omega}) + \Lambda_1 |X(\bar{\Omega})| + \Lambda_2 |\mathcal{L}^n(\bar{\Omega}) - \omega| + \Lambda_3(V(\bar{\Omega}\setminus B_{\rho+\delta}) + V(B_{\rho-\delta}\setminus \bar{\Omega}))
\]
\[
= P(\bar{\Omega}) - \gamma_\omega V(\bar{\Omega}) + \Lambda_1 |X(\bar{\Omega})| + \Lambda_2 |\mathcal{L}^n(\bar{\Omega}) - \omega| + \Lambda_3(V(\bar{\Omega}\setminus B_{\rho+\delta}) + V(B_{\rho-\delta}\setminus \bar{\Omega}))
\]
\[
\leq P(\Omega) - \gamma_\omega V(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 |\mathcal{L}^n(\Omega) - \omega| + \Lambda_3(V(\Omega\setminus B_{\rho+\delta}) + V(B_{\rho-\delta}\setminus \Omega))
\]
\[
+ (C_7 C_8 + C_8 \gamma_\omega + \Lambda_1 \gamma_{10} + C_8 \Lambda_2 - \Lambda_3)(V(\Omega\setminus B_{\rho+\delta}) + V(B_{\rho-\delta}\setminus \Omega))
\]
\[
< J_0(\Omega) + \Lambda_1 |X(\Omega)| + \Lambda_2 |\mathcal{L}^n(\Omega) - \omega| + \Lambda_3(V(\Omega\setminus B_{\rho+\delta}) + V(B_{\rho-\delta}\setminus \Omega)),
\]
provided \(\Lambda_3\) is large enough. Once again \(\Lambda_3\) can be expressed in terms of \(n, \omega, \delta_0, \Lambda_1\) and \(\Lambda_2\).

Therefore, if a minimizer of (34) exists then it is necessarily bounded. Since a minimizing sequence of equibounded sets for (34) certainly exists, by compactness and semicontinuity existence of minimizers trivially follows.

Summing up Claim 1, Claim 2 and Claim 3, it is therefore possible to choose positive constants \(\delta_0, \Lambda_1, \Lambda_2\) and \(\Lambda_3\), in such a way that (31) admits minimizers whenever \(0 < \delta < \delta_0\) and every minimizer is also a minimizer of (29).

Finally for the same choice of the constants \(\delta_0, \Lambda_1, \Lambda_2\) and \(\Lambda_3\), all minimizers of (29) for some \(0 \leq \delta < \delta_0\) are also a minimizers of (31). \(\square\)

**Proposition 3.3.** Let \(\omega > 0\) and let \(\delta_0, \Lambda_1, \Lambda_2, \Lambda_3\) be the positive constants given in Proposition 3.2. There exists \(\delta\) with \(0 < \delta < \delta_0\) such that any minimizer in the class of the sets with finite measure and perimeter of the functional \(J\) defined in (31) is a ball.

**Proof.** Let \(\Omega\) be a minimizer of the functional \(J\) defined in (31) for a fixed value of \(\delta\) with \(0 < \delta < \delta_0\). Our first aim is to prove that \(\Omega\) is an almost-minimizer for the perimeter in the sense of (11).

Since \(\Omega\) is a minimizer of \(J\), in view of Proposition 3.2, we have \(J(\Omega) = J_0(\Omega)\), the barycenter of \(\Omega\) is the origin, and \(B_{\rho-\delta} \subset \Omega \subset B_{\rho+\delta}\). Let now \(0 < r_0 < 1\), we consider a set \(\tilde{\Omega}\) such that \(\Omega \Delta \tilde{\Omega} \subset \subset B_r(x_0)\) for some \(0 < r < r_0\). If \(|x_0| > \rho + \delta + 1\), then
\[
P(\Omega) \leq P(\tilde{\Omega}),
\]
while if \(|x_0| \leq \rho + \delta + 1\), then
\[
J(\Omega) \leq J(\tilde{\Omega}) \leq P(\tilde{\Omega}) - \gamma_\omega V(\tilde{\Omega}) + K_1 \mathcal{L}^n(B_r(x_0)),
\]
for some \(K_1\) which depends only on \(n\) and \(\omega\). Therefore we can say that \(\Omega\) is an almost-minimizer for the weighted perimeter \(P(\cdot)\), in the sense that there exist two positive constants \(K_1\) and \(r_0\), which depend only on \(n\) and \(\omega\), such that
\[
P(\Omega) \leq P(\tilde{\Omega}) + K_1 r^n,
\]
whenever \(\Omega \Delta \tilde{\Omega} \subset \subset B_r(x_0)\) and \(0 < r < r_0\). Then, arguing as in [27], observing that the weight \((|x|^1 - \frac{2}{2} \mathcal{L}_{2-1}(|x|))^2\) which defines \(P(\cdot)\) is locally a Lipschitz function, (35) implies (11). Using the results quoted in the introduction the reduced boundary \(\partial^* \Omega\) is a \(C^{1,1/2}\) hypersurface. Let us now consider a vanishing decreasing sequence of positive numbers \(\delta_k, k \in \mathbb{N}\), with \(\delta_k < \delta_0\), and let \(\Omega_k\) be a sequence of minimizers for the functional \(J\) with \(\delta = \delta_k\). Then \(\Omega_k\) is a sequence of almost-minimizers for the Euclidean perimeter, with
uniform constants $K$ and $r_0$, converging to $B_\rho$ in $L^1$. It follows that there exists $\tilde{k} \in \mathbb{N}$ such that, for $k > \tilde{k}$, the boundary of $\Omega_k$ can be represented in polar coordinates as

\begin{equation}
    r_k(\xi) = \rho(1 + v_k(\xi)),
\end{equation}

where $v_k \to 0$ in $C^{1,\alpha}([0,1])$ for all $\alpha \in (0,1/2)$. Therefore, for $n \in \mathbb{N}$ and $\omega > 0$ let $\varepsilon = \varepsilon(n,\omega)$ be the positive constant given in Theorem 1.4. In view of what we have proved so far, there exists $\bar{\delta} \in (0,\delta_0)$ such that for every $\delta \in (0,\bar{\delta})$ any minimizer $\Omega$ of $J$ is a $(n,\omega,\varepsilon)$-NS set. Employing Theorem 1.4 we deduce that $\Omega$ must be a ball.

\phantomsection
\ref*{proof:cor11}

**Proof of Corollary 1.1.** In order to prove Corollary 1.1 it is enough to show that Conjecture 2 is equivalent to Conjecture 1, so that it follows from Theorem 1.1. Here we only prove that Conjecture 1 implies Conjecture 2 as the reverse follows using similar arguments.

Slightly modifying the notation given in the Introduction, for a given open bounded Lipschitz set $\Omega$ we consider

\begin{equation}
    \lambda_{R,\alpha}(\Omega) = \min_{\substack{w \in H^1(\Omega) \\
 w \neq 0}} \frac{\int_{\Omega} |Dw|^2 \, dx + \alpha \int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1}}{\int_{\Omega} w^2 \, dx},
\end{equation}

with $\alpha < 0$. Observing that $\lambda_{R,\alpha}(\Omega)$, as a function of $\alpha$, is Lipschitz continuous, monotone increasing from $]-\infty,0]$ onto $]-\infty,0]$, for every fixed $\alpha < 0$ there exists $\bar{\alpha} < 0$ such that:

\begin{equation}
    \lambda_{R,\alpha}(\Omega) = \lambda_{R,\bar{\alpha}}(\Omega).
\end{equation}

Our aim is to show that $\bar{\alpha} \leq \alpha$.

Indeed, denoting by $u$ and $v$ two extremal functions in (37) relative to $\Omega$ and $\Omega^\sharp$, we have:

\begin{align*}
    \int_{\Omega} |Dw|^2 \, dx + \alpha \int_{\partial \Omega} w^2 \, d\mathcal{H}^{n-1} &= \lambda_{R,\alpha}(\Omega) \int_{\Omega} w^2 \, dx \quad \forall w \in H^1(\Omega) \\
    \int_{\Omega} |Dv|^2 \, dx + \alpha \int_{\partial \Omega} v^2 \, d\mathcal{H}^{n-1} &= \lambda_{R,\alpha}(\Omega^\sharp) \int_{\Omega^\sharp} v^2 \, dx \quad \forall v \in H^1(\Omega^\sharp)
\end{align*}

and

\begin{align*}
    \int_{\Omega} |Dz|^2 \, dx + \alpha \int_{\partial \Omega^\sharp} z^2 \, d\mathcal{H}^{n-1} &= \lambda_{R,\bar{\alpha}}(\Omega^\sharp) \int_{\Omega^\sharp} z^2 \, dx \quad \forall z \in H^1(\Omega^\sharp) \\
    \int_{\kappa \Omega^\sharp} |Dz|^2 \, dx + \alpha \int_{\partial (\kappa \Omega^\sharp)} z^2 \, d\mathcal{H}^{n-1} &= \lambda_{R,\bar{\alpha}}(\Omega^\sharp) \int_{\kappa \Omega^\sharp} z^2 \, dx \quad \forall z \in H^1(\kappa \Omega^\sharp)
\end{align*}

By a rescaling with $\kappa = \sqrt{|\lambda_{R,\alpha}(\Omega)|}$ we have

\begin{equation}
    \int_{\kappa \Omega} |Dw|^2 \, dx + \int_{\kappa \Omega} w^2 \, dx \geq \frac{|\alpha|}{\kappa} \int_{\partial (\kappa \Omega)} w^2 \, d\mathcal{H}^{n-1} \quad \forall w \in H^1(\kappa \Omega)
\end{equation}

and

\begin{equation}
    \int_{\kappa \Omega^\sharp} |Dz|^2 \, dx + \int_{\kappa \Omega^\sharp} z^2 \, dx \geq \frac{\bar{\alpha}}{\kappa} \int_{\partial (\kappa \Omega^\sharp)} z^2 \, d\mathcal{H}^{n-1} \quad \forall z \in H^1(\kappa \Omega^\sharp).
\end{equation}
with equality holding in (39) and (40) for \( w(x) = u(x/\kappa) \) and \( z(x) = v(x/\kappa) \). It follows that

\[
\lambda(\kappa \Omega) = \frac{|\alpha|}{\kappa} \quad \text{and} \quad \lambda(\kappa \Omega^\#) = \frac{|\bar{\alpha}|}{\kappa}
\]

and then using Conjecture 1 we have \( \bar{\alpha} \leq \alpha \).

\[\square\]

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