STRUCTURAL STABILITY OF SOLUTIONS TO THE Riemann Problem for a Scalar Nonconvex CJ Combustion Model

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Abstract. The Riemann problem for the simplest scalar nonconvex CJ combustion model is considered. A set of entropy conditions are summarized, including pointwise entropy conditions and global entropy conditions. The later is according to the requirement of the structural stability. The unique self-similar entropy solution of the Riemann problem is constructed case by case. Transition from deflagration to detonation is shown, which dose not occur for the convex model.

1. Introduction. W. Fickett (1979, [4]) and A. Majda (1981, [8]) proposed the simplest Chapman-Jouguet (CJ) combustion model

\[
(u + q)_t + f(u)_x = 0, \tag{1}
\]

\[
q(x, t) = \begin{cases}
q(x, 0), & \sup_{0 \leq \tau \leq t} u(x, \tau) \leq u_i, \\
0, & \text{otherwise},
\end{cases} \tag{2}
\]

for the Euler equations of ideal combustible gas, where (1) is a conservation law with flux function \( f(u) \), \( x \) the Lagrangian coordinate, \( u \) a lumped quantity representing the density, pressure, energy, velocity, temperature and everything except binding energy \( q \). Combustion is nothing but releasing the binding energy with infinite speed once the temperature \( u \) is higher than the ignition temperature \( u_i \) (we take \( u_i = 0 \) for simplicity), which is characterized by the reacting equation (2).

The Riemann problem of (1) and (2) with

\[
(u, q)\big|_{t=0} = (u_\pm, q_\pm), \quad \pm x > 0, \tag{3}
\]

is solved in the case of \( f(u) \) being convex by Ying and Teng ([11]), Liu and Zhang([7]). The former proved constructively the existence and uniqueness of the

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\]

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\]

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Lax entropy solution of the Riemann problem (3) of the Zeldovich-von Neumann-Döring (ZND) combustion model (1) and the reacting equation

\[ q_t = -k \varphi(u)q, \quad (4) \]

where \( k \) is the rate of reaction and Heaviside function \( \varphi(u) = 0 \) as \( u \leq 0 \), \( \varphi(u) = 1 \) as \( u > 0 \). Furthermore, they obtained limits of the solutions of the Riemann problem of ZND model, and defined the limits as the solutions of the Riemann problem of CJ model. Based on Ying and Teng’s results, Liu and Zhang summarized a set of entropy conditions, including pointwise and global entropy conditions, which characterize the effect of \( k \to \infty \). It is just as the Lax’s entropy condition characterizing the effect of viscous vanishing. With these entropy conditions, they constructed the unique entropy solution of the Riemann problem for CJ model.

In fact, the Euler equations are not convex because one of its three characteristic fields is linear degenerate ([5], [1]), furthermore the rest two nonlinear characteristic fields are nonconvex in the case of van der Waals gas ([9]). Mathematically speaking, a genuine two dimensional conservation law must be nonconvex in certain directions ([1], [15]). So, it is necessary to investigate a scalar combustion model with a nonconvex flux \( f(u) \). There is another motivation to study the nonconvex model (1). A well-known phenomenon in combustion theory is the transition from deflagration to detonation. However, this phenomenon does not occur for a scalar convex conservation law because detonation and deflagration waves do not propagate in the same direction (forward or backward). In the nonconvex case, both deflagration wave and detonation wave may propagate in the same direction, and the phenomenon may appear.

For the simplest nonconvex system (1), in 1995, Zhang and Zhang ([12]) gave an entropy condition that mimics those in [7]. In 2003, Li and Zhang ([6]) proved that the Riemann solutions in [12] are the limit of the Riemann solutions for the nonconvex self-similar ZND combustion model

\[
\begin{align*}
(u + q)_t + f(u)_x &= 0, \\
q_t &= -\frac{k}{v} \varphi(u)q,
\end{align*}
\]

as the rate of reaction goes to infinity. However, through the study of the structural stability of combustion solutions in this paper, we find the global entropy conditions in [6] do not guarantee the uniqueness in some cases, which were missed there. We clarify a set of complete global entropy conditions by supplementing a condition: the structural stability of Riemann solutions. By using phase portrait (the flux function plane \( (f, u) \)) analysis method ([13] and [7]), we construct the unique self-similar entropy solution for the Riemann problem (3) of (1-2) case by case. By the way, we note that the solution in Case 3 in [7] is not structurally stable and a structurally stable solution is delivered in section 4 in this paper.

The present paper is organized as follows. In section 2, some preliminaries are given. In section 3, a set of entropy conditions are delivered, including pointwise entropy conditions and global entropy conditions. We complete the global entropy condition, then, the definition of a self-similar entropy solution of the Riemann problem is delivered. In section 4, according to the requirement of the structural stability, the solution is constructed uniquely case by case. The main theorem of this paper is summarized. In section 5, by using the Riemann solutions, we construct some solutions, which reveal transition from deflagration to detonation.
2. Preliminaries. We seek weak self-similar solutions of the Riemann problem (3) of (1) and (2). The Riemann solutions are piecewise smooth. It is easy to show that \( q(x, t) \) is piecewise constant, 0 or \( q_0 \) (total binding energy). Smooth solutions \( u(x, t) \) are constant states or self-similar rarefaction waves (R).

\[
\frac{dx(t)}{dt} = \sigma, \quad (6)
\]

where \([f] = f(u_l) - f(u_r), u_l = u(x(t) - 0, t), u_r = u(x(t) + 0, t)\), etc. There are several kinds of jumps:

A. noncombustion waves:
   a. \([q] = 0, [u] \neq 0 \Rightarrow \sigma = \frac{[f]}{[u]}, \) it is a shock (S\((u_l, u_r)\)) supplemented by Oleinik-type entropy condition;
   b. \([q] \neq 0, [u] = 0 \Rightarrow \sigma = 0, \) it is a contact discontinuity (J);
   c. \([q] \neq 0, [u] \neq 0, \sigma = 0, \) it is a combination of S and J (SJ\((u_l, u_r)\)).

B. combustion jumps:
   d. \([q] \neq 0, [u] \neq 0, \sigma < 0, \) backward jump.
   e. \([q] \neq 0, [u] \neq 0, \sigma > 0, \) forward jump.

The backward combustion jumps can be classified into nine types according to that the characteristic lines ahead of or behind the combustion jumps are incoming, parallel or outgoing followed by [3], which are shown in Fig.2.1. So do the forward jumps.

3. Entropy conditions. Without loss of generality, we consider the backward combustion wave only, i.e., \( u_\pm \leq 0 < u_+, q_+ = 0 < q_- = q_0 \). Let \( x = x(t) \) be a combustion wave and denote \( u_f = u(x(t) - 0, t) \) (wave front) and \( u_b = u(x(t) + 0, t) \) (wave back). The combustion jump is a combustion wave after supplementing pointwise entropy conditions, which are given as follows.
Pointwise entropy conditions: (see Fig 3.1)

a. If there exists a \( u_R \in [u_f, u_b) \), such that for all \( u \in (u_f, u_R) \),
\[
\frac{dx(t)}{dt} = \sigma = \frac{f(u_0) - f(u_f)}{u_0 - (u_f + q_0)} = \frac{f(u_R) - f(u_f)}{u_R - u_f} \leq \frac{f(u) - f(u_f)}{u - u_f},
\]
the discontinuity line \( x = x(t) \) is called deflagration. Furthermore, it can be divided into three subcases:

1. \( f(u_f) = \sigma < f(u_b) \): CJ deflagration \( (\text{CJDF}(u_f, q_0; u_b, 0)) \);
2. \( f(u_f) > \sigma < f(u_b) \): weak deflagration \( (\text{WDF}(u_f, q_0; u_b, 0)) \);
3. \( f(u_f) = \sigma = f(u_b) \): double contact combustion \( (\text{DCC}(u_f, q_0; u_b, 0)) \).

b. If there exists a \( u_R \in [u_b, +\infty) \) satisfying (7) for \( u \in (u_f, u_R), x = x(t) \) is called detonation. Also, it can be divided into three subcases:

4. \( f(u_f) > \sigma = f(u_b) \): CJ detonation \( (\text{CJDT}(u_f, q_0; u_b, 0)) \);
5. \( f(u_f) > \sigma > f(u_b) \): strong detonation \( (\text{SDT}(u_f, q_0; u_b, 0)) \);
6. \( f(u_f) = \sigma > f(u_b) \): contact detonation \( (\text{CDT}(u_f, q_0; u_b, 0)) \).

We call these six kinds of combustion waves are admissible.

All in all, the six pointwise entropy conditions can be summarized as one: the characteristic line in front of the combustion wave is not outgoing.

For the forward combustion waves, the pointwise entropy conditions can be easily defined by means of transformation \( \tilde{x} = -x, \tilde{f} = -f \).

We call \( R, S, J, SJ, \text{CJDF}, \text{WDF}, \text{DCC}, \text{CJDT}, \text{SDT} \) and \( \text{CDT} \) elementary waves for (1) and (2) without convexity.

The aforementioned entropy conditions can not guarantee the uniqueness and structural stability of the solutions to the Riemann problem (3) for (1) and (2). For this purpose, global entropy condition is needed.

Global entropy conditions:

In the case of that the Riemann problem (3) for (1) and (2) has several solutions satisfying the pointwise entropy conditions, we choose the one which satisfies following rules:

a. the propagating speed of combustion wave is as low as possible;

b. all elementary waves are structurally stable: the strength \( |u_l - u_r + q_l - q_r| \) of each elementary wave and the propagating speeds of combustion wave and shock wave vary continuously with respect to the variation of \( u_{\pm} \).

Remark 1. The structural stability of Riemann solution is needed obviously both in physics and computation, and has never been considered before as we know.
**Definition 3.1.** A piecewise smooth function \((u, q)(\xi)\) is called a self-similar entropy solution of the Riemann problem \((3)\) to \((1-2)\), if

(i) \((1)\) is satisfied when \((u, q)\) is smooth, and \(q\) is piecewise constant: 0 or \(q_0\), which satisfies \((2)\);

(ii) \((6)\) as well as the pointwise conditions and global entropy conditions are satisfied when \((u, q)\) is discontinuous.

(iii) \((3)\) is satisfied.

**Remark 2.** A self-similar entropy solution is a weak solution composed of a concatenation of elementary waves.

4. Structurally stable solutions to the Riemann problem.

4.1. A modification to some Riemann solutions for convex case.

By the pointwise entropy conditions and global entropy conditions, we give the unique self-similar entropy solution to the Riemann problem of a scalar convex CJ combustion model. The results are the modification to some Riemann solutions for convex case ([13], [7]).

**Theorem 4.1.** (see Fig. 4.1) The Riemann solutions of the CJ combustion model for a scalar conservation laws with \(f''(u) > 0\) and \(f'(0) < 0\) as well as \(u_- < 0 < u_+, q = q_0 > q_+ = 0\) are as follows: Let \(u_*\) be determined by

\[
\frac{f(u_*) - f(0)}{u_+ - q_0}.
\]

1) When \(u_+ < u_*\),

\[
(u, q) = \begin{cases} 
(u_-, q_0), & \frac{\xi}{\tau} < f'(u_-), \\
(f'^{-1}(\xi), q_0), & f'(u_-) \leq \frac{\xi}{\tau} \leq f'(0), \\
(u_*, 0), & f'(0) < \frac{\xi}{\tau} < \frac{f(u_*) - f(u_+)}{u_* - u_+}, \\
(u_+, 0), & \frac{\xi}{\tau} > \frac{f(u_*) - f(u_+)}{u_* - u_+};
\end{cases} \tag{8}
\]

2) When \(u_+ > u_*\),

\[
(u, q) = \begin{cases} 
(u_-, q_0), & \frac{\xi}{\tau} < f'(u_-), \\
(f'^{-1}(\xi), q_0), & f'(u_-) \leq \frac{\xi}{\tau} \leq f'(0), \\
(f'^{-1}(\xi), 0), & f'(u_+) < \frac{\xi}{\tau} \leq f'(u_+), \\
(u_+, 0), & \frac{\xi}{\tau} > f'(u_+); \tag{9}
\end{cases}
\]

These solutions can be briefly shown as follows ("+" means followed by):

\[
R(u_-, 0) + \text{CJDF}(0, q_0; u_*, 0) + \begin{cases} \text{S}(u_*, u_+), \\
\text{or} \quad R(u_*, u_+). \end{cases} \tag{10}
\]

In Figure 4.1, as in later figures, we draw the graph of \(f(u)\) by interchanging the \(u\) and \(f(u)\) coordinates, as is commonly done, so that the tangent of the graph is parallel to characteristic lines \(dx/dt = f'(u)\) in the physical \((x, t)\)-plane. To satisfy the Rankine-Hugoniot condition \((6)\) for a combustion wave, the graph of \(f(u)\) is lifted upward by \(q_0\). State \(u_-\) is lifted to \(u_- + q_0\).
Theorem 4.2. (see Fig.4.2-4.3) The Riemann solutions of CJ combustion model for a scalar conservation laws with \( f''(u) < 0 \) and \( f(0) > f(u_-) \) as well as \( u_- < 0 < u_+, q_0 = q_0 > q_+ = 0 \) are as follows: Let \( q_1 > 0 \) be determined by \( f(u_-) = f(u_- + q_1) \).

1) \( q_0 \leq q_1, 0 < u_+ \leq u_- + q_1 \) (Fig.4.2). The solution is a noncombustion wave composed of a J \((q_0, 0)\): \( x = 0 \) for \( q \) and a shock wave \( S(u_-, u_+): \frac{\dot{\tau}}{\tau} = \frac{f(u_+)-f(u_-)}{u_+ - u_-} \) for \( u \), and

\[
(u, q) = \begin{cases} 
(u_-, q_0), & x < 0, \\
(u_-, 0), & 0 < x < \frac{f(u_+)-f(u_-)}{u_+ - u_-}, \\
(u_+, 0), & x > \frac{f(u_+)-f(u_-)}{u_+ - u_-}.
\end{cases}
\] (11)

2) \( q_0 \leq q_1, u_+ \geq u_- + q_1 \) (Fig.4.3). The solution is a combustion wave \( SDT(u_-, q_0; u_+, 0): \frac{\dot{\tau}}{\tau} = \frac{f(u_+)-f(u_-)}{u_+ - u_- - q_0} \) and

\[
(u, q) = \begin{cases} 
(u_-, q_0), & x < \frac{f(u_+)-f(u_-)}{u_+ - u_- - q_0}, \\
(u_+, 0), & x > \frac{f(u_+)-f(u_-)}{u_+ - u_- - q_0}.
\end{cases}
\] (12)

3) \( q_0 > q_1 \) (Fig.4.3). The solution is as follows: \( CJDT(u_-, q_0; u_+, 0) + R(u_+, u_+)(u_+ \leq u_+) \), where \( u_* \) is determined by \( f'(u_*) = \frac{f(u_+)-f(u_-)}{u_+ - u_- - q_0} \) and Fig.4.5, and

\[
(u, q) = \begin{cases} 
(u_-, q_0), & \frac{\dot{\tau}}{\tau} < f'(u_*), \\
(f^{-1}((\dot{\tau})^+), 0), & f'(u_* \leq \frac{\dot{\tau}}{\tau} \leq f'(u_+)), \\
(u_+, 0), & \frac{\dot{\tau}}{\tau} > f'(u_+),
\end{cases}
\] (13)

or \( SDT(u_-, q_0; u_+, 0) \) \((u_+ > u_*)\), which is the same as (11).
The results are the same as that in [13] and [7] except for case 3) in Theorem 4.2. In this case: $q_0 > q_1$, by the entropy condition $b$: the structural stability, the solution must be a combustion wave $CJDT(u_-, q_0; u_+, 0)$ followed by a rarefaction wave $R(u_+, u_+)$ when $u_+ < u_*$. In fact, if we take the noncombustion solution: a shock wave $S(u_-, u_+)$ when $u_+ < u_- + q_1$ as in [7], the strength of rarefaction wave is 0 ($u_+ < u_- + q_1$, no rarefaction wave) or $u_* - u_+ (u_+ > u_- + q_1)$ as $u_+$ perturbed near $u_- + q_1$. So, it is not continuous. Therefore, the structure is unstable.

4.2. Structurally stable solutions to the Riemann problems for nonconvex case. For simplicity, we consider the simplest nonconvex case, i.e., the flux function $f$ has only one inflection point with two assumptions: $f'(\pm\infty) = -\infty$ or $f'(\pm\infty) = +\infty$ (see Fig.4.4.).
As a preparation, we show a theorem on the Riemann problem for a noncombustion scalar conservation laws without convexity

\[
\begin{align*}
&u_t + f(u)_x = 0, \\
&u|_{t=0} = u_\pm, \quad \pm x > 0. 
\end{align*}
\]

(14)

**Theorem 4.3.** ([1]) The Riemann solution of a noncombustion scalar conservation laws with the simplest nonconvex flux function \(f(u)\) and \(u_- < \tilde{u} < u_+\) is as follows (see Fig.4.5):

\[
\text{RJ}(u_-, u_+, u_+) : \begin{cases} 
  u = u_\pm, \\
  \tilde{u} \notin (g'(u_-; u_+, u_-), g'(u_+; u_-, u_+)), \\
  \tilde{u} = g'(u; u_+, u_-), \quad \tilde{u} \in (g'(u_-; u_+, u_-), g'(u_+; u_-, u_+)),
\end{cases}
\]

(15)

where \(g(u; u_+, u_-)\) is the convex hull of \(f(u)\): passing through points \((f(u_-), u_-)\) and \((f(u_+), u_+)\) and satisfies \(g'(u_-; u_+, u_-) \leq g'(u_+; u_-, u_+)\).
Combining the methods used in the theorem and Theorem 4.1-4.2 on the Riemann problems of a scalar convex CJ combustion model, we will construct the Riemann solutions in nonconvex combustion case in this paper.

4.2.1. Solutions for \( f'(\pm \infty) = -\infty \).

Denote \( \tilde{u} \) as the inflection point of \( f(u) \). We assume \( f'(\tilde{u}) > 0 \) (Fig.4.4 and 4.6). For \( f'(\tilde{u}) \leq 0 \), it can be discussed in the similar way. In this case, there are \( u_1, u_2, u'_1 \) and \( u'_2 (u'_2 < u_1 < u_2 < u'_1) \) such that \( f'(u_1) = f'(u_2) = 0, f(u_1) = f(u'_1), f(u_2) = f(u'_2) \). For the nonconvex combustion model, we need only consider the case \( u_- < \tilde{u} \).

We construct the solutions in two cases: 1. \( u_- \in (-\infty, u_1] \); 2. \( u_- \in (u_1, \tilde{u}) \).

Each of them consists of several subcases according to the value of \( u_+ \) as well as \( q_0 \) and position of ignition point \( u = 0 \) on the graph of \( f(u) \).

Case 1. \( u_- \in (-\infty, u_1] \). We discuss this case into three subcases due to the value of \( u_+ \).

\[ u_- \in (u_-, u_1) \] or \( 0 \in [u_1, u'_1) \) or \( 0 \in [u'_1, +\infty) \).

Fig.4.6. Lift upward the unburnt arc by \( q \) to get \( q_1, q_2, \) and \( q_3 \).

Subcase 1.1. \( 0 \in (u_-, u_1) \). The ignition point is lower than \( u_1 \). Lifting upward the arc \([u_-, f(u_-), (0, f(0))]\) on the graph of \( f(u) \) (we call it an unburnt arc) by \( q \) (see Fig.4.6), we get \( q_3 > q_2 > q_1 > 0 \) satisfying:

1. \( q_1 > 0 : f(q_1) = f(0), f'(q_1) < 0 \);
2. \( q_2 > q_1 \) : there is a common tangent point \((\tilde{u} + q_2, f(\tilde{u}))(\tilde{u} \in (u_-, 0))\) satisfying \( f'(\tilde{u}) = f'(\tilde{u} + q_2) \);
3. \( q_3 > q_2 \) : there is a common tangent from \((u_- + q_3, f(u_-))\) on the lifted arc to \( f(u) \). Therefore, there exists \( \tilde{u} > u_2 \) satisfying \( f'(\tilde{u}) = f'(u_-) = \frac{f(u_-) - f(q_2)}{u_- - q_2} \).

According to the different value of binding energy \( q_0 \), there are three subcases.

Subcase 1.1.1. \( 0 < q_0 < q_1 \). In this case, there exist \( u_+, u_{++}, u_{**}, \) and \( u_{***} \) satisfying \( u_{***} > u_+ > u_2 > u_* > u_1 \), \( f'(0) = \frac{f(u_+)}{u_+ - q_0} = \frac{f(u_{**}) - f(0)}{u_{**} - q_0} \), and \( f'(u_-) = \frac{f(u_{**}) - f(u_-)}{u_{**} - u_- - q_0} \) (Fig.4.7-4.8).

We construct the Riemann solutions according to the entropy conditions for the different value of \( u_+ \in (0, u_*] \cup (u_{**}, u_{***}) \cup (u_{***}, +\infty) \).
(a) $u_+ \in (0, u_*)$. Look at Fig.4.7. Firstly, by the Rankine-Hugoniot condition (6), the solution must involve a combustion wave. So, we lift upward the unburnt arc by $q_0$. From the global entropy condition (a) and (b), the unique solution consists of a rarefaction wave $R(u-, 0)$ for $u$, a CJ deflagration wave $\text{CJDF}(0, q_0; u_*, 0)$, and a shock (or rarefaction) wave $S(u_+, u_+)$ (or $R(u_+, u_+)$). When the value of $u_+$ changes from 0 to $u_*$, it can be easily found that the solution is structurally stable. In fact, the noncombustion wave at burnt region in the solution changes from $S(u_+, u_+) \to R(u_+, u_+) \to RJ(u_+, u_+) \to S(u_*, u_+)$ and their strengths change continuously, while the other waves $R(u-, 0) + \text{CJDF}(0, q_0; u_*, 0)$ are unchanged. It can be briefly shown as follows:

$$R(u-, 0) + \text{CJDF}(0, q_0; u_*, 0) + \begin{cases} \text{S}(u_+, u_+), \\ \text{R}(u_+, u_+), \\ \text{RJ}(u_+, u_+). \end{cases}$$

(b) $u_+ \in (u_*, u_*)$. Look at Fig.4.8. In this case, the state at combustion wave front is $(u_f, q_0)$ ($u_+ \leq u_f \leq 0$). The solution is shown as follows:

$$R(u_-, u_f) + \text{CDT}(u_f, q_0; u_+, 0).$$

(c) $u_+ \in (u_*, +\infty)$. In this case, the solution consists of the strong detonation wave $\text{SDT}(u_-, q_0; u_+, 0)$.

**Subcase 1.1.2.** $q_1 < q_0 < q_2$. Look at Fig.4.9. In this case, there is a intersection point of the unburnt arc and $f(u)$, denote it as $(\bar{u} + q_0, f(\bar{u})) (\bar{u} \in [u_-, 0])$. We construct the Riemann solutions in two subcases for the different value of $u_+ \in (0, \bar{u} + q_0] \cup (\bar{u} + q_0, +\infty)$.

(a) $u_+ \in (0, \bar{u} + q_0]$. Look at Fig.4.9. The solution is similar to the subcase 1.1.1(a), but $u$ at the combustion wave front is taken as $\bar{u}$ according to the global entropy condition. In fact, if take the value of $u$ as the ignition point 0 at the front of combustion wave from the condition a, the strength and the propagating speed of the combustion wave do not vary continuously when $u$ is perturbed at $\bar{u}$. The Riemann solution can be briefly shown as follows:

$$R(u_-, \bar{u}) + \text{CJDF}(\bar{u}, q_0, u_*, 0) + \begin{cases} \text{S}(u_+, u_+), \\ \text{R}(u_+, u_+), \\ \text{RJ}(u_+, u_+). \end{cases}$$

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Fig.4.7. Subcase 1.1.1. (a) Solution containing $\text{CJDF}(0, q_0; u_*, 0)$. 

Fig.4.8. In this case, the state at combustion wave front is $(u_f, q_0)$ ($u_+ \leq u_f \leq 0$). The solution is shown as follows:

$$R(u_-, u_f) + \text{CDT}(u_f, q_0; u_+, 0).$$

Fig.4.9. In this case, there is a intersection point of the unburnt arc and $f(u)$, denote it as $(\bar{u} + q_0, f(\bar{u})) (\bar{u} \in [u_-, 0])$. We construct the Riemann solutions in two subcases for the different value of $u_+ \in (0, \bar{u} + q_0] \cup (\bar{u} + q_0, +\infty)$.
Subcase 1.1.1. $q_2 \leq q_0 \leq q_3$. It has two subcases due to $u_+ \in (0, u_{**}] \cup (u_{**}, +\infty)$, where $u_{**}$ is determined by Fig.4.10.

(a) $u_+ \in (0, u_{**}]$. Look at Fig.4.10. In this case, the Riemann solution contains a double contact combustion wave. It can be briefly shown as follows:

$$R(u_-, u_f) + DCC(u_f, q_0; u_{**}, 0) + \begin{cases} R(u_{**}, u_+), \\ \text{or} \quad RJ(u_{**}, u_+) \end{cases}$$

(b) $u_+ \in (u_{**}, +\infty)$. This case is similar to the cases (b) and (c) in subcase 1.1.1.

Subcase 1.1.4. $q_0 \geq q_3$. It has two subcases according to $u_+ \in (0, u_{**}] \cup (u_{**}, +\infty)$, where $u_{**}$ is determined by Fig.4.11.

(a) $u_+ \in (0, u_{**}]$. Look at Fig.4.11. In this case, the Riemann solution contains a CJ detonation wave. It can be briefly shown as follows:

$$CJDT(u_-, q_0; u_{**}, 0) + \begin{cases} R(u_{**}, u_+), \\ \text{or} \quad RJ(u_{**}, u_+) \end{cases}$$

(b) $u_+ \in (u_{**}, +\infty)$. The solution consists of the strong detonation wave $SDT(u_-, q_0; u_+, 0)$.
Subcase 1.2. $0 \in [u_1, u'_1]$. The ignition point is equal or higher than $u_1$ and lower than $u'_1$. The solution can be constructed in the following three cases according to $u_+ \in (0, u'_1] \cup (u'_1, u_{\text{sm}}] \cup (u_{\text{sm}}, +\infty)$, where $u_{\text{sm}}$ and $u_{\text{ss}}$ are determined by Fig.4.12.
(a) $u_+ \in (0, u'_1]$. Obviously, the solution doesn’t contain combustion wave, which is $R(u_-, u_+) \text{ or } RJ(u_-, u_{\text{ss}}, u_+)$. (b) $u_+ \in (u'_1, u_{\text{sm}}]$. Look at Fig.4.12. Similar to above subcase 1.1, we give the solution in the case of $q_0$ is small only. It can be shown as $R(u_-, u_f) + CDT(u_f, q_0; u_+, 0)$.
(c) $u_+ \in (u_{\text{sm}}, +\infty)$. The solution can be briefly shown $SDT(u_-, q_0; u_+, 0)$.

Subcase 1.3. $0 \in [u'_1, +\infty)$. The ignition point is equal or higher than $u'_1$. This case is similar to (ii) of the above subcase 1.2. We omit them.

Case 2. $u_- \in (u_1, \bar{u})$. This case is similar to the convex case in ([7]).

Fig.4.10. Subcase 1.1.2. (b) Solution containing $DCC(u_f, q_0; u_{\text{ss}}, 0)$.

Fig.4.11. Subcase 1.1.4. (a) Solution containing $CJDT(u_-, q_0; u_{\text{ss}}, 0)$. 
4.2.2. Solutions for $f'(\pm \infty) = +\infty$.

As in the above subsection 4.1, $\bar{u}$ is the inflection point of $f(u)$. If $f'(\bar{u}) \geq 0$, it is clear that for any $u_- \leq 0 < u_+$, there exists a unique noncombustion wave of the Riemann problem 3 of (1-2) due to the entropy conditions. Next we assume $f'(\bar{u}) < 0$. In this case, there are $u_1$ and $u_2$ ($u_1 < u_2$) such that $f'(u_1) = f'(u_2) = 0$ (Fig.4.14(b)). For the nonconvex combustion model, we need only consider the case $u_- < \bar{u}$.

We construct the solutions in two cases: 1. $u_- \in (-\infty, u_2^1]$; 2. $u_- \in (u_2^1, \bar{u})$. Each of them consists of several subcases according to the different value of $u_+$, and $q_0$ as well as position of ignition point $u = 0$ on the graph of $f(u)$.

**Case 1.** $u_- \in (-\infty, u_2^1]$. It is easy to see that the Riemann problem has the unique noncombustion solution for $u_- \leq u_2^1$.

**Subcase 2.1.** $0 \in (u_-, u_3)$. The ignition point is lower than $u_3$. We classify the value of $u_+$ into three cases: 1) $u_+ \in (0, u_3)$ and $f(u_+) \geq f(u_-)$; 2) $u_+ \in (0, \bar{u})$ and $f(u_+) < f(u_-)$; 3) $u_+ \in (\bar{u}, u_3)$ and $f(u_+) < f(u_-)$; 4) $u_+ \in [u_3, +\infty)$.

**Subcase 2.1.1.** $u_+ \in (0, u_3)$ and $f(u_+) \geq f(u_-)$. It appears only when $u_- < u_2$. The noncombustion wave solution is done for this case.

**Subcase 2.1.2.** $u_+ \in (0, \bar{u}]$ and $f(u_+) < f(u_-)$. This case is the same as the convex case.

**Subcase 2.1.3.** $u_+ \in (\bar{u}, u_3)$ and $f(u_-) < f(u_+)$. Let $q_1 > 0$ satisfies $f(u_-) = f(u_- + q_1)$, $u_+, u_q$ satisfy $f'(u_3) = \frac{f(u_- + q_0) - f(u_-)}{u_- + q_0 - u_-}$ for $q_0 > 0$ (see Fig.4.13). We classify binding energy $q_0$ into two subcases: (a) $q_0 \in (0, q_1)$; (b) $q_0 \in [q_1, +\infty)$.

(a) $q_0 \in (0, q_1)$. It has two subcases due to $u_+ \in (\bar{u}, u_q) \cup (u_q, u_3)$.

(a1) $\bar{u} < u_+ < u_q$. From the entropy conditions, considering the structural stability, we can obtain the solution containing a strong detonation, which is shown as $\text{SDT}(u_-, q_0; u_+, 0)$ (see Fig.4.13 and 4.14 (a1)).

The solutions in cases (a1) and (a2) are structural stability when $u_+$ is perturbed at $u_q$. In fact, the propagating speed of combustion wave and the strength of other waves vary continuously with respect to $u_+$.

(b) $q_0 \in [q_1, +\infty)$. In this case, the solution is similar to the case (a2) (see Fig.4.13 and Fig.4.14 (a2)).
Subcase 2.1.4. $u_+ \in [u_3, +\infty)$. This case is similar to above case (a2) and (b).

Subcase 2.2. $0 \in [u_3, +\infty)$. The ignition point is larger than $u_3$. Then, we get $u_+ > u_3$. The solution can be obtained similarly.

![Fig.4.13. Sketch of $q_1; u_3, u_*, u_q$.](image)

(a1) Solution for $\bar{u} < u_+ < u_q$.

(a2) Solution for $u_+ \geq u_q$.

![Fig.4.14. Solutions for $q_0 \in (0, q_1)$.](image)

Remark 3. In the similar way, we can obtain the structural stability of the Riemann solution for the variation of the values of $u_-$ when fixed $u_+$ and $q_0$.

(a2) $u_+ \in (u_q, u_3)$. The solution is shown as follows (see Fig.4.10 (a2)):

$$ WDF(u_-, q_0, u_*, 0) + \left\{ \begin{array}{ll}
S(u_*, u_+) , & \text{or} \\
R(u_*, u_+) . & \end{array} \right. $$

Summarizing the above results, we obtain our main theorem.

**Theorem 4.4.** There exists a unique self-similar entropy solution for the Riemann problem (3) of (1-2) in the simplest nonconvex case, i.e., the flux function $f$ has only one inflection point with two assumptions: $f'(\pm \infty) = -\infty$ or $f'(\pm \infty) = +\infty$. Considering the pointwise and global entropy conditions, the Riemann solutions can be constructed uniquely case by case.

5. **Transition from deflagration to detonation.** Based on the Riemann solutions constructed above, we investigate transitions between combustion waves in three cases.
Firstly, we focus our attention on the case that a CJ deflagration wave \( \text{CJDF}(0, q_0; u_m, 0) \) is overtaken by a shock \( S(u_m, u_+) \). We assume that the ignition point \( 0 < u_1, u_+ > u_{**}, q_0 < q_1 \) and \( u_m \in (u_1, u_2) \) satisfying \( f'(0) = \frac{f(u_m) - f(0)}{u_m - q_0} \), see Fig.4.1(a) and Fig.5.1. The shock wave \( S(u_m, u_+) \) will overtake CJ deflagration wave \( \text{CJDF}(0, q_0; u_m, 0) \) at a finite time \( t = t_1 \), i.e., a point \( (x_1, t_1) \) according to the entropy conditions and thus. Then, a new generalized Riemann problem is formed with \( R(u_-, q_0, 0; 0, q_0) \) and \( (u_+, 0) \) as its left-hand side state and right-hand side state, respectively. They will interact each other. From the results in above subcase 1.1.1 (a)(Fig.4.3) and (c), the solution is strong detonation \( \text{SDT}(u, q_0; u_+, 0) : x = x(t) \) satisfying that

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(u_+) - f(u) \quad u_+ - u - q_0, \\
x &= f'(u)t, \quad u \in (u_-, 0), \\
x(t_1) &= x_1.
\end{align*}
\]

Because of \( f(u)(u \in [u_-, 0]) \) is concave, we know that the integral curve \( x = x(t) \) is convex up to a finite time \( t = t_2 > t_1 \) when the detonation wave \( \text{SDT}(u, q_0; u_+, 0) \) penetrates \( R(u_-, 0) \) completely. Finally, \( \text{SDT}(u_-, q_0; u_+, 0) \) is formed. This can be expressed in the following (see Fig.5.1):

\[
R(u_-, 0) + \text{CJDF}(0, q_0; u_m, 0) + S(u_m, u_+) \rightarrow R(u_-, u) + \text{SDT}(u, q_0; u_+, 0) \rightarrow \text{SDT}(u_-, q_0; u_+, 0).
\]

We find that the shock speeds up the deflagration.

Secondly, we study the case that a weak deflagration wave \( \text{WDF}(u_-, q_0; u_m, 0) \) is overtaken by a shock \( S(u_m, u_+) \). We assume that \( u_- < u_1, q_0 < q_1, u_- + q_1 < u_+ < u_q \) and \( u_m > u_2 \). The shock wave \( S(u_m, u_+) \) will overtake the weak deflagration wave \( \text{WDF}(u_-, q_0; u_m, 0) \) at a finite time according to the entropy conditions and thus a new Riemann problem is formed with \( (u_-, q_0) \) and \( (u_+, 0) \) as its left-hand side state and right-hand side state, respectively. From the result in above subcase 2.1.3 (a1)(Fig.4.10(a1)), the Riemann solution is strong detonation \( \text{SDT}(u_-, q_0; u_+, 0) \).

This can be expressed in the following (see Fig.5.2):

\[
\text{WDF}(u_-, q_0; u_m, 0) + S(u_m, u_+) \rightarrow \text{SDT}(u_-, q_0; u_+, 0).
\]

We obtain that the shock slows down the deflagration.

At last, we discuss the case that a strong detonation wave \( \text{SDT}(u_-, q_0; u_m, 0) \) is overtaken by a shock \( S(u_m, u_+) \). We assume that \( q_0 < q_1, u_- < 0 < u_1 \) and \( u_+ > u_m > u_{**} \). The shock wave \( S(u_m, u_+) \) will overtake the strong detonation wave \( \text{SDT}(u_-, q_0; u_m, 0) \) at a finite time according to the entropy conditions and thus a new Riemann problem is formed with \( (u_-, q_0) \) and \( (u_+, 0) \) as its left-hand side state and right-hand side state, respectively. Similarly, the Riemann solution is strong detonation \( \text{SDT}(u_-, q_0; u_+, 0) \). This can be expressed in the following (see Fig.5.3):

\[
\text{SDT}(u_-, q_0; u_m, 0) + S(u_m, u_+) \rightarrow \text{SDT}(u_-, q_0; u_+, 0).
\]

**Remark 4.** When a CJ detonation is overtaken by a shock, the result is similar to the above third case.

In summary, we have
Theorem 5.1. When a shock overtakes a combustion wave, it transforms the deflagration to a detonation, speeds up the CJ deflagration and slows down the weak deflagration, and it speeds up the detonation.

Fig.5.1. Transition from CJDF to SDT.

Fig.5.2. Transition from WDF to SDT.

Fig.5.3. Transition from SDT to SDT.
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