THE K-THEORY TYPE OF QUANTUM CW-COMPLEXES

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ABSTRACT. The multipullback quantization of complex projective spaces lacks the naive quantum CW-complex structure because the quantization of an embedding of the n-skeleton into the (n + 1)-skeleton does not exist. To overcome this difficulty, we introduce the framework of cw-Waldhausen categories, which includes the concept of weak equivalences leading to the notion of a finite weak quantum CW-complex in the realm of unital C*-algebras. Here weak equivalences are unital ∗-homomorphisms that induce an isomorphism on K-theory. Better still, we construct a noncommutative counterpart of the cup product in K-theory, which is equivalent to its standard version in the classical case. To this end, we define k-topology, a noncommutative version of Grothendieck topology with covering families given by compact principal bundles and bases related by continuous maps, which leads to the much desired idea of multiplicative K-theory for noncommutative C*-algebras. Combining this with cw-Waldhausen structure on the category of compact quantum spaces, we arrive at the multiplicative K-theory type of finite weak quantum CW-complexes. We show that non-isomorphic quantizations of the standard CW-complex structure of a complex projective space enjoy the same multiplicative K-theory type admitting a noncommutative generalization of the Atiyah–Todd calculation of the K-theory ring in terms of truncated polynomials.

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1. Introduction

The main goal of this paper is to create and develop a conservative extension of the theory of finite CW-complexes, together with their K-theory, to the framework of noncommutative topology. To substantiate and exemplify our theory we show that the resulting extension covers some well-known quantizations of important finite CW-complexes.

The problem with the state of the art of noncommutative topology, mostly based on KK-theory, is that when applied to classical spaces it identifies very different homotopy types (e.g. disconnected and connected ones, see the discussion in Section 2). This means that the present noncommutative topology is not a conservative extension of classical topology, making its name a misnomer. Our objective here is to complete the existing approach with structures and methods using the richness of noncommutative geometry, without changing the content of the classical theory.

The source of the main difficulties with extending classical topology to the noncommutative setting is lack of the diagonal of a noncommutative space, which classically is necessary to construct a cup product in K-theory, a fundamental tool to obtain many important results. Our new approach to a multiplicative K-theory consists in the structure of a module over the K-ring structure on the plain abelian K-group, equipped with a distinguished element given by the class of the algebra itself. Notice that we can identify the K-ring with its (cyclic) module spanned by that distinguished element of its underlying abelian K-group. Therefore in the classical setting our new approach is equivalent to the standard one. Notice also that even in the classical case the K-ring structure has to be weakened sometimes to the plain abelian group structure. For example, the Mayer-Vietoris relates K-groups by maps not always preserving cup products.

Untill the present paper, among the two interrelated parts of the above structure, only the plain abelian part had been extended to the noncommutative setting (one can see e.g. [6]). In order to extend the full structure we introduce the notion of \textit{k-topology}. The idea is that, although the balanced tensor product of right modules over a noncommutative algebra doesn’t make sense, the balanced tensor product of noncommutative vector bundles associated with a given compact quantum principal bundle makes perfect sense since they are bimodules in a natural way [35, Chap. 2.5]. In classical topology, it changes nothing since every vector bundle is associated to some compact principal bundle (e.g. its bundle of unital frames) and the tensor product of the corresponding modules of sections over the commutative algebra of functions coincides with the tensor product of the corresponding bimodules since they are symmetric by construction. In the noncommutative setting, however, only the second option is viable. Since symmetric bimodules correspond to sheaves supported on the diagonal, the tensor product of associated noncommutative vector bundles is a replacement for a non-existent diagonal.

Classically, \textit{k-topology} is a version of Grothendieck topology with covering families given by compact principal bundles, whose bases are related by continuous maps. In the noncommutative setting, when one tries to use compact quantum principal bundles instead, one faces the problem that the customary use of balanced tensor products to model fibered products and hence pullbacks, is not based on a universal property in the category of associative unital algebras, hence it must be made a part of the structure. We define the pullback of compact quantum principal bundles independently, so to make
it satisfy a universal property in the commutative case. Its definition is so natural that many aspects of the theory of compact quantum principal bundles can be equivalently rewritten in terms of it.

Maps along which one can pullback compact quantum principal bundles, in the above sense, are called \textit{k-continuous}. We show that compact quantum principal bundles as covering families and k-continuous maps between bases form a noncommutative version of a \textit{coverage} (for the classical case see [26]), which we call k-topology. We stress the fact that in the classical case, thanks to the existence of pullbacks of (compact) principal bundles along continuous maps between (compact) spaces, k-continuity means simply continuity and k-topology depends only on topology. However, using noncommutative associated vector bundles, we successfully extend our new approach to multiplicative structure of K-theory from the classical to the noncommutative setting. This extension provides a presheaf on k-topology taking values in the category of modules with a distinguished element. The above discussion about K-theory of compact quantum spaces can be summarized succinctly as follows.

\begin{center}
\textbf{k-topology $\sim$ multiplicative topological K-theory}
\end{center}

While the classical CW-structure is a tool to compute topological K-theory, some quantizations of CW-complexes lack of a CW-structure in a strict sense. We will see in some examples that while the quantization of an embedding of one skeleton into the next one is impossible, it can be done for its compact tubular neighbourhood. Classically, this doesn’t change the situation a lot, since the compact tubular neighbourhood can be deformation retracted onto the skeleton of lower dimension. However, in the quantized setting, this property doesn’t always hold (think for example of a family of quantum disks, each known to be non-contractible, transversal to the skeleton of lower dimension). To overcome this difficulty, we introduce a framework of \textit{cw-Waldhausen categories} which allows a calculus of fractions [17] leading to a notion of \textit{weak cofibration} and finally of \textit{weak CW-structure} (for standard Waldhausen categories see [42]). This weak CW-structure is an equally good tool to compute K-theory. Moreover, it exists in some examples lacking a naively quantized \textit{strict} CW-structure.

Notice that our construction seems to generalize Eilers–Lorin–Pedersen’s notion of noncommutative CW-complex [14] by allowing more general noncommutative balls and spheres as building blocks, and by loosening gluing maps by formally inverting K-equivalences (i.e. morphisms inducing isomorphisms of K-groups, that are weak equivalences in our cw-Waldhausen structure). On the other hand, since formal inversion of weak equivalences consisting of maps inducing isomorphisms in a (co)homology theory is a standard procedure in Algebraic Topology [8], we do not change much the classical picture when extending it to the noncommutative setting. However, although it would be extremely interesting to see how far the Bousfield localization with respect to K-theory can be extended to quantum spaces, we postpone this study to the future since we have only fairly trivial examples of K-local quantum spaces.
The latter part of discussion can be summarized in the following diagram:

\[
\text{cw-Waldhausen category} \rightsquigarrow \text{weak CW-structure}
\]

After formal inversion of K-equivalences, we can group different quantizations of CW-complexes in a single K-theory type (we do not insist on this class to be maximal). The K-theory can be computed using any model in a given K-theory type, e.g. using weak CW-structures which can turn out to be strict for some models. For example, we will show that both the equatorial Podleś sphere and mirror quantum sphere (which we show to have a weak CW-structure) are K-equivalent to the standard Podleś sphere (which has an obvious strict CW-structure). Furthermore, a class of models admitting k-topologies related by k-continuous K-equivalences defines a multiplicative K-theory type.

Other examples of quantized CW-structures are the multipullback quantum real projective plane and Toeplitz compact quantum surfaces. However, we show that, in general, there are (even strict) quantum CW-structures which are not quantizations of any classical one. A family of examples is given by quantum weighted complex projective lines, also known as quantum teardrops.

Our main example of a multiplicative K-theory type is based on our result that there exists a k-continuous K-equivalence relating the weak CW-structures of a multipullback and a $q$-deformed quantum complex projective space. The existence of such a K-equivalence is quite unexpected, since the C*-algebras of the above quantum projective spaces are not only non-isomorphic, but also come from very different constructions: one is obtained by a multipullback procedure, the other is a quantum homogeneous space of a compact quantum group (as well as a graph C*-algebra), although both are groupoid C*-algebras. Moreover, the CW-structure of the former is only weak while the one of the latter is strict.

To prove effectiveness of our machinery, we obtain a noncommutative version of the Atiyah–Todd result for such a multiplicative K-theory type of weak CW-structures. As in the classical case, we compute the system of multiplicative K-theories of (weak) hyperplane skeleta, equivalent to the inverse system of rings of truncated polynomials over the integers.

2. Multiplicative K-theory of compact quantum spaces

2.1. Going beyond K-groups and KK-equivalence. For the sake of simplicity and the correspondence between classical and quantized compact Hausdorff spaces, we denote a C*-algebra corresponding to an object $X$ of $(\text{C}^*\text{-Alg})^{\text{op}}$ by $C(X)$. We call an object $X$ of this category finite if its corresponding C*-algebra $C(X)$ is finite dimensional. In this convention, we write

\[
K^*(X) := K_*(C(X))
\]

(2.1)

to emphasize that the K-theory of unital C*-algebras extends the topological K-theory of compact Hausdorff spaces. For the relative K-theory, we write

\[
K^*(X, Z) := K_*(C(X), J_Z).
\]

(2.2)
Here \( J_Z \) := \( \ker (C(X) \to C(Z)) \triangleleft C(X) \), for \( C(X) \) commutative being the ideal of the functions vanishing on the closed subset \( Z \subseteq X \). Note that, by the Gelfand-Naimark duality, every closed \(*\)-ideal in a unital commutative \( C^* \)-algebra is of this form, what motivates our notation [6, Section 5.4]. Similarly, for compact metrizable spaces we can define the bivariant theory using the \( \mathbb{Z}/2\mathbb{Z} \)-graded Kasparov KK-theory as follows

\[
KK_s(X, Y) := K^*(C(Y), C(X)),
\]

and we extend this notation to (metrizable, meaning corresponding to separable \( C^* \)-algebras) compact quantum spaces. Note that in this notation K-homology and K-theory can be expressed by KK-theory and the one point space, denoted by \( \ast \), as follows.

\[
K_s(X) = KK_s(\ast, X), \quad K^*(X) = KK_s(X, \ast).
\]

Classically, K-equivalence being implemented by a continuous map, it preserves the ring structure of topological K-theory, absent in the noncommutative setting: if two compact Hausdorff spaces \( X \) and \( Y \) are K-equivalent, then \( K^*(X) \) and \( K^*(Y) \) are isomorphic as rings. As an immediate corollary, there are many KK-equivalent classical spaces that are not K-equivalent, e.g. take any two spaces \( X \) and \( Y \) such that \( K^*(X) \) and \( K^*(Y) \) are isomorphic as abelian groups but not as rings. Since their \( C^* \)-algebras are UCT, they are KK-equivalent, but they cannot be K-equivalent by the above argument. For instance, the complex projective space \( \mathbb{C}P^n \) and the discrete space \( P_{n+1} \) consisting of \( n + 1 \) points have the K-theories

\[
K^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1}), \quad K^*(P_{n+1}) \cong \mathbb{Z}^x/(x^{n+1}),
\]

that are isomorphic as abelian groups but not as rings, obviously. Another example showing the usefulness of the ring structure of K-theory is in the proof that classically, for \( n \geq 3 \), there is no retraction from \( \mathbb{C}P^n \) its projective hyperplane \( \mathbb{C}P^{n-1} \). Indeed, such a retraction would induce an injective ring homomorphism

\[
f : K^*(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[x]/(x^n) \to K^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1}).
\]

Let \( f(x) = a + by + cy^2 + \ldots \). Since \( 0 = f(x^n) = f(x)^n = a^n + na^{n-1}by + \ldots \), where the dots are terms of degree at least 2 in \( y \), one has \( a = b = 0 \). But then

\[
f(x^{n-1}) = f(x)^{n-1} = c^{n-1}y^{2n-2} + \ldots = 0,
\]

since \( 2n - 2 \geq n + 1 \), contradicting injectivity of \( f \). It is clear that our argument with \( \mathbb{Z}/2\mathbb{Z} \)-graded K-theory does not work for \( n = 2 \), since in that case \( x \mapsto y^2 \) is an injective ring homomorphism, but one can prove the same result using the \( \mathbb{Z} \)-grading on integral cohomology.

In the quantum setting, all these arguments could not be applied because of the lack of ring structure of K-theory (and, of course, we don’t have a replacement for integral cohomology). Moreover, in the example of complex projective spaces, not only the proof above cannot be reproduced, but the statement is even false: there is indeed a retraction from \( \mathbb{C}P^n \) to \( \mathbb{C}P^{n-1} \), for every \( n \geq 1 \) (see the proof of Theorem 4.2 in [4]).
2.2. From k-topology to multiplicative K-theory. For a general unital separable C*-algebra $A$, the $\mathbb{Z}/2\mathbb{Z}$-graded K-group $K_*(A)$ is a module over the unital ring $KK^*(A,A)$, with both ring and module structure given by the Kasparov product. However, the ring $KK^*(A,A)$ cannot be a replacement for the multiplicative K-theory for two reasons. First, contrary to the K-theory functor, the association $A \mapsto KK^*(A,A)$ is not a functor, and second, for commutative $A$ it is usually much bigger than $K_*(A)$.

Consider a compact quantum principal $G$-bundle $E \to X$, with $G$ a compact quantum group. Every finite-dimensional unitary representation $V$ of $G$ defines an associated vector bundle, given at the C*-algebraic level by the cotensor product $C(E) \square^{C(G)} V$. Explicitly, denoting by $\delta_L$ the coaction of $C(G)$ on $V$ and by $\delta_R$ the coaction on $C(E)$, the cotensor product is defined by

$$C(E) \square^{C(G)} V := \ker \{ \delta_R \otimes \text{id}_V - \text{id}_{C(E)} \otimes \delta_L : C(E) \otimes V \to C(E) \otimes C(G) \otimes V \}, \quad (2.6)$$

where the tensor product between C*-algebras is the minimal one, and the tensor product with a finite-dimensional vector space $V$ doesn’t need to be completed.

Observe that (2.6) is not just a one-sided $C(X)$-module, but a $C(X)$-bimodule that is finitely generated and projective as a right module. The Hermitian scalar product of $V$ induces a canonical right pre-Hilbert module structure on the right $C(X)$-module $C(E) \square^{C(G)} V$, given by:

$$\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle := a_1^* a_2 \langle v_1, v_2 \rangle \quad \forall \ a_1, a_2 \in C(E), v_1, v_2 \in V.$$

Consider the monoid of isomorphism classes in the category $\text{Rep}(G)$ of finite-dimensional unitary representations of $G$, with operations given by direct sum and tensor product. Denote by $R(G)$ the associated Grothendieck ring, which from now on we will call representation ring of $G$. This is a unital ring, with unit given by $\mathbb{C}$ regarded as a trivial one-dimensional representation of $G$.

Lemma 2.1. The map

$$R(G) \to KK_*(X,X), \quad V \mapsto C(E) \square^{C(G)} V,$$

is a homomorphism of unital rings.

Proof. The cotensor product trivially preserves the direct sums. Next, it was proved by Baum and Hajac [5] that for any compact quantum principal $G$-bundle $E \to X$ there exists a dense unital $\ast$-subalgebra $PW(C(E)) \subseteq C(E)$ and a Hopf $\ast$-algebra $PW(C(G)) \subseteq C(G)$, such that

$$C(E) \square^{C(G)} V = PW(C(E)) \square^{PW(C(G))} V$$

for every finite-dimensional unitary representation $V$. Furthermore, by a result of Schauenburg and Schneider [36], the functor $PW(C(E)) \square^{PW(C(G))} (-)$ from $\text{Rep}(G)$ to the category of Hilbert $C(X)$-bimodules is strong monoidal. But, from the first observation, $PW(C(E)) \square^{PW(C(G))} (-) = C(E) \square^{C(G)} (-)$.

Finally, since we can make any Hilbert $C(X)$-bimodule into an even Kasparov $C(X)$-bimodule with a trivial odd part, and hence a trivial operator $F := 0$, we get a map $\text{Rep}(G) \to KK_*(X,X)$ transforming direct sum and tensor product into addition and the Kasparov product, respectively. Such a map factors uniquely through a ring homomorphism $R(G) \to KK_*(X,X)$, by the universal property of Grothendieck group completion. \hfill $\Box$
Combining previous result with the action of $KK_*(X,X)$ on $K^*(X)$ we immediately get the following corollary.

**Corollary 2.2.** For any compact quantum principal $G$-bundle $E \to X$, the group $K^*(X)$ is a module over $R(G)$, with action induced by the tensor product with the bimodule associated to a corepresentation.

Our goal now is to use the above construction to obtain an invariant (i.e. a functor) of quantum spaces which is an enhancement of K-theory as plain abelian group, which does not allow for example to distinguish between such different spaces as in (2.5). As a motivation, let us remind first the classical context. Consider the category whose objects are pairs $(G, E \to X)$ of a compact Hausdorff group $G$ and a compact principal $G$-bundle $E \to X$, and whose morphisms

$$(G', E' \to X') \to (G, E \to X)$$

are pairs consisting of a morphism $\gamma: G' \to G$ of groups and a commutative diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{\tilde{f}} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
X' & \xrightarrow{f} & X
\end{array}
$$

where $\tilde{f}$ is $G'$-equivariant. It is not difficult to check that the map:

$$(e', g) \mapsto (\pi'(e'), \tilde{f}(e')g),$$

(2.7)

is well-defined and it is an isomorphism in the category of right $G$-spaces equipped with a map to $X'$ and a $G$-equivariant map to $E$. Indeed, the map (2.7) is a morphism of principal $G$-bundles over $X' —$ the induction of the principal $G'$-bundle $E' \to X'$ to a principal $G$-bundle over $X'$ along $\gamma$ and the pullback of the principal $G$-bundle $E$ over $X$ to a principal $G$-bundle over $X'$ along $f —$ and it is well known that any morphism principal bundles with the same base and structure group is an isomorphism. The map (2.7) is not only a morphism of principal $G$-bundles over $X'$, but it can be also checked that it commutes with the canonical $G$-equivariant maps to $E$.

The main idea of our construction relies on the identification of the K-theory $K^*(X)$ of the base of a principal $G$-bundle $E \to X$ with the $G$-equivariant K-theory $K^*_G(E)$ of the $G$-space $E$, and the natural ring homomorphism

$$R(G) \cong K^*(BG) \to K^*_G(E).$$

Since the tensor product of vector bundles induces the ring structure of K-theory and every vector bundle is associated to a principal bundle, and the K-theory ring admits a distinguished element $1$ given by the class of the trivial line bundle, the tensoring of $1$ with associated vector bundles fills the whole $K^0(X)$. We have the following diagram

$$
\begin{array}{ccc}
R(G) & \xrightarrow{\text{ring hom}} & K^0(X) \\
\downarrow{\text{action}} & & \downarrow{\text{action}} \\
[X,G] & \xrightarrow{\text{module map}} & K^1(X)
\end{array}
$$
where \([X, G]\) are homotopy classes of continuous maps \(X \to G\).

Consider now the category of module over unital rings: objects are pairs \((R, M)\) of a unital ring \(R\) and an \(R\)-module \(M\); morphisms are pairs of a morphism \(R \to R'\) of unital rings and a morphism \(M \to M'\) of \(R\)-modules. Then we have a functor from the first category to the second, sending \((G, E \to X)\) to the pair \((R(G), K^*(X))\).

Observe that \(K^*(X)\) has a distinguished element \(1\) given by the class of the trivial \(C(X)\)-module. Called \(I\) the kernel of the action of \(R(G)\) on \(1\), we get a quotient ring \(R(G)/I\) and an injective \(R(G)\)-module map:

\[
R(G)/I \to K^*(X)
\]

which for classical spaces is an (injective) ring homomorphism. If \(G = U(1)\) and \(E \to X = S^{2n+1} \to \mathbb{C}P^n\), then \(I = 0\) and the ring homomorphism

\[
R(U(1)) = \mathbb{Z}[t, t^{-1}] \to K^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})
\]

is onto.

However, if we restrict our attention to classical spaces over \(BG\), for a fixed compact group \(G\), we can equip the K-theory of such spaces with a structure of a module over the representation ring \(R(G)\). Then, continuous maps over \(BG\) induce \(R(G)\)-module maps in K-theory. In particular, if two objects \(X\) and \(Y\) in this category are \(K\)-equivalent through a map over \(BG\), then \(K^*(X)\) and \(K^*(Y)\) are isomorphic as \(R(G)\)-modules. Therefore, we could use the module structure instead of the ring structure to distinguish spaces over \(BG\).

This approach can be translated into the language of principal \(G\)-bundles. Indeed, consider a morphism in this category of spaces over \(BG\):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow c_X & & \downarrow c_Y \\
BG & & BG
\end{array}
\]

Pulling back the universal principal \(G\)-bundle \(EG \to BG\) we get a commutative diagram

\[
\begin{array}{ccc}
c_X^*EG & \xrightarrow{\tilde{f}} & c_Y^*EG \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

Since \(c_X^* = f^*c_Y^*\), the latter is a pullback diagram of principal \(G\)-bundles, where \(\tilde{f}\) is \(G\)-equivariant.
Conversely, suppose we have a morphism of principal $G$-bundles, that means a commutative diagram:

\[
\begin{array}{ccc}
E_X & \xrightarrow{\tilde{f}} & E_Y \\
\pi_X & \downarrow & \pi_Y \\
X & \xrightarrow{f} & Y
\end{array}
\] (2.9)

where the columns are principal $G$-bundles and $\tilde{f}$ a $G$-equivariant map. Notice that $\tilde{f}$ factors through a pullback:

\[
\begin{array}{ccc}
E_X & \xrightarrow{\tilde{f}} & E_Y \\
\pi_X & \downarrow & \pi_Y \\
X & \xrightarrow{\pi} & Y
\end{array}
\]

and a morphism $g$ of principal $G$-bundles over the same base space must be an isomorphism. Thus, (2.9) is a pullback diagram as well.

Let $c_Y : Y \to BG$ be a classifying map of the bundle $E_Y \to Y$. Define $c_X := c_Y \circ \pi$. Since (2.9) is a pullback diagram, one easily proves that such a $c_X : X \to BG$ is a classifying map of the bundle $E_X \to X$, and we get a morphism of the type (2.8).

We can further simplify the picture by observing that in (2.9) the diagram is uniquely determined by the equivariant map $\tilde{f}$. We can thus speak of free actions of $G$ related by a $G$-equivariant map $\tilde{f}$. Again, given a free $G$-action on a compact space $E$, $K^*(E/G)$ is an $R(G)$-module, with module structure induced by tensoring by associated vector bundles, and every $G$-equivariant map induces an $R(G)$-modules map between the K-theories of quotient spaces.

**Proposition 2.3.** Consider the standard embedding $\mathbb{S}^{2n-1} \hookrightarrow \mathbb{S}^{2n+1}$. If $n \geq 2$, then there is no $U(1)$-equivariant retraction $\mathbb{S}^{2n+1} \to \mathbb{S}^{2n-1}$.

**Proof.** By contradiction, such an $U(1)$-equivariant retraction would induce a retraction of $\mathbb{C}P^n$ onto $\mathbb{C}P^{n-1}$, and then a section of the restriction map induced in integral cohomology. Such a section is an injective morphism of graded rings from

\[
f : H^*(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}[x]/(x^n) \to H^*(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}).
\]

Now, $f(x) = \lambda y$ must be a scalar multiple of $y$, since $x$ is of degree 2 and $y$ is a generator of $H^2(\mathbb{C}P^n, \mathbb{Z})$. But then, $\lambda^n y^n = f(x)^n = f(x^n) = 0$, hence $\lambda = 0$ and $f(x) = 0$, contradicting injectivity. \qed

3. **Cofibration-Weakening-Waldhausen categories**

The customary framework for homotopy theory of CW-complexes uses model category structures. The question of the existence of a model category structure on C*-algebras
has been raised by Schochet [37]. In our approach, however, inverting not only homotopy equivalences but also maps inducing isomorphism on K-theory, we extend the Mayer–Vietoris principle for K-theory to the context of a specific structure, which we call cofibration-weakening-Waldhausen structure (cw-Waldhausen, in short). We apply this framework to the category of compact quantum spaces (meaning the opposite category of unital C*-algebras). Since general noncommutative C*-algebras rarely admit characters, we are forced to consider an unpointed version of the Waldhausen structure [34] with an initial and a terminal object separated. This does not change the classical situation much because the fact that for such categories there is always a canonical functor into the Waldhausen category of pointed objects. In the classical situation of the category of compact spaces, this functor is faithful and reconstructs the K-theory of compact spaces as the reduced K-theory of compact spaces with a distinguished point added as a disjoint connected component.

Below we provide a categorical framework of cw-Waldhausen categories allowing weakening of the notion of CW-complex structure with respect to K-equivalences. In the case of quantized complex projective spaces, it will allow constructing their weak filtration by skeleta by composing weak hyperplane embeddings of multipullback quantum complex projective spaces as hyperplanes into ones of higher dimension.

**Definition 3.1.** An unpointed Waldhausen category $\mathcal{C}$ is a category with an initial object $\emptyset$ and a terminal object $\star$, with distinguished two classes of maps, $\text{Cof}$ of cofibrations, depicted as $\hookrightarrow$, and $\text{Weq}$ of weak equivalences, depicted as $\sim$, such that

- (Cof1) all isomorphisms and compositions of cofibrations are cofibrations,
- (Cof2) for any object $X$ the unique morphism $\emptyset \rightarrow X$ is a cofibration,
- (Cof3) if $X \rightarrow Y$ is a cofibration and $X \rightarrow \tilde{X}$ any morphism, then the pushout $\tilde{X} \rightarrow \tilde{X} \cup_X Y$ is a cofibration,
- (Weq1) all isomorphisms are weak equivalences,
- (Weq2) weak equivalences are closed under composition,
- (Weq3) “glueing for weak equivalences”: Given any commutative diagram of the form

\[
\begin{array}{ccc}
Z & \xleftarrow{X} & Y \\
\downarrow & & \downarrow \\
\tilde{Z} & \xleftarrow{\tilde{X}} & \tilde{Y}
\end{array}
\]

in which the vertical arrows are weak equivalences and right horizontal maps cofibrations, the induced map $Z \cup_X Y \rightarrow \tilde{Z} \cup_{\tilde{X}} \tilde{Y}$ is a weak equivalence.

**Definition 3.2.** We call an unpointed Waldhausen category a cw-Waldhausen category (cofibration-weakening-Waldhausen category) iff (cw-W) for every pushout diagram

\[
\begin{array}{ccc}
\tilde{j} & \tilde{Z} & \tilde{h} \\
\tilde{Y} & \tilde{Z} & \end{array}
\]

with $j$ being a cofibration, $\tilde{h}$ is a weak equivalence if and only if so is $g$. 
Note that, by $\text{Cof}3$, the map $\tilde{j}$ is necessarilly a cofibration. Introducing the condition $\text{cw-W}$ is motivated by the fact that uncontrolled inverting weak equivalences could lead, in principle, to unwanted collapses of the homotopy category. The latter condition prevents this and allows one to work within the left fraction calculus of the form $\text{Weq}^{-1} \circ \text{Cof}$ in the homotopy category $\text{Ho}(\mathcal{C}) := \mathcal{C}[\text{Weq}^{-1}]$. We call the morphisms in the class $\text{Weq}^{-1} \circ \text{Cof}$ in $\text{Ho}(\mathcal{C})$ weak cofibrations. Under the $(\text{cw-W})$ assumption we can represent them as cospans, which we think of as generalized maps from $X$ to $Y$,

\[
\begin{array}{c}
\tilde{Y} \\
\downarrow & \nearrow \\
X & \leftarrow & Y,
\end{array}
\]

depict them as $X \rightarrow Y$, and compose in the homotopy category $\text{Ho}(\mathcal{C})$ as follows

\[
\begin{array}{c}
j \rightarrow \tilde{Z} \\
\downarrow & \circ & \downarrow \\
Y & \circ & Z \\
\downarrow & \nearrow & \downarrow \\
X & \leftarrow & Y \\
\downarrow & \nearrow & \downarrow \\
\tilde{Z} & \circ & Z
\end{array}
\]

where $\tilde{j}$ and $\tilde{h}$ are the arrows completing the pushout square in the diagram below

3.1. A cw-Waldhausen category of compact quantum spaces. Recall that the pullback [6, Definition 15.3.1] of two morphisms of C*-algebras $A_1 \xrightarrow{\pi_1} A_{12} \leftarrow \xrightarrow{\pi_2} A_2$ can be realized as the fiber-product C*-algebra

\[
A_1 \times_{A_{12}} A_2 := \{(a_1, a_2) \in A_1 \times A_2 \mid \pi_1(a_1) = \pi_2(a_2)\}
\]  

(3.1)

with morphisms $A_1 \leftarrow A_1 \times_{A_{12}} A_2 \rightarrow A_2$ given by the projections on the two factors.

**Theorem 3.3.** The opposite of the category of unital C*-algebras, with unital *-homomorphisms as opposite morphisms, zero C*-algebra as an initial object, complex numbers as a terminal object, surjective unital *-homomorphisms as cofibrations and unital *-homomorphisms inducing an isomorphism on K-theory as weak equivalences, is a cw-Waldhausen category.

**Proof.** It is obvious that the zero algebra (resp. complex numbers) is a terminal (resp. initial) object in the category of unital C*-algebras. (In the proof, we assume that all *-homomorphisms are unital.)
(Cof1) Every *-isomorphism of unital C*-algebras is surjective and a composite of unital surjective *-homomorphisms is such as well.

(Cof2) The *-homomorphism into the zero algebra is surjective.

(Cof3) If \( \hat{C} \to B \) is a surjective unital *-homomorphism and \( \hat{B} \to B \) any *-homomorphism then the pullback *-homomorphism \( \hat{C} := \hat{B} \times_B \hat{C} \to \hat{B} \) is unital surjective.

(Weq1) and (Weq2) are obvious by functoriality of K-theory.

(Weq3) After inverting directions of all arrows, this is verbatim [15, Theorem 3.1]. Suppose we have a commutative diagram (of C*-algebras and morphisms):

\[
\begin{array}{ccc}
A & \overset{\phi}{\rightarrow} & B \\
\phi_1 & & \phi_2 \\
\phi_2 & & \phi_1 \\
A_1 & \overset{\pi_1}{\rightarrow} & A_2 & \overset{\pi_2}{\rightarrow} & B_1 & \overset{q_1}{\rightarrow} & B_2 \\
\phi_1 & & \phi_2 & & q_1 & & q_2 \\
\phi_2 & & \phi_1 & & \pi_1 & & \pi_2 \\
A_{12} & \overset{\phi_{12}}{\rightarrow} & B_{12}
\end{array}
\] (3.2)

with \( \pi_2 \) and \( \rho_2 \) surjective, and suppose the two squares are pullback diagrams. Assume also that the morphisms \( \phi_i \) and \( \phi_{12} \) induce isomorphisms on K-groups:

\[
\phi_{i*} : K_* (A_i) \xrightarrow{\cong} K_* (B_i), \quad \phi_{12*} : K_* (A_{12}) \xrightarrow{\cong} K_* (B_{12}).
\]

Then the morphism \( \phi \) also induces an isomorphism on K-theory:

\[
\phi_* : K_* (A) \xrightarrow{\cong} K_* (B).
\]

(cw-W) Assume that in the following pullback diagram of C*-algebras \( \tau \) is a surjective. Then \( \pi \) is surjective as well and \( \beta \) is a K-equivalence if and only if \( \delta \) is such. Indeed, since surjective *-homomorphisms are regular epimorphisms, they are stable under all pullbacks. This proves surjectivity of \( \pi \). Thanks to surjectivity of \( \tau \) the Mayer-Vietoris theorem provides the six-term exact sequence

\[
\begin{array}{ccc}
K_0(C) & \xrightarrow{(\pi_\ast, -\tau_\ast)} & K_0(A) \oplus K_0(B) & \xrightarrow{(\delta_\ast, -\tau_\ast)} & K_0(D) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(D) & \xrightarrow{(\delta_\ast, -\tau_\ast)} & K_1(A) \oplus K_1(B) & \xrightarrow{(\pi_\ast, -\beta_\ast)} & K_1(C)
\end{array}
\]

which we use as follows.
First, if $\delta$ is a $K$-equivalence, we have a section of the map $(\delta_*,-\tau_*)$, provided by composing the standard embedding $i_A : K_*(A) \to K_*(A) \oplus K_*(B)$ with $\delta_*^{-1}$. This cuts the six-term exact sequence into split short exact sequences

$$0 \to K_*(C) \xrightarrow{(\pi_* \beta_*)} K_*(A) \oplus K_*(B) \xrightarrow{(\delta_*,-\tau_*)} K_*(D) \to 0.$$ 

This splitting produces idempotent endomorphisms of $K_*(A) \oplus K_*(B)$

$$p := i_A \circ \delta_*^{-1} \circ (\delta_*,-\tau_*) = p \circ p, \quad p^\perp := \text{id} - p$$

such that

$$\ker p = \ker(\delta_*,-\tau_*) = \text{im} \left( \frac{\pi_*}{\beta_*} \right), \quad \text{coim} p^\perp = \text{coim} p_B,$$

where $p_B$ is the canonical projection from $K_*(A) \oplus K_*(B)$ onto $K_*(B)$, and the following diagram is commutative

$$\begin{array}{ccc}
K_*(C) & \xrightarrow{\beta_*} & K_*(B) \\
\downarrow{\cong} & & \downarrow{\cong} \\
K_*(A) \oplus K_*(B) & \xrightarrow{p_B} & \text{coim} p^\perp
\end{array}$$

which proves that $\beta_*$ is an isomorphism, i.e. $\beta$ is a $K$-equivalence as well.

Now, to prove the opposite implication, assume in turn that $\beta$ is a $K$-equivalence. This implies that we have a map provided by composing $\beta_*^{-1}$ with the standard projection $p_B : K_*(A) \oplus K_*(B) \to K_*(B)$, for which the map $(\delta_*,-\tau_*)$ is a section. This cuts the six-term exact sequence into split short exact sequences

$$0 \to K_*(C) \xrightarrow{(\pi_* \beta_*)} K_*(A) \oplus K_*(B) \xrightarrow{(\delta_*,-\tau_*)} K_*(D) \to 0.$$ 

This splitting produces idempotent endomorphisms of $K_*(A) \oplus K_*(B)$

$$q := \left( \frac{\pi_*}{\beta_*} \right) \circ \beta_*^{-1} \circ p_B = q \circ q, \quad q^\perp := \text{id} - q$$

such that

$$\text{im} q^\perp = \text{im} i_A, \quad \text{coker} q = \text{coker} \left( \frac{\pi_*}{\beta_*} \right) = \text{coim} (\delta_*,-\tau_*),$$

and the following diagram is commutative

$$\begin{array}{ccc}
K_*(C) & \xrightarrow{\beta_*} & K_*(B) \\
\downarrow{\cong} & & \downarrow{\cong} \\
K_*(A) \oplus K_*(B) & \xrightarrow{\text{coim} p^\perp} & \text{ker} p
\end{array}$$
which proves that $\delta_*$ is an isomorphism, i.e. $\delta$ is a $K$-equivalence as well. □

Keeping in mind opposite directions of arrows in the opposite category, the composition of weak cofibrations in the homotopy category $Ho((C^*-Alg_1)^{op})$, depicted as spans in $C^*-Alg_1$ but understood as morphisms in $Ho((C^*-Alg_1)^{op})$ from the left foot to the right foot, reads as follows

![Diagram]

where $\tilde{\rho}$ and $\tilde{\gamma}$ are the arrows completing the pullback square in the diagram below in the category $C^*-Alg_1$

![Diagram]

**Corollary 3.4.** The category of compact Hausdorff spaces with embeddings as cofibrations and $K$-equivalences as weak equivalences is a cw-Waldhausen category.

**Proof.** First, using the Gelfand-Naimark duality and the fact that the topological K-theory of compact Hausdorff spaces expresses as K-theory of corresponding unital commutative $C^*$-algebras, we can think about the full subcategory of commutative objects of the category of unital associative $C^*$-algebras. Since the pushouts, initial and final objects, cofibrations and K-equivalences of that subcategory are inherited from the category of associative $C^*$-algebras, the cw-Waldhausen structure of the latter restricts to that of this full subcategory. □

Notice that the axioms $Cof1$-$Cof3$ and $Weq1$-$Weq3$ allow us to create new cofibrations and weak equivalences, respectively, from previously given ones. Similarly, our axiom $cw-W$ allows us to create new weak equivalences and weak cofibrations. In particular, the
following pushout diagram of compact quantum spaces, describing a collapse of a closed subspace $Z \subset X$ to a point, reads as a pullback diagram of unital C*-algebras as follows

$$
\begin{array}{ccc}
X/Z & \rightarrow & I^+ \\
\downarrow & \swarrow & \downarrow \\
X & \rightarrow & A \\
\downarrow & \nearrow & \downarrow \\
Z & \rightarrow & A/I
\end{array}
$$

where in the pullback diagram on the right hand side, equivalent to an extension of C*-algebras

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0, \quad (3.3)$$

$I^+$ denotes the minimal unitization of the ideal $I$. Now, thanks to the $cw-W$ axiom, if $K_*(A/I) \cong \mathbb{Z}$ is generated by the $K_0$-class of the unit, the map of unital C*-algebras $I^+ \rightarrow A$ induces an isomorphism in K-theory. Later on we will provide other examples of K-equivalences induced by a K-theory preserving collapse of a compact quantum subspace.

Note also that the K-theory six-term exact sequence of the extension (3.3) is equivalent to the Mayer-Vietoris sequence for the above pullback diagram, which makes K-theory of C*-algebra extensions a special case of our cw-Waldhausen approach.

Similarly, assuming that an augmentation $A \rightarrow \mathbb{C}$ is a K-equivalence, using the axiom $(cw-W)$ as follows

$$
\begin{array}{ccc}
X \vee Y & \rightarrow & A \times_\mathbb{C} B \\
\downarrow & \swarrow & \downarrow \\
X & \rightarrow & A \\
\downarrow & \nearrow & \downarrow \\
Y & \rightarrow & B \\
\downarrow & \searrow & \downarrow \\
* & \rightarrow & \mathbb{C}
\end{array}
$$

we infer that $A \times_\mathbb{C} B \rightarrow B$ is a K-equivalence as well. This means that the canonical map $Y \rightarrow X \vee Y$ is a K-equivalence provided the embedding $\star \rightarrow X$ is a K-equivalence. Note that the latter two simple constructions of K-equivalences work thanks to the axiom $(cw-W)$, that is absent in the original Waldhausen setting.

### 3.2. Finite CW-complexes and K-theory types of their quantizations.

A **finite CW-complex** is a topological space $X$ which is a union of a system of closed embeddings

$$
X^{d_0} \hookrightarrow X^{d_1} \hookrightarrow \cdots \hookrightarrow X^{d_{n-1}} \hookrightarrow X^{d_n} = X \quad (3.4)
$$

with $X^{d_0}$ finite discrete, arising by a finite sequence of pushouts, called **attaching cells**,
where \( \partial \) is the boundary map between disjoint unions of spheres and balls, and \( 0 = d_0 < d_1 < \ldots < d_n \) is the dimension sequence. A system (3.4) is referred to as \textit{filtration by skeleta}.

Keeping the same notation, we generalize this notion to quantum spaces.

**Definition 3.5.** By \textit{boundary map from a K-sphere to a K-ball} we mean a cofibration \( \partial : S^{d-1} \to \mathbb{B}^d \) in the cw-Waldhausen category \( (C^*\text{-Alg}_1)^{\text{op}} \) inducing short exact sequences

\[
0 \to K^0(\mathbb{B}^d) \to K^0(S^{d-1}) \to 0 \to 0,
\]

for \( d \) even, and

\[
0 \to K^1(\mathbb{B}^d) \to K^1(S^{d-1}) \to \mathbb{Z} \to 0,
\]

for \( d \) odd, as classical boundary maps from spheres to balls do.

**Remark 3.6.** Note that, by exactness of the six-term exact sequence of a pair \( (\mathbb{B}^d, S^{d-1}) \), the quotients in the above short exact sequences, by [6], can be canonically identified with the relative K-theory

\[
\tilde{K}^*(S^{d-1})/K^*(\mathbb{B}^d) \cong K^{*+1}(\mathbb{B}^d, S^{d-1})
\]

which classically is isomorphic to the reduced K-theory \( \tilde{K}^{*+1}(S^d) \).
**Definition 3.7.** A finite quantum $K$-weak CW-complex is an object $X$ of the category $\text{Ho}((C^*\text{-Alg}_1)^{op})$ admitting a finite sequence of weak cofibrations

$$X^{d_0} \hookrightarrow X^{d_1} \hookrightarrow \cdots \hookrightarrow X^{d_n-1} \hookrightarrow X^{d_n} = X$$

(3.10)

of the form

$$\tilde{X}^{d_k} \rightarrow X^{d_{k-1}} \rightarrow X^{d_k}$$

(3.11)

Here $X^{d_0}$ is finite, and the above sequence (referred to as weak filtration by skeleta) is compatible with pushouts in $(C^*\text{-Alg}_1)^{op}$ (called attaching cells)

$$X^{d_k} \rightarrow X^{d_{k-1}} \rightarrow \coprod_{i=1}^{i_k} B^{d_k}_i \coprod_{i=1}^{i_k} S^{d_k-1}_i \rightarrow \partial.$$

If in all presentations (3.11) of the weak cofibrations (3.10) the K-equivalences are identities, then we either suppress the adjective “weak” or we use the term “strict” instead.

**Remark 3.8.** Due to Bott periodicity of topological K-theory, the only sensible notion of dimension in Definition 3.5 is modulo 2. Therefore, the classical notion of increasing sequence of dimensions of skeleta in our framework doesn’t make sense. This means that our notion of a CW-complex admits more general examples even classically. For an exotic quantum example see Section 4.3.

The system (3.10) induces a system of $\mathbb{Z}/2\mathbb{Z}$-graded K-groups

$$K^* (X^{d_0}) \leftarrow K^* (X^{d_1}) \leftarrow \cdots \leftarrow K^* (X^{d_n-1}) \leftarrow K^* (X^{d_n})$$

(3.12)

whose arrows are related by the Mayer–Vietoris six-term exact sequence

$$\begin{array}{ccc}
K^0 (X^{d_k}) & \longrightarrow & K^0 (X^{d_{k-1}}) \oplus \bigoplus_{i=1}^{i_k} K^0 (B^{d_k}_i) \\
\downarrow_{d_{10}} & & \downarrow_{d_{10}} \\
\bigoplus_{i=1}^{i_k} K^1 (S^{d_k-1}_i) & \xrightarrow{(a^*_i,-\partial^*_i)} & K^1 (X^{d_{k-1}}) \oplus \bigoplus_{i=1}^{i_k} K^1 (B^{d_k}_i) \leftarrow K^1 (X^{d_k}).
\end{array}$$

Since every homomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups $\partial^*_i : K^* (B^{d_k}_i) \rightarrow K^* (S^{d_k-1}_i)$ is an embedding onto a direct summand, by (3.9) the latter sequence boils down to the
following six-term exact sequence

\[
\begin{array}{c}
\delta_{10} \\
\oplus_{i=1}^{i_k} K^0(\mathbb{S}^{d_k-1}_i) / K^0(\mathbb{B}^{d_k}_i) \\
K^0(X^{d_k}) \longrightarrow K^0(X^{d_k-1}) \longrightarrow \bigoplus_{i=1}^{i_k} K^0(\mathbb{S}^{d_k-1}_i) / K^0(\mathbb{B}^{d_k}_i) \longrightarrow K^1(X^{d_k}) \longrightarrow K^1(X^{d_k-1}) \longrightarrow \ker \delta_{01}\end{array}
\]

allowing inductive calculations of K-theory of quantized CW-complexes, provided the corresponding images and kernels of the \(\delta\) maps are known, according to the following theorem.

**Theorem 3.9.** In the above context we have the system of noncanonical isomorphisms

\[
K^1(X^{d_k-1}) \cong K^1(X^{d_k}) \oplus \ker \delta_{10}
\]  
for \(d_k\) even, and

\[
K^0(X^{d_k-1}) \cong K^0(X^{d_k}) \oplus \ker \delta_{01}
\]  
for \(d_k\) odd. In addition, provided \(K^0(X^{d_k-1})\) is free, we have an additional noncanonical isomorphism

\[
K^0(X^{d_k}) \cong K^0(X^{d_k-1}) \oplus \text{im} \delta_{10}
\]  
for \(d_k\) even, and similarly, provided \(K^1(X^{d_k-1})\) is free, another additional noncanonical isomorphism

\[
K^1(X^{d_k}) \cong K^1(X^{d_k-1}) \oplus \text{im} \delta_{01}
\]  
for \(d_k\) odd.

**Proof.** By (3.5)-(3.8), the above six-term exact sequence is equivalent to the following system of short exact sequences

\[
\begin{array}{c}
0 \longrightarrow \text{im} \delta_{10} \longrightarrow K^0(X^{d_k}) \longrightarrow K^0(X^{d_k-1}) \longrightarrow 0, \quad (3.17) \\
0 \longrightarrow K^1(X^{d_k}) \longrightarrow K^1(X^{d_k-1}) \longrightarrow \ker \delta_{10} \longrightarrow 0, \quad (3.18)
\end{array}
\]

for \(d_k\) even, and

\[
\begin{array}{c}
0 \longrightarrow K^0(X^{d_k}) \longrightarrow K^0(X^{d_k-1}) \longrightarrow \ker \delta_{01} \longrightarrow 0, \quad (3.19) \\
0 \longrightarrow \text{im} \delta_{01} \longrightarrow K^1(X^{d_k}) \longrightarrow K^1(X^{d_k-1}) \longrightarrow 0, \quad (3.20)
\end{array}
\]

for \(d_k\) odd. Since a subgroup of a free abelian group is free abelian, kernels of the \(\delta\) maps are free abelian and in general we have noncanonical isomorphisms (3.13) and (3.14).

For \(d_k\) even (resp. odd), if \(K^0(X^{d_k-1})\) (resp. \(K^1(X^{d_k-1})\)) is free, the short exact sequence (3.17) (resp. (3.20)) splits and we have an additional noncanonical isomorphism (3.15) (resp. (3.16)).
By a quantization of a locally compact Hausdorff space $X$ we mean a fiber in a field of $C^*$-algebras, in the sense of Kasparov [27, 1.5. Definition], i.e. a central algebra map from a $C^*$-algebra of bounded continuous functions on a locally compact, $\sigma$-compact Hausdorff space of parameters to a total $C^*$-algebra, with a fixed central fiber isomorphic to the algebra of continuous functions $C(X)$. Since we allow quite general models for K-balls, e.g. Toeplitz polydiscs, it is natural to encompass quantizations of cartesian products. It is widely accepted to understand them as appropriate fibered tensor products of fields of $C^*$-algebras, which in the version of Kasparov satisfy all expected properties [27, 1.6. Proposition], contrary to the class of continuous fields of $C^*$-algebras in the sense of Dixmier [13, Chapter 10], which is not closed under these tensor products [28, 7]. The case of the Vaksman-Soibelman complex projective spaces is covered by results of [30] providing a structure of a strict quantization in the sense of Dixmier’s continuous fields of $C^*$-algebras.

It is convenient to pass to the the opposite category of unital $C^*$-algebras understood as the category of quantum compact Hausdorff spaces, so that a quantization can be regarded as a fiber of a family of quantum spaces with the central fiber isomorphic to the classical space $X$. Since we want to allow gluings of local pieces like quantum polydiscs, the notion of a continuous field of $C^*$-algebras due to Rieffel is not sufficient for the notorious problem with tensor products [7, 29].

Note that from the very definition it is by no means obvious that that one can have two different quantizations connected by a sequence of morphisms inducing isomorphism in K-theory or their inverses. If $X$ is a finite CW-complex, similar problems arise in connection with the CW-structure and it is clear how to define the $K$-theory type of a quantization of the latter.

Among all CW-complexes probably the most important for mathematics and mathematical physics is the direct limit of the system of hyperplane embeddings of complex projective spaces. It serves as a canonical model for the classifying space $BU(1)$ and bridges between topology and representation theory. Its K-theory ring is the inverse limit of the tower of K-theory rings of the skeleta, the rings of truncated polynomials, and is isomorphic with the adic completion of the representation ring $R(U(1))$ with respect to the augmentation ideal. In particular, the tower of the K-theory rings of the skeleta is a a surjective inverse system of the quotient rings of $R(U(1))$. The kernels of the restriction morphisms in this tower have two equivalent descriptions, a purly representation-theoretic one, in terms of associated vector bundles, and a purly topological one, in terms of Milnor idempotents.

The natural question arising at this point is whether this beautiful picture survives quantization. The aim of the present paper is to create and apply a framework to answer this question in a mathematically rigorous way.

4. Two-dimensional examples

In this section we present some two-dimensional examples of finite quantized CW complexes. These include all Podleś quantum spheres [32], the mirror quantum spheres [23], the multipullback quantum real projective plane [25], Toeplitz compact quantum surfaces [18], which include the quantum real projective plane [22, 19], and quantized
weighted complex projective lines [9]. In the example of 2-spheres, we will show that they are of the same K-theory type. Let us stress that all these finite quantum CW-complexes are strict, except $S^2_{q,\infty}$, which however is a finite quantum K-weak CW-complex, with a weak filtration which is K-equivalent to the strict filtration by skeleta of $S^2_q$.

### 4.1. A K-theory type of quantized 2-spheres.

Probably the easiest non-trivial examples of finite quantum K-weak CW-complexes are given by Podleś [32] and mirror quantum spheres [23]. We denote by $C(S^2_q)$ the C*-algebra of the standard Podleś quantum sphere, by $C(S^2_{q,\pm})$ the equatorial Podleś quantum sphere and by $C(S^2_{q,\infty})$ the mirror quantum sphere. The standard, equatorial and mirror quantum spheres have the following pushout structure:

\[
\begin{array}{ccc}
S^2_q & \xrightarrow{\sigma^+} & B^2_q \\
\downarrow & & \downarrow \\
S^1 & \xrightarrow{\partial} & B^2_q
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
S^2_{q,\pm} & \xrightarrow{\sigma^\pm} & B^2_q \\
\downarrow & & \downarrow \\
S^1 & \xrightarrow{\partial} & B^2_q
\end{array}
\]

The boundary map $\partial : S^1 \to B^2_q$ is dual to the symbol map $\sigma : \mathcal{T} = C(B^2_q) \to C(S^1)$ and it satisfies, as it is well-known, the hypothesis of Definition 3.5 with $d = 2$. From the diagram on the left, we read the CW-complex structure of the standard Podleś quantum sphere, given by the filtration

$$\{\ast\} \hookrightarrow S^2_q.$$

We will not write down the filtration for the other two spheres, but rather prove that they are all K-equivalent.

We can connect the two pushout diagrams in (4.2) as follows. Written it in terms of C*-algebras:

\[
\begin{array}{ccc}
C(S^2_q) & \xrightarrow{\sigma^+} & C(S^2_{q,\pm}) \\
\downarrow & & \downarrow \\
C(S^1) & \xrightarrow{\eta} & \mathcal{T}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
C(S^2_{q,\pm}) & \xrightarrow{\sigma^-} & C(S^2_{q,\infty}) \\
\downarrow & & \downarrow \\
C(S^1) & \xrightarrow{\eta} & \mathcal{T}
\end{array}
\]

where $\eta$ is the unit map, $\sigma^+ = \sigma$ is the standard symbol map, and $\sigma^-$ is the $*$-homomorphism uniquely defined by $\sigma^+(t) = u^*$, where $t$ is the unilateral right shift on $l^2(\mathbb{N})$, generator of $\mathcal{T}$, and $u$ is the standard unitary generator of $C(S^1)$. Observe that it follows from the “glueing for weak equivalences” axiom ($\text{Weq} \ 3$), cf. (3.2), that there exists $*$-homomorphisms $\alpha^\pm$ making the diagram commute. Moreover, we have the following proposition.
Proposition 4.1. $\alpha^\pm$ are $K$-equivalences.

Proof. The map $\eta$ is a $K$-equivalence for trivial reasons: it is a unital $^*$-homomorphism and both domain and codomain have $K_0 = \mathbb{Z}[1]$ and $K_1 = 0$. It follows then from Theorem 3.3 (Weq 3) that $\alpha^\pm$ induce isomorphisms in $K$-theory. \Box

In the case of equatorial and mirror quantum sphere, consider the pushout diagram

\[ \begin{array}{ccc}
\tilde{S}^2_{q,\infty} & \to & S^2_{q,\infty} \\
\downarrow & & \downarrow \\
\{\ast\} & \to & S^2_{q,\infty}
\end{array} \]

where $\mathbb{B}^2_q \overset{\sim}{\to} \{\ast\}$ is the pullback of the unit map in (4.2). Axiom $(cw-W)$ guarantees that the dashed part of this diagram defines a weak embedding $\{\ast\} \hookrightarrow S^2_{q,\infty}$.

It follows from the top square of (4.2) that we have a commutative diagram:

\[ \begin{array}{ccc}
\{\ast\} & \to & S^2_{q,\infty} \\
\downarrow & & \downarrow \\
\mathbb{B}^2_q & \to & S^2_q
\end{array} \]  \hspace{1cm} (4.3)

This means that we have a commutative diagram:

\[ \begin{array}{ccc}
\{\ast\} & \to & S^2_{q,\infty} \\
\downarrow & & \downarrow \\
\{\ast\} & \to & S^2_q
\end{array} \]  \hspace{1cm} (4.4)

which we interpret as a $K$-equivalence of $K$-weak filtrations by skeleta.

Let us note that in (4.3) we can view $\mathbb{B}^2_q$ as tubular neighbourhood of a point weakly embedded in $S^2_{q,\infty}$.

4.2. Toeplitz compact quantum surfaces. A class of examples of finite quantum CW-complexes is given by the noncommutative compact surfaces of [18]. There, the Authors define orientable compact surfaces of genus $g \geq 0$, here denoted $T_{g,q}$, and non-orientable of demi-genus $g \geq 1$, here denoted $P_{g,q}$, which reduces to the quantum real projective plane of [22, 19] when $g = 1$. They are given by pushout diagrams:
The maps $(\pi_g^+)^*$ and $(\pi_g^-)^*$ are a suitable embeddings of $C(\vee^{2g} S^1)$ and $C(\vee^g S^1)$ in $C(S^1)$, respectively. The $C^*$-algebras $C(T_{g,q})$ and $C(P_{g,q})$ are preimages inside the Toeplitz algebra of $C(\vee^{2g} S^1)$ and $C(\vee^g S^1)$, respectively, under the symbol map $\sigma = \partial^*$ [18, Section 2]. It is then clear that the diagrams

\[
\begin{align*}
C(T_{g,q}) &= \sigma^{-1}(\pi_g^+)^* C(\vee^{2g} S^1) \\
C(\vee^{2g} S^1) &\leftarrow C(S^1) \\
(\pi_g^+)^* &\leftarrow \sigma
\end{align*}
\]

\[
\begin{align*}
C(P_{g,q}) &= \sigma^{-1}(\pi_g^-)^* C(\vee^g S^1) \\
C(\vee^g S^1) &\leftarrow C(S^1) \\
(\pi_g^-)^* &\leftarrow \sigma
\end{align*}
\]

are pullbacks. Indeed, by definition of preimage, they are pullbacks in the category of vector spaces. But since all the maps are *-homomorphisms, they are pullbacks in the category of $C^*$-algebras as well.

From the above pushout diagrams, one deduces the following filtrations by skeleta:

\[
\{ \ast \} \hookrightarrow \vee^{2g} S^1 \hookrightarrow T_{g,q} , \quad \{ \ast \} \hookrightarrow \vee^g S^1 \hookrightarrow P_{g,q} .
\]

### 4.3. An exotic example of a quantum CW-complex.

In this section we present an example with no classical analogue. Let $\ell \geq 1$. The quantum weighted complex projective line $WP^1_q(1, \ell)$ is defined as a quotient of $SU_q(2)$ by a suitable weighted action of $U(1)$, see e.g. [9]. Observe that $WP^1_q(1, 1) = S^2_q$.

For $\ell \geq 2$, there is a pushout diagram [2, Proposition 3.6]:

\[
\begin{align*}
WP^1_q(1, \ell) &\leftarrow WP^1_q(1, \ell - 1) \leftarrow S^1 \leftarrow \mathbb{B}^2
\end{align*}
\]
that no classical analogue. Since the boundary map \( \partial \) is a cofibration, it follows from \((\text{Cof} \ 3)\) of Definition 3.1 that the map \( \wp^1_q(1, \ell - 1) \rightarrow \wp^1_q(1, \ell) \) is a cofibration as well. Therefore, we have a strict filtration by skeleta (in the sense of Definition 3.7)

\[ \{ * \} \rightarrow S^2_q \rightarrow \wp^1_q(1, 2) \rightarrow \ldots \rightarrow \wp^1_q(1, \ell - 1) \rightarrow \wp^1_q(1, \ell) . \]

This provides an exotic example of a finite quantum CW-complex, since \( K^0(\wp^1_q(1, \ell)) = \mathbb{Z}^{\ell+1} \) depends on \( \ell \) [9, Corollary 5.3], while classically weighted quantum projective lines corresponding to different values of \( \ell \) are all homeomorphic to \( S^2 \).

5. A weak CW-complex structure of \( \mathbb{C}P^n_q \)

5.1. Multipullback quantum odd spheres and complex projective spaces. Given a family of \(*\)-homomorphisms of \( C^* \)-algebras:

\[ \{ \pi^i_j : A_i \rightarrow A_{ij} = A_{ji} \}_{i,j \in I, i \neq j} , \]

with \( I = \{ 1, \ldots, k \} \) a finite set, one can similarly define the canonical multi-pullback \( A^\pi \) as a suitable limit, or concretely as

\[ A^\pi := \left\{ (a_1, \ldots, a_k) \in A_1 \times \ldots \times A_k \mid \pi^i_j(a_i) = \pi^j_i(a_j) \ \forall \ i, j \in I, i \neq j \right\} . \]

For \( n \geq 0 \), the choice

\[ A_i := T^{\otimes i} \otimes C(S^1) \otimes T^{\otimes n-i} , \quad \forall \ i = 0, \ldots, n, \]

\[ A_{ij} = A_{ji} := T^{\otimes i} \otimes C(S^1) \otimes T^{\otimes j-i-1} \otimes C(S^1) \otimes T^{\otimes n-j} , \quad \forall \ 0 \leq i < j \leq n, \]

\[ \pi^i_j := \text{id}^{\otimes j} \otimes \sigma \otimes \text{id}^{\otimes (n-j)} , \quad \forall \ i, j = 0, \ldots, n : i \neq j, \]

(where \( \sigma : T \rightarrow C(S^1) \) is the symbol map) defines the \( 2n + 1 \)-dimensional multipullback quantum sphere [24, Section 4.1].

This turns out to be isomorphic to the universal \( C^* \)-algebra \( C(S^{2n+1}_n) \) defined in terms of generators and relations as follows (this is a special case of Theorem 2.3 of [24]).

**Definition 5.1.** For \( n \geq 0 \), we denote by \( C(S^{2n+1}_n) \) the universal \( C^* \)-algebra with generators \( s_0, \ldots, s_n \) satisfying the relations:

\[ [s_i, s_j] = [s_i, s_j^*] = 0 \quad \forall \ 0 \leq i \neq j \leq n \]

\[ s_i^* s_i = 1 \quad \forall \ i = 0, \ldots, n, \]

\[ \prod_{i=0}^n (1 - s_i s_i^*) = 0 . \]

Recall that the \( C^* \)-algebra \( \mathcal{T} \) of Toeplitz operators is the \( C^* \)-subalgebra of \( B(\ell^2(\mathbb{N})) \) generated by the unilateral right shift \( t \) on \( \ell^2(\mathbb{N}) \), which is the operator given on the canonical basis \( \{ \xi_n : n \in \mathbb{N} \} \) of \( \ell^2(\mathbb{N}) \) by \( t \xi_n = \xi_{n+1} \). It can be equivalently defined in a more abstract way as universal \( C^* \)-algebra generated by a partial isometry \( t \). We will denote by

\[ t_i := \underbrace{1 \otimes \ldots \otimes 1}_{i \text{ times}} \otimes t \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-i \text{ times}}, \quad i = 0, \ldots, n, \]

the generators of \( \mathcal{T}^{\otimes n+1} \).
We think of $C(S^1)$ – the $C^*$-algebra of continuous complex-valued functions on the unit circle – as a compact quantum group with standard coproduct dual to the group multiplication. Thus, $C(S^1)$ is the universal $C^*$-algebra generated by a unitary $u$ with coproduct defined by $\Delta u = u \otimes u$.

Right coactions of $C(S^1)$ on $C(S^{2n+1}_H)$ and $T^\otimes n + 1$, all denoted by $\delta$, are defined on generators by $\delta(s_i) = s_i \otimes u$ and $\delta(t_i) = t_i \otimes u$ respectively (for $i = 0, \ldots, n$).

Lemma 5.2 ([24, Eq. (4.5)]). For all $n \geq 0$, there is the $U(1)$-equivariant short exact sequence $0 \rightarrow K(\ell^2(N^\otimes n + 1)) \rightarrow T^\otimes n + 1 \xrightarrow{\sigma_n} C(S^{2n+1}_H) \rightarrow 0$, where $\sigma_n$ is defined explicitly on generators by $\sigma_n(t_i) := s_i$, $i = 0, \ldots, n$.

Multipullback quantum spheres admit a presentation as higher rank graph $C^*$-algebras [24], while such a presentation for multipullback projective spaces is not known.

5.2. A tubular-neighbourhood lemma. For every $n \geq 1$ there is a pullback diagram [24, Lemma 5.2]:

\[
\begin{array}{ccc}
C(S^{2n+1}_H) & \xrightarrow{p_1} & C(S^{2n-1}_H) \otimes T \\
| & & | \\
| & \downarrow{\pi_1} & | \\
| & \uparrow{\pi_2} & | \\
C(S^{2n-1}_H) \otimes C(S^1) & \xleftarrow{p_2} & T^\otimes n \otimes C(S^1)
\end{array}
\]

(5.1)

which is $U(1)$-equivariant with respect to the diagonal action on each vertex. Let us give the explicit definition of the four maps and state the result in the form of a theorem.

With a slight abuse of notation, we denote by the same symbol the generators of $C(S^{2n+1}_H)$ for different values of $n$, and similar for $T^n$.

Lemma 5.3. Define the maps in (5.1) by $\pi_1 = \text{id} \otimes \sigma_0$, $\pi_2 = \sigma_{n-1} \otimes \text{id}$, and the remaining two in terms of generators by:

\[
\begin{align*}
p_1(s_i) & := \begin{cases} 
  s_i \otimes 1 & \forall \ i = 0, \ldots, n - 1 \\
  1 \otimes t & \text{if } i = n
\end{cases} \\
p_2(s_i) & := \begin{cases} 
  t_i \otimes 1 & \forall \ i = 0, \ldots, n - 1 \\
  1 \otimes u & \text{if } i = n
\end{cases}
\end{align*}
\]

Then (5.1) is a pullback diagram, $U(1)$-equivariant w.r.t. the diagonal $U(1)$-action on each vertex. Furthermore, all four maps in the diagram are surjective.

We need one more preliminary lemma.

Lemma 5.4. For all $k \geq 0$ and all $n \geq 1$, we have a $U(1)$-equivariant pullback diagram:
where the $U(1)$-action is diagonal on the top and left vertices, and only on the $C(S^1)$ factor on the bottom and right vertices. The map $\pi_2^k$ is given by $\pi_2^k = \sigma_{n-1} \otimes \text{id}_{C(S^1)} \otimes \text{id}_{T^k}$.

Proof. Given a pullback diagram, if we tensor each vertex with a fixed unital C*-algebra and each map with the identity map, the new diagram we get is still a pullback diagram [31, Theo. 3.9]. We get (5.2) from (5.1) first tensoring all algebras with $T^k$ and all maps with the identity. This gives a $U(1)$-equivariant pullback diagram:

$$
C(S_H^{2n+1}) \otimes T^k
$$

where $p_1, p_2, \pi_1, \pi_2$ are the maps in Lemma 5.3 and a dot denotes the factors where $U(1)$ acts. Now we use Remark 5.7 to move the actions on the bottom and right vertices on the $C(S^1)$ factor. To both vertices, of the form $A \otimes C(S^1) \otimes B$, we apply the automorphism

$$
a \otimes f \otimes b \mapsto a_{(0)} \otimes a_{(1)} f b_{(-1)} \otimes b_{(0)}
$$

where $a \mapsto a_{(0)} \otimes a_{(1)}$ and $b \mapsto b_{(-1)} \otimes b_{(0)}$ are the coactions of $C(S^1)$ on $A$ and $B$ dual to the $U(1)$-action. This gives us an equivariant map:

$$
\phi : A_* \otimes C(S^1)_* \otimes B_* \to A \otimes C(S^1)_* \otimes B
$$

as one can easily checks, and we get a new $U(1)$-equivariant pullback diagram:
where with a slight abuse of notation (two different maps are both denoted by $\phi$):

\[
p^k_1 := p_1 \otimes \text{id}_{T^k}, \quad \pi^k_1 := \phi \circ (\pi_1 \otimes \text{id}_{T^k}),
\]

\[
\pi^k_2 := \phi \circ (\pi_2 \otimes \text{id}_{T^k}) \circ \phi^{-1}, \quad p^k_2 := \phi \circ (p_2 \otimes \text{id}_{T^k}).
\]

Equivariance of $\pi_2$ implies that $\pi^k_2 = \pi_2 \otimes \text{id}_{T^k}$. \hfill \square

For $k = 0$, the $U(1)$-equivariant part of (5.2) gives the pullback diagram:

\[
\begin{array}{ccc}
C(\mathbb{C}P^n) & \xrightarrow{\phi \circ p_2} & C(S^{2n-1}_H) \\
p_1 & & \phi \circ (\text{id} \otimes \sigma_0) & \sigma_{n-1} & C(S^{2n-1}_H) \\
(C(S^{2n-1}_H) \otimes T)^{U(1)} & \xrightarrow{\phi \circ (\pi_2 \otimes \text{id}_{T^k})} & T^\otimes n
\end{array}
\]  

(5.3)

In order to compute recursively the K-theory of multipullback quantum projective spaces, we need to relate the K-theory of $(C(S^{2n-1}_H) \otimes T)^{U(1)}$ to that of $C(\mathbb{C}P^n)$. This is obtained from the following tubular neighbourhood lemma.

Geometrically, we can think of the former algebra as describing a bundle of closed quantum disks of the normal bundle (isomorphic to the Hopf line bundle) of the quantum hyperplane $\mathbb{C}P^n_T$ in $\mathbb{C}P^n_T$. Note that classically, the bundle of normal discs is homeomorphic to a tubular neighbourhood.

**Lemma 5.5.** For all $k \in \mathbb{N}$ and all $n \geq 1$, the $U(1)$-equivariant map (w.r.t. the diagonal $U(1)$-action):

\[
- \otimes 1_T : C(S^{2n-1}_H) \otimes T^\otimes k \to C(S^{2n-1}_H) \otimes T^\otimes k+1
\]

restricted and corestricted to the $U(1)$-fixed point algebras induces an isomorphism in K-theory:

\[
K_*\left((C(S^{2n-1}_H) \otimes T^\otimes k)^{U(1)}\right) \xrightarrow{\cong} K_*\left((C(S^{2n-1}_H) \otimes T^\otimes k+1)^{U(1)}\right).
\]

(5.4)

**Proof.** Let us consider then the following commutative diagram:

\[
\begin{array}{ccc}
C(S^{2n+1}_H) \otimes T^k & \xrightarrow{\phi} & C(S^{2n+1}_H) \otimes T^{k+1} \\
C(S^{2n-1}_H) \otimes T^{k+1} & \xrightarrow{T^n \otimes C(S^1) \otimes T^k} & C(S^{2n-1}_H) \otimes T^{k+2} \\
C(S^{2n-1}_H) \otimes C(S^1) \otimes T^k & \xrightarrow{T^n \otimes C(S^1) \otimes T^{k+1}} & C(S^{2n-1}_H) \otimes C(S^1) \otimes T^{k+1}
\end{array}
\]

The left diamond is (5.2), the right diamond is (5.2) with $k$ replaced by $k+1$, the horizontal arrows are all given by $\text{id} \otimes 1_T$. Passing to fixed point algebras we get the commutative
where the two diamonds are still pullback diagrams. We now prove by induction on $n \geq 0$ that, for all $k \geq 0$, the map $\phi_{n,k}$ induces an isomorphism in K-theory.

Let us start with $n = 0$ and consider the $U(1)$-equivariant commutative diagram:

$$
\begin{align*}
\xymatrix{ & (C(S^{2n-1}_H) \otimes \mathcal{T}^{k}) U(1) \ar[rd]^{\phi_{n,k}} \ar[rd] & & (C(S^{2n+1}_H) \otimes \mathcal{T}^{k+1}) U(1) \ar[ld] & \\
(C(S^{2n-1}_H) \otimes \mathcal{T}^{k+1}) U(1) \ar[ru] & \mathcal{T}^n \otimes \mathcal{T}^k \ar[ru] & (C(S^{2n-1}_H) \otimes \mathcal{T}^{k+2}) U(1) \ar[ru] & \mathcal{T}^n \otimes \mathcal{T}^{k+1} \ar[ru] & \\
C(S^{2n-1}_H) \otimes \mathcal{T}^k \ar[ru] & & C(S^{2n-1}_H) \otimes \mathcal{T}^{k+1} \ar[ru] & & }
\end{align*}
$$

where the vertical arrows are the isomorphisms $a \otimes b \mapsto ab(-1) \otimes b(0)$. The $U(1)$-invariant part gives:

$$
\begin{align*}
\xymatrix{ & C(S^1)_\ast \otimes \mathcal{T}^k \ar[r]^{\otimes 1_T} \ar[d] & C(S^1)_\ast \otimes \mathcal{T}^{k+1} \ar[d] & \\
C(S^1)_\ast \otimes \mathcal{T} \ar[r]^{\otimes 1_T} & C(S^1)_\ast \otimes \mathcal{T}^{k+1} & & }
\end{align*}
$$

Suppose $A$ and $B$ are two unital $C^*$-algebras both with $K_0 = \mathbb{Z}[1]$ and $K_1 = 0$. Then any unital $\ast$-homomorphism $A \to B$ induces an isomorphism in K-theory. In particular, $\mathcal{T}^k \otimes 1_T \to \mathcal{T}^{k+1}$ induces an isomorphism in K-theory and, since the vertical arrows in (5.6) are isomorphisms, $\phi_{0,k}$ induces an isomorphism in K-theory as well.

Let us now assume by inductive hypothesis that $\phi_{n-1,k}$ induces an isomorphism in K-theory for all $k \geq 0$. Let us look again at diagram (5.5). The unital $\ast$-homomorphisms $C(S^{2n-1}_H) \otimes \mathcal{T}^k \to C(S^{2n-1}_H) \otimes \mathcal{T}^{k+1}$ and $\mathcal{T}^n \otimes \mathcal{T}^k \to \mathcal{T}^n \otimes \mathcal{T}^{k+1}$ induce isomorphisms in K-theory by the same argument as before (all algebras have $K_0 = \mathbb{Z}[1]$ and $K_1 = 0$); $\phi_{n-1,k+1}$ induces an isomorphism in K-theory by inductive hypothesis. It follows from Theorem 3.3 (Weq 3) that the top arrow $\phi_{n,k}$ induces an isomorphism in K-theory as well, thus completing the proof. \hfill \Box
Remark 5.6. Given a commutative diagram of $C^*$-algebras and morphisms:

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
| & | & | \\
E & \longrightarrow & F
\end{array}
\quad \begin{array}{ccc}
| & | & | \\
B & \longrightarrow & C \\
| & | & | \\
F & \longrightarrow & G
\end{array}
\]

(5.7)

one proves by diagram chasing that if the two squares are pullbacks, then so is the outer rectangle; if the right square and the outer rectangles are pullbacks, so is the left square.

Given a $U(1)$-equivariant pullback diagram, restricting/corestricting all maps to the corresponding $U(1)$-fixed point subalgebras we get a new pullback diagram. The following observation will be useful to "gauge" $U(1)$-actions.

Remark 5.7. Suppose we have a commutative diagram:

\[
\begin{array}{ccc}
P_1 & \longrightarrow & P_2 \\
| & | & | \\
P_1' & \longrightarrow & P_2'
\end{array}
\quad \begin{array}{ccc}
P_1 & \longrightarrow & P_2 \\
| & | & | \\
P_1' & \longrightarrow & P_2'
\end{array}
\]

and the maps $\phi_1$ and $\phi_2$ are isomorphisms. Using Remark 5.6 twice one proves that the upper-left square is a pullback diagram if and only if the outer rectangle is a pullback.

Consider the two pushout squares

\[
\begin{array}{ccc}
\widetilde{C}\mathbb{P}_T^n & \longrightarrow & \mathbb{C}\mathbb{P}_T^n \\
\psi & \longrightarrow & \phi \\
\widetilde{I}_{C\mathbb{P}_T^n} & \longrightarrow & \mathbb{C}\mathbb{P}_T^n
\end{array}
\quad \begin{array}{ccc}
\mathbb{C}\mathbb{P}_T^{n-1} & \longrightarrow & \mathbb{C}\mathbb{P}_T^n \\
\pi & \longrightarrow & \epsilon \\
\mathbb{C}\mathbb{P}_T^{n-1} & \longrightarrow & \mathbb{C}\mathbb{P}_T^n
\end{array}
\]

(5.8)
Here the lower diagram is the diagram (5.3) understood in the opposite category, and the upper diagram is constructed using the axiom \((cw-W)\) (see Definition 3.2) and the fact that the collapsing map \(\pi : \text{Tub}_{\mathbb{CP}^n_T}(\mathbb{CP}^{n-1}_T) \rightarrow \mathbb{CP}^{n-1}_T\) is a weak equivalence in our cw-Waldhausen category. Furthermore, applying the axiom \((cw-W)\) to the lower pushout square of (5.8), we see that \(\epsilon\) is a cofibration. Therefore, as \(\pi\) is a weak equivalence, applying the axiom \((cw-W)\) now to the upper pushout square of (5.8), we infer that \(\psi\) is a weak equivalence as well.

The quantum space \(\widetilde{\mathbb{C}P}^n_T\) is then the result of collapsing the tubular neighborhood \(\text{Tub}_{\mathbb{CP}^n_T}(\mathbb{CP}^{n-1}_T)\) of the hyperplane \(\mathbb{C}P^{n-1}_T\) in \(\mathbb{CP}^n_T\) to such a hyperplane. Concatenating the pushout squares in (5.8), we obtain the outer pushout square

\[
\begin{array}{c}
\widetilde{\mathbb{C}P}^n_T \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathbb{C}P^{n-1}_T \\
\uparrow \quad \uparrow \quad \uparrow \\
\partial \quad \partial \quad \partial \\
\end{array}
\]

Here \(h := \pi \circ \chi\) is a quantum Hopf fibration, \(\tilde{\phi} := \psi \circ \phi\), where \(\psi : \mathbb{CP}^n_T \rightarrow \widetilde{\mathbb{CP}^n_T}\) is a weak equivalence, and the map \(\partial : S^{2n-1}_H \rightarrow \mathbb{D}^n_T\) is a boundary map from a K-sphere to a K-ball, in the sense of Definition 3.5.

Next, since \(\psi\) is a weak equivalence, the cospan

\[
\begin{array}{c}
\widetilde{\mathbb{C}P}^n_T \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathbb{C}P^{n-1}_T \\
\uparrow \quad \uparrow \quad \uparrow \\
\mathbb{C}P^n_T \\
\end{array}
\]

defines a morphism \(\iota := \psi^{-1} \circ \tilde{i} : \mathbb{C}P^{n-1}_T \rightarrow \mathbb{C}P^n_T\) in the homotopy category. This morphism is a weak cofibration, which we can understand as a weak replacement of the classical hyperplane embedding. Therefore, since we now can compose weak cofibrations, we can form a weak filtration by skeleta

\[
\mathbb{C}P^0_T \rightarrow \mathbb{C}P^1_T \rightarrow \cdots \rightarrow \mathbb{C}P^{n-1}_T \rightarrow \mathbb{C}P^n_T,
\]

where at every step we attach a single quantum cell, much as in the classical case. Hence, we can conclude that:

**Proposition 5.8.** For every \(n \geq 0\), the multipullback quantum complex projective space \(\mathbb{C}P^n_T\) is a finite quantum K-weak CW-complex (in the sense of Definition 3.7).

5.3. **A Mayer–Vietoris computation of K-groups.** In this section, as an application of general theory from Section 3, we show that \(K^0(\mathbb{C}P^n_T) \cong \mathbb{Z}^{n+1}\) and \(K^1(\mathbb{C}P^n_T) \cong 0\).
Our proof is independent of and simpler than the proof used in [24]. Better still, Proposition 5.10 provides us with the Milnor clutching picture, which leads to an inductive construction of basis elements of $K^0(\mathbb{CP}^n_T)$.

For starters, as an application of Theorem 3.9, we compute the K-groups:

**Theorem 5.9.** For all $n \geq 0$, one has $K^0(\mathbb{CP}^n_T) \cong \mathbb{Z}^{n+1}$ and $K^1(\mathbb{CP}^n_T) \cong 0$.

**Proof.** We prove the theorem by induction on $n$. It is trivially true for $n = 0$. Assume that it is true for $n - 1$ with $n \geq 1$. Since $K^0(\mathbb{CP}^{n-1}_T)$ is free, from (3.13) and (3.15) we obtain, respectively,

$$K^1(\mathbb{CP}^n_T) \oplus \ker \delta_{10} \cong K^1(\mathbb{CP}^{n-1}_T) \cong 0,$$

$$K^0(\mathbb{CP}^n_T) \cong K^0(\mathbb{CP}^{n-1}_T) \oplus \im \delta_{10} \cong \mathbb{Z}^n \oplus \im \delta_{10}.\tag{5.13}$$

Here $\delta_{10}$ is the Milnor connecting homomorphism from the six-term exact sequence associated to the pushout diagram (5.9). From (5.12), we deduce that both $K^1(\mathbb{CP}^n_T) \cong 0$ and $\ker \delta_{10} \cong 0$. Hence,

$$\delta_{10}: \mathbb{Z} \cong K^1(S^{2n-1}_H) \longrightarrow K^0(\mathbb{CP}^n_T)$$

is injective, so $\im \delta_{10} \cong \mathbb{Z}$. Finally, from (5.13), we obtain $K^0(\mathbb{CP}^n_T) \cong \mathbb{Z}^n \oplus \mathbb{Z} \cong \mathbb{Z}^{n+1}$, thus proving the inductive step. \qed

In the run of the proof of the preceding theorem, we obtain the following multipullback analogue of [3, Proposition 3.3]:

**Proposition 5.10.** For all $n \geq 1$, one has a non-canonical isomorphism

$$(\kappa, \delta_{10}): K^0(\mathbb{CP}^{n-1}_T) \oplus K^1(S^{2n-1}_H) \longrightarrow K^0(\mathbb{CP}^n_T).\tag{5.15}$$

Here $\delta_{10}$ is the Milnor connecting homomorphism from the six-term exact sequence associated to the pushout diagram (5.9) and $\kappa$ is a section of the split epimorphism

$$K^0(\iota): K^0(\mathbb{CP}^n_T) \longrightarrow K^0(\mathbb{CP}^{n-1}_T),\tag{5.16}$$

where $\iota$ is the weak cofibration defined by (5.10).

In the present example, the tower (3.12) of $\mathbb{Z}/2\mathbb{Z}$-graded K-groups becomes

$$K^*(\mathbb{CP}^0_T) \leftarrow K^*(\mathbb{CP}^1_T) \leftarrow \cdots \leftarrow K^*(\mathbb{CP}^{n-1}_T) \leftarrow K^*(\mathbb{CP}^n_T).\tag{5.17}$$

Here all arrows are surjective and split by (5.15). Better still, denoting by $g$ a generator of $K^1(S^{2n-1}_H) \cong \mathbb{Z}$, we call $\delta_{10}(g) \in K^0(\mathbb{CP}^n_T)$ the Milnor generator. Now, inductively, splittings of (5.16) taken in all dimensions up to $n$ provide a basis of $K^0(\mathbb{CP}^n_T)$ consisting of the last Milnor generator and liftings of the Milnor generators from previous skeleta.

To end with, we compare our proof of Theorem 5.9 with its proof in [24]. In the latter proof, the starting point is the decomposition of the odd spheres into disjoint components (see [24, p. 847]):

$$S^{2n+1} \cong \mathbb{B}^{2n} \times S^1 \coprod S^{2n-1} \times D_0.\tag{5.18}$$

Here $D_0$ is an open disc, and we look at the spaces up to homeomorphism. Now, (5.18) leads to the auxiliary non-unital C*-algebra $(C(S^{2n-1}_H) \otimes \mathcal{K})^{U(1)}$ (of the open quantum
tubular neighbourhood of $\mathbb{C}P^{n-1}_{T}$ in $\mathbb{C}P^n_T$, which is K-equivalent to $C(\mathbb{C}P^{n-1}_T)$. This K-equivalence yields a short exact sequence of $C^*$-algebras that allows one to complete the proof by induction using the six-term exact sequence in K-theory.

A key difference between this proof and our proof is that our starting point is the pushout decomposition of the odd spheres:

$$S^{2n+1}_+ \cong \mathbb{B}^{2n} \times S^1 \bigoplus_{S^{2n-1} \times S^1} S^{2n-1} \times \mathbb{D}. \quad (5.19)$$

Thus, instead of decomposing $S^{2n+1}$ into a closed and an open subset that are disjoint, we decompose it into two closed subsets that intersect non-trivially. Our decomposition leads to the auxiliary unital $C^*$-algebra $(C(S^{2n-1}_+) \otimes \mathcal{T})^{U(1)}$ (of the closed quantum tubular neighbourhood of $\mathbb{C}P^{n-1}_T$ in $\mathbb{C}P^n_T$), which is K-equivalent to $C(\mathbb{C}P^{n-1}_T)$. Again, this K-equivalence allows one to complete the proof by induction. However, this time one uses the Mayer–Vietoris six-term exact sequence in K-theory, which is deeply built into our cw-Waldhausen-categorical framework.

6. A CW-complex structure of $\mathbb{C}P^n_q$

6.1. Vaksman–Soibelman quantum spheres. Let $E = (E_0, E_1, s, r)$ be a directed graph, with $s, r : E_1 \to E_0$ the source and range maps. Recall that $E$ is row-finite if $s^{-1}(v)$ is a finite set for all $v \in E_0$. A sink is a vertex $v$ that emits no edges, i.e. $s^{-1}(v) = \emptyset$. A path is a sequence $e_0 e_1 \ldots e_n$ of edges with $r(e_{i-1}) = s(e_i)$ for all $i = 1, \ldots, n$; such a path is a cycle if $r(e_n) = s(e_0)$. A loop is an edge $e$ with $r(e) = s(e)$ (a cycle with one edge).

**Definition 6.1.** The graph $C^*$-algebras $C^*(E)$ of a row-finite graph $E$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\{P_v : v \in E_0\}$ and partial isometries $\{S_e : e \in E_1\}$ with relations (Cuntz-Krieger relations):

$$S_e^* S_e = P_r(e) \quad \text{for all } e \in E_1$$
$$\sum_{e \in E_1 : s(e) = v} S_e S_e^* = P_v \quad \text{for all } v \in E_0 \text{ that are not sinks}.$$

We give now a slight reformulation of the gauge-invariant uniqueness theorem [33, Theorem 2.2], that is more suitable for the purposes of this work.

**Theorem 6.2.** Let $E$ be a row-finite graph, $A$ a $C^*$-algebra with a continuous action of $U(1)$ and $\omega : C^*(E) \to A$ a $U(1)$-equivariant $\ast$-homomorphism. If $\omega(P_v) \neq 0$ for all $v \in E_0$, then $\omega$ is injective.
Let $\Sigma^0$ and $\Gamma^1$ be the graphs in Figure 1. It is well known that [33]: (i) there is an isomorphism $C^*(\Sigma^1) \to C(S^1)$ defined on generators by

$$S_{e_{00}} \mapsto u, \quad P_{v_0} \mapsto 1,$$

with $u$ the unitary generator of $C(S^1)$; (ii) there is an isomorphism $C^*(\Gamma^1) \to \mathcal{T}$ defined on generators by

$$S_{e_{00}} \mapsto t^2 t^* , \quad S_{e_{01}} \mapsto t(1 - tt^*), \quad P_{v_0} \mapsto tt^*, \quad P_{v_1} \mapsto 1 - t^* .$$

where $t$ is the right unilateral shift. These isomorphisms intertwine the $U(1)$ gauge action on the graph $C^*$-algebras with the natural action on $C(S^1)$ resp. $\mathcal{T}$.

The $C^*$-algebra $C(S_{q}^{2n+1})$ [41] can be presented as graph $C^*$-algebra of the graph in Figure 2. Such a graph has $n + 1$ vertices $v_0, \ldots, v_n$ and an edge $e_{ij} : v_i \to v_j$ for all $i \leq j$. Note that all vertices here are targets of edges. Therefore the corresponding graph $C^*$-algebra $C(S_{q}^{2n+1})$ is fully generated by partial isometries corresponding to the edges.

By removing the edge $e_{nn}$ from the graph in Figure 2 we obtain the graph $\Gamma^n$ of the $2n$-dimensional noncommutative closed ball $C(B_{q}^{2n})$ (see Figure 3). We denote the edges and vertices of $\Gamma^n$ by the same symbols as edges and vertices of $\Sigma^n$, only with tildes over them. By removing $\tilde{v}_n$ and all the edges with target $\tilde{v}_n$ from $\Gamma^n$ we obtain the graph $\Sigma^{n-1}$ of $S_{q}^{2n-1}$.

We can view $B_{q}^{2n}$ has half equator of $S_{q}^{2n+1}$, and $S_{q}^{2n-1}$ as boundary of $B_{q}^{2n}$. Two $C^*$-algebra morphisms

$$r_n : C(S_{q}^{2n+1}) \to C(B_{q}^{2n}) \quad \text{and} \quad \partial_n : C(B_{q}^{2n}) \to C(S_{q}^{2n-1}) ,$$

where $S_{q}^{2n+1}$ and $S_{q}^{2n-1}$ are the $2n$-dimensional and $(2n-1)$-dimensional spheres, respectively.
the restriction to half equator and boundary map, can be defined in terms of the projections and partial isometries in Definition 6.1 as follows: \( r_n \) sends the generators \( S_e, P_v \) of \( C(\mathbb{S}^{2n+1}_q) \) to the similarly named generators \( S_{\tilde{e}}, P_{\tilde{v}} \) of \( C(\mathbb{B}^{2n}_q) \) for all \( v \in \Sigma_n, e \in \Sigma^n_1 \setminus \{e_{nn}\} \), and sends \( S_{e_{nn}} \in C(\mathbb{S}^{2n+1}_q) \) to \( P_{\tilde{v}_n} \in C(\mathbb{B}^{2n}_q) \); \( \partial_n \) sends \( S_{\tilde{e}}, P_{\tilde{v}} \) to \( S_e, P_v \) in \( C(\mathbb{S}^{2n-1}_q) \) for all \( \tilde{v} \in \Gamma^0_n \setminus \{\tilde{v}_n\} \), \( \tilde{e} \in \Gamma^1_n \setminus r^{-1}(\tilde{v}_n) \), sends \( P_{\tilde{v}_n} \) to 0 and \( S_{\tilde{e}} \) to 0 if \( r(\tilde{e}) = \tilde{v}_n \).

Observe that \( \partial_n \) is \( U(1) \)-equivariant while \( r_n \) is not. The composition \( \partial_n \circ r_n \) is \( U(1) \)-equivariant as well.

In parallel to (5.1), it was proved in [3] that we have a \( U(1) \)-equivariant pullback diagram

6.2. A pullback structure of the Hong–Szymański quantum even balls. In this section we are going to prove that, for all \( n \geq 1 \), there is a pullback diagram:

\[
\begin{array}{ccc}
C(\mathbb{B}^{2n}_q) & \xleftarrow{\rho_n} & \mathcal{T}^\otimes \nabla \\
\downarrow \partial_n & & \downarrow \sigma_{n-1} \\
C(\mathbb{S}^{2n-1}_q) & \xleftarrow{\omega_{n-1}} & C(\mathbb{S}^{2n-1}_H)
\end{array}
\]  

(6.4)

Here \( \partial_n \) is the boundary map in Section 6.1 and \( \sigma_{n-1} \) the map in Lemma 5.2. We now construct the remaining two maps. The vertical maps are surjective, while the horizontal maps are going to be injective.

**Proposition 6.3.** Two \( U(1) \)-equivariant injective *-homomorphism \( \rho_n : C(\mathbb{B}^{2n}_q) \to \mathcal{T}^\otimes \) and \( \omega_{n-1} : C(\mathbb{S}^{2n-1}_q) \to C(\mathbb{S}^{2n-1}_H) \) are defined by.\(^1\)

\[
\rho_n(S_{e_{ij}}) := t_i t_j t^*_k \prod_{k=0}^{j-1} (1 - t_k t_k^*) \quad \forall 0 \leq i \leq j < n, \tag{6.5a}
\]

\[
\rho_n(S_{e_{nn}}) := t_i \prod_{k=0}^{n-1} (1 - t_k t_k^*) \quad \forall 0 \leq i < n, \tag{6.5b}
\]

\[
\omega_{n-1}(S_{e_{ij}}) := s_i s_j s^*_k \prod_{k=0}^{j-1} (1 - s_k s_k^*) \quad \forall 0 \leq i \leq j < n. \tag{6.5c}
\]

With \( \rho_n \) and \( \omega_{n-1} \) defined as above, (6.4) is a commutative diagram.

\(^1\)By convention, an empty sum is 0 and an empty product is 1.
Proof. One checks with an explicit computation that Cuntz-Krieger relations are satisfied, so that \( \rho_n \) and \( \omega_{n-1} \) define *-homomorphisms. For all \( 0 \leq i \leq j < n \) one has
\[
\rho_n(S_{e_{ij}})^* \rho_n(S_{e_{ij}}) = t_j t_j^* \prod_{k=0}^{j-1} (1 - t_k t_k^*) =: \rho_n(P_{e_{ij}}),
\]
and for all \( 0 \leq i < n \):
\[
\rho_n(S_{e_{ii}}) \rho_n(S_{e_{ii}}) = \prod_{k=0}^{n-1} (1 - t_k t_k^*) =: \rho_n(P_{v_i}).
\]
Since the \( t_i \)'s are commuting isometries, and \( (1 - t_k t_k^*) t_i = 0 \), it follows that the projections \( \rho_n(P_{e_{ij}}) \) are mutually orthogonal. Since \( \rho_n(S_{e_{ij}}) \rho_n(S_{e_{ij}})^* = t_i \rho_n(P_{e_{ij}}) t_i^* \) one also has:
\[
\sum_{j=1}^{n} \rho_n(S_{e_{ij}}) \rho_n(S_{e_{ij}})^* = t_i \left( \sum_{j=1}^{n} \rho_n(P_{e_{ij}}) \right) t_i^*,
\]
for all \( i \neq n \). By induction on \( i \) from \( n \) to lower values one proves that
\[
\sum_{j=i}^{n} \rho_n(P_{e_{ij}}) = \prod_{k=0}^{i-1} (1 - t_k t_k^*), \tag{6.6}
\]
that means
\[
\sum_{j=i}^{n} \rho_n(S_{e_{ij}}) \rho_n(S_{e_{ij}})^* = t_i \prod_{k=0}^{i-1} (1 - t_k t_k^*) t_i^* = t_i \prod_{k=0}^{i-1} (1 - t_k t_k^*) = \rho_n(P_{v_i})
\]
for all \( i \neq n \). Cuntz-Krieger relations are then satisfied by the elements \( \rho_n(S_{e_{ij}}), \rho_n(P_{v_i}) \), proving that the *-homomorphism \( \rho_n \) is well-defined. Similarly one proves that \( \omega_{n-1} \) is well-defined.

Equivariance of \( \rho_n \) and \( \omega_{n-1} \) is obvious, while injectivity follows from Theorem 6.2. Finally, one explicitly checks on generators that the diagram is commutative (in particular one may notice that \( \sigma_{n-1} \rho_n(S_{e_{ij}}) = 0 \) since \( \prod_{k=0}^{n-1} (1 - s_k s_k^*) = 0 \) in \( C(S_H^{2n-1}) \)).

Lemma 6.4. \( \text{Im}(\rho_n) \supset K(\ell^2(\mathbb{N}^n)) \) for all \( n \geq 1 \).

Proof. \( \text{Im}(\rho_n) \) is a C*-subalgebra of \( T^{\otimes n} \). It contains a non-zero compact operator, \( \rho(P_{v_n}) = \prod_{k=0}^{n-1} (1 - t_k t_k^*) \in K(\ell^2(\mathbb{N}^n)) \). We now prove that it is irreducible, so that from [12, Corollary I.10.4] it will follow that \( \text{Im}(\rho_n) \supset K(\ell^2(\mathbb{N}^n)) \).

Let \( a \in B(\ell^2(\mathbb{N}))^{\otimes n} \). We must show that \( a \) is in the commutant of \( \text{Im}(\rho_n) \) iff it is proportional to the identity. It follows from (6.6) that
\[
x_i := \sum_{j=1}^{n} \rho_n(S_{e_{ij}}) = t_i \sum_{j=1}^{n} \rho_n(P_{e_{ij}}) = t_i \prod_{k=0}^{i-1} (1 - t_k t_k^*)
\]
for all \( 0 \leq i < n \). Since \( a \) commutes \( x_0 = t_0 \), it follows that its first leg proportional to the identity. By the same argument, since \( a \) commutes with \( x_1 = (1 - t_0 t_0^*) t_1 \), its second leg must be proportional to the identity as well. By repeating the argument \( n \) times one reaches the conclusion.

Proposition 6.5. (6.4) is a pullback diagram.
Proof. It is enough to prove that [31, Prop. 3.1]:

(i) $\ker \partial_n \cap \ker \rho_n = \{0\}$, (ii) $\text{Im}(\rho_n) = \sigma_{n-1}^{-1} \omega_{n-1}(C(S_q^{2n-1}))$, (iii) $\partial_n(\ker \rho_n) = \ker \omega_n - 1$.

Points (i) and (iii) are trivial, since $\rho_n$ and $\omega_{n-1}$ are both injective. In (ii), the inclusion $\text{Im}(\rho_n) \subset \sigma_{n-1}^{-1} \omega_{n-1}(C(S_q^{2n-1}))$ follows from the commutativity of the diagram (6.4). We have to prove the opposite inclusion.

Take any $x \in \sigma_{n-1}^{-1} \omega_{n-1}(C(S_q^{2n-1}))$. Then $\sigma_n^{-1}(x) = \omega_{n-1}(y)$ for some $y \in C(S_q^{2n-1})$. Since $\partial_n$ is surjective, $y = \partial_n z$ for some $z \in C(B_q^{2n})$. From

$$\sigma_{n-1}(\rho_n(z)) = \omega_{n-1}(\partial_n(z)) = \omega_{n-1}(y) = \sigma_n^{-1}(x)$$

we deduce that $x - \rho_n(z) \in \ker \sigma_{n-1}$. From Lemma 5.2 and Lemma 6.4, $\ker \sigma_{n-1} = K(\ell^2(S^n)) \subset \text{Im}(\rho_n)$. Thus $x - \rho_n(z) = \rho_n(t)$ for some $t \in C(B_q^{2n})$. But then $x = \rho_n(z + t)$ is in the image of $\rho_n$, thus proving the thesis. □

As a byproduct of previous proposition, we get a pullback realization of Vaksman–Soibelman quantum spheres and projective spaces in terms of non-spherical balls.

**Proposition 6.6.** There exists pullback diagrams

![Diagram](attachment:image.png)

**Proof.** Recall that in (5.7) the outer rectangle is a pullback diagram if both inner squares are pullbacks. The $U(1)$-invariant part of (6.3) gives the pullback diagram:

![Diagram](attachment:image.png)

If we attach it to the pullback diagram (6.4) we get the second diagram in the Proposition. To get the first one we attach (6.3) to the diagram obtained by tensoring (6.4) everywhere with $C(S^1)$ and tensoring all maps with the identity on $C(S^1)$ (this is a pullback diagram by [31, Theorem 3.9]). □
6.3. The Milnor connecting homomorphism in terms of graphs. Using results in [10], In general, the $K_1$ class of a unitary

\[ a = S_{e_{n-1}-1} + \sum_{i=0}^{n-2} \sum_{j=i}^{n-1} S_{e_{ij}} S_{e_{ij}}^* = S_{e_{n-1}-1} + \sum_{i=0}^{n-2} P_{\tilde{v}_i} \]

is the generator of $K_1(C(S_{q}^{2n-1}))$. We consider the following lift of $a$ to $C(\mathbb{P}_{q}^{2n})$ and its adjoint:

\[ c = S_{e_{n-1}-1} + S_{e_{n-1}-1} + \sum_{i=0}^{n-2} P_{\tilde{v}_i}, \]

\[ c^* = S_{e_{n-1}-1}^* + S_{e_{n-1}-1}^* + \sum_{i=0}^{n-2} P_{\tilde{v}_i}. \]

Notice that

\[ c^* c = \left( S_{e_{n-1}-1}^* + S_{e_{n-1}-1}^* + \sum_{i=0}^{n-2} P_{\tilde{v}_i} \right) \left( S_{e_{n-1}-1} + S_{e_{n-1}-1} + \sum_{k=0}^{n-2} P_{\tilde{v}_i} \right) = 1, \]

\[ c c^* = \left( S_{e_{n-1}-1} + S_{e_{n-1}-1} + \sum_{i=0}^{n-2} P_{\tilde{v}_i} \right) \left( S_{e_{n-1}-1}^* + S_{e_{n-1}-1}^* + \sum_{k=0}^{n-2} P_{\tilde{v}_i} \right) = \sum_{i=0}^{n-1} P_{\tilde{v}_i}. \]

We consider the following 2 by 2 matrix with entries in $C(\mathbb{C}P^{n+1})$ (recall the pullback structure of projective spaces):

\[ p = \begin{bmatrix} (1, c(2 - c^*c)c^*) & (0, c(2 - c^*c)(1 - c^*c)) \\ (0, (1 - c^*c)c^*) & (0, (1 - c^*c)^2) \end{bmatrix} = \begin{bmatrix} (1, \sum_{i=0}^{n-1} P_{\tilde{v}_i}) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}. \]

We know that $[p] - [I_2]$ is the generator of $K_0(C(\mathbb{C}P_{q}^{n}))$ coming from the Milnor connecting homomorphism [11].

7. A K-theory type of quantized $\mathbb{C}P^n$

In Section 7.1 we construct a $U(1)$-equivariant *-homomorphisms from $q$-sphere to multipullback quantum spheres that induces an isomorphism between the K-theory of $U(1)$-fixed point algebras. This will be used to describe generators of $K_0(C(\mathbb{C}P_{q}^{n}))$ in terms of associated vector bundles.

\[ \text{2The computations in this section were carried out by Mariusz Tobolski.} \]
7.1. A K-equivalence from $\mathbb{CP}^n_T$ to $\mathbb{CP}^n_q$. A comparison theorem:

**Theorem 7.1.** The map $\omega_n : C(S^{2n+1}_q) \to C(S^{2n+1}_H)$ in Prop. 6.3 induces an isomorphism in K-theory. Its restriction to $U(1)$-fixed point algebras induces an isomorphism

$$K_*(C(\mathbb{CP}^n_q)) \to K_*(C(\mathbb{CP}^n_T))$$

as well.

**Proof.** Let us begin by constructing a commutative diagram:

\[
\begin{array}{cccccc}
C(S^{2n+1}_q) & \xrightarrow{\omega_n} & C(S^{2n+1}_H) \\
\downarrow{(r_n \otimes \text{id}) \delta} & & \downarrow{\phi \circ p_2} \\
C(\mathbb{B}^{2n}_q) \otimes C(S^1) & \xrightarrow{\rho_n \otimes \text{id}} & T^\otimes n \otimes C(S^1) \\
\downarrow{\partial_n \circ r_n} & & \downarrow{p_1} \\
C(S^{2n-1}_q) \otimes C(S^1) & \xrightarrow{\omega_n \otimes 1_T} & C(S^{2n-1}_H) \otimes T \\
\downarrow{\omega_n \otimes \text{id}} & & \downarrow{\omega_n \otimes \text{id}} \\
C(S^{2n-1}_q) \otimes C(S^1) & \xrightarrow{\partial_n \otimes \text{id}} & T^\otimes n \otimes C(S^1) \\
\downarrow{\omega_n \otimes \text{id}} & & \downarrow{\omega_n \otimes \text{id}} \\
C(S^{2n-1}_q) \otimes C(S^1) & \xrightarrow{\phi \circ (\text{id} \otimes \sigma_0)} & C(S^{2n-1}_q) \otimes C(S^1)
\end{array}
\]

where the left diamond is the pullback diagram (6.3) and the right diamond is the pullback diagram (5.2) for $k = 0$. In order to check commutativity, let us rewrite the four faces:

- $C(S^{2n+1}_q)$
- $C(S^{2n+1}_H)$
- $C(S_q^{2n-1})$
- $C(S^{2n-1}_H)$
- $C(\mathbb{B}^{2n}_q) \otimes C(S^1)$
- $T^\otimes n \otimes C(S^1)$
- $C(S^{2n-1}_q) \otimes C(S^1)$
- $C(S^{2n-1}_q) \otimes C(S^1)$

where $p_1$ and $p_2$ are the maps in Lemma 5.3 and $\phi$ is the map $a \otimes f \mapsto a_{(0)} \otimes a_{(1)} f$. The third diagram is simply (6.4) tensored everywhere by $C(S^1)$. Commutativity of the other three diagrams can be explicitly checked on generators. Let us start with the first diagram. Firstly

$$\phi \circ p_2(s_i) = \begin{cases} t_i \otimes u & \forall i = 0, \ldots, n-1 \\ 1 \otimes u & \text{if } i = n \end{cases}$$

Then one checks using (6.5) that

$$\phi \circ p_2 \circ \omega_n(S_{e_{ij}}) = \begin{cases} \rho_n(S_{e_{ij}}) \otimes u & \text{if } i \neq n \\ \rho_n(P_{e_{n}}) \otimes u & \text{if } i = n \end{cases}$$
Since
\[(r_n \otimes \text{id}) \circ \delta(S_{e_{ij}}) = \begin{cases} S_{e_{ij}} \otimes u & \text{if } i \neq n \\ P_{e_n} \otimes u & \text{if } i = n \end{cases} \]
clearly \(\phi \circ p_2 \circ \omega_n = (\rho_n \otimes \text{id}) \circ (r_n \otimes \text{id}) \circ \delta\). We now pass to the second diagram. Here
\[p_1 \circ \omega_n(S_{e_{ij}}) = \begin{cases} \rho_n(S_{e_{ij}}) \otimes 1 & \forall 0 \leq i \leq j < n \\ 0 & \text{if } j = n \end{cases} \]
(for \(j = n\) one has \(\prod_{k=0}^{n-1}(1 - s_k s_k^*) = 0\) in \(C(\mathbb{S}_{H}^{2n-1})\)). On the other hand
\[\partial_n \circ r_n(S_{e_{ij}}) = \begin{cases} S_{e_{ij}} & \forall 0 \leq i \leq j < n \\ 0 & \text{if } j = n \end{cases} \]
hence \(p_1 \circ \omega_n = \omega_{n-1} \circ \partial_n \circ r_n \otimes 1_T\).
Finally the fourth diagram. Firstly we notice that
\[\phi \circ (\text{id} \otimes \sigma_0)(s_i \otimes 1) = s_i \otimes u \text{ for all } i = 0, \ldots, n - 1, \text{ and then} \]
\[\phi \circ (\text{id} \otimes \sigma_0) \circ (\omega_{n-1} \otimes 1_T) = \phi \circ \omega_{n-1} \otimes 1_{C(\mathbb{S}^1)} \]
the latter is equal to \((\omega_{n-1} \otimes \text{id})\delta\) by equivariance of \(\omega_{n-1}\). This can also be checked explicitly on generators:
\[\phi \circ \omega_{n-1}(S_{e_{ij}}) \otimes 1_{C(\mathbb{S}^1)} = \omega_{n-1}(S_{e_{ij}}) \otimes u = (\omega_{n-1} \otimes \text{id})\delta(S_{e_{ij}}) \]
for all \(0 \leq i \leq j < n\).

Now that we proved that (7.1) is commutative, we can use Theorem 3.3 (Weq 3) and induction on \(n\) to prove that \(\omega_n\) induces an isomorphism in K-theory. For \(n = 0\), \(\omega_0 : C(\Sigma^0) \to C(\mathbb{S}^1)\) is the isomorphism sending \(S_{e_{00}}\) to \(u\). Assume by inductive hypothesis that, for some \(n \geq 1\), \(\omega_{n-1}\) induces an isomorphism in K-theory. The map \(\rho_n\) induces an isomorphism for trivial reasons: it is a unital \(*\)-homomorphism and both domain and codomain have \(K_0 = \mathbb{Z}[1]\) and \(K_1 = 0\).

Recall that if \(f : A \to B\) and \(g : C \to D\) are \(*\)-homomorphisms, then \(K_*(A \otimes B) \cong K_*(A) \otimes K_*(B)\) by Kunneth formula (the one on the right hand side is the graded tensor product of graded abelian groups) and under this isomorphism \((f \otimes g)_* = f_* \otimes g_*\). It follows that \(\omega_{n-1} \otimes \text{id}\) and \(\rho_n \otimes \text{id}\) induce isomorphisms in K-theory.

Finally, note that both \(\omega_{n-1}\) and the map \(1_T : \mathbb{C} \to \mathcal{T}\) induce isomorphisms in K-theory (the latter because \(K_0(\mathcal{T}) = \mathbb{Z}[1]\) and \(K_1(\mathcal{T}) = 0\), hence \(\omega_{n-1} \otimes 1_T\) induces an isomorphism in K-theory as well. It follows from Theorem 3.3 (Weq 3) that \(\omega_n\) induces an isomorphism in K-theory, thus completing the inductive step.
Concerning the fixed point algebras, from (7.1) we get the commutative diagram:

\[
\begin{array}{ccc}
C(\mathbb{C}P^n_q) & \xrightarrow{\omega_n} & C(\mathbb{C}P^n_T) \\
\downarrow & & \downarrow \\
C(\mathbb{C}P^{n-1}_q) & \xrightarrow{\omega_{n-1} \otimes 1_T} & C(S^{2n-1}_q) \\
\downarrow & & \downarrow \\
C(B^{2n}) & \xrightarrow{C(\mathbb{S}^{2n-1}_H) \otimes T^{(1)}} & C(S^{2n-1}) \\
\downarrow & & \downarrow \\
C(S^{2n-1}_q) & \xrightarrow{\omega_{n-1}} & C(S^{2n-1}) \\
\end{array}
\]

(7.2)

where now \(\omega_n\) and \(\omega_{n-1} \otimes 1_T\) are restricted and corestricted to the fixed point algebras. We can use again Theorem 3.3 (Weq 3) to prove that \(\omega_n : C(\mathbb{C}P^n_q) \rightarrow C(\mathbb{C}P^n_T)\) induces an isomorphism in K-theory. For \(n = 0\), \(\omega_0 : C(\Gamma^0_0) \rightarrow \mathbb{C}\) is the isomorphism sending \(P_{v_0}\) to 1. Assume the claim is true for \(\omega_{n-1}\), \(n \geq 1\).

We already proved that the maps \(\omega_{n-1} : C(S^{2n-1}_q) \rightarrow C(S^{2n-1}_H)\) and \(\rho_n : C(B^{2n}) \rightarrow T^{\otimes n}\) in (7.2) induce an isomorphism in K-theory. The map \(\omega_{n-1} \otimes 1_T : C(\mathbb{C}P^{n-1}_q) \rightarrow (C(S^{2n-1}_H) \otimes T)^{(1)}\) is the composition of the map \(\omega_{n-1} : C(\mathbb{C}P^n_q) \rightarrow C(\mathbb{C}P^n_T)\), which induces an isomorphism in K-theory by inductive hypothesis, and the restriction-corestriction to \(U(1)\)-fixed point algebras of the map \(\dashv \otimes 1_T : C(S^{2n-1}_H) \rightarrow C(S^{2n-1}_H) \otimes T\), that gives an isomorphism in K-theory by Lemma 5.5. From Theorem 3.3 (Weq 3), we conclude that \(\omega_n : C(\mathbb{C}P^n_q) \rightarrow C(\mathbb{C}P^n_T)\) induces an isomorphism in K-theory.

\[ \square \]

8. The Atiyah–Todd picture

8.1. The classical case revisited. The classical result of Atiyah–Todd says that the \(K_0\)-group \(K^0(\mathbb{C}P^n) = K_0(C(\mathbb{C}P^n))\) equipped with the ring structure defined via the tensor product of vector bundles over \(\mathbb{C}P^n\) (or, equivalently, the tensor product of finitely generated projective left \(C(\mathbb{C}P^n)\)-modules, which can be regarded as symmetric \(C(\mathbb{C}P^n)\)-bimodules) fits into the following commutative square of rings:

\[
\begin{array}{ccc}
\mathbb{Z}[t, t^{-1}] & \xrightarrow{\cong} & R(U(1)) \\
\downarrow & & \downarrow \\
\mathbb{Z}[x]/(x^{n+1}) & \xrightarrow{\cong} & K^0(\mathbb{C}P^n). \\
\end{array}
\]  

(8.1)

Here the left vertical arrow is given by \(t \mapsto 1 + x\), the right vertical arrow is induced by the associated vector bundle construction, the top isomorphism maps \(t\) into the fundamental representation of \(U(1)\) in the representation ring \(R(U(1))\), and the bottom isomorphism maps \(x\) to the K-theory element \([L_1] - [1]\), where \(L_1\) denotes the Hopf line bundle on \(\mathbb{C}P^n\) associated with the fundamental representation of \(U(1)\). Below, for any \(k \in \mathbb{Z}\), we denote by \(L_k\) the \(k\)-th tensor power of \(L_1\) when \(k\) is non-negative, and the \([k]\)-th tensor power of \(L_{-1}\) when \(k\) is negative and where \(L_{-1}\) is the Hopf line bundle on \(\mathbb{C}P^n\) associated with the dual of the fundamental representation of \(U(1)\). Equivalently, \(L_k\) is the Hopf line bundle on \(\mathbb{C}P^n\) associated with the \(k\)-th tensor power of the fundamental representation.
of $U(1)$, where negative tensor powers refer to tensor powers of the dual of the fundamental representation of $U(1)$.

Since the elements $(1 + x)^k, k = 0, \ldots, n,$ form a basis of the free $\mathbb{Z}$-module $\mathbb{Z}[x]/(x^{n+1})$ and the assignment $(1 + x) \mapsto [L_i]$ gives an isomorphism of rings, the classes

$$[L_0], \ldots, [L_n]$$

form a basis of the free $\mathbb{Z}$-module $K^0(\mathbb{C}P^n)$. We call this basis the shifted Atiyah–Todd basis of $K^0(\mathbb{C}P^n)$ whose name refers to the standard Atiyah–Todd basis $x^k, k = 0, \ldots, n$ replaced by the basis $(1 + x)^k, k = 0, \ldots, n$.

Our next step is to unravel how the classes $[L_k]$, for $k = -1$ or $k = n + 1$, can be expressed in the shifted Atiyah–Todd basis.

Note first that the equality

$$0 = x^{n+1} = ((1 + x) - 1)^{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} (1 + x)^k$$

in $\mathbb{Z}[x]/(x^{n+1})$ translates to the equality

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} [L_k] = 0$$

in $K^0(\mathbb{C}P^n)$. Thus we obtain

$$[L_{n+1}] = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} [L_k],$$

which we will refer to as the primary shifted Atiyah–Todd identity.

Furthermore, since $(1 + x)$ is invertible in $\mathbb{Z}[x]/(x^{n+1})$ and the initial equality (8.3) can be rewritten as

$$(1 + x)^{n+1} = \sum_{k=1}^{n+1} (-1)^{1-k} \binom{n+1}{k} (1 + x)^{k-1} = 1,$$

we obtain

$$(1 + x)^{-1} = \sum_{k=1}^{n+1} (-1)^{1-k} \binom{n+1}{k} (1 + x)^{k-1} = \sum_{k=0}^{n} (-1)^{k} \binom{n+1}{k+1} (1 + x)^k$$

in $\mathbb{Z}[x]/(x^{n+1})$. This equality translates to $K^0(\mathbb{C}P^n)$ as

$$[L_{-1}] = \sum_{k=0}^{n} (-1)^{k} \binom{n+1}{k+1} [L_k].$$

We will refer to (8.7) as the secondary shifted Atiyah–Todd identity.

In [1, Prop. 3.3 and 3.4], only the additive version of the bottom isomorphism in the diagram (8.1) was established for the Vaksman–Soibelman quantum complex projective spaces $\mathbb{C}P^n_q$. It yields a noncommutative version of the Atiyah–Todd basis (8.2). All this seems interesting because Atiyah–Todd’s method to prove the existence of the commutative diagram (8.1) uses the ring structure of K-theory, which is missing in the noncommutative setting. In the forthcoming subsection devoted to the multipullback
noncommutative deformation of the complex projective spaces, not only we obtain an analogue of the Atiyah–Todd basis (8.2), but also we establish analogues of the Atiyah–Todd identities (8.5) and (8.7), which are lacking in [1, Prop. 3.3 and 3.4].

8.2. The multipullback noncommutative deformation. Although the $K_0$-group of a noncommutative C*-algebra does not have an intrinsic ring structure, it turns out that, much as in the diagram (8.1), the abelian group $K_0(C(CP^n_T))$ is a free module of rank one over the representation ring $R(U(1))$ divided by the ideal generated by the $(n+1)$-st power of the formal difference between the fundamental representation and the trivial one-dimensional representation. The basis of this free module is the $K_0$-class of $C(CP^n_T)$ itself. The module structure comes from tensoring finitely generated projective $C(CP^n_T)$-modules by the bimodules associated with the quantum Hopf $U(1)$-principal bundle $S^{2n+1}_H \rightarrow CP^n_T$. Moreover, we will show that, despite the aforementioned lack of an intrinsic ring structure, we still enjoy analogs of the (shifted) Atiyah–Todd identities (8.4) and (8.7).

Recall that we denote by $S^{2n+1}_H$ the multipullback $(2n+1)$-dimensional quantum sphere [?] and by $CP^n_T$ the corresponding multipullback quantum complex projective space [20], whose C*-algebra we identify with a $U(1)$-fixed-point subalgebra of $C(S^{2n+1}_H)$ (see [?]). In the same fashion, denoting by $\alpha$ the aforementioned $U(1)$-action, we define the section modules

$$L_k := \{ a \in C(S^{2n+1}_H) | \alpha_\lambda(a) = \lambda^k a \text{ for all } \lambda \in U(1) \}, \quad k \in \mathbb{Z}, \quad (8.8)$$

of the associated noncommutative line bundles bundles over $CP^n_T$.

Next, let

$$\partial_{n+1} : T^{\otimes(n+1)} \longrightarrow T^{\otimes(n+1)}/K^{\otimes(n+1)} \cong C(S^{2n+1}_H) \quad \text{(see [?, Lemma 5.1])} \quad (8.9)$$

be the canonical quotient map, and let

$$P_k := \sum_{i=1}^k e_{ij} \in K \subset T, \quad P_k^+ := I - P_k \in K^+ \subset T, \quad k \in \mathbb{N}. \quad (8.10)$$

Here $e_{ij}$ with $i, j \in \mathbb{N}$ represents a matrix unit in $K$ which we identify with $K(\ell^2(\mathbb{N}))$, and $K^+$ stands for the minimal unitization of $K$. Note that, according to the standard summation-over-the-empty-set convention, $P_0 := 0$, so $P_0^+ = I$. For finite square matrices $P, Q \in M_\infty(A)$ with entries in a unital C*-algebra $A$, we use the notion $P \sim_A Q$ to denote that they are unitarily equivalent over $A$, and use $P \sqcup Q$ to denote their diagonal direct sum.

Furthermore, for $0 \leq j \leq n$ and $k \geq 0$, we define the projections

$$E_k^j := \partial_{n+1}((\otimes^j P_1) \otimes P_k^+ \otimes (\otimes^{n-j} I)) \in C(CP^n_T). \quad (8.11)$$

Note that $E_k^n = \partial_{n+1}((\otimes^n P_1) \otimes P_k^+) = \partial_{n+1}((\otimes^n P_1) \otimes I)$ since $\partial_{n+1}((\otimes^n P_1) \otimes P_k) = 0$. For the sake of forthcoming recursive formulas, we adopt the convention $E_k^{n+1} := 0$ and $0^0 := 1$. Now, recall from [39, Theorem 4] and the remark therein just after this theorem that, for $j = 0, \ldots, n$, the classes $[E_0^j]$ form a basis of the free $\mathbb{Z}$-module $K_0(C(CP^n_T)) \cong \mathbb{Z}^{n+1}$:

$$K_0(C(CP^n_T)) = \bigoplus_{j=0}^n \mathbb{Z}[E_0^j]. \quad (8.12)$$
Next, remembering that $E^j_k \in C(\mathbb{CP}^n)$ (they are all $U(1)$-invariant), we will follow an argument used in [39] to establish $\{[\partial_n((\otimes^j I) \otimes (\otimes^{n-j} P_1)))]_{0 \leq j \leq n}$ as a basis of $K_0(C(\mathbb{CP}^{n-1}))$, to prove the recursive relation

$$[E^j_{k+1}] = [E^j_k] - [E^{j+1}_k]$$

in $K_0(C(\mathbb{CP}^n))$. To this end, we need the following lemma:

**Lemma 8.1.** Let $S$ be the generating isometry of the Toeplitz algebra $\mathcal{T}$ identified with the unilateral shift on the Hilbert space $\ell^2(\mathbb{N})$. For any $k \geq 0$ and $n \geq 1$,

$$U_k := \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} \in M_2(\mathcal{T}^\otimes 2)$$

is a self-adjoint unitary conjugating $(e_{kk} \otimes I) \boxplus 0$ to $0 \boxplus (P_1 \otimes P_k^\perp)$.

**Proof.** First, we verify that the self-adjoint element $U_k \in M_2(\mathcal{T}^\otimes 2)$ is unitary:

$$\begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} = \begin{pmatrix} P_k \otimes I + S^k(S^k)^* \otimes (S^k)^*S^k & P_kS^k \otimes (S^k)^* + S^k \otimes (S^k)^*P_k \\ (S^k)^*P_k \otimes S^k + (S^k)^* \otimes P_kS^k & (S^k)^*S^k \otimes S^k(S^k)^* + I \otimes P_k \end{pmatrix} = \begin{pmatrix} I \otimes I & 0 \\ 0 & I \otimes I \end{pmatrix}. \tag{8.14}$$

Next, $U_k$ conjugates $(e_{kk} \otimes I) \boxplus 0$ to $0 \boxplus (P_1 \otimes P_k^\perp)$ because

$$\begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} e_{kk} \otimes I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_{0k} \otimes S^k & 0 \end{pmatrix} \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & e_{00} \otimes P_k^\perp \end{pmatrix}. \tag{8.15}$$

□

**Lemma 8.2.** In $K_0(C(\mathbb{CP}^n))$, we have the following recursive relation

$$[E^j_{k+1}] = [E^j_k] - [E^{j+1}_k]$$

for all $0 \leq j \leq n$ and $k \geq 0$.

**Proof.** First, note that the statements are true for $j = n$ because

$$E^n_k = \partial_{n+1}((\otimes^n P_1) \otimes I) = E^n_{k+1} \tag{8.16}$$

is independent of $k$, and $E^{n+1}_k := 0$. Hence, we can assume $0 \leq j < n$. 
Furthermore, since $P^\perp_k = P^\perp_{k+1} + e_{kk}$ and the summands are orthogonal projections, we obtain

$$E^j_k = \partial_{n+1}((\otimes^j P_1) \otimes P^\perp_k \otimes (\otimes^{n-j} I))$$

$$\sim_{C(\mathbb{CP}_2^n)} \partial_{n+1}((\otimes^j P_1) \otimes P^\perp_{k+1} \otimes (\otimes^{n-j} I)) \oplus \partial_{n+1}((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I))$$

$$= E^j_{k+1} \oplus \partial_{n+1}((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I)).$$

(8.17)

Therefore, to finish the proof, it suffices to show the following auxiliary identity

$$[\partial_{n+1}((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I))] = [E^j_{k+1}].$$

(8.18)

To this end, we take advantage of Lemma 8.1 to conclude that $(\otimes^j P_1) \otimes U_k \otimes (\otimes^{n-j-1} I)$ conjugates $((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I)) \oplus 0$ to

$$0 \oplus ((\otimes^j P_1) \otimes P_1 \otimes P^\perp_k \otimes (\otimes^{n-j-1} I)) = 0 \oplus E^{j+1}_k.$$

(8.19)

Here the tensor product $(\otimes^j P_1) \otimes U_k \otimes (\otimes^{n-j-1} I)$ is understood entrywise with respect to the matrix $U_k$.

Finally, since $\partial_{n+1}(a_{ij})$ is $U(1)$-invariant for each entry $a_{ij}$ of $(\otimes^j P_1) \otimes U_k \otimes (\otimes^{n-j-1} I)$, we have $\partial_{n+1}(a_{ij}) \in C(\mathbb{CP}_2^n)$, so

$$\partial_{n+1}(((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I)) \oplus 0) \sim_{C(\mathbb{CP}_2^n)} 0 \oplus E^{j+1}_k.$$

(8.20)

Passing to the $K_0$-classes, we obtain (8.18), as needed.

Having shown the recursive relation (8.13), we are ready to prove:

**Lemma 8.3.** For any $k \geq 0$,

$$[L_k] = \sum_{j=0}^{k} (-1)^j \binom{k}{j} [E^j_0].$$

Proof. It is known that, for $k \geq 0$, the modules $L_k$ are represented, respectively, by the projections $\partial_{n+1}(P^\perp_k \otimes (\otimes^n I)) =: E^0_k$ (see [39, Theorem 6]). Starting from $l = 0$, we prove inductively, for $0 \leq l \leq k$ with $k \geq 0$ fixed, that

$$[L_k] = \sum_{j=0}^{l} (-1)^j \binom{l}{j} [E^j_{k-l}].$$

(8.21)

Equation (8.21) is clearly true for $l = 0$. Now, for $0 < l \leq k$, taking advantage of the induction hypothesis and the recursive relation (8.13) in Lemma 8.2, we compute:

$$[L_k] = \sum_{j=0}^{l-1} (-1)^j \binom{l-1}{j} [E^j_{k-l+1}]

= \sum_{j=0}^{l-1} (-1)^j \binom{l-1}{j} ([E^j_{k-l}] - [E^{j+1}_{k-l}])

= \sum_{j=0}^{l-1} \left((-1)^j \binom{l-1}{j} [E^j_{k-l}] + (-1)^{j+1} \binom{l-1}{j} [E^{j+1}_{k-l}]\right)$$
Theorem 8.4. For any \( n \in \mathbb{N} \), we have noncommutative analogs of the shifted Atiyah–Todd basis and identities:

\[
K_0(C(\mathbb{CP}^n_\mathbb{C})) = \bigoplus_{k=0}^{n} \mathbb{Z}[L_k],
\]

(8.24)

\[
[L_{n+1}] = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} [L_k],
\]

(8.25)

\[
[L_{-1}] = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} [L_k].
\]

(8.26)

Proof. To begin with, note that (8.24) follows immediately from Lemma 8.3 and (8.12) because the expansion coefficients \((-1)^j \binom{k}{j}\) in Lemma 8.3 form a matrix in \( GL_{n+1}(\mathbb{Z}) \). (The matrix is lower-triangular of determinant \( \pm 1 \).)

Next, to prove (8.25), we will show an equivalent identity reflecting the classical case equality (8.4):

\[
\begin{align*}
\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} [L_k] &= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \left( \sum_{j=0}^{k} (-1)^j \binom{k}{j} [E^j_0] \right) \\
&= \sum_{j=0}^{n+1} \sum_{k=j}^{n+1} (-1)^{n+1+j-k} \binom{n+1}{k} \binom{k}{j} [E^j_0] \\
&= \sum_{j=0}^{n+1} \left( \sum_{k=j}^{n+1} (-1)^{n+1+j-k} \frac{(n+1)!}{k!(n+1-k)!j!(k-j)!} \right) [E^j_0] \\
&= \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} \left( \sum_{k=j}^{n+1} (-1)^{n+1+j-k} \frac{(n+1-j)!}{(n+1-k)!(k-j)!} \right) [E^j_0] \\
&= \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} (-1)^j \left( \sum_{k=0}^{n+1-j} (-1)^{n+1-j-k} \frac{(n+1-j)!}{(n+1-j-k)!k!} \right) [E^j_0]
\end{align*}
\]
\[
\sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!}(-1)^j(1+(-1))^{n+1-j}[E_0^j] = 0.
\] (8.27)

Finally, to prove (8.42), we recall from [39] that the class \([L_{-1}]\) can be represented by the projection \(\prod_{j=0}^n E_0^j\). Thus (8.42) becomes
\[
\sum_{j=0}^n (-1)^k \binom{n+1}{k+1} [L_k] = \sum_{j=0}^n [E_0^j].
\] (8.28)

The left-hand-side can be computed as follows:
\[
\sum_{k=0}^n (-1)^k \binom{n+1}{k+1} [L_k] = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \left( \sum_{j=0}^k (-1)^j \binom{k}{j} [E_0^j] \right)
\]
\[
= \sum_{j=0}^n \sum_{k=j}^n \frac{(-1)^{k+j} (n+1)!}{(k+1)! (n-k)!} \frac{k!}{j! (k-j)!} [E_0^j]
\]
\[
= \sum_{j=0}^n \frac{(n+1)!}{j! (n-j)!} \left( \sum_{k=j}^n \frac{(-1)^{k+j} (n-j)!}{(n-k)! (k-j)!} \frac{1}{k+j+1} \right) [E_0^j]
\]
\[
= \sum_{j=0}^n \frac{(n+1)!}{j! (n-j)!} \left( \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \frac{1}{k+j+1} \right) [E_0^j].
\] (8.29)

Hence it remains to show that, for all \(0 \leq j \leq n\),
\[
\frac{(n+1)!}{j! (n-j)!} \left( \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \frac{1}{k+j+1} \right) = 1.
\]

To this end, we introduce auxiliary polynomials over \(\mathbb{Q}\):
\[
f_j(x) := \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} \frac{1}{k+j+1} x^{k+j+1},
\] (8.30)

which can be evaluated at rationals and formally differentiated and integrated. Now our goal can be rephrased as follows:
\[
f_j(1) = \int_0^1 (1-x)^j (x-1)^{n-j} dx
\]

To compute this, note first that
\[
f_j'(x) = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} x^{k+j} = (-1)^{n-j} x^j (x-1)^{n-j}.
\] (8.31)

Therefore, as \(f_j(0) = 0\) because \(k, j \geq 0\), we obtain:
\[
f_j(1) = \int_0^1 (-1)^{n-j} x^j (x-1)^{n-j} dx
\]
realized as a groupoid $C^*$-algebra is a subgroupoid of $\mathcal{F}$ for all $(\sim_i)$. It is well known that the groupoid C*-algebra $C^*_\mathcal{F}$ of $\mathcal{F}$ can be described quantum line bundles over $\mathbb{CP}_q^n$ as concrete groupoid C*-algebras. For starters, recall that the graph $\Gamma$ of the graph $C^*$-algebra $C^*_\mathcal{F}$ acts on $\mathbb{CP}_q^n$ of its unit space $\mathbb{Z}^n$, where $\mathbb{Z}^n$ is the one-point compactification of $\mathbb{Z}$, and $\mathbb{Z}^n$ acts on $\mathbb{CP}_q^n$ by componentwise addition in the canonical way. More explicitly,

$$\mathcal{F}_0^n := \{(x, w) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid x + w \in \mathbb{N}^n\} \cong (\mathcal{F}_0^n)^n.$$ 

8.3. The Vaksman–Soibelman noncommutative deformation. In this section, we adapt the groupoid method used in the previous section to establish analogous Atiyah–Todd identities for the Vaksman–Soibelman noncommutative deformation $\mathbb{CP}_q^n$ of $\mathbb{CP}_n$. For starters, recall that the graph $\Gamma$ of the graph $C^*$-algebra $C^*_\mathcal{F}$ is neither row finite nor sinkless, so one cannot directly apply the standard Deaconu–Renault construction to $\Gamma$ to present $C^*_\mathcal{F}$ as a groupoid $C^*$-algebra [40, Example 8.4.7]. However, we can use the groupoid presentation from [38] to describe quantum line bundles over $\mathbb{CP}_q^n$ in terms of concrete elementary projections.

Let $\mathcal{F}_0^n := \{(\mathbb{Z}^n \times \mathbb{Z}^n) |_{\mathbb{N}^n}\}$ denote the transformation groupoid $\mathbb{Z}^n \times \mathbb{Z}^n$ restricted to the positive cone $\mathbb{N}^n$ of its unit space $\mathbb{Z}^n$, where $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$ is the one-point compactification of $\mathbb{Z}$, and $\mathbb{Z}^n$ acts on $\overline{\mathbb{Z}}^n$ by componentwise addition in the canonical way. More explicitly,

$$\mathcal{F}_0^n \cong \{(x, w) \in \mathbb{Z}^n \times \mathbb{N}^n \mid x + w \in \mathbb{N}^n\} \cong (\mathcal{F}_0^n)^n.$$ 

It is well known that the groupoid $C^*$-algebra $C^*_\mathcal{F}$ of $\mathcal{F}_0^n$ is isomorphic to the groupoid $C^*$-algebra $C^*_\mathcal{F}$ of the quotient groupoid $(\mathcal{F}_0^n) := (\mathcal{F}_0^n)/\sim$. Here

$$(\mathcal{F}_0^n) := \{(x, w) \in \mathcal{F}_0^n \subset \mathbb{Z}^n \times \mathbb{N}^n \mid w_i = \infty \text{ with } 1 \leq i \leq n, \text{ then } x_i = -x_1 - x_2 - \cdots - x_{i-1} \text{ and } x_{i+1} = \cdots = x_n = 0\}$$

is a subgroupoid of $\mathcal{F}_0^n$, and $\sim$ is the equivalence relation on $(\mathcal{F}_0^n)$ generated by

$$(x, w) \sim (x, w_1, \ldots, w_{i-1}, \infty, \ldots, \infty)_{n-i+1 \text{ copies}}$$

for all $(x, w)$ with $w_i = \infty$ for an $1 \leq i \leq n$. This realization of $C^*_\mathcal{F}$ as the concrete groupoid $C^*$-algebra $C^*(\mathcal{F}_0^n)$ turns out to be useful in analyzing generators of $K_0(C^*_\mathcal{F})$ as summarized below.

Via the faithful representation of $C^*(\mathcal{F}_0^n)$ on $\ell^2(\mathbb{N}^n)$ determined by the dense open invariant subset $\mathbb{N}^n$ of the unit space $\overline{\mathbb{N}}^n$ of $(\mathcal{F}_0^n)$, we view $C^*(\mathcal{F}_0^n)$ as a concrete
operator subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}^n))$. Furthermore, the $n + 1$ unitary equivalence classes of the projections $$(\otimes^j P_1) \otimes (\otimes^{n-j} I) \in C^* (\mathfrak{F}_n \cap \mathfrak{N}) \subset \mathcal{B}(\ell^2(\mathbb{N}^n))$$ with $0 \leq j \leq n$ form a basis of $K_0 \left( C(\mathbb{C}P^n_q) \right) \cong \mathbb{Z}^{n+1}$. Also, denoting by $\tilde{\alpha}$ the diagonal $U(1)$-action on $C(S^2_q)$, we define the section modules $\tilde{L}_k := \{ a \in C(S^2_q) \mid \tilde{\alpha}_\lambda (a) = \lambda^k a \text{ for all } \lambda \in U(1) \}, \ k \in \mathbb{Z}$, (8.34) of the associated noncommutative line bundles over $\mathbb{C}P^n_q$ (cf. (8.8)). They are identified with the projections $P_k \otimes (\otimes^{n-1} I)$ for $k \geq 0$.

Let $T_k \in \mathcal{B}(\ell^2(\mathbb{N}))$ be the canonical partial isometry shifting $\ell^2(\{0, \ldots, k-1\})$ onto $\ell^2(\{k, \ldots, 2k-1\})$ while annihilating $\ell^2(\{0, \ldots, k-1\})^\perp$. Then $T_k^*$ is the canonical partial isometry shifting $\ell^2(\{k, \ldots, 2k-1\})$ back onto $\ell^2(\{0, \ldots, k-1\})$. Note that the self-adjoint operator $W_k := \begin{pmatrix} P_k \otimes I & T_k \otimes (S^k)^* \\ T_k^* \otimes S^k & I \otimes P_k \end{pmatrix}$ is a partial isometry because

$$W_k^* W_k = \begin{pmatrix} P_k \otimes I & T_k \otimes (S^k)^* \\ T_k^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} P_k \otimes I & T_k \otimes (S^k)^* \\ T_k^* \otimes S^k & I \otimes P_k \end{pmatrix} = (P_{2k} \otimes I) \oplus (P_k \otimes P_k^\perp + I \otimes P_k)$$

is a projection. Consequently, denoting by $I_2$ the unit of $M_2(\mathcal{B}(\ell^2(\mathbb{N}^2)))$, from

$$W_k (I_2 - W_k^2) = 0 = (I_2 - W_k^2) W_k,$$

we conclude that the self-adjoint operator $U_k := W_k + (I_2 - W_k^2)$ is unitary.

Next, we derive an analogue of Lemma 8.1.

**Lemma 8.5.** The self-adjoint unitary $U_k$ intertwines $(e_{kk} \otimes I) \oplus 0$ and $0 \oplus (P_1 \otimes P_k^\perp)$, i.e.

$$U_k ((e_{kk} \otimes I) \oplus 0) U_k^* = 0 \oplus (P_1 \otimes P_k^\perp).$$

**Proof.** Note first that $W_k ((e_{kk} \otimes I) \oplus 0) W_k^* = 0 \oplus (P_1 \otimes P_k^\perp)$, which can be verified by the direct computation

$$\begin{pmatrix} P_k \otimes I & T_k \otimes (S^k)^* \\ T_k^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} e_{kk} \otimes I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_k \otimes I & T_k \otimes (S^k)^* \\ T_k^* \otimes S^k & I \otimes P_k \end{pmatrix} = \begin{pmatrix} P_k \otimes I & T_k \otimes (S^k)^* \\ T_k^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} 0 & e_{k0} \otimes (S^k)^* \\ 0 & 0 \end{pmatrix}.$$

\[^3\text{The module } \tilde{L}_k \text{ is denoted by } L_{-k} \text{ in [38].}\]
for all \( j > n \).

Lemma 8.6. Naturally, we take \( E_k \) as the operator representing the characteristic function \( \chi_{B_k} \in C^* \) of the set \( B_k \) consisting of points \( [(x, w)] \in (\mathcal{F}_n) \) with

\[
\begin{align*}
x_1 = \cdots = x_j & = 0, \quad x_{j+1} = k, \quad x_{j+2} = -k, \quad x_{j+3} = \cdots = x_n = 0, \\
w_1 = \cdots = w_j & = 0 \leq w_{j+1} \leq k - 1 < w_{j+2} \leq \infty, \quad \text{and} \quad w_{j+3}, \ldots, w_n \in \mathbb{N}.
\end{align*}
\]

Note that, in particular for the case of \( w_{j+2} = \infty \), the condition

\[
x_{j+2} = -k = 0 - \cdots - 0 - k = -x_1 - \cdots - x_j - x_{j+1}
\]

is satisfied.

Now for \( k \geq 0 \) and \( 0 \leq j \leq n - 1 \), we introduce the projections

\[
\tilde{E}_k^j := (\otimes^j P_1) \otimes P_{k+1}^j \otimes (\otimes^{n-j-1} I) \in C(\mathbb{C}P^n_q).
\]

Naturally, we take \( \tilde{E}_k^n := \otimes^n P_1 \in C(\mathbb{C}P^n_q) \) for all \( k \geq 0 \) in case \( j = n \), and we also take \( \tilde{E}_k^j := 0 \) for all \( k \geq 0 \) and all \( j \geq n + 1 \).

In the following lemma, we derive properties of \( \tilde{E}_k^j \) analogous to properties of \( E_k^j \in C(\mathbb{C}P^n_q) \) established earlier.

**Lemma 8.6.** In \( K_0(C(\mathbb{C}P^n_q)) \), we have the recursive relation

\[
[\tilde{E}_{k+1}^j] = [\tilde{E}_k^j] - [\tilde{E}_k^{j+1}]
\]

for all \( 0 \leq j \leq n \) and \( k \geq 0 \).

**Proof.** For starters, the relation is automatically satisfied for \( j \geq n \) because \( \tilde{E}_k^j := 0 \) for all \( j > n \) and \( k \geq 0 \), and \( \tilde{E}_k^n := \otimes^n P_1 \) for all \( k \geq 0 \). Therefore, we can focus on the case of \( 0 \leq j \leq n - 1 \).

Since \( P_{k+1}^j = P_{k+1}^j \oplus e_{kk} \) with \( P_{k+1}^j \) and \( e_{kk} \) orthogonal to each other, we get

\[
[\tilde{E}_k^j] = [(\otimes^j P_1) \otimes P_{k+1}^j \otimes (\otimes^{n-j-1} I)] + [(\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j-1} I)]
\]

\[
= [\tilde{E}_{k+1}^j] + [(\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j-1} I)].
\]

Therefore, to conclude the proof, it suffices to show that

\[
[(\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j-1} I)] = [\tilde{E}_{k+1}^j].
\]

(8.35)
To this end, note that, if $0 \leq j \leq n - 2$, the entry-wise tensor product
\[ P^j_1 \otimes U_k \otimes I^{\otimes n-j-2} \in M_2 \left( C \left( \mathbb{CP}^n \right) \right) \subseteq M_2 \left( \mathcal{B} \left( \ell^2 (\mathbb{N}^n) \right) \right) \]
is a partial isometry intertwining \((\otimes^j P_1) \otimes (e_{kk} \boxplus 0) \otimes (\otimes^{n-j-1} I)\) and
\[
(\otimes^j P_1) \otimes (0 \boxplus (P_1 \otimes P^j_k)) \otimes (\otimes^{n-j-2} I) = 0 \boxplus \tilde{E}^{j+1}_k
\]
according to Lemma 8.5. If \(j = n - 1\), then \((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j-1} I) = (\otimes^{n-1} P_1) \otimes e_{kk}\) and \(\tilde{E}^{j+1}_k = \tilde{E}^n_k = \otimes^n P_1\) are intertwined by the self-adjoint partial isometry
\[
(\otimes^{n-1} P_1) \otimes (I - e_{00} - e_{kk} + e_{0k} + e_{k0}) \in C \left( \mathbb{CP}^n \right).
\]
This proves the desired Equation (8.35). \(\square\)

As recalled above, the classes \([\tilde{E}^0_0] = [(\otimes^j P_1) \otimes (\otimes^{n-j} I)]\) with \(0 \leq j \leq n\) form a basis of the free \(\mathbb{Z}\)-module \(K_0(C(\mathbb{CP}^n)) \cong \mathbb{Z}^{n+1}\), i.e.
\[
K_0(C(\mathbb{CP}^n)) = \bigoplus_{j=0}^n \mathbb{Z}[\tilde{E}^j_0],
\]
while the section modules \(\tilde{L}_k\) are represented by the projections \(\tilde{E}^0_k = P^j_k \otimes (I^{\otimes n-1})\) in \(C(\mathbb{CP}^n)\). Combining this with the recursive relation \([\tilde{E}^j_k] = [\tilde{E}^j_{k+1}] - [\tilde{E}^{j+1}_k]\), the following lemma and theorem can be proved in the same way as Lemma 8.3 and Theorem 8.4, respectively.

**Lemma 8.7.** For any \(k \geq 0\),
\[
[\tilde{L}_k] = \sum_{j=0}^k (-1)^j \binom{k}{j} [\tilde{E}^j_0].
\]

**Theorem 8.8.** For any \(n \in \mathbb{N}\), we have noncommutative analogues of the shifted Atiyah–Todd basis and identities:
\[
K_0(C(\mathbb{CP}^n)) = \bigoplus_{k=0}^n \mathbb{Z}[\tilde{L}_k],
\]
\[
[\tilde{L}_{n+1}] = \sum_{k=0}^n (-1)^{n-k} \binom{n+1}{k} [\tilde{L}_k],
\]
\[
[\tilde{L}_{-1}] = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} [\tilde{L}_k].
\]

### 8.4. The \(R(U(1))\)-module structure of \(K^0(\mathbb{CP}^n)\).

Assume that a free action of a compact quantum group \(G\) on a \(C^*\)-algebra \(A\) is given, with the subalgebra \(B = A^G\) of invariants. Let \(H = \mathcal{O}(G)\) be the the Peter-Weyl Hopf dense \(*\)-subalgebra in the \(C^*\)-algebra \(C(\mathcal{G})\) “of continuous functions on \(G\)” and \(\mathcal{A}\) be the Peter-Weyl dense \(H\)-comodule \(*\)-\(B\)-subalgebra in \(A\). Given a representation \(V\) of \(G\) equivalent to a finite dimensional left \(H\)-comodule \(V\) one has a finitely generated projective from either side associated \(B\)-bimodule \(A \boxtimes H V\). This \(B\)-bimodule defines an endofunctor \((-) \otimes_B (A \boxtimes H V)\) on the exact category of finitely generated projective right \(B\)-modules. Since the association is a strong monoidal functor from the category of left \(H\)-comodules to the category of \(B\)-bimodules,
it defines an action of the representation ring \( R(\mathbb{G}) \) of \( \mathbb{G} \) on the topological \( K \)-theory of \( B \). The action on the distinguished class \([B] \in K_*(B)\) defines a right \( R(\mathbb{G}) \)-module map

\[
R(\mathbb{G}) \to K_*(B),
\]

essentially being forgetting of the left \( B \)-module structure of an associated finitely generated projective \( B \)-bimodule. In our case, when \( A = C(S^2_{[x]}/H), \mathbb{G} = U(1), B = C(\mathbb{C}P^n_T) \)

we obtain a map of right \( R(U(1)) \)-modules

\[
R(U(1)) \to K^*(\mathbb{C}P^n_T), \tag{8.36}
\]

The above considerations together with Theorem 8.4 lead to the following theorem.

**Theorem 8.9.** The map (8.36) of right \( R(U(1)) \)-modules and the left-hand-side map being a ring map induced by \( t \mapsto 1 + x \) fit into the following diagram of right \( \mathbb{Z}[t,t^{-1}] \)-modules

\[
\begin{array}{ccc}
\mathbb{Z}[t,t^{-1}] & \xrightarrow{t} & R(U(1)) \\
\downarrow & & \downarrow \\
\mathbb{Z}[x]/(x^{n+1}) & \xrightarrow{t} & K^0(\mathbb{C}P^n_T).
\end{array}
\tag{8.37}
\]

In particular, \( K^*(\mathbb{C}P^n_T) \) is a rank one free right \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \mathbb{Z}[x]/(x^{n+1}) \)-module, where \( x \) is even, generated by the class \([L_0]\).

**Proof.** By Theorem 8.4 the canonical right \( \mathbb{Z}[t,t^{-1}] \)-module structure on the free \( \mathbb{Z} \)-module

\[
K^0(\mathbb{C}P^n_T) = \bigoplus_{k=0}^n \mathbb{Z}[L_k] \tag{8.38}
\]

is uniquely determined by shifting the winding number by one

\[
[L_k]t = [L_{k+1}], \quad \text{for } k = 0, \ldots, n-1, \tag{8.39}
\]

\[
[L_n]t = \sum_{k=0}^n (-1)^{n-k} \binom{n+1}{k} [L_k], \tag{8.40}
\]

and by minus one

\[
[L_0]t^{-1} = \sum_{k=0}^n (-1)^{k} \binom{n+1}{k+1} [L_k], \tag{8.41}
\]

\[
[L_k]t^{-1} = [L_{k-1}], \quad \text{for } k = 1, \ldots, n. \tag{8.42}
\]

The fact that the minimum polynomial of the matrix of the right action of \( t \) is equal to \((t-1)^{n+1}\) and the isomorphism of rings

\[
\mathbb{Z}[t,t^{-1}]/((t-1)^{n+1}) \xrightarrow{\sim} \mathbb{Z}[x]/(x^{n+1})
\]

\[
t \mapsto 1 + x, \quad t^{-1} \mapsto 1 - x + x^2 - \ldots + (-1)^n x^n
\tag{8.43}
\]

prove the bottom isomorphism fitting into the diagram (8.37) of \( \mathbb{Z}[t,t^{-1}] \)-modules. \( \square \)
Since K-theory of a noncommutative ring lacks an intrinsic ring structure, the above \(\mathbb{Z}[x]/(x^{n+1})\)-module structure is best one could expect about the structure of K-theory of \(\mathbb{C}P^n_T\). Another good feature of this module structure is compatibility with the “weak filtration by skeleta” (5.11), namely the tower of the K-theories of the “weak skeleta” becomes the tower of truncated polynomials

\[
0 \hookrightarrow K^*(\mathbb{C}P^n_T) \hookrightarrow K^*(\mathbb{C}P^{n+1}_T) \hookrightarrow \cdots \hookrightarrow K^*(\mathbb{C}P^{n+1}_T) \hookrightarrow K^*(\mathbb{C}P^0_T)
\]

(8.44)

giving the generators of the kernels of successive restriction morphisms of K-theory to lower weak skeleta in terms of the distinguished basis

\[
[L_n]x^n = [L_0](t - 1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [L_k].
\]

To end with, let us observe:

**Corollary 8.10.** The isomorphism induced on K-theory by the \(U(1)\)-equivariant map \(\omega_n : C(S_q^{2n+1}) \to C(S_H^{2n+1})\) identifies the set of distinguished generators coming from noncommutative line bundles associated with the corresponding quantum \(U(1)\)-principal bundles \(S_q^{2n+1}\) and \(S_H^{2n+1}\) over quantum complex projective spaces \(\mathbb{C}P^n_q\) and \(\mathbb{C}P^n_T\), respectively.

**Proof.** By virtue of Proposition 6.3 and Proposition 7.1 it follows immediately from the Pushforward commutes with association theorem from [21]. \(\square\)

We know [3, Theorem 2.3] that \(K_1(C(S_q^{2n+1})) \cong \mathbb{Z}\) is generated by the class of the unitary \(S_{n+1} + (1 - S_{n+1}^* S_{n+1})\). Its image through \(\omega_n\), as a corollary of previous proposition, gives a generator of \(K_1(C(S_H^{2n+1}))\). Using \(s_n s_n^* \prod_{k=0}^{n-1} (1 - s_k s_k^*) = \prod_{k=0}^{n-1} (1 - s_k s_k^*)\) in \(C(S_H^{2n+1})\) such a unitary can be written in the following form:

**Corollary 8.11.** \(K_1(C(S_H^{2n+1})) \cong \mathbb{Z}\) is generated by the unitary \(U := s_n \prod_{k=0}^{n-1} (1 - s_k s_k^*)\).

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References

[1] Arici, F., Brain, S., and Landi, G. The Gysin sequence for quantum lens spaces. *J. Noncommut. Geom.* 9 (2015), 1077–1111.

[2] Arici, F., D'Andrea, F., Hajac, P. M., and Tobolski, M. An equivariant pullback structure of trimmable graph C*-algebras. *arXiv preprint arXiv:1712.08010* (2017).

[3] Arici, F., D'Andrea, F., Hajac, P. M., and Tobolski, M. An equivariant pullback structure of trimmable graph C*-algebras. *to appear in J. Noncomm. Geom.*, *arXiv preprint arXiv:1712.08010* (2017).

[4] Arici, F., and Mikkelsen, S. E. Split extensions and KK-equivalences for quantum projective spaces. *arXiv preprint arXiv:2108.11360* (2021).

[5] Baum, P. F., and Hajac, P. M. Local proof of algebraic characterization of free actions. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 10 (2014), 060.

[6] Blackadar, B. *K-Theory for Operator Algebras*, vol. 5. Cambridge University Press, 1998.

[7] Blanchard, E. Tensor products of $C(X)$-algebras over $C(X)$. In *Recent advances in operator algebras - Orléans, 1992*, no. 232 in *Astérisque*. Société mathématique de France, 1995.

[8] Bousfield, A. K. The localization of spaces with respect to homology. *Topology* 14, 2 (1975), 133–150.

[9] Brzeziński, T., and Fairfax, S. A. Quantum teardrops. *Communications in Mathematical Physics* 316, 1 (2012), 151–170.

[10] Carlsen, T. M., Eilers, S., and Tomforde, M. Index maps in the K-theory of graph algebras. *Journal of K-Theory* 9, 2 (2012), 385–406.

[11] Dabrowski, L., Hadfield, T., Hajac, P. M., Matthes, R., and Wagner, E. Index pairings for pullbacks of C*-algebras. *Banach Center Publications* 98 (2012), 67–84.

[12] Davidson, K. R. *C*-algebras by example*, vol. 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[13] Dixmier, J. *C*-algebras, vol. 15 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[14] Eilers, S., Loring, T. A., and Pedersen, G. K. Stability of anticommutation relations: an application of noncommutative CW-complexes. *J. Reine Angew. Math.* 99 (1998), 101–143.

[15] Farsi, C., Hajac, P. M., Maszczyk, T., and Zielinski, B. Rank-two Milnor idempotents for the multipullback quantum complex projective plane. *arXiv preprint arXiv:1708.04426* (2017).

[16] Fillmore, P. A. *A User’s Guide to Operator Algebras*, vol. 14. Wiley-Interscience, 1996.

[17] Gabriel, P., and Zisman, M. *Calculus of fractions and homotopy theory*, vol. 35. Springer Science & Business Media, 2012.

[18] García, F. D., and Wagner, E. A spectral triple for noncommutative compact surfaces. *arXiv preprint arXiv:2002.10624* (2020).

[19] Hajac, P. M. Strong connections on quantum principal bundles. *Communications in mathematical physics* 182, 3 (1996), 579–617.

[20] Hajac, P. M., Kaygun, A., and Zielinski, B. Quantum complex projective spaces from Toeplitz cubes. *J. of Noncomm. Geom.* 6, 3 (2012), 603–621.

[21] Hajac, P. M., and Maszczyk, T. Pullbacks and nontriviality of associated noncommutative vector bundles. *J. Noncomm. Geom.* 12, 4 (2018), 1341–1358.
[22] Hajac, P. M., Matthes, R., and Szymański, W. Quantum real projective space, disc and spheres. *Algebras and Representation Theory* 6, 2 (2003), 169–192. 19, 21

[23] Hajac, P. M., Matthes, R., and Szymański, W. Noncommutative index theory for mirror quantum spheres. *Comptes Rendus Mathematique* 343, 11-12 (2006), 731–736. 19, 20

[24] Hajac, P. M., Nest, R., Pask, D., Sims, A., and Zieliński, B. The K-theory of twisted multipullback quantum odd spheres and complex projective spaces. *J. Noncomm. Geom.* 12, 3 (2018), 823–863. 23, 24, 30

[25] Hajac, P. M., Rudnik, J., and Zielinski, B. Reductions of piecewise-trivial principal comodule algebras. *arXiv preprint arXiv:1101.0201* (2021). Accepted in Communications in Mathematical Physics. 19

[26] Johnstone, P. *Sketches of an Elephant – A Topos Theory Compendium*. 2002. 3

[27] Kasparov, G. Equivariant KK-theory and the novikov conjecture. *Inventiones mathematicae* 91, 1 (1988), 147–202. 19

[28] Kirchberg, E., and Wassermann, S. Operations on continuous bundles of C*-algebras. *Mathematische Annalen* 303 (1995), 677–698. 19

[29] Nagy, G., and Nica, A. On the “quantum disk” and a “non-commutative circle”. In *Algebraic Methods in Operator Theory* (Boston, MA, 1994), R. E. Curto and P. E. T. Jorgensen, Eds., Birkhäuser Boston, pp. 276–290. 19

[30] Neshveyev, S., and Tuset, L. Quantized algebras of functions on homogeneous spaces with poisson stabilizers. *Commun. Math. Phys.* 312 (2012), 223–250. 19

[31] Pedersen, G. K. Pullback and pushout constructions in C*-algebra theory. *J. Funct. Anal.* 167 (1999), 243–344. 25, 35

[32] Podleś, P. 19, 20

[33] Raeburn, I. *Graph algebras*. No. 103 in CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, RI, 2005. 31, 32

[34] Raptis, G., and Steimle, W. A cobordism model for Waldhausen K-theory. *J. London Math. Soc.* 99 (2019), 516–534. 10

[35] Schauenburg, P. Hopf-galois and bi-galois extensions. In *Galois Theory, Hopf Algebras, and Semisabelian Categories* (2004), vol. 43, Fields Institute Communications. 2

[36] Schauenburg, P., and Schneider, H.-J. On generalized Hopf-Galois extensions. *Journal of Pure and Applied Algebra* 202, 1-3 (Nov 2005), 168–194. 6

[37] Schochet, C. Topological methods for C*-algebras. III. Axiomatic homology. *Pacific J. Math.* 114, 2 (1984), 399–445. 10

[38] Sheu, A. J.-L. Projections over quantum homogeneous odd-dimensional spheres. *Journal of Functional Analysis* 277, 10 (2019), 3491–3512. 46, 47

[39] Sheu, A. J.-L. Vector bundles over multipullback quantum complex projective spaces. *J. Noncomm. Geom.* 15, 1 (2021), 305–345. 41, 42, 43, 45

[40] Sims, A., Szabó, G., and D., W. *Operator Algebras and Dynamics: Groupoids, Crossed Products, and Rokhlin Dimension*. Birkhäuser, 2002. 46

[41] Vaksman, L. L., and Soibel’man, Y. S. Algebra of functions on the quantum group SU(n + 1) and odd-dimensional quantum spheres. *Algebra i analiz* 2, 5 (1990), 101–120. 3

[42] Waldhausen, F. Algebraic K-theory of topological spaces. i. In *In R. J. Milgram, editor, Algebraic and geometric topology (Stanford Univ., Stanford, Calif., 1976)* (Providence, R.I., 1978), vol. Part 1, Proc. Sympos. Pure Math., XXXII, American Mathematica Society, pp. 35–60. 3
