Skyrmionic Beams

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Abstract

Vector vortex beams possess a topological property that derives both from the spatially varying amplitude of the field and also from its varying polarization. This property arises as a consequence of the inherent Skyrmion nature of such beams and is quantified by the associated Skyrmion number. We illustrate this idea for some of the simplest vector beams and discuss the physical significance of the Skyrmion number in this context. © 2019 Optical Society of America under the terms of the OSA Open Access Publishing Agreement

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Recent developments have highlighted the growing utility of structured light, that is optical fields in which the spatial variation of the field amplitude and/or the polarization are specifically designed for a given task [1–5]. Important examples include the formation of optical beams carrying orbital angular momentum [6–10], polarization or helicity patterns [11–16] and the vector vortex beams and their relatives [17–25]. We show that there is a Skyrmion field associated in particular with vector vortex beams and that the associated Skyrmion number is readily identified with a simple property of the beam. It is noteworthy that this property is explicitly a feature of vector beams; there is Skyrmion field only if both the polarization and the field amplitude are spatially varying.

Skyrmions were first proposed for the study of mesons [26, 27], but the idea has since found wide application in many areas of physics including quantum liquids [28–30], in magnetic materials [31–33], in 2D photonic materials [34] and in the study of fractional statistics [35]. Recently they have been observed in optics by the controlled interference of plasmon polaritons [36, 37]. We show here that a wide range of freely propagating optical beams also possess a non-trivial Skyrmion field and with it a Skyrmion number, the value of which is simply related to a topological property of the beam.

We consider a paraxial beam of either light [38, 39] or electrons [40–42] and express the local polarization or spin direction in the form

\[
|\Psi(r)\rangle = |0\rangle + e^{i\theta_0}u_1(r)|1\rangle.
\]

Here \(|0\rangle\) and \(|1\rangle\) represent any two orthogonal polarization (or electron spin) states, while \(u_0(r)\) and \(u_1(r)\) are two orthogonal spatial modes and the global phase difference between the two modes is denoted by \(\theta_0\). That this decomposition is always possible follows from the familiar Schmidt decomposition [43]. The Skyrmion field and number depend only on the spatial variation of the polarization or spin direction and for this reason it is convenient to work with a locally-normalized state in the form

\[
|\psi(r)\rangle = |0\rangle + e^{i\theta_0}v(r)|1\rangle \sqrt{1 + |v(r)|^2},
\]

where \(v(r) = u_1(r)/u_0(r)\).

The Skyrmion field is most readily defined in terms of an effective magnetization \(\mathbf{M}\), which is the local direction of the Poincaré vector for light or the Bloch vector for an electron beam as shown in Fig. 1. In terms of our locally normalized state it is

\[
\mathbf{M} = \langle \psi(r) | \sigma | \psi(r) \rangle,
\]

where \(\sigma\) is a vector operator with the familiar Pauli matrices as Cartesian components. The \(i\)th component of the associated Skyrmion field is

\[
\Sigma_i = \frac{1}{2} \epsilon_{ijk} \epsilon_{pq} M_p \frac{\partial M_q}{\partial x_j} \frac{\partial M_r}{\partial x_k},
\]

where \(\epsilon_{ijk}\) is the alternating or Levi-Civita symbol and we employ the summation convention. The Skyrmion number is

\[
n = \frac{1}{4\pi} \int \Sigma_z \, dx \, dy,
\]

where we have chosen the \(z\)-axis to define the direction of propagation and the integral runs over the whole of the transverse plane. The form of the Skyrmion field ensures that it is transverse (\(\nabla \cdot \Sigma = 0\)). This means that there are no sources or sinks for the Skyrmion field and the associated field lines can only
s polarization electron beams. However, our results apply both to classical and quantum states of light and also to waves that have the same wavelength \( \lambda \) and focus at the same focal position \( z_0 \). These modes have a vortex of strength \( \ell \) on the \( z \)-axis, which is associated with a \( z \)-component of the orbital angular momentum of \( \ell \hbar \) per photon (or electron) [6–10]. Modes with different angular momentum numbers \( \ell \) are orthogonal and if we choose two such modes for our two complex amplitudes \( u_0 \) and \( u_1 \) in (1) then the function \( v(r) \) in (2) for the locally normalized state \( |\psi(r)\rangle \) has the general form

\[
v(r) = f(\rho, z) e^{i\theta(\rho, z)} e^{i\Delta \ell \phi},
\]

where \( \Delta \ell = \ell_1 - \ell_0 \), \( f \) and \( \theta \) are real functions of the coordinates \( \rho \) and \( z \) and \( \phi \) incorporates all phase terms including \( \theta_0 \), the phase difference between the modes. It is straightforward to calculate the Skyrmion field and from this the Skyrmion number for our vector vortex beam. We find the simple result that for such beams the Skyrmion number is

\[
n = \Delta \ell \left( \frac{1}{1 + f^2(0, z)} - \frac{1}{1 + f^2(\infty, z)} \right), \tag{8}
\]

the value of which is determined only by which of the two modes \( u_0(r) \) and \( u_1(r) \) dominates on the \( z \)-axis, the location of the vortex, and which dominates as \( \rho \) tends to infinity.

\[ u_p(\rho, \phi, z) = \sqrt{\frac{2}{\pi (p + |\ell|)!}} \frac{1}{w(z)} (\rho \sqrt{2} w(z))^{|\ell|} \exp \left( -\rho^2 w(z) \right)
\]

form loops or extend to infinity. It follows that the flux of the Skyrmion field through any closed surface is zero, \( \oint \Sigma \cdot dS = 0 \).

Optical vector vortex beams typically have a rotating polarization pattern that originates from the differential orbital angular momentum of the contributing modes [4, 18] and exhibit intriguing topological [44–47] and focussing properties [48, 49]. We consider the simplest case of such beams in which the two orthogonal modes, with amplitudes \( u_0(r) \) and \( u_1(r) \), are Laguerre-Gaussian (LG) modes

\[ u_p(\rho, \phi, z) = \sqrt{\frac{2}{\pi (p + |\ell|)!}} \frac{1}{w(z)} (\rho \sqrt{2} w(z))^{|\ell|} \exp \left( -\rho^2 w(z) \right)
\]

familiar from the study of orbital angular momentum [6–10]. Here \( z_R = \pi w_0^2 / \lambda \) is the Rayleigh range and \( w(z) = w_0 \sqrt{1 + (z - z_0)^2 / z_R^2} \) is the beam width on propagation. We assume that the modes have the same wavelength \( \lambda \), but they may differ in the beam parameters \( \ell, p, w_0 \) and the focal position \( z_0 \). These modes have a vortex of strength \( \ell \) on the \( z \)-axis, which is associated with a \( z \)-component of the orbital angular momentum of \( \ell \hbar \) per photon (or electron) [6–10]. Modes with different angular momentum numbers \( \ell \) are orthogonal and if we choose two such modes for our two complex amplitudes \( u_0 \) and \( u_1 \) in (1) then the function \( v(r) \) in (2) for the locally normalized state \( |\psi(r)\rangle \) has the general form

\[
v(r) = f(\rho, z) e^{i\theta(\rho, z)} e^{i\Delta \ell \phi}, \tag{7}
\]

where \( \Delta \ell = \ell_1 - \ell_0 \), \( f \) and \( \theta \) are real functions of the coordinates \( \rho \) and \( z \) and \( \phi \) incorporates all phase terms including \( \theta_0 \), the phase difference between the modes. It is straightforward to calculate the Skyrmion field and from this the Skyrmion number for our vector vortex beam. We find the simple result that for such beams the Skyrmion number is

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the value of which is determined only by which of the two modes \( u_0(r) \) and \( u_1(r) \) dominates on the \( z \)-axis, the location of the vortex, and which dominates as \( \rho \) tends to infinity.
point and with orbital angular momentum differing by one. In this case (7) simplifies to \( v(r) = A(z) \rho^\ell \phi \) (where \( A \) is generally complex and includes the overall phase difference \( \theta \)) and one polarization dominates at the position of the vortex with the orthogonal polarization appearing as \( \rho \to \infty \). We provide two examples of such polarization patterns in Figs. 2c and 2e. The local Bloch or Poincaré vector, representing the local spin or polarization direction, is clearly reminiscent of the spiral and hedgehog Skyrmions, familiar from the study of magnetic Skyrmions [32]; the former arises when \( A \) is real and the latter when it is imaginary. We note that the natural propagation of the beam will cause the polarization pattern to evolve continuously from one of these forms into the other by virtue of the relative Gouy phase [39] that changes as the beam propagates.

We can design Skyrmionic beams with more structure than the simple hedgehog and spiral Skyrmions by superposing LG beams with orbital angular momentum numbers that differ by more than one. In Fig. 3 we plot the local Bloch or Poincaré vector for a pair of modes with \( \Delta \ell = 2 \). We see that, both in this case and in Fig. 2, these vectors rotate as we traverse a path around the vortex and, moreover, that along such a path the vector completes two rotations if \( \Delta \ell = 2 \) but only one rotation if \( \Delta \ell = 1 \). These are examples of a more general result that for a superposition of modes with a difference in orbital angular momentum number of \( \Delta \ell \), the Bloch or Poincaré vector rotates \( \Delta \ell \) times on a path enclosing the vortex. The corresponding polarization ellipse rotates by only half the amount. This behavior persists when we consider modes with radial indices different from zero, although the polarization structure becomes more intricate because of the additional nodal lines. The resulting Skyrmion number is nevertheless governed by the difference in dominating behavior described in (8).

**Fig. 3.** The vector field \( \mathbf{M} \) of a propagating paraxial beam when the spatial components are two LG modes with a difference of 2 in their winding numbers \( \ell_0 \) and \( \ell_1 \): a) an achiral Skyrmion, i.e. no phase difference between \( u_0 \) and \( u_1 \); b) a chiral Skyrmion, when the phase difference is \( \pi \).

The corresponding Skyrmion number is \( \Delta \ell \) if the spin or polarization states at the vortex position and at infinity are orthogonal but will be zero if they are the same. This dependence of the Skyrmion number on both the difference in the orbital angular momentum or vortex charges, \( \Delta \ell \), and the change in the polarization between the position of the vortex and at large distances from it clearly demonstrates that the Skyrmion field and number are topological properties of both the spin and orbital angular momenta.

The fact that the Skyrmion field, \( \Sigma \), is divergenceless suggests that the Skyrmion number, as we have defined it, should be conserved on propagation. This is not true, however, and the simplest way to demonstrate this is to consider a superposition of LG beams that are focussed at different positions along the z-axis. The effect of this is that the polarization behavior at large values of \( \rho \) changes as the beam propagates and the Skyrmion number changes from \( \Delta \ell \) to 0 (or from 0 to \( \Delta \ell \)). This behavior is depicted in Fig. 4, where we see that the polarization at large distances from the central vortex changes abruptly at one transverse plane and with it the Skyrmion number. To see how this happens, consider a circular-cylindrical surface of radius \( R \) centered on the position of the vortex extending from \(-z_0\) to \(z_0\). In general, there will be a flux through this surface of the Skyrmion field, \( \Sigma \):

\[
\int_0^{2\pi} d\phi \int_{-z_0}^{z_0} dz \Sigma_r = \Delta \ell \left( \frac{1}{1 + f^2(R,-z_0)} - \frac{1}{1 + f^2(R,z_0)} \right).
\]

Clearly, this will take a non-zero value if \( f^2(R,-z_0) \neq f^2(R,z_0) \). If we allow \( R \) to tend to infinity then a non-zero value of this flux corresponds to a situation in which the polarization at large distances from the vortex differs at \(-z_0\) and \(z_0\). This is what happens if the Skyrmion number changes between \(-z_0\) and \(z_0\).

We have shown that vector vortex beams, either of light or electrons, possess a topological property that can be identified with a Skyrmion number. The associated Skyrmion field is transverse (or divergenceless) and this means that there are no sinks or sources of this field. The Skyrmion number for a beam can change on free space propagation, however, if Skyrmion field

![Fig. 4.](image-url)
lines escape radially out of the beam towards regions of negligible intensity. Demonstrating these properties requires the preparation of vector vortex beams and measurement of the polarization or spin in planes perpendicular to the beam axis [50]. We shall report on such experiments elsewhere.

We close by emphasising that the Skyrmionic property of vector beams is distinct from the familiar spin and orbital angular momentum of optical beams [6–10, 51]. It is true that the beams we consider here combine optical vortices and polarization, commonly associated with orbital and spin angular momentum respectively, but the Skyrmion number is a topological rather than a mechanical property of the beam. To see this we note that the Skyrmion number is unchanged if we apply a global transformation of the polarization. Moreover, we have seen that it is possible for the Skyrmion number to change if the two superimposed modes are focussed in different places. The total spin and angular momentum passing through each transverse plane, however, remains unchanged.

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