A temperature correction to the tachyon, using the Casimir effect

Y. Koohsarian

Department of Physics, Ferdowsi University of Mashhad
P.O.Box 1436, Mashhad, Iran
yo.koohsarian@stu-mail.um.ac.ir

Abstract

We find the free energy of the string by applying the known Matsubara formalism. Then through the Casimir effect we offer a temperature correction to the tachyon mass of the string. We see that for the fermionic part the temperature correction is precisely the opposite of that of bosonic part, so the quantum ground state of the superstring would remain massless, as expected.

Keywords: Tachyon, Thermodynamic of the string theory, Vacuum energy, Casimir effect, Temperature correction to the tachyon mass

1 Introduction

The zero- and nonzero-temperature Casimir effect has been studied theoretically and experimentally for many different systems (see for example [1, 2, 3, 4, 5, 6]). As we know the Casimir energy is actually the renormalized (quantum) vacuum energy of a system having physical boundaries. Here we want to study the nonzero-temperature vacuum for the string theory. Applying the Matsubara formalism, we find the free energy of the string as a summation over the free energies of independent oscillatory modes of the string transverse coordinates. We see that the free energy of the string can be interpreted as the nonzero-temperature vacuum energy of the string. As we know the tachyon comes from the quantum vacuum state of the string. So through the Casimir effect, we will offer a relation for the temperature correction to the tachyon mass. In section 2 we study the vacuum state of the bosonic as well as fermionic string at nonzero temperature by finding the free energy of the string. In section 3 we analyze the tachyon in the framework of the Casimir effect, and find a temperature correction to tachyon mass.

2 Nonzero-temperature vacuum energy of the string

2.1 Bosonic string

The string sigma model action with flat world-sheet metric and for the light-cone gauge, can be written as

\[ S = -\frac{1}{4\pi\alpha'} \int d\tau \int d\sigma X^I \left( \partial_{\tau}^2 - \partial_\sigma^2 \right) X^I, \]  

(1)
in which $\alpha'$ is the Regge slope parameter, $\sigma$ and $\tau$ are coordinates of the world-sheet, and $X^I$ are the transverse coordinates of the space. By implementing a Wick rotation for the coordinate $\tau$, the Euclidean action of the string would be obtained as

$$S_E(X^I) = i \frac{1}{4\pi\alpha'} \int d\tau \int d\sigma X^I \left(-\partial^2_\tau - \partial^2_\sigma\right) X^I.$$  \hfill (2)

Now through the Matsubara formalism one can write the coordinate modes (of an open string with Neumann boundary condition) as

$$X^I(\sigma, \tau) \sim \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} b^n_I j \exp(-i\xi_j \tau) \cos(n\sigma); \quad \xi_j \equiv 2\pi j T,$$  \hfill (3)

in which $b_I^n$ are the operators of bosonic oscillators, $\xi_j$ are the Matsubara frequencies, and "$T$" is the temperature parameter of the string. So using the functional integral representation, the partition function would be obtained as

$$Z = \int DX^I \exp \left[iS_E(X^I)\right] \sim \prod_{I=1}^{d-1} \left[ \det \left(-\partial^2_\tau - \partial^2_\sigma\right) \right]^{-\frac{1}{2}},$$  \hfill (4)

in which $d$ is the dimension of the space. Therefore the free energy is obtained as

$$F = -T \ln Z = \frac{d-1}{2} T \sum_{j=-\infty}^{\infty} \sum_{n=1}^{\infty} \ln \left[ \xi_j^2 + n^2 \right].$$  \hfill (5)

One can simplify the above expression as a sum of two terms: temperature-dependent and temperature-independent part (see for example chapter 5 of Ref. [1]). Then the free energy of the open bosonic string (denoted as OB) takes the form

$$F_{OB}(T) = \frac{d-1}{2} \sum_{n=1}^{\infty} n + (d-1)T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left(-\frac{n}{T}\right) \right].$$  \hfill (6)

The above calculation has a simple physical explanation. As we know, the transverse coordinates $X^I$ can be written as a superposition of independent harmonic oscillatory modes (see Eq. (3)). For a harmonic oscillator of temperature $T$, the partition function is well known:

$$\Theta_n = \frac{\exp(-\omega_n/2T)}{1 - \exp(-\omega_n/T)}$$

in which $\omega_n$ is frequency of the oscillator. If we similarly assign a partition function to each oscillator mode of the transverse coordinate, the free energy of the fields $X^I$ can then be obtained just by summing over the free energies of these independent oscillatory modes. Now, for the harmonic oscillatory modes of the open bosonic string we have $\omega_n = n$, so the free energy for each transverse direction of an oscillator mode takes the form

$$F_n = -T \ln \Theta_n = \frac{n}{2} + T \ln \left[ 1 - \exp \left(-\frac{n}{T}\right) \right].$$  \hfill (7)
Therefore the total free energy of the transverse coordinate fields of the open bosonic string is obtained as

\[
F_{\text{tot}} = \sum_{I=1}^{d-1} \sum_{n=1}^{\infty} F_n = \frac{d-1}{2} \sum_{n=1}^{\infty} n + (d-1)T \sum_{n=1}^{\infty} \ln \left[1 - \exp \left(-\frac{n}{T}\right)\right],
\]  

(8)

that equals just to the free energy (6) obtained through the Matsubara formalism. The series in the first term of the free energy (6), can be regularized simply by applying the known Riemann zeta function \(\zeta_R\) so that

\[
\sum_{n=1}^{\infty} n = \zeta_R(-1) = -\frac{1}{12}.
\]

Hence the free energy of the open bosonic string is obtained as

\[
F_{\text{OB}}(T) = -\frac{d-1}{24} + (d-1)T \sum_{n=1}^{\infty} \ln \left[1 - \exp \left(-\frac{n}{T}\right)\right].
\]  

(9)

Note that at \(T = 0\) the logarithmic term of the above free energy would be zero, so the first term (i.e. the temperature-independent term) is actually the zero-temperature part of the free energy

\[
F_{\text{OB}}(0) = -\frac{d-1}{24}.
\]  

(10)

But as we know, by summing over zero-point energy of transverse oscillatory modes, we find just the vacuum state energy of the string:

\[
E_{\text{OB}} = \sum_{I} \sum_{n=1}^{\infty} \left(\frac{1}{2}n\right) = -\frac{d-1}{24},
\]  

(11)

that is just equal to the zero-temperature free energy (10). This equivalence has a meaningful interpretation through the Casimir effect. In fact in the framework of the Casimir effect the free energy of a system (in the presence of physical boundaries) is taken as its \textit{nonzero-temperature Casimir energy} (see e.g. the section 5 of [2] or the chapter 5 of [1]). Recall that the Casimir energy is actually the renormalized vacuum energy of the system. For example for conducting planes the temperature-dependent part of the vacuum energy gives an observable temperature correction to the Casimir force on planes [3, 4, 5, 6]. Here similarly we could take the free energy (9) as the \textit{nonzero-temperature vacuum energy} of the open string:

\[
E_{\text{OB}}(T) = E_{\text{OB}} + (d-1)T \sum_{n=1}^{\infty} \ln \left[1 - \exp \left(-\frac{n}{T}\right)\right].
\]  

(12)

As we see in section 3 the temperature-dependent part of the above vacuum energy can be interpreted as a temperature correction to the tachyon mass. At high temperature using the known Abel-Plana formula [7] we find

\[
\sum_{n=1}^{\infty} \ln \left[1 - \exp \left(-\frac{n}{T}\right)\right] = -\frac{\pi^2}{6} T + O(\ln T) \quad T \gg 1.
\]  

(13)
On the other hand, at low temperature one can approximate the series in Eq. (9) with its first term
\[
\sum_{n=1}^{\infty} \ln [1 - \exp (-n/T)] \approx \ln [1 - \exp (-1/T)] \quad T \ll 1.
\] (14)

So the high- and low-temperature vacuum energy of the open bosonic string can be written as
\[
\mathcal{E}_{OB}(T) \approx \mathcal{E}_{OB} - \frac{(d-1)\pi^2}{6} T^2 \quad T \gg 1,
\]
\[
\mathcal{E}_{OB}(T) \approx \mathcal{E}_{OB} + (d-1) T \ln [1 - \exp (-1/T)] \quad T \ll 1.
\] (15)

Corresponding results for the closed string can be similarly obtained by taking \(\omega_n = 2n\) in Eqs. (7) and (11), and so on. So we find
\[
\mathcal{E}_{CB}(T) = \mathcal{E}_{CB} - \frac{d-1}{12} \quad T \ll 1,
\]
\[
\mathcal{E}_{CB}(T) = \mathcal{E}_{CB} + (d-1) T \sum_{n=1}^{\infty} \ln [1 - \exp (-2n/T)] ,
\] (16)

and so
\[
\mathcal{E}_{CB}(T) \approx \mathcal{E}_{CB} - \frac{(d-1)\pi^2}{12} T^2 \quad T \gg 1,
\]
\[
\mathcal{E}_{CB}(T) \approx \mathcal{E}_{CB} + (d-1) T \ln [1 - \exp (-2/T)] \quad T \ll 1.
\] (17)

Note that the low- as well as high-temperature free energy of the bosonic string has a negative value.

### 2.2 Fermionic string

The Euclidean action of the fermionic string can be written as
\[
S_E(\psi^I) = i \frac{1}{2\pi} \int d\tau \int d\sigma \left[ \psi_1^I (-i\partial_\tau - \partial_\sigma) \psi_1^I + \psi_2^I (-i\partial_\tau + \partial_\sigma) \psi_2^I \right],
\] (18)

in which \(\psi_{1,2}\) are the components of the spinor \(\psi\). Then the partition function would be given as
\[
\mathcal{Z} = \prod_I \left[ \det (-\partial_\tau^2 - \partial_\sigma^2) \right]^{1/2}.
\] (19)

For an open fermionic string with Ramond boundary condition, the spinor fields are given as
\[
\psi^I(\tau, \sigma) \sim \sum_{n \in \mathbb{Z}} f_n^I \exp [-in(\tau - \sigma)],
\] (20)

in which \(\mathbb{Z}\) denotes the set of integers, and \(f_n^I\) are operators of fermionic oscillators. Therefore after some calculations the free energy of the open fermionic string would be obtained as
\[
\mathcal{F}_{OF}(T) = -\frac{d-1}{2} \sum_{n=1}^{\infty} n - (d-1) T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{n}{T} \right) \right].
\] (21)
Similarly as for the bosonic string, the above free energy can also be obtained by summing over the free energies of all fermionic oscillatory modes of the string, if we regard each oscillator mode of the fermionic string as a fermionic oscillator with frequency $\omega_n = n$. Recall that the free energy of a fermionic oscillator with frequency $\omega_n$ is given as

$$F_n(T) = -\frac{1}{2}\omega_n - T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{\omega_n}{T} \right) \right].$$  \hfill (22)

The first term of the free energy \[21\] can be regularized using Riemann zeta function, so

$$F_{OF}(T) = \frac{d-1}{24} - (d-1)T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{n}{T} \right) \right].$$  \hfill (23)

But the vacuum state energy of the open fermionic string is simply given as

$$\mathcal{E}_{OF} = \sum_I \sum_{n=1}^{\infty} \left( -\frac{n}{2} \right)$$

$$= \frac{d-1}{24} = F_{OF}(0),$$ \hfill (24)

where the minus sign of the zero-point energy (in the first line) is from the anti-commutative feature of operators of the fermionic oscillator. So the nonzero-temperature vacuum energy of the open fermionic string can be written as

$$\mathcal{E}_{OF}(T) = \mathcal{E}_{OF} - (d-1)T \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{n}{T} \right) \right],$$  \hfill (25)

and the low- and high-temperature limits can be obtained as

$$\mathcal{E}_{OF}(T) \approx \mathcal{E}_{OF} \pm \frac{(d-1)\pi^2}{6} T^2 \quad T \gg 1,$$

$$\mathcal{E}_{OF}(T) \approx \mathcal{E}_{OF} - (d-1)T \ln \left[ 1 - \exp \left( -1/T \right) \right] \quad T \ll 1.$$  \hfill (26)

Similar results for the closed fermionic string can be obtained taking $\omega_n = 2n$ in Eq. \[22\] and so on.

### 3 A temperature correction to the tachyon mass

As we know the quantum mass formula of the open string is given as

$$M^2 = \frac{1}{\alpha'} (a + N),$$ \hfill (27)

in which $N$ is the number operator, and $a$ is the ordering constant. Taking $N = 0$ one finds the mass formula of the open string tachyon

$$m^2 = \frac{1}{\alpha'} a.$$ \hfill (28)
As we know, the constant $a$ is obtained as the summation over the zero-point energy of transverse oscillatory modes, i.e. just the vacuum state energy of the string:

$$a = \mathcal{E}. \quad (29)$$

So we can write

$$m^2 = \frac{1}{\alpha'} \mathcal{E}. \quad (30)$$

But as we previously realized, $\mathcal{E}$ is actually the zero-temperature vacuum energy of the string (see Eqs (12) and (25)). So Eq. (30) is actually the zero-temperature mass formula of open string tachyon. Therefore we can generalize the mass formula of the open string tachyon to the nonzero temperature as

$$m^2(T) = \frac{1}{\alpha'} \mathcal{E}(T). \quad (31)$$

So using Eq. (12), (24) and (30) we can take the nonzero-temperature mass formula of the open bosonic and fermionic tachyon as

$$m^2_{OB}(T) = \frac{1}{\alpha'} \mathcal{E}_{OB}(T) = m^2_{OB} + \frac{(d-1)T}{\alpha'} \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{n}{T} \right) \right]$$

$$m^2_{OF}(T) = \frac{1}{\alpha'} \mathcal{E}_{OF}(T) = m^2_{OF} - \frac{(d-1)T}{\alpha'} \sum_{n=1}^{\infty} \ln \left[ 1 - \exp \left( -\frac{n}{T} \right) \right], \quad (32)$$

in which by using Eqs. (11), (24) and (30) we have

$$m^2_{OF} = -m^2_{OB} = \frac{d-1}{24\alpha'}. \quad (33)$$

Note that for bosonic as well as fermionic string at $T = 0$ we obviously have $m^2(0) = m^2$, hence the temperature-dependent terms of the mass formulas (32) can be considered as temperature corrections to the tachyon mass of the bosonic and fermionic strings. The high- and low-temperature limits of the mass formulas (32) can be simply obtained using Eqs. (15) and (26). For example at high temperature we have

$$m^2_{OB}(T) \approx m^2_{OB} - \frac{(d-1)\pi^2}{6\alpha'} T^2$$

$$m^2_{OF}(T) \approx m^2_{OF} + \frac{(d-1)\pi^2}{6\alpha'} T^2. \quad (34)$$

As is seen the high-temperature correction to the tachyon of the bosonic string, contrary to that of fermionic string, has a negative value. Eventually for an open superstring we find

$$m^2_{OS} = \frac{1}{\alpha'} [\mathcal{E}_{OB} + \mathcal{E}_{OF}] = 0$$

$$m^2_{OS}(T) = \frac{1}{\alpha'} [\mathcal{E}_{OB}(T) + \mathcal{E}_{OF}(T)] = 0, \quad (35)$$

where we have used Eqs (11), (12), (24) and (25). As we see from the above equation, the quantum ground state of the superstring at nonzero (as well as zero) temperature is massless, as expected. This is due to the relative minus sign between the vacuum energy of the fermionic and bosonic parts at any temperature. Similar results can be obtained for the closed string using its tachyon mass formula

$$m^2 = \frac{2}{\alpha'} \mathcal{E}, \quad (36)$$

and so on.
Acknowledgements

I thank Mohammadreza Garousi and also Mohammad Moghadassi for their valuable discussions.

References

[1] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect*, Oxford Science Publications, 2009.

[2] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rep. **353**, 1205 (2001).

[3] J. Mehra Physica **37**, 145 (1967).

[4] L.S. Brown and G.J. Maclay, Phys. Rev. **184**, 1272 (1969).

[5] J. Schwinger, L.L. de Raad and K.A. Milton, Ann. Phys. **115**, 1 (1978).

[6] C. Genet, A. Lambrecht and S. Reynaud, Phys. Rev. A **62**, 012110 (2000)

[7] A. Erdélyi et al., *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.