Lectures on Integrable Structures in Quantum Field Theory and Massive ODE/IM Correspondence

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Abstract

This review was born as notes for a lecture given at the YRIS school on integrability in Durham, in the summer of 2015. It deals with a beautiful method, developed in the mid-nineties by V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov and, as such, called BLZ. This method can be interpreted as a field theory version of the quantum inverse scattering (QIS), also known as algebraic Bethe ansatz (ABA). Starting with the case of conformal field theories (CFT) we show how to build the field theory analogues of commuting transfer $T$ matrices and Baxter $Q$-operators of integrable lattice models. These objects contain the complete information of the integrable structure of the theory, viz. the integrals of motion, and can be used, as we will show, to derive the thermodynamic Bethe ansatz (TBA) and non-linear integral (NLIE) equations. This same method can be easily extended to the description of integrable structures of certain particular massive deformations of CFTs, these, in turn, can be described as quantum group reductions of the quantum sine-Gordon model and it is an easy step to include this last theory in the framework of BLZ approach. Finally we show an interesting and surprising connection of the BLZ structures with classical objects emerging from the study of classical integrable models via the inverse scattering transform method. This connection goes under the name of ODE/IM correspondence and we will present it for the specific case of quantum sine-Gordon model only.

This review is part of a collection of papers [33, 38, 35, 26, 13] all of which were born out of lectures given at YRIS school in Durham.
The history of Integrable Systems is as old as that of Classical Mechanics and the two were, for the largest part of 18th century, more or less coinciding. Following the formulation of Isaac Newton’s laws of motion, for more than a century, eminent mathematicians and physicists such as J. B. d’Alembert, L. Euler, J. L. Lagrange, C. G. J. Jacobi and Sir W. R. Hamilton devoted many works to the problem of finding exact solutions to Newton’s equations. These efforts brought about a striking amount of results, which condensed in the theory of Lagrangian mechanics first and that of Hamiltonian mechanics then, culminating in the first definition of integrability, as given by J. E. É. Bour and J. Liouville. Although the discovery of many new integrable systems quickly followed, by the end of the 19th century a fundamental result of J. H. Poincaré doused the excitement of the mathematical and physical community, effectively deeming the integrable systems as exceptions amongst the Hamiltonian ones. From that moment the theory of integrable system laid more or less dormant until the second half of the ‘60, when the idea of integrability resurfaced thanks to the efforts of C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura, P. D. Lax, L. D. Faddeev, V. E. Zakharov and many other. From that pivotal half-decade, the theory of integrable systems began growing more and more, incorporating results obtained in other branches of theoretical physics, first and foremost Bethe’s method for the study of quantum spin chains as well as Baxter’s approach to 2D statistical lattice models. The number of publications devoted to the study of integrability grew steadily, especially after the introduction of Conformal Field Theories by A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov and the “first superstring revolution”. Finally a last breakthrough came just before the end of the century, thanks to J. Maldacena: the “AdS-CFT” correspondence. This discovery “opened the floodgates” (to borrow the words of Polyakov) and stimulated an impressive amount of work, especially in the field of integrable models.

As the title explicitly reveals, this review deals with the analysis of the integrable structures in field theories. What is meant with this denomination are not simply the fundamental objects that are seen appearing in all integrable theories: the integrals of motion and their “generating functions”, the $T$ and $Q$ operators. The expression “integrable structures” encompasses the whole algebraic skeleton which allows for the building of integrability to stand. The study of integrable structures in field theory was first exhaustively addressed to by VV. Bazhanov, S.L. Lukyanov and...
A.B. Zamolodchikov in a remarkable series of papers \[1, 2, 3, 4\] and the goal of these 50 odd pages is essentially to go through their work and present it in a uniform and coherent way, providing details and insights in the definitions that we hope will help the reader to understand this beautiful approach. It is, nonetheless, impossible to include in a single review all the implications of the method introduced by Bazhanov, Lukyanov and Zamolodchikov (hereafter addressed to as the BLZ method), as his connections with the theory of quantum integrable systems and CFTs are deep and widespread. For this reason we included a list of references which will be addressed to in the text when a certain topic will be simply cited.

Given the length and the complexity of the subject we decided to keep the main body of the notes as clean as possible by separating heavier calculations and in-depth analyses, not strictly essential to the progression of the review, to boxed sections. These ”in-depth boxes” are interleaved with the main body and the reader can, depending on its necessities and the level of its knowledge, skip them without missing anything fundamental about the method. We believe, however, that they can be extremely useful in getting a deeper understanding of the subject and of its many relations to other topics of integrability.

We would like to address a final word of caution to students and young researchers first approaching this subject: do not feel discouraged if you cannot grasp every aspect presented here. The BLZ method requires the use of diverse advanced mathematical concepts and the computations sketched here are often very technical and demanding; insisting on understanding everything at a first reading would be foolish. Instead we suggest multiple readings so that, at each step, it would be possible to go through the concepts, references and calculations in more and more detail. We especially suggest the readers willing to spend time learning this method to go, at some point, through the computations outlined here as this will often force them to explore the references and think about the very meaning of the objects into play: the reward will surely be a deeper and broader understanding of the concepts exposed here.

This review is organised as follows. In the first section, after a brief review of Conformal Field Theories (CFTs) and of classical KdV hierarchy, we will begin building the integrability objects for the \( c < 1 \) CFTs from scratch. We first introduce the quantum transfer matrices \( T_j \) and show how they can be interpreted as sort of generating functions for the quantum integrals of motion. We will then broaden our view, generalising the algebraic setting and constructing the Baxter operators \( Q \). While doing so we will also show how these objects can be used to derive the useful TBA, Bethe Ansatz and NLIE equations, making a connection to the other reviews in this volume. Following this will be a section devoted to the extension of the previous results to the massive integrable deformations of CFTs. This section contains a brief account on the theory of integrable CFT deformations which can be skipped by the readers already familiar with the concept. Finally, in the last section we present a completely different yet, we believe, really interesting approach to the construction of the integrable structures in the particular case of sine–Gordon model. This method, bearing the name ODE/IM correspondence, reveal an intimate and still poorly understood connection between the theory of classical and quantum integrable models. Finally, given the large number of parameters appearing in this review, we thought it would be useful to collect the most relevant ones and the relationship amongst them here, in Table 1.

### Table 1: Relationship amongst parameters

| Central charge | Conformal dimension | Spectral parameter |
|----------------|---------------------|--------------------|
| \( c \)        | \( h \)             | \( \lambda \)      |
| \( \beta = \sqrt{1-c} - \sqrt{\frac{25-c}{24}} \) |                      | \( \theta = (1 + \xi) \log (\lambda) \) |
| \( q = e^{i\pi \beta^2} = -e^{-i\pi \frac{1}{\beta^2}} \) |                      | \( p : h = \left( \frac{p}{\beta^2} \right)^2 + \frac{c-1}{24} \) |
| \( \xi = \frac{2\pi}{1-\beta^2} \) |                      | \( y = \frac{\Gamma(1-\beta^2)}{\beta^2} \lambda \) |
| \( \kappa = -i \frac{\pi}{\sin(\pi \beta^2)} \lambda \) |                      | |

2 Integrable Structures of Conformal Field Theory

The 2D CFTs are the perfect and probably best known example of exactly solvable quantum field theories. From the year 1984, when the concept of CFT was first introduced in an article of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [6], up to our days they received a great deal of attention and most of their features are now known, to the point of making them a self-contained theory which is very often subject of advanced courses in
The goal of the following section is to recall the basics of 2D CFT and to set up the notation, without any pretense of completeness. It is intended for readers having already a good knowledge of the subject; for those who are less familiar with it, there exists a plethora of, often very good, reviews and books dealing with 2D CFT, offering a wide range of different points of view. I suggest [8] for a straightforward introduction and the references therein for a more in-depth study; the most daring might consider the “Big Yellow Book” by P. Di Francesco, P. Mathieu and D. Sénéchal [9].

2.1 Brief overview of CFT basic concepts

The Virasoro algebra A Conformal Field Theory (CFT) in $D$ Euclidean dimensions, that is a local, isotropic field theory possessing no characteristic length scale, is invariant under the global conformal group $SO(D + 1, 1)$, a non-compact Lie group of dimension $\frac{1}{2}(D + 1)(D + 2)$. Although bigger than the Galilei group $\mathbb{R}^D \times SO(D)$, whose dimension is $\frac{1}{2}D(D + 1)$, it is still not sufficient to grant integrability to the CFT: we need an infinite number of symmetries to perform this task and that is exactly what we find if we look at the particular case $D = 2$. In fact when we consider the conformal transformations of a plane, even if the special orthogonal group $SO(3, 1)$ has dimension 6, we can find an infinite number of conformal coordinate transformations: the holomorphic mappings from the complex plane (or part of it) onto itself. This discrepancy between the finiteness of the conformal group and the infinite amount of independent conformal coordinate transformations is easily resolved by remarking that most of these last are not globally well-defined. The set of infinitesimal conformal transformations form an infinite dimensional local algebra, the renowned Virasoro algebra $\hat{\text{Vir}}$, which exponentiate to the Virasoro group $\text{Diff} S^1$ (the centrally extended group of diffeomorphisms of the unit circle) [10, 11]. This last contains, as a subgroup, the Mobius group $\text{SL}(2, \mathbb{C})/\mathbb{Z}_2 \approx SO(3, 1)$ of global conformal transformations. Since a local field theory should be sensitive to local symmetries, even if the related transformations are not globally defined, the behaviour of a $(1 + 1)$-dimensional is indeed constrained by the full algebra $\hat{\text{Vir}}$. It is precisely the local conformal invariance that, being infinite-dimensional, allows for exact solutions of 2D CFTs to exist. The Virasoro algebra $\hat{\text{Vir}}$ is generated by the operators $\{L_n\}_{n \in \mathbb{Z}}$, obeying the famous commutation relations

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^2 - 1) \delta_{n, -m},$$

1. Note that this characterisation is not unique! One can choose different massless particle bases of the Hilbert space, with different particle content and different scattering amplitudes. These different choices reflect the possibility of reaching a certain CFT as massless limit of different massive field theories.

2. The intuitive reason for that comes from Liouville’s definition of an integrable system. This states that a system is integrable if it possesses the same number of conserved quantities and degrees of freedom. By Noether theorem conserved charges correspond to symmetries and so we need a set of symmetries having dimension equal to the degrees of freedom of our system. But a field theory has “infinite degrees of freedom”! Hence we need an infinite number of symmetries to make it integrable.

3. Actually the algebra of infinitesimal conformal mappings is the Witt algebra. However in the quantum theory this symmetry is anomalous and the Witt algebra gets extended to $\hat{\text{Vir}}$ by the addition of a central charge $c$, also called conformal anomaly.
where \( c \) is a number called central charge or conformal anomaly.

Let us now consider a CFT on a flat Euclidean plane, having coordinates \((x,y)\). The presence of scale invariance means that we are dealing with massless theories. Were we considering a Minkowski geometry \((x^0, x^1)\), it would be then natural to describe the system using light-cone coordinates \((x^+, x^-) = (x^0 + x^1, x^1 - x^0)\), so that left/right-moving massless fields depend uniquely on, respectively, \(x^+\) or \(x^-\). Here, in the same spirit, it turns out to be extremely useful to introduce the complex coordinates \(w, \bar{w} = (x + iy, x - iy)\) and the notion of left/right-moving fields turns into that of purely holomorphic/antiholomorphic Euclidean fields. The algebra of symmetries of a CFT will thus be the direct sum of two Virasoro algebras: \(\text{Vir} \oplus \overline{\text{Vir}}\). From now on we will consider the complex coordinates \(w\) and \(\bar{w}\) to be independent, so that \(\text{Vir} \oplus \overline{\text{Vir}}\) naturally acts on \(\mathbb{C}^2\) and we can treat each term in the direct sum independently and effectively work only with holomorphic fields. When the time comes to compute physical quantities we will "remember" to add the antiholomorphic contributions and impose the "real slice" condition \(\bar{w} = w^*\).

Since we are dealing with massless fields, we must pay attention to infrared divergencies. For this reason we invest one of the dimensions, say \(y\), with the rôle of spatial dimension and compactify it on a unit circle: \(y + 2\pi \equiv y\). This procedure defines our theory on a cylinder \(\mathbb{R} \times S^1\). Next we can perform the following conformal map

\[
 w \rightarrow z = e^w = e^{x+iy} ,
\]

which "squashes" the cylinder on the \(z\)-complex plane as shown pictorially in Figure 2.1. It is easy to see that the "time" direction \(x\) is mapped in the radial one \(\rho = \sqrt{z\bar{z}},\) while the "space" direction \(y\) is sent in the angular one \(\phi = \frac{1}{2i} \log (\frac{z}{\bar{z}})\). The infinite past and infinite future \(x = \pm\infty\) are sent to the points 0 and \(\infty\), respectively, of the \(z\)-plane. The usual procedure of quantisation in this setup is called radial quantisation.

The subalgebra of \(\text{Vir} \times \overline{\text{Vir}}\) generated by \(\{L_i, \bar{L}_i\}_{i=-1}^1\) is associated with the global conformal group \(SL(2, \mathbb{C})/\mathbb{Z}_2\) and is anomaly free. It is useful for characterising physical states. In fact suppose that we are working, as we will, with eigenstates of the operators \(L_0\) and \(\bar{L}_0\) and denote the eigenvalues, called conformal weights, as \(h\) and \(\bar{h}\).

Consider the following two particular operators of \(SL(2, \mathbb{C})/\mathbb{Z}_2\):

- \(L_0 + \bar{L}_0\): on the cylinder it generates the translations along the time direction and gets mapped, on the plane, to the generator of dilatations \((z, \bar{z}) \rightarrow \gamma(z, \bar{z})\). In radial quantisation, it corresponds to the Hamiltonian of the system;

- \(i(L_0 - \bar{L}_0)\): on the cylinder it generates the translations along the space direction and gets mapped, on the plane, to the generator of rotations \((z, \bar{z}) \rightarrow (e^{i\alpha}z, e^{-i\alpha}\bar{z})\). In radial quantisation, it corresponds to the momentum of the system.
The eigenvalues of these two operators are the \textit{scaling dimension} \( \Delta \doteq h + \tilde{h} \) and the \textit{conformal spin} \( s \doteq h - \tilde{h} \). In the context of radial quantisation they correspond to the energy and the momentum of the state.

**The energy-momentum tensor** The energy-momentum tensor \( T^{\mu \nu} \) is defined as the conserved current associated to the invariance of the system with respect to coordinate transformations \( \epsilon_\mu(x) \):

\[
\delta_c \mathcal{A} = \frac{1}{2} \int d^2x T^{\mu \nu}(x) \left( \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) \right),
\]

where \( \mathcal{A} \) is the action of the system. The energy-momentum tensor is symmetric in its indices \( T^{\mu \nu} = T^{\nu \mu} \) and for a CFT it is traceless \( T^\mu_\mu = 0 \) (this is actually valid in any dimension \( D \)). If \( D = 2 \), then, we only have one component for each chirality:

\[
T_{\text{plane}}(z) = -2\pi T^{zz}, \quad \overline{T}_{\text{plane}}(\overline{z}) = -2\pi T^{\overline{z}\overline{z}}.
\]

These two components are the generating functions of the Virasoro generators

\[
T_{\text{plane}}(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad \overline{T}_{\text{plane}}(\overline{z}) = \sum_{n \in \mathbb{Z}} \overline{z}^{-n-2} \overline{L}_n,
\]

which, in turn, can be expressed in terms of \( T \) by means of Cauchy theorem:

\[
L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T_{\text{plane}}(z), \quad \overline{L}_n = \frac{1}{2\pi i} \oint d\overline{z} \overline{z}^{n+1} \overline{T}_{\text{plane}}(\overline{z}).
\]

These formulae allow to build the energy-momentum tensor also for those CFT who do not possess an action (or if that action is not known). In the following we will be considering CFTs defined on the cylinder and is useful to have an expression of the energy-momentum tensor in this geometry:

\[
T(w) = -\frac{c}{24} + \sum_{n \in \mathbb{Z}} e^{in\theta} L_n, \quad \overline{T}(\overline{w}) = -\frac{c}{24} + \sum_{n \in \mathbb{Z}} e^{-in\theta} \overline{L}_n.
\]

The corresponding expressions of the Virasoro generators take the form

\[
L_n = \frac{1}{2\pi} \int_0^{2\pi} dw e^{in\theta} T(w) , \quad \overline{L}_n = \frac{1}{2\pi} \int_0^{2\pi} d\overline{w} e^{-in\theta} \overline{T}(\overline{w}).
\]

It is useful to remark here that on the cylinder \( T(w + 2\pi) = T(w) \) and \( \overline{T}(\overline{w} + 2\pi) = \overline{T}(\overline{w}) \).

**Primary fields** Let us consider a transformation of coordinates \((z, \overline{z}) \to (\omega(z), \overline{\omega}(\overline{z}))\); any field in our CFT which transforms as follows

\[
\varphi_{h, \overline{h}}(z, \overline{z}) \to \varphi'_{h, \overline{h}}(\omega(z), \overline{\omega}(\overline{z})) = \left( \frac{d\omega}{dz} \right)^{-h} \left( \frac{d\overline{\omega}}{d\overline{z}} \right)^{-\overline{h}} \varphi_{h, \overline{h}}(z, \overline{z}),
\]

is named \textit{primary} field, while the quantities \( h \) and \( \overline{h} \) are the holomorphic and anti-holomorphic conformal weights introduced above. All the fields which are not primary will be called \textit{descendants}. The energy-momentum tensor is an example of a particular descendant field, called \textit{quasi-primary} as it transforms as a primary only under global conformal transformations:

\[
T(z) \to T'(\omega(z)) \doteq \left( \frac{d\omega}{dz} \right)^{-h} T(z) + \frac{c}{12} \{ z; \omega(z) \},
\]

where \( \{ z; \omega(z) \} \) is the Schwartzian derivative of \( \omega(z) \), which vanishes iff \( \omega \in SL(2, \mathbb{C})/\mathbb{Z}_2 \). A consequence of the form of primary fields and energy-momentum tensor is the particular operator product expansion (OPE) that they satisfy:

\[
T(z) \varphi_{h, \overline{h}}(z', \overline{z'}) \sim \frac{h}{(z-z')^2} \varphi_{h, \overline{h}}(z', \overline{z'}) + \frac{1}{z-z'} \partial_{z'} \varphi_{h, \overline{h}}(z', \overline{z'}) ,
\]

\[
T(z) T(z') \sim \frac{c/2}{(z-z')^4} + \frac{2}{(z-z')^2} T(z') + \frac{1}{z-z'} \partial_{z'} T(z') .
\]
The Hilbert space \( \mathcal{H}_{ph} \) of our CFT is built on some vacuum state \( |0\rangle \), which must be invariant under \( SL(2, \mathbb{C})/\mathbb{Z}_2 \) and satisfy

\[
L_n |0\rangle = 0, \quad T_n |0\rangle = 0, \quad \forall n \geq -1,
\]

which is a consequence of the request that \( T(z) |0\rangle \) and \( \overline{T}(\overline{z}) |0\rangle \) be well-defined as \( (z, \overline{z}) \to (0, 0) \). The action of a primary field on this vacuum generates eigenstates of the Hamiltonian:

\[
|h, \overline{h}\rangle \doteq \varphi_{h, \overline{h}}(0, 0) |0\rangle.
\]

The fact that these are eigenstates is easily obtained from the OPE properties of the primary fields:

\[
[L_0, \varphi_{h, \overline{h}}(0, 0)] = \frac{1}{2\pi i} \oint_0 dz T(z) \varphi_{h, \overline{h}}(0, 0) = h \varphi_{h, \overline{h}}(0, 0).
\]

Similar properties are valid for other operators \( L_n \) and one finds

\[
L_0 \left| h, \overline{h}\right\rangle = h | h, \overline{h}\rangle, \quad L_n | h, \overline{h}\rangle = 0, \quad \forall n > 0.
\]

So these states, which we call primary like the fields generating them, are highest-weight vectors for \( \text{Vir} \times \overline{\text{Vir}} \). We can thus generate a subset \( \mathcal{V}_h \otimes \mathcal{V}_{\overline{h}} \) of the Hilbert space by the free action of \( \{ L_n, T_n \}_{n=-\infty}^{1} \) on the primary state \( | h, \overline{h}\rangle \):

\[
\mathcal{V}_a \equiv \mathcal{V}_{h_a} \doteq \left\{ L_{-k_1} L_{-k_2} \cdots L_{-k_n} | h, \overline{h}\rangle \mid 1 \leq k_1 \leq \cdots \leq k_n, \forall n \geq 0 \right\}.
\]

The vector field \( \mathcal{V}_a \) is closed under the action of \( \text{Vir} \) and it’s called Verma module.

The Hilbert space of a CFT is embedded in some suitable way (we will not cover this topic here) into \( \mathcal{H}_{ch} \otimes \overline{\mathcal{H}}_{ch} \), where \( \mathcal{H}_{ch} \doteq \bigoplus_a \mathcal{V}_a \) is the space of “right-chiral” states. The index \( a \) of the direct sum runs on the admissible conformal dimension of our CFT; this number depends on \( c \) and is usually infinite. There are however some notable exceptions where the structure of the Verma modules becomes degenerate and the number of allowed highest weights becomes finite. Let us list three important categories of CFTs:

- **Unitary non degenerate**: when \( c \geq 1 \), the conformal dimension may take a continuum of positive values, for any of these, \( \mathcal{V}_h \) is unitary. An example in this class is the free boson, corresponding to \( c = 1 \);

- **Minimal models**: when

\[
c = 1 - \frac{(m - m')^2}{mm'}, \quad m, m' \in \mathbb{N} \setminus \{0\},
\]

the allowed conformal dimensions are restricted to take a finite number of discrete values, indexed by two integers \( (r, s) \):

\[
h_{r,s} = \frac{(mr - m's)^2 - (m - m')^2}{4mm'}, \quad 1 \leq r < m',
\]

\[
1 \leq s < m.
\]

These models are denoted as \( \mathcal{M}_{m,m'} \). An example in this class is the Yang-Lee model, corresponding to the choice \( m = 5 \) and \( m' = 2 \), meaning \( c = -\frac{22}{5} \).

- **Unitary minimal models**: amongst the above models, the unitary ones are those with \( m' = m + 1 \) and they usually are denoted as \( \mathcal{M}_m \). An example in this class is the Ising model, corresponding to \( m = 3 \) and \( c = \frac{1}{2} \).

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\footnote{This request contain in itself that of invariance with respect to \( SL(2, \mathbb{C})/\mathbb{Z}_2 \).}
Integrals of motion  The first step towards the unveiling of the integrable structures of CFT was performed by R. Sasaki and I. Yamanaka [12]. They considered a CFT on a cylinder and the algebra \( \mathcal{U}(\text{Vir}) \) generated by the energy-momentum tensor \( T(w) \) along with composite fields built as (normal ordered) powers of \( T(w) \) and its derivatives. What they found is that there exists an infinite dimensional abelian subalgebra \( \mathcal{I} \subset \mathcal{U}(\text{Vir}) \) spanned by "local integrals of motion" (IM):

\[
\mathcal{I} = \{ I_{2k-1} \}_{k=1}^{\infty}, \quad I_{2k-1} \equiv \int_0^{2\pi} \frac{dw}{2\pi} T_{2k}(w), \quad [I_{2k-1}, I_{2l-1}] = 0,
\]

where the cylinder radius is fixed at \( R = 1 \). The densities are some regularised polynomials of \( T(w) \) and its derivatives; for example

\[
T_2(w) \equiv T(w), \quad T_4(w) \equiv :T^2(w):, \quad T_6(w) \equiv :T^3(w): + \frac{c+2}{12} :\partial T(w)^2:,
\]

where the regularised product is defined as

\[
:T^2(w):\equiv \oint \frac{dw'}{2\pi i} \mathcal{T}(T(w')T(w)) /
\]

and \( \mathcal{T} \) is the "chronological product"\(^5\)

\[
\mathcal{T}(A(w')B(w)) = \begin{cases} A(w')B(w) & \text{if } \Im(w) > \Im(w') \\ B(w)A(w') & \text{if } \Im(w') > \Im(w) \end{cases}.
\]

Although there is no known closed formula for the densities \( T_{2k}(w) \), they are uniquely determined by the requirement of commutativity of local IMs and by the spin assignment rule:

\[
\oint \frac{dw'}{2\pi i} (w'-w) \mathcal{T}(T(w')T_{2k}(w)) = 2k T_{2k}(w),
\]

which can be simply implemented by requiring \( T_{2k}(w) \) to be a polynomial of total grade \( 2k \) and assigning grade 2 to \( T \) and grade 1 to derivatives. The first few IMs are "easily" computed

\[
I_1 = L_0 - \frac{c}{24}, \quad \text{This is the (chiral part of the) Hamiltonian!},
\]

\[
I_3 = 2 \sum_{n=1}^{\infty} L_{-n} L_n + L_0^2 - \frac{c+2}{12} L_0 + \frac{5c+22}{4 \times 6!},
\]

\[
I_5 = \sum_{n_1+n_2+n_3=0} :L_{n_1} L_{n_2} L_{n_3}:+ \sum_{n=1}^{\infty} \left[ \frac{c+11}{6} n^2 - 1 - \frac{c}{4} \right] L_{-n} L_n + \frac{3}{2} \sum_{n=1}^{\infty} L_{-2n} L_{2n-1} +
\]

\[
- \frac{c+4}{8} L_0^2 + 5(c+2) \frac{3c+20}{4 \times 6!} L_0 - 5c \frac{3c+14}{4 \times 9!} (7c+68)
\]

\[\text{(2.4)}\]

2.2 Brief overview of classical KdV

In this section we will briefly present some very basic concepts and facts about the classical KdV hierarchy. The reader interested in this topic can find a good starting point for the study of classical integrability in the review [13]; for a more advanced read we suggest the beautiful book [14].

Why being concerned with the classical KdV? The reason is very simple. It is known [12] that the CFTs with \( c < 1 \) are, in some sense, a quantum version of the classical KdV; in fact if we consider the "classical limit" \( c \to -\infty \) and perform the following substitutions:

\[
T(w) \to -\frac{c}{6} T(w), \quad [\cdot, \cdot] \to \frac{6\pi}{1+c} \{\cdot, \cdot\}_P,
\]

\[\text{This is the corresponding on the cylinder of the radial ordering on the } z\text{-plane.}\]
where \{ , \}_P are the Poisson brackets, the Virasoro algebra reduces to the following Poisson algebra

\[ \{ U(w), U(w') \}_P = 2( U(w) + U(w')) \delta'(w - w') + \delta'''(w - w'), \]

which is known to describe the second Hamiltonian structure of KdV, provided the Hamiltonian is chosen amongst the classical IM:

\[
\begin{align*}
I_1^{cl} &= \int_0^{2\pi} \frac{dw}{2\pi} U(w) \\
I_3^{cl} &= \int_0^{2\pi} \frac{dw}{2\pi} U^3(w) \\
I_5^{cl} &= \int_0^{2\pi} \frac{dw}{2\pi} \left[ U^3(w) - \frac{1}{2} (\partial_w U(w))^2 \right]
\end{align*}
\]

We choose the field \( U(w) \) to be periodic \( U(w + 2\pi) = U(w) \), just like \( T(w) \). These classical IMs, which form a commutative Poisson algebra \{I_{2k-1}, I_{2l-1}\}_P = 0, are clearly the classical versions of the operators (2.2-2.4).

Different choices of Hamiltonian bring us to different equation of motion:

\[
\begin{align*}
I_1^{cl} &: \partial_1 U = \partial_w U \\
I_3^{cl} &: \partial_3 U = \partial^3_w U + 6U \partial_w U \quad \text{The 'canonical' KdV} \\
I_5^{cl} &: \partial_5 U = -\partial^5_w U - 2U \partial^3_w U + 5\partial_w U \partial^2_w U + 20U^2 \partial_w U \\
\end{align*}
\]

This infinite sequence of partial differential equations is called KdV hierarchy and it can be shown to be equivalent to a description of the isospectral deformations of the following second order differential operator depending on a spectral parameter \( \lambda \):

\[ L(w|\lambda) = \partial_w^2 + U(w) - \lambda^2 , \]

called Lax operator. The connection between this operator and the tower of differential equations (2.5) relies on the existence of an infinite set of operators \( M_{2n-1}(w) \), such that

\[ \frac{d}{dt_{2n-1}} L(w|\lambda) = [M_{2n-1}(w), L(w|\lambda)] \iff \text{KdV equation associated to } I_{2n-1}^{cl} \text{ is satisfied.} \]

For example the canonical KdV equation is obtained from the operator

\[ M_3(w) = 4\partial^3_w + 6U(w) \partial_w + 3U'(w) . \]

Associated to each Lax operator, there exists a differential equation, called usually auxiliary equation (or system if one has to deal with matrix Lax operators). In our case the equation has the form (from here on the prime ' will denote differentiation)

\[ L(w|\lambda) \psi(w|\lambda) = \psi'''(w) - (\lambda^2 - U(w)) \psi(w) = 0 . \]

This is a second order differential equation and, as such, possesses two linearly independent solutions \( \psi_1(w|\lambda) \) and \( \psi_2(w|\lambda) \). Very important characteristics of differential equations are the monodromy properties of the solutions; these can be encoded in the monodromy matrix, defined as

\[ (\psi_1(w|\lambda), \psi_2(w|\lambda)) M(\lambda) = (\psi_1(w + 2\pi|\lambda), \psi_2(w + 2\pi|\lambda)) . \]

Out of the monodromy matrix is then possible to define the \( T \)-function. This is a central object in integrable systems and is defined most simply as the trace of \( M \)

\[ T(\lambda) = tr M(\lambda) . \]
Although an explicit expression of $T$ can be complicated to obtain, we can express it as an asymptotic series at large $\lambda$:

$$
\frac{1}{2\pi} \log [T(\lambda)] \sim \lambda \left[ 1 - \sum_{n=1}^{\infty} c_n I_{2n-1} \lambda^{-2n} \right],
$$

where $c_1 = 1/2$ and $c_n = \frac{(2n-3)!!}{2^n n!}$.

### WKB expansion of the Lax auxiliary equation

It is instructive to compute explicitly the expression of the monodromy matrix. In order to do so we need to find a representation of the solutions to the differential equation

$$
\psi''(w) = (\lambda^2 - U(w)) \psi(w),
$$

and a standard procedure which allows us to do so is the WKB method [15]. The first step consists in introducing a small parameter $\epsilon^2$ in front of the second derivative:

$$
\epsilon^2 \psi''(w) = (\lambda^2 - U(w)) \psi(w),
$$

and search for solutions of the form

$$
\psi(w) \sim \exp \left[ \frac{1}{\epsilon} S(w) + A_0(w) + \sum_{n=1}^{\infty} \epsilon^n A_n(w) \right], \text{ as } \epsilon \to 0,
$$

where the sign $\sim$ reminds us that the right-hand side is an asymptotic series. By inserting this form in the differential equation and isolating each power of $\epsilon$, we find

$$
\epsilon^0 : \quad S'(w)^2 = \lambda^2 - U(w) \quad \Rightarrow \quad S(w) = \pm \int_{w_0}^{w} \sqrt{\lambda^2 - U(w')} dw',
$$

$$
\epsilon^1 : \quad S''(w) + 2S'(w)A_0(w) = 0 \quad \Rightarrow \quad A_0(w) = k - \frac{1}{4} \log (\lambda^2 - U(w)),
$$

$$
\epsilon^n : \quad 2S'(w)A'_{n-1}(w) + A''_{n-2}(w) + \sum_{k=0}^{n-2} A_k(w)A'_{n-k}(w) = 0, \quad \forall n > 1.
$$

This is a triangular system of differential equations, which allows us to obtain the $n$-th term by the simple integration of a first-order differential equation. What’s more, each even order equation happens to be the difference of total derivatives; for example

$$
\epsilon^3 : \quad A_2(w) = \partial_w \left[ \frac{(\partial_w S'(w))^2 - 2\partial^2 w \log S'(w)}{16 S'(w)^2} \right].
$$

The odd-order equations, on the other hand, are proper first-order differential equations, as an example, the first term reads

$$
\epsilon^2 : \quad A_1(w) = \int_{w_0}^{w} \frac{2S''(w')S'(w') - 3S''(w')^2}{8S'(w')^3} dw'.
$$

Now, in order to obtain an expression for $M$, we need to see what happens to our solution when we shift $w \to w + 2\pi$. This is easily computed remembering that $U(w + 2\pi) = U(w)$, so that

$$
A_{2n}(w + 2\pi) = A_{2n}(w), \quad \forall n \geq 0,
$$
while
\[ A_{2n-1}(w + 2\pi) = \int_{w_0}^{w_0 + 2\pi} A_{2n-1} [U(w')] dw' = \int_{w_0}^{w_0 - 2\pi} A_{2n-1} [U(w')] dw' + \int_{w_0}^{w} A_{2n-1} [U(w')] dw' , \]

so that
\[ A_{2n-1}(w + 2\pi) = A_{2n-1}(w) + \int_{0}^{2\pi} A_{2n-1} [U(w')] dw' , \]

where \( A_{2n-1} [U(t)] \) is some functional of \( U(t) \) and \( A_{-1}(w) \equiv S(w) \). From this we easily infer that the monodromy matrix is diagonal with eigenvalues
\[ \exp \left[ \pm \int_{0}^{2\pi} \left( \sqrt{\lambda^2 - U(w')} + \sum_{n=1}^{\infty} A_{2n-1} [U(w')] \right) dw' \right] . \]

The first two terms in the large-\( \lambda \) expansion of \( T \) (2.8) come from the expansion of the square root for large \( \lambda \),
\[ \int_{0}^{2\pi} \frac{dw'}{2\pi} \sqrt{\lambda^2 - U(w')} \sim \int_{0}^{2\pi} \frac{dw'}{2\pi} \lambda \left( 1 - \frac{U(w')}{2\lambda^2} - \frac{U(w')^2}{8\lambda^4} \ldots \right) = \lambda \left( 1 - \frac{1}{2\lambda^2} I_1^{cl} - \frac{1}{8\lambda^4} I_3^{cl} \right) . \]

On the other hand, higher order terms require the computation of more and more \( A_n(w) \) in the WKB expansion; as a simple example, let us take in consideration \( A_1 \)
\[ A_1(w) = - \frac{4U''(w)(\lambda^2 - U(w)) - 5U'(w)^2}{32(\lambda^2 - U(w))^\frac{3}{2}} \sim \frac{U''(w)}{8\lambda^3} - \frac{5U'(w)^2 + 6U(w)U''(w)}{32\lambda^5} . \]

When integrating the above term between 0 and 2\( \pi \), all total derivatives vanish and we can perform integration by parts, so that the coefficient of \( \lambda^{-5} \) reads, as expected
\[ - \frac{1}{16} \int_{0}^{2\pi} \frac{dw'}{2\pi} \left( U'(w')^2 - \frac{1}{2} U''(w')^2 \right) = - \frac{1}{16} I_3^{cl} . \]

We have thus shown that the \( T \)-function (2.7) serves as a sort of generating function for the classical IMs (2.8). However we can do more, much more: in fact we can construct an infinite tower of Poisson commuting \( T \)-functions! This is a consequence of a deep connection between KdV hierarchy and the Lie algebra \( sl(2) \) as we are going to briefly hint at. A close look at the differential operator (2.6) shows that it can be factorised:
\[ L(w|\lambda) = (\partial_w + \phi'(w))(\partial_w - \phi'(w)) - \lambda^2 , \]

with the field \( \phi(w) \) being the Miura transform of \( U(w) \) [16]:
\[ U(w) = (\phi'(w))^2 + \phi''(w) , \]

having canonical Poisson brackets
\[ \{ \phi(w), \phi(w') \}_P = \epsilon(w - w') , \quad \epsilon(x) = n , \quad 2\pi n < x \leq 2\pi(n + 1) . \]

Note that since the field \( U(w) \) is periodic, the Miura field has to be taken, in full generality, quasiperiodic
\[ \phi(w + 2\pi) = \phi(w) + 2\pi i p . \]
We can now reduce the second order differential equation $L(w|\lambda)\psi(w|\lambda) = 0$ to a system of first order equations:

$$
\begin{align*}
(\partial_w - \phi'(w)) \psi(w|\lambda) &= \lambda \tilde{\psi}(w|\lambda) \\
(\partial_w + \phi'(w)) \tilde{\psi}(w|\lambda) &= \lambda \psi(w|\lambda)
\end{align*}
$$

which can be written in matrix form

$$(\partial_w - \phi'(w)\sigma^3 - \lambda \sigma^1) \Psi(w|\lambda) = 0,$$

with $\sigma^i$ being the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now comes the generalisation: since the connection between $L$ and the hierarchy of KdV equations (2.5) uniquely relies on the commutation relations between $L$ and the operators $M_{2n-1}$, we can think of defining an abstract Lax operator

$$\mathcal{L}^j(w|\lambda) \doteq \partial_w - \phi^j(w) H - \lambda (E + F),$$

where $H, E$ and $F$ are the generators of $sl(2)$ Lie algebra:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = 2H.$$

The commutation properties of this operator are exactly the same as those of $L$ and, moreover, it reduces to this last when the 2-dimensional representation of $sl(2)$ is chosen. We thus expect the monodromy properties of this operator to encode information on the KdV hierarchy. Now, let $\pi_j$, with $j \in \frac{1}{2}\mathbb{N}$, denote the $(2j + 1)$-dimensional representation of $sl(2)$, such that $\pi_j[H] = \text{diag}(2j, 2j - 2, \ldots, -2j + 2, -2j)$. Consider the matrix equation

$$\pi_j[\mathcal{L}^j(w|\lambda)] \Psi_j(w|\lambda) = 0,$$

where $\Psi_j(w|\lambda)$ is a $(2j + 1)$-dimensional vector, and let us repeat what has been done just above. In order to obtain a nice form of the solution to this equation, we rewrite it as follows (we omit $\pi_j$ in the next few equations, for clarity):

$$(\partial_w - \phi'(w) H) \Psi(w) = e^{\phi(w)H} \partial_w e^{-\phi(w)H} \Psi(w) = \lambda (E + F) \Psi(w),$$

where the first passage is allowed, since $H$ is diagonal. Now define $\tilde{\Psi}(w) = e^{-\phi(w)H} \Psi(w)$, so that it satisfies the equation

$$\partial_w \tilde{\Psi}(w) = \lambda e^{-\phi(w)H} (E + F) e^{\phi(w)H} \tilde{\Psi}(w) = \lambda \left( e^{-2\phi(w)E} + e^{2\phi(w)F} \right) \tilde{\Psi}(w),$$

where we used the property of any Lie algebra element $A$: $e^{2\alpha H} A e^{-\alpha H} = e^{\alpha \text{ad}_H(A)} A$, where the adjoint action is defined by $[H, A] = \text{ad}_H(A) A$. The general solution of a first-order matrix equation can be written as a path-ordered exponential:

$$\tilde{\Psi}(w) = \mathcal{P} \exp \left[ \lambda \int_0^w dw' \left( e^{-2\phi(w')} E + e^{2\phi(w')} F \right) \right] \Psi^0,$$

with $\Psi^0$ being an arbitrary constant vector, representing the integration constants. A path-ordered exponential $\mathcal{P} \exp \left[ \int_0^w a(w') dw' \right]$ is defined as the following series expansion

$$\mathcal{P} \exp \left[ \int_0^w a(w') dw' \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^w \cdots \int_0^w \mathcal{P} [a(w'_1) \cdots a(w'_n)] dw'_1 \cdots dw'_n,$$

and the path-ordering $\mathcal{P}$ forces an ordering of decreasing argument from left to right:

$$\mathcal{P} [a(w_1)a(w_2)] = \begin{cases} a(w_1)a(w_2) & w_1 > w_2 \\
a(w_2)a(w_1) & w_1 < w_2 \end{cases}.$$
Re-expressing $\tilde{\Psi}$ in terms of $\Psi$, we obtain

$$\Psi_j(w|\lambda) = \pi_j \left\{ e^{\phi(w)H} \Phi \exp \left[ \lambda \int_0^w dw' \left( e^{-2\phi(w')} E + e^{2\phi(w')} F \right) \right] \right\} \Psi_j^0,$$

Now, for each representation $\pi_j$ we can define a monodromy matrix:

$$\mathbf{M}_j(\lambda) = \pi_j \left\{ e^{2\pi ipH} \Phi \exp \left[ \lambda \int_0^{2\pi} dw \left( e^{-2\phi(w)} E + e^{2\phi(w)} F \right) \right] \right\}, \quad (2.10)$$

and a corresponding $L$-matrix:

$$\mathbf{L}_j(\lambda) = \pi_j \left[ e^{-\pi ipH} \right] \mathbf{M}_j(\lambda). \quad (2.11)$$

This last matrix can be shown to satisfy the $r$-matrix Poisson relation:

$$\{ \mathbf{L}_j(\lambda) \otimes \mathbf{L}_{j'}(\lambda') \}_P = [r_{j,j'}(\lambda/\lambda'), \mathbf{L}_j(\lambda) \otimes \mathbf{L}_{j'}(\lambda')], \quad (2.12)$$

where the $r$-matrix is defined as

$$r_{j,j'}(\lambda) \equiv (\pi_j \otimes \pi_{j'}) [\mathbf{r}(\lambda)] , \quad \mathbf{r}(\lambda) \equiv \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}} \frac{H \otimes H}{2} + \frac{2}{\lambda - \lambda^{-1}} (E \otimes F + F \otimes E).$$

The $r$-matrix Poisson algebra tells us immediately that the quantities

$$\mathbf{T}_j(\lambda) = tr \mathbf{M}_j(\lambda),$$

are in involution with respect with the Poisson brackets

$$\{ \mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda') \}_P = 0,$$

and are expected to generate the classical IMs in their asymptotic limit. Note that $\mathbf{T}_{\frac{j}{2}}(\lambda) = \mathbf{T}(\lambda)$.

We will not be showing the explicit proof of the relation (2.12), however we wish to close this section suggesting an approach to the computation which we think gives an intuitive interpretation of the form of $\mathbf{M}_j$. The core of this approach resides in the following expansion of the path-ordered exponential as a *continuum limit* of an ordered product:

$$\Phi \exp \left[ \int_0^{2\pi} a(w') dw' \right] = \lim_{N \to \infty} e^{a(w_N)\Delta w} e^{a(w_{N-1})\Delta w} \cdots e^{a(w_0)\Delta w},$$

where $w_j = j\Delta w$ and $\Delta w = \frac{2\pi}{N}$. This expression allows us to write $\mathbf{L}_j$ as (here too we omit $\pi_j$ for clarity)

$$\mathbf{L}(\lambda) = e^{\pi ipH} \lim_{N \to \infty} \prod_{j=0}^N I(w_N|\lambda)I(w_{N-1}|\lambda) \cdots I(w_0|\lambda),$$

where the matrices $\mathbf{I}_j$ are

$$I(w|\lambda) = \sum_{n=0}^\infty \frac{\lambda^n \Delta w^n}{n!} \left( e^{-2\phi(w)} E + e^{2\phi(w)} F \right)^n.$$

It is now sufficient to show that $\mathbf{I}$ satisfies the relation (2.12), for any value of $w$ and $\lambda$. This is reminiscent of the approach to lattice models, in which we have a matrix $\mathbf{I}$ on each site and the full transfer matrix of the system is built as a trace of the product of these matrices for each site. In fact we can interpret the matrix $\mathbf{L}(\lambda)$ as the *continuum limit* of the monodromy matrix of a lattice model, where $e^{\pi ipH}$ plays the role of twist.

---

6 In the literature this object is sometimes called Lax matrix.

7 The operator denoted here with $\mathbf{T}$ are not to be confused with the densities of IM (2.1) introduced above.
2.3 The quantum monodromy matrix and the $T$-operators

Let us now concentrate on our goal: we are going to reproduce in the $c < -2$ CFTs what has been sketched above for the classical KdV hierarchy. Namely we are going to address the problem of simultaneous diagonalisation of the local IMs (2.2-2.4) with a method that can be interpreted as a version of the Quantum Inverse Scattering (QIS) [17] for field theories. Just as it was suggested in the previous section, all the object we are going to introduce have a counterpart in lattice models and it is a good idea to keep in mind this parallelism. On the other hand these objects will be the quantised version of those introduced above for the classical KdV hierarchy and we are going to use the same symbols to denote them. Note that, from now on, we will consider the right chirality only.

In order to proceed to the construction of the objects $T_j$, we first need the quantum version of Miura transformation: the Feigin-Fuchs free field representation [18]

\[
- \beta^2 T(w) =: (\varphi'(w))^2 : + (1 - \beta^2) \varphi''(w) + \frac{\beta^2}{24}, \quad \beta \doteq \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}}, \quad (2.13)
\]

where $\varphi(w)$ is a free field

\[
\varphi(w) = iQ + iPw + \sum_{n \neq 0} \frac{a_n}{n} e^{inw},
\]

and the normal ordering $:\cdot:\$ consist in placing the $a_n$ oscillators in increasing $n$ from left to right. The operators $Q, P$ and $\{a_n\}_{n \neq 0}$ generate an Heisenberg algebra:

\[
[Q, P] = i\beta^2, \quad [a_n, a_m] = \frac{n}{2} \beta^2 \delta_{n+m,0}.
\]

It is easy to see that this transformation becomes exactly (2.9) as $c \to -\infty$. With this expression for $T(w)$ we are able to give a description of the Hilbert space $\mathcal{H}_{ch}$ in terms of Fock spaces $\mathcal{F}_p$, defined as highest-weight modules over the Heisenberg algebra; the highest-weight vector $|p\rangle \in \mathcal{F}_p$ obeys to the following relations

\[
P |p\rangle = p |p\rangle, \quad a_n |p\rangle = 0, \quad \forall n > 0.
\]

The Fock space thus defined is isomorphic to the Verma module $\mathcal{V}_h$, where

\[
h = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24},
\]

and we can describe the Hilbert space as

\[
\mathcal{H}_{ch} = \bigoplus_a \mathcal{F}_a, \quad \mathcal{F}_a \equiv \mathcal{F}_{pa},
\]

where the direct sum runs over the values of $p$ corresponding to the allowed Virasoro highest-weights. These Fock spaces are naturally graded under the action of $L_0$:

\[
\mathcal{F}_p = \bigoplus_{\ell=0}^{\infty} \mathcal{F}_p^{(\ell)}, \quad L_0 \mathcal{F}_p^{(\ell)} = (h+\ell) \mathcal{F}_p^{(\ell)}.
\]

With some simple algebraic manipulation we can express the Virasoro generators $\{L_n\}$ in terms of the Heisenberg algebra as

\[
\beta^2 L_n = \beta^2 \frac{c-1}{24} + 2 \sum_{j \neq 0, n} a_j a_{n-j} + a_n \left(2P - n(1 - \beta^2)\right),
\]

\[
\beta^2 L_0 = \beta^2 \frac{c-1}{24} + 2 \sum_{j=1}^{\infty} a_{-j} a_j + P^2.
\]

The restriction to this domain will be clearer later.
Since in theory we know how to express the local IMs \( \{ I_{2k-1} \} \) in terms of the local densities \( T_{2k}(w) \), the formers can be re-expressed in terms of polynomials in the free field \( \varphi(w) \) and its derivatives:

\[
I_{2k-1} = (-1)^k \beta^{-2k} \int_0^{2\pi} \frac{dw}{2\pi} \left[ (\varphi'(w))^{2k} + \cdots \right].
\]

As it is evident from their definition (remember the spin assignment request), each term in a local IM, as complicated as it might be, is nevertheless a product of operators \( I_{n_i} \), where the sum of indices vanishes: \( \sum_n n_i = 0 \). As a consequence \( [L_0, I_{2k-1}] = 0 \) and the local IMs act invariantly on the level subspaces \( \mathcal{F}_p^{(\ell)} \). The full diagonalisation of the integrals of motion is thus reduced to their diagonalisation on each level subspace, which requires a finite number of algebraic manipulations; these, however, become rapidly extremely involved and so far the result is known only for some simple cases, e.g. for the vacuum \( |0\rangle \):

\[
I_1^{(\text{vac})}(h, c) = h - \frac{c}{24}, \\
I_3^{(\text{vac})}(h, c) = h^2 - \frac{c + 2}{12} h + \frac{5c + 22}{4 \times 6!}, \\
I_5^{(\text{vac})}(h, c) = h^3 - \frac{c + 4}{8} h^2 + \frac{5(c + 2)}{4 \times 6!} h - \frac{5c + 20}{4 \times 6!} h + \cdots 
\]

where \( I_{2k-1} \big| p \rangle = I_{2k-1}^{(\text{vac})} \big| p \rangle \).

In the setting provided by the Fock description of the Hilbert space, we can easily follow the footprints of section 2.2 and define quantum counterparts of the monodromy matrices (2.10) and of the \( L \)-matrices (2.11). In order to do so, we consider the quantum enveloping algebra \( \mathcal{U}_q(\mathfrak{sl}(2)) \) [19, 20] generated by the elements \( E, F \) and \( H \):

\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}},
\]

with \( q = e^{i\pi \beta^2} \), and let \( \pi_j \) denote the \( (2j + 1) \)-dimensional representation of this algebra. The "quantum monodromy matrices" are then defined as the following operator-valued matrices:

\[
\mathbf{M}_j(\lambda) = \pi_j \left\{ e^{2\pi iPH} \mathcal{P} \exp \left[ \frac{2\pi}{\lambda} \int_0^{2\pi} dw \left( V_-(w)q^{\frac{H}{2}}E + V_+(w)q^{-\frac{H}{2}}F \right) \right] \right\}, \quad (2.14)
\]

where

\[
V_\pm(w) \equiv e^{\pm 2\varphi(w)} := \exp \left[ \pm 2\sum_{n=1}^{\infty} \frac{a_n}{n} e^{inw} \right] e^{\pm 2i(Q + PW)} \exp \left[ \mp 2\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-inw} \right],
\]

are called vertex operators; they have conformal dimension \( \beta^2 \) and act on the Fock spaces by shifting the highest weight:

\[
V_+(w) : \quad \mathcal{F}_p \rightarrow \mathcal{F}_{p + \beta^2}.
\]

The reason why in the path ordered exponent the combinations \( q^{\frac{H}{2}} E \) and \( q^{-\frac{H}{2}} F \) appear is related to the fact that the correct construction of \( \mathbf{M}_j \) should start from the quantum affine enveloping algebra \( \mathcal{U}_q(\mathfrak{sl}(2)) \); we will return to this point in section 2.5.

The \( L \)-operators are defined in the same way as in the classical case:

\[
\mathbf{L}_j(\lambda) = \pi_j \left[ e^{-\pi iPH} \right] \mathbf{M}_j(\lambda). \quad (2.15)
\]

Note that these objects can be informally interpreted as the monodromy matrices of the solution \( \Psi_j \) to the operator-matrix equation \( \pi_j \left[ \mathcal{L}'(w|\lambda) \right] \Psi_j(w|\lambda) = 0 \), where \( \mathcal{L}'(w|\lambda) = \partial - \varphi'(w) H - \lambda \left( q^{\frac{H}{2}} E + q^{-\frac{H}{2}} F \right) \). Here one has to take care of normal ordering when defining the corresponding of \( \hat{\Psi} := e^{-\varphi} : \Psi \). This gives rise to the presence of vertex operators in the path-ordered integral.
Both the quantum monodromy matrices and the $L$-operators are $(2j+1) \times (2j+1)$ matrices whose elements are operators acting on the space

$$\mathcal{F}_p \cong \bigotimes_{n=-\infty}^{\infty} \mathcal{F}_{p+n\beta^2}.$$  

They have to be understood as power series in $\lambda$:

$$L_j(\lambda) = \pi_j \left[ e^{piPH} \sum_{k=0}^{\infty} \lambda^k \int_{0}^{2\pi} dw_1 \cdots dw_k \mathcal{K}(w_1) \cdots \mathcal{K}(w_k) \right],$$

where we introduced

$$\mathcal{K}(w) = V_-(w)q^{\frac{H}{2}} E + V_+(w)q^{-\frac{H}{2}} F.$$  \hspace{1cm} (2.16)

These series converge for any $\lambda$ if $-\infty < c < -2$, outside this region the definition of $M_j$ and $L_j$ necessitate a proper regularisation. In these notes we will limit ourselves to the cases $c < -2$.

The operators $L_j$ are tailored in such a way that the following RLL relation is satisfied

$$R_{j,j'}(\lambda/\lambda') (L_j(\lambda) \otimes I) (I \otimes L_{j'}(\lambda')) = (I \otimes L_{j'}(\lambda')) (L_j(\lambda) \otimes I) R_{j,j'}(\lambda/\lambda'),$$ \hspace{1cm} (2.17)

where $R_{j,j'}(\lambda)$ is the trigonometric $R$-matrix of $U_q(\mathfrak{sl}(2))$, acting on $\pi_j \otimes \pi_{j'}$. The fundamental–fundamental case $j = j' = \frac{1}{2}$ reads as follows\(^11\)

$$R_{\frac{1}{2},\frac{1}{2}}(\lambda) = \begin{pmatrix} \frac{\lambda}{q} - \frac{q}{\lambda} & \lambda - \lambda^{-1} & q^{-1} - q & \lambda^{-1} - q \\ \frac{\lambda}{q} - \frac{q}{\lambda} & q^{-1} - q & \lambda - \lambda^{-1} & \lambda^{-1} - q \\ \frac{\lambda}{q} - \frac{q}{\lambda} & q^{-1} - q & \lambda - \lambda^{-1} & \lambda^{-1} - q \\ \frac{\lambda}{q} - \frac{q}{\lambda} & q^{-1} - q & \lambda - \lambda^{-1} & \lambda^{-1} - q \end{pmatrix}.$$  \hspace{1cm} (2.18)

In order to check the validity of the RLL relation it is possible to adopt a brute force method, that is, discretise the $\mathcal{P}$ exponential and compute the two sides of the relation, or interpret $L_j$ and $R_{j,j'}$ as particular realisations of universal objects of the algebra $U_q\left(\mathfrak{sl}(2)\right)$; this last approach is sketched in section 2.5.

We can finally define the “quantum transfer matrices” as traces of the quantum monodromy matrices (2.14):

$$T_j(\lambda) = \text{tr}_{\pi_j} (M_j(\lambda)).$$ \hspace{1cm} (2.19)

As a direct consequence of the RLL relation, these matrices form a commuting family:

$$[T_j(\lambda), T_{j'}(\lambda')] = 0,$$

moreover they commute with the operator $P$ and, as such, act invariantly\(^12\) on each $\mathcal{F}_p$. Finally through some tedious computation [4] it is possible to show that, with the definition (2.19), the quantum transfer matrices commute with all the local IMs\(^13\):

$$[T_j(\lambda), I_{2k-1}] = 0,$$

which means that the level subspaces $\mathcal{F}_p^{(\ell)}$ are the eigenspaces of $T_j(\lambda)$.

Before explicitly presenting the simple case of $\pi_j = \pi_\frac{1}{2}$, we wish to underline the connection with lattice models. Just as in the classical case, we can express the $L$-operators as a continuum limit of a product:

$$L(\lambda) = e^{\pi iPH} \lim_{N \to \infty} I(w_N|\lambda)I(w_{N-1}|\lambda) \cdots I(w_0|\lambda).$$

\(^10\) This is most easily inferred from the fact that the vertex operators $V_{\pm}$ have conformal dimension $\beta^2$ and, thus, $V_+(w)V_-(w') \sim (w-w')^{-2\beta^2} (1 + O(w-w'))$. So, for the integrals to converge we must impose $\beta^2 < \frac{1}{4}$, which is equivalent to $c < -2$.

\(^11\) Note that this is the same exact matrix as for the 6-vertex model [21].

\(^12\) An informal way to see this is to notice that the operators $\mathcal{K}(w)$ are traceless, moreover the only terms in $T_j$ having non-vanishing trace are those containing an equal number of operators $E$ and $F$. Thus, only products of elements of the type $V_-(w)V_+(w')$ appear and these act on Fock spaces as $\mathcal{F}_p \to \mathcal{F}_{p+n\beta^2} \to \mathcal{F}_p$.

\(^13\) Note that the proof is limited to some low order IM, a full proof of the commutativity is still lacking.
Here the “local” operators \( l \) are expressed as
\[
l(w|\lambda) = \exp[\lambda \mathcal{K}(w) \Delta w] \sim 1 + \lambda \mathcal{K}(w) \Delta w.
\]
The form of \( \mathcal{K}(w) \) is exactly that which we would expect from a lattice model:
\[
\mathcal{K}(w) = \sum_{j=\pm} V_j(w) \omega_j,
\]
where \( \omega_{\pm} \) are generators of the \( U_q(\hat{sl}(2)) \) algebra in matrix realisation (see section 2.5 for more details), and \( V_{\pm} \) are a vertex operator realisation of the same algebra. In fact the operators
\[
V_0(w) = \sqrt{2} \partial_w \varphi(w), \quad V_{\pm}(w) = e^{\pm 2\varphi(w)},
\]
satisfy the \( \hat{sl}(2) \) subalgebra at level 1 [9]. So we can interpret \( l \) as being the tensor product of two operators acting on two different spaces: one, corresponding to the matrices \( \omega_j \), is the auxiliary space; the other, corresponding to the vertex operators \( V_j \), is the quantum space. A pictorial representation is given in Figure 2.2. We wish to stress that this connection is by no means mathematically precise, but rather an intuitive interpretation of the physical meaning of the operators introduced above.

![Graphical representation of operators](image)

**Fig. 2.2:** Graphical representation of operators \( l(w_j) \) and \( L \); the horizontal green line represents the auxiliary space, while the vertical blue ones correspond to the quantum spaces.

**The basic representation** All that has been said until now is rather general and abstract. In order to make things more concrete, let us concentrate on the simplest amongst the quantum transfer matrices, namely \( T(\lambda) \equiv T_{2}(\lambda) \). The 2-dimensional representation of \( U_q(sl(2)) \) can be chosen as
\[
\pi_{\hat{T}}[H] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi_{\hat{T}}[E] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi_{\hat{T}}[F] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
and the operator \( T(\lambda) \) can thus be written as a power series in \( \lambda^2 \) (due to the tracelessness of the matrices \( E \) and \( F \)):
\[
T(\lambda) = 2 \cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} G_{2n}, \quad \text{(2.20)}
\]
where
\[
G_{2n} = q^n \int_{w_1 \geq \cdots \geq w_{2n}} d\omega_1 \cdots d\omega_{2n} \ e^{2\pi i P} V_-(w_1) V_+(w_2) \cdots V_-(w_{2n-1}) V_+(w_{2n}) +
\]
\[
+ e^{-2\pi i P} V_+(w_1) V_-(w_2) \cdots V_+(w_{2n-1}) V_-(w_{2n}), \quad \text{(2.21)}
\]
are operators commuting amongst themselves and with the local IMs

\[ [G_{2n}, G_{2m}] = 0, \quad [G_{2n}, I_{2k-1}] = 0, \]

and, for this reason, are called non-local integrals of motion. Just as their local counterpart, they act invariantly on each level subspace \( \mathcal{H}_p \) and, in particular, the highest weight vector \(| p \rangle\) is one of their common eigenstates:

\[ G_{2n} | p \rangle = G_{2n}^{(vac)} | p \rangle. \]

The vacuum eigenvalues can be calculated rather straightforwardly from the definition (2.21):

\[
G_{2n}^{(vac)} (p) = \int dw_1 dw_2 \cdots dw_n \prod_{j>1} \left[ 4 \sin \left( \frac{w_i - w_j}{2} \right) \sin \left( \frac{\tilde{w}_i - \tilde{w}_j}{2} \right) \right]^{2\beta^2} \times \\
\times \prod_{i,j=1}^n \left[ 2 \sin \left( \frac{w_i - \tilde{w}_j}{2} \right) \right]^{-2\beta^2} 2 \cos \left[ 2p \left( \pi + \sum_{j=1}^n (\tilde{w}_j - w_j) \right) \right],
\]

and when \( n = 1 \), this expression greatly simplifies to

\[
G_2^{(vac)} (p) = \int dw dw \int d\tilde{w} \left[ 2 \cos \left( 2\pi p + 2p(\tilde{w} - w) \right) \right]^{2\beta^2} = \frac{4\pi^2 \Gamma (1 - \beta^2)}{\Gamma (1 - 2p - \beta^2) \Gamma (1 + 2p - \beta^2)}. 
\]

In order to obtain these results one has to use the following property of vertex operators:

\[
\langle p | V_\epsilon (w) V_{\tilde{\epsilon}} (\tilde{w}) | p \rangle = e^{-2ip(\epsilon w + \tilde{\epsilon} \tilde{w})} \left[ 2 \sin \left( \frac{w - \tilde{w}}{2} \right) \right]^{-2\epsilon \tilde{\epsilon} \beta^2}, \quad \epsilon, \tilde{\epsilon} = \pm 1,
\]

and the Wick theorem.

All the operators \( T_j (\lambda) \) are entire functions of \( \lambda^2 \), possessing an essential singularity at infinity due to the accumulation of zeroes on the negative \( \lambda^2 \)-axis. This can be shown by comparison with a result obtained in [22], where a series similar to (2.20) was analysed and shown to converge in the whole complex plane, defining an entire function with an essential singularity at infinity. As it turns out, the coefficients of this series are larger in absolute value than (2.21), meaning that \( T \) is entire as well. Finally the entirety of the operators \( T_j \) directly descends from this result, thanks to the \( T \)-system we are going to present just below (2.25).

We are interested in obtaining an asymptotic series expansion of \( T (\lambda) \) since, recalling equation (2.8), we expect the integrals of motion to appear there as coefficients. In fact, as pointed out just above, it is possible to examine the discretised version of \( M_\lambda (\lambda) \). Then, by means of standard Algebraic Bethe Ansatz (ABA), and, subsequently, taking the continuum limit back to \( M_\lambda (\lambda) \) one obtains the following expression for the quantum transfer matrix:

\[ T (\lambda) = \Lambda (q \lambda) + \Lambda^{-1} (q^{-1} \lambda), \]

where

\[
\log \Lambda (q \lambda) \sim m \lambda^{1+\xi} - \sum_{k=1}^{\infty} C_k I_{2k-1} \lambda^{(1+\xi)(1-2k)}, \quad \xi = \frac{\beta^2}{1 - \beta^2}.
\]

The constants in the asymptotic expansion are

\[
m = 2\sqrt{\pi} \frac{\Gamma \left( \frac{1}{2} - \frac{\xi}{2} \right)}{\Gamma \left( 1 - \frac{\xi}{2} \right)} \left[ \Gamma \left( \frac{1}{1 + \xi} \right) \right]^{1+\xi}, \quad \xi = \frac{\beta^2}{1 - \beta^2}.
\]

(2.22)

\[
C_k = \frac{1 + \xi}{k!} \left( \frac{\pi \xi}{1 + \xi} \right)^k \left( \frac{2 \Gamma \left( \frac{1}{2} - \frac{\xi}{2} \right)}{\Gamma \left( 1 - \frac{\xi}{2} \right)} \right)^{2k-1} \frac{\Gamma \left[ (1 + \xi)(k - \frac{1}{2}) \right]}{\Gamma \left[ 1 + (k - \frac{1}{2}) \xi \right]}.
\]

(2.23)
2.4 T-system, Y-system and Thermodynamic Bethe Ansatz equations

Let us return to the analysis of the quantum monodromy matrices \( T_j(\lambda) \) associated highest dimensional representations of \( \mathcal{U}_q(sl(2)) \). It is easily deduced from their definition that then too are power series in \( \lambda^2 \):

\[
T_j(\lambda) = \frac{\sin(2(2j + 1) \pi P)}{\sin(2 \pi P)} + \sum_{n=1}^{\infty} \lambda^{2n} G_{2n}^{(j)}. \]

The surprising fact about these expansions is that they are deeply interrelated; in fact the non-local IMs \( G_{2n}^{(j)} \) with \( j > \frac{1}{2} \) can all be written as polynomials in \( G_{2n}^{(j)} \equiv G_{2n} \), e.g.

\[
\begin{align*}
G_2^{(j)} &= A_j(2\pi P, \pi \beta^2)G_2, \\
G_4^{(j)} &= A_j(2\pi P, 2\pi \beta^2)G_4 + B_j(2\pi P, \pi \beta^2)G_2, \\
&\quad \ldots,
\end{align*}
\]

where

\[
\begin{align*}
A_j(a, b) &= \frac{1}{4 \sin a \sin b} \left[ \frac{\sin[(2j + 1)(a - b)]}{\sin(a)} - \frac{\sin[(2j + 1)(a + b)]}{\sin(a + b)} \right], \\
B_j(a, b) &= \frac{1}{16 \sin a \sin b \sin 2b} \left[ \frac{\sin[(2j + 1)(a - 2b)]}{\sin(a)} - \frac{\sin[(2j + 1)(a + 2b)]}{\sin(a + 2b)} \right] + \\
&\quad -2 \cos b \left[ \frac{\sin[(2j + 1)a]}{\sin(a)} + \frac{\sin[(2j + 1)b]}{\sin(b)} \right].
\end{align*}
\]

These polynomial relations suggest that there might exist a algebraic relation between quantum transfer matrices belonging to different representations \( \pi_j \). This is indeed the case, as the operators \( T_j \) satisfy the following system of finite-difference functional equations

\[
T_j(q^{\pm \lambda})T_j(q^{-\pm \lambda}) = 1 + T_{j+\frac{1}{2}}(\lambda)T_{j-\frac{1}{2}}(\lambda), \tag{2.25}
\]

known as T-system or Hirota bilinear equations [23, 24]. This system of equations is a direct consequence of the RLL relation (2.17) and can be obtained by using a procedure called R-matrix fusion\(^{14}\), well known in lattice theory [19]. It is worth noticing that equations (2.17), (2.25) and the R-matrix (2.18) are essentially the same as the corresponding ones in the integrable XXZ model [21]. This, clearly, is not just a coincidence as the underlying algebraic structure of the latter is the same as that of the CFIs we are studying here; this structure knows nothing about the discrete or continuous nature of the system and is thus expected that the equations arising from purely algebraic considerations (such as the RLL relation above or, as we will see, the TQ equation) have the same structure, no matter what is the model under study. The information on the different nature of the models will be contained then in the analytical properties of the objects involved in these relations. These considerations will be precious later, when we will extend this setting to massive theories.

For generic values of the central charge \( c \), we have an infinite hierarchy of quantum transfer matrices which, thanks to the system (2.25), can all be expressed in terms of the fundamental one \( T(\lambda) \):

\[
\begin{align*}
T_1(\lambda) &= T(q^{\pm \lambda})T(q^{-\pm \lambda}) - 1, \\
T_2(\lambda) &= T(q\lambda)T(\lambda)T(q^{-1}\lambda) - T(q^{\pm \lambda}) - T(q^{-\pm \lambda}), \\
&\quad \ldots
\end{align*}
\]

\(^{14}\) In fact, while the expressions (2.24) can be obtained by brute force computation, we prefer to consider it as a consequence of (2.25).
With some algebraic effort, we can also recast the $T$-system in the following form
\[ T(\lambda)T_j(q^{\frac{2j+1}{2}}\lambda) = T_{j-\frac{1}{2}}(q^{\frac{2j+2}{2}}\lambda) + T_{j+\frac{1}{2}}(q^{\frac{2j+2}{2}}\lambda). \] (2.26)

Let us now introduce the $Y$-operators as follows [25]:

\[ Y_j(\theta) \doteq T_{j-\frac{1}{2}}(\lambda)T_{j+\frac{1}{2}}(\lambda), \quad \lambda^{1+\xi} = e^\theta, \]

with the convention $T_0 = 1$ and $T_{-\frac{1}{2}} = 0$; then it is easily showed that they satisfy the $Y$-system equations

\[ Y^+ Y^- = \left(1 + Y_{j+\frac{1}{2}}\right)\left(1 + Y_{j-\frac{1}{2}}\right), \] (2.27)

where we have introduced the short-hand notation for shifts: $Y^\pm \doteq Y(\theta \pm i\pi \frac{\xi}{2})$. This last infinite system of finite difference equation can be further recast in an infinite set of non-linear integral equations, known as Thermodynamic Bethe Ansatz equations whose general form is the following

\[ \epsilon_j^{(\ell)}(\theta) = Z^{(\ell)}(\theta) - \sum_k \int_{-\infty}^{\infty} d\theta' \varphi_{jk}(\theta - \theta') \log \left[1 + e^{-\epsilon_k^{(\ell)}(\theta')}\right], \] (2.28)

where the pseudo-energies $\epsilon_j^{(\ell)}$ are the logarithms of the $Y$-operators eigenvalues:

\[ \epsilon_j^{(\ell)}(\theta) \doteq \log \left[Y_j^{(\ell)}(\theta)\right], \quad Y_j(\theta) |\ell\rangle = Y_j^{(\ell)}(\theta) |\ell\rangle, \]

and $\ell$ labels the eigenstate under consideration. The function $Z^{(\ell)}(\theta)$ is called driving term and depends on the particular eigenstate, while the kernel $\varphi_{jk}(\theta)$ only depends on the algebraic structure of the $Y$-system. The procedure to go from (2.27) to (2.28) is intuitively simple, however one has to take great care to the analytic properties of the functions involved. More specifically one has to know the asymptotic behaviour of the $Y$-functions, which will be encoded into the function $Z^{(\ell)}$. Moreover the presence of poles and zeroes in the functions $Y_j^{(\ell)}$ might create a great deal of problems. All these questions are addressed in the [26] and we recommend interested readers to refer to that review.

### Truncation and the minimal models $\mathcal{M}_{2,2n+1}$ TBA

The relations we derived just above, the $T$-system, the $Y$-system and the TBA equations, although very simple-looking and fancy, still consists of an infinity of equations for an infinite set of functions; for this reason, dealing with them is, to use an euphemism, complicated. However there are situations in which the number of equations and functions involved reduce to a finite number; this phenomenon is called truncation. The parallel we traced above with the lattice model helps us identify these cases: it is known that, for some particular values of the parameters, the $XXZ$ system can be reduced to the RSOS model [27] and the $T$-system collapses to a finite set of equations for a finite number of functions $\mathcal{M}_{2,2n+1}$. This phenomenon of truncation in $XXZ$ can be traced back to a purely algebraic fact: when $q$ is a $N$-th root of unity the $(N+1)$-dimensional representation $\pi_N$ of $\mathcal{U}_q(sl(2))$ becomes reducible, while all the representations with $\frac{1}{N} \leq j < \frac{N}{2}$ remain irreducible. In particular

\[ \pi_N = \pi_{N-1} + \vartheta_+^N + \vartheta_-^N, \]

where $\vartheta_{\pm}^N$ are two particular one-dimensional representations such that

\[ \vartheta_{\pm}^N[E] = \vartheta_{\pm}^N[F] = 0, \quad \vartheta_{\pm}^N[H] = \pm N. \]

Being purely algebraic and, as such, blind to the particular theory overlying the algebra structure, we expect the phenomenon of truncation to happen for CFTs as well. Indeed, considering the decomposition above, and applying it to the abstract definition 2.14 we immediately obtain that

\[ T_N^-(\lambda) = 2 \cos(2\pi NP) + T_{N-1}^-(\lambda), \]
which makes (2.25) a closed set of equations for the operators \( \{ T_j \}_{j=0}^{N/2 - 1} \).

In this case too is convenient to introduce the \( Y \)-operators, with a slight modification with respect to the general case, due to the finiteness of the system:

\[
Y_j(\theta) \equiv T_{j - \frac{1}{2}}(\lambda)T_{j + \frac{1}{2}}(\lambda), \quad j = \frac{1}{2}, 1, \ldots, \frac{N}{2} - 1, \\
Y_0(\theta) \equiv 0, \\
Y(\theta) \equiv T_{\frac{N}{2} - 1}.
\]

The \( Y \)-system is then immediately seen to be as follows

\[
Y_j^+Y_j^- = \left(1 + Y_{j - \frac{1}{2}}\right)\left(1 + Y_{j + \frac{1}{2}}\right), \quad j = \frac{1}{2}, 1, \ldots, \frac{N}{2} - \frac{3}{2}, \\
Y_{\frac{N}{2} - 1}^+Y_{\frac{N}{2} - 1}^- = \left(1 + Y_{\frac{N}{2} - \frac{1}{2}}\right)\left(1 + e^{2\pi i N P Y}\right)\left(1 + e^{-2\pi i N P \bar{Y}}\right), \\
Y^+Y^- = \left(1 + Y_{\frac{N}{2} - 1}\right).
\] (2.29)

This \( Y \)-system is called of type \( D_N \) [30] as it can be nicely encoded in the Dynkin diagram of said type. Indeed let us associate each \( Y \)-operator with a node of a graph and draw lines between these whenever the corresponding \( Y \)'s appear in the same equation. What we obtain is the diagram shown in the picture below: a Dynkin diagram of type \( D_N \).

\[\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array}\]

\[\begin{array}{c}
Y_{\frac{1}{2}} \\
Y_1 \\
Y_{\frac{3}{2}} \\
\cdots \\
Y_{\frac{N}{2} - 1}
\end{array}\]

\[\begin{array}{c}
e^{2\pi i N P \bar{Y}} \\
e^{-2\pi i N P \bar{Y}}
\end{array}\]

\[\begin{array}{c}
\mathbb{Z}_2
\end{array}\]

Note that for \( q \) to be a root of unity, we must require \( \beta^2 \) to be a rational number, say \( \beta^2 = \frac{m'}{m} \) and, by virtue of (2.13), the central charge becomes

\[
c = 13 - 6 \left(\beta^2 + \beta^2\right) = 1 - 6 \frac{(m - m')^2}{m m'},
\]

identifying our CFT as the minimal model \( \mathcal{M}_{m,m'} \). Thanks to this identification we now realize that the truncation of the \( T \)-system is a clear reflection of the finite number of primary fields of these CFTs.

We can get a further simplification of (2.29) by considering the particular minimal models \( \mathcal{M}_{2,2n+3} \)

\[
c = 1 - 3 \frac{(2n + 1)^2}{2n + 3}, \quad \beta^2 = \frac{2}{2n + 3}, \quad p_k = \frac{2k - 2n - 3}{2(2n + 3)}, \quad k = 1, \ldots, n + 1,
\]

for which the Kač table is composed of a single line of \( 2(n + 1) \) boxes, symmetric along the middle. This symmetry reflects itself in the \( T \)-system and the following relation is valid

\[
T_{n + \frac{1}{2} - 1}(\lambda) = T_j(\lambda), \quad j = 0, \frac{1}{2}, \ldots, n + \frac{1}{2} \implies T_{n + \frac{1}{2}}(\lambda) = 1,
\]
thanks to this which, the $Y$-system simplifies to
\[
Y_j^{\pm} Y_j^{-} = \left( 1 + Y_{j-\frac{1}{2}} \right) \left( 1 + Y_{j+\frac{1}{2}} \right), \quad j = \frac{1}{2}, 1, \ldots, n
\]
which corresponds to the Dynkin diagram of $A_{2n}$ type \[30, 31\], depicted in the following figure.

Let us now focus on the particular eigenvalues $Y_{gr.st.}^j(\theta)$ corresponding to the ground state $|p_{n+1}\rangle$ (this is the state with lowest $L_0$ eigenvalue). We know that the functions $T_{gr.st.}^j(\lambda)$ are entire functions of $\lambda^2$, with asymptotic behaviour
\[
T_{gr.st.}^j(\lambda) \sim m_j \lambda^{1+\xi}, \quad m_j = \frac{2m}{\pi} \cot \left( \frac{\pi}{2} \xi \right) \sin \left( \pi j \xi \right),
\]
with $m$ given in (2.22). This information is sufficient to pass from the $Y$-system to the TBA equations:
\[\varepsilon_j(\theta) = \pi m_j e^{\theta} - \sum_{j'=-\infty}^{\infty} \int \frac{d\theta'}{2\pi} \varphi_{j j'}(\theta - \theta') \log \left[ 1 + e^{-\varepsilon_{j'}(\theta')} \right], \quad (2.30)\]
where we introduced the pseudo-energies $\varepsilon_j$ as
\[
Y_{gr.st.}^j(\theta) = e^{\varepsilon_j(\theta)}.
\]
The kernel $\varphi_{j j'}(\theta)$ is defined from the equation
\[
(1 - \varphi)^{-1} = 1 - s \hat{I}, \Rightarrow \varphi_{j k}(\theta) = -s(\theta) \hat{I}_{j k} + \sum_i s(\theta + \theta') \varphi_{i k}(\theta') \int_{-\infty}^{\infty} d\theta',
\]
with $\hat{I}_{j k}$ being the incidence matrix of $A_{2n}$ and $s(\theta) = \frac{1}{\xi \cosh(\theta/\xi)}$ the inverse of the shift operator: $s^{-1} : f \rightarrow f^+ + f^-$. Taking the Fourier transform of the above relation, with some effort, is possible to show that the kernel $\varphi$ can be expressed as the logarithmic derivative of the "massless $S$-matrix" $S_{j j'}(\theta)$ \[31\]
\[
\varphi_{j j'}(\theta) \equiv -i \partial_\theta \log \left( S_{j j'}(\theta) \right),
\]
whose explicit form is known:
\[
S_{j j'}(\theta) = F_{j + j'}(\theta) F_{j - j'}(\theta) \prod_{k=1}^{2 \min(j, j')} F_{j-j'+k}^2(\theta), \quad F_j(\theta) = \frac{\sinh \theta + i \sin \left( \pi j \xi \right)}{\sinh \theta - i \sin \left( \pi j \xi \right)}.
\]
Notice that this matrix appears directly from the algebraic properties of the truncated $Y$-system. It is a consequence of the internal consistency of this setting that $S_{j j'}$ happens to be exactly the two-particle element of the factorisable scattering matrix proposed in \[32\] for the $S$-matrix\[15\] description of minimal models of the type $M_{2,2n+1}$. This little "miracle" gives us a strong confirmation of the correctness of the BLZ approach.
2.5 Baxter Q-operators

The construction of the Q-operators follows very closely that of the T-operators presented above. Like these last they are defined as traces of some particular monodromy matrix built out of vertex operators and the generators of some algebra. The difference between the two stands exactly in the choice of the algebra. For the construction of Q-operators it turns out that we need the quantum oscillator algebra \( \text{osc}_q \) generated by \( \{ \mathcal{H}, \mathcal{E}_+, \mathcal{E}_- \} \) with commutation relations

\[
[\mathcal{H}, \mathcal{E}_\pm] = \pm 2 \mathcal{E}_\pm, \quad q \mathcal{E}_+ \mathcal{E}_- - q^{-1} \mathcal{E}_- \mathcal{E}_+ = \frac{1}{q - q^{-1}}.
\]

The appearance of this algebra might seem strange as, at first sight, it does not seem to be related to the sl(2) algebraic structure we have been using to construct everything else. Truth is, \( \text{osc}_q \) and sl(2) really are intimately related and the following in-depth box explains this relation. We encourage the reader not familiar with this fact to go through this explanation to better understand the profound relation between the T- and Q-operators.

Quantum affine sl(2) and universal operators In many cases, the right way to delve deeper in the core of a theory is to generalise the mathematical setting; this not only opens the way for further achievements but almost always cleans up the table and bring about a great simplification of the structures: complicating to clarify. It turns out that the most natural starting point for the construction of the L- and T-operators is a slight generalisation of the algebra \( \mathcal{U}_q (\text{su}(2)) \): the quantum Kač-Moody affine algebra \( \mathcal{U}_q (\hat{\text{sl}}(2)) \). Using this as a starting point we will obtain in one fell swoop a natural description of both T- and Q-operators, displaying explicitly their deep connection, as well as a setting in which the algebraic relations introduced in the previous section can be easily demonstrated. Let us thus introduce the algebra \( \mathcal{U}_q (\hat{\text{sl}}(2)) \): it generated by the six elements \( \{ x_i, y_i, h_i \} \) which satisfy the commutation relations

\[
[h_i, x_j] = -a_{ij} x_j, \quad [h_i, h_j] = 0, \quad i, j = 0, 1, \quad (i \neq j),
\]

\[
[h_i, y_j] = a_{ij} y_j, \quad [y_i, x_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_j}}{q - q^{-1}}, \quad i, j = 0, 1,
\]

and the quantum Serre relations

\[
x_i^2 x_j - [3]_q x_j x_i x_i + [3]_q x_i x_j x_i^2 - x_j x_i^3 = 0, \quad (i \neq j),
\]

\[
y_i^3 y_j - [3]_q y_j y_i y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0, \quad (i \neq j),
\]

where we define the \( q \)-numbers as

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q \xrightarrow{q \rightarrow 1} n.
\]

The matrix \( a_{ij} \) is the Cartan matrix of the affine algebra \( \hat{\text{su}}(2) \):

\[
a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}_{ij}.
\]

In order to be consistently defined this algebra necessitate the further introduction of the grade operator \( d \)

\[
[d, h_0] = [d, h_1] = 0, \quad [d, x_1] = x_1, \quad [d, x_0] = [d, y_0] = 0, \quad [d, y_1] = -y_1,
\]

and the central charge (obviously not the same thing as the central charge \( c \) of the CFT) \( k = h_0 + h_1 \). The algebra thus defined is a \textit{quasi}triangular Hopf algebra \cite{20, 34, 35} whose co-multiplication

\[
\Delta : \mathcal{U}_q (\hat{\text{sl}}(2)) \longrightarrow \mathcal{U}_q (\hat{\text{sl}}(2)) \otimes \mathcal{U}_q (\hat{\text{sl}}(2))
\]

\footnote{More information on the \( - \)-matrix approach to integrable models can be found in \cite{33}.}
is defined by its action on the generators:
\[ \Delta(x_i) = x_i \otimes 1 + q^{-h_i} \otimes x_i , \quad \Delta(y_i) = y_i \otimes q^{h_i} + 1 \otimes y_i , \]
\[ \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i , \quad \Delta(d) = d \otimes 1 + 1 \otimes d . \]

There exists a second possible choice for comultiplication:
\[ \Delta' = \sigma \circ \Delta , \quad \sigma (A \otimes B) = B \otimes A , \quad \forall A, B \in U_q(\mathfrak{sl}(2)) , \]
and the property of quasitriangularity makes sure \[36\] there exists an object called universal \( R \)-matrix
\[ R \in B_+ \otimes B_- , \]
intertwining between these two co-multiplications
\[ \Delta'(A)R = R \Delta(A) , \quad \forall A \in U_q(\mathfrak{sl}(2)) . \quad (2.31) \]

Here \( B_+ \) and \( B_- \) are the Borel subalgebras of \( U_q(\mathfrak{sl}(2)) \) generated by the elements \( \{h_0, h_1, y_0, y_1\} \) and \( \{h_0, h_1, x_0, x_1\} \), respectively. This universal \( R \)-matrix satisfies the Yang-Baxter equation
\[ R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} , \quad \begin{cases} R^{12} = R \otimes 1 \\ R^{23} = 1 \otimes R \\ R^{13} = (\sigma \otimes 1) R^{23} \end{cases} , \]
as is obtained straightforwardly from the action of the comultiplication
\[ (\Delta \otimes 1) R = R^{13}R^{23} , \quad (1 \otimes \Delta) R = R^{13}R^{12} , \]
and the relation (2.31). An explicit form of the universal \( R \)-matrix for \( U_q(\mathfrak{sl}(2)) \) can be found in \[37\].

Using the generators introduced above, we can build the following abstract operator
\[ \mathcal{L} = e^{i \pi P_h} \mathcal{P} \exp \left[ \int_0^{2 \pi} K(w) dw \right] , \quad K(w) = V_-(w)y_0 + V_+(w)y_1 , \]
where the operators \( P \) and \( V_\pm \) are the same ones we defined in Sect. 2.3 and we set \( h_0 = -h_1 = h \), which corresponds to choosing the central charge \( k \) to be zero. It is evident that \( \mathcal{L} \in B_+ \) and, with some simple computation, one can show that
\[ \Delta (\mathcal{L}) = (\mathcal{L} \otimes 1) (1 \otimes \mathcal{L}) , \quad \Delta' (\mathcal{L}) = (1 \otimes \mathcal{L}) (\mathcal{L} \otimes 1) . \]

For this reason, the \( RLL \) relation follows automatically from the definition of \( \mathcal{L} \) and \( R \)
\[ R (\mathcal{L} \otimes 1) (1 \otimes \mathcal{L}) = (1 \otimes \mathcal{L}) (\mathcal{L} \otimes 1) R . \quad (2.32) \]

This is an extremely important relation; being completely abstract, contains in itself the \( RLL \) relation for any possible representation of \( U_q(\mathfrak{sl}(2)) \). In particular if we are able to map \( \mathcal{L} \) into \( L_j \), then the (2.17) will be automatically demonstrated: we need to find an homomorphism between \( U_q(\mathfrak{sl}(2)) \) and \( U_q(\mathfrak{sl}(2)) \). More precisely we want a whole family of homomorphisms, since we need to introduce the spectral parameter \( \lambda \) which, in this abstract setting, is absent. These homomorphisms actually exist and are named evaluation representations:
\[ \text{ev}_\lambda : U_q(\mathfrak{sl}(2)) \to U_q(\mathfrak{sl}(2)) , \quad \begin{cases} x_0 \mapsto \lambda^{-1} F q^{-\frac{H}{2}} \\ y_0 \mapsto \lambda q^{\frac{H}{2}} E \\ h_0 \mapsto H \end{cases} , \quad \begin{cases} x_1 \mapsto \lambda^{-1} E q^{\frac{H}{2}} \\ y_1 \mapsto \lambda q^{\frac{H}{2}} F \\ h_1 \mapsto -H \end{cases} , \]
Consider now any representation $\rho$ of $\text{osc}_q$, such that the following object
\[ Z(p) \doteq tr_\rho \left[ e^{2\pi i p H} \right] , \]
exists and do not vanish for $\Im(p) < 0$. Then we can construct the following operators
\[ L_\pm(\lambda) \doteq \rho \left( e^{\pm \pi i p H} \cdot \text{exp} \left[ \frac{2\pi}{\lambda} \int_0^\lambda dw \left( V_-(w) q^{+\frac{\lambda}{2}} e_\pm + V_+(w) q^{-\frac{\lambda}{2}} e_\mp \right) \right] \right) , \]
where
\[ \rho_\pm(\lambda) \doteq \rho \circ \omega^{\pm}_\lambda , \]
and the corresponding realisation of $T$\(^{16}\)
\[ A_\pm(\lambda) \doteq Z^{-1}(\pm P) tr_\rho \left[ e^{\pm \pi i P H} L_\pm(\lambda) \right] . \]
Notice how, allowing analytic continuation from the $\Im(p) < 0$ half-plane, these operators enjoy the symmetry
\[ A_\pm(\lambda) \mid_{(p, \varphi(w)) \to (-p, -\varphi(w))} \rightarrow A_\mp(\lambda) . \]
These operators, as much as the other we introduced, have to be understood as power series in $\lambda^2$ (since, here too, the odd terms vanish under the trace)
\[ A_\pm(\lambda) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{\sigma_1, \ldots, \sigma_2n \mid \pm P \atop \sum_i \sigma_i = 0 \atop \sum_i \sigma_i = \pm 1}} \lambda^{2n} a_{2n} (\sigma_1, \ldots, \sigma_2n) J_{2n}(\pm \sigma_1, \pm \sigma_2, \ldots, \pm \sigma_{2n}) , \quad (2.34) \]

\(^{16}\) Note that in this case we need to add a regularising factor to the trace. This is due to infinite dimensionality of the algebra $\text{osc}_q$. 

where $\{E, F, H\}$ are the usual generators of $U_q \left( \mathfrak{sl}(2) \right)$. It is immediate to verify that
\[ \pi_j(\lambda) [L] = L_j(\lambda) , \quad \pi_j(\lambda) \doteq \pi_j \circ ev_\lambda , \]
while it is less obvious but still verifiable that, starting from the general definition of [37], we obtain
\[ (\pi_j(\lambda) \otimes \pi_{j'}(\lambda')) [R] = \rho_{jj'}(\lambda/\lambda') R_{jj'}(\lambda/\lambda') , \]
with $\rho_{jj'}(\lambda)$ being an uninteresting scalar factor. In this light the operators $T_j(\lambda)$ are nothing but a specific representation of the following more general "universal $T$-operator"
\[ T \doteq tr_{U_q \left( \mathfrak{sl}(2) \right)} \left[ e^{\pi i P H} L \right] , \quad (2.33) \]
and one can think of defining new operators from $T$, by choosing different representations of $U_q \left( \mathfrak{sl}(2) \right)$. Any two of these operators will commute amongst themselves, as a direct consequence of the universal $RLL$ relation (2.32), and this is precisely a property that we want for our $Q$-operators. However with the family $\pi(\lambda)$, we have exhausted all the finite dimensional evaluation representations: we need to look elsewhere. As mentioned above, the correct choice of algebra for the construction of $Q$-operators happens to be $\text{osc}_q$. Even though it might look rather different from $U_q \left( \mathfrak{sl}(2) \right)$, it is an easy exercise to show that the two following maps
\[ \omega^{\pm}_\lambda : U_q \left( \mathfrak{sl}(2) \right) \rightarrow \text{osc}_q , \quad \left\{ \begin{array}{c} h \mapsto \pm H \\ y_0 \mapsto \lambda e_\pm \\ y_1 \mapsto \lambda e_\mp \end{array} \right. \]
are homomorphisms of $B_+$ into the quantum oscillator algebra $\text{osc}_q$. 

Consider now any representation $\rho$ of $\text{osc}_q$, such that the following object
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where we introduced the two functions

\[ J_{2n}(\sigma_1, \ldots, \sigma_{2n}) \doteq q^n \int_{w_1 \geq \cdots \geq w_{2n}} dw_1 \cdots dw_{2n} V_{\sigma_1}(w_1) \cdots V_{\sigma_{2n}}(w_{2n}) , \]

\[ a_{2n}(\sigma_1, \ldots, \sigma_{2n}|P) \doteq Z^{-1}(P) \text{tr}_P [ \xi^{2\pi i P \mathcal{H}_\sigma} \cdots \xi_{\sigma_{2n}}] . \]

The really interesting fact about this decomposition is that the dependence on the chosen representation does not depend on the chosen representation \( \rho \) of \( \text{osc}_q \). Clearly the coefficients \( J_{2n} \) are closely related to the non-local integrals of motion \( G_{2n} \), however a more neat expansion of the operators \( A_{\pm} \) is the following

\[ \log (A_{\pm}(\lambda)) = - \sum_{n=1}^{\infty} y^{2n} H_{2n} , \quad y = \frac{\beta (1 - \beta^2)}{\beta^2} \lambda , \quad (2.35) \]

where the \( H_{2n} \) are a set of non-local integrals of motion alternative to the \( G_{2n} \) ones and, obviously, algebraically related to these, e.g.

\[ H_1 = \frac{\beta^4 \Gamma(\beta^2)}{4\pi \Gamma(1 - \beta^2) \sin(2\pi P + \pi \beta^2)} G_1 . \quad (2.36) \]

Finally we can introduce the Baxter \( Q \) operators as

\[ Q_{\pm}(\lambda) \doteq \lambda^{\pm 2} \hat{e} \cdot A_{\pm}(\lambda) . \]

Just as the operators \( T_j(\lambda) \), they act invariantly on \( \mathcal{F}_P \), which is an immediate consequence of the representation (2.34) of \( A_{\pm} \). Below is a list of the properties of \( Q \) operators, which descend from from their definition as representations of \( T \) (2.33) and from the structure of the representations involved:

1. they commute amongst themselves and with all the \( T \)-operators

\[ [Q_{\pm}(\lambda), Q_{\pm}(\lambda')] = [Q_{\pm}(\lambda), Q_{\mp}(\lambda')] = [Q_{\pm}(\lambda), T_j(\lambda')] = 0 . \]

Consequently they commute with all the IMs, local and nonlocal

\[ [Q_{\pm}(\lambda), I_{2k-1}] = [Q_{\pm}(\lambda), G_{2n}] = [Q_{\pm}(\lambda), H_{2n}] = 0 ; \]

2. they satisfy the Baxter \( T - Q \) relation

\[ T(\lambda)Q_{\pm}(\lambda) = Q_{\pm}(q\lambda) + Q_{\pm}(q^{-1}\lambda) . \]

This relation is a second order finite-difference equation whose “potential” \( T(\lambda) \) is a periodic function of \( \log (\lambda^2) \), for this reason the two solutions \( Q_+ \) and \( Q_- \) can be interpreted as Bloch wave solutions to the \( T - Q \) relation.

3. they satisfy the quantum wronskian relation

\[ Q_+(q^{1/2}\lambda)Q_-(q^{-1/2}\lambda) - Q_+(q^{-1/2}\lambda)Q_-(q^{1/2}\lambda) = 2i \sin (2\pi P) . \quad (2.37) \]

This relation guarantees the independence of the functions \( Q_+ \) and \( Q_- \), solutions to the \( T - Q \) relation.

4. Wronskian expression of \( T_j \)

\[ 2i \sin (2\pi P) T_j(\lambda) = Q_+(q^{j+\frac{1}{2}}\lambda)Q_-(q^{-j-\frac{1}{2}}\lambda) - Q_+(q^{-j-\frac{1}{2}}\lambda)Q_-(q^{j+\frac{1}{2}}\lambda) . \quad (2.38) \]
2.6 Bethe ansatz and non-linear integral equation

In this section we will concentrate on the eigenvalue $Q(\lambda) \equiv Q^+_\lambda(\lambda)$ of $Q_+(\lambda)$ on the state $|\alpha\rangle \in \mathcal{C}_p$; similar considerations can be obtained for the eigenvalues of $Q_-(\lambda)$.

Let us denote $T(\lambda)$ and $A(\lambda)$ the eigenvalues of $T(\lambda)$ and $A_+(\lambda)$, respectively, on the state $|\alpha\rangle$. Then the following two equations descend directly from Baxter $T - Q$ relation

\[
T(\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda),
\]
\[
T(\lambda)A(\lambda) = e^{2\pi i p} A(q\lambda) + e^{-2\pi i p} A(q^{-1}\lambda).
\]

As we will shortly see, provided the analytic properties of the functions $A$ and $T$, these equations impose severe restrictions on the allowed solutions. Let us recall the properties of $A$ and $T$ we agree on

- Analyticity: both functions $A(\lambda)$ and $T(\lambda)$ are entire in $\lambda^2 \in \mathbb{C}$;
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- **Asymptotic behaviour:**

\[
A(\lambda) \sim \exp \left( M \left( -\lambda^2 \right)^{\frac{1}{2-2\beta^2}} \right),
\]

\[
T(\lambda) \sim \exp \left( m \left( \lambda^2 \right)^{\frac{1}{2-2\beta^2}} \right),
\]

where \( m \) is given by the formula (2.22) and \( M \) is presented below in (2.42).

- **Location of zeroes:** the zeroes \( \{ \lambda_k^2 \}_{k=0}^{\infty} \) of \( A(\lambda) \) are either real or pairs of complex conjugates. For any eigenvalue \( A(\lambda) \), the number of zeroes on the positive real \( \lambda^2 \)-axis accumulate towards \( +\infty \), while the number of other zeroes remains finite. For the vacuum, if \( 2p > -\beta^2 \), the only zeroes are those on the positive real \( \lambda^2 \)-axis. We avoid those values of the highest weight (e.g. \( 2p = -\beta^2 - n \)) for which \( \lambda_k^2 = 0 \).

These properties allow us to use Hadamard factorisation theorem: if \( 0 < \beta^2 < \frac{1}{2} \) then the asymptotic behaviour of \( A(\lambda) \) tells us that its order \( \rho_A \), as a function \( \lambda^2 \), is \( \frac{1}{2} < \rho_A < 1 \), meaning that we can write the very simple product

\[
A(\lambda) = \prod_{k=0}^{\infty} \left( 1 - \frac{\lambda^2}{\lambda_k^2} \right), \quad A(0) = 1.
\]

Now, let us take the "\( T - A \) relation", which we can write as

\[
e^{2\pi i p} \frac{A(\lambda)}{A(q^{-1}\lambda)} = a(\lambda) + 1, \quad a(\lambda) = e^{4\pi i p} \frac{A(q\lambda)}{A(q^{-1}\lambda)},
\]

and evaluate it at \( \lambda^2 = \lambda_k^2 \), recalling that \( T(\lambda) \) is devoid of singularities at finite \( \lambda^2 \); what we obtain is an infinite set of coupled algebraic equations of Bethe ansatz-type\(^{18}\)

\[
a(\lambda_k) = -1 \implies \prod_{k=1}^{\infty} \frac{\lambda_k^2 - q^2 \lambda_k^2}{\lambda_k^2 - q^{-2} \lambda_k^2} = -e^{-4\pi i p}, \quad \forall k \in \mathbb{N}^0.
\]

As always the infinity of equations and variables at hand might sound slightly scary, however there exists a beautiful procedure which allow us to "resum" these equations turning them into a single non-linear integral equation (NLIE) paired with a finite set of Bethe ansatz equations for a finite set of variables. The introduction of this method in the context of QFT is due to C. Destri and H.J. de Vega [39], although the non-linear integral equation first appeared in a work of A. Klümper, M.T. Batchelor and P.A. Pearce [40], where it was used to compute the central charge of 6- and 19-vertex models. We will not present the derivation of the equations, as it follows the same exact lines of the original article, we limit ourselves to displaying the result:

\[
\begin{aligned}
i \log \left( a(\theta) \right) &= -2\pi \frac{\beta}{\beta^2} + 2M \cos \left( \pi \frac{\beta^2}{2-2\beta^2} \right) e^\theta + \\
&\quad + i \sum_a \log \left( S(\theta - \theta_a) \right) - 2\mathcal{G} * \Im \left[ \log (1 + a(\theta - i0)) \right], \quad (2.40)
\end{aligned}
\]

where, as before, \( \lambda^{1+\epsilon} \equiv \lambda^\frac{1-\beta}{2-2\beta} = e^\theta \) and \( \{ \theta_a \}_a \) corresponds to the set of those zeroes \( \{ \lambda_a^2 = e^{2\theta_a(1-\beta)} \}_a \) which lie outside the real positive \( \lambda^2 \)-axis. We used the sign \( * \) to denote the convolution of two functions:

\[
f * g(\theta) \doteq \int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta') \equiv \int_{-\infty}^{\infty} d\theta' f(\theta') g(\theta - \theta'),
\]

and we introduced the kernel

\[
\mathcal{G}(\theta) \doteq \delta(\theta) + \frac{1}{2\pi i} \partial_\theta \log \left( S(\theta) \right),
\]

\(^{17}\) We will not make use of the knowledge about the zeroes of \( T(\lambda) \).

\(^{18}\) For more information on Bethe Ansatz method of solution for integrable models see [38]
and the function

\[ S(\theta) \doteq \exp \left[ -i \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \sin (\nu \theta) \left( \frac{\sinh (\pi \nu \frac{1+\xi}{2})}{\cosh (\pi \nu \frac{1}{2})} \right) \right], \]

which coincides precisely with the soliton-soliton scattering amplitude for the sine-Gordon model \[41\]. Given a solution \(a(\lambda)\) of the NLIE (2.40) above, one can recover the function \(A(\lambda)\) with the following formula

\[ \log (A(\lambda)) = -i \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \cos (\pi \nu \frac{1+\xi}{2}) \sinh (\pi \nu \frac{1}{2}) \left( -\lambda^2 \right)^{i\nu \frac{1+\xi}{2}}, \]

\[ g(\nu) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} 3 \log (1 + a(\theta - i\theta)) e^{-i\nu \theta}. \]

(2.41)

It is possible to use this formulae in the case of the vacuum eigenvalue \(A^{(vac)}(\lambda)\) in the limit \(p \rightarrow \infty\) to compute the exact form of the coefficient \(M\) in (2.39). This turns out to be

\[ M = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{\xi}{2} \right) \Gamma \left( \frac{1-\xi}{2} \right) \Gamma \left( \frac{1}{2} \right) \left( \frac{1}{1+\xi} \right) \right)^{1+\xi}. \]

(2.42)

We will not present the computations here and refer the interested reader to the original article \[2\].

3 Integrable structures of massive integrable field theories

Having extracted and analysed the integrable structures of conformal field theories, a natural question arises: are these results “exportable” in massive field theories? The answer, at least for what concerns theories obtained as integrable deformations of CFTs, is positive. Actually, as it turns out, this extension is rather straightforward: the equations keep the same exact form they have in the massless case. As we already noticed above, this is expected since the algebraic structure governing massless theories survives unscathed to the integrable deformation. On the other hand, the analytic properties of the various objects we introduced undergo a radical change as a consequence of the interplay between the two chiralities which, in presence of a mass scale, is no more trivial.

In the following we will first briefly review A.B. Zamolodchikov results concerning integrable deformations of CFTs \[42\] and then construct the \(T\)-operators for a particular class of these.

3.1 Brief overview of CFT integrable deformations

Remember how in a CFT there exists an infinite set of integrals of motion, which can be constructed from normal ordered products of the energy momentum tensor \(T(u)\) and its derivatives:

\[ I_{2k-1} = \int_{0}^{2\pi} \frac{dw}{2\pi} T_{2k}(w), \quad T_{2k-1} = \int_{0}^{2\pi} \frac{d\bar{w}}{2\pi} T(\bar{w}). \]

All these IMs are in involution: they form an abelian subalgebra \(I\) of \(U(\text{Vir})\) (same goes for the left chirality, obviously).

Given a CFT with Hamiltonian \(H_{\text{CFT}}\), we can think of deforming it by a relevant field \(\Phi\) (clearly belonging to that same CFT), obtaining thus a massive field theory

\[ H_{\Phi} = H_{\text{CFT}} + \mu^2 \int \Phi(x) dx^2. \]

In general these theories are not integrable: the appearance of a mass scale inevitably destroys the conformal symmetry and, along with it, the abelian subalgebra containing the IMs. Only some of these survive and can be written as

\[ I_s = \int_{0}^{2\pi} \left[ T_{s+1} + \Theta_{s-1} d\bar{w} \right], \quad \bar{I}_{\bar{s}} = \int_{0}^{2\pi} \left[ \Theta_{s} dw + T_{s+1} \bar{w} \right]. \]
where \( T_s, \overline{T}_s, \Theta_s \) and \( \overline{\Theta}_s \) are some local fields satisfying the current conservation law
\[
\overline{\partial} T_{s+1} = \partial \Theta_{s-1}, \quad \overline{\partial} \overline{T}_{s+1} = \partial \overline{\Theta}_{s-1},
\]
and the index \( s \) takes values in a finite set: \( s \in \mathcal{X} \), \( C(\mathcal{X}) < \infty \), where \( C \) denotes the cardinality of a set. It turns out, however, that with the right choice of perturbing field this set \( \mathcal{X} \) becomes infinite; in other words there exist particular perturbations for which the abelian subalgebra \( \mathcal{T} \) survives in its entirety\(^{19}\). As a consequence the massive theory obtained through these deformations is integrable inheriting in toto, with suitable modifications, the integrable structure of the corresponding CFT. In \([42]\) A.B. Zamolodchikov showed\(^{20}\) that this phenomenon happens in CFTs with \( c < 1 \) if one chooses \( \Phi(1,3), \Phi(1,2) \) or \( \Phi(2,1) \) as perturbations, where the bracketed indices stands for \( (r, s) \), identifying the fields on the Kač Table \([8]\). In the following we will concentrate on \( \Phi(1,3) \) perturbations only.

The \( \Phi(1,3) \) perturbations of CFT Consider the massive field theory defined by the Hamiltonian
\[
H_{(1,3)} = H_{\text{CFT}} + \mu^2 \int \Phi_{(1,3)}(x) dx^2,
\]
where \( \Phi_{(1,3)} \) is a primary field of \( H_{\text{CFT}} \) with conformal dimensions
\[
h_{1,3} = h_{1,3} = 2\beta^2 - 1 \equiv \frac{\xi - 1}{\xi + 1},
\]
which satisfies the null-vector equations
\[
\begin{align*}
\left[ 2(1 + 2\beta^2)L_{-3} - 4L_{-1}L_{-2} + \frac{1}{\beta^2} L_{-3}^3 \right] \Phi_{(1,3)}(w, \overline{w}) &= 0, \\
\left[ 2(1 + 2\beta^2)T_{-3} - 4T_{-1}T_{-2} + \frac{1}{\beta^2} T_{-3}^3 \right] \Phi_{(1,3)}(w, \overline{w}) &= 0,
\end{align*}
\]
and is assumed to have the following canonical normalisation
\[
\langle \Phi_{(1,3)}(w, \overline{w}) \Phi_{(1,3)}(w', \overline{w}') \rangle_{\text{CFT}} \sim |w - w'|^{-4h_{1,3}}.
\]
From dimensional analysis one immediately sees that \([\mu^2] = \text{[length]}^{h_{1,3} - 1}\) and \( \mu \) can be thought of carrying an anomalous dimension of \( h_{1,3} = \overline{h}_{\mu} = \frac{1 - h_{1,3}}{2} = \frac{1}{1 + \xi} \). Notice how this is exactly the opposite of the anomalous dimension carried by the spectral parameter \( \lambda \) in the analysis of the above section.

Any field theory is completely characterised by the infinite-dimensional vector space of local fields \( \Omega = \{ \mathcal{O}_j \}_{j \in \mathbb{N}_i} \), along with the totality of their correlation functions. Generically, when taking a perturbation of a CFT, the fields of this last require an infinite number of renormalisation parameters in order to cancel the ultraviolet divergencies which arise in correlation functions. In the case under consideration, however, the massive theory turns out to be "super-renormalisable" meaning only a finite number of counter-terms is needed (for \( \beta^2 < \frac{1}{2} \), that is \( c < -2 \), there are actually no UV divergencies at all). If, furthermore, we are in a finite size geometry, as we are, the infrared divergencies are completely under control, thanks to the natural cutoff \( R < \infty \). For these reasons we can safely assume that the Hilbert space of the perturbed theory has the same structure of that of the original CFT
\[
\mathcal{H}_{(1,3)} \simeq \mathcal{H}_{\text{CFT}},
\]
and there exists a one-to-one correspondence between fields of the two theories. In other words one can assign to fields in the massive theory the roles they played in the CFT; in this sense there exists in the massive theory a concept of primary fields, of descendants and also of Virasoro operators.

Consider the subspace \( \Lambda_{\text{CFT}} \subset \Omega_{\text{CFT}} \), consisting of all the composite fields built out of \( T(w) \) and its derivatives. Now take the quotient of this subspace by the action of \( L_{-1} \):
\[
\hat{\Lambda}_{\text{CFT}} \equiv \Lambda_{\text{CFT}} / (L_{-1} \Lambda_{\text{CFT}}) \subset \Lambda_{\text{CFT}}.
\]
\(^{19}\) Actually this is not strictly a consequence of \( |\mathcal{X}| = \infty \), but for the models we are going to analyse it is conjectured to be so.
\(^{20}\) Actually he made conjectures based on strong physical assumptions. His conjectures were never disproved since then and are assumed to be true.
This further subset $\hat{\Lambda}_{\text{CFT}}$ consists of all those composite fields which are not total derivatives (remember that $L_{-1}$ acts on local fields as a derivative). Then it is immediate to notice that all the integrals of motion are generated as integrals of some element of $\hat{\Lambda}_{\text{CFT}}$, informally

$$\mathcal{I} = \int \hat{\Lambda}_{\text{CFT}}.$$ 

Now, while clearly $\overline{\partial} \hat{\Lambda}_{\text{CFT}} = 0$, this is no more true when considering the corresponding subspace in the deformed theory; in general one will have

$$\overline{\partial} T_s = \sum_{n=1}^{\infty} \mu^{2n} R_{s-1}^{(n)} , \quad T_s \in \Lambda , \quad R_{s-1}^{(n)} \in \Omega.$$ 

By simple dimensional analysis, we see that the fields $R_{s-1}^{(n)}$ must have dimensions $(h, \overline{h}) = (s - n + nh_{1,3}, 1 - n + nh_{1,3})$; however, no field in $\Omega$ is allowed to have negative left conformal dimension, meaning that the series above must truncate\(^{21}\) for $n$ bigger than some integer $N \geq 1$. Moreover, since $\Phi_{1,3}$ is the most relevant field in its OPE subalgebra, we conclude that $N = 1$

$$\overline{\partial} T_s = \mu^2 R_{s-1}.$$ 

Denoting as usual with $\mathcal{V}_{1,3}$ the Verma module with highest weight $h_{1,3}$, we have

$$L_0 \mathcal{V}_{1,3}^{(s)} = (h_{1,3} + s) \mathcal{V}_{1,3}^{(s)} , \quad \overline{L}_0 \mathcal{V}_{1,3}^{(s)} = h_{1,3} \mathcal{V}_{1,3}^{(s)} , \quad \mathcal{V}_{1,3} = \bigoplus_{s=0}^{\infty} \mathcal{V}_{1,3}^{(s)},$$

and, clearly\(^{22}\)

$$R_{s-1} \in \mathcal{V}_{1,3}^{(s-1)}.$$ 

We can thus interpret the anti-holomorphic partial derivative $\overline{\partial}$ as a map between subspaces of $\Omega$:

$$\overline{\partial} : \hat{\Lambda} \rightarrow \mathcal{V}_{1,3}^{(s-1)}.$$ 

By taking first-order corrections to correlation functions involving the field $T_s$, one can show that

$$\overline{\partial} T_s(w, \overline{w}) = \left[ T_s(w, \overline{w}) , \mu^2 \int_0^{2\pi} dw' e^{i n (w' - w)} \Phi_{1,3}(w', \overline{w}) \right],$$

which is a usual formula of perturbation theory. As a consequence

$$[\overline{\partial}, L_{-1}] = 0,$$

and we can define a set of operators $D_n : \Lambda \rightarrow \mathcal{V}_{1,3}$ from their action on the vectors of $\Lambda$, informally:

$$D_n \Lambda \doteq \left[ \Lambda , \mu^2 \int_0^{2\pi} dw' e^{i n (w' - w)} \Phi_{1,3}(w', \overline{w}) \right].$$

Clearly we have $\overline{\partial} \equiv D_0$ and it is fairly easy to prove the following relations

$$[L_n, D_m] = -\left[ (1 - h_{1,3})(n + 1) + m\right] D_{n+m} ,$$

$$D_{-n-1} \cdot 1 = \frac{1}{n!} L_{-1}^{n+1} \Phi_{1,3}(w, \overline{w}).$$

\(^{21}\) Remember that $h_{1,3} = 2\beta^2 - 1$ and, if $c < 1$ then $h_{1,3} < 1$.

\(^{22}\) Note that we care-freely apply the CFT concept of Verma modules to the deformed theory. This can actually be done thanks to the isomorphism (3.3), since we logically expect that the decomposition of the CFT Hilbert space into Verma modules survives to the deformation along with the other structures. Therefore it should exist in the massive theory a decomposition of $\mathcal{H}_{1,3} = \bigoplus_n \left( \mathcal{V}_n \otimes \mathcal{T}_n \right)$ into spaces $\mathcal{V}_n \simeq \mathcal{V}_0$. We use the same notation as in the CFT, hoping this note will be sufficient to avoid confusion.
Finally we are at a point where we can explicitly compute the action of $\mathcal{T}$ on the elements of $\hat{A}$. Let us begin with the energy-momentum tensor $T_2 \equiv T = L_{-2} \cdot 1$ itself

$$\mathcal{T}T = \hat{\mu}^2 D_0 L_{-2} \cdot 1 = \hat{\mu}^2 (h_{1,3} - 1) D_{-2} \cdot 1 = \hat{\mu}^2 (h_{1,3} - 1) L_{-1} \Phi_{(1,3)}.$$  

Introducing the local field $\Theta_0 \equiv \Theta \doteq \hat{\mu}^2 (h_{1,3} - 1) \Phi_{(1,3)}$, we see that we can write

$$\mathcal{T}T_2 = \partial \Theta_0,$$

which is a current conservation equation and, as such, defines an integral of motion $I_1$.

Let us now try and see if there's a current conservation law also for the next element of $\hat{A}$: $T_4 \equiv L_{-2}^2 \cdot 1$. In this case we obtain

$$\mathcal{T}T_4 = \hat{\mu}^2 D_0 L_{-2}^2 \cdot 1 = \hat{\mu}^2 (h_{1,3} - 1) (D_{-2} L_{-2} + L_{-2} D_{-2}) \cdot 1$$

$$= \hat{\mu}^2 (h_{1,3} - 1) \left(2L_{-2} L_{-1} + \frac{h_{1,3} - 3}{6} L_{-1}^2\right) \Phi_{(1,3)},$$

and we cannot write $\mathcal{T}T_4$ as the holomorphic derivative of a local field! As a consequence there seem to be no conservation law.

**Degenerate fields and integrals of motion** We have seen that there seems to be no hope of recovering the continuity equations $\mathcal{D}T_{2n} = \partial \Theta_{2n-2}$ for $n > 1$ and, as such, we have only two integrals of motion: $I_1$ and $\bar{I}_1$, which is to say the energy $\bar{I}_1 + I_1$ and the momentum $I_1 - \bar{I}_1$ of our system. However, we forgot that we have an ace in our sleeve: we chose the perturbing field to be $\Phi_{(1,3)}$, a degenerate field which satisfies the level-3 null vector equations (3.1) and (3.2)! Recalling that $L_{-2} L_{-1} = L_{-1} L_{-2} - L_{-3}$, we can rewrite $\mathcal{T}T_4 = \partial \Theta_2$ where

$$\Theta_2 \doteq \hat{\mu}^2 \frac{h_{1,3} - 1}{h_{1,3} + 2} (2h_{1,3} L_{-2} + (h_{1,3} - 2) \frac{(h_{1,3} - 1)(h_{1,3} + 3)}{6(h_{1,3} + 1)} L_{-1}^2) \Phi_{(1,3)},$$

which is a proper conservation law and give rise to the integral of motion $I_s$.

The conjecture of Al. Zamolodchikov is that the phenomenon illustrated just above, happens on every level subspace $V_{(1,3)}^{(s-1)}$ with odd $s$. A nice way to see this for $s \leq 7$ is the following. Consider the operator $B_s$ defined as

$$B_s \doteq \Pi_{s-1} \mathcal{T} : \hat{\Lambda}_s \rightarrow \hat{V}_{(1,3)}^{(s-1)}, \quad \hat{V}_{(1,3)}^{(s-1)} \doteq V_{(1,3)}^{(s-1)} \bigg/ \left(L_{-1} V_{(1,3)}^{(s-1)}\right),$$

where $\Pi_s$ is the projector onto $V_{(1,3)}^{(s-1)}$:

$$\Pi_s : V_{(1,3)}^{(s-1)} \rightarrow \hat{V}_{(1,3)}^{(s-1)}.$$

By definition, if $B_{s+1} T_{s+1} = 0$, then we are assured that there exists a field $\Theta_{s-1}$ such that $\mathcal{T}T_{s+1} = \partial \Theta_{s-1}$. This means that a conservation law is present at the level $s$ iff $B_s$ has a non-vanishing kernel. The two conservation laws we found above appear for a very simple reason

$$\dim \left(V_{(1,3)}^{(1)}\right) = \dim \left(V_{(1,3)}^{(3)}\right) = 0.$$

This suggests that we can try to compare the dimensions of the spaces $\hat{A}_s$ and $\hat{V}_{(1,3)}^{(s-1)}$, we can do this by using the character formulae [9]:

$$\sum_{s=0}^{\infty} q^s \dim \left(V_{(1,3)}^{(s-1)}\right) = (1 - q) \chi_{(1,3)}(q), \quad \chi_{(1,3)}(q) \doteq \prod_{n=1}^{\infty} (1 - q^n)^{-1},$$

$$\sum_{s=0}^{\infty} q^s \dim \left(\hat{A}_s\right) = (1 - q) \chi_0(q) + q, \quad \chi_0(q) \doteq (1 - q) \prod_{n=2}^{\infty} (1 - q^n)^{-1} + q.$$
Moreover in a minimal model, it not justified to assume that
As we already mentioned above, important facts of deformed CFTs are the presence of a mass scale
meaning that the integrals
existence of higher spin conservation laws is a conjecture which finds a partial justification in the classical limit
(where their existence is guaranteed to be true).
As a last note, one should pay particular attention when
Here
these integrals are in involution
Let us consider the theory defined by the following formal action
\( \mathcal{A}_{(1,3)} \equiv \mathcal{A}_{\text{CFT}} + \mu^2 \int du dv \Phi_{(1,3)}(u, v) , \quad [\mu] = [\text{length}]^{2\beta^2 - 2} , \) (3.4)
defined on a cylinder of radius\(^2\) \( R: \{(u, v) \mid u + R = 0\} \), with \( u \) playing the role of space and \( v \) that of time.
Here \( \mathcal{A}_{\text{CFT}} \) is the formal action of a conformal field theory with \( c < 1 \) and \( \Phi_{(1,3)} \) is its primary field of conformal dimension \( h_{1,3} = 2\beta^2 - 1 \). We recall the relation between \( \beta \) and the central charge:
\( \beta = \sqrt{\frac{1 - c}{24}} - \sqrt{\frac{25 - c}{24}} \implies c = 13 - 6\left(\beta^2 + \beta^{-2}\right) . \)
As we already mentioned above, important facts of deformed CFTs are the presence of a mass scale \( m \propto \mu^{-\frac{1}{1-\beta^2}} \) and the breaking of conformal invariance, which translates into the non-holomorphicity of the energy-momentum tensor. As we have seen above, however, in a \( \Phi_{(1,3)} \)-deformed CFT, the local fields \( T_{2k}(w, \bar{w}) \) satisfy the continuity equations
\( \overline{\partial} T_{2k}(w, \bar{w}) = \partial \Theta_{2k-2}(w, \bar{w}) , \quad \partial T_{2k}(w, \bar{w}) = \overline{\partial} \Theta_{2k-2}(w, \bar{w}) , \)
meaning that the integrals
\[ \mathbb{I}_{2k-1} \equiv \int_0^R \frac{du}{2\pi} \left[ T_{2k}(w, \bar{w}) + \Theta_{2k-2}(w, \bar{w}) \right] , \quad (3.5) \]
\[ \mathbb{I}_{2k-1} \equiv \int_0^R \frac{du}{2\pi} \left[ T_{2k}(w, \bar{w}) + \overline{\Theta}_{2k-2}(w, \bar{w}) \right] , \quad (3.6) \]
do not depend on the "time" \( v \) and are, as such, integrals of motion. It is not hard to verify, at least for small \( k \), that these integrals are in involution
\[ [\mathbb{I}_{2k-1}, \mathbb{I}_{2l-1}] = [\mathbb{I}_{2k-1}, \mathbb{I}_{2l-1}] = [\mathbb{I}_{2k-1}, \mathbb{I}_{2l-1}] = 0 . \]
Notice that \( \mathbb{H} \equiv \mathbb{I}_1 + \mathbb{I}_1 \) and \( \mathbb{P} = \mathbb{I}_1 - \mathbb{I}_1 \) are, respectively, the Hamiltonian and the momentum operators of \( \mathcal{A}_{(1,3)} \).

\(^{23}\) We slightly change the geometry here by introducing the radius as a new parameter of the system. We hope this will not be confusing.
The left chirality In order to describe the integrable structures of our deformed CFT, we have to consider both chiralities at the same time. To this end, let us construct the left-chiral integrable structure of the CFT, which we will then join with the right-chiral part when moving to the massive model.

Remember that the starting point of our construction was the Feigin-Fuchs free field representation of the energy-momentum tensor (2.13), which allows us to express the right-chiral Hilbert space \( \mathcal{H}_{\text{ch}} \) as a direct sum of Fock spaces \( \mathcal{H}_{\text{ch}} = \bigoplus_n \mathcal{F}_{p_n} \), with the latter generated by the free action of the Heisenberg operators \( \{ a_{-n} \}_{n=1}^{\infty} \) on the "vacuum" \( |p\rangle \), such that \( P | p\rangle = p | p\rangle \) and \( a_n | p\rangle = 0, \forall n > 0 \). We will need to repeat these steps in the left chirality, so let us introduce the following free field\(^{23}\)

\[
\mathcal{V}(\pi) = iQ - \frac{2\pi}{R} P + \sum_{n \neq 0} \frac{\pi_n}{n} e^{-i \frac{2\pi}{R} n \pi}
\]

where the operators \( \{ Q, P; \pi_n \}_{n \neq 0} \) span an Heisenberg algebra

\[
[Q, P] = \frac{i}{2} \beta^2, \quad [\pi_n, \pi_m] = \frac{n}{2} \beta^2 \delta_{n+m, 0}.
\]

The full Hilbert space can then be expressed as

\[
\mathcal{H}_{\text{CFT}} = \bigoplus_a \left( \mathcal{F}_{p_a} \otimes \mathcal{F}_{-p_a} \right),
\]

where \( \mathcal{F}_{p_a} \otimes \mathcal{F}_{-p_a} \) is an irreducible representation of \( \text{Vir} \otimes \text{Vir} \) with highest weights \( (h(p), h(p)) \) and highest-weight vector \( |p\rangle \otimes |-p\rangle \).

Now, we most simply, define the left-chiral \( L \)-operator as

\[
\mathcal{L}_j(\lambda) = \pi_j \left[ e^{-\pi \mathcal{T}H} \mathcal{P} \exp \left( \lambda \int_0^R \pi \mathcal{V}_- (\pi) q^{\frac{3}{2}} E + \mathcal{V}_+ (\pi) q^{-\frac{3}{2}} F \right) \right],
\]

where \( \{ H, E, F \} \) are \( \mathcal{U}_q(\mathfrak{sl}(2)) \) generators and \( \mathcal{V}_\pm (\pi) \) are the left-chiral vertex operators:

\[
\mathcal{V}_\pm (\pi) = e^{\pm \mathcal{V}(\pi)}.
\]

Just as for the right-chiral \( L \)-operators they satisfy a relation with the \( \mathcal{U}_q(\mathfrak{sl}(2)) \) trigonometric \( R \)-matrix which we will call "\( LLR \) relation":

\[
(\mathcal{L}_j(\lambda) \otimes 1) (1 \otimes \mathcal{L}_j'(\lambda')) = \mathcal{R}_j, j'(\lambda/\lambda') \left( 1 \otimes \mathcal{L}_j'(\lambda') \right) (\mathcal{L}_j(\lambda) \otimes 1).
\]

Notice the different ordering of this relation with respect to the corresponding one for the right chirality (2.17), which can be traced down to the minus sign in front of \( \mathcal{P} \) in the definition of \( \mathcal{V} \). From the knowledge of the left-chiral \( L \)-operator, one can repeat exactly what has been done for the right chirality and obtain the operators \( \mathcal{T}_j, \mathcal{Q}_\pm \), and so on. We will not go into details as these constructions are essentially identical as the ones for the right chirality.

The massive integrable structure Now that we have the \( L \)-operators of both right and left chiralities, we need to fuse them into one single object. The "vault key" holding the two pieces together will have to be the deformation parameter \( \mu^2 \) which, we recall, carries a dimension \( \text{[length]}^{3\beta^2 - 2} \). Since the spectral parameter carries a dimension of \( [\lambda] = [\text{length}]^{3\beta^2 - 1} \), the most natural way (and, as it turns out, the correct one) to couple the chiralities is

\[
\mathcal{L}_j(\mu/\lambda) = \mathcal{L}_j(\lambda) \mathcal{L}_j(\mu/\lambda),
\]

where \( \mu \propto \mu^2 \). As usual, the \( T \)-operators are obtained taking the trace over the \( \mathcal{U}_q(\mathfrak{sl}(2)) \) representation\(^{25}\)

\[
\mathcal{T}_j(\mu/\lambda) = \text{tr}_{\pi_j} [\mathcal{L}_j(\mu/\lambda)].
\]

---

\(^{23}\) Note the appearance of the ratio \( \frac{2\pi}{R} \) due to the rescaling on a cylinder with radius \( R \).

\(^{25}\) Notice the missing exponential factor in front of the \( L \)-operator. In fact \( \mathcal{T}_j(\lambda) = \text{tr}_{\pi_j} (e^{-\pi \mathcal{T}H} \mathcal{T}_j(\lambda)) \) and, when combining the two chiralities, the exponential factors in front of \( \mathcal{L}_j \) and \( \mathcal{L}_j \) cancel each other.
The commutativity of these operators
\[ [T_j(\mu|\lambda), T_{j'}(\mu|\lambda)] = 0 , \]
is easily inferred from the RLL and \( \mathcal{LR} \) relations, the commutativity of the chiralities
\[ [L_j(\lambda), L_{j'}(\lambda')] = 0 , \]
and from the unitarity of the \( R \)-matrices
\[ R_{j j'}(\lambda) R_{j j'}(\lambda^{-1}) = 1 . \]
The following two properties immediately descend from the definition of \( T_j \) and from the properties of \( T_j \) and \( T_j' \):

- **Massless limit**: in the limit \( \mu \to 0 \) we recover the right and left chiral \( T \)-operators as
  \[ T_j(\mu|\lambda) \xrightarrow{\mu \to 0} T_j(\lambda) \otimes 1 , \]
  \[ T_j(\mu|\mu/\lambda) \xrightarrow{\mu \to 0} 1 \otimes T_j(\lambda) , \]
  where the tensor notation is employed here to specify that \( T \) acts as the identity in the left (right) chirality space.

- **Analytic properties**: the operators \( T \) are single valued functions of \( \lambda^2 \), regular everywhere except \( \lambda^2 = 0, \infty \). Moreover they inherit the asymptotic behaviour of the \( T \) and \( T' \) functions:
  \[ \log (T_j(\mu|\lambda)) \xrightarrow{\lambda \to \infty} m_j \frac{R}{2\pi} \lambda^{1+\xi} , \]
  \[ \log (T_j(\mu|\lambda)) \xrightarrow{\lambda \to 0} m_j \frac{R}{2\pi} \left( \frac{\mu}{\lambda} \right)^{1+\xi} , \]
  where
  \[ m_j = \frac{\sin (j\pi \xi)}{\sin (\pi \frac{j}{2})} m , \]
  and \( m \) is given in (2.22). Clearly the massive \( T \)-operators have two essential singularities: one at \( \infty \) (like the chiral operators) and the other at \( 0 \). It is natural to expect the integrals of motion (3.5-3.6) to appear in the asymptotic expansion around these singular points.

As we already said above, all the purely algebraic relations that we displayed in the previous section transfer directly here, without any change in their appearance. So the \( T \)-system reads:
\[ T_j(\mu|q^{\frac{1}{2}}\lambda) \cdot T_j(\mu|q^{-\frac{1}{2}}\lambda) = 1 + T_{j+\frac{1}{2}}(\mu|\lambda) \cdot T_{j-\frac{1}{2}}(\mu|\lambda) , \]
and, when \( q \) is a root of unity, it truncates to a finite system, exactly as in the massive case, which can then be recast in a \( Y \)-system, from which a TBA equation can be extracted (under some suitable analyticity assumptions).

The difference between the massive and the massless case emerges in the analytic properties of the \( T \)-operators. As we have seen just above, the \( T_j \) possess two singularities instead of just one, which corresponds to the two different chiralities of the massive theory. Here below we list the conjectures about the massive \( T \)-operators:

- **Asymptotic expansion**: the integrals of motion (3.5-3.6) govern the asymptotic expansions of \( T = T_{\frac{1}{2}} \) around its singular points \( \lambda^2 = 0, \infty \) as follows:
  \[ \log (T(\mu|\lambda)) = m \frac{R}{2\pi} \lambda^{1+\xi} - \sum_{n=1}^{\infty} C_n \lambda^{(1-2n)(1+\xi)} T_{2n-1} , \]
  \[ \log (T(\mu|\lambda)) = m \frac{R}{2\pi} \left( \frac{\mu}{\lambda} \right)^{1+\xi} - \sum_{n=1}^{\infty} C_n \left( \frac{\mu}{\lambda} \right)^{(1-2n)(1+\xi)} T_{2n-1} , \]
  where the constants \( C_j \) are given in (2.23).
\begin{itemize}
  \item **Commutation with IMs**: The massive transfer matrices commute with all the massive IMs
  \[ [T_j(\mu|\lambda), \mathbb{I}_{2k-1}] = [T_j(\mu|\lambda), \mathbb{I}_{2k-1}] = 0. \]
  \item **Relation between \( \mu \) and \( \tilde{\mu} \)**: the parameter entering in \( T \) is related to the deformation parameter \( \tilde{\mu} \) as follows
  \[ \tilde{\mu}^2 = \frac{\Gamma^2 (1 - \beta^2)}{\pi (1 - 2\beta^2) (3\beta^2 - 1)} \left[ \frac{\Gamma (3\beta^2) \Gamma (\beta^2)}{\Gamma (1 - 3\beta^2) \Gamma (1 - \beta^2)} \right]^{\frac{1}{2}} {\mu}^2. \]
\end{itemize}

The last of these conjectures might appear a bit outlandish, however it becomes natural when one considers \( T \) to actually be the transfer matrix of the sine-Gordon model. The reasons leading us to conjecture that the operators \( T_j \) introduced above can be interpreted as \( T \)-operators of sine-Gordon model is briefly discussed in the following subsection.

### 3.3 Moving on to sine-Gordon model

The quantum sine-Gordon model on a cylinder of radius \( R \) is a massive integrable QFT described by the following action
\[ \mathcal{A}_{sG} = \int_0^R du \int_{-\infty}^{\infty} dv \left[ \frac{1}{16\pi} (\partial_u \phi(u, v))^2 + 2\tilde{\mu} \cos (\beta \phi(u, v)) \right], \]
where \( \phi(w, \overline{w}) \) is a scalar field, \( \beta \) is the coupling and \( \tilde{\mu} \) is a parameter with dimensions \( [\text{length}]^{2\beta^2 - 2} \). It possesses an infinite set of integrals of motion whose form is exactly the same as (3.5-3.6), where now \( T_{2k}, \Theta_{2k-2}, \mathbb{T}_{2k} \) and \( \mathbb{S}_{2k-2} \) are local fields of sine-Gordon model and \( \tilde{\mu} \) has to be replaced by \( \tilde{\mu} \). The spectrum of the model contains two topologically charged particles (the soliton and the anti-soliton) with mass \( \mathfrak{M} \)
\[ \mathfrak{M} = \frac{2}{\sqrt{\pi}} \frac{\Gamma \left( \frac{\beta}{2 \pi} \right)}{\Gamma \left( \frac{\beta}{2 \pi} + \frac{\beta}{2 \pi} \right)} \left[ \pi \tilde{\mu} \left( \frac{\beta}{\pi \xi} \right) \right]^{-\frac{1+i}{2}}, \quad \xi = \frac{\beta^2}{1 - \beta^2}, \tag{3.7} \]
and a set of neutral particles (bound-states), whose number depends on the coupling, with masses
\[ m_j = 2\mathfrak{M} \sin (j\pi \xi), \quad j = \frac{1}{2}, 1, \ldots, n \text{ s.t. } n < \frac{1}{2\xi}. \]

The connection between sine-Gordon model and the perturbed CFTs we considered above is not evident at first. However, as it was unveiled in [43, 44] these latter can be obtained as ‘quantum group reductions’ of the former, as we will recall very briefly. When considered in infinite volume, sine-Gordon model exhibits a symmetry with respect to the quantum group \( U_q (SL(2)) \), where
\[ \tilde{q} = e^{i \frac{\beta}{2\pi}}. \]

This means that the soliton and anti-soliton transform in the two-dimensional representation of this quantum group and that the local IMs and \( S \)-matrix commute with the generators \( \{ \hat{H}, \hat{E}, \hat{F} \} \) of the associated quantum algebra \( U_q (sl(2)) \). Now, the Hilbert space \( \mathcal{H}_{sG}^{\infty} \) of sine-Gordon model in infinite volume, contains a subspace \( \mathcal{H}_{sG}^{\infty, \text{singlet}} \) consisting of those states annihilated by the \( U_q (sl(2)) \) generators. What is remarkable is that this last Hilbert space can be interpreted as the space of states of a certain local QFT, which was called restricted sine-Gordon model. As it turns out, this model coincides exactly with the perturbed CFT (3.4) (considered in infinite volume) where the parameters \( \tilde{\mu} \) and \( \tilde{\mu} \) are related as
\[ \tilde{\mu}^2 = \frac{(1 - 2\beta^2) (3\beta^2 - 1)}{\pi} \left[ \frac{\Gamma^3 (\beta^2) \Gamma (1 - 3\beta^2)}{\Gamma^3 (1 - \beta^2) \Gamma (3\beta^2)} \right]^{\frac{1}{2}} {\mu}^2. \]
Although in finite volume the quantum group symmetry breaks down, it is still possible to define singlet states and their Hilbert sub-space \( \mathcal{H}_{sG}^{R, \text{singlet}} \) and these still allow an interpretation in terms of deformed CFTs. In particular

\[
\mathcal{H}_{sG}^{R, \text{singlet}} \simeq \mathcal{H}_{(1,3)} \simeq \mathcal{H}_{\text{CFT}} .
\]

It is not difficult to verify that the action of massive \( T \)-operators and their properties, defined in the previous sub-section naturally extend to the full sine–Gordon Hilbert space \( \mathcal{H}_{sG}^{R} \) so that they can be interpreted as the transfer matrices of the unrestricted sine–Gordon model. In this optics, the Feigin–Fuchs fields are naturally identified with sine–Gordon one as

\[
\phi(w) = \frac{\beta}{2} \phi(w, 0), \quad \varphi(\overline{w}) = -\frac{\beta}{2} \phi(0, R - \overline{w}).
\]

It can moreover be shown that the truncation of the \( T \)-system happens in sine–Gordon as well, meaning that it is possible employ the methods of subsection 2.4 to obtain TBA-like equations for the ground state of the system\(^{26}\). It is as well possible to proceed as in sub-sections 2.5 and 2.6, constructing the operators \( Q \) and \( A \) and recovering the Destri–de Vega equation. This path is actually more rewarding than the \( Y \)-system one, since it works for any value of the coupling \( \beta^2 \) (in the region \((0,1)\), as discussed for the CFTs). In order to obtain the \( Q \)-operators, one proceeds in the same exact way as for the operators \( T \), that is by combining the right and left-chiral \( L \)-operators in the \( q \)-oscillator representation into a single operator \( \mathbb{L}_k \) and then taking its trace. We will not delve in the detail of this construction as, really, it is a simple variation on the theme of what has been done in the CFT case. We wish instead to present a different approach to the integrability structure of sine–Gordon model, which relies on a surprising and still not completely understood connection between classical and quantum worlds.

4 \ Sine-Gordon model and the massive ODE/IM correspondence

In this section we wish to briefly present an approach to quantum sine-Gordon\(^{27}\) model in finite geometry which was proposed in [5] by S.L. Lukyanov and A.B. Zamolodchikov. This approach relies on older studies on the so-called ODE/IM correspondence [45, 46, 47] (see [48] for a review) which related the integrals of motion of certain CFTs to the spectral properties of specific ordinary differential equations (ODE). This setting was extended by Lukyanov and Zamolodchikov to the massive case [5] with the study of sine- and sinh-Gordon cases and later this method was generalised, first to the Tzitzéica-Bullough-Dodd model [49], corresponding to the affine Lie algebra \( A^{(2)}_1 \), then to the Toda theories associated to the affine algebras \( A^{\infty}_n \) [50]. Finally, the ODE/IM was applied to the whole set Toda field theories, associated both to simply-laced [51, 52] and non-simply-laced [53, 54] algebras.

4.1 \ Quantum sine-Gordon \( T \) and \( Q \) operators

Before venturing in the description of the ODE/IM correspondence for the sine-Gordon model, we wish to recapitulate the properties (proved or conjectured) satisfied by the operators \( T \), as discussed for the CFTs). In order to obtain the \( Q \)-operators, one proceeds in the same exact way as for the operators \( T \), that is by combining the right and left-chiral \( L \)-operators in the \( q \)-oscillator representation into a single operator \( \mathbb{L}_k \) and then taking its trace. We will not delve in the detail of this construction as, really, it is a simple variation on the theme of what has been done in the CFT case. We wish instead to present a different approach to the integrability structure of sine–Gordon model, which relies on a surprising and still not completely understood connection between classical and quantum worlds.

Consider the quantum sine–Gordon model as defined by the Lagrangian\(^{28}\)

\[
\mathcal{L}_{sG} = \frac{1}{16\pi} \left[ (\partial_t \phi)^2 - (\partial_u \phi)^2 \right] + 2\mu \cos (\beta \phi),
\]

which is obviously invariant under shifts of the field \( \phi \rightarrow \phi + 2\pi/\beta \). As a consequence, the Hilbert space \( \mathcal{H}_{sG} \) splits into orthogonal subspaces \( \mathcal{H}_k \) characterised by the quasi–momentum \( k \). Denote \( \mathbb{U} \) the operator performing the shift of \( \phi \), then:

\[
\mathbb{U} : \quad \phi \rightarrow \phi + 2\pi \frac{k}{\beta}, \quad \forall \left| \Phi_k \right> \in \mathcal{H}_k.
\]

Let us also introduce the charge and parity operators as:

\[
\mathbb{C}(u,v) = -\phi(u,v), \quad \mathbb{P}(u,v) = \phi(-u,v).
\]

\(^{26}\) Actually, there exists a method [25] which, in theory, allows one to recover non-linear integral equations from the \( Y \)-system for all the eigenvalues.

\(^{27}\) This approach can actually be extended to sinh-Gordon model without too many difficulties.

\(^{28}\) Note that here we set the parameter in the Lagrangian as \( \mu \), while before it was \( \tilde{\mu} \). We hope this will not create too much confusion.
This model possesses an infinite set of $T$-operators $\{T_j\}_{j=1}^\infty$ and two Baxter $Q$-operators $Q_\pm$. Their properties, which are mostly conjectured on the basis of massless, classical and discrete limits analysis, are listed below. From this moment on we will drop the explicit dependence of $T$ and $Q$ on the parameter $\mu$. We will also use the variable $\theta = \log(\lambda^2 + \xi)$ instead of the spectral parameter.

**Properties of $T$-operators**

- **Mutual commutativity:**
  \[[T_j(\theta), T_{j'}(\theta')] = 0,\]

- **Invariance under discrete shift:**
  \[[T_j(\theta), U] = 0,\]

- **Invariance under charge conjugation:**
  \[[T_j(\theta), C] = 0,\]

- **Parity conjugation:**
  \[PT_j(\theta)P = T_j(-\theta),\]

- **Analytic properties:** the functions $T_j(\theta)$ are entire functions of the variable $\theta$ with essential singularities at $\theta \to \pm \infty$.

- **Hermiticity:**
  \[T^\dagger_j(\theta) = T_j(\theta^*),\]

- **Periodicity:**
  \[T_j(\theta + i\pi(1 + \xi)) = T_j(\theta),\]
  note that this property is the translation in terms of $\theta$ of property of single-valuedness of $T_j$ as a function of $\lambda^2$.

- **Fusion relation ($T$-system):**
  \[T_\pm\left(\theta + i\pi \frac{2j + 1}{2}\right) = T_j(\theta) e^{\frac{i\pi\xi(2j + 1)}{2}} T_j(\theta) e^{\frac{2j\pi i}{2}},\]
  or, equivalently
  \[T_j\left(\theta + i\pi \frac{\xi}{2}\right) T_j\left(\theta - i\pi \frac{\xi}{2}\right) = 1 + T_{j+\frac{1}{2}}(\theta) T_{j-\frac{1}{2}}(\theta),\]

- **Asymptotic behaviour on the real line:**
  \[\log(T_j(\theta)) \to \sum_{n=0}^\infty 2(-1)^n \sin(\pi \xi \frac{n}{2}) C_n \cdot 2n! e^{(1-2n)\theta},\]
  \[\log(T_{\pm j}(\theta)) \to \sum_{n=0}^\infty 2(-1)^n \sin(\pi \xi \frac{n}{2}) C_n \cdot 2n+1! e^{(2n+1)\theta},\]
  where we set $I_{-1} = I_{-1} = \frac{\mu}{2\pi}$ and $C_0 = m.$
Properties of $Q$-operators

- **Commutativity:**
  $$[Q_{±}(θ), T_j(θ')] = [Q_{±}(θ), Q_{±}(θ')] = [Q_{+}(θ), Q_{−}(θ')] = 0 ,$$

- **Invariance under discrete shift:**
  $$[Q_{±}(θ), U] = 0 ,$$

- **Charge conjugation:**
  $$\mathcal{C}Q_{±}(θ)\mathcal{C} = Q_{±}(θ) ,$$

- **Parity conjugation:**
  $$PQ_{±}(θ)P = Q_{±}(−θ) ,$$

- **Analytic properties:** the functions $Q_{±}(θ)$ are entire functions of the variable $θ$ with essential singularities at $θ → ±∞$.

- **Hermiticity:**
  $$Q_{±}^†(θ) = Q_{±}(θ^*) ,$$

- **Baxter $T$-Q relation:**
  $$T_{±}(θ)Q_{±}(θ) = Q_{±}(θ + iπξ) + Q_{±}(θ − iπξ) ,$$

- **Shift property:**
  $$Q_{+} (θ + iπ(ξ + 1)) = UQ_{+}(θ) ,$$
  $$Q_{−} (θ + iπ(ξ + 1)) = U^{-1}Q_{−}(θ) ,$$

  this property, along with the $T$-$Q$ relation, can be regarded as defining the $Q$-operators as “Bloch-wave” solutions to a second order finite difference equation,

- **Quantum Wronskian:**
  $$Q_{+} \left(θ + iπ\frac{ξ}{2} \right)Q_{−} \left(θ − iπ\frac{ξ}{2} \right) − Q_{+} \left(θ − iπ\frac{ξ}{2} \right)Q_{−} \left(θ + iπ\frac{ξ}{2} \right) = U^{-1} − U ,$$

- **Wronskian expression of operators $T_j$:**
  $$T_j(θ) (U^{-1} − U) = \left[ Q_{+} \left(θ + iπ\frac{2j + 1}{2} \right)Q_{−} \left(θ − iπ\frac{2j + 1}{2} \right) + \right.$$
  $$\left. − Q_{+} \left(θ − iπ\frac{2j + 1}{2} \right)Q_{−} \left(θ + iπ\frac{2j + 1}{2} \right) \right] ,$$

- **Leading asymptotic:**
  $$Q_{±}(θ) \underset{R(θ) → ∞}{\sim} U^{±\frac{1}{2}}S^{±\frac{1}{2}} \exp \left[ \frac{2MR e^{θ±iπ\frac{ξ}{2} + 1}}{4 \cos \left( π\frac{ξ}{2} \right)} \right] , \quad θ ∈ H_{±} ,$$

  $$Q_{±}(θ) \underset{R(θ) → −∞}{\sim} U^{±\frac{1}{2}}S^{−\frac{1}{2}} \exp \left[ \frac{2MR e^{θ±iπ\frac{ξ}{2} + 1}}{4 \cos \left( π\frac{ξ}{2} \right)} \right] , \quad θ ∈ H_{±} ,$$

  where $H_{±} = \{ θ ∈ \mathbb{C} \mid 0 < ±\Im(θ) < π(ξ + 1) \}$ and $S$ is some operator such that
  $$[S, P] = [S, U] = 0 , \quad \mathcal{C}SC = S^{−1} , \quad S^{†} = S .$$

---

29 This is a consequence of the periodicity of $T_j$. 
4.2 The modified sinh-Gordon equation and its linear problem

We begin our study of quantum sine–Gordon starting from an apparently far away point. Indeed let us consider the following classical partial differential equation

\[ \partial \bar{\partial} \eta(z, \overline{z}) - e^{2\eta(z, \overline{z})} + p(z)e^{-2\eta(z, \overline{z})} = 0, \quad p(z) = z^{2\alpha} - s^{2\alpha}, \]  

(4.2)

where \( z \) and \( \overline{z} \) are formal complex variables in no way related to the space–time of the quantum model. On the other hand, the real and positive parameters \( \alpha \) and \( s \) will be, later, related to parameters of the quantum model. This equation, whose name is modified sinh–Gordon (MshG) equation, arise in the context of differential geometry (see e.g. [55]) where it describes the conformal metric of certain surfaces with smooth constant mean curvature immersed in \( \mathbb{R}^3 \). The recent interest in this equation was sparked by its appearance in the computation of gluon scattering amplitudes in \( \mathcal{N} = 4 \) super Yang-Mills at strong coupling [56]; these can be analysed in terms of classical strings in \( AdS_5 \) which, in turn, lead to the study of minimal surfaces, whence the MshG equation arises. This equation is integrable, as we will see, for any choice of the function \( p \). In our case, the MshG equation (4.2) possesses an evident discrete symmetry

\[ (z, \overline{z}) \longrightarrow (e^{i\alpha} z, e^{-i\alpha} \overline{z}), \]  

(4.3)

and we will restrict our attention to solutions which respect this symmetry. More in detail, it is not difficult to verify that there exists a family of solutions to (4.2), parametrised by the real number \( l \in [\frac{\pi}{2}, \frac{\pi}{4}] \), satisfying the following properties (here we sit on the real slice of \( \mathbb{C}^2 \), by setting \( z = \rho e^{i\phi} \) and \( \overline{z} = \rho e^{-i\phi} \)):

1. **Periodicity:**

\[ \eta(\rho, \phi) = \eta(\rho, \phi + \frac{\pi}{\alpha}), \]

or, in other words, we consider the MshG equation restricted on the cone of apex angle \( \frac{\pi}{\alpha} \)

\[ \mathcal{C}_{\frac{\pi}{\alpha}} \doteq \{(\rho, \phi) \in \mathbb{R}^+ \times [-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}] \mid \phi + \frac{\pi}{\alpha} \sim \phi\}, \]

2. **Analyticity:** the solution \( \eta(\rho, \phi) \) is single-valued, real and finite everywhere on \( \mathcal{C}_{\frac{\pi}{\alpha}} \) with the sole exception of \( \rho = 0, \infty \),

3. **Large-\( \rho \) asymptotics:**

\[ \eta(\rho, \phi) \sim \alpha \log \rho + o(1), \]

4. **Small-\( \rho \) asymptotics:**

\[ \eta(\rho, \phi) \sim \begin{cases} 2l \log \rho + O(1) & |l| < \frac{1}{2} \\ \pm \log \rho + O(\log(-\log \rho)) & l = \pm \frac{1}{2} \end{cases}. \]

Starting from the above small-\( \rho \) asymptotics we can iteratively construct a \((z, \overline{z}) \rightarrow (0, 0)\) expansion of the following form

\[ \eta(z, \overline{z}) = l \log (z\overline{z}) + \eta_0 + \sum_{k=1}^{\infty} \gamma_k (z^{2\alpha k} + \overline{z}^{2\alpha k}) - s^{4\alpha} \frac{e^{-2\eta_0}}{(1 - 2l)^2} (z\overline{z})^{1-2l} + \\
\]

\[ \frac{e^{2\eta_0}}{(1 - 2l)^2} (z\overline{z})^{1+2l} + \cdots, \]

where \( \eta_0 \) and \( \{\gamma_k\}_{k=1}^{\infty} \) are integration constants that are to be determined by imposing the properties listed above on this expansion. The utility of this expression is that it remains valid on the whole \( \mathbb{C}^2 \), which means that we can safely fix \( z \) to some finite value and send only \( \overline{z} \rightarrow 0 \), obtaining

\[ \eta(z, \overline{z}) \sim l \log (z\overline{z}) + \eta_0 + \gamma(z), \quad \gamma(z) = \sum_{k=1}^{\infty} \gamma_k z^{2\alpha k}. \]

\[ ^{30} \text{Note that here } \phi \text{ denotes the argument of } z \text{ and not the sine–Gordon field! Hopefully this will not cause confusion.} \]
As hinted at above, the MshG equation is integrable and, as such, possesses a Lax pair \( \{ \mathcal{D}, \mathcal{F} \} \) and an associated linear problem (from here on, we will omit the explicit dependence on the complex variables unless necessary and denote \( \mathcal{P} \equiv p(\mathcal{P}) \))

\[
\begin{align*}
\mathcal{D} \Psi &= 0 \ , \quad \mathcal{D} \equiv \partial + \frac{1}{2} \partial \eta \sigma^3 - e^\theta (\sigma^+ e^\eta + \sigma^- e^{-\eta}) \\
\mathcal{F} \Psi &= 0 \ , \quad \mathcal{F} \equiv \partial - \frac{1}{2} \partial \eta \sigma^3 - e^{-\theta} (\sigma^- e^\eta + \sigma^+ e^{-\eta}),
\end{align*}
\]  

(4.4)

where \( \Psi \) is a 2D vector function and \( \{ \sigma^3, \sigma^\pm \} \) are the usual Pauli matrices

\[
\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

The parameter \( \tilde{\theta} = \log \tilde{\lambda} \) is called spectral parameter, just as \( \theta = \log \lambda \), even though the two will turn out to be one and the same thing, we prefer to denote them differently to stress the difference in their origin.

### Analysis of the linear problem around \((z, \bar{z}) = 0\)

Clearly the general solution to linear problem are not known, however the knowledge of the asymptotic behaviour of the function \( \eta \) allow us to perform a local analysis at the singular points which, as easily verified, are the same as those of MshG equation. Let us start by analysing the behaviour of solutions around zero. Notice how the linear problem (4.4) is not invariant under the discrete symmetry (4.3); instead it is invariant under a joint rotation of \( \phi \) and \( \tilde{\theta} \):

\[
\tilde{\Omega} : (\phi, \tilde{\theta}) \longrightarrow (\phi + \frac{\pi}{\alpha}, \tilde{\theta} - \frac{\pi}{\alpha}).
\]

This property suggests that a solution \( \Psi \) of the linear problem should be represented, in the neighborhood of zero\(^{32}\), by a function of \( i \phi + \tilde{\theta} \). Another easily verified symmetry is the parity

\[
\tilde{\Pi} : \quad \tilde{\Pi} : \quad \Psi \longrightarrow e^{i \pi \tau} \sigma^3 \Psi , \quad \text{for some } \tau \in \mathbb{R}.
\]

With these facts in mind, we define two particular solutions \( \Psi_{\pm} (\rho, \phi; \tilde{\theta}) \) to the linear problem specifying their asymptotic behaviour (note that we assume \(|l| < \frac{1}{2}\))

\[
\Psi_+ \underset{\rho \to 0}{\sim} \frac{1}{\sqrt{\cos \left( \frac{\pi l}{2} \right)}} \begin{pmatrix} 0 \\ e^{i(\phi + \tilde{\theta}) l} \end{pmatrix}, \quad \Psi_- \underset{\rho \to 0}{\sim} \frac{1}{\sqrt{\cos \left( \frac{\pi l}{2} \right)}} \begin{pmatrix} e^{-i(\phi + \tilde{\theta}) l} \\ 0 \end{pmatrix}.
\]

Here follows a list of easily proven properties of these two solutions

- Analyticity: the solutions \( \Psi_{\pm} (\rho, \phi; \tilde{\theta}) \) are entire functions of \( \tilde{\theta} \) for any \( \phi \in \left[ -\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha} \right] \) and any \( \rho > 0 \).

- \( \tilde{\Omega} \)-invariance

\[
\Psi_{\pm} (\rho, \phi + \frac{\pi}{2\alpha}, \tilde{\theta} - \frac{\pi}{2\alpha}) = \Psi_{\pm} (\rho, \phi - \frac{\pi}{2\alpha}, \tilde{\theta} + \frac{\pi}{2\alpha})
\]

- \( \tilde{\Pi} \)-transformation

\[
\Psi_+ (\rho, \phi; \tilde{\theta} \pm i\pi) = -e^{\pm i\pi l} \sigma^3 \Psi_+ (\rho, \phi; \tilde{\theta}) , \quad \Psi_- (\rho, \phi; \tilde{\theta} \pm i\pi) = e^{\mp i\pi l} \sigma^3 \Psi_- (\rho, \phi; \tilde{\theta})
\]

\(^{31}\) The rotation in \( \tilde{\theta} \) is actually a hyperbolic rotation

\(^{32}\) This argument works around zero, since this is a Fuchsian singularity of the linear problem. The same reasoning does not hold around \( \infty \), since the presence of the potential \( p \) causes the Stokes phenomena to arise. We will return on this later, but, in short words, this means that solutions to (4.4) can be represented correctly as asymptotic expansions only inside wedges of the complex plane; as one tries to move outside these, the control on the asymptotic behavior gets lost.
is uniquely specified by its asymptotic behaviour.

Thus, the line $\pi l < |\phi| < \pi n$, which are entire functions of $\theta$. This immediately means that $\Sigma_-$ as well is entire in $\theta$ and we can thus perform analytic continuation in it. In particular we can exploit the existence of the symmetry $\Omega$, which allows us to generate a new solution starting from $\Sigma_-$:

\[
\Xi_+(\rho, \phi | \tilde{\theta}) = \Omega \left[ \Xi_-(\rho, \phi | \tilde{\theta}) \right] = \Xi_-(\rho, \phi + \frac{\pi}{\alpha} | \tilde{\theta} - i \frac{\pi}{\alpha}) .
\]

One immediately obtain the asymptotic expansion of this function

\[
\Xi_+ \sim -i \left( e^{i \Theta_0} \right) \exp \left[ 2 \frac{e^{\frac{\phi}{\alpha}}}{\alpha + 1} \cosh \left( \tilde{\theta} + i(\alpha + 1)\phi \right) \right] , \quad |\phi| < \frac{\pi}{2(\alpha + 1)} ,
\]

which allows us to compute the determinant

\[
\det |(\Xi_-, \Xi_+)| = -2i .
\]

Thus $\{\Xi_\pm\}$ is another basis of the space of solutions to the linear problem (4.4). This, however, is not the end as we can in fact repeatedly apply the transformation $\Omega$ on $\Xi_-$ and generate an infinite set of solutions

\[
\Xi_n(\rho, \phi | \tilde{\theta}) = \Omega^n \left[ \Xi_-(\rho, \phi | \tilde{\theta}) \right] = \Xi_-(\rho, \phi + \frac{n}{\alpha} | \tilde{\theta} - i \frac{n}{\alpha}) .
\]

Note that these solutions can be interpreted as decaying solutions in the wedge $-\frac{\pi}{2} \frac{1 + 2n}{\alpha - 1} < \phi < \frac{\pi}{2} \frac{1 - 2n}{\alpha - 1}$

**Spectral determinants** Since the two solutions $\Psi_\pm$ are linearly independent, they form a basis of the space of solutions of the linear problem (4.4), meaning that we can expand the solution $\Sigma_-$ as

\[
\Xi_-(\rho, \phi | \tilde{\theta}) = Q_-(\tilde{\theta}) \Psi_+(\rho, \phi | \tilde{\theta}) + Q_+(\tilde{\theta}) \Psi_-(\rho, \phi | \tilde{\theta}) ,
\]

where the connection coefficients

\[
Q_\pm \equiv \pm \cos (\pi l) \det |(\Xi_-, \Psi_\pm)| ,
\]
are functions of $\tilde{\theta}$ and $l$ only. They are also known as spectral determinants for the central problem of (4.4). This last denomination simply means that the zeroes of these functions are precisely the eigenvalues of the linear problem considered on functions in $L^2(0, \infty)$: those values of the spectral parameters for which the function $\Psi_{\pm}$ decays at infinity. The notation chosen for these functions is not random: in the following sub-section we will see how these determining solutions to (4.4) are spectral determinants, this time for the lateral problems of our linear system. These problems consist in determining solutions to (4.4) decaying in both wedges

$$T_n \left( \tilde{\theta} - i\pi \frac{n + 1}{2\alpha} \right) = \frac{1}{2i} \det \left( |\Xi_{2n+1, \Xi_2}| \right) \quad (4.6)$$

are spectral determinants, this time for the lateral problems of our linear system. These problems consist in determining solutions to (4.4) decaying in both wedges

$$-\frac{\pi}{2\alpha} - \frac{1}{2} < \phi < -\frac{\pi}{2\alpha} \quad \text{and} \quad -\frac{\pi}{2\alpha} - \frac{1}{2} < \phi < -\frac{\pi}{2\alpha} - \frac{1}{2}, \quad \text{for } n \in \mathbb{Z},$$

$$-\frac{\pi}{2\alpha} + \frac{1}{2} < \phi < -\frac{\pi}{2\alpha} + \frac{1}{2} \quad \text{and} \quad -\frac{\pi}{2\alpha} + \frac{1}{2} < \phi < -\frac{\pi}{2\alpha} + \frac{1}{2}, \quad \text{for } n \in \mathbb{Z} + \frac{1}{2},$$

and its particular eigenvalues corresponds to the zeroes of the functions $T_j$. Again, the choice of notation for these spectral determinants hints at the fact that they can be interpreted as eigenvalues of the operators $T_j$, as it will be shown later.

4.3 From spectral determinants to the quantum $Q$-operators

Before beginning to unveil the correspondence between the linear problem (4.4) and the quantum world, let us list the properties of the spectral determinants $Q_{\pm}$ introduced just above

- **Analyticity:** $Q_{\pm}(\tilde{\theta})$ are entire functions of $\tilde{\theta}$, note also that the functions are defined for $l = \pm \frac{1}{2}$ by continuity

$$\lim_{l \to \pm \frac{1}{2}} \left( Q_+ (\tilde{\theta}) - Q_- (\tilde{\theta}) \right) = 0$$

- **Quasi-periodicity:**

$$Q_{\pm} \left( \tilde{\theta} + i\pi \frac{\alpha + 1}{2\alpha} \right) = e^{\pm i\pi (l + \frac{1}{2})} Q_{\pm} \left( \tilde{\theta} - i\pi \frac{\alpha + 1}{2\alpha} \right)$$

- **Complex conjugation:**

$$Q_{\pm}^* (\tilde{\theta}) = Q_{\pm} (\tilde{\theta}^*), \quad \forall \tilde{\theta} \in \mathbb{R},$$

- **Parity symmetry:**

$$Q_{\pm} (\tilde{\theta}) = Q_{\mp} (-\tilde{\theta}), \quad \forall \tilde{\theta} \in \mathbb{R},$$

- **Quantum Wronskian:**

$$Q_+ \left( \tilde{\theta} + i\frac{\pi}{2\alpha} \right) Q_- \left( \tilde{\theta} - i\frac{\pi}{2\alpha} \right) - Q_- \left( \tilde{\theta} + i\frac{\pi}{2\alpha} \right) Q_+ \left( \tilde{\theta} - i\frac{\pi}{2\alpha} \right) = -2i \cos (\pi l) \quad (4.7)$$

In order to proceed, it is convenient to define the following single function of two variables

$$Q(\tilde{\theta}, \tilde{k}) \doteq \begin{cases} Q_+ (\tilde{\theta}) \Big|_{l = 2k - \frac{1}{2}} & 0 < k < \frac{1}{2} \\ Q_- (\tilde{\theta}) \Big|_{l = -2k - \frac{1}{2}} & -\frac{1}{2} < k < 0 \end{cases}$$
where \( \tilde{k} = 0 \) is treated by continuity. Since, obviously, \( Q(\tilde{\theta}, \tilde{k}) = Q(\tilde{\theta}, \tilde{k} + 1) \), this function admits an analytic extension to all \( \tilde{k} \in \mathbb{R} \). The properties above become in term of this function

\[
Q \left( \tilde{\theta} + i\pi \frac{\alpha}{2\alpha} + \frac{1}{\alpha}, \tilde{k} \right) = e^{2i\pi k} Q(\tilde{\theta}, \tilde{k}) ,
\]

\[
Q^*(\tilde{\theta}, \tilde{k}) = Q(-\tilde{\theta}, \tilde{k}) ,
\]

\[
Q(\tilde{\theta} + i\pi \frac{\alpha}{2\alpha} + \frac{1}{\alpha} - \tilde{k}) = Q \left( \tilde{\theta} - i\pi \frac{\alpha}{2\alpha}, \tilde{k} \right) - Q \left( \tilde{\theta} - i\pi \frac{\alpha}{2\alpha}, \tilde{k} \right) - Q \left( \tilde{\theta} + i\pi \frac{\alpha}{2\alpha}, -\tilde{k} \right) - 2i \sin \left( 2\pi \tilde{k} \right) .
\]

The similarity of the above properties with the one listed at the beginning of this section for the operators \( Q_{\pm} \) are striking, however, in order to univocally fix the function \( \tilde{\theta} \) we still need to determine its asymptotic behaviour and the distribution of its zeroes. Thanks to the property of periodicity we can concentrate on the strip \( \tilde{H} : \tilde{H} = \tilde{H} \cup \tilde{H} \) where \( \tilde{H} = \{ \tilde{\theta} \in \mathbb{C} \setminus 0 < \pm \Re(\tilde{\theta}) < \pi \} \). With a careful and thorough WKB analysis of the solutions of the linear problem, it is possible to establish the following behaviours

\[
Q \sim \mathcal{J}(\tilde{k}) \exp \left[ r \frac{e^{\tilde{\theta} \pm i\pi \frac{\alpha}{2\alpha}}}{4 \cos \left( \frac{\pi}{2\alpha} \right)} \right] , \quad \tilde{\theta} \in \tilde{H} ,
\]

where we introduced \( r = B s^{1+\alpha} \) with \( B = 2 \sqrt{\pi \Gamma \left( \frac{1+\alpha}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right)} \) and the function

\[
\mathcal{J}(\tilde{k}) = \frac{\Gamma \left( \frac{2\tilde{k}}{\alpha} \right)}{\Gamma \left( 1 - \frac{2\tilde{k}}{\alpha} \right)} 2^{\frac{4\tilde{k} - 1}{2\alpha}} e^{\frac{2\pi}{\alpha}}, \quad 0 \leq \tilde{k} \leq \frac{1}{2} ,
\]

which enjoys the symmetries

\[
\mathcal{J}(\tilde{k}) \mathcal{J}(-\tilde{k}) = 1 , \quad \mathcal{J}(\tilde{k} + 1) = \mathcal{J}(\tilde{k}) .
\]

Now that we verified that the asymptotic of the function \( Q \) has exactly the same form of the operator \( Q_{\pm} \) (for \( \tilde{k} > 0 \)), the fact that we are actually dealing with an eigenvalue of this last in disguise is becoming more than a simple suspect. What is left to do is to identify which is this specific eigenvalue by studying the pattern of the zeroes of \( Q(\tilde{\theta}) \): this is done again by thorough WKB analysis (remember that they are the eigenvalues of a central problem for the system (4.4)). As it turns out, for any \( \tilde{k} \in \mathbb{R} \), these zeroes are real, simple, symmetrically disposed with respect to the origin and accumulating at the singularities \( \tilde{\theta} \rightarrow \pm \infty \), which identifies the spectral determinants \( Q(\tilde{\theta}, \pm \tilde{k}) \) with the eigenvalues \( Q_{\pm}(\tilde{\theta}) \) of the operator \( Q_{\pm}(\tilde{\theta}) \) on the vacuum state with quasi-momentum \( k \). The parameters on the two sides of this correspondence have to be identified as

\[
\alpha = \frac{1}{\xi} = \frac{1}{\beta^2} - 1 , \quad \tilde{k} = k , \quad (4.8)
\]

\[
r = 2M R \Rightarrow s = \left( \frac{R}{\pi \beta^2} \right)^{\beta^2} \left[ \frac{\mu \pi}{\Gamma (\beta)} \right]^{\frac{2}{1 - 2\beta^2}} , \quad (4.9)
\]

where \( M \) is the soliton mass of quantum sine-Gordon (3.7) and \( \mu \) is the parameter appearing in the Lagrangian (4.1).
The NLIE equation

Just as we did in the previous section, we wish to use the analytic properties of the spectral determinants/$Q$-functions to derive a NLIE equation, as a further verification of the identification we performed. From now on we will drop the tildas on $\theta$ and $k$ and use equivalently $\alpha$, $\xi$ or $\beta$, depending on notational reasons. Consider the following function

$$
\varepsilon(\theta) = i \log \left[ \frac{Q(\theta + i\pi \xi, k)}{Q(\theta - i\pi \xi, k)} \right],
$$

with the branch of the logarithm fixed so that

$$
\varepsilon(\theta) - r \frac{e^\theta}{2 \cos \left( \pi \xi \frac{2}{2} \right)} \underset{\theta \to \infty}{\sim} -2\pi k, \quad |\Im(\theta)| < \frac{\pi}{2}.
$$

Thanks to this function we can label univocally the zeroes of $Q$ by integers $n \in \mathbb{Z}$ in such a way that

$$
\theta_n < \theta_{n+1}, \quad \varepsilon(\theta_n) = \pi(2n + 1),
$$

where the last relation descends from the Quantum Wronskian relation. The zeroes $\{\theta_n\}_{n \in \mathbb{Z}}$ are more conveniently represented in the following form

$$
e^{2\theta_n \pi^{-1}} = \begin{cases} 
    s^{-2\alpha} E_n(k) & n \geq 0 \\
    s^{2\alpha} E_{-n-1}(-k) & n < 0
\end{cases}, \quad 0 \leq k \leq \frac{1}{2},
$$

which makes more explicit the symmetry $n \to -n - 1$. The *zeroes* $\{E_n\}_{n=0}^\infty$ are functions of $k$ and satisfy the following relations

$$
E_n(k + 1) = E_{n+1}(k), \quad E_0(-k)E_0(k) = s^{4\alpha},
$$

and have the following asymptotic behaviour

$$
E_n(\pm k) \underset{n \to \infty}{\sim} \left[ \frac{2\pi}{B} (2n \pm 2k + 1) \right]^{2\pi^{-1}},
$$

which, again, can be obtained by thorough WKB analysis of the linear problem. We have now all the informations needed to express the function $Q$ as a Hadamard product

$$
Q(\theta, k) = \mathcal{C}(k) e^{2k\theta \pi^{-1}} \prod_{n=0}^\infty \left( 1 - s^{2\alpha} e^{2\theta \pi^{-1}} E_n(k) \right) \left( 1 - s^{-2\alpha} e^{-2\theta \pi^{-1}} E_n(-k) \right),
$$

which converges only for $\alpha > 1$; if one wishes to extend the above product to the region $0 < \alpha \leq 1$, then a Weierstrass prime multiplier has to be added in order to regulate the divergency of the product [58]. The normalisation in front of the product satisfies the following relations

$$
\mathcal{C}(k) = \mathcal{C}(-k), \quad \mathcal{C}(k) = -s^{-2\alpha} E_0(k) \mathcal{C}(k + 1).
$$

We can finally write down the NLIE equation for the function $\varepsilon$ by combining the Hadamard representation of the $Q$, its analytic properties, its asymptotic behaviour and the equation $\varepsilon(\theta_n) = \pi(2n + 1)$, obtaining

$$
\varepsilon(\theta) = -2\pi k + r \sinh \theta - \frac{2}{\infty} \int_{-\infty}^\infty d\theta' G(\theta - \theta') \Im \left[ \log \left( 1 + e^{-i\varepsilon(\theta' - i0)} \right) \right],
$$

where we introduced the kernel

$$
G(\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{\sinh \left( \pi \nu \frac{1}{2} \right)}{2 \cosh \left( \pi \nu \frac{2}{2} \right)} \sinh \left( \pi \nu \frac{\theta}{2\alpha} \right) e^{i\theta\nu}.
$$
As was expected, this equation coincides exactly with the NLIE equation for the ground state of quantum sine–Gordon model [39], given the identifications (4.8–4.9) are made. Once \( \varepsilon \) is known, one can then recover the function \( Q \) from the following formula

\[
\log \left( Q(\theta + i \pi \frac{\alpha + 1}{2\alpha}, k) \right) = \frac{r \cosh \theta}{2 \cos \frac{\pi}{2\alpha}} + i \pi k + \frac{1}{2} \log (\mathcal{I}(k)) + 2i \int_{-\infty}^{\infty} d\theta' \Im \left[ F(\theta - \theta' - i0) \log \left( 1 + e^{-i\varepsilon(\theta' - i0)} \right) \right],
\]

(4.10)

where the following kernel was introduced

\[
F(\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{e^{i\nu\theta}}{4 \cosh \left( \frac{\pi \nu}{2\alpha} \right) \sinh \left( \frac{\pi \nu}{2\alpha} \right)}.
\]

The formula (4.10) is actually valid for \( \Im(\theta) = 0 \) only; however it is possible to suitably modify it so that it provides \( Q \) in the whole strip \( H_+ \). The function \( \mathcal{I}(k) \) can also be recovered from the solution to the NLIE equation

\[
\log \left( \mathcal{I}(k) \right) = \alpha \int_{-\infty}^{\infty} \frac{d\theta}{\pi} \Im \left[ \log \left( 1 + e^{-i\varepsilon(\theta - i0)} \right) \right].
\]

Before closing this in–depth box we “close the circle” by recovering the integrals of motion from the asymptotic expansion of the function \( Q \). For this task the equation (4.10) is perfect, since it allows straightforward evaluation of the large-|\( \theta \)| expansion. This most simply involves writing \( F(\theta) \) as an sum over his residues in the correct half plane and plugging it in the formula for the function \( Q \). The result is

\[
\log \left( Q(\theta + i \pi \frac{\alpha + 1}{2\alpha}, k) \right) \sim -\frac{r}{4 \cos \left( \frac{\pi}{2\alpha} \right)} \frac{(-1)^{n+1}}{\sin \left( \frac{2n\pi}{2\alpha} \right)} \int_{-\infty}^{\infty} d\theta \Im \left[ e^{\pm(2n-1)\theta} - G_{\pm 2n} e^{\mp 2n\alpha \theta} \right],
\]

where we defined the following objects

\[
\mathcal{I}_{\pm(2n-1)} \equiv -\frac{r}{4 \cos \left( \frac{\pi}{2\alpha} \right)} \delta_{n,1} \pm \frac{(-1)^{n+1}}{\sin \left( \frac{2n\pi}{2\alpha} \right)} \int_{-\infty}^{\infty} d\theta \Im \left[ e^{\mp(2n-1)i0} \log \left( 1 + e^{-i\varepsilon(\theta - i0)} \right) \right],
\]

\[
G_{\pm 2n} \equiv \pm \frac{\alpha(-1)^{2n}}{\cos \left( 2\pi \alpha n \right)} \int_{-\infty}^{\infty} d\theta \Im \left[ e^{\pm 4\alpha n(\theta - i0)} \log \left( 1 + e^{-i\varepsilon(\theta - i0)} \right) \right].
\]

which are related to the local and non–local integrals of motion, e.g.

\[
\mathcal{I}_{2n-1} = \mathcal{C}_n I_{2n-1}(k), \quad \forall n > 0,
\]

\[
\mathcal{F}_{2n-1} = \mathcal{C}_n I_{2n-1}(k),
\]

\[
\mathcal{I}_{2n-1} \left| \Phi^{(\text{vac})}_k \right. = I_{2n-1} \left| \Phi^{(\text{vac})}_k \right. ,
\]

\[
\mathcal{F}_{2n-1} \left| \Phi^{(\text{vac})}_k \right. = I_{2n-1} \left| \Phi^{(\text{vac})}_k \right. ,
\]

with

\[
\mathcal{C}_n \equiv \left( -\frac{\alpha^2}{\alpha + 1} \right)^{n-1} \frac{\Gamma \left( \frac{2n-1}{2\alpha} \right)}{2 \sqrt{\pi} n!} \frac{2 \sin \left( \frac{\pi}{2\alpha} \right)}{\sqrt{\alpha}} \Gamma \left( \frac{\alpha + 1}{2\alpha} \right) \Gamma \left( \frac{1}{2\alpha} \right)^{n-1}.
\]

Similar formulae exist for the non–linear integrals of motion.
4.4 The $T$-functions

We wish to conclude this review by pointing out the connection of the spectral determinants (4.6) to the $T$-functions of Quantum sine-Gordon. In order to bring it into light, we can expand the $\Xi$ functions entering the definition of $T_j$ in terms of the basis $\Psi_\pm$, remembering that $\hat{\Omega}[\Psi_\pm] = \Psi_\pm$ and then use the quantum Wronskian relation (4.7). The result is, as expected

$$T_j(\theta) = \frac{i}{2 \cos(\pi l)} \left[ Q_+ \left( \theta + i \pi \frac{2j+1}{2\alpha} \right) Q_- \left( \theta - i \pi \frac{2j+1}{2\alpha} \right) + \right. $$

$$ \left. - Q_+ \left( \theta - i \pi \frac{2j+1}{2\alpha} \right) Q_- \left( \theta + i \pi \frac{2j+1}{2\alpha} \right) \right], \quad (4.11)$$

which is precisely the Wronskian expression for $T$-functions of the quantum sine-Gordon model. We easily see that this relation directly implies the validity of the $T$-system

$$T_{-\frac{l}{2}}(\theta) = 0, \quad T_0(\theta) = 1,$$

To perform the precise identification between a spectral determinant $T$ and the $k$-vacuum eigenvalue of the operator $T_j$, one has to analyse the analytic properties of the first and compare them with the last's ones. At this point it is not a surprise anymore to find out that these properties match exactly, allowing us a perfect identification of the results given in this section with those presented in Secs 2 and 3. We will not present these calculations here, but encourage the interested reader to go through them.

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