Polar transform of Spacelike isothermic surfaces in 4-dimensional Lorentzian space forms

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Dedicated to Professor Udo Simon on the occasion of his 70th birthday

Abstract

The conformal geometry of spacelike surfaces in 4-dimensional Lorentzian space forms has been studied by the authors in a previous paper, where the so-called polar transform was introduced. Here it is shown that this transform preserves spacelike conformal isothermic surfaces. We relate this new transform with the known transforms (Darboux transform and spectral transform) of isothermic surfaces by establishing the permutability theorems.

Keywords: Spacelike isothermic surfaces; polar transform; Darboux transform; spectral transform; permutability theorem.

1 Introduction

Isothermic surfaces are classical objects in differential geometry. The most beautiful results about them are those transforms producing new isothermic surfaces, such as the dual isothermic surface (also named the Christoffel transform), the spectral transform (also known as the T-transform, the Bianchi transform or the Calapso transform), and the

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Darboux transform. In particular, people established the permutability theorems relating them (see [9] for an overview). These facts indicate that there is a structure of integrable system underlying the theory about isothermic surfaces, which was revealed only in the past 20 years [6, 4, 3, 2].

For Lorentzian space forms there is also a parallel theory of conformal geometry [1]. Thus it is natural to study isothermic surfaces in this context [7, 8]. Zuo et al [12] generalized the Darboux transform of isothermic surfaces to the pseudo-Riemannian space forms using the methods developed by Burstall in [3] and Bruck et al in [2]. Their methods mainly concerned the integrable system aspect of the theory.

In [11] we studied spacelike surfaces in $Q^4_1$, the conformal compactification of the 4-dimensional Lorentzian space forms $R^4_1$, $S^4_1$, and $H^4_1$. The key observation is that in this codim-2 case, the normal plane at any point is Lorentzian. The two null lines $[L]$, $[R]$ in this plane define two conformal maps into $Q^4_1$, called the left and the right polar surface, respectively. Conversely, $\hat{Y}$ is also the right polar surface of $[L]$, and the left polar surface of $[R]$ (when $[L]$ and $[R]$ are immersions). Applying these transforms successively, we obtain a sequence of conformal immersions. We proved in [11] that these transforms preserve the Willmore property.

The first main result in this paper is that the isothermic property is also invariant under the polar transform (Theorem 5.3). A new isothermic surface produced in this way is neither the spectral transform nor the Darboux transform of $[Y]$. Hence it turns out to be a new transform for isothermic surfaces. (The authors note that similar results hold for timelike Willmore surfaces and for timelike isothermic surfaces in $Q^4_2$, which might be treated in another paper.)

It is natural to wonder about the relationship between this new transform and the old ones. In particular, does the polar transform commute with the spectral transform or the Darboux transform? The answer is affirmative. Two of such permutability theorems are established at here. See Theorem 6.1 and Theorem 7.1.
This paper is organized as follows. In Section 2 and Section 3 we review the main theory about the Lorentzian conformal space \( Q^4_1 \) and spacelike surfaces in it. The definition and examples of isothermic surfaces are discussed in Section 4. Then we introduce the polar transform of spacelike isothermic surfaces in Section 5. Finally, after describing the spectral transform and Darboux transform of an isothermic surface, we establish the commutability between them and the polar transform in Section 6 and Section 7 separately.

## 2 The Lorentzian conformal space \( Q^4_1 \)

Let \( \mathbb{R}^n_s \) denote the space \( \mathbb{R}^n \) equipped with the quadratic form

\[
\langle x, x \rangle = \sum_{i=1}^{n-s} x_i^2 - \sum_{i=n-s+1}^{n} x_i^2.
\]

In this paper we will mainly work with \( \mathbb{R}^6_2 \), whose light cone is denoted as \( C^5 \). The quadric

\[
Q^4_1 = \{ [x] \in \mathbb{R}P^5 \mid x \in C^5 \setminus \{0\}\}
\]

is exactly the projective light cone with the projection map \( \pi : C^5 \setminus \{0\} \rightarrow Q^4_1 \). It is easy to see that \( Q^4_1 \) is equipped with a Lorentzian metric \( h \) induced from projection \( S^3 \times S^1 \rightarrow Q^4_1 \). Here

\[
S^3 \times S^1 = \{ x \in \mathbb{R}^6 \mid \sum_{i=1}^{4} x_i^2 = x_5^2 + x_6^2 = 1 \} \subset C^5 \setminus \{0\} \quad (1)
\]

is endowed with the Lorentzian metric \( g(S^3) \oplus (-g(S^1)) \), where \( g(S^3) \) and \( g(S^1) \) are standard metrics on \( S^3 \) and \( S^1 \). The conformal group of \( (Q^4_1, [h]) \) is exactly the orthogonal group \( O(4,2)/\{\pm1\} \), which keeps the inner product of \( \mathbb{R}^6_2 \) invariant and acts on \( Q^4_1 \) by

\[
T([x]) = [xT], \quad T \in O(4,2). \quad (2)
\]

As in Moebius geometry, \( Q^4_1 \) serves as the common conformal compactification of the three 4-dimensional Lorentzian space forms given below,
each with constant sectional curvature $c = 0, +1, -1$:

\[ R^4_1, \ c = 0; \]
\[ S^4_1 := \{ x \in \mathbb{R}^5_1 \mid \langle x, x \rangle = 1 \}, \ c = 1; \]
\[ H^4_1 := \{ x \in \mathbb{R}^5_2 \mid \langle x, x \rangle = -1 \}, \ c = -1. \]

The conformal embedding into $Q^4_1$ for each of them is

\[ \varphi_0 : R^4_1 \to Q^4_1, \quad \varphi_0(x) = \left[ \left( \frac{-1 + \langle x, x \rangle}{2}, \frac{1 + \langle x, x \rangle}{2} \right) \right]; \]
\[ \varphi_+ : S^4_1 \to Q^4_1, \quad \varphi_+(x) = [(x, 1)]; \]
\[ \varphi_- : H^4_1 \to Q^4_1, \quad \varphi_-(x) = [(1, x)]. \]

Thus $Q^4_1$ is the proper space to study the conformal geometry of these Lorentzian space forms.

We also have round spheres as the most important conformally invariant objects in $Q^4_1$. Here we only discuss round 2-spheres (they were named conformal 2-spheres in [1]). Each of them could be viewed as a geodesic 2-sphere in a 3-dim Lorentzian space form. Alternatively, a round 2-sphere is identified with a 4-dim Lorentzian subspace in $\mathbb{R}^6_2$.

Given such a 4-space $V$, the round 2-sphere is given by

\[ S^2(V) := \{ [v] \in Q^4_1 \mid v \in V \}. \]

## 3 Spacelike surfaces in $Q^4_1$

For a surface $y : M \to Q^4_1$ and any open subset $U \subset M$, a local lift of $y$ is just a map $Y : U \to C^5 \setminus \{0\}$ such that $\pi \circ Y = y$. Two different local lifts differ by a scaling, so the metric induced from them are conformal to each other.

Let $M$ be a Riemann surface. An immersion $y : M \to Q^4_1$ is called a conformal spacelike surface if $\langle Y_z, Y_{\bar{z}} \rangle = 0$ and $\langle Y_z, Y_{\bar{z}} \rangle > 0$ for any local lift $Y$ and any complex coordinate $z$ on $M$. For such a surface there is a decomposition $M \times \mathbb{R}^6_2 = V \oplus V^\perp$, where

\[ V = \text{Span}\{Y, \text{Re}(Y_z), \text{Im}(Y_z), Y_{z\bar{z}}\} \]

is a Lorentzian rank-4 subbundle independent to the choice of $Y$ and $z$. $V^\perp$ is also a Lorentzian subbundle, which might be identified with the
normal bundle of $y$ in $Q_4^1$. Their complexifications are denoted separately as $V_C$ and $V_C^\perp$.

Fix a local coordinate $z$. There is a local lift $Y$ satisfying $|dY|^2 = |dz|^2$, called the canonical lift (with respect to $z$). Choose a frame $\{Y, Y_z, Y_{\bar{z}}, N\}$ of $V_C$, where $N \in \Gamma(V)$ is uniquely determined by

$$\langle N, Y_z \rangle = \langle N, Y_{\bar{z}} \rangle = \langle N, N \rangle = 0, \langle N, Y \rangle = -1.$$  \hspace{1cm} (5)

For $V^\perp$ which is a Lorentzian plane at every point of $M$, a natural frame is $\{L, R\}$ such that

$$\langle L, L \rangle = \langle R, R \rangle = 0, \langle L, R \rangle = -1.$$  \hspace{1cm} (6)

Given frames as above, we note that $Y_{zz}$ is orthogonal to $Y$, $Y_z$ and $Y_{\bar{z}}$. So there must be a complex function $s$ and a section $\kappa \in \Gamma(V_C^\perp)$ such that

$$Y_{zz} = -\frac{s^2}{2} Y + \kappa.$$  \hspace{1cm} (7)

This defines two basic invariants $\kappa$ and $s$ dependent on $z$. Similar to the case in Möbius geometry, $\kappa$ and $s$ are called the conformal Hopf differential and the Schwarzian derivative of $y$, respectively (see [5],[10]).

Decompose $\kappa$ as

$$\kappa = \lambda_1 L + \lambda_2 R.$$  \hspace{1cm} (8)

Let $D$ denote the normal connection, i.e. the induced connection on the bundle $V^\perp$. We have

$$D_z L = \alpha L, \quad D_z R = -\alpha R$$

for the connection 1-form $\alpha dz$. Denote

$$\langle \kappa, \bar{\kappa} \rangle = -\beta, \quad D_{\bar{z}} \kappa = \gamma_1 L + \gamma_2 R,$$  \hspace{1cm} (9)

where

$$\left\{ \begin{array}{l}
\beta = \lambda_1 \bar{\lambda}_2 + \lambda_2 \bar{\lambda}_1, \\
\gamma_1 = \lambda_1 \bar{\alpha} + \lambda_\bar{\alpha}, \\
\gamma_2 = \lambda_2 \bar{\alpha} - \lambda_\bar{\alpha}.
\end{array} \right.$$  \hspace{1cm} (10)

The structure equations are given as follows:

$$\begin{pmatrix}
Y_{zz} = -\frac{s^2}{2} Y + \lambda_1 L + \lambda_2 R, \\
Y_{\bar{z}z} = \beta Y + \frac{1}{2} N, \\
N_{\bar{z}} = 2\beta Y_{\bar{z}} - sY_{\bar{z}} + 2\gamma_1 L + 2\gamma_2 R, \\
L_z = \alpha L - 2\gamma_2 Y + 2\lambda_2 Y_{\bar{z}}, \\
R_z = -\alpha R - 2\gamma_1 Y + 2\lambda_1 Y_{\bar{z}}.
\end{pmatrix}$$  \hspace{1cm} (11)
The conformal Gauss, Codazzi and Ricci equations as integrable conditions are:

\[
\begin{align*}
  s_z &= -2\beta_z - 4\lambda_1 \gamma_2 - 4\lambda_2 \bar{\gamma}_1, \\
  \text{Im}(\gamma_1 z + \gamma_1 \bar{\alpha} + \frac{i}{2} \lambda_1) &= 0, \\
  \text{Im}(\gamma_2 z - \gamma_2 \bar{\alpha} + \frac{i}{2} \lambda_2) &= 0, \\
  \alpha_z - \bar{\alpha}_z &= 2(\lambda_1 \lambda_2 - \lambda_2 \lambda_1).
\end{align*}
\] (12)

\section{Spacelike isothermic surfaces}

\textbf{Definition 4.1.} Let \( y : M \to Q^4_1 \) be a conformal spacelike surface without umbilic points. It is called isothermic if around each point of \( M \) there exists a complex coordinate \( z \) and canonical lift \( Y \) such that the Hopf differential \( \kappa \) is real-valued. Such a coordinate \( z \) is called an adapted coordinate.

Since \( \kappa \) is real-valued, from the conformal Ricci equations in (12) we see that its normal bundle is flat. This is an important property of isothermic surfaces, which guarantees that all shape operators commute and the curvature lines could still be defined. Indeed we can equivalently define \( y \) to be isothermic if it has flat normal bundle and if it has conformal curvature line parameters. Put differently, the two fundamental forms of an isothermic surface are of the form

\[
I = e^{2\omega}(du^2 + dv^2), \quad II = (b_1 du^2 + b_2 dv^2)e_3 + (b_3 du^2 + b_4 dv^2)e_4 \tag{13}
\]

with respect to some parallel normal frame \( \{e_3, e_4\} \). Then \((u, v)\) are curvature line parameters and \( z = u + iv \) is an adapted complex coordinate.

Our definition generalizes the notion of isothermic surfaces in 3-dim space forms and includes them as special cases. In the following we provide more examples of isothermic surfaces in \( Q^4_1 \).

\textbf{Example 4.2.} Rotational surfaces in \( \mathbb{R}^3 \) are isothermic as well known.

To generalize this construction, consider a spacelike curve \( \gamma(u) = (0, f(u), g(u), h(u)) : \mathbb{R} \to \mathbb{R}_1^4 \) such that \( f(u) \neq 0, f'(u) \neq 0, g'(u) \neq h'(u). \) A rotational surface \( x : \mathbb{R} \times [0, 2\pi] \to R_1^4 \) generated by \( \gamma \) is just

\[
x(u, v) = \left( f(u) \cos v, f(u) \sin v, g(u), h(u) \right).
\]

It is easy to verify that (13) is satisfied when \( u \) is reparameterized suitably.
Example 4.3. In \[11\] we constructed a class of homogenous spacelike tori as below, which are both Willmore and isothermic. Set $\psi = \psi(t, \theta) = \theta/\sqrt{t^2 - 1}$. Then $Y_t(\theta, \phi) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_2^6$ is given by

$$Y_t(\theta, \phi) = \left( \cos(t\psi) \cos \phi, \cos(t\psi) \sin \phi, \sin(t\psi) \cos \phi, \sin(t\psi) \sin \phi, \cos \psi, \sin \psi \right).$$

Note that the period condition is satisfied if $t$ is a rational number; hence after projection $\pi$ we obtain an immersed torus. For the details see \[11\].

5 Polar transform of isothermic surfaces

For a conformal spacelike surface $y : M \to Q^4_1$ with canonical lift $Y : M \to \mathbb{R}^6_2$ with respect to complex coordinate $z = u + iv$, its normal plane at any point is spanned by two lightlike vectors $L, R$, determined up to a real factor around each point. Suppose that $\mathbb{R}^6_2$ is endowed with a fixed orientation and that

$$\{Y, Y_u, Y_v, N, R, L\}$$

form a positively oriented frame. $\{R, L\}$ might also be viewed as a frame of the normal plane compatible with the orientation of $M$ and that of the ambient space. Since $\langle L, R \rangle = -1$ has been fixed in \[11\], either one of the null lines $[L]$ ($[R]$) is well-defined.

Definition 5.1. The two maps $[L], [R] : M^2 \to Q^4_1$ are named the left and the right polar surface of $y = [Y]$, respectively.

Alternatively sometimes we call $[L], [R]$ the (left and right) polar transforms of $[Y]$. An interesting fact showed in \[11\] is that they again produce conformal mappings; Moreover we have a duality in this construction:

Proposition 5.2 (\[11\]). The polar surfaces $[L], [R] : M^2 \to Q^4_1$ are both conformal maps. $[L]$ ($[R]$) is degenerate if, and only if, $\lambda_2 = 0$ ($\lambda_1 = 0$); it is a spacelike immersion otherwise. The original surface $[Y]$ is the left polar surface of $[R]$ (the right polar surface of $[L]$) when $[R]$ ($[L]$) is not degenerate.

In \[11\] we have shown that the polar transforms of a spacelike Willmore surface are again Willmore. Here we want to show that a similar result holds true for isothermic surfaces.
Theorem 5.3. Let \( y : M \to Q^4_1 \) be a spacelike isothermic surface. Then its left and right polar surfaces \([L], [R] : M \to Q^4_1\) are also spacelike isothermic surfaces when they are not degenerate. In particular they share the same adapted coordinate \( z \).

Proof. Let \( Y : M \to \mathbb{R}_2^6 \) be the canonical lift and \( \kappa \) be the real-valued conformal Hopf differential for an adapted isothermic coordinate \( z \). We show the conclusion for \([L]\). For \([R]\) the proof is similar.

Assume that the left polar surface \([L]\) is an immersion, i.e. \( \lambda_2 \neq 0 \). Choosing \( L \) such that \( \kappa = \lambda_1 L + \lambda_2 R \) with \( \lambda_2 = \frac{1}{2} \), by (11) we have
\[
L_z = \alpha L + \bar{\alpha} Y + Y_{\bar{z}}. \tag{14}
\]
Thus \( L \) is the canonical life of \([L] : M \to Q^4_1\) as desired. To determine the normal bundle of \([L]\), we differentiate once more and invoke (11), obtaining
\[
L_{zz} = (\alpha_{\bar{z}} + \lambda_1)L + \frac{1}{2} \left[ R + 2\alpha Y_{\bar{z}} + 2\bar{\alpha} Y_z + 2|\alpha|^2 L + 2(\bar{\alpha}_{\bar{z}} + \alpha^2 - \frac{s}{2})Y \right]. \tag{15}
\]
We point out that each of \( \alpha_{\bar{z}}, \lambda_1 \) and \( \alpha_{\bar{z}} - \alpha^2 - \frac{s}{2} \) is real valued (or by the Codazzi and Ricci equations (12)). Now we can verify directly that \( Y \) and
\[
\hat{Y} = N + 2\alpha Y_{\bar{z}} + 2\bar{\alpha} Y_z + 2|\alpha|^2 Y - 2(\alpha_{\bar{z}} - \alpha^2 - \frac{s}{2})L \tag{16}
\]
are two lightlike vectors in the orthogonal complement of \( \text{Span}\{L, L_u, L_v, L_{\bar{z}\bar{z}}\} \) with \( \langle Y, \hat{Y} \rangle = -1 \). Differentiate (14) at both sides. After simplification we get
\[
L_{zz} = (2\alpha_{\bar{z}} - \frac{s}{2})L + \frac{1}{2} \hat{Y} + (\bar{\alpha}_{\bar{z}} + \lambda_1)Y. \tag{17}
\]
By definition, the conformal Hopf differential of \( L \) is given by \( \kappa_L = -\frac{1}{2} \hat{Y} - (\bar{\alpha}_{\bar{z}} + \lambda_1)Y \), which is obviously real-valued. This shows that \([L]\) is isothermic with the same adapted coordinate. \( \square \)

Note that \( \{L, L_u, L_v, L_{\bar{z}\bar{z}}, Y, \hat{Y}\} \) is again a positively oriented frame. So \([Y]\) and \([\hat{Y}]\) is the right and the left polar surface of \([L]\), respectively. This proves the conclusion of Proposition 5.2 in this special case. On the other hand, \([\hat{Y}]\) is the left polar surface of \([L]\), hence the 2-step left polar transform of \([Y]\).
6 Permutability with spectral transform

Let $y : M \to Q_1^4$ be an immersed spacelike isothermic surface with canonical lift $Y : M \to \mathbb{R}^6_2$ with respect to an adapted coordinate $z$. The conformal Gauss, Codazzi, and Ricci equations are still satisfied under the deformation

$$s^c = s + c, \quad \lambda_1^c = \lambda_1, \quad \lambda_2^c = \lambda_2, \quad \alpha^c = \alpha,$$

where $c \in \mathbb{R}$ is a real parameter. By the integrable conditions, there are an associated family of non-congruent isothermic surfaces $[Y^c]$ with corresponding invariants. Similar to the case of Möbius geometry, they are called the spectral transforms of the original surface (see [5]). Observe that they are conformal and share the same adapted coordinate $z$.

Now we have two transforms, the polar transform and the spectral transform, associated with an isothermic surface. The permutability between them is established as below.

**Theorem 6.1.** Let $y^c$ be a spectral transform (with parameter $c$) of $y : M \to Q_1^4$, both being spacelike isothermic surfaces. Denote their canonical lift as $[Y], [Y^c]$ for the same adapted coordinate $z$. If the left polar surface $[L]$ and $[L^c]$ corresponding to them are non-degenerate, then $[L^c]$ is also a spectral transform (with parameter $c$) of $[L]$, i.e., we have the commuting diagram:

$$
\begin{array}{ccc}
[Y] & \longrightarrow & [Y^c] \\
\downarrow & & \downarrow \\
[L] & \longrightarrow & [L^c]
\end{array}
$$

A similar result holds between the right polar transform and the spectral transform.

**Proof.** Set $Y_{zz} = -\frac{s}{2}Y + \lambda_1 L + \lambda_2 R$, and $D_2 L = \alpha L$. Choose $L$ such that $\lambda_2 = \frac{1}{2}$. By assumption, for $[Y^c]$ the corresponding frame $\{Y^c, Y^c_z, Y^c_z, N^c, L^c, R^c\}$ has the same inner product matrix and satisfies

$$Y^c_{zz} = -\frac{s + c}{2}Y^c + \lambda_1 L^c + \frac{1}{2}R^c, \quad D^c_2 L^c = \alpha L^c.$$
Recall that we have computed out (17)(16):

\[
L_{zz} = -\left( \frac{s}{2} - 2\alpha_z \right)L + \frac{1}{2} \hat{Y} + (\bar{\alpha}_z + \lambda_1)Y,
\]

\[
\hat{Y} = N + 2\alpha Y_z + 2\bar{\alpha} Y_z + 2|\alpha|^2 Y - 2(\alpha_z - \alpha^2 - \frac{s}{2})L,
\]

where \{Y, \hat{Y}\} form a basis of the normal plane of \(L\) at any point. The same result applys to \(Y^c\) and \(L^c\), hence

\[
L^c_{zz} = -\left( \frac{s + c}{2} - 2\alpha_z \right)L^c + \frac{1}{2} \hat{Y}^c + (\bar{\alpha}_z + \lambda_1)Y^c,
\]

where \(\hat{Y}^c, Y^c\) span the normal bundle of \([L^c]\). Comparison shows that the Schwarzian derivative of \([L]\) is \(s - 4\alpha_z\), while that of \([L^c]\) differs from it by \(c\) as we expected. Their conformal Hopf differential has the same components \(\frac{1}{2}\) and \(\bar{\alpha}_z + \lambda_1\). Finally, the normal connection of \([L]\) is given by \(\langle Y_z, \hat{Y} \rangle = \alpha\), exactly the same as \([Y]\). So \([L^c]\) also share the same normal connection as \([Y^c]\), which is again \(\alpha\). We conclude that \([L^c]\) is exactly a spectral transform of \([L]\) with parameter \(c\).

\[\square\]

**Remark 6.2.** For a spacelike Willmore surface there is also an associated family of Willmore surfaces, called the Willmore spectral transform. One could show that this transform commutes with the left/right polar transform. We did not notice this result in [11], yet the proof is similar and easy.

### 7 Permutability with Darboux transform

The most important transform of isothermic surfaces in \(\mathbb{R}^n\) is the Darboux transform. It is a second isothermic surface obtained from the original one by integration, depending on the choice of initial values and a real parameter. (So there are many of them.) For such a pair of isothermic surfaces forming Darboux transform to each other, a geometric characterization is that they envelop one and the same 2-sphere congruence at corresponding points, and that their conformal curvature lines are preserved by this correspondence (see [3], [9], [10]). This description is easy to adapted to our case:
Definition and Proposition Let \( y : M \to Q^4_1 \) denote a spacelike isothermic surface with canonical lift \([Y]\) with respect to the adapted coordinate \( z \). A spacelike immersion \( y^\ast : M \to Q^4_1 \) is called a Darboux transform of \( y \) if its local lift \( Y^\ast \) satisfies
\[
Y_z^\ast \in \text{Span}_C\{Y^\ast, Y, \bar{Y}_z\}.
\]
(18)
Note that this is well-defined, where \( Y^\ast \) is not necessarily the canonical lift. We have the following conclusions:

1) \( y, y^\ast \) are conformal; they envelop one and the same round 2-sphere congruence given by \( \text{Span}\{Y, Y^\ast, dY\} = \text{Span}\{Y, Y^\ast, dY^\ast\} \).

2) Set \( \langle Y, Y^\ast \rangle = -1 \). Then \( Y_z^\ast = \frac{\mu}{2} Y^\ast + \theta(Y_z + \frac{\bar{\mu}}{2} Y) \), where \( \theta \) is a non-zero real constant. This Darboux transform is specified as \( D^\theta \)-transform.

3) \( Y^\ast \) is an isothermic surface sharing the same adapted coordinate \( z \). Hence the curvature lines of \( y, \tilde{y} \) do correspond.

Proof. The conclusion 1) is obvious under the assumption (18). (Recall that a round 2-sphere in \( \mathbb{R}^4_2 \) is identified to a 4-dimensional Lorentzian subspace in \( \mathbb{R}^6_2 \). See Section 2.)

The normalization \( \langle Y, Y^\ast \rangle = -1 \) ensures \( \langle Y_z, Y^\ast \rangle = -\langle Y_z^\ast, Y \rangle = \mu/2 \). Then \( Y_z^\ast \in \text{Span}_C\{Y^\ast, Y, \bar{Y}_z\} \) is explicitly expressed by
\[
Y_z^\ast = \frac{\mu}{2} Y^\ast + \theta \left( \frac{\bar{\mu}}{2} Y + \frac{\mu \bar{\mu}}{4} \right) Y.
\]
(19)
Differentiate (19). We obtain
\[
Y_{zz}^\ast = \left( \frac{\mu}{2} \theta \right) Y_z + \left( \frac{\mu}{2} \theta + \theta \bar{\kappa} \right) Y_z + \theta \bar{\kappa} + \left( \frac{\mu \bar{\mu}}{2} + \frac{\mu \bar{\mu}}{4} \right) Y^\ast + (\cdots) Y.
\]
Since \( \kappa \) is real and non-zero by assumption, comparison shows that \( \theta \) is real-valued with \( \theta \bar{z} = 0 \). Hence \( \theta \) must be a real constant. It is non-zero since \([Y^\ast] \) is an immersion. This verifies 2). Note that \( \mu \bar{z} \) is real by the same argument.

Now \( \frac{1}{\theta} Y^\ast \) is the canonical lift of \([Y^\ast]\). To show conclusion 3) we need only to show that \( Y_{zz}^\ast \) is real modulo the components of \( Y^\ast, Y_z^\ast, \bar{Y}_z^\ast, Y_{zz}^\ast \). Differentiate (19). The result is
\[
Y_{zz}^\ast = \theta(Y_{zz} + \frac{\mu}{2} Y_z + \frac{\bar{\mu}}{2} Y) + (\cdots) Y^\ast + (\cdots) Y_{zz}^\ast
\]
\[
= \theta Y_{zz} + \left( \frac{\mu \bar{\mu}}{2} - \frac{\mu \bar{\mu}}{4} \right) Y \quad \text{(mod } Y^\ast, Y_z^\ast, \bar{Y}_z^\ast),
\]
(19)
which is real as desired. Finally, for \( z = u + iv \) with \( u, v \) real, the \( u \)-curves and \( v \)-curves are exactly the curvature lines on both of \( y \) and \( y^* \). This completes the proof. \( \square \)

Write out \( Y^* \) explicitly:

\[
Y^* = N + \mu Y_z + \mu Y_{\bar{z}} + \left( \frac{1}{2} |\mu|^2 - 4f_1f_2 \right) Y + 2f_1L + 2f_2R.
\]

Set

\[
P = Y_z + \frac{\mu}{2} Y, \quad \xi = L - 2f_2Y, \quad \eta = R - 2f_1Y. \tag{20}
\]

Note that both of \( \xi, \eta \) are lightlike and \( \langle \xi, \eta \rangle = -1 \). The orthogonal complement of \( \text{Span}\{\xi, \eta\} \) gives the Ribaucour 2-sphere congruence enveloped by \( [Y], [Y^*] \). The structure equations (11) of \( Y \) can be rewritten with respect to the frame \( \{Y, Y^*, P, \bar{P}, \xi, \eta\} \) as below.

\[
\begin{align*}
Y_z &= -\frac{\mu}{2} Y + P, \\
Y_{\bar{z}}^* &= \frac{\mu}{2} Y^* + \theta \bar{P}, \\
P_z &= \frac{\mu}{2} P + \frac{\theta}{2} Y + \lambda_1 \xi + \lambda_2 \eta, \\
\bar{P}_z &= -\frac{\mu}{2} \bar{P} + \frac{1}{2} Y^* - f_1 \xi - f_2 \eta, \\
\xi_z &= -2f_2P + 2\lambda_2 \bar{P}, \\
\eta_z &= -2f_1P + 2\lambda_1 \bar{P}. \tag{21}
\end{align*}
\]

Now let us find out the left polar transform of \( Y^* \). From (21), we have

\[
Y_{z\bar{z}}^* = \frac{\mu_z}{2} Y^* + \frac{\mu}{2} Y_{\bar{z}}^* + \theta \bar{P}_z = 2f_1f_2Y^* + \frac{\theta^2}{2} N^*,
\]

where

\[
N^* = Y + \frac{\mu}{\theta} P + \frac{\bar{\mu}}{\theta} \bar{P} + \frac{|\mu|^2}{2\theta^2} Y^* + \frac{2\lambda_1}{\theta} \xi + \frac{2\lambda_2}{\theta} \eta - \frac{4\lambda_1 \lambda_2}{\theta} Y^*.
\]

Set \( L^* = \xi - \frac{2\lambda_1}{\theta} Y^*, R^* = \eta - \frac{2\lambda_2}{\theta} Y^* \). Plus the orientation restriction, it is easy to see that \([L^*], [R^*]\) are just the left and right polar transform of \( Y^* \). Suppose that \([L], [L^*]\) are both non-degenerate. Computation shows

\[
L_z^* = -\frac{f_2}{\lambda_2} L_{\bar{z}} - \frac{2\lambda_{2z} + \mu \lambda_2}{\lambda_2} (L - L^*). \tag{22}
\]
Note that by Theorem 5.3 and (11), $[L]$ has the same adapted coordinate $z$ with canonical lift $\frac{1}{2\lambda_2}L$. Next, $-\frac{\theta}{2f_2}L^*$ is a lift of $[L^*]$ such that

$$\left\langle \frac{1}{2\lambda_2}L, -\frac{\theta}{2f_2}L^* \right\rangle = -\frac{1}{2\lambda_2} \cdot \frac{\theta}{2f_2} \left\langle \xi + 2f_2Y, \xi - \frac{2\lambda_2}{\theta}Y^* \right\rangle = -1,$$

$$\left( -\frac{\theta}{2f_2}L^* \right)_z = \theta \cdot \left( \frac{1}{2\lambda_2}L \right)_z + (\cdots)L + (\cdots)L^*.$$  

This proves that $[L^*]$ is just a Darboux transform of $[L]$ with the same parameter $\theta$. So we have estalished

**Theorem 7.1.** Let $y : M \to Q^4_1$ be a spacelike isothermic surface and $[y^*]$ be a $D^\theta$-transform of $y$. If both of their left polar surfaces $[L]$ and $[L^*]$ are not degenerate, $[L^*]$ is also a $D^\theta$-transform of $L$, i.e., we have the commuting diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{D^\theta\text{-transform}} & Y^* \\
\text{left polar} \downarrow & & \downarrow \text{left polar} \\
L & \xrightarrow{D^\theta\text{-transform}} & L^*
\end{array}$$  

(23)

A similar result holds between the right polar transform and the $D^\theta$-transform.

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