The Lawrence-Krammer representation is a quantization of the symmetric square of the Burau representation

A.V. Kosyak

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
Institute of Mathematics, Ukrainian National Academy of Sciences, 3 Tereshchenkivs’ka Str., Kyiv, 01601, Ukraine

Abstract

We show that the Lawrence–Krammer representation can be obtained as the quantization of the symmetric square of the Burau representation. This construction allows us to find new representations of the braid groups.

Keywords: braid group, Lawrence-Krammer representation, Burau representation, symmetric tensor power, quantization, quantum group, $q$–Pascal triangle, knot invariant

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Email address: ksyak02@gmail.com (A.V. Kosyak)

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1. Introduction

In 1936 W. Burau introduced the family of representations of the braid group \( B_n \) as the deformation of the standard representation of the symmetric group \( S_n \). The Burau representation for \( n = 2, 3 \) has been known to be faithful for some time. Indeed, in 1969 Magnus and Peluso showed that the Burau and Gasner representations are faithful for \( n = 2, 3 \). See also J. Birman [8, Theorem 3.15]. Moody in 1991 showed that the Burau representation is not faithful for \( n \geq 9 \); this result was improved to \( n \geq 6 \) by Long and Paton in 1992. The non-faithfulness for \( n = 5 \) was shown by Bigelow in 1999. The faithfulness of the Burau representation when \( n = 4 \) is an open problem.

In 1990 R. Lawrence have constructed two-parameter family of representation of the braid group \( B_n \), using the homological methods. In 2000 D. Krammer using completely algebraic methods have constructed the
same representations and showed that this representations is faithful for $B_4$. One year later in 2001 S. Bigelow showed that the Lawrence-Krammer representation is faithful for all $B_n$. Next year D. Krammer [15] proved the same result by a different method.

The Burau representation appears as a summand of the Jones representation [11], and for $n = 4$, the faithfulness of the Burau representation is equivalent to that of the Jones representation, which on the other hand is related to the question of whether or not the Jones polynomial is an unknot detector [7].

2. Connection between the Lawrence-Krammer and the Burau representations

2.1. The Braid group

The braid group $B_n$ is defined by Artin, 1926 in [2] as follows:

$$B_n = \langle (\sigma_i)_{i=1}^{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2 \rangle. \quad (2.1)$$

2.2. The Burau representation

The Burau representation $\rho : B_n \mapsto \text{GL}_n(\mathbb{Z}[t, t^{-1}])$ is defined for a non-zero complex number $t$ by

$$\sigma_i \mapsto \beta_i = I_{i-1} \oplus (1 - t \quad t) \oplus I_{n-i-1}$$

where $1 - t$ is the $(i, i)$ entry. For $t = 1$ this is the standard representation of the symmetric group $S_n$ interchanging two neighbors basic elements in the space $\mathbb{C}^n$. The representation $\rho$ splits into 1-dimensional and an $n-1$-dimensional irreducible representations, known as reduced Burau representation $\rho_n^{(t)} : B_n \mapsto \text{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$

$$\sigma_1 \mapsto b_1 = \begin{pmatrix} -t & 0 \\ 1 & -t \end{pmatrix} \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto b_{n-1} = I_{n-3} \oplus \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad (2.2)$$

$$\sigma_i \mapsto b_i = I_{i-2} \oplus \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \oplus I_{n-i-2}, \quad 2 \leq i \leq n - 2. \quad (2.3)$$

We use the following equivalent form of the reduced Burau representation $\rho_n^{(t)} : B_n \mapsto \text{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$, compare with (2.2) and (2.3):

$$\sigma_1 \mapsto b_1 = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto b_{n-1} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.4)$$

$$\sigma_i \mapsto b_i = I_{i-2} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus I_{n-i-2}, \quad 2 \leq i \leq n - 2. \quad (2.5)$$
2.3. The Krammer representation of the braid group $B_n$

Let the Krammer representation $K_n^{(t,q)}$ be the following action of $B_n$ on a free module $V$ of rank $n = (\frac{n}{2})$ with basis $\{F_{i,j} : 1 \leq i < j \leq n\}$.

The Krammer representation introduced by Krammer in [14] is defined by Bigelow in [5] and after appropriate corrections in [6] as follows:

\[
\sigma_i(F_{j,k}) = \begin{cases} 
F_{j,k}, & j \notin \{j-1, j, k-1, k\}, \\
qF_{i,k} + q(q-1)F_{i,j} + (1-q)F_{j,k}, & i = j-1, \\
F_{j+1,k}, & i = j \neq k-1, \\
qF_{j,k} + (1-q)F_{j,k} + q(1-q)tF_{i,k}, & i = k-1 \neq j, \\
F_{j,k+1}, & i = k, \\
-tq^2F_{j,k}, & i = j = k - 1.
\end{cases}
\]

From [6]: “This action is given in [14], except that Krammer’s $-t$ is my $t$. It is also in [5], but with a sign error. The name “Krammer representation” was chosen because Krammer seems to have initially found this independently of Lawrence and without any use of homology”.

2.4. Construction of the braid group representations $B_{n+1}$ via representations of the Lie algebra $\mathfrak{sl}_n$

We can generalize the reduced Burau representation of $B_n$ just observing (see [13]) that it is exp of the natural or fundamental representations of the Lie algebra $\mathfrak{sl}_n$. To be more precise, we show that the reduced Burau representation for $B_3$

\[
\sigma_1 \mapsto \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 1 \\ -t & t \end{pmatrix},
\]

can be rewritten as follows:

\[
\sigma_1 \mapsto \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \exp(E_{12}) \exp(sE_{11}),
\]

\[
\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \exp(sE_{22}) \exp(-E_{21}),
\]

where $E_{kn}$ are matrix unities and $s = \ln t$ for positive $t$.

For $n = 4$ the reduced Burau representation is defined as follows:

\[
\rho_4^{(t)} : B_4 \mapsto \text{GL}_3(\mathbb{Z}[t, t^{-1}]),
\]

\[
\sigma_1 \mapsto \begin{pmatrix} -t & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t & 0 \end{pmatrix},
\]

(2.10)
The Krammer representation is the quantization of the symmetric square of the reduced Burau representation:

\[ K_n^{(q^{-t})} = S^2(\rho_n^{(t)})_q. \]  

(2.14)
Proof. We use two evidence to prove the theorem:
1) compare dimensions of two representations,
2) compare the spectrums of two representations.

We note that because of the braid relations (2.1)
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
the spectrum of all generators are the same, i.e., for any representations \( \pi \) of the group \( B_n \) holds \( \text{Sp}(\pi(\sigma_k)) = \text{Sp}(\pi(\sigma_{k+1})), \) \( 1 \leq k \leq n - 2, \) that is why we can speak about the spectrum of the representation.

1) The dimension of the Burau representation \( \rho_n^{(t)}(\sigma) \) equals to \( n - 1. \) Therefore, its symmetric square by (2.17) has dimension \( \left( \begin{array}{c} n \\ 2 \end{array} \right) = \frac{n(n-1)}{2}, \) which coincides with the dimension of the Krammer representation \( K_n^{(t,q^2)} \).

2) The spectrum of the reduced Burau representation is as follows:
\[ \text{Sp}(\rho_n^{(t)}(\sigma_k)) = \{ -t, 1, \ldots, 1 \}. \]
The spectrum of its symmetric square is:
\[ \text{Sp} S^2(\rho_n^{(t)}(\sigma_k)) = \{ t^2, -t, \ldots, -t, 1, \ldots, 1 \}. (2.15) \]

In [21, Lemma 5] the following fact is proved

Lemma 2.3. The spectra of the Krammer representation of the group \( B_n, \) i.e., \( \text{Sp}(\sigma^K) \) is as follows:
\[ \text{Sp} (\sigma^K) = \{ -tq^2, -q, \ldots, -q, 1, \ldots, 1 \}. (2.16) \]

If we set \( t = -1 \) and \( q = t \) in (2.16) we get (2.15).

Remark 2.1. The same fact was mentioned in [20, p.38]: “Under the specialization \( t = 1 \) the Lawrence-Krammer module becomes the symmetric square of the Burau module”.

We show more, i.e., that the quantization of the symmetric square of the Burau representation gives us the Krammer representation.

Remark 2.2. The following procedure of quantisation (“Quant”) is based on the quantization of the Pascal triangle explained in [1] and [13], formulas (10) and (11). To be more precise we change the binomial coefficients \( \binom{n}{k} \) by \( q \)-binomial coefficients defined as follows: \( \binom{n}{k}_q := \frac{\binom{n}{k}}{(k)_q^{(n-k)}} \), where \( q \)-natural numbers \( (n)_q \), are defined by \( (n)_q := \frac{q^n-1}{q-1} \), for complex number \( q \in \mathbb{C}^\times \).
Let an operator $A$ acts in a space $V$. Denote by $A \otimes A$ its tensor product in a space $V \otimes V$ and by $S^2(A)$ its symmetric square, i.e., the subspace $S^2(V) \subset V \otimes V$ generated by the symmetric basis $e^s_{k,r}$ defined as follows:

$$e^s_{k,k} = e_k \otimes e_k, \quad 1 \leq k \leq n, \quad e^s_{k,r} = e_k \otimes e_r + e_r \otimes e_k, \quad 1 \leq k < r \leq n,$$  \hspace{1cm} (2.17)

where $(e_k)_{k=1}^n$ is a basis in the space $V$ (we use both notations $S^2 = \Sym^2$).

**Remark 2.3.** Using (2.8) and (2.9) we get for the Burau representation $\rho^B_{\lambda}$:

$$\sigma_1 \mapsto \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right) \mapsto \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right) \mapsto \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right).$$

Similarly, we get

$$\sigma_1 \mapsto \sigma^B_1 := \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right), \quad \sigma_1 \mapsto \sigma^B_2 := \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right).$$  \hspace{1cm} (2.18)

The Krammer representation for $B_3$ is as follows:

$$\sigma_1 \mapsto \sigma^K_1 := \left( \begin{array}{ccc} -tq^2 & q(q-1) \\ 0 & 1 & 1-q \end{array} \right), \quad \sigma_2 \mapsto \sigma^K_2 := \left( \begin{array}{ccc} 0 & q & 0 \\ 1 & 1-q & 0 \\ 0 & tq(1-q) & -tq^2 \end{array} \right).$$  \hspace{1cm} (2.19)

**Remark 2.3.** Using [23], see also [13, n7, n8], we conclude that all representation of $B_3$ in dimension 3 are given by the formulas:

$$\sigma^A_1(q, 2) = \left( \begin{array}{ccc} \lambda_0 q & (1+q)\lambda_1 & \lambda_2 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_2 \end{array} \right), \quad \sigma^A_2(q, 2) = \left( \begin{array}{ccc} \lambda_2 & 0 & 0 \\ -\lambda_1 & \lambda_1 & 0 \\ -\lambda_0 (1+q) & \lambda_0 q \end{array} \right),$$  \hspace{1cm} (2.20)

where $\lambda_0 \lambda_2 = \lambda_1^2$. In particular, for $\Lambda = \text{diag}(\lambda^2, \lambda, 1)$ and $q = Q$ we get

$$\sigma^B_1 = \sigma^A_1(Q, 2) = \left( \begin{array}{ccc} \lambda Q & 0 & 0 \\ \lambda^2 & -\lambda^2 (1+Q) & \lambda Q \end{array} \right), \quad \sigma^B_2 = \sigma^A_2(Q, 2) = \left( \begin{array}{ccc} \lambda Q & (1+Q) \lambda & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{array} \right).$$  \hspace{1cm} (2.21)
To compare the formulas \((2.18)\) and \((2.19)\) we calculate the eigenvalues \(\nu_k\) of \(\sigma^K_1\)

\[
\det(\sigma^K_1 - \nu I) = \begin{vmatrix}
-tq^2 - \nu & 0 & q(q-1) \\
0 & -\nu & q \\
1 & -q & -\nu
\end{vmatrix} = (-tq^2 - \nu)(\nu^2 - (1 - q)\nu - q) = 0.
\]

Hence, the eigenvalues of \(\sigma^K_1\) are the following: \(\nu_0 = -tq^2\), \(\nu_1 = -q\), \(\nu_2 = 1\), i.e., \(\text{Sp}(\sigma^K_1) = \{-tq^2, -q, 1\}\), compare with \((2.16)\). Respectively, the eigenvalues of \(\sigma^B_1\) are \(\nu_0 = \lambda^2 Q\), \(\nu_1 = \lambda\), \(\nu_2 = 1\). Finally, we get

\[
\text{Sp}(\sigma^K_1) = \{-tq^2, -q, 1\} \quad \text{and} \quad \text{Sp}(\sigma^B_1) = \{\lambda^2 Q, \lambda, 1\}.
\]

Representations \((2.18)\) and \((2.19)\) are equivalent if and only if \(\text{Sp}(\sigma^K_1) = \text{Sp}(\sigma^B_1)\), i.e., when

\[
\lambda^2 Q = -tq^2, \quad \lambda = -q \quad \text{or} \quad q = -\lambda, \quad t = -Q. \tag{2.22}
\]

Finally, representations given by \((2.19)\) and \((2.20)\) are equivalent, i.e.,

\[
C_3^{-1}\sigma^B_1 C_3 = \sigma^K_1, \quad C_3^{-1}\sigma^B_2 C_3 = \sigma^K_2 \tag{2.23}
\]

for an appropriate \(C_3 \in \text{Mat}(3, \mathbb{C})\).

We calculate explicitly \(C_3\) and prove the second formula in \((2.23)\).

**Remark 2.4.** Let \(A \in \text{Mat}(n, \mathbb{C})\) and \(\lambda_k, e_k, 1 \leq k \leq n\), be its eigenvalues and eigenvectors, i.e., \(A e_k = \lambda_k e_k\). Denote by \(e(A) \in \text{Mat}(n, \mathbb{C})\) the matrix which columns are eigenvectors of the matrix \(A\), i.e., \(e(A) = (e^t_k)_{k=1}^n\), then we get

\[
A e(A) = e(A) \Lambda \quad \text{and} \quad (e(A))^{-1} A e(A) = \Lambda, \tag{2.24}
\]

when \(e(A)\) is invertible.

Let \(e(B)\) and \(e(K)\) be matrices which columns are eigenvectors of the matrix \(\sigma^K_1\) and \(\sigma^B_1\) respectively defined by \((2.18)\) and \((2.19)\) corresponding to the eigenvalues \((\lambda_0, \lambda_1, \lambda_2) = (\lambda^2 Q, \lambda, 1)\) of the matrix \(\Lambda = \text{diag}(\lambda^2 Q, \lambda, 1)\). By definition we have

\[
\sigma^K_1 e(B) = e(B) \Lambda, \quad \sigma^B_1 e(K) = e(K) \Lambda. \tag{2.25}
\]

Then we get

\[
e^{-1}(B)\sigma^K_1 e(B) = \Lambda = e^{-1}(K)\sigma^K_1 e(K) \quad \text{therefore,} \quad C_3^{-1}\sigma^K_1 C_3 = \sigma^K_1, \tag{2.26}
\]

\[
C_3^{-1}\sigma^B_1 C_3 = \sigma^B_1.
\]
where
\[ C_3 = e(B)e^{-1}(K). \]  

We can verify that
\[ e(B) = \begin{pmatrix} 1 & 1+Q & 1+\lambda Q \\ 0 & 1-\lambda Q & 1-\lambda^2 Q \\ 0 & 0 & (1-\lambda)(1-\lambda^2 Q) \end{pmatrix}, \quad e(K) = \begin{pmatrix} 1 & 1-q & q(q-1) \\ 0 & 1-tq & (1+tq^2)/tq \\ 0 & tq-1 & 1+tq^2 \end{pmatrix}, \]
\[ e^{-1}(B) = \begin{pmatrix} 1 & -1+Q & -1+\lambda Q \\ 0 & 1-\lambda Q & 1-\lambda^2 Q \\ 0 & 0 & (1-\lambda)(1-\lambda^2 Q) \end{pmatrix}, \quad e^{-1}(K) = \begin{pmatrix} 1 & -1-q & 1 \\ 0 & 1-tq & (1+tq^2)/tq \\ 0 & tq-1 & 1+tq^2 \end{pmatrix}, \]
and \( C_3^{-1}\sigma_2^B C_3 = \sigma_2^K \).

The symmetric square of the reduced Burau representation \( \rho_4^{(-\lambda)} \) for \( B_4 \) is as follows (see (2.10) and (2.11)):
\[ S^2(\sigma_1) \mapsto \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} q \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \]
\[ S^2(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \]
\[ S^2(\sigma_3) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]

Using Remark 2.2 we get
\[ S^2(\sigma_1) \quad \text{Quant} \rightarrow \begin{pmatrix} Q & 1+Q & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 Q & \lambda(1+Q) & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \]
\[ S^2(\sigma_3) \quad \text{Quant} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]

In what follows we should be more attentive and use \( Q \) on the diagonal only in the third factors, to be sure that the spectrum of the image of \( \sigma_2 \) is correct, i.e., \( \text{Sp}(\sigma_2^B) = \{ \lambda^2 Q, \lambda, \lambda, 1, 1, 1 \} \):
\[ S^2(\sigma_2) \quad \text{Quant} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}. \]
Finally, we get

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 (1+q) \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-\lambda \\
\lambda^2 - \lambda^2 (1+q) \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-\lambda & 0 & 0 & 0 & 0 & 0 \\
\lambda^2 & -\lambda^2 (1+q) & \lambda^2 Q & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

If we compare the spectrum of \(\sigma\) in (2.29) and (2.31) we get:

\[
\{\lambda^2 Q, \lambda, 1, 1, 1, 1\} = \text{Sp}(\sigma_1) = \{-tq^2, -q, -q, 1, 1, 1\}.
\]

Using Remark 2.31 we can find \(e(B)\) and \(e(K)\) such that

\[
e^{-1}(B)\sigma^B_1 e(B) = \Lambda = e^{-1}(K)\sigma^K_1 e(K).
\]

Define \(C_4 = e(B)e^{-1}(K)\) and prove that

\[
C_4^{-1}\sigma^B_r C_4 = \sigma^K_r \text{ for } r = 2, 3.
\]

For \(n = 5\) the reduced Burau representation is defined as follows:

\[
\rho_4^{(t)} : B_5 \mapsto \text{GL}_4(\mathbb{Z}[t, t^{-1}]),
\]

\[
\sigma_1 \mapsto \begin{pmatrix}
-t & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
t & -t & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \sigma_3 \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -t & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -t
\end{pmatrix}.
\]
If we set \( s = \ln t \) for \( t > 0 \), we get

\[
\begin{align*}
\sigma_1 & \mapsto \exp(X_1) \exp(-sE_{11}), \\
\sigma_2 & \mapsto \exp(X_2) \exp(-sE_{22}) \exp(-Y_1), \\
\sigma_3 & \mapsto \exp(X_3) \exp(-sE_{33}) \exp(-Y_2), \\
\sigma_3 & \mapsto \exp(-sE_{44}) \exp(-Y_3).
\end{align*}
\]

(2.35)

(2.36)

Using Theorem 2.1, formulas (2.13), symmetric square of the natural representations of the Lie algebras \( \mathfrak{sl}_{n-1} \) and the quantization, as it was explained before, we obtain the general formulas for \( B_n \).

\[ \square \]

**Remark 2.5.** See [9, n 4.6, p.71] for more details about connections between the Lawrence-Krammer and other representations of braid groups.

### 3. Some polynomial invariants for knots

#### 3.1. Some polynomial invariants for the frefoil, see [3] and [24]

The Alexander polynomial \( \Delta(x) \), BLM/Ho polynomial \( Q(x) \), Conway polynomial \( \nabla(x) \), HOMFLY polynomial \( P(l, m) \) Jones polynomial \( V(t) \), and Kauffman polynomial \( F(a, z) \) of the trefoil knot are

\[
\begin{align*}
\Delta(x) &= x - 1 + x^{-1}, \\
Q(x) &= 2x^2 + 2x - 3, \\
\nabla(x) &= x^2 + 1, \\
P(l, m) &= -l^4 + m^2l^2 - 2l^2, \\
V(t)_{31} &= t + t^3 - t^4, \\
V(t)_{32} &= t^{-1} + t^{-3} - t^{-4}, \\
F(a, z) &= -a^4 - 2a^2 + (a^4 + a^2)z^2 + (a^5 + a^3)z.
\end{align*}
\]

(3.1)

(3.2)

(3.3)

(3.4)

(3.5)

(3.6)

#### 3.2. The Burau representation and the Alexander polynomial

Let \( K \) be some link, by Alexander’s theorem this link can be obtained as the closure of some braid \( X \in B_n \), notation \( K = \text{Cl}(X) \). The **Alexander polynomial** \( \Delta_K(t) \) can be obtained using the reduced Burau representation by the following formula, [9, (4.3), p.64]:

\[
\Delta_{\text{Cl}(X)}(t) = \frac{(1-t)\det(\rho_n(t)(X) - I_{n-1})}{1 - t^n} = \frac{\det(\rho_n(t)(X) - I_{n-1})}{\det(\rho_n(t)(\sigma_1 \ldots \sigma_{n-1}) - I_{n-1})}.
\]

(3.7)
3.3. The Krammer representation and the corresponding polynomial

**Definition 3.1.** By analogy with the definition of the Alexander polynomial we can introduce the following polynomial $K(t, q)$ connected with the Krammer representation

$$K_{C_l(X)}(t, q) = \frac{\det(K_{n(t,q)}^*(X) - I_{n(n-1)/2})}{\det(K_{n(t,q)}^{(t,q)}(\sigma_1 \ldots \sigma_{n-1}) - I_{n(n-1)/2})}.$$  

(3.8)

**Theorem 3.1.** The polynomial $K_{C_l(X)}(t, q)$ defined by (3.8) is the knot invariant.

**Proof.** It is sufficient to verify (see, e.g., [12, 2.5. Markov’s theorem, p.67]) that the polynomial $K_{C_l(X)}(t, q)$ respect the first and the second Markov moves, i.e., for any $X \in B_n$ the following properties hold:

$$K_{C_l(wXw^{-1})}(t, q) = K_{C_l(X)}(t, q) \quad \text{for all} \quad w \in B_n,$$

(3.9)

$$K_{C_l(X\sigma_1^{\pm 1})}(t, q) = K_{C_l(X)}(t, q).$$

(3.10)

Indeed, by definition we get, since $\det(ABC) = \det(CAB)$

$$\det(K_{n(t,q)}^{(t,q)}(wXw^{-1}) - I_{n(n-1)/2}) = \det(K_{n(t,q)}^{(t,q)}(wXw^{-1}) - K_{n(t,q)}^{(t,q)}(ww^{-1}))$$

$$= \det(K_{n(t,q)}^{(t,q)}(w)K_{n(t,q)}^{(t,q)}(X) - I_{n(n-1)/2})K_{n(t,q)}^{(t,q)}(w^{-1})$$

$$= \det(K_{n(t,q)}^{(t,q)}(X) - I_{n(n-1)/2}),$$

that proves (3.9).

The proof of (3.10) should be similar to the proof that the Alexander polynomial respect the second Markov move: $\Delta_{C_l(X\sigma_1^{\pm 1})}(t) = \Delta_{C_l(X)}(t)$. □
3.4. Symmetric square of the reduced Burau representation and quantization

The reduced Burau representation $\rho_3^{(t)}$ for $B_3$ we take as follows:

$$b_1 = (-t \ 0 \ 1 \ 0 \ 1 \ 0) \ (1 \ 0 \ 1 \ 0) = (-t \ 0 \ 1 \ 0 \ 1 \ 0), \quad b_2 = (1 \ 0 \ -t \ 1 \ 0 \ 1).$$

The symmetric powers $b_i \otimes b_i \ |_{S_i}$, $i = 1, 2$ in the symmetric basis $e_{11}^1, e_{12}^1, e_{22}^1$ defined by (2.17) in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ are as follows:

$$b_1 \otimes b_1 \mid_{S_1} = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & -t & 0 \end{pmatrix}, \quad b_2 \otimes b_2 \mid_{S_1} = \begin{pmatrix} 1 & 2t & 1 \\ 0 & -t & t^2 \end{pmatrix}.$$ 

The $q$-version are as follows, we replace $\frac{3}{2} = 2$ by $\frac{(q}}{2} = (2)q = \frac{q-1}{q-1} = 1+q$:

$$\sigma_1^t(q,2) = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & -t & 0 \end{pmatrix} \begin{pmatrix} 0, 0, 1 \\ 1, 1, 0 \end{pmatrix}, \quad \sigma_2^t(q,2) = \begin{pmatrix} 1 & 2t & 1 \\ 0 & -t & t^2 \end{pmatrix} \begin{pmatrix} 0, 0, 1 \\ 1, 1, 0 \end{pmatrix}.$$ 

3.5. Alexander polynomial for the trefoil

To calculate the Alexander polynomial for the trefoil knot $3_1 = Cl(\sigma_1^3)$ we get for the representation $\rho_2^{(t)}(\sigma_1) = -t$ of the group $B_2$

$$\Delta_3(t) = \frac{\det(\rho_2^{(t)}(\sigma_1) - I_1)}{\det(\rho_2^{(t)}(\sigma_1) - I_1)} = \frac{-t^3 - 1}{-(t+1)} = (t^2 - t + 1).$$

The second Markov move (see (3.10)). For the Burau representation $\rho_3^{(t)}$ of the group $B_3$

$$\rho_3^{(t)}(\sigma_1) = (-t \ 0 \ 1 \ 0 \ 1 \ 0), \quad \rho_3^{(t)}(\sigma_2) = (1 \ 0 \ -t)$$

we have

$$\rho_3^{(t)}(\sigma_1^3) = (-t \ 0 \ 1 \ 0 \ 1 \ 0)^3 \ = \begin{pmatrix} -t^3 & 0 & 0 \\ 0 & -t & t^2 \end{pmatrix} \begin{pmatrix} 1 \ 0 \ -t \end{pmatrix} = \begin{pmatrix} -t^3 & t^4 \\ -t^2 & t^2 \end{pmatrix}.$$ 

Hence,

$$\det(\rho_3^{(t)}(\sigma_1^3) - I_2) = \det(-t^3 - t^4) = -(t^3 + 1)(t^3 - t^2 - 1) + t^4(t^2 - t + 1) = t^4 + t^2 + 1$$

and we obtain the same result as in (3.13):

$$\Delta_3(t) = \frac{\det(\rho_3^{(t)}(\sigma_1^3) - I_2)}{\det(\rho_3^{(t)}(\sigma_1^3) - I_2)} = \frac{t^4 + t^2 + 1}{t^2 + t + 1} = t^2 - t + 1.$$
3.6. Krammer representations and the trefoil

We can calculate the polynomial $K(t, q)$ for the trefoil knot using Theorem 2.2. Indeed, we have (see (3.11))

$$\det(\sigma_1^t(2, q)\sigma_2^t(2, q) - I_3) = \begin{vmatrix} t^2q^{t-1} - t^3q^{t+1} & t^4 \\
q & -t^2q^{t-1} \end{vmatrix} = t^6q^2 - 1,$$

$$\det(\sigma_1^t(2, q)^3\sigma_2^t(2, q) - I_3) = t^{12}q^4 - 1,$$

hence, by (2.14) we get

$$K_{3_1}(q, t) = t^6q^2 + 1 \text{ or } K_{3_1}(t, q) = t^2q^6 + 1. \quad (3.14)$$

Indeed, we have

$$K_{3_1}(q, t) = (t^{12}q^4 - 1)(t^6q^2 - 1)^{-1} = t^6q^2 + 1.$$ 

For $t = i$ we get as a factor the Alexander polynomial $\Delta(q) = q^2 - q + 1$:

$$K_{3_1}(i, q) = -q^6 + 1 = -(q + 1)(q - 1)(q^2 + q + 1)(q^2 - q + 1),$$

for $q = 1$ we get the Conway polynomial:

$$K_{3_1}(t, 1) = t^2 + 1 = \nabla(t).$$

To calculate $K_{3_2}(t, q)$ for the mirror image of the trefoil $3_2 = Cl(\sigma_1^{-3}\sigma_2)$, for first simple knots $4_1, 5_1, 5_2$ and to compare with the known polynomials.

**Open problem:** Whether the reduced Burau representation for $B_4$

$$\rho_4^{(t)} : B_4 \mapsto \text{GL}_3(\mathbb{Z}[t, t^{-1}])$$

is faithful:

$$b_1 = \begin{pmatrix} -t & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & -t & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & -t \end{pmatrix}. \quad (3.15)$$
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