ON THE MAXIMALLY CLUSTERED ELEMENTS OF COXETER GROUPS

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Abstract. We continue the study of the maximally clustered elements for simply laced Coxeter groups which were recently introduced by Losonczy. Such elements include as a special case the freely braided elements of Losonczy and the author, which in turn constitute a superset of the $iji$-avoiding elements of Fan. Our main result is to classify the MC-finite Coxeter groups, namely those Coxeter groups having finitely many maximally clustered elements. Remarkably, any simply laced Coxeter group having finitely many $iji$-avoiding elements also turns out to be MC-finite.

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1. Introduction.

Let $W$ be a simply laced Coxeter group with set $S$ of Coxeter generators. Recently, Losonczy [7] introduced the notion of maximally clustered elements for $W$, and studied applications of maximally clustered elements to Schubert varieties.

Maximally clustered elements are defined in terms of certain triples of root vectors. We explain this briefly as follows. Let $w \in W$. Every reduced expression for $w$ determines a sequence whose terms are the positive roots sent negative by $w$. If such a “root sequence” has a consecutive subsequence of the form $\alpha, \alpha + \beta, \beta$, then the set containing these three vectors is called a “contractible triple” of $w$.

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Some previously studied classes of group elements can be characterized in terms of contractible triples. For example, the $iji$-avoiding elements of Fan [3] are precisely those elements of $W$ having no contractible triples, and the freely braided elements of the author and Losonczy [4, 5] are the elements of $W$ with pairwise disjoint contractible triples.

A maximally clustered element $w \in W$ is one with the property that if $T$ and $T'$ are contractible triples of $w$ and $T \cap T' \neq \emptyset$, then the highest roots of $T$ and $T'$ agree. Maximally clustered elements exist in abundance: for example, in the Coxeter group of type $A_3$, 21 of the 24 elements are maximally clustered. Maximally clustered elements have convenient reduced expressions called “contracted reduced expressions”, and every reduced expression is short braid equivalent (commutation equivalent) to a contracted one. A main result of [7] is a criterion for a Schubert variety indexed by a maximally clustered element to be smooth, and this is much simpler than the corresponding situation for an arbitrary group element.

The main result of this paper is a classification of simply laced Coxeter groups $W$ having finitely many maximally clustered elements; we call these MC-finite Coxeter groups for short. It is not hard to show that for simply laced Coxeter groups, the maximally clustered elements are a superset of the freely braided elements, and that in turn, the freely braided elements are a superset of the fully commutative elements of Stembridge [9], or equivalently the $iji$-avoiding elements of Fan [3]. In order for $W$ to be MC-finite, it is therefore necessary for $W$ to have finitely many fully commutative elements. Our main result is that this condition is also sufficient to ensure that $W$ is MC-finite. (It is possible for $W$ to be both infinite and MC-finite.)

Our main result generalizes one of the two main results in [5], which proves that a simply laced Coxeter group has finitely many fully commutative elements if and only if it has finitely many freely braided elements. Remarkably, the proof presented here is simpler than the proof of the less general result in [5], although the argument of this paper relies on the classification of Coxeter groups having finitely
many fully commutative elements, whereas the argument of [5] is conceptual.

The paper is organized as follows. Sections 2–4 recall the necessary background from the theory of Coxeter groups. The main result is stated in Section 5. Section 6 develops the theory of certain operators, \( \pi_i \), on reduced expressions, and these are used to prove the main result in Section 7.

2. Preliminaries.

Let \( W \) be a simply laced Coxeter group with set \( S = \{ s_i : i \in I \} \) of distinguished generators and Coxeter matrix \( (m_{ij})_{i,j \in I} \). The Coxeter graph, \( \Gamma \), of \( (W, S) \) has vertices indexed by \( S \), and an edge between distinct vertices \( s_i \) and \( s_j \) if and only if \( m_{ij} = 3 \). The basic facts about Coxeter groups needed for this paper can be found in [2, 6]. We are primarily interested in this paper in the case where \( \Gamma \) is a subgraph of a Coxeter graph of type \( E_n \) (for arbitrary \( n \geq 6 \)).

**Figure 1.** Coxeter graph of type \( E_n \)

\[ \begin{array}{ccccccc}
1 & 2 & 3 & 4 & \cdots & n & \bullet \\
\bullet & | & | & | & | & | & \bullet \\
0 & & & & & & \\
\end{array} \]

The full subgraph of \( \Gamma \) omitting vertex 0 is called the Coxeter graph of type \( A_{n-1} \), and the full subgraph of \( \Gamma \) omitting vertex 1 is called the Coxeter graph of type \( D_{n-1} \).

Denote by \( I^* \) the free monoid on \( I \). We call the elements of \( I \) *letters* and those of \( I^* \) *words*. The *length* of a word \( i \in I^* \) is the number of factors used to express \( i \) as a product of letters. A *subword* of a given word \( i_1i_2\cdots i_n \) (each \( i_l \in I \)) is any word of the form \( i_pi_{p+1}\cdots i_q \) where \( 1 \leq p \leq q \leq n \). Let \( \phi : I^* \rightarrow W \) be the surjective morphism of monoid structures determined by the equalities \( \phi(i) = s_i \) for all \( i \in I \). We say that a word \( i \in I^* \) represents its image \( \phi(i) \in W \). If the length of a word \( i \in I^* \) is as small as possible over all words representing \( w = \phi(i) \in W \), then we call \( i \) a *reduced word*, or a *reduced expression* for \( w \). The *length* of \( w \in W \), denoted by \( \ell(w) \), is the length of any reduced expression for \( w \).

Let \( V \) be a real vector space with basis \( \{ \gamma_i : i \in I \} \) in one-to-one correspondence
with \( I \). Let \( B \) denote the Coxeter form on \( V \) associated with \((m_{ij})_{i,j \in I}\). This is the symmetric bilinear form on \( V \) satisfying \( B(\gamma_i, \gamma_j) = -\cos \frac{\pi}{m_{ij}} \) for all \( i, j \in I \).

Throughout the paper, we view \( V \) as the underlying space of a particular reflection representation of \( W \), determined by the equalities \( s_i \gamma_j = \gamma_j - 2B(\gamma_j, \gamma_i) \gamma_i \) for all \( i, j \in I \). Note that \( B \) is preserved by \( W \) relative to this representation.

Given \( u, v \in V \), we say that \( u \) is orthogonal to \( v \), and write \( u \perp v \), if \( B(u, v) = 0 \).

Define \( \Phi = \{ w \gamma_i : i \in I \} \). This is the root system of \( W \). The elements of \( \Phi \), i.e., the roots, all have unit length relative to \( B \). The basis vectors \( \gamma_i \) are called simple roots. Let \( \Phi^+ \) (respectively, \( \Phi^- \)) denote the set of all roots expressible as a linear combination of simple roots with nonnegative (respectively, nonpositive) coefficients. This is the set of positive (respectively, negative) roots. We have \( \Phi = \Phi^+ \cup \Phi^- \) (disjoint).

Let \( w \in W \). Denote by \( \Phi(w) \) the set of all \( \alpha \in \Phi^+ \) such that \( w \alpha \in \Phi^- \). The cardinality of \( \Phi(w) \) is \( \ell(w) \). Given any reduced expression \( i_1 \cdots i_n \) for \( w \), we have \( \Phi(w) = \{ r_1, \ldots, r_n \} \), where \( r_1 = \gamma_{i_n} \) and \( r_q = s_{i_n} \cdots s_{i_{n-q+2}}(\gamma_{i_{n-q+1}}) \) for all \( q \in \{2, \ldots, n\} \). The sequence \( r = (r_1, \ldots, r_n) \) is called the root sequence of \( i_1 \cdots i_n \), or a root sequence for \( w \). Note that for any \( 1 \leq q \leq n \), the initial segment \( (r_1, \ldots, r_q) \) of \( r \) is the root sequence of \( i_{n-q+1} \cdots i_n \). Observe also that every reduced word is uniquely determined by its root sequence, so that the map \( i_1 \cdots i_n \mapsto r \) is a bijection from the set of all reduced expressions for \( w \) to the set of all root sequences for \( w \).

3. Braid moves and contractible triples.

For any letters \( i, j \in I \) and any positive integer \( q \), we define \( (i,j)_q \) to be the length \( q \) word \( iji \cdots \in I^* \).

Let \( i, j \in I^* \) and let \( i, j, k \in I \) with \( i \neq j \). Denote the length of \( j \) by \( n \). We call the substitution \( i(i,j)_{m_{ij}} j \rightarrow i(j,i)_{m_{ji}} j \) a braid move, qualifying it short or long according as \( m_{ij} = 2 \) or 3, respectively. Braid moves can be described in terms of root sequences [4, Proposition 3.1.1], as follows. Suppose that the word \( ii jj \) is reduced, and let \( r = (r_q) \) be its root sequence. Then \( m_{ij} = 2 \) if and only if
Let $w \in W$. We shall find useful a theorem of Matsumoto [8] and Tits [10] (stated in [1, Theorem 3.3.1]), which states that every reduced expression for $w$ can be obtained from any other through a sequence of braid moves. A similar statement holds for root sequences, on account of the aforementioned bijection.

We say that two words in $I^*$ are short braid equivalent if one can be obtained from the other by a sequence of short braid moves. There is a corresponding notion for root sequences.

The height of any $\alpha \in \Phi$ is the sum of the coefficients used to express $\alpha$ as a linear combination of simple roots. The following definition comes from [7, §3.1].

**Definition 3.1.** Let $w \in W$. We call any subset of $\Phi(w)$ of the form $\{\alpha, \beta, \alpha + \beta\}$ an inversion triple of $w$. If $T$ is an inversion triple of $w$ such that there is a root sequence for $w$ in which the elements of $T$ appear consecutively (in some order), then we say that $T$ is contractible. We denote by $N(w)$ the number of contractible inversion triples of $w$, and by $\tilde{N}(w)$ the number of roots $\alpha$ such that $\alpha$ is the highest root of at least one contractible inversion triple of $w$.

For brevity, we usually write “contractible triple” instead of “contractible inversion triple”. Observe that $\tilde{N}(w) \leq N(w)$ for all $w \in W$.

The following basic properties of inversion triples will be used freely in the sequel. (Further details may be found in [7, §3.1].)
Lemma 3.2.

(i) In the type A setting, every inversion triple is contractible, but this does not hold in general.

(ii) Suppose that \((\ldots, \alpha, \gamma, \beta, \ldots)\) is a root sequence for some \(w \in W\), and the following long braid move can be applied: \((\ldots, \alpha, \gamma, \beta, \ldots) \rightarrow (\ldots, \beta, \gamma, \alpha, \ldots)\). Then \(\{\alpha, \beta, \gamma\}\) is a contractible triple of \(w\) with highest root \(\gamma\).

(iii) Suppose that \(\{\alpha, \beta, \alpha + \beta\}\) is an inversion triple of \(w\). In every root sequence for \(w\), the root \(\alpha + \beta\) must appear between \(\alpha\) and \(\beta\).

(iv) If \(T\) and \(T'\) are inversion triples of the same element of \(W\) and \(\#(T \cap T') > 1\), then \(T = T'\). \(\square\)

Definition 3.3 [7, Definition 3.1.6]. Let \(w \in W\). Suppose that for any pair of intersecting contractible triples \(T\) and \(T'\) of \(w\), the highest roots in \(T\) and \(T'\) agree. Then we say that \(w\) is maximally clustered. If \(W\) has finitely many maximally clustered elements, then we will call \(W\) MC-finite.

4. Contracted reduced expressions.

We are almost ready to define a key concept for the proof of our main result, namely that of a contracted reduced expression for a maximally clustered element.

Definition 4.1. Let \(w \in W\) be maximally clustered. Let \(C\) be a collection of pairwise-intersecting contractible triples of \(w\). Suppose that \(C\) is nonempty and not properly contained in another set of pairwise-intersecting contractible triples of \(w\). Then we call \(C\) a maximal set of triples for \(w\).

Definition 4.2. Let \(w \in W\) be maximally clustered, and let \(r\) be a root sequence for \(w\). Suppose that \(C\) is a maximal set of triples for \(w\), and that

\[r = (\ldots, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n, \gamma, \beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1, \ldots),\]

where \(\{\{\alpha_1, \beta_1, \gamma\}, \ldots, \{\alpha_n, \beta_n, \gamma\}\} = C\). Then we say that \(r\) is contracted for \(C\). We say that a root sequence for \(w\) is contracted if it is contracted for every maximal
set of triples for \( w \). We call a reduced expression for \( w \) contracted if its root sequence is contracted.

**Definition 4.3.** Suppose that \( i \in I^* \) is of the form \( i = i_1i_2\cdots i_ni_{n+1}i_n\cdots i_2i_1 \) \((n \geq 1)\), where \( i_1, i_2, \ldots, i_{n+1} \in I \) are distinct and where, for each \( 1 \leq q \leq n \), there is a unique \( q < r \leq n+1 \) such that \( m_{i_qi_r} = 3 \). We call \( i \) a braid cluster.

The following result can be used to reduce questions about maximally clustered elements to questions about braid clusters.

**Proposition 4.4 ([7, Corollary 4.3.3]).** Let \( w \in W \) be maximally clustered. The following statements hold:

(i) Every reduced expression for \( w \) is short braid equivalent to a contracted reduced expression.

(ii) If \( i \) is a contracted reduced expression for \( w \), then \( i = i_0c_1i_1c_2i_2\cdots c_{\tilde{N}(w)}i_{\tilde{N}(w)} \), where each \( c_q \) is a braid cluster, of length \( 2n_q + 1 \) say, and \( N(w) = \sum q n_q \).

The next lemma will be useful in the sequel.

**Lemma 4.5.**

(i) Suppose that \( w \in W \) is maximally clustered and that \( i \in I \) satisfies \( \ell(ws_i) < \ell(w) \). Then \( ws_i \) is maximally clustered.

(ii) Suppose that \( w \in W \) is maximally clustered and that \( i \in I \) satisfies \( \ell(s_iw) < \ell(w) \). Then \( s_iw \) is maximally clustered.

(iii) If \( i = i_1\cdots i_n \) is a reduced expression for a maximally clustered element of \( w \), then every subword of \( i \) is a reduced expression for some maximally clustered element.

**Proof.** Part (ii) may be seen to hold by extending any root sequence for \( s_iw \) to a root sequence for \( w \) by appending a single element. It then follows that the contractible triples for \( s_iw \) are a subset of the contractible triples for \( w \).

Observe that (iii) follows immediate from (i) and (ii) using an inductive argument, so it remains to prove (i).
Suppose that $w \in W$ is maximally clustered. If $r$ is a root sequence for $ws_i$, then $(\alpha_i, s_i(r))$ is a root sequence for $w$. Combining this with the fact that $s_i|_V$ is linear and preserves the Coxeter form, we find that every contractible triple $T$ of $ws_i$ gives rise to a contractible triple $s_i(T)$ of $w$. Thus, $N(ws_i) \leq N(w)$, and the inequality is strict if $\alpha_i$ lies in a contractible triple of $w$. Further, as $s_i|_V$ is injective, the fact that $w$ is maximally clustered implies that $ws_i$ is maximally clustered. □

5. Main results.

In order to understand the context for the main result of this paper, we now recall the notions of freely braided elements (introduced by the author and Losonczy in [4]) and fully commutative elements (introduced by Stembridge in [9]).

**Definition 5.1.** An element $w \in W$ is said to be **freely braided** if both (a) $w$ is maximally clustered and (b) the contractible triples of $w$ are disjoint.

An element $w \in W$ is said to be **fully commutative** if all reduced expressions for $w$ are short braid equivalent.

In the simply laced case, the fully commutative elements agree with the $iji$-avoiding elements introduced by Fan [3].

**Lemma 5.2.** Let $w \in W$.

(i) If $w$ is fully commutative, then $w$ is freely braided.

(ii) If $w$ is freely braided, then $w$ is maximally clustered.

*Proof.* It was shown in [5, Proposition 1.2.2] that $w$ is fully commutative if and only if $N(w) = 0$, from which it follows vacuously that $w$ is freely braided. This proves (i), and (ii) is immediate from the definitions. □

Our main result, which will be proved in §7, is as follows.

**Theorem 5.3.** Let $W$ be a simply laced Coxeter group. Then the following are equivalent:

(i) $W$ has finitely many maximally clustered elements;

(ii) $W$ has finitely many freely braided elements;
(iii) $W$ has finitely many fully commutative elements.

Remark 5.4. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are immediate from Lemma 5.2.

6. The operators $\pi_j$ on braid clusters.

Throughout §6, we assume that $W$ is a Coxeter group of type $E_n$, whose Coxeter graph is as shown in Figure 1.

Definition 6.1. Let $c = i_1 i_2 \cdots i_{n-1} i_n i_{n-1} \cdots i_2 i_1$ be a braid cluster. We define $\Gamma_c$ (the subgraph of $\Gamma$ induced by $c$) to be the full subgraph whose vertices are the letters $i_j$ appearing in a(ny) reduced expression for $c$. If $1 \leq j < k \leq n$ and we have $m(i_j, i_k) = 3$, then we orient $\Gamma_c$ by adding an arrow pointing from $i_j$ to $i_k$.

The next result follows from the above definition and [7, Definition 4.1.1].

Lemma 6.2. The orientation of $\Gamma_c$ given by Definition 6.1 assigns a unique arrow to each edge. The orientation is characterized by the fact that the vertex $i_n$ is the unique sink of $\Gamma_c$. □

Lemma 6.3. If $c$ is a braid cluster, then $\Gamma_c$ is a Coxeter graph of type $A$, $D$ or $E$.

Proof. It follows from Definition 4.3 that $\Gamma_c$ is connected and has no circuits. The remaining assertions follow from the fact that $\Gamma_c$ is a subgraph of $\Gamma$, which is of type $E_n$. □

Definition 6.4. Let $c$ be a braid cluster with middle letter $i_n$. We say that $c$ is normalized if either (i) $\Gamma_c$ has no branch point and $i_n$ is an extremal vertex of $\Gamma_c$ or (ii) $i_n$ is the (unique) branch point of $\Gamma_c$.

We define a weak braid cluster $\overline{c}$ to be any reduced expression for a group element represented by a braid cluster $c$. If $w \in W$ is maximally clustered, we define a weakly contracted reduced expression for $w$ to be an expression of the form

$$\overline{1} = i_0 \overline{c}_1 i_1 \overline{c}_2 i_2 \cdots \overline{c}_{N(w)} i_{N(w)}$$
such that (a) for each $i$, $\overline{c}_i$ and $c_i$ are reduced expressions for the same element and
(b) $i = i_0 c_1 i_1 c_2 i_2 \cdots c_{N(w)} \overline{i}_{N(w)}$ is a contracted reduced expression for $w$.

**Definition 6.5.** Let $w \in W$ be maximally clustered, and let $i$ be a weakly contracted reduced expression for $w$ with

$$i = i_0 \overline{c}_1 i_1 \overline{c}_2 i_2 \cdots \overline{c}_{N(w)} \overline{i}_{N(w)};$$

where each $\overline{c}_q$ is a maximal weak braid cluster, of length $2n_q + 1$ say, and $N(w) = \sum_q n_q$. If $c_j = \overline{c}_j = c'_{ij} c''_{ij}$ is a normalized braid cluster with middle letter $i_j$ and induced subgraph $\Gamma_j = \Gamma_{c_j}$, then we define $\pi_j(i) = i' c'_j i''$, where $i' = i_0 c_1 i_1 \cdots i_{j-1}$, $i'' = i_j c_{j+1} \cdots \overline{i}_{N(w)}$, and

$$c'_j = \begin{cases} c''_{ij} & \text{if } \Gamma_j \text{ has a branch point and } i' i_j \text{ is not reduced,} \\ i_j c''_{ij} & \text{otherwise.} \end{cases}$$

**Lemma 6.6.** Let $w \in W$ be maximally clustered, and let $c$ be a maximal normalized braid cluster in a weakly contracted reduced expression $i = i' c i''$ for $w$.

(i) If $i$ is a letter of $c$ that is not a branch point of the induced subgraph $\Gamma_c$, then $\forall i$ and $ii''$ are reduced.

(ii) If $s$ is a letter and we can parse $i' = v_1 s v_2$ and $i'' = v_3 s v_4$ so that $v_2$ and $v_3$

each consists of letters commuting with $s$, then $s$ is a branch point and $s$ occurs in $c$.

**Proof.** We will prove (i) for the case of $i' i'$; the other case follows similarly. Let $c = i_1 i_2 \cdots i_{n-1} i_n i_{n-1} \cdots i_2 i_1$, and define $k$ to be the unique integer $1 \leq k \leq n$ such that $i_k = i$. Since $i_k$ is not a branch point, there is at most one vertex $i'$ in $\Gamma_c$ such that there is an arrow from $i'$ to $i$. If no such $i'$ exists, then $i c$ is not reduced, and (i) follows by the exchange condition.

If such an $i'$ does exist, then $i' = i_j$ for some $j < k$, and we have

$$i c = i i_1 i' i_2 i_3,$$

where $i_1$ and $i_2$ consist entirely of generators distinct from $i$ that commute with $i$. It follows that the letter $i'$ is involved in two contractible inversion triples: one
involving to the occurrences of $i, i', i$ listed above, and one involving the occurrences of $i_j, i_n, i_j$ in $c$. Since $i'$ is the highest root in the first triple but not in the second, we conclude that $ic$ is not maximally clustered. Since $w$ is maximally clustered, all its subwords must be as well, by Lemma 4.5 (iii), and (i) follows by the exchange condition.

For (ii), let us first suppose that $s$ does not occur in $c$. Since $scs$ is reduced, $c$ must contain a letter not commuting with $s$. Suppose that we have $1 \leq j < k \leq n$ such that $i_j$ and $i_k$ both fail to commute with $s$. Since $\Gamma_c$ is a tree (by Lemma 6.3) and $s$ does not occur in $s$, $s$ completes the unique path in $\Gamma_c$ from $i_j$ to $i_k$ into a circuit in the Coxeter graph of $W$, which is a contradiction. We conclude that there is a unique $i_j$ not commuting with $s$. It now follows from Definition 4.3 that $scs$ is a braid cluster, contradicting the maximality of $c$. This proves that $s$ occurs in $c$, and an application of (i) shows that $s$ must be a branch point. □

**Lemma 6.7.** Let $w \in W$ be maximally clustered, let $i$ be a weakly contracted reduced expression for $w$ in which cluster $c_j$ is normalized, and let $\pi_j(i) = i'c_ji''$ be as in Definition 6.5, so that $i = i'c_ji''$. If we have

$$\pi_j(i) = u_1su_2su_3$$

for some letter $s$, then $u_2$ must contain a letter not commuting with $s$.

**Proof.** Suppose that $u_2$ consists entirely of letters commuting with $s$.

If the right-hand occurrence of $s$ shown comes from $i'$, then this would contradict the fact that $u_1$ is reduced.

If the right-hand occurrence of $s$ comes from $i''$, then the left-hand occurrence of $s$ must come from $i'$, otherwise $c'_ji''$ and $c_ji''$ would not be reduced. In this case, we must have $i' = v_1sv_2$ and $i'' = v_3sv_4$, where $v_2$ and $v_3$ consist of generators commuting with $s$, which implies that $v_1v_2(sc_js)v_3v_4$ is a reduced expression for $w$, and that $c_j$ contains a generator not commuting with $s$. If $c_j$ and $c'_j$ contain the same set of generators (disregarding multiplicities), then $c'_j$ contains a generator
not commuting with $s$ and the conclusion follows. The only other possibility is that we are in the case $c'_j = c''$ of Definition 6.5, and the preceding argument still works unless $s$ is adjacent to the branch point of $\Gamma_j$. Since all elements of the Coxeter graph $\Gamma$ adjacent to the branch point lie in $\Gamma_j$, it follows that $s$ occurs in the braid cluster $c_j$, but that $s$ is not itself a branch point, and this contradicts Lemma 6.6 (ii).

We may now assume that the rightmost occurrence of $s$ shown lies in the subword $c'_j$. Since $c'_j$ consists of distinct generators, the leftmost occurrence of $s$ must come from $i'$, and $w$ has a reduced expression of the form $vs c'_j i''$. Lemma 6.6 (i) now forces $s$ to be the branch point of $\Gamma_j$. Since $vs$ is reduced, $i'$'s cannot be reduced, and Definition 6.5 shows that we are in the case $c'_j = c''$, and this is a contradiction because $s$ does not occur in $c''$. □

**Lemma 6.8.** Let $w \in W$ be maximally clustered, and let $i$ be as in Lemma 6.7, with $\pi_j(i) = i' c'_j i''$ and $i = i' c_j i''$. Suppose that

$$\pi_j(i) = u_1 s u_2 t u_3 s u_4$$

for some noncommuting letters $s$ and $t$, and that $u_2$ and $u_3$ consist of generators commuting with $s$. Then the indicated occurrences of $s$ and $t$ come from the same weak braid cluster $c_i$ (where $i \neq j$).

**Note.** Note that $i$ itself satisfies the condition claimed for $\pi_j(i)$, by properties of weakly contracted reduced expressions.

**Proof.** Assume for a contradiction that the indicated occurrences of $s$ and $t$ do not all come from the same braid cluster.

If the right-hand occurrence of $s$ shown comes from $i'$, then the conclusion follows from the observations that $i'$ is maximally clustered and its braid clusters are a subset of those of $i$.

If the right-hand occurrence of $s$ comes from $i''$, then the left-hand occurrence of $s$ must come from $i'$, as in the proof of Lemma 6.7. Assume that the two occurrences
of $s$ do not come from the same cluster. By the Note above, these occurrences must be separated in $i$ by at least two generators not commuting with $s$, and it follows that one of these two (an occurrence of $u$, say) must have been deleted by $\pi_j$, and that $u = t$. In turn, this means that we have both $i' = v_1sv_2$ and $i'' = v_3sv_4$ so that $v_2$ and $v_3$ each consists of letters commuting with $s$. By Lemma 6.6 (ii), $s$ occurs in $c$, and $s$ is a branch point of $\Gamma_j$. This means that the occurrences of $s$ in $\pi_j(i)$ are separated by occurrences of three different generators not commuting with $s$, a contradiction.

We may now assume that the rightmost occurrence of $s$ comes from $c'_j$. Suppose first that $s$ is not the leftmost letter of $c'_j$; in particular, $s$ is not a branch point of $\Gamma_j$. Let $u$ be the unique letter of $c_j$ such that there is an arrow from $s$ to $u$ in $\Gamma_j$. Since $u$ lies between the two occurrences of $s$ in $\pi_j(i)$, we must have $u = t$, so that in particular, $t$ lies in $c'_j$. This implies that $i' = vsv'$, where $v'$ consists of generators commuting with $s$, which contradicts Lemma 6.6 (i).

From now on, we may assume that $s$ is the leftmost letter of $c'_j$, and therefore that $i' = v_1sv_2tv_3$, where $v_2$ and $v_3$ consist of generators commuting with $s$.

Suppose first that we are in the case $c'_j = c''$ of Definition 6.5, and let $u \neq s$ be the middle letter of $c_j$. Then $u$ is a branch node, $s$ is not, and there are precisely two generators in $S$, $u$ and $u'$, that do not commute with $s$. We have $c_j = c'suc''$, where $c'$ contains no occurrences of $u$, and the expression for $i$ in the previous paragraph shows that $i'c_j$ is short braid equivalent to a reduced expression containing the subword $stsus$. If $t = u$, this is not reduced, and if $t = u'$, this is not maximally clustered, as the middle $s$ is involved in two contractible inversion triples without corresponding to the highest root of either.

We must therefore have $c'_j = sc''$ in Definition 6.5, so that $s$ is the middle letter of the cluster $c_j$. We also have $i' = v_1tv_2$, where every letter of $v_2$ commutes with $t$.

Suppose first that $s$ is a branch point of $\Gamma_j$. In this case, $t$ is not a branch point, but since $t$ is adjacent to $s$ (both in $\Gamma$ and in $\Gamma_j$), it must be the case that $t$ occurs
in $c_j$. The fact that the expression $i' = v_1 tv_2$ is reduced now contradicts Lemma 6.6 (i).

The only other possibility is that $s$ is an endpoint of $\Gamma_j$. In this case, there is a unique generator $u$ appearing in $c_j$ such that $su \neq us$, so that we have $c_j = v_3 us u v_4$, where every letter of $v_3$ commutes with $s$. Since $c_j$ is normalized, $u$ is not a branch point of $\Gamma_j$, and Lemma 6.6 (i) implies that $i'u$ is reduced, meaning that $t \neq u$ and $t$ does not occur in $c_j$. Since the Coxeter graph contains no circuits and $t$ is adjacent to $s$, Definition 4.3 shows that $t$ commutes with every generator in $c_j$ apart from $s$. It follows that $i$ is short braid equivalent to a reduced expression with $stusu$ as a subword, but $stusu = stsus$ is not maximally clustered, a contradiction. □

**Lemma 6.9.** Let $w \in W$ be maximally clustered, and let $i$ be as in Lemma 6.7, with $\pi_j(i) = i'c_j'i''$ and $i = i'_c j_i''$. Then $\pi_j(i)$ is a reduced expression for some element $w' \in W$, and every reduced expression for $w'$ is short braid equivalent to $\pi_j(k)$ for some weakly contracted reduced expression $k$ for $w$.

**Proof.** We apply the well known theorem on braid moves due to Matsumoto and Tits, as stated in [1, Theorem 3.3.1], starting with the expression $\pi_j(i)$.

By Lemma 6.7, it is not possible to apply a sequence of short braid relations followed by removal of a consecutive pair $ss$ to $\pi_j(i)$. Suppose instead that we apply a sequence of short braid relations followed by a long braid relation to $\pi_j(i)$. By Lemma 6.8, this long braid relation involves three letters from a weak braid cluster $c_i$, where $i \neq j$. Applying this long braid relation to $i$ results in another weakly contracted reduced expression in which the cluster $c_j$ is still normalized. It follows that application of the long braid relation commutes (up to short braid equivalence) with the map $\pi_j$, as required. The procedure may now be iterated.

By [1, Theorem 3.3.1], any reduced expression for $i$ can be obtained by application of short and long braid relations, together with excision of any consecutive pairs $ss$. Iterating the procedure of the last paragraph shows that we never have
an opportunity to remove a pair of the form $ss$, which means that $\pi_j(i)$ is reduced. The claim about short braid equivalence also follows immediately from the proof in the previous paragraph. \hfill \Box

Lemma 6.10. Let $w \in W$ be maximally clustered, and let $i$ be a weakly contracted reduced expression for $w$. Then $\pi_1(i)$ is a reduced expression for $w' \in W$, where $w'$ is maximally clustered and $\tilde{N}(w') = \tilde{N}(w) - 1$.

Proof. We know from Lemma 6.9 that $w'$ is reduced. If $T$ is a contractible inversion triple for $w'$, then there is a reduced expression $\mathbf{vsts}w'$ for $w'$ in which $s$ and $t$ are noncommuting generators and the subword $sts$ corresponds to the triple $T$. By Lemma 6.9, there is a weakly contractible reduced expression

$$\mathbf{i} = i_0c_1i_1c_2i_2 \cdots c_{\tilde{N}(w)}i_{\tilde{N}(w)}$$

for $w$ such that $\mathbf{vsts}w'$ is short braid equivalent to $\pi_1(\mathbf{i})$. By Lemma 6.8, the indicated occurrences of $s$ and $t$ come from the same weak braid cluster, $\mathbf{c}_i$, where $i > 1$. Since $w$ and $w'$ both have reduced expressions ending in

$$i_1c_2i_2 \cdots c_{\tilde{N}(w)}i_{\tilde{N}(w)},$$

it follows that the triple $T$ is also a contractible inversion triple for $w$, and conversely that every contractible inversion triple of $w$ not arising from the cluster $\mathbf{c}_1$ is a contractible inversion triple of $w'$. The assertions now follow. \hfill \Box

Lemma 6.11. Let $w \in W$ be maximally clustered, and let $i$ be a contracted reduced expression for $w$. Applying the operator $\pi_1 \tilde{N}(w)$ times to $i$, we obtain a reduced expression for a fully commutative element.

Proof. By Lemma 6.10, the element obtained is a maximally clustered element $w'$ with $\tilde{N}(w') = 0$. Since $w'$ has no contractible inversion triples, it follows that no reduced expression for $w'$ can have a subword of the form $iji$, so that $w'$ is fully commutative. \hfill \Box
7. Proof of Theorem 5.3.

A key ingredient of the proof of our main result is the following

**Proposition 7.1.** Let $W$ be a simply laced Coxeter group. Then the following are equivalent:

(i) $W$ has finitely many fully commutative elements;

(ii) the connected components of the Coxeter graph of $W$ are subgraphs of the Coxeter graph of type $E_n$ (see Figure 1).

**Proof.** This is a special case of Stembridge’s result [9, Theorem 4.1]. □

**Theorem 7.2.** A Coxeter group $W$ of type $E_n$ (for any $n \geq 6$) has finitely many maximally clustered elements.

**Proof.** For each maximally clustered element $w \in W$, choose a contracted reduced expression $i(w)$. By Lemma 6.11, there is a map $f$ from the set $\{i(w) : w \in W\}$ to the set of reduced expressions for fully commutative elements of $W$ given by

$$f(i(w)) = \pi_1^{\tilde{N}(w)}(i(w)).$$

By Proposition 7.1, there are only finitely many fully commutative elements of $W$, and since each one has only finitely many reduced expressions, the problem reduces to showing that the fibres of $f$ are finite.

If we write

$$i(w) = i_0 c_1 i_1 c_2 i_2 \cdots c_{N(w)} i_{N(w)};$$

then

$$f(i(w)) = i_0 c'_1 i_1 c'_2 i_2 \cdots c'_{N(w)} i_{N(w)},$$

where $c'_i$ is as in Definition 6.5. Given any reduced expression in the image of $f$, there are only finitely many ways to select the subwords $c'_i$, and for each such subword, there are at most two possible $c_i$ that could have given rise to it. It follows that the fibres of $f$ are finite, as required. □
Proof of Theorem 5.3. By Remark 5.4, it only remains to prove the implication (iii) \( \Rightarrow \) (i), and the problem immediately reduces to the case where the Coxeter graph of \( W \) is connected. Suppose that \( W \) has finitely many fully commutative elements. By Proposition 7.1, the Coxeter graph of \( W \) is a subgraph of the Coxeter graph of type \( E_n \), so it is enough to deal with the case of \( W \) being of type \( E_n \). The result now follows from Theorem 7.2. \( \square \)

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