Simulation on Natural Convection of a Nanofluid along an Isothermal Inclined Plate

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Abstract: A numerical algorithm is presented for studying laminar natural convection flow of a nanofluid along an isothermal inclined plate. By means of similarity transformation, the original nonlinear partial differential equations of flow are transformed to a set of nonlinear ordinary differential equations. Subsequently they are reduced to a first order system and integrated using Newton Raphson and adaptive Runge-Kutta methods. The computer codes are developed for this numerical analysis in Matlab environment. Dimensionless velocity, temperature profiles and nanoparticle concentration for various angles of inclination are illustrated graphically. The effects of Prandtl number, Brownian motion parameter and thermophoresis parameter on Nusselt number are also discussed. The results of the present simulation are then compared with previous one available in literature with good agreement.

Keywords: Natural Convection, Nano Fluid; Brownian Motion, Thermophoresis, Isothermal Inclined Plate,

I. Introduction
Natural convection flow past a vertical plate is a classical problem in convective heat transfer analysis [1-4]. Natural convective boundary-layer flow of a nanofluid past an isothermal vertical plate was studied by Kuznetsov and Nield [5]. Afterwards, Khan and Aziz [6] studied the Kuznetsov and Nield problem for uniform heat flux boundary condition. The aim of the present paper is to extend the work of Kuznetsov and Nield to study the natural convection flow of a nanofluid along an inclined isothermal plate. In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow and heat transfer over an isothermal horizontal plate is developed. The paper is organized as follows: Mathematical model of the problem, its solution procedure, and development of codes in Matlab, results and discussion, conclusion.

II. Mathematical Model
We assume the natural convection flow to be steady, laminar, two-dimensional. We take the direction along the plate to be x, and the direction normal to surface to be y. The plate is inclined at an angle of γ from the vertical plane. At the plate the temperature T and nanoparticle fraction φ take constant values Tw and φw respectively. Far away from the plate, the ambient values are T∞ and φ∞ respectively. Following Oberbeck-Boussinesq approximation, the equations governing the flow are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\rho_f \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} + [(1-\phi_w) \rho_f \beta_f g (T - T_w) - (\rho_p - \rho_f) g (\phi - \phi_w)] \cos \gamma
\]

\[
(\rho c)_f \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} + (\rho c)_f \left[ D_\phi \frac{\partial \phi}{\partial y} + \left( \frac{D_T}{T_w} \right) \frac{\partial T}{\partial y} \right]
\]
\[ u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = D_B \frac{\partial^2 \phi}{\partial y^2} + \left( \frac{D_c}{T_{\infty}} \right) \frac{\partial^2 T}{\partial y^2} \]

where, velocity components along \( x \) and \( y \) are represented by \( u \) and \( v \), respectively, \( \rho_f \) is the density of the base fluid and \( \rho_p \) is the density of the nanoparticles, \( k \) and \( \beta \) are the viscosity, thermal conductivity and volumetric expansion coefficient of the nanofluid, \( g \) is the acceleration due to gravity, \( (\rho c)_f \) is the heat capacity of the fluid, \( (\rho c)_p \) is the effective heat capacity of the nanoparticle material, \( D_B \) and \( D_T \) are the Brownian diffusion coefficient and thermophoretic diffusion coefficient, respectively.

The boundary conditions on the solution are:

At \( y = 0 \): \( u = v = 0 \), \( T = T_w \), \( \phi = \phi_w \)

For large \( y \): \( u = v = 0 \), \( T = T_\infty \), \( \phi = \phi_\infty \)

The continuity equation (1) is automatically satisfied through introduction of the stream function:

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

With the introduction of the following similarity variables

\[ \eta = \left( \frac{y}{x} \right) Ra_x^{\frac{1}{4}} \]

\[ s(\eta) = \frac{\psi}{\alpha Ra_x^{\frac{1}{4}}} \]

\[ \theta (\eta) = \frac{T - T_w}{T_e - T_w} \]

\[ f (\eta) = \frac{\phi - \phi_w}{\phi_\infty - \phi_w} \]

equations (2), (3) and (4) may be rewritten as (with a prime denoting differentiation with respect to \( \eta \))

\[ s''' + \frac{1}{4 \Pr} (3 s s'' - 2 s') + (\Theta - N r f) \cos \gamma = 0 \]

\[ \Theta''' + \frac{3}{4} s \Theta' + N b f \Theta' + N t \Theta' r^2 = 0 \]

\[ f''' + \frac{3}{4} L e s f' - \frac{3}{4 N_b} s \Theta' - N f \Theta' - \frac{N t^2}{N_b} \Theta' r^2 = 0 \]

The appropriate boundary conditions are:

\[ s = 0, \quad s' = 0, \quad \Theta = 1, \quad f' = 1, \quad at \quad \eta = 0 \]

\[ s' = 0, \quad \Theta = 0, \quad f = 0, \quad at \quad \eta = \infty \]

where Local Rayleigh number \( Ra_x \), the buoyancy-ratio parameter (\( N_r \)), Brownian motion parameter (\( N_b \)), thermophoresis parameter (\( N_t \)), Prandtl number (\( \Pr \)) and Lewis number (\( Le \)) are defined as follows:

\[ Ra_x = \frac{(1 - \phi_w) \beta g (T_w - T_e) x^3}{v \alpha} \]

\[ N_r = \frac{(\rho - \rho_{\infty})(\phi_w - \phi_w)}{\rho_{\infty} \beta (T_w - T_e)(1 - \phi_w)} \]

\[ N_b = \frac{(\rho c)_p D_B (\phi_w - \phi_w)}{(\rho c) \beta} \]

\[ N_t = \frac{(\rho c)_p D_c (T_w - T_e)}{(\rho c) \alpha T_\infty} \]
Pr = \frac{c_p \mu}{k} = \frac{\mu}{\rho k} = \frac{V}{\alpha}

Le = \frac{\alpha}{D_b}

Local Nusselt number is defined as

\[ Nu_s = \frac{q_w x}{k (T_w - T_v)} \]

where \( q_w \) is the wall heat flux. The reduced local Nusselt number is

\[ Nu_r = \frac{Nu}{Ra} = -\theta' (0) \]

III. Solution Procedure

Eqs (8), (10) and (10) are coupled nonlinear ordinary differential equations. There are three unknown initial values at the wall: \( s''(0), \theta'(0) \) and \( f'(0) \).

**Reduction of Equations to First-order System**

This is done easily by defining new variables:

\[ z_1 = s \]
\[ z_2 = z'_1 = s' \]
\[ z_3 = z''_1 = z'_2 = s'' \]
\[ z_4 = \theta \]
\[ z_5 = z'_4 = \theta' \]
\[ z_6 = f \]
\[ z_7 = z'_6 = f' \]

Therefore from eqs (8), (9) and (10), we get the following set of differential equations

\[ z'_1 = z'_2 \]
\[ z'_2 = z''_1 = z_3 \]
\[ z'_3 = z''_1 = z'_1 = z_3 \]
\[ z'_4 = \theta' = z_5 \]
\[ z'_5 = \theta'' = \theta'' = -\frac{3}{4} z_1 z_5 - Na z_4 z_7 - Nt z'_5 \]
\[ z'_6 = f' = z_7 \]
\[ z'_7 = f'' = z'_6 = \frac{3}{4} Le z_7 + \frac{3}{4} Nt z_5 z_7 + Nt z_5 z_7 + Nt^2 z_5 \]

Eq (8) is third-order and is replaced by three first-order equations, whereas eqs (9) and (10) are second-order each and are replaced with two first-order equations each. The boundary conditions are then:
Solution to Initial Value Problems

To solve eqs (17), we denote the three unknown initial values by $a_1$, $a_2$ and $a_3$, the set of initial conditions is then:

$$
\begin{align*}
    z_1(0) &= s(0) = 0 \\
    z_2(0) &= s'(0) = 0 \\
    z_3(0) &= s''(0) = a_1 \\
    z_4(0) &= \theta(0) = 1 \\
    z_5(0) &= \theta'(0) = a_2 \\
    z_6(0) &= f(0) = 1 \\
    z_7(0) &= f'(0) = a_3
\end{align*}
$$

If eqs (17) are solved with Runge-Kutta method using the initial conditions in eq (19), the computed boundary values at $\eta = \infty$ depend on the choice of $a_1$, $a_2$ and $a_3$ respectively. We express this dependence as

$$
\begin{align*}
    z_2(\infty) &= s'(\infty) = f_1(a_1) \\
    z_4(\infty) &= \theta(\infty) = f_2(a_2) \\
    z_6(\infty) &= f(\infty) = f_3(a_3)
\end{align*}
$$

The correct choice of $a_1$, $a_2$ and $a_3$ yields the given boundary conditions at $\eta = \infty$; that is, it satisfies the equations

$$
\begin{align*}
    f_1(a_1) &= 0 \\
    f_2(a_2) &= 0 \\
    f_3(a_3) &= 0
\end{align*}
$$

These nonlinear equations can be solved by the Newton-Raphson method. A value of 15 is fine for infinity, even if we integrate further nothing will change.

Program Details

Using Newton Raphson and adaptive Runge-Kutta methods a set of Matlab routines for the solution of eqs (17) along with the boundary conditions (19) are developed and are shown in Table 1.

| Matlab code          | Brief Description                                                                 |
|----------------------|----------------------------------------------------------------------------------|
| deqs.m               | Defines the differential equations (17)                                          |
| incond.m             | Describes initial values for integration, $a_1$, $a_2$ and $a_3$ are guessed values, eq (19) |
| runKut5.m            | Integrates as initial value problem using adaptive Runge-Kutta method            |
| residual.m           | Provides boundary residuals and approximate solutions                           |
| newtonraphson.m      | Provides correct values $a_1$, $a_2$ and $a_3$ using approximate solutions from residual.m |
| runKut5.m            | Again integrates eqs (17) using correct values of $a_1$, $a_2$ and $a_3$        |
The final output of these codes give the tabulated values of \( s, \ s', \ s'', \ \theta, \ \theta', \ \phi, \ \phi' \) as functions of \( \eta \).

**IV. Results and Discussion**

**Agreement with previous works.**
The numerical results obtained from the abovementioned codes are validated by comparing with previously published results [5] for the case of vertical plate in Table 2 and are found in very good agreement.

Table 2. Comparison of the present reduced Nusselt number (Nur) values with previously published results [1] when \( Le = 10, \ Nr = Nb = Nt = 10^{-5} \) and \( \gamma = 0^\circ \).

| Pr   | Nur[5] | Nur [present] |
|------|--------|---------------|
| 1    | 0.401  | 0.401         |
| 10   | 0.463  | 0.465         |
| 100  | 0.481  | 0.482         |
| 1000 | 0.484  | 0.486         |

**Effect of angles of inclination (\( \gamma \) on Dimensionless velocity (\( S' \)), temperature profiles (\( \theta \)) and nanoparticle concentration (\( f \)).**
Variations of dimensionless velocity, temperature profiles and nanoparticle concentration with \( \eta \) for various angles of inclination are shown graphically in figures 1 to 3 for \( Pr = 0.71, \ Le = 10, \ Nr = Nb = Nt = 0.2 \).

Figure 1 Velocity profiles at different inclination angle
Figure 2 Temperature profiles at different inclination angle

Figure 3 Nanoparticle concentration profiles at different inclination angle

Effect of angle of inclination ($\gamma$), Brownian motion parameter ($Nb$) and thermophoresis parameter ($Nt$) on reduced Nusselt number ($Nur$).

Variations of reduced Nusselt number ($Nur$) with Prandtl ($Pr$) for various parameters (angle of inclination, Brownian motion parameter and thermophoresis parameter) are illustrated graphically in figures 4 to 6.

Figure 4 Nusselt number variation with $Pr$ at different inclination angle
Figure 5 Nusselt number variation with Pr for different values of Brownian motion parameter

Figure 6 Nusselt number variation with Pr for different values of thermophoresis parameter

V. Conclusions
In the present numerical simulation, steady, laminar, two-dimensional flow of a nanofluid along an isothermal inclined plate is presented. Details of the solution procedure of the nonlinear partial differential equations of flow are discussed. The computer codes are developed in Matlab environment. Using these codes, dimensionless velocity, temperature profiles, nanoparticle concentration and reduced Nusselt number are computed and illustrated graphically for various values of the parameters (angles of inclination, Brownian motion parameter and thermophoresis parameter). The computed values are in very good agreement with results published in literatures.

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