Evaluating Thin Flat Surfaces

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Abstract: We consider recognizable evaluations for a suitable category of oriented two-dimensional cobordisms with corners between finite unions of intervals. We call such cobordisms thin flat surfaces. An evaluation is given by a power series in two variables. Recognizable evaluations correspond to series that are ratios of a two-variable polynomial by the product of two one-variable polynomials, one for each variable. They are also in a bijection with isomorphism classes of commutative Frobenius algebras on two generators with a nondegenerate trace fixed. The latter algebras of dimension \( n \) correspond to points on the dual tautological bundle on the Hilbert scheme of \( n \) points on the affine plane, with a certain divisor removed from the bundle. A recognizable evaluation gives rise to a functor from the above cobordism category of thin flat surfaces to the category of finite-dimensional vector spaces. These functors may be non-monoidal in interesting cases. To a recognizable evaluation we also assign an analogue of the Deligne category and of its quotient by the ideal of negligible morphisms.

Contents

1. Introduction ................................. 1836
2. The Category of Thin Flat Surfaces .............. 1838  
\hspace{1em} 2.1 Category TFS .............................. 1838  
\hspace{1em} 2.2 Classification of thin flat surfaces ............ 1840  
\hspace{1em} 2.3 Endomorphisms of 1 and homs between 0 and 1 in TFS .... 1844
3. Linearizations of the Category TFS ................. 1846  
\hspace{1em} 3.1 Categories kTFS and VTFS\( \alpha \) for recognizable \( \alpha \) ............... 1847  
\hspace{1em} 3.2 Skein category STFS\( \alpha \) .......................... 1850  
\hspace{1em} 3.3 Quotient by negligible morphisms and Karoubi envelopes. ............ 1854  
\hspace{1em} 3.4 Summary of the categories and functors ................ 1856
4. Hilbert Scheme and Recognizable Series ........... 1857
5. Modifications ................................ 1861
1. Introduction

Universal constructions of topological theories [1,13,21] that are not necessarily multiplicative [9] are interesting even in dimension two [14,16], providing examples somewhat different from commutative Frobenius algebras for the invariants of two-dimensional cobordisms. In this note we consider the analogue of the latter construction for oriented two-dimensional cobordisms with corners. For simplicity we restrict to cobordisms between finite unions of intervals; boundary points of the intervals give rise to corners of cobordisms. Furthermore, we require that each connected component of a cobordism has non-empty boundary, which is a natural condition when excluding cobordisms with corners that have circles as some boundary components.

Cobordisms that we consider can be “thinned” to consist of ribbons glued to disks and can be depicted in the plane as regular neighbourhoods of immersed graphs, see Fig. 2 below for an example. For this reason we refer to these cobordisms as thin flat surfaces or tf-surfaces throughout the paper. When viewed as a morphism in the appropriate category TFS of thin flat cobordisms, a particular immersion of the surface into the plane is inessential, and morphisms are equivalence classes of such cobordisms modulo diffeomorphisms that fix the boundary.

The category TFS admits an analogue of $\alpha$-evaluations from [14–16]. This time closed connected morphisms $S$ (connected endomorphisms of the unit object 0, the empty union of intervals) are parametrized by two non-negative integers $(\ell, g)$, where $\ell + 1$ is the number of boundary components of $S$ and $g$ is the genus. Assigning an element $\alpha_{\ell, g}$ of the ground field $k$ (or a ground commutative ring $R$) to such a component and extending multiplicatively to disjoint unions gives an evaluation $\alpha$ on endomorphisms of the unit object 0. Evaluation $\alpha$ can be conveniently encoded as power series

$$Z_\alpha(T_1, T_2) = \sum_{k, g \geq 0} \alpha_{k, g} T_1^k T_2^g, \quad \alpha = (\alpha_{k, g})_{k, g \in \mathbb{Z}_+}, \quad \alpha_{k, g} \in k,$$

where the degree of the first variable $T_1$ counts “holes” in a cobordism (a disk has no holes and an annulus has one hole) and $T_2$ keeps track of the genus.

With $\alpha$ as above and $n \geq 0$, one can define a bilinear form on a $k$-vector space with a basis given by equivalence classes of thin flat surfaces with $n$ boundary intervals. The quotient by the kernel of the bilinear form is a vector space $A_\alpha(n)$. The collection of quotient spaces $\{A_\alpha(n)\}_{n \geq 0}$ is what we refer to as the universal construction for the category TFS, given $\alpha$.

The spaces $A_\alpha(n)$ rarely satisfy the Atiyah factorization axiom, that is, the relation

$$A_\alpha(m + n) \cong A_\alpha(m) \otimes A_\alpha(n)$$

does not hold. From the quantum field theory (QFT) perspective, this violation may happen if the 2-dimensional QFT is embedded as a 2-dimensional defect inside a higher-dimensional QFT.

It is straightforward to see that $A_\alpha(n)$ is finite-dimensional for all $n$ iff $A_\alpha(1)$ is finite-dimensional iff the series (1) is recognizable or rational (terms from the control...
Recognizable power series in this case have the form
\[ Z_\alpha(T_1, T_2) = \frac{P(T_1, T_2)}{Q_1(T_1)Q_2(T_2)}, \]
that is, \( Z_\alpha \) can be written as a ratio of a polynomial in \( T_1, T_2 \) and two one-variable polynomials, see Proposition 3.1 and Fliess [7].

Constructions of [16] go through for the category of thin flat surfaces and any recognizable \( \alpha \) as above. We define the category \( \text{STFS}_\alpha \) (skein thin flat surfaces) where homs are finite linear combinations of cobordisms, closed components evaluate to coefficients of \( \alpha \), and there are skein relations given by adding holes and handles to a component of a cobordism and equating to zero linear combinations corresponding to elements of the kernel ideal \( I_\alpha \subset k[T_1, T_2] \) associated to \( \alpha \) and also known as the syntactic ideal of rational series \( \alpha \). A two variable polynomial \( z = z(T_1, T_2) \) is in \( I_\alpha \) if \( \alpha(zf) = 0 \) for any polynomial \( f \in k[T_1, T_2] \), with \( \alpha(T_1^\ell T_2^g) = \alpha_{\ell,g} \) extended to a linear map \( k[T_1, T_2] \rightarrow k \).

For the rest of the paper we change our terminology and call connected components of a thin flat surface that have neither top nor bottom boundary intervals floating components instead of closed components, since they otherwise have boundary, what we call side boundary, that is present inside the cobordism but not at its top or bottom. This avoid possible confusion with the usual notion of a closed surface. A non-empty thin flat surface is never closed in the latter sense.

The category \( \text{STFS}_\alpha \) has finite-dimensional hom spaces. Taking the additive Karoubi envelope of this category to form \( \text{DTFS}_\alpha := \text{Kar}(\text{STFS}_\alpha) \) gives an idempotent-complete \( k \)-linear rigid symmetric monoidal category \( \text{DTFS}_\alpha \) which is the analogue of the Deligne category [3,4,6] for TFS and recognizable series \( \alpha \) in two variables.

Once we pass to \( k \)-linear combinations of cobordisms, and \( \alpha \) is available to evaluate floating cobordisms, there is a trace map on endomorphisms of any object \( n \). It is given by closing each term in the linear combination of tf-surfaces describing the endomorphism via \( n \) strips into a floating tf-surface and evaluating it via \( \alpha \). Consequently, one can form the ideal \( J_\alpha \) of negligible morphisms [3,4,6,16] and quotient the category by that ideal.

We call the quotient category glibigle quotient to avoid the awkward-sounding word “non-negligible quotient” and mirroring the terminology from [15]. The glibigle quotient \( \text{TFS}_\alpha \) of the skein category \( \text{STFS}_\alpha \) carries non-degenerate bilinear forms on its hom spaces and otherwise shares key properties of \( \text{STFS}_\alpha \): objects are non-negative integers, category \( \text{TFS}_\alpha \) is rigid symmetric tensor, and the hom spaces are finite-dimensional over \( k \).

Likewise, the Deligne category \( \text{DTFS}_\alpha \) has the glibigle quotient \( \text{DTFS}_\alpha \) by the ideal of negligible morphisms. The same category can be recovered as the additive Karoubi closure of \( \text{TFS}_\alpha \).

Section 3.4 and diagram (12) contain a summary of these categories and key functors relating them.

Similar to [5,6,15], it is natural to ask under what conditions will \( \text{DTFS}_\alpha \) be semisimple. Unlike [14,15], we do not work out any specific examples of these categories here and leave that to an interested reader or another time.

Our evaluation \( \alpha \) is encoded by a power series \( Z_\alpha \) in two variables (1), and the recognizable series \( Z_\alpha \) gives rise to a finite-codimension ideal \( I_\alpha \) in \( k[T_1, T_2] \), the largest ideal...
contained in the hyperplane ker(α). Such an ideal defines a point on the Hilbert scheme of the affine plane \( \mathbb{A}^2 \). We discuss the relation to the Hilbert schemes in Section 4 and explain a bijection between recognizable power series with the ideal \( I_\alpha \) of codimension \( k \) and points in the complement \( T_k^\vee \setminus D_k \) of the dual tautological bundle \( T_k^\vee \) on the Hilbert scheme and a suitable divisor \( D_k \) on it.

It is not clear whether the appearance of the Hilbert scheme of \( \mathbb{A}^2 \) is a bug or a feature. In Section 5 we explain two generalizations of our construction. One of them involves “coloring” side boundary components of a thin flat surface into \( r \) colors. For the resulting category, recognizable series depends on \( r + 1 \) parameters (generalizing from 2 parameters for \( r = 1 \)), and one would get a generalization of our construction from the Hilbert scheme of \( \mathbb{A}^2 \) to that of \( \mathbb{A}^{r+1} \), with the appropriate divisor removed from the dual tautological bundle in both cases. Of course, the Hilbert scheme has vastly different properties and uses in the case of algebraic surfaces versus higher dimensional varieties.

The other generalization considered in that section is given by extending the TFS (thin flat surfaces) category by allowing closed components and circles as boundaries. This corresponds to the usual category of two-dimensional oriented cobordisms with boundary and corners studied in [2, 17, 19, 23] and other papers. Objects of that category are finite disjoint unions of intervals and circles. We briefly touch on this generalization and explain encoding of recognizable series via certain rational power series in this case as well.

Relations between Frobenius algebras, recognizable power series, codes and two-dimensional TFTs are considered in Friedrich [8], which is quite close in spirit to this paper.

A possible relation between moduli spaces of \( SU(m) \) instantons on \( \mathbb{R}^4 \) (the Hilbert scheme of \( \mathbb{C}^2 \) corresponds to \( U(1) \) case) and control theory is explored in [12, 22] and the follow-up papers. We do not know how to connect it to the constructions in the present paper.

2. The Category of Thin Flat Surfaces

2.1. Category TFS. We introduce the category TFS of thin flat surfaces. Its objects are non-negative integers \( n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). An object is represented by \( n \) intervals \( I_1, \ldots, I_n \) placed along the \( x \)-axis in the \( xy \)-plane. A morphism from \( n \) to \( m \) is a “thin” surface \( S \) immersed in \( \mathbb{R} \times [0, 1] \) connecting \( n \) intervals on the line \( \mathbb{R}^2 \times \{0\} \) with \( m \) intervals on the line \( \mathbb{R}^2 \times \{1\} \). The immersion map \( S \rightarrow \mathbb{R}^2 \times [0, 1] \) is a local diffeomorphism, but the image of \( S \) may have overlaps, that can be thought of as virtual overlaps and ignored. The surface \( S \) inherits an orientation from its immersion into \( \mathbb{R} \times [0, 1] \). Restricting to the complement of the boundary of \( S \), the immersion is open.

Alternatively, the immersion can be perturbed to an embedding of \( S \) into \( \mathbb{R}^2 \times [0, 1] \) by turning overlaps into over- and under-crossings of strips of a surface. This can be done just for aesthetic purposes, and whether one chooses an over- or an under-crossing does not matter for the morphism associated to the surface.

The boundary of \( S \) consists of several circles (at least one circle unless \( S \) is the empty surface) and decomposes into \( n + m \) disjoint intervals that constitute horizontal boundary \( \partial_h S \) and \( n + m \) intervals and some number of circles that constitute side, or vertical, or inner boundary \( \partial_v S \) of \( S \):

\[
\partial S = \partial_h S \cup \partial_v S.
\]

Horizontal intervals that constitute \( \partial_h S \) are the intersections of \( S \) with \( \mathbb{R} \times \{0, 1\} \subset \mathbb{R} \times [0, 1] \). Vertical boundary \( \partial_v S \) is the closure of the intersection of \( \partial S \) with \( \mathbb{R} \times (0, 1) \).
The intersection $\partial h S \cap \partial v S$ consists of $2(n + m)$ boundary points of the horizontal intervals. These are also the corners of the surface $S$.

In the graphical depictions of thin flat surfaces below, we will draw horizontal boundary segments as red intervals, and vertical boundary components as green arcs for better visualization (Figs. 1 and 2 right), but the figures can also be viewed and carry full information in greyscale. Starting from Fig. 2, we depict tf-surfaces in light aquamarine.

Another way to describe $S$ is to immerse a finite unoriented graph $\Gamma$, possibly with multiple edges and loops, into the strip $\mathbb{R} \times [0, 1]$, via the immersion $j : \Gamma \to \mathbb{R} \times [0, 1]$. The graph $\Gamma$ may have several boundary vertices $v$ of valency 1 such that $j(v) \in \mathbb{R} \times \{0, 1\}$. The remaining vertices are mapped inside the strip. The immersion is disjoint on vertices. Edges of $\Gamma$ may intersect in $\mathbb{R} \times [0, 1]$. We consider these virtual intersections and not vertices. An example of $\Gamma$ and $j$ is shown in Fig. 2 left.

Taking a regular neighbourhood $N(\Gamma, j)$ of $\Gamma$ under $j$, locally in $\Gamma$, results in a thin flat surface $N(\Gamma)$. Vice versa, any thin flat surface $S$ can be deformed to the surface $N(\Gamma)$ for some $\Gamma$.

Take a thin flat surface $S$ and forget the embedding into $\mathbb{R} \times [0, 1]$, only remembering boundary intervals and their order, on both top and bottom lines. In this way we view $S$ as a cobordism between ordered collections of oriented intervals (induced by the orientation
A set of generating morphisms. From left to right: \( \iota, \epsilon, m, \Delta \) are morphisms from 0 to 1, from 1 to 0, from 2 to 1 and from 1 to 2, respectively. The rightmost morphism \( P \) is the permutation morphism on \( 1 \otimes 1 = 2 \). Identity morphism \( \text{id}_1 \) of object 1 is shown for completeness.

Self-duality morphisms \( \epsilon m : 1 \otimes 1 \to 0 \) and \( \Delta \iota : 0 \to 1 \otimes 1 \) for the object 1 of \( \mathbb{R} \), say from left to right. The cobordism \( S \) has corners (unless \( n = m = 0 \)) and two types of boundary, as discussed. By definition, two cobordisms \( S_1, S_2 \) represent the same morphism if they are diffeomorphic rel horizontal boundary, that is, keeping all horizontal boundary points fixed.

The category \( \text{TFS} \) is symmetric monoidal, and a possible set of generating morphisms is shown in Fig. 3. We have included the identity morphism \( \text{id}_1 \) into the Figure to emphasize that the identity morphism \( \text{id}_n \) is represented by the surface which is the direct product of the disjoint union of \( n \) intervals (representing object \( n \)) and \([0, 1]\). The permutation morphism \( P \) of \( 2 = 1 \otimes 1 \), shown on the right, is part of the symmetric monoidal structure on \( \text{TFS} \) and squares to the identity (Fig. 3).

The elements \( \iota, \epsilon, m, \Delta, P \) constitute a set of monoidal generators of \( \text{TFS} \). Together with the identity morphism \( \text{id}_1 \) they can be used to build any morphism in \( \text{TFS} \), via horizontal and vertical compositions. In particular, from these generators we can build the self-duality morphisms for the object 1, see Fig. 4.

Some relations in the category \( \text{TFS} \) are shown in Fig. 5.

We call a surface \( S \) representing a morphism from \( n \) to \( m \) in \( \text{TFS} \) a thin flat cobordism from \( n \) to \( m \). A thin flat cobordism \( S \) is a disjoint union of its connected components \( S_1, \ldots, S_k \). Consider one such component \( S' \). It necessarily has non-empty boundary, and we can assign to \( S' \) non-negative integers \( \ell, g \), where \( \ell + 1 \) is the number of boundary components and \( g \geq 0 \) is the genus of \( S' \). The surface \( S' \) carries an orientation, inherited via an immersion from the orientation of the plane.

We will also call a thin flat surface a tf-surface and, when viewed as a cobordism, a tf-cobordism.

The morphisms \( \iota : 0 \to 1, \epsilon : 1 \to 0, m : 1 \otimes 1 \to 1, \Delta : 1 \to 1 \otimes 1 \)

and relations on them show that the object 1 is a symmetric Frobenius algebra object in \( \text{TFS} \) (top left relation in Fig. 5 shows that the trace map is symmetric). It’s not a commutative algebra object, since the two morphisms in Fig. 6 left are not equal in \( \text{TFS} \).

2.2. Classification of thin flat surfaces. By a closed or floating tf-surface \( S \) we mean one without horizontal boundary. A floating tf-surface necessarily has side boundary, unless
it is the empty surface. Diffeomorphism equivalence classes of floating tf-surfaces are in a bijection with endomorphisms of the object 0 of TFS. Such a surface is a disjoint union of its connected components, and a component is uniquely determined by its pair \((\ell + 1, g), \ell, g \in \mathbb{Z}_+,\) the number of boundary components and the genus, respectively. Any such pair is realized by some surface, since pairs \((0 + 1, 0), (1 + 1, 0),\) and \((0 + 1, 1)\) are realized by a disk, an annulus, a flat punctured torus, see Fig. 7, and taking band-connected sum of surfaces with invariants \((\ell_1 + 1, g_1), (\ell_2 + 1, g_2)\) yields a surface with the invariant \((\ell_1 + \ell_2 + 1, g_1 + g_2).\) Choose a closed connected tf-surface \(S_{\ell+1,g},\) one for each value of these parameters.

Connected morphisms from 0 to 0 in TFS are in a bijection with \(S_{\ell+1,g}\) as above. Endomorphisms of 0 in TFS is a free commutative monoid on generators \(S_{\ell+1,g},\) over all \(\ell, g \in \mathbb{Z}_+,\)

\[ \text{End}_{\text{TFS}}(0) \cong \langle S_{\ell+1,g} \rangle_{\ell, g \geq 0}. \]
Fig. 7. Examples of closed connected tf-surfaces $S_{\ell+1, g}$ for small values of $\ell$ and $g$. We explicitly write $\ell+1$ to remember that a surface always have at least one boundary component.

An element $a \in \text{End}_{\text{TFS}}(0)$ has a unique presentation as a finite product of $S_{\ell+1, g}$’s with positive integer multiplicities,

$$a = \prod_{i=1}^{k} S_{\ell_i+1, g_i}^{r_i}, \quad r_i \in \{1, 2, \ldots\}.$$

Consider a tf-surface $S$ describing a morphism from $n$ to $m$ in TFS. It may have some floating connected components, that is, those that are disjoint from the horizontal boundary of $S$. Each of these components is homeomorphic to $S_{\ell+1, g}$ as above for a unique $\ell, g$. Components of $S$ that have non-empty horizontal boundary are called viewable or visible components. Any component of $S$ is either floating or viewable. We call $S$ viewable if it has no floating components. The empty cobordism is viewable.

The commutative monoid $\text{End}_{\text{TFS}}(0)$ acts on the set $\text{Hom}_{\text{TFS}}(n, m)$ by taking a cobordism to its disjoint union with a floating cobordism. Any morphism $S \in \text{Hom}_{\text{TFS}}(n, m)$ has a unique presentation $S = S_0 \cdot S_1$ where $S_0 \in \text{End}_{\text{TFS}}(0)$, $S_1$ is a viewable cobordism in $\text{Hom}_{\text{TFS}}(n, m)$ and dot denotes the monoid action. In particular, $\text{Hom}_{\text{TFS}}(n, m)$ is a free $\text{End}_{\text{TFS}}(0)$-set with a “basis” of viewable cobordisms.

Let us specialize to viewable cobordisms $S$. All connected components of $S$ are viewable and determine a set-theoretic partition of $n + m$ horizontal boundary intervals of $S$. Let us label these boundary intervals from left to right by $1, 2, \ldots, n$ for the bottom intervals and $1', 2', \ldots, m'$ for the top intervals.

Each viewable component contains a non-empty subset of this set of intervals and together viewable components give a decomposition $\lambda$ of this set into disjoint sets. We denote by $D_n^m$ the set of partitions of these $n + m$ intervals, so that $\lambda \in D_n^m$. To further understand the structure of morphisms, we restrict to the case of connected $S$, thus a surface with one viewable connected component. All horizontal intervals are in $S$.

The surface $S$ and its horizontal boundary segments inherit orientation from $\mathbb{R} \times [0, 1]$ and from induced orientations of the top and bottom boundary of $\mathbb{R} \times [0, 1]$, see Fig. 8.

We use the convention of reversing the orientation on the source (bottom) part of the boundary of a cobordism, see Fig. 8. Consequently, bottom intervals $I_1, \ldots, I_n$ in $\partial S$ are oppositely oriented from the rest of the boundary, while top intervals $I_1', \ldots, I_m'$ are oriented compatibly with the side boundary orientations, inherited from that of $S$ and in turn inherited from the orientation of $\mathbb{R} \times [0, 1]$. In Fig. 8 right we shrank “tentacles” of $S$ into the “core” of $S$ to make it easier to see compatible and reverse orientations of the horizontal boundary segments of $S$.

We can now classify isomorphism classes of connected tf-cobordisms $S$ from $n$ to $m$. Such a cobordism has $\ell + 1$ boundary circles and genus $g$. On $\ell + 1$ boundary circles
choose $n + m$ non-overlapping intervals and label them $1, \ldots, n, 1', \ldots, m'$. Choose an orientation of the interval $1'$ or, if $m = 0$, orientation of interval 1.

The orientation of the interval $1'$ induces an orientation of that boundary component of $S$ and hence of $S$ itself. One then gets induced orientations for all boundary components of $S$. Horizontal parts of $\partial S$ for the intervals $2', \ldots, m'$ are then oriented compatibly with the boundary, while those corresponding to the intervals $1, \ldots, n$ in the opposite way from that for the boundary.

Horizontal intervals on the $\ell + 1$ boundary components determine a partition of

$$\mathbb{N}_n^m := \{1, \ldots, n, 1', \ldots, m'\}$$

into $\ell + 1$ disjoint subsets, possibly with some subsets empty. Orientations of boundary components induce a cyclic order on elements of each subset, where one goes along a component in the direction of its orientation and records horizontal intervals that one encounters. We call an instance of this data a locally cyclic partition of $\mathbb{N}_n^m$ together with a cyclic order on each subset. To such data we can assign a connected thin flat surface $S(\lambda, g)$ of genus $g$ with $\partial S$ the horizontal boundary these $n + m$ intervals, $\ell + 1$ boundary components, and horizontal intervals placed according to the cyclic order for the subset along each component.

For the example in Fig. 8 we have $n = 3, m = 2$, the set of horizontal intervals is $\{1, 2, 3, 1', 2'\}$, there are two components ($\ell = 1$), and the cyclic orders are $(1', 1, 2', 3)$ and $(2)$.

Vice versa, suppose given $(\ell, g)$ as above, a locally cyclic partition $\lambda \in D_{n,cyc}^m(\ell)$ of $\mathbb{N}_n^m$ into $\ell + 1$ subsets, possibly with some subsets empty, with a cyclic order on each subset. To such data we can assign a connected thin flat surface $S(\lambda, g)$ of genus $g$ with the horizontal boundary these $n + m$ intervals, $\ell + 1$ boundary components, and horizontal intervals placed according to the cyclic order for the subset along each component.

For another example, for $n = 4, m = 3$, the partition $\{(1', 4, 2', 2), (3', 1, 3), ()\}$, which includes one copy of the empty set, with cyclic orders as indicated and genus $g = 2$ the resulting tf-surface is shown in Fig. 9.

This bijection between connected morphisms from $n$ to $m$ and elements of the set $D_{n,cyc}^m \times \mathbb{Z}_+$ leads to a classification of morphisms in TFS. An arbitrary morphism
S ∈ Hom_{TFS}(n, m) is the union of the viewable subcobordism of S and the floating subcobordism. The latter are classified by elements of Hom_{TFS}(0, 0) and admit a very explicit description, via pairs \((ℓ, g)\) of the number of circles minus one and the genus of each connected component. The viewable subcobordism \(S'\) of S determines a partition of \(\mathbb{N}_n^m\) by with the set of horizontal intervals for each component of \(S'\) being a part of that partition. Each part of this partition is non-empty.

Next, for each part of the partition, remove the connected components of \(S'\) for all other parts, downsizing to just one component \(S''\). Relabel the horizontal intervals for \(S''\) into 1, 2, \ldots, \(n''\) and 1, 2, \ldots, \(m''\). Then such components \(S''\) are classified by data as above: a locally cyclic partition of \(\mathbb{N}_n^{m''}\) (possibly with empty subsets included) and a choice of genus \(g \geq 0\).

Putting the steps of this algorithm together gives a classification of morphisms from \(n\) to \(m\) in TFS.

2.3. Endomorphisms of 1 and homs between 0 and 1 in TFS. The category TFS is rigid symmetric monoidal, with the unit object 0 and the generating self-dual object 1, with all objects being tensor powers of the generating object, \(n = 1^\otimes n\).

In the rest of this section, since we only consider the category TFS, we may write \(\text{Hom}(n, m)\) instead of \(\text{Hom}_{TFS}(n, m)\), \(\text{End}(n)\) instead of \(\text{End}_{TFS}(n)\), etc.

Connected endomorphisms of 1: Endomorphisms \(\text{End}(1) = \text{End}_{TFS}(1)\) of the object 1 in the category TFS constitute a monoid. Consider the submonoid \(\text{End}^c(1)\) of \(\text{End}(1)\) that consists of connected endomorphisms of 1. Define endomorphisms \(b_1, b_2, b_3 \in \text{End}^c(1)\) via diagrams in Fig. 10.

Note that \(b_2\) has equivalent presentations, as shown in Fig. 11. The last diagram is not a tf-surface, but describes a diffeomorphism class of one (rel boundary). The tf-cobordism \(b_2\) has genus one and one boundary component, with two horizontal segments labelled 1 and 1′ on it, which uniquely determines it as an element of \(\text{End}(1)\).
We refer to $b_1$ as the “hole” cobordism, $b_2$ as the “handle” cobordism, $b_3$ as the “cross” cobordism.

**Proposition 2.1.** The endomorphisms $b_1, b_2, b_3 \in \text{End}_{\text{TFS}}(1)$ pairwise commute:

\[ b_1 b_2 = b_2 b_1, \quad b_1 b_3 = b_3 b_1, \quad b_2 b_3 = b_3 b_2. \]

*Proof.* Note that the product with $b_1$ just adds a hole with no horizontal segments on it to a connected cobordism. Product with $b_2$ adds a handle to a connected cobordism. \(\square\)

**Proposition 2.2.**

- $\text{End}^c(1)$ is a commutative monoid generated by commuting elements $b_1, b_2, b_3$ with an additional defining relation

\[ b_3^2 = b_1 b_3. \]

- $\text{End}^c(1)$ consists of the following distinct elements:

\[ b_1^n b_2^m, \; b_1^n b_2^m b_3, \; n, m \geq 0. \]

*Proof.* A cobordism $S \in \text{End}^c(1)$ is a connected surface with $\ell + 1$ boundary circles, genus $g$, and two horizontal intervals on it. If the intervals are on the same connected component of the boundary, $S = b_1^\ell b_2^g$. If the intervals lie on distinct boundary components then $\ell \geq 1$ and $S = b_1^{\ell-1} b_2^g b_3$. \(\square\)

Spaces $\text{Hom}(0, 1)$ and $\text{Hom}(1, 0)$: An element $y \in \text{Hom}(0, 1)$ is a tf-cobordism with one horizontal interval, at the top. It is a product $y_1 y_0$ of one viewable component $y_1 \in \text{Hom}(0, 1)$ and a closed cobordism $y_0 \in \text{Hom}(0, 0)$. Assume that $y$ is viewable, thus connected, since it has a unique horizontal segment. Then $y$ is determined by the number $\ell + 1$ of its boundary components and the genus $g$ and can be written as

\[ y = b_1^\ell b_2^g \tau, \]

where $\tau$ is the morphism $0 \to 1$ shown in Fig. 3 on far left. Note that $b_3 \tau = b_1 \tau$. 
**Proposition 2.3.** A morphism $y \in \text{Hom}_{TFS}(0, 1)$ has a unique presentation $y = b_1^\ell b_2^g \cdot y_0$, where $y_0 \in \text{End}(0)$ is a floating cobordism.

Reflecting cobordisms about the horizontal line, we obtain a classification of elements in $\text{Hom}_{TFS}(1, 0)$.

**Proposition 2.4.** A morphism $y \in \text{Hom}_{TFS}(1, 0)$ has a unique presentation $y = y_0 \cdot \epsilon b_1^\ell b_2^g$, where $y_0 \in \text{End}(0)$ is a floating cobordism.

*Endomorphism monoid $\text{End}(1)$.* Recall that we continue with a minor abuse of notation, where we denote by 1 the generating object of TFS, also use it as the label for the bottom left horizontal interval of a cobordism in $\text{Hom}(n, m)$, and use it conventionally as the label for the first natural number.

An element $y$ of $\text{End}_{TFS}(1)$ may be one of the two types:

1. Horizontal intervals 1 and $1'$ belong to the same connected component of $y$.
2. Intervals 1 and $1'$ belong to different connected components of $y$.

Denote by $U_i$ the set of elements of type $i \in \{1, 2\}$, so that

$$\text{End}(1) = U_1 \sqcup U_2.$$  \hfill (3)

The set $U_2$ is closed under left and right multiplication by elements of $\text{End}(1)$, thus constitutes a 2-sided ideal in this monoid. The set $U_1$ is a unital submonoid in $\text{End}(1)$. These maps

$$U_1 \rightarrow \text{End}(1) \leftarrow U_2$$

upgrade decomposition (3). The monoid $U_1$ is commutative and naturally decomposes

$$U_1 \cong \text{End}^c(1) \times \text{End}(0)$$

into the direct product, both terms of which we have already described. The direct product corresponds to splitting an element of $U_1$ into the viewable connected component and a floating cobordism.

Likewise, an element $y$ of $U_2$ splits into a floating cobordism $y_0$ and a viewable one $y_1$. A viewable element $y_1$ of $U_2$ consists of two connected components, one bounding horizontal interval 1, the other bounding $1'$. Such an element can be written as

$$y_1 = b_1^\ell_1 b_2^g_1 \cdot \epsilon b_1^\ell_2 b_2^g_2,$$

with a general $y \in U_2$ given by

$$y = b_1^\ell_1 b_2^g_1 \cdot y_0 \cdot \epsilon b_1^\ell_2 b_2^g_2.$$  

Multiplication of two viewable elements as above produces an additional connected component, see Fig. 12, where by the $(\ell, g)$ coupon we denote the endomorphism $b_1^\ell b_2^g$ of 1.

*Remark.* Unlike the monoids $\text{End}(0), \text{End}^c(1)$, and their direct product $U_1$, monoid $\text{End}(1)$ and its subsemigroup $U_2$ are not commutative.

### 3. Linearizations of the Category TFS

In this section we work over a field $k$, but the construction and some results may be generalized to an arbitrary commutative ring $R$ (or a commutative ring with additional conditions, such as being noetherian). A definitive starting reference for recognizable series with coefficients in commutative rings is Hazewinkel [11].
Fig. 12. Product of two viewable elements of $U_2$ produces a floating component $\epsilon h_{12}^{\ell_2+\ell_3} h_{23}^{g_2+g_3} = S_{\ell_2+\ell_3+1,g_2+g_3}$, in addition to the components bounding intervals 1 and 1'.

3.1. Categories $kTFS$ and $VTFS_\alpha$ for recognizable $\alpha$. Category $kTFS$. Starting with TFS we can pass to its preadditive closure $kTFS$. Objects of $kTFS$ are the same as those of TFS, that is, non-negative integers $n \in \mathbb{Z}_+$. A morphism in $kTFS$ from $n$ to $m$ is a finite $k$-linear combination of morphisms from $n$ to $m$ in TFS. In particular, Hom$_{kTFS}(n, m)$ is a $k$-vector space with a basis Hom$_{TFS}(n, m)$. Composition of morphisms is defined in the obvious way.

Category $kTFS$ is a $k$-linear preadditive category. It is also a rigid symmetric monoidal category.

Power series $\alpha$. The ring Hom$_{kTFS}(0, 0)$ of endomorphisms of the unit object 0 of $kTFS$ is naturally isomorphic to the monoid algebra of Hom$_{TFS}(0, 0)$. The latter is a free commutative monoid on generators $S_{\ell+1,g}$, over all $\ell, g \in \mathbb{Z}_+$, so that

$$\text{Hom}_{kTFS}(0, 0) \cong k[S_{\ell+1,g}]_{\ell, g \in \mathbb{Z}_+}$$

is the polynomial algebra on countably many generators, parametrized by pairs $(\ell, g)$ of non-negative integers. Homomorphisms of $k$-algebras

$$\text{Hom}_{kTFS}(0, 0) \longrightarrow k$$

are in a bijection with doubly-infinite sequences

$$\alpha = (\alpha_{\ell,g})_{\ell, g \in \mathbb{Z}_+}, \quad \alpha_{\ell,g} \in k.$$ 

The bijection associates to a sequence $\alpha$ the homomorphism, also denoted $\alpha$,

$$\text{Hom}_{TFS}(0, 0) \cong k[S_{\ell+1,g}]_{\ell, g \in \mathbb{Z}_+} \xrightarrow{\alpha} k, \quad \alpha(S_{\ell+1,g}) = \alpha_{\ell,g}.$$ 

Sequences $\alpha$ are also in a bijection with multiplicative $k$-valued evaluations of floating cobordisms in TFS. These evaluations are maps from the set of floating cobordisms (endomorphisms of object 0) in TFS to $k$ that take disjoint union of cobordisms to the product of evaluations,

$$\alpha(S \sqcup S') = \alpha(S) \cdot \alpha(S').$$

Thus, $\alpha$ is a map of sets

$$\alpha : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow k.$$
that we can think of a \( \mathbb{Z}_+ \times \mathbb{Z}_+ \)-matrix with coefficients in \( k \)

\[
\alpha = \begin{pmatrix}
\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} & \ldots \\
\alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \ldots \\
\alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \ldots \\
\alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We encode \( \alpha \) into power series in two variables \( T_1, T_2 \):

\[
Z_\alpha(T_1, T_2) = \sum_{k,g \geq 0} \alpha_{k,g} T_1^k T_2^g, \quad \alpha = (\alpha_{k,g})_{k,g \in \mathbb{Z}_+}, \quad \alpha_{k,g} \in k.
\]  

(4)

A doubly-infinite sequence \( \alpha \) can also be thought of as a linear functional on the space of polynomials in two variables:

\[
\alpha \in k[T_1, T_2]^* := \text{Hom}_k(k[T_1, T_2], k).
\]

We assume that \( \alpha \) is not identically zero (the theory is trivial otherwise). Then \( \ker(\alpha) \subset k[T_1, T_2] \) is a codimension one subspace.

**Category VTFS\( _{\alpha} \).** Given \( \alpha \), we can form the quotient \( \text{VTFS}_{\alpha} \) of category \( k\text{TFS} \) by adding the relation that a floating surface \( S_{\ell+1,g} \) of genus \( g \) with \( \ell + 1 \) boundary components evaluates to \( \alpha_{\ell,g} \in k \). Objects of \( \text{VTFS}_{\alpha} \) are still non-negative integers \( n \). Morphisms from \( n \) to \( m \) are finite \( k \)-linear combinations of \( \text{viewable} \) cobordisms from \( n \) to \( m \). Composition of cobordisms from \( n \) to \( m \) and from \( m \) to \( k \) is a cobordism from \( n \) to \( k \) which may have floating components. These components are removed simultaneously with multiplying the \( \text{viewable} \) cobordism that remains by the product of \( \alpha_{\ell,g} \)'s, for every component \( S_{\ell+1,g} \).

The space of homs from \( n \) to \( m \) in this category has a basis of \( \text{viewable} \) cobordisms from \( n \) to \( m \). Letter \( V \) in the notation \( \text{VTFS}_{\alpha} \) stands for \textit{viewable}.

**Recognizable series.** Borrowing terminology from control theory [7,10], we say that a linear functional or series \( \alpha \) is \textit{recognizable} if \( \ker(\alpha) \) contains an ideal \( I \in k[T_1, T_2] \) of finite codimension.

**Proposition 3.1.** \( \alpha \) is recognizable iff the power series \( Z_\alpha \) has the form

\[
Z_\alpha(T_1, T_2) = \frac{P(T_1, T_2)}{Q_1(T_1) Q_2(T_2)},
\]

(5)

where \( Q_1(T_1), Q_2(T_2) \) are one-variable polynomials and \( P(T_1, T_2) \) is a two-variable polynomial, all with coefficients in the field \( k \).

We assume that \( Q_1(0) \neq 0, Q_2(0) \neq 0 \), otherwise at least one of these polynomials is not coprime with \( P(T_1, T_2) \) and either \( T_1 \) or \( T_2 \) cancels out from the numerator and denominator. With the denominator not zero at \( T_1, T_2 = 0 \) the power series expansion makes sense.

**Proof.** See [7] for a proof. This result is also mentioned in [10, Remark 2]. To prove it, assume that \( \alpha \) is recognizable. We start with the case when \( k \) is algebraically closed. A finite codimension ideal \( I \subset k[T_1, T_2] \) necessarily contains a sum,

\[
I_1 \otimes k[T_2] + k[T_1] \otimes I_2 \subset I \subset k[T_1, T_2]
\]

(6)
for some finite codimension ideals $I_1 \subset k[T_1]$ and $I_2 \subset k[T_2]$. To see this, note that the finite affine scheme $\text{Spec}(k[T_1, T_2]/I)$ is supported over finitely many points of the affine plane $\mathbb{A}^2$. Projecting these points onto the coordinate lines and counting them with multiplicities produces two one-variable polynomials $U_1(T), U_2(T)$ such that $I$ contains the ideal $(U_1(T_1)) + (U_2(T_2))$ of $k[T_1, T_2]$. We can now take principal ideals $I_i = (U_i(T_i)), i = 1, 2$ to get the inclusion on the LHS of (6). This also gives a quotient map

$$k[T_1]/(U_1(T_1)) \otimes k[T_2]/(U_2(T_2)) \rightarrow k[T_1, T_2]/I$$

lifting to the identity map on $k[T_1, T_2]$. Existence of such finite codimension ideals $I_1, I_2$ over an arbitrary field $k$ follows as well.

Hence, recognizable series $\alpha$ has the property that $\alpha(U_1(T_1)T_1^kT_2^m) = 0$ for any $k, m \geq 0$. We can assume that $U_1(T)$ is a polynomial of some degree $r$ with the lowest degree term $u_r T^r$ for $s \leq r$ and write

$$U_1(T) = u_r T^r + u_{r-1} T^{r-1} + \cdots + u_{s+1} T^{s+1} + u_s T^s, \quad 0 \leq s \leq r, \quad u_r, u_s \neq 0, \quad u_j \in k.$$ 

Then, for any $k, m \geq 0$

$$u_r \alpha_{r+k,m} + u_{r-1} \alpha_{r-1+k,m} + \cdots + u_{s+1} \alpha_{s+k,m} + u_s \alpha_{s+k,m} = 0. \quad (7)$$

We obtain a similar relation on the coefficients with $U_2$ and $T_2$ in place of $U_1$ and $T_1$ and varying the second index. Let us write

$$U_2(T) = v_r T^{r'} + v_{r'-1} T^{r'-1} + \cdots + v_{s'+1} T^{s'+1} + v_{s'} T^{s'}, \quad 0 \leq s' \leq r', \quad v_r, v_{s'} \neq 0, \quad v_j \in k.$$ 

Then, for any $k, m \geq 0$

$$v_r \alpha_{r'+k,m} + v_{r'-1} \alpha_{r'-1+k,m} + \cdots + v_{s'+1} \alpha_{s'+k,m} + v_{s'} \alpha_{s'+k,m} = 0. \quad (8)$$

Consequently, $\alpha$ is eventually recurrent in both $T_1$ and $T_2$ directions and its values are determined by $\alpha_{i,j}$ with $0 \leq i < r, 0 \leq j < r'$.

Consider polynomials

$$\hat{Q}_1(T) = T^r U_1(T^{-1}) = u_s T^r + u_{s+1} T^{r-1} + u_{s+2} T^{r-2} + \cdots + u_r T^{r-s},$$

$$\hat{Q}_2(T) = T^{r'} U_2(T^{-1}) = v_s T^{r'} + v_{s'+1} T^{r'-1} + v_{s'+2} T^{r'-2} + \cdots + v_{r'} T^{r'-s'}.$$ 

Form the product

$$\hat{P}(T_1, T_2) := Z_\alpha(T_1, T_2) \hat{Q}_1(T_1) \hat{Q}_2(T_2) = \sum_{i,j \geq 0} w_{i,j} T_1^i T_2^j,$$

and examine coefficients of its power series expansion. Formulas (7), (8) show that $w_{i,j} = 0$ if $i \geq r$ or $j \geq r'$. Therefore, $\hat{P}(T_1, T_2)$ is a polynomial with $T_1, T_2$ degrees bounded by $r - 1, r' - 1$, respectively. We can then form the quotient

$$\frac{\hat{P}(T_1, T_2)}{\hat{Q}_1(T_1) \hat{Q}_2(T_2)}.$$ 

The numerator and denominator may share common factors, including $T_1^{r-s} T_2^{r'-s'}$. After canceling those out, we arrive at the presentation (5) for $Z_\alpha(T_1, T_2)$.

We leave the proof of the opposite implication of the proposition to the reader or refer to [7].

Note that the proof works for any finite number of variables $T_1, \ldots, T_c$, not only for two. □
The condition that $\alpha$ is recognizable can also be expressed via its Hankel matrix $H_\alpha$. The latter matrix has rows and columns enumerated by pairs $(m,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, equivalently by the monomial basis elements $T_1^m T_2^k$. The $(m_1, k_1, (m_2, k_2))$-entry of $H_\alpha$ is $\alpha_{m_1+k_2, k_1+k_2}$. The following result is proved in [7].

**Proposition 3.2.** The series $\alpha$ is recognizable iff the Hankel matrix $H_\alpha$ has finite rank.

Note that $H_\alpha$ has finite rank iff there exists $M$ such that any $M \times M$ minor of $H_\alpha$ has determinant zero. The rank is $M - 1$ if in addition there is an $(M - 1) \times (M - 1)$ minor with a non-zero determinant.

3.2. Skein category $\text{STFS}_\alpha$. Recognizable series and commutative Frobenius algebras.

Assume that $\alpha$ is recognizable. Among all finite-codimension ideals $I \subset \ker(\alpha)$ there is a unique largest ideal $I_\alpha$, given by the sum over all such $I$. Equivalently, it can be described as follows. There is a homomorphism of $k[T_1, T_2]$-modules

$$h : k[T_1, T_2] \rightarrow k[T_1, T_2]^*$$

given by sending 1 to $\alpha$ and $z \in k[T_1, T_2]$ to $z\alpha \in k[T_1, T_2]^*$ with $(z\alpha)(f) = \alpha(zf)$. The ideal $I_\alpha$ is the kernel of $h$.

Notice that $\alpha$ descends to a nondegenerate bilinear form on the quotient algebra

$$A_\alpha := k[T_1, T_2]/I_\alpha.$$  

In particular, $A_\alpha$ is a commutative Frobenius algebra on two generators $T_1, T_2$ with a nondegenerate trace form $\alpha$.

Vice versa, assume given a commutative Frobenius $k$-algebra $B$ with the nondegenerate trace form $\beta : B \rightarrow k$ and a pair of generators $g_1, g_2$. To such data we can associate a surjective homomorphism

$$\psi : k[T_1, T_2] \rightarrow B, \ \psi(T_i) = g_i, \ i = 1, 2,$$

the trace map $\alpha = \beta \circ \psi$ on $k[T_1, T_2]$ given by composing $\psi$ with $\beta$, and recognizable series

$$\alpha = \sum_{\ell, g \geq 0} \beta(g_1^\ell g_2^g) T_1^\ell T_2^g.$$

Thus, recognizable power series on $k[T_1, T_2]$ are classified by isomorphism classes of data $(B, g_1, g_2, \beta)$: a commutative Frobenius algebra $B$ generated by $g_1, g_2 \in B$ and a non-degenerate trace $\beta$.

**Category STFS$_\alpha$.** We can now define the category $\text{STFS}_\alpha$ (where first S stands for “skein”) to be a quotient of $\text{VTFS}_\alpha$ by the skein relations in the ideal $I_\alpha$. The category $\text{STFS}_\alpha$ has the same objects as all the other cobordism categories we’ve considered so far, that is, nonnegative integers $n$. Morphisms from $n$ to $m$ are $k$-linear combinations of viewable cobordisms modulo the relations in $I_\alpha$. Precisely, let

$$p(T_1, T_2) = \sum_{i, j} p_{ij} T_1^i T_2^j \in I_\alpha$$

be a polynomial in the ideal $I_\alpha$. Given a viewable cobordism $x$ choose a component $c$ of $x$ and denote by $x_c(i, j)$ the cobordism given by inserting $i$ holes and adding $j$ handles
to \( x \) at the component \( c \). We now mod out the hom space \( \text{Hom}_{\text{VTFS}_\alpha}(n, m) \), which is a \( \mathbb{k} \)-vector space with a basis of all viewable cobordisms from \( n \) to \( m \), by the relations

\[
\sum_{i,j} p_{i,j} x_c(i, j) = 0,
\]

one for each component \( c \) of \( x \), over all viewable cobordisms \( x \).

It is easy to see that these “skein” relations are compatible with \( \alpha \)-evaluation of floating cobordisms. Namely, if instead of a viewable cobordism \( x \) we consider a floating cobordism \( y \) and choose a component \( c \) of \( y \) to add holes and handles, resulting in cobordisms \( y_c(i, j) \), then

\[
\sum_{i,j} p_{i,j} \alpha(y_c(i, j)) = 0.
\]

This compatibility condition, immediate from our definition of \( I_\alpha \) as the kernel of the module map (9), ensures non-triviality of this quotient and its compatibility with the composition of morphisms.

Viewing \( \text{VTFS}_\alpha \) as a tensor category, it is enough to write down corresponding relations on homs from 0 to 1 and then mod out by them in the tensor category (by gluing each term in the resulting linear combination of products of holes and handles on a disk to any component along a segment on its side boundary). Choose a generating set \( v_1, \ldots, v_r \) of \( I_\alpha \) viewed as \( \mathbb{k}[T_1, T_2] \)-module. Specializing to a single basis element \( v_j \), assume that it is given by the polynomial \( p \) on the right hand side of (11). Form the element

\[
b(v_j) := \sum_{i,j} p_{i,j} b_1^i b_2^j t \in \text{Hom}(0, 1).
\]

The skein category \( \text{STFS}_\alpha \) can be defined as the quotient of \( \text{VTFS}_\alpha \) by the tensor ideal generated by elements \( b(v_1), \ldots, b(v_r) \). Figure 13 shows an example of an element \( b(v) \).

**Remark.** For a recognizable series \( \alpha \) there are unique minimal degree monic polynomials \( q_{\alpha,1}, q_{\alpha,2} \):

\[
q_{\alpha,1}(x) = x^t + a_{t-1}x^{t-1} + \cdots + a_0, \quad q_{\alpha,2}(x) = x'^t + a'_{t'-1}x'^{t'-1} + \cdots + a'_0,
\]

such that

\[
q_{\alpha,1}(T_1) \in I_\alpha, \quad q_{\alpha,2}(T_2) \in I_\alpha.
\]
Among skein relations associated to elements of $I_\alpha$ in $\text{STFS}_\alpha$ there is a polynomial relation that utilizes only adding holes to a component of the cobordism. This relation is given by the polynomial $q_{\alpha,1}(T_1)$:

$$b_1^t + a_{t-1} b_1^{t-1} + \cdots + a_0 = 0,$$

describing an equality in the ring of endomorphisms of object 1 of $\text{STFS}_\alpha$, where $b_1$ is the hole cobordism, see Fig. 11. Equivalently, it can be rewritten as a relation in $\text{Hom}(0, 1)$:

$$(b_1^t + a_{t-1} b_1^{t-1} + \cdots + a_0)\iota = 0,$$

Likewise, there is a skein relation on cobordisms that differ only by genus of a given component. The relation is given by the polynomial $q_{\alpha,2}(T_2)$:

$$b_2^{t'} + a_{t'-1} b_2^{t'-1} + \cdots + a_0' = 0,$$

where $b_2$ is the handle morphism, see Fig. 11.

Minimal viewable cobordisms, $B_\alpha$-companions, and bases of hom spaces of $\text{STFS}_\alpha$.

Consider a connected viewable cobordism $x$. We say that $x$ is minimal if it has genus zero and no holes, that is, each boundary component of $x$ contains at least one horizontal segment. Equivalently $x$ is minimal if it cannot be factored into $x' b_1 x''$ or $x' b_2 x''$ for some morphisms $x', x''$. Note that if such a factorization exists, then there exists one with $x''$ the identity cobordism and one with $x'$ the identity cobordism. Any viewable connected cobordism $x$ from $n$ to $m$ with $m > 0$ can be written as $(b_1^i b_2^j \otimes \text{id}_{m-1}) y$ for some minimal $y$ and, if $n > 0$, as $y(b_1^i b_2^j \otimes \text{id}_{n-1})$ for the same $y$, see Fig. 14. If one of $n$ or $m$ is zero, only one of these two presentations exist.

Equivalently, a connected viewable cobordism $x$ is minimal if it is handless and has no holes.

A viewable cobordism $y$ is called minimal if each connected component of $y$ is minimal. A viewable cobordism $x$ factors into a product of a minimal cobordism and “coupons” carrying powers of $b_1$, $b_2$, one for each connected component of $x$. That is, for each connected component $c$ of $x$ count holes and handles on it and then remove them to get a minimal connected component $c'$. The original component can be recovered by inserting holes and handles back anywhere along $c'$. For instance, they may be inserted at one of its top or bottom legs by multiplying $c'$ by the corresponding powers of $b_1$ and $b_2$ there.

To any viewable $x$ we can associate its minimal counterpart $y$ by removing holes and handles from each connected component of $x$. Given $y$, we can recover $x$ by multiplying
by appropriate powers of $b_1$ and $b_2$ at horizontal intervals for different components of $y$.

Denote by $\mathcal{M}(n, m)$ the set of minimal viewable cobordisms from $n$ to $m$.

**Proposition 3.3.** $\mathcal{M}(n, m)$ is a finite set.

*Proof.* From our classification of morphisms in TFS it is clear that minimal cobordisms from $n$ to $m$ are in bijection with partitions $\lambda$ of the set $\mathbb{N}^m$ of $n+m$ horizontal intervals, together with a choice of a partition $\mu_i$ of each part $\lambda_i$ of $\lambda$ and a cyclic order on each part of $\mu_i$. $\square$

Recall finite codimension ideal $I_\alpha$ (the syntactic ideal) associated with recognizable series $\alpha$. Let

$$d_\alpha = \dim(\mathbf{k}[T_1, T_2]/I_\alpha).$$

Choose a set of pairs

$$P_\alpha = \{(i_t, j_t)\}_{t=1}^{d_\alpha}, \quad i_t, j_t \in \mathbb{Z}_+$$

such that monomials $T_1^{i_1}T_2^{j_1}$ constitute a basis of the algebra $\mathbf{k}[T_1, T_2]/I_\alpha$. Denote this basis by $B_\alpha$. It is well-known [18] that a basis can always be chosen so that the exponents $(i_t, j_t)$ of the monomials, when placed into corresponding points of the square lattice, constitute a partition of $d_\alpha$, but we do not need this result here.

Choose a minimal cobordism $y$ and assign an element $v_c \in B_\alpha$ to each connected component $c$ of $y$. This assignment gives rise to a cobordism $x$ obtained from $y$ by inserting cobordisms $b(v_c)$ at all components $c$ of $y$. For $v_c = T_1^{i_1}T_2^{j_1}$ we add $i$ holes and $j$ handles to the component $c$ or, equivalently, multiply it at one of its horizontal boundary intervals by $b_i^ib_j^j$.

In this way to $y \in \mathcal{M}(n, m)$ there are assigned $d_\alpha^r$ cobordisms $x$, where $r$ is the number of components of $y$. These $x$ are called $B_\alpha$-companions of $y$. Denote the set of such $x$ by $B_\alpha(y)$.

**Proposition 3.4.** Elements of sets $B_\alpha(y)$, over all $y \in \mathcal{M}(n, m)$, constitute a basis of $\text{Hom}_{\text{STFS}_\alpha}(n, m)$.

In other words, to get a basis of homs from $n$ to $m$ in the skein category $\text{STFS}_\alpha$ we take all minimal cobordisms $y$ from $n$ to $m$ and insert a basis element from $B_\alpha$ into each component of $y$.

*Proof.* The proposition follows immediately from our construction of $\text{STFS}_\alpha$. One needs to check consistency, that our rules do not force additional relations when composing cobordisms. This is straightforward. $\square$

**Corollary 3.5.** $\text{Hom}$ spaces in the category $\text{STFS}_\alpha$ are finite dimensional.

*Remark.* In a seeming discrepancy, object $1$ of the category TFS is a symmetric Frobenius object but not a commutative Frobenius object, see Fig. 6 left, since the multiplication map $1 \otimes 1 \rightarrow 1$ does not commute with the permutation endomorphism of $1 \otimes 1$. Yet, in the category $\text{STFS}_\alpha$ the state space $\text{Hom}(0, 1)$ of the interval is a commutative Frobenius algebra $A_\alpha$, defined in (10), with the multiplication on $\text{Hom}(0, 1)$ given by the thin flat pants cobordism in Fig. 15 left. This is explained by the observation that the thin flat pants multiplication is commutative in the categories we consider, including TFS and $\text{VTFS}_\alpha$ and $\text{STFS}_\alpha$. Indeed, viewable morphisms from $0$ to $1$ in TFS have the form $b^n_1b^n_2\iota$, and the product of two such morphisms does not depend on their order, see Fig. 15 right. Adding floating components (or passing to linear combinations, or taking quotients) does not break commutativity.

Later, in Section 5.1, in a similar situation we also denote $b^n_1b^n_2\iota$ by $b^n_1b^n_2$.
3.3. Quotient by negligible morphisms and Karoubi envelopes. Category $\text{TFS}_\alpha$. Consider the ideal $J_\alpha \subset \text{STFS}_\alpha$ of negligible morphisms, relative to the trace form $\text{tr}_\alpha$ associated with $\alpha$, and form the quotient category

$$\text{TFS}_\alpha := \text{STFS}_\alpha / J_\alpha.$$ 

The trace form is given on a cobordism $x$ from $n$ to $n$ by closing it via $n$ annuli connecting $n$ top with $n$ bottom circles of the horizontal boundary of $x$ into a floating cobordism $\hat{x}$ and applying $\alpha$,

$$\text{tr}_\alpha(x) := \alpha(\hat{x}).$$

This operation is depicted in Fig. 16.

A morphism $y \in \text{Hom}(n, m)$ is called negligible if $\text{tr}_\alpha(zy) = 0$ for any morphism $z \in \text{Hom}(m, n)$. Negligible morphisms constitute a two-sided ideal in the pre-additive category $\text{STFS}_\alpha$.

The quotient category $\text{TFS}_\alpha$ has finite-dimensional hom spaces, as does $\text{STFS}_\alpha$ (recall that $\alpha$ is recognizable). The trace is nondegenerate on $\text{TFS}_\alpha$ and defines perfect bilinear pairings

$$\text{Hom}(n, m) \otimes \text{Hom}(m, n) \rightarrow \mathbf{k}$$

on its hom spaces. We may call $\text{TFS}_\alpha$ the negligible quotient of $\text{STFS}_\alpha$, having modded out by the ideal of negligible morphisms.

Let us go back to the category $\text{TFS}$ and its linear version $\mathbf{kTFS}$. Fix the number $n$ of intervals and consider the vector space $V_n$ with a basis of all viewable tf-surfaces with that boundary, that is viewable cobordisms in $\text{TFS}$ from 0 to $n$. Given $\alpha$, define a bilinear form on $V_n$ via its values on pairs of basis vectors:

$$(x, y) = \alpha(\overline{y}x) \in \mathbf{k},$$

where $\overline{y}$ is given by reflecting $y$ about a horizontal line to get a cobordism from $n$ to 0, and $\overline{y}x$ is a floating cobordism from 0 to 0 given by composing $\overline{y}$ and $x$. This bilinear
form on $V_n$ is symmetric. Define $A_\alpha(n)$ as the quotient of $V_n$ by the kernel of this bilinear form. Then there is a canonical isomorphism

$$A_\alpha(n) \cong \text{Hom}_{TFS_\alpha}(0, n)$$

as well as isomorphisms

$$A_\alpha(n + m) \cong \text{Hom}_{TFS_\alpha}(0, n + m) \cong \text{Hom}_{TFS_\alpha}(m, n)$$

given by moving $m$ invervals from bottom to top via the duality morphism.

The symmetric group $S_n$ acts by permutation cobordisms on $A_\alpha(n)$. Furthermore, at each circle there is an action of the endomorphism algebra $\text{End}(1) = \text{End}_{TFS_\alpha}(1)$. Consequently, the cross-product algebra $kS_n \rtimes \text{End}(1)^{\otimes k}$ acts on $A_\alpha(n)$.

Multiplication maps

$$A_\alpha(n) \otimes A_\alpha(m) \longrightarrow A_\alpha(n + m)$$

turn the direct sum

$$A_\alpha := \bigoplus_{n \geq 0} A_\alpha(n)$$

into a unital commutative associative graded algebra, with $A_\alpha(0) \cong k$. All of this data, including the power series $\sum_{n \geq 0} \dim A_\alpha(n) z^n$ encoding dimensions of $A_\alpha(n)$, are invariants of recognizable series $\alpha$.

In the diagram of five categories and four functors

$$\text{TFS} \longrightarrow k\text{TFS} \longrightarrow V\text{TFS}_\alpha \longrightarrow \text{STFS}_\alpha \longrightarrow \text{TFS}_\alpha$$

one can get from $k\text{TFS}$ to $\text{TFS}_\alpha$ in one step, bypassing $V\text{TFS}_\alpha$ and $\text{STFS}_\alpha$, by taking the ideal of negligible morphisms in $k\text{TFS}$ (for essentially the same trace map, shown in Fig. 16) and modding out by it. It is convenient to introduce those intermediate categories, though. For instance, $\text{STFS}_\alpha$ already has finite-dimensional hom spaces and allows to define the analogue of the Deligne category in our case.

The Deligne category $DTFS_\alpha$ and its negligible quotient $DTFS_\alpha$. The skein category $\text{STFS}_\alpha$ is a rigid symmetric monoidal $k$-linear category with objects $n \in \mathbb{Z}_+$ and finite-dimensional hom spaces. We form the additive Karoubi closure

$$DTFS_\alpha := \text{Kar}(\text{STFS}_\alpha^{\oplus})$$

by allowing formal finite direct sums of objects in $\text{STFS}$, extending morphisms correspondingly, and then adding idempotents to get a Karoubi-closed category. Category $DTFS_\alpha$ plays the role of the Deligne category in our construction.

In the Deligne category $DTFS_\alpha$ endomorphisms of an object $(n, e)$, where $e$ is an idempotent endomorphism of $n$, inherit the trace map $\text{tr}_\alpha$ into the ground field. Consequently, category $DTFS_\alpha$ also has a two-sided ideal of negligible morphisms $J_{D,\alpha}$. Taking the quotient by this ideal

$$DTFS_\alpha := DTFS_\alpha / J_{D,\alpha}$$

gives us an idempotent-complete category with non-degenerate symmetric bilinear forms on hom spaces $\text{Hom}(0, (n, e))$, where $(n, e)$ is an object as above, and more generally non-degenerate bilinear pairings on hom spaces

$$\text{Hom}((n, e), (m, e')) \otimes \text{Hom}((m, e'), (n, e)) \longrightarrow k$$
where $e'$ is an idempotent endomorphism of object $m$. Due to the symmetry between homs given by the contravariant involution on all categories that we have considered so far (reflection about a horizontal line), the above bilinear pairings can be converted into non-degenerate symmetric bilinear forms on $\text{Hom}((n, e), (m, e'))$ in $\text{DTFS}_\alpha$.

Category $\text{DTFS}_\alpha$ is also equivalent to the additive Karoubi closure of the category $\text{TFS}_\alpha$, see the commutative square in (12).

3.4. Summary of the categories and functors. Below is a summary for each category that has been considered.

- **TFS**: the category of thin flat surfaces (tf-surfaces). Its objects are non-negative integers and morphisms are thin flat surfaces.
- **$k\text{TFS}$**: this category has the same objects as TFS; its morphisms are formal finite $k$-linear combinations of morphisms in TFS.
- **$\text{VTFS}_\alpha$**: in this quotient category of $k\text{TFS}$ we reduce morphisms to linear combinations of viewable cobordisms. Floating connected components are removed by evaluating them via $\alpha$.
- **$\text{STFS}_\alpha$**: to define this category, specialize to recognizable $\alpha$ and add skein relations by modding out by elements of the ideal $I_\alpha$ in $k[T_1, T_2]$ along each connected component of a surface ($T_1$ is a hole, $T_2$ a handle). Hom spaces in this category are finite-dimensional.
- **$\text{TFS}_\alpha$**: the quotient of $\text{STFS}_\alpha$ by the ideal $J_\alpha$ of negligible morphisms. This category is also equivalent (even isomorphic) to the quotients of $k\text{TFS}$ and $\text{VTFS}_\alpha$ by the corresponding ideals of negligible morphisms in them. The trace pairing in $\text{TFS}_\alpha$ between $\text{Hom}(n, m)$ and $\text{Hom}(m, n)$ is perfect.
- **$\text{DTFS}_\alpha$**: the quotient of $\text{DTFS}_\alpha$ by the two-sided ideal of negligible morphisms. This category is equivalent to the additive Karoubi closure of $\text{TFS}_\alpha$ and sits in the bottom right corner of the commutative square below.

We arrange these categories and functors, when $\alpha$ is recognizable, into the following diagram:

\[
\begin{array}{cccccc}
\text{TFS} & \longrightarrow & k\text{TFS} & \longrightarrow & \text{VTFS}_\alpha & \longrightarrow & \text{STFS}_\alpha & \longrightarrow & \text{DTFS}_\alpha \\
& & & & \downarrow & & \downarrow & & \\
& & & & \text{TFS}_\alpha & \longrightarrow & \text{DTFS}_\alpha & & \\
\end{array}
\]

(12)

All seven categories are rigid symmetric monoidal. All but the leftmost category TFS are $k$-linear. Except for the two categories on the far right, the objects of each category are non-negative integers. The four categories on the right all have finite-dimensional hom spaces. The two categories on the far right are additive and Karoubi-closed. The four categories in the middle of the diagram are pre-additive but not additive.

The arrows show functors between these categories considered in the paper. The square is commutative. An analogous diagram of functors and categories can be found in [16] for the category of oriented 2D cobordisms in place of TFS.

It is possible to go directly from $k\text{TFS}$ to $\text{TFS}_\alpha$ by modding out by the ideal of negligible morphisms in the former category. We found it convenient to get to this quotient in several steps, introducing categories $\text{VTFS}_\alpha$ and $\text{STFS}_\alpha$ along the way.
Remark. For possible future use, it may be convenient to relabel the categories above using shorter strings. For instance, writing $S$ (for “surfaces”) in place of TFS we can rename the categories as follows:

$$
\begin{array}{cccc}
S & \longrightarrow & kS & \longrightarrow \\
\downarrow & & \downarrow & \\
S_\alpha & \longrightarrow & DS_\alpha \\
\end{array}
$$

(13)

For convenience we wrote below short reminders of what these categories are:

- cobordisms $\longrightarrow$ $k$-linear $\longrightarrow$ viewable $\longrightarrow$ skein $\longrightarrow$ Deligne (Karoubian)
- gligible $\longrightarrow$ gligible and Karoubian

(14)

If $\alpha$ is not recognizable, we can still define categories $VTFS_\alpha$, $TFS_\alpha$ and $DTFS_\alpha$ (in the streamlined notation, categories $VS_\alpha$, $S_\alpha$ and $DS_\alpha$), but it is not clear whether these categories may be interesting for some such $\alpha$.

4. Hilbert Scheme and Recognizable Series

Recognizable series and points on the dual tautological bundle. Recognizable series $\alpha$ gives rise to the ideal $I_\alpha$ in $k[T_1, T_2]$ of finite codimension $k = d_\alpha$ and the quotient algebra $A_\alpha$ by this ideal, see formula (10) in Section 3.2. This algebra is commutative Frobenius and comes with two generators $T_1, T_2$ and a non-degenerate trace. The ideal $I_\alpha$ describes a point in the Hilbert scheme of codimension $k$ ideals in $A^2 = \text{Spec } k[T_1, T_2]$, where $k = d_\alpha = \dim A_\alpha$.

Let us specialize to the ground field $k = \mathbb{C}$. Denote by $\text{Rec}_k$ the set of recognizable series with the syntactic ideal $I_\alpha$ of codimension $k$ and refer to $\alpha \in \text{Rec}_k$ as a recognizable series of codimension $k$. Let also

$$\text{Rec} := \bigsqcup_{k \geq 0} \text{Rec}_k, \quad \text{Rec}_{\leq n} := \bigsqcup_{k \leq n} \text{Rec}_k.$$

Consider the Hilbert scheme $H^k = \text{Hilb}^k(\mathbb{C}^2)$ of $k$ points in $\mathbb{C}^2$ or, equivalently, the scheme of codimension $k$ ideals in $\mathbb{C}[T_1, T_2]$, see [20].

Denote by $T_k$ the tautological bundle over $H^k$ whose fiber over the point associated to an ideal $I$ of codimension $k$ is the space $A_I = \mathbb{C}[T_1, T_2]/I$. Points of the dual bundle $T_k^\vee$ above a point $I \in H^k$ describe elements of $A_I^* = \text{Hom}_{\mathbb{C}}(A_I, \mathbb{C})$, that is, linear functionals on $A_I$. Let

$$\pi : T_k^\vee \longrightarrow H^k$$

be the projection of the bundle onto its base. For a point $p \in T_k^\vee$, the element $\pi(p) \in H^k$ is the projection of $p$ onto the base of the bundle, and we denote by $I_{\pi(p)}$ the corresponding codimension $k$ ideal of $\mathbb{C}[T_1, T_2]$.

The point $p$ also defines a linear functional $\alpha_p$ on

$$A_{\pi(p)} := \mathbb{C}[T_1, T_2]/I_{\pi(p)}, \quad \alpha_p : A_{\pi(p)} \longrightarrow \mathbb{C},$$
associated to \( p \). This functional lifts to a functional on \( \mathbb{C}[T_1, T_2] \), which is recognizable, contains \( I_{\pi(p)} \) in its kernel, and has codimension at most \( k \). The latter functional (equivalently, recognizable power series) is also denoted \( \alpha_p \).

This functional has the associated ideal \( I_p = I_{\alpha_p} \subset \mathbb{C}[T_1, T_2] \) of finite codimension, the largest ideal in the kernel of functional \( \alpha_p \) on \( \mathbb{C}[T_1, T_2] \). There is an inclusion of ideals

\[
I_{\pi(p)} \subset I_p.
\]

For a generic point \( p \) on \( T_k^\vee \) this inclusion is the equality \( I_{\pi(p)} = I_p \), but for some points \( p \) the inclusion is proper.

Another way to describe the ideal \( I_p \) is to consider the symmetric bilinear form \( (,)_p \) on \( A_{\pi(p)} \) given by

\[
(x, y)_p := \alpha_p(xy), \quad x, y \in A_{\pi(p)}.
\]

The kernel of the form \( (,)_p \) is an ideal \( I'_p \) in \( A_{\pi(p)} \) that lifts to the above ideal \( I_p \) in \( \mathbb{C}[T_1, T_2] \), and there is an isomorphism \( I'_p \cong I_p/I_{\pi(p)} \). The inclusion \( I_{\pi(p)} \subset I_p \) is proper precisely when \( I'_p \) is a nonzero ideal, that is, when the bilinear form \( (,)_p \) is degenerate.

These ideals are shown in the diagram below, where the two squares on the left are pull-backs. The bottom sequence is short exact, and the top row becomes exact upon replacing \( I_{\pi(p)} \) by 0.

\[
\begin{array}{cccccc}
I_{\pi(p)} & \subset & I_p & \cong & \mathbb{C}[T_1, T_2] & \longrightarrow & A_p & \longrightarrow & 0 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \uparrow \\
0 & \subset & I'_p & \longrightarrow & A_{\pi(p)} & \longrightarrow & A_p & \longrightarrow & 0
\end{array}
\]

Denote by \( D_k \) the subset of \( T_k^\vee \) that consists of points \( p \) such that the inclusion \( I_{\pi(p)} \subset I_p \) is proper:

\[
D_k := \{ p \in T_k^\vee | I_{\pi(p)} \neq I_p \}.
\]

Recognizable power series \( \alpha_p \) for \( p \in T_k^\vee \) has codimension \( k \) (in our notations, \( \alpha_p \in \text{Rec}_k \)) precisely when \( p \in T_k^\vee \setminus D_k \).

If \( p \in D_k \) so that

\[
\text{codim}(I_p) = m < k = \text{codim}(I_{\pi(p)}),
\]

then recognizable power series \( \alpha_p \) has codimension \( m \) strictly less than \( k \) and \( \alpha_p \in \text{Rec}_m \).

For example, if \( p \in H^k \subset T_k^\vee \) is a point in the zero section of \( T_k^\vee \), so that the linear map \( \alpha_p \) is identically zero, the ideal \( I_p = \mathbb{C}[T_1, T_2] \) has zero codimension and \( m = 0 \). A mild confusion exists in our notations in this case (and in this case only), for then \( p = \pi(p) \).

Going the other way, to a recognizable series \( \alpha \) with the associated ideal \( I_\alpha \) of codimension \( d_\alpha = k \) as above we associate a point \( p_\alpha \) of \( T_k^\vee \). It is the point in the fiber of \( T_k^\vee \) over the ideal \( I_\alpha \) which describes functional \( \alpha \) on \( \mathbb{C}[T_1, T_2] \) and the induced functional on the quotient algebra \( A_\alpha = A_{I_\alpha} \).

The above discussion implies the proposition below.

**Proposition 4.1.** Assigning \( p_\alpha \) to \( \alpha \in \text{Rec}_k \) and \( \alpha_p \) to \( p \in T_k^\vee \setminus D_k \) establishes a bijection

\[
\text{Rec}_k \cong T_k^\vee \setminus D_k.
\]
In particular, $p_{\alpha p} = p$ and $\alpha_{p_{\alpha}} = \alpha$ for $p$ and $\alpha$ as in the proposition, so the two assignments are mutually-inverse bijections. □

Note that the two ideals coincide, $I_{\pi(p)} = I_p$, precisely when $\alpha_{p}$ is a nondegenerate trace map on $A_{\pi(p)}$. In particular, in this case $A_{\pi(p)}$ is Frobenius. We obtain the following statement.

**Proposition 4.2.** Points $p \in T_k^\vee \setminus D_k$ classify isomorphism classes of data $(A, \epsilon, t_1, t_2)$: a commutative Frobenius algebra $A$ over $\mathbb{C}$ of dimension $k$ with a non-degenerate trace $\epsilon$ and generators $t_1, t_2$ of $A$.

Not every commutative Frobenius algebra can be generated by two elements, of course.

Taking codimension $m \leq k$ of $I_p$ into account, one gets the following statement.

**Proposition 4.3.** Associating $\alpha_p$ to $p \in T_k^\vee$ gives a surjective map

$$T_k^\vee \longrightarrow \bigsqcup_{m=0}^{k} \text{Rec}_m.$$

Restricting this map to $D_k$ gives a surjective map

$$D_k \longrightarrow \bigsqcup_{m=0}^{k-1} \text{Rec}_m,$$

while on the complement to $D_k$ this map is the bijection in Proposition 4.1.

**Example.** The set $\text{Rec}_0$ is a single point corresponding to the zero series $\alpha$, $\alpha_{i, j} = 0$, $i, j \in \mathbb{Z}_+$. The ideal for this point is the entire algebra $\mathbb{C}[T_1, T_2]$. Points of $\text{Rec}_1$ correspond to hyperplanes (codimension one subspaces) that are ideals $J = (T_1 - \lambda_1, T_2 - \lambda_2)$ together with a nonzero functional on $\mathbb{C} \cong \mathbb{C}[T_1, T_2]/J$, determined by its value $\lambda$ on 1. Consequently, we can identify $\text{Rec}_1 \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}^\times$ by sending a point in $\text{Rec}_1$ to the triple $(\lambda_1, \lambda_2, \lambda)$.

**Set-theoretic divisor** $D_k$. Quasi-projective variety $H^k$ admits an open cover by affine sets $U_{\lambda}$, over all partitions $\lambda$ of $k$, see Theorem 18.4 in [18, Section 18.4], for example. Place partition $\lambda$ in the corner of the first quadrant of the plane so that it covers nodes $(i, j)$ of the square lattice with $0 \leq i < \lambda_{j+1}$. In particular, it covers $\lambda_1$ nodes on the $x$-axis.

Let $T_{\lambda}$ be the set of monomials $T_1^iT_2^j$ with $(i, j) \in \lambda$ (in particular, $|T_{\lambda}| = k$) and $T_{\lambda}'$ be the set of complementary monomials, for pairs $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \setminus \lambda$. Open set $U_{\lambda} \subset H^k$ consists of ideals $I$ such that monomials in $T_{\lambda}$ descend to a basis of $A_I = \mathbb{C}[T_1, T_2]/I$, see [18, Section 18.4] for details.

The vector bundle $T_k^\vee \longrightarrow H^k$ can be trivialized over $U_{\lambda}$, being naturally isomorphic to the trivial bundle of functions on the set $T_{\lambda}$. A functional $p$ on $\mathbb{C}[T_1, T_2]/I_{\pi(p)}$ is determined by its values on the basis elements $t \in T_{\lambda}$ of this quotient space.

To describe the points $p \in T_k^\vee$ with $\pi(p) \in U_{\lambda}$ consider an arbitrary linear functional $\alpha \in (\mathbb{C}T_{\lambda})^*$, given by its values

$$\alpha(T_1^iT_2^j) \in \mathbb{C}, \text{ for } T_1^iT_2^j \in T_{\lambda},$$
and an ideal $I \in U_{\lambda}$. Such pair $(\alpha, I)$ trivializes a pair $(p, \pi(p))$ with $\pi(p) \in U_{\lambda}$. For a pair $u, v \in T_{\lambda}$ take the product $uv$, view it as an element of $A_I = \mathbb{C}[T_1, T_2]/I$, and then write it as a linear combination of elements in $T_{\lambda}$, allowing to apply $\alpha$ to it explicitly.

Consider a matrix $M_\alpha$ where rows and columns are labelled by elements of $T_{\lambda}$ and put $\alpha(uv)$ as the entry at the intersection of row $u$ and column $v$.

**Proposition 4.4.** Point $p$ with $\pi(p) \in U_{\lambda}$ is in the subset $D_k$ iff $\det(M_\alpha) = 0$.

**Proof.** Matrix $M_\alpha$ is the Gram or Hankel matrix of the bilinear form $(x, y) = \alpha(xy)$ on the associative algebra $A_I$ in the basis $T_{\lambda}$. A bilinear form on a finite-dimensional algebra $B$ given by the composition of the multiplication with a fixed linear functional on $B$ is non-degenerate exactly when its Hankel matrix with respect to some (equivalently, any) basis is non-degenerate, that is, has a non-zero determinant. □

Condition that $\det(M_\alpha) = 0$ is locally a codimension one condition (given by a single equation), unless the determinant is identically zero on points $(p, \pi(p))$ with $\pi(p)$ on some irreducible component of the open subset $U_{\lambda}$ of the Hilbert scheme. To see that the latter case does not happen, observe that a “generic” point $I$ on the Hilbert scheme $H^k$ corresponds to a semisimple quotient (no nilpotent elements in $\mathbb{C}[T_1, T_2]/I$), with the quotient algebra isomorphic to the product of $k$ fields,

$$\mathbb{C}[T_1, T_2]/I \cong \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}.$$ 

On this quotient an open subset of linear functionals are non-degenerate, with the associated bilinear forms having trivial kernels. Indeed, a functional $\alpha$ on the algebra $\prod_{i=1}^k \mathbb{C}$ is non-degenerate iff each of its $k$ coefficients is non-zero.

These observations imply the following result.

**Proposition 4.5.** $D_k$ is a set-theoretic divisor on the variety $T_k^\vee$.

It is straightforward to check that $D_k$ comes from an actual divisor on $T_k^\vee$. For a finite-dimensional $\mathbb{C}$-vector space $V$ define a one-dimensional vector space

$$\det V := (\Lambda^{\text{top}} V)^\vee = \Lambda^{\text{top}} (V^\vee).$$

The determinant $\det \widehat{\alpha}$ of a bilinear form $\widehat{\alpha} : V \otimes V \to \mathbb{C}$ is an element $\det \widehat{\alpha} \in (\det V)^{\otimes 2}$ defined as the determinant of the matrix of $\widehat{\alpha}$. Namely, if $e_1, \ldots, e_k$ is a basis of $V$ and $e^1, \ldots, e^k$ is the dual basis in $V^\vee$, then $e^1 \wedge \cdots \wedge e^k$ is a basis in the one-dimensional space $\det V$ and

$$\det \widehat{\alpha} := \det |\widehat{\alpha}(e_i, e_j)| (e^1 \wedge \cdots \wedge e^k)^{\otimes 2}.$$ 

A point $p \in T_k^\vee$ defines a symmetric bilinear form $\widehat{\alpha}_p(x, y) := \alpha_p(xy)$ on the fiber $T_{\pi(p)} = I_{\pi(p)}$ of the tautological bundle. The determinant of this form is an element of $(\det T_{\pi(p)})^{\otimes 2}$. Hence the pullback line bundle

$$\pi^* \left((\det T)^{\otimes 2}\right) \to T_k^\vee$$

over $T_k^\vee$ has a canonical section $\sigma_{\det}$ given by $\sigma_{\det}(p) := \det \widehat{\alpha}_p$. The set $D_k$ is the divisor of zeroes of this section.

**Corollary 4.6.** $D_k$ is the divisor of zeros of the section $\sigma_{\det}$.
Each point of $T^\vee_k \setminus D_k$ gives rise to recognizable series $\alpha$ in two variables and to the corresponding rigid symmetric monoidal categories, as discussed in the Section 3 and summarized in Section 3.4, including category $\text{TFS}_\alpha$, the Deligne category $\text{DTFS}_\alpha$ and its eligible quotient $\text{DTFS}_\alpha$. It may be interesting to understand these categories for various $\alpha$‘s, including finding the analogue of the classification result from [15] on when the category $\text{DTFS}_\alpha$ is semisimple.

5. Modifications

5.1. Adding closed surfaces. Category TFS can be enlarged to a category $C$ with morphisms – oriented 2D cobordisms (surfaces) with corners between oriented 1D manifolds with corners. Extensions of 2D TQFTs to this category have been widely studied [2, 17, 19, 23]. An oriented 1D manifold with corners is diffeomorphic to a disjoint union of finitely-many oriented intervals and circles. We adopt a minimalist approach and choose one manifold for each such diffeomorphism class. Consequently, objects of $C$ are pairs $n = (n_1, n_2)$ of non-negative integers, and an object $n$ is represented by a fixed disjoint union $W(n) = W(n_1, n_2)$ of $n_1$ intervals and $n_2$ circles. Morphisms from $n$ to $m = (m_1, m_2)$ are compact oriented 2D cobordisms $M$, possibly with corners, with both horizontal and side boundary and corners where these two different boundary types meet:

$$\partial M = \partial_h M \cup \partial_v M, \quad \partial_h M = W(m_1, m_2) \sqcup (-W(n_1, n_2)).$$

Cobordisms that are diffeomorphic rel boundary define the same morphisms. Category $C$ contains TFS as a subcategory.

$C$ is a rigid symmetric monoidal category, with self-dual objects. The unit object $1$ is the empty one-manifold $W(0, 0)$. Its endomorphism monoid is freely generated by diffeomorphism types of compact connected surfaces with boundary. The latter are classified by surfaces $S_{\ell, g}$ with $\ell$ boundary components and of genus $g$, one for each pair $(\ell, g)$, $\ell, g \in \mathbb{Z}_+$. The difference with endomorphisms of the unit object of TFS is that in $C$ closed surfaces are allowed, which corresponds to $\ell = 0$ and surfaces $S_{0, g}$, over all $g \in \mathbb{Z}_+$.

Multiplicative evaluations $\beta$ of endomorphisms of the unit object are again encoded by a power series

$$\tilde{Z}_\beta(T_1, T_2) = \sum_{k, g \geq 0} \beta_{k, g} T_1^k T_2^g, \quad \beta = (\beta_{k, g})_{k, g \in \mathbb{Z}_+}, \quad \alpha_{k, g} \in \mathbb{k}, \quad (15)$$

with the first index shifted by 1 compared to evaluations for TFS. We changed the label from $\alpha$ in evaluations in TFS to $\beta$ in $C$ to make it easier to compare evaluations in these two categories. Now the coefficient

$$\beta_{k, g} = \beta(S_{k, g})$$

is the evaluation of connected genus $g$ surface with $k$ boundary components rather than with $k + 1$ components as in the TFS case, see earlier.

To relate these two power series encodings, in formulas (1) and (4) versus (15), start with $Z_\alpha(T_1, T_2)$ as in (4) and also form a one-variable power series

$$Z_\gamma(T_2) = \sum_{k \geq 0} \gamma_k T_2^k, \quad \gamma_k \in \mathbb{k}.$$
To the pair \((Z_\alpha, Z_\gamma)\) assign the series
\[
\tilde{Z}_\beta(T_1, T_2) = T_1 Z_\alpha(T_1, T_2) + Z_\gamma(T_2).
\] (16)

Adding coefficients of \(Z_\gamma\) to the data provided by \(Z_\alpha\) precisely means that we now include evaluations of closed surfaces, via coefficients \(\gamma_k\) (for a closed surface genus \(k\)). The scaling factor \(T_1\) in the formula is needed to match the discrepancy in the evaluation conventions in the two categories TFS and \(\mathcal{C}\). Formula (16) gives a bijection between series encoded by \(\beta\) and those encoded by \((\alpha, \gamma)\). Starting from \(\tilde{Z}_\beta\), one recovers \(Z_\alpha\) and \(Z_\gamma\) as
\[
Z_\gamma(T_2) = \tilde{Z}_\beta(0, T_2)
\]
\[
Z_\alpha(T_1, T_2) = \left(\tilde{Z}_\beta(T_1, T_2) - \tilde{Z}_\beta(0, T_2)\right)/T_1.
\]

From \(\mathcal{C}\) pass to its \(k\)-linearization \(k\mathcal{C}\) by allowing finite \(k\)-linear combinations of morphisms in \(\mathcal{C}\). Given series \(\beta\), we can define analogues of categories \(\mathcal{V}\text{TFS}_\alpha\) and \(\text{TFS}_\alpha\) in (12). Denote these new categories by \(\mathcal{V}\mathcal{C}_\beta\) and \(\mathcal{C}_\beta\):

- In \(\mathcal{V}\mathcal{C}_\beta\) one evaluates floating components to elements of \(k\) via \(\beta\). A connected component is floating if it has no horizontal boundary.
- To form category \(\mathcal{C}_\beta\) we quotient category \(\mathcal{V}\mathcal{C}_\beta\) (alternatively, category \(k\mathcal{C}\)) by the two-sided ideal of negligible morphisms, defined in the same way as for TFS.

We say that evaluation \(\beta\) (or series \(\tilde{Z}_\beta\)) is recognizable if category \(\mathcal{C}_\beta\) have finite-dimensional hom spaces.

**Proposition 5.1.** \(\beta\) is recognizable iff the power series \(\tilde{Z}_\beta\) has the form
\[
\tilde{Z}_\beta(T_1, T_2) = \frac{\tilde{P}(T_1, T_2)}{Q_1(T_1)Q_2(T_2)},
\] (17)
where \(\tilde{Q}_1(T_1), \tilde{Q}_2(T_2)\) are one-variable polynomials and \(\tilde{P}(T_1, T_2)\) is a two-variable polynomial, all with coefficients in the field \(k\).

It is easy to see that series \(\beta\) is recognizable iff hom spaces

\[\text{Hom}(1, (1, 0))\] and \(\text{Hom}(1, (0, 1))\)

in \(\mathcal{C}_\beta\) are finite-dimensional. These are the hom spaces from the empty 1-manifold \(W(0, 0)\) (representing the unit object \(1\)) to an interval \(W(1, 0)\) and a circle \(W(0, 1)\), respectively. Necessity of this condition is obvious. Vice versa, if these homs are finite-dimensional, by analogy with the proof of Proposition 3.1, there are skein relations allowing to reduce some large number of handles (respectively, holes) on a connected component to a linear combination of otherwise identical cobordisms but with fewer handles (respectively holes). The rest of the proof of Proposition 5.1 follows that of Proposition 3.1. □

**Corollary 5.2.** Series \(\beta\) is recognizable iff the corresponding series \(\alpha\) and \(\gamma\) are both recognizable.
Note that, when $\alpha$ and $\gamma$ are recognizable, their rational function presentation may have very different denominators,

$$Z_\alpha(T_1, T_2) = \frac{P(T_1, T_2)}{Q_1(T_1) Q_2(T_2)}, \quad Z_\gamma(T_2) = \frac{P_\gamma(T_2)}{Q_\gamma(T_2)},$$

so that

$$\tilde{Z}_\beta(T_1, T_2) = \frac{T_1 P(T_1, T_2)}{Q_1(T_1) Q_2(T_2)} + \frac{P_\gamma(T_2)}{Q_\gamma(T_2)} = \frac{T_1 P(T_1, T_2) Q_\gamma(T_2) + Q_1(T_1) Q_2(T_2) P_\gamma(T_2)}{Q_1(T_1) Q_2(T_2) Q_\gamma(T_2)}.$$

For generic polynomials, there are no cancellations and

$$\tilde{Q}_1(T_1) = Q_1(T_1), \quad \tilde{Q}_2(T_2) = Q_2(T_2) Q_\gamma(T_2)$$

are the denominators in the minimal rational presentation (17) for $\tilde{Z}_\beta$.

For recognizable $\beta$, the state spaces

$$A_\beta(1, 0) := \text{Hom}_{C_\beta} (\mathbf{1}, (1, 0)), \quad A_\beta(0, 1) := \text{Hom}_{C_\beta} (\mathbf{1}, (0, 1)),$$

of homs from the unit object $\mathbf{1} = (0, 0)$ to the interval and the circle objects, respectively, are both commutative Frobenius algebras. Annuli, viewed as morphisms between $(1, 0)$ and $(0, 1)$, see Fig. 17, give linear maps

$$\delta_0 : A_\beta(1, 0) \rightarrow A_\beta(0, 1), \quad \delta_1 : A_\beta(0, 1) \rightarrow A_\beta(1, 0)$$

between the underlying vector spaces.

Consider the **hole** and **handle** endomorphisms $b_1, b_2$ of the interval and $c_1, c_2$ of the circle, respectively, in Fig. 18 top.

Multiplications in algebras $A_\beta(1, 0)$ and $A_\beta(0, 1)$ are given by pants and flat pants cobordisms, see Fig. 19, where the cobordisms for the unit and trace morphisms on $A_\beta(1, 0)$ and $A_\beta(0, 1)$ are shown as well.

Take endomorphisms $b_1, b_2, c_1, c_2$ of the interval and circle and cap them off at the bottom with the unit morphisms $\iota$ and $\iota'$ for the interval and circle (see Fig. 19) to get elements $\overline{b}_1 = b_1 \iota, \overline{b}_2 = b_2 \iota$ in $A_\beta(1, 0)$ and elements $\overline{c}_1 = c_1 \iota', \overline{c}_2 = c_2 \iota'$ in $A_\beta(0, 1)$, shown in Fig. 18.

The analogue of Proposition 2.3 holds in $C$, and the “interval” Frobenius algebra $A_\beta(1, 0)$ is generated by commuting elements $\overline{b}_1, \overline{b}_2$ (the hole and handle elements).
Fig. 18. Endomorphisms $b_1, b_2$ of the interval, endomorphisms $c_1, c_2$ of the circle and corresponding elements $\beta_1, \beta_2$ of $A_\beta(1, 0) = \text{Hom}(1, (1, 0))$ and elements $\xi_1, \xi_2 \in A_\beta(0, 1) = \text{Hom}(1, (0, 1))$.

Fig. 19. Flat pants and pants cobordisms, together with the other structure maps $\iota, \epsilon$ and $\iota', \epsilon'$ (units $\iota, \iota'$ and counits $\epsilon, \epsilon'$) of commutative Frobenius algebras $A_\beta(1, 0)$ and $A_\beta(0, 1)$.

Likewise, the “circle” Frobenius algebra $A_\beta(0, 1)$ is generated by commuting hole and handle elements $\xi_1$ and $\xi_2$.

Endomorphisms $b_1, b_3$ of the interval in the category $\mathcal{C}$ are different (endomorphism $b_3$ is also shown in Fig. 11), but they induce the same map on $A_\beta(1, 0)$, see Fig. 20. There, $x \in A_\beta(1, 0)$ can be written as a linear combination of monomials $b_1^n b_2^m$, with $b_3$ acting by

$$b_3 b_1^n b_2^m = b_1^{n+1} b_2^m = b_1 b_1^n b_2^m.$$

Trace maps

$$\epsilon : A_\beta(0, 1) \longrightarrow \mathbf{k}, \quad \epsilon' : A_\beta(1, 0) \longrightarrow \mathbf{k},$$

(18)
given by capping off the interval with a disk, respectively the circle with a cap, turn these two commutative algebras into Frobenius algebras (for recognizable $\beta$).

Compositions of $\delta_0$ and $\delta_1$ are endomorphisms of the interval and the circle in $\mathcal{C}$ (and in $\mathcal{C}_\beta$) and satisfy

$$\delta_1 \delta_0 = b_3, \quad \delta_0 \delta_1 = c_1,$$

$$\delta_1 c_1 = b_1 \delta_1, \quad \delta_1 c_2 = b_2 \delta_1,$$

$$\delta_0 b_1 = c_1 \delta_0, \quad \delta_0 b_2 = c_2 \delta_0.$$
In particular, maps $\delta_0, \delta_1$ intertwine the hole endomorphisms $b_1, c_1$ of the interval and the circle. They also intertwine the handle endomorphisms $b_2, c_2$ of the interval and the circle.

Their two compositions produce the hole endomorphisms of the interval and the circle.

The map $\delta_1$ is a surjective unital homomorphism of commutative algebras, while the map $\delta_0$ is an injective homomorphism of cocommutative coalgebras, with comultiplications given by the dual of the multiplications on these Frobenius algebras. In particular, $\delta_0$ respects traces, in the sense that $\epsilon' \delta_0 = \epsilon$.

A recognizable power series $\beta$ is encoded by a commutative Frobenius algebra (the state space of a circle) $A_\beta(0, 1)$ with generators $c_1, c_2$ and non-degenerate trace map $\epsilon'$ such that

$$\beta_{\ell, g} = \epsilon'(c_1^{\ell} c_2^g), \quad \ell, g \in \mathbb{Z}_+. \quad (19)$$

Further unwrapping this data, to a recognizable power series $\beta$ we can associate

- Two commutative Frobenius algebras $A(1, 0) = A_\beta(1, 0)$ and $A(0, 1) = A_\beta(0, 1)$ with generators $b_1, b_2$ and $c_1, c_2$, respectively (hole and handle elements).
- Non-degenerate traces $\epsilon$ and $\epsilon'$ as in (18), subject to (19) and

$$\beta_{\ell+1, g} = \epsilon(b_1^{\ell} b_2^g), \quad \ell, g \in \mathbb{Z}_+. \quad (19)$$

- Linear maps $\delta_0, \delta_1$:

$$A_\beta(1, 0) \xrightarrow{\delta_0} A_\beta(0, 1) \xrightarrow{\delta_1} A_\beta(1, 0)$$

that intertwine the action of handle elements $b_2$ and $c_2$. The hole elements are given by

$$b_1 = \delta_1 \delta_0(1), \quad c_1 = \delta_0 \delta_1(1).$$

- $\delta_1$ is a surjective unital homomorphism of commutative algebras.

The reader may want to constrast the data coming from a recognizable series $\beta$ as above, with both algebras $A_\beta(0, 1)$ and $A_\beta(1, 0)$ commutative Frobenius, with that given by a 2-dimensional TQFT with corners $[2, 17, 19, 23]$ where the Frobenius algebra $B$ associated to the interval is not necessarily commutative and the algebra associated to the circle is related to the center of $B$.

To a recognizable series $\beta$ there is associated a finite codimension ideal $I_\beta \subset k[T_1, T_2]$ describing relations on the hole and handle endomorphisms along any component of a surface. Starting with the viewable category $\mathcal{VC}_\beta$, described earlier, where

$\text{Fig. 20. } b_3 x = b_1 x \text{ for any } x \in A_\beta(1, 0). \quad b_3 \neq b_1 \text{ as } \text{End}((1, 0)) \text{ in } C \text{ (and in } C_\beta, \text{ in general})$
Fig. 21. Left: A morphism in TFS\(^{(r)}\) from the colored interval \((3, 1)\) to the union \((3, 2)\cup(2, 1)\) of two colored intervals. Middle: the dual of object \((i, j)\) is the object \((j, i)\). Right: a connected floating component of genus 1 and the sequence \((1, 2, 1)\). It has one boundary circle of colors 1 and 3 each and two circles of color 2.

Floating components are evaluated via \(\beta\), we impose relations in \(I_\beta\) on hole and handle endomorphisms along any component. The resulting category, denoted \(SC_\beta\) (the skein category) has finite-dimensional hom spaces.

From the skein category we can pass to the already defined negligible quotient \(C_\beta\) by taking the quotient of \(SC_\beta\) by the ideal of negligible morphisms. This ideal comes from the trace on \(SC_\beta\) or, equivalently, from the bilinear form given by pairing of cobordisms.

Taking the additive Karoubi closure of \(SC_\beta\) results in the Deligne category \(DC_\beta\).

Taking the quotient of \(DC_\beta\) by the ideal of negligible morphisms produces the category \(DC_\beta\). Alternatively, this category is equivalent to the additive Karoubi closure of \(C_\beta\), and the square below is commutative.

The following diagram summarizes these categories and functors (compare with (12), (14), and [16]).

\[
\begin{array}{cccc}
\mathcal{C} & \longrightarrow & k\mathcal{C} & \longrightarrow & \mathcal{VC}_\beta & \longrightarrow & SC_\beta & \longrightarrow & DC_\beta \\
& & & & \downarrow & & \downarrow & & \\
& & & & C_\beta & \longrightarrow & DC_\beta & & \\
\end{array}
\]  

(20)

Each of the four categories in the vertices of the commutative square has finite-dimensional hom spaces between its objects.

5.2. Coloring side boundaries of cobordisms. Fix a natural number \(r \geq 1\) and consider a modification TFS\(^{(r)}\) of the category TFS where side boundaries of cobordisms are colored by numbers from 1 to \(r\). Let \(\mathbb{N}_r = \{1, \ldots, r\}\) be the set of colors. A morphism in TFS\(^{(r)}\) is a tf-surface \(x\), up to rel boundary diffeomorphisms, such that each side (or vertical) boundary component of \(x\) carries a label from \(\mathbb{N}_r\). Coloring of \(x\) induces a coloring on the set of corners of \(x\), that is, on endpoints of the one-manifold \(\partial_hx\) which is the horizontal boundary of \(x\), see Fig. 21.

Consequently, each boundary interval \(I\) of \(x\), being oriented, gets an induced ordered sequence \((r_1(I), r_0(I))\) of two colors. We consider a skeletal version of TFS\(^{(r)}\), choosing only one object for each isomorphism class. An object \(a\) then is determined by the \(r \times r\) matrix \(M = M(a)\) with the \((i, j)\)-entry the number of intervals in \(a\) colored \((i, j)\).

Thus, objects \(a\) are described by \(r \times r\) matrices of non-negative integers counting number of colored intervals in \(a\). We can call these objects \(r\)-colored or \(r\)-labelled thin one-manifolds or \(r\)-boundary colored thin one-manifolds. An object can also be described by a list of colored intervals in it.
This skeletal version is still rigid tensor, with the obvious tensor product. The unit object $\mathbf{1} = \emptyset$ corresponds to the matrix of size $0 \times 0$.

The notion of a connected component, floating and viewable components of a morphism are defined as for TFS. Commutative monoid $\operatorname{End}(\emptyset)$ of endomorphisms of the empty one-manifold $\emptyset$ is a free abelian monoid generated by diffeomorphism classes of connected floating $r$-colored tf-surfaces. Such a surface $S$ is classified by its genus $g \geq 0$ and a sequence of $r$ non-negative integers $\mathbf{n} = (n_1, \ldots, n_r)$, where $n_i$ is the number of boundary components of color $i$. Denote such component by $S_{\mathbf{n},g}$. Figure 21 right shows the component $S_{(1,2,1),1}$.

For each color $i \leq r$ there is an embedding of TFS into $\operatorname{TFS}(r)$ by coloring each side boundary of morphisms in TFS by $i$. Each horizontal interval is then an $(i,i)$-interval.

For a morphisms between two objects in $\operatorname{TFS}(r)$ to exist, there must exist a suitable matching between the colorings of their endpoints. For instance, there are no morphisms from the empty object $\emptyset$ to $(i,j)$ interval if $i \neq j$, since the $i$ and $j$ endpoints must belong to the same side interval and have the same coloring. There are morphisms from $\emptyset$ to $(i,j) \sqcup (j,i)$ but no morphisms from $\emptyset$ to $(i,j) \sqcup (i,j)$ for $i \neq j$, since matching the two $i$’s via a side interval is not possible with our orientation setup.

As usual, denote by $k\operatorname{TFS}(r)$ the $k$-linear version of $\operatorname{TFS}(r)$, with the same objects as $\operatorname{TFS}(r)$ and morphisms – $kk$-linear combinations of morphisms in $\operatorname{TFS}(r)$.

The construction of evaluation categories and recognizable (or rational) series can be extended from TFS to $\operatorname{TFS}(r)$ in a direct way.

An evaluation $\alpha$ is a multiplicative homomorphism from the monoid $\operatorname{End}(\emptyset)$ of floating colored tf-surfaces to a field $k$. Such an evaluation is determined by its values on connected floating surfaces $S_{\mathbf{n},g}$. Let

$$Z_{\alpha}(T_0, \ldots, T_r) = \sum_{\mathbf{n},g} \alpha_{\mathbf{n},g} T_0^g T_1^{n_1} \cdots T_r^{n_r}, \quad \alpha_{\mathbf{n},g} \in k$$

be a formal power series in $r + 1$ variables, with

$$T^n := T_1^{n_1} \cdots T_r^{n_r}, \quad \mathbf{n} = (n_1, \ldots, n_r), \quad n_i \in \mathbb{Z}_+$$

where $T^n$ is a monomial in $T_1, \ldots, T_r$. Thus, $T_0$ is the genus variable and $T_1, \ldots, T_r$ are color variables. Coefficient $\alpha_{\mathbf{n},g}$ at $T_0^g T_1^{n_1} \cdots T_r^{n_r}$ encodes the evaluation of floating connected surface $S_{\mathbf{n},g}$.

Since each component of a tf-surface has non-empty boundary, coefficients at $T_0^g$, with $\mathbf{n} = \mathbf{0} = (0, \ldots, 0)$ do not appear in this formal sum. We set them to zero and extend the sum to these indices by setting

$$\alpha_{\mathbf{0},g} = 0, \quad g \in \mathbb{Z}_+.$$  \hspace{1cm} (22)

Thus, our power series has the property that

$$Z_{\alpha}(T_0, 0, \ldots, 0) = 0.$$  \hspace{1cm} (23)

We can also view $\alpha$ as a linear map of vector spaces

$$\alpha : k[T_0, \ldots, T_r] \longrightarrow k$$

subject to condition (22), that is, $\alpha(T_0^g) = 0, g \geq 0$. 

To \( \alpha \) we assign the category \( \text{VTFS}_\alpha^{(r)} \), the quotient of \( \text{VTFS}^{(r)} \) by the relations that a connected floating component diffeomorphic to \( S_{n,g} \) evaluates to \( \alpha_{n,g} \). This is the category of viewable \( r \)-colored tf-surfaces with the \( \alpha \)-evaluation.

Category \( \text{VTFS}_\alpha^{(r)} \) carries a natural trace form given on an endomorphism \( x \) of an object \( a \) by closing \( x \) into a floating surface \( \hat{x} \) and evaluating this surface via \( \alpha \), see Fig. 16, where now side boundaries are \( r \)-colored. If \( x \) is not a single cobordism but a linear combination, we use linearity of the trace to define \( \text{tr}_\alpha(x) = \alpha(\hat{x}) \).

Denote by \( J_\alpha \) the two-sided ideal of negligible morphisms in \( \text{VTFS}_\alpha^{(r)} \) for this trace map. Define the gligible cobordism category \( \text{TFS}_\alpha^{(r)} \) as the quotient of \( \text{VTFS}_\alpha^{(r)} \) by the ideal \( J_\alpha \):

\[
\text{TFS}_\alpha^{(r)} := \frac{\text{VTFS}_\alpha^{(r)}}{J_\alpha}.
\]

We say that evaluation \( \alpha \) is rational or recognizable if category \( \text{TFS}_\alpha^{(r)} \) has finite-dimensional hom spaces.

**Proposition 5.3.** The following properties are equivalent.

1. \( \alpha \) is recognizable.
2. Hom spaces \( \text{Hom}(\emptyset, (i, i)) \) from the empty one-manifold to the \((i, i)\)-interval are finite-dimensional in \( \text{TFS}_\alpha^{(r)} \) for all \( i = 1, \ldots, r \).
3. Power series \( Z_\alpha \) has the form

\[
Z_\alpha(T_0, \ldots, T_r) = \frac{P(T_0, \ldots, T_r)}{Q_0(T_0)Q_1(T_1)\ldots Q_r(T_r)},
\]

where \( P \) is a polynomial in \( r + 1 \) variables and \( Q_0, \ldots, Q_r \) are one-variable polynomials, with \( Q_i(0) \neq 0 \), \( i = 0, \ldots, r \).

Polynomials \( Q_i \) can be normalized so that \( Q_i(0) = 1 \) for all \( i \). Power series \( Z_\alpha \) also satisfies equation (23).

**Proof.** The proof is essentially the same as in \( r = 1 \) case, when all side components carry the same color and there is no need to mention colors. Proof of Proposition 3.1 carries directly to the case of arbitrary \( r \). \( \square \)

Take any floating component \( S \) and a monomial \( T = T_0^nT_1^{n_1}\ldots T_r^{n_r} \). Define \( S(T) \) as the surface \( S \) with additional \( g \) handles and additional \( n_i \) holes with boundary colored \( i \), for \( i = 1, \ldots, r \).

Given a linear combination \( y = \sum \mu_i T_i \) of monomials, define \( S(y) = \sum \mu_i S(T_i) \) as the linear combination of corresponding floating surfaces. Evaluation \( \alpha(S(y)) \) is an element of the ground field \( k \).

Given \( \alpha \), we can then define the syntactic ideal \( I_\alpha \subset k[T_0, \ldots, T_r] \). Namely, \( y \in I_\alpha \) if \( \alpha(S(y)) = 0 \) for any floating \( S \).

**Proposition 5.4.** \( \alpha \) is recognizable iff the ideal \( I_\alpha \) has finite codimension in \( k[T_0, \ldots, T_r] \).

Thus, for recognizable \( \alpha \), one can define the skein category \( \text{STFS}_\alpha^{(r)} \) as the quotient of \( \text{VTFS}_\alpha^{(r)} \) by the relations that inserting any \( y \in I_\alpha \) into a cobordism is zero. Category \( \text{STFS}_\alpha^{(r)} \) has finite-dimensional hom spaces. It also has the ideal of negligible morphisms, with the quotient category isomorphic to \( \text{TFS}_\alpha^{(r)} \). One can then define the analogue of the Deligne category for \( \text{STFS}_\alpha^{(r)} \) by taking its additive Karoubi closure and define the glibigle quotient of the latter. The resulting diagram of categories and functors...
below mirrors diagrams (12), (20), and the corresponding diagram in [16]. The square is commutative.

\[
\begin{array}{cccccc}
\text{TFS}^{(r)} & \longrightarrow & \text{kTFS}^{(r)} & \longrightarrow & \text{VTFS}_{\alpha}^{(r)} & \longrightarrow & \text{STFS}_{\alpha}^{(r)} & \longrightarrow & \text{DTFS}_{\alpha}^{(r)} \\
\downarrow & & & & \downarrow & & \downarrow & \downarrow & \\
\text{TFS}_{\alpha}^{(r)} & \longrightarrow & \text{DTFS}_{\alpha}^{(r)}
\end{array}
\] (24)

Condition (23) on the power series \( Z_{\alpha} \) seems rather unnatural. It can be removed by passing to the larger category, as in Section 5.1, where now closed components are allowed. Objects of the new category that extends \( \text{TFS}^{(r)} \) are disjoint unions of oriented intervals (with endpoints colored by elements of \( \mathbb{N}_r \)) and circles. Morphisms are two-dimensional oriented cobordisms between these collections, with side boundary intervals and side circles colored by elements of \( \mathbb{N}_r \). In the definition of evaluation \( \alpha \) we can now omit condition (22) or, equivalently, restriction (23) on the power series \( Z_{\alpha} \).

Definition and basic properties or recognizable series now work as in the \( \text{TFS}^{(r)} \) case. In the analogue of Proposition 5.3 for this modification, property (2) is replaced by the condition that the state space of the circle is finite-dimensional (hom space \( \text{Hom}(\emptyset, S^1) \)) is finite-dimensional). This is due to the surjection from the state space of the circle to that of the interval \((i, i)\) induced by the map \( \delta_1 \) in Fig. 18 with the side (vertical) interval colored \( i \). It is straightforward to set up the analogue of the diagram (24) of categories and functors for this case as well, for recognizable \( \alpha \).

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