Simultaneous two-dimensional best Diophantine approximations in the Euclidean norm

Evgeny V. Ermakov

1. Introduction

This paper is devoted to the exponents of growth of denominators of best simultaneous Diophantine approximations. Consider $\mathbb{R}^n$ with a norm $\| \cdot \|$. For any vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \setminus \mathbb{Q}^n$ and any $q \in \mathbb{Z}$ define the following value:

$$\delta_q = \min_{p = (p_1, \ldots, p_n) \in \mathbb{Z}^n} \| q \cdot \alpha - p \|.$$

Let $p(q) \in \mathbb{Z}^n$ be the vector, where the minimum is attained; let $r(q) = q \cdot \alpha - p(q)$, so $\delta_q = \| r(q) \|$. Given a norm $\| \cdot \|$ in $\mathbb{R}^n$ and a vector $\alpha \in \mathbb{R}^n \setminus \mathbb{Q}^n$ we can define the sequence of best approximations (with respect to this norm) as a sequence $(q_k)_{k=1}^\infty$, such that $q_1 = 1$ and $\forall q < q_k \quad \delta_q > \delta_{q_k}$. Now we can define following values:

$$g(\alpha, \| \cdot \|) = \liminf_{k \to \infty} (q_k)^{1/k},$$

$$G(n, \| \cdot \|) = \inf_{\alpha \in \mathbb{R}^n \setminus \mathbb{Q}^n} g(\alpha, \| \cdot \|).$$

J. Lagarias [1] has proved the following statement:

**Theorem 1.** For any norm $\| \cdot \|$ on $\mathbb{R}^n$ and a vector $\alpha$, that has at least one irrational coordinate, the inequality $q_{k+2^{n+1}} \geq 2q_{k+1} + q_k$ holds for all $k \geq 1$. So $G(n, \| \cdot \|) \geq \theta$, where $\theta$ is the maximal positive root of $\theta^{2^{n+1}} = 2\theta + 1$.

In this paper we consider $\mathbb{R}^2$ with the Euclidian norm. From Theorem 1 it follows that for the Euclidian norm in $\mathbb{R}^2$, and any vector $\alpha$ one has $q_{k+8} \geq q_{k+1} + q_k$.

There is another well known statement that holds for any norm. Given a norm $\| \cdot \|$ in $\mathbb{R}^n$ consider the contact number $K(n, \| \cdot \|)$. This number is defined as the maximal number of unit balls with respect to the norm $\| \cdot \|$ without interior common points that can touch another unit ball.

**Theorem 2.** For any norm $\| \cdot \|$ on $\mathbb{R}^n$ with the contact number $K = K(n, \| \cdot \|)$ and a vector $\alpha$, that has at least one irrational coordinate, we have the inequality $q_{k+K} \geq q_{k+1} + q_k$, and so $G(\| \cdot \|) \geq \theta$, where $\theta$ is maximum positive root of $\theta^K = \theta + 1$.

For the Euclidian norm in $\mathbb{R}^2$ we have $K = 6$. So Theorem 2 gives the inequality

$$q_{k+6} \geq q_{k+1} + q_k. \quad (1)$$

It follows that $G(2, \| \cdot \|_e) \geq \theta$, where $\theta$ is maximum positive root of $\theta^6 = \theta + 1$ and $\| \cdot \|_e$ is the Euclidian norm.
Theorem 2 is a well known result, one can find a proof of it in M. Romanov paper [3]. M. Romanov
proved a stronger result that the inequality
\[ q_{k+4} \geq q_{k+1} + q_k. \]  
(2)

is valid for any \( k \geq 1 \). From inequality (2) it follows that \( G(2, \| \cdot \|) \geq \theta_0 \) where \( \theta_0 \) is a positive root of \( \theta_0^6 = \theta_0 + 1 \), \( \theta_0 = 1.220744 \) The main result of the present paper is an improvement of Romanov’s result.

**Theorem 3.** For the Euclidian norm in \( \mathbb{R}^2 \) and any vector \( \alpha \), that has at least one irrational coordinate one has \( G(2, \| \cdot \|) \geq \theta_0 \).

The proof of Theorem 3 is based on following geometric statement that together with the inequality (2) and some numerical calculations gives the lower bound.

**Theorem 4.** Suppose that \( \alpha \in \mathbb{R}^2 \) has at least one irrational coordinate. Let \( q_k \ldots q_{k+4} \) be consecutive denominators from the sequence of best approximations in Euclidian norm for vector \( \alpha \). Then for every \( k \geq 1 \) at least one of two following inequalities are valid:
\[ q_{k+3} + q_{k+2} \geq 2q_{k+1} + q_k \]  
(3)
\[ q_{k+4} \geq q_{k+2} + q_k \]  
(4)

Moreover, among any two successive values of \( k \) for at least one value the inequality (3) holds.

A.Brentjes [2] gave the following example. Let \( \eta \) be the maximal root of the equation \( \eta^3 = \eta + 1 \), \( \eta = 1.3248 \ldots \) Then for \( \alpha = (\alpha_1, \alpha_2) = (\eta, \eta^2) \) one has \( g(\alpha, \| \cdot \|) = \eta \). J.Lagarias [1] made a conjecture, that \( G(2, (\| \cdot \|)) = \eta \).

In Sections 2, 3 below we give a complete proof of Theorem 4. In Section 4 we deduce Theorem 3 from Theorem 4 and Romanov’s theorem. There we describe all necessary computer calculations.

### 2. Geometric lemmas

**Lemma 1.** Consider a convex hexagon \( A_1A_2A_3A_4A_5A_6 \). Suppose that its opposite sides are equal and parallel. Suppose that \( O \) is an interior point of the hexagon. Let all the distances \( |A_1O|, |A_2O|, |A_3O|, |A_5O| \) are different. Then there exists \( i \in \{1, 2, 3, 5\} \) such that
\[ |A_iO| > \min_{j=1,2,3} |A_jA_{j+1}|. \]

**Proof.** Let \( a = \min(|A_1A_2|, |A_2A_3|) \). Without loss of generality suppose that \( a = |A_1A_2| \).

Consider circles \( \omega_1 \) and \( \omega_2 \) with radiuses \( a \) and centers in \( A_1 \) and \( A_3 \) correspondingly. Let \( \kappa_1 \) and \( \kappa_2 \) be closed disks bounded by \( \omega_1 \) and \( \omega_2 \). Define \( \Omega = \kappa_1 \cap \kappa_2 \). (See fig.1.)

Suppose that the conclusion of Lemma 1 is not true, that is there exists an interior point \( O \) of hexagon \( A_1A_2A_3A_4A_5A_6 \) such that \( |A_1O|, |A_2O|, |A_3O|, |A_5O| \) are different and
\[ |A_iO| \leq \min_{j=1,2,3} |A_jA_{j+1}|, \quad i = 1, 2, 3, 5. \]
So there exist \( i \in \{1, 2, 3, 5\} \) such that
\[
\|OA_i\| \leq \min_{j=1,2,3} |A_jA_{j+1}| \leq a.
\]
By the condition \( \max(|OA_1|,|OA_3|) \leq a \) we see that \( O \in \Omega \). So \( \Omega \neq \emptyset \) and circles \( \omega_1 \) and \( \omega_2 \) have common points. If \( \omega_1 \) and \( \omega_2 \) have the unique common point \( O \) then \( \|A_1O\| = \|A_3O\| \). This contradicts to the conditions of Lemma 1. So we see that circles \( \omega_1 \) and \( \omega_2 \) have two different common points.

The line \( A_1A_3 \) divides the plane into two different half-planes. Define \( Q \) to be that point of the intersection \( \omega_1 \) and \( \omega_2 \) such that \( A_2Q \) and \( Q \) belong to different half-planes. Let \( M \) be the point symmetric to \( A_2 \) with respect to the center of the segment \( A_1A_3 \). So \( MA_3A_4A_5 \) is a parallelogram and \( M \in \omega_2 \). Consider the disk \( \Theta \) with center in \( A_5 \) and radius \( \|A_5M\| = \|A_3A_4\| \).

By the construction \( O \in \Omega \cap \Theta \). But if \( \Omega \) and \( \Theta \) have a common point, it is the unique point \( Q = M \) as the distance from \( Q \) to the line \( A_1A_3 \) is less or equal to the distance from \( M \) to the line \( A_1A_3 \). So \( M \) belongs to \( \omega_2 \) but does not belong to \( \Omega \) if it is not point of intersection of \( \omega_1 \) and \( \omega_2 \). So if such point \( O \) exists it is equal to \( Q \). This contradicts to the condition that \( \|A_1O\| \neq \|A_3O\| \). Lemma 1 is proved.

Suppose that \( q_{k+3} < q_{k+1} + q_k \), otherwise we at once get \[\text{[3]}\] as the sequence \((q_k)\) increases.

Consider remainder vectors \( r(q_k), r(q_{k+1}), r(q_{k+2}), r(q_{k+3}) \). There exist a substitution of four indices \( s = (s(1), s(2), s(3), s(4)) \) such that \( r(q_{k-1+i}) = OR_s(i) \) and \( R_1R_2R_3R_4 \) is a tetragon without
self intersections.

**Lemma 2.** 1. The tetragon \( R_1R_2R_3R_4 \) is convex, point \( O \) lies inside it.
2. All of its sides and diagonals are not less then the longest remainder vector \( |r(q_k)| \).
3. Angles between vectors \( \overrightarrow{OR_i} \) and \( \overrightarrow{OR_j} \) \((i \neq j)\) are greater than \( \frac{\pi}{3} \).

**Proof.** Suppose, that \(|R_iR_j| < |r(q_k)|\) for any \(i \neq j\). Let \( R_i, R_j \) are the endpoints of vectors \( r(q_s) \) and \( r(q_l) \) correspondingly. Then \(|r(|q_s - q_l|)| < |r(q_k)|\). From \( q_{k+3} < q_{k+1} + q_k \) it follows that \( 0 < |q_s - q_l| < q_{k+1} \). Last inequalities contradict to the fact that \( q_k \) and \( q_{k+1} \) are denominators of consecutive best approximations. The second statement of Lemma 2 is proved.

In any triangle \( OR_iR_j, i \neq j \) the side \( R_iR_j \) is the greatest one. Lengths of \( r(q_k) \) decrease strictly, so those triangles can not have three equal sides and angles between vectors \( \overrightarrow{OR_i} \) are greater then \( \frac{\pi}{3} \). Other angles in these triangles are less or equal to \( \frac{\pi}{3} \). We see that \( R_1R_2R_3R_4 \) is convex, and the point \( O \) lies inside it. Lemma 2 is proved.

### 3. Proof of Theorem 4

We need two more lemmas.

**Lemma 3.** If tetragon \( R_1R_2R_3R_4 \) is not a parallelogram, then the inequality \([3]\) holds.

**Proof.** If \( R_1R_2R_3R_4 \) has no parallel sides, then we can make a convex hexagon by building parallelograms on two pairs of its sides. (See fig.2.) Without loss of generality we may suppose that the hexagon vertex \( R_4 \) lies between the vertices \( X_1 \) and \( X_2 \). So we have constructed the hexagon \( R_1R_2R_3X_2R_4X_1 \).

Consider the segment \( R_3X_1 \) (it is equal and parallel to segment \( R_1X_2 \)). Put \( x = |R_3X_1| \). By the construction the length of the remainder vector for the denominator \( q = |q_1 + q_3 - q_2 - q_4| \) is not greater then \( x \).

As the sequence \((q_k)\) increases strictly, we have three possible values of \( q \). So we should consider three cases.

**Case 1.** \( q = |q_{k+3} + q_k - q_{k+2} - q_{k+1}| \). Here \( 0 < q < q_k \), and the length of the remainder vector for \( q \) is not less then \(|r(q_{k-1})|\). So \( x \geq |r(q_k)|\).

**Case 2.** \( q = q_{k+3} + q_{k+1} - q_{k+2} - q_k \). Here \( 0 < q < q_{k+1} \). The length of the remainder vector for \( q \) is not less then \(|r(q_k)|\) (\( q \) is the denominator of the next best approximation). So \( x \geq |r(q_k)|\).

**Case 3.** \( q = q_{k+3} + q_{k+2} - q_{k+1} - q_k \). Then \( q > 0 \) and we have 2 subcases:

3a. \( q = q_{k+3} + q_{k+2} - q_{k+1} - q_k < q_{k+1} \). Here as in cases 1 and 2 we have \( x \geq |r(q_k)|\).
3b \( q = q_{k+3} + q_{k+2} - q_{k+1} - q_k \geq q_{k+1} \) Here we get the inequality \([3]\).

In cases 1, 2, 3a we have the following situation. As \(|\mathbf{r}(q_k)| > |\mathbf{r}(q_{k+1})| > |\mathbf{r}(q_{k+2})| > |\mathbf{r}(q_{k+3})|\) we see that the hexagon \(R_1R_2R_3X_2R_4X_1\) and the zero point \(O\) satisfy the conditions of Lemma 1. By Lemma 1 we see that

\[
\max_{i=1,2,3,4} R_iO > \min\{|X_1R_1|, |R_1R_2|, |R_2R_3| \}.
\]

As in our cases \(x \geq |\mathbf{r}(q_k)|\) we see that

\[
\max_{i=1,2,3,4} R_iO > \min\{|R_1R_1|, |R_1R_2|, |R_2R_3|, |R_3R_4| \}.
\]

This contradicts to Lemma 2. So the cases are 1, 2, 3a are not possible.

But in the remaining case 3b we have the inequality \([3]\).

To finish the proof of Lemma 3 we must consider the case when \(R_1R_2R_3R_4\) has a pair of parallel sides. Then the hexagon \(R_1R_2R_3X_2R_4X_1\) is a degenerate one (two its angles are equal to \(\pi\)). Now the proof follows the steps of the proof in non-degenerate case. The only difference is that we apply Lemma 1 for the degenerate hexagon. Lemma 3 is proved.

**Lemma 4.** If \(R_1R_2R_3R_4\) is a parallelogram and \(q_{k+3} < q_{k+1} + q_k\), then endpoints of the next four remainder vectors (for \(k + 1, k + 2, k + 3, k + 4\)) do not form a parallelogram.

**Proof.** Suppose they do. Let \(\mathbf{r}(q_{k+1}) = \overline{OR_5}\), \(\mathbf{r}(q_k) = \overline{OR}\). This parallelogram has three common vertices with \(R_1R_2R_3R_4\). So one of the vertices of the hexagon \(R_1R_2R_3R_4\) is the center of the segment \(RR_5\). This vertex we denote by \(R_6\).

As \(|\overline{OR}| = |\mathbf{r}(q_k)| > |\mathbf{r}(q_{k+4})| = |\overline{OR_5}|\), we see that the zero point \(O\) lies closer to \(R_5\) than to \(R\). So in the triangle \(\overline{ORR_6}\) the angle in the vertex \(R_6\) is greater than \(\frac{\pi}{3}\) and the length of the remainder vector \(\mathbf{r}(q_k) = \overline{OR}\) is greater than the length of the parallelogram’s side \(RR_2\). We get the contradiction to Lemma 2. Lemma 4 is proved.

**Proof of Theorem 4.**

1. If points \(R_1, R_2, R_3, R_4\) do not form a parallelogram, then using Lemma 3 we get inequality \([3]\).
2. If the inequality \(q_{k+3} < q_{k+1} + q_k\) do not holds, we again get inequality \([3]\).
3. We may suppose that \(R_1, R_2, R_3, R_4\) do form a parallelogram and \(q_{k+3} < q_{k+1} + q_k\). Then by Lemma 4 the endpoints of the next four remainder vectors (for \(k + 1, k + 2, k + 3, k + 4\)) do not form a parallelogram. So for the approximations \(k + 1, k + 2, k + 3, k + 4\) the inequality \([3]\) is valid. We see that

\[
q_{k+4} + q_{k+3} \geq 2q_{k+2} + q_{k+1}.
\]

Let the endpoints of vectors

\[
\mathbf{r}(q_k), \mathbf{r}(q_{k+1}), \mathbf{r}(q_{k+2}), \mathbf{r}(q_{k+3})
\]

form a parallelogram in the order

\[
\mathbf{r}(\hat{q}_1), \mathbf{r}(\hat{q}_2), \mathbf{r}(\hat{q}_3), \mathbf{r}(\hat{q}_4).
\]

Then the remainder vector for the denominator \(p = |\hat{q}_1 + \hat{q}_3 - \hat{q}_2 - \hat{q}_4|\) is equal to zero. As \(\alpha\) is not a rational vector we see that \(p = 0\). As the sequence of denominators of best approximations increases strictly we get \(0 = p = q_k + q_{k+3} - q_{k+2} - q_{k+1}\). 5
The last equality together with (5) implies (8). Theorem 4 is proved.

4. Proof of Theorem 3

From Theorem 4 we immediately obtain

**Proposition 1.** Let \( l \in \mathbb{R} \). Let \( \alpha \in \mathbb{R}^2 \setminus \mathbb{Q}^2 \). Let \( \in \mathbb{R} \). Then for every \( k \geq 1 \) for five consecutive denominators \( q_k, \ldots, q_{k+4} \) we have at least one of three following inequalities

\[
q_{k+2} \geq lq_{k+1} \tag{6}
\]
\[
q_{k+3} \geq (2 - l)q_{k+1} + q_k \tag{7}
\]
\[
q_{k+4} \geq q_{k+2} + q_k \tag{8}
\]

Moreover, for any two successive values of \( k \) for at least one value the inequality (6) or the inequality (7) holds.

For further proof we need to use some computer calculations.

Let \( 0 < l < 2 \). Let \( m = 1, \ldots, 7 \).

Put \( r_0 = r_1 = 31, \ r_2 = r_3 = r_4 = 33, \ r_5 = 34, \ r_6 = 35, \)
\( l_0 = \ldots = l_3 = 1.298, \ l_4 = l_5 = l_6 = 1.293, \)
\( \theta_0 = 1.2207, \ \theta_1 = 1.2272, \ \theta_2 = 1.2275, \ \theta_3 = 1.22779, \)
\( \theta_4 = 1.2278, \ \theta_5 = 1.22785, \ \theta_6 = 1.22791, \ \theta_7 = 1.228043. \)

Consider a sequence \( I = (i_0, \ldots, i_{r-1}) \), \( i_\nu \in \{1, 2, 3\} \) such that in any couple \( i_\nu, i_{\nu+1} \) at least one element is not equal to 3. For such \( I \) we construct a sequence \( \{Q_k(I, m)\}, \ 0 \leq k \leq r + 3 \) by the following procedure.

First of all we define three rules for obtaining the vector

\[
(Q^{j+1}_{j+1}, Q^{j+1}_{j+2}, Q^{j+1}_{j+3}, Q^{j+1}_{j+4})
\]

from the vector

\[
(Q^j, Q^j_{j+1}, Q^j_{j+2}, Q^j_{j+3})
\]

rule \( \mathcal{R}_1 \), rule \( \mathcal{R}_2 \) and rule \( \mathcal{R}_3 \). These rules correspond to different inequalities in Proposition 1.

**Rule \( \mathcal{R}_1 \):**

\[
\begin{align*}
Q^{j+1}_{j+1} &= Q^j_{j+1}, \\
Q^{j+1}_{j+2} &= \max\{lQ^j_{j+2}, Q^j_{j+2}\}, \\
Q^{j+1}_{j+3} &= \max\{lQ^j_{j+3}, Q^j_{j+3}\}, \\
Q^{j+1}_{j+4} &= \max\{lQ^j_{j+4}, Q^j_{j+4}, Q^j + Q^j_{j+1}\}.
\end{align*}
\]

**Rule \( \mathcal{R}_2 \):**

\[
\begin{align*}
Q^{j+1}_{j+1} &= Q^j_{j+1}, \\
Q^{j+1}_{j+2} &= Q^j_{j+2}, \\
Q^{j+1}_{j+3} &= \max\{(2 - l)Q^j_{j+3} + Q^j, Q^j_{j+3}\}, \\
Q^{j+1}_{j+4} &= \max\{(2 - l)Q^j_{j+4} + Q^j, Q^j_{j+4}, Q^j + Q^j_{j+1}\}.
\end{align*}
\]
Rule $\mathcal{R}_3$:

$$
\begin{cases}
Q_{j+1}^{j+1} = Q_{j+1}^j, \\
Q_{j+2}^{j+1} = Q_{j+2}^j, \\
Q_{j+3}^{j+1} = Q_{j+3}^j, \\
Q_{j+4}^{j+1} = \max\{Q_{j+3}^j, Q_j^j + Q_{j+2}^j\}.
\end{cases}
$$

For a sequence $I = (i_0, \ldots, i_{r-1})$ we take a sequence of rules $(\mathcal{R}_{i_0}, \ldots, \mathcal{R}_{i_{r-1}})$ and construct a sequence $\{Q_j(I, m)\}$, $j = 0, \ldots, r + 3$ in the following way.

For $j = 0$ put

$$
Q_0(m) = Q_0^0(m) = 1, \quad Q_t^0(m) = \theta_t^m, \quad t = 1, 2, 3.
$$

For $j \geq 0$ given

$$(Q_j^j(m), Q_{j+1}^j(m), Q_{j+2}^j(m), Q_{j+3}^j(m))$$

we construct

$$(Q_{j+1}^{j+1}(m), Q_{j+2}^{j+1}(m), Q_{j+3}^{j+1}(m), Q_{j+4}^{j+1}(m))$$

by the rule $\mathcal{R}_{i_j}$ with $l = l_m$.

Now we define $Q_j(I, m) = Q_j^j(m)$ for $j \leq r$ and $Q_{r+t, I}^r(m) = Q_{r+t, I}^r(m)$, $t = 1, 2, 3$.

The following proposition presents a result of computer calculation.

**Proposition 2.** Let $m = 0, \ldots, 6$. For any considered sequence of rules $I$ and defined sequence $\{Q_k(I, m)\}$ one has

$$(Q_{r+j}(I, m))^{\frac{1}{r+j}} \geq \theta_{m+1}, \quad j = 0, 1, 2, 3.$$

Remind that the increasing sequence of remainders of best approximations $\{q_k\}$ satisfies (2) and Proposition 1. When $l \in (0, 2)$ all coefficients in inequalities (6), (7), (8) are positive.

So we immediately deduce from Proposition 1 and Proposition 2 the following statement:

**Proposition 3.** Suppose that

$$q_{i+j} \geq \lambda \theta_m^j, \quad j = 0, 1, 2, 3, \quad \lambda > 0.$$ Then

$$q_{r+j} \geq \lambda \theta_{m+1}^{r+j}, \quad j = 0, 1, 2, 3.$$ From (2) it follows that for some positive $\lambda$ one has $q_i \geq \lambda \theta_0^j$. By Proposition 3 we see that $q_{j+4r_0+2r_1} \geq \lambda \theta_7^{j+4r_0+2r_1}$ for any $j$. Theorem 3 is proved.
References

[1] J.C. Lagarias, Best simultaneous diophantine approximation I. Growth rates of best approximation denominators. Trans. Am. Math. Soc., 272:545-554, 1980.

[2] A.J. Brentjes, Multidimensional continued fraction algorithms, volume 145 of Mathematical Center Tracts. Mathematisch Centrum Amsterdam, 1982.

[3] M.V. Romanov, Simultaneous two-dimensional best Diophantine approximations in the Euclidean norm. Moscow Univ. Math. Bull. 61 (2006), no. 2, 34-37.

Author’s address:
Dept. of Number Theory
Fac. Mathematics and Mechanics
Moscow State University
119992 Moscow
Russia
e-mail: zzremi@gmail.com