Wigner’s little group as a gauge generator in linearized gravity theories

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Abstract

We show that the translational subgroup of Wigner’s little group for massless particles in 3+1 dimensions generate gauge transformation in linearized Einstein gravity. Similarly a suitable representation of the 1-dimensional translational group $T(1)$ is shown to generate gauge transformation in the linearized Einstein-Chern-Simons theory in 2+1 dimensions. These representations are derived systematically from appropriate representations of translational groups which generate gauge transformations in gauge theories living in spacetime of one higher dimension by the technique of dimensional descent. The unified picture thus obtained is compared with a similar picture available for vector gauge theories in 3+1 and 2+1 dimensions. Finally, the polarization tensor of Einstein-Pauli-Fierz theory in 2+1 dimensions is shown to split into the polarization tensors of a pair of Einstein-Chern-Simons theories with opposite helicities suggesting a doublet structure for Einstein-Pauli-Fierz theory.

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1 Introduction

The concept of Wigner’s little group \textsuperscript{1} is of great importance in modern theoretical physics. One of the best known applications of the little group is its role in the classification of elementary particles according to their spin and helicity quantum numbers. Also, the transformation properties of state vectors in Hilbert space under Poincare transformation are obtained from their transformation properties under the action of little group, using the method of induced representation \textsuperscript{2}. More recently, the little group is shown to have nontrivial implications in studying the naturality of ghosts \textsuperscript{3}. Apart from its various roles mentioned above, Wigner’s little group is also known to be acting as generator of gauge transformations in various types of gauge theories \textsuperscript{2, 4} including the topologically massive gauge theories \textsuperscript{2, 4}. It was shown by Han et al. \textsuperscript{4} that the $E(2)$ like little group for massless particles, or more precisely the translational subgroup $T(2) \subset E(2)$ generates gauge transformations for free Maxwell theory in 3+1 dimensions. It was shown in \textsuperscript{4} that the same little group generates gauge transformations

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for Kalb-Ramond theory \[7\] while the translational group \(T(3)\) does the job in the topologically massive \(B \wedge F\) theory \[8\]. These theories involve 2-form gauge fields whereas Maxwell theory is a vector gauge theory, i.e., involves 1-form gauge field. Now coming to lower dimensional case, it is again proved that appropriate representations of translational group \(T(1)\) generate gauge transformations for Maxwell theory and topologically massive Maxwell-Chern-Simons(MCS) theory in 2+1 dimensions \[9\]. It is interesting to note that all these representations of the little group that generate gauge transformations in topologically massive theories can be systematically obtained from that in one higher dimension by a method of dimensional descent \[9\]. For instance, the representations that generate gauge transformation for MCS theories in 2+1 dimensions can be obtained by dimensional descent from the representation of \(T(2)\) that generate the gauge transformation in 3+1 dimensional Maxwell theory which involves splitting of the representation of \(T(2)\) in to a pair of \(T(1)\) representations appropriate for the doublet of MCS theories. Interestingly, Proca theory in 2+1 dimensions, which is not a gauge theory, can be regarded as a doublet of MCS theories \[10, 11\] which are \(U(1)\) gauge theories by themselves. These \(T(1)\) groups generate gauge transformations in each of these MCS theories and are related by complex conjugation. Also, the representation of \(T(3)\) that generates gauge transformation \(B \wedge F\) theory in 3+1 dimensions can be similarly obtained by the same method of dimensional descent from the little group for massless particles in 4+1 dimensions as shown in \[9\]. (The \(B \wedge F\) theory however does not form a doublet.) The 4+1 dimensional little group generates gauge transformations in various gauge theories inhabiting this higher dimensional spacetime. All the above mentioned theories in whose context the role of little group in generating gauge transformations were studied are either vector or antisymmetric 2-form gauge fields. These studies established that the various representations of the little group (or more precisely, the translational subgroup of the little group) which generate gauge transformations in the theories, massless or topologically massive, are in fact related. The connection between Wigner’s little group for massless particles and gauge transformation for 3+1 dimensional linearized Einstein gravity, which is a 2nd rank symmetric tensor gauge theory, was studied in \[13\]. The tensor analogue of MCS theory is the linearized version of the 2+1 dimensional Einstein-Chern-Simons(ECS) theory obtained by coupling a CS term to pure Einstein gravity. The ECS theory is a topologically massive gauge theory possessing a single degree of freedom and is described a 2nd rank symmetric tensor field. (Unlike the Maxwell case, the bare Einstein gravity in 2+1 dimensions is devoid of any propagating degree of freedom.) Also it was suggested in \[12, 16\] that the 2+1 dimensional Einstein-Pauli-Fierz(EPF) theory is a doublet of a pair of ECS theories just as 2+1 dimensional Proca theory is a doublet of MCS theories. At this stage one can ask whether the Wigner’s little group generate gauge transformations in the ECS theories also. If the answer is yes, are these representations the same as the ones obtained in the case of vector/2-form gauge fields and can one employ the method of dimensional descent in this case too? Analogous to Proca theory in 2+1 dimensions, does the suggested doublet structure of EPF theory enables one to construct the representations that generates gauge transformations in ECS theories? Many of these questions are yet to be answered and the purpose of this paper is to undertake a study of these problems. Here we study the little group and dimensional descent from 3+1 dimensional linearized Einstein gravity to linearized ECS theory in 2+1 dimensions. However, in the present work our main intention is to study the role of Wigner’s
little group as a generator of gauge transformation in topologically massive ECS theory, where
gauge invariance coexists with massive excitations, like the corresponding MCS theory involving
vector gauge fields. Our approach to the problem consists of considering the momentum
space equation of motion using which we obtain explicit form of the polarization vector/tensor
of the theory under consideration. Using the expression of the polarization vector/tensor thus
obtained, we study the role of the respective Wigner’s little group in generating gauge trans-
formation in the theory. This approach, therefore, is different from the one used in [13]. It
should be noticed that, on account of the fact that unlike other field theories in whose contexts
the translational group was shown to generate gauge transformations, in the linearized gravity
theories the gauge transformation and spacetime diffeomorphisms are intertwined. Thus the
problem is interesting in its own right.

The paper is organized as follows. Section 2 provides a brief review of the role of little group
as a gauge generator in the usual and topologically massive gauge theories. In the context of
topologically massive gauge theories like the MCS and $B \wedge F$ theories, we provide a brief
outline of the method of dimensional descent. In section 3, we study the role of little group as a
generator of gauge transformation in linearized Einstein gravity in 3+1 dimensions and in
2+1 dimensional ECS theory. Section 4 discusses the doublet structure of EPF theory in 2+1
dimensions and applies the method of dimensional descent from 3+1 dimensional linearized
gravity to 2+1 dimensional EPF theory. We also explain in section 4, how the representations
of $T(1)$ that generates gauge transformation in the doublet of ECS theories can be obtained
by a careful study of the polarization tensors of EPF and ECS theories. We finally conclude in
section 5.

2 Wigner’s little group and dimensional descent in vector and 2-form gauge theories

Wigner’s little group is defined as the subgroup of Lorentz group that preserves the energy-
momentum vector of a particle. We begin by reviewing the role of Wigner’s little group as a
generator of gauge transformation in Maxwell and Kalb-Ramond (KR) theories in 3+1 dimen-
sions and also of the method of dimensional descent. Our notation is as follows. We use $\mu, \nu$ etc
to denote the spacetime indices in 3+1 dimensions and $a, b$ etc for 2+1 dimensions. Signature
of the metric is ‘mostly negative’.
2.1 Wigner’s little group and gauge transformation in Maxwell and KR theories

An element of the little group that leaves the four-momentum \( k^\mu = (\omega, 0, 0, \omega) \) of a massless particle moving in the \( z \)-direction invariant is given by [4]

\[
W_4(p, q; \phi) = \begin{pmatrix}
1 + \frac{p^2 + q^2}{2} & p \cos \phi - q \sin \phi & q \sin \phi + p \cos \phi & -\frac{p^2 + q^2}{2} \\
p & \cos \phi & \sin \phi & -p \\
q & -\sin \phi & \cos \phi & -q \\
\frac{p^2 + q^2}{2} & p \cos \phi - q \sin \phi & q \sin \phi + p \cos \phi & 1 - \frac{p^2 + q^2}{2}
\end{pmatrix} .
\]

(1)

Here \( p, q \) are any real numbers. This little group can be written as

\[
W_4(p, q; \phi) = W(p, q)R(\phi)
\]

(2)

where

\[
W(p, q) = W_4(p, q; 0) = \begin{pmatrix}
1 + \frac{p^2 + q^2}{2} & p & q & -\frac{p^2 + q^2}{2} \\
p & 1 & 0 & -p \\
q & 0 & 1 & -q \\
\frac{p^2 + q^2}{2} & p & q & 1 - \frac{p^2 + q^2}{2}
\end{pmatrix}
\]

(3)

is a particular representation of the translational subgroup of \( T(2) \) of the little group. We first discuss how this representation of \( T(2) \) generates gauge transformation in Maxwell theory. The Lagrangian for Maxwell theory is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \quad F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

(4)

and the corresponding equation of motion is

\[
\partial_\mu F^{\mu\nu} = 0
\]

(5)

For a single mode, without loss of generality, the gauge field \( A^\mu(x) \) can be written as

\[
A^\mu(x) = \varepsilon^\mu(k)e^{ikx}
\]

(6)

suppressing the positive frequency part for simplicity. Here \( \varepsilon^\mu \) stands for the polarization vector of the photon\(^3\). In terms of the polarization vector, the gauge transformation \( A^\mu(x) \rightarrow A'^\mu = A^\mu + \partial_\mu f(x) \) (where \( f(x) \) is an arbitrary scalar function) is expressed as

\[
\varepsilon_\mu(k) \rightarrow \varepsilon'_\mu = \varepsilon_\mu(k) + i f(k)k_\mu
\]

(7)

where \( f(x) \) has been written as \( f(x) = f(k)e^{ikx} \). The equation of motion, in terms of the polarization vector, will now be given by

\[
k^2 \varepsilon^\mu - k_\nu k_\mu \varepsilon^\nu = 0.
\]

---

\(^3\)This method [4], henceforth referred to as “plane wave method”, was used in [5, 10] to find masses of the excitations and structures of the maximally reduced form of the polarization vectors/tensors in various theories.
The massive excitations corresponding to $k^2 \neq 0$ leads to the solution $\varepsilon^\mu \propto k^\mu$ which can therefore be gauged away. For massless excitations ($k^2 = 0$), the Lorentz condition $k_\mu \varepsilon^\mu = 0$ follows immediately from (8). Taking $k^\mu = (\omega, 0, 0, \omega)^T$, corresponding to a photon of energy $\omega$ propagating in the $z$ direction, one can easily show that $\varepsilon^\mu(k)$ takes the form

$$\varepsilon^\mu(k) = (0, \varepsilon^1, \varepsilon^2, 0)^T$$

up to a gauge transformation. Note that (9) is the maximally reduced form of $\varepsilon^\mu$ displaying the two transverse degrees of freedom $\varepsilon^1$ and $\varepsilon^2$. Under the action (3) of the translational group $T(2)$ this polarization vector transforms as follows:

$$\varepsilon^\mu \rightarrow \varepsilon'^\mu = W^\mu_\nu(p, q)\varepsilon^\nu = \varepsilon^\mu + \left(\frac{p\varepsilon^1 + q\varepsilon^2}{\omega}\right) k^\mu .$$

Clearly, using (7), this can be identified as a gauge transformation by choosing $f(k)$ suitably.

By similar methods it can shown that gauge transformations are generated by $T(2)$ in the 3+1 dimensional Kalb-Ramond(KR) theory [5] described by the Lagrangian

$$\mathcal{L} = \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda}$$

where the 3rd rank field strength tensor $H_{\mu\nu\lambda}$ is given by

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$$

with $B_{\mu\nu} = -B_{\nu\mu}$ being the rank-2 antisymmetric gauge field. The equation of motion for the KR theory is given by

$$\partial_\mu H^{\mu\nu\lambda} = 0 .$$

Here the model is invariant under the reducible gauge transformation given by

$$B_{\mu\nu} \rightarrow B'_{\mu\nu} = B_{\mu\nu} + \partial_\mu f_\nu - \partial_\nu f_\mu .$$

Analogous to Maxwell case, we follow the plane wave method and adopt for KR theory, the ansatz

$$B^{\mu\nu}(x) = \varepsilon^{\mu\nu}(k)e^{ik\cdot x}$$

where $\varepsilon^{\mu\nu}(k)$ is the antisymmetric polarization tensor of KR theory. In terms of the polarization tensor, the equation of motion (13) can be written as

$$k_\mu \left[ k^\mu \varepsilon^{\nu\lambda} + k^\nu \varepsilon^{\lambda\mu} + k^\lambda \varepsilon^{\mu\nu} \right] = 0$$

and the gauge transformation (14) as

$$\varepsilon_{\mu\nu} \rightarrow \varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + i(k_\mu f_\nu(k) - k_\nu f_\mu(k)).$$

This is reducible, since by the choice $f_\mu = \partial_\mu \Lambda$, of the arbitrary functions $f_\mu$, the gauge variation can be made to vanish.
For the case $k^2 \neq 0$, from (16) one has
\[ \varepsilon^{\nu\lambda} = \frac{1}{k^2} [k^{\nu}(k_\nu \varepsilon^{\mu\lambda}) - k^\lambda(k_\mu \varepsilon^{\mu\nu})]. \] (18)

These (massive) solutions can be gauged away by choosing \( f^{\lambda}(k) = \frac{i}{k^2} k_\mu \varepsilon^{\mu\lambda} \). Similar to Maxwell case, here too the massive excitations are gauge artifacts. As for \( k^2 = 0 \), we have from (16) \( k_\mu \varepsilon^{\mu\nu} = 0 \) which is the similar to the Lorentz condition in Maxwell theory. Using this condition and by choosing a frame where \( k^\mu = (\omega, 0, 0, \omega) \) just as in Maxwell theory, the polarization tensor reduces to the maximally reduced form
\[ \mathcal{E} \equiv \{ \varepsilon^{\mu\nu} \} = \varepsilon^{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] (19)
after a suitable gauge transformation that does away with the spurious degrees of freedom.

In the maximally reduced form of \( \mathcal{E} \) (13) one is left with the only physical degree of freedom corresponding to \( \varepsilon^{12} \). Since under Lorentz transformation \( \varepsilon^{\mu\nu} \) transforms as \( \varepsilon^{\mu\nu} \rightarrow \varepsilon'^{\mu\nu} = \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} \varepsilon^{\rho\sigma} \) the transformation of \( \mathcal{E} \) matrix under the action of the little group \( W(p, q) \), which in this case is subgroup of Lorentz group, is given by
\[ \mathcal{E} \rightarrow \mathcal{E}' = W(p, q) \mathcal{E} W^T(p, q) = \varepsilon^{12} \begin{pmatrix} 0 & -q & p & 0 \\ q & 0 & 1 & q \\ -p & -1 & 0 & -p \\ 0 & -q & p & 0 \end{pmatrix} = \mathcal{E} + \varepsilon^{12} \begin{pmatrix} 0 & -q & p & 0 \\ q & 0 & 0 & q \\ -p & 0 & 0 & -p \\ 0 & -q & p & 0 \end{pmatrix}. \] (20)

With the choice \( f^1 = \frac{i}{\omega} \varepsilon^{12}, f^2 = -\frac{\omega}{\omega} \varepsilon^{12} \) and \( f^3 = f^0 \), one can write (20) as a gauge transformation (17). Thus, the translational group \( W(p, q) \) generates gauge transformation in KR theory also.

By similar methods it was shown that the translational group \( T(3) \) generates gauge transformation in the topologically massive \( B \wedge F \) theory which has the Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{m}{6} \varepsilon^{\mu\nu\lambda\rho} H_{\mu\nu\lambda} A_\rho, \quad m > 0 \] (21)
which is obtained by topologically coupling the \( B_{\mu\nu} \) field of Kalb-Ramond theory(11) with the Maxwell field \( A_\mu \) so that the last term in (21) does not contribute to the energy-momentum tensor. The \( B \wedge F \) Lagrangian (21) can be regarded either as a massive Maxwell(i.e., Proca) theory or a massive KR theory [15]. For the sake of completeness, we summarize here the essential points concerning the role of \( T(3) \) in generating gauge transformation in \( B \wedge F \) theory [4]. Following the plane wave method, the polarization vector and tensor for this theory, in the maximally reduced form (analogous to (9) and (13)), are found to be
\[ \varepsilon^\mu = -i \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c & -b \\ 0 & -c & 0 & a \\ 0 & b & -a & 0 \end{pmatrix} \] (22)
where \(a, b, c\) being three arbitrary real parameters corresponding to three degrees of freedom and we have chosen the rest frame where

\[
k^\mu = (m, 0, 0, 0)^T.
\]

Note that the number of degrees of freedom for \(B \wedge F\) theory can be obtained by adding the number of degrees of freedom for Maxwell and KR theories together \(\text{[5]}\). Clearly, unlike the Maxwell or KR theories, the 2-dimensional translational group \(T(2)\) cannot generate gauge transformation in \(B \wedge F\) theory having three degrees of freedom. It is the 3-dimensional translational group \(T(3)\) represented by

\[
D(p, q, r) = 1 + p T_1 + q T_2 + r T_3 =
\begin{pmatrix}
1 & p & q & r \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

that generate gauge transformation in this case. Here \(T_1 = \frac{\partial D}{\partial p}, T_2 = \frac{\partial D}{\partial q}, T_2 = \frac{\partial D}{\partial r}\) are the Lie algebra generators of the abelian algebra of \(T(3)\). One can immediately see that

\[
\delta \varepsilon^\mu = D^\mu_v(p, q, r) \varepsilon^\nu - \varepsilon^\mu = \frac{i}{m} (pa + qb + rc) k^\mu = (p T_1 + q T_2 + r T_3) \varepsilon^\mu
\]

and

\[
\delta \mathcal{E} = D(p, q, r) \mathcal{E} D^T(p, q, r) - \mathcal{E} = \begin{pmatrix}
0 & (rb - qc) & (pc - ra) & (qa - pb) \\
-(rb - qc) & 0 & 0 & 0 \\
-(pc - ra) & 0 & 0 & 0 \\
-(qa - pb) & 0 & 0 & 0 \\
\end{pmatrix}
\]

so that using (\(\text{[6]}\)) and (\(\text{[17]}\)) one can cast (\(\text{[23]}\)) and (\(\text{[26]}\)) respectively in the form of gauge transformations. Further it has been shown in \(\text{[5]}\) that the representation \(\text{[24]}\) of \(T(3) \subset E(3)\) can be constructed from one higher i.e., 4+1 dimensional spacetime by a method of ‘dimensional descent’ which involves the projection operator \(\mathcal{P} = \text{diag}(1,1,1,1,0)\). This is expected, as \(E(3)\) is the Wigner’s little group for massless particles in 4+1 dimensions and under this projection, the polarization vector (tensor) of free Maxwell (KR) theory and the momentum 5-vector for a massless photon moving in the 4th direction in 4+1 dimensional spacetime maps to the polarization vector \(\varepsilon^\mu\) (tensor \(\varepsilon^{\mu\nu}\)) \(\text{[22]}\) and momentum 4-vector \(k^\mu\) \(\text{[23]}\) of a massive \(B \wedge F\) quanta at rest in 3+1 dimensions. The same feature of “dimensional descent” goes through from 3+1 dimensions to 2+1 dimensions so that starting from usual photon in 3+1 dimensions and using \(\mathcal{P} = \text{diag}(1,1,1,0)\) one obtain the polarization vector and momentum 3-vector of a Proca quanta governed by the Proca theory

\[
\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab} + \frac{\omega^2}{2} A^a A_a
\]

in 2+1 dimensions. In order to discuss the gauge transformation properties it is essential to provide a \(3 \times 3\) representation of \(T(2)\) (denoted by \(\tilde{D}(p, q)\)) which now amounts to deleting the
last row and column of $D(p, q, r)$ in (24),

\[
D(p, q) = \begin{pmatrix}
1 & p & q \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (28)

The corresponding generators are given by,

\[
\bar{T}_1 = \frac{\partial \bar{D}}{\partial p} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \bar{T}_2 = \frac{\partial \bar{D}}{\partial q} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (29)

Just as the Proca theory in 3+1 dimensions maps to the $B \wedge F$ theory, where gauge transformations were discussed, the Proca theory in 2+1 dimensions is actually a doublet of Maxwell-Chern-Simons theories,\[10, 15, 16\]

\[
\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-
\] (30)

where

\[
\mathcal{L}_\pm = -\frac{1}{4} F^{ab} F_{ab} \pm \frac{\theta}{2} \epsilon^{abc} A_a \partial_b A_c
\] (31)

with each of $\mathcal{L}_+$ or $\mathcal{L}_-$ being a topologically massive gauge theory. The mass of the MCS quanta is $\omega = |\theta|$, where $\omega$ is the parameter entering in (27). We can therefore study the gauge transformation generated in this doublet. The polarization vector for $\mathcal{L}_\pm$, with only one degree of freedom for each of $\mathcal{L}_+$ and $\mathcal{L}_-$, has been found to be\[10, 13\]

\[
\bar{\epsilon}_\pm^a = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
\pm i
\end{pmatrix}
\] (32)

while the 3-momentum $k^a = (\omega, 0, 0)^T$ obviously takes the same form as in the Proca model. In analogy with (25), here also one can write,

\[
\delta \bar{\epsilon}^a = (p \bar{T}_1 + q \bar{T}_2)^a_b \bar{\epsilon}^b
\] (33)

where $\bar{\epsilon}^a = (0, a_1, a_2)^T$ is the polarization vector for the Proca theory in 2+1 dimensions. Had the Proca theory been a gauge theory, (33) would have represented a gauge transformation, as it can be written as

\[
\delta \bar{\epsilon}^a = \frac{p a_1 + q a_2}{\omega} k^a.
\] (34)

But since Proca theory is not a gauge theory, we can only study the gauge transformation properties of each of the doublet $\mathcal{L}_\pm$ (31) individually. First note that the Proca polarization vector $\bar{\epsilon}^a$ is just a linear combination of the two real orthonormal canonical vectors $\epsilon_1$ and $\epsilon_2$ where,

\[
\bar{\epsilon}^a = a_1 \epsilon_1 + a_2 \epsilon_2; \quad \epsilon_1 = (0, 1, 0)^T, \epsilon_2 = (0, 0, 1)^T.
\] (35)
Correspondingly the generators $\bar{T}_1$ and $\bar{T}_2$ (29), form an orthonormal basis as they satisfy 
\[
\text{tr}(\bar{T}_a^\dagger \bar{T}_b) = \delta_{ab}.
\]
Furthermore,
\[
\bar{T}_1 \varepsilon_1 = \bar{T}_2 \varepsilon_2 = (1, 0, 0)^T = \frac{k^a}{\omega}, \quad \bar{T}_1 \varepsilon_2 = \bar{T}_2 \varepsilon_1 = 0
\]
(36)
On the other hand, the polarization vectors $\bar{\varepsilon}_a^+$ and $\bar{\varepsilon}_a^-$ (32) also provide an orthonormal basis (complex) in the plane as
\[
(\bar{\varepsilon}_a^+)^\dagger (\bar{\varepsilon}_a^-) = 0; \quad (\bar{\varepsilon}_a^+)^\dagger (\bar{\varepsilon}_a^+) = (\bar{\varepsilon}_a^-)^\dagger (\bar{\varepsilon}_a^-) = 1.
\]
(37)
Here we note that spatial part $\bar{\varepsilon}_\pm$ of $\varepsilon_\pm$ can be obtained from the space part of the above mentioned canonical ones by appropriate $SU(2)$ transformation. That is, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SU(2)$ when acts on $\bar{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ yields respectively the vectors $\bar{\varepsilon}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\bar{\varepsilon}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ (up to an irrelevant factor of $i$):  
\[
\bar{\varepsilon}_+ = U \bar{\varepsilon}_1, \quad \bar{\varepsilon}_- = i U \bar{\varepsilon}_2.
\]
This suggests that we consider the following orthonormal basis for the Lie algebra of $T(2)$:
\[
\bar{T}_\pm = \frac{1}{\sqrt{2}} (\bar{T}_1 \mp i \bar{T}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & \mp i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
(39)
instead of $\bar{T}_1$ and $\bar{T}_2$. Note that they also satisfy relations similar to the (1-2) basis,
\[
\text{tr}(\bar{T}_+^\dagger \bar{T}_+) = \text{tr}(\bar{T}_-^\dagger \bar{T}_-) = 1; \text{tr}(\bar{T}_+^\dagger \bar{T}_-) = 0
\]
(40)
One can now easily see that
\[
\bar{T}_+ \varepsilon_+ = \bar{T}_- \varepsilon_- = \frac{k^a}{\omega}, \quad \bar{T}_+ \varepsilon_- = \bar{T}_- \varepsilon_+ = 0
\]
(41)
analogous to (36). Furthermore,
\[
\delta \bar{\varepsilon}_\pm = \alpha_\pm \bar{T}_\pm \bar{\varepsilon}_\pm = \frac{\alpha_\pm}{\omega} k^a.
\]
(42)
This indicates that $\bar{T}_\pm$ - the generators of the Lie algebra of $T(2)$ in the rotated (complex) basis - generate independent gauge transformations in $L_\pm$ respectively. One therefore can understand how the appropriate representation of the generator of gauge transformation in the doublet of MCS theory can be obtained from higher 3+1 dimensional Wigner’s group through dimensional descent. A finite gauge transformation is obtained by integrating (42) i.e., exponentiating the
\[\text{This ambiguity of } i \text{ factor is related to the } U(1) \text{ phase arbitrariness of the polarization vector [10].}\]
corresponding Lie algebra element. This gives two representations of Wigner’s little group for massless particles in 2 + 1 dimensions, which is isomorphic to \( \mathcal{R} \times \mathbb{Z}_2 \), although here we are just considering the component which is connected to the identity,

\[
G_\pm(\alpha_\pm) = e^{\alpha_\pm T_\pm} = 1 + \alpha_\pm T_\pm = \left( \begin{array}{ccc} 1 & \frac{\alpha_\pm}{\sqrt{2}} & \mp i \frac{\alpha_\pm}{\sqrt{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)
\] (43)

Note that \( G_\pm(\alpha_\pm) \) generates gauge transformation in the doublet \( L_\pm \),

\[
G_\pm^{a} Z_\pm^{b} = \tilde{\varepsilon}_\pm^{a} + \frac{\alpha_\pm k^a}{|\mu|}
\] (44)

and are related by complex conjugation. This complex conjugation is also a symmetry of the doublet \([10]\).

Therefore, it is clear that suitable representations of translational groups in different dimensions act as generators of gauge transformations in ordinary as well as topologically massive gauge theories. We now turn our attention to the case of linearized gravity.

### 3 Role of little group in linearized gravity theories

Gravity (linearized) in \( d \) spacetime dimensions is described by a symmetric second rank tensor gauge field and has \( \frac{1}{2} d(d-3) \) degrees of freedom\(^6\). Therefore general relativity in 3+1 dimensions has two degrees of freedom and in 2+1 dimensions it has none. However, 2+1 dimensional gravity coupled to a Chern-Simons topological term, with gauge group being the Lorentz group itself, possesses a single propagating massive degree of freedom \([12]\). Just like the MCS theory, the gauge invariance coexists with mass in the linearized version of this theory too where the gauge group reduces to abelian group \( T(1) \). In this section we study the role of translational group in generating gauge transformations in the linearized versions of Einstein theory in 3+1 dimensions and gravity coupled to Chern-Simons term 2+1 dimensions.

Following the same conventions as of \([12]\), we write the pure Einstein action in 3+1 dimensions as

\[
I^E = - \int d^4x \mathcal{L}^E, \quad \mathcal{L}^E = \sqrt{g} R = \sqrt{g} g^{\mu\nu} R_{\mu\nu}
\] (45)

where \( \mathcal{L}^E \) is the Einstein Lagrangian and \( R_{\mu\nu} \) is the Ricci tensor. In the linearized approximation the metric \( g_{\mu\nu} \) is assumed to be close to the flat background part \( \eta^{\mu\nu} \) and therefore\\footnote{The degree of freedom counting can be done by following Weinberg \([17]\). To start with, note that a symmetric second rank tensor in \( d \) dimensions has \( \frac{1}{2} d(d-1) \) independent components. Analogous to the Lorentz gauge condition \( \partial^\mu A_\mu = 0 \) of Maxwell theory, in general relativity we have the harmonic gauge condition \( g^{\mu\nu} \mathcal{L}^E_{\mu\nu} = 0 \) which amounts to \( d \) constraints on the components of \( g_{\mu\nu} \). These along with the \( d \) independent components of the gauge parameter (which by itself is a \( d \)-vector now; see the ensuing discussion below, particularly \([51]\)), in the linearized version of the theory, reduces the number of independent components of the tensor field to \( \frac{1}{2} d(d+1) - 2d = \frac{1}{2} d(d-3) \).}

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.
\] (46)

\footnote{We have set the parameter \( \kappa \) appearing in \([12]\) equal to unity.}
where $h_{\mu\nu}$ is the deviation such that $|h_{\mu\nu}| << 1$. When the deviation is small one considers only terms up to first order in $h_{\mu\nu}$. The raising and lowering of indices is done using $\eta_{\mu\nu}$.

### 3.1 Linearized Einstein gravity in 3+1 dimensions

The linearized version of Einstein-Hilbert Lagrangian is

$$\mathcal{L}_E^L = \frac{1}{2} h_{\mu\nu} \left[ R^{\mu\nu}_L - \frac{1}{2} \eta^{\mu\nu} R_L \right].$$  \hspace{1cm} (47)

Here $R^{\mu\nu}_L$ is the linearized Ricci tensor given by

$$R^{\mu\nu}_L = \frac{1}{2} (\Box h^{\mu\nu} + \partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu} - \partial^\mu \partial^\nu h)$$ \hspace{1cm} (48)

with $h = h^{\alpha\alpha}$. Similarly $R_L = R^{\alpha\alpha}_L$. The field equations for $h^{\mu\nu}$ following from this Lagrangian is given by

$$-\Box h^{\mu\nu} + \partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu} - \partial^\mu \partial^\nu h + \eta^{\mu\nu} (\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}) = 0.$$ \hspace{1cm} (49)

The above equation is invariant under the following gauge transformation:

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \zeta_\nu(x) + \partial_\nu \zeta_\mu(x)$$ \hspace{1cm} (50)

which is just the symmetric counterpart of the gauge transformation (14) for KR and $B \wedge F$ theories. However, unlike KR and $B \wedge F$ theories, linearized gravity is not a reducible gauge system. Here $\zeta_\mu(x)$ are completely arbitrary except that they are considered to be small.

Following the plane wave method, we now adopt the ansatz

$$h_{\mu\nu} = \epsilon_{\mu\nu}(k)e^{ik.x} + c.c.$$ \hspace{1cm} (51)

where $\epsilon_{\mu\nu}$ is the symmetric polarization tensor and is the counterpart of the antisymmetric polarization tensor appearing in (15) for KR theory. With the choice

$$\zeta_\mu(x) = -i\zeta_\mu(k)e^{ik.x} + c.c.$$ \hspace{1cm} (52)

the gauge transformation in $h_{\mu\nu}$ can be written in terms of the polarization tensor as

$$\epsilon_{\mu\nu}(k) \rightarrow \epsilon'_{\mu\nu}(k) = \epsilon_{\mu\nu}(k) + k_\mu \zeta_\nu(k) + k_\nu \zeta_\mu(k).$$ \hspace{1cm} (53)

Just as in the Maxwell case, hereafter we will consider only the the negative frequency part for simplicity. Substituting the ansatz in the equation of motion yields

$$k^2 \epsilon^{\mu\nu} - k^\mu k_\alpha \epsilon^{\alpha\nu} - k^\nu k_\alpha \epsilon^{\alpha\mu} + k^\mu k^\nu \epsilon + \eta^{\mu\nu} (-k^2 \epsilon + k_\alpha k_\beta \epsilon^{\alpha\beta}) = 0.$$ \hspace{1cm} (54)

As was done in the previous cases, we separately consider the two possibilities $k^2 \neq 0$ and $k^2 = 0$. Choosing the massive ($k^2 \neq 0$) case first, we contract the equation of motion (54) with $\eta^{\mu\nu}$ to obtain

$$k^2 \epsilon - k_\alpha k_\beta \epsilon^{\alpha\beta} = 0; \quad \epsilon = \epsilon^{\nu}_{\mu}.$$ \hspace{1cm} (55)
A general solution to this equation is given by

$$\varepsilon^{\mu\nu} = k^\mu f^\nu(k) + k^\nu f^\mu(k)$$  \hfill (56)

with $f^\nu(k)$ being arbitrary functions of $k$. Therefore, it is easily seen this solution can be ‘gauged away’ by appropriate choice of the variables $\zeta_\mu(k)$ in (53) as this corresponds to pure gauge. Thus, analogous to Maxwell theory, the massive excitations of linearized Einstein gravity are gauge artefacts.

For massless ($k^2 = 0$) excitations, the equation of motion (54) reduces to

$$-k^\mu k_\alpha \varepsilon^{\alpha\nu} - k^\nu k_\alpha \varepsilon^{\alpha\mu} + k^\mu k^\nu \varepsilon + \eta^{\mu\nu} k_\alpha k_\beta \varepsilon^{\alpha\beta} = 0.$$  \hfill (57)

In a frame of reference where $k^\mu = (\omega, 0, 0, \omega)^T$ the above equation can be written as

$$-\omega[k^\mu(\varepsilon^{0\nu} - \varepsilon^{3\nu}) + k^\nu(\varepsilon^{0\mu} - \varepsilon^{3\mu})] + k^\mu k^\nu \varepsilon + \omega^2 \eta^{\mu\nu}(\varepsilon^{00} + \varepsilon^{33} - 2\varepsilon^{03}) = 0.$$  \hfill (58)

Now with the choice $\zeta^0 = -\varepsilon^{00}_\omega$, $\zeta^2 = -\varepsilon^{02}_\omega$, $\zeta^3 = -\varepsilon^{03}_\omega$, the polarization tensor $\{\varepsilon_{\mu\nu}\}$ can be written in the maximally reduced form (similar to (9) of Maxwell theory) as follows:

$$\mathcal{E} \equiv \{\varepsilon_{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hfill (60)

(which is gauge equivalent to (59)) where $a$ and $b$ are arbitrary parameters representing the two degrees of freedom for the theory. (This form of the polarization tensor of linearized Einstein gravity in 3+1 dimensions is derived in [17] following a different approach.) In this form (60), the polarization tensor $\mathcal{E}$ satisfies the harmonic gauge condition,

$$k_\mu \varepsilon^{\mu}_{\nu} = \frac{1}{2} k_\nu \varepsilon^{\mu}_{\mu}$$  \hfill (61)

in momentum space, automatically. Using the maximally reduced form (60) of the polarization tensor, it is now straightforward to show that the group $W(p, q)$ in (3) generate gauge transformations in linearized Einstein gravity also. For this purpose consider the action of $W(p, q)$ on $\mathcal{E}$ in (60), just as in KR case,

$$\mathcal{E} \rightarrow \mathcal{E}' = W(p, q) \mathcal{E} W^T(p, q)$$
\[
E + \begin{pmatrix}
(p^2 - q^2)a + 2pqb & (pa + qb) & (pb - qa) & (p^2 - q^2)a + 2pqb \\
(pa + qb) & 0 & 0 & (pa + qb) \\
(pb - qa) & 0 & 0 & (pb - qa) \\
((p^2 - q^2)a + 2pqb) & (pa + qb) & (pb - qa) & ((p^2 - q^2)a + 2pqb)
\end{pmatrix}.
\] (62)

The above transformation can be cast in the form of a gauge transformation (53) with the following choice for the arbitrary functions \(\zeta^\mu(k)\):

\[
\begin{align*}
\zeta^0 &= \frac{(p^2 - q^2)a + 2pqb}{\omega}, \\
\zeta^1 &= \frac{pa + qb}{\omega}, \\
\zeta^2 &= \frac{pb - qa}{\omega}.
\end{align*}
\] (63-65)

We therefore conclude that the translational subgroup \(T(2)\) of the Wigner’s little group \(E(2)\), in its original representation \(W(p, q)\) (3), acts as a gauge generator in linearized gravity in 3+1 dimensions.

It is important to notice that, unlike vector gauge (Maxwell) fields, the translational subgroup of Wigner’s little group acts as gauge generator for linearized gravity only when the spacetime has the dimension \(d = 4\) and not in any other dimension. This can be seen immediately from the following simple reasoning. In \(d\) dimensions, the translational subgroup of the little group (for massless particles) has \(d - 2\) generators which, incidentally, is the number of (transverse) degrees of freedom in Maxwell theory. On the other hand, the number of degrees of freedom for linearized gravity in \(d\) dimensions is \(\frac{1}{2}d(d - 3)\). Therefore, the number of generators of the translational subgroup equals the number of degrees of freedom of the theory, if and only if \(d = 4\).

### 3.2 Linearized Einstein-Chern-Simons (ECS) theory in 2+1 dimensions

The full action of the topologically massive gravity in 2+1 dimensions is

\[
I^{ECS} = I^E + I^{CS}
\] (66)

where the 2+1 dimensional Einstein action here is

\[
I^E = \int d^3x \sqrt{g} R
\] (67)

and the Chern-Simons action \(I^{CS}\) is given by

\[
I^{CS} = -\frac{1}{4\mu} \int d^3x \epsilon^{def}[R_{deab}\omega_{f}^{ab} + \frac{2}{3}\omega_{db}^{c}\omega_{ec}^{a}\omega_{fa}^{b}].
\] (68)
Here the indices $a, b, c$ etc. stand for local Lorentz indices in 2+1 dimensions. The $\omega_{dab}$ are the components of the spin connection one-form and are related to the curvature two-form by the second Cartan’s equation of structure ($R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$). Note that the sign in front of the Einstein action is now opposite to the conventional one [13] and is required to make the full theory free of ghosts [12]. The linearization, $g_{ab} = \eta_{ab} + h_{ab}$, of the ECS theory (56) results in the abelian theory given by

$$I_{ECS}^L = \int d^3 x L_{ECS}^L$$

where

$$L_{ECS}^L = L_E^L + L_{CS}^L.$$  

Here

$$L_E^L = -\frac{1}{2} h_{ab} \left[ R^b_{L} - \frac{1}{2} \eta^{ab} R_L \right]$$

now is the Lagrangian for linearized version of pure gravity in 2+1 dimensions and is the same as (47) except that it has the opposite sign and the indices, in this case, vary over 0, 1, 2. Similarly,

$$L_{CS}^L = -\frac{1}{2\mu} \epsilon^{abc} \partial_c (R^b_{L} \partial^a h^c_d)$$

is the linearized Chern-Simons term with the Chern-Simons parameter $\mu$. Under the gauge transformation $h_{ab} \rightarrow h'_{ab} = h_{ab} + \partial_a \xi_b(x) + \partial_b \xi_a(x)$, the Chern-Simons part $L_{CS}^L$ changes by a total derivative:

$$\delta L_{CS}^L = \frac{1}{\mu} \epsilon_{abc} \partial_d \left( R^b_{L} \partial^a \xi^c \right)$$  

The equation of motion corresponding to $L_{ECS}^L$ is given by

$$-\Box h^{ab} + \partial^a \partial_b h + \partial^b \partial_c k^c_a - \partial^a \partial^b h + \eta^{ab} (\Box h - \partial_c \partial_d h^{cd})$$

$$- \frac{1}{2\mu} \epsilon^{acd} \partial_c (\Box h^b_d - \partial_e \partial^a h^e_d) - \frac{1}{2\mu} \epsilon^{bcd} \partial_c (\Box h^a_d - \partial_e \partial^a h^e_d) = 0.$$  

With the ansatz $h_{ab} = \chi_{ab}(k) e^{ik}.x$, the above equation of motion can be written in terms of the symmetric polarization tensor $\chi_{ab}(k)$ and the 3-momentum $k^a$ as follows

$$k^2 \chi^{ab} - k^a k^c \chi^{cb} - k^b k^c \chi^{ca} + k^a k^b \chi + \eta^{ab} (-k^2 \chi + k_c k^c d^e)$$

$$- \frac{i}{2\mu} \left[ \epsilon^{acd} k_c (-k^2 \chi^a_d + k^c b \chi^e_d) + \epsilon^{bcd} k_c (-k^2 \chi^a_d + k^c b \chi^e_d) \right] = 0.$$  

Analogous to (53), the expression for the gauge transformation for ECS theory in terms of its polarization tensors $\chi_{ab}(k)$ is given by

$$\chi_{ab}(k) \rightarrow \chi'_{ab}(k) = \chi_{ab}(k) + k_a \xi_b(k) + k_b \xi_a(k)$$

where $\xi_a(k)$ are small arbitrary functions of $k$. Depending on whether the excitations are massless or massive, we have two options for $k^2$:
\( (i) \ k^2 = 0 \) or \( (ii) \ k^2 \neq 0. \)

**case (i):** \( k^2 = 0 \)

Contracting the (75) with \( \eta_{ab} \) gives
\[
k_a k_b \chi^{ab} = 0.
\] (77)

A general solution to this equation consistent with the equation of motion (75) is
\[
\chi^{ab} = k^a f^b(k) + k^b f^a(k)
\] (78)

where \( f^a(k) \) are arbitrary functions of \( k \). However, with \( \xi^a = -f^a \) we can ‘gauge away’ these solutions. Therefore, massless excitations of ECS theory are pure gauge artefacts. We now proceed to the other option:

**case (ii) \( k^2 \neq 0. \)**

Let \( k^2 = m^2 \). On contraction with \( \eta_{ab} \) (75) gives
\[
k_a k_b \chi^{ab} = m^2 \chi\]
(79)

where \( \chi = \chi^a_a \). With \( k^a = (m, 0, 0)^T \), this yields
\[
\chi_{11} + \chi_{22} = 0.
\] (80)

By considering the spatial part of (75) one can show that the mass \( m \) of the excitations can be identified with the Chern-Simons parameter \( \mu \) as follows. The spatial part of (75) is
\[
m^2 \chi^{ij} - k^i k_a \chi^{aj} - k^j k_a \chi^{ai} + k^i k^j \chi + \eta^{ij}(-k^2 \chi + k_a k_b \chi^{ab})
- \frac{i}{2\mu} \left[ \epsilon^{iab} k_a (-k^2 \chi_b + k_c k^j \chi^c_b) + \epsilon^{jab} k_a (-k^2 \chi^i_b + k_c k^i \chi^c_b) \right] = 0.
\] (81)

In this equation \( i, j \) takes values 1 and 2. On passing to the rest frame the above equation simplifies to
\[
\chi^{ij} - \chi^{kk} - \frac{im}{2\mu} \left[ \epsilon^{ik} \chi^j_k + \epsilon^{jk} \chi^i_k \right] = 0
\] (82)

from which we obtain (for \( i = j = 1 \) and \( i = j = 2 \) respectively)
\[
\chi_{11} = -\frac{im}{\mu} \chi_{12}; \quad \chi_{22} = \frac{im}{\mu} \chi_{12}.
\] (83)

With \( i = 1 \) and \( j = 2 \) we have
\[
\chi_{12} = -\frac{im}{\mu} \chi_{22}.
\] (84)

This relation together with (83) implies that \( m^2 = \mu^2 \). The remaining components can be made to vanish by a suitable gauge choice. Finally, for the Chern-Simons parameter \( \mu > 0 \), the polarization tensor \( \chi_+ \) of the gravity coupled to Chern-Simons theory in 2+1 dimensions in the rest frame can be written as
\[
\chi_+ = \{ \chi_+^{ab} \} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix} \tau
\] (85)
where $\tau$ is an arbitrary real parameter. Notice that the ECS theory has only a single degree of freedom corresponding to the parameter $\tau$. Similarly, the rest frame polarization tensor for an ECS theory having the Chern-Simons parameter $\mu < 0$ is

$$\chi_- = \{\chi^{ab}_-\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & -1 \end{pmatrix} \tau.$$  \hspace{1cm} (86)

It is important to note that these rest frame polarization tensors $\chi_\pm$ of ECS theories (with $\tau = \frac{1}{2}$) can be obtained as direct products of the rest frame polarization vectors $\bar{\varepsilon}^a_\pm$ (32) of MCS theories. i.e.,

$$\chi^{ab}_\pm = \bar{\varepsilon}_\pm^a \bar{\varepsilon}_\pm^b.$$  \hspace{1cm} (87)

This suggests that we adopt orthonormality conditions for $\chi_\pm$ which will are similar to (37) that are used for $\bar{\varepsilon}_\pm^a$. Hence we require

$$tr \left( (\chi_+)^\dagger (\chi_-) \right) = 0; \quad tr \left( (\chi_-)^\dagger (\chi_+) \right) = 1.$$  \hspace{1cm} (88)

Therefore, analogous to (32) of MCS theory, we have the following maximally reduced form for the polarization tensors of a pair of ECS theories with opposite helicities

$$\chi_\pm = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \pm i \\ 0 & \pm i & -1 \end{pmatrix}.$$  \hspace{1cm} (89)

Note that these polarization matrices of ECS theories are traceless and singular. We are now equipped to study the role played by the translational group in generating the gauge transformation in this theory. The representations of $T(1)$ that generates gauge transformation in pair of MCS theories with opposite helicities is given by $G_\pm(\alpha)$ (43). On account of the relation (87) it is expected that the same representation will generate gauge transformations in ECS theories also. Indeed one can easily see that $G_\pm(\tau_\pm)$ are the gauge generators in ECS theories:

$$\chi_\pm \rightarrow \chi'_\pm = G_\pm(\tau_\pm)\chi_\pm G^T_\pm(\tau_\pm) = \chi_\pm + \begin{pmatrix} \tau_\pm^2 & \frac{\tau_\pm}{\sqrt{2}} & \pm i \tau_\pm \sqrt{2} \\ \frac{\tau_\pm}{\sqrt{2}} & 0 & 0 \\ \pm i \tau_\pm \sqrt{2} & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (90)

This transformation can be cast in the form of the gauge transformation (76) with the following choice of $\xi$'s;

$$\xi_0 = \frac{\tau_\pm^2}{2|\mu|}, \quad \xi_1 = \frac{\tau_\pm}{\sqrt{2}|\mu|}, \quad \xi_2 = \frac{\pm i \tau_\pm}{\sqrt{2}|\mu|}.$$  \hspace{1cm} (91)

One can obtain the moving frame expression for polarization tensors $\chi_\pm(k)$ from the above rest frame results by applying appropriate Lorentz boost as follows:

$$\chi_\pm(k) = \Lambda^T(k)\chi_\pm(0)\Lambda(k)$$
\[
\begin{align*}
&= \frac{1}{2 \mu^2} \left( \begin{array}{ccc}
\frac{k_0^2 - \mu^2}{k_0^2 + i \mu k_1} & \frac{k_0 k_1 \mp i \mu k_1}{(k_0^2 + i \mu k_1)^2} & \frac{k_0 k_2 \pm i \mu k_1}{(k_0^2 + i \mu k_1)^2} \\
\frac{k_0 k_1 \mp i \mu k_1}{k_0^2 + i \mu k_1} & \frac{\gamma/\beta^1}{(\gamma-1)(\beta^1)^2} & \frac{\gamma/\beta^2}{(\gamma-1)(\beta^2)^2} \\
\frac{k_0 k_2 \pm i \mu k_1}{k_0^2 + i \mu k_1} & \frac{\gamma/\beta^1}{(\gamma-1)(\beta^1)^2} & \frac{\gamma/\beta^2}{(\gamma-1)(\beta^2)^2}
\end{array} \right) e^{\pm 2i \phi(k)}
\end{align*}
\]

where \( \phi(k) = \arctan \left( \frac{k^2}{\mu} \right) \) and the momentum space boost matrix

\[
\Lambda(k) = \begin{pmatrix}
\gamma & \gamma/\beta^1 & \gamma/\beta^2 \\
\gamma/\beta^1 & 1 + \frac{(\gamma-1)(\beta^1)^2}{(\beta^1)^2} & \frac{(\gamma-1)(\beta^2)^2}{(\beta^2)^2} \\
\gamma/\beta^2 & \frac{(\gamma-1)(\beta^1)^2}{(\beta^1)^2} & 1 + \frac{(\gamma-1)(\beta^2)^2}{(\beta^2)^2}
\end{pmatrix}
\]

with \( \beta = \frac{k_0}{k_1} \) and \( \gamma = \frac{k_0}{\mu} \). Identical results were obtained in other contexts \[18\] by different methods. Here, we have shown how these results can be obtained in a simpler and straightforward manner just by considering the momentum space expression of the equation of motion in the rest frame using the plane wave method with a subsequent boost transformation. Obviously, the relation (87) holds true in the moving frame also.

### 4 Dimensional descent and linearized gravity theories

In order to discuss dimensional descent from 3+1 dimensions to 2+1 dimensions for symmetric 2nd rank tensor fields it is essential to discuss the relevant aspects of 2+1 dimensional Einstein-Pauli-Fierz (EPF) theory whose action is given by

\[
I_{\text{EPF}} = \int d^3 x \left( -\sqrt{g} R - \frac{\mu^2}{4} (h_{ab}^2 - \eta_{ab} h^2) \right).
\]

Note that the usual sign in front of Einstein action has been restored to avoid ghosts and tachyons, as has been observed recently by Deser and Tekin \[19\]. As noted in \[13\], both the relative and and overall signs of the two terms in (94) have to be of conventional Einstein and Pauli-Feirz mass terms in order to have a physically meaningful theory. Upon linearization, \[94\] reduces to

\[
\mathcal{L}_{L_{\text{EPF}}} = \frac{1}{2} h_{ab} \left[ R_{ab} - \frac{\eta_{ab} R_L}{2} - \frac{\mu^2}{2} (h_{ab}^2 - h^2) \right].
\]

Analogous to the doublet structure of Proca theory discussed above, the EPF theory is a doublet, as was suggested in \[12\], comprising of a pair of ECS theories having opposite helicities. And just like the Proca theory, EPF theory is a gauge noninvariant theory. The equation of motion following from the EPF Lagrangian is given by

\[
- \square h_{ab} + \partial^c \partial_c h_{ab} + \partial^a \partial_b h^{ca} - \partial^a \partial_b h^{ac} + \eta_{ab} (\square h - \partial_c \partial_d h^{cd}) - \mu^2 (h_{ab}^2 - \eta_{ab} h) = 0.
\]

\(^8\)On the other hand in the ECS theory, sign of the Einstein term has to be opposite to that of the conventional one for the theory to be viable. Therefore, if one attempts to couple the ECS theory with a Pauli-Fierz term, one is faced with an unavoidable conflict of signs.
With the ansatz \( h^{ab} = \mathcal{X}^{ab} e^{ik_x} \), where \( \mathcal{X}^{ab} \) is the polarization tensor in this case, this equation can be written as

\[
k^2 \mathcal{X}^{ab} - k^a k_c \mathcal{X}^{cb} - k^b k_c \mathcal{X}^{ca} + \eta^{ab} (-k^2 \mathcal{X} + k_c k_d \mathcal{X}^{cd}) - \mu^2 (\mathcal{X}^{ab} - \eta^{ab} \mathcal{X}) = 0 \tag{97}
\]

We now proceed along the same lines as was done in the previous cases to arrive at the physical polarization tensor of EPF theory. If we choose \( k^2 = 0 \), the equation of motion (97) leads, upon contraction with \( k_a \), to the condition

\[
k_a \mathcal{X}^{ab} = k^b \mathcal{X}. \tag{98}
\]

On the other hand, the contraction of (97) with \( \eta_{ab} \) leads to

\[
\mathcal{X} + k_a k_b \mathcal{X}^{ab} = 0. \tag{99}
\]

A solution of the above pair of equations (98, 99) is given by \( \mathcal{X}^{ab} = k_a f_b(k) + k_b f_a(k) \) where \( f \)'s are arbitrary functions of \( k \). This solution automatically satisfies the condition \( k.f = 0 \). However, such a solution is compatible with the equation of motion (97) if and only if the \( f \)'s vanish identically. Thus the EPF theory does not have massless excitations.

Now for \( k^2 \neq 0 \), in the rest frame \( k^\mu = (m, 0, 0)^T \) and so the (00) component of the equation of motion yields \( \mu^2 (\mathcal{X}^{00} - \mathcal{X}) = 0 \) which in turn gives

\[
\mathcal{X}^{11} + \mathcal{X}^{22} = 0. \tag{100}
\]

Therefore, one is free to arbitrarily choose either \( \mathcal{X}^{11} \) or \( \mathcal{X}^{22} \). Similarly, the (0i) component in the rest frame becomes \( \mu^2 \mathcal{X}^{0i} = 0 \) implying

\[
\mathcal{X}^{0i} = 0. \tag{101}
\]

The space part \((ij)-\text{components}) of the equation of motion with \( i = j = 1 \) and \( i = j = 2 \), respectively yields in the rest frame,

\[
- \mu^2 (\mathcal{X}^0 + \mathcal{X}^2) + m^2 \mathcal{X}^1 = 0 \tag{102}
\]

\[
- \mu^2 (\mathcal{X}^0 + \mathcal{X}^1) + m^2 \mathcal{X}^2 = 0. \tag{103}
\]

Adding the above to equation gives,

\[
\mathcal{X}^{00} = 0. \tag{104}
\]

Finally, the \((ij)\) component of (97) for \( i \neq j \) in the rest frame becomes

\[
(m^2 - \mu^2) \mathcal{X}^1 = 0. \tag{105}
\]

which can be satisfied if either \( m^2 - \mu^2 = 0 \) (in which case \( \mathcal{X}^1 \) may remain arbitrary) or \( \mathcal{X}^1 = 0 \). The latter case implies that we will have only one degree of freedom in the theory corresponding to \( \mathcal{X}^1 \) (or \( \mathcal{X}^2 \)). However, as is well known, the EPF theory has two degrees of freedom [12]. Therefore, we must have

\[
m^2 = \mu^2 \tag{106}
\]
thus establishing that the mass of the EPF excitation to be $|\mu|$ and leaving the $\chi^{12}$ component arbitrary. The two massive degrees of freedom of EPF model correspond to $\chi_{11} = -\chi_{22} = a$ and $\chi^{12} = \chi^{21} = b$ where $a, b$ are arbitrary parameters. Therefore, from (101, 104, 106) it is obvious that we can write the polarization tensor of EPF model in the rest frame as

$$\mathcal{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a \end{pmatrix}.$$  \hspace{1cm} (107)

With the aid of the expressions (107) for $\mathcal{X}$ of EPF theory and (89) for $\chi^{\pm}$ of a pair of ECS theories, we now embark on a discussion of dimensional descent for the case of 2nd rank symmetric tensor gauge fields emphasising the near exact parallel with case of vector gauge fields. One can obtain the momentum 3-vector $k^a$ and polarization tensor $\mathcal{X}$ (107) of EPF model in 2+1 dimensions from those of linearized gravity in 3+1 dimensions as follows. By applying the projection operator $\mathcal{P} = \text{diag}(1, 1, 1, 0)$ on momentum 4-vector $k^\mu = (\omega, 0, 0, \omega)^T$ of a massless graviton moving in the $z$-direction of 3+1 dimensional linearized Einstein gravity and subsequently deleting the last row of the resulting vector, one get momentum 3-vector $k^a$ of 2+1 dimensional EPF quanta at rest. By a similar application of $\mathcal{P}$ on $E$ (60) and deleting the last row and column, one gets the polarization tensor $\mathcal{X}$ (107) in the rest frame of the EPF quanta. Next we notice that, just like the way Proca polarization vector $\vec{\varepsilon}^a$ is written as a linear combination of two orthonormal canonical vectors (35), one can write the EPF polarization tensor $\mathcal{X}$ as

$$\mathcal{X} = a\mathcal{X}_1 + b\mathcal{X}_2 = a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \hspace{1cm} (108)$$

We may consider the above equation to be the EPF analogue of (35) in the case Proca theory. Notice that the space parts of the matrices appearing in the above linear combination are nothing but the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hspace{1cm} (109)$$

Clearly, the space part $\{\chi^{\pm ij}\}$ of the ECS polarization tensors $\chi^{\pm}$ (39) can be expressed in terms of $\sigma_1$ and $\sigma_3$ as follows:

$$\{\chi^{\pm ij}\} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \sigma_3 \pm \frac{i}{\sqrt{2}} \sigma_1 \right). \hspace{1cm} (110)$$

This amounts to an SU(2) transformation in the 2-dimensional subspace (of the SU(2) Lie algebra in an orthonormal basis) spanned by $\frac{1}{\sqrt{2}} \sigma_1$ and $\frac{1}{\sqrt{2}} \sigma_3$. It should be noticed that (110) differs from the one obtained by SU(2) rotation by an irrelevant $i$ factor just as in the vector case. 

\footnote{Note that the $\{\chi^{\pm ij}\}$ obtained in (110) differs from the one obtained by SU(2) rotation by an irrelevant $i$ factor just as in the vector case.}
is the analogue of (38) in the case of Proca and MCS theories. In the case of vector (Proca and MCS) field theories, the basis vectors \( \varepsilon_1 \) and \( \varepsilon_2 \) (35) of the Proca polarization vector, when transformed by a suitable SU(2) transformation yield the polarization vectors \( \varepsilon_\pm \) of a pair of MCS theories. Similarly, in the case of tensor (EPF and ECS) field theories, the same SU(2) transformation when acted on the \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) provides the polarization tensors \( \chi_\pm \) of a doublet of ECS theories with opposite helicities just as Proca theory is a doublet of MCS theories having opposite spins. This corroborates the proposition that EPF theory is consisted of a doublet of ECS theories with opposite spins at least at the level of polarization tensor. Moreover, as we have discussed earlier, the polarization tensor and momentum vector of (2+1 dimensional) EPF theory can be obtained from those of linearized Einstein gravity (in 3+1 dimensions) by applying suitable projection operator. This relationship between EPF and ECS theories resemble the one between Proca and MCS theories. Therefore we expect that the procedure of dimensional descent to be valid here as well. As described in section 2, the generator of the representations of \( T(1) \), obtained by dimensional descent, that generate gauge transformation in a pair of MCS theories with opposite helicities are given by \( \bar{T}_\pm \) (39). Also, the ECS polarization tensors \( \chi_\pm \) can be made to satisfy the orthonormality relations (88) similar to (87) for MCS case owing to the fact that the former is a tensor product of MCS polarization vectors. Hence it is natural to expect that the \( T(1) \) group representation \( G_\pm(\tau) \) (13) obtained by exponentiation of \( \bar{T}_\pm \) generates gauge transformation in ECS doublet, which in fact it does, as we have shown in (59). Therefore, it is evident that by a dimensional descent from 3+1 dimensional linearized gravity one could obtain the representations of \( T(1) \) that generate gauge transformations in the doublet of topologically massive ECS theories in 2+1 dimensions. This is similar to the dimensional descent from 3+1 dimensional Maxwell theory to 2+1 dimensional MCS theory discussed in section 2.

5 Conclusion

In this paper we have studied the role of translational subgroup of Wigner’s little group for massless particles, as a generator of gauge transformations in 3+1 dimensional linearized (pure) Einstein gravity and 2+1 dimensional ECS theory which are 2nd rank symmetric tensor field theories. This property of translational group was earlier shown to hold for vector gauge theories and 2-form gauge fields in different dimensions. In those cases the theories considered included ordinary gauge fields as well as topologically massive gauge fields. In section 2, a review of the role of translational group in generating gauge transformations in the vector gauge (Maxwell and MCS) theories and in 2nd rank antisymmetric tensor gauge (KR and \( B \wedge F \)) theories is provided. Taking the example of 3+1 dimensional Maxwell and 2+1 dimensional Proca theory (which is a doublet of MCS theories with opposite helicities), we have illustrated the method of dimensional descent by which one obtains the representation of the little group that generates gauge transformation in a topologically massive theory. In section 3, we studied the linearized Einstein gravity in 3+1 dimensions and by a straightforward analysis of its the equation of motion, using the plane wave method in momentum space, we obtained the maximally reduced form of the polarization tensor of the theory and the result agrees with the
expression of the polarization tensor obtained by other methods [17]. Using this maximally reduced polarization tensor, we have shown explicitly that \( T(2) \), the translational subgroup of the Wigner’s little group \( E(2) \), generates gauge transformations in 3+1 dimensional linearized gravity just as it happens in the cases of vector or Kalb-Ramond theory involving antisymmetric tensor gauge fields. In the next section we consider the topologically massive ECS theory, obtained by coupling a Chern-Simons mass term to the linearized gravity in 2+1 dimensions and show that the polarization tensor of ECS theory is actually a tensor product of the polarization vectors of a pair of MCS theories with the same Chern-Simons parameter. We have further shown that the same representation of \( T(1) \) that generate gauge transformation in MCS theories acts as generator of gauge transformation in ECS theory also. Finally, in section 4 we obtain the polarization tensor of EPF theory (obtained by coupling the linearized gravity to a Pauli-Fierz mass term) in 2+1 dimensions and show that it splits into ECS polarization tensors with opposite helicities thus suggesting a doublet structure for EPF theory at the level of polarization tensors. This is very similar to Proca theory which is a doublet of MCS theories. Drawing this analogy further, we have been able to extend the method of dimensional descent to the case of second rank symmetric tensor field theories and show that one can obtain the EPF polarization tensor from that of linearized Einstein gravity in 3+1 dimensions. Further more, one obtains the representation of \( T(1) \) that generate gauge transformation in ECS theory by following exactly the same procedure as that in the case of MCS theory.

However, it should be mentioned that the analogy between EPF and Proca theories with their respective doublet structures breaks down if one considers the fact that sign of the Einstein term flips from EPF to ECS theories in contrast to Proca theory where the sign of the Maxwell term remains unchanged irrespective of whether it is coupled to a Chern-Simons term or a usual mass term. Therefore, further investigations are necessary in order to rigorously establish the doublet structure, if any, of EPF theory beyond the level of polarization tensors.

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