LIE ALGEBRAS OF DIFFERENTIAL OPERATORS II: THE UNIVERSAL RING

HELGE ÖYSTEIN MAAKESTAD

Abstract. The aim of this note is to introduce the notion of a D-Lie algebra and to prove some general properties of the category of D-Lie algebras, connections on D-Lie algebras, and universal enveloping algebras of D-Lie algebras. We also define cohomology and homology of a connection on a D-Lie algebra. A D-Lie algebra \( \tilde{L} \) is a Lie-Rinehart algebra over \( A/k \) equipped with an \( A \otimes_k A \)-module structure and a canonical central element \( D \in Z(\tilde{L}) \) satisfying a compatibility property with the Lie-structure. Given a D-Lie algebra \( \tilde{L} \) and a connection \( (\rho, E) \) we construct the universal enveloping ring \( \tilde{U} \otimes (\tilde{L}, \rho) \) of \( (\rho, E) \). The associative unital ring \( \tilde{U} \otimes (\tilde{L}, \rho) \) is a quotient of the associative ring \( \tilde{U} \otimes (\text{End}(\tilde{L}, E)) \) corresponding to the non-abelian extension \( \text{End}(\tilde{L}, E) \) of the D-Lie algebra \( \tilde{L} \), and is a sub ring of \( \text{Diff}(E) \) - the ring of differential operators on \( E \). In the case when \( A \) is Noetherian and \( E \) and \( \tilde{L} \) are finitely generated as left \( A \)-modules it follows the ring \( \tilde{U} \otimes (\tilde{L}, \rho) \) is an almost commutative Noetherian ring. The ring \( \tilde{U} \otimes (\tilde{L}, \rho) \) is a quotient of the associative ring \( U \otimes (\text{End}(\tilde{L}, E)) \) of the non-abelian extension \( \text{End}(\tilde{L}, E) \) and \( U \otimes (\text{End}(\tilde{L}, E)) \) is non-noetherian in general. If \( E \) is a finitely generated \( A \)-module it follows the non-flat connection \( (\rho, E) \) is a finitely generated \( \tilde{U} \otimes (\tilde{L}, \rho) \) module, hence we may speak of the characteristic variety \( SS(\rho, E) \) of \( (\rho, E) \) in the sense of \( D \)-modules. We may define the notion of holonomicity for non-flat connections using the universal ring \( \tilde{U} \otimes (\tilde{L}, \rho) \). This was previously done for flat connections.

Contents

1. Introduction 1
2. Functorial properties of D-Lie algebras and connections 4
3. Functorial properties of universal rings of D-Lie algebras 14
4. The universal ring is an almost commutative Noetherian ring 32

References 39

1. Introduction

The aim of this note is to introduce and prove various general properties of Lie-Rinehart algebras and a generalization of a Lie-Rinehart algebra - a D-Lie algebra. A D-Lie algebra \( \tilde{L} \) is a Lie-Rinehart algebra over \( A/k \) equipped with an...
A \otimes_k A\text{-}module structure that is compatible with the Lie-structure. There is a central element \( D \in \tilde{\mathcal{L}} \) satisfying a compatibility property with the Lie product. In the special case when \( \tilde{\mathcal{L}} \) as a left \( A \)\text{-}module is an abelian extension of \( A \) by some 2-cocycle \( f \in \mathbb{Z}^2(L, A) \) we may view \( \tilde{\mathcal{L}} \) as an Atiyah algebra with an additional \( A \otimes_k A \)-structure. Hence \( \tilde{\mathcal{L}} \) with the underlying left \( A \)\text{-}module structure may be viewed as a simultaneous generalization of a Lie-Rinehart algebra and an Atiyah algebra with additional structure. We introduce the category \( D^1(A, f)_{\text{Lie}} \) of \( D \)-Lie algebras, connections on \( D \)-Lie algebras and prove various general properties of this construction. We also correct some mistakes in an earlier paper on this subject related to the universal algebra \( U^{ua}(L(f\alpha)) \) of a Lie-Rinehart algebra \( (L, \alpha) \) (see [7]). The main results in the paper are the following theorems:

Given a \( D \)-Lie algebra \( \tilde{\mathcal{L}}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D) \) and the categories of \( \tilde{\mathcal{L}} \)-connections \( \text{Mod}(\tilde{\mathcal{L}}, \text{Id}) \) and \( \text{Conn}(\tilde{\mathcal{L}}, \text{Id}) \), we construct two associative unital rings \( U^\otimes(\tilde{\mathcal{L}}) \) and \( U^\rho(\tilde{\mathcal{L}}) \) with the following property (see Theorem 3.5 and 3.18):

**Theorem 1.1.** There are covariant functors

\[
(1.1.1) \quad U^\otimes : D^1(A, f)_{\text{Lie}} \rightarrow \text{Rings}
\]
\[
(1.1.2) \quad U^\rho : D^1(A, f)_{\text{Lie}} \rightarrow \text{Rings}.
\]

with the following property: For any \( D \)-Lie algebra \( \tilde{\mathcal{L}} \) there are exact equivalences of categories

\[
(1.1.3) \quad F_1 : \text{Mod}(\tilde{\mathcal{L}}, \text{Id}) \cong \text{Mod}(U^\otimes(\tilde{\mathcal{L}}))
\]
\[
(1.1.4) \quad F_2 : \text{Conn}(\tilde{\mathcal{L}}, \text{Id}) \cong \text{Mod}(U^\rho(\tilde{\mathcal{L}}))
\]

with the property that \( F_1 \) and \( F_2 \) preserves injective and projective objects.

We use the associative rings \( U^\otimes(\tilde{\mathcal{L}}) \) and \( U^\rho(\tilde{\mathcal{L}}) \) in Definition 3.20 to define the cohomology and homology of an arbitrary connection \((\rho, E)\). Previously the notion of cohomology and homology was defined for flat connections. By Theorem 1.1 it follows the associative rings \( U^\otimes(\tilde{\mathcal{L}}) \) and \( U^\rho(\tilde{\mathcal{L}}) \) may be viewed as universal enveloping algebras for non-flat connections. The rings \( U^\otimes(\tilde{\mathcal{L}}) \) and \( U^\rho(\tilde{\mathcal{L}}) \) are non-Noetherian in general. If \( A \) is a Noetherian ring and the connection \((\rho, E)\) has the property that \( E \) is a finitely generated \( A \)-module it follows from Proposition 3.24 the quotient ring \( U^\otimes(\tilde{\mathcal{L}})/\text{ann}(\rho, E) \) is Noetherian. A similar property holds for \( U^\rho(\tilde{\mathcal{L}}) \). Hence even though the rings \( U^\otimes(\tilde{\mathcal{L}}) \) and \( U^\rho(\tilde{\mathcal{L}}) \) are non-Noetherian in general, we may always pass to Noetherian quotients when studying connections \( E \) that are finitely generated as \( A \)-modules (see Example 3.22).

To illustrate how the associative rings \( U^\otimes(\tilde{\mathcal{L}}) \) and \( U^\rho(\tilde{\mathcal{L}}) \) can be used in the study of the classical curvature we construct in Example 3.36 the following: For any \( A/k \)-Lie-Rinehart algebra \((L, \alpha)\) and any 2-cocycle \( f \in \mathbb{Z}^2(L, A) \) we construct a 2-sided ideal \( I(f) \subseteq U^\rho(L(0)) \) where \( L(0) \) is the abelian extension of \( L \) with the zero cocyle. There is an equivalence of categories

\[
\text{Mod}(U^\rho(L(0))/I(f)) \cong \text{Mod}(U(A, L, f))
\]

where \( U(A, L, f) \) is the generalized universal enveloping algebra studied in [7]. The associative ring \( U(A, L, f) \) has the property that left \( U(A, L, f)\)-modules correspond
to \( L \)-connections of curvature type \( f \). Hence any left \( U^\rho(L(0)) \)-module \((\rho, E)\) annihilated by the ideal \( I(f) \) corresponds to an \( L \)-connection \((\rho, E)\) with curvature type \( f \). Hence we may use one fixed ring \( U^\rho(L(0)) \) and the set of 2-sided ideals in \( U^\rho(L(0)) \) to study the curvature \( R_\rho \) of a connection \((\rho, E)\) on \( L \). Hence the study of the set of 2-sided ideals in the rings \( U^\otimes(\tilde{L}) \) and \( U^\rho(\tilde{L}) \) have applications in the study of the curvature of a classical connection. The algebra \( U(A, L, f) \) is a local version of a much studied object in the field \( D \)-modules.

The two associative unital rings \( U^\otimes(\tilde{L}) \) and \( U^\rho(\tilde{L}) \) are equipped with 2-sided ideals \( I^\otimes \subseteq U^\otimes(\tilde{L}) \) and \( I^\rho \subseteq U^\rho(\tilde{L}) \) such that the following holds for the quotient rings \( \tilde{U}^\otimes(\tilde{L}) := U^\otimes(\tilde{L})/I^\otimes \) and \( \tilde{U}^\rho(\tilde{L}) := U^\rho(\tilde{L})/I^\rho \) (see Theorem 3.17):

**Theorem 1.2.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) be a \( D \)-Lie algebra where \( A \) is Noetherian and \( \tilde{L} \) is finitely generated as left \( A \)-module. It follows the rings \( \tilde{U}^\otimes(\tilde{L}) \) and \( \tilde{U}^\rho(\tilde{L}) \) are almost commutative unital Noetherian rings.

Hence we get many non-trivial examples of Noetherian quotients of the non-Noetherian rings \( \tilde{U}^\otimes(\tilde{L}) \) and \( \tilde{U}^\rho(\tilde{L}) \).

Given an arbitrary \( D \)-Lie algebra \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) and an arbitrary connection \((\rho, E)\) in \( \text{Mod}(\tilde{L}, Id) \) we may construct the non-abelian extension \( \text{End}(\tilde{L}, E) \) of \( \tilde{L} \) by the \( \tilde{L} \)-connection \( \text{End}_A(E) \) as done in [11]. We use this construction to construct the universal ring \( \tilde{U}^\otimes(\tilde{L}, \rho) \) of the connection \((\rho, E)\). In Theorem 1.10 we prove the following:

**Theorem 1.3.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) be a \( D \)-Lie algebra and let \((\rho, E)\) be an \( \tilde{L} \)-connection. There is a canonical map

\[
\rho^! : \text{End}(\tilde{L}, E) \to \text{Diff}^1(E)
\]

and \( \rho^! \) is a map of \( B := A \otimes_k A \)-modules and \( k \)-Lie algebras. The map \( \rho^! \) induce a map \( T(\rho^!) : \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \to \text{Diff}(E) \) of associative rings. Let \( \tilde{U}^\otimes(\tilde{L}, \rho) := \text{Im}(T(\rho^!)) \) be the image. We get an exact sequence of rings

\[
0 \to \ker(T(\rho^!)) \to \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \to \tilde{U}^\otimes(\tilde{L}, \rho) \to 0
\]

where \( \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) := U^\otimes(\text{End}(\tilde{L}, E))/I \) where \( I \) is the 2-sided ideal generated by the elements \( u \otimes v - v \otimes u - [u, v] \) for \( u, v \in \text{End}(\tilde{L}, E) \). The rings \( \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \) and \( \tilde{U}^\otimes(\tilde{L}, \rho) \) are almost commutative. If \( A \) is noetherian and \( \tilde{L}, E \) finitely generated as left \( A \)-modules it follows \( \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \) and \( \tilde{U}^\otimes(\tilde{L}, \rho) \) are Noetherian rings.

Hence given an arbitrary connection \((\rho, E)\) on a \( D \)-Lie algebra \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) we may construct the universal ring \( \tilde{U}^\otimes(\tilde{L}, \rho) \) of \((\rho, E)\) and in the case when \( A \) is Noetherian and \( \tilde{L}, E \) finitely generated \( A \)-modules it follows \( \tilde{U}^\otimes(\tilde{L}, \rho) \) is an almost commutative Noetherian subring of \( \text{Diff}(E) \). The ring \( \tilde{U}^\otimes(\tilde{L}, \rho) \) is defined for an arbitrary connection \( \rho \) and one may use \( \tilde{U}^\otimes(\tilde{L}, \rho) \) to define the characteristic variety \( \text{SS}(\rho, E) \) of \((E, \rho)\) and holonomiticy for non-flat connections. Previously notions such as holonomicity and characteristic variety have been studied for flat connections on holomorphic vector bundles on complex manifolds (see Example 4.12).
2. Functorial properties of D-Lie algebras and connections

In this section we introduce the notion of an D-Lie algebra - a generalization of a Lie-Rinehart algebra. It is a Lie-Rinehart algebra equipped with the structure of an $A \otimes_k A$-module that is compatible with the Lie-structure. Given any 2-cocycle $f \in Z^2(\text{Der}_k(A), A)$ we construct in Theorem 2.4 a functor

$$F_f : \text{LR}(A/k) \to D^1(A,f) - \text{Lie}$$

from the category of $A/k$-Lie-Rinehart algebras to the category of D-Lie algebras. We also consider connections on D-Lie algebras and curvature of connections.

Let in the following $k \to A$ be an arbitrary map of unital commutative rings. Let $P^1 := P^1_{A/k}$ be the module of principal parts. Its dual $D^1(A,0) := \text{Hom}_A(P^1, A) = A \oplus \text{Der}_k(A)$ has a canonical structure as a $k$-Lie algebra and $(A,A)$-module. There is an inclusion $D^1(A,0) \subseteq \text{End}_k(A)$ and we may define for any element $u = aI + x, v = bI + y$ with $I$ the identity endomorphism of $A$ and $x, y$ derivations and $a, b \in A$ the following:

$$[u,v] := u \circ v - v \circ u = (x(b) - y(a))I + [x,y].$$

One checks this gives $D^1(A,0)$ the structure of a $k$-Lie algebra. It is the Lie algebra struture induced by the inclusion $D^1(A,0) \subseteq \text{End}_k(A)$. Given any 2-cocycle $f \in Z^2(\text{Der}_k(A), A)$ we may construct the following structure as $k$-Lie algebra on $A \oplus \text{Der}_k(A)$:

$$(2.0.1) \quad [aI + x, bI + y] = (x(b) - y(a) + f(x,y))I + [x,y] \in A \oplus \text{Der}_k(A).$$

The abelian group $A \oplus \text{Der}_k(A)$ equipped with the $k$-Lie algebra structure $[,]$ in $[2.0.1]$ is denoted $D^1(A,f)$. The abelian group $D^1(A,f)$ has two natural $A$-module structures:

$$(2.0.2) \quad cu = c(aI + x) = (ca)I + cx$$

$$(2.0.3) \quad uc = (aI + x)c = (ac + x(c))I + cx.$$  

The structures in $[2.0.2]$ and $[2.0.3]$ are induced by the left and right $A$-module structure on $\text{End}_k(A)$. It follows $D^1(A,f)$ is an $A \otimes_k A$-module and a $k$-Lie algebra. There is an endomorphism

$$\pi : D^1(A,f) \to D^1(A,f)$$

defined by

$$\pi(u) = \pi(aI + x) = x.$$  

We get

$$\pi([u,v]) = \pi((x(b) - y(a) + f(x,y))I + [x,y]) = [x,y] = [\pi(x), \pi(y)],$$

hence $\pi$ is a morphism of $k$-Lie algebras. We may think of $\pi$ as a map of $k$-Lie algebras and $A \otimes_k A$-modules

$$\pi : D^1(A,f) \to \text{Der}_k(A).$$

Here we give $\text{Der}_k(A)$ the trivial right $A$-module structure. Let $\overline{f} := (1,0) \in D^1(A,f)$. It follows $[\overline{f},u] = 0$ for all elements $u \in D^1(A,f)$ hence $\overline{f}$ is a central element.
Lemma 2.1. The following holds for every $u = aI + x, v = bI + y \in D^1(A, f)$ and $c \in A$:

\begin{align}
(2.1.1) & \quad [u, cv] = c[u, v] + \pi(u)(c)v. \\
(2.1.2) & \quad uc = cu + \pi(u)(c)\pi
\end{align}

Proof. we get

\[ [u, cv] = [aI + x, (cb)I + cy] = (x(cb) - cy(a) + f(x, cy)I + [x, cy]) = \]

\[ (cx(b) + x(c)b - cy(a) + cf(x, y))I + c[x, y] + x(c)y = \]

\[ c((x(b) - y(a) + f(x, y))I + [x, y]) + x(c)(bI + y) = c[u, v] + \pi(u)(c)v. \]

we get

\[ uc := (aI + x)c := (ac + x(c))I + cx = a(cI + x) + x(c)\pi = cu + \pi(u)(c)\pi \]

The Lemma follows.

Let us sum this up in a Proposition:

Proposition 2.2. Let $k \to A$ be a unital map of commutative rings and let $D^1(A, f) := A \oplus \text{Der}_k(A)$ with $f \in Z^2(\text{Der}_k(A), A)$ a 2-cocycle. Let $u := aI + x, v := bI + y \in D^1(A, f)$. Define the following left and right $A$-module structure on $D^1(A, f)$:

\[ c(aI + x) := (ca)I + cx \]

and

\[ (aI + x)c := (ac + x(c))I + cx \]

for $c \in A$. Define the following product $[,]$ on $D^1(A, f)$:

\[ [u, v] = [aI + x, bI + y] := (x(b) - y(a) + f(x, y))I + [x, y]. \]

Define the map $\pi : D^1(A, f) \to \text{Der}_k(A)$ by $\pi(aI + x) := x$ and give $\text{Der}_k(A)$ the trivial right $A$-module structure. It follows $D^1(A, f)$ is an $A \otimes_k A$-module and a $k$-Lie algebra. The map $\pi$ is a map of $A \otimes_k A$-modules and $k$-Lie algebras. The product $[,]$ satisfies

\[ [u, cv] = c[u, v] + \pi(u)(c)v \]

and

\[ uc = cu + \pi(u)(c)\pi \]

for all $u, v \in D^1(A, f)$ and $c \in A$.

Proof. The proof follows from the calculations above. \qed

Hence the underlying left $A$-module of the pair $(D^1(A, f), \pi)$ is an ordinary Lie-Rinehart algebra. It is the abelian extension of $\text{Der}_k(A)$ with the 2-cocycle $f \in Z^2(\text{Der}_k(A), A)$. It follows $D^1(A, f)$ is an Atiyah algebra in the sense of \cite{19}.

Note: We see that it is impossible to construct a non-trivial right $A$-module structure on $\text{Der}_k(A)$ induced by the inclusion $\text{Der}_k(A) \subseteq \text{End}_k(A)$. To get a non-trivial right $A$-module structure we must consider the abelian extension $D^1(A, f)$ for some 2-cocycle $f$.

We may define the following:
Definition 2.3. A 5-tuple $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)$ where $\tilde{L}$ is an $A \otimes_k A$-module and $k$-Lie algebra and

$$\tilde{\alpha} : \tilde{L} \to D^1(A, f)$$

is a map of $A \otimes_k A$-modules and $k$-Lie algebras is a $D$-Lie algebra if the following holds: The element $D \in Z(\tilde{L})$ is a central element. The map $\tilde{\pi} : \tilde{L} \to \text{Der}_k(A)$ is a map of $A \otimes_k A$-modules and $k$-Lie algebras with $\tilde{\pi}(D) = 0$ and $\pi \circ \tilde{\alpha} = \tilde{\pi}$. Here $\text{Der}_k(A)$ has the trivial right $A$-module structure. For all $u, v \in \tilde{L}$ and $c \in A$ the following holds:

\begin{align*}
(2.3.1) & \quad uc = cu + \tilde{\pi}(u)(c)D \\
(2.3.2) & \quad [u, cv] = c[u, v] + \tilde{\pi}(u)(c)v
\end{align*}

A 4-tuple $(\tilde{L}, \tilde{\pi}, [\cdot, \cdot], D)$ (we remove the element $\tilde{\alpha}$ from the definition of a $D$-Lie algebra) satisfying the above criteria is a pre-$D$-Lie algebra. Let $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)$ and $(\tilde{M}, \tilde{\beta}, \tilde{\gamma}, [\cdot, \cdot], D')$ be $D$-Lie algebras. A map $\tilde{\psi} : \tilde{L} \to \tilde{M}$ of $k$-Lie algebras and $A \otimes_k A$-modules is a map of $D$-Lie algebras if $\tilde{\beta} \circ \tilde{\psi} = \tilde{\alpha}$ and $\tilde{\psi}(D) = D'$. Let $D^1(A, f)$-Lie denote the category of $D$-Lie algebras and morphisms. Let $E$ be a left $A$-module. An $\tilde{L}$-connection on $E$ is an $A \otimes_k A$-linear map

$$\rho : \tilde{L} \to \text{End}_k(E).$$

The module $\text{End}_k(E)$ has the $A \otimes_k A$-module structure defined by $a \otimes b \psi := a(\psi b)$ for $a \otimes b \in A \otimes_k A$ and $\psi \in \text{End}_k(E)$. Given two connections $(E, \rho_E)$ and $(F, \rho_F)$, a morphism $\phi : (E, \rho_E) \to (F, \rho_F)$ is a map of $A$-modules $\phi : E \to F$ such that for any element $u \in \tilde{L}$ we get a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\rho_E(u)} & E \\
\downarrow{\phi} & & \downarrow{\phi} \\
F & \xrightarrow{\rho_F(u)} & F
\end{array}
$$

A $D$-Lie algebra is also referred to as a Lie algebra of differential operators acting on $A/k$. Let $D^1(A, f)$-Lie denote the category of $D$-Lie algebras and morphisms of $D$-Lie algebras. Let $\text{Mod}(\tilde{L})$ denote the category of $\tilde{L}$-connections and morphisms of connections. Let $\text{Mod}(\tilde{L}, Id)$ denote the category of $\tilde{L}$-connections $(\rho, E)$ where $\rho(D) = Id_E$. We define similar notions for a pre-$D$-Lie algebra.

Note: By definition $(D^1(A, f), id, \pi, [,], z)$ where $\pi := (1, 0)$ is a $D$-Lie algebra.

Note: A $D$-Lie algebra is an $A$-$A$-module in the sense of $[5]$ and non-commutative geometry and such objects are much studied in this field.

Lemma 2.4. Let $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)$ be a $D$-Lie algebra. The following formula holds for all $u, v \in \tilde{L}$ and $c \in A$:

$$[u, vc] = c[u, v] + \tilde{\pi}(u)(c)v + \tilde{\pi}(u)\tilde{\pi}(v)(c)D.$$

Proof. We get

$$[u, vc] = [u, cv + \tilde{\pi}(v)(c)D] = c[u, v] + \tilde{\pi}(u)(c)v + \tilde{\pi}(u)\tilde{\pi}(v)(c)D$$

and the Lemma is proved. \qed
Lemma 2.5. Let $(L, \alpha)$ be an $A/k$-Lie-Rinehart algebra and let $f \in Z^2(Der_k(A), A)$ be a 2-cocycle. We get in a natural way a 2-cocycle $f^\alpha \in Z^2(L, A)$ defined by $f^\alpha(x, y) := f(\alpha(x), \alpha(y))$.

Proof. Let $u := x \wedge y \wedge z \in \wedge^3 \text{Der}_k(A)$. We get

$$d^2(f^\alpha)(u) = d^2(f)(\alpha(x) \wedge \alpha(y) \wedge \alpha(z)) = 0$$

since $d^2(f) = 0$. The Lemma follows. □

Let $k \to A$ be a unital map of commutative rings and let $\alpha : L \to \text{Der}_k(A)$ be a Lie-Rinehart algebra. Let $f \in Z^2(\text{Der}_k(A), A)$ be a 2-cocycle. Let $L(f^\alpha) := Az \oplus L$ with the following left and right $A$-module structure: Let $u := az + x, v := bz + y \in L(f^\alpha)$.

$$b(az + x) := (ba)z + bx$$

and

$$(az + x)b := (ab + \alpha(x)(b))z + ax.$$ Define the product $[u, v]$ as follows:

$$[u, v] := (\alpha(x)(b) - \alpha(y)(a) + f^\alpha(x, y))z + [x, y] \in (L f^\alpha).$$

Define the map $\alpha_f : L(f^\alpha) \to D^1(A, f)$ by $\alpha_f(u) := aI + \alpha(x)$. Define $\pi_f : L(f^\alpha) \to \text{Der}_k(A)$ by $\pi_f(u) = \alpha(x)$. Assume $\phi : (L, \alpha) \to (L', \alpha')$ is a map of Lie-Rinehart algebras. Define the map $\phi_f : L(f^\alpha) \to L'(f'^\alpha)$ by

$$\phi_f(az + x) := az + \phi(x).$$

One checks that for any $u \in L(f^\alpha)$ and $c \in A$ the following holds:

$$uc = cu + \pi_f(u)(c)z$$

and that $[z, u] = 0$ hence $z$ is a central element in $L(f^\alpha)$.

Lemma 2.6. The abelian group $(L(f^\alpha), [\,])$ is an $A \otimes_k A$-module and $k$-Lie algebra. The map $\alpha_f$ is a map of $A \otimes_k A$-modules and $k$-Lie algebras. The product $[\,]$ satisfies

$$[u, cv] = c[u, v] + \pi_f(u)(c)v$$

and

$$uc = cu + \pi_f(u)(c)z.$$ for all $u, v \in L(f^\alpha)$ and $c \in A$. The map $\phi_f : L(f^\alpha) \to L'(f'^\alpha)$ is a map of $A \otimes_k A$-modules and $k$-Lie algebras. There is an equality $\alpha_f' \circ \phi_f = \alpha_f$. Hence the 5-tuple $(L(f^\alpha), \alpha_f, \pi_f, [\,], z)$ is a D-Lie algebra.

Proof. One checks $L(f^\alpha)$ is an $A \otimes_k A$-module and $k$-Lie algebra and that $\alpha_f$ is a map of $A \otimes_k A$-modules and $k$-Lie algebras. One also checks that

$$[u, cv] = c[u, v] + \pi_f(u)(c)v$$

for all $u, v \in L(f^\alpha)$ and $c \in A$. Finally one checks that $\phi_f$ is a map of $A \otimes_k A$-modules and $k$-Lie algebras and that $\alpha_f' \circ \phi_f = \alpha_f$. The Lemma follows. □

Theorem 2.7. Let $k \to A$ be an arbitrary map of unital commutative rings and let $f \in Z^2(\text{Der}_k(A), A)$ and let $f^\alpha \in Z^2(L, A)$ be the pull back of $f$ via $\alpha$. There is a covariant functor

$$F : LR(A/k) \to D^1(A, f) - \text{Lie}$$

defined by

$$F(L, \alpha) := (L(f^\alpha), \alpha_f, \pi_f, [\,], z).$$
A map \( \phi : (L, \alpha) \to (L', \alpha') \) of Lie-Rinehart algebras gives a map
\[
\phi_f : L(f^\alpha) \to L'(f'^{\alpha'})
\]
of D-Lie algebras. Hence \( \phi_f(z) = z' \) with \( z, z' \) the central elements of \( L(f^\alpha) \) and \( L'(f'^{\alpha'}) \). The construction is functorial in the sense that if \( \phi' : (L', \alpha') \to (L'', \alpha'') \) is another map of Lie-Rinehart algebras it follows
\[
(\phi' \circ \phi)_I = \phi' \circ \phi_f.
\]

**Proof.** The proof follows from Lemma 2.6. \( \Box \)

**Example 2.8.** Atiyah algebras and D-Lie algebras.

Given a Lie-rinehart algebra \( \alpha : L \to \text{Der}_k(A) \) and a 2-cocycle \( f \in Z^2(\text{Der}_k(A), A) \) it follows we get an exact sequence of \( A \otimes_k A \)-modules
\[
0 \to Az \to L(f^\alpha) \to L \to 0
\]
and the left \( A \)-module \( L(f^\alpha) \) is an Atiyah algebra in the sense of [3] if \( L := \text{Der}_k(A) \). Hence \( L(f^\alpha) \) is an Atiyah algebra equipped with a canonical right \( A \)-module structure and a marked central element \( z \in L(f^\alpha) \) satisfying
\[
uc = cu + \pi_f(u)(c)z
\]
for all \( u \in L(f^\alpha) \) and \( c \in A \). A general D-Lie algebra \( \tilde{L} \) is not an extension of \( \text{Der}_k(A) \) by a rank one free \( A \)-module in general. Hence the underlying left \( A \)-module of \( \tilde{L} \) is not an Atiyah algebra in general.

Note: In [18] Tortella constructs a canonical Hodge structure on the cohomology of the Atiyah algebra \( \text{At}(\mathcal{L}) \) of a holomorphic line bundle \( \mathcal{L} \) on a complex projective manifold \( X \). If \( \mathcal{L} \) is a holomorphic line bundle on \( X \) and
\[
0 \to \mathcal{L} \to \text{At}(\mathcal{L}) \to \Theta_X \to 0
\]
an extension it follows \( X \) has an open cover \( \{U_i\}_{i \in I} \) where \( \mathcal{L} \) trivialize. Hence we get exact sequences
\[
0 \to \mathcal{O}_{U_i} \to \text{At}(\mathcal{O}_{U_i}) \to \Theta_{U_i} \to 0
\]
where there is a 2-cocycle \( f_i \in Z^2(\Theta_{U_i}, \mathcal{O}_{U_i}) \) and an isomorphism \( \text{At}(\mathcal{O}_{U_i}) \cong \Theta_{U_i}(f_i) \) of sheaves of Lie-Rinehart algebras. Hence \( \text{At}(\mathcal{L}) \) is locally the abelian extension of \( \Theta_{U_i} \) by \( \mathcal{O}_{U_i} \). Hence Tortella’s Atiyah algebra is a global version of an abelian extension of Lie-Rinehart algebras valid for arbitrary complex manifolds.

**Definition 2.9.** Let \( (\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [,], D) \) be an D-Lie algebra and let \( (L, \alpha) \) be an \( A/k \)-Lie-Rinehart algebra. Let \( E \) be an \( A \)-module and let \( \psi \in \text{End}_A(E) \). An \( (\tilde{L}, \psi) \) – *connection* on \( E \) is a map of left \( A \)-modules
\[
\nabla : \tilde{L} \to \text{End}_k(E)
\]
with the property that
\[
\nabla(u)(ae) = a\nabla(u)(e) + \tilde{\pi}(u)(a)\psi(e)
\]
for all \( u \in \tilde{L}, a \in A \) and \( e \in E \). An \( (L, \psi) \) – *connection* on \( E \) is a map of left \( A \)-modules
\[
\nabla : L \to \text{End}_k(E)
\]
with the property that
\[
\nabla(x)(ae) = a\nabla(x)(e) + \tilde{\pi}(x)(a)\psi(e)
\]
for all \( x \in L, a \in A \) and \( e \in E \). Let \( \text{Conn}(\tilde{L}, \text{End}) \) be the category of \((\tilde{L}, \psi)\)-connections and morphisms where we let \( \psi \in \text{End}_A(E) \) and \( E \) vary. Let \( \text{Conn}(\tilde{L}, \text{Id}) \) be the category of \((\tilde{L}, \text{Id})\)-connections \((E, \rho)\). Let \( \text{Conn}(L, \text{End}) \) denote the category of \((L, \psi)\)-connections \( \rho : L \to \text{End}_k(E) \) where \( \psi \in \text{End}_A(E) \) may vary. Let \( \text{Conn}(L, \text{Id}) \) denote the category of \((L, \text{Id})\)-connections \( \rho : L \to \text{End}_k(E) \).

Note: It follows there are inclusions of categories
\[
\text{Conn}(\tilde{L}, \text{Id}) \subseteq \text{Conn}(\tilde{L}) \subseteq \text{Conn}(L, \text{End})
\]
Recall that \( \text{Conn}(\tilde{L}) \) is the category of maps of \( A \otimes_k A \)-modules
\[
\rho : \tilde{L} \to \text{End}_k(E)
\]
and morphisms.

Note: The morphisms in \( \text{Conn}(\tilde{L}, \text{End}) \) \( \phi : (\rho, E) \to (\rho', F) \) are maps of left \( A \)-modules \( \phi : E \to F \) such that
\[
\rho'(u) \circ \phi = \phi \circ \rho(u)
\]
for all \( u \in \tilde{L} \). The morphisms \( \phi : (\rho, E, \psi) \to (\rho', F, \psi') \) in \( \text{Conn}(L, \text{End}) \) are by definition maps of \( A \)-modules
\[
\phi : E \to F
\]
with the property that
\[
\psi' \circ \phi = \phi \circ \psi \text{ and } \rho'(x) \circ \phi = \phi \circ \rho(x)
\]
for all \( x \in L \).

**Lemma 2.10.** A \( \psi \)-connection \( \nabla : \tilde{L} \to \text{End}_k(E) \) gives a map
\[
\nabla : \tilde{L} \to \text{Diff}^l_k(E)
\]
where \( \text{Diff}^l_k(E) := \text{Hom}_A(\text{P}^1 \otimes_A E, E) \) is the module of first order differential operators of \( E \).

**Proof.** We must show that for any \( b \in A \) and \( x \in L \) it follows \([\nabla(x), b \text{Id}_E] \in \text{End}_A(E)\). We get
\[
[\nabla(x), b \text{Id}_E] = \tilde{\pi}(x)(b)\psi \in \text{End}_A(E)
\]
and the Lemma follows. \(\square\)

**Lemma 2.11.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) be a \( D \)-Lie algebra and let \( E \) be a left \( A \)-module. Any \( \tilde{L} \)-connection
\[
\rho : \tilde{L} \to \text{End}_k(E)
\]
is an \((\tilde{L}, \psi)\)-connection with \( \psi := \rho(D) \in \text{End}_A(E) \). It follows we get a map
\[
\rho : \tilde{L} \to \text{Diff}^l_k(E)
\]
of \( A \otimes_k A \)-modules.

**Proof.** Assume \( \rho : \tilde{L} \to \text{End}_k(E) \) is an \( A \otimes_k A \)-linear map. We get for any \( x \in \tilde{L}, a \in A \) and \( e \in E \) the following calculation:
\[
\rho(x)(ae) = \rho(xa)(e) = \rho(ax + \tilde{\pi}(x)(a)D)(e) = a\rho(x)(e) + \tilde{\pi}(x)(a)\rho(D)(e).
\]
We get
\[
\rho(D)(ae) = \rho(Da)(e) = \rho(aD + \tilde{\pi}(D)(a)D)(e) = \rho(aD)(e) = a\rho(D)(e)
\]
since \( \tilde{\pi}(D) = 0 \). It follows \( \psi := \rho(D) \in \text{End}_A(E) \). It follows \( \rho \) is a \( \rho(D) \)-connection with \( \rho(D) \in \text{End}_A(E) \). \(\square\)
Lemma 2.12. Let $f \in Z^2(Der_k(A), A)$ and let $F(L, \alpha) = (L(f^\alpha), \alpha_f, \pi_f, [,], z)$ be the D-Lie algebra from Theorem $2.7$. Let $E$ be a left $A$-module. There is a one-to-one correspondence between $A \otimes_k A$-linear maps $\rho : L(f^\alpha) \to \text{End}_k(E)$ and $\psi$-connections $\nabla : L \to \text{End}_k(E)$ with $\psi \in \text{End}_A(E)$.

Proof. Assume $\rho : L(f^\alpha) \to \text{End}_k(E)$ is an $A \otimes_k A$-linear map. Let $i : L \to L(f^\alpha)$ be the canonical inclusion map and define $\nabla := \rho \circ i$. Since $\rho$ is $A \otimes_k A$-linear it follows $\nabla$ is left $A$-linear. Let $x \in L, a \in A$ and $e \in E$. We get

$$\nabla(x)(ae) = \rho(i(x))(ae) = \rho(xa)(e) = \rho(\alpha(x)(a)z + ax)(e) = a\rho(i(x))(e) + \alpha(x)(a)\rho(z)(e).$$

Put $\rho(z) := \psi$. It follows $\psi \in \text{End}_A(E)$ and we get

$$\nabla(x)(ae) = a\nabla(x)(e) + \alpha(x)(a)\psi(e)$$

hence $\nabla$ is a $\psi$-connection. Assume $\nabla : L \to \text{End}_k(E)$ is a $\psi$-connection where $\psi \in \text{End}_A(E)$. Define $\rho : L(f^\alpha) \to \text{End}_k(E)$ by

$$\rho(u) = \rho(az + x) := a\psi + \nabla(x).$$

It follows $\rho$ is a left $A$-linear map. It is right $A$-linear for the following reason:

$$\rho(uc) := \rho((az + x)c) = \rho((ac + \alpha(x)(c))z + cx) =$$

$$(ac + \alpha(x)(c))\psi + \nabla(cx) =$$

$$ac\psi + c\nabla(x) + \alpha(x)(c)\psi =$$

$$ac\psi + \nabla(x)c = a\psi c + \nabla(x)c = (a\psi + \nabla(x))c = \rho(u)c.$$ 

Hence

$$\rho(uc) = \rho(u)c.$$

It follows $\rho$ is a map of $A \otimes_k A$-modules. This gives a one-to-one correspondence as claimed and the Lemma follows.

Note: The definition of a connection $\rho : L(f^\alpha) \to \text{End}_k(E)$ depends on the $A \otimes_k A$-module structure on $L(f^\alpha)$ and not on the Lie-algebra structure. Since for any 2-cocycles $f, g \in Z^2(Der_k(A), A)$ it follows $L(f^\alpha) \cong L(g^\alpha)$ as $A \otimes_k A$-modules it follows there is a one-to-one correspondence between $L(f^\alpha)$-connections and $L(g^\alpha)$-connections. In fact there is an equivalence of categories

$$\text{Conn}(L(f^\alpha), \text{End}) \cong \text{Conn}(L(g^\alpha), \text{End})$$

for any pair of 2-cocycles $f, g \in Z^2(Der_k(A), A)$.

Example 2.13. D-Lie algebras, connections and $(A, A)$-vector bundles.

Note: An ordinary $L$-connection $\nabla : L \to \text{End}_k(E)$ corresponds by Lemma 2.12 to an $A \otimes_k A$-linear map $\rho : L(f^\alpha) \to \text{Diff}^1_k(E)$ with $\rho(z) = Id_E$. Usually a connection is a $k$-linear map $\nabla : E \to \Omega^1_{A/k} \otimes_A E$.
satisfying $\nabla(ae) = a\nabla(e) + d(a) \otimes e$. Hence $\text{ker}(\nabla)$ and $\text{Im}(\nabla)$ are $k$-vector spaces. The vector spaces $\text{ker}(\nabla)$, $\text{Im}(\nabla)$ are infinite dimensional in general.

If $A$ is a finitely generated and regular ring over a field of characteristic zero, $E$ is finitely generated and projective as $A$-module and $L(f^n) = \text{Diff}^1(A,f)$ it follows $\text{Diff}^1_k(E)$ and $L(f^n)$ are locally trivial $A$-modules of finite rank as left and right $A$-modules separately. Hence the map $\rho$ is a more "geometric" object: One of the reasons to define a connection as a map $\rho$ of $A \otimes_k A$-modules is because we want to study the kernel $\text{ker}(\rho)$ and image $\text{Im}(\rho)$ and these modules are "geometric" objects since they are vector bundles from the left and right in many cases. With $\nabla$ the kernel $\text{ker}(\nabla)$ and image $\text{Im}(\nabla)$ are infinite dimensional $k$-vector spaces and not $A$-modules, and such objects are "more difficult" to handle.

**Example 2.14.** Left and right $A$-module structures on modules of principal parts.

Note: An $A \otimes_k A$-module $W$ that is finitely generated and projective as left and right $A$-module separately is called an $(A,A)$-vector bundle. The module of principal parts $\text{P}^l(E)$ is an $(A,A)$-vector bundle in many cases. There are examples where the left structure on $\text{P}^l(E)$ is different from the right structure (see [12]). Similar results hold for the module of differential operators $\text{Diff}^l(E,E)$. From [12] we get the following example. Let $C := \mathbb{P}^1$ be the projective line over a field of characteristic zero and let $\mathcal{O}(n)$ be the invertible sheaf with $n \in \mathbb{Z}$ an integer. The module of $l$th order differential operators $\text{Diff}^l(\mathcal{O}(n))$ from $\mathcal{O}(n)$ to $\mathcal{O}(n)$ has a left and right structure as $\mathcal{O}_C$-module and we get the following classification:

**Theorem 2.15.** Let $k \geq 1$ and $n \in \mathbb{Z}$ be integers. The following holds:

$$\text{Diff}^l(\mathcal{O}(n))^{\text{right}} \cong \mathcal{O}_C \oplus \mathcal{O}(l+1)^k \text{ for all } l \geq 1 \text{ and } n \in \mathbb{Z}.$$  

$$\text{Diff}^l(\mathcal{O}(n))^{\text{left}} \cong \mathcal{O}(l+1)^{l+1} \text{ for all } 1 \leq l \leq n.$$  

$$\text{Diff}^l(\mathcal{O}(n))^{\text{left}} \cong \mathcal{O}_C^{n+1} \oplus \mathcal{O}(l+1)^{n-l} \text{ for all } n < l \text{ and } l \geq 1.$$

**Proof.** The proof follows from [12] since the sheaf of differential operators $\text{Diff}^l(\mathcal{O}(n))$ is the dual of the sheaf of principal parts. \hfill $\square$

Hence $\text{Diff}^l(\mathcal{O}(n))^{\text{left}} \neq \text{Diff}^l(\mathcal{O}(n))^{\text{right}}$ as $\mathcal{O}_C$-module in general.

We denote the left and right $A$-module structure on $\text{P}^l(E)$ as $\text{P}^l(E)^{\text{left}}$ and $\text{P}^l(E)^{\text{right}}$. The class

$$\gamma^l(E) := [\text{P}^l(E)^{\text{left}}] - [\text{P}^l(E)^{\text{right}}] \in K_0(A)$$

is zero in most cases. This is because $\text{P}^l(E)$ is an extension $\text{P}^l-1(E)$ with $\text{Sym}^l(\Omega^1)$ and $\text{Sym}^l(\Omega^1)^{\text{left}} \cong \text{Sym}^l(\Omega^1)^{\text{right}}$ as $A$-modules for all $l \geq 1$.

**Lemma 2.16.** Let $k \to A$ be a commutative ring that is a finitely generated and regular $k$-algebra where $k$ is a field of characteristic zero and let $E$ be a finitely generated and projective $A$-module. There is for every $l \geq 1$ an exact sequence of left and right $A$-modules

$$0 \to \text{Diff}^{l-1}(E) \to \text{Diff}^l(E) \to \text{Sym}^l(\text{Der}_k(A)) \otimes \text{End}_A(E) \to 0.$$  

We let $\text{Diff}^0(E) := \text{End}_A(E)$. 
Proof. There is an exact sequence of left and right $A$-modules

$$0 \to \text{Sym}^l(\Omega^1_{A/K}) \otimes_A E \to P^l(E) \to P^{l-1}(E) \to 0$$

where $P^l(E)$ is the $l$'th module of principal parts. Since $A$ is regular it follows $P^l(E)$ is a projective $A$-module of finite rank as left and right $A$-module. When we apply the functor $\text{Hom}_A(-, E)$ to the sequence we get the claimed sequence and the Lemma follows.

Lemma 2.17. Let $A$ be a commutative ring satisfying the hypothesis from Lemma 2.16. Let $\text{Diff}^l(E)^{\text{left}}$ and $\text{Diff}^l(E)^{\text{right}}$ denote the left and right $A$-module structure on $\text{Diff}^l(E)$. The following holds in the Grothendieck group $K_0(A)$ of $A$:

$$[\text{Diff}^l(E)^{\text{left}}] = [\text{End}_A(\text{Diff}^l(E))^{\text{left}}](1 + [\text{Der}_k(A)]^{\text{left}}] + [\text{Sym}^2([\text{Der}_k(A)]^{\text{left}}] + \cdots + [\text{Sym}^l([\text{Der}_k(A)]^{\text{left}}].$$

A similar formula holds when we consider the right $A$-module structure.

Proof. The Lemma follows from Lemma 2.16 and an induction on $l$.

Theorem 2.18. Let $A$ be a commutative ring satisfying the hypothesis in Lemma 2.16. Let $E$ be a finitely generated and projective $A$-module and let

$$\eta^l(E) := [\text{Diff}^l(E)^{\text{left}}] - [\text{Diff}^l(E)^{\text{right}}] \in K_0(A).$$

It follows $\eta^l(E) = 0$.

Proof. The Theorem follows from Lemma 2.17 since there is for every $l \geq 1$ an isomorphism $\text{Sym}^l([\text{Der}_k(A)]^{\text{left}}] \cong \text{Sym}^l([\text{Der}_k(A)]^{\text{right}}$ of $A$-modules.

Hence the Grothendieck group $K_0(A)$ does not detect that $P^l(E)^{\text{left}} \neq P^l(E)^{\text{right}}$ and $\text{Diff}^l(E)^{\text{left}} \neq \text{Diff}^l(E)^{\text{right}}$ in general (see [10] for a more detailed discussion).

The aim of this study is to construct "generalized jet bundles" where the classes $\gamma^l(E)$ and $\eta^l(E)$ are non trivial in $K_0(A)$ and to apply this in the study of Chern classes and Hodge theory. One want to construct non-trivial classes in $H^{2*}(L, A)$ coming from $K_0(L)$.

Lemma 2.19. Let $k \to A$ be a map of unital commutative rings and let $f \in Z^2(\text{Der}_k(A), A)$ be a 2-cocycle. Let $(L, \alpha)$ be an $(A/k)$-Lie-Rinehart algebra and let $f^\alpha := Z^2(L, A)$ be the pull back of $f$. Assume $\rho : L(f^\alpha) \to \text{End}_k(E)$ is an $A \otimes_k A$-linear map, with $E$ a left $A$-module and let $i : L \to L(f^\alpha)$ be the canonical injective map. Let $u = az + x, v = bz + y \in L(f^\alpha)$ and define $R_\rho(u, v) := [\rho(u), \rho(v)] - \rho([u, v])$. The following holds:

$$R_\rho(u, v)(e) = R_{\rho_0}(x, y)(e) - f^\alpha(u, v)e$$

where

$$R_{\rho_0}(x, y) := [\rho(i(x)), \rho(i(y))] - \rho(i([x, y])).$$

Proof. Let $\rho : L(f^\alpha) \to \text{End}_k(E)$ be an $A \otimes_k A$-linear map. We get

$$R_\rho(u, v)(e) = [\rho(u), \rho(v)](e) - \rho([u, v])(e).$$

We get

$$\rho([u, v])(e) = \rho((\alpha(x)(b) - \alpha(y)(a) + f(\alpha(x), \alpha(y)))z + [x, y])(e) = (\alpha(x)(b) - \alpha(y)(a) + f(\alpha(x), \alpha(y)))e + \rho(i([x, y]))(e).$$
Similarly we get
\[ [\rho(u), \rho(v)](e) = [\rho(i(x)), \rho(i(y))](e). \]
We get
\[ R_\rho(u, v)(e) = [\rho(i(x)), \rho(i(y))](e) - \rho([u, v])(e) = \rho(f(\alpha(x), \alpha(y))e = R_{\rho \alpha}(x, y)(e) - f(\alpha(x), \alpha(y))e. \]

The Lemma is proved. \(\square\)

**Example 2.20. Families of connections.**

Given two 2-cocycles \(f, g \in Z^2(Der_k(A), A)\). It follows from Lemma 2.12 we get for any \(\psi\)-connection \(\nabla : L \to \text{End}_k(A)\) two connections
\[ \rho_{f\alpha} : L(f^\alpha) \to \text{End}_k(E) \]
and
\[ \rho_{g\alpha} : L(g^\alpha) \to \text{End}_k(E). \]
If \(g = f + d^3 \phi\) for an element \(\phi \in C^1(L, A)\), there is an isomorphism of Lie-Rinehart algebras \(L(f^\alpha) \cong L(g^\alpha)\). Hence the construction gives for a fixed \(A\)-module \(E\), a family of connections \(\rho_{f\alpha} : L(f^\alpha) \to \text{End}_k(E)\) parametrized by the set of 2-cocycles \(Z^1(Der_k(A), A)\) and from Lemma 2.11 we see the curvature \(R_{\rho_{f\alpha}}\) varies with the 2-cocycle \(f\). One may ask if it is possible to use the family \((E, \rho_{f\alpha})\) to study the original connection \(\nabla\) and its characteristic classes. If \(g = f + d^3 \phi\) it follows there is a canonical isomorphism
\[ L(f^\alpha) \cong L(g^\alpha) \]
of Lie-Rinehart algebras, hence we may for any cohomology class \(c := \overline{f^\alpha} \in H^2(L, A)\) define \(L(c) := L(f^\alpha)\). We get for any \(\psi\)-connection \(\nabla\) a family of connections
\[ \rho_c : L(c) \to \text{End}_k(E) \]
parametrized by the cohomology group \(H^2(L, A)\). If \(B := \text{Sym}_k^*(H^2(L, A)^*)\) we get in a canonical way a connection
\[ q^*(\rho_c) : q^*L(c) \to \text{End}_B(B \otimes_k E) \]
On \(B \otimes_k A\). The element \(c\) gives in a canonical way a \(k\)-rational point \(x(c) \in \text{Spec}(B)(k)\). Hence we may view \(\rho_c\) as the restriction of \(q^*(\rho_c)\) to the fiber \(p^{-1}(x(c))\) where \(x(c) \in \text{Spec}(B)\). Here the maps \(p, q\) are the canonical maps \(q : A \to B \otimes_k A\) and \(p : B \to B \otimes_k A\). We may ask if there is a globally defined \(D(B \otimes_k A, g)\)-Lie algebra
\[ \beta : T \to D^1(B \otimes_k A, g) \]
for some 2-cocycle \(g \in Z^2(Der_k(B \otimes_k A), B \otimes_k A)\) with \(T_{p^{-1}(x(c))} = L(c)\).

**Example 2.21. The curvature of a \(\psi\)-connection with \(\psi = 1\).**

The map \(R_{\rho \alpha}(x, y)\) from Lemma 2.14 is not in \(\text{End}_A(E)\) in general. We always have \(R_{\rho \alpha}(x, y) \in \text{End}_k(E)\). In general it follows we get a map
\[ \nabla := \rho \circ i : L \to \text{End}_k(E) \]
and \(R_{\nabla}(x, y) = R_{\rho \alpha}(x, y) \in \text{End}_A(E)\) if \(\rho(z) = \psi = 1\). In this case \(\nabla\) is an ordinary connection.
3. Functorial properties of universal rings of D-Lie algebras

In this section we construct for any D-Lie algebra $\tilde{L}$ two associative rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$ with the property that there are exact equivalences of categories

\begin{align}
F_1 : \text{Mod}(\tilde{L}, Id) & \cong \text{Mod}(U^\otimes(\tilde{L})) \\
F_2 : \text{Conn}(\tilde{L}, Id) & \cong \text{Mod}(U^\rho(\tilde{L}))
\end{align}

such that $F_1$ and $F_2$ preserve injective and projective objects. The categories $\text{Mod}(\tilde{L}, Id)$ and $\text{Conn}(\tilde{L}, Id)$ are categories of non-flat connections and the rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$ are non-Noetherian in general. Hence the rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$ may be viewed as universal enveloping algebras for non-flat connections. The rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$ contain 2-sided ideals $I^\otimes$ and $I^\rho$ with the property that the quotient rings $\tilde{U}^\otimes(\tilde{L})$ and $\tilde{U}^\rho(\tilde{L})$ are almost commutative rings. If $A$ is Noetherian and $\tilde{L}$ is a finitely generated left $A$-module it follows $\tilde{U}^\otimes(\tilde{L})$ and $\tilde{U}^\rho(\tilde{L})$ are Noetherian (see Theorem 3.17 and Theorem 3.18). We use $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$ to construct cohomology and homology groups for non-flat connections. This was previously done for flat connections.

Let in the following $(\tilde{L}, \alpha, \pi, [\cdot,\cdot], D)$ be a D-Lie algebra. A connection on $\tilde{L}$ is by Definition 2.3 a $B := A \otimes_k A$-linear map

$$\rho : \tilde{L} \to \text{End}_k(E)$$

where $E$ is a left $A$-module.

**Definition 3.1.** Let $T^i_k(\tilde{L}) := \tilde{L} \otimes_k \cdots \otimes_k \tilde{L} = \tilde{L}^\otimes i$ be the tensor product of $\tilde{L}$ with itself $j$ times over the ring $k$. Let $\tilde{T}_k(\tilde{L}) := \oplus_{i \geq 0} T^i_k(\tilde{L})$ be the tensor algebra of $\tilde{L}$ over $k$. Let $T^i_k(\tilde{L})^j := \oplus_{i \geq j} T^i_k(\tilde{L})$ for an integer $j \geq 1$. Let $\tilde{T}_k(\tilde{L})_j := \oplus_{i=1}^j T^i_k(\tilde{L})$.

Note: It follows $T^i_k(\tilde{L})^j \subseteq T^i_k(\tilde{L})$ is a 2-sided ideal for every integer $j \geq 1$. There is moreover a filtration

$$T^i_k(\tilde{L})_1 \subseteq \cdots \subseteq T^i_k(\tilde{L})_j \subseteq T^i_k(\tilde{L})^1$$

that is compatible with the multiplication on $T^i_k(\tilde{L})^1$.

**Lemma 3.2.** Let $R$ be an associative $k$-algebra where $k$ is in the center of $R$. For each $k$-linear map $\rho : \tilde{L} \to R$ there is a canonical map

$$\rho^i : T^i_k(\tilde{L}) \to R$$

defined by

$$\rho^i(u_1 \otimes \cdots \otimes u_i) := \rho(u_1) \cdots \rho(u_i).$$

The abelian group $T^*_{k}(\tilde{L})$ is an associative $k$-algebra with $k$ in its center. There is a functorial equality of sets

$$\text{Hom}_k(\tilde{L}, R) \cong \text{Hom}_{k-\text{alg}}(T^*_{k}(\tilde{L}), R).$$

**Proof.** The proof is straight forward and is left to the reader. \qed

Recall that $\text{Mod}(\tilde{L})$ (resp. $\text{Mod}(L(f^\alpha))$) denote the categories of $\tilde{L}$-connections and morphisms (resp. $L(f^\alpha)$-connections and morphisms) and let for an associative ring $R$, $\text{Mod}(R)$ denote the category of left $R$-modules and maps of $R$-modules.
Note: In [7], Definition A.7 we defined the universal algebra \( U^\text{ua}(L) \) of a Lie-Rinehart algebra \((L, \alpha)\) using the tensor algebra \( T_k(L(f^\alpha)) \) where \( L(f^\alpha) \) was the abelian extension of \( L \) with the 2-cocycle \( f^\alpha \in \mathbb{Z}^2(L, A) \). Definition A.7 in [7] is not correct: The ring \( B := A \otimes_k A \) is not in the center of the associative ring \( \text{End}_k(E) \) where \( E \) is any \( A \)-module, and this is needed for Definition A.7 to make sense. In the following give a correct construction of the universal ring for any \( D \)-Lie algebra.

There is an inclusion of abelian groups
\[
T^*_k(\tilde{L})^1 := \tilde{L} \oplus \tilde{L} \otimes A^2 \oplus \cdots \oplus \tilde{L} \otimes_k \cdots \subseteq T^*_k(\tilde{L})
\]
and the multiplication on \( T^*_k(\tilde{L}) \) induces a multiplication on \( T^*_k(\tilde{L})^1 \) making \( T^*_k(\tilde{L})^1 \) an associative non-unital ring. Define the following two 2-sided ideals in the associative non-unital ring \( T^*_k(\tilde{L})^1 \):
\[
J_1 := \{ au - (aD) \otimes u \text{ and } vb - v \otimes (bD) : \text{ such that } a, b \in A \text{ and } u, v \in \tilde{L} \}
\]
and
\[
J_2 := \{ au - (aD) \otimes u \text{ and } v \otimes (bD) - (bD) \otimes v - \tilde{\pi}(v)(bD) : \text{ such that } a, b \in A \text{ and } u, v \in \tilde{L} \}
\]

**Definition 3.3.** Let \( U^\otimes(\tilde{L}) := \tilde{T}^*_k(\tilde{L})^1 / J_1 \) and \( U^\rho(\tilde{L}) := \tilde{T}^*_k(\tilde{L})^1 / J_2 \). When we speak of the universal ring of the \( D \)-Lie algebra \( \tilde{L} \) we refer to \( U^\otimes(\tilde{L}) \) or \( U^\rho(\tilde{L}) \).

**Lemma 3.4.** Let \( (\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D) \) be a \( D^1(A, f) \)-Lie algebra. The abelian groups \( U^\otimes(\tilde{L}) \) and \( U^\rho(\tilde{L}) \) are associative unital rings with the element \( \overline{D} := 1 \) as unit.

**Proof.** The abelian subgroup \( T^*_k(\tilde{L})^1 \subseteq T^*_k(\tilde{L}) \) inherits a multiplication which is associative and distributive over addition. Passing to the quotient \( U^\rho(\tilde{L}) \) we get the following. Let \( a \in A \), \( D \in \tilde{L} \) be the central element and let \( u \in \tilde{L} \). Write \( \bullet \) for the multiplication in \( U^\rho(\tilde{L}) \). We get
\[
(aD) \bullet u := au
\]
hence when \( a = 1 \) we get
\[
(1D) \bullet u := 1u = u
\]
hence \( D \) is a left identity. We get
\[
u \bullet (1D) = (1D) \bullet u + \tilde{\pi}(u)(1D) = (1D) \bullet u = u
\]
hence \( D \) is a left and right identity for the multiplication and it follows \( U^\rho(\tilde{L}) \) is an associative unital ring. A similar argument proves that \( U^\otimes(\tilde{L}) \) is an associative unital ring and the Lemma follows. \( \square \)

**Theorem 3.5.** Let \( \text{Rings} \) denote the category of associative unital rings and morphisms. Definition 3.3 gives rise to two covariant functors
\[
(3.5.1) \quad U^\otimes : D^1(A, f) \text{-Lie} \to \text{Rings}
\]
\[
(3.5.2) \quad U^\rho : D^1(A, f) \text{-Lie} \to \text{Rings}.
\]

**Proof.** Given a map of \( D \)-Lie algebras \( \phi : \tilde{L} \to \tilde{L}' \) define the following map
\[
T(\phi) : T^*_k(\tilde{L})^1 \to T^*_k(\tilde{L}')^1
\]
by
\[
T(\phi)(u_1 \otimes \cdots \otimes u_i) := \phi(u_1) \otimes \cdots \otimes \phi(u_i) \in \tilde{L}'^\otimes_k.
\]
It follows
\[ T(\phi)(au - (aD) \otimes u) = a\phi(u) - a\phi(D) \otimes \phi(u) = a\phi(u) - (aD') \otimes \phi(u) \]
and
\[ T(\phi)(ua - u \otimes (Da)) = \phi(u)a - \phi(u) \otimes \phi(D)a = \phi(u)a - \phi(u) \otimes (D'a) \]
hence \( T(\phi) \) induce a canonical map of rings
\[ U^\otimes(\phi) : U^\otimes(\bar{L}) \to U^\otimes(\bar{L}). \]
If \( \phi, \psi \) are maps in \( D^1(A, f) \) it follows \( U^\otimes(\phi \circ \psi) = U^\otimes(\phi) \circ U^\otimes(\psi) \) hence \( U^\otimes(\phi) \) define by Lemma 3.4 a covariant functor as claimed. A similar result holds for \( U^\phi(\phi) \) and the Theorem follows. \[ \square \]

**Lemma 3.6.** Let \( A \) be a not necessarily unital associative ring and let \( I, J \) be 2-sided ideals in \( A \). Let \( p_I : A \to A/I \) and \( p_J : A \to A/J \) be the canonical projection maps. There is a canonical isomorphism
\[ A/I/p_I(J) \cong A/J/p_J(I) \]
of associative rings.

**Proof.** Let \( p_I : A \to A/I \) be the canonical projection map. It follows \( p_I^{-1}(p_I(J)) = I + J \). Clearly \( I + J \subseteq p_I^{-1}(p(J)) \). Assume \( x \in p_I^{-1}(p(J)) \). It follows \( p_I(x) \in p_I(J) \). Hence there is an element \( y \in J \) with \( p_I(x) = p_I(y) \) and it follows \( p_I(x - y) = 0 \) hence \( x - y \in ker(p_I) = I \). It follows \( x = y + u \) with \( u \in I \) and hence \( x \in I + J \). It follows we get a canonical isomorphism
\[ A/I + J \cong A/I/p_I(J) \]
and this gives rise to a canonical isomorphism
\[ A/I/p_I(J) \cong A/I + J \cong A/J/p_J(I) \]
of associative rings. The Lemma follows. \[ \square \]

**Lemma 3.7.** Let \( (U, U_i) \) be an almost commutative ring and let \( I \subseteq U \) be a 2-sided ideal. Let \( V := U/I \) be the quotient. It follows \( V \) is an almost commutative ring.

**Proof.** There is a canonical projection map
\[ p : U \to V \]
and we define a filtration on \( V \) with \( V_i := p(U_i) \). We get an increasing filtration \( V_i \) on \( V \) and it follows \( Gr(V) \) is commutative: If \( x \in V_i := p(U_i) \) and \( y \in V_j := p(U_j) \) with \( x = p(u), y = p(v) \) it follows \( uv - vu \in U_{i+j-1} \) hence \( xy - yx \in V_{i+j-1} \). The Lemma follows. \[ \square \]

We state a general result on properties of modules on and associative ring \( A \). Note: The following holds for non-unital rings as well.

**Lemma 3.8.** Let \( A \) be an associative unital ring and let \( M' \subseteq M \) be a submodule with \( p : M \to M/M' \) the canonical projection map. Assume \( N_1 \subseteq N_2 \subseteq M \) are sub
modules. The module \(M\) is Noetherian if and only if all sub modules \(M' \subseteq M\) are finitely generated.

(3.8.1) If \(p(N_1) = p(N_2)\) and \(N_1 \cap M' = N_2 \cap M'\) it follows \(N_1 = N_2\).

(3.8.2) If \(M'\) and \(M/\!\!/M'\) are Noetherian it follows \(M\) is Noetherian.

(3.8.3) If \(M \neq 0\) is Noetherian it follows \(A/\text{ann}_A(M)\) is a Noetherian ring.

Proof. In [14] Theorem 3.1 and 3.5 the Lemma is stated and proved for commutative unital rings. Note that the proof in [14] is valid for an arbitrary associative non-unital ring.

Consider the following 2-sided ideal in \(T_k^+(\tilde{L})^1\):

\[ I := \{ u \otimes v - v \otimes u - [u, v] : \text{such that } u, v \in \tilde{L} \} \subseteq T_k^+(\tilde{L})^1. \]

Let \(I_i := T_k^+(\tilde{L})^1 \cap I\). It follows \(I_i \subseteq I\) is a filtration of the 2-sided ideal \(I\) that is compatible with the multiplication on \(T_k^+(\tilde{L})^1\).

Lemma 3.9. Let \(T := T_k^+(\tilde{L})^1\) and let \(T_j := T_k^+(\tilde{L})\). For all \(i, j \geq 1\) let \(x := x_1 \otimes \cdots \otimes x_j, y := y_1 \otimes \cdots \otimes y_j\) with \(x_a, y_b \in \tilde{L}\) for all \(1 \leq a \leq i\) and \(1 \leq b \leq j\). The following holds: We may write

\[ x \otimes y - y \otimes x = \eta + \omega \]

with \(\eta \in T_{i+j-1}\) and \(\omega \in I_{i+j}\).

Proof. Assume \(i = j = 1\). We get

\[ x \otimes y - y \otimes x = [x, y] + (x \otimes y - y \otimes x - [x, y]) = \eta + \omega \]

with \(\eta = [x, y]\) and \(\omega = x \otimes y - y \otimes x - [x, y]\). Here \(\eta \in T_1\) and \(\omega \in I_2\). Hence the claim is true for \(i = j = 1\). Assume the claim is true for \(i + j \leq k\) with \(k \geq 1\) an integer. Assume \(i + j = k + 1\). Let

\[ x := x_1 \otimes \cdots \otimes x_i \in T_i \]

and

\[ y := y_1 \otimes \cdots \otimes y_j \in T_j. \]

We get by the induction hypothesis

\[ x \otimes y = x \otimes y_1 \otimes \cdots \otimes y_{j-1} \otimes y_j = (y_1 \otimes \cdots \otimes y_{j-1} \otimes x + \eta_1 + \omega_1) \otimes y_j = y_1 \otimes \cdots \otimes y_{j-1} \otimes x \otimes y_j + \eta_1 \otimes y_j + \omega_1 \otimes y_j \]

with \(\eta_1 \otimes y_j \in T_{i+j-1}\) and \(\omega_1 \otimes y_j \in I_{i+j}\). Again by induction we get

\[ y_1 \otimes \cdots \otimes y_{j-1} \otimes x \otimes y_j = y_j \otimes x + \eta_2 + \omega_2 \]

with \(y_1 \otimes \cdots \otimes y_{j-1} \otimes \eta_2 \in T_{i+j-1}\) and \(y_1 \otimes \cdots \otimes y_{j-1} \otimes \omega_2 \in I_{i+j}\). We get

\[ x \otimes y = y \otimes x + \eta + \omega \]

with

\[ \eta = y_1 \otimes \cdots \otimes y_{j-1} \otimes \eta_2 + \eta_1 \otimes y_j \in T_{i+j-1} \]

and

\[ y_1 \otimes \cdots \otimes y_{j-1} \otimes \omega_2 + \omega_1 \otimes y_j \in I_{i+j}. \]
The Lemma follows. \hfill \Box

**Lemma 3.10.** Use the notation in Lemma 3.9 For any elements \( u \in T_i, v \in T_j \) we may write

\[
u \otimes v - v \otimes u = \eta + \omega
\]

with \( \eta \in T_{i+j-1} \) and \( \omega \in I_{i+j} \).

**Proof.** We may write \( u = x_1 + x \) with \( x_1 \in T_{i-1} \) and \( x \in \tilde{L}^\otimes \) and similar \( v = y_1 + y \) with \( y_1 \in T_{j-1}, y \in \tilde{L}^\otimes \). We get

\[
u \otimes v - v \otimes u = (x_1 + x) \otimes (y_1 + y) - (y_1 + y) \otimes (x_1 + x) =
\]

\[
\begin{align*}
x_1 \otimes y_1 + x_1 \otimes y + x \otimes y_1 + x \otimes y - (y_1 \otimes x_1 + y_1 \otimes x + y \otimes x_1 + y \otimes x) = \\
x_1 \otimes y_1 + x_1 \otimes y + x \otimes y_1 - (y_1 \otimes x_1 + y_1 \otimes x + y \otimes x_1) + x \otimes y - y \otimes x =
\end{align*}
\]

with

\[
z := x_1 \otimes y_1 + x_1 \otimes y + x \otimes y_1 - (y_1 \otimes x_1 + y_1 \otimes x + y \otimes x_1) \in T_{i+j-1}.
\]

By Lemma 3.9 we get

\[
x \otimes y - y \otimes x = \eta + \omega
\]

with \( \eta \in T_{i+j-1} \) and \( \omega \in I_{i+j} \). It follows

\[
u \otimes v - v \otimes u = \eta + z + \omega
\]

where \( \eta + z \in T_{i+j-1} \) and \( \omega \in I_{i+j} \) and the Lemma follows. \hfill \Box

**Definition 3.11.** Use the notation in Lemma 3.9. Let \( I \subseteq T \) be the 2-sided ideal

\[
I := \{ u \otimes v - v \otimes u - [u, v] : u, v \in \tilde{L} \}
\]

Define \( U(\tilde{L}) := T / I = T_k^*(\tilde{L})^2 / I \). Let \( p : T \to U(\tilde{L}) \) be the canonical projection map and let \( U_i (\tilde{L}) := p( T_k^*(\tilde{L})_i ) \) for \( i \geq 1 \).

There are canonical projection maps \( p^\otimes : T \to U^\otimes (\tilde{L}) \) and \( p^\rho : T \to U^\rho (\tilde{L}) \). Define

\[
\tilde{U}^\otimes (\tilde{L}) := U^\otimes (\tilde{L}) / p^\otimes (I)
\]

and

\[
\tilde{U}^\rho (\tilde{L}) := U^\rho (\tilde{L}) / p^\rho (I).
\]

Since \( \tilde{U}^\otimes (\tilde{L}) \) and \( \tilde{U}^\rho (\tilde{L}) \) are quotients of \( T \) we get filtrations \( \tilde{U}_i^\otimes (\tilde{L}) \subseteq U^\otimes (\tilde{L}) \) and \( \tilde{U}_i^\rho (\tilde{L}) \subseteq U^\rho (\tilde{L}) \) for all integers \( i \geq 1 \).

We get a filtration on the associative ring \( U(\tilde{L}) \):

\[
U_1 (\tilde{L}) \subseteq U_2 (\tilde{L}) \subseteq \cdots \subseteq U_i (\tilde{L}) \subseteq U(\tilde{L})
\]

compatible with the multiplication: For any elements \( x \in U_i (\tilde{L}), y \in U_j (\tilde{L}) \) it follows \( xy \in U_{i+j} (\tilde{L}) \). A similar property holds for the filtrations \( \tilde{U}_i^\otimes (\tilde{L}) \) and \( \tilde{U}_i^\rho (\tilde{L}) \).

**Proposition 3.12.** The associative ring \( U(\tilde{L}) \) is an almost commutative ring.
Proof. Use the notation from Lemma 3.9. There is a canonical projection map
\[ p : T \to U(\hat{L}) \]
with \( U_i(\hat{L}) := p(T_i) \). Let \( x \in U_i(\hat{L}), y \in U_j(\hat{L}) \) with \( x = p(u), y = p(v) \). It follows from Lemma 3.10 we may write
\[ u \otimes v - v \otimes u = \eta + \omega \]
with \( \eta \in T_{i+j-1}, \omega \in I_{i+j} \). It follows \( xy -yx = p(u \otimes v - v \otimes u) = p(\eta + \omega) = p(\eta) \in U_{i+j-1}(\hat{L}) \) hence \( U(\hat{L}) \) is almost commutative. The Proposition follows. \( \square \)

**Corollary 3.13.** The rings \( \tilde{U}^{\otimes}(\hat{L}) \) and \( \tilde{U}^\rho(\hat{L}) \) are almost commutative associative unital rings.

Proof. By Lemma 3.9 we may do the following: There is by definition inclusions \( I, J_1 \subseteq T_k^*(\hat{L})^1 \) and there is a canonical quotient map
\[ p^\otimes : T_k^*(\hat{L})^1 \to U^\otimes(\hat{L}). \]
By definition we have
\[ \tilde{U}^{\otimes}(\hat{L}) := U^{\otimes}(\hat{L})/p^\otimes(I). \]
By Lemma 3.9 we may write
\[ U^{\otimes}(\hat{L}) \cong U(\hat{L})/p(J_1) \]
where
\[ p : T_k^*(\hat{L})^1 \to U(\hat{L}) \]
is the canonical projection map. It follows \( \tilde{U}^{\otimes}(\hat{L}) \) is a quotient of \( U(\hat{L}) \) which by Proposition 3.12 is almost commutative. It follows \( \tilde{U}^{\otimes}(\hat{L}) \) is almost commutative. A similar argument shows \( \tilde{U}^\rho(\hat{L}) \) is almost commutative. The rings \( \tilde{U}^{\otimes}(\hat{L}) \) and \( \tilde{U}^\rho(\hat{L}) \) are quotients of associative unital rings by 2-sided ideals. Hence it follows \( \tilde{U}^{\otimes}(\hat{L}) \) and \( \tilde{U}^\rho(\hat{L}) \) are almost commutative associative unital rings. The Corollary follows. \( \square \)

Let in the following \( \hat{L} := A\{u_1, \ldots, u_n\} \) be a finite generating set of the D-Lie algebra \( \hat{L} \) as left \( A \)-module. Make the following definitions:

**Definition 3.14.** Let \( B_i := A\{u_{j(1)} \otimes \cdots \otimes u_{j(i)} \} \) such that \( u_{j(a)} \in \{u_1, \ldots, u_n\} \). Let \( B^i := B_1 \oplus \cdots \oplus B_i \subseteq T_k^*(\hat{L})^1 \).

**Lemma 3.15.** Let \((\hat{L}, \alpha, \pi, [\cdot, \cdot], D)\) be a D-Lie algebra with generating set \( \{u_1, \ldots, u_n\} \) as left \( A \)-module. The following holds: For any element \( x \in T_k^*(\hat{L})_i \) we may write \( x = x_1 + x_2 \) where \( x_1 \in B^i \subseteq T_k^*(\hat{L})_i \) and \( x_2 \in I_i \subseteq T_k^*(\hat{L})_i \).

Proof. The claim is obvious for \( i = 1 \). Let \( i = 2 \) and let \( au_i \otimes bu_j \in \hat{L}^{\otimes_2} \). We get
\[ au_i \otimes bu_j = a(u_i \otimes bu_j - bu_j \otimes u_i - [u_i, bu_j] + bu_j \otimes u_i + [u_i, bu_j]) = abu_j \otimes u_i + a[u_i, bu_j] + a\omega \]
where
\[ \omega := u_i \otimes bu_j - bu_j \otimes u_i - [u_i, bu_j] \in I_2. \]
Define \( v_1 := a[u_i, bu_j] \in \hat{L} \) and \( x_2 := a\omega \in I_2 \). We may write \( v_1 = \sum c_i u_i \) and define \( x_1 := abu_j \otimes u_i + v_1 \in B^2 \). We get
\[ au_i \otimes bu_j = x_1 + x_2 \]
with $x_1 \in B^2$ and $x_2 \in I_2$ and the claim is true for $i = 2$. Assume the claim is true for $x \in T_k^*(\hat{L})_{i-1}$. Let $x \in T_k^*(\hat{L})_i$, we may write $x = u + x_1$ with $u \in T_k^*(\hat{L})_{i-1}$ and $x_1 \in \hat{L} \otimes v$. By induction we get $u = u_1 + u_2$ where $u_1 \in B^{i-1}$ and $u_2 \in I_{i-1} \subseteq I_i$. we may write $x_1$ as a sum of elements on the form $a_1v_1 \otimes \cdots a_lv_i$ with $a_j \in A$ and $v_j \in \{v_1, \ldots, v_n\}$. We get

$$a_{i-1}v_{i-1} \otimes a_iv_i = a_{i-1}(v_{i-1} \otimes a_iv_i + a_iv_i \otimes v_{i-1} - [v_{i-1}, a_iv_i]) +$$

$$a_{i-1}(a_iv_i \otimes v_{i-1} + [v_{i-1}, a_iv_i]) =$$

$$a_{i-1}a_iv_1 \otimes v_{i-1} + a_{i-1}v_1 + a_{i-1}[v_{i-1}, a_iv_i]$$

with

$$w_i := v_{i-1} \otimes a_iv_i + a_iv_i \otimes v_{i-1} - [v_{i-1}, a_iv_i] \in I_2.$$ 

It follows

$$x_1 = a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}(a_iv_i \otimes v_{i-1} + w_i + [v_{i-1}, a_iv_i]).$$

Hence

$$x_1 = a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}a_iv_i \otimes v_{i-1} +$$

$$a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}w_i + a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}[v_{i-1}, a_iv - i] =$$

$$a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}a_iv_i \otimes v_{i-1} + \omega_1 + \eta_1$$

where

$$\omega_1 := a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}w_i$$

and

$$\eta_1 := a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}[v_{i-1}, a_iv - i].$$

It follows $\omega_1 \in I_i$ and $\eta_1 \in T_k^*(\hat{L})_{i-1}$. We may write as follows:

$$a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}a_iv_i \otimes v_{i-1} = \eta \otimes v_{i-1}$$

with $\eta' \in T_k^*(\hat{L})_{i-1}$ hence it follows $\eta' = \eta'_1 + \eta'_2$ where $\eta'_1 \in B^{i-1}$ and $\eta'_2 \in I_{i-1}$. It follows

$$a_1v_1 \otimes \cdots a_{i-2}v_{i-2} \otimes a_{i-1}a_iv_i \otimes v_{i-1} = \eta'_1 \otimes v_{i-1} + \eta'_2 \otimes v_{i-1}$$

with $\eta'_1 \otimes v_{i-1} \in B^i$ and $\eta'_2 \otimes v_{i-1} \in I_i$. It follows

$$x_1 = \eta'_1 \otimes v_{i-1} + \eta'_2 \otimes v_{i-1} + \omega_1 + \eta_1$$

with $\eta_1 \in T_k^*(\hat{L})_{i-1}$. we may write $\eta_1 = a_1 + a_2$ with $a_1 \in B^{i-1}$ and $a_2 \in I_{i-2}$. It follows

$$x_1 = \eta + \omega$$

where

$$\eta := \eta'_1 \otimes v_{i-1} + a_1 \in B^i$$

and

$$\omega := \eta'_2 \otimes v_{i-1} + a_2 \in I_i.$$ 

For a general $x_1 \in \hat{L} \otimes v^i$ we may do something similar: $x_1 = x'_1 + x'_2$ with $x'_1 \in B^i$ and $x'_2 \in I_i$. It follows

$$x = u_1 + x'_1 + u_2 + x'_2$$

with $u_1 + x'_1 \in B^i$ and $u_2 + x'_2 \in I_i$ and the claim of the Lemma follows. \[\square\]
Corollary 3.16. Let $(\tilde{L}, \tilde{\alpha}, tp, [,], D)$ be a D-Lie algebra where $A$ is a Noetherian ring and where $\tilde{L}$ is finitely generated as left $A$-module. Let $T := T^*_k(\tilde{L})^1$ and let $I := \{u \otimes v - v \otimes u - [u, v] : u, v \in \tilde{L}\} \subseteq T^*_k(\tilde{L})^1$. Let $\rho : T \to U(\tilde{L}) := T/I$. Let $U_i(\tilde{L}) \subseteq U(\tilde{L})$ be the canonical filtration. It follows the quotient $U_i(\tilde{L})/U_{i-1}(\tilde{L})$ is a finitely generated left $A$-module for all $i \geq 1$.

Proof. Let $x \in U_i(\tilde{L})/U_{i-1}(\tilde{L})$ be an element. We may write $x = x_1 + x_2$ where $x_1 = p(u)$ with $u \in B^i$ and $x_2 = p(v)$ with $v \in I_i$. We get

$$x = p(u + v) = p(u) + \sum_{(j(1), \ldots, j(i))} a_{(j(1), \ldots, j(i))} u_{j(1)} \cdots u_{j(i)}$$

hence $U_i(\tilde{L})/U_{i-1}(\tilde{L})$ is generated by the finite set

$$\{u_{j(1)} \cdots u_{j(i)} : j(a) \in \{1, \ldots, n\}\}.$$

The claim follows.

Theorem 3.17. Let $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [,], D)$ be a D-Lie algebra where $A$ is Noetherian and $\tilde{L}$ is finitely generated as left $A$-module. It follows the rings $\tilde{U}^{\otimes}(\tilde{L})$ and $\tilde{U}^{\rho}(\tilde{L})$ are Noetherian rings.

Proof. Let $J_1, J_2 \subseteq T^*_k(\tilde{L})^1$ be the ideals defining the rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$. Let $p : T := T_k(\tilde{L}) \to T/I = U(\tilde{L})$ where $I$ is the 2-sided ideal generated by the elements $u \otimes v - v \otimes u - [u, v]$ for $u, v \in \tilde{L}$. Let $\tilde{U}^{\otimes}(\tilde{L}) \subseteq U(\tilde{L})$ be the canonical filtration for $i \geq 1$. We get since the map $p$ is surjective a canonical surjective map of left $A$-modules

$$U^\otimes(\tilde{L})/U_{i-1}^\otimes(\tilde{L}) \to \tilde{U}^\otimes_i(\tilde{L})/\tilde{U}^\otimes_{i-1}(\tilde{L})$$

and since $U^\otimes_i(\tilde{L})/U_{i-1}^\otimes(\tilde{L})$ is a finitely generated left $A$-module it follows the module $\tilde{U}^\otimes_i(\tilde{L})/\tilde{U}^\otimes_{i-1}(\tilde{L})$ is finitely generated as left $A$-module. The associated graded ring $Gr(\tilde{U}^\otimes(\tilde{L}))$ is generated by $\tilde{U}_1^\otimes(\tilde{L})$ as $A$-algebra. Hence $Gr(\tilde{U}^\otimes(\tilde{L}))$ is a Noetherian ring. By [3], Proposition 1.1.6 it follows $U^\otimes(\tilde{L})$ is a Noetherian ring. A similar argument proves $U^\rho(\tilde{L})$ is a Noetherian ring and the Theorem follows.

Recall the following: Let $\text{Mod}(\tilde{L}, Id)$ denote the category of pairs $(\rho, E)$ where $\rho$ is a $B := A \otimes_k A$-linear map $\rho : \tilde{L} \to \text{End}_k(E)$ such that $\rho(D) = Id_E$ and where morphisms are defined as follows: Given two element $(\rho, E), (\rho', E') \in \text{Mod}(\tilde{L}, Id)$, a morphism $\phi : (\rho, E) \to (\rho', E')$ is an $A$-linear map $\phi : E \to E'$ such that for all elements $u \in \tilde{L}$ it follows $\rho'(u) \circ \phi = \phi \circ \rho(u)$.

Let $\text{Conn}(\tilde{L}, Id)$ denote the category of pairs $(\rho, E)$ where $\rho : \tilde{L} \to \text{End}_k(E)$ is a left $A$-linear map such that the following holds for all $u \in \tilde{L}, a \in A$ and $e \in E$:

$$\rho(u)(ae) = a\rho(u)(e) + \tilde{\pi}(u)(a)e.$$ A morphism $\phi : (\rho, E) \to (\rho', E')$ in $\text{Conn}(\tilde{L}, Id)$ is an $A$-linear map $\phi : E \to E'$ such that for any element $u \in \tilde{L}$ it follows $\rho'(u) \circ \phi = \phi \circ \rho(u)$.

Theorem 3.18. There are exact equivalence of categories

\begin{align*}
(3.18.1) & \quad F_1 : \text{Mod}(\tilde{L}, Id) \cong \text{Mod}(U^\otimes(\tilde{L})) \\
(3.18.2) & \quad F_2 : \text{Conn}(\tilde{L}, Id) \cong \text{Mod}(U^\rho(\tilde{L}))
\end{align*}

with the property that $F_1$ and $F_2$ preserves injective and projective objects.
Proof. Let us prove claim 3.18.1. Let $(\rho, E) \in \text{Mod}(\hat{L}, \text{Id})$ be an object. It follows $\rho$ is a $B := A \otimes_k A$-linear map with $\rho(D) = \text{Id}_E$. Define the following map:

$$T(\rho)^1 : T_k^*(\hat{L})^1 \rightarrow \text{End}_k(E)$$

by

$$T(\rho)^1(u_1 \otimes \cdots \otimes u_i) := \rho(u_1) \circ \cdots \circ \rho(u_i).$$

If

$$x := w(bu - (bD) \otimes u)v$$

with $w, v \in T_k^*(\hat{L})$. We get

$$T(\rho)^1(x) = T(\rho)^1(w)(\rho(bu) - \rho(bD)\rho(u)) T(\rho)^1(v) = T(\rho)(w)(b\rho(u) - b\rho(u)) T(\rho)^1(v) = 0$$

hence $T(\rho)^1(x) = 0$. If

$$y := w(ub - u \otimes (bD))v$$

with $w, v \in T_k^*(\hat{L})$ it follows $T(\rho)^1(y) = 0$, hence the map $T(\rho)^1$ satisfies $T(\rho)^1(x) = 0$ for all $x \in J$. It follows $T(\rho)^1$ defines a canonical map

$$U(\rho) : U^\otimes(\hat{L}) \rightarrow \text{End}_k(E)$$

of associative unital rings since $\rho(D) = \text{Id}_E$. Define the functor $F_1$ by $F_1(\rho, E) := (U(\rho), E) \in \text{Mod}(U^\otimes(\hat{L}))$.

Conversely, given a map of associative unital rings $U : U^\otimes(\hat{L}) \rightarrow \text{End}_k(E)$ we get a canonical map

$$\rho_U : \hat{L} \rightarrow \text{End}_k(E)$$

defined by

$$\rho_U(u) := U(u)$$

for $u \in \hat{L}$. There is a map $\rho_A : A \rightarrow \text{End}_k(E)$ defined by $\rho_A(a) := U(aD)$ and $ae := U(aD)(e)$ defines the $A$-module structure on $E$ for every $e \in E$. We get for all $u, v \in \hat{L}$ the following:

$$\rho_U(u + v) := U(u + v) = U(u) + U(v) = \rho_U(u) + \rho_U(v),$$

$$\rho_U(au) = U(au) = U(aDu) = U(aD)U(u) = a\rho_U(u)$$

and

$$\rho_U(ua) = U(uaD) = U(u)(U(aD)) = U(u)a = \rho_U(u)a$$

hence the map $\rho_U$ is a $B$-linear map $\rho_U : \hat{L} \rightarrow \text{End}_k(E)$. Since $U$ is unital it follows $\rho_U(D) := U(D) = \text{Id}_E$ and hence the pair $(\rho_U, E)$ is an object in $\text{Mod}(\hat{L}, \text{Id})$. We define the inverse functor $G_1 : \text{Mod}(U^\otimes(\hat{L})) \rightarrow \text{Mod}(\hat{L}, \text{Id})$ by $G_1(U, E) := (\rho_U, E)$. Assume $\phi : (\rho, E) \rightarrow (\rho', E')$ is a map in $\text{Mod}(\hat{L}, \text{Id})$. Define $F_1(\phi) := \phi$. Let $x \in U^\otimes(\hat{L})$ be an element where

$$x = \sum_i x_i = \sum_i u_1 \otimes \cdots \otimes u_{d(i)}$$

with $x_i = u_1 \otimes \cdots \otimes u_{d(i)} \in \hat{L}^{\otimes i}$. We get for any element $e \in E$ the following calculation:

$$\phi(x_i e) := \phi(\rho(u_1) \cdots \rho(u_{d(i)}) e) = \rho'(u_1) \cdots \rho'(u_{d(i)}) \phi(e) = x_i \phi(e).$$

Hence

$$\phi(x e) = \phi(\sum_i x_i e) = \sum_i \phi(x_i e) = \sum_i x_i \phi(e) = x \phi(e)$$
hence the map $\phi$ is $U(\tilde{L})$-linear. It follows for two maps $\phi, \psi \in \text{Mod}(\tilde{L}, \text{Id})$ we get $\phi = F_1(\phi) = F_1(\psi) = \psi$ hence the functor $U$ is an injection at the level of maps. we get an inclusion

$$\text{Hom}_{\text{Mod}(L, \text{Id})}(\rho, E) \subseteq \text{Hom}_{U(\tilde{L})}(E, F).$$

Assume $\phi : E \to F$ is an $U(\tilde{L})$-linear map. It follows similarly $G_1(\phi) = \phi$ is a map in $\text{Mod}(\tilde{L}, \text{Id})$ with $F_1(G_1(\phi)) = F_1(\phi) = \phi$ hence there is an equality of sets

$$\text{Hom}_{\text{Mod}(L, \text{Id})}(\rho, E) = \text{Hom}_{U(\tilde{L})}(E, F).$$

It follows the functors $F_1$ and $G_1$ are exact functors: They map exact sequences to exact sequences. One checks $F_1$ and $G_1$ maps projective and injective objects to projective and injective objects and it follows the first claim is proved. A similar proof proves that $F_2$ is an exact equivalence preserving projective and injective objects and the Theorem follows. □

Recall that $\text{Conn}(L, \text{Id})$ denotes the category of pairs $(\rho, E)$ wher $\rho : L \to \text{End}_k(E)$ such that $\rho$ is left $A$-linear and for all $x \in L, a \in A$ and $e \in E$ we have

$$\rho(x)(ae) = a(\rho(x)(e)) + \alpha(x)(a)e.$$

A map of connections $\phi : (\rho, E) \to (\rho', E')$ is a map of left $A$-modules $\phi : E \to E'$ such that for all $x \in L$ the following holds: $\rho'(x) \circ \phi = \phi \circ \rho(x)$. Let $f \in Z^2(L, A)$ be a 2-cocycle and let $L(f) := Az \oplus L$ be the abelian extension of $L$ with $f$. It follows $L(f)$ is a $B := A \otimes_k A$-modules and $k$-Lie algebra with a map $\alpha_f : L(f) \to \text{Der}_k(A)$ defined by $\alpha_f(ax + z) := \alpha(x) \in \text{Der}_k(A)$. Let $\text{Mod}(L(f), \text{Id})$ denote the category of pairs $(\rho, E)$ where $\rho$ is a $B$-linear map $\rho : L(f) \to \text{End}_k(E)$ such that $\rho(z) = \text{Id}_E$. A map $\phi : (\rho, E) \to (\rho', E')$ in $\text{Mod}(\tilde{L}, \text{Id})$ is a map $\phi : E \to E'$ such that for all $x \in L(f)$ the following holds: $\rho'(x) \circ \phi = \phi \circ \rho(x)$.

**Lemma 3.19.** There is an exact equivalence of categories

$$F : \text{Conn}(L, \text{Id}) \to \text{Mod}(L(f), \text{Id})$$

where $F$ preserve injective and projective objects.

**Proof.** Given a connection $(\rho, E)$ define $F(\rho, E) := (\tilde{\rho}, E)$ where $\tilde{\rho}(az + x) := aI + \rho(x) \in \text{End}_k(E)$. Define the inverse functor $G$ by $G(\tilde{\rho}, E) := (\rho \circ i, E)$ where $i : L \to L(f)$ is the canonical injection. The result follows with an argument similar to the one in Theorem 3.15. □

**Definition 3.20.** Let $(\tilde{L}, \tilde{\alpha}, \tilde{\beta}, [, ,])$, $D$ be a D-Lie algebra and let $(L, \alpha)$ be a classical Lie-Rinehart algebra. Let $f \in Z^2(\text{Der}_k(A), A)$ be a 2-cocycle. Let furthermore $(\rho_U, U), (\rho_V, V)$ be objects in $\text{Mod}(\tilde{L}, \text{Id})$ and let $(\rho_W, W), (\rho_Z, Z)$ be objects in $\text{Conn}(L(f^o), \text{Id})$. By Theorem 3.15 there are exact equivalences of categories

$$\text{Mod}(\tilde{L}, \text{Id}) \cong \text{Mod}(U(\tilde{L}))$$

and

$$\text{Conn}(L(f^o), \text{Id}) \cong \text{Mod}(U^o(L(f^o)))$$

preserving injective and projective objects. Since the categories $\text{Mod}(U(\tilde{L}))$ and $\text{Mod}(U^o(L(f^o)))$ have enough injectives we may define the Ext and Tor-groups of $U, V, W, Z$. Let in the following $U(\tilde{L}) := U(\tilde{L})$ and $U^o(L(f^o))$. We may define the groups

$$\text{Ext}_U^i(U, V), \text{Tor}_U^i(U, V).$$
and
\[ \text{Ext}^i_{U^\otimes}(Z, W), \text{Tor}^U_i(Z, W). \]

Let
\[ H_i(\tilde{L}, V) := \text{Tor}^U_i(A, V) \]
and
\[ H^i(\tilde{L}, V) := \text{Ext}^i_{U^\otimes}(A, V). \]

**Example 3.21. Hochschild cohomology of left and right \( U^\otimes(\tilde{L}) \) and \( U^\sigma(\tilde{L}) \)-modules.**

If \( B \) is an algebra over a field or a commutative ring \( R \) such that \( B \) is projective as left \( R \)-module, it follows the Ext and Tor-groups may be calculated by the Hochschild cohomology and homology groups of certain left and right modules. If \((E, \rho)\) and \((E', \rho')\) are left \( U^\otimes(\tilde{L}) \)-modules it follows from [20], Lemma 9.1.9 that
\[ \text{Ext}^i_{U^\otimes(\tilde{L})/k}((E, \rho), (E', \rho')) \cong H^i(U^\otimes(\tilde{L}), \text{Hom}_k((E, \rho), (E', \rho'))) \]
where \( \text{Hom}_k((E, \rho), (E', \rho')) \) is the left and right \( U^\otimes(\tilde{L}) \)-module of \( k \)-linear maps between \((E, \rho)\) and \((E', \rho')\). Here \( H^i \) is the Hochschild cohomology of the \( U^\otimes(\tilde{L}) \)-bimodule \( \text{Hom}_k((E, \rho), (E', \rho')) \). Hence there is an explicit complex calculating the Ext-groups from Definition 3.20.

**Example 3.22. Cohomology and homology of a Noetherian module**

**Definition 3.23.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [,], D)\) be a D-Lie algebra and let \((\rho, E) \in \text{Mod}(\tilde{L}, \text{Id})\).

Let \( a^\otimes(\rho, E) \subseteq U^\otimes(\tilde{L}) \) be the annihilator ideal of \((\rho, E)\) and define \( U^\otimes_E(\tilde{L}) := U^\otimes(\tilde{L})/a^\otimes(\rho, E) \). Define similarly \( U^\sigma_E(\tilde{L}) := U^\sigma(\tilde{L})/a^\sigma(\rho, E) \) where \( a^\otimes(\rho, E) \) is the annihilator ideal of \((\rho, E)\) as \( U^\otimes(\tilde{L}) \)-module.

**Proposition 3.24.** Assume \( A \) is a Noetherian ring and \((\rho, E) \in \text{Mod}(\tilde{L}, \text{Id})\) a connection with the property that \( E \) is a finitely generated \( A \)-module. It follows \((\rho, E)\) is a Noetherian \( U^\otimes(\tilde{L}) \)-module. The ring \( U^\otimes_E(\tilde{L}) \) is Noetherian.

*Proof.* Let \((\rho', E') \subseteq (\rho, E)\) be an \( U^\otimes(\tilde{L}) \)-sub module. It follows \( E' \subseteq E \) is a sub-\( A \)-module and since \( A \) is Noetherian it follows \( E' \) is a finitely generated \( A \)-module. It follows \((\rho', E')\) is a finitely generated \( U^\otimes(\tilde{L}) \)-module, hence by Lemma 3.2 it follows \((\rho, E)\) is a Noetherian \( U^\otimes(\tilde{L}) \)-module. Again from Lemma 3.2 it follows the ring \( U^\otimes_E(\tilde{L}) := U^\otimes(\tilde{L})/a^\otimes(\rho, E) \) is Noetherian. The Proposition follows.

The rings \( U^\otimes(\tilde{L}) \) and \( U^\sigma(\tilde{L}) \) from Theorem 3.18 are non-Noetherian in general but from Proposition 3.24 we can in many cases reduce to the Noetherian case when studying cohomology and homology of a connection \((\rho, E)\) over a Noetherian ring \( A \) when \( E \) is a finitely generated \( A \)-module, We may study \((\rho, E)\) as \( U^\otimes_E(\tilde{L}) \) or \( U^\sigma_E(\tilde{L}) \)-module and the rings \( U^\otimes_E(\tilde{L}) \) and \( U^\sigma_E(\tilde{L}) \) are by Proposition 3.24 Noetherian rings. There is moreover an explicit complex calculating the Ext and Tor-groups in Definition 3.20.

**Note:** If \( A \) is Noetherian and we are given a finite family \((\rho_i, E_i)_{i \in I}\) of \( \tilde{L} \)-connections with \( E_i \) a finitely generated \( A \)-module for all \( i \) and we let \( E := \oplus E_i \) it follows \( \text{ann}(\rho, E) \subseteq \text{ann}(\rho, E_i) \) for all \( i \). The \( A \)-module \( E \) is a finitely generated \( A \)-module hence \( U^\otimes_E(\tilde{L}) \) and \( U^\sigma_E(\tilde{L}) \) are by Proposition 3.24 Noetherian and \( E_i \) are.
left $U^0_E(\tilde{L})$ and $U^l_E(\tilde{L})$-modules for all $i$. Hence when studying cohomology or homology a finite set of connections $E_i$ we may always work over a fixed Noetherian ring $\mathcal{U}$.

**Example 3.25.** Differential operators, connections and projective modules.

Recall the notion of a projective basis for a finitely generated projective $A$-module $E$, where $A$ is a commutative unital ring over a base ring $k$ (see [9]). A set of $r$ elements $x_1, ..., x_r \in E^*$ and $e_1, ..., e_r \in E$ satisfying the formula

$$\sum_i x_i(e_i)e_i = e$$

for all $e \in E$ is a projective basis. The $A$-module $E$ has a projective basis if and only if it is finitely generated and projective as $A$-module. If $A$ is a finitely generated algebra over a field and $E$ is a finitely generated and projective $A$-module, a projective basis for $E$ may be calculated using a Gröbner basis.

**Definition 3.26.** Let $k \to A$ be an arbitrary map of unital commutative rings and let $E$ be a left $A$-module. Define $Diff^0(A) := A$ and $Diff^1(A) := End_A(E)$. An operator $D \in End_k(E)$ is a differential operator of order $\leq l$ if for all sequences of $l + 1$ elements $a_1, ..., a_{l+1} \in A$ the following holds:

$$[\cdots [D, a_1Id_E] \cdots , a_{l+1}Id_E] = 0.$$ 

Here $[,]$ is the Lie product on $End_k(E)$. We let $Diff^l_k(E)$ (or for short $Diff^l(E)$) be the set of all differential operators of order $\leq l$. Let $Diff(E) := \cup_{l \geq 0} Diff^l(E)$.

It follows the $k$-vector space $Diff(E)$ has a filtration of $k$-vector spaces

$$Diff^0(E) = End_A(E) \subseteq Diff^1(E) \subseteq \cdots \subseteq Diff^l(E) \subseteq \cdots \subseteq Diff(E).$$

The composition of operators gives for all integers $l, l' \geq 0$ an associative product

$$Diff^l(E) \times Diff^{l'}(E) \to Diff^{l+l'}(E)$$

making $Diff(E)$ into an associative ring. Similar properties hold for the ring $A$: $Diff(A)$ is an associative ring containing $A$. The ring $A$ does not lie in the centre of $Diff(A)$. The product on $Diff(A)$ respects the filtration, hence we get a well defined product

$$Diff^l(A) \times Diff^{l'}(A) \to Diff^{l+l'}(A).$$

It follows $Diff^l(A)$ and $Diff^l(E)$ are left and right $A$-modules. It follows $Diff^l(A)$ and $Diff^l(E)$ are left $A \otimes_k A$-modules for all integers $l \geq 0$. Given an operator $D \in Diff^l(A)$ or $Diff^l(E)$ the action of $A \otimes_k A$ is defined as follows:

$$a \otimes b, D := \phi_a \circ D \circ \phi_b$$

where $\phi_a$ is multiplication with the element $a$.

Define the following map:

$$\rho : Diff_k(A) \to End_k(E)$$

by

$$\rho(D)(e) := \sum_i D(x_i(e))e_i.$$ 

**Lemma 3.27.** The map $\rho$ is a map of $A \otimes_k A$-modules. It induces a map

$$\rho^l : Diff^l(A) \to Diff^l(E).$$
There is an explicit formula for the curvature $\text{R}_{\phi}$ of an idempotent $\phi$. One checks $\rho(D) \in \text{End}_k(E)$. Moreover $\rho(D + D') = \rho(D) + \rho(D')$. Let $a \otimes b \in A \otimes_k A$. We get
\[
\rho(a \otimes b.D)(e) := \sum_i aD(bx_i(e))e_i = \\
\sum_i aD(x_i(be))e_i = a\rho(D)(be) := (a \otimes b.\rho(D))(e).
\]
Hence $\rho(a \otimes b.D) = a \otimes b.\rho(D)$. It follows $\rho$ gives an $A \otimes_k A$-linear map as claimed. Assume $D \in \text{Diff}^l(A)$. We need to prove that $\rho(D) \in \text{Diff}^l(E)$. Let $a_1, \ldots, a_{l+1} \in A$ be $l + 1$ arbitrary elements. We get
\[
[\cdots[\rho(D), a_1 Id_E]\cdots], a_{l+1} Id_E](e) = \\
\sum_{i}[\cdots[D, a_1 Id_A]\cdots], a_{l+1} Id_A](x_i(e))e_i
\]
and since $D \in \text{Diff}^l(A)$ it follows
\[
[\cdots[\rho(D), a_1 Id_A]\cdots], a_{l+1} Id_A] = 0.
\]
It follows
\[
[\cdots[\rho(D), a_1 Id_E]\cdots], a_{l+1} Id_E](e) = 0
\]
for all $e \in E$. It follows $\rho(D) \in \text{Diff}^l(E)$. The Lemma follows. 

We get for any pair of integers $k, l \geq 0$ a map of $k$-vector spaces
\[
R_{\rho}^{k,l} : \text{Diff}^k(A) \otimes_k \text{Diff}^l(A) \to \text{Diff}^{k+l}(E)
\]
defined by
\[
R_{\rho}^{k,l}(D, D') := \rho^{k+l}(D \circ D') - \rho^{k}(D)\rho^{l}(D').
\]

**Definition 3.28.** Let the map $\rho^1$ from Lemma 3.27 be an $l$-connection on $E$. Let $\rho$ be an $\infty$-connection on $E$. Let $R_{\rho}^{k,l}$ be the $(k, l)$-curvature of $\rho$.

Note: An ordinary connection $\nabla : \text{Der}_k(A) \to \text{Diff}^1(E)$ has curvature
\[
R_{\nabla} \in \text{Hom}_A(\bigwedge^2 \text{Der}_k(A), \text{End}_A(E)).
\]
The $(k, l)$-connection $R_{\rho}^{k,l}$ does not satisfy a similar property.

In the case when $k = l = 1$ we get a map of $A \otimes_k A$-modules
\[
\rho^1 : \text{D}^1(A, 0) \to \text{Diff}^1(E).
\]
We also get a map of left $A$-modules
\[
\nabla : \text{Der}_k(A) \to \text{End}_k(E).
\]
The map $\nabla$ is a connection on $E$.

Note: By the paper [6] it follows the connections $\rho^1$ and $\nabla$ are non-flat in general. There is an explicit formula for the curvature $R_{\phi^1}$ of $\rho^1$ and $\nabla$ in terms of an idempotent $\phi$ for the module $E$. One uses the projective basis $x_i, e_i$ for $E$ to define an idempotent and the formula for the curvature involves the idempotent $\phi$. See Theorem 2.14 in [6] for a proof of the formula and some explicit examples. Given a projective basis $x_1, \ldots, x_r, e_1, \ldots, e_r$ for $E$ we get a surjection $p : A\{u_1, \ldots, u_r\} \to E$. 

**Proof.** The action of $A \otimes_k A$ on $\text{Diff}(E)$ is as follows $a \otimes b \psi := \phi_a \circ \psi \circ \phi_b$ where $\phi_a$ is multiplication with $a$. One checks $\rho(D) \in \text{End}_k(E)$. Moreover $\rho(D + D') = \rho(D) + \rho(D')$. Let $a \otimes b \in A \otimes_k A$. We get
\[
\rho(a \otimes b.D)(e) := \sum_i aD(bx_i(e))e_i = \\
\sum_i aD(x_i(be))e_i = a\rho(D)(be) := (a \otimes b.\rho(D))(e).
\]
defined by \( p(u_i) = e_i \). It follows \( A^r/\ker(p) \cong E \). Define the following matrix \( \phi \in \text{End}_A(A^r) \):
\[
\phi := \begin{pmatrix}
x_1(e_1) & x_1(e_2) & \cdots & x_1(e_r) \\
x_2(e_1) & x_2(e_2) & \cdots & x_2(e_r) \\
\vdots & \vdots & \ddots & \vdots \\
x_r(e_1) & x_r(e_2) & \cdots & x_r(e_r)
\end{pmatrix}
\]

Given two derivations \( \delta, \eta \in \text{Der}_A(A) \) we may consider the matrix \( \delta(\phi) := (\delta(x_i(e_j))) \in \text{End}_A(A^r) \). The Lemma follows. The following holds:

**Theorem 3.29.** The following holds:

\[
(3.29.1) \quad R_\nabla(\delta, \eta) = [\delta(\phi), \eta(\phi)] \in \text{End}_A(E).
\]

**Proof.** For a proof see [1], Theorem 2.14. \( \square \)

From formula (3.29.1) we see that given a projective basis \( x_i, e_j \) it follows the corresponding connection \( \nabla \) is seldom flat since the Lie product \( [\delta(\phi), \eta(\phi)] \) is seldom zero as an element of \( \text{End}_A(E) \).

**Lemma 3.30.** By Lemma 2.19 it follows the map \( \rho^1 : D^1(A, 0) \to \text{Diff}^1(E) \) has curvature \( R_{\rho^1} \) defined by \( R_{\rho^1}(u, v) = \rho^1([u, v]) - [\rho^1(u), \rho^1(v)] \) for \( u, v \in D^1(A, 0) \). It follows
\[
R_{\rho^1}(u, v) = R_{\rho^1}^1(u, v) - R_{\rho^1}^{1,1}(v, u).
\]

**Proof.** We get since \( \rho \) is defined for operators in \( \text{Diff}^2(A) \) the following calculation:
\[
R_{\rho^1}(u, v) = \rho^1([u, v]) - [\rho^1(u), \rho^1(v)] = \rho^2(uv - vu) - (\rho^1(u)\rho^1(v) - \rho^1(v)\rho^1(u)) = \\
\rho^2(uv) - \rho^1(u)\rho^1(v) - (\rho^2(vu) - \rho^1(v)\rho^1(u)) = R_{\rho^1}^1(u, v) - R_{\rho^1}^{1,1}(v, u).
\]

The Lemma follows. \( \square \)

Hence \( R_{\rho} \) is determined by \( R_{\rho^1}^{1,1} \).

**Lemma 3.31.** The map \( \rho \) is a morphism of rings if and only if \( R_{\rho}^{k,l} = 0 \) for all pairs of integers \( k, l \geq 0 \). If \( \rho \) is a map of rings it follows the connection \( R_{\rho} \) is a flat connection.

**Proof.** Assume \( \rho \) is a map of rings. It follows for \( D \in \text{Diff}^k(A), D' \in \text{Diff}^l(A) \) we get
\[
R_{\rho}^{k,l}(D, D') = \rho^k(D)\rho^l(D') - \rho^{k+l}(D \circ D') = \rho(D)\rho(D') - \rho(D \circ D') = 0.
\]

It follows \( R_{\rho}^{k,l} = 0 \) for all \( k, l \). The converse is proved in a similar fashion. If \( \rho \) is a map of rings it follows \( R_{\rho}^{1,1} = 0 \) hence by Lemma 2.19 it follows \( R_{\rho} \) is a flat connection. The Lemma follows. \( \square \)

Note: A map of rings \( \rho : \text{Diff}(A) \to \text{Diff}(E) \) is sometimes referred to as a stratification in the literature (see [1]). Hence the maps \( R_{\rho}^{k,l} \) are obstructions for \( (E, \rho) \) to be a stratification. The following may happen: For a given choice of projective basis \( x_i, e_j \) for \( E \), it might be the corresponding \( \infty \)-connection \( (E, \rho) \) is not a stratification. It might still be there exists another projective basis \( x_i', e_j' \) for \( E \) such that the corresponding \( \infty \)-connection \( (E, \rho') \) is a stratification. To determine
if a module $E$ has a stratification is a difficult problem in general. It is well known that if $A$ is a finitely generated regular algebra over a field $k$ of characteristic zero and $E$ is a coherent $A$-module it is necessary that $E$ is projective for $E$ to have a stratification $\rho$. Given a stratification $\rho$ on a coherent module $E$ we get a connection $\nabla : \text{Der}_k(A) \to \text{End}_k(E)$ and one uses the connection $\nabla$ to prove that $E$ is locally free, hence projective. The proof does not use the flatness of the connection $\nabla$.

Note: One would like to realize the category of $l$-connections $(E,\rho^l)$ and morphisms of $l$-connections as a module category over "some universal enveloping algebra" $U^{la}(\text{Diff}^l(A))$ of the module of $l$th order differential operators $\text{Diff}^l(A)$ as done for 1-connections, and to define the notion of cohomology and homology of an $l$-connection $(E,\rho^l)$ as done in Definition 3.20 for 1-connections.

When $A$ is a finitely generated and regular commutative ring over a field $k$ of characteristic zero, it follows $\text{Diff}(A)$ is generated by $D^1(A,0)$: Every higher order differential operator $D$ is a sum of products of first order differential operators. Hence to give a ring homomorphism $\rho : \text{Diff}(A) \to \text{Diff}(E)$ is equivalent to give a flat connection $\nabla : D^1(A,0) \to \text{Diff}^1(E)$. This property does not hold in characteristic $p > 0$.

**Example 3.32. Rings of differential operators, annihilator ideals and polynomial relations between Chern classes of a connection.**

The universal algebras $U^\otimes(L(f^\alpha))$ and $U^\rho(L(f^\alpha))$ may have applications in the theory of characteristic classes. Recall the following results from from [7]:

**Lemma 3.33.** Let $A$ be a commutative ring containing the field of rational numbers $k$ and let $(\rho,E)$ be an $L$-connection of curvature type $f$ where $f \in Z^2(L,A)$ is a 2-cocycle and where $E$ is a projective $A$-module of rank $rk(E)$. The following formula holds in $\text{C}^{2k}(L,\text{End}_A(E))$:

$$R^k_\rho(x_1,\ldots,x_{2k}) = f^k(x_1,\ldots,x_{2k})I_E$$

where $I_E \in \text{End}_A(E)$ is the identity endomorphism of $E$.

**Proof.** See [2], Lemma 5.14. \qed

The graded ring $\text{H}^2(L,A) := \bigoplus_{k=0,\ldots,\ell} \text{H}^{2k}(L,A)$ is a commutative ring. Let $k[x_1,\ldots,x_l]$ be the polynomial ring in the independent variables $x_1,\ldots,x_l$. Given a polynomial $P(x_1,\ldots,x_l)$ and a connection $(\rho,E)$ where $E$ is a projective $A$-module of finite rank we may evaluate the polynomial $P$ in the Chern classes $c_k(E)$ to get a cohomology class

$$P(c_1(E),c_2(E),\ldots,c_l(E)) \in \text{H}^{2k}(L,A).$$

Given a set of connections $(\rho_i,E_i)_{i=1,\ldots,k}$ where $E_i$ is a finitely generated and projective $A$-module we may for any polynomial $P(x_1,\ldots,x_k) \in \mathbb{Q}[x_1,\ldots,x_k]$ consider the class

$$P(c_1(E_1),\ldots,c_l(E_k)) \in \text{H}^{2k}(L,A).$$

When we vary the polynomial $P$ and the number $k$ of independent variables, we get a subring $R(c_1)$ of the cohomology ring $\text{H}^{2k}(L,A)$ - the subring (over the field of rational numbers) generated by the first Chern classes $c_1(E) \in \text{H}^2(L,A)$ of all finitely generated and projective $A$-modules $E$.

Define for $k \geq 2$ the polynomial
\[ P_k(x_1, ..., x_l) := \text{rk}(E)^{k-1}x_k - x_1^k \in k[x_1, ..., x_l]. \]

Lemma 3.33 has consequences for the Chern class \( c_k(E) \).

**Corollary 3.34.** Let \( A \) be a commutative ring containing the field of rational numbers \( k \) and let \((\rho, E)\) be an \( L\)-connection of curvature type \( f \) where \( f \in Z^2(L, A) \) is a 2-cocycle and where \( E \) is a projective \( A \)-module of rank \( \text{rk}(E) \). The following holds:
\[ P_k(c_1(E), ..., c_l(E)) = 0 \]
for all \( k \geq 2 \).

**Proof.** By Lemma 3.33 we get
\[ c_k(E) = \text{tr}(R^k_\rho) = \text{tr}(f^k I_E) = \text{rk}(E)f^k \in H^2(L, A). \]

By definition
\[ c_1(E) = \text{tr}(R_\rho) = \text{rk}(E)f \in H^2(L, A). \]

It follows
\[ c_1(E)^k = \text{rk}(E)^k f^k \]
hence
\[ P_k(c_1(E), ..., c_l(E)) := \text{rk}(E)^{k-1}c_k(E) - c_1(E)^k = 0 \]
and the Corollary follows. \( \square \)

Hence from Corollary 3.34 the following holds: For a connection \((\rho, E)\) where \( \rho \) has curvature type \( f \) for a 2-cocycle \( f \in Z^2(L, A) \) we get an equality
\[ c_k(E) = \frac{1}{\text{rk}(E)^{k-1}}c_1(E)^k. \]

Hence the \( k \)'th Chern class \( c_k(E) \) is determined by the first Chern class and hence \( c_k(E) \in R(c_1) \) for all \( k \geq 1 \).

Hence the annihilator ideal \( \text{ann}(\rho, E) \) can be used to detect if there are polynomial relations between the Chern classes of \( E \). If the connection \((\rho, E)\) has curvature type \( f \in Z^2(L, A) \), this puts strong conditions on the Chern classes of \( E \): Hence the annihilator ideal \( \text{ann}(\rho, E) \) detects if the Chern class \( c_k(E) \) is interesting from the point of view of Hodge theory. The Hodge conjecture is known for \( H^2(L, A) \) calculates the 2'nd singular cohomology of \( X_C \) with complex coefficients.

**Example 3.35.** 2-sided ideals in the universal ring \( U^\otimes(L(f^\alpha)) \) and Morita equivalence.

Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [,], D)\) be a D-Lie algebra and let \( \rho : \tilde{L} \to \text{End}_k(E) \) be an object in \( \text{Mod}(\tilde{L}, Id) \). There is an equivalence of categories
\[ \text{Mod}(\tilde{L}, Id) \cong \text{Mod}(U^\otimes(\tilde{L})). \]

Let in the following \( f \in Z^2(\text{Der}_k(A), A) \) and let \((L, \alpha)\) be an \( A/k \)-Lie-Rinehart algebra. Let \((L(f^\alpha), \alpha_f, \pi_f, [,], z)\) be the corresponding D-Lie algebra. Assume \((\rho, E)\) is an object in \( \text{Mod}(L(f^\alpha), Id) \) with the property that \( R_\rho(u, v) = 0 \) for all \( u, v \in L(f^\alpha) \). It follows \((\rho, E)\) is an object in \( \text{Mod}(U^\otimes(L(f^\alpha))) \) since \( U^\otimes(L(f^\alpha)) \) is the quotient of \( U^\otimes(L(f^\alpha)) \) by the 2-sided ideal generated by elements on the form \( u \otimes v - v \otimes u - [u, v] \) for \( u, v \in L(f^\alpha) \). The category \( \text{Mod}(L(f^\alpha), Id) \) is equivalent
to the category of $L$-connection of curvature type $f^\alpha$ hence there is an equivalence of categories
\[
\text{Mod}(U^\otimes(L(f^\alpha))) \cong \text{Mod}(U(A, L, f^\alpha)).
\]
Hence the two associative rings $U^\otimes(L(f^\alpha))$ and $U(A, L, f^\alpha)$ are Morita equivalent. They are not isomorphic in general but they both have isomorphic centres equal to the base ring $k$.

**Example 3.36. Families of 2-sided ideals in $U^\rho(L(0))$ and families of connections**

Let consider the abelian extension $L(0)$ of $L$ by $A_z$ and the canonical map $\hat{\alpha} : L(0) \to \text{Der}_k(A)$ defined by $\hat{\alpha}(az + x) := \alpha(x)$. It follows $(L(0), \hat{\alpha})$ is an $A/k$-Lie-Rinehart algebra. There is an equivalence of categories
\[
\text{Conn}(L(0), Id) \cong \text{Mod}(U^\rho(L(0)))
\]
where $\text{Conn}(L(0), Id)$ is the category of connections $\hat{\rho} : L(0) \to \text{End}_k(E)$ such that $\hat{\rho}(z) = \text{Id}_E$. The category $\text{Conn}(L(0), Id)$ is equivalent to the category $\text{Conn}(L)$ of ordinary connections $\rho : L \to \text{End}_k(E)$ and morphisms. We get an equivalence of categories
\[
\text{Conn}(L) \cong \text{Mod}(U^\rho(L(0))).
\]
For any 2-cocycle $f \in Z^2(L, A)$ there is an associated 2-cocycle $\tilde{f} \in Z^2(L(0), A)$ and we may consider the 2-sided ideal
\[
I(f) := \{u \otimes v - v \otimes u - [u, v] - \tilde{f}(u, v)z \text{ such that } u, v \in L(0).\} \subseteq U^\rho(L(0)).
\]
Here $\tilde{f}(u, v) := f(x, y)$.

Let $U^\rho_f(L(0)) := U^\rho(L(0))/I(f)$ be the quotient. There is an equivalence of categories between the category $\text{Mod}(U^\rho_f(L(0)))$ and the category of connections $(\hat{\rho}, E)$ in $\text{Mod}(L(0), Id)$ such that
\[
\hat{\rho}(u)\hat{\rho}(v) - \hat{\rho}(v)\hat{\rho}(u) - \hat{\rho}([u, v]) - \tilde{f}(u, v)\text{Id}_E = 0 \text{ in } \text{End}_k(E).
\]
It follows
\[
R_{\hat{\rho}}(u, v) = \tilde{f}(u, v)\text{Id}_E
\]
hence the induced connection $\rho := \hat{\rho} \circ i$ on $L$ has curvature type $f$. It follows there is an equivalence of categories
\[
\text{Mod}(U^\rho_f(L(0))) \cong \text{Mod}(U(A, L, f))
\]
hence $U^\rho_f(L(0))$ and $U(A, L, f)$ are Morita equivalent rings. They are not isomorphic in general but have $k$ as centre. Hence we may construct the categories $\text{Mod}(U(A, L, f))$ using quotients of one fixed ring $U^\rho(L(0))$ by the family of 2-sided ideals $I(f)$ for $f \in Z^2(L, A)$. In the case when $A$ is noetherian and $L$ a finitely generated an projective $A$-module it follows $U(A, L, f)$ is Noetherian. Since being Noetherian is Morita invariant it follows the ring $U^\rho_f(L(0))$ is Noetherian for any 2-cocycle $f \in Z^2(L, A).

We observe that an ordinary $L$-connection $(\rho, E)$ gives rise to a connection $\hat{\rho} : L(f^\alpha) \to \text{End}_k(E)$. The map $\hat{\rho}$ is an $A \otimes_k A$-linear map. It follows $(\rho, E)$ is a left $U^\otimes(L(f^\alpha))$-module. If the annihilator ideal $\text{ann}(\rho, E)$ contains the ideal $J(f^\alpha)$ generated by elements on the form $u \otimes v - v \otimes u - [u, v]$ for $u, v \in L(f^\alpha)$ it follows the connection $\rho$ has curvature type $f^\alpha$. It follows from Corollary 3.34 the $k$-th Chern class $c_k(E)$ is determined by $c_1(E)$. Hence the structure of the set of 2-sided ideals in $U^\otimes(L(f^\alpha))$ can be used to study properties of the Chern classes $c_k(E)$. 


One wants to study the correspondence between 2-sided ideals in $U \otimes \tilde{L}$ such that cohomology classes in $H^2(U)$ are related to the study of algebraicity of cohomology classes in singular cohomology. For the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional semi simple Lie algebra $\mathfrak{g}$ it follows the set of 2-sided ideals in $U(\mathfrak{g})$ corresponding to finite dimensional irreducible $\mathfrak{g}$-modules is a discrete set. The enveloping algebras $U(\mathfrak{g})$ and $U(A, L, f)$ are Morita equivalent to quotients of the universal algebra $U^\otimes(L(f^\alpha))$. One wants to study the correspondence between 2-sided ideals in $U^\otimes(L(f^\alpha))$ and cohomology classes in $H^2(L, A)$ and use this correspondence to give interesting.

Example 3.37. Moduli spaces of $\Gamma$-modules and connections

When studying moduli spaces of connections many authors use the Hilbert scheme and Quot scheme to construct parameter spaces of connections and these spaces are large and complicated. The rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$ are non-Noetherian in general but they have as shown in this paper many Noetherian quotients. It could be one gets an alternative to the study of the Hilbert and Quot schemes by studying parameter spaces of 2-sided ideals in Noetherian quotients of the rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$. Instead of studying large parameter spaces of pairs $(\rho, E)$ where $\rho$ is an $\tilde{L}$-connection, we study the parameter space of annihilator ideals $\text{ann}(\rho, E)$ in a Noetherian quotient one of the rings $U^\otimes(\tilde{L})$ and $U^\rho(\tilde{L})$.

In [16] the author constructs for any smooth projective complex variety $X$, any sheaf of filtered algebras $\Gamma$ on $X$ and any numerical polynomial $P$, a quasi-projective scheme $M(X, \Gamma, P)$ parametrizing semi stable $\Gamma$-modules with Hilbert polynomial $P$. The construction of $M(X, \Gamma, P)$ uses the Hilbert scheme and GIT quotients. Assume we are given a parameter space $M(d, \Gamma, P)$ parametrizing locally trivial $\Gamma$-modules $(\rho, E)$ with Hilbert polynomial $P$, such that $E$ is a locally trivial $\mathcal{O}_X$-modules of rank $d$. Given two isomorphic $\Gamma$-modules $(\rho, E)$ and $(\rho', E')$ where $E$ and $E'$ are isomorphic locally free $\mathcal{O}_X$-modules corresponding to different points in the parameter space $M(d, \Gamma, P)$. The sheaves of annihilator ideals $\text{ann}(\rho, E)$ and $\text{ann}(\rho', E')$ in $\Gamma$ will be equal. Hence in the parameter space $M(d, \Gamma, P)$ we get two different points corresponding to $(\rho, E)$ and $\rho'$, $E'$). In the parameter space of sheaves of 2-sided ideals we get one point corresponding to $\text{ann}(\rho, E) = \text{ann}(\rho', E')$. Hence we should expect the parameter space of sheaves of 2-sided ideals in $\Gamma$ to have fewer points than the parameter space $M(d, \Gamma, P)$. In the case of a holomorphic Lie algebroid $\mathcal{L}$ on a complex projective manifold $X$, it follows from [15] that the moduli spaces $M_{\mathcal{L}, \mathcal{Q}}(P)$ are in many cases empty. Hence one should take care when studying such moduli spaces in general: If one is unable to write down explicit non-trivial examples, this may indicate they are empty. In the affine situation as shown in this paper, it is relatively easy to write down explicit non-trivial examples of the theory as shown in Theorem 3.29. One may moreover implement computer algorithms calculating such examples.

If $A$ is finitely generated and regular over the complex numbers $\mathbb{C}$ and $L := \text{Der}_\mathbb{C}(A)$ it follows $H^1(L, A) \cong H^1_{\text{sing}}(X_C, \mathbb{C})$ is singular cohomology of $X_C$ with complex coefficients. Here $X_C$ is the underlying complex manifold of $X := \text{Spec}(A)$. The Hodge conjecture is known to hold for $H^2_{\text{sing}}(X_C, \mathbb{C})$ hence if one wants to study cohomology classes in $H^2_{\text{sing}}(X_C, \mathbb{C})$ that are not coming from $H^2_{\text{sing}}(X_C, \mathbb{C})$ under the cup product, one needs to study other types of connections. One has to study left $U^\otimes(L(f^\alpha))$ modules $(\rho, E)$ that are finitely generated and projective over $A$, such that $\text{ann}(\rho, E)$ does not contain the ideal $J(f^\alpha)$ for a 2-cocycle $f^\alpha$ - we get a computable criteria on $(\rho, E)$ which can be used to determine if the Chern class $c_k(E)$ is "interesting". Hence the structure of the set of 2-sided ideals in $U^\otimes(L(f^\alpha))$ is related to the study of algebraicity of cohomology classes in singular cohomology.

For the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional semi simple Lie algebra $\mathfrak{g}$ it follows the set of 2-sided ideals in $U(\mathfrak{g})$ corresponding to finite dimensional irreducible $\mathfrak{g}$-modules is a discrete set. The enveloping algebras $U(\mathfrak{g})$ and $U(A, L, f)$ are Morita equivalent to quotients of the universal algebra $U^\otimes(L(f^\alpha))$. One wants to study the correspondence between 2-sided ideals in $U^\otimes(L(f^\alpha))$ and cohomology classes in $H^2(L, A)$ and use this correspondence to give interesting.
examples of algebraic and non-algebraic classes in the singular cohomology of a complex algebraic manifold.

4. THE UNIVERSAL RING IS AN ALMOST COMMUTATIVE NOETHERIAN RING

In this section we construct in Theorem 4.10 for any D-Lie algebra $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)$ and any $\tilde{L}$-connection $(\rho, E)$ the universal ring $\tilde{U}^\otimes(\tilde{L}, \rho)$ of $(\rho, E)$. In the case when $A$ is Noetherian and $\tilde{L}, E$ finitely generated as left $A$-modules it follows the associative unital ring $\tilde{U}^\otimes(\tilde{L}, \rho)$ is an almost commutative Noetherian sub ring of $\text{Diff}(E)$ - the ring of differential operators on $E$. The non-flat connection $(\rho, E)$ is a finitely generated left $\tilde{U}^\otimes(\tilde{L}, \rho)$-module and we may use $\tilde{U}^\otimes(\tilde{L}, \rho)$ to construct the characteristic variety $SS(\rho, E)$ of $(\rho, E)$. We may use the variety $SS(\rho, E)$ to define holonomicity for non-flat connections. Previously this was defined for flat connections (see Example 4.12).

**Proposition 4.1.** Let $f \in Z^2(\text{Der}_k(A), A)$ be a 2-cocycle and let $(L, \alpha)$ be an $A/k$-Lie-Rinehart algebra. Let $\alpha_f : L(f^\alpha) \to \mathcal{D}^1(A, f)$ be the corresponding D-Lie algebra. Let $\rho : L(f^\alpha) \to \text{End}_k(E)$ be an object in $\mathsf{Mod}(L(f^\alpha))$. It follows $\rho$ is an $A \otimes_k A$-linear map. We get an induced map

$$T(\rho) : U^\otimes(L(f^\alpha)) \to \text{Diff}(E)$$

and

$$T(\rho)_i : U^\otimes_i(L(f^\alpha)) \to \text{Diff}^i(E)$$

for all $i \geq 1$.

**Proof.** Let $u := ax + x, v := bz + y \in L(f^\alpha)$ and let $c \in A$. Since $\rho$ is $A \otimes_k A$-linear we get the following:

$$\rho(cu) = c\rho(u) \text{ and } \rho(u + v) = \rho(u) + \rho(v).$$

Moreover

$$\rho(uc) = \rho(u)c.$$

We get for $e \in E$ the following:

$$\rho(u)(ce) = \rho(uc)(e) = \rho(cu + \alpha_f(u)(c)z)(e) = \rho(u)(e) + \alpha_f(u)(c)\rho(z)(e).$$

We get

$$\rho(z)(ce) = \rho(zc)(e) = \rho(cz)(e) = c\rho(z)(e)$$

hence $\rho(z) \in \text{End}_A(E)$. It follows

$$[\rho(u), cId_E] = \rho(u)c - c\rho(u) = \alpha_f(u)(c)Id_E \in \text{End}_A(E)$$

hence

$$\rho(u) \in \text{Diff}^1(E).$$

It follows we get an induced map

$$T(\rho) : U^\otimes(L(f^\alpha)) \to \text{Diff}(E)$$

and

$$T(\rho)_i : U^\otimes_i(L(f^\alpha)) \to \text{Diff}^i(E)$$

and the Lemma follows. \qed

**Lemma 4.2.** Let $k \to A$ be a map of commutative unital rings. Let $\text{Diff}_k(E) := \text{Diff}(E)$ be the ring of $k$-linear differential operators on $E$. The ring $\text{Diff}(E)$ is almost commutative.
Proof. We must prove that for any differential operators \( x \in \text{Diff}^i(E) \) and \( y \in \text{Diff}^j(E) \) it follows \( xy - yx \in \text{Diff}^{i+j-1}(E) \). By definition it follows for any element \( a \in A \) that

\[
[x, a Id_E] \in \text{Diff}^{i-1}(E)
\]

and

\[
[y, a Id_E] \in \text{Diff}^{j-1}(E).
\]

It follows we may write \( xa = ax + \eta \) with \( \eta \in \text{Diff}^{i-1}(E) \) and \( ya = ay + \eta' \) with \( \eta' \in \text{Diff}^{j-1}(E) \). We get

\[
[xy - yx, a Id_E] = yxa - axa - a(xy - yx) = x(ay + \eta') - y(ax + \eta) - a(xy - yx) = a(xy - yx) + \eta y + \eta' x - y\eta - a(xy - yx) = \eta y + \eta' x - \eta x - y\eta \in \text{Diff}^{i+j-1}(E).
\]

The Lemma follows.

\[\square\]

Lemma 4.3. Let \( f \in Z(\text{Der}_k(A), A) \) be a 2-cocycle, \((L, \alpha)\) an \( A/k\)-Lie-Rinehart algebra and let \( \alpha_f : \text{L}(f^\alpha) \to \text{D}^1(A, f) \) be the associated D-Lie algebra. There is a one-to-one correspondence between the set of \( A \otimes_k \) \( \alpha \)-linear maps \( \rho : \text{L}(f^\alpha) \to \text{End}_k(E) \) and the set of \( \psi \)-connections \( \nabla : L \to \text{End}_k(E) \) for varying \( \psi \in \text{End}_A(E) \). If \( \nabla : L \to \text{End}_k(E) \) is a \( \psi \)-connection it follows the corresponding \( A \otimes_k \) \( \alpha \)-linear map map \( \rho : \text{L}(f^\alpha) \to \text{End}_k(E) \) has curvature

\[
R_\rho(u, v) = R_\nabla(x, y) - f(x, y) Id_E + b[\nabla(x), \psi] - a[\nabla(y), \psi]
\]

with \( u = az + x, v = bz + y \in \text{L}(f^\alpha) \). If \( \psi = Id_E \) it follows

\[
R_\rho(u, v) = R_\nabla(x, y) - f(x, y) Id_E.
\]

Conversely, if \( \rho : \text{L}(f^\alpha) \to \text{End}_k(E) \) is an \( A \otimes_k \) \( \alpha \)-linear map and \( \nabla \) the corresponding \( \psi \)-connection, it follows the curvature of \( \nabla \) satisfies

\[
R_\nabla(x, y) = R_\rho(i(x), i(y)) + f(x, y) \psi
\]

where \( i : L \to \text{L}(f^\alpha) \) is the canonical inclusion map.

Proof. The first statement is proved earlier in this paper. Let \( \nabla : L \to \text{End}_k(E) \) be a \( \psi \)-connection and let \( \rho : \text{L}(f^\alpha) \to \text{End}_k(E) \) be the corresponding \( A \otimes_k \) \( \alpha \)-linear map. By definition \( \rho(az + x) := a\psi + \nabla(x) \). We get for any elements \( u := az + x, v := bz + y \in \text{L}(f^\alpha) \) the following calculation:

\[
R_\rho(u, v) := [\rho(u), \rho(v)] - \rho([u, v]) = [a\psi + \nabla(x), b\psi + \nabla(y)] - (a(x)(b) - a(y)(a) + f(x, y))z + \nabla([x, y]) = R_\nabla(x, y) + b[\nabla(x), \psi] - a[\nabla(y), \psi] - f(x, y) \psi
\]

and the second claim follows. The third claim follows since \([\nabla(x), Id_E] = [\nabla(y), Id_E] = 0 \). Let \( i : L \to \text{L}(f^\alpha) \) be the left \( A \)-linear canonical inclusion map. It follows for any elements \( x, y \in L \) we get

\[
[i(x), i(y)] - i([x, y]) = f(x, y)z \in \text{L}(f^\alpha).
\]

We get

\[
R_\nabla(x, y) := [\rho(i(x)), \rho(i(y))] - \rho(i([x, y])).
\]

We get

\[
i([x, y]) = [i(x), i(y)] - f(x, y)z
\]
hence
\[ R_\nabla(x, y) = [\rho(i(x)), \rho(i(y))] - \rho([i(x), i(y)]) + \rho(f(x, y)z) = \]
\[ R_\rho(i(x), i(y))f(x, y)\rho(z) = R_\rho(i(x), i(y)) + f(x, y)\psi. \]

The Lemma is proved. \hfill \Box

**Lemma 4.4.** Let \( \nabla : L \to \text{End}_k(E) \) be a \( \psi \)-connection where \( \psi \in \text{End}_A(E) \). It follows
\[ R_\nabla(x, y)(ae) = aR_\nabla(x, y)(e) + \alpha(y)(a)[\nabla(x), \psi](e) - \alpha(x)(b)[\nabla(y), \psi](e) \]
for all \( a \in A \) and \( e \in E \).

**Proof.** The proof is a straightforward calculation. \hfill \Box

**Lemma 4.5.** Let \( f \in Z^2(\text{Der}_k(A), A) \) and let \( \nabla : L \to \text{End}_k(E) \) be a \( \psi \)-connection with \( \psi \in \text{End}_A(E) \). Let \( \rho : L(f^\alpha) \to \text{End}_k(E) \) be the associated \( A \otimes_k A \)-linear map. It follows the curvature \( R_\rho(u, v) \in \text{Diff}^1(E) \). If \( \psi = I_{DE} \) it follows \( R_\rho(u, v) \in \text{Diff}^0(E) := \text{End}_A(E) \).

**Proof.** We have seen that if \( u := az + x, v := bz + y \) it follows
\[ R_\rho(u, v) = R_\nabla(x, y) - f(x, y)\psi + b[\nabla(x), \psi] - a[\nabla(y), \psi]. \]
The following holds for \( R_\nabla \).
\[ R_\nabla(x, y)(ae) = aR_\nabla(x, y)(e) + \alpha(y)(a)[\nabla(x), \psi](e) - \alpha(x)(b)[\nabla(y), \psi](e) \]
for all \( a \in A, e \in E \). One checks that
\[ f(x, y)\psi + b[\nabla(x), \psi] - a[\nabla(y), \psi] \in \text{End}_A(E) \]
for all \( u, v \). We get for any \( a \in A \) the following:
\[ [R_\nabla(x, y), aI_{DE}] = R_\nabla(x, y)a - aR - \nabla(x, y) = \alpha(y)(a)[\nabla(x), \psi] - \alpha(x)(a)[\nabla(y), \psi] \in \text{End}_A(E) \]
hence
\[ R_\nabla(x, y) \in \text{Diff}^1(E). \]
It follows
\[ R_\rho(u, v) \in \text{Diff}^1(E). \]
One checks that if \( \psi = I_{DE} \) it follows \( R_\rho(u, v) \in \text{End}_A(E) \) and the Lemma follows. \hfill \Box

**Lemma 4.6.** Let \( (\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D) \) be a D-Lie algebra and let \( \rho : \tilde{L} \to \text{End}_k(E) \) be an \( A \otimes_k A \)-linear map. Let \( u, v \in \tilde{L} \). The following holds:
\[ \rho : \tilde{L} \to \text{Diff}^1(E). \]

For any set of elements \( u_1, \ldots, u_i \in \tilde{L} \) it follows \( \rho(u_1) \circ \cdots \circ \rho(u_i) \in \text{Diff}^i(E) \). Moreover
\[ R_\rho(u, v) \in \text{Diff}^2(E). \]
If \( \rho(D) = I_{DE} \) it follows
\[ R_\rho(u, v) \in \text{Diff}^1(E). \]
**Example 4.7.** The universal ring of an L-extension.

Let \( \tilde{\mathcal{L}}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D \) be a D-Lie algebra and let \( \rho : \tilde{\mathcal{L}} \to \text{End}_k(E) \) be an object in \( \text{Mod}(\tilde{\mathcal{L}}, \text{Id}) \). This means \( \rho \) is \( A \otimes_k A \)-linear and \( \rho(D) = \text{Id}_E \).

**Lemma 4.8.** The connection \( \rho \) induce a connection

\[ \rho : \tilde{\mathcal{L}} \to \text{Der}_k(\text{End}_A(E)) \]

defined by

\[ \tilde{\rho}(u)(\phi) := [\rho(u), \phi] \]

with curvature

\[ R_{\tilde{\rho}}(u, v)(\phi) := [R_{\rho}(u, v), \phi]. \]

**Proof.** Let \( a \in A, e \in E, u \in \tilde{\mathcal{L}} \) and \( \phi \in \text{End}_A(E) \). We get

\[ \tilde{\rho}(u)(\phi)(ae) = \rho(u)(\phi(ae)) - \phi(\rho(u)(ae)) = a\rho(u)(\phi(e)) + \tilde{\pi}(u)(a)\rho(D)(\phi(e)) - a\phi(\rho(u)(e)) - \tilde{\pi}(u)(a)\phi(\rho(D)(e)) = a[\rho(u), \phi](e) = a\rho(u)(\phi)(e) \]

hence \( \tilde{\rho}(\phi) \in \text{End}_A(E) \). One checks that for any two \( \phi, \psi \in \text{End}_A(E) \) we get

\[ \tilde{\rho}(u)((\phi, \psi)) = [\tilde{\rho}(u)(\phi), \psi] + [\phi, \tilde{\rho}(u)(\psi)]. \]

One checks that \( R_{\tilde{\rho}}(u, v)(\phi) = [R_{\rho}(u, v), \phi] \) and the Lemma follows. \( \square \)

We get by (4.8.1) a non-abelian extension of \( A/k \)-Lie-Rinehart algebras

\[ 0 \to \text{End}_A(E) \to \text{End}(\tilde{\mathcal{L}}, E) \to \tilde{\mathcal{L}} \to 0 \]

where \( \text{End}(\tilde{\mathcal{L}}, E) := \text{End}_A(E) \oplus \tilde{\mathcal{L}} \) with the following Lie product. For any \( z := (\phi, u), z' := (\psi, v) \in \text{End}(\tilde{\mathcal{L}}, E) \) define

\[ [z, z'] := ([\phi, \psi] + [\rho(u), \psi] - [\rho(v), \phi] + R_{\rho}(u, v), [u, v]). \]

Define the central element \( \tilde{D} := (0, D) \in \text{End}(\tilde{\mathcal{L}}, E) \). There is a canonical map

\[ \alpha_E : \text{End}(\tilde{\mathcal{L}}, E) \to \text{D}^1(A, f) \]

defined by

\[ \alpha_E(\phi, u) := \tilde{\alpha}(u). \]

There is a map

\[ \pi_E : \text{End}(\tilde{\mathcal{L}}, E) \to \text{Der}_k(A) \]
defined by
\[ \pi_E(\phi, u) := \tilde{\pi}(u). \]

**Proposition 4.9.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [,], \tilde{D})\) be a \(D\)-Lie algebra and let \((\rho, E)\) be an object in \(\text{Mod}(\tilde{L}, \text{Id})\). It follows the 5-tuple \((\text{End}(\tilde{L}, E), \alpha_E, \pi_E, [,], \tilde{D})\) constructed above is a \(D\)-Lie algebra.

**Proof.** Let \(u \in \tilde{L}\) and \(\phi \in \text{End}_A(E)\). We get
\[ [\rho(u), \phi] a = \rho(u) \circ \phi a - \phi \circ \rho(u) a = \]
\[ a \rho(u) \circ \phi + \tilde{\pi}(u)(a) \phi - a \phi \circ \rho(u) - \phi \tilde{\pi}(u)(a) = \]
hence \(\tilde{\rho}(u)(\phi) := [\rho(u), \phi] \in \text{End}_A(E)\). We get a map
\[ \tilde{\rho} : \tilde{L} \to \text{End}_k(\text{End}_A(E)) \]
and one checks that this map induce a map
\[ \tilde{\rho} : \tilde{L} \to \text{Der}_k(\text{End}_A(E)). \]
One moreover checks that
\[ R_{\tilde{\rho}}(u, v)(\phi) = [R_{\rho}(u, v), \phi]. \]
Hence we get by [11] a non-abelian extension of \(A/k\)-Lie-Rinehart algebras
\[ 0 \to \text{End}_A(E) \to \text{End}(\tilde{L}, E) \to \tilde{L} \to 0 \]
where \(\text{End}(\tilde{L}, E) := \text{End}_A(E) \oplus \tilde{L}\) with the given Lie product. The maps are the canonical maps. Define \(\tilde{D} := (0, D) \in \text{End}(\tilde{L}, E)\). Let \(z := (\phi, u) \in \text{End}(\tilde{L}, E)\) and let \(c \in A\). Define
\[ z c = (\phi, u) c := (\phi c, u c) = (c \phi, cu + \tilde{\pi}(u)(c) D) = c(\phi, u) + \tilde{\pi}(u)(0, D). \]
It follows
\[ z c = c z + \pi_E(z)(c) \tilde{D} \]
where we have defined
\[ \pi_E : \text{End}(\tilde{L}, E) \to \overline{D^1}(A, f) \]
by
\[ \pi_E(\phi, u) := \tilde{\pi}(u) \in \overline{D^1}(A, f). \]
It follows \(\text{End}(\tilde{L}, E)\) is a left and right \(A\)-module and the map \(f : \text{End}(\tilde{L}, E) \to \tilde{L}\) is a map of \(A \otimes_k A\)-modules with \(f(\tilde{D}) = D\). Define
\[ \alpha_E : \text{End}(\tilde{L}, E) \to \text{Der}_k(A) \]
by
\[ \alpha_E(\phi, u) := \tilde{\alpha}(u) \in \text{Der}_k(A). \]
It follows \(\alpha_E\) is a map of \(A \otimes_k A\)-modules and \(k\)-Lie algebras. The element \(\tilde{D}\) is in the center of \(\text{End}(\tilde{L}, E)\) hence the 5-tuple \((\text{End}(\tilde{L}, E), \alpha_E, \pi_E, [,], \tilde{D})\) is a \(D\)-Lie algebra. The sequence [4.8.1] is a non-abelian extension of \(\tilde{L}\) by \(\text{End}_A(E)\) and the Lemma follows. \(\square\)
Theorem 4.10. Let $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)$ be a D-Lie algebra and let $(\rho, E)$ be an $\tilde{L}$-connection with $\rho(D) = Id_E$. There is a canonical map

$$\rho^1 : \text{End}(\tilde{L}, E) \to \text{Diff}^1(E)$$

and $\rho^1$ is a map of $B := A \otimes_k A$-modules and $k$-Lie algebras. The map $\rho^1$ induce a map $T(\rho^1) : \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \to \text{Diff}(E)$ of associative rings. Let $\tilde{U}^\otimes(\tilde{L}, \rho) := \text{Im}(T(\rho^1))$ be the image. We get an exact sequence of rings

$$0 \to \text{ker}(T(\rho^1)) \to \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \to \tilde{U}^\otimes(\tilde{L}, \rho) \to 0$$

where $\tilde{U}^\otimes(\text{End}(\tilde{L}, E)) := U^\otimes(\text{End}(\tilde{L}, E))/I$ where $I$ is the 2-sided ideal generated by the elements $u \otimes v - v \otimes u - [u, v]$ for $u, v \in \text{End}(\tilde{L}, E)$. The rings $\tilde{U}^\otimes(\text{End}(\tilde{L}, E))$ and $\tilde{U}^\otimes(\tilde{L}, \rho)$ are almost commutative. If $A$ is noetherian and $\tilde{L}, E$ finitely generated as left $A$-modules it follows $\tilde{U}^\otimes(\text{End}(\tilde{L}, E))$ and $\tilde{U}^\otimes(\tilde{L}, \rho)$ are Noetherian rings.

Proof. By Proposition 4.9 we may do the following: Define the map

$$\rho^1 : \text{End}(\tilde{L}, E) \to \text{Diff}^1(E)$$

by

$$\rho^1(\phi, u) := \phi + \rho(u).$$

Since $\rho(u) \in \text{Diff}^1(E)$ it follows $\rho^1(\phi, u) \in \text{Diff}^1(E)$. By definition it follows $\rho^1$ is an $A \otimes_k A$-linear map. Let $z := (\phi, u), z' := (\psi, v) \in \text{End}(\tilde{L}, E)$. We get

$$\rho^1([z, z']) = \rho^1([\phi, \psi] + [\rho(u), \psi] + [\phi, \rho(v)] + R_{\rho}(u, v) + \rho([u, v]) = [\phi, v] + [\rho(u), \psi] + [\phi, \rho(v)] + [\rho(u), \rho(v)] = [\phi + \rho(u), \psi + \rho(v)] = [\rho^1(z), \rho^1(z')]$$

and the map $\rho^1$ is a map of $k$-Lie algebras. We get an induced map

$$T(\rho^1) : \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \to \text{Diff}(E)$$

Since $T(\rho^1)(u \otimes v - v \otimes u - [u, v]) = 0$ it follows we get an induced exact sequence

$$0 \to \text{ker}(T(\rho^1)) \to \tilde{U}^\otimes(\text{End}(\tilde{L}, E)) \to \tilde{U}^\otimes(\tilde{L}, \rho) \to 0$$

By Lemma 3.7 and Proposition 3.12 it follows $\tilde{U}^\otimes(\text{End}(\tilde{L}, E))$ and $\tilde{U}^\otimes(\tilde{L}, \rho)$ are almost commutative associative unital rings. If $A$ is Noetherian and $\tilde{L}, E$ are finitely generated as left $A$-modules it follows from Theorem 3.17 $\text{Gr}(\tilde{U}^\otimes(\text{End}(\tilde{L}, E)))$, $\text{Gr}(\tilde{U}^\otimes(\tilde{L}, \rho))$, $\tilde{U}^\otimes(\text{End}(\tilde{L}, E))$ and $\tilde{U}^\otimes(\tilde{L}, \rho)$ are Noetherian rings . The Theorem follows.

Definition 4.11. Let $\tilde{U}^\otimes(\tilde{L}, \rho) := \text{Im}(T(\rho^1)) \subseteq \text{Diff}(E)$ be the universal ring of the connection $(\rho, E)$.

Example 4.12. Methods from the theory of rings of differential operators and D-modules.

There is an extensive theory of flat connections on complex manifolds, the Riemann-Hilbert correspondence and modules on rings of differential operators and D-modules. See [2] and [3] for an introduction to the subject with references.

The universal ring $\tilde{U}^\otimes(\tilde{L}, \rho)$ defined above is an almost commutative Noetherian ring in many cases. Left and right $\tilde{U}^\otimes(\tilde{L}, \rho)$-modules that are finitely generated over $\tilde{U}^\otimes(\tilde{L}, \rho)$ have many properties similar to D-modules as studied in [3]. We can define
the characteristic variety of \((\rho, E)\) using filtrations coming from \(\tilde{U}^\circ (\tilde{L}, \rho)\). We may use such filtrations to define the notion of holonomy for non-flat connections. Definition 1.1.11 in [3] is algebraic and the only assumption is that \(\tilde{U}^\circ (\tilde{L}, \rho)\) is defined over a field of characteristic zero. Once the characteristic variety \(SS(\rho, E)\) is defined we may define holonomicity as in [3]. It remains to find out if this leads to a reasonable theory where one can calculate explicit examples. Since the universal ring \(\tilde{U}^\circ (\tilde{L}, \rho)\) is Noetherian when \(E, \tilde{L}\) are finitely generated as \(A\)-modules it might be worthwhile to investigate this.

In [3] one uses localization for non-commutative rings to prove that results for modules on almost commutative Noetherian rings globalize to give constructions valid for modules on sheaves of differential operators on complex manifolds. Similar methods can be used to prove that the construction of the universal ring \(\tilde{U}^\circ (\tilde{L}, \rho)\) globalize to give a construction for arbitrary schemes. Since the cohomology and homology of a connection is defined using the theory of modules over associative rings, the theory uses methods from non-commutative algebra/algebraic geometry and the theory of sheaves of rings of differential operators and jet bundles.

Recall the following theorem

**Theorem 4.13.** Let \(U\) be a Noetherian almost commutative associative unital ring and \(M\) a finitely generated \(U\)-module. The characteristic variety

\[ SS(M) \subseteq \text{Spec}(Gr(U)) \]

is coisotropic with respect to the Poisson structure on \(Gr(U)\).

**Proof.** See [3], Theorem 1.2.5. \(\square\)

In the case above where \(A\) is Noetherian, \(\tilde{L}, E\) finitely generated as left \(A\)-modules, it follows \(\tilde{U}^\circ (\text{End}(\tilde{L}, E))\) and \(\tilde{U}^\circ (\tilde{L}, \rho)\) are almost commutative Noetherian rings. Hence Theorem [4][13][15][18] applies to any finitely generated \(\tilde{U}^\circ (\tilde{L}, \rho)\) or \(\tilde{U}^\circ (\text{End}(\tilde{L}, E))\)-module. The connection \((\rho, E)\) is a finitely generated left \(\tilde{U}^\circ (\tilde{L}, \rho)\)-module and we may construct the characteristic variety \(SS(\rho, E)\). One wants to study the ring \(\tilde{U}^\circ (\tilde{L}, \rho)\), the variety \(SS(\rho, E)\) and its relationship to the connection \(\rho\) and the Chern classes \(c_i(E)\) for non-flat connections \((\rho, E)\).

Sridharan studied the enveloping algebra \(U(k, g, f)\) in [17] for \(k\) a fixed commutative ring, \(g\) a Lie algebra over \(k\) and \(f\) a 2-cocycle for \(g\). He gave a complete description of the deformation groupoid of \(g\) in the case when \(g\) is a \(k\)-Lie algebra with a basis as \(k\)-module. Rinehart studied in [15] the enveloping algebra \(U(A, L)\) for an arbitrary Lie-Rinehart algebra and proved the PBW-theorem for \(U(A, L)\) in the case when \(L\) is a projective \(A\)-module. He used this theorem to study cohomology and homology of \(L\)-connections. Tortella gave in [18] a simultaneous generalization of the construction of Sridharan and Rinehart for holomorphic Lie-algebroids on complex manifolds and proved a PBW-theorem for the sheaf of enveloping algebras of such holomorphic Lie algebroids. In [7] I gave an algebraic construction of the enveloping algebra \(U(A, L, f)\) for any Lie-Rinehart algebra \(L\) and any 2-cocycle \(f\). I also gave some algebraic proofs of results in Tortella’s paper and a proof of the PBW-theorem for \(U(A, L, f)\) in the case when \(L\) is a projective \(A\)-module. I used the PBW-theorem to give a complete determination of the deformation groupoid of \((L, \alpha)\) in the case when \(L\) is a projective \(A\)-module. Note that the paper [7] contains some errors in the section on families of connections (Definition 5.1). In [11] I give
a classification of all non-abelian extensions of a D-Lie algebra \((\tilde{\mathcal{L}}, \tilde{\alpha}, \tilde{\pi}, [~,~], \mathcal{D})\) by an \(A\)-Lie algebra \((W, [~,~])\) with \(aw = wa\) for all \(a \in A\) and \(w \in W\) such that \(\tilde{\mathcal{L}}\) is projective as left \(A\)-module. This classification generalize the classification given in \[19\] to D-Lie algebras. In \[19\] the classification is done for Lie-Rinehart algebras.

**References**

[1] P. Berthelot, A. Ogus, Notes on crystalline cohomology, *Princeton University Press* (1978)
[2] A. Borel, Algebraic D-modules, *Academic Press Perspectives in Mathematics* (1987)
[3] V. Ginzburg, Lectures on D-modules, *University of Chicago Lecture Notes* (1998)
[4] R. Hartshorne, Algebraic geometry, *Springer Verlag*, GTM. no. 52 (1983)
[5] M. Kontsevich, A. Rosenberg, Non commutative geometry, *The Gelfand Mathematical Seminars* (1996-1999)
[6] H. Maakestad, Algebraic connections on ellipsoid surfaces, *arXiv preprint*, https://arxiv.org/abs/1208.2806 (2012)
[7] H. Maakestad, Algebraic connections on projective modules with prescribed curvature, *J. of Algebra* no. 436, 161-227 (2015)
[8] H. Maakestad, Cohomology of \(t\)-connections, *preprint in progress*
[9] H. Maakestad, Differential operators on projective modules, *arXiv preprint*, https://arxiv.org/abs/1110.4966
[10] H. Maakestad, Jet bundles on projective space: New examples, *arXiv preprint*, https://arxiv.org/abs/1206.1175 (2012)
[11] H. Maakestad, Lie algebras of differential operators I: Extensions, *arXiv preprint* https://arxiv.org/abs/1512.02967 (2019)
[12] H. Maakestad, Principal parts on the projective line over arbitrary rings, *Manuscr. Math.* 126, no. 4 (2008)
[13] H. Maakestad, The Chern-character for Lie-Rinehart algebras, *Ann. Inst. Fourier* no. 55, 2551-2574 (2005)
[14] H. Matsumura, Commutative ring theory, *Cambridge Studies in Advanced Mathematics* no. 8 (1986)
[15] G. S. Rinehart, Differential forms on general commutative algebras, *Trans. Am. Math. Soc.* no. 108 (1963)
[16] C. T. Simpson, Moduli spaces of representations of the fundamental group of a smooth projective variety I, *Publ. Math. IHES* no. 79 (1994)
[17] R. Sridharan, Filtered algebras and representations of Lie algebras, *Trans. Amer. Math. Soc.* no. 100 (1961)
[18] P. Tortella, \(\Gamma\)-modules and holomorphic Lie algebroid connections, *Centr. Eur. J. Math.* no. 10 (2012)
[19] U. Bruzzo, I. Mencattini, V. Rubtsov, P. Tortella, Nonabelian holomorphic Lie algebroid extensions, *arXiv* https://arxiv.org/abs/1305.2377 (2013)
[20] C. Weibel, Homological algebra, *Cambridge Studies in Advanced Math.* no. 38 (1994)

E-mail address: h.maakestad@hotmail.com