IMPROVING ROTH’S THEOREM IN THE PRIMES

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Abstract. Let $A$ be a subset of the primes. Let
$$
\delta_P(N) = \frac{|\{n \in A : n \leq N\}|}{|\{n \text{ prime} : n \leq N\}|}.
$$
We prove that, if
$$
\delta_P(N) \geq C \frac{\log \log \log N}{(\log \log N)^{1/3}}
$$
for $N \geq N_0$, where $C$ and $N_0$ are absolute constants, then $A \cap [1, N]$ contains a non-trivial three-term arithmetic progression.

1. Introduction

1.1. History and statement. In 1953, K. Roth [Ro] proved that any subset of positive integers of positive density contains infinitely many non trivial three-term arithmetic progressions. More precisely, his result is as follows. Given a set $A \subset \mathbb{Z}^+$, we define the density of $A \cap [1, N]$ by $\delta(N) = \frac{1}{N} |\{n \in A : n \leq N\}|$. (We write $|S|$ for the number of elements of a set $S$.) Roth proved that, given any set of integers $A \subset \mathbb{Z}^+$ such that $\delta(N) \geq C/\log \log N$ for some $N \geq N_0$ (where $C$ and $N_0$ are absolute constants) there must be at least one non-trivial three-term arithmetic progression in $A \cap [1, N]$. (By a non-trivial arithmetic progression we mean one with non-zero modulus, i.e., $(a, a + d, a + 2d)$ with $d \neq 0$.)

Much later, Heath-Brown [HB] (1987) and Szemerédi [Sz] (1990) improved this result by showing that it is enough to require that $\delta(N) \geq C/(\log \log N)^c$ for some small positive $c$. By considering Bohr sets where previous arguments had used arithmetic progressions, Bourgain relaxed the condition to $\delta(N) \geq C \sqrt{\log \log N / \log N}$ in [Bo2] (1999) and to $\delta(N) \geq C(\log \log N)^2(\log N)^{-2/3}$ in [Bo3] (2006).

Van der Corput proved [vdC] that the primes contain infinitely many non trivial 3-term arithmetic progressions. The question then arises – is Roth’s theorem true in the primes? That is – must a subset of primes of positive relative density $1$ contain a non-trivial 3-term arithmetic progression?\footnote{Given a subset $A$ of the set $P$ of all primes, we define the relative density $\delta_P(N)$ of $A$ to be $\delta_P(N) = |\{n \in A : n \leq N\}|/|\{n \text{ prime} : n \leq N\}|$. We are asking whether, given $A \subset P$ such that $\delta_P(N) > \delta_0$ ($\delta_0 > 0$) for some sufficiently large $N$, the set $A$ contains a non-trivial 3-term arithmetic progression.}
In [Gr], B. Green showed that the answer is “yes”. He proved that, given any subset $A$ of the primes such that $A \cap [1,N]$ has relative density $\delta_P(N) \geq C(\log \log \log \log \log N / \log \log \log N)^{1/2}$ for some $N \geq N_0$, where $C$ and $N_0$ are absolute constants, there exists a 3-term arithmetic progression in $A$.

We prove the following result.

**Theorem 1.1.** Let $A$ be a subset of the primes. Assume that $A \cap [1,N]$ is of relative density $\delta_P(N) \geq C \frac{\log \log N}{(\log \log N)^{1/3}}$ for some $N \geq N_0$, where $C$ and $N_0$ are absolute constants. Then $A$ contains a non-trivial 3-term arithmetic progression.

In other words, we gain two logs over what was previously known. One of the two logs gained is ultimately due to an enveloping use of a sieve; this idea is by now familiar to the specialists, and, indeed, it will come into our proof via a restriction theorem from [GT] (based partially on work on sieves in [Ra]). The other gain of a log stems from a more essential change in approach.

Our overall procedure is as follows. The first step is to replace the characteristic function $a$ of $A$ by a smoothed-out version $a_1$ whose Fourier transform is close to that $a$ (and thus, as can be easily shown, $a_1$ behaves like $a$ does when it comes to the number of 3-term progressions). This is much the same as in [Gr, §6]; it is in accord with the general strategy (the “uniformity strategy”) described in [Ta, §6]. We then show that the $\ell_2$-norm of $a_1$ is actually small enough that one can find a set $A'$ of large density in the integers such that $a_1$ is large on $A'$. This reduces the problem over the primes to the problem over the integers.

1.2. **Notation.** Let $N'$ be a positive integer. Let $f : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{C}$ be a function in $l^1(\mathbb{Z}/N'\mathbb{Z})$. We define the Fourier transform of $f$ as the function

$$\hat{f} : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{C}$$

$$b \mapsto \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} f(n)e(-nb/N'),$$

where we write $e(x)$ for $e^{2\pi ix}$. We write $\pi$ for the reduction map $\pi : \mathbb{Z} \to \mathbb{Z}/N'\mathbb{Z}$. Given $x \in \mathbb{R}$, we define $\{x\}$ to be the distance of $x$ to the nearest integer. We define $\|n\| = \{n/N\}$; this works because $\{x\}$ depends only on $x \mod 1$.

Given a finite or countable set $S$, a function $f : S \to \mathbb{C}$ and a parameter $0 < r < \infty$, we define the $\ell_r$-norm $|f|_r$ of $f$ by $|f|_r = (\sum_{x \in S} |f(x)|^r)^{1/r}$.

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2. FROM THE PRIMES TO THE INTEGERS

2.1. FROM THE PRIMES TO THE SET \( \{ n : b + nM \text{ is prime} \} \). Let us first show that we can focus on the intersection of the primes with an arithmetic progression of large modulus, rather than work on all the primes.

**Lemma 2.1.** Let \( \alpha, z \) be positive real numbers and \( N \) be a large integer. We define \( M = \prod_{p \leq z} p \). Let \( A \) be a subset of the primes less than \( N \) such that \( |A| \geq \alpha N/\log N \). Then there exists some arithmetic progression \( P(b) = \{ b + nM : 1 \leq n \leq N/M \} \) such that

\[
|P(b) \cap A| \gg \alpha \frac{\log z}{\log N} \frac{N}{M} - \log z,
\]

where the implied constant is absolute.

**Proof.** If \( (b, M) \neq 1 \), the set \( \{ m \in P(b) : m \text{ prime} \} \) is empty. Since the progressions \( P(b) \) with \( (b, M) = 1 \) are distinct, we have

\[
\sum_{b : (b, M) = 1} |A \cap P(b)| = |A| - |A \cap [1, M - 1]| \geq \alpha \frac{N}{\log N} - M.
\]

But \( |\{ b \leq M : (b, M) = 1 \}| \sim M/\log z \sim e^z/\log z \). Therefore there exists some progression \( P(b) \) such that

\[
|A \cap P(b)| \gg \left( \frac{\alpha N}{\log N} - M \right) \frac{\log z}{M} \gg \alpha \frac{\log z}{\log N} \frac{N}{M} - \log z.
\]

\( \square \)

The passage to an arithmetic progression \( b + nM \) of large modulus is exactly what Green and Tao [GT2, p. 484] call the "W-trick" (due to Green's use of the letter \( W \) for \( M \) in [Gr]). Green uses the fact that such a passage removes all but the largest peaks in the Fourier transform of the primes, whereas we simply use in a more direct way the fact that the elements of \( \{ n : b + nM \text{ prime} \} \) are not forbidden from having small divisors. Of course, these are two manifestations of the same idea.

Now, we fix \( z = \frac{1}{3} \log N, M = \prod_{p \leq z} p \), and let \( N' \) be the least prime larger than \( \lceil 2N/M \rceil \). (The requirement \( N' > \lceil 2N/M \rceil \) will ensure that no new three-term arithmetic progressions are created when we apply the reduction map \( \pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z} \) to a set contained in \([1, N/M]\).) By Bertrand's postulate, \( N' \ll N/M \). Let \( A \) be a subset of the primes less than \( N \) such that \( |A| \geq \alpha N/\log N \). We assume \( \alpha \geq (\log N)N^{-1/2} \) (say) and obtain from Lemma 2.1 that there is an arithmetic progression \( P(b) \) such that \( |P(b) \cap A| \gg \alpha (\log z/\log N)N' \). We define \( A_0 \) to be

\[
A_0 = \pi \left( \left\{ n = \frac{m - b}{M} : m \in P(b) \cap A \right\} \right).
\]

This is a subset of \( \pi(\{ n \in [1, N'] : b + nM \text{ is prime} \}) \) satisfying

\[
|A_0| \gg \alpha \frac{\log z}{\log N} \frac{N'}{N}.
\]
Our task is to show that there is a non-trivial three-term arithmetic progression in \( A_0 \subset \mathbb{Z}/N'\mathbb{Z} \). It will follow immediately that there is a non-trivial three-term arithmetic progression in \( A \subset \mathbb{Z} \).

2.2. From the set \( \{ n : b + nM \text{ is prime} \} \) to the set of integers. Let \( a \) be the normalised characteristic function of \( A_0 \), i.e., \( a = (\log N/(N' \log z))1_{A_0} \). Fixing \( \delta > 0 \) and \( \epsilon \in (0, 1/4) \) to be chosen later, define \( R := \{ x \in \mathbb{Z}/N'\mathbb{Z} : |\hat{a}(x)| \geq \delta \} \cup \{1\} \) and the Bohr set

\[
B := \{ n \in \mathbb{Z}/N'\mathbb{Z} : \forall x \in R, \|nx\| \leq \epsilon \} .
\]

We also define on \( \mathbb{Z}/N'\mathbb{Z} \) the functions \( \sigma = \frac{1}{|B|}1_B \) and \( a_1 = a \ast \sigma \).

To begin with, we remark that \(|a_1| = (\log N/(N' \log z)|A_0| \gg \alpha \text{ and } |a_1|_1 = |a_1|_1|\sigma|_1 = |a|_1 \). Thus \( |a_1|_1 \gg \alpha \), i.e., \( a_1 \) is large in \( \ell_1 \)-norm. We will later show that \( a_1 \) is small in \( \ell_2 \)-norm. These bounds on the \( \ell_1 \)-norm and the \( \ell_2 \)-norm will enable us to find a large set of integers on which \( a_1 \) is \( \gg \frac{1}{N} \). This will enable us to reduce the problem for large subsets of the primes to Roth’s theorem for large subsets of the integers.

We first have to show that \( a_1 \) is “close” to \( a \) in the sense that we care about, namely – we must show that \( a_1 \) is large on all three terms of many three-term arithmetic progressions if and only if the same is true of \( a \) (i.e., if and only if \( A \) contains many three-term arithmetic progressions). More precisely, our aim is to bound from above the quantity

\[
(2.2) \quad \Delta = N' \cdot \left| \sum_{n_1, n_2, n_3 \text{ in AP}} a(n_1)a(n_2)a(n_3) - \sum_{n_1, n_2, n_3 \text{ in AP}} a_1(n_1)a_1(n_2)a_1(n_3) \right|
\]

where the sums \( \sum_{n_1, n_2, n_3 \text{ in AP}} \) are over all triples \( (n_1, n_2, n_3) \) of elements of \( \mathbb{Z}/N'\mathbb{Z} \) in arithmetic progression. Since \( (n_1, n_2, n_3) \) is an arithmetic progression if and only if \( n_1 + n_3 = 2n_2 \),

\[
\Delta = \left| \sum_m \hat{a}(-2m)(\hat{a}(m))^2 - \sum_m \hat{a}_1(-2m)(\hat{a}_1(m))^2 \right| ,
\]

as we can see simply by replacing all Fourier transforms by their definitions and using the fact that \( \sum_m e((n_1 + n_3 - 2n_2)m/N') = 0 \) when \( n_1 + n_3 - 2n_2 \neq 0 \).

We will show that \( \Delta \) is small, namely, \( \Delta \ll \epsilon + \delta \). First note that, since \( a_1 = a \ast \sigma \) and so \( \hat{a}_1 = \hat{a}\hat{\sigma} \),

\[
\Delta \leq \sum_m |\hat{a}(-2m) - (\hat{a}(m))^2||1 - \hat{\sigma}(-2m)(\hat{\sigma}(m))^2| .
\]
For $x \in R$, since $\sigma$ is supported on $B$ and $\sum_n \sigma(n) = 1$, we have

$$|\hat{\sigma}(x) - 1| = \left| \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) e(nx/N') - 1 \right|$$

$$= \left| \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) - 1 + \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \sigma(n) (e(nx/N') - 1) \right|$$

$$\leq \sum_{n \in B} \sigma(n) |e(nx/N') - 1| \ll \sum_{n \in B} \sigma(n) \|nx\| \ll \varepsilon.$$

Similarly, for $x \in R$,

$$|\hat{\sigma}(-2x) - 1| \ll \sum_{n \in B} \sigma(n) \| -2nx\| \ll \sum_{n \in B} \sigma(n) \varepsilon = \varepsilon$$

and so

$$|1 - \hat{\sigma}(-2x)\hat{\sigma}(x)^2| \ll \varepsilon$$

for $x \in R$, i.e., when $|\hat{a}(x)| \geq \delta$.

Before we proceed further, we need to bound $\hat{a}$ in an average sense.

**Lemma 2.2.** For $p > 2$,

$$\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^p \ll_p 1.$$

This is the same as [Gr, Lemma 6.6]; the only difference is that our function $a$ was defined with a much larger modulus $M$ than in [Gr], and thus we must use a restriction theorem for an upper-bound sieve, rather than a restriction theorem for the primes (such as [Bo, (4.39)]). 

**Proof.** Applying [GT, Prop. 4.2] with $F(n) = b + nM$ and $R = N'^{1/10}$, we obtain that, for $p > 2$ and any complex sequence $(b_n)_n$,

$$\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} \left| \frac{1}{N'} \sum_{n=1}^{N'} b_n \beta(n) e(-mn/N') \right|^p \ll_p \left( \frac{1}{N'} \sum_{n=1}^{N'} |b_n|^2 \beta(n) \right)^{p/2},$$

where $\beta$ is an enveloping sieve function with $R = N'^{1/10}$. This means that, according to [GT] Prop. 3.1, $\beta : \mathbb{Z}^+ \to \mathbb{R}$ is a non-negative function satisfying the majorant property

$$\beta(n) \gg \mathcal{S}_F^{-1} \cdot \log R \cdot 1_{X_R}(n)$$

with

$$\mathcal{S}_F = \prod_p \frac{\gamma(p)}{1 - 1/p},$$

$$\gamma(p) = \frac{1}{p} |\{ n \in \mathbb{Z}/p\mathbb{Z}, (p, b + nM) = 1\}| = \begin{cases} (1 - 1/p) & \text{if } p > z \\ 1 & \text{if } p \leq z \end{cases}$$
and

\[ X_R = \{ n \in \mathbb{Z} : \forall d \leq R \ (b + nM, d) = 1 \}. \]

In particular, for any integer \( n \in A_0 \), we have \( n \in X_R \) and

\[ \beta(n) \gg (\log R) \prod_{p \leq z} (1 - 1/p)^{-1} \gg \frac{\log N}{\log z}. \]

We apply (2.5) to the sequence \((b_n)_n\) defined by

\[ b_n = \begin{cases} \frac{1}{\beta(n)}a(n) & \text{if } n \in A_0 \\ 0 & \text{otherwise} \end{cases} \]

and get

\[
\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^p = (N')^p \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} \left| \frac{1}{N'} \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} a(n)e(-mn/N) \right|^p \\
\ll_p (N')^{p/2} \left( \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \frac{1}{\beta(n)}a(n)^2 \right)^{p/2} \\
\ll_p (N')^{p/2} \left( \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} \frac{\log z}{\log N} \left( \frac{\log N}{N' \log z} \right) a(n) \right)^{p/2} \\
\ll_p \left( \sum_{n \in \mathbb{Z}/N'\mathbb{Z}} a(n) \right)^{p/2} \ll_p 1,
\]

since \( a(n) \) was normalised so that \( \sum_n a(n) \ll 1 \).

By Hölder’s inequality and Lemma 2.2 we have

\[
\sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(-2m)\hat{a}(m)|^2 \leq \left( \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/2} \right)^{2/5} \left( \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{10/3} \right)^{3/5} \ll 1.
\]

Hence, by (2.3),

\[
\sum_{m : |\hat{a}(m)| \geq \delta} |\hat{a}(-2m)\hat{a}(m)|^2 \ll \varepsilon,
\]
On the other hand (again by Hölder, and again by Lemma 2.2),
\[
\sum_{m : |\hat{a}(m)| < \delta} |\hat{a}(-2m)\hat{a}(m)|^2 \leq 2 \sum_{m : |\hat{a}(m)| < \delta} |\hat{a}(-2m)\hat{a}(m)|^2
\]
\[
\leq 2 \left( \sum_{m : |\hat{a}(m)| < \delta} |\hat{a}(m)|^{5/2} \right)^{2/5} \left( \sum_{m : |\hat{a}(m)| < \delta} |\hat{a}(m)|^{10/3} \right)^{3/5}
\]
\[
\leq 2 \left( \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/2} \right)^{2/5} \left( \delta^{5/3} \sum_{m \in \mathbb{Z}/N'\mathbb{Z}} |\hat{a}(m)|^{5/3} \right)^{3/5} \ll \delta.
\]
Thus
\[
(2.6) \quad \Delta \ll (\varepsilon + \delta).
\]

2.3. An upper bound for the \( \ell_2 \)-norm of \( a_1 \). Our aim in this subsection is to bound from above the \( \ell_2 \)-norm of of the function \( a_1 = a * \sigma \). (This will later enable us to show that \( a_1 \) is in some sense close to the characteristic function of a set of large density in the integers.) We will prove that \(|a_1|_2 \ll 1/\sqrt{N'}\), where the implied constant is absolute.

Recall that we write \( \pi \) for the reduction map \( \pi : \mathbb{Z} \rightarrow \mathbb{Z}/N'\mathbb{Z} \). Given a function \( f : \mathbb{Z}/N'\mathbb{Z} \rightarrow \mathbb{C} \), we can lift it to a function \( \tilde{f} : \mathbb{Z} \rightarrow \mathbb{C} \) supported on the interval \([-(N' - 1)/2, (N' - 1)/2]\):
\[
\tilde{f}(n) = \begin{cases} f(n \mod N') & \text{if } n \in \left[ -\frac{N' - 1}{2}, \frac{N' - 1}{2} \right], \\ 0 & \text{otherwise.} \end{cases}
\]
By the definition of \( A_0 \) and \( a \), we see that \( A_0 \subset \pi ([1, (N' - 1)/2]) \), and thus \( a \) is supported on \( \pi ([1, (N' - 1)/2]) \). By the definition of \( R, B \) and \( \sigma \) and the assumption \( \varepsilon < 1/4 \), we see that \( \sigma \) is supported on \( \pi ([-N'/4, N'/4]) \). Thus \(|a * \sigma|_2 = |\hat{a} * \hat{\sigma}|_2|\).

By the definition of \( a \), we have \( 0 \leq \hat{a}(n) \leq \lambda(n) \), where \( \lambda : \mathbb{Z} \rightarrow \mathbb{R} \) is defined by
\[
\lambda(n) = \begin{cases} \log N / N' \log z & \text{if } 1 \leq n \leq N' \text{ and } b + nM \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}
\]
Recall that \( \sigma \) is non-negative, and thus \( \hat{\sigma} \) is non-negative. Hence \( |a * \hat{\sigma}|_2 \leq |\lambda * \hat{\sigma}|_2 \).

We conclude that
\[
|a_1|_2 = |a * \sigma|_2 = |\hat{a} * \hat{\sigma}|_2 \leq |\lambda * \hat{\sigma}|_2.
\]
It is thus our task to prove that \(|\lambda * \hat{\sigma}|_2 \ll 1/\sqrt{N'}\).

We proceed as follows:
\[
(2.8) \quad \sum_n |\hat{\sigma} * \lambda(n)|^2 = \sum_n \left| \sum_m \hat{\sigma}(m) \lambda(n - m) \right|^2
\]
\[
= \sum_{m_1} \sum_{m_2} \hat{\sigma}(m_1) \hat{\sigma}(m_2) \sum_n \lambda(n + m_1) \lambda(n + m_2),
\]
where we recall that \( \sigma(m) = \sigma(-m) \) (by the definition of \( B \) and \( \sigma \)).
Lemma 2.3. Let $\lambda$ be as in (2.7). Then, for any integers $m_1$, $m_2$,

$$(2.9) \sum_n \lambda(n + m_1) \lambda(n + m_2) \ll \begin{cases} \frac{1}{N} \prod_{p \mid (m_1 - m_2), \, p > z} \frac{p}{p - 1} & \text{if } m_1 = m_2, \\ \log N / (N' \log z) & \text{if } m_1 \neq m_2, \end{cases}$$

where the implied constant is absolute.

Proof. The case $m_1 = m_2$ follows from Brun-Titchmarsh:

$$\sum_n \lambda^2(n + m) = \left( \frac{\log N}{N' \log z} \right)^2 \{m \leq n \leq N' + m : b + (n - m)M \text{ is prime }\} \ll \left( \frac{\log N}{N' \log z} \right)^2 \frac{N'M}{\varphi(M) \log N'} \ll \left( \frac{\log N}{N' \log z} \right)^2 \frac{N'}{\log N'} \prod_{p \mid M} (1 - 1/p)^{-1} \ll \log N / N' \log z.$$ 

To obtain the case $m_1 \neq m_2$, we will use a result based on Selberg’s sieve. (This is a familiar type of application of upper-bound sieves, similar to the proof that the number of twin primes up to $N$ is at most a constant times its conjectured value.)

It is clear that

$$(2.10) \quad \{1 \leq n \leq N' : b + nM \text{ and } b + (n + m_2 - m_1)M \text{ are primes}\}.$$ 

By [HR] Thm. 5.7,

$$(2.10) \ll \prod_p \left( 1 - \frac{\rho(p) - 1}{p - 1} \right) \left( 1 - \frac{1}{p} \right)^{-1} \frac{N'}{\log N'}^2,$$

where the implied constant is absolute. (We are implicitly using the fact that $\log M \ll \log N'$, and thus the term in the third line of [HR] (8.3) is $= 1 + o(1)$.)

Here $\rho(p)$ is the number of solutions $x \in \mathbb{Z}/p\mathbb{Z}$ to

$$(b + xM)(b + (x + m_2 - m_1)M) \equiv 0 \mod p$$

for $p$ prime. It is easy to see that $\rho(p) = 0$ if $p \mid M$ (i.e., if $p \leq z$), $\rho(p) = 1$ if $p > z$ and $p \mid (m_2 - m_1)$, and $\rho(p) = 2$ if $p > z$ and $p \nmid (m_2 - m_1)$. Hence

$$(2.10) \ll \prod_{p \leq z} \left( 1 - \frac{1}{p} \right)^{-2} \prod_{p > z} \left( 1 - \frac{1}{p} \right)^{-1} \frac{N'}{\log N'}^2 \ll \prod_{p > z} \frac{p}{p - 1} \frac{(\log z)^2}{\log N'}.$$ 

The statement follows. \qed
Let us now evaluate the last line of (2.8), with Lemma 2.3 in hand. The contribution of the diagonal terms \((m_1 = m_2)\) in (2.8) is \(\ll \log N / (|B|N' \log z)\). The contribution of the non-diagonal terms \((m_1 \neq m_2)\) is

\[
\ll \frac{1}{N'} \sum_{m_1 \neq m_2} \hat{\sigma}(m_1) \hat{\sigma}(m_2) \prod_{p > z \atop p \mid |m_1 - m_2|} \frac{p}{p - 1}.
\]

(2.11)

Recall that \(\tilde{\sigma}\) is supported on \([-N'/4, N'/4]\), and thus \(|m_2 - m_1| \leq N'/2 < N'\) whenever \(\tilde{\sigma}(m_1) \tilde{\sigma}(m_2) \neq 0\).

Now, a non-zero integer \(m\) with \(|m| \leq N'\) cannot have more than \(\log N' / \log z\) prime factors \(p > z\). Since \(x \mapsto x/(x - 1)\) is decreasing on \(x\), this means that

\[
\prod_{p > z \atop p \mid m} p \frac{1}{p - 1} \leq \left(\frac{z}{z - 1}\right)^{(\log N') / (\log z)}.
\]

Now \((z/(z - 1))^2 \ll 1\) (because \(\lim_{n \to \infty} (1 + 1/n)^n = e\)) and

\[
\frac{\log N'}{\log z} \ll \frac{\log N}{\log \log N} < \log N \ll z.
\]

Hence

\[
\prod_{p > z \atop p \mid m} p \frac{1}{p - 1} \ll 1
\]

for any \(m \neq 0\) with \(|m| \leq N'\). Thus

\[
(2.11) \ll \frac{1}{N'} \sum_{m_1 \neq m_2} \hat{\sigma}(m_1) \hat{\sigma}(m_2) \ll \frac{1}{N'}.
\]

Putting everything together, we conclude that

\[
\sum_n |\tilde{\sigma} * \lambda(n)|^2 \ll \frac{1}{N'} \left(\frac{\log N}{|B| \log z} + 1\right).
\]

The right side is \(\ll 1/N'\) as long as \(|B| \gg \log N / \log z\).

Now, as is well-known (see, e.g., [TV Lem. 4.20]),

\[
|B| \gg \varepsilon^r N',
\]

where \(r = |R|\). (The proof of this is a simple pigeonhole argument.) Since by (2.4) we have \(\sum_m |\hat{a}(m)|^{5/2} \ll 1\), we know that that the set of \(x \in \mathbb{Z}/N'\mathbb{Z}\) with \(|\hat{a}(x)| \geq \delta\) has at most \(\ll \delta^{-5/2}\) elements. Thus, \(r \ll \delta^{-5/2}\).

Hence all that we need for \(|B| \geq \log N / \log z\) to hold is that \(\varepsilon \delta^{-5/2} \geq N^{-1/2}\) (say). In other words, we need \(\log |\varepsilon| \cdot \delta^{-5/2} \leq \frac{1}{2} \log N\). We will recall that we need to satisfy this condition at the end.
2.4. Extracting a dense set from $a_1$. We now have a function $a_1: \mathbb{Z}/N\mathbb{Z} \to \mathbb{R}_0^+$ of $\ell_2$ norm $\ll 1/\sqrt{N'}$. Its $\ell_1$ norm is $\gg \alpha$, where $\alpha$ is the density of our original set $A$ on the primes. We must show that there is a large set on which $a_1$ is large.

Lemma 2.4. Let $S$ be a set with $N'$ elements. Let $a: S \to \mathbb{R}_0^+$ and $0 < \alpha < 1$ be such that
\begin{enumerate}[(a)]  
  \item $\|a\|_1 \geq \alpha$;  
  \item $\|a\|_2^2 \leq c/N'$.  
\end{enumerate}
Then there exists a subset $A'$ of $S$ such that
\begin{enumerate}[(a)]  
  \item $|A'| \geq \alpha^2 N'/ \left(4cN'\right)$;  
  \item $\forall n \in A'$, $a(n) \geq \alpha/(2N')$.  
\end{enumerate}

Proof. If $A' = \{ n : a(n) \geq \alpha/(2N') \}$, then
\[
\alpha \leq \sum_n a(n) \leq \frac{\alpha}{2N'}(N' - |A'|) + \sum_{n \in A'} a(n) 
\leq \frac{\alpha}{2N'}(N' - |A'|) + \sqrt{|A'|} \sqrt{\frac{c}{N'}},
\]
by $\|a\|_2 \leq c/N'$ and Cauchy’s inequality. In other words, $f(\sqrt{|A'|}) \leq 0$, where $f(x) = \frac{\sqrt{cN'}}{2N'} \left( x^2 - 2\frac{\sqrt{cN'}}{\alpha} x + N' \right)$. Completing the square, we see that $f(x) \leq 0$ implies $x \geq \frac{\sqrt{cN'}}{\alpha} - \sqrt{\left( \frac{\sqrt{cN'}}{\alpha^2} - 1 \right) N'}$. Hence
\[
|A'| \geq N' \cdot \left( \frac{\sqrt{c}}{\alpha} - \sqrt{\frac{c}{\alpha^2} - 1} \right)^2 \geq \frac{\alpha^2 N'}{4c}.
\]
\[\square\]

We apply Lemma 2 to $a_1$ with the bound $\|a_1\|_2^2 \leq c/N'$ being provided by our work in §2.3. We get a subset $A'$ of $\mathbb{Z}/N\mathbb{Z}$ such that $|A'| \geq \alpha^2 N'/ (4c) \gg \alpha^2 N'$ and $a_1(n) \geq \alpha/(2N')$ for every $n \in A'$. Hence
\[
(2.12) \quad \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{\alpha^3 Z}{8N'^3},
\]
where $Z$ is the number of 3-term arithmetic progressions in $A'$.

Lemma 2.5. Let $A' \subset \mathbb{Z}/N'\mathbb{Z}$, where $N'$ is a prime. Assume $|A'| \geq \eta N'$, $\eta > 0$. The number of 3-term arithmetic progressions in $A'$ is then at least
\[
\frac{\eta N'^2}{c_0 \exp \left( c_1 \eta^{-3/2}(\log(1/\eta))^{3/2} \right)},
\]
where $c_0$ and $c_1$ are absolute constants.

Proof. We will proceed much as in [Gr, Lem. 6.8]; the basic argument goes back to Varnavides [Va]. Bourgain’s best result on three-term arithmetic progressions in the integers [Bo3, Thm. 1] states that, for given $L$ and $\eta \gg (\log \log L)^2(\log L)^{-2/3}$, every
subset of \( \{1, 2, \ldots, L\} \) with \( \geq \eta L \) elements contains at least one non-trivial three-term arithmetic progression. This can be rephrased as follows: there are constants \( c_0 \) and \( c_1 \) such that, if \( L \geq c_0 \exp \left( c_1 \eta^{-3/2} (\log(1/\eta))^3 \right) \), then any subset of \( \{1, \ldots, L\} \) of density at least \( \eta/2 \) contains a non-trivial three-term arithmetic progression. (Here we are simply expressing \( L \) in terms of the density, rather than the density in terms of \( L \).)

It follows that, given an arithmetic progression \( S_{a,d} = \{a + d, a + 2d, a + 3d, \ldots, a + Ld\} \) in \( \mathbb{Z}/N'\mathbb{Z} \) (\( a, d \in \mathbb{Z}/N'\mathbb{Z} \), \( d \neq 0 \), \( L \leq N' \)) whose intersection with \( A' \) has at least \( (\eta/2)L \) elements, there is at least one non-trivial three-term arithmetic progression in \( A' \cap S \subset \mathbb{Z}/N'\mathbb{Z} \). (Note that there is no need for the progression \( S \) to be the reduction mod \( N' \) of a progression in the integers \( \{1, 2, \ldots, N'\} \); the argument works regardless of this.) If we consider all arithmetic progressions of length \( L \) and given modulus \( d \neq 0 \) in \( \mathbb{Z}/N'\mathbb{Z} \), we see that each element of \( A' \) is contained in exactly \( L \) of them. Hence, \( \sum_{d \neq 0} |S_{a,d} \cap A'| = |A'| \geq \eta N'L \), and so (for \( d \neq 0 \) fixed) \( |S_{a,d} \cap A'| \geq (\eta/2)L \) for at least \( (\eta/2)N' \) values of \( a \). Varying \( d \), we get that \( |S_{a,d} \cap A'| \geq (\eta/2)L \) for at least \( (\eta/2)N'(N' - 1) \) arithmetic progressions \( S_{a,d} \). By the above, each such intersection \( S_{a,d} \cap A' \) contains at least one non-trivial three-term arithmetic progression.

Each non-trivial three-term arithmetic progression \( a_1, a_2, a_3 \) in \( \mathbb{Z}/N'\mathbb{Z} \) can be contained in at most \( L(L - 1) \) arithmetic progressions \( \{a + d, a + 2d, \ldots, a + Ld\} \) of length \( L \) (the indices of \( a_1 \) and \( a_2 \) in the progression of length \( L \) determine the progression). Hence, when we count the three-term arithmetic progressions coming from the intersections \( S_{a,d} \cap A' \), we are counting each such progression at most \( L(L - 1) \) times. Thus we have shown that \( A' \) contains at least

\[
\frac{\eta}{2} \frac{N'(N' - 1)}{L(L - 1)} \geq \frac{\eta}{2} \frac{N^2}{L^2}
\]

distinct non-trivial three-term arithmetic progressions for

\[
L = \left\lfloor c_0 \exp \left( c_1 \eta^{-3/2} (\log(1/\delta))^3 \right) \right\rfloor,
\]

provided that \( L \leq N' \). If \( L > N' \), the bound in the statement of the lemma is trivially true (as there is always at least one trivial three-term arithmetic progression in \( A' \)). \( \square \)

From (2.12) and Lemma 2.5 we conclude that

\[
(2.13) \quad \sum_{m,d} a_1(m)a_1(m + d)a_1(m + 2d) \geq \frac{\alpha^3}{8N'^3} \frac{\alpha^2}{8c} \frac{(N')^2}{c_0 \exp(c_1 (\alpha^2/4c)^{-3/2} (\log(4c/\alpha^2))^3)} \geq \frac{1}{N' c_2 \exp(c_3 \alpha^{-3} (\log(1/\alpha))^3)},
\]

where \( c_2, c_3 > 0 \) are absolute constants.

3. Conclusion

Assume that \( A \) contains no non-trivial three-term arithmetic progressions. Then \( A_0 \) (defined in (2.11)) contains no non-trivial three term arithmetic progressions, and
so
\[ \sum_{m,d} a(m)a(m+d)a(m+2d) = \sum_{m,d} a(m)^3 \ll \left( \frac{\log N}{N' \log z} \right)^2. \]

We also have
\[ \Delta = N' \left| \sum_{m,d} a(m)a(m+d)a(m+2d) - \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \right| \ll (\varepsilon + \delta), \]
by the definition (2.2) and (2.6). Lastly, we have just shown that
\[ \sum_{m,d} a_1(m)a_1(m+d)a_1(m+2d) \geq \frac{1}{N' c_2 \exp \left( c_3 \alpha^{-3}(\log(1/\alpha))^3 \right)} \]
(see (2.13)). We conclude that
\[ (3.1) \frac{1}{c_2 \exp \left( c_3 \alpha^{-3}(\log(1/\alpha))^3 \right)} \ll \varepsilon + \delta + \frac{1}{N' \left( \log N \log z \right)^2}. \]

Recall that \( z = (\log N)/3 \). There are constants \( c_4, c_5 \) such that, for
\[ \delta = \varepsilon = \frac{1}{c_4} \exp \left( -c_5 \alpha^{-3} \log^3(1/\alpha) \right), \]
we get a contradiction with (3.1), provided that \( N \) is larger than an absolute constant and \( \alpha \geq (\log N)^{-1/4} \), say. These values of \( \delta \) and \( \varepsilon \) satisfy \( |\log \varepsilon| \delta^{-2.5} \leq (\log N)/2 \) as long as
\[ (\log(c_4) + c_5 \alpha^{-3} \log^3(1/\alpha)) \cdot c_4^{2.5} \exp \left( 2.5 c_5 \alpha^{-3} \log^3(1/\alpha) \right) \leq \frac{1}{2} \log N. \]

Therefore we have a contradiction if \( \alpha \geq C \log \log \log N \log \log N \) is larger than an absolute constant. Theorem 1.1 is thereby proven.

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