GELFAND PAIRS AND SPHERICAL FUNCTIONS FOR IWAHORI-HECKE ALGEBRAS

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Abstract. We introduce the notion of Gelfand pairs and zonal spherical functions for Iwahori-Hecke algebras.

1. Introduction

This short note is a spin off of the collaboration work [HHY] with Masahito Hayashi and Akihito Hora, and it was originally planned to be included as Appendix D of that paper. The aim is to give an analogue of the theory of spherical functions for Iwahori-Hecke algebras. To explain the motivation, let us briefly explain the parts of [HHY] which concern this note.

In [HHY], a certain discrete probability distribution is introduced, which originates in the irreducible decomposition of the SU(2)-$\mathfrak{S}_n$-bimodule

$$(\mathbb{C}^2)^{\otimes n} = \sum_{x=0}^{[n/2]} U_{(n-x,x)} \otimes V_{(n-x,x)},$$

i.e., the tensor space appearing in the classical Schur-Weyl duality of the special unitary group SU(2) and the symmetric group $\mathfrak{S}_n$. Here we denote by $V_{(n-x,x)}$ the irreducible $\mathfrak{S}_n$-representation associated to the partition $(n-x,x)$, and by $U_{(n-x,x)}$ the corresponding irreducible SU(2)-representation. The distribution has four non-negative integer parameters $n, m, k, l$, and the probability mass function is denoted by $p(x) = p(x \mid n, m, k, l)$ for $x = 0, 1, \ldots, [n/2]$.

The function $p(x)$ has surprisingly rich property. For example, it can be written as a summation of Hahn polynomials [HHY, Theorem 2.2.1]. Hahn polynomials are $3F_2$-hypergeometric orthogonal polynomials, and appear in the computation as values of zonal spherical functions for the Gelfand pair $(\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$. In addition, the function $p(x)$ can be written by a single Racah polynomial [HHY, Theorem 2.2.2], which is a $4F_3$-hypergeometric orthogonal polynomial sitting in the first line of the Askey scheme. Moreover, the cumulative distribution function $\sum_{u=0}^{x} p(u)$ can also be written by a terminating $4F_3$-hypergeometric series [HHY, Theorem 2.2.4]. In the course of deriving these theorems, some hypergeometric summation formulas are also obtained [HHY, Corollaries 2.2.3 and 2.2.5].

These arguments have natural $q$-analogues, which are discussed in [HHY, Appendix C]. There is introduced a $q$-analogue of $p(x)$ [HHY, (C.2.11)], and are obtained the corresponding basic hypergeometric summation formulas [HHY, Corollaries C.3.2 and C.3.4]. Although we may say that those computations are natural $q$-analogues, the setting is a little bit artificial in the viewpoint of representation theory. In [HHY, Appendix C], instead of considering the $n$-tensor space (1.1), we consider

$$(\mathbb{C}^2)^{\otimes n} = (\mathbb{C}^2)^{[n]_q},$$

where $\mathbb{F}_q$ denotes the finite field of order $q$, $\mathbb{P}^1(\mathbb{F}_q^n)$ denotes the projective line for the linear space $\mathbb{F}_q^n$, and $[n]_q = 1 + q + \cdots + q^{n-1}$ is the $q$-integer. In other words, we take the formula $|[\mathbb{P}^1(\mathbb{F}_q^n)]| = [n]_q$ of the number of points as a clue of $q$-deformation. As for the group action, instead of the permutation action of $\mathfrak{S}_n$ on $(\mathbb{C}^2)^{\otimes n}$, we consider the action of the Chevalley group $\text{GL}(n, \mathbb{F}_q)$ on $(\mathbb{C}^2)^{\otimes n}$ which is induced by the natural action of that on $\mathbb{F}_q^n$. The usage of Hahn polynomials as zonal spherical functions for the Gelfand pair $(\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$ is then replaced by $q$-Hahn polynomials as zonal spherical functions for the pair $(\text{GL}(n, \mathbb{F}_q), P(m, n-m, \mathbb{F}_q))$, where $P(m, n-m, \mathbb{F}_q)$ denotes the maximal parabolic subgroup of $\text{GL}(n, \mathbb{F}_q)$ with block sizes $m$ and $n-m$.

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As mentioned at the end of the introduction of [HHY, Appendix C], we have another candidate of $q$-analog for (1.1): the $q$-Schur-Weyl duality between the quantum group $U_q(\mathfrak{sl}_2)$ and the Iwahori-Hecke algebra $H_q(\mathfrak{S}_n)$ acting on the same tensor space $(\mathbb{C}^2)^\otimes n$ as (1.1), discovered by Jimbo [J86]. This one looks more natural in the representation theoretic viewpoint.

However, there are several missing items in this $q$-Schur-Weyl duality to make a $q$-analogue of the arguments of [HHY]. One of them is the theory of Gelfand pairs and spherical functions for Iwahori-Hecke algebras. As sketched above, we used in [HHY] the corresponding theory for finite groups, for which we refer [M95, §7.1]. Since Iwahori-Hecke algebras are deformation of group algebras of Coxeter groups, it is natural to expect such a theory exists, although we could not find it in literature.

The purpose of this note is to give the start line of the theory of Gelfand pairs and zonal spherical functions for Iwahori-Hecke algebras, which will be demonstrated in §3. The main objects introduced are Gelfand pairs in Definition 3.1 and zonal spherical functions in Definition 3.2. The main Theorem 3.3 gives fundamental properties of our zonal spherical functions, which can be regarded as a Hecke-analogue of the properties for finite groups [M95, VII.1, (1.4)].

Below is the list of notations used in the main text.

1. We denote by $\mathbb{N} := \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ the set of non-negative integers.
2. For a finite set $S$, we denote by $|S|$ the number of elements in $S$.
3. For a group $G$, we denote by $e \in G$ the unit element.

2. Recollection on Iwahori-Hecke algebra

As a preliminary of the main §3, we recall several basic notions on the structure and representation theories of Hecke algebras. Our main reference is [GP00]. Hereafter, Iwahori-Hecke algebras will be just called Hecke algebras.

2.1. Generic Hecke algebra. Here we explain our notation for the Iwahori-Hecke algebra, borrowing terminology and symbols from [GP00]. We make one major change on symbols: we denote the generic Hecke algebra by $H$, whereas it is denoted by $KH$ in [GP00].

Let $W$ be a finite Coxeter group with generating set $S$. We denote by $\ell(w)$ the length of a minimal expression of $w \in W$ with respect to $S$. Let $A$ be a unital commutative ring, and take a parameter set $\{a_s \mid s \in S\} \subset A$ such that $a_s = a_t$ if $s, t \in S$ are conjugate in $W$. We denote by

$$H = H_A(W, S, \{a_s \mid s \in S\})$$

the Iwahori-Hecke algebra associated to the Coxeter system $(W, S)$ over $A$, and simply call it the Hecke algebra. It is a unital associative $A$-algebra with generators $\{T_s \mid s \in S\}$ and relations

$$\begin{align*}
T_s^2 &= (a_s - 1)T_s + a_s & (s \in S), \\
(T_sT_t)^{m_{st}} &= 1 & (s, t \in S, s \neq t, m_{st} < \infty).
\end{align*}$$

The first relation is called the quadratic relation. In the second relation we denote by $m_{st}$ the order of the element $st$ in $W$. Let us recall some basic properties of $H$. For the detail and proofs we refer [GP00, 4.4.3, 4.4.6].

1. For each $w \in W$, take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. Then

$$T_w := T_{s_{i_1}} \cdots T_{s_{i_l}} \in H$$

is independent of the choice of the reduced expression. In particular, we have $T_e = 1$ for the unit $e \in W$. Also the quadratic relation is rewritten as the factorized form $(T_s + 1)(T_s - w) = 0$.

2. For $s \in S$ and $w \in W$, we have

$$T_sT_w = \begin{cases} T_{sw} & (\ell(sw) > \ell(w)), \\
(a_s - 1)T_{sw} + uT_w & (\ell(sw) < \ell(w)).
\end{cases}$$

3. As an $A$-module, $H$ is free and finitely generated, and has an $A$-basis $\{T_w \mid w \in W\}$. We always assume $a_s$ are invertible in $A$. Then the basis element $T_w$ is invertible in $H$ for any $w \in W$ by [GP00, 8.1.1].

We recall two types of modules over the Hecke algebra $H$. The first one is the index representation given by

$$\text{ind}: H \rightarrow A, \ T_w \mapsto a_w \ (w \in W),$$

(2.1)
and the second one is the sign representation given by

\[ \text{sgn}: H \rightarrow A, \quad T_w \mapsto (-1)^{f(w)} \quad (w \in W). \]  

(2.2)

The well-definedness of these modules are shown by the quadratic relation \((T_e + 1)(T_e - u) = 0\).

Next we cite from [GP00, §8.1] the specialization argument of the Hecke algebra. Let \(\{u_s \mid s \in S\}\) be a set of indeterminates over \(\mathbb{C}\) such that \(u_s = u_t\) for \(s, t \in S\) conjugate in \(W\). We set \(A := \mathbb{Z}[u^{\pm 1}]\). The generic Hecke algebra \(H_A\) is defined to be the Hecke algebra over \(A\):

\[ H_A := H_A(W, S, \{u_s \mid s \in S\}). \]

Now assume that there is a ring map \(\theta: A \rightarrow \mathbb{C}\) with \(\theta(u_s) = q \in \mathbb{C} \setminus \{0\}\), which will be called a specialization map. The map \(\theta\) induces a \(\mathbb{C}\)-algebra, which is denoted by

\[ H_q := H_A \otimes_A \mathbb{C}. \]

Then by [GP00, 8.1.7], there exists a finite Galois extension \(K \supset \mathbb{C}(u)\) such that the \(K\)-algebra

\[ H := H_A \otimes_A K \]

is semisimple and isomorphic to the group algebra \(K[W]\). Abusing the terminology, we also call \(H\) the generic Hecke algebra. Moreover, if the parameter \(q \in \mathbb{C} \setminus \{0\}\) is taken such that \(H_q\) is semisimple, then the specialization map \(H \rightarrow H_q\) induces a bijection

\[ \text{Irr}(H) \overset{\sim}{\longrightarrow} \text{Irr}(H_q) \]  

(2.3)

between the sets of isomorphism classes of finite-dimensional simple modules over \(H\) and \(H_q\) respectively. In particular, for a semisimple \(H_q\), the set \(\text{Irr}(H_q)\) is independent of \(q\).

It is known that for a generic complex number \(q\), the algebra \(H_q\) is semisimple. A particular case is \(q = 1\), where we have \(H_1 = \mathbb{C}[W]\), and thus isomorphism classes of finite-dimensional simple \(H\)-modules correspond bijectively to those of finite-dimensional irreducible representations of \(W\). Also note that the index representation (2.1) and the sign representation (2.2) correspond respectively to the trivial and sign representations of the Coxeter group \(W\) under the specialization \(q = 1\).

The generic Hecke algebra \(H\) over \(K\) is split semisimple in the sense of [GP00, Chap. 7]. Let us explain what it means. Since \(H\) is a semisimple \(K\)-algebra, there is a direct sum decomposition \(H = \bigoplus V H(V), \) where \(V\) runs over the simple \(H\)-modules, and each \(H(V)\) is a simple \(K\)-algebra. Thus \(H(V)\) is isomorphic to a matrix algebra \(M_{n_V}(D_V)\), where \(n_V\) is the multiplicity of \(V\) in the decomposition of \(H\) as a module over itself, and \(D_V\) is a division algebra over \(K\). We also have \(D_V \simeq \text{End}_H(V)\) and \(\dim_K V = n_V \dim_K D_V\). Now every simple \(H\)-module \(V\) is split simple. i.e., \(\dim_K D_V = 1\). This is the defining condition of split semisimplicity.

Let us close this subsection by the case \(W = S_n\) with \(S = \{s_1, \ldots, s_{n-1}\}\) being the set of transpositions \(s_i = (i, i+1)\). All of the generators \(s_i\) are conjugate in \(W\), and the associated Hecke algebra \(H\) has a unique parameter \(q\). Similarly, the generic Hall algebra \(H\) has a unique parameter \(u\). The Hecke algebra \(H_q(S_n)\) mentioned in §1 is nothing but \(H_2(W, S, q)\). As for the simple modules, the set \(\text{Irr}(H)\) of finite-dimensional simple \(H\)-modules sits in a series of bijections

\[ \text{Irr}(H) \overset{\sim}{\longrightarrow} \text{Irr}(H_1) \overset{\sim}{\longrightarrow} \{\lambda \vdash n\}, \]

(2.4)

where \(\lambda \vdash n\) means that \(\lambda\) is a partition of \(n\). We will denote by \(V_\lambda\) the simple \(H\)-module corresponding to \(\lambda \vdash n\) under (2.4). Note that the index representation (2.1) is \(V_\lambda\) and the sign representation sgn (2.2) is \(V_{(1^n)}\).

2.2. Schur elements. We continue to use the symbols in the previous subsection. Thus \(H\) denotes the Hecke algebra \(H\) associated to a finite Coxeter system \((W, S)\) defined over a commutative ring \(A\).

We recall trace functions and their relation to the center \(Z(H)\) of \(H\), referring [GP00, §7.1] for the detail. An \(A\)-linear map \(f \in H^* := \text{Hom}_A(H, A)\) is called a trace function on \(H\) if \(f(hh') = f(h'h)\) for any \(h, h' \in H\). We denote the \(A\)-module of trace functions on \(H\) by

\[ \text{TF}(H) := \{f: H \rightarrow A \mid \text{trace functions on } H\}. \]

\(H\) is a symmetric algebra in the sense of [GP00, 7.1.1], i.e., it is equipped with a trace function \(\tau: H \rightarrow A\) such that the \(A\)-bilinear form

\[ H \otimes_A H \rightarrow A, \quad h \otimes h' \mapsto \tau(hh') \]  

(2.5)

is non-degenerate. Such a map \(\tau\) is explicitly given by

\[ \tau: H \rightarrow A, \quad \tau(T_e) = 1, \quad \tau(T_w) = 0 \quad (w \neq e). \]
We sometimes denote it by $\tau_H$ for the distinction.

For $w \in W$ with a reduced expression $w = s_1 \cdots s_l$, $s_i \in S$, we set $a_w := a_{s_1} \cdots a_{s_l} \in A$. By [GP00, 8.1.1], we have

$$
\tau(T_w^wT_{w'}) = \delta_{w,w'}, \quad T_w^w := a_w^{-1}T_{w^{-1}} \in H \quad (w, w' \in W).
$$

Then for $f \in H^*$, we set

$$
f^* := \sum_{w \in W} f(T_w)T_w^w \in H.
$$

By [GP00, 7.1.7], we have

$$
f \in TF(H) \iff f^* \in Z(H).
$$

An important class of trace functions is given by the character of a $\mathbf{K}$-module. Let $V$ be an $H$-module which is finitely generated and free as an $\mathbf{A}$-module, and $\rho_V : H \to \text{End}_\mathbf{K}(V)$ be the corresponding $\mathbf{A}$-algebra homomorphism. The character of $V$ is defined to be the $\mathbf{A}$-linear map

$$
\chi_V : H \to \mathbf{A}, \quad \chi_V(h) := \text{tr}_V \rho_V(h),
$$

where $\text{tr}_V$ denotes the usual matrix trace. By the property of matrix trace, the character is a trace function, i.e., $\chi_V \in TF(H)$.

Next we introduce Schur elements, following [GP00, §7.2]. Hereafter we consider the Hecke algebra $H$ defined over a field $\mathbf{K}$, instead of a commutative ring $\mathbf{A}$. For $H$-modules $V, V'$ and $\varphi \in \text{Hom}_\mathbf{K}(V, V')$, we define $I(\varphi) \in \text{Hom}_\mathbf{K}(V, V')$ by

$$
I(\varphi)(v) := \sum_{w \in W} \varphi(vT_w)T_w^w \quad (v \in V).
$$

By [GP00, 7.1.10], we have $I(\varphi) \in \text{Hom}_H(V, V')$. Now assume that $V$ is a split simple $H$-module as explained in the previous subsection. Then by [GP00, 7.2.1], there is a unique element $c_V \in \mathbf{K}$ such that

$$
I(\varphi) = c_V \text{tr}_V(\varphi) \text{id}_V
$$

for any $\varphi \in \text{End}_\mathbf{K}(V)$, depending only on the isomorphism class of $V$. The element $c_V$ is called the Schur element associated to $V$.

Let us explain representation theoretic meaning of Schur elements. Let $V, V'$ be finite-dimensional simple $H$-modules. Then by [GP00, 7.2.2], we have

$$
\sum_{w \in W} \chi_V(T_w)\chi_V'(T_w^w) = \delta_{V,V'}c_V \dim_\mathbf{K} V,
$$

where $\delta_{V,V'} = 1$ if $V \simeq V'$, and $\delta_{V,V'} = 0$ otherwise. Following [GP00, 8.1.8], we define the Poincaré polynomial $P_V$ of the Coxeter group $W$ to be the Schur element $c_{\text{ind}}$ associated to the index representation $\text{ind}(2.1)$ of the generic Hecke algebra $H$. By (2.9) and (2.6), it is given by

$$
P_V := c_{\text{ind}} = \sum_{w \in W} \text{ind}(T_w)\text{ind}(T_w^w) = \sum_{w \in W} \text{ind}(T_w) \in \mathbb{N}[u_s \mid s \in S].
$$

Schur elements are also related to idempotents of the generic Hecke algebra $H$, as shown in [GP00, 7.2.7]. Since $H$ is split semisimple as explained in the previous §2.1, it has a decomposition

$$
H = \bigoplus_{V \in \text{Irr}(H)} H(V)
$$

into simple $\mathbf{K}$-algebras $H(V)$, where $V$ runs over finite-dimensional simple $H$-modules, and $H(V) \simeq M_{n_V}(\mathbf{K})$ with $n_V := \dim_\mathbf{K} V$. We have the corresponding decomposition of the unit 1 of $H$:

$$
1 = \sum_{V \in \text{Irr}(H)} e_V, \quad e_V \in H(V).
$$

Then $e_V$ are central primitive idempotents and mutually orthogonal, i.e., $e_V e_{V'} = 0$ if $V \nless V'$. We have $e_V H \simeq V$ as $H$-modules. Moreover, by [GP00, 7.2.6], the Schur element $c_V$ is non-zero for any $V \in \text{Irr}(H)$, and using the character $\chi_V$ we have

$$
e_V = \frac{1}{c_V} \sum_{w \in W} \chi_V(T_w)T_w^w.
$$
In particular, using (2.6) and (2.10), we can compute the idempotent $e_{\text{ind}}$ corresponding to the index representation $\text{ind} \ (2.1)$ as

$$e_{\text{ind}} = \frac{1}{p_W} \sum_{w \in W} T_w. \quad (2.12)$$

For a finite-dimensional simple $H$-module $V$, let $\chi_V \in H$ be the element associated to the character $\chi_V$ by the correspondence (2.7). Since $\chi_V$ is a trace function, we have $\chi_V \in Z(H)$ by the equivalence (2.8). Moreover, by [GP00, 7.2.8], the elements $\{\chi_V \mid V \in \text{Irr}(H)\}$ form a $K$-basis of $Z(H)$, and

$$\chi_V^* = c_V \chi_V. \quad (2.13)$$

The idempotence and orthogonality of $c_V$ yields

$$\chi_V^* \chi_{V'} = \delta_{V,V'} c_V \chi_{V'} \quad (V, V' \in \text{Irr}(H)). \quad (2.14)$$

Let us consider the behavior of objects discussed so far under the specialization $H \rightarrow H_{q=1} = \mathbb{C}[W]$ induced by $K \rightarrow \mathbb{C}$, $u \mapsto 1$. Obviously we have $T_w \mapsto w$ and $T_w^\vee \mapsto w^{-1}$ for each $w \in W$. For a finite-dimensional $H$-module $V$, we denote by $V_{q=1}$ the representation of $W$ induced by the bijection (2.3) of finite-dimensional simple modules. Then by [GP00, Example 7.2.5], the specialization yields

$$c_V \mapsto \frac{|W|}{\dim V_{q=1}}.$$

Thus we may say that the Schur element $c_V$ is a “$q$-analogue” of the quantity $|W|/\dim V_{q=1}$. The relation (2.9) reduces to the orthogonal relation for characters $\chi_{V_{q=1}}$ of the finite group $W$:

$$\frac{1}{|W|} \sum_{w \in W} \chi_{V_{q=1}}(w) \chi_{V_{q=1}}^\vee(w^{-1}) = \delta_{V,V'}.$$

The Poincaré polynomial $P_W = e_{\text{ind}} \ (2.10)$ reduces to $|W|$, and the idempotent $c_V \ (2.11)$ for a simple $H$-module $V$ reduces to

$$e_{V_{q=1}} = \frac{\dim V_{q=1}}{|W|} \sum_{w \in W} \chi_{V_{q=1}}(w) w^{-1},$$

which is the primitive idempotent associated to the irreducible representation $V_{q=1}$ of $W$. In particular, the idempotent $e_{\text{ind}}$ for the index representation reduces to $|W|^{-1} \sum_{w \in W} w$, the idempotent of the group algebra $\mathbb{C}[W]$ corresponding to the trivial representation of $W$.

We close this subsection by an explicit formula of Schur elements in the case $W = S_n$. Recall (2.4) that finite-dimensional simple $H$-modules are parametrized by partitions of $n$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be such a partition. We denote by $\ell = \ell(\lambda)$ the length of $\lambda$, i.e., the number of non-zero $\lambda_i$’s. Let $V_\lambda$ be the simple $H$-module corresponding to $\lambda$, and $e_\lambda \in K$ be the Schur element associated to $V_\lambda$. By [GP00, 9.4.5, 10.5.1, 10.5.2], we have

$$c_\lambda^{-1} = u^{n(\lambda)} \prod_{1 \leq i < j \leq \ell} [\tilde{\lambda}_j - \tilde{\lambda}_i]. \quad (2.15)$$

with $n(\lambda) := \sum_{i=1}^{\ell} (i-1)\lambda_i$ and $\tilde{\lambda}_i := \lambda_i + \ell - i$. We also used the symbols of $q$-integers and $q$-factorials:

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q [2]_q \cdots [n]_q.$$  

Note that under the specialization $u \rightarrow 1$ we have

$$\left. c_\lambda^{-1}\right|_{u \rightarrow 1} = \prod_{1 \leq i < j \leq \ell} (\tilde{\lambda}_i - \tilde{\lambda}_j).$$

Then (2.9) is rewritten by the pairing (2.15) as

$$\langle \chi_V, \chi_V \rangle_{H^*} = c_V \delta_{V,V'},$$
3. Gelfand pairs and zonal spherical functions for Iwahori-Hecke algebras

We give a Hecke analogue of the theory of Gelfand pairs and zonal spherical functions for finite groups. We refer [M95, VII.1] for the theory of finite groups. We continue to use the symbols in the previous §2. So $H$ is the generic Hecke algebra associated to a finite Coxeter system $(W, S)$ defined over the finite Galois extension $K \supseteq \mathbb{C}(u_s \mid s \in S)$.

For a subset $J \subseteq S$, we call the subgroup $W_J := \langle J \rangle \subseteq W$ generated by $J$ the parabolic subgroup, following the terminology in [GP00, 1.2]. The corresponding subalgebra

$$H_J := \{ T_w \mid w \in W_J \} \subseteq H$$

is called the parabolic subalgebra of $H$ associated to $J$.

We denote by $e_J \in Z(H_J)$ the idempotent (2.12) corresponding to the index representation of $H_J$. We regard it as an element of $H$. Thus we have

$$e_J = \frac{1}{P_J} \sum_{w \in W_J} T_w \in H,$$

where $P_J := P_{W_J} \in \mathbb{N}[u_s \mid s \in J] \subseteq K$ denotes the Poincaré polynomial (2.10) of $W_J$.

Now we consider the $H$-module $e_J H$. It is isomorphic to the induced module

$$\text{Ind}_H^{H_J}(e_J H_J) = e_J H_J \otimes_{H_J} H$$

of the index representation $e_J H_J$. Since $e_J$ is an idempotent, we have a $K$-algebra isomorphism

$$\text{End}_H(e_J H) \xrightarrow{\sim} e_J H e_J, \quad \varphi \mapsto \varphi(e_J).$$

By [GP00, 9.1.9], the multiplicities $m(V)$ of the decomposition

$$e_J H = \bigoplus_{V \in \text{Irr}(H)} V^{\otimes m(V)}$$

into finite-dimensional simple $H$-modules $V$ are preserved under the specialization $H \rightarrow H_{q=1}$, $u_s \mapsto 1$. In other words, the multiplicity $m(V)$ is the same as that in the irreducible decomposition of the representation $H_W^{W_J}$ of the finite group $W$ obtained by inducing the trivial one-dimensional representation of the subgroup $W_J$. Now let us recall the notion of Gelfand pair for finite groups [M95, VII.1, (1.1)]: A pair $(G, K)$ of a finite group $G$ and its subgroup $K$ is called a Gelfand pair of finite groups if the induced representation $H_K^G$ is multiplicity-free, i.e., each of the non-zero multiplicities is one. Thus the following definition seems to be natural.

**Definition 3.1.** Let $(W, S)$ be a finite Coxeter system and $J \subseteq S$ be a subset. The pair $(S, J)$ or $(H, H_J)$ is called a Gelfand pair if the $H$-module $e_J H$ is multiplicity-free, i.e., the pair $(W_S, W_J)$ is a Gelfand pair of finite groups.

Hereafter we assume $(S, J)$ is a Gelfand pair, and denote by

$$e_J H = \bigoplus_{i=1}^r V_i$$

the decomposition into simple $H$-modules. Let $\chi_i$ be the character of $V_i$ and $\chi_i^* \in H$ be the element given by (2.7). Then we introduce:

**Definition 3.2.** For $i = 1, 2, \ldots, r$, we define an element $\omega_i \in H$ by

$$\omega_i := \chi_i^* e_J = e_J \chi_i^*,$$

where the equality follows from $\chi_i^* \in Z(H)$, explained in the paragraph of (2.13). We call $\omega_i$'s the zonal spherical functions of the Gelfand pair $(S, J)$.

The zonal spherical functions $\omega_i$ satisfy similar properties as the zonal spherical functions in the standard sense. The following lemma is a “Hecke-analogue” of [M95, VII.1, (1.4)].

**Theorem 3.3.** The elements $\omega_i$ ($i = 1, 2, \ldots, r$) satisfy the following properties.

1. Let us express the expansion of $\omega_i$ in terms of the basis $\{ T_w \mid w \in W \}$ as $\omega_i = \sum_{w \in W} \omega_i(w) T_w$. Then $\omega_i(e) = 1$.

2. $\omega_i \omega_j = \delta_{ij} c_i \omega_i$, where $c_i := c_{V_i} \in K$ is the Schur element associated to $V_i$ (see §2.2).

3. $\omega_i \in e_J H c_j \cap V_i$.

4. Zonal spherical functions $\omega_i$, $i = 1, \ldots, r$, form a $K$-basis of $e_J H e_J$. 
Proof. (1) By definition we have \( \omega_i = (P_j^{-1} \sum_{v \in W_j} T_v) \left( \sum_{w \in W} \chi_i(T_w)T_w^\vee \right) \). Since \( \tau(T_w T_w^\vee) = \delta_{v,w} \) by (2.6), we have \( \omega_i(e) = P_j^{-1} \sum_{v \in W_j} \chi_i(T_v) \). Since the module \( V_i \) lies in the induced module \( e_jH \) on which \( T_v \) acts by the index representation (2.1), we have \( \sum_{v \in W_j} \chi_i(T_v) = \sum_{w \in W} a_w = P_j \). Thus we have the result.

(2) Denote by \( e_i := e_{V_i} \in Z(H) \) the idempotent for \( V_i \). By (2.13) we have \( \chi_i^* = c_i e_i \). Then the idempotence of \( e_j \) and (2.14) yields the conclusion.

(3) We continue to use the symbol \( e_i \). Since \( e_iH \simeq V_i \) as \( H \)-modules, we have \( \omega_i = c_i e_i \chi_i^* \in V_i \). (2) yields \( \omega_i = e_j(\chi_i^*)^2 e_j \), and since \( c_i \) is non-zero as explained in the paragraph of (2.11), we have \( \omega_i \in e_jHe_j \). Thus we have the statement.

(4) It follows from (3) that \( \omega_i \)'s are linearly independent in \( e_jH \). On the other hand, Schur's lemma yields \( \dim_K e_jHe_j = \dim_K \text{End}_H(e_jH) = r \). Thus we have the conclusion.

\[ \square \]

For the zonal spherical function \( \omega_i \in H \) and \( w \in W \), we define \( \omega_i(T_w) = K \) by the expansion

\[ \omega_i = \sum_{w \in W} \omega_i(T_w)T_w^\vee. \]

The limit \( \omega_i(T_w) \equiv 1 \) coincides with the value of the zonal spherical function on \( (W, W_j) \) at the double coset \( W_jwW_j \subset W_j \setminus W/W_j \). Unfortunately, \( \omega_i(T_w) \) is not constant on the double coset \( W_jwW_j \), and at this point we have a discrepancy between Hecke- and classical settings.

As a closing remark, let us recall that there are some explicit formulas of the values of spherical functions for the Gelfand pairs of \( \mathcal{S}_n \) and its subgroups, and for the pairs of \( GL(n, F_q) \) and its parabolic subgroups, in terms of terminating hypergeometric and basic hypergeometric series, respectively. See [Du79] and [M95, Chap. VII] for the detail. However, in the case of our zonal spherical functions for Hecke algebras, we don’t have such nice formulas at this moment. We invite the reader to tackle this problem.

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References

[Du79] C. F. Dunkl, Orthogonal functions on some permutation groups, in Relations between combinatorics and other parts of mathematics, 129–147, Proc. Sympos. Pure Math., XXXIV, Amer. Math. Soc., Providence, R.I., 1979.

[HHY] M. Hayashi, A. Hora, S. Yanagida, Asymmetry of tensor product of asymmetric and invariant vectors arising from Schur-Weyl duality based on hypergeometric orthogonal polynomial, preprint, arXiv:2104.12635.

[GP00] M. Geck, G. Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Mathematical Society Monographs. New Series, 21, The Clarendon Press, Oxford University Press, New York, 2000.

[J86] M. Jimbo, A \( q \)-analogue of \( U(gl(N + 1)) \), Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys., 11 (1986), no. 3, 247–252.

[M95] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, 1995.

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