Unitals in Projective Planes of Order 25

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Abstract In this paper, results of a non-exhaustive computer search for unitals in the known planes of order twenty-five are reported. The 2-(126, 6, 1) designs associated with newly found unitals are studied in detail. 938 non-isomorphic unital designs are discovered and we show that three of the unital designs are embeddable in two non-isomorphic planes and 239 of them are resolvable. The findings of this study improve some well-known lower bounds on the number of such designs and provide new connections between some pairs of planes. A conjecture concerning the \( p \)-ranks of unital designs embedded in planes of order \( q^2 \) is formulated.

Keywords Unital · Projective plane · Steiner design

Mathematics Subject Classification 05B05 · 51E10 · 51E20

1 Introduction

We assume familiarity with the basic notions from combinatorial design theory and finite geometries [7, 11, 13, 16, 30].

A 2-(\( v, k, \lambda \) ) design is a pair \( D = (X, B) \) of a set \( X \) of \( v \) points and a collection \( B \) of subsets of \( X \) of size \( k \) called blocks, such that every two points appear together in exactly \( \lambda \) blocks. A 2-(\( v, k, 1 \) ) design is called a Steiner design.

Two designs are isomorphic if there is a one-to-one and onto map between their point sets that maps every block of one design to a block of the other design.

A parallel class of a design \( D \) is a collection of blocks that partitions the point set of \( D \). A partial parallel class of \( D \) is a set of blocks that contain no point of \( D \) more than once. A resolution of \( D \) is a partition of the collection of blocks of \( D \) into disjoint parallel classes. A design having at least one resolution is called a resolvable design.

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A set of type \((a, b)\) in a projective plane \(\Pi\) of order \(q^2\) is a subset \(M\) of the points of \(\Pi\) that meets every line of \(\Pi\) in either \(a\) or \(b\) points (assume that \(a < b\)). The quantities \(a\) and \(b\) are called the intersection numbers of \(M\). If \(a = 0\), these sets are known as maximal arcs. It can be shown that a set of type \((1, 1 + q)\) has either \(q^2 + q + 1\) or \(q^3 + 1\) points. The former set is called a Baer subplane and latter is called a unital. If \(M\) is a unital in \(\Pi\), then the lines meeting \(M\) in one point is a unital in \(\Pi^\perp\). The point set of a unital with the line intersections of size \(q + 1\) form a \(2 \times (q^3 + 1, q + 1, 1)\) design, called unital design associated with \(M\).

A classical example of a unital is the Hermitian unital \(H(q)\) defined by the absolute points of a unitary polarity in \(PG(2, q^2)\). Since \(PG(2, q^2)\) admits unitary polarities for each prime power \(q\), unitals exist in these planes. In [10], Buekenhout provided two methods for constructing parabolic and hyperbolic unitals. In [21], Metz used Buekenhout’s first method to construct a non-classical unital in a Desarguesian plane of order \(q^2\). Later, Barwick showed that any unitals constructed by the second method of Buekenhout is a classical unital [5]. Rosati constructed a unital in Hughes planes of order \(q\), for every odd prime power \(q\) [24], and later Kestenband generalized this construction [18]. Some other studies of unitals can be found in [2–4,8,9,14,17,25–29,31]. The reader is referred to the book [6] for more details on unitals.

In this article, some results of a non-exhaustive computer search for unitals in the known planes of order twenty-five are given. An algorithm developed by the first author was used to find such sets, and some computer programs were developed by the second author to find the computational results given in the rest of the paper.

The paper is organized in the following way. In Sect. 2, we provide a brief description of the known projective planes of order 25. For each plane, the total number of unitals found by the algorithm used in this study is given. A link to a website containing point sets of all found unitals is provided. In Sect. 3, we first give a brief history of the known connections within the projective planes of orders 9 and 16. It is discovered that three of the unital designs embedded in planes of order 25 are embeddable in two different planes. The computations show that some (non-isomorphic) pairs of planes of order 25 share a binary (and a ternary) linear code generated by the block by point incidence matrices of the unital designs. In Sect. 4, some statistics related to 2-(126, 6, 1) designs associated with the new unitals are outlined. Lower bounds on the number of such designs given in [11] are improved. In Sect. 5, we conclude our results and a conjecture concerning the \(p\)-ranks of unital designs embedded in projective planes of order \(q^2\) is formulated. In the “Appendix”, the automorphism group orders, 2-ranks, 3-ranks, number of parallel classes and partial parallel classes, and resolvability of the unital designs embedded in planes of order 25 are listed.

2 Unitals in Projective Planes of Order 25

Up to isomorphism, 193 projective planes of order 25 are known to exist [22]. Twenty-one of these planes are translation planes, which are classified by Czerwinski and Oakden [12], and denoted by \(a1, \ldots, a8, b1, \ldots, b8\), and \(s1, \ldots, s5\). In this list, \(s1\) is \(PG(2, 25)\) and \(a2\) is Dickson nearfield plane. There are also other planes of order 25: Hughes plane \((h1)\), exceptional Hughes plane \((h2)\), Wyoming planes \((w1\) and \(w2)\), and the planes constructed from these planes by derivation [22]. Five of these planes are self-dual, namely \(s1, s2a, h1, h1b\) and \(h2\), the rest have non-isomorphic duals.

An algorithm developed by the first author is used to find unitals in the known planes of order 25. The algorithm is based on searching union of orbits that satisfy the unital condition and it contains the following steps:

1: Find the (bipartite) graph of plane with parts \(P_1\) and \(P_2\)
2: for each vertex \(X\) of \(P_1\) do
3: find the generators of the stabilizer \(H\) of \(X\)
4: for each combination of generators of \(H\) do
5: find the subgroups \(H_i, i \geq 1\)
6: find the orbits of each \(H_i\) (orbits from parts \(P_1\) and \(P_2\))
7: find the relative degrees between orbits of \(P_1\) and \(P_2\)
8: find all possible combinations of sets having length 126 (in \(P_1\))
9: save the 126-length set satisfying the unital condition
10: print non-isomorphic 126-length set if it is not isomorphic to previously found unitals
11: end for
12: end for

More details of the algorithm can be found in [25–28], and the results of the execution of the algorithm on all known planes of order 25 are listed in [27].

Our computations show that there are at least 938 unitals in the known projective planes of order 25. Table 1 shows the number of found unitals for each plane. The name of the planes starts with one of the five letters \( a, b, s, h \), and \( w \), following by a number and a letter from \( a \) to \( g \), which are listed in the first column, the first row, and the second column of Table 1, respectively. The remaining entries are for the number of known unitals for each known plane of order 25.

In Table 1, an entry 0 means that no unitals are known to exist in that plane, an entry with * indicates that the plane is self-dual, and an entry without a number implies that there is no known plane with that name. For example, we have not found any unitals in plane \( a5c \) yet, the plane \( s2a \) is a self-dual plane, and no plane with the name \( s6c \) exists.

The projective planes of order 25 given at 
https://euniversite.nku.edu.tr/kullanicidosyalari/3705/files/pl25.txt
were used for the computations carried out in this study. The point sets of the unitals found with the algorithm used in this article can be found at 
https://euniversite.nku.edu.tr/kullanicidosyalari/3705/files/un25.txt

Table 1 shows that unitals exist in 144 projective planes of order 25, and no unitals are known to exist in the remaining 49 of the planes.

3 New Connections Between Projective Planes

A unital design in a projective plane of order 9 is a 2-(28, 4, 1) design. In [9], Brouwer constructed 138 non-isomorphic such designs and showed that twelve of them could be embedded as a unital in planes of order 9: two in the Desarguesian plane, three in the Hall plane, three in the dual of Hall plane, and four in the Hughes plane. The remaining of the 126 designs were not embedded as a unital in planes of order 9. In the same paper, Brouwer also showed that the Hall plane and the dual of the Hall plane share a unital design. In [23], Penttila and Royle classified all unital designs embedded in planes of order 9 and they showed that there are exactly 17 such designs (up to isomorphism).

Even though only the Hall plane of order 9 and its dual share a unital design, our computations show that many of the unitals in planes of order 9 have isomorphic copies which are unitals (as a point set, not as a design) in the other planes. In order to see this observation, we define the following sets:

\[
\begin{align*}
  u_1 &= \{1, 2, 7, 9, 12, 14, 19, 20, 25, 27, 38, 41, 44, 53, 54, 55, 57, 61, 62, 68, 69, 70, 75, 78, 81, 83, 87, 91\}, \\
  u_2 &= \{1, 12, 14, 19, 20, 25, 27, 29, 34, 36, 40, 42, 44, 49, 51, 53, 56, 61, 63, 67, 69, 71, 74, 79, 81, 85, 87, 89\}, \\
  u_3 &= \{5, 6, 8, 10, 12, 13, 14, 17, 22, 30, 38, 41, 42, 44, 50, 56, 57, 58, 61, 65, 68, 71, 73, 74, 75, 76, 81, 89\}, \\
  u_4 &= \{5, 7, 9, 10, 15, 16, 18, 19, 21, 22, 24, 26, 30, 31, 32, 37, 39, 47, 49, 51, 55, 56, 67, 74, 83, 84, 87, 91\}, \\
  u_5 &= \{1, 2, 3, 4, 23, 24, 25, 35, 36, 37, 39, 42, 46, 48, 52, 54, 57, 59, 62, 67, 69, 73, 76, 77, 80, 85, 88, 90\}, \\
  u_6 &= \{1, 2, 3, 4, 11, 12, 13, 35, 36, 37, 39, 40, 44, 47, 50, 52, 61, 62, 64, 68, 72, 73, 75, 76, 81, 83, 86, 88\}, \\
  u_7 &= \{1, 2, 3, 5, 11, 14, 15, 32, 33, 37, 39, 43, 44, 47, 54, 55, 56, 57, 63, 70, 71, 73, 78, 80, 81, 84, 86, 88\}, \\
  u_8 &= \{5, 6, 7, 10, 14, 15, 17, 19, 27, 34, 38, 47, 51, 52, 55, 57, 61, 63, 64, 65, 76, 78, 79, 81, 83, 87, 90, 91\}.
\end{align*}
\]

One can show that the sets \( u_1, u_2, u_3, \) and \( u_4 \) are unitals in all four planes, \( u_5 \) is a unital in the Hall Plane, the dual of Hall plane and the Hughes plane, \( u_6 \) and \( u_8 \) are unitals in the dual of Hall plane and the Hughes plane, and \( u_7 \) is a unital in the Hughes plane. The specific incidence matrices of the planes of order 9 we used for our computations can be found at 
https://web.math.pmf.unizg.hr/~krcko/results/steiner.html
Further computations show that the above list of eight sets provide all unitals in these planes: the two unitals in \( PG(2, 9) \) are the sets \( u_1 \) and \( u_2 \) (or \( u_3, u_4 \)), the sets \( u_1, u_2, u_3 \) and \( u_4 \) are the four non-isomorphic unitals in the Hall plane and the dual of Hall plane of order 9, the sets \( u_1 \) (or \( u_2 \)), \( u_3 \) (or \( u_4 \)), \( u_5, u_6, u_7 \) and \( u_8 \) are the six unitals in the Hughes plane of order 9. The remaining two unitals in the Hughes plane are the dual unitals of the unitals \( u_6 \) and \( u_8 \).

In [23], Pentila and Royle used the notation \( U1 \) – \( U14 \) for the unitals in projective planes of order 9. The correspondence between the sets provided in this paper and unitals \( U1 \) – \( U14 \) is summarized in Table 2.

Table 2 shows that all unitals except \( u_7 \) have isomorphic copies which are unitals in more than one plane of order 9. We can summarize above discussion as in the following theorem:

**Theorem 1** All non-isomorphic pairs of projective planes of order 9 share a unital.

In [15], Grüning used Buekenhout’s second method [10] to show that for every prime power \( q \), the Hall plane and the dual of the Hall plane of order \( q^2 \) share a unital design. It was reported that three of the planes of order 16 contain unitals which are unitals in more than one plane [19, 20]. Recently, the authors showed that the total number of unitals shared by the known non-isomorphic pairs of planes of order 16 is greater than or equal to 39, and many of these shared unitals are unitals in more than two non-isomorphic planes [28].

As discussed above, shared unitals were found in the planes of orders 9 and 16. We developed an algorithm to examine the isomorphic copies of unitals in planes of order 25. As the number of known planes is 193 and the number of known unital designs in these planes is 938, one may expect more shared unitals within these planes.
Table 2  Correspondence between $u_1 - u_8$ and $U_1 - U_{14}$

|     | $PG(2, 9)$ | Hall(9) | Hall(9)$^\perp$ | Hughes(9) |
|-----|------------|---------|-----------------|------------|
| $u_1$ | U2         | U6      | $U_6^\perp$     | U14        |
| $u_2$ | U1         | U5      | $U_5^\perp$     | U14        |
| $u_3$ | U1         | U4      | $U_4^\perp$     | U8         |
| $u_4$ | U1         | U3      | $U_3^\perp$     | U8         |
| $u_5$ | –          | U4      |                 | U13        |
| $u_6$ | –          | –       |                 | U9         |
| $u_7$ | –          | –       |                 | U7         |
| $u_8$ | –          | –       | $U_3^\perp$     | U11        |

However, our computations show that none of the known unitals nor their isomorphic copies are unitals in more than one plane, that is, any unital (and its isomorphic copies) found by the algorithm used in this study is a unital only in one of the known planes of order 25.

All previously known connections between known planes of order 25 can be found in [22]. An isomorphism check was performed between the unital designs found in the previous section:

– the 2-(126, 6, 1) design associated with the unital 1 of $a2b$ plane is embeddable in $a6a$ plane;
– the 2-(126, 6, 1) design associated with the unital 3 of $a3a$ plane is embeddable in $b5b$ plane;
– the 2-(126, 6, 1) design associated with the unital 2 of $s2$ plane is embeddable in its dual plane.

This observation implies that three new connections based on unital designs occur between the known planes of order 25.

The data given in the “Appendix” shows that the majority of the unital designs considered in this study have 2-rank 125, implying that many of the dual codes of the linear codes over GF(2) spanned by the block by point incidence matrix of the 2-(126, 6, 1) designs have only 2-elements, but sharing such a code does not provide any useful information. One may also get a similar conclusion for the ternary codes generated by the unital designs embedded in these planes.

4 2-(126, 6, 1) Designs

In this section, we provide some statistics about unital designs embedded in the known projective planes of order 25. A unital design in these planes is a 2-(126, 6, 1) design. In Table 3, Column 1 states the name of the planes in alphabetical order. Columns 2 and 3 indicate the unital number and its group order, respectively. Columns 4 and 5 states the 2-ranks and the 3-ranks of the unital designs, respectively. For a unital $U$, Columns 6 and 7 gives the number of parallel classes and the number of partial parallel classes of $D(U)$ and $D(U^\perp)$, respectively. Column 8 indicates whether the 2-(126, 6, 1) designs are resolvable or not.

Table 3 shows that the number of pairwise non-isomorphic 2-(126, 6, 1) designs is at least 938, and the number of non-isomorphic resolvable such designs is at least 239. Prior to this study, these bounds were 2 and 1, respectively [11]. We summarize this observation in the following theorem:

**Theorem 2** The number of pairwise non-isomorphic 2-(126, 6, 1) designs is greater than or equal to 938, and the number of resolvable such designs is greater than or equal to 239.

We can get the following fact from the data given in the “Appendix” as well:
Theorem 3  The distribution of the group orders, the 2-ranks and the 3-ranks of the known 2-(126, 6, 1) designs embedded in the known planes of order 25 are

\{10^{568}, 12^{32}, 15^{12}, 20^{147}, 24^{20}, 30^{12}, 40^{64}, 48^{3}, 50^{4}, 60^{8}, 72^{8}, 80^{16}, 96^{4}, 100^{4}, 120^{16},

144^{8}, 200^{3}, 240^{5}, 1000^{1}, 1440^{1}, 2000^{1}, 756000^{1}\},

\{105^{1}, 109^{2}, 114^{1}, 115^{2}, 117^{6}, 119^{4}, 121^{9}, 122^{7}, 123^{23}, 124^{2}, 125^{884}\},

\{105^{1}, 108^{2}, 115^{3}, 116^{6}, 118^{2}, 119^{2}, 120^{8}, 121^{10}, 122^{4}, 123^{21}, 124^{619}, 125^{263}\},

respectively.

5 Conclusion

In [1], Andriamanalimanana conjectured that \( p \)-rank of classical unital is \( q^3 - q^2 + q \) (here, \( q \) is a prime power, and \( p \) is a prime divisor of \( q + 1 \)). The minimum 2-rank of the unital designs embedded in planes of order 9 is the 2-rank of the design associated with the classical unital [9,23]. Meanwhile, in the known unital designs embedded in the known projective planes of order 16, the design associated with the classical unital is the only design having minimum 5-rank (up to isomorphism) [28]. Table 3 shows that Andriamanalimanana’s conjecture holds for the classical unital in planes of order 25 as well: For \( p \in \{2, 3\} \), Theorem 3 shows that there is only one design having \( p \)-rank 105, and from Table 3 we see that this design is the unital design associated with the classical unital. The remaining of the unital designs presented in this paper have \( p \)-rank greater than 105. These observations make us believe that this situation holds in general:

Conjecture 1  Let \( q \) be a prime power, \( s \in \mathbb{Z}^+ \), \( \Pi = \{\pi_1, \pi_2, \ldots, \pi_s\} \) be the set of all projective planes of order \( q^2 \) and \( U = \{u_1, u_2, \ldots, u_s\} \) be the set of all unital sets in the planes in \( \Pi \). For a prime divisor \( p \) of \( q + 1 \), the \( p \)-rank of the 2-(\( q^3 + 1, q + 1, 1 \)) design associated with a unital \( u \in u_i \) in \( \pi_i \) is greater than or equal to \( q(q^2 - q + 1) \), \( i \in \{1, 2, \ldots, s\} \). The equality holds if and only if the design associated with \( u \) is isomorphic to the design associated with the classical unital.

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A Some Statistics Related to Unital Designs Embedded in Projective Planes of Order 25

See Table 3.
### Table 3  Unital designs embedded in the known planes of order 25

| Plane No. | Unital | $|\text{Aut}(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-----------|--------|-----------------|--------|--------|---------------------------------|----------------------------------|------------|
| a1        | 1      | 144             | 125    | 121    | 240/32                          | 7200/1232                        | No         |
|           | 2      | 24              | 125    | 125    | 5/3                             | 1934/618                         | No         |
|           | 3      | 20              | 125    | 125    | 25/25                           | 1050/1050                        | Yes        |
|           | 4      | 20              | 125    | 125    | 25/25                           | 1090/1090                        | Yes        |
|           | 5      | 15              | 125    | 125    | 70/30                           | 2110/1215                        | Yes        |
|           | 6      | 10              | 125    | 125    | 30/30                           | 1155/1155                        | Yes        |
|           | 7      | 10              | 125    | 125    | 25/25                           | 1060/1060                        | Yes        |
|           | 8      | 10              | 125    | 125    | 25/25                           | 1050/1050                        | Yes        |
|           | 9      | 10              | 125    | 125    | 30/30                           | 1155/1155                        | Yes        |
| a1a       | 1      | 20              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 2      | 20              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 3      | 10              | 125    | 124    | 6/6                             | 723/723                          | No         |
|           | 4      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 5      | 10              | 125    | 124    | 6/6                             | 723/723                          | No         |
|           | 6      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
| a1b       | 1      | 15              | 125    | 124    | 6/31                            | 763/1318                         | No         |
|           | 2      | 10              | 125    | 124    | 1/1                             | 674/658                          | No         |
|           | 3      | 10              | 125    | 124    | 6/6                             | 779/743                          | No         |
|           | 4      | 10              | 125    | 125    | 25/25                           | 1130/1075                        | Yes        |
| a1c       | 1      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 2      | 10              | 125    | 124    | 1/1                             | 628/628                          | No         |
|           | 3      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 4      | 10              | 125    | 124    | 1/1                             | 628/628                          | No         |
|           | 5      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 6      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 7      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 8      | 10              | 125    | 124    | 1/1                             | 628/628                          | No         |
|           | 9      | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 10     | 10              | 125    | 124    | 1/1                             | 618/618                          | No         |
|           | 11     | 10              | 125    | 124    | 1/1                             | 658/653                          | No         |
|           | 12     | 10              | 125    | 124    | 1/1                             | 719/663                          | No         |
|           | 13     | 10              | 125    | 124    | 11/11                           | 887/848                          | No         |
|           | 14     | 10              | 125    | 124    | 11/11                           | 887/853                          | No         |
| a2        | 1      | 120             | 125    | 125    | 295/265                         | 4990/6310                        | Yes        |
|           | 2      | 72              | 122    | 125    | 519/393                         | 12876/9132                       | No         |
|           | 3      | 80              | 125    | 125    | 25/25                           | 1570/1050                        | Yes        |
| a2a       | 1      | 10              | 125    | 124    | 11/11                           | 848/848                          | No         |
|           | 2      | 40              | 125    | 124    | 11/11                           | 878/848                          | No         |
|           | 3      | 120             | 125    | 124    | 241/181                         | 5878/4558                        | No         |
|           | 4      | 80              | 125    | 124    | 1/1                             | 618/738                          | No         |
| Plane | Unital No. | $|\text{Aut}(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-------|------------|----------------|--------|--------|---------------------------------|-----------------------------------------------|------------|
| 5     | 12         | 123            | 125    | 15/35  | 890/1291                        | No                                             |            |
| a2b   | 1          | 40             | 125    | 125    | 25/25                           | 1300/1050                                     | Yes        |
|       | 2          | 40             | 125    | 125    | 35/35                           | 1310/1260                                     | Yes        |
|       | 3          | 20             | 125    | 124    | 11/11                           | 828/828                                       | No         |
|       | 4          | 10             | 125    | 124    | 1/1                             | 638/638                                       | No         |
|       | 5          | 10             | 125    | 124    | 1/1                             | 628/628                                       | No         |
|       | 6          | 10             | 125    | 124    | 6/6                             | 733/733                                       | No         |
|       | 7          | 10             | 125    | 124    | 6/6                             | 743/743                                       | No         |
|       | 8          | 10             | 125    | 124    | 1/1                             | 618/778                                       | No         |
|       | 9          | 10             | 125    | 125    | 25/25                           | 1075/1065                                     | Yes        |
|       | 10         | 10             | 125    | 125    | 30/30                           | 1190/1170                                     | Yes        |
| a3    | 10         | 10             | 125    | 124    | 11/11                           | 838/838                                       | No         |
|       | 2          | 120            | 125    | 125    | 40/25                           | 2460/1050                                     | Yes        |
|       | 3          | 50             | 125    | 125    | 25/25                           | 1250/1100                                     | Yes        |
| a3a   | 1          | 40             | 125    | 124    | 1/6                             | 666/748                                       | No         |
|       | 2          | 50             | 125    | 125    | 25/25                           | 1100/1125                                     | Yes        |
|       | 3          | 200            | 125    | 125    | 25/25                           | 1300/1050                                     | Yes        |
|       | 4          | 100            | 125    | 125    | 25/25                           | 5878/4558                                     | No         |
|       | 5          | 100            | 125    | 125    | 25/25                           | 1180/1050                                     | Yes        |
|       | 6          | 10             | 125    | 124    | 1/1                             | 618/618                                       | No         |
| a3b   | 1          | 40             | 125    | 124    | 1/1                             | 666/748                                       | No         |
|       | 2          | 10             | 125    | 124    | 1/1                             | 666/698                                       | No         |
|       | 3          | 10             | 125    | 124    | 1/1                             | 618/618                                       | No         |
|       | 4          | 10             | 125    | 124    | 1/1                             | 618/618                                       | No         |
|       | 5          | 10             | 125    | 124    | 1/1                             | 618/618                                       | No         |
|       | 6          | 10             | 125    | 124    | 1/1                             | 618/618                                       | No         |
| a4    | 1          | 72             | 122    | 125    | 519/393                         | 12876/9132                                    | No         |
|       | 2          | 12             | 124    | 125    | 282/195                         | 7792/4824                                     | No         |
|       | 3          | 30             | 125    | 125    | 205/265                         | 4990/6310                                     | Yes        |
|       | 4          | 10             | 125    | 125    | 25/25                           | 1180/1050                                     | Yes        |
| a4a   | 1          | 12             | 125    | 125    | 15/35                           | 888/1291                                      | No         |
|       | 2          | 12             | 123    | 125    | 15/35                           | 890/1291                                      | No         |
|       | 3          | 12             | 125    | 125    | 15/35                           | 900/1303                                      | No         |
|       | 4          | 12             | 125    | 125    | 15/35                           | 888/1291                                      | No         |
|       | 5          | 30             | 125    | 124    | 241/181                         | 5878/4558                                     | No         |
|       | 6          | 10             | 125    | 124    | 6/31                            | 723/1288                                      | No         |
|       | 7          | 10             | 125    | 124    | 1/1                             | 618/648                                       | No         |
|       | 8          | 10             | 125    | 124    | 1/1                             | 618/648                                       | No         |
| Plane | Unital No. | $|\text{Aut}(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-------|-----------|-----------------|--------|--------|-------------------------------|---------------------------------|------------|
| a4b   | 1         | 10              | 125    | 124    | 1/1                           | 628/628                         | No         |
|       | 2         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 3         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 4         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 5         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 6         | 10              | 125    | 124    | 1/1                           | 628/628                         | No         |
|       | 7         | 10              | 125    | 124    | 1/1                           | 628/628                         | No         |
|       | 8         | 10              | 125    | 124    | 11/11                         | 828/828                         | No         |
|       | 9         | 10              | 125    | 124    | 1/1                           | 628/628                         | No         |
|       | 10        | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
| a4c   | 1         | 10              | 125    | 124    | 1/1                           | 623/618                         | No         |
|       | 2         | 10              | 125    | 124    | 11/11                         | 828/828                         | No         |
|       | 3         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 4         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 5         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 6         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 7         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 8         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 9         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 10        | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
| a4d   | 1         | 10              | 125    | 124    | 1/1                           | 768/638                         | No         |
|       | 2         | 10              | 125    | 124    | 6/6                           | 777/738                         | No         |
|       | 3         | 10              | 125    | 124    | 1/1                           | 667/618                         | No         |
| a5    | 1         | 24              | 125    | 125    | 5/3                           | 1790/618                        | No         |
|       | 2         | 10              | 125    | 125    | 25/25                         | 1050/1050                       | Yes        |
|       | 3         | 10              | 125    | 125    | 35/35                         | 1260/1260                       | Yes        |
| a5a   | 1         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 2         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 3         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 4         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 5         | 10              | 125    | 124    | 6/6                           | 723/723                         | No         |
|       | 6         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 7         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 8         | 10              | 125    | 124    | 11/11                         | 828/828                         | No         |
|       | 9         | 10              | 125    | 124    | 1/1                           | 618/618                         | No         |
|       | 10        | 10              | 125    | 124    | 11/11                         | 828/828                         | No         |
| a5b   | 1         | 10              | 125    | 124    | 1/1                           | 634/638                         | No         |
| a6    | 1         | 144             | 125    | 121    | 240/32                        | 6768/1232                       | No         |
|       | 2         | 10              | 125    | 125    | 25/25                         | 1050/1050                       | Yes        |
|       | 3         | 10              | 125    | 125    | 35/35                         | 1300/1270                       | Yes        |
Table 3 continued

| Plane | Unital No. | | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolable? |
|-------|-------------|---|---|---|---|---|---|
|       |             | $|Aut(U)|$ |        |        |                                  |                                  |           |
| 4     |             | 15 | 125 | 125 | 55/30 | 1750/1185 | Yes |
| 5     |             | 10 | 125 | 125 | 25/25 | 1050/1050 | Yes |
| a6a   | 1           | 40 | 125 | 125 | 25/25 | 1300/1050 | Yes |
|       | 2           | 20 | 125 | 124 | 1/1   | 748/628   | No  |
|       | 3           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 4           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 5           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 6           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
| a6b   | 1           | 15 | 125 | 124 | 6/31  | 753/1318  | No  |
| a6c   | 1           | 10 | 125 | 124 | 1/1   | 637/648   | No  |
|       | 2           | 10 | 125 | 124 | 1/1   | 689/653   | No  |
|       | 3           | 20 | 125 | 124 | 1/1   | 693/658   | No  |
|       | 4           | 10 | 125 | 124 | 6/6   | 738/738   | No  |
|       | 5           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 6           | 10 | 125 | 124 | 11/11 | 838/838   | No  |
|       | 7           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 8           | 10 | 125 | 124 | 6/6   | 723/738   | No  |
|       | 9           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 10          | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 11          | 10 | 125 | 124 | 6/6   | 723/738   | No  |
|       | 12          | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 13          | 10 | 125 | 124 | 6/6   | 738/738   | No  |
| a7    | 1           | 120| 125 | 125 | 16/17 | 1980/970  | No  |
|       | 2           | 20 | 125 | 125 | 25/25 | 1050/1050 | Yes |
|       | 3           | 20 | 125 | 125 | 25/25 | 1050/1050 | Yes |
|       | 4           | 20 | 125 | 125 | 25/25 | 1050/1050 | Yes |
|       | 5           | 20 | 125 | 125 | 25/25 | 1050/1050 | Yes |
|       | 6           | 10 | 125 | 125 | 25/25 | 1050/1050 | Yes |
|       | 7           | 10 | 125 | 125 | 30/30 | 1165/1165 | Yes |
|       | 8           | 10 | 125 | 125 | 30/30 | 1165/1165 | Yes |
|       | 9           | 10 | 125 | 125 | 25/25 | 1050/1050 | Yes |
| a7a   | 1           | 20 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 2           | 20 | 125 | 124 | 1/1   | 618/618   | No  |
| a7b   | 1           | 20 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 2           | 20 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 3           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 4           | 10 | 125 | 124 | 1/1   | 618/618   | No  |
|       | 5           | 10 | 125 | 124 | 6/6   | 733/733   | No  |
|       | 6           | 10 | 125 | 124 | 6/6   | 723/723   | No  |
| a8    | 1           | 120| 125 | 125 | 205/265 | 5030/6310 | Yes |
|       | 2           | 20 | 125 | 125 | 25/25 | 1050/1050 | Yes |
|       | 3           | 40 | 125 | 125 | 25/25 | 1050/1050 | Yes |
### Table 3 continued

| Plane | Unital No. | \(|\text{Aut}(U)|\) | 2-rank | 3-rank | Par. Clas. of \(D(U)/D(U^\perp)\) | Part. Par. Clas. of \(D(U)/D(U^\perp)\) | Resolvable? |
|-------|-----------|----------------|--------|--------|----------------------------------|--------------------------------------|------------|
| 4     | 20        | 125            | 125    | 25/25  | 1180/1050                        | Yes                                   |
| 5     | 20        | 125            | 125    | 25/25  | 1180/1050                        | Yes                                   |
| 6     | 60        | 125            | 125    | 205/265| 4990/6310                        | Yes                                   |
| 7     | 20        | 125            | 125    | 25/25  | 1050/1050                        | Yes                                   |
| a8a   | 1         | 120            | 125    | 124    | 241/181                          | 5878/4598                           | No         |
|       | 2         | 20             | 125    | 124    | 1/1                              | 618/618                             | No         |
|       | 3         | 40             | 125    | 124    | 1/1                              | 618/618                             | No         |
|       | 4         | 12             | 125    | 124    | 15/35                            | 888/1291                           | No         |
| a8b   | 1         | 20             | 125    | 124    | 1/1                              | 618/618                             | No         |
|       | 2         | 60             | 125    | 124    | 241/181                          | 5878/4558                           | No         |
|       | 3         | 10             | 125    | 124    | 1/1                              | 618/618                             | No         |
|       | 4         | 12             | 125    | 125    | 15/35                            | 888/1291                           | No         |
| a8c   | 1         | 20             | 125    | 124    | 1/1                              | 618/648                             | No         |
|       | 2         | 20             | 125    | 124    | 1/1                              | 618/648                             | No         |
|       | 3         | 20             | 125    | 124    | 1/1                              | 618/648                             | No         |
|       | 4         | 10             | 125    | 124    | 6/31                             | 723/1288                           | No         |
| a8e   | 1         | 10             | 125    | 124    | 1/1                              | 721/673                            | No         |
| b3    | 1         | 15             | 125    | 125    | 30/30                            | 1190/1190                          | Yes        |
| b3a   | 1         | 15             | 125    | 124    | 6/6                              | 768/758                            | No         |
| b3b   | 1         | 10             | 125    | 124    | 1/1                              | 739/633                            | No         |
| b3d   | 1         | 10             | 125    | 124    | 1/1                              | 714/638                            | No         |
| b4    | 1         | 30             | 125    | 125    | 50/60                            | 1650/1860                          | Yes        |
| b4a   | 1         | 10             | 125    | 124    | 21/21                            | 1068/1068                          | No         |
|       | 2         | 10             | 125    | 124    | 6/6                              | 723/733                            | No         |
| b4b   | 1         | 30             | 125    | 124    | 36/26                            | 1443/1223                          | No         |
|       | 2         | 10             | 125    | 124    | 1/1                              | 618/618                            | No         |
|       | 3         | 10             | 125    | 124    | 1/1                              | 618/618                            | No         |
|       | 4         | 10             | 125    | 124    | 1/1                              | 618/618                            | No         |
|       | 5         | 10             | 125    | 124    | 1/1                              | 618/618                            | No         |
|       | 6         | 10             | 125    | 124    | 11/11                            | 858/863                            | No         |
|       | 7         | 10             | 125    | 124    | 6/6                              | 723/723                            | No         |
|       | 8         | 10             | 125    | 124    | 6/6                              | 723/723                            | No         |
|       | 9         | 10             | 125    | 124    | 11/11                            | 853/853                            | No         |
|       | 10        | 10             | 125    | 124    | 6/6                              | 733/733                            | No         |
| b5    | 1         | 80             | 125    | 125    | 25/25                            | 1050/1050                          | Yes        |
|       | 2         | 40             | 125    | 125    | 25/25                            | 1050/1050                          | Yes        |
|       | 3         | 40             | 125    | 125    | 25/30                            | 1050/1160                          | Yes        |
| b5a   | 1         | 40             | 125    | 124    | 1/1                              | 618/618                            | No         |
|       | 2         | 40             | 125    | 124    | 6/1                              | 728/618                            | No         |
|       | 3         | 80             | 125    | 124    | 1/1                              | 618/618                            | No         |
| Plane No. | Unital No. | $|Aut(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-----------|-----------|---------|--------|--------|-------------------------------|-----------------------------------|-----------|
| 4         | 1         | 20      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 5         | 2         | 20      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 6         | 3         | 10      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 7         | 4         | 10      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| b5b       | 1         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 2         | 2         | 40      | 125    | 124    | 1/1                           | 816/816                           | No        |
| 3         | 3         | 200     | 125    | 125    | 25/25                         | 1300/1600                         | Yes       |
| 4         | 4         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 5         | 5         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 6         | 6         | 10      | 125    | 124    | 1/1                           | 628/628                           | No        |
| 7         | 7         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| b6        | 1         | 20      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 2         | 2         | 20      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 3         | 3         | 10      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 4         | 4         | 10      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 5         | 5         | 10      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| 6         | 6         | 10      | 125    | 125    | 25/25                         | 1050/1050                         | Yes       |
| b6a       | 1         | 20      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 2         | 2         | 20      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 3         | 3         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 4         | 4         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 5         | 5         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 6         | 6         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| b6b       | 1         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 2         | 2         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 3         | 3         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 4         | 4         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 5         | 5         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 6         | 6         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 7         | 7         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| b6c       | 1         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 2         | 2         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 3         | 3         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 4         | 4         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 5         | 5         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 6         | 6         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
| 7         | 7         | 10      | 125    | 124    | 1/1                           | 618/618                           | No        |
Table 3 continued

| Plane No. | Unital No. | \(|\text{Aut}(U)\)| | 2-rank | 3-rank | Part. Clas. of \(D(U)/D(U^\perp)\) | Part. Par. Clas. of \(D(U)/D(U^\perp)\) | Resolvable? |
|-----------|------------|-----------|--------|--------|----------------|----------------|-------------|
| 8         | 10         | 125       | 124    | 1/1    | 618/618        | No             |
| 9         | 10         | 125       | 124    | 1/1    | 618/618        | No             |
| 10        | 10         | 125       | 124    | 6/6    | 723/723        | No             |
| b7        | 1          | 20        | 125    | 125    | 25/25          | 1050/1050      | Yes          |
|           | 2          | 20        | 125    | 125    | 25/25          | 1050/1050      | Yes          |
|           | 3          | 20        | 125    | 125    | 25/25          | 1050/1050      | Yes          |
|           | 4          | 20        | 125    | 125    | 25/25          | 1050/1050      | Yes          |
|           | 5          | 40        | 125    | 125    | 35/35          | 1260/1260      | Yes          |
|           | 6          | 40        | 125    | 125    | 35/35          | 1260/1260      | Yes          |
|           | 7          | 10        | 125    | 125    | 25/25          | 1060/1060      | Yes          |
|           | 8          | 20        | 125    | 125    | 25/25          | 1070/1070      | Yes          |
|           | 9          | 10        | 125    | 125    | 25/25          | 1050/1050      | Yes          |
|           | 10         | 10        | 125    | 125    | 25/25          | 1050/1050      | Yes          |
|           | 11         | 20        | 125    | 125    | 25/25          | 1130/1130      | Yes          |
|           | 12         | 10        | 125    | 125    | 25/25          | 1090/1090      | Yes          |
| b7a       | 1          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 2          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 3          | 20        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 4          | 40        | 125    | 124    | 11/11          | 828/828        | No           |
|           | 5          | 20        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 6          | 20        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 7          | 20        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 8          | 40        | 125    | 124    | 11/11          | 828/828        | No           |
|           | 9          | 10        | 125    | 124    | 6/6            | 723/723        | No           |
|           | 10         | 10        | 125    | 124    | 6/6            | 723/723        | No           |
| b7b       | 1          | 10        | 125    | 124    | 1/1            | 658/658        | No           |
|           | 2          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 3          | 10        | 125    | 124    | 1/1            | 628/628        | No           |
|           | 4          | 20        | 125    | 124    | 1/1            | 678/678        | No           |
|           | 5          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 6          | 20        | 125    | 124    | 1/1            | 618/618        | No           |
| b7c       | 1          | 40        | 125    | 125    | 25/25          | 1100/1050      | Yes          |
|           | 2          | 40        | 125    | 124    | 1/1            | 716/638        | No           |
|           | 3          | 10        | 125    | 124    | 11/11          | 828/828        | No           |
|           | 4          | 10        | 125    | 124    | 11/11          | 828/828        | No           |
|           | 5          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 6          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
|           | 7          | 10        | 125    | 124    | 7/5            | 780/686        | No           |
|           | 8          | 10        | 125    | 124    | 1/1            | 618/618        | No           |
| b7d       | 1          | 10        | 125    | 124    | 1/1            | 662/638        | No           |
|           | 2          | 10        | 125    | 124    | 1/1            | 772/618        | No           |
| Plane | Unital No. | $|\text{Aut}(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-------|------------|----------------|--------|--------|-------------------------------|-----------------------------------|------------|
| b8    | 1          | 40             | 125    | 125    | 45/105                        | 1470/2790                         | Yes        |
|       | 2          | 40             | 125    | 125    | 45/45                         | 1510/1530                         | Yes        |
|       | 3          | 20             | 125    | 125    | 25/25                         | 1050/1050                         | Yes        |
|       | 4          | 20             | 125    | 125    | 25/25                         | 1050/1050                         | Yes        |
|       | 5          | 20             | 125    | 125    | 25/25                         | 1070/1070                         | Yes        |
|       | 6          | 20             | 125    | 125    | 25/25                         | 1050/1050                         | Yes        |
|       | 7          | 10             | 125    | 125    | 25/25                         | 1050/1050                         | Yes        |
|       | 8          | 10             | 125    | 125    | 25/25                         | 1050/1050                         | Yes        |
|       | 9          | 20             | 125    | 125    | 25/25                         | 1090/1090                         | Yes        |
|       | 10         | 10             | 125    | 125    | 25/25                         | 1050/1050                         | Yes        |
|       | 11         | 20             | 125    | 125    | 25/25                         | 1110/1110                         | Yes        |
|       | 12         | 10             | 125    | 125    | 25/25                         | 1060/1060                         | Yes        |
| b8a   | 1          | 10             | 125    | 124    | 16/16                         | 943/943                           | No         |
|       | 2          | 10             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 3          | 40             | 125    | 124    | 21/21                         | 1078/1078                         | No         |
|       | 4          | 20             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 5          | 20             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 6          | 20             | 125    | 124    | 1/1                           | 638/638                           | No         |
|       | 7          | 20             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 8          | 40             | 125    | 124    | 81/21                         | 2358/1038                         | No         |
|       | 9          | 10             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 10         | 10             | 125    | 124    | 6/6                           | 728/723                           | No         |
| b8c   | 1          | 20             | 125    | 124    | 1/1                           | 678/678                           | No         |
|       | 2          | 20             | 125    | 124    | 1/1                           | 638/638                           | No         |
|       | 3          | 10             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 4          | 10             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 5          | 10             | 125    | 124    | 1/1                           | 618/618                           | No         |
|       | 6          | 10             | 125    | 124    | 1/1                           | 618/618                           | No         |
| b8d   | 1          | 10             | 125    | 124    | 21/16                         | 1048/933                          | No         |
|       | 2          | 10             | 125    | 124    | 11/11                         | 853/848                           | No         |
|       | 3          | 10             | 125    | 124    | 11/11                         | 868/868                           | No         |
|       | 4          | 10             | 125    | 124    | 21/16                         | 1048/933                          | No         |
|       | 5          | 10             | 125    | 124    | 11/11                         | 843/843                           | No         |
|       | 6          | 10             | 125    | 124    | 6/6                           | 723/723                           | No         |
|       | 7          | 10             | 125    | 124    | 6/6                           | 723/723                           | No         |
|       | 8          | 10             | 125    | 124    | 16/16                         | 938/953                           | No         |
|       | 9          | 10             | 125    | 124    | 6/6                           | 723/723                           | No         |
|       | 10         | 10             | 125    | 124    | 16/16                         | 943/958                           | No         |
| b8e   | 1          | 10             | 125    | 124    | 1/1                           | 658/638                           | No         |
| b8f   | 1          | 10             | 125    | 124    | 1/1                           | 677/653                           | No         |
Table 3 continued

| Plane | Unital No. | $|\text{Aut}(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-------|------------|-----------------|--------|--------|---------------------------------|---------------------------------|-------------|
| s1    | 1          | 756000          | 105    | 105    | 78025                           | 1716550                        | Yes         |
|       | 2          | 1000            | 125    | 125    | 25                              | 1050                           | Yes         |
|       | 3          | 2000            | 121    | 125    | 25                              | 1050                           | Yes         |
| s2    | 1          | 144             | 115    | 115    | 2209/802                        | 52234/18040                    | No          |
|       | 2          | 1440            | 109    | 108    | 13801/13801                     | 307078/307078                  | No          |
|       | 3          | 80              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 4          | 40              | 125    | 124    | 1/1                             | 618/618                        | No          |
| s2a   | 1          | 40              | 125    | 125    | 45                              | 1530                           | Yes         |
|       | 2          | 40              | 125    | 125    | 45                              | 1470                           | Yes         |
|       | 3          | 10              | 125    | 124    | 1                               | 618                            | No          |
|       | 4          | 10              | 125    | 124    | 296                             | 7483                           | No          |
|       | 5          | 20              | 125    | 124    | 31                              | 1288                           | No          |
| s2b   | 1          | 10              | 125    | 124    | 131/301                         | 3468/7628                      | No          |
|       | 2          | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 3          | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 4          | 10              | 125    | 124    | 121/216                         | 3263/5913                      | No          |
|       | 5          | 10              | 125    | 124    | 121/301                         | 3248/7588                      | No          |
|       | 6          | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
| s2c   | 1          | 12              | 121    | 119    | 421/167                         | 10351/4160                     | No          |
|       | 2          | 24              | 123    | 122    | 164/80                          | 4142/2312                      | No          |
|       | 3          | 12              | 125    | 124    | 11                              | 898                            | No          |
|       | 4          | 24              | 123    | 122    | 97/86                           | 2692/2440                      | No          |
|       | 5          | 60              | 125    | 120    | 436/496                         | 10188/11908                    | No          |
|       | 6          | 20              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 7          | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 8          | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 9          | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 10         | 24              | 123    | 123    | 99/65                           | 2736/1952                      | No          |
|       | 11         | 24              | 123    | 123    | 202/121                         | 4978/3196                      | No          |
|       | 12         | 24              | 123    | 123    | 50/30                           | 1610/1175                      | Yes         |
|       | 13         | 20              | 125    | 125    | 25/35                           | 1050/1270                      | Yes         |
|       | 14         | 10              | 125    | 124    | 1/1                             | 628/628                        | No          |
|       | 15         | 10              | 125    | 124    | 1/1                             | 618/618                        | No          |
|       | 16         | 10              | 125    | 124    | 1/1                             | 623/643                        | No          |
|       | 17         | 10              | 125    | 124    | 1/1                             | 623/643                        | No          |
|       | 18         | 10              | 125    | 124    | 1/1                             | 623/643                        | No          |
| Plane | Unital No. | $|\text{Aut}(U)|$ | 2-rank | 3-rank | Par. Clas. of $D(U)/D(U^\perp)$ | Part. Par. Clas. of $D(U)/D(U^\perp)$ | Resolvable? |
|-------|-----------|----------------|--------|--------|--------------------------------|--------------------------------|-----------|
| s3    | 1         | 72             | 117    | 116    | 1030/772                       | 24430/17482                    | No        |
|       | 2         | 120            | 121    | 121    | 525/405                        | 12750/9330                     | Yes       |
|       | 3         | 40             | 125    | 125    | 25/25                          | 1050/1050                      | Yes       |
|       | 4         | 20             | 125    | 125    | 25/25                          | 1050/1050                      | Yes       |
| s3a   | 1         | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 2         | 10             | 125    | 124    | 6/6                            | 723/733                        | No        |
|       | 3         | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 4         | 10             | 125    | 124    | 11/11                          | 828/858                        | No        |
|       | 5         | 120            | 125    | 120    | 321/321                        | 7638/7638                      | No        |
|       | 6         | 40             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 7         | 20             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 8         | 24             | 123    | 122    | 164/91                         | 4172/2518                      | No        |
|       | 9         | 24             | 123    | 123    | 196/98                         | 4846/2648                      | No        |
|       | 10        | 12             | 125    | 123    | 96/48                          | 2670/1946                      | No        |
|       | 11        | 12             | 125    | 123    | 100/59                         | 2758/1822                      | No        |
| s3b   | 1         | 20             | 125    | 124    | 1/11                           | 618/868                        | No        |
|       | 2         | 10             | 125    | 124    | 1/1                            | 628/648                        | No        |
|       | 3         | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 4         | 10             | 125    | 124    | 76/116                         | 2258/3288                      | No        |
|       | 5         | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 6         | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 7         | 10             | 125    | 124    | 6/6                            | 728/753                        | No        |
|       | 8         | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 9         | 10             | 125    | 124    | 71/141                         | 2148/3823                      | No        |
|       | 10        | 10             | 125    | 124    | 1/1                            | 618/618                        | No        |
|       | 11        | 10             | 125    | 124    | 21/36                          | 1048/1403                      | No        |
|       | 12        | 10             | 125    | 124    | 1/1                            | 622/638                        | No        |
| s4    | 1         | 144            | 117    | 116    | 1261/718                       | 29326/16336                    | No        |
|       | 2         | 96             | 122    | 125    | 654/83                         | 15708/22282                    | No        |
|       | 3         | 240            | 121    | 121    | 345/375                        | 8790/8730                      | Yes       |
|       | 4         | 80             | 125    | 125    | 25/25                          | 1130/1130                      | Yes       |
|       | 5         | 10             | 125    | 125    | 25/25                          | 1050/1050                      | Yes       |
|       | 6         | 10             | 125    | 125    | 25/25                          | 1060/1060                      | Yes       |
|       | 7         | 10             | 125    | 125    | 55/30                          | 1730/1165                      | Yes       |
|       | 8         | 10             | 125    | 125    | 25/25                          | 1060/1060                      | Yes       |
|       | 9         | 20             | 125    | 125    | 55/30                          | 1720/1155                      | Yes       |
|       | 10        | 20             | 125    | 125    | 200/80                         | 4940/2235                      | Yes       |
| s4a   | 1         | 10             | 125    | 124    | 1/1                            | 623/643                        | No        |
|       | 2         | 10             | 125    | 124    | 36/41                          | 1363/1538                      | No        |
|       | 3         | 240            | 125    | 120    | 351/321                        | 8298/7638                      | No        |
Table 3 continued

| Plane No. | Unital No. | $|Aut(U)|$ | 2-rank | 3-rank | Par. Clas. of \(D(U)/D(U^\perp)\) | Part. Par. Clas. of \(D(U)/D(U^\perp)\) | Resolvable? |
|-----------|------------|----------|--------|--------|-----------------|-------------------------------|-----------|
| 4         | 80         | 125      | 124    | 1/1    | 698/698         | No                            |           |
| 5         | 40         | 125      | 124    | 1/1    | 618/618         | No                            |           |
| 6         | 24         | 123      | 123    | 84/53  | 4214/2000       | No                            |           |
| 7         | 12         | 125      | 123    | 4/8    | 2436/1712       | No                            |           |
| s4b       | 1          | 10       | 125    | 124    | 1/1             | 688/688                       | No        |
| 2         | 10         | 125      | 124    | 26/46  | 1198/1808       | No                            |           |
| 3         | 20         | 125      | 124    | 36/66  | 1353/2058       | No                            |           |
| 4         | 20         | 125      | 124    | 6/31   | 723/1288        | No                            |           |
| 5         | 20         | 125      | 124    | 1/6    | 753/743         | No                            |           |
| 6         | 10         | 125      | 124    | 1/1    | 628/628         | No                            |           |
| 7         | 10         | 125      | 124    | 1/1    | 618/618         | No                            |           |
| 8         | 10         | 125      | 124    | 1/1    | 628/628         | No                            |           |
| 9         | 10         | 125      | 124    | 6/31   | 723/1288        | No                            |           |
| 10        | 10         | 125      | 124    | 41/41  | 1478/1503       | No                            |           |
| 11        | 10         | 125      | 124    | 6/6    | 723/723         | No                            |           |
| 12        | 10         | 125      | 124    | 6/6    | 723/728         | No                            |           |
| 13        | 10         | 125      | 124    | 31/61  | 1253/1953       | No                            |           |
| 14        | 10         | 125      | 124    | 6/6    | 723/733         | No                            |           |
| 15        | 10         | 125      | 124    | 31/56  | 1253/1834       | No                            |           |
| s5        | 1          | 96       | 123    | 124    | 630/43          | 15132/1498                    | No        |
| 2         | 30         | 125      | 125    | 25/25  | 3310/1720       | Yes                           |           |
| 3         | 10         | 125      | 125    | 25/25  | 1050/1050       | Yes                           |           |
| 4         | 20         | 125      | 125    | 40/40  | 1365/1365       | Yes                           |           |
| 5         | 10         | 125      | 125    | 25/25  | 1050/1050       | Yes                           |           |
| 6         | 10         | 125      | 125    | 25/25  | 1050/1050       | Yes                           |           |
| 7         | 10         | 125      | 125    | 55/30  | 1750/1185       | Yes                           |           |
| 8         | 10         | 125      | 125    | 55/30  | 1750/1185       | Yes                           |           |
| 9         | 20         | 125      | 125    | 40/40  | 1365/1365       | Yes                           |           |
| 10        | 20         | 125      | 125    | 75/50  | 2160/1595       | Yes                           |           |
| 11        | 10         | 125      | 125    | 25/25  | 1060/1060       | Yes                           |           |
| 12        | 10         | 125      | 125    | 25/25  | 1050/1050       | Yes                           |           |
| s5a       | 1          | 20       | 125    | 124    | 16/26           | 933/1498                      | No        |
| 2         | 10         | 125      | 124    | 1/1    | 628/628         | No                            |           |
| 3         | 10         | 125      | 124    | 1/1    | 618/618         | No                            |           |
| 4         | 10         | 125      | 124    | 21/21  | 1058/1058       | No                            |           |
| 5         | 10         | 125      | 124    | 26/26  | 1163/1168       | No                            |           |
| 6         | 10         | 125      | 124    | 11/11  | 843/863         | No                            |           |
| 7         | 10         | 125      | 124    | 16/16  | 943/953         | No                            |           |
### Table 3 continued

| Plane | Unital No. | \(|\text{Aut}(U)|\) | 2-rank | 3-rank | Par. Clas. of \(D(U)/D(U^\perp)\) | Part. Par. Clas. of \(D(U)/D(U^\perp)\) | Resolable? |
|-------|------------|--------------------|--------|--------|---------------------------------|-----------------------------------|-----------|
| s5b   | 1          | 30                 | 125    | 124    | 31/26                           | 1308/1208                         | No        |
|       | 2          | 10                 | 125    | 124    | 6/6                             | 723/723                           | No        |
|       | 3          | 10                 | 125    | 124    | 16/16                           | 933/933                           | No        |
|       | 4          | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
|       | 5          | 10                 | 125    | 124    | 21/21                           | 1038/1068                         | No        |
|       | 6          | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
|       | 7          | 10                 | 125    | 124    | 16/16                           | 933/933                           | No        |
|       | 8          | 10                 | 125    | 124    | 6/6                             | 723/723                           | No        |
|       | 9          | 10                 | 125    | 124    | 21/46                           | 1038/1613                         | No        |
|       | 10         | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
| s5c   | 1          | 10                 | 125    | 124    | 11/11                           | 828/828                           | No        |
|       | 2          | 10                 | 125    | 124    | 6/6                             | 723/723                           | No        |
|       | 3          | 20                 | 125    | 124    | 16/16                           | 933/933                           | No        |
|       | 4          | 20                 | 125    | 124    | 16/16                           | 933/933                           | No        |
|       | 5          | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
|       | 6          | 10                 | 125    | 124    | 16/16                           | 938/963                           | No        |
|       | 7          | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
|       | 8          | 10                 | 125    | 124    | 6/31                            | 733/1298                          | No        |
|       | 9          | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
|       | 10         | 10                 | 125    | 124    | 1/1                             | 628/628                           | No        |
|       | 11         | 10                 | 125    | 124    | 16/16                           | 938/968                           | No        |
|       | 12         | 10                 | 125    | 124    | 1/1                             | 618/618                           | No        |
|       | 13         | 10                 | 125    | 124    | 6/6                             | 723/723                           | No        |
|       | 14         | 10                 | 125    | 124    | 11/11                           | 828/828                           | No        |
| h1    | 1          | 240                | 114    | 115    | 1035                            | 23250                             | No        |
|       | 2          | 40                 | 122    | 125    | 5                               | 630                               | No        |
| h1a   | 1          | 24                 | 119    | 120    | 125/50                          | 3458/1618                         | No        |
|       | 2          | 20                 | 125    | 125    | 55/30                           | 1720/1165                         | Yes       |
| h2    | 1          | 48                 | 123    | 123    | 33                              | 1314                              | No        |
| h2a   | 1          | 48                 | 123    | 124    | 57/57                           | 1746/1746                         | Yes       |
|       | 2          | 40                 | 125    | 125    | 35/35                           | 1270/1260                         | Yes       |
| w1    | 1          | 80                 | 125    | 125    | 25/25                           | 1050/1090                         | Yes       |
|       | 2          | 40                 | 125    | 125    | 25/25                           | 1050/1090                         | Yes       |
|       | 3          | 40                 | 125    | 125    | 265/205                         | 6310/4990                         | Yes       |

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