UNIQUENESS THEOREMS FOR INVERSE OBSTACLE SCATTERING PROBLEMS IN LIPSCHITZ DOMAINS*

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Abstract. For the Neumann and Robin boundary conditions the uniqueness theorems for inverse obstacle scattering problems are proved in Lipschitz domains. The role of non-smoothness of the boundary is analyzed.

1. Introduction.

Let \( D \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded domain with a sufficiently smooth boundary \( \Gamma \), not necessarily connected, but consisting of a finitely many connected components. Let \( D' := \mathbb{R}^n \setminus D \) be the exterior domain, \( k > 0 \) a fixed wavelength, \( \alpha \in S^{n-1} \) a given unit vector, \( S^{n-1} \) the unit sphere. It is well known that the obstacle scattering problem:

\[
\nabla^2 u + k^2 u = 0 \text{ in } D', \\
u_N = 0 \text{ on } \Gamma, \\
u = u_0 + v, \quad u_0 := \exp(ika \cdot x),
\]

where \( v \) satisfies the radiation condition

\[
\lim_{r \to \infty} \int_{|x|=r} |v_r - ikv|^2 ds = 0,
\]

and \( N \) is the exterior unit normal to \( \Gamma \) has been studied intensively and there are many ways known for proving the existence and uniqueness of its solution which is called the scattering solution [3]. The function \( v \) has the following asymptotics

\[
v = A(\alpha', \alpha, k)\gamma(r) + o(\frac{1}{r}) \text{ as } r \to \infty, \quad \alpha' := x/r.
\]

The coefficient \( A(\alpha', \alpha, k) \) is called the scattering amplitude.

We also consider the Robin boundary condition in place of (2):

\[
u_N + \sigma(s)v = 0 \text{ on } \Gamma,
\]

where \( \sigma \) is a continuous real-valued function on \( \Gamma \).

In what follows, we denote by a subindex zero the quantity which is fixed. The inverse obstacle scattering problems (IOSP1-5) can be stated as follows:

1) Given \( A(\alpha', \alpha_0, k) \) \( \forall \alpha' \in S^{n-1}, \forall k \in [a, b], \) \( 0 \leq a < b, \) find \( \Gamma \), or, if Robin’s condition is assumed, find \( \Gamma \) and \( \sigma \):

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2) Given $A(\alpha', \alpha, k_0) \forall \alpha' \in S^{n-1}$, find $\Gamma$, or, if Robin’s condition is assumed, find $\Gamma$ and $\sigma$;

3) Given $A(\alpha', \alpha_0, k_0) \forall \alpha' \in S^{n-1}$, find $\Gamma$, or, if Robin’s condition is assumed, find $\Gamma$ and $\sigma$;

4) Given $A(-\alpha, \alpha, k_0) \forall \alpha \in S^{n-1}$, find $\Gamma$, or, if Robin’s condition is assumed, find $\Gamma$ and $\sigma$; (backscattering data)

5) Given $A(-\alpha, \alpha, k) \forall \alpha \in S^{n-1}, \forall k \in [a, b], 0 \leq a < b$, find $\Gamma$, or, if Robin’s condition is assumed, find $\Gamma$ and $\sigma$;

Of course, if IOSP4 is solved then IOSP5 is solved.

In all these problems one can assume for uniqueness studies that the data are given on open subsets of $S^{n-1}$, however small, since such data allow one to uniquely recover the data on all of $S^{n-1}$ [3].

In this paper we discuss only IOSP1-2. Uniqueness of the solution to other three problems has been (and still is) an open problem for several decades, although for IOSP5 uniqueness for convex obstacles follows from the results in [3]. The reconstruction of $\Gamma$ from the scattering data is not discussed here, see [3],[4] and references therein.

The history and various proofs of the uniqueness theorems for IOSP1,2 are given in [3],[4] and references therein, and a new method of proof and its applications are given in [5]-[8].

Uniqueness of the solution for IOSP1 was proved by M.Schiffer (1962) for the Dirichlet boundary condition, while for IOSP2 it was proved by A.G.Ramm (1985) for the Dirichlet, Neumann and Robin boundary conditions (see [3] for these proofs). In [5]-[7] a new method of proof was given.

In this paper we discuss the technical question: the role of smoothness of the boundary in the various proofs of the uniqueness theorems for IOSP1-2. We justify the applicability of Green’s formula in the Schiffer’s and other proofs and point out that the question of whether the Neumann Laplacian has a discrete spectrum in a certain domain with non-smooth boundary can be avoided completely. This question arises in the Schiffer’s type of proofs.

Furthermore, we generalize the uniqueness results for Lipschitz domains, i.e., for domains with Lipschitz boundaries.

In section 2 the Schiffer’s type proof and the proof from [6],[7] are presented, and the role of the non-smoothness of some of the domains, used in these proofs, is analyzed. An important role is played by the sets of finite perimeter and Green’s formula for such sets. The related theory is discussed in [1],[2] and [9].

We first assume in this paper that the boundary $\Gamma$ is sufficiently smooth and then show that our argument is valid in Lipschitz domains. So, this paper deals with the technical problems.

Recall that a Lipschitz domain is a bounded domain each point of whose boundary has a neighborhood in which the equation of the boundary in the local coordinates is given by a function satisfying a Lipschitz condition. Lipschitz domains are denoted as $C^{0,1}$ domains. In [10] the potential theory results are given for Lipschitz domains.

The definition of the solution to problem (1)-(3) in non-smooth domains is as follows:

A function $u \in H^2_{loc} \cap H^1(D'_R)$ solves (1)-(3), iff it satisfies conditions (3) and (4), and the following identity:

$$\int_{D'} (k^2 u \phi - \nabla u \nabla \phi) dx = 0 \quad \forall \phi \in H^2_{loc} \cap H^1_c(D').$$

Here $H^1$ is the Sobolev space, $H^2_{loc}$ is the space of functions which are in $H^2(D')$ for any compact strictly inner subdomain $D'$ of $D'$, $H^1_c(D')$ is the space of functions which vanish
near infinity (but not necessarily near $\Gamma$), and $H^1(D'_R)$ is the space of functions which for any sufficiently large $R$ belong to $H^1(D' \cap B_R)$, where $B_R$ is the ball of radius $R$, centered at the origin. This definition does not require any smoothness of the boundary.

The proof of this claim goes along the usual line. The new point is the discussion of the points which belong to the intersection of $\Gamma$. 

The solution to (1), (6), (3) is a function in $H^2_{loc} \cap H^1(D'_R)$ which satisfies conditions (3) and (4), and the identity

$$\int_{D'} (k^2 u\phi - \nabla u \nabla \phi) dx + \int_{\Gamma} \sigma u\phi ds = 0 \quad \forall \phi \in H^2_{loc} \cap H^1(D').$$

Here the Lipschitz boundary $\Gamma$ is admissible because the imbedding theorem holds for such a boundary.

In this paper we use the following notations: $D_{12} := D_1 \cup D_2$, $D^{12} := D_1 \cap D_2$, $\Gamma_{12}$ is the boundary of $D_{12}$, $\Gamma^{12}$ is the boundary of $D^{12}$, $\Gamma'_1$ is the part of $\Gamma_1$ which lies outside of $D_2$, and $\Gamma'_2$ is defined likewise, $D_1$ is a connected component of $D_1 \setminus D^{12}$, $D_3 := D_{12} \setminus D^{12}$.

2. Uniqueness results for IOSP with the Neumann and Robin boundary conditions.

2.1. Uniqueness for IOSP1.

Consider IOSP1 first. Let us outline a variant of the Schiffer’s type of proof, which allows us to deal with non-smooth boundaries of the domains arising in the proof. Assume that there are two different obstacles, $D_j$, $j = 1, 2$, which generate the same scattering data for IOSP1. Let $w := u_1 - u_2$, where $u_j$ are the corresponding scattering solutions. The function $w$ solves equation (1) in $D'_1$ and $w = o(1/r)$ because the scattering data are the same for $D_1$ and $D_2$. Thus, lemma [3,p.25] implies $w = 0$ in $D'_{12}$. Let $U := u_1 = u_2$ in $D'_1$. Then $U$ can be continued analytically, as a solution to (1), to the domains $D_3$ and $(D^{12})'$, because either $u_1$ or $u_2$ are defined in these domains and solve (1) there. We assume that $(D^{12})'$ is not empty. If it is, the argument is even simpler: $V := U - u_0$ solves equation (1) in $\mathbb{R}^n$ and satisfies the radiation condition; thus, $V = 0$ and $U = u_0$ in $\mathbb{R}^n$. Since $u_0$ does not satisfy the boundary condition (2), we have got a contradiction. This contradiction proves that the assumption ($(D^{12})'$ is empty) is wrong.

The domain $D_3$ is bounded since both $D_j$ are. The function $U$ solves equation (1) and satisfies the homogeneous boundary condition (2) on its boundary $\Gamma_3$, except for, possibly, the set of $(n-1)$-dimensional Hausdorff measure, namely, except for the set of points which belong to the intersection of $\Gamma_1$ and $\Gamma_2$. Since the scattering solutions in domains with smooth boundaries are uniformly bounded functions whose first derivatives are smooth (Lipschitz are sufficient for our argument), the function $U$ has the same properties. Therefore, for any $k \in [a, b]$, $U$ is in $L^2(D_3)$, and, as we prove below, the functions $U$, corresponding to different $k$, are orthogonal in $L^2(D_3)$. Since this Hilbert space is separable, we arrive at a contradiction: existence of a continuum of orthogonal non-trivial elements in the separable Hilbert space $L^2(D_3)$. This contradiction proves that $D_1 = D_2$, and the uniqueness theorem is proved for IOSP1. The original Schiffer’s argument presented in the literature, uses discreteness of the spectrum of the Laplacian, corresponding to a boundary condition, in a bounded domain. The discreteness of the spectrum holds for any bounded domain for the Dirichlet Laplacian, but not necessarily for the Neumann one. This is why we want to avoid the reference to the discreteness of the spectrum of the Neumann Laplacian or the Robin Laplacian. To complete the proof, it is sufficient to prove the claim about the orthogonality of $U$ with different $k$. The proof of this claim goes along the usual line. The new point is the discussion of the applicability of Green’s formula, used in the argument, for non-smooth domains. Let
\( U_j := U(x, \alpha_0, k_j) \), \( L := \nabla^2 \), and let the overline denote complex conjugate. Then:

\[
I := \int_{D_3} (U_1 \nabla U_2 - \overline{U}_2 \nabla U_1) dx = (k_1^2 - k_2^2) \int_{D_3} U_1 \overline{U}_2 dx. \tag{9}
\]

We wish to prove that the right-hand side vanishes. This follows if \( I = 0 \). The integral \( I \) can be transformed formally by Green’s formula and, using the boundary condition, one concludes that \( I = 0 \). The problem is to justify the applicability of Green’s formula in the domain \( D_3 \) with non-smooth boundary. The remaining part of the proof contains such a justification.

Our starting point is the known (see [1], [2], [9]) result:

Green’s formula holds for the domains with finite perimeter and functions whose first derivatives are in the space \( BV \), provided that their rough traces are summable on the reduced boundary of the domain (in our case \( D_3 \) is the domain) with respect to \((n - 1)\)-dimensional Hausdorff measure.

Let \( \Omega \in \mathbb{R}^n \) be a domain. Recall that the space \( BV(\Omega) \) consists of functions whose first derivatives are signed measures locally in \( \Omega \) ([2], [9]). A set \( D \) has finite perimeter if \( \chi_j \), the characteristic function of this set, belongs to \( BV(\mathbb{R}^n) \). The reduced boundary, denoted by \( \Gamma^* \), is the set of points at which the exterior normal in the sense of Federer exists (see [1], [2], or [9] for the definition of this normal and [2] for that of the rough trace). It is proved in [9], that for the sets with finite perimeter the reduced boundary has full \((n - 1)\)-dimensional Hausdorff measure, so that the normal in the sense of Federer is defined almost everywhere on \( \Gamma \) with respect to \((n - 1)\)-dimensional Hausdorff measure (we will write \( s \)-almost everywhere for brevity). What we need is to check that:

1) the set \( D_3 \) has finite perimeter,
2) the function \( \nabla \cdot \psi \) is a measure in \( D_3 \), where \( \psi := U_1 \nabla \overline{U}_2 - \overline{U}_2 \nabla U_1 \), and
3) \( \psi \) has a summable rough trace on \( \Gamma^* \), the reduced boundary of \( D_3 \).

Note that the integrand in the first integral in formula (9) is of the form \( \nabla \cdot \psi \), and \( \nabla \cdot \psi = (k_1^2 - k_2^2)U_1 \overline{U}_2 \).

First, let us prove that \( D_3 \) has finite perimeter. Note, that \( D_3 \) is not necessarily a Lipschitz domain, although \( D_1 \) and \( D_2 \) are. Let us denote by \( P(D) \) the perimeter of \( D \) and by \( ||\nabla \chi|| \) the norm of \( \chi \) in the space \( BV(\mathbb{R}^n) \), that is, the total variation of the vector measure \( \nabla \chi \). By definition, \( P(D) = ||\nabla \chi|| \).

Let \( s(\Gamma) \) denote the \( n - 1 \)-dimensional Hausdorff measure of \( \Gamma \). It is known that \( P(D) \leq s(\Gamma) \) and the strict inequality is possible for non-smooth \( \Gamma \). Also, it can happen that \( P(D) < \infty \), but \( s(\Gamma) = \infty \). If \( P(D) < \infty \), then \( \Gamma^* \) is \( s \)-measurable and \( s(\Gamma^*) = P(D) \), see [9, p.193].

The set \( D_3 \) has finite perimeter iff \( ||\nabla \chi_3|| < \infty \). Clearly:

\[
\chi_3 = \chi_{12} - \chi^{12}, \tag{10}
\]

\[
\chi_{12} = \chi_1 + \chi_2 - \chi^{12}, \tag{11}
\]

\[
\chi^{12} = \chi_1 \chi_2, \tag{12}
\]

where \( \chi_j^{12} \), e.g., is the characteristic function of the domain \( D^{12} \). By the assumption, \( ||\nabla \chi_j|| < \infty \), \( j = 1, 2 \). The space \( BV \) is linear. Therefore, by formulas (10)-(12), it follows that \( P(D_3) < \infty \), if one checks that the function \( \chi_1 \chi_2 \in BV(\mathbb{R}^n) \). This, however, is a direct consequence of the known formula [9,p.189] for the derivative of the product of bounded \( BV \) functions: \( \nabla(\chi_1 \chi_2) = \hat{\chi}_2 \nabla \chi_1 + \hat{\chi}_1 \nabla \chi_2 \), where \( \hat{\chi} \) denotes the averaged value.
Theorem 1. Assume that the obstacles \( D_j, j = 1, 2 \), have the following properties:

1) they are Lipschitz domains,
2) \( A_1(\alpha', \alpha_0, k) = A_2(\alpha', \alpha_0, k) \forall \alpha' \in S^{n-1}, \forall k \in [a, b], 0 \leq a < b.\)

Then \( D_1 = D_2 \) and, in the case of Robin's boundary condition, \( \sigma_1 = \sigma_2. \)

Proof. Only the last statement is not yet proved. However, since we have already established that \( D_1 = D_2 := D \) and \( u_1 = u_2 \) in \( D' \), it follows that

\[
\sigma_1 = -\frac{u_{1N}}{u_1} = -\frac{u_{2N}}{u_2} = \sigma_2 \quad \text{on} \quad \Gamma. \quad \square
\]

Another proof can be given. It is based on formula (13) and on the method, developed in section 2.2 below.

If \( \Gamma_j \) are Lipschitz boundaries, then the existence and uniqueness of the scattering solutions can be established as in [3] with the help of the potential theory for domains with Lipschitz boundaries [10]. The details of this theory will be published elsewhere.

In the next subsection we consider IOSP2 and use the method developed in [5]-[7] for the uniqueness proof.

2.2. Uniqueness for IOSP2.

The starting point is the identity first established in [5]:

\[
4\pi(A_1 - A_2) = \int_{\Gamma_{12}} |u_1 u_{2N} - u_{1N} u_2| ds,
\]

(13)

where \( u_1 := u_1(x, \alpha, k), u_2 := u_2(x, -\alpha', k), u_N \) denotes the normal derivative, as before, \( u_j \) and \( A_j := A_j(\alpha', \alpha, k) \) are, respectively, the scattering solution and scattering amplitude, corresponding to the obstacle \( D_j, j = 1, 2 \). Applications of this useful formula are given in [5]-[8].

If \( A_1 = A_2 \) for the fixed energy data in IOSP2, then (13) yields:

\[
0 = \int_{\Gamma_{12}} [u_1(s, \alpha) u_{2N}(s, -\alpha') - u_{1N}(s, \alpha) u_2(s, -\alpha')] ds, \quad \forall \alpha, \alpha' \in S^{n-1},
\]

(14)
where we have dropped the dependence on the fixed energy \( k_0 \).

Let \( G_j := G_j(x,y,k) \) denote Green’s function for the problem (1)-(3), or the exterior problem with the Robin boundary condition. It is proved in [3,p.46], that

\[
G_j = \gamma(r)[u_j(x,\alpha,k) + O\left(\frac{1}{r}\right)], \quad r \to \infty, \quad y/r := -\alpha, \quad r := |y|,
\]

where \( \gamma(r) \) is a known function (e.g., \( \gamma = \frac{\exp(ikr)}{4\pi r} \) if \( n = 3 \)), and the coefficient \( u_j \) in (15) is the scattering solution.

**Lemma 1.** Equation (14) implies:

\[
0 = \int_{\Gamma_{12}} [G_1(s,x)G_2N(s,y) - G_{1N}(s,x)G_2(s,y)]ds, \quad \forall x,y \in D'_{12}.
\]

**Proof.** We give a proof for \( n = 3 \). For other \( n \) the proof is similar. First, let us derive the equation:

\[
W(y) := \int_{\Gamma_{12}} [u_1(s,\alpha)G_{2N}(s,y) - u_{1N}(s,\alpha)G_2(s,y)]ds = 0, \quad \forall y \in D'_{12}, \forall \alpha \in S^{n-1}.
\]

Indeed, \( W(y) \) solves equation (1) in \( D'_1 \) and \( W = o(1/r) \), as follows from (14) and (15). Thus, \( W = 0 \) in \( D'_1 \), see [3,p.25].

Let us prove (16) now. Fix any \( y \in D'_{12} \) and let \( w \) denote the integral in (16). Then \( w \) solves (1) in \( D'_1 \) and \( w = o(1/r) \), as follows from (15) and (17). Thus, (16) follows and Lemma 1 is proved. \( \square \)

We want to derive a contradiction from (16). This contradiction will prove that \( D_1 = D_2 \). According to the argument given in the section 2.1, the set \( D_{12} \) has finite perimeter, Green’s formula is applicable to (16) in the domain \( D'_{12} \), and we get the following equation:

\[
0 = G_1(y,x) - G_2(x,y) \quad \forall x,y \in D'_{12}, \quad \forall \alpha,\alpha' \in S^{n-1}.
\]

This completes the discussion of the uniqueness theorem for IOSP2 for the case of Neumann and Robin boundary conditions.
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