Resonant Spheres: Multifrequency Detectors of Gravitational Waves

M. Angeles Serrano and J. Alberto Lobos
Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain
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We discuss the capabilities of spherical antennae as single multifrequency detectors of gravitational waves. A first order theory allows us to evaluate the coupled spectrum and the resonators readouts when the first and the second quadrupole bare sphere frequencies are simultaneously selected for layout tuning. We stress the existence of non-tuning influences in the second mode coupling causing drags in the frequency splittings. These URF effects are relevant to a correct physical description of resonant spheres, still more if operating as multifrequency appliances like our PHCA proposal.

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A spherical antenna is particularly well adapted to sense metric tidal gravitational wave excitations as a consequence of the perfect matching between the GW radiation patterns and the sphere’s vibration eigenmodes. When suitably monitored, it can act as an individual multimode detector generating potential knowledge on the incident direction of the signal, its quadrupole or monopole amplitudes, or its polarizations.

Spherical geometry also offers another substantial ability which optimizes the capabilities of resonant bars currently in operation. By comparison, on equality of building material and mass, the estimated sphere’s absorption cross section for the fundamental mode is over a factor of 4 better than that of a Weber’s cylinder in its most favorable orientation. In addition, the second sphere’s quadrupole still shows a rather large decrease in its most favorable orientation. In addition, the second sphere’s quadrupole still shows a rather large decrease in its most favorable orientation. Hence, spheres show good sensitivity not only at the first but also at the second quadrupole harmonic, and therefore can efficiently operate at two frequencies as single multifrequency detectors.

The challenge becomes how to take advantage of this potentiality. A preliminary task before multifrequency implementation design is to acquire a clear understanding of the detector’s performance when working at one resonant frequency – see for instance or –.

The theories are proposed for practical multimode devices consisting of a perfect elastic sphere of radius $R$, mass $M$, density $\rho$ and Lamé coefficients $\lambda$ and $\mu$, monitored by a suitable readout system, commonly a set of $J$ identical resonators modelled as linear springs of mass $m \equiv \eta M, \eta << 1$, and endowed at certain positions $n_a \equiv x_a/R$ on the sphere’s surface. Each resonator amplifies the elastic radial deformation $u_a(t)$ of the sphere at its attachment point caused by an external force and supplies an essential increase in the coupling by measuring the springs’ deformations $q_a(t)$ relative to the sphere’s undeformed surface. Resonant tuning requires the sensors to be typically built to possess a natural frequency $\Omega$ equal to one of the eigenfrequencies of the free sphere’s spectrum, say $\omega_{NL}$. This is always taken to be a monopole or a quadrupole frequency $-\omega_{N\theta}$ or $\omega_{N2}$ respectively, since only monopole or quadrupole sphere’s spheroidal modes can be excited by an incoming wave.

In the universally assumed ideal approximation, the tuning frequency $\omega_{NL}$ is thought to be an isolated resonance frequency (IRF): no other frequency $\omega_{nt}$ of the free sphere’s spectrum interferes in the resonance. For instance, this assumption is correct when the resonators couple to $\omega_{12}$, in practice the most interesting coupling for unifrequency quadrupole radiation sensing.

We follow the perturbative approach in naturally opened to refined analysis, and recall that the responses of the coupled device to an incident GW can be inferred from the system

$$0 \frac{\partial^2 u_a}{\partial t^2} \equiv \mu \nabla^2 u_a - (\lambda + \mu) \nabla (\nabla \cdot u_a) = f_{\text{res}} + f_{\text{GW}}$$

$$\ddot{q}_a(t) + \Omega^2_a q_a(t) = -\ddot{u}_a(t) + \zeta_{a,\text{GW}}(t) \quad a = 1, \ldots, J, \quad (1)$$

where $f_{\text{res}}(x_a, t)$ and $f_{\text{GW}}(x_a, t)$ are respectively due to the resonators’ back action on the sphere and to the incident GW which also causes a tidal acceleration $\zeta_{a,\text{GW}}(t)$ on resonator $a$.

After implementation of Green function formalism, the equations can be rewritten in the $s$-Laplace domain as

$$\sum_{b=1}^{J} M_{ab} \hat{q}_b(s) = \frac{s^2}{s^2 + \omega_{NL}^2} \hat{\ddot{u}}_a(s) + \frac{\zeta_{a,\text{GW}}(s)}{s^2 + \omega_{NL}^2}, \quad a = 1, \ldots, J, \quad (2)$$

with $\hat{\ddot{u}}_a(s)$ the bare sphere’s radial responses to the signal at the resonators’ locations.

In the IRF circumstance, matrix $M_{ab}$ is of the form

$$M_{ab} \equiv \left[ \delta_{ab} + \eta \frac{s^2 \omega_{NL}^2}{s^2 + \omega_{NL}^2} P_{L,ab} \right]. \quad (3)$$
The geometric properties of a particular resonator arrangement are displayed by the $J \times J$ symmetric real matrix $P_L$ associated to the $L$-mode selected for tuning. It basically has as element $ab$ the Legendre polynomial of order $L$, a sum of products of spherical harmonics:

$$P_{L,ab} = |A_{NL}(R)|^2 \sum_{m=-L}^{m=L} Y^*_{LM}(n_a) Y_{LM}(n_b)$$

with $A_{NL}(R)$ radial components in the spheroidal normal modes of the free sphere. It is highly remarkable that all the information determining the distinctive readout of a given configuration is just concentrated the eigenvalues $\xi_n^2$ and eigenvectors $\mathbf{v}^{(a)}$ of $P_L$.

First order resolution for the spectrum of coupled-mode eigenfrequencies from $\text{det} M_{ab} = 0$ shows that the attachment of resonators causes the tuning frequency $\omega_{NL}$ to split into $J$ symmetric pairs around the original value,

$$\omega_{a,\pm}^2 = \omega_{NL}^2 \left( 1 \pm \xi_a \eta^2 \right) + O(\eta) \quad a = 1, \ldots, J, \quad (5)$$

whereas resonators amplitudes amount to be linear combinations of the GW amplitudes $\hat{g}^{(2m)}(s)$:

$$\hat{q}_a(s) = \eta^{\pm 2} \sum_{b=1}^{J} \sum_{\pm \xi_b \neq 0} \left\{ F_L(\pm \xi_c) \frac{1}{(s^2 + \omega_{c,\pm}^2)} \mathbf{v}_a^{(c)} \mathbf{v}_b^{(c)*} \right\} \times$$

$$\times \sum_{m=-2}^{m=2} Y_{2m}(n_b) \hat{g}_{2m}^I(s) + O(0) \quad a = 1, \ldots, J, \quad (6)$$

where $F_L(\pm \xi_c) = [a_{NL} A_{NL}(R)] (-1)^l (\xi_c)^{-1}$

with $a_{NL}$ non zero overlap coefficients only if $L = 0$ or $L = 2$.

| Layout | IRF Eigenfrequencies around $\omega_{12}$ | IRF Eigenfrequencies around $\omega_{22}$ |
|--------|----------------------------------------|----------------------------------------|
| PHCA   | $\omega_{0,\pm} = \omega_{12}(1 + 0.589 \eta^2)$ | $\omega_{0,\pm} = \omega_{22}(1 + 0.03 \eta^2)$ |
|        | $\omega_{1,\pm} = \omega_{12}(1 + 0.88 \eta^2)$ | $\omega_{1,\pm} = \omega_{22}(1 + 0.05 \eta^2)$ |
|        | $\omega_{2,\pm} = \omega_{12}(1 + 1.07 \eta^2)$ | $\omega_{2,\pm} = \omega_{22}(1 + 0.06 \eta^2)$ |
| TIGA   | $\omega_{\pm} = \omega_{12}(1 + 1.00 \eta^2)$ | $\omega_{\pm} = \omega_{22}(1 + 0.05 \eta^2)$ |

| TABLE I. IRF frequencies around $\omega_{12}$ and $\omega_{22}$ for the PHCA and TIGA arrangements. |

However, this is not always the case that the interesting frequency for tuning is an isolated resonant frequency. For example, the second quadrupole $\omega_{22}$ is in fact a suitable sphere’s frequency for a second resonator set to be tuned to in order to exploit the antenna as a multifrequency device. A careful examination of the spectrum of a typical planned full scale aluminium sphere $\eta_s \approx 1/40000$, $R = 1.5m$ around $\omega_{22}$ shows that $\omega_{14}$ is merely within a distance of order $\eta^{3/2}$:

$$\omega_{14}^2 = \omega_{22}^2 \left( 1 + K \eta^{3/2} \right), \quad (7)$$

where the dimensionless parameter $K$ takes the negative value $K = -2.21$ for $\eta = \eta_s$.

The expectation is that this nearness alters in some way the above IRF results. The effects of such unisolation –URF effects must be accurately considered to reach a faithful description of the detector’s real behaviour.

Let us analyse the general case when $\omega_{NL}$ is the single selected frequency for tuning and $\omega_{NL'}$ is in its neighbourhood. Again, we start from expressions (1) and (2) which remains unchanged, although a new contribution appears in (3):

$$M_{ab} = \left[ \delta_{ab} + \eta \frac{\omega_{NL}^2}{(s^2 + \omega_{NL}^2)^2} P_{L,ab} + \right. \left. \eta \frac{s^2 \omega_{NL}^2}{(s^2 + \omega_{NL}^2)^2} P_{L',ab} \right]. \quad (8)$$

We can still go further in drawing generic conclusions valid for any resonator distribution whenever it allows $P_L$ and $P_{L'}$ to commute: $[P_L, P_{L'}] = 0$. Then, we name $\mathbf{v}^{(a)}$ the orthogonal basis which simultaneously diagonalises the two matrices, with eigenvalues $\xi_{n,L}^2$ and $\xi_{n,L'}^2$, respectively, so that in this basis $M_{ab}$ is also diagonal. Then, the URF resonances around $\omega_{NL}$ can be written in $J$ triplets

$$\omega_{a,T}^2 = \omega_{NL}^2 \left( 1 + U_a \eta^{3/2} \right) + O(\eta) \quad a = 1, \ldots, J, \quad (9)$$

where for each resonator index $a$, the upper label $\{T\}$ represents the group $\{u, c, d\}$ which refers to the three different solutions of the cubic equation

$$U_a^3 - K \cdot U_a^2 - \chi^2 \cdot U_a + K \xi_{a,2} = 0$$

$$\chi^2 = \xi_{a,2}^2 + 2 \xi_{a,L'},$$

with parameters which are univocally determined by fixing the layout, $\omega_{NL}$ and $\omega_{NL'}$. Inspection of orders of magnitude in (10) for $\omega_{NL} = \omega_{22}$ and $\omega_{NL'} = \omega_{14}$ leads to the conclusion that the triplets present a general pattern with independence of the resonator distribution: one of the three frequencies will always be located very close to the original tuning frequency $\omega_{22}$ to assess how much near this frequency actually is one needs to restrict to particular cases and numerical evaluations–. The remaining pair forms a non-symmetric doublet around it,
in good agreement with the idea that the presence of $\omega_{14}$ causes a perturbation of the ideal IRF situation.

Then, the URF effect results in a dragging effect breaking the symmetry of the IRF doublets which approach the disturbing frequency $\omega_{14}$, and moreover induces the appearance of a third central component near the resonant $\omega_{22}$.

Near actually means really near, at least for two known proposals: PHCA $\{1,1\}$ and TIGA $\{3,1\}$. It can be seen from the numbers in Table II that in each a-group the central $U^c$ is itself of order $\eta^0$ or smaller, so that these central URF resonances actually differ from $\omega_{22}$ in terms of order $\eta$. Reproduction of the amplitudes evaluation process demonstrates that the contribution of such modes are not at leading order $\eta^{-2}$ but at higher terms. Therefore, they are referred to as weakly coupled.

The result is that both PHCA and TIGA URF triplets are composed of a a weakly coupled singlet plus a strongly coupled doublet (triplet named of the SCD+WCS type) of span comparable to that of the IRF pairs. The only exception is the sixth mode in TIGA which corresponds to a triplet composed of a strongly coupled singlet plus a weakly coupled doublet (SCS+WCD type). Also, the degeneracy pattern of the IRF triplets is maintained: for PHCA, three different triplets, two of them doubly degenerated, and for TIGA one triplet five-fold degenerate plus one single weakly coupled triplet.

As said, only strongly coupled frequencies (SC) contribute to the amplitudes:

$$q_a(s) = \eta^{-\frac{5}{2}} \sum_{b=1}^{J} \sum_{SC} \left\{ F_{LL'}(U^c_{SC}) \frac{1}{(s^2 + \omega^2_{c,SC})} v^{(c)}_a v^{(c)*}_b \right\} \times \sum_{m=2}^{m=-2} Y_{2m}(\mathbf{n}_b) \hat{g}^{(2m)}(s) + O(0) \quad a = 1, \ldots, J, \quad (11)$$

with frequency weights

$$F_{LL'}(U^c_{SC}) = \left[ a_{NL} A_{NL} + a_{NL'} A_{NL'} \frac{U_{SC}^c}{U^c_{SC} - K} \right] \times \frac{U_{SC}^c - K}{U^c_{SC} - U^c_{SC'}(U^c_{SC'} - U^c_{SC})}. \quad (12)$$

The high degree of symmetry showed by both PHCA and TIGA explains why these configurations fulfill the property $[P_L, P_{L'}] = 0$ for $L = 2$ and $L' = 4$, so that $P_2$ and $P_4$ simultaneously diagonalise. After algebraic calculus, the eigenvectors of the common basis are finally found to be arrangements of spherical harmonics of order two, precisely the eigenvectors of $P_2$ when diagonalised separately in the IRF situation:

$$v^{(m)}_a = \sqrt{\frac{4 \pi}{5}} \xi_{m,(2)} Y_{2m}(\mathbf{n}_a). \quad (13)$$

By inserting these functions in (12), we arrive to simplified expressions for the readouts of the PHCA layout:

$$\hat{q}_{a,PHCA}(s) = \eta^{-\frac{5}{2}} \sum_{SC} \left\{ F_{24}(U^c_0) \frac{Y_{20}(\mathbf{n}_a) \hat{g}^{(20)}(s)}{(s^2 + \omega^2_{0,SC})} ight.\left. + F_{24}(U^c_{14}) \frac{Y_{21}(\mathbf{n}_a) \hat{g}^{(21)}(s) + Y_{2-1}(\mathbf{n}_a) \hat{g}^{(2-1)}(s)}{(s^2 + \omega^2_{1,SC})} \right\} + F_{24}(U^c_{25}) \frac{Y_{22}(\mathbf{n}_a) \hat{g}^{(22)}(s) + Y_{2-2}(\mathbf{n}_a) \hat{g}^{(2-2)}(s)}{(s^2 + \omega^2_{2,SC})} \right\}; \quad (14)$$

and the readouts of the TIGA layout:

$$\hat{q}_{a,TIGA}(s) = \eta^{-\frac{5}{2}} \sum_{SC} F_{24}(U^c_{-5}) \frac{m=2}{m=-2} \sum_{m=2}^{m=-2} Y_{2m}(\mathbf{n}_a) \hat{g}^{(2m)}(s). \quad (15)$$

The weight functions $F_{24}(U^c_{-5})$ in (14) and (15) can be settled from (12). It is here important to recall that the overlap coefficient $\mathbf{a}_{14}$ is zero, which is in accordance with the idea that only monopole or quadrupole sphere’s modes can be excited by an impinging GW. One has to take also into account that just up and down frequencies in the triplets correspond to SC values, and that the the

| Layout | Strongly Coupled | Weakly Coupled |
|--------|------------------|----------------|
| PHCA   | $\omega_{0,u} = \omega_{22}(1 + 0.06\eta^{\frac{5}{2}})$ | $\omega_{0,c} = \omega_{22}(1 - 3.3 \cdot 10^{-3}\eta^{\frac{5}{2}})$ |
|        | $\omega_{0,d} = \omega_{22}(1 - 1.16\eta^{\frac{5}{2}})$ | $\omega_{1,c} = \omega_{22}(1 - 1.5 \cdot 10^{-4}\eta^{\frac{5}{2}})$ |
|        | $\omega_{1,u} = \omega_{22}(1 + 1.41\eta^{\frac{5}{2}})$ | $\omega_{1,d} = \omega_{22}(1 - 2.51\eta^{\frac{5}{2}})$ |
|        | $\omega_{2,u} = \omega_{22}(1 + 1.07\eta^{\frac{5}{2}})$ | $\omega_{2,c} = \omega_{22}(1 - 3.5 \cdot 10^{-4}\eta^{\frac{5}{2}})$ |
|        | $\omega_{2,d} = \omega_{22}(1 - 2.18\eta^{\frac{5}{2}})$ | $\omega_{6,d} = \omega_{22}(1 - 1.05\eta^{\frac{5}{2}})$ |
|        | $\omega_{6,u} = \omega_{22}(1 + 1.42\eta^{\frac{5}{2}})$ | $\omega_{6,c} = \omega_{22}(1 - 2.5 \cdot 10^{-4}\eta^{\frac{5}{2}})$ |
| TIGA   | $\omega_{1-5,u} = \omega_{22}(1 + 2.2\eta^{\frac{5}{2}})$ | $\omega_{1-5,c} = \omega_{22}(1 - 2.5 \cdot 10^{-4}\eta^{\frac{5}{2}})$ |
|        | $\omega_{1-5,d} = \omega_{22}(1 - 2.32\eta^{\frac{5}{2}})$ | $\omega_{6,d} = \omega_{22}(1 - 1.05\eta^{\frac{5}{2}})$ |
|        | $\omega_{6,S} = \omega_{22}(1 - 1.105\eta^{\frac{5}{2}})$ | $\omega_{6,d} = \omega_{22}(1 - 2.18\eta^{\frac{5}{2}})$ |

| TABLE II. SCD+WCS URF triplets around $\omega_{22}$. Calculations have been performed for the proposals PHCA and TIGA, and for $\eta$, a theoretical value for a full scale future spherical detector. Subindex $u$ (up) labels the values which are above the resonance frequency $\omega_{22}$, whereas $d$ (down) labels those underneath, and $c$ (central) those practically at $\omega_{22}$. |
remaining $U^c_c$ give place to weakly coupled resonances which do not contribute at leading order. Hence,

$$F_{24}(U_c^{SC}) = F_{24}(U_c^{u,d}) = a_{22} A_{22} \left( \frac{1}{U_{c,d,u} - U_{c,u,d}} \right) \frac{U_{d,u}}{U_{c,u,d}}.$$

(16)

In Figure 1 we display numerical simulations for the IRF and URF qualitative responses caused by the action of a GW burst travelling down the symmetry axis of the PHCA pentagonal layout. In both cases the practical outputs become beats, although differing in modulation and modulated frequencies. The IRF output exactly cancels at the nodes because the weights affecting the contributions to the amplitudes of $\omega_+$ and $\omega_-$ equal each other whereas the URF readout shows a certain thickness at those same points caused by the difference between the weights associated to $\omega_u$ and $\omega_d$. The occurrence of different weights is in direct relation to the dragging induced by the URF effect which breaks the symmetry of the IRF-doublets. Nevertheless, the similarity of the patterns again corroborates the weakness of the new third central frequency in the URF triplets.

Hence, the analysis of the URF effect is essential for a complete study of any spherical multifrequency GW detector, but we are specially interested in it with respect to our PHCA proposal.

It is actually idealized as a multifrequency antenna with two sets of supplementary pentagonal layouts, one tuned to the first quadrupole harmonic $\omega_{12}$ and another coupled to the second quadrupole harmonic $\omega_{22}$, which in fact has $\omega_{14}$ very close to it. The pentagonal hexacontahedron polyhedric shape with more than enough number of faces for 10 resonators matching in non parallel pentagonal positions (and even an eleventh resonator position for monopole sensing) guarantees technical feasibility, also supported by the absence of cross interactions between the outputs of each set at order $\eta^{-\frac{1}{2}}$. Our model ensures a correct description of the multifrequency PHCA coupled spectrum. We find two main groups of frequencies, the first composed of pairs symmetrically distributed around $\omega_{12}$ and the second arising as a non-symmetrical splitting of $\omega_{22}$ in triplets dragged towards $\omega_{14}$.

The PHCA monitored at their first two quadrupole modes can for instance be advantageously used to detect chirp signals from coalescing binary systems, and even to determine some of their characteristic parameters by means of a robust double passage method [12]. This potentiality, unthinkable for currently operating bars, shows that resonant spherical antennae are abreast of projected broadband large laser interferometers with respect to their predicted ability in monitoring gravitational waves from these sources [13].

In any case, the URF effect plays per se an essential part in a rigorous theoretical analysis of any single multifrequency GW detector. Its features are concisely reported by our developments without severely complicating the evaluations. We conclude by emphasizing that the philosophy underlying these algorithms is easily extensible to the study of other practical real situations
departing from ideality.

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* E-mail address: marian@hermes.ffn.ub.es
† E-mail address: lobo@hermes.ffn.ub.es

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