Exchangeable pairs on Wiener chaos

Ivan Nourdin and Guangqu Zheng
Université du Luxembourg

Abstract: In [14], Nourdin and Peccati combined the Malliavin calculus and Stein’s method of normal approximation to associate a rate of convergence to the celebrated fourth moment theorem [19] of Nualart and Peccati. Their analysis, known as the Malliavin-Stein method nowadays, has found many applications towards stochastic geometry, statistical physics and zeros of random polynomials, to name a few. In this article, we further explore the relation between these two fields of mathematics. In particular, we construct exchangeable pairs of Brownian motions and we discover a natural link between Malliavin operators and these exchangeable pairs. By combining our findings with E. Meckes’ infinitesimal version of exchangeable pairs, we can give another proof of the quantitative fourth moment theorem. Finally, we extend our result to the multidimensional case.

Key words: Stein’s method; Exchangeable pairs; Brownian motion; Malliavin calculus.

Dedicated to the memory of Charles Stein, in remembrance of his beautiful mind and of his inspiring, creative, very original and deep mathematical ideas, which will, for sure, survive him for a long time.

1. Introduction

At the beginning of the 1970s, Charles Stein, one of the most famous statisticians of the time, introduced in [24] a new revolutionary method for establishing probabilistic approximations (now known as Stein’s method), which is based on the breakthrough application of characterizing differential operators. The impact of Stein’s method and its ramifications during the last 40 years is immense (see for instance the monograph [3]), and touches fields as diverse as combinatorics, statistics, concentration and functional inequalities, as well as mathematical physics and random matrix theory.

Introduced by Paul Malliavin [7], Malliavin calculus can be roughly described as an infinite-dimensional differential calculus whose operators act on sets of random objects associated with Gaussian or more general noises. In 2009, Nourdin and Peccati [14] combined the Malliavin calculus and Stein’s method for the first time, thus virtually creating a new domain of research, which is now commonly known as the Malliavin-Stein method. The success of their method relies crucially on the existence of integration-by-parts formulae on both sides: on one side, the Stein’s lemma is built on the Gaussian integration-by-parts formula and it is one of the cornerstones of the Stein’s method; on the other side, the integration-by-parts formula on Gaussian space is one of the main tools in Malliavin calculus. Interested readers can refer to the constantly updated website [13] and the monograph [15] for a detailed overview of this active field of research.

A prominent example of applying Malliavin-Stein method is the obtention (see also (1.1) below) of a Berry-Esseen’s type rate of convergence associated to the celebrated fourth moment theorem [19] of Nualart and Peccati, according to which a standardized sequence of multiple Wiener-Itô integrals converges in law to a standard Gaussian random variable if and only if its fourth moment converges to 3.

Theorem 1.1. (i) (Nualart, Peccati [19]) Let \( (F_n) \) be a sequence of multiple Wiener-Itô integrals of order \( p \), for some fixed \( p \geq 1 \). Assume that \( E[F_n^2] \to \sigma^2 > 0 \) as \( n \to \infty \). Then, as \( n \to \infty \), we have the following equivalence:

\[
F_n \xrightarrow{\text{law}} N(0,\sigma^2) \iff E[F_n^4] \to 3\sigma^4.
\]
(ii) (Nourdin, Peccati [14, 15]) Let $F$ be any multiple Wiener-Itô integral of order $p \geq 1$, such that
\[ E[F^2] = \sigma^2 > 0. \]
Then, with $N \sim N(0, \sigma^2)$ and $d_{TV}$ standing for the total variation distance,
\[ d_{TV}(F, N) \leq \frac{2}{\sigma^2} \sqrt{\frac{p-1}{3p}} \sqrt{E[F^4]} - 3\sigma^2. \]

Of course, (ii) was obtained several years after (i), and (ii) implies ‘$\ll$’ in (i). Nualart and Peccati’s fourth moment theorem has been the starting point of a number of applications and generalizations by dozens of authors. These collective efforts have allowed one to break several long-standing deadlocks in several domains, ranging from stochastic geometry (see e.g. [3, 21, 23]), to statistical physics (see e.g. [3, 9, 10]), and zeros of random polynomials (see e.g. [11, 2, 4]), to name a few. At the time of writing, more than two hundred papers have been written, which use in one way or the other the Malliavin-Stein method (see again the webpage [13]).

Malliavin-Stein method has become a popular tool, especially within the Malliavin calculus community. Nevertheless, and despite its success, it is less used by researchers who are not specialists of the Malliavin calculus. A possible explanation is that it requires a certain investment before one is in a position to be able to use it, and doing this investment might refrain people who are not originally trained in the Gaussian analysis. This paper takes its root from this observation.

During our attempt to make the proof of Theorem 1.1(ii) more accessible to readers having no background on Malliavin calculus, we discovered the following interesting fact for exchangeable pairs of multiple Wiener-Itô integrals. When $p \geq 1$ is an integer and $f$ belongs to $L^2([0,1]^p)$, we write $I^B_p(f)$ to indicate the multiple Wiener-Itô integral of $f$ with respect to Brownian motion $B$, see Section 2 for the precise meaning.

**Proposition 1.2.** Let $(B, B^t)_{t \geq 0}$ be a family of exchangeable pairs of Brownian motions (that is, $B$ is a Brownian motion on $[0, 1]$ and, for each $t$, one has $(B, B^t) \overset{law}{=} (B^t, B)$). Assume moreover that
\[ \text{for any integer } p \geq 1 \text{ and any } f \in L^2([0,1]^p), \]
\[ \lim_{t \downarrow 0} \frac{1}{t} E \left[ I^B_p(f) - I^B_p(f) \sigma(B) \right] = -p I^B_p(f) \quad \text{in } L^2(\Omega). \]

Then, for any integer $p \geq 1$ and any $f \in L^2([0,1]^p)$,
\[ \text{(b)} \lim_{t \downarrow 0} \frac{1}{t} E \left[ (I^B_p(f) - I^B_p(f))^2 \sigma(B) \right] = 2p^2 \int_0^1 I^B_{p-1}(f(x, \cdot))^2 \, dx \quad \text{in } L^2(\Omega); \]
\[ \text{(c)} \lim_{t \downarrow 0} \frac{1}{t} E \left[ (I^B_p(f) - I^B_p(f))^4 \right] = 0. \]

Why is this proposition interesting? Because, as it turns out, it combines perfectly well with the following result, which represents the main ingredient from Stein’s method we will rely on and which corresponds to a slight modification of a theorem originally due to Elizabeth Meckes (see [11] Theorem 2.1).

**Theorem 1.3 (Meckes [11]).** Let $F$ and a family of random variables $(F_t)_{t \geq 0}$ be defined on a common probability space $(\Omega, \mathcal{F}, P)$ such that $F_t \overset{law}{=} F$ for every $t \geq 0$. Assume that $F \in L^3(\Omega, \mathcal{G}, P)$ for some $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and that in $L^1(\Omega)$,
\[ \text{(a)} \lim_{t \downarrow 0} \frac{1}{t} E[F_t - F^3] = -\lambda F \quad \text{for some } \lambda > 0, \]
\[ \text{(b)} \lim_{t \downarrow 0} \frac{1}{t} E[(F_t - F)^2] = (2\lambda + S) \text{Var}(F) \quad \text{for some random variable } S, \]
\[ \text{(c)} \lim_{t \downarrow 0} \frac{1}{t} (F_t - F)^3 = 0. \]

Then, with $N \sim N(0, \text{Var}(F))$,
\[ d_{TV}(F, N) \leq \frac{E[S]}{\lambda}. \]
To see how to combine Proposition 1.2 with Theorem 1.3 (see also point (ii) in Remark 1.4), consider indeed a multiple Wiener-Itô integral of the form $F = \int_{\mathbb{R}}^B(f)$, with $\sigma^2 = E[F^2] > 0$. Assume moreover that we have at our disposal a family $\{(B, B')\}_{t \geq 0}$ of exchangeable pairs of Brownian motions, satisfying the assumption (a) in Proposition 1.2. Then, putting Proposition 1.2 and Theorem 1.3 together immediately follows.

It is then easy and straightforward to check that, for any $t \geq 0$, this new Brownian motion $B'$, together with $\check{B}$, forms an exchangeable pair (see Lemma 3.1). Moreover, we will compute below (see Section 3) that $E[\int_{\mathbb{R}}^B(f) | \sigma \{ B \}] = e^{-pt} \int_{\mathbb{R}}^B(f)$ for any $p \geq 1$ and any $f \in L^2([0, 1]^p)$, from which (a) in Proposition 1.2 immediately follows.

For the second construction, we consider two independent Gaussian white noise $W$ and $W'$ on $[0, 1]$ with Lebesgue intensity measure. For each $n \in \mathbb{N}$, we introduce a uniform partition $\{\Delta_1, \ldots, \Delta_n\}$ and a uniformly distributed index $I_n \sim \mathcal{U}(1, \ldots, n)$, independent of $W$ and $W'$. For every Borel set $A \subset [0, 1]$, we define $W^n(A) = W(A \cap \Delta_{I_n}) + W'(A \setminus \Delta_{I_n})$. This will give us a new Gaussian white noise $W^n$, which will form an exchangeable pair with $W$. This construction is a particular Gibbs sampling procedure. The analogue of (a) in Proposition 1.2 is satisfied, namely, if $f \in L^2([0, 1]^p)$, $F = \int_{\mathbb{R}}^B(f)$ is the $p$th multiple integral with respect to $W$ and $F^{(n)} = \int_{\mathbb{R}}^B(f)$, we have

$$nE[F^{(n)} - F | \sigma \{ W \}] \to -pF \quad \text{in} \ L^2(\Omega) \quad \text{as} \ n \to \infty.$$
where $\| \cdot \|$ stands for the usual Euclidean $L^2$-norm of $\mathbb{R}^d$. Combining the main findings of [16] and [17] yields the following quantitative version associated to Theorem [1.4], which we are able to recover by means of our elementary exchangeable approach.

**Theorem 1.5** (Nourdin, Peccati, Réveillac, Rosiński [16] [17]). Let $F = (F^1, \ldots, F^d)$ be a vector composed of multiple Wiener-Itô integrals $F^k$, $1 \leq k \leq d$. Assume that the covariance matrix $\Sigma$ of $F$ is invertible. Then, with $N \sim N(0, \Sigma)$,

\begin{equation}
    d_W(F, N) \leq \sqrt{d} \| \Sigma \|_{op}^{-1} \sqrt{E[\|F\|^4] - E[\|N\|^4]},
\end{equation}

where $d_W$ denotes the Wasserstein distance and $\| \cdot \|_{op}$ the operator norm of a matrix.

The currently available proof of (1.4) relies on two main ingredients: (i) simple manipulations involving the product formula (2.7) and implying that

\[
    \sum_{i,j=1}^d \text{Var} \left( \int_0^1 I_{p_i-1}(f_i(x, \cdot)) I_{p_j-1}(f_j(x, \cdot)) dx \right) \leq E[\|F\|^4] - E[\|N\|^4],
\]

(see [17] Theorem 4.3 for the details) and (ii) the following inequality shown in [16] Corollary 3.6 by means of the Malliavin operators $D$, $\delta$ and $L$:

\begin{equation}
    d_W(F, N) \leq \sqrt{d} \| \Sigma \|_{op}^{-1} \sqrt{\sum_{i,j=1}^d \text{Var} \left( \int_0^1 I_{p_i-1}(f_i(x, \cdot)) I_{p_j-1}(f_j(x, \cdot)) dx \right)},
\end{equation}

Here, in the spirit of what we have done in dimension one, we also apply our elementary exchangeable pairs approach to prove (1.3), with slightly different constants.

The rest of the paper is organized as follows. Section 2 contains preliminary knowledge on multiple Wiener-Itô integrals. In Section 3 (resp. 4), we present our first (resp. second) construction of exchangeable pairs of Brownian motions, and we give the main associated properties. Section 5 is devoted to the proof of Proposition [1.2] whereas in Section 6 we offer a simple proof of Meckes’ Theorem [1.3]. Our new, elementary proof of Theorem [1.1(ii)] is given in Section 7. In Section 8, we further investigate the connections between our exchangeable pairs and the Malliavin operators. Finally, we discuss the extension of our approach to the multidimensional case in Section 9.

**Acknowledgement.** We would like to warmly thank Christian Döbler and Giovanni Peccati, for very stimulating discussions on exchangeable pairs since the early stage of this work.

## 2. Multiple Wiener-Itô integrals: definition and elementary properties

In this subsection, we recall the definition of multiple Wiener-Itô integrals, and then we give a few soft properties that will be needed for our new proof of Theorem [1.1(ii)]. We refer to the classical monograph [15] for the details and missing proofs.

Let $f : [0, 1]^p \to \mathbb{R}$ be a square-integrable function, with $p \geq 1$ a given integer. The $p$th multiple Wiener-Itô integral of $f$ with respect to the Brownian motion $B = (B(x))_{x \in [0, 1]}$ is formally written as

\begin{equation}
    \int_{[0, 1]^p} f(x_1, \ldots, x_p) dB(x_1) \ldots dB(x_p).
\end{equation}

To give a precise meaning to (2.6), Itô’s crucial idea from the fifties was to first define elementary functions that vanish on diagonals, and then to approximate any $f$ in $L^2([0, 1]^p)$ by such elementary functions.

Consider the diagonal set of $[0, 1]^p$, that is, $D = \{ (t_1, \ldots, t_p) \in [0, 1]^p : \exists i \neq j, t_i = t_j \}$. Let $E_p$ be the vector space formed by the set of elementary functions on $[0, 1]^p$ that vanish over $D$, that is, the
set of those functions $f$ of the form

$$f(x_1, \ldots, x_p) = \sum_{i_1, \ldots, i_p = 1}^{k} \beta_{i_1 \ldots i_p} \mathbf{1}_{[\tau_1, \ldots, \tau_p]}(x_1, \ldots, x_p),$$

where $k \geq 1$ and $0 = \tau_0 < \tau_1 < \ldots < \tau_k$, and the coefficients $\beta_{i_1 \ldots i_p}$ are zero if any two of the indices $i_1, \ldots, i_p$ are equal. For $f \in \mathcal{E}_p$, we define (without ambiguity with respect to the choice of the representation of $f$)

$$I^B_p(f) = \sum_{i_1, \ldots, i_p = 1}^{k} \beta_{i_1 \ldots i_p} (B(\tau_i) - B(\tau_{i-1})) \ldots (B(\tau_p) - B(\tau_{p-1})).$$

We also define the symmetrization $\tilde{f}$ of $f$ by

$$\tilde{f}(x_1, \ldots, x_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} f(x_{\sigma(1)}, \ldots, x_{\sigma(p)}),$$

where $\mathfrak{S}_p$ stands for the set of all permutations of $\{1, \ldots, p\}$. The following elementary properties are immediate and easy to prove.

1. If $f \in \mathcal{E}_p$, then $I^B_p(f) = I^B_p(\tilde{f})$.
2. If $f \in \mathcal{E}_p$ and $g \in \mathcal{E}_q$, then $E[I^B_p(f)I^B_q(g)] = 0$ and $E[I^B_p(f)I^B_q(g)] = \begin{cases} 0 & \text{if } p \neq q \\ 1 & \text{if } p = q \end{cases}$.
3. The space $\mathcal{E}_p$ is dense in $L^2([0,1]^p)$. In other words, to each $f \in L^2([0,1]^p)$ one can associate a sequence $(f_n)_{n \geq 1} \subset \mathcal{E}_p$ such that $\|f - f_n\|_{L^2([0,1]^p)} \to 0$ as $n \to \infty$.
4. Since $E[(I^B_p(f_n) - I^B_p(f_m))^2] = p!\|f_n - f_m\|_{L^2([0,1]^p)}^2 \leq p!\|f_n - f_m\|_{L^2([0,1]^p)^2} \to 0$ as $n, m \to \infty$ for $f$ and $(f_n)_{n \geq 1}$ as in the previous point 3, we deduce that the sequence $(I^B_p(f_n))_{n \geq 1}$ is Cauchy in $L^2(\Omega)$ and, as such, it admits a limit denoted by $I^B_p(f)$. It is easy to check that $I^B_p(f)$ only depends on $f$, not on the particular choice of the approximating sequence $(f_n)_{n \geq 1}$, and that points 1 to 3 continue to hold for general $f \in L^2([0,1]^p)$ and $g \in L^2([0,1]^q)$.

We will also crucially rely on the following product formula, whose proof is elementary and can be made by induction. See, e.g., [13, Proposition 1.1.3].

5. For any $p, q \geq 1$, and if $f \in L^2([0,1]^p)$ and $g \in L^2([0,1]^q)$ are symmetric, then

$$I^B_p(f)I^B_q(g) = \sum_{r=0}^{p\land q} \mathcal{C}^p_r \mathcal{C}^q_r \mathcal{C}^{p+q-2r} \|f \otimes r g\|_{L^2([0,1])}^2,$$

where $f \otimes r g$ stands for the $r$th-contraction of $f$ and $g$, defined as an element of $L^2([0,1]^{p+q-2r})$ by

$$(f \otimes r g)(x_1, \ldots, x_{p+q-2r}) = \int_{[0,1]^r} f(x_1, \ldots, x_{p-r}, u_1, \ldots, u_r)g(x_{p-r+1}, \ldots, x_{p+q-2r}, u_1, \ldots, u_r)du_1 \ldots du_r.$$

Product formula (2.7) has a nice consequence, the inequality below. It is a very particular case of a more general phenomenon satisfied by multiple Wiener-Itô integrals, the hypercontractivity property.

6. For any $p \geq 1$, there exists a constant $c_{4,p} > 0$ such that, for any (symmetric) $f \in L^2([0,1]^p)$,

$$E[I^B_p(f)^4] \leq c_{4,p} E[I^B_p(f)^2]^2.$$
Indeed, thanks to (2.9) one can write $I_p^B(f)^2 = \sum_{r=0}^{p} r! \left( \frac{p}{r} \right)^2 I_{2p-2r}^B(f \otimes_r f)$ so that

$$E[I_p^B(f)^4] = \sum_{r=0}^{p} r! \left( \frac{p}{r} \right)^4 (2p-2r)! \|f \otimes_r f\|_{L^2(\{0,1\}^{2p-2r})}^2.$$ 

The conclusion (2.8) follows by observing that

$$p!^2 \|f \otimes_r f\|_{L^2(\{0,1\}^{2p-2r})}^2 \leq p!^2 \|f \otimes_r f\|_{L^2(\{0,1\}^{2n-2r})}^2 \leq p!^2 \|f\|_{L^2(\{0,1\}^n)}^4 = E[I_p^B(f)^2]^2.$$

Furthermore, for each $n \geq 2$, using (2.7) and induction, one can show that, with $c_{2^n,p}$ a constant depending only on $p$ but not on $f$,

$$E[I_p^B(f)^{2^n}] \leq c_{2^n,p} E[I_p^B(f)^2]^{2n-1}.$$

So for any $r > 2$, there exists an absolute constant $c_{r,p}$ depending only on $p, r$ (but not on $f$) such that

$$E[I_p^B(f)^r] \leq c_{r,p} E[I_p^B(f)^2]^{r/2}.$$ 

3. Exchangeable pair of Brownian motions: a first construction

As anticipated in the introduction, for this construction we consider two independent Brownian motions on $[0,1]$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, namely $B$ and $\tilde{B}$, and we interpolate between them by considering, for any $t \geq 0$, $B^t = e^{-t}B + \sqrt{1-e^{-2t}}\tilde{B}$.

**Lemma 3.1.** For each $t \geq 0$, the pair $(B, B^t)$ is exchangeable, that is, $(B, B^t) \overset{\text{law}}{=} (B^t, B)$. In particular, $B^t$ is a Brownian motion.

**Proof.** Clearly, the bi-dimensional process $(B, B^t)$ is Gaussian and centered. Moreover, for any $x, y \in [0,1]$,

$$E[B^t(x)B^t(y)] = e^{-2t}E[B(x)B(y)] + (1-e^{-2t})E[\tilde{B}(x)\tilde{B}(y)] = E[B(x)B(y)]$$

$$E[B(x)B^t(y)] = e^{-t}E[B(x)B(y)] = E[B^t(x)B(y)].$$

The desired conclusion follows. \qed

We can now state that, as written in the introduction, our exchangeable pair indeed satisfies the crucial property (a) of Proposition 1.2.

**Theorem 3.2.** Let $p \geq 1$ be an integer, and consider a kernel $f \in L^2([0,1]^p)$. Set $F = I_p^B(f)$ and $F_t = I_p^{B^t}(f)$, $t \geq 0$. Then,

$$E[F_t|\sigma(B)] = e^{-pt}F.$$ 

In particular, convergence (a) in Proposition 1.2 takes place:

$$\lim_{t \downarrow 0} \frac{1}{t} E \left[ I_p^{B^t}(f) - I_p^B(f) | \sigma(B) \right] = -p I_p^B(f) \text{ in } L^2(\Omega).$$

**Proof.** Consider first the case where $f \in E_p$, that is, $f$ has the form

$$f(x_1, \ldots, x_p) = \sum_{i_1, \ldots, i_p=1}^{k} \beta_{i_1} \ldots \beta_{i_p} 1_{[\tau_{i_1}-1, \tau_{i_1}) \times \ldots \times [\tau_{i_p}-1, \tau_{i_p})}(x_1, \ldots, x_p),$$
with \( k \geq 1 \) and \( 0 = \tau_0 < \tau_1 < \ldots < \tau_k \), and the coefficients \( \beta_{i_1 \ldots i_p} \) are equal if any two of the indices \( i_1, \ldots, i_p \) are equal. We then have

\[
F_t = \sum_{i_1, \ldots, i_p=1}^k \beta_{i_1 \ldots i_p} (B^t(\tau_{i_1}) - B^t(\tau_{i_1-1})) \ldots (B^t(\tau_{i_p}) - B^t(\tau_{i_p-1})) \\
= \sum_{i_1, \ldots, i_p=1}^k \beta_{i_1 \ldots i_p} [e^{-t}(B(\tau_{i_1}) - B(\tau_{i_1-1})) + \sqrt{1 - e^{-2t}} (\hat{B}(\tau_{i_1}) - \hat{B}(\tau_{i_1-1}))] \\
\times \ldots \times [e^{-t}(B(\tau_{i_p}) - B(\tau_{i_p-1})) + \sqrt{1 - e^{-2t}} (\hat{B}(\tau_{i_p}) - \hat{B}(\tau_{i_p-1}))].
\]

Expanding and integrating with respect to \( \hat{B} \) yields (3.10) for elementary \( f \). Thanks to point 4 in Section 2 we can extend it to any \( f \in L^2([0,1]^p) \). We then deduce that

\[
\frac{1}{t} E[F_t - F|\sigma(B)] = \frac{e^{-pt} - 1}{t} F,
\]

from which (3.11) now follows immediately. \( \square \)

4. EXCHANGEABLE PAIR OF BROWNIAN MOTIONS: A SECOND CONSTRUCTION

In this section, we present yet another construction of exchangeable pairs via Gaussian white noise. We believe it is of independent interest, as such a construction can be similarly carried out for other additive noises. This part may be skipped in a first reading, as it is not used in other sections. And we assume that the readers are familiar with the multiple Wiener-Itô integrals with respect to the Gaussian white noise, and refer to [18, Page 8-13] for all missing details.

Let \( W \) be a Gaussian white noise on \([0,1]\) with Lebesgue intensity measure \( \nu \), that is, \( W \) is a centered Gaussian process indexed by Borel subsets of \([0,1]\) such that for any Borel sets \( A, B \subset [0,1] \), \( W(A) \sim N(0,\nu(A)) \) and \( E[W(A)W(B)] = \nu(A \cap B) \). We denote by \( \mathcal{G} \) the \( \sigma \)-algebra generated by \( \{W(A): A \text{ a Borel subset of } [0,1]\} \). Now let \( W' \) be an independent copy of \( W \) (denote by \( \mathcal{G}' = \sigma(W') \) the \( \sigma \)-algebra generated by \( W' \)) and \( I_n \) be a uniform random variable over \( \{1, \ldots, n\} \) for each \( n \in \mathbb{N} \) such that \( I_n, W, W' \) are independent. For each fixed \( n \in \mathbb{N} \), we consider the partition \([0,1] = \bigcup_{j=1}^n \Delta_j \) with \( \Delta_1 = [0, \frac{1}{n}], \Delta_2 = (\frac{1}{n}, \frac{2}{n}], \ldots, \Delta_n = (1 - \frac{1}{n}, 1] \).

Definition 4.0. Set \( W^n(A) := W'(A \cap \Delta_{I_n}) + W(A \setminus \Delta_{I_n}) \) for any Borel set \( A \subset [0,1] \).

Remark 4.1. One can first treat \( W \) as the superposition of \( \{W|\Delta_j, j = 1, \ldots, n\} \), where \( W|\Delta_j \) denotes the Gaussian white noise on \( \Delta_j \). Then according to \( I_n = j \), we (only) replace \( W|\Delta_j \) by an independent copy \( W'|\Delta_j \), so that we get \( W^n \). This is nothing else but a particular Gibbs sampling procedure (see [5, A.2]), hence heuristically speaking, the new process \( W^n \) shall form an exchangeable pair with \( W \).

Lemma 4.2. \( W \) and \( W^n \) form an exchangeable pair with \( W \), that is, \( (W, W^n) \overset{\text{law}}{=} (W^n, W) \). In particular, \( W^n \) is a Gaussian white noise on \([0,1]\) with Lebesgue intensity measure.

Proof. Let us first consider \( m \) mutually disjoint Borel sets \( A_1, \ldots, A_m \subset [0,1] \). Given \( D_1, D_2 \) Borel subsets of \( \mathbb{R}^m \), we have

\[
P\left( (W(A_1), \ldots, W(A_m)) \in D_1, (W^n(A_1), \ldots, W^n(A_m)) \in D_2 \right) \\
= \sum_{v=1}^n P\left( (W(A_1), \ldots, W(A_m)) \in D_1, (W^n(A_1), \ldots, W^n(A_m)) \in D_2, I_n = v \right) \\
= \frac{1}{n} \sum_{v=1}^n P\left( g(X_v, Y_v) \in D_1, g(X'_v, Y_v) \in D_2 \right),
\]

where for each \( v \in \{1, \ldots, n\} \),
• \(X_v := (W(A_1 \cap \Delta_v), \ldots, W(A_m \cap \Delta_v)), X'_v := (W'(A_1 \cap \Delta_v), \ldots, W'(A_m \cap \Delta_v))\).

• \(Y_v := (W(A_1 \setminus \Delta_v), \ldots, W(A_m \setminus \Delta_v))\), and \(g\) is a function from \(\mathbb{R}^{2m}\) to \(\mathbb{R}^m\) given by 
\[
(x_1, \ldots, x_m, y_1, \ldots, y_m) \mapsto g(x_1, \ldots, x_m, y_1, \ldots, y_m) = (x_1 + y_1, \ldots, x_m + y_m)
\]

It is clear that for each \(v \in \{1, \ldots, n\}\), \(X_v, X'_v\) and \(Y_v\) are independent, therefore \(g(X_v, Y_v)\) and \(g(X'_v, Y_v)\) form an exchangeable pair. It follows from the above equalities that
\[
P\left(\{W(A_1), \ldots, W(A_m)\} \in D_1, \{W^n(A_1), \ldots, W^n(A_m)\} \in D_2\right) = \frac{1}{n} \sum_{v=1}^n P\left(g(X'_v, Y_v) \in D_1, g(X_v, Y_v) \in D_2\right) = P\left(\{W^n(A_1), \ldots, W^n(A_m)\} \in D_1, \{W(A_1), \ldots, W(A_m)\} \in D_2\right).
\]

This proves the exchangeability of \((W(A_1), \ldots, W(A_m))\) and \((W^n(A_1), \ldots, W^n(A_m))\).

Now let \(B_1, \ldots, B_m\) be Borel subsets of \([0,1]\), then one can find mutually disjoint Borel sets \(A_1, \ldots, A_p\) (for some \(p \in \mathbb{N}\)) such that each \(B_j\) is a union of some of \(A_i\)’s. Therefore we can find some measurable \(\phi : \mathbb{R}^p \to \mathbb{R}^m\) such that 
\[
(W(B_1), \ldots, W(B_m)) = \phi(W(A_1), \ldots, W(A_p)).
\]

Accordingly, 
\[
(W^n(B_1), \ldots, W^n(B_m)) = \phi(W^n(A_1), \ldots, W^n(A_p)),
\]

hence \((W(B_1), \ldots, W(B_m))\) and \((W^n(B_1), \ldots, W^n(B_m))\) are exchangeable. Now our proof is complete. 

\(\square\)

**Remark 4.3.** For each \(t \in [0,1]\), we set \(B(t) := W([0,t])\) and \(B^n(t) := W^n([0,t])\). Modulo continuous modifications, one can see from Lemma 4.2 that \(B, B^n\) are two Brownian motions that form an exchangeable pair. An important difference between this construction and the previous one is that \((B, B')\) is bi-dimensional Gaussian process whereas \(B, B^n\) are not jointly Gaussian.

Before we state the analogous result to Theorem 3.2, we briefly recall the construction of multiple Wiener-Itô integrals in white noise setting.

(1) For each \(p \in \mathbb{N}\), we denote by \(\mathcal{E}_p\) the set of simple functions of the form
\[
f(t_1, \ldots, t_p) = \sum_{i_1, \ldots, i_p = 1}^m \beta_{i_1 \ldots i_p} \mathbf{1}_{A_{i_1} \times \ldots \times A_{i_p}}(t_1, \ldots, t_p),
\]
where \(m \in \mathbb{N}\), \(A_1, \ldots, A_m\) are pair-wise disjoint Borel subsets of \([0,1]\), and the coefficients \(\beta_{i_1 \ldots i_p}\) are zero if any two of the indices \(i_1, \ldots, i_p\) are equal. It is known that \(\mathcal{E}_p\) is dense in \(L^2([0,1]^p]\).

(2) For \(f\) given as in (4.12), the \(p\)th multiple integral with respect to \(W\) is defined as
\[
I^W_p(f) := \sum_{i_1, \ldots, i_p = 1}^m \beta_{i_1 \ldots i_p} W(A_{i_1}) \ldots W(A_{i_p}),
\]
and one can extend \(I^W_p\) to \(L^2([0,1]^p]\) via usual approximation argument. Note \(I^W_p(f)\) is nothing else but \(I^B_p(f)\) with the Brownian motion \(B\) constructed in Remark 4.3.

**Theorem 4.4.** If \(F = I^W_p(f)\) for some symmetric \(f \in L^2([0,1]^p]\) and we set \(F^{(n)} := I^{W^n}_p(f)\), then in \(L^2(\Omega, \mathcal{G}, P)\) and as \(n \to +\infty\), \(n E[F^{(n)} - F]\mathcal{G} \rightarrow -pF\).

**Proof.** First we consider the case where \(f \in \mathcal{E}_p\), we assume moreover that \(F = \prod_{j=1}^p W(A_j)\) with \(A_1, \ldots, A_p\) mutually disjoint Borel subsets of \([0,1]\), and accordingly we define \(F^{(n)} = \prod_{j=1}^p W^n(A_j)\).
Then, (we write \([p] = \{1, \ldots, p\}\), \(A^v = A \cap \Delta_v\) for any \(A \subset [0,1]\) and \(v \in \{1, \ldots, n\}\))

\[
\begin{align*}
n E[F^{(n)} | \mathcal{G}] &= n E \left\{ \sum_{v=1}^n 1_{(J_n = v)} \prod_{j=1}^p \left[ W'(A^v_j) + W(A_j \setminus \Delta_v) \right] | \mathcal{G} \right\} \\
&= \sum_{v=1}^n E \left\{ \prod_{j=1}^p \left[ W'(A^v_j) + W(A_j \setminus \Delta_v) \right] | \mathcal{G} \right\} = \sum_{v=1}^n \prod_{j=1}^p W(A_j \setminus \Delta_v) \\
&= \sum_{v=1}^n \left\{ \sum_{\ell=2}^p \left( \prod_{j=1}^p W(A_j) - \sum_{k=1}^p W(A^v_k) \left( \prod_{j \in [p] \setminus \{k\}} W(A_j) \right) \right) \right. \\
&\quad \left. + \sum_{\ell=2}^p (-1)^{\ell} \sum_{k_1, \ldots, k_{\ell} \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} W(A_j) \right) W(A_{k_1}^v) \cdots W(A_{k_{\ell}}^v) \right\} \\
&= n F - p F + R_n(F),
\end{align*}
\]

where \(R_n(F) = \sum_{\ell=2}^p (-1)^{\ell} \sum_{k_1, \ldots, k_{\ell} \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} W(A_j) \right) \sum_{v=1}^n W(A_{k_1}^v) \cdots W(A_{k_{\ell}}^v)\).

Then \(R_n(F)\) converges in \(L^2(\Omega, \mathcal{G}, P)\) to 0, due to the fact that \(\sum_{v=1}^n \prod_{k=1}^q W(A_k^v)\) converges in \(L^2(\Omega)\) to 0, as \(n \to +\infty\), if \(q \geq 2\) and all \(k_i\)'s are distinct numbers. This proves our theorem when \(f \in E_p\).

By the above computation, we can see that if \(F = I_p^W(f)\) with \(f\) given in (1.12), then

\[
\begin{align*}
R_n(F) &= \sum_{i_1, \ldots, i_p=1}^m \beta_{i_1 i_2 \ldots i_p} \sum_{v=1}^n \sum_{\ell=2}^p (-1)^{\ell} \sum_{k_1, \ldots, k_{\ell} \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} W(A_j) \right) W(A_{k_1}^v) \cdots W(A_{k_{\ell}}^v).
\end{align*}
\]

Therefore, using Wiener-Itô isometry, we can first write \(\|R_n(F)\|_{L^2(\Omega)}^2\) as

\[
p! \sum_{i_1, \ldots, i_p=1}^m (\beta_{i_1 i_2 \ldots i_p})^2 \sum_{v=1}^n \left\| \sum_{\ell=2}^p (-1)^{\ell} \sum_{k_1, \ldots, k_{\ell} \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} W(A_j) \right) W(A_{k_1}^v) \cdots W(A_{k_{\ell}}^v) \right\|_{L^2(\Omega)}^2,
\]

and then using the elementary inequality \((a_1 + \ldots + a_m)^\beta \leq m^{\beta - 1} \sum_{i=1}^m |a_i|^\beta\) for \(a_i \in \mathbb{R}\), \(\beta > 1\), \(m \in \mathbb{N}\), we have

\[
\begin{align*}
\left\| \sum_{\ell=2}^p (-1)^{\ell} \sum_{k_1, \ldots, k_{\ell} \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} W(A_j) \right) W(A_{k_1}^v) \cdots W(A_{k_{\ell}}^v) \right\|_{L^2(\Omega)}^2 \\
\leq \Theta_1 \sum_{\ell=2}^p \sum_{k_1, \ldots, k_{\ell} \in [p]} \left\| \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} W(A_j) \right) W(A_{k_1}^v) \cdots W(A_{k_{\ell}}^v) \right\|_{L^2(\Omega)}^2 \\
= \Theta_2 \sum_{\ell=2}^p \sum_{k_1, \ldots, k_{\ell} \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, \ldots, k_{\ell}\}} \nu(A_j) \right) \nu(A_{k_1}^v) \cdots \nu(A_{k_{\ell}}^v) \\
\leq \Theta_2 \sum_{k_1, k_2 \in [p]} \left( \prod_{j \in [p] \setminus \{k_1, k_2\}} \nu(A_j) \right) \nu(A_{k_1}^v) \nu(A_{k_2}^v),
\end{align*}
\]
where $\Theta_1, \Theta_2$ (and $\Theta_3$ in the following) are some absolute constants that do not depend on $n$ or $F$. Note now for $k_1 \neq k_2$, $\sum_{i=1}^n \nu(A_{t_1}^{k_1}) \cdot \nu(A_{t_2}^{k_2}) \leq \nu(A_{t_1}^{k_1}) \sum_{i=1}^n \nu(A_{t_2}^{k_2}) = \nu(A_{t_1}^{k_1}) \cdot \nu(A_{t_2}^{k_2})$, thus,

$$\|R_n(F)\|_{L^2(\Omega)} \leq p! \sum_{i_1, \ldots, i_p=1}^m (\beta_{i_1, i_2, \ldots, i_p})^2 \Theta_2 \sum_{k_1, k_2 \in [p]} \prod_{j \in [p] \setminus \{k_1, k_2\}} \nu(A_i) \nu(A_{i_1}) \nu(A_{i_2})$$

$$\leq p! \sum_{i_1, \ldots, i_p=1}^m (\beta_{i_1, i_2, \ldots, i_p})^2 \Theta_3 \prod_{j \in [p]} \nu(A_i) = \Theta_3 \cdot \|F\|_{L^2(\Omega)}^2.$$ 

Since $\{I_p^W(f) : f \in C_0^\infty\}$ is dense in the $p$th Wiener chaos $\mathcal{H}_p$, $R_n : \mathcal{H}_p \to L^2(\Omega)$ is a bounded linear operator with operator norm $\|R_n\|_{op} \leq \sqrt{\Theta_3}$ for each $n \in \mathbb{N}$. Note the linearity follows from its definition $R_n(F) := n E[F^{(n)} - F|\mathcal{G}] + pF, F \in \mathcal{H}_p$.

Now we define

$$\mathcal{C}_p := \left\{ F \in \mathcal{H}_p : R_\infty(F) := \lim_{n \to +\infty} R_n(F) \text{ is well defined in } L^2(\Omega) \right\}.$$ 

It is easy to see that $\mathcal{C}_p$ is a dense linear subspace of $\mathcal{H}_p$ and for each $f \in \mathcal{C}_p$, $I_p^W(f) \in \mathcal{C}_p$ and $R_\infty(I_p^W(f)) = 0$. As 

$$\sup_{n \in \mathbb{N}} \|R_n\|_{op} \leq \sqrt{\Theta_3} < +\infty,$$

$R_\infty$ has a unique extension to $\mathcal{H}_p$ and by density of $\{I_p^W(f) : f \in \mathcal{C}_p\}$ in $\mathcal{H}_p$, $R_\infty(F) = 0$ for each $F \in \mathcal{C}_p$. In other words, for any $F \in \mathcal{H}_p$, $n E[F^{(n)} - F|\mathcal{G}]$ converges in $L^2(\Omega)$ to $-pF$, as $n \to +\infty$. 

5. Proofs of Proposition 1.2

We now give the proof of Proposition 1.2 which has been stated in the introduction. We restate it for the convenience of the reader.

**Proposition 1.2** Let $(B, B')_{t \geq 0}$ be a family of exchangeable pairs of Brownian motions (that is, $B$ is a Brownian motion on $[0, 1]$ and, for each $t$, one has $(B, B') \overset{law}{=} (B^t, B)$). Assume moreover that

(a) for any integer $p \geq 1$ and any $f \in L^2([0, 1]^p)$,

$$\lim_{t \downarrow 0} \frac{1}{t} E\left[I_p^B(f) - I_p^B(f)\sigma(B)\right] = -p I_p^B(f) \text{ in } L^2(\Omega).$$

Then, for any integer $p \geq 1$ and any $f \in L^2([0, 1]^p)$,

(b) $\lim_{t \downarrow 0} \frac{1}{t} E\left[(I_p^B(f) - I_p^B(f))^2|\sigma(B)\right] = 2p^2 \int_0^1 I_p^B(f(x, \cdot))^2 dx \text{ in } L^2(\Omega)$;

(c) $\lim_{t \downarrow 0} \frac{1}{t} E\left[(I_p^B(f) - I_p^B(f))^4\right] = 0$.

**Proof.** First we concentrate on the proof of (b). Fix $p \geq 1$ and $f \in L^2([0, 1]^p)$, and set $F = I_p^B(f)$ and $F_t = I_p^B(f)$. First, we observe that

$$\frac{1}{t} E[(F_t - F)^2|\sigma(B)] = \frac{1}{t} E[F_t^2 - F^2|\sigma(B)] - \frac{2}{t} F E[F_t - F|\sigma(B)].$$

Also, an immediate consequence of the product formula (2.7) and the definition of $f \otimes_r f$, we have

$$p^2 \int_0^1 I_{p-1}^B(f(x, \cdot))^2 dx = \sum_{r=1}^p r! \left(\frac{p}{r}\right)^2 I_{2p-2r}(f \otimes_r f).$$
Dividing by as well, that and the exchangeability of when which is exactly (5.13). The proof of (b) is complete.

In particular, it appears that the limit of (5.14)

The product formula used for multiple integrals with respect to \( B^t \) (resp. \( B \)) yields

\[
F_t^2 = \sum_{r=0}^{p-1} r! \binom{p}{r}^2 I_{2p-2r}(f \otimes f) \quad \text{(resp. } F^2 = \sum_{r=0}^{p-1} r! \binom{p}{r}^2 I_{2p-2r}(f \otimes f)\).
\]

Hence it follows from (a) that

\[
\frac{1}{t} E[F_t^2 - F^2|\sigma(B)] = \sum_{r=0}^{p-1} r! \binom{p}{r}^2 \frac{1}{t} E[I_{2p-2r}(f \otimes f) - I_{2p-2r}(f \otimes f)|\sigma(B)] \\
\rightarrow \sum_{r=0}^{p-1} r! \binom{p}{r}^2 (2r - 2p)I_{2p-2r}(f \otimes f) \\
= -2p(F^2 - E[F^2]) + 2 \sum_{r=1}^{p-1} r! \binom{p}{r}^2 I_{2p-2r}(f \otimes f),
\]

which is exactly (5.13). The proof of (b) is complete.

Let us now turn to the proof of (c). Fix \( p \geq 1 \) and \( f \in L^2([0,1]^p) \), and set \( F = I^B(f) \) and \( F_t = I^B_t(f) \), \( t \geq 0 \). We claim that the pair \((F,F_t)\) is exchangeable for each \( t \). Indeed, thanks to point 4 in Section 2, we first observe that it is enough to check this claim when \( f \) belongs to \( \mathcal{E}_p \), that is, when \( f \) has the form

\[
f(x_1, \ldots, x_p) = \sum_{i_1, \ldots, i_p = 1}^k \beta_{i_1 \ldots i_p} \mathbf{1}_{[\tau_{i_1-1}, \tau_{i_1}) \times \cdots \times [\tau_{i_p-1}, \tau_{i_p})}(x_1, \ldots, x_p),
\]

with \( k \geq 1 \) and \( 0 = \tau_0 < \tau_1 < \cdots < \tau_k \), and the coefficients \( \beta_{i_1 \ldots i_p} \) are zero if any two of the indices \( i_1, \ldots, i_p \) are equal. But, for such an \( f \), one has

\[
F = I^B(f) = \sum_{i_1, \ldots, i_p = 1}^k \beta_{i_1 \ldots i_p}(B(\tau_{i_1}) - B(\tau_{i_1-1})) \cdots (B(\tau_{i_p}) - B(\tau_{i_p-1}))
\]

\[
F_t = I^B_t(f) = \sum_{i_1, \ldots, i_p = 1}^k \beta_{i_1 \ldots i_p}(B^t(\tau_{i_1}) - B^t(\tau_{i_1-1})) \cdots (B^t(\tau_{i_p}) - B^t(\tau_{i_p-1})),
\]

and the exchangeability of \((F,F_t)\) follows immediately from those of \((B,B^t)\). Since the pair \((F,F_t)\) is exchangeable, we can write

\[
E[(F_t - F)^4] = E[F_t^4 + F^4 - 4F_t^3F_t - 4F_t^3F + 6F_t^2F^2]
\]

\[
= 2E[F^4] - 8E[F^3F_t] + 6E[F^2F_t^2] \quad \text{by exchangeability;}
\]

\[
= 4E[F^3(F_t - F)] + 6E[F^2(F_t - F)^2] \quad \text{after rearrangement;}
\]

\[
= 4E[F^3E([F_t - F]|\sigma(B))] + 6E[F^2E([F_t - F]^2|\sigma(B))].
\]

Dividing by \( t \) and taking the limit \( t \downarrow 0 \) into the previous identity, we deduce, thanks to (a) and (b) as well, that

\[
\lim_{t \downarrow 0} E\left[ (F_t - F)^4 \right] = -4pE[F^4] + 12p^2 E \left[ F^2 \int_0^1 I_{p-1}(f(x, \cdot))^2 dx \right].
\]

In particular, it appears that the limit of \( \frac{1}{t} E[(F_t - F)^4] \) is always the same, irrespective of the choice of our exchangeable pair of Brownian motions \((B,B^t)\) satisfying (a). To compute it, we can then choose
the pair \((B, B')\) we want, for instance, the pair constructed in Section 3. This is why, starting from now and for the rest of the proof, \((B, B')\) refers to the pair defined in Section 3 (which satisfies (a), that is, \((3.11)\)). What we gain by considering this particular pair is that it satisfies a hypercontractivity-type inequality. More precisely, there exists \(c_p > 0\) (only depending on \(p\)) such that, for all \(t \geq 0\),

\[
E[(F_t - F)^4] \leq c_p E[(F_t - F)^2]^2.
\]

Indeed, going back to the definition of multiple Wiener-Itô integrals as given in Section 2 (first for elementary functions and then by approximation for the general case), we see that \(F_t - F\) is a multiple Wiener-Itô integral of order \(p\) with respect to the two-sided Brownian motion \(B = (\overline{B}(s))_{s \in [-1,1]}\), defined as

\[
\overline{B}(s) = B(s)1_{[0,1]}(s) + \tilde{B}(-s)1_{[-1,0]}(s).
\]

But product formula (2.7) is also true for a two-sided Brownian motion, so the claim (5.15) follows from (2.8) applied to \(\overline{B}\). On the other hand, it follows from (b) that \(\frac{1}{t} E[(F_t - F)^2]\) converges to a finite number, as \(t \downarrow 0\). Hence, combining this fact with (5.15) yields

\[
\frac{1}{t} E\left[ (F_t - F)^4 \right] \leq c_p t \left( \frac{1}{t} E\left[ (F_t - F)^2 \right] \right)^2 \to 0,
\]
as \(t \downarrow 0\). \(\square\)

**Remark 5.1.**

(i) A byproduct of (5.14) in the previous proof is that

\[
\frac{1}{3} E[F^4] - 3\sigma^4 = E\left[ F^2 \left( \int_0^1 I_0^B (f(x, \cdot))^2 dx - \sigma^2 \right) \right].
\]

Note [10] was originally obtained by chain rule, see [15] equation (5.2.9).

(ii) As a consequence of (c) in Proposition 1.2, we have \(\lim_{t \downarrow 0} \frac{1}{t} E\left[ |I_p^B(f) - I_p^B(f)|^3 \right] = 0\). Indeed,

\[
\frac{1}{t} E\left[ |I_p^B(f) - I_p^B(f)|^3 \right] \leq \left( \frac{1}{t} E\left[ (I_p^B(f) - I_p^B(f))^2 \right] \right)^{\frac{3}{2}} \left( \frac{1}{t} E\left[ (I_p^B(f) - I_p^B(f))^4 \right] \right)^{\frac{1}{2}} \to 0, \quad \text{as } t \downarrow 0.
\]

(iii) For any \(r > 2\), in view of (2.9) and (5.15), there exists an absolute constant \(c_{r,p}\) depending only on \(p, r\) (but not on \(f\)) such that

\[
E\left[ |I_p^B(f) - I_p^B(f)|^r \right] \leq c_{r,p} E\left[ (I_p^B(f) - I_p^B(f))^2 \right]^{r/2}.
\]

Moreover, if \(F \in L^2(\Omega, \sigma\{B\}, P)\) admits a finite chaos expansion, say, (for some \(p \in \mathbb{N}\))

\(F = E[F] + \sum_{q=1}^p I_q^B(f_q)\),

and we set \(F_t = E[F] + \sum_{q=1}^p I_q^B(f_q)\), then there exists some absolute constant \(C_{r,p}\) that only depends on \(p\) and \(r\) such that

\[
E\left[ |F - F_t|^r \right] \leq C_{r,p} E\left[ (F - F_t)^2 \right]^{r/2}.
\]

6. Proof of E. Meckes’ Theorem 1.3

In this section, for sake of completeness and because our version slightly differs from the original one given in [11] Theorem 1.1, we provide a proof of Theorem 1.3 which we restate here for convenience.

**Theorem 1.3 (Meckes [11]).** Let \(F\) and a family of random variables \((F_t)_{t \geq 0}\) be defined on a common probability space \((\Omega, \mathcal{F}, P)\) such that \(F_t \overset{law}{=} F\) for every \(t \geq 0\). Assume that \(F \in L^4(\Omega, \mathcal{G}, P)\) for some \(\sigma\)-algebra \(\mathcal{G} \subset \mathcal{F}\) and that in \(L^1(\Omega),\)

(a) \(\lim_{t \downarrow 0} \frac{1}{t} E[F_t - F|\mathcal{G}] = -\lambda F\) for some \(\lambda > 0\),

(b) \(\lim_{t \downarrow 0} \frac{1}{t} E[(F_t - F)^2|\mathcal{G}] = (2\lambda + S) \text{Var}(F)\) for some random variable \(S\),
Then, with $N \sim N(0, \text{Var}(F))$, 
\[ d_{TV}(F, N) \leq \frac{E|S|}{\lambda}. \]

Proof. Without loss of generality, we may and will assume that $\text{Var}(F) = 1$. It is known that
\[ d_{TV}(F, N) = \frac{1}{2} \sup E[\varphi(F) - \varphi(N)], \]
where the supremum runs over all smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support and such that $\|\varphi\|_\infty \leq 1$. For such a $\varphi$, recall (see, e.g. [3, Lemma 2.4]) that
\[ g(x) = e^{x^2/2} \int_{-\infty}^{x} (\varphi(y) - E[\varphi(N)])e^{-y^2/2} \, dy, \quad x \in \mathbb{R}, \]
satisfies
\[ g'(x) - xg(x) = \varphi(x) - E[\varphi(N)] \]
as well as $\|g\|_\infty \leq \sqrt{2\pi}$, $\|g'\|_\infty \leq 4$ and $\|g''\|_\infty \leq 2\|\varphi\|_\infty < +\infty$. In what follows, we fix such a pair $(\varphi, g)$ of functions. Let $G$ be a differentiable function such that $G' = g$, then due to $F_t \overset{\text{law}}{=} F$, it follows from the Taylor formula in mean-value form that
\[ 0 = E[G(F_t) - G(F)] = E[g(F)(F_t - F)] + \frac{1}{2} E[g'(F)(F_t - F)^2] + E[R], \]
with remainder $R$ bounded by $\frac{1}{6} \|g''\|_\infty |F_t - F|^3$.

By assumption (c) and as $t \downarrow 0$,
\[ \left| \frac{1}{t} E[R] \right| \leq \frac{1}{6} \|g''\|_\infty \frac{1}{7} \|F_t - F|^3 \to 0. \]

Therefore as $t \downarrow 0$, assumptions (a) and (b) imply that
\[ \lambda E[g'(F) - Fg(F)] + \frac{1}{2} E[g'(F)S] = 0. \]

Plugging this into Stein’s equation [6.18] and then using [6.17], we deduce the desired conclusion, namely,
\[ d_{TV}(F, N) \leq \frac{1}{2} \|g'\|_\infty \frac{E|S|}{2\lambda} \leq \frac{E|S|}{\lambda}. \]

Remark 6.1. Unlike the original Meckes’ theorem, we do not assume the exchangeability condition $(F_t, F) \overset{\text{law}}{=} (F, F_t)$ in our Theorem 1.3. Our consideration is motivated by [22].

7. Quantitative fourth moment theorem revisited via exchangeable pairs

We give an elementary proof to the quantitative fourth moment theorem, that is, we explain how to prove the inequality of Theorem 1.1(ii) by means of our exchangeable pairs approach. For sake of convenience, let us restate this inequality: for any multiple Wiener-Itô integral of order $p \geq 1$ such that $E[F^2] = \sigma^2 > 0$, we have, with $N \sim N(0, \sigma^2)$,
\[ d_{TV}(F, N) \leq \frac{2}{\sigma^2} \sqrt{\frac{p-1}{3p}} \sqrt{E[F^4] - 3\sigma^2}. \]
To prove (7.19), we consider, for instance, the exchangeable pairs of Brownian motions \(\{(B, B')\}_{t \geq 0}\) constructed in Section 6. We deduce, by combining Proposition 4.2 with Theorem 1.3 and Remark 5.1(ii), that

\[
\begin{align*}
    d_{TV}(F, N) & \leq \frac{2}{\sigma^2} E \left[ \left| \int_0^1 I_{p-1}^B(f(x, \cdot))^2 \, dx - \sigma^2 \right| \right].
\end{align*}
\]

To deduce (7.19) from (7.20), we are thus left to prove the following result.

**Proposition 7.1.** Let \(p \geq 1\) and consider a symmetric function \(f \in L^2([0,1]^p)\). Set \(F = I_p^B(f)\) and \(\sigma^2 = E[F^2]\). Then

\[
    E \left[ \left( p \int_0^1 I_{p-1}^B(f(x, \cdot))^2 \, dx - \sigma^2 \right)^2 \right] \leq \frac{p-1}{3p} (E[F^4] - 3\sigma^4).
\]

**Proof.** Using the product formula (2.4), we can write

\[
    F^2 = \sum_{r=0}^{p} r! \left( \frac{p}{r} \right)^2 I_{2p-2r}^B(f \otimes_r f) = \sigma^2 + \sum_{r=0}^{p-1} r! \left( \frac{p}{r} \right)^2 I_{2p-2r}^B(f \otimes_r f),
\]

as well as

\[
    p \int_0^1 I_{p-1}^B(f(x, \cdot))^2 \, dx = p \sum_{r=0}^{p-1} r! \left( \frac{p-1}{r} \right)^2 I_{2p-2r-2}^B \left( \int_0^1 f(x, \cdot) \otimes_r f(x, \cdot) \, dx \right)
    \]

\[
    = p \sum_{r=1}^{p} (r-1)! \left( \frac{p-1}{r-1} \right)^2 I_{2p-2r}^B(f \otimes_r f) = \sigma^2 + \sum_{r=1}^{p-1} r! \left( \frac{p}{r} \right)^2 I_{2p-2r}^B(f \otimes_r f).\]

Hence, by the isometry property (point 2 in Section 2),

\[
    E \left[ \left( p \int_0^1 I_{p-1}^B(f(x, \cdot))^2 \, dx - \sigma^2 \right)^2 \right] = \sum_{r=1}^{p} r!^2 \left( \frac{p}{r} \right)^4 (2p - 2r)! \|f \otimes_r f\|_{L^2([0,1]^{2p-2r})}^2.
\]

On the other hand, one has from (5.10) and the isometry property again that

\[
    \frac{1}{3} (E[F^4] - 3\sigma^4) = E \left[ F^2 \left( p \int_0^1 I_{p-1}^B(f(x, \cdot))^2 \, dx - \sigma^2 \right) \right]
    \]

\[
    = \frac{1}{3} (E[F^4] - 3\sigma^4) = \sum_{r=1}^{p} r! \left( \frac{p}{r} \right)^4 (2p - 2r)! \|f \otimes_r f\|_{L^2([0,1]^{2p-2r})}^2.
\]

The desired conclusion follows. \(\square\)

### 8. Connections with Malliavin operators

Our main goal in this paper is to provide an elementary proof of Theorem 1.1(ii). Nevertheless, in this section we further investigate the connections we have found between our exchangeable pair approach and the operators of Malliavin calculus. This part may be skipped in a first reading, as it is not used in other sections. It is directed to readers who are already familiar with Malliavin calculus. We use classical notation and so do not introduce them in order to save place. We refer to [18] for any missing detail.

In this section, to stay on the safe side we only consider random variables \(F\) belonging to

\[
    \mathcal{A} := \bigcup_{p \in \mathbb{N}} \bigoplus \mathcal{H}_p^r,
\]

where \(\mathcal{H}_p^r\) is the \(r\)th chaos associated to the Brownian motion \(B\). In other words, we only consider random variables that are \(\sigma(B)\)-measurable and that admit a finite chaotic expansion. Note that \(\mathcal{A}\) is an algebra (in view of product formula) that is dense in \(L^2(\Omega, \sigma(B), P)\).
As is well-known, any \( \sigma(B) \)-measurable random variable \( F \) can be written \( F = \psi_F(B) \) for some measurable mapping \( \psi_F : \mathbb{R}^d \to \mathbb{R} \) determined \( P \circ B^{-1} \) almost surely. For such an \( F \), we can then define \( F_t = \psi_F(B^t) \), with \( B^t \) defined in Section 3. Another equivalent description of \( F_t \) is to define it as \( F_t = E[F] + \sum_{r=1}^p I_t^B(f_r) \), if the family \( (f_r)_{1 \leq r \leq p} \) is such that \( F = E[F] + \sum_{r=1}^p I_t^B(f_r) \).

Our main findings are summarized in the statement below.

**Proposition 8.1.** Consider \( F, G \in \mathcal{A} \), and define \( F_t, G_t \) for each \( t \in \mathbb{R}_+ \) as is done above. Then, in \( L^2(\Omega) \),

\[
\begin{align*}
&\text{(a)} \lim_{t \downarrow 0} \frac{1}{t} E\left[ F_t - F \mid \{B \} \right] = LF, \\
&\text{(b)} \lim_{t \downarrow 0} \frac{1}{t} E\left[ (F_t - F)(G_t - G) \mid \{B \} \right] = LFG - FGL - GLF = 2(DF, DG).
\end{align*}
\]

**Proof.** The proof of (a) is an immediate consequence of \( L \), the linearity of conditional expectation, and the fact that \( L I_t^B(f_r) = -r I_t^B(f_r) \) by definition of \( L \). Let us now turn to the proof of (b). Using elementary algebra and then (a), we deduce that, as \( t \downarrow 0 \) and in \( L^2(\Omega) \),

\[
\begin{align*}
\frac{1}{t} E\left[ (F_t - F)(G_t - G) \mid \{B \} \right] &= \frac{1}{t} E[F_t G_t - F G_t - F_t G + F G] - \frac{1}{t} G E[F_t - F] \mid \{B \}] \\
&\to LFG - FGL - GLF.
\end{align*}
\]

Using \( L = -\delta D, D(FG) = FGD + GDF \) (Leibniz rule) and \( \delta(FDG) = F\delta(DG) - (DF, DG) \) (see Proposition 1.3.3), it is easy to check that \( L(FG) - FGL - GLF = 2(DF, DG) \), which concludes the proof of Proposition 8.1.

**Remark 8.2.** The expression appearing in the right-hand side of (b) is nothing else but \( 2 \Gamma(F,G) \), the (doubled) carré du champ operator.

To conclude this section, we show how our approach allows to recover the diffusion property of the Ornstein-Uhlenbeck operator.

**Proposition 8.3.** Fix \( d \in \mathbb{N} \), let \( F = (F_1, \ldots, F_d) \in \mathcal{A}^d \) (with \( \mathcal{A} \) given in \( 3.21 \)), and \( \Psi : \mathbb{R}^d \to \mathbb{R} \) be a polynomial function. Then

\[
L\Psi(F) = \sum_{j=1}^d \partial_j \Psi(F)LF_j + \sum_{i,j=1}^d \partial_{ij} \Psi(F)(DF_i, DF_j).
\]

**Proof.** We first define \( F_t = (F_{1,t}, \ldots, F_{d,t}) \) as explained in the beginning of the present section. Using classical multi-index notations, Taylor formula yields that

\[
\Psi(F_t) - \Psi(F) = \sum_{j=1}^d \partial_j \Psi(F)(F_{j,t} - F_j) + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} \Psi(F)(F_{j,t} - F_j)(F_{i,t} - F_i)
\]

\[
+ \sum_{|\beta|=3} \frac{3}{!} \beta_1! \cdots \beta_d! (F_t - F)^\beta \int_0^1 (1 - s)^k (\partial_1^{\beta_1} \cdots \partial_d^{\beta_d} \Psi)(F + s(F_t - F)) \, ds.
\]

In view of the previous proposition, the only difficulty in establishing (8.22) is about controlling the last term in (8.23) while passing \( t \downarrow 0 \). Note first \( (\partial_1^{\beta_1} \cdots \partial_d^{\beta_d} \Psi)(F + s(F_t - F)) \) is polynomial in \( F \) and \( (F_t - F) \), so our problem reduces to show

\[
\lim_{t \downarrow 0} \frac{1}{t} E\left[ |F^\alpha(F_t - F)^\beta| \right] = 0,
\]

for \( \alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in (\mathbb{N} \cup \{0\})^d \) with \( |\beta| \geq 3 \).
Indeed, (assume $\beta_j > 0$ for each $j$)
\[
\frac{1}{t} E[|F^n(F_i - F)|] \leq \frac{1}{t} E[|F^n|^{1/2}] E[(F_i - F)^{\beta}]^{1/2} \quad \text{by Cauchy-Schwarz inequality;}
\]
\[
\leq E[|F^n|^{1/2}] \frac{1}{t} \left( \prod_{j=1}^d E[(F_{j,t} - F_j)^{2,\beta}]^{\beta_j} \right)^{1/2} \quad \text{by Hölder inequality;}
\]
\[
\leq C E[|F^n|^{1/2}] \frac{1}{t} \prod_{j=1}^d \frac{1}{\beta_j} E[(F_{j,t} - F_j)^{2,\beta}]^{\beta_j} \quad ,
\]
where the last inequality follows from point-(iii) in Remark 5.1 with $C > 0$ independent of $t$. Since $F^n \in A$ and $|\beta| \geq 3$, (8.24) follows immediately from the above inequalities. \hfill \square

9. Peccati-Tudor theorem revisited too

In this section, we combine a multivariate version of Meckes’ abstract exchangeable pairs [12] with our results from Section 3 to prove (1.5), thus leading to a fully elementary proof of Theorem 1.5 as well.

First, we recall the following multivariate version of Meckes’ theorem (see [12, Theorem 4]). Unlike in the one-dimensional case, it seems inevitable to impose the exchangeability condition in the following proposition, as we read from its proof in [12].

**Proposition 9.1.** For each $t > 0$, let $(F, F_1)$ be an exchangeable pair of centered $d$-dimensional random vectors defined on a common probability space. Let $\mathcal{G}$ be a $\sigma$-algebra that contains $\sigma\{F\}$. Assume that $\Lambda \in \mathbb{R}^{d \times d}$ is an invertible deterministic matrix and $\Sigma$ is a symmetric, non-negative definite deterministic matrix such that

(a) $\lim_{t \downarrow 0} \frac{1}{t} E[F_i - F_i|\mathcal{G}] = -\Lambda Y$ in $L^1(\Omega)$,

(b) $\lim_{t \downarrow 0} \frac{1}{t} E[(F_i - F)(F_i - F)^T|\mathcal{G}] = 2\Lambda \Sigma + S$ in $L^1(\Omega, \| \cdot \|_{HS})$ for some matrix $S = S(F)$, and with $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm

(c) $\lim_{t \downarrow 0} \sum_{i=1}^d \frac{1}{t} E[F_{i,t} - F_i^2] = 0$, where $F_{i,t}$ (resp. $F_i$) stands for the $i$th coordinate of $F_i$ (resp. $F$).

Then, with $N \sim N_d(0, \Sigma)$,

1. for $g \in C^2(\mathbb{R}^d)$,
   \[
   |E[g(F)] - E[g(N)]| \leq \frac{\|\Lambda^{-1}\|_{op} \sqrt{7} M_2(g)}{4} E \left[ \sum_{i,j=1}^d S_{ij}^2 \right] ,
   \]
   where $M_2(g) := \sup_{x \in \mathbb{R}^d} \|D^2 g(x)\|_{op}$ with $\| \cdot \|_{op}$ the operator norm.

2. if, in addition, $\Sigma$ is positive definite, then
   \[
   d_{W}(F, N) \leq \frac{\|\Lambda^{-1}\|_{op} \|\Sigma^{-1}\|_{op} \sqrt{2\pi}}{\sqrt{2\pi}} E \left[ \sum_{i,j=1}^d S_{ij}^2 \right] .
   \]

**Remark 9.2.** Constant in (2) is different from Meckes’ paper [12]. We took this better constant from Christian Döbler’s dissertation [5], see page 114 therein.
By combining the previous proposition with our exchangeable pairs, we get the following result, whose point 2 corresponds to (3.10).

**Theorem 9.3.** Fix \( d \geq 2 \) and \( 1 \leq p_1 \leq \ldots \leq p_d \). Consider a vector \( F := (I_{p_1}^B(f_1), \ldots, I_{p_d}^B(f_d)) \) with \( f_i \in L^2([0,1]^p) \) symmetric for each \( i \in \{1, \ldots, d\} \). Let \( \Sigma = (\sigma_{ij}) \) be the covariance matrix of \( F \), and \( N \sim N_d(0, \Sigma) \). Then

1. for \( g \in C^2(\mathbb{R}^d) \),
   
   \[
   \left| E[g(F)] - E[g(N)] \right| \leq \frac{\sqrt{d} M_2(g)}{2p_1} \left( \sum_{i,j=1}^d \var \left( p_i p_j \int_0^1 \left. I_{p_i-1}(f_i(x, \cdot)) I_{p_j-1}(f_j(x, \cdot)) \right| dx \right),
   \]

   where \( M_2(g) := \sup_{x \in \mathbb{R}^d} \| D^2 g(x) \|_{op} \).

2. if in addition, \( \Sigma \) is positive definite, then
   
   \[
   d_W(F, N) \leq \frac{2 \| \Sigma^{-1/2} \|_{op}}{q_1 \sqrt{2\pi}} \left( \sum_{i,j=1}^d \var \left( p_i p_j \int_0^1 \left. I_{p_i-1}(f_i(x, \cdot)) I_{p_j-1}(f_j(x, \cdot)) \right| dx \right).
   \]

**Proof.** We consider \( F_t = (I_{p_1}^B(f_1), \ldots, I_{p_d}^B(f_d)) \), where \( B^t \) is the Brownian motion constructed in Section 3. We deduce from (3.10) that

\[
\frac{1}{t} E[F_t - F|\sigma(B)] = \left( \frac{e^{-pt} - 1}{t} I_{p_1}^B(f_1), \ldots, \frac{e^{-pt} - 1}{t} I_{p_d}^B(f_d) \right)
\]

implying in turn that, in \( L^2(\Omega) \) and as \( t \downarrow 0 \),

\[
\frac{1}{t} E[F_t - F|\sigma(B)] \rightarrow -\Lambda F,
\]

with \( \Lambda = \text{diag}(p_1, \ldots, p_d) \) (in particular, \( \|\Lambda^{-1}\|_{op} = p_1^{-1} \)). That is, assumption (a) in Proposition 9.1 is satisfied (with \( \mathcal{G} = \sigma(B) \)). That assumption (c) in Proposition 9.1 is satisfied as well follows from Proposition 1.2(c). Let us finally check that assumption (b) in Proposition 9.1 takes place too. First, using the product formula (2.7) for multiple integrals with respect to \( B^t \) (resp. \( B \)) yields

\[
F_i F_j = \sum_{r=0}^{p_i \wedge p_j} r! \binom{p_i}{r} \binom{p_j}{r} I_{p_i+p_j-2r}(f_i \otimes_r f_j)
\]

\[
F_{i,t} F_{j,t} = \sum_{r=0}^{p_i \wedge p_j} r! \binom{p_i}{r} \binom{p_j}{r} I_{p_i+p_j-2r}(f_i \otimes_r f_j).
\]

Hence, using (3.10) for passing to the limit,

\[
\frac{1}{t} E[(F_{i,t} - F_i)(F_{j,t} - F_j)|\sigma(B)] - \frac{1}{t} E[F_{i,t} F_{j,t} - F_i F_j|\sigma(B)]
\]

\[
= -\frac{1}{t} F_i E[F_{j,t} - F_j|\sigma(B)] - \frac{1}{t} F_j E[F_{i,t} - F_i|\sigma(B)]
\]

\[
\rightarrow (p_i + q_j) F_i F_j = \sum_{r=0}^{p_i \wedge p_j} r! \binom{p_i}{r} \binom{p_j}{r} (p+q) I_{p_i+p_j-2r}(f_i \otimes_r f_j) \quad \text{as} \ t \downarrow 0.
\]
Now, note in $L^2(\Omega)$,
\[
\frac{1}{t} E\left[ F_{i,t} F_{j,t} - F_i F_j \right] = \sum_{\pi, \lambda \in \mathcal{P}_j} \left( \frac{p_i}{r} \right) \left( \frac{p_j}{r} \right) \frac{1}{t} E\left[ I_{\pi + \lambda}^B \left( f_i \otimes r f_j \right) - I_{\pi + \lambda}^B \left( f_i \otimes r f_j \right) \right] \rightarrow \sum_{\pi, \lambda \in \mathcal{P}_j} \left( \frac{p_i}{r} \right) \left( \frac{p_j}{r} \right) \frac{1}{t} E\left[ (2r - p_i - p_j) I_{\pi + \lambda}^B \left( f_i \otimes r f_j \right) \right], \quad \text{as } t \downarrow 0,
\]
where $p_i \wedge p_j$.

Thus, as $t \downarrow 0$,
\[
\frac{1}{t} E\left[ (F_{i,t} - F_i)(F_{j,t} - F_j) \right] = \frac{2}{\pi} \sum_{\pi, \lambda \in \mathcal{P}_j} \left( \frac{p_i}{r} \right) \left( \frac{p_j}{r} \right) \int_{0}^{1} I_{\pi - 1}^B \left( f_i(x, \cdot) \right) I_{\lambda - 1}^B \left( f_j(x, \cdot) \right) dx.
\]

By the isometry property (point 2 in Section 2), it is straightforward to check that
\[
P_j \int_{0}^{1} E\left[ I_{\pi - 1}^B \left( f_i(x, \cdot) \right) I_{\lambda - 1}^B \left( f_j(x, \cdot) \right) \right] dx = \sigma_{ij}.
\]

Therefore,
\[
E\left[ \sum_{i,j=1}^{d} S_{ij}^2 \right] = \sum_{i,j=1}^{d} E\left[ S_{ij}^2 \right] = 2 \sum_{i,j=1}^{d} \text{Var}\left( P_j \int_{0}^{1} I_{\pi - 1}^B \left( f_i(x, \cdot) \right) I_{\lambda - 1}^B \left( f_j(x, \cdot) \right) dx \right).
\]

Hence the desired results in (1) and (2) follow from Proposition 9.1. \( \square \)

**References**

[1] J.-M. Azaïs, F. Dalmao and J.R. León. CLT for the zeros of classical random trigonometric polynomials, Ann. Inst. H. Poincaré Probab. Statist. Volume 52, Number 2 (2016), 804-820.

[2] J.-M. Azaïs and J.R. León. CLT for crossing of random trigonometric polynomials, Electron. J. Probab. Volume 18, no. 68 (2013), 1-17.

[3] L.H.Y. Chen, L. Goldstein and Q.M. Shao. Normal approximation by Stein’s method. Probability and Its Applications, 2011 Springer-Verlag Berlin Heidelberg.

[4] F. Dalmao. Asymptotic variance and CLT for the number of zeros of Kostlan Shub Smale random polynomials, Comptes Rendus Mathematique, Volume 353, Issue 12 (2015), Pages 1141-1145.

[5] C. Döbler. New developments in Stein’s method with applications. Ph.D dissertation, Ruhr-Universität Bochum (2012). Available online at [http://www-brs.ub.ruhr-uni-bochum.de/netahtml/HSS/Diss/DoeblerChristian/](http://www-brs.ub.ruhr-uni-bochum.de/netahtml/HSS/Diss/DoeblerChristian/)

[6] D. Hug, G. Last, and M. Schulte. Second-order properties and central limit theorems for geometric functionals of Boolean models, Ann. Appl. Probab. Volume 26, Number 1 (2016), 73-135.

[7] P. Malliavin. Stochastic calculus of variations and hypoelliptic operators. In: Proc. Inter. Symp. on Stoch. Diff. Equations, Kyoto 1976, Wiley 1978, 195-263.

[8] D. Marinucci and G. Peccati. Ergodicity and Gaussianity for Spherical Random Fields, J. Math. Phys. 51, (2010)

[9] D. Marinucci, G. Peccati, M. Rossi, I. Wigman. Non-Universality of Nodal Length Distribution for Arithmetic Random Waves, Geom. Funct. Anal. (2016).
[10] D. Marinucci, M. Rossi. Stein-Malliavin approximations for nonlinear functionals of random eigenfunctions on $S^d$. J. Funct. Anal. 268 (8) (2015), 2379-2420.

[11] E. Meckes. An Infinitesimal Version of Stein’s Method of Exchangeable Pairs. Ph.D dissertation, Stanford University (2006)

[12] E. Meckes. On Stein’s method for multivariate normal approximation. IMS collections, High dimensional Probability V: The Luminy Volume, Vol. 5 (2009) 153-178.

[13] I. Nourdin. Malliavin-Stein approach: a webpage maintained by Ivan Nourdin. http://tinyurl.com/kvpdgcy

[14] I. Nourdin and G. Peccati. Stein’s method on Wiener chaos, Probab. Theory Relat. Fields (2009), Vol. 145, Issue 1, p. 75-118.

[15] I. Nourdin and G. Peccati. Normal approximations with Malliavin calculus: from Stein’s method to universality, Cambridge tracts in Mathematics, Vol. 192, 2012, Cambridge University Press.

[16] I. Nourdin, G. Peccati and A. Réveillac. Multivariate normal approximation using Stein’s method and Malliavin calculus. Ann. Instit. H. Poincaré - Probab. Statist. 46, no. 1 (2010), 45-58.

[17] I. Nourdin and J. Rosiński. Asymptotic independence of multiple Wiener-Itô integrals and the resulting limit laws. Ann. Probab. 42, no. 2 (2014), 497-526

[18] D. Nualart. The Malliavin Calculus and Related Topics, second edition. Probability and Its Applications, Springer-Verlag Berlin Heidelberg (2006)

[19] D. Nualart and G. Peccati. Central limit theorems for sequences of multiple stochastic integrals. Ann. Probab. 33(1), 177-193 (2005).

[20] G. Peccati and C.A. Tudor. Gaussian limits for vector-valued multiple stochastic integrals, Séminaire de Probabilités XXXVIII, 2005

[21] M. Reitzner and M. Schulte. Central Limit Theorems for U-Statistics of Poisson Point Processes, Ann. Probab.41, no. 6 (2013), 3879-3909

[22] A. Röllin. A note on the exchangeability condition in Stein’s method. Statist. Probab. Lett. 78 (2008), 1800-1806.

[23] M. Schulte. A Central Limit Theorem for the Poisson-Voronoi Approximation, Adv. Appl. Math. 49, no. 3-5 (2012), 285-306

[24] C. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables, Berkeley Symp. on Math. Statist. and Prob. Proc. Sixth Berkeley Symp. on Math. Statist. and Prob., Vol. 2 (Univ. of Calif. Press, 1972), 583-602.