Is Pseudo-Hermitian Quantum Mechanics an Indefinite-Metric Quantum Theory?

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Abstract

With a view to eliminate an important misconception in some recent publications, we give a brief review of the notion of a pseudo-Hermitian operator, outline pseudo-Hermitian quantum mechanics, and discuss its basic difference with the indefinite-metric quantum mechanics. In particular, we show that the answer to the question posed in the title is a definite No.

1 Introduction

The theory of pseudo-Hermitian operators, as formulated in[1, 2, 3, 4], owes its existence to the author’s efforts to elucidate the origin of the reality of the spectrum of the $PT$-symmetric non-Hermitian Hamiltonians considered in [5]. It emerged in trying to respond to the question: “What is the necessary and sufficient conditions for the reality of the spectrum of a linear operator?,” and indeed turned out to achieve its basic goal of understanding the mathematical structure of the $PT$-symmetric quantum mechanics [6]. It also found remarkable applications in other areas of theoretical physics [7, 8, 9, 10, 11].

Since the initial announcement of the results of [1] in July 2001, there have appeared dozens of publications exploring various aspects of the pseudo-Hermitian operators. Among these are

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a number of articles \cite{12,11} reflecting the view that the notion of a pseudo-Hermitian operator has indeed been known since as early as the 1940’s through the works of Dirac, Pauli, Gupta, Bleuler, Sudarshan, Lee and Wick \cite{13} and other authors who developed quantum theories based on a vector space with an indefinite-metric \cite{14}. This view seems to support the claim that pseudo-Hermitian quantum mechanics can be reduced to the well-known indefinite-metric quantum theories. The main purpose of the present article is to show that this is actually not true. It will be shown that there is a subtle difference between the notion of a Hermitian operator in an indefinite-metric vector space (that is admittedly a ‘pseudo-Hermitian operator’) and the notion of a pseudo-Hermitian operator as defined in \cite{1}. This difference which has also been overlooked in a number of recent publications \cite{12} has important conceptual and technical ramifications. These will also be alluded to here.

2 Pseudo-, $\eta$-pseudo-, and quasi-Hermitian operators

We begin our discussion by recalling some basic definitions.

**Def. 1:** Let $\mathcal{H}$ be a separable Hilbert space. Then a linear operator $H : \mathcal{H} \to \mathcal{H}$ is said to be *pseudo-Hermitian* \cite{1} if there exists an invertible, self-adjoint, linear operator $\eta : \mathcal{H} \to \mathcal{H}$ satisfying $H^\dagger = \eta H \eta^{-1}$, where $\dagger$ denotes the adjoint of the corresponding operator.

According to Def. 1, the pseudo-Hermiticity of an operator is not sensitive to the particular form of the operators $\eta$ satisfying $H^\dagger = \eta H \eta^{-1}$ but to the existence of such operators. In fact, it is not difficult to see that either such an operator $\eta$ does not exist and $H$ is not pseudo-Hermitian or there are infinitely many $\eta$’s fulfilling $H^\dagger = \eta H \eta^{-1}$ and subsequently $H$ is pseudo-Hermitian. In the latter case, we shall denote the set of all such $\eta$’s by $\mathcal{E}(H)$. For a given diagonalizable pseudo-Hermitian operator with a discrete spectrum, the problem of the construction of the most general $\eta \in \mathcal{E}(H)$ is addressed in \cite{3}.

**Def. 2:** Let $\mathcal{H}$ be a separable Hilbert space and $\eta : \mathcal{H} \to \mathcal{H}$ be a given invertible, self-adjoint, linear operator. Then a linear operator $H : \mathcal{H} \to \mathcal{H}$ satisfying $H^\dagger = \eta H \eta^{-1}$ is called *$\eta$-pseudo-Hermitian* \cite{1}.

It is essential to observe that, unlike Def. 1, Def. 2 involves a fixed operator $\eta$. Clearly, $\eta$-pseudo-Hermitian operators are pseudo-Hermitian, but not every pseudo-Hermitian operator is $\eta$-pseudo-Hermitian. This is simply because $\eta$ may happen not to belong to $\mathcal{E}(H)$. In summary,
the set of $\eta$-pseudo-Hermitian operators is just a proper subset of the set of pseudo-Hermitian operators [15].

**Def. 3:** Let $\mathcal{H}$ be a separable Hilbert space. Then a pseudo-Hermitian operator $H : \mathcal{H} \to \mathcal{H}$ is said to be quasi-Hermitian if $\mathcal{E}(H)$ includes a positive operator $\eta_+$. 

Recall that a positive operator is a self-adjoint operator with a nonnegative spectrum. As the elements of $\mathcal{E}(H)$ are by definition invertible, $\eta_+$ is actually a positive-definite operator, i.e., it has a strictly positive spectrum. Furthermore, given a quasi-Hermitian operator $H$, the positive operator $\eta_+$ belonging to $\mathcal{E}(H)$ is not unique. But any two such operators $\eta_+$ and $\eta'_+$ are related according to $\eta'_+ = A^\dagger \eta_+ A$ where $A$ is some invertible linear operator commuting with $H$ [3]. It is also worth emphasizing that there are always infinitely many non-positive elements of $\mathcal{E}(H)$; a quasi-Hermitian operator is always $\eta$-pseudo-Hermitian for some indefinite operator $\eta$. Here by the indefiniteness of $\eta$, we means that neither $\eta$ nor $-\eta$ are positive-definite operators.

A given pseudo-Hermitian operator may or may not be quasi-Hermitian. Hence, quasi-Hermitian operators form a proper subset of the set of pseudo-Hermitian operators. Similarly, the set of Hermitian operators is a proper subset of the set of quasi-Hermitian operators:

$$\text{Hermitian} \subset \text{Quasi-Hermitian} \subset \text{Pseudo-Hermitian}.$$ 

The main results of the theory of pseudo-Hermitian operators are the following spectral characterization theorems.

**Thm. 1:** Let $\mathcal{H}$ be a separable Hilbert space and $H : \mathcal{H} \to \mathcal{H}$ be a diagonalizable linear operator with a discrete spectrum. Then the following are equivalent. 1.a) $H$ is pseudo-Hermitian; 1.b) The complex-conjugate of every eigenvalue of $H$ is also an eigenvalue; 1.c) $H$ commutes with an invertible antilinear operator.

**Thm. 2:** Let $\mathcal{H}$ be a separable Hilbert space and $H : \mathcal{H} \to \mathcal{H}$ be a linear operator with a discrete spectrum. Then the following are equivalent. 2.a) $H$ is quasi-Hermitian; 2.b) $H$ is Hermitian with respect to some positive-definite inner product on $\mathcal{H}$; 2.c) $H$ is a diagonalizable operator with a real spectrum; 2.d) $H$ may be mapped to a Hermitian operator by a similarity transformation.

The proof of Thm. 1 and a slightly stronger variant of Thm. 2 is given in [1] [2]. Note also that in view of Thm. 2, Def. 3 is equivalent to the definition of a quasi-Hermitian operator given in [16].
3  Pseudo-Hermiticity and the indefinite-metric vector spaces

The operators \( \eta \) entering the discussion of the pseudo-Hermitian operators are sometimes called metric operators. This is because they may be used to define an inner product, namely \( \langle \cdot, \cdot \rangle_\eta := \langle \cdot, \eta \cdot \rangle \), which is a genuine positive-definite inner product if and only if \( \eta \) is a positive-definite operator. If \( H \) happens not to be quasi-Hermitian, then a positive-definite \( \eta \) does not exist, and the inner product \( \langle \cdot, \cdot \rangle_\eta \) with respect to which \( H \) is Hermitian (i.e., \( \langle \cdot, H \cdot \rangle_\eta = \langle H \cdot, \cdot \rangle_\eta \)) is necessarily indefinite. This means that \( \langle \cdot, \cdot \rangle_\eta \) satisfies all the defining relations of an inner product except that there are \( \psi \in H \) such that \( \| \psi \|_\eta^2 := \langle \psi, \psi \rangle_\eta \leq 0 \).

Endowing a Hilbert space \( H \) with a self-adjoint, invertible, linear operator \( \eta : H \rightarrow H \) turns \( H \) into an indefinite-metric vector space \( H_\eta \). The linear operators that act in this space and are Hermitian with respect to its indefinite inner product \( \langle \cdot, \cdot \rangle_\eta \) are precisely the \( \eta \)-pseudo-Hermitian operators acting in \( H \). It is the properties of the \( \eta \)-pseudo-Hermitian operators for a fixed \( \eta \) that have been studied within the context of the indefinite-metric vector spaces [13, 14].

The main difference between the approach pursued in the theory of pseudo-Hermitian operators [1, 2, 3] over that of the above-mentioned studies of the indefinite-metric vector spaces [13, 14] is that the former does not involve fixing a metric operator \( \eta \) from the outset. This apparently minor difference has remarkable conceptual as well as practical implications. The superiority of the former approach over the latter is reminiscent of that of General Relativity over Special Relativity. This also reminds one of the following important lessons of the history of modern physics: 1. The geometrical structures underlying physical theories must not be fixed according to one’s wishes or for mere mathematical convenience; 2. Having an arbitrariness in a construction is an indication of the presence of a symmetry, a quality that is almost always desirable and often useful. Ironically, the tendency to fix a metric operator from the outset and delve in the intricacies of a fixed indefinite-metric vector space, that has been the predominant attitude for the past 75 years, is in clear violation of both these principles.

In contrast to the historical developments leading to the indefinite-metric quantum theories, the theory of pseudo-Hermitian operators has been formulated in a way as to incorporate the freedom in the choice of the metric operator into the basic structure of the theory. Indeed, the recent application of pseudo-Hermitian operators in addressing some of the outstanding open problems of relativistic quantum mechanics [8, 9] and quantum cosmology [7, 8] makes a direct use of the arbitrariness of the choice of the metric operator. It is perhaps the most recent confirmation of the validity of the views reflected in the above-mentioned lessons of the history.
of modern physics.

The particular problems referred to in the preceding paragraph have to do with the existence and construction of a conserved positive-definite inner product on the solution space of the Klein-Gordon equation and its generalizations that arise as the Wheeler-DeWitt equations in quantum cosmology [17]. These are conveniently called Klein-Gordon-type equations [7, 8]. It is well-known that the solution space of such a field equation admits an invariant indefinite inner product, namely the Klein-Gordon inner product. The indefiniteness of the latter has been the fundamental obstacle for devising a probabilistic interpretation for relativistic quantum mechanics and quantum cosmology. The theory of pseudo-Hermitian operators provides a simple and explicit solution for this problem. In the following we include a brief outline of this solution for the free Klein-Gordon equation: \[ \partial_t^2 + D \psi(t, \vec{x}) = 0, \]
where
\[ D := -\nabla^2 + m^2, \]
m is the mass, and \( c = \hbar = 1 \).

First, we write the Klein-Gordon equation as a Schrödinger equation for a two-component field. It is well-known [18] that the corresponding Hamiltonian \( H \) is a \( \sigma_3 \)-pseudo-Hermitian operator acting in \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \), i.e., \( \sigma_3 \in \mathcal{E}(H) \), where \( \sigma_3 \) is the diagonal Pauli matrix with diagonal entries \( \pm 1 \). The Klein-Gordon inner product corresponds to \( \langle \cdot, \cdot \rangle_{\sigma_3} \) which is manifestly indefinite. However, it can be easily shown that \( H \) is diagonalizable and has a real spectrum [7, 8, 9]. Hence according to Thm. 2, \( \mathcal{E}(H) \) includes positive elements \( \eta_+ \) that can be used to define a positive-definite inner product \( \langle \cdot, \cdot \rangle_{\eta_+} \). The latter leads to an explicit expression for a class of positive-definite inner products on the solution space \( \mathcal{S} \) of Klein-Gordon fields [7]. Because all positive-definite inner products on \( \mathcal{S} \) are unitarily equivalent [8], one may choose any one of the inner products obtained in this way to develop a probabilistic quantum theory of first-quantized scalar fields. A particularly appealing example is the relativistically invariant inner product [9]:

\[
(\psi_1, \psi_2) := \frac{1}{2\mu} \int_{\mathbb{R}^3} d^3x \left[ \psi_1^*(\vec{x}, t)D^{1/2}\psi_2(\vec{x}, t) + \partial_t \psi_1^*(\vec{x}, t)D^{-1/2}\partial_t \psi_2(\vec{x}, t) \right].
\]

Endowing \( \mathcal{S} \) with this inner product and completing the resulting inner product space via Cauchy completion [19], one obtains a separable Hilbert space that turns out to be most conveniently modelled as \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \). This allows for an explicit construction of a novel set of relativistic position operators and the associated localized and coherent states [9].

It is remarkable that although this problem was formulated in the late 1920s and examined by some of the founding fathers of both quantum mechanics (QM) and its extension to indefinite-metric theories such as Dirac in as early as the 1930s, its solution only appeared recently. The lack of progress in solving this problem during the past 75 years may be traced back to the fact that all the workers on the subject preferred to use the indefinite metric operator
σ_3 which looked simple and could be related to the electric charge conservation. It was the recent formulation of the theory of the pseudo-Hermitian operators (and perhaps the fortunate ignorance of the author about the early literature on the subject at the time) that allowed for considering other metric operators that were unlike σ_3 positive-definite.

4 Pseudo-Hermitian and indefinite-metric QM

Pseudo-Hermitian QM is defined by an auxiliary pseudo-Hermitian Hamiltonian H' acting in a separable Hilbert space ℱ such that the span ℶ of the eigenvectors of H' with real eigenvalues is nonempty. The physical Hilbert space ℱ and the Hamiltonian H are obtained as follows. First, one considers the restriction K of H' onto ℶ as a linear operator acting in the complete closure K̄ of ℶ. Then, by construction K is a densely defined quasi-Hermitian operator and ℰ(K) includes a positive operator η+. Next, one makes a choice for η+ (noting that all choices are unitarily/physically equivalent) and endows ℶ with the inner product ⟨·,·⟩_{η+} so that K may be viewed as a Hermitian operator acting in this inner product space. ℱ and K are respectively the Cauchy completion of ℶ and the closed self-adjoint extension of K to ℱ. [19]

One can use the auxiliary Hamiltonian H' and the Hilbert space ℱ to formulate an indefinite-metric quantum system. This is simply done by choosing an arbitrary indefinite η ∈ ℰ(H') and defining the nonphysical Hilbert space ℱ_η to be the indefinite-metric vector space obtained by endowing ℱ (viewed as a complex vector space) with the indefinite inner product ⟨·,·⟩_{η}. [18] The physical Hilbert space is then identified with the subspace of ℱ_η that includes besides the zero vector the elements that have a positive real norm ||·||_η.

Performing the constructions of the indefinite-metric QM for the Klein-Gordon equation, one arrives at a physical Hilbert space that consists of positive-energy solutions. In contrast, the constructions of the pseudo-Hermitian QM lead to a physical Hilbert space that includes the positive-, zero-, as well as negative-energy solutions. This is a concrete evidence that pseudo-Hermitian QM is not just the well-known indefinite-metric QM.

Acknowledgment: This work has been supported by the Turkish Academy of Sciences in the framework of the Young Researcher Award Program (EA-TÜBA-GEBİP/2001-1-1).

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