Bell’s theorem states that local realistic theories cannot give account of all the quantum mechanical correlations. The best known proofs are based on the Clauser-Horne [3] or the Clauser-Horne-Shimony-Holt [4] inequality. These inequalities are consequences of local realism, thus experimental demonstrations of their violation [3], [4] imply the necessity of a revision of at least one fundamental physical concept.

This far reaching consequence motivates the continued search for new methods for formulating Bell’s theorem and testing local realism. A recent direction is to exploit the full angular dependence of the quantum mechanical correlations, which is equivalent with the case of infinitely many apparatus settings at each side of the experiment [3], [8]. A standard Bell type experiment is assumed, i.e., a source emits two spin-half particles in a maximally entangled (singlet) state, so that the particles do not interact after the emission. Subsequently, the projection of the spin at a direction \( \mathbf{a} \) is measured on the first particle and the projection of the spin at a direction \( \mathbf{b} \) is measured on the second particle (\( \mathbf{a}, \mathbf{b} \) are unit vectors). In a real experiment one has losses and thus the joint probability of having the results \( m \) and \( m' (= \pm 1) \) is

\[
P_{QM}(m, m'; \mathbf{a}, \mathbf{b}) = \frac{1}{4} (1 - mm'V \mathbf{a} \cdot \mathbf{b})
\]  

Here \( 0 \leq V \leq 1 \) stands for the visibility. It is an important question which visibility suffices to falsify the premises of local realism in case of infinitely many apparatus settings. In a recent paper [9] Zukowski showed that the threshold visibility is not larger than \( 3/4 \). This improved the previous estimates \( 8/\pi^2 \) (derived for the case of coplanar settings) and \( \pi/4 \) [10]. In the present paper we argue that the threshold visibility is actually \( 1/3 \), and no lower value is possible. Throughout we adhere to the notations of Ref. [10].

Local realism implies that the above joint probability is of the form

\[
P_{HV}(m, m'; \mathbf{a}, \mathbf{b}) = \int_\Lambda d\rho(\lambda)P_A(m|\mathbf{a}, \lambda)P_B(m'|\mathbf{b}, \lambda)
\]  

where \( \lambda \) stands for the 'hidden variable' characterizing the common past of the particles and \( \Lambda \) for the space of its allowed values. The question is whether the actually observed joint probability (1) admits the representation (2). We shall present an analytical and a numerical argument. The analytical treatment gives a precise value for the threshold visibility, but involves some additional assumptions concerning the nature of the hidden variable. In case of the numerical treatment we do not rely upon these additional assumptions, but the resulting value of the threshold visibility is determined only up to a finite accuracy. The analytical result coincides with the numerical one within this accuracy.

In case of the analytical argument we make the following three assumptions:

1. As we have identical particles,
   \[
P_A(m|m, \lambda) = P_B(-m|m, \lambda).
   \]

2. As \( P_A(m|m, \lambda) \) is a scalar, rotational invariance suggests that \( \lambda \) is actually a unit vector, \( \lambda \) and
   \[
P_A(m|m, \lambda) = f(m \mathbf{n} \lambda)
   \]

3. Rotational invariance also requires that there is no distinguished direction, thus
   \[
   \rho(\lambda)d\lambda = \frac{1}{4\pi}d\Omega,
   \]

where \( d\Omega \) stands for an infinitesimal spatial angle around the direction \( \lambda \).

Let us represent the function \( f(x) \) (cf. Eq.(6), note that \( -1 \leq x \leq 1 \)) as a series of Legendre polynomials:

\[
f(x) = \sum_{j=0}^{\infty} c_j P_j(x)
\]

Here \( c_j \)'s are real coefficients.

Inserting Eqs.(3)-(6) into Eq.(2) and using the identity

\[
\int P_j(\alpha \lambda) P_k(\beta \lambda) d\Omega = \delta_{jk} \frac{4\pi}{2j+1} P_j(\alpha \beta)
\]

we have
\[ P_{HV}(m, m'; \vec{a}, \vec{b}) = \sum_{j=0}^{\infty} \frac{c_j^2}{2j+1} P_j(-mm'\vec{a}\vec{b}) \]  
(8)

On the other hand, Eq. (4) can be written as
\[ P_{QM}(m, m'; \vec{a}, \vec{b}) = \frac{1}{4} P_0(-mm'\vec{a}\vec{b}) + \frac{V}{4} P_1(-mm'\vec{a}\vec{b}) \]  
(9)

Comparing this with Eq. (8) we get that \( c_0 \) and \( c_1 \) are the only nonzero coefficients and
\[ c_0 = \pm \frac{1}{2} \]  
(10)
\[ c_1 = \pm \sqrt{3V} \]  
(11)

If \( \vec{a} \) and \( \vec{\lambda} \) are perpendicular, \( P_1(-mm'\vec{a}\vec{\lambda}) \) is zero, thus the positivity of \( P_\lambda(m|\vec{a}, \lambda) \) requires (cf. Eqs. (3), (4)) that in Eq. (10) the positive sign must be chosen. If \( V = \pm \vec{a} \), then \( P_1(-mm'\vec{a}\vec{\lambda}) = \pm 1 \), thus the positivity of \( P_\lambda(m|\vec{a}, \lambda) \) requires that \( c_0 \pm c_1 \geq 0 \), i.e., by Eqs. (10), (11)
\[ V \leq \frac{1}{3} \]  
(12)

As \( |P_1(x)| \leq 1 \), Eq. (12) ensures the positivity of \( P_\lambda(m|\vec{a}, \lambda) \), \( P_B(m'|\vec{b}, \lambda) \) for any directions \( \vec{a}, \vec{b}, \vec{\lambda} \). This readily implies that the threshold visibility is 1/3, as below this value the observed joint probability (1) admits a local realistic representation (2), while above this value the positivity of the conditional probabilities \( P_\lambda(m|\vec{a}, \lambda) \), \( P_B(m'|\vec{b}, \lambda) \) cannot be achieved.

Although the assumptions (3)-(5) look reasonable, one might suspect that the above result hinges upon these assumptions and perhaps without them the joint probability (1) admits a local realistic representation even at higher visibilities. We present a numerical argument that it is not the case.

The numerical procedure begins with choosing \( N \) directions for \( \vec{a} \) and \( N \) directions for \( \vec{b} \) and then equating Eqs. (1) and (2). In the latter the integral is estimated by a sum, i.e., the equation
\[ \frac{1}{4}(1 - mm'V\vec{a}_j\vec{b}_k) = \sum_{n=1}^{M} \rho_n P_A(m|\vec{a}_j, n)P_B(m'|\vec{b}_k, n) \]  
(13)

is to be solved. This is equivalent with
\[ V\vec{a}_j\vec{b}_k = \sum_{n=1}^{M} \rho_n A_{j,n}B_{k,n} \]  
(14)

where
\[ A_{j,n} = 1 - 2 P_A(1|\vec{a}_j, n) \]
\[ B_{k,n} = 2 P_B(1|\vec{b}_k, n) - 1 \]  
(15)

which satisfy
\[ |A_{j,n}| \leq 1 \]
\[ |B_{k,n}| \leq 1 \]  
(16)

and
\[ \sum_{n=1}^{M} \rho_n A_{j,n} = 0 \]
\[ \sum_{n=1}^{M} \rho_n B_{k,n} = 0 \]  
(17)

At the solution of Eq. (14) one can utilize the singular value decomposition
\[ \vec{a}_j\vec{b}_k = \sum_{i=1}^{3} \rho_i U_{j,i}V_{k,i} \]  
(18)

where \( U_{j,i} \) and \( V_{k,i} \) are orthogonal matrices. Note that there are only three nonzero singular values \( \rho_n \), owing to the three dimensionality of the vectors \( \vec{a}_j, \vec{b}_k \). A solution of Eqs. (14)-(17) can be then obtained in the following way:

1. One chooses three \( M \) dimensional vectors \( \vec{\tilde{a}}_i \) and another three \( M \) dimensional vectors \( \vec{\tilde{t}}_i \) and makes them orthogonal to each other,
\[ \vec{\tilde{a}}_i\vec{\tilde{t}}_j = \delta_{i,j} \]  
(19)

and to the vector \( (\sqrt{\rho_1}, \sqrt{\rho_2}, \ldots, \sqrt{\rho_n})^T \), i.e.
\[ \sum_{n} \sqrt{\rho_n}(\vec{\tilde{a}}_i)_n = 0 \]
\[ \sum_{n} \sqrt{\rho_n}(\vec{\tilde{t}}_i)_n = 0 \]  
(20)

2. Calculate
\[ A'_{j,n} = \sum_{i=1}^{3} U_{j,i}\sqrt{\rho_i}(\vec{\tilde{a}}_i)_n/\sqrt{\rho_n} \]
\[ B'_{k,n} = \sum_{i=1}^{3} V_{k,i}\sqrt{\rho_i}(\vec{\tilde{t}}_i)_n/\sqrt{\rho_n} \]  
(21)

3. A solution of Eq. (14) is given by
\[ A_{j,n} = \sqrt{V} A'_{j,n} \]
\[ B_{k,n} = \sqrt{V} B'_{k,n} \]  
(22)

4. Eq. (18) implies that
\[ 1/\sqrt{V} = \max_{j,n} \{ |A'_{j,n}|, |B'_{j,n}| \} \]  
(23)
Applying this scheme, a Monte-Carlo simulation has been used to find the threshold visibility. The steps of the simulation are the following:

1. Choose the unit vectors $\vec{a}_j, \vec{b}_k$ at random
2. Choose the vectors $\vec{q}_i, \vec{t}_i$ at random and normalize them according to Eqs. (19), (20).
3. Choose $\rho_n$ at random and normalize by
   \[ \sum_n \rho_n = 1 \]  
   (24)

4. Calculate $V$ from Eqs. (21), (23).
5. Change a component of $\vec{q}_i$, $\vec{t}_i$ or $\rho_n$ at random and repeat the calculation of $V$. If the resulting value is larger than the previous one, keep the changes, otherwise discard them.
6. After having found the maximal $V$ for fixed $\vec{a}_j, \vec{b}_k$, change these vectors at random and find (by repeating the previous steps) the corresponding $V$ again. The minimum of these $V$ values is selected, because at the corresponding setting for $\vec{a}_j, \vec{b}_k$ a local realistic representation of the observed joint probabilities at higher visibility cannot be given.

Obviously, one is interested in the limit $N \to \infty$. For a finite $N$ one has less restrictions, thus the resulting estimate for the threshold visibility is higher. By increasing $N$ the estimates decrease monotonously. As for the number $M$, numerically it turned out that increasing its value makes the convergence slower but does not influence the results for $V$. Hence one may set it to the minimum $M = 4$. The values for $N$ have been changed from 3 to $10^4$. The convergence proved to be rather slow, e.g., the estimate for the threshold visibility at $N = 1000$ is still $0.37 \pm 0.001$. The final numerical result (extrapolation for $N \to \infty$) is $0.33 \pm 0.03$. This is consistent with the previous analytical result.

It is instructive to compare these results with those one may obtain from Bell’s inequality and the Clauser-Horne-Shimony-Holt inequality. Since these inequalities do not exploit the full angular dependence, one gets a higher estimate for the threshold visibility. Note, however, that at the derivation of Bell’s inequality

\[ P_{HV}(1, 1; \vec{a}, \vec{b}) + P_{HV}(1, 1; \vec{b}, \vec{c}) \geq P_{HV}(1, 1; \vec{a}, \vec{c}) \]  
(25)

strict anticorrelation (i.e., $P_{HV}(1, 1; \vec{b}, \vec{b}) = 0$) implied by the singlet state is also assumed. Inserting Eq. (1) into Eq. (25) we have

\[ \frac{V}{2} \left(3 - (\vec{a} + \vec{c} - \vec{b})^2\right) \leq 1. \]  
(26)

The l.h.s. will be the largest when $\vec{a} + \vec{c} - \vec{b} = 0$. This implies that the threshold visibility $V$ is not larger than $2/3 \approx 0.667$, as below this value Eq. (26) is always satisfied, while for a larger value of $V$ it may be violated for suitably chosen directions $\vec{a}, \vec{b}, \vec{c}$. This value is smaller than the previously known smallest value $3/4 = 0.75$, but here the additional information about the properties of the singlet state has also been utilized. This is equivalent with the assumption that in the special case of $\vec{a} = \vec{b}$ (cf. Eq. (1)) the visibility is $100\%$. Using only the requirements of local realism together with the full angular dependence we obtained above the much lower threshold visibility $1/3 \approx 0.33$.

Let us consider now the Clauser-Horne-Shimony-Holt inequality

\[ P_{HV}(1, 1; \vec{a}, \vec{b}) - P_{HV}(1, 1; \vec{a}, \vec{b}') + P_{HV}(1, 1; \vec{a}', \vec{b}) \]
\[ + P_{HV}(1, 1; \vec{a}', \vec{b}') - P_{HV}(1, 1; \vec{a}'' \vec{b}'') \leq 0 \]  
(27)

Here $P_{HV}(1; \vec{a}'') = \sum_m' P_{HV}(1, m'; \vec{a}', \vec{b})$ and $P_{HV}(1; \vec{b}) = \sum_m P_{HV}(m; 1, \vec{a}', \vec{b})$. Inserting Eq. (1) into Eq. (27), the resulting inequality can be cast to the form

\[ \frac{V}{2} \left(\vec{a} + \vec{b}' - \vec{b}^2 + (\vec{a}' - \vec{b}' - \vec{b})^2 - 6\right) \leq 2. \]  
(28)

For fixed $V$, $\vec{b}$ and $\vec{b}'$ the l.h.s of inequality (28) is the largest if $\vec{a}$ is parallel to $\vec{b}' - \vec{b}$ and $\vec{a}'$ is antiparallel to $\vec{a} + \vec{b} - \vec{b}'$ (similarly in case of $\vec{a}'$). Thus, denoting the angle between $\vec{b}$ and $\vec{b}'$ by $\varphi$, we get from Eq. (28) the inequality

\[ 2\sqrt{2}V \sin \left(\frac{\varphi}{2} + \frac{\pi}{4}\right) \leq 2 \]  
(29)

This readily implies that the l.h.s. is maximal (for a fixed $V$) if $\varphi = \frac{\pi}{2}$ and the threshold visibility is not larger than $1/\sqrt{2} \approx 0.707$. Note that even this value is smaller than the previously obtained smallest value $3/4 = 0.75$. The CHSH inequality relies only upon the assumption of local realism, but does not exploit the full angular dependence. As we have seen, taking into account the latter, too, a much lower value for the threshold visibility can be deduced.

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