Slopes of a Hypergeometric System
Associated to a Monomial Curve

Francisco Jesús Castro-Jiménez* and Nobuki Takayama

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1 Introduction

Let \( A_n = \mathbb{C}(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n) \) be the Weyl algebra of order \( n \) over the complex numbers \( \mathbb{C} \) and \( \mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n] \) the subring of \( A_n \) of linear differential operators with constant coefficients.

Let \( A = (a_{ij}) \) be an integer \( d \times n \)-matrix of rank \( d \). We denote by \( I_A \subset \mathbb{C}[\partial] \) the toric ideal associated to \( A \): i.e., \( I_A \) is the ideal generated by the set

\[
\{ \partial^u - \partial^v \mid u, v \in \mathbb{N}^n, \, Au = Av \}
\]

where \( (\cdot)^T \) means “transpose”.

We denote by \( \theta \) the vector \( (\theta_1, \ldots, \theta_n)^T \) with \( \theta_i = x_i \partial_i \). For a given \( \beta = (\beta_1, \ldots, \beta_d)^T \in \mathbb{C}^d \) we consider the column vector \((\mathbb{A}^d \theta) A \theta - \beta \) and we denote by \( \langle A \theta - \beta \rangle \) the left ideal of \( A_n \) generated by the entries of \( A \theta - \beta \).

Following Gel’fand, Kapranov and Zelevinsky [3], we denote by \( H_A(\beta) \) the left ideal of \( A_n \) generated by \( I_A \cup \langle A \theta - \beta \rangle \). It is called the GKZ-hypergeometric system associated to the pair \((A, \beta)\). The quotient \( H_A(\beta) = A_n / H_A(\beta) \) is a holonomic \( A_n \)-module (see e.g. [17]).

If the toric ideal \( I_A \) is homogeneous, i.e., if the \( \mathbb{Q} \)-row span of \( A \) contains \((1, \ldots, 1)\), it is known ([3], see also [17]) that \( H_A(\beta) \) is regular holonomic and the book [17] is devoted to an algorithmic study of such systems. Especially, the book gives an algorithmic method to construct series solutions around singular points of the system.

In this article, we would start a study of singularities of GKZ-hypergeometric systems for non homogeneous toric ideals \( I_A \) by treating the “first” case when \( d = 1 \), \( A = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \), \( a_1 = 1 \). We evaluate the geometric slopes of \( H_A(\beta) \) by successive restrictions of the number of variables. The slopes characterize Gevrey class solutions around singular locus.

Our evaluation of the slopes is done as follows: (1) We translate Laurent and Mebkhout’s theorem [12] on restrictions and slopes of \( \mathcal{D} \)-modules

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into an algorithm to evaluate the slopes by utilizing the results \[2\] and \[13\].

(2) Apply our general algorithm to the hypergeometric system associated to 
\[A = (1, a_2, \ldots, a_n)\]. This system has a lot of nice properties and our algorithm
outputs the slopes without computation on computers.

2 Micro-characteristic varieties

In this section following Laurent \[10\], we describe micro-characteristic varieties
for a given \(\mathcal{D}\)-module. We will state a result of Laurent and Mebkhout \[12\]
Corollaire 2.2.9 (see also \((14), p. 125)\) and \((11), p. 42)\), allowing to reduce
our general problem of evaluating the slopes to less variables.

In this section \(X = \mathbb{C}^n\) and \(\mathcal{D}_X = \mathcal{D}\) is the sheaf of linear differential
operators with holomorphic function coefficients. Let \(M\) be a coherent \(\mathcal{D}_X\)
-module. Recall that the characteristic variety of \(M\) (denoted by \(\text{Ch}(M)\)) is a
analytic sub-variety of the cotangent bundle \(T^*X\).

Suppose that \(Y \subset X\) is a smooth hypersurface. We say that \(Y\) is non-characteristic
for \(M\) if \(T^*_Y X \cap \text{Ch}(M) \subset T^*_X X\). Here \(T^*_Y X\) is the conormal
bundle to \(Y\) in \(X\) and \(T^*_X X\) is the zero section of \(T^*X\).

Now, following Laurent \[9, 10\], we shall define the notion of non-micro-characteristic variety
for \(M\). To simplify the presentation we will assume that
\((x_1, x_2, \ldots, x_n)\) are local coordinates in \(X\) and \(Y\) is defined by \(x_n = 0\). We
denote by \((x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)\) local coordinates in \(T^*X\). Sometimes it will
be useful to write \(x_1 = y_1, \ldots, x_{n-1} = y_{n-1}, x_n = t, \xi_1 = \eta_1, \ldots, \xi_{n-1} = \eta_{n-1}\)
and \(\xi_n = \tau\).

Let us denote by \(\Lambda\) the conormal bundle of \(Y\) in \(T^*X\) (i.e. \(\Lambda = T^*_Y X\)). So, in
local coordinates \(\Lambda = \{(y, t, \eta, \tau) \in T^*X \mid t = \eta = 0\}\). Here, \(\eta = (\eta_1, \ldots, \eta_{n-1})\)
and \(y = (y_1, \ldots, y_{n-1})\). We denote by \((y, \tau, y^*, \tau^*)\) local coordinates on the
cotangent bundle \(T^*\Lambda\).

We denote by \(V_\bullet(\mathcal{D})\) (or simply by \(V\)) the Malgrange-Kashiwara filtration
associated to \(Y\) on \(\mathcal{D}\) and by \(F_\bullet(\mathcal{D})\) (or simply by \(F\)) the order filtration on
\(\mathcal{D}\). For a given rational number \(p/q \geq 0\) we denote by \(L_{p/q}\) the filtration on \(\mathcal{D}\)
defined by \(pF + qV\). The \(L_{p/q}\)-order of a monomial \(y^a t^b \partial_y^j \partial_t^k\) is equal to
\(p(|\beta| + k) + q(k - l)\), where \(|\beta| = \beta_1 + \cdots + \beta_{n-1}\). We will simply write \(L = L_{p/q}\)
if no confusion arises.

For \(p > 0\) the associated graded ring \(\text{gr}^L(\mathcal{D})\) is canonically isomorphic to
\(\pi_* \mathcal{O}_{T^*\Lambda}\) \[16\], p. 407\], where \(\pi : T^*\Lambda \to \Lambda\) is the canonical projection
and \(\mathcal{O}_{T^*\Lambda}\) denotes holomorphic functions on \(T^*\Lambda\) which are polynomials on the
fibers of \(\pi\). In local coordinates, \(\text{gr}^L(\mathcal{D}_0)\) is expressed as \(\mathbb{C}[y][\tau, y^*, \tau^*]\) where
\(\mathcal{D}_0\) is the stalk of \(\mathcal{D}\) at the origin.

Given a differential operator
\(P = \sum_{a_1, \ldots, a_k} p_{a_1, \ldots, a_k} y^{a_1} t^{a_2} \partial_y^{a_3} \partial_t^{a_4}\)
the \(L\)-order of \(P\) is the maximum value of \(p(|\beta| + k) + q(k - l)\) over the monomials
of $P$. For $p > 0$ we define the $L$-principal symbol of $P$ by
\[ \sigma^L(P) = \sum p_{\alpha \beta k} y^\alpha (\tau^*)^k (y^*)^\beta (-\tau)^k \]
where the sum is taken over monomials with maximal $L$-order. The $L$-principal symbol of $P$ is an element of $\text{gr}^L(D)$ and then is a function on $T^*\Lambda$. In the classical case, i.e., for $L = F$, $\text{gr}^F(D)$ is identified with $\mathbb{C}\{x\}[\xi_1, \ldots, \xi_n] = \mathbb{C}\{y, t\}[\eta, \tau]$ and the $F$-principal symbol of $P$ is simply denoted by $\sigma^F(P) = \sum p_{\alpha \beta k} y^\alpha t^\beta \eta^k$ where the sum is taken for $|\beta| + k$ maximum.

For each left ideal $I \subset D$ we denote by $\sigma^L(I)$ the ideal of $\text{gr}^L(D)$ generated by the set of $\sigma^L(P)$ for $P \in I$.

For each $L$-filtration on $D$ we associate a “good” $L$-filtration on $M$, by means of a finite presentation. The associated $\text{gr}^L(D)$-module $\text{gr}^L(M)$ is coherent (see \cite{[12]} 3.2.2). The radical of the annihilating ideal $\text{Ann}_{\text{gr}^L(D_X)}(\text{gr}^L(M))$, which is independent of the “good” filtration on $M$, defines an analytic sub-variety of $T^*\Lambda$. This variety is called the $L$-characteristic variety of $M$ and it is denoted by $\text{Ch}^L(M)$.

Suppose now $Z \subset X$ is a smooth hypersurface transverse to $Y$. Suppose for simplicity $Z$ is defined in local coordinates by $y_1 = 0$. The conormal space $\Lambda' := T'_Y \cap Z$ is a smooth subvariety of $\Lambda = T'_X X$ defined in local coordinates by $y_1 = 0$. So $T'_X \Lambda$ is the subvariety of $T^*\Lambda$ defined in local coordinates by $y_1 = y'^* = \tau^* = 0$, where $y' = (y_2, \ldots, y_{n-1})$.

**Definition 2.1** We say that $Z$ is non-micro-characteristic of type $L$ for $M$ if $T'_X \Lambda \cap \text{Ch}^L(M)$ is contained in $T'_X \Lambda$. Sometimes we will say that, if this condition holds, $Z$ is non-$L$-micro-characteristic for $M$.

The sheaf of rings $\text{gr}^L(D)$ is endowed with two graduations: First one is induced by the $F$-filtration and the second one is induced by $V$-filtration. Recall Laurent’s definition of slope of a coherent $D$-module $M$.

**Definition 2.2** The rational number $-p/q$ is said to be a slope of $M$ w.r.t. $Y$ at the origin if and only if the radical of the ideal $\text{Ann}_{\text{gr}^L(D_X)}(\text{gr}^L(M))$ is not bi-homogeneous for $F$ nor $V$ graduations.

An important consequence of the work \cite{[11]} (see Th. 2.4.2) is what follows: An holonomic $D_X$-module $M$ is regular w.r.t. $Y$ at the origin if and only if $M$ has no slope w.r.t. $Y$ at the origin. We will use this fact freely in the text.

**Remark:** In \cite{[12]}, $\infty$ and 0 ($F$ and $V$) are included in the set of the slopes. We do not include them in the set of the slopes in this paper.

Finally the following result by Laurent and Mebkhout allows induction on the number of variables to calculate slopes.

**Theorem 2.3** \cite{[12]} Corollaire 2.2.9 Let $M$ be a holonomic $D_X$-module. Let $Z$ and $Y$ be transverse smooth hypersurfaces on $X$ such that $Z$ is non $L$-micro-characteristic (for all $p > 0, q > 0$) for $M$. Then the slopes of $M$ w.r.t. $Y$ equals the slopes of $M'$ w.r.t. $Y' = Z \cap Y$ where $M'$ is the restriction of $M$ to $Z$.  

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This is a deep result in $\mathcal{D}$-module theory. Its proof uses the algebraic-analytic comparison theorem ([11, Th. 2.4.2]) and a Cauchy-Kowalewska theorem for Gevrey functions w.r.t. $\mathcal{Z}$ ([12, Corollaire 2.2.4]; see also [14, Th. 6.3.4].)

3 Computing slopes by reducing the number of variables

We have introduced the notion of the slopes and the invariance of them under restrictions satisfying a condition on $L$. We have assumed that our ideal is that of the Weyl algebra $A_n$. Constructions in sheaves such as restrictions, micro-characteristic varieties in the previous section can be done via constructions in the Weyl algebra as we usually see in the computational $\mathcal{D}$-module theory.

We are interested in computation of the slopes. The slopes of $A_n/I$ (at the origin) along $x_n = 0$ can be computed by the ACG algorithm introduced in [4]. In this section, translating Laurent and Mebkhout’s result into computer algebra algorithms, we will give a preprocessing method for the ACG algorithm to accelerate the original. The preprocessing is useful for a class of inputs including GKZ hypergeometric ideals as we will see in Section 4. Let us firstly recall the ACG algorithm.

A weight vector is an element $W = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ such that $u_i + v_i \geq 0$ for all $i$. This weight vector $W = (u, v)$ induces a natural filtration on $A_n$ and it is called the $W$-filtration. The associated graded ring is denoted by $\text{gr}_W(A_n)$ and for each left ideal $I \subset A_n$ the associated graded ideal is denoted by $\text{in}_W(I)$ or $\text{in}_{(u,v)}(I)$. Here, the initial ideal $\text{in}_{(u,v)}(I)$ is the ideal generated by $\text{in}_{(u,v)}(f), f \in I$ in $\text{gr}_W(A_n)$. When $u_i + v_i > 0$, it is an ideal in the polynomial ring of $2n$ variables: $\text{gr}_W(A_n) = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. The initial ideal of $I$ with respect to the weight $(u, v)$ is generated by the $(u, v)$-initial terms of a Gröbner basis of $I$ by an order which refines the partial order defined by $(u, v)$. See, e.g., [4, Theorem 1.1.6].

Consider the filtration $L = pF + qV, p > 0, q > 0$ introduced in the previous section. The ideal $\sigma^L(I)$, which gives the $L$-characteristic variety, can be expressed in terms of the initial ideal as follows:

$$\sigma^L(I) = \text{gr}^L(D) \cdot \text{in}_\ell(I)|_{x_1 \rightarrow y_1, \ldots, x_{n-1} \rightarrow y_{n-1}, x_n \rightarrow \tau, \xi_1 \rightarrow y_1', \ldots, \xi_{n-1} \rightarrow y_{n-1}', \xi_n \rightarrow \tau'}$$

$$\ell = p(0, \ldots, 0, 1, \ldots, 1) + q(0, \ldots, 0, -1, 0, \ldots, 0, 1).$$

For two weight vectors $W$ and $W'$ and a term order $<$, we denote by $<_{W, W'}$ the order

$$x^\alpha \partial^\beta <_{W, W'} x^\alpha \partial^\beta$$

$$\Leftrightarrow \quad W \cdot (\alpha, \beta) < W \cdot (a, b)$$

or $W \cdot (\alpha, \beta) = W \cdot (a, b)$ and $W' \cdot (\alpha, \beta) < W' \cdot (a, b)$

or $W^* \cdot (\alpha, \beta) = W^* \cdot (a, b)$ for both $W^* = W, W'$ and $x^\alpha \partial^\beta < x^a \partial^b$. 

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To each differential operator $P = \sum p_{\alpha \beta} x^\alpha \partial^\beta \in A_n$ we associate the Newton polygon $N(P)$ of $P$ (w.r.t. $x_n = 0$) defined as the convex hull of the subset of $\mathbb{Z}^2$ \[ \bigcup_{p_{\alpha \beta} \neq 0} (|\beta|, \beta_n - \alpha_n) + (-N)^2. \]

Let $I$ be a left ideal in $A_n$. As we said in Definition 2.2 the notion of slope of a differential system was introduced by Y. Laurent [10]. Let us give here a slight different but equivalent definition: the number $r$, $-\infty < r < 0$ is a geometric slope of $I$ (or of $A_n/I$) w.r.t. $x_n = 0$ if and only if $\sqrt{\sigma(-r)F+V(I)}$ is not bihomogeneous with respect to the weight vectors $F = (0, \ldots, 0, 1, \ldots, 1)$ and $V = (0, \ldots, 0, -1, 0, \ldots, 0, 1)$. Following [2], we say that the number $r$, $-\infty < r < 0$ is an algebraic slope of $I$ (or of $A_n/I$) if and only if $\sigma(-r)F+V(I)$ is not bihomogeneous with respect to the weight vectors $F$ and $V$. The geometric slope is simply called the slope in this paper if confusion does not arise. Note that we may consider in $L(I)$ instead of $\sigma L(I)$ as far as we are concerning about homogeneity. For algebraic or geometric slope $r$, the weight vector $L = (-r)F + V$ lies on a face of the Gröbner fan of $I$ ([3], [17]), which yields the following algorithm.

**Algorithm 3.1** ([2], ACG algorithm)

**Input:** $G = \{P_1, \ldots, P_m\}$ (generators of an ideal $I$)

**Output:** All algebraic and geometric slopes of $A_n/I$ w.r.t. $x_n = 0$ at the origin.

```
geometric_slope = \emptyset; algebraic_slope = \emptyset;
F = (0, \ldots, 0, 1, \ldots, 1); V = (0, \ldots, 0, -1, 0, \ldots, 0, 1);
p = 1; q = 0; slope = -\infty; previous_slope = slope;
while (slope! = 0) {
    L = pF + qV;
    G = a Gröbner basis of $I$ with respect to the order $<_{L,V}$;
    slope = the minimum of 0 and
    \{ the slopes $r$ of the Newton polygon $N(P)|P \in G, r > previous_slope$\}
    if slope = 0, then return(algebraic_slope and geometric_slope).
    if $\sigma^L(G)$ is not homogeneous for $F$ nor $V$ then {
      algebraic_slope = algebraic_slope \cup \{slope\}
    }
    if $\sqrt{\sigma^L(G)}$ is not homogeneous for $F$ nor $V$ then {
      geometric_slope = geometric_slope \cup \{slope\}
    }
    p = numerator(|slope|); q = denominator(|slope|);
    previous_slope = slope;
}
return (algebraic_slope and geometric_slope)
```

Here, we use the convention $F := \infty F + V$. Note that the ideal $J = \langle f_1, \ldots, f_m \rangle \subset \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ is homogeneous for the weight $(u, v) \in \mathbb{Z}^{2n}$ if and only if all $(u, v)$-homogeneous subsums of $f_i$ belong to the ideal $J$. 

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For ordinary differential equations, the ACG algorithm is nothing but the well-known Newton polygon method; see two examples below.

**Example 3.2** Put \( G = \{ x_1^{p+1} \partial_1 + p \} \), \( n = 1, p \in \mathbb{N} \). Then, the ACG algorithm returns \( \{-p\} \). ( \( \exp(x_1^{-p}) \) is a classical solution of \( G \).)

**Example 3.3** Put \( G = \{ 2x_1(x_1 \partial_1)^2 + x_1^2 \partial_1 + 1 \} \), \( n = 1 \). Then, the ACG algorithm returns \( \{-1/2\} \). ( \( \exp(x_1^{-1/2}) \) is a classical solution of \( G \).)

Next example of two variables is generated by a computer algebra system **kan/k0**.

**Example 3.4** We consider the GKZ hypergeometric ideal \( I \) associated to the matrix \( A = (1, 3) \) and \( \beta = -3 \). We will compute the slopes of \( A_2/I \) at the origin along \( x_2 = 0 \) by the ACG algorithm. The Gröbner basis of \( I \) for the weight \((x_1 \, x_2 \, \partial_1 \, \partial_2)\) is

\[
\begin{align*}
&x_1 \partial x_1 + 3 x_2 \partial x_2 + 3, \\
&-2x_1 \partial x_1^3 + x_2 \partial x_2, \\
&-3x_2 x_1 \partial x_1^2 \partial x_2 - 5 \partial x_1^2 - 2 - 3 x_1 \partial x_2, \\
&-9x_2^2 \partial x_1 \partial x_2^2 - 36x_2 \partial x_1 + 4x_2 \partial x_1^2 - 10x_2 - 20x_1, \\
&-27x_2^3 \partial x_2^3 - 189x_2^2 \partial x_2 - x_1^3 \partial x_2 - 276x_2^2 \partial x_2 - 60
\end{align*}
\]

Here, \( \partial x_i \) and \( x_i \) stand for \( \partial_i \) and \( x_i \) respectively. The Newton polygons \( N(P) \)'s are figured below.

![Newton Polygons](image)

From the Newton polygons, \(-1/2\) is the candidate of the first slope. Next, we compute the Gröbner basis for the weight \((0, -2, 1, 3) = (0, 0, 1, 1) + 2(0, -1, 0, 1)\). The Gröbner basis is
The radical is generated by

\[
\begin{align*}
&[\, -x_1^*D_{x_1} - 3x_2^*D_{x_2}, -D_{x_1} - 3x_2^*D_{x_2}, 2x^2D_{x_2}^2 + x_1^*D_{x_2},
& \qquad -9x_2^*D_{x_2}^2 + x_1^*D_{x_2}, 27x_2^*D_{x_2}^3 + x_1^*D_{x_2}^2 \,]
\end{align*}
\]

Here, \( x_1 = y_1^*, x_2 = \tau^*, D_{x_1} = y_1^*, D_{x_2} = -\tau \). It is not bi-homogeneous and then \(-1/2\) is a geometric and algebraic slope.

By looking at the two Newton polygons of \( x_1^*D_{x_1} + 3x_2^*D_{x_2} + 3, D_{x_2} - D_{x_1}^3 \), we see that there is no more slope that is larger than \(-1/2\). Then, the ACG algorithm terminates here.

The ACG algorithm requires a repetition of Gröbner basis computations in the Weyl algebra of \( 2n \) variables to evaluate the slopes of \( A_n/I \). However, if \( x_i = 0 \) is non-micro-characteristic of type \( L = (-r)F + V \) for all \(-\infty < r < 0\) and the restriction of \( A_n/I \) to \( x_i = 0 \) is singly generated, we can preprocess the input so that the input ideal for the ACG algorithm lies in the Weyl algebra of \( 2(n - 1) \) variables. The correctness of the following algorithm can be shown by Laurent and Mebkhout’s theorem 2.3.

**Algorithm 3.5** (Computing slopes with a preprocessing)

Step 1: check if \( x_i = 0 \) is non-micro-characteristic of \( A_n/I \) for all types \( L \) by calling Algorithm 3.6.

Step 2: Compute the restriction of \( A_n/I \) to \( x_i = 0 \) and check if it is expressed as \( D'/I' \) where \( D' \) is the Weyl algebra of \( 2(n - 1) \) variables.

Step 3: If we failed either in Step 1 or in Step 2, then apply the ACG algorithm for \( I \).

If we succeeded both in step 1 and in step 2, then try to reduce more variables or apply the ACG algorithm for \( I' \).

We can compute restrictions of a given \( \mathcal{D} \)-module by using Oaku’s algorithm [13]. This algorithm is implemented in computer algebraic systems Macaulay2 and Kan [1], [13], [13]. Therefore, the remaining algorithmic question for the preprocessing is to determine the range of type \( L = (-r)F + V \) for which \( x_i = 0 \) (\( i \leq n - 1 \)) is non-micro-characteristic. It follows from the definition of non-micro-characteristic that the question is nothing but to find the segment \(( -\infty, r_1 ) \) such that

\[
\mathcal{V}(\sigma^{(-r)F+V}(I), y_1^*, \ldots, y_{i-1}^*, y_i, y_{i+1}^*, \ldots, y_{n-1}^*, \tau^*) \subseteq \mathcal{V}(y_1^*, \ldots, y_{n-1}^*, \tau^*) = T_{A'}^* \Lambda
\]

for \( r \in (-\infty, r_1) \). Here, \( \mathcal{V}(f_1, \ldots, f_m) \) is the affine variety defined by the polynomials \( f_1, \ldots, f_m \). The inclusion condition can algebraically rephrased as

\[
\sqrt{\text{in}(-r)F+V}(I), \xi_1, \ldots, \xi_{i-1}, x_i, \xi_{i+1}, \ldots, \xi_n \ni \xi_i.
\]

Since the Gröbner fan is a finite union of Gröbner cones [2], the range can be determined by a similar method with the ACG algorithm.
Algorithm 3.6
\textbf{range\_of\_nonMC}(H, r_0)
\begin{itemize}
\item \textbf{Input:} \( H \) is a finite set in \( A_n \), \( r_0 \) is a negative number or \(-\infty\)
\item \textbf{Output:} \( r_1 \) such that \( x_1 = 0 \) is non-micro-characteristic of type \((-r)F + V\)
for \( A_n/A_n \cdot \{H\} \), for \( r \in [r_0, r_1) \).
\end{itemize}
\begin{itemize}
\item previous\_slope = slope = \( r_0 \);
\item \( F = (0, \ldots, 0, 1, \ldots, 1); \ V = (0, \ldots, 0, -1, 0, \ldots, 0, 1); \)
\item \textbf{while} (slope = 0) { \}
\begin{itemize}
\item \( p = \text{numerator}(|\text{slope}|); q = \text{denominator}(|\text{slope}|); \)
\item \( L = pF + qV; \)
\item \( G = \text{a Gröbner basis of} \ H \text{ with respect to } <_L; \)
\item \( \sqrt{(\sigma^L(H), y_1, y_2, \ldots, y_{n-1}, \tau^*) \ni y^*_1} \) and
\( \sqrt{(\sigma pF + (q + \varepsilon)V(H), y_1, y_2, \ldots, y_{n-1}, \tau^*) \ni y^*_1} \) \)
\end{itemize}
then { \}
\begin{itemize}
\item previous\_slope = slope;
\item slope = the minimum of 0 and \{ the slopes \( r \) of the Newton polygon \( N(P) | P \in G, r > \text{previous\_slope} \} \}
\end{itemize}
\} else { \}
\begin{itemize}
\item return(slope);
\end{itemize}
\} \}
\textbf{return}(0); \)
\end{itemize}

Here, \( \varepsilon \) is a sufficiently small positive rational number so that \( pF + (q + \varepsilon)V \)
lies in the interior of a Gröbner cone.

When \( r_0 = -\infty \), we use the convention \( \text{numerator}(|r_0|) = 1 \) and \( \text{denominator}(|r_0|) = 0 \) and \( F\text{-non-micro-characteristic} \) means that it is non-characteristic in the classical sense.

**Example 3.7** Suppose \( n \geq 2 \). \( \text{range\_of\_nonMC}(\{\partial^2_2 - \partial_2\}, -\infty) \) returns 0.

If the function \( \text{range\_of\_nonMC}(I, -\infty) \) returns 0, then \( x_1 = 0 \) is non-micro-characteristic of type \( pL + qV \) \( (p > 0, q \geq 0) \) for \( A_n/I \). We note that it is not always a clever strategy to call the function with the full set of generators. In fact, if \( x_1 = 0 \) is non-micro-characteristic of type \( L \) for \( A_n/J \), then it is non-micro-characteristic of type \( L \) for \( A_n/I \) for any \( I \supset J \). Therefore, for step 1, it is sometimes more efficient to call the function \( \text{range\_of\_nonMC} \) for a subset of the generators of the input ideal as we will see in case of GKZ hypergeometric ideals in the next section.

**4 Computing slopes of \( \mathcal{H}_{(a_1, a_2, \ldots, a_n)}(\beta) \)**

Put \( A = (1, a_2, \ldots, a_n), a_1 = 1 < a_2 < \cdots < a_n \). We will evaluate the slopes of the GKZ hypergeometric ideal \( H_A(\beta) \) associated to the \( 1 \times n \) matrix \( A \) and \( \beta \in \mathbb{C} \) by using the general algorithm given in Section 3. To apply this algorithm, we need to find non-microcharacteristic varieties and compute the
restrictions of $\mathcal{H}_A(\beta)$ to these varieties. When $f_1, \ldots, f_m$ are polynomials in $\mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$, we denote by $\mathcal{V}(f_1, \ldots, f_m)$ the affine subvariety in $\mathbb{C}^{2n}$ defined by the $f_i$.

The following theorem can be shown by a standard method of Koszul complex ([1], [2]).

**Theorem 4.1** The characteristic variety of $\mathcal{H}_A(\beta)$ is $\mathcal{V}(\xi_1, \ldots, \xi_{n-1}, x_n \xi_n)$. In particular, the singular locus of $\mathcal{H}_A(\beta)$ is $x_n = 0$.

Note that there is no slope along $x_i = 0$, $1 \leq i \leq n - 1$, which can be shown easily.

Recall that $F = (0, \ldots, 0, 1, \ldots, 1)$ and $V = (0, \ldots, 0, -1, 0, \ldots, 0, 1)$. For positive number $p$ and a non-negative number $q$, we define a weight vector $L = pF + qV$. In section [3], we explained the notion of non-micro-characteristic.

When the variety is $y_i = x_i = 0$, this notion is rephrased as follows: for a given left $A_n$-module $A_n/I$, the hyperplane $y_i = 0$ ($1 \leq i \leq n - 1$) is called non-micro-characteristic of type $L$ when

$$\sqrt{\langle \sigma^L(I), y_i, y_i^* \rangle_{(j \neq i), \tau^s} } \ni y_i^*.$$ 

**Proposition 4.2** For the hypergeometric $A_n$-module $\mathcal{H}_A(\beta)$, the variety $y_i = 0$, $(1 \leq i \leq n-2)$ is non-micro-characteristic of type $L$ for all $L = pF + qV$, $p > 0$.

Proof: Consider $\partial_i^{\alpha_j} - \partial_j^{\alpha_i} \in H_A(\beta)$. For all $L$ and for $i < j \leq n - 1$, we have $\sigma^L(\partial_i^{\alpha_j} - \partial_j^{\alpha_i}) = (y_i^*)^{\alpha_j}$, which implies that $y_i = 0$ is non-micro-characteristic of type $L$. [7]

Now, let us apply the second step of the algorithm to evaluate the slopes, i.e., we will compute the restriction of $\mathcal{H}_A(\beta)$ to $y_i = x_i = 0$, $(1 < i \leq n - 2)$. Let $s$ be an indeterminate. Consider the ideal $H_A[s]$ in $A_n[s]$ generated by $A\theta - s$ and $I_A$.

**Theorem 4.3** We have a left $D'[s]$-module isomorphism

$$A_n[s]/(A_n[s]H_A[s] + x_i A_n[s]) \cong D'[s]/D'[s]H_{A'}[s], \quad i \neq 1$$

(1)

Here, $D' = \mathbb{C}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \partial_1, \ldots, \partial_{i-1}, \partial_{i+1}, \ldots, \partial_n)$ and $A' = (1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

Proof: Fix an order $\prec$ such that $\partial_i \succ \partial_n \succ \partial_{n-1} \succ \cdots \succ \partial_1$. Then,

$$\partial_n - \partial_1^{a_n}, \partial_{n-1} - \partial_1^{a_{n-1}}, \ldots, \partial_2 - \partial_1^{a_2}$$

is the reduced Gröbner basis of $I_A$ with respect to $\succ$.

For each $i$ ($2 \leq i \leq n$), by applying the same method with the proof of [17] Th 3.1.3], we can prove that $\text{in}_{(-e_i, e_i)}(H_A(s))$ is generated by $A\theta - s$ and $\text{in}_{e_i}(\partial_j - \partial_1^{e_i})$, $(j = 2, \ldots, n)$.
Define the $V_i$-filtration $F_k[s]$ of $A_n[s]$ by

$$F_k[s] = \left\{ \sum a_{\alpha\beta\gamma}x^\alpha\partial^\beta s^\gamma \mid (\beta - \alpha) \cdot e_i \leq k \right\}.$$ 

We compute the restriction of $A_n[s]/H_A(s)$ to $x_i = 0$ by Oaku’s algorithm (see, e.g., [17, Theorem 5.2.6, Alg. 5.2.8]). Since $\text{in}_{(x_i, x_j)}(x_i \partial_i - x_j \partial_j^a_i) = x_i \partial_i$, the $b$-function of $H_A(s)$ along $x_i = 0$ is $b(p) = p$. Therefore, by [17, Theorem 5.2.6], we have the following isomorphism as left $D'[s]$-modules:

$$A_n[s]/(H_A(s) + x_i A_n[s]) \simeq F_0[s]/(F_0[s](A\theta - s) + \sum_{j=2, j \neq i} F_0[s](\partial_j - \partial_1^{a_j} + F_{-1}[s](\partial_i - \partial_1^{a_i}) + x_i F_1[s])$$

$$\simeq D'[s]/\left( D'[s](A\theta - s) + \sum_{j=2, j \neq i} D'[s](\partial_j - \partial_1^{a_j}) \right).$$

We can specialize $s$ to any complex number $\beta$. So, we have

$$A_n/(A_n H_A(\beta) + x_i A_n) \simeq D'/D'H_A(\beta),$$

which means that the restriction of $H_A(\beta)$ to $x_i = 0$ can be exactly expressed in terms of the GKZ system for smaller $A$. By applying our algorithm [15] of computing slopes by reduction of number of variables to variables $x_2, \ldots, x_{n-2}$, we obtain the following theorem by Proposition [12] and Theorem [13].

**Theorem 4.4** The geometric slopes of $H_A(\beta)$ along $x_n = 0$ at the origin and $H(1, a_{n-1}, a_n)(\beta)$ along $x_3 = 0$ at the origin coincide.

**Example 4.5** We note that (the algebraic slopes) $\neq$ (the geometric slopes) in general. For example, let us apply the ACG algorithm to get the algebraic slopes of $H_A(-30)$ for $A = (1, 3, 7)$. This ideal is generated by

$$x_1 \partial_1 + 3x_2 \partial_2 + 7x_3 \partial_3 + 30, \partial_1^3 - \partial_2, -\partial_1^2 \partial_2 + \partial_3, \partial_2^3 - \partial_1^3 \partial_3.$$

The output is

$$\{-1, -3/4, -1/2\}.$$

On the other hand, if we apply the ACG algorithm to get the geometric slopes, the output is $\{-3/4\}$.

**Example 4.6** (the slopes of $H(1, a_{n-1}, a_n)(\beta)$) $\neq$ (the slopes of $H(1, a_n)(\beta)$) in general.

Let us take the example: $A = (1, 3, 7)$. Consider the hypergeometric $A_3$-module $A_3/I$ where $I = H_{(1, 3, 7)}(-30)$. As we have seen, the slope of this system along $x_3 = 0$ is $\{-3/4\}$. 

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Consider \( \text{in}_L(I) \) for \( L = F + 4V \). By computing Gröbner basis with respect to \( L \), we can see that
\[
\mathcal{V}(\text{in}_L(I)) = \mathcal{V}(x_2, \xi_1, \xi_3) \cup \mathcal{V}(\xi_1, \xi_2, \xi_3).
\]
It is not included in \( \mathcal{V}(\xi_2) \) Hence \( x_2 = 0 \) is micro-characteristic of type \( L \) and we cannot apply for the restriction criterion.

The condition “non-microcharacteristic for all the filtration \( pF + qV \)” cannot be taken a way to evaluate the slopes by the restriction. In fact, it can be easily checked by the ACG algorithm that the set of the geometric slopes of \( H_{(1,a_n)}(\beta) \) is equal to \( \{1/(1-a_n)\} \). Hence, the set of the slopes of \( H_{(1,7)}(\beta) \) is \( \{-1/6\} \), which is not equal to \( \{-3/4\} \).

We have shown that the computation of the slopes of \( H_A(\beta) \) is reduced to the three variables case. The slopes of this case are as follows.

**Theorem 4.7**

\[
\text{the slopes of } H_{(1,a_{n-1},a_n)}(\beta) = \{a_{n-1}/(a_{n-1} - a_n)\}
\]

**Proof:** We fix some notation:

1. \( A = (1, a, b) \in \mathbb{Z}^3 \) and \( 1 < a < b \).
2. \( P_1 = \partial_1^a - \partial_2, \ P_2 = \partial_1^b - \partial_3, \ P_3 = \partial_2^b - \partial_3^3, \ P_4 = x_1\partial_1 + ax_2\partial_2 + bx_3\partial_3 - \beta \)
3. Let \( \Lambda \) be the linear form with slope \(-a/(b-a)\) (i.e. \( L = aF + (b-a)V \)).
4. Let \( L, L' \) be linear forms. We say that \( L > L' \) if \( \text{slope}(L) > \text{slope}(L') \).
5. We will write \( y_1 = x_1, \ y_2 = x_2, \ t = x_3 \).

The operators \( P_1, P_2, P_3, P_4 \) are in \( H = H_A(\beta) \). Let \( \Lambda = a_{n-1}/(a_n - a_{n-1}) \). Then, we have the following claims.

1. For all linear form \( L \) we have \( \sigma^L(P_1) = (\eta_1^a)^a \) and so \( \eta_1^a \in \sqrt{\sigma^L(H)} \) for all \( L \).
2. For all linear form \( L \) we have \( \sigma^L(P_4) = y_1\eta_1^a + a\eta_2^a + b\tau^*(-\tau) \)
3. For all linear form \( L > \Lambda \) we have \( \sigma^L(P_3) = (\tau)^a \) and so \( \tau \in \sqrt{\sigma^L(H)} \) for all \( L > \Lambda \).
4. So, for all \( L > \Lambda \) we have \( \text{Ch}^L(H) \subset T_{y_1=0}^* \mathbb{C}^3 \cup T_\mathbb{C}^* \mathbb{C}^3 \) and then \( \sqrt{\sigma^L(H)} \) is bi-homogeneous and \( L \) is not a geometric slope of \( H \).
5. On the other hand we have, for \( L < \Lambda, \ \sigma^L(P_3) = (\eta_2^a)^b \). Then \( \eta_2^b \in \sqrt{\sigma^L(H)} \) and \( \text{Ch}^L(H) \subset T_{t=0}^* \mathbb{C}^3 \cup T_\mathbb{C}^* \mathbb{C}^3 \). So \( L \) is not a geometric slope of \( H \).
6. So, the only possible geometric slope of \( H \) is \( \Lambda \).
Now, suppose that Λ is not a slope. Then, there is no slope, which implies that the $L$-characteristic variety $\text{Ch}^L(\mathcal{H}_A(\beta))$ is the same for all $L = pF + qV$, $p, q > 0$ by \[ \] and \[ \text{Th 3.4.1} \]. It follows from
\[ [P_1, P_2] = 0, \quad [P_1, P_3] = aP_1, \quad [P_2, P_3] = bP_2 \]
and the Buchberger algorithm that \{ $P_1, P_2, P_3$ \} is a Gröbner basis for the order defined by the weight vector $L = (0, 0, -N, 1, 1, N+1), N \geq b$ and a tie-breaking term order such that $x_2 \succ \partial_3 \succ \partial_1 \succ \partial_2$. Therefore, the initial ideal $I^L_L(\mathcal{H}_A(\beta))$ is generated by $\xi^n_1, \xi_3, ax_1\xi_1 + ax_2\xi_2 + bx_3\xi_3$ and hence the $L$-characteristic variety is equal to $T_{y_2=0}C^3 \cup T_{y_2=0}^*C^3$. This fact contradicts that the $L$-characteristic variety is the same for all $L$. \[ \]

Finally, let us remark on an analytic meaning of slopes. Let $X = C^n$ and $Y = \{x|\alpha_n = 0\} \subset X$. We denote by $\mathcal{O}_{X|Y} = C[x_1, \ldots, x_{n-1}]$ the formal completion of $\mathcal{O}_X$ along $Y$. For each real number $s \in [1, +\infty)$ we denote by $\mathcal{O}_{X|Y}(s)$ the subsheaf of $\mathcal{O}_{X|Y}$ of Gevrey functions of order $s$ (along $Y$). The sheaf $\mathcal{O}_{X|Y}(1)$ is the restriction $\mathcal{O}_{X|Y}$ and, by definition, we write $\mathcal{O}_{X|Y}(+\infty) = \mathcal{O}_{X|Y}$. For any holonomic $D_X$-module $M$, Mebkhout associates in \[ \] the sheaf $\text{Irr}_Y(s)(M)$ as the solution sheaf $\text{RHom}_D(M, \mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y})$. One fundamental result of $D$-modules is the fact that $\text{Irr}_Y(s)(M)$ is a perverse sheaf, for any $s$ (see \[ \text{[14]} \]). These sheaves define a filtration of the irregularity of $M$ along $Y$, i.e. $\text{Irr}_Y(M) := \text{Irr}_Y(+\infty)(M)$. The main result of \[ ] is that $1/(1-s)$ is a slope of $M$ w.r.t. $Y$ if and only if $s$ is a gap of the graduation defined by the filtration on the irregularity. In other words, $1/(1-s)$ is a slope if and only if $\text{Irr}_Y(s)(M)/\text{Irr}_Y(<s)(M) \neq 0$.

## 5 Rational Solutions and Reducibility

Our ultimate aim of studying the slopes of $H_A(\beta)$ is to get a better understanding on solutions of this system. We are far from the goal, but to this end, it will be useful to present some facts on classical solutions and a relation to generalized confluent hypergeometric functions.

In case of the hypergeometric ideal associated to homogeneous monomial curves, Cattani, D’Andrea, Dickenstein \[ \] studied rational solutions and reducibility of the system. We will study rational solutions and reducibility of our system.

**Theorem 5.1** Any rational solution of the hypergeometric system $H_{(1,a_2,\ldots,a_n)}(\beta)$ is a polynomial. It has a polynomial solution if and only if $\beta \in \mathbb{N} = \{0, 1, \ldots, \}$. The polynomial solution is the residue of $\exp(\sum x_i t^{a_i}) t^{-\beta}$ at the origin $t = 0$:
\[
\int_C \exp\left(\sum x_i t^{a_i}\right) t^{-\beta} \frac{dt}{t}
\]
Here, $C$ is a circle that encircles the origin in the positive direction.
Proof: Since the singular locus of $H_A(\beta)$, $(A = (a_1, a_2, \ldots, a_n), \ a_1 = 1)$ is $x_n = 0$, any rational solution $f$ is a Laurent polynomial with poles on $x_n = 0$. Take a weight vector $w = (0, 1, 1, \ldots, 1)$. Then, we have $\text{in}_w(I_A) = (\partial_2, \ldots, \partial_n)$. The initial term $\text{in}_w(f)$ is annihilated by

$$\sum a_i \theta_i - \beta, \ \partial_2, \ldots, \partial_n.$$ 

Therefore, $x^\beta_1 = \text{in}_w(f)$. This implies $\beta \in \mathbb{N}$, because $f$ has a pole only on $x_n = 0$.

Take a $\mathbb{Z}$-basis of $\text{Ker} (\mathbb{Z}^n \rightarrow \mathbb{Z})$ as

$$(-a_2, 1, 0, \ldots, 0), (-a_3, 0, 1, 0, \ldots, 0), \ldots, (-a_n, 0, \ldots, 0, 1)$$

to construct series solutions. Since $[\mathbb{Z}]$ Prop. 3.4.1 holds for non-homogeneous $A$ as well, formal series solution $g$ of $H_A(\beta)$ satisfying $\text{in}_w(g) = x^\beta_1$ can be uniquely expressed as

$$\sum_{m \in \mathbb{N}^{n-1}} \frac{\beta(\beta - 1) \cdots (\beta - \sum m_k a_k + 1) }{m_2! \cdots m_n!} \left( \frac{x_2}{x_1} \right)^{m_2} \cdots \left( \frac{x_n}{x_1} \right)^{m_n} x^\beta_1. \quad (1)$$

When $\beta \in \mathbb{N}$, it is a polynomial. The rest part of the theorem is easy to show.

Let $R$ be the ring of differential operators of $n$ variables with rational function coefficients over $k = \mathbb{C}$. A left ideal $J$ of $R$ is called irreducible when $J$ is a maximal ideal in $R$. We will study the reducibility of $R \cdot H_A(\beta)$.

We assume that $J$ is zero-dimensional, i.e., $r = \text{dim}_{\mathbb{C}(x)} R/J < +\infty$. Let $V = V(J)$ be the vector space of holomorphic solutions of $J$ on a simply connected open set contained in the non-singular domain of $J$. It is known that $\text{dim}_{\mathbb{C}} V = r$. Define $I(V)$ by $R \cdot \{ \ell \in R | \ell \cdot f = 0 \text{ for all } f \in V \}$. If $J \subset I(V)$, $J \neq I(V)$, then we have $\text{dim}_{\mathbb{C}(x)} R/J < \text{dim}_{\mathbb{C}} V = r$ because of the zero-dimensionality of $J$. Therefore, we have $J = I(V(J))$.

Under this correspondence of ideals and solutions, a zero-dimensional ideal $J$ of $R$ is reducible if and only if there exists a proper subspace $W$ of the solution space of $V(J)$ such that $0 < \text{dim}_{\mathbb{C}(x)} R/I(W) < \text{dim}_{\mathbb{C}(x)} R/J$. In case of one variable, the reducibility is equivalent to saying that the generator of the ideal can be factored in $R$.

**Theorem 5.2** The systems of differential equations $R \cdot H_A(\beta)$ is reducible if and only if $\beta \in \mathbb{Z}$.

Proof: Any curve is Cohen-Macaulay. By applying the theorem of Adolphson [3], the holonomic rank of $H_A(\beta)$ is $a_n$ for all $\beta$. 

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Put \( M(\beta) = A_n/H_A(\beta) \). Consider the left \( A_n \)-morphism
\[
\partial_1 : M(\beta) \to M(\beta + 1).
\]
(2)
It has the inverse when \( \beta \neq -1 \). Therefore, we have \( M(-1) \simeq M(-2) \simeq M(-3) \simeq \cdots \) and \( M(0) \simeq M(1) \simeq M(2) \simeq \cdots \).
When \( \beta \in \mathbb{N} \), the system admits polynomial solution, then it is reducible.

It is also easy to see that when \( \beta \in \mathbb{Z}_{<0} \), the equation is reducible. In fact, consider the left \( D \)-morphism
\[
\partial_1 : M(-1) \to M(0).
\]
It induces a morphism to the solutions by \( f \to \partial_1 \circ f \).

The solution \( f = 1 \) of \( M(0) \) is sent to zero, so the image of \( \partial_1 \) gives a proper subspace of solutions in the solution space of \( M(-1) \). To find differential equations for the subspace, take all \( \ell \) such that \( \ell \partial_1 \in H_A(0) \). Then, \( \{ \ell \} \subset H_A(-1) \). By the isomorphism (2), we conclude that when \( \beta \in \mathbb{Z}_{<0} \), the system is reducible.

Let us prove that the system is irreducible when \( \beta \notin \mathbb{Z} \) by applying the result of Beukers, Brownawell, and Heckman [4]. For this purpose, we first construct convergent series solution of \( H_A(\beta) \). Take \( w = (1,1,\ldots,1,0) \). Then the degree of \( \in_w(I_A) \) is equal to \( a_n \). Since \( I_A \) contains the elements of the form \( \partial_1^n - \partial_n^a \), the radical of \( \in_w(I_A) \) is \( \langle \partial_1, \ldots, \partial_{n-1} \rangle \). Therefore, the top dimensional standard pairs have the form \( (\partial_1^n, \{ n \}) \).

Let \( v \) be the zero of the indicial ideal associated to \( (\partial_1^n, \{ n \}) \):
\[
v_1 = b_1, \ldots, v_{n-1} = b_{n-1}, v_n = \frac{\beta - \sum a_i b_i}{a_n}.
\]
Assume \( \beta \notin \mathbb{Z} \) or \( \beta \gg 0 \). Taking the lattice basis \((a_2, -1, 0, \ldots, 0), \ldots, (a_n, 0, \ldots, 0, -1)\), we have the following \( a_n \) linearly independent convergent series solutions
\[
\sum_{m \in \mathbb{N}^{n-1}} \frac{v_2(v_2-1) \cdots (v_2-m_2+1) \cdots v_n(v_n-1) \cdots (v_n-m_n+1)}{(v_1+1)(v_1+2) \cdots (v_1+\sum ak m_k)} \cdot \left( \frac{x_1^{a_2}}{x_2} \right)^{m_2} \cdots \left( \frac{x_2}{x_n} \right)^{m_n} x^n.
\]
(3)
We consider the change of variables:
\[
y_1 = x_1, y_2 = x_1^{a_2}/x_2, \ldots, y_n = x_1^{a_n}/x_n.
\]
The inverse of this change of variables is also rational and the change of variables induces that of \( \partial_1 \). We denote by \( \Phi \) the operation of these change of variables of \( x_i \) and \( \partial_1 \). Since the irreducibility is invariant under any birational change of
variables, we will prove the irreducibility of the ideal \( J = R \cdot \Phi(x^{-v}H_A(\beta)x^v) \) where \( R = C(y)(\partial_{y_1}, \ldots, \partial_{y_n}) \).

Let \( V \) be the solution space of \( J \) spanned by the series \( \Phi((3)^{\cdot} x^{-v}) \) near \( y = 0 \). If \( J = I(V) \) is reducible, then there exists a proper subspace \( W \) of \( J \) such that \( 0 < \dim C(y)R/I(W) < a_n \). We consider the vector space \( W' = \{ f(0, \ldots, 0, y_n) \mid f \in W \} \). It is easy to see that \( \dim C(y)R'/I(W') \leq \dim C W' = a_n \). Let us prove that \( \dim C W' = a_n \) when \( \beta \notin \mathbb{Z} \), which implies the irreducibility of \( J \) by a contradiction.

We restrict the series \( \Phi((3)^{\cdot} x^{-v}) \) to \( y_1 = x_1 = 0, y_2 = x_1^{a_2}/x_2 = 0, \ldots, y_{n-1} = x_1^{a_{n-1}}/x_n = 0 \) and replace \( y_n = x_1^{a_n}/x_n \) by \( z \). Without a loss of generality, we may assume \( v_1 = 0 \). Then the restricted series has the form

\[
\sum_{m=0}^{\infty} \frac{v_n(v_n-1)\cdots(v_n-m+1)}{(a_n m)!} (-z)^m. \tag{4}
\]

It is annihilated by the ordinary differential operator

\[
(a_n\theta_z)(a_n\theta_z - 1) \cdots (a_n\theta_z - a_n + 1) - z(\theta_z + v_n).
\]

By replacing \( z/a_n^m \) by \( x \), we obtain the generalized hypergeometric ordinary differential equation

\[
\theta_x(\theta_z - 1/a_n) \cdots (\theta_z - (a_n - 1)/a_n) - x(\theta_z + v_n). \tag{5}
\]

By \( \frac{a}{b} \), this ordinary differential equation of rank \( a_n \) is reducible if and only if \( v_n - k/a_n \notin \mathbb{Z} \) for all \( k = 0, 1, \ldots, a_n - 1 \). If one of them is an integer, \( \beta \) becomes an integer. Therefore, the ideal \( I(W') \) contains the principal ideal generated by \( \frac{a}{b} \), which is maximal when \( \beta \notin \mathbb{Z} \). We conclude that \( I(W') \) is generated by \( \frac{a}{b} \) and hence \( \dim C W' = a_n \).

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