Logical Interpretation of a Reversible Measurement in Quantum Computing

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Abstract

We give the logical description of a new kind of quantum measurement that is a reversible operation performed by a hypothetical insider observer, or, which is the same, a quantum measurement made in a quantum space background, like the fuzzy sphere. The result is that the non-contradiction and the excluded middle principles are both invalidated, leading to a paraconsistent, symmetric logic. Our conjecture is that, in this setting, one can develop the adequate logic of quantum computing. The role of standard quantum logic is then confined to describe the projective measurement scheme.

1 Introduction

Since a quite long time, it was assumed that standard quantum logic [4] was the right logic for the quantum world. However, standard quantum logic fails when trying to describe a closed quantum system, like a quantum computer during the computational process. In the context of quantum computing [], standard quantum logic is only able to describe the standard (projective) quantum measurement, not the whole computational process (which in fact looks like a “black box” to an external observer). This “fallacy” of standard quantum logic has been already recognized in the literature, as in [6], where paraconsistency [20] and linearity [10] play relevant roles. Also Basic Logic [21] can be considered a promising alternative to standard quantum logic, in the context of quantum computing. The main aim of our work is to look for a logic
which describes the whole quantum computational process, comprising the measurement, from “inside”. The first step we do in this direction is to illustrate, in logical terms, a reversible quantum measurement, with no hidden quantum information, performed by a hypothetical “insider observer” [23]. In the literature, the problem of a reversible quantum measurement has been considered, up to now, from a true physical point of view, see for example: [12]; [17]; [18]; [15]. Instead, our kind of reversible quantum measurement is a purely theoretical tool to investigate the internal computational state. The “insider observer” is a fictitious being, who is used to describe the quantum measurement scheme in a quantum space-background. Actually, we discuss three different kinds of reversible measurements applied to one qubit: the mirror measurement, which does not change the probabilities, the fuzzy measurement, which mixes up the probabilities, and the Liar measurement, which interchanges the probabilities. The first one is conceived in a discrete, but still classical space, a ordered 2-points lattice, embedded in the fuzzy sphere [13] with two cells, and corresponds to an anti-clockwise rotation about the z-axis of the Bloch sphere. The second one is conceived in the fuzzy sphere with two cells, and corresponds to a generic rotation of the Bloch sphere. The third one, is conceived in the same 2-points lattice as the first one, but with the two points interchanged. Then, our reversible quantum measurement is a kind of “thought experiment”, which is useful to tune our reasoning with the internal logic of quantum computation. Moreover, this new kind of quantum measurement might give some fresh insights in the foundations of quantum mechanics. In fact, the reversible measurement performed in this way, offers an interpretation of quantum mechanics which is very much on line with that of Mermin [16], as it attributes objectivity to the probabilities of the superposed state of one qubit, and separates this objective reality from the external observer and his knowledge. This theoretical construction has been used to build up a computational model for Loop Quantum Gravity and quantum black holes [24]. Moreover, it lead to a new approach to quantum computability and to a quantum interpretation of Godel’s first incompleteness theorem [25]. A preliminary treatment of the logical aspects arising from the reversible quantum measurement scheme, can be found in [1]. In this paper, we develop the arguments introduced in [23] and [1] and we illustrate the geometrical origin of logical symmetry which is due to the symmetry between the mirror measurement and the liar measurement. Also, we show how the logic of the insider observer is related to the standard quantum logic of the external observer, and to classical logic. Here, as in [23] and [1], we consider the toy-model of one qubit. Due to the reversible measurement scheme, we get two new axioms (symmetric to each other) which are the opposite of the excluded middle and the non-contradiction principles. Thus, the logic of the insider observer is paraconsistent and symmetric. However, if an external observer performs a standard quantum measurement, the excluded middle and the non-contradiction principles can be recovered. This paper is organized as follows: In Sect.2, we illustrate the reversible quantum measurement, more specifically, the mirror measurement, the fuzzy measurement, and the Liar measurement; in Sect. 3 we describe the logic of the mirror and of the
Liar measurements and we discuss some philosophical implications of the mirror measurement; in Sect. 4, we illustrate, in logical terms, the border of the black box, which is equivalent to the standard (projective) quantum measurement. Sect. 5 is devoted to the conclusions.

2 The Reversible Measurement Scheme

As it is well known, the interpretational problem of quantum measurement, when performed by an external observer, is one of the hardest in the foundations of Quantum Mechanics. However, we believe that it is not a problem due to our lack of understanding (or knowledge), but it is just a consequence of the fact that Quantum Mechanics is a quantum theory settled on a classical background. If the background were quantum as well, the interpretational problem would disappear. Perhaps, the issue of simulations might clarify our point of view, as follows.

2.1 Simulations

The issue of simulations of a quantum system has been at the heart of the discovery of quantum computers. All that started in 1981, when Feynman [9] proposed for the first time to use quantum phenomena to simulate quantum systems. A classical computer can simulate a quantum system perfectly, as both the computer’s memory (bits) and the quantum spectra are discrete, but very slowly (in exponential time). Instead, a quantum computer can simulate a quantum system perfectly because both the quantum register (qubits) and the spectra of the quantum system are discrete, and efficiently, because of quantum parallelism [7]. However, both the quantum system to be simulated and the quantum computer lie on a classical space-time background. The classical background is present before the start of the simulation, when a classical input is provided, and at the end, when the observation takes place: at this point a large amount of quantum information becomes hidden. This is a kind of inconsistency of the whole simulation process, and is due to the fact that Quantum Mechanics is a quantum theory formulated on a classical background. During the very computational process, the classical background is not taken into account, and one could guess what is going on inside the “black box” by performing a new kind of measurement which is unitary (reversible) and does not destroy the superposed state. To do so, the observer should be internal, that is, she should enter a quantum space whose states are in a one-to-one correspondence with the machine states [23].

2.2 The Standard (Projective) Quantum Measurement

We start by reminding the modalities of the standard quantum measurement (of one qubit). Let us consider a qubit in the superposed state:

\[ |\psi\rangle = a|0\rangle + b|1\rangle \] (1)
Where $|0\rangle$ and $|1\rangle$ form an orthonormal basis, called the computational basis, and $a$ and $b$, called probability amplitudes, are complex numbers such that the probabilities sum up to one: $|a|^2 + |b|^2 = 1$.

In vector notation we have:

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

The standard quantum measurement of the qubit $|\psi\rangle$ in (1) gives either $|0\rangle$ with probability $|a|^2$, or $|1\rangle$ with probability $|b|^2$. This is achieved by the use of projector operators. A projector operator $P$ is defined by:

$$P^2 = P \quad P^+ = P$$  \hspace{1cm} (2)

(Where $P^+$ is the Hermitian adjoint of $P$, that is, the conjugate transpose: $P^+ \equiv P^T \ast$). Thus a projector $P$ is idempotent and Hermitian.

Let us consider a general superposed quantum state: $\psi = \sum_{i=1}^{n} c_i |\psi_i\rangle$ in the Hilbert space $C^n$, with $\sum_{i=1}^{n} |c_i|^2 = 1$. The probability $p_r(i)$ of finding the state $\psi$ in one of the basis states $|\psi_i\rangle$ is, after a measurement: $p_r(i) = |P_i |\psi\rangle|^2$. After the measurement, the state $|\psi\rangle$ has “collapsed” to the state $|\psi\prime\rangle = \frac{P_i |\psi\rangle}{\sqrt{p_r(i)}}$. The $n$ projectors $P_i$ ($i = 1, 2, \ldots, n$) are orthogonal: $P_i P_j = \delta_{ij} P_i$ and sum up to 1:

$$\sum_{i=1}^{n} P_i = 1$$  \hspace{1cm} (3)

In our case, $i = 0, 1$. We have the two 2-dimensional projectors:

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For which it holds:

$$P_0 P_1 = P_1 P_0 = 0, \quad P_0^2 = P_0, \quad P_1^2 = P_1, \quad P_0 + P_1 = 1$$

The actions of $P_0$ and $P_1$ on the basis states are, respectively:

$$P_0 |0\rangle = |0\rangle, \quad P_0 |1\rangle = 0 P_1 |0\rangle = 0, \quad P_1 |1\rangle = |1\rangle$$

From which it follows that their action on the superposed state $|\psi\rangle$ is, respectively:

$$P_0 |\psi\rangle = a |0\rangle, \quad P_1 |\psi\rangle = b |1\rangle.$$ 

The probability of finding the qubit state $|\psi\rangle$ in the state $|0\rangle$ is, for example:

$$p_r(0) = |P_0 |\psi\rangle|^2 = |a|^2 = |a|^2.$$ 

After the measurement, the qubit $|\psi\rangle$ has “collapsed” to the state:

$$|\psi\prime\rangle = \frac{P_0 |\psi\rangle}{\sqrt{p_r(0)}} = \frac{a |0\rangle}{\sqrt{|a|^2}} = |0\rangle.$$
Then, a lot of quantum information that was encoded in $|1\rangle$ is made hidden by the standard quantum measurement. As a projector is not a unitary transformation, the standard quantum measurement is not a reversible operation. This means that the hidden quantum information will never be recovered (i.e., we will not be able to get back the superposed state $|1\rangle$).

### 2.3 The Bloch Sphere

We believe that the irreversibility of the standard quantum measurement is strictly related to the classical geometry of the space-time background. To see why, let us consider the Bloch sphere, which is the sphere with unit radius:

$$S^2 = \{x_i \in \mathbb{R}^3 | \sum_{i=1}^{3} x_i^2 = 1\}$$

Any generic 1-qubit state in $|\psi\rangle$ can be rewritten as:

$$\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

Where the Euler angles $\theta$ and $\phi$ define a point on the unit sphere $S^2$. Thus, any 1-qubit state can be visualized as a point on the Bloch sphere, the two basis states being the poles. See Fig.1. A standard quantum measurement of one qubit is then equivalent to the projection of one of the poles of the Bloch sphere, resulting in one point in $\mathbb{R}^3$, where the external observer is placed. We wish to remind that any transformation on a qubit during a computational process is a reversible operation, as it is performed by a unitary operator $U$ such that $U^+U = I$. This can be seen geometrically as follows. Any unitary $2 \times 2$ matrix $U_2$ on $\mathbb{C}^2$, (which is an element of the group SU(2) multiplied by a global phase factor):

$$U_2 = e^{i\phi} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

(4)

(where $\alpha^*$ is the complex conjugate of $\alpha$ and $|\alpha|^2 + |\beta|^2 = 1$), can be rewritten in terms of a rotation of the Bloch sphere:

$$U_2 = e^{i\phi} R_\hat{n}(\theta)$$

Where $R_\hat{n}(\theta)$ is the rotation matrix of the Bloch sphere by an angle $\theta$ about an axis $\hat{n}$.

However, a projector is not a unitary operator, and it cannot be rewritten in terms of a rotation of the Bloch sphere. This means that the observer, who has performed the standard quantum measurement, is not able to recover the original state by a rotation of the sphere. In fact, what the external observer sees, is just one pole.
The question is now whether a reversible measurement could be feasible, at least in principle. Of course, the projector should be replaced by a unitary operator, but this means that the reversible measurement should be performed “from inside”. Or, in other words, the hypothetical observer should be placed in a quantum space whose states are in a one-to-one correspondence with the quantum computational states. This quantum space will be a discrete topological space associated, by the non-commutative version of the Gelfand-Naimark theorem [11], with the algebra of quantum logic gates. Now, $n$-dimensional quantum logic gates are unitary $n \times n$ complex matrices, with $n = 2^N$, where $N$ is the number of qubits in the quantum register. For example, in the case of one qubit, the quantum logic gates are $2 \times 2$ unitary matrices. Thus, quantum logic gates form a subset of the set of $V$ complex matrices, whose algebra is a non-commutative $C^*$-algebra [8].

To this algebra it is associated, by the non-commutative version of the Gelfand-Naimark theorem, a quantum space which is the fuzzy sphere [13] with $n$ elementary cells. This means that the computational state of a quantum computer with $N$ qubits can be geometrically viewed as a fuzzy sphere with $2^N$ cells.

We recall here that the fuzzy sphere is constructed replacing the algebra of polynomials on the (unit) sphere $S^2$ by the non-commutative algebra of complex matrices, which is obtained by quantizing the coordinates $x_i$ ($i = 1, 2, 3$): $x_i \rightarrow X_i = kJ_i$, where the $J_i$ form the $n$-dimensional irreducible representation of SU(2): $[J_i, J_j] = i\epsilon_{ijk}J_k$ and the non-commutative parameter $k$ is, for a unit radius:

$$k = \frac{1}{\sqrt{n^2 - 1}}.$$

Then, the ensemble of rotations of the Bloch sphere (unitary transformations of one qubit) can be viewed as a fuzzy sphere in the $n = 2$ case (two elementary cells), the $x_i$ being replaced by: $x_i \rightarrow X_i = \frac{1}{\sqrt{3}}\sigma_i$, where the $\sigma_i$ are the Pauli matrices.

### 2.5 The Mirror Measurement

In what follows, we will generalize the standard quantum measurement of one qubit by using complex matrices that are not projectors, and we will analyze the associated geometries.

To start, let us consider the diagonal matrices on the complex numbers (they form a commutative $C^*$-algebra, which is a sub algebra of the non-commutative $C^*$-algebra of $2 \times 2$ matrices on the complex field). Recall, however, that we shall require unitarity, so that we should consider only matrices of the kind:

$$U_2 = e^{i\phi} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}$$ (5)
With $|\alpha|^2 = 1$. Where $U^D_2$ in $\text{(5)}$ is the particular case of $U_2$ in $\text{(4)}$, with $\beta = 0$. The action of $U^D_2$ on the qubit state $\text{(1)}$ gives:

$$U^D_2|\psi\rangle = a'|0\rangle + b'|1\rangle$$

Which is still a superposed state with: $a' = e^{i\phi} a, b' = e^{i\phi} \alpha^* b$, and:

$$|a'|^2 = |a|^2, |b'|^2 = |b|^2.$$ 

That is, the probabilities are unchanged. Notice that geometrically, this is equivalent to project both the poles of the Bloch sphere at the same time. The associated space is a 2-points lattice (which is a discrete, but still classical, space). In fact, $U^D_2$ can be rewritten as:

$$U^D_2 = e^{i\phi}(\alpha P_0 + \alpha^* P_1)$$

Which is a linear superposition of the two projectors $P_0$ and $P_1$. Then, $U^D_2$ is the reversible origin of a standard (irreversible) quantum measurement. The application of $U^D_2$ to the state $|\psi\rangle$ in $\text{(1)}$ is a superposition of two standard quantum measurements made simultaneously. We will call this new kind of quantum measurement "mirror-measurement" because the qubit remains in a superposed state, and the probabilities are unchanged.

After the mirror-measurement, the state $|\psi\rangle$ is left in the state:

$$|\psi\rangle \rightarrow |\psi'\rangle = \frac{U^D_2|\psi\rangle}{\sqrt{|a'|^2 + |b'|^2}} = e^{i\phi}(\alpha a|0\rangle + \alpha^* b|1\rangle)$$

Where one has considered the total probability.

The state $|\psi'\rangle$ is still a superposed state, and, from it, one can recover the original state by performing the inverse operation:

$$(U^D_2)^{-1}|\psi'\rangle = e^{-i\phi} \left( \begin{array}{cc} \alpha^* & 0 \\ 0 & \alpha \end{array} \right) \left( \begin{array}{c} e^{-i\phi} \alpha a \\ e^{-i\phi} \alpha^* b \end{array} \right) = \left( \begin{array}{c} a \\ b \end{array} \right) = |\psi\rangle$$

In summary, the mirror measurement does not destroy the superposition, since it is reversible, and does not change the probabilities, but just changes the probability amplitudes.

It should be noticed that the internal observer (who will be called $P$ in the following) uses the projectors $P_0$ and $P_1$ and at the same time. She can do so as she lives in a discrete space, namely a 2-points lattice, which is in a one-to-one correspondence with the two basis states, as it is the space associated with the algebra of diagonal unitary $2 \times 2$ matrices $U^D_2$. If an external observer ($G$) should try to do the same, she would fail, as she lives in a classical, continuous space, namely, $R^3$. Or, $G$ could try to achieve the same result of $P$ by using $P_0$ and $P_1$ in parallel on two copies of the same state $|\psi\rangle$, but that is forbidden by the no-cloning theorem [22], which states that an unknown quantum state cannot be copied. Then, the only thing that $G$ can do, is to use either $P_0$ or $P_1$. 

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that is, to perform a standard quantum measurement. The action of $G$ then breaks the superposition of $P_0$ and $P_1$, used by $P$.

Finally, it can be seen that the mirror measurement performed by $P$ is not in contradiction with the standard quantum measurement made by $G$. In fact, if $G$ uses $P_0$ on the state $|\psi'\rangle = U_2^D|\psi\rangle$, (that is, she performs a standard measurement after a mirror measurement), she gets:

$$P_0U_2^D|\psi\rangle = P_0[e^{i\phi}(\alpha P_0 + \alpha^* P_1)]|\psi'\rangle = a'|0\rangle$$

(9)

With probability $|a'|^2 = |a|^2$. After this “composed” measurement, the qubit is left in the state:

$$|\psi'\rangle = \frac{P_0U_2^D|\psi\rangle}{\sqrt{p_r(0)}} = \frac{a'|0\rangle}{\sqrt{|a'|^2}} = |0\rangle$$

(10)

In other words, the result obtained by $G$ in the classical world is not influenced by the result obtained by $P$ in the quantum world. $P$’s operation gives a result that is consistent with the standard quantum measurement. Or, in other words, she does not create contradiction to the external observer. This is very important not only in the physical sense, but also in the logical sense, as we will see in the next sections. In passing from the mirror measurement to the standard quantum measurement, the associated geometry has changed: from the 2-points lattice (the two poles of the Bloch sphere) to one point (one pole of the Bloch sphere). In the context of the mirror measurement, it should be noticed, however, that the 2-points lattice breaks rotational invariance, so that, from this space, $P$ cannot reach any other 1-qubit state of the Bloch sphere, by a rotation. She can just make a quite limited operation that, up to a global phase factor, is just a phase shift, as showed below.

Any 1-qubit unitary transformation $U_2$ can be written as:

$$U_2 = e^{i\phi}R_Z(\gamma)R_\gamma(\theta)R_\delta$$

with:

$$R_Y(\theta) = e^{-i\theta Y/2} = \begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

$$R_Z(\delta) = e^{-i\delta Z/2} = \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$$

Where $Y$ and $Z$ are the Pauli matrices:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With the choice $\theta = 0$ and $\gamma = \delta$ one gets:

$$U_2 = e^{i\phi} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$$

(11)
Which is our $U^D_2$ matrix in (5) with $\alpha = e^{-i\delta}$ (recall that $\alpha$ is a complex number with unit modulus). Finally, the diagonal matrix in (11) can be written as:

$$U_2 = e^{i\phi'} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}$$

(12)

With: $\phi' = \phi + \delta$, and $\lambda = 2\delta$. The matrix in (12) is, up to the global phase factor $e^{i\phi}$, the quantum gate “phase shift”.

By Eq. (12), then, the mirror measurement corresponds to an anti-clockwise rotation about the z-axis of the Bloch sphere. See Fig. 2.

This is equivalent to the fact that $P$ stands on a 2-points lattice embedded in the fuzzy sphere.

On the other side, Eq. (6) provides the geometrical interpretation of the mirror measurement as conceived by $G$, from outside: an ordered 2-points lattice, without the embedding in the fuzzy sphere. Of course, it is also possible to perform a mirror measurement in a different basis, for example, in the dual basis, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which is obtained by applying the Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

to the computational basis states $|0\rangle$ and $|1\rangle$ respectively.

In the dual basis, the two orthogonal projectors are:

$$P^+ = H P_0 H^{-1}, \quad P^- = H P_1 H^{-1}.$$ 

The diagonal unitary operator in (5) is transformed as:

$$U^D_2 \rightarrow (U^D_2)' = H U^D_2 H^{-1}$$

Which can be written as a linear superposition of $P^+$ and $P^-:$

$$U^D_2 = e^{i\phi}(\alpha P^+ + \alpha^* P^-)$$

That is, the mirror measurement in the dual basis can be viewed as the simultaneous actions of two orthogonal projective measurements in that basis.

### 2.6 The Fuzzy Measurement

If $P$ wished to reach any other point of the Bloch sphere, she should not limit herself to the diagonal 2×2 unitary matrices, but should consider the full algebra of 2×2 unitary matrices, which is a non-commutative C*-algebra.

Notice that the 2-point lattice considered above, is a sub-space [14] of the fuzzy sphere [13]. When instead of considering the unitary diagonal 2×2 matrices as in [5], one considers the full algebra of unitary 2×2 matrices, the two points of the lattice are “smeared out” into two cells of a fuzzy sphere. See Fig.3. Then, the original qubit has not just been “phase shifted” but has been rotated, so that its original probability amplitudes have been “mixed up”. In fact, the action of a unitary 2×2 matrix on the qubit state [11] is:

$$U_2|\phi\rangle = e^{i\phi} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\phi} \begin{pmatrix} \alpha a + \beta b \\ -\beta^* a + \alpha^* b \end{pmatrix} = |\phi\rangle'$$

(13)
To summarize, P should place herself in a fuzzy sphere if she wishes to follow the whole computational process from inside the quantum computer, but she can just stand in a subspace of the fuzzy sphere, namely the 2-points lattice, if she wants to perform a mirror measurement. Notice that when P is performing a generic $L_2$-operation on the qubit, she has lost any contact with the external world, and cannot communicate anymore with G, as now G can only interpret the world of P as a “black box”. This will be shown in logical terms in the next sections.

### 2.7 The Liar Measurement

We finally introduce a particular kind of reversible measurement, that we will call “Liar measurement”. Algebraically, it consists of an off-diagonal unitary matrix:

$$L = e^{i\phi} \begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix}$$

Which is obtained by applying a NOT gate after a diagonal unitary matrix:

$$L = \text{NOT} U^D_2$$

Geometrically, it corresponds to a clockwise rotation around the z-axis of the Bloch sphere, where the North pole is $|0\rangle$ and the South pole is $|1\rangle$, as usual. As it is well known, a clockwise rotation is equivalent to an anti-clockwise rotation with the poles interchanged. See Fig.4. Both these geometrical pictures are relative to the internal observer P when she stands on the 2-points lattice inside the fuzzy sphere. Moreover, since from Eq. 6 we know that $U^D_2$ can be written as:

$$U^D_2 = e^{i\phi}(\alpha P_0 + \alpha^* P_1),$$

we get:

$$L = e^{i\phi}(\alpha Q_0 + \alpha^* Q_1)$$

Where:

$$Q_0 = \text{NOT} P_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad Q_1 = \text{NOT} P_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Obviously, $Q_0$ and $Q_1$ exchange the truth-values:

$$Q_0|0\rangle = |1\rangle \quad Q_0|1\rangle = 0 Q_1|0\rangle = 0 \quad Q_1|1\rangle = |0\rangle$$

Eq. 15 gives the interpretation of the Liar measurement by the external observer who conceives the ordered 2-points lattice with the two elements interchanged without the embedding in the fuzzy sphere. The action of $L$ on the qubit $|\psi\rangle$ is:

$$L|\psi\rangle = a'|0\rangle + b'|1\rangle$$

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where: $a' = e^{i\phi} \alpha^* b$ and $b' = e^{i\phi} \alpha a$. Then, the two probabilities are interchanged.

If the external observer $G$ measures $|\psi\rangle$ after $P$ has performed a Liar measurement, for example, by means of the projector $P_0$:

$$P_0 L |\psi\rangle = a' |0\rangle$$

She gets $|0\rangle$ with probability $|b|^2$.

### 3 Logical Interpretation of the Mirror and Liar Measurements

In this paper, we have defined “black box” the computational state of a quantum computer, which cannot be known by an external observer $G$. In fact, $G$ can only achieve a small part of the quantum information being processed, when she performs a standard quantum measurement. The aim of this section is to provide a logical interpretation of quantum computation, from the point of view of an observer $P$ who is inside the black box. Or, which is the same, $G$ is in a quantum space like the fuzzy sphere, described in the previous section, whose states are in a one-to-one correspondence with the machine states.

#### 3.1 How the Insider Observer Gets Rid of the Non-Contradiction Principle

We confine ourselves to the simplest case, a toy model-quantum computer based on a quantum register of one qubit. The qubit is a vector of $\mathbb{C}^2$. Let us consider an orthonormal basis of $\mathbb{C}^2$, that is a couple of orthonormal states $A, A^\perp$. The state of the qubit is a superposition of the two basis states: we will see now the effect of performing a measurement on it in logical terms. We first consider a standard quantum measurement, performed by the external observer $G$. For example, let us suppose that the projector $P_0$ with respect to the first element of the orthonormal basis is applied, obtaining the state $A$. Then the external observer puts the judgement $\vdash A$, meaning: “the qubit is in the state $A$” [1].

Analogously, by the application of the projector $P_1$, the observer would obtain the state $A^\perp$, and then she would put the judgement $\vdash A^\perp$.

Now, we consider the reversible measurement discussed in Sect. 2. Eq. (6) means that the external observer can interpret the mirror measurement as the superposition of two orthogonal projectors. Then, in logical terms, the reversible measurement gives back a couple of judgements:

$$\vdash A \quad \vdash A^\perp$$

Which are given by the simultaneous actions of $P_0$ and $P_1$. Now, as in [1], we follow the reflection principle in [21]. By the reflection principle, the logical connectives are the result of importing some existent meta-linguistic links between assertions into the formal language. Let us consider the physical link of superposition between orthogonal states, which is possible only inside the quantum
computer. This becomes a logical link between opposite judgements once the superposition has been measured, obtaining the couple of judgements above. We interpret the juxtaposition of two judgements by the additive conjunction \( \& \) and put:

\[
\vdash A \& A^\perp \equiv \vdash A \vdash A^\perp
\]

(18)

The external observer interprets the reversible measurement by means of (18), or by the inference derived from it:

\[
\begin{align*}
\vdash A & \vdash A^\perp \\
\vdash & \vdash A \& A^\perp
\end{align*}
\]

(19)

We remind that the internal observer is unaware of the possibility of writing \( U^D_2 \) as a linear superposition of projectors, as to her, projectors are meaningless, being non-unitary, thus not belonging to the quantum network. Then, the couple of judgements (17) is not available to her, so that she cannot perform the derivation (19); she just interprets the measurement of the superposed state as the axiom:

\[
\vdash A \& A^\perp
\]

(20)

Which is the opposite of the non-contradiction principle.

Axiom (20) coincides with the conclusion of the external observer in (19). To the internal observer, the axiom (20) means precisely the following: “The superposed state has been measured by a mirror measurement”. Then, the proposition “\( \vdash A \& A^\perp \)” represents, in logical terms, the superposition of the two orthogonal states, in a qualitative way, without taking into account the probability amplitudes. As a diagonal quantum logic gate, which does not change the truth-values, performs the mirror measurement, we say that a superposed state is “measurable” if and only if the following axiom holds:

\[
A \& A^\perp \vdash A \& A^\perp
\]

(21)

Where here the sequent symbol \( \vdash \) must be read: the transition is done by means of a \( U^D_2 \) (a generalized sequent calculus for quantum computing, comprising non-diagonal unitary transformations, is under study [3].

### 3.2 How the Internal Observer Gets Rid of the Excluded Middle

Let us suppose now that the external observer applies a standard quantum measurement to the qubit and then decides to apply a classical NOT gate on the result. If she had obtained, for example after the measurement, she would get \( A^\perp \) after the NOT, but now she cannot assert \( A^\perp \) as true, since it is not the result of her measurement! In fact, the correct assertion is that \( A^\perp \) is false. We write such assertion as in [21]: \( A^\perp \vdash \), that is a primitive way to say that \( A^\perp \) is false. The composite operation just discussed, converted the original judgement \( \vdash A \) into \( A^\perp \vdash \). So, we put the equivalence:

\[
(\vdash A)^\perp \equiv A^\perp \vdash
\]
If instead she measured $A^\perp$, and performed a NOT afterwards, she would get the judgement: $A \vdash$. If by absurd, she could perform the two measurements simultaneously, and apply a NOT afterwards, she would obtain both the judgements:

$$A^\perp \vdash A \vdash$$

This is a logical link between two “falsity judgements”, and, as in [21], it is solved as:

$$A \oplus A^\perp \vdash \equiv A \vdash A^\perp \vdash$$  \hspace{1cm} (22)

Where $\oplus$ is the additive logical disjunction. Let us recall, however, that the external observer $G$ cannot perform the two measurements simultaneously, while the internal observer $P$ can. Then, $P$ puts the following falsity judgement that is an axiom:

$$A^\perp \oplus A \vdash$$  \hspace{1cm} (23)

which implies that the excluded middle principle does not hold inside the quantum computer. Axiom [23] states that the superposed state has been measured by a Liar measurement. In logical terms, the Liar measurement means that inside the quantum computer the judgement $\vdash A \& A^\perp$ has been reversed: $A^\perp \oplus A \vdash$. Axiom (23) is so recovered from axiom (20) by symmetry, a fundamental feature of Basic Logic [21]. The connective $\oplus$ is the symmetric of $\&$ since propositions ”$A^\perp \oplus A$ and ”$A \& A^\perp$” both represent the superposed state. In conclusion, $\oplus$ and $\&$ are logical connectives corresponding to the same meta-linguistic link: superposition.

The axiom:

$$A^\perp \oplus A \vdash A^\perp \oplus A$$  \hspace{1cm} (24)

Is obtained by making the symmetric of axiom [21], and must be interpreted in the following way: a superposed state, which has already been measured by a Liar measurement, can be measured by a mirror measurement performed by $U_D^P$. Axioms (20) and (23) state that, in the black box, both the non-contradiction and the excluded middle principles do not hold. This means that inside the black box the adequate logic is a paraconsistent and symmetric logic, like Basic Logic. The additive connectives $\&$ and $\oplus$ are the only ones we can consider, when we deal with a 1-qubit model.

### 3.3 The Qubit and the Mirror: Some Philosophical Implications

To us, the philosophical meaning of axiom [21] is the following. It looks like the superposed state reflects itself in a slightly deformed mirror, that is, the diagonal unitary operator, which just changes the probability amplitudes, but leaves unchanged the truth-values (the identity operator being the perfect mirror). This analogy would suggest that the qubit has gone through a self-measurement. The act of “looking at itself in the mirror” confirms the existence of the qubit as it stands. This is what we would call **objectivity** of an elementary quantum system. This is on the same line of thought followed by Mermin in [16].
Among his six desiderata for an interpretation of quantum mechanics, the first concerns objective reality which should be separated from external observers and their “knowledge”, and this is in fact our case, as we do not add external judgements to the logical interpretation of the insider observer. The fourth desideratum is that quantum mechanics should not require the existence of a classical domain, which is in fact our case, as the internal observer just represents a quantum mechanical system in a quantum domain (the fuzzy sphere). The sixth desideratum, that the probabilities should be objective properties of individual systems, is fulfilled in the mirroring of the qubit. We think that this philosophical interpretation might be useful in the case of two entangled qubits, as the “spooky action at a distance”, as seen by an external observer might be related to a reversible measurement of the entangled state.

4 The Border between the Black Box and the Classical World

In Sect. 3 we gave a logical description of the reversible quantum measurement performed in the black box. To be exhaustive, however, we should also understand the right way to pass through the border between the black box and the classical world, (the infamous measurement problem) in logical terms. To this aim, we will exploit axioms 20 and 23. To illustrate the border of the black box, we present two possible schemes: the first is physical, the second is logical.

4.1 The Physical Scheme

The physical scheme is the following: a classical input is provided to the quantum computer from the classical world; the quantum computation is the black box. In the black box, an insider observer P performs a reversible quantum measurement. A classical output is obtained by the action of an external observer G, who, from outside, opens the quantum system by performing a standard quantum measurement. In the physical scheme, then, the border coincides with the standard (projective) quantum measurement process. See Fig. 5.

4.2 The Logical Scheme

The logical scheme is the following: if the black box is to be interpreted by an external logician, it can be considered embedded in the classical world at the left and right sides (past and future). On the left side, from where the classical input is provided, we can imagine an external observer A (Aristotle) who can reason by classical logic. In the black box, we have a new kind of quantum logic which is the logic of the insider observer P. She can manage the two new axioms 20 and 23 as far as the border. On the right side of the black box, however, one cannot immediately place A reading the classical output. As, in that case, A would receive the two new axioms that for him, who is a classical logician, are
contradictions. In fact, $A$ can identify $A^\perp$ as follows:

$$A^\perp \equiv A \rightarrow \perp \quad (25)$$

Also, he can use the modus ponens rule:

$$\frac{A}{A \rightarrow \perp} \perp \quad (26)$$

The identity $25$ and the rule $26$ would then allow to derive the falsum from the premises $A$ and $A^\perp$.

Therefore, prior to $A$, there must be another external observer, who is a quantum logician: $G$. In fact, $G$ cannot apply $25$ and $26$ as her logic is too weak. She is able to drop the two new axioms in the following way: she applies the cut rule to $23$, obtaining the following derivation, valid in quantum logic:

$$\frac{\vdash A \& A^\perp}{\vdash A \& L} \quad \frac{A^\perp \vdash A \& L}{\vdash A} \quad \text{cut}$$

Then, from $27$ one can see that the cut rule, when applied to the axiom $20$, is equivalent to the physical demolition of the superposed state, as it "creates a projector".

It should be noticed that the insider observer $P$ can provide one new axiom $20$ only once, because of the no-cloning theorem $??$, which forbids the copy of an arbitrary quantum state. This is a very useful theorem, otherwise $G$ would not be able to drop the axioms (as she does using $27$), which will then provided to $A$, as a contradiction! In fact, let us suppose, by absurd, that the no-cloning theorem did not hold, and $P$ could give the axiom $20$ to $G$ twice. Then $G$ could derive first $\vdash A$ and then $\vdash A^\perp$, and finally she would derive again $\vdash A \& A^\perp$, as follows: she would perform the derivation $27$ by using the axiom $20$ for the first time. Then, she would perform the same derivation for the second time, by replacing the sequent calculus axiom $A \vdash A$ by $A^\perp \vdash A$ and obtaining then $\vdash A^\perp$. At this point, $G$ could apply $27$ and thus get the conclusion $\vdash A \& A^\perp$. That is, she would get again the axiom $20$ (which is false in the external world) just by logical reasoning, not by a mirror measurement.

However, $G$, having measured $A$ in the standard way, can logically conclude $\vdash A \oplus A^\perp$, and $A \& A^\perp \vdash$. They are instances of the excluded middle and of the non-contradiction principles. In this case such instances are like empirical laws, as they follow from a measurement, not from physical reasoning. So, $G$ has a logical point of view opposite to that of $P$. Notice finally that, if the no-cloning theorem did not hold, $G$, besides providing contradiction to $A$ as shown above, would also fall in contradiction herself. In fact, she could derive axioms $21$ and $22$ which are the opposite of her results, that is non-contradiction and excluded middle. In summary, the physical border of the black box corresponds, in logical terms, to the site of an intermediate logic (between black box logic and classical logic), which is standard quantum logic. See Fig. 6. To conclude this section, we wish to illustrate the main differences among observers $A$, $G$, and $P$, in both the physical and logical ways. Let us consider first the differences in the
physical sense. $A$ is an external observer, who does not perform any kind of quantum measurement. He lives in a classical world, without any interaction with the quantum world. He just provides the classical input to the quantum computer, and considers the classical output as an element of the classical world (for example one bit of a classical computer) without any link to the quantum computer. $P$ is an insider observer: she is in a quantum space whose states are in a one-to-one correspondence with the machine states, so that she can perform a reversible measurement, without hidden information. $G$ is also an external observer like $A$, but she can perform a standard quantum measurement. However, $PG$ lives in a classical space-time, so that she breaks the isomorphism (created by $P$) between observer and machine. For this reason, $G$ can only perform a standard quantum measurement (which is irreversible). She opens the black box, and destroys the superposed state, that is, she is responsible of the hidden quantum information. Now, let us consider the differences in the logical sense. $A$ (Aristotle) is a classical logician, who believes in the excluded middle and in the non-contradiction principles, only by logical reasoning, in a formal way, as he never does any kind of quantum measurement. He is in total contradiction with $P$, so he can never communicate with her. However, $A$ can communicate with $G$, because they are not in contradiction; simply, $G$ has a weaker logic than $A$. $G$ can communicate with both $A$ and $P$. $G$ believes in the excluded middle and non-contradiction, but not in the formal way like $A$, as she relies her beliefs on standard quantum measurement. The logic of $P$ is still under study (see [?] for the two-qubits model, in particular the Bell’s states); at present we only know it is paraconsistent and symmetric when $P$ stands on a 1-dimensional subspace of the fuzzy sphere. However, we foresee the following. Having at her disposal such a big amount of quantum information (encoded in a couple of very strong axioms), $P$ will need fewer structural rules than $G$. Finally, as we have seen, $P$ can communicate with $G$ because of the no-cloning theorem, but $P$ cannot, by no means, communicate directly with $A$. In meta-language, let us introduce the new meta-connective $@$ (communicate). For the previous arguments, we have: $A@G$, $G@P$ but not $A@P$. That is, the transitive rule does not hold for this communication process.

5 Conclusions

In this paper, we have considered quantum computation from the point of view of three different observers: $P$, the insider one, and $G$ and $A$, the external ones. $P$ can reason in terms of a new quantum logic, which is paraconsistent and symmetric, $G$ in terms of standard quantum logic, and $A$ in terms of classical logic. What made possible this logical distinction is the physical distinction between two different kinds of quantum measurement: the reversible one, from inside, and the irreversible one, from outside. The physical distinction, moreover, is based on the geometrical distinction between quantum and classical backgrounds, which is based itself on the distinction between non commutative and commutative $C^*$-algebras. In this way, we put together quantum computing
(and quantum physics), geometry, algebra and logic, as summarized in Table 1. We are aware of the fact that our toy model is, up to now, limited to the case of only one qubit and two qubits, for the Bell’s states \[?\]. To conclude, we would like to make the following remark: we believe that our logical approach to quantum computing might be useful in conceiving quantum control in quantum information theory.

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Fig. 1

The Bloch Sphere

$|0\rangle + b|1\rangle$
Fig. 2

The Mirror Measurement
Fig. 3

The Fuzzy Measurement
Fig. 4

The Liar Measurement
Fig. 5

The Physical Scheme

BLACK BOX
Insider observer P

reversible measurement

border

standard quantum measurement

classical output

External observer G

classical input
The Logical Scheme

![Diagram showing classical logic, black box logic, quantum logic, and classical logic connected by an arrow labeled "corresponds to the physical border"]