Degree three spin Hurwitz numbers

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Abstract

Recently, Gunningham [G] calculated all spin Hurwitz numbers in terms of combinatorics of Sergeev algebra. In this paper, we use a spin curve degeneration to obtain a recursion formula for degree three spin Hurwitz numbers.

Let $D$ be a complex curve of genus $h$ and $N$ be a theta characteristic on $D$, i.e. $N^2 = K_D$. The pair $(D, N)$ is called a spin curve of genus $h$ with parity $p \equiv h^0(N) \pmod{2}$. For $i = 1, \cdots, k$, let $m^i = (m^i_1, \cdots, m^i_{\ell_i})$ be an odd partition of $d > 0$, namely all components $m^i_j$ are odd. Fix $k$ points $q^1, \cdots, q^k$ in $D$ and consider degree $d$ maps $f : C \to D$ from possibly disconnected domains $C$ of Euler characteristic $\chi$ that are ramified only over the fixed points $q^i$ with ramification data $m^i$. Observe that the Riemann-Hurwitz formula shows

$$2d(1-h) - \chi + \sum_{i=1}^k (\ell(m^i) - d) = 0 \quad (0.1)$$

where $\ell(m^i) = \ell_i$ is the length of $m^i$. By the Hurwitz formula, the twisted line bundle

$$L_f = f^*N \otimes \mathcal{O}(\sum_{i,j} \frac{1}{2}(m^i_j - 1)x^i_j) \quad (0.2)$$

is a theta characteristic on $C$ where $f^{-1}(q^i) = \{x^i_j\}_{1 \leq j \leq \ell_i}$ and $f$ has multiplicity $m^i_j$ at $x^i_j$. We define the parity $p(f)$ of a map $f$ by

$$p(f) \equiv h^0(L_f) \pmod{2}. \quad (0.3)$$

Given odd partitions $m^1, \cdots, m^k$ of $d$, the spin Hurwitz number of genus $h$ and parity $p$ is defined as a (weighted) sum of (ramified) covers $f$ satisfying (0.1) with sign determined by the parity $p(f)$:

$$H^{h,p}_{m^1, \cdots, m^k} = \sum_f \frac{(-1)^{p(f)}}{|\text{Aut}(f)|} \quad (0.4)$$

Eskin, Okounkov and Pandharipande [EOP] calculated the genus $h = 1$ and odd parity spin Hurwitz numbers in terms of characters of Sergeev group. Recently, Gunningham [G] calculated all spin Hurwitz numbers in terms of combinatorics of Sergeev algebra.

The trivial partition $(1^d)$ of $d$ is a partition whose components are all one. If $m^k = (1^d)$, then $f$ has no ramification points over the fixed point $q^k$ and hence we have

$$H^{h,p}_{m^1, \cdots, m^{k-1}, (1^d)} = H^{h,p}_{m^1, \cdots, m^{k-1}}. \quad (0.5)$$
When all partitions \(m^i = (1^d)\), denote the spin Hurwitz numbers (0.4) by \(H^{h,p}_d\). These are dimension zero local GW invariants \(GT_d^{loc,h,p}\) of spin curve \((D,N)\) that give all dimension zero GW invariants of Kähler surfaces with a smooth canonical divisor (cf. [KL1], [KL2], [LP1], [MP]). For notational simplicity, we set \(H^{h,p}_3 = H^{h,p}_3\) and for \(k \geq 1\) write
\[
H^{h,p}_{(3)^k}
\]
for the spin Hurwitz numbers \(H^{h,p}_{(3),(3),\cdots,(3)}\) with the same \(k\) partitions \((3)\). Since there are two odd partitions \((1^3)\) and \((3)\) of \(d = 3\), by (0.5) it suffices to compute \(H^{h,p}_{(3)^k}\) for \(k \geq 0\). The aim of this paper is to use a spin curve degeneration to obtain the following recursion formula.

**Theorem 0.1.** If \(h = h_1 + h_2\) and \(p \equiv p_1 + p_2 \pmod{2}\) then for \(k_1 + k_2 = k\)
\[
H^{h,p}_{(3)^k} = 3! H^{h_{1},p_1}_{(3)^{k_1}} \cdot H^{h_{2},p_2}_{(3)^{k_2}} + 3 H^{h_{1},p_1}_{(3)^{k_1+1}} \cdot H^{h_{2},p_2}_{(3)^{k_2+1}}. 
\]  

(0.6)

One can use Theorem 0.1 and the result of [EOP] to explicitly compute the degree \(d = 3\) spin Hurwitz numbers. In Proposition 7.1, we show that
\[
H^{h,\pm}_{(3)^{k}} = 3^{2h-2} (-1)^{k} 2^{h-1} \pm 1
\]
where \(+\) and \(−\) denote the even and odd parities. When the degree \(d = 1, 2\), the dimension zero local GW invariants are given by the formulas
\[
GT_1^{loc,h,\pm} = \pm 1 \quad \text{and} \quad GT_2^{loc,h,\pm} = \pm 2^{h-1}
\]
(cf. Lemma 2.6 of [L]). Since \(GT_d^{loc,h,p} = H^{h,p}_d\) as mentioned above, the formula (0.7) shows
\[
GT_3^{loc,h,\pm} = 3^{2h-2} (2^{h-1} \pm 1).
\]
This calculation is, in fact, the main motivation for the paper.

In Section 1, we express the degree \(d\) spin Hurwitz numbers (0.4) in terms of relative GW moduli spaces. We can then apply a degeneration method for a family of curves \(D \to \Delta\) where the central fiber \(D_0\) is a nodal curve and the general fiber \(D_\lambda (\lambda \neq 0)\) is a smooth curve. Section 2 describes the relative moduli space \(M_0\) of maps \(f\) into the nodal curve \(D_0\). In Section 3, we show that the union over \(\lambda \in \Delta\) of relative moduli spaces \(M_\lambda\) of maps into \(D_\lambda\) consists of connected components \(Z_{m,f} \to \Delta\) containing \(f \in M_0\). Here \(m\) is the ramification data of \(f\) over nodes of \(D_0\) such that \(d - \ell(m)\) is even.

The (ordinary) Hurwitz numbers are sums of (ramified) maps modulo automorphism without sign. One can easily obtain a recursion formula for Hurwitz numbers by counting maps in the general fiber of \(Z_{m,f} \to \Delta\). For spin Hurwitz numbers, one needs to calculate parities of maps induced from a fixed spin structure on the family of curves \(D\).

The novelty of our approach is to apply a Schiffer variation for the parity calculation. The space \(Z_{m,f}\) is, in general, not smooth. In Section 4, we construct a smooth model for \(Z_{m,f}\) by Schiffer variation. In Section 5, we use the smooth model to twist the pull-back of the spin structure on \(D\). When the degree \(d = 3\), the partition \(m\) is odd, either \((1^3)\) or \((3)\). In this case, a suitable twisting immediately yields a required parity calculation. We prove Theorem 0.1 in Section 6 and the formula (0.7) in Section 7.

For higher degree \(d \geq 4\), the partition \(m\) may not to be odd! A new parity calculation is needed. In [LP2], we generalized the recursion formula (0.6) for higher degree spin Hurwitz numbers by employing additional geometric analysis approach for parity calculation.
1 Dimension zero relative GW moduli spaces

In this section, we express the spin Hurwitz numbers (0.4) in terms of dimension zero relative GW moduli spaces. We will follow the definitions of [IP2] for the relative GW theory.

Let $D$ be a smooth curve of genus $h$ and let $V = \{q^1, \ldots, q^k\}$ be a fixed set of points on $D$. Given partitions $m^1, \ldots, m^k$ of $d$, a degree $d$ holomorphic map $f : C \rightarrow D$ from a possibly disconnected curve $C$ is called $V$-regular with contact vectors $m^1, \ldots, m^k$ if $f^{-1}(V)$ consists of $\sum \ell(m^i)$ contact marked points $x^i_j$ $(1 \leq j \leq \ell(m^i))$ with $f(x^i_j) = q^i$ such that $f$ has ramification index (or multiplicity) $m^i_j$ at $x^i_j$. Two $V$-regular maps $(f, C; \{x^i_j\})$ and $(\tilde{f}, \tilde{C}; \{\tilde{x}^i_j\})$ are equivalent if they are isomorphic, i.e., there is a biholomorphism $\sigma : C \rightarrow \tilde{C}$ with $\tilde{f} \circ \sigma = f$ and $\sigma(x^i_j) = \tilde{x}^i_j$ for all $i, j$. The relative moduli space $\mathcal{M}^V_{\chi, m^1, \ldots, m^k}(D, d)$ (1.1)

consists of equivalence classes of $V$-regular maps $(f, C; \{x^i_j\})$ with the Euler characteristic $\chi(C) = \chi$ and with contact vectors $m^1, \ldots, m^k$. Since no confusion can arise, we will regard a point in the space (1.1) as a $V$-regular map $(f, C; \{x^i_j\})$. For simplicity, we will often write a $V$-regular map $(f, C; \{x^i_j\})$ simply as $f$.

The (formal) complex dimension of the space (1.1) is given by the left-hand side of the Riemann-Hurwitz formula (0.1):

$$2d(1 - h) - \chi - \sum_{i=1}^{k} \left( d - \ell(m^i) \right).$$

Suppose this dimension is zero. Then, for each $V$-regular map $(f, C; \{x^i_j\})$ in (1.1), forgetting the contact marked points $x^i_j$ gives a (ramified) cover $f$ that is ramified only over fixed points $q^i$ and satisfies (0.1). The automorphism group $\text{Aut}(f)$ of a (ramified) cover $f$ consists of automorphisms $\sigma \in \text{Aut}(C)$ with $f \circ \sigma = f$. The automorphism group $\text{Aut}(f, V)$ of a $V$-regular map $(f, C; \{x^i_j\})$ consists of automorphisms $\sigma \in \text{Aut}(f)$ with $\sigma(x^i_j) = x^i_j$ for all $i, j$.

For a partition $m$ of $d$, let $\text{Aut}(m)$ be the subgroup of symmetric group $S_{\ell(m)}$ permuting equal parts of the partition $m$.

Lemma 1.1. Let $m^1, \ldots, m^k$ be as above and suppose the dimension (1.2) is zero.

(a) If $m^i = (1^d)$ for some $1 \leq i \leq k$, then $\text{Aut}(f, V)$ is trivial for all $f$ in (1.1).

(b) If $m^1, \ldots, m^k$ are all odd partitions, then

$$H^{h, p}_{m^1, \ldots, m^k} = \frac{1}{\prod_{i=1}^{k} |\text{Aut}(m^i)|} \sum_{f} \frac{(-1)^{p(f)}}{|\text{Aut}(f, V)|}$$

where the sum is over all $f$ in (1.1) and $p(f)$ is the parity (0.3).

Proof. Let $(f, C; \{x^i_j\})$ be a $V$-regular map in (1.1) and $\sigma \in \text{Aut}(f, V)$. If $m^i = (1^d)$, then the set of branch points $B$ of $f$ is a subset of $V \setminus \{q^i\}$ and the restriction of $\sigma$ to $C \setminus f^{-1}(B)$ is a
covering transformation that fixes contact marked points $x_j^1, \ldots, x_j^d$. Noting $f^{-1}(B)$ is finite, we conclude that $\sigma$ is an identity map on $C$. This proves (a).

As mentioned above, forgetting contact marked points $x_j$ gives a (ramified) cover $f$ satisfying $(0.1)$. Conversely, given a (ramified) cover $f$ satisfying $(0.1)$, one can mark a point over $q_i$ with ramification index $m_i^j$ as a contact marked point $x_j^i$. Such marking gives $V$-regular maps $(f, C; \{x_j^i\})$ in $\prod_{i=1}^{k} |\text{Aut}(m)|$ ways. Observe that $(f, C; \{x_j^i\})$ and $(f, C; \{\sigma(x_j^i)\})$ are isomorphic for each $\sigma \in \text{Aut}(f)$ and that $\text{Aut}(f, V)$ is a normal subgroup of $\text{Aut}(f)$. Consequently, the quotient group $G = \text{Aut}(f)/\text{Aut}(f, V)$ acts freely on the set of $V$-regular maps $(f, C; \{x_j^i\})$ obtained by the (ramified) cover $f$. Its orbits give $\prod_{i=1}^{k} |\text{Aut}(m)|/|G|$ points (i.e. equivalence classes of $V$-regular maps) in the space $(1.1)$, each of which has the same automorphism group $\text{Aut}(f, V)$. Now, (b) follows from counting maps with the parity of map modulo automorphisms.

\[ \square \]

2 \ Maps into a nodal curve

Let $D_0 = D_1 \cup E \cup D_2$ be a connected nodal curve of (arithmetic) genus $h$ with two nodes $p^1$ and $p^2$ such that for $i = 1, 2$, $E = \mathbb{P}^1$ meets $D_i$ at node $p^i$ and $D_i$ has genus $h_i$ with $h_1 + h_2 = h$.

In this section, we consider maps into $D_0$ that are relevant to our subsequent discussion.

In the below, we fix $d, h, \chi$ and odd partitions $m_1, \ldots, m_k$ of $d$ so that the Riemann-Hurwitz formula $(0.1)$ holds, or equivalently, the dimension formula $(1.2)$ is zero. For each partition $m$ of $d$, consider the product space

$$\mathcal{P}_m = \mathcal{M}_{\chi, (1^d), m_1, \ldots, m_k}(D_1, d) \times \mathcal{M}_{\chi, (1^d), m}(E, d) \times \mathcal{M}_{\chi, (1^d), m_1, \ldots, m_k}(D_2, d)$$

where $V_1 = \{q^{k+1}, q^1, \ldots, q^{k_i}, p^1\}$, $V_0 = \{p^1, q^{k+2}, p^2\}$, $V_2 = \{p^2, q^{k+1}, \ldots, q^{k_k}, q^{k+3}\}$ and

$$\chi_1 + \chi_0 + \chi_2 - 4\ell(m) = \chi.$$  \hspace{1cm} (2.1)

For simplicity, let $\mathcal{M}_m^1, \mathcal{M}_m^0$ and $\mathcal{M}_m^2$ denote the first, the second and the third factors of $\mathcal{P}_m$.

**Lemma 2.1.** If $\mathcal{P}_m \neq \emptyset$, then the spaces $\mathcal{M}_m^1, \mathcal{M}_m^0$ and $\mathcal{M}_m^2$ have dimension zero. Consequently, $\chi_0 = 2\ell(m)$ and $d - \ell(m)$ is even.

**Proof.** Each $\mathcal{M}_m^i$ $(0 \leq i \leq 2)$ has nonnegative dimension by the Riemann-Hurwitz formula. The formula (2.1) and our assumption that the dimension $\chi_0 = 2\ell(m)$ is zero thus imply that each $\mathcal{M}_m^i$ has dimension zero. The dimension formulas for $\mathcal{M}_m^0$ and $\mathcal{M}_m^i$ $(i = 1, 2)$ then show that $\chi_0 = 2\ell(m)$ and $d - \ell(m)$ is even because $d - \ell(m^i) = \sum (m_j^i - 1)$ is even for all $1 \leq i \leq k$. \hspace{1cm} \( \square \)

Let $|A|$ denote the cardinality of a set $A$.

**Lemma 2.2.** $|\mathcal{M}_m^0| = \frac{d! |\text{Aut}(m)|}{\prod m_j^i}$.

**Proof.** Let $f \in \mathcal{M}_m^0$. Since $\chi_0 = 2\ell(m)$, we have
the domain of \( f \) is a disjoint union of smooth rational curves \( E_j \) for \( 1 \leq j \leq \ell(m) \),

- each restriction \( f_j = f|_{E_j} \) has exactly one contact marked point over \( p^i \) (\( i = 1, 2 \)) with multiplicity \( m_j \), so \( f_j \) has degree \( m_j \).

Consequently, forgetting contact marked points of maps in \( \mathcal{M}^0_m \) gives exactly one map (as a cover) with automorphism group of order \( |\text{Aut}(m)| \prod m_j \). Here the factor \( |\text{Aut}(m)| \) appears because we can relabel maps \( f_j \) in \( |\text{Aut}(m)| \) ways and the factor \( \prod m_j \) appears because each restriction map \( f_j \) (as a cover) has an automorphism group of order \( m_j \). The argument in the proof of Lemma 1.1 then shows the lemma.

For each \( (f_1, f_0, f_2) \in P_m \), by identifying contact marked points over \( p_i \) \( (i = 1, 2) \), one can glue the domains of \( f_1 \) and \( f_0 \) to obtain a map \( f : C \to D_0 \) with \( \chi(C) = \chi \). For notational convenience, we will often write the glued map \( f \) as \( f = (f_1, f_0, f_2) \). Denote by

\[
\mathcal{M}_{m,0}
\]

the space of such glued maps \( f = (f_1, f_0, f_2) \). Contact marked points are labeled, but nodal points of \( C \) are not labeled. Thus, we have:

**Lemma 2.3.** \( P_m \) is a degree \( |\text{Aut}(m)|^2 \) cover of \( \mathcal{M}_{m,0} \).

### 3 Limiting and gluing

Following [IP2], this section describes limiting and gluing arguments under a degeneration of target curves. Let \( D_0 = D_1 \cup E \cup D_2 \) be the nodal curve with fixed points \( q^1, \cdots, q^{k+3} \) as in Section 2. In Section 3, we will construct a family of curves together with \( k + 3 \) sections:

\[
\begin{array}{c}
\pi \\
Q^i \downarrow \\
\Delta
\end{array}
\]

Here the total space \( \mathcal{D} \) is a smooth complex surface, \( \Delta \subset \mathbb{C} \) is a disk with parameter \( \lambda \), the central fiber is \( D_0 \), the general fiber \( D_\lambda \) (\( \lambda \neq 0 \)) is a smooth curve of genus \( h \) and \( Q^i(0) = q^i \) for \( 1 \leq i \leq k + 3 \). By Gromov Convergence Theorem, a sequence of holomorphic maps into \( D_\lambda \) with \( \lambda \to 0 \) has a map into \( D_0 \) as a limit. For notational simplicity, for \( \lambda \neq 0 \) we set

\[
\mathcal{M}_\lambda = \mathcal{M}^{V_\lambda} \left( D_\lambda, d \right) \quad \text{where} \quad V_\lambda = \{Q^1(\lambda), \cdots, Q^{k+3}(\lambda)\},
\]

and denote the set of limits of sequences of maps in \( \mathcal{M}_\lambda \) as \( \lambda \to 0 \) by

\[
\lim_{\lambda \to 0} \mathcal{M}_\lambda.
\]

**Lemma 3.1** below shows that limit maps in (3.3) lie in the union of spaces (2.2), namely

\[
\lim_{\lambda \to 0} \mathcal{M}_\lambda \subset \bigcup_m \mathcal{M}_{m,0}
\]

(3.4)
where the union is over all partitions $m$ of $d$ with $d - \ell(m)$ even.

Conversely, by the Gluing Theorem of [IP2], the domain of each map in $M_{m,0}$ can be smoothed to produce maps in $M_{\lambda}$ for small $|\lambda|$. Shrinking $\Delta$ if necessary, for $\lambda \in \Delta$, one can assign to each $f_{\lambda} \in M_{\lambda}$ a partition $m$ of $d$ by (3.3). Let $M_{m,\lambda}$ be the set of all pairs $(f_{\lambda}, m)$.

For each $f \in M_{m,0}$, let

$$Z_{m,f} \to \Delta$$

be the connected component of $\bigcup_{\lambda \in \Delta} M_{m,\lambda} \to \Delta$ that contains $f$ and let

$$Z_{m,f,\lambda}$$

denote the fiber of (3.5) over $\lambda \in \Delta$. It follows that for $\lambda \neq 0$

$$M_{\lambda} = \bigsqcup_{f \in M_{m,0}} Z_{m,f,\lambda}.$$  

For $f = (f_1, f_0, f_2) \in M_{m,0}$ where $m = (m_1, \ldots, m_{\ell})$, let $y_{f_j}^i$ be the node mapped to $p^i$ at which $f_i$ and $f_0$ have multiplicity $m_j$. The Gluing Theorem shows that one can smooth each node $y_{f_j}^i$, in $m_j$ ways, to produce $(\prod m_j)^2$ maps in $Z_{m,f,\lambda}$, so we have

$$|Z_{m,f,\lambda}| = (\prod m_j)^2 \quad (\lambda \neq 0).$$

In order to prove (3.4), we will use the following fact on stable maps. An irreducible component of a stable holomorphic map $f$ is a ghost component if its image is a point. Write the domain of $f$ as $C^g \cup C$ where $C^g$ is a connected curve whose irreducible components are all ghost components. Then the stability of $f$ implies that

$$\chi(C^g) - \ell^g - n \leq -1$$

where $\ell^g = |C^g \cap C|$ and $n$ is the number of marked points on $C^g$. 

Lemma 3.1. Let $M_r$ and $M_{m,0}$ be as above. Then we have

$$\lim_{\lambda \to 0} M_{\lambda} \subset \bigcup_m M_{m,0}$$

where the union is over all partitions $m$ of $d$ with $d - \ell(m)$ even.

Proof. Let $f$ be a limit map in (3.3). The domain $C$ of $f$ can be written as

$$C = C_1 \cup C_0 \cup C_2 \cup \bigcup_{i=1}^{k+3} C_i^g \cup C^g \cup \hat{C}^g$$

where $C_0$ maps to $E$, $C_1$ and $C_2$ map to $D_1$ and $D_2$, $C_i^g$ is the union of all ghost components over $q^i$ where $i = 1, \ldots, k+3$, $C^g$ is the union of all ghost components over points in $D_0 \setminus (V_1 \cup V_0 \cup V_2)$ and $\hat{C}^g$ is the union of all ghost components over $\{p^1, p^2\}$. Let $f_j = f_{|C_j}$ for $j = 0, 1, 2$. Observe that $f_j$ is $V_j$-regular because $C_j$ has no ghost components. Let $\hat{m}^i$ be a contact vector over $q^i$,
\( \tilde{m}^1 \) and \( \tilde{m}^2 \) be contact vectors of \( f_1 \) and \( f_2 \) over \( p^1 \) and \( p^2 \), and \( \tilde{m}^{0;1} \) and \( \tilde{m}^{0;2} \) be contact vectors of \( f_0 \) over \( p^1 \) and \( p^2 \). The Riemann-Hurwitz formulas for \( f_0, f_1 \) and \( f_2 \) give

\[
\sum_{j=0}^{2} \chi(C_j) \leq 2d(1-h) + \sum_{i=1}^{k+3} \left( \ell(\tilde{m}^i) - d \right) + \sum_{i=1}^{2} \left( \ell(\tilde{m}^i) + \ell(\tilde{m}^{0;i}) \right). \tag{3.11}
\]

For \( i = 1, \cdots, k + 3 \), let \( \ell_i = |C_1 \cup C_0 \cup C_2 \cap C_i^g| \) and let \( n_i \) be the number of marked points on \( C_i^g \). Since all marked points are limits of marked points, we have

\[
\ell(\tilde{m}^i) = \ell(m^i) - n_i + \ell_i. \tag{3.12}
\]

For \( j = 0, 1, 2 \), let \( \tilde{\ell}_j = |C_j \cap \tilde{C}^g| \). Counting the number of nodes mapped to \( p^1 \) and \( p^2 \) shows

\[
\sum_{i=1}^{2} \left( \ell(\tilde{m}^i) - \tilde{\ell}_i \right) = \sum_{i=1}^{2} |C_i \cap C_0| = \sum_{i=1}^{2} \ell(\tilde{m}^{0;i}) - \tilde{\ell}_0. \tag{3.13}
\]

Let \( \ell^g = |C_1 \cup C_0 \cup C_2 \cap C^g| \). Since \( \chi(C) = \chi \), by (3.10) and (3.13) we have

\[
\chi = \sum_{j=0}^{2} \chi(C_j) + \sum_{i=1}^{k+3} \left( \chi(C_i^g) - 2\ell_i \right) + \chi(C^g) - 2\ell^g + \chi(\tilde{C}^g) - \tilde{\ell} - \sum_{i=1}^{2} \left( \ell(\tilde{m}^i) + \ell(\tilde{m}^{0;i}) \right) \tag{3.14}
\]

where \( \tilde{\ell} = \tilde{\ell}_0 + \tilde{\ell}_1 + \tilde{\ell}_2 \). By our assumption that the formula (0.1) holds, it follows from (3.11), (3.12) and (3.14) that

\[
\chi \leq \chi + \sum_{i=1}^{k+3} \left( \chi(C_i^g) - \ell_i - n_i \right) + \chi(C^g) - 2\ell^g + \chi(\tilde{C}^g) - \tilde{\ell}. \tag{3.15}
\]

Noting \( C^g \) and \( \tilde{C}^g \) have no marked points, by (3.9) and (3.15) we conclude that the domain \( C \) of \( f \) has no ghost components. Consequently,

- \( f_j \) is \( V_j \)-regular for \( j = 0, 1, 2 \),
- \( \tilde{m}^i = \tilde{m}^{0;i} \) for \( i = 1, 2 \) (cf. Lemma 3.3 of [IP2]) and \( \tilde{m}^i = m^i \) for \( i = 1, \cdots, k + 3 \).

In particular, the equality in (3.11) holds; otherwise we have a strict inequality in (3.15). So, we have \( \chi(C_0) = \ell(\tilde{m}^1) + \ell(\tilde{m}^2) \). But \( \chi(C_0) \leq 2 \min\{\ell(\tilde{m}^1), \ell(\tilde{m}^2)\} \). It follows that

- \( C_0 \) has \( \ell(\tilde{m}^1) = \ell(\tilde{m}^2) \) connected components \( E_j \) with \( \chi(E_j) = 2 \) for all \( j \),
- \( \tilde{m}_j^1 = \deg(f_0|_{E_j}) = \tilde{m}_j^2 \) for all \( j \), i.e., \( \tilde{m}^1 = \tilde{m}^2 \).

It follows that the Euler characteristics of \( C_0, C_1 \) and \( C_2 \) satisfy (2.1) by (3.14). Therefore, \( f \in \mathcal{M}_{m,0} \) for \( m = \tilde{m}^1 = \tilde{m}^2 \) and \( d - \ell(m) \) is even by Lemma 2.1. \( \square \)
4 Smooth model by Schiffer variation

A Schiffer Variation of a nodal curve (cf. pg. 184 of [ACG]) is obtained by gluing deformations $uv = \lambda$ near nodes with the trivial deformation away from nodes. In this section, we use the method of Schiffer variation to construct a smooth model for the space $Z_{m,f}$ in (3.5) which has several branches intersecting at $f$ unless $m$ is trivial.

Throughout this section, we fix an odd partition $m = (n\ell)$, i.e. $m = (m_1, \cdots, m_\ell)$ with
\[
m_1 = \cdots = m_\ell = n \quad \text{where } n = d/\ell \text{ is odd.} \tag{4.1}
\]
Let $f = (f_1, f_0, f_2)$ be a map in $\mathcal{M}_{m,0}$ in (2.2). As described in Section 2, the central fiber of $\rho : D \to \Delta$ is the nodal curve $D_0 = D_1 \cup E \cup D_2$ with two nodes $p_1 \in D_1 \cap E$ and $p_2 \in D_2 \cap E$ where $E = \mathbb{P}^1$. The domain of $f$ is a nodal curve
\[C = C_1 \cup C_0 \cup C_2 \quad \text{where } C_0 = \cup_{j=1}^{\ell} E_\ell\]
with $2\ell$ nodes such that for $i = 1, 2$ and $j = 1, \cdots, \ell$,
- $f^{-1}(p_i)$ consists of the $\ell$ nodes $y^i_j \in C_i \cap E_j$,
- $C_i$ is smooth and $f|_{C_i} = f_i$ has ramification index $m_j = n$ at the node $y^i_j$,
- $E_j = \mathbb{P}^1$ and $f|_{E_j} = f_0|_{E_j} : E_j \to E$ has ramification index $m_j = n$ at the node $y^i_j$.

The following is a main result of this section.

**Proposition 4.1.** Let $f$ be as above. Then, for each vector $\zeta = (\zeta_1^1, \zeta_1^2, \cdots, \zeta_1^1, \zeta_2^2)$ where $\zeta_j^i$ is a $n$-th root of unity, there are a family of curves $\varphi_\zeta : C_\zeta \to \Delta$, with smooth total space $C_\zeta$, over a disk $\Delta$ (with parameter $s$) and a holomorphic map $F_\zeta : C_\zeta \to D$ satisfying:

(a) The central fiber $C_{\zeta,0} = C$ and the restriction map $F_\zeta|_C = f$.

(b) The general fiber $C_{\zeta,s}$ ($s \neq 0$) is smooth and for $\lambda = s^n \neq 0$
\[
\bigcup \{ f_{\zeta,s} \} = Z_{m,f,\lambda} \tag{4.2}
\]
where the union is over all $\zeta$, $f_{\zeta,s} = F_\zeta|_{C_{\zeta,s}}$ and $Z_{m,f,\lambda}$ is the space (3.7).

**Proof.** The proof consists of 4 steps.

**Step 1:** We first show how to construct the family of curves $\rho : D \to \Delta$ with $k + 3$ sections. For $i = 1, 2$, a neighborhood of the node $p^i \in D_i \cap E$ can be regarded as the union $U^i \cup V^i$ of the two disks
\[U^i = \{ w^i \in \mathbb{C} : |w^i| < 1 \} \subset D_i \quad \text{and} \quad V^i = \{ v^i \in \mathbb{C} : |v^i| < 1 \} \subset E\]
with their origins identified. We may assume that the fixed points $q^1, \ldots, q^{k+3}$ in $D_0$ described above (2.1) lie outside these sets. Consider the regions
\[A^i = \{ (u^i, v^i, \lambda) \in U^i \times V^i \times \Delta : u^i v^i = \lambda \}, \]
\[B = \bigcup_{i=1}^{2} G^i \cup \left( D_0 \setminus \bigcup_{i=1}^{2} (U^i \cup V^i) \right) \times \Delta \]

8
where
\[ G^i = \{ (u^i, \lambda) \in U^i \times \Delta : |u^i| > \sqrt{|\lambda|} \} \cup \{ (v^i, \lambda) \in V^i \times \Delta : |v^i| > \sqrt{|\lambda|} \}. \]

We obtain a smooth complex surface \( \mathcal{D} \) by gluing \( A^1, A^2 \) and \( B_0 \) using the maps
\[ G^i \to A^i \text{ defined by } (u^i, \lambda) \to \left( u^i, \frac{1}{\lambda}, \lambda \right) \text{ and } (v^i, \lambda) \to \left( \frac{1}{\lambda}, v^i, \lambda \right). \] (4.3)
Let \( \rho : \mathcal{D} \to \Delta \) be the projection to the last factor and define \( k + 3 \) sections \( Q^i \) of \( \rho \) by
\[ Q^i(\lambda) = (q^i, \lambda). \]

**Step 2:** We can similarly construct a family of curves over a \( 2\ell \)-dimensional polydisk:
\[ \varphi_{2\ell} : \mathcal{X} \to \Delta_{2\ell} = \{ t = (t_1^1, t_1^2, \ldots, t_{\ell}^1, t_{\ell}^2) \in \mathbb{C}^{2\ell} : |t_j^i| < 1 \}. \] (4.4)

For each node \( y_j^i \in C_i \cap E_j \), choose a neighborhood obtained from two disks
\[ U_j^i = \{ u_j^i \in \mathbb{C} : |u_j^i| < 1 \} \subset C_i \quad \text{and} \quad V_j^i = \{ v_j^i \in \mathbb{C} : |v_j^i| < 1 \} \subset E_j \]
by identifying the origins. Consider the regions
\[ A_j^i = \{ (u_j^i, v_j^i, t) \in U_j^i \times V_j^i \times \Delta_{2\ell} : |u_j^i| = t_j^i \}, \]
\[ B_{2\ell} = \bigcup_{i,j} G_j^i \cup (C \setminus \bigcup_{i,j} (U_j^i \cup V_j^i)) \times \Delta_{2\ell} \]
where
\[ G_j^i = \{ (u_j^i, t) \in U_j^i \times \Delta_{2\ell} : |u_j^i| > \sqrt{|t_j^i|} \} \cup \{ (v_j^i, t) \in V_j^i \times \Delta_{2\ell} : |v_j^i| > \sqrt{|t_j^i|} \}. \]

We can then obtain a smooth complex manifold \( \mathcal{X} \) of dimension \( 2\ell + 1 \) by gluing \( \cup A_j^i \) and \( B_{2\ell} \) with the maps
\[ G_j^i \to A_j^i \text{ defined by } (u_j^i, t) \to (u_j^i, v_j^i, t) \text{ and } (v_j^i, t) \to (\sqrt[2\ell]{t_j^i}, v_j^i, t). \] (4.5)

Let \( \varphi_{2\ell} : \mathcal{X} \to \Delta \) be the projection to the factor \( t \).

**Step 3:** Since \( f_i \) and \( f_0|_{E_j} \) have ramification index \( m_j = n \) at \( y_j^i \), we may assume (after coordinates change) that on \( U_j^i \) and \( V_j^i \) the map \( f \) can be written as
\[ U_j^i \to U^i \text{ by } u_j^i \to (u_j^i)^n \quad \text{and} \quad V_j^i \to V^i \text{ by } v_j^i \to (v_j^i)^n. \] (4.6)

For each \( i, j \), define a map
\[ G_j^i \to G^i \text{ by } (u_j^i, t) \to ((u_j^i)^n, t_j^i)^n \text{ and } (v_j^i, t) \to ((v_j^i)^n, t_j^i)^n. \] (4.7)

On the other hand, for each \( i, j \), we have a map
\[ A_j^i \to A^i \text{ defined by } (u_j^i, v_j^i, t) \to ((u_j^i)^n, (v_j^i)^n, (t_j^i)^n). \] (4.8)
These two maps (4.7) and (4.8) are glued together under the maps (4.3) and (4.5). The glued map extends to a holomorphic map \( f_\ell : \mathcal{X}_\ell \rightarrow D_\lambda \) if and only if
\[
(t_1^1)^n = (t_1^2)^n = \cdots = (t_\ell^1)^n = (t_\ell^2)^n = \lambda.
\]

There are \( n^{2\ell} \) solutions \( t \) of (4.9) and the extension map \( f_\ell \) is given by
\[
(x, t) \rightarrow (f(x), \lambda) \quad \text{on} \quad \mathcal{X}_\ell - \bigcup A^i_j.
\]

**Step 4:** For each vector \( \zeta = (\zeta_1^1, \zeta_1^2, \ldots, \zeta_\ell^1, \zeta_\ell^2) \) where each \( \zeta_j^i \) is an \( n \)-th root of unity, define
\[
\delta_\zeta : \Delta \rightarrow \Delta_{2\ell} \quad \text{by} \quad s \rightarrow (\zeta_1^1 s, \zeta_1^2 s, \zeta_2^1 s, \zeta_2^2 s, \ldots, \zeta_\ell^1 s, \zeta_\ell^2 s).
\]

The pull-back \( \delta_\zeta^* \mathcal{X} \) gives a family of curves:
\[
\begin{align*}
\delta_\zeta^* \mathcal{X} & \xrightarrow{\varphi_\zeta} \mathcal{X} \\
\Delta & \xrightarrow{\delta_\zeta} \Delta_{2\ell}
\end{align*}
\]

The central fiber is \( C_{\zeta, 0} = C \) and the general fiber \( C_{\zeta, s} (s \neq 0) \) is smooth. A neighborhood of the node \( y_j^i \) of \( C \) in \( C_{\zeta} \) can be viewed as
\[
\hat{A}_j^i = \{(u_j^i, v_j^i, s) \in \mathbb{C}^3 : |u_j^i| < 1, |v_j^i| < 1, u_j^i v_j^i = \zeta_j^i s\}.
\]

It follows that the total space \( C_\zeta \) is a complex smooth surface. Noting \( \delta_\zeta(s) \) is a solution of (4.9) for \( \lambda = s^n \), we obtain a holomorphic map \( \mathcal{F}_\zeta : C_\zeta \rightarrow \mathcal{D} \) given by
\[
\begin{align*}
(u_j^i, v_j^i, s) & \rightarrow ((u_j^i)^n, (v_j^i)^n, s^n) \quad \text{on} \quad \hat{A}_j^i, \\
(x, s) & \rightarrow (f(x), s^n) \quad \text{on} \quad C_{\zeta} - \bigcup \hat{A}_j^i.
\end{align*}
\]

Since the restriction \( \mathcal{F}_\zeta|_{C} = f \) by (4.6) and (4.12), it remains to show (4.2). By our choice of fixed points \( q^j \) on \( D_0 \), each contact marked point \( x_j^i \) of \( f \) lies in \( C_{\zeta} - \bigcup \hat{A}_j^i \). Thus, by (4.12), the pull-back \( \mathcal{F}_\zeta^* Q^i \) of the section \( Q^i \) of \( \rho \) gives a section \( X_j^i \) of \( \varphi_\zeta \) given by \( X_j^i(s) = (x_j^i, s) \). After marking the points \( X_j^i(s) \) in \( C_{\zeta, s} \), the restriction map
\[
f_{\zeta, s} = \mathcal{F}_\zeta|_{C_{\zeta, s}} : C_{\zeta, s} \rightarrow D_\lambda \quad \text{where} \quad \lambda = s^n \neq 0
\]
has contact marked points \( X_j^i(s) \) over \( Q^i(\lambda) \) with multiplicity \( m_j^i \). This means \( f_{\zeta, s} \) lies in the space \( \mathcal{M}_\lambda \) in (3.2) for \( \lambda = s^n \). Therefore, noting (i) \( f_{\zeta, s} \rightarrow f \) as \( s \rightarrow 0 \) and (ii) \( |Z_{m, f, \lambda}| = n^{2\ell} \) by (3.3), we conclude (4.2). This completes the proof. \qed
5 Spin structure and parity

The aim of this section is to use a spin structure on a family of nodal curves [C] to show parity calculation in Proposition 5.4 below. Twisting bundle as in (5.6) below is a key idea for parity calculation.

We first introduce a spin structure on a family of nodal curves that is relevant to our discussion. We refer to [C] for the definition of spin structure and more details. The relative dualizing sheaf \( \omega_\rho \) of the family of curves \( \rho : \mathcal{D} \to \Delta \) in (3.1) is the canonical bundle \( K_{\mathcal{D}} \) on the total space \( \mathcal{D} \) since \( \mathcal{D} \) is smooth and \( K_\Delta \) is trivial. For each \( \lambda \neq 0 \), the restriction \( K_{\mathcal{D}^0} |_{D_\lambda} \) is the canonical bundle \( K_{D_\lambda} \) on \( D_\lambda \) and the restriction \( K_{\mathcal{D}^0} |_{D_0} \) is the dualizing sheaf \( \omega_{D_0} \) of the nodal curve \( D_0 = D_1 \cup E \cup D_2 \). As described in Section 4, \( D_0 \) is locally given by \( u^i v^i = 0 \) near each node \( p_i \) in \( D_i \) for \( i = 1, 2 \). Then the local generators of \( \omega_{D_0} \) are \( du^i / u^i \) and \( dv^i / v^i \) with a relation \( du^i / u^i + dv^i / v^i = 0 \) (cf. page 82 of [HM]). This implies the restriction \( \omega_{D_0} |_{D_i} = K_{D_i} \otimes \mathcal{O}(p_i) \). On the other hand, \( 1 / u^i \) is a local defining function for the divisor \(-E \) on \( \mathcal{D} \) near \( p_i \). By restricting \( 1 / u^i \) to \( D_i \), one can see that \( \mathcal{O}(-E) |_{D_i} = \mathcal{O}(-p_i) \). Consequently, for \( i = 1, 2 \)

\[
K_{\mathcal{D}^0} |_{D_i} \otimes \mathcal{O}(-E) |_{D_i} = \omega_{D_0} |_{D_i} \otimes \mathcal{O}(-p_i) = K_{D_i}. \tag{5.1}
\]

From Cornalba’s construction (cf. pg. 570 of [C]), there are a line bundle \( \mathcal{N} \to \mathcal{D} \) and a homomorphism \( \Phi : \mathcal{N}^2 \to \omega_\rho = K_{\mathcal{D}^0} \) satisfying:

- \( \Phi \) vanishes identically on the exceptional component \( E \) and \( \mathcal{N} |_E = \mathcal{O}_E(1) \).
- Since \( \Phi |_E \equiv 0 \), there is an induced homomorphism \( \hat{\Phi} : \mathcal{N}^2 \to K_{\mathcal{D}^0} \otimes \mathcal{O}(-E) \) such that \( \Phi \) is the composition of \( \hat{\Phi} \) and tensoring with \( \eta \):

\[
\Phi : \mathcal{N}^2 \overset{\hat{\Phi}}{\longrightarrow} K_{\mathcal{D}^0} \otimes \mathcal{O}(-E) \overset{\otimes \eta}{\longrightarrow} K_{\mathcal{D}^0} \tag{5.2}
\]

where \( \eta \) is a section of \( \mathcal{O}(E) \) with zero divisor \( E \). Then, for \( i = 1, 2 \), the restriction

\[
\hat{\Phi} |_{D_i} : (\mathcal{N} |_{D_i})^2 \to K_{\mathcal{D}^0} |_{D_i} \otimes \mathcal{O}(-E) |_{D_i} = K_{D_i}
\]

is an isomorphism so that the restriction \( N_i = \mathcal{N} |_{D_i} \) is a theta characteristic on \( D_i \).

- For each \( \lambda \neq 0 \), the restriction \( \Phi |_{D_\lambda} : (\mathcal{N} |_{D_\lambda})^2 \to K_{D_\lambda} \) is an isomorphism so that the restriction \( N_\lambda = \mathcal{N} |_{D_\lambda} \) is a theta characteristic on \( D_\lambda \).

The pair \((\mathcal{N}, \Phi)\) is a spin structure on \( \rho : \mathcal{D} \to \Delta \) and the restriction \( \mathcal{N} |_{D_0} \) is a theta characteristic on the nodal curve \( D_0 \).

Remark 5.1. Atiyah [A] and Mumford [M] showed that the parity of a theta characteristic on a smooth curve is a deformation invariant. Cornalba used the homomorphism \( \Phi \) to extend Mumford’s proof to the case of spin structure on a family of nodal curves (see pg. 580 of [C]). Thus, if \( p_1, p_2 \) and \( p \) are the parities of \( N_1, N_2 \) and \( N_\lambda \) (\( \lambda \neq 0 \)), then we have

\[
p \equiv p_1 + p_2 \pmod{2}.
\]
Let \( \varphi : C_\xi \to \Delta \) be the family of curves in Proposition 4.1. Recall that the central fiber of \( \varphi \) is \( C = C_1 \cup C_0 \cup C_2 \) where \( C_0 = \cup j E_j \) is a disjoint union of \( \ell \) exceptional components \( E_j \) and \( C_i \cap E_j = \{ y^i_j \} \) for \( i = 1, 2 \) and \( 1 \leq j \leq \ell \). Similarly as for (5.1), by restricting local defining functions, we have

\[
\mathcal{O}(\pm C_0)|_{C_i} = \mathcal{O}(\pm \sum_j y^i_j) \quad (i = 1, 2) \quad \text{and} \quad \mathcal{O}(\pm C_0)|_{C_{\xi,s}} = \mathcal{O} \quad (s \neq 0).
\]

Since any fiber of \( \varphi \) is a principal divisor on \( C_\xi \), \( \mathcal{O}(C) = \mathcal{O} \) and hence \( \mathcal{O}(C_0) = \mathcal{O}(-C_1 - C_2) \). We also have

\[
\mathcal{O}(\pm C_0)|_{E_j} = \mathcal{O}(\mp (C_1 + C_2))|_{E_j} = \mathcal{O}(\mp (y^1_j + y^2_j)) = \mathcal{O}(\mp 2) \quad (1 \leq j \leq \ell).
\]

Let \( f = (f_1, f_0, f_2) \) and \( \mathcal{F}_\xi : C_\xi \to \mathcal{D} \) be the maps in Proposition 4.1. The ramification divisor \( R_{\mathcal{F}_\xi} \) of \( \mathcal{F}_\xi \) has local defining functions given by the Jacobian of \( \mathcal{F}_\xi \), so 4.12 shows

\[
R_{\mathcal{F}_\xi} = \mathcal{O}(X_\xi + (n - 1)C) = \mathcal{O}(X_\xi)
\]

where \( X_\xi = \sum_{i,j} (m^i_j - 1)X^i_j \) and \( X^i_j \) is the section of \( \varphi_\xi \) defined below 4.12. Note that

(i) the ramification divisor of \( f_i = \mathcal{F}_\xi|_{C_i} \) (\( i = 1, 2 \)) is \( R_{f_i} = X_\xi|_{C_i} + \sum_j (n - 1)y^i_j \);

(ii) the ramification divisor of \( f_{\xi,s} = \mathcal{F}_\xi|_{C_{\xi,s}} \) (\( s \neq 0 \)) is \( R_{f_{\xi,s}} = X_\xi|_{C_{\xi,s}} \).

Now, noting \( n \) is odd, we twist the pull-back bundle \( \mathcal{F}_\xi^* \mathcal{N} \) by setting

\[
\mathcal{L}_\xi = \mathcal{F}_\xi^* \mathcal{N} \otimes \mathcal{O}\left( \frac{1}{2} X_\xi + \frac{(n-1)}{2} C_0 \right).
\]

The lemma below shows that the twisted line \( \mathcal{L}_\xi \) restricts to a theta characteristic on each fiber of \( \varphi_\xi \), including the central fiber \( C \).

**Lemma 5.2.** Let \( \mathcal{L}_\xi \) be as above. Then, we have

(a) \( \mathcal{L}_\xi|_{E_j} = \mathcal{O}(1) \) for \( 1 \leq j \leq \ell \),

(b) \( \mathcal{L}_\xi|_{C_i} = L_{f_1}, \mathcal{L}_\xi|_{C_2} = L_{f_2} \) and \( \mathcal{L}_\xi|_{C_{\xi,s}} = L_{f_{\xi,s}} \) for \( s \neq 0 \)

where \( L_{f_1}, L_{f_2} \) and \( L_{f_{\xi,s}} \) are the theta characteristics on \( C_1, C_2 \) and \( C_{\xi,s} \) defined by (0.3).

**Proof.** (a) follows from (5.4) and the fact that each restriction map \( \mathcal{F}_\xi|_{E_j} \) has degree \( n \). (b) follows from (5.3), (i) and (ii). \( \square \)

Observe that the relative dualizing sheaf \( \omega_{\varphi_\xi} \) is the canonical bundle \( K_{C_\xi} \) since \( C_\xi \) is smooth. The Hurwitz formula and (5.5) thus imply that

\[
\omega_{\varphi_\xi} = K_{C_\xi} = \mathcal{F}_\xi^* K_\mathcal{D} \otimes \mathcal{O}(X_\xi).
\]

Define a homomorphism

\[
\hat{\Psi}_\xi : \mathcal{L}_\xi^2 = \mathcal{F}_\xi^* \mathcal{N}^2 \otimes \mathcal{O}(X_\xi + (n - 1)C_0) \to \mathcal{F}_\xi^* (K_\mathcal{D} \otimes \mathcal{O}(-E)) \otimes \mathcal{O}(X_\xi + (n - 1)C_0)
\]
by \( \hat{\Psi}_\zeta = F_\zeta^* \hat{\Phi} \otimes Id \) where \( \hat{\Phi} \) is the induced homomorphism in (5.2). Noting \( O(C) = O \) and \( O(D_0) = O \), by (4.12) we have

\[
F_\zeta^* O(-E) = F_\zeta^* O(D_1 + D_2) = O(n(C_1 + C_2)) = O(-nC_0).
\]

Together with (5.7), this implies that the right-hand side of (5.8) is \( K_{C_\zeta} \otimes O(-C_0) \). Now, define a homomorphism \( \Psi_\zeta : L_\zeta^2 \rightarrow K_{C_\zeta} \) to be the composition

\[
\Psi_\zeta : L_\zeta^2 \xrightarrow{\hat{\Psi}_\zeta} K_{C_\zeta} \otimes O(-C_0) \xrightarrow{\otimes \xi} K_{C_\zeta}
\]

where \( \xi \) is a section of \( O(C_0) \) with zero divisor \( C_0 \).

**Lemma 5.3.** \((L_\zeta, \Psi_\zeta)\) is a spin structure on \( \varphi_\zeta : C_\zeta \rightarrow \Delta \).

**Proof.** First, \( L_\zeta|_E = O(1) \) by Lemma 5.2 (a) and \( \Psi_\zeta \) vanishes identically on each exceptional component \( E_j \) since \( \xi = 0 \) on \( C_0 = \sqcup_j E_j \). Second, since \( \hat{\Phi}|_{D_i} \) is an isomorphism, (5.3) and (i) show that for \( i = 1, 2 \) the restriction

\[
\hat{\Psi}_i|_{C_1} = f_i^* (\hat{\Phi}|_{D_i}) \otimes Id : (L_\zeta|_{C_i})^2 = f_i^* N_i^2 \otimes O(R_{f_i}) \rightarrow f_i^* K_{D_i} \otimes O(R_{f_i}) = K_{C_i}
\]

is an isomorphism. Lastly, let \( \lambda = s^n \neq 0 \). Since \( \hat{\Phi}|_{D_\lambda} \) is an isomorphism, so is \( \hat{\Phi}|_{D_\lambda} \). Thus, by (5.3), (ii) and the facts \( K_P|_{D_\lambda} = K_{D_\lambda} \) and \( O(-E)|_{D_\lambda} = O \), the restriction

\[
\hat{\Psi}_\zeta|_{C_\zeta,s} = f_\zeta,s^* \hat{\Phi}|_{D_\lambda} \otimes Id : (L_\zeta|_{C_\zeta,s})^2 = f_\zeta,s^* N_\zeta^2 \otimes O(R_{f_\zeta,s}) \rightarrow f_\zeta,s^* K_{D_\lambda} \otimes O(R_{f_\zeta,s}) = K_{C_\zeta,s}
\]

is an isomorphism. This implies that the restriction

\[
\Psi_\zeta|_{C_\zeta,s} : (L_\zeta|_{C_\zeta,s})^2 \rightarrow K_{C_\zeta}|_{C_\zeta,s} = K_{C_\zeta,s}
\]

is also an isomorphism. Therefore, we conclude that \((L_\zeta, \Psi_\zeta)\) is a spin structure on \( \varphi_\zeta \). \( \square \)

The following is a key fact for the proof of Theorem 0.1 in the Introduction.

**Proposition 5.4.** Let \( f = (f_1, f_0, f_2) \) and \( f_\zeta,s \) be maps in Proposition 4.1. Then, for all \( s \neq 0 \)

\[
p(f_\zeta,s) \equiv p(f_1) + p(f_2) \pmod{2}.
\]

**Proof.** Since \((L_\zeta, \Psi_\zeta)\) is a spin structure on \( \varphi_\zeta \), the Cornalba's proof, mentioned in Remark 5.1 shows that for all \( s \neq 0 \)

\[
h^0(L_\zeta|_{C_\zeta,s}) \equiv h^0(L_\zeta|_{C_1}) + h^0(L_\zeta|_{C_2}) \pmod{2}.
\]

This and Lemma 5.2(b) prove (5.10). \( \square \)
6 Proof of Theorem 0.1

Proof of Theorem 0.1: Fix a spin structure \((N, \Phi)\) on \(\rho : D \to \Delta\) given in Section 5. Consider the space \(M_{m,0}\) in (2.2) where \(m\) is a partition of \(d = 3\). In this case, by Lemma 2.1 either \(m = (1^3)\) or \(m = (3)\). Note that both of them satisfy (4.1). Fix \(\lambda \neq 0\) and let \(f = (f_1, f_0, f_2)\) be a map in \(M_{m,0}\). Then (4.2) and (5.10) show that for all \(f_\mu \in Z_{m,f,\lambda}\)

\[ p(f_\mu) \equiv p(f_1) + p(f_2) \pmod{2}. \]  (6.1)

Lemma 1.1 and (3.7) show that

\[ H_{h,p}^{(3)^0} = H_{h,p}^{(3)^0} = 1 \]

\[ H_{h,p}^{(3)^3} = -1 \]

\[ H_{h,p}^{(3)^1} = 2 \]

(6.2)

By (3.8) and (6.1), (6.2) becomes

\[ H_{h,p}^{(3)^0} = \sum_{f = (f_1, f_0, f_2) \in M_{(1^3),0}} \frac{(-1)^{p(f_1) + p(f_2)}}{(3!)^3} + \sum_{f = (f_1, f_0, f_2) \in M_{(3),0}} \frac{3^2(-1)^{p(f_1) + p(f_2)}}{(3!)^3} \]  (6.3)

It then follows from Lemma 2.3 and (6.3) that

\[ H_{h,p}^{(3)^k} = \sum_{f_1 \in M_{(1^3)^1}} \frac{(-1)^{p(f_1)} + \sum_{f_2 \in M_{(3)^1}} (-1)^{p(f_2)}}{(3!)^3} + \sum_{f_2 \in M_{(3)^2}} \frac{3^2(-1)^{p(f_1) + p(f_2)}}{(3!)^3} \]

\[ = 3! H_{h,p_1}^{(3)^{k_1}} \cdot H_{h,p_2}^{(3)^{k_2}} + 3 H_{h,p_1}^{(3)^{k_1+1}} \cdot H_{h,p_2}^{(3)^{k_2+1}} \]

where the second equality follows from Lemma 2.2 and the last from Lemma 1.1.

7 Calculation

The aim of this section is to show:

Proposition 7.1. \( H_{h}^{(3)^k} = 3^{2h-2} \left[ (-1)^k 2^{k+h-1} + 1 \right]. \)

Proof. The proof consists of four steps.

Step 1: We first show the following facts which we use in the computation below.

Lemma 7.2.

(a) \( H_{(3)^0}^{0,0} = H_{(3)^0}^{0,0} = \frac{1}{3!} \)
(b) \( H_{(3)^3}^{0,0} = -\frac{1}{3} \)
(c) \( H_{(3)^0}^{1,0} = H_{(3)^0}^{1,0} = 2 \)
Proof. Consider the dimension zero space $M^V_{\chi}(\mathbb{P}^1, 3)$ where $V = \emptyset$. The Euler characteristic $\chi = 6$ by (11) and hence the space contains only one map $f : C \to \mathbb{P}^1$ where $C$ is a disjoint union of three rational curves and $|\text{Aut}(f)| = 3!$. This shows (a). Let $(f, C)$ be a map in the dimension zero space $M^V_{\chi(3), (3), (3)}(\mathbb{P}^1, 3)$. Then $C$ is a connected curve of genus one and the theta characteristic $L_f$ on $C$ defined by (0.1) is

$$L_f = \mathcal{O}(-2x_1 + x_2 + x_3) = \mathcal{O}(x_1 - 2x_2 + x_3) = \mathcal{O}(x_1 + x_2 - 2x_3)$$

where $x_1, x_2$ and $x_3$ are ramification points of $f$. This implies $L^3_f = \mathcal{O}$ and hence $L_f = \mathcal{O}$ because $L^2_f = L^3_f = \mathcal{O}$. We have $p(f) = 1$. Therefore,

$$H^{0,+}_{(3)^3} = -H^{0}_{(3)^3} = -\frac{1}{3}$$

where $H^{0}_{(3)^3}$ denotes the (ordinary) Hurwitz number which is calculated by using the character formula (cf. (0.10) of [OP]). By Proposition 9.2 of [LP1], the spin Hurwitz numbers $H^{h,p}_d$ are the dimension zero local invariants of spin curve that count maps from possibly disconnected domains. Let $GW^{h,p}_d$ denote the dimension zero local invariants of spin curve that count maps from connected domains. Then $H^{h,p}_d$ and $GW^{h,p}_d$ are related as follows:

$$1 + \sum_{d>0} H^{h,p}_d t^d = \exp \left( \sum_{d>0} GW^{h,p}_d t^d \right).$$

Now, (c) follows from: $GW^{1+,+}_1 = 1$, $GW^{1+,+}_2 = \frac{1}{2}$ and $GW^{1+,+}_3 = \frac{4}{3}$ (see Section 10 of [LP1]).

Step 2 : In this step, we compute $H^{1,-}_{(3)^k}$. For a spin curve of genus one with trivial theta characteristic, it follows from the formula (3.12) of [EOP] that

$$H^{1,-}_{(3)^k} = 2^{-k} \left[ (f(3)(21))^k - (f(3)(3))^k \right].$$

Here the so-called central character $f(3)$ can be written as $f(3) = \frac{1}{7} p_3 + a_2 p_1^2 + a_1 p_1 + a_0$ for some $a_i \in \mathbb{Q}$ $(0 \leq i \leq 2)$ and the supersymmetric functions $p_1$ and $p_3$ are defined by

$$p_1(m) = d - \frac{1}{24} \quad \text{and} \quad p_3(m) = \sum_j m_j^3 - \frac{1}{240}$$

where $m = (m_1, \cdots, m_\ell)$ is a partition of $d$. For $k = 0, 1, (7.1)$ shows

$$H^{1,-}_{(3)^0} = 0 \quad \text{and} \quad H^{1,-}_{(3)} = -3. \quad (7.2)$$

Lemma 7.2b, (7.2) and the formula (0.6) give

$$H^{1,-}_{(3)^2} = 3 H^{1,-}_{(3)} \cdot H^{0,+}_{(3)^3} = 3. \quad (7.3)$$

By (7.1), (7.2) and (7.3) we conclude

$$f(3)(21) = -4 \quad \text{and} \quad f(3)(3) = 2. \quad (7.4)$$

Consequently, by (7.1) and (7.4), for $k \geq 0$ we have

$$H^{1,-}_{(3)^k} = (-1)^k 2^k - 1. \quad (7.5)$$
Step 3: In this step, we compute $H_{(3)k}^{h,1+}$ for $h = 0, 1$. For $k \geq 1$, (7.2) and the formula (0.6) give

$$H_{(3)k-1}^{1-} = 3 H_{(3)}^{1-} \cdot H_{(3)k}^{0,1+} = -3^2 H_{(3)k}^{0,1+}. \tag{7.6}$$

Combining Lemma 2.2a and (7.6) yields that for $k \geq 0$

$$H_{(3)k}^{0,1+} = -\frac{1}{2^k} \left( (-1)^k 2^{k-1} - 1 \right). \tag{7.7}$$

Lemma 2.2c, (7.3), (7.7) and the formula (0.6) show

$$H_{(3)^0}^{2,1+} = 3! H_{(3)^0}^{1-} \cdot H_{(3)^0}^{1-} + 3 H_{(3)^0}^{1-,1-} \cdot H_{(3)}^{1-} = 27,$$

$$H_{(3)}^{2,1+} = 3! H_{(3)^0}^{1-,1-} \cdot H_{(3)^0}^{1-,1-} + 3 H_{(3)}^{1-,1-,1-} \cdot H_{(3)^0}^{1-,1-} = -27,$$

$$H_{(3)^0}^{2,1+} = 3! H_{(3)^0}^{1-,1-} \cdot H_{(3)^0}^{1-,1-} + 3 H_{(3)^0}^{1-,1-,1-} \cdot H_{(3)^0}^{1-,1-,1-} = 24 + 3 H_{(3)^0}^{1-,1-,1-} \cdot H_{(3)^0}^{1-,1-,1-},$$

$$H_{(3)^0}^{2,1+} = 3! H_{(3)^0}^{1-,1-} \cdot H_{(3)^0}^{1-,1-} + 3 H_{(3)^0}^{1-,1-,1-} \cdot H_{(3)^0}^{1-,1-,1-} = 12 H_{(3)^0}^{1-,1-,1-} + 3 H_{(3)^0}^{1-,1-,1-},$$

$$H_{(3)^0}^{2,1+} = 3! H_{(3)^0}^{1-,1-} \cdot H_{(3)^0}^{1-,1-} + 3 H_{(3)^0}^{1-,1-,1-} \cdot H_{(3)^0}^{1-,1-,1-} = 4 - H_{(3)^0}^{1-,1-,1-}.$$

It follows that $H_{(3)^0}^{1-,1-} = -1$. Consequently, Lemma 2.2c, (7.7) and the formula (0.6) give

$$H_{(3)k}^{1+,1+} = 3! H_{(3)^0}^{1-,1-} \cdot H_{(3)^0}^{1-,1-} + 3 H_{(3)^0}^{1-,1-,1-} \cdot H_{(3)^0}^{1-,1-,1-} = (-1)^k 2^k + 1. \tag{7.8}$$

Step 4: It remains to compute $H_{(3)k}^{h,p}$ for $h \geq 2$. The formula (0.6) gives

$$H_{(3)k}^{h,p} = 3! H_{(3)^0}^{h-1,p} \cdot H_{(3)^0}^{1+,1+} + 3 H_{(3)^0}^{h-1,p} \cdot H_{(3)^0}^{1+,1+}.$$ From this, we can deduce that for $h \geq 2$

$$\begin{pmatrix} H_{(3)k}^{h,p} \\ H_{(3)k+1}^{h,p} \end{pmatrix} = \begin{pmatrix} 3! H_{(3)^0}^{h-1,p} \\ 3! H_{(3)^0}^{h-1,p} \end{pmatrix} \begin{pmatrix} 3 H_{(3)^0}^{1+,1+} \\ 3 H_{(3)^0}^{1+,1+} \end{pmatrix} \begin{pmatrix} H_{(3)^0}^{1+p} \\ H_{(3)^0}^{1+p} \end{pmatrix} \begin{pmatrix} 3 H_{(3)^0}^{1+,1+} \\ 3 H_{(3)^0}^{1+,1+} \end{pmatrix} \begin{pmatrix} H_{(3)^0}^{h-1,p} \\ H_{(3)^0}^{h-1,p} \end{pmatrix} = \begin{pmatrix} 3! H_{(3)^0}^{h-1,p} \\ 3! H_{(3)^0}^{h-1,p} \end{pmatrix} \begin{pmatrix} 3 H_{(3)^0}^{1+,1+} \\ 3 H_{(3)^0}^{1+,1+} \end{pmatrix} \begin{pmatrix} H_{(3)^0}^{h-1,p} \\ H_{(3)^0}^{h-1,p} \end{pmatrix} \begin{pmatrix} 3! H_{(3)^0}^{h-1,p} \\ 3! H_{(3)^0}^{h-1,p} \end{pmatrix} \begin{pmatrix} 3 H_{(3)^0}^{1+,1+} \\ 3 H_{(3)^0}^{1+,1+} \end{pmatrix} \begin{pmatrix} H_{(3)^0}^{h-1,p} \\ H_{(3)^0}^{h-1,p} \end{pmatrix}. \tag{7.9}$$

Therefore, (7.5), (7.8) and (7.9) complete the proof. 

\[\square\]

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