Towards Lagrangian construction for infinite half-integer spin field

I.L. Buchbinder¹,², S. Fedoruk³, A.P. Isaev³,⁴, V.A. Krykhtin¹

¹Department of Theoretical Physics, Tomsk State Pedagogical University, 634041 Tomsk, Russia, joseph@tspu.edu.ru, krykhtin@tspu.edu.ru

²National Research Tomsk State University, Lenin Av. 36, 634050 Tomsk, Russia

³Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia, fedoruk@theor.jinr.ru, isaevap@theor.jinr.ru

⁴St.Petersburg Department of Steklov Mathematical Institute of RAS, Fontanka 27, 191023 St. Petersburg, Russia

Abstract

We formulate the conditions for the generalized fields in the space with additional commuting Weyl spinor coordinates which define the infinite half-integer spin representation of the four-dimensional Poincaré group. Using this formulation we develop the BRST approach and derive the Lagrangian for the half-integer infinite spin fields.

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1 Introduction

Various aspects of massless infinite spin irreducible representations of the Poincaré group [1–3] attract much attention last time (see, e.g., [4]–[38]). Such representations contain an infinite number of states with all possible integer or half-integer helicities, in contrast to the usual massless representations describing the fields of fixed helicity. New approaches to the Lagrangian description of such fields have been recently developed in [17], [18], [19], [23], [33] combining the appropriate number of free massless fields with definite helicities. Also, we note the BRST approach [27] to Lagrangian construction for bosonic massless infinite spin fields. However, many of interesting points related to the Lagrangian formulation for massless infinite spin fields still deserve further study. In this paper, we consider the BRST approach to Lagrangian formulations of the fermionic massless infinite spin fields.

As known, it is convenient to realize a field description of the above representations in terms of space-time fields depending on the additional coordinates, so that the expansion in these coordinates gives an infinite number of the helicity states. Beginning with the pioneer papers [1–3], the space-time vector-like quantities are usually used as such additional coordinates (see review [21]).

However, there is another possibility for the field description of infinite spin particles, which uses the commuting Dirac or Majorana spinors as the additional coordinates. For the first time, this type of fields was considered in [4]. In our recent papers [26, 27, 31, 37] (see also [32]) we constructed the new infinite (continuous) spin fields depending on spinor additional coordinates. It was shown that such fields are obtained as a result of a quantization of the special twistor particle models.

Following [26, 27, 31, 37], the infinite integer spin representation is described by the field

\[ \Psi(x; \xi, \bar{\xi}) , \]  

which depends on the space-time coordinates \( x^m \) and additional commuting Weyl spinor \( \xi^\alpha, \bar{\xi}^{\dot{\alpha}} = (\xi^\alpha)^* \). The conditions that such field describes the infinite spin representation are written in the form\(^1\)

\[ P^2 \Psi = 0 , \]
\[ (\xi^\alpha P_{\alpha\beta} \bar{\xi}^{\dot{\beta}}) \Psi = \mu \Psi , \]
\[ \left( \frac{\partial}{\partial \xi^{\dot{\alpha}}} P^{\alpha\beta} \frac{\partial}{\partial \xi^\beta} \right) \Psi = -\mu \Psi , \]
\[ \left( \xi^\alpha \frac{\partial}{\partial \xi^\alpha} - \bar{\xi}^{\dot{\alpha}} \frac{\partial}{\partial \xi^{\dot{\alpha}}} \right) \Psi = 0 , \]

where \( P_m = -i\partial/\partial x^m \) and \( \mu \) is a real dimensional parameter. Infinite integer-spin field (1.1) does not have any external vector or spinor indices (see the detailed analysis of helicity content of infinite integer- and half-integer-spin fields in Appendix B). Further, we will call the relations (1.2)-(1.5) the basic conditions.

In works [26, 31], in contrast to the field (1.1), it was proposed to describe the massless infinite half-integer spin representation by the field with an external spinor Dirac index \( A = 1, 2, 3, 4 \):

\[ \Psi_A(x; \xi, \bar{\xi}) . \]  

\(^1\)We use the notation as in the monograph [39] (see also the Appendix A).
In addition to the basic conditions (1.2)-(1.5) (written for the field \( \Psi_A(x; \xi, \bar{\xi}) \)), the field (1.6) should satisfy also the additional condition in form of Dirac equation
\[
P_m(\gamma^m)_A^B \Psi_B = 0. \tag{1.7}
\]

The Klein-Gordon equation (1.2) is evident consequence of the Dirac equation (1.7). In the representation \( \gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \bar{\sigma}_m & 0 \end{pmatrix} \) (see (A.9)), the field (1.6) is represented in terms of the Weyl spinors
\[
\Psi_A(x; \xi, \bar{\xi}) = \begin{pmatrix} \Psi_\alpha(x; \xi, \bar{\xi}) \\ \bar{\Upsilon}_\dot{\alpha}(x; \xi, \bar{\xi}) \end{pmatrix}. \tag{1.8}
\]

For Weyl components of the field (1.6), the Dirac equation (1.7) are written as follows\(^2\)
\[
P_m \bar{\sigma}_m^\alpha \psi_\gamma = 0, \quad P_m \sigma^m_\alpha \bar{\Upsilon}_\dot{\gamma} = 0. \tag{1.10}
\]

Taking into account the basic conditions (1.2)-(1.5) and the condition (1.7), it is natural to assume that there exists the Lagrangian formulation where all the above conditions are the consequences of the Lagrangian equations of motion. Diverse approaches to the Lagrangian description of the infinite spin fields with vector additional coordinate were considered in works [9-12, 14, 21, 23-25, 28, 30, 33]. One of the powerful general methods for studying the equations of motions and Lagrangian formulations in higher spin theories is the BRST construction which was applied to the massless infinite spin field theory in refs. [13, 17, 18, 24, 25, 29, 30].

In the recent paper [27] the BRST construction was used to derive the Lagrangian for the infinite integer spin fields (1.1) with additional spinorial coordinate. This approach is some generalization of the BRST construction which was used for finding the Lagrangians of the free fields of different types in flat and AdS spaces (see e.g. [42-46] and the references therein, see also the review [47]).

In the present paper we develop the generalization of the BRST method used in [27] to derive the Lagrangian for the infinite half-integer spin fields.

The paper is organized as follows. In Sect. 2, we describe the component structure of the space-time generalized fields (1.6), (1.8), depending on additional commuting spinor variables \( \xi^\alpha, \bar{\xi}^{\dot{\alpha}} \), and present the equations of motion for these component fields. In Sect. 3, we introduce the extended Fock space in terms of additional bosonic creation and annihilation operators and ghost operators. Then we construct the Hermitian BRST charge and the corresponding equation of motion which reproduce the conditions for the component fields. Taking into account this BRST charge, we derive the space-time Lagrangian for fermionic infinite spin field. The resulting Lagrangian contains both physical fields and auxiliary and gauge fields. In Sect. 4, we discuss the results and open issues. In Appendix A, we fix the spinor notations used in this paper. In

\(^2\)In case of infinite half-integer spin the equations (1.2)-(1.5), (1.7) are not sufficient to describe the irreducible massless infinite spin representation. For irreducible representation we must put additional constraints
\[
\frac{\partial}{\partial \xi^\gamma} \Psi_\gamma(x; \xi, \bar{\xi}) = 0, \quad \frac{\partial}{\partial \bar{\xi}^{\dot{\gamma}}} \bar{\Upsilon}_{\dot{\gamma}}(x; \xi, \bar{\xi}) = 0. \tag{1.9}
\]

These equations have been encoded in twistor formulation developed in [26, 31, 37]. In the Appendix C we demonstrate this statement. Further we derive the Lagrangian without considering these constraints, assuming that they can be somehow taken into account in the final result.
Appendix B, we describe in details the component decomposition of the infinite spin fields which are considered in the paper. Appendix C is devoted to construction of the solution to the equations \((1.9)\) on the base of the twistor formalism and description of the irreducible representation of the infinite half-integer spin field.

\section{Spin-tensor representation for fermionic continuous spin field}

First of all we extend the generalized coordinate space \((x^m, \xi^\alpha, \bar{\xi}^{\dot{\alpha}})\) by additional commuting Weyl spinor \(\bar{\zeta}^{\dot{\alpha}} = (\zeta^\alpha)^*\). It allows us to replace the four-component spinor field \((1.8)\) by the scalar field

\[ \hat{\Psi}(x; \xi, \bar{\xi}; \zeta, \bar{\zeta}) = \zeta^\alpha \Psi_{\alpha}(x; \xi, \bar{\xi}) + \bar{\zeta}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}}(x; \xi, \bar{\xi}). \]  

(2.1)

In this case the conditions \((1.2)-(1.5)\) for the field \(\hat{\Psi}(x; \xi, \bar{\xi}; \zeta, \bar{\zeta})\) are rewritten in the form

\[ \left( \zeta^\alpha P_{\alpha\beta} \bar{\zeta}^\beta - \mu \right) \hat{\Psi} = 0, \quad \left( \frac{\partial}{\partial \xi^\alpha} P_{\alpha\beta} \frac{\partial}{\partial \bar{\xi}^{\beta}} + \mu \right) \hat{\Psi} = 0, \quad \left( \zeta^\alpha \frac{\partial}{\partial \xi^\alpha} - \bar{\zeta}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\xi}^{\dot{\alpha}}} \right) \hat{\Psi} = 0, \]  

(2.2)

whereas the massless Dirac equations \((1.10)\) look like

\[ P^{\dot{\alpha}\gamma} \frac{\partial}{\partial \zeta^\gamma} \hat{\Psi} = 0, \quad \frac{\partial}{\partial \bar{\zeta}^{\dot{\gamma}}} P^\dot{\alpha}_{\dot{\gamma}} \hat{\Psi} = 0. \]  

(2.3)

The equations \((2.3)\) imply the equation

\[ P^2 \hat{\Psi} = 0. \]  

(2.4)

In this representation for the field \(\hat{\Psi}(x; \xi, \bar{\xi}; \zeta, \bar{\zeta})\), the Hermitian angular momentum operator \(M_{mn}\) is written as follows

\[ M_{mn} = \sigma_{mn}^{\alpha\beta} M_{\alpha\beta} - \bar{\sigma}_{mn}^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}, \]  

(2.5)

where

\[ M_{\alpha\beta} = i \xi_{(\alpha} \frac{\partial}{\partial \xi^{\beta)}} + i \xi_{(\alpha} \frac{\partial}{\partial \bar{\xi}^{\beta)}}, \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = i \bar{\xi}_{(\dot{\alpha}} \frac{\partial}{\partial \bar{\xi}^{\dot{\beta})}} + i \bar{\xi}_{(\dot{\alpha}} \frac{\partial}{\partial \xi^{\dot{\beta})}}. \]  

(2.6)

One can prove that the operator \((2.5)\) satisfies the standard commutation relations for the Lorentz group generators.

In spinor notation, the Pauli-Lubanski pseudovector \(W_m = \frac{1}{2} \varepsilon_{mnkl} M^{nk} P^l\) takes the form

\[ W_{\dot{\alpha} \dot{\beta}} = i M_{\alpha\beta} P_{\dot{\alpha} \dot{\beta}} - i \bar{M}_{\dot{\alpha}\dot{\beta}} P^\dot{\alpha}, \]  

(2.7)

and the second Casimir operator \(W^2\) is

\[ W^2 = M_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}} P_{\dot{\alpha} \dot{\beta}} P_{\alpha\beta} - \frac{1}{2} \left( M_{\alpha\beta} M^{\alpha\beta} + \bar{M}_{\dot{\alpha}\dot{\beta}} \bar{M}^{\dot{\alpha}\dot{\beta}} \right) P^2. \]  

(2.8)

Using the expression \((2.6)\) for the angular momentum operator, we obtain one of the possible forms for the second Casimir operator

\[ W^2 = -\xi P \bar{\xi} \left( \frac{\partial}{\partial \xi^{\beta}} P_{\beta\dot{\beta}} \frac{\partial}{\partial \bar{\xi}^{\dot{\beta}}} \right) - (\zeta P \bar{\zeta}) \left( \frac{\partial}{\partial \zeta^\beta} P^\beta_{\dot{\beta}} \frac{\partial}{\partial \bar{\zeta}^{\dot{\beta}}} \right) \]  

(2.9)

\[ -\xi P \bar{\xi} \left( \frac{\partial}{\partial \zeta^\beta} P^{\beta\dot{\beta}} \frac{\partial}{\partial \bar{\zeta}^{\dot{\beta}}} \right) - (\zeta P \bar{\zeta}) \left( \frac{\partial}{\partial \xi^{\beta}} P^{\beta\dot{\beta}} \frac{\partial}{\partial \bar{\xi}^{\dot{\beta}}} \right) + \mathcal{D} \cdot P^2, \]
where the operator \( \mathcal{D} \) in the last line is
\[
\mathcal{D} := -\frac{1}{2} \left( M_{\alpha \beta} M^{\alpha \beta} + M_{\dot{\alpha} \dot{\beta}} M^{\dot{\alpha} \dot{\beta}} \right) - \frac{1}{2} \left( \xi^\alpha \partial_{\xi^\alpha} + \zeta^\alpha \partial_{\zeta^\alpha} \right) \left( \xi^{\dot{\beta}} \partial_{\xi^{\dot{\beta}}} + \zeta^{\dot{\beta}} \partial_{\zeta^{\dot{\beta}}} \right)
\]
and \((\xi P \dot{\xi}) := \xi^\alpha P_{\alpha \beta} \dot{\xi}^\beta, (\xi P \dot{\zeta}) := \xi^\alpha P_{\alpha \beta} \dot{\zeta}^\beta, \) etc.

Due to the first two equations in (2.2) and the equation (2.3), the Casimir operators of the Poincare group \( P^2 \) and \( W^2 \), defined in (2.9), act on the field (2.1) as following
\[
P^2 \dot{\Psi} = 0, \quad W^2 \dot{\Psi} = \mu^2 \dot{\Psi}.
\]
Hence, the field (2.1) and therefore the fields (1.8) describe the infinite half-integer spin representation. The homogeneity operator \((U(1)\text{-charge})\), given by the last equation in (2.2), commutes with all Poincaré generators and is the superselection operator.

Next, we solve the first equation in (2.2) (or the equations (1.3)) as
\[
\Psi_\alpha = \delta \left( \xi P \dot{\xi} - \mu \right) \Phi_\alpha(x, \xi, \dot{\xi}), \quad \tilde{\Psi}^{\dot{\alpha}} = \delta \left( \xi P \dot{\zeta} - \mu \right) \tilde{X}^{\dot{\alpha}}(x, \xi, \dot{\zeta}).
\]
After that, the second equation in (2.2) (or the equations (1.3)) and the equations (2.3) (or the equations (1.10)) for the fields \( \Phi_\alpha, \tilde{X}^{\dot{\alpha}} \) take the form
\[
\left( \frac{\partial}{\partial \xi^\alpha} P^{\dot{\alpha} \beta} \frac{\partial}{\partial \xi^\beta} + \mu \right) \Phi_\gamma = 0, \quad P^{\dot{\alpha} \gamma} \Phi_\gamma = 0, \quad (2.12)
\]
\[
\left( \frac{\partial}{\partial \xi^\alpha} P^{\dot{\alpha} \beta} \frac{\partial}{\partial \xi^\beta} + \mu \right) \tilde{X}^\dot{\gamma} = 0, \quad P_{\alpha \dot{\gamma}} \tilde{X}^\dot{\gamma} = 0. \quad (2.13)
\]
We consider the solution of last equation in (2.2) (or the equations (1.3)) for the fields \( \Phi_\alpha, \tilde{X}^{\dot{\alpha}} \)
\[
\left( \xi^\alpha \frac{\partial}{\partial \xi^\alpha} - \dot{\xi}^{\dot{\alpha}} \frac{\partial}{\partial \xi^\alpha} \right) \Phi_\gamma = 0, \quad \left( \xi^\alpha \frac{\partial}{\partial \xi^\alpha} - \dot{\zeta}^{\dot{\alpha}} \frac{\partial}{\partial \xi^\alpha} \right) \tilde{X}^\dot{\gamma} = 0. \quad (2.14)
\]
in form of power expansion in \( \xi \) and \( \dot{\xi} \):
\[
\Phi_\gamma(x, \xi, \dot{\xi}) = \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\gamma \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(x) \xi^{\alpha_1} \ldots \xi^{\alpha_s} \dot{\xi}^{\beta_1} \ldots \dot{\xi}^{\beta_s}, \quad (2.15)
\]
\[
\tilde{X}^\dot{\gamma}(x, \xi, \dot{\xi}) = \sum_{s=0}^{\infty} \frac{1}{s!} \tilde{X}^{\dot{\gamma} \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(x) \xi^{\alpha_1} \ldots \xi^{\alpha_s} \dot{\zeta}^{\beta_1} \ldots \dot{\zeta}^{\beta_s}. \quad (2.16)
\]
The component fields \( \varphi_{\gamma \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(x) \) and \( \tilde{X}^{\dot{\gamma} \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(x) \) are symmetric with respect \( \alpha \) and separately \( \beta \) indices. In this way we use the shortened notations for them:
\[
\varphi_{\gamma \alpha(s) \beta(s)} := \varphi_{\gamma \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}, \quad \tilde{X}^{\dot{\gamma} \alpha(s) \beta(s)} := \tilde{X}^{\dot{\gamma} \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}. \quad (2.17)
\]
The equations (2.12), (2.13) lead to the following equations for the component fields:
\[
\partial^\dot{\beta} \varphi_{\gamma \alpha(s) \beta(s)}(x) = 0, \quad \mu \partial^\beta \varphi_{\gamma \alpha(s) \beta(s)} = \mu \varphi_{\gamma \alpha(s-1) \beta(s-1)}, \quad (2.18)
\]
\[
\partial^\dot{\gamma} \tilde{X}^{\dot{\gamma} \alpha(s) \beta(s)}(x) = 0, \quad \mu \partial^\gamma \tilde{X}^{\dot{\gamma} \alpha(s) \beta(s)} = \mu \tilde{X}^{\dot{\gamma} \alpha(s-1) \beta(s-1)}. \quad (2.19)
\]
where we have used $P_m = -i\partial/\partial x^m$, $\partial^{\dot{\alpha}} = (\hat{\sigma}^m)^{\dot{\alpha}}_\alpha i P_m$.

It is convenient to combine the fields $\varphi_{\gamma(\alpha(\beta(s))}$ and $\bar{\chi}_\gamma^\alpha(\beta(s))$ in a single four component object $\varphi_{\gamma(\alpha(\beta(s))}$ of the form

$$
\varphi_{\gamma(\alpha(\beta(s))} = \begin{pmatrix}
\varphi_{\gamma(\alpha(\beta(s))} \\
\bar{\chi}_\gamma^\alpha(\beta(s))
\end{pmatrix},
$$

(2.20)

and define the Dirac adjoint (with respect indices $C = (\gamma, \dot{\gamma})$ and $C = (\gamma, \dot{\gamma})$) field as

$$
\varphi^C_{\beta(\alpha(\beta(s))} = \begin{pmatrix}
\bar{\chi}_\gamma^\alpha(\beta(s)) \\
\varphi_\gamma^\alpha(\beta(s))
\end{pmatrix}.
$$

(2.21)

Hereafter, we will often omit the index $C$ in four component field $\varphi_{\gamma(\alpha(\beta(s))} (2.20)$.

Similarly to the expression (2.1) for the fields $\Psi_\alpha$, $\bar{\Upsilon}_\dot{\alpha}$, $\hat{\Psi}$, we construct the expression

$$
\hat{\Phi}(x; \xi, \bar{\xi}, \zeta, \bar{\zeta}) = \bar{\zeta}_\alpha \Phi_\alpha(x; \xi, \bar{\xi}) + \bar{\zeta}_\dot{\alpha} \bar{X}_\dot{\alpha}(x; \xi, \bar{\xi})
$$

(2.22)

for the fields $\Phi_\alpha$ and $\bar{X}_\dot{\alpha}$ defined by (2.15) and (2.16).

In the next section we will derive the Lagrangian formulation for the fields under consideration using the BRST approach in terms of the four component field (2.20).

3 BRST Lagrangian construction

3.1 Generalized Fock space

In the previous section we have used the spinor variables $\xi^\alpha$, $\bar{\xi}^\dot{\alpha}$ and the corresponding momenta given by the derivatives with respect to $\xi$, $\bar{\xi}$. For this reason, we introduce the operators

$$
a_\alpha, \ b^\alpha,
$$

(3.1)

which satisfy the algebra

$$
[a_\alpha, b^\beta] = \delta^\beta_\alpha.
$$

(3.2)

Hermitian conjugation yield the operators

$$
\bar{a}_{\dot{\alpha}} = (a_\alpha)^\dagger, \ \bar{b}^\dot{\alpha} = (b^\alpha)^\dagger,
$$

(3.3)

with the commutation relation

$$
[\bar{b}^\dot{\alpha}, \bar{a}_{\dot{\beta}}] = \delta^\dot{\alpha}_{\dot{\beta}}.
$$

(3.4)

Below, similar to (2.17), we use the notations

$$
a_{\alpha(\beta(s))} := a_{\alpha_1} \ldots a_{\alpha_s}, \ \bar{a}_{\dot{\alpha}(s)} := \bar{a}_{\dot{\alpha}_1} \ldots \bar{a}_{\dot{\alpha}_s}, \ b^{\alpha(\beta(s))} := b^{\alpha_1} \ldots b^{\alpha_s}, \ \bar{b}^{\dot{\alpha}(s)} := \bar{b}^{\dot{\alpha}_1} \ldots \bar{b}^{\dot{\alpha}_s}.
$$

(3.5)

Following (3.2) and (3.4) we consider the operators $a_\alpha$ and $\bar{b}^\dot{\alpha}$ as annihilation operators and define the "vacuum" state

$$
|0\rangle, \quad \langle 0| = (|0\rangle)^\dagger, \quad \langle 0|0\rangle = 1
$$

(3.6)

by the relations

$$
a_\alpha |0\rangle = \bar{b}^\dot{\alpha} |0\rangle = 0, \quad \langle 0|\bar{a}_{\dot{\alpha}} = \langle 0|\bar{b}^\dot{\alpha} = 0.
$$

(3.7)
Let us define the auxiliary Fock space with the vectors of the form
\[
|\varphi_C\rangle = \sum_{s=0}^{\infty} |\varphi_{C,s}\rangle, \quad |\varphi_{C,s}\rangle := \frac{1}{s!} \varphi_{C,\alpha}^{\beta(s)}(x) b^{\alpha(s)} a_{\beta(s)}(x) |0\rangle.
\] (3.8)

Then the conjugate vector to (3.8) is written as follows
\[
\langle \bar{\varphi}_C | = \sum_{s=0}^{\infty} \langle \bar{\varphi}_{s} C |, \quad \langle \bar{\varphi}_{s} C | := \frac{1}{s!} \langle 0 | \bar{b}^{\dot{\alpha}(s)} a_{\beta(s)} \varphi_{C,\beta(s)}^{\alpha(s)}(x).
\] (3.9)

These expansions (3.8) and (3.9) contain an equal number of operators with undotted and dotted indices, like the expressions (2.15) and (2.16). It is natural to consider that the creation and annihilation operators are realized in the space of the vectors (3.8) and (3.9) with external Dirac index \(C = 1, 2, 3, 4\).

Let us introduce the following \((4 \times 4)\) matrix operators
\[
(T_0)_C^D := i \varphi_C^D, \quad (L_0)_C^D := l_0 \delta_C^D, \quad (L_1)_C^D := (l_1 - \mu) \delta_C^D, \quad (L_1^+)_C^D := (l_1^+ - \mu) \delta_C^D.
\] (3.10) - (3.13)

where
\[
l_0 := \partial^2 = \Box, \quad l_1 := i a^{\alpha} \bar{b}^{\dot{\beta}} \partial_{\alpha\dot{\beta}}, \quad l_1^+ := i b^{\alpha} \bar{a}^{\dot{\beta}} \partial_{\alpha\dot{\beta}}.
\] (3.14)

In what follows we will often omit the four-component indices \(C, D\) in the operators (3.10)-(3.13) also. The nonzero (anti)commutators of the above matrix operators are
\[
[L_1^+, L_1] = (N + \bar{N} + 2) L_0, \quad \{T_0, T_0\} = 2 L_0,
\] (3.15)

where \(\{L, T\} \equiv L \cdot T + T \cdot L\), and
\[
N = b^{\alpha} a_{\alpha}, \quad \bar{N} = \bar{a}_{\dot{\alpha}} \bar{b}^{\dot{\alpha}}.
\] (3.16)

All other (anti)commutators among the operators (3.10)-(3.13) vanish.

One can show that the vector \(|\varphi_C\rangle\) (3.8) reproduces the fermionic infinite spin equations (2.18), (2.19) if the constraints
\[
(T_0)_C^D |\varphi_D\rangle = 0, \quad (L_1)_C^D |\varphi_D\rangle = 0 \tag{3.17}
\]
on the vector \(|\varphi_D\rangle\) are imposed. Further, by using BRST procedure, we will construct the Lagrangian, which reproduces the conditions (3.17) as the equations of motion.

### 3.2 BRST charge

Let us consider the operators \(F_a = (T_0, L_0, L_1, L_1^+)\), defined in (3.10)-(3.13), as operators of the constraints of some yet unknown Lagrangian theory. Since these operators form closed (super)algebra \([F_a, F_b] = f_{ab}^c F_c\) (3.15) we can build BRST charge in a standard way as
\[
Q = c^a F_a + \frac{1}{2} (-1)^{n_a+n_b} f_{ab}^c c^a b^c P_c, \quad Q^2 = 0, \tag{3.18}
\]
where $c^a$ and $P_a$ are the ghosts and their momenta and $n_a = 0$ or 1 is the parity of the operator $F_a$.

The next step is to construct such a vector that contains the physical fields under the equations (3.17). The constraint on this vector, stipulated by the operator $L_1^+$, is not imposed. The BRST procedure for such systems was studied in papers [40] – [48] and we apply it to the infinite spin system under consideration.

Thus, using the operators $F_a = (T_0, L_0, L_1, L_1^+)$ and the corresponding ghosts $c_a = (q_0, \eta_0, \eta_1, \eta_1^+)$ we construct Hermitian BRST charge $Q = Q^+$ in the form

$$Q = q_0 T_0 + \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_1^+ \eta_1 (N + \tilde{N} + 2) P_0 - q_0^2 P_0,$$

(3.19)

which is nilpotent by definition

$$Q^2 = 0.$$  

(3.20)

The BRST-charge acts in the extended Fock space, where the action of fermionic $\eta_0, \eta_1, \eta_1^+$ and bosonic $q_0$ ghost “coordinates”, as well as the corresponding ghost “momenta” $P_0, P_1^+$, $P_1$ and $q_0$, are defined earlier. These ghost operators obey the (anti)commutation relations

$$[q_0, p_0] = i, \quad \{\eta_1, P_1^+\} = \{P_1, \eta_1^+\} = \{\eta_0, P_0\} = 1$$

(3.21)

and act on the "vacuum" vector as follows

$$p_0|0\rangle = \eta_1|0\rangle = P_1|0\rangle = P_0|0\rangle = 0.$$  

(3.22)

They possess the standard ghost numbers, $gh(\text{"coordinates"}) = - gh(\text{"momenta"}) = 1$, providing the property $gh(Q) = 1$.

The operator (3.19) acts in the extended Fock space of the vectors

$$|\Phi_C\rangle = |\varphi_C\rangle + \eta_0 P_1^+ |\varphi_{1C}\rangle + \eta_1^+ P_1^+ |\varphi_{2C}\rangle + q_0 P_1^+ |\varphi_{3C}\rangle.$$  

(3.23)

The equation of motion of this BRST-field is postulated in the form

$$Q_{CD}^+ |\Phi_D\rangle = 0.$$  

(3.24)

Due to the nilpotency of the BRST charge the field (3.23) is defined up to the gauge transformations

$$|\Phi_C'\rangle = |\Phi_C\rangle + Q_{CD}^+ |\Lambda_D\rangle,$$

(3.25)

where the gauge parameter $|\Lambda_D\rangle$ has (since $gh(Q) = 1$ and $gh(P_1^+) = -1$) the form

$$|\Lambda_D\rangle = P_1^+ |\lambda_D\rangle.$$  

(3.26)

The fields $|\varphi_C\rangle, |\varphi_{1C}\rangle, |\varphi_{2C}\rangle, |\varphi_{3C}\rangle$ and the gauge parameter $|\lambda_C\rangle$ in (3.23) and (3.26) have the decompositions similar with $|\varphi_C\rangle$ in (3.8).

We emphasize that we take “the momentum representation” with respect to the canonical pair of ghost variables $(\eta_1, P_1^+)$, in contrast to “the coordinate representation” for other canonical pairs of ghosts. This prescription leads to the possibility to consider the corresponding constraint $L_1^+$ by using gauge symmetry, as it is given in Appendix B. Description of such a treatment to use the constraints in the BRST approach was given in [24, 29].
The equation of motion $Q|\Phi\rangle = 0$ (3.24) can be rewritten in the form (we omit here the Dirac indices $C,...$ in all quantities)

$$T_0|\varphi\rangle + (l_1^+ - \mu)|\varphi_3\rangle = 0, \quad \text{(3.27)}$$

$$l_0|\varphi\rangle - (l_1^+ - \mu)|\varphi_1\rangle = 0, \quad \text{(3.28)}$$

$$(l_1 - \mu)|\varphi\rangle + (N + \bar{N} + 2)|\varphi_1\rangle - (l_1^+ - \mu)|\varphi_2\rangle = 0, \quad \text{(3.29)}$$

$$T_0|\varphi_1\rangle + l_0|\varphi_3\rangle = 0, \quad \text{(3.30)}$$

$$T_0|\varphi_2\rangle + (l_1 - \mu)|\varphi_3\rangle = 0, \quad \text{(3.31)}$$

$$-(l_1 - \mu)|\varphi_1\rangle + l_0|\varphi_2\rangle = 0, \quad \text{(3.32)}$$

$$-|\varphi_1\rangle - T_0|\varphi_3\rangle = 0. \quad \text{(3.33)}$$

In this case the gauge transformations $\delta|\Phi\rangle = Q|\Lambda\rangle$ (3.25) look like

$$\delta|\varphi\rangle = (l_1^+ - \mu)|\lambda\rangle, \quad \delta|\varphi_1\rangle = l_0|\lambda\rangle, \quad \delta|\varphi_2\rangle = (l_1 - \mu)|\lambda\rangle, \quad \delta|\varphi_3\rangle = -T_0|\lambda\rangle. \quad \text{(3.34)}$$

Making use of equation (3.33), one can express field $|\varphi_1\rangle$ in the form $|\varphi_1\rangle = -T_0|\varphi_3\rangle$ and then substitute it to other equations. As a result we obtain only three independent equations

$$T_0|\varphi\rangle + (l_1^+ - \mu)|\varphi_3\rangle = 0, \quad \text{(3.35)}$$

$$(l_1 - \mu)|\varphi\rangle - (N + \bar{N} + 2)T_0|\varphi_3\rangle - (l_1^+ - \mu)|\varphi_2\rangle = 0, \quad \text{(3.36)}$$

$$T_0|\varphi_2\rangle + (l_1 - \mu)|\varphi_3\rangle = 0. \quad \text{(3.37)}$$

Residual gauge transformations, which follow from (3.34), have the form

$$\delta|\varphi\rangle = (l_1^+ - \mu)|\lambda\rangle, \quad \delta|\varphi_2\rangle = (l_1 - \mu)|\lambda\rangle, \quad \delta|\varphi_3\rangle = -T_0|\lambda\rangle. \quad \text{(3.38)}$$

### 3.3 Construction of the Lagrangian

It is easy to see that the equations (3.35)–(3.37) are Lagrangian equations for the following Lagrangian

$$\mathcal{L} = \langle \varphi | \left\{ T_0|\varphi\rangle + (l_1^+ - \mu)|\varphi_3\rangle \right\} - \langle \varphi_2 | \left\{ T_0|\varphi_2\rangle + (l_1 - \mu)|\varphi_3\rangle \right\} + \langle \varphi_3 | \left\{ (l_1 - \mu)|\varphi\rangle - (N + \bar{N} + 2)T_0|\varphi_3\rangle - (l_1^+ - \mu)|\varphi_2\rangle \right\}. \quad \text{(3.39)}$$

Then we calculate the inner products in (3.39). After that we convert the Weyl spinor indices of the component fields into vector ones, leaving Dirac indices intact. This way we obtain the following tensor Dirac spinor fields

$$\varphi_{A m(s)} := \varphi_{A m_1...m_s}, \quad \text{(4.40)}$$

which are symmetric with respect to all vector indices and defined by

$$\varphi_{A m(s)} = \frac{(-1)^s}{2^s} \tilde{\sigma}^{\delta_1\alpha_1}_{m_1} \cdots \tilde{\sigma}^{\delta_s\alpha_s}_{m_s} \varphi_{A \alpha(s)\beta(s)} \quad \text{(4.41)}$$
We use this Rarita-Schwinger-like fields in the expansions of all “physical” fields $\varphi$ and gauge field $\lambda$. By construction all “physical” and gauge fields are totally symmetric traceless tensor Dirac spinors

$$\eta^{m_1m_2}\varphi_{A_{m(s)}} = 0.$$  \hfill (3.42)

One can check that

$$\langle \varphi^A_s | \chi_{A s} \rangle = (-1)^s \varphi^A_{\alpha(s)}\hat{\beta}(s)\chi_{A\alpha(s)}\hat{\beta}(s) = 2^s \varphi^{A m(s)}\chi_{A m(s)}, \hfill (3.43)$$

$$\langle \varphi^A_s | l_1^+ | \chi_{A s+1} \rangle = 2^s \varphi^{A m(s)}(-2i)(s + 1)\partial^n\chi_{A nm(s)}, \hfill (3.44)$$

$$\langle \varphi^A_s | l_1^+ | \chi_{A s-1} \rangle = 2^s \varphi^{A m(s)}(-i)s\partial_m\chi_{A m(s-1)}. \hfill (3.45)$$

As a result the BRST Lagrangian \((3.39)\) yields the following component Lagrangian

$$\mathcal{L} = \sum_{s=0}^{\infty} 2^s \varphi^{m(s)} \left[ i \partial \varphi_{m(s)} - is \partial_m \varphi_{3m(s-1)} - \mu \varphi_{3m(s)} \right]$$

$$- \sum_{s=0}^{\infty} 2^s \varphi^{m(s)} \left[ i \partial \varphi_{2m(s)} - 2is(s + 1)\partial^n\varphi_{3nm(s)} - \mu \varphi_{3m(s)} \right]$$

$$+ \sum_{s=0}^{\infty} 2^s \varphi^{m(s)} \left[ -2is(s + 1)\partial^n\varphi_{nm(s)} - \mu \varphi_{m(s)} + is \partial_m \varphi_{2m(s-1)} + \mu \varphi_{2m(s)} - 2is(s + 1)\partial \varphi_{3m(s)} \right]. \hfill (3.46)$$

Gauge transformations \((3.38)\) take the form

$$\delta \varphi_{m(s)} = -is \partial_m \lambda_{m(s-1)} + \frac{i(s - 1)}{2} \eta_{m_{s-1}m_{s}} \partial^n \lambda_{m(s-2)n} - \mu \lambda_{m(s)} \hfill (3.47)$$

$$\delta \varphi_{3m(s)} = -i\partial \lambda_{m(s)} \hfill (3.48)$$

$$\delta \varphi_{2m(s)} = -2is(s + 1)\partial^n \lambda_{nm(s)} - \mu \lambda_{m(s)} \hfill (3.49).$$

The relations \((3.46)\) and \((3.47)-(3.49)\) are the final results.

Let us show that the Lagrangian \((3.46)\) reproduces the conditions \((2.18)\) and \((2.19)\) after the appropriate gauge fixing. Equations of motion following from Lagrangian \((3.46)\) in component form have the form

$$i\partial \varphi_{m(s)} - is \partial_m \varphi_{3m(s-1)} + \frac{i(s - 1)}{2} \eta_{m_{s-1}m_{s}} \partial^n \varphi_{3m(s-2)n} - \mu \varphi_{3m(s)} = 0, \hfill (3.50)$$

$$i\partial \varphi_{2m(s)} - 2is(s + 1)\partial^n \varphi_{3nm(s)} - \mu \varphi_{3m(s)} = 0, \hfill (3.51)$$

$$-2is(s + 1)\partial^n \varphi_{nm(s)} - \mu \varphi_{m(s)} + is \partial_m \varphi_{2m(s-1)} - \frac{i(s - 1)}{2} \eta_{m_{s-1}m_{s}} \partial^n \varphi_{2m(s-2)n}$$

$$+ \mu \varphi_{2m(s)} - 2is(s + 1)\partial \varphi_{3m(s)} = 0. \hfill (3.52)$$

Here we taken into account that the equations of motion are constructed for the traceless fields. We can remove the fields $\varphi_{3m(s)}$ using their gauge transformations and after that we can make the gauge transformations using restricted gauge parameters subjected to the conditions $\partial \lambda_{m(s)} = 0$.  

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Note that the equations (3.51) on fields $\varphi_{2m(s)}$ take the same form as the equations on the gauge
parameters $\bar{\theta} \varphi_{2m(s)} = 0$. Therefore we have enough gauge freedom to remove fields $\varphi_{2m(s)}$. Thus,
after removing the fields $\varphi_{2m(s)}$ and $\varphi_{3m(s)}$, the equations of motion (3.50), (3.52) take the form

$$\bar{\theta} \varphi_{m(s)} = 0, \quad -2i(s + 1)\partial^n \varphi_{nm(s)} - \mu \varphi_{m(s)} = 0$$

and coincide with (2.18) and (2.19). As a result, we have shown that the Lagrangian (3.46) describes the fermionic infinite spin field. We emphasize that this Lagrangian (3.46) has consistently derived in the framework of the general BRST construction. In fact, it is a direct consequence of the basic conditions (1.2)-(1.5). The only assumption we made was a homogeneity condition (1.5) (see also (2.14)) for the fields $\Phi_{\alpha}(x, \xi, \bar{\xi})$ and $\bar{X}^{\dot{\alpha}}(x, \xi, \bar{\xi})$ in (2.11).

We see that the Lagrangian (3.46) depends on three sets of traceless Dirac fields $\varphi_{m(s)}$, $\varphi_{2m(s)}$, $\varphi_{3m(s)}$ (3.42) and each traceless field can be decomposed into two $\gamma$-traceless fields thus Lagrangian (3.46) depends on six sets of Dirac $\gamma$-traceless fields. We emphasize that just such a Lagrangian corresponds to the fields satisfying the basic conditions (1.2)-(1.5).

Recently the Lagrangian for fermionic infinite spin field has been proposed in [18], [23] by combining the free massless fermionic fields with definite helicities and assuming the special gauge symmetry. This Lagrangian depends on one set of Dirac triple $\gamma$-traceless fields and each field also can be decomposed into three Dirac $\gamma$-traceless fields. Thus, one can say that the set of the fields of our Lagrangian (3.46) (as well as gauge parameters and, respectively, degrees of freedom) is twice as large as that of Lagrangian proposed in [18], [23]. Nevertheless we emphasize once more that Lagrangian (3.46) was consistently derived only on the base of the basic conditions (1.2)-(1.5) including the homogeneity condition. At present it is not clear how our Lagrangian (3.46) relates to the Lagrangian obtained in the works [18], [23].

4 Summary and outlook

We have constructed the Lagrangian for the infinite half-integer spin fields. This construction is characterized by the following:

- Irreducible infinite half-integer spin representation is described by the fields (2.11), which depend on additional even spinor variables. The fields (2.11) contain the fields (2.15), (2.16) that satisfy the conditions (2.12), (2.13) and have the power expansion.

- The second Casimir operator (2.9), which acts in the space of infinite spin fields with additional spinor variables is derived.

- Without the presence of the $\delta$-function in (2.11), the fields (2.15), (2.16) with the component field equations (2.18), (2.19) describe reducible infinite half-integer spin representation.

- The fermionic infinite spin equations (2.18), (2.19) are reproduced by the constraints (3.17) imposed on the vector (3.8).

- The constraints (3.17) are obtained from the BRST equation (3.24) for the vector (3.23), where BRST operator is defined in (3.19). After the elimination of some auxiliary states, we stay with the physical and gauge states, that are described by the equations of motion (3.35), (3.36), (3.37) and the gauge transformations (3.38).
• The Lagrangian (3.46) is invariant under the gauge transformations (3.47), (3.48), (3.49) of component fields. The corresponding equations of motion of the component fields have the form (3.50), (3.51), (3.52).

Let us note some comments on the constructed Lagrangian.

i) It is interesting to generalize the BRST approach for obtaining the Lagrangian for the irreducible representation of infinite spin field.

ii) It is interesting to generalize the BRST approach for obtaining field Lagrangian to supersymmetric infinite spin field theory.

iii) It would also be interesting to obtain the Lagrangian of such type for the infinite spin fields in the AdS space.

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Appendix A: Notations

In this Appendix we present the notations used in this paper.

The space-time metric is \( \eta_{mn} = \text{diag}(-1, +1, +1, +1) \). The totally antisymmetric tensor \( \varepsilon_{mnkl} \) has the component \( \varepsilon_{0123} = -1 \). The two-component Weyl spinor indices are raised and lowered by \( \epsilon^\alpha_\beta, \epsilon_\alpha^\beta, \epsilon^{\dot{\alpha}\dot{\beta}}, \epsilon_{\dot{\alpha}}^{\dot{\beta}} \) with the non-vanishing components \( \epsilon_{12} = -\epsilon_{21} = \epsilon^{21} = -\epsilon^{12} = 1 \):

\[
\psi_\alpha = \epsilon_\alpha^\beta \psi^\beta, \quad \bar{\psi}_\dot{\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}},
\]

and so on. Relativistic \( \sigma \)-matrices are

\[
(\sigma_m)_{\alpha\dot{\beta}} = (1_2; \sigma_1, \sigma_2, \sigma_3)_{\alpha\dot{\beta}},
\]

where \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices. The matrices

\[
(\tilde{\sigma}_m)^\dot{\alpha}_\beta = \epsilon^{\dot{\alpha}\dot{\gamma}} (\sigma_m)_{\gamma\delta} = (1_2; -\sigma_1, -\sigma_2, -\sigma_3)^{\dot{\alpha}_\beta}
\]

satisfy the relations

\[
\sigma_n^m \tilde{\sigma}_n^{\dot{\alpha}_\beta} + \sigma_n^m \tilde{\sigma}_n^{\dot{\gamma}_\delta} = -2 \eta^{mn} \delta^\alpha_\gamma, \quad \sigma_n^m \tilde{\sigma}_n^{\dot{\alpha}\beta} = -2 \delta_m^\beta.
\]

The link between the Minkowski four-vector \( A_m \) and bi-spinor \( A_{\alpha\dot{\beta}} \) is given by \( A_{\alpha\dot{\beta}} = A_m (\sigma_m)_{\alpha\dot{\beta}}, \quad A^{\dot{\alpha}\dot{\beta}} = A_m (\tilde{\sigma}_m)^{\dot{\alpha}\dot{\beta}}, \quad A_m = -\frac{1}{2} A_{\alpha\dot{\beta}} (\tilde{\sigma}_m)^{\dot{\alpha}\dot{\beta}}, \) so that \( A_m B_m = -\frac{1}{2} A_{\alpha\dot{\beta}} B^{\dot{\alpha}\dot{\beta}} \).

The \( \sigma \)-matrices with two vector indices are defined by

\[
(\sigma_{mn})^\alpha_\beta = -\frac{1}{4} (\sigma_m \sigma_n - \sigma_n \sigma_m)^\alpha_\beta, \quad (\tilde{\sigma}_{mn})^{\dot{\alpha}_\beta} = -\frac{1}{4} (\tilde{\sigma}_m \sigma_n - \tilde{\sigma}_n \sigma_m)^{\dot{\alpha}_\beta}.
\]
They satisfy the identities
\[ \varepsilon^{mnkl}\sigma_{kl} = -2i \sigma^{mn}, \quad \varepsilon^{mnkl}\bar{\sigma}_{kl} = 2i \bar{\sigma}^{mn}. \] (A.6)

Using the \( \sigma \)-matrices (A.5), we represent the antisymmetric second rank vector tensor in the form
\[ X_{mn} = X_{[mn]} = (\sigma_{mn})_{\alpha\beta}X^{\alpha\beta} - (\bar{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}}X^{\dot{\alpha}\dot{\beta}}, \] (A.7)

where the inverse expressions for the symmetric second rank spinor tensors are
\[ X_{\alpha\beta} = X_{(\alpha\beta)} = \frac{1}{2}(\sigma_{mn})_{\alpha\beta}X_{mn}, \quad X_{\dot{\alpha}\dot{\beta}} = X_{(\dot{\alpha}\dot{\beta})} = -\frac{1}{2}(\bar{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}}X_{mn}. \] (A.8)

In the Weyl representation, the Dirac matrices \((\gamma_m)_A^B, A, B = 1, 2, 3, 4\) have the form
\[ \gamma_m = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\beta} \\ (\bar{\sigma}_m)_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix}, \quad \{\gamma_m, \gamma_n\} = -2\eta_{mn}. \] (A.9)

We use the following notations:
\[ P_A^B := P_m(\gamma_m)_A^B, \quad \partial_A^B := \partial_m(\gamma_m)_A^B. \] (A.10)

Four-component Dirac spinor \( \Psi_A \) is represented by two Weyl spinors
\[ \Psi_A = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^\alpha \end{pmatrix}. \] (A.11)

Dirac conjugate spinor \( \bar{\Psi} = \Psi^\dagger\gamma_0 \) has the components
\[ \bar{\Psi}^A = (\chi^\alpha, \bar{\psi}_\alpha), \quad \bar{\psi}_\alpha = (\psi_\alpha)^*, \quad \chi_\alpha = (\bar{\chi}^\alpha)^*. \] (A.12)

In case of the Majorana spinor, the equality \( \chi_\alpha = \psi_\alpha \) holds.

**Appendix B: Free infinite spin fields with additional spinor coordinates in the space-time description**

To analyze the field contents let us consider the light-cone reference system, where \( p^+ := p^0 + p^3 = 2E, p^- := p^0 - p^3 = 0, p^1 = p^2 = 0 \) and the four-momentum has the form
\[ p^m = (E, 0, 0, E), \quad p_{\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}, \quad p^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}. \] (B.1)

In this system the helicity operator takes the form
\[ h = \frac{W^0}{E} = -\frac{1}{2} \varepsilon_{0mnk}J^{mn}P_k^k \frac{E}{E} = \frac{i}{2} \varepsilon_{0mn3}M_{mn} = (\sigma_{03})^{\alpha\beta}M_{\alpha\beta} + (\bar{\sigma}_{03})^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}. \] (B.2)

Taking into account the relations (see (2.35), (2.6))
\[ (\sigma_{03})^{\alpha\beta}M^{\alpha\beta} = \frac{1}{2} \left( \xi^1 \frac{\partial}{\partial \xi^1} - \xi^2 \frac{\partial}{\partial \xi^2} + \xi^1 \frac{\partial}{\partial \xi^2} - \xi^2 \frac{\partial}{\partial \xi^1} \right), \]
\[ (\bar{\sigma}_{03})^{\dot{\alpha}\dot{\beta}}\bar{M}^{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \left( \bar{\xi}^1 \frac{\partial}{\partial \bar{\xi}^1} - \bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}^2} + \bar{\xi}^1 \frac{\partial}{\partial \bar{\xi}^1} - \bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}^2} \right), \] (B.3)

we obtain the following expression for helicity operator in the light-cone system
\[ h = \frac{1}{2} \left( \xi^1 \frac{\partial}{\partial \xi^1} - \xi^2 \frac{\partial}{\partial \xi^2} - \bar{\xi}^1 \frac{\partial}{\partial \bar{\xi}^1} + \bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}^2} + \xi^1 \frac{\partial}{\partial \xi^1} - \xi^2 \frac{\partial}{\partial \xi^2} - \bar{\xi}^1 \frac{\partial}{\partial \bar{\xi}^1} + \bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}^2} \right). \] (B.4)

Note, to obtain this expression for helicity operator we do not use the equations of motion.
B.1 Integer spins

B.1.1 Field without \( \delta \)-function

Let us consider the generalized field in momentum representation

\[
\Phi(p, \xi, \bar{\xi}) = \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\alpha_1...\alpha_s\beta_1...\beta_s}(p) \xi^{\alpha_1}...\xi^{\alpha_s} \bar{\xi}^{\beta_1}...\bar{\xi}^{\beta_s},
\]

which satisfy the equations of motion

\[
P^2 \Phi = 0, \quad \left( \frac{\partial}{\partial \xi^\alpha} P^{\dot{\alpha}\beta} \frac{\partial}{\partial \xi^\beta} + \mu \right) \Phi = 0. \tag{B.6}
\]

For the component fields

\[
\varphi_{\alpha(s)\beta(s)} := \varphi_{\alpha_1...\alpha_s\beta_1...\beta_s},
\]

these equations have the form

\[
P^2 \varphi_{\alpha(s)\beta(s)} = 0, \quad s \ P^{\dot{\alpha}s} \varphi_{\alpha(s)\beta(s)} = -\mu \varphi_{\alpha(s-1)\beta(s-1)}. \tag{B.8}
\]

First equation in (B.8) is massless Klein-Gordon equation, which can be written in the light-cone system (B.1). Second set of the equations in (B.8)

\[
P^{\dot{\alpha}\alpha} \varphi_{\alpha\beta} = -\mu \varphi_0, \quad P^{\dot{\beta}{\alpha}} \varphi_{\alpha\beta} = -\frac{\mu}{2} \varphi_{\alpha\beta_1}, \quad P^{\dot{\beta}{\alpha}} \varphi_{\alpha\alpha_1\beta_2\beta_2} = -\frac{\mu}{3} \varphi_{\alpha\alpha_2\beta_1\beta_2}, \quad \ldots
\]

in the system (B.1) take the form

\[
2E \varphi_{22} = -\mu \varphi_0, \quad 2E \varphi_{(2\alpha_1)(2\beta_1)} = -\frac{\mu}{2} \varphi_{\alpha\beta_1}, \quad 2E \varphi_{(2\alpha_1\alpha_2)(2\beta_1\beta_2)} = -\frac{\mu}{3} \varphi_{\alpha\alpha_2\beta_1\beta_2}, \quad \ldots
\]

As we see, the independent fields (for below fields we point out their helicities, calculated by formula (B.4)) are

\[
\varphi_0, \quad \varphi_{01}, \quad \varphi_{12}, \quad \varphi_{21}, \quad \varphi_{111}, \quad \varphi_{112}, \quad \varphi_{121}, \quad \varphi_{122}, \quad \varphi_{221}, \quad \ldots \tag{B.9}
\]

We indicate the structure of this set of states: on \( s \)-th step in the expansion, the physical states have the helicities from 0 to \( \pm s \), on \((s+1)\)-th step they have the helicities from 0 to \( \pm(s+1) \) etc.

That is, at each step there arise the states with the same helicities as in the previous step, and the additional states with helicities which are one more modulo larger. So, in the spectrum, there are all helicities, and there is an infinite number of states with an arbitrary fixed helicity. That is, this representation of the infinite spin is not irreducible, it is infinitely degenerate.

We get the same result using a slightly different procedure.

In the light-cone system (B.1) second equation in (B.6)

\[
\left( \frac{\partial}{\partial \xi^2} \frac{\partial}{\partial \bar{\xi}^2} + \frac{\mu}{2E} \right) \Phi(E; \xi^1, \xi^2, \bar{\xi}^1, \bar{\xi}^2) = 0 \tag{B.10}
\]
has the general polynomial solution
\[
\Phi(\xi^1, \xi^2, \bar{\xi}^1, \bar{\xi}^2) = \mathcal{G}^{(0)}(E; \xi^2, \bar{\xi}^2) F^{(0)}(E; \xi^1, \bar{\xi}^1) + \sum_{n=1}^{\infty} (\xi^2)^n \mathcal{G}^{(n)}(E; \xi^2, \bar{\xi}^2) F^{(-n)}(E; \xi^1, \bar{\xi}^1) + \sum_{n=1}^{\infty} (\bar{\xi}^2)^n \mathcal{G}^{(n)}(E; \xi^2, \bar{\xi}^2) F^{(n)}(E; \xi^1, \bar{\xi}^1),
\]
where the multipliers are modified Bessel functions.

\[
G^{(0)}(E; \xi^2, \bar{\xi}^2) := I_0 \left( \sqrt{-\frac{2\mu \xi^2 \bar{\xi}^2}{E}} \right) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( -\frac{\mu \xi^2 \bar{\xi}^2}{2E} \right)^k,
\]
and the polynomial functions with respect to the variable \(\xi^2\):

\[
G^{(n)}(E; \xi^2, \bar{\xi}^2) := \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} \left( -\frac{\mu \xi^2 \bar{\xi}^2}{2E} \right)^k.
\]

In the case of a general field with zero degree of homogeneity, the fields \(F^{(k)}(E; \xi^1, \bar{\xi}^1), -\infty < k < \infty\) in the expansion have the following degrees of homogeneity (\(U(1)\)-charges):

\[
\left( \frac{\partial}{\partial \xi^1} - \frac{\partial}{\partial \bar{\xi}^1} \right) F^{(k)}(E; \xi^1, \bar{\xi}^1) = k F^{(k)}(E; \xi^1, \bar{\xi}^1).
\]

As result, the solutions of the equations have the form

\[
F^{(k)}(E; \xi^1, \bar{\xi}^1) = \begin{cases} 
(\xi^1)^k \sum_{l=0}^{\infty} (\xi^1 \bar{\xi}^1)^l f_l^{(k)}(E), & k \geq 0, \\
(\bar{\xi}^1)^{-k} \sum_{l=0}^{\infty} (\xi^1 \bar{\xi}^1)^l f_l^{(k)}(E), & k < 0.
\end{cases}
\]

In these expansions, the infinite number of the functions \(f_l^{(k)}(E), l = 0, 1, \ldots, \infty\) describe the infinite number of massless states with helicities \(k\). In particle, the helicity-zero fields \(f_l^{(0)}, l = 0, 1, \ldots, \infty\) correspond to the fields \(\varphi_0, \varphi_{11}, \varphi_{111}, \ldots\) in (B.9), helicity-one fields \(f_l^{(1)}, l = 0, 1, \ldots, \infty\) correspond to the fields \(\varphi_{12}, \varphi_{1112}, \varphi_{111112}, \ldots\) in (B.9), helicity-minus-one fields \(f_l^{(-1)}, l = 0, 1, \ldots, \infty\) correspond to the fields \(\varphi_{21}, \varphi_{121}, \varphi_{1121}, \ldots\) in (B.9), etc.

**B.1.2 Field with \(\delta\)-function**

Now we consider the field

\[
\Psi(p, \xi, \bar{\xi}) = \delta(\xi p \bar{\xi} - \mu) \Phi(p, \xi, \bar{\xi}) = \delta(\xi p \bar{\xi} - \mu) \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\alpha_1 \ldots \alpha_s \bar{\beta}_1 \ldots \bar{\beta}_s} (p) \xi^{\alpha_1} \ldots \xi^{\alpha_s} \bar{\xi}^{\bar{\beta}_1} \ldots \bar{\xi}^{\bar{\beta}_s},
\]

(B.16)
which satisfies also the equations (B.6):

\[ P^2 \Psi = 0, \quad \left( \frac{\partial}{\partial \xi^\alpha} P^{\alpha \beta} \frac{\partial}{\partial \bar{\xi}^\beta} + \mu \right) \Psi = 0. \] (B.17)

In addition, due to the presence of \( \delta \)-function in the expression (B.16), this field satisfies the equation

\[ \left( \xi^\alpha P_{\alpha \beta} \bar{\xi}^\beta - \mu \right) \Psi = 0. \] (B.18)

Note that all component fields \( \varphi_{\alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s} \) in the expansion (B.9) are independent up to solution of the equation of motion. But this is not so for the field (B.16) containing the \( \delta \)-function.

The presence of \( \delta \)-function in (B.16) implies that the equality \( \xi p \bar{\xi} = \mu \) is fulfilled in this expression. We pass on as before to the light-cone system (B.1). Then this equality takes the form

\[ \xi^1 \bar{\xi}^1 = \frac{\mu}{2E} = \text{const}. \] (B.19)

Consequently, the terms

\[ \xi^1 \bar{\xi}^i \varphi_{1i} + \frac{1}{2} \xi^1 \xi^1 \bar{\xi}^i \bar{\xi}^i \varphi_{11ii} + \frac{1}{3!} \xi^1 \xi^1 \xi^1 \bar{\xi}^i \bar{\xi}^i \bar{\xi}^i \varphi_{1111ii} + \ldots \] (B.20)

are absorbed by the field \( \varphi_0 \) after its redefinition, the terms

\[ \frac{1}{2} \xi^1 \xi^1 \bar{\xi}^1 \bar{\xi}^2 \varphi_{11i2} + \frac{1}{3!} \xi^1 \xi^1 \xi^1 \bar{\xi}^1 \bar{\xi}^1 \bar{\xi}^2 \varphi_{1111i2} + \ldots \] (B.21)

are absorbed by the term \( \xi^1 \bar{\xi}^2 \varphi_{12} \), the terms

\[ \frac{1}{2} \xi^1 \xi^1 \xi^2 \bar{\xi}^i \varphi_{12ii} + \frac{1}{3!} \xi^1 \xi^1 \xi^2 \bar{\xi}^1 \bar{\xi}^i \varphi_{1121ii} + \ldots \] (B.22)

are absorbed by the field \( \xi^2 \bar{\xi}^i \varphi_{2i} \), etc.

Thus, if we leave independent fields in the expansion (B.16), then the fields present in the expressions (B.20), (B.21), (B.22) can be set equal to zero:

\[ \varphi_{1i} = \varphi_{11ii} = \varphi_{1111ii} = \ldots = 0, \quad \varphi_{1i2} = \varphi_{11i2} = \ldots = 0, \quad \varphi_{12ii} = \varphi_{1121ii} = \ldots = 0, \ldots \] (B.23)

But these fields are precisely those fields in (B.9) that led to the (infinite) multiplicativity of the spectrum. As a result, in this case, with the \( \delta \)-function, the spectrum consists of states

\[ \overset{\varphi_0}{0}, \quad \overset{\varphi_{12}}{1}, \quad \overset{\varphi_{2i}}{-1}, \quad \overset{\varphi_{1122}}{2}, \quad \overset{\varphi_{22ij}}{-2}, \quad \ldots \] (B.24)

Thus, in the spectrum, all helicities are present once and we get an irreducible representation of the infinite spin.

Note that the analysis of the second equation in (B.17) (second equation in (B.6)) is carried out in exactly the same way as in the case without the \( \delta \)-function. In particular, for convenience, we can impose the conditions (B.23) in the solution presented in (B.11).
B.1.3 Gauged field

From the definition \( \text{(B.10)} \) we get that the field \( \Psi(p, \xi, \bar{\xi}) \) is not changed under the following transformations of the field \( \Phi(p, \xi, \bar{\xi}) \):

\[
\delta \Phi(p, \xi, \bar{\xi}) = (\xi p \bar{\xi} - \mu) \Lambda(p, \xi, \bar{\xi}),
\]

where the field \( \Lambda(p, \xi, \bar{\xi}) \) satisfies the equations \( \text{(B.6)} \) or, the same, \( \text{(B.17)} \).

We pass on as before to the light-cone system \( \text{(B.1)} \). Then the field \( \Phi(p, \xi, \bar{\xi}) \) in \( \text{(B.25)} \) is represented by the formulae \( \text{(B.11)}, \text{(B.12)}, \text{(B.13)}, \text{(B.15)} \) due to the equation \( \text{(B.17)} \). The field \( \Lambda(p, \xi, \bar{\xi}) \) has the same expression with replacement the fields \( f \) with \( \bar{\xi} \), in full accordance with the conditions \( \text{(B.23)} \). As result, residual physical fields are \( f_0^{(\pm k)}(E) \), as in \( \text{(B.24)} \).

Due to the gauge transformation we can eliminate the field \( f_l^{(\pm k)}(E), l \geq 1 \) by the gauge field \( \lambda_{l-1}^{(\pm k)}(E) \), in full accordance with the conditions \( \text{(B.23)} \). As result, residual physical fields are \( f_0^{(\pm k)}(E) \), as in \( \text{(B.24)} \).

Now we describe how the field towers \( f_l^{(\pm k)}(E), l \neq 0 \), preserving the first component fields \( f_0^{(\pm k)}(E) \) in the expansions are eliminated by gauge transformations. Let us demonstrate it on the tower \( f_l^{(0)}(E), l = 0, 1, \ldots, \infty \), which are grouped in the field

\[
F^{(0)}(E; \xi^1, \bar{\xi}^1) = f_0(E) + \rho f_1(E) + \rho^2 f_2(E) + \rho^3 f_3(E) + \cdots,
\]

where \( \xi^1 = \sqrt{\rho} \exp(i\theta) \) (we omit index \( 0 \) in \( f_l^{(0)}(E) \)). First terms of the gauge field are given by expansion:

\[
\Lambda^{(0)}(E; \xi^1, \bar{\xi}^1) = \lambda_0(E) + \rho \lambda_1(E) + \rho^2 \lambda_2(E) + \rho^3 \lambda_3(E) + \cdots.
\]

The transformation \( \text{(B.25)} \) gives

\[
\delta f_0 + \rho \delta f_1 + \rho^2 \delta f_2 + \rho^3 \delta f_3 + \cdots = -\mu \lambda_0 + \rho(2E \lambda_0 - \mu \lambda_1) + \rho^2(2E \lambda_1 - \mu \lambda_2) + \cdots.
\]

The quadratic integration of the fields \( f_l(E) = f_l^{(0)}(E) \) with respect to the integration measure \( \delta(p^-)dp^- \delta(2E \rho - \nu) d\rho \frac{dE}{E} = \delta(p^-)dp^- \delta(\rho - \frac{\nu}{2E}) d\rho \frac{dE}{2E^2} \) implies the following asymptotic behavior of these fields in the lower energy boundary

\[
f_l(E)|_{E \to 0} = o(E^{l+1}).
\]

At following asymptotic behavior of gauge fields

\[
\lambda_l(E)|_{E \to 0} = o(E^l),
\]

the transformations \( \text{(B.28)} \) kill all the fields \( f_l(E), l \geq 1 \), except the field \( f_0(E) \).

Gauge removal of the fields \( f_l^{(\pm k)}(E), l \neq 0 \) at \( k \neq 0 \) is performed in the similar way.

B.2 Half-integer spins

The difference between the case of half-integer spins and the case of integer spins is the presence of an external spinor index in the generalized field, which is described by additional spinor \( \xi \). As a result, in the case of the half-integer spin, the extraction of an irreducible representation requires the use of the additional conditions for the infinite spin field.
So, let us consider the field

$$
\Phi(p, \zeta, \xi, \bar{\xi}) := \zeta^\gamma \Phi_\gamma(p, \zeta, \xi) = \zeta^\gamma \sum_{s=0}^{\infty} \frac{1}{s!} \varphi_{\gamma, \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(p) \xi^{\alpha_1} \ldots \xi^{\alpha_s} \bar{\xi}^{\beta_1} \ldots \bar{\xi}^{\beta_s}, \quad (B.31)
$$

which is subjected by the equations

$$
P^{\alpha \gamma} \frac{\partial}{\partial \zeta^\gamma} \Phi = 0, \quad \left( \frac{\partial}{\partial \xi^\alpha} P^{\alpha \beta} \frac{\partial}{\partial \xi^\beta} + \mu \right) \Phi = 0. \quad (B.32)
$$

The component fields \( \varphi_{\gamma, \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(p) \) in (B.31) do not have definite symmetry property with respect the index \( \gamma \) and the indices \( \alpha \)-s.

As first step in the solution of the equation (B.32) we extract in (B.31) the fully symmetric component fields by the following expansion:

$$
\Phi(p, \zeta, \xi, \bar{\xi}) = \Phi(p, \zeta, \xi, \bar{\xi}) + \zeta^\gamma \tilde{X}(p, \zeta, \xi), \quad (B.33)
$$

$$
\Phi(p, \zeta, \xi, \bar{\xi}) := \zeta^\gamma \sum_{s=0}^{\infty} \frac{1}{s!} \phi_{(\gamma, \alpha_1 \ldots \alpha_s) (\beta_1 \ldots \beta_s)}(p) \xi^{\alpha_1} \ldots \xi^{\alpha_s} \bar{\xi}^{\beta_1} \ldots \bar{\xi}^{\beta_s}, \quad (B.34)
$$

$$
\tilde{X}(p, \zeta, \xi) := \sum_{s=0}^{\infty} \frac{1}{s!} \psi_{(\alpha_1 \ldots \alpha_s)}(\beta_1 \ldots \beta_{s+1}) \xi^{\alpha_1} \ldots \xi^{\alpha_s} \bar{\xi}^{\beta_1} \ldots \bar{\xi}^{\beta_{s+1}}, \quad (B.35)
$$

i.e. in (B.31) the fields are written as follows

$$
\varphi_{\gamma, \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(p) = \phi_{(\gamma, \alpha_1 \ldots \alpha_s) (\beta_1 \ldots \beta_s)}(p) + s \epsilon_{\gamma(\alpha_1 \psi_{\alpha_2 \ldots \alpha_s})}(\beta_1 \ldots \beta_s)(p). \quad (B.36)
$$

Similarly to the solution of the integer spin equations (B.6), the equations (B.32) take the form

$$
P^{\beta s} \varphi_{\gamma, \alpha(s)}(\beta(s)) = 0, \quad s P^{\beta s} \varphi_{\gamma, \alpha(s)}(\beta(s)) = -\mu \varphi_{\gamma, \alpha(s-1)}(\beta(s-1)). \quad (B.37)
$$

in terms of the component fields \( \varphi_{\gamma, \alpha(k)}(\beta(s)) := \varphi_{\gamma, \alpha_1 \ldots \alpha_k}(\beta_k) \).

First equations in (B.37) is the massless Dirac equation and in the light-cone system (B.1) it leads to

$$
\varphi_{2, \alpha(s)}(\beta(s)) = \phi_{(2 \alpha_1 \ldots \alpha_s)}(\beta_1 \ldots \beta_s)(E) + s \epsilon_{2(\alpha_1 \psi_{\alpha_2 \ldots \alpha_s})}(\beta_1 \ldots \beta_s)(E) = 0, \quad s = 0, 1, 2, \ldots
$$

These equations express the fields \( \phi \) with at least one undotted index 2 in terms of the fields \( \psi \).

The second equations in (B.37), which are also written in the light-cone system (B.1) in the form

$$
2E s \phi_{(\alpha_1 \ldots \alpha_s-12)}(\beta_1 \ldots \beta_{s-1} \beta_s)(E) + 2E s^2 \epsilon_{1(\alpha_1 \psi_{\alpha_2 \ldots \alpha_s-12})}(\beta_1 \ldots \beta_{s-1} \beta_s)(E)
$$

$$
+ \mu \phi_{(\alpha_1 \ldots \alpha_s-1)}(\beta_1 \ldots \beta_{s-1})(E) + s \epsilon_{1(\alpha_1 \psi_{\alpha_2 \ldots \alpha_s-1})}(\beta_1 \ldots \beta_{s-1})(E) = 0, \quad s = 1, 2, \ldots
$$

express the fields \( \psi \) with at least one dotted index 2 in terms of other fields.

Thus, in the solution of the equations (B.37), the following fields

$$
\phi_{1(\alpha_1 \ldots \alpha_s)}(\beta_1 \ldots \beta_s)(E), \quad \chi_{(\alpha_1 \ldots \alpha_s)}(1 \ldots s)(E), \quad s = 0, 1, 2, \ldots \quad (B.38)
$$
are independent.

Now consider the field \( \tilde{\Psi} \) defined by the following expression

\[
\tilde{\Psi}(p, \zeta, \xi, \bar{\xi}) := \delta(\xi p \bar{\xi} - \mu) \tilde{\Phi}(p, \zeta, \xi, \bar{\xi}), 
\] (B.39)

where the field \( \tilde{\Phi} \) is defined by (B.31), (B.33). Due to the presence of \( \delta \)-function in the expression (B.31), this field satisfies the equation

\[
\left( \xi^\alpha P_{\alpha\beta} \bar{\xi}^\beta - \mu \right) \tilde{\Psi} = 0 
\] (B.40)
in addition to the equations (B.32).

Analysis of the solutions of the equation (B.40) has already been done in Subsection B.1.2. It was shown there that the fields having pairs of indices \( 1\dot{1} \) are not independent. Thus, the following fields

\[
\phi(11\ldots1)(2\ldots2)(s+1)(E), \quad \psi(22\ldots2)(11\ldots1)(s+1)(E), \quad s = 0, 1, 2, \ldots 
\] (B.41)

are independent ones among the fields (B.38) of a generalized field (B.39). Helicities of these fields are equal to

\[
\frac{1}{2} + s, \quad \frac{1}{2} - s, \quad s = 0, 1, 2, \ldots 
\] (B.42)

Second term in (2.1) is described by generalized field

\[
\tilde{\Upsilon}(p, \tilde{\zeta}, \tilde{\xi}, \bar{\xi}) := \delta(\xi p \bar{\xi} - \mu) \tilde{\chi}(p, \tilde{\zeta}, \tilde{\xi}, \bar{\xi}), 
\] (B.43)

where

\[
\tilde{\chi}(p, \tilde{\zeta}, \tilde{\xi}, \bar{\xi}) := \tilde{\chi} \tilde{\chi} \chi(p, \xi, \bar{\xi}) = \tilde{\chi} \sum_{s=0}^{\infty} \frac{1}{s!} \chi_{\gamma, \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_s}(p) \chi^{\alpha_1} \ldots \chi^{\alpha_s} \xi^{\beta_1} \ldots \xi^{\beta_s}. 
\] (B.44)

The field (B.43) is subjected by the equations

\[
\frac{\partial}{\partial \tilde{\zeta}^\gamma} P^{\gamma\alpha} \tilde{\Upsilon} = 0, \quad \left( \frac{\partial}{\partial \tilde{\xi}^\alpha} P^{\alpha\beta} \frac{\partial}{\partial \zeta^\beta} + \mu \right) \tilde{\Upsilon} = 0, \quad \left( \xi^\alpha P_{\alpha\beta} \bar{\xi}^\beta - \mu \right) \tilde{\Upsilon} = 0. 
\] (B.45)

Performing a similar analysis, we find that this field describes the helicities

\[
-\frac{1}{2} - s, \quad \frac{1}{2} + s, \quad s = 0, 1, 2, \ldots 
\] (B.46)

Thus, the field (2.1), consisting two fields (B.39), (B.43), describes a reducible representation of an infinite spin that contains each helicity twice.

We emphasize that the condition (B.40) can be taken into account due to local symmetry, as in the case of integer helicities considered in the Subsection B.1.3.
Appendix C: Irreducible massless infinite half-integer spin representation

In the space-time description, the irreducible infinite half-integer spin representation is described by the field \([31, 37]\):

\[
\Psi_{\alpha}(x; \xi, \bar{\xi}) = \int d^4\pi \ e^{i\pi_{\beta}\bar{\pi}_{\beta} x^{\beta\beta}} \pi_{\alpha} \Psi_{tw}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}),
\]

where the twistor field of the infinite half-integer spin particle is

\[
\Psi_{tw}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) = \delta \left((\pi\xi)(\bar{\xi}\bar{\pi}) - \mu\right) e^{-i\eta_0/p_0} \Psi_{tw}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}),
\]

\[
\Psi_{tw}^{(-1/2)} = \psi^{(-1/2)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\xi\bar{\pi})^k \psi^{(1/2-k)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\pi\xi)^k \psi^{(-1/2-k)}(\pi, \bar{\pi}).
\]

Here

\[
\frac{\eta_0}{p_0} = \sqrt{\frac{\sum_{\alpha=\alpha} (\pi_{\alpha}\xi_{\bar{\alpha}} + \xi_{\alpha}\bar{\pi}_{\bar{\alpha}})}{\sum_{\beta=\beta} \pi_{\beta}\bar{\pi}_{\bar{\beta}}}}.
\]

Twistor field \([C.2]\) satisfies the equations

\[
i\pi_{\alpha} \frac{\partial}{\partial \pi_{\alpha}} \Psi_{tw}^{(-1/2)} = \sqrt{\mu} \Psi_{tw}^{(-1/2)}, \quad i\bar{\pi}_{\bar{\alpha}} \frac{\partial}{\partial \bar{\pi}_{\bar{\alpha}}} \Psi_{tw}^{(-1/2)} = \sqrt{\mu} \Psi_{tw}^{(-1/2)},
\]

\[
\left(\pi_{\alpha} \frac{\partial}{\partial \pi_{\alpha}} - \bar{\pi}_{\bar{\alpha}} \frac{\partial}{\partial \bar{\pi}_{\bar{\alpha}}} + \xi_{\alpha} \frac{\partial}{\partial \xi_{\bar{\alpha}}} - \bar{\xi}_{\bar{\alpha}} \frac{\partial}{\partial \bar{\xi}_{\bar{\alpha}}} \right) \Psi_{tw}^{(-1/2)} = -\Psi_{tw}^{(-1/2)}.
\]

Last equation \([C.5]\) is equivalent to the invariance property of the twistor field

\[
\Psi_{tw}^{(-1/2)}(e^{i\pi_{\alpha}}, e^{-i\bar{\pi}_{\bar{\alpha}}}; e^{i\xi_{\alpha}}, e^{-i\bar{\xi}_{\bar{\alpha}}}) = e^{-i\eta} \Psi_{tw}^{(-1/2)}(\pi_{\alpha}, \bar{\pi}_{\bar{\alpha}}; \xi_{\alpha}, \bar{\xi}_{\bar{\alpha}}).
\]

In \([31]\) it was shown that the field \([C.1]\) is the solution to the equations of motion \([1.2, 1.3, 1.4, 1.7]\).

The field, defined by the expression \([C.1]\), automatically satisfies the additional irreducibility condition

\[
\frac{\partial}{\partial \xi_{\bar{\alpha}}} \Psi_{\alpha}(x; \xi, \bar{\xi}) = 0.
\]

To prove the equation \([C.7]\) we consider

\[
\frac{\partial}{\partial \xi_{\bar{\alpha}}} \Psi_{\alpha}(x; \xi, \bar{\xi}) = \int d^4\pi e^{i\pi_{\beta}\bar{\pi}_{\beta} x^{\beta\beta}} \pi_{\alpha} \frac{\partial}{\partial \xi_{\bar{\alpha}}} \Psi_{tw}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi})
\]

\[
= -i\sqrt{\mu} \int d^4\pi e^{i\pi_{\beta}\bar{\pi}_{\beta} x^{\beta\beta}} \Psi_{tw}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}),
\]

where the first equation in \([C.4]\) was used. But in the integrand of \([C.8]\), the field \(\Psi_{tw}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi})\) has the property \([C.6]\), while the rest quantity \(d^4\pi e^{i\pi_{\beta}\bar{\pi}_{\beta} x^{\beta\beta}}\) is invariant. Therefore, the integral \([C.8]\) is equal to zero identically, that proves the the equation \([C.7]\).
Note that the equation (C.7) zeroes the field (B.35) in the expansion (B.33).

It should also be emphasized that in this paper we consider the fields which are the power series in the additional spinor variable $\xi$, while in the paper [31] there was considered different class of space-time fields (see Appendix B in [31]).

References

[1] E.P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Annals Math. 40 (1939) 149.

[2] E.P. Wigner, *Relativistische Wellengleichungen*, Z. Physik 124 (1947) 665.

[3] V. Bargmann, E.P. Wigner, *Group theoretical discussion of relativistic wave equations*, Proc. Nat. Acad. Sci. US 34 (1948) 211.

[4] G.J. Iverson, G. Mack, *Quantum fields and interactions of massless particles - the continuous spin case*, Annals Phys. 64 (1971) 253.

[5] L. Brink, A.M. Khan, P. Ramond, X.-Z. Xiong, *Continuous spin representations of the Poincaré and superPoincaré groups*, J. Math. Phys. 43 (2002) 6279, arXiv:hep-th/0205145.

[6] G.K. Savvidy, *Tensionless strings: Physical Fock space and higher spin fields*, Int. J. Mod. Phys. A 19 (2004) 3171, arXiv:hep-th/0310085.

[7] J. Mourad, *Continuous spin particles from a string theory*, arXiv:hep-th/0504118.

[8] X. Bekaert, N. Boulanger, *The unitary representations of the Poincaré group in any spacetime dimension*, Lectures presented at 2nd Modave Summer School in Theoretical Physics, 6-12 Aug 2006, Modave, Belgium, arXiv:hep-th/0611263.

[9] X. Bekaert, J. Mourad, *The continuous spin limit of higher spin field equations*, JHEP 0601 (2006) 115, arXiv:hep-th/0509092.

[10] P. Schuster, N. Toro, *On the theory of continuous-spin particles: wavefunctions and soft-factor scattering amplitudes*, JHEP 1309 (2013) 104, arXiv:1302.1198 [hep-th].

[11] P. Schuster, N. Toro, *On the theory of continuous-spin particles: helicity correspondence in radiation and forces*, JHEP 1309 (2013) 105, arXiv:1302.1577 [hep-th].

[12] P. Schuster, N. Toro, *A gauge field theory of continuous-spin particles*, JHEP 1310 (2013) 061, arXiv:1302.3225 [hep-th].

[13] A.K.H. Bengtsson, *BRST Theory for Continuous Spin*, JHEP 1310 (2013) 108, arXiv:1303.3799 [hep-th].

[14] P. Schuster, N. Toro, *A CSP field theory with helicity correspondence*, Phys. Rev. D91 (2015) 025023, arXiv:1404.0675 [hep-th].

[15] V.O. Rivelles, *Gauge theory formulations for continuous and higher spin fields*, Phys. Rev. D91 (2015) 125035, arXiv:1408.3576 [hep-th].
[16] X. Bekaert, M. Najafizadeh, M.R. Setare, *A gauge field theory of fermionic Continuous-Spin Particles*, Phys. Lett. **B760** (2016) 320, [arXiv:1506.00973][hep-th].

[17] R.R. Metsaev, *Continuous spin gauge field in (A)dS space*, Phys. Lett. **B767** (2017) 458, [arXiv:1610.00657][hep-th].

[18] R.R. Metsaev, *Fermionic continuous spin gauge field in (A)dS space*, Phys. Lett. **B773** (2017) 135, [arXiv:1703.05780][hep-th].

[19] Yu.M. Zinoviev, *Infinite spin fields in d=3 and beyond*, Universe **3** (2017) 63, [arXiv:1707.08832][hep-th].

[20] M. Najafizadeh, *Modified Wigner equations and continuous spin gauge field*, Phys. Rev. D **97** (2018) 065009, [arXiv:1708.00827][hep-th].

[21] X. Bekaert, E.D. Skvortsov, *Elementary particles with continuous spin*, Int. J. Mod. Phys. **A32** (2017) 1730019, [arXiv:1708.01030][hep-th].

[22] X. Bekaert, J. Mourad, M. Najafizadeh, *Continuous-spin field propagator and interaction with matter*, JHEP **1711** (2017) 113, [arXiv:1710.05788][hep-th].

[23] M.V. Khabarov, Yu.M. Zinoviev, *Infinite (continuous) spin fields in the frame-like formalism*, Nucl. Phys. **B928** (2018) 182, [arXiv:1711.08223][hep-th].

[24] K.B. Alkalaev, M.A. Grigoriev, *Continuous spin fields of mixed-symmetry type*, JHEP **1803** (2018) 030, [arXiv:1712.02317][hep-th].

[25] R.R. Metsaev, *BRST-BV approach to continuous-spin field*, Phys. Lett. **B781** (2018) 568, [arXiv:1803.08421][hep-th].

[26] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, A. Rusnak, *Model of massless relativistic particle with continuous spin and its twistorial description*, JHEP **1807** (2018) 031, [arXiv:1805.09706][hep-th].

[27] I.L. Buchbinder, V.A. Krykhtin, H. Takata, *BRST approach to Lagrangian construction for bosonic continuous spin field*, Phys. Lett. **B785** (2018) 315, [arXiv:1806.01640][hep-th].

[28] V.O. Rivelles, *A gauge field theory for continuous spin tachyons*, [arXiv:1807.01812][hep-th].

[29] K. Alkalaev, A. Chekmenev, M. Grigoriev, *Unified formulation for helicity and continuous spin fermionic fields*, JHEP **1811** (2018) 050, [arXiv:1808.09385][hep-th].

[30] R.R. Metsaev, *Cubic interaction vertices for massive/massless continuous-spin fields and arbitrary spin fields*, JHEP **1812** (2018) 055, [arXiv:1809.09075][hep-th].

[31] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, *Twistorial and space-time descriptions of massless infinite spin (super)particles and fields*, Nucl. Phys. B **945** (2019) 114660, [arXiv:1903.07947][hep-th].

[32] I.L. Buchbinder, S.J. Gates, K. Koutrolikos, *Superfield continuous spin equations of motion*, Phys. Lett. **B793** (2019) 445, [arXiv:1903.08631][hep-th].
[33] I.L. Buchbinder, M.V. Khabarov, T.V. Snegirev, Y.M. Zinoviev, *Lagrangian formulation for the infinite spin N=1 supermultiplets in d=4*, Nucl. Phys. B 946 (2019) 114717, arXiv:1904.05580 [hep-th].

[34] M. Khabarov and Y. Zinoviev, *Massive higher spin fields in the frame-like multispinor formalism*, Nucl. Phys. B 948 (2019), 114773 arXiv:1906.03438 [hep-th].

[35] M. Khabarov and Y. Zinoviev, *Massive higher spin supermultiplets unfolded*, Nucl. Phys. B 953 (2020), 114959 arXiv:2001.07903 [hep-th].

[36] R.R. Metsaev, *Light-cone continuous-spin field in AdS space*, Phys. Lett. B 793 (2019) 134; arXiv:1903.10495 [hep-th].

[37] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, *Massless infinite spin (super)particles and fields*, contribution to the Volume dedicated to the 80-th Anniversary Jubilee of A.A. Slavnov, arXiv:1911.00362 [hep-th].

[38] N. Najafizadeh, *Supersymmetric continuous spin gauge theory*, JHEP 2003 (2020) 027, arXiv:1912.12310 [hep-th].

[39] I.L. Buchbinder, S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, IOP Publ., 1998, 656 pages.

[40] I.L. Buchbinder, V.A. Krykhtin, A. Pashnev, *BRST approach to Lagrangian construction for fermionic massless higher spin fields*, Nucl. Phys. B711 (2005) 367, arXiv:hep-th/0410215.

[41] I.L. Buchbinder, V.A. Krykhtin, *Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimensions*, Nucl. Phys. B727 (2005) 537, arXiv:hep-th/0505092.

[42] I.L. Buchbinder, A. Pashnev, M. Tsulaia, *Lagrangian formulation of the massless higher integer spin fields in the AdS background*, Phys. Lett. B523 (2001) 338, arXiv:hep-th/0109067.

[43] I.L. Buchbinder, V.A. Krykhtin, A. Pashnev, *BRST approach to Lagrangian construction for fermionic massless higher spin fields*, Nucl. Phys. B711 (2005) 367, arXiv:hep-th/0410215.

[44] I.L. Buchbinder, V.A. Krykhtin, *Gauge invariant Lagrangian construction for massive bosonic higher spin fields in D dimensions*, Nucl. Phys. B727 (2005) 537, arXiv:hep-th/0505092.

[45] I.L. Buchbinder, V.A. Krykhtin, L.L. Ryskina, H. Takata, *Gauge invariant Lagrangian construction for massive higher spin fermionic fields*, Phys. Lett. B641 (2006) 386, arXiv:hep-th/0603212.

[46] I.L. Buchbinder, V.A. Krykhtin, P.M. Lavrov, *Gauge invariant Lagrangian formulation of higher spin massive bosonic field theory in AdS space*, Nucl. Phys. B762 (2007) 344, arXiv:hep-th/0608005.

[47] A. Fotopoulos, M. Tsulaia, *Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation*, Int. J. Mod. Phys. A24 (2009) 1, arXiv:0805.1346 [hep-th].
[48] I.L. Buchbinder, K. Koutrolikos, \textit{BRST Analysis of the Supersymmetric Higher Spin Field Models}, JHEP \textbf{1512} (2015) 106, \href{https://arxiv.org/abs/1510.06569}{arXiv:1510.06569} [hep-th].

[49] H. Bateman, A. Erdélyi, \textit{Higher transcendental functions. Volume II}, New-York Toronto London McGraw-Hill Book Company, inc. 1953.