ON LYUBEZNIK’S INVARIANTS AND ENDOMORPHISMS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let \((R, m)\) denote an \(n\)-dimensional Gorenstein ring. For an ideal \(I \subset R\) of height \(c\) we are interested in the endomorphism ring \(B = \operatorname{Hom}_R(H^c_I(R), H^c_I(R))\). It turns out that \(B\) is a commutative ring. In the case of \((R, m)\) regular local ring containing a field \(B\) is a Cohen-Macaulay ring. Its properties are related to the highest Lyubeznik number \(l = \dim_k \operatorname{Ext}^d_R(k, H^c_I(R))\). In particular \(R \cong B\) if and only if \(l = 1\). Moreover, we show that the natural homomorphism \(\operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow k\) is non-zero.

1. Introduction

Let \((R, m, k)\) denote a local Noetherian ring. For an ideal \(I \subset R\) let \(H^i_I(R), i \in \mathbb{Z}\), denote the local cohomology modules of \(R\) with respect to \(I\) (cf. [2] for the definition). They carry several information about \(I\) and \(R\). Their Bass numbers \(\dim_k \operatorname{Ext}^j_R(k, H^i_I(R)), i, j \in \mathbb{Z}\), are in various directions important invariants (cf. for instance [12], [13], [11]). In the case of a regular local ring they are investigated by Lyubeznik (cf. [12]) known as Lyubeznik invariants.

On the other hand, in recent research there are sufficient conditions when the endomorphism ring of \(H^c_I(R), c = \text{height} I\), isomorphic to \(R\) (cf. for instance [7]). Note that it is not clear whether it is a commutative ring in general. The endomorphism ring \(B := \operatorname{Hom}_R(H^c_I(R), H^c_I(R))\) is the main subject of our investigations here, in particular when \((R, m)\) is a Gorenstein ring. It carries a lot of interesting properties. For an ideal \(I \subset R\) let \(I_d\) denote the intersection of the highest dimensional primary components of \(I\).

Theorem 1.1. Let \((R, m)\) denote an \(n\)-dimensional Gorenstein ring. Let \(I \subset R\) be an ideal with \(c = \text{height} I\) and \(d = \dim R/I\). Then:

(a) The endomorphism ring \(B = \operatorname{Hom}_R(H^c_I(R), H^c_I(R))\) is commutative. There is a natural isomorphism

\[\operatorname{Hom}_R(H^c_I(R), H^c_I(R)) \cong \operatorname{Ext}^d_R(H^c_I(R), R).\]

Moreover \(B\) is \(m\)-adically complete provided \(R\) is a complete local ring.

(b) The ring extension \(R \subset B\) is module-finite and \(B\) is a Noetherian ring.

(c) \(B\) is a free \(R\)-module of rank \(\dim_k \operatorname{Ext}^d_R(k, H^c_I(R))\) and therefore \(B\) is a Cohen-Macaulay module.

(d) \(B\) is a local Noetherian ring if and only if \(V(I_d)\) is connected in codimension one.

(e) \(R \to \operatorname{Hom}_R(H^c_I(R), H^c_I(R)) = B\) is an isomorphism if and only if the completion of the strict Henselization of \(R/I_d\) is connected in codimension one.

By the aid of the result shown by Huneke and Lyubeznik (cf. [10], Theorem 2.9) about the second Vanishing Theorem of Hartshorne (cf. [5], Theorem 7.5) we get as a consequence of Theorem 1.1.

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Corollary 1.2. Let \((R, \mathfrak{m})\) is a complete \(n\)-dimensional regular local ring containing a field. Let \(I \subset R\) denote an ideal and \(d = \dim R/I \geq 2\). Then the following conditions are equivalent:

(i) The natural homomorphism \(R \rightarrow \operatorname{Hom}_R(H^c_I(R), H^c_I(R))\) is an isomorphism.

(ii) \(H^c_i(R) = 0\) for all \(i \geq n - 1\).

In the general case of a Gorenstein ring it is not true (cf. 3.3) true that \(B\) is a module finite extension, while – in this example – it is a Noetherian ring. We conjecture that \(B\) is always a Noetherian ring (cf. 3.5).

Another feature of our considerations here is the the following natural map

\[
\phi : \operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow k,
\]

where \(I \subset R\) is an ideal of height \(I = c\) and \(d = \dim R/I\). The homomorphisms \(\phi\) occurs as an edge homomorphism of a certain spectral sequence resp. as a homomorphism in the construction of the truncation complex (cf. Definition 5.2 and Lemma 6.1). In [7] Conjecture 2.7 Hellus and the author conjectured that it is always non-zero. Here we show that \(\phi\) is related to the homomorphism \(\lambda : \operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow H^c_m(H^c_I(R))\). As an application of our techniques there is the following result:

Theorem 1.3. Let \((R, \mathfrak{m})\) be an \(n\)-dimensional Gorenstein ring. Let \(I \subset R\) denote an ideal with \(c = \text{height } I = d = \dim R/I\). Then:

(a) If \(\phi : \operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow k\) is non-zero, then \(\lambda : \operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow H^c_m(H^c_I(R))\) is non-zero.

Suppose that \((R, \mathfrak{m})\) is a complete regular local ring containing a field. Then:

(b) The homomorphism \(\phi : \operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow k\) is non-zero.

In a certain sense (cf. Remark 6.3) one might consider the homomorphism \(\lambda\) as the version of the homomorphism \(\operatorname{Ext}^d_R(k, \mathcal{K}(R/I^\alpha)) \rightarrow H^c_m\mathcal{K}(R/I^\alpha)\) as it was studied by Hochster (cf. 8 Section 4). Here \(\mathcal{K}(R/I^\alpha)\) denotes the canonical module of \(R/I^\alpha, \alpha \in \mathbb{N}\). For the proof of Theorem 1.3 we refer to Section 6 and Theorem 6.2.

As another application of our technique we proof a slight sharpening of Blickle’s result (cf. 11 Theorem 1.1) about a certain duality for the Lyubeznik numbers. In the second Section of the paper there are preliminaries and auxiliary results needed in the sequel. In the third section we investigate the endomorphism ring of \(H^c_I(R), c = \text{height } I\), in the case of \(R\) a Gorenstein ring and in Section four in the case of \(R\) a regular ring containing a field. To this end we make use of the deep results of Huneke and Sharp (cf. 11) in the case of prime characteristic \(p > 0\) and of Lyubeznik (cf. 12) in characteristic zero. In Section 5 there are some additional results about the so-called Lyubeznik numbers. In Section 6 we study the natural homomorphism \(\phi : \operatorname{Ext}^d_R(k, H^c_I(R)) \rightarrow k\). In the final section 7 we discuss some additional examples.

2. Preliminaries and auxiliary results

In this section we will summarize a few results about completions of local rings and inverse limits as well as some statements about the canonical module of a module. They are needed for further constructions related to local cohomology modules and their asymptotic behavior. With our notation we follow the textbook 11.

Let \(\{I_i\}_{i \in \mathbb{N}}\) denote a family of ideals of a Noetherian ring \(R\) such that \(I_{i+1} \subset I_i\) for all \(i \geq 1\). With the natural epimorphism \(R/I_{i+1} \rightarrow R/I_i\) the family \(\{R/I_i\}_{i \in \mathbb{N}}\) forms an inverse system. Its inverse limit \(\varprojlim R/I_i\) is given by

\[
\varprojlim R/I_i \simeq \{(b_i + I_i)_{i \in \mathbb{N}} : b_i \in R, b_{i+1} - b_i \in I_i \text{ for all } i \in \mathbb{N}\}.
\]
In the following result there is a summary of some technical constructions.

**Lemma 2.1.** Let \((R, \mathfrak{m})\) denote a complete local ring. Let \(\{I_i\}_{i \in \mathbb{N}}\) denote a descending sequence of ideals of \(R\).

(a) Assume that \(\bigcap_{i \in \mathbb{N}} I_i = 0\). Then for any \(i \in \mathbb{N}\) there exists an integer \(j = j(i) \geq i\) such that \(j(i + 1) > j(i)\) and \(I_{j(i)} \subseteq \mathfrak{m}^i\).

(b) Suppose that for all \(i \in \mathbb{N}\) there is an integer \(j = j(i)\) with \(j \geq i, j(i + 1) > j(i)\), and \(I_j \subset \mathfrak{m}^i\). Then \(\varprojlim R/I_i \simeq R\).

**Proof.** The statement in (a) is a consequence of Chevalley’s Lemma (cf. e.g. [15 Theorem 2.1]).

For the proof of (b) let \((b_i + I_i)_{i \in \mathbb{N}} \subseteq \varprojlim R/I_i\) denote a given element. Therefore \(b_{i+1} - b_i \in I_i\) for all \(i \in \mathbb{N}\). By view of the assumption it turns out that for every \(i \in \mathbb{N}\) there is an \(j = j(i)\) such that \(I_{j(i)} \subset \mathfrak{m}^i\). Therefore we see that \(b_j - b_i \in I_{j(i)} \subset \mathfrak{m}^i\) for all \(i \in \mathbb{N}\). That is,

\[
(b_j(i) + \mathfrak{m}^i)_{i \in \mathbb{N}} \subseteq \varprojlim R/\mathfrak{m}^i \simeq R
\]

because \(R\) is complete with respect to the \(\mathfrak{m}\)-adic topology. That is, \((b_j(i) + \mathfrak{m}^i)_{i \in \mathbb{N}}\) defines an element \(b \in R\). Consequently \(b_{j(i)} - b \in \mathfrak{m}^i\) for all \(i \in \mathbb{N}\). Now fix \(l \in \mathbb{N}\) and choose \(i \geq l\). Then \(b_{j(i)} - b_{j(l)} \in I_{j(l)}\) as it follows by induction since \(j(i) \geq j(l)\). So there is the following relation \(b - b_{j(l)} \in (\mathfrak{m}^i, I_{j(l)})\) for all \(i \geq l\). By the Krull Intersection Theorem it provides that \(b - b_{j(l)} \in I_{j(l)}\) for any \(l \geq 1\). But this means nothing else but

\[
b = (b + I_{j(l)})_{i \in \mathbb{N}} = (b_{j(i)} + I_{j(i)})_{i \in \mathbb{N}} \subseteq \lim R/I_{j(l)} \simeq \lim R/I_i,
\]

so that \((b_i + I_i)_{i \in \mathbb{N}} \subseteq R\) and \(\varprojlim R/I_i \simeq R\), as required.

For an application we have to know whether the inverse limit of an inverse system of rings is a quasi-local ring. Here a commutative ring is called quasi-local provided there is a unique maximal ideal.

**Lemma 2.2.** Let \(\{(B_i, n_i)\}_{i \in \mathbb{N}}\) denote an inverse system of local rings such that

\[
\phi_{i+1} : (B_{i+1}, n_{i+1}) \to (B_i, n_i), i \in \mathbb{N},
\]

is a local homomorphism. Then

\[
B := \varprojlim (B_i, n_i) \simeq \{(b_i)_{i \in \mathbb{N}} : b_i \in B_i, \phi_{i+1}(b_{i+1}) = b_i \text{ for all } i \in \mathbb{N}\}
\]

is a quasi-local ring.

**Proof.** It is easily seen that \(B\) admits the structure of a commutative ring with identity element \((1)_{i \in \mathbb{N}} \subseteq B\). Let \((b_i)_{i \in \mathbb{N}} \subseteq B\) denote a unit. By definition there exists an element \((a_i)_{i \in \mathbb{N}} \subseteq B\) such that

\[
(a_i b_i)_{i \in \mathbb{N}} = (1)_{i \in \mathbb{N}}.
\]

This means \(a_i b_i = 1\) for all \(i \in \mathbb{N}\), so that \(b_i\) is a unit in \((B_i, n_i)\) for all \(i \in \mathbb{N}\). On the other hand let \((b_i)_{i \in \mathbb{N}} \subseteq B\) denote an element such that for all \(i \in \mathbb{N}\) the element \(b_i \in B_i\) is a unit. Then there is a sequence \(a_i \in B_i, i \in \mathbb{N}\), of elements such that \(a_i b_i = 1\). We claim that \((a_i)_{i \in \mathbb{N}} \subseteq B\). To this end we have to show that \(\phi_{i+1}(a_{i+1}) = a_i\) for all \(i \in \mathbb{N}\). Because \(\phi_{i+1}\) is a homomorphism of rings

\[
1 = \phi_{i+1}(a_{i+1}) \phi_{i+1}(b_{i+1}) = \phi_{i+1}(a_{i+1})b_i \text{ and } 1 = a_i b_i,
\]

so that \(\phi_{i+1}(a_{i+1}) = a_i\) because \(b_i\) is a unit in \(B_i\).
Now let \((b_i)_{i \in \mathbb{N}} \in B\) be an element such that \(b_i \in B_j\) is not a unit for some \(j \in \mathbb{N}\). Then \(b_j \in \mathfrak{n}_j\) and \(b_i \in \mathfrak{n}_i\) for all \(i \in \mathbb{N}\) as easily seen since \(\phi_i\) is a local homomorphism of local rings for all \(i \in \mathbb{N}\). Therefore
\[
B \setminus B^* = \{(b_i)_{i \in \mathbb{N}} \in B : b_i \in \mathfrak{n}_i \text{ for all } i \in \mathbb{N}\},
\]
where \(B^*\) denotes the set of units of \(B\). Moreover, \(\{(b_i)_{i \in \mathbb{N}} \in B : b_i \in \mathfrak{n}_i \text{ for all } i \in \mathbb{N}\}\) is an ideal of \(B\). Because \(B \setminus B^*\) is equal to the union of maximal ideals of \(B\) it follows that \(B \setminus B^* \) itself is a maximal ideal. But this means nothing else but \(B\) is a quasi-local ring.

Now let \((R, \mathfrak{m})\) denote a local Gorenstein ring with \(n = \dim R\). Let \(M\) be a finitely generated \(R\)-module and \(d = \dim M\). Define
\[
K^i(M) := \operatorname{Ext}^{n-i}_R(M, R), i \in \mathbb{Z},
\]
the \(i\)-th module of deficiency. For \(i = d\) let
\[
K(M) = K^d(M) = \operatorname{Ext}^c_R(M, R), c = n - d,
\]
denote the canonical module of \(M\). Let \(H^i_{\mathfrak{m}}(\cdot)\) denote the \(i\)-th local cohomology functor with support in \(\mathfrak{m}\). By the Local Duality Theorem (see e.g. [2] or [15, Theorem 1.8]) there are the following functorial isomorphisms
\[
H^i_{\mathfrak{m}}(M) \simeq \operatorname{Hom}_R(K^i(M), E), i \in \mathbb{Z},
\]
where \(E = E_R(R/\mathfrak{m})\) denotes the injective hull of \(k = R/\mathfrak{m}\), the residue field.

For some basic properties about the modules of deficiency we refer to [2] and [16, Lemma 1.9]. In particular, \(\operatorname{Ann}_R K(M) = (\operatorname{Ann}_R M)_d\), the intersection of all the \(p\)-primary components of \(\operatorname{Ann}_R M\) such that \(\dim R/p = d\). Moreover \(K(M)\) satisfies Serre’s condition \(S_2\).

For an \(R\)-module \(M\) we will consider \(K(K(M))\). To this end we need the following construction.

**Proposition 2.3.** There is a canonical isomorphism
\[
K(K(M) \otimes M) \simeq \operatorname{Hom}(K(K(M)), K(M)) \simeq \operatorname{Hom}(M, K(K(M)))
\]
for a finitely generated \(R\)-modules \(M\).

**Proof.** Let \(R \to E^c\) denote a minimal injective resolution of \(R\). Then it induces an exact sequence
\[
0 \to K(M) \to \operatorname{Hom}(M, E^c) \to \operatorname{Hom}(M, E^c)_{c+1},
\]
where \(c = \dim R - \dim M\). To this end recall that \(\operatorname{Hom}_R(M, E^c)^i = 0\) for all \(i < c\). By applying the functor \(\operatorname{Hom}(K(M), \cdot)\) it yields an exact sequence
\[
0 \to \operatorname{Hom}(K(M), K(M)) \to \operatorname{Hom}(K(M), \operatorname{Hom}(M, E^c))^c \to \operatorname{Hom}(K(M), \operatorname{Hom}(M, E^c))_{c+1}.
\]
By the adjunction formula and the definition it provides that
\[
K(K(M) \otimes M) \simeq \operatorname{Hom}(K(M), K(M)).
\]
Because of \(\dim K(M) = \dim M\) a similar argument provides the proof of the second isomorphism.

Of a particular interest of (2.3) is the case of \(R/I = A\) for an ideal \(I \subset R\).

**Proposition 2.4.** Let \(I\) denote an ideal of the Gorenstein ring \(R\). For \(A = R/I\) there are the following results:

(a) \(K(K(A)) \simeq \operatorname{Ext}^c_R(\operatorname{Ext}^d_A(A, R), R), c = \dim R - \dim A\).

(b) \(K(K(A))\) is isomorphic to the endomorphism ring \(\operatorname{Hom}(K(A), K(A))\) which is commutative.
(c) There is an injection $A/0_d \to \text{Hom}(K(A), K(A))$.
(d) $\text{Hom}(K(A), K(A))$ is the $S_2$-ification of $A/0_d$.

For the proof we refer to [9, 16] and the previous Proposition 2.3. Next we recall a definition introduced by Hochster and Huneke (cf. [9 (3.4)]).

**Definition 2.5.** Let $R$ denote an commutative Noetherian ring with finite dimension. We denote by $G_R$ the undirected graph whose vertices are primes $p$ of $R$ such that $\dim R = \dim R/p$, and two distinct vertices $p, q$ are joined by an edge if and only if $(p, q)$ is an ideal of height one.

By view of the definition 2.5 observe the following: For a commutative ring $R$ with $\dim R < \infty$ the variety $V(0_d)$ is connected in codimension one if and only if $G_R$ is connected. For the notion of connectedness in codimension one we refer to Hartshorne’s paper [3, Proposition 1.1 and Definition].

As an application here we describe when the endomorphism ring $K(K(A)) \simeq \text{Hom}(K(A), K(A))$ of the canonical module $K(A)$ is a local ring, see [9, 3.6].

**Lemma 2.6.** With the notion of Proposition 2.4 and assuming that $R$ is complete the following conditions are equivalent:

(i) $K(A)$ is indecomposable.
(ii) $V(0_d)$ is connected in codimension one.
(iii) $K(K(A))$ is a local ring.

3. ON A FORMAL RING EXTENSION

Let $(R, m)$ denote an $n$-dimensional Gorenstein ring. Then it is well-known (cf [3]) that the minimal injective resolution $R \longrightarrow E^\cdot$ is a dualizing complex. In particular, this provides a natural homomorphism

$$M \to \text{Ext}_R^c(\text{Ext}_R^c(M, R), R), \quad c = n - \dim M,$$

for a finitely generated $R$-module $M$. Let $I \subset R$ denote an ideal of $R$ and put $A = R/I$. So there is a natural homomorphism

$$A \to \text{Ext}_R^c(\text{Ext}_R^c(A, R), R) \simeq \text{Hom}_R(K(A), K(A)), \quad c = n - d.$$

It is well-known (cf. e.g. [16]) that the kernel coincides with the ideal $I_d$, that is the intersection of all primary components of $I$ whose dimension is $d = \dim R/I$. In other words, $I_d = IR_S \cap R$, where $S = \bigcap_{p \in \text{Assh} R/I} R \setminus p$ and $\text{Assh} M = \{ p \in \text{Ass} M \mid \dim R/p = \dim M \}$ for an $R$-module $M$. Moreover, $\text{Hom}_R(K(A), K(A))$ is a commutative ring, the $S_2$-ification of $A$ and the natural homomorphism above is a homomorphism of commutative rings.

Now let $R/I^{\alpha+1} \to R/I^\alpha, \alpha \in \mathbb{N}$, denote the natural homomorphism. Then there is a commutative diagram

$$
\begin{array}{ccc}
R/I^{\alpha+1} & \to & \text{Ext}_R^c(\text{Ext}_R^c(R/I^{\alpha+1}, R), R) \\
\downarrow & & \downarrow \\
R/I^\alpha & \to & \text{Ext}_R^c(\text{Ext}_R^c(R/I^\alpha, R), R).
\end{array}
$$

Clearly, for each $\alpha \in \mathbb{N}$ the vertical homomorphisms are homomorphisms of commutative rings.

**Theorem 3.1.** With the previous notation there is a homomorphism

$$\phi : \hat{R} \to \varinjlim \text{Ext}_R^c(\text{Ext}_R^c(R/I^\alpha, R), R) =: B,$$

where $\hat{R}$ denotes the $I$-adic completion of $R$. Moreover, there are the following properties:

(a) $B$ admits the structure of a commutative ring so that $\phi$ is a homomorphism of rings.
admits the structure of a commutative ring such that \( \phi(a) \).

inverse system are homomorphism of rings (cf. [9]). Then it is easy to check that

\[ \lim_{\leftarrow} \text{map to zero.} \]

definition of \( H \)

passing to the inverse limit – the homomorphism \( \phi \). As discussed above

\[ \text{Ext}_{R}^{c}(R/I^{\alpha}, R) \simeq \text{Hom}_{R}(K(R/I^{\alpha}), K(R/I^{\alpha})) \]

admits the structure of a commutative ring such that the corresponding homomorphisms in the inverse system are homomorphism of rings (cf. [9]). Then it is easy to check that

\[ \lim_{\leftarrow} \text{Hom}_{R}(K(R/I^{\alpha}), K(R/I^{\alpha})) \]

admits the structure of a commutative ring such that \( \phi \) is a homomorphism of rings. This proves (a).

For the proof of (b) recall that the kernel of \( R/I^{\alpha} \rightarrow \text{Hom}_{R}(K(R/I^{\alpha}), K(R/I^{\alpha})) \) is equal to \( (I^{\alpha})_{d} \). Then \( \ker \phi = \cap_{\alpha \geq N}(I^{\alpha})_{d} \), as follows by elementary properties of the inverse limit. Moreover \( \ker \phi = 0_S \) by the Krull Intersection Theorem and the discussion above.

For the proof of (c) notice that the homomorphisms \( R \rightarrow \hat{R} \) as well as \( \phi : \hat{R} \rightarrow B \) are ring homomorphisms. That is, they respect the identity. Therefore, the residue class \( 1 + m \) does not map to zero.

Now let us relate the structure of the ring \( B = \lim_{\leftarrow} \text{Ext}_{R}^{c}(\text{Ext}_{R}^{c}(R/I^{\alpha}, R), R) \) to the local cohomology of \( R \) with respect to \( I \). Surprisingly the Matlis dual of \( H_{m}^{d}(H_{I}^{c}(R)) \) admits the structure of a commutative ring.

**Theorem 3.2.** Let \( (R, m) \) denote a Gorenstein ring. Let \( I \subset R \) be an ideal with \( d = \dim R/I \) and \( c = \text{height } I \).

(a) There are natural isomorphisms

\[ B \simeq \text{Ext}_{R}^{c}(H_{I}^{c}(R), R) \simeq \text{Hom}_{R}(H_{I}^{c}(R), H_{I}^{c}(R)). \]

(b) If \( R \) is in addition complete, then

\[ \lim_{\leftarrow} \text{Ext}_{R}^{c}(\text{Ext}_{R}^{c}(R/I^{\alpha}, R), R) \simeq \text{Hom}_{R}(H_{m}^{d}(H_{I}^{c}(R), E)). \]

(c) If \( R \) is complete, then \( B \) is also \( m \)-adically complete.

(d) If \( V(I_d) \) is connected in codimension one, then \( (B, n) \) is a quasi-local ring.

**Proof.** By definition \( H_{I}^{c}(R) \simeq \lim_{\leftarrow} \text{Ext}_{R}^{c}(R/I^{\alpha}, R) \), so that \( B \simeq \text{Ext}_{R}^{c}(H_{I}^{c}(R), R) \). To this end recall that \( \text{Ext}_{R}(\cdot, R) \) transforms a direct system into an inverse system. Let \( R \rightarrow E^{c} \) denote a minimal injective resolution. Then there is an exact sequence \( 0 \rightarrow H_{I}^{c}(R) \rightarrow \Gamma_{I}(E)^{c} \rightarrow \Gamma_{I}(E^{c})^{c+1} \). It induces a natural commutative diagram with exact rows

\[
0 \rightarrow \text{Hom}_{R}(H_{I}^{c}(R), H_{I}^{c}(R)) \rightarrow \text{Hom}_{R}(H_{I}^{c}(R), \Gamma_{I}(E)^{c}) \rightarrow \text{Hom}_{R}(H_{I}^{c}(R), \Gamma_{I}(E^{c})^{c+1})
\]

\[
0 \rightarrow \text{Ext}_{R}^{c}(H_{I}^{c}(R), R) \rightarrow \text{Hom}_{R}(H_{I}^{c}(R), E^{c}) \rightarrow \text{Hom}_{R}(H_{I}^{c}(R), E^{c+1})
\]

because \( \Gamma_{I}(E^{c})^{c+1} \) is a subcomplex of \( E^{c} \). The two last vertical homomorphisms are isomorphisms. This follows because \( \text{Hom}_{R}(X, E_{R}(R/p)) = 0 \) for an \( R \)-module \( X \) with \( \text{Supp}_{R} X \subset V(I) \) and \( p \notin V(I) \). Therefore the first vertical map is also an isomorphism.

For the proof of (b) recall that the local cohomology commutes with direct limits. So, by the definition of \( H_{I}^{c}(R) \) there is an isomorphism

\[ \text{Hom}(H_{m}^{d}(H_{I}^{c}(R)), E) \simeq \lim_{\leftarrow} \text{Hom}(H_{m}^{d}(\text{Ext}_{R}^{c}(R/I^{\alpha}, R)), E) \]
and the Local Duality Theorem provides the statement in (b).

By virtue of (a) there is the natural isomorphism
\[ \lim B/m^\alpha B \simeq \lim(R/m^\alpha \otimes \text{Hom}(H^d_m(H^r_f(R)), E)). \]

Since \( E \) is an injective \( R \)-module there are the following natural isomorphisms
\[ R/m^\alpha \otimes \text{Hom}(H^d_m(H^r_f(R)), E) \simeq \text{Hom}(\text{Hom}(R/m^\alpha, H^d_m(H^r_f(R))), E) \]
for all \( \alpha \in \mathbb{N} \). As a consequence there is the isomorphism
\[ \lim B/m^\alpha B \simeq \text{Hom}(H^0_m(H^d_m(H^r_f(R))), E). \]

But \( H^d_m(H^r_f(R)) \) is a module whose support is contained in \( V(m) \) so that
\[ H^0_m(H^d_m(H^r_f(R))) \simeq H^d_m(H^r_f(R)). \]

But this means nothing else but \( \lim B/m^\alpha B \simeq B \). So, (c) is true.

For the proof of (d) let \( \alpha \in \mathbb{N} \) be an integer. Define
\[ B_\alpha = \text{Ext}_R^\alpha(\text{Ext}_R^\alpha(R/I^\alpha, R), R) = \text{Hom}_R(K(R/I^\alpha), K(R/I^\alpha)) \]
the endomorphism ring of the canonical module of \( R/I^\alpha \). Since \( V(I) = V(I^\alpha), \alpha \in \mathbb{N} \), is connected in codimension one we know (cf. Lemma 2.6) that \( (B_\alpha, n_\alpha), \alpha \in \mathbb{N} \), is a local ring. By view of Lemma 2.2 it follows that \( (B, n) \simeq \lim(B_\alpha, n_\alpha) \) is a quasi-local ring. To this end note \( (B_{\alpha+1}, n_{\alpha+1}) \rightarrow (B_\alpha, n_\alpha) \) is a local homomorphism. This follows by contracting the maximal ideal \( n_\alpha \) along the commutative diagram before Theorem 3.1. On the left it provides an injection \( k = k \hookrightarrow B_\alpha/n_\alpha \). Therefore, on the right it yields \( k \hookrightarrow B_{\alpha+1}/n_\alpha \cap B_{\alpha+1} \). Because \( B_{\alpha+1} \) is finitely generated over \( R/I^{\alpha+1} \) it follows that \( n_\alpha \cap B_{\alpha+1} = n_{\alpha+1} \). \( \square \)

**Problem 3.3.** It is a natural question to ask whether the commutative ring \( B \) constructed in Theorem 3.1 is a Noetherian ring. We do not know an answer in general. The stronger question whether \( B \) is a finitely generated \( R \)-module is not true.

To this end observe the following: Suppose that \( R \) is complete. Because of Theorem 5.1 \( B \) is a finitely generated \( R \)-module if and only if \( \dim B/mB < \infty \) (cf. e.g. [14, Theorem 8.4]).

**Lemma 3.4.** Fix the notation of Theorem 7.2. Assume in addition that \( R \) is complete. Then the following conditions are equivalent:

(i) \( \text{Ext}_R^\alpha(H^r_f(R), \hat{R}) \) is a finitely generated \( R \)-module.
(ii) \( \dim \text{Hom}(k, H^r_m(H^r_f(R))) < \infty \).
(iii) \( H^r_m(H^r_f(R)) \) is an Artinian \( R \)-module.

**Proof.** The proof is clear by view of Matlis duality and the previous observation. \( \square \)

In the following there is an example of an ideal \( I \subset R \) in a local complete Gorenstein ring \( R \) such that \( B \) is not a finitely generated \( R \)-module.

**Example 3.5.** Let \( k \) be a field and let \( A = k[[u, v, x, y]] \) be the formal power series ring in four variables. Put \( R = A/IA \), where \( f = xv - yu \). Let \( I = (x, y)R \). We will show that
\[ B = \lim \text{Ext}_R^1(\text{Ext}_R^1(R/I^\alpha, R), R) \]
is not a finitely generated \( R \)-module, while it is a Noetherian ring.

To this end put \( A_\alpha = R/I^\alpha \simeq A/((x, y)^\alpha, f) \) and \( B_\alpha = k[[u, v]][a]/(a^\alpha) \), where \( a \) denotes a variable over \( k[[u, v]] \). Consider the ring homomorphism \( A \rightarrow B_\alpha \) induced by \( x \mapsto ua, y \mapsto va \). As
it is easily seen it induces an injection \( A_\alpha \rightarrow B_\alpha, \alpha \in \mathbb{N} \). Clearly \( B_\alpha, \alpha \in \mathbb{N} \), is a two-dimensional Cohen-Macaulay ring. The cokernel of this embedding is
\[
k[[u, v]][a]/(k[[u, v]][ua, va] + \alpha^a k[[u, v]][a]),
\]
which is a finite dimensional \( k \)-vector space. Whence \( \dim B_\alpha/A_\alpha = 0 \) and \( B_\alpha \) is the \( S_n \)-ification of \( A_\alpha \), that is \( B_\alpha \simeq \text{Ext}^1_R(\text{Ext}^1_R(R/I^\alpha, R), R) \) (cf. Proposition 2.4).
Therefore there are short exact sequences
\[
0 \rightarrow A_\alpha \rightarrow B_\alpha \rightarrow H^1_{m}(A_\alpha) \rightarrow 0
\]
for all \( \alpha \in \mathbb{N} \). By passing to the inverse limit it induces a short exact sequence
\[
0 \rightarrow R \rightarrow B \rightarrow \lim_{\alpha} H^1_{m}(R/I^\alpha) \rightarrow 0.
\]
Moreover it provides that \( B \simeq k[[u, v, a]] \), which is clearly a Noetherian ring. Moreover \( B \) is not a finitely generated \( R \)-module as easily seen.

Furthermore, by the local duality theorem there is the isomorphism
\[
\lim_{\alpha} H^1_{m}(R/I^\alpha) \simeq \text{Hom}_R(H^2_I(R), E).
\]
Therefore, \( \text{Hom}_R(H^2_I(R), E) \) is not a finitely generated \( R \)-module. By Matlis duality it follows that \( H^2_I(R) \) is not an Artinian \( R \)-module and therefore the socle \( \text{Hom}_R(k, H^2_I(R)) \) is not of finite dimension. Originally this was shown by Hartshorne (cf. [6, Section 3]). In fact, the analysis of Hartshorne’s example inspired the above construction.

The proof of Theorem 1.1 (a) follows by Theorem 3.1 (a) and Theorem 3.2 (b). In the following sections we shall discuss some particular cases in which \( B \) is a finitely generated \( R \)-module.

4. The case of regular local rings

In this section let \((R, m)\) denote a regular local ring. Let \( I \subset R \) be an ideal of \( R \). Huneke and Sharp (cf. [11]) in the case of prime characteristic \( p > 0 \) resp. Lyubeznik (cf. [12]) in the case of characteristic zero proved the following result:

**Remark 4.1.** Let \((R, m)\) be a regular local ring containing a field. Let \( I \subset R \) be an ideal. For all \( i, j \in \mathbb{Z} \) the following (cf. [11] and [12]) was shown:

(a) \( H^j_m(H^i_I(R)) \) is an injective \( R \)-module,

(b) \( \text{injdim}_R H^j_m(H^i_I(R)) \leq \dim_R H^j_I(R) \leq \dim R - i \).

(c) \( \text{Ext}^j_R(k, H^i_I(R)) \simeq \text{Hom}_R(k, H^j_m(H^i_I(R))) \) and \( \dim_k \text{Ext}^j_R(k, H^j_I(R)) < \infty \).

As we shall see the previous result applies in an essential way in order to describe properties of the ring
\[
B = \lim \text{Ext}^c_R(\text{Ext}^c_R(R/I^j, R), R), \ c = \dim R - \dim R/I,
\]
introduced in Section 4. But before let us recall the Bass numbers in Remark 4.1 (c) were introduced by Lyubeznik (cf. [12, 4.1]). In fact Lyubeznik has shown that they only depend upon \( R/I \). With the above results in mind we shall describe the structure of the ring \( B \) in case \((R, m)\) is a complete regular local ring containing a field.

**Lemma 4.2.** With the notation as above assume that \((R, m)\) is a complete regular local ring containing a field. There is an isomorphism
\[
\lim \text{Ext}^c_R(\text{Ext}^c_R(R/I^j, R), R) \simeq R^l,
\]
where \( l = \dim \text{Ext}^d_R(k, H^c_I(R)), d = \dim R/I, c = \dim R - \dim R/I. \)
Proof. By virtue of Remark 4.1 it turns out that \( l \) is a finite number. As a consequence of Lemma 3.4 it follows that \( B \) is a finitely generated \( R \)-module. Moreover, by virtue of Remark 4.1 (a) and the definition of \( B \) as the Matlis dual of \( H^d_m(H^d_I(R)) \) (cf. Theorem 3.2) we see that \( B \) is flat as an \( R \)-module. Therefore \( B \) is a free \( R \)-module and \( B \simeq R^l \), the direct sum of \( l \) copies of \( R \).

In the following we will give an interpretation of the rank \( l \) in topological terms. To this end we use results of Lyubeznik (cf. [13]) and Zhang (cf. [19]).

Remark 4.3. Let \((R, m)\) be an \( n \)-dimensional regular local ring containing a field. Let \( I \subset R \) denote an ideal with \( c = \text{height } I \) and \( d = \dim R/I \). Let \( A \) denote the completion of the strict Henselization of the completion of \( R/I \). Let \( t \) denote the number of connected components of the graph \( G_A \). Then

\[
\dim_k \text{Hom}_R(k, H^d_m(H^d_I(R))) = \dim_k \text{Ext}^t_R(k, H^d_I(R)) = t
\]

(cf. [13] and [19]).

It was pointed out by Lyubeznik (cf. [13]) that the graph \( G_A \), where \( A \) the completion of the strict Henselization of the completion of \( R/I \), is realized by a smaller ring. Namely, let \( k \subset \hat{A} \) denote a coefficient field. It follows (cf. [10, Theorem 4.2]) that there exists a finite separable field extension \( k \subset K \) such that the graphs \( G_{K\otimes_k R/I} \) and \( G_A \) are isomorphic.

Here we want to give another description of the invariant \( t = \dim_k \text{Ext}^d_R(k, H^d_I(R)) \). As a first step in this direction there is the following result:

Theorem 4.4. Let \((R, m)\) denote an \( n \)-dimensional complete regular ring containing a field. Let \( I \) be an ideal of \( R \) and \( c = \text{height } I \). Then the following is equivalent:

(i) \( V(I) \) is connected in codimension one.

(ii) \( B = \lim_{\leftarrow} \text{Ext}^t_R(\text{Ext}^t_k(R/I^n, R), R) \) is a local ring.

Proof. Because of \( B \simeq \text{Hom}_R(H^d_m(H^d_I(R)), E) \) we may replace \( I \) by \( I_d \). The implication (i) \( \implies \) (ii) is a consequence of the results shown in Theorem 3.2 (d) and Lemma 4.2 where it follows that \( B \) is a Noetherian ring.

In order to prove (ii) \( \implies \) (i) suppose that \( V(I) \) is not connected in codimension one. Let \( G_1, \ldots, G_t, t > 1 \), denote the connected components of \( G_{R/I} \). Moreover, let \( I_i, i = 1, \ldots, t \), denote the intersection of all minimal primes of \( V(I) \) that are vertices of \( G_i \). Then

\[
H^d_I(R) \simeq \bigoplus_{i=1}^t H^d_{I_i}(R)
\]

as it is a consequence of the Mayer-Vietoris sequence for local cohomology (cf. [13] Proposition 2.1 for the details). Clearly \( c = \text{height } I_i, i = 1, \ldots, t \), and \( H^e_{I_i}(R) \neq 0 \). Moreover

\[
\text{Ext}^e_R(H^d_{I_i}(R), R) \simeq \bigoplus_{i=1}^t \text{Ext}^e_R(H^d_{I_i}(R), R)
\]

and therefore \( B \simeq B_1 \times \ldots \times B_t \), where \( B_i = \text{Ext}^e_R(H^d_{I_i}(R), R), i = 1, \ldots, t \), are local rings as shown by (i) \( \implies \) (ii). Because of \( t > 1 \) it follows that \( B \) is not a local ring.

As a corollary of the previous statement we are able to describe the number of connected components of \( G_{R/I} \) in terms of the ring structure of \( B = \text{Ext}^d_R(\text{Ext}^d_K(R_I^d, R), R) \).

Corollary 4.5. With the notation of Theorem 4.4 the ring \( B \) is a semi-local ring. The number of maximal ideals of \( B \) is equal to the number of connected components of \( G_{R/I} \).
Proof. First $B$ is a semi-local ring as follows because it is a finitely generated $R$-module. If $G_{R/I}$ is connected in codimension one, then $B$ is a local ring (cf. Theorem 4.4). Then the claim follows as in the proof of Theorem 4.4 by $H^i_I(R) \cong \oplus_{i=1}^t H^i_{R}(R)$. Here $I_i, i = 1, \ldots, t$, denotes the intersection of all minimal primes of $V(I)$ that are vertices of $G_i$, the connected components of $G_{R/I}$.

\[ \square \]

**Remark 4.6.** The proof of the statements (b) and (c) of Theorem 1.1 follows by Lemma 4.2 and Remark 4.6. The claim (d) of Theorem 1.1 is shown in Theorem 4.4. Finally (c) is a consequence of the results by Lyubeznik (cf. [13]) and Zhang (cf. [19]).

5. **On Lyubeznik numbers**

In this section let $(R, \mathfrak{m})$ be a regular local ring containing a field. Let $I \subset R$ be an ideal of $R$. We will add a few results concerning the Lyubeznik numbers

\[ \dim_k \Hom_R(k, H^j_{\mathfrak{m}}(H^i_I(R))) = \dim_R \Ext^j_R(k, H^i_I(R)) \]

in general. That means, we are interested in them for all pairs $(j, i)$ not necessary for $(j, i) = (d, c)$. As a first step towards this direction we improve the estimate in Remark 4.1 (b). To this end we use the Hartshorne-Lichtenbaum Vanishing Theorem. It yields that $H^j_{\mathfrak{m}}(R) = 0, n = \dim R$, whenever $(R, \mathfrak{m})$ is a complete local domain and $I$ is an ideal with $\dim R/I > 0$ (cf. [5, Theorem 3.1] or [15, Theorem 2.20]).

**Lemma 5.1.** Let $(R, \mathfrak{m})$ denote a regular local ring with $\dim R = n$. Let $I$ be an ideal of $R$ and $c = \text{height } I < n$. Then

\[ \dim H^j_I(R) \leq n - i - 1 \]

for all $c < i \leq n$ and $\dim H^j_I(R) = n - c$.

**Proof.** Let $p \in V(I)$ denote a prime ideal such that $\dim R_p = c$. Then

\[ 0 \neq H^j_{IR_p}(R_p) \cong H^j_I(R) \otimes_R R_p \]

because $c = \dim R_p$ and $\text{Rad } IR_p = pR_p$. Recall that $p \in V(I)$ is a minimal prime ideal. Therefore $\dim H^j_I(R) \geq \dim R/p = n - c$. The equality is true since $\text{Supp } H^j_I(R) \subseteq V(I)$.

Now let $c < i \leq n$. First of all note that $H^j_{\mathfrak{m}}(R) = 0$ as follows by the Hartshorne-Lichtenbaum Vanishing Theorem (cf. [5] or [15, Theorem 2.20]). Now suppose the contrary to the claim. That is $\dim H^j_I(R) \geq n - i$ for a certain $c < i < n$. Then there is a prime ideal $p \in \text{Supp } H^j_I(R)$ such that $\dim R/p \geq n - i$. Therefore $H^j_{IR_p}(R_p) \neq 0$ and $i < \dim R_p$ as follows again by the Hartshorne-Lichtenbaum Vanishing Theorem. Moreover $\dim R/p = n - \dim R_p \geq n - i$, and therefore $\dim R_p \geq i$, which is a contradiction. \[ \square \]

For a better understanding of the Lyubeznik numbers we need an auxiliary construction of a certain complex $C^*(I)$ for an ideal $I$ of $R$. Here we assume that $(R, \mathfrak{m})$ is a Gorenstein ring. Let $R \xrightarrow{\sim} E^*$ denote a minimal injective resolution of $R$. Because of $\Gamma_I(E^*)^i = 0$ for all $i < c = \text{height } I < n$ there is a homomorphism of complexes

\[ 0 \rightarrow H^j_I(R)[-c] \rightarrow \Gamma_I(E^*) \]

where $H^j_I(R)$ is considered as a complex concentrated in homological degree zero.

**Definition 5.2.** The cokernel of the embedding $H^j_I(R)[-c] \rightarrow \Gamma_I(E^*)$ is defined as $C^*_R(I)$, the truncation complex with respect to $I$. So there is a short exact sequence of complexes of $R$-modules

\[ 0 \rightarrow H^j_I(R)[-c] \rightarrow \Gamma_I(E^*) \rightarrow C^*_R(I) \rightarrow 0. \]

We observe that $H^j(C^*_R(I)) \cong H^j_I(R)$ for all $i \neq c$ while $H^c(C^*_R(I)) = 0$. 

By applying the derived functor $\mathrm{R}\Gamma_m$ of the section functor it induces (in the derived category) a short exact sequence of complexes

$$0 \rightarrow \mathrm{R}\Gamma_m(H^j_I(R))[-c] \rightarrow E[-n] \rightarrow \mathrm{R}\Gamma_m(C^\cdot(I)) \rightarrow 0.$$  

Recall that $\mathrm{R}\Gamma_m(\Gamma_I(E^\cdot)) \simeq \Gamma_m(\Gamma_I(E^\cdot)) \simeq \Gamma_m(E^\cdot) \simeq E[-n]$, where $E = E_R(R/m)$ denotes the injective hull of the residue field.

In order to compute the hypercohomology $H^i_m(C^\cdot(I))$ there is the following $E_2$-term spectral sequence (cf. [17] for the details)

$$E_2^{p,q} = H^p_m(H^q(C^\cdot(I))) \Rightarrow E_\infty^{p+q} = H^{p+q}_m(C^\cdot(I)).$$

Now recall that $H^q(C^\cdot(I)) = H^q_I(R)$ for $c < q < n$, and $H^q(C^\cdot(I)) = 0$ for $q \leq c$ resp. $q \geq n$. We notice that $H^q_I(R) = 0$ as a consequence of the Hartshorne-Lichtenbaum Vanishing Theorem.

**Proposition 5.3.** Let $(R,m)$ be a regular local ring. With notation of Definition 5.2 there is a short exact sequence

$$0 \rightarrow H^{n-1}_m(C^\cdot(I)) \rightarrow H^d_m(H^j_I(R)) \rightarrow E \rightarrow 0$$

and isomorphisms $H^{j-1}_m(C^\cdot(I)) \simeq H^{j-c}_m(H^j_I(R))$ for all $j < n$ and all $j > n = \dim R$.

**Proof.** The proof follows by the long exact cohomology sequence of the above short exact sequence of complexes in the derived category. The only claim we have to show is the vanishing of $H^q_m(C^\cdot(I))$. To this end we apply the previous spectral sequence. By virtue of Lemma 5.1 we know that $\dim H^q_I(R) < n - q$ for all $q \neq c$. Therefore $E_2^{n-q,c} = H^q_m(H^j_I(R)) = 0$ for all $q \neq c$. But this provides the vanishing of $H^q_m(C^\cdot(I))$, as required.

As an application of our investigations we prove a slight improvement of a duality result shown by Blickle (cf. [1] Theorem 1.1).

**Corollary 5.4.** Let $I \subset R$ denote an ideal of a regular local ring $(R,m)$ containing a field. Suppose that $\text{height} \ c < n$ and $\text{Supp} H^j_I(R) \subset V(m)$ for all $i \neq c$. Then the following is true:

(a) There is a short exact sequence

$$0 \rightarrow H^{n-1}_I(R) \rightarrow H^d_m(H^j_I(R)) \rightarrow E \rightarrow 0.$$

(b) For $j < n$ there are isomorphisms $H^{j-1}_I(R) \simeq H^{j-c}_m(H^j_I(R)).$

**Proof.** Because of $\text{Supp} H^j_I(R) \subset V(m)$ for all $i \neq c$ it follows that $H^p_m(H^q(C^\cdot(I)) = 0$ for all $p \neq 0$. So the previous spectral sequence degenerates to isomorphisms

$$H^q_m(C^\cdot(I)) \simeq H^0_m(H^q(C^\cdot(I))$$

for all $q \in \mathbb{Z}$. Then the claim is a consequence of the statements in Proposition 5.3. Recall that $H^q_m(H^q(C^\cdot(I)) \simeq H^q(C^\cdot(I))$ for all $q \in \mathbb{Z}$ because $\text{Supp} H^j_I(R) \subset V(m)$ for all $i \neq c$ by the assumption.

Under the assumption of Corollary 5.4 Blickle (cf. [1] Theorem 1.1) proved the following equalities

$$\dim_k \text{Hom}_R(k, H^0_m(H^{j-1}_I(R))) = \dim_k \text{Hom}_R(k, H^{j-c}_m(H^j_I(R))) - \delta_{j-c,d}$$

for all $j \in \mathbb{N}$. In fact, this is a consequence of the present Corollary 5.4. The assumption $\text{Supp} H^j_I(R) \subset V(m)$ for all $i \neq c$ is fulfilled whenever $cd IR_p = \text{height} I$ for all $p \in V(I) \setminus \{m\}$.

Another result of this type is the following.

**Corollary 5.5.** With the notation of 5.4 let $I$ denote an ideal of $R$. Suppose there is an integer $c < a < n$ such that $H^j_I(R) = 0$ for all $i \neq c, a$. 

(a) **There is a short exact sequence**

\[ 0 \to H_m^{n-a-1}(H_I^a(R)) \to H_m^d(H_I^c(R)) \to E \to 0. \]

(b) **For** \( j < n \) **there are isomorphisms**

\[ H_m^{j-a-1}(H_I^a(R)) \simeq H_m^{j-c}(H_I^c(R)). \]

**Proof.** By the assumption on the vanishing of \( H^i_I(R) \) it follows that \( C \cdot (I) \to H_I^a(R)[-a] \) in the derived category. Thus, the statement is an immediate consequence of Proposition 5.3. □

It would be of some interest to get an understanding of the Lyubeznik numbers in general.

### 6. On a Trace Map

Let \((R, \mathfrak{m})\) denote a Gorenstein ring and \( n = \dim R \). Let \( I \subset R \) denote an ideal of height \( I = c \) and \( \dim R/I = d \).

**Lemma 6.1.** **With the previous notation there is a natural homomorphism**

\[ \phi : \text{Ext}^d_R(k, H_I^c(R)) \to k. \]

**Proof.** Apply the derived functor \( R \text{Hom}_R(k, \cdot) \) to the short exact sequence as it is defined in the definition of the truncation complex (cf. 5.2). Then there is the following short exact sequence of complexes in the derived category

\[ 0 \to R \text{Hom}_R(k, H_I^c(R))[-c] \to R \text{Hom}_R(k, \Gamma_m(E')) \to R \text{Hom}_R(k, C_R(I)) \to 0. \]

Now we consider the complex in the middle. It is represented by \( \text{Hom}_R(k, \Gamma_m(E')) \) since \( \Gamma_m(E') \) is a complex of injective modules. Moreover there are the following isomorphisms

\[ \text{Hom}_R(k, \Gamma_m(E')) \simeq \text{Hom}_R(k, E') \simeq k[-n]. \]

By virtue of the long exact cohomology sequence it yields the natural homomorphism of the statement. □

In [7, Conjecture 2.7] Hellus and the author conjectured that the homomorphism in Lemma 6.1 is in general non-zero. In the following we shall confirm this question in the case of \((R, \mathfrak{m})\) a regular local ring containing a field. To this end we need a few auxiliary constructions.

By virtue of Proposition 5.3 and Lemma 6.1 there is the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^d_R(k, H_I^c(R)) & \xrightarrow{\phi} & k \\
\downarrow \lambda & & \downarrow \\
H_m^d(H_I^c(R)) & \xrightarrow{\psi} & E.
\end{array}
\]

Here the vertical homomorphism \( k \to E \) is – by construction – the natural inclusion. Therefore, \( \lambda \) is not zero, provided \( \phi \) is not zero.

**Theorem 6.2.** **Let** \((R, \mathfrak{m})\) **denote a regular local ring containing a field with** \( \dim R = n \). **Let** \( I \subset R \) **denote an ideal of height** \( I = c \) **and** \( \dim R/I = d \). **Then the homomorphism**

\[ \phi : \text{Ext}^d_R(k, H_I^c(R)) \to k \]

**is non-zero.**

**Proof.** We may assume that \( R \) is a complete local ring. By applying \( \text{Hom}_R(k, \cdot) \) to the above diagram it provides the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^d_R(k, H_I^c(R)) & \xrightarrow{\phi} & k \\
\downarrow \lambda & & \| \\
\text{Hom}_R(k, H_m^d(H_I^c(R))) & \xrightarrow{\tilde{\psi}} & k.
\end{array}
\]
Because of Remark 4.1 the vertical homomorphism $\lambda$ is an isomorphism. Therefore it will be enough to show that $\psi$ is not zero. By Matlis duality it follows that

$$B = \text{Ext}_R^d(H_f^i(R), R) \simeq \text{Hom}_R(H_m^d(H_f^i(R)), E).$$

So, $\tilde{\psi}$ is the Matlis dual of the natural homomorphism $k \otimes R \to k \otimes B$ which is non-zero as shown in Theorem 5.1 (c). This proves that $\psi$ is non-zero. By the previous observation it follows that $\phi$ is non-zero too.

**Remark 6.3.** Let $(R, m)$ be local ring that is the factor ring of a Gorenstein ring. Let $M$ be a finitely generated $R$-module with $d = \dim_R M$. In connection to his canonical element conjecture (cf. [8, Section 4]) Hochster has studied the natural homomorphism $\text{Ext}_R^d(k, K(M)) \to H_m^d(K(M))$, where $K(M)$ denotes the canonical module of $M$. In particular, he considered the problem whether this map is non-zero.

In our situation here the natural homomorphism $\lambda : \text{Ext}_R^d(k, H_f^i(R)) \to H_m^d(H_f^i(R))$ is the direct limit of the natural homomorphisms $\lambda_\alpha : \text{Ext}_R^d(k, K(R/I^\alpha)) \to H_m^d(K(R/I^\alpha)), \alpha \in \mathbb{N}$. Recall that $H_f^i(R) \simeq \lim \limits_{\leftarrow} K(R/I^\alpha)$ and that local cohomology commutes with direct limits. So in a certain sense, $\lambda$ is the stable value of all of the $\lambda_\alpha$. It would be of some interest to relate the non-vanishing of $\lambda$ to other problems in commutative algebra.

7. Examples

Let $I \subset R$ denote an ideal with $c = \text{height } I$. The following example shows that the property $\text{Hom}_R(H_f^i(R), H_f^j(R)) \simeq R$ is not preserved by passing to the localization with respect to a prime ideal.

**Example 7.1.** Let $k$ denote an algebraically closed field. Let $R = k[[a, b, c, d, e]]$ denote the formal power series ring in five variables. Let $I \subset R$ denote the prime ideal with the parametrization

$$a = su^2, b = stu, c = tu(t - u), d = t^2(t - u), e = u^3.$$ 

It is easy to see that $I = (ad - bc, a^2c + abc - b^2e, c^3 + cde - d^2e, ade - bde + ac^2)$. Moreover $\dim R/I = 3, n = 5$ and $H_f^i(R) = 0$ for all $i \neq 2, 3$ as it is a consequence of the Second Vanishing Theorem. Clearly $V(I)$ is connected in codimension one because $I$ is a prime ideal. So it follows (cf. Remark 4.3) $\dim_k \text{Ext}_R^2(k, H_f^2(R)) = 1$. Therefore the endomorphism ring $\text{Hom}_R(H_f^2(R), H_f^2(R))$ is isomorphic to $R$ (cf. Lemma 4.2).

Let $p = (a, b, c, d)R$. Then $\text{dim}_{R_p} I_{R_p} = 2$. The ideal $I_{R_p}$ corresponds to the parametrization $(x, xy, y(y - 1), y^2(y - 1))$. It follows that $V(I_{R_p}) \setminus \{p_{R_p}\}$ is not formally connected (cf. [3, 3.4.2]). Therefore $V(I_{R_p})$ has two connected components. Whence $\text{Hom}_{R_p}(H_{I_{R_p}}^2(R_p), H_{I_{R_p}}^2(R_p)) \simeq \widehat{R_p}^2$, because $\dim_{k_{(p)}} \text{Ext}_{R_p}^2(k(p), H_{I_{R_p}}^2(R_p)) = 1$.

The following example (invented by Hochster) shows that the Bass numbers $\dim_k \text{Ext}_R^1(k, H_f^i(R))$ depend upon the characteristic of the ground field $k$.

**Example 7.2.** Let $R = k[[x_1, \ldots, x_6]]$ denote the formal power series rings in six variables over the basic field $k$. Let $I$ denote the ideal generated by the two by two minors of the matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$ 

Then $R/I$ is a four dimensional Cohen-Macaulay ring and $c = \text{height } I = 2$. It follows that $H_f^i(R) = 0$ for all $i \neq 2$ provided $k$ is a field of characteristic $p > 0$. Furthermore $H_f^i(R) = 0$ for all $i \neq 2, 3$ and $H_f^3(R) \neq 0$ in the case of $k$ a field of characteristic zero. This is shown via
the Reynolds operator. Clearly $R/I$ has an isolated singularity, so that $\text{Supp} \, H^3_\omega(R) \subset \{m\}$. By virtue of Corollary 5.3 and Remark 4.1 it follows that $\text{Ext}^0_R(k, H^3_\omega(R)) = 0$ for all $i < 4$ if $k$ is of positive characteristic, while $\text{Ext}^2_R(k, H^3_\omega(R)) \neq 0$ if $k$ is of characteristic zero.

Clearly $H^3_\omega(R) \sim H^3_\Omega(R)$, and therefore $H^3_\Omega(R)$ is an injective $R$-module (cf. Remark 4.1). Now we apply the derived functor $R \text{Hom}_R(k, \cdot)$ to the short exact sequence of the truncation complex (cf. Definition 5.2). Because of $H^3(C_R(I)) \sim H^3_\Omega(R)$ and $H^3(C_R(I)) = 0$ for all $i \neq 3$ it induces an isomorphism $\text{Ext}^2_R(k, H^3_\Omega(R)) \sim \text{Hom}_R(k, H^3_\Omega(R))$.

Finally Uli Walther (cf. [18, Example 6.1]) has computed that $H^3_\Omega(R) \simeq E_R(k)$. Therefore $\dim \text{Ext}^2_R(k, H^3_\Omega(R)) = 1$ in the case of characteristic zero.

The Example 7.3 shows that the number of maximal ideals of $B = \text{Hom}_R(H^3_\Omega(R), H^3_\Omega(R))$ does not coincide with the multiplicity of $B$.

**Example 7.3.** Let $\mathbb{Q}$ denote the field of rational numbers. Consider $\mathbb{Q}(i)$ denote the field extension of $\mathbb{Q}$ by the imaginary unit. Let $R = \mathbb{Q}[[w, x, y, z]]$ and $S = \mathbb{Q}(i) [[w, x, y, z]]$ denote the formal power series ring in four variables over $\mathbb{Q}$ and $\mathbb{Q}(i)$ respectively. Let $J = (w - ix, y - iz) \cap (w + ix, y + iz) \subset S$ and $I = J \cap R$. Then $I = (w^2 + x^2, y^2 + z^2, wy + xz, wz - xy)$ is a two-dimensional prime ideal. Therefore (cf. Theorem 4.4) $B_R = \text{Hom}_R(H^2_\Omega(R), H^2_\Omega(R))$ is a local ring. Moreover

$$B_S = \text{Hom}_S(H^2_\Omega(S), H^2_\Omega(S)) \simeq S/(w - ix, y - iz) \oplus S/(w + ix, y + iz)$$

as it follows by the Mayer-Vietoris sequence for local cohomology. It is easily seen that $B_R \simeq \mathbb{Q}[a]/(a^2 + 1)[[w, x, y, z]]$. In fact, $\dim \text{Ext}^2_R(R/m, H^3_\Omega(R)) = 2$.

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