**MODULAR COVARIANTS OF CYCLIC GROUPS OF ORDER $p$**

JONATHAN ELMER

**Abstract.** Let $G$ be a cyclic group of order $p$ and let $V, W$ be $kG$-modules. We study the modules of covariants $k[V]^G(W) = (S(V^*) \otimes W)^G$. For $V$ indecomposable with dimension 2, and $W$ an arbitrary indecomposable module, we show $k[V]^G(W)$ is a free $k[V]^G$-module (recovering a result of Broer and Chuai) and we give an explicit set of covariants generating $k[V]^G(W)$ freely over $k[V]^G$. For $V$ indecomposable with dimension 3 and $W$ an arbitrary indecomposable module, we show that $k[V]^G(W)$ is a Cohen-Macaulay $k[V]^G$-module (again recovering a result of Broer and Chuai) and we give an explicit set of covariants which generate $k[V]^G(W)$ freely over a homogeneous set of parameters for $k[V]^G$.

1. Introduction

Let $G$ be a group, $k$ a field, and $V$ and $W$ $kG$-modules. Then $G$ acts on the set of functions $V \to W$ according to the formula

$$g \cdot \phi(v) = g\phi(g^{-1}v)$$

for all $g \in G$ and $v \in V$.

Classically, a **covariant** is a $G$-equivariant polynomial map $V \to W$. An **invariant** is the name given to a covariant $V \to k$ where $k$ denotes the trivial indecomposable $kG$-module. If the field $k$ is infinite, then the set of polynomial maps $V \to W$ can be identified with $S(V^*) \otimes W$, where the action on the tensor product is diagonal and the action on $S(V^*)$ is the natural extension of the action on $V^*$ by algebra automorphisms. Then the natural pairing $S(V^*) \times S(V^*) \to S(V^*)$ is compatible with the action of $G$, and makes the invariants $S(V^*)^G$ a $k$-algebra, and the covariants $(S(V^*) \otimes W)^G$ a $S(V^*)^G$-module.

If $G$ is finite and the characteristic of $k$ does not divide $|G|$, then Schur’s lemma implies that every covariant restricts to an isomorphism of some direct summand of $S(V^*)$ onto $W$. Thus, covariants can be viewed as “copies” of $W$ inside $S(V^*)$. Otherwise, the situation is more complicated.

The algebra of polynomial maps $V \to k$ is usually written as $k[V]$. In this article we will write $k[V]^G$ for the algebra of $G$-invariants on $V$, and $k[V]^G(W)$ for the module of $V - W$ covariants. We are interested in the structure of $k[V]^G(W)$ as a $k[V]^G$-module. Throughout, $G$ denotes a finite group.

This question has been considered by a number of authors over the years. For example, Chevalley and Sheppard-Todd [2], [10] showed that if the characteristic of $k$ does not divide $|G|$ and $G$ acts as a reflection group on $V$, then $k[V]^G$ is a polynomial algebra and $k[V]^G(W)$ is free. More generally, Eagon and Hochster [7] showed that if the characteristic of $k$ does not divide $|G|$ then $k[V]^G(W)$ is a Cohen-Macaulay module (and $k[V]^G$ a Cohen-Macaulay ring in particular). In the modular case, Hartmann [5] and Hartmann-Shepler [6] gave necessary and sufficient
conditions for a set of covariants to generate $k[V]^G(W)$ as a free $k[V]^G$-module, provided that $k[V]^G$ is polynomial and $W \cong V^*$. Broer and Chuai [1] remove the restrictions on both $W$ and $k[V]^G$.

2. Preliminaries

The purpose of this article is to investigate modular covariants for cyclic groups of order $p$. We are particularly interested in what happens when $k[V]^G(W)$ is not a free $k[V]^G$-module. Accordingly, from this point onwards we let $G$ be a cyclic group of order $p$ and $k$ a field of characteristic $p$. We fix a generator $\sigma$ of $G$. Recall that, up to isomorphism, there are exactly $p$ indecomposable $kG$-modules $V_1, V_2, \ldots, V_p$, where the dimension of $V_i$ is $i$. The isomorphism class of $V_i$ is usually represented by a module of column vectors on which $\sigma$ acts as left-multiplication by a single Jordan block of size $i$. Suppose $V \cong V_m$ and $W \cong W_n$. We will choose bases $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ of $V$ and $W$ respectively for which the action of $G$ is given by

\[
\begin{align*}
\sigma v_1 &= v_1 \\
\sigma v_2 &= v_2 - v_1 \\
\sigma v_3 &= v_2 - w_2 + v_1 \\
&\vdots \\
\sigma v_n &= w_n - w_{n-1} + w_{n-2} - \ldots \pm w_1
\end{align*}
\]

with analogous formula for the action on $V$ (thus, the action of $\sigma^{-1}$ is given by left-multiplication by a upper-triangular Jordan block). Now let $x_1, \ldots, x_m$ be the dual basis to $v_1, v_2, \ldots, v_m$; the action of $G$ on these variables is then given by

\[
\begin{align*}
\sigma x_1 &= x_1 + x_2, \\
\sigma x_2 &= x_2 + x_3, \\
\sigma x_3 &= x_3 + x_4, \\
&\vdots \\
\sigma x_m &= x_m.
\end{align*}
\]

Note that $k[V] = k[x_1, x_2, \ldots, x_m]$, and a general element of $k[V](W) = k[V] \otimes W$ is given by

\[
\phi = f_1 w_1 + f_2 w_2 + \ldots + f_n w_n
\]

where each $f_i \in k[V]$. We define the **support** of $\phi$ by

\[
\text{Supp}(\phi) = \{i : f_i \neq 0\}.
\]

The operator $\Delta = \sigma - 1 \in kG$ will play a major role in our exposition. $\Delta$ is a $\sigma$-twisted derivation on $k[V]$; that is, it satisfies the formula

(1) \quad $\Delta(fg) = f\Delta(g) + \Delta(f)\sigma(g)$

for all $f, g \in k[V]$.

Further, using induction and the fact that $\sigma$ and $\Delta$ commute, one can show $\Delta$ satisfies a Leibniz-type rule

(2) \quad $\Delta^k(fg) = \sum_{i=0}^{k} \binom{n}{i} \Delta^i(f)\sigma^{n-i}(\Delta^{n-i}(g))$. 

For any $f \in k[V]$ we define the **weight** of $f$:

$$\text{wt}(f) = \min\{i > 0 : \Delta^i(f) = 0\}.$$  

Notice that $\Delta^{\text{wt}(f)-1} \in \ker(\Delta) = k[V]^G$ for all $f \in k[V]$.

Note also the following crucial observation: for all $i = 1, \ldots, n-1$ we have

$$\Delta(w_{i+1}) + \sigma(w_i) = 0$$

and $\Delta(w_1) = 0$.

From this we obtain a simple characterisation of covariants:

**Proposition 1.** Let

$$\phi = f_1 w_1 + f_2 w_2 + \ldots + f_n w_n \in k[V]^G(W).$$

Then there exists $f \in k[V]$ with weight $\leq n$ such that $f_i = \Delta^{i-1}(f)$ for all $i = 1, \ldots, n$.

**Proof.** We have

$$0 = \Delta \left( \sum_{i=1}^{n} f_i w_i \right)$$

$$= \sum_{i=1}^{n} (f_i \Delta(w_i) + \Delta(f_i) \sigma(w_i))$$

$$= \sum_{i=1}^{n-1} (\Delta(f_i) - f_{i+1}) \sigma(w_i)) + \Delta(f_n) \sigma(w_n)$$

where we used (3) in the final step. Now note that

$$\sigma(w_i)) = w_i \text{ (terms in } w_{i+1}, w_{i+2}, \ldots, w_n)$$

for all $i = 1, \ldots, n$. Thus, equating coefficients of $w_i$, for $i = 1, \ldots, n$ gives

$$\Delta(f_1) = f_2, \Delta(f_2) = f_3, \ldots, \Delta(f_{n-1}) = f_n, \Delta(f_n) = 0.$$  

Putting $f = f_1$ gives $f_i = \Delta^{i-1}(f)$ for all $i = 1, \ldots, n$ and $0 = \Delta^n(f)$ as required. \qed

Note that the support of any covariant is therefore of the form $\{1, 2, \ldots, i\}$ for some $i \leq n$. Where no confusion will arise, we write

$$\text{Supp}(\phi) = i$$

if $\phi$ is a covariant and $\text{Supp}(\phi) = \{1, 2, \ldots, i\}$.

Introduce notation

$$K_n := \ker(\Delta^n)$$

and

$$I_n := \text{im}(\Delta^n).$$

These are $k[V]^G$-modules lying inside $k[V]$.

Now we can define a map

$$\Theta : K_n \rightarrow k[V]^G(W)$$

$$(4) \quad \Theta(f) = \sum_{i=1}^{n} \Delta^{i-1}(f) w_i.$$  

Clearly $\Theta$ is injective, and Proposition \ref{prop:1} implies it is also surjective. Therefore $\Theta$ is a degree-preserving bijective map.
3. Hilbert series

Let $k$ be a field and let $S = \bigoplus_{i \geq 0} S_i$ be a positively graded $k$-vector space. The dimension of each graded component of $S$ is encoded in its Hilbert Series

$$H(S, t) = \sum_{i \geq 0} \dim(S_i) t^i.$$ 

The observations following Proposition 1 imply that $H(k[V]^G(W), t) = H(K_n, t)$. In this section we will outline a method for computing $H(K_n, t)$.

Each homogeneous component $k[V]^G_i$ of $k[V]^G$ is a $kG$-module. As such, each one decomposes as a direct sum of modules isomorphic to $V_k$ for some values of $k$. Write $\mu_k(k[V]^G)$ for the multiplicity of $V_k$ in $k[V]^G_i$ and define the series

$$H_k(k[V]^G) = \sum_{i \geq 0} \mu_k(k[V]^G_i) t^i.$$ 

The series $H_k(k[V]^G)$ were studied by Hughes and Kemper in [8]. They can also be used to compute the Hilbert series of $k[V]^G_m$; since $\dim(V_k^G) = 1$ for all $k = 1, \ldots, p$ we have

$$H_k(k[V]^G_m, t) = \sum_{k=1}^{p} H_k(k[V]^G_m),$$ 

(5)

Now observe that

$$\dim(\ker(\Delta^n|_{V_k})) = \begin{cases} n & n \geq k \\ k & \text{otherwise}. \end{cases}$$

Therefore

$$H(K_n, t) = \sum_{k=1}^{n-1} kH_k(k[V]^G, t) + \sum_{k=n}^{p} nH_k(k[V]^G, t).$$

We can also write this as a series not depending on $p$:

$$H(K_n, t) = nH(k[V]^G, t) - \sum_{k=1}^{n-1} (n - k)H_k(k[V]^G, t),$$ 

(6)

using equation (5).

We will need the Hilbert Series of $I_n^G = k[V]^G \cap I_n$ in a later section. For all $k = 1, \ldots, p$ we have

$$\dim(\Delta^n(V_k))^G = \begin{cases} 1 & n > k \\ 0 & \text{otherwise}. \end{cases}$$

Therefore

$$H(I_n^G, t) = \sum_{k=n+1}^{p} H_k(k[V]^G, t),$$

which we can write independently of $p$ as

$$H(I_n^G, t) = H(k[V]^G, t) - \sum_{k=1}^{n} H_k(k[V]^G, t).$$ 

(7)

Note that

$$H(K_{n-1}, t) + H(I_{n-1}^G, t) = H(K_n, t).$$ 

(8)

We will make use of this in the final section.
4. Decomposition Theorems

In this section we will compute the series $H_k(k[V_2], t)$ and $H_k(k[V_3], t)$ for all $k = 1, \ldots, p - 1$.

Hughes and Kemper [8, Theorem 3.4] give the formula

$$H_k(k[V_m], t) = \sum_{\gamma \in M_{2p}} \left( \frac{\gamma - \gamma^{-1}}{2p} \gamma^{-k} \frac{1 - \gamma^{p(m-1)t} \gamma^{m-1-2j} (1 - \gamma^{m-1-2j}t)^{-1}}{1 - t^p} \prod_{j=0}^{m-1} \right),$$

where $M_{2p}$ represents the set of $2p$th roots of unity in $\mathbb{C}$. A similar formula is given for $H_p(k[V], t)$ but we will not need this. The following result can be derived from the formula above, but follows more easily from [4, Proposition 3.4]:

Lemma 2. $H_k(k[V_2], t) = \frac{t^{k-1}}{1 - t^p}$.

For $V_3$ we will have to use Equation (9). This becomes

$$H_k(k[V_3], t) = \frac{1}{2p(1 - t)} \sum_{\gamma \in M_{2p}} \left( \frac{(\gamma - \gamma^{-1})^{k+2}}{(1 - \gamma^2t)(\gamma^2 - t)} \right).$$

Lemma 3.

$$H_k(k[V_3], t) = \begin{cases} \frac{t^{-l} - t^{-(l+1)} + t^{l+1} - t^l}{(1 - r)(1 - r^2)} & \text{if } k = 2l + 1 \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Proof. We evaluate

$$\frac{(\gamma - \gamma^{-1})^{k+2}}{(1 - \gamma^2t)(\gamma^2 - t)} = \frac{A}{\gamma - \frac{t}{\gamma^2}} + \frac{B}{\gamma - \frac{1}{\gamma^2}} + \frac{C}{\gamma + \frac{1}{\gamma^2}} + \frac{D}{\gamma + \frac{t}{\gamma^2}}$$

using partial fractions, finding

$$A = \frac{t^{-l+1} - t^{-l}}{(2t^{1/2})(1 - t^2)};$$

$$B = (-1)^{-k+3} \frac{t^{-l+1} - t^{-l}}{(-2t^{1/2})(1 - t^2)};$$

$$C = \frac{t^{l-1} - t^l}{2(t^2 - 1)};$$

$$D = (-1)^{-k+3} \frac{t^{l-1} - t^l}{2(t^2 - 1)}.$$

Now we compute:

$$\sum_{\gamma \in M_{2p}} \left( \frac{1}{\gamma - \frac{t}{\gamma^2}} \right) = \sum_{\gamma \in M_{2p}} \left( \frac{-t^{-1/2}}{1 - \gamma t^{-1/2}} \right)$$

$$= -t^{-1/2} \sum_{i=0}^{\infty} \sum_{\gamma \in M_{2p}} (\gamma t^{-1/2})^i$$

$$= -t^{-1/2} 2p \sum_{i=0}^{\infty} (t^{-1/2})^{2pi}$$

$$= -t^{-1/2} 2p \frac{1}{1 - (t^{-1/2})^{2p}}$$

$$= -t^{1/2} 2p \frac{1}{1 - t^p}$$

$$= 2p \frac{t^{p-1/2}}{1 - t^p}$$
Similarly we have

$$\sum_{\gamma \in M_{2p}} \frac{1}{\gamma + t^{1/2}} = -2p t^{-1/2} \frac{1}{1 - t^p}$$

while

$$\sum_{\gamma \in M_{2p}} \frac{1}{1 - \gamma t^{1/2}} = \sum_{i=0}^{\infty} \sum_{\gamma \in M_{2p}} (\gamma t^{1/2})^i
= 2p \sum_{i=0}^{\infty} (t^{1/2})^{2pi}
= 2p \sum_{i=0}^{\infty} (t^{pi})
= 2p \frac{1}{1 - t^p}$$

and similarly

$$\sum_{\gamma \in M_{2p}} \frac{1}{1 + \gamma t^{1/2}} = 2p \frac{1}{1 - t^p}$$

as \{-\gamma : \gamma \in M_{2p}\} = M_{2p}.

It follows that

$$H_k(k[V_3], t) = \frac{1}{2p(1-t)} \left( \frac{(A-B)2pt^{-1/2} + 2p(C+D)}{1-t^p} \right)
= \frac{1}{(1-t)(1-t^p)} \left( \frac{(1 + (-1)^{-k+3})(t^{p-1} - t^{p-1-1})}{2(1-t^2)} + \frac{(1 + (-1)^{-k+3})(t^{l-1} - t^l)}{2(t^{-1} - t)} \right)
= \begin{cases} \frac{e^{p-1-p^{l-1}-1+t^{l+1}-l^i}}{(1-t)(1-t^2)(1-t^p)}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

as required. \(\square\)

5. Constructing Covariants

To prove our main results we will need to construct covariants with specified degree and weight. We present two methods in this section, one using monomials and one using differentiation. Throughout we set \(V = V_m\) and \(W = V_n\).

Lemma 4. Let \(z = x_1^{e_1} x_2^{e_2} \ldots x_m^{e_m}\). Let \(d = \sum_{i=1}^{m} e_i(m-i)\) and assume \(d < p\). Then

$$\text{wt}(z) = d + 1$$

and \(\Delta^d(z) = \lambda x_m^{\text{deg}(z)}\) for some nonzero constant \(\lambda\).

Proof. We prove this by induction on \(\text{deg}(z)\). When \(\text{deg}(z) = 1\) the result is clear. Suppose \(\text{deg}(z) > 1\), then in particular \(e_j > 0\) for some \(j = 1, \ldots, m\). By equation (2) we have

$$\Delta^d(z) = \sum_{j=0}^{d} \binom{d}{i} \Delta^{d-i}(y) \sigma^i(\Delta^i(x_j)),$$

$$= \sum_{j=0}^{m-j} \binom{d}{i} \Delta^{d-i}(y) \sigma^i(x_{j+i})$$

where \(y = z/x_j\). Now \(\text{deg}(y) < \text{deg}(z)\) so by induction

$$\Delta^{d-i}(y) = 0$$
for all $i < m - j$ and
\[
\Delta^{d-(m-j)}(y) = \mu x_{m}^{\deg(y)}
\]
for some nonzero constant $\mu \in k$. Hence
\[
\Delta^{d}(z) = \binom{d}{m-j} \mu x_{m}^{\deg(y)} x_{m}^{m-j} = \lambda x_{m}^{\deg(z)}
\]
for some nonzero constant $\lambda \in k$ as required. □

The following is well-known. We include a proof for completeness’ sake.

**Proposition 5.** Let $G$ be any group, and $V$ a $kG$-module. Write $k[V] = k[x_{1}, \ldots, x_{m}]$. Let $g \in G$ and write $gx_{i} = \sum_{j} a_{ij}x_{j}$. Let $A$ be the $m \times m$ matrix whose values are $a_{ij}$, and let $B = A^{-1}$. with entries $b_{ij}$. Then
\[
g \left( \frac{\partial f}{\partial x_{i}} \right) = \sum_{j} b_{ij} \frac{\partial}{\partial x_{j}}(gf).
\]

**Proof.** By considering the action of $g$ on a general monomial and using the product rule, one can show, for any $f \in k[V]$
\[
\frac{\partial}{\partial x_{j}}(gf) = \sum_{k} a_{kj} \left( g \frac{\partial f}{\partial x_{k}} \right).
\]
Now by the definition of $B$ we have $x_{i} = \sum_{j} b_{ij}(gx_{j})$. Multiplying both sides by $b_{ji}$ and summing over $j$ gives the result we want. □

We now return to the notation of the previous Lemma. For each $j = 1, \ldots, m$ write $\partial_{j}$ for the linear operator $\frac{\partial}{\partial x_{j}}$ on $k[V]$. Let $\Omega$ be the $k$-vector space of linear maps with basis $\partial_{1}, \partial_{2}, \ldots, \partial_{m}$ and define a linear map $\Delta : \Omega \to \Omega$ by the formula
\[
\Delta(\partial_{j}) = \sum_{i=1}^{j-1} (-1)^{i}\partial_{j-i}.
\]

Proposition 5 now implies
\[
\Delta(\partial_{j}f) = \sum_{i=0}^{j}((-1)^{i}\partial_{j-i}(\sigma f)) - \partial_{j}f
\]
\[
= \partial_{j}(\Delta f) + (\Delta \partial_{j})(\sigma f)
\]
for all $f \in k[V]$, hence by linearity we obtain

(10) \[
\Delta(\partial f) = \partial(\Delta f) + (\Delta \partial)(\sigma f)
\]
for all $f \in k[V]$ and $\partial \in \Omega$. One may then show, using induction on $d$ and the fact that $\sigma$ and $\Delta$ commute on $k[V]$

(11) \[
\Delta^{d}(\partial f) = \sum_{k=0}^{d} \binom{d}{k} (\Delta^{k} \partial)(\Delta^{d-k}(\sigma f))
\]
for all $d \in \mathbb{N}$, $\partial \in \Omega$ and $f \in k[V]$. 
Lemma 6. Let $f \in \mathbb{k}[V]^G$ and let $e_1, e_2, \ldots, e_m$ be positive integers. Write $e = \sum_{i=1}^m e_i$ and suppose that that $\sum_{i=1}^m ie_i := d < p$. Then
\[
\text{wt}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_m^{e_m} f) = d + 1
\]
and
\[
\Delta^{d}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_m^{e_m} f) = \lambda \partial_1^f
\]
for some nonzero constant $\lambda \in \mathbb{k}$.

Proof. The proof is by induction on $e$, the case $e = 0$ being clear. Suppose $e_j > 0$, then we have
\[
\Delta^{d}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_m^{e_m} f) = \Delta^{d}(\partial_j(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_j^{e_j-1} \cdots \partial_m^{e_m} f))
\]
\[
= \sum_{k=0}^{d} \binom{d}{k} (\Delta^{d-k}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_j^{e_j-1} \cdots \partial_m^{e_m} f))
\]
where we used Equation (11) in the last step. Now clearly $\Delta^{k}(\partial_j) = 0$ for $k > j - 1$, and $\Delta^{j-1}(\partial_j) = (-1)^j \partial_1$. Moreover, by induction $\text{wt}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_j^{e_j-1} \cdots \partial_m^{e_m} f) = d + 1 - (j - 1)$. So
\[
\Delta^{d-j}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_j^{e_j-1} \cdots \partial_m^{e_m} f) = 0
\]
for $k < j - 1$ and again by induction
\[
\Delta^{d-j}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_j^{e_j-1} \cdots \partial_m^{e_m} f) = \mu \partial_1^{e_1} f
\]
for some nonzero constant $\mu \in \mathbb{k}$. Hence
\[
\Delta^{d}(\partial_1^{e_1} \partial_2^{e_2} \cdots \partial_m^{e_m} f) = \binom{d}{j-1} (-1)^j \partial_1 (\mu \partial_1^{e_1} f)
\]
\[
= \lambda \partial_1^f
\]
for some nonzero constant $\lambda \in \mathbb{k}$ as required. \hfill \Box

6. Main results: $V_2$.

We are now in a position to state our main results. First, suppose $V = V_2$ and $W = V_n$ where $n \leq p$. Then it’s well known that $\mathbb{k}[V]^G$ is a polynomial ring, generated by $x_2$ and
\[
N = \prod_{i=0}^{p-1} \sigma^i(x_1) = x_1^p - x_1 x_2^{p-1}.
\]
Therefore we have
\[
(12) \quad H(\mathbb{k}[V]^G, t) = \frac{1}{(1-t)(1-t^p)}.
\]

Proposition 7. We have
\[
H(\mathbb{k}[V]^G(W), t) = \frac{1 + t + t^2 + \ldots + t^{n-1}}{(1-t)(1-t^p)}.
\]

Proof. Using equations (11) and (12) and Lemma 2 we have
\[
H(K_n, t) = \frac{n}{(1-t)(1-t^p)} - \sum_{k=1}^{n-1} \frac{(n-k)t^{k-1}}{1-t^p} = \frac{1 + t + t^2 + \ldots + t^{n-1}}{(1-t)(1-t^p)}.
\]
The result now follows from the observations following Proposition 11. \hfill \Box
Theorem 8. The module of covariants $k[V]^G(W)$ is generated freely over $k[V]^G$ by
\[ \{ \Theta(x_k^i) : k = 0, \ldots, n - 1 \} \]
where $\Theta(x_k^i) = \Theta(1) = w_1$.

Note that, by Proposition 6(i), $k[V]^G(W)$ is free over $k[V]^G$ and we could use Theorem 3 (loc. cit.) to check our proposed module generators. However, we prefer a more direct approach.

Proof. It follows from Lemma 4 that $\text{wt}(x_k^i) = k + 1$. Therefore $\text{Supp}(\Theta(x_k^i)) = k + 1$, and so it’s clear that the $k[V]^G$-submodule $M$ of $k[V]^G(W)$ generated by the proposed generating set is free. Moreover, as $\text{deg}(\Theta(x_k^i)) = k$, $M$ has Hilbert series
\[ \frac{1 + t + t^2 + \ldots + t^{n-1}}{(1 - t)(1 - t^p)}. \]

But by Proposition 9, this is the Hilbert series of $k[V]^G(W)$. Therefore $M = k[V]^G(W)$ as required. □

7. Main results: $V_3$

In this section let $p$ be an odd prime and $V = V_3$. Let $W = V_n$ for some $n \leq p$. In this case $k[V]^G$ is a Cohen-Macaulay ring, with a homogeneous system of parameters given by
\[ a_1 = x_3, a_2 = x_3^2 - 2x_1x_3 - x_2x_3, a_3 = N_1 = \prod_{i=0}^{p-1} \sigma^i(x_1). \]

Let $A = k[a_1, a_2, a_3]$. Then $k[V]^G$ is a free $A$-module generated by 1 and
\[ N_2 = \prod_{i=0}^{p-1} \sigma^i(x_2), \]
see [3] for details. Therefore the Hilbert series for $k[V]^G$ is
\[ \frac{1 + t^p}{(1 - t)(1 - t^2)(1 - t^p)}. \]

Note that by Proposition 6(ii), $k[V]^G(W)$ is a Cohen-Macaulay $k[V]^G$-module and hence free over $A$. However, [1] provides no method of finding or testing an $A$-generating set.

Our first task is to compute the Hilbert Series of $k[V]^G(W)$. Once more we use equation (13) and the bijection $\Theta$ to do this. We omit the details.

Proposition 9. Let $l = \frac{1}{2}n$ if $n$ is even, with $l = \frac{1}{2}(n - 1)$ if $n$ is odd. Then
\[ H(k[V]^G(W), t) = \frac{1 + 2t + 2t^2 + \ldots + 2t^l + 2t^{p-l} + 2t^{p-l+1} + \ldots + t^p}{(1 - t)(1 - t^2)(1 - t^p)}, \]
if $n$ is odd, while
\[ H(k[V]^G(W), t) = \frac{1 + 2t^l + \ldots + 2t^{l-1} + t^{l+1} + t^{p-l} + 2t^{p-l+1} + \ldots + 2t^{p-1} + t^p}{(1 - t)(1 - t^2)(1 - t^p)}, \]
if $n$ is even.

We will also need:

Proposition 10. Retain the notation of the previous proposition. Then
\[ H(I_{n-1}) = \frac{t^l + t^{p-l}}{(1 - t)(1 - t^2)(1 - t^p)}. \]
Proof. This follows from equations (7) and (13), along with Lemma 3. □

We will need the following technical result:

Lemma 11. Let \( l \leq \frac{p-1}{2} \). Then \( \partial_1^l(N_1) \) is not divisible by \( x_1^l \) in \( \mathbb{k}[V] \).

Proof. We use the expansion of \( N_1 \) as a polynomial in \( x_3 \) found in [9, Lemma 6.1]. In our notation, this states that
\[
N_1 = A_0 + A_1 x_3 + A_2 x_3^2 + \ldots A_p x_3^p
\]
where
\[
A_0 = x_1^p - x_1 x_2^{p-1}; \quad A_p = A_{p-1} = 0
\]
and
\[
A_i = \left\{ \sum_{k=1}^{i+1} \xi_{i,k} x_1^k x_2^{i-k} : 1 \leq i \leq \frac{p-1}{2} \right\}, \quad \sum_{k=1}^{i+1} \xi_{i,k} x_1^k x_2^{i-k} \quad x_2^{i+1} \leq i \leq p - 2
\]
where
\[
\xi_{i,k} = \frac{(-1)^i}{2^i (p-k)} \binom{p-2k+1}{i-k+1} \binom{p-k}{k-1}.
\]
Only the first line in Equation (14) is relevant to us. We have
\[
\partial_1^l A_{l-1} = \xi_{l-1,k} l! x_2^{p-2l+1} \neq 0.
\]
This shows that the coefficient of \( x_3^{l-1} \) in \( \partial_1^l N_1 \) is nonzero, and consequently \( \partial_1^l N_1 \) is not divisible by \( x_3^{l} \) as required. □

Corollary 12. \( I_{n-1}^G \) is generated freely over \( A \) by \( x_1^l \) and \( \partial_1^l N_1 \).

Proof. If \( n \) is even then by Lemma 3 we have
\[
\text{wt}(x_1^{l-1} x_2) = 1 + 2(l-1) + 1 = n
\]
and
\[
\Delta^{n-1}(x_1^{l-1} x_2) = \lambda x_3^l
\]
for some nonzero \( \lambda \in \mathbb{k} \). Therefore \( x_3^l \in I_{n-1}^G \). Similarly by Lemma 6 we have
\[
\text{wt}(\partial_3^{l-1} \partial_2 N_1) = 2(l-1) + 1 + 1 = n
\]
and
\[
\Delta^{n-1}(\partial_3^{l-1} \partial_2 N_1) = \mu \partial_1^l N_1
\]
for some nonzero \( \mu \in \mathbb{k} \). Therefore \( \partial_1^l N_1 \in I_{n-1}^G \).

On the other hand if \( n \) is odd we have
\[
\text{wt}(x_1^l) = 1 + 2(l-1) = n
\]
and
\[
\Delta^{n-1}(x_1^l) = \lambda x_3^l
\]
for some nonzero \( \lambda \in \mathbb{k} \). Therefore \( x_3^l \in I_{n-1}^G \). Similarly by Lemma 6 we have
\[
\text{wt}(\partial_3^{l-1} N_1) = 2(l) + 1 + 1 = n
\]
and
\[
\Delta^{n-1}(\partial_3^{l-1} N_1) = \mu \partial_1^l N_1
\]
for some nonzero \( \mu \in \mathbb{k} \). Therefore \( \partial_1^l N_1 \in I_{n-1}^G \).

Since \( \deg(\partial_1^l N_1) = p - l < p \), \( \partial_1^l N_1 \in \mathbb{k}[a_1, a_2] \subset A \). Now Lemma 3 shows that \( x_1^l \) and \( \partial_1^l N_1 \) are coprime in \( A \), and therefore generate a free \( A \)-submodule of \( I_{n-1}^G \).

This module has Hilbert series
\[
\frac{t^l + p^{n-l}}{(1-t)(1-t^2)(1-t^p)}
\]
But we showed in Proposition 10 this is the Hilbert series of $I_{n-1}^G$. Hence, $I_{n-1}^G = A(x_1, \partial_1^l N_1)$ as required. 

We are now ready to prove our main result:

**Theorem 13.** Let $V = V_3$ and $W = V_n$. Then $k[V]^G(W)$ is a Cohen-Macaulay $k[V]^G$-module, generated freely over $A$ by the following set $S_n$ of covariants:

\[
\{ \Theta(x_1^i) : i = 0, \ldots, l \} \cup \{ \Theta(x_1^{i}x_2) : i = 0, \ldots, l - 1 \} \cup \{ N_2w_1 \} \\
\cup \{ \Theta(\partial_3^i N_1) : i = 1, \ldots, l \} \cup \{ \partial_3^i \partial_2 N_1 : i = 1, \ldots, l - 1 \}
\]

if $n$ is odd, and

\[
\{ \Theta(x_1^i) : i = 0, \ldots, l - 1 \} \cup \{ \Theta(x_1^{i}x_2) : i = 0, \ldots, l - 1 \} \cup \{ N_2w_1 \} \\
\cup \{ \Theta(\partial_3^i N_1) : i = 1, \ldots, l - 1 \} \cup \{ \partial_3^i \partial_2 N_1 : i = 1, \ldots, l - 1 \}
\]

if $n$ is even.

**Proof.** The proof is by induction on $n$. If $n = 1$ then $k[V]^G(W) = (k[V]^G)w_1$ and the claim is that this is generated freely over $A$ by $w_1$ and $N_2w_1$. This is clear, because $k[V]^G$ is generated freely over $A$ by 1 and $N_2$.

Now suppose $n > 1$. Let $\phi \in k[V]^G(W)$. Then as $\Theta$ is a bijection, we have $\phi = \Theta(f)$ for some $f \in k[V]$.

Note that $k[V]^G(V_{n-1})$ is embedded in $k[V]^G(W)$ as the set of covariants with support at most $n - 1$. If $\phi$ has support $\leq n - 1$, we have by induction that $\phi$ lies in the $A$-submodule of $k[V]^G(W)$ generated by $S_{n-1}$, which is a subset of the claimed generating set.

So we may assume $\phi$ has support $n$. Therefore $f$ has weight $n$. Write

\[
\phi = \Theta(f) = \sum_{i=1}^{n} \Delta^{i-1}(f) w_i = f_1w_1 + f_2w_2 + \ldots + f_nw_n.
\]

Then $f = f_1$ and $f_n = \Delta^{n-1}(f) \in I_{n-1}^G$. Therefore by Corollary 12 there exist $\alpha, \beta \in A$ such that

\[
f_n = \alpha x_1^l + \beta \partial_3^l N_1.
\]

Let

\[
\phi - \overline{\phi} = \begin{cases} 
\alpha \Theta(x_1^{l-1}x_2) + \beta \Theta(\partial_3^{l-1} \partial_2 N_1) & \text{if } n \text{ is even}, \\
\alpha \Theta(x_1^l) + \beta \Theta(\partial_3^l N_1) & \text{if } n \text{ is odd}.
\end{cases}
\]

In either case, the coefficient of $w_n$ in $\phi$ is $f_n$. Therefore, $\text{Supp}(\phi - \overline{\phi}) \leq n - 1$. By induction, $\phi - \overline{\phi}$ lies in the $A$-submodule of $k[V]^G(W)$ generated by $S_{n-1}$. Since

\[
S_n = \begin{cases} 
S_{n-1} \cup \{ \Theta(x_1^{l-1}x_2), \Theta(\partial_3^{l-1} \partial_2 N_1) \} & \text{if } n \text{ is even}, \\
S_{n-1} \cup \{ \Theta(x_1^l), \Theta(\partial_3^l N_1) \} & \text{if } n \text{ is odd},
\end{cases}
\]

we get that $\phi$ lies in the $A$-submodule of $k[V]^G(W)$ generated by $S_n$ as required.

To see that $k[V]^G(W)$ is a free $A$-module generated by $S_n$ we use induction on $n$ once more. This is known to be true if $n = 1$. We showed above that submodule of $k[V]^G(W)$ generated by $S_{n-1}$ is the set of covariants with support $\leq n - 1$, and we may assume inductively that this is free, i.e. there are no $A$-linear relations between the elements of $S_{n-1}$. We know that $S_n \setminus S_{n-1}$ consists of two covariants of degrees $l$ and $p - l$ respectively, whose coefficients of $w_n$ generate freely the $A$-module $I_{n}^G$, so there are no $A$-linear relations between these two covariants. So, the $A$-module generated by $S_n$ (which we have shown is $k[V]^G(W)$) has Hilbert series bounded above by
\[ H(I_{n-1}^G, t) + H(\mathbb{k}[V]^G(V_{n-1}, t) \]

with equality if and only if it is freely generated by \( S_n \). But by Equation (8), this is the Hilbert series of \( K_n \), which in turn is the Hilbert series of \( \mathbb{k}[V]^G(W) \). So \( S_n \) generates \( \mathbb{k}[V]^G(W) \) freely as required.

\[ \square \]

**References**

[1] Abraham Broer and Jianjun Chuai. Modules of covariants in modular invariant theory. *Proc. Lond. Math. Soc. (3)*, 100(3):705–735, 2010.

[2] Claude Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955.

[3] Leonard Eugene Dickson. *On invariants and the theory of numbers*. Dover Publications, Inc., New York, 1966.

[4] Jonathan Elmer. Symmetric powers and modular invariants of elementary abelian \( p \)-groups. *J. Algebra*, 492:157–184, 2017.

[5] J. Hartmann. Transvection free groups and invariants of polynomial tensor exterior algebras. *Transform. Groups*, 6(2):157–164, 2001.

[6] Julia Hartmann and Anne V. Shepler. Reflection groups and differential forms. *Math. Res. Lett.*, 14(6):955–971, 2007.

[7] M. Hochster and John A. Eagon. Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *Amer. J. Math.*, 93:1020–1058, 1971.

[8] Ian Hughes and Gregor Kemper. Symmetric powers of modular representations, Hilbert series and degree bounds. *Comm. Algebra*, 28(4):2059–2088, 2000.

[9] Müfit Sezer and R. James Shank. On the coinvariants of modular representations of cyclic groups of prime order. *J. Pure Appl. Algebra*, 205(1):210–225, 2006.

[10] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.