MAXIMAL IDEALS IN MODULE CATEGORIES AND APPLICATIONS

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Abstract. We study the existence of maximal ideals in preadditive categories defining an order \( \preceq \) between objects, in such a way that if there do not exist maximal objects with respect to \( \preceq \), then there is no maximal ideal in the category. In our study, it is sometimes sufficient to restrict our attention to suitable subcategories. We give an example of a category \( \mathcal{C}_F \) of modules over a right noetherian ring \( R \) in which there is a unique maximal ideal. The category \( \mathcal{C}_F \) is related to an indecomposable injective module \( F \), and the objects of \( \mathcal{C}_F \) are the \( R \)-modules of finite \( F \)-rank.

INTRODUCTION

This paper is related to the study of ideals in preadditive categories. Recall that an ideal in a preadditive category \( \mathcal{C} \) is an additive subfunctor \( I \) of the additive bifunctor \( \text{Hom}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab} \), where \( \text{Ab} \) is the category of abelian groups.

Let us mention two motivations for our study. The first is related to extensions of the classical Krull-Schmidt theorem to additive categories. In [8], the second author proved that the class of all uniserial right modules over a ring \( R \) does not satisfy the Krull-Schmidt theorem, thus answering a question posed by Warfield in 1975, but that nevertheless a weak version of the Krull-Schmidt theorem for uniserial modules holds [8, Theorem 1.9]. This weak version of the Krull-Schmidt theorem was extended as follows, in [6, Theorem 6.4], to any additive category \( \mathcal{A} \) with a pair of ideals \( \mathcal{I} \) and \( \mathcal{J} \) satisfying suitable conditions: if \( U_1, \ldots, U_n \) and \( V_1, \ldots, V_n \) are objects in \( \mathcal{A} \) with local endomorphism rings in the quotient categories \( \mathcal{A}/\mathcal{I} \) and \( \mathcal{A}/\mathcal{J} \), then \( U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_n \) if and only if \( n = m \) and there exist two permutations \( \sigma \) and \( \tau \) of \( \{1, \ldots, n\} \) such that \( U_i \) and \( V_{\sigma(i)} \) are isomorphic in \( \mathcal{A}/\mathcal{I} \), and \( U_i \) and \( V_{\tau(i)} \) are isomorphic in \( \mathcal{A}/\mathcal{J} \), for every \( i = 1, \ldots, n \).

Our second motivation is related to the problem of approximating objects by morphisms belonging to some ideal. This idea first appeared in [12], where the author introduced phantom maps in module categories, considered the ideal consisting of all such maps and proved that each module \( M \) has a phantom cover (that is, a phantom map \( \varphi: P \to M \) such that every phantom map \( \psi: Q \to M \) factors through \( \varphi \), and minimal with respect to this property). This particular situation was extended in [11], where it was characterized when an ideal \( \mathcal{I} \) in an exact category provides approximations in this sense. Notice that this theory contains, as a particular case, the classical one about precovers and covers by objects, see [1].
As in the case of ideals of rings, one can consider minimal and maximal ideals in a preadditive category $\mathbf{C}$. In [5, Theorem 3.1], it is proved that the minimal ideals in a module category are in one-to-one correspondence with the simple modules. Hence we have a complete description of the minimal ideals of the category. A similar description of maximal ideals is not known (the best description of maximal ideals is Prihoda’s result [9, Lemma 2.1]). One of the main results of our paper is now that there do not exist maximal ideals in module categories Mod-$R$ (actually, in Grothendieck categories). The idea of the proof is to define a order $\preceq$ in the class of objects and relate the existence of maximal ideals with the existence of non-zero maximal objects with respect to this order. More precisely, we prove (Theorem 3.1) that if for each object $A$ in the category there exists an object $B$ such that $A \prec B$, then there do not exist maximal ideals. Since a Grothendieck category has this property (Proposition 2.4), we conclude that there are no maximal ideals in Grothendieck categories.

If $\mathbf{C}$ is a preadditive category, we can consider the full subcategory $\mathbf{M}(\mathbf{C})$ of $\mathbf{C}$ consisting of all objects $C$ of $\mathbf{C}$ for which there do not exist objects $B$ in $\mathbf{C}$ with $C \prec B$. Then the maximal ideals of $\mathbf{M}(\mathbf{C})$ determine those of $\mathbf{C}$ (Proposition 3.9). Using these ideas, the last part of the paper is devoted to describing the maximal ideals in a full subcategory $\mathbf{C}_F$ constructed starting from an indecomposable injective module $F$ over a right noetherian ring.

All rings in this paper are associative with unit and not necessarily commutative. If $R$ is such a ring, module will mean right $R$-module and we will denote by Mod-$R$ the category whose objects are all right $R$-modules.

1. PRELIMINARIES

By a preadditive category, we mean a category together with an abelian group structure on each of its hom-sets such that composition is bilinear. An additive category is a preadditive category with finite products. Let $\mathbf{C}$ be a preadditive category and $A$ an object of $\mathbf{C}$. We will denote by add$(A)$ the class of objects $X$ of $\mathbf{C}$ for which there exist an integer $n > 0$ and morphisms $f_1, \ldots, f_n \in \text{Hom}_\mathbf{C}(A, X)$ and $g_1, \ldots, g_n \in \text{Hom}_\mathbf{C}(X, A)$ such that $1_X = \sum_{i=1}^n f_i g_i$. If $\mathbf{C}$ is additive and idempotents split in $\mathbf{C}$, then $X \in \text{add}(A)$ if and only if $X$ is isomorphic to a direct summand of $A^n$ for some integer $n \geq 0$. If, moreover, $\mathbf{C}$ has arbitrary direct sums, we will denote by Add$(A)$ the class of all objects that are isomorphic to direct summands of arbitrary direct sums of copies of $A$.

An ideal in $\mathbf{C}$ is an additive subfunctor $\mathcal{I}$ of the additive bifunctor Hom$_\mathbf{C}$: $\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. Thus $\mathcal{I}$ associates to every pair $A$ and $B$ of objects in $\mathbf{C}$ a subgroup $\mathcal{I}(A, B)$ of Hom$_\mathbf{C}(A, B)$ so that if $f: X \to A$ and $g: B \to Y$ are morphisms in $\mathbf{C}$ and $i \in \mathcal{I}(A, B)$, then $g i f \in \mathcal{I}(X, Y)$. An ideal in $\mathbf{C}$ is maximal if it is proper, that is, it is not equal to Hom$_\mathbf{C}$, and is not properly contained in any other proper ideal. For instance, it is easy to see that the zero ideal is a maximal ideal in the full subcategory of Mod-$K$ whose objects are all finite-dimensional vector spaces over a field $K$.

Given an object $A$ in $\mathbf{C}$ and any two-sided ideal $I$ of End$_\mathbf{C}(A)$, we will denote by $\mathcal{A}_I$ the ideal of the category $\mathbf{C}$ defined, for each pair of objects $X, Y \in \mathbf{C}$, by

$$\mathcal{A}_I(X, Y) = \{ f \in \text{Hom}_\mathbf{C}(X, Y) : \beta f \alpha \in I \text{ for all } \alpha \in \text{Hom}_\mathbf{C}(A, X) \text{ and } \beta \in \text{Hom}_\mathbf{C}(Y, A) \}.$$
This ideal is called the \textit{ideal associated to} \( I \) ([11] Section 2] and [10] Section 3]). The ideal \( \mathcal{A}_I \) contains any ideal \( I \) in \( C \) satisfying \( I(A,A) \subseteq I \). As proved in [9] Lemma 2.4, there is a strong relation between ideals associated to maximal ideals of the endomorphism ring of an object, and maximal ideals in the preadditive category. For instance, the same argument as [9] Proposition 2.5 gives:

\textbf{Example 1.1.} Let \( C \) be an additive category in which idempotent splits and \( C \) any object of \( C \). Then the maximal ideals in the category \( \text{add}(C) \) are the ideals associated to maximal ideals of \( \text{End}_C(C) \).

The following easy lemma will be useful to compute ideals in the endomorphism ring of a finite direct sum of objects.

\textbf{Lemma 1.2.} Let \( C \) be an additive category, \( A \) an object of \( C \) and \( I \) an ideal in \( \text{End}_C(A) \). Given any finite family \( B_1, \ldots, B_n \) of objects of \( C \), denote by \( \iota_l \) and \( \pi_1 \) the inclusion and the projection corresponding to the \( l \)-th component of \( B = \bigoplus_{i=1}^n B_i \) for each \( l = 1, \ldots, n \). Then
\[ \mathcal{A}_I(B,B) = \{ f \in \text{End}_C(B) : \pi_m f \iota_l \in \mathcal{A}_I(B_l,B_m) \text{ for every } l, m = 1, 2, \ldots, n \}. \]

Note that, as a consequence of this result, if \( M_1 \) and \( M_2 \) are objects in an additive category \( C \) and \( I \) is an ideal in the endomorphism ring of an object \( A \) of \( C \), then \( \mathcal{A}_I(M_1 \oplus M_2, M_1 \oplus M_2) = \text{End}_R(M_1 \oplus M_2) \) if and only if \( \mathcal{A}_I(M_i, M_i) = \text{End}_R(M_i) \) for \( i = 1, 2 \).

\section{The strict order \( \prec \) and its corresponding partial order \( \preceq \).}

The existence of maximal ideals in preadditive categories is related to an order \( \preceq \) between objects. In this section, we define the partial order \( \preceq \) and give a number of examples.

\textbf{Definition 2.1.} Let \( C \) be a preadditive category and \( A, B \) objects of \( C \). Set \( A \prec B \) if there exists an infinite subset \( E \subseteq \text{Hom}_C(B,A) \times \text{Hom}_C(A,B) \) with the following properties:
\begin{enumerate}
\item \( fg = 1_A \) for every \((f,g) \in E\).
\item For each \( \varphi \in \text{Hom}_C(A,B) \), \(|\{(f,g) \in E : f \varphi \neq 0\}| < |E|\).
\end{enumerate}

We shall write \( A \preceq B \) if either \( A \prec B \) or \( A = B \).

Here we are using the well known one-to-one correspondence between strict orders and partial orders. For any partial order \( \preceq \), the corresponding strict order \( \prec \) is defined by \( A \prec B \) if \( A \preceq B \) and \( A \neq B \).

Let \( C \) be a preadditive category, \( A \) a subcategory of \( C \) and \( A \) and \( B \) objects of \( A \). Notice that it can occur that \( A \prec B \) in \( C \) but not in \( A \). However, if \( A \) is full, \( A \prec B \) in \( C \) if and only if \( A \prec B \) in \( A \).

\textbf{Example 2.2.} Let \( C \) be any preadditive category and \( A, B \in C \) objects. If both \( \text{Hom}_C(A,B) \) and \( \text{Hom}_C(B,A) \) are finite, then \( A \neq B \). In particular, if \( C \) has a zero object \( 0 \), then \( 0 \neq B \) and \( B \neq 0 \) for every object \( B \).

Let us see some properties of the order \( \preceq \).

\textbf{Lemma 2.3.} Let \( C \) be a preadditive category and \( A, B \) and \( C \) objects of \( C \).
(1) If $A \prec B$ and $B$ is a retract of $C$, then $A \prec C$.
(2) If $B \in \text{add}(A)$, then $A \not\prec B$.

Proof. (1) Denote by $\iota_B : B \to C$ and $\pi_B : C \to B$ the morphisms satisfying $\pi_B \iota_B = 1_B$. Since $A \prec B$, there exists a set $E \subseteq \text{Hom}_G(B, A) \times \text{Hom}_G(A, B)$ satisfying the conditions of Definition 2.1. Then $E' = \{(f \pi_B, \iota_B g) : (f, g) \in E\}$ is a subset of $\text{Hom}_G(B \oplus C, A) \times \text{Hom}_G(A, B \oplus C)$ that has cardinality equal to $|E|$ and that trivially verifies the conditions of Definition 2.1. Thus $A \prec C$.

(2) Let $n > 0$ be an integer and

$$f_1, \ldots, f_n \in \text{Hom}_C(A, B), \quad g_1, \ldots, g_n \in \text{Hom}_C(B, A)$$

be such that $\sum_{i=1}^n f_i g_i = 1_B$. Suppose, in order to get a contradiction, that $A \prec B$. Let $E \subseteq \text{Hom}_C(B, A) \times \text{Hom}_C(A, B)$ be the set satisfying the conditions of Definition 2.1. By Definition 2.1(2), the set

$$E_k := \{(f, g) \in E : ff_k \neq 0\}$$

has cardinality smaller than $|E|$ for each $k = 1, \ldots, n$. But, for each morphism $\varphi : B \to A$, $\varphi \neq 0$ if and only if $\varphi f_k \neq 0$ for some $k = 1, \ldots, n$. This implies that $E = \bigcup_{k=1}^n E_k$ as $f \neq 0$ for each $(f, g) \in E$. Since $E$ is infinite, we conclude that at least one of the sets $E_k$ has the same cardinality as $E$, which is a contradiction. \qed

Let $C$ be a preadditive category. The main consequence of the preceding result is that the relation $\prec$ is a strict order, since it is irreflexive by (2) and transitive by (1). As we have already said, we denote by $\preceq$ the partial order associated to the strict order $\prec$.

Now we will consider a relation between large direct sums of copies of a non-zero object in a Grothendieck category and the strict order $\prec$ of Definition 2.1. Let $G$ be a Grothendieck category, $A$ an object of $G$ and $\kappa$ an infinite regular cardinal. Recall that $A$ is said to be $\kappa$-generated [1, Definition 1.67] if $\text{Hom}_G(A, \cdotp)$ commutes with $\kappa$-directed colimits with all morphisms in the direct system being monomorphisms (a $\kappa$-directed colimit is the colimit of a $\kappa$-system in $G$, $(A_i, f_{ij})_i$, the latter meaning that each subset of $I$ of cardinality smaller than $\kappa$ has an upper bound [1 Definition 1.13]).

**Proposition 2.4.** Let $G$ be a Grothendieck category and $\kappa$ an infinite regular cardinal.

(1) Let $A$ be a non-zero $\kappa$-generated object of $G$. Then $A \prec A^{(\kappa)}$.
(2) For each non-zero object $A$ of $G$, there exists an object $B$ of $G$ such that $A \prec B$.

Proof. (1) Denote by $\iota_\alpha : A \to A^{(\kappa)}$ and $\pi_\alpha : A^{(\kappa)} \to A$ the injection and the projection corresponding to the $\alpha$-component of $A^{(\kappa)}$ for each $\alpha < \kappa$. Consider the subset $\{\pi_\alpha \iota_\alpha : \alpha < \kappa\}$ of $(A^{(\kappa)}, A) \times \text{Hom}_G(A, A^{(\kappa)})$. Then $E$ satisfies (1) of Definition 2.1 since $\pi_\alpha \iota_\alpha = 1_A$ for each $\alpha < \kappa$.

In order to prove condition (2) of Definition 2.1, note that $A^{(\kappa)}$ is the colimit of the $\kappa$-direct system $(A^{(\alpha)}, \iota_{\alpha \beta})_\kappa$, where $\iota_{\alpha \beta} : A^{(\alpha)} \to A^{(\beta)}$ is the inclusion for each $\alpha < \beta$ in $\kappa$. The colimit maps are the inclusions $\iota_\alpha : A^{(\alpha)} \to A^{(\kappa)}$ for each $\alpha < \kappa$. Let $\varphi : A \to A^{(\kappa)}$ be any morphism. Since $A$ is $\kappa$-generated and the morphism $\iota_{\alpha \beta}$ is monic for every $\alpha < \beta$ in $\kappa$, there exists $\alpha_0 < \kappa$ and $\overline{\varphi} : A \to A^{(\alpha_0)}$ such that
$\varphi = \iota_\alpha \overline{\varphi}$. In particular, we get that
\[|\{(\pi_\alpha, \iota_\alpha) : \pi_\alpha \varphi \neq 0\}| \leq |\alpha_0| < \kappa = |E|.
\]

(2) Notice that, for each object $A$ in $C$, there exists an infinite regular cardinal $\kappa$ such that $A$ is $<\kappa$-generated [13, Lemma A.1].

Using these results, we can characterize when $V \prec W$ for vector spaces $V$ and $W$.

**Corollary 2.5.** Let $V$ and $W$ be two vector spaces over a field $K$. Then $V \prec W$ if and only if $W$ is infinite dimensional and $0 \neq \dim(V) < \dim(W)$.

**Proof.** Suppose $V \prec W$. First of all, note that $\dim(V) < \dim(W)$ since, otherwise, there would exist an epimorphism $\varphi : V \to W$. This would imply that, for any subset $E$ of Hom$_K(W,V) \times$ Hom$_K(V,W)$, $\{(f,g) \in E : f\varphi \neq 0\} = E$. That is, $V \not\prec W$. Furthermore, $W$ has to be infinite dimensional, since finite dimensional vector spaces belong to add($V$) and, by Lemma 2.3, for any vector space $W$ in add($V$), we have that $V \not\prec W$.

Conversely, suppose that $W$ is infinite dimensional and that $0 \neq \dim(V) < \dim(W)$. Set $\dim W = \kappa$ and $\dim V = \lambda$ and take an infinite regular cardinal $\mu$ with $\lambda < \mu \leq \kappa$ (if $\kappa$ is regular, take $\mu = \kappa$; otherwise, set $\mu = \lambda^+$, the successor cardinal of $\lambda$). By Proposition 2.4 and Lemma 2.3, $V \prec V^{(\mu)} \prec V^{(\mu)} \oplus V^{(\kappa)}$. Since $\lambda < \mu \leq \kappa$, $\dim (V^{(\mu)} \oplus V^{(\kappa)}) = \kappa$ and $V^{(\mu)} \oplus V^{(\kappa)} \cong W$. Consequently, $V \prec W$.

Let $R$ be a ring and $A$ and $B$ right $R$-modules. If $A \prec B$ and $E$ is the set of Definition 2.4, then, for each $(f,g) \in E$, Im $g$ is a direct summand of $B$ isomorphic to $A$. That is, $B$ contains many direct summands isomorphic to $A$. In view of the preceding result, a natural question arises: is $B$ isomorphic to a direct sum of copies of $A$? The following example shows that the answer to this question is negative in general.

**Example 2.6.** Let $\kappa$ be an infinite regular cardinal and consider the abelian group $M = \mathbb{Z}^{(\kappa)} \oplus \mathbb{Z}_2$. Then $\mathbb{Z} \prec M$ by Proposition 2.4 and Lemma 2.3 while $M$ is not isomorphic to a direct sum of copies of $\mathbb{Z}$ since it is not free.

3. **MAXIMAL IDEALS**

In this section, using the order $\preceq$, we prove that there do not exist maximal ideals in Grothendieck categories. We will prove a more general result: if $C$ is a preadditive category such that there is no non-zero maximal object with respect to $\preceq$, then $C$ does not have maximal ideals. The proof is based on the following theorem:

**Theorem 3.1.** Let $C$ be a preadditive category, $A$ and $B$ objects of $C$ such that $A \prec B$, and $I$ a proper ideal of End$_C(A)$. Then $A_I(B,B)$ is a proper ideal of End$_C(B)$ which is not maximal.

**Proof.** Since $A \prec B$, there exists $E \subseteq$ Hom$_C(B,A) \times$ Hom$_C(A,B)$ satisfying the conditions of Definition 2.4. Let $J$ be the ideal of End$_C(B)$ generated by the set of all the endomorphisms of $B$ that factors through $A$. We claim that $J + A_I(B,B)$ is a proper ideal of End$_C(B)$ strictly containing $A_I(B,B)$.
First of all, note that $A_1(B, B)$ is not equal to $J + A_1(B, B)$. In fact, for each $(f, g) \in E$, $gf$ is an element of $J$ not belonging to $A_1(B, B)$, since $fgfg = 1_A \notin I$ because $I$ is proper.

Now we will prove that $J + A_1(B, B)$ is a proper ideal. Fix any element $\psi \in J + A_1(B, B)$. Let $\varphi \in J$ and $\varphi' \in A_1(B, B)$ be such that $\psi = \varphi + \varphi'$. Since $\varphi \in J$, $\varphi = \sum_{i=1}^{n} f_i g_i$ for morphisms $f_1, \ldots, f_n \in \text{Hom}_C(A, B)$ and $g_1, \ldots, g_n \in \text{Hom}_C(B, A)$. The set $\{(f, g) \in E : f \psi g \notin I\}$ is contained in the set $\{(f, g) \in E : f \varphi g \notin I\}$, which is contained in \[ \bigcup_{i=1}^{n} \{(f, g) \in E : f f_i \neq 0\}. \]

Since $A \prec B$ and $E$ is infinite, this set has cardinality smaller than $|E|$. The conclusion is that, for each $\psi \in J + A_1(B, B)$, the set $\{(f, g) \in E : f \psi g \notin I\}$ has cardinality smaller than $|E|$. But this implies that $1_B$ does not belong to $J + A_1(B, B)$, as $\{(f, g) \in E : f 1_B g \notin I\} = E$. Consequently, $J + A_1(B, B)$ is a proper ideal.

This theorem has a number of consequences.

**Corollary 3.2.** Let $C$ be a preadditive category such that there do not exist non-zero maximal objects with respect to $\preceq$. Then $C$ does not have maximal ideals.

**Proof.** Let $I$ be any proper ideal in $C$. Then there exists an object $A$ such that $I(A, A) \neq \text{End}_C(A)$. Set $I = I(A, A)$. Let $B$ be an object such that $A \prec B$. Then $I(B, B) \subseteq A_1(B, B)$ which, as a consequence of the previous result, is properly contained in a proper ideal of $\text{End}_C(B)$. This means that $I(B, B)$ is not a maximal ideal of $\text{End}_C(B)$, and $I$ is not a maximal ideal in $C$ by [9, Lemma 2.4].

Combining this result with Proposition 2.4 we obtain that maximal ideals do not exist in any Grothendieck category (in particular, in any module category).

**Corollary 3.3.** Let $G$ be a Grothendieck category. Then there do not exist maximal ideals in $G$.

Another remarkable consequence of Theorem 3.1 is the following.

**Corollary 3.4.** Let $C$ be a preadditive category, $M$ a maximal ideal of $C$ and $A$ an object of $C$. If $M(A, A) \neq \text{End}_R(A)$, then $A$ is maximal with respect to $\preceq$.

**Proof.** Set $I = M(A, A)$. Suppose that there exists an object $B$ such that $A \prec B$. Then $M(B, B) = A_1(B, B)$ by [9, Lemma 2.4]. But, by Theorem 3.1, $A_1(B, B)$ is a proper ideal that is not maximal. This contradicts the maximality of $M$.

**Remark 3.5.** Notice that, equivalently, if $A$ is an object of a preadditive category $C$ and there exists an object $B$ with $A \prec B$, then $M(A, A) = \text{End}_C(A)$ for every maximal ideal $M$ in $C$.

As a consequence of Corollary 3.2 if a preadditive category $C$ has maximal ideals, then there exist maximal objects with respect to the partial order $\preceq$. However, not all objects have to be maximal. That is, there can exist objects $A$ for which there are objects $B$ with $A \prec B$. For example, let $\kappa$ a cardinal and $\kappa^+$ be its successor cardinal. Let $K$ be a field and $C$ the full subcategory of $\text{Mod-}K$ whose objects are all vector spaces of dimension smaller than $\kappa^+$. As is proved in [9, Example 4.1],
C has one maximal ideal. However, for each vector space \( M \) of dimension smaller than \( \kappa \), there exists spaces \( V \) with \( M \prec V \) by Proposition 2.5.

Despite this observation, we are going to see that, in order to determine if a preadditive category has maximal ideals, we can restrict our attention to a full subcategory in which each object is maximal with respect to \( \preceq \).

**Definition 3.6.** Let \( C \) be a preadditive category.

1. We will denote by \( M(C) \) the full subcategory of \( C \) consisting of all maximal objects with respect to \( \preceq \), that is,
   \[
   \{ C \in C : \text{there does not exist } A \in C \text{ with } C \prec A \}
   \]
2. We will denote by \( S(C) \) the full subcategory of \( C \) whose class of objects is
   \[
   \{ C \in C : \text{there exists } A \in M(C) \text{ with } C \prec A \}
   \]

Let \( C \) be a preadditive category and \( D \) a full subcategory of \( C \). We now define how to restrict an ideal of \( C \) to \( D \) and, conversely, how to extend a maximal ideal of \( D \) to \( C \).

**Definition 3.7.** Let \( C \) be a preadditive category and \( D \) a full subcategory of \( C \).

1. Given \( I \) an ideal of \( C \), define its restriction \( I' \) to \( D \) by
   \[
   I'(D, D') = I(D, D')
   \]
   for every \( D, D' \in D \).
2. Given any maximal ideal \( M \) of \( D \), there exists an object \( D \in D \) such that
   \( M(D, D) \neq \text{End}_D(D) \). Define the extension \( M^e \) of \( M \) to \( C \) to be the ideal of \( C \) associated to \( M(D, D) \).

**Lemma 3.8.** Let \( C \) be a preadditive category, \( D \) a full subcategory of \( C \) and \( M \) a maximal ideal in \( D \).

1. Let \( D \) and \( D' \) be objects of \( D \) such that \( I := M(D, D) \) and \( I' := M(D', D') \) are maximal ideals of \( \text{End}_C(D) \) and \( \text{End}_C(D') \) respectively. Then the ideals \( A_I \) and \( A_{I'} \) coincide in \( C \). In particular, the definition of \( M^e \) does not depend on the choice of the object \( D \) with \( M(D, D) \neq \text{End}_D(D) \).
2. For any objects \( D, D' \) of \( D \), \( M^e(D, D') = M(D, D') \).

**Proof.** (1) By [9] Lemma 2.4], \( M = A_I \) in \( D \). Then \( A_I(D', D') \subseteq I' \), which implies that \( A_I \) is contained in \( A_{I'} \) (in \( C \)). Using the same argument, \( M = A_{I'} \) in \( D \) and, consequently, \( A_{I'}(D, D) \subseteq I \). Thus \( A_{I'} \) is contained in \( A_I \) (in \( C \)).

(2) By [9] Lemma 2.4]. \( \square \)

Now we can establish, for any preadditive category \( C \), the relation between the maximal ideals of \( C \) and those of \( M(C) \).

**Theorem 3.9.** Let \( C \) be a preadditive category. Then the assignments \( M \mapsto M^e \) and \( M \mapsto M^e \) define bijective correspondences between the following classes of ideals:

1. Maximal ideals of \( C \).
2. Maximal ideals \( M \) of \( M(C) \) satisfying \( M^e(C, C) = \text{End}_C(C) \) for each object \( C \) not belonging to \( M(C) \cup S(C) \), that is, for each object \( C \) with no maximal \( N \) with \( C \preceq N \).
Proof. Let $\mathcal{M}$ be a maximal ideal of $C$. We will now prove that $\mathcal{M}^e$ is a maximal ideal of $M(C)$ satisfying $\mathcal{M}^e(C, C) = \text{End}_C(C)$ for each object $C$ not belonging to $M(C) \cup S(C)$. Since $\mathcal{M}$ is proper, there exists an object $C_0$ of $C$ such that $\mathcal{M}(C_0, C_0) \neq \text{End}_C(C_0)$. By Theorem 3.1, $C_0$ must belong to $M(C)$. This means that $\mathcal{M}^e$ is a proper ideal in $M(C)$, which is trivially maximal, as $\mathcal{M}$ is maximal in $C$. Moreover, note that $\mathcal{M}^{e*}$ is the ideal of $C$ associated to $\mathcal{M}(C_0, C_0)$, which is equal to $\mathcal{M}$ by \cite[Lemma 2.4]{ref}. Then, again by Theorem 3.1, $\mathcal{M}^{e*}(C, C) = M(C, C) = \text{End}_C(C)$ for every object $C$ not belonging to $M(C)$.

Conversely, let $\mathcal{M}$ be a maximal ideal of $M(C)$ satisfying $\mathcal{M}^e(C, C) = \text{End}_C(C)$ for each object $C$ not belonging to $M(C) \cup S(C)$. We claim that $\mathcal{M}^e(C, C) = \text{End}_C(C)$ for each object $C$ belonging to $S(C)$. To prove the claim, let $C$ be an object of $S(C)$ and suppose that $D \in M(C)$ satisfies $C \preceq D$. If $\mathcal{M}^e(C, C) \neq \text{End}_C(C)$, then $\mathcal{M}^e(D, D)$ is a proper ideal of $\text{End}_C(D)$, which is not maximal by Theorem 3.1. By Lemma 3.2, $\mathcal{M}^e(D, D) = M(D, D)$ and, consequently, $\mathcal{M}(D, D)$ is a proper ideal of $\text{End}_C(M(C))(D)$ that is not maximal. Since $\mathcal{M}$ is maximal in $M(C)$, this contradicts \cite[Lemma 2.4]{ref}. The contradiction proves the claim.

Now let $\mathcal{N}$ be an ideal of $C$ properly containing $\mathcal{M}^e$. We will prove that $\mathcal{N} = \text{Hom}_C$. Since $\mathcal{N}(C, C) = \text{End}_C(C)$ for each object $C$ not belonging to $M(C)$, it follows that $\mathcal{N}^e$ properly contains $\mathcal{M}^{e*}$. But $\mathcal{M}^{e*} = \mathcal{M}$ by Lemma 3.3 and, since $\mathcal{M}$ is maximal in $M(C)$, we get that $\mathcal{N}^e = \text{Hom}_{M(C)}$. This fact with the previous claim gives that $\mathcal{N} = \text{Hom}_C$.

Finally, it is easy to see that the two assignments are mutually inverse. \hfill $\square$

Remark 3.10. Let $C$ be a preadditive category, $A$ and $C$ objects of $C$ and $I$ an ideal of $\text{End}_C(C)$. Then, $\mathcal{A}_I(A, A) = \text{End}_C(A)$ if and only if each endomorphism of $C$ factoring through $A$ belongs to $I$. Consequently, if $\mathcal{M}$ is a maximal ideal in $M(C)$ and $C$ is an object with no maximal $N$ satisfying $C \preceq N$, then the following conditions are equivalent:

1. $\mathcal{M}^e(C, C) = \text{End}_C(C)$,
2. There exists an object $A \in M(C)$ with $\mathcal{M}(A, A) \neq \text{End}_C(A)$ such that each endomorphism of $A$ factoring through $C$ belongs to $\mathcal{M}(A, A)$,
3. For each object $A \in M(C)$ with $\mathcal{M}(A, A) \neq \text{End}_C(A)$, every endomorphism of $A$ factoring through $C$ belongs to $\mathcal{M}(A, A)$.

Proposition 3.9 says that, in order to compute the maximal ideals in a category $C$, we can (1) determine the subcategories $M(C)$ and $S(C)$, and (2) find the maximal ideals $\mathcal{M}$ of $M(C)$ such that $\mathcal{M}^e(C, C) = \text{End}_C(C)$ for each object $C$ with no maximal $N$ satisfying $C \preceq N$. We will use this procedure in the following example.

Example 3.11. Let $R$ be a simple non-artinian ring with $\text{Soc}(R_R)$ non-projective as a right $R$-module. Then there exists a non projective simple right module $S$ contained in $R$. Consider the full subcategory $C = \text{add}(R_R) \cup \text{Add}(S)$ of $\text{Mod-}R$. Then $M(C) = \text{add}(R_R)$ and $S(C) = \emptyset$. Since $R$ is simple, Example 3.1 says that the unique maximal ideal of $M(C)$ is the ideal $0_0$ associated to the zero ideal of $R$. However, $A_0(S, S) \neq \text{End}_R(S)$ since, if we take $f : R_R \rightarrow S$ an epimorphism and we denote by $g : S \rightarrow R_R$ the inclusion, we have that $g1_Sf \neq 0$, which means that $1_S \not\in A_0(S, S)$. Then, by Theorem 3.1, $C$ does not have maximal ideals.

Remark 3.12. Let $C$ be a preadditive category. As the preceding example shows, there does not exist a bijective correspondence between maximal ideals in $C$ and maximal ideals in $M(C)$. This is due to the fact that there can exist maximal ideals
This implies that $f$ trivially satisfies $F$ the desired property.

then, by the previous lemma, there exists $C$ injective endomorphism ring of an indecomposable injective module. of the previous sections to describe the ideal associated to a maximal ideal in the endomorphism ring [2, Theorem 25.4].

Lemma 4.1. Let $F$ be an indecomposable injective module, and let $I$ be the maximal ideal of $\text{End}_R(F)$. The following conditions are equivalent for modules $A, B$ and $f \in \text{Hom}_R(A, B)$:

(1) $f \notin A_I(A, B)$.

(2) There exists $\alpha: F \to A$ and $\beta: B \to F$ such that $\beta f \alpha = 1_F$.

(3) There exists a submodule $C$ of $A$ such that $C \cong F$ and $C \cap \ker f = 0$.

Proof. (1) $\Rightarrow$ (2). If $f \notin A_I$, there exists $\alpha: F \to A$ and $\beta: A \to F$ such that $\beta f \alpha \notin I$. Since the endomorphisms of $F$ not belonging to $I$ are isomorphisms, there exists an inverse $\gamma \in \text{End}_R(F)$ of $\beta f \alpha$. Then $\gamma \beta f \alpha = 1_F$.

(2) $\Rightarrow$ (1) is trivial.

(2) $\Rightarrow$ (3). Set $C = \alpha(F)$, which is isomorphic to $F$ as $\alpha$ is monic. Then $A = C \oplus \ker(\beta f)$. In particular, $C \cap \ker f \subseteq C \cap \ker(\beta f) = 0$.

(3) $\Rightarrow$ (2). Let $\alpha: F \to A$ be a monomorphism with image $C$. Since $C \cap \ker f = 0$, $f \alpha$ is a monomorphism. Since $F$ is an injective, this implies the existence of $\beta: B \to F$ with $\beta f \alpha = 1_F$, as desired. \hfill $\square$

As a byproduct of this result we get:

Corollary 4.2. Let $F$ be an indecomposable injective module, and let $I$ be the maximal ideal of $\text{End}_R(F)$. Then, for any pair $A, B$ of modules, $A_I(A, B) \neq \text{Hom}_R(A, B)$ if and only if both $A$ and $B$ contain a submodule isomorphic to $F$.

Proof. If $A_I(A, B) \neq \text{Hom}_R(A, B)$ and $f: A \to B$ does not belong to $A_I(A, B)$, then, by the previous lemma, there exists $C \leq A$ with $C \cong F$ and $C \cap \ker f = 0$. This implies that $f(C)$ is isomorphic to $F$. Thus $C$ and $f(C)$ are submodules with the desired property.

Conversely, assume $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1 \cong B_1 \cong F$. Let $f: A_1 \to B_1$ be an isomorphism and let $F: A \to B$ be the morphism $f \oplus 0$. Then $F$ trivially satisfies (3) of the previous lemma and, consequently, $F$ does not belong to $A_I(A, B)$. \hfill $\square$
It follows that, in order to determine $A_I$, we only have to look at the modules $A$ containing isomorphic copies of $F$. These modules have a nice description if $R$ is right noetherian as we will prove next. We shall use the following well known facts about an injective module $F$:

(1) If $K$ is a non-zero submodule of $F$, then there exists an injective submodule $G$ of $F$ containing $K$ such that the inclusion $K \leq G$ is an injective envelope. In particular, if $F$ is indecomposable, then the inclusion $K \leq F$ is an injective envelope of $K$.

(2) $F$ satisfies the exchange property, which means that for each module $N$ and each decomposition $F \oplus N = \bigoplus_{\alpha < \kappa} A_{\alpha}$, there exists a submodule $B_{\alpha} \leq A_{\alpha}$ for each $\alpha < \kappa$ such that $F \oplus N = F \oplus \left( \bigoplus_{\alpha < \kappa} B_{\alpha} \right)$.

**Theorem 4.3.** Suppose $R$ right noetherian. Let $F$ be an indecomposable injective module. Then every module $M$ has a decomposition $M = M_1 \oplus M_2$ where:

1. $M_1 \cong F^{(\Gamma)}$ for some set $\Gamma$.
2. $M_2$ does not contain submodules isomorphic to $F$.

Moreover, $\Gamma$ is uniquely determined up to cardinality and $M_2$ is uniquely determined up to isomorphism.

**Proof.** If $M$ does not have submodules isomorphic to $F$, there is nothing to prove. So suppose that $M$ has submodules isomorphic to $F$ and consider the non-empty family of submodules

$$S = \{ N \leq M : N \cong F^{(\Gamma)} \text{ for some set } \Gamma \}$$

Let us show that $S$ is inductive. Take a chain $S' = \{ M_\lambda : \lambda \in \Lambda \}$ in $S$. We will prove that $M' := \bigcup_{\lambda \in \Lambda} M_\lambda \in S$. Since $R$ is right noetherian, directed colimits of injective modules are injective [3, Exercise 8 of Chapter I], so that $M'$ is injective. Hence, $M'$ has a direct-sum decomposition, $M' = \bigoplus_{i \in I} F_i$, where the submodules $F_i$ of $M'$ are injective and indecomposable [13, Theorem 3.48, p. 82]. Let $i \in I$ and $x$ be a non-zero element of $F_i$; note that $F_i$ is the injective envelope of $xR$. Since $x \in M_\lambda$ for some $\lambda \in \Lambda$, and $M_\lambda \in S$, $x$ belongs to a direct summand of $M_\lambda$ isomorphic to $F^n$ for some $n$. But, as $F_i$ is the injective envelope of $xR$, $F^n$ must contain a direct summand isomorphic to $F_i$. This implies that $F_i \cong F$ because all indecomposable direct summands of $F^n$ are isomorphic to $F$. Consequently, $M' \in S$.

The first part of the statement now follows taking a maximal element $M_1$ of $S$ and a submodule $M_2$ of $M$ with $M_1 \oplus M_2 = M$.

In order to prove the last part of the statement, suppose that $M = M_1' \oplus M_2'$ is another decomposition of $M$ satisfying (1) and (2). Write $M_1 = \bigoplus_{\beta < \kappa} G_\beta$ and $M_1' = \bigoplus_{\alpha < \lambda} F_\alpha$ for suitable families of submodules $\{ G_\beta : \beta < \kappa \}$ and $\{ F_\alpha : \alpha < \lambda \}$ of $M_1$ and $M_1'$ respectively, and cardinals $\kappa$ and $\lambda$, satisfying $G_\beta \cong F_\alpha \cong F$ for each $\beta < \kappa$ and $\alpha < \lambda$.

Since $M_1$ satisfies the exchange property, there exist submodules $H_\alpha \leq F_\alpha$ for each $\alpha < \lambda$ and $N_2' \leq M_2'$ such that $M = M_1 \oplus \left( \bigoplus_{\alpha < \lambda} H_\alpha \right) \oplus N_2'$. Since $F_\alpha$ is indecomposable for each $\alpha < \lambda$, it follows that $H_\alpha = 0$ or $H_\alpha = F_\alpha$. But, as $\left( \bigoplus_{\alpha < \lambda} H_\alpha \right) \oplus N_2' \cong M_2$ and $M_2$ does not contain any submodule isomorphic to $F$, we get that $H_\alpha = 0$ for each $\alpha < \lambda$. That is, $M = M_1 \oplus N_2'$.
Applying the modular law, we see that \( M'_2 = N'_2 \oplus (M'_2 \cap M_1) \). We claim that \( M'_2 \cap M_1 = 0 \). Assume the contrary, i.e., that \( M_1 \cap M'_2 \neq 0 \). Since \( M_1 \cap M'_2 \) is a direct summand of \( M \), it is a direct summand of \( M_1 \) and, consequently, it is injective. As it is non-zero, there exists \( \beta < \kappa \) such that \( G_\beta \cap M_1 \cap M'_2 \neq 0 \). Let \( x \) be a non-zero element in this intersection. Notice that \( xR \leq G_\beta \) is an injective envelope. Since \( M_1 \cap M'_2 \) is injective, there exists an injective envelope \( C \) of \( xR \) contained in \( M_1 \cap M'_2 \). But \( C \) is isomorphic to \( F \) and \( M'_2 \) does not contain any submodule isomorphic to \( F \), a contradiction. This proves the claim.

As a consequence, \( M = M_1 \oplus M'_2 \). Then \( M_1 \cong M'_1 \) and \( M_2 \cong M'_2 \). By Azumaya’s Theorem [2, Theorem 12.6], \( \kappa = \lambda \) and we are done. \( \square \)

We can use this result to define the \( F \)-rank of a module \( M \), for any indecomposable injective module \( F \) and any module \( M \) over a right noetherian ring.

**Definition 4.4.** Suppose \( R \) is right noetherian. Let \( F \) be an indecomposable injective module. Given any module \( M \) and any cardinal \( \kappa \), we say that \( M \) has \( F \)-rank equal to \( \kappa \) (written \( r_F(M) = \kappa \)) if \( M = M_1 \oplus M_2 \), where \( M_1 \cong F^{(\kappa)} \) and \( M_2 \) has no direct summand isomorphic to \( F \). We will denote by \( C_F \) the full subcategory of Mod-\( R \) whose objects are all modules of finite \( F \)-rank.

Now we can compute the maximal ideals in the category \( C_F \) for an indecomposable injective module \( F \). First of all, we compute the subcategories \( M(C_F) \) and \( S(C_F) \).

**Proposition 4.5.** Suppose \( R \) is right noetherian. Let \( F \) be an indecomposable injective module. Then:

1. \( M(C_F) = \{ M \in C_F : r_F(M) > 0 \} \).
2. \( S(C_F) = \{ M \in C_F : r_F(M) = 0 \} \).

**Proof.** (1) Let \( M \) be a module in \( C_F \) with \( r_F(M) = 0 \). We can find an infinite cardinal \( \kappa \) such that \( M \) is \( \prec \kappa \)-generated in Mod-\( R \). By Proposition 2.4, \( M \cong M^{(\kappa)} \) in Mod-\( R \). Since \( C_F \) is a full subcategory of Mod-\( R \) and \( M^{(\kappa)} \in C_F \), we get that \( M \cong M^{(\kappa)} \) in \( C_F \). Consequently, \( M \) does not belong to \( M(C_F) \). This proves the inclusion \( M(C_F) \subseteq \{ M \in C_F : r_F(M) > 0 \} \).

In order to prove the inverse inclusion, let \( M \) be any module with \( r_F(M) > 0 \) and suppose, by contradiction, that \( M \notin M(C_F) \). Then there exists \( N \in C_F \) such that \( M \prec N \). Let \( M = M_1 \oplus M_2 \) and \( N = N_1 \oplus N_2 \) be the decompositions given by Theorem 1.2, and let \( E \) be the set of Definition 2.1. Write \( M_1 = \bigoplus_{i=1}^{n} G_i \) and \( N_1 = \bigoplus_{j=1}^{m} F_j \) for modules \( G_i \) and \( F_j \) isomorphic to \( F \) for each \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

We claim that \( f(N_1) \neq 0 \) for each \( (f, g) \in E \). Given \( i = 1, \ldots, n \), \( g(G_i) \) is isomorphic to \( F \). Then \( g(E_i) \cap N_1 \neq 0 \) since, otherwise, \( N_1 \oplus g(E_i) \) would be a direct summand of \( N_2 \), and \( N_2 \) would contain a submodule isomorphic to \( F \). Now, taking \( y \in g(E_i) \cap N_1 \) non-zero and \( x \in E_i \) with \( g(x) = y \), we have that \( f(y) = x \neq 0 \). This proves the claim.

For each \( i = 1, \ldots, m \), let \( q_i : G_1 \to F_1 \) be an isomorphism. Then \( q_i \) extends to a morphism \( p_i : M \to N \). The preceding claim says that for each \( (f, g) \in E \), \( fp_i \neq 0 \) for some \( i = 1, \ldots, m \). Consequently, if \( E_i = \{(f, g) \in E : fp_i \neq 0 \} \)
for each \( i = 1, \ldots, m \), we conclude that \( E \subseteq \bigcup_{i=1}^{m} E_i \). This is a contradiction, because the second set has cardinality smaller than \(|E|\) by Definition 2.1(2).

The conclusion is that there is no object \( N \in \mathcal{C}_F \) with \( M \not\leq N \), so that \( M \in \mathcal{M}(\mathcal{C}_F) \).

(2) As a direct consequence of (1) we have \( \mathbf{S}(\mathcal{C}_F) \subseteq \{ M \in \mathcal{C}_F : r_F(M) = 0 \} \). In order to see the other inclusion, fix \( M \in \mathcal{C}_F \) with \( r_F(M) = 0 \). Then, as in the proof of (1), there exists an infinite cardinal \( \kappa \) such that \( M \not\leq M^{(\kappa)} \) in \( \mathcal{C}_F \). By Lemma 4.2 \( M \not\leq F \oplus M^{(\kappa)} \). Then \( M \in \mathbf{S}(\mathcal{C}_F) \) because \( F \oplus M^{(\kappa)} \in \mathcal{M}(\mathcal{C}_F) \) by (1). \( \square \)

Finally, we can determine all maximal ideals of the category \( \mathcal{C}_F \) for an indecomposable injective module \( F \) over a right noetherian ring.

**Proposition 4.6.** Suppose \( R \) is right noetherian. Let \( F \) be an indecomposable injective module and \( I \) be the maximal ideal of \( \text{End}_R(F) \). Then \( \mathcal{A}_I \) is the unique maximal ideal of \( \mathcal{C}_F \).

*Proof.* By Theorem 5.4 we only have to compute the maximal ideals of \( \mathcal{M}(\mathcal{C}_F) \). First, we prove that \( \mathcal{A}_I \) is a maximal ideal of \( \mathcal{M}(\mathcal{C}_F) \). Given \( M \in \mathcal{M}(\mathcal{C}_F) \), since \( \mathcal{A}_I(M, M) \neq \text{End}_R(M) \) by Corollary 4.2 we have to see, applying [9, Lemma 2.4], that

\[
\begin{align*}
(a) & \quad \mathcal{A}_I(M, M) \text{ is maximal in } \text{End}_R(M) \text{ and,} \\
(b) & \quad \text{if } J_0 = \mathcal{A}_I(M, M), \text{ then } \mathcal{A}_I = \mathcal{A}_{J_0}.
\end{align*}
\]

Let \( M = M_1 \oplus M_2 \) be the decomposition of \( M \) given in Theorem 4.3. Note that, by Lemma 4.2 and Corollary 4.2,

\[
\mathcal{A}_I(M, M) = \{ f \in \text{End}_R(M) : \pi_1 f \iota_1 \in \mathcal{A}_I(M_1, M_1) \},
\]

where \( \pi_i : M \to M_i \) and \( \iota_i : M_i \to M \) are the corresponding projections and inclusions for \( i = 1, 2 \). As \( M_1 \in \text{add}(F) \) and \( \mathcal{A}_I \) is a maximal ideal in this category by Example 4.1 \( \mathcal{A}_I(M_1, M_1) \) is a maximal ideal in \( \text{End}_R(M_1, M_1) \). In order to see that \( \mathcal{A}_I(M, M) \) is maximal, let \( J \) be an ideal of \( \text{End}_R(M) \) strictly containing \( \mathcal{A}_I(M, M) \). Let \( f \in J \) not belonging to \( \mathcal{A}_I(M, M) \). Then \( \pi_1 f \iota_1 \) does not belong to \( \mathcal{A}_I(M_1, M_1) \) and, by the maximality of this ideal in \( \text{End}_R(M_1, M_1) \), there exist \( g \in \mathcal{A}_I(M_1, M_1) \) and \( \alpha, \beta \in \text{End}_R(M_1) \) such that \( 1_{M_1} = g + \alpha \pi_1 f \iota_1 \beta \). Then we have the identity

\[
1_{M_1} \oplus 0 = g \oplus 0 + (\alpha \oplus 0) f (\beta \oplus 0)
\]

in \( \text{End}_R(M) \), with both \( g \) and \( (\alpha \oplus 0) f (\beta \oplus 0) \) in \( J \). Consequently, \( 1_{M_1} \oplus 0 \in J \). Now use \( 0 \oplus 1_{M_2} \in J \) to get that \( 1_M = 1_{M_1} \oplus 0 + 0 \oplus 1_{M_2} \in J \) and that \( J = \text{End}_R(M) \).

Let us prove (b). Since \( \mathcal{A}_{J_0} \) is the greatest of all the ideals \( \mathcal{I}' \) of \( \mathcal{C}_F \) such that \( \mathcal{I}'(M, M) \leq J_0 \), we conclude that \( \mathcal{A}_I \subseteq \mathcal{A}_{J_0} \). In order to prove the other inclusion, we only have to see, by the same argument, that \( \mathcal{A}_{J_0}(F, F) \leq I \). Let \( f \in \mathcal{A}_{J_0}(F, F) \). Fix a monomorphism \( \alpha_1 : F \to M_1 \), which, as \( \text{Im} \alpha_1 \) is a direct summand, has an splitting \( \beta_1 : M_1 \to F \). Then we that \( \iota_1 \alpha_1 \beta_1 \pi_1 \in J_0 \), because \( f \in \mathcal{A}_{J_0}(F, F) \). Then \( \pi_1 \iota_1 \alpha_1 \beta_1 \iota_1 \alpha_1 \in \mathcal{A}_I(M_1, M_1) \) and, consequently, \( \beta_1 \pi_1 \iota_1 \alpha_1 \beta_1 \iota_1 \alpha_1 \in \mathcal{A}_I(F, F) = I \). Since

\[
f = \beta_1 \pi_1 \iota_1 \alpha_1 \beta_1 \iota_1 \alpha_1,
\]

we conclude that \( f \in I \).

To finish the proof, we will see that \( \mathcal{A}_I \) is the unique maximal ideal of \( \mathcal{M}(\mathcal{C}_F) \). Let \( M \) be any maximal ideal of \( \mathcal{M}(\mathcal{C}_F) \) and \( M \in \mathcal{M}(\mathcal{C}_F) \) be such that \( M(M, M) \neq \).
End\(_R(M)\). If \(J = M(M, M)\), then \(M = A_J\) by \([9\), Lemma 2.4.\] Let \(M = M_1 \oplus M_2\) be the decomposition of \(M\) given by Theorem 4.3. By Lemma 1.2, either \(M(M_1, M_1)\) or \(M(M_2, M_2)\) have to be proper. But \(M_2 \in \mathcal{S}(C_F)\) by Proposition 4.5, so that \(M(M_2, M_2) = \text{End}_R(M_2)\) by Remark 3.5. Thus \(M(M_1, M_1) \neq \text{End}_R(M_1)\) which implies, again by Lemma 1.2, that \(M(F, F) \neq \text{End}_R(F)\). Since \(I\) is the unique maximal ideal of \(\text{End}_R(F)\) and \(M(E, E)\) is maximal, we conclude that \(M(F, F) = I\) Now \(M = A_I\) by \([9\), Lemma 2.4\], which concludes the proof. □

Example 4.7. The category \(C_F\) has maximal ideals and objects \(M, N\) with \(M \prec N\) since, if \(M\) is an object in \(C_F\) with \(F\)-rank 0, then each direct sum of copies of \(M\) belongs to \(C_F\). By Proposition 2.4, there exist objects \(N\) in \(C_F\) with \(M \prec N\).

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