Data-driven stabilization of switched and constrained linear systems

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Abstract—We consider the design of state feedback control laws for both the switching signal and the continuous input of an unknown switched linear system, given past noisy input-state trajectories measurements. Based on Lyapunov-Metzler inequalities, we derive data-dependent bilinear programs whose solution directly returns a provably stabilizing controller and ensures $H_\infty$ performance. We further present relaxations that considerably reduce the computational cost, still without requiring stabilizability of any of the switching modes. Finally, we showcase the flexibility of our approach on the constrained stabilization problem for a perturbed linear system.

Index Terms—Data-driven control, linear matrix inequalities, robust control, sliding mode control, switched systems.

1. INTRODUCTION

DIRECT data-driven control refers to the design of controllers based only on observed trajectories generated by an unknown dynamical system, without explicitly identifying a model of the system [1]–[3]. Bypassing the modeling step comes with important advantages. First, the computational complexity of system identification is mitigated. Second, uncertainty propagation is avoided. In fact, measurement noise potentially leads to inaccurate models: while it is possible to bound the modeling error and resort to robust controllers, this two-step procedure is typically conservative. Third, in general control synthesis requires less information on the system (hence, less data) than full identification of its dynamics [4].

Literature review: Direct data-driven control traces back to the work of Ziegler and Nichols [5] on proportional-integral-derivative (PID) controllers. Classical methods are also reference-model algorithms (virtual reference feedback tuning [6] and iterative feedback tuning [7]) and reinforcement learning [8]. Later alternatives include intelligent PID controllers [9] and model-free adaptive control [10]; methods based on Willems’ fundamental lemma are currently enjoying renewed popularity as well [1], [11]–[14].

In this paper, we focus on a recent, robust-control approach, of interest for both linear [4] and nonlinear systems [15].

In simple terms, when the true plant is not known, the goal is to find a controller that provably guarantees closed-loop stability (or performance/optimality [16], safety [17]) for all the systems compatible with (a) a few, finite, open-loop measured trajectories (possibly corrupted by noise) and (b) prior knowledge on the model class (e.g., polynomial dynamics in [15]) and noise bounds. The controller is typically built by solving a data-dependent optimization problem, for instance a linear matrix inequality (LMI) in [4] or a polynomial program in [15]. In this stream of literature, much effort is devoted to providing tractable conditions for stabilization of several classes of nonlinear systems and under various noise assumptions [18]–[23]. In particular, the works [21]–[24] focus on the data-driven design of the continuous input for switching linear systems [25], where the dynamics switches freely among a set of LTI plants (also called “modes”). In [21], [22], the switching signal is arbitrary but measured. The authors of [23], [24] study the case of an unknown switching signal, with stability ensured under a sufficiently large dwell time. Both cases require stabilizability of each subsystem.

On the contrary, in this paper, we consider switched linear systems—where the active mode is chosen by the controller. In particular, we are interested in “stabilization by switching”, namely controlling the system by opportunistically choosing the discrete switching signal: famously, this can be achieved even when all the modes are unstable [25]. Not only switched dynamics naturally arise in prominent engineering applications (robotics, embedded systems, traffic control, power electronics, just to name a few [26], [27]), but even for a single plant switching among different compensators can considerably improve performance, or achieve otherwise impossible control goals [28, Ch. 9]. For this reason, stabilization by switching has been extensively investigated (for the case of a known model) [25], [29]–[31]. The major challenge remains finding good tradeoffs between complexity and conservatism: for instance, tight conditions for stabilizability of discrete-time (DT) switched linear systems are known, but computationally prohibitive [30]. In this spirit, the so-called Lyapunov-Metzler (LM) inequalities, introduced by Geromel and Colaneri in [32], [33], provide sufficient conditions for stability in the form of a bilinear matrix inequality (BMI). The success of the LM conditions has witnessed many reformulations, relaxations and generalizations [34], [35].

In contrast, data-driven stabilization of switched linear dynamics is essentially unexplored. To the best of our knowledge, the only work to address this problem (under additional
dwell-time constraints) is [36]. Yet, the design is particularly restrictive (e.g., a specific class of linear systems is considered, and at least one of the modes must be stable) and the assumption on the noiseless data implies that each subsystem can be uniquely identified. [37] also treats switched systems, but rather considers a finite-horizon control task (via a reinforcement learning approach).

Contributions: In this paper, we consider the direct data-driven stabilization of an unknown switched linear system. We mainly focus on the continuous-time case, but we also discuss the discrete-time case. We start by observations of finite input-state-derivative trajectories generated by the system; differently from [36], the measurements are subject to disturbance, obeying an energy-type bound. We provide a method to design a controller (for the discrete switching signal, and for the continuous input if present) that provably stabilizes the unknown dynamics, with guaranteed performance. Our results are complementary to those in [21]–[24], about stabilization of systems with uncontrolled switching. The novelties of our work can be summarized as follows:

- **Data-driven LM inequalities**: We derive a BMI, dependent only on measurements and prior structural knowledge, whose solution directly supplies a stabilizing controller. The condition is obtained by applying the S-procedure [16] to a dual version of the LM inequalities of [32]; crucially, we handle the additional complexity (variable inversion) induced by the dualization. Our formulation is nonconservative (namely, data efficient), i.e., it is solvable if and only if there exists a solution to the LM inequalities common to all the systems that can explain the data (subject to a Slater’s qualification, as in [16]) (§III).

- **Relaxations**: As for the case of a known model, our data-driven LM inequalities are nonconvex, and thus challenging to solve. We propose two relaxations that, while being possibly conservative, considerably reduce the computational burden; furthermore, we provide sufficient conditions, in terms of the system properties, for the existence of a solution. In particular, our second relaxation is an LMI when a scalar variable is fixed (hence, the inequality can be efficiently solved via standard semidefinite programming (SDP) methods and line-search), and directly improves on that proposed in [32], by being both easier to solve and less restrictive (§IV).

- **Performance**: We generalize known results on $H_2$ and $H_\infty$ control of switched linear systems to take into account sliding motions (Appendix I) and we address the problem in a data-driven fashion (§V).

- **Discrete-time**: We extend our results to DT switched linear systems, where the main challenge is the occurrence of additional nonlinearities in the matrix conditions (§VI).

- **Safe stabilization**: We show that the tools developed in this paper are of interest even when the original system is linear time invariant (LTI). In particular, we design, solely based on data, a switched controller and a robustly invariant set that guarantee not only robust satisfaction of state (and input) constraints, but also –differently from the results in [38], [39]– closed-loop asymptotic stability (§VII).

**Notation**: We denote $\Delta := \{ \lambda \in \mathbb{R}^N_{\geq 0} \mid 1_N^T \lambda = 1 \}$ and $\Delta_+ := \{ \lambda \in \mathbb{R}^N_{\geq 0} \mid 1_N^T \lambda = 1 \}$; let $S$ ($S_+$) be the set of matrices whose columns belong to $\Delta$ ($\Delta_+$). We define the sets of Metzler matrices $M := \{ \Pi \in \mathbb{R}^{N \times N} \mid \pi_{i,j} \geq 0 \ \forall i \neq j; \sum_{i=1}^N \pi_{i,j} = 0 \ \forall j \}$ and $M_+ := \{ \Pi \in \mathbb{R}^{N \times N} \mid \pi_{i,j} > 0 \ \forall i \neq j; \sum_{i=1}^N \pi_{i,j} = 0 \ \forall j \}$, where $\pi_{i,j}$ is the element of $\Pi$ in row $i$ and column $j$. We also use the compact notation $P := P^{-1}$ for the inverse of a matrix $P$. For a signal $s : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, its 2-norm is $\|s\|_2 = (\int_{t=0}^\infty s(t)^2 dt)^{1/2}$; we denote by $L_2$ the set of signals with bounded 2-norm. $e_i \in \mathbb{R}^m$ is the $i$-th column of the identity matrix $I$ of appropriate dimension $m$. $\delta(t)$ denotes the continuous-time unitary impulse. $P > 0$ ($P \succeq 0$) denotes a symmetric positive (semi-)definite matrix; $\lambda_{\text{min}}(P)$ and $\lambda_{\text{max}}(P)$ denote the minimum and maximum eigenvalue of $P$. We may replace the elements of a matrix that can be deduced by symmetry with the shorthand notation “$\ast$”.

**LM inequalities**: We review in Appendix I some results on stabilization of switched systems and refine them to cope with sliding motions.

**Lemma 1 (Matrix S-lemma [16, Th. 9])**: Let $G,H \in \mathbb{R}^{(k+n) \times (k+n)}$ be symmetric. Consider the following:

(a) $\exists \theta \in \mathbb{R}_{\geq 0}$ such that $G - \alpha H \succeq 0$;

(b) $\begin{bmatrix} I & G \\ Z & I \end{bmatrix} \succeq 0, \forall \theta \in \mathbb{R}^n$ such that $\begin{bmatrix} I \\ Z \end{bmatrix} H \begin{bmatrix} I \\ Z \end{bmatrix} \succeq 0$.

Then (a) $\Rightarrow$ (b); if in addition $\exists \theta \in \mathbb{R}^{n \times k}$ such that $\begin{bmatrix} I & Z \\ Z & I \end{bmatrix} \succeq 0$, then also (b) $\Rightarrow$ (a).

**II. Problem statement**

We consider a switched linear system

$$\dot{x} = \bar{A}_i x + \bar{B}_i u,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is a controlled continuous input, $\sigma \in \mathcal{I} := \{1,2,\ldots,N\}$ is a controlled discrete input (switching signal), and $\{\bar{A}_i\}_{i \in \mathcal{I}}, \{\bar{B}_i\}_{i \in \mathcal{I}}$ are the system matrices. Here the bar indicates the true system matrices, which are unknown.

Instead, we assume that some experimental data are available, generated by applying inputs $u$ and $\sigma$, measuring the state $x$ and obtaining an estimate of the derivative, $\dot{x} + w$, for an unknown disturbance $w$. In particular, $T_s \geq 0$ samples have been collected for each subsystem $i \in \mathcal{I}$ (at non switching instants $\{t_k\}_{k=1}^T$, not necessarily from a unique trajectory), and organized in the following matrices:

$X_i := \begin{bmatrix} x(t_1) & x(t_2) & \ldots & x(t_T) \end{bmatrix}$

$\dot{X}_i := \begin{bmatrix} \dot{x}(t_1) + w(t_1) & \dot{x}(t_2) + w(t_2) & \ldots & \dot{x}(t_T) + w(t_T) \end{bmatrix}$

$W_i := \begin{bmatrix} w(t_1) & w(t_2) & \ldots & w(t_T) \end{bmatrix}$

that satisfy

$$\dot{X}_i = \bar{A}_i X_i + \bar{B}_i U_i + W_i,$$

where the matrix $\bar{W}_i$ is unknown.

**Assumption 1 (Disturbance model)**: For each $i \in \mathcal{I}, \bar{W}_i \in \mathcal{W}_i$, where

$$\mathcal{W}_i := \left\{ W_i \in \mathbb{R}^{n \times T_s} \mid \begin{bmatrix} I & \Phi_{1,1} & \Phi_{1,2} \\ W_i^T & \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \begin{bmatrix} I \\ W_i^T \end{bmatrix} \succeq 0 \right\}$$

$:= \Phi^+$

$\geq$
for some known matrices $\Phi_{1,1}^i = \Phi_{1,1}^i \top$, $\Phi_{1,2}^i$, $\Phi_{2,2}^i = \Phi_{2,2}^i \top < 0$.

The unknown-but-bounded model in Assumption 1 is common in the literature [16, Asm. 1], [18, Asm. 1]. It represents an energy-like bound on the disturbance, and captures prior knowledge on the noise-to-signal ratio or sample covariance.

By replacing (2) in Assumption 1, we infer that, for any $i \in I$, a pair of matrices $(A_i, B_i)$ is consistent with the experiment (i.e., can explain the data) if and only if
\[
\begin{bmatrix}
I & \dot{X}_i \\
A_i^\top & 0 \\
B_i^\top & 0
\end{bmatrix}
\begin{bmatrix}
\Phi_{1,1}^i \\
\Phi_{1,2}^i \\
\Phi_{2,2}^i
\end{bmatrix}
\begin{bmatrix}
I & \dot{X}_i \\
A_i^\top & 0 \\
B_i^\top & 0
\end{bmatrix}^\top \succ 0,
\]
where the matrix $H_i$ depends on known quantities only. Let us define, for each $i \in I$, the compatibility set
\[
C_i := \{(A_i, B_i) : \exists W_i \in W_i: \dot{X}_i = A_i X_i + B_i U_i + W_i\} \tag{4}
\]
with $C_i := \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ if $T_i = 0$. Clearly, $(\tilde{A}_i, \tilde{B}_i) \in C_i$.

However, the true system matrices cannot be discerned via the available data. To ensure stability of the true system, we consider the following control objective.

**Problem 1:** Under Assumption 1, find state-feedback laws for the input $u$ and the switching signal $\sigma$ that stabilize the system in (1) (with $A_\sigma, B_\sigma$ replaced by $A_\sigma, B_\sigma$) for any set of matrices compatible with the measured data in (2), i.e., for all $(A_i, B_i) \in C_i$.

**Remark 1 (Alternative disturbance model):** Although with some conservatism, our results also apply if the prior on the disturbance is given by instantaneous bounds on the norm of $w(t)$, i.e., $\sup_{t \in [0,T]} \|w(t)\| \leq \pi$. In fact, in this case Assumption 1 holds with $\Phi_{1,1} = \pi_0 \pi_1$, $\Phi_{1,2} = 0$, $\Phi_{2,2} = -I$. A tighter upper bound on the compatibility sets (in the form (3), for some data-dependent matrix $H_i$) can also be computed as in [39, Eq. 18], by solving a data-dependent program. Moreover, Assumption 1 also comprises the case of zero disturbance (in turn, $C$ can be a singleton if the data are rich enough, i.e., the system (1) is identifiable). Finally, we interpreted $w$ as noise on the derivative estimate; however, it is actually irrelevant how $w$ is generated, as long as (2) and Assumption 1 hold. On this basis, we can also account for process disturbance or state measurement error.

**III. DATA-DRIVEN STABILIZATION**

For the continuous input $u$, we restrict our attention to linear controllers of the form $u = K_\sigma x$, where $\{K_i\}_{i \in I}$ are feedback gains to be designed. For brevity, we define
\[
A_i^{cl} := A_i + B_i K_i.
\]

To solve the stabilization problem in a data-driven fashion, we aim at leveraging Proposition A1 in Appendix I. In particular, if we can find $\{K_i\}_{i \in I}$, $\{P_i > 0\}_{i \in I}$, $Q \succ 0$ and $\Pi \in \mathcal{M}$ such that for all $i \in I$, for all $(A_i, B_i) \in C_i$,
\[
A_i^{cl} P_i + P_i A_i^{cl} + \sum_{j \in I} \Pi_{ij} P_j + Q \prec 0, \tag{6}
\]
then Proposition A1 ensures that the controller
\[
\sigma(x) = \min \left\{ \arg \min_{i \in I} x^\top P_i x \right\}, \quad u(x) = K_\sigma(x) x, \tag{7}
\]
asymptotically stabilizes (1), with guaranteed performance $\int_0^\infty x^\top Q x \ dt < \min_{i \in I} x(0)^\top P_i x(0)$. Note that we look for a common weight matrix $\Pi$ for all the systems in the compatibility set $C$, akin to most methods in the literature, that seek a common Lyapunov function [16], [18] (instead, $K_i$ and $P_i$ must be common to all systems just in $C_i$ to implement (7)).

In place of (6), for technical reasons that will be clear later on, we consider the following related problem:

**Problem 2:** Find $\{K_i\}_{i \in I}$, $\{P_i > 0\}_{i \in I}$, $Q > 0$, $\Pi \in \mathcal{M}_+$ such that for all $i \in I$, for all $(A_i, B_i) \in C_i$,
\[
A_i^{cl} P_i + P_i A_i^{cl} + \sum_{j \in I} \Pi_{ij} P_j + Q \preceq 0. \tag{8}
\]

Note that a solution to (8) immediately provides a solution to (6) (e.g., with $Q = 0$ in (6)). Thus implementing (7) with matrices $\{P_i, K_i\}_{i \in I}$ that satisfy (8) does stabilize (1); this solution assures that
\[
\int_0^\infty x^\top Q x \ dt \leq \min_{i \in I} x(0)^\top P_i x(0). \tag{9}
\]

Conversely, we wonder if feasibility of (6) implies feasibility of (8)? When the compatibility set $C$ is a singleton (i.e., if the real system is known), or more generally when $C$ is compact, this is immediately true (due to the strict inequality in (6)). The following result shows that (8) is nonrestrictive (i.e., (6) can be bounded away from zero) even in case of unbounded compatibility sets $C_i$’s.

**Lemma 2 (Equivalent LM conditions):** The system of inequalities in (6) is feasible if and only if the system of inequalities in (8) is feasible.

The proof of this result is given in Appendix II-A.

**Remark 2 (Problems with unbounded C):** If, for some $i$, $C_i$ is quadratically stabilizable\footnote{Namely, if there are $K_i, P_i > 0$ such that $A_i^{cl} P_i + P_i A_i^{cl} < 0$ for all $(A_i, B_i) \in C_i$; we require $P_i$ independent of $(A_i, B_i)$ because this condition can be easily checked from data [16, Th. 14].}, then Problem 1 admits a trivial solution with $\sigma(t) \equiv \hat{i}$; however, the system in (6) might still be unfeasible when $C_j$ is unbounded, for some $j \neq i$. In fact, the proof of Lemma 2 shows that if (8) admits a solution, then for all $j \in I$, there exists $K_j$ that makes the set $\{A_j^{cl} = A_j + B_j K_j \mid (A_j, B_j) \in C_j\}$ bounded, or equivalently
\[
\text{rank}(X_j) = n. \tag{10}
\]

This necessary condition on the data is well known for the case $N = 1$ (i.e., LTI systems) [4, Lem. 15]. If the data collected do not verify (10) for some $j$, then that mode must be ignored in the design (by removing $j$ from $I$); this simple expedient automatically avoids pathological unfeasibility of the kind described above. Finally, note that (10) and (4) imply

\[
\Pi_{ij} = 0 \quad \forall j = 1, \ldots, N, \quad i \neq j.
\]
boundedness of $C_j$ if (1) is autonomous (i.e., $m = 0$), but not in general.

To impose (8) for all systems in the compatibility set, we use the S-procedure [16]. First, we need to “dualize” (8) to have $A_i, B_i$ as factors on the left, and their transpose on the right, as in (3). By left- and right-multiplying both sides of (8) by $P_i = P_i^{-1} > 0$, we obtain the equivalent inequality

\[
\begin{bmatrix}
I^T & A_i^T \\
B_i^T & P_i 
\end{bmatrix} = 0. 
\]

Then the S-lemma (cf. Lemma 1) shows that (11) holds for all $(A_i, B_i)$ satisfying (3) if there is a scalar $\alpha_i \geq 0$ such that

\[
G_i - \alpha_i H_i \succ 0; 
\]

It is necessary also holds under the mild Slater’s condition

\[
\exists \tilde{z}_i \in \mathbb{R}^{n+m} \times n \text{ s.t. } [I \tilde{Z}_i^T]^T H_i [I \tilde{Z}_i^T]^T > 0. 
\]

Note that dualization generates an additional nonlinearity in the top-left corner of (11), since $G_i$, $P_i$, and $P_j$ and $\pi_{i,j}$ are all variables; we cope with this complication using the Schur complement. The following is our main result of this section; let us define, throughout the paper,

\[
P_{-i} := \text{diag}((P_i)_{i \in \mathcal{I} \setminus \{i\}}), \quad \pi_{-i} := \text{diag}((\pi_{i,j} I_n)_{i \in \mathcal{I} \setminus \{i\}}). \tag{14a}
\]

\[
\pi_{-i} := \text{diag}((\pi_{i,j} I_n)_{i \in \mathcal{I} \setminus \{i\}}). \tag{14b}
\]

**Theorem 1 (Data-driven LM inequalities):** For all $i \in \mathcal{I}$, let the Slater’s condition in (13) hold. Then, \{\{P_i \succ 0, K_i\}_{i \in \mathcal{I}}, \Pi \in \mathcal{M}_+, Q \succ 0\} solve Problem 2 if and only if there exist scalars \{\alpha_i \geq 0\}_{i \in \mathcal{I}} such that $(P_i, L_i := K_i P_i)_{i \in \mathcal{I}}, Q, \{\alpha_i\}_{i \in \mathcal{I}}$ verify the inequality (C1), for all $i \in \mathcal{I}$.

**Proof:** By taking the Schur complement with respect to the 2-by-2 block upper-left matrix in (C1), and by definition of $L_i$, we retrieve (12).

Sufficiency in Theorem 1 holds also without the assumption in (13). Therefore, by recalling that the unknown real system belongs to the consistency set, we have the following.

**Corollary 1 (Data-driven stabilization):** Assume that \{\{P_i \succ 0\}_{i \in \mathcal{I}}, \{\alpha_i \geq 0\}_{i \in \mathcal{I}}, \{L_i \}_{i \in \mathcal{I}}, Q \succ 0, \Pi \in \mathcal{M}_+\} satisfy (C1), for all $i \in \mathcal{I}$. Then the controller (7) with $K_i = L_i P_i$ globally asymptotically stabilizes the switched system in (1). Furthermore, (9) holds true.

**Remark 3 (Nonnegative weight matrix):** Using (8) in place of (6) is crucial for Theorem 1 as it allows inversion of $\pi_{-i}$ and $Q$. However, if $\Pi$ is given and known a priori, it is easy to accommodate for $\Pi \in \mathcal{M}$ instead of $\Pi \in \mathcal{M}_+$ (i.e., $\pi_{i,j}$ might be zero – note that stability of the closed-loop is not affected; this case will be useful in the following, see Examples 1 and 2); it is enough to remove in $\pi_{-i}, P_{-i}$ any block $j$ such that $\pi_{i,j} = 0$ and cancel the corresponding rows and columns in (C1).

**Remark 4 (Batch sizes):** The number of variables and the dimension in (C1) grows with $N$, but do not depend on the dataset lengths $\{T_i\}_{i \in \mathcal{I}}$, a convenient feature inherited from [16].

**Remark 5 (Linear convergence):** The proof of Proposition A1 in Appendix I shows that stability in Corollary 1 is exponential, with rate $\gamma_{\min}(Q)/\max_{i \in \mathcal{I}}(\gamma_{\max}(P_i))$.

The inequalities (C1) depend on experimental data and prior on the disturbance only; hence, their solution provides a direct data-driven criterion to seek a controller that stabilizes (1). Note that the problem is nonconvex; it can be cast as a BMI in the variables $\{P_i\}_{i \in \mathcal{I}}, Q, \{L_i\}_{i \in \mathcal{I}}, \Pi, \{\pi_{i,j}\}_{i,j \in \mathcal{I}}$, via the additional constraints

\[
\forall i \in \mathcal{I}, \forall j \in \mathcal{I} \setminus \{i\}, \quad \pi_{i,j} \pi_{j,i} = 1, \tag{15}
\]

which is necessary to enforce $\Pi \in \mathcal{M}_+$. Due to their ubiquity in control systems, significant effort has been devoted to the development of algorithms [40] and software [41] for the solution of BMIs, typically resorting to sequential convex (often, SDP) relaxations. These tools can be readily used to solve (C1).

Nevertheless, the problem remains computationally expensive and poorly scalable. Let us recall that the model-based condition in [32] (see Proposition A1) is also a BMI; more recently, bilinear conditions have been also proposed in the context of data-driven control, e.g., [15], [21], [39]. All these formulations suffer the same limitations.

To address this issue, in Section IV, we propose two relaxations of (C1) that, at the price of some conservatism, result in substantial complexity reduction. However, first let us note that, if $\Pi$ is fixed, then (C1) reduces to an LMI, which is convex and hence can be efficiently solved. This case is still relevant, as exemplified next.

**Example 1 (LTI systems):** Consider the system $\dot{x} = Ax + Bu$ (corresponding to (1) for $N = 1$), set $\Pi = 0$ (justified by Remark 3), and note that (6) recovers the standard Lyapunov inequality. Then, under (13), Theorem 1 proves that (C1) is equivalent to

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} = 0. \tag{16}
\]

2These additional nonconvex constraints can be avoided via auxiliary variables $\{Y_i\}_{i \in \mathcal{I}} \in \mathbb{R}^{n \times n}$ subject to $Y_i \succ P_i P_i - P_i$ (which are LMIs after application of the Schur complement), and by replacing the term $-\sum_{i,j \in \mathcal{I}}(\pi_{i,j})^2 Y_{i,j}$ in the top-left block of (11) by $-\sum_{i,j \in \mathcal{I}}(\pi_{i,j})^2 Y_{i,j}$, which leads to an equivalent formulation. We do not introduce this (large number of) auxiliary variables in view of the relaxations in the next section, which do not require (15).
necessary and sufficient for quadratic stabilization of all the systems in \( \mathcal{C} \). Differently from [16, Th. 14] (formulated in discrete-time), we used the nonstrict version of the S-lemma in our derivation, as Lemma 2 already bounds (6) away from zero over \( \mathcal{C} \) at a solution –in fact, \( Q \succ 0 \) can be used to certify the worst-case convergence rate. \( \square \)

Example 2 (Markov jump linear systems): Consider the system \( \dot{x} = (A_s + B_s K_s)x \), where \( \sigma(t) \) is an (uncontrolled, but measured) continuous-time Markov chain with known infinitesimal transition matrix \( \Pi^T \). The system is stochastically stable (i.e., \( \int_0^\infty (E(\|x\|^2)) dt < \infty \) for any initial condition) if and only if (6) admits a solution [42, Th. 3.25]. In turn, the system of LMIs (C1) provides nonconservative conditions for stabilization if the system matrices are unknown, but open-loop experiments have been recorded. A numerical example is in Section VIII-A. \( \square \)

IV. RELAXATIONS VIA STRUCTURED WEIGHT MATRIX

In this section, we provide two conditions, in the form of matrix inequalities, that are sufficient for the satisfaction of (C1): the first is still a BMI, but of reduced dimension; the second is more conservative, but only requires solving an LMI with line-search on a scalar parameter. Both are obtained by assuming some extra structure on the weight matrix \( \Pi \).

A. BMI of reduced order

Let us impose, for all \( i \in \mathcal{I} \), that

\[
\pi_{j,i} = \gamma_{i}, \quad \forall j \neq i; \quad \pi_{i,i} = -(N-1)\gamma_{i},
\]

where \( \gamma_{i} > 0 \) is a variable to be determined. The rationale is that, in the known-model case, [32, Eq. 39] ensures that the LM inequalities in (6) admit a solution satisfying (16) if there is a Hurwitz convex combination of the matrices \( A_i^k \)'s.

Interestingly, although the same argument does not carry on when dealing with sets of possible systems, an analogous result holds for the data-driven case if the convex combination of the compatibility sets is quadratically stabilizable (i.e., with common quadratic Lyapunov function). The condition \( \lambda_{i} > 0 \) in the next result is actually nonrestrictive (similarly to Lemma 2).

Lemma 3 (A condition for solvability): Assume that there exist \( \lambda := (\lambda_{i})_{i \in \mathcal{I}} \in \Lambda_{+}, \{K_{i}\}_{i \in \mathcal{I}}, P \succ 0 \) such that

\[
(\sum_{i \in \mathcal{I}} \lambda_{i}A_{i}^{1})^{T}P + P(\sum_{i \in \mathcal{I}} \lambda_{i}A_{i}^{1}) \prec 0
\]

for all \((A_{i}, B_{i})_{i \in \mathcal{I}} \in \mathcal{C}\). Then, (8) is solvable with \( \Pi \) as in (16), with \( \gamma_{i} = \mu \lambda_{i}^{-1} \) and large enough \( \mu > 0 \). \( \square \)

The proof is given in Appendix II-B.

For a more efficient solution under (16), we replace the variable \( Q \succ 0 \) in (8) with \( \gamma_{i}Q, \ Q \succ 0 \), and divide both sides by \( \gamma_{i} \); hence we obtain: for all \( i \in \mathcal{I} \), for all \( (A_{i}, B_{i})_{i \in \mathcal{I}} \in \mathcal{C}_{i} \),

\[
A_{i}^{1T}(\gamma_{i}P_{i}) + (\gamma_{i}P_{i})A_{i}^{1} + \sum_{j \in \mathcal{I}\setminus\{i\}}(P_{j} - P_{i}) + Q \preceq 0.
\]

(18)

Note that this is not restrictive, in terms of feasibility: a solution to (18) also provides a solution to (8) (with \( Q = Q \min_{i \in \mathcal{I}} \gamma_{i} \)) and vice versa (in fact, one could even fix \( Q = I \) or \( Q = I \) because of homogeneity).\( \square \)

Remark 6 (Comparison with classical conditions): In the model-based case, assuming for simplicity that \( m = 0 \), the nonlinearity in (18) arises only via the product of the variables \( \gamma_{i} \) (scalar) and \( P_{i} \), for each \( i \in \mathcal{I} \). Similarly, the inequality (17) is bilinear in \( \lambda_{i} \)'s and \( P_{i} \); it is known [25] that, if a solution exists, then the controller

\[
\sigma(x) = \arg \min_{i \in \mathcal{I}} x^{T}A_{i}^{T}PA_{i}x
\]

stabilizes (1). Let us remark some advantages of (18) with controller (7) over (17) with controller (19). First, existence of a Hurwitz convex combination is sufficient, but not necessary, for solvability of (18). Second, in the data-driven case, it is prohibitive to check (17) –in a non-conservative way– when \( \mathcal{C} \) is not a singleton. Technically speaking, we would need to deal with the convex combination of the ellipsoids \( C_{i} \)'s, which is not an ellipsoid. Third and most important, the controller (19) cannot be implemented (in contrast to (7)), because the true system matrices are not known. \( \square \)

By repeating the derivation in Theorem 1, we can show that, under the Slater’s condition (13) and via the changes of variable \( L_{i} = \tilde{\gamma}_{i}K_{i}P_{i} \), (18) is equivalent to (C2) on the next page.

Theorem 2 (Relaxed data-driven stabilization): Assume the convex combination of the ellipsoids \( C_{i} \)'s, which is not an ellipsoid. Third and most important, the controller (19) cannot be implemented (in contrast to (7)), because the true system matrices are not known. \( \square \)

By repeating the derivation in Theorem 1, we can show that, under the Slater’s condition (13) and via the changes of variable \( L_{i} = \tilde{\gamma}_{i}K_{i}P_{i} \), (18) is equivalent to (C2) on the next page.

Theorem 2 (Relaxed data-driven stabilization): Assume that \( \{P_{i} > 0, \alpha_{i} \geq 0, \tilde{\gamma}_{i} > 0, L_{i} \}_{i \in \mathcal{I}}, Q \succ 0 \) satisfy (C2), for all \( i \in \mathcal{I} \). Then the controller in (7), with \( K_{i} = \gamma_{i}L_{i}P_{i} \), globally asymptotically stabilizes (1). Furthermore, (9) holds with \( Q = \tilde{Q} \min_{i \in \mathcal{I}} \gamma_{i} \).

Although this relaxation is still a BMI, (C2) is computationally much easier to solve than (C1), where the bilinearity involves significantly many more variables. For instance, (C2) could be solved via SDP and line-search over \( N \) scalar variables instead of \( N(N - 1) \). As another example, let \( M = \text{diag}((M_{i})_{i \in \mathcal{I}}) \),

\[
M_{i} := \begin{bmatrix} \tilde{\gamma}_{i} \text{vec}(P_{i})^{T} & 1 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{i} \text{vec}(P_{i})^{T} \end{bmatrix},
\]

and note that (18) is an LMI in the elements of \( M \) and the variables \( \{L_{i}\}_{i \in \mathcal{I}}, Q, \{\alpha_{i}\}_{i \in \mathcal{I}} \). Therefore, the problem can be recast as the following rank constrained LMI: find \( Q, \{L_{i}\}_{i \in \mathcal{I}}, \{\alpha_{i}\}_{i \in \mathcal{I}}, M \) such that:

\[
\begin{cases}
M \succ 0, \quad \text{rank}(M) = N, \\
\forall i \in \mathcal{I}: \quad [M_{i}]_{1,1} = 1, \quad (C2), \quad P_{i} > 0, \quad \tilde{\gamma}_{i} > 0
\end{cases}
\]

(21)

where the constraints on \( M \) enforce the structure in (20). This class of problems are for instance solved via recursive algorithms, where each iteration requires solving an SDP in all the variables (see [43] for an overview on the topic; we use these techniques in Section VIII-B.2). In the case of (C2), the square matrix \( M \) has \( O(Nn^4) \) nonzero entries (variables); the corresponding number for the case of (C1) is \( O(N^3n^4) \) (and additional constraints are required).

B. LMI with line-search

Let us discuss a more conservative choice for the weight matrix \( \Pi \). We consider again the structure in (16), but we further assume the parameters \( \gamma_{i} \)'s to be identical, so that

\[
\Pi = \gamma(I_{N}1_{N}^{T} - NI_{N}),
\]

(22)
for some scalar variable \(\gamma > 0\). The advantage is computational: for \(\gamma\) fixed, (C1) (or (C2)) reduces to an LMI. Thus, a solution can be sought via standard convex solvers and line-search over one scalar variable \(\gamma\).

Lemma 3 provides a sufficient condition for existence of a solution, i.e., quadratic stabilizability of the “average” compatibility set (once again, otherwise dealing with this set is not trivial). If a solution is found, then Theorem 2 specifies how to build a stabilizing controller.

Remark 7 (Comparison with the literature): [32, Th. 4] (see also [35], [44]) proposes the following relaxation to the LM inequalities (6):

\[
A_i^T P_j + P_j A_i + \gamma (P_j - P_i) + Q < 0, \quad j \neq i \in \mathcal{I},
\]

with \(\gamma > 0\). These conditions are also LMIs for fixed \(\gamma\). For any solution \(\{P_i, K_i\}_{i \in \mathcal{I}}, Q, \gamma\) of (23) and for any matrix \(\Pi \in \mathcal{M}\) such that \(\pi_{i,j} = -((N - 1)\gamma)\) (including (22)), multiplying both sides of (23) by \(\pi_{j,i}/\gamma\) and summing over \(j\) shows that (23) implies \(A_i^T P_j + P_j A_i + \sum_{j \in \mathcal{I}} \pi_{j,i} P_j + Q < 0\), i.e., (6). From this, we conclude that fixing (22) in (6) is (strictly, see an example in Section VIII-B.1) less restrictive than (23) – as computationally convenient: the number of LMIs in (23) is quadratic in \(N\).

To conclude this section, we note that optimizing some performance metric over the solutions of LM inequalities is particularly simple under (16). For instance

\[
\min_{\{\dot{P}_i > 0, \alpha_i \geq 0, L_i \in \mathcal{L}, \dot{\varrho} > 0, \Pi_i \in \mathcal{M}_+\}} - \sum_{i \in \mathcal{I}} \tr(\dot{P}_i)
\]

s.t. (C1), \(\tr(\dot{\varrho}) = 1\) (24)

promotes matrices \(P_i\)’s with eigenvalues far from 0 (once the homogeneous inequality in (C2) is normalized via the trace constraint), see (9). Under (16) and for fixed \(\gamma > 0\), (24) is an SDP.

V. PERFORMANCE SPECIFICATIONS

In this section we study quantitative performance specifications. We consider a model like (1) with disturbances in the dynamics and outputs,

\[
\dot{x} = A_ix + B_iu + E_{i}\varphi
\]

\[
z = C_{i}x + D_{i}u + F_{i}\varphi,
\]

where \(\varphi(t) \in \mathbb{R}^{q}\) is an exogenous disturbance, \(z \in \mathbb{R}^{p}\) is a performance output, the matrices \(\{C_{i}\}_{i \in \mathcal{I}}\) and \(\{D_{i}\}_{i \in \mathcal{I}}\) are chosen by the designer, and the known matrices \(\{E_{i}\}_{i \in \mathcal{I}}\) and \(\{F_{i}\}_{i \in \mathcal{I}}\) measure the influence of the disturbance signal on the state evolution and output, respectively. Furthermore, the only information available on the matrices \(\{A_{i}, B_{i}\}_{i \in \mathcal{I}}\) is a set of experiments satisfying (2).

\[\text{Remark 8 (Refined disturbance model): We can regard (2) as generated by (25a), where the quantities } \tilde{W}_{i} \text{ in (2) are due to the unknown process disturbance } \varphi \text{ in (25a): in short, } w(t) = E_{i}\varphi(t). \text{ In particular, if } \tilde{\Psi} := [\varphi(t_1) \ldots \varphi(t_T)] \text{ satisfies}
\]

\[
\begin{pmatrix}
I & \tilde{\Phi}_{1,1} & \tilde{\Phi}_{1,2} \\
\tilde{\Phi}_{1,1}^T & \tilde{\Phi}_{2,2} & \tilde{\Phi}_{2,2} \\
\tilde{\Phi}_{1,2} & \tilde{\Phi}_{2,2} & 0
\end{pmatrix} \geq 0,
\]

with \(\tilde{\Phi}_{2,2} < 0\) (e.g., an energy-type prior on the process noise \(\varphi\)), then Assumption 1 is satisfied with \(\Phi_{1,1} = E_{i}\hat{\Phi}_{1,1}E_{i}^T\), \(\Phi_{1,2} = E_{i}\hat{\Phi}_{1,1}\), \(\Phi_{2,2} = E_{i}\hat{\Phi}_{2,2}\) (see [16, Rem. 2]), which allows us to incorporate the prior knowledge on the matrix \(E_{i}\). Additional measurement noise/error on the estimate of \(\dot{x}\) could also be taken into account by properly choosing \(\Phi_{i}\). \(\Box\)

We address \(\mathcal{H}_2\) and \(\mathcal{H}_{\infty}\) stabilization problems for (25). We refer to Appendix I-B for a formal description of the performance indices \(J_2\) and \(J_{\infty}\); analogously to the LTI case, these can be defined in terms of the system response to impulsive and integrable disturbances, respectively. The problems are well known to be hard to solve for switched linear systems, even when a model of the system is perfectly known [44]-[46]. The additional complication of our data-driven formulation is that we need to ensure closed-loop performance for all the systems in the compatibility set \(C\). For the continuous input, we again only focus on linear switched controllers \(u = K_{\pi}x\).

A. Data-driven \(\mathcal{H}_2\) control

Following Proposition A2 in Appendix Section I, to design a controller with guaranteed \(\mathcal{H}_2\) performance, we impose: for all \(i\), for all \(\{A_i, B_i\} \in C_i\),

\[
A_i^T P_i + P_i A_i + \sum_{j \in \mathcal{I}} \pi_{i,j} P_j + C_i^T C_i < 0,
\]

where \(C_i^T := C_i + D_i K_i\). Ideally, we should introduce no additional nonlinearity with respect to (C1) (we recall that \(K_i\) is a variable, and further the matrix \(C_i^T C_i\) might be singular, thus not positive definite). This goal is achieved in the following result.

\[\text{Theorem 3 (Data-driven } \mathcal{H}_2 \text{ stabilization): Let } \{\{P_i > 0\}, \{\alpha_i \geq 0\}_{i \in \mathcal{I}}, \{L_i\}_{i \in \mathcal{I}}, \{Y_i > 0\}_{i \in \mathcal{I}}, \Pi \in \mathcal{M}_+\} \text{ satisfy } (\mathcal{H}_2), \text{ where the matrix } \Phi_i \text{ is given in Assumption 1, for all } i \in \mathcal{I}. \text{ Then the controller in (7) with } K_i = L_i P_i \text{ globally asymptotically stabilizes the switched system in (1). Furthermore, the closed-loop system satisfies}
\]

\[
J_2(s, u) < \min_{i \in \mathcal{I}} \tr(E_i^T P_i E_i).
\]

\[\Box\]

\[\text{Proof: By taking the Schur complement with respect to the upper-left block and by Lemma 1, we deduce that } (\mathcal{H}_2)\]
defines (actually, it is equivalent to if (13) holds) for all \( i \in \mathcal{I} \), for all \( (A_i, B_i) \in \mathcal{C}_i \),

\[
-P_i A_i^T \preceq A_i^T P_i - \sum_{j \in \mathcal{I}} \pi_{ij} P_j P_i - Y_i \succeq 0, \quad (27a)
\]

\[
Y_i > (C_i \tilde{P}_i + D_i L_i)^T (C_i \tilde{P}_i + D_i L_i) \quad (27b)
\]

where \( K_i := L_i P_i \). Multiplying both sides of the previous inequalities by \( P_i \), we retrieve (26), because \( C_i \tilde{P}_i + D_i L_i = C_i^d \tilde{P}_i \). The conclusion follows by Proposition A2.

**B. Data-driven \( \mathcal{H}_\infty \) control**

The following is the data-driven counterpart of Proposition A3 in Appendix I.

**Theorem 4 (Data-driven \( \mathcal{H}_\infty \) stabilization):** Let \( \{ (P_i > 0), \{ \lambda_i \geq 0 \} \}_{i \in \mathcal{I}}, \{ L_i \}_{i \in \mathcal{I}}, \{ Y_i > 0 \} \), \( \Pi \in \mathcal{M}_+, \rho > 0 \) satisfy (\( \mathcal{H}_\infty \)), where the matrix \( \Phi_{i} \) is given in Assumption 1, for all \( i \in \mathcal{I} \). Then the controller in (7) with \( K_i = L_i P_i \) globally asymptotically stabilizes the switched system in (1). Furthermore, the closed-loop systems satisfies

\[
J_{\infty}(\sigma, u) < \rho.
\]

**Proof:** By applying the Schur complement with respect to the two bottom-right blocks, the second inequality of (\( \mathcal{H}_\infty \)) is equivalent to

\[
Y_i \succ \begin{bmatrix} E_i^T & -F_i^T \end{bmatrix} \begin{bmatrix} P_i & I \\ C_i^d \tilde{P}_i \end{bmatrix}^{-1} \begin{bmatrix} E_i^T \\ C_i^d \tilde{P}_i \end{bmatrix}^T
\]

where we used the definition \( C_i \tilde{P}_i + D_i L_i = C_i^d \tilde{P}_i \). By replacing the previous inequality in (27a), by multiplying both sides by \( P_i \), and by a Schur complement argument, we obtain: for all \( i \in \mathcal{I} \), for all \( (A_i, B_i) \in \mathcal{C}_i \),

\[
A_i^T P_i + P_i A_i^T + \sum_{j \in \mathcal{I}} \pi_{ij} P_j P_i E_i \succ 0.
\]

The conclusion follows by Proposition A3.

The relaxations in Section IV can also be applied to reduce the computational cost in (\( \mathcal{H}_2 \)) and (\( \mathcal{H}_\infty \)). Let us note that, for a fixed matrix \( \Pi \), optimizing the cost \( \min_{i \in \mathcal{I}} \text{tr}(E_i^T P_i E_i) \) over the solutions of (\( \mathcal{H}_2 \)) reduces to \( N \) SDP problems (where the cost in the \( i \)-th problem is \( \text{tr}(E_i^T P_i E_i) \)). Instead the upper bound \( \rho \) in Theorem 4 is conveniently linear, thus its minimization corresponds to one SDP.

**VI. DISCRETE-TIME SWITCHED LINEAR SYSTEMS**

Data-driven stabilization of discrete-time switched linear systems via LM inequalities presents some additional technical difficulties, which we address in this section. We consider the (nominal) plant

\[
x(k+1) = A_s x(k) + B_s u(k),
\]

where \( \sigma \in \mathcal{I} \) is the switching signal, and \( \{ A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m} \}_{i \in \mathcal{I}} \) are the unknown true system matrices. Analogously to the continuous-time case, we assume that for each mode \( i \in \mathcal{I} \), \( T_i > 0 \) samples have been collected for each subsystem at sampling times \( \{ t_s^i \}_{s=1,2,...,T_i} \subset \mathbb{N} \), satisfying

\[
X_i^+ := \tilde{A}_i X_i + \tilde{B}_i U_i + \tilde{W}_i,
\]

and the other quantities in (28) are defined as in (2), with \( w \) being an unknown process noise satisfying Assumption 1. Note that, for each mode, this setup retrieves exactly that considered for LTI systems, e.g., in [1], [16].

Our first result provides a data-driven version of the LM inequalities in [32, Lemma 2], which read: for all \( i \in \mathcal{I} \),

\[
A_i^d^T \left( \sum_{j \in \mathcal{I}} \lambda_{ij} P_j \right) A_i^d - P_i + Q \lesssim 0,
\]

where \( \{ P_i > 0 \}_{i \in \mathcal{I}}, \Lambda = [\lambda_{ij}]_{i,j \in \mathcal{I}} \in \mathbb{S}, Q \succeq 0 \) and \( A_i^d := A_i + B_i K_i \). We wish to impose these conditions for all the matrices compatible with the experiments. Similarly to Lemma 2, we can consider without loss of generality the variant: for all \( i \in \mathcal{I} \), for all \( (A_i, B_i) \in \mathcal{C}_i \),

\[
A_i^d^T \left( \sum_{j \in \mathcal{I}} \lambda_{ij} P_j \right) A_i^d - P_i + Q \preceq 0,
\]

with \( \{ P_i > 0 \}_{i \in \mathcal{I}}, \quad Q > 0 \) and \( \Lambda \in \mathbb{S}, \quad P_i - Q \succeq 0 \).

Note that in (31), the quantity \( A_i^d^T \) is a factor on the left, which prevents us from directly applying the S-lemma. A standard dualization argument (i.e., double application of the Schur complement) gives the equivalent inequality

\[
\left( \sum_{j \in \mathcal{I}} \lambda_{ij} P_j \right)^{-1} - A_i^d^T (P_i - Q)^{-1} A_i^d^T \succeq 0.
\]

Differently from the continuous-time case, the first term of (32) is the inverse of the sum of variables, a nonlinearity that cannot be dealt with by using the Schur complement. To cope
with this complication, we leverage the Woodbury identity, which we hereby recall:

\[(D + UCV)^{-1} = \tilde{D} - \tilde{D}U(\tilde{C} + V\tilde{D})^{-1}V\tilde{D},\]  

(33)

for any suitably invertible and sized matrices \(C, D, U, V\). The following result provides a DT counterpart of Theorem 1 and it is the main result of this section.

**Theorem 5 (DT data-driven LM inequalities)**: Assume that the Slater’s condition in (13) holds, and let \(P_i := \text{diag}((P_i)_{j\in\mathcal{I}})\), \(\Lambda_i := \text{diag}(\{\lambda_{j,i}I\}_{j\in\mathcal{I}})\), for all \(i \in \mathcal{I}\). Then, \(((P_i \succ 0)_{i \in \mathcal{I}}, \{K_i\}_{i \in \mathcal{I}}, \Lambda \in \mathcal{S}_+; Q > 0\) solve (32) if and only if there exist \(\{\alpha_i \geq 0, Y_i > 0\}_{i \in \mathcal{I}}\) such that \(((P_i, Y_i, L_i := K_iY_i, \alpha_i)_{i \in \mathcal{I}})\in\mathcal{L}(\Lambda, Q)\) verify the inequality (C3), where \(\Phi_i\) is as in Assumption 1, for all \(i \in \mathcal{I}\). □

**Proof**: By applying the Schur complement with respect to the top-left block in the first inequality of (C3), we get a (4-by-4 block) inequality, where the component of the top-right block derived by the matrix \(i\) (i.e., excluding the matrix multiplied by \(\alpha_i\) in (C3)) is, with \(* = \tilde{\lambda}_{i,i}P_i\) for brevity,

\[* = *(1 \otimes I)^T (\tilde{A}_{-i}P_i + (1 \otimes I) \ast (1 \otimes I)^T)^{-1} (1 \otimes I)\ast = (\tilde{\lambda}_{i,i}P_i + (1 \otimes I)^T \pi_{-i}P_i(1 \otimes I))^{-1} = (\sum_{j\in\mathcal{I}} \lambda_{j,i}P_j)^{-1}\]

where the first equality is (33). Subsequently, by applying the Schur complement to the bottom right block we get a (3-by-3 block) inequality where the component of the bottom-right block derived by the first matrix is \(-L_iY_iL_i^T\). Then, by Lemma 1, the first inequality of (C3) is equivalent to: for all \((A_i, B_i) \in C_i\)

\[
\begin{bmatrix}
I
0
0
\end{bmatrix}^T \begin{bmatrix}
\sum_{j\in\mathcal{I}} \lambda_{j,i}P_j
0
0
\end{bmatrix}^{-1} \begin{bmatrix}
0
0
\end{bmatrix} \begin{bmatrix}
0
0
0
\end{bmatrix} \begin{bmatrix}
I
A_i^T
B_i^T
\end{bmatrix} \succ 0.
\]

(34)

The second inequality in (C3) forces \(P_i - Q > 0\), while the third, by Schur complement, is

\[Y_i \succ \tilde{P}_i - \tilde{P}_i(\tilde{P}_i - \tilde{Q})^{-1}\tilde{P}_i = (P_i - Q)^{-1},\]

(35)

where the equality is again (33). Together with (34) and by definition of \(L_i\), this implies (32); conversely, if (32) holds, there exists \(\alpha_i\) that satisfies (C3) with \(Y_i = (P_i - Q)^{-1}\). □

As an immediate consequence of Theorem 5 and [33, Lem. 2], we have the following result.

**Corollary 2 (DT stabilization)**: Assume that \(((P_i \succ 0, \alpha_i \geq 0, L_i, Y_i > 0, \Lambda, Q \succ 0, \Lambda \in \mathcal{S}_+)\) solve (C3) for all \(i \in \mathcal{I}\). Then the controller (7) with \(K_i = L_i\tilde{Y}_i\) globally asymptotically stabilizes the discrete-time switched system in (28). Furthermore, it holds that

\[
\sum_{k=0}^{\infty} x(k)^T Q x(k) \leq \min_{i \in \mathcal{I}} x(0)^T P_i x(0).
\]

Casting (C3) as a BMI requires some auxiliary variables and constraints to impose \(\Lambda \in \mathcal{S}_+\), analogously to (1)-(21). Nonetheless, as in (C2), this complication can be avoided when employing a relaxation analogous to (16) (i.e., imposing \(\lambda_{j,i} = \gamma_i\), for all \(j \neq i\) and some \(0 < \gamma_i < \frac{1}{\lambda_i}\)). Moreover, (C3) is an LMI when \(\Lambda \) is fixed; this is relevant, for instance because existence of a solution to (30) is equivalent to stability of a DT Markov jump linear system with transition matrix \(\Lambda^0\) (and system matrices \(\Lambda^0\)), cf. Example 2.

Finally, similar techniques to Theorem 5 can be leveraged to address data-driven \(H_2\) and \(H_\infty\) problems for DT switched linear systems, for example based on the results in [47].

**VII. SWITCHED COMPENSATORS FOR ROBUST CONSTRAINED STABILIZATION OF LTI SYSTEMS**

In this section, we build upon the design ideas, developed above for switched systems, to design a switched controller for a fixed linear plant. In particular, we study a robust constrained control problem, motivated by applications where it is paramount to keep a plant in safe operating conditions, despite the presence of disturbances. Let us consider the perturbed LTI system

\[
\dot{x} = \tilde{A}x + \tilde{B}u + E\psi,
\]

(36)

\(x \in \mathbb{R}^n, u \in \mathbb{R}^m\), where the (persistent) disturbance \(\psi(t) \in \mathbb{R}^q\) satisfies \(\psi(t)^T \psi(t) \leq 1\) for all times, subject to polyhedral state constraints described by \(x \in X\),

\[X := \{x \mid c_j^T x \leq 1, \forall j \in \mathcal{J} := \{1, 2, \ldots, M\}\},
\]

(37)

for some \(c_j \in \mathbb{R}^n\). One fundamental challenge is to find an (as large as possible) set \(X_0 \subseteq X\), together with a feedback controller that makes the set \(X_0\) invariant for the closed-loop dynamics (36) for any possible \(\psi\) and ensures asymptotic stability for the nominal system in the absence of disturbance. Here we follow an approach by [48], that leverages the Lyapunov function \(v(x) = \max \{x^T P_0 x, x^T c_1 c_1^T x, \ldots, x^T c_M c_M^T x\}\) (for some \(P_0 > 0\)) for the estimation of maximal invariant sets. We depart from [48] by designing a switched controller with guaranteed invariance region, directly from data and without requiring identification of the system (36).

In particular, we assume that the matrix \(E \in \mathbb{R}^{n \times q}\) is known, but the only information available on the matrices
(\hat{A}, \hat{B}) is a set of data satisfying (2). Let us start by defining
\[ P_j := c_j c_j^\top + \nu I \succ 0, \quad (\forall j \in J) \] (38)
where \( \nu > 0 \) is a fixed design constant, and
\[ v_{\text{max}}(x) := \max \{ x^\top P_j x \mid j \in J_0 := J \cup \{0\} \}, \] (39)
where \( P_0 > 0 \) is to be designed. Furthermore let \( X_0 \) be the 1-sublevel set of \( v_{\text{max}} \), i.e.,
\[ X_0 := \{ x \in \mathbb{R}^n \mid v_{\text{max}}(x) \leq 1 \} \subset X. \] (40)

**Theorem 6 (Data-driven safe stabilization):** Let \( \{P_j \succ 0, L_j, \gamma_j \geq 0, \alpha_j \geq 0\}_{j \in J_0}, \Pi = [\pi_{j,k}]_{j,k \in J_0} \in M_+ \) satisfy (C4), for all \( j \in J_0 \). Consider the controller
\[ \sigma(x) = \arg\max_{j \in J_0} x^\top P_j x \] (41a)
and
\[ u = K_\sigma x, \] (41b)
where \( K_j := L_j P_j \) (and any selection rule when the argmax is not unique). Then, any Carathéodory solution \( x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) of the closed-loop system (36), (41), with \( x(0) \in X_0 \), satisfies \( x(t) \in X_0 \) for all \( t \geq 0 \). Furthermore, if \( \beta_j > 0 \) for all \( j \in J_0 \) and \( \psi(t) \to 0 \) for \( t \to \infty \), then also \( x(t) \to 0 \).

**Proof:** By taking the Schur complement with respect to the top-left block and by Lemma 1, we deduce that (C4) implies: for all \( j \in J_0 \), for all \((A, B) \in C, \)
\[ -\hat{P}_j A_j^\top - A_j^\top \hat{P}_j - \beta_j \hat{P}_j E E^\top + \sum_{k \in J_0} \pi_{j,k} \hat{P}_k \hat{P}_j \succ 0, \]
where \( A_j^\top := A + BK_j \). Multiplying both sides by \( P_j \), we obtain via Schur complement that
\[ \begin{bmatrix} P_j + P_j A_j^\top - P_j - \sum_{k \in J_0} \pi_{j,k} (P_k - P_j) & * \\ E^\top P_j & -\beta_j I \end{bmatrix} \preceq 0 \] (42)
for all \( j \in J_0 \). The proof then follows [48, Prop. 1] and it is omitted.

Some remarks on the conditions in (C4) are necessary. First, the matrices \( \{P_j\}_{j \in J} \) are fixed a priori—they are not variables. We can reformulate (C4) as a BMI
\[ \{ \beta_j, L_j, \alpha_j \}_{j \in J_0}, \Pi, \tilde{P}_0, P_0, \tilde{Y}_0, Y_0 \], where the auxiliary positive definite variables \( P_0, Y_0, \tilde{Y}_0 \in \mathbb{R}^{n \times n} \) satisfy
\[ P_0 \tilde{P}_0 = I, \quad Y_0 \tilde{Y}_0 = I, \quad \begin{bmatrix} \tilde{Y}_0 & P_0 \end{bmatrix} \begin{bmatrix} \sum_{k \neq 0} \pi_{k,0} P_k \end{bmatrix} \geq 0; \]
the last inequality equals \( Y_0 \geq \tilde{P}_j (\sum_{k \neq 0} P_k) \tilde{P}_j \), and provides an upper bound for the trilinear term in the block (2.2) in (C4) (only arising for \( j = 0 \)). This nonlinear term, due to dualization, is not present in the model-based formulation (42); it results in a nonconvex condition (so it cannot dealt with via LMIs) and it is in general impossible to avoid (because of tightness of the S-lemma); it does not arise in our data-driven formulations of the LM inequalities, where the coupling term has the opposite sign (cf. (42), (6)). This also means that (C4) is bilinear even when \( \Pi \) is fixed.

If we further impose the structure in (16) for the weight matrix \( \Pi \). (C4) simplifies as in (C5) (via changes of variable \( \beta_j = \gamma_j \beta_j \), \( Y_j = \gamma_j Y_j \), \( L_j = \gamma_j L_j \), \( \alpha_j = \gamma_j \alpha_j \)), which is already a BMI. Besides being easier to solve (see Section VIII-B), an advantage of (C5) is that the number of bilinear terms does not depend on the number of constraints \( M \).

**Remark 9 (Maximizing \( X_0 \)):** The volume of the set \( X_0 \) can be enlarged in different ways. For instance, we can fit inside \( x \in \mathbb{R}^n \) | \( x^\top Q x \leq 1 \) | of maximal volume (by imposing \( Q \) \( \geq 0 \) for all \( j \in J_0 \), and minimizing the convex cost \( -\log(\det(Q)) \), where \( Q \) is a new variable). Similarly, \( X_0 \) can be maximized with respect to a reference shape \( [48, \text{Eq. 20}] \). Note that in (38) we include a positive regularization weighted by \( \nu \), which allows for the inversion of \( P_j \). A smaller value of this parameter reduces conservatism and can result in a larger guaranteed invariant set \( X_0 \).

**Remark 10 (Input saturation):** Input constraints could be included in the design via additional sufficient LMIs. For instance, we can enforce \( \|u\|_{\infty} \leq 1 \) by imposing, for each \( j \in J_0 \), with \( L_{j,l} \) being the \( l \)-th row of \( L_j \),
\[ \forall l \in \{1, 2, \ldots, m\}, \quad \begin{bmatrix} 1 \\ L_{j,l} \end{bmatrix} \begin{bmatrix} L_{j,l}^\top \\ P_j \end{bmatrix} \geq 0; \] (43)
this ensures \( \|K_j x\|_{\infty} \leq 1 \) for any \( x \in \mathcal{E}_{P_j} := \{ x \mid x^\top P_j x \leq 1 \} \supseteq X_0 \). Interestingly, for \( j \neq 0 \), this condition is more restrictive (and possibly conservative) if \( \nu \) is small.

**Remark 11 (On chattering and alternative approaches):** While the results in Section III-Section VI are based on the Lyapunov function \( v_{\text{min}} \) in (54) (Appendix I), Theorem 6

---

3Since \( N = 1 \), we omit the subscript \( i = 1 \) in this section, see also (C4).
latures \( v_{\text{max}} \) in (39). An important difference, already noted in [25, p. 71], is that the switching rule (41a) cannot guarantee invariance or stability if sliding mode occurs—in fact, only Carathéodory solutions are considered in Theorem 6. This is not a problem when \( m = 0 \), i.e., the goal is simply to estimate a maximal invariant set, in the spirit of [48]. Otherwise, chattering could be avoided by employing a linear controller—but with some disadvantages. First, imposing \( K_j = K \) for all \( j \)'s in (C4) results in nonlinear constraints (efficiently enforcing this condition is an interesting topic for future research). Second, a switched controller is more powerful than a linear gain in achieving invariance. Alternatively, [49, Th. 6] constructs a nonlinear (continuous) controller, by solving the following BMIs: find \( \{ Q_j < 0 \}_{j \in J}, \beta > 0, \Pi \in \mathbb{M} \) such that for all \( j \in J \),

\[
A^c j Q_j + Q_j A^c + \beta Q_j - \sum_{j \neq k} \pi_{k,j} (Q_k - Q_j) \preceq 0, \tag{44}
\]

with \( c_j^T Q_j c_j \leq 1 \) for all \( \ell \in J \). As (44) is already in “dual” form, a data-driven version can be obtained via Lemma 1 without introducing additional nonlinearities; on the other hand, (44) has more variables than (C4), as the matrices \( Q_j \)'s are not fixed.

Remark 12 (Beyond LM inequalities): The conditions in (6), (30), (42), (44) all belong to the class of coupled Lyapunov functions as \( v_{\text{min}} \) and \( v_{\text{max}} \) [50]. These inequalities are pervasive in applications related to switched systems, constrained stabilization and stabilization of differential inclusions (including switching systems) [51]. Our results here provide the tools to recast such conditions in terms of experimental data only and in a form that allows for efficient solution. □

VIII. NUMERICAL EXAMPLES

Here we illustrate our results in the data-driven stabilization of Markov jump, switched and constrained linear systems.

A. Data-driven control of Markov jump linear systems

We consider a Markov jump linear system (MJLS) as in Example 2, with \( N = 3, n = 3, m = 2 \).

\[
\tilde{A}_1 = \begin{bmatrix}
0.5 & 0.5 & 0.3 \\
0.1 & 0.5 & 0.3 \\
0 & 0.4 & 0.3
\end{bmatrix}, \tilde{A}_2 = \begin{bmatrix}
0.3 & 0.2 & 0 \\
0 & 0.2 & 0.5 \\
0 & 0.1 & 0.3
\end{bmatrix}, \tilde{A}_3 = \begin{bmatrix}
0 & 0.1 & 0.2 \\
0 & 0.1 & 0.3
\end{bmatrix},
\]

\[
\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \tilde{B}_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \Pi = \begin{bmatrix} -3 & 4 & 5 \\
3 & -7 & 0 \\
0 & 3 & -5
\end{bmatrix}.
\]

It is easily proven that the stochastic system is open-loop unstable (cf. red plot in Figure 1). We assume that the system matrices are unknown; we simulate a trajectory with random input \( u \) and measure 20 data points for each subsystem, as in (2), with disturbance generated to satisfy Assumption 1 with \( \Phi_{1,i} = 10 I \), \( \Phi_{1,2} = 0 \), \( \Phi_{2,2} = -I \) for all \( i \in \mathbb{I} \). We want to find a switching controller \( u = K_{x,x} \) that stabilizes the system by solving the LMIs in (C1); we use MATLAB equipped with YALMIP [52] (with solver Mosek). Note that the subsystem \( i = 2 \) is not affected by the continuous input, and we assume that this information is available by first principles. To incorporate the prior knowledge in the design, we allow for different input dimensions for the modes, i.e., \( m_1 = m_3 = 2, m_2 = 0 \) (this simply correspond to changing the dimension of \( L_i \) with \( i \in \{1, 3\} \)). The program returns two stabilizing gains \( K_1, K_3 \); Figure 1 show the resulting closed-loop behavior.

B. Data-driven stabilization of switched linear systems

1) Stable average and solvers: We consider a switched linear system as (1), with \( N = 3, n = 2, m = 0 \),

\[
\tilde{A}_1 = \begin{bmatrix} 2 & 0.1 \\
0.1 & -0.2 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} -10 & 0.1 \\
0.1 & 0 \end{bmatrix}, \tilde{A}_3 = \begin{bmatrix} 0.1 & 0 \\
0 & 0.1 \end{bmatrix}.
\]

Each mode is unstable. We note that, even in the case of perfectly known model, the inequality in (23) (i.e., the relaxation proposed in [32]) does not admit a solution\(^4\). However, \( \frac{1}{\gamma} (\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3) \) is Hurwitz; therefore, the LM inequalities (6) are solvable with \( \Pi \) as in (22), and so must be (C1) for sufficiently informative data/small disturbances, by Lemma 3.

We perform simulations for several noise bounds; in particular, we choose

\[
\Phi_{1,i} = \epsilon I, \Phi_{1,2} = 0, \Phi_{2,2} = -I, \quad (\forall i \in \mathbb{I}), \tag{45}
\]

for different values of \( \epsilon \in \{0.1, 1, 10, 20, 40, 80\} \). For each disturbance level, we generate 100 datasets satisfying (2), with \( T_i = 20 \) for all \( i \); the Slater’s condition (13) is verified for all datasets. Our goal is to find suitable matrices \( P_i \)'s to implement the stabilizing controller (7); in particular, we investigate feasibility of (C2) for \( \Pi \) restricted to be as in (22) and as in (16), respectively. In both cases, we solve (C2) on Matlab, using YALMIP (solver Mosek) and line-search (over one scalar \( \gamma \) in the first case, over three scalars \( \{\gamma_i\}_{i \in \mathbb{I}} \) in the second; in both cases, between 1 and 100). We record the percentage of experiences for which we could find a solution, and the corresponding solver time on a commercial laptop. The results are shown in Table I (excluding the fourth and seventh columns).

As expected, both conditions admit a solution in all the experiments, for small noise levels. The second condition is computationally more expensive by a large margin, but also less restrictive. In fact, for larger noise bounds (i.e., for larger compatibility set) imposing (22) becomes too conservative,\(^4\)

\(^4\)In fact, due to the particular form of \( \tilde{A}_3 \), for \( i = 3 \) and \( j = 1 \), (23) would imply that \( (P_j - P_i) \) is negative definite; in turn, (23) for \( i = 1 \) and \( j = 3 \) reduces to the Lyapunov inequality and would imply that \( A_{\ell} \) is Hurwitz, which is false.
resulting in unfeasibility—even though the average of the true system matrices is Hurwitz.

Intuitively, the gap between the two conditions, in terms of both complexity and conservatism, is bound to further grow if the number of modes $N$ (equivalently, the number of free parameters in (16)) increases. This is a fundamental tradeoff, which also arises in the model-based case.

Alternatively, we also seek a solution to (C2) under (16) via a rank-minimization (RM) heuristic that, while giving up on theoretical guarantees, exhibits good empirical performance. In particular, we exploit the reformulation (10), and we attempt to solve it via the reweighted nuclear norm iteration in [53] recently successfully employed in the context of data-driven control in [15], [21]. Note that, although computing a solution to (C2) is hard in general, verifying a solution (from data only) is straightforward. The results are shown in the remaining columns of Table I. The average solver time improves remarkably with respect to the brute-force line-search approach (even if we compare it with the more conservative case (22)). However, for the largest noise level, the RM method fails to return a solution for some experiments where line-search or rank-minimization (RM) are employed.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$\varepsilon$ & Feasibility of (C2) (%) & Average solver time (s) & & & & & & \\
\hline
 & (22) & (16) & (16)w.RM & (22) & (16) & (16)w.RM & & \\
\hline
0.1 & 100 & 100 & 100 & 0.67 & 3.64 & 0.25 & & \\
1 & 100 & 100 & 100 & 0.60 & 3.45 & 0.25 & & \\
10 & 100 & 100 & 100 & 0.68 & 12.31 & 0.44 & & \\
20 & 71 & 100 & 100 & 0.92 & 21.81 & 0.30 & & \\
40 & 0 & 100 & 100 & 47.57 & 0.25 & & & \\
80 & 0 & 69 & 57 & 102.14 & 3.45 & & & \\
\hline
\end{tabular}
\caption{Feasibility and computational time of (C2), for the two relaxations (16), (22) and several noise levels $\varepsilon$. Line-search or rank-minimization (RM) are employed.}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=0.9\textwidth]{fig2}
\caption{$H_\infty$ stabilization of a switched linear system: state evolution (top), and switching signal (bottom-left, and a detail in the bottom-right axes).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.9\textwidth]{fig3}
\caption{$H_\infty$ stabilization of a switched linear system: state evolution (top), and switching signal (bottom-left, and a detail in the bottom-right axes).}
\end{figure}

Two relaxations (16), (22) and several noise levels $\varepsilon$. Line-search or rank-minimization (RM) are employed.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$\varepsilon$ & & & & & & \\
\hline
0.1 & 100 & 100 & 100 & 0.67 & 3.64 & 0.25 & \\
1 & 100 & 100 & 100 & 0.60 & 3.45 & 0.25 & \\
10 & 100 & 100 & 100 & 0.68 & 12.31 & 0.44 & \\
20 & 71 & 100 & 100 & 0.92 & 21.81 & 0.30 & \\
40 & 0 & 100 & 100 & 47.57 & 0.25 & & \\
80 & 0 & 69 & 57 & 102.14 & 3.45 & & \\
\hline
\end{tabular}
\caption{Feasibility and computational time of (C2), for the two relaxations (16), (22) and several noise levels $\varepsilon$. Line-search or rank-minimization (RM) are employed.}
\end{table}

If the number of modes $N$ is relatively small, and, for all $\varepsilon > 0$, the LM inequalities (6) do not hold. However, for larger $N$, we can achieve fast convergence even with large noise levels $\varepsilon$.

We collect datasets from open-loop experiments, with $N = 3$, $n = 3$, $m = 0$, $\bar{A}_1 = \begin{bmatrix} -1 & -0.1 & 0.1 \\ 0.1 & -0.1 & 0.1 \\ -0.1 & -0.1 & 0.1 \end{bmatrix}$, $\bar{E}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\bar{A}_2 = \begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.1 & 0 \\ -0.1 & 0 & -0.1 \end{bmatrix}$, $\bar{E}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\bar{A}_3 = \begin{bmatrix} -0.1 & -0.1 & -0.1 \\ 0.1 & 0.1 & 0.1 \\ -0.1 & -0.1 & -0.1 \end{bmatrix}$, $\bar{E}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and, for all $i \in I$, $F_i = 0$ and $C_i = \text{diag}(1, 3, 1)$, corresponding to a larger penalization for the second state.

We collect 100 datasets from open-loop experiments, with $T_i = 20$ for all $i \in I$, and the samples of the disturbance $\psi$ satisfying an energy bound as in (45) with $\varepsilon = 0.01$; the corresponding compatibility sets and matrices $\Phi_i$’s are computed as in Remark 8. For each dataset, we aim at designing a controller to optimize the $H_\infty$ performance of the system, by solving the data-driven program (H\(_\infty\)).

Note that each $A_i$ is unstable and that $\bar{A}_1 + \bar{A}_2 + \bar{A}_3$ also has an eigenvalue with positive real part; moreover, even in the case of known model, the condition (22) results in unfeasibility.

\footnote{The algorithm consists of a sequence of SDPs, which we solve with YALMIP [52] and Mosek.}
APPENDIX I
LM INEQUALITIES AND SLIDING MOTION

We consider a continuous-time switched linear system
\[ \dot{x}(t) = A_{\sigma(t)}x(t), \]  
where \( x \in \mathbb{R}^n \), \( \sigma(t) \in \mathcal{I} = \{1, 2, \ldots, N\} \) is a controlled switching signal, \( \{A_i\}_{i \in \mathcal{I}} \) are the system matrices. We are interested in the *min-switching* feedback law
\[ \sigma(x) = \min \{\argmin_{i \in \mathcal{I}} x^T P_i \}, \]  
(where the min selects the minimum index when the argmin is set-valued; any other selection rule can also be chosen), where the matrices \( \{P_i \in \mathbb{R}^{n \times n}\}_{i \in \mathcal{I}} \) solve the following LM inequalities problem [32]: find \( \{P_i \succ 0\}_{i \in \mathcal{I}}, \{Q_i \geq 0\}_{i \in \mathcal{I}}, \Pi = [\pi_{ij}], i, j \in \mathcal{M} \) such that
\[ \forall i \in \mathcal{I}, \quad A_i^T P_i + P_i A_i + \sum_{j \in \mathcal{I}} \pi_{ij} P_j + Q_i < 0. \]  

A. Solution concept and stability

It was shown in [32] that matrices \( \{P_i\}_{i \in \mathcal{I}} \) satisfying (48) ensure asymptotic stability for all Carathéodory solutions of the closed loop system (46)-(47). However, the switching rule (47) can results in *chattering*. For this reason, we instead consider the Filippov\(^6\) solutions of (46), namely absolutely continuous trajectories \( x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) such that, for almost all \( t \),
\[ \dot{x}(t) = \sum_{i \in \mathcal{I}(x)} \alpha_i(x) A_i x, \]  
for some \( \{\alpha_i(x)\}_{i \in \mathcal{I}(x)} \in \Delta \), where
\[ \mathcal{I}(x) := \{i \in \mathcal{I} \mid \forall V \in \mathcal{N}(x), \exists y \in V \text{ s.t. } y \in \mathcal{X}_i\} \]  
\[ \mathcal{X}_i := \{x \in \mathbb{R}^n \mid \sigma(x) = i\}, \]  
where \( \mathcal{N}(x) \) is the set of neighborhoods of \( x \). When \( x \in \text{int}(\mathcal{X}_i) \), \( \mathcal{I}(x) = \{i\} \) is a singleton and \( \dot{x} = A_i x_i \). A solution can also cross the boundary between two regions \( \mathcal{X}_i \) and \( \mathcal{X}_j \). Finally, \( x(t) \) can evolve along the boundaries between two or more regions, in a direction specified not by one of the modes \( i \in \mathcal{I} \), but by a convex combination of the matrices \( A_i \) with \( i \in \mathcal{I}(x) \); in this case we talk about “sliding mode”. Although ideal sliding mode would not happen in practice (due to discretized controllers, hysteresis, time-delay), it provides a close approximation of the behavior of the real system under fast switching.

It is known that the LM inequalities do not ensure stability for all closed-loop Filippov solutions; in fact, *repulsive* sliding motion [54] can cause instability [55, Ex. 1.1], [34, Rem. 6]. On the other hand, repulsive sliding mode would not appear in practice, e.g., if the controller is discretized, (and further implies the existence of alternative solutions to (46)). For the case of \( N = 2 \), asymptotic stability of all solutions with attractive sliding mode was shown in [25, p. 70]. The result

\(^6\)According to the standard definition, (49) actually defines a superset of the Filippov solutions, as we do not exclude sets of measure zero (which would require \( \mathcal{I}(x) := \{i \in \mathcal{I} \mid \forall V \in \mathcal{N}(x), \exists U \subset V \cap \mathcal{X}_i \text{ s.t. } \mu(U) > 0\} \); in this way we also include all Carathéodory solutions.)
was generalized in [32, Rem. 2] and [56] to all Filippov solutions such that, for almost all $t$
\[ \dot{v}_i(x) \left( x, \sum_{i \in I(x)} \alpha_i(x) A_i x \right) \leq \sum_{i \in I(x)} \dot{v}_i(x, \alpha_i(x) A_i x) \] (52)
with (possibly zero) $\alpha_i$’s as in (49), and
\[ \dot{v}_i(x, \xi) := x^T P_i \xi + \xi^T P_i x \] (53)
is the directional derivative of $v_i(x) := x^T P_i x$ at $x$ along $\xi$.
We also restrict our attention to this class: throughout the paper, by solution of (46) we mean a trajectory satisfying (49)–(52), wherever needed.

Remark 13 (On the solution concept): Condition (52) virtually always holds in practice: for example, it is verified for any attractive sliding motion involving only two modes [25, Eq. 3.22] – the most relevant case and often the only considered [25], [54]. Indeed, the proof of [50, Prop. 1] argues that (52) is necessary for the occurrence of chattering if (47) is discretized with arbitrarily small sampling time. Nonetheless, examples can be constructed where the continuous-time system (46) does not admit a solution satisfying (52) (at the boundary between three or more regions $X_i$). These pathological cases are excluded from our analysis: like the related literature [32], [50], we assume throughout existence of a solution satisfying (49)–(52), wherever needed.

Following [32], we study the stability of (46) via the non-convex, non-differentiable Lyapunov function
\[ v_{\min}(x) := \min_{i \in I} x^T P_i x = \min_{i \in I} \dot{v}_i(x), \] (54)
with matrices $\{P_i\}_{i \in I}$ solving (48).

Proposition A1 (Switched stabilization): If there exist $\{P_i \succ 0, Q_i \succeq 0\}_{i \in I}$ and $\Pi = [\pi_{i,j}]_{i,j \in I} \in M$ satisfying the Lyapunov-Metzler inequalities (48), then the switched feedback control law (47) makes $x^* = 0$ globally asymptotically stable for the system (46). Moreover, it holds that
\[ \int_0^\infty x^T Q_{\infty}(x) x \, dt < \min_{i \in I} x(0)^T P_i x(0), \] (55)
with $Q_{\infty}(x) := \sum_{i \in I} \alpha_i(x) Q_i \{ \alpha_i \}_{i \in I}$ satisfying (49)\(^7\). □

Proof: For any $x \in \mathbb{R}_n$, it holds that
\[ \dot{v}_{\min}(x, \xi) := \lim_{h \to 0^+} \frac{v_{\min}(x + h \xi) - v_{\min}(x)}{h} \leq \dot{v}_{\infty}(x, \xi) \] [50, Lem. 2]; therefore, by (52), along any solution satisfying (49), it holds, for almost all $t$, that
\[ \dot{v}_{\min}(x, \dot{x}) \leq \sum_{i \in I(x)} \alpha_i(x) x^T (A_i^T P_i + P_i A_i) x \] (56)
\[ < - \sum_{i \in I(x)} \alpha_i(x) x^T Q_i x, \] (57)
where the last inequality follows by (48), because $x^T P_i x \leq x^T P_i x$ for $i \in I(x), j \in I$, and thus $x^T (\sum_{j \in I} \pi_{i,j} P_j x) x \leq 0$.

\(^7\) [32], [56] assume the stronger condition $v_{\infty}(x, \alpha_i(x) A_i x) \leq \dot{v}_i(x, \alpha_i(x) A_i x)$ \forall i \in I(x), but the proof holds under the weaker (52).

With analogous definition when considering systems with input or disturbances: if (46) is replaced by $\dot{x}(t) = \tilde{x}_\infty(x)(x, t)$ for some (time dependent) mappings $\{\tilde{\xi}_j\}_{j \in I}$, then the term $A_i x$ shall be replaced by $\tilde{\xi}_i(x, t)$ in (49) and (52). The considerations in Remark 13 are still valid.

Without loss of generality, we can take $t \to \alpha_i(x(t))$ almost everywhere continuous (because $\dot{x}$ in (46) is), which ensures integrability in (55).

for all $i \in I(x)$. Stability follows by (57) because $v_{\min}$ is radially unbounded; the inequality (55) holds by integrating (57) over time, since $v_{\min}(x(t)) \to 0$ as $t \to \infty$.

While the stability in Proposition A1 was already established in [56], with respect to [44, Th. 1] we refined the performance guarantee (55) to cope with the possible occurrence of sliding motions. We next leverage the result to review $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems for switched linear systems.

Remark 14 (Solvability of LM inequalities): A sufficient condition for (48) to admit a solution is that a convex combination of the matrices $\{A_i\}_{i \in I}$ is Hurwitz [32], meeting the classical stabilizability condition given, e.g., in [25]. Nonetheless, finding a solution is not an easy task, as the problem is nonconvex – due to the bilinear terms in [25]. Nonetheless, finding a solution is not an easy task, as the problem is nonconvex – due to the bilinear terms in (59).

\[ \begin{align*}
\mathcal{H}_2 & \text{ index: Let } F_i = 0, E_i = E \text{ for all } i \in I; \text{ denote by } x_k : \mathbb{R}_{>0} \to \mathbb{R}^n \text{ and } z_k : \mathbb{R}_{>0} \to \mathbb{R}^p \text{ state and modified output trajectories generated with the the disturbance } \psi_k(t) := e_k \delta(t) \text{ (i.e., with zero disturbance and } x(0) = E e_k). \text{ Then}
\end{align*} \]
\[ J_2 := \sum_{k=1}^q \| z_k \|_2^2 ; \] (60)

\[ \begin{align*}
\mathcal{H}_\infty & \text{ index: Let } \tilde{x} : \mathbb{R}_{>0} \to \mathbb{R}^n \text{ and } \tilde{z} : \mathbb{R}_{>0} \to \mathbb{R}^p \text{ state and modified output trajectories generated with an arbitrary disturbance } \tilde{\psi} \in L_2. \text{ We define}
\end{align*} \]
\[ J_\infty := \sup_{0 \neq \theta \in L_2} \| \tilde{z}_\theta \|_2^2 \] (61)
Intuitively, if $N = 1$, the definitions recover the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices for LTI systems. For switched
linear systems, the quantities $J_2$ and $J_\infty$ were defined analogously in the literature, but in terms of the output $z$ [44], [46]. In fact, these works only consider Carathéodory systems. We remedy in the following two propositions, by considering the modified output $\hat{z}_n$ and by refining the proof of [44, Th. 3], [46, Th. 2] to account for the presence of sliding motions. Let us note that, in contrast to the LTI case, the quantities (60)-(61) are hard to compute, and in practice suitable upper bounds must be employed [45], [46].

**Proposition A2 (H2 control):** If there exist $\{P_i > 0\}_{i \in I}$ and $\Pi = \{\pi_{i,j}\}_{i,j \in I} \in M$ satisfying the Lyapunov-Metzler inequalities (48) with $\{Q_i = C_i^T C_i\}_{i \in I}$, then the switched feedback control law (47) makes $x^* = 0$ globally asymptotically stable for the system (58) and ensures that $J_2 < \min_{i \in I} \text{tr}(E^T P_i E)$.

**Proof:** Proposition A1 implies stability and that $J_2 = \sum_{k=1}^\infty \pi_k Q_k x(t) dt < \sum_{k=1}^\infty \min_{i \in I}(E_k)^T P_i (E_k) \leq \min_{i \in I} \sum_{k=1}^\infty (E_k)^T P_i (E_k)$; the conclusion follows by definition of $e_k$.

**Proposition A3 (H\infty control):** If there exist $\{P_i > 0\}_{i \in I}$, a scalar $\rho > 0$ and $\Pi = \{\pi_{i,j}\}_{i,j \in I} \in M$ such that

$$\begin{bmatrix} A_i P_i + P_i A_i + \sum_{j \in I} \pi_{i,j} P_j & P_i E_i C_i^T \\ * & -\rho I & F_i^T \\ * & * & -I \end{bmatrix} < 0,$$

then the switched feedback control law (47) makes $x^* = 0$ globally asymptotically stable for the system (58) and ensures that $J_\infty < \rho$.

**Proof:** Stability holds by Proposition A1 and negative definiteness of the upper-left block. As in the proof of Proposition A1, we have for all $t \geq 0$

$$v_{\min}(x, \dot{x}) \leq \dot{v}_{\sigma(x)} \left( (\sum_{i \in I}(\alpha_i(x))(A_i x + E_i \psi)) \right)$$

$$= (\sum_{i \in I}(\alpha_i(x))v_i(x, A_i x + E_i \psi)$$

and note that

$$\sum_{i \in I}(\alpha_i(x))v_i(x, A_i x + E_i \psi)$$

where $(a)$ is the analogous of condition (52) (see Footnote 8) and $(b)$ follows via a Schur complement argument by (62) (see [46, Eq. 14]). The result follows by integrating, since $v_{\min}(x(0)) = 0$ and $v_{\min}(x(t)) \to 0$ as $t \to \infty$.

**APPENDIX II PROOFS**

**A. Proof of Lemma 2**

We only need to show that feasibility of (6) implies feasibility of (8). Let $S = (\{P_i \}_{i \in I}, \{K_i \}_{i \in I}, \Pi, Q)$ be a solution for (6). We prove that, for all $i \in I$, the set $C_i := \{ (A_i + B_i K_i)^T \mid (A_i, B_i) \in C_i \}$ is compact; then there exists a solution to (8) in any neighborhood of $S$ due to the strict inequality in (6).

In the equation (4) in affine, we can write $C_i = \{ (A_i, B_i) \mid (A_i, B_i) = E_i^T V_i + A_i^0 B_i^0, V_i \in \mathcal{V}_i, \{A_i^0, B_i^0\} \in C_i \}$, with $E_i = [X_i^T \ U_i^T]$, $E_i^T$ its pseudo-inverse, $V_i := \{ (X_i, W_i) \mid E_i \alpha | \alpha \in \mathbb{R}^{(m+n) \times m} \}$, $C_i = \{ (A_i^0, B_i^0) \mid \{A_i^0 X_i + B_i^0 U_i = 0 \}$ (in simple terms, $E_i^T V_i$ is a particular solution and the $\{\text{disturbance independent- homogeneous solutions}\}$.

Consider any $(A_i^0, B_i^0) \in C_i$. We define $A'_i := (A_i^0 + B_i^0 K_i)^T$ and note that $(A'_i, B'_i) \in C_i$. We claim that the symmetric matrix $M := ((A'_i + B_i K_i)^T, P_i + P_i (A'_i + B_i K_i))$ is nilpotent, hence equal 0. If not, take any $V_i \in \mathcal{V}_i$, let $[A_i^0 B_i^0] := E_i^T V_i + \beta [A'_i B'_i]^T$, and note that $(A_i^0, B_i^0) \in C_i$ for all $\beta \in \mathbb{R}$; yet this pair violates (6) for $\beta > 0$ or $\beta < 0$ large enough, providing a contradiction. This also means that the symmetric matrix $A'_i + B_i K_i = (A_i^0 + B_i K_i)^T (A_i^0 + B_i K_i)$ is nilpotent, hence equals 0 if not, let $\mathbb{R} \not\ni \lambda \not\ni m \neq 0$ be an eigenvalue-eigenvector couple, and note that $m^T \lambda m = 0$ contradicts $P_i > 0$. Therefore, we finally have $A_i^0 + B_i K_i = 0$.

We conclude that $C_i = \{ [I \ K_i^T] E_i^T V_i \mid V_i \in \mathcal{V}_i \}$. The proof follows because $V_i$ is compact (due to $\Phi_{i,2,2} \times 0 \in$ Assumption 1), and so must be $V_i$ and in turn $C_i$.

**B. Proof of Lemma 3**

Let $M_i := A_i^T P_i + P_i A_i$. With II as in the statement, $P_i = N P + \mu^{-1} \lambda_i M_i$, $Q = 0$, and recalling that $\sum_{j \in I} \pi_{i,j} P = 0$, the left-hand side of (6) is

$$A_i^T (NP + \mu \lambda_i M_i) + \sum_{j \in I} \pi_{i,j} \lambda_j (\lambda_i M_j - \lambda_i M_i),$$

($\ast$ is the transpose of the first addend), which is negative definite for $\mu$ large enough, as taking its limit $\mu \to \infty$ gives

$$A_i^T P_i + P_i A_i \ast \sum_{j \in I} \lambda_j M_j < 0$$

where the inequality is (17). Thus we constructed a solution to (6) based on (17); a solution to (8) with the same $P_i$'s, and some $Q > 0$ then exists as per Lemma 2.

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