CONIC BUNDLES AND ITERATED ROOT STACKS

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Abstract. We generalize a classical result by V. G. Sarkisov about conic bundles to the case of a not necessarily algebraically closed perfect field, using iterated root stacks, destackification, and resolution of singularities. More precisely, we prove that whenever resolution of singularities is available, over a general perfect base field, any conic bundle is birational to a standard conic bundle.

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1. INTRODUCTION

In this paper, we study the geometry of conic bundles, that is, fibrations whose generic fiber is a smooth conic. They have been widely studied in the context of rationality problems, notably the classic result of Artin and Mumford, who computed their Brauer groups in [5] to produce unirational non-rational conic bundles over rational surfaces. In order to better understand these bundles, it is desirable to bring a conic bundle into a standard form where the locus of degeneration can be controlled.

Over an algebraically closed field $k$ and assuming resolution of singularities, a classical result by Sarkisov ([17]) states every conic bundle $\pi : V \to S$ of irreducible varieties such that $S$ is smooth and $\pi$ is projective can be brought into a standard form. Concretely, this means that there exist smooth varieties $\tilde{V}$ and $\tilde{S}$ and a projective morphism $\tilde{\pi} : \tilde{V} \to \tilde{S}$ fitting

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into a commutative square

\[
\begin{array}{ccc}
\tilde{V} & \longrightarrow & V \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\tilde{S} & \longrightarrow & S
\end{array}
\]

such that the rational map \( \tilde{V} \to V \) and the projective morphism \( \tilde{S} \to S \) are birational, and such that

- the generic fiber of \( \tilde{\pi} \) is a smooth conic,
- the discriminant divisor of \( \tilde{\pi} \) is a simple normal crossing divisor,
- the general fiber of \( \tilde{\pi} \) along every irreducible component of the discriminant divisor is a singular irreducible reduced conic, and
- the fibers of \( \tilde{\pi} \) over the singular locus of the discriminant divisor are non-reduced conics, i.e. double lines.

We use root stacks, resolution of singularities, and the destackification procedure ([7]) to generalize Sarkisov’s result to a general perfect base field in Theorem 4. An analogous result for Brauer-Severi surface bundles, i.e. fibrations whose generic fiber is a form of \( \mathbb{P}^2 \), has been proven by Kresch and Tschinkel in [12], also using root stack techniques.

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2. Preliminaries

Throughout this section, when \( X \) is a Noetherian Deligne-Mumford (DM) stack, we will let \( n \) be a positive integer, invertible in the local rings of an étale atlas of \( X \). A sheaf on \( X \) is a sheaf of abelian groups on the étale site of \( X \); all cohomology will be étale cohomology.

2.1. Conic bundles.

Definition 1. Let \( S \) be a regular scheme such that \( 2 \) is invertible in its local rings. A regular conic bundle over \( S \) is a flat, projective morphism \( \pi : V \to S \) from a regular scheme \( V \) such that the generic fiber over every irreducible component is smooth and such that all fibers are isomorphic to a conic in \( \mathbb{P}^2 \). A regular conic bundle is called standard if \( \pi \) is relatively minimal, i.e. if the preimage of an irreducible divisor under \( \pi \) is an irreducible divisor; equivalently, if there exists a reduced divisor \( D \subset S \) whose singular locus is regular, such that

- The morphism \( \pi \) is smooth over \( S \setminus D \) and the generic fiber over every irreducible component is a smooth conic.
- The generic fiber of \( \pi \) over every irreducible component of \( D \) is a singular reduced irreducible conic, i.e. the union of two lines with conjugate slopes.
• The fiber of $\pi$ over every point of $D^{\text{sing}}$ is non-reduced, i.e. a double line.

We remark that we put no further requirements on $D$, although our construction actually produces standard conic bundles with simple normal crossing discriminant divisor.

2.2. Gerbes. Let $X$ be a Noetherian Deligne-Mumford stack.

**Definition 2.** A gerbe over $X$ banded by $\mu_n$, or simply a $\mu_n$-gerbe over $X$, is the data of a Deligne-Mumford stack $H$ and a morphism $H \to X$ that is étale locally isomorphic to a product with $B\mu_n$, together with compatible identifications of the automorphism groups of local sections with $\mu_n$.

We can classify $\mu_n$-gerbes by their class in $H^2(X, \mu_n)$. We use the notion of the residual gerbe $G_x$ of $X$ at a point $x \in |X|$, an étale gerbe over the residue field $\kappa(x)$ satisfying certain universal properties ([16, App. B]).

**Lemma 1.** Let $x \in |X|$. Then $H^1(G_x, \mathbb{Z}) = 0$.

*Proof.* The Leray spectral sequence for $f : G_x \to \text{Spec} \kappa(x)$ gives a monomorphism

$$H^1(G_x, \mathbb{Z}) \to H^0(\text{Spec} \kappa(x), R^1 f_* \mathbb{Z}).$$

Let $K/\kappa(x)$ be a finite separable extension such that the gerbe $Y := \text{Spec} K \times_{\text{Spec} \kappa(x)} G_x$ has a section. This implies that $Y \cong BG$ for a finite étale group scheme $G$ over $K$. But then

$$H^1(Y, \mathbb{Z}) = H^1(BG, \mathbb{Z}) = \text{Hom}(G, \mathbb{Z}) = 0.$$

Hence, we have $R^1 f_* \mathbb{Z} = 0$. \qed

We will also frequently make use of the *Kummer sequence*

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 0,$$

which is an exact sequence of sheaves on $X$.

Suppose now that $X$ is regular and integral, with trivial stabilizer at the generic point $\iota_\eta : \text{Spec} \eta \to X$. By [10, (2)-(3)], the following is an exact sequence of sheaves on $X$:

$$(1) \quad 0 \to \mathbb{G}_m \to (\iota_\eta)_* \mathbb{G}_m \to \bigoplus_{x \in X^{(1)}} (\iota_x)_* \mathbb{Z} \to 0.$$

By Lemma 1, this implies that $H^2(X, \mathbb{G}_m) \to H^2(X, (\iota_\eta)_* \mathbb{G}_m)$ is injective. The Leray spectral sequence for $\iota_\eta$ and Hilbert’s Theorem 90 imply that

$$H^2(X, (\iota_\eta)_* \mathbb{G}_m) \to H^2(\text{Spec} \eta, \mathbb{G}_m)$$

is injective. Hence, we can infer that $H^2(X, \mathbb{G}_m)$ is a torsion group, which we call the *Brauer group* of $X$. This motivates the following definition.
Definition 3. The **Brauer group** $\text{Br}(X)$ of a Noetherian DM stack $X$ is defined to be the torsion subgroup of $H^2(X, \mathbb{G}_m)$.

This reduces to the classical definition of the Brauer group if $X$ is the spectrum of a field.

2.3. **Root stacks and the Brauer group.** Given effective Cartier divisors $D_1, \ldots, D_\ell$ on a Noetherian scheme or Deligne-Mumford stack $X$, we can define the *iterated $n$-th root stack* of $X$ along those divisors ([8]), denoted by $\sqrt[n]{(X, \{D_1, \ldots, D_\ell\})} \to X$.

This construction adds stacky structure along the divisors, and is an isomorphism outside of the union of divisors. It should be noted that when any intersection $D_i \cap D_j$ with $i \neq j$ is nonempty, this is not isomorphic to the root stack along the union of the divisors. For any $i \in \{1, \ldots, \ell\}$, the iterated root stack $\sqrt[n]{(X, \{D_1, \ldots, \widehat{D}_i, \ldots, D_\ell\})}$ is a relative coarse moduli space for $\sqrt[n]{(X, \{D_1, \ldots, D_\ell\})}$ in the sense of [1, §3]. In particular, a locally free sheaf on $\sqrt[n]{(X, \{D_1, \ldots, D_\ell\})}$ such that the associated linear $\mu_n$-representation is trivial at a general point of any irreducible component of $D_i$ descends to $\sqrt[n]{(X, \{D_1, \ldots, \widehat{D}_i, \ldots, D_\ell\})}$.

To prove this, we can apply a relative version of [2, Thm. 10.3] for good moduli space morphisms, which are relative versions of good moduli spaces. Note that

1. the condition of trivial stabilizer actions at closed points is replaced by trivial action of the relative inertia stack at closed points;
2. due to the construction of root stacks, this action will be trivial if it is trivial at a general point of every irreducible component of $D_i$;
3. since the stacks involved are tame, the relative coarse moduli space mentioned above is a good moduli space morphism.

We will use this several times throughout.

Lemma 2. Let $S$ be a regular integral Noetherian scheme of dimension 1, let $n$ be invertible in the local rings of $S$, and let $D_1, \ldots, D_\ell$ be distinct closed points of $S$. Then we have

$$\text{Br}(\sqrt[n]{(S, \{D_1, \ldots, D_\ell\})})[n] \cong \text{Br}(S \setminus \{D_1, \ldots, D_\ell\})[n].$$

Proof. Let $X$ be any regular integral Noetherian DM stack with trivial generic stabilizer such that $\dim(X) = 1$, and such that $n$ is invertible in the local rings of an étale atlas of $X$. Then the results from [11, 2.] and
the Leray spectral sequence for $\iota_\eta$ imply that $H^2(X, (\iota_\eta)_* \mathbb{G}_m)[n] \cong \text{Br}(\eta)[n]$.

Moreover, if $x \in X^{(1)}$, the vanishing of $H^1(\mathcal{G}_x, \mathbb{Z})$ implies that

$$H^1(\mathcal{G}_x, \mathbb{Z}/n\mathbb{Z}) \cong H^2(\mathcal{G}_x, \mathbb{Z})[n].$$

The long exact sequence of cohomology of (1) then gives rise to an exact sequence

$$0 \to \text{Br}(X)[n] \to \text{Br}(\eta)[n] \to \bigoplus_{x \in X^{(1)}} H^1(\mathcal{G}_x, \mathbb{Z}/n\mathbb{Z}).$$

Taking $X = \sqrt[n]{(S, \{D_1, \ldots, D_\ell\})}$, with codimension 1 point $x_i$ over $D_i$ for all $i$, we compare the exact sequence (2) with the analogous exact sequence for $S$ (loc. cit.) to obtain the vanishing of the right-hand map in (2) after projection to the factor $x_i$ for any $i$ [13, §3.2]. Comparison with the exact sequence for $S \setminus \{D_1, \ldots, D_\ell\}$ gives the result. 

**Lemma 3.** Let $k$ be a field and let $X$ be integral, smooth and of finite type over $k$ with trivial generic stabilizer. For any positive integer $n$ with $\text{char}(k) \nmid n$ and open substack $U \subset X$ whose complement has codimension at least 2, we have $H^2(X, \mu_n) \cong H^2(U, \mu_n)$, and therefore $\text{Br}(X)[n] \cong \text{Br}(U)[n]$.

**Proof.** By [14, Rem II.3.17] there is no loss of generality in assuming that $k$ is perfect. By shrinking $X$ if necessary and iterating the process for large open substacks of $X$, we can assume that the complement $Y = X \setminus U$ is smooth and of constant codimension $c \geq 2$ everywhere. In this situation, we know by [4, §XVI.3] that $\overline{H}^i_Y(X, \mu_n) = 0$ for $i \neq 2c$. Combining this with the exact sequence for cohomology with support ([14, Prop III.1.25]) and the local-to-global spectral sequence ([14, §VI.5]) gives the result; cf. [11, Cor 6.2]. 

**3. Proof the of the main result**

Here we state and prove the main result.

**Theorem 4.** Let $k$ be a perfect field of characteristic different from 2 and $S$ a smooth projective algebraic variety over $k$. Assume that embedded resolution of singularities for reduced subschemes of $S$ of pure codimension 1 and desingularization of reduced finite-type Deligne-Mumford stacks of pure dimension equal to $\dim(S)$ are known. Let

$$\pi : V \to S$$

be a morphism of projective varieties over $k$ whose generic fiber is a smooth conic. Then there exists a commutative diagram

$$\begin{array}{ccc}
\tilde{V} & \xrightarrow{\rho_\pi} & V \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\tilde{S} & \xrightarrow{\rho_S} & S
\end{array}$$


where $\rho_S$ is a projective birational morphism, $\rho_V$ is a birational map, and $\pi$ is a standard conic bundle with simple normal crossing discriminant divisor.

Remark 1. Embedded resolution of singularities is known in characteristic 0 for all dimensions by Hironaka’s celebrated result. In positive characteristic, embedded resolution of singularities for both curves and surfaces is known (cf. [9]). Since the resolutions commute with smooth base change, the assumptions about desingularization of reduced finite-type Deligne-Mumford stacks are also true in all of these cases (apply resolution of singularities to a presentation).

Before embarking on the proof we make several observations of a general nature. Let $k$ be a perfect field, let $S$ be a smooth projective algebraic variety over $k$, let $n$ be a positive integer such that $\text{char}(k) \nmid n$, and let $\alpha \in \text{Br}(k(S))[n]$. Then there exists a dense open $U \subset S$ and $\beta \in \text{Br}(U)$ such that $\alpha$ is the restriction of $\beta$. Taking $U$ to be maximal, by Lemma 3, the complement $S \setminus U$ is a finite union of divisors.

Assume embedded resolution of singularities for reduced subschemes of $S$ of pure codimension 1.

Then, upon replacing $S$ by a smooth projective variety with birational morphism to $S$, we may suppose that the complement of $U$ is a simple normal crossing divisor $D_1 \cup \cdots \cup D_\ell$. Let

$$X := \sqrt[n]{(S, \{D_1, \ldots, D_\ell\)},$$

the iterated root stack of $S$ along the divisors $D_i$. We apply Lemma 2 to the scheme obtained by gluing the local rings at the generic points of the $D_i$ along the generic point of $S$. Then [3, §VII.5, Thm 5.7] implies that $\alpha$ extends to an open neighborhood of the root stack over this scheme. Hence, by Lemma 3, there is a unique $\beta \in \text{Br}(X)[n]$ that restricts to $\alpha$.

Now suppose that $\alpha$ is the class of a central simple algebra $A$ of dimension $d^2$ as a $k(S)$-vector space, with $n \mid d$. Let $\beta_0 \in H^2(X, \mu_n)$ be a lift of $\beta \in \text{Br}(X)[n]$ with corresponding gerbe $G_0$ banded by $\mu_n$. Now $A$ is the fiber at the generic point of some sheaf of Azumaya algebras $\mathcal{A}$ on an open $W \subset X$ with complement of codimension at least 3 ([10, Thm 2.1]). The Brauer class of the pullback of $\mathcal{A}$ to $W \times_X G_0$ is trivial, hence this pullback is the endomorphism algebra of a locally free coherent sheaf of rank $d$, which is the restriction of a coherent sheaf $E_0$ on $G_0$.

Assume resolution of singularities for reduced Noetherian DM stacks of finite type over $k$ of pure dimension $\dim(S)$.

The identity of $E_0[W \times_X G_0]$ induces a morphism to the Grassmannian of rank $d$ quotients of $E_0$ ([15]). Apply resolution of singularities to the closure of the image to obtain a smooth DM stack $Y$ with a projective morphism to $X$ that restricts to an isomorphism over $W$, a gerbe $H_0 := Y \times_X G_0$, and a locally free coherent sheaf $F_0$ on $H_0$ whose restriction over $W$ is isomorphic to the restriction of $E_0$. In this situation, let $\gamma_0 := \beta_0|_Y$.

In the proof of Theorem 4 we specialize the above to $n = d = 2$. 


Proof. The proof begins with a series of reductions steps, starting with $Y$ as above, equipped with a sheaf of Azumaya algebras, restricting to the quaternion algebra $A$ over $k(S)$, associated with the generic fiber of $\pi$.

Step 1. We may suppose that $Y \cong \sqrt{(T, \{E_1, \ldots, E_m\})}$ for some smooth projective variety $T$ with birational morphism to $S$ and irreducible divisors $E_i$ such that $E_1 \cup \cdots \cup E_m$ is a simple normal crossing divisor and such that at the generic point of each $E_i$, the projective representation $\mu_2 \to \text{PGL}_2$ given by the sheaf of Azumaya algebras is nontrivial. Indeed, the destackification program ([7, Thm 1.2]) yields a morphism $\tilde{Y} \to Y$ that is a composition of blow-ups with smooth centers, such that

$$\tilde{Y} \cong \sqrt{(T, \{E_1, \ldots, E_m\})}$$

for a smooth projective variety $T$ and irreducible divisors $E_i$ such that $E_1 \cup \cdots \cup E_m$ is a simple normal crossing divisor. We pull back the sheaf of Azumaya algebras to $\tilde{Y}$. If there is an $i$ such that the projective representation $\mu_2 \to \text{PGL}_2$ over a general point of $E_i$ is trivial, the sheaf of Azumaya algebras descends to $\sqrt{(T, \{E_1, \ldots, \hat{E}_i, \ldots, E_m\})}$.

Step 2. We may suppose, additionally, that generically along every component of $E_i \cap E_{i'}$ for $i \neq i'$ the projective representation of $\mu_2 \times \mu_2$ is faithful. Let $F \subset E_i \cap E_{i'}$ be an irreducible component with non-faithful representation. Let $\tilde{T}$ be the blow-up of $T$ along $F$. For every $j \in \{1, \ldots, m\}$, we denote the proper transform of $E_j$ by $\tilde{E}_j$, and we denote the exceptional divisor of the blow-up by $E'$. We let $\tilde{Y}$ be the normalization of $\tilde{T} \times_T Y$. Then $\tilde{Y}$ is isomorphic to the blow-up of $Y$ at the corresponding component of the fiber product of the gerbes of the root stacks, which is itself isomorphic to the root stack

$$\sqrt{(\tilde{T}, \{\tilde{E}_1, \ldots, \tilde{E}_m, E'\})}.$$ 

The projective representation over a general point of $E'$ is trivial, so the sheaf of Azumaya algebras descends to $\sqrt{(\tilde{T}, \{\tilde{E}_1, \ldots, \tilde{E}_m\})}$.

Step 3. We may suppose, furthermore, that all triple intersections $E_i \cap E_i' \cap E_i''$ are empty, where $i, i'$, and $i''$ are distinct. Since there can never be more than 2 independent commuting subgroups of order 2 in $\text{PGL}_2$ ([6]), the projective representation $(\mu_2)^3 \to \text{PGL}_2$ over a general point of $E_i \cap E_i' \cap E_i''$ has kernel equal to the diagonal $\mu_2$. We blow up $T$ along $E_i \cap E_i' \cap E_i''$ and proceed as in Step 2.

Step 4. We may suppose, furthermore, that the Brauer class $[A] \in \text{Br}(k(S))$ does not extend across the generic point of $E_i$ for any $i$. Assume that it does, for some $i$. Let

$$\tilde{Y} := \sqrt{(T, \{E_1, \ldots, \hat{E}_i, \ldots, E_m\})}.$$
Then by Lemma 3, the Brauer class is restriction of an element \( \delta \in \text{Br}(\tilde{Y}) \). Let \( \varepsilon \in H^2(\tilde{Y}, \mu_2) \) denote an arbitrary lift of \( \delta \), with corresponding gerbe \( H_1 \), such that if we let \( H_0 \) denote the base-change

\[ Y \times_{\tilde{Y}} H_1, \]

then on \( H_0 \) the sheaf of Azumaya algebras is identified with endomorphism algebra of some locally free coherent sheaf \( \mathcal{F}_0 \). Notice that \( H_0 \) is a root stack over \( H_1 \). The relative stabilizer acts with eigenvalues 1 and \(-1\) on fibers of \( \mathcal{F}_0 \). The \((-1)\)-eigensheaf is a quotient sheaf \( \mathcal{L}_{-1} \) supported on the gerbe of the root stack, such that the kernel \( \mathcal{F}_1 \) in

\[ 0 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{L}_{-1} \to 0 \]

is again locally free and descends to \( H_1 \), yielding a sheaf of Azumaya algebras on \( \tilde{Y} \).

We now have \( Y = \sqrt{(T, \{ E_1, \ldots, E_m \})} \), equipped with a sheaf of Azumaya algebras \( \mathcal{A} \), such that the projective representations at a general point of every \( E_i \) are nontrivial, the projective representations at a general point of every intersection is faithful, there are no triple intersections, and such that the Brauer class does not extend over any of the generic points of the \( E_i \).

Let \( P \to Y \) be the smooth \( \mathbb{P}^1 \)-fibration associated with \( \mathcal{A} \).

Let \( T_0 \) denote the complement of the intersections of pairs of divisors,

\[ T_0 := T \setminus \bigcup_{1 \leq i < i' \leq m} E_i \cap E_{i'}. \]

We apply [12, Proposition 3.1] to \( T_0 \times_T P \) to obtain a regular conic bundle

\[ \pi_0: V_0 \to T_0. \]

This factors canonically through \( \mathbb{P}(\pi_{0*}(\omega_{\mathcal{V}/T_0}^\vee)) \). Let \( i: T_0 \to T \) denote the inclusion. We claim that \( i_* (\pi_{0*}(\omega_{\mathcal{V}/T_0}^\vee)) \) is a locally free coherent sheaf and, denoting this by \( \mathcal{E} \), the closure \( V = \overline{V_0} \) of \( V_0 \) in \( \mathbb{P}(\mathcal{E}) \) is a regular conic bundle over \( T \). It suffices to verify these assertions after passing to an algebraic closure of \( k \). Then there is a unique faithful projective representation \( \mu_2 \times \mu_2 \to \text{PGL}_2 \) (up to conjugacy), cf. [6]. So, by [12, Lemma 2.8] after base change to a suitable affine étale neighborhood \( T' = \text{Spec}(B') \to T \) of a given point of an intersection \( E_i \cap E_{i'} \), we have

\[ Y' \cong \sqrt{(T', \{ E_i, E_{i'} \})} = [\text{Spec}(B'[t, t']/(t^2 - x, t^2 - x'))/\mu_2 \times \mu_2], \]

where \( x \) and \( x' \) are the respective defining equations for the preimage in \( T' \) of \( E_i \) and \( E_{i'} \), with \( B' \) obtained by pulling back \( [\mathbb{P}^1/\mu_2 \times \mu_2] \). Here, on \( B'[t, t']/(t^2 - x, t^2 - x') \), the action of the factors of \( \mu_2 \times \mu_2 \) is by respective scalar multiplication of \( t \) and \( t' \), while the action on \( \mathbb{P}^1 \) corresponds to the faithful projective representation \( \mu_2 \times \mu_2 \to \text{PGL}_2 \). Over \( T' \) we compute \( \overline{V_0} \cong \text{Proj}(B'[u, v, w]/(xu^2 + x'v^2 - w^2)) \), which is regular.
The fact that the Brauer class does not extend across the generic point of any $E_i$ ensures that the conic bundle $V \rightarrow T$ is standard. □

Remark 2. While the general destackification process outlined in [7] requires stacky blow-ups, it is never necessary to take root stacks when all stabilizers are powers of $\mu_2$.

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