Simulating oscillations of the system with piecewise-linear characteristics of elastic elements excited by self-synchronizing unbalance vibration exciters

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Abstract. Oscillations of a system of three solids with piecewise-linear characteristics of the elastic element between the bodies on which unbalanced vibration exciters of limited power are installed are considered in the paper. The modes of oscillations of the system near the second resonant frequency are analyzed. It is shown that intermittent interaction between system elements leads to a change in the dynamic characteristics of the system, including a change in the frequency range of stability of anti-phase synchronization of debalance rotation.

1. Introduction
In the development of vibrating machines with self-synchronizing unbalance exciters, the issues of accounting for the interaction of the machine’s working bodies with the processed material and its effect on the dynamics of vibrating machine are of particular importance [1]. In engineering practice, in order to assign operating modes, the models in which this interaction is taken into account in the form of equivalent viscous friction forces are usually used [1, 2]. It is assumed that the vibrations are excited far from the resonant frequencies. However, the forces of interaction of the machine’s working bodies with the medium being processed can have a substantially non-linear character and can have a significant effect on the dynamic properties of the machine [1,3,4]. Ultimately, this can cause a violation of the required synchronization of vibration exciters and, accordingly, the required vibration mode of the machine.

In order to identify the possible effect of the processed medium properties on the dynamic properties of vibrating machines, in this work, we consider the oscillations of a system of three solids with piecewise linear characteristics of the elasticity and viscosity of the element between the bodies on which unbalance vibration exciters of limited power are installed. It is assumed that the machine’s vibrations can be with the separation of the processed medium from its working bodies. The dynamic scheme under consideration can, for example, be used to study the dynamics of vibrating jaw crushers.

2. Mathematical model and simulation methods
A design scheme of a vibromachine’s model with translational vibrations of its elements is shown in figure 1. The model consists of a rigid frame modeled by a solid body of mass \( m_1 \), and two working bodies modeled by solid bodies with the same mass \( m \). The frame is attached to an immovable base using a linear spring with stiffness and viscosity coefficients \( c_1 \) and \( b_1 \), respectively. The left and right
working bodies are elastically fixed to the frame using the identical linear springs with stiffness and viscosity coefficients $c$ and $b$, respectively. The displacements of each of the bodies are described by the displacements $x_1$, $x_2$, $x_3$ of their centers of mass along the horizontal axis $Ox$ with respect to the equilibrium positions. The interaction with the processed medium is modeled by a linear spring with stiffness coefficients $c_j$ and viscosity $b_j$, fixed on the right working body and set with a gap $\Delta$ relative to the left working body. Thus, the interaction of the machines’ working bodies with the processed medium is described by a piecewise linear function of the form:

$$\Phi^* = \begin{cases} 
0, & \text{for } x_2 - x_3 - \Delta \leq 0, \\
(c_2(x_2 - x_3 - \Delta) + b_2(x_2 - x_3)), & \text{for } x_2 - x_3 - \Delta > 0.
\end{cases}$$  \hspace{1cm} (1)$$

Oscillations of the system are excited by two identical unbalance vibration exciters with an imbalance mass $m_e$ and an eccentricity $r$, rigidly attached to the working bodies. The rotation of each of the debalances is carried out by an induction motor, with the moment of inertia $J_0$ reduced to the shaft. The torque $L$ of each motor is described by its static characteristic. Friction in the bearings of the debalance shafts is taken into account in the form of moments $R$ of dry friction forces (not shown in Figure 1). The debalance positions are described by the rotation angles $\varphi_j$ ($j = 1,2$ is the debalance number), counted from the negative direction of the $Ox$ axis.

The equations of the system’s motion are:

$$\begin{align*}
M\ddot{X} + CX = -B\dot{X} + \Phi + P, \\
J\ddot{\varphi}_1 = L_1(\dot{\varphi}_1) - R_1(\dot{\varphi}_1) + m_e r \ddot{x}_2 \sin \varphi_1, \\
J\ddot{\varphi}_2 = L_2(\dot{\varphi}_2) - R_2(\dot{\varphi}_2) + m_e r \ddot{x}_2 \sin \varphi_2,
\end{align*}$$  \hspace{1cm} (2)

where:

$$M = \begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_e
\end{bmatrix}, \quad C = \begin{bmatrix}
c_1 + 2c & -c & -c \\
-c & c & 0 \\
-c & 0 & c
\end{bmatrix}, \quad B = \begin{bmatrix}
b_1 + 2b & -b & -b \\
-b & b & 0 \\
-b & 0 & b
\end{bmatrix}, \quad X = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix},$$

$$\Phi = \begin{bmatrix}
-c \\
-c \\
0
\end{bmatrix}, \quad P = \begin{bmatrix}
m_e r (\dot{\varphi}_1 \sin \varphi_1 + \dot{\varphi}_1^2 \cos \varphi_1) \\
m_e r (\dot{\varphi}_2 \sin \varphi_2 + \dot{\varphi}_2^2 \cos \varphi_2)
\end{bmatrix},$$

where $m_2 = m + m_e$, $J = J_0 + m_e r^2$. The presented system of equations allows to analyze the modes of movement of the vibromachine taking into account the possible separation of the processed from its working bodies.

Consider the problem of simple self-synchronization of vibration exciters in the case of oscillations excitation near the second resonance. The problem’s solution will be obtained by the averaging
method [5-7]. Substituting the variables $x_j = \sum_{i=1}^n u_{ij} q_i$, where $u_{ij}$ are the elements of the matrix $U$ composed of eigenvectors for the generating system (2), we obtain the system of equations of motion with respect to the main coordinates $q_i$:

$$
\begin{align*}
&M^* \ddot{q} + C^* \dot{q} = -B^* \dot{q} + U^T \Phi(q) + U^T p \\
&\dot{\phi}_1 = L_1(q_1) - R_1(q_1) + m_\sigma u \sum_{i=1}^n u_{1i} \dot{q}_i \sin \phi_1, \\
&\dot{\phi}_2 = L_2(q_2) - R_2(q_2) + m_\sigma u \sum_{i=1}^n u_{2i} \dot{q}_i \sin \phi_2
\end{align*}
$$

(3)

where $M^* = U^T MU$, $B^* = U^T BU$, $C^* = U^T CU$, $Q = [q_1 \quad q_2 \quad q_3]^T$.

In the vicinity of the second resonance one can take $q_1 = q_3 = 0$. Then, following the averaging method and introducing a small parameter $\epsilon$, the system under consideration (3) become:

$$
\begin{align*}
&\ddot{q}_2 + p_2^2 q_2 = \epsilon [-b_2^2 \dot{q}_2 + (u_{23} - u_{22}) \Phi^*(q_2, \dot{q}_2) + u_{22} P_2 + u_{23} P_3]/m_2^2, \\
&\dot{\phi}_1 = \epsilon [L_1(q_1) - R_1(q_1) + m_\sigma u \sum_{i=1}^n u_{1i} \dot{q}_i \sin \phi_1]/
\end{align*}
$$

(4)

Let us look for solutions to system (4) that correspond to the periodic modes of the machine oscillation close to harmonic with one contact interaction of the working bodies during the period at which the vibration exciters rotate almost uniformly with the same angular velocity $\omega$. The generating system of equations (4) can be solved in the form of: $\varphi = \sigma_1(\omega t + \varphi_1)$ and $q_2 = \nu \cos(\omega t + \psi)$, where $\sigma = 1$ when the debalance rotates in positive direction of $\varphi$ and $\sigma = -1$ when it rotates in opposite direction. Let’s introduce variable substitutions as follows: $\tilde{q}_2 = -\nu \nu \sin(\omega t + \psi)$, $\varphi = \frac{\nu}{\nu} \sigma_1 \varphi_1 + \sigma_2 \nu \nu_2$, $\bar{\nu} = \frac{\nu}{\nu}(\sigma_1 \varphi_1 - \sigma_2 \nu \nu_2)$, $\psi = \omega t$, $\tilde{\nu} = \nu$. After substituting these expressions in (4) taking into account $u_{22} = -u_{23} = u$ we obtain a system of equations in standard form:

$$
\begin{align*}
&\ddot{q} = -\frac{\epsilon}{p_2 m_{22}^2} [-b_2^2 \dot{q}_2 - 2u \Phi^* + u(P_2 - P_3)]\sin(\omega t + \psi) \\
&\dot{\psi} = \epsilon \omega - \frac{\epsilon}{q \nu_{22}^2} [-b_2^2 \dot{q}_2 - 2u \Phi^* + u(P_2 - P_3)]\cos(\omega t + \psi) \\
&\dot{\omega} = \frac{\nu}{\nu} \epsilon [L_1(\omega + \nu) + L_2(\omega - \nu) - R_1(\omega + \nu) - R_2(\omega - \nu) + \\
&\quad + m_\sigma u \sum_{i=1}^n u_{1i} \dot{q}_i (\sin(\omega t + \psi) \cos(\alpha_1 - \psi) - \cos(-\alpha_2 - \psi))) + \\
&\quad + \cos(\omega t + \psi) (\sin(\alpha_1 - \psi) + \sin(-\alpha_2 - \psi))]]/
\end{align*}
$$

(5)

where

$$\begin{align*}
P_2 - P_3 &= m_\sigma r [\omega + \nu (\sin(\omega t + \psi) \cos(\alpha_1 - \psi) + \cos(\omega t + \psi) \sin(\alpha_1 - \psi))] + \\
&\quad + (\omega + \nu)^2 \cos(\omega t + \psi) \sin(\alpha_1 - \psi) - \sin(\omega t + \psi) \sin(\alpha_1 - \psi)) - \\
&\quad - (\omega + \nu)^2 \cos(\omega t + \psi) \cos(-\alpha_2 - \psi) - \cos(\omega t + \psi) \sin(-\alpha_2 - \psi)) - \\
&\quad - (\omega - \nu)^2 \cos(\omega t + \psi) \cos(-\alpha_2 - \psi) + \sin(\omega t + \psi) \sin(-\alpha_2 - \psi))] = 0. \quad \text{for } 2\nu \nu \cos(\omega t + \psi) - \Delta \leq 0,
\end{align*}$$

and the small parameters are denoted as $\epsilon \omega = \nu = \nu$, $\nu = \nu$.

Averaging the right-hand sides in a first approximation, we obtain a system of five interconnected equations:

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\[
\begin{align*}
\dot{q} &= -\frac{1}{p_2m_{22}} \left[ b_{2z}^* q p_2 + \Phi^* - \frac{1}{2} u m_e r((\omega + \nu)^2 \sin(\alpha_1 - \psi)) + \right. \\
&\quad \left. + \frac{1}{2} u m_e \nu (\omega - \nu)^2 \sin(-(\alpha_2 - \psi)) \right] \\
\dot{\psi} &= p_2 - \omega - \frac{1}{q p_2 m_{22}} \left[ \Phi^{**} + \frac{1}{2} u m_e r((\omega + \nu)^2 \cos(\alpha_1 - \psi)) - \right. \\
&\quad \left. - \frac{1}{2} u m_e \nu (\omega - \nu)^2 \cos(-(\alpha_2 - \psi)) \right] \\
\dot{\omega} &= \frac{1}{2} \left( L_1 (\omega + \nu) + L_2 (\omega - \nu) - R_1 (\omega + \nu) - R_2 (\omega - \nu) - \right. \\
&\quad \left. - \frac{1}{2} u m_e r u p_2 \nu q \sin(\psi) + \sin(-(\alpha_2 - \psi)) \right) / \Omega \\
\dot{\psi} &= \frac{1}{2} \left( L_1 (\omega + \nu) - L_2 (\omega - \nu) - R_1 (\omega + \nu) + R_2 (\omega - \nu) - \right. \\
&\quad \left. - \frac{1}{2} u m_e r u p_2 \nu q (\sin(\psi) - \sin(-(\alpha_2 - \psi))) \right) / \Omega \\
\dot{\gamma} &= \nu
\end{align*}
\]

where

\[
\Phi^{**} = \begin{cases} 
0, \text{ for } q \leq \Delta/2u, \\
\frac{2 b_2 u^2}{\pi} q \psi, \text{ for } q > \Delta/2u.
\end{cases}
\]

\[
\Phi^{***} = \begin{cases} 
0, \text{ for } q \leq \Delta/2u, \\
-\frac{2 c_2 u}{\pi} (u q (\psi) + \frac{1}{2} \nu^2 (2 \psi)), \text{ for } q > \Delta/2u.
\end{cases}
\]

Equations (5) allows to study the stationary resonance modes of oscillations of the system under consideration. Equating the right-hand sides of system (6) to zero, we obtain a system of equations for determining stationary values \( q, \psi, \omega, \nu, \gamma \). Note that when studying stationary modes from the last equation of system (6) it follows that \( \nu = 0 \). Thus, the remaining four equations contain five unknowns, including two initial phases \( \alpha_1 \) and \( \alpha_2 \) of vibration extractors rotation, determined by the initial conditions. Due to the appropriate choice of the time reference, it is possible to achieve that one of the initial phases of debalance rotation is equal to zero, for example, \( \alpha_1 = 0 \). Then from the remaining four equations we can determine the remaining four unknowns \( q, \psi, \omega, \alpha \).

Consider, for example, the system vibrations at the same characteristics of the driving moments of electric motors and the moments of resistance to debalance rotation equal to \( L(\omega) \) and \( R(\omega) \), respectively. Then the fourth equation of system (6) is:

\[
-\frac{1}{2} u m_e r u p_2 \nu q (\sin(-\psi) + \sin(-\alpha_2 - \psi)) = 0.
\]

Solution of this equation gives two possible modes of the debalance rotation with mutual phase shift \( \Delta \alpha = \alpha_2 - \alpha_1 = \alpha_2 = 2 \psi \) and \( \alpha_2 = \pi \).

The value \( \alpha_2 = \pi \) corresponds to the antiphase rotation of debalances. In this case, the dependences of the amplitude and phase of the oscillations on the frequency are:

\[
q = \frac{u m_e r \omega^2}{p_2^2 \sqrt{((1 - \omega^2) m_{22} + \frac{2 c_2 u^2}{\pi p_2^2} (\psi))^2 + \left( \frac{1}{2} b_2^* + \frac{2 b_2 u^2}{\pi r^2} g(\alpha) \right)^2}}, \quad \sin(\psi) = \frac{-\left( \frac{1}{2} b_2^* + \frac{2 b_2 u^2}{\pi r^2} g(\alpha) \right)}{\sqrt{((1 - \omega^2) m_{22} + \frac{2 c_2 u^2}{\pi p_2^2} (\psi))^2 + \left( \frac{1}{2} b_2^* + \frac{2 b_2 u^2}{\pi r^2} g(\alpha) \right)^2}}.
\]

With the expression found for \( q \), the rotational frequency of the debalances is determined from the nonlinear equation \( 2 (L(\omega) - R(\omega)) \omega - (\frac{1}{2} b_2^* + b_2 u^2) p_2^2 q^2 = 0 \) representing equation of power balance in the system in a first approximation.

The value \( \alpha_2 = 2 \psi \) corresponds to the only possible oscillation mode with frequency \( \omega = p_2 + (c_2 u^2 / p_2 m_{22}^2) g(q) \), which corresponds to the resonant oscillations of the system. Note that this
expression is an equation for determining the eigenfrequency of a system depending on $\Delta$ and the oscillation amplitude $q$. In this case, $q$ and $\psi$ defined by the expressions

$$q^2 = \frac{2(L(\omega)-R(\omega)) (p_2 + \frac{c_2 \omega^2}{\pi p_2 m_2})}{(\frac{b_1^2}{2} + \frac{b_3 \pi^2}{\pi^2 - q^2}) p_2^2}, \sin(\psi) = \frac{L(\omega)-R(\omega)}{2 \mu m r \omega q}.$$

3. Simulation results

Figure 2 shows the dependences of the oscillations amplitude $q$ on the frequency $\omega$ for various values of the gap $\Delta$, for $u m c \pi r / m = \rho = 0.1$, $b/2mp_2 = \lambda = 0.03$, $c_2/c = n_1 = 2$, $b_2/b = n_2 = 10$, $\alpha_2 = \pi$. One can see that in the frequency range near the second resonance, a change in $\Delta$ leads to a significant distortion of the frequency characteristics of the system. At the same oscillation excitation frequency, a decrease in the gap leads to a decrease in the oscillation amplitudes.

![Figure 2. Oscillations amplitude $q$ depending on the frequency $\omega$.](image)

The stability of the obtained solutions was studied based on the analysis of the roots of the characteristic polynomial for the system in variations:

$$
\begin{align*}
\delta q &= \left( -\lambda p_2 - \frac{2\lambda n_2 p_2 (g(q) + q \frac{\partial g(q)}{\partial q})}{\pi} \right) \delta q + \left( -\frac{0.5\rho \Delta \omega^2 \cos(\psi)}{u p_2} \right) \delta \psi + \left( -\frac{\rho \Delta \omega \sin(\psi)}{u p_2} \right) \delta \omega \\
\delta \psi &= \left( \frac{0.5\rho \Delta \omega^2 \cos(\psi)}{q^2 u p_2} - \frac{p_2 n_2}{\pi} \frac{\partial g(q)}{\partial q} \right) \delta q + \left( \frac{0.5\rho \Delta \omega^2 \sin(\psi)}{q u p_2} \right) \delta \psi + \left( -1 - \frac{\rho \Delta \omega \sin(\psi)}{q u p_2} \right) \delta \omega \\
\delta \omega &= \frac{1}{f} (0.5 \rho m p_2 u \Delta \omega \sin(\psi)) \delta q + \frac{1}{f} (0.5 \rho m p_2 u \Delta \omega \cos(\psi)) \delta \psi + \cdots \\
&+ \frac{1}{f} \left( -0.5 (k_1 + k_2) + 0.5 \rho m p_2 u \Delta \sin(\psi) \right) \delta \omega + \frac{1}{f} \left( -0.5 (k_1 - k_2) \right) \delta \nu \\
\delta \nu &= \frac{1}{f} \left( -0.5 (k_1 - k_2) \right) \delta \omega + \frac{1}{f} \left( -0.5 (k_1 + k_2) \right) \delta \nu - \frac{1}{f} \left( 0.5 \rho m p_2 u \Delta \omega \cos(\psi) \right) \delta \gamma \\
\delta \gamma &= \delta \nu
\end{align*}
$$

where $k_1$ and $k_2$ – coefficients determining the slope of the torque characteristics of the motors of the left and right debalances [3, 4].

Figure 3 shows the dependences of the amplitude of oscillations $q$ on the frequency $\omega$, at which stable modes with synchronous antiphase rotation of unbalances are highlighted in red bold points. The results were obtained for $p=62.8$ rad/s, $m=1$ kg, $J=1 \cdot 10^{-3}$ kg m$^2$, $k_1=k_2=10$ N m s rad. Thin blue
lines correspond to unstable modes. It is seen that a decrease in the gap leads to the expansion of the frequency range of stable synchronous antiphase rotation of the unbalances. Note that for the same $\Delta$ several solutions can correspond to the same excitation frequency – for example points $A$, $B$ and $C$ at $\omega^*$ in Figure 3. Moreover, point $B$, as it is known, corresponds to a fundamentally unstable solution. Point $A$ corresponds to a stable solution with another (in-phase) form of synchronization of the debalance rotation, which could be found only taking into account vibration modes caused by the coordinates $q_1$ and $q_3$.

![Diagram](image)

**Figure 3.** Stable and unstable modes of debalance synchronization.

### 4. Conclusion

The presented mathematical model of a vibrating machine in the form of a three-mass oscillatory system with self-synchronizing unbalance vibration exciters and a nonlinear elastic-viscous element between its working bodies allows one to take into account the intermittent nature of the interaction of working bodies with the processed material and to analyze its effect on synchronous modes of machine vibrations. It is shown that taking into account intermittent interaction leads to a significant change in the dynamic characteristics of the system, including a change in the frequency range of stability of antiphase vibrations of the vibromachine working bodies and antiphase synchronization of the debalance rotation. The presented model can be used, for example, to analyze the dynamics of vibrating jaw crushers.

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