EXPONENTIAL MIXING PROPERTY FOR AUTOMORPHISMS OF COMPACT KÄHLER MANIFOLDS

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ABSTRACT. Let $f$ be a holomorphic automorphism of a compact Kähler manifold. Assume moreover that $f$ admits a unique maximal dynamic degree $d_p$ with only one eigenvalue of maximal modulus. Let $\mu$ be its equilibrium measure. In this paper, we prove that $\mu$ is exponentially mixing for all d.s.h. test functions.

1. Introduction and main results

Let $(X, \omega)$ be a compact Kähler manifold of dimension $k$ and let $f$ be a holomorphic automorphism of $X$. Denote by $f^*$ the pull-back operator acting on the Hodge cohomology groups $H^{p,q}(X, \mathbb{C})$. Recall that the dynamic degree of order $q$ of $f$ is the spectral radius of $f^*$. Khovanskii-Teissier-Gromov [9] proved that the function $q \mapsto \log d_q$ is concave. Thus there are integers $0 \leq p \leq p' \leq k$ such that

$$1 = d_0 < \cdots < d_p = \cdots = d_{p'} > \cdots > d_k = 1.$$  

When $p = p'$ and in addition, when $f^*$ acting on $H^{p,p}(X, \mathbb{C})$, admits only one eigenvalue of maximal modulus (necessary equal to $d_p$), there is a unique invariant probability measure $\mu := T_+ \wedge T_-$, where $T_+$ is the Green $(p,p)$-current of $f$ and $T_-$ is the Green $(k-p,k-p)$-current of $f^{-1}$. They satisfy $f^*(T_+) = d_p T^+$ and $f_*(T_-) = d_{k-p} T_-$. Moreover, for any positive closed $(p,p)$-current (resp. $(k-p,k-p)$-current) $S$ of mass 1, we have $d_p^{-n} (f^n)^* S$ (resp. $d_{k-p}^{-n} (f^{-n})_* S$) converge to $T_+$ (resp. $T_-$). And $T_+$ (resp. $T_-$) is the unique positive closed current in the class $\{T_+\}$ (resp. $\{T_-\}$) (see [8]). The measure $\mu$ is called the equilibrium measure of $f$.

Recall that a function is quasi-plurisubharmonic (quasi-p.s.h for short) if locally it is the difference of a plurisubharmonic (p.s.h. for short) function and a smooth one. The following theorem is our first main result.

**Theorem 1.1.** Let $f$ be a holomorphic automorphism on a compact Kähler manifold $X$ of dimension $k$ and let $\mu$ be its equilibrium measure. Let $d_q$ be the dynamic degree of order $q$, $0 \leq q \leq k$. Assume that there is a integer $p$ such that $d_p$ is strictly large than other dynamic degrees and $d_p$ admits only one eigenvalue of maximal modulus $d_p$. Then $\mu$ is exponentially mixing for bounded quasi-p.s.h. observables. More precisely, if $\delta$ is a constant such that $\max \{d_{p-1}, d_{p+1} \} < \delta < d_p$ and all the eigenvalues of $f^*$, acting on $H^{p,p}(X, \mathbb{C})$, except $d_{p'}$ are strictly smaller than $\delta$, then there exists a constant $c > 0$, such that

$$\left| \int (\varphi \circ f^n) \psi \, d\mu - \left( \int \varphi \, d\mu \right) \left( \int \psi \, d\mu \right) \right| \leq c \left( \frac{d_p}{\delta} \right)^{-n/2} \|\varphi\|_{L^\infty} \|\psi\|_{L^\infty}$$

for all $n \geq 0$ and all bounded quasi-p.s.h. functions $\varphi$ and $\psi$ satisfy $d\varphi \geq -\omega, d\psi \geq -\omega$. 

Another version of Theorem 1.1 has been proved in [7] for $\varphi, \psi \in C^2$ and it can be extended to $C^\alpha$ case, $0 < \alpha \leq 2$, using interpolation theory between Banach spaces. In this case, one considers the new system $(z, w) \mapsto (f^{-1}(z), f(w))$ on $X \times X$ and the test function $\varphi(z)\psi(w)$, which plays a linear “role” in the new system. Since $\varphi(z)\psi(w)$ is of class $C^2$ and in particular, it is Hölder continuous, some estimates of super-potentials on the currents with Hölder continuous super-potentials imply the desired result.

However, when $\varphi$ and $\psi$ are not of class of $C^2$, then same idea can not be directly applied since $\varphi(z)\psi(w)$ is not Hölder continuous in this case. In this paper, we do some regularization of quasi-p.s.h. functions, then we combine the idea above with some techniques in [11].

Recall that a function $u$ on $X$ with values in $\mathbb{R} \cup \{\pm \infty\}$ is said to be d.s.h. if outside a pluripolar set it is equal to a difference of two quasi-p.s.h. functions. Two d.s.h. functions are identified when they are equal out of a pluripolar set. Denote the set of d.s.h. functions by $\text{DSH}(X)$. Clearly it is a vector space and equips with a norm

$$
\|u\|_{\text{DSH}} := \left| \int_X u \omega^2 \right| + \min \|T^\pm\|,
$$

where the minimum is taken on all positive closed $(1, 1)$-currents $T^\pm$ such that $dd^c u = T^+ - T^-$. A positive measure $\nu$ on $X$ is said to be moderate if $\nu$ has no mass on pluripolar sets and for any bounded family $\mathcal{F}$ of d.s.h. functions on $X$, there are constants $\alpha > 0$ and $c > 0$ such that

$$
\nu\{z \in X : |\psi(z)| > M\} \leq ce^{-\alpha M}
$$

for $M \geq 0$ and $\psi \in \mathcal{F}$ (see [3, 4, 6]). The papers [3, 8] show that if $f$ is a holomorphic automorphism of a compact Kähler surface or more generally, on a compact Kähler manifold, then the equilibrium measure $\mu$ of $f$ is moderate. Using the moderate property of $\mu$ and following the same approach as in the proof of Corollary 1.3 in [11], we get the following theorem, which removes the boundedness conditions of $\varphi$ and $\psi$.

**Theorem 1.2.** Let $f$ be a holomorphic automorphism of positive entropy on a compact Kähler surface $X$ and let $d_1$ be the dynamic degree of order of 1. Then the equilibrium measure $\mu$ is exponentially mixing for all d.s.h observables. More precisely, if $\delta$ is a constant satisfies $1 < \delta < d_1$, then for any two d.s.h. functions $\varphi$ and $\psi$, there exists a constant $c > 0$ such that

$$
\left| \int (\varphi \circ f^n) \psi d\mu - \left( \int \varphi d\mu \right) \left( \int \psi d\mu \right) \right| \leq c(d_1/\delta)^{-n/2}
$$

for all $n \geq 0$.

Now we consider a particular case. When $X$ is a compact Kähler surface and $f$ is of positive entropy, Gromov [10] and Yomdin [12] showed that the entropy is equal to $\log d_1$. Thus in this case, $d_1 > 1$. Moreover, Cantat [11] proved that the eigenvalues of $f^*$, acting on $H^{1,1}(X, \mathbb{C})$, are $d_1, 1/d_1$ and others with modulus 1. Thus we get the following corollary.
Corollary 1.3. Let $f$ be a holomorphic automorphism of positive entropy on a compact Kähler surface $X$. Then the equilibrium measure $\mu$ is exponentially mixing for all d.s.h. observables.

In this paper, the symbols $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant.

Acknowledgements: This work was supported by the grant: AcRF Tier 1 R-146-000-248-114 from National University of Singapore.

2. Super-potentials of currents

In this section, we will introduce the notion called super-potential. The readers may refer to [5, 8] for details. Some estimates of super-potentials on a family of currents with Hölder continuous super-potentials also be obtained at the end of this section.

Denote by $D^q$ the real space that generated by all positive closed $(q,q)$-currents on $X$.

Define a norm $\| \cdot \|_*$ on $D^q$ by

$$\| \Omega \|_* := \min \{ \| \Omega^+ \| + \| \Omega^- \| \},$$

where $\| \Omega^\pm \| := \langle \Omega^\pm, \omega^{k-q} \rangle$ are the mass of $\Omega^\pm$, and the minimum is taken over all the positive closed currents $\Omega^\pm$ with $\Omega = \Omega^+ - \Omega^-$. Observe that $\| \Omega^\pm \|$ only depend on the cohomology classes of $\Omega^\pm$ in $H^{q,q}(X, \mathbb{R})$. We have the following lemma.

Lemma 2.1. Let $\Omega$ be a real $ddc$-exact $(q,q)$-current on $X$ and assume $\Omega \geq -S$ for some positive closed $(q,q)$-current $S$, then $\| \Omega \|_* \leq 2 \| S \|$.

Proof. Note that $\Omega + S$ is a positive closed current and we can write $\Omega$ as

$$\Omega = (\Omega + S) - S.$$

The mass of $\Omega + S$ is $\| S \|$ because $\Omega$ is $ddc$-exact. \qed

We introduce the $*$-topology on $D_q$: for a sequence of currents $S_n$ in $D_q$, we say $S_n$ converge to a current $S$ in $D_q$ if $S_n$ converge to $S$ in the sense of currents and if $\| S_n \|_*$ are uniformly bounded. Note that smooth forms are dense in $D_q$ for this topology.

Let $D^0_q$ be the subspace of $D_q$ which contains all the currents of class $\{0\}$ in $H^{q,q}(X, \mathbb{R})$. It is not hard to see $D^0_q$ is closed under the above topology.

Now we define the super-potential of a current $S \in D_q$. Fix a basis of $H^{q,q}(X, \mathbb{R})$, denoted by $\{ \alpha \} := \{ \{ \alpha_1 \}, \ldots, \{ \alpha_t \} \}$. We can assume all the $\alpha_j$ are smooth forms. For any $R \in D^0_{k-q+1}$, there exists a real $(k-q, k-q)$-current $U_R$ such that $ddc U_R = R$. We call $U_R$ a potential of $R$. After adding some closed form to $U_R$ we can assume $\langle U_R, \alpha_j \rangle = 0$ for all $1 \leq j \leq t$, after that we say $U_R$ is $\alpha$-normalized. If in addition, $R$ is smooth, then we can choose $U_R$ smooth.

The $\alpha$-normalized super-potential $U_S$ of $S$ is a linear functional on the smooth forms in $D^0_{k-q+1}$, and it is defined by

$$U_S(R) := \langle S, U_R \rangle,$$
where $U_R$ is a smooth $\alpha$-normalized potential of $R$. Note that $\mathcal{U}_S(R)$ does not depend on the choice of $U_R$.

If $\mathcal{U}_S$ can be extended continuously to a linear functional on $\mathcal{D}_{k-q+1}$ for the $\star$-topology we defined above, then we say $S$ has a continuous super-potential. If $S \in \mathcal{D}_q$, then $\mathcal{U}_S$ does not depend on the choice of $\alpha$. If $S$ is smooth, then it has a continuous super-potential and we have $\mathcal{U}_S(R) = \mathcal{U}_R(S)$, where $\mathcal{U}_R$ is the super-potential of $R$. The equality still holds when we only assume $S$ has a continuous super-potential (see [8]).

For any $0 < l < \infty$, we define a norm $\| \cdot \|_{\mathcal{E}^{-l}}$ and a distance $\text{dist}_l$ on $\mathcal{D}_p$ by

$$\| \Omega \|_{\mathcal{E}^{-l}} := \sup_{\| \Phi \|_{\mathcal{E}^{-1}} \leq 1} |\langle \Omega, \Phi \rangle| \quad \text{and} \quad \text{dist}_l(\Omega, \Omega') := \| \Omega - \Omega' \|_{\mathcal{E}^{-l}},$$

where $\Phi$ is a smooth test form of bidegree $(k - q, k - q)$ on $X$. For $0 < l < l' < \infty$, on any $\| \cdot \|_*$-bounded subset of $\mathcal{D}_p$, we have

$$\text{dist}_{l'} \leq \text{dist}_l \leq c_{l,l'}(\text{dist})^{l/l'}$$

for some positive constant $c_{l,l'}$ (see [8]).

For $S \in \mathcal{D}_q$ and constants $l > 0, 0 < \lambda \leq 1, M \geq 0$, a super-potential $\mathcal{U}_S$ of $S$ is said to be $(l, \lambda, M)$-Hölder continuous if it is continuous and it satisfies

$$|\mathcal{U}_S(R)| \leq M \| R \|_{\mathcal{E}^{-l}}^\lambda$$

for all $R \in \mathcal{D}_{k-q+1}$ with $\| R \|_* \leq 1$. If $l' > 0$ is another constant, the above identity for $\text{dist}_l$ and $\text{dist}_{l'}$ implies that $\mathcal{U}_S$ is also $(l',\lambda,M')$-Hölder continuous for some constants $\lambda'$ and $M'$ independent of $S$. And this definition does not depend on the normalization of the super-potential. We need the following two lemmas (see [7]).

**Lemma 2.2.** Let $R \in \mathcal{D}_{k-p+1}$ with $\| R \|_* \leq 1$ and $\mathcal{U}_R$ is $(2, \lambda, M)$-Hölder continuous. There is a constant $A > 0$ independent of $R, \lambda$ and $M$ such that the super-potential $\mathcal{U}_S$ of $S$ satisfies

$$|\mathcal{U}_S(R)| \leq A(1 + \lambda^{-1} \log^+ M),$$

for any $S \in \mathcal{D}_p$ with $\| S \|_* \leq 1$, where $\log^+ := \max\{0, \log\}$.

**Lemma 2.3.** Let $f, p$ be as in Theorem [7] and let $R \in \mathcal{D}_{k-p+1}$ whose super-potential $\mathcal{U}_R$ is $(2, \lambda, M)$-Hölder continuous. Then there is a constant $A_0 \geq 1$ independent of $R, \lambda, M$ such that the super-potential $\mathcal{U}_{f(R)}$ of $f(R)$ is $(2, \lambda, A_0 M)$-Hölder continuous.

We will use the above two lemmas to show the following result. A simple version was shown in [7], Proposition 3.1], which is crucial in the proof of exponential mixing theorem for $\mathcal{C}^\alpha$ observables for $0 < \alpha \leq 2$. Since $T_+$ is the unique positive current in $\{T_+ \}$, if $S \in \mathcal{D}_p$, then $d_p(f^n)^*(S)$ converge to a multiple of $T_+$.

**Proposition 2.4.** Let $f, d_p, \delta$ be as in Theorem [7] and $S \in \mathcal{D}_p$. Let $r$ be the constant such that $d_p(f^n)^*(S)$ converge to $rT_+$. Let $R_\epsilon, 0 < \epsilon \leq 1/2$ be a family of currents in $\mathcal{D}_{k-p+1}$ with $\| R_\epsilon \|_* \leq 1$ whose super-potentials $\mathcal{U}_{R_\epsilon}$ are $(2, \lambda, \epsilon^{-2})$-Hölder continuous. Let $\mathcal{U}_n$ and $\mathcal{U}_+$ be the $\alpha$-normalized super-potential of $d_p^{-n}(f^n)^*(S)$ and $T_+$ respectively. Then there exists a constant $c > 0$ such that

$$|\mathcal{U}_n(R_\epsilon) - r\mathcal{U}_+(R_\epsilon)| \leq -c\log \epsilon(d_p/\delta)^{-n}$$

for all $n$ and $\epsilon$. 

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Proof. It was shown in [7, Section 3] and [8, Section 4] that for $S \in D^0_p$ smooth and closed, we have $|\mathcal{U}_n(R) - r^k \mathcal{U}_n(R)| \lesssim (d_p/\delta)^{-n}$ for all $R \in D^0_{k-p+1}$ with $||R||_* \leq 1$. So we can subtract a smooth closed $(p, p)$-form from $S$ and assume that $S \in D^0_p$ and $r = 0$.

Fix a constant $\delta_0$ such that $\max\{d_{p-1}, d_{p+1}\} < \delta_0 < \delta$. Then $\delta_0$ satisfies the same properties of $\delta$ as in Theorem [1.1]. By Poincaré duality, the dynamic degree $d_\delta$ of $f$ is equal to the dynamic degree $d_{k-p}(f^{-1})$ of $f^{-1}$. Since the mass of a positive current can be computed cohomologically, we have $|\mathcal{U}_n(R_\epsilon)| \lesssim \delta_0^n ||R_\epsilon||_*$. 

Define $R_{n, \epsilon} := c^{-1} \delta_0^{-n} (f^n)_\ast (R_\epsilon)$ where $c \geq 1$ is a fixed constant large enough such that $||R_{n, \epsilon}||_* \leq 1$ for all $n$ and $\epsilon$. By Lemma [2.3], the super-potential of $R_{n, \epsilon}$, denoted by $\mathcal{U}_{R_{n, \epsilon}}$, is $(2, \lambda, A_0^n \epsilon^{-2})$-Hölder continuous for some $A_0 \geq 1$. On the other hand, since $S \in D^0_p$, by definition we have $\mathcal{U}_n(R_\epsilon) = \mathcal{U}_S((f^n)_\ast (R_\epsilon)) = c(d_p/\delta_0)^{-n} \mathcal{U}_S(R_{n, \epsilon})$.

Finally, applying Lemma [2.2], we obtain

$$|\mathcal{U}_n(R_\epsilon)| = c(d_p/\delta_0)^{-n} |\mathcal{U}_S(R_{n, \epsilon})| \lesssim (d_p/\delta_0)^{-n} \left(1 + \lambda^{-1} \log^+ (A_0^n \epsilon^{-2})\right) \lesssim - \log (d_p/\delta)^{-n}.$$ 

This finishes the proof. \qed

3. Exponentially mixing of $\mu$

From now on, let $f$ and $d_p$ be as in Theorem [1.1] and let $S$ be a positive closed $(p, p)$-current of mass 1 on $X$. Define a sequence of currents $S_n$ by $S_n := d_p^{-n} (f^n)_\ast (S)$. We know that $S_n$ converge to $T_\ast$. Denote by $\mathcal{U}_n$ and $\mathcal{U}_\ast$ the be the $\alpha$-normalized super-potentials of $d_p^{-n} (f^n)_\ast (S)$ and $T_\ast$ respectively.

Fix a bounded quasi-p.s.h. function $\phi$ on $X$ such that $dd^c \phi \geq -\omega$. We consider the same regularization of $\phi$ as in [2, Theorem 2.1], with minor modifications of the proofs, which shows that there exists a family of smooth functions $\phi_\epsilon, 0 < \epsilon \leq 1/2$ such that $dd^c \phi_\epsilon \geq -\omega$, and $\phi_\epsilon$ decreases to $\phi_0 := \phi$ when $\epsilon$ decreases to 0. And $\phi_\epsilon$ satisfies the following two estimates:

$$||\phi_\epsilon - \phi||_{L^1(\omega^k)} \lesssim \epsilon \quad \text{and} \quad ||\phi_\epsilon||_{L^2} \lesssim \epsilon^{-2}. \quad (3.1)$$

Notice that the first estimate may not hold if we remove the boundedness condition of $\phi$.

We define a sequence of functions $h_\epsilon$ and $h$ on $[0, 1/2]$ by

$$h_\epsilon(\epsilon) = \mathcal{U}_n(\ddbar \phi_\epsilon \wedge T_-) \quad \text{and} \quad h(\epsilon) = \mathcal{U}_\ast(\ddbar \phi_\epsilon \wedge T_-).$$

By definition, $h_\epsilon(\epsilon) = \langle S_\epsilon \wedge T_-, \phi_\epsilon \rangle + \gamma_\epsilon$ and $h(\epsilon) = \langle T_\ast \wedge T_-, \phi_\epsilon \rangle = \langle \mu, \phi_\epsilon \rangle + \gamma_\epsilon$, where $\gamma_\epsilon$ is a constant depends on $\epsilon$ such that $(\phi_\epsilon + \gamma_\epsilon) \wedge T_-$ is the $\alpha$-normalized potential of $dd^c \phi_\epsilon \wedge T_-$. Observe that $h_\epsilon$ converge pointwise to $h$ on $[0, 1/2]$.

On the other hand, note that $\{\omega^k\}$ is a basis of $H^{k,k}(X, \mathbb{R})$. We consider the $\{\omega^k\}$-normalized super potential of $\mu = T_\ast \wedge T_-$. Define the function $g(\epsilon) := \mathcal{U}_\mu(\ddbar \phi_\epsilon) = \langle T_\ast \wedge T_-, \phi_\epsilon \rangle - \langle \omega^k, \phi_\epsilon \rangle$.

We have the following lemma.

Lemma 3.1. The function $g$ is Hölder continuous at 0, i.e. $|g(\epsilon) - g(0)| \lesssim \epsilon^\alpha$ for some $\alpha > 0.$
Proof. Dinh-Sibony [8] showed that $T_+ \wedge T_-$ has a Hölder continuous super-potential. Thus by definition, we have
\[ |g(\epsilon) - g(0)| \leq M \text{dist}_2(\mathcal{F}_{\epsilon}, \mathcal{F})^\alpha \]
for some constants $\alpha, M > 0$.

Since $\phi_\epsilon$ is decreasing when $\epsilon$ decreases, by definition and estimates (3.1) we have
\[ \text{dist}_2(\mathcal{F}_{\epsilon}, \mathcal{F}) = \sup_{\|\Phi\|_{\mathcal{F}} \leq 1} \langle \mathcal{F}_{\epsilon} - \mathcal{F}, \Phi \rangle = \sup_{\|\Phi\|_{\mathcal{F}} \leq 1} \langle \phi_\epsilon - \phi, d\Phi \rangle \]
\[ \leq \|\phi_\epsilon - \phi\|_{L^1(\omega^k)} \lesssim \epsilon. \]

Therefore,
\[ |g(\epsilon) - g(0)| \leq M \text{dist}_2(\mathcal{F}_{\epsilon}, \mathcal{F})^\alpha \lesssim \epsilon^\alpha. \]

We complete the proof of this lemma. \(\square\)

Since $\phi_\epsilon$ is smooth for every $\epsilon \neq 0$, in particular it is Hölder continuous. We can easily get the estimates of $h_n(\epsilon)$ for $\epsilon \neq 0$ by using Proposition 2.4. Combining with the above lemma we have the following key proposition.

**Proposition 3.2.** Let $h_n, h, S, S_n$ and $\phi$ be as above. We have
\[ \langle S_n \wedge T_-, \phi \rangle - \langle T_+ \wedge T_-, \phi \rangle \lesssim (d_p/\delta)^{-n}. \]

**Proof.** Again, we fix a constant $\delta_0$ such that $\max\{d_{p-1}, d_{p+1}\} < \delta_0 < \delta$. By Lemma 2.4, $\|dd^c\phi_\epsilon\|_* \leq 2$ for all $\epsilon$, thus $\|dd^c\phi_\epsilon \wedge T_-\|_*$ are uniformly bounded for $1 < \epsilon \leq 1/2$. Since $\|\phi_\epsilon\|_{\mathcal{Y}} \lesssim \epsilon^{-2}$ and $T_-$ has a Hölder continuous super-potential (see [8]), by [8, Proposition 3.4.2], $dd^c\phi_\epsilon \wedge T_-$ has a $(2, \lambda, M\epsilon^{-2})$-Hölder continuous super-potential for some constant $\lambda$ and $M$. Multiplying $\phi$ by some constant allows us to assume $M = 1$ and $\|dd^c\phi_\epsilon \wedge T_-\|_* \leq 1$ for all $0 < \epsilon \leq 1/2$. Applying Proposition 2.4 to the family $dd^c\phi_\epsilon \wedge T_-$ instead of $R_\epsilon$, we get that for $0 < \epsilon \leq 1/2,
\[ h_n(\epsilon) - h(\epsilon) \lesssim -\log \epsilon (d_p/\delta_0)^{-n}. \]

Combine this with estimates (3.1) and Lemma 3.1, we have
\[ h_n(0) - h(0) = \langle S_n \wedge T_-, \phi \rangle - \langle T_+ \wedge T_-, \phi \rangle \leq \langle S_n \wedge T_-, \phi_\epsilon \rangle - \langle T_+ \wedge T_-, \phi_\epsilon \rangle \]
\[ = \langle S_n \wedge T_-, \phi_\epsilon \rangle - \langle T_+ \wedge T_-, \phi_\epsilon \rangle + \langle T_+ \wedge T_-, \phi_\epsilon \rangle - \langle T_+ \wedge T_-, \phi \rangle \]
\[ = h_n(\epsilon) - h(\epsilon) + g(\epsilon) + \langle \omega^k, \phi_\epsilon \rangle - \langle \omega^k, \phi \rangle \]
\[ \lesssim -\log \epsilon (d_p/\delta_0)^{-n} + \epsilon^\alpha + \epsilon \]

Finally, since $\alpha \leq 1$, by taking $\epsilon := (d/\delta_0)^{-n/\alpha}$, we get
\[ h_n(0) - h(0) \lesssim n \log (d_p/\delta_0) (d_p/\delta_0)^{-n} + (d_p/\delta_0)^{-n} \lesssim (d_p/\delta)^{-n}. \]

The proof is complete. \(\square\)

Now we can start to prove Theorem 1.1.

Proof of Theorem 1.1: Multiplying $\varphi$ and $\psi$ by some constant allows us to assume $\|\varphi\|_{L^\infty} \leq 1/2$ and $\|\psi\|_{L^\infty} \leq 1/2$. It is sufficient to prove Theorem 1.1 for $n$ even because
applying it to $\varphi$ and $\psi \circ f$ gives the case of odd $n$. Using the invariance of $\mu$, it is enough to show that

$$|\langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n}) \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle| \leq c(d_p/\delta)^{-n}$$

for some $c > 0$. It is equivalent to prove

$$\langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n}) \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle \leq c(d_p/\delta)^{-n}$$

and

$$\langle \mu, (\varphi \circ f^n)(-\psi \circ f^{-n}) \rangle - \langle \mu, \varphi \rangle \langle \mu, -\psi \rangle \leq c(d_p/\delta)^{-n}.$$  

For $j = 1, 2$, we define

$$\varphi_j^+ = \varphi^2 + j\varphi + A, \quad \varphi_j^- = \varphi^2 + j\varphi - A, \quad \psi_j^+ = \psi^2 + j\psi + A, \quad \psi_j^- = -\psi^2 - j\psi + A,$$

where $A$ is a positive constant whose value will be determined later. Consider the following eight functions on $X \times X$:

$$\Phi_{jl}^+(z, w) = \varphi_j^+(z)\psi_l^+(w), \quad \Phi_{jl}^-(z, w) = \varphi_j^-(z)\psi_l^-(w),$$

where $j, l = 1, 2$.

**Lemma 3.3.** The functions $\Phi_{jl}^\pm$ are quasi-p.s.h. on $X \times X$ for $A$ large enough.

**Proof.** We only show $\Phi_{11}^+$ is quasi-p.s.h. because the other cases can be obtained in the same way. By a direct computation (see also Lemma 3.1 in [11]), we have

$$i\partial\bar{\partial}\Phi_{11}^+ = (\varphi^2 + \psi + A)((2\varphi + 1)i\partial\varphi + 2i\partial\varphi \wedge \bar{\partial}\varphi) + (2\varphi + 1)(2\psi + 1)i\partial\varphi \wedge \bar{\partial}\psi + (2\varphi + 1)(2\psi + 1)i\partial\psi \wedge \bar{\partial}\varphi + (\varphi^2 + \varphi + A)((2\psi + 1)i\partial\varphi \wedge \bar{\partial}\psi + 2i\partial\psi \wedge \bar{\partial}\varphi)$$

Combining with the identity

$$i\partial\varphi \wedge \bar{\partial}\varphi + i\partial\psi \wedge \bar{\partial}\psi + i\partial\psi \wedge \bar{\partial}\varphi + i\partial\varphi \wedge \bar{\partial}\psi = i\partial(\varphi + \psi) \wedge \bar{\partial}(\varphi + \psi) \geq 0,$$

we get

$$i\partial\bar{\partial}\Phi_{11}^+ \geq (\varphi^2 + \psi + A)(2\varphi + 1)^2i\partial\varphi \wedge (\varphi^2 + \varphi + A)(2\psi + 1)\partial\bar{\partial}\psi$$

$$+ (2\varphi^2 + 2\varphi + 2A - (2\varphi + 1)(2\psi + 1))i\partial\varphi \wedge \bar{\partial}\varphi$$

$$+ (2\varphi^2 + 2\varphi + 2A - (2\varphi + 1)(2\psi + 1))i\partial\psi \wedge \bar{\partial}\psi.$$  

Recall that we assume $\|\varphi\|_{L^\infty} \leq 1/2$ and $\|\psi\|_{L^\infty} \leq 1/2$. So $2\varphi + 1 \geq 0, 2\psi + 1 \geq 0$. We take $A$ large enough such that $\varphi^2 + \psi + A, \varphi^2 + \varphi + A, 2\varphi^2 + 2\varphi + 2A - (2\varphi + 1)(2\psi + 1), 2\varphi^2 + 2\varphi + 2A - (2\varphi + 1)(2\psi + 1)$ are all positive. Since $\varphi$ and $\psi$ are quasi-p.s.h. on $X$ and $i\partial\varphi \wedge \bar{\partial}\varphi$, $i\partial\psi \wedge \bar{\partial}\psi$ are positive, we get that $\Phi_{11}^+$ is quasi-p.s.h. on $X \times X$.  

We choose $A$ large enough such that all the $\Phi_{jl}^\pm$ are bounded and quasi-p.s.h. on $X \times X$. Note that the choice of $A$ is independent of $\varphi$ and $\psi$. Define $\overline{\omega} := \pi_1^*\omega + \pi_2^*\omega$, where $\pi_1, \pi_2$ are the two canonical projections of $X \times X$ onto its factors. Then $\overline{\omega}$ is the canonical Kähler form on $X \times X$. From the computation in Lemma 3.3, we deduce that $dd^c\Phi_{11}^+ \geq -3A\overline{\omega}$. By Lemma 2.1, we get $\|dd^c\Phi_{11}^+\|_* \leq 6A$ on $X \times X$.

Next we consider the automorphism $F$ of $X \times X$ which is defined by

$$F(z, w) := (f^{-1}(z), f(w)).$$

By using Kähler formula, we obtain that the dynamic degree of order $k$ of $F$ is equal to $d^*_p$ (see also [7, Section 4]), and the dynamical
degrees and the eigenvalues of $F^n$ on $H^{k,k}(X \times X, \mathbb{R})$, except $d_p^2$, are strictly smaller than $d_p\delta$. Hence $F$ satisfy the conditions of $f$ in Theorem 1.1.

It is not hard to see that the Green $(k,k)$-currents of $F$ and $F^{-1}$ are $T_+ \otimes T_+$ and $T_+ \otimes T_-$ respectively, and they satisfies $F^*(T_+ \otimes T_-) = d_p^2(T_- \otimes T_+), F_*(T_+ \otimes T_-) = d_p^2(T_+ \otimes T_-).$ In particular, they have Hölder continuous super-potentials. Let $\Delta$ denote the diagonal of $X \times X$. Then $[\Delta]$ is a positive closed $(k,k)$-current on $X \times X$. With the help of $F$, we get the following estimates.

**Lemma 3.4.** There exists a constant $c > 0$ such that

\[
\langle \mu, (\varphi_j^+ \circ f^n)(\psi_i^+ \circ f^{-n}) \rangle - \langle \mu, \varphi_j^+ \rangle \langle \mu, \psi_i^+ \rangle \leq c(d_p/\delta)^{-n}
\]

and

\[
\langle \mu, (\varphi_j^- \circ f^n)(\psi_i^- \circ f^{-n}) \rangle - \langle \mu, \varphi_j^- \rangle \langle \mu, \psi_i^- \rangle \leq c(d_p/\delta)^{-n}
\]

for all $j, l$ and $n$.

**Proof.** We only show the lemma holds for $\varphi^+_1$ and $\psi^+_1$, the proofs of others are similar. For the automorphism $F$, consider the sequence of currents $d_p^{-n}(F^n)^*\mu$, which are currents of mass 1. Apply Proposition 3.2 to $d_p^{-2n}(F^n)^*[\Delta], T_+ \otimes T_-$ and $\Phi_{11}^+$ instead of $S_n, T_-$ and $\phi$, we get

\[
\langle d_p^{-2n}(F^n)^*[\Delta] \wedge (T_+ \otimes T_-), \Phi_{11}^+ \rangle - \langle (T_- \otimes T_+) \wedge (T_+ \otimes T_-), \Phi_{11}^+ \rangle \lesssim (d^2/(d\delta))^{-n}.
\]

On the other hand, by definition, we have

\[
\langle d_p^{-2n}(F^n)^*[\Delta] \wedge (T_+ \otimes T_-), \Phi_{11}^+ \rangle = \langle [\Delta], d_p^{-2n}(F^n)*[(T_+ \otimes T_-) \wedge \Phi_{11}^+] \rangle
\]

\[
= \langle [\Delta] \wedge (T_+ \otimes T_-), \Phi_{11}^+ \rangle
\]

\[
= \langle (T_+ \wedge T_-), (\varphi_1^+ \circ f^n)(\psi_1^+ \circ f^{-n}) \rangle,
\]

and

\[
\langle (T_- \otimes T_+) \wedge (T_+ \otimes T_-), \Phi_{11}^+ \rangle = \langle \mu \otimes \mu, \Phi_{11}^+ \rangle = \langle \mu, \varphi_1^+ \rangle \langle \mu, \psi_1^+ \rangle.
\]

This finishes the proof of this lemma. \qed

**End of the proof of Theorem 1.1** Consider $\alpha_{11}^+ = 2, \alpha_{22}^+ = \alpha_{11}^- = \alpha_{22}^- = 1$ and $\alpha_{21}^+ = \alpha_{12}^+ = \alpha_{21}^- = \alpha_{12}^- = 0$. A direct computation gives

\[
\sum_{j,l=1,2} \left( \alpha_{jl}^+(\varphi_j^+ \circ f^n)(\psi_l^+ \circ f^{-n}) + \alpha_{jl}^-(\varphi_j^- \circ f^n)(\psi_l^- \circ f^{-n}) \right)
\]

\[
= (\varphi \circ f^n)(\psi \circ f^{-n}) + \beta_1\varphi^2 \circ f^n + \beta_2\psi^2 \circ f^{-n} + \beta_3\varphi \circ f^n + \beta_4\psi \circ f^{-n} + \beta_5
\]

for some constants $\beta_t, 1 \leq t \leq 5$. We now apply this identity and Lemma 3.4. Observe that the invariance of $\mu$ implies that the terms involving $\beta_t$ cancel each other out. We obtain

\[
\langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n}) \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle \leq \left( \sum_{j,l=1,2} (\alpha_{jl}^+ + \alpha_{jl}^-) \right) c d^{-n} = 6 c d^{-n}.
\]

Similarly, taking $\gamma_{11}^- = 2, \gamma_{11}^+ = \gamma_{21}^- = \gamma_{12}^+ = \gamma_{22}^- = 1$ and $\gamma_{22}^+ = \gamma_{21}^+ = \gamma_{12}^- = 0$, we get

\[
\langle \mu, (\varphi \circ f^n)(-\psi \circ f^{-n}) \rangle - \langle \mu, \varphi \rangle \langle \mu, -\psi \rangle \leq \left( \sum_{j,l=1,2} (\gamma_{jl}^+ + \gamma_{jl}^-) \right) c d^{-n} = 6 c d^{-n}.
\]
The above two inequalities imply inequality (3.2) and finish the proof of Theorem 1.1.

Using the moderate property of $\mu$ and the technical of replacing $\delta$ by $\delta_0$, we can prove Theorem 1.2.

**Proof of Theorem 1.2.** It is enough to prove this theorem for all negative quasi-p.s.h. functions $\varphi$ and $\psi$. Multiplying them by some constant allows us to assume $dd^c\varphi \geq -\omega, dd^c\psi \geq -\omega$ and $\langle \mu, |\varphi| \rangle \leq 1, \langle \mu, |\psi| \rangle \leq 1$. Define
\[
\varphi_1 := \max\{\varphi, -M\}, \quad \psi_1 := \max\{\psi, -M\},
\]
and
\[
\varphi_2 := \varphi - \varphi_1, \quad \psi_2 := \psi - \psi_1.
\]

Then $\varphi_1$ and $\psi_1$ are bounded quasi-p.s.h. functions which satisfy $dd^c\varphi_1 \geq -\omega, dd^c\psi_1 \geq -\omega$. Fix a constant $\delta_0$ such that $1 < \delta_0 < \delta$. Applying Theorem 1.1 to $\varphi_1$ and $\psi_1$, we get
\[
\left| \int (\varphi_1 \circ f^n) \psi_1 \, d\mu - \left( \int \varphi_1 \, d\mu \right) \left( \int \psi_1 \, d\mu \right) \right| \leq \left( \frac{d_1}{\delta_0} \right)^{-n/2} M^2.
\]

On the other hand, since $\mu$ is moderate, we can repeat the same proof of Corollary 1.3 in [11] and get for some $\alpha > 0$,
\[
\|\varphi_2\|_{L^1(\mu)} \lesssim e^{-\alpha M/2}, \quad \|\psi_2\|_{L^1(\mu)} \lesssim e^{-\alpha M/2}, \quad \|\varphi_2\|_{L^2(\mu)} \lesssim e^{-\alpha M/2}, \quad \|\psi_2\|_{L^2(\mu)} \lesssim e^{-\alpha M/2}.
\]

From the invariance of $\mu$, we have that $\|\varphi_2 \circ f^n\|_{L^p(\mu)} = \|\varphi_2\|_{L^p(\mu)}$ and $\|\psi_2 \circ f^n\|_{L^p(\mu)} = \|\psi_2\|_{L^p(\mu)}$ for $1 \leq p \leq \infty$. We do the following direct computation,
\[
\left| \langle \mu, (\varphi \circ f^n) \psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle \right| = \left| \langle \mu, (\varphi_1 \circ f^n + \varphi_2 \circ f^n)(\psi_1 + \psi_2) \rangle - \langle \mu, \varphi_1 \rangle \langle \mu, \psi_1 \rangle + \langle \mu, \varphi_2 \rangle \langle \mu, \psi_2 \rangle \right|
\]
\[
\leq \left| \langle \mu, (\varphi_1 \circ f^n) \psi_1 \rangle - \langle \mu, \varphi_1 \rangle \langle \mu, \psi_1 \rangle \right| + \left| \langle \mu, (\varphi_1 \circ f^n) \psi_2 \rangle - \langle \mu, \varphi_1 \rangle \langle \mu, \psi_2 \rangle \right| + \left| \langle \mu, (\varphi_2 \circ f^n) \psi_1 \rangle - \langle \mu, \varphi_2 \rangle \langle \mu, \psi_1 \rangle \right| + \left| \langle \mu, (\varphi_2 \circ f^n) \psi_2 \rangle - \langle \mu, \varphi_2 \rangle \langle \mu, \psi_2 \rangle \right|
\]
\[
\leq \langle \mu, \varphi_1 \rangle \langle \mu, \psi_1 \rangle + M \|\varphi_2\|_{L^1(\mu)} + M \|\psi_2\|_{L^1(\mu)} + M \|\varphi_2\|_{L^2(\mu)} \|\psi_2\|_{L^2(\mu)} + \frac{M}{\delta_0} \|\varphi_2\|_{L^1(\mu)} \|\psi_2\|_{L^1(\mu)}
\]
\[
\leq \left( \frac{d_1}{\delta_0} \right)^{-n/2} M^2 + (2M + 2)e^{-\alpha M/2} + 2e^{-\alpha M}.
\]

Taking $M = \left( n \log \left( \frac{d_1}{\delta_0} \right) \right)/\alpha$, we obtain the estimate
\[
\left( \frac{d_1}{\delta_0} \right)^{-n/2} M^2 + (2M + 2)e^{-\alpha M/2} + 2e^{-\alpha M} \lesssim n^2 \left( \frac{d_1}{\delta_0} \right)^{-n/2} \lesssim \left( \frac{d_1}{\delta} \right)^{-n/2}.
\]

Therefore,
\[
\left| \int (\varphi \circ f^n) \psi \, d\mu - \left( \int \varphi \, d\mu \right) \left( \int \psi \, d\mu \right) \right| \lesssim \left( \frac{d_1}{\delta} \right)^{-n/2}.
\]

The proof is finished.
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