Global existence and optimal decay rates for three-dimensional compressible viscoelastic flows

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Abstract

In this paper, we are concerned with the global existence and optimal rates of strong solutions for three-dimensional compressible viscoelastic flows. We prove the global existence of the strong solutions by the standard energy method under the condition that the initial data are close to the constant equilibrium state in $H^2$-framework. If additionally the initial data belong to $L^1$, the optimal convergence rates of the solutions in $L^p$-norm with $2 \leq p \leq 6$ and optimal convergence rates of their spatial derivatives in $L^2$-norm are obtained.

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1 Introduction

In this paper, we are interested in three-dimensional compressible viscoelastic flows [3,5,12,23]:

\begin{align}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P(\rho) &= \alpha \text{div}(\rho FF^T), \\
F_t + u \cdot \nabla F &= \nabla u F,
\end{align}

for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$. Here $\rho, u \in \mathbb{R}^3$, $F \in M^{3 \times 3}$ (the set of $3 \times 3$ matrices with positive determinants) denote the density, the velocity, and the deformation gradient, respectively. The Lamé coefficients $\mu$ and $\lambda$ are satisfied the physical condition:

$\mu > 0, \quad 2\mu + 3\lambda > 0,$

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which ensures that the operator $-\mu \Delta - (\lambda + \mu)\nabla \text{div}$ is a strongly elliptic operator. The pressure term $P(\rho)$ is an increasing and convex function of $\rho$ for $\rho > 0$. The symbol $\otimes$ denotes the Kronecker tensor product, $F^T$ means the transpose matrix of $F$, and the notation $u \cdot \nabla F$ is understood to be $(u \cdot \nabla)F$. For system (1.1), the corresponding elastic energy is chosen to be the special form of the Hookean linear elasticity:

$$W(F) = \frac{\alpha}{2} |F|^2 + \frac{1}{\rho} \int_0^\rho P(s)ds, \quad \alpha > 0,$$

which, however, does not reduce the essential difficulties for analysis. Indeed, all the results we describe here can be generalized to a more general cases.

In this paper, we investigate the Cauchy problem of system (1.1) with the initial condition:

$$(\rho, u, F)|_{t=0} = (\rho_0(x), u_0(x), F_0(x)), \quad x \in \mathbb{R}^3.$$ (1.2)

We also assume that

$$\text{div}(\rho F^T) = 0, \quad F^{il}(0)\nabla_i F^{lj}(0) = F^{ij}(0)\nabla_i F^{lk}(0).$$ (1.3)

It is standard that the condition (1.3) is preserved by the flow, which has been proved in [7,21].

For the incompressible viscoelastic flows and related models, there are many important progress on classical solutions, refer to [1,2,9,13,16] and references therein. On the other hand, the global existence of weak solutions to the incompressible viscoelastic flows with large initial data is still an outstanding open question, although there are some progress in that direction [15,17,18]. For the compressible viscoelastic flows, to our knowledge, there are few results on the dynamics of global solutions to compressible viscoelastic flows, especially on the large time behavior. The local existence of multi-dimensional strong solution was obtained in [6], and the global existence of strong solution with the lowest regularity was shown in [7,21]. For the initial boundary value problem, global in time solution was proved to exist uniquely near the equilibrium state in [8,22].

In this paper, we firstly study the optimal time-decay rate of the global strong solutions to the Cauchy problem (1.1)-(1.2). To be more precise, the main purpose of this paper is to study the existence and uniqueness of global strong solutions and in particular the asymptotic behavior on the Cauchy problem of compressible viscoelastic flows. We prove the global existence of strong solutions by the standard energy method in spirit of Matsumura and Nishida [19,20]. In order to obtain the linear time-decay estimates, we need to analysis the properties of the semigroup, as in [10,11,14,24]. Unfortunately, it seems untractable, since the system (1.1) has thirteen equations. To overcome this difficulty, we take Hodge decomposition of the linear system, then it becomes two similar systems, each of those only involves two variables, which makes us be able to obtain the optimal time-decay estimates.

Our main results are formulated in the following theorem:
Theorem 1.1. Assume that the initial value $(\rho_0 - 1, u_0, F_0 - I) \in H^2(\mathbb{R}^3)$ satisfies the constraints (1.3), then there exists a constant $\delta_0$ such that if

$$|(\rho_0 - 1, u_0, F_0 - I)|_{H^2} \leq \delta_0,$$  

(1.4)

then there exists a unique globally strong solution $(\rho, u, F)$ of the Cauchy problem (1.1) – (1.2) such that for any $t \in [0, \infty)$,

$$|(\rho - 1, u, F - I)(\cdot, t)|_{H^2}^2 + \int_0^t |\nabla(\rho, F)|_{H^1}^2 + |\nabla u|_{H^2}^2 \leq C|(\rho_0 - 1, u_0, F_0 - I)|_{H^2}. \quad (1.5)$$

Moreover, if $(\rho_0 - 1, u_0, F_0 - I) \in L^1(\mathbb{R}^3)$, then

$$|(\rho - 1, u, F - I)(t)|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p})}, \quad \forall \ p \in [2, 6], \quad (1.6)$$

$$|\nabla(\rho - 1, u, F - I)(t)|_{H^1} \leq C(1 + t)^{-\frac{3}{4}}. \quad (1.7)$$

Finally, denote

$$(\varrho_0, m_0, \mathcal{F}_0) = (\rho_0 - 1, \rho_0 u_0, \rho_0 F_0 - I)$$

and assume that the Fourier transform $(\hat{\varrho}_0, \hat{m}_0, \hat{\mathcal{F}}_0)$ satisfies

$$|\hat{\varrho}_0| \geq c_0, \quad |\hat{m}_0| \leq |\xi|^\eta, \quad |\hat{\mathcal{F}}_0 - \hat{\mathcal{F}}_0| \leq |\xi|^\eta, \quad \text{for} \quad 0 \leq |\xi| \ll 1, \quad (1.8)$$

where $c_0$ and $\eta$ are two positive constants. Then we also have the lower bound time decay rate as

$$|(\rho - 1)(t)|_{L^2} \geq c_1(1 + t)^{-\frac{1}{4}}, \quad \quad (1.9)$$

$$|u(t)|_{L^2} \geq c_1(1 + t)^{-\frac{1}{4}}, \quad \quad (1.10)$$

$$|(F - I)(t)|_{L^2} \geq c_1(1 + t)^{-\frac{1}{4}}, \quad \quad (1.11)$$

where $c_1$ is a positive constant independent of time.

Notations. We denote by $L^p$, $W^{m,p}$ the usual Lebesgue and Sobolev spaces on $\mathbb{R}^3$ and $H^m = W^{m,2}$, with norms $|\cdot|_{L^p}$, $|\cdot|_{W^{m,p}}$ and $|\cdot|_{H^m}$ respectively. For the sake of conciseness, we do not distinguish functional space when scalar-valued or vector-valued functions are involved. We denote $\nabla = \partial_x = (\partial_1, \partial_2, \partial_3)$, where $\partial_i = \partial_{x_i}$, $\nabla_i = \partial_i$ and put $\partial_f = \nabla f = \nabla(\nabla^{-1} f)$. We assume $C$ be a positive generic constant throughout this paper that may vary at different places and the integration domain $\mathbb{R}^3$ will be always omitted without any ambiguity. Finally, $\langle \cdot, \cdot \rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$.

The rest of this paper is devoted to prove Theorem 1.1. In Section 2, we first reformulate the system and do some careful a priori estimates for the strong solutions. Then the global existence of the strong solutions is established by the standard continuity argument. In Section 3 we will derive the decay-in-time estimates for the linearized system and use the energy method to derive a Lyapunov-type energy inequality of all
the derivatives controlled by the first order derivatives, then we utilize the decay-in-time estimates for the linearized system to control the first order derivatives by the higher order derivatives. Hence, the optimal decay rates of the global strong solutions follow from these two kinds of estimates. In section 4, we establish the lower bound time decay rate for the global solution.

2 Global existence

2.1. Reformulation

In this subsection, we first reformulate the system (1.1). Without loss of generality, we assume \( P'(1) > 0 \), and denote \( \chi_0 = (P'(1))^{-\frac{1}{2}} \). For \( \rho > 0 \), system (1.1) can be rewritten as

\[
\begin{align*}
\rho_t + \rho \text{div } u + u \nabla \rho &= 0, \quad (2.1a) \\
u_t^i + u \cdot \nabla u^i - \frac{1}{\rho} (\mu \Delta u^i + (\lambda + \mu) \nabla_i \text{div } u) + \frac{P'(\rho)}{\rho} \nabla_i \rho &= \alpha F^j k \nabla_j F^{ik}, \quad (2.1b) \\
F_t + u \cdot \nabla F &= \nabla u F, \quad (2.1c)
\end{align*}
\]

where we used the condition \( \text{div}(\rho F^T) = 0 \) for all \( t \geq 0 \) which ensures that the i-th component of the vector \( \text{div}(\rho F^T) \) is

\[
\nabla_j (\rho F^{ik} F^{jk}) = \rho F^{jk} \nabla_j F^{ik} + F_{ik} \nabla_j (\rho F^{jk}) = \rho F^{jk} \nabla_j F^{ik}.
\]

Denote

\[
n(t, x) = \rho(\chi_0^2 t, \chi_0 x) - 1, \quad v(t, x) = \chi_0 u(\chi_0^2 t, \chi_0 x), \quad E(t, x) = F(\chi_0^2 t, \chi_0 x) - I,
\]

then

\[
\begin{align*}
n_t + \text{div } v &= f - v \cdot \nabla n, \quad (2.2a) \\
v_t^i + \mu \Delta v^i - (\lambda + \mu) \nabla_i \text{div } v + \nabla_i n - a \nabla_j E^{ij} &= g, \quad (2.2b) \\
E_t - \nabla v &= h - v \cdot \nabla E, \quad (2.2c)
\end{align*}
\]

where

\[
f = -n \nabla \cdot v, \quad h = \nabla v E, \quad a = \frac{1}{P'(1)}, \quad g^i = a E^{jk} \nabla_j F^{ik} - \frac{n}{1+n} (\mu \Delta v^i + (\lambda + \mu) \nabla_i \text{div } v) - v \cdot \nabla v^i - \left( \frac{P'(n+1)}{(1+n)P'(1)} - 1 \right) \nabla_i n.
\]

Again, without loss of generality, we will assume that \( a = 1 \) for the rest of this paper.

2.2. A priori estimate

As a classical argument, the global existence of solutions will be obtained by combining the local existence result with a priori estimates. Since the local strong solutions can
be proven by standard argument of Lax-Milgram theorem and the Schauder-Tychonoff fixed-point as [6] whose details we omit, global solutions will follow in a standard continuity argument after we establish (1.5) a priori. Therefore, we assume a priori that

\[ |(\rho - 1, u, F - I)|_{H^2} \leq \delta_0 \ll 1, \tag{2.3} \]

which is equivalent to

\[ |(n, v, E)|_{H^2} \leq \delta \ll 1. \tag{2.4} \]

Here \( \delta_0 \sim \delta \) is small enough. This, together with Sobolev's inequality, Hölder's inequality and Moser-type's inequality, implies in particular that

\[ |(n, v, E)|_{L^\infty} \leq C\delta. \tag{2.5} \]

This should be kept in mind in the rest of this paper.

For later use we first estimate the norm of \( f, g, h \). By (2.4), (2.5), together with Sobolev's inequality, Hölder's inequality and Moser-type's inequality, we easily deduce that

\[
|f, h|_{L^2} \leq |n, E|_{L^\infty}|\nabla v|_{L^2} \leq C\delta|\nabla v|_{L^2},
\]

\[
|\nabla (f, h)| \leq |(n, E)|_{L^\infty}|\nabla^2 v|_{L^2} + |\nabla (n, E)|_{L^3}|\nabla v|_{L^6} \leq C\delta|\nabla^2 v|_{L^2},
\]

\[
|\nabla^2 (f, h)| \leq C(|(n, E)|_{L^\infty}|\nabla^3 v|_{L^2} + |\nabla^2 (n, E)|_{L^2}|\nabla v|_{L^\infty}) \leq C\delta|\nabla^2 v|_{H^1},
\]

\[
|g|_{L^2} \leq C(|v|_{L^3}|\nabla v|_{L^6} + |n|_{L^\infty}|\nabla^2 v|_{L^2} + |(n, E)|_{L^\infty}|\nabla (n, E)|_{L^2})
\leq C\delta(|\nabla^2 v|_{L^2} + |\nabla (n, E)|_{L^2}),
\]

where we used the fact

\[
\frac{P'(n+1)}{(1+n)P'(1)} - 1 \sim O(1)n.
\]

In what follows, a series of lemmas on the energy estimates is given. Firstly the energy estimate of lower order for \((n, u, E)\) is obtained in the following lemma.

**Lemma 2.1.** Under the priori assumption (2.4), we have

\[
\frac{1}{2} \frac{d}{dt} |(n, v, E)|_{L^2}^2 + C|\nabla v|^2 \leq C\delta|\nabla (n, E)|_{L^2}^2. \tag{2.6}
\]

**Proof.** Multiply (2.2a), (2.2b), (2.2c) by \( n, v, E \) respectively and then integrating them over \( \mathbb{R}^3 \), we have

\[
\frac{1}{2} \frac{d}{dt} |(n, v, E)|_{L^2}^2 + \mu|\nabla v|_{L^2}^2 + (\mu + \lambda)|\nabla \cdot v|_{L^2}^2 = \langle f - v \cdot \nabla n, n \rangle + \langle g, v \rangle + \langle h - v \cdot \nabla E, E \rangle. \tag{2.7}
\]

The three terms on the right hand side of the above equation can be estimated as follows.
First, it holds that
\[ \langle f - v \cdot \nabla n, n \rangle = \langle -n \nabla \cdot v - v \cdot \nabla n, n \rangle = \langle v \cdot \nabla n, n \rangle. \]

It follows from Sobolev's inequality, Hölder's inequality and (2.4) that
\[ |\langle f - v \cdot \nabla n, n \rangle| \leq |n|_{L^\infty} |v|_{L^6} |\nabla n|_{L^2} \leq C |n|_{H^1} |\nabla v|_{L^2} |\nabla n|_{L^2} \leq C\delta (|\nabla n|_{L^2}^2 + |\nabla v|_{L^2}^2). \tag{2.8} \]

Similar to the proof of (2.8), we have
\[ |\langle h - v \cdot \nabla E, E \rangle| \leq C\delta (|\nabla E|_{L^2}^2 + |\nabla v|_{L^2}^2). \tag{2.9} \]

For the second term, we have
\[ |\langle g^i, v^i \rangle| = C (|\langle E^{jk} \nabla_j E^{ik}, v^i \rangle| + |\langle \frac{n}{1+n} \Delta v^i, v^i \rangle| + |\langle \frac{n}{1+n} \nabla \text{div} v, v^i \rangle| + |\langle v \cdot \nabla v^i, v^i \rangle| + |\langle \frac{P'(n+1)}{(1+n)P'(1)} - 1 \rangle \nabla_i n, v^i \rangle|). \tag{2.10} \]

As the proof of (2.8), it follows from Sobolev's inequality, Hölder's inequality and (2.4) that
\[ |\langle E^{jk} \nabla_j E^{ik}, v^i \rangle| \leq C\delta (|\nabla E|_{L^2}^2 + |\nabla v|_{L^2}^2), \tag{2.11} \]
\[ |\langle v \cdot \nabla v, v \rangle| \leq |v|_{L^3} |v|_{L^6} |\nabla v|_{L^2} \leq C |v|_{L^6} |\nabla v|_{L^2}^2 \leq C\delta |\nabla v|_{L^2}^2, \tag{2.12} \]
\[ |\langle \frac{n}{1+n} \Delta v^i, v^i \rangle| = |\langle \nabla \left( \frac{n}{1+n} \Delta v^i \right), v^i \rangle| + |\frac{n}{1+n} \nabla v^i, \nabla v^i \rangle \leq C (|\nabla n|_{H^1} |\nabla v|_{L^2} + |n|_{L^\infty} |\nabla v|_{L^2}) \leq C\delta |\nabla v|_{L^2}^2, \tag{2.14} \]
\[ |\langle \frac{n}{1+n} \nabla \text{div} v^i, v^i \rangle| \leq C\delta |\nabla v|_{L^2}^2. \tag{2.15} \]

Substituting (2.11)-(2.15) into (2.10) gives that the second term is bounded by
\[ |\langle g, v \rangle| \leq C\delta (|\nabla (n, E)|_{L^2}^2 + |\nabla v|_{L^2}^2). \tag{2.16} \]

Hence combining (2.7), (2.8), and (2.16) yields (2.6) since \( \delta > 0 \) is sufficiently small. This completes the proof of the lemma.

In the following lemma we give the energy estimate of the higher order for \((n, v, E)\).

**Lemma 2.2.** Under the assumption (2.4), we have
\[ \frac{1}{2} \frac{d}{dt} |\nabla (n, v, E)|_{H^1}^2 + C |\nabla^2 v|_{H^1}^2 \leq C\delta |\nabla (n, E)|_{H^1}^2. \tag{2.17} \]
Proof. Applying \( \nabla \) to (2.2a), (2.2b), (2.2c) and multiplying by \( \nabla n, \nabla v, \nabla E \) respectively, integrating over \( \mathbb{R}^3 \), we have

\[
\frac{1}{2} \frac{d}{dt} |\nabla(n, v, E)|_{L^2}^2 + \mu |\nabla^2 v|_{L^2}^2 + (\mu + \lambda) |\nabla(\nabla \cdot v)|_{L^2}^2
= \langle \nabla(f - v \cdot \nabla n), \nabla n \rangle + \langle \nabla g, \nabla v \rangle + \langle \nabla(h - v \cdot \nabla E), \nabla E \rangle.
\] (2.18)

Now let us estimate the right-hand side term by term. First of all, by Hölder’s inequality and Sobolev’s inequality, we have

\[
|\langle \nabla f, \nabla n \rangle| + |\langle \nabla h, \nabla E \rangle| \leq |\nabla f|_{L^2}|\nabla(n, E)|_{L^2} \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla(n, E)|_{L^2}^2).
\]

Next, integrating by parts, we get

\[
|\langle \nabla g, \nabla v \rangle| = |\langle g, \nabla^2 v \rangle| \leq C|g|_{L^2}|\nabla^2 v|_{L^2} \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla(n, E)|_{L^2}^2).
\]

Finally by symmetry, we have

\[
|\langle \nabla(v \cdot \nabla n), \nabla n \rangle| + |\langle \nabla(v \cdot \nabla E), \nabla E \rangle| = |\langle \nabla v \cdot \nabla n, \nabla n \rangle + \langle v \cdot \nabla^2 n, \nabla n \rangle + |\langle \nabla v \cdot \nabla E, \nabla E \rangle + \langle v \cdot \nabla E, \nabla E \rangle|
= |\langle \nabla v \cdot \nabla n, \nabla n \rangle + \frac{1}{2}\langle \text{div } v, |\nabla n|^2 \rangle + |\langle \nabla v \cdot \nabla E, \nabla E \rangle + \frac{1}{2}\langle \text{div } v, |\nabla E|^2 \rangle|
\leq C|\nabla v|_{L^\infty}|\nabla(n, E)|_{L^2}^2 \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla(n, E)|_{L^2}^2).
\]

Substituting these results into (2.18), we conclude

\[
\frac{1}{2} \frac{d}{dt} |\nabla(n, v, E)|_{L^2}^2 + C|\nabla^2 v|_{L^2}^2 \leq C\delta(|\nabla(n, E)|_{L^2}^2 + |\nabla^3 v|_{L^2}^2). \tag{2.19}
\]

Similarly, applying \( \nabla^2 \) to (2.2a), (2.2b), (2.2c) and multiplying by \( \nabla^2 n, \nabla^2 v, \nabla^2 E \) respectively, integrating over \( \mathbb{R}^3 \), we have

\[
\frac{1}{2} \frac{d}{dt} |\nabla^2(n, v, E)|_{L^2}^2 + \mu |\nabla^3 v|_{L^2}^2 + (\mu + \lambda) |\nabla(\nabla^2 \cdot v)|_{L^2}^2
= \langle \nabla^2(f - v \cdot \nabla n), \nabla^2 n \rangle + \langle \nabla^2 g, \nabla^2 v \rangle + \langle \nabla^2(h - v \cdot \nabla E), \nabla^2 E \rangle. \tag{2.20}
\]

To estimate the right-hand side of the above equation, we note, by Hölder’s inequality and Sobolev’s inequality, that

\[
|\langle \nabla^2 f, \nabla^2 n \rangle| + |\langle \nabla^2 h, \nabla^2 E \rangle| \leq |\nabla^2 f|_{L^2}|\nabla^2(n, E)|_{L^2} \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla^2(n, E)|_{L^2}^2).
\]

Integrating by parts, we have

\[
|\langle \nabla^2 g, \nabla^2 v \rangle| = |\langle \nabla^2 g, \nabla^3 v \rangle| \leq |\nabla^2 g|_{L^2}|\nabla^3 v|_{L^2} \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla^2(n, E)|_{L^2}^2).
\]

Finally, by symmetry, we have

\[
|\langle \nabla^2(v \cdot \nabla n), \nabla^2 n \rangle|
= |\langle \nabla^2 v \cdot \nabla n, \nabla^2 n \rangle + \langle \nabla v \cdot \nabla^2 n, \nabla^2 n \rangle + \langle v \cdot \nabla^3 n, \nabla^2 n \rangle|
= |\langle \nabla^2 v \cdot \nabla n, \nabla^2 n \rangle + \langle v \cdot \nabla^2 n, \nabla^2 n \rangle - \frac{1}{2}\langle \text{div } v, |\nabla^2 n|^2 \rangle|
\leq |\nabla^2 v|_{L^6}|\nabla n|_{L^3}|\nabla^2 n|_{L^2} + |\nabla v|_{L^\infty}|\nabla^2 n|_{L^2}^2
\leq C(|\nabla^3 v|_{L^2}|\nabla^2 n|_{H^1} + |\nabla^2 n|_{L^2}^2 + |\nabla^2 v|_{H^1} |\nabla^3 n|_{L^2})
\leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla^2(n, E)|_{L^2}^2).
Similarly, we have

\[ |\langle \nabla^2(v \cdot \nabla E), \nabla^2 E \rangle| \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla^2 E|_{L^2}^2) \]

Putting these estimates into (2.20), we get

\[
\frac{1}{2} \frac{d}{dt} |\nabla^2(n, v, E)|_{L^2}^2 + C|\nabla^3 v|_{L^2}^2 \leq C\delta(|\nabla^2 v|_{H^1}^2 + |\nabla^2 (n, E)|_{L^2}^2). \tag{2.21}
\]

Combining (2.19) and (2.21) yields (2.17) if \( \delta \) is small enough. This completes the proof of the lemma.

In the following lemma we give the dissipation on \(|\nabla n|_{H^1}|.

**Lemma 2.3.** Under the assumption (2.4), we have

\[
- \frac{d}{dt} \langle \text{div} \, v, n \rangle + C|\nabla n|_{L^2}^2 \leq C|\nabla v|_{H^1}^2 + C\delta|\nabla E|_{L^2}^2, \tag{2.22}
\]

\[
\frac{d}{dt} \langle \text{div} \, v, \Delta n \rangle + C|\nabla^2 n|_{L^2}^2 \leq C|\nabla v|_{H^2}^2 + C\delta|\nabla^2 E|_{L^2}^2. \tag{2.23}
\]

**Proof.** Notice that the condition \( \text{div}(\rho F^T) = 0 \) for all \( t \geq 0 \) gives

\[
\text{divdiv}[(1 + n)(E + I)^T] = 0, \quad \forall \ t \geq 0.
\]

Thus we have

\[
\frac{\partial^2 (E^i)}{\partial x_i \partial x_j} = \text{divdiv}(E^T)
\]

\[
= \text{divdiv}[(1 + n)(E + I)^T] - \text{divdiv}(nI + E^T)
\]

\[
= -\Delta n - \text{divdiv}(nE). \tag{2.24}
\]

Thus by applying \( \nabla_i \) to (2.2b) and summing over \( i \), we have

\[
\langle \text{div} \, v \rangle_t - (2\mu + \lambda)\Delta \text{div} \, v + 2\Delta n = \text{div} \, g_1, \tag{2.25}
\]

where

\[
g_1 = g - \text{div}(nE).
\]

Multiplying the above equation by \( n \), and then integration over \( \mathbb{R}^3 \), we have

\[
|\nabla n|_{L^2}^2 = \langle \langle \text{div} \, v \rangle_t, n \rangle - \langle (2\mu + \lambda)\Delta \text{div} \, v, n \rangle - \langle \text{div} \, g_1, n \rangle
\]

\[
= \frac{d}{dt} \langle \text{div} \, v, n \rangle - \langle \text{div} \, v, n_t \rangle + \langle (2\mu + \lambda)\Delta v, \nabla n \rangle + \langle g_1, \nabla n \rangle
\]

\[
= \frac{d}{dt} \langle \text{div} \, v, n \rangle - \langle \text{div} \, v, f \rangle + \langle \text{div} \, v, v \cdot \nabla n \rangle + |\text{div} \, v|_{L^2}^2
\]

\[
+ \langle (2\mu + \lambda)\Delta v, \nabla n \rangle + \langle g_1, \nabla n \rangle,
\]

where we use the the continuity equation (2.2a). By Sobolev’s, Hölder’s and Cauchy’s inequalities, we obtain

\[
- \frac{d}{dt} \langle \text{div} \, v, n \rangle + |\nabla n|_{L^2}^2 \leq C(|\nabla v|_{L^2}^2|f|_{L^2} + |\nabla v|_{L^2}^2|v|_{L^3} |\nabla n|_{L^3} + |\nabla v|_{L^2}^2 + |\nabla^2 v|_{L^2}^2 |\nabla n|_{L^2}^2
\]

\[
+ |g_1|_{L^2} |\nabla n|_{L^2}^2 \leq C|\nabla v|_{H^1}^2 + \frac{1}{2}|\nabla n|_{L^2}^2 + C\delta|\nabla (n, E)|_{L^2}^2,
\]

\[
\leq C|\nabla v|_{H^1}^2 + \frac{1}{2}|\nabla n|_{L^2}^2 + C\delta|\nabla (n, E)|_{L^2}^2,
\]

\[
\leq C|\nabla v|_{H^1}^2 + \frac{1}{2}|\nabla n|_{L^2}^2 + C\delta|\nabla (n, E)|_{L^2}^2,
\]
which gives (2.22) if $\delta$ is sufficiently small.

Multiplying (2.25) by $\triangle n$, and then integrating over $\mathbb{R}^3$, we have

$$2|\triangle n|_{L^2}^2 = -\langle (\text{div } v)_t, \triangle n \rangle + \langle (2\mu + \lambda)\triangle \text{div } v, \triangle n \rangle + \langle \text{div } g_1, \triangle n \rangle$$

$$= -\frac{d}{dt}\langle \text{div } v, \triangle n \rangle + \langle \text{div } v, \triangle n_t \rangle + \langle (2\mu + \lambda)\triangle \text{div } v, \triangle n \rangle$$

$$+ \langle \text{div } g_1, \triangle n \rangle$$

$$= -\frac{d}{dt}\langle \text{div } v, \triangle n \rangle + \langle \text{div } v, \triangle f \rangle - \langle \text{div } v, \nabla (v \cdot \nabla n) \rangle$$

$$- \langle \text{div } v, \triangle \text{div } v \rangle + \langle (2\mu + \lambda)\triangle \text{div } v, \triangle n \rangle + \langle \text{div } g_1, \triangle n \rangle$$

$$= -\frac{d}{dt}\langle \text{div } v, \triangle n \rangle - \langle \nabla \text{div } v, \nabla f \rangle - \langle \triangle \text{div } v, v \cdot \nabla n \rangle$$

$$+ \langle \nabla \text{div } v, \nabla \triangle n \rangle + \langle (2\mu + \lambda)\triangle \text{div } v, \triangle n \rangle + \langle \text{div } g_1, \triangle n \rangle.$$

By Sobolev’s, Hölder’s and Cauchy’s inequalities, we have

$$\frac{d}{dt}\langle \text{div } v, \triangle n \rangle + 2|\nabla^2 n|^2_{L^2} \leq C(\langle \nabla^2 v|_{L^2} \rangle |\nabla f|_{L^2} + |\nabla^3 v|_{L^2} \rangle |v|_{L^6} |\nabla n|_{L^3} + |\nabla^2 v|_{L^2}^2$$

$$+ |\nabla^2 v|_{L^2} |\triangle n|_{L^2} + |\nabla g_1|_{L^2} |\triangle n|_{L^2})$$

$$\leq C|\nabla v|_{H^3}^2 + \frac{1}{2}|\triangle n|_{L^2}^2 + C\delta|\nabla^2 (n, E)|_{L^2}^2,$$

which gives (2.23) if $\delta$ is small enough. This completes the proof of lemma.

In the following lemma we give the dissipation on $|\nabla (E^T - E)|_{H^1}$.

**Lemma 2.4.** Under the assumption (2.4), we have

$$-\frac{d}{dt}\langle \mathcal{W}, E^T - E \rangle + C|\nabla (E^T - E)|_{L^2}^2 \leq C|\nabla v|_{H^1}^2 + C\delta|\nabla (n, E)|_{L^2}^2,$$

(2.26)

$$\frac{d}{dt}\langle \mathcal{W}, \triangle (E^T - E) \rangle + C|\nabla^2 (E^T - E)|_{L^2}^2 \leq C|\nabla v|_{H^2}^2 + C\delta|\nabla^2 (n, E)|_{L^2}^2,$$

(2.27)

where $\mathcal{W} = \nabla u - (\nabla u)^T = \text{curl } u$.

**Proof.** Taking (2.2c)$^T - (2.2c)$, we have

$$(E^T - E)_t + \mathcal{W} = h^T - h - v \cdot \nabla (E^T - E).$$

(2.28)

Note the condition $F^{ijl} \nabla_i F^{ijkl} = F^{ijkl} \nabla_i F^{ijkl}$ for all $t \geq 0$, which means that

$$\nabla_k E^{ijkl} + E^{kl} \nabla_i E^{ijkl} = \nabla_j E^{ik} + E^{ij} \nabla_k E^{ijkl}, \quad \forall \ t \geq 0.$$

(2.29)

Thus we have

$$\nabla_j \nabla_k E^{ijkl} - \nabla_k \nabla_j E^{ijkl}$$

$$= \nabla_k \nabla_j E^{ik} - \nabla_j \nabla_k E^{ijkl}$$

$$= \nabla_k \nabla_j E^{ij} - \nabla_k \nabla_j E^{ij} + \nabla_k (E^{ik} \nabla_i E^{ijkl} - E^{ij} \nabla_i E^{ijkl})$$

$$- \nabla_j (E^{ik} \nabla_i E^{ijkl} - E^{ij} \nabla_i E^{ijkl})$$

$$= \nabla_j (E^{ij} - E^{ij}) + \nabla_k (E^{ik} \nabla_i E^{ijkl} - E^{ij} \nabla_i E^{ijkl}) - \nabla_j (E^{ik} \nabla_i E^{ijkl} - E^{ij} \nabla_i E^{ijkl}).$$

(2.30)
Thus by applying \( \text{curl} \) to (2.2b), we have

\[
W_t - \mu \Delta W + \Delta (E^T - E) = \text{curl } g + S,
\]

where the antisymmetric matrix \( S \) is defined as

\[
S_{ij} = \nabla_k (E^{lk} \nabla_l E_{ij} - E^{lj} \nabla_l E_{ik}) - \nabla_k (E^{lk} \nabla_l E_{ji} - E^{li} \nabla_l E_{jk}).
\]

Notice that the system (2.28)-(2.31) takes a similar form as the system (2.2a)-(2.25). Thus after a similar argument as Lemma 2.3, (2.26) and (2.27) follows. The proof of lemma is completed.

Finally, in the following lemma we give the dissipation on \( |\nabla E|_{H^1} \).

**Lemma 2.5.** Under assumption (2.4), we have

\[
|\nabla E|_{L^2}^2 \leq C|\nabla (n, E^T - E)|_{L^2}^2,
\]

(2.32)

\[
|\nabla^2 E|_{L^2}^2 \leq C|\nabla^2 (n, E^T - E)|_{L^2}^2.
\]

(2.33)

**Proof.** Combining (2.24) and (2.30), we have

\[
\Delta \text{div } E = \nabla \text{div } \text{div } E - \text{curl } \text{curl } \text{div } E
\]

\[
= -\Delta \nabla n - \nabla \text{div } \text{div } (nE) + \text{curl } (E - E^T) + \text{curl } S.
\]

(2.34)

Thus using the property of Riesz potential, (2.4) and (2.5), we arrive at

\[
|\text{div } E|_{L^2}^2 \leq C(|\nabla n|_{L^2}^2 + |\nabla (E^T - E)|_{L^2}^2 + |\nabla nE|_{L^2}^2 + |\nabla \text{div } E|_{L^2}^2)
\]

\[
\leq C|\nabla (n, E^T - E)|_{L^2}^2 + C\delta|\nabla^2 E|_{L^2}^2,
\]

and

\[
|\nabla \text{div } E|_{L^2}^2 \leq C(|\nabla^2 n|_{L^2}^2 + |\nabla^2 (E^T - E)|_{L^2}^2 + |\nabla^2 nE|_{L^2}^2 + |\nabla (E \nabla E)|_{L^2}^2)
\]

\[
\leq C|\nabla^2 (n, E^T - E)|_{L^2}^2 + C\delta|\nabla^2 E|_{L^2}^2.
\]

Under the above estimate, we may deduce from (2.29) that

\[
|\nabla E|_{L^2}^2 \leq |\text{div } E|_{L^2}^2 + |\text{curl } E|_{L^2}^2
\]

\[
\leq C|\nabla (n, E^T - E)|_{L^2}^2 + C\delta|\nabla^2 E|_{L^2}^2 + |\nabla \text{div } E|_{L^2}^2
\]

\[
\leq C|\nabla (n, E^T - E)|_{L^2}^2 + |\nabla \text{div } E|_{L^2}^2,
\]

and

\[
|\nabla^2 E|_{L^2}^2 \leq |\nabla \text{div } E|_{L^2}^2 + |\nabla \text{curl } E|_{L^2}^2
\]

\[
\leq C|\nabla^2 (n, E^T - E)|_{L^2}^2 + C\delta|\nabla^2 E|_{L^2}^2 + |\nabla (E \nabla E)|_{L^2}^2
\]

\[
\leq C|\nabla^2 (n, E^T - E)|_{L^2}^2 + C\delta|\nabla^2 E|_{L^2}^2.
\]

This proves (2.32) and (2.33), and the proof lemma is completed.
Now we are in a position to verify (2.4). Since $\delta > 0$ is sufficiently small, from Lemma 2.1-Lemma 2.5, we can choose a constant $D_1 > 0$ suitably large such that
\[
\frac{d}{dt} \{ D_1 |(n, v, E)|_{H^2}^2 + \langle \text{div} v, \triangle n - n \rangle + \langle W, \triangle(E^T - E) - (E^T - E) \rangle \} \\
+ C(\|\nabla(n, E)\|_{H^1}^2 + |\nabla v|_{H^2}^2) \leq 0,
\]
for any $t \geq 0$, which implies
\[
|(n, v, E)|_{H^2}^2 + \int_0^t (\|\nabla(n, E)\|_{H^1}^2 + |\nabla v|_{H^2}^2) \leq C|(n_0, v_0, E_0)|_{H^2}^2, \tag{2.35}
\]
since
\[
D_1 |(n, v, E)|_{H^2}^2 + \langle \text{div} v, \triangle n - n \rangle + \langle W, \triangle(E^T - E) - (E^T - E) \rangle \sim |(n, v, E)|_{H^2}^2.
\]
Then (2.35) gives (2.4). Thus we prove the global existence result of Theorem 1.1.

3 Convergence rate of the solution

In this section we shall prove the decay rates of the solution to finish the proof of Theorem 1.1. In Section 3.1, we list some elementary conclusion on the decay-in-time estimates for the linearized system and a useful inequality. In Section 3.2, we shall first obtain the energy inequality for the derivatives of the orders from the first to the third, and then we show a decay-in-time estimate for the first order derivatives, where the error is related to the derivatives of the higher order. Finally, by combining these estimates we get the optimal decay rates.

3.1 Spectral analysis and linear $L^2$ estimates

We first note that the linearized system (2.2a)-(2.25) depends only on $(n, \text{div} v)$ while the linearized system (2.28)-(2.31) also depends only on $(W, E^T - E)$. Denote by $\Lambda^s$ the pseudo differential operator defined by
\[
\Lambda^s u = \mathcal{F}^{-1}(|\xi|^s \hat{u}(\xi)),
\]
and let
\[
m = \Lambda^{-1} \text{div} v
\]
be the “compressible part” of the velocity, and
\[
\omega = \Lambda^{-1} W = \Lambda^{-1} \text{curl} v
\]
be the “incompressible part” of the velocity. We finally obtain
\[
n_t + \Lambda d = f - v \cdot \nabla n, \tag{3.1a}
\]
\[
d_t - (2\mu + \lambda) \Delta d - 2\Lambda n = \Lambda^{-1} \text{div} \ g_1, \quad \text{(3.1b)}
\]
\[
(E^T - E)_t + \Lambda \omega = h^T - h - v \cdot \nabla (E^T - E), \quad \text{(3.2a)}
\]
\[
\omega_t - \mu \Delta \omega - \Lambda (E^T - E) = \Lambda^{-1} \text{curl} \ g + \Lambda^{-1} \mathcal{S}. \quad \text{(3.2b)}
\]

Indeed, as the definition of \(d\) and \(\omega\), and the relation

\[
v = -\Lambda^{-1} \nabla d + \Lambda^{-1} \text{curl} \ \omega
\]

involve pseudo-differential operators of degree zero, the estimates in space \(H^l(\mathbb{R}^3)\) for the original function \(v\) will be the same as for \((d, \omega)\).

Here, we just discuss the system (3.1) for example, since the system (3.2) is the same as system (3.1). To use the \(L^p - L^q\) estimates of the linear problem for the nonlinear system (3.1) and system (3.2), we rewrite the solution of (3.1) as

\[
U(t) = K(t) U_0 + \int_0^t K(t - \tau) G(\tau) d\tau \quad t \geq 0,
\]

where we use the notations

\[
U = [n, d]^T, \quad U_0 = [n_0, d_0]^T, \quad G = [f - v \cdot \nabla n, \Lambda^{-1} \text{div} \ g_1]^T,
\]

and \(K(t)\) is the solution semigroup defined by \(K(t) = e^{tB}, t \geq 0\), with \(B\) being a matrix-valued differential operator given by

\[
B := \begin{pmatrix} 0 & -\Lambda \\ 2\Lambda & (2\mu + \lambda) \Delta \end{pmatrix}.
\]

Now we aim to analyze the differential operator \(B\) in terms of its Fourier expression \(A\) and to show the long time properties of the semigroup \(K(t)\). For this purpose, we need to consider the following linearized system

\[
U_t = BU. \quad \text{(3.3)}
\]

Applying the Fourier transform to system (3.3), we have

\[
\partial_t \hat{U} = A(\xi) \hat{U}, \quad \hat{U}(0) = \hat{U}_0,
\]

where \(\hat{U}(t) = \hat{U}(\xi, t) = \mathcal{F} U(\xi, t), \xi = (\xi_1, \xi_2, \xi_3)\) and \(A(\xi)\) is defined as

\[
A(\xi) := \hat{B} = \begin{pmatrix} 0 & -|\xi| \\ 2|\xi| & -(2\mu + \lambda)|\xi|^2 \end{pmatrix}.
\]

The characteristic polynomial of \(A(\xi)\) is \(\kappa^2 + (2\mu + \lambda) \kappa + 2|\xi|^2\), which implies the eigenvalues are

\[
\kappa_{\pm} = -(\mu + \frac{1}{2} \lambda)|\xi|^2 \pm \frac{1}{2} i \sqrt{8|\xi|^2 - (2\mu + \lambda)^2|\xi|^4}.
\]
The semigroup $e^{tA}$ is expressed as
\[
e^{tA} = e^{\kappa_+ t A + \kappa_- I} + \frac{\kappa_+ - \kappa_-}{2\xi (\kappa_+ e^{\kappa_+ t} - \kappa_- e^{\kappa_- t})} \left( \kappa_+ e^{\kappa_+ t} - \kappa_- e^{\kappa_- t} - \frac{\xi (e^{\kappa_+ t} - e^{\kappa_- t})}{2(\mu + \lambda)} \right) + \frac{\kappa_+ - \kappa_-}{\kappa_+ - \kappa_-} \left( e^{\kappa_+ t} - e^{\kappa_- t} \right).
\]

Thus the semigroup $K(t)$ has the following properties on the decay in time, which can be found in [10,11,14].

**Lemma 3.1.** Let $k \geq 0$ be an integer and $1 \leq l \leq 2$. Then for any $t \geq 0$, the solution $U(t) = (n(t), d(t))$ of system (3.6) satisfies
\[
|\nabla^k K(t) U_0|_{L^2} \leq C(1 + t)^{-\sigma(l,2;k)}(|U_0|_{L^2} + |\nabla^k U_0|_{L^2}) \leq C(1 + t)^{-\sigma(l,2;k)}(n, v)|_{U \cap H^k},
\]
where the decay rate is measured by
\[
\sigma(l,2;k) = \frac{3}{2} \frac{1}{l} - \frac{1}{2} + \frac{k}{2}.
\]
Moreover, if $|\hat{n}_0| \geq c_0, \hat{d}_0 = 0$ for $0 \leq |\xi| \ll 1$, then there exists a positive constant $c_2$ such that
\[
|n(t)|_{L^2} \geq c_2 (1 + t)^{-\frac{3}{4}},
\]
\[
|d(t)|_{L^2} \geq c_2 (1 + t)^{-\frac{3}{4}}.
\]
Finally, if $|(\hat{n}_0, \hat{d}_0)| \leq |\xi|^\eta$ for $0 \leq |\xi| \ll 1$, then there exists a positive constant $C$ such that
\[
|(n, d)(t)|_{L^2} \leq C(1 + t)^{-\frac{3}{2} - \frac{3}{4} \eta} |(n_0, d_0)|_{L^2}.
\]

We finish this subsection by listing an elementary but useful inequality [4]:

**Lemma 3.2.** If $r_1 > 1$ and $r_2 \in [0, r_1]$, then it holds that
\[
\int_0^r (1 + t - \tau)^{-r_1 (1 + \tau)^{-r_2}} \leq C(r_1, r_2)(1 + t)^{-r_2}.
\]

3.2. Convergence rates

Now we will show the energy inequality as follows:

**Lemma 3.3.** Under the assumption (2.4), let $(n, v, E)$ be the solution to the initial value problem (2.2), then there are two positive constants $C$ and $D_2$ such that if $\delta > 0$ in (2.4) is small enough, it holds
\[
\frac{d}{dt} M(t) + D_2 M(t) \leq C |\nabla(n, v, E)(t)|_{L^2}^2,
\]
where
\[
M(t) = C(n_0, v_0, E_0) = \int \frac{1}{2} |n_0|^2 + \frac{1}{2} |v_0|^2 + \frac{1}{2} |E_0|^2.
\]
where the energy function $M(t)$ defined by (3.7) is equivalent to $|\nabla (n, v, E)|^2_{H^1}$, that is, there exists a positive constant $C_1 > 0$ such that

$$\frac{1}{C_1}|\nabla (n, v, E)(t)|^2_{H^1} \leq M(t) \leq C_1|\nabla (n, v, E)(t)|^2_{H^1}.$$ 

**Proof.** Since $\delta > 0$ is sufficiently small, from Lemma 2.2-Lemma 2.5, we can choose a constant $D_2 > 0$ suitably large such that

$$\frac{d}{dt}\{D_2|\nabla (n, v, E)|^2_{H^1} + \langle \text{div } v, \Delta n \rangle + \langle W, \Delta (E^T - E) \rangle\} + C|\nabla^2 (n, E)|^2_{L^2} \leq C\delta|\nabla (n, v, E)|_{L^2}. \tag{3.6}$$

Define the energy functional

$$M(t) = D_2|\nabla (n, v, E)|^2_{H^1} + \langle \text{div } v, \Delta n \rangle + \langle W, \Delta (E^T - E) \rangle, \tag{3.7}$$

for any $t \geq 0$, where it is noticed that $M(t)$ is equivalent to $|\nabla (n, v, E)|^2_{H^1}$ since $D_2$ can be large enough. Adding $|\nabla (n, v, E)|$ to both sides of (3.6) gives (3.5). This completes the proof of the lemma.

If we define

$$N(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}}M(\tau), \tag{3.8}$$

then

$$|\nabla (n, v, E)(t)|_{H^1} \leq C\sqrt{M(t)} \leq C(1 + t)^{-\frac{3}{4}}\sqrt{N(t)}. \tag{3.9}$$

To close the estimate (3.5), we shall estimate the decay rate of the first order derivatives, this will be based on Lemma 3.1 about the decay estimates on the semigroup $K(t)$. Precisely, we have the following lemma.

**Lemma 3.4.** Under the assumption (2.4), let $(n, v, E)$ be the solution to the initial value problem (2.2). Then we have

$$|\nabla (n, v, E)(t)|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}(K_0 + \delta\sqrt{N(t)}), \tag{3.10}$$

where $K_0 = |(n_0, v_0, E_0)|_{L^1 \cap H^2}$.

**Proof.** From the Duhamel’s principle, it holds that

$$\left(\begin{array}{c} n \\ d \end{array}\right) = K(t)\left(\begin{array}{c} n_0 \\ d_0 \end{array}\right) + \int K(t - \tau)G(\tau) d\tau.$$

Thus from Lemma 3.1, we have

$$|(n, d)| \leq CK_0(1 + t)^{-\sigma(1,2,1)} + C \int_0^t (1 + t - \tau)^{-\sigma(1,2,1)}(|\hat{G}(\tau)|_{L^\infty} + |G(\tau)|_{H^1}) d\tau, \tag{3.11}$$

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where $\sigma(1,2;1) = \frac{2}{5}$ by (3.4). By Hölder’s inequality and Sobolev’s inequality, the nonlinear source terms can be estimated as follows:

$$|\dot{G}(t)|_{L^\infty} \leq C|(f - v \cdot \nabla n, g_1)|_{L^1} \leq C\delta(|\nabla(n, E)|_{L^2} + |\nabla v|_{H^1}),$$

$$|G(t)|_{H^1} \leq C|(f - v \cdot \nabla n, g_1)|_{H^1} \leq C\delta|\nabla(n, v, E)|_{H^1} + C|\nabla n|_{H^1}|\nabla^3 v|_{L^2}).$$

Putting these estimates into (3.11), by (2.35), (3.9), Lemma 3.2 and Hölder’s inequality, we arrive at

$$|\nabla(n,d)(t)|_{L^2} \leq CK_0(1 + t)^{-\frac{5}{2} + C\delta \int_0^t (1 + t - \tau)^{-\frac{5}{2}}(1 + \tau)^{-\frac{5}{2}} \sqrt{N(\tau)} \, d\tau}$$

$$+ C\int_0^t (1 + t - \tau)^{-\frac{5}{2}}(1 + \tau)^{-\frac{5}{2}} \sqrt{N(\tau)} |\nabla^3 v(\tau)|_{L^2} \, d\tau}$$

$$\leq C(1 + t)^{-\frac{5}{2}}(K_0 + \delta \sqrt{N(t)})$$

$$+ C\sqrt{N(t)} \int_0^t (1 + t - \tau)^{-\frac{5}{2}}(1 + \tau)^{-\frac{5}{2}} \sqrt{N(\tau)} |\nabla^3 v(\tau)|_{L^2} \, d\tau}$$

$$\leq C(1 + t)^{-\frac{5}{2}}(K_0 + \delta \sqrt{N(t)})$$

Similarly, we have

$$|\nabla(\omega, E^T - E)(t)|_{L^2} \leq C(1 + t)^{-\frac{5}{2}}(K_0 + \delta \sqrt{N(t)}).$$

Combining the above two inequalities, Lemma 2.5, and the relation of $v$ and $(d, \omega)$, we get (3.10). This completes the proof of the lemma.

Now we are in a position to prove (1.6)-(1.7) in Theorem 1.1. Applying the Gronwall’s inequality to the Lyapunov-type inequality (3.5), by (3.10), we get

$$M(t) \leq M(0)e^{-\frac{5}{2}K_0(1 + t)^{-\frac{5}{2}}(K_0 + \delta^2 N(t))} = C(1 + t)^{-\frac{5}{2}}(M(0) + K_0^2 + \delta^2 N(t)).$$

In view of (3.8), we have

$$N(t) \leq C(M(0) + K_0^2 + \delta^2 N(t)),$$

which implies

$$N(t) \leq C(M(0) + K_0^2) \leq CK_0^2,$$

since $\delta > 0$ is sufficiently small. Thus (3.9) gives

$$|\nabla(n,v,E)(t)|_{H^1} \leq CK_0(1 + t)^{-\frac{5}{2}}.$$  (3.12)

This proves (1.7). Now for (1.6), first by Sovolev’s inequality and (3.12), we have

$$|(n,v,E)(t)|_{L^6} \leq C|\nabla(n,v,E)(t)|_{L^2} \leq CK_0(1 + t)^{-\frac{5}{2}}.$$  (3.13)
Hence, by the interpolation, it follows from (3.13), (3.18) that for any $2 \leq p \leq 6$

\[
\left| (n, d)(t) \right|_{L^2} \leq CK_0(1 + t)^{-\frac{3}{4}} + C f_0(1 + t - \tau)^{-\frac{3}{4}} |(f - \nu \cdot \nabla n, g_1)(\tau)|_{L^1 \cap L^2 d\tau}
\]

\[
\leq CK_0(1 + t)^{-\frac{3}{4}} + C\delta \int_0^t (1 + t - \tau)^{-\frac{3}{4}} |\nabla (n, v, E)\tau|_{H^1} d\tau \tag{3.14}
\]

\[
\leq CK_0(1 + t)^{-\frac{3}{4}} + C\delta K_0 \int_0^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau
\]

\[
\leq CK_0(1 + t)^{-\frac{3}{4}}.
\]

Similarly, we have

\[
\left| (\omega, E^T - E)(t) \right|_{L^2} \leq CK_0(1 + t)^{-\frac{3}{4}}.
\]

Finally, we derive the time-decay-rate on $|E(t)|_{L^2}$. From (2.29) and (2.34), we have

\[
|\Lambda^{-1} \text{curl } E(t)|_{L^2} \leq C|\Lambda^{-1}(E\nabla E)(t)|_{L^2} \leq C|E(t)\nabla E(t)|_{L^\frac{6}{5}} \leq CK_0(1 + t)^{-\frac{2}{4}}, \tag{3.15}
\]

\[
|\Lambda^{-1} \text{div } E(t)|_{L^2} \leq C\left| (n, E - E^T)(t) \right|_{L^2} + |n(t)E(t)|_{L^2} + |\Lambda^{-2}S(t)|_{L^2}
\]

\[
\leq CK_0(1 + t)^{-\frac{3}{4}} + |\Lambda^{-1}(E\nabla E)(t)|_{L^2}
\]

\[
\leq CK_0(1 + t)^{-\frac{3}{4}}.
\]

The above two inequalities give

\[
|E(t)|_{L^2} \leq CK_0(1 + t)^{-\frac{3}{4}}. \tag{3.17}
\]

Combining (3.14), (3.15), (3.17) and the relation of $v$ and $(d, \omega)$, we have

\[
\left| (n, v, E)(t) \right|_{L^2} \leq CK_0(1 + t)^{-\frac{3}{4}}. \tag{3.18}
\]

Hence, by the interpolation, it follows from (3.13), (3.18) that for any $2 \leq p \leq 6$

\[
\left| (n, v, E)(t) \right|_{L^p} \leq \left| (n, v, E)(t) \right|_{L^2}^{\theta} \left| (n, v, E)(t) \right|_{L^6}^{1-\theta} \leq C(1 + t)^{-\frac{3}{4}(1-\frac{2}{p})},
\]

where $\theta = \frac{6 - p}{2p}$. The proof of (1.6)-(1.7) is completed.

### 4 Lower bound time decay rate

In this section, we investigate the lower bound time decay for global solutions.

Define

\[
g(t, x) = \rho(t, x) - 1, \quad m(t, x) = \rho u, \quad \mathcal{F} = \rho F - I.
\]

Then the condition $\text{div} \mathcal{F}^T = 0$ ensures that

\[
\text{div}(\rho FF^T) = \text{div}[(\mathcal{F} + I)(\mathcal{F}^T + I)]
\]

\[
= \text{div} \mathcal{F} - \nabla g + \text{div}(-g\mathcal{F} + \mathcal{F}^T - \mathcal{F}^T g + g^2 I),
\]

and

\[
\text{div} \text{div} \mathcal{F} = \text{div} \text{div} \mathcal{F}^T = 0.
\]
Thus we have the following system which only depends on \((\varrho, \text{div } m)\)

\[ \varrho_t + \text{div } m = 0, \]

\[ (\text{div } m)_t - (2\mu + \lambda)\Delta(\text{div } m) + (1 + \alpha)\Delta \varrho = G_1, \]

where

\[ G_1 = \alpha \text{divdiv} \left( \frac{-\varrho \mathcal{F} + \mathcal{F} \mathcal{F}^T - \varrho \mathcal{F}^T + \varrho^2 I}{1 + \varrho} \right) + \Delta(\varrho - P(1 + \varrho)) \\
+ (2\mu + \lambda)\Delta(\text{div } m) - \text{divdiv}(\rho u \otimes u). \]

By Hölder’s inequality and Sobolev’s inequality, it is easy to verify that

\[ (|\Lambda^{-2} G_1(t)|_{L^\infty} + |\Lambda^{-1} G_1(t)|_{L^2}) \leq C|((\varrho, m, \mathcal{F})|^2_{H^2} \leq C(1 + t)^{-\frac{3}{2}}. \]

Thus by Duhamel’s principle, Lemma 3.1 and the condition (1.8), we have

\[ \frac{|(\varrho, \Lambda^{-1} \text{div } m)(t)|_{L^2}}{C} \geq |K(t)(\varrho_0, \Lambda^{-1} \text{div } m_0)|_{L^2} - \int_0^t |K(t - \tau)(0, \Lambda^{-1} G_1(\tau))|_{L^2} d\tau \\
\geq c_1 (1 + t)^{-\frac{3}{4}} - C \int_0^t (1 + t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{3}{2}} d\tau \]

\[ \geq c_2 (1 + t)^{-\frac{3}{4}}. \]

Hence (1.9) is proved.

On the other hand, the condition \(F^{lk} \nabla_l F^{ij} = F^{ij} \nabla_l F^{ik}\) means that

\[ \nabla_k F^{ij} + \mathcal{F}^{lk} \nabla_l \left( \frac{F^{ij} + \delta_{ij}}{1 + \varrho} \right) + \nabla_k \left( \frac{-\varrho \mathcal{F}^{ij} + \delta_{ik}}{1 + \varrho} \right) \]

\[ = \nabla_j F^{ik} + \mathcal{F}^{lj} \nabla_l \left( \frac{F^{ik} + \delta_{ik}}{1 + \varrho} \right) + \nabla_j \left( \frac{-\varrho \mathcal{F}^{ik} + \delta_{ik}}{1 + \varrho} \right), \]

where

\[ \delta = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j.
\end{cases} \]
Using the fact \( \text{div} \mathcal{F}^T = 0 \), we have
\[
\nabla_j \nabla_k F^{jk} - \nabla_j \nabla_k F^{jk} = \nabla_j \nabla_k F^{ij} - \nabla_k \nabla_j F^{ij} + \nabla_k (\mathcal{F}^{lk} \nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta})) - \nabla_k (\mathcal{F}^{lj} \nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta})) + \nabla_k (\mathcal{F}^{lj} \nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta})))
\]
\[
= \nabla_k (\mathcal{F}^{lj} \nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta})) - \nabla_k (\mathcal{F}^{lj} \nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta}))) + \nabla_k (\mathcal{F}^{lj} \nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta})))
\]
Thus by applying \( \text{curl} \) to (1.1b) we have
\[
(\text{curl } m)_t - \mu \Delta (\text{curl } m) + \alpha \Delta (\mathcal{F}^T - \mathcal{F}) = H_1 \tag{4.2}
\]
where
\[
H_1^{ij} = \nabla_j (\mu \Delta \frac{\rho m}{1 + \theta} - \nabla_i (\mu \Delta \frac{\rho m}{1 + \theta})) + \nabla_j (\text{div} (\rho u \otimes u)) - \nabla_i (\text{div} (\rho u \otimes u)) + \nabla_k (\nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta}))) - \nabla_k (\nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta}))) + \nabla_l (\nabla_k (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta}))) + \nabla_i (\nabla_l (\frac{\mathcal{F}^{ij} + \delta_{ij}}{1 + \theta} + \nabla_k (\frac{-\rho \mathcal{F}^{ij} + \delta_{ij}}{1 + \theta}))).
\]
We also note that
\[
\nabla \times (u \times \rho \mathcal{F}^T) = \rho \mathcal{F}^T \cdot \nabla u - \rho \mathcal{F}^T \text{div } u - u \cdot \nabla (\rho \mathcal{F}^T) + u \text{div} (\rho \mathcal{F}^T) = \rho \mathcal{F}^T (\nabla u)^T - \rho \mathcal{F}^T \text{div } u - (u \cdot \nabla \rho) \mathcal{F}^T - \rho (u \cdot \nabla F) = \rho \mathcal{F}^T + \rho_t F = (\mathcal{F}^T)_t,
\]
where we used the condition \( \text{div} (\rho \mathcal{F}^T) = 0 \). Thus we have
\[
(\mathcal{F}^T - \mathcal{F})_t + \text{curl } m = H_2, \tag{4.3}
\]
where
\[
H_2 = \nabla \times (u \times \mathcal{F}^T) + \text{curl } (qu) - (\nabla \times (u \times \mathcal{F}^T) + \text{curl } (qu))^T.
\]
By Hölder’s inequality and Sobolev’s inequality, it is easy to verify that
\[
|\langle \Lambda^{-2}H_1, \Lambda^{-1}H_2 \rangle(t)|_{L^\infty} + |\langle \Lambda^{-1}H_1, H_2 \rangle(t)|_{L^2} \leq C|\langle \varrho, m, \mathcal{F} \rangle|_{H^2}^2 \leq C(1 + t)^{-\frac{3}{4}}.
\]
Duhamel’s principle, Lemma 3.1 and the condition (1.8), we have the following estimates of system (4.2)-(4.3):
\[
|\langle \mathcal{F}_T - \mathcal{F}, \Lambda^{-1} \text{curl } m \rangle(t)|_{L^2} \\
\leq |K(t)(\mathcal{F}_T^0 - \mathcal{F}_0, \Lambda^{-1} \text{curl } m_0)|_{L^2} - \int_0^t |K(t - \tau)(H_2(\tau), \Lambda^{-1}H_1(\tau))|_{L^2} d\tau \\
\leq C + C \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \times (|\langle \Lambda^{-2}H_1, \Lambda^{-1}H_2 \rangle(\tau)|_{L^\infty} + |\langle \Lambda^{-1}H_1, H_2 \rangle(\tau)|_{L^2}) d\tau \\
\leq C(1 + t)^{-\frac{3}{4} - \frac{3}{4}} + C \int_0^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau \\
\leq C(1 + t)^{-\min(\frac{3}{4}, \frac{3}{4})}.
\]
Combining (4.1) and (4.4) gives
\[
|m(t)|_{L^2} \geq |\Lambda^{-1} \text{div } m(t)|_{L^2} - |\Lambda^{-1} \text{curl } m(t)|_{L^2} \\
\geq c_3(1 + t)^{-\frac{3}{4}} - C(1 + t)^{-\min(\frac{3}{2}, \frac{3}{4})} \\
\geq c_4(1 + t)^{-\frac{3}{4}}.
\]
Hence (1.10) is proved.

By (4.4), we also have
\[
|(E^T - E)(t)|_{L^2} \leq C|\langle \mathcal{F}_T - \mathcal{F} \rangle(t)|_{L^2} \leq C(1 + t)^{-\min(\frac{3}{2}, \frac{3}{2}, \frac{3}{4})}.\tag{4.5}
\]
Thus from (2.34), (4.1) and (4.5) , we obtain
\[
|\Lambda^{-1} \text{div } E(t)|_{L^2} \geq |n(t)|_{L^2} - |n(t)E(t)|_{L^2} - |(E^T - E)(t)|_{L^2} - |\Lambda^{-2}S(t)|_{L^2} \\
\geq c_2(1 + t)^{-\frac{3}{4}} - C(1 + t)^{-\min(\frac{3}{2}, \frac{3}{2}, \frac{3}{4})} \\
\geq c_5(1 + t)^{-\frac{3}{4}}.
\]
Combining the above inequality with (3.16), we arrive at
\[
|E(t)|_{L^2} \geq |\Lambda^{-1} \text{div } E(t)|_{L^2} - |\Lambda^{-1} \text{curl } E(t)|_{L^2} \geq c_6(1 + t)^{-\frac{3}{4}}.
\]
Thus, (1.11) is proved and this completes the proof of Theorem 1.1.

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