LATTICE SIMPLICES WITH A FIXED POSITIVE NUMBER
OF INTERIOR LATTICE POINTS:
A NEARLY OPTIMAL VOLUME BOUND

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October 25, 2017

Abstract

We give an explicit upper bound on the volume of lattice simplices with fixed positive number of interior lattice points. The bound differs from the conjectural sharp upper bound only by a linear factor in the dimension. This improves significantly upon the previously best results by Pikhurko from 2001.

1 Introduction

Throughout, the positive integer $d \geq 1$ stands for the dimension of the ambient space $\mathbb{R}^d$. We consider problems for lattice polytopes in $\mathbb{R}^d$ and fix our underlying lattice to be $\mathbb{Z}^d$. We call the elements of $\mathbb{Z}^d$ lattice points or lattice vectors. A map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is called unimodular transformation if $\phi$ is an affine bijection satisfying $\phi(\mathbb{Z}^d) = \mathbb{Z}^d$. By $o$ we denote the zero vector and $e_1, \ldots, e_d$ are the standard unit vectors in $\mathbb{R}^d$.

A polytope $P \subseteq \mathbb{R}^d$ is called a lattice polytope (with respect to the lattice $\mathbb{Z}^d$) if all vertices of $P$ belong to $\mathbb{Z}^d$. We will discuss the relationship of the Euclidean volume $\text{vol}(P)$ and the number of interior lattice points for lattice polytopes $P$. Both these values are invariant up to unimodular transformations of $P$.

Given positive integers $d, k \geq 1$, let $\mathcal{P}^d(k)$ be the family of all $d$-dimensional lattice polytopes in $\mathbb{R}^d$ that have exactly $k$ interior lattice points and let $\mathcal{S}^d(k)$ be the family of all simplices belonging to $\mathcal{P}^d(k)$. It is known since the work of Hensley [Hen83] that, for all positive integers $d, k$, the values

$$p(d, k) := \max \left\{ \text{vol}(P) : P \in \mathcal{P}^d(k) \right\} \quad \text{and} \quad s(d, k) := \max \left\{ \text{vol}(S) : S \in \mathcal{S}^d(k) \right\}$$

are finite. Exact determination of $p(d, k)$ and $s(d, k)$ is a long-standing problem [Hen83, LZ91, Pik01, Ave12, AKN15] with additional motivation from integer optimization [AKN15, Section 2.7] and toric geometry [AKN15, Section 2.6], cf. [Nil07, Amb16].

In 1982, Zaks, Perles and Wills [ZPW82] introduced the simplex

$$S_{d,k} := \text{conv}(o, s_1 e_1, \ldots, s_{d-1} e_{d-1}, (k + 1)(s_d - 1)e_d), \quad (1.1)$$
arising from the Sylvester sequence \((s_i)_{i=1,2,\ldots}\),

\[
s_i := \begin{cases} 
2, & \text{for } i = 1, \\
1 + s_1 \ldots s_{i-1} & \text{for } i \geq 2.
\end{cases}
\]  

(1.2)

They observed that \(S_{d,k}\) belongs to \(\mathcal{S}^d(k)\) and has the volume

\[
\operatorname{vol}(S_{d,k}) = (k+1)\frac{(s_d-1)^2}{d!}
\]

of the asymptotic order \(k^{2\Theta(d)}\). This example shows that both \(p(d,k)\) and \(s(d,k)\) grow doubly-exponentially in the dimension \(d\). By our main result, we want to partially verify the plausibility of the following conjecture that can be traced back to [ZPW82]. This explicit version can be found in [BK16, Conj. 1.5].

**Conjecture 1.1.** For every \(d \geq 3\) and \(k \geq 1\), one has

\[
p(d,k) = s(d,k) = (k+1)\frac{(s_d-1)^2}{d!}.
\]

Furthermore, unless \(d = 3\) and \(k = 1\), every polytope \(P \in \mathcal{P}^d(k)\) satisfying \(\operatorname{vol}(P) = p(d,k)\) coincides with \(S_{d,k}\) up to unimodular transformations.

The conjecture claims that, in most cases, \(S_{d,k}\) is an essentially unique volume maximizer in the family of all \(d\)-dimensional lattice polytopes with \(k\) interior lattice points. It was recently verified for lattice simplices with \(k = 1\) [AKN15]. It is also known to hold in dimension \(d = 3\) for \(k \leq 2\) [Kas10, BK16]. The previously best upper volume bounds were achieved by Pikhurko in 2001: [Pik01, Equation (10)] gives \(s(d,k) \leq k2^{2d-2}15^{(d-1)2^{d+1}}/d!\). Improving upon his results, our main theorem confirms that the values \(s(d,k)\) and \(\operatorname{vol}(S_{d,k})\) are indeed very close to each other.

**Theorem 1.2.** Let \(d, k\) be positive integers. Then

\[
s(d,k) \leq k(d+1)\frac{(s_d-1)^2}{d!}.
\]

Theorem 1.2 implies \(\operatorname{vol}(S_{d,k}) \leq s(d,k) \leq (d+1)\operatorname{vol}(S_{d,k})\). This determines \(s(d,k)\) up to a linear factor in the dimension \(d\).

To prove Theorem 1.2 we use the following result of Pikhurko.

**Theorem 1.3 ([Pik01, Lem. 5]).** Let \(S\) be a \(d\)-dimensional simplex in \(\mathbb{R}^d\), let \(x\) be an interior lattice point of \(S\) with barycentric coordinates \(\beta_1 \geq \ldots \geq \beta_{d+1}\) and let \(k = |\mathbb{Z}^d \cap \text{int}(S)|\). Then

\[
\operatorname{vol}(S) \leq \frac{k}{d!\beta_1 \ldots \beta_d}.
\]  

(1.3)

In view of Theorem 1.3 for an arbitrary \(S \in \mathcal{S}^d(k)\), we need to provide a lower bound on \(\beta_1 \ldots \beta_d\) for some interior lattice point \(x\) of \(S\). Following Pikhurko, we choose \(x\) to be the interior lattice point that maximizes the minimum barycentric coordinate of \(x\). In Section 2 we derive so-called generalized product-sum inequalities for the barycentric
coordinates of a point \( x \) chosen in this way. They are the key tools that allow to bound \( s(d,k) \) by considering purely analytical optimization problems. These inequalities are derived by adapting ideas from \cite{Ave12} from the case \( k = 1 \) (where they led to sharp volume bounds, see \cite{AKN15}) to the case of an arbitrary \( k \geq 1 \). In Section \( \text{3} \) we formulate an optimization problem that involves generalized product-sum inequalities as constraints and which can be used for deriving lower bounds on \( \beta_1 \cdots \beta_d \). In Sections \( \text{4} \) and \( \text{5} \) we study this optimization problem and eventually prove Theorem \( \text{1.2} \).

Along the way, we improve two other results by Pikhurko \cite[Theorem 2 and Equation (9)]{Pik01}. 

**Theorem 1.4.** Let \( d, k \) be positive integers.

(a) For every \( S \in S^d(k) \) there exists an interior lattice point of \( S \) all of whose all barycentric coordinates are at least \( \frac{1}{(d+1)(s_{d+1}-1)} \).

(b) \( p(d,k) \leq (d(2d+1)(s_{2d+1}-1))^{d} \).

The proof can be found at the end of Section \( \text{4} \). We expect that the order of the bound in (a) is sharp up to the reciprocal of a linear factor in the dimension.

Let us finish with two remarks. First, the authors wonder whether it is possible to use the present methods to replace in Theorem \( \text{1.2} \) the constant \( d + 1 \) by a constant independent of the dimension \( d \). The best such constant theoretically achievable would be 2 (as one sees from \( S_{d,1} \)). Any further improvement would have to take the number \( k \) of interior lattice points into account. However, it seems unclear how to introduce \( k \) into the product-sum inequalities without significantly weakening the resulting bounds. Moreover, it was checked by Gabriele Balletti in the database of 3-simplices with \( k = 2 \) \cite{BK16} that (in contrast to the case \( k = 1 \)) it is impossible to use Theorem \( \text{1.3} \) to get a sharp volume bound – hence, for sharpness a new approach would be needed. Secondly, the by far major challenge in this area of research is to replace \( 2d \) in Theorem \( \text{1.4(b)} \) by a linear function in \( d \). This doubling of the dimension results from a clever trick by Pikhurko \cite[proof of Theorem 4]{Pik01} to reduce the problem from one for polytopes to one for simplices. However, it is a wide open question how such a reduction could be done without increasing the dimension.

**Basic notation and terminology.** By \( \mathbb{N} = \{1, 2, 3, \ldots \} \) we denote the set of (positive) natural numbers. For a non-negative integer \( t \), let \( \lfloor t \rfloor := \{1, \ldots, t\} \) if \( t > 0 \) and \( \lfloor t \rfloor := \emptyset \) if \( t = 0 \). The cardinality of a set \( X \) is denoted by \( |X| \). For \( X \subseteq \mathbb{R}^d \), the dimension \( \dim(X) \) of \( X \) is defined as the dimension of the affine hull of \( X \). The volume (that is, the Lebesgue measure) of a Lebesgue measurable set \( X \) is denoted by \( \text{vol}(X) \). We use the standard scaling of \( \text{vol} \) with \( \text{vol}([0,1]^d) = 1 \).

The interior and the convex hull of \( X \subseteq \mathbb{R}^d \) are denoted by \( \text{int}(X) \) and \( \text{conv}(X) \), respectively. The standard basis vectors of \( \mathbb{R}^d \) are denoted by \( e_1, \ldots, e_d \). For \( a, b \in \mathbb{R}^d \) we use \([a,b] \) to denote the convex hull of \( \{a,b\} \). If \( a \neq b \), the set \( [a,b] \) is the line segment joining \( a \) and \( b \). A polytope in \( \mathbb{R}^d \) is the convex hull of a finite subset of \( \mathbb{R}^d \). For more information on polytopes see \cite{Zie95}. The set of all vertices of a polytope \( P \) will be denoted by \( \text{vert}(P) \). A polytope \( S \) is called a simplex if the vertices of \( S \) are affinely independent.
If $S$ is a $d$-dimensional simplex in $\mathbb{R}^d$ with vertices $v_1, \ldots, v_{d+1}$, then every $x \in \mathbb{R}^d$ has a unique representation as the affine combination of $v_1, \ldots, v_{d+1}$, that is, there exist $\beta_1, \ldots, \beta_{d+1} \in \mathbb{R}$ uniquely determined by $S$ and $x$ such that

$$x = \beta_1 v_1 + \cdots + \beta_{d+1} v_{d+1},$$

$$1 = \beta_1 + \cdots + \beta_{d+1}.$$  

The values $\beta_1, \ldots, \beta_{d+1}$ are called the barycentric coordinates of $x$ with respect to $S$. Clearly, $x \in S$ if and only if all the barycentric coordinates are non-negative and $x \in \text{int}(S)$ if and only if all the barycentric coordinates are strictly positive.

## 2 Generalized product-sum inequalities

In [Ave12] it was shown that, for every $S \in \mathcal{S}^d(1)$, the barycentric coordinates $\beta_1 \geq \ldots \geq \beta_{d+1}$ of the unique interior lattice point of $S$ satisfy the inequalities

$$\prod_{i=1}^t \beta_i \leq \sum_{j=t+1}^{d+1} \beta_j \quad \forall t \in [d]. \quad (2.1)$$

We call (2.1) the product-sum inequalities. In [AKN15], $s(d, 1)$ was determined using the product-sum inequality and Theorem 1.3.

For $\mathcal{S}^d(k)$ with an arbitrary $k \geq 1$ the situation is more difficult, because there is a freedom in the choice of $x \in \mathbb{Z}^d \cap \text{int}(S)$ and it is crucial to choose $x$ appropriately. Pikhurko [Pik01] suggested to choose a point $x \in \mathbb{Z}^d \cap \text{int}(S)$ with barycentric coordinates $\beta_1, \ldots, \beta_{d+1}$ maximizing $\gamma := \min\{\beta_1, \ldots, \beta_{d+1}\}$. He used a lemma of Lagarias and Ziegler [LZ91, Lemma 2.1] to get lower bounds on $\gamma$ in terms of $d$.

Our approach to generating inequalities for barycentric coordinates combines ideas from [Ave12] and [Pik01]. We use an interior lattice point maximizing the minimum barycentric coordinate as in [Pik01] and then, for this point, we derive generalized product-sum inequalities by adapting the arguments from [Ave12]. For a vector $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ ($n \in \mathbb{N}$), we introduce its maximum norm $\|y\|_\infty := \max\{|y_1|, \ldots, |y_n|\}$.

**Lemma 2.1** (Determinant lemma; see [Ave12]). Let $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n \times n}$. If $0 < |\det(A)| < 1$, then there exists $y \in \mathbb{Z}^n \setminus \{o\}$ satisfying $\|Ay\|_\infty < 1$.

**Theorem 2.2** (Generalized product-sum inequalities). Let $S \subseteq \mathbb{R}^d$ be a $d$-dimensional lattice simplex in $\mathbb{R}^d$ and let $x$ be an interior lattice point of $S$ with barycentric coordinates $\beta_1 \geq \ldots \geq \beta_{d+1}$ having the property that all barycentric coordinates of every other interior lattice point of $S$ are not smaller than $\beta_{d+1}$. Then, for every $t \in [d]$, one has

$$\prod_{i=1}^t (\beta_i - \beta_{d+1}) \leq \sum_{j=t+1}^{d+1} \beta_j. \quad (2.2)$$

**Proof.** If $\beta_i = \beta_{d+1}$ for some $i \in [t]$, the assertion is trivially fulfilled. So, we assume that $\beta_i > \beta_{d+1}$ holds for every $i \in [t]$. Let $v_1, \ldots, v_{d+1}$ be the vertices of $S$ such
that $x = \beta_1 v_1 + \cdots + \beta_{d+1} v_{d+1}$. Throughout the proof, we consider integer variables $m, m_1, \ldots, m_t \in \mathbb{Z}$. If

$$m > 0 \quad \text{and} \quad m = m_1 + \cdots + m_t, \quad (2.3)$$

then

$$r = \sum_{i=1}^{t} \frac{m_i}{m} v_i$$

is a point in the affine hull of the points $v_1, \ldots, v_t$ and

$$q = (m + 1)x - mr$$

is a lattice point on the line passing through $r$ and $x$. Our proof approach is to assume that (2.2) is not fulfilled. Under this assumption we derive a contradiction to the choice of $x$ by showing that for some $m, m_1, \ldots, m_t$ satisfying (2.3), the minimum barycentric coordinate of the point $q$ is strictly larger than $\beta_{d+1}$ (necessarily, $q$ is then an interior lattice point of $S$).

For this, let us note that whenever (2.3) and

$$(m + 1)\beta_i - m_i > \beta_{d+1} \quad \forall i \in [t] \quad (2.4)$$

hold, the minimum barycentric coordinate of $q$ is strictly larger than $\beta_{d+1}$. Indeed, $(m + 1)\beta_i - m_i$ with $i \in [t]$ are the barycentric coordinates of $q$ with respect to the vertices $v_1, \ldots, v_t$. They are strictly larger than $\beta_{d+1}$. Since $m \geq 1$, one has $(m + 1)\beta_j \geq 2\beta_{d+1} > \beta_{d+1}$, and so the remaining barycentric coordinates $(m + 1)\beta_j$ with $j > t$ are also strictly larger than $\beta_{d+1}$. Summing up, if (2.3) and (2.4) hold, we get a contradiction.

We reformulate (2.4) as $\frac{1}{\beta_i - \beta_{d+1}} (m_i - \beta_i m) < 1$. Since we want to use Lemma 2.1, we will consider the stronger condition $|\frac{1}{\beta_i - \beta_{d+1}} (m_i - \beta_i m)| < 1$ on the variables $m_1, \ldots, m_t, m$. Taking into account that $m, m_1, \ldots, m_t$ are integer variables, the condition $m = m_1 + \cdots + m_t$ can be reformulated as the strict inequality $| - m_1 - m_2 - \cdots - m_t + m | < 1$. Altogether, we have introduced a system of $t + 1$ strict inequalities for the $t + 1$ integer variables $m_1, \ldots, m_t, m$, which can be formulated as

$$\begin{pmatrix}
\frac{1}{\beta_1 - \beta_{d+1}} & \cdots & \frac{1}{\beta_t - \beta_{d+1}} \\
\vdots & \ddots & \vdots \\
-1 & \cdots & -1
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_2 \\
\vdots \\
m_t \\
m\end{pmatrix}
< 1. \quad (2.5)$$

$= A$
Performing elementary row operations, we compute \( \det(A) \):

\[
\begin{vmatrix}
1 & -\beta_1 \\
\vdots & \vdots \\
-1 & -1 \\
\end{vmatrix}
\]

\[
= \frac{1}{\prod_{i=1}^{t}(\beta_i - \beta_{d+1})} \begin{vmatrix}
1 & -\beta_1 \\
\vdots & \vdots \\
0 & -1 \\
\end{vmatrix}
= \frac{1 - \sum_{i=1}^{t} \beta_i}{\prod_{i=1}^{t}(\beta_i - \beta_{d+1})}.
\]

From this we see that \( \det(A) > 0 \) and, since we assumed that (2.2) is not fulfilled, we also have \( \det(A) < 1 \). Thus, by Lemma 2.1 applied to (2.5), there exists \((m_1, \ldots, m_t, m) \in \mathbb{Z}^{t+1} \setminus \{0\} \) satisfying (2.5). The value \( m \) cannot be zero, since otherwise (2.5) would imply that \( m_1, \ldots, m_t \) are all zero, too. Since \( m \neq 0 \), possibly replacing \((m_1, \ldots, m_t, m)\) by \(- (m_1, \ldots, m_t, m)\), we can assume \( m > 0 \). We have found \( m_1, \ldots, m_t, m \) satisfying (2.3) and (2.4). This contradicts the choice of \( x \), as described above.

\[ \text{Remark 2.3.} \] Another generalization of the product-sum inequalities was obtained in the master thesis of Brunink [Bru16, Satz 4.1]. Brunink’s inequalities are valid with respect to every \( x \in \mathbb{Z}^d \cap \text{int}(S) \) and so they are necessarily weaker than our inequalities, which are valid for \( x \) that maximizes the minimum barycentric coordinate.

### 3 An optimization problem for bounding \( s(d, k) \)

We weaken (2.2) to the more convenient inequality

\[
\prod_{i=1}^{t}(\beta_i - \beta_{d+1}) \leq \sum_{j=t+1}^{d+1} (\beta_j - \beta_{d+1}) + (d + 1)\beta_{d+1}
\]  

(3.1)

and use this inequality to introduce the following optimization problem on a compact subset of \( \mathbb{R}^{d+1} \):

\[
\tau_d := \min \left\{ \beta_1 \cdots \beta_d : (\beta_1, \ldots, \beta_{d+1}) \in \mathbb{R}^{d+1}, \begin{array}{l}
\beta_1 \geq \cdots \geq \beta_{d+1} \geq 0, \\
\beta_1 + \cdots + \beta_d = 1, \\
\prod_{i=1}^{t}(\beta_i - \beta_{d+1}) \leq \sum_{j=t+1}^{d+1} (\beta_j - \beta_{d+1}) + (d + 1)\beta_{d+1} \quad \forall t \in [d] \end{array} \right\}.
\]  

(3.2)

During the analysis of this problem, we use the standard optimization terminology, such as feasible solution, optimal solution and optimal value. Inequality \( \beta_t \geq \beta_{t+1} \)
for a given $t \in [d]$ will be denoted by $ORD(t)$, where $ORD$ stands for ‘ordering’, while inequality (3.1) for a given $t \in [d]$ will be denoted by $PS(t)$, where $PS$ stands for ‘product-sum’. We will use $ORD(0)$ to denote the inequality $1 \geq \beta_1$.

We start our analysis of (3.2) by observing some basic properties of feasible solutions.

**Lemma 3.1.** Let $\beta = (\beta_1, \ldots, \beta_{d+1})$ be a feasible solution of the optimization problem (3.2). Then the following hold:

(a) For every $t \in [d+1]$, $\beta_t > 0$.

(b) For every $t \in [d]$, $ORD(t)$ or $PS(t)$ is strict.

(c) If for some $\ell \in [d]$, $PS(i)$ holds with equality for every $i \in [\ell]$, then $\beta_i = \frac{1}{s_i} + \beta_{d+1}$ for every $i \in [\ell]$.

**Proof.** (a): If (a) was false, we would have $\beta_1 \geq \cdots \geq \beta_\ell > \beta_{\ell+1} = \cdots = \beta_{d+1} = 0$ for some $\ell \in [d]$. With this choice of $\ell$, the right-hand side of $PS(\ell)$ would be zero and the left-hand side would be strictly positive, which is a contradiction.

(b): If $ORD(t)$ holds with equality, then $\beta_t = \beta_{t+1}$ and so $\beta_t - \beta_{d+1} = \beta_{t+1} - \beta_{d+1}$. Then, in $PS(t)$ the left hand side is at most $\beta_t - \beta_{d+1}$, while the right hand side is strictly larger than $\beta_{t+1} - \beta_{d+1} = (\beta_1 - \beta_{d+1})$ (as $\beta_{d+1} > 0$ by (a)). Thus, (3.1) is strict.

(c): The equality $\beta_i = \frac{1}{s_i} + \beta_{d+1}$ can be reformulated as $\frac{1}{\beta_i - \beta_{d+1}} = s_i$. We can prove (c) by induction on $i \in [\ell]$. For $i = 1$, the equality in $PS(1)$ gives $\beta_1 - \beta_{d+1} = (1 - \beta_1) + \beta_{d+1}$, so $\frac{1}{\beta_1 - \beta_{d+1}} = 2 = s_1$. Assume that $\ell \geq i > 1$ and $\frac{1}{\beta_j - \beta_{d+1}} = s_j$ for every $j < i$. Since $PS(i)$ holds with equality we have

$$\prod_{j=1}^{i}(\beta_j - \beta_{d+1}) + \sum_{j=1}^{i}(\beta_j - \beta_{d+1}) = 1.$$  

Factoring out $\beta_i - \beta_{d+1}$ and using the induction hypothesis, we arrive at

$$(\beta_i - \beta_{d+1}) \left(\frac{1}{s_1 \cdots s_{i-1}} + 1\right) + \frac{1}{s_1} + \cdots + \frac{1}{s_{i-1}} = 1.$$  

Using the well-known equality $\frac{1}{s_1} + \cdots + \frac{1}{s_{i-1}} = 1$, we get $\frac{1}{\beta_i - \beta_{d+1}} = s_1 \cdots s_{i-1} + 1 = s_i$, as desired.

**Theorem 3.2.** Let $d, k \in \mathbb{N}$. Then $\tau_d > 0$ and

$$s(d, k) \leq \frac{k}{d! \tau_d}.$$  

**Proof.** By Lemma 3.1(c), $\tau_d > 0$. The asserted bound on $s(d, k)$ follows by combining Theorems 1.3 and 2.2. 

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4 Localizing optimal solutions

Our aim now is to provide a possibly exact description of optimal solutions of (3.2), in order to determine $\tau_d$ as precisely as possible.

Lemma 4.1. Let $\beta = (\beta_1, \ldots, \beta_{d+1})$ be an optimal solution of (3.2). Then there exists $\ell \in \{0, \ldots, d\}$ such that:

(a) ORD($t$) holds with equality for $\ell + 1 \leq t \leq d$,

(b) PS($t$) holds with equality for $1 \leq t \leq \ell - 1$.

Proof. We set $\beta_0 := 1$. Choose $\ell \in \{0, \ldots, d\}$ such that $\beta_\ell > \beta_{\ell + 1} = \cdots = \beta_{d+1}$.

With this choice of $\ell$, (a) clearly holds and it remains to show (b).

We first show that ORD($t$) is also strict for each $t \leq \ell - 1$. If the latter is not the case, then we can find indices $a$ and $b$ with $1 \leq a < b \leq \ell$ such that $\beta_a - 1 > \beta_a = \cdots = \beta_b > \beta_{b + 1}$.

We can perturb $\beta$ by changing $\beta_a$ to $\beta_a + \varepsilon$ and $\beta_b$ to $\beta_b - \varepsilon$ for a small $\varepsilon > 0$. This change makes the value of the objective function smaller because the product $\beta_a\beta_{b - 1}$ in the objective function gets smaller in view of the inequality

$$(\beta_a + \varepsilon)(\beta_{b - 1} - \varepsilon) < \beta_a\beta_b. \quad (4.1)$$

On the other hand, our perturbation keeps $\beta$ a feasible solution. All ORD(1), \ldots, ORD($d$) remain fulfilled. Inequalities PS($t$) with $t < a$ remain fulfilled because, both sides of these inequalities remain unchanged. Inequalities PS($t$) with $a \leq t < b$ remain valid if $\varepsilon > 0$ is small, because in view of Lemma 3.1(b) they were all strict. Inequalities PS($t$) with $t \geq b$ remain valid, because by (4.1) the left hand side of these inequalities gets smaller, while the right hand side remains unchanged. This is a contradiction to the choice of $\beta$, because we have verified that a small perturbation of $\beta$ yields a feasible solution with a strictly smaller value of the objective function. This proves

$$\beta_1 > \cdots > \beta_{\ell + 1} = \cdots = \beta_{d+1}. \quad (4.2)$$

To verify (b), we choose an arbitrary $t$ with $1 \leq t \leq \ell - 1$ and show that PS($t$) holds with equality. If PS($t$) is strict, we can perturb $\beta$ by changing $\beta_t$ to $\beta_t + \varepsilon$ and $\beta_{t + 1}$ to $\beta_{t + 1} - \varepsilon$ for a small $\varepsilon > 0$. This perturbation makes the value of the objective function strictly smaller. Indeed, the objective function contains the product $\beta_t\beta_{t + 1}$, which gets smaller in view of

$$(\beta_t + \varepsilon)(\beta_{t + 1} - \varepsilon) < \beta_t\beta_{t + 1}. \quad (4.3)$$

Clearly, ORD(1), \ldots, ORD($d$) remain valid. PS(1), \ldots, PS($t - 1$) remain valid because both sides of the inequality are unchanged. PS($t$) remains valid because it was strict. PS($t + 1$), \ldots, PS($d$) remain valid because the left-hand side gets smaller in view of (4.2) and the right-hand side does not change. This implies that a small perturbation of $\beta$ is a feasible solution with a strictly smaller value of the objective function. This contradicts the choice of $\beta$ and shows that PS($t$) holds with equality. \qed
For \( \ell \in [d] \) let us define the following auxiliary function:

\[
f_\ell(\alpha) := \left( \prod_{i=1}^{\ell-1} \left( \frac{1}{s_i} + \alpha \right) \right) \left( \frac{1}{s_\ell - 1} - d\alpha \right) \alpha^{d-\ell} \quad \text{for } \alpha \in \mathbb{R}.
\]

**Lemma 4.2.** Let \( \beta = (\beta_1, \ldots, \beta_{d+1}) \) be an optimal solution of (3.2). Then there exists some \( \ell \in [d] \) such that the following conditions hold:

(a) The value \( \beta_1 \cdots \beta_d \) of the objective function can be expressed using \( \beta_{d+1} \) as

\[
\beta_1 \cdots \beta_d = f_\ell(\beta_{d+1}).
\]

(b) The value \( \beta_{d+1} \) satisfies

\[
\frac{1}{(d+1)(s_{\ell+1} - 1)} \leq \beta_{d+1} \leq \frac{1}{(d+1)(s_\ell - 1)}.
\]

**Proof.** If \( \beta_1 = \cdots = \beta_{d+1} \), then \( \beta_1 = \cdots = \beta_{d+1} = \frac{1}{d+1} \) and the assertions hold with \( \ell = 1 \) (note that \( f_1(\alpha) = (1 - d\alpha)\alpha^{d-1} \)). Otherwise, we choose \( \ell \in \{0, \ldots, d\} \) as in Lemma 4.1. Since \( \beta_1, \ldots, \beta_{d+1} \) are not all equal, we have \( \ell > 0 \). Using Lemma 4.1 and Lemma 3.1(c), we arrive at the equalities

\[
\beta_i = \frac{1}{s_i} + \beta_{d+1} \quad \text{for } i \leq \ell - 1,
\]

\[
\beta_i = \beta_{d+1} \quad \text{for } i \geq \ell + 1.
\]

This expresses all \( \beta_i \) with \( i \in [d+1] \setminus \{\ell\} \) in terms of \( \beta_{d+1} \). From \( \beta_1 + \cdots + \beta_{d+1} = 1 \), we also obtain a representation of \( \beta_\ell \) using \( \beta_{d+1} \):

\[
\beta_\ell = 1 - \sum_{i=1}^{\ell-1} \left( \frac{1}{s_i} + \beta_{d+1} \right) - (d+1-\ell)\beta_{d+1} = 1 - \sum_{i=1}^{\ell-1} \frac{1}{s_i} - d\beta_{d+1} = \frac{1}{s_\ell - 1} - d\beta_{d+1}.
\]

This yields (a). For verifying (b), it suffices to observe that in view of the above representations of \( \beta_1, \ldots, \beta_d \) in terms of \( \beta_{d+1} \), inequality ORD(\( \ell \)) and PS(\( \ell \)) amount after some straightforward computation to \( \beta_{d+1} \leq \frac{1}{(d+1)(s_{\ell+1} - 1)} \) and \( \beta_{d+1} \geq \frac{1}{(d+1)(s_\ell - 1)} \), respectively.

**Proof of Theorem 1.4.** Assertion (a) follows from Theorem 2.2 and Lemma 4.2(b).

We prove (b). Let \( \gamma > 0 \) be such that every lattice simplex of dimension at most \( 2d \) having interior lattice points contains an interior lattice point all of whose barycentric coordinates are at least \( \gamma \). In the proof of Theorem 4 of [Pik01], Pikhurko shows that every \( d \)-dimensional lattice polytope \( P \) having interior lattice points contains an interior lattice point \( w \) with \( P - w \subseteq \sigma(w - P) \), where \( \sigma := \frac{d}{\gamma} - 1 \). By (a), we can fix \( \gamma := \frac{1}{(2d+1)(s_{2d+1} - 1)} \). Lagarias and Ziegler [LZ91, Thm. 2.5] observe that, if \( P \in \mathcal{P}_d(k) \), then \( \text{vol}(P) \leq (1 + \sigma)^d k \) (see also [Pik01 Eq. (8)])]. This yields the desired estimate. □

9
5 Univariate optimization problems and the conclusion

By Lemma 4.2,

\[ \tau_d \geq \min \{ f^*_\ell : \ell \in [d] \}, \]

where

\[ f^*_\ell := \min \left\{ f_\ell(\alpha) : \frac{1}{(d+1)(s_{\ell+1} - 1)} \leq \alpha \leq \frac{1}{(d+1)(s_{\ell} - 1)} \right\}. \]

Thus, we have relaxed optimization problem (3.2) with \( d + 1 \) variables to \( d \) univariate optimization problems. We will determine a common lower bound on the optimal values \( f^*_1, \ldots, f^*_d \).

**Lemma 5.1.** Let \( d \geq 4 \). Then

\[ \tau_d \geq \frac{1}{(d+1)(s_d - 1)^2}. \]

**Proof.** By Lemma 4.2, it suffices to verify

\[ f^*_\ell \geq \frac{1}{(d+1)(s_d - 1)^2} \]

for every \( \ell \in [d] \).

Each \( f_\ell(\alpha) \) with \( \ell \in [d] \) is a polynomial of degree \( d \) in \( \alpha \), whose all roots are real and exactly one root is positive. Applying Rolle’s theorem it is straightforward to see that all roots of the derivative of \( f_\ell(\alpha) \) are real too and that the derivative has at most one positive root. Taking into account the asymptotics of \( f_\ell(\alpha) \) at infinity, the latter observations show that \( f_\ell(\alpha) \) is unimodal on the segment \( \left[ \frac{1}{(d+1)(s_{\ell+1} - 1)}, \frac{1}{(d+1)(s_{\ell} - 1)} \right] \) and so it attains its minimum over this segment at one of its endpoints. This allows us to compute \( f^*_\ell \) for concrete values of \( d \). It is thus straightforward to check that our assertion is true for \( d = 4 \). Assume \( d \geq 5 \).

For \( \frac{1}{(d+1)(s_{\ell+1} - 1)} \leq \alpha \leq \frac{1}{(d+1)(s_{\ell} - 1)} \), one has

\[ f_\ell(\alpha) \geq \frac{1}{s_1 \cdots s_{\ell-1}} \left( \frac{1}{s_{\ell} - 1} - d\alpha \right) \alpha^{d-\ell} \]

\[ = \frac{1}{s_{\ell} - 1} \left( \frac{1}{s_{\ell} - 1} - d\alpha \right) \alpha^{d-\ell} \]

\[ \geq \frac{1}{s_{\ell} - 1} \left( \frac{1}{s_{\ell} - 1} - \frac{d}{(d+1)(s_{\ell} - 1)} \right) \left( \frac{1}{(d+1)(s_{\ell+1} - 1)} \right)^{d-\ell} \]

\[ = \frac{1}{(s_{\ell} - 1)(d+1)(s_{\ell} - 1)} \left( \frac{1}{(d+1)(s_{\ell+1} - 1)} \right)^{d-\ell}. \]

We have thus derived the inequality \( f^*_\ell \geq \frac{1}{y_\ell} \) where

\[ y_\ell := (s_{\ell} - 1)^2(d+1)^{d-\ell+1}(s_{\ell+1} - 1)^{d-\ell}. \]

Note that

\[ \frac{1}{y_d} = \frac{1}{(d+1)(s_d - 1)^2} \]
Hence, we have to show $f_\ell^*(y) \geq \frac{1}{yd}$ for $\ell \in [d-1]$.

We first verify $y_1 \leq \cdots \leq y_{d-2}$. That is, for $\ell \in [d-3]$, we need to check $y_\ell \leq y_{\ell+1}$. One can check using the relation $s_{i+1} - 1 = s_i(s_i - 1)$ valid for all $i \geq 1$ that the latter inequality is equivalent to
\[
(s_\ell - 1)(d + 1) \leq s_{\ell+1}^{d-\ell-1} s_\ell
\]
We will verify the slightly stronger inequality
\[
d + 1 \leq s_{\ell+1}^{d-\ell-1}
\]
for each $\ell \in [d-3]$. For this, let us first observe that the right hand side of (5.1) is increasing in $\ell$ in the range $1 \leq \ell \leq d-3$. Indeed, if $\ell \in [d-4]$, one has $s_{\ell+1}^{d-\ell-1} \leq s_{\ell+1}^{d-\ell-2}$, which can be verified using the estimate $s_{\ell+2} \geq s_{\ell+1}(s_{\ell+1} - 1)$. This implies that it suffices to check (5.1) for $\ell = 1$, in which case it amounts to $d + 1 \leq 3^{d-2}$, which can be easily verified by induction on $d \geq 4$.

We have shown $y_1 \leq \cdots \leq y_{d-2}$. Let us now check $f_\ell^* \geq \frac{1}{yd}$ for all $\ell \in [d-2]$ by showing $y_{d-2} \leq y_d$. Inequality $y_{d-2} \leq y_d$ amounts to $(s_{d-2} - 1)(d + 1) \leq s_{d-1}$. The stronger inequality $d + 1 \leq s_{d-2}$ can be checked by induction on $d \geq 5$.

To conclude the proof, it remains to verify $f_{d-1}^* \geq \frac{1}{yd}$. For this, we estimate $f_{d-1}(\alpha)$ again from below by a quadratic function in $\alpha$:
\[
f_{d-1}(\alpha) \geq \frac{1}{s_{d-1} - 1} \left( \frac{1}{s_{d-1} - 1} - d\alpha \right) \alpha.
\]
The quadratic function on the right hand side is a concave downward parabola. Hence, it takes its minimum on the segment $\left( \frac{1}{(d+1)(s_{d-1})}, \frac{1}{(d+1)(s_{d-1} - 1)} \right]$ at one of the two endpoints. So, we are left with checking $f_{d-1} \left( \frac{1}{(d+1)(s_{d-1})} \right) \geq \frac{1}{(d+1)(s_{d-1})}$ and $f_{d-1} \left( \frac{1}{(d+1)(s_{d-1} - 1)} \right) \geq \frac{1}{(d+1)(s_{d-1} - 1)^2}$. This can be easily verified.

**Proof of Theorem 3.2** For $d = 1$, the inequality is trivial. For $d = 2$, the inequality is true in view of Scott’s result [Sco76] which implies $s(2, k) = 2(k + 1)$ for $k \geq 2$ and $s(2, 1) = 4.5$. For $d = 3$, the asserted inequality is weaker than Pikhurko’s inequality $s(d, k) \leq 14.106k$; see [Pik01]. In the case $d \geq 4$, the asserted inequality follows from Theorem 3.2 and Lemma 5.1.

**Acknowledgements**

The second author was supported by a scholarship of the state of Sachsen-Anhalt, Germany. The third author is an affiliated researcher of Stockholm University and partly supported by the Vetenskapsrådet grant NT:2014-3991. We thank Christian Haase, Martina Juhnke-Kubitzke and Noleen Köhler for their interest. We are also grateful to Gabriele Balletti for checking some conjectures in the database [BK10]. Both authors are PI’s in the Research Training Group Mathematical Complexity Reduction funded by the German Research Foundation (DFG-GRK 2297).
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