QUANTITATIVE HOMOGENIZATION OF THE DISORDERED $\nabla \phi$ MODEL

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Abstract. We study the $\nabla \phi$ model with uniformly convex Hamiltonian $H(\phi) := \sum V(\nabla \phi)$ and prove a quantitative rate of convergence for the properly rescaled partition function as well as a quantitative rate of convergence for the field $\phi$ subject to affine boundary condition in the $L^2$ norms. One of our motivations is to develop a new toolbox for studying this problem that does not rely on the Helffer-Sjöstrand representation. Instead, we make use of the variational formulation of the partition function, the notion of displacement convexity from the theory of optimal transport, and the recently developed theory of quantitative stochastic homogenization.

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Date: October 16, 2018.
1. Introduction, notation and main results

Let $\mathbb{Z}^d$, $\mathbb{B}_d$ be the graph $\mathbb{Z}^d$ in dimension $d \geq 2$ with its nearest-neighbor unoriented edges. Let $\lambda \in (0,1)$ be a fixed parameter, and let $(V_i)_{i=1,...,d}$ be a family of functions in $C^2(\mathbb{R})$ satisfying

1. $V_i(x) = V_i(-x)$
2. $0 < \lambda \leq V''(x) \leq \frac{1}{\lambda} < \infty$.

In this article, we wish to study the Ginzburg-Landau or $\nabla \phi$ model associated to this potential. Specifically, we fix a bounded discrete set $U \subseteq \mathbb{Z}^d$ and define the boundary $\partial U$ of $U$ to be the set of vertices of $U$ which are connected to $\mathbb{Z}^d \setminus U$. With this notation, we define for each function $\phi$ from $U$ to $\mathbb{R}$ the following convex Hamiltonian

$$H(\phi) := \sum_{e \subseteq U} V_e(\nabla \phi(e)),$$

where $\nabla \phi(e) = \phi(y) - \phi(x)$ if the edge is given by $e = (x,y)$ and the function $V_e$ is equal to $V_i$ if the edge $e$ is of the form $(x,x+\epsilon_i)$, for $x \in \mathbb{Z}^d$. The goal of this article is to derive some quantitative information about the large scale behaviour of the Gibbs measure associated to this Hamiltonian and defined by, for some $p \in \mathbb{R}^d$,

$$d\mu_U = Z_U^{-1} \exp\left(-\sum_{e \subseteq U} V_e(\nabla \phi(e))\right) \prod_{x \in U \setminus \partial U} d\phi(x) \prod_{x \in \partial U} \delta_{\lambda p}(\phi(x)),$$

as well as some quantitative information about the normalization factor $Z_U$, also referred to as the partition function, which makes $\mu_U$ a probability measure.

This model and its large scale behaviour have already been studied in several works. A common tool to study the $\nabla \phi$ model is the Helffer-Sjöstrand PDE representation which originates in [23]. Naddaf and Spencer in [27] were able to obtain a central limit theorem for this model by homogenizing the infinite dimensional elliptic PDE obtained from the Helffer-Sjöstrand representation. Funaki and Spohn in [16] studied the dynamics of this model. Deuschel Giacomin and Ioffe in [14] established the large scale $L^2$ convergence of the surface shape to some deterministic function, which can be characterized as the solution of an elliptic equation, as well as a large deviation principle. These result were later extended by Funaki and Sakagawa in [15]. In 2001, Giacomin, Olla and Spohn established in [17] a central limit theorem for the Langevin dynamic associated to this model. More recently Miller in [26] proved a central limit theorem for the fluctuation field around a macroscopic tilt.

The Helffer-Sjöstrand representation is a very powerful tool, but may also face some limitations. The PDE operator arising in this representation contains a divergence-form part whose coefficients are given by $V''$. In case when $V''$ is singular, or of a varying sign, then it is rather unclear how to proceed (see however [11, 9, 10]). Besides the specific results to be proved in this paper, we are interested in developing new tools to study the $\nabla \phi$ model that completely forego any reference to the Helffer-Sjöstrand representation. We will rely instead on the variational formulation of the free energy, and of the displacement convexity of the associated functional. To the best of our knowledge, it is the first time that tools from optimal transport are being used to study this model.

The mechanism by which we will obtain a rate of convergence, as opposed to a qualitative homogenization result, is inspired by recent developments in the homogenization of divergence-form operators with random coefficients. The first results in this context date back to the early 1980s, with the results of Kozlov [24], Papanicolaou-Varadhan [28] and Yurinskiǐ [30] who were able to prove qualitative homogenization for linear elliptic equations under very general assumptions on the coefficient field. These results were later extended by Dal Maso and Modica in [12, 13] to the nonlinear setting. Obtaining quantitative rates of convergence has been the subject of much recent study over the past few years. Some notable progress were achieved by Gloria, Neukamm and Otto [21, 22, 20] and by Armstrong, Kuusi, Mourrat and Smart [3, 4, 5, 7, 6].
While most of the theory developed to understand stochastic homogenization focuses on linear elliptic equations, the closest analogy with the $\nabla \phi$ interface model is the stochastic homogenization of nonlinear equations. In this setting, the results are more sparse: one can mention the work of Armstrong, Mourrat and Smart [6, 7] who quantified the work of Dal Maso and Modica [12, 13]. More recently, Armstrong, Ferguson and Kuusi [2] were able to adapt part of the theory developed in the linear setting to the nonlinear setting.

The main results of this article are a sub-optimal algebraic rate of convergence for the logarithm of the partition function, cf Theorem 1.1, and to deduce from the previous result an algebraic rate of convergence of the field $\phi$ with affine boundary conditions in the $L^2$ norm to an affine function, cf Theorem 1.2. The analysis relies on the study of two subadditive quantities, denoted by $\nu$ and $\nu^*$, which are approximately convex dual to one another. These quantities are reminiscent of those used in stochastic homogenization (cf [5, Chapters 1 and 2]) and which were key ingredients to develop this theory.

1.1. Notations and assumptions.

1.1.1. Notations for the lattice and cubes. In dimension $d \geq 2$, let $\mathbb{Z}^d$ be the standard $d$-dimensional hypercubic lattice, $\mathbb{B}_d := \{(x, y) : x, y \in \mathbb{Z}^d, |x - y| = 1\}$ the set of unoriented nearest neighbors and $E_d$ be the set of oriented nearest neighbors. We denote the canonical basis of $\mathbb{R}^d$ by $\{e_1, \ldots, e_d\}$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if $x$ and $y$ are nearest neighbors. For $x \in \mathbb{Z}^d$ and $r > 0$, we denote by $B(x, r) \subseteq \mathbb{Z}^d$ the discrete ball of center $x$ and of radius $r$. We usually denote a generic edge by $e$. For a given subset $U$ of $\mathbb{Z}^d$, we denote by $\mathbb{B}_d(U)$ the unoriented edges of $U$, i.e,

$$\mathbb{B}_d(U) := \{(x, y) \in \mathbb{B}_d : x \in U, y \in U \text{ and } x \sim y\}.$$ 

If we wish to talk about an oriented edge we will use an arrow to distinguish them from the unoriented edge. We denote by $\vec{e}$ a generic oriented edge. Similarly, we denote by $E_d$ the oriented edges of $U$,

$$E_d(U) := \{\overrightarrow{xy} \in E_d : x \in U, y \in U \text{ and } x \sim y\}.$$ 

We also denote by $\partial U$ the discrete boundary of $U$, defined by

$$\partial U := \{x \in U : \exists y \in \mathbb{Z}^d, y \sim x \text{ and } y \notin U\}$$

and by $U^\circ$ the discrete interior of $U$,

$$U^\circ := U \setminus \partial U.$$ 

We also denote by $|U|$ the cardinality of $U$, we may refer to this quantity as the (discrete) volume of $U$. For $N \in \mathbb{N}$, we write $N\mathbb{Z}^d$ to refer to the set $\{N x : x \in \mathbb{Z}^d\} \subseteq \mathbb{Z}^d$. A cube of $\mathbb{Z}^d$ is a set of the form

$$\mathbb{Z}^d \cap (z + [0, N]^d), \ z \in \mathbb{Z}^d,\ N \in \mathbb{N}.$$ 

We define the size of a cube given in the previous display above to be $N + 1$. For $n \in \mathbb{N}$, we denote by $\square_n$ the discrete triadic cube of size $3^n$,

$$\square_n := \left(-\frac{3^n}{2}, \frac{3^n}{2}\right)^d \cap \mathbb{Z}^d.$$ 

We say that a cube $\square$ is a triadic cube if it can be written

$$\square = z + \square_n, \text{ for some } n \in \mathbb{N}, \text{ and } z \in 3^n\mathbb{Z}^d.$$ 

Note that two triadic cubes are either disjoint or included in one another. Moreover for each $n \in \mathbb{N}$, the family of triadic cubes of size $3^n$ forms a partition of $\mathbb{Z}^d$. A caveat must be mentioned here, the family of triadic cubes $(z + \square_n)_{z \in 3^n\mathbb{Z}^d}$ forms a partition of $\mathbb{Z}^d$ but the family of edges $(\mathbb{B}(z + \square_n))_{z \in 3^n\mathbb{Z}^d}$ does not form a partition of $\mathbb{B}_d$, indeed we are missing the edges connecting two triadic cubes, i.e., the edges of the set

$$\{(x, y) \in \mathbb{B}_d : \exists z \in 3^n\mathbb{Z}^d, x \in z + \square_n \text{ and } y \notin z + \square_n\}.$$
We mention that the volume of a discrete triadic cube of the form \( z + \Box_n \) is \( 3^{dn} \) since this will be extensively used in this article.

Given two integers \( m, n \in \mathbb{N} \) with \( m < n \), we denote by
\[
Z_{m,n} := 3^m \mathbb{Z}^d \cap \Box_n,
\]
we also frequently use the shortcut notation
\[
Z_n := Z_{n,n+1} = 3^n \mathbb{Z}^d \cap \Box_{n+1}.
\]
These sets have the property that \( (z + \Box_m)_{z \in Z_{m,n}} \) is a partition of \( \Box_m \). In particular \( (z + \Box_n)_{z \in Z_n} \) is a partition of \( \Box_{n+1} \).

1.1.2. Notations for functions. For a bounded subset \( U \subseteq \mathbb{Z}^d \), and a function \( \phi : U \to \mathbb{R} \), we denote by \((\phi)_U\) its mean value, that is to say
\[
(\phi)_U := \frac{1}{|U|} \sum_{x \in U} \phi(x).
\]
We let \( h_0(U) \) and \( \hat{h}^1(U) \) be the set of functions from \( U \) to \( \mathbb{R} \) with value zero on the boundary of \( U \) and mean value zero respectively, i.e
\[
h_0(U) := \{ \phi : U \to \mathbb{R} : u = 0 \text{ on } \partial U \}
\]
and
\[
\hat{h}^1(U) := \{ \psi : U \to \mathbb{R} : (\psi)_U = 0 \}.
\]
These spaces are finite dimensional and their dimension is given by the formulas
\[
\dim h_0(U) = |U \setminus \partial U| \quad \text{and} \quad \dim \hat{h}^1(U) = |U| - 1.
\]
We sometimes need to restrict functions, to this end we introduce the following notation, for any subsets \( U, V \subseteq \mathbb{Z}^d \) satisfying \( V \subseteq U \), and any function \( \phi : U \to \mathbb{R} \), we denote by \( \phi|_V \) the restriction of \( \phi \) to \( V \).

Let \( U \subseteq \mathbb{Z}^d \), a vector field \( G \) on \( U \) is a function
\[
G : E_d(U) \to \mathbb{R}
\]
which is antisymmetric, that is, \( G(\bar{x}\bar{y}) = -G(\bar{y}\bar{x}) \) for each \( x, y \in E_d(U) \). Given a function \( \phi : U \to \mathbb{R} \), we define its gradient by, for each \( \bar{e} = \bar{x}\bar{y} \in E_d(U) \),
\[
\nabla \phi(\bar{e}) = \phi(y) - \phi(x).
\]
The divergence of a vector field \( G \) is the function from \( U \) to \( \mathbb{R} \) defined by, for each \( x \in U \)
\[
\text{div} G(x) = \sum_{y \in U ; y \sim x} G(\bar{x}\bar{y}).
\]
We also define the Laplacian \( \Delta \) of a function \( \phi : U \to \mathbb{R} \) to be, for each \( x \in U \)
\[
\Delta \phi(x) = \sum_{y \in U ; y \sim x} (\phi(y) - \phi(x)).
\]
For \( p \in \mathbb{R}^d \), we also denote by \( p \) the constant vector field given by
\[
(1.2) \quad p(x,y) := p \cdot (x - y).
\]
Given two vector fields \( F \) and \( G \), we define their product to be the function defined on the set of unoriented edges by
\[
F \cdot G(x,y) = F(\bar{x}\bar{y})G(\bar{x}\bar{y}).
\]
This notation will be frequently applied when \( F \) is a constant vector \( q \) and when \( G \) is the gradient of a function \( \nabla \psi \), so we frequently write, for each \( (x,y) \in \mathbb{B}_d \)
\[
q \cdot \nabla \psi(x,y) = q(x,y) \nabla \psi(x,y).
\]
We also often use the shortcut notation
\[ \sum_{e \in U} \text{to mean} \sum_{e \in B_d(U)}. \]

If one assumes additionally that $U$ is bounded, then for any vector field $F : E_d(U) \to \mathbb{R}$, we denote by $(F)_{U}$ the unique vector in $\mathbb{R}^d$ such that, for each $p \in \mathbb{R}^d$
\[ p \cdot (F)_{U} = \frac{1}{|U|} \sum_{e \subseteq U} p \cdot F(e). \]

1.1.3. Notations for vector spaces and scalar products. Let $V$ be a finite dimensional real vector space equipped with a scalar product $(\cdot, \cdot)_V$, this space can be endowed with a canonical Lebesgue measure denoted by Leb$_V$. This measure will be simply denoted by $dx$ when we are integrating on $V$, i.e., we write, for any measurable, integrable or nonnegative, function $f : V \to \mathbb{R}$,
\[ \int_V f(x) \, dx \text{ to mean } \int_V f(x) \, \text{Leb}_V(dx). \]
For any linear subspace $H \subseteq V$, we denote by $H^\perp$ the orthogonal complement of $H$. Given $H, K \subseteq V$, we use the notation
\[ V = H \perp K \text{ if } V = H \perp K \text{ and } \forall (h, k) \in H \times K, (h, k)_V = 0. \]
Note that from the scalar product on $V$, one can define scalar products on $H$ and $H^\perp$ naturally by restricting the scalar product on $V$ to these spaces. Consequently the spaces $H$ and $H^\perp$ are equipped with Lebesgue measures denoted by Leb$_H$ and Leb$_{H^\perp}$. These measures are related to the Lebesgue measure on $V$ by the relation
\[ \text{Leb}_V = \text{Leb}_H \otimes \text{Leb}_{H^\perp}, \]
where the notation $\otimes$ is used to denote the standard product of measures. Note that we used in the previous notation the equality $V = H \perp H^\perp$ to obtain a canonical isomorphism between the spaces $V$ (on which Leb$_V$ is defined), and the space $H \times H^\perp$ (on which Leb$_H \otimes \text{Leb}_{H^\perp}$ is defined).

Given a bounded subset $U \subseteq \mathbb{Z}^d$, we equip any linear space $V$ of functions from $U$ to $\mathbb{R}$ with the standard $L^2$ scalar product, i.e., for any $\phi, \psi \in V$, we define
\[ (\phi, \psi)_V := \sum_{x \in U} \phi(x) \psi(x). \]
This in particular applies to the spaces $h^2(U)$ and $h^1(U)$. From now on, we consider that these spaces are equipped with a scalar product and consequently with Lebesgue measure denoted by Leb$_{h^2(U)}$ and Leb$_{h^1(U)}$, or simply by $dx$ when we use the notation convention (1.3).

1.1.4. Notations for measures and random variables. For any finite dimensional real vector space $V$, we denote by $\mathcal{P}(V)$ the set of probability measures on $V$ equipped with its Borel $\sigma$-algebra denoted by $\mathcal{B}(V)$. For a pair of finite dimensional real vector spaces $V$ and $W$, and a measure $\pi \in \mathcal{P}(V \times W)$, the first marginal of $\pi$ is the probability measure $\mu \in \mathcal{P}(V)$ defined by, for each $A \in \mathcal{B}(V)$,
\[ \mu(A) := \pi(A \times W), \]
we similarly define the second marginal as a measure in $\mathcal{P}(W)$. Given two probability measures $\mu, \nu \in \mathcal{P}(V)$, we denote by $\Pi(\mu, \nu)$ the set of probability measures of $\mathcal{P}(V \times W)$ whose first marginal is $\mu$ and second marginal is $\nu$, i.e.,
\[ \Pi(\mu, \nu) := \{ \pi \in \mathcal{P}(V \times W) : \forall (A, B) \in (\mathcal{B}(V), \mathcal{B}(W)), \pi(A \times W) = \mu(A) \text{ and } \pi(V \times B) = \nu(B) \}. \]
We define a coupling between two probability measures $\mu \in \mathcal{P}(V)$ and $\nu \in \mathcal{P}(W)$ to be a measure in $\Pi(\mu, \nu)$. For a generic random variable $X$, we denote by $\mathbb{P}_X$ its law. Given two random variables $X$ and $Y$, a coupling between $X$ and $Y$ is a random variable whose law belongs to $\Pi(\mathbb{P}_X, \mathbb{P}_Y)$. 
An important caveat must be mentioned here, in this article we do not assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all the random variables are defined. In short, we use the random variables as proxy for their laws to simplify the notations. A consequence of this is that given two random variables $X$ and $Y$, we have to be careful to first fix a coupling between $X$ and $Y$ if one wants to define the random variables $X + Y$, $XY$ etc.

Given a measurable space $(X, \mathcal{F})$ and two $\sigma$-finite measures $\mu$ and $\nu$ on $X$, we write $\mu \ll \nu$ to mean that $\mu$ is absolutely continuous with respect to $\nu$, and denote by $\frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of $\mu$ with respect to $\nu$. If we are given a second measurable space $(Y, \mathcal{F}')$ and a measurable map $T : X \to Y$, we denote by $T_* \mu$ the pushforward of the measure $\mu$ by the map $T$.

1.1.5. Notations for the $\nabla \phi$ model. We consider a family of function $(V_i)_{i=1, \ldots, d} \in C^2(\mathbb{R})$ satisfying the following assumptions, for each $i \in \{1, \ldots, d\}$

1. Symmetry: for each $x \in \mathbb{R}$, $V_i(x) = V_i(-x)$,
2. Uniform convexity: there exists $\lambda \in (0, 1)$ such that $\lambda < V_i'' < \frac{1}{\lambda}$,
3. Normalisation: The value of $V_i$ at 0 are fixed: $V_i(0) = 0$

Assumption (2) implies, for each $p_1, p_2 \in \mathbb{R}$,

\[
(1.5) \quad \lambda |p_1 - p_2|^2 \leq V_i(p_1) + V_i(p_1) - 2V_i\left(\frac{p_1 + p_2}{2}\right) \leq \frac{1}{\lambda} |p_1 - p_2|^2.
\]

From the first and second assumptions, we see that the functions $V_i$ have a unique minimum achieved in 0. The third assumption is not necessary and can be easily removed, but thanks to this assumption we have the convenient inequality, for each $x \in \mathbb{R}$,

\[
\lambda |x|^2 \leq V_i(x) \leq \frac{1}{\lambda} |x|^2.
\]

For each edge $e \in \mathbb{B}_d$, we define $V_e := V_i$ where $i$ is the unique integer in $\{1, \ldots, d\}$ such that $e$ can be written $(x, x + e_i)$ for some $x \in \mathbb{Z}^d$. We then define the partition function, for each bounded subset $U \subseteq \mathbb{Z}^d$ and for each $p \in \mathbb{R}^d$,

\[
(1.6) \quad Z_p(U) := \int_{h^1_U} \exp\left(-\sum_{e \subseteq U} V_e(p(e) + \nabla \phi(e))\right) d\phi,
\]

where we recall that the notation $p(e)$ denotes the constant vector field introduced in (1.2) and $d\phi$ stands for the Lebesgue measure on $h^1_U$. Note that thanks to the symmetry of the functions $V_i$, even if the vector field $p + \nabla \phi$ is defined for oriented edges, the quantity $V_e(p(e) + \nabla \phi(e))$ can be defined for unoriented edges. From this one can define

\[
(1.7) \quad \nu(U, p) := -\frac{1}{|U|} \ln Z_p(U)
\]

and the probability measure on $h^1_U$

\[
\mathbb{P}_{U,p}(d\phi) := \frac{\exp\left(-\sum_{e \subseteq U} V_e(p(e) + \nabla \phi(e))\right) d\phi}{Z_p(U)}.
\]

We also denote by $\phi_{U,p}$ a random variable of law $\mathbb{P}_{U,p}$. These objects will frequently be used with triadic cubes, we thus define the shortcut notations, for $n \in \mathbb{N}$,

\[
\mathbb{P}_{n,p} := \mathbb{P}_{\square_{n,p}}
\]

and by $\phi_{n,p}$ a random variable of law $\mathbb{P}_{n,p}$. We also define the quantity, for each $q \in \mathbb{R}^d$,

\[
(1.8) \quad Z_q^*(U) := \int_{h^1(U)} \exp\left(-\sum_{e \subseteq U} (V_e(\nabla \psi(e)) - q \cdot \nabla \psi(e))\right) d\psi,
\]
as well as the quantity
\begin{equation}
\nu^*(U, q) := \frac{1}{|U|} \ln Z_q^*(U)
\end{equation}
and the probability measure
\[ P_{U, q}^*(d\phi) := \frac{\exp \left( - \sum_{e \in U} (V_e(\nabla \psi(e)) - q \cdot \nabla \psi(e)) \right) d\psi}{Z_q^*(U)}. \]

We denote by \( \psi_{U, p} \) a random variable of law \( P_{U, p} \) and we frequently write \( P_{n, q}^* \) and \( \psi_{n, p} \) instead of \( P_{\Box_n, q}^* \) and \( \psi_{\Box_n, p} \).

1.1.6. Convention for constants and exponents. Throughout this article, the symbols \( c \) and \( C \) denote positive constants which may vary from line to line. These constants may depend solely on the parameters \( d \), the dimension of the space, and \( \lambda \), the ellipticity bound on the second derivative of the function \( V_e \). Similarly we use the symbols \( \alpha \) and \( \beta \) to denote positive exponents which may vary from line to line and depend only on \( d \) and \( \lambda \). Usually, we use \( C \) for large constants (whose value is expected to belong to \([1, \infty)\)) and \( c \) for small constants (whose value is expected to be in \((0, 1]\)). The values of the exponents \( \alpha \) and \( \beta \) are always expected to be small.

We also frequently write \( C := C(d, \lambda) < \infty \) to mean that the constant \( C \) depends only on the parameters \( d, \lambda \) and that its value is expected to be large. We may also write \( C := C(d) < \infty \) or \( C := C(\lambda) < \infty \) if the constant \( C \) depends only on \( d \) (resp. \( \lambda \)). For small constants or exponents we use the notations \( c := c(d, \lambda) > 0 \), \( \alpha := \alpha(d, \lambda) > 0 \), \( \beta := \beta(d, \lambda) > 0 \).

1.2. Main result. The goal of this article is to show the quantitative convergence of the quantities \( \nu(\Box_n, p) \) and \( \nu^*(\Box_n, q) \). This is stated in the following theorem.

**Theorem 1.1** (Quantitative convergence to the Gibbs state). **There exists a constant \( C := C(d, \lambda) < \infty \) and an exponent \( \alpha := \alpha(d, \lambda) > 0 \) such that for each \( p, q \in \mathbb{R}^d \), there exist two real numbers \( \varphi(p) \) and \( \varphi^*(q) \) such that**
\begin{equation}
|\nu(\Box_n, p) - \varphi(p)| \leq C 3^{-\alpha n} (1 + |p|^2)
\end{equation}
and
\begin{equation}
|\nu^*(\Box_n, q) - \varphi^*(q)| \leq C 3^{-\alpha n} (1 + |q|^2).
\end{equation}

Moreover the functions \( p \to \varphi(p) \) and \( q \to \varphi^*(q) \) are uniformly convex, there exists a constant \( C := C(d, \lambda) < \infty \) such that for each \( p_1, p_2 \in \mathbb{R}^d \)
\begin{equation}
\frac{1}{C} |p_1 - p_2|^2 \leq \varphi(p_1) + \varphi(p_2) - 2\varphi \left( \frac{p_1 + p_2}{2} \right) \leq C |p_1 - p_2|^2,
\end{equation}
for each \( q_1, q_2 \in \mathbb{R}^d \),
\begin{equation}
\frac{1}{C} |q_1 - q_2|^2 \leq \varphi^*(q_1) + \varphi^*(q_2) - 2\varphi^* \left( \frac{q_1 + q_2}{2} \right) \leq C |q_1 - q_2|^2
\end{equation}
and are dual convex, i.e we have for each \( q \in \mathbb{R}^d \)
\begin{equation}
\varphi^*(q) = \sup_{p \in \mathbb{R}^d} (-\varphi(p) + p \cdot q) .
\end{equation}

From this, we deduce that the random variables \( \phi_{n, p} \) and \( \psi_{n, q} \) are close to affine functions in the expectation of the \( L^2 \) norm. Note that (1.12) and (1.13) imply that the functions \( p \to \varphi(p) \) and \( q \to \varphi^*(q) \) are \( C^{1,1} \) and we denote their gradients by \( \nabla_p \varphi \) and \( \nabla_q \varphi^* \).
Theorem 1.2 (L^2 contraction of the Gibbs measure). There exist a constant \( C := C(d, \lambda) < \infty \) and an exponent \( \alpha := \alpha(d, \lambda) > 0 \) such that for each \( n \in \mathbb{N}, p, q \in \mathbb{R}^d \),

\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} \left( |\phi_{n,p}(x)|^2 + |\psi_{n,q}(x) - \nabla_q \nu^*(q) \cdot x|^2 \right) \right] \leq C 3^n (2 - \alpha) (1 + |p|^2 + |q|^2).
\]

1.3. Strategy of the proof. The strategy of the proof is to use the ideas from the theory of quantitative stochastic homogenization, and in particular the ideas developed by Armstrong, Kuusi, Mourrat and Smart in [5] and [7] to the setting of the \( \nabla \phi \) model. To this end, we introduce the two quantities \( \nu \) and \( \nu^* \), which are in some sort equivalent to the subadditive quantities with the same notation used in [5, Chapters 1 and 2].

The idea is then to find a variational formulation for these quantities to rewrite them as a minimization problem of a convex functional, this in done in the Subsection 2.2. Nevertheless this functionnal involves a term of entropy and it is not a priori clear that the functional is convex. To solve this issue, we appeal to optimal transport and more specifically to the notion of displacement convexity to obtain some sort of convexity for the entropy.

Once this is done, we are able to collect some properties about \( \nu \) and \( \nu^* \) which match the basic properties of the equivalent quantities in stochastic homogenization, to see this, one can for instance compare Proposition 3.1 with Lemma 1.1 of [5]. One can then exploit the convex dualiy between \( \nu \) and \( \nu^* \) to obtain a quantitative rate of convergence for these quantities as it is done in Section 4.

1.4. Outline of the paper. The rest of the article is organised as follows. In Section 2, we collect some preliminary results which will be useful to prove the main theorems. Specifically, we introduce the differential entropy of a measure and state some of its properties, we also record some definitions from the theory of optimal transport and state the main result we need to borrow from this theory, namely the displacement convexity of the entropy, Proposition 2.10. We also introduce the variational formulation for \( \nu \) and \( \nu^* \) in Subsection 2.2. In Subsection 2.4 we state and prove a technical lemma which allows to construct suitable coupling between random variables. We then complete Section 2 by stating some functional inequalities on the lattice \( \mathbb{Z}^d \), in particular the multiscale Poincaré inequality which is an important ingredient in the theory of stochastic homogenization.

In Section 3, we use the tools from Section 2 to prove a series of properties on the quantities \( \nu \) and \( \nu^* \), summed up in Proposition 3.1. These estimates, though not particularly difficult, are technical and the details are many.

In Section 4, we combine the tools collected in Section 2 with the result proved in Section 3 to first prove that the variance of the random variable \( (\psi_{n,q})_{\square_n} \) contracts, this is done in Lemma 4.2. We then deduce from this result and the multiscale Poincaré inequality that the random variable \( \psi_{n,q} \) is close to an affine function in the \( L^2 \) norm, this is Proposition 4.3. We then use these results combined with a patching construction, reminiscent to the one performed in [7], to prove Theorems 1.1 and 1.13.

Appendix A is devoted to the proof of some technical estimates useful in Sections 2 and 3.

Appendix B is devoted to the proof of some inequalities from the theory of elliptic equations adapted to the setting of the \( \nabla \phi \) model. Namely we prove a version of the Caccioppoli inequality, the reverse Hölder inequality and the Meyers estimate for the \( \nabla \phi \) model.

1.5. Acknowledgements. I would like to thank Jean-Christophe Mourrat for helpful discussions and comments.


2. Preliminaries

2.1. The entropy and some of its properties. In this section, we define one of the main tools used in this article, the differential entropy, we then collect a few properties of this quantity which will be useful in the rest of the article. We first give the definition of the entropy.

Definition 2.1 (Differential entropy). Let $V$ be a finite dimensional vector space equipped with a scalar product. Denote by $\mathcal{B}_V$ the associated Borel set. Consider the Lebesgue measure on $V$ and denote it by $\text{Leb}$. For each probability measure $\mathbb{P}$ on $V$, we define its entropy according to

$$H(\mathbb{P}) := \begin{cases} \int_V \frac{d\mathbb{P}}{d\text{Leb}}(x) \ln \left( \frac{d\mathbb{P}}{d\text{Leb}}(x) \right) \, dx \text{ if } \mathbb{P} \ll \text{Leb} \text{ and } \frac{d\mathbb{P}}{d\text{Leb}} \ln \left( \frac{d\mathbb{P}}{d\text{Leb}} \right) \in L^1(V, \mathcal{B}(V), \text{Leb}), \\ +\infty \text{ otherwise}. \end{cases}$$

Remark 2.2.

• In this article we implicitly extend the function $x \to x \ln x$ by 0 at 0.

• We emphasize that the usual definition of the differential entropy is stated with the function $x \to -x \ln x$ instead of the function $x \to x \ln x$. Adopting the other sign convention is more meaningful in this article because we want the entropy to be convex in the sense of displacement convexity as it will be explained in the following subsections.

We now record a few properties about the entropy. We first study how the entropy behaves under translation and affine change of variable. These properties are standard and fairly simple to prove, the details are thus omitted.

Proposition 2.3 (Translation and linear change of variable of the entropy). Let $X$ be a random variable taking values in a finite dimensional real vector space $V$ equipped with a scalar products. Denote by $\mathbb{P}_X$ the law of $X$. For each $a \in V$, if we denote by $\mathbb{P}_{X+a}$ the law of the random variable $X + a$ then we have

$$H(\mathbb{P}_{X+a}) = H(\mathbb{P}_X) \quad (2.1)$$

Now consider $L$ a linear map from $V$ to $V$. Then if we denote by $\mathbb{P}_{L(X)}$ the law of the random variable $L(X)$, then we have

$$H(\mathbb{P}_{L(X)}) = H(\mathbb{P}_X) - \ln |\det L|. \quad (2.2)$$

In particular if $L$ is non invertible then $\det L = 0$ and $H(\mathbb{P}_{L(X)}) = \infty$.

Let $V, W$ be two finite dimensional real vector spaces equipped with scalar product denoted by $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$. Denote by $\text{Leb}_V$ and $\text{Leb}_W$ the Lebesgue measure on $V$ and $W$. Consider the space $V \times W$. Define a scalar product on this space by, for each $v, v' \in V$ and each $w, w' \in W$,

$$(v, w, v', w')_{V \times W} = (v, v')_V + (w, w')_W.$$ 

Then the Lebesgue measure on $V \oplus W$ satisfies

$$\text{Leb}_{V \times W} = \text{Leb}_V \otimes \text{Leb}_W.$$

The following proposition gives a property about the entropy of a pair of random variables.

Proposition 2.4. Let $V, W$ be two finite dimensional real vector spaces equipped with scalar product. Consider the space $V \times W$ equipped with the scalar product defined above. Let $X$ and $Y$ be two random variables valued in respectively $V$ and $W$. Assume that $H(\mathbb{P}_X) < \infty$, $H(\mathbb{P}_Y) < \infty$ and that we are given a coupling $(X, Y)$ between $X$ and $Y$, then we have

$$H(\mathbb{P}_{(X,Y)}) \geq H(\mathbb{P}_X) + H(\mathbb{P}_Y),$$

with equality if and only if $X$ and $Y$ are independent.

Remark 2.5. This inequality states that the entropy of two random variable is minimal when $X$ and $Y$ are independent while the reader may be used for the entropy to be maximal when the random variables are independent. This is due to the sign convention adopted in Definition 2.1.
Proof. These estimates can be obtained using the convexity of the function $x \mapsto x \ln x$ and the Jensen inequality. The proof is standard and the details are omitted. \hfill \square

Frequently in this article, the previous proposition will be used with the following formulation

**Proposition 2.6.** Let $U$ be a finite dimensional vector space equipped with a scalar product and assume that we are given two linear spaces of $U$, denoted by $V$ and $W$ such that

$$U = V \oplus W.$$ 

Assume moreover that we are given two random variables $X$ and $Y$ taking values respectively in $V$ and $W$. Assume that $H(\mathbb{P}_X) < \infty$, $H(\mathbb{P}_Y) < \infty$ and that we are given a coupling $(X, Y)$ between $X$ and $Y$, then we have

$$H(\mathbb{P}_{X+Y}) \geq H(\mathbb{P}_X) + H(\mathbb{P}_Y),$$

where the entropy of $X + Y$ (resp. $X$ and $Y$) is computed with respect to the Lebesgue measure on $U$ (resp. $V$ and $W$). Moreover there is equality in the previous display if and only if $X$ and $Y$ are independent.

**Proof.** This proposition is a consequence of Proposition 2.4 and the fact that there exists a canonical isometry between $U$ and $V \times W$ given by

$$(2.3) \quad \Phi : \left\{ \begin{array}{c}
V \times W \to U \\
(v, w) \mapsto v + w.
\end{array} \right.$$ 

\hfill \square

### 2.2. Variational formula for $\nu$ and $\nu^\ast$.

One of the key ideas of this article is to introduce a convex functional $\mathcal{F}_{n,p}$ (resp. $\mathcal{F}_{n,q}^\ast$) defined on set of the probability measures on the space $h^1_0(\square_n)$ (resp. $\hat{h}^1(\square_n)$) such that $\mathbb{P}_{n,p}$ (resp. $\mathbb{P}_{n,q}^\ast$) is the minimizer of $\mathcal{F}_{n,p}$ (resp. $\mathcal{F}_{n,q}^\ast$). The convexity of $\mathcal{F}_{n,p}$ (resp. $\mathcal{F}_{n,q}^\ast$) allows to perform a perturbative analysis around its minimizer, i.e., the measure $\mathbb{P}_{n,p}$ (resp. $\mathbb{P}_{n,q}^\ast$), and to obtain quantitative estimates which will turn out to be crucial in the proof of Theorem 1.1.

**Definition 2.7.** For each $n \in \mathbb{N}$ and each $p, q \in \mathbb{R}^d$, we define

$$\mathcal{F}_{n,p} : \mathcal{P}(h^1_0(\square_n)) \to \mathbb{R}$$

$$\mathbb{P} \mapsto \mathbb{E} \left[ \sum_{e \subseteq \square_n} V_e (p \cdot e + \nabla \phi_e) \right] + H(\mathbb{P}),$$

where $\phi$ is a random variable of law $\mathbb{P}$. Similarly, we define

$$\mathcal{F}_{n,q}^\ast : \mathcal{P}(\hat{h}^1(\square_n)) \to \mathbb{R}$$

$$\mathbb{P}^\ast \mapsto \mathbb{E} \left[ \sum_{e \subseteq \square_n} (V_e (\nabla \psi(e)) - q \nabla \psi(e)) \right] + H(\mathbb{P}^\ast),$$

where $\psi$ is a random variable of law $\mathbb{P}^\ast$.

The main property about this functional is stated in the following proposition.

**Proposition 2.8.** Let $V$ be a finite dimensional real vector space equipped with a scalar product. We denote by $\mathcal{B}_V$ be the Borel set associated to $V$. For any measurable function $f : V \to \mathbb{R}$ bounded from below, one has the formula

$$(2.4) \quad - \log \int_V \exp (-f(x)) \, dx = \inf_{\mathbb{P} \in \mathcal{P}(V)} \left( \int f(x) \mathbb{P}(dx) + H(\mathbb{P}) \right)$$

where the integral in the left-hand side is computed with respect to the Lebesgue measure on $V$. 

\hfill \square
As a consequence, one has the following formula, for each \( n \in \mathbb{N} \) and each \( p \in \mathbb{R}^d \),

\[
\nu(\square_n, p) = \inf_{P \in \mathcal{P}(h_0^1(\square_n))} \left( \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} V_e (p \cdot e + \nabla \phi(e)) \right] + \frac{1}{|\square_n|} H(P) \right),
\]

where in the previous formula, \( \phi \) is a random variable of law \( P \). Moreover the minimum is attained for the measure \( P_{n,p} \). Similarly, one has, for each \( q \in \mathbb{R}^d \),

\[
\nu^*(\square_n, q) = \sup_{P^* \in \mathcal{P}(h^1(\square_n))} \left( \frac{1}{|\square_n|} \mathbb{E} \left[ - \sum_{e \subseteq \square_n} (V_e (\nabla \psi(e)) - q \cdot \nabla \psi(e)) \right] - \frac{1}{|\square_n|} H(P^*) \right),
\]

where in the previous formula, \( \psi \) is a random variable of law \( P^* \). Moreover the minimum is attained for the measure \( P^*_{n,q} \).

**Proof.** We first prove (2.4) and decompose the proof into two steps.

**Step 1.** Let \( P \) be a probability measure on \( V \), we want to show that

\[
\int_V f(x) P(dx) + H(P) \geq - \log \int_V \exp(-f(x)) \, dx.
\]

First note that if \( H(P) = \infty \), then the term on the left-hand side is equal to \( \infty \) and the inequality is satisfied. Thus one can assume that \( H(P) < \infty \), this implies that \( P \) is absolutely continuous with respect to the Lebesgue measure on \( V \) and we denote by \( h \) its density. In particular, we have

\[
H(P) = \int_V h(x) \ln h(x) \, dx.
\]

Similarly, one can assume that \( \int_V f(x) h(x) \, dx < \infty \) otherwise the estimate (2.7) is verified. Using that \( h \) is a probability density and the Jensen inequality, one obtains

\[
\exp \left( - \int_V f(x) h(x) \, dx - H(P) \right) \leq \int_V \exp \left( -f(x) - \ln h(x) \right) h(x) \, dx.
\]

We then denote

\[
A := \{ x \in V : h(x) > 0 \} \in \mathcal{B}_V,
\]

so that

\[
\exp \left( - \int_V f(x) h(x) \, dx - H(P) \right) \leq \int_V 1_A(x) \exp \left( -f(x) \right) h(x)^{-1} h(x) \, dx
\]

\[
\leq \int_V 1_A(x) \exp \left( -f(x) \right) \, dx
\]

\[
\leq \int_V \exp \left( -f(x) \right) \, dx.
\]

This is precisely (2.7).

**Step 2.** We assume that

\[
\int_V \exp \left( -f(x) \right) \, dx < \infty \quad \text{and} \quad \int_V |f(x)| \exp \left( -f(x) \right) \, dx < \infty.
\]

and construct a measure \( P \in \mathcal{P}(V) \) satisfying

\[
\int_P f(x) P(dx) + H(P) = - \log \int_V \exp \left( -f(x) \right) \, dx.
\]

In this case, we define

\[
P := \frac{\exp \left( -f(x) \right)}{\int_V \exp \left( -f(x) \right) \, dx} dx.
\]
It is clear that $\mathbb{P} \ll \text{Leb}_V$ and, from the assumption required on the function $f$ for this step, that $H(\mathbb{P}) < \infty$. An explicit computation gives

$$
\int_V f(x)\mathbb{P}(dx) + H(\mathbb{P}) = \int_V f(x) - f(x) - \ln \left( \int_V \exp(-f(x)) \, dx \right) \mathbb{P}(dx) = \ln \left( \int_V \exp(-f(x)) \, dx \right) \mathbb{P}(dx).
$$

This completes the proof of (2.4).

**Step 3.** In this step, we assume that

$$
\int_V \exp(-f(x)) \, dx = \infty \quad \text{or} \quad \int_V |f(x)| \exp(-f(x)) \, dx = \infty
$$

and we construct a sequence of probability measures $\mathbb{P}_n$ such that

$$
\int_V f(x)\mathbb{P}_M(dx) + H(\mathbb{P}_M) \xrightarrow{n \to \infty} -\log \int_V \exp(-f(x)) \, dx,
$$

where we used the convention

$$
-\log \int_V \exp(-f(x)) \, dx = -\infty \quad \text{if} \quad \int_V \exp(-f(x)) \, dx = \infty.
$$

To this end, we define, for each $n \in \mathbb{N}$,

$$
\mathbb{P}_n := \frac{\exp(-f(x)) \mathbb{1}_{\{|x| \leq n \text{ and } f(x) \leq n\}}}{\int_V \exp(-f(x)) \mathbb{1}_{\{|x| \leq n \text{ and } f(x) \leq n\}} \, dx}.
$$

With this definition, we compute

$$
\int_V f(x)\mathbb{P}_n(dx) + H(\mathbb{P}_n) = -\ln \left( \int_V \exp(-f(x)) \mathbb{1}_{\{|x| \leq n \text{ and } f(x) \leq n\}} \, dx \right).
$$

Sending $n \to \infty$ gives the result.

**Step 4.** In this step, we prove (2.5). First note that by the bound $V_e(x) \leq \frac{1}{2} |x|^2$, there exists a constant $C := C(d, \lambda) < \infty$ such that for each $n \in \mathbb{N}$ and each $\phi \in h_0^1(\square_n)$,

$$
\sum_{e \subseteq \square_n} V_e (p \cdot e + \nabla \phi(e)) \leq C |p|^2 + C \sum_{x \in \square_n} |\phi(x)|^2.
$$

(2.9)

Using this inequality, we deduce that one can apply (2.4) with $V = h_0^1(\square_n)$ equipped with the standard scalar product and $f(\phi) = \sum_{e \subseteq \square_n} V_e (p \cdot e + \nabla \phi(e))$. Moreover, using the estimate (2.9), one sees

$$
\int_{h_0^1(\square_n)} \exp(-f(\phi)) \, d\phi < \infty \quad \text{and} \quad \int_{h_0^1(\square_n)} |f(\phi)| \exp(-f(\phi)) \, d\phi < \infty.
$$

Thus applying the result proved in Steps 1 and 2, one has

$$
\nu(\square_n, p) = \inf_{\mathbb{P} \in \mathcal{P}(h_0^1(\square_n))} \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} V_e (p \cdot e + \nabla \phi(e)) \right] + \frac{1}{|\square_n|} H(\mathbb{P})
$$

and the minimum is attained for the measure

$$
\mathbb{P}_{n,p}(d\phi) = \frac{\exp \left( - \sum_{e \subseteq U} V_e (p \cdot e + \nabla \phi(e)) \right) \, d\phi}{\int_{h_0^1(\square_n)} \exp \left( - \sum_{e \subseteq U} V_e (p \cdot e + \nabla \phi(e)) \right) \, d\phi}.
$$

The proof of (2.6) is similar and the details are left to the reader. \qed
2.3. **Optimal transport and displacement convexity.** In this section, we introduce a few definitions about optimal transport and state one of the main tools of this article, namely the displacement convexity. We first give a definition of the optimal coupling. The existence of this coupling is rather standard and the uniqueness is more involved and is a byproduct of Brenier’s theorem. We refer to [29, Proposition 2.1 and Theorem 2.12] for this definition.

**Definition 2.9.** Let $U$ be a finite dimensional real vector space equipped with a scalar product. We denote by $|\cdot|$ the norm associated to this scalar product. Let $X$ and $Y$ be two random variables taking values in $U$ and denote their laws by $\mathbb{P}_X$ and $\mathbb{P}_Y$ respectively. Assume additionally that $\mathbb{P}_X$ and $\mathbb{P}_Y$ have a finite second moment, i.e,  

$$
\mathbb{E} \left[ |X|^2 \right] < \infty \text{ and } \mathbb{E} \left[ |Y|^2 \right] < \infty, 
$$

and that they are absolutely continuous with respect to the Lebesgue measure on $V$, then the minimization problem

$$
\inf_{\mu \in \Pi(\mathbb{P}_X,\mathbb{P}_Y)} \int_U |x - y|^2 \mu(dx,dy)
$$

admits a unique minimizer denoted by $\mu_{(X,Y)}$ and called the optimal coupling between $X$ and $Y$.

For $t \in [0, 1]$, we denote by $T_t$ the mapping

$$
T_t := \begin{cases} 
U \times U & \to U \\
(x, y) & \mapsto (1 - t)x + ty, 
\end{cases}
$$

and for two random variables $X$ and $Y$ taking values in $U$ with finite second moment, we denote by

$$
\mu_t := (T_t)_* \mu_{(X,Y)},
$$

this is the law of $(1-t)X + ty$ when the coupling between $X$ and $Y$ is the optimal coupling. The main property we need to use is called the displacement convexity and stated in the following proposition. We refer to [29] for this theorem but it is mostly due to McCann [25].

**Proposition 2.10 (Displacement convexity, Theorem 5.15 and Remark 5.16 of [29]).** Let $U$ be a finite dimensional real vector space equipped with a scalar product, let $X$ and $Y$ be two random variables taking values in $U$ with finite second moment, i.e satisfying (2.10), then the function $t \to H(\mu_t)$ is convex, i.e., for each $t \in [0, 1]$,

$$
H(\mu_t) \leq (1-t)H(\mathbb{P}_X) + tH(\mathbb{P}_Y).
$$

**Remark 2.11.** In [29, Remark 5.16], the theorem is proved when the measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ are absolutely continuous with respect to the Lebesgue measure and have finite second moment. The proposition can be deduced from [29, Remark 5.16] since we assumed $H(\mu) = \infty$ if the measure $\mu$ is not absolutely continuous with respect to the Lebesgue measure.

2.4. **Coupling lemmas.** Thanks to optimal transport theory particularly thanks to Definition 2.9, we are able to couple two random variables. The next question which arises, and which needs to be answered to prove Theorem 1.1, is to find a way to couple three random variables. Broadly speaking, the question we need to answer is the following: assume that we are given three random variables $X$, $Y$ and $Z$, a coupling between $X$ and $Y$ and another coupling between $Y$ and $Z$, can we find a coupling between $X$, $Y$ and $Z$? This question can be positively answered thanks to the following proposition.

**Proposition 2.12.** Let $(E_1, \mathcal{B}_1)$, $(E_2, \mathcal{B}_2)$, $(E_3, \mathcal{B}_3)$ be three Polish spaces equipped with their Borel $\sigma$-algebras. Assume that we are given three probability measures $\mathbb{P}_X$ on $E_1$, $\mathbb{P}_Y$ on $E_2$ and $\mathbb{P}_Z$ on $E_3$
as well as a coupling \( \mathbb{P}_{(X,Y)} \) between \( \mathbb{P}_X, \mathbb{P}_Y \) and a coupling \( \mathbb{P}_{(Y,Z)} \) between \( \mathbb{P}_Y, \mathbb{P}_Z \), that is to say two measures on \( (E_1 \times E_2, \mathcal{B}_1 \otimes \mathcal{B}_2) \) and \( (E_2 \times E_3, \mathcal{B}_2 \otimes \mathcal{B}_3) \) satisfying, for each \( (B_1, B_2, B_3) \in (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \),

\[
\mathbb{P}_{(X,Y)}(B_1 \times E_2) = \mathbb{P}_X(B_1), \quad \mathbb{P}_{(X,Y)}(E \times B_2) = \mathbb{P}_Y(B_2) \quad \text{and} \quad \mathbb{P}_{(Y,Z)}(B_2 \times E) = \mathbb{P}_Y(B_2), \quad \mathbb{P}_{(Y,Z)}(E \times B_3) = \mathbb{P}_Z(B_3),
\]

then there exists a probability measure \( \mathbb{P}_{(X,Y,Z)} \) on \( (E_1 \times E_2 \times E_3, \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3) \) such that for each \( B_{12} \in \mathcal{B}_1 \otimes \mathcal{B}_2 \) and each \( B_{23} \in \mathcal{B}_2 \otimes \mathcal{B}_3 \),

\[
(2.11) \quad \mathbb{P}_{(X,Y,Z)}(B_{12} \times E_3) = \mathbb{P}_{(X,Y)}(B_{12}) \quad \text{and} \quad \mathbb{P}_{(X,Y,Z)}(E_1 \times B_{23}) = \mathbb{P}_{(Y,Z)}(B_{23}).
\]

**Remark 2.13.** As was said earlier, in this article we think of random variables as laws and we do not assume that there is an underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which all the random variables are already defined. For instance, we will say that we are given two random variables \((X, Y)\) and \((Y, Z)\) to mean that we are given two measures \( \mathbb{P}_{(X,Y)} \) and \( \mathbb{P}_{(Y,Z)} \) such that the marginals of \( \mathbb{P}_{(X,Y)} \) are \( \mathbb{P}_X \) and \( \mathbb{P}_Y \) and the marginals of \( \mathbb{P}_{(Y,Z)} \) are \( \mathbb{P}_Y \) and \( \mathbb{P}_Z \) without assuming that there exists an implicit probability space on which \( X, Y \) and \( Z \) are defined, indeed in that case the statement of the proposition would be trivial.

This convention allows to to simplify the notation in the proofs and has the following consequence: when we are given two random variables \( X \) and \( Y \), we need to be careful to always construct a coupling between \( X \) and \( Y \) before introducing the random variables \( X + Y, XY \) or any other display involving both \( X \) and \( Y \).

The proof of Proposition 2.12 relies on the existence of the conditional law which is recalled below.

**Proposition 2.14 (Existence of the conditional law, Theorem 33.3 and Theorem 34.5 of [8]).** Let \((E, \mathcal{E})\) and \((F, \mathcal{F})\) be two Polish spaces equipped with their Borel \( \sigma \)-algebras. Assume that we are given two probability measures \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) on \( E \) and \( F \) respectively. Let \( \mathbb{P}_{12} \) be a probability measure on \((E \times F, \mathcal{E} \otimes \mathcal{F})\) whose first and second marginals are \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) respectively, then there exists a mapping \( \nu : E_1 \times \mathcal{F} \to \mathbb{R}_+ \) such that

1. for each \( x \in E \), \( \nu(x, \cdot) \) is a probability measure on \((F, \mathcal{F})\)
2. for each \( A \in \mathcal{F} \), the mapping

\[
\nu(\cdot, A) \left\{ \begin{array}{ll}
(\varepsilon, \mathcal{E}) & \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\
 x & \mapsto \nu(x, A)
\end{array} \right.
\]

is measurable.
3. For each \( A_1 \in \mathcal{E} \) and each \( A_2 \in \mathcal{F} \),

\[
\mathbb{P}_{12}(A_1 \times A_2) = \int_{A_1} \nu(x, A_2) \mathbb{P}_1(dx).
\]

We can now prove Proposition 2.12.

**Proof of Proposition 2.12.** The idea is to apply Proposition 2.14 to the two laws \( \mathbb{P}_{(X,Y)} \) and \( \mathbb{P}_{(Y,Z)} \). This gives the existence of two conditional laws denoted by \( \nu_X \) and \( \nu_Z \) such that for each \( B_1, B_2, B_3 \in (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \),

\[
\mathbb{P}_{(X,Y)}(B_1 \times B_2) = \int_{B_2} \nu_X(y, B_1) \mathbb{P}_Y(dy) \quad \text{and} \quad \mathbb{P}_{(Y,Z)}(B_2 \times B_3) = \int_{B_2} \nu_Z(y, B_3) \mathbb{P}_Y(dy).
\]

We can then define for each \( B_1, B_2, B_3 \in (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) \),

\[
\mathbb{P}_{(X,Y,Z)}(B_1 \times B_2 \times B_3) = \int_{B_2} \nu_X(y, B_1) \nu_Z(y, B_3) \mathbb{P}_Y(dy).
\]

Using standard tools from measure theory, one can then extend \( \mathbb{P}_{(X,Y,Z)} \) into a measure on \( \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3 \) and verify that this measure satisfies (2.11). \( \square \)
2.5. Functional inequalities on the lattice. In this section, we want to prove a few functional inequalities for functions on the lattice $\mathbb{Z}^d$, namely the Poincaré inequality, and the multiscale Poincaré inequality. These inequalities are known on $\mathbb{R}^d$ so the strategy of the proof is to extend functions defined on $\mathbb{Z}^d$ to $\mathbb{R}^d$, to apply the inequalities to the extended functions and then show that the inequality obtained for the extended function is enough to prove the inequality for the discrete function.

The second inequality presented, called the multiscale Poincaré inequality, is a convenient tool to control the $L^2$ norm of a function by the spatial average of its gradient. It is proved in [5, Proposition 1.7 and Lemma 1.8]. The philosophy behind it comes from the theory of stochastic homogenization and roughly states that the usual Poincaré inequality can be refined by estimating the $L^2$ norm of a function by the spatial average of the gradient. This inequality is useful when one is dealing with rapidly oscillating functions, which frequently appears in homogenization. Indeed for these functions, the oscillations cancel out in the spatial average of the gradient, as a result these spatial averages are much smaller than the $L^2$ norm of the gradient. The resulting estimate is thus much more precise than the standard Poincaré inequality. We recall the definition of $\mathbb{Z}_{m,n}$ given in (1.1).

**Proposition 2.15** (Poincaré and Multiscale Poincaré inequalities). Let $\Box$ be a cube of $\mathbb{Z}^d$ of size $R$ and $u : \Box \to \mathbb{R}$, then one has the inequality, for some $C := C(d) < \infty$

$$\sum_{x \in \Box} |u(x) - (u)_{\Box}|^2 \leq CR^2 \sum_{e \subseteq \Box} |\nabla u(e)|^2, \tag{2.12}$$

if one assumes that $u = 0$ on $\partial \Box$, then one has

$$\sum_{x \in \Box} |u(x)|^2 \leq CR^2 \sum_{e \subseteq \Box} |\nabla u(e)|^2. \tag{2.13}$$

For each $n \in \mathbb{N}$, there exists a constant $C := C(d) < \infty$ such that for each $u : \Box_n \to \mathbb{R}$,

$$\frac{1}{|\Box_n|} \sum_{x \in \Box_n} |u(x) - (u)_{\Box_n}|^2 \leq C \sum_{e \subseteq \Box_n} |\nabla u(e)|^2 + C^3 n \sum_{k=1}^n 3^k \left( \frac{1}{|Z_{k,n}|} \sum_{y \in Z_{k,n}} |\langle \nabla u \rangle_{z+\Box_k}|^2 \right). \tag{2.14}$$

If one assume that $u \in h^1_0(\Box_n)$, then one has

$$\frac{1}{|\Box_n|} \sum_{x \in \Box_n} |u(x)|^2 \leq C \sum_{e \subseteq \Box_n} |\nabla u(e)|^2 + C^3 n \sum_{k=1}^n 3^k \left( \frac{1}{|Z_{k,n}|} \sum_{y \in Z_{k,n}} |\langle \nabla u \rangle_{z+\Box_k}|^2 \right). \tag{2.15}$$

**Proof.** The idea is to construct a smooth function $\tilde{u}$ which is close to $u$ by first extending it to be piecewise constant on the cubes $z + (-\frac{1}{2}, \frac{1}{2})^d$, where $z \in \Box_m$. We then make this function smooth by convolving against a smooth approximation of the identity supported in the ball $B_{1/2}$. It follows that $\tilde{u}(z) = u(z)$ for each $z \in \Box_m$ and that the following estimate on holds: for each $z \in \Box_m$,

$$\sup_{z + (-\frac{1}{2}, \frac{1}{2})^d} |\nabla \tilde{u}(x)| \leq C \sum_{y \sim x} |u(y) - u(x)|. \tag{2.16}$$

One can then apply the Poincaré inequalities to the function $\tilde{u}$ and then check that this is enough to obtain (2.12) and (2.13). The proof of (2.14) follows the same lines, a proof of this inequality can be found in [1, Proposition A2]. Note that the version stated here is a slight modification of the one which can be found there but can be deduced from it by applying the Cauchy-Schwarz inequality.

The version of the multiscale Poincaré inequality with 0 boundary condition given in (2.15) cannot be found in [1, Proposition A2]. Nevertheless the continous version of this inequality is a consequence of [5, Proposition 1.7 and Lemma 1.8]. The transposition to the discrete setting is identical to the proof given in [1, Proposition A2].
3. Subadditive quantities and their basic properties

The goal of this section is to study the quantities \( \nu \) and \( \nu^* \) introduced in the previous sections. We prove a series of result about these quantities. These results are reminiscent of the basic properties of \( \nu \) and \( \nu^* \) in stochastic homogenization, see [5, Lemma 1.1 and Lemma 2.2]. We first state most of these properties in the same proposition, then split this proposition into a number of shorter propositions and prove these propositions one by one.

**Proposition 3.1** (Properties of \( \nu \) and \( \nu^* \)). There exists a constant \( C := C(d, \lambda) < \infty \) such that, the following properties hold

- **Subadditivity.** For each \( n \in \mathbb{N} \) and each \( p \in \mathbb{R}^d \),
  \[
  \nu(\square_{n+1}, p) \leq \nu(\square_n, p) + C \left( 1 + |p|^2 \right) 3^{-n}.
  \]
  Similarly, for each \( q \in \mathbb{R}^d \),
  \[
  \nu^*(\square_{n+1}, q) \leq \nu^*(\square_n, q) + C \left( 1 + |q|^2 \right) 3^{-n}.
  \]

- **One-sided convex duality.** For each \( p, q \in \mathbb{R}^d \) and each \( n \in \mathbb{N} \),
  \[
  \nu(\square_n, p) + \nu^*(\square_n, q) \geq p \cdot q - C3^{-n}.
  \]

- **Quadratic bounds.** there exists a small constant \( c := c(d, \lambda) > 0 \) such that for each \( n \in \mathbb{N} \) and each \( p \in \mathbb{R}^d \)
  \[
  -C + c|p|^2 \leq \nu(\square_n, p) \leq C \left( 1 + |p|^2 \right),
  \]
  and for each \( q \in \mathbb{R}^d \),
  \[
  -C + c|q|^2 \leq \nu^*(\square_n, q) \leq C \left( 1 + |q|^2 \right).
  \]

- **Uniform convexity of \( \nu \).** For each \( p_0, p_1 \in \mathbb{R}^d \),
  \[
  \frac{1}{C}|p_0 - p_1|^2 \leq \frac{1}{2} \nu(\square_n, p_0) + \frac{1}{2} \nu(\square_n, p_1) - \nu \left( \square_n, \frac{p_0 + p_1}{2} \right) \leq C|p_0 - p_1|^2.
  \]

- **Convexity of \( \nu^* \).** The mapping \( q \rightarrow \nu^*(q) \) is convex.

- **\( L^2 \) bounds for the minimizers.** For each \( p \in \mathbb{R}^d \)
  \[
  \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subset \square_n} |\nabla \phi_{n,p}(e)|^2 \right] \leq C(1 + |p|^2).
  \]
  Similarly, for each \( q \in \mathbb{R}^d \), one has
  \[
  \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subset \square_n} |\nabla \psi_{n,q}(e)|^2 \right] \leq C(1 + |q|^2).
  \]

**Remark 3.2.**

1. There is a technical reason why we can obtain a strong uniform convexity result for the quantity \( \nu \) and only a mild convexity result for \( \nu^* \), this is explained in Remark 3.12.

2. From the subadditivity property and the quadratic bounds, we obtain that for each \( p, q \in \mathbb{R}^d \), the quantities \( \nu(\square_n, p) \) and \( \nu^*(\square_n, q) \) converge as \( n \rightarrow \infty \). Moreover, by passing to the limit, we see that the limit satisfies the convexity and one-sided duality properties. This is summarized in the following proposition.

**Proposition 3.3.** For each \( p \in \mathbb{R}^d \) and \( q \in \mathbb{R}^d \), the quantities \( \nu(\square_n, p) \) and \( \nu^*(\square_n, q) \) converge as \( n \rightarrow \infty \). We denote by \( \overline{\nu}(p) \) and \( \overline{\nu}^*(q) \) their respective limits. Moreover, there exists a constant \( C := C(d, \lambda) > \infty \) such that the following properties hold
• One-sided convex duality. For each \( p, q \in \mathbb{R}^d \),
\[
\nabla(p) + \nabla^*(q) \geq p \cdot q.
\]

• Quadratic bounds. There exists a small constant \( c := c(d, \lambda) > 0 \) such that for each \( p \in \mathbb{R}^d \),
\[
-C + c|p|^2 \leq \nabla(p) \leq C \left(1 + |p|^2\right),
\]
and for each \( q \in \mathbb{R}^d \),
\[
-C + c|q|^2 \leq \nabla^*(q) \leq C \left(1 + |q|^2\right).
\]

• Convexity and uniform convexity. The mapping \( q \mapsto \nabla^*(q) \) is convex and for each \( p_1, p_2 \in \mathbb{R}^d \),
\[
\frac{1}{C}|p_0 - p_1|^2 \leq \frac{1}{2} \nabla(p_0) + \frac{1}{2} \nabla(p_1) - \nabla \left(\frac{p_0 + p_1}{2}\right) \leq C|p_0 - p_1|^2.
\]

Proof. The properties mentioned in the proposition are valid for the quantities \( \nu(\square_n, p) \) and \( \nu^*(\square_n, q) \) and pass to the pointwise limit as \( n \to \infty \).

Remark 3.4. The previous proposition proves the estimate (1.12) of Theorem 1.1. Also by the previous proposition, to prove (1.14), there remains to show the upper bound
\[
\nabla^*(q) \leq \sup_{p \in \mathbb{R}^d} -\nabla(p) + p \cdot q,
\]
since the lower bound follows from the one-sided convex duality. This upper bound will be proved later in the article. The uniform convexity of \( \nabla^* \) stated in (1.13) will then be deduced from (1.14) and (1.12).

3.1. Subadditivity. We first prove the subadditivity property for \( \nu \). This subadditivity property as well as its proof can be found in [16, Appendix 2], we rewrite it for completeness and because we need to quantify the error we made by going from one scale to another in the setting of triadic cubes.

Proposition 3.5 (Subadditivity for \( \nu \)). There exists a constant \( C := C(d, \lambda) < \infty \) such that for each \( n \in \mathbb{N} \) and for each \( p \in \mathbb{R}^d \),
\[
\nu(\square_{n+1}, p) \leq \nu(\square_n, p) + C \left(1 + |p|^2\right) 3^{-n}.
\]

Proof. We split the proof into two steps.

• In Step 1, we prove that for any real number \( x_0 \in \mathbb{R} \) and any family of real numbers \( y_1, \ldots, y_k \in \mathbb{R} \), there exists a constant \( c := c(k, \lambda) > 0 \) such that for any \( V \) satisfying the uniform convexity property (1.5),
\[
\int_{\mathbb{R}} \exp \left(-\sum_{i=1}^{k} V(y_i - x)\right) dx \geq c \exp \left(-\sum_{i=1}^{k} V(y_i - x_0)\right)
\]
\[
(3.1)
\]

• In Step 2, we deduce the subadditivity of \( \nu \) from (3.1).

Step 1. Using the assumption (1.5) and performing a Taylor expansion of \( V \) at \( y_i - x_0 \), gives, for each \( x \in \mathbb{R} \),
\[
-V(y_i - x) \geq -V(y_i - x_0) + (x - x_0)V'(y_i - x_0) - \frac{1}{2\lambda} |x - x_0|^2.
\]

Summing the previous inequality and taking the exponential yields,
\[
\exp \left(-\sum_{i=1}^{k} V(y_i - x)\right) \geq \exp \left(\sum_{i=1}^{k} -V(y_i - x_0) + (x - x_0)\sum_{i=1}^{k} V'(y_i - x_0) - \frac{k}{2\lambda} |x - x_0|^2\right)
\]
We then fix another point $x$. As a consequence, by integrating over the variable $x$ setting $c := \sqrt{2\lambda \pi / k}$ completes the proof of (3.1).

**Step 2.** For $n \in \mathbb{N}$, we denote by $\partial_{int} \Box_n$ the points of $\mathbb{Z}^d$ which lies in the interior of $\Box_{n+1}$ and on the boundary of the subcubes $z + \Box_n$, for $z \in \mathcal{Z}_n$, that is to say

$$\partial_{int} \Box_{n+1} := \left( \bigcup_{z \in \mathcal{Z}_n} \partial (z + \Box_n) \right) \setminus \partial \Box_{n+1}.$$  

The idea of the proof is to apply consecutively the inequality of Step 1 to the points of $\partial_{int} \Box_n$. First note that one can estimate the cardinal of $\partial_{int} \Box_{n+1}$ by

$$|\partial_{int} \Box_{n+1}| \leq C 3^n (d-1),$$  

for some constant $C := C(d) < \infty$. Note also that the edges of $\Box_{n+1}$ can be partitioned into three set of edges: the edges lying in the subcubes $z + \Box_n$, $z \in \mathcal{Z}_n$ and the edges of $\partial_{int} \Box_{n+1}$.

Fix a point $x_0 \in \partial_{int} \Box_{n+1}$ and denote $y_1, \ldots, y_{2d}$ its $2d$ neighbours in $\mathbb{Z}^d$ (which all belongs to $\Box_{n+1}$ by definition of $\partial_{int} \Box_{n+1}$). Applying the result of Step 1 shows, for any $\phi(y_1), \ldots, \phi(y_{2d}) \in \mathbb{R}$,

$$\int_{\mathbb{R}} \exp \left( - \sum_{i=1}^{2d} V_{y_i, x_0} (p \cdot (y_i - x_0) + \phi(y_i) - \phi(x_0)) \right) d\phi(x_0) \geq c \exp \left( - \sum_{i=1}^{2d} V_{y_i, x_0} (p \cdot (y_i - x_0) + \phi(y_i)) \right).$$  

As a consequence, by integrating over the variable $h_0^1 (\Box_{n+1})$ and applying the estimate (3.3), we obtain the inequality

$$\int_{h_0^1 (\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e (p(e) + \nabla \phi(e)) \right) d\phi \geq c \int_{h_0^1 (\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e (p(e) + \nabla \phi(e)) \right) \delta_0 (d\phi(x_0)) \prod_{x \neq x_0} d\phi(x).$$

We then fix another point $x_1 \in \partial_{int} \Box_{n+1}$, by Step 1, we have, exactly as in (3.3),

$$\int_{\mathbb{R}} \exp \left( - \sum_{y \sim x_1} V_{y, x_1} (p \cdot (y - x_1) + \phi(y) - \phi(x_1)) \right) d\phi(x_1) \geq c \exp \left( - \sum_{y \sim x} V_{y, x_1} (p \cdot (y - x_1) + \phi(y)) \right).$$
Consequently, we obtain,
\[
\int_{h_0(\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e(p(e) + \nabla \phi(e)) \right) \delta_0(d\phi(x_0)) \otimes \prod_{x \neq x_0} d\phi(x) \\
\geq c \int_{h_0(\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e(p(e) + \nabla \phi(e)) \right) \delta_0(d\phi(x_0)) \otimes \delta_0(d\phi(x_1)) \otimes \prod_{x \notin \{x_0,x_1\}} d\phi(x).
\]

We then iterate this reasoning to every points \( x \in \partial_{int}\Box_{n+1} \) to obtain
\[\int_{h_0(\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e(p(e) + \nabla \phi(e)) \right) d\phi \geq c |\partial_{int}\Box_{n+1}| \int_{h_0(\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e(p(e) + \nabla \phi(e)) \right) \prod_{x \in \partial_{int}\Box_{n+1}} \delta_0(d\phi(x)) \prod_{x \notin \partial_{int}\Box_{n+1}} d\phi(x).\]

The term on the right-hand side can be written in a more convenient form. Indeed, we have
\[
\int_{h_0(\Box_{n+1})} \exp \left( - \sum_{e \subseteq \Box_{n+1}} V_e(p(e) + \nabla \phi(e)) \right) d\phi = \prod_{z \in \mathbb{Z}^n} \left( \int_{h_0(z+\Box_{n})} \exp \left( - \sum_{e \subseteq z+\Box_{n}} V_e(p(e) + \nabla \phi(e)) \right) d\phi(x) \right) \times \exp \left( - \sum_{e \subseteq \partial_{int}\Box_{n+1}} V_e(p(e)) \right)
\]
Combining (3.4), (3.5) and taking the logarithm yields
\[
\nu(\Box_{n+1},p) \leq \sum_{z \in \mathbb{Z}^n} \left| \frac{z+\Box_{n}}{|\Box_{n+1}|} \right| \nu(z+\Box_{n},p) - \frac{|\partial_{int}\Box_{n+1}|}{|\Box_{n+1}|} \ln c + \frac{1}{|\Box_{n+1}|} \sum_{e \subseteq \partial_{int}\Box_{n+1}} V_e(p \cdot e).
\]

Using that, for each \( z \in \mathbb{Z}^n \), for each \( p \in \mathbb{R}^d \), \( \nu(z+\Box_{n},p) = \nu(\Box_{n},p) \) and that for each \( x \in \mathbb{R} \), \( V_e(x) \leq \Lambda x^2 \), we obtain
\[
\nu(\Box_{n+1},p) \leq \nu(\Box_{n},p) - \frac{|\partial_{int}\Box_{n+1}|}{|\Box_{n+1}|} \ln c + \frac{\Lambda |p|^2 |\partial_{int}\Box_{n+1}|}{|\Box_{n+1}|}.
\]

Using (3.2), we eventually obtain
\[
\nu(\Box_{n+1},p) \leq \nu(\Box_{n},p) + C \left( 1 + |p|^2 \right) 3^{-n}.
\]

We now state and prove the subadditivity property for \( \nu^* \).

**Proposition 3.6** (Subadditivity of \( \nu^* \)). There exists a constant \( C := C(d, \Lambda) < \infty \) such that for each \( n \in \mathbb{N} \) and for each \( q \in \mathbb{R}^d \), we have the inequality
\[
\nu^*(\Box_{n+1},q) \leq \nu^*(\Box_{n},q) + C \left( 1 + |q|^2 \right) 3^{-n}.
\]

**Proof.** The proof is split into two steps

- In Step 1, we prove a subadditivity inequality for general domains partitioned into two subdomains. More precisely, we prove that there exists a constant \( C := C(d, \Lambda) < \infty \) such that, for any \( q \in \mathbb{R}^d \), for any bounded connected domain \( U \subseteq \mathbb{Z}^d \) and any partition of \( U \) into two connected subdomains \( U_1 \) and \( U_2 \), we have
\[
\nu^*(U,q) \leq \frac{|U_1|}{|U|} \nu^*(U_1,q) + \frac{|U_2|}{|U|} \nu^*(U_2,q) + C \left( 1 + |q|^2 \right) \frac{|\partial_{int}U|}{|U|} + \log |U|.
\]
where $\partial_{int} U$ is defined by
\[
\partial_{int} U := \{ x \in U_1 : \exists y \in U_2 \text{ such that } x \sim y \} \cup \{ x \in U_2 : \exists y \in U_1 \text{ such that } x \sim y \}.
\]

- In Step 2, we deduce (3.7) from Step 1.

**Step 1.** The idea to prove (3.8) is to use the following orthogonal decomposition of the space $\hat{h}^1(U)$,
\[
(3.9) \quad \hat{h}^1(U) = \hat{h}^1(U_1) \oplus \hat{h}^1(U_2) \oplus \mathbb{R}v
\]
where $\hat{h}^1(U_1)$ and $\hat{h}^1(U_2)$ are defined by
\[
\hat{h}^1(U_1) := \{ \psi \in \hat{h}^1(U) : \psi|_{U_2} = 0 \} \quad \text{and} \quad \hat{h}^1(U_2) := \{ \psi \in \hat{h}^1(U) : \psi|_{U_1} = 0 \},
\]
with a slight abuse of notation. The function $v$ mentioned in the previous decomposition is given by
\[
(3.10) \quad v = \frac{1}{\sqrt{|U|}} \left( \sqrt{\frac{|U_2|}{|U_1|}} \mathbb{1}_{U_1} - \sqrt{\frac{|U_1|}{|U_2|}} \mathbb{1}_{U_2} \right).
\]
It is easy to check that
\[
\sum_{x \in U} v(x) = 0 \quad \text{and} \quad \sum_{x \in U} (v(x))^2 = 1,
\]
and that the decomposition (3.9) is indeed an orthogonal decomposition of $\hat{h}^1(U)$. Consequently, any functions $\psi \in \hat{h}^1(U)$ can be written
\[
\psi = \psi_1 + \psi_2 + tv,
\]
for some (unique) $\psi_1 \in \hat{h}^1(U_1)$, $\psi_2 \in \hat{h}^1(U_2)$ and $t \in \mathbb{R}$. Moreover, using the property (1.4) mentioned in the introduction, we have the equality
\[
(3.11) \quad \int_{\hat{h}^1(U)} \exp \left( - \sum_{e \subseteq U} \left( V_e (\nabla \psi(e)) - q \cdot \nabla \psi(e) \right) \right) \, d\phi
\]
\[
= \int_{\hat{h}^1(U_1)} \int_{\hat{h}^1(U_2)} \int_{\mathbb{R}} \exp \left( - \sum_{e \subseteq U} V_e (\nabla \psi_1(e) + \nabla \psi_2(e) + t \nabla v(e)) - q \cdot \nabla (\psi_1 + \psi_2 + tv)(e) \right) \, dt \, d\psi_2 \, d\psi_1.
\]

We then notice that the gradient of $v$ is supported on the edges of $\partial_{int} U$. We now prove the intermediate result: there exists a constant $C := C(d, \lambda) < \infty$ such that for any $\psi_1 \in \hat{h}^1(U_1)$ and $\psi_2 \in \hat{h}^1(U_2),$
\[
\int_{\mathbb{R}} \exp \left( - \sum_{e \subseteq \partial_{int} U} V_e (\nabla (\psi_1 + \psi_2)(e) + t \nabla v(e)) - q \cdot \nabla (\psi_1 + \psi_2 + tv)(e) \right) \, dt \leq C \exp \left( C |\partial_{int} U| (1 + |q|^2) \right).
\]
To simplify the notation, we denote
\[
V_{q, \psi} : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad t \mapsto \sum_{e \subseteq \partial_{int} U} V_e (\nabla (\psi_1 + \psi_2)(e) + t \nabla v(e)) - q \cdot \nabla (\psi_1 + \psi_2 + tv)(e).
\]
Computing the second derivative of \( V_{q,\psi} \) and using that \( V''_e \geq \lambda \), we obtain, for each \( t \in \mathbb{R} \),
\[
V''_{q,\psi}(t) \geq \lambda \sum_{e \subseteq \partial_{int} U} |\nabla v(e)|^2.
\]
Moreover, the definition of \( v \) (3.10), we have, for each \( e \subseteq \partial_{int} U \),
\[
|\nabla v(e)|^2 = \frac{1}{|U|} \left( \sqrt{\frac{|U_2|}{|U_1|}} + \sqrt{\frac{|U_1|}{|U_2|}} \right)^2 \geq \frac{4}{|U|},
\]
where we used the inequality \( x + \frac{1}{x} \geq 2 \). We thus obtain
\[
V''_{q,\psi}(t) \geq 4\lambda \frac{|\partial_{int} U|}{|U|} > 0.
\]
In particular, this implies that there exists a real number \( t_{min} \in \mathbb{R} \) which achieves the minimum of \( V_{q,\psi} \) and that for each \( t \in \mathbb{R} \),
\[
V_{q,\psi}(t) \geq 4\lambda \frac{|\partial_{int} U|}{|U|} (t - t_{min})^2 + V_{q,\psi}(t_{min}).
\]
Since \( V_e \) satisfies \( V_e(t) \geq \lambda t^2 \), we have the bound
\[
V_{q,\psi}(t_{min}) \geq -C|\partial_{int} U| (1 + |q|^2)
\]
for some constant \( C := C(d, \lambda) < \infty \). Combining the two previous results shows
\[
V_{q,\psi}(t) \geq 4\lambda \frac{|\partial_{int} U|}{|U|} (t - t_{min})^2 - C|\partial_{int} U| (1 + |q|^2).
\]
We can thus compute, for any \( \psi_1 \in \check{h}^1(U_1) \) and any \( \psi_2 \in \check{h}^1(U_2) \),
\[
\int_{\mathbb{R}} \exp \left( - \sum_{e \subseteq \partial_{int} U} V_e(\nabla(\psi_1 + \psi_2)(e) + t\nabla v(e)) - q \cdot \nabla(\psi_1 + \psi_2 + tv)(e) \right) dt
\]
\[
= \int_{\mathbb{R}} \exp (-V_{q,\psi}(t)) dt
\]
\[
\leq \int_{\mathbb{R}} \exp \left( -4\lambda \frac{|\partial_{int} U|}{|U|} (t - t_{min})^2 + C|\partial_{int} U| (1 + |q|^2) \right) dt
\]
\[
\leq \exp \left( C|\partial_{int} U| (1 + |q|^2) \right) \int_{\mathbb{R}} \exp \left( -4\lambda \frac{|\partial_{int} U|}{|U|} (t - t_{min})^2 \right) dt.
\]
Using that \( |\partial_{int} U| \geq 1 \) (indeed \( |\partial_{int} U| \) is an integer and if we have \( |\partial_{int} U| = 0 \) then \( U_1 = \emptyset \) or \( U_2 = \emptyset \) and the inequality (3.8) is valid), we obtain
\[
\int_{\mathbb{R}} \exp \left( - \sum_{e \subseteq \partial_{int} U} V_e(\nabla(\psi_1 + \psi_2)(e) + t\nabla v(e)) - q \cdot (\nabla(\psi_1 + \psi_2)(e) + t\nabla v(e)) \right) dt
\]
\[
\leq C \exp \left( C|\partial_{int} U| (1 + |q|^2) \right) \sqrt{\frac{|U|}{|\partial_{int} U|}}.
\]
We then notice that the sum in the integrand of the right-hand term of (3.11) can be split as follows
\[
\sum_{e \leq U} (V_e(\nabla \psi_1(e) + \nabla \psi_2(e) + t \nabla v(e)) - q \cdot (\nabla \psi_1 + \nabla \psi_2 + t \nabla v)(e))
\]
\[
= \sum_{e \leq U_1} (V_e(\nabla \psi_1(e)) - q \cdot \nabla \psi_1(e))
\]
\[
+ \sum_{e \leq \partial_{int} U} (V_e(\nabla \psi_1(e) + \nabla \psi_2(e) + t \nabla v(e)) - q \cdot (\nabla \psi_1 + \nabla \psi_2 + t \nabla v)(e))
\]
\[
+ \sum_{e \leq U_2} (V_e(\nabla \psi_2(e)) - q \cdot \nabla \psi_2(e)),
\]
because the gradient of \(\psi_1\) is supported in the edges of \(U_1 \cup \partial_{int} U_1\), the gradient of \(\psi_2\) is supported in the edges of \(U_2 \cup \partial_{int} U_2\) and the gradient of \(v\) is supported in the edges of \(\partial_{int} U\). In particular, using the two previous displays and combining them with (3.11), we obtain
\[
\int_{\hat{h}^1(U)} \exp \left( - \sum_{e \leq U} (V_e(\nabla \psi(e)) - q \cdot \nabla \psi(e)) \right) d\psi
\]
\[
\leq C \exp \left( C |\partial_{int} U| (1 + |q|^2) \right) \sqrt{\frac{|U|}{|\partial_{int} U|}}
\]
\[
\times \int_{\hat{h}^1(U_1)} \exp \left( - \sum_{e \leq U_1} V(\nabla \psi_1(e)) - q \cdot \nabla \psi_1(e) \right) d\psi_1
\]
\[
\times \int_{\hat{h}^1(U_2)} \exp \left( - \sum_{e \leq U_2} V(\nabla \psi_2(e)) - q \cdot \nabla \psi_2(e) \right) d\psi_2.
\]
Taking the logarithm and dividing by \(|U|\) yields
\[
\nu^*(U, q) \leq \frac{|U_1|}{|U|} \nu^*(U_1, q) + \frac{|U_2|}{|U|} \nu^*(U_2, q) + C \frac{1 + \log \left( \frac{|U|}{|\partial_{int} U|} \right) + |\partial_{int} U|(1 + |q|^2)}{|U|}.
\]
This implies in particular (3.8).

**Step 2.** Fix an enumeration \((z_i)_{i=1,\ldots,3^d}\) of \(\mathbb{Z}^d\) such that for each \(k \in \{1, \ldots, 3^d\}\), the set \(\cup_{i=1}^k (z_i + \Box_n)\) is connected. We prove by induction for each \(k \in \{1, \ldots, 3^d\}\) that there exists a constant \(C_k := C(k, d, \lambda) < \infty\) such that
\[
\nu^* \left( \cup_{i=1}^k (z_i + \Box_n), q \right) \leq \nu^* (\Box_n, q) + C_k 3^{-n}(1 + |q|^2).
\]
We proceed by induction on \(k\). If \(k = 1\) then
\[
\nu^* (z_1 + \Box_n, q) = \nu^* (\Box_n, q)
\]
and the result is valid with \(C_0 = 0\).

If we assume the result to be true for \(k\) and want to prove it for \(k + 1\). The idea is to apply the main result (3.8) of Step 1 with \(U = \cup_{i=1}^{k+1} (z_i + \Box_n), U_1 = z_{k+1} + \Box_n\) and \(U_2 = \cup_{i=1}^{k} (z_i + \Box_n)\). Note that in that case the volume of \(\partial_{int} U\) can be estimated (and even explicitly computed if we know how the \(z_i\) are chosen). In particular, we have the following inequality, for some constant \(C := C(d) < \infty\),
\[
\frac{|\partial_{int} U|}{|U|} \leq C 3^{-n}.
\]
Applying (3.8) yields (here we use the assumption that $U$ is connected),

$$\nu^*\left(\bigcup_{i=1}^{k+1} (z_i + \Box_n), q\right) \leq \frac{1}{k+1} \nu^* (\Box_{k+1}, q) + \frac{k}{k+1} \nu^* \left(\bigcup_{i=1}^{k+1} (z_i + \Box_n), q\right) + C3^{-n}(1 + |q|^2).$$

The induction hypothesis then yields

$$\nu^* \left(\bigcup_{i=1}^{k+1} (z_i + \Box_n), q\right) \leq \frac{1}{k+1} \nu^* (\Box_n, q) + \nu^* \left(\bigcup_{i=1}^{k+1} (z_i + \Box_n), q\right) + C3^{-n}(1 + |q|^2),$$

where we defined $C_{k+1} := \frac{k}{k+1} C_k + C$. The proof of the induction is complete. The inequality obtained in the case $k = 3d$ gives

$$\nu^* (\Box_{n+1}, q) \leq \nu^* (\Box_n, q) + C3^{-n}(1 + |q|^2),$$

which is precisely (3.7). \qed

3.2. Convex duality: lower bound. We now turn to the proof of the convex duality for $\nu$ and $\nu^*$.

**Proposition 3.7** (Convex duality). There exists a constant $C := C(d, \lambda) < \infty$ such that for each $p, q \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,

$$\nu(\Box_n, p) + \nu^*(\Box_n, q) \geq p \cdot q - C3^{-n}.$$

**Proof.** We recall the notation $\partial\Box_n$ and $\Box^0_n$ to denote respectively the boundary and the interior of the cube $\Box_n$. As in the proof of Proposition 3.6, we decompose the space $\hat{h}^1(\Box_n)$ into three orthogonal subspaces

$$\hat{h}^1(\Box_n) = \hat{h}^1 (\partial\Box_n) \perp \hat{h}^1 (\Box^0_n) \perp \mathbb{R}v,$$

where as in Proposition 3.6 we use a slight abuse of notation and denote by

$$\hat{h}^1 (\partial\Box_n) := \left\{ \psi \in \hat{h}^1(\Box_n) : \psi|_{\partial\Box_n} = 0 \right\} \text{ and } \hat{h}^1 (\Box^0_n) := \left\{ \psi \in \hat{h}^1(\Box_n) : \psi|_{\partial\Box_n} = 0 \right\},$$

and where $v$ is the function defined by

$$v = \frac{1}{|\Box_n|} \left( \sqrt{\frac{|\partial\Box_n|}{|\Box^0_n|}} \mathbf{1}_{\Box^0_n} - \sqrt{\frac{|\Box^0_n|}{|\partial\Box_n|}} \mathbf{1}_{\partial\Box_n} \right),$$

so that

$$\sum_{x \in \Box_n} v(x) = 0 \text{ and } \sum_{x \in \Box_n} v(x)^2 = 1.$$

Since it will be important later in the proof, we note that, for each $n \in \mathbb{N}$,

$$\dim \hat{h}^1 (\partial\Box_n) = |\partial\Box_n| - 1 \leq C3^{(d-1)n},$$

for some $C := C(d) < \infty$.

We split the proof into 4 steps.

- In Step 1, we show that, for some $C := C(d, \lambda) < \infty$

$$\nu^*(\Box_n, q) \geq \frac{1}{|\Box_n|} \log \left( \int_{\hat{h}^1(\Box^0_n) \perp \mathbb{R}v} \exp \left( - \sum_{e \subseteq \Box_n} V_e(\nabla \psi(e)) d\psi \right) \right) - C3^{-n}. \quad (3.14)$$

- In Step 2, we show that, for some $C := C(d) < \infty$

$$\nu(\Box_n, 0) \geq - \frac{1}{|\Box_n|} \log \left( \int_{\hat{h}^1(\Box^0_n) \perp \mathbb{R}v} \exp \left( - \sum_{e \subseteq \Box_n} V_e(\nabla \psi(e)) d\psi \right) \right) - Cn3^{-dn}. \quad (3.15)$$
• In Step 3, we combine the results of Steps 1 and 2 to obtain that there exists \( C := C(d, \lambda) < \infty \) such that for each \( p \in \mathbb{R}^d \),
\[
\nu^* (\square_n, q) + \nu (\square_n, 0) \geq -C3^{-n}.
\]

• In Step 4, we remove the assumption \( p = 0 \) and prove for each \( p, q \in \mathbb{R}^d \),
\[
\nu^* (\square_n, q) + \nu (\square_n, 0) \geq p \cdot q - C3^{-n}.
\]

**Step 1.** First, by (3.12), any function \( \psi \in \dot{h}^1 (\square_n) \) can be uniquely decomposed according to
\[
\psi = \psi_1 + \psi_2 + tv
\]
with \( \psi_1 \in \dot{h}^1 (\partial \square_n) \), \( \psi_2 \in \dot{h}^1 (\square_n^0) \) and \( t \in \mathbb{R} \). Note that each function \( \psi_2 \) in \( \dot{h}^1 (\square_n^0) \) is equal to 0 on the boundary of \( \square_n \), thus we have, for each \( q \in \mathbb{R}^d \),
\[
\sum_{e \subseteq \square_n} q \cdot \nabla \psi_2 (e) = 0.
\]
Since the function \( v \) is constant on \( \partial \square_n \), we also have
\[
\sum_{e \subseteq \square_n} q \cdot \nabla v (e) = 0.
\]
To prove (3.14), it is sufficient to prove, for each \( \psi_2 \in \dot{h}^1 (\square_n^0) \) and each \( t \in \mathbb{R} \),
\[
(3.16) \quad \int_{\dot{h}^1 (\partial \square_n)} \exp \left( -\sum_{e \subseteq \square_n} (V_e (\nabla (\psi_1 + \psi_2 + tv) (e)) - q \cdot \nabla \psi_1 (e)) \right) d\psi_1
\]
\[
\geq c^{3(d-1)n} \exp \left( -\sum_{e \subseteq \square_n} V_e (\nabla (\psi_2 + tv) (e)) \right),
\]
for some \( c := c(d, \lambda) > 0 \). Indeed, the estimate (3.14) is then obtained by integrating the previous inequality over \( \dot{h}^1 (\square_n^0) \oplus \mathbb{R}v \). To prove (3.16), we use the following Taylor expansion
\[
V_e (\nabla (\psi_1 + \psi_2 + tv) (e)) \leq V_e (\nabla (\psi_2 + tv) (e)) + V_e' (\nabla (\psi_2 + tv) (e)) \nabla \psi_1 (e) + \frac{1}{2\lambda} |\nabla \psi_1 (e)|^2.
\]
This implies
\[
\exp \left( -\sum_{e \subseteq \square_n} (V_e (\nabla (\psi_1 + \psi_2 + tv) (e)) - q \cdot \nabla \psi_1 (e)) \right)
\]
\[
\geq \exp \left( -\sum_{e \subseteq \square_n} \left( V_e (\nabla (\psi_2 + tv) (e)) + (V_e' (\nabla (\psi_2 + tv) (e)) - q) \nabla \psi_1 (e) + \frac{1}{2\lambda} |\nabla \psi_1 (e)|^2 \right) \right).
\]
Using the crude inequality for an edge \( e = (x, y) \subseteq \square_n \)
\[
|\nabla \psi_1 (e)|^2 = |\psi_1 (x) - \psi_1 (y)|^2 \leq 2 |\psi_1 (x)|^2 + 2 |\psi_1 (y)|^2,
\]
and summing over all the edges of \( \square_n \) yields
\[
\sum_{e \subseteq \square_n} |\nabla \psi_1 (e)|^2 \leq 2d \sum_{x \in \partial \square_n} \psi_1 (x)^2.
\]
But note that, as in Step 1 of Proposition 3.5, for each \( a \in \mathbb{R} \),
\[
\int_{\mathbb{R}} \exp \left( ax - \frac{d}{\lambda} x^2 \right) dx \geq \sqrt{\frac{\lambda \pi}{d}}.
\]
With the previous estimate and \((3.13)\), one proves
\[
\int_{h^1(\partial \Box_n)} \exp \left( - \sum_{e \subseteq \Box_n} \left( V'_e (\nabla (\psi_2 + tv)(e)) - q \cdot \nabla \psi_1(e) \right) + \frac{d}{\lambda} \sum_{x \in \partial \Box_n} |\psi_1(x)|^2 \right) \geq c^{3(d-1)n},
\]
for some \(c := c(d, \lambda) > 0\). Combining the few previous displays gives
\[
\int_{\hat{h}^1(\partial \Box_n)} \exp \left( - \sum_{e \subseteq \Box_n} V_e (\nabla (\psi_1 + \psi_2 + tv)(e)) - q \cdot \nabla \psi_1(e) \right) \geq c^{3(d-1)n} \exp \left( - \sum_{e \subseteq \Box_n} V_e (\nabla (\psi_2 + tv)(e)) \right).
\]
This is precisely \((3.16)\).

**Step 2.** We denote by
\[
\tilde{v} := \frac{1}{\|\Box_o\|} \mathbf{1}_{\Box_o}
\]
so that \(\sum_{x \in \Box_n} \tilde{v}(x)^2 = 1\). Note that the two functions \(v\) and \(\tilde{v}\) are linked by
\[
v + \frac{1}{|\Box_n|} \sqrt{\frac{|\Box_o|}{|\partial \Box_n|}} \mathbf{1}_{\Box_n} = \frac{1}{|\Box_n|} \left( \sqrt{\frac{|\partial \Box_n|}{|\Box_o|}} + \sqrt{\frac{|\Box_o|}{|\partial \Box_n|}} \right) \mathbf{1}_{\Box_n}
= \sqrt{\frac{|\Box_o|}{|\Box_n|}} \left( \sqrt{\frac{|\partial \Box_n|}{|\Box_o|}} + \sqrt{\frac{|\Box_o|}{|\partial \Box_n|}} \right) \tilde{v}.
\]
To shorten the notation, we denote by
\[
\alpha_n := \sqrt{\frac{|\Box_o|}{|\Box_n|}} \left( \sqrt{\frac{|\partial \Box_n|}{|\Box_o|}} + \sqrt{\frac{|\Box_o|}{|\partial \Box_n|}} \right).
\]
Combining the two previous displays we obtain, for each \(e \subseteq \Box_n\),
\[
\nabla v(e) = \alpha_n \nabla \tilde{v}(e).
\]
Note also that, there exists \(c := c(d) > 0\) and \(C := C(d) < \infty\) such that
\[
(3.18) \hspace{1cm} c^{3 - \frac{(d-1)n}{2}} \leq \alpha_n \leq C^{3 - \frac{(d-1)n}{2}}.
\]
We then use the orthogonal decomposition \(h^1_0(\Box_n) = \hat{h}^1(\Box_o^\perp) \oplus \mathbb{R} \tilde{v}\) and the decomposition of the Lebesgue measure explained in \((1.4)\) to obtain
\[
\int_{h^1_0(\Box_n)} \exp \left( - \sum_{e \subseteq \Box_n} V_e(\nabla \phi(e)) \right) d\phi = \int_{\mathbb{R}} \int_{\hat{h}^1(\Box_o^\perp)} \exp \left( - \sum_{e \subseteq \Box_n} V_e(\nabla \phi(e) + t\nabla \tilde{v}(e)) \right) d\phi dt.
\]
Using (3.17), we obtain
\[
\int_{h_0^1(\square_n)} \exp \left( - \sum_{e \subseteq \square_n} V_e(\nabla \phi(e)) \right) d\phi
\]
\[
= \int_\mathbb{R} \int_{h_1^1(\square_n)} \exp \left( - \sum_{e \subseteq \square_n} V_e(\nabla \phi(e) + \frac{t}{\alpha_n} \nabla v(e)) \right) d\phi dt
\]
\[
= \alpha_n \int_\mathbb{R} \int_{h_1^1(\square_n)} \exp \left( - \sum_{e \subseteq \square_n} V_e(\nabla \phi(e) + t \nabla v(e)) \right) d\phi dt
\]
\[
= \alpha_n \int_{h_1^1(\square_n) \oplus \mathbb{R} v} \exp \left( - \sum_{e \subseteq \square_n} V_e(\nabla \phi(e)) \right) d\phi.
\]
Taking the logarithm, dividing by $|\square_n|$ and using (3.18), we obtain (3.15).

**Step 3.** Combining the main results of Steps 1 and 2 gives
\[
\nu^*(\square_n, q) + \nu(\square_n, 0) \geq -C3^{-n} - Cn3^{-dn} \geq -C3^{-n}.
\]

**Step 4.** Let $p \in \mathbb{R}^d$, define $\tilde{V}_e := V_e(p(e) + \cdot)$ and denote by
\[
\tilde{\nu}(\square_n, 0) := -\frac{1}{|\square_n|} \log \int_{h_0^1(\square_n)} \exp \left( - \sum_{e \subseteq U} \tilde{V}_e(\nabla \phi(e)) \right) d\phi,
\]
and, for every $q \in \mathbb{R}^d$,
\[
\tilde{\nu}^*(\square_n, q) := \frac{1}{|\square_n|} \log \int_{h_1^1(\square_n)} \exp \left( - \sum_{e \subseteq \square_n} \left( \tilde{V}_e(\nabla \phi(e)) - q \cdot \nabla \phi(e) \right) \right) d\phi.
\]
The functions $\tilde{V}_e$ satisfy the same property of uniform convexity property (1.5) as $V_e$, thus one can apply the result of Steps 1, 2 and 3 with these functions. This gives, for every $q \in \mathbb{R}^d$,
\[
\tilde{\nu}(\square_n, 0) + \tilde{\nu}^*(\square_n, q) \geq -C3^{-n}.
\]
But note that
\[
\tilde{\nu}(\square_n, 0) = -\frac{1}{|\square_n|} \log \int_{h_0^1(\square_n)} \exp \left( - \sum_{e \subseteq U} \tilde{V}_e(\nabla \phi(e)) \right) d\phi
\]
\[
= -\frac{1}{|\square_n|} \log \int_{h_0^1(\square_n) \oplus \mathbb{R} v} \exp \left( - \sum_{e \subseteq U} V_e(\nabla \phi(e)) \right) d\phi
\]
\[
= \nu(\square_n, p).
\]
Note also that, by translation invariance of the Lebesgue measure on $h_1^1(\square_n)$, one can perform the change of variable $\phi := \phi - l_p$, where $l_p \in h_1^1(\square_n)$ is the function defined by $l_p(x) = p \cdot x$. This gives,
for each \( q \in \mathbb{R}^d \),
\[
\tilde{\nu}^*(\square, q) = \frac{1}{|\square|} \log \int_{h_1(\square)} \exp \left( - \sum_{e \subseteq \square} \left( \tilde{V}_e(\nabla \phi(e)) - q \cdot \nabla \phi(e) \right) \right) d\phi
\]
\[
= \frac{1}{|\square|} \log \int_{h_1(\square)} \exp \left( - \sum_{e \subseteq \square} \left( V_e(p \cdot e + \nabla \phi(e)) - q \cdot \nabla \phi(e) \right) \right) d\phi
\]
\[
= \frac{1}{|\square|} \log \int_{h_1(\square)} \exp \left( - \sum_{e \subseteq \square} \left( V_e(\nabla \psi(e)) + (q \cdot e)(p \cdot e) - q \cdot \nabla \psi(e) \right) \right) d\psi
\]
\[
= \nu^*(\square, q) + p \cdot q.
\]
Combining the few previous displays yields, for each \( p, q \in \mathbb{R}^d \),
\[
\nu(\square, p) + \nu^*(\square, q) \geq p \cdot q - C3^{-n},
\]
for some \( C := C(d, \lambda) < \infty \). The proof of Proposition 3.7 is complete.

3.3. Quadratic bounds. We now deal with the property about quadratic bounds. For later purposes, we state and prove it for general cubes for \( \nu \).

**Proposition 3.8** (Quadratic bounds on \( \nu \) and \( \nu^* \)). There exist two constants \( c := c(d, \lambda) > 0 \) and \( C := C(d, \lambda) < \infty \) such that, for each \( p \in \mathbb{R}^d \) and each cube \( \square \subseteq \mathbb{Z}^d \),
\[
(3.19) \quad - C + c|p|^2 \leq \nu(\square, p) \leq C(1 + |p|^2).
\]
For each \( q \in \mathbb{R}^d \) and each \( n \in \mathbb{N} \),
\[
(3.20) \quad - C + c|q|^2 \leq \nu^*(\square, q) \leq C(1 + |q|^2).
\]

**Remark 3.9.** Since it will be useful later and does not impact the proof, we prove the quadratic bounds for \( \nu \) for general cubes. Since such a statement is not required for \( \nu^* \) in the proof of Theorem 1.1 and since using triadic cubes allows a simpler proof, we only write the proof for triadic cubes.

**Proof.** Since we have for each \( z \in \mathbb{Z}^d \) and each bounded subset \( U \subseteq \mathbb{Z}^d \), \( \nu(z + U, p) = \nu(U, p) \) and that the same property is valid for \( \nu^* \), one can assume, without loss of generality, that the cube \( \square \) is of the form, for some \( N \in \mathbb{N} \),
\[
[0, N - 1]^d \cap \mathbb{Z}^d.
\]
We recall the following result from (1.5),
\[
(3.21) \quad \lambda|x|^2 \leq V_\epsilon(x) \leq \frac{1}{\lambda}|x|^2.
\]
Using these inequalities, one obtains
\[
\nu(\square, p) \leq \frac{1}{\lambda}|p|^2 - \frac{1}{|\square|} \log \int_{h_1(\square)} \exp \left( - \frac{1}{\lambda} \sum_{e \subseteq \square} |\nabla \phi(e)|^2 \right) d\phi
\]
and
\[
\nu(\square, p) \geq \lambda|x|^2 - \frac{1}{|\square|} \log \int_{h_1(\square)} \exp \left( - \lambda \sum_{e \subseteq \square} |\nabla \phi(e)|^2 \right) d\phi
\]
To prove (3.19), it is sufficient to derive an exact formula for
\[
(3.22) \quad \frac{1}{|\square|} \log \int_{h_1(\square)} \exp \left( - \sum_{e \subseteq \square} |\nabla \phi(e)|^2 \right) d\phi.
\]
This is performed by computing the eigenvalues of the discrete laplacian with Dirichlet boundary condition on the box $\Box$ and we obtain the following explicit formula.

$$\frac{1}{|\Box|} \log \int_{h_0^d(\Box)} \exp \left( -\sum_{e \subseteq \Box} |\nabla \phi(e)|^2 \right) d\phi = \frac{(N - 1)^d}{2N^d} \log \pi + \frac{1}{2 \cdot N^d} \sum_{k \in \{1, \ldots, N-1\}^d} \log \left( \sum_{l=1}^d \left( 1 - \cos \left( \frac{\pi k_l}{N-1} \right) \right) \right).$$

The second term on the right-hand side converges to $\int_{(0,1)^d} \log \left( \sum_{l=1}^d \left( 1 - \cos (\pi x_l) \right) \right)$ and thus is bounded in $n$. In particular, combining this with the previous displays shows, for each $p \in \mathbb{R}^d$,

$$\lambda |p|^2 - C \leq \nu(\Box, p) \leq \frac{1}{\lambda} |p|^2 + C$$

for some constant $C := C(d, \lambda) < \infty$. This completes the proof of (3.19).

We now prove (3.20). We start with the upper bound. By Proposition 3.6, we have for each cube $\Box \subseteq \mathbb{Z}^d$ and each $q \in \mathbb{R}^d$,

$$\nu^*(\Box_n, q) \leq \nu^*(\Box_1, q) + C(1 + |q|^2),$$

and a straightforward computation gives the bound

$$\nu^*(\Box_1, q) \leq C(1 + |q|^2),$$

for some $C := C(d, \lambda) < \infty$. Combining the two previous displays gives the upper bound of (3.20).

We then prove the lower bound, the idea is to use the convex duality proved in Proposition 3.7 combined with the upper bound estimate (3.19). By Proposition 3.7, for each $p, q \in \mathbb{R}^d$,

$$\nu(\Box_n, p) + \nu^*(\Box_n, q) \geq p \cdot q - C3^{-n}.$$

Using (3.19) and the crude bound $3^{-n} \leq 1$, the previous estimate becomes

$$\nu^*(\Box_n, q) \geq p \cdot q - C(1 + |p|^2).$$

Picking $p = q/2C$ gives

$$\nu^*(\Box_n, q) \geq \frac{|q|^2}{4C} - C.$$

This is the desired lower estimate.

3.4. Convexity and uniform convexity. In this subsection, we use the tools developed in Section 2 to prove the uniform convexity of $\nu$. We also prove the convexity of $\nu^*$ using the Cauchy-Schwarz inequality.

**Proposition 3.10** (Uniform convexity of $\nu$). For each $n \in \mathbb{N}$, the mapping $p \rightarrow \nu(\Box_n, p)$ is uniformly convex. Precisely, there exists a constant $C := C(d, \lambda) < \infty$ such that for each $p_0, p_1 \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,

$$\frac{1}{C} |p_0 - p_1|^2 \leq \frac{1}{2} \nu(\Box_n, p_0) + \frac{1}{2} \nu(\Box_n, p_1) - \nu \left( \Box_n, \frac{p_0 + p_1}{2} \right) \leq C|p_0 - p_1|^2.$$

**Proof.** The main idea in this proof is to use the variational formula for $\nu$,

$$\nu(\Box_n, p) = \inf_{\mathbb{P} \in \mathcal{P}(\mathbb{P}_{\Box_0}(\Box_n))} \mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{e \subseteq \Box_n} V_e \left( p \cdot e + \nabla \phi(e) \right) \right] + \frac{1}{|\Box_n|} H(\mathbb{P}).$$

We recall that the probability measures $\mathbb{P}_{n, p_0}$ and $\mathbb{P}_{n, p_1}$ achieve the infimum in the variational formulation of $\nu(\Box_n, p_0)$ and $\nu(\Box_n, p_1)$. Denote by $\mu$ the probability measure on $h_0^d(\Box_n) \times h_0^d(\Box_n)$ which is the optimal coupling between $\mathbb{P}_{n, p_0}$ and $\mathbb{P}_{n, p_1}$ as described in Section 2. For $t \in [0, 1]$,
we denote by $\mathbb{P}_t$ the law of $(1 - t)\phi_{n,p_0} + t\phi_{n,p_1}$ under the optimal coupling. By the property of displacement convexity described in Proposition 2.10, the function $t \to H(\mathbb{P}_t)$ is convex.

We first prove the lower bound of (3.23). Using (3.24), we have

\begin{equation}
\nu \left( \square_n, \frac{p_0 + p_1}{2} \right) \leq E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e \left( \frac{p_0 + p_1}{2} - \frac{\nabla \phi_{n,p_0} + \nabla \phi_{n,p_1}}{2} \right) e \right] + \frac{1}{|\square_n|} H \left( \mathbb{P}_{\frac{1}{2}} \right).
\end{equation}

We then estimate the first term on the right-hand side by using the uniform convexity of $V_e$,

\begin{align*}
2E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e \left( \frac{p_0 + p_1}{2} - \frac{\nabla \phi_{n,p_0} + \nabla \phi_{n,p_1}}{2} \right) e \right] &\leq E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e \left( p_0(e) - \frac{\nabla \phi_{n,p_0}}{2} \right) + \nabla \phi_{n,p_0}(e) \right] + E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e \left( p_1(e) - \frac{\nabla \phi_{n,p_1}}{2} \right) + \nabla \phi_{n,p_1}(e) \right] \\
&\quad - \lambda E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |p_0 - p_1| e + \nabla (\phi_{n,p_0} - \phi_{n,p_1}) e \right]^2.
\end{align*}

But for each function $\phi \in h_0^1(\square_n)$, and each $p \in \mathbb{R}^d$,

\begin{equation}
\frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |p(e) + \nabla \phi e| \geq |p|^2.
\end{equation}

Applying the previous estimate with $p = (p_0 - p_1), \phi = \phi_{n,p_0} - \phi_{n,p_1}$ and taking the expectation gives

\begin{equation}
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |(p_0 - p_1)(e) + \nabla (\phi_{n,p_0} - \phi_{n,p_1}) e|^2 \right] \geq |p_0 - p_1|^2.
\end{equation}

Combining the few previous displays gives

\begin{align*}
2E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e \left( \frac{p_0 + p_1}{2} - \frac{\nabla \phi_{n,p_0} + \nabla \phi_{n,p_1}}{2} \right) e \right] &\leq E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e (p_0(e) + \nabla \phi_{n,p_0}(e)) \right] + E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e (p_1(e) + \nabla \phi_{n,p_1}(e)) \right] - \lambda |p_0 - p_1|^2
\end{align*}

We then estimate the second term on the right-hand side of (3.25) by using the convexity of $t \to H(\mathbb{P}_t)$,

\begin{equation}
H \left( \mathbb{P}_{\frac{1}{2}} \right) \leq \frac{1}{2} H(\mathbb{P}_{n,p_0}) + \frac{1}{2} H(\mathbb{P}_{n,p_1}).
\end{equation}

Combining the two previous displays with (3.25) gives

\begin{align*}
2\nu \left( \square_n, \frac{p_0 + p_1}{2} \right) &\leq E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e (p_0(e) + \nabla \phi_{n,p_0}(e)) \right] + \frac{1}{|\square_n|} H (\mathbb{P}_{n,p_0}) \\
&\quad + E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e (p_1(e) + \nabla \phi_{n,p_1}(e)) \right] + \frac{1}{|\square_n|} H (\mathbb{P}_{n,p_1}) - \lambda |p_0 - p_1|^2.
\end{align*}
But note that by definition of $\mathbb{P}_{n,p_0}$ and $\mathbb{P}_{n,p_1}$, we have

$$\nu(\square_n, p_0) = \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e(p_0(e) + \nabla \phi_{n,p_0}(e)) \right] + \frac{1}{|\square_n|} H(\mathbb{P}_{n,p_0})$$

and

$$\nu(\square_n, p_1) = \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} V_e(p_1(e) + \nabla \phi_{n,p_1}(e)) \right] + \frac{1}{|\square_n|} H(\mathbb{P}_{n,p_1}).$$

Combining the few previous displays gives

$$\nu\left(\square_n, \frac{p_0 + p_1}{2}\right) \leq \frac{1}{2} \nu(\square_n, p_0) + \frac{1}{2} \nu(\square_n, p_1) - c|p_0 - p_1|^2.$$
Remark 3.12. This proof of the convexity of $\nu^*$ does not use the convexity of $V$. Moreover, we cannot easily obtain a uniform convexity statement for $\nu^*$ as it was the case for $\nu$ in Proposition 3.10. A reason for that is that if we compute the Hessian of $\nu^*$, we obtain

$$\nabla^2 \nu^*(\Box_n, q) = \frac{1}{|\Box_n|} \text{Var} \left[ \sum_{e \subseteq \Box_n} \nabla \psi_{n,q}(e) \right].$$

In particular proving the uniform convexity of $\nu^*$ would prove a bound on the variance of the spatial average of the gradient of $\psi_{n,q}$ with the scaling of the CLT, which is, in a way, far stronger than the main result of this article where we obtain a small gain $\alpha > 0$.

3.5. $L^2$ bounds on the gradient of minimizers. The last property that remains to be proved is the bound on the $L^2$ norm of the minimizers, this is done in the following proposition.

Proposition 3.13 ($L^2$ bounds on the gradients of the minimizers). There exists a constant $C := C(d, \lambda) < \infty$ such that for each $n \in \mathbb{N}$, for each $p \in \mathbb{R}^d$

$$(3.27) \quad \mathbb{E} \left[ \sum_{e \subseteq \Box_n} |\nabla \phi_{n,p}(e)|^2 \right] \leq C(1 + |p|^2).$$

Similarly, for each $q \in \mathbb{R}^d$, one has

$$(3.28) \quad \mathbb{E} \left[ \sum_{e \subseteq \Box_n} |\nabla \psi_{n,q}(e)|^2 \right] \leq C(1 + |q|^2).$$

Proof. We first prove (3.27). Let $\kappa$ be a random variable valued in $h^1_0(\Box_n)$ satisfying

- for each $x \in \Box_n$, the law of the random variable $\kappa(x)$ is the uniform law on $[0, 1]$,
- the random variables $(\kappa(x))_{x \in \Box_n}$ are independent.

With these properties, one can compute the entropy of the law $\mathbb{P}_\kappa$. Indeed using that the entropy of the uniform law on $[0, 1]$ is equal to 0, combined with the independence of the $\kappa(x)$, one obtains

$$H(\mathbb{P}_\kappa) = 0.$$  

We then consider the optimal coupling between the random variables $\kappa$ and $\phi_{n,p}$. By the displacement convexity of the entropy, one has

$$H\left(\frac{\mathbb{P}_{\phi_{n,p} + \kappa}}{2}\right) \leq \frac{1}{2} H(\mathbb{P}_{\phi_{n,p}}) + \frac{1}{2} H(\mathbb{P}_\kappa)$$

$$\leq \frac{1}{2} H(\mathbb{P}_{\phi_{n,p}}).$$

Using the uniform convexity of $V_e$, we further obtain

$$\lambda \mathbb{E} \left[ \sum_{e \subseteq \Box_n} |\nabla \phi_{n,p}(e) - \nabla \kappa(e)|^2 \right] \leq \mathbb{E} \left[ \sum_{e \subseteq \Box_n} V_e(p(e) + \nabla \phi_{n,p}(e)) \right] + \mathbb{E} \left[ \sum_{e \subseteq \Box_n} V_e(p(e) + \nabla \kappa(e)) \right] - 2 \mathbb{E} \left[ \sum_{e \subseteq \Box_n} V_e \left( p(e) + \frac{\nabla \phi_{n,p}(e) + \nabla \kappa(e)}{2} \right) \right]$$

Since the law of of $\phi_{n,p}$ is $\mathbb{P}_{n,p}$, we have

$$\mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{e \subseteq \Box_n} V_e(p(e) + \nabla \phi_{n,p}(e)) \right] + \frac{1}{|\Box_n|} H(\mathbb{P}_{n,p}) = \nu(\Box_n, p)$$
and we also have by (2.5)
\[ \| \nabla \| \nu(\square_n, p) = \inf_{P \in \mathcal{P}(\hat{h}^1(\square_n))} \mathbb{E} \left[ \sum_{e \sub \square_n} V(p \cdot e + \nabla \phi(e)) \right] + H(P) \]
\[ \leq \mathbb{E} \left[ \sum_{e \sub \square_n} V \left( p \cdot e + \frac{\nabla \phi_{n, p}(e) + \nabla \kappa(e)}{2} \right) \right] + H \left( \mathbb{P}_{\phi_{n, p} + \kappa} \right). \]
Combining the few previous displays then gives
\[ \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \sub \square_n} |\nabla \phi_{n, p}(e) - \nabla \kappa(e)|^2 \right] \leq C \left( \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \sub \square_n} V_e (p \cdot e + \nabla \kappa(e)) \right] - \nu(\square_n, p) \right). \]
We then use that, by definition of \( \kappa \), its gradient is bounded by 1. Thus using the inequality \( V_e(x) \leq \frac{1}{x} |x|^2 \) and Proposition 3.8, we obtain the following estimate
\[ \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \sub \square_n} |\nabla \phi_{n, p}(e) - \nabla \kappa(e)|^2 \right] \leq C(1 + |p|^2), \]
for some \( C := C(d, \lambda) < \infty \). We then estimate the term on the left-hand side, using again that the gradient of \( \kappa \) is bounded by 1 to obtain
\[ \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \sub \square_n} |\nabla \phi_{n, p}(e)|^2 \right] \leq \frac{2}{|\square_n|} \mathbb{E} \left[ \sum_{e \sub \square_n} |\nabla \phi_{n, p}(e) - \nabla \kappa(e)|^2 \right] + C, \]
for some \( C := C(d) < \infty \). Combining the few previous displays then proves
\[ \frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \sub \square_n} |\nabla \phi_{n, p}(e)|^2 \right] \leq C(1 + |p|^2). \]

The proof of (3.27) is then complete.

The proof of (3.28) is similar. Let \( (h_i)_{i=1, ..., 3^nd - 1} \) be an orthogonal basis of \( \hat{h}^1(\square_n) \) and let \( \chi \) be a random variable valued in \( \hat{h}^1(\square_n) \) satisfying
- for each \( i \in \{1, \ldots, 3^nd - 1\} \), the law of \( \langle \chi, h_i \rangle_{\square_n} \) is the uniform law on \([0, 1]\),
- the random variables \( \langle \chi, h_i \rangle_{\square_n} \) are independent.

Note that these two conditions completely characterize the law of \( \chi \) as a random variable in \( \hat{h}^1(\square_n) \). Using that for each edge \( e = (x, y) \) of \( \square_n \), we have the inequality \( |\nabla \chi(e)|^2 = |\chi(x) - \chi(y)|^2 \leq 2\chi(x)^2 + 2\chi(y)^2 \), we obtain the estimate on the \( L^2 \) of the gradient of \( \chi \),
\[ \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \sub \square_n} |\nabla \chi(e)|^2 \right] \leq C \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{x \sub \square_n} |\chi(x)|^2 \right] \]
\[ \leq C \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{i=1}^{3^nd - 1} \langle \chi, h_i \rangle_{\square_n}^2 \right] \]
\[ \leq C, \]
for some \( C := C(d) < \infty \). Additionally, using the independence of the family \( \langle \chi, h_i \rangle_{\square_n} \) and the fact that there law is the uniform law on \([0, 1]\), we have
\[ H(\mathbb{P}_\chi) = 0. \]
With this in mind, we consider the optimal coupling between $\chi$ and $\psi_{n,q}$, using the displacement convexity of the entropy and the uniform convexity of $V_e$, we obtain, for some $C := C(d, \lambda) < \infty$,

$$
\frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e) - \nabla \chi(e)|^2 \right] \leq C \left( \frac{1}{|\square_n|} \mathbb{E} \left[ - \sum_{e \subseteq \square_n} (V_e(\nabla \chi(e)) + q \cdot \nabla \chi(e)) \right] + \nu^*(\square_n, q) \right).
$$

Using the bound $V(x) \leq \frac{1}{x}|x|^2$, we obtain

$$
\frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e) - \nabla \chi(e)|^2 \right] \leq C \left( \frac{1}{|\square_n|} \mathbb{E} \left[ - \sum_{e \subseteq \square_n} \frac{1}{\lambda} |\nabla \chi(e)|^2 + q \cdot \nabla \chi(e) \right] + \nu^*(\square_n, q) \right).
$$

We use (3.29), and apply Proposition 3.8 to get

$$
\frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e) - \nabla \chi(e)|^2 \right] \leq C(1 + |q|^2).
$$

Another application of (3.29) eventually yields

$$
\frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e)|^2 \right] \leq C(1 + |q|^2),
$$

for some $C := C(d, \lambda) < \infty$. The proof of (3.28) is complete. \hfill \Box

**Remark 3.14.** Using the explicit formula for $\nu$ and $\nu^*$, one can get the much stronger bounds, with a computation similar to [14, Lemma 2.11],

$$
\mathbb{E} \left[ \exp \left( \varepsilon \sum_{e \subseteq \square_n} |\nabla \phi_{n,q}|^2 \right) \right] \leq \exp \left( C(1 + |p|^2) \right)
$$

and

$$
\mathbb{E} \left[ \exp \left( \varepsilon \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}|^2 \right) \right] \leq \exp \left( C(1 + |p|^2) \right).
$$

for some $\varepsilon := \varepsilon(d, \lambda) > 0$ and $C := C(d, \lambda) < \infty$. Nevertheless we only need a bound on the $L^2$ norm for the purposes of this paper and the proof given here uses the variational formulation of $\nu$ and $\nu^*$, which is one of the main tool of this article.

### 3.6. Two scales comparison.

We have now proved all the properties of $\nu$ and $\nu^*$ presented in Proposition 3.1. The next goal of this section is to construct, for $m < n$, a coupling between the random variables $\phi_{m,p}$ and $\psi_{n,p}$ (resp. $\psi_{m,p}$ and $\psi_{n,p}$) such that the $L^2$ norm of their difference is controlled by $\nu(\square_m, p) - \nu(\square_n, p)$. Since we know that the sequence $\nu(\square_n, p)$ converges, the quantity $\nu(\square_m, p) - \nu(\square_n, p)$ will be small when $m$ and $n$ are large and thus the random variables $\phi_{m,p}$ and $\phi_{n,p}$ will be close in the $L^2$ norm.

For any pair of integers $m, n \in \mathbb{N}$ with $m < n$, the triadic cube $\square_n$ can be split into $3^{(n-m)d}$ cubes of the form $z + \square_m$, with $z \in \mathcal{Z}_{m,n}$. Denote by $(\phi_z)_{z \in \mathcal{Z}_{m,n}}$ a family of random variables such that

- For each $z \in \mathcal{Z}_{m,n}$, $\phi_z$ takes value in $\mathcal{H}_0^1(z + \square_m)$ and the law of $\phi_z(\cdot - z)$ is $\mathbb{P}_{m,p}$
- The random variables $\phi_z$ are independent.

We can then, for each $z \in \mathcal{Z}_{m,n}$, see $\phi_z$ as a random variable taking value in $\mathcal{H}_0^1(\square_n)$, by extending it to be 0 on $\square_n \setminus (z + \square_m)$. We also denote by $\phi := \sum_{z \in \mathcal{Z}_{m,n}} \phi_z$. 

**Remark 3.14.** Using the explicit formula for $\nu$ and $\nu^*$, one can get the much stronger bounds, with a computation similar to [14, Lemma 2.11],

$$
\mathbb{E} \left[ \exp \left( \varepsilon \sum_{e \subseteq \square_n} |\nabla \phi_{n,q}|^2 \right) \right] \leq \exp \left( C(1 + |p|^2) \right)
$$

and

$$
\mathbb{E} \left[ \exp \left( \varepsilon \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}|^2 \right) \right] \leq \exp \left( C(1 + |p|^2) \right).
$$

for some $\varepsilon := \varepsilon(d, \lambda) > 0$ and $C := C(d, \lambda) < \infty$. Nevertheless we only need a bound on the $L^2$ norm for the purposes of this paper and the proof given here uses the variational formulation of $\nu$ and $\nu^*$, which is one of the main tool of this article.
the random variable $\phi$ defined in the previous paragraph and taking values in $h^1_0(\Box_n)$. There exists a coupling between $\phi$ and $\phi_{n,p}$ and a constant $C := C(d, \lambda) < \infty$ such that,

$$
\left(3.30\right) \quad \frac{1}{|\Box_n|} \mathbb{E} \left[ \sum_{e \subseteq \Box_n} \left| \nabla \phi(e) - \nabla \phi_{n,p}(e) \right|^2 \right] \leqslant C \left( \nu(\Box_m, q) - \nu(\Box_n, q) \right) + C^{3-d} \left( 1 + |p|^2 \right).
$$

**Proof.** We first introduce the set of vertices $\partial_{\text{int}} \Box_{m,n} \subseteq \Box_m$ which are on the boundary of one of the $z + \Box_m$ but not on the boundary of $\Box_n$,

$$
\left(3.31\right) \quad \partial_{\text{int}} \Box_{m,n} := \left( \bigcup_{z \in 3^n \mathbb{Z} \cap \Box_n + 1} \partial (z + \Box_n) \right) \setminus \partial \Box_{n+1}.
$$

Note that the cardinality of this set is bounded by $C3^{dn-m}$, for some $C := C(d) < \infty$.

An idea to obtain (3.30) would be to consider the optimal coupling between $\phi$ and $\phi_{n,p}$, then use the displacement convexity of the entropy and the uniform convexity of $V_e$. Unfortunately this does not work since, for each $x \in \partial_{\text{int}} \Box_{m,n}$, $\phi(x) = 0$ and thus the law of $\phi$ is not absolutely continuous with respect to the Lebesgue measure on $h^1_0(\Box_n)$. Nevertheless, this is the only obstruction and to remedy this we introduce a random variable $X$ taking values in $h^1_0(\Box_n)$ and satisfying

- for each $x \in \partial_{\text{int}} \Box_{m,n}$, the law of $X(x)$ is uniform on $[0, 1]$ and for each $x \in \Box_{n+1} \setminus \partial_{\text{int}} \Box_{m,n}$, $X(x) = 0$,
- the $\mathbb{R}$-valued random variables $(X(x))_{x \in \partial_{\text{int}} \Box_{m,n}}$ are independent,
- the random variables $X$ and $\phi$ are independent.

We then consider the random variable $\phi' := \phi + X$. It is a random variable taking value in $h^1_0(\Box_n)$. Moreover, by construction, we see that this random variable is absolutely continuous with respect to the Lebesgue measure on $h^1_0(\Box_n)$. We denote by $\mathbb{P}_{\phi'}$ its law. The idea to keep in mind is that the random variable $\phi'$ is a small perturbation of the random variable $\phi$ and thanks to that it will be possible to obtain estimates on $\phi$ from estimates on $\phi'$. This is what is done in Steps 3 and 4 below.

We then split the proof into 6 steps.

- In Step 1, we compute the entropy of $\mathbb{P}_{\phi'}$ and prove that

$$
\left(3.32\right) \quad H \left( \mathbb{P}_{\phi'} \right) = 3^{(n-m)d} H \left( \mathbb{P}_{m,p} \right),
$$

where the entropy on the left-hand side is computed with respect to the Lebesgue measure on $h^1_0(\Box_n)$ and the entropy on the right-hand side is computed with respect to the Lebesgue measure on $h^1_0(\Box_m)$.

- In Step 2, we consider the optimal coupling between $\phi'$ and $\phi_{n,p}$ and prove that, with this coupling,

$$
\left(3.33\right) \quad \mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{e \subseteq \Box_n} \left| \nabla \phi'(e) - \nabla \phi_{n,p}(e) \right|^2 \right] \leqslant C \left( \frac{1}{|\Box_n|} \mathbb{E} \left[ \sum_{e \subseteq \Box_n} V_e \left( p(e) + \nabla \phi'(e) \right) \right] + \frac{1}{|\Box_n|} \mathbb{E} \left[ H \left( \mathbb{P}_{\phi'} \right) - \nu(\Box_n, p) \right] \right),
$$

for some constant $C := C(d, \lambda) < \infty$.

- In Step 3, we estimate the term on the right-hand side of (3.33) and prove

$$
\left(3.34\right) \quad \frac{1}{|\Box_n|} \mathbb{E} \left[ \sum_{e \subseteq \Box_n} V_e \left( p(e) + \nabla \phi'(e) \right) \right] \leqslant \frac{1}{|\Box_n|} \mathbb{E} \left[ \sum_{e \subseteq \Box_n} V_e \left( p(e) + \nabla \phi(e) \right) \right] + C^{3-d} \left( 1 + |p| \right).
$$
for some $C := C(d, \lambda) < \infty$.

- In Step 4, we combine the main results of Steps 1 and 3 to obtain

$$\frac{1}{|Q_n|} \mathbb{E} \left[ \sum_{e \subseteq Q_n} V_e \left( p(e) + \nabla \phi'(e) \right) \right] + \frac{1}{|Q_n|} H \left( \mathbb{P}_{\phi'} \right) \leq \nu(Q_n, p) + C 3^{-\frac{m}{d}} (1 + |p|^2),$$

for some $C := C(d, \lambda) < \infty$.

- In Step 5, we estimate the term on the left-hand side of (3.33) and remove the "prime" of $\phi'$. This yields

$$\mathbb{E} \left[ \frac{1}{|Q_n|} \sum_{e \subseteq Q_n} \left| \nabla \phi(e) - \nabla \phi_{n,p}(e) \right|^2 \right] \leq C \mathbb{E} \left[ \frac{1}{|Q_n|} \sum_{e \subseteq Q_n} \left| \nabla \phi'(e) - \nabla \phi_{n,p}(e) \right|^2 \right] + C 3^{-\frac{m}{d}} (1 + |p|),$$

for some $C := C(d, \lambda) < \infty$. Note that to compute the expectation on the left-hand side, we used the coupling between $\phi$ and $\phi_{n,p}$ which is induced by the coupling between $\phi'$ and $\phi_{n,p}$.

- In Step 6, the conclusion, we combine the main results of Steps 2, 3, 4 and 5 to obtain (3.30).

**Step 1.** The idea to obtain (3.32) is to use Proposition 2.4 about the entropy of a pair of random variables. To this end, note that we have the orthogonal decomposition of $h_0(Q_n)$

$$h_0(Q_n) = \bigoplus_{z \in \mathbb{Z}_{m,n}} h_0(z + Q_m) \bigoplus \mathbb{R}|\partial_{m}Q_{m,n}|,$$

where $\mathbb{R}|\partial_{m}Q_{m,n}|$ stands for the set of functions from $\partial_{m}Q_{m,n}$ to $\mathbb{R}$.

Using the previous remark, we can applying Proposition 2.6 with $Y := \phi$, $Z := X$ and consequently $Y + Z = \phi'$. We obtain

$$H \left( \mathbb{P}_{\phi'} \right) = H \left( \mathbb{P}_{\phi} \right) + H \left( \mathbb{P}_X \right),$$

where the entropy of $\phi'$ (resp. $\phi$ and $X$) is computed with respect to the Lebesgue measure on $h_0(Q_n)$ (resp. $\bigoplus_{z \in \partial_{m}Q_{m,n}} h_0(z + Q_n)$ and $\mathbb{R}|\partial_{m}Q_{m,n}|$).

On the one hand, since the random variables $(X(z))_{z \in \partial_{m}Q_{m,n}}$ are independent and of law uniform on $[0, 1]$, we have

$$H \left( \mathbb{P}_X \right) = 0.$$

On the other hand, we have $\phi' = \sum_{z \in \partial_{m}Q_{m,n}} \phi_z$. Using the independence of the family $(\phi_z)_{z \in \mathbb{Z}_{m,n}}$, that for each $z \in \mathbb{Z}_{m,n}$, $\phi_z(\cdot - z)$ has law $\mathbb{P}_{m,p}$ and Proposition 2.6, with $3^{(n-m)d}$ random variables instead of two, we have

$$H \left( \mathbb{P}_\phi \right) = \sum_{z \in \mathbb{Z}_{m,n}} H \left( \mathbb{P}_{\phi_z} \right) = 3^{(n-m)d} H \left( \mathbb{P}_{m,p} \right).$$

Combining the few previous displays yields (3.32) and completes the proof of Step 1.

**Step 2.** Consider the optimal coupling with respect to the $L^2$ scalar product on $h_0(Q_n)$ between $\phi'$ and $\phi_{n,p}$ and denote by $\mathbb{P}_{\phi' + \phi_{n,p}}$ the law of the random variable $\frac{\phi' + \phi_{n,p}}{2}$ with this coupling. Using Proposition 2.10 about the displacement convexity of the entropy, one has

$$H \left( \mathbb{P}_{\phi' + \phi_{n,p}} \right) \leq \frac{1}{2} H \left( \mathbb{P}_{\phi'} \right) + \frac{1}{2} H \left( \mathbb{P}_{n,p} \right).$$
Also by the uniform convexity of $V$, one has
\[
\lambda E \left[ \sum_{e \subseteq \Box \ k} |\nabla \phi'(e) - \nabla \phi_{n,p}(e)|^2 \right] \\
\leq E \left[ \sum_{e \subseteq \Box \ k} V(p(e) + \nabla \phi'(e)) \right] + E \left[ \sum_{e \subseteq \Box \ k} V(p(e) + \nabla \phi_{n,p}(e)) \right] \\
- 2E \left[ \sum_{e \subseteq \Box \ k} V \left( p(e) + \frac{\nabla \phi_{n,p}(e) + \nabla \phi'(e)}{2} \right) \right].
\]

But we know that
\[
E \left[ \sum_{e \subseteq \Box \ k} V(p \cdot e + \nabla \phi_{n,p}(e)) \right] + H(\mathbb{P}_{n,p}) = \nu(\Box \ k, p)
\]
and we also have by the variational formulation for $\nu$ given in Proposition 2.8,
\[
\nu(\Box \ k, p) \leq E \left[ \sum_{e \subseteq \Box \ k} V \left( p(e) + \frac{\nabla \phi_{n,p}(e) + \nabla \phi'(e)}{2} \right) \right] + H\left( \mathbb{P}_{\phi' + \phi_{n,p}} \right).
\]

Combining the few previous displays then gives
\[
E \left[ \sum_{e \subseteq \Box \ k} |\nabla \phi'(e) - \nabla \phi_{n,p}(e)|^2 \right] \\
\leq C \left( \frac{1}{|\Box |} E \left[ \sum_{e \subseteq \Box \ k} V_e(p(e) + \nabla \phi'(e)) \right] + \frac{1}{|\Box |} H(\mathbb{P}_\phi) - \nu(\Box \ k, p) \right).
\]

This is precisely (3.33) and the proof of Step 2 is complete.

**Step 3.** The main goal of this step is to prove the following estimate
\[
\frac{1}{|\Box |} E \left[ \sum_{e \subseteq \Box \ k} V_e(p(e) + \nabla \phi'(e)) \right] \leq \frac{1}{|\Box |} E \left[ \sum_{e \subseteq \Box \ k} V_e(p(e) + \nabla \phi(e)) \right] + C 3^{-\frac{n}{2}}(1 + |p|).
\]

To do this, we recall that $\phi'$ is defined from $\phi$ according to the formula
\[
\phi' := \phi + X,
\]
and that the random variable $X$ is supported in the vertices of $\partial_{int} \Box \ k$. We denote by $B_{m,n}^{(1)}$ the set of edges of $\Box \ k$ where $\nabla X$ is supported, i.e
\[
B_{m,n}^{(1)} := \{ (x,y) \in \mathbb{B}_d(\Box \ k) \mid x \in \partial_{int} \Box \ k \text{ or } y \in \partial_{int} \Box \ k \}.
\]

One can estimate the cardinality of $B_{m,n}^{(1)}$ according to
\[
|B_{m,n}^{(1)}| \leq C 3^{dn-m},
\]
for some constant $C := C(d) < \infty$. 
We then split the sum and use that $\nabla X$ is supported on $B_{m,n}$ to get

$$
\sum_{e \subseteq B_{m,n}} V_e (p(e) + \nabla \phi'(e)) = \sum_{e \subseteq \square_n \setminus B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)) + \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e))
$$

$$
= \sum_{e \subseteq \square_n \setminus B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi(e)) + \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)).
$$

We then estimate the second term on the right-hand side of (3.37) using the uniform convexity of $V_e$ and a Taylor expansion,

$$
\sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)) = \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi(e) + \nabla X(e))
$$

$$
\leq \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi(e)) + V_e' (p(e) + \nabla \phi(e)) \nabla X(e) + \frac{1}{\lambda} |\nabla X(e)|^2.
$$

But we know that, by definition of $X$, its gradient is bounded by 1 and by the assumption we made on $V_e$ that for each $x \in \mathbb{R}$,

$$
|V_e'(x)| \leq \frac{1}{\lambda} |x|.
$$

Using these ideas, the previous estimate can be rewritten

$$
\sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)) \leq \sum_{e \in B_{m,n}^{(1)}} (V_e (p(e) + \nabla \phi(e)) + |\nabla \phi(e)|) + C \left| B_{m,n}^{(1)} \right| (1 + |p|).
$$

Using the estimate on the cardinality of $B_{m,n}^{(1)}$ given in (3.36) and the Cauchy-Schwarz inequality, one has

$$
\sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)) \leq \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi(e)) + 3^{\frac{dn-m}{2}} \left( \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e)|^2 \right) + C 3^{dn-m} (1 + |p|).
$$

Taking the expectation and dividing by $|\square_n|$ gives

$$
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)) \right] \leq E \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi(e)) \right]
$$

$$
+ 3^{-\frac{dn+m}{2}} E \left[ \left( \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e)|^2 \right)^{\frac{1}{2}} \right] + C 3^{-m} (1 + |p|).
$$

By the Cauchy-Schwarz inequality, we further obtain

$$
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi'(e)) \right] \leq E \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}^{(1)}} V_e (p(e) + \nabla \phi(e)) \right]
$$

$$
+ 3^{-\frac{dn+m}{2}} E \left[ \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e)|^2 \right]^{\frac{1}{2}} + C 3^{-m} (1 + |p|).
$$
Combining the previous display with (3.37) gives

\[ E \left[ \frac{1}{\square_n} \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi'(e)) \right] \leq E \left[ \frac{1}{\square_n} \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi(e)) \right] + 3^{-\frac{dm + m}{4}} E \left[ \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e)|^2 \right]^{\frac{1}{2}} + C3^{-m}(1 + |p|). \]

The proof of (3.34) is almost complete, there only remains to prove an estimate on the second term of the right-hand side of the previous display. More precisely we will prove

\[ E \left[ \frac{1}{\square_n} \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 \right] \leq C(1 + |p|^2), \]

for some \( C := C(d, \lambda) < \infty \). This immediately implies (3.34) since we have the estimate

\[ 3^{-\frac{dm + m}{4}} E \left[ \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e)|^2 \right]^{\frac{1}{2}} \leq 3^{-\frac{dm}{4}} E \left[ \frac{1}{\square_n} \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 \right]^{\frac{1}{2}} \leq C3^{-m}(1 + |p|), \]

for some \( C := C(d, \lambda) < \infty \). We thus turn to the proof of (3.38). To do so we let \( B_{m,n} \) be the set of edges linking two subcubes of the form \( z + \square_m \), i.e.,

\[ B_{m,n} := \{(x, y) : \exists z, z' \in Z_{m,n}, z \neq z' \text{ such that } x \in z + \square_m \text{ and } y \in z' + \square_m \}. \]

In particular, we have the following partition of edges

\[ e \subseteq \square_n \implies \exists z \in Z_{m,n}, e \subseteq z + \square_m \text{ or } e \in B_{m,n}. \]

That is to say: an edge \( e \) of \( \square_n \) either belongs to one of the subcubes \( z + \square_m \) or belongs to \( B_{m,n} \). This will be useful in the proof because it gives the following splitting of the sum

\[ \sum_{e \subseteq \square_n} = \sum_{z \in Z_{m,n}} \sum_{e \subseteq (z + \square_m)} + \sum_{e \in B_{m,n}}. \]

Using this we decompose the sum

\[ \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 = \sum_{z \in Z_{m,n}} \sum_{e \subseteq (z + \square_m)} |\nabla \phi(e)|^2 + \sum_{e \in B_{m,n}} |\nabla \phi(e)|^2. \]

But by definition, we have \( \phi(x) = 0 \) for each \( x \in \partial_{int} \square_{m,n} \), this implies \( \nabla \phi(e) = 0 \) for each \( e \in B_{m,n} \). This yields the following simplification

\[ \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 = \sum_{z \in Z_{m,n}} \sum_{e \subseteq (z + \square_m)} |\nabla \phi(e)|^2. \]

Then using the decomposition \( \phi = \sum_{z \in Z_{m,n}} \phi_z \), we rewrite the previous equality

\[ \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 = \sum_{z \in Z_{m,n}} \sum_{e \subseteq (z + \square_m)} |\nabla \phi_z (e)|^2. \]

But we know that the law of \( \phi_z (\cdot - z) \) is \( P_{m,p} \). We can thus apply Proposition 3.13 to obtain that there exists a constant \( C := C(d, \lambda) < \infty \) such that for each \( z \in Z_{m,n} \),

\[ E \left[ \frac{1}{\square_m} \sum_{e \subseteq (z + \square_m)} |\nabla \phi_z (e)|^2 \right] \leq C(1 + |p|^2). \]
Combining the two previous displays yields

(3.39) \[ E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 \right] \leq C(1 + |p|^2), \]

for some \( C := C(d, \lambda) < \infty \). This is precisely (3.38).

**Step 4.** With the same proof as in Step 2, we can decompose

\[
\frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi(e)) \right] = \frac{1}{|\square_n|} E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+m)} V_e (p(e) + \nabla \phi_z(e)) + \sum_{e \in B_{m,n}} V_e (p(e)) \right].
\]

Using the estimate \( V(x) \leq \frac{1}{3} |x|^2 \) and the bound on the cardinality of \( B_{m,n} \)

\[ |B_{m,n}| \leq C3^{dn-m}, \]

one obtains

\[
\frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi(e)) \right] \leq \sum_{z \in \mathbb{Z}_{m,n}} E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq (z+m)} V_e (p(e) + \nabla \phi_z(e)) \right] + C3^{-m}(1 + |p|^2).
\]

Combining this inequality with the main result (3.34) of Step 3, we obtain

\[
\frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi'(e)) \right] \leq \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} E \left[ \sum_{e \subseteq (z+m)} V_e (p(e) + \nabla \phi_z(e)) \right] + C3^{-m}(1 + |p|^2) + C3^{-\frac{m}{2}}(1 + |p|).
\]

The error term can be simplified according to

\[
\frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi'(e)) \right] \leq \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} E \left[ \sum_{e \subseteq (z+m)} V_e (p \cdot e + \nabla \phi_z(e)) \right] + C3^{-\frac{m}{2}}(1 + |p|^2).
\]

Adding the entropy of \( P_{\phi'} \) and using the equality proved in Step 1, we obtain

\[
\frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} V_e (p(e) + \nabla \phi'(e)) \right] + \frac{1}{|\square_n|} H \left( P_{\phi'} \right) \leq 3^{-d(n-m)} \sum_{z \in \mathbb{Z}_{m,n}} \left( E \left[ \frac{1}{|\square_m|} \sum_{e \subseteq (z+m)} V_e (p(e) + \nabla \phi_z(e)) \right] + \frac{1}{|\square_m|} H \left( P_{m,p} \right) \right) + C3^{-\frac{m}{2}}(1 + |p|^2).
\]
Since the law of $\phi_z$ in $z + \square_m$ is $\mathbb{P}_{m,p}$, we have, for each $z \in Z_{m,n}$,

$$
\mathbb{E} \left[ \frac{1}{|\square_m|} \sum_{e \subseteq z + \square_m} V_e (p(e) + \nabla \phi_z(e)) \right] + \frac{1}{|\square_m|} H (\mathbb{P}_{m,p}) = \nu(\square_m, p).
$$

Combining the two previous displays shows

$$
\frac{1}{|\square_n|} \mathbb{E} \left[ \sum_{e \subseteq \square_n} V_e \left( p(e) + \nabla \phi'(e) \right) \right] + \frac{1}{|\square_n|} H (\mathbb{P}_{\phi'}) \leq \nu(\square_m, p) + C3^{-\frac{m}{2}} (1 + |p|^2)
$$

and the proof of Step 4 is complete.

**Step 5.** We proceed as in Step 3 and use that the gradient of $X$ is supported in $B_{m,n}^{(1)}$ to decompose the sum

$$
\sum_{e \subseteq \square_n} |\nabla \phi'(e) - \nabla \phi_{n,p}(e)|^2 = \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) + \nabla X(e) - \nabla \phi_{n,p}(e)|^2
+ \sum_{e \not\in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2.
$$

We then expand the first term on the right-hand side

$$
\sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) + \nabla X(e) - \nabla \phi_{n,p}(e)|^2
= \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 + 2 \sum_{e \in B_{m,n}^{(1)}} (\nabla \phi(e) - \nabla \phi_{n,p}(e)) \nabla X(e) + \sum_{e \in B_{m,n}^{(1)}} |\nabla X(e)|^2.
$$

Using that the gradient of $X$ is bounded by 1, the Cauchy-Schwarz inequality and the upper bound on the cardinality of $B_{m,n}^{(1)}$, one further obtains

$$
\sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) + \nabla X(e) - \nabla \phi_{n,p}(e)|^2
\geq \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 - C3^{\frac{dn-m}{2}} \left( \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right)^{\frac{1}{2}} - C3^{dn}.
$$

Dividing by $|\square_n|$ on both sides of the previous inequality and taking the expectation gives

$$
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) + \nabla X(e) - \nabla \phi_{n,p}(e)|^2 \right] \geq \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right]
- C3^{\frac{dn-m}{2}} \mathbb{E} \left[ \left( \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right)^{\frac{1}{2}} \right] - C3^{-m}.
$$
By (3.40) and the Cauchy-Schwarz inequality, one obtains
\[
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi'(e) - \nabla \phi_{n,p}(e)|^2 \right] \\
\geq E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right] - C3^{-\frac{dn+m}{2}}E \left[ \sum_{e \in B_{m,n}^{(1)}} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right]^{\frac{1}{2}} - C3^{-m}.
\]

There remains to estimate the second term on the right-hand side of the previous equation. To do so, first note that we have already proved in (3.39), for some $C := C(d, \lambda) < \infty$,
\[
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi(e)|^2 \right] \leq C(1 + |p|^2),
\]
and by Proposition 3.13, we have
\[
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi_{n,p}(e)|^2 \right] \leq C(1 + |p|^2).
\]
Combining the three previous displays shows
\[
E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi'(e) - \nabla \phi_{n,p}(e)|^2 \right] \geq E \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right] - C3^{-\frac{m}{2}}(1 + |p|),
\]
for some $C := C(d, \lambda) < \infty$. The proof of Step 5 is complete.

**Step 6.** The conclusion. First, combining the main results of Steps 2 and 4 gives,
\[
E \left[ \sum_{e \subseteq \square_n} |\nabla \phi'(e) - \nabla \phi_{n,p}(e)|^2 \right] \leq C (\nu(\square_m, p) - \nu(\square_n, p)) + C3^{-\frac{m}{2}}(1 + |p|^2).
\]
Then by the main result of Step 5, we obtain
\[
E \left[ \sum_{e \subseteq \square_n} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right] \leq C (\nu(\square_m, p) - \nu(\square_n, p)) + C3^{-\frac{m}{2}}(1 + |p|^2).
\]
This is exactly (3.30) and the proof of Proposition 3.15 is complete.

We now want to prove a version of the two scales comparison for $\nu^*$. To this end, fix two integers $m, n \in \mathbb{N}$ such that $m < n$ and define a family of random variables $\psi_z$ for $z \in \mathbb{Z}_{m,n}$ according to

- For each $z \in \mathbb{Z}_{m,n}$, $\psi_z$ takes value in $\tilde{h}^1(\square_n)$, is equal to 0 in $\square_n \setminus (z + \square_m)$ and the law of $\psi_z(\cdot - z)$ is $P_m^{\nu}$.
- The random variables $\psi_z$, for $z \in \mathbb{Z}_{m,n}$ are independent.

We also denote by $\psi' := \sum_{z \in \mathbb{Z}_{m,n}} \psi_z$ and by $P_\psi$ its law, which is a probability measure on $\tilde{h}^1(\square_n)$.

**Proposition 3.16** (Two scales comparison for $\nu^*$). Given $n \in \mathbb{N}$ and $q \in \mathbb{R}^d$, consider the random variables $\psi_z$ and $\psi'$ defined in the previous paragraph. There exists a coupling between $\psi'$ and $\psi_{n,q}$.
and a constant $C := C(d, \lambda) < \infty$ such that,

$$
(3.41) \quad \mathbb{E} \left[ \frac{1}{|\mathbb{B}_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \mathbb{B}_m)} |\nabla \psi_z(e) - \nabla \psi_{n,q}(e)|^2 \right] \leq C (\nu^*(\mathbb{B}_m, q) - \nu^*(\mathbb{B}_n, q)) + C(1 + |q|^2) 3^{-m}.
$$

Proof. The first idea of the proof is to note that the space $\mathbb{H}^1(\mathbb{B}_n)$ can be decomposed

$$
(3.42) \quad \mathbb{H}^1(\mathbb{B}_n) = \bigoplus_{z \in \mathbb{Z}_{m,n}} \mathbb{H}^1(z + \mathbb{B}_n) \oplus H
$$

where we denote by, for $z \in \mathbb{Z}_{m,n}$ with a slight abuse of notation,

$$
\mathbb{H}^1(z + \mathbb{B}_n) := \left\{ \psi \in \mathbb{H}^1(\mathbb{B}_n) : \psi|_{\mathbb{B}_n \setminus (z + \mathbb{B}_m)} = 0 \right\}
$$

and where $H$ is the space of functions of $\mathbb{H}^1(\mathbb{B}_n)$ which are constant on the subcubes $z + \mathbb{B}_n$, for $z \in \mathbb{Z}_{m,n}$. It is a space of dimension $3^{d(n-m)} - 1$ and each function $h \in H$ can be written in the following form

$$
h = \sum_{z \in \mathbb{Z}_{m,n}} \lambda_z \mathbb{1}_{z + \mathbb{B}_m},
$$

for some real constants $(\lambda_z)_{z \in \mathbb{Z}_{m,n}}$ satisfying $\sum_{z \in \mathbb{Z}_{m,n}} \lambda_z = 0$.

Consider the random variable $\psi_{n,q}$ and, for $z \in \mathbb{Z}_{m,n}$, denote by $\psi^z_{n,q}$, its orthogonal projections on the spaces $\mathbb{H}^1(z + \mathbb{B}_m)$. This defines a random variable taking values in $\mathbb{H}^1(z + \mathbb{B}_m)$. We also denote by $h$ the orthogonal projection of $\psi_{n,q}$ on the space $H$. We also introduce the notation

$$
\psi'_{n,q} := \sum_{z \in \mathbb{Z}_{m,n}} \psi^z_{n,q}.
$$

This is a random variable taking values in $\bigoplus_{z \in \mathbb{Z}_{m,n}} \mathbb{H}^1(z + \mathbb{B}_m) \subseteq \mathbb{H}^1(\mathbb{B}_n)$. We denote by $\mathbb{P}_{\psi'_{n,q}}$ its law, and see it as a probability measure on $\bigoplus_{z \in \mathbb{Z}_{m,n}} \mathbb{H}^1(z + \mathbb{B}_m)$.

As in the proof of the previous proposition, we introduce $B_{m,n}$ the set of edges linking two subcubes of the form $z + \mathbb{B}_m$, i.e.,

$$
(3.43) \quad B_{m,n} := \left\{ (x, y) : \exists z, z' \in \mathbb{Z}_{m,n}, z \neq z' \text{ such that } x \in z + \mathbb{B}_m \text{ and } y \in z' + \mathbb{B}_m \right\},
$$

so as to have the decomposition of the sum

$$
(3.44) \quad \sum_{e \subseteq \mathbb{B}_n} = \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \mathbb{B}_m)} + \sum_{e \in B_{m,n}}.
$$

Note also that for every $h \in H$, the gradient $\nabla h$ is supported on the edges of $B_{m,n}$ and

$$
(3.45) \quad \text{for each } z, z' \in \mathbb{Z}_{m,n} \text{ with } z \neq z' \text{ and for each } e \subseteq (z' + \mathbb{B}_m), \nabla \psi^z_{n,q}(e) = 0.
$$

This implies, for each $z \in \mathbb{Z}_{m,n}$ and each $e \subseteq z + \mathbb{B}_m$,

$$
\nabla \psi^z_{n,q}(e) = \nabla \psi^z_{n,q}(e).
$$

The same result is true for $\psi'$: for each $z \in \mathbb{Z}_{m,n}$ and each $e \subseteq z + \mathbb{B}_m$,

$$
\nabla \psi'(e) = \nabla \psi_z(e).
$$

We now split the proof into 5 Steps. In Steps 1 to 4, we assume assume $q = 0$. We then remove this additional assumption in Step 5.
• In Step 1, we show that the law of $\psi'$ is the minimizer of the problem

$$
\inf_{\mathbb{P}} \mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} V_e(\nabla \psi(e)) \right] + H(\mathbb{P}),
$$

where the infimum is chosen over all the probability measures on $\oplus \mathbb{h}^1 (z + \Box_m)$ and the entropy is computed with respect to the Lebesgue measure on this space.

• In Step 2, we consider the optimal coupling between $\psi'$ and $\psi_{n,0}'$ and derive the following inequality, for some constant $C := C(d, \lambda) < \infty$

$$
\mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} |\nabla \psi_z(e) - \nabla \psi_{n,0}'(e)|^2 \right] 
\leq C \left( \nu^*(\Box_m, 0) + \mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} V_e(\nabla \psi_{n,0}'(e)) \right] + \frac{1}{|\Box_n|} H(\mathbb{P}_{\psi_{n,0}'}) \right).
$$

• In Step 3, we prove the following estimate, for some $C := C(d, \lambda) < \infty$

$$
(3.46) \quad \sum_{z \in \mathbb{Z}_{m,n}} \mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{e \subseteq (z+\Box_m)} V(\nabla \psi_{n,0}'(e)) \right] + \frac{1}{|\Box_n|} H(\mathbb{P}_{\psi_{n,0}'}) \leq -\nu^*(\Box_m, 0) + C m 3^{-dm}.
$$

• In Step 4, we complete the proof and show that there exists a coupling between $\psi_{n,0}'$ and $\psi'$ such that

$$
\mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} |\nabla \psi_z(e) - \nabla \psi_{n,0}(e)|^2 \right] 
\leq C(\nu^*(\Box_m, 0) - \nu^*(\Box_m, 0)) + C m 3^{-dm};
$$

• In Step 5, we remove the assumption $q = 0$ and prove the more general result, for each $q \in \mathbb{R}^d$, there exists a coupling between the random variable $\psi'$ and $\psi_{n,q}'$ such that

$$
\mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} |\nabla \psi_z(e) - \nabla \psi_{n,q}(e)|^2 \right] 
\leq C(\nu^*(\Box_m, q) - \nu^*(\Box_m, q)) + C(1 + |q|^2) 3^{-m}.
$$

**Step 1.** By Proposition 2.8, we have

$$
(3.47) \quad \inf_{\mathbb{P}} \left( \mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} V_e(\nabla \psi(e)) \right] + H(\mathbb{P}) \right)
\quad = -\log \int_{\bigoplus_{z \in \mathbb{Z}_{m,n}} \mathbb{h}^1 (z + \Box_m)} \exp \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z+\Box_m)} V_e(\nabla \psi(e)) \right) d\psi,
$$

where the infimum is considered over all probability measure on $\bigoplus_{z \in \mathbb{Z}_{m,n}} \mathbb{h}^1 (z + \Box_m)$. 
But on the one hand, we have the equality

\[
\int_{\mathbb{Z}_{m,n}} \exp \left( - \sum_{z \in \mathbb{Z}_{m,n}, e \subseteq (z + \Box_{m})} V_e(\nabla \psi(e)) \right) \, d\phi = \left( \int_{\mathbb{Z}_{m,n}} \exp \left( - \sum_{e \subseteq \Box_{m}} V_e(\nabla \psi(e)) \right) \, dv \right)^{3d(n-m)}.
\]

On the other hand, since by assumption the random variables \( \psi_z \), for \( z \in \mathbb{Z}_{m,n} \) are independent, one has, by Proposition 2.4,

\[
H(\mathbb{P}_{\psi}) = \sum_{z \in \mathbb{Z}_{m,n}} H(\mathbb{P}_{\psi_z}) = 3^{d(n-m)} H(\mathbb{P}_{m,0}^*).
\]

As a remark, note that the entropy of \( \mathbb{P}_{\psi} \) is computed with respect to the Lebesgue measure on \( \oplus_{z \in \mathbb{Z}_{m,n}} \mathbb{H}^1(z + \Box_{m}) \), while the entropies of \( \mathbb{P}_{\psi_z} \) and \( \mathbb{P}_{m,0}^* \) is computed with respect to the Lebesgue measure on \( \mathbb{H}^1(z + \Box_{m}) \) and on \( \mathbb{H}^1(\Box_{m}) \) respectively.

Moreover, one can compute

\[
\mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_{m})} V_e(\nabla \psi_z(e)) \right] = \sum_{z \in \mathbb{Z}_{m,n}} \mathbb{E} \left[ \sum_{e \subseteq (z + \Box_{m})} V_e(\nabla \psi_z(e)) \right] = 3^{d(n-m)} \mathbb{E} \left[ \sum_{e \subseteq \Box_{m}} V_e(\nabla \psi_{n,0}(e)) \right],
\]

and thus

\[
\mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_{m})} V_e(\nabla \psi_z(e)) \right] + H(\mathbb{P}_{\psi}) = 3^{d(n-m)} \left( \mathbb{E} \left[ \sum_{e \subseteq \Box_{m}} V_e(\nabla \psi_{m,0}(e)) \right] + H(\mathbb{P}_{m,0}) \right).
\]

But we know that \( \psi_{m,0} \) has law \( \mathbb{P}_{m,0}^* \) and thus, by definition of \( \mathbb{P}_{m,0}^* \),

\[
\mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_{m})} V_e(\nabla \psi_z(e)) \right] + H(\mathbb{P}_{\psi}) = -3^{d(n-m)} \log \left( \int_{\mathbb{H}^1(\Box_{m})} \exp \left( - \sum_{e \subseteq \Box_{m}} V_e(\nabla \psi(e)) \right) \, dv \right) \cdot
\]

Combining the previous display with (3.47) and (3.48), we obtain

\[
\mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_{m})} V_e(\nabla \psi_z(e)) \right] + H(\mathbb{P}_{\psi}) = \inf_{\mathbb{P}} \left( \mathbb{E} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_{m})} V_e(\nabla \psi(e)) \right] + H(\mathbb{P}) \right).\]

The proof of Step 1 is complete.

**Step 2.** Consider the optimal coupling between the random variables \( \psi' \) and \( \psi'_{n,0} \). In particular, by Proposition 2.10, the law of \( \frac{\psi' + \psi'_{n,0}}{2} \) satisfies the following convexity inequality

\[
H\left( \mathbb{P}_{\psi' + \psi'_{n,0}} \right) \leq \frac{H(\mathbb{P}_{\psi'}) + H(\mathbb{P}_{\psi'_{n,0}})}{2}.
\]
Moreover, by the uniform convexity of $V_e$, one has

$$
(3.50) \qquad E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e \left( \frac{\nabla \psi'(e) + \nabla \psi'_{n,0}(e)}{2} \right) \right] 
\leq E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e \left( \nabla \psi'(e) \right) \right] + \lambda E \left[ \left\| \left( \frac{\nabla \psi'(e)}{2} \right) \right\| \right].
$$

We can then use Step 1 to compute

$$
E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e(\nabla \psi'(e)) \right] + H \left( \mathbb{P}_\psi \right) = \inf_{\mathbb{P}} E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e(\nabla \psi(e)) \right] + H \left( \mathbb{P} \right)
\leq E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e \left( \frac{\nabla \psi'(e) + \nabla \psi'_{n,0}(e)}{2} \right) \right] + H \left( \mathbb{P} \right).
$$

One can then apply (3.49) and (3.50) to deduce

$$
E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e(\nabla \psi'(e)) \right] + H \left( \mathbb{P}_\psi \right)
\leq \frac{1}{2} \left( E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e(\nabla \psi'(e)) \right] + H \left( \mathbb{P}_\psi \right) \right)
\quad + \frac{1}{2} \left( E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e(\nabla \psi'_{n,0}(e)) \right] + H \left( \mathbb{P}_{\psi,0} \right) \right)
\quad - \frac{\lambda}{4} \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} \left| \nabla \psi'(e) - \nabla \psi'_{n,0}(e) \right|^2 \right].
$$

Note that from Step 1, one also has

$$
\frac{1}{|\Box_n|} \left( E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{\nu(z+\Box_m)} V_e(\nabla \psi'(e)) \right] + H \left( \mathbb{P}_\psi \right) \right) = - \frac{3d(n-m)|\Box_m|}{|\Box_n|} \nu^* (\Box_m, 0) = - \nu^* (\Box_m, 0).
$$
Combining the two previous displays and dividing by $|\square_n|$ gives
\[
\frac{\lambda}{4} \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} |\nabla \psi'(e) - \nabla \psi'_{n,0}(e)|^2 \right]
\leq \frac{1}{2} \nu^* (\square_m, 0) + \frac{1}{2} \left( \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi'_{n,0}(e)) \right] + \frac{1}{|\square_n|} H \left( P^*_{\psi_{n,0}} \right) \right).
\]
Dividing both sides of the previous inequality by $\frac{1}{4}$ and setting $C := \frac{2}{3}$ completes the proof of Step 2.

**Step 3.** Before starting this step, we recall the notation for the edges between two subcubes $B_{m,n}$ introduced in (3.43) and the decomposition of the sum (3.44). With this in mind, to prove the main result of this step, we first show
\[
E \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}} V_e (\nabla \psi_{n,0}(e)) \right] + \frac{1}{|\square_n|} H \left( P^*_{\psi_{n,0}} \right) \geq \frac{1}{|\square_n|} H \left( P_{\psi_{n,0}} \right) - C m 3^{-dm}.
\]
Indeed, once this inequality has been established, we have
\[
E \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi_{n,0}(e)) \right] + E \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi_{n,0}(e)) \right] + \frac{1}{|\square_n|} H \left( P_{\psi_{n,0}} \right) \geq E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi_{n,0}(e)) \right] + \frac{1}{|\square_n|} H \left( P_{\psi_{n,0}} \right) - C m 3^{-dm}.
\]
By (3.44), the term on the left-hand side can be rewritten
\[
E \left[ \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi_{n,0}(e)) \right] + E \left[ \sum_{e \in B_{m,n}} V_e (\nabla \psi_{n,0}(e)) \right] = E \left[ \sum_{e \subseteq \square_n} V_e (\nabla \psi_{n,0}(e)) \right].
\]
We then use the equality
\[
E \left[ \sum_{e \subseteq \square_n} V_e (\nabla \psi_{n,0}(e)) \right] + H \left( P^*_{\psi_{n,0}} \right) = -|\square_n| \nu^* (\square_n, 0).
\]
Combining the few previous results and dividing by $|\square_n|$ then yields
\[
- \nu^* (\square_n, 0) \geq E \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi_{n,0}(e)) \right] + \frac{1}{|\square_n|} H \left( P_{\psi_{n,0}} \right) - C m 3^{-dm}.
\]
This is (3.46) up to two small details. The first one is that there should be a $\psi_{n,0}^{z}$ instead of a $\psi_{n,0}$ on the right-hand side. But by definition $\psi_{n,0}^{z}$ is the orthogonal projection of $\psi_{n,0}$ on the space $\hat{h}^1 (z + \square_m)$, using the property (3.45), we see that
for each $z \in \mathbb{Z}_{m,n}$ and each $e \subseteq z + \square_m$, $\nabla \psi_{n,0}(e) = \nabla \psi_{n,0}^{z}(e)$.
With this, the inequality (3.52) can be rewritten
\[
- \nu^* (\square_n, 0) \geq E \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi_{n,0}^{z}(e)) \right] + \frac{1}{|\square_n|} H \left( P_{\psi_{n,0}} \right) - C m 3^{-dm}.
\]
The second detail needs to fix is the entropy, but thanks to Proposition 2.6, one has

\[
H \left( \mathbb{P}_{\psi_{n,0}} \right) \geq \sum_{z \in Z_{m,n}} H \left( \mathbb{P}_{\psi_{n,0}^z} \right).
\]

Combining the few previous displays gives (3.46).

We now turn to the proof of (3.51). We recall the notation \(Z_q^* (\square_n)\) introduced in (1.8). We also let \(\rho\) be the density associated to the law \(\mathbb{P}_{n,0}^*\), it is defined on \(\hat{h}^1 (\square_n)\) by

\[
\rho : \begin{cases} 
\hat{h}^1 (\square_n) \to \mathbb{R} \\
\psi \mapsto \frac{1}{Z_q (\square_n)} \exp \left( - \sum_{e \subseteq \square_n} V_e (\nabla \psi (e)) \right).
\end{cases}
\]

Using the orthogonal decomposition (3.42), and the definition of \(\psi'\), we can compute the density \(\rho'\) of the random variable \(\psi'\). It is defined on the space \(\oplus_{z \in Z_{m,n}} \hat{h}^1 (z + \square_m)\) according to

\[
\rho' : \begin{cases} 
\oplus_{z \in Z_{m,n}} \hat{h}^1 (z + \square_m) \to \mathbb{R} \\
\psi \mapsto \frac{1}{Z_q^* (\square_n)} \int_H \exp \left( - \sum_{e \subseteq B_{m,n}} V_e (\nabla \psi (e) + \nabla h (e)) \right) dh,
\end{cases}
\]

where we integrate with respect to the Lebesgue measure on the space \(H\) defined in (3.42). Using that for every \(h \in H\), \(\nabla h\) is supported in \(B_{m,n}\), we can use (3.44) to obtain, for each \(\psi \in \oplus_{z \in Z_{m,n}} \hat{h}^1 (z + \square_m)\),

\[
\rho' (\psi) = \frac{\exp \left( - \sum_{z \in Z_{m,n}} \sum_{e \subseteq z + \square_m} V (\nabla \psi (e)) \right)}{Z_q^* (\square_n)} \int_H \exp \left( - \sum_{e \in B_{m,n}} V_e (\nabla \psi (e) + \nabla h (e)) \right) dh.
\]

The next idea of the proof is to prove the following estimate, there exists \(C := C(d, \lambda) < \infty\) such that, for each \(\psi \in \oplus_{z \in Z_{m,n}} \hat{h}^1 (z + \square_m)\),

\[
\log \int_H \exp \left( - \sum_{e \in B_{m,n}} V_e (\nabla \psi (e) + \nabla h (e)) \right) dh \leq C m 3^{d(n-m)}.
\]

The proof of this estimate is technical and postponed to Appendix A, Proposition A.1. We now prove how to deduce (3.51) from (3.54). We first compute the entropy of the law \(\mathbb{P}_{n,0}^*\), this gives

\[
H (\mathbb{P}_{n,0}^*) = \int_{\hat{h}^1 (\square_n)} \rho (\psi) \log \rho (\psi) \, d\psi
= \int_{\hat{h}^1 (\square_n)} \rho (\psi) \left( - \sum_{e \subseteq \square_n} V_e (\nabla \psi) \right) \, d\psi - \log Z_q^* (\square_n).
\]
Adding the term $\mathbb{E} \left[ \sum_{e \in B_{m,n}} V_e (\nabla \psi(e)) \right]$ and using (3.44) gives

$$
E \left[ \frac{1}{\mathbb{P}^*_{n,0}} \sum_{e \in B_{m,n}} V_e (\nabla \psi_{n,0}(e)) \right] + H \left( \mathbb{P}^*_{n,0} \right)
$$

$$
= \int_{\hat{h}^1(\square_n)} \rho(\psi) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi) \right) d\psi - \log Z_{\triangle}^* (\square_n).
$$

We focus on the integral on the right-hand side, using the decomposition (3.42), we can apply Fubini’s Theorem and first integrate over $H$ then over $\hat{h}^1 (z + \square_m)$. This gives

$$
\int_{\hat{h}^1(\square_n)} \rho(\psi) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi) \right) d\psi
$$

$$
= \int_{\hat{h}^1(\square_n)} \rho(\psi + h) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi) \right) dh d\psi.
$$

Note that here we have used that the gradient of an element of $H$ is supported on $B_{m,n}$ combined with the property (3.44). Using the definition of $\rho'$, the previous equality can be rewritten

$$
\int_{\hat{h}^1(\square_n)} \rho(\psi) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi) \right) d\psi
$$

$$
= \int_{\hat{h}^1(\square_n)} \rho'(\psi) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi) \right) d\psi.
$$

Combining the few previous displays then gives

$$
(3.55) \quad \mathbb{E} \left[ \sum_{e \in B_{m,n}} V_e (\nabla \psi_{n,0}(e)) \right] + H \left( \mathbb{P}^*_{n,0} \right)
$$

$$
= \int_{\hat{h}^1(\square_n)} \rho'(\psi) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e (\nabla \psi) \right) d\psi - \log Z_{\triangle}^* (\square_n).
$$

But note that by the definition of $\rho'$ in (3.53) and the technical estimate (3.54), we have, for each $\psi \in \oplus \hat{h}^1 (z + \square_m)$,

$$
\log \rho'(\psi) \leq - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V (\nabla \psi(e)) d\psi - \log Z_{\triangle}^* (\square_n) + C m^3 (n - m).
$$
This allows the following computation
\begin{equation}
H (P_{\psi'}) = \int_{z \in \mathbb{Z}_{m,n}} h^1(z + \Box_m) \rho'(\psi) \log \rho'(\psi) d\psi
\end{equation}
\begin{equation}
\leq \int_{z \in \mathbb{Z}_{m,n}} h^1(z + \Box_m) \rho'(\psi) \left( - \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_m)} V(\nabla \psi(e)) \right) d\psi
- \log Z^*_{\psi}(\Box_n) + C m 3^{d(n-m)}.
\end{equation}
Combining (3.55) and (3.56) gives
\begin{equation}
H (P_{\psi'}) \leq E \left[ \sum_{e \in B_{m,n}} V_e(\nabla \psi_{n,0}(e)) \right] + H (P^*_{n,0}) + C m 3^{d(n-m)}.
\end{equation}
This is exactly (3.51) and completes the proof of Step 3.

**Step 4.** Combining the main results of Steps 2 and 3, we obtain that there exists a coupling between \( \psi' \) and \( \psi'_{n,0} \), such that
\begin{equation}
E \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_m)} |\nabla \psi_z(e) - \nabla \psi_{n,0}(e)|^2 \right] \leq C \left( \nu^*(\Box_m, 0) - \nu^*(\Box_n, 0) \right) + C m 3^{-dm},
\end{equation}
for some \( C := C(d, \lambda) < \infty \). The main difficulty of this step is thus to find a coupling between the random variables \( \psi_{n,0} \) and \( \psi' \), instead of \( \psi'_{n,0} \) and \( \psi' \). Recall that we denoted by \( h \) the orthogonal projection of \( \psi_{n,0} \) on the space \( H \). Denote by \( P_h \) its law, it is a probability measure on \( H \).

By Lemma 2.12, there exists a coupling between the three probability measures \( P_{\psi'}, P_{\psi'_{n,0}} \) and \( P_h \) such that, with this coupling, the law of \( (\psi', \psi'_{n,0}) \) is the optimal coupling between \( P_{\psi'} \) and \( P_{\psi'_{n,0}} \) and the law of \( (\psi'_{n,0}, h) \) is \( P^*_{n,0} \). Note that in the previous sentence, we used the canonical bijection between the spaces
\[ \bigoplus_{z \in \mathbb{Z}_{m,n}} h^1(z + \Box_m) \times H \quad \text{and} \quad \bigoplus_{z \in \mathbb{Z}_{m,n}} h^1(z + \Box_m) \times H, \]
given by the sum as in (2.3) to say that \( P^*_{n,0} \), which is defined as a probability measure on \( h^1(\Box_n) \), can be seen as a law on \( \bigoplus_{z \in \mathbb{Z}_{m,n}} h^1(z + \Box_m) \times H \), on which the random variable \( (\psi'_{n,0}, h) \) is defined.

This provides a coupling between the random variables \( \psi' \) and \( \psi_{n,0} \) satisfying (3.57) and completes the proof fo Step 4.

**Step 5.** We remove the assumption \( q = 0 \). This can be achieved by applying the result obtained for \( q = 0 \) with
\[ V_{e,q}(x) := V_e(x) - q(e)x. \]
Applying the same proof with \( V_{e,q} \) instead of \( V_e \) gives
\begin{equation}
E \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_m)} |\nabla \psi'(e) - \nabla \psi_{n,q}(e)|^2 \right] \leq C \left( \nu^*(\Box_m, q) - \nu^*(\Box_n, q) \right) + C m 3^{d(n-m)} + C(1 + |q|^2) 3^{-m}.
\end{equation}
This can be rewritten
\begin{equation}
E \left[ \frac{1}{|\Box_n|} \sum_{z \in \mathbb{Z}_{m,n}} \sum_{e \subseteq (z + \Box_m)} |\nabla \psi'(e) - \nabla \psi_{n,q}(e)|^2 \right] \leq C \left( \nu^*(\Box_m, q) - \nu^*(\Box_n, q) \right) + C(1 + |q|^2) 3^{-m}.
\end{equation}
The proof of Proposition 3.16 is complete. □

4. Convergence of the subadditive quantities

The main goal of this section is to use the tools developed in Section 3 to prove the main result of this article, namely Theorem 1.1. To this end we first introduce the following notation.

**Definition 4.1.** For each $p \in \mathbb{R}^d$ and each $n \in \mathbb{N}$, we define

$$\tau_n(p) := \nu(\Box_n, p) - \nu(\Box_{n+1}, p)$$

and for each $q \in \mathbb{R}^d$,

$$\tau_n^*(q) := \nu^*(\Box_n, q) - \nu^*(\Box_{n+1}, q).$$

We also recall the following notation from the introduction. For a bounded subset $U \subseteq \mathbb{Z}^d$ and a vector field $F : E_d(U) \to \mathbb{R}$, we denote by $(F)_U$ the unique vector in $\mathbb{R}^d$ such that, for each $p \in \mathbb{R}^d$

$$p \cdot (F)_U = \frac{1}{|U|} \sum_{e \in U} p \cdot F(e).$$

In the rest of this section, this will be applied when $U$ is a triadic cube and when $F$ is the gradient of a function. We may also refer to the quantity $(\nabla \phi)_U$ as the slope of the function $\phi$ over the set $U$.

This section is organised as follows, we first prove, using the Proposition 3.16 that the variance of the slope of the random variable $\psi_{n,q}$ over the cube $\Box_n$ contracts as $n$ goes to $\infty$. More precisely we control the variance of the slope by the quantity $\tau_n^*(q)$ which is expected to be small as $n$ goes to $\infty$. This is done in Proposition 4.2. Once the slope is controlled, we apply the multiscale Poincaré inequality, stated in Proposition 2.15 to prove that $\psi_{n,q}$ is in fact close, in the expectation of the $L^2$-norm of the cube $\Box_n$, to an affine fonction. With all these tools at hand, we prove the technical estimate of this article, Proposition 4.4, thanks to a patching construction. This technical lemma, combined with the convex duality property proved in Proposition 3.7, shows that on a large scale, the functions $p \to \nu(\Box_n, p)$ and $q \to \nu^*(\Box_n, q)$ are approximately convex dual to one another, i.e, they satisfy up to a small error

$$\nu^*(\Box_n, q) \simeq \sup_{p \in \mathbb{R}^d} -\nu(\Box_n, p) + p \cdot q.$$ 

Once this technical lemma is proved, we are able to prove Theorem 1.1.

4.1. Contraction of the variance of the slope of $\psi_{n,q}$. We first prove the contraction of the variance of the slope

**Proposition 4.2 (Contraction of the slope of $\psi_{n,q}$).** There exists a constant $C := C(d, \lambda) < \infty$ such that for each $n \in \mathbb{N}$, and each $q \in \mathbb{R}^d$,

$$\text{Var} \left[ (\nabla \psi_{n+1,q})_{\Box_{n+1}} \right] \leq C(1 + |q|^2)3^{-n} + C \sum_{m=0}^{n} \frac{3^{(m-n)/2}}{\tau_m^*(q)}.$$

**Proof.** Let $q \in \mathbb{R}^d$. Consider the family of random variables $\psi_z$, for $z \in \mathcal{N}_{n,n+1}$ and the random variable $\psi' := \sum_{z \in \mathcal{N}_{n,n+1}} \psi_z$ which were introduced before the statement of Proposition 3.16. We recall that it satisfies the following properties

- for each $z \in \mathcal{N}_{n,n+1}$, $\psi_z$ is a random variable valued in $\mathcal{N}^1(\Box_{n+1})$ is equal to 0 outside $z + \Box_n$ and has law $\mathbb{P}^*_n$ in $z + \Box_n$,
- the $\psi_z$ are independent.

We also consider the coupling between $\psi'$ and $\psi_{n+1,q}$ which was introduced in Proposition 3.16. In particular estimate (3.41) holds.

We then split the proof into 2 steps
We then estimate the three terms on the right-hand side of the previous estimate separately. First to estimate the second term we use the independence of the random variables \( \psi_z \) to compute
\[
\var \left[ 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_z \rangle_{z + \Box_n} \right] = 3^{-2d} \sum_{z \in \mathbb{Z}_n} \var \left[ \langle \nabla \psi_z \rangle_{z + \Box_n} \right].
\]
Using that the law of \( \psi_z \) on the subcube \( z + \Box_n \) is \( \mathbb{P}_{n,q}^z \), one obtains
\[
\var \left[ 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_z \rangle_{z + \Box_n} \right] = 3^{-d} \var \left[ \langle \nabla \psi_{n,q} \rangle_{\Box_n} \right].
\]

In Step 1, we use Proposition 3.16 to prove
\[
\var_{\frac{1}{2}} \left[ \langle \nabla \psi_{n+1,q} \rangle_{\Box_{n+1}} \right] \leq 3^{-\frac{d}{2}} \var_{\frac{1}{2}} \left[ \langle \nabla \psi_{n,q} \rangle_{\Box_n} \right] + C \tau_n^* (q) + C 3^{-\frac{d}{2}} (1 + |q|).
\]

In Step 2, we iterate the inequality obtained in Step 1 to get (4.1).

\textbf{Step 1.} First we recall the definition of the set of edges linking two subcubes of the form \( z + \Box_n \),
\[
B_n := \{(x, y) : \exists z, z' \in 3^n \mathbb{Z}^d \cap \Box_{n+1} \text{ such that } z \neq z', x \in z + \Box_n \text{ and } y \in z' + \Box_n\}
\]
and the decomposition of the sum (3.44)
\[
\sum_{e \subseteq \Box_{n+1}} = \sum_{z \in \mathbb{Z}_n} \sum_{e \subseteq z + \Box_n} + \sum_{e \in B_n}.
\]
This set is equal to the set \( B_{n,n+1} \) from the previous section, but since it depends only on one parameter, we use the shortcut notation \( B_n \). From the previous decomposition of the sum, we see that we have the identity
\[
\left| \langle \nabla \psi_{n+1,q} \rangle_{\Box_{n+1}} - 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_{n+1,q} - \nabla \psi_z \rangle_{z + \Box_n} - 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_z \rangle_{z + \Box_n} \right| \leq \frac{1}{|\Box_{n+1}|} \sum_{e \in B_n} |\nabla \psi_{n+1,q}(e)|.
\]
Taking the square-root of the variance and using the triangle inequality, one obtains
\[
\var_{\frac{1}{2}} \left[ \langle \nabla \psi_{n+1,q} \rangle_{\Box_{n+1}} \right] \leq \var_{\frac{1}{2}} \left[ 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_{n+1,q} - \nabla \psi_z \rangle_{z + \Box_n} \right]
\]
\[
+ \var_{\frac{1}{2}} \left[ 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_z \rangle_{z + \Box_n} \right] + \mathbb{E} \left[ \left( \frac{1}{|\Box_{n+1}|} \sum_{e \in B_n} |\nabla \psi_{n+1,q}(e)| \right)^2 \right]^{\frac{1}{2}}.
\]
We then estimate the three terms on the right-hand side of the previous estimate separately. First by Proposition 3.16, one has
\[
\var \left[ 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_{n+1,q} - \nabla \psi_z \rangle_{z + \Box_n} \right] \leq \mathbb{E} \left[ \frac{1}{|\Box_{n+1}|} \sum_{z \in \mathbb{Z}_n} \sum_{e \subseteq z + \Box_n} |\nabla \psi_{n+1,q}(e) - \nabla \psi_z(e)|^2 \right] \leq C \tau_n^* (q) + C (1 + |q|^2) 3^{-n}.
\]
To estimate the second term we use the independence of the random variables \( \psi_z \) to compute
\[
\var \left[ 3^{-d} \sum_{z \in \mathbb{Z}_n} \langle \nabla \psi_z \rangle_{z + \Box_n} \right] = 3^{-2d} \sum_{z \in \mathbb{Z}_n} \var \left[ \langle \nabla \psi_z \rangle_{z + \Box_n} \right].
\]
The third term is estimated thanks to the Cauchy-Schwarz inequality combined with the bound \(|B_n| \leq C3^{-n} |\square_{n+1}| \) on the cardinality of \(B_n\),

\[
\mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_n} \left| \nabla \psi_{n+1}(e) \right|^2 \right] \leq \mathbb{E} \left[ \frac{|B_n|}{|\square_{n+1}|^2} \sum_{e \in B_n} \left| \nabla \psi_{n+1}(e) \right|^2 \right] 
\leq C3^{-n} \mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_n} \left| \nabla \psi_{n+1}(e) \right|^2 \right].
\]

The bound on the \(L^2\) norm of \(\nabla \psi_{n+1}\) obtained in Proposition 3.13, this yields

\[
\mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_n} \left| \nabla \psi_{n+1}(e) \right|^2 \right] \leq C3^{-n} (1 + |q|^2),
\]

for some \(C := C(d, \lambda) < \infty\). Combining the few previous displays gives the estimate

\[
\text{Var} \frac{1}{|\square_{n+1}|} \left[ \langle \nabla \psi_{n+1,q} \rangle_{\square_{n+1}} \right] \leq 3^{-\frac{n}{2}} \text{Var} \frac{1}{|\square_{n}|} \left[ \langle \nabla \psi_{n}(e) \rangle_{\square_{n}} \right] + C \tau_n^*(q) \frac{1}{2} + C3^{-\frac{n}{2}} (1 + |q|).
\]

**Step 2.** Iteration and conclusion. We denote by \(\sigma_n := \text{Var} \frac{1}{|\square_{n}|} \left[ \langle \nabla \psi_{n} \rangle_{\square_{n}} \right] \).

The main estimate of Step 1 can be rewritten with this new notation

\[
\sigma_{n+1} \leq 3^{-\frac{n}{2}} \sigma_n + C \tau_n^*(q) \frac{1}{2} + C3^{-\frac{n}{2}} (1 + |q|).
\]

An iteration of the previous display gives

\[
\sigma_n \leq 3^{-\frac{dn}{2}} \sigma_0 + C \sum_{m=0}^{n} 3^{-\frac{d(n-m)}{2}} \tau_m^*(q) \frac{1}{2} + C (1 + |q|) \sum_{m=0}^{n} 3^{-\frac{d(n-m)}{2}} 3^{-\frac{m}{2}} 
\leq 3^{-\frac{dn}{2}} \sigma_0 + C \sum_{m=0}^{n} 3^{-\frac{d(n-m)}{2}} \tau_m^*(q) \frac{1}{2} + C (1 + |q|) 3^{-\frac{n}{2}}.
\]

Note that \(3^{-\frac{1}{2}} \geq 3^{-\frac{d}{2}}\). Also note that by Proposition 3.13, we have the bound \(\sigma_0 \leq C (1 + |q|)\). From the previous estimate, we obtain

\[
\sigma_n \leq C (1 + |q|) 3^{-\frac{n}{2}} + C \sum_{m=0}^{n} 3^{-\frac{(m-n)}{2}} \tau_m^*(q) \frac{1}{2},
\]

for some \(C := C(d, \lambda) < \infty\). Squaring the previous inequality gives

\[
\sigma_n^2 \leq C (1 + |q|^2) 3^{-n} + C \left( \sum_{m=0}^{n} 3^{-\frac{(m-n)}{2}} \tau_m^*(q) \right)^2 \leq C (1 + |q|^2) 3^{-n} + C \sum_{m=0}^{n} 3^{-\frac{(m-n)}{2}} \tau_m^*(q).
\]

The proof of Step 2 is complete. \(\square\)

We now want to deduce from the previous proposition that the random variable \(\psi_{n,q}\) is close to an affine function. To this end, note that, using the explicit formula for \(\nu^*\) given in (1.9) and computing explicitly the gradient of this quantity with respect to the \(q\) variable gives the formula

\[
\nabla_q \nu^*(\square_n, q) = \mathbb{E} \left[ \langle \nabla \psi_{n,q} \rangle_{\square_n} \right].
\]
We also note that, for each \( q \in \mathbb{R}^d \),

\[
|\nabla q^{\nu^*}(\square_n, q)| \leq \left( \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e)|^2 \right] \right)^{\frac{1}{2}} \leq C(1 + |q|).
\]

For future reference that, we record the following estimate, for each \( m \leq n \) and each \( q \in \mathbb{R}^d \)

\[
|\nabla q^{\nu^*}(\square_m, q) - \nabla q^{\nu^*}(\square_n, q)|^2 \leq C \sum_{k=m}^{n} \tau_k^*(q) + C(1 + |q|^2)3^{-m}.
\]

To prove this, consider the coupling between the random variables \( \psi_{n,q} \) and \( \psi_z \) introduced in Proposition 3.16. Using this one has

\[
|\nabla q^{\nu^*}(\square_m, q) - \nabla q^{\nu^*}(\square_n, q)|^2 = \left| \sum_{e \subseteq \square_n} \mathbb{E} \left[ \langle \nabla \psi_{n,q}(e) \rangle_{\square_n} - \frac{1}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \langle \nabla \psi_z \rangle_{z + \square_m} \right] \right|^2.
\]

We then introduce the following set of edges

\[
B_{m,n} := \{ e = (x, y) \subseteq \square_n : \exists z, z' \in 3^m \mathbb{Z}^d \cap \square_n, z \neq z', x \in z + \square_m \text{ and } y \in z' + \square_m \}.
\]

It represents the set of edges linking two triadic cubes of size \( 3^m \) contained in the larger cube \( \square_n \). Also the cardinality of this set can be estimated by

\[
|B_{m,n}| \leq C3^{-m} |\square_n|.
\]

We can then partition the set of edges of \( \square_n \) according to

\[
e \subseteq \square_n \implies \exists z \in 3^m \mathbb{Z}^d \cap \square_n, e \subseteq z + \square_m \text{ or } e \in B_{m,n}.
\]

As a consequence, we obtain the splitting of the sum

\[
\sum_{e \subseteq \square_n} = \sum_{e \in B_{m,n}} + \sum_{z \in Z_{m,n}} \sum_{e \subseteq z + \square_m}.
\]

Combining this with (4.5), one obtains

\[
|\nabla q^{\nu^*}(\square_m, q) - \nabla q^{\nu^*}(\square_n, q)|^2 \leq 2\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in Z_{m,n}} \sum_{e \subseteq z + \square_m} |\nabla \psi_{n,q}(e) - \nabla \psi_z(e)|^2 \right] + 2\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}} |\nabla \psi_{n,q}(e)|^2 \right].
\]

By Proposition 3.16, the first term on the right-hand side can be estimated by

\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in Z_{m,n}} \sum_{e \subseteq z + \square_m} |\nabla \psi_z(e) - \nabla \psi_{n,q}(e)|^2 \right] \leq C \sum_{k=m}^{n-1} \tau_k^*(q) + C(1 + |q|^2)3^{-m}.
\]

By the Cauchy-Schwarz inequality and Proposition 3.13, we can estimate the second term on the right-hand side according to

\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \in B_{m,n}} |\nabla \psi_{n,q}(e)|^2 \right] \leq \mathbb{E} \left[ \frac{|B_{m,n}|}{|\square_n|^2} \sum_{e \in B_{m,n}} |\nabla \psi_{n,q}(e)|^2 \right] \leq C(1 + |q|^2)3^{-m}.
\]
Combining the three previous displays gives
\[
|\nabla q \nu^*(\square_m, q) - \nabla q \nu^*(\square_n, q)|^2 \leq C \sum_{k=m}^{n-1} \tau_k^*(q) + C (1 + |q|^2) 3^{-m},
\]
which is the desired estimate.

4.2. \(L^2\) contraction of the field \(\psi_{n,q}\) to an affine function. Combining the multiscale Poincaré inequality with the contraction of the variance of the slope of \(\psi_{n,q}\) proved in the previous subsection, we obtain that the field \(\psi_{n,q}\) is close in the \(L^2\) norm to an affine function. The right-hand side of the estimate still depends on the quantities \(\tau_k^*(q)\) which are expected to be small when \(n\) is large.

**Proposition 4.3.** There exists a constant \(C := C(d, \lambda) < \infty\) such that, for every \(n \in \mathbb{N}\) and every \(q \in \mathbb{R}^d\),
\[
(4.6) \quad \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} |\psi_{n,q}(x) - \nabla q \nu^*(\square_n, q) \cdot x|^2 \right] \leq C 3^{2n} \left( (1 + |q|^2) 3^{-\frac{n}{2}} + \sum_{m=0}^{n} 3^{\frac{(n-m)}{2}} \tau_m^*(q) \right).
\]

**Proof.** By the discrete version of the multiscale Poincaré inequality, one has
\[
(4.7) \quad \frac{1}{|\square_n|} \sum_{x \in \square_n} |\psi_{n,q}(x) - \nabla q \nu^*(\square_n, q) \cdot x|^2 \leq C \frac{1}{|\square_n|} \sum_{e \subseteq \square_{n+1}} |\nabla \psi_{n,q}(e) - \nabla q \nu^*(\square_n, q) \cdot e|^2
\]
\[
+ C 3^n \sum_{m=0}^{n} 3^m \left( \frac{1}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} |\nabla \psi_{n,q}(z)_{z+\square_m} - \nabla q \nu^*(\square_n, q)|^2 \right).
\]

By Proposition 3.13 and (4.3), we can bound the expectation of the first term on the right-hand side
\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e) - \nabla q \nu^*(\square_n, q) \cdot e|^2 \right] \leq \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e)|^2 + |\nabla q \nu^*(\square_n, q)|^2 \right]
\]
\[
\leq C (1 + |q|^2).
\]

We then split the proof into 2 steps
- In Step 1, we estimate the expectation of the second term on the right-hand side and prove the estimate, for any integer \(m \leq n\),
\[
(4.8) \quad \frac{1}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \mathbb{E} \left[ |\nabla \psi_{n,q}(z)_{z+\square_m} - \nabla q \nu^*(\square_n, q)|^2 \right]
\]
\[
\leq C (1 + |q|^2) 3^{-m} + C \sum_{k=0}^{m} 3^{\frac{(k-m)}{2}} \tau_k^*(q) + C \sum_{k=m}^{n} \tau_k^*(q).
\]
- In Step 2, we deduce (4.6) from the previous display.

**Step 1.** To prove (4.8), we need to apply Proposition 3.16. We first recall the notations which were used in this proposition. We consider the family of random variables \(\psi_z\), for \(z \in Z_{m,n}\) whose law is defined by
- For each \(z \in Z_{m,n}\), \(\psi_z\) takes value in \(\hat{\mathbb{H}}^1(z + \square_m)\), is equal to 0 in \(\square_n \setminus (z + \square_m)\) and the law of \(\psi_z(z - z)\) is \(\mathbb{P}_{m,q}^*.\)
- The random variables \(\psi_z\), for \(z \in Z_{m,n}\) are independent.

Then from Proposition 3.16, there exists a coupling between \(\psi_{n,q}\) and \(\psi_z\) such that
\[
(4.9) \quad \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in Z_{m,n}} \sum_{e \subseteq z + \square_m} |\nabla \psi_z(e) - \nabla \psi_{n,q}(e)|^2 \right] \leq C \sum_{k=m}^{n} \tau_k^*(q) + C (1 + |q|^2) 3^{-m}.
\]
We can then split (4.8)
\[
\frac{1}{|Z_{m,n}|} \mathbb{E} \left[ \sum_{z \in Z_{m,n}} \left| \langle \nabla \psi_{n,q} \rangle_{z+\square_m} - \nabla q \nu^*(\square_n, q) \right|^2 \right]
\]
\[
\leq \frac{3}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \frac{1}{|\square_m|} \mathbb{E} \left[ \sum_{e \subseteq z+\square_m} \left| \nabla \psi_{n,q}(e) - \nabla \psi_z(e) \right|^2 \right]
+ \frac{3}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \mathbb{E} \left[ \left| \langle \nabla \psi_z \rangle_{z+\square_m} - \nabla q \nu^*(\square_m, q) \right|^2 \right]
+ 3 \left| \nabla q \nu^*(\square_n, q) - \nabla q \nu^*(\square_m, q) \right|^2.
\]
We then estimate the first term on the right-hand side thanks to (4.9) and the third term thanks to (4.4). This gives
\[
\frac{1}{|Z_{m,n}|} \mathbb{E} \left[ \sum_{z \in Z_{m,n}} \left| \langle \nabla \psi_{n,q} \rangle_{z+\square_m} - \nabla q \nu^*(\square_n, q) \right|^2 \right]
\leq \frac{3}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \frac{1}{|\square_m|} \mathbb{E} \left[ \sum_{e \subseteq z+\square_m} \left| \nabla \psi_z(e) - \nabla q \nu^*(\square_m, q) \cdot e \right|^2 \right]
+ C \sum_{k=m}^n \tau^*_k(q) + C(1+|q|^2)3^{-m}
\]
and since \( \nabla q \nu^*(\square_m, q) := \mathbb{E} \left[ \langle \nabla \psi_{m,q} \rangle_{\square_m} \right] \), we have by Lemma 4.2,
\[
\frac{1}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \mathbb{E} \left[ \left| \langle \nabla \psi_z \rangle_{z+\square_m} - \nabla q \nu^*(\square_m, q) \right|^2 \right] = \text{Var} \left[ \langle \nabla \psi_{n,q} \rangle_{\square_m} \right]
\leq C(1+|q|^2)3^{-m} + C \sum_{k=0}^m 3^{(k-m)/2} \tau^*_k(q).
\]
Combining the previous displays yields
\[
\frac{1}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \mathbb{E} \left[ \left| \langle \nabla \psi_{n,q} \rangle_{z+\square_m} - \nabla q \nu^*(\square_n, q) \right|^2 \right] 
\leq C(1+|q|^2)3^{-m} + C \sum_{k=0}^m 3^{(k-m)/2} \tau^*_k(q) + C \sum_{k=m}^n \tau^*_k(q).
\]
This is (4.8). The proof of Step 1 is complete.

Step 2. To ease the notation, we denote by, for each \( m \in \{1, \ldots, n\} \),
\[
X_m := \frac{1}{|Z_{m,n}|} \sum_{z \in Z_{m,n}} \left| \langle \nabla \psi_{n,q} \rangle_{z+\square_m} - \nabla q \nu^*(\square_n, q) \right|^2.
\]
By the main result of Step 1 is equivalent to
\[
\mathbb{E} [X_m] \leq C(1+|q|^2)3^{-m} + C \sum_{k=0}^m 3^{(k-m)/2} \tau^*_k(q) + C \sum_{k=m}^n \tau^*_k(q).
\]
and by (4.7), one has
\[
\frac{1}{|\square_n|} \sum_{x \in \square_n} |\nabla q \nu^*(\square_n, q) \cdot x|^2 \leq C(1+|q|^2) + C 3^m \sum_{m=0}^n 3^m X_m.
\]
Taking the expectation gives
\[
\mathbb{E}\left[3^n \sum_{m=0}^{n} 3^m X_m\right] \leq C3^n \sum_{m=0}^{n} 3^m \left( C(1 + |q|^2)3^{-m} + C \sum_{k=0}^{m} 3^{(k-m)/2} \tau_k(q) + C \sum_{k=m}^{n} \tau_k^*(q) \right)
\]
\[
\leq C3^n \left( (1 + |q|^2)n 3^{-n} + \sum_{k=0}^{n} 3^{(k-n)/2} \tau_k(q) + \sum_{k=0}^{n} 3^{(k-n)/2} \tau_k^*(q) \right).
\]
To simplify the previous display, we appeal to the crude estimates \( n \leq C3^{\frac{n}{2}} \) and \( 3^{(k-n)/2} \leq \frac{3}{2} \). This gives
\[
\mathbb{E}\left[\left(\frac{1}{\|n\|} \sum_{x \in \|n\|} |\psi_{n,q}(x) - \nabla_q \nu^*(\|n\|, q) \cdot x|^2\right)\right] \leq C3^{2n} \left( (1 + |q|^2)3^{-\frac{n}{2}} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q) \right).
\]
The proof of Proposition 4.3 is complete.

4.3. Convex duality: upper bound. We now turn to the main technical lemma of this article. First, we introduce the following notation. For each \( n \in \mathbb{N} \), we denote by \( \square_n^+ \) the triadic cube \( \square_n \) to which one has added a boundary layer of size 1, i.e.,
\[
\square_n^+ := \left( -\frac{3^n + 1}{2} , \frac{3^n + 1}{2} \right)^d.
\]
It is a cube of size \( 3^n + 2 \) and satisfies the following convenient property
\[
\left( \square_n^+ \right)^o = \square_n.
\]
The statement of the next lemma can be formulated as follows: if the \( \tau_n^*(q) \) are small, then for each \( q \in \mathbb{R}^d \), there exists a \( p \in \mathbb{R}^d \) such that
\[
\nu \left( \square_{2n}^+, p \right) + \nu^* \left( \square_n, q \right) - p \cdot q \quad \text{is small.}
\]
Moreover we have an explicit value of \( p \) which is \( \nabla_q \nu^* \left( \square_n, q \right) \). In a later statement, we will remove the condition \( \square_{2n}^+ \) and prove that for each \( q \in \mathbb{R}^d \), there exists \( p \in \mathbb{R}^d \) such that
\[
\nu \left( \square_n, p \right) + \nu^* \left( \square_n, q \right) - p \cdot q \quad \text{is small.}
\]
Combining this result with the lower bound on the convex duality proved in Proposition 3.7, we obtain
\[
\nu^* \left( \square_n, q \right) = \inf_{p \in \mathbb{R}^d} \left( -\nu \left( \square_n, p \right) + p \cdot q \right) \quad \text{up to a small error.}
\]
The main argument in the proof of Proposition 4.4 is a patching construction: we need to patch functions of laws \( \mathbb{P}_{n,q}^* \) in the cube \( \square_{2n}^+ \), then contract from this patching construction a law on the space \( h_0^k \left( \square_{2n} \right) \) and test it in the variational formulation for \( \nu \).

**Proposition 4.4.** There exist a constant \( C := C(d, \lambda) < \infty \) and an exponent \( \beta := \beta(d, \lambda) > 0 \) such that for each \( q \in \mathbb{R}^d \) and each \( n \in \mathbb{N} \),
\[
\nu \left( \square_{2n}^+, \nabla_q \nu^* \left( \square_n, q \right) \right) + \nu^* \left( \square_n, q \right) - \nabla_q \nu^* \left( \square_n, q \right) \cdot q \leq C \left( 1 + |q|^2 \right)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q) \right).
\]

**Proof.** Fix \( q \in \mathbb{R}^d \) and for each \( p \in \mathbb{R}^d \), we denote by \( l_p \) the linear function of slope \( p \), defined according to, for each \( x \in \mathbb{R}^d \),
\[
l_p(x) = p \cdot x.
\]
To simplify the notation, we also write, for each \( q \in \mathbb{R}^d \) and each \( n \in \mathbb{N} \),
\[
\nabla \nu_n^*(q) := \nabla_q \nu^* \left( \square_n, q \right) \in \mathbb{R}^d
\]
We also recall the notation (1.2) introduced in Section 1 which will be used with \( p = \nabla \nu_n^*(q) \) frequently in the proof.
The strategy of the proof is the following. We will construct a random variable taking values in \( h_0^1 (\mathbb{R}^+_{2n}) \), denoted by \( \kappa_{2n}^+ \) in the proof. This random variable is essentially constructed by patching together independent random variables, which are defined on the triadic cubes \( z + \Box_n \), for \( z \in \mathbb{Z}_{n,2n} \) and whose laws are the law of \( \psi_{n,q} - I (\nabla \nu_{n}^*(q)) \). The technical details are carried out in Step 1 below.

We then prove that \( \kappa_{2n}^+ \) satisfies

\[
E \left[ \frac{1}{|\Box_{2n}^+|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \nu_{n}^*(q)(e) + \nabla \kappa_{2n}^+(e)) \right] + \frac{1}{|\Box_{2n}^+|} H \left( \mathbb{P}_{\kappa_{2n}^+} \right) \leq -\nu^* (\Box_n, q) + \nabla \nu_{n}^*(q) \cdot q
\]

\[
+ C \left( (1 + |q|^2)3^{-\beta_n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m(q) \right).
\]

We finally use \( \kappa_{2n}^+ \) as a test function in the variational formulation of \( \nu (\Box_{2n}^+, \nabla \nu_{n}^*(q)) \), this gives

\[
\nu (\Box_{2n}^+, \nabla \nu_{n}^*(q)) \leq E \left[ \frac{1}{|\Box_{2n}^+|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \nu_{n}^*(q) + \nabla \kappa_{2n}^+(e)) \right] + \frac{1}{|\Box_{2n}^+|} H \left( \mathbb{P}_{\kappa_{2n}^+} \right).
\]

Combining the two previous displays will complete the proof.

We split the proof into 4 steps.

- In Step 1, we construct the random variable \( \kappa_{2n}^+ \) taking values in \( h_0^1 (\mathbb{R}^+_{2n}) \).
- In Step 2, we show that the entropy of \( \kappa_{2n}^+ \) is controlled by the entropy of \( \mathbb{P}_{n,q} \). Precisely, we prove

\[
(4.10) \quad \frac{1}{|\Box_{2n}^+|} H \left( \mathbb{P}_{\kappa_{2n}^+} \right) \leq \frac{1}{|\Box_n|} H \left( \mathbb{P}_{n,q} \right) + C n 3^{-n},
\]

where the entropy on the left-hand side is computed with respect to the Lebesgue measure on \( h_0^1 (\mathbb{R}^+_{2n}) \) and the entropy on the right-hand side is computed with respect to the Lebesgue measure on \( h_0^1 (\Box_n) \).

- In Steps 3 and 4, we show that the energy of the random variable \( \kappa_{2n}^+ \) is controlled by the energy of \( \psi_{n,q} \). Precisely, we prove

\[
(4.11) \quad E \left[ \frac{1}{|\Box_{2n}^+|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \psi_{n,q}^*(q) + \nabla \kappa_{2n}^+(e)) \right] \leq E \left[ \frac{1}{|\Box_n|} \sum_{e \subseteq \Box_n} V_e (\nabla \psi_{n,q}(e)) \right]
\]

\[
+ C \left( (1 + |q|^2)3^{-\beta_n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m(q) \right).
\]

- In Step 5, we combine the results of Steps 3 and 4 to prove

\[
\nu (\Box_{2n}^+, \nabla \nu_{n}^*(q)) - \nu^* (\Box_n, q) + q \cdot \nabla \nu_{n}^*(q) \leq C \left( (1 + |q|^2)3^{-\beta_n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m(q) \right).
\]

**Step 1.** Denote by \( h^1 (\mathbb{R}^+_{2n}) \) the set of functions from \( \mathbb{R}^+_{2n} \) to \( \mathbb{R} \). There is a canonical bijection between \( h^1 (\mathbb{R}^+_{2n}) \) and \( h_0^1 (\mathbb{R}^+_{2n}) \) obtained by extending the functions of \( h^1 (\mathbb{R}^+_{2n}) \) to be 0 on the boundary of \( \mathbb{R}^+_{2n} \). We first explain the strategy to construct \( \kappa_{2n}^+ \) and then write the details. First consider a family \( (\psi_z)_{z \in \mathbb{Z}_{n,2n}} \) of random variables satisfying

- for each \( z \in \mathbb{Z}_{n,2n} \), \( \psi_z \) is valued in \( \hat{h}^1 (z + \Box_n) \) and its law is \( \mathbb{P}_{n,q}^* \).
- the random variables \( \psi_z \) are independent.
We then recall the definition of the set of edges linking two triadic cubes of size $3^n$ in $\Box_{2^n}$:

$$B_{n,2n} := \{ e = (x,y) \subseteq \Box_{2^n} : \exists z, z' \in Z_{n,2n}, z \neq z', x \in z + \Box_n and y \in z' + \Box_n \}.$$ 

Note that one can partition the set edges of $\Box_{2^n}$ according to

$$e \subseteq \Box_{2^n} \implies \exists z \in 3^n \mathbb{Z}^d \cap \Box_{2^n}, e \subseteq z + \Box_n or e \in B_{n,2n}.$$ 

With this in mind, we construct a random vector field $f$ defined on the edges $\Box_{2^n}$ by patching together the vector fields $\nabla \psi_z - \nabla \nu^*_n(q)$ defined on the edges of $z + \Box_n$. In details, the vector field $f$ is defined as follows, for each $e \subseteq \Box_{2^n}$,

$$f(e) = \begin{cases} 
\nabla \psi_z(e) - \nabla \nu^*_n(q)(e) & \text{if } e \subseteq z + \Box_n, \text{ for some } z \in Z_{n,2n}, \\
0 & \text{if } e \in B_{n,2n}.
\end{cases}$$

(4.12)

The idea is then to construct $\kappa^+_{2n}$ so that, for each $e \subseteq \Box_{2^n}$,

$$\nabla \kappa^+_{2n}(e) \approx f(e).$$

The first obstruction is that the vector field $f(e)$ is not necessarily the gradient of a function in $h_0^1(\Box^+_{2n})$. To remedy this, we project the vector field $f$ on the space of gradients of function in $h_0^1(\Box^+_{2n})$. This is equivalent to solving the Dirichlet problem

$$\begin{cases}
\Delta \kappa = \text{div} f \text{ in } \Box_{2^n}, \\
\kappa \in h_0^1(\Box^+_{2n}).
\end{cases}$$

(4.13)

This solves the first obstruction. The second obstruction is that if we denote by $\kappa$ the random variable which is defined by (4.13), this random variable almost surely belongs to a strict linear subspace of $h_0^1(\Box^+_{2n})$, consequently its law is not absolutely continuous with respect to the Lebesgue measure on $h_0^1(\Box^+_{2n})$ and its entropy will be infinite. To remedy this we add some independent random variables with law uniform on $[0,1]$, as was done in Proposition 3.15.

We now turn to the details of the construction. Consider the orthogonal decomposition with respect to the standard $L^2$ scalar product

$$h_1^1(\Box_{2n}) = \bigoplus_{z \in Z_{n,2n}} \hat{h}_1^1(z + \Box_n) \perp H,$$

(4.14)

where $H := \left( \bigoplus_{z \in Z_{n,2n}} \hat{h}_1^1(z + \Box_n) \right) \perp$. It is characterised as the vector space of functions which are constant on the subcubes $z + \Box_n$ and its dimension is $3^{3dn}$.

Then consider $L$ the linear operator defined on $\bigoplus_{z \in Z_{n,2n}} \hat{h}_1^1(z + \Box_n)$ valued in $h_0^1(\Box^+_{2n})$ defined by the following construction.

For each $\psi \in \bigoplus_{z \in Z_{n,2n}} \hat{h}_1^1(z + \Box_n)$, let $L(\psi)$ be the unique solution to

$$\begin{cases}
\Delta L(\psi) = \text{div} f \text{ in } \Box_{2^n}, \\
L(\psi) \in h_0^1(\Box^+_{2n}).
\end{cases}$$

(4.15)

where $f$ is the vector field defined according to, for each $e \subseteq \Box_{2^n}$,

$$f(e) = \begin{cases} 
\nabla \psi(e) & \text{if } e \subseteq z + \Box_n, \text{ for some } z \in Z_{n,2n}, \\
0 & \text{if } e \in B_{2n,n}.
\end{cases}$$

We first verify that the operator $L$ is injective. To this end, we check that the kernel of $L$ is reduced to $\{0\}$. Let $\psi \in \bigoplus_{z \in Z_{n,2n}} \hat{h}_1^1(z + \Box_n)$ such that $L(\psi) = 0$, we want to prove that $\psi = 0$. 

We then recall the definition of the set of edges linking two triadic cubes of size $3^n$ in $\Box_{2^n}$.
First by definition of $L$, the condition $L(\psi) = 0$ implies $\text{div } f = 0$. But the function $\text{div } f$ can be computed explicitly and we have, for each $z \in \mathcal{Z}_{n,2n}$,

$$\Delta_{z+\square_n} \psi = 0 \text{ in } z + \square_n,$$

where $\Delta_{z+\square_n}$ is the Laplacian on the graph $z + \square_n$ and is defined by, for each $x \in \square_n$,

$$\Delta_{z+\square_n} \psi(x) = \sum_{y \sim x, y \in z+\square_n} (\psi(y) - \psi(x)).$$

Note that this Laplacian is different from the standard Laplacian on $\square_{2n}^+$ which is used in (4.15) and defined by, for each $x \in \square_{2n}$,

$$\Delta \psi(x) = \sum_{y \sim x} (\psi(y) - \psi(x)).$$

This difference comes from the fact that $f$ was set to be $0$ on the edges of $B_{2n,n}$. Now note that by the maximum principle, we have, for each $z \in \mathcal{Z}_{n,2n}$,

$$\Delta_{z+\square_n} \psi = 0 \text{ in } z + \square_n \implies \psi \text{ is constant in } z + \square_n.$$

Combining the previous remark with the assumption $\psi \in \oplus \mathcal{H}^1(z + \square_n)$ gives $\psi = 0$ and thus $\ker L = \{0\}$. In particular, if we denote by $\text{im } L$ the image of $L$, one has

$$\dim(\text{im } L) = \dim \left( \bigoplus_{z \in \mathcal{Z}_{n,2n}} \mathcal{H}^1(z + \square_n) \right) = 3^{2dn} - 3^{dn}.$$

We now wish to extend $L$ to $\mathcal{H}^1(\square_{2n})$. Recall that we have the orthogonal decomposition

$$\mathcal{H}^1(\square_{2n}) = \bigoplus_{z \in \mathcal{Z}_{n,2n}} \mathcal{H}^1(z + \square_n) \perp H$$

and consider an orthonormal basis $h_1, \ldots, h_{3^{dn}}$ of $H$. Consider now the $L^2$ orthogonal decomposition (4.16)

$$\mathcal{H}^1(\square_{2n}) = \text{im } L \perp (\text{im } L)\perp.$$

By the injectivity of $L$, we have $\dim(\text{im } L)\perp = 3^{dn}$. Let $\tilde{h}_1, \ldots, \tilde{h}_{3^{dn}}$ be an orthonormal basis of $(\text{im } L)\perp$. We extend $L$ to $\mathcal{H}^1(\square_{2n})$ by setting

$$L(h_i) = \tilde{h}_i, \forall i \in \{1, \ldots, 3^{dn}\}.$$}

The linear mapping $L$ is then an isomorphism between $\mathcal{H}^1(\square_{2n})$ and $\mathcal{H}^1(\square_{2n}^+)$. We now construct the random variable $\kappa_{2n}$ using the operator $L$. To this end consider two families $(\psi_z)_{z \in \mathcal{Z}_{n,2n}}$ and $(X_i)_{i = 1, \ldots, 3^{nd}}$ of random variables satisfying

- for each $z \in \mathcal{Z}_{n,2n}$, $\psi_z$ is valued in $\mathcal{H}^1(z + \square_n)$ and its law is $\mathcal{P}_n^{\text{Unif}[0,1]}$. We extend it by $0$ outside $z + \square_n$ so that it can be seen as a random variable taking values in $\mathcal{H}^1(\square_{2n})$,
- for each $i \in \{1, \ldots, 3^{nd}\}$, $X_i$ is valued in $[0,1]$ and its law is $\text{Unif}[0,1]$,
- the random variables $\psi_z$ and $X_i$ are independent.

We also define for each $z \in \mathcal{Z}_{n,2n}$ the random variable $\sigma_z$ taking values in $\mathcal{H}^1(\square_{2n})$ defined by substracting the affine function of slope $\nabla \nu^*(q)$ to $\psi_z$, i.e for each $x \in \square_{2n}$,

$$\sigma_z(x) := \begin{cases} 
\psi_z(x) - \nabla \nu^*(q) \cdot (x - z) & \text{if } x \in z + \square_n, \\
0 & \text{otherwise}.
\end{cases}$$

Let $\kappa$ and $\kappa_{2n}^+$ be the random variable valued in $\mathcal{H}^1(\square_{2n})$ defined by

$$\kappa := L \left( \sum_{z \in \mathcal{Z}_{n,2n}} \sigma_z \right) \quad \text{and} \quad \kappa_{2n}^+ := L \left( \sum_{z \in \mathcal{Z}_{n,2n}} \sigma_z + \sum_{i=1}^{3^{nd}} X_i h_i \right).$$
Step 2. Using the canonical bijection between \( h^1(\square_{2n}) \) and \( h_0^1(\square_{2n}^+) \), one can see \( L \) as an automorphism of \( h_0^1(\square_{2n}^+) \). Using this and the change of variable formula for the differential entropy, one has

\[
H(\mathbb{P}_{2n}^+) = H\left(\mathbb{P} \sum_{z \in \mathbb{Z}_{2n}} \sigma_z + \sum_{i=1}^{3^d} X_i h_i\right) - \ln |\det L|.
\]

We first focus on the first term on the right-hand side. By construction of \( \sigma_z, X_i, h_i \) and using the formula to compute the entropy of two independent random variables given in Proposition 2.4, one has

\[
H\left(\mathbb{P} \sum_{z \in \mathbb{Z}_{2n}} \sigma_z + \sum_{i=1}^{3^d} X_i h_i\right) = \sum_{z \in \mathbb{Z}_{2n}} H(\mathbb{P}_{\sigma_z}) + \sum_{i=1}^{3^d} H(\mathbb{P}_{X_i}).
\]

Using that the law of \( X_i \) is uniform on \([0, 1]\) and since the entropy of a random variable is translation invariant the previous display can be rewritten

\[
H\left(\mathbb{P} \sum_{z \in \mathbb{Z}_{2n}} \sigma_z + \sum_{i=1}^{3^d} X_i h_i\right) = \sum_{z \in \mathbb{Z}_{2n}} H(\mathbb{P}_{\psi_z}).
\]

Since the law of \( \psi_z \) is \( \mathbb{P}_{\sigma_n}^* \), one obtains from the previous display

\[
H\left(\mathbb{P}_{2n}^\ast\right) = 3^d H(\mathbb{P}_{\sigma_n}^*) - \ln |\det L|.
\]

We now focus on the second term on the right-hand side of (4.20). More precisely, we prove

\[
|\ln |\det L|| \leq C3^{(2d-1)n} n.
\]

Combining (4.21) with (4.22) gives the main result (4.10) of Step 2.

To prove (4.22), note that the dimension of the vector space \( h_0^1(\square_{2n}^+) \) is \( 3^{2dn} \). Denote by \( (l_1, \ldots, l_{3^{2dn}}) \) the (potentially complex) eigenvalues of \( L \). Note that since \( L \) is bijective, none of these eigenvalues is equal to 0. With this notation the previous display can be rewritten

\[
\ln |\det L| = \sum_{i=1}^{3^{2dn}} \ln |l_i|.
\]

To prove an estimate on the previous display, we first prove that most of the eigenvalues of \( L \) are actually equal to 1. To this end, recall the definition of the trimmed cube \( \square^-_n \),

\[
\square^-_n := \left(-\frac{3^n - 2}{2}, \frac{3^n - 2}{2}\right)^d \cap \mathbb{Z}^d = \square_n \setminus \partial \square_n.
\]

In particular, one has \( \square^-_n \subseteq \square_n \) and thus \( \hat{h}^1(\square^-_n) \) is a linear subspace of \( \hat{h}^1(\square_n) \). This implies that \( \bigoplus_{z \in \mathbb{Z}_{2n}} \hat{h}^1(z + \square^-_n) \) is a linear subspace of \( \bigoplus_{z \in \mathbb{Z}_{2n}} \hat{h}^1(z + \square_n) \). The important observation is that for each \( \psi \in \bigoplus_{z \in \mathbb{Z}_{2n}} \hat{h}^1(z + \square^-_n) \) and each edge \( e \in B_{2n,n} \),

\[
\nabla \psi(e) = 0,
\]

this is due to the fact that, by definition of \( \psi \), for each \( z \in \mathbb{Z}_{2n} \), \( \psi \) is equal to 0 on \( \partial (z + \square_n) \). Consequently the vector field \( \hat{f} \) defined from \( \psi \) according to

\[
f(e) = \begin{cases} \nabla \psi(e) & \text{if } e \subseteq z + \square_n, \text{ for some } z \in \mathbb{Z}_{2n}, \\ 0 & \text{if } e \in B_{2n,n} \end{cases}
\]

satisfy

\[
f = \nabla \psi.
\]
Thus $L(\psi)$ is the solution of
\[ \begin{cases} \Delta L(\psi) = \Delta \psi \text{ in } \square_{2n}, \\ L(\psi) \in h_0^1(\square_{2n}). \end{cases} \]
This implies $L(\psi) = \psi$. We have thus proved that for each $\psi \in \bigoplus_{z \in \mathbb{Z}_{n,2n}} \tilde{h}^1(z + \square^{-}_n)$, $L(\psi) = \psi$. Consequently, the vector space $\bigoplus_{z \in \mathbb{Z}_{n,2n}} \tilde{h}^1(z + \square^{-}_n)$ is an eigenspace for $L$ associated to the eigenvalue 1, its dimension can be estimated thanks to the following computation
\[
\dim \left( \bigoplus_{z \in \mathbb{Z}_{n,2n}} \tilde{h}^1(z + \square^{-}_n) \right) = \sum_{z \in \mathbb{Z}_{n,2n}} \dim \left( \tilde{h}^1(z + \square^{-}_n) \right) = 3^{dn} \dim \left( \tilde{h}^1(\square^{-}_n) \right) = 3^{dn} (|\square^{-}_n| - 1)
\]
The volume of $\square^{-}_n$ can then be estimated according to
\[ |\square^{-}_n| \geq 3^{dn} - C3^{(d-1)n}, \]
for some $C := C(d) < \infty$. Combining the two previous displays gives
\[ \text{(4.25)} \quad \dim \left( \bigoplus_{z \in \mathbb{Z}_{n,2n}} \tilde{h}^1(z + \square^{-}_n) \right) \geq 3^{2dn} - C3^{(2d-1)n}. \]
Thus we can, without loss of generality, assume that for each $i \geq C3^{(2d-1)n}, l_i = 1$. Using this, the equality (4.23) can be rewritten
\[ \ln |\det L| = \sum_{i=1}^{C3^{(2d-1)n}} \ln |l_i|. \]
We then use the inequalities
\[ |||L^{-1}||| \leq \inf_{i \in \{1, \ldots, 3^{2dn}\}} |l_i| \leq \sup_{i \in \{1, \ldots, 3^{2dn}\}} |l_i| \leq |||L|||, \]
where $|||L|||$ (resp. $|||L^{-1}|||$) denotes the operator norm of $L$ (resp. $L^{-1}$) with respect to the $L^2$ norm on $h_0^1(\square_{2n}^+)$. A combination of the two previous display gives
\[ \text{(4.26)} \quad |\ln |\det L|| \leq C3^{(2d-1)n} \max (|\ln |||L|||, |\ln |||L^{-1}|||)). \]
To complete the proof, there remains to prove an estimate on the operator norms of $L$ and $L^{-1}$. Specifically, we are going to prove, for some $C := C(d) < \infty$,
\[ \text{(4.27)} \quad |||L||| \leq C3^{2n} \quad \text{and} \quad |||L^{-1}||| \leq C3^{2n}. \]
We first focus on the first estimate. Let $\phi \in h_0^1(\square_{2n}^+)$ such that $\sum_{x \in \square_{2n}^+} \phi(x)^2 \leq 1$, we want to prove
\[ \text{(4.28)} \quad \sum_{x \in \square_{2n}^+} |L(\phi)(x)|^2 \leq C3^{4n}. \]
To this end, we first decompose $\phi$ according to the orthogonal decomposition (4.14), this gives
\[ \phi = \psi + h, \text{ with } \psi \in \bigoplus_{z \in \mathbb{Z}_{n,2n}} \tilde{h}^1(z + \square_n) \text{ and } h \in H. \]
In particular,
\[ \sum_{x \in \square_{2n}^+} |\psi(x)|^2 + \sum_{x \in \square_{2n}^+} |h(x)|^2 = \sum_{x \in \square_{2n}^+} |\phi(x)|^2 \leq 1. \]
By definition of $L$, $L(\psi)$ and $L(h)$ are orthogonal in $h^1_0(\square^+_{2n})$ and
\[
\sum_{x \in \square^+_{2n}} |L(h)(x)|^2 = \sum_{x \in \square^+_{2n}} |h(x)|^2
\]
From this we deduce
\[
\sum_{x \in \square^+_{2n}} |L(\psi)(x)|^2 = \sum_{x \in \square^+_{2n}} |L(\psi)(x)|^2 + \sum_{x \in \square^+_{2n}} |L(h)(x)|^2 = \sum_{x \in \square^+_{2n}} |L(\psi)(x)|^2 + \sum_{x \in \square^+_{2n}} |h(x)|^2 \\
\leq \sum_{x \in \square^+_{2n}} |L(\psi)(x)|^2 + 1.
\]
Thus to prove (4.28), it is sufficient to prove for each $\psi \in \oplus \hat{h}^1(z + \square_{n})$ satisfying
\[
\sum_{x \in \square^+_{2n}} |\psi(x)|^2 \leq 1,
\]
\[
\sum_{x \in \square^+_{2n}} |L(\psi)(x)|^2 \leq C3^{4n}.
\]
For each $\psi \in \oplus \hat{h}^1(z + \square_{n})$, we know that $L(\psi)$ is a solution to
\[
(4.29) \quad \begin{cases}
\Delta L(\psi) = \text{div } f \text{ in } \square_{2n}, \\
L(\psi) \in h^1_0(\square^+_{2n}).
\end{cases}
\]
where $f$ is defined by
\[
f(e) = \begin{cases}
\nabla \psi(e) \text{ if } e \subseteq z + \square_{n}, \text{ for some } z \in \mathbb{Z}_{n,2n}, \\
0 \text{ if } e \in B_{2n,n}.
\end{cases}
\]
Consequently testing $L(\psi)$ against itself in (4.29) shows
\[
\sum_{e \subseteq \square^+_{2n}} |\nabla L(\psi)(e)|^2 = \sum_{e \subseteq \square^+_{2n}} \nabla L(\psi)(e)f(e).
\]
By the Cauchy-Schwarz inequality, this implies
\[
\sum_{e \subseteq \square^+_{2n}} |\nabla L(\psi)(e)|^2 \leq \sum_{e \subseteq \square^+_{2n}} |f(e)|^2
\]
and by definition of $f$, one obtains
\[
(4.30) \quad \sum_{e \subseteq \square^+_{2n}} |\nabla L(\psi)(e)|^2 \leq \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla \psi(e)|^2.
\]
Using the crude inequality, for each $e = (x, y) \subseteq \square_{2n}$
\[
|\nabla \psi(e)|^2 = |\psi(x) - \psi(y)|^2 \leq 2|\psi(x)|^2 + 2|\psi(y)|^2,
\]
we obtain the estimate, for some $C := C(d) < \infty$,
\[
\sum_{z \in \mathbb{Z}^n \cap \square_{2n}} \sum_{e \subseteq z + \square_{n}} |\nabla \psi(e)|^2 \leq C \sum_{x \in \square_{2n}} |\psi(x)|^2.
\]
Combining the previous displays and using the assumption $\sum_{x \in \square_{2n}} |\psi(x)|^2 \leq 1$ shows
\[
\sum_{e \subseteq \square^+_{2n}} |\nabla L(\psi)(e)|^2 \leq C \sum_{x \in \square_{2n}} |\psi(x)|^2 \leq C.
\]
Also, since \( L(\psi) \in h_0^1(\mathbb{Z}^d) \), one has by the Poincaré inequality
\[
\sum_{x \in \mathbb{Z}^d} |L(\psi)(x)|^2 \leq 3^{4n} \sum_{e \subseteq \mathbb{Z}^d} |\nabla L(\psi)(e)|^2.
\]
Combining the two previous displays gives
\[
\sum_{x \in \mathbb{Z}^d} |L(\psi)(x)|^2 \leq C3^{4n}.
\]
This is the desired result. We now turn to the bound on the operator norm of \( L^{-1} \), we aim to prove
\[
\|\|L^{-1}\|| \leq C3^{2n}.
\]
To this end, let \( \psi \in h_0^1(\mathbb{Z}^d) \), we will prove
\[
(4.31) \quad \sum_{x \in \mathbb{Z}^d} |\psi(x)|^2 \leq C3^{4n} \sum_{x \in \mathbb{Z}^d} |L(\psi)(x)|^2.
\]
First, using the same idea as in the proof of the bound for the operator norm of \( L \), we see that it is enough to prove (4.31) with the additional assumption \( \psi \in \bigoplus_{z \in \mathbb{Z}^d} h^1(z + \mathbb{Z}_{2n}) \). In this case, one has
\[
\begin{cases}
\Delta L(\psi) = \text{div } f \text{ in } \mathbb{Z}^d, \\
L(\psi) \in h_0^1(\mathbb{Z}^d).
\end{cases}
\]
where \( f \) is defined by
\[
f = \begin{cases}
\nabla \psi(e) & \text{if } e \subseteq z + \mathbb{Z}_{2n}, \text{ for some } z \in 3^n \mathbb{Z}^d \cap \mathbb{Z}_{2n}, \\
0 & \text{if } e \in B_{2n,n}.
\end{cases}
\]
Testing this equation against \( \psi \) gives
\[
\sum_{x \in \mathbb{Z}^d} \Delta L(\psi)(x)\psi(x) = \sum_{x \in \mathbb{Z}^d} \text{div } f(x)\psi(x) = \sum_{e \subseteq \mathbb{Z}^d} f(e)\nabla \psi(e).
\]
We then use the definition of \( \psi \) to get
\[
\sum_{e \subseteq \mathbb{Z}^d} f(e)\nabla \psi(e) = \sum_{z \in \mathbb{Z}^d} \sum_{e \subseteq z + \mathbb{Z}_{2n}} |\nabla \psi(e)|^2.
\]
Since \( \psi \) belongs to \( \bigoplus_{z \in \mathbb{Z}^d} h^1(z + \mathbb{Z}_{2n}) \), it has mean 0 on each of the subcubes \( z + \mathbb{Z}_{2n} \). We can thus apply the Poincaré inequality on each of the subcubes \( z + \mathbb{Z}_{2n} \) to get, for some \( C := C(d) < \infty \),
\[
\sum_{x \in z + \mathbb{Z}_{2n}} |\psi(x)|^2 \leq C3^{2n} \sum_{e \subseteq z + \mathbb{Z}_{2n}} |\nabla \psi(e)|^2, \quad \forall z \in \mathbb{Z}^d.
\]
Summing the previous inequality over each \( z \in \mathbb{Z}^d \) gives
\[
\sum_{x \in \mathbb{Z}^d} |\psi(x)|^2 \leq C3^{2n} \sum_{z \in \mathbb{Z}^d} \sum_{e \subseteq z + \mathbb{Z}_{2n}} |\nabla \psi(e)|^2.
\]
Combining the few previous displays gives
\[
\sum_{x \in \mathbb{Z}^d} |\psi(x)|^2 \leq C3^{2n} \sum_{x \in \mathbb{Z}^d} \Delta L(\psi)(x)\psi(x).
\]
By the Cauchy-Schwarz inequality, we further obtain
\[
\sum_{x \in \mathbb{Z}^d} |\psi(x)|^2 \leq C3^{4n} \sum_{x \in \mathbb{Z}^d} |\Delta L(\psi)(x)|^2.
\]
But by definition of the Laplacian, we have, for each \( x \in \Box_{2n} \)
\[
|\Delta L(\psi)(x)|^2 = \left| \sum_{y \sim x} (\psi(y) - \psi(x)) \right|^2 \leq C \sum_{y \sim x} |\psi(y)|^2 + C|\psi(x)|^2.
\]
From this obtain, for some \( C := C(d) < \infty \),
\[
\sum_{x \in \Box_{2n}} |\Delta L(\psi)(x)|^2 \leq C \sum_{x \in \Box_{2n}} |L(\psi)(x)|^2
\]
and consequently
\[
\sum_{x \in \Box_{2n}} |\psi(x)|^2 \leq C3^{ln} \sum_{x \in \Box_{2n}} |L(\psi)(x)|^2.
\]
This is (4.31).

We now complete the proof of of the bound \(| \ln |\det L||\) stated in (4.22). Indeed combining (4.26) and (4.27) gives
\[
|\ln |\det L|| \leq C3^{(2d-1)n} \max (|\ln |\|L|||, |\ln |\|L^{-1}|||)
\leq C3^{(2d-1)n} \ln (C3^{2n})
\leq C3^{(2d-1)n} n.
\]
The proof of Step 2 is complete.

**Step 3.** The goal of this step is to show the following estimate
(4.32) \[
E \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}^+} |f(e) - \nabla \kappa_{2n}^+(e)|^2 \right] \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m(q) \right),
\]
where we recall that \( f \) is the random vector field defined in (4.12). To achieve this, we proceed as follows
- We first remove the extra random variable \( L \left( \sum_{i=1}^{3^{nd}} X_i h_i \right) \). Precisely we are going to prove
\[
E \left[ \frac{1}{|\Box_{2n}|} \sum_{x \subseteq \Box_{2n}^+} |\kappa_{2n}^+(x) - \kappa(x)|^2 \right] \leq 3^{-dn}.
\]
- Then we construct a random function \( \Psi \) taking values in \( h_0^\ast (\Box_{2n}^+) \) such that
(4.33) \[
E \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}^+} |f(e) - \nabla \Psi(e)|^2 \right] \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m(q) \right).
\]
- We deduce from (4.33) that
\[
E \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}^+} |f(e) - \nabla \kappa(e)|^2 \right] \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m(q) \right).
\]
We first prove that we can remove the extra random variable \( L \left( \sum_{i=1}^{3^{nd}} X_i h_i \right) \) which was added to \( \kappa \) to obtain \( \kappa_{2n}^+ \). These random variables were added so that the law of \( \kappa_{2n}^+ \) was absolutely continuous with respect to the Lebesgue measure on \( h_0^\ast (\Box_{2n}^+) \), so that we would not obtain an infinite entropy.
They were also chosen in a way that their role in the energy is negligible. More precisely we will prove the following statement

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{x \in \square_{2n}} \left| \kappa_{2n}^+ (x) - \kappa(x) \right|^2 \right] \lesssim 3^{-dn}.
\]

We first recall the definition of \( L \) on the subvector space \( H \) given in (4.17). The previous estimate is then a consequence of the following computation

\[
\begin{align*}
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{x \in \square_{2n}} \left| \kappa_{2n}^+ (x) - \kappa(x) \right|^2 \right] &= \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{x \in \square_{2n}} \left( \sum_{i=1}^{2^nd} X_i \bar{h}_i(x) \right)^2 \right] \\
&= \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{i=1}^{2^nd} |X_i|^2 \right],
\end{align*}
\]

Since the family \( \bar{h}_i \), for \( i \in \{1, \ldots, 2^nd\} \) is orthonormal with respect to the standard \( L^2 \) scalar product in \( h_0^\prime (\square_{2n}^+ \cdot) \). Since the random variables \( (X_i)_{i \in \{1, \ldots, 2^nd\}} \) are i.i.d of law uniform in \([0,1]\), we can complete the computation

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{x \in \square_{2n}} \left| \kappa_{2n}^+ (x) - \kappa(x) \right|^2 \right] = \frac{3^{nd}}{3 |\square_{2n}|} \lesssim C \exp^{-dn}.
\]

Using this and the inequality, for each \( e = (x,y) \subseteq \square_{2n} \)

\[
|\nabla (\kappa_{2n}^+ - \kappa) (e)|^2 = |(\kappa_{2n}^+ - \kappa) (x) - (\kappa_{2n}^+ - \kappa) (y)|^2 \lesssim 2 |(\kappa_{2n}^+ - \kappa) (x)|^2 + 2 |(\kappa_{2n}^+ - \kappa) (y)|^2,
\]

we obtain

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}} |\nabla \kappa_{2n}^+(e) - \nabla \kappa(e)|^2 \right] \lesssim C \exp^{-dn}.
\]

We now prove (4.33) and construct the random variable \( \Psi \). We recall the definition of the family \( (\psi_z) \), for \( z \in Z_{n,2n} \), and extend it to each \( z \in 3^n \mathbb{Z}^d \) according to

(i) for each \( z \in 3^n \mathbb{Z}^d \), \( \psi_z \) is a function form \( Z^d \) to \( \mathbb{R} \) equals to 0 outside \( z + \square_n \) and the law of \( \psi_z (\cdot - z) \) restricted to \( \square_n \) is \( P_{n,q}^* \),

(ii) the random variables \( \psi_z \) are independent.

It is the same family as in Step 1, except that it was extended for each \( z \in 3^n \mathbb{Z}^d \) and not only for \( z \in Z_{n,2n} \). The reason behind this extension will become clear later in the proof.

Then for each \( z \in Z_{n,2n} \), we let \( \psi_{z,n+1} \) be a random variable such that

\[
\psi_{z,n+1} \text{ is valued in } h^1 (z + \square_{n+1}) \text{ and the law of } \psi_{z,n+1} (\cdot - z) \text{ is } P_{n+1,q}^*.
\]

As usual we extend this function by 0 outside \( z + \square_{n+1} \) so that one can see \( \psi_{z,n+1} \) as a random function from \( \mathbb{Z}^d \) to \( \mathbb{R} \).

The goal of the following argument is to construct a suitable coupling between the random variables \( \psi_{z,n+1} \), for \( z \in Z_{n,2n} \).

For some fixed \( z \in Z_{n,2n} \), we apply Proposition 3.16 and Proposition 2.12, with the random variables \( X = \sum_{z' \in 3^n \mathbb{Z}^d \cap (z + \square_{n+1})} \psi_{z'} \), \( Y = \sum_{z' \in 3^n \mathbb{Z}^d \setminus (z + \square_{n+1})} \psi_{z'} \) and \( Z = \psi_{z,n+1} \), we obtain that
there exists a coupling between the random variables \( \psi \) and \( \psi_{z,n+1} \) such that

\[
E \left[ \frac{1}{|n|} \sum_{z' \in 3^n \mathbb{Z}^d \cap (n+\Box_{n+1})} \sum_{e \subseteq z'+\Box_n} |\nabla \psi_{z,n+1}(e) - \nabla \psi_{z'}(e)|^2 \right] \leq C r_n^*(q) + C(1 + |q|^2)3^{-n}.
\]

This is where we used that \( \psi_{z'} \) is defined for some \( z' \) outside \( \Box_{2n} \). Indeed for some \( z \in 3^n \mathbb{Z}^d \cap \Box_{2n} \), close to the boundary of \( \Box_{2n} \), the set \( 3^n \mathbb{Z}^d \cap (z + \Box_{n+1}) \) is not included in \( \Box_{2n} \).

Thanks to the previous argument, we have constructed, for each \( z \in \mathbb{Z}_{n,2n} \), a coupling between \( \psi \) and \( \psi_{z,n+1} \). Let \((z_1, \ldots, z_{3^n})\) be an enumeration of the elements of \( \mathbb{Z}_{n,2n} \).

Applying Proposition 2.12 with \( X = \psi \), \( Y = \psi_{z_1,n+1} \) and \( Z = \psi_{z_2,n+1} \) constructs a coupling between \( \psi \), \( \psi_{z_1,n+1} \) and \( \psi_{z_2,n+1} \) such that (4.36) is satisfied.

We then apply Proposition 2.12 a second time, with this time the random variables \( X = \psi \), \( Y = (\psi_{z_1,n+1}, \psi_{z_2,n+1}) \) and \( Z = \psi_{z_3,n+1} \) constructs a coupling between \( \psi \), \( \psi_{z_1,n+1} \) and \( \psi_{z_2,n+1} \) and \( \psi_{z_3,n+1} \) such that (4.36) is satisfied.

Iterating this construction \( 3^n \) times construct a coupling between the random variables \( \psi \) and \( \psi_{z,n+1} \), for \( z \in \mathbb{Z}_{n,2n} \), such that (4.36) is satisfied.

From the previous construction, we obtain a coupling between the random variables \( \psi_{z,n+1} \), for \( z \in \mathbb{Z}_{n,2n} \), which is what we wanted to construct. Moreover this coupling satisfies (4.36), which will be a key ingredient later in the proof.

We now construct the function \( \Psi \) by patching together the random variables \( \psi_{z,n+1} \). To do so, we build the following partition of unity. Let \( \chi_0 \in h^1_0(\Box_n) \) be a function satisfying

\[
0 \leq \chi_0 \leq C 3^{-dn}, \quad \sum_{x \in \mathbb{Z}^d} \chi_0(x) = 1, \quad |\nabla \chi_0| \leq C 3^{-(d+1)n}, \quad \text{supp} \, \chi_0 \subseteq \frac{1}{2} \Box_n.
\]

We then define, for each \( z \in \mathbb{Z}^d \)

\[
\chi(y) := \sum_{x \in \Box_n} \chi_0(y - x).
\]

Note that \( \chi \) is supported in \( \frac{3}{4} \Box_{n+1} \), satisfies \( 0 \leq \chi \leq 1 \) and the translates of \( \chi \) form a partition of unity:

\[
\sum_{z \in 3^n \mathbb{Z}^d} \chi(\cdot - z) = 1,
\]

and we have the bound on the gradient of \( \chi \)

\[
|\nabla \chi| \leq C 3^{-n}.
\]

We next consider the cutoff function \( \zeta \in h^1_0(\Box_{2n}) \) satisfying

\[
0 \leq \zeta \leq 1, \quad \zeta = 1 \text{ on } \{ x \in \Box_{2n} : \text{dist}(x, \partial \Box_{2n}) \geq 3^n \}, \quad |\nabla \zeta| \leq C 3^{-n}.
\]

We also define the following discrete set

\[
\mathbb{Z}_{n,2n} := \left\{ z \in 3^n \mathbb{Z}^d : z \in \mathbb{Z}_{n,2n} \text{ or dist}(z, \partial \Box_{2n}) \leq 3^n \right\},
\]

it represents the set \( \mathbb{Z}_{n,2n} \) with an extra boundary layer of size 1 of points in \( 3^n \mathbb{Z}^d \) around it. We are going to use this set because it satisfies the following property

\[
\forall y \in \Box_{2n}^+, \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) = 1
\]

We then define the function \( \Psi \) by

\[
\Psi(x) = \zeta(x) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(x - z) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot (x - z) \right).
\]
Now that $\Psi$ has been constructed, we prove \((4.33)\). The main ingredients to prove this estimate are \((4.36)\), Proposition 4.3 and also the interior Meyers estimate, Proposition B.5 stated in Appendix B, Proposition B.5.

We first compute the derivative of $\Psi$. An explicit computation gives, for each $e = (x, y) \subset \Box_{2n}$,

\[
(4.39) \quad \nabla \Psi(e) = \zeta(y) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right)
+ \zeta(y) \sum_{z \in \mathbb{Z}_{n,2n}^+} \nabla \chi(y - z)(e) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x \right)
+ \nabla \zeta(e) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(x - z) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x \right)
\]

The $L^2$ norm second term can be estimated thanks to the bound \((4.37)\) on the gradient of $\chi$,

\[
E \left[ \frac{1}{|2n|} \sum_{e = (x, y) \subset \Box_{2n}^+} \zeta(y) \sum_{z \in \mathbb{Z}_{n,2n}^+} \nabla \chi(y - z)(e) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x \right) \right]^2 
\leq C3^{-2n}E \left[ \frac{1}{|2n|} \sum_{z \in \mathbb{Z}_{n,2n}^+} \sum_{x \in \mathbb{Z}^{+n+1}} |\psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x|^2 \right] 
\leq C3^{-2n}E \left[ \frac{1}{|2n|} \sum_{x \in \Box_{n+1}} |\psi_{n+1,q}(x) - \nabla \nu_{n+1}^*(q) \cdot x|^2 \right].
\]

We then apply Proposition 4.3 to obtain

\[
E \left[ \frac{1}{|2n|} \sum_{e = (x, y) \subset \Box_{2n}^+} \zeta(y) \sum_{z \in \mathbb{Z}_{n,2n}^+} \nabla \chi(y - z)(e) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x \right) \right]^2 
\leq C \left( 1 + |q|^2 \right)^{3 - \frac{q}{2}} + \sum_{m=0}^{n} 3^{(m-1)2} \tau_{m}(q).
\]

The third term of \((4.39)\) can be estimated in a similar manner, using \((4.38)\) this time,

\[
E \left[ \frac{1}{|2n|} \sum_{e = (x, y) \subset \Box_{2n}^+} \nabla \zeta(e) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(x - z) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x \right) \right]^2 
\leq C3^{-2n}E \left[ \frac{1}{|2n|} \sum_{z \in \mathbb{Z}_{n,2n}^+} \sum_{x \in \mathbb{Z}^{+n+1}} |\psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x|^2 \right] 
\leq C3^{-2n}E \left[ \frac{1}{|2n|} \sum_{x \in \Box_{n+1}} |\psi_{n+1,q}(x) - \nabla \nu_{n+1}^*(q) \cdot x|^2 \right].
\]
We then apply Proposition 4.3 again to obtain
\[
\mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} \nabla \zeta(e) \sum_{z \in \mathbb{Z}_{n+2}^+} \chi(x-z) \left( \psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot x \right) \right]^2 \leq C \left( 1 + |q|^2 \right)^{3^{-\frac{n}{2}}} + \sum_{m=0}^{n} 3^{(m-n)} \tau_m^*(q). \]

Combining the few previous displays the yields
\[
\mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} \nabla \Psi(e) - \zeta(y) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 \leq C \left( 1 + |q|^2 \right)^{3^{-\frac{n}{2}}} + \sum_{m=0}^{n} 3^{(m-n)} \tau_m^*(q). \]

Thus to prove (4.33), it is sufficient to prove
\[
(4.40) \quad \mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} \left| f(e) - \zeta(y) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right|^2 \right] \leq C \tau_n^*(q) + C \left( 1 + |q|^2 \right)^{3^{-\beta n}},
\]
for some small exponent \( \beta := \beta(d, \lambda) > 0 \). We simplify the previous display by removing the function \( \zeta \). Note that if we denote \( \partial \mathbb{Z}_{n,2n} := \{ z \in \mathbb{Z}_{n+2}^+ : \operatorname{dist}(z, \partial \square_{2n}) \leq 2 \cdot 3^n \} \), we have the following computation, using the properties of the functions \( \zeta \) and \( \chi \),
\[
\mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} (1 - \zeta(y)) \sum_{z \in \mathbb{Z}_{n+2}^+} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 \leq \mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} \sum_{z \in \partial \mathbb{Z}_{n,2n}} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 \leq \mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} \sum_{z \in \partial \mathbb{Z}_{n,2n}} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right)^2 \right].
\]

Using that, by definition, the function \( \chi \) is supported in \( \frac{3}{2} \square_{n+1} \), we can simplify the previous display and obtain
\[
\mathbb{E} \left[ \frac{1}{|2^n|} \sum_{e=(x,y) \subseteq \square_{2^n}} (1 - \zeta(y)) \sum_{z \in \mathbb{Z}_{n+2}^+} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 \leq \mathbb{E} \left[ \frac{1}{|2^n|} \sum_{z \in \partial \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n+1}} \left| \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right|^2 \right].
\]
Using that all the $\psi_{z,n+1}$ have the same law, which is $\mathbb{P}^x_{n+1}$, we have

$$
E \left[ \frac{1}{\square_{2n}^+} \sum_{e=(x,y) \subseteq \square_{2n}^+} \left( 1 - \zeta(y) \right) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 
\leq \frac{\partial \mathbb{Z}_{n,2n}}{\square_{2n}^+} E \left[ \sum_{e \subseteq \square_{n+1}} \left| \nabla \psi_{n+1,q}(e) - \nabla \nu_{n+1}^*(q)(e) \right|^2 \right].
$$

But by Proposition 3.13, the term on the right-hand side is bounded and we have

$$
E \left[ \frac{1}{\square_{2n}^+} \sum_{e=(x,y) \subseteq \square_{2n}^+} \left( 1 - \zeta(y) \right) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 
\leq \frac{\partial \mathbb{Z}_{n,2n}}{\square_{2n}^+} \cdot |\square_{n+1}| C(1 + |q|^2).
$$

But one has the estimates

$$|\partial \mathbb{Z}_{n,2n}| \leq C3^{(d-1)n} \quad \text{and} \quad |\square_{n+1}| = 3^{d(n+1)}.$$

Consequently, one has

$$
E \left[ \frac{1}{\square_{2n}^+} \sum_{e=(x,y) \subseteq \square_{2n}^+} \left( 1 - \zeta(y) \right) \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right]^2 
\leq C3^{-n}(1 + |q|^2).
$$

By the previous display and (4.40), it is enough to prove (4.33) to show

$$
(4.41) \quad E \left[ \frac{1}{\square_{2n}^+} \sum_{e=(x,y) \subseteq \square_{2n}^+} \left| f(e) - \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right|^2 \right] 
\leq C\tau_n^*(q) + C(1 + |q|^2)3^{-\beta n},
$$

for some small exponent $\beta := \beta(d, \lambda) > 0$. We now prove this estimate, using that $\chi$ is a partition of unity, we rewrite

$$
E \left[ \frac{1}{\square_{2n}^+} \sum_{e=(x,y) \subseteq \square_{2n}^+} \left| f(e) - \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right|^2 \right] 
= E \left[ \frac{1}{\square_{2n}^+} \sum_{e=(x,y) \subseteq \square_{2n}^+} \sum_{z \in \mathbb{Z}_{n,2n}^+} \chi(y - z) \left( f(e) - \nabla \psi_{z,n+1}(e) + \nabla \nu_{n+1}^*(q)(e) \right) \right]^2.
$$
Using that the function $\chi$ is supported in $\frac{3}{4}B_{n+1}$, one obtains

$$
E \left[ \frac{1}{|\Box_{2n}|} \sum_{e=(x,y) \subseteq \Box_{2n}^+} \left| \sum_{z \in \mathbb{Z}_{n,2n}} \chi(y-z) \left( f(e) - \nabla \psi_{z,n+1}(e) + \nabla \nu_{n+1}^*(q)(e) \right) \right|^2 \right]
$$

\[ \leq E \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq (z+\frac{1}{4}B_{n+1}) \cap \Box_{2n}} \left| f(e) - \nabla \psi_{z,n+1}(e) + \nabla \nu_{n+1}^*(q)(e) \right|^2 \right]. \]

Using the definition of $f$ given in (4.24), and splitting the sum according to the partition of edges,

$$
e \subseteq \Box_{2n} \implies e \in B_{2n,n} \quad \text{or} \quad \exists z \in \mathbb{Z}_{n,2n}, \ e \subseteq z + \Box_n,$$

we obtain

$$
(4.42) \quad E \left[ \frac{1}{|\Box_{2n}|} \sum_{e=(x,y) \subseteq \Box_{2n}^+} \left| f(e) - \sum_{z \in \mathbb{Z}_{n,2n}} \chi(y-z) \left( \nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q)(e) \right) \right|^2 \right]
$$

\[ \leq E \left[ \frac{1}{|\Box_{2n}|} \sum_{z,z' \in \mathbb{Z}_{n,2n}, z \in z' + \Box_{n+1}} \sum_{e \subseteq z' + \Box_n} \left| \nabla \psi_{z'}(e) - \nabla \psi_{z,n+1}(e) \right|^2 \right]
$$

\[ + E \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \in B_{2n,n} \cap (z+\frac{1}{4}B_{n+1})} \left| \nabla \psi_{z,n+1}(e) \right|^2 \right]
$$

\[ + C \left| \nabla \nu_{n+1}^*(q) - \nabla \nu_n^*(q) \right|^2. \]

The first term on the right-hand side is estimated thanks to (4.36). this gives

$$
(4.43) \quad E \left[ \frac{1}{|\Box_{2n}|} \sum_{z,z' \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z' + \Box_n} \left| \nabla \psi_{z'}(e) - \nabla \psi_{z,n+1}(e) \right|^2 \right]
$$

\[ = \frac{1}{|\Box_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} E \left[ \sum_{z' \in \mathbb{Z}_{2n} \cap (z+\Box_{n+1})} \sum_{e \subseteq z' + \Box_n} \left| \nabla \psi_{z,n+1}(e) - \nabla \psi_{z'}(e) \right|^2 \right]
$$

\[ \leq C \tau_n^*(q) + C(1 + |q|^2)3^{-n}. \]

The third term can be estimated thanks to (4.4),

$$
\left| \nabla \nu_{n+1}^*(q) - \nabla \nu_n^*(q) \right|^2 \leq C \tau_n^*(q) + C(1 + |q|^2)3^{-n}. \]
To estimate the second term on the right-hand side of (4.42), we first use that all the \( \psi_{z,n+1} \) have the same law, which is \( \mathbb{P}^*_{n+1,q} \). This gives

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_n} \sum_{e \in B_{2n,n} \cap (z + \frac{1}{2} \square_{n+1})} |\nabla \psi_{z,n+1}(e)|^2 \right]
\]

\[
= \frac{1}{|\square_{n+1}|} \sum_{e \in B_{2n,n} \cap \frac{3}{4} \square_{n+1}} \mathbb{E} \left[ \sum_{e \in B_{2n,n} \cap \frac{3}{4} \square_{n+1}} |\nabla \psi_{n+1,q}(e)|^2 \right]
\]

\[
\leq C \mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_{2n,n} \cap \frac{3}{4} \square_{n+1}} |\nabla \psi_{n+1,q}(e)|^2 \right].
\]

We then estimate this term thanks to the Meyers estimate, Proposition B.5 with \( \alpha = \frac{3}{4} \). We denote by \( \delta \) the exponent of Proposition B.5 and compute

\[
\mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_{2n,n} \cap \frac{3}{4} \square_{n+1}} |\nabla \psi_{n+1,q}(e)|^2 \right] \leq C \mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_{2n,n} \cap \frac{3}{4} \square_{n+1}} |\nabla \psi_{n+1,q}(e)|^2 \right]^{\frac{3}{3+\delta}} (1 + |q|^2).
\]

We then use that

\[
\frac{|B_{2n,n} \cap \frac{3}{4} \square_{n+1}|}{|\frac{3}{4} \square_{n+1}|} \leq C 3^{-n},
\]

we obtain

\[
\mathbb{E} \left[ \frac{1}{|\square_{n+1}|} \sum_{e \in B_{2n,n} \cap \frac{3}{4} \square_{n+1}} |\nabla \psi_{n+1,q}(e)|^2 \right] \leq C 3^{-\frac{4}{3+\delta}} (1 + |q|^2).
\]

Combining the few previous displays gives the following estimate for the second term on the right-hand side of (4.42)

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_n} \sum_{e \in B_{2n,n} \cap (z + \frac{1}{2} \square_{n+1})} |\nabla \psi_{z,n+1}(e)|^2 \right] \leq C 3^{-\frac{4}{3+\delta}} (1 + |q|^2).
\]
Combining (4.42) with (4.43), the previous displays and setting $\beta := \frac{\delta}{1 + \delta}$ yields

$$
E \left[ \frac{1}{|D|} \sum_{e=(x,y) \subseteq D} \left| f(e) - \sum_{z \in \mathbb{Z}^d} \chi(y - z) (\nabla \psi_{z,n+1}(e) - \nabla \nu_{n+1}^*(q) \cdot e) \right|^2 \right] \leq C \tau_n^*(q) + C(1 + |q|^2) 3^{-\beta n}.
$$

This is precisely (4.41). The proof of (4.33) is complete.

We now deduce from (4.33) that

$$
E \left[ \frac{1}{|D|} \sum_{e \subseteq D} |f(e) - \nabla \kappa(e)|^2 \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^n 3^{(m-n)} r_m^*(q) \right).
$$

We recall that $\kappa$ was defined as the solution of the problem

$$
\Delta \kappa = \text{div } f \text{ in } D,
\kappa \in h^1_0(D).
$$

This implies the almost sure inequality

$$
\sum_{e \subseteq D} |f(e) - \nabla \kappa(e)|^2 = \inf_{\kappa' \in h^1_0(D)} \sum_{e \subseteq D} |f(e) - \nabla \kappa'(e)|^2 \leq \sum_{e \subseteq D} |f(e) - \nabla \Psi(e)|^2.
$$

Taking the expectation and using (4.33) gives

$$
E \left[ \frac{1}{|D|} \sum_{e \subseteq D} |f(e) - \nabla \kappa(e)|^2 \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^n 3^{(m-n)} r_m^*(q) \right).
$$

Combining the previous display with (4.34) proves the estimate

$$
E \left[ \frac{1}{|D|} \sum_{e \subseteq D} |f(e) - \nabla \kappa^+_{2n}(e)|^2 \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^n 3^{(m-n)} r_m^*(q) \right).
$$

**Step 4.** The goal of this step is to use the main result (4.45) of Step 3 to prove

$$
E \left[ \frac{1}{|D|} \sum_{e \subseteq D} V_e \left( \nabla \nu_n^*(q)(e) + \nabla \kappa^+_{2n}(e) \right) \right] \leq E \left[ \frac{1}{|D|} \sum_{e \subseteq D} V_e \left( \nabla \psi_{n,q}(e) \right) \right] + C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^n 3^{(m-n)} r_m^*(q) \right).
$$

We recall the definition of the random variable $\sigma_z(x) := \psi_z(x) - \nabla \nu_n^*(q) \cdot (x - z)$ introduced in (4.18). The proof relies on the following technical estimate, the proof of which is postponed to
Appendix A, Lemma A.2.

\begin{equation}
\frac{1}{|\mathbb{Z}_n|} \sum_{z \in \mathbb{Z}_n} \sum_{e \leq z + \square_n} V_e (\nabla \nu^*_n(q)(e) + \nabla \kappa(e)) \leq \mathbb{E} \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{z \in \mathbb{Z}_n} \sum_{e \leq z + \square_n} V_e (\nabla \psi_z(e)) \right] + C(1 + |q|^2)^{3/2} + CE \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{z \in \mathbb{Z}_n} \sum_{e \leq z + \square_n} |\nabla \kappa(e) - \nabla \sigma_z(e)|^2 \right].
\end{equation}

We now show how to deduce (4.46) from the previous estimate. First, since all the \( \psi_z \) have the same law which is \( \mathbb{P}_{n,q} \), we can simplify the first term on the right-hand side,

\begin{equation}
\frac{1}{|\mathbb{Z}_n|} \sum_{z \in \mathbb{Z}_n} \sum_{e \leq z + \square_n} V_e (\nabla \psi_z(e)) = \mathbb{E} \left[ \frac{|\mathbb{Z}_n| - 2^n}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} V_e (\nabla \psi_{n,q}(e)) \right] = \mathbb{E} \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} V_e (\nabla \psi_{n,q}(e)) \right].
\end{equation}

We now estimate the last term on the right-hand side of (4.47). One has

\begin{equation}
\frac{1}{|\mathbb{Z}_n|} \sum_{z \in \mathbb{Z}_n} \sum_{e \leq z + \square_n} |\nabla \sigma_z(e) - \nabla \kappa(e)|^2 \leq \mathbb{E} \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} |f(e) - \nabla \kappa(e)|^2 \right] + \left( \frac{|\mathbb{Z}_n| - |\mathbb{Z}_n|}{|\mathbb{Z}_n| \cdot |\mathbb{Z}_n|} \right) \mathbb{E} \left[ \sum_{e \leq \square_n} |f(e) - \nabla \kappa(e)|^2 \right].
\end{equation}

We first estimate the second term on the right-hand side of the previous display. Note that

\begin{equation}
\frac{|\mathbb{Z}_n| - |\mathbb{Z}_n|}{|\mathbb{Z}_n| \cdot |\mathbb{Z}_n|} \leq 3^{-2n} \frac{1}{|\mathbb{Z}_n|}
\end{equation}

and using (4.44) and the fact that all the \( \psi_z \) have the same law, one has

\begin{equation}
\frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} |f(e) - \nabla \kappa(e)|^2 \leq \mathbb{E} \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} |f(e)|^2 \right] \leq \mathbb{E} \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{z \in \mathbb{Z}_n, 2n e \leq z + \square_n} |\nabla \sigma_z(e)|^2 \right] \leq CE \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} |\nabla \psi_{n,q}(e)|^2 \right].
\end{equation}

We can then bound the last term on the right-hand side thanks to Proposition 3.13. This gives

\begin{equation}
\mathbb{E} \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} |f(e) - \nabla \kappa(e)|^2 \right] \leq CE \left[ \frac{1}{|\mathbb{Z}_n|} \sum_{e \leq \square_n} |\nabla \psi_{n,q}(e)|^2 \right] \leq C(1 + |q|^2).
\end{equation}
Combining the few previous displays shows
\[ \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} |\nabla \sigma_z(e) - \nabla \kappa(e)|^2 \right] \leq \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{e \subseteq \Box_{2n}^+} |f(e) - \nabla \kappa(e)|^2 \right] + C 3^{-2n} (1 + |q|^2). \]

We then use (4.32) to complete the estimate
\[ \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} |\nabla \kappa(e) - \nabla \sigma_z(e)|^2 \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{(m-n)/2} \tau_m(q) \right). \]

Combining this estimate with (4.47) shows
\[ \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right] \leq \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} V_e (\nabla \psi_z(e)) \right] + C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{(m-n)/2} \tau_m(q) \right). \]

To complete the proof of (4.46), it is thus sufficient to prove
\[ (4.49) \]
\[ \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n}^+(e)) \right] \leq \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right] + C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{(m-n)/2} \tau_m(q) \right), \]

for some constant \( C := C(d, \lambda) < \infty \) and some exponent \( \beta := \beta(d, \lambda) > 0 \). To this end, we prove the two following inequalities

(1) We first prove that
\[ (4.50) \]
\[ \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right] \leq \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right] + C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{(m-n)/2} \tau_m(q) \right). \]

(2) We then prove
\[ (4.51) \]
\[ \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n}^+(e)) \right] \leq \mathbb{E} \left[ \frac{1}{|D_{2n}|} \sum_{e \subseteq \Box_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right] + C 3^{-\beta n} (1 + |q|). \]

Proof of (1). We define \( B_{n,2n}^+ \) to be the set of edges of \( \Box_{2n}^+ \) which do not belong to a cube of the form \( z + \Box_n \), for \( z \in \mathbb{Z}_{n,2n} \), i.e.,
\[ B_{n,2n}^+ := \{ e \subseteq \Box_{2n}^+ : \forall z \in \mathbb{Z}_{n,2n}, e \not\subseteq z + \Box_n \}. \]
This set has been defined to have the following splitting of the sum

\[
\sum_{e \in \square_{2n}} = \sum_{z \in Z_{n, 2n}} \sum_{e \subseteq \square_{2n}} + \sum_{e \in B_{n, 2n}}.
\]

Note also that the set \(B_{n, 2n}^+\) is almost equal to the set \(B_{n, 2n}\), the only difference is that we have added the edges which belong to \(\square_{2n}^+\) but not \(\square_{2n}\), which is a small boundary layer of edges. Additionally, one has the estimate on the cardinality of \(B_{n, 2n}^+\),

\[
|B_{n, 2n}^+| \leq C 3^{-n} |\square_{2n}|.
\]

We use the splitting of the sum mentioned earlier to prove the estimate

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} V_e (\nabla \nu_n^* (q)(e) + \nabla \kappa(e)) \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m^* (q) \right).
\]

To prove this inequality we first use that for each \(x \in \mathbb{R}\), \(V_e(x) \leq \frac{1}{\lambda} x^2\) to prove

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} V_e (\nabla \nu_n^* (q)(e) + \nabla \kappa(e)) \right] \leq C \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} |\nabla \kappa(e)|^2 \right] + C \frac{|B_{2n,n}|}{|\square_{2n}|} |\nabla \nu_n^* (q)|^2
\]

\[
\leq C \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} |\nabla \kappa(e)|^2 \right] + C (1 + |q|^2) 3^{-2n}.
\]

Since for each \(e \in B_{2n,n}\), we have \(f(e) = 0\), we have the estimate

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} |\nabla \kappa(e)|^2 \right] \leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} |f(e) - \nabla \kappa(e)|^2 \right].
\]

Using the exact same computation as in (4.48), we obtain

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} |f(e) - \nabla \kappa(e)|^2 \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m^* (q) \right).
\]

Combining the few previous displays shows

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} V_e (\nabla \nu_n^* (q)(e) + \nabla \kappa(e)) \right] \leq C \left( (1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m^* (q) \right)
\]
and consequently

$$\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right] = \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} \sum_{z \in \mathcal{Z}_{n,2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right]$$

$$+ \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B_{2n,n}} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right]$$

$$\leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} \sum_{z \in \mathcal{Z}_{n,2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) \right]$$

$$+ C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{(\frac{m-n}{2})} \tau_m(q) \right).$$

This is (4.50).

Proof of (2). The main tool is the estimate (4.35), which we recall

$$\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} |\nabla \kappa_{2n}^+(e) - \nabla \kappa(e)|^2 \right] \leq C3^{-dn}.$$

Using this inequality and a Taylor expansion, with the assumption $V''_e \leq \frac{1}{\lambda}$, one obtains

$$\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n}^+(e)) \right]$$

$$\leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n})(e) \right]$$

$$+ \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V'_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) (\nabla \kappa_{2n}^+(e) - \nabla \kappa(e)) \right]$$

$$+ \frac{1}{2\lambda} \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} |\nabla \kappa_{2n}^+(e) - \nabla \kappa(e)|^2 \right].$$

First combining the two previous displays, one has

$$(4.52)$$

$$\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n}^+(e)) \right] \leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V_e (\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n})(e) \right] + C3^{-dn}$$

$$+ \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \leq \square_{2n}^+} V'_e (\nabla \nu_n^*(q)(e) + \nabla \kappa(e)) (\nabla \kappa_{2n}^+(e) - \nabla \kappa(e)) \right].$$
so that there only remains to study the last term on the right-hand side of the previous display. This is achieved thanks to the Cauchy-Schwarz inequality

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} V'_e (\nabla \nu^*_n(q) + \nabla \kappa) \left( \nabla \kappa^e_{2n} - \nabla \kappa \right) \right] \\
\leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} \left| V'_e (\nabla \nu^*_n(q) + \nabla \kappa) \right|^2 \right] \frac{1}{2} \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} \left| \nabla \kappa^e_{2n} - \nabla \kappa \right|^2 \right] \frac{1}{2} \\
\leq C 3^{-\frac{3}{2}n} \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} \left| V'_e (\nabla \nu^*_n(q) + \nabla \kappa) \right|^2 \right] \frac{1}{2}.
\]

We then use that for each \( x \in \mathbb{R} \), \( |V'_e (x) | \leq \frac{1}{2} |x| \) to get

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} \left| V'_e (\nabla \nu^*_n(q) + \nabla \kappa) \right|^2 \right] \leq C \left| \nabla \nu^*_n(q) \right|^2 + C \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} \left| \nabla \kappa \right|^2 \right]
\]

and by definition of \( \kappa \) given in (4.19), we have

\[
\sum_{e \subseteq \square_{2n}^+} \left| \nabla \kappa \right|^2 \leq \sum_{e \subseteq \square_{2n}^+} \left| f(e) \right|^2 \\
\leq \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z+\square_{n}} \left| \nabla \psi_z(e) \right|^2.
\]

Taking the expectation and using Proposition 3.13, we have

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} \left| \nabla \kappa \right|^2 \right] \leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z+\square_{n}} \left| \nabla \psi_z(e) \right|^2 \right] \\
\leq \mathbb{E} \left[ \frac{1}{|\square_{n}|} \sum_{e \subseteq \square_{n}} \left| \nabla \psi_{n,q}(e) \right|^2 \right] \\
\leq C(1 + |q|^2).
\]

Combining the few previous displays and the bound \( \left| \nabla \nu^*_n(q) \right|^2 \leq C(1 + |q|^2) \) proved in (4.3) gives

\[
\left| \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} V'_e (\nabla \nu^*_n(q) + \nabla \kappa) \left( \nabla \kappa^e_{2n} - \nabla \kappa \right) \right] \right| \leq C 3^{-\frac{3}{2}n} (1 + |q|).
\]

Combining this with (4.52) gives

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} V_e (\nabla \nu^*_n(q) + \nabla \kappa^e_{2n}) \right] \leq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square_{2n}^+} V_e (\nabla \nu^*_n(q) + \nabla \kappa_{2n}) \right] + C(1 + |q|)3^{-\frac{3}{2}n}.
\]
We complete the proof of (4.51) by noting that $V_e$ is positive and that $|\Box_{2n}| \leq |\Box_{2n}^+|$ so that
\[
\mathbb{E} \left[ \frac{1}{|\Box_{2n}^+|} \sum_{e \subseteq \Box_{2n}^+} V_e \left( \nabla \nu_n^*(q) + \nabla \kappa_{2n}^+(e) \right) \right] \leq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} V_e \left( \nabla \nu_n^*(q) + \nabla \kappa_{2n}(e) \right) \right]
\leq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} V_e \left( \nabla \nu_n^*(q) + \nabla \kappa(e) \right) \right] + C3^{-\frac{d}{2}n}(1 + |q|).
\]

This completes the proof of (4.51).

We can now conclude this step. Combining (4.50) and (4.51) implies (4.49) and thus completes the proof of (4.46). Step 4 is complete.

Step 5. The conclusion. Combining the main results (4.10) of Step 2 and (4.11) of Steps 3 and 4, we obtain
\[
\mathbb{E} \left[ \frac{1}{|\Box_{2n}^+|} \sum_{e \subseteq \Box_{2n}^+} V_e \left( \nabla \nu_n^*(q) + \nabla \kappa_{2n}^+(e) \right) \right] + \frac{1}{|\Box_{2n}|} H \left( \mathbb{P}_{\kappa_{2n}^+} \right) \leq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} V_e \left( \nabla \psi_{n,q}(e) \right) \right]
\leq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} V_e \left( \nabla \nu_n^*(q) + \nabla \kappa_{2n}^+(e) \right) \right] + \frac{1}{|\Box_{2n}|} H \left( \mathbb{P}_{\kappa_{2n}^+} \right).
\]

But we know that
\[
\nu \left( \Box_{2n}^+, \nabla \nu_n^*(q) \right) = \inf_{\mathbb{P} \in \mathcal{P}(\{2^n\})} \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} V_e \left( \nabla \nu_n^*(q) + \nabla \phi(e) \right) \right] + \frac{1}{|\Box_{2n}|} H \left( \mathbb{P} \right)
\leq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} V_e \left( \nabla \nu_n^*(q) + \nabla \kappa_{2n}^+(e) \right) \right] + \frac{1}{|\Box_{2n}|} H \left( \mathbb{P}_{\kappa_{2n}^+} \right).
\]

Moreover, by the definition of $\mathbb{P}_{n,q}$ and the equality $\nabla \nu_n^*(q) = \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{e \subseteq \Box_{2n}} \nabla \psi_{n,q}(e) \right]$, we also know that
\[
\nu^* \left( \Box_n, q \right) = -\mathbb{E} \left[ \frac{1}{|\Box_n|} \sum_{e \subseteq \Box_n} V_e \left( \nabla \psi_{n,q}(e) \right) \right] + q \cdot \nabla \nu_n^*(q) - \frac{1}{|\Box_n|} H \left( \mathbb{P}_{n,q}^* \right).
\]

Combining the three previous displays shows
\[
\nu \left( \Box_{2n}^+, \nabla \nu_n^*(q) \right) + \nu^* \left( \Box_n, q \right) - q \cdot \nabla \nu_n^*(q) \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^n 3 \frac{(m-n)}{2} \tau_m(q) \right).
\]

The proof of Proposition 4.4 is complete. \[\square\]

4.4. Quantitative convergence of the partitions functions. Now that Proposition 4.4 is proved, we can deduce the main result of the article from it. The theorem is recalled below.

**Theorem 1.1** (Quantitative convergence to the Gibbs state). There exist a constant $C := C(d, \lambda) < \infty$ and an exponent $\alpha := \alpha(d, \lambda) > 0$ such that for each $p, q \in \mathbb{R}^d$
\[
|\nu(\Box_n, p) - \mathcal{P}(p)| \leq C3^{-\alpha n}(1 + |p|^2)
\]
and
\[
|\nu^*(\Box_n, q) - \mathcal{P}^*(q)| \leq C3^{-\alpha n}(1 + |q|^2).
\]
Proof. From Proposition 3.3, we know that, for each $p,q \in \mathbb{R}^d$, the sequences $(\nu(\square_{n},p))_{n \in \mathbb{N}}$ and $(\nu^*(\square_{n},p))_{n \in \mathbb{N}}$ converge and that, for each $p,q \in \mathbb{R}^d$,

$$\nu(\square_{n},p) \leq \nu(\square_{2n}^+,p) + C3^{-n}(1 + |p|^2).$$

(4.53) \hfill $\mathbb{P}(p) + \mathbb{P}^*(q) \geq p \cdot q.$

We split the proof into 5 steps

- In Step 1, we show the estimate, for each $p \in \mathbb{R}^d$,

$$\nu(\square_{3n},p) \leq C3^{-n}(1 + |p|^2).$$

(4.54) \hfill \(\nu^*(\square_{n},q) - \mathbb{P}^*(q)\) \leq C3^{-an}.

- In Step 2, we show that for each $n \in \mathbb{N}$ and each $q \in \mathbb{R}^d$

$$|\nu^*(\square_{n},q) - \mathbb{P}^*(q)| \leq C3^{-an}.$$

(4.55) \hfill \|\nu^*(\square_{n},q) - \mathbb{P}^*(q)\| \leq C3^{-an}.

- In Step 3, we show that there exists an exponent $\alpha := \alpha(d, \lambda) > 0$ such that,

$$|\nu^*(\square_{n},q) - \mathbb{P}^*(q)| \leq C3^{-an}.$$

(4.56) \hfill \|\nu^*(\square_{n},q) - \mathbb{P}^*(q)\| \leq C3^{-an}.

- In Step 4, we show that for each $\mathbb{P}$ and $\mathbb{P}^*$ are linked together by the relation,

$$\mathbb{P}^*(q) = \sup_{p \in \mathbb{R}^d} -\mathbb{P}(p) + p \cdot q.$$

- In Step 5, we show that there exists an exponent $\alpha := \alpha(d, \lambda) > 0$ such that,

$$|\nu(\square_{n},q) - \mathbb{P}(q)| \leq C3^{-an}.$$

(4.57) \hfill |\nu(\square_{n},q) - \mathbb{P}(q)| \leq C3^{-an}.

Step 1. The main idea of this step is to consider a cube $\square \subseteq \mathbb{Z}^d$ satisfying the two following properties

1) The cube $\square$ is included in $\square_{3n}$ and is almost as large as $\square_{3n}$ in the sense that it satisfies the volume estimate

$$|\square_{3n}| \leq |\square| + C3^{3dn} \times 3^{-n},$$

(4.58) \hfill \text{for some } C := C(d) < \infty.

2) The cube $\square$ can be decomposed as a disjoint union of cubes of the same size than $\square_{2n}^+$. i.e. the union of disjoint translated of $\square_{2n}^+$.

More precisely, the cube $\square$ can be constructed as follows: denote by $\mathcal{Z}$ the set

$$\mathcal{Z} := \left\{ z \in (3^{2n} + 2)\mathbb{Z}^d : z + \square_{2n}^+ \subseteq \square_{3n} \right\}$$

and then define

$$\square = \bigcup_{z \in \mathcal{Z}} (z + \square_{2n}^+).$$

With this definition, it is clear that the cube $\square$ satisfies the two properties (1) and (2). We then apply the same proof as in Step 2 of Proposition 3.5, with the partition of $\square_{3n}$ given by $(z + \square_{2n}^+), z \in \mathcal{Z}$ and $\square_{3n} \setminus \square$. With this partition, one obtains a formula which is similar to (3.6), with the set

$$\left\{ x \in \square_{3n} \setminus \square : \exists y \in \square, x \sim y \right\} \cup \bigcup_{z \in \mathcal{Z}} \partial (z + \square_{2n}^+)$$

instead of $\partial_{\text{int}} \square_{n+1}$. Nevertheless the cardinality of this set can be estimated according to

$$\frac{|\{ x \in \square_{3n} \setminus \square : \exists y \in \square, x \sim y \}| + \sum_{z \in \mathcal{Z}} |\partial (z + \square_{2n}^+)|}{|\square_{3n}|} \leq C3^{-2n}.$$

So we obtain the inequality

$$\nu(\square_{3n},p) \leq \frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{3n} \setminus \square, p) + \sum_{z \in \mathcal{Z}} \frac{|z + \square_{2n}^+|}{|\square_{3n}|} \nu(z + \square_{2n}^+, p) + C3^{-2n}(1 + |p|^2).$$

(4.59) \hfill \nu(\square_{3n},p) \leq \frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{3n} \setminus \square, p) + \sum_{z \in \mathcal{Z}} \frac{|z + \square_{2n}^+|}{|\square_{3n}|} \nu(z + \square_{2n}^+, p) + C3^{-2n}(1 + |p|^2).
Using Proposition A.3, proved in Appendix A, we know that \( \nu(\square_{3n} \setminus \square, p) \) is bounded by \( C(1 + |p|^2) \), thus by (4.58), we can estimate
\[
|\square_{3n} \setminus \square| \nu(\square_{3n} \setminus \square, p) \leq C3^{-n}(1 + |p|^2).
\]

Thus from (4.59) and the previous display, we obtain
\[
\nu(\square_{3n}, p) \leq \sum_{z \in \mathbb{Z}} \frac{|z + \square_{2n}|}{|\square_{3n}|} \nu(z + \square_{2n}, p) + C3^{-n}(1 + |p|^2).
\]

But, we have for each \( z \in \mathbb{Z}, \nu(z + \square_{2n}, p) = \nu(\square_{2n}, p), \) thus
\[
\sum_{z \in \mathbb{Z}} \frac{|z + \square_{2n}|}{|\square_{3n}|} \nu(z + \square_{2n}, p) = \frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{2n}, p).
\]

Using Proposition 3.8, we have \( \nu(\square_{2n}, p) \geq -C + \lambda|p|^2 \). Combining this bound with (4.58), we can refine the previous display
\[
\frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{2n}, p) \leq \nu(\square_{2n}, p) + C(1 + |p|^2)3^{-n}.
\]

Combining the few previous displays eventually shows
\[
\nu(\square_{3n}, p) \leq \nu(\square_{2n}, p) + C3^{-n}(1 + |p|^2),
\]
which is the desired result. The proof of Step 2 is complete.

**Step 2.** First, by the formula, for each \( q \in \mathbb{R}^d \),
\[
\nabla_q \nu^* (\square, q) = E \left[ \frac{1}{|\square|} \sum_{e \subseteq \square_n} \nabla \psi_{n,q}(e) \right]
\]
and by Proposition 3.13, we have
\[
|\nabla_q \nu^* (\square, q)| \leq C(1 + |q|).
\]

As a consequence, by the main result (4.54) of Step 1, we have
\[
\nu(\square_{3n}, \nabla_q \nu^* (\square, q)) \leq \nu(\square_{2n}, \nabla_q \nu^* (\square, q)) + C3^{-n}(1 + |q|^2).
\]

Combining the previous display with Proposition 4.4, one obtains
\[
(4.60) \quad \nu(\square_{3n}, \nabla_q \nu^* (\square, q)) + \nu^*(\square, q) + q \cdot \nabla q \nu^* (\square, q) \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m+n)}{2}} r_m(q) \right).
\]

Moreover using the inequality (4.53) applied with \( p = \nabla_q \nu^* (\square, q) \) and \( q \) gives
\[
0 \leq \nabla \nu^* (\square, q) + \nabla^* (q) - \nabla_q \nu^* (\square, q) \cdot q.
\]

A combination of the two previous displays gives
\[
(4.61) \quad (\nu(\square_{3n}, \nabla_q \nu^* (\square, q)) - \nabla \nu^* (\square, q)) + (\nu^*(\square, q) - \nabla^* (q)) \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} r_m(q) \right).
\]

But by Proposition 3.5, we see that for some constant \( C := C(d, \lambda) < \infty \) (in particular larger than the one appearing in the proposition), the sequence \( n \to \nu(\square_n, p) + C(1 + |p|^2)3^{-n} \) is decreasing. As a consequence, for each \( n \in \mathbb{N}, \) and each \( p \in \mathbb{R}^d, \)
\[
\nu(\square_n, p) \geq \nabla (p) - C(1 + |p|^2)3^{-n}.
\]
This implies, for each \( q \in \mathbb{R}^d \),
\[
\nu(\square_{3n}^n, \nabla q^* (\square_{n}, q)) - \nu(\nabla q^* (\square_{n}, q)) \geq -C(1 + |q|^2)3^{-3n}.
\]
Combining the previous inequality with (4.61) shows
\[
(4.62) \quad \nu^*(\square_{n}, q) - \nu^*(q) \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{(m-n)} r_m(q) \right).
\]
The proof of Step 3 is complete.

**Step 3.** Let \( C := C(d, \lambda) < \infty \) be a constant large enough so that the sequence \( \nu^*(\square_{n}, q) + C(1 + |q|^2)3^{-n} \) is decreasing. To shorten the notation, we denote by, for \( q \in \mathbb{R}^d \),
\[
F_n(q) := \nu^*(\square_{n}, q) + C(1 + |q|^2)3^{-n} - \nu^*(q),
\]
so that \( F_n(q) \) is decreasing and tends to 0 as \( n \to \infty \). Moreover, we have the following inequality
\[
\tau_n(q) \leq F_n(q) - F_{n+1}(q).
\]
We can then rewrite the main result (4.62) of Step 3 with this notation
\[
(4.64) \quad F_n(q) \leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{(m-n)} (F_m(q) - F_{m+1}(q)) \right).
\]
We then define
\[
\widetilde{F}_n(q) := 3^{-\frac{\beta}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} F_m(q).
\]
We next show that there exist \( \theta := \theta(d, \lambda) \in (0, 1) \), \( C := C(d, \lambda) < \infty \) and \( \beta := \beta(d, \lambda) > 0 \) such that for every \( n \in \mathbb{N} \),
\[
(4.65) \quad \widetilde{F}_{n+1}(q) \leq \theta \widetilde{F}_n(q) + C3^{-\beta n}.
\]
Using \( F_0(q) \leq C(1 + |q|^2) \), we have
\[
(4.66) \quad \widetilde{F}_n(q) - \widetilde{F}_{n+1}(q) \geq 3^{-\frac{\beta}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} (F_m(q) - F_{m+1}(q)) - C(1 + |q|^2)3^{-\frac{\beta}{4}}.
\]
Since \( F_n(q) \) is a decreasing sequence, we deduce from the previous display that, for each \( n \in \mathbb{N} \),
\[
\widetilde{F}_{n+1}(q) \leq \widetilde{F}_n(q) + C(1 + |q|^2)3^{-\frac{\beta}{4}}.
\]
By (4.64) and reducing the exponent \( \beta \) if necessary, we deduce
\[
\widetilde{F}_{n+1}(q) \leq \widetilde{F}_n(q) + C(1 + |q|^2)3^{-\frac{\beta}{4}}
\]
\[
\leq 3^{-\frac{\beta}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} F_m(q) + C(1 + |q|^2)3^{-\frac{\beta}{4}}
\]
\[
\leq C3^{-\frac{\beta}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} \left( (1 + |q|^2)3^{-\beta m} + \sum_{k=0}^{m} 3^{(k-m)} (F_k(q) - F_{k+1}(q)) \right) + C(1 + |q|^2)3^{-\frac{\beta}{4}}
\]
\[
\leq C3^{-\frac{\beta}{4}} \sum_{m=0}^{n} \sum_{k=0}^{m} 3^{-\frac{m-k}{4}} 3^{\frac{k}{4}} (F_k(q) - F_{k+1}(q)) + C(1 + |q|^2)3^{-\beta n}
\]
\[
\leq C3^{-\frac{\beta}{4}} \sum_{k=0}^{n} \sum_{m=k}^{n} 3^{-\frac{m-k}{4}} 3^{\frac{k}{4}} (F_k(q) - F_{k+1}(q)) + C(1 + |q|^2)3^{-\beta n}
\]
\[
\leq C3^{-\frac{\beta}{4}} \sum_{k=0}^{n} 3^{\frac{k}{4}} (F_k(q) - F_{k+1}(q)) + C(1 + |q|^2)3^{-\beta n}.
\]
Comparing the previous display with \((4.66)\) gives
\[
\overline{F}_{n+1}(q) \leq C \left( \overline{F}_n(q) - \overline{F}_{n+1}(q) \right) + C(1 + |q|^2)3^{-\beta n}.
\]
A rearrangement of this inequality gives \((4.65)\). We then iterate \((4.65)\), we obtain
\[
\overline{F}_n(q) \leq \theta^n \overline{F}_0 + C(1 + |q|^2) \sum_{k=0}^{n} \theta^{k}3^{-\beta(n-k)}
\]
By making \(\theta\) closer to 1 if necessary, we have
\[
\sum_{k=0}^{n} \theta^{k}3^{-\beta(n-k)} \leq C\theta^n.
\]
Combining the few previous displays thus gives
\[
\overline{F}_n(q) \leq C(1 + |q|^2)\theta^n.
\]
Setting \(\alpha := -\frac{\ln\theta}{\ln3}\), so that \(\theta = 3^{-\alpha}\) gives the bound \(\overline{F}_n(q) \leq C3^{-\alpha n}\). But by the definition of \(\overline{F}_n(q)\), we also have the clear inequality
\[
F_n(q) \leq \overline{F}_n(q)
\]
and thus
\[
F_n(q) \leq C(1 + |q|^2)3^{-\alpha n}.
\]
We conclude the proof by noting that \(F_n(q)\) was defined so that it is decreasing and tends to 0. In particular, these two properties implies the \(F_n(q)\) is positive. By the explicit formula \((4.63)\) for \(F_n\) and the previous display, we obtain
\[
-C(1 + |q|^2)3^{-n} \leq \nu^*(\square_n, q) - \overline{\nu}^*(q) \leq C(1 + |q|^2)3^{-\alpha n},
\]
for some \(C := C(d, \lambda) < \infty\) and \(\alpha := \alpha(d, \lambda) > 0\). By making \(\alpha\) closer to 0 if necessary, we eventually obtain
\[
|\nu^*(\square_n, q) - \overline{\nu}^*(q)| \leq C(1 + |q|^2)3^{-\alpha n}.
\]
The proof of Step 4 is complete.

**Step 4.** First note that, by \((4.53)\), for each \(p, q \in \mathbb{R}^d\)
\[
0 \leq \overline{\nu}(p) + \overline{\nu}^*(q) - p \cdot q.
\]
This implies
\[
\overline{\nu}^*(q) \geq \sup_{p \in \mathbb{R}^d} -\overline{\nu}(p) + p \cdot q
\]
The main idea of this step is to use Proposition 4.4 to show the two following result

1. For each \(q \in \mathbb{R}^d\), the sequence \(\nabla_q \nu^*(\square_n, q)\) converges as \(n \in \mathbb{N}\). We denote its limit by \(\overline{\nu}(q)\). Moreover one has that the following quantitative estimate holds
\[
|\nabla_q \nu^*(\square_n, q) - \overline{\nu}(q)| \leq C(1 + |q|^2)3^{-\alpha n}.
\]

   **Remark 4.5.** We would like to say that the limit is in fact \(\nabla_q \overline{\nu}^*(q)\) but as for now we only know that the function \(q \to \overline{\nu}^*(q)\) is convex and in particular we do not know that it is derivable everywhere. We will prove later that \(\overline{\nu}^*(q)\) is in fact \(C^1\) and this will imply \(\overline{\nu}(q) = \nabla_q \overline{\nu}^*(q)\).

2. We deduce from (1) that for each \(q \in \mathbb{R}^d\), one has the following quantitative convergence
\[
|\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \overline{\nu}(\overline{\nu}(q))| \leq C(1 + |q|^2)3^{-\alpha n}
\]
We first prove (1). From the main result of the previous setup (4.56), we deduce that, for each $q \in \mathbb{R}^d$,
\[ \tau_n^*(q) = \nu^*(\Box_n, q) - \nu^*(\Box_{n+1}, q) \leq C(1 + |q|^2)3^{-\alpha n}. \]
Combining this result with (4.4) gives, for each $q \in \mathbb{R}^d$,
\[ |\nabla_q \nu^*(\Box_{n+1}, q) - \nabla_q \nu^*(\Box_n, q)| \leq C(1 + |q|^2)3^{-\alpha n}. \]
The previous display implies that the sequence $\nabla_q \nu^*(\Box_{n+1}, q)$ converges for each $q \in \mathbb{R}^d$ together with the quantitative rate of convergence (4.68). We denote by $\overline{P}(q)$ its limit. The proof of (1) is complete.

We now prove (2). We first split (4.69) thanks to the triangle inequality
\[ |\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) - \overline{P}(q)| \leq |\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) - \nu(\Box_{3n}, \overline{P}(q))| + |\nu(\Box_{3n}, \overline{P}(q)) - \overline{P}(q)|. \]
Since one has the bound, for each $n \in \mathbb{N}$, $|\nabla_q \nu^*(\Box_n, q)| \leq C(1 + |q|)$, we obtain by passing to the limit $n \to \infty$,
\begin{equation}
|\overline{P}(q)| \leq C(1 + |q|). \tag{4.70}
\end{equation}
Combining the previous bound with (4.55) gives
\[ |\nu(\Box_{3n}, \overline{P}(q)) - \overline{P}(q)| \leq C(1 + |q|^2)3^{-\alpha n}. \]
To prove (4.69), it is sufficient to prove
\[ |\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) - \nu(\Box_{3n}, \overline{P}(q))| \leq C(1 + |q|^2)3^{-\alpha n}. \]
To this end, we first prove the following property: for each $p, p' \in \mathbb{R}^d$, and each $n \in \mathbb{N}$,
\[ |\nu(\Box_n, p) - \nu(\Box_n, p')| \leq C(|p| + |p'|)|p - p'|. \]
The idea to prove the previous inequality is to compute the gradient of $p \to \nu(\Box_n, p)$. A straightforward computation gives
\[ |\nabla_p \nu(\Box_n, p)| \leq \mathbb{E} \left[ \frac{1}{\Box_n} \sum_{e \in \Box_n} |V_e'(p \cdot e + \nabla \phi_{n,p}(e))| \right]. \]
Using the bound, for each $x \in \mathbb{R}$ $V_e'(x) \leq \frac{1}{\lambda} |x|$ and the Jensen inequality, we obtain
\[ |\nabla_p \nu(\Box_n, p)| \leq |p| + \mathbb{E} \left[ \frac{1}{\Box_n} \sum_{e \in \Box_n} |\nabla \phi_{n,p}(e)| \right] \leq |p| + \mathbb{E} \left[ \frac{1}{\Box_n} \sum_{e \in \Box_n} |\nabla \phi_{n,p}(e)|^2 \right]^{\frac{1}{2}}. \]
We then apply Proposition 3.13 to obtain the bound
\begin{equation}
|\nabla_p \nu(\Box_n, p)| \leq C(1 + |p|). \tag{4.71}
\end{equation}
This implies that, for each $n \in \mathbb{N}$ and each $p, p' \in \mathbb{R}^d$, $\nu(\Box_n, \cdot)$ is $C(1 + |p| + |p'|)$-Lipschitz in the ball $B(0, |p| + |p'|)$. Since both $p$ and $p'$ belongs to $B(0, |p| + |p'|)$, we have
\begin{equation}
|\nu(\Box_n, p) - \nu(\Box_n, p')| \leq C(1 + |p| + |p'|)|p - p'|. \tag{4.72}
\end{equation}
This is the desired result. Applying the previous estimate with $p = \nabla_p \nu^*(\Box_n, q)$ and $p' = \overline{P}(q)$ gives
\[ |\nu(\Box_{3n}, \nabla_p \nu^*(\Box_n, q)) - \nu(\Box_{3n}, \overline{P}(q))| \leq C(1 + |\nabla_q \nu^*(\Box_n, q)| + |\overline{P}(q)|)|\nabla_q \nu^*(\Box_n, q) - \overline{P}(q)|. \]
By (4.70), for each $q \in \mathbb{R}^d$,
\[ |\overline{P}(q)| \leq C(1 + |q|). \]
Combining the three previous display with (4.68) gives

\[(4.73) \quad \| \nu(\Box_{3n}, \nabla_p \nu^*(\Box_n, q)) - \nu(\Box_{3n}, \nabla_p \nu^*(\Box_n, q)) \| \leq C(1 + |q|^2)3^{-\alpha n}\]

and completes the proof of (2).

We now prove the main result (4.56) of Step 5. By (4.60) and the main result (4.55) of Step 4, we have, for each \( q \in \mathbb{R}^d \),

\[\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) + \nu^*(\Box_n, q) - \nabla_q \nu^*(\Box_n, q) \cdot q \leq C(1 + |q|^2)3^{-\alpha n}.\]

By (4.68) and (4.69), we have the convergence

\[\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) + \nu^*(\Box_n, q) - \nabla_q \nu^*(\Box_n, q) \cdot q \xrightarrow{n \to \infty} \nabla(\nabla_p \nu^*(\Box_n, q)) + \nabla^* (q) - \nabla(\nabla_p \nu^*(\Box_n, q)) \cdot q.\]

A combination of the two previous displays gives

\[\nabla(\nabla_p \nu^*(\Box_n, q)) + \nabla^* (q) \cdot q \leq 0.\]

Together with (4.67), the previous estimate gives in particular

\[\nabla(\nabla_p \nu^*(\Box_n, q)) + \nabla^* (q) \cdot q = 0\]

and thus

\[\nabla^* (q) = \sup_{p \in \mathbb{R}^d} -\nabla(p) + p \cdot q.\]

This is precisely (4.56) and the proof of Step 5 is complete.

**Step 5.** The main result (4.56) of Step 5 asserts that \( \nabla^* \) is the Legendre-Fenchel transform of \( \nabla \). But by Proposition 3.3, we know that for each \( p_1, p_2 \in \mathbb{R}^d \),

\[\frac{1}{C} |p_0 - p_1|^2 \leq \frac{1}{2} \nabla(\Box_n, p_0) + \frac{1}{2} \nabla(\Box_n, p_1) - \nabla(\Box_n, \frac{p_0 + p_1}{2}) \leq C |p_0 - p_1|^2.\]

With the two previous ideas, one deduces that \( \nabla^* \) is also uniformly convex. As a consequence, it is \( C^{1,1}(\mathbb{R}^d) \) and one has the following equalities, for each \( p, q \in \mathbb{R}^d \),

\[(4.74) \quad \nabla_p \nabla_q \nabla^*(\Box_n, q) = q, \quad \nabla_q \nabla^*(\nabla_p \nu^*(\Box_n, q)) = q, \quad \text{and} \quad \nabla(\nabla_p \nu^*(\Box_n, q)) = \nabla_q \nabla^*(\nabla_p \nu^*(\Box_n, q)).\]

We are now ready to prove (4.57). We start from (4.61), which reads for each \( q \in \mathbb{R}^d \),

\[(4.75) \quad (\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) - \nabla(\nabla_q \nu^*(\Box_n, q))) + (\nu^*(\Box_n, q) - \nabla^*(q))\]

\[\leq C \left( (1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m+n)}{2}} r_m(q) \right).\]

We then apply (4.55) which allows to estimate most of the terms in the previous display. Precisely, we have the inequalities

\[\nu^*(\Box_n, q) - \nabla^*(q) \leq C(1 + |q|^2)3^{-\alpha n}\]

and

\[\sum_{m=0}^{n} 3^{\frac{(m+n)}{2}} r_m(q) \leq C(1 + |q|^2)3^{-\alpha n}.\]

With these estimates, the estimate (4.75) becomes

\[\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) - \nabla(\nabla_q \nu^*(\Box_n, q)) \leq C(1 + |q|^2)3^{-\alpha n}.\]

Then by (4.73), we have

\[|\nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q)) - \nu(\Box_{3n}, \nabla_q \nu^*(\Box_n, q))| \leq C(1 + |q|^2)3^{-\alpha n}.\]

Then by sending \( n \to \infty \) in (4.72), we obtain for each \( p, p' \in \mathbb{R}^d \)

\[(4.76) \quad |\nabla(p) - \nabla(p')| \leq C(|p| + |p'|)|p - p'|.\]
With the same proof as the one which gives (4.73), we obtain
\[
\left| \mathcal{P}(\nabla q^{n^*}(\square_n, q)) - \mathcal{P}(\mathcal{P}(q)) \right| \leq C(1 + |q|^2)3^{-\alpha n}.
\]
Combining the few previous displays, for each \( q \in \mathbb{R}^d \),
\[
\nu(\square_{3n}, \mathcal{P}(q)) - \mathcal{P}(\mathcal{P}(q)) \leq C(1 + |q|^2)3^{-\alpha n}.
\]
Applying the previous inequality with \( q = D\mathcal{P}(p) \) gives, thanks to (4.74),
\[
\nu(\square_{3n}, p) - \mathcal{P}(p) \leq C(1 + |\nabla_p \mathcal{P}(p)|^2)3^{-\alpha n}.
\]
We then simplify the term on the right-hand side. Thanks to (4.76), one obtains the bound on the gradient of \( \mathcal{P} \), for each \( p \in \mathbb{R}^d \),
\[
|\nabla_p \mathcal{P}(p)| \leq C(1 + |p|).
\]
A combination of the two previous displays gives, for each \( n \in \mathbb{N} \), and each \( p \in \mathbb{R}^d \),
\[
\nu(\square_{3n}, p) - \mathcal{P}(p) \leq C(1 + |p|^2)3^{-\alpha n}.
\]
We now want to remove the \( 3n \) term. To this end, we use the subadditivity of \( \nu \), Proposition 3.5, to obtain, for each \( p \in \mathbb{R}^d \) and each \( n \in \mathbb{N} \),
\[
\nu(\square_{3n+2}, p) - \mathcal{P}(p) \leq \nu(\square_{3n+1}, p) - \mathcal{P}(p) + C(1 + |p|^2)3^{-n} \leq \nu(\square_{3n}, p) - \mathcal{P}(p) + C(1 + |p|^2)3^{-n} \leq C(1 + |p|^2)3^{-\alpha n}.
\]
From the previous display and by making \( \alpha \) smaller, we obtain for each \( n \in \mathbb{N} \) and each \( p \in \mathbb{R}^d \),
\[
\nu(\square_n, p) - \mathcal{P}(p) \leq C(1 + |p|^2)3^{-\alpha n}.
\]
The proof of (4.57) is almost complete, there only remains to prove a lower bound for \( \nu(\square_n, p) - \mathcal{P}(p) \). But by Proposition 3.5, we now that there exists a constant \( C := C(d, \lambda) < \infty \) such that the sequence
\[
\nu(\square_n, p) + C(1 + |p|^2)3^{-n}
\]
is decreasing and converges to \( \mathcal{P}(p) \). This implies in particular that, for each \( n \in \mathbb{N} \) and each \( p \in \mathbb{R}^d \),
\[
\nu(\square_n, p) - \mathcal{P}(p) \geq -C(1 + |p|^2)3^{-n}
\]
and provides the lower bound. Indeed a combination of the two previous displays gives
\[
|\nu(\square_n, p) - \mathcal{P}(p)| \leq C(1 + |p|^2)3^{-\alpha n}
\]
and completes the proof of Step 6 and of Theorem 1.1. \( \square \)

4.5. **Quantitative contraction of the fields \( \phi_{n,p} \) and \( \psi_{n,q} \) to affine functions.** Now that Theorem 1.1 is proved, we deduce the \( L^2 \) estimate on the random variables \( \phi_{n,p} \) and \( \psi_{n,q} \) stated in Theorem 1.2. The theorem is recalled below.

**Theorem 1.2** (\( L^2 \) contraction of the Gibbs measure). There exists a constant \( C := C(d, \lambda) < \infty \) and an exponent \( \alpha := \alpha(d, \lambda) > 0 \) such that for each \( n \in \mathbb{N} \), \( p, q \in \mathbb{R}^d \),
\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} \left( |\phi_{n,p}(x)|^2 + |\psi_{n,q}(x) - \nabla q^{n^*}(q) \cdot x|^2 \right) \right] \leq C3^{n(2-\alpha)} (1 + |p|^2 + |q|^2).
\]

**Proof.** We first prove the estimate for the random variable \( \psi_{n,q} \), i.e
\[
(4.77) \quad \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} |\psi_{n,q}(x) - \nabla q^{n^*}(q) \cdot x|^2 \right] \leq C3^{n(2-\alpha)} (1 + |q|^2).
\]
Indeed in that case all the tools have already been developed and this allows for a short proof. First by Theorem 1.1, we know that for each \( q \in \mathbb{R}^d \),
\[
\tau_n^*(q) \leq 3^{-\alpha n} (1 + |q|^2).
\]
Using the previous display together with Proposition 4.3, we obtain
\[ E \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} |\psi_{n,q}(x) - \nabla q \nu^*(\square_n, q) \cdot x|^2 \right] \leq C 3^{(2-\alpha)n} (1 + |q|^2). \]

But by (4.68) and (4.74), we also have
\[ |\nabla q \nu^*(\square_n, q) - \nabla q \nu^*(q)| \leq C 3^{-\alpha n} (1 + |q|^2). \]

A combination of the two previous displays gives (4.77) and completes the proof.

We now want to prove the estimate with the random variable \( \phi_{n,p} \), i.e,
\[ E \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} \left( |\phi_{n,p}(x) - p \cdot x|^2 + |\psi_{n,q}(x) - \nabla q \nu^*(q) \cdot x|^2 \right) \right] \leq C 3^{n(2-\alpha)} (1 + |p|^2). \]

The proof follows the same lines as the proof of (4.77) except that we have not proved an equivalent of Proposition 4.3 and thus have to prove it. The proof is split into 2 steps.

- In Step 1, we show that, for each \( m \in \mathbb{N} \) with \( m \leq n \),
\[ E \left[ \frac{1}{|\square_{m,n}|} \sum_{z \in \square_{m,n}} \left( |\phi_{n,p}(z) - p \cdot z|^2 + |\psi_{n,q}(z) - \nabla q \nu^*(q) \cdot z|^2 \right) \right] \leq C (1 + |p|^2) 3^{-\alpha m}. \]

- In Step 2, we deduce from the previous step and the multiscale Poincaré inequality
\[ E \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} |\phi_{n,p}|^2 \right] \leq C (1 + |p|^2) 3^{n-\alpha n}. \]

**Step 1.** Consider the random variable \( \phi = \sum_{z \in \square_{m,n}} \phi_z \) introduced in Proposition 3.15 as well as the coupling between \( \phi \) and \( \phi_{n,p} \) introduced in the same proposition, such that the following estimate holds
\[ \frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right] \leq C (\nu(\square_{m,n}, q) - \nu(\square_n, q)) + C 3^{-\alpha n} (1 + |p|^2). \]

Using Theorem 1.1, the previous estimate can be refined and one obtains
\[ \frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right] \leq C (1 + |p|^2) 3^{-\alpha m}. \]

Using this estimate, one has
\[ \frac{1}{|\square_{m,n}|} \sum_{z \in \square_{m,n}} E \left[ |\langle \nabla \phi_{n,p} \rangle_{z+\square_m}|^2 \right] \]
\[ \leq \frac{1}{|\square_{m,n}|} \sum_{z \in \square_{m,n}} E \left[ |\langle \nabla \phi \rangle_{z+\square_m}|^2 \right] + \frac{1}{|\square_n|} E \left[ \sum_{e \subseteq \square_n} |\nabla \phi(e) - \nabla \phi_{n,p}(e)|^2 \right] \]
\[ \leq \frac{1}{|\square_{m,n}|} \sum_{z \in \square_{m,n}} E \left[ |\langle \nabla \phi \rangle_{z+\square_m}|^2 \right] + C (1 + |p|^2) 3^{-\alpha m}. \]

We then note that for each \( z \in \square_{m,n} \), \( \langle \nabla \phi \rangle_{z+\square_m} = \langle \nabla \phi \rangle_{z+\square_m} \) and that, since \( \phi_z \in h^1_0(z+\square_m) \), one has \( \langle \nabla \phi \rangle_{z+\square_m} = 0 \). Consequently, the previous display can be simplified and one obtains
\[ \frac{1}{|\square_{m,n}|} \sum_{z \in \square_{m,n}} E \left[ |\langle \nabla \phi_{n,p} \rangle_{z+\square_m}|^2 \right] \leq C (1 + |p|^2) 3^{-\alpha m}. \]
Step 2. We now apply the multiscale Poincaré inequality stated in Proposition 2.15 for functions in $h_0^1(\square_n)$, this gives
\[
\frac{1}{|\square_n|} \sum_{x \in \square_n} |\phi_{n,p}(x)|^2 \leq C \sum_{e \subseteq \square_n} |\nabla \phi_{n,p}(e)|^2 + C3^p \sum_{m=1}^n 3^m \left( \frac{1}{|\mathbb{Z}_{m,n}|} \sum_{y \in \mathbb{Z}_{m,n}} \left| \langle \nabla \phi_{n,p} \rangle_{z+\square_m} \right|^2 \right).
\]
Taking the expectation and using Proposition 3.8 to estimate the first term on the right-hand side and the main result of Step 1 to estimate the second term gives
\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{x \in \square_n} |\phi_{n,p}(x)|^2 \right] \leq C(1 + |p|^2) + (1 + |p|^2)3^n \sum_{m=1}^n 3^{3-\alpha m} \leq C(1 + |p|^2)3^{(2-\alpha)n}.
\]
This is the desired result. The proof of Theorem 1.2 is complete.

**Appendix A. Technical estimate**

Before stating the first proposition of this appendix, we recall that $H$ is the space of functions of $\dot{h}^1(\square_n)$ which are constant on the subcubes $z + \square_m$, for $z \in \mathbb{Z}_{m,n}$. It is a space of dimension $3^{d(n-m)} - 1$ and each function $h \in H$ can be written in the following form
\[
h = \sum_{z \in \mathbb{Z}_{m,n}} \lambda_z \mathbf{1}_{z + \square_m},
\]
for some constants $(\lambda_z)_{z \in \mathbb{Z}_{m,n}}$ satisfying $\sum_{z \in \mathbb{Z}_{m,n}} \lambda_z = 0$.

**Proposition A.1.** There exists a constant $C := C(d, \lambda) < \infty$ such that the following estimate holds, for each $\psi \in \dot{h}^1(\square_n)$,
\[
\log \int_H \exp \left( - \sum_{e \in B_{m,n}} V_e (\nabla \psi(e) + \nabla h(e)) \right) dh \leq C m 3^{d(n-m)}.
\]

**Proof.** By the uniform convexity of $V_e$, we have, for each $x \in \mathbb{R}$,
\[
V_e(x) \geq \lambda x^2.
\]
This gives the following estimate, for each $\psi \in \bigoplus_{z \in \mathbb{Z}_{m,n}} \dot{h}^1(z + \square_m)$,
\[
- \sum_{e \in B_{m,n}} V_e (\nabla \psi(e) + \nabla h(e)) \leq - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2.
\]
For the rest of the proof, we introduce the following notation: for $z, z' \in \mathbb{Z}_{m,n}$, we write
\[
z \sim z' \text{ if and only if } |z - z'|_1 = 3^m.
\]
That is to say, we write $z \sim z'$ if and only if $z$ and $z'$ are neighbours in the rescaled lattice $3^m \mathbb{Z}^d$. With this notation, the set $B_{m,n}$ can be partitioned according to
\[
B_{m,n} = \bigcup_{z, z' \in \mathbb{Z}_{m,n}, z \sim z'} F_{z, z'},
\]
where we introduce the notation
\[
F_{z, z'} := \{ e = (x, y) \subseteq \square_{n+1} : x \in z + \square_n \text{ and } y \in z' + \square_n \}.
\]
With this notation, the right-hand side of (A.1) can be rewritten
\[
\sum_{e \in B_{m,n}} (\nabla \psi(e) + \nabla h(e))^2 = \sum_{z, z' \in \mathbb{Z}_{m,n}, z \sim z'} \sum_{e \in F_{z, z'}} (\nabla \psi(e) + \lambda z' - \lambda z)^2.
\]
Expanding the square gives, for each $z, z' \in \mathbb{Z}_{m,n}$ satisfying $z \sim z'$,
\[
\sum_{e \in F_{z,z'}} (\nabla \psi(e) + \lambda'_z - \lambda_z)^2 = \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2 + 2\nabla \psi(e) (\lambda_{z'} - \lambda_z) + |\lambda_{z'} - \lambda_z|^2
\]
\[
= |F_{z,z'}| \left( \lambda_{z'} - \lambda_z + \frac{1}{|F_{z,z'}|} \sum_{e \in F_{z,z'}} \nabla \psi(e) \right)^2
\]
\[
- \frac{1}{|F_{z,z'}|} \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2 + \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2.
\]
But we have
\[
- \frac{1}{|F_{z,z'}|} \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2 \geq 0
\]
thus we obtain
\[
\sum_{e \in F_{z,z'}} (\nabla \psi(e) + \lambda'_z - \lambda_z)^2 \geq |F_{z,z'}| \left( \lambda_{z'} - \lambda_z + \frac{1}{|F_{z,z'}|} \sum_{e \in F_{z,z'}} \nabla \psi(e) \right)^2.
\]
Note that the cardinality $|F_{z,z'}|$ is the same for each $z, z' \in \mathbb{Z}_{m,n}$ such that $z \sim z'$. It is indeed the cardinality of a face of the cube $\Box_m$ and is equal to $3^{(d-1)m}$. This cardinality will be simply denoted by $|F_m|$ in the rest of the proof.

The next step of the proof is to construct an isometry between the spaces $H$ and $\hat{h}^1(\Box_{n-m})$. To do so, note that we have the equality
\[
\mathbb{Z}_{m,n} = 3^m \Box_{n-m}.
\]
in particular if $z \in \mathbb{Z}_{m,n}$ then $z/3^m \in \Box_{n-m}$. From this we obtain that there exists an isometry between the spaces $H$ and $\hat{h}^1(\Box_{n-m})$ given by
\[
(A.2) \quad \Phi : \begin{cases} \quad H : \mathbb{Z}_{m,n} \ni h := \sum_{z \in \mathbb{Z}_{m,n}} \lambda_z \mathbb{1}_{\{z+\Box_m\}} \mapsto \hat{h}^1(\Box_{n-m}) \ni \Phi(h) = 3^\frac{d}{m} \sum_{z \in \mathbb{Z}_{m,n}} \lambda_z \delta_{z/3^m}, \end{cases}
\]
where $\delta_z$ is the function defined by $\delta_z(z') = 1$ if $z = z'$ and $\delta_z(z') = 0$ otherwise. The scalar $3^\frac{d}{m}$ is here to ensure that
\[
\sum_{x \in \Box_n} h(x)^2 = \sum_{z \in \mathbb{Z}_{m,n}} 3^{dm} |\lambda_z|^2
\]
is equal to
\[
\sum_{x \in \Box_n, z \in \Box_{n-m}} \Phi(h)(x)^2 = \sum_{z \in \mathbb{Z}_{m,n}} 3^{\frac{d}{m}} |\lambda_z|^2.
\]
We also denote by $X_\psi$ the vector field defined on the edges of $\Box_{n-m}$ by
\[
X_\psi \left( \frac{z}{3^m}, \frac{z'}{3^m} \right) = \frac{1}{|F_m|} \sum_{e \in F_{z,z'}} \nabla \psi(e).
\]
Performing the change of variable by the isometry $\Phi$ and using the few previous displays gives

$$
\int_H \exp \left( - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2 \right) \, dh
\leq \int_{H^1(\square_{n-m})} \exp \left( - \sum_{e \in \square_{n-m}} \lambda |F_m| \left( 3^{-\frac{dm}{2}} \nabla h(e) + X_\psi(e) \right)^2 \right) \, dh.
$$

Using the equality $|F_m| = 3^{(d-1)m}$, we obtain

$$(A.3) \quad \int_H \exp \left( - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2 \right) \, dh
\leq \int_{H^1(\square_{n-m})} \exp \left( - \sum_{e \subseteq \square_{n-m}} \lambda 3^{(d-1)m} \left( 3^{-\frac{dm}{2}} \nabla h(e) - X_\psi(e) \right)^2 \right) \, dh.
$$

We denote by $V(\square_{n-m})$ the vector space of vector field of $\square_{n-m}$ and equip it with the standard $L^2$ scalar product. The idea is then to consider the following orthogonal decomposition

$$
V(\square_{n-m}) = \nabla h^1(\square_{n-m}) \perp \left( \nabla h^1(\square_{n-m}) \right)^\perp.
$$

We can then consider the orthogonal decomposition of $X_\psi$,

$$
X_\psi = \nabla h_\psi + X_\psi^\perp,
$$

where $h_\psi \in \nabla h^1(\square_{n-m})$ and $X_\psi^\perp \in \left( \nabla h^1(\square_{n-m}) \right)^\perp$. Using this decomposition, we have

$$
\sum_{e \subseteq \square_{n-m}} \left( 3^{-\frac{dm}{2}} \nabla h(e) - X_\psi(e) \right)^2 = \sum_{e \subseteq \square_{n-m}} \left( 3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi \right)^2 + \sum_{e \subseteq \square_{n-m}} \left( X_\psi^\perp \right)^2
\geq \sum_{e \subseteq \square_{n-m}} \left( 3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi \right)^2.
$$

Using the previous inequality, the estimate (A.3) becomes

$$
\int_H \exp \left( - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2 \right) \, dh
\leq \int_{H^1(\square_{n-m})} \exp \left( - \sum_{e \subseteq \square_{n-m}} \lambda 3^{(d-1)m} \left( 3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi \right)^2 \right) \, dh.
$$

We then use the translation invariance of the Lebesgue measure to prove

$$
\int_{H^1(\square_{n-m})} \exp \left( - \sum_{e \subseteq \square_{n-m}} \lambda 3^{(d-1)m} \left( 3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi \right)^2 \right) \, dh
= \int_{H^1(\square_{n-m})} \exp \left( - \sum_{e \subseteq \square_{n-m}} \lambda 3^{-m} \nabla h(e)^2 \right) \, dh.
$$
Combining the two previous displays yields

\[
\int_H \exp \left( - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2 \right) \, dh \leq \int_{h^1(\Box_{n-m})} \exp \left( - \sum_{e \in \Box_{n-m}} \lambda 3^{-m} \nabla h(e)^2 \right) \, dh.
\]

We then perform a change of variable to get, for some \( C := C(d, \lambda) < \infty \),

\[
\int_H \exp \left( - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2 \right) \, dh 
\leq \left( \frac{3^m}{\lambda} \right) \frac{3^{d(n-m)} - 1}{\lambda^2} \int_{h^1(\Box_{n-m})} \exp \left( - \sum_{e \in \Box_{n-m}} \nabla h(e)^2 \right) \, dh,
\]

since \( \dim h^1(\Box_{n-m}) = 3^{d(n-m)} - 1 \). Taking the logarithm and applying Proposition 3.8, we see that

\[
\log \int_H \exp \left( - \sum_{e \in B_{m,n}} \lambda (\nabla \psi(e) + \nabla h(e))^2 \right) \, dh \leq Cm \left( 3^{d(n-m)} - 1 \right) + C|\Box_{n-m}|
\leq Cm3^{d(n-m)}.
\]

This completes the proof of Proposition A.1.

We now turn to the proof of the second technical lemma of the appendix which allows to use the uniform convexity of \( V \) to obtain an \( L^2 \) estimate when perturbing around \( \psi_z \). This lemma and the notation are used in Step 4 of the proof of Proposition 4.4.

**Lemma A.2.** There exists a constant \( C := C(d, \lambda) < \infty \) such that for each \( n \in \mathbb{N} \),

\[
(A.4) \quad \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in Z_{n,2n}} \sum_{e \subseteq z + \Box_{n}} V_e (\nabla \nu^*_n(q)(e) + \nabla \kappa(e)) \right] 
\leq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in Z_{n,2n}} \sum_{e \subseteq z + \Box_{n}} V_e (\nabla \psi_z(e)) \right]
+ C(1 + |q|^2)3^{-\frac{n}{2}} + C\mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in Z_{n,2n}} \sum_{e \subseteq z + \Box_{n}} |\nabla \kappa(e) - \nabla \sigma_z(e)|^2 \right].
\]

**Proof.** Denote by \( \sigma \) the random variable taking values in \( \oplus_{z \in Z_{n,2n}} \hat{h}^1(z + \Box_n) \) given by

\[
\sigma = \sum_{z \in Z_{n,2n}} \sigma_z,
\]

we also recall that the random variables \( \phi_z \) and \( \psi_z \) are linked together by the relation

\[
\forall z \in Z_{n,2n}, \forall x \in z + \Box_n, \; \psi_z(x) = \sigma_z(x) - \nabla \nu^*_n(q) \cdot x.
\]

Let \( P \) be the orthogonal projection from \( h^1(\Box_{2n}) \) to \( \oplus_{z \in Z_{n,2n}} \hat{h}^1(z + \Box_n) \). Note the operator \( P \) satisfies the following property

\[
(A.5) \quad \forall g \in h^1(\Box_{2n}), \forall z \in Z_{n,2n}, \forall e \subseteq z + \Box_n, \; \nabla Pg(e) = \nabla g(e).
\]

Denote by \( \xi \) the random variable taking values in \( \oplus_{z \in Z_{n,2n}} \hat{h}^1(z + \Box_n) \), defined according to

\[
\xi := 2\sigma - P\kappa,
\]
We first simplify a bit the previous display by removing the linear terms in the left and right-hand side of the previous estimate. Using that \( \sigma(z + \square_n) \) is a random variable \( \sum_{z \in \mathbb{Z}^2} \psi_z \) is the minimizer of the problem

\[
\inf_{\mathbb{P}} \mathbb{E} \left[ \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} (V_e(\nabla \psi') - q \cdot \nabla \psi'(e)) \right] + H(\mathbb{P}),
\]

where the infimum is taken over all the probability measures on \( \Theta \) and the random variable of law \( \mathbb{P} \). In particular, using the translation invariance of the entropy, this gives

\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} V_e(\nabla \nu_n^*(q)(e) + \nabla \xi(e)) - q \cdot (\nabla \nu_n^*(q) + \nabla \xi)(e) \right] + \frac{1}{|\square_n|} H(\mathbb{P}^\xi)
\]}

\[
\geq \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} V_e(\nabla \psi_z(e)) - q \cdot \nabla \psi_z(e) \right] + \frac{1}{|\square_n|} H(\mathbb{P}^\xi).
\]

We first simplify a bit the previous display by removing the linear terms in the left and right-hand side of the previous estimate. Using that \( \nabla \nu_n^*(q) = \mathbb{E} \left[ \langle \nabla \psi_n, q, \square_n \rangle \right] \), we obtain

\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} q \cdot \nabla \psi_z(e) \right] = q \cdot \nabla \nu_n^*(q),
\]

and, using the definition of \( \sigma \),

\[
\mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} q \cdot (\nabla \nu_n^*(q) + \nabla \xi)(e) \right] = q \cdot \nabla \nu_n^*(q) + \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} q \cdot \nabla \nu_n^*(q) \right] + \mathbb{E} \left[ \frac{1}{|\square_n|} \sum_{z \in \mathbb{Z}^2} \sum_{e \subseteq z + \square_n} q \cdot \nabla \nu_n^*(q) \right].
\]
We denote by $B^+_{n,2n}$ the set of edges of $\square^+_{2n}$ which do not belong to a cube of the form $z + \square_n$, for $z \in \mathbb{Z}_{n,2n}$, i.e.,

$$B^+_{n,2n} := \{ e \subseteq \square^+_{2n} : \forall z \in \mathbb{Z}_{n,2n}, e \not\subseteq z + \square_n \} .$$

Using this set, one can split the sum

$$\sum_{e \subseteq \square_{2n}} = \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_n} + \sum_{e \in B^+_{n,2n}} .$$

Note also that the set $B^+_{n,2n}$ is almost equal to the set $B_{n,2n}$, the only difference is that we have added the edges which belong to $\square^+_{2n}$ but not $\square_{2n}$, which is a small boundary layer of edges. Additionally, one has the estimate on the cardinality of $B^+_{n,2n}$:

$$|B^+_{n,2n}| \leq C3^{-n} |\square_{2n}| .$$

Using the partition of the sum, the fact that $\kappa \in h_0^1(\square^+_{2n})$ and the property (A.5), one obtains,

$$\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_n} q \cdot (\nabla P\kappa)(e) \right] = \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_n} q \cdot \nabla \kappa(e) \right] = \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B^+_{n,2n}} q \cdot \nabla \kappa(e) \right] .$$

We then apply the Cauchy-Schwarz inequality as well as the definition of $\kappa$ given in (4.19) to obtain

$$\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B^+_{n,2n}} q \cdot \nabla \kappa(e) \right] \leq |q| \left( \mathbb{E} \left[ \frac{B^+_{n,2n}}{|\square_{2n}|} \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \in B^+_{n,2n}} |\nabla \kappa(e)|^2 \right] \right)^{\frac{1}{2}} \leq C|q|3^{-\frac{n}{2}} \left( \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square^+_{2n}} |\nabla \kappa(e)|^2 \right] \right)^{\frac{1}{2}} \leq C|q|3^{-\frac{n}{2}} \left( \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square^+_{2n}} |f(e)|^2 \right] \right)^{\frac{1}{2}} \leq C|q|3^{-\frac{n}{2}} (1 + |q|) .$$

But using the definition of $f$ given in (4.12) together with Proposition 3.13, one has

$$\left( \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{e \subseteq \square^+_{2n}} |f(e)|^2 \right] \right)^{\frac{1}{2}} \leq C(1 + |q|) .$$

A combination of the previous displays gives

$$\left| \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_n} q \cdot (\nabla P\kappa)(e) \right] \right| \leq C3^{-\frac{n}{2}} (1 + |q|^2) .$$
Using the previous estimate in (A.7), one obtains the simpler display

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} V_e (\nabla \psi_z(e) + \nabla \xi(e)) \right] + \frac{1}{|\square_{2n}|} H (\mathbb{P}_\xi) \geq \mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} V_e (\nabla \psi_z(e)) \right] + \frac{1}{|\square_{2n}|} H (\mathbb{P}_\psi) - C3^{-\frac{n}{2}} (1 + |q|^2).
\]

We now show the following estimate comparing the entropy of \( \mathbb{P}_\xi \) and \( \mathbb{P}_\psi \),

\[
\mathbb{E} \left[ \frac{1}{|\square_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} V_e (\nabla \psi_z(e)) \right] + \frac{1}{|\square_{2n}|} H (\mathbb{P}_\psi) - C3^{-n}.
\]

First we recall the definition of the linear operator \( L \) given in (4.15) and that the random variable \( \kappa \) is defined by

\[
\kappa := L \left( \sum_{z \in \mathbb{Z}_{n,2n}} \sigma_z \right).
\]

We consequently have

\[\xi = (2\text{Id} - P \circ L)\psi,\]

where \( 2\text{Id} - P \circ L \) is seen as a linear operator from \( \oplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \square_{n}) \) into itself. Using the change of variable formula for the entropy, one obtains

\[
\frac{1}{|\square_{2n}|} H (\mathbb{P}_\xi) = \frac{1}{|\square_{n}|} H (\mathbb{P}_{\xi_{n,q}}) - \ln |\det(2\text{Id} - P \circ L)|.
\]

Since the dimension of \( \oplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \square_{n}) \) is \( 3^{2dn} - 3^{dn} \), we denote by \( l_1, \ldots, l_{3^{2dn} - 3^{dn}} \) the eigenvalues (potentially complex and with repetition) of \( P \circ L \). We thus have

\[
\ln |\det(2\text{Id} - P \circ L)| = \sum_{i=0}^{3^{2dn} - 3^{dn}} \ln |2 - l_i|.
\]

We now prove the two following statements on \( l_i \)

1. for each \( i \in \{1, \ldots, 3^{2dn} - 3^{dn}\}, |l_i| \leq 1.
2. There exists a constant \( C := C(d) < \infty \) such that at least \( 3^{2dn} - C3^{dn} \) eigenvalues \( l_i \) are equal to 1.

To prove the first fact, note that, by (A.5), for each \( \psi \in \oplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \square_{n}), \)

\[
\sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla P \circ L(\psi)(e)|^2 = \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla L(\psi)(e)|^2.
\]

Moreover by (4.30), one has

\[
\sum_{e \subseteq \square_{2n}^+} |\nabla L(\psi)(e)|^2 \leq \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla \psi(e)|^2.
\]

Since one clearly has

\[
\sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla L(\psi)(e)|^2 \leq \sum_{e \subseteq \square_{2n}^+} |\nabla L(\psi)(e)|^2,
\]

we obtain

\[
\sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla P \circ L(\psi)(e)|^2 \leq \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \square_{n}} |\nabla \psi(e)|^2.
\]
Thus if \( l_i \) is an eigenvalue of \( P \circ L \), consider \( \psi_i \) an eigenvector (which may be complex) associated to this eigenvalue, then we have
\[
|l_i|^2 \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} |\nabla \psi_i(e)|^2 \leq \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} |\nabla \psi_i(e)|^2,
\]
which implies \( |l_i| \leq 1 \) as soon as \( \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} |\nabla \psi_i(e)|^2 \neq 0 \), but since \( \psi_i \in \bigoplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \Box_n) \), we have
\[
\sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} |\nabla \psi_i(e)|^2 = 0 \iff \psi_i = 0.
\]
This completes the proof of the first fact.

We now show the second fact. To this end we proceed exactly as in Step 1 of Proposition 4.4, where we proved that if we define the trimmed cube \( \Box_n^- \),
\[
\Box_n^- := \left( -\frac{3^n}{2}, \frac{3^n}{2} \right) \cap \mathbb{Z}^d = \Box_n \setminus \partial \Box_n,
\]
then for each \( \psi \in \bigoplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \Box_n^-) \) we have
\[
L(\psi) = \psi.
\]
Moreover, from (4.25) one has the estimate on the dimension of \( \bigoplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \Box_n^-) \),
\[
\dim \left( \bigoplus_{z \in \mathbb{Z}_{n,2n}} \hat{h}^1(z + \Box_n^-) \right) \geq 3^{2dn} - C3^{(2d-1)n}.
\]
This implies that among the \( l_i \), at least \( 3^{2dn} - C3^{(2d-1)n} \) of them are equal to 1. Without loss of generality, we can thus assume that for each \( i \in \{1, \ldots, 3^{2dn} - C3^{(2d-1)n}\} \), \( l_i = 1 \).

Combining (1) and (2), we obtain
\[
\frac{1}{|\Box_{2n}|} \ln |\det (2\text{Id} - P \circ L)| \leq \sum_{i=0}^{3^{2dn} - 3^{dn}} |\ln |2 - l_i|| \leq \sum_{i=3^{2dn} - C3^{(2d-1)n}}^{3^{2dn} - 3^{dn}} |\ln |2 - l_i|| \leq C3^{(2d-1)n}.
\]
Thus
\[
\frac{1}{|\Box_{2n}|} \ln |\det (2\text{Id} - P \circ L)| \leq C3^{-n}.
\]
Combining this estimate with (A.10) gives
\[
\frac{1}{|\Box_{2n}|} H(\mathbb{P}_0) \leq \frac{1}{|\Box_n|} H(\mathbb{P}_{n,q}^*) + C3^{-n}.
\]
This is precisely (A.9)

We then combine (A.8) and (A.9) to obtain
\[
\mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} V_e(\nabla \xi(e)) \right] \geq \mathbb{E} \left[ \frac{1}{|\Box_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \Box_n} V_e(\nabla \psi_z(e)) \right] - C(1 + |q|^2)3^{-\frac{n}{2}}.
\]
Using this inequality together with \((A.6)\) gives
\[
\mathbb{E} \left[ \frac{1}{|\mathbb{Z}_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \mathbb{Z}_n} V_e (\nabla \psi(e)) \right] \geq \mathbb{E} \left[ \frac{1}{|\mathbb{Z}_{2n}|} \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \mathbb{Z}_n} V_e (\nabla \kappa(e)) \right] - C(1 + |q|^2)3^{-\frac{d}{2}}
\]
\[
- C \mathbb{E} \left( \sum_{z \in \mathbb{Z}_{n,2n}} \sum_{e \subseteq z + \mathbb{Z}_n} \left| \nabla \psi(e) - \nabla \kappa(e) \right|^2 \right).
\]
This is \((A.4)\) and thus the proof of Lemma \((A.2)\) is complete.

We then prove the last lemma of this appendix. It gives a quadratic upper bound for the value of \(\nu(U,p)\) for any bounded domain \(U \subseteq \mathbb{Z}^d\).

**Proposition A.3.** There exists a constant \(C := C(d, \lambda) < \infty\) such that for each bounded domain \(U \subseteq \mathbb{Z}^d\) and each \(p \in \mathbb{R}^d\),
\[
\nu(U,p) \leq C(1 + |p|^2).
\]

**Remark A.4.** This statement is a more general version of the upper bound for \(\nu\) than the one given in Proposition 3.8, since it is valid for any bounded domain \(U \subseteq \mathbb{Z}^d\), nevertheless the argument presented here does not give a lower bound as the one we computed in the case of cubes. It also does not give bounds on \(\nu^\star\), this is why this statement is presented in the appendix.

**Proof.** Consider a random variable \(X\), taking values in \(h^1_0(U)\) whose law is defined by
- for each \(x \in U\), the law of \(X(x)\) is uniform in \([0,1]\)
- the random variables \(X(x)\), for \(x \in U\) are independent.

Using that the entropy of the law uniform in \([0,1]\) is equal to 0 together with Proposition 2.4, one obtains
\[
H(\mathbb{P}_X) = 0.
\]

Then by Proposition 2.8, one has the following computation
\[
\nu(U,p) \leq \mathbb{E} \left[ \frac{1}{|U|} \sum_{e \subseteq U} V_e (p(e) + \nabla X(e)) \right] + \frac{1}{|U|} H(\mathbb{P}_X)
\]
\[
\leq \mathbb{E} \left[ \frac{1}{|U|} \sum_{e \subseteq U} V_e (p(e) + \nabla X(e)) \right].
\]

We then use the bound \(V_e(x) \leq \frac{1}{4} |x|^2\) combined with the estimate \(|\nabla X(e)| \leq 1\) for each \(e \subseteq U\) to obtain
\[
\mathbb{E} \left[ \frac{1}{|U|} \sum_{e \subseteq U} V_e (p(e) + \nabla X(e)) \right] \leq C(1 + |p|^2).
\]

A combination of the two previous displays completes the proof of the proposition. \(\square\)

**Appendix B. Functional inequalities**

The goal of this appendix is to prove some classic inequalities in the theory of elliptic equations in the setting of the \(\nabla \phi\) model. These inequalities are proved with the random variable \(\psi_{n,q}\) associated to the law \(\mathbb{P}_{n,q}\) because it is needed in the proof of Theorem 1.2, nevertheless similar statements, with similar proofs, should exist for the random variable \(\phi_{n,p}\) associated to the law \(\mathbb{P}_{n,p}\).
Proposition B.1 (Interior Caccioppoli inequality). There exists a constant $C := C(d, \lambda) < \infty$ such that for every integer $n \geq 1$, every $x \in \square_n$ and every $r \geq 1$ such that $B(x, 2r) \subseteq \square_n$

$$
\mathbb{E} \left[ \sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq \frac{C^r}{r^2} \mathbb{E} \left[ \sum_{x \in B(x,2r)} |\psi_{n,q}(y) - (\psi_{n,q})_{B(x,2r)}|^2 \right] + Cr^d.
$$

Proof. Let $\eta$ be a cutoff function defined on the discrete lattice $\square_n$ valued in $\mathbb{R}$ satisfying

$$
1_{B(x,r)} \leq \eta \leq 1_{B(x,2r)} \quad \text{and} \quad \forall e = (x,y) \subseteq \square_n, |\nabla \eta(e)|^2 \leq Cr^{-2} (\eta(x) + \eta(y)),
$$

for some $C := C(d) < \infty$. For $t \geq 0$, denote by $L_t$ the following linear operator

$$
L_t := \left\{ \begin{array}{ll}
\hat{h}^1(\square_n) & \rightarrow \ \hat{h}^1(\square_n) \\
\psi & \mapsto \ \psi + t\eta \left( \psi - \left( \psi \right)_{B(x,2r)} \right) - \left( \psi + t\eta \left( \psi - \left( \psi \right)_{B(x,2r)} \right) \right)_{\square_n}.
\end{array} \right.
$$

As a remark, note that the last term on the right-hand side can be rewritten

$$
\left( \psi + t\eta \left( \psi - \left( \psi \right)_{B(x,2r)} \right) \right)_{\square_n} = t \left( \eta \left( \psi - \left( \psi \right)_{B(x,2r)} \right) \right)_{\square_n},
$$

since $\psi \in \hat{h}^1(\square_n)$. Note that $L_0$ is the identity of $h^1(\square_n)$. We now show the following inequality which estimates the distance between $L_t$ and the identity of $h^1(\square_n)$, in the $L^2$ operator norm:

$$
\forall \psi \in \hat{h}^1(\square_n), \quad \sum_{x \in \square_n} |\psi(x) - L_t(\psi)(x)|^2 \leq |t|^2 \sum_{x \in \square_n} |\psi(x)|^2.
$$

This is a consequence of the computation

$$
\sum_{x \in \square_n} |\psi(x) - L_t(\psi)(x)|^2 \leq \sum_{x \in \square_n} |t\eta(x) \left( \psi(x) - \left( \psi \right)_{B(x,2r)} \right)|^2
\leq |t|^2 \sum_{x \in B(x,2r)} |\psi(x) - \left( \psi \right)_{B(x,2r)}|^2
\leq |t|^2 \sum_{x \in B(x,2r)} |\psi(x)|^2.
$$

This implies in particular that for each $t \in (-1,1)$, the operator $L_t$ is bijective. Note also that by definition of $L_t$,

$$(B.1) \quad \forall \psi \in \hat{h}^1(\square_n), \forall e \subseteq \square_n, \nabla L_t(\psi)(e) = \nabla \psi(e) + t\nabla \left( \eta \left( \psi - \left( \psi \right)_{B(x,2r)} \right) \right)(e)
$$

Fix $q \in \mathbb{R}$. We use the random variable $L_t(\psi_{n,q})$ as a test random variable in Proposition 2.8. This yields

$$
\mathbb{E} \left[ - \sum_{e \subseteq \square_n} \left( V_e \left( \nabla \psi_{n,q}(e) \right) - q \cdot \nabla \psi_{n,q}(e) \right) \right] - H(\mathbb{P}_{n,q})
\geq \mathbb{E} \left[ - \sum_{e \subseteq \square_n} \left( V_e \left( \nabla L_t(\psi_{n,q})(e) \right) - q \cdot \nabla L_t(\psi_{n,q})(e) \right) \right] - H \left( \mathbb{P}_{L_t(\psi_{n,q})} \right).
$$

First note that, since $\eta$ is supported in $B(x,2r) \subseteq \square_n$,

$$
\langle \nabla L_t(\psi_{n,q}) \rangle_{\square_n} = \langle \nabla \psi_{n,q} \rangle_{\square_n},
$$

consequently,

$$
\mathbb{E} \left[ \sum_{e \subseteq \square_n} q \cdot \nabla \psi_{n,q}(e) \right] = \mathbb{E} \left[ \sum_{e \subseteq \square_n} q \cdot \nabla L_t(\psi_{n,q})(e) \right].
$$
We first deal with the term coming from the entropy. By the chain rule, we have the formula

\[ E \left[ - \sum_{e \subseteq \square_n} V_e (\nabla \psi_{n,q}(e)) \right] - H(\mathbb{P}^*_n) \geq E \left[ - \sum_{e \subseteq \square_n} V_e (\nabla L_t (\psi_{n,q})(e)) \right] - H(\mathbb{P}_{L_t(\psi_{n,q})}). \]

By Proposition 2.3,

\[ H(\mathbb{P}_{L_t(\psi_{n,q})}) = H(\mathbb{P}^*_n) - \ln \det L_t. \]

Using the previous display and the formula for \( L_t \), we obtain, for each \( t \in (-1,1) \),

\[ E \left[ \sum_{e \subseteq \square_n} V_e (\nabla \psi_{n,q}(e) + t \nabla (\eta - (\psi_{n,q})_{B(x,2r)}) \psi_{n,q}(e)) \right] - \ln \det L_t \geq 0. \]

It is clear that the function \( t \rightarrow \ln \det L_t \) is smooth for \( t \in (-1,1) \). In particular, dividing the previous display by \( t \) and sending \( t \) to 0 gives

\[ \frac{d}{dt|_{t=0}} \ln \det L_t = 0. \]

We first deal with the term coming from the entropy. By the chain rule, we have the formula

\[ \frac{d}{dt|_{t=0}} \ln \det L_t = \text{tr} \ L'_0, \]

where \( L'_0 \) denote the derivative of the operator \( L_t \) at \( t = 0 \), it is given by the explicit formula

\[ L'_0 := \left\{ \begin{array}{ll}
\hat{h}^1(\square_n) & \rightarrow \hat{h}^1(\square_n) \\
\psi & \mapsto \eta (\psi - (\psi)_{B(x,2r)}) - \left( \eta (\psi - (\psi)_{B(x,2r)}) \right)_{\square_n}.
\end{array} \right. \]

In particular, for each \( \psi \in \hat{h}^1(\square_n) \),

\[ \sum_{x \in \square_n} |L'_0(\psi)(x)|^2 \leq \sum_{x \in \square_n} |\eta(x)(\psi(x) - (\psi)_{B(x,2r)})|^2 \leq \sum_{x \in B(x,2r)} |\psi(x) - (\psi)_{B(x,2r)}|^2 \leq \sum_{x \in B(x,2r)} |\psi(x)|^2. \]

This implies that every function \( \psi \) supported in \( \square_n \setminus B(x,2r) \) is in the kernel of \( L'_0 \) and thus we have

\[ \dim \ker L'_0 \geq |\square_n| - C r^d, \]

for some \( C := C(d) < \infty \). We now combine the two previous displays

\[ |\text{tr} L'_0| \leq \dim \hat{h}^1(\square_n) - \dim \ker L'_0 \leq C r^d. \]

We now turn to the first term on the right-hand side of (B.2). To simplify the notation in the following computation, we use the notation

\[ \chi_{n,q} := \psi_{n,q} - (\psi_{n,q})_{B(x,2r)} \]
and compute
\[\sum_{e \subseteq B(x, 2r)} V'(\nabla \chi_{n,q})(e) \nabla (\eta \chi_{n,q})(e)\]
\[= \sum_{x,y \in B(x, 2r), x \sim y} \left( \eta(x) \chi_{n,q}(x) - \eta(y) \chi_{n,q}(y) \right) \nabla \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right)\]
\[= \sum_{x,y \in B(x, 2r), x \sim y} \eta(x) \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right) V^{\prime}_{(x,y)} \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right)\]
\[+ \sum_{x,y \in B(x, 2r), x \sim y} \chi_{n,q}(y) \left( \eta(x) - \eta(y) \right) V^{\prime}_{(x,y)} \left( \chi_{n,q}(x) - \chi_{n,q}(x) \right) .\]

Using the uniform convexity of \( V \) and (B.2), we obtain, by taking the expectation,
\[\lambda \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \eta(x) \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right)^2 \right] \leq \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \eta(x) \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right) V^{\prime}_{(x,y)} \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right) \right] \]
\[\leq \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \left| \chi_{n,q}(y) \right| \left| \eta(x) - \eta(y) \right| \left| V^{\prime}_{(x,y)} \left( \chi_{n,q}(x) - \chi_{n,q}(x) \right) \right| \right] + | \text{tr} L^\prime | .\]

We then use the bound \( V^{\prime}_{e}(x) \leq \lambda |x| \). This yields
\[\lambda \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \eta(x) \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right)^2 \right] \leq \lambda \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \frac{|\eta(x) - \eta(y)|^2}{\eta(x) + \eta(y)} |\chi_{n,q}(y)|^2 \right] \]
\[+ \lambda \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} (\eta(x) + \eta(y)) \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right)^2 \right] + | \text{tr} L^\prime | \]
\[\leq 3^{-2n} \mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \left| \chi_{n,q}(y) \right|^2 \right] + \frac{\lambda}{2} \mathbb{E} \left[ \sum_{x,y \in U, x \sim y} \eta(x) \left| \chi_{n,q}(x) - \chi_{n,q}(y) \right|^2 \right] + | \text{tr} L^\prime | .\]

Absorbing the second term on the right back on the left side and using the estimate (B.3) gives, for some \( C := C(d, \lambda) < \infty \),
\[\mathbb{E} \left[ \sum_{x,y \in B(x, 2r), x \sim y} \eta(x) \left( \chi_{n,q}(x) - \chi_{n,q}(y) \right)^2 \right] \leq C r^{-2} \mathbb{E} \left[ \sum_{x} \left| \chi_{n,q}(x) \right|^2 \right] + Cr^d .\]

Now we replace \( \chi_{n,q} \) by \( \psi_{n,q} - (\psi_{n,q})_{B(x, 2r)} \) to eventually obtain
\[\mathbb{E} \left[ \sum_{e \subseteq B(x, r)} \left| \nabla \chi_{n,q}(e) \right|^2 \right] \leq C r^{-2} \mathbb{E} \left[ \sum_{e \subseteq B(x, 2r)} \left| \psi_{n,q}(e) - (\psi_{n,q})_{B(x, 2r)} \right|^2 \right] + Cr^d .\]

This is the desired result. \( \square \)
The next statement we wish to obtain is a reverse Hölder inequality on the random variable \( \psi_{n,q} \). It is obtained by combined the Caccioppoli inequality proved in the previous proposition with the Sobolev inequality which is recalled below, since we only need to apply this inequality to balls, we only state it for balls.

**Proposition B.2** (Sobolev inequality on \( \mathbb{Z}^d \)). There exists a constant \( C := C(d) < \infty \) such that for each \( x \in \mathbb{Z}^d \), each \( r \geq 1 \), each exponent \( s \in \left( \frac{d}{d-1}, \infty \right) \) and each function \( f : B(x,r) \to \mathbb{R} \) satisfying

\[
\sum_{x \in B(x,r)} f(x) = 0,
\]

we have the estimate

\[
\left( \sum_{x \in B(x,r)} |f(x)|^s \right)^{\frac{1}{s}} \leq C \left( \sum_{e \subseteq B(x,r)} |\nabla f(e)|^{s_*} \right)^{\frac{1}{s_*}},
\]

where \( s_* \) is the Sobolev conjugate defined from \( s \) by the formula

\[
s_* := \frac{sd}{s + d}.
\]

This inequality can be deduced from the the continuous Sobolev inequality (on \( \mathbb{R}^d \)) thanks to an interpolation argument. From the Sobolev inequality and the Caccioppoli inequality, we deduce the following reverse Hölder property

**Proposition B.3** (Reverse Hölder inequality for \( \mathbb{P}_{n,q}^* \)). There exists a constant \( C := C(d, \lambda) < \infty \) such that for every integer \( n \geq 1 \), every \( x \in \square_n \), every \( r \geq 1 \) such that \( B(x,2r) \subseteq \square_n \), and every \( q \in \mathbb{R}^d \),

\[
\mathbb{E} \left[ \frac{1}{|B(x,r)|} \sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq C \left( \frac{1}{|B(x,2r)|} \sum_{e \subseteq B(x,2r)} \mathbb{E} \left[ |\nabla \psi_{n,q}(e)|^2 \right] \right)^{\frac{d+2}{d}} + C.
\]

**Proof.** Fix \( q \in \mathbb{R}^d \), an integer \( n \geq 1 \), and a ball \( B(x,r) \) with \( x \in \square_n \) such that \( B(x,2r) \subseteq \square_n \). By Proposition B.1, we have the inequality

\[
\mathbb{E} \left[ \sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq C \left( \sum_{x \in B(x,2r)} \mathbb{E} \left[ |\psi_{n,q}(y) - (\psi_{n,q})_{B(x,2r)}|^2 \right] \right)^{\frac{1}{2}} + C r^d.
\]

We then apply Proposition B.2 with \( s = 2 \) and \( s_* = \frac{2d}{d+2} \) and we obtain

\[
\mathbb{E} \left[ \sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq C \left( \sum_{e \subseteq B(x,r)} \mathbb{E} \left[ |\nabla \psi_{n,q}(e)|^2 \right] \right)^{\frac{d+2}{d}} + C r^d.
\]

Then for each \( N \in \mathbb{N} \) denote by

\[
\mathbb{R}_+^N := \{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 \geq 0, \ldots, x_N \geq 0 \}.
\]

With this notation, note that the mapping

\[
F := \begin{cases} \mathbb{R}_+^N & \to \mathbb{R} \\ (x_1, \ldots, x_N) & \mapsto \left( \sum_{e \subseteq B(x,r)} |x_i|^{\frac{d}{d+2}} \right)^{\frac{d+2}{d}} \end{cases}
\]
is concave. Then we pick $N$ to be the number of edges of $\Box_n$ and we apply Jensen’s inequality to the random variable $\left(\left|\nabla \psi_{n,q}(e)\right|^2\right)_{e \subseteq \mathbb{R}^N}$, which is valued in $\mathbb{R}_+^N$, to obtain

$$
\mathbb{E}\left[ \left( \sum_{e \subseteq B(x,r)} \left|\nabla \psi_{n,q}(e)\right|^2 \right)^{\frac{d+2}{d}} \right] \leq \left( \sum_{e \subseteq B(x,r)} \mathbb{E}\left[ \left|\nabla \psi_{n,q}(e)\right|^2 \right]^{\frac{d+2}{d}} \right)^{\frac{d}{d+2}}.
$$

Combining this estimate with (B.4) and dividing by $r^d$ completes the proof of Proposition B.3. □

We then combine the previous estimate with the discrete version of the Gehring Lemma, which is stated in the following proposition. The continuous version of this result can be found in [19]. The discrete version stated below can be deduced by interpolation.

**Proposition B.4** (Discrete Gehring Lemma). Fix $q < 1$, $K \geq 1$ and $R > 0$. Suppose that we are given two (discrete) functions $f, g : B(0, R) \to \mathbb{R}$, and that $f$ satisfies the following reverse Hölder inequality, for each $z \in \mathbb{Z}^d$ and each $r \geq 1$ such that $B(z, 2r) \subseteq B(0, R)$,

$$
\frac{1}{|B(x, r)|} \sum_{x \in B(x, r)} |f(x)| \leq K \left( \frac{1}{|B(x, 2r)|} \sum_{x \in B(x, 2r)} |f(x)|^q \right)^{\frac{1}{q}} + \frac{K}{|B(x, 2r)|} \sum_{x \in B(x, 2r)} |g(x)|,
$$

then there exist an exponent $\delta := \delta(q, K, d) > 0$ and a constant $C := C(q, K, d) < \infty$ such that

$$
\left( \frac{1}{|B(x, R)|} \sum_{x \in B(x, \frac{R}{2})} |f(x)|^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq C \left( \frac{1}{|B(x, R)|} \sum_{x \in B(x, R)} |f(x)| \right) + C \left( \frac{1}{|B(x, R)|} \sum_{x \in B(x, R)} |g(x)|^{1+\delta} \right)^{\frac{1}{1+\delta}}.
$$

From this estimate, one obtains the following version of the interior Meyers estimate for the $\nabla \phi$ model. The idea is to combine the reverse Hölder inequality and the Gehring’s Lemma to improve the integrability of the expectation of the field $\psi_{n,q}$ (seen as a function from $\Box_n$ to $\mathbb{R}$) form $L^2$ to $L^{2+\delta}$. The continuous version of this estimate can be found in [18].

**Proposition B.5** (Interior Meyers estimate for $\mathbb{P}^n_{n,q}$). For each $\alpha \in (0, 1]$ and each $n \in \mathbb{N}$, denote by $\alpha \Box_n$ the cube

$$
\alpha \Box_n := \left( -\frac{\alpha 3^n}{2}, \frac{\alpha 3^n}{2} \right)^d \cap \mathbb{Z}^d.
$$

Fix $q \in \mathbb{R}^d$. For each $\alpha \in (0, 1)$, each $n \in \mathbb{N}$, There exist an exponent $\delta := \delta(d, \lambda) > 0$ and a constant $C := C(d, \lambda, \alpha) < \infty$ such that

$$
\left( \frac{1}{|\alpha \Box_n|} \sum_{e \subseteq \alpha \Box_n} \mathbb{E}\left[ \left|\nabla \psi_{n,q}(e)\right|^2 \right]^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq C \left\| \sum_{e \subseteq \Box_n} \mathbb{E}\left[ \left|\nabla \psi_{n,q}(e)\right|^2 \right] \right\| + C.
$$

**Proof.** The main idea of the proof is to apply the Gehring Lemma, Proposition B.4, with the following choice of functions

$$
\forall x \in \Box_n, \ f(x) := \mathbb{E}\left[ \sum_{y \in \Box_n, x \sim y} \left|\psi_{n,q}(y)\right|^2 \right] \text{ and } g(x) = 1.
$$
By Proposition B.3, we have the following reverse Hölder inequality: there exists $C := C(d, \lambda) < \infty$ such that for each $z \in \mathbb{Z}^d$ and each $r \geq 1$ satisfying $B(z, 2r) \subseteq \Box_n$, 

$$\frac{1}{|B(x, r)|} \sum_{x \in B(x, r)} |f(x)| \leq C \left( \frac{1}{|B(x, 2r)|} \sum_{x \in B(x, 2r)} |f(x)|^{\frac{d+2}{d}} \right)^\frac{d}{d+2} + \frac{C}{|B(x, 2r)|} \sum_{x \in B(x, 2r)} |g(x)|.$$ 

Applying Proposition B.4, there exist an exponent $\delta := \delta(d, \lambda) > 0$ and a constant $C := C(d, \lambda) < \infty$, for each $z \in \Box_n$, $R \geq 1$ such that $B(x, 2R) \subseteq \Box_n$, 

$$\left( \frac{1}{|B(x, R)|} \sum_{x \in B(x, R)} |f(x)|^{1+\delta} \right)^\frac{1}{1+\delta} \leq \frac{C}{|B(x, 2R)|} \sum_{x \in B(x, 2R)} |f(x)|$$

$$+ C \left( \left( \frac{1}{|B(x, 2R)|} \sum_{x \in B(x, 2R)} |g(x)|^{1+\delta} \right)^\frac{1}{1+\delta} \right).$$

Which can be rewritten

$$\left( \frac{1}{|B(x, R)|} \sum_{x \in B(x, R)} \mathbb{E} [\| \nabla \psi_{n,q}(x) \|^2]^{1+\delta} \right)^\frac{1}{1+\delta} \leq \frac{C}{|B(x, 2R)|} \sum_{x \in B(x, 2R)} \mathbb{E} [\| \nabla \psi_{n,q}(x) \|^2] + C.$$

We then conclude that, for each $\alpha \in [0, 1)$, the cube $\alpha \Box_n$ can be covered by finitely many balls of the form $B(x, R)$ such that $B(x, 2R)$ is included in $\Box_n$. The cardinality of this covering family can be bounded from above by a constant depending only on $d$ and $\alpha$. This implies that, for each $\alpha \in (0, 1)$, there exist an exponent $\delta := \delta(d, \lambda) > 0$ and a constant $C := C(d, \lambda, \alpha) < \infty$ such that

$$\left( \frac{1}{|\alpha \Box_n|} \sum_{x \in \alpha \Box_n} \mathbb{E} [\| \nabla \psi_{n,q}(x) \|^2]^{1+\delta} \right)^\frac{1}{1+\delta} \leq \frac{C}{|\Box_n|} \sum_{x \in \Box_n} \mathbb{E} [\| \nabla \psi_{n,q}(x) \|^2] + C.$$ 

\[\square\]

Remark B.6. Combining the Meyers estimate with Proposition 3.13, one obtains

$$\left( \frac{1}{|\alpha \Box_n|} \sum_{x \in \alpha \Box_n} \mathbb{E} \left[ \| \nabla \psi_{n,q}(x) \|^2 \right]^{1+\delta} \right)^\frac{1}{1+\delta} \leq C(1 + |q|^2).$$

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