An Explicit Construction of Universally Decodable Matrices

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Abstract

Universally decodable matrices can be used for coding purposes when transmitting over slow fading channels. These matrices are parameterized by positive integers \( L \) and \( n \) and a prime power \( q \). Based on Pascal’s triangle we give an explicit construction of universally decodable matrices for any non-zero integers \( L \) and \( n \) and any prime power \( q \) where \( L \leq q + 1 \). This is the largest set of possible parameter values since for any list of universally decodable matrices the value \( L \) is upper bounded by \( q + 1 \), except for the trivial case \( n = 1 \). For the proof of our construction we use properties of Hasse derivatives, and it turns out that our construction has connections to Reed-Solomon codes, Reed-Muller codes, and so-called repeated-root cyclic codes. Additionally, we show how universally decodable matrices can be modified so that they remain universally decodable matrices.

Index terms — Universally decodable matrices, coding for slow fading channels, Pascal’s triangle, rank condition, linear independence, Reed-Solomon codes, Reed-Muller codes, repeated-root cyclic codes, Hasse derivative.

1 Introduction

Let \( L \) and \( n \) be non-zero integers, let \( q \) be a prime power, let \([M] \triangleq \{0, \ldots, M-1\}\) for any positive integer \( M \), and let \([M] \triangleq \{ \} \) for any non-positive integer \( M \). While studying slow fading channels (c.f. e.g. [2]), Tavildar and Viswanath [3] introduced the communication system shown in Fig. 1 which works as follows. An information (column) vector \( \mathbf{u} \in \mathbb{F}_q^n \) is encoded into vectors \( \mathbf{x}_\ell \triangleq A_\ell \cdot \mathbf{u} \in \mathbb{F}_q^n \), \( \ell \in [L] \), where \( A_0, \ldots, A_{L-1} \) are \( L \) matrices over \( \mathbb{F}_q \) of size \( n \times n \). Upon sending \( \mathbf{x}_\ell \) over the \( \ell \)-th channel we receive \( \mathbf{y}_\ell \in (\mathbb{F}_q \cup \{\}^n) \),

\[ \mathbf{x}_0 \quad \text{0-th Channel} \quad \mathbf{x}_{L-1} \quad \text{L-th Channel} \]

\[ \mathbf{y}_0 \quad \text{Decoder} \quad \mathbf{y}_{L-1} \quad \text{Sink} \]

Figure 1: Communication system with \( L \) parallel channels.

**Index terms** — Universally decodable matrices, coding for slow fading channels, Pascal’s triangle, rank condition, linear independence, Reed-Solomon codes, Reed-Muller codes, repeated-root cyclic codes, Hasse derivative.
where the question mark denotes erasures. The channels are such that the received vectors $\mathbf{y}_1, \ldots, \mathbf{y}_{L-1}$ can be characterized as follows: there are integers $k_0, \ldots, k_{L-1}$, $0 \leq k_\ell \leq n$, $\ell \in [L]$ (that can vary from transmission to transmission) such that the first $k_\ell$ entries of $\mathbf{y}_\ell$ are non-erased and agree with the corresponding entries of $\mathbf{x}_\ell$ and such that the last $n - k_\ell$ entries of $\mathbf{y}_\ell$ are erased.

Based on these non-erased entries we would like to reconstruct $\mathbf{u}$. The obvious decoding approach works as follows: construct a $((\sum_{\ell \in [L]} k_\ell) \times n)$-matrix $\mathbf{A}$ that stacks the $k_0$ first rows of $\mathbf{A}_0$, ..., the $k_{L-1}$ first rows of $\mathbf{A}_{L-1}$; then construct a length-$((\sum_{\ell \in [L]} k_\ell) \times n)$ vector $\mathbf{y}$ that concatenates the $k_0$ first entries of $\mathbf{y}_0$, ..., the $k_{L-1}$ first entries of $\mathbf{A}_{L-1}$; finally, the vector $\hat{\mathbf{u}}$ is given as the solution of the linear equation system $\mathbf{A} \cdot \hat{\mathbf{u}} = \mathbf{y}$. Since $\mathbf{u}$ is arbitrary in $\mathbb{F}_q^n$, a necessary condition for successful decoding is that $\sum_{\ell \in [L]} k_\ell \geq n$. Because we would like to be able to decode correctly for all $L$-tuples $(k_0, \ldots, k_{L-1})$ that satisfy this necessary condition, we must guarantee that the matrix $\mathbf{A}$ has full rank for all possible $L$-tuples $(k_0, \ldots, k_{L-1})$ with $\sum_{\ell \in [L]} k_\ell \geq n$. Matrices that fulfill this condition are called universally decodable matrices (UDMs).

Given this setup there are two immediate questions. First, for what values of $L$, $n$, and $q$ do such matrices exist? Secondly, how can one construct such matrices? In [3] a construction is given for $L = 3$, any $n$, and $q = 2$. Doshi [4] gave a construction for $L = 4$, $n = 3$, and $q = 3$ and conjectured a construction for $L = 4$, $n$ any power of 3, and $q = 3$. Ganesan and Boston [5] showed that for any $n \geq 2$ the value $L$ is upper bounded by $L \leq q + 1$. In this paper we will give an explicit construction that works for any positive integers $L$ and $n$ and any prime power $q$ as long as $L \leq q + 1$, in other words, this construction achieves for any $n \geq 2$ any prime power $q$ the above-mentioned upper bound on $L$. To the best of our knowledge this is the first construction of universally decodable matrices that covers all possible parameter values. As a side result, our construction shows that the above-mentioned conjecture is indeed true.

The above problem is reminiscent of the following well-know problem. An information vector $\mathbf{u} \in \mathbb{F}_q^n$ is encoded into the vector $\mathbf{x} \triangleq \mathbf{G} \cdot \mathbf{u} \in \mathbb{F}_q^{n'}$ where $\mathbf{G}$ is an $n' \times n$-matrix $\mathbf{G}$. Upon sending $\mathbf{x}$ over an erasure channel we receive $\mathbf{y} \in (\mathbb{F}_q \cap \{?\})^{n'}$: the $i$-th entry of $\mathbf{y}$ is either equal to the $i$-th entry of $\mathbf{x}$ or equal to the question mark. Since $\mathbf{u} \in \mathbb{F}_q^n$ is arbitrary, a necessary condition for successful reconstruction is that the number of non-erased entries is at least $n$. Because we would like to be able to decode successfully whenever the number of non-erased entries is at least $n$ this implies that all sub-matrices of $\mathbf{G}$ of size $n \times n$ must have full rank. This problem is well-studied and leads to so-called maximum-distance separable (MDS) codes like Reed-Solomon codes [5, 6]. As was noted in [3, Sec. 4.5.5], for $L$, $n$, and $q$ such that $q \geq Ln - 1$ the problem of constructing UDMs can be reduced to the problem of constructing MDS codes. However, the required field size $(q \geq Ln - 1)$ is much larger than the field size that is required by our UDMs construction $(q \geq L - 1)$.

The paper is structured as follows. In Sec. [2] we properly define UDMs and in Sec. [3] we show how UDMs can be modified to obtain new UDMs. Sec. [4] is the main section where an explicit construction of UDMs is presented. In Sec. [5] we offer some conclusions, Sec. [6] contains the longer proofs, and Sec. [7] collects some results on Hasse derivatives which are the main tool for the proof of our UDMs construction.
2 Universally Decodable Matrices

The notion of universally decodable matrices (UDMs) was introduced by Tavildar and Viswanath [3]. Before we give the definition of UDMs, let us agree on some notation. For any positive integer $n$, we let $I_n$ be the $n \times n$ identity matrix and we let $J_n$ be the $n \times n$ matrix where all entries are zero except for the anti-diagonal entries that are equal to one. Row and column indices of matrices will always be counted from zero on and the entry in the $i$-th row and $j$-th column of $A$ will be denoted by $[A]_{i,j}$. Similarly, indices of vectors will be counted from zero on and the $i$-th entry of $a$ will be denoted by $[a]_i$. For any positive integer $L$ and any non-negative integer $n$ we define the sets

$$K_L^n = \left\{ (k_0, \ldots, k_{L-1}) \mid 0 \leq k_\ell \leq n, \ell \in [L], \sum_{\ell \in [L]} k_\ell = n \right\},$$
$$K_L^{\geq n} = \left\{ (k_0, \ldots, k_{L-1}) \mid 0 \leq k_\ell \leq n, \ell \in [L], \sum_{\ell \in [L]} k_\ell \geq n \right\}.$$

**Definition 1** Let $n$ and $L$ be some positive integers and let $q$ be a prime power. The $L$ matrices $A_0, \ldots, A_{L-1}$ over $\mathbb{F}_q$ and size $n \times n$ are $(L,n,q)$-UDMs, or simply UDMs, if for every $(k_0, \ldots, k_{L-1}) \in K_L^n$ they fulfill the UDMs condition which says that the $(\sum_{\ell \in [L]} k_\ell) \times n$ matrix composed of the first $k_0$ rows of $A_0$, the first $k_1$ rows of $A_1$, $\ldots$, the first $k_{L-1}$ rows of $A_{L-1}$ has full rank.

We list some immediate consequences of the above definition.

- To assess that some matrices $A_0, \ldots, A_{L-1}$ are UDMs, it is sufficient to check the UDMs condition only for every $(k_0, \ldots, k_{L-1}) \in K_L^n$. There are $(n+L-1)$ such $L$-tuples.

- If the matrices $A_0, \ldots, A_{L-1}$ are UDMs then these matrices are all invertible.

- If the matrices $A_0, \ldots, A_{L-1}$ are $(L,n,q)$-UDMs then they are $(L,n,q')$-UDMs for any $q'$ that is a power of $q$.

- Let $\sigma$ be any permutation of $[L]$. If the matrices $A_0, \ldots, A_{L-1}$ are $(L,n,q)$-UDMs then the matrices $A_{\sigma(0)}, \ldots, A_{\sigma(L-1)}$ are also $(L,n,q)$-UDMs.

- If the matrices $A_0, \ldots, A_{L-1}$ are $(L,n,q)$-UDMs then the matrices $A_0, \ldots, A_{L'}$ are $(L',n,q)$-UDMs for any positive $L'$ with $L' \leq L$.

- If the matrices $A_0, \ldots, A_{L-1}$ are $(L,n,q)$-UDMs and $B$ is an invertible $n \times n$-matrix over $\mathbb{F}_q$ then the matrices $A_0 \cdot B, \ldots, A_{L-1} \cdot B$ are $(L,n,q)$-UDMs. Without loss of generality, we can therefore assume that $A_0 = I_n$.

- For $n = 1$ we see that for any positive integer $L$ and any prime power $q$, the $L$ matrices $(1), \ldots, (1)$ are $(L,n=1,q)$-UDMs. Because of the trivial-ness of the case $n = 1$, the rest of the paper focuses on the case $n \geq 2$.

**Example 2** Let $n$ be any positive integer, let $q$ be any prime power, and let $L \triangleq 2$. Let $A_0 \triangleq I_n$ and let $A_1 \triangleq J_n$. It can easily be checked that $A_0, A_1$ are $(L=2,n,q)$-UDMs.
Indeed, let for example $n \triangleq 5$. We must check that for any non-negative integers $k_1$ and $k_2$ such that $k_1 + k_2 = 5$ the UDMs condition is fulfilled. E.g. for $(k_1, k_2) = (3, 2)$ we must show that the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

has rank 5, which can easily be verified. □

**Example 3** In order to give the reader a feeling how UDMs might look like for $L > 2$, we give here a simple example for $L = 4$, $n = 3$, and $q = 3$, namely

\[
\begin{align*}
A_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & A_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & A_2 &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, & A_3 &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

One can verify that for all $(k_0, k_1, k_2, k_3) \in \mathcal{K}^3_4$ (there are 20 such four-tuples) the UDMs condition is fulfilled and hence the above matrices are indeed UDMs. For example, for $(k_0, k_1, k_2, k_3) = (0, 0, 1, 2)$, and $(k_0, k_1, k_2, k_3) = (1, 1, 0, 1)$ the UDMs condition means that we have to check if the matrices

\[
\begin{align*}
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}
\end{align*}
\]

have rank 3, respectively, which is indeed the case. Before concluding this example, let us remark that the above UDMs are the same UDMs that appeared in [4] and [3] Sec. 4.5.4. □

### 3 Modifying UDMs

**Lemma 4** Let $A_0, \ldots, A_{L-1}$ be $(L, n, q)$-UDMs. For any $\ell \in [L]$ and $i \in [n]$ we can replace the $i$-th row of $A_\ell$ by any non-zero multiple of itself without violating any UDMs condition. Moreover, for any $\ell \in [L]$ and $i, i' \in [n]$, $i > i'$, we can add any multiples of the $i'$-th row of $A_\ell$ to the $i$-th row of $A_\ell$ without violating any UDMs condition. More generally, the matrix $A_\ell$ can be replaced by $C_\ell \cdot A_\ell$ without violating any UDMs condition, where $C_\ell$ is an arbitrary lower triangular $n \times n$-matrix over $\mathbb{F}_q$ with non-zero diagonal entries.

**Proof:** Follows from well-known properties of determinants. □

**Lemma 5** Let $A_0, \ldots, A_{L-1}$ be $(L, n, q)$-UDMs for which we know that the tensor powers $A_0^\otimes m, \ldots, A_{L-1}^\otimes m$ are $(L, n^m, q)$-UDMs for some positive integer $m$. For all $\ell \in [L]$, let $A'_\ell \triangleq A_\ell \cdot B$, where $B$ is an arbitrary invertible $n \times n$ matrix over $\mathbb{F}_q$. Then $A'_0, \ldots, A'_{L-1}$ are $(L, n, q)$-UDMs and $(A'_0)^\otimes m, \ldots, (A'_{L-1})^\otimes m$ are $(L, n^m, q)$-UDMs. On the other hand, if for all $\ell \in [L]$ we define $A'_\ell \triangleq C_\ell \cdot A_\ell$, where $C_\ell, \ell \in [L]$, are lower-triangular matrices with non-zero diagonal entries, then $A'_0, \ldots, A'_{L-1}$ are $(L, n, q)$-UDMs and $(A'_0)^\otimes m, \ldots, (A'_{L-1})^\otimes m$ are $(L, n^m, q)$-UDMs.

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Lemma 6 Let $A_0, \ldots, A_{L-1}$ be $(L,n,q)$-UDMs. Then there exist matrices $A_0', \ldots, A_{L-1}'$ that are $(L,n,q)$-UDMs and where for all $\ell' \in [L/2]$ the matrix $A_{2\ell'+1}'$ is the same as $A_{2\ell'}'$ except that the rows are in reversed order, i.e. $A_{2\ell'+1}' = J_n \cdot A_{2\ell'}'$. 

Proof: See Sec. A.4.

From Lemma 6 we see that when considering $(L,n,q)$-UDMs $A_0, \ldots, A_{L-1}$ we can without loss of generality assume that $A_0 = I_n$ and that $A_1 = J_n$. Indeed, if $A_0$ and $A_1$ are not of this form then the algorithm in the proof of Lemma 6 allows us to replace these two matrices by two matrices where $A_1$ is the same as $A_0$ except that the rows are in reversed order, i.e. $A_1 = J_n \cdot A_0$. Let $B \triangleq A_0^{-1}$. Replacing for all $\ell \in [L]$ the matrix $A_\ell$ by the matrix $A_\ell \cdot B$ we obtain the desired result.

Lemma 7 Let the matrices $A_0, \ldots, A_{L-1}$ be $(L,n,q)$-UDMs with $A_0 = I_n$ and $A_1 = J_n$. The matrices $A_0', \ldots, A_{L-1}'$ are $(L,n-1,q)$-UDMs if $A_\ell'$ is obtained as follows from $A_\ell$: if $\ell = 1$ then delete the first column and last row of $A_\ell$, otherwise delete the last column and last row of $A_\ell$.

Proof: See Sec. A.5.

Lemma 8 (1) The above results imply the following bound: if $n \geq 2$ then $(L,n,q)$-UDMs can only exist for $L \leq q + 1$. (Note that this upper bound on $L$ is independent of $n$ as long as $n \geq 2$.)

Proof: See Sec. A.6.

4 An Explicit Construction of UDMs

We introduce some conventions and notations that will be used in this section. First, whenever necessary we use the natural mapping of the integers into the prime subfield1 of $\mathbb{F}_q$. Secondly, we define the binomial coefficients in the usual way, i.e. for any integers $a$ and $b$ we let

$$\binom{a}{b} \triangleq \frac{a \cdot (a-1) \cdots (a-b+2) \cdot (a-b+1)}{b \cdot (b-1) \cdots 2 \cdot 1} \quad (1)$$

if $b$ is positive, $\binom{a}{b} \triangleq 1$ if $b$ equals zero, and $\binom{a}{b} \triangleq 0$ if $b$ is negative. One can check that for any integers $a$ and $b$ this yields $\binom{a}{b} \in \mathbb{Z}$ and the well-known relationship $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ among different binomial coefficients.2

1When $q = p^s$ for some prime $p$ and some positive integer $s$ then $\mathbb{F}_p$ is a subfield of $\mathbb{F}_q$ and is called the prime subfield of $\mathbb{F}_q$. $\mathbb{F}_p$ can be identified with the integers where addition and multiplication are modulo $p$.

2It is probably the best to think of $\binom{a}{b}$ as a function $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$. The relationship $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ holds obviously over any $\mathbb{F}_p$ where $q$ is a prime power. Note that this is the only fact we need about binomial coefficients, i.e. we do not need the “internal structure” on the right-hand side of of (1).
Let \( n \) be some positive integer, let \( q \) be some prime power, and let \( \alpha \) be a primitive element in \( \mathbb{F}_q \), i.e. \( \alpha \) is an \((q-1)\)-th primitive root of unity. If \( L \leq q + 1 \) then the following \( L \) matrices over \( \mathbb{F}_q \) of size \( n \times n \) are \((L,n,q)\)-UDMs:

\[
A_0 \triangleq I_n, \quad A_1 \triangleq J_n, \quad A_2, \ldots, \ A_{L-1},
\]

where \([A_{\ell+2}]_{ij} \triangleq \binom{\ell}{i} \alpha^{(t-i)}, \ (\ell,i,t) \in [L-2] \times [n] \times [n]\).

Note that \( \binom{\ell}{i} \) is to be understood as follows: compute \( \binom{\ell}{i} \) over the integers and apply only then the natural mapping to \( \mathbb{F}_q \).

**Proof:** See Sec. A.3. However, before looking at the proof we recommend to first study Ex. 10 and secondly to familiarize oneself with Hasse derivatives, cf. Sec. B. Especially Lemma 14 and Cor. 15 in Sec. B are interesting since they will provide the key for proving the proposition.

**Example 10** For \( n \triangleq 3, \ p \triangleq 3, \) and \( \alpha \triangleq 2, \) we obtain the \( L = 3 + 1 = 4 \) matrices that were shown in Ex. B. Note that \( A_3 \) is nearly the same as \( A_2 \): it differs only in that the main diagonal is multiplied by \( \alpha^0 = 1 \), the first upper diagonal is multiplied by \( \alpha^1 = 2 \), the second upper diagonal is multiplied by \( \alpha^2 = 1 \), the first lower diagonal is multiplied by \( \alpha^{-1} = 2 \), and the second lower diagonal is multiplied by \( \alpha^{-2} = 1 \).

We collect some remarks about the UDMs constructed in Prop. 9.

- All matrices \( A_{\ell} \), \( 2 \leq \ell < L \), are upper triangular matrices with non-zero diagonal entries. This follows from the fact that \( \binom{\ell}{i} = 1 \) if \( t = i \) and \( \binom{\ell}{i} = 0 \) if \( t < i \).

- The matrix \( A_2 \) is an upper triangular matrix where the non-zero part equals Pascal’s triangle (modulo \( p \)), see e.g. \( A_2 \) in Ex. B. However, whereas usually Pascal’s triangle is depicted such that the lines correspond to the upper entry in the binomial coefficient, here the vertical lines of the matrix correspond to the upper entry in the binomial coefficient.

- For \( t \in [n] \), let us define the matrix \( \Delta_t \) of size \( n \times n \): all entries are zero except \( [\Delta_t]_{t',t'} = +1 \) for all \( t' \in [n] \) and \( [\Delta_t]_{t'-1,t'} = -1 \) for all \( t < t' \leq n-1 \). (Note that \( \Delta_{n-1} = I_n \).) Because \( \Delta_t \) is an upper triangular matrix with non-zero diagonal entries it is an invertible matrix. One can show that \( A_2 \cdot \Delta_0 \cdot \cdots \cdot \Delta_{n-1} = I_n \). Therefore, \( A_2 = \Delta_1^{-1} \cdots \Delta_{n-1}^{-1} \). Without going into the details, these \( \Delta_t \) matrices can be used (as part of the matrices needed) to solve the equation system \( A \cdot \hat{u} = y \) in Sec. 1 with a type of Gaussian elimination.

- Applying Lemma 7 to \((q+1,n,q)\)-UDMs as constructed in Prop. 9 yields \((q+1,n-1,q)\)-UDMs as constructed in Prop. 9.

- The setup in Sec. 1 can be generalized as follows. Instead of sending vectors \( x_t \) of length \( n \) we can also send vectors of length \( n' \) where \( n' \) is any positive integer. Obviously, the matrices \( A_{\ell} \) are then of size \( n' \times n \). Essentially all results in this paper also hold for this setup, except for statements that involve the invertibility of the \( A_{\ell} \) matrices. Moreover, the sets \( K_L^{\leq n} \) and \( K_L^{\geq n} \) have to be modified to account for the fact that \( 0 \leq k_{\ell} \leq n' \).
Let us briefly focus on the case $n' = 1$, which results in the problem mentioned in Sec. 11 whose solution used MDS codes, in particular Reed-Solomon codes. We let $x$ be the the stacked version of all $x_\ell$ vectors. Because $x_\ell$ has length one, the vector $x$ has length $L$. Similarly, we define the length-$L$ vector $y$. It is not difficult to see that for the construction in Prop. 9 the vector $x$ is an element of a doubly-extended Reed-Solomon code [5,6] of length $L$, dimension $n$, and minimum distance $d_{\text{min}} = L - n + 1$.

Note that $k_\ell$ can only be zero or one and that the sum $\sum_{\ell \in [L]} k_\ell$ equals the number of non-erased symbols in $y$. In this case the proof of the construction in Prop. 9 is very simple since we do not have to worry if a root has multiplicity one or higher. Indeed, let us show that if all non-erased entries of $y$ are equal to zero then we must have $u(L) = 0$. If $k_1 = 1$ then $\deg(u(L)) \leq n - 2$. However, the other non-erased entries of $y$ require that $u(L)$ has at least $\sum_{\ell \in [L]\setminus\{1\}} k_\ell = n - k_1 = n - 1$ roots. This is a contradiction. If $k_1 = 1$ then $\deg(u(L)) \leq n - 1$. However, the other non-erased entries of $y$ require that $u(L)$ has at least $\sum_{\ell \in [L]\setminus\{1\}} k_\ell = n - k_1 = n$ roots. Again, this is a contradiction and so $u(L) = 0$ as desired. This argument is essentially equivalent to the proof used for showing that $d_{\text{min}} \geq L - n + 1$ for the above-mentioned doubly-extended Reed-Solomon code. (Together with the Singleton bound $d_{\text{min}} \leq L - n + 1$ we get $d_{\text{min}} = L - n + 1$.)

- Besides the generalization mentioned in the previous paragraph, the setup in Sec. 11 can also be be generalized in the following way. Instead of requiring that decoding is uniquely possible for any $(k_0, \ldots, k_{L-1}) \in K_L^{\geq n}$ one may ask that decoding is uniquely possible for any $(k_0, \ldots, k_{L-1}) \in K_L^{\geq n''}$ where $n'' \geq n$. Of course, UDMs designed for $n'' = n$ can be used for any $n'' \geq n$, however, for suitably chosen UDMs the required field size might be smaller, i.e. $L \leq q + 1$ (cf. Lemma 8) might not be a necessary condition anymore. Indeed, in the same way as Goppa codes / algebraic-geometry codes [7] are generalizations of Reed-Solomon codes, one can construct UDMs that are generalizations of the UDMs in Prop. 9. The generalization goes as follows: instead of obtaining the entries of the $x_\ell$, $\ell \in [L]$, by evaluating the information polynomial (see Sec. A.4 for notation) at the rational points of the curve $L^q - L = 0$ (projectively: $L^q \bar{L} - L \bar{L}^q = 0$), they are obtained by evaluating the information polynomial at the rational places of a projective, geometrically irreducible, non-singular algebraic curve of genus $g \leq n'' - n$. The proof for this setup is very similar to the proof in Sec. A.4, however instead of the fundamental theorem of algebra one needs the Riemann-Roch theorem [7]. Using Hasse-Weil-Serre bound [7] one can generalize the result in Lemma 8 to the necessary condition $L \leq q + 1 + \lceil 2\sqrt{g} \rceil g$. (Obviously, better bounds than the Hasse-Weil-Serre bound, cf. e.g. 8, lead to better necessary conditions on $L$.)

- There is some connection between the construction in Prop. 9 and so-called repeated-root cyclic codes [9, 10, 11, 12]. Namely, the “$= 0$” part of Lemma 14 is used to construct parity-check equations (and therefore a parity-check matrix) for a repeated-root cyclic code whose generator polynomial is known [10].

**Corollary 11** Consider the setup of Prop. 9. Let $p$ be the characteristic of $\mathbb{F}_q$, let $m$ be the smallest integer such that $n \leq p^m$, and let

$$i = i_{m-1}p^{m-1} + \cdots + i_1p + i_0, \quad 0 \leq i_h < p, \ h \in [m] \quad \text{and}$$

$$t = t_{m-1}p^{m-1} + \cdots + t_1p + t_0, \quad 0 \leq t_h < p, \ h \in [m]$$

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be the radix-\(p\) representations of \(i \in [n]\) and \(t \in [n]\), respectively. Then the entries of \(A_{\ell+2}, \ell \in [L-2]\), can be written as

\[
[A_{\ell+2}]_{i,t} = \prod_{h \in [m]} \binom{t_h}{i_h} \alpha^{t_h(i_h) p^h}.
\]

This shows that in the case \(n = p^m\) the matrices \(A_\ell, \ell \in [L]\) can be written as tensor products of some \(p \times p\) matrices. In the special case \(q = p\) (i.e. \(q\) is a prime) we can say more. Namely, letting \(A_0, \ldots, A_{L-1}\) be the \((p+1, p, p)\)-UDMs as constructed in Prop. 9 we see that \(A_\ell = (A_\ell')^{\otimes m}\) for all \(\ell \in [L]\).

Proof: Note that \(\binom{t}{h}\) is an integer and therefore (by the natural mapping) an element of the prime subfield \(\mathbb{F}_q\) of \(\mathbb{F}_q\). Using the Lucas correspondence theorem which states that \(\binom{t}{h} = \prod_{h \in [m]} \binom{t_h}{i_h}\) in \(\mathbb{F}_p\) (and therefore also in \(\mathbb{F}_q\)), we obtain the reformulation. The last statement in the corollary follows from the fact that \(\alpha^p = \alpha\) if \(q = p\). (Note that for \(A_0 = I_p^m\) and \(A_1 = J_p^m\) it is trivial to verify that they can be written as tensor product and tensor powers of \(p \times p\) matrices.) \(\square\)

Consider the same setup as in Cor. 11. Because \(0 \leq i_h < p\), we observe that \(\binom{t_h}{i_h}\) is a polynomial function of degree \(i_h\) in \(t\). Using Lemma 4, the matrices can therefore be modified so that the entries are

\[
[A_{\ell+2}]_{i,t} = \prod_{h \in [m]} \binom{t_h}{i_h} \alpha^{t_h(i_h) p^h}, \quad (\ell, i, t) \in [L-2] \times [n] \times [n].
\]

Letting \(q = p \triangleq 2\), \(n = 2^m\), \(L \triangleq q + 1 = 3\), and \(\alpha \triangleq 1\) we have \([A_2]_{i,t} = \prod_{h \in [m]} \binom{t_h}{i_h}\), which recovers the \((L=3, n=2^m, q=2)\)-UDMs in \([3\text{ Sec. 4.5.3}]\) since the latter matrix is a Hadamard matrix. In general (i.e. not just in the case \(q = 2\)), the fact that the entries of \([A_2]_{i,t}\) can be written as \([A_2]_{i,t} = \prod_{h \in [m]} \binom{t_h}{i_h}\), reminds very strongly of Reed-Muller code \([5\text{ 6}]\). In the former case, the rows of \(A_2\) are the evaluation of the multinomial function \((t_0, \ldots, t_{m-1}) \mapsto \prod_{h \in [m]} t_h^{i_h}\); in the latter the rows of the generator matrix can be seen as the evaluation of multinomials at various places.

Recall the \((L=4, n=3, q=3)\)-UDMs \(A_0, \ldots, A_3\) from Ex. 3. The authors of \([3\text{ 4}]\) conjecture that the tensor powers \(A_0^{\otimes m}, \ldots, A_3^{\otimes m}\) are \((4, 3^m, 3)\)-UDMs for any positive integer \(m\). This is indeed the case and can be shown as follows. From Ex. 11 we know that \(A_0, \ldots, A_3\) can be obtained by the construction in Prop. 9. Because \(q = 3\) is a prime, Cor. 11 yields the desired conclusion that the tensor powers \(A_0^{\otimes m}, \ldots, A_3^{\otimes m}\) are \((4, 3^m, 3)\)-UDMs for any positive integer \(m\).

### 5 Conclusions

We have presented an explicit construction of UDMs for all parameters \(L, n, q\) for which UDMs can potentially exist. They are essentially based on Pascal's triangle (and modifications thereof) and the proof was heavily based on properties of Hasse derivatives. We have also pointed out connections to Reed-Solomon codes, Reed-Muller codes, and repeated-root cyclic codes. One wonders if there are also other UDMs constructions that are not simply reformulations of the present UDMs.
A Proofs

A.1 Proof of Lemma 6

It is sufficient to show how $A_0$ and $A_1$ can be used to construct matrices $A'_0$ and $A'_1$ such that $A'_0, A'_1, A_2, \ldots, A'_{L-1}$ are $(L, n, q)$-UDMs and such that $A'_1$ is the same as $A'_0$ except that the rows are in reversed order. We use the following algorithm:

- Assign $A'_0 := A_0$ and $A'_1 := A_1$.
- For $i$ from 0 to $n - 1$ do
  - Let $B'_0$ be the $(i + 1) \times n$ matrix that contains the rows 0 to $i$ from $A'_0$. Similarly, let $B'_1$ be the $(n - i) \times n$ matrix that contains the rows 0 to $n - i - 1$ from $A'_0$.
  - Build the $(n + 1) \times n$-matrix $B$ by stacking $B'_0$ and $B'_1$.
  - Because of the size of $B$, the left null space of $B$ is non-empty. (In fact, because of the UDMs conditions the matrix $B$ must have rank $n$ which implies that the left null space is one-dimensional.) Pick a non-zero (row) vector $b^T$ in this left null space, i.e. $b^T$ fulfills $b^T \cdot B = 0^T$. Write $b^T = (b'_0^T \mid b'_1^T)$ where $b'_0$ is of length $i + 1$ and $b'_1$ is of length $n - i$.
  - Because of the UDMs conditions it can be seen that neither $[b]_i$ nor $[b]_{n+1}$ can be zero, i.e. neither the last component of $b'_0$ nor the last component of $b'_1$ is zero. Replace the $i$-th row of matrix $A'_0$ by the vector $b'_0^T B'_0$. Similarly, replace the $(n - i - 1)$-th row of matrix $A'_1$ by the vector $-b'_1^T B'_1$. We see that the $i$-th row of $A'_0$ equals the $(n - i - 1)$-th row of $A'_1$ and because of Lemma 4 the matrices $A'_0, A'_1, A_2, \ldots, A'_{L-1}$ are still $(L, n, q)$-UDMs

Applying the algorithm to $A_2$ and $A_3$, then to $A_4$ and $A_5$, ... yields the desired result.

A.2 Proof of Lemma 7

It is clear that $A'_0 = I_{n-1}$ and $A'_1 = J_{n-1}$. It is enough to focus on the case $L > 2$ since for $L \leq 2$ the lemma statement is already verified.

So, fix some $L > 2$. We know that for any $(k_0, \ldots, k_{L-1}) \in \mathbb{K}_L^{n^L}$ the UDMs condition is fulfilled for the matrices $A_0, \ldots, A_{L-1}$. We have to show that for any $(k'_0, \ldots, k'_{L-1}) \in \mathbb{K}_L^{n-1}$ the UDMs condition is also fulfilled for the matrices $A'_0, \ldots, A'_{L-1}$.

Take such an $L$-tuple $(k'_0, \ldots, k'_{L-1}) \in \mathbb{K}_L^{n-1}$. If $k'_e = 0$ for $2 \leq e \leq L$ then $k'_0 + k'_1 = n$ and it is clear that the UDMs condition is fulfilled. So, assume that there is at least one $e$ with $2 \leq e \leq L$ such that $k'_e > 0$ (which implies among other things that $k'_0 + k'_1 < n$). The $(n - 1) \times (n - 1)$-matrix $A'$ for which we have to check the full-rank condition looks like

$$A' = \begin{pmatrix}
I_{k'_0} & 0 & 0 \\
0 & 0 & J_{k'_1} \\
B' & B'' & B'''
\end{pmatrix},$$

where $B'$, $B''$, and $B'''$ are matrices of size $(n-k'_0-k'_1) \times k'_0$, $(n-k'_0-k'_1) \times (n-k'_0-k'_1)$, and $(n-k'_0-k'_1) \times k'_1$, respectively, and where $[B', B'', B''']$ consists of rows from $A'_e$, $2 \leq e < L$. 


It can easily be seen that the \((n - 1) \times (n - 1)\)-matrix \(A'\) has full rank if and only if the \(n \times n\)-matrix

\[
A = \begin{pmatrix}
I_{k_0'} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & J_{k_1'} & 0 \\
B' & B'' & B''' & b
\end{pmatrix} = \begin{pmatrix}
I_{k_0'} & 0 & 0 \\
0 & 0 & J_{k_1'+1} \\
B' & B'' & B''' 
\end{pmatrix}
\]

has full rank, where \(b\) is an arbitrary length-\((n-k_0'-k_1')\) vector and where \(B''' = [B'' \mid b]\).

Let \(k_\ell \triangleq k_\ell'\) for \(\ell \in [L] \setminus \{1\}\) and let \(k_1 \triangleq k_1' + 1\). (Because \(\sum_{\ell \in [L]} k_\ell = n - 1\) we have \(\sum_{\ell \in [L]} k_\ell = n\).) Choosing \(b\) such that the first \(k_2\) entries of \(b\) equal the top \(k_2\) entries of the \((n-1)\)-th column of \(A_2\), \(\ldots\), the last \(k_{L-1}\) entries of \(b\) equal the top \(k_{L-1}\) entries of the \((n-1)\)-th column of \(A_{L-1}\), we see that \(A\) represents the matrix that we have to look at when checking the UDMs property for \((k_0, \ldots, k_{L-1})\) for \(A_0, \ldots, A_{L-1}\). However, by assumption we know that \(A\) has full rank and so the matrix \(A'\) has also full rank.

**A.3 Proof of Lemma 8**

Assume that \(A_0, \ldots, A_{L-1}\) are \((L, n, q)\)-UDMs. The comments after Lemma 6 allows to assume without loss of generality that \(A_0 = I_n\) and that \(A_1 = J_n\).

First, we want to show that all entries in the first row of \(A_\ell\), \(2 \leq \ell < L\), must be non-zero. Indeed, for \(2 \leq \ell < L\) and \(m \in [n]\) the UDMs condition for \(k_0 = m\), \(k_1 = n - m - 1\), and \(k_2 = 1\) (all other \(k_{\ell'}\) are zero) shows that \([A_\ell]_{0,m} \neq 0\). Using Lemma 4 we can therefore without loss of generality assume that \([A_\ell]_{0,n-1} = 1\) for all \(2 \leq \ell < L\).

Secondly, the UDMs condition for \(k_0 = n - 2\), \(k_1 = 1\), and \(k_2 = 1\) (all other \(k_{\ell'}\) are zero) implies that the matrix

\[
\begin{pmatrix}
[A_\ell]_{0,n-2} & [A_\ell]_{0,n-1} \\
[A_{\ell'}]_{0,n-2} & [A_{\ell'}]_{0,n-1}
\end{pmatrix}
\]

must have rank 2 for any distinct \(\ell\) and \(\ell'\) fulfilling \(2 \leq \ell < L\) and \(2 \leq \ell' < L\). It is not difficult to see that this implies that \([A_\ell]_{0,n-2}\) must be distinct for all \(2 \leq \ell < L\). Since \([A_\ell]_{0,n-2}\) must be non-zero and since \(\mathbb{F}_q\) has \(q-1\) non-zero elements we see that \(L-2 \leq q-1\), i.e. \(L \leq q+1\).

**A.4 Proof of Proposition 9**

We will use the following notation. We set \(\beta_0 \triangleq 0\), \(\beta_{\ell+2} \triangleq \alpha^\ell\), \(\ell \in [L-2]\). Because \(\alpha\) is a primitive element of \(\mathbb{F}_q\), all \(\beta_i\)'s are distinct. (Note that \(\beta_1\) has not been defined.) Let \(u_\ell \triangleq [u]_\ell\), \(t \in [n]\), where \(u\) is the information vector, cf. Sec. 1 and let the information polynomial be the polynomial

\[
u(L) \triangleq \sum_{\ell \in [n]} u_\ell L^\ell \in \mathbb{F}_q[L],
\]

whose degree \(\deg(u(L))\) is at most \(n-1\). Moreover, we let \(X \triangleq [x_0] \cdots [x_{L-1}]\) be a matrix of size \(n \times L\) with entries \(x_{i,\ell} \triangleq [X]_{i,\ell} = [x_\ell]_i\) for \((i, \ell) \in [n] \times [L]\). The \(n \times L\)-matrix \(Y\) is defined similarly.
Lemma 12 Using Hasse derivatives, the elements of $A_{\ell}$, $\ell \in [L] \setminus \{1\}$, can be expressed as

$$[A_{\ell}]_{i,t} = D^{(i)}_{L} (L^t) \bigg|_{L=\beta_{\ell}} \quad (\ell \in [L] \setminus \{1\}, \ i \in [n], \ t \in n).$$

Proof: Applying the definition of the Hasse derivative (cf. Sec. 11) we get $D^{(i)}_{L} (L^t) = \binom{t}{i} L^{t-i}$, where $\binom{t}{i} L^{t-i}$ is the zero polynomial if $t < i$. Upon substituting $L = \beta_{\ell}$ we obtain

$$D^{(i)}_{L} (L^t) \bigg|_{L=\beta_{\ell}} = \binom{t}{i} \beta_{\ell}^{t-i} = \begin{cases} 1 & (\ell = 0, \ t = i) \\ 0 & (\ell = 0, \ t \neq i) \\ \binom{t}{i} \alpha^{(\ell-2)(t-i)} & (\ell \in \{2, \ldots, L - 1\}) \end{cases} = [A_{\ell}]_{i,t}.$$

Lemma 13 The elements of $X$ can be expressed as

$$x_{i,\ell} = \begin{cases} D^{(i)} (u(L)) \bigg|_{L=\beta_{\ell}} & (i \in [n], \ \ell \in [L] \setminus \{1\}) \\ u_{n-1-i} & (i \in [n], \ \ell = 1) \end{cases}.$$

Proof: Remember that in Sec. 10 we defined $x_{\ell} = A_{\ell} \cdot u$. The result for $\ell = 1$ is clear. For $\ell \in [L] \setminus \{1\}$ we use the results of Lemma 12 and the linearity of the Hasse derivative to obtain

$$x_{i,\ell} = [X]_{i} = \sum_{t \in [n]} [A_{\ell}]_{i,t} [u]_{t} = \sum_{t \in [n]} u_{t} D^{(i)} (L^t) \bigg|_{L=\beta_{\ell}} = D^{(i)} \left( \sum_{t \in [n]} u_{t} L^t \right) \bigg|_{L=\beta_{\ell}}$$

for all $(i, \ell) \in [n] \times ([L] \setminus \{1\})$. \hfill $\square$

After these preliminary lemmas, let us turn to task of checking the UDMs condition for all $(k_{0}, \ldots, k_{L-1}) \in K_{L}^{\infty}$. Fix such a tuple $(k_{0}, \ldots, k_{L-1}) \in K_{L}^{\infty}$ and let $\psi$ be the mapping of the vector $u$ to the non-erased entries of the matrix $Y$; it is clear that $\psi$ is a linear mapping. Reconstructing $u$ is therefore nothing else than applying the mapping $\psi^{-1}$ to the non-erased positions of $Y$. However, this gives a unique vector $u$ only if $\psi$ is an injective function. Because $\psi$ is linear, showing injectivity of $\psi$ is equivalent to showing that the kernel of $\psi$ contains only the vector $u = 0$, or equivalently, only the polynomial $u(L) = 0$.

So, let us show that the only possible pre-image of

$$y_{i,\ell} = 0, \ i \in [k_{\ell}], \ \ell \in [L],$$

or, equivalently, of

$$x_{i,\ell} = 0, \ i \in [k_{\ell}], \ \ell \in [L],$$

is $u(L) = 0$. Using Lemma 13 this is equivalent to showing that

$$D^{(i)} (u(L)) \bigg|_{L=\beta_{\ell}} = 0 \quad (i \in [k_{\ell}], \ \ell \in [L] \setminus \{1\}) \quad (2)$$

$$u_{n-1-i} = 0 \quad (i \in [k_{1}]) \quad (3)$$

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implies that \( u(L) = 0 \). In a first step, (2) together with Cor. 15 tells us that \( \beta_\ell, \ell \in [L] \setminus \{1\} \), must be a root of \( u(L) \) of multiplicity at least \( k_\ell \). Adding up and using the fundamental theorem of algebra we get
\[
\text{deg}(u(L)) \geq \sum_{\ell \in [L] \setminus \{1\}} k_\ell = n - k_1 \quad \text{or} \quad u(L) = 0. \tag{4}
\]

In a second step, (3) tells us that we must have \( \text{deg}(u(L)) \leq n - 1 - k_1 \). Combining this with (1), we obtain the desired result that \( u(L) = 0 \).

In our proof, the matrix \( A_1 \) and the vector \( x_1 \) had a special position. On wonders if it is possible to homogenize the setup so as to compactify the notation. Something like this is indeed possible. Letting \( P \equiv (y) \) and the vector \( L \) there can be quite a gap between these two derivatives since \( i \) is a non-negative integer.

For any prime power, \( L \) or \( 1 \) there is not a big difference between these two derivatives since \( i \) is always non-zero, however for finite fields there can be quite a gap between these two derivatives since \( i! \) can be zero or non-zero.

\[ A \]

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The Hasse derivative was introduced in [13]. Throughout this appendix, let \( q \) be some prime power. For any non-negative integer \( i \), the \( i \)-th Hasse derivative of a polynomial \( \sum_{k=0}^{d} a_k X^k \in \mathbb{F}_q[X] \) is defined to be\(^3\)
\[
\mathcal{D}^{(i)}_{X} \left( \sum_{k=0}^{d} a_k X^k \right) = \sum_{k=0}^{d} \binom{k}{i} a_k X^{k-i}. \]

\(^3\)The \( i \)-th formal derivative equals \( i! \) times the Hasse derivative: so, for fields with characteristic zero there is not a big difference between these two derivatives since \( i! \) is always non-zero, however for finite fields there can be quite a gap between these two derivatives since \( i! \) can be zero or non-zero.
Note that when \( i > k \) then \( \binom{i}{k}X^{k-i} = 0 \), i.e. the zero polynomial. We list some well-know properties of the Hasse derivative:

\[
D_X^{(i)}(\gamma f(X) + \eta g(X)) = \gamma D_X^{(i)}(f(X)) + \eta D_X^{(i)}(g(X)),
\]

\[
D_X^{(i)}(f(X)g(X)) = \sum_{i'=0}^{d} D_X^{(i')}(f(X))D_X^{(i-i')}(g(X)),
\]

\[
D_X^{(i)} \left( \prod_{h \in [M]} f_h(X) \right) = \sum_{(i_0,\ldots,i_{M-1}) \in \mathcal{K}_M} \prod_{h \in [M]} D_X^{(i_h)}(f_h(X)),
\]

\[
D_X^{(i)}((X - \gamma)^k) = \binom{k}{i} (X - \gamma)^{k-i},
\]

where \( k \) and \( i \) are some non-negative integers, \( M \) is some positive integer, and where \( \gamma, \eta \in \mathbb{F}_q \). Be careful that \( D_X^{(i_1)}D_X^{(i_2)} \neq D_X^{(i_1+i_2)} \) in general. However, it holds that \( D_X^{(i_1)}D_X^{(i_2)} = D_X^{(i_1+i_2)} \).

**Lemma 14** Let \( q \) be some prime power and let us denote the elements of \( \mathbb{F}_q \) by \( \gamma_r, r \in [q] \), i.e. \( \mathbb{F}_q \cong \{\gamma_0, \ldots, \gamma_{q-1}\} \). If \( m_0, \ldots, m_{q-1} \) are some non-negative integers then for any \( r \in [q] \) we have

\[
D_X^{(i)} \left( \prod_{r' \in [q]} (X - \gamma_{r'})^{m_{r'}} \right) \bigg|_{X=\gamma_r} = \begin{cases} 0 & (0 \leq i < m_r) \\ \neq 0 & (i = m_r) \end{cases}
\]

**Proof:** Using properties of the Hasse derivative we see that

\[
D_X^{(i)} \left( \prod_{r' \in [q]} (X - \gamma_{r'})^{m_{r'}} \right) \bigg|_{X=\gamma_r} = \sum_{(i_0,\ldots,i_{q-1}) \in \mathcal{K}_{q^{m_r}}} \prod_{r' \in [q]} D_X^{(i_{r'})}((X - \gamma_{r'})^{m_{r'-i_{r'}}}) \bigg|_{X=\gamma_r}
= \sum_{(i_0,\ldots,i_{q-1}) \in \mathcal{K}_{q^{m_r}}} \prod_{r' \in [q]} \binom{m_{r'}}{i_{r'}} (X - \gamma_{r'})^{m_{r'-i_{r'}}} \bigg|_{X=\gamma_r}
= \sum_{(i_0,\ldots,i_{q-1}) \in \mathcal{K}_{q^{m_r}}} \prod_{r' \in [q]} \binom{m_{r'}}{i_{r'}} (\gamma_{r'} - \gamma_{r'})^{m_{r'-i_{r'}}}
\]

The polynomial \( \binom{m_{r'}}{i_{r'}} (X - \gamma_{r'})^{m_{r'-i_{r'}}} \) is the zero polynomial for \( i_r > m_r \) and so \( \binom{m_{r'}}{i_{r'}} (\gamma_{r'} - \gamma_{r'})^{m_{r'-i_{r'}}} \) is interpreted to be zero if \( i_r > m_r \). If \( 0 \leq i < m_r \) then \( 0 \leq i_r < m_r \) and so all summands are zero, yielding a zero sum. However, if \( i = m_r \) there is exactly one summand that is non-zero, namely when \( i_{r'} = 0, r' \in [q] \setminus \{r\} \) and \( i_r = m_r \), and so

\[
D_X^{(m_r)} \left( \prod_{r' \in [q]} (X - \gamma_{r'})^{m_{r'}} \right) \bigg|_{X=\gamma_r} = \binom{m_r}{m_r} (\gamma_{r'} - \gamma_{r'})^{m_r-m_{r'}} \prod_{r' \in [q] \setminus \{r\}} \binom{k_{r'}}{0} (\gamma_{r'} - \gamma_{r'})^{k_{r'}-0}
= \prod_{r' \in [q] \setminus \{r\}} (\gamma_{r'} - \gamma_{r'})^{k_{r'}-0} \neq 0.
\]

\[ \square \]
Corollary 15 Let \( p(X) \in \mathbb{F}_q[X] \). If for some \( \beta \in \mathbb{F}_q \) and some non-negative integer \( m \) it holds that
\[
D_X^{(i)}(p(X)) \big|_{X=\beta} = 0 \quad \text{for all } i \in [m],
\]
then \( \beta \) is a root of \( p(X) \) with multiplicity at least \( m \).

Proof: Let \( q' \) be a power of \( q \) such that the polynomial \( p(X) \) splits in \( \mathbb{F}_{q'} = \{ \gamma_0, \ldots, \gamma_{q'} \} \), i.e. so that all roots of \( p(X) \) are in \( \mathbb{F}_{q'} \). (The theory of finite fields tells us that such a \( q' \) always exists.) Then there are non-negative integers \( m_0, \ldots, m_{q'} \) and a non-zero \( \eta \in \mathbb{F}_q \) such that
\[
p(X) = \eta \prod_{r' \in [q']} (X - \gamma_{r'})^{m_{r'}} \tag{5}
\]
and such that \( \sum_{r' \in [q']} m_{r'} = \deg(p(X)) \). Let \( r \in [q'] \) be such that \( \beta = \gamma_r \). The proof will be by contradiction. So, assume for the moment that \( m_r < m \), i.e. that \( \beta \) is a root of \( p(X) \) of multiplicity \( m_r \) smaller than \( m \). Using Lemma 14 and Eq. 5 we see that
\[
D_L^{(m_r)}(p(X)) \big|_{X=\beta} \neq 0,
\]
which is a contradiction to the assumption made in the corollary statement. This proves the corollary. \( \square \)

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