Bilinear equations and Bäcklund transformation for a generalized ultradiscrete soliton solution

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Abstract

Ultradiscrete soliton equations and Bäcklund transformation for a generalized soliton solution are presented. The equations include the ultradiscrete Korteweg–de Vries (KdV) equation or the ultradiscrete Toda equation in a special case. We also express the solution by the ultradiscrete permanent, which is defined by ultradiscretizing the signature-free determinant, that is, the permanent. Moreover, we discuss a relation between Bäcklund transformations for discrete and ultradiscrete KdV equations.

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1. Introduction

The soliton equation has explicit N-soliton solutions and generally an infinite number of conserved quantities. In the beginning of the development of the soliton theory, continuous or semi-discrete soliton equations were mainly studied. For example, the Korteweg–de Vries (KdV) equation is a continuous soliton equation of the PDE type, and the Toda equation is a semi-discrete soliton equation with continuous and discrete independent variables. There are two types of soliton solutions to the bilinear equations derived from these equations. One is expressed by a sum of a finite number of exponential functions, which was first proposed by Hirota [1, 2]. We call this type of expression type I. The other is expressed by Wronski determinant [3, 4]. We call this type of expression type II.

After the discovery of various continuous or semi-discrete soliton equations, discrete soliton equations of which independent variables are all discrete were proposed [5, 6]. The discrete soliton equation is also transformed into the bilinear equation and has multi-soliton solutions. It has also two types of expressions, type I and II, where the determinant of type II is generally the Casorati determinant for discrete soliton equations.
Discretization process is completed if dependent and independent variables are all discretized. In the 1990s, Tokihiro et al. proposed the ultradiscretization method to discretize dependent variables [7]. The key formula in the method is

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{a/\varepsilon} + e^{b/\varepsilon}) = \max(a, b). \quad (1)$$

Usual addition, multiplication and division for the real values in the original discrete equation are replaced with max operation, addition and subtraction respectively by this method. Due to these replacements, dependent variables can be discrete in the ultradiscrete equation if we use appropriate constants and initial values. Many ultradiscrete soliton equations or cellular automata have been proposed and the integrability is shown even for the digitized equations [8, 9]. Moreover, an ultradiscrete system has also been studied in terms of the tropical geometry recently [10, 11].

However, the operation in the ultradiscrete equation corresponding to the subtraction in the discrete equation is not well defined. Thus, we cannot ultradiscretize a discrete equation automatically. This obstruction is called a ‘negative problem’ [12, 13]. Thus, the above soliton solution of type II cannot be ultradiscretized directly since the antisymmetry is crucial for the determinant. On the other hand, the solution of type I can be ultradiscretized generally choosing the appropriate parameters included in the solution.

The imbalance between the two types of expressions for the ultradiscrete soliton solution is partially solved. One of the authors (Takahashi) and Hirota proposed the ultradiscrete analog of determinant solution for the ultradiscrete KdV (uKdV) equation [12]. One of the authors (Nagai) proposed a similar type of solution for the ultradiscrete Toda (uToda) equation [13]. This analog is called an ‘ultradiscrete permanent’ (UP) defined by

$$\max[a_{ij}] \equiv \max_{\pi} \sum_{1 \leq i \leq N} a_{i\pi_i}, \quad (2)$$

where $[a_{ij}]$ denotes an arbitrary $N \times N$ matrix and $\pi = [\pi_1, \pi_2, \ldots, \pi_N]$ an arbitrary permutation of $1, 2, \ldots, N$. The $N$-soliton solution to the ultradiscrete bilinear equation of the uKdV equation is expressed by the following two forms:

$$f_n^i = \max_{\mu_j = 0, 1} \left( \sum_{1 \leq j \leq N} \mu_1 s_j(n, i) - 2 \sum_{1 \leq j < j' \leq N} \mu_j \mu_{j'} p_{j'}, \right), \quad (3)$$

and

$$\tilde{f}_n^i = \frac{1}{2} \max \left[ \begin{array}{cccc}
|s_1(n, i) + (-N + 1)p_1| & |s_1(n, i) + (-N + 3)p_1| & \cdots & |s_1(n, i) + (N - 1)p_1| \\
\cdots & \cdots & \cdots & \cdots \\
|s_N(n, i) + (-N + 1)p_N| & |s_N(n, i) + (-N + 3)p_N| & \cdots & |s_N(n, i) + (N - 1)p_N| 
\end{array} \right], \quad (4)$$

where

$$s_j(n, i) = p_j n - q_j i + c_j, \quad 0 \leq p_1 \leq p_2 \leq \cdots \leq p_N. \quad (5)$$

Each $q_j$ satisfies a dispersion relation $\min(p_j, 1)$, and $\max_{\mu_j = 0, 1} X(\mu_1, \mu_2, \ldots, \mu_N)$ denotes the maximum value of $X$ in $2^N$ possible cases $\{\mu_1, \mu_2, \ldots, \mu_N\}$ replacing each $\mu_j$ by 0 or 1. We call the form (3) type I and (4) type II respectively in this paper.

Bäcklund transformation is an important object in the soliton theory since it gives the links among equations or solutions [5, 6]. The ultradiscrete version of the Bäcklund transformation
is discussed in [14] or [15]. The equations treated in the references are the ultradiscrete 
Kadomtsev–Petviashvili equation and the uKdV equation.

In this paper, we consider a generalized solution of both types:

\[ f_i^n = \max_{\mu_j=0,1} \left( \sum_{1 \leq j \leq N} \mu_j s_j(n, i) - \sum_{1 \leq j < j' \leq N} \mu_j \mu_{j'} r_{j'} \right) \]

and

\[ \tilde{f}_i^n = \max \left[ s_1(n, i) + \left( -N + 1 \right) \frac{r_1}{2}, \ldots, s_N(n, i) + \left( -N + 1 \right) \frac{r_N}{2} \right], \]

where

\[ s_j(n, i) = p_j n - q_j i + c_j, \quad 0 \leq p_1 \leq p_2 \leq \cdots \leq p_N, \]

\[ r_j = k p_j + l q_j, \quad k, l: \text{non-negative constants}. \]

Obviously this solution is a generalization of the soliton solution for the uKdV equation. We discuss ultradiscrete soliton equations and a Bäcklund transformation for this solution.

The contents of this paper are as follows. In section 2, we give equations where \( f_i^n \) and \( \tilde{f}_i^n \) satisfy under some conditions. In section 3, we show a Bäcklund transformation between \( N \)- and \( (N + 1) \)-soliton solutions. In section 4, we ultradiscretize the Bäcklund transformations for the discrete KdV equation and obtain those for the uKdV equation. The results suggest an algebraic correspondence between the solutions of determinant and of UP. In section 5, we give concluding remarks.

The proofs in this paper are given under the condition \( p_j \geq 0 \) for simplicity. However, note that these results can be easily extended to the case of arbitrary \( p_j \)'s by replacing (6) with

\[ f_i^n = \max_{\mu_j=0,1} \left( \sum_{1 \leq j \leq N} \mu_j s_j(n, i) - \sum_{1 \leq j < j' \leq N} \mu_j \mu_{j'} a_{j'j} \right), \]

where

\[ a_{j'j} = \min(\max(r_j, -r_j'), \max(-r_j, r_j')). \]

2. Bilinear equations for the generalized soliton solution

First we give the following propositions.

**Proposition 1.** The generalized solution (6) with a dispersion relation

\[ q_j = \min(p_j, L), \]

where \( L (\geq 0) \) is a constant, satisfies a bilinear equation

\[ f_i^n + f_{i+n_{k+1}} = \max \left( f_{i-k-1}^{n_{k+1}} + f_{i+1}^{n_{k+1}}, f_{i-1}^{n_{k+1}} + f_{i+1}^{n_{k+1}} - L \right) \quad (k \geq 2). \]

Similarly, the following propositions hold.
Proposition 2. The generalized solution (6) with a dispersion relation

\[ q_j = \max(0, p_j - L), \]  \hspace{1cm} (13)

where \( L \geq 0 \) is a constant, satisfies

\[ f^n_i + f^{n+k-1}_{i-1} = \max \left( f^{n+k-1}_{i-1} + f^n_{i-1}, f^{n+k}_{i-1} + f^{n-1}_{i-1} - L \right) \]  \hspace{1cm} (k, l \geq 1).  \hspace{1cm} (14)

Proposition 3. The generalized solution (6) with a dispersion relation

\[ q_j = \begin{cases} 0 & (0 \leq p_j \leq M) \\ L & (M < p_j) \end{cases}, \] \hspace{1cm} (15)

where \( M \geq L \geq 0 \) are constants, satisfies

\[ f^n_i + f^{n+k}_{i-1} = \max \left( f^n_{i-1} + f^{n+k}_{i+1}, f^n_{i-1} + f^{n+k}_{i-1} - L \right) \]  \hspace{1cm} (l \geq 3).  \hspace{1cm} (16)

Note that (12) reduces to the uKdV equation in the case \((k, l) = (2, 0)\), and (14) reduces to the uToda equation in the case \((k, l) = (1, 1)\), respectively.

Since the proof of the propositions is long, let us show the outline. First, considering \( f_{yy}^{n+x} + f_{yw}^{n+z} \) with arbitrary constants \( x, y, z \) and \( w \), from (6) we have

\[ f_{yy}^{n+x} + f_{yw}^{n+z} = \max_{\mu_j, v_j = 0, 1} \left( \sum_{1 \leq j \leq N} (\mu_j + v_j) x_j + \sum_{1 \leq j \leq N} (\mu_j x_j + v_j z_j) - \sum_{1 \leq j < f \leq N} (\mu_j \mu_f + v_j v_f) r_{jf} \right), \] \hspace{1cm} (17)

where \( s_j \) denotes \( s_j(n, i) \) for short. Using new parameters \( \lambda_j \) and \( \sigma_j \) defined by \( \lambda_j = \mu_j + v_j \), \( \sigma_j = \mu_j - v_j \) \((1 \leq j \leq N)\), (17) reduces to

\[ f_{yy}^{n+x} + f_{yw}^{n+z} = \max_{(\lambda_j, \sigma_j)} \left( \sum_{1 \leq j \leq N} \lambda_j s_j + \frac{1}{2} \sum_{1 \leq j \leq N} \lambda_j ((x + z) p_j - (y + w) q_j) + \frac{1}{2} \sum_{1 \leq j < f \leq N} (\lambda_j \lambda_f + \sigma_j \sigma_f) r_{jf} \right). \] \hspace{1cm} (18)

Note that the pair \((\lambda_j, \sigma_j)\) can be one of the following:

\[ (0, 0), \hspace{1cm} (1, 1), \hspace{1cm} (1, -1), \hspace{1cm} (2, 0), \] \hspace{1cm} (19)

and \( \max_{(\lambda_j, \sigma_j)} X(\lambda_1, \ldots, \lambda_N, \sigma_1, \ldots, \sigma_N) \) denotes the maximum value of \( X \) in \( 4^N \) possible cases \( \{\lambda_1, \ldots, \lambda_N, \sigma_1, \ldots, \sigma_N\} \) replacing each \((\lambda_j, \sigma_j)\) by one of the above four pairs. Substituting (18) into (12), we can show that proposition 1 holds if the following is proved for any \( N \) satisfying \( 1 \leq N' \leq N \) [12]:

\[ \max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j (k p_j + (l - 1) q_j) - \sum_{1 \leq j < f' \leq N'} \sigma_j \sigma_{j'} r_{j'} \right) = \max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j ((k - 2) p_j + (l + 1) q_j) - \sum_{1 \leq j < f' \leq N'} \sigma_j \sigma_{j'} r_{j'} \right) - 2L. \] \hspace{1cm} (20)
Using the dispersion relation (11), the maxima of the LHS and of the first argument on the RHS of (20) are both given by the case \( \sigma_j = (-1)^{N-j} (1 \leq j \leq N') \) [13]. About the second argument on the RHS, the case \( \sigma_{N'} = 1 \) and \( \sigma_j = (-1)^{N-j+1} (1 \leq j \leq N' - 1) \) gives the maximum. Therefore, (20) becomes

\[
\max(-p_{N'} + q_{N'} + p_{N'-1} - q_{N'-1} - p_{N'-2} + q_{N'-2} + \cdots, q_{N'} - L) = 0, \tag{21}
\]
after the above evaluation. In particular, the dispersion relation (11) derives

\[
\begin{cases}
-p_{N'} + q_{N'} + p_{N'-1} - q_{N'-1} - p_{N'-2} + q_{N'-2} + \cdots \leq 0 & \text{if } p_{N'} \geq L, \\
p_{N'} - q_{N'} - L = 0 & \text{if } p_{N'} \geq L,
\end{cases}
\]
and

\[
\begin{cases}
-p_{N'} + q_{N'} + p_{N'-1} - q_{N'-1} - p_{N'-2} + q_{N'-2} + \cdots = 0 & \text{if } p_{N'} < L, \\
p_{N'} - q_{N'} - L < 0 & \text{if } p_{N'} < L.
\end{cases}
\tag{22}
\]

Thus (21) holds for any \( 1 \leq N' \leq N \) and proposition 1 is proved.

Substituting (18) into (14), we can show that proposition 2 holds if the following is proved for any \( 1 \leq N' \leq N \):

\[
\max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j ((k - 1)p_j + (l + 1)q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_r r_j \right) = \max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j ((k - 1)p_j + (l - 1)q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_r r_j \right). \tag{23}
\]

Using the dispersion relation (13), the case \( \sigma_j = (-1)^{N-j} (1 \leq j \leq N') \) gives the maxima of the LHS and of the first argument on the RHS. The case \( \sigma_{N'} = 1 \) and \( \sigma_j = (-1)^{N-j+1} (1 \leq j \leq N' - 1) \) gives the maximum of the second argument on the RHS. Therefore, the above equation becomes

\[
\max(-q_{N'} + q_{N'-1} - q_{N'-2} + q_{N'-3} - \cdots, p_{N'} - q_{N'} - L) = 0, \tag{24}
\]
after the above evaluation. This equation holds since

\[
\begin{cases}
-q_{N'} + q_{N'-1} - q_{N'-2} + q_{N'-3} - \cdots \leq 0 & \text{if } p_{N'} \geq L, \\
p_{N'} - q_{N'} - L = 0 & \text{if } p_{N'} \geq L,
\end{cases}
\]
and

\[
\begin{cases}
-q_{N'} + q_{N'-1} - q_{N'-2} + q_{N'-3} - \cdots = 0 & \text{if } p_{N'} < L, \\
p_{N'} - q_{N'} - L < 0 & \text{if } p_{N'} < L.
\end{cases}
\tag{25}
\]

Thus proposition 2 is proved.

Substituting (18) into (16), we can show that proposition 3 holds if the following is proved for any \( 1 \leq N' \leq N \):

\[
\max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j (kp_j + (l - 1)q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_r r_j \right) = \max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j (kp_j + (l - 3)q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_r r_j \right), \tag{26}
\]

Using the dispersion relation (15), the case \( \sigma_j = (-1)^{N-j} (1 \leq j \leq N') \) gives the maxima of the LHS and of the first argument on the RHS. The case \( \sigma_{N'} = 1 \) and

\[
\begin{cases}
-kp_{N'} + (l - 1)q_j < L & \text{if } (l - 3)p_j + (l - 1)q_j \leq L, \\
kp_{N'} + (l - 3)q_j > L & \text{if } (l - 3)p_j + (l - 1)q_j > L.
\end{cases}
\]
\[ \sigma_j = (-1)^{N-j+1} (1 \leq j \leq N' - 1) \] gives the maximum of the second argument on RHS. Therefore, the above equation becomes

\[ \max(-q_{N'} + q_{N'-1} - q_{N'-2} + q_{N' - 3} - \cdots, q_{N'} - L) = 0, \] (27)

after the above evaluation. This equation holds since

\[
\begin{cases}
-q_{N'} + q_{N'-1} - q_{N'-2} + q_{N' - 3} - \cdots \leq 0 & \text{if } p_N = M, \\
-q_{N'} + q_{N'-1} - q_{N'-2} + q_{N' - 3} - \cdots > 0 & \text{if } p_N < M.
\end{cases}
\] (28)

Thus proposition 3 is proved.

Moreover these propositions lead the following propositions.

**Proposition 4.** The generalized solution (7) satisfies

\[ \tilde{f}^n_i + \tilde{f}^{n+k}_{i-k+1} = \max \left( \tilde{f}^{n+k-1}_{i-k} + \tilde{f}^{n+1}_{i-k+1}, \tilde{f}^{n+k}_{i-k} + \tilde{f}^n_{i-k+1} - 2L \right) \] (29)

under the dispersion relation (11).

**Proposition 5.** The generalized solution (7) satisfies

\[ \tilde{f}^n_i + \tilde{f}^{n+k-1}_{i-k+1} = \max \left( \tilde{f}^{n+k-1}_{i-k} + \tilde{f}^n_{i-k+1}, \tilde{f}^{n+k}_{i-k} + \tilde{f}^{n-1}_{i-k+1} - 2L \right) \] (30)

under the dispersion relation (13).

**Proposition 6.** The generalized solution (7) satisfies

\[ \tilde{f}^n_i + \tilde{f}^{n+k}_{i-k+1} = \max \left( \tilde{f}^n_{i-k+1} + \tilde{f}^{n+k+1}_{i-k}, \tilde{f}^{n+1}_{i-k+1} + \tilde{f}^{n+1}_{i-k-1} - 2L \right) \] (31)

under the dispersion relation (15).

These propositions are proved by reducing (29), (30) and (31) to (12), (14) and (16), respectively. Using the formula [12]

\[
\begin{align*}
\max \begin{bmatrix}
|y_1 + (-N + 1)r_1| & |y_1 + (-N + 3)r_1| & \cdots & |y_1 + (N - 1)r_1| \\
\vdots & \vdots & \ddots & \vdots \\
|y_N + (-N + 1)r_N| & |y_N + (-N + 3)r_N| & \cdots & |y_N + (N - 1)r_N|
\end{bmatrix}
\end{align*}
\]

\[ = \max \left( \sum_{1 \leq j \leq N} \rho_j y_j - \sum_{1 \leq j < j' \leq N} \rho_j \rho_{j'} r_{j'}, \sum_{1 \leq j < j' \leq N} r_{j'} \right), \] (32)

where

\[ 0 \leq r_1 \leq r_2 \leq \cdots \leq r_N, \]

and the transformation \( \rho_j = 2\mu_j - 1 \), we have

\[
\begin{align*}
\tilde{f}^n_i = \max_{\rho_j = \pm 1} & \left( \sum_{1 \leq j \leq N} \rho_j y_j - \frac{1}{2} \sum_{1 \leq j < j' \leq N} \rho_j \rho_{j'} r_{j'} + \frac{1}{2} \sum_{1 \leq j < j' \leq N} r_{j'} \right) \\
\geq & \max_{\mu_j = 0, 1} \left( 2 \sum_{1 \leq j \leq N} \mu_j y_j - 2 \sum_{1 \leq j < j' \leq N} \mu_j \mu_{j'} r_{j'} + \sum_{1 \leq j < j' \leq N} (\mu_j + \mu_{j'}) r_{j'} \right) - \sum_{1 \leq j \leq N} y_j \\
\geq & 2 \max_{\mu_j = 0, 1} \left( \sum_{1 \leq j \leq N} \mu_j \left( y_j + \frac{N - j}{2} r_j + \frac{1}{2} \sum_{1 \leq j' \leq N-1} r_{j'} \right) - \sum_{1 \leq j < j' \leq N} \mu_j \mu_{j'} r_{j'} \right) - \sum_{1 \leq j \leq N} s_j.
\end{align*}
\]
Here \( f_i^n \approx g_i^n \) denotes that \( f_i^n \) and \( g_i^n \) give the same solution of (29) or (30). Hence, using a replacement
\[
c_j + \frac{N - j}{2} r_j + \frac{1}{2} \sum_{1 \leq j \leq j-1} r_j \rightarrow c_j,
\]
we have
\[
\tilde{f}_{i+1}^{n+1} \approx 2f_{i+1}^{n+1} - \sum_{1 \leq j \leq N} s_j(n + x_i + y_j).
\]
Thus, (29) is reduced to (12) by adding \( \sum_{1 \leq j \leq N} (2s_j + kp_j + (l - 1)q_j) \) to both sides, (30) is reduced to (14) by adding \( \sum_{1 \leq j \leq N} (2s_j + (k - 1)p_j + (l + 1)q_j) \) and (31) is reduced to (16) by adding \( \sum_{1 \leq j \leq N} (2s_j + kp_j + (l - 1)q_j) \), respectively.

3. Bäcklund transformation for the generalized soliton solution

We have the following proposition about a Bäcklund transformation for the generalized soliton solution.

**Proposition 7.** The generalized solution (6) with (8) and the following additional conditions:
\[
0 \leq q_1 \leq q_2 \leq \cdots \leq q_{N+1},
\]
\[
0 \leq p_1 - q_1 \leq p_2 - q_2 \leq \cdots \leq p_{N+1} - q_{N+1},
\]
(35)
satisfies a Bäcklund transformation
\[
f_i^n + g_i^{n+\alpha} = \max \left( f_i^{n+\alpha} + g_{i+1}^{n+\beta}, f_i^{n-\alpha} + g_{i-1}^{n+\beta} - A \right),
\]
(36)
where \( g_i^n \) is the \((N + 1)\)-soliton solution defined by
\[
g_i^n = \max_{\mu_j=0,1} \left( \sum_{1 \leq j \leq N+1} \mu_j s_j(n, i) - \sum_{1 \leq j < j' \leq N+1} \mu_j \mu_j' r_j \right),
\]
(37)
and the parameters \( A, \alpha, \beta \) satisfy
\[
A = (k - \alpha)p_{N+1} + (l + \beta)q_{N+1},
\]
(38)
\[
0 \leq \alpha \leq k, \quad -k - l + \alpha \leq \beta \leq \alpha.
\]
Note that the dispersion relations (11), (13) and (15) satisfy the additional condition (35). Therefore the solutions given in the previous section satisfy the Bäcklund transformation (36) as a special case.

To prove proposition 7, we rewrite \( g_i^n \) by
\[
g_i^n = \max_{\mu_j=0,1} \left( \sum_{1 \leq j \leq N} \mu_j s_j - \sum_{1 \leq j < j' \leq N} \mu_j \mu_j' r_j, \right.
\]
\[
\left. \sum_{1 \leq j \leq N} \mu_j (s_j - r_j) + s_{N+1} - \sum_{1 \leq j < j' \leq N} \mu_j \mu_j' r_j \right),
\]
(39)
Substituting (6) and (39) into the LHS of (36), we obtain

\[ f_{n_1} + g_{n_2} = \max_{\mu_j, \nu_j=0,1} \left( \sum_{1 \leq j \leq N} (\mu_j + v_j)s_j + \sum_{1 \leq j \leq N} v_j(\alpha p_j - \beta q_j) \right. \]

\[ - \sum_{1 \leq j < j' \leq N} (\mu_j \mu_j' + v_j v_j')r_{j'} + \sum_{1 \leq j \leq N} (\mu_j + v_j)s_j + s_{N+1} + \alpha p_{N+1} - \beta q_{N+1} \]

\[ + \sum_{1 \leq j \leq N} v_j((-k + \alpha)p_j - (l + \beta)q_j) - \sum_{1 \leq j < j' \leq N} (\mu_j \mu_j' + v_j v_j')r_{j'} \left. \right). \]

(40)

Using new parameters \( \lambda_i \) and \( \sigma_i \) defined by

\[ \lambda_j = \mu_j + \nu_j, \quad \sigma_j = -\mu_j + \nu_j \]

(1 \( \leq j \leq N \)), (40) reduces to

\[ f_{n_1} + \alpha \mu_i + \beta \nu_i = \max_{(\lambda_j, \sigma_j)} \left( \sum_{1 \leq j \leq N} \lambda_j s_j + \frac{1}{2} \sum_{1 \leq j \leq N} (\lambda_j + \sigma_j)(\alpha p_j - \beta q_j) \right. \]

\[ - \frac{1}{2} \sum_{1 \leq j < j' \leq N} (\lambda_j \lambda_j' + \sigma_j \sigma_j')r_{j'} + \sum_{1 \leq j \leq N} \lambda_j s_j + s_{N+1} + \alpha p_{N+1} - \beta q_{N+1} \]

\[ + \frac{1}{2} \sum_{1 \leq j \leq N} (\lambda_j + \sigma_j)((-k + \alpha)p_j - (l + \beta)q_j) \]

\[ - \frac{1}{2} \sum_{1 \leq j < j' \leq N} (\lambda_j \lambda_j' + \sigma_j \sigma_j')r_{j'} \left. \right). \]

(41)

Similarly, we have

\[ f_{n_1} - k + \alpha \mu_i + l + \beta \nu_i = \max_{(\lambda_j, \sigma_j)} \left( \sum_{1 \leq j \leq N} \lambda_j s_j + \frac{1}{2} \sum_{1 \leq j \leq N} (\lambda_j + \sigma_j)(\alpha p_j - \beta q_j) \right. \]

\[ - \frac{1}{2} \sum_{1 \leq j < j' \leq N} (\lambda_j \lambda_j' + \sigma_j \sigma_j')r_{j'} + \sum_{1 \leq j \leq N} \lambda_j s_j + s_{N+1} + \alpha p_{N+1} - \beta q_{N+1} \]

\[ + \frac{1}{2} \sum_{1 \leq j \leq N} (\lambda_j + \sigma_j)((-k + \alpha)p_j - (l + \beta)q_j) \]

\[ - \frac{1}{2} \sum_{1 \leq j < j' \leq N} (\lambda_j \lambda_j' + \sigma_j \sigma_j')r_{j'} \left. \right). \]

(42)

\[ f_{n_1}^{k+1} + g_{n_2}^{n+1} - A = \max_{(\lambda_j, \sigma_j)} \left( \sum_{1 \leq j \leq N} \lambda_j s_j + \frac{1}{2} \sum_{1 \leq j \leq N} \lambda_j(\alpha p_j - \beta q_j) - A \right. \]

\[ + \frac{1}{2} \sum_{1 \leq j \leq N} \sigma_j((-2k + \alpha)p_j - (2l + \beta)q_j) \]

\[ - \frac{1}{2} \sum_{1 \leq j < j' \leq N} (\lambda_j \lambda_j' + \sigma_j \sigma_j')r_{j'} + \sum_{1 \leq j \leq N} \lambda_j s_j + s_{N+1} + \alpha p_{N+1} - \beta q_{N+1} \]

\[ + \frac{1}{2} \sum_{1 \leq j \leq N} (\lambda_j + \sigma_j)((-k + \alpha)p_j - (l + \beta)q_j) - \frac{1}{2} \sum_{1 \leq j < j' \leq N} (\lambda_j \lambda_j' + \sigma_j \sigma_j')r_{j'} \left. \right). \]

(43)
Note that \( \sigma_j \) is redefined by \( \sigma_j = \mu_j - \nu_j \) in (42) and (43). The former argument of (41) is equal to the former of (42), and the latter to the latter of (43). Hence, (36) holds if both of the following inequalities hold for any \( 1 \leq N' \leq N \):
\[
\max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j (\alpha p_j - \beta q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_j' r_j \right) 
\geq \max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j ((-2k + \alpha) p_j - (2l + \beta) q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_j' r_j \right) - 2A, \tag{44}
\]
\[
\max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j ((k - \alpha) p_j + (l + \beta) q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_j' r_j \right) 
\geq \max_{\sigma_j = \pm 1} \left( \sum_{1 \leq j \leq N'} \sigma_j ((k + \alpha) p_j - (-l + \beta) q_j) - \sum_{1 \leq j < j' \leq N'} \sigma_j \sigma_j' r_j \right) 
- 2ap_{N+1} + 2\beta q_{N+1}. \tag{45}
\]

Since (44) is equivalent to (45) through the transformations \( \alpha \to k - \alpha \) and \( \beta \to -l - \beta \), we only need to prove (45). The maximum of the LHS of (45) is given by the case \( \sigma_j = (-1)^{N'-j} \), and that of the RHS is given by the case \( \sigma_j = (-1)^{N'-j+1} \) (\( 1 \leq j \leq N' - 1 \)), respectively [13]. Thus, the difference between the LHS and the RHS of (45) is \( 2(\alpha(p_{N+1} - p_N) - \beta(q_{N+1} - q_N)) \) and it satisfies
\[
2(\alpha(p_{N+1} - p_N) - \beta(q_{N+1} - q_N)) \geq 2\alpha(p_{N+1} - p_N - (q_{N+1} - q_N)) \geq 0. \tag{46}
\]

Hence, we have proved the proposition.

Next, we give the Bäcklund transformation in type II.

**Proposition 8.** The generalized solution (7) with conditions (8) and (35) satisfies the Bäcklund transformation
\[
\tilde{g}^n_{l,i} + \tilde{g}^{n+\alpha}_{l,i} = \max \left( \tilde{g}^{n+\alpha}_{l+1,i} + \tilde{g}^{n}_{l,i+1} - B, \tilde{g}^{n-k+\alpha}_{l,i+1} + \tilde{g}^{n+k}_{l,i-1} - A \right), \tag{47}
\]
where \( \tilde{g}^n_{l,i} \) is the \((N+1)\)-soliton solution defined by
\[
\tilde{g}^n_{l,i} = \max \left[ \begin{array}{cccc}
\left| s_1 + (-N - 1) \frac{r_1}{2} \right| & \left| s_1 + (-N + 1) \frac{r_1}{2} \right| & \cdots & \left| s_1 + (N - 1) \frac{r_1}{2} \right| \\
\cdots & \cdots & \cdots & \cdots \\
\left| s_{N+1} + (-N - 1) \frac{r_{N+1}}{2} \right| & \left| s_{N+1} + (-N + 1) \frac{r_{N+1}}{2} \right| & \cdots & \left| s_{N+1} + (N - 1) \frac{r_{N+1}}{2} \right|
\end{array} \right].
\]
\tag{48}

The parameters \( \alpha, \beta \) and \( A \) are the same as in (38) and \( B \) is defined by \( \alpha p_{N+1} - \beta q_{N+1} \).

Proposition 8 is proved after the manner of propositions 4 and 5. In particular, using a property of UP,
\[
\max \left[ \begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1N} \\
a_{21} & a_{22} & \cdots & a_{2N} \\
\cdots & \cdots & \cdots & \cdots \\
a_{N1} & a_{N2} & \cdots & a_{NN}
\end{array} \right] + \sum_{1 \leq j \leq N} b_j = \max \left[ \begin{array}{cccc}
a_{11} + b_1 & a_{12} + b_1 & \cdots & a_{1N} + b_1 \\
a_{21} + b_2 & a_{22} + b_2 & \cdots & a_{2N} + b_2 \\
\cdots & \cdots & \cdots & \cdots \\
a_{N1} + b_N & a_{N2} + b_N & \cdots & a_{NN} + b_N
\end{array} \right]. \tag{49}
\]
and formula (32), \( \hat{g}_j^n \) reduces to

\[
\hat{g}_j^n = \max_{\rho_j = \pm 1} \left( \max \left[ \rho_1 \left( s_1 + \frac{(-N - 1) \rho_1}{2} \right) \left( \begin{array}{cccc}
\rho_1 & \cdots & \rho_1 \\
\rho_{N+1} & \cdots & \rho_{N+1} \\
\rho_{N+1} & \cdots & \rho_{N+1}
\end{array} \right) \right] \right),
\]

\[
= \max_{\rho_j = \pm 1} \left( \max \left[ \rho_1 \left( s_1 + \frac{N \rho_1}{2} \right) \left( \begin{array}{cccc}
\rho_1 & \cdots & \rho_1 \\
\rho_{N+1} & \cdots & \rho_{N+1} \\
\rho_{N+1} & \cdots & \rho_{N+1}
\end{array} \right) \right] - \frac{1}{2} \sum_{1 \leq j \leq N+1} \rho_j r_j \right)
\]

\[
= \max_{\rho_j = \pm 1} \left( \sum_{1 \leq j \leq N+1} \rho_j \left( s_j - \frac{1}{2} r_j \right) - \frac{1}{2} \sum_{1 \leq j < j' \leq N+1} \rho_j \rho_j r_j \right) + \frac{1}{2} \sum_{1 \leq j \leq N+1} r_j.
\]

(50)

Thus, we can derive through the transformation \( \rho_j = 2 \mu_j - 1 \),

\[
\tilde{\phi}_{j+1}^{n+1} + \tilde{\phi}_{j+1}^{n+2} = 2 \phi_{j+1}^{n+1} + 2 \phi_{j+1}^{n+2} - \sum_{1 \leq j \leq N} s_j (n + x, i + y) - \sum_{1 \leq j \leq N+1} s_j (n + z, i + w).
\]

(51)

Hence, (47) is equivalent to (36) with a difference of the negligible term \( \sum_{1 \leq j \leq N} s_j (n, i) + \sum_{1 \leq j \leq N+1} s_j (n + \alpha, i + \beta) \).

### 4. Relation between Bäcklund transformations for discrete and ultradiscrete KdV equations

The \( N \)-soliton solution of type II to the discrete KdV equation [5, 15],

\[
F_i^{n+1} F_i^{n-1} - \delta F_i^{n+1} F_i^{n-1} = (1 - \delta) F_i^{n+1} F_i^{n} = 0,
\]

is expressed by

\[
F_i^n = \left| \begin{array}{cccc}
\eta_1(n, i) & \eta_1(n+2, i) & \cdots & \eta_1(n+2(N-1), i) \\
\vdots & \ddots & \ddots & \vdots \\
\eta_N(n, i) & \eta_N(n+2, i) & \cdots & \eta_N(n+2(N-1), i)
\end{array} \right|,
\]

(53)

where \( \eta_j(n, i) \) is defined by

\[
\eta_j(n, i) = c_j \omega_j^n k_j + \frac{1}{c_j \omega_j^n k_j},
\]

(54)

with the dispersion relation

\[
k_j^2 = \frac{1 + \omega_j^n \delta}{\omega_j^n + \delta}.
\]

(55)

The Bäcklund transformations for the discrete KdV equation are expressed by

\[
F_i^n G_i^{n+1} = F_i^{n+1} G_i^n / D + F_i^{n-1} G_i^{n+2} / D',
\]

\[
F_i^n G_i^{n+2} = \delta F_i^{n+1} G_i^n / D' + F_i^{n-1} G_i^{n+2} / D',
\]

(56)

where \( G_i^n \) is the \((N + 1)\)-soliton solution:

\[
G_i^n = \left| \begin{array}{cccc}
\eta_1(n-2, i) & \eta_1(n, i) & \cdots & \eta_1(n+2(N-1), i) \\
\vdots & \ddots & \ddots & \vdots \\
\eta_{N+1}(n-2, i) & \eta_{N+1}(n, i) & \cdots & \eta_{N+1}(n+2(N-1), i)
\end{array} \right|.
\]

(57)
and $D$ and $D'$ are defined by

$$D = \omega_{N+1} + 1/\omega_{N+1}, \quad D' = k_{N+1}(\omega_{N+1} + \delta/\omega_{N+1}).$$  \hfill (58)

On the other hand, proposition 8 gives the Bäcklund transformations for the uKdV equation,

$$\tilde{f}_i^n + \tilde{g}_i^n = \max \left( \tilde{f}_i^{n+1} - p_{N+1}, \tilde{f}_i^{n-1} + \tilde{g}_i^{n+2} - p_{N+1} \right),$$
$$\tilde{f}_i^n + \tilde{g}_i^{n+1} = \max \left( \tilde{f}_i^{n+1} - q_{N+1}, \tilde{f}_i^{n-1} + \tilde{g}_i^{n+2} - p_{N+1} + q_{N+1} \right).$$ \hfill (59)

by setting $(k, l, \alpha, \beta) = (2, 0, 1, 0), (2, 0, 1, -1)$ and the dispersion relation

$$q_j = \min(p_j, 1).$$ \hfill (60)

In particular, we can rewrite $\tilde{f}_i^n$ and $\tilde{g}_i^n$ as

$$\tilde{f}_i^n = \max \left[ \begin{array}{cccc}
|s_1(n, i)| & |s_1(n + 2, i)| & \cdots & |s_1(n + 2(N - 1), i)| \\
\vdots & \vdots & \ddots & \vdots \\
|s_{N}(n, i)| & |s_{N}(n + 2, i)| & \cdots & |s_{N}(n + 2(N - 1), i)| \end{array} \right],$$ \hfill (61)

$$\tilde{g}_i^n = \max \left[ \begin{array}{cccc}
|s(n - 2, i)| & |s_1(n, i)| & \cdots & |s_1(n + 2(N - 1), i)| \\
\vdots & \vdots & \ddots & \vdots \\
|s_{N+1}(n - 2, i)| & |s_{N+1}(n, i)| & \cdots & |s_{N+1}(n + 2(N - 1), i)| \end{array} \right].$$ \hfill (62)

Let us discuss the ultradiscretization of (56) and its correspondence to (59). Introducing the transformations $\delta = e^{-2i\varepsilon}, \omega_j = e^{p_j/\varepsilon}, k_j = e^{-q_j/\varepsilon}$ and $c_j = e^{c_j/\varepsilon}$, we obtain the ultradiscrete analogs of (54), (55) and (58) as

$$\eta_j(n, i) = e^{p_ja - q_jb + c_j} + e^{-p_ja + q_jb - c_j} \to |p_jn - q_ji + c_j|,$$ \hfill (63)

$$e^{-2q_j/\varepsilon} = \frac{e^{(1-p_j)/\varepsilon} + e^{(-1+p_j)/\varepsilon}}{e^{(-1-p_j)/\varepsilon} + e^{(1+p_j)/\varepsilon}} \to q_j = \frac{1}{2}(|p_j + 1| - |p_j - 1|)$$ \hfill (64)

and

$$D \to |p_{N+1}|, \quad D' \to -q_{N+1} + \max(p_{N+1}, -p_{N+1} - 2).$$ \hfill (65)

If $p_j$'s are all positive, $q_j, D$ and $D'$ in (64) and (65) are reduced to $q_j = \min(p_j, 1), p_{N+1}$ and $p_{N+1} - q_{N+1}$. However, determinants in (53) and (57) cannot be ultradiscretized directly due to the negative problem. To avoid this missing link, let us assume that ultradiscretization replaces determinant with UP. By this assumption, the ultradiscrete analogs of $F_i^n$ and $G_i^n$ are replaced with $\tilde{f}_i^n$ and $\tilde{g}_i^n$ respectively. Hence, we have the ultradiscrete analog of (56),

$$\tilde{f}_i^n + \tilde{g}_i^{n+1} = \max \left( \tilde{f}_i^{n+1} - p_{N+1}, \tilde{f}_i^{n-1} + \tilde{g}_i^{n+2} - p_{N+1} \right),$$
$$\tilde{f}_i^n + \tilde{g}_i^{n+1} = \max \left( \tilde{f}_i^{n+1} + \tilde{g}_i^n - p_{N+1} + q_{N+1} - 2, \tilde{f}_i^{n-1} + \tilde{g}_i^{n+2} - p_{N+1} + q_{N+1} \right).$$ \hfill (66)

The latter equation of (66) seems not to coincide with that of (59). However, in the case $p_{N+1} > 1, -p_{N+1} - q_{N+1}$ is equivalent to $-p_{N+1} + q_{N+1} - 2$ by the dispersion relation. In the
case $p_{N+1} \leq 1$, we can prove
\[
\tilde{f}_{n+1}^i + \tilde{g}_i^{n+1} - p_{N+1} + q_{N+1} - 2 < \tilde{f}_{n-1}^i + \tilde{g}_i^n - p_{N+1} - q_{N+1} \leq \tilde{f}_{n+1}^i + \tilde{g}_i^{n+2} - p_{N+1} + q_{N+1}
\]
(67)

after the proof in the previous section. Thus (59) and (66) are equivalent each other.

5. Concluding remarks

In this paper, we consider the generalized soliton solution of two types and propose the ultradiscrete soliton equations and the Bäcklund transformation. The equations are equivalent to the uKdV and the uToda equation in a special case, which are shown in propositions 1 and 2. We have another concrete example of the equation and its solution shown in proposition 3. However, the dispersion relation (15) is expressed by a step function. This type of function cannot be found by ultradiscretizing the discrete expression. In this sense, we have not yet found the corresponding set of difference equation and its solution. Moreover, though we obtain the solutions to the ultradiscrete bilinear equation, we have not found the adequate transformation of dependent variable from the bilinear one to show the soliton behavior of solutions. This is one of the future problems.

The known discrete bilinear equations are expressed by the Plücker relation and some dispersion relations. We consider that the generalized equations in this paper are the important realization of the ultradiscrete Plücker relation. It is another future problem to clarify the relation between the obtained equations in this paper and the ultradiscrete Plücker relation.

As for the Bäcklund transformation, we derive (36) under condition (35). The dispersion relations (11), (13) and (15) satisfy the condition. This means that the Bäcklund transformations for the uKdV and the uToda equations are also obtained as a special case. Furthermore, we discuss the ultradiscretization of the Bäcklund transformations for the discrete KdV equation assuming that determinant is replaced with UP. Although determinant cannot be ultradiscretized directly in general, their counterpart gives the Bäcklund transformations for the uKdV equation.

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