Data-Driven Stochastic Optimal Control Using Kernel Gradients

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Abstract—We present an empirical, gradient-based method for solving data-driven stochastic optimal control problems using the theory of kernel embeddings of distributions. By embedding the integral operator of a stochastic kernel in a reproducing kernel Hilbert space (RKHS), we can compute an empirical approximation of stochastic optimal control problems, which can then be solved efficiently using the properties of the RKHS. Existing approaches typically rely upon finite control spaces or optimize over policies with finite support to enable optimization. In contrast, our approach uses kernel-based gradients computed using observed data to approximate the cost surface of the optimal control problem, which can then be optimized using gradient descent. We apply our technique to the area of data-driven stochastic optimal control, and demonstrate our proposed approach on a linear regulation problem for comparison and on a nonlinear target tracking problem.

I. INTRODUCTION

The advent of autonomous systems, and the increasing complexity of real-world autonomy stemming from human interactions and learning-enabled components, obviates the need for algorithms which can accommodate real-world stochasticity. In such scenarios, model-based approaches may simply fail or hinge upon unrealistic assumptions such as linearity or Gaussianity, which can lead to questionable outcomes or unpredictable behaviors. One approach to dealing with such systems is data-driven control, which has proven to be useful for systems which may be resistant to traditional modeling techniques, or for which finding a simple mathematical model is simply impossible. In order to circumvent the problems faced by traditional model-based approaches, data-driven control uses empirical modeling techniques to synthesize implicit models which are amenable to analysis and control. Nevertheless, these data-driven representations present new challenges for controller synthesis and optimization, which require the development of new tools and techniques to enable their use.

We present a method for computing data-driven solutions to stochastic optimal control problems using an empirical, gradient-based approach. Our approach is based on Hilbert space embeddings of distributions, a nonparametric statistical learning technique that uses data collected from system observations to construct an implicit model of the dynamics as an element in a high-dimensional function space known as a reproducing kernel Hilbert space (RKHS). Hilbert space embeddings of distributions have been applied to Markov models [1]–[3], policy synthesis [4, 5], state estimation and filtering [6]–[8], and also for solving stochastic optimal control problems [9]–[11]. Additionally, these techniques admit finite sample bounds, which show convergence in probability as the sample size increases [12]. Reproducing kernel Hilbert spaces and kernel embeddings of distributions, specifically, are broadly used in the area of nonparametric statistical inference and estimation. However, these techniques have not yet seen widespread adoption for controls.

The use of kernel methods for control has been explored in literature, and is closely related to the theory of Gaussian processes, Koopman operators, and support vector machines, in that they rely upon kernels or operators in high-dimensional function spaces. The use of functional gradients in an RKHS for motion planning and policy synthesis have been used in [4, 13]. However, these techniques either impose a particular problem structure, which limits their use more broadly, or rely upon a specific policy representation to compute the functional gradient. Methods applying kernel embeddings to optimal control problems have been explored previously in [2, 10, 11] for MDPs and chance-constrained control. In addition, [9, 10] show that a kernel-based approximation of the stochastic optimal control problem can be solved as a linear program, but relies upon finite or discrete control spaces, which may be restrictive in practical control scenarios.

Our main contribution is a technique for computing solutions to stochastic optimal control problems using empirical, kernel-based stochastic gradient descent in an RKHS. Unlike existing functional gradient approaches such as [4], our approach does not rely upon explicit parameterizations of the policy in an RKHS. Instead, we use the partial derivative reproducing property of kernels presented in [14] to compute an empirical gradient of the cost using observed data. Our approach is based on the RKHS control framework presented in [9], which optimizes over a finite set of user-specified admissible control actions. However, our proposed approach improves upon the techniques in [9] by eliminating the need for the control designer to strategically pre-select the policy support, at the cost of increased computation time due to the iterative nature of gradient descent. Our approach optimizes directly within the Hilbert space, which opens the door for future development of optimization tools which can be used for the purpose of kernel-based controller synthesis.
The rest of the paper is outlined as follows. In Section II, we define the stochastic optimal control problem using kernel embeddings. Then, in Section III, we describe the gradient-based optimization approach. In Section IV we demonstrate our approach on a double integrator system for comparison to existing approaches and then on a nonholonomic vehicle system to demonstrate the capabilities of the approach. Concluding remarks are presented in Section V.

II. PRELIMINARIES & PROBLEM FORMULATION

A. System Model

Let \((\mathcal{X}, \mathcal{B}_X)\) be a Borel space called the state space and \((\mathcal{U}, \mathcal{B}_U)\) be a compact Borel space called the control space. Consider a discrete-time stochastic system,

\[ x_{t+1} = f(x_t, u_t, w_t), \]

where \(x_t \in \mathcal{X} \subseteq \mathbb{R}^n\), \(u_t \in \mathcal{U} \subseteq \mathbb{R}^m\), and \(w_t\) are independent and identically distributed (i.i.d.) random variables representing a stochastic disturbance. As shown in [15], the dynamics in (1) can equivalently be represented by a stochastic kernel \(Q: \mathcal{B}_X \times \mathcal{X} \times \mathcal{U} \to [0, 1]\) that assigns a probability measure \(Q(\cdot | x, u)\) on \((\mathcal{X}, \mathcal{B}_X)\) to every \((x, u) \in \mathcal{X} \times \mathcal{U}\).

The system evolves from an initial condition \(x_0 \in \mathcal{X}\) (which may be drawn from an initial distribution \(\mathbb{P}_0\) on \(\mathcal{X}\)) over a finite time horizon \(t = 0, 1, \ldots, N, N \in \mathbb{N}\).

B. Stochastic Optimal Control Problem

Consider the following problem with additive cost,

\[ \min_{u_0, u_1, \ldots, u_{N-1}} \mathbb{E}\left[ g_N(x_N) + \sum_{t=0}^{N-1} g_t(x_t, u_t) \right], \]

where \(g_N\) is the terminal cost and \(g_t\) is the convex cost at time \(t\) of taking an action \(u_t\) while in state \(x_t\). The problem in (2) can be written via the Chapman-Kolmogorov identity and the Markov property as a sequence of sub-problems, where the problem is solved via backward recursion [15]. We define the value functions \(V_t: \mathcal{X} \to \mathbb{R}, t = 0, 1, \ldots, N-1,\)

\[ V_t(x) = \min_{u \in \mathcal{U}} g_t(x, u) + \int_{\mathcal{X}} V_{t+1}(y)Q(dy | x, u), \]

which we assume are measurable and bounded functions that lie in a Hilbert space of functions \(\mathcal{H}\).

In many practical control scenarios, such as those involving human-in-the-loop systems or unknown stochastic disturbances, we do not have access to a measurable and bounded function of the system dynamics. Thus, we assume that the stochastic kernel \(Q\) is unknown, meaning we do not have direct information of the dynamics in (1) or the structure of the stochastic disturbance. Instead, we assume that we have access to a sample \(S = \{(x_i, u_i, y_i)\}_{i=1}^M\) of observations taken i.i.d. from \(Q\), where \(x_i\) and \(u_i\) are taken randomly from \(\mathcal{X}\) and \(\mathcal{U}\) and \(y_i \sim Q(\cdot | x_i, u_i)\) is the resulting state after taking action \(u_i\) at state \(x_i\). In practice, such a sample may be generated from historical observations of multiple system trajectories, taking every other observation to ensure conditional independence, for instance, \(\{(x_1, u_1, x_2), (x_3, u_3, x_4), \ldots\}\).

Because the stochastic kernel \(Q\) is unknown, we cannot solve (3) directly since the integral in (3) is intractable. Instead, as shown in [9], we can use \(S\) to approximate the intractable integral with respect to \(Q\) in (3) as an empirical embedding in a high-dimensional space of functions known as a reproducing kernel Hilbert space (RKHS). Then, we can solve an approximation of the original problem in (3) in order to compute an approximately optimal control. We outline the procedure below, but refer the reader to [9] for more details.

C. Approximate Problem Using Kernel Embeddings

Define a positive definite kernel \(k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}\) [16, Definition 4.15]. According to the Moore-Aronszajn theorem [17], given a positive definite kernel \(k\), there exists a unique corresponding RKHS \(\mathcal{H}\) with \(k\) as its reproducing kernel.

**Definition 1.** A Hilbert space \(\mathcal{H}\) of functions from \(\mathcal{X}\) to \(\mathbb{R}\) is called a reproducing kernel Hilbert space if there exists a positive definite function \(k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}\) called the reproducing kernel that satisfies the following properties: (i) For every \(x \in \mathcal{X}\), \(k(x, \cdot) \in \mathcal{H}\), and (ii) For every \(x \in \mathcal{X}\) and \(f \in \mathcal{H}\), \(f(x) = \langle f, k(x, \cdot) \rangle_{\mathcal{H}}\), which is known as the reproducing property.

Similarly, we define the RKHS \(\mathcal{H}\) of functions from \(\mathcal{U}\) to \(\mathbb{R}\) with \(l: \mathcal{U} \times \mathcal{U} \to \mathbb{R}\) as its associated reproducing kernel. According to [1, 18], assuming the kernel \(k\) is measurable and bounded, and given a probability measure \(Q(\cdot | x, u)\), then by the Riesz representation theorem there exists a corresponding element \(m(x, u) \in \mathcal{H}\) called the kernel distribution embedding, such that by the reproducing property, \(\langle V_{t+1}, m(x, u) \rangle_{\mathcal{H}} = \int_{\mathcal{X}} V_{t+1}(y)Q(dy | x, u)\). This means that by representing the integral operator with respect to \(Q\) as an element in the RKHS, we can compute the expectation of any function \(f \in \mathcal{H}\) as an RKHS inner product.

Using a sample \(S\), we can compute an empirical estimate \(\hat{m}(x, u)\) of \(m(x, u)\) as the solution to a regularized least-squares problem [18]. The solution is given by

\[ \hat{m}(x, u) = \Phi^T (G + \lambda M I)^{-1} K(x, u) \]

where \(\Phi\) is a feature vector with elements \(\Phi_i = k(y_i, \cdot)\), \(G \in \mathbb{R}^{M \times M}\) is a real matrix with \(G_{ij} = k(x_i, x_j)(u_i, u_j)\), and \(K(x, u) \in \mathbb{R}^M\) is a vector that depends on \(x\) and \(u\) with elements \([K(x, u)]_i = k(x_i, x_j)(u_i, u_j)\). For simplicity of notation, let \(W = (G + \lambda M I)^{-1}\).

Using the estimate \(\hat{m}(x, u)\), we can approximate the intractable integrals with respect to \(Q\) in (3) via an RKHS inner product,

\[ \int_{\mathcal{X}} V_{t+1}(y)Q(dy | x, u) \approx \langle V_{t+1}, \hat{m}(x, u) \rangle_{\mathcal{H}} \]

\[ = V_{t+1}^T W K(x, u), \]

where \(V_{t+1}\) is a vector with elements \([V_{t+1}]_i = V_{t+1}(y_i)\).

This representation is key to our approach, since it means we can approximate the previously intractable problem in (3) using data comprised of system observations.
D. Problem Statement

Following [9], we can approximate the value functions in (3) using the estimate \( \hat{m}(x, u) \) and (6) as

\[
\min_{u \in \mathcal{U}} g_t(x, u) + \mathbf{V}_{t+1}^T W K(x, u).
\]

(7)

Theoretically, we could optimize for \( u \) directly. However, this is a non-convex problem in general, which makes solving (7) difficult. For example, it may be exceptionally difficult to solve (7) for common kernel choices such as the Gaussian kernel function, since optimizing a linear combination of Gaussians is a non-convex problem. Thus, finding a control action \( u \in \mathcal{U} \) that minimizes the approximate problem presents a significant challenge.

One possible approach is given in [9, 10], where a stochastic control policy with finite support is represented as an embedding in the RKHS \( \mathcal{V} \). Then, the policy can be obtained as the solution to a linear program, giving a set of probability values over a user-specified set of control actions \( \mathcal{A} \subseteq \mathcal{U} \). However, a significant drawback of this approach is the need to strategically select \( \mathcal{A} \) such that it contains controls which are close to the true solution, and typically only finds a sub-optimal solution to the approximate problem.

We propose to compute the control input via a kernel-based stochastic gradient descent method. Using the properties of reproducing kernel Hilbert spaces, we can compute the gradient by taking the partial derivative of the kernel, rather than explicitly computing the gradient with respect to \( u \). This allows us to optimize the control input by directly optimizing within the RKHS and avoids the problem of optimizing over a linear combination of kernels. An illustration of this idea is depicted in Figure 1.

III. COMPUTING CONTROLS USING GRADIENT DESCENT IN AN RKHS

We seek to compute the partial derivative of the objective in (7) with respect to \( u \). We first define the notation used to describe the partial derivative of a bivariate function.

**Definition 2** (Partial Functional Derivative Notation). Given a bivariate function \( l : \mathcal{U} \times \mathcal{U} \to \mathbb{R} \), \( u, u' \in \mathcal{U} \) and multi-indices \( p, q \), we denote the partial derivative as

\[
\partial^{p,q} l(u, u') := \frac{\partial^{p_1+\cdots+p_m+q_1+\cdots+q_m} l(u, u')}{\partial (u_{1})^{p_1} \cdots \partial (u_{m})^{p_m} \partial (u'_{1})^{q_1} \cdots \partial (u'_{m})^{q_m}}.
\]

As shown in [14], we can compute the partial derivative of any function \( h \in \mathcal{V} \) via the reproducing property as

\[
\partial^{p} h(u) = \langle h, \partial^{p,0} l(u, \cdot) \rangle_{\mathcal{V}}.
\]

In short, this means that we do not need to directly compute the partial derivative of the cost function \( g \) with respect to \( u \) (which may be unknown if we are only given points \( g(y_t) \)), and we may compute the empirical gradient by taking the partial derivative of the kernel \( l \).

Let \( \hat{J}(u) = g_t(x, u) + \mathbf{V}_{t+1}^T W K(x, u) \) be the objective of the approximate optimal control problem in (7). As a function of \( u \), note that \( \hat{J}(u) \) can be written as a linear combination of kernel functions and is an element of the RKHS \( \mathcal{V} \), such that via the reproducing property, \( \hat{J}(u) = \langle \hat{J}, l(u, \cdot) \rangle_{\mathcal{V}} \), which can be written as

\[
\hat{J}(u) = \langle g_t(x, \cdot) + \mathbf{V}_{t+1}^T W K(x, \cdot), l(u, \cdot) \rangle_{\mathcal{V}}.
\]

(9)

Then, using (8), the partial derivative of \( \hat{J}(u) \) with respect to the control \( u \) can be computed as

\[
\partial^{1} \hat{J}(u) = \langle g_t(x, \cdot) + \mathbf{V}_{t+1}^T W K(x, \cdot), \partial^{1} l(u, \cdot) \rangle_{\mathcal{V}}.
\]

(10)

This approach has a significant advantage, most notably that most popular kernels are easy to differentiate, meaning we can quickly compute the empirical gradient for an arbitrary cost function in \( \mathcal{H} \). In addition, the empirical cost gradient can be computed as a simple matrix multiplication.

As a practical example, consider the Gaussian kernel, \( l(u, u') = \exp(-\|u - u'\|^2/2\sigma^2) \), \( \sigma > 0 \) (assuming \( u \) is a scalar variable for simplicity). The partial derivative of the
Gaussian kernel is given by
\[ \partial_{1,0}^{l}(u, u') = -|u - u'|\sigma^{-2} \exp(-\|u - u'\|^2/2\sigma^2). \] (11)

Then the partial derivative of the value function expectation in (10) can be computed as \( V_{t+1}^\top W(K(x, u) \odot \Delta) \), where \( \odot \) denotes the Hadamard (or element-wise) product and \( \Delta \in \mathbb{R}^{M} \) is a vector with elements \( \Delta_i = -|u_i - u'|/\sigma^2 \).

We use the empirical gradient of the value function computed using (10) in order to compute the gradient direction for stochastic gradient descent. Then, by traversing the approximate cost surface using the empirical gradient, we can find as the solution to a linear program,
\[
\begin{align*}
\min_{\gamma(x) \in \mathbb{R}^P} & \quad (g_i(x))^\top \gamma(x) + V_{t+1}^\top W R(x) \gamma(x) \\
\text{s.t.} & \quad 1^\top \gamma(x) = 1, \quad 0 \preceq \gamma(x)
\end{align*}
\] (12a)

Importantly, this also means that the solution to the approximate cost surface also converges in probability to the true cost surface. Hence, as the sample size increases, we obtain a closer approximation of the true cost surface. Therefore, we can view the coefficients \( \gamma(x) \) as a set of probabilities that weight the user-specified control actions in \( \mathcal{A} \), which we can find as the solution to a linear program,
\[
\min_{\gamma(x) \in \mathbb{R}^P} \quad (g_i(x))^\top \gamma(x) + V_{t+1}^\top W R(x) \gamma(x) \\
\text{s.t.} \quad 1^\top \gamma(x) = 1, \quad 0 \preceq \gamma(x)
\] (12b)

where \( g_i(x) \in \mathbb{R}^P \) is a vector that depends on \( x \) with elements \( g_i(x) = g_i(x, \bar{u}_j) \), \( R(x) \in \mathbb{R}^{M \times P} \) is a real matrix that depends on \( x \), such that \( [R(x)]_{ij} = k(x_i, x)(u_i, \bar{u}_j) \). In (12), because we minimize a weighted sum by choosing non-negative weights, the solution is to allocate the maximum weight to the smallest term. Thus, according to [19] the solution is given by a vector of all zeros except at the index \( j = \arg \min_i \{C_i(x)^\top \} \), where it is 1. See [9] for details.

By choosing control actions in \( \mathcal{A} \) that are good candidate solutions to the optimal control problem in (3), we obtain a good initial guess for the gradient-based learning algorithm. However, unlike the approach in [9], we do not require the solution to lie within \( \mathcal{A} \), and we further improve the solution of the LP using stochastic gradient descent.

### IV. Numerical Results

We demonstrate our approach on a regulation problem using a discrete-time stochastic chain of integrators for verification, and on a target tracking problem using nonholonomic vehicle dynamics to demonstrate the utility of the approach. For all problems, we use a Gaussian kernel for \( k \) and \( l \), which has the form \( k(x, x') = \exp(-\|x - x'\|^2/2\sigma^2) \), where \( \sigma > 0 \). Following [1], we fix the regularization parameter to be \( \lambda = 1/M^2 \), where \( M \in \mathbb{N} \) is the sample size used to construct the estimate \( \hat{m} \). In practice, \( \sigma \) and \( \lambda \) are typically found via cross-validation, where \( \sigma \) is chosen according to the relative spacing of the data points (usually the median distance) and \( \lambda \) is selected such that \( \lambda \to 0 \) as \( M \to \infty \). A more detailed discussion of parameter selection is outside the scope of the current work (see [12] for recent results on regularization rates). Numerical experiments were performed in Python on an AWS cloud computing instance. Code for all analysis and experiments is available as part of the stochastic optimal control using kernel methods (SOCKS) toolbox [20].

### A. Regulation of a Double Integrator System

We consider the problem of regulation for a 2D stochastic chain of integrators system, with dynamics given by
\[
x_{t+1} = \begin{bmatrix} 1 & T_s \ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} T_s^2/2 \ T_s \end{bmatrix} u_t + w_t,
\] (13)

where \( x_t \in \mathbb{R}^2 \) is the state, \( u_t \in \mathbb{R} \) is the control input, which we constrain to be within \([ -1, 1] \), \( w_t \) is a random variable with distribution \( \mathcal{N}(0, 0.01I) \), and \( T_s \) is the sampling time. We seek to compute a control input \( u \) as the solution to the following stochastic optimal control problem,
\[
\min_u \quad \int x g(y)Q(dy | x, u),
\] (14)

where \( Q \) is a representation of the dynamics as a stochastic kernel. We use the cost function \( g(x) = \|x\|_2 \), which serves as a kernel embedding
\[
U \quad \pi \quad (7) \quad \text{by representing a stochastic policy} \quad \pi : \mathcal{B}_U \times \mathcal{X} \to [0, 1]
\]
as a kernel embedding \( p(x) = \sum_{j=1}^{P} \gamma_j(x)(\bar{u}_j, \cdot) \) in the RKHS \( \mathcal{H} \), where \( \gamma(x) \in \mathbb{R}^P \) are real-valued coefficients that depend on the state \( x \in \mathcal{X} \), and the points \( \mathcal{A} = \{\bar{u}_j\}_{j=1}^{P} \) are a set of user-specified control actions that we want to optimize over. The problem then becomes finding the coefficients \( \gamma(x) \) that optimize the approximate control problem. According to

### Algorithm 1 Kernel-Based Gradient Descent

1: given embedding estimate \( \hat{m} \), initial guess \( u_0 \)
2: repeat
3: \( \Delta u_n \leftarrow \langle g(x, \cdot) + V_{t+1}^\top W K(x, \cdot), \partial_{1,0}^{l}(u_n, \cdot) \rangle \eta \)
4: choose step size \( \eta \)
5: \( u_{n+1} \leftarrow u_n - \eta \Delta u_n \)
6: until stopping criterion satisfied
7: return \( u_n \)

Since the estimate \( \hat{m} \) converges in probability to the true embedding \( m \) at a minimax optimal rate of \( O(M^{-1/2}) \) [12], the approximate cost surface also converges in probability to the true cost surface. Hence, as the sample size increases, we obtain a closer approximation of the true cost surface. However, it is important to note that the empirical cost surface is generally not convex, even if the original function is convex, meaning we are only guaranteed to find a locally optimal solution to the approximate problem. This is obvious, since the noise of the data also adds noise to the empirical cost surface. Importantly, this also means that the solution to the approximate problem is primarily heuristic. Nevertheless, we can use more advanced gradient descent methods (e.g. using momentum or a “temperature” in place of the learning rate) to mitigate the issues of optimizing over an empirical cost surface. This also motivates the need to choose an initial guess as close as possible to the optimal solution, which is detailed in the next section.

**A. Initialization**

Initializing the gradient descent algorithm close to the true solution ensures that we obtain an optimally approximated solution in fewer gradient steps. One possibility is to compute a sub-optimal initial guess for Algorithm 1 using [9].

As shown in [9], we can compute a solution to the (unconstrained) approximate stochastic optimal control problem in (7) by representing a stochastic policy \( \pi : \mathcal{B}_U \times \mathcal{X} \to [0, 1] \) as a kernel embedding \( p(x) = \sum_{j=1}^{P} \gamma_j(x)(\bar{u}_j, \cdot) \) in the RKHS \( \mathcal{H} \), where \( \gamma(x) \in \mathbb{R}^P \) are real-valued coefficients that depend on the state \( x \in \mathcal{X} \), and the points \( \mathcal{A} = \{\bar{u}_j\}_{j=1}^{P} \) are a set of user-specified control actions that we want to optimize over. The problem then becomes finding the coefficients \( \gamma(x) \) that optimize the approximate control problem. According to
to drive the system to the origin. Note that here we seek to optimize only over a single time step for the purpose of comparing against a known solution.

We consider a sample \( S = \{(x_i, u_i, y_i)\}_{i=1}^M \) of size \( M = 1600 \) taken i.i.d. from \( Q \). The states \( x_i \) were taken uniformly in the region \([-1,1] \times [-1,1]\), the control inputs \( u_i \) were taken uniformly from \([-1,1]\), and the resulting states were generated according to \( y_i \sim Q(\cdot \mid x_i, u_i) \). We then presumed no knowledge of the system dynamics or the stochastic disturbance for the purpose of computing the approximately optimal control action using our proposed method.

Using the sample \( S \), we then computed an estimate \( \hat{m} \) of the kernel embedding \( m \) as in (4) using Gaussian kernels with bandwidth parameter \( \sigma = 3 \), chosen via cross-validation. We then selected \( R = 121 \) evaluation points \( \{x_j\}_{j=1}^R \) spaced uniformly in the region \([-1,1] \times [-1,1]\), from which to compute the optimal control actions.

To provide a basis for comparison, we computed the optimal control actions using CVX from the evaluation points \( \{x_j\}_{j=1}^R \) using the deterministic dynamics. We then propagate the dynamics forward in time using the optimal inputs to obtain the state at the next time instant. The vector field of the closed-loop dynamics under the optimal control inputs is shown in Figure 2 (blue). We then computed the approximately optimal control actions using Algorithm 1 with the sample \( S \) taken from the stochastic dynamics to minimize the cost at each point \( \{x_j\}_{j=1}^R \) over a single time step. For Algorithm 1, we used a step size of \( \eta = 0.01 \), a stopping threshold of \( 10^{-5} \), and limited the number of iterations to 100. The vector field of the closed-loop dynamics using the approximately optimal solution computed using our proposed method is shown in Figure 2 (orange).

We see that the gradient based algorithm computes approximately optimal control inputs which closely match the solution via CVX. This demonstrates the effectiveness of the gradient-based algorithm to compute approximately optimal control actions with no prior knowledge of the dynamics or the stochastic disturbance. Note that the quality of the empirical approximation of the cost surface depends on the sample size \( M \). As \( M \) increases, the approximation improves, and we obtain a closer approximation of the optimal solution using our method. To demonstrate this, we computed the mean and maximum error between our gradient-based solution and the solution via CVX using the deterministic dynamics for varying sample sizes \( M \in [250,2500] \) averaged over 20 iterations (Figure 3). We see that the error of the approximate solution decreases as \( M \) increases. However, we also see that the error does not decrease appreciably as the sample size increases, which is due to the asymptotic convergence of the estimate \( \hat{m} \) to the true embedding \( m \). This presents a tradeoff between computation time and numerical accuracy, since the computational complexity scales polynomially with the sample size.

B. Target Tracking Using a Nonholonomic Vehicle

We consider the problem of target tracking for a nonholonomic vehicle system as in [9]. The dynamics are given by

\[
\begin{align*}
\dot{x}_1 &= u_1 \sin(x_3), & \dot{x}_2 &= u_1 \cos(x_3), & \dot{x}_3 &= u_2
\end{align*}
\]

(15)

where \( x = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \) are the states, \( u = [u_1, u_2]^T \in \mathbb{R}^2 \) are the control inputs. The control inputs are constrained such that \( u_t \in [0.5, 1.2] \times [-10,1,10,1] \). We discretize the dynamics in time using a zero-order input hold and apply an affine disturbance \( w \sim \mathcal{N}(0,0.01I) \). We define a target trajectory as a sequence of position coordinates indexed by time, shown in Figure 4 (blue). We choose an initial condition of \( x_0 = [-1,-0.2, \pi/2]^T \), and evolve the system over the time horizon \( N = 20 \). Working backward in time, at each time step we seek to compute the control inputs \( u_t \) by iteratively optimizing a sequence of value functions, and then selecting the control input which has the highest value,

\[
V_t(x_t) = \min_{u_t} g_t(x_t) + \int_X V_{t+1}(y)Q(dy \mid x_t, u_t),
\]

(16)
where the cost function $g_i$ seeks to minimize the squared Euclidean distance of the system’s position to the target trajectory’s position at each time step.

We consider a sample $\mathcal{S} = \{(x_i, u_i, y_i)\}_{i=1}^M$, with sample size $M = 2000$. The states $x_i$ were taken uniformly within $[-1.2, 1.2] \times [-1.2, 1.2] \times [-\pi, \pi]$, the controls $u_i$ were taken uniformly within $[0.5, 1.2] \times [-10.1, 10.1]$, and the resulting states were drawn according to $y_i \sim Q(\cdot | x_i, u_i)$. Using the sample $\mathcal{S}$, we then computed an estimate $\hat{m}$ of the kernel embedding $m$ as in (4) using Gaussian kernels with bandwidth parameter $\sigma = 3$, chosen via cross-validation.

In order to compute a baseline for comparison, we then computed the control actions using [9], which computes a stochastic policy embedding $p_\theta$ at every time step over a finite set $\mathcal{A} = \{\tilde{u}_j\}_{j=1}^{210}$, as in (12). We chose the controls $\tilde{u}_j$ in the set $\mathcal{A}$ uniformly in the range $\tilde{u}_j \in [0.5, 1.2] \times [-10.1, 10.1]$. We then computed the control actions by solving (12) at each time step backward in time. The resulting trajectory is plotted in Figure 4 (green), and had a total cost of $\sum_{t=0}^{N} g_t(x_t) = 2.507$, and the computation time was 2.056 seconds.

We then simulate the system using the approximately optimal control action selected via the kernel-based gradient descent algorithm (Algorithm 1), and initializing using the solution to (12) using $\mathcal{A}$ as above. For Algorithm 1, we chose a step size of $\eta = 0.1$, a stopping threshold of $10^{-5}$, and limited the number of gradient iterations to 100. The resulting trajectory is plotted in Figure 4 (orange), and has a total cost of $\sum_{t=0}^{N} g_t(x_t) = 2.360$. The total computation time was 12.206 seconds. As expected, we see that the trajectory computed using our method more closely follows the target trajectory (has a lower overall cost). This shows that the gradient-based algorithm is able to compute the approximately optimal control actions for a nonlinear system, using only data collected from system observations.

V. Conclusions & Future Work

In this paper, we presented a method for computing the approximately optimal control action for stochastic optimal control problems using a data-driven approach. Our proposed method leverages kernel gradient-based methods and achieves more optimal control solutions than existing sample-based approaches. We plan to explore methods to compute control solutions using the geometric properties of the RKHS, e.g. via projections, to adapt the algorithm to constrained stochastic optimal control problems, and to explore adaptations to online data scenarios.

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