QUASIFIBRATIONS IN CONFIGURATION LIE GROUPOIDS
AND ORBIFOLD BRAID GROUPS

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Abstract. In [19] we studied a Fadell-Neuwirth type fibration theorem for orbifolds, and gave a short exact sequence of fundamental groups of configuration Lie groupoids of Lie groupoids corresponding to the genus zero 2-dimensional orbifolds with cone points, and at least one puncture. In this paper we extend this work to all genus \( \geq 1 \), 2-dimensional orbifolds with cone points. As a consequence, we prove the Farrell-Jones Isomorphism conjecture for the fundamental groups of the associated configuration Lie groupoids. This answers a substantial part of a question we posed in [[18], Problem]. In [19] we also showed that for all global quotient type orbifolds, the fibration theorem does not hold. Here, we give some nontrivial examples of orbifolds where a Fadell-Neuwirth type quasifibration theorem holds. Finally, we state an Asphericity conjecture and a Quasifibration conjecture for orbifolds.

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1. Introduction

Let \( M \) be a connected smooth manifold of dimension \( \geq 2 \). Let \( PB_n(M) \) be the configuration space of ordered \( n \)-tuples of pairwise distinct points of \( M \). Then, the Fadell-Neuwirth fibration theorem ([7]) says that the projection map \( M^n \to M^{n-1} \) to the first \( n - 1 \) coordinates defines a fibration \( f(M) : PB_n(M) \to PB_{n-1}(M) \), with fiber homeomorphic to \( M - \{ (n - 1) \text{ points} \} \). It is an important subject to study the homotopy groups, especially the fundamental groups of the configuration spaces of a manifold. Since in dimension \( \geq 3 \), the space \( PB_n(M) \) and the product manifold \( M^n \) have isomorphic fundamental groups, the dimension 2 case
is of much interest. Using the above fibration, there are results to compute the higher homotopy groups of the configuration space also. See [7] for more details. For $M = \mathbb{C}$, the fundamental group of the configuration space $PB_n(M)$ is known as the classical pure braid group. The braid groups appear in a wide range of areas in both Mathematics and Physics.

In [19] we studied the possibility of extending the Fadell-Neuwirth fibration theorem for orbifolds, to understand a certain class of Artin groups. Orbifolds are also of fundamental importance in algebraic and differential geometry, topology and string theory. However, to define a fibration between orbifolds, we had to consider the category of Lie groupoids. Since an orbifold can be realized as a Lie groupoid ([13]), and there are enough tools in this category to define a fibration. There, we defined two notions ($a$ and $b$-types) of a fibration ([19, Definition 2.4]) and the corresponding ($a$ and $b$-types) configuration Lie groupoids of a Lie groupoid to enable us to state a Fadell-Neuwirth type theorem. For an orbifold $M$, the $b$-type configuration Lie groupoid is the correct model to induce the orbifold structure on $PB_n(M)$. We proved that the Fadell-Neuwirth fibration theorem extends in this generality, under some strong hypothesis ($c$-groupoid). We also showed that this is the best possible extension. For this, we deduced that the map $f(M)$ is not a $a$($or$ $b$)-type fibration for the $a$($or$ $b$)-type configuration Lie groupoids of Lie groupoids, corresponding to global quotient compact orbifolds of dimension $\geq 2$, with non-empty singular set (See [[19], Proposition 2.11]). However, there we proved the existence of a short exact sequence of fundamental groups of the $b$-type configuration Lie groupoids ([19, Theorem 2.14, Remark 2.15]) corresponding to all genus zero 2-dimensional orbifolds with cone points and at least one puncture. This shows that, $f(M)$ behaves like a quasi-fibration in low degree. This result had multiple applications, for example, poly-freeness of certain class of Artin groups (partially answering [4, Question 3]) and the Farrell-Jones Isomorphism conjecture for the affine Artin group of type $\tilde{D}_n$ ([18, Problem]).

In this paper we establish the above mentioned short exact sequence for all genus $\geq 1$, 2-dimensional orbifolds with cone points (Theorem 2.2). Then we apply it to prove the Farrell-Jones Isomorphism conjecture for the fundamental groups (called surface orbifold (pure) braid groups) of the associated $b$-type configuration Lie groupoids (Theorem 2.5). For some more applications of Theorem 2.2 see [20]. Also, we show that $f(M)$ is a quasifibration (see Definition 4.2) for some nontrivial 2-dimensional orbifolds (Proposition 2.4). From this, we state some conjectures for orbifolds, and some problems in the category of Lie groupoids, which are of independent interest. The main ingredient of the proof of Theorem 2.2 is the presentation of the surface pure braid groups of a surface of genus $\geq 1$, from [3].

2. MAIN RESULTS AND THE CONJECTURES

Let $M$ be a connected 2-dimensional orbifold. Then $PB_n(M)$ is again an orbifold. Consider the action of the symmetric group $S_n$ on $PB_n(M)$, by permuting coordinates. The quotient orbifold $PB_n(M)/S_n$ is denoted by $B_n(M)$.

**Definition 2.1.** The orbifold fundamental group of $B_n(M)(PB_n(M))$ is called the surface orbifold (pure) braid group of $M$ of $n$ strings.

Let $C^0$ be the class of all connected, genus zero 2-dimensional orbifolds with cone points and at least one puncture, and let $C^1$ be the class of all connected, genus $\geq 1$,
2-dimensional orbifolds with cone points. If \( M \in C^0 \cup C^1 \) has nonempty boundary, then we replace each boundary component of \( M \) by a puncture.

We use the presentation of surface pure braid groups from [3], to produce an explicit set of generators and relations of the surface orbifold pure braid group of \( M \in C^1 \). Then, using this presentation and the Fadell-Neuwirth fibration theorem for surfaces, we prove the following theorem.

**Theorem 2.2.** There is a short exact sequence of surface orbifold pure braid groups of \( M \in C^1 \), as follows.

\[
1 \longrightarrow \pi_1^{orb}(\mathcal{F}) \longrightarrow \pi_1^{orb}(PB_n(M)) \xrightarrow{f(M)_*} \pi_1^{orb}(PB_{n-1}(M)) \longrightarrow 1.
\]

Here \( \mathcal{F} = M - \{(n-1) \text{ smooth points}\} \).

In [19], we gave a proof of this short exact sequence, for \( M \in C^0 \) using a stretching technique. In fact, there we did not need to appeal to the Fadell-Neuwirth fibration theorem for punctured complex plane. However, the stretching technique is not applicable for genus \( \geq 1 \) orbifolds, as in this case the movement of the strings of a braid are more complicated and a pictorial description is not possible.

The following corollary can be deduced easily from the above result.

**Corollary 2.3.** Let \( M \in C^1 \). Assume that \( M \) has finitely generated orbifold fundamental group and has at least one puncture. Then, the surface orbifold pure braid group of \( M \) is poly-virtually finitely generated free. That is, it has a normal series whose successive quotients are virtually finitely generated free.

A surjective map \( f : X \to Y \) is called a quasifibration if for all \( y \in Y \), \( f : (X, f^{-1}(y)) \to (Y, y) \) is a weak homotopy equivalence ([6]), and hence a quasifibration induces a long exact sequence of homotopy groups, similar to the one induced by a fibration. We say a homomorphism between two Lie groupoids is a quasifibration, if the induced map on their classifying spaces induces a long exact sequence of homotopy groups, similar to the one induced by a fibration. This definition applies to orbifolds as well. In [19], Theorem 2.14, Remark 2.15 and in Theorem 2.2, we see that for a large class of 2-dimensional orbifolds, the map \( f(M) \) has a quasifibration type property in low degree. This together with the following proposition, motivate us to state an Asphericity conjecture and a Fadell-Neuwirth type quasifibration conjecture. In [20] we give some further evidence to the Asphericity conjecture. See Definitions 3.1 and 3.12 for a precise definition of aspherical Lie groupoids and in particular, of aspherical orbifolds. As in the standard terminology, it says that the higher homotopy groups should vanish.

**Proposition 2.4.** Let \( M \) be either an aspherical 2-manifold or the complex plane with at most two cone points of order 2. Then \( PB_n(M) \) is aspherical and hence \( f(M) \) is a quasifibration.

The two cone points case in Proposition 2.4 answers the test case of the last problem posed in [18], Problem. The proof needs the \( D_n \)-type affine Artin group case, of the recent proof of the \( K(\pi,1) \)-conjecture for affine Artin groups ([15]), and some technique from [2] to construct an aspherical orbifold covering of \( PB_n(M) \).

**Asphericity conjecture.** (Problem 4.9 and [18], Problem) Let \( M \) be a connected aspherical 2-dimensional orbifold, then \( PB_n(M) \) is aspherical.

**Quasifibration conjecture.** (Problem 4.10) Let \( M \) be a connected orbifold, then \( f(M) \) is a quasifibration.
We end this section with an application of Theorem 2.2.

Before we give the statement of the result, let us recall that the Farrell-Jones Isomorphism conjecture is an important conjecture in Geometry and Topology. The conjecture implies some of the classical conjectures in Topology, like Borel and Novikov conjectures. It provides an understanding of the $K$- and $L$-theory of the group ring of a group from the group ring of its virtually cyclic subgroups. A lot of works have been done in recent times on this conjecture. See [9] for more on this subject. Relevant to this work, are [18] and [19] where we deduced the conjecture for a large class of Artin groups, and for the surface orbifold braid groups of $M \in C^0$. Also see [8] and [24] for some recent works in this direction.

**Theorem 2.5.** Let $M \in C^1$. Then, the Farrell-Jones Isomorphism conjecture with coefficients and with finite wreath product is true for the surface orbifold braid groups of $M$.

Note that, the Asphericity and the Quasifibration conjectures, together imply Theorem 2.2 and 2.5 for all connected aspherical 2-dimensional orbifolds.

The paper is organized as follows. In Section 3, we give a brief introduction to Lie groupoids and orbifolds. Some background materials are recalled in Section 4, to enable us to state Theorem 2.2 and the two conjectures above in the category of Lie groupoids. In Section 5 we give the proofs of the results of this section and of Theorem 4.3.

3. **Lie groupoids and orbifolds**

In this paper by a ‘manifold’ we mean a ‘Hausdorff smooth manifold’ and by a ‘group’ we mean a ‘discrete group’, unless mentioned otherwise. A ‘map’ is either continuous or smooth, which will be clear from the context. And a ‘fibration’ will mean a ‘Hurewicz fibration’.

We now recall some basics on Lie groupoids and orbifolds. See [1], [13] or [14] for more details.

3.1. **Lie groupoids.** Let $\mathcal{G}$ be a Lie groupoid with object space $\mathcal{G}_0$ and morphism space $\mathcal{G}_1$. Let $s, t: \mathcal{G}_1 \to \mathcal{G}_0$ be the source and the target maps defined by $s(f) = x$ and $t(f) = y$, for $f \in \text{mor}_\mathcal{G}(x, y) \subset \mathcal{G}_1$. Recall that $s$ and $t$ are smooth and submersions.

A homomorphism $f$ between two Lie groupoids is a smooth functor which respects all the structure maps. $f_0$ and $f_1$ denote the object and morphism level maps of $f$, respectively. For any $x \in \mathcal{G}_0$, the set $t(s^{-1}(x))$ is called the orbit of $x$. The space $|\mathcal{G}|$ of all orbits with respect to the quotient topology is called the orbit space of the Lie groupoid. If $f: \mathcal{G} \to \mathcal{H}$ is a homomorphism between two Lie groupoids, then $f$ induces a map $|f|: |\mathcal{G}| \to |\mathcal{H}|$, making the following diagram commutative.

\[
\begin{array}{ccc}
\mathcal{G}_0 & \xrightarrow{f} & \mathcal{H}_0 \\
|\mathcal{G}| & \xrightarrow{|f|} & |\mathcal{H}|
\end{array}
\]

We define $\mathcal{G}$ to be Hausdorff if $|\mathcal{G}|$ is Hausdorff, and it is called a $c$-groupoid if the quotient map $\mathcal{G}_0 \to |\mathcal{G}|$ is a covering map. Hence a $c$-groupoid is Hausdorff.

Given a Hausdorff Lie groupoid $\mathcal{G}$, we defined in [[19], Definition 2.8] the $b$-configuration Lie groupoid $\text{PB}^b_n(\mathcal{G})$. In this paper we do not use the superscript $b$. 
as we consider only this configuration Lie-groupoid. Recall that, its object space $PB_n(G)_0$ is the $n$-tuple of objects of $G$ with mutually distinct orbits.

$$PB_n(G)_0 = \{(x_1, x_2, \ldots, x_n) \in G_0^n \mid t(s^{-1}(x_i)) \neq t(s^{-1}(x_j)), \text{ for } i \neq j\}.$$ The morphism space $PB_n(G)_1$ is $(s^n, t^n)^{-1}(PB_n(G)_0 \times PB_n(G)_0)$. We also showed in [[19], Lemma 2.9] that the projection to the first $n-1$ coordinates on both the object and morphism spaces define a homomorphism $f(G) : PB_n(G) \to PB_{n-1}(G)$.

Let $H$ and $G$ be two Lie groupoids and $f : H \to G$ be a homomorphism, such that $f_0 : H_0 \to G_0$ is a covering map. Then, $f$ is called a covering homomorphism of Lie groupoids if $H_0$ is a left $G$-space with $f_0$ equal to the action map, $H_1 = G_1 \times_{G_0} H_0$ and $f_1$ is the first projection. The source and the target maps of $H$ coincide with the second projection and the action map, respectively.

Next we recall the important concept of the classifying space of a Lie groupoid, which is required to define algebraic invariants of the Lie groupoid. For a Lie groupoid $G$, the classifying space $BG$ is defined as the geometric realization of the simplicial manifold $G_*$ defined by the following iterated fibered products.

$$G_0 = G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1.$$ See [[1], p. 25] or [13] for some discussion on this matter, in the context of orbifold Lie groupoids (Definition 3.3).

**Definition 3.1.** The $k$-th homotopy group of $G$ is defined as the $k$-th ordinary homotopy group of $BG$. That is, $\pi_k(G, \bar{x}_0) := \pi_k(BG, \bar{x}_0)$ for $\bar{x}_0 \in G_0$. A Lie groupoid $G$ is called aspherical if $\pi_k(G, \bar{x}) = 0$ for all $k \geq 2$ and for all $\bar{x} \in G_0$.

Note that, a homomorphism $f : G \to H$ induces a map $Bf : BG \to BH$. Also see [1] or [13] for some more on homotopy groups of Lie groupoids.

Now, we recall the concept of equivalence between two Lie groupoids. One consequence of this concept is that an equivalence $f : G \to H$ between two Lie groupoids induces a weak homotopy equivalence $Bf : BG \to BH$.

A more appropriate notion of equivalence between Lie groupoids is Morita equivalence.

**Definition 3.2.** Let $f : G \to H$ be a homomorphism between Lie groupoids. $f$ is called an equivalence if the following conditions are satisfied.

- The following composition is a surjective submersion.
  $$H_1 \times_{G_0} G_0 \xrightarrow{t} H_1 \xrightarrow{\pi_1} H_0.$$
  Here $H_1 \times_{G_0} H_0$ is the fibered product, defined by $f_0$ and $s$.
  $$\begin{array}{ccc}
  H_1 \times_{G_0} G_0 & \xrightarrow{\pi_1} & H_1 \\
  \downarrow \pi_2 & & \downarrow s \\
  G_0 & \xrightarrow{f_0} & H_0.
  \end{array}$$

- The following commutative diagram is a fibered product of manifolds.
  $$G_1 \xrightarrow{f_1} H_1 \xrightarrow{(s,t)} H_0 \\
  G_0 \times G_0 \xrightarrow{f_0 \times f_0} H_0 \times H_0.$$

$G$ and $H$ are called Morita equivalent if there is a third Lie groupoid $K$ and two equivalences as follows.
Lemma 3.6. \( f : \mathcal{G} \rightarrow \mathcal{H} \).

Next, we recall a standard example of a Lie groupoid which is relevant for us.

**Example 3.3.** Let \( \widetilde{M} \) be a manifold, and a Lie group \( H \) is acting on \( \widetilde{M} \) smoothly. Out of this information we construct a Lie groupoid \( \mathcal{G}(\widetilde{M}, H) \) as follows, and call it the *translation Lie groupoid*. Define \( \mathcal{G}(\widetilde{M}, H)_0 = \widetilde{M} \), \( \mathcal{G}(\widetilde{M}, H)_1 = \widetilde{M} \times H \), \( s(x, h) = x \), \( t(x, h) = h(x) \), \( u(x) = (x, 1) \), \( i(x, h) = (x, h^{-1}) \) and \( (h(x), h') \circ (x, h) = (x, h'h) \), for \( h, h' \in H \) and \( x \in \widetilde{M} \). When \( H \) is the trivial group then \( \mathcal{G}(\widetilde{M}, H) \) is called the *unit groupoid*, denoted by \( \mathcal{G}(\widetilde{M}) \) and is identified with \( \widetilde{M} \). In this paper we always consider \( H \) to be discrete (unless explicitly mentioned) and is acting effectively and properly discontinuously on \( M \) ([[23], Definition 3.7.1]). Then \( \mathcal{G}(\widetilde{M}, H) \) is an example of an (effective) *orbifold Lie groupoid*. In this case \( \mathcal{G}(\widetilde{M}, H) \) is also called an orbifold Lie groupoid inducing the orbifold structure on \( M = \widetilde{M}/H \). For the more general definition of orbifold Lie groupoid see [13] or [[1], Definition 1.38].

**Definition 3.4.** We call an effective orbifold Lie groupoid of type \( \mathcal{G}(\widetilde{M}, H) \) as in Example 3.3, a *translation orbifold Lie groupoid*.

**Example 3.5.** Let \( H \) and \( \widetilde{M} \) be as in Example 3.3. Let \( H' \) be a subgroup of \( H \) and \( i : H' \rightarrow H \) be the inclusion map. Then the maps \( f_0 := id : \widetilde{M} \rightarrow \widetilde{M} \) and \( f_1 := (id, i) : \widetilde{M} \times H' \rightarrow \widetilde{M} \times H \) together define a homomorphism \( f : \mathcal{G}(\widetilde{M}, H') \rightarrow \mathcal{G}(\widetilde{M}, H) \).

Frequently, in this paper we will be using the following lemma.

**Lemma 3.6.** Let \( \mathcal{G} \) and \( \mathcal{H} \) be two orbifold Lie groupoids, and \( f : \mathcal{G} \rightarrow \mathcal{H} \) be a covering homomorphism. Then \( f \) induces isomorphisms on higher homotopy groups and an injection on the fundamental groups.

**Proof.** It is easy to see that a covering homomorphism between two orbifold Lie groupoids induces a covering map on their classifying spaces. The Lemma now follows from standard covering space theory. See [[1], Proposition 2.17]. \( \Box \)

3.2. **Orbifolds as Lie groupoids.** An orbifold ([[23]]) or a V-manifold as in [21], is defined as follows.

**Definition 3.7.** Let \( M \) be a paracompact Hausdorff topological space. Assume for each \( x \in M \), there is a connected open neighborhood \( U_x \subset M \) of \( x \) satisfying the following conditions.

- There is a connected open set \( \widetilde{U}_x \) in some \( \mathbb{R}^n \) and a finite group \( G_x \) of diffeomorphisms of \( \widetilde{U}_x \). Furthermore, there is a \( G_x \)-equivariant map \( \phi_x : \widetilde{U}_x \rightarrow M \) such that the induced map \( \phi_x : \widetilde{U}_x/G_x \rightarrow M \) is a homeomorphism onto \( U_x \).

Then, \( (\widetilde{U}_x, G_x, \phi_x) \) is called a chart and \( M \) is called an orbifold, with underlying space \( M \). Given a chart \( (U_x, G_x, \phi_x) \), the group \( G_x \) can be shown to be unique, and is called the local group at \( x \). If the local group at \( x \) is trivial, then \( x \) is called a smooth or a regular point, otherwise it is a singular point or a singularity.

Assume dimension of \( M \) is 2. Let \( (U_x, G_x, \phi_x) \) be a chart such that, \( G_x \) is finite cyclic of order \( q \), acting by rotation around the origin \((0, 0) \in \widetilde{U}_x \subset \mathbb{R}^2 \) by an angle \( \frac{2\pi}{q} \) and \( \phi_x((0,0)) = x \). Then, \( x \) is called a cone point of order \( q \). Also, there are two other types of singularities, called reflector lines and corner reflectors. In
this dimension, it is known that the underlying space is homeomorphic to a 2-
dimensional manifold. The genus of the underlying space $M$ is called the genus of
the orbifold $M$. See [22] for more details.

**Example 3.8.** An orbifold Lie groupoid $\mathcal{G}$ gives an orbifold structure on $|\mathcal{G}|$. See [[1], Proposition 1.44].

We now have the following useful lemma.

**Lemma 3.9.** Let $\mathcal{G}$ and $\mathcal{H}$ be two orbifold Lie groupoids and $f : \mathcal{G} \to \mathcal{H}$ be a homomorphism. Then, $f$ is a covering homomorphism of Lie groupoids if and only if $|f| : |\mathcal{G}| \to |\mathcal{H}|$ is an orbifold covering map.

**Proof.** See [[1], p. 40] for a proof. □

**Example 3.10.** If a group $H$ acts effectively and properly discontinuously on a manifold $\tilde{M}$, and $H'$ is a subgroup of $H$, then the homomorphism $\mathcal{G}(\tilde{M}, H') \to \mathcal{G}(\tilde{M}, H)$ (Example 3.5) is a covering homomorphism. In particular, if a finite group $H$ acts effectively on a manifold $\tilde{M}$, then $\mathcal{G}(\tilde{M}) \to \mathcal{G}(\tilde{M}, H)$ is a covering homomorphism.

It is well known that two orbifold Lie groupoids induce equivalent orbifold structures on $M$ if and only if they are Morita equivalent ([14], [13]).

In our situation of translation orbifold Lie groupoids we see in the following lemma, that when two translation orbifold Lie groupoids are Morita equivalent then in the Morita equivalence, the third orbifold Lie groupoid also can be chosen to be a translation orbifold Lie groupoid. We need this lemma for the proof of Theorem 4.3.

**Lemma 3.11.** Let two translation orbifold Lie groupoids $\mathcal{G}(M_1, H_1)$ and $\mathcal{G}(M_2, H_2)$ are inducing equivalent orbifold structures on $M$. Then there is a third translation orbifold Lie groupoid $\mathcal{G}(M_3, H_3)$, which is equivalent to both $\mathcal{G}(M_1, H_1)$ and $\mathcal{G}(M_2, H_2)$.

**Proof.** Let $p_1 : (M_1, m_1) \to (M, m)$ and $p_2 : (M_2, m_2) \to (M, m)$ be the orbifold covering projections, with groups of covering transformations $H_1$ and $H_2$, respectively. Here $m \in M$ is a smooth point. Consider the orbifold covering $M_3$ of $M$ corresponding to the subgroup

$$K := (p_1)_*(\pi_1(M_1, m_1)) \cap (p_2)_*(\pi_1(M_2, m_2)) < \pi_1^{orb}(M, m).$$

Let $H_3$ be the group of covering transformation of the orbifold covering map $p_3 : M_3 \to M$.

Then, we will now establish the following diagram.

$$\mathcal{G}(M_1, H_1) \quad \mathcal{G}(M_3, H_3) \quad \mathcal{G}(M_2, H_2).$$

We will first define the arrows and then show that they are, in fact, homomorphisms and equivalences.

We just need to check it for one of these arrows, since the same proof will work for the other one as well.

Denote $\mathcal{G}(M_3, H_3)$ by $\mathcal{G}$ and $\mathcal{G}(M_1, H_1)$ by $\mathcal{H}$. Let $p : (M_3, m_3) \to (M_1, m_1)$ be the covering map corresponding to the subgroup $(p_1)_*(K)$ of $\pi_1(M_1, m_1)$, and let

$$\rho : H_3 = \pi_1^{orb}(M, m)/p_3*(\pi_1(M_3, m_3)) \quad \pi_1^{orb}(M, m)/p_1*(\pi_1(M_1, m_1)) = H_1$$

This completes the proof.
be the quotient homomorphism. Note that, \( p \) is a genuine covering map of manifolds.

Then define \( f_0 = p \) and \( f_1 = (p, \rho) : M_3 \times H_3 \to M_1 \times H_1 \). The following commutative diagram shows that \( f : \mathcal{G} \to \mathcal{H} \) is a homomorphism, where \( h_3 \in H_3 \).

Now we check that \( f \) is an equivalence. See Definition 3.2.

The first condition in the definition of an equivalence says that the composition \( H_1 \times H_0 \xrightarrow{\pi_1} H_1 \xrightarrow{t} H_0 \) should be a surjective submersion. In our situation it takes the following form.

The fibered product is with respect to \( s \) (first projection) and \( p \). Since \( p \) and \( t \) are both surjective submersions, it follows that \( t \circ \pi_1 \) is a surjective submersion.

Next, we have to check the second condition in the definition of an equivalence, that is, we have to show that the following diagram is a fibered product in the category of manifolds.

In this case, the diagram takes the form.

Recall that, here \((s, t)(m_3, h_3) = (m_3, h_3(m_3))\) for \((m_3, h_3) \in M_3 \times H_3\), and similarly, \((s, t)(m_1, h_1) = (m_1, h_1(m_1))\) for \((m_1, h_1) \in M_1 \times H_1\).

To check that the above diagram is a fibered product, we have to complete the following commutative diagram by defining the dashed arrow \( l \), such that the whole diagram commutes.

Here \( A \) is a manifold and the maps \( i \), \( j \) and \( k \) are smooth. The two lower right hand side triangles are given to be commutative, that is, \((s, t) \circ i = k = (p, p) \circ j\).
For \( a \in A \) let \( i(a) = (m_1(a), h_1(a)) \) and \( j(a) = (m_3(a), m'_3(a)) \). Then, we get
\[
k(a) = (p(m_3(a)), p(m'_3(a))) = (m_1(a), h_1(a)(m_3(a))).
\]
Next, consider the following diagram. The unique map \( h_3(a) \) is obtained by lifting the composition of the horizontal maps using lifting criterion of covering space theory.

\[
\begin{array}{c}
(M_3, m_3(a)) \xrightarrow{h_3(a)} (M_3, m'_3(a)) \\
\downarrow p \\
(M_1, m_1(a)) \xrightarrow{h_1(a)} (M_1, p(m_3(a))).
\end{array}
\]

Now, we define \( l(a) = (m_3(a), h_3(a)) \). This completes the proof that \( f \) is an equivalence of Lie groupoids.

This completes the proof of the lemma.

We are now ready to recall the following definition.

**Definition 3.12.** Let \( G \) be an orbifold Lie groupoid inducing an orbifold structure on \( M \). Then, the \( k \)-th orbifold homotopy group \( \pi_k^{orb}(M, x_0) \) of \( M \) is defined as the homotopy group \( \pi_k(G, \tilde{x}_0) \) for some \( \tilde{x}_0 \in G_0 \) lying above a base point \( x_0 \in M \). \( M \) is called aspherical if \( G \) is aspherical.

The fundamental group of an orbifold Lie groupoid \( G \) inducing the orbifold structure on \( M \), is also called the orbifold fundamental group of the orbifold \( M \). This group is identified with the standard definition of orbifold fundamental group of \( M \) (see [23]). Hence, by Lemmas 3.6 and 3.9, we get the following lemma.

**Lemma 3.13.** An orbifold covering map induces isomorphisms on higher homotopy groups, and an injection on orbifold fundamental groups.

An useful immediate corollary of the lemma is the following.

**Corollary 3.14.** Let \( M \) be a connected orbifold and \( p : \tilde{M} \to M \) be an orbifold covering map. Assume that \( \tilde{M} \) is connected and has no singular points. Then \( p_* : \pi_k(\tilde{M}) \to \pi_k^{orb}(M) \) is an isomorphism for all \( k \geq 2 \).

In general an orbifold need not be the quotient of a manifold by an effective and properly discontinuous action of a discrete group. For example, the sphere with one cone point and the sphere with two cone points of different orders are examples of closed 2-dimensional orbifolds, which are not covered by manifolds. See [23]. Also see Proposition 1.54 and Conjecture 1.55 in [1] for some more general discussion.

It is standard to call a connected orbifold \( M \) good or developable if there is a manifold \( \tilde{M} \) and an orbifold covering map \( \tilde{M} \to M \).

We will need the following extension of the main theorem of [17], which says that a connected 2-dimensional orbifold with finitely generated and infinite orbifold fundamental group is good.

**Proposition 3.15.** Let \( M \) be a connected 2-dimensional orbifold, with infinite orbifold fundamental group. Then \( M \) is good.

**Proof.** First, recall that a 2-dimensional orbifold has three different types of singular sets: cone points, reflector lines and corner reflectors (See [22], p. 422). The points on the reflector lines and corner reflectors contributes to the boundary of the underlying space, called orbifold boundary. Hence, after taking a double of the
underlying space along this orbifold boundary components, and then applying [[17], Lemma 2.1], we get an orbifold double cover of $M$ which has only cone points.

Therefore, we can assume that the orbifold $M$ has only cone points. Also, we replace each manifold boundary component with a puncture. Clearly, this does not affect the orbifold fundamental group of $M$.

Note that, if $\pi_1^{\text{orb}}(M)$ is finitely generated, then the proposition follows from [[17], Theorem 1.1]. If it is infinitely generated, then $M$ either has infinite genus or has infinitely many punctures or infinitely many cone points.

Hence, we can write $M$ as an infinite increasing union of orbifolds of the type $M(r_i, s_i), i \in \mathbb{N}$. Each $M(r_i, s_i)$ has finite genus, $r_i$ number of punctures and $s_i$ number of cone points. Furthermore, we can assume that $M(r_i, s_i)$ has infinite orbifold fundamental group. Then clearly, $M(r_i, s_i)$ is aspherical, since they are all good orbifolds with infinite orbifold fundamental groups. Hence, a direct limit argument shows that $M$ is also aspherical, since the orbifold homotopy groups are covariant functors. Therefore, by Lemma 3.13, the universal orbifold covering $\tilde{M}$ of $M$ has all the orbifold homotopy groups trivial. We now apply [10] to conclude that $\tilde{M}$ is a manifold. Hence $M$ is a good orbifold. □

4. QUASIFIBRATIONS AND ASPHERICITY

In this section we recall some background materials, so that we can state the Asphericity and the Quasifibration conjectures and a functorial formulation of Theorem 2.2 of Section 2 in the general set up of the category of Lie groupoids. Also, here we will justify the term ‘quasifibration’ in this category, which in particular applies to orbifolds as well.

Given a topological space $X$, the configuration space of ordered $n$-tuples of pairwise distinct points of $X$ is defined by the following topological space.

$$PB_n(X) = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j, \text{ for } i \neq j\}.$$

Now, if $M$ is an orbifold, then $PB_n(M)$ is an orbifold, since it is an open set in the product orbifold $M^n$.

Let $H$ be a group acting effectively and properly discontinuously on a connected manifold $\tilde{M}$ with quotient $M = \tilde{M}/H$. Consider the space

$$PB_n(\tilde{M}, H) = \{(x_1, x_2, \ldots, x_n) \in \tilde{M}^n \mid Hx_i \neq Hx_j, \text{ for } i \neq j\},$$

of $n$-tuples of points of $\tilde{M}$ with pairwise distinct orbits. Then, $H^n$ acts effectively and properly discontinuously on $PB_n(\tilde{M}, H)$, with quotient, the orbifold $PB_n(M)$. Hence, $PB_n(\mathcal{G}(\tilde{M}, H))$, the configuration Lie groupoid of $n$ points of $\mathcal{G}(\tilde{M}, H)$ is the corresponding translation orbifold Lie groupoid $\mathcal{G}(PB_n(\tilde{M}, H), H^n)$. Recall that, for a good orbifold $M$, we can have many configuration Lie groupoids associated to different regular orbifold coverings of $M$. Also, clearly given two such orbifold coverings, the corresponding configuration Lie groupoids will induce equivalent orbifold structures on $PB_n(M)$.

We need to make the following definitions now.

**Definition 4.1.** Let $k \geq 1$ be an integer. For two path-connected topological spaces $X$ and $Y$, a surjective continuous map $f : X \rightarrow Y$ is called a $k$-quasifibration for some $y \in Y$, if for all $x \in F_y := f^{-1}(y)$, there exists homomorphisms $\partial : \pi_q(Y, y) \rightarrow \pi_{q-1}(F_y, x)$, for $q = 1, 2, \ldots, k$, making the following sequence exact.
horizontals maps are isomorphisms. Lie groupoid $H$.

Theorem 4.3. In a wider context. The advantage of this statement is that we will have the first part of the following theorem is equivalent to Theorem 2.2, but stated in the category of Lie groupoids. The best possible class of Lie groupoids to which this fibration result can be proven. Recall that, we denoted by $C^0$, the class of all genus zero connected 2-dimensional orbifolds with cone points and with at least one puncture, and $C^1$ was the class of all connected 2-dimensional orbifolds of genus $\geq 1$ with cone points and empty boundary.

Let $M \in C^0 \cup C^1$. Hence, $\pi^{orb}_1(M)$ is infinite. Then, by Proposition 3.15, $M$ is a good orbifold. For convenience we denote by $G_M$, a translation orbifold Lie groupoid $G(\tilde{M}, H)$, inducing the orbifold structure on $M$. Since $M$ is good, there are many such translation orbifold Lie groupoids.

We now consider the homomorphism

$$f(G_M) : PB_n(G_M) \to PB_{n-1}(G_M).$$

The first part of the following theorem is equivalent to Theorem 2.2, but stated in the category of Lie groupoids. The advantage of this statement is that we will have a scope of stating the Asphericity and Quasifibration conjecture of Section 2 in a wider context.

**Theorem 4.3.** Let $M \in C^0 \cup C^1$. Then, for a translation orbifold Lie groupoid $G_M$, $f(G_M)$ is a $1$-quasifibration. Furthermore, for another translation orbifold Lie groupoid $H_M$, we have the following commutative diagram for all $q$, where the horizontals maps are isomorphisms.

$$\begin{array}{ccc}
\pi_q(PB_n(G_M)) & \longrightarrow & \pi_q(PB_n(H_M)) \\
\downarrow f(G_M)_* & & \downarrow f(H_M)_* \\
\pi_q(PB_{n-1}(G_M)) & \longrightarrow & \pi_q(PB_{n-1}(H_M)).
\end{array}$$

The following lemma can be deduced easily by induction, since $M \in C^0 \cup C^1$ is aspherical.

**Lemma 4.4.** Let $M \in C^0 \cup C^1$ and consider a translation orbifold Lie groupoid $G(\tilde{M}, H)$, such that $M = \tilde{M}/H$. Then, $PB_n(M)$ is aspherical if and only if the
manifold $PB_n(\tilde{M}, H)$ is aspherical if and only if $f(\mathcal{G}(\tilde{M}, H))$ is a $k$-quasifibration for all $k \geq 2$.

**Proof.** Note that, by our convention (Example 3.3) $H$ is acting on $\tilde{M}$ effectively and properly discontinuously, so that $M = \tilde{M}/H$. Therefore, the quotient map $PB_n(\tilde{M}, H) \to PB_n(M)$ is an orbifold covering map. Hence by Corollary 3.13, $PB_n(\tilde{M}, H)$ is aspherical if and only if $PB_n(M)$ is aspherical. The other statement follows by induction on $n$ and using Theorem 2.2 and ([19], Theorem 2.14, Remark 2.15). □

We see that the homomorphism $f(\mathcal{G}_M)$ behaves like a quasifibration in low degree. Proposition 2.4 shows that in many instances, it is in fact a quasifibration in the stronger sense, that is, in addition the configuration Lie groupoid is aspherical.

For a translation orbifold Lie groupoid $\mathcal{G}$, the following proposition gives a sufficient condition for the homomorphism $f(\mathcal{G})$ to be a quasifibration. This is an application of ([11], Theorem 12.7).

**Proposition 4.5.** Let $\mathcal{G} = \mathcal{G}(\tilde{M}, H)$ be a translation orbifold Lie groupoid. If the object level projection $f(\mathcal{G})_0 : PB_n(\mathcal{G})_0 \to PB_{n-1}(\mathcal{G})_0$ is a fibration, then the induced map $Bf(\mathcal{G}) : BPB_n(\mathcal{G}) \to BPB_{n-1}(\mathcal{G})$ is a quasifibration, that is $f(\mathcal{G})$ is a quasifibration.

**Remark 4.6.** The conclusion of Proposition 4.5 should be stronger, that is, $Bf(\mathcal{G})$ is, in fact, a fibration. However, I did not find the appropriate reference for this.

**Proof of Proposition 4.5.** Note that $PB_n(\mathcal{G})_1 = PB_n(\mathcal{G})_0 \times H^n$ and the morphism level map $f(\mathcal{G})_1 : PB_n(\mathcal{G})_1 \to PB_{n-1}(\mathcal{G})_1$ is the projection to the first $n - 1$ coordinates on both the factors. Hence it is a fibration, since by hypothesis $f(\mathcal{G})_0$ is a fibration. It now follows, by induction, using the following lemma, that the map $f(\mathcal{G})_k : PB_n(\mathcal{G})_k \to PB_{n-1}(\mathcal{G})_k$ induced by $f(\mathcal{G})_1$ is a fibration, for all $k$. Hence $f(\mathcal{G})_\bullet : PB_n(\mathcal{G})_\bullet \to PB_{n-1}(\mathcal{G})_\bullet$ is a fibration of simplicial manifolds. It now follows from ([11], Theorem 12.7), that the map $Bf(\mathcal{G})$ is a quasifibration. □

**Lemma 4.7.** Consider the following commutative diagram of topological spaces and maps. Assume that the dotted maps are fibrations, then the map

$$A_1 \times_{C_1} B_1 \to A_2 \times_{C_2} B_2$$

is a fibration.

**Proof.** For any topological space $X$, we have to find a map on the dashed line in the following commutative diagram.
Since the dotted maps are fibrations, and the diagram is commutative, one easily uses the universal property of the fibered product \( A_1 \times C_1 \), to get a map on the dashed line. We produce one instance below.

Using the fibration on the dotted line in the above diagram, we get a map on the dashed line. Similarly, we will get maps from \( X \times I \) to \( A_1 \) and \( C_1 \) making the respective triangles commutative. This proves the lemma.

Here we recall that in this paper we are considering \( b \)-configuration Lie groupoids and \( b \)-fibrations. Proposition 4.5 is true for \( a \)-fibrations and \( a \)-configuration Lie groupoids as well. See [[19], Definition 2.4] and [[19], Lemmas 2.7 and 2.9].

In the following remark we show that the converse of Proposition 4.5 is not true. On the other hand, note that for an orbifold Lie groupoid \( G \), the association \( G \to B G \) induces an equivalence between the categories of covering Lie groupoids of \( G \) and covering spaces of \( B G \). See [[1], Proposition 2.17].

For the rest of the paper, we denote the cyclic group of order \( k \) by \( C_k \) and the complex plane with an order \( k \) cone point at the origin by \( \mathbb{C}_c^k \). We also assume \( C_k \) is acting on \( \mathbb{C} \) by rotation about the origin, by an angle \( \frac{2\pi}{k} \), to get the orbifold \( \mathbb{C}_c^k \) as the quotient.

**Remark 4.8.** Although, for \( M \in C^0 \cup C^1 \) and for a translation orbifold Lie groupoid \( G_M \), we have shown in Theorem 4.3, that the homomorphism \( f(G_M) \) is a 1-quasifibration, this need not imply that the object level map of this homomorphism, that is the projection \( f(G_M)_0 : PB_n(G_M)_0 \to PB_{n-1}(G_M)_0 \) need not be a 1-quasifibration for all points in \( PB_{n-1}(G_M)_0 \). The simplest example is when \( M = \mathbb{C}_c^2 \). In this case consider \( \mathbb{C}_c^2 = \mathbb{C} \) and \( H = C_2 \). Note that, \( PB_n(\mathbb{C}_c^2, C_2) \) is the hyperplane arrangement complement of the finite Artin group of type \( D_n \).

Explicitly,

\[
PB_n(\mathbb{C}_c^2, C_2) = \{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq \pm z_j \text{ if } i \neq j \}.
\]

It is easy to see that in this case, the above projection is not a fibration. In fact, it is not even a quasifibration, since, there are fibers of \( f(G_M)_0 \) which have non-isomorphic fundamental groups. Using Proposition 2.4 we conclude that it is not a 1-quasifibration for all points of \( PB_{n-1}(\mathbb{C}_c^2, C_2) \).

In fact we proved a more general statement in [[19], Proposition 2.11]. Let \( M \) be a global quotient orbifold with non-empty singular set. Consider a finite group
and a manifold \( \tilde{M} \) on which \( H \) acts effectively, so that \( \tilde{M}/H = M \). Then we showed that

\[
f(\mathcal{G}(\tilde{M}, H))_0 : PB_n(\tilde{M}, H) \to PB_{n-1}(\tilde{M}, H)
\]

is not a quasifibration in the sense of [6]. Here we make a correction that for the proof of [[19], Proposition 2.11] to work, we further need that either \( \tilde{M} \) is compact or it has finitely generated integral homology groups.

The above examples are the primary reasons why the Fadell-Neuwirth fibration theorem does not extend to orbifolds, and more generally to Lie groupoids.

Now, we state the two conjectures of Section 2, in a broader context. Recall that, we defined a Lie groupoid \( \mathcal{G} \) to be Hausdorff if \( |\mathcal{G}| \) is Hausdorff.

First, we state a general version of the Asphericity conjecture.

**Problem 4.9.** If a Hausdorff Lie groupoid \( \mathcal{G} \) is aspherical, then show that \( PB_n(\mathcal{G}) \) is also aspherical.

For Lie groupoids, the Quasifibration conjecture can be generalized as follows.

**Problem 4.10.** For a Hausdorff Lie groupoid \( \mathcal{G} \), show that \( f(\mathcal{G}) : PB_n(\mathcal{G}) \to PB_{n-1}(\mathcal{G}) \) is a quasifibration.

Using Remark 5.2, there is a version of Problem 4.10 for any \( k < n \) in place of \( n - 1 \).

5. Proofs

We begin with the proof of Theorem 4.3, which says that for any \( M \in C^0 \cup C^1 \), and for any orbifold Lie groupoid \( \mathcal{G}_M \), the homomorphism \( f(\mathcal{G}_M) \) is a 1-quasifibration. Recall that, we denote by \( \mathcal{G}_M \), a translation orbifold Lie groupoid inducing the orbifold structure on \( M \).

**Proof of Theorem 4.3.** Note that, by Definition 3.12, the statement that \( f(\mathcal{G}_M) \) is a 1-quasifibration for some orbifold Lie groupoid \( \mathcal{G}_M \), is equivalent to Theorem 2.2 in the case \( M \in C^1 \) and to [[19], Theorem 2.14, Remark 2.15] in the case \( M \in C^0 \), respectively.

The second statement is to relate the identifications of homotopy groups of the different translation orbifold Lie groupoids inducing the orbifold structure on \( PB_n(M) \), and the corresponding homomorphism \( f(\mathcal{G}_M) \), via a commutative diagram.

Consider two translation orbifold Lie groupoids of the form \( \mathcal{G}_M \) and \( \mathcal{H}_M \). Since they both induce the same orbifold structure on \( M \), by Lemma 3.11 there is another translation orbifold Lie groupoid \( \mathcal{K}_M \) and a diagram of equivalences.

\[
\begin{array}{ccc}
\mathcal{G}_M & \xrightarrow{\sim} & \mathcal{K}_M \\
& \searrow & \downarrow \\
& & \mathcal{H}_M
\end{array}
\]

Since all the orbifold Lie groupoids we are considering are of translation type, it is easy to see that the above diagram induces the following diagram of equivalences. For, the morphism space of \( PB_n(\mathcal{G}(M, H)) \) is nothing but \( PB_n(\mathcal{G}(M, H))_0 \times H^n \), for any translation Lie groupoid \( \mathcal{G}(M, H) \). See Definition 3.2.

\[
\begin{array}{ccc}
PB_n(\mathcal{G}_M) & \xrightarrow{\sim} & PB_n(\mathcal{K}_M) \\
& \searrow & \downarrow \\
& & PB_n(\mathcal{H}_M)
\end{array}
\]

Hence, we get the following commutative diagram of homomorphisms.
We call it a braid loop. A braid loop is a map $F$.

Note that composition between braid loops $\gamma$ satisfying the following conditions.

In the diagram all the horizontal homomorphisms are equivalences, and hence induce weak homotopy equivalences on the classifying spaces.

Now, note that the homomorphisms $f(G_M)$, $f(K_M)$ and $f(H_M)$ all induce the same projection map $PB_n(M) \to PB_{n-1}(M)$.

The second statement now follows by applying the homotopy functor on the above diagram.

Next, we give the proof of our main result Theorem 2.2, which gives the short exact sequence of surface orbifold pure braid groups of $M \in C^1$.

**Proof of Theorem 2.2.** Let us assume that the underlying space of the orbifold is orientable. The proof for the non-orientable case is similar. We will make a remark on this at the end of the proof (Remark 5.1). Also, we assume that the orbifold has at least one cone point, since otherwise the Fadell-Neuwirth fibration theorem will be applicable.

First we give the proof in the case when $\pi_1^{orb}(M)$ is finitely generated.

**Case 1: $\pi_1^{orb}(M)$ is finitely generated:** Under this hypothesis, we can assume that $M$ is of the type $M(r, s)$, where $M(r, s)$ is a 2-dimensional orientable orbifold of genus $g \geq 1$, with $r$ number of punctures and $s \geq 1$ number of cone points of orders $q_1, q_2, \ldots, q_n$.

The plan of the proof is as follows. We start with the direct way to define the orbifold fundamental group of an orbifold from [23]. Then, we give an explicit presentation of the orbifold fundamental group of $PB_n(M)$, using the presentation of the surface pure braid group of a surface, from [3]. The homomorphism $\pi_1^{orb}(PB_n(M)) \to \pi_1^{orb}(PB_{n-1}(M))$ is then described using this presentation. Finally, we apply the Fadell-Neuwirth fibration theorem for surfaces and some group theoretic arguments, to conclude the proof in this case.

Let $P = \{1, 2, \ldots, n\}$ and $\{x_1, x_2, \ldots, x_n\} \subset M(k, 0)$ be a fixed subset of $n$ distinct points. Consider the set $S$ of all continuous maps $\gamma : P \times I \to M(k, 0)$, satisfying the following conditions.

- $\gamma(i, t) \neq \gamma(j, t)$ for all $t \in I$ and for $i \neq j$.
- $\gamma(i, 0) = \gamma(i, 1) = x_i$ for all $i \in P$.

Note that $\gamma$ is a loop in $PB_n(M(k, 0))$ based at $(x_1, x_2, \ldots, x_n) \in PB_n(M(k, 0))$. We call it a braid loop.

Given two braid loops $\gamma_0$ and $\gamma_1$, a homotopy between them, fixing end points, is a map $F : P \times I \times I \to M(k, 0)$ satisfying the following conditions.

- For each $t \in I$, $F|_{P \times I \times \{t\}} \in S$.
- $F|_{P \times I \times \{0\}} = \gamma_0$ and $F|_{P \times I \times \{1\}} = \gamma_1$.

Composition between the braid loops $\gamma_0$ and $\gamma_1$ is defined as $\gamma = \gamma_0 * \gamma_1$, where for $i \in P$, the following are satisfied.

- $\gamma(i, t) = \gamma_0(i, 2t)$ for $0 \leq t \leq \frac{1}{2}$.
- $\gamma(i, t) = \gamma_1(i, 2t - 1)$ for $\frac{1}{2} \leq t \leq 1$. 
Clearly, the homotopy classes under this composition law of braid loops, gives \( \pi_1(\mathcal{P}B_n(M(k,0))) \), with base point \((x_1, x_2, \ldots, x_n)\). We are not including the base point in the notation of the fundamental group, as it will remain fixed during the proof. For \( \mathcal{P}B_{n-1}(M(k,0)) \) the base point will be \((x_1, x_2, \ldots, x_{n-1})\).

A presentation of the group \( \pi_1(\mathcal{P}B_n(M(k,0))) \), in terms of generators and relations is given in \([3], \text{Theorem 5.1}\). In Figure 1 we show all the generators of this group. Let the list of relations be \( R \). We do not reproduce the list of relations here, as we will not need its explicit descriptions. That is, we have the following.

\[
\pi_1(\mathcal{P}B_n(M(k,0))) = \langle A_i^l, B_i^j, C_i^m, P_i^p; i = 1, 2, \ldots, n; \\
j = 1, 2, \ldots, g; m < l, m = 1, 2, \ldots, n - 1; p = 1, 2, \ldots, k \mid R \rangle.
\]

Figure 1: Generators of the surface pure braid group of \( M(k,0) \).

Now let \( k = r + s \) and \( s \geq 1 \). Next, we replace the punctures from \( r + 1 \) to \( r + s \) by cone points of orders \( q_1, q_2, \ldots, q_s \), respectively, as shown in Figure 2. We also rename the generators with a ‘bar’ to avoid confusion.

Then, a presentation of the surface pure orbifold braid group \( \pi_1^{orb}(\mathcal{P}B_n(M(r,s))) \) is obtained from the presentation of \( \pi_1(\mathcal{P}B_n(M(r+s,0))) \) by adding the extra relations, \((F_i^{r+j})^{q_i} \), for \( i = 1, 2, \ldots, n; j = 1, 2, \ldots, s \). Since, by definition of orbifold fundamental group (see [2] or [23]), if a loop \( \eta \) circles around a cone point of order \( q \), then \( \eta^q = 1 \) appears as a relation. The only difference between a puncture and a cone point on a 2-dimensional orbifold, which reflects on the orbifold fundamental group, is this finite order relation. Hence, we have the following.

\[
\pi_1^{orb}(\mathcal{P}B_n(M(r,s))) = \\
\langle A_i^l, B_i^j, C_i^m, P_i^p; i = 1, 2, \ldots, n; j = 1, 2, \ldots, g; m < l, m = 1, 2, \ldots, n - 1; p = 1, 2, \ldots, r + s \mid R \cup \{(F_i^{r+j})^{q_i}, \text{ for } i = 1, 2, \ldots, n, j = 1, 2, \ldots, s \} \rangle.
\]

Here \( R \) is the same set of relations as in \( R \) but the generators are replaced with a ‘bar’.
Figure 2: Generators of the surface pure orbifold braid group.

Clearly, now there is the following surjective homomorphism, which sends a generator \(X\) of \(\pi_1(PB_n(M(r+s,0)))\) to \(\overline{X}\),

\[ g_n : \pi_1(PB_n(M(r+s,0))) \to \pi^{orb}_1(PB_n(M(r,s))). \]

This homomorphism is also obtained by applying the \(\pi^{orb}_1\) functor on the inclusion \(PB_n(M(r+s,0)) \to PB_n(M(r,s))\).

Therefore, we have the following commutative diagram.

\[
\begin{array}{ccc}
\pi_1(PB_n(M(r+s,0))) & \xrightarrow{f_n} & \pi_1(PB_{n-1}(M(r+s,0))) \\
g_n & & g_{n-1} \\
\pi^{orb}_1(PB_n(M(r,s))) & \xrightarrow{f_n^*} & \pi^{orb}_1(PB_{n-1}(M(r,s))).
\end{array}
\]

Here, \(f_n = f(M(r+s,0))\), which is induced by the projection to the first \(n-1\) coordinates. Recall that, \(f(M(r+s,0))\) is a fibration by the Fadell-Neuwirth fibration theorem.

\(f_n^*\) is also induced by a similar projection. However, we have seen in ([19], Proposition 2.11) that the corresponding homomorphism of this projection, on an associated configuration Lie groupoid is not a fibration.

Since \(f_n\) and \(g_{n-1}\) are both surjective, together with the Fadell-Neuwirth fibration theorem we get the following commutative diagram.

\[
\begin{array}{ccc}
\pi_1(F) & \xrightarrow{f_n} & \pi_1(PB_n(M(r+s,0))) \\
g_n & & g_{n-1} \\
\pi^{orb}_1(PB_n(M(r,s))) & \xrightarrow{f_n^*} & \pi^{orb}_1(PB_{n-1}(M(r,s))).
\end{array}
\]

Here \(F = M(r+s+n-1,0)\) is the fiber of the projection map over the point \((x_1, x_2, \ldots, x_{n-1})\). The points \(x_1, x_2, \ldots, x_{n-1}\) are replaced by punctures in \(F\).

It is clear that, the kernel of \(f_n\) is normally generated by the following generators of \(\pi_1(PB_n(M(r+s,0)))\). See Figure 3.

\[ \{A^j, B^j, C^m_n, P^p_n; j = 1, 2, \ldots, g; m = 1, 2, \ldots, n-1; p = 1, 2, \ldots, r+s\}. \]

The Fadell-Neuwirth fibration theorem implies that \(f_n\) is induced by a fibration, and hence \(\pi_2(PB_k(M(r+s,0))) = 0\) for all \(k\) (Proposition 2.4). Therefore, the above generators, which generate \(\pi_1(F)\), generate a normal subgroup of \(\pi_1(PB_n(M(r+s,0)))\).
On the other hand, the kernel of $f^o_n$ is normally generated by the following generators of $\pi^{orb}_1(PB_n(M(r, s)))$. See Figure 4.

$$\{A^j_n, B^g_n, C^m_n, P^p_n; j = 1, 2, \ldots, g; m = 1, 2, \ldots, n - 1; p = 1, 2, \ldots, r + s\}.$$

Next, note that the image of $\pi^{orb}_1(F)$ is obtained from the above presentation of $\pi_1(F)$, by adding the finite order relations, $(F_n^{j+1})^q$, for $j = 1, 2, \ldots, s$. But this is the presentation of $\pi^{orb}_1(F)$, where $F = M(r + n - 1, s)$. Here also, $x_1, x_2, \ldots, x_{n-1}$ are replaced by punctures in $F$. Therefore, the homomorphism $\pi^{orb}_1(F) \rightarrow \pi^{orb}_1(PB_n(M(r, s)))$ is injective.

Since $g_n$ is a surjective homomorphism, and $g_n$ sends the image of $\pi_1(F)$ onto the image of $\pi^{orb}_1(F)$, we see that the above generators generate a normal subgroup of $\pi^{orb}_1(PB_n(M(r, s)))$. Hence, the kernel of $f^o_n$ is exactly the image of $\pi^{orb}_1(F)$.

Therefore, from the above argument we get the following commutative diagram.

$$\begin{array}{c}
\pi_1(F) \xrightarrow{f_n} \pi_1(PB_n(M(r + s, 0))) \\
\pi^{orb}_1(F) \xrightarrow{f_n} \pi^{orb}_1(PB_n(M(r + s, 0)))
\end{array}$$

This completes the proof of Theorem 2.2 in the finitely generated case.

Next, we give the proof in the case, when $\pi^{orb}_1(M)$ is infinitely generated. We will use the previous finitely generated case in a direct limit argument.
Case 2: \( \pi_1^{\text{orb}}(M) \) is infinitely generated: We write \( M \) as an increasing union \( \bigcup_{i \in \mathbb{N}} M(r_i, s_i) \) of suborbifolds \( M(r_i, s_i) \). Here \( r_1 \leq r_2 \leq \cdots, s_1 \leq s_2 \leq \cdots \) and \( M(r_i, s_i) \) has finite \((\geq 1)\) genus. Clearly, \( \pi_1^{\text{orb}}(M(r_i, s_i)) \) is infinite and finitely generated.

To simplify notation, we denote \( M(r_i, s_i) \) by \( M_i \).

Now, we have inclusions \( PB_n(M_i) \subset PB_n(M_j) \) for \( i \leq j \), and \( PB_n(M) = \bigcup_{i \in \mathbb{N}} PB_n(M_i) \). Then, there is the following commutative diagram.

\[
P B_n(M_i) \xrightarrow{f(M_i)} P B_{n-1}(M_i)
\]

Using Case 1, the above diagram induces the following commutative diagram.

\[
1 \xrightarrow{\pi_1^{\text{orb}}(\mathcal{F}_i)} \pi_1^{\text{orb}}(PB_n(M_i)) \xrightarrow{f_n} \pi_1^{\text{orb}}(PB_{n-1}(M_i)) \xrightarrow{f_n} 1
\]

\[
1 \xrightarrow{\pi_1^{\text{orb}}(\mathcal{F}_j)} \pi_1^{\text{orb}}(PB_n(M_j)) \xrightarrow{f_n} \pi_1^{\text{orb}}(PB_{n-1}(M_j)) \xrightarrow{f_n} 1.
\]

Next, we take the direct limit of this directed system of short exact sequences, and complete the proof of Theorem 2.2. \( \square \)

See [2] or [[19], Section 4] for a different way of looking at elements of the surface pure orbifold braid groups. But that way of presentation is useful for the surface pure orbifold braid groups for members of \( \mathcal{C}^0 \).

Remark 5.1. In the non-orientable case the proof of Theorems 2.2 (and hence of Theorem 4.3) are exactly the same. We have to consider the presentation of the surface pure braid groups in the non-orientable case from [[3], Theorem A.2] and replicate the above proof.

Remark 5.2. Recall that for a manifold \( M \) of dimension \( \geq 2 \), the Fadell-Neuwirth fibration theorem also says that the projection to the first \( k \)-coordinates, for \( k < n \), gives a fibration \( PB_n(M) \to PB_k(M) \), with typical fiber homeomorphic to \( PB_{n-k}(M - \{k \text{ points}\}) \). Our proof can be suitably modified to give a proof of the following exact sequence, where \( M \in \mathcal{C}^1 \) and \( \mathcal{F} = M - \{k \text{ smooth points}\} \).

\[
1 \xrightarrow{\pi_1^{\text{orb}}(PB_{n-k}(\mathcal{F}))} \pi_1^{\text{orb}}(PB_n(M)) \xrightarrow{f_n} \pi_1^{\text{orb}}(PB_{k}(M)) \xrightarrow{f_n} 1.
\]

Next, we give the proof of Theorem 2.5, that is, the Farrell-Jones Isomorphism conjecture for the surface orbifold braid group of \( M \in \mathcal{C}^1 \).

We will not recall the exact statement of the Isomorphism conjecture, as we will not need the terms used there. The statement and many of the hereditary properties of the conjecture are widely published. See [9] or [16].

Proof of Theorem 2.5. The proof is short and falls in the same line of the proof of [[19], Theorem 2.20], with some modification.

First let us recall the definition of the following class of groups from [[19], Definition 5.6].

Let \( \mathcal{D} \) denote the smallest class of groups satisfying the following conditions.

1. \( \mathcal{D} \) contains the fundamental group of any connected manifold of dimension \( \leq 3 \).
2. If $H$ is a subgroup of a group $G$, then $G \in \mathcal{D}$ implies $H \in \mathcal{D}$. The reverse implication is also true if $H$ is of finite index in $G$.

3. If $G_1, G_2 \in \mathcal{D}$ then $G_1 \times G_2 \in \mathcal{D}$.

4. If $\{G_\alpha\}_{\alpha \in J}$ is a directed system of groups and $G_\alpha \in \mathcal{D}$ for each $\alpha \in J$, then the colim$_{\alpha \in J} G_\alpha \in \mathcal{D}$.

5. Let \[ 1 \longrightarrow K \longrightarrow G \xrightarrow{p} H \longrightarrow 1 \]
be a short exact sequence of groups. If $K$, $H$ and $p^{-1}(C)$, for any infinite cyclic subgroup $C$ of $H$, belong to $\mathcal{D}$ then $G$ also belongs to $\mathcal{D}$.

6. If a group $G$ has a normal subgroup $H$, such that $H$ is free and $G/H$ is infinite cyclic, then $G$ belongs to $\mathcal{D}$.

We noted in the proof of [[19], Theorem 2.20], that the Farrell-Jones Isomorphism conjecture is true for any member of $\mathcal{D}$. Therefore, it is enough to prove that $\pi_1^{orb}(B_n(M)) \in \mathcal{D}$. Note that, by 2 we just need to prove that $\pi_1^{orb}(PB_n(M)) \in \mathcal{D}$, since $\pi_1^{orb}(B_n(M))$ contains $\pi_1^{orb}(PB_n(M))$ as a subgroup of finite index.

Now, if $M = M(r, s)$, that is, if $\pi_1^{orb}(M)$ is finitely generated, then by Theorem 2.2 we have the following exact sequence of finitely generated groups, where $\overline{F} = M - (n - 1)$-smooth points.

\[ 1 \longrightarrow \pi_1^{orb}(\overline{F}) \longrightarrow \pi_1^{orb}(PB_n(M)) \xrightarrow{f(M)_*} \pi_1^{orb}(PB_{n-1}(M)) \longrightarrow 1. \]

Since $M$ is good, it has a finite sheeted orbifold covering, which is a manifold. Hence, we can use 1 and 2, to deduce that $\pi_1^{orb}(PB_1(M)) = \pi_1^{orb}(M) \in \mathcal{D}$. Next, we use 5 inductively, as follows. Let $C$ be an infinite cyclic subgroup of $\pi_1^{orb}(PB_{n-1}(M))$. Then, since $\pi_1^{orb}(\overline{F})$ is finitely generated and virtually free, $(f(M)_*)^{-1}(C)$ contains a finite index free-by-cyclic subgroup. Hence we can now invoke 6 and 2, to see that $(f(M)_*)^{-1}(C) \in \mathcal{D}$. This completes the proof of the finitely generated case.

When $\pi_1^{orb}(M)$ is infinitely generated, we write $M = \bigcup_{i \in \mathbb{N}} M(r_i, s_i)$ as in Case 2 of the proof of Theorem 2.2. To complete the argument we apply 4 to the directed system $\{\pi_1^{orb}(PB_n(M_i))\}_{i \in \mathbb{N}}$.

This completes the proof of Theorem 2.5.

Finally, we give the proof of Proposition 2.4 which together with Lemma 4.4 claim the asphericity of a certain non-trivial class of configuration Lie groupoids.

Proof of Proposition 2.4. We divide the proof in the three different cases. We first prove that $PB_n(M)$ is aspherical. The proof of $f(M)$ is a quasifibration is then a consequence of [[19], Theorem 2.14, Remark 2.15].

Case 1. Let $M$ be an aspherical 2-manifold. In this case, the Fadell-Neuwirth fibration theorem and the long exact sequence of homotopy groups are used inductively, to prove the asphericity of $PB_n(M)$.

Case 2. $M = \mathbb{C}_x$, that is, $M$ is the complex plane with a cone point of order 2 at the origin. First note that, in this case $\tilde{M} = \mathbb{C}$ and $H = C_2$. Next, recall from Remark 4.8, that $PB_n(\tilde{M}, H)$ is the $D_n$-type hyperplane arrangement complement.

\[ PB_n(\tilde{M}, H) = \{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_i \neq \pm z_j, i \neq j \}. \]

Also, in [5] it was shown that $PB_n(\tilde{M}, H)$ fibers over an aspherical manifold $PZ_{n-1}(\mathbb{C}^*)$ with aspherical fiber. The explicit definition of this target space is
given below.

\[ PZ_{n-1}(\mathbb{C}^*) = \{(y_1, y_2, \ldots, y_{n-1}) \in (\mathbb{C}^*)^n \mid y_i \neq y_j, \; i \neq j\}. \]

And the map \( B : P\bar{B}_n(\tilde{M}, H) \to PZ_{n-1}(\mathbb{C}^*) \) is defined by \( y_j = z_{n-j}^2 - z_j^2 \), for \( j = 1, 2, \ldots, n-1 \). \( PZ_{n-1}(\mathbb{C}^*) \) is aspherical by Case 1, and a fiber of \( B \) is clearly a non-compact 2-manifold and hence is aspherical. Therefore, again using the long exact sequence of homotopy groups, and by an induction on \( n \), we get that \( P\bar{B}_n(\tilde{M}, H) \) is aspherical. Hence, by Corollary 3.14 \( P\bar{B}_n(M) \) is aspherical.

**Case 3.** \( M = \mathbb{C} \) with two cone points of order 2. We can assume that the cone points are at 0 and \( \frac{1}{2} \). Consider the hyperplane arrangement complement corresponding to the affine Artin group of type \( \widetilde{D}_n \) ([5]).

\[ P\widetilde{D}_n(\mathbb{C}) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \pm z_j \notin \mathbb{Z}, \; i \neq j\}. \]

We proceed to show that there is an orbifold covering map \( P\widetilde{D}_n(\mathbb{C}) \to P\bar{B}_n(M) \). We define this map in the following diagram.

\[ \begin{array}{ccc}
P\widetilde{D}_n(\mathbb{C}) & \xrightarrow{E} & X \\
& \searrow & \downarrow Q \\
& & P\bar{B}_n(M)
\end{array} \]

Here \( X = \{(w_1, w_2, \ldots, w_n) \in (\mathbb{C}^*)^n \mid z_i \neq z_j^{\pm 1}, \; i \neq j\} \). The map \( E \) is the restriction of the \( n \)-fold product of the exponential map \( z \mapsto \exp(2\pi iz) \), and \( Q \) is the restriction of the \( n \)-fold product of the map \( q : \mathbb{C}^* \to M \), defined by

\[ q(w) = \frac{1}{4} \left( 1 - \frac{1+w^2}{2w} \right). \]

\( E \) is a genuine covering map and \( Q \) is a \( 2^n \)-sheeted orbifold covering map. Since \( q \) is a 2-fold orbifold covering map as \( q \) sends the branch point +1 to 0 and \(-1\) to \( \frac{1}{2} \), and it is of degree 2 around these points. Therefore, \( Q \circ E : P\widetilde{D}_n(\mathbb{C}) \to P\bar{B}_n(M) \) is an orbifold covering map.

On the other hand recently, in [15], it was proved that \( P\widetilde{D}_n(\mathbb{C}) \) is aspherical. Therefore, by Corollary 3.14 \( P\bar{B}_n(M) \) is also aspherical.

This completes the proof of Proposition 2.4. \( \square \)
References

[1] A. Adem, J. Leida and Y. Ruan, Orbifolds and stringy topology, Cambridge Tracts in Mathematics, 171, Cambridge University Press, Cambridge, 2007. xii+149 pp.

[2] D. Allcock, Braid pictures of Artin groups, Trans. Amer. Math. Soc. 354 (2002), no. 9, 3455-3474.

[3] P. Bellingeri, On presentations of surface braid groups, J. Algebra 274 (2004), no. 2, 543-563.

[4] M. Bestvina, Non-positively curved aspects of Artin groups of finite type, Geom. Topol. 3 (1999), 269-302.

[5] E. Brieskorn, Sur les groupes de tresses [d’apr`es V.I. Arnol’d] S´ eminaire Bourbaki, 24` eme ann´ee (1971/1972), Exp. No. 401, pp.21-44. Lecture Notes in Math., Vol 317, Springer, Berlin, 1973.

[6] A. Dold and R. Thom, Quasifaserrungen und unendliche symmetrische Produkte, Ann. of Math., Second Series, 67 (1958), 239-281.

[7] E. Fadell and L. Neuwirth, Configuration spaces, Math. Scand. 10 (1962), 111-118.

[8] J. Huang and D. Osajda, Helly meets Garside and Artin, Inventiones Mathematicae (2021), doi:10.1007/s00222-021-01030-8.

[9] W. Lück, Assembly maps, Handbook of homotopy theory (Haynes R. Miller, ed.), CRC Press/Chapman & Hall, 2019, 853-892.

[10] A. Lytchak, On contractible orbifolds, Proc. of Amer. Math. Soc. 141 (2013), 3303-3304.

[11] J.P. May, The geometry of iterated loop spaces, Lecture Notes in Math., Vol 271, Springer-Verlag, 1972.

[12] J.P. May, Weak equivalences and quasifibrations, Groups of self-equivalences and related topics (Montreal, PQ, 1988), 91-101, Lecture Notes in Math., 1425, Springer, Berlin, 1990.

[13] I. Moerdijk, Orbifolds as groupoids: an introduction, Orbifolds in mathematics and physics (Madison, WI, 2001), 205-222, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[14] I. Moerdijk and D.A. Pronk, Orbifolds, sheaves and groupoids, K-Theory 12 (1997), no. 1, 3-21.

[15] G. Paulini and M. Salvetti, Proof of the $K(\pi, 1)$-conjecture for affine Artin groups, Inventiones Mathematicae (2020). doi:10.1007/s00222-020-01016-y.

[16] S.K. Roushon, The isomorphism conjecture for groups with generalized free product structure, Handbook of Group Actions, Vol II, ALM 32 (2014), 77-119. Higher Education Press and International Press, Beijing-Boston.

[17] S.K. Roushon, A sufficient condition for a 2-dimensional orbifold to be good, Math. Student. 88 (2019), no. 1-2, 165-171.

[18] S.K. Roushon, A certain structure of Artin groups and the isomorphism conjecture, Canad. J. Math., 73 (2021), no. 4, 1153-1170. Erratum: A certain structure of Artin groups and the isomorphism conjecture, Canadian Journal of Mathematics, 1 (2020), 1. doi:10.4153/S0008414X20000802.

[19] S.K. Roushon, Configuration Lie groupoids and orbifold braid groups, Bull. Sci. math. 171 (2021), 103028, 35pp. doi:https://doi.org/10.1016/j.bulsci.2021.103028.

[20] S.K. Roushon, Almost-quasifibrations and fundamental groups of orbit configuration spaces, https://doi.org/10.48550/arXiv.2111.06159.

[21] I. Satake, On a generalization of the notion of manifolds, Proc. Nat. Acad. Sci. USA 42 (1956) 359-363.

[22] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 no. 5 (1983), 401-487.

[23] W. P. Thurston, Three-dimensional Geometry & Topology, December 1991 version, Mathematical Sciences Research Institute Notes, Berkeley, California.

[24] X. Wu, Poly-freeness of Artin groups and the Farrell-Jones Conjecture, arXiv:1912.04350.

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