SCHRÖDINGER FUNCTIONAL OF A QUANTUM SCALAR FIELD IN STATIC SPACE-TIMES FROM PRECANONICAL QUANTIZATION

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Abstract

The functional Schrödinger representation of a scalar field on an n-dimensional static space-time background is argued to be a singular limiting case of the hypercomplex quantum theory of the same system obtained by the precanonical quantization based on the space-time symmetric De Donder-Weyl Hamiltonian theory. The functional Schrödinger representation emerges from the precanonical quantization when the ultraviolet parameter $\kappa$ introduced by precanonical quantization is replaced by $\gamma_0\delta^{\text{inv}}(0)$, where $\gamma_0$ is the time-like tangent space Dirac matrix and $\delta^{\text{inv}}(0)$ is an invariant spatial $(n-1)$-dimensional Dirac’s delta function whose regularized value at $x=0$ is identified with the cutoff of the volume of the momentum space. In this limiting case, the Schrödinger wave functional is expressed as the trace of the product integral of Clifford-algebra-valued precanonical wave functions restricted to a certain field configuration and the canonical functional derivative Schrödinger equation is derived from the manifestly covariant Dirac-like precanonical Schrödinger equation which is independent of a choice of a codimension-one foliation.

Keywords: Quantum field theory in curved space-time, De Donder-Weyl formalism, precanonical quantization, canonical quantization, functional Schrödinger representation, Clifford algebra, product integral.

1 Introduction

The canonical quantization of scalar field theory in curved space-time using the functional Schrödinger picture of QFT [1–7]) leads to the description of the corresponding quantum field in terms of the Schrödinger wave functional $\Psi([\phi(x)], t)$ which satisfies the functional derivative Schrödinger equation

$$i\hbar \partial_t \Psi = \int dx \sqrt{-g} \left( \frac{\hbar^2}{2} g^{\mu\nu} \frac{\delta^2}{\delta \phi(x)^2} - \frac{1}{2} g^{ij} \partial_i \phi(x) \partial_j \phi(x) + V(\phi) \right) \Psi,$$

where $g_{\mu\nu} = g_{\mu\nu}(x)$ denotes the metric tensor, $g$ is the determinant of $g_{\mu\nu}$, and $x^\mu = (t, x^i) = (t, x)$ are space-time coordinates. In writing this equation, one chooses the
space-time coordinates adapted to the codimension-one space-like foliation with the lapse and shift functions $N = \sqrt{g_{00}}$ and $N_i = g_{0i} = 0$, respectively, and the induced metric $g_{ij}$ on the space-like leaves of the foliation. A detailed treatment of the functional Schrödinger picture of QFT in flat space-time can be found in [8, 9].

Instead, precanonical quantization of a scalar field $\phi(x)$ on a curved space-time background (cf. [10,11]) given by the metric tensor $g_{\mu\nu}(x)$ leads to the description in terms of a Clifford-algebra-valued wave function $\Psi(\phi, x^\mu)$ which satisfies the partial derivative precanonical Schrödinger equation on the finite-dimensional bundle with the coordinates $(\phi, x^\mu)$:

$$i\hbar \gamma^\mu(x) \nabla_\mu \Psi = \left( -\frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \phi^2} + \frac{1}{\kappa} V(\phi) \right) \Psi =: \frac{1}{\kappa} \hat{H} \Psi ,$$  \tag{2}$$

where $\gamma^\mu(x)$ are the Dirac matrices in curved space time which factorize the metric tensor $g^{\mu\nu}(x)$:

$$\gamma^\mu(x) \gamma^\nu(x) + \gamma^\nu(x) \gamma^\mu(x) = 2 g^{\mu\nu}(x) ,$$  \tag{3}$$

$\nabla_\mu := \partial_\mu + \frac{i}{4} \omega_\mu(x)$ is the covariant derivative acting on Clifford-algebra-valued wave functions, $\omega_\mu(x) = \omega_{\mu AB}(x) \gamma^{AB}$ are the spin-connection matrices (Fock–Ivanenko coefficients [12,13]), and $\gamma_A$ are the constant tangent space Dirac matrices: $\gamma_A \gamma_B + \gamma_B \gamma_A = 2 \eta_{AB}$ corresponding to the Minkowski metric $\eta_{AB}$ with the signature of our choice $++--$.

The operator $\hat{H}$ on the right-hand side of (2) is called the De Donder-Weyl (DW) Hamiltonian operator and it is constructed according to the procedure of precanonical quantization [10,11]. It contains an ultraviolet parameter $\kappa$ of the dimension of the inverse spatial volume which typically appears in the representations of precanonical quantum operators already in flat space-time [10,14–16]. It is interesting to note that the DW Hamiltonian operator $\hat{H}$ coincides with the expression in flat space-time (cf. [10,14,16]): the metric tensor components appear in the classical expression of the DW Hamiltonian function but they disappear in the quantum operator corresponding to it. The only manifestation of curved space-time is through the curved space-time Dirac matrices (3) and the spin-connection on the left-hand side of Eq. (2).

Obviously, the precanonical description contrasts with a familiar description of quantum fields derived from the canonical quantization, in particular, with the description using the functional Schrödinger picture outlined above, which explicitly distinguishes the role of space variables $x$ as the labels of degrees of freedom and the time variable $t$ along which the quantum evolution proceeds. In this paper, we will show that the description of quantum scalar fields on a static metric background in the functional Schrödinger picture of QFT can be understood as a singular limiting case of a quantum scalar theory in curved space-time derived from precanonical quantization. This limiting case corresponds to the value of the ultraviolet parameter $\kappa$ introduced by precanonical quantization going to the regularized value of the Dirac delta function at equal spatial points (which is the ultraviolet cutoff of the volume of the momentum space). This claim generalizes the similar result for quantum scalar fields and quantum YM theory in flat space-time [20, 22].
(see also [17, 18] for an earlier treatment) to curved space-time. The insights and notations from [20, 22] will be closely followed in the present consideration.

Let us recall that precanonical quantization [14, 16, 19] is based on a generalization of the Hamiltonian formalism to field theory known as the De Donder-Weyl (DW) Hamiltonian theory [23, 26], which treats space-time variables on an equal footing. In this formulation, the Poisson brackets are defined on differential forms representing the dynamical variables. The construction of the brackets uses the polysymplectic structure (whose integration over the initial data yields the standard symplectic 2-form on an infinite-dimensional phase space of a field theory) related to the Poincaré-Cartan form in the calculus of variations and it leads to the Poisson-Gerstenhaber algebra structure generalizing the usual Poisson algebra known in the canonical Hamiltonian formalism [19, 27, 28] (for further generalizations see also [29, 31]). The existence of a Hamilton-Jacobi theory corresponding to the DW Hamiltonian formulation [24, 26] inevitably raises the question as to which formulation of the quantum theory of fields would reproduce the (partial derivative!) DW Hamilton-Jacobi equation in the classical limit. The precanonical quantization aims at clarifying this question. Instead of quantizing the full Poisson-Gerstenhaber algebra, it is based on quantization of its small Heisenberg-like subalgebra, that leads to a hypercomplex generalization of the formalism of quantum theory where both operators and wave functions are Clifford-algebra-valued, and the precanonical Schrödinger equation includes the space-time Dirac operator which generalises the time derivative in the standard Schrödinger equation [14, 16, 19]. One of the features of this formulation of quantized fields is that it allows us to reproduce the classical field equations as the equations of expectation values of operators defined by and evolving according to the equations of precanonical quantization [10, 11]. Moreover, by treating the space-time variables on an equal footing and leading to a formulation on a finite-dimensional space of field and space-time variables the precanonical quantization approach provides a natural and promising framework for quantization of gravity [35, 36] and even for the consideration of the mass gap problem in quantum gauge theory [37]. On the classical level, the DW Hamiltonian formulation and the underlying polysymplectic structure have been used recently in order to construct highly effective numerical integrators for PDEs and general relativity [38, 39].

In this paper, in Sec. 2 we assume that the Schrödinger wave functional $\Psi$ is a functional of precanonical wave function $\Psi$ restricted to a field configuration $\phi(x)$ at time $t$. This implies a global space+time decomposition in the precanonical Schrödinger equation [2]. The time evolution of the wave functional $\Psi$ is then controlled by the time evolution of the precanonical wave function $\Psi$ restricted to the subspace $\Sigma$ representing the above mentioned field configuration: $\Psi_\Sigma$. We will confine ourselves to the case of static space-times and present a restriction of the precanonical Schrödinger equation [2] to $\Sigma$, which controls the time evolution of the precanonical wave function restricted to $\Sigma$. Then we will use it to write an equation for the time evolution of an arbitrary wave functional of $\Psi_\Sigma$. By comparing the terms in this equation with the expressions of variational derivatives of the composite functional $\Psi([\Psi_\Sigma(\phi(x), t)])$ with respect to $\phi(x)$ we demonstrate how
the terms in the functional Schrödinger equation are emerging from the terms in the precanonical Schrödinger equation when the parameter \( \kappa \) in the latter is replaced by a singular scalar combination of Dirac \( \delta \)-function at equal spatial points, \( \gamma_0 \) matrix and the density \( \sqrt{-g} \). In this limiting case, we will be able to obtain an expression of the Schrödinger wave functional in terms of the precanonical wave function and to argue that the contributions from the additional terms in the equation for \( \Psi_{\Sigma} \), which have no counterparts in the functional derivative Schrödinger equation (1), are vanishing. In this way we derive (1) from (2). Our conclusions and some problems for further work are presented in Sec. 3.

2 Schrödinger wave functional from precanonical wave function

In the case of interacting scalar fields in flat space-time, the connection between the functional Schrödinger representation and precanonical quantization was established in [20,21]. It was further extended to quantum YM theory in flat space-time in [22]. These papers improve our earlier considerations in [17] and [18]. The conclusion is that the standard QFT in functional Schrödinger representation can be derived from the precanonical quantization in the limiting case of the infinite parameter \( \kappa \) or, more precisely, when the combination \( \gamma_0 \kappa \) is replaced by \( \delta(0) \), a regularized value of Dirac delta function \( \delta(x - x') \) at coinciding spatial points, i.e. the cutoff of the momentum space volume introduced by regularization.

Here we explore a similar connection between the functional Schrödinger representation of a quantum scalar field theory in curved space-time and precanonical quantization of the same system. Specifically, we restrict ourselves to static space-times. In this case, \( \partial_t g^{\mu\nu} = 0 \), all curved space-time Dirac matrices \( \gamma^\mu \) are \( x \)-dependent, and the consistency with the choice of the adapted coordinate system used in (1) implies for the Christoffel symbols and the spin-connection coefficients \( \Gamma^\nu_{\mu\nu} = 0 = \Gamma^\mu_{\mu\nu} \) and \( \omega^i_{0J} = 0 = \omega^i_{0J} \), respectively. An extension of the consideration below to more general space-times will be presented elsewhere [40].

Intuitively, the relation between the Schrödinger wave functional and the precanonical wave function is suggested by the probabilistic interpretation. While the former has the meaning of the probability amplitude of finding a field configuration \( \phi(x) \) at some moment of time \( t \), the latter can be interpreted as the probability amplitude of observing the field value \( \phi \) at the space-time point \( x \). This allows us to expect that the time-dependent complex functional probability amplitude \( \Psi([\phi(x)], t) \) is a composition of space-time dependent Clifford-valued probability amplitudes given by the precanonical wave function \( \Psi(\phi, x) \).

A relation between the functional Schrödinger picture and the precanonical description implies that the Schrödinger wave functional \( \Psi([\phi(x)], t) \) can be expressed as a functional of precanonical wave functions \( \Psi(\phi, x) \) restricted to a specific field configuration \( \Sigma: \phi = \phi(x) \) at time \( t \):

\[
\Psi([\phi(x)], t) = \Psi([\Psi_{\Sigma}(x, t), \phi(x)]) ,
\]

where we have denoted the restriction of precanonical wave function \( \Psi(\phi, x) \) to \( \Sigma \) as \( \Psi_{\Sigma}(x, t) := \Psi(\phi = \phi(x), x, t) \). Thus the time dependence of the wave functional \( \Psi \)
is totally controlled by the time dependence of precanonical wave function restricted to Σ and the time derivative of Ψ is obtained by the chain rule differentiation
\[ i\partial_t \Psi = \text{Tr} \int dx \left\{ \frac{\delta \Psi}{\delta \Psi^T(x,t)} i\partial_t \Psi(x,t) \right\}, \]

where Ψ^T is the transpose matrix of Ψ. In what follows, for brevity, we will denote \( \Psi_\Sigma(x,t) \) simply as \( \Psi_\Sigma(x) \) or even \( \Psi_\Sigma \) when appropriate.

The time derivative \( i\partial_t \Psi_\Sigma(x,t) \) is determined by the restriction of precanonical Schrödinger equation (2) to Σ, which takes the form
\[ i\partial_t \Psi_\Sigma = -i\gamma_0 \gamma^i \left( \frac{d}{dx^i} - \partial_i \phi(x) \frac{\partial}{\partial \phi} + \partial_i \phi_k(x) \frac{\partial}{\partial \phi_k} + \ldots \right) \Psi_\Sigma - \frac{i}{4} \gamma_0 \gamma^i [\omega_i, \Psi_\Sigma] + \gamma_0 \frac{1}{\kappa} \hat{H}_\Sigma \Psi_\Sigma, \]

where \( \frac{d}{dx^i} \) is the total derivative along Σ:
\[ \frac{d}{dx^i} := \partial_i + \partial_i \phi(x) \frac{\partial}{\partial \phi} + \partial_i \phi_k(x) \frac{\partial}{\partial \phi_k} + \ldots, \]
\( \phi_k \) denote the fiber coordinates of the first-jet bundle of the bundle of field variables \( \phi \) over space-time (cf. [43, 44]) and \( \hat{H}_\Sigma \) is the restriction of the DW Hamiltonian operator to Σ:
\[ \frac{1}{\kappa} \hat{H}_\Sigma = \frac{1}{\kappa} \hat{H} = -\frac{\kappa}{2} \frac{\partial}{\partial \phi^2} + \frac{1}{\kappa} V(\phi). \]

In (6) we already explicitly specify the action of the spin-connection on Clifford-valued functions via a commutator, though the necessity of it will be revealed only later in Eq. (31). The first two terms in (6) can be rewritten in terms of the total covariant derivative \( \nabla^\text{tot} \) of \( \Psi \) restricted to Σ,
\[ i\partial_t \Psi_\Sigma = -i\gamma_0 \gamma^i \nabla^\text{tot}_i \Psi_\Sigma + i\gamma_0 \gamma^i \partial_i \phi(x) \partial_\phi \Psi_\Sigma + \frac{1}{\kappa} \gamma_0 \hat{H}_\Sigma \Psi_\Sigma, \]

where the superscript \( \text{tot} \) has two meanings: the first meaning is that the derivative is total in the sense of being taken of a composite function \( \Psi(\phi(x), x) \), and the second meaning is that the covariant derivative of a Clifford-valued tensor function \( T^{\mu_1 \mu_2 \ldots} \) is total in the sense that it includes both the spin-connection \( \omega_\mu^{\alpha \beta} \) and the Christoffel symbols \( \Gamma^\alpha_{\beta \gamma} \):
\[ \nabla^\text{tot}_\alpha T^{\mu_1 \mu_2 \ldots} := \frac{d}{dx^\alpha} T^{\mu_1 \mu_2 \ldots} + \frac{1}{4} [\omega_\alpha, T^{\mu_1 \mu_2 \ldots}] + [\Gamma, T]^{\mu_1 \mu_2 \ldots}, \]

where \([\Gamma, T]^{\mu_1 \mu_2 \ldots}\) is just a short-hand notation for \( \Gamma^{\mu_1 \alpha \beta}_{\alpha \beta \nu_1 \nu_2 \ldots} + \Gamma^{\mu_2 \alpha \beta}_{\alpha \beta \nu_1 \nu_2 \ldots} + \ldots - \Gamma^{\beta \mu_1 \nu_1 \nu_2 \ldots}_{\alpha \beta \nu_1 \nu_2 \ldots} - \Gamma^{\beta \mu_2 \nu_1 \nu_2 \ldots}_{\alpha \beta \nu_1 \nu_2 \ldots} - \ldots \) and the commutator with the spin-connection matrix in the second term ensures that the covariant derivative satisfies the Leibniz rule when acting on the Clifford (matrix) product of two Clifford-valued tensor quantities. This property will be very important for our integrations by parts below. The last term in (10) vanishes when acting on the scalar Clifford-valued function \( \Psi \).
However, it appears in the condition of the covariant constancy of Dirac matrices $\gamma^\mu$ and their antisymmetric products $\gamma^{\mu_1\mu_2\cdots}$, which in terms of $\nabla_\alpha^{\text{tot}}$ reads
\[ \nabla_\alpha^{\text{tot}} \gamma^{\mu_1\mu_2\cdots} = 0. \] (11)

Obviously, when acting on $x$-dependent $\gamma$-matrices, only the first partial derivative term in (7) is non-vanishing.

Now, from (5), (6) and (8) the time evolution of the wave function of the quantum scalar field in static space-time is given by
\[ \begin{align*}
    i\partial_t \Psi & = \int d^4x \ Tr \left\{ \Phi(x) \left[ -i\gamma_0^i \frac{d}{dx^i} \Psi_\Sigma(x) + i\gamma_0^i \partial_i \phi(x) \partial_\phi \Psi(x) ight. \\ & \quad - \frac{i}{4} \gamma_0^i [\omega_i, \Psi_\Sigma(x)] - \frac{\kappa}{2} \gamma_0 \partial_\phi \Psi_\Sigma(x) + \frac{1}{\kappa} \gamma_0 V(\phi(x)) \Psi_\Sigma(x) \bigg] \right\},
\end{align*} \] (12)

where the notation
\[ \Phi(x) := \frac{\delta \Psi}{\delta \Psi_\Sigma^T(x)} \] (13)
is introduced. We would like to see how this equation can reproduce the functional derivative Schrödinger equation (1).

In order to compare (12) with (1), let us calculate the first and the second total functional derivatives of $\Psi$ in (11) with respect to $\phi(x)$:
\[ \begin{align*}
    \frac{\delta \Psi}{\delta \phi(x)} & = \text{Tr} \left\{ \Phi(x) \partial_\phi \Psi_\Sigma(x) \right\}, \\
    \frac{\delta^2 \Psi}{\delta \phi(x)^2} & = \text{Tr} \left[ \Phi(x) \delta(0) \partial_\phi \Psi_\Sigma(x) + 2 \frac{\delta \Phi(x)}{\delta \phi(x)} \partial_\phi \Psi_\Sigma(x) \right] + \frac{\delta^2 \Psi}{\delta \phi(x)^2} \\
    & \quad + \text{Tr} \text{Tr} \left\{ \frac{\delta^2 \Psi}{\delta \Psi_\Sigma(x)^T \otimes \delta \Psi_\Sigma^T(x)} \partial_\phi \Psi_\Sigma(x) \otimes \partial_\phi \Psi_\Sigma(x) \right\}.
\end{align*} \] (15)

where the double trace notation in the last line refers to the fact that the trace is taken of each of the matrices in the direct product. Here and in what follows, $\delta$ denotes the partial functional derivative with respect to $\phi(x)$, and $\delta(0)$ is the $(n - 1)$-dimensional delta-function at $x = 0$ which results from the variational differentiation of the function $\Psi_\Sigma(x)$ with respect to itself at the same spatial point. Henceforth, when writing $\delta(0)$, we imply that a proper regularization like a point splitting or a lattice one has been applied in order to make sense of this singular expression. This is the regularization which is usually implied when the second variational derivative is used in the functional Schrödinger equation in QFT.

Let us start from the observation that the potential energy term in the canonical functional derivative Schrödinger equation for the quantum scalar field (11) should be obtained from the potential energy term $V$ in (12), i.e.
\[ \int d^4x \ Tr \left\{ \Phi(x) \frac{1}{\kappa} \gamma_0 V(\phi(x)) \Psi_\Sigma(x) \right\} \mapsto \int d^4x \sqrt{-g} \ V(\phi(x)) \Psi, \] (16)
where a more precise meaning of the symbol \( \mapsto \) will be clarified below. To accomplish that, the following relation should be fulfilled at any spatial point \( x \):

\[
\text{Tr} \left\{ \Phi(x) \frac{1}{\gamma_0} \gamma_0 \Psi(x) \right\} \mapsto \sqrt{-g} \Psi. \tag{17}
\]

By functionally differentiating both sides of Eq. (17) with respect to \( \Psi^T x \), we obtain

\[
\text{Tr} \left\{ \frac{\delta^2 \Psi}{\delta \Psi^T(x) \otimes \delta \Psi(x)} \frac{1}{\gamma_0} \gamma_0 \Psi(x) \right\} + \Phi(x) \frac{1}{\gamma_0} \gamma_0 \delta(0) \mapsto \sqrt{-g} \Phi(x), \tag{18}
\]

where \( \delta(0) := \delta \Psi(x) / \delta \Psi(x) \). This type of relation is possible only if the second variational derivative of \( \Psi \) with respect to \( \Psi(x) \) vanishes:

\[
\frac{\delta^2 \Psi}{\delta \Psi^T(x) \otimes \delta \Psi(x)} = 0, \tag{19}
\]

and

\[
\frac{1}{\gamma_0} \gamma_0 (x) \delta(0) - \sqrt{-g}(x) \mapsto 0 \tag{20}
\]

(here we explicitly recall that both \( \gamma_0 \) and \( \sqrt{-g} \) depend on \( x \)-s). The latter can be understood as the condition

\[
\gamma^0 \sqrt{-g} \mapsto \delta(0). \tag{21}
\]

By noticing that \( \sqrt{-g} = \sqrt{-g_{00}} h \), where \( h := \det |g_{ij}| \) is the determinant of the spatial part of the metric tensor, and \( \gamma^0 \sqrt{g_{00}} = \gamma_0 \) is the time-like component of the tangent Minkowski space Dirac matrices, we can rewrite (21) in the form

\[
\gamma_0 \mapsto \delta(0) / \sqrt{-h} = \delta^\text{inv}(0), \tag{22}
\]

where in the last equality the invariant \((n-1)\)-dimensional delta function appears which is defined by the property \( \int d x \sqrt{-h}(x) \delta^\text{inv}(x) = 1 \). We, therefore, see that the curved space-time generalization of the limiting map \( \gamma_0 \mapsto \delta(0) \) found earlier in flat space-time \([20, 21]\) just replaces the \((n-1)\)-dimensional delta function by the invariant one, at least in static space-times.

Similarly, in the limiting case \([21]\), the term \( IV \) in (12) reproduces the first term on the right-hand side of (15):

\[
IV : -\frac{\gamma_0}{2} \partial_{\phi_0} \Psi \mapsto -\frac{1}{\sqrt{-g}} g_{00} \delta(0) \partial_{\phi_0} \Psi. \tag{23}
\]

Then, by comparing with (15) we conclude that the term \( IV \) in (12) leads to the following expression in variational derivatives of \( \Psi \):

\[
IV : \text{Tr} \left\{ \frac{1}{2} \Phi(x) \gamma_0 \partial_{\phi_0} \Psi(x) \right\} \mapsto \frac{1}{2} \frac{g_{00}}{\sqrt{-g}} \left( \frac{\delta^2 \Psi}{\delta \phi(x)^2} - 2 \text{Tr} \left\{ \frac{\delta \Phi(x)}{\delta \phi(x)} \partial_{\phi} \Psi(x) \right\} - \frac{\delta^2 \Psi}{\delta \phi(x)^2} \right). \tag{24}
\]
While the first term on the right-hand side of (24) correctly reproduces the first term in the functional derivative Schrödinger equation (1), two other terms need further investigation, which follows.

Since we expect Eq. (12) to lead to a description solely in terms of the wave functional $\Psi$, as in the functional Schrödinger equation (1), the term $II$ in (12) with $\partial_\phi \Psi_\Sigma$ should be cancelled by some other term with $\partial_\phi \Psi_\Sigma(x)$. Now we can do so with the help of the second term in (24). Thus the requirement of mutual cancellation of the terms with $\partial_\phi \Psi_\Sigma(x)$ leads to the relation

$$II + \text{part of } IV : \Phi(x) i \gamma_0 \gamma^i \partial_i \delta \phi(x) + \frac{g_{00}}{\sqrt{-g}} \frac{\delta \Phi(x)}{\delta \phi(x)} \mapsto 0.$$  

By taking the variational derivative of (25) with respect to $\phi(x')$ we can easily see that (25) with $\mapsto$ replaced by the equality sign is not an integrable equation in variational derivatives. Nevertheless, by taking into account that it should be valid only under the limiting map (21), we can write the solution for $\Phi(x)$ in the form

$$\Phi(x) = \Xi([\Psi_\Sigma]; \hat{x}) e^{-i \phi(x) \gamma^i \partial_i \phi(x)/\kappa},$$

where $\Xi([\Psi_\Sigma]; \hat{x})$ denotes a functional of $\Psi_\Sigma(x')$ at $x' \neq x$ (i.e. on a punctured space with the removed point $x$), i.e. $\frac{\delta \Xi([\Psi_\Sigma]; \hat{x})}{\delta \phi(x)} \equiv 0$, which plays the role of the integration constant here. Indeed, by differentiating (24) with respect to $\phi(x)$, replacing $\kappa$ according to the limiting map (21), and taking into account that $\partial_i \delta(0) = 0$ and $\gamma_0(x) = g_{00}(x)$, we conclude that (25) is fulfilled by (26) under the condition (21). Moreover, (26) by construction fulfills

$$\frac{\delta \Phi(x)}{\delta \Psi^T_\Sigma(x)} = \frac{\delta^2 \Psi}{\delta \Psi^T_\Sigma(x) \otimes \delta \Psi^T_\Sigma(x)} \equiv 0,$$

which is consistent with (19) as it should. Let us emphasize that the cancellation of the terms with $\partial_\phi \Psi_\Sigma(x)$ can be only achieved in the limiting case (21).

Now, based on (13), (17) and (26) we can write the wave functional $\Psi$ in the following form valid at any point $x$:

$$\Psi \sim \text{Tr} \left\{ \Xi([\Psi_\Sigma]; \hat{x}) e^{-i \phi(x) \gamma^i \partial_i \phi(x)/\kappa} \frac{\gamma_0}{\sqrt{-g}} \Psi_\Sigma(x) \right\} \mapsto \frac{\gamma_0}{\sqrt{-g}} \Psi_\Sigma(x),$$

where $\sim$ denotes the equality up to a normalization factor, which will also include $\kappa$ and $\sqrt{-g}$, and the notation $\{...\} \mapsto \gamma_0 \delta(0)/\sqrt{-g}$ means that every appearance of $\kappa$ in the expression inside braces is replaced by $\gamma_0 \delta(0)/\sqrt{-g}$ with a regularized value of $\delta(0)$ in accord with the limiting map (21).

Using this expression for the wave functional $\Psi$ we can now evaluate the last term in (24) in the limit (21):

$$\text{part of } IV : \frac{1}{2} \frac{g_{00}}{\sqrt{-g}} \frac{\delta^2 \Psi}{\delta \phi(x)^2} \mapsto -\frac{1}{2} g_{ij} \partial_i \phi(x) \partial_j \phi(x) \Psi.$$
We see that it correctly reproduces the second term in the functional derivative Schrödinger equation (1) which is responsible for the inherent non-ultralocality (adapting Klauder’s terminology [45]) of quantum relativistic scalar field theory.

Thus, in the limiting case (21), we have successfully derived all terms in (1) from our precanonical Schrödinger equation restricted to Σ, Eq. (6). However, there are still two terms \( I \) and \( III \) left in (12) which have not played any role yet:

\[
I + III : -i \int dx \, \text{Tr} \left\{ \Phi(x) \gamma_0 \gamma^i \nabla^\text{tot}_i \Psi_\Sigma \right\}.
\] (30)

In our previous discussions in flat space-time [20–22] it has been always easy to show in the end of the calculation that the corresponding term with the total derivative \( d/dx \Psi_\Sigma(x) \) is vanishing (under the boundary condition that \( \Psi_\Sigma(x) \) is vanishing at the spatial infinity). Let us see if this property extends to the case of static curved space-times which we consider here.

By integration by parts using the covariant Stokes theorem and the Leibniz property of the covariant derivative \( \nabla^\text{tot}_i \) acting on Clifford-valued functions, (30) can be transformed as follows:

\[
I + III : -i \int dx \sqrt{-h} \left( \text{Tr} \left\{ \frac{1}{\sqrt{-h}} \Phi(x) \gamma_0 \gamma^i \nabla^\text{tot}_i \Psi_\Sigma \right\} \right)
= -i \int dx \sqrt{-h} \text{Tr} \left\{ \nabla^\text{tot}_i \left( \frac{1}{\sqrt{-h}} \Phi(x) \gamma_0 \gamma^i \right) \Psi_\Sigma \right\}
+ i \int dx \left( \sqrt{-h} \text{Tr} \left\{ \nabla^\text{tot}_i \left( \frac{1}{\sqrt{-h}} \Phi(x) \gamma_0 \gamma^i \right) \Psi_\Sigma \right\} \right)
= -i \oint_{\partial \Sigma} dx_i \text{Tr} \left\{ \Phi \gamma_0 \gamma^i \Psi_\Sigma \right\} + i \int dx \text{Tr} \left\{ \Phi \left( \nabla^\text{tot}_i (\gamma_0 \gamma^i) \right) \Psi_\Sigma \right\}
+ i \int dx \left( \frac{-\nabla_i \sqrt{-h}}{\sqrt{-h}} \text{Tr} \left\{ \Phi \gamma_0 \gamma^i \Psi_\Sigma \right\} + \text{Tr} \left\{ \left( \nabla^\text{tot}_i \Phi(x) \right) \gamma_0 \gamma^i \Psi_\Sigma \right\} \right). \] (31)

Here,

(i) the first boundary term on the right-hand side of (31) follows from the covariant Stokes theorem; \( dx_i = d^{n-2}x |_{\partial \Sigma} n_i(x) \) is the measure of \((n-2)\)-dimensional integration on the boundary \( \partial \Sigma \) with the normal vector \( n_i(x) \) tangent to \( \Sigma \). Under the assumption that \( \Psi \) vanishes on the boundary \( \partial \Sigma \) the boundary term vanishes too.

(ii) Next three terms on the right-hand side of (31) follow from the Leibniz rule with respect to the Clifford product fulfilled by the total covariant derivative \( \nabla^\text{tot}_i \) acting on tensor Clifford-algebra-valued functions. This is where the fact that the spin-connection matrix \( \omega_i \) acts on \( \Psi \) by means of the commutator product \([\omega_i, \Psi]\) is essential.

(iii) The second term with \( \nabla^\text{tot}_i (\gamma_0 \gamma^i) \) vanishes due to the covariant constancy of Dirac matrices (11).
(iv) The third term vanishes because the covariant derivative of the density $\sqrt{-h}$ vanishes: $\nabla_i \sqrt{-h} = 0$, as a consequence of the covariant constancy of $g_{\mu\nu}$.

(v) In the fourth term on the right-hand side of (31), using the explicit formula for $\Phi(x)$ in (26) and the Leibniz rule for $\nabla^\text{tot}_i$, we obtain

$$\nabla^\text{tot}_i \Phi(x) = \frac{-i}{\kappa} \Phi(x) \left( \partial_i \phi \gamma^l \partial_l \phi + \phi \gamma^l \partial_i \phi + \phi (\nabla^\text{tot}_i \gamma^l) \partial_l \phi \right). \quad (32)$$

The last term in (32) vanishes due to the covariant constancy of Dirac matrices (11). By substituting (32) into the last term in (31), integrating by parts (assuming the field configurations $\phi(x)$ vanish at the spatial infinity) and using the covariant Stokes theorem again, we obtain

$$\int dx \; \text{Tr} \left\{ ||\Psi|| \sqrt{-g} g^{il} \partial_i \phi \partial_l \phi + \phi g^{il} \partial_i \phi \right\} = -\Psi \int dx \; \sqrt{-h} \nabla_i \left( \sqrt{g_{00}} g^{il} \right) \frac{1}{2} \partial_l \phi^2 = 0, \quad (33)$$

where $||\Psi|| := \Phi(x) \frac{-1}{x \sqrt{-g}} \gamma_0 \Psi_{\Sigma}(x)$ such that $\text{Tr} ||\Psi|| = \Psi$ (c.f. (17)) is independent of $x$. Again, the right-hand side of (33) vanishes because the covariant derivative of the metric tensor vanishes.

Thus we have shown that all four terms on the right-hand side of (31) vanish and, therefore, the terms $I$ and $III$ in (12) in the limiting case (21) yield a vanishing contribution in the equation for the functional $\Psi$.

Note that this result is a consequence of the properties of the pseudo-Riemannian geometry of the space-time background, the assumed boundary conditions that the values of $\Psi_{\Sigma}(x)$ and $\phi(x)$ at $x \to \infty$ are vanishing, and the particular form of the functional $\Phi(x)$ which was established earlier in (26). The covariant Stokes theorem and the Leibniz property of the covariant derivative with respect to the Clifford product have been instrumental. The latter property is guaranteed only when the spin-connection matrix acts on the Clifford-algebra-valued wave functions by the commutator product.

Finally, we can specify the functional $\Xi([\Psi_{\Sigma}(x)], \dot{x})$ in (28) by combining all the above observations together and noticing that the formula Eq. (28) is valid at any given point $x$. This can be accomplished only if the functional $\Psi$ has the structure of the continuous product of identical terms at all points $x$, i.e. up to a normalization factor which includes $\kappa$ and $\sqrt{-h}$,

$$\Psi \sim \text{Tr} \left\{ \prod_x e^{-i\phi(x) \gamma^i \partial_i \phi(x)} \sqrt{-g} \gamma_0 \Psi_{\Sigma}(x) \right\} \right|_{x \to \gamma_0 \delta(0)/\sqrt{-g}}. \quad (34)$$

Thus we have obtained the expression of the Schrödinger wave functional in terms of precanonical wave functions. This formal continuous product expression can be understood as a multidimensional product integral [41, 42] with an invariant measure $\sqrt{-h} dx$ (c.f. (22)) denoted as $\int_x f(x) \sqrt{-h} dx$.

$$\Psi \sim \text{Tr} \left\{ \prod_x e^{-i\phi(x) \gamma^i \partial_i \phi(x)} \gamma_0 \Psi_{\Sigma}(x) \right\} \right|_{x \to \gamma_0 \delta(0)/\sqrt{-h}}. \quad (35)$$
This result looks remarkably similar to the result in flat space-time [21], differing only in that the Dirac matrices are now \( x \)-dependent and the spatial integration measure \( d\mathbf{x} \) is replaced by the invariant one.

Summarizing the results of the above consideration of (12), we have shown that:

- the potential term \( V \) reproduces the potential term in (1) in the limiting case [21];
- in the same limiting case, the term \( IV \) reproduces the second functional derivative term in the canonical Hamiltonian operator in (1) up to some additional terms in [21];
- the required cancellation of those additional terms together with the term \( II \) in (12) allows us to obtain an explicit product integral formula for the Schrödinger wave functional in terms of the precanonical wave function, Eq. (35), and to reproduce the second term in (1) which is responsible for non-ultralocality;
- the terms \( I \) and \( III \) do not contribute to the functional Schrödinger equation (1) if the fields \( \phi(x) \) and \( \Psi_\Sigma(\phi(x), x, t) \) are vanishing at the spatial infinity.

Thus, in static space-times, we have demonstrated that the canonical functional derivative Schrödinger equation (1) and the explicit product integral formula (35) relating the Schrödinger wave functional with the Clifford-valued precanonical wave function follow from the precanonical Schrödinger equation (2) in the symbolic limiting case when \( \gamma_0/\kappa \) is replaced by (a regularized) invariant delta-function \( \delta(0)/\sqrt{-h} \) at equal spatial points, i.e. essentially, by the UV cutoff of the total volume of the momentum space.

3 Conclusion

We investigated a connection between the description of quantum scalar theory in curved space-time based on precanonical quantization and the standard description using the functional Schrödinger representation resulting from the canonical quantization. Conceptually, there is a huge gap between the canonical description in terms of a Schrödinger wave functional of field configurations \( \phi(x) \) on fixed-time hypersurfaces labelled by \( t \) and the precanonical description in terms of a precanonical wave function, which is a section of the Clifford bundle [46] over the finite-dimensional bundle with the fiber coordinates \( \phi \) and the base coordinates \( x^\mu \). Nevertheless, we demonstrated in the case of static space-times that the latter can be derived from the former in the limiting case of an infinitesimal value of \( 1/\kappa \) when the Clifford algebra element \( \gamma_0/\kappa \) can be replaced by or mapped to the differential form representing an infinitesimal invariant spatial volume element \( \sqrt{-h} d\mathbf{x} \). In this (symbolic) limiting case, we were able to derive the standard functional derivative Schrödinger equation for the quantum scalar field in static curved space-times from the precanonical Schrödinger equation for the same physical system,
and also to obtain, up to a normalization factor, an expression of the Schrödinger wave functional of quantum scalar field theory in terms of a multiple product integral of precanonical wave functions restricted to a field configuration $\phi = \phi(x)$ at a fixed moment of time, Eq. (35).

Our result confirms and generalizes to static space-times the statement from our previous papers [20–22] that the standard functional Schrödinger representation of quantum field theory is a certain (symbolic) limiting case of the theory of quantum fields obtained by precanonical quantization. While the former, in order to be a well-defined theory at least on the physical level of rigour, is known to require an ad hoc regularization (e.g. a point-splitting in the second variational derivative in the functional derivative Schrödinger equation (1)), which typically introduces a UV cutoff scale $\Lambda$ as a necessary additional element of the theory removed by a subsequent renormalization, the precanonical formulation is “already regularized” since the ultraviolet scale $\kappa$ is an inherent element of the precanonical quantization procedure.

One can still wonder if $\kappa$ is a fundamental scale or an auxiliary element of precanonical quantization of fields which should be removed from the physical results by a procedure similar to the usual renormalization. The latter point of view is supported by the observation that $\kappa$ actually disappears from the observable quantities of the theory in the case of free fields. On the other hand, the fundamental nature of the scale $\kappa$ is supported by our recent estimation of the mass gap in the precanonical formulation of quantum pure SU(2) gauge theory [37] and a naive estimation of the cosmological constant from the precanonically quantized pure Einstein gravity [36]. Surprisingly, both estimations point to a subnuclear scale of $\kappa$, thus a further research is required to understand this fact.

The consideration in this paper has also allowed us to realize that the spin-connection matrix in the curved space-time generalization of precanonical Schrödinger equation, Eq. (2), acts on the Clifford-algebra-valued precanonical wave function via a commutator. This guarantees the Leibniz property when the corresponding covariant derivative acts on the Clifford product of two Clifford-algebra-valued quantities, and this property is crucial for the proof in Eq. (31) of the vanishing contribution of the terms $I$ and $III$ in (12) to the functional Schrödinger equation (1). This observation implies that some details of the earlier demonstrations of the Ehrenfest theorem in curved space-time [10] and precanonical quantum gravity [11] will have to be modified, which we hope to address in a future paper.

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