Abstract

We make a first step to extend to the supersymmetric arena the effective action method, which is used to covariantly deduce the low energy dynamics of topological defects directly from their parent field theory. By focussing on two-dimensional supersymmetric theories we are able to derive the appropriate change of variables that singles out the low energy degrees of freedom. These correspond to super-worldline embeddings in superspace which are subject to a geometrical constraint. We obtain a supersymmetric and \( \kappa \)–invariant low energy expansion, with the standard superparticle action as the leading term, which can be used for the determination of higher-order corrections. Our formulation fits quite naturally with the present geometrical description of \( \kappa \)–symmetry in terms of the so-called geometrodynamical constraints. It also provides a basis for the exploration of these issues in higher-dimensional supersymmetric theories.
1 Introduction

It is a well known fact that, at sufficiently low energies, the dynamics of topological defects of bosonic relativistic field theories is governed by Dirac-Nambu-Goto–type effective actions [1]. Adding supersymmetry amounts to the replacement of such actions by their supersymmetric extensions, the so-called super $p$-brane actions, with the common feature of being invariant under a fermionic gauge transformation, the (kappa) $\kappa$–symmetry [2]. Together with the standard reparametrization invariance, the $\kappa$–symmetry allows to gauge away the redundant degrees of freedom that are necessarily present for a manifestly Poincaré invariant description of the effective dynamics of these objects.

Two different, although complementary, lines have been pursued in order to identify the low energy dynamics of topological defects. A first approach—which rests on general principles of effective actions and symmetries—consists essentially in identifying the most general action, with the lowest possible number of derivatives, compatible with the required field content and symmetries of the low energy regime of the theory. In this way, a complete classification of the possible super $p$–brane actions is presently at our disposal [3].

In the second approach instead one attempts to deduce the low energy effective action directly from the underlying field theory by trying to single out the degrees of freedom which are relevant in this regime and integrating the rest. This will generally be a lengthier and more intricate way to proceed because of the difficulty involved in this splitting procedure. Yet one expects as a reward to be able to understand better the origin of the low energy symmetries—the $\kappa$–symmetry in the case of supersymmetric theories—and to get hints from it on how to characterize these symmetries in a fully geometrical way. Furthermore, this method will furnish a perturbative expansion that can be used to get higher-order corrections to the basic Dirac-Nambu-Goto–type actions.

The origin of this approach can be traced back to the covariant method developed by Förster [4] to obtain the effective dynamics of the Nielsen-Olesen vortices [5]. A key ingredient of this method consists in coordinating spacetime by means of a set of curvilinear coordinates adapted to the topological defect. These new coordinates are composed of the worldsheet parameters, giving the spacetime location of the topological defect, plus normal coordinates parametrizing orthogonal displacements with respect to it. While for bosonic topological defects the covariant derivation of the low energy effective actions through this method is fairly well understood, preliminary proposals in supersymmetric cases [6], however, have been able to produce only gauge-fixed versions of the $\kappa$–invariant super $p$–brane actions. Clearly thus, the supersymmetric extension of this procedure still lacks a proper parametrization of the low energy degrees of freedom, i.e. a proper supersymmetry-invariant generalization of the above defect-adapted coordination.

In fact, this central obstruction for a systematic derivation of supersymmetric low energy effective actions is closely related to another long-standing problem in this framework: the search for a tensor calculus for the local $\kappa$–symmetry, known to be a guiding ingredient in the construction of higher-order “curvature” corrections to the basic super $p$–brane actions. Several attempts have been made in order to develop such a calculus [7, 8, 9, 10]. The common trend has
been to impose a so-called \textit{geometrodynamical constraint} on the basic superembedding describing the location of the topological defect in superspace. In this way $\kappa$-symmetry transformations arise as remnants, from within the larger set of general superdiffeomorphisms, that preserve the geometrodynamical constraint. This approach has definitely provided an important insight over the nature of $\kappa$–symmetry. However, it is also clear that a full understanding of the subject would involve explaining the origin of this constraint and studying its relevance in the derivation of the effective action in terms of the underlying field theory.

In this paper, we make a first step to answer the above questions by addressing them in the framework of a generic supersymmetric scalar field theory in two dimensions. We review in Sect. 2 the main features of the covariant effective action method in the simpler bosonic case. In Sect. 3 we introduce the supersymmetric theory and in Sect. 4 we extend the method to this case. A suitable generalization is derived of the change of variables that we use to single out the low energy modes. Consistency of this change, however, requires an extra condition on the superembedding, implying that only a special kind of superembeddings will be able to represent a super domain wall. This condition turns out to coincide with the normal projection of the geometrodynamical constraint. Imposing the tangent projection as well simply amounts to a gauge-fixing reducing the field content and gauge symmetry (superdiffeomorphisms) down to the standard one, \textit{i.e.} reparametrizations and $\kappa$-symmetry. In the end, by straightforward application of the derived change of variables, the lowest-order effective action is directly obtained from the original field theory. We conclude in Sect. 5 with a few remarks on the extension of this work to higher-dimensional supersymmetric models.

2 Effective actions for bosonic topological defects

In this section we briefly describe the method used to obtain effective actions describing the low energy regime of topological defects in the simplest situation, \textit{i.e.} that of soliton-like configurations in a 2$d$ scalar field theory. This will help us illustrate the basic steps to follow later in the derivation of the effective action in the supersymmetric case.

Consider a bosonic two–dimensional model described by a generic action of the form

$$S[\varphi] = \int d^2x \left[ \frac{1}{2} (\partial_{\mu} \varphi)^2 - \frac{1}{2} (U'(\varphi))^2 \right].$$

(2.1)

Such a model features soliton configurations whenever the positive semi-definite potential given by $U'(\varphi)^2/2$ possesses a non-trivial vacuum structure, \textit{i.e.} when there are two or more constant field configurations for which the potential assumes its minimum zero value. Typical examples are, for instance, the $\varphi^4$ kink, where $U' = \frac{1}{2} \left( \varphi^2 - \frac{m^2}{\lambda} \right)$ gives rise to two different vacua, $\varphi = \pm m/\lambda$, and the sine-Gordon soliton, where $U' = \frac{2m^2}{\lambda} \sin \left( \frac{\lambda \varphi}{2m} \right)$, with an infinite set of vacua \cite{11}.

Lowest energy solutions of this kind can be obtained by looking for static configurations interpolating between two different vacua, say $\varphi_{0+}$ and $\varphi_{0-}$, \textit{i.e.} satisfying the boundary conditions

$$\varphi \to \varphi_{0\pm} \quad \text{for} \quad x \to \pm \infty.$$  

(2.2)
Then, it is clear that this sort of solutions will be stable against decay into any of the vacua of the model because such transitions would involve an infinite amount of energy.

For a static soliton the equation of motion derived from (2.1) reduces to

\[- \frac{d^2 \varphi}{dx^2} + U'U'' = 0, \quad \Rightarrow \quad \varphi' = \pm U'(\varphi), \]  

(2.3)
together with boundary conditions of the form (2.2). Due to the time-independence of its solutions, \( \varphi_{S,A}(x - a) \) (corresponding to soliton (+) and antisoliton (−), respectively), it is said that they break only half of the translational symmetries of the model. The space coordinate is the broken direction because of the explicit \( x \)-dependence, while time remains as the unbroken direction.

The specific boundary conditions obeyed by these soliton configurations give them the common feature of dividing space in two zones such that in each of them \( \varphi_{S,A}(x - a) \) is basically in one of the vacua. Both zones will meet in a narrow region around \( x = a \) where the field rapidly evolves from one vacuum to the other and where its energy is effectively confined—a generically called domain wall. These stable energy lumps can thus be associated with particle-like objects, the width of the transition region between vacua being proportional to the inverse of the particle mass \( M_S \). In a low energy regime, characterized by \( E \ll M_S \), non-zero modes will scarcely be excited because their typical scale is of the order \( M_S \). This implies that an effective description of the dynamics of the system in this sector can naturally be achieved by focussing on the dynamics of the zero modes alone, i.e., in terms of the location of the soliton regarded as a point particle. To study such dynamics, one may consider small perturbations around the above soliton solutions which will generally make the domain wall fluctuate while approximately preserving the shape of the \( \varphi \)-field configuration.

Let us now review the process of derivation of the effective action \( S_{\text{eff}}[x(t)] \), where \( x^\mu(t) \) is the worldline of the soliton center, and \( t \) represents an arbitrary parametrization of this worldline. Essentially, one has to perform a splitting in the degrees of freedom contained in \( \varphi \) between the zero modes, describing the location \( x^\mu(t) \) of the soliton, and the other (massive) modes,

\[ \varphi \rightarrow (x^\mu(t), \phi), \]  

(2.4)
and integrate then over the massive modes \( \phi \).

A common way to do this splitting consists in first making a change of variables from standard spacetime coordinates \( x^\mu \) to a new pair of coordinates \( (t, \rho) \), such that both sets are related by

\[ x^\mu = x^\mu(t) + \rho n^\mu(t), \]  

(2.5)
where \( x^\mu(t) \) is the worldline describing the location of the soliton and \( n^\mu(t) \) the unit normal to the unit tangent vector \( v^\mu(t) \). The splitting (2.4) is afterwards implemented by defining \( \phi(t, \rho) = \varphi(x^\mu) \) by means of this change of variables.

In fact, the rationale behind the change (2.5) is precisely to provide a covariant splitting between instantaneous broken and unbroken symmetry directions. With this idea in mind, the form (2.5) of the change of variables can be directly guessed but it can also be derived with
the help of a simple method. The procedure consists in, first, finding the transformation from the \((x^\mu)\) spacetime coordinates to the coordinates \((z^\mu)\) of the instantaneous co-moving frame at, say, \(t = t_0\); and, second, writing the instantaneous space direction in terms of the original \((x^\mu)\) coordinates. Both frames \((x^\mu)\) and \((z^\mu)\) will be related by a Poincaré transformation

\[
x^\mu = \Lambda^\mu_\nu(t_0) z^\nu + a^\mu(t_0),
\]

where the values of \(\Lambda^\mu_\nu(t_0)\) and \(a^\mu(t_0)\) are determined by imposing that \(z(t)\) —the expression of the worldline \(x^\mu(t)\) in the \((z^\mu)\) coordinates— should have an expansion around \(t = t_0\)

\[
z^\mu(t) = z^\mu(t_0) + \dot{z}^\mu(t_0)(t - t_0) + O((t - t_0)^2),
\]

such that, at first order, the particle is at rest in the origin, \(i.e.\)

\[
z^\mu(t_0) = 0, \quad \dot{z}^\mu(t_0) \propto \delta^{\mu 0}.
\]

It is a simple exercise to check that these conditions determine the parameters of the Poincaré transformation (2.6) to be

\[
(\Lambda^\mu_\nu(t_0)) = (v^\mu(t_0) \ n^\mu(t_0)), \quad a^\mu(t_0) = x^\mu(t_0).
\]

Now, locally around \(t = t_0\), the instantaneous broken symmetry direction is described by the pure space direction in the \((z^\mu)\) coordinates, \(i.e.\) the spacetime points of the form \(z^\mu = \rho \delta^{\mu 1}\). Using (2.6) and (2.7) these points can be described in the original \((x^\mu)\) coordinates as follows

\[
x^\mu = \rho n^\mu(t_0) + x^\mu(t_0).
\]

By dropping the subindex of \(t_0\) we get precisely the coordinate transformation (2.5). This method can thus provide a systematic way to derive the appropriate change of variables in more complex situations, where it is less simple to guess it. In fact, we will use a completely analogous procedure later on in order to derive the proper generalization of (2.5) for supersymmetric two-dimensional models with solitonic solutions.

After the splitting between massless and massive modes one should “integrate” the massive modes, \(i.e.\) eliminate them by means of their equation of motion:

\[
\frac{\partial S}{\partial \phi}[x, \phi] = 0, \quad \Rightarrow \quad \phi = \phi[x(s)].
\]

In this way, after introducing this solution back into the original action we will get an effective action describing the motion of the soliton as a point particle

\[
S_{\text{eff}}[x] = S[x, \phi[x]].
\]

Furthermore, any solution \(x^*(s)\) of (2.9) will also provide a solution of the original equations of motion through the assignment \(\phi^* = \phi[x^*]\).

The explicit realization of this programme requires some care in order to avoid overcounting of degrees of freedom (see [12] for a detailed analysis of this problem). Basically, one should make
sure when deriving the equation of motion for $\phi$ that the appropriate boundary conditions are satisfied. For example, if we assume the worldline $x^\mu(t)$ to be the locus of zero-field spacetime points, i.e. those satisfying

$$\phi(t, 0) = \varphi(x^\mu(t)) = 0,$$  

(2.10)

(the so-called core of the field), we should take care of using the variational principle subject to the constraint $\delta \phi|_{\rho=0} = 0$, which can be enforced by adding to the action a Lagrange multiplier times the condition (2.10). This term, however, leads to delta-type contributions to the equation of motion, giving rise to a non-analytic behavior for $\phi$ at the location of the core [13]. Other definitions for the worldline representing the soliton, i.e. other constraints different than (2.10), are also possible and, in fact, preferable. The simplest amongst them is given by

$$\chi[\phi] = \int d\rho \varphi'_S(\rho)(\phi(t, \rho) - \varphi_S(\rho)) = 0,$$  

(2.11)

which ensures that $\delta \phi = \phi(t, \rho) - \varphi_S(\rho)$ is orthogonal to the translational zero mode $\varphi'_S(\rho)$. Thus, it guarantees that the dynamics of this zero mode is no longer described by $\phi(t, \rho)$ but by the worldline variables $x^\mu(t)$. This condition can be regarded as a “smoother” version of the core condition (2.10), in the sense that it avoids the presence of non-analyticities at $\rho = 0$. Yet, both conditions (2.10) and (2.11) can be seen to coalesce for large values of the soliton mass. So, in summary, the action used to derive $S_{\text{eff}}[x]$ can be written, using the proper time parametrization, as

$$S[x, \phi, g] = \int ds d\rho \Delta \left[ \frac{1}{2\Delta^2} (\partial_s \phi)^2 - \frac{1}{2} (\partial_\rho \phi)^2 - \frac{1}{2} (U'(\phi))^2 \right] + \int ds g(s) \chi[\phi],$$  

(2.12)

where $\Delta = 1 + \rho k$ is the determinant of the coordinate change (2.3) and $k$ is the (signed) curvature of the worldline.

By writing the equation of motion for $\phi$ derived from (2.12) it is simple to show that an exact solution is given by

$$\phi(s, \rho) = \varphi_S(\rho),$$  

(2.13)

where $\varphi_S$ stands for the static soliton solution centered at the origin. Because of the $\rho$ dependence, this solution needs not represent in principle a static configuration, but one that looks like a static configuration in the co-moving frame of $x^\mu(s)$. However, plugging it into (2.12) we find an effective action describing just a free particle motion

$$S_{\text{eff}}[x] = -M_S \int ds,$$

with a mass given by the soliton mass

$$M_S = \int d\rho (\varphi'_S(\rho))^2.$$  

(2.14)

The effect of fluctuations around the soliton-like configurations described by (2.13) can be studied with the help of the expansion

$$\phi(t, \rho) = \varphi_S(\rho) + \zeta(t, \rho),$$

where $\zeta(t, \rho)$ represents the fluctuations around the static solution $\varphi_S(\rho)$. These fluctuations can be treated as quantum corrections to the classical solution, and the theory can be further developed by including quantum field theory techniques.
by solving for the small fluctuations $\zeta(t, \rho)$ by means of their own equations of motion. This will generally produce new contributions to the effective action, making it depart from the trivial free propagation.

As a summary of the procedure described in this section; in order to explicitly derive the effective action for topological defects we have to go through the following steps: (i) perform a change of variables in the base spacetime (2.5) so as to get a covariant splitting between broken and unbroken directions, (ii) introduce an explicit parametrization of the topological defect as an embedding in spacetime, $x^\mu(t)$, covariantly describing the zero modes of the theory, and eliminate from the field $\phi$ the dependency on these zero modes by constraining it (2.11), and (iii) solve the $\phi$ field equation (2.8) and, plugging the solution into the action, get the effective action for the zero modes.

3 Solitons in two-dimensional supersymmetric models

In this section and the following one we shall generalize the above analysis to two-dimensional supersymmetric models. First, we will provide an appropriate parametrization of the static (super) domain-wall solutions. Then, after studying the relevant geometric properties of this kind of superembeddings we will go on to discuss the explicit derivation of the effective action.

The generic form of a supersymmetric scalar field theory in two dimensions\cite{14}

\begin{equation}
S = \int d^2x \frac{1}{2} \left[ (\partial_\mu \varphi)^2 - (U'(\varphi))^2 + \bar{\psi} i \partial_\mu \psi - U''(\varphi) \bar{\psi} \psi \right], \tag{3.1}
\end{equation}

coupling a Majorana fermion $\psi$ and a boson $\varphi$ by means of an arbitrary function $U(\varphi)$, exhibits supersymmetric soliton configurations whenever the associated bosonic field theory itself gives rise to soliton configurations as well. Indeed, a simple inspection to the equations of motion arising from (3.1)

\begin{equation}
\partial^2 \varphi + U' U'' + \frac{1}{2} U''' \bar{\psi} \psi = 0,
\psi = 0,
\end{equation}

shows that the static soliton and antisoliton solutions of the purely bosonic theory,

\begin{equation}
\varphi'_{S,A} = \pm U'(\varphi_{S,A}), \quad \psi = 0, \tag{3.3}
\end{equation}

provide also lowest-energy solutions of the supersymmetric model. New lowest-energy solutions can now be obtained by simply applying a SUSY transformation to the above configuration

\begin{equation}
\delta \varphi = \bar{\alpha} \bar{\psi} = 0, \\
\delta \psi = -i [\partial_{\varphi} \varphi_{S,A} - i U'(\varphi_{S,A})] \alpha = -2U'(\varphi_{S,A}) \alpha \pm, \tag{3.4}
\end{equation}

where $\alpha \pm = \frac{1}{2} (1 \pm i \gamma^1) \alpha$.

\footnote{Our conventions for the metric and gamma matrices are: $(g_{\mu\nu}) = \text{diag}(+-)$ and $\gamma^0 = \sigma_2$, $\gamma^1 = i \sigma_1$, $\gamma_5 = \sigma_3$.}
By looking at these transformations it is clear that only half of the supersymmetries are actually effective when acting on a given solution. This implies that the initial bosonic solution—and, in fact, the new solutions as well—will still be invariant under (the other) half of the supersymmetries (3.4). In particular, splitting them into the $\delta_\pm$ parts, generated by $\alpha_\pm$ respectively, the soliton (antisoliton) configurations are seen to maintain the $-\ (+)$ part of (3.4). Then, by applying a broken SUSY transformation onto the soliton configuration, we will get a new lowest-energy solution. In other words, the flat directions of the superpotential are parametrized by the coordinate of the soliton center of mass, $a$, and its SUSY counterpart, $\alpha_+$, which together describe, albeit in a non-covariant way, the location of the supersoliton in superspace.

All these features are in fact most conveniently described in superspace. Indeed, introducing the scalar superfield

$$\Phi(x, \theta) = \varphi(x) + \bar{\theta} \psi(x) + \frac{1}{2} \bar{\theta} \theta F(x),$$

and the standard covariant derivative

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i (\gamma^\mu \theta)_{\alpha} \partial_\mu,$$

one can make up the manifestly SUSY-invariant action

$$S = -i \int d^2xd^2\theta \left[ \frac{1}{4} D\Phi D\Phi + U(\Phi) \right],$$

which boils down to (3.1) after integration over $\theta$ and elimination of the auxiliary field $F$. The above lowest-energy SUSY soliton solutions can now be written in a compact form as

$$\Phi_S(x^\mu, \theta) = \varphi_S (x - a - \bar{\theta}_-(\theta_+ - \alpha_+) \rangle.$$ (3.6)

Such a form indicates that by setting $a = 0$ and $\alpha_+ = 0$ it will be describing a supersymmetric soliton sitting at $x = 0$ and $\theta_+ = 0$. On the other hand, it is direct to show that the coordinate combination $(x - a - \bar{\theta}_-(\theta_+ - \alpha_+))$ present in (3.6) is invariant under both $t$ and $\theta_-$ translations ($\delta_-$ SUSY transformations), meaning that the tangents to the $t$ and $\theta_-$ superspace coordinates remain as the unbroken symmetry directions.

It is thus natural to parametrize these solutions by means of an embedding $(X^\mu(t, \tau), \Theta(t, \tau))$, representing the location of the domain wall in superspace. In the case of $a = \alpha_+ = 0$ this embedding can be described in the simple form

$$X^\mu(t, \tau) = (t - t_0) \delta^\mu_0, \quad \Theta(t, \tau) = (\tau - \tau_0) \eta_-,$$ (3.7)

where $\eta_-$ is a constant chiral bosonic spinor.

One might reasonably question why it is necessary to describe the topological defect with a superembedding of the form $(X^\mu(t, \tau), \Theta(t, \tau))$, since it is quite clear that representing it as

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2We will hereafter concentrate, unless otherwise stated, on the soliton background. Of course, everything goes through in the same way for the antisoliton solution, after keeping track of a few changes of sign.
\((x^\mu(t), \theta(t))\) would already suffice for a low-energy covariant description of the system. The benefit of the former description over a standard worldline parametrization lies in the fact that the geometric picture of the derivation of the effective action is much more transparent in terms of the super-worldline coordinates \((t, \tau)\). This is because, in this way, we will be able to mimic very closely the splitting process between broken and unbroken directions that we have used for the bosonic model. Moreover, the standard constraints that are usually imposed for a geometrical derivation of \(\kappa\)-symmetry will appear very naturally in this setting.

4 Supersymmetric covariant effective action

4.1 Geometry of domain wall superembeddings

The presence of a second (fermionic) direction of unbroken symmetry means that the soliton can be viewed as an extended object in superspace, spanning along a fermionic direction. Hence, an effective description for it can naturally be made in terms of an embedding describing the location of the defect in superspace:

\[
X^\mu = X^\mu(t, \tau), \quad \Theta = \Theta(t, \tau),
\]

where the super-worldline coordinates \((t, \tau)\) parametrize the two unbroken symmetry directions.

The remaining task should then be trying to find the appropriate change from the field-theoretical degrees of freedom of the superfield \(\Phi\) to the superembedding \((X^\mu(t, \tau), \Theta(t, \tau))\) plus the other (massive) modes. In complete analogy with the bosonic case, we should first perform a change of coordinates in superspace, \(i.e.\) to find a covariant splitting between “instantaneous” broken and unbroken superspace directions. The unbroken directions are given just by the tangents to \(t\) and \(\tau\) coordinates whereas the broken ones will be a suitable generalization of the normal vector, parametrized by \(\rho\), and a further fermionic coordinate, related to the broken supersymmetry direction.

To proceed we will consider a super-Poincaré transformation \((x^\mu, \theta) \rightarrow (z^\mu, \xi)\) parametrized as follows

\[
x^\mu = \Lambda^\mu_\nu(z^\nu - i\xi^\nu\alpha) + a^\mu, \\
\theta = S(\Lambda)(\xi + \alpha),
\]

where \(S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu_\nu\gamma^\nu\). We want to choose it in such a way that the embedding \((X^\mu(t, \tau), \Theta(t, \tau))\), when written in the new coordinates \((z^\mu, \xi)\), be (locally around a chosen point \((t_0, \tau_0)\)) represented as a flat superembedding of the form (3.7) obtained in the previous section. In other words, such super-Poincaré transformation should implement the change to a super-comoving frame for the point \((t_0, \tau_0)\) of the superembedding. Then, the supersymmetric analogous of the change (2.5) will come out as relating a superspace point of coordinates \(z^\mu = \rho^{\mu 11}, \xi = \eta_\tau\), at the “instant” \((t_0, \tau_0)\) to its \((x^\mu, \theta)\) coordinates.

In order to get the explicit form of the transformation (4.2) we can expand \((Z^\mu(t, \tau), \Xi(t, \tau))\) —the expression of the superembedding in the \((z^\mu, \xi)\) coordinates— to first order around \((t_0, \tau_0)\)

\[
Z^\mu(t, \tau) = Z^\mu(t_0, \tau_0) + \dot{Z}^\mu(t_0, \tau_0)(t - t_0) + \ddot{Z}^\mu(t_0, \tau_0)(\tau - \tau_0) + \ldots,
\]
\[ \Xi(t, \tau) = \Xi(t_0, \tau_0) + \Xi'(t_0, \tau_0)(t - t_0) + \Xi''(t_0, \tau_0)(\tau - \tau_0) + \ldots. \]

Since we want to match (3.7) we should require \( Z^\mu(t_0, \tau_0) \) and \( \Xi(t_0, \tau_0) \) to vanish. This condition fixes the translations and supersymmetry part of the super-Poincaré transformation (4.2) to be

\[ a^\mu = X^\mu(t_0, \tau_0), \quad \alpha = S^{-1}(\Lambda)\Theta(t_0, \tau_0). \]  

(4.3)

Then, a further Lorentz transformation will also get \( \dot{z}^\mu(t_0, \tau_0) \) to point in the time direction. This is achieved by choosing

\[ (\Lambda_\nu^\mu) = (V^\mu(t_0, \tau_0) \quad N^\mu(t_0, \tau_0)), \]  

(4.4)

where the vector \( V^\mu(t, \tau) \) is defined as the unitarization of the SUSY-invariant tangent \( W^\mu(t, \tau) \)

\[ V^\mu = \frac{W^\mu}{\sqrt{W^2}}, \quad W^\mu = \dot{X}^\mu - i\bar{\Theta}\gamma^\mu\dot{\Theta}, \]

and \( N^\mu(t, \tau) \) is the unit normal, given by \( N^\mu = e^{\mu}_\nu V^\nu \).

We have already fixed completely the super-Poincaré transformation, but we are still far from having the superembedding to look like (3.7) at first order around \((t_0, \tau_0)\). In addition to super-Poincaré, we can also resort to an arbitrary super-reparametrization of the superembedding. However, a simple counting suffices to convince oneself that this is not enough freedom in order to bring, even to first order, an arbitrary superembedding to the desired form (3.7).

Since we are interested in deformations of the flat domain wall solution, our focus should be on superembeddings that locally resemble a flat solution. This is an indication that superembeddings describing super-domain walls cannot be completely general ones but will be restricted by some condition. Actually, for the sake of derivation of the change of variables, we do not need the explicit form of this condition but only the expression of the super-Poincaré transformation above. So we will just go on by assuming that the superembedding \((X^\mu(t, \tau), \Theta(t, \tau))\) does obey the required condition. Once we analyze the properties of the change of variables it will be quite straightforward to find this condition explicitly.

Following exactly the same steps as for the derivation of the bosonic change of variables (2.5), we can get the appropriate form of the superspace change of coordinates from the expression of the superspace points of the form \((z^\mu = \rho \delta^\mu_1, \xi = \eta_+)\) (those spanning the instantaneous broken symmetry directions), in terms of the original \((x^\mu, \theta)\) coordinates. Indeed, if we explicitate the transformation (4.2), with (4.3) and (4.4), we get for these points

\[ x^\mu = X^\mu(t, \tau) + \rho N^\mu(t, \tau) + i\bar{\Theta}(t, \tau)\gamma^\mu\epsilon_+, \]
\[ \theta = \Theta(t, \tau) + \epsilon_+, \]  

(4.5)

where we have already dropped the subindex ‘0’ to identify the point \((t, \tau)\) of the superembedding. Here \( \epsilon_+ \equiv S(\Lambda)\eta_+ \) parametrizes the broken supersymmetry direction and, because of the relation \((1 - i\gamma^1)\eta_+ = 0\), it satisfies the constraint

\[ (1 + i\bar{N}(t, \tau))\epsilon_+ = 0. \]  

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Equation (4.5) is thus a change from the superspace coordinates \((x^\mu, \theta)\) to a new set of (curved) coordinates \((t, \tau, \rho, \epsilon_+)\).

Let us describe a few properties of this change of coordinates. First of all, it is simple to check that both \(\rho\) and \(\epsilon_+\) are invariant under supersymmetry transformations. We can also study how (4.5) is affected by a superworldline reparametrization \((t, \tau) \rightarrow (\tilde{t}, \tilde{\tau})\). In other words, given a fixed superspace point \((x^\mu, \theta)\), we want to study the relation between the \((t, \tau, \rho, \epsilon_+)\) coordinates obtained from (4.5) with the superembedding \((X^\mu(t, \tau), \Theta(t, \tau))\) and the coordinates \((\tilde{t}, \tilde{\tau}, \tilde{\rho}, \tilde{\epsilon}_+)\) obtained after a superworldline reparametrization (i.e. with \(\tilde{X}^\mu(\tilde{t}, \tilde{\tau}) = X^\mu(t, \tau)\) and \(\tilde{\Theta}(\tilde{t}, \tilde{\tau}) = \Theta(t, \tau)\)).

A quick inspection shows that we must have \(\tilde{\epsilon}_+ = \epsilon_+, \tilde{\rho} = \rho\) and that \(N^\mu\), and hence \(V^\mu\) as well, must be scalar
\[
\tilde{V}^\mu(\tilde{t}, \tilde{\tau}) = V^\mu(t, \tau), \quad \tilde{N}^\mu(\tilde{t}, \tilde{\tau}) = N^\mu(t, \tau).
\] (4.6)

This is quite reasonable, since these quantities should depend only on the extrinsic geometry of the superembedding and not on the way it is parametrized. Yet, this simple observation will suffice to determine the condition that we alluded to above.

With this idea in mind, let us first check the transformation of various objects under super-reparametrizations. Consider the SUSY-invariant generalization of the fermionic tangent vector, \(\partial_\tau X(t, \tau)\). It is given by
\[
U^\mu = DX^\mu + i\bar{\Theta} \gamma^\mu D\Theta,
\]
where \(D = \partial_\tau - i\tau \partial_t\). A direct computation shows that \(W^\mu\) and \(U^\mu\) transform under super-reparametrizations according to
\[
W^\mu(t, \tau) = (\partial_t \tilde{t} - i\tilde{\tau} \partial_\tilde{t}) \tilde{W}^\mu(\tilde{t}, \tilde{\tau}) + \partial_\tau \tilde{\tau} \tilde{U}^\mu(\tilde{t}, \tilde{\tau}),
U^\mu(t, \tau) = (D \tilde{t} + i\tilde{\tau} D\tilde{\tau}) \tilde{W}^\mu(\tilde{t}, \tilde{\tau}) + D\tilde{\tau} \tilde{U}^\mu(\tilde{t}, \tilde{\tau}).
\] (4.7)

Since \(W^\mu\) mixes in general with \(U^\mu\) this implies that the \(V^\mu\) and \(N^\mu\) vectors of an arbitrary superembedding (4.1) will not be scalar under general super-reparametrizations. By writing \(U^\mu\) in the 2d vector basis formed by \((V^\mu, N^\mu)\) it is clear from (4.7) that only when \(U^\mu \propto W^\mu\) will \(V^\mu\) and \(N^\mu\) be scalar. We get in this way the following (super-reparametrization invariant) condition for the superembedding
\[
U \cdot N = 0.
\] (4.8)

This condition is already obeyed by the flat domain wall superembedding (3.7), and it is in fact the condition we have discussed about when deriving the change of variables. Indeed, one can explicitly check that arbitrary superembeddings satisfying (4.8) look, locally around any point \((t_0, \tau_0)\), like the flat domain wall superembedding (3.7) (i.e., it is possible to bring the superembedding to the form (3.7), at first order around \((t_0, \tau_0)\), with a suitable choice of super-Poincaré and super-reparametrization transformations). This is why we will call them domain wall superembeddings and we will hereafter assume this condition to be satisfied.

At this point, it is possible to check that a domain wall superembedding contains precisely the degrees of freedom that one expects, i.e. those describing a (super)particle moving in superspace. To show it, we can first reduce the field content of the superembedding by restricting the general...
super-reparametrization invariance with a (partial) gauge-fixing \( U \cdot V = 0 \) which, together with (4.8), implies that
\[
U^{\mu} = DX^{\mu} + i\bar{\Theta}\gamma^{\mu}D\Theta = 0.
\]
(4.9)

It is clear from (4.7) that this condition is preserved by those restricted superdiffeomorphisms \((t, \tau) \rightarrow (\tilde{t}, \tilde{\tau})\) satisfying
\[
D\tilde{t} + i\tilde{\tau}D\tilde{\tau} = 0,
\]
(4.10)
which can be regarded as a one-dimensional analogue of 2d-superconformal transformations.

Using (4.9) we can show that the whole superembedding \((X^{\mu}(t, \tau), \Theta(t, \tau))\) is completely determined in terms of the worldline supercoordinates \((x^{\mu}(t), \theta(t))\). Indeed, expanding the embedding superfields as
\[
X^{\mu}(t, \tau) = x^{\mu}(t) + i\tau\psi^{\mu}(t),
\]
\[
\Theta(t, \tau) = \theta(t) + \tau\lambda(t),
\]
(4.11)
and imposing the constraint (4.9) we are led to the following conditions
\[
\psi^{\mu} = -\bar{\lambda}\gamma^{\mu}\theta,
\]
\[
\bar{\lambda}\gamma^{\mu}\lambda = \dot{x}^{\mu} - i\bar{\theta}\gamma^{\mu}\dot{\theta}.
\]
(4.12)

Multiplying (4.12) with \(n^{\mu}(t)\), the unit vector orthogonal to \(w^{\mu}(t) \equiv \dot{x}^{\mu} - i\bar{\theta}\gamma^{\mu}\dot{\theta}\), we find that \(\lambda\) has to be chiral with respect to the splitting \(\lambda_{\pm} = 1/2 (1 \mp i \gamma)\lambda\). We get the correct chirality for \(\lambda\) by noting that (4.11) should reduce to (3.7) in the limit of a static soliton, which implies that \(\lambda = \bar{\lambda}^{-}\).

To complete the analysis of the low energy degrees of freedom we should prove that the gauge freedom generated by the restricted superdiffeomorphisms \((t, \tau) \rightarrow (\tilde{t}, \tilde{\tau})\) satisfying (4.10) corresponds to just worldline reparametrizations plus \(\kappa\)-symmetry. Expanding the infinitesimal transformations as
\[
\delta t = a(t) - i\tau\beta(t),
\]
\[
\delta \tau = \alpha(t) + \tau b(t),
\]
we see that condition (4.10) implies \(\beta = -\alpha\) and \(b = -\dot{a}/2\). Being \(X^{\mu}(t, \tau)\) and \(\Theta(t, \tau)\) scalar superfields, these restricted superdiffeomorphisms induce the following transformations on the component fields \(x^{\mu}(t)\) and \(\theta(t)\)
\[
\delta x^{\mu} = a \dot{x}^{\mu} + i\bar{\theta}\gamma^{\mu}\alpha\lambda,
\]
\[
\delta \theta = a \dot{\theta} + \alpha \lambda.
\]

We thus identify \(a\) and \(\kappa \equiv \alpha\lambda\) respectively as the generators of infinitesimal worldline reparametrizations and \(\kappa\)-symmetry transformations.

This sort of geometrical interpretation for \(\kappa\)-symmetry has been previously observed by various authors \([1, 2, 3]\) and further pursued in \([4]\). The starting point in these papers is to consider a superembedding satisfying some sort of geometrical constraint — the so-called geometrodynamical constraint — from which one finds, as we have reproduced above, that invariance
under those restricted superdiffeomorphisms satisfying the constraint is equivalent to worldline reparametrizations plus \( \kappa \)-symmetry invariance. We have seen that our analysis on the effective description of the soliton dynamics naturally accommodates within this formulation. Here we regard the superembedding \((X^\mu(t, \tau), \Theta(t, \tau))\) as the locus of a topological defect and give a neat interpretation of the constraint (4.9) as the result of, first, selecting a definite type of superembeddings—the domain wall superembeddings, which locally in an appropriate frame resemble a static solution—and, second, reducing invariance under general super-reparametrizations to the restricted superdiffeomorphisms satisfying equation (4.10). In this sense, we have traced here the origin of the geometrodynamical constraints leading to \( \kappa \)-symmetry by studying the interplay between the effective model and the underlying field theory.

Before we go on to explicitly obtain the effective action it will be useful to derive a few geometric relations obeyed by domain wall superembeddings in the gauge \( U \cdot V = 0 \). Taking the covariant derivative of equation (4.9), \( U^\mu(t, \tau) = 0 \), we immediately get

\[
V^\mu = \nabla \bar{\Theta} \gamma^\mu \nabla \Theta, \tag{4.13}
\]

where we have introduced the scalar covariant derivative

\[
\nabla = \frac{1}{E^{1/2}} D,
\]

with \( E = \sqrt{W^2} \). Now, multiplying (4.13) by \( N^\mu \) and using the orthogonality condition \( N \cdot V = 0 \) we find that \( \nabla \Theta \) must be a chiral spinor. Since it has to tend, in addition, to the static solution (3.7) in the flat superembedding limit, we conclude that it has to satisfy the chirality constraint

\[
(1 - i \bar{\theta}) D \Theta = 0. \tag{4.14}
\]

It is now simple to derive the following generalization of Frenet equations:

\[
\nabla V(t, \tau) = \mathcal{K} N(t, \tau),
\]

\[
\nabla N(t, \tau) = \mathcal{K} V(t, \tau),
\]

where \( \mathcal{K}(t, \tau) \) plays the role of a (fermionic) curvature characterizing the superembedding. Owing to the relations (4.9) and (4.14) we can express all these objects in terms of \( \Theta(t, \tau) \) alone:

\[
E = -i D \bar{\Theta} \gamma_5 D \Theta,
\]

\[
\mathcal{K} = 2i \nabla^2 \bar{\Theta} \nabla \Theta,
\]

\[
V^\mu = \nabla \bar{\Theta} \gamma^\mu \nabla \Theta,
\]

\[
N^\mu = \nabla \bar{\Theta} \gamma^\mu \gamma_5 \nabla \Theta.
\]

From equation (4.14) we notice that \( D \Theta \) and \( \epsilon_+ \), the spinor parameter appearing in the superspace change of variables (4.3), have opposite chiralities. Since \( \epsilon_+ \) describes only one degree of freedom, it is possible to rewrite it with the help of \( \nabla \Theta \) as

\[
\epsilon_+ = \gamma_5 \nabla \Theta \sigma, \tag{4.15}
\]
where \( \sigma \) is a scalar SUSY-invariant fermionic parameter.

Taking all of this into account, the final form of the change of variables in superspace is

\[
x^{\mu} = X^{\mu}(t, \tau) + \rho N^{\mu}(t, \tau) + i \bar{\Theta}(t, \tau) \gamma^{\mu} \gamma^{5} \nabla \Theta(t, \tau) \sigma,
\]

\[
\theta = \Theta(t, \tau) + \gamma^{5} \nabla \Theta(t, \tau) \sigma,
\]

(4.16)

where \( \rho \) is inert under supersymmetry, this is a SUSY-invariant decomposition. It is also simple to show that \( \rho \) tends to the coordinate combination \((x - \bar{\theta}, \theta)\) as \((X^{\mu}(t, \tau), \Theta(t, \tau))\) approaches the flat superembedding \((3.7)\), implying that \( \varphi_{S}(\rho) \) tends to the flat solution \((3.6)\) with \( a = \alpha_{+} = 0 \).

We thus have a splitting of the action in powers of the perturbation superfield \( \zeta \)

\[
S = S_{0} + S_{1} + S_{2} + \ldots,
\]

(4.17)

where

\[
S_{0} = -i \int d^{2}xd^{2}\theta \left( \frac{1}{4} D_{\bar{x}} \bar{D} \varphi_{S} \varphi_{S} + U(\varphi_{S}) \right),
\]

\[
S_{1} = -i \int d^{2}xd^{2}\theta \left( \frac{1}{2} D_{\bar{x}} \bar{D} \zeta + U'(\varphi_{S}) \zeta \right),
\]

\[
S_{2} = -i \int d^{2}xd^{2}\theta \left( \frac{1}{4} D_{\bar{x}} \bar{D} \zeta + \frac{1}{2} U''(\varphi_{S}) \zeta^{2} \right).
\]

(4.18)

We should now perform the change of variables (4.16) in these integrals. There is, however, an important point that has to be taken into account when making a change of variables involving fermionic coordinates and which is relevant to our derivation of the effective action. It refers to the fact that the computation of a superspace integral can produce a wrong answer if, prior to integration, one performs a change of variables involving a generic mixing of bosonic and fermionic degrees of freedom. This is a known problem and it was already encountered by Ivanov and Kapustnikov \[3\] in their search for a non-covariant gauge-fixed version of the effective action. We can illustrate this point with the following simple example. Consider an integral of the form \( \int dxd\theta_{1}d\theta_{2} F(x - \theta_{1}\theta_{2}) \), where \( x \) is a single real variable and \( \theta_{1}, \theta_{2} \) are a pair of real Grassmann variables. If we perform the integration directly we get

\[
\int dxd\theta_{1}d\theta_{2} F(x - \theta_{1}\theta_{2}) = \int dx \ F'(x) = F(\infty) - F(-\infty).
\]
However, if we would make a change of variable \( x \rightarrow \rho = x - \theta_1 \theta_2 \), with a unit Jacobian, before the integration, we would get
\[
\int dp d\theta_1 d\theta_2 F(\rho) = 0,
\]
in contradiction with the correct result whenever \( F(\infty) \neq F(-\infty) \).

This sort of problems arise in integrals involving functions which do not fall to zero at infinity and the discrepancy between both calculations is always a boundary term. This is precisely the situation with our soliton configurations \( \varphi_S \), which tend to non-zero vacuum values at space infinity. Because of this, changing to the new variables \( (t, \tau, \rho, \sigma) \) and integrating over \( \rho \) and \( \sigma \) to get the effective action at the lowest (\( \zeta \)-independent) order, \( S_0 \), would not produce the right answer for this term. On the other hand, it is clear that this problem will not appear for the other (\( \zeta \)-dependent) contributions, \( S_1, S_2, \ldots \), because the \( \zeta \)-field boundary conditions force these integrals to tend to zero at large distances, thus preventing the presence of any boundary-related terms.

The way to solve this problem is to re-express, in the \( S_0 \) part, the coordinate \( \rho \) in terms of a new coordinate \( \tilde{\rho} \) which tends to the pure space coordinate, \( x \), as the superembedding approaches the flat configuration (3.7). The expression of \( \tilde{\rho} \) can be obtained by rewriting the bosonic part of the change of variables (4.16) as
\[
x^\mu = X^\mu(t, \tau) + \beta V^\mu(t, \tau) + \tilde{\rho} N^\mu(t, \tau),
\]
which implies the relation
\[
\tilde{\rho} = \rho + \Theta \gamma_5 \nabla \Theta \sigma.
\]
Expressing the \( S_0 \) part of the action (4.18) in terms of \( \tilde{\rho} \) we immediately see that the whole contribution comes from the potential term. Explicitly,
\[
S_0 = i \int dt d\tau d\tilde{\rho} d\sigma \ E^{1/2} \left[ 1 + \sigma \mathcal{K} + i(\tilde{\rho} - \Theta \gamma_5 \nabla \Theta \sigma) \nabla \mathcal{K} \right] U (\varphi_S (\tilde{\rho} - \Theta \gamma_5 \nabla \Theta \sigma)),
\]
where we have used the expression of the super-Jacobian for the change of variables (4.16) which is found to be
\[
J = -E^{1/2}(1 + \sigma \mathcal{K} + i\rho \nabla \mathcal{K}).
\]

After integrating over \( \sigma \) and \( \tilde{\rho} \) we find that all curvature-dependent terms pack up in a total time derivative, so that they do not contribute at this order. We get in this way the correct expression for the lowest-order contribution to the effective action. It is given by
\[
S_0 = M_S \int dt d\tau \ E^{1/2} \nabla \Theta \gamma_5 \Theta,
\]
(4.19)
where \( M_S \) is the soliton mass defined in (2.14). This action is manifestly invariant under supersymmetry and under restricted super-diffeomorphisms. It has been previously obtained by Gauntlett [8] and by Ivanov and Kapustnikov [9] as the simplest model that can be constructed in this geometrical setting for \( \kappa \)-symmetry.
One can resort to its component form in order to identify the nature of this action. Plugging the superfield expansions (4.11) and the constraints (4.12) into $S_0$ we can write everything in terms of $x^\mu(t)$ and $\theta(t)$. After integration over $\tau$ we find

$$S_0[x,\theta] = -M_S \int dt \left( \sqrt{\left(\dot{x} - i\bar{\theta}\gamma\dot{\theta}\right)^2 + \bar{\theta}\gamma\dot{\theta}} \right), \quad (4.20)$$

which is nothing but the free massive superparticle action, invariant under $\kappa$–symmetry by virtue of the Wess-Zumino term $\bar{\theta}\gamma\dot{\theta}$.

The terms $S_1$ and $S_2$ in the expansion of the action (4.17), which are respectively linear and quadratic in $\zeta$, can be found after some tedious but straightforward algebra. After integration over $\sigma$ we get the following expression for $S_1$

$$S_1[x,\theta,\eta] = \int dt d\tau E^{1/2} \int d\rho i\varphi_S'(\rho)\eta(t,\tau,\rho),$$

where the superfield $\eta$ comes from the $\sigma$ expansion of the superfield $\zeta$

$$\zeta(t,\tau,\rho,\sigma) = \eta(t,\tau,\rho) + \sigma \varepsilon(t,\tau,\rho).$$

The form of the linear term is giving us a hint of an appropriate definition for the location of the domain wall associated with a given $\Phi$ configuration. A possible choice would be the core of $\Phi$, satisfying $\Phi(X(t,\tau),\Theta(t,\tau)) = 0$ but a probably better one is to take the associated superembedding as the one satisfying

$$\int d\rho \ i\varphi_S'(\rho)\eta(t,\tau,\rho) = 0. \quad (4.21)$$

One can check that this condition provides a unique assignment of a superembedding to a given superfield configuration, $\Phi(x,\theta) \rightarrow (X(t,\tau),\Theta(t,\tau))$. Moreover, this condition eliminates the zero modes from $\eta$, since they are already described by the superembedding. It is also manifestly invariant under restricted superdiffeomorphisms. Thus, it preserves worldline and $\kappa$–symmetry invariance. It can be shown, in addition, that this choice merges with the core definition as the mass $M_S$ of the soliton tends to infinity.

Condition (4.21) should be enforced in the effective action before attempting to solve for the massive modes and obtain the effective action for the zero modes. We have already discussed this issue for the simpler bosonic case and the same reasoning applies here as well. A thorough discussion of this technical point can be found in reference [12]. The bottom line is that we may absorb the linear part of the action $S$, i.e. the whole of $S_1$, in a Lagrange multiplier term enforcing the condition (4.21).

Taking all of this into account and including also the contribution from $S_2$ we find the following form of the action to second order in the perturbations

$$S = -M_S \int dt \left( \sqrt{\left(\dot{x} - i\bar{\theta}\gamma\dot{\theta}\right)^2 + \bar{\theta}\gamma\dot{\theta}} \right) + \int dt d\tau \Lambda(t,\tau) \int d\rho i\varphi_S'(\rho)\eta(t,\tau,\rho)$$

$$+ \int dt d\tau E^{1/2} \int d\rho \left[ -\frac{1}{2} \nabla\eta\nabla^2\eta + (1 + i\rho\nabla K) \left( -\frac{1}{2} \varepsilon\nabla\varepsilon - i\varepsilon\eta' + iU''(\varphi_S)\eta \right) - K \left( \frac{1}{2} \varepsilon\nabla\eta - \frac{i}{2} U''(\varphi_S)\eta^2 - \frac{i\rho}{2(1 + i\nabla K\rho)}(\nabla^2\eta)^2 + \frac{i\rho\varepsilon\nabla^2\varepsilon}{2} \right) \right],$$

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where $\Lambda(t, \tau)$ is a Lagrange multiplier superfield.

Just as in the bosonic case, we can see that $\Phi(x, \theta) = \varphi_S(\rho)$, that is $\zeta = 0$, is already a solution of the equations of motion for the massive modes, leaving out the superparticle action (4.20) as the only contribution to the low energy effective action. Non-trivial boundary conditions, however, are expected to generate other (non-trivial) solutions for $\zeta$, giving rise to higher-order $\kappa$-invariant corrections to the basic superparticle term.

5 Conclusions and outlook

We have extended in this paper the effective action method to the domain of $2d$ supersymmetric scalar field theories. In this way, we have obtained a SUSY- and $\kappa$-invariant expansion for the low energy action that can be used to derive higher-order corrections to the basic superparticle action for topological defects. Beyond the interest of such an expansion, we would like to draw attention to other potentials of this deductive approach. Thus, for example, in our method we are able to trace very closely the origin of $\kappa$-symmetry from the underlying field theory action. We have connected with the geometrical interpretation of this symmetry previously proposed by several authors. The geometrodynamical constraint, $U^\mu = 0$ in our model, which is at the root of this interpretation, has been understood in a very natural way from within our approach. Part of this constraint, $U \cdot N = 0$, serves to select the so-called domain wall superembeddings—those that locally resemble a flat domain wall. The rest of the constraint, $U \cdot V = 0$, acts as a gauge-fixing, reducing the general super-reparametrization invariance down to a subset giving rise to the $\kappa$-tensor calculus.

The above analysis raises our hope that a similar construction can be used for higher-dimensional supersymmetric theories, aiming also at a general $\kappa$-tensor calculus in arbitrary dimensions. Unfortunately, a straightforward generalization of these techniques to the closest higher-dimensional objects, that is, the $4d$ Nielsen-Olesen supervortex and the domain wall of the $4d$ Wess-Zumino model—respectively described by the Green-Schwarz superstring and the supermembrane action—is presently failing to produce the expected answers, due to further subtleties not present in the $2d$ model just analyzed.

Thus, for instance, when trying to describe the four-dimensional supermembrane it is simple to check that the appropriate generalization of equation (4.10), $D\tilde{t} + i\tilde{\tau}D\tilde{\tau} = 0$, does the job and provides the required $3d$ reparametrizations and $\kappa$-transformations. However, the geometrodynamical constraint alone—which comes out in an analogous way as in two dimensions—is not enough to constraint the field contents of the theory to the required one. This fact seems to indicate the existence of further geometrical constraints, yet unknown, in order to properly characterize $4d$ domain wall superembeddings.

In conclusion, although our method provides a fairly complete picture of the geometry of the $2d$ domain wall superembeddings and of its use in deriving low energy effective actions for SUSY topological defects, it will still require further investigation before being generalized to higher dimensions. As a byproduct of this extension, one should expect a better understanding of the geometry of higher-dimensional superembeddings and a general formulation of the long-standing problem of $\kappa$-tensorial calculus. Work in the directions just sketched is already in progress.
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References

[1] P.A.M. Dirac, Proc. Roy. Soc. A268 (1962) 57.
[2] P.K. Townsend, Phys. Lett. 202B (1988) 53.
[3] A. Achúcarro, J.M. Evans, P.K. Townsend and D.L. Wiltshire, Phys. Lett. 198B (1987) 411.
[4] D. Förster, Nucl. Phys. B81 (1974) 84.
[5] H.B. Nielsen and P. Olesen, Nucl. Phys. B61 (1973) 45.
[6] E.A. Ivanov and A.A. Kapustnikov, Phys. Lett. 252B (1990) 212.
[7] D.P. Sorokin, V.I. Tkach and D.V. Volkov, Mod. Phys. Lett. A4 (1989) 901.
[8] J.P. Gauntlett, Phys. Lett. 272B (1991) 25.
[9] E.A. Ivanov and A.A. Kapustnikov, Phys. Lett. 267B (1991) 175.
[10] I.A. Bandos, D. Sorokin, M. Tonin, P. Pasti and D.V. Volkov, Nucl. Phys. B446 (1995) 79.
[11] R. Rajaraman, Solitons and Instantons, North-Holland Publishing Company, 1982.
[12] J. París and J. Roca, ‘A Remark on the Effective Description of Topological Defects’, preprint UB-ECM-PF 97/06 [hep-th/9705085]. To be published in Nucl. Phys. B (1998).
[13] B. Carter and R. Gregory, Phys. Rev. D51 (1995) 5839.
[14] P. diVecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93; J. Hruby, Nucl. Phys. B131 (1977) 275.