Hurwitz Algebras and the Octonion Algebra

Čestmir Burdik\textsuperscript{1,2,†}, Sultan Catto\textsuperscript{3,4,††},

\textsuperscript{1}Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Prague, Trojanova 13, CZ-120 00 Czech Republic
\textsuperscript{2}Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia
\textsuperscript{3}Physics Department, The Graduate School, City University of New York, New York, NY 10016-4309
\textsuperscript{4}Theoretical Physics Group, Rockefeller University, 1230 York Avenue, New York, NY 10021-6399

\textsuperscript{†}Grant No. SG15/215/OHK4/3T/14 Czech Technical University in Prague
\textsuperscript{††}Work supported in DOE contracts No. DE-AC-0276 ER 03074 and 03075; and NSF Grant No. DMS-8917754.

Abstract.

We explore some consequences of a theory of internal symmetries for elementary particles constructed on exceptional quantum mechanical spaces based on Jordan algebra formulation that admit exceptional groups as gauge groups.

1. Introduction

Soon after the discovery of Quantum Mechanics which brought miraculous light and established law and order in the hitherto mysterious world of atomic physics, physicists were confronted with the puzzling phenomena of the nuclear frontier. This happened in the early thirties with the emergence of the neutron, the riddle of beta decay and Heisenberg’s concept of isotopic spin to describe the symmetrical proton-neutron doublet. Emboldened with the striking success of radical ideas which had replaced classical mechanics by quantum mechanics, some of the founding fathers of Quantum Theory were already looking beyond its boundaries to an even more revolutionary form of mechanics, hoping that this new structure would be more appropriate to the description of the nuclear world. Now to go from classical to Quantum Mechanics we represent observables by hermitian matrices (which can be infinite) which obey an associative but non-commutative algebra. The elements of these matrices are in general complex numbers. Thus we drop both the real numbers and the commutativity of the algebra of observables in the transition from classical to quantum theory. What further algebraic properties can we change if we want to construct an “ultra mechanics”? Clearly we can drop associativity and replace complex numbers by the two remaining normed algebras of quaternions and octonions.

Now, in a reformulation of Quantum Mechanics, Jordan introduced an algebra of observables in which the product (anticommutator) leads again to an observable. This is the Jordan product, which can be infinite) which obey an associative but non-commutative algebra. The elements of these matrices are in general complex numbers. Thus we drop both the real numbers and the commutativity of the algebra of observables in the transition from classical to quantum theory. What further algebraic properties can we change if we want to construct an “ultra mechanics”? Clearly we can drop associativity and replace complex numbers by the two remaining normed algebras of quaternions and octonions.

Now, in a reformulation of Quantum Mechanics, Jordan introduced an algebra of observables in which the product (anticommutator) leads again to an observable. This is the Jordan product, which is symmetrical but not associative. In this formulation states are not represented by column matrices in Hilbert space but by projection operators which are again Hermitian matrices but are also idempotent.
Thus we can trade associativity for commutativity and still remain within the Quantum Mechanical framework.

In 1934, Jordan, von Neumann and Wigner\(^2\) put the algebra of observables on an axiomatic basis and found solutions which are not complex matrices. In fact they are \(3 \times 3\) octonionic hermitian matrices. Hence these observables correspond to a new mechanics with finite degrees of freedom. Such observables were called exceptional observables by the authors who hoped they would have some relevance to nuclear physics.

Thanks to a wealth of new experimental discoveries Nuclear Theory developed in the late thirties along the lines of conventional quantum mechanics. The key ideas were the Fermi theory of the beta decay using Pauli’s neutrinos and Yukawa’s idea of new bosons (mesons) mediating strong interactions. Hence the exceptional observables associated with exceptional Jordan algebras dropped out of physics to be captured, studied and developed further by mathematicians.

Meanwhile the theory of Lie groups as formulated by Cartan was taking new shapes. It was realized that the three infinite series of semi simple classical groups, namely the orthogonal groups (\(B_r\) and \(C_r\)), the unitary groups (\(A_r\)) and the symplectic groups (\(C_r\)) were respectively associated with real, complex and quaternionic numbers. In fact they arise as automorphism groups of Jordan algebras over these three Hurwitz algebras. On the other hand, it was already known to Cartan that the exceptional group \(G_2\) is the automorphism group of the non-associative octonion algebra. We had to wait until 1949 before Chevalley and Schafer\(^2\) realized that the exceptional group \(F_4\) is the automorphism group of the Jordan exceptional algebra of \(3 \times 3\) hermitian octonionic matrices. The same year Jordan\(^3\) showed that the homogeneous space \(F_4/\text{SO}(9)\) coincides with the projective Moufang plane where a point is represented by a pair of octonions.

These discoveries initiated a flurry of activity by Borel, Rozenfeld, Freudenthal, Springer and others, culminating in the work of Tits\(^4\) who gave a unified treatment of exceptional groups and some of their subgroups that fit in Freudenthal’s magic square. There followed a unification of the new projective geometries which generalize the Moufang plane and have non-Desarguesian properties. Since there is a correspondence between projective geometries and ray representations of physical states in Hilbert spaces, (Weyl, von Neumann, Varadarajan) the new geometries associated with exceptional groups provided new exceptional Quantum Mechanical spaces extending the Jordan-von Neumann-Wigner (JNW) construction and combining exceptional with ordinary observables in well determined patterns.

During the decades (from the fifties to the seventies) in which the new geometries were developed, a parallel explosion of modern particle physics took place. Besides the electric charge and isotopic spin of the pre-war years, the space of internal quantum numbers expanded to include strangeness, which combined with isospin within an \(SU(3)\) group, and charm which extended \(SU(3)\) to \(SU(4)\). These groups of the strong interactions had to be broken spontaneously. A field theory of strong interactions based on elementary quarks necessitated yet a new and a fundamental exact \(SU(3)\) group, namely the color group \(SU(3)^c\) (Greenberg, Nambu, Gell Mann)\(^5\). On the other hand a unification of weak and electromagnetic interactions had to be based on \(SU(2) \times U(1)\) (the Weinberg-Salam group) which acted differently on left handed fermions (leptons and quarks), and right handed fermions. Thus a unified theory of fundamental interactions (excluding gravitation) had to be reformulated in a large but finite charge space in which the invariance group would have at least rank five. A local, spontaneously broken gauge field theory based on such a group would be a good candidate for a fundamental theory of elementary particles.

At this point we are naturally led to the speculative identification of the phenomenological finite space of quantum numbers with the exceptional quantum mechanical spaces associated with octonions. The simplest is the space of JNW exceptional observables with the associated invariance group \(F_4\), and the others are the geometrical spaces related to the groups \(E_6\), \(E_7\) and \(E_8\). If this were so we would not only get a natural explanation of the color group \(SU^c(3)\), but we would also be in a position to predict the nature of the "flavor groups" like \(SU(3)\) of Gell-Mann and Neeman. Finally the new non-Desarguesian
geometries would be realized in nature as the Hilbert spaces of generalized charges. So far this view has not been refuted by experiment. Hence we can take advantage of this possibility and explore the consequences of a theory of internal symmetries for elementary particles based on exceptional quantum mechanical spaces that admit exceptional groups as gauge groups.

Before this view was proposed, exceptional groups have made only sporadic appearances on the stage of physics. Racah introduced the group $G_2$ in connection with the level degeneracies of rare earth atoms. $G_2$ has also been proposed as a possibility for the symmetry of strong interactions before the supremacy of $SU(3)$ was established. Gamba pointed out the $SU(3)$ structure of the $3 \times 3$ Jordan algebras without mentioning the groups $F_4$ and $E_6$ with which they are associated.

2. Speculation concerning the origin of internal symmetries of quarks and leptons

There are some unique exceptional geometries which are connected with octonions and correspond to a finite number of degrees of freedom that cannot be extended. This is the new Hilbert space that we want tentatively to identify with the Hilbert space of internal symmetries carrying color and flavor quantum numbers.

One of the fundamental questions in particle physics is the understanding of the quark substructure of hadrons. We would like to understand the emergence of the hierarchy of interactions, the mass spectrum of quarks and leptons and the wider mass spread of the fundamental gauge bosons through the spontaneous breaking of a local symmetry based on a unifying group or supergroup associated with a gauge field theory.

The concept of a field theory that unifies both the various kinds of interactions and the great variety of symmetries of fundamental processes is philosophically and aesthetically more satisfying than the old division of labor, concepts and methods in the main areas of physics; geometry for gravitation, field theory for electromagnetism, effective field theory for weak interactions and S-matrix methods (based on analyticity and unitarity) for strong interactions. The only trouble with the new natural philosophy is that it deals largely with an unobservable substructure (quarks, color quantum numbers, superheavy gauge bosons, essentially undetectable gravitinos, etc.) which makes physicists rather uncomfortable and reminds them of their imponderable past (the caloric fluid, the aether) ridiculed by later generations.

The confrontation of the new theory with experiment is by its very nature indirect, in contrast to the foundations of the twentieth century physics (special relativity and quantum mechanics) that had their roots in the principle of dealing only with the observables. We are now back to a view more akin to the one held by Einstein in his old age, spinning out geometrical structures far removed from space-time geometry based on equivalence principle, trying to unify gravity with other fields that seemed to have non-geometrical properties. It is ironical that Heisenberg also in his old age rejected his positivist S-matrix approach in favor of underlying (non-linear but non-renormalizable) field theory for the unification of electromagnetism and strong interactions.

In defense of the new philosophy it might be said that other past substructures that were regarded imponderable or too abstract by contemporary physicists did prevail in the end and were accepted as physical reality. Atoms, fields (both electromagnetic and gravitational), antimatter, the neutrino, the particle-wave dual nature of matter and radiation, Hilbert space, the complex wave function, etc., might provide a few examples.

In this note we shall move to a still more abstract plane, in fact the octonionic planes that may provide the geometrical foundation for the existence of internal symmetries like color and flavor. These octonionic geometries allow us to construct new finite Hilbert spaces with unique properties. The non-Desargues’ian geometric property makes them non-embeddeble in higher spaces, hence essentially finite. It also leads to peculiarities in the superposition principle in the color sector of the Hilbert space. This theory of the charge space, if correct, suggests a new geometric picture for the substructure of the material world, that of an octonionic geometry attached at each point of Einstein’s Riemannian manifold for space time with local symmetry that leaves the properties of charge space invariant.

There are deep connections between the lattices of certain groups that arise in superstrings and
Let $\alpha$. Hurwitz algebras and the octonion algebra

The unit is $e_0 = (1, 0)$, so that $\alpha \bar{\alpha} = (a_1 \bar{a}_1 + a_2 \bar{a}_2) e_0$ (6)

and the inverse is $\alpha^{-1} = (\alpha \bar{\alpha})^{-1} \bar{\alpha}$, (7)

so that $\alpha^{-1} \alpha = \alpha \bar{\alpha}^{-1} = e_0$.

(a) Complex numbers:

Take $(0, 1) = i$

$\alpha = a_1 + ia_2 = a_1 (1, 0) + a_2 (0, 1),$ (8)

$\bar{\alpha} = a_1 - ia_2.$ (9)

Here the field $\mathbb{A}$ so that $a_1, a_2 \in \mathbb{A}$ is commutative and associative. So is the field $\mathbb{E} = (a_1, a_2)$.

(b) Quaternions:
Writing

\[(i, 0) = j_1, \quad (0, 1) = j_2, \quad (0, -i) = j_3\]  \hspace{1cm} (10)

so that

\[j_1j_2 = (i.0 - 1.0, -i.1 + 0.1) = (0, -i) = j_3,\]  \hspace{1cm} (11)

similarly

\[j_2j_1 = (0, 1)(i, 0) = (0, i) = -j_3.\]  \hspace{1cm} (12)

Quaternions are associative but not commutative.

\textbf{(c) Octonions}

Writing

\[(j_1, 0) = e_i, \quad (0, 1) = e_7, \quad (0, j_i) = e_{i+3}, \quad (i = 1, 2, 3)\]  \hspace{1cm} (13)

we get, for example

\[e_1e_7 = (j_1, 0)(0, 1) = (0, j_1, 1) = (0, -j_1) = -e_4,\]  \hspace{1cm} (14)

\[e_7e_1 = (0, 1)(j_1, 0) = (0, j_1) = e_4\]  \hspace{1cm} (15)

and

\[(j_1, 0)(j_2, 0) = (j_1j_2, 0) = j_3.\]  \hspace{1cm} (16)

Thus we can write

\[e_ae_b = -\delta_{ab} + f_{abc}e_c\]  \hspace{1cm} (17)

with the involution

\[\bar{e}_a = -e_a.\]  \hspace{1cm} (18)

The tensor \(f_{abc}\) takes values 1, 0, -1 and is antisymmetrical in all three indices \(a, b, c\), taking the value 1 for \((a, b, c) = (1, 2, 3)\) and its cyclic permutations with \(e_1, e_2, e_4, e_6, e_5, e_7\) arranged clockwise on a circle shown in our previous paper \(\cite{12}\).

These values correspond to the successive columns of the seven dimensional finite projective plane with the standard arrangement

\[
\begin{array}{ccccccc}
1 & 2 & 4 & 3 & 6 & 5 & 7 \\
2 & 4 & 3 & 6 & 5 & 7 & 1 \\
3 & 6 & 5 & 7 & 1 & 2 & 4
\end{array}
\]  \hspace{1cm} (19)

In geometrical language each column represents lines with 3 points on them so that we have seven lines (seven triangles on the circle) and 7 points. The same number always occurs in 3 different columns so that 3 lines pass through each point.

The corresponding algebra is non associative as the associator

\[[e_a, e_b, e_c] = 2\tilde{f}_{abcd}e_d\]  \hspace{1cm} (20)

where

\[\tilde{f}_{abcd} = \frac{1}{3!}e_{abcdklm}f_{klm}\]  \hspace{1cm} (21)

is the dual tensor for \(f_{abc}\) of the multiplication table, \(\epsilon\) being the completely anti symmetrical Levi-Civita symbol in 7 dimensions. Due to alternativity the octonions form a power associative algebra.

Defining the conjugates

\[\bar{e}_a = -e_a, \quad \bar{e}_0 = e_0\]  \hspace{1cm} (22)

the octonionic conjugate \(\bar{x}\) is written as

\[\bar{x} = x_0 - x_\alpha e_\alpha = x_i \bar{e}_i\]  \hspace{1cm} (23)
The scalar part of the octonion \((Sc(x))\) and its vector part \((Vec(x))\) are
\[
Sc(x) = \frac{1}{2}(x + \bar{x}) = x_0
\]
and
\[
Vec(x) = \frac{1}{2}(x - \bar{x}) = x_\alpha e_\alpha
\]
Conjugate of the product of two octonions \(x\) and \(y\) is
\[
(xy) = \bar{y}\bar{x}
\]
and their scalar product is defined by
\[
<x, y> = x_i y_i = \frac{1}{2} (x\bar{y} + y\bar{x}) = \frac{1}{2} (\bar{x}y + \bar{y}x)
\]
which, in terms of octonion units gives
\[
<e_i, e_j> = \frac{1}{2} (\bar{e_i}e_j + \bar{e_j}e_i) = \frac{1}{2} (e_i \bar{e_j} + e_j \bar{e_i}) = \delta_{ij}
\]
The norm \(N(x)\) of an octonion is
\[
N(x) = \bar{x} x = x \bar{x} = x_i x_i
\]
and is zero if \(x = 0\), and is always positive otherwise. For a nonzero octonion \(x\), its inverse is given by
\[
x^{-1} = \frac{\bar{x}}{N(x)}
\]
with
\[
(xy)^{-1} = y^{-1} x^{-1}
\]
The norm defined above satisfies
\[
N(xy) = N(x) N(y)
\]
In analogous way to the quaternionic case in \(\mathbb{R}^4\), we now make the following definitions of vectorial products of octonion units, namely two new antisymmetric tensors:
\[
e_{ij} = \frac{1}{2} (\bar{e_i}e_j + \bar{e_j}e_i)
\]
and
\[
e'_{ij} = \frac{1}{2} (e_i \bar{e_j} - e_j \bar{e_i})
\]
Componentwise they read as
\[
e_{\alpha\beta} = e'_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} e_\gamma
\]
and
\[
e_{0\alpha} = -e'_{0\alpha} = e_\alpha
\]
These octonionic tensors naturally enter into covariant formulation of various cross-products in \(\mathbb{R}^8\). We can now write
\[
\bar{e_i} e_j = \frac{1}{2} (\bar{e_i} e_j + \bar{e_j} e_i) + \frac{1}{2} (\bar{e_i} e_j - \bar{e_j} e_i) = \delta_{ij} + e_{ij}
\]
\[ e_i \bar{e}_j = \frac{1}{2} (e_i \bar{e}_j + e_j \bar{e}_i) + \frac{1}{2} (e_i \bar{e}_j - e_j \bar{e}_i) = \delta_{ij} + e'_{ij} \]  

(38)

Using the definition of commutator of two octonions

\[ [x, y] = 0 \]  

(39)

the product rule implies

\[ [e_\alpha, e_\beta] = 2\epsilon_{\alpha\beta\gamma}e_\gamma \]  

(40)

and

\[ [e_0, e_\alpha] = 0 \]  

(41)

so that

\[ [x, y] = 2x_\alpha y_\beta \epsilon_{\alpha\beta\gamma}e_\gamma \]  

(42)

We see the relation between the commutator and \( e_{ij} \) and \( e'_{ij} \) through the triality relation

\[ [e_i, e_j] + e_{ij} + e'_{ij} = 0 \]  

(43)

We now define the associator \([x, y, z]\) of three octonions by

\[ [x, y, z] = (xy)z - x(yz) \]  

(44)

which is completely antisymmetric:

\[ [x, y, z] = [y, z, x] = [z, x, y] \]  

(45)

and

\[ [x, y, z] = -[y, x, z] = -[x, z, y] = -[z, y, x] \]  

(46)

It is also purely vectorial since

\[ [x, y, z] = -[x, y, z] \]  

(47)

Since \( e_0 \) commutes and associates with other octonion units, only the purely vectorial parts of \( x, y, z \) contribute to the associator \([x, y, z]\).

We now define a completely antisymmetric 4-index object \( \psi_{\alpha\beta\mu\nu} \) related to the associator as

\[ [e_\alpha, e_\beta, e_\mu] = 2\psi_{\alpha\beta\mu\nu} e_\nu \]  

(48)

By means of the associator and the product rules, we find

\[ \psi_{\alpha\beta\mu\nu} = \frac{1}{2} (\delta_{\beta\mu}[\delta_{\alpha}\nu] + \epsilon_{\beta\gamma}[\mu\epsilon_{\alpha}\gamma\nu]) \]  

(49)

Explicit calculation of the values of \( \psi_{\alpha\beta\mu\nu} \) show it is dual to \( \epsilon_{\lambda\sigma\rho} \) in \( R^7 \). \( \psi_{\alpha\beta\mu\nu} \) take value 1 for the following combinations:

\[ (\alpha\beta\mu\nu) = (1346), (2635), (4567), (3751), (6172), (5214), (7423) \]  

(50)

Duality property between \( \epsilon_{\lambda\sigma\rho} \) and \( \psi_{\alpha\beta\mu\nu} \) in \( R^7 \) is best seen in the following construction:

\[
\begin{bmatrix}
2 & 4 & 3 & 6 & 5 & 7 & 1 \\
5 & 7 & 1 & 2 & 4 & 3 & 6 \\
7 & 1 & 2 & 4 & 3 & 6 & 5 \\
1 & 2 & 4 & 3 & 6 & 5 & 7 \\
3 & 6 & 5 & 7 & 1 & 2 & 4 \\
4 & 3 & 6 & 5 & 7 & 1 & 2 \\
6 & 5 & 7 & 1 & 2 & 4 & 3 \\
\end{bmatrix}
= \epsilon_{\lambda\sigma\rho}
\]

(51)

\[
\begin{bmatrix}
2 & 4 & 3 & 6 & 5 & 7 & 1 \\
5 & 7 & 1 & 2 & 4 & 3 & 6 \\
7 & 1 & 2 & 4 & 3 & 6 & 5 \\
1 & 2 & 4 & 3 & 6 & 5 & 7 \\
3 & 6 & 5 & 7 & 1 & 2 & 4 \\
4 & 3 & 6 & 5 & 7 & 1 & 2 \\
6 & 5 & 7 & 1 & 2 & 4 & 3 \\
\end{bmatrix}
= \psi_{\alpha\beta\mu\nu}
\]
This table is read as
\[ \frac{1}{2}[e_2, e_5] = e_7, \quad \frac{1}{2}[e_4, e_7] = e_1, \ldots \] (52)

etc., for the triads, and
\[ \frac{1}{2}[e_1, e_3, e_4] = -e_6, \quad \frac{1}{2}[e_2, e_6, e_3] = -e_5, \ldots \] (53)

etc., for the associators.

We see that there are seven associative planes and their transverse non-associative planes. While an associative plane is closed under commutation, a non-associative plane closes under the associator.

From the definition of the involution for octonions we get
\[ w = w_0 + e_\alpha w_\alpha \quad (\alpha = 1, \ldots, 7) \] (54)
\[ \bar{w} = w_0 - e_\alpha w_\alpha \] (55)
\[ |w|^2 = [N(w)]^2 = w\bar{w} = \bar{w}w = w_0^2 + w_1^2 + \ldots + w_7^2 \] (56)

which is positive definite if \( w_\alpha (\alpha = 0, \ldots, 7) \) are real.

The inverse octonion is
\[ w^{-1} = |w|^{-2} \bar{w}. \] (57)

It is also convenient to introduce the scalar part
\[ s(w) = \frac{1}{2}(w + \bar{w}) = \frac{1}{2}t(w) = w_0 \] (58)
where \( t(w) \) is the trace part. We have the norm property
\[ |ab| = |a||b|. \] (59)

Because of non associativity the octonion units can not be represented by matrices.

The following trilinear identities
\[ a(ab) = a^2b, \quad a(ba) = (ab)a, \quad ba^2 = (ba)a \] (60)

which follow from alternativity so that
\[ [a, a, b] = [a, b, a] = [b, a, a] = 0 \] (61)

These lead to
\[ a(\bar{a}x) = (a\bar{a})x, \quad a(x\bar{a}) = (ax)\bar{a}, \quad (xa)\bar{a} = x(a\bar{a}) \] (62)

so that \( axa \) and \( axa^{-1} \) have an unambiguous meaning.

The following quadrilinear Moufang identities also hold
\[ (xax)b = x[a(xb)] \] (63)
\[ b(xax) = [(bx)a]x. \] (64)
\[ (xa)(bx) = x(ab)x \] (65)

We also have
\[ [\bar{x}a, \bar{x}, \bar{b}] = -x[x, a, b] = x[x, \bar{a}, \bar{b}]. \] (66)

If we continue the Cayley-Dickson process we use alternativity, hence power associativity and the norm property.
We shall also use the split octonionic units by introducing an imaginary unit $i$ that commute with $e_a$.

We define

\[ u_0 = \frac{1}{2}(1 + ie_7), \quad u_j = \frac{1}{2}(e_j + ie_{j+3}) \quad (j = 1, 2, 3) \]  

\[ u_0^* = \frac{1}{2}(1 - ie_7), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}). \]  

A real octonion can be written as

\[ w = \psi_0^* u_0 + \psi_j^* u_j + \psi_0 u_0^* + \psi_j u_j^* = \psi^* + \psi \]  

where

\[ \psi = \psi_0 u_0^* + \psi_j u_j^* \]  

and

\[ \psi_0 = w_0 + iw_7, \quad \psi_j = w_j + iw_{j+3} \]  

A complex octonion has the form

\[ \xi = \chi + \zeta^* = \chi_0 u_0^* + \chi_j u_j^* + \zeta_0 u_0 + \zeta_j u_j. \]  

Complex octonions, unlike real octonions can have zero norm so that they do not form a division algebra. Nevertheless complex octonions are important in constructing new exceptional quantum mechanical spaces.

The split units obey the following multiplication table:

\[ u_i u_j = \epsilon_{ijk} u_k^* \quad (i, j = 1, 2, 3) \]  

\[ u_i u_j^* = -\delta_{ij} u_0 \]  

\[ u_0 u_i = u_i u_0^* = u_i \]  

\[ u_i u_0 = u_0 u_i^* = 0 \]  

\[ u_0 u_i^* = 0 \]  

\[ u_0^2 = u_0 \]  

and the complex conjugate equations obtained by changing the sign of $i$. We can anticipate from this table the $SU(3)$ structure of the split units.

Now, we put together the compactified multiplication table for the split octonion units:

|       | $u_0$ | $u_0^*$ | $u_k$  | $u_k^*$  |
|-------|-------|---------|--------|----------|
| $u_0$ | $u_0$ | 0       | $u_k$  | 0        |
| $u_0^*$ | 0    | $u_0$  |       | $u_k^*$  |
| $u_j$ | 0     | $u_j$  |       | $\epsilon_{jki} u_k^*$ |
| $u_j^*$ | 0    | 0      | $-\delta_{jk} u_0$ | $\epsilon_{jki} u_i$ |

It is worth noting that $u_i$ and $u_j^*$ behave like fermionic annihilation and creation operators:

\[ \{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ij} \]
This fermionic Heisenberg algebra shows the three split units $u_i$ to be Grassmann numbers. Being non-associative, these split units give rise to an exceptional Grassmann algebra.

Operators $u_i$, unlike ordinary fermion operators, are not associative. We also have $\frac{1}{2}[u_i, u_j] = \epsilon_{ijk} u_k^*$. The Jacobi identity does not hold since $[u_i, [u_j, u_k]] = -i\epsilon_{ijk} \neq 0$, where $\epsilon_{ijk}$ anticommute with $u_i$ and $u_i^*$. We note that, like the imaginary units $e_\alpha$, the split units cannot be represented by matrices. Unlike the octonion algebra, the split octonion algebra contains zero divisors and is therefore not a division algebra.

The associators of split octonion units are given below:

\[
[u_i, u_j, u_k] = \epsilon_{ijk} (u_0^* - u_0) \quad (80)
\]
\[
[u_i^*, u_j^*, u_k^*] = \epsilon_{ijk} (u_0 - u_0^*) \quad (81)
\]
\[
[u_i, u_j, u_0] = -\epsilon_{ijk} u_k^* \quad (82)
\]
\[
[u_i, u_j, u_0^*] = \epsilon_{ijk} u_k \quad (83)
\]
\[
[u_i, u_j^*, u_k^*] = \delta_{jk} u_i - \delta_{ik} u_j \quad (84)
\]
\[
[u_i, u_j^*, u_k^*] = \delta_{kj} u_i^* - \delta_{ij} u_k^* \quad (85)
\]
\[
[u_i^*, u_j^*, u_0] = \epsilon_{ijk} u_k \quad (86)
\]
\[
[u_i^*, u_j^*, u_0^*] = -\epsilon_{ijk} u_k \quad (87)
\]
\[
[u_i, u_j^*, u_0] = 0 \quad (88)
\]
\[
[u_i, u_j^*, u_0^*] = 0 \quad (89)
\]

Defining hermitian conjugation as both complex and octonionic conjugation we write

\[
u_i^\dagger = \bar{u}_i^* = -u_i^*, \quad u_i^{\dagger 0} = \bar{u}_0^* = u_0
\]

We also make new definitions:

\[
u_{\mu\nu} = \frac{1}{2}(u_{\mu}^\dagger u_{\nu} - u_{\nu}^\dagger u_{\mu}) \quad (91)
\]

and

\[
u'_{\mu\nu} = \frac{1}{2}(u_{\mu} u_{\nu}^\dagger - u_{\nu} u_{\mu}^\dagger) \quad (92)
\]

and see that the left handed product

\[
u'_{\mu\nu} = 0 \quad (93)
\]

and, in the component form the right handed product $\nu'_{\mu\nu}$ survives only as $u_{0i} = \frac{1}{2}e_i$, i.e.

\[
u_{ij} = 0 \quad (94)
\]

with

\[
u_{0i} = \frac{1}{2}(u_i + u_i^*) = \frac{1}{2}e_i \quad (95)
\]

thereby reducing the octonions to purely vectorial quaternions.

Now, through $w = \Re \psi$ we can associate with each real octonion a special complex octonion $\psi$ so that

\[
\psi = \begin{pmatrix}
\psi_0 \\
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix} = \Psi^T u^*
\]

(96)
where

\[ u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \]

(97)

and \( T \) denotes transposition. The multiplication table gives

\[ uu^T = \begin{pmatrix} u_0^* & -u_1^* & -u_2^* & -u_3^* \end{pmatrix} \begin{pmatrix} 0 & -u_1 & -u_2 & -u_3 \\ u_1 & 0 & -u_3^* & u_2^* \\ u_2 & u_3^* & 0 & -u_1^* \\ u_3 & -u_2^* & u_1^* & 0 \end{pmatrix} \]

(98)

\[ uu^\dagger = u\bar{u}^T = \begin{pmatrix} u_0^* \\ u_1^* \\ u_2^* \\ u_3^* \end{pmatrix} \begin{pmatrix} u_0 & 0 & 0 & 0 \\ 0 & u_0 & 0 & 0 \\ 0 & 0 & u_0 & 0 \\ 0 & 0 & 0 & u_0 \end{pmatrix} \]

(99)

\[ \psi^2 = (\psi_0 u_0^* + \psi_j u_j^*)(\psi_0 u_0^* + \psi_k u_k^*) \]

(100)

\[ = \psi_0^2 u_0^* + \psi_j \psi_k \epsilon_{jkl} u_l + \psi_0 \psi_k u_k^* \]

(101)

\[ = \psi_0^2 u_0^* + \psi_0 \psi_k u_k = \psi_0 (\psi_0 u_0^* + \psi_k u_k^*) \]

(102)

so that

\[ \psi^2 = \psi_0 \psi \]

(103)

or

\[ \frac{\psi^2}{\psi_0^2} = \frac{\psi}{\psi_0}. \]

(104)

Thus \( \frac{\psi}{\psi_0} \) is idempotent.

Hence a real octonion is characterized by a complex multiple of an idempotent complex octonion.

4. Construction of octonionic Hilbert spaces
(a) Jordan algebra formulation of Quantum Mechanics

Consider finite \( n \)-dimensional quantum mechanical spaces.

In the usual formulation of quantum mechanics an observable is represented by a \( n \times n \) hermitian matrix over complex numbers

\[ \Omega = \Omega^\dagger = \Omega^{*T}. \]

(105)

A state is a ket vector with \( n \) components \( |\alpha> = a \) so that

\[ \alpha|\alpha> = a^\dagger a = 1. \]

(106)

Another state \( b \) is orthogonal to it if

\[ <\beta|\alpha> = b^\dagger a = 0 \]

(107)

A complete set of states will obey

\[ <\alpha_i|\alpha_j> = \delta_{ij} \]

(108)

Thus for the state \( \alpha \) we can write \( \alpha = \alpha_i|\alpha_i> \). We have

\[ \Omega = \sum_{i=1}^n w_i |\alpha_i> <\alpha_i| \]

(109)
so that
\[ \Omega |\alpha_i > = w_i |\alpha_i > \]  
(110)
and \( \alpha_i \) are the real eigenvalues of \( w_i \). They are the quantities that are measured for the observable \( \Omega \).

The hermitian matrices
\[ P_i = |\alpha_i > < \alpha_i | = a_i a_i^\dagger \]  
(111)
are in one-to-one correspondence with the kets \( |\alpha_i > \).

The \( P_i \) are idempotent hermitian operators since
\[ P_i^2 = P_i, \]
\[ Tr P_i = 1, \]  
(112)
and
\[ \Omega = \sum w_i P_i. \]  
(114)

We define
\[ (\Omega_1, \Omega_2) = Tr(\omega_1 \Omega_2). \]  
(115)

The transition amplitude between states \( \alpha \) and \( \beta \) is
\[ <\alpha|\beta > = (\alpha_i^* \beta_i)(\alpha_j^* \beta_j^*) = <\alpha|\beta > <\beta|\alpha > \]  
(116)
and the transition probability is
\[ \Pi_{\alpha\beta} = |<\alpha|\beta >|^2 = (\alpha_i^* \beta_i)(\alpha_j^* \beta_j^*) = <\alpha|\beta > <\beta|\alpha > \]  
(117)

We consider the density matrices
\[ P_\alpha = |\alpha > <\alpha |, \quad P_\beta = |\beta > <\beta | \]  
(118)
so that
\[ P_\alpha P_\beta = |\alpha > <\alpha | <\alpha | <\beta | = <\alpha|\beta > <\alpha| <\beta | \]  
(119)
\[ (P_\alpha, P_\beta) = Tr P_\alpha P_\beta = <\alpha|\beta > <\alpha| <\beta | = <\alpha|\beta > <\beta|\alpha > \]  
(120)

Thus
\[ (P_\alpha, P_\beta) = \Pi_{\alpha\beta} = Tr P_\alpha P_\beta \]  
(121)

where
\[ P_\alpha P_\beta = \frac{1}{2}(P_\alpha P_\beta + P_\beta P_\alpha) \]  
(122)
is the Jordan product of \( P_\alpha \) and \( P_\beta \) under which hermitian matrices form a closed algebra.

The observable \( \Omega = \sum w_i P_i \) is a density matrix in general.

The matrix elements of the observable \( \Omega \) are given by
\[ x <\alpha_i|\Omega|\alpha_j > = \Omega_{ij} \]  
(123)
and
\[ |\Omega_{ij}|^2 = Tr (\Omega P_i \Omega P_j) = Tr (\Omega P_i^\dagger \Omega P_j^\dagger).\Omega_{ij} \]  
(124)
The transition probability \( |<\alpha|\beta >|^2 \) has a simple geometrical meaning. Let
\[ d^2_{\alpha\beta} = \frac{1}{2} Tr (P_\alpha - P_\beta)^2 \]  
(125)
where \( P_\alpha \) and \( P_\beta \) are idempotents. Then
\[
d^2_{\alpha\beta} = \frac{1}{2} \left( 1 + 1 - 2Tr P_\alpha P_\beta \right) = 1 - \Pi_{\alpha\beta}
\] (126)
or
\[
\Pi_{\alpha\beta} = 1 - d^2_{\alpha\beta}
\] (127)

Now in a projective geometry the normalized idempotent \( P_\alpha \) represents a point with \( n \) homogeneous and \((n - 1)\) inhomogeneous coordinates.

The state \( \lambda|\alpha> \) is equivalent to the state \(|\alpha> \) in a ray representation. The corresponding projection operator is \( |\lambda|^2 P_\alpha \) which is equivalent to \( P_\alpha \) as we can take \( |\lambda| = 1 \) by normalization. This corresponds to homogeneous coordinates \( \alpha_i \) and \( \lambda \alpha_i \) representing the same point \( P_\alpha \).

Thus a point is in a one-to-one correspondence with a state or an idempotent hermitian matrix (projection operator).

The transition probabilities are invariant under the unitary transformation \((U \in U(n))\).
\[
|\alpha> \rightarrow U|\alpha>, \quad |\beta> \rightarrow U|\beta>
\] (128)
or
\[
P_\alpha \rightarrow UP_\alpha U^{-1}, \quad P_\beta \rightarrow UP_\beta U^{-1} \quad (U \in SU(n))
\] (129)
The invariant distance \( d_{\alpha\beta} \) between \( P_\alpha \) and \( P_\beta \) is defined as
\[
Tr P_\alpha P_\beta = d_{\alpha\beta}^2.
\] (130)

When \( \alpha = \beta \) their relative distance vanishes while the transition probability becomes unity. Thus
\[
\Pi_{\alpha\beta} = 1 - d^2_{\alpha\beta}
\] (131)
is the complement of the square of the relative distance.

A point is left invariant by the subgroup of \( U(n) \) which leaves the idempotent invariant. Hence it is \( U(n - 1) \times U(1) \). This is the stability group of \( U(n) \).

Note that
\[
|\Omega_{ij}|^2 = Tr (\Omega P_i \Omega P_j)
\] (132)
can also be expressed by means of the Jordan product alone. The Jordan product is commutative and non associative. It is not alternative but power associative. We have
\[
(A, B, C) = (A.B).C - A.(B.C) = -(C, B, A)
\] (133)
so that
\[
(A, B, A) = 0
\] (134)

Writing
\[
(A, B, A^2) = (AB + BA)A^2 + A^2(AB + BA) - A(BA^2 + A^2B) - (BA^2 + A^2B)A
\]
\[
= ABA^2 + BA^3 + A^3B + A^2BA - ABA^2 - A^3B - BA^3 - A^2BA = 0
\] (136)
and

\[(A, B, A^{-1}) = (A.B).A^{-1} - A.(B.A^{-1})\]

\[= \frac{1}{4}[(AB + BA)A^{-1} + A^{-1}(AB + BA)\]
\[\quad - A(BA^{-1} + A^{-1}B) - (BA^{-1} + A^{-1}B)A]\]
\[= \frac{1}{4}[ABA^{-1} + B + B + A^{-1}BA\]
\[\quad - ABA^{-1} - B - B - A^{-1}BA = 0\]

(137)

We have shown

\[(A, B, A^2) = 0\]

(138)

and for a special Jordan algebra of matrices

\[(A, B, A^{-1}) = 0\]

(139)

as well.

We also have

\[(P, A, A) = \frac{1}{4}[(PA + A.P)A + A(PA + AP) - 2PA^2 - 2A^2P]\]

\[= \frac{1}{4}[PA^2 + APA + APA + A^2P - 2PA^2 - 2A^2P]\]

\[= \frac{1}{4}(2APA - A^2P - PA^2)\]

\[= \frac{1}{2}APA - \frac{1}{4}(A^2P + PA^2)\]

\[= \frac{1}{2}APA - \frac{1}{2}A^2.P\]

(140)

Hence

\[\frac{1}{2}APA = \frac{1}{2}A^2.P + (P, A, A)\]

(141)

can be expressed by means of the Jordan product

\[APA = A^2P + 2(P, A, A)\]

\[= A^2.P + 2(P.A).A - 2P.A^2\]

(142)

simply

\[APA = 2(P.A).A - A^2.P,\]

(143)

or

\[\Omega^\frac{1}{2} = 2(P_1 \Omega^\frac{1}{2} \Omega^\frac{1}{2} - \Omega.P_i\]

(144)

Inverse matrix \(A^{-1}\) satisfies

\[A^{-1}.A = I, \quad A^{-1}.A^2 = A\]

(145)

where \(I\) is the identity matrix.

We now express the unitary transformation in infinitesimal form as

\[U = 1 + iA, \quad P'_\alpha = UP_\alpha U^{-1} = (1 + iA)P_\alpha(1 - iA) = P_\alpha + i[A, P_\alpha]\]

(146)
Let

\[-4iA = [H_1, H_2]\]  \tag{147}\]

where $H_1$ and $H_2$ are hermitian. Then

\[P'_\alpha = P_\alpha - \frac{1}{4}[ [H_1, H_2], P_\alpha]. \tag{148}\]

Now

\[-\frac{1}{4}[ [H_1, H_2], P_\alpha] = (H_1, P_\alpha)H_2 - H_1, (P_\alpha, H_2) = (H_1, P_\alpha, H_2) \tag{149}\]

for matrices, so that

\[P'_\alpha = P_\alpha + (H_1, P_\alpha, H_2) \tag{150}\]

By iteration we can express the finite unitary transformation as

\[TP_\alpha = E_{H_1, H_2}P_\alpha = P_\alpha + (H_1, P_\alpha, H_2) + \frac{1}{2!}(H_1, (H_1, P_\alpha, H_2)H_2 + \cdots \tag{151}\]

which is an expression involving multiple associators. For the usual matrices we have

\[TP_\alpha = e^{-\frac{1}{2}[[h_1, H_2], P_\alpha]}e^{\frac{1}{2}[[h_1, H_2]} \tag{152}\]

If $|\alpha_i>\$ is an orthogonal set $U|\alpha_i> = |\alpha'_i>$ is another.
If $P_i$ form a complete set $\sum P_i$, then $UP_iU^{-1} = P'_i$ are also complete.

Thus, states, transition probabilities, unitary transformations in the algebraic formulation of quantum mechanics involves only hermitian matrices and their Jordan product.

(b) Exceptional observables:

We start from the axioms of Jordan algebras.

\[A.B = B.A \tag{153}\]

gives

\[[A, B, C] = [C, B, A] \tag{154}\]

giving $[A, B, A] = 0$. We do require

\[[A, B, A^2] = 0 \tag{155}\]

proven above.

This system is also satisfied by $3 \times 3$ octonionic hermitian matrices (Jordan, von Neumann, Wigner)

\[J = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix} \tag{156}\]

which gives rise to a power associative algebra. Entries $a, b, c$ are real octonions.

Now, these matrices are closed under the Jordan product. What fails is the identity

\[-\frac{1}{4}[[H_1, H_2], J] = (H_1, J, H_2) = (H_1, J).H_2 - H_1.(J.H_2) \tag{157}\]
Instead we have

\[-\frac{1}{4}[[H_1, H_2], J] = (H_1, J, H_2) - \frac{1}{4}[J, H_1, H_2] + \frac{1}{4}[J, H_2, H_1]
\]

\[-\frac{1}{4}[H_1, J, H_2] + \frac{1}{4}[H_2, J, H_1]
\]

\[-\frac{1}{4}[H_1, H_2, J] + \frac{1}{4}[H_2, H_1, J]
\]

(158)

note that the first term on the left is not a hermitian matrix, and the first term on the right, namely \((H_1, J, H_2) = J'\) is a hermitian matrix.

These extra terms vanish for an ordinary matrix algebra as they refer to associators with respect to the usual matrix product

\[[H_1, J, H_2] = (H_1, J)H_2 - H_1(JH_2)
\]

(159)

etc. They vanish for ordinary matrices but not for the octonionic matrices. It is with the Jordan associators that we obtain an automorphism. Because the left hand side is in general not hermitian.

We define the scalar product

\((J, J) = \frac{1}{2}TrJ^2\),

(160)

also the Freudenthal product by

\(J \times J = (J^2 - JT \cdot J) - \frac{1}{2}ITr(J^2 - JT \cdot J)
\)

(161)

with

\(Det J = \frac{1}{3}TrJ.(J \times J) = \frac{2}{3}(J, J \times J)
\)

(162)

If \(DetJ \neq 0\), then \(J \times J\) has been defined so that

\(J^{-1} = (DetJ)^{-1}(J \times J)
\)

(163)

and

\(J \times J = J^{-1}DetJ
\)

(164)

We have

\(J^{-1}.J = I
\)

(165)

and

\(J^{-1}.J^2 = J
\)

(166)

so that \(J^{-1}\) is the inverse matrix.

Also from \(J \times J = J^{-1}DetJ\) we find

\(J.(J \times J) = (J,J^{-1})DetJ = IDetJ
\)

(167)

Now we have for any two non singular JNW matrices \(J\) and \(K\)

\((J, K, J^2) = 0
\)

(168)

also

\((J, K, \lambda J) = 0
\)

(169)

and

\((J, K, \kappa J) = 0
\)

(170)
Thus
\[(J, K, J \times J) = 0\] (171)

Because we have a 3 \times 3 matrix we find
\[(J, K, J^{-1}) = 0\] (172)

This ensures power associativity for non singular matrices. For \(K = J\) we have
\[(J, J, J^2) = 0\] (173)
or
\[J^2, J^2 = J, J^3\] (174)
\[(J, J, J^{-1}) = 0\] (175)
or
\[J^2, J^{-1} = J\] (176)

Also
\[(J, J^n, J^2) = 0\] (177)
gives
\[(J, J^n, J)^2 = (J, J^{n+2}\] (178)
\[(J, J^n, J^{-1}) = 0\] (179)
\[(J, J^n, J^{-1}) = J, (J^{n+1})\] (180)

Note that we have the identity
\[(J \times J, J) = IDetJ = J^3 = J^3 Tr J + \frac{1}{2} J(Tr J)^2 - Tr J^2\] (181)

where \(J^3 = J^2 J\).

Explicitly we find
\[(J, J) = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) + |a|^2 + |b|^2 + |c|^2\] (182)

where \(|a|^2 = a\bar{a}\), etc.

\[Det J = \alpha \beta \gamma - \alpha |a|^2 - \beta |b|^2 \gamma |c|^2 + (ab)c + c(\bar{b}a)\]
\[= \alpha \beta \gamma - \alpha |a|^2 - \beta |b|^2 = \gamma |c|^2 + t[(abc)]\] (183)

This expression is unambiguous since we have
\[(ab)c + c(\bar{b}a) = a(bc) + (c\bar{b})\bar{a}.\] (184)

The proof is simple. The above equation is equivalent to
\[[a, b, c] = [\bar{c}, \bar{b}, \bar{a}]\] (185)

Because
\[\bar{c} = -c + 2c_0, \quad \bar{b} = -b + 2b_0, \quad \bar{a} = -a + 2a_0,\] (186)
the right hand side takes the form
\[[\bar{c}, \bar{b}, \bar{a}] = -[c, b, a]\] (187)
Using alternativity we find
\[-[c, b, a] = [a, b, c]\] (188)
so that
\[ t[(ab)c] = t[a(bc)] = t(abc) \] (189)
We also have
\[ J \times J = \begin{pmatrix} \beta \gamma - a\bar{a} & b\bar{a} - \gamma c & c\bar{a} - \beta \bar{b} \\ ab - \gamma \bar{c} & \gamma a - b\bar{b} & \bar{c}b - \alpha a \\ \bar{a}c - \beta \bar{b} & bc - \alpha \bar{a} & \alpha \beta - c\bar{c} \end{pmatrix} \] (190)
Now in ordinary quantum mechanics a $3 \times 3$ projection operator has the form
\[ P = UEU^{-1} \] (191)
where $E$ is diagonal, idempotent with trace unity. For instance
\[ E = E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \] (192)
It satisfies
\[ E_3 \times E_3 = 0 \] (193)
Similarly
\[ E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \] (194)
\[ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \] (195)
\[ E_1 + E_2 + E_3 = I \] (196)
For $E_i$ we have $E_i \times E_i = 0$, so that
\[ P \times P = 0 \] (197)
with
\[ TrP = I, \quad DetP = 0 \] (198)
Hence, in the octonionic case we can define a projection operator by
\[ TrP = I, \quad P \times P = 0 \] (199)
and with
\[ P = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix}, \quad \alpha + \beta + \gamma = 1 \] (200)
we find
\[ \beta \gamma = a\bar{a}, \quad \gamma \alpha = b\bar{b}, \quad \alpha \beta = c\bar{c}, \quad bc = \alpha \bar{a}, \quad ca = \beta \bar{b}, \quad ab = \gamma \bar{c} \] (201)
This is satisfied by
\[ P_3 = \lambda_3 \bar{\lambda}_3^T \] (202)
where

\[ \lambda_3 = \begin{pmatrix} \gamma^{-\frac{1}{2}}b \\ \frac{1}{2} \end{pmatrix} \]  

(203)

so that

\[ P_3 = \begin{pmatrix} \gamma^{-1}b \bar{b} & \gamma^{-1}b \bar{a} & \bar{b} \\ \gamma^{-1}a \bar{b} & \gamma^{-1}a \bar{a} & \bar{a} \\ b & a & \gamma \end{pmatrix} \]  

(204)

giving

\[ \alpha = \gamma^{-1}b \bar{b}, \quad \beta = \gamma^{-1}a \bar{a}, \quad c = \gamma^{-1}\bar{b} \bar{a} \]  

(205)

with the constraint

\[ \gamma^2 - \gamma + |a|^2 + |b|^2 = 0 \]  

(206)

or

\[ \gamma = \gamma(a, b) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4(|a|^2 + |b|^2)} \]  

(207)

Reality of \( \gamma \) requires

\[ 0 \leq (|a|^2 + |b|^2) < \frac{1}{4} \]  

(208)

then \( \gamma > 0 \).

Thus an idempotent JNW matrix is determined by two octonions \( a, b \) with

\[ \sqrt{|a|^2 + |b|^2} < \frac{1}{2} \]  

(209)

and corresponds to a point on a generalized sphere. We can represent a physical state by such a point.

We have

\[ \gamma(0, 0) = 1, \quad P_3 = E_3. \]  

(210)

Now we turn to an infinitesimal transformation that leaves \( J, TrJ, TrJ^2 \) and \( DetJ \) invariant. Let

\[ \delta J = (H_1, J, H_2) = L_{H_1, H_2}J \]  

(211)

Then we find

\[ \delta TrJ = 0 \]  

(212)

\[ \delta(J, J) = (\delta J, J) = 0 \]  

(213)

and

\[ \delta DetJ = (\delta J, J \times J) + (J, \delta J \times J) + (J, J \times \delta J) = 0 \]  

(214)

or

\[ (\delta J, J \times J) + 2(J, \delta J \times J) = 0 \]  

(215)

We also find that if

\[ J \rightarrow J' = J + \delta J, \quad K' = K + \delta K \]  

(216)

then

\[ (J', K', J' \times J') = 0 \]  

(217)
or
\[(J + \delta J, K, J \times J) + (J, K + \delta K, J \times J) + 2(J, K, J + \delta J) = 0\] (218)
so that \(J \rightarrow J'\) is an automorphism of the algebra.

It has \(H_1, H_2\) with \(Tr H_1 = 0\), \(Tr H_2 = 0\) so that the automorphism group is defined by 26 parameters. It is the group \(F_4\).

The projection operator transforms as
\[P' = \lambda' \lambda'^\dagger = \lambda \lambda^\dagger + (H_1, \lambda \lambda^\dagger, H_2) + \frac{1}{2!} (H_1, (H_1, \lambda \lambda^\dagger, H_2), H_2) + \cdots\] (219)
In the non octonionic case this can be written
\[\lambda' \lambda'^\dagger = e^{-\frac{1}{4}[H_1, H_2]} \lambda \lambda^\dagger e^{\frac{1}{4}[H_1, H_2]}\] (220)
which is equivalent to
\[\lambda' = e^{-\frac{1}{4}[H_1, H_2]} \lambda = U \lambda\] (221)
so that \(\lambda\) transforms linearly under the group and therefore forms a representation of the automorphism group. In the octonionic case such a small representation does not exist as
\[U(\lambda \lambda^\dagger) U^\dagger \neq (U \lambda)(\lambda^\dagger U^\dagger)\] (222)
so that there is no ket representation of the state, only a projection operator representation.

Also note that in this case, besides the states \(P_1, P_2, P_3\) we can also define observables \(H_1, H_2\) associated with the "charges" carried by the states.

Another remark contains the transformation property of \(c\). The \(c\) transforms like
\[\left\{ \frac{1}{2} \pm \sqrt{\frac{1}{4} - (|a|^2 + |b|^2)^{-1} b a} \right\}\] (223)
If we choose a subgroup of \(F_4\) which preserve \(|a|\) and \(|b|\), it belongs to \(SO(8)\). In that case
\[t[ab(ba)]\] (224)
is invariant, and similarly
\[t[a(\bar{a}c)c]\] (225)
and
\[t[(\bar{b}b)bc]\] (226)
are invariant.

(c) Octonionic form of \(G_2\)

The norm group of the octonions is the group that preserves
\[[N(w)]^2 = w_0^2 + w_1^0 + \cdots + w_7^2\] (227)
Hence it is \(SO(8)\).

The norm group for traceless octonions is \(SO(7)\). A subgroup of \(SO(7)\) is the automorhism group
\[u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix} \rightarrow e^{i \lambda_3 \alpha_7} u_1, \quad e^{i \lambda_8 \beta_7} u_0\] (228)
leave $u_0$ invariant and preserve the commutation relations.

Let $S$ be the discrete symmetry that gives a cyclic permutation to $e_a$ Then $S^7 = 1$

Consider $S$ with $7 \to 1$.

\[
S u \to e^{i\lambda_3 \alpha_1} S u, \quad S u \to e^{i\lambda_8 \beta_1} S u
\]

(229)

(230)

thus is also an automorphism.

Hence automorphism group has $7 \times 2 = 14$ parameters.

\[w \to awa \in SO(8)\]  

(231)

7 parameters in $SO(8)/SO(7)$

\[w \to bwb^{-1} \in SO(7)\]  

(232)

in $SO(7)/G_2$

\[(ed)^{-1}[d(cwc_{-1})d^{-1}](ed)\]  

(233)

is a 14 parameter subgroup of $SO(7) \in G_2$.

Finally, the infinitesimal form is

\[w' = w + \frac{1}{4}[[\tilde{\alpha}, \tilde{\beta}], \tilde{w}] - \frac{3}{4}[\tilde{\alpha}, \tilde{\beta}, \tilde{w}].\]  

(234)

We also note that $G_2$ is also the automorphism group of the anti-symmetrical algebra defined by

\[e_a \wedge e_b = f_{abc} e_c\]  

(235)

which is not a Lie algebra.

Invariance principles of exceptional quantum mechanics based on $F_4$ and its generalization to the case of complex Jordan algebra $E_6$ were worked out before.\cite{14} It is clear that the fundamental symmetries of elementary particles seem to point out to field theories based on local internal symmetries connected with structures that can be constructed by using octonions (see some of the references at the end of the paper). Octonionic observables form a Jordan algebra whose automorphism group is an exceptional group $F_4$ or $E_6$ and we are led into a gauge field theory of quarks and leptons based on exceptional groups.

It is also possible to consider super Jordan algebras for generalized Jordan algebras involving both bosonic and fermionic observables. In this case the automorphism group is a supergroup. If the generalized Jordan algebra is octonionic then its automorphism is given by an exceptional supergroup, it could be $F_4$ for example\cite{15} which has 40 parameters and its Lie subgroup is $SO(7) \times SU(2)$ which admits $SU(2) \times U(1) \times SU(3)^c$ as a subgroup. Instead of leptoquarks and diquarks\cite{16}, the super $F_4$ would introduce spin $\frac{3}{2}$ gauge particles alongside with gauge bosons, so it would be similar to supergravity when used as a local supersymmetry group.

There remain many unsolved problems. The outstanding ones are concerned with the building of Fock space (tensor products with exceptional structures), the understanding of color confinement and realization of the ultimate synthesis with supergravity. At this point, for many reasons it seems impossible to imbed a successful grand unified theory based on $E_6$ with extended supergravity. A way out may be the reformulation of supergravity in superspace consisting of space-time manifold and Grassmann manifold. Grassmann numbers are obtained by using direct products of quaternions. Exceptional Grassmann numbers are based on octonions, hence it is possible that the invariance properties of such exceptional superspace will involve new supersymmetries that generalize the exceptional groups.
5. Acknowledgments

The development of ideas presented here is due in large part to stimulating discussions with our colleagues and friends including Vladimir Akulov, Robert L. Blumenblatt, Vladimir Dobrev, Yasemin Gürcan, Yuan K. Ha, Francesco Iachello, Amish Khalfan, Alexander Kheyfits, Levent Kurt, David Tepper and Francesco Toppan.

Bibliography

[1] Jordan P, von Neumann J and Wigner E P 1934 Ann. Math. 35, 29.
[2] Chevalley C and Schafer R D 1950 Proc. Natl. Acad. Sci. U.S. 36 137
[3] Jordan P 1933 Nachr. Ges. Wiss. Göttingen 209
[4] Tits J 1966 Nederl. Akad. Wetensch. Proc. Ser. A 69, 223
[5] Greenberg 1964 Phys. Rev. Lett. 13, 598
[6] Nambu Y 1966 in “Preludes in Theoretical Physics” eds. A. de Shalit, H. Fesbach and L. van Hove, p.133 (Amsterdam)
[7] Fritzsch H, Gell-Mann M and Leutwyler H 1973 Phys. Lett. 47B,
[8] Freudenthal H 1965 Adv. in Math. 1 145
[9] Gürsey F and Tze C H 1966 “On the Role of Division, Jordan and Related Algebras in Particle Physics,” World Scientific
[10] Barton C H 2000 Magic Squares of Lie Algebras,” Ph.D. Thesis, University of York
[11] Barton C H and Sudbery A 2002 Mat. RA/0001803, and RA/0203010
[12] Burdik C, Catto S, Gürcan Y, Khalfan A and Kurt L 2017 Physics of Particles and Nuclei Letters 14 390
[13] Gınaydin M and Gürsey F 1973 J. Math. Phys. 14, 1651
[14] Catto S., Choun Y S and Kurt L 2013 “Invariance Properties of the Exceptional Quantum Mechanics (F4) and its Generalization to Complex Jordan Algebras (E6)’” in Lie Theory and Its Applications in Physics, Ed. V. Dobrev (Springer), Japan 469
[15] Gürsey F and Marchildon L 1977 J. Math. Phys. 19, 942
[16] Catto S 1993 ”Colored Supersymmetry of Mesons and Baryons Based on Octonionic Algebras,” in ”Symmetries in Science VI: From the Rotation Group to Quantum Algebras.” Ed. B. Gruber, Plenum Publishers 129
[17] Gınaydin M 1977 ”Non-associative Algebras in Quantum Mechanics and Particle Physics.” In Proceedings of the Charlottesville Conference on Non Associative Algebras, U. Virginia
[18] Jacobson N 1951 Trans. Amer. Math. Soc. 70 509
[19] Albert A A 1934 Ann. Math. 35, 65
[20] Mostow G D 1973 ”Strong Rigidity of Locally Symmetric Spaces.” (Princeton U. Press)
[21] Moufang R 1933 Abh. Math. Sem. Univ. Hamburg 9 207
[22] Gınaydin M, Piron C and Ruegg H 1978 Comm. Math. Phys. 61 69
[23] Gınaydin M and Gürsey F 1974 Phys. Rev. D9 3387 (1974)
[24] G.B. Seligman, ”Modular Lie Algebras.” Springer-Verlag, Berlin, Heidelberg, New York, 1967.
[25] Schafer R D 1966 ”An Introduction to Non-associative Algebras.” (Academic Press, New York
[26] Gürsey F , Ramond P and Sikivie P 1976 Phys. Lett. 60B 177
[27] Faulkner J R 1970 Memoirs of the Am. Math. Soc. No.104 (Providence, RI).
[28] Kac V 1977 it Comm. Math. Phys. 53, 31.
[29] Pati J and Salam A 1973 Phys. Rev. D8, 1240