On the imbedding of a finite family of closed disks into $\mathbb{R}^2$ or $S^2$

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Abstract

Let $\{V_i\}_{i=1}^n$ be a finite family of closed subsets of a plane $\mathbb{R}^2$ or a sphere $S^2$, each homeomorphic to the two-dimensional disk. In this paper we discuss the question how the boundary of connected components of a complement $\mathbb{R}^2 \setminus \bigcup_{i=1}^n V_i$ (accordingly, $S^2 \setminus \bigcup_{i=1}^n V_i$) is arranged.

It appears, if a set $\bigcup_{i=1}^n \text{Int} V_i$ is connected, that the boundary $\partial W$ of every connected component $W$ of the set $\mathbb{R}^2 \setminus \bigcup_{i=1}^n V_i$ (accordingly, $S^2 \setminus \bigcup_{i=1}^n V_i$) is homeomorphic to a circle (see theorems 1, 2 below).

Let $U \in \mathbb{R}^2$ be an open area (subset of a plane, homeomorphic to the two-dimensional disk). One of the classical problems of complex analysis is the question of a possibility of an extention of conformal mapping defined in $U$ out of this area. The answer to this question is tightly connected with the structure of the boundary $\partial U$ of $U$ and depends on how much the closure $\text{Cl} U$ differs from the closed two-dimensional disk. As a rule, it is known only the local information about a structure of the set $\partial U$ (accessibility of points of the boundary from area $U$ and so on).

In works [P1, P2] the criterion is given for a compact subset of a plane to be homeomorphic to the closed two-dimensional disk, which uses only local information about the boundary of this set (see theorem 3 below). This criterion enables to investigate the problems connected to a mutual disposition of closed disks on a plane.

Let $\{V_i\}_{i=1}^n$ be a finite family of closed subsets of a plane $\mathbb{R}^2$ or a sphere $S^2$, each homeomorphic to the two-dimensional disk. In this paper we discuss the question how the boundary of connected components of a complement $\mathbb{R}^2 \setminus \bigcup_{i=1}^n V_i$ (accordingly, $S^2 \setminus \bigcup_{i=1}^n V_i$) is arranged.

It appears, if a set $\bigcup_{i=1}^n \text{Int} V_i$ is connected, that the boundary $\partial W$ of every connected component $W$ of the set $\mathbb{R}^2 \setminus \bigcup_{i=1}^n V_i$ (accordingly, $S^2 \setminus \bigcup_{i=1}^n V_i$) is homeomorphic to a circle (see theorems 1, 2 below).

**Theorem 1** Let $V_1, \ldots, V_n$ be a finite collection of the closed subsets of $\mathbb{R}^2$, each homeomorphic to the two-dimensional disk. Suppose the set $\bigcup_{i=1}^n \text{Int} V_i$ is connected.
Let $W$ be the unlimited connected component of the set $\mathbb{R}^2 \setminus \bigcup_{i=1}^{n} V_i$. Then the set $\mathbb{R}^2 \setminus W$ is homeomorphic to the closed two-dimensional disk.

**Theorem 2** Let $V_1, \ldots, V_n$ be a finite collection of the closed subsets of $S^2$, each homeomorphic to the two-dimensional disk. Suppose the set $\bigcup_{i=1}^{n} \text{Int} V_i$ is connected and $S^2 \setminus \bigcup_{i=1}^{n} V_i \neq \emptyset$.

Let $W$ be a connected component of the set $S^2 \setminus \bigcup_{i=1}^{n} V_i$. Then the set $\text{Cl} W$ is homeomorphic to the closed two-dimensional disk.

The following definitions and statements will be useful for us in what follows.

**Definition 1** [ZVC] Let $D$ be an open set. The point $x \in \partial D$ is called accessible from $D$ if there exists a continuous injective mapping $\varphi : I \to \text{Cl} D$, such that $\varphi(1) = x$ and $\varphi([0,1)) \subset \text{Int} D$ (this map is named a cut).

**Definition 2** [ZVC] Let $E$ be a subset of a topological space $X$ and $a \in X$ be a point. The set $E$ is called locally arcwise connected in $a$, if any neighbourhood $U$ of $a$ contains such neighbourhood $V$ of $a$ that any two points from $V \cap E$ can be connected by a path in $U \cap E$.

**Proposition 1** [ZVC] Let $D$ be an area with a nonempty interior in $\mathbb{R}^2$ or $S^2$. If $D$ is locally arcwise connected in a point $a \in \partial D$ then $a$ is accessible from $D$.

**Theorem 3** [P1, P2] Let $D$ be a compact subset of a plane $\mathbb{R}^2$ with a nonempty interior. Then $D$ is homeomorphic to the closed two-dimensional disk if and only if the following conditions holds:

1) the set $\text{Int} D$ is connected;
2) the set $\mathbb{R}^2 \setminus D$ is connected;
3) any point $x \in \partial D$ is accessible from $\text{Int} D$;
4) any point $x \in \partial D$ is accessible from $\mathbb{R}^2 \setminus D$.

**Theorem 4 (Shönflies)** [ZVC] Let $\gamma$ be a simple closed curve in $S^2$ (respectively, in $\mathbb{R}^2$). There exists a homeomorphism $f$ of $S^2$ onto itself (respectively, of $\mathbb{R}^2$ onto itself) mapping the curve $\gamma$ onto the unit circle.

**Proof of theorem** Let us show, that the compact set $D = \mathbb{R}^2 \setminus W$ complies with the conditions of theorem. We will divide our argument into several steps.
1. Since $∂D \subset \bigcup_{i=1}^{n} ∂V_i$ then for any $x \in ∂D$ we can find $i \in \{1, \ldots, n\}$ such that $x \in ∂V_i$. Theorem 3 states that the point $x$ is accessible from $\text{Int} V_i$. Hence $x$ is accessible from $\text{Int} D$ because $\text{Int} V_i \subset \text{Int} D$.

2. Let us show, that any point $a \in ∂D$ is accessible from $W = \mathbb{R}^2 \setminus D$. Without loss of generality we can assume that the origin of coordinates lies in $\text{Int} D$.

We fix $a \in ∂D$. The set of all points accessible from $W$ is dense in $∂W = ∂D$ [ZVC], therefore there exists a point $x_0 \in ∂D$ accessible from $W$ which do not coincide with $a$.

All compact subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, are known to be limited. Therefore there exists $R > 0$ such that

$$\bigcup_{i=1}^{n} V_i \subset \{x \in \mathbb{R}^2 | d(0, x) < R\}.$$ 

We fix a point $x' \in W$ which meets an equality $|x'| = R$. It is known (see [ZVC]) that there exists a cut

$$\gamma_0 : I \to \mathbb{R}^2,$$ 

$$\gamma_0(0) = x_0, \gamma_0(1) = x', \gamma_0((0, 1]) \subset W.$$ 

Let

$$\tau = \min\{t \in I | |\gamma_0(t)| = R\}.$$ 

According to the conditions of theorem $\tau > 0$. Let $\gamma_0(\tau) = x''$. Denote a polar angle of $x''$ by $\varphi$.

Consider continuous injective mapping

$$\gamma_1 : \mathbb{R}^+ \to \mathbb{R}^2,$$ 

$$\gamma_1(t) = \begin{cases} 
\gamma_0(t) & \text{when } t \in [0, \tau), \\
(\varphi, R + t - \tau) & \text{when } t \in [\tau, +\infty). 
\end{cases}$$

This map is an imbedding of $\mathbb{R}^+$ into $\mathbb{R}^2$, moreover $\gamma_1(0) \in ∂W$, $\gamma_1(\mathbb{R}^+ \setminus \{0\}) \subset W$.

2.1. Let us show, that the open set $W \setminus \gamma_1(\mathbb{R}^+) \subset W$ is connected.

Consider an involution

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\},$$ 

$$f(r, \varphi) = (r^{-1}, \varphi).$$

This map is known to be a homeomorphism. Under the action of $f$ the area $W$ will pass to an open connected set $\widetilde{W} = f(W)$. Mark that the origin of the coordinates is an isolated point of the boundary $∂\widetilde{W}$ because

$$\{ (r, \varphi) \in \mathbb{R}^2 | r > R \} \subset W,$$

$$\{ (r, \varphi) \in \mathbb{R}^2 | 0 < r < R^{-1} \} \subset f(W).$$

Therefore, $\widetilde{W}_0 = \widetilde{W} \cup \{0\}$ appears to be the open connected set and the map

$$\widetilde{\gamma} : I \to \mathbb{R}^2,$$
\[ \tilde{\gamma}(t) = \begin{cases} f \circ \gamma_1(t^{-1} - 1) & \text{when } t \in (0, 1], \\ 0 & \text{for } t = 0. \end{cases} \]

is a cut of the set \( \tilde{W}_0 \). Moreover \( \tilde{W}_0 \setminus \tilde{\gamma}(I) = f(W \setminus \gamma_1(\mathbb{R}_+)) \).

So, for a proof of connectivity of the set \( W \setminus \gamma_1(\mathbb{R}_+) \) it is sufficient to check the validity of the following statement.

**Lemma 1** Let \( U \subset \mathbb{R}^2 \) be an open connected set, point \( z \in \partial U \) be accessible from \( U \), \( \alpha : I \to \mathbb{R}^2 \) be a cut of \( U \) with the end in \( z \) (a continuous injective mapping such that \( \alpha(0) = z \) and \( \alpha((0, 1]) \subset U \)).

Then the set \( U \setminus \alpha(I) \) is connected.

Let us prove this statement. Let \( y = \alpha(t) \) for some \( t > 0 \). According to propositions 6.4.6 and 6.5.1 from [ZVC] there exists a homeomorphism \( h \) of \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \), such that the map

\[ h \circ \alpha = \tilde{\alpha} : I \to \mathbb{R}^2 \]

complies the relation

\[ \tilde{\alpha}(t) = (t, 0) \in \mathbb{R}_+ \times \{0\} \subset \mathbb{R}^2, \quad t \in I. \]

Since \( \alpha([t, 1]) \subset U \), there exists an \( \varepsilon > 0 \) such that

\[ \tilde{U}_t = \{ x \in \mathbb{R}^2 \mid d(x, \tilde{\alpha}([t, 1])) < \varepsilon \} \subset h(U). \]

Obviously, \( U_t = h^{-1}(\tilde{U}_t) \) is a neighbourhood of a point \( \alpha(t) \) in \( U \) and the set \( U_t \setminus \alpha(I) \) is connected. Besides, a set \( (U_{t_1} \cap U_{t_2}) \setminus \alpha(I) \) is not empty for any \( t_1, t_2 \in (0, 1] \).

Therefore

\[ \bigcup_{t \in (0,1)} U_t \]

is a connected open neighbourhood of a set \( \alpha(I) \) in \( U \), hence \( U \setminus \alpha(I) \) is a connected set. \( \square \)

So, the set \( W \setminus \gamma_1(\mathbb{R}_+) \) is connected.

2.2. Select a point \( x_i \in \partial V_i, x_i \neq a \) for each \( i \in \{1, \ldots, n\} \). The set

\[ \bigcup_{i=1}^n \text{Int } V_i \]

is connected by the condition of theorem and the point \( x_i \) is accessible from \( \text{Int } V_i \) for any \( i \). Therefore we can find a continuous map

\[ \beta : [1, n + 1] \to \bigcup_{i=1}^n V_i \]

which meets the following conditions

\[ \beta((i, i + 1)) \subset \bigcup_{i=1}^n \text{Int } V_i \subset \text{Int } D, \quad i = 1, \ldots, n; \]
\[ \beta(i) = x_i, \quad i = 1, \ldots, n; \quad \beta(n + 1) = x_0. \]

Consider a continuous map
\[
\gamma : \mathbb{R}_+ \to \mathbb{R}^2, \\
\gamma(t) = \begin{cases} 
\beta(t + 1) & \text{for } t \in [0, n), \\
\gamma_1(t - n) & \text{for } t \in [n, +\infty).
\end{cases}
\]

Since the relations
\[
\gamma(\mathbb{R}_+) \subset (\beta([1, n + 1]) \cup \gamma_0(I) \cup \gamma_1([\tau, +\infty))) , \\
\gamma_1([\tau, +\infty)) \subset \{ z \in \mathbb{R}^2 | d(z, 0) \geq R \}, \\
a \in \partial D \subset \{ z \in \mathbb{R}^2 | d(z, 0) < R \}
\]
hold and a compact set \( \beta([1, n + 1]) \cup \gamma_0(I) \) does not contain a point \( x \) on a construction, there exists \( \varepsilon_0 > 0 \) which complies the inequality
\[
d(a, z) > \varepsilon_0 \quad \text{for all } z \in \gamma(\mathbb{R}_+).
\]

Now we are ready for proof of local linear connectivity of the area \( W \) in the point \( a \in \partial W = \partial D \).

2.3. Let \( U \) be a curtain neighbourhood of the point \( a \). Find \( \varepsilon > 0 \) which meets the conditions
\[
U_\varepsilon(a) = \{ x \in \mathbb{R}^2 | d(a, x) < \varepsilon \} \subset U, \quad U_\varepsilon(a) \cap \gamma(\mathbb{R}_+) = \emptyset.
\]

Fix imbeddings
\[
f_i : S^1 \to \partial V_i, \quad i = 1, \ldots, n.
\]
Here \( S^1 = \{ (r, \varphi) \in \mathbb{R}^2 | r = 1 \} \). The metric on \( S^1 \) we shall define as follows:
\[
d_s((1, \varphi_1), (1, \varphi_2)) = \min_{k \in \mathbb{Z}} | \varphi_1 - \varphi_2 + 2\pi k |.
\]

Mark that maps \( f_i, i = 1, \ldots, n \) are uniformly continuous.
Fix \( \delta_1 > 0 \) such that an inequality \( d_s(\tau_1, \tau_2) < \delta_1 \) implies
\[
d(f_i(\tau_1), f_i(\tau_2)) < \min(\varepsilon_0/2, \varepsilon/3)
\]
for any \( i = 1, \ldots, n \) and \( \tau_1, \tau_2 \in S^1 \).
Find also \( \delta_2 > 0 \), such that \( d(z_1, z_2) < \delta_2 \) has as a consequence an inequality
\[
d_s(f_i^{-1}(z_1), f_i^{-1}(z_2)) < \min(\delta_1/2, \pi/4)
\]
for every \( i = 1, \ldots, n \) and any \( z_1, z_2 \in \partial V_i \).
Assume \( \delta = \min(\delta_2/2, \varepsilon/3) \).

2.4. Let us show, that for any \( a_1, a_2 \in U_\delta(a) \cap W \) there exists a continuous map \( g : I \to U_\varepsilon(a) \cap W \) such that \( g(0) = a_1, g(1) = a_2 \).
The inequality \( d(z_1, z_2) < \delta_2 \) is fulfilled for all \( z_1, z_2 \in U_\delta(a) \), hence
\[
d \left( f^{-1}(\partial V_i \cap U_\delta(a)) \right) < \min (\delta_1/2, \pi/4)
\]
for every \( i \in \{1, \ldots, n\} \) and in the case \( \partial V_i \cap U_\delta(a) \neq \emptyset \) the circle \( S^1 \) could be decomposed into two not intersecting intervals \( J'_i \) and \( J''_i \) with common endpoints in such a way that the following relations are fulfilled
\[
f^{-1}_i(\partial V_i \cap U_\delta(a)) \subset J'_i, \\
\text{diam}(J'_i) = \max_{t_1, t_2 \in J'_i} d_s(t_1, t_2) < \min (\delta_1, \pi/2).
\]
In the case \( \partial V_i \cap U_\delta(a) = \emptyset \) set \( J''_i = S^1, J'_i = \emptyset \).

Therefore, \( f_i(J''_i) \cap U_\delta(a) = \emptyset \) and \( f_i(J'_i) \subset U_{2\varepsilon/3}(a) \).

**Lemma 2** Let \( B \) be a closed disk satisfying the following conditions:
\[
\partial B \cap \left( \bigcup_{i=1}^n \partial V_i \right) \subset U_\delta(a), \\
(\partial B \setminus U_\delta(a)) \subset (W \setminus \gamma(\mathbb{R}_+)).
\]
Then \( B \cap \partial V_i \subset f_i(J'_i) \subset U_{2\varepsilon/3}(a), \ i = 1, \ldots, n. \)

On the condition of lemma \( \partial B \cap f_i(J''_i) = \emptyset \) for every \( i = 1, \ldots, n. \), Therefore, \( f_i(J''_i) \subset \text{Int} B \) or \( f_i(J''_i) \subset (\mathbb{R}^2 \setminus B) \). By a construction \( x_i \in f_i(J''_i) \) and \( x_i \in \gamma(\mathbb{R}_+) \subset (\mathbb{R}^2 \setminus B) \), hence \( f_i(J''_i) \subset (\mathbb{R}^2 \setminus B), \ i = 1, \ldots, n. \)

Let \( a_1, a_2 \in (U_\delta(a) \cap W). \) Since \( U_\delta(a) \cap \gamma(\mathbb{R}_+) = \emptyset \), then \( a_1, a_2 \in (U_\delta(a) \cap (W \setminus \gamma(\mathbb{R}_+))). \) From connectivity of the set \( W \setminus \gamma(\mathbb{R}_+) \) follows, that there exists an injective continuous map
\[
\tilde{\mu} : I \rightarrow (W \setminus \gamma(\mathbb{R}_+)),
\]
complying the equalities \( \tilde{\mu}(0) = a_1, \tilde{\mu}(1) = a_2 \) (the concepts of connectivity and linear connectivity coincide for open subsets of \( \mathbb{R}^n \)).

Find smooth imbeddings
\[
\eta_1 : S^1 \rightarrow U_\delta(a), \\
\eta_2 : S^1 \rightarrow (U_\varepsilon(a) \setminus \text{Cl} U_{2\varepsilon/3}(a))
\]
such that the points \( a_1, a_2 \) lie inside disks bounded by curves \( \eta_1, \eta_2. \)

It is known that an imbedding of a segment or circle into \( \mathbb{R}^2 \) could be as much as desired precisely approximated by a smooth imbedding. It is known as well that any two one-dimensional smooth compact submanifolds of \( \mathbb{R}^2 \) could be reduced to the general position by a small perturbation fixed on their boundary.

Therefore, there exists smooth imbedding
\[
\mu : I \rightarrow W \setminus \gamma(\mathbb{R}_+), \ a_1 = \mu(0), \ a_2 = \mu(1)
\]
such that the sets $\mu(I) \cap \eta_1(S^1)$ and $\mu(I) \cap \eta_2(S^1)$ consist of final number of points.

For every $z \in \mu(I) \cap \eta_2(S^1)$ there exist $t', t'' \in I$, $t' < t''$, which comply with the following conditions

$$z \in \mu((t', t'')),$$
$$\mu(t'), \mu(t'') \in \eta_1(S^1),$$
$$\mu((t', t'')) \cap \eta_1(S^1) = \emptyset.$$ 

We receive a finite family of nonintersecting intervals

$$(t_{j,1}, t_{j,2}) \subset I \quad j = 1, \ldots, k$$ 

satisfying to relations

$$\mu((t_{j,1}, t_{j,2})) \cap \eta_1(S^1) = \emptyset, \quad \mu(t_{j,1}), \mu(t_{j,2}) \in \eta_1(S^1), \quad j = 1, \ldots, k,$$
$$\mu\left(I \setminus \bigcup_{j=1}^{k} (t_{j,1}, t_{j,2})\right) \subset U_\varepsilon(a).$$

Now for each $j = 1, \ldots, k$ we fix an arc $\Theta_j : I \to \eta_1(S^1)$ with the endpoints $\mu(t_{j,1})$ and $\mu(t_{j,2})$. A set

$$\Theta_j(i) \cup \mu((t_{j,1}, t_{j,2}))$$

is homeomorphic to a circle, therefore it bounds a closed disk $B_j$ such that

$$\left(\partial B_j \cap \bigcup_{i=1}^{n} \partial V_i\right) \subset U_\delta(a),$$
$$\left(\partial B_j \setminus U_\delta(a)\right) \subset (W \setminus \gamma(\mathbb{R}^+)) .$$

By lemma $\mathbb{Z}$ these relations has as a consequence following inclusion

$$\left( B_j \cap \bigcup_{i=1}^{n} \partial V_i\right) \subset U_{2\varepsilon/3}(a).$$

Since $\eta_2(S^1) \subset (U_\varepsilon(a) \setminus \text{Cl} U_{2\varepsilon/3}(a))$, then

$$B_j \cap \eta_2(S^1) = \bigcup_{s=1}^{m_j} \chi_s .$$

Here $\{\chi_s\}_{s=1}^{m_j}$ is a final family of nonintersecting arcs of the circle $\eta_2(S^1)$. In addition $\chi_s \subset (W \setminus \gamma(\mathbb{R}^+))$, $s = 1, \ldots, m_j$.

A set

$$\left(\text{Int } B_j\right) \setminus \left(\bigcup_{s=1}^{m_j} \chi_s\right)$$

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represents a final union of connected components homeomorphic to the two-dimensional disk, lying either inside or outside the closed disk limited by a circle \( \eta_2(S^1) \). Select that from components, which bounds with an arc \( \Theta_j \). Designate by \( \tilde{B}_j \) a closure of this component. Obviously,

\[
\tilde{B}_j \subset U_\varepsilon(a), \quad (\partial \tilde{B}_j \setminus \Theta_j(I)) \subset (W \setminus \gamma(\mathbb{R}^+)).
\]

Let

\[
g_j : I \to (\partial \tilde{B}_j \setminus \Theta_j((0, 1)))
\]

be an arc of a circle \( \partial \tilde{B}_j \) with the endpoints \( \mu(t_{j,1}), \mu(t_{j,2}) \). As we already have shown, it complies with the relation

\[
g_j(I) \subset (U_\varepsilon(a) \cap (W \setminus \gamma(\mathbb{R}^+))).
\]

A continuous curve

\[
g : I \to (W \setminus \gamma(\mathbb{R}^+)),
\]

\[
g(t) = \begin{cases} 
\mu(t) & \text{if } t \in \left(I \setminus \bigcup_{j=1}^k (t_{j,1}, t_{j,2})\right), \\
g_j((t_{j,2} - t_{j,1})t + t_{j,1}) & \text{if } t \in (t_{j,1}, t_{j,2}).
\end{cases}
\]

represents a continuous path in \( U_\varepsilon(a) \cap (W \setminus \gamma(\mathbb{R}^+)) \), connecting points \( a_1 \) and \( a_2 \).

Therefore, the point \( a \) is accessible from \( W \setminus \gamma(\mathbb{R}^+)) \) and all the more it is accessible from \( W = \mathbb{R}^2 \setminus D \). Then each point of \( \partial D \) is accessible from \( \mathbb{R}^2 \setminus D \) because of the arbitrary rule we selected the point \( a \in \partial D \).

3. The set \( W \) is connected on a condition of theorem.

4. Let us show that the set \( \text{Int} D \) is connected. The set \( \bigcup_{i=1}^n V_i \) is connected since any point of \( \bigcup_{i=1}^n \partial V_i \) is accessible from a connected set \( \bigcup_{i=1}^n \text{Int} V_i \), therefore it is sufficient to show that the boundary \( \partial \tilde{W} \) does not lie in the set \( \partial D \) for any connected component \( \tilde{W} \) of the set \( \mathbb{R}^2 \setminus (\bigcup_{i=1}^n V_i) \), different from \( W \).

Assume that \( \partial \tilde{W} \subset \partial D \). The set \( \partial \tilde{W} \) divides \( \mathbb{R}^2 \), consequently it has dimension not less than one (see [G-W]). Therefore, we can find three different points \( z_1, z_2, z_3 \in \partial \tilde{W} \). Each of these points is accessible from the connected sets \( W \) and \( \bigcup_{i=1}^n \text{Int} V_i \).

There exists a continuous injective mapping (see [ZVC])

\[
\varphi : I \to \mathbb{R}^2
\]

which satisfies the conditions

\[
\varphi(0) = z_1, \quad \varphi(1) = z_2, \quad \varphi((0, 1)) \subset \bigcup_{i=1}^n \text{Int} \ V_i.
\]

Let \( z = \varphi(1/2) \). There exists a continuous injective mapping

\[
\tilde{\varphi} : I \to \mathbb{R}^2,
\]
\[ \varphi(0) = z_3, \quad \varphi(1) = z, \quad \varphi((0, 1]) \subset \bigcup_{i=1}^{n} \text{Int} V_i. \]

Let \( t_1 = \min \{ t \in I \mid \tilde{\varphi}(t) \in \varphi(I) \} \). We have \( t_1 > 0 \) since \( z_3 = \tilde{\varphi}(0) \notin \varphi(I) \). Denote \( z' = \varphi(t_1) \).

Then \( t_2 \in (0, 1) \) is uniquely defined, such that \( z' = \varphi(t_2) \).

Consider continuous injective mappings

\[
\begin{align*}
\varphi_1 : I &\to \mathbb{R}^2, \quad \varphi_1(t) = \varphi(t_2(1 - t)) ; \\
\varphi_2 : I &\to \mathbb{R}^2, \quad \varphi_2(t) = \varphi((1 - t_2)t + t_2) ; \\
\varphi_3 : I &\to \mathbb{R}^2, \quad \varphi_3(t) = \tilde{\varphi}(t_1(1 - t)).
\end{align*}
\]

which comply with the relations

\[
\begin{align*}
\varphi_s(0) &= z_s, \quad \varphi_s(1) = z', \quad \varphi_s((0, 1]) \subset \bigcup_{i=1}^{n} \text{Int} V_i, \quad s = 1, 2, 3 ; \\
\varphi_{s_1}([0, 1)) \cap \varphi_{s_2}([0, 1)) &= \emptyset \quad \text{when } s_1 \neq s_2.
\end{align*}
\]

Similarly, there exists a point \( z'' \in W \) and continuous injective mappings \( \psi_s : I \to \mathbb{R}^2, s = 1, 2, 3, \) such that

\[
\begin{align*}
\psi_s(0) &= z_s, \quad \psi_s(1) = z'', \quad \psi_s((0, 1]) \subset W, \quad s = 1, 2, 3 ; \\
\psi_{s_1}([0, 1)) \cap \psi_{s_2}([0, 1)) &= \emptyset \quad \text{when } s_1 \neq s_2.
\end{align*}
\]

Since

\[
W \cap \left( \bigcup_{i=1}^{n} \text{Int} V_i \right) = \emptyset,
\]

the equality

\[
\left( \left( \bigcup_{s=1}^{3} \varphi_s(I) \right) \cap \left( \bigcup_{s=1}^{3} \psi_s(I) \right) \right) = \bigcup_{s=1}^{3} \{ z_s \}
\]

is valid. Therefore, everyone from the sets

\[
\varphi_{s_1}(I) \cup \varphi_{s_2}(I) \cup \psi_{s_1}(I) \cup \psi_{s_2}(I), \quad s_1 \neq s_2
\]

is homeomorphic to a circle.

The set

\[
\mathbb{R}^2 \setminus \left( \bigcup_{s=1}^{3} (\varphi_s(I) \cup \psi_s(I)) \right)
\]

falls into the three connected components \( U_1, U_2, U_3 \), two of which are homeomorphic to the open two-dimensional disk and third is not limited.
then there exists \( j \in \{1, 2, 3\} \) such that \( \widehat{W} \subset U_j \). But it is impossible because everyone from the sets \( \text{Cl} U_s, s = 1, 2, 3 \) contains exactly two from the points \( z_1, z_2, z_3 \).

So, we have proved that the set \( \partial \widehat{W} \cap \partial D \) consists not more than from two points. Therefore, \( z \in \partial \widehat{W} \) and \( \varepsilon > 0 \) could be found to comply the inclusion \( U_\varepsilon(z) \subset \text{Int} D \).

The set
\[
\widehat{W} \cup U_\varepsilon(z) \cup \left( \bigcup_{i=1}^n \text{Int} V_i \right) \subset \text{Int} D
\]
is connected since \( \partial \widehat{W} \subset \bigcup_{i=1}^n \partial V_i \) and the sets \( \widehat{W}, U_\varepsilon(z), \bigcup_{i=1}^n \text{Int} V_i \) are connected.

By virtue of arbitrariness in a choice of \( \widehat{W} \), the set \( \text{Int} D \) is connected.

Applying to \( D \) theorem \( \mathbf{3} \) we conclude that this set is homeomorphic to the closed two-dimensional disk. \( \blacksquare \)

**Proof of theorem \( \mathbf{2} \).** Let \( i_n : I^2 \to S^2, i = 1, \ldots , n \) be the inclusion maps, \( i_n(I^2) = V_i \).

Without loss of a generality, it is possible to assume that a North Pole \( s_0 \) of \( S^2 \) lies in \( W \).

Consider a stereographic projection
\[
f : S^2 \setminus \{s_0\} \to \mathbb{R}^2.
\]
As is known, this map is a homeomorphism. Since \( V_i \subset S^2 \setminus \{s_0\}, i = 1, \ldots , n \) and the set \( S^2 \setminus \{s_0\} \) is open in \( S^2 \), the compositions
\[
In_i = f \circ i_n : I^2 \to \mathbb{R}^2, \quad i = 1, \ldots , n
\]
are continuous and are one-to-one. The set \( I^2 \) is compact, therefore maps \( In_i, i = 1, \ldots , n \) are imbeddings. Sign \( \widehat{V}_i = f(V_i) = In_i(I^2), i = 1, \ldots , n \).

From a mutual uniqueness of map \( f \) follows that
\[
f\left( \bigcup_{i=1}^n \text{Int} V_i \right) = \bigcup_{i=1}^n f(\text{Int} V_i) = \bigcup_{i=1}^n \text{Int} \widehat{V}_i.
\]
The set \( \bigcup_{i=1}^n \text{Int} \widehat{V}_i \) is connected as an image of a connected set at a continuous map.

So, family \( \widehat{V}_1, \ldots , \widehat{V}_n \) satisfies to conditions of theorem \( \mathbf{1} \).

Consider an open set \( W' = W \setminus \{s_0\} \subset S^2 \). It is easy to see that \( \partial W' = \partial W \cup \{s_0\} \) and \( s_0 \) is an isolated point of the boundary of \( W' \).

Denote \( \widehat{W} = f(W') \subset \mathbb{R}^2 \). Obviously, \( \widehat{W} \) is the unique unlimited connected component of a set \( \mathbb{R}^2 \setminus \bigcup_{i=1}^n \widehat{V}_i \). Applying theorem \( \mathbf{3} \), we conclude that a set \( \mathbb{R}^2 \setminus \widehat{W} \) is homeomorphic to the closed two-dimensional disk, and it’s boundary \( \partial(\mathbb{R}^2 \setminus \widehat{W}) = \partial \widehat{W} \) is homeomorphic to a circle \( S^1 \). From this immediately follows, that the set \( \partial W = f^{-1}(\partial \widehat{W}) \) of the limit points of \( W \) is homeomorphic to a circle.

From theorem \( \mathbf{1} \) it immediately follows that the set \( \partial W \) divides \( S^2 \) into two opened connected components and for each of these components it’s closure is homeomorphic to the closed two-dimensional disk. Consequently, the set \( \text{Cl} W \) is homeomorphic to the closed two-dimensional disk. \( \blacksquare \)
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