FULLY NON-LINEAR PARABOLIC EQUATIONS ON COMPACT MANIFOLDS WITH A FLAT HYPERKÄHLER METRIC

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Abstract. Our recent work about fully non-linear elliptic equations on compact manifolds with a flat hyperkähler metric is hereby extended to the parabolic setting. This approach will help us to study some problems arising from hyperhermitian geometry.

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1. Introduction

After Yau’s solution [68] of the Calabi conjecture [14], Cao [15] was able to provide a parabolic proof, using what is now called the Kähler-Ricci flow. Ever since then, it is now a well-established practice to design parabolic geometric flows as an alternative way to solve fully non-linear elliptic equations (see e.g. [10, 19, 22, 23, 33, 34, 47, 48, 54, 55, 56, 69, 70, 71]).

Following this line of thoughts, we extend to the parabolic setting the investigation of our previous work [29], in which we studied a class of fully non-linear elliptic equations on hyperhermitian manifolds. More precisely we took into account equations that are symmetric in the eigenvalues of the quaternionic Hessian of the unknown. In [29] we were inspired by the work of Székelyhidi [58], while here we develop the corresponding parabolic theory in the same spirit as Phong-Tô [45].

Let $(M, I, J, K, g, \Omega_0)$ be a compact locally flat hyperhermitian manifold where $\Omega_0$ is the $(2,0)$-form induced by $g$, i.e. $\Omega_0 = g(J \cdot, \cdot) + ig(K \cdot, \cdot)$ (see Section 2 for further details on the relevant definitions). Here, and throughout the paper, the type decomposition is taken with respect to $I$. The assumption of local flatness allows us to represent locally in quaternionic coordinates every $q$-real $(2,0)$-form $\Omega$ by a hyperhermitian matrix $(\Omega_{rs})$. The same is true for...
a hyperhermitian metric \(g\), whose corresponding matrix we denote \((g_{rs})\). Fix one such form \(\Omega\), which does not need to coincide with \(\Omega_0\). Consider the operator \(\partial J := J^{-1}\partial J\), where \(\partial\) (as well as \(\overline{\partial}\)) is taken with respect to \(I\) everywhere in the paper. For a smooth real function \(\varphi\) on \(M\) the \((2, 0)\)-form \(\partial \overline{\partial} J \varphi\) is \(q\)-real. Then we may associate a hyperhermitian matrix to the form

\[
\Omega_{\varphi} := \Omega + \partial \overline{\partial} J \varphi
\]

let us denote it by \((\Omega_{\varphi})_{rs}\). Set \(A^c[\varphi] = g^{ijr}\Omega^c_{js},\) where \((g^{ijr})\) is the inverse matrix of \((g_{jr})\). The matrix \((A^c[\varphi])\) defines a hyperhermitian endomorphism of \(TM\) with respect to the metric \(g\) and this makes it meaningful to speak about the \(n\)-tuple of its eigenvalues \(\lambda(A[\varphi])\).

The class of parabolic equations that we take into account in the present paper is the following:

\[
\partial_t \varphi = F(A[\varphi]) - h, \quad \varphi(x, 0) = \varphi_0, \quad t \in [0, \infty),
\]

where \(h \in C^\infty(M, \mathbb{R})\) is the datum and \(F(A[\varphi]) = f(\lambda(A[\varphi]))\) is a smooth symmetric operator of the eigenvalues of \(A[\varphi]\) satisfying certain assumptions. More precisely, let \(\Gamma\) be a proper convex open cone in \(\mathbb{R}^n\) with vertex at the origin, containing the positive orthant

\[
\Gamma_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_i > 0, \ i = 1, \ldots, n\},
\]

and assume that \(\Gamma\) is symmetric, i.e. it is invariant under permutations of the \(\lambda_i\)'s. We require that \(f: \Gamma \to \mathbb{R}\) satisfies the following assumptions:

1. \(f_i := \frac{\partial f}{\partial \lambda_i} > 0\) for all \(i = 1, \ldots, n\) and \(f\) is a concave function.
2. \(\sup_{\partial \Gamma} f < \inf_M h\), where \(\sup_{\partial \Gamma} f = \sup_{\lambda_0 \in \partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda)\).
3. For any \(\sigma < \sup_{\Gamma} f\) and \(\lambda \in \Gamma\) we have \(\lim_{t \to \infty} f(t \lambda) \geq \sigma\).

Assumption C1 implies parabolicity of equation (1.1) over the space of \(\Gamma\)-admissible functions, where a function \(\varphi \in C^{1,1}(M \times [0, T])\) is \(\Gamma\)-admissible if

\[
\lambda(A[\varphi]) \in \Gamma, \quad \text{for all } (x, t) \in M \times [0, T).
\]

In particular, from standard parabolic theory, equation (1.1) admits a unique maximal smooth solution. Assumption C2 guarantees that the level sets of \(f\) do not intersect the boundary of \(\Gamma\), this yields non-degeneracy of (1.1) and entails uniform parabolicity, once we obtain the \(C^{1,1}\) estimate. We also remark that the assumptions on \(\Gamma\) imply the inclusion

\[
\Gamma \subseteq \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0 \right\}.
\]

We now project \(\Gamma\) onto a new cone in \(\mathbb{R}^{n-1}\):

\[
\Gamma_\infty = \{ \lambda' = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1} \mid \text{there exists } \lambda_n \in \mathbb{R} \text{ such that } (\lambda', \lambda_n) \in \Gamma \}.
\]

Therefore, for every \(\lambda' \in \Gamma_\infty\), there exists a constant \(s_0\) such that for each \(s \geq s_0\), we have \((\lambda', s) \in \Gamma\). Let \(f_\infty(\lambda') = \lim_{s \to \infty} f(\lambda', s)\). It is an observation of Trudinger [62] that, since \(f\) is concave on \(\Gamma\), there is a dichotomy:

(i) Either \(f_\infty\) is unbounded at any point in \(\Gamma_\infty\) and we will refer to this case by saying that \(f\) is unbounded over \(\Gamma\);

(ii) Or \(f_\infty\) is bounded on \(\Gamma_\infty\) and we will simply say that \(f\) is bounded over \(\Gamma\).

Before stating our main results, we need to recall the terminology of parabolic \(C\)-subsolutions introduced in [45].
**Definition 1.** We say that a function \( \varphi \in C^{1,1}(M \times [0,T]) \) is a parabolic \( C \)-subsolution for equation (1.1) if there exist uniform constants \( \delta, R > 0 \), such that on \( M \times [0,T) \),
\[
f(\lambda(A[\varphi]) + \mu) - \partial_t \varphi + \tau = h, \quad \mu + \delta 1 \in \Gamma_n \quad \text{and} \quad \tau > -\delta
\]
implies that \( |\mu| + |\tau| < R \), where \( 1 = (1,1,\ldots,1) \).

In the unbounded case, as we shall show, any \( \Gamma \)-admissible function is a parabolic \( C \)-subsolution, and we have the following result:

**Theorem 1.** Suppose \( f \) is unbounded on \( \Gamma \). Let \((M,I,J,K,g,\Omega_0)\) be a compact flat hyperkähler manifold. Then for any \( \Gamma \)-admissible initial datum \( \varphi_0 \), the solution \( \varphi \) to (1.1) exists for all time.

Moreover, if we let
\[
\tilde{\varphi} = \varphi - \int_M \varphi \Omega_0^n \wedge \Omega_0^n,
\]
then \( \tilde{\varphi} \) converges smoothly to some function \( \tilde{\varphi}_\infty \in C^\infty(M,\mathbb{R}) \) as \( t \to \infty \), and there exists a constant \( b \in \mathbb{R} \) such that
\[
F(A[\tilde{\varphi}_\infty]) = h + b.
\]

Before we discuss the bounded case we present two of the many possible applications provided by Theorem 1, namely we show the convergence of the quaternionic Hessian flow and of the \((n-1)\)-quaternionic plurisubharmonic flow on compact flat hyperkähler manifolds.

Let \((M,I,J,K,g,\Omega_0)\) be a compact hyperhermitian manifold. Let \( 1 \leq k \leq n \) and fix a q-real \( k \)-positive \((2,0)\)-form \( \Omega \), that is
\[
\frac{\Omega^i \wedge \Omega_0^{n-i}}{\Omega_0^n} > 0 \quad \text{for every } i = 1,\ldots,k.
\]
Then the quaternionic Hessian flow can be written as
\[
\partial_t \varphi = \log \frac{\Omega_k^i \wedge \Omega_0^{n-k}}{\Omega_0^n} - H, \quad \varphi \in \text{QSH}_k(M,\Omega),
\]
where \( H \in C^\infty(M,\mathbb{R}) \) is the datum and \( \text{QSH}_k(M,\Omega) \) is the space of continuous functions \( \varphi \) such that \( \Omega_\varphi \) is a \( k \)-positive q-real \((2,0)\)-form in the sense of currents.

**Theorem 2.** Let \((M,I,J,K,g,\Omega_0)\) be a compact flat hyperkähler manifold and \( \Omega \) a q-real \( k \)-positive \((2,0)\)-form. Then for any smooth initial datum \( \varphi_0 \in \text{QSH}_k(M,\Omega) \),

1. the solution \( \varphi \) to (1.6) exists for all time;
2. the normalization \( \tilde{\varphi} \) (defined as in (1.4)) converges smoothly as \( t \to \infty \) to a function \( \tilde{\varphi}_\infty \in \text{QSH}_k(M,\Omega) \), and there exists a constant \( b \in \mathbb{R} \) such that
\[
\frac{\Omega_k^i \wedge \Omega_0^{n-k}}{\Omega_0^n} = be^H.
\]

We remark that the constant \( b \) in (1.7) is uniquely determined by
\[
b = \frac{\int_M \Omega_k^i \wedge \Omega_0^{n-k} \wedge \Omega_0^n}{\int_M e^H \Omega_0^n \wedge \Omega_0^n}.
\]

Flow (1.6) provides the quaternionic counterpart of the complex Hessian flow (see e.g. [47]). For \( k = 1 \) equation (1.6) is the parabolic Poisson equation, while for \( k = n \) it becomes the parabolic quaternionic Monge-Ampère equation. Thus, Theorem 2 generalizes the main result of [10, 70], which was inspired by the investigation of the quaternionic Monge-Ampère equation.
on compact HKT manifolds proposed in [6] as an analogue of the Calabi-Yau Theorem. Broadly speaking, HKT geometry constitutes a promising and interesting quaternionic analogue of Kähler geometry (see e.g. [5, 8, 9, 16, 26, 31, 32, 38, 40, 41, 53, 57, 64, 65] and the reference therein). As shown in [66], the solvability of such a “quaternionic Calabi conjecture” would lead to an interesting geometric application as it would imply the existence of a balanced HKT metric on compact HKT manifolds with holomorphically trivial canonical bundle. Despite the equation not yet being entirely solved, there are some partial results available (see [2, 3, 4, 10, 21, 27, 28, 29, 52, 70]).

Our second aforementioned application is the \((n - 1)\)-quaternionic plurisubharmonic flow. Let \((M, I, J, K, g, \Omega_0)\) be a compact hyperhermitian manifold and \(\Omega_1\) be a positive \(q\)-real \((2, 0)\)-form. Denote with \(\Delta_g\) the quaternionic Laplacian with respect to \(g\). The \((n - 1)\)-quaternionic plurisubharmonic flow is encoded in the following parabolic equation:

\[
\partial_t \varphi = \log \left( \frac{\Omega_1 + \frac{1}{n-1} [(\Delta_g \varphi) \Omega_0 - \partial \partial_J \varphi]}{\Omega_0^n} \right)^n - H, \quad \varphi \in \text{QPSH}_{n-1}(M, \Omega_1, \Omega_0),
\]

where \(\text{QPSH}_{n-1}(M, \Omega_1, \Omega_0)\) denotes the space of continuous functions \(\varphi\) that are \((n - 1)\)-quaternionic plurisubharmonic with respect to \(\Omega_1\) and \(\Omega_0\), i.e. \(\Omega_1 + \frac{1}{n-1} [(\Delta_g \varphi) \Omega_0 - \partial \partial_J \varphi] > 0\) in the sense of currents.

**Theorem 3.** Let \((M, I, J, K, g, \Omega_0)\) be a compact flat hyperkähler manifold and \(\Omega_1\) a \(q\)-real positive \((2, 0)\)-form. Then for any smooth initial datum \(\varphi_0 \in \text{QPSH}_{n-1}(M, \Omega_1, \Omega_0)\),

1. the solution \(\varphi\) to (1.8) exists for all time;
2. the normalization \(\tilde{\varphi}\) of \(\varphi\) (defined as in (1.4)) converges smoothly as \(t \to \infty\) to a function \(\tilde{\varphi}_\infty \in \text{QPSH}_{n-1}(M, \Omega_1, \Omega_0)\), and there exists a constant \(b \in \mathbb{R}\) such that

\[
\left( \Omega_1 + \frac{1}{n-1} [(\Delta_g \tilde{\varphi}_\infty) \Omega_0 - \partial \partial_J \tilde{\varphi}_\infty] \right)^n = b e^H \Omega_0^n.
\]

The constant \(b\) in (1.9) is uniquely determined by

\[
b = \frac{\int_M (\Omega_1 + \frac{1}{n-1} [(\Delta_g \tilde{\varphi}_\infty) \Omega_0 - \partial \partial_J \tilde{\varphi}_\infty])^n \wedge \Omega_0^n}{\int_M e^H \Omega_0^n \wedge \Omega_0^n}.
\]

The complex version of flow (1.8) was studied by Gill [34] as a parabolic approach to the complex Monge-Ampère equation for \((n - 1)\)-plurisubharmonic functions, which originally arose from superstring theory in the works of Fu, Wang and Wu [24, 25], and was then solved by Tosatti and Weinkove [60, 61] (see also [20, 39]). As proven in [29], the solvability of (1.9) leads to Calabi-Yau–type theorems for quaternionic balanced, quaternionic Gauduchon, and quaternionic strongly Gauduchon metrics. Therefore, convergence of flow (1.8) results to be an interesting tool in the search of special metrics.

Going back to the discussion of the bounded case we observe that, unfortunately, under this assumption \(\Gamma\)-admissible functions might not be \(C\)-subsolutions. Compared to Theorem 1 the main result in the bounded case looks a little bit more artificial, as it requires some additional assumptions.

**Theorem 4.** Suppose \(f\) is bounded on \(\Gamma\). Let \((M, I, J, K, g, \Omega_0)\) be a compact flat hyperkähler manifold. For any \(\Gamma\)-admissible initial datum \(\varphi_0\), let \(\varphi \in C^\infty(M \times [0, T], \mathbb{R})\) be the maximal solution of flow (1.1). Assume further that
(i) either it holds
\[ \partial_t \phi \geq \sup_M (F(A[\phi_0]) - h) ; \]  
\hspace{2cm} (1.10)

(ii) or there exists a non-increasing function \( \Phi \) of class \( C^1 \) on \( \mathbb{R} \) such that
\[ \begin{align*}
\sup_M (\phi(\cdot, t) - \phi(\cdot, t) - \Phi(t)) &\geq 0, \\
\sup_M (\phi(\cdot, t) - \Phi(t)) &\leq -C \inf_M (\phi(\cdot, t) - \Phi(t)) + C
\end{align*} \]  
\hspace{2cm} (1.11)

for all \( t \in (0, T) \) and a time-independent positive constant \( C \). Then \( T = \infty \), i.e. the solution \( \phi \) exists for all times, and the normalization \( \tilde{\phi} \) converges smoothly to a function \( \tilde{\phi}_\infty \in C^\infty(M, \mathbb{R}) \) as \( t \to \infty \), which solves (1.5) for some \( b \in \mathbb{R} \).

Within the bounded case various equations can be included, for instance, parabolic quaternionic Hessian quotient equations, parabolic quaternionic mixed Hessian equations. We limit ourselves to prove the following general result.

**Theorem 5.** Suppose \( f \) is bounded on \( \Gamma \). Let \((M, I, J, K, g, \Omega_0)\) be a compact flat hyperkähler manifold. If there exists a \( \Gamma \)-admissible function \( \phi_0 \) and a \( C \)-subsolution of the equation
\[ F(A[\phi]) = h \]  
in the sense of [29]. Then there exists a smooth solution of the equation
\[ F(A[\phi]) = h + b \]  
for some constant \( b \in \mathbb{R} \).

The organization of the paper is the following. Our main results (Theorems 1 and 4) are proved via a fairly standard technique, that requires a priori estimates for the solution of our flow and its normalization. After a brief discussion of preliminaries in Section 2, we prove estimates of order zero in Section 3. Section 4 is then devoted to finding a bound for the quaternionic Laplacian of the solution, in terms of the norm of its gradient. This type of bound is suitable to apply an interpolation argument that gives an estimate for the gradient; this is explained in Section 5. The aim of Section 6 is to apply an Evans-Krylov–type Theorem to deduce a \( C^{2,\alpha} \) estimate, which is readily improved to higher order estimates via a standard bootstrapping argument. Long-time existence is also proved here. The convergence of flow (1.1) is then showed in Section 7. Finally, in the last Section, we employ these estimates to present a proof of Theorems 1 and 4 as well as Theorems 2, 3 and 5 which readily follow.

**Notation.** Let \( Q \subseteq M \times [0, \infty) \) and fix \( \alpha \in (0, 1] \). We write
\begin{itemize}
  \item \( u \in C^{0,\alpha}(Q) \) if there exists a positive constant \( C \) such that for \((x, s), (y, t) \in Q\), the following holds
    \[ |u(x, s) - u(y, t)| \leq C(|x - y|^\alpha + |t - s|^\frac{\alpha}{2}) \].
  \item \( u \in C^{1,\alpha}(Q) \) if \( u \) is \( C^{\alpha+1} \) in time and \( \nabla u \) is \( C^\alpha \) in space.
  \item \( u \in C^{2,\alpha}(Q) \) if \( \partial_t u \) is \( \frac{\alpha}{2} \)-Hölder in time and \( \nabla^2 u \) is \( C^\alpha \) in space.
\end{itemize}

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2. Preliminaries

In this section we describe some basic useful preliminaries in hyperhermitian geometry and investigate a little bit the notion of parabolic \( \mathcal{C} \)-subsolution.

A smooth manifold \( M \) of real dimension \( 4n \) is called hypercomplex if it is equipped with three complex structures \( I, J, K \) that behave like the standard quaternion units:

\[
IJ = -JI = K,
\]

we call the triple \( (I, J, K) \) a hypercomplex structure on \( M \).

A hypercomplex structure is said to be locally flat if \( M \) is locally isomorphic to the flat space \( \mathbb{H}^n \) of \( n \)-tuples of quaternions. By this, we mean that, locally, \( (I, J, K) \) is the pull-back of the standard hypercomplex structure on \( \mathbb{H}^n \) given by the unit quaternions \( i, j, k \). By abuse of language if \( (I, J, K) \) is locally flat we also say that the hypercomplex manifold \( M \) is locally flat. This class of manifolds was originally studied by Sommese [49]. As explained e.g. in [29], on a locally flat hypercomplex manifold \( (M, I, J, K) \) we can introduce quaternionic coordinates \( (q^1, \ldots, q^n) \) around any point. This is a major difference with respect to complex and real manifold which always admit neighborhoods isomorphic to open subsets of \( \mathbb{C}^n \) and \( \mathbb{R}^n \) respectively.

Let \( (M, I, J, K, g) \) be a hyperhermitian manifold, which is a hypercomplex manifold endowed with a hyperhermitian metric \( g \), i.e. is a Riemannian metric that is Hermitian with respect to each of \( I, J, K \). To the hyperhermitian metric is naturally associated a \( (2,0) \)-form

\[
\Omega_0 = \omega_I + \iota \omega_K
\]

where \( \omega_I \) and \( \omega_K \) are the fundamental forms of \( (g, J) \) and \( (g, K) \) respectively. The form \( \Omega_0 \) will be said to be induced by the hyperhermitian metric \( g \) and satisfies some nice properties, namely it is \( q \)-real and positive. A \( (2p,0) \)-form \( \alpha \) on a hypercomplex manifold \( (M, I, J, K) \) is called \( q \)-real if \( J\alpha = \alpha \), where the action of \( J \) is given by \( J\alpha = \alpha(J \cdot \cdot \cdot J) \); and it is called positive if it is \( q \)-real and \( \alpha(Z_1, JZ_1, \ldots, Z_{2p}, JZ_{2p}) > 0 \) for every vector fields \( Z_1, \ldots, Z_{2p} \) on \( M \) of type \((1,0)\).

A hyperhermitian manifold \( (M, I, J, K, g) \) is called hyperkähler if for any induced complex structure \( L \in \mathbb{H} \) with \( L^2 = -1 \), the corresponding Hermitian manifold \( (M, L, g) \) is Kähler. It is well-known that \( (M, I, J, K, g) \) is hyperkähler if and only if \( d\Omega_0 = 0 \). The weaker assumption \( \partial\delta\Omega_0 = 0 \) is the defining condition for the realm of HKT geometry.

Let \( (M, I, J, K, g) \) be a locally flat hyperhermitian manifold and \( \Omega \) a \( q \)-real \((2,0)\)-form. Equation (1.1) can be expressed in terms of the matrix

\[
A^r_s[\varphi] = g^{3r} \Omega^r_{js} = g^{3r} (\Omega_{js} + \varphi_{js})
\]

where \( (g^{3r}) \) is the inverse matrix of \( (g_{js}) \) and \( (\varphi_{js}) \) denotes the hyperhermitian matrix associated to \( \partial\delta r \varphi \). Recall that a quaternionic matrix \( H \in \mathbb{H}^{n,n} \) is called hyperhermitian if \( t^H = H \). With respect to quaternionic local coordinates \( (q^1, \ldots, q^n) \) it is well-known that

\[
\varphi_{js} = \frac{1}{4} \partial_{\varphi^r} \partial_{q^s} \varphi = : \text{Hess}_{q^s} \varphi,
\]

where the so-called Cauchy-Riemann-Fueter operators \( \partial_{q^r} \) and \( \partial_{q^s} \) act on smooth \( \mathbb{H} \)-valued functions as follows

\[
\partial_{q^r} u := \sum_{i=0}^3 c_i \partial_{x^i} u, \quad \partial_{q^s} u := \partial_{x^0} u e_0 - \sum_{i=1}^3 \partial_{x^i} u e_i.
\]
here we are denoting the unit quaternions $1, i, j, k$ with $e_0, e_1, e_2, e_3$ and we are taking the derivatives with respect to the real coordinates underlying the quaternionic ones, according to the relation $q' = x'_0 + x'_1i + x'_2j + x'_3k$. The operators $\partial_q^r$ and $\partial_q^s$ commute, but in general they do not satisfy the Leibniz or the chain rule, so care must be taken during computations. Taking the real part of the trace of the quaternionic Hessian $\text{Hess}_H u$ with respect to the metric $g$ we have a second order linear elliptic operator called the quaternionic Laplacian
\[
\Delta_g u := \text{Re } \text{tr}_g (\text{Hess}_H u) = \text{Re } \left( g_0^{jr} u^j_r \right)
\]

More generally, when the manifold is not necessarily locally flat, the quaternionic Laplacian allows an intrinsic definition as
\[
\Delta_g u := n \frac{\partial \partial_f u \wedge \Omega_0^{n-1}}{\Omega_0^n},
\]
where $\Omega_0$ is the $(2,0)$-form induced by $g$. By [10, Lemma 3], when the manifold is locally flat we recover the previous definition. Note that in quaternionic local coordinates $\Delta_g u$ is the sum of the eigenvalues of $\text{Hess}_H u$ with respect to $g$.

Now we briefly discuss the notion of $C$-subsolution. Székelyhidi introduced it in [58] for elliptic equations. His definition is also shown to be a relaxation of that given by Guan [35]. As for the parabolic case, Guan, Shi and Sui [36] worked on Riemannian manifolds with the classical notion of a subsolution, while Phong and Tô provided in [45] the extension to the parabolic case of Székelyhidi’s definition. Of course, as we shall see in a moment with a characterization of $C$-subsolutions, what happens in hyperhermitian geometry is entirely parallel to the Hermitian case. Thus, Definition 1 is the right extension of the one given in [29] for the elliptic case. We shall refer to $C$-subsolutions in the sense of [29] as elliptic ones.

**Lemma 6.** Let $\varphi \in C^{1,1}(M \times [0, +\infty))$ be such that $\|\varphi\|_{C^{1,1}} < +\infty$. Then $\varphi$ is a parabolic $C$-subsolution if and only if there exists a uniform constant $\rho > 0$ such that
\[
\lim_{s \to \infty} f(\lambda [\varphi(x,t)] + se_i) - \partial_t \varphi(x,t) > \rho + h(x)
\]
for each $i = 1, \ldots, n$, where $e_i$ is the $i^{th}$ standard basis vector of $\mathbb{R}^n$. In particular when $\varphi$ is time-independent it is a $C$-subsolution in the parabolic sense if and only if it is such in the elliptic sense.

**Proof.** The proof can be reproduced almost verbatim from [45, Lemma 8].

This lemma in particular implies that when $f$ is unbounded over $\Gamma$, every $\Gamma$-admissible function is a parabolic $C$-subsolution.

We conclude this section by fixing some notations. Unless otherwise stated we shall always denote by $\varphi, \tilde{\varphi}$ and $\varphi$ the maximal solution to flow (1.1) with initial datum $\varphi_0$, its normalization as in (1.4) and a parabolic $C$-subsolution in the sense of Definition 1, respectively. All these functions are assumed to be defined over $M \times [0, T)$, where $(M, I, J, K, g)$ is a compact locally flat hyperhermitian manifold and $T$ is the maximal time of existence of $\varphi$.

From here on, we will always denote with $C$ a positive constant that only depends on background data (not on time!), including the initial datum $\varphi_0$. Occasionally we might say that $C$ is uniform, to stress that it is time-independent. As it is customary, the constant $C$ may change value from line to line.
3. $C^0$ estimates

In this section we achieve estimates of order zero for the solution $\varphi$ and its normalization $\tilde{\varphi}$. We start by bounding their time derivatives, then, in order to treat the bounded case we need an additional inequality proved in Lemma 9. Such lemma follows as an application of the parabolic version of the Alexandroff-Bakelman-Pucci (ABP) inequality due to Tso [63, Proposition 2.1] by adapting the argument of Phong-Tô [45, Lemma 1].

3.1. Bounds on $\partial_t \varphi$ and $\partial_t \tilde{\varphi}$.

**Lemma 7.** We have

$$\inf_M \left( F(A[\varphi_0]) - h \right) \leq \partial_t \varphi \leq \sup_M \left( F(A[\varphi_0]) - h \right)$$

(3.1)

and

$$|\partial_t \tilde{\varphi}| \leq C,$$  

(3.2)

for a uniform constant $C > 0$ depending only on $h$ and the initial datum $\varphi_0$.

**Proof.** Differentiating the flow (1.1) along $\partial_t$ we see that $\partial_t \varphi$ satisfies the following heat type equation

$$\partial_t (\partial_t \varphi) = \frac{1}{4} \text{Re} \left( F^{rs} \partial_{q^r} \partial_{q^s} (\partial_t \varphi) \right),$$

(3.3)

where $F^{rs} := \frac{\partial F}{\partial A_{rs}}$. By the parabolic maximum principle for (3.3), we know that $\partial_t \varphi$ hits its extremum at $t = 0$. Thus,

$$\inf_{M \times \{0\}} \partial_t \varphi \leq \partial_t \varphi \leq \sup_{M \times \{0\}} \partial_t \varphi, \quad \partial_t \varphi(\cdot, 0) = F(A[\varphi_0]) - h$$

and we then obtain (3.1). The bound (3.2) on $|\partial_t \tilde{\varphi}|$ follows immediately.  

We remark that a direct consequence of the previous lemma is the following short-time estimate:

$$|\varphi| \leq C\delta,$$  

(3.4)

on $M \times [0, \delta]$.

3.2. Intermediate bounds.

**Lemma 8.** Let $\psi$ be a smooth function on $M \times [0, T)$ satisfying

$$\Delta_g \psi \geq c_0$$

(3.5)

for a uniform constant $c_0 \in \mathbb{R}$, then there exist $p, C > 0$, depending only on the background data, such that

$$\|\psi - \sup_M \psi\|_{L^p(M)} \leq C.$$  

**Proof.** The proof can be found in [29, Lemma 5], for convenience of the reader we shall also sketch it here. Take an open cover of $M$ made of coordinate balls $B_{2r_i}(x_i)$ such that $\{B_i = B_{r_i}(x_i)\}$ still covers $M$. Set $\Psi = \psi - \sup_M \psi$ for simplicity. By the elliptic inequality (3.5) we can apply the weak Harnack inequality [30, Theorem 9.22] deducing

$$\|\Psi\|_{L^p(B_i)} \leq C \left( \inf_{B_i} (-\Psi) + 1 \right)$$

(3.6)

where $p, C > 0$ depend only on the cover and the background metric. Since $\Psi$ is non-positive there is at least one coordinate ball $B_j$ such that $\inf_{B_j} (-\Psi) = 0$, and thus $\|\Psi\|_{L^p(B_j)} \leq C$. The bound on $\|\Psi\|_{L^p(B_j)}$ also gives a bound for $\inf_{B_i} (-\Psi)$ on all coordinate balls intersecting $B_j$. 


Using again (3.6) and repeating the argument we obtain an upper bound on each ball of the cover.

Lemma 9. If there exists a non-increasing function \(\Phi \in C^1([0, T), \mathbb{R})\) satisfying
\[
\sup_M (\varphi(\cdot, t) - \varphi(\cdot, t) - \Phi(t)) \geq 0,
\]
then there exists a constant \(C > 0\), depending only on \(\Omega, g, \varphi, \|\varphi_0\|_{C^0}\) such that
\[
\varphi(x, t) - \varphi(x, t) - \Phi(t) \geq -C \quad \text{for all } (x, t) \in M \times [0, T).
\]

Proof. First, observe that the requirement \(\Phi' \leq 0\) implies that \(\varphi + \Phi\) is still a parabolic \(C\)-subsolution of (1.1), therefore, as long as the involved constants do not depend on the time derivative of \(\varphi\), we may assume \(\Phi \equiv 0\).

Choose \(\delta \in (0, 1)\) and \(R > 0\) such that (1.3) holds for the subsolution \(\varphi\). By (3.4), it suffices to estimate \(v = \varphi - \varphi\) on \(M \times [\delta, T]\). Fix an arbitrary \(T' < T\) and assume \(v\) achieves its minimum \(S\) at a point \((x_0, t_0) \in M \times [\delta, T']\), i.e.,
\[
S = v(x_0, t_0) = \min_{M \times [\delta, T']} v.
\]

Now we are reduced to prove that if \(\sup_M v \geq 0\) for all \(t \in [\delta, T]\), then \(S\) is bounded from below by a constant depending only on \(\Omega, g, \varphi, \|\varphi_0\|\) and independent of \(T'\).

Consider quaternionic local coordinates \((q^1, \ldots, q^n)\) centered at the point \(x_0\). We may identify such coordinate neighborhood with the open ball of unit radius \(B_1 = B_1(0) \subseteq \mathbb{H}^n\) centered at the origin. Let
\[
w(x, t) = v(x, t) + \frac{\delta^2}{4}|x|^2 + (t - t_0)^2,
\]
be a function defined on \(\mathcal{B} = B_1 \times [t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}]\). Observe that \(\inf_{\mathcal{B}} w = w(0, t_0) = v(0, t_0) = S\) and \(\inf_{\partial \mathcal{B}} w \geq w(0, t_0) + \frac{\delta^2}{8}\). These conditions allow us to apply the parabolic ABP method of Tso [63, Proposition 2.1] to obtain
\[
C_0 \delta^{8n+2} \leq \int_{\mathcal{P}} |\partial_t w| \det(D^2 w),
\]
where \(C_0 > 0\) is a dimensional constant,
\[
\mathcal{P} = \left\{ (x, t) \in \mathcal{B} \mid \begin{array}{l}
w(x, t) \leq S + \frac{\delta^2}{4}, \quad |Dw(x, t)| < \frac{\delta^2}{8}, \\
w(y, s) \geq w(x, t) + Dw(x, t) \cdot (y - x), \quad \forall y \in B_1, \quad s \leq t
\end{array} \right\}
\]
is the parabolic contact set of \(w\) on \(\mathcal{B}\) and \(Dw, D^2 w\) are the gradient and the (real) Hessian of \(w\) on \(M\) with respect to the variable \(x\).

Claim: both \(|\partial_t w|\) and \(\det(D^2 w)\) are bounded on \(\mathcal{P}\).

Let \(\tau = -\partial_t \varphi + \partial_t \varphi - \partial_t v + \mu = \lambda(A(\varphi) - \lambda(A(\varphi)))\). Observe that \(D^2 w \geq 0\) and \(\partial_t w \leq 0\) on \(\mathcal{P}\). Thus,
\[
\tau = -\partial_t w + 2(t - t_0) \geq -\delta, \quad \mu + \delta \mathbb{1} \in \Gamma_n.
\]
Now by Definition 1 we conclude that \(|\tau| + |\mu| \leq R\), then \(|\partial_t w| \leq R\) and \(\text{Hess}_w w\) is a bounded matrix. But then we are done as we have
\[
\det(D^2 w) \leq 2^{4n} \det(\text{Hess}_w w)^4 \quad \text{on } \mathcal{P},
\]
which follows from a computation in [11] and [51, Lemma 2], (see also the proof of [3, Proposition 2.1]). Here, on the right-hand side, “det” denotes the Moore determinant, introduced in [44] (see also e.g. [1, 7, 53]). This confirms the claim.
With this claim at hand, by (3.7) we have
\[ C_0 \delta^{8n+2} \leq C \text{Vol}(P). \] (3.8)
From (1.2) we readily obtain \( \text{Re} \text{tr}_g(\Omega \varphi) > 0 \), where \( \Omega \varphi = \Omega + \partial_t \varphi \), which in turn yields a uniform lower bound for the quaternionic Laplacian of \( \varphi \):
\[ \Delta_g \varphi = \text{Re} \text{tr}_g(\Omega \varphi) - \text{Re} \text{tr}_g(\Omega) \geq -C. \]
This also gives a uniform lower bound for \( \Delta_g v \). Then by Lemma 8, we see that
\[ \| v - \sup_M v \|_{L^p(M)} \leq C. \] (3.9)
The definition of \( P \) and our assumption that \( \sup_M v \geq 0 \) on \([0, T)\) yields
\[ v - \sup_M v \leq v \leq w + \frac{\delta^2}{4} \] on \( P \).
We may further assume \( S + \frac{\delta^2}{4} < 0 \), otherwise we are done. As a consequence for any \( p > 0 \)
\[ \left| S + \frac{\delta^2}{4} \right|^p \text{Vol}(P) \leq \int_P |v - \sup_M v|^p \text{dx} \text{dt} \leq \int_{[t_0 - \frac{\delta^2}{4} + \frac{\delta^2}{4}, t_0 + \frac{\delta^2}{4}]} \| v - \sup_M v \|^p_{L^p(M)} \text{dt} \leq C \delta, \]
where we have used (3.9). This, together with (3.8), gives the uniform lower bound of \( S \) we were after. \( \Box \)

### 3.3. Bounds on \( \varphi \) and \( \tilde{\varphi} \).

As it often happens for solutions to flows, we only manage to control the oscillation and not the full \( C^0 \) norm. On the other hand, once the oscillation is under control, we immediately achieve the \( C^0 \) estimate for the normalization of the solution.

**Proposition 10.** Let \( f \) be either bounded or unbounded. In case \( f \) is bounded on \( \Gamma \) assume that it satisfies either one of the two conditions expressed in Theorem 4. Then there exists a uniform constant \( C > 0 \), depending only on the background data such that
\[ \text{osc}_M \varphi(\cdot, t) := \sup_M \varphi(\cdot, t) - \inf_M \varphi(\cdot, t) \leq C, \] (3.10)
and
\[ \| \tilde{\varphi} \|_{C^0} \leq C. \] (3.11)

**Proof.** First, we observe that (3.11) follows from (3.10). Indeed, by the normalization of \( \tilde{\varphi} \), for any \((x, t) \in M \times [0, T)\) we can find \( y(x) \in M \) such that \( \tilde{\varphi}(y(x), t) = 0 \), therefore
\[ \| \tilde{\varphi} \|_{C^0} = \sup_{(x, t) \in M} \left| \tilde{\varphi}(x, t) - \tilde{\varphi}(y(x), t) \right| = \sup_{(x, t) \in M} \left| \varphi(x, t) - \varphi(y(x), t) \right| \leq \text{osc}_M \varphi(\cdot, t). \]

We will prove (3.10) by rewriting the flow (1.1) as
\[ F(A[\varphi]) = h + \partial_t \varphi, \] (3.12)
and interpreting it for every fixed time as an elliptic equation with datum \( h + \partial_t \varphi \). We split the argument into two cases according as \( f \) is bounded or unbounded.

- **Case 1.** \( f \) is unbounded on \( \Gamma \). In this case any \( \Gamma \)-admissible function is a parabolic \( C \)-subsolution, therefore we can take the initial datum \( \varphi_0 \) as such. Since \( \varphi_0 \) is time-independent, it can be regarded as an elliptic \( C \)-subsolution. Furthermore, by Lemma 7 we know that the right-hand side of (3.12) is uniformly bounded, therefore we may apply [29, Proposition 6] to obtain (3.10).
• Case 2. $f$ is bounded on $\Gamma$. We consider two subcases. Assume that condition (i) of Theorem 1 holds, then (1.10) and Lemma 7 imply that $\partial_t \varphi \geq \partial_t \varphi$, this entails that $\varphi$ is a $C$-subsolution of (3.12) in the elliptic sense. Again (3.10) follows from [29, Proposition 6].

If, instead, condition (ii) of Theorem 1 is satisfied, then there exists $\Phi \in C^\infty([0, T), \mathbb{R})$ with $\Phi' \leq 0$ satisfying (1.11) and we can readily apply Lemma 9 to conclude. □

4. QUATERNIONIC LAPLACIAN ESTIMATE

Here we adopt the technique of [17, 37] which allows to find a Laplacian bound in terms of the norm of the gradient.

Before we tackle the proof, we recall the following preliminary lemma given in Phong-Tô [45, Lemma 3], which was inspired by the elliptic version of [58, Proposition 6]. We will use the following derivatives of $F$

$$F_{rs} := \frac{\partial F}{\partial A_{rs}}, \quad F_{rs,tt} := \frac{\partial^2 F}{\partial A_{rs} \partial A_{tt}}.$$  

Lemma 11. Let $\delta, R$ be uniform constants such that on $M \times [0, T)$, if $(\mu, \tau) \in \mathbb{R}^n \times \mathbb{R}$ satisfy (1.3), then $|\mu| + |\tau| < R$. There exists a uniform constant $\kappa > 0$ depending on $\delta$ and $R$ such that if $|\lambda(A[\varphi]) - \lambda(A[\varphi])| > R$, we have

$$\text{either} \quad \text{Re} \, F^{rs}(A[\varphi]) \left( A_{rs}[\varphi] - A_{rs}[\varphi] \right) - (\partial_t \varphi - \partial_t \varphi) > \kappa \sum_{r=1}^{n} F^{rr}(A[\varphi]),$$

or

$$F^{ss}(A[\varphi]) > \kappa \sum_{r=1}^{n} F^{rr}(A[\varphi]), \quad \text{for all } s = 1, \ldots, n.$$  

Proof. Since the quaternionic analogue of the Schur-Horn theorem holds, see e.g. [29, Lemma 8], the proof of the lemma can be adapted from [45, Lemma 3]. □

Proposition 12. Suppose $(M, I, J, K, g)$ is a compact flat hyperkähler manifold. Then there is a constant $C > 0$, depending only on $(M, I, J, K)$, $\|g\|_{C^2}$, $\|h\|_{C^2}$, $\|\Omega\|_{C^2}$, $\|\varphi\|_{C^{1,1}}$, $\|\partial_t \varphi\|_{C^0}$ and $\|\varphi\|_{C^0}$, such that

$$\|\Delta_g \varphi\|_{C^0} \leq C (\|\nabla \varphi\|_{C^0} + 1).$$

Proof. By (1.2) we already know that the quaternionic Laplacian is uniformly bounded from below, therefore it is enough to obtain a bound of the form

$$\frac{\lambda_1}{\|\nabla \varphi\|_{C^0} + 1} \leq C,$$

where $\lambda_1$ is the largest eigenvalue of $A[\varphi]$. Let $T' < T$, all computations will be performed in quaternionic local coordinates around some fixed point $p_0 = (x_0, t_0) \in M \times [0, T']$ which we will specify in a moment. As pointed out by Székelyhidi [58] in order for $\lambda_1: M \rightarrow \mathbb{R}$ to define a smooth function at $p_0$ we need the eigenvalues to be distinct; to be sure of that, we perturb the matrix $A$ as follows. Using the assumption that $g$ is a flat hyperkähler metric we may take quaternionic coordinates such that $(g_{rs})$ is the identity in the whole neighborhood of $p_0$ and $(\Omega^r_{rs})$ is diagonal at $p_0$. In particular $A[\varphi]$ is diagonal with ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $D$ be a constant diagonal matrix with entries satisfying $0 = D_{11} < D_{22} < \cdots < D_{nn}$. The matrix $\tilde{A} = A[\varphi] - D$ has distinct eigenvalues $\tilde{\lambda}_r$ by construction, and its largest eigenvalue $\tilde{\lambda}_1$ coincides with $\lambda_1$ at $p_0$.  

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Choose $p_0 \in M \times [0, T']$ to be a maximum point of the function
\[ \hat{G} = 2\sqrt{\lambda_1} + \alpha(|\nabla \varphi|^2) + \beta(\tilde{v}) \]
where
\[
\alpha(s) = -\frac{1}{2} \log \left(1 - \frac{s}{2N}\right), \quad N = \|\nabla \varphi\|_{C^0}^2 + 1,
\]
\[
\beta(s) = -2Ss + \frac{1}{2}s^2, \quad S > \|\tilde{v}\|_{C^0}, \text{ large constant to be chosen later},
\]
and $\tilde{v}$ is the normalization of $v = \varphi - \varphi_1$. As said, to avoid smoothness issues we shall not work with $\lambda_1$. Therefore, in a small neighborhood of $p_0$, instead of working with $\hat{G}$ we consider the function
\[ G = 2\sqrt{\lambda_1} + \alpha(|\nabla \varphi|^2) + \beta(\tilde{v}). \]

It will be useful to observe that
\[
\frac{1}{4N} < \alpha'(|\nabla \varphi|^2) < \frac{1}{2N}, \quad \alpha'' = 2(\alpha')^2, \quad (4.1)
\]
\[
S \leq -\beta'(\tilde{v}) \leq 3S, \quad \beta'' = 1. \quad (4.2)
\]

We also remark that, as in [45], at the point $p_0$ there exists a constant $\tau > 0$ depending only on $\|h\|_{C^0}$ and $\|\partial_t \varphi\|_{C^0}$ such that
\[ F := \sum_{a=1}^n F^{aa}(A[\varphi]) = \sum_{a=1}^n f_a(\lambda(A[\varphi])) > \tau. \]

Indeed, for any $\sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)$ and by [58], there exists a constant $\tau' > 0$ depending only on $\sigma$ such that $\sum_a f_a(\lambda) > \tau'$ for any $\lambda \in \partial \Gamma^\sigma$. Now for $f(\lambda(A[\varphi])) = \partial_t \varphi + h$, $\sigma$ lies a compact set bounded by $\|h\|_{C^0} + \|\partial_t \varphi\|_{C^0}$ and whence $F$ is bounded below by some $\tau$. This will be useful to absorb some constants during our computations.

The linearized operator $L$ is defined by
\[ L(u) = 4 \sum_{a,b=1}^n F^{ab} g^{\bar{a} \bar{b}} u_{\bar{a} \bar{b}} - 4\partial_t u, \]
where $u_{\bar{a} \bar{b}} = \frac{1}{4} \partial_{\bar{q}^a} \partial_{\bar{q}^b} u$. In particular, at $p_0$ the linearized operator has the simpler expression $L(u) = 4(F^{aa} u_{\bar{a} \bar{a}} - \partial_t u)$. We emphasize here that the terms $F^{aa}$ are real, a fact that we shall use in all the computations to come.

At the maximum point $p_0$ we have $L(G) \leq 0$ i.e.
\[ 0 \geq L\left(2\sqrt{\lambda_1}\right) + L\left(\alpha(|\nabla \varphi|^2)\right) + L\left(\beta(\tilde{v})\right). \quad (4.3) \]

4.1. **Bound for** $L(2\sqrt{\lambda_1})$.

We claim that
\[ L\left(2\sqrt{\lambda_1}\right) \geq -\frac{F^{aa} \Omega_{11,a}^\varphi}{2\lambda_1 \sqrt{\lambda_1}} - \frac{CF}{\sqrt{\lambda_1}}, \quad (4.4) \]
where $\Omega_{11,a}^\varphi = \partial_{\bar{q}^a} \Omega_1^\varphi$ and $C > 0$ is a positive uniform constant.
We clearly have

\[
L\left(2\sqrt{\tilde{\lambda}_1}\right) = 8F^{aa}\left(\sqrt{\tilde{\lambda}_1}\right)_{aa} - 8\partial_t\left(\sqrt{\tilde{\lambda}_1}\right)
= 2F^{aa}\sum_{p=0}^{3}\left(\sqrt{\tilde{\lambda}_1}\right)_{x^p,x^p} - 8\partial_t\left(\sqrt{\tilde{\lambda}_1}\right)
= \frac{1}{\sqrt{\tilde{\lambda}_1}}\left(3F^{aa}\sum_{p=0}^{3}\tilde{\lambda}_{1,x^p,x^p} - 4\partial_t\tilde{\lambda}_1\right) - F^{aa}\sum_{p=0}^{3}\left|\lambda_{1,x^p}\right|^2
\]

where the subscripts \(x_a^p\) denote the real derivative with respect to the corresponding real coordinates underlying the chosen quaternionic local coordinates. Using the formulas for the derivatives of the eigenvalues (see [29]) and the fact that \(D\) is a constant matrix we obtain at \(p_0\)

\[
\tilde{\lambda}_{1,x^p} = \tilde{\lambda}_1^{rs}\tilde{\lambda}_{r,s,x^p} = \Omega_{11,x^p}^\varphi
\]

\[
\tilde{\lambda}_{1,x^p,x^p} = \tilde{\lambda}_1^{rs,lt}\tilde{\lambda}_{r,s,x^p}\tilde{\lambda}_{lt,x^p} + \tilde{\lambda}_1^{rs}\tilde{\lambda}_{r,s,x^p,x^p} = 2\sum_{r>1}^{3}\frac{\left|\Omega_{r1,x^p}\right|^2}{\lambda_1 - \lambda_r} + \frac{\left|\Omega_{11,x^p}\right|^2}{\lambda_1 - \lambda_r}.
\]

Observe that

\[
\sum_{p=0}^{3}\Omega_{11,x^p}^\varphi = \sum_{p=0}^{3}\left(\Omega_{11,x^p}^\varphi + \varphi_{11,x^p}\right) = 4\Omega_{11,aa} + 4\varphi_{aa,11}
\]

\[
= 4\Omega_{11,aa} - 4\Omega_{aa,11} + 3\Omega_{aa,x^p,x^1}
\]

which implies

\[
F^{aa}\tilde{\lambda}_{1,x^p,x^p} \geq F^{aa}\sum_{p=0}^{3}\Omega_{aa,x^p,x^1} - C\mathcal{F}.
\]

Differentiating the equation \(\partial_t\varphi = F(A[\varphi]) - h\) twice with respect to \(x^1\) gives, at \(p_0\),

\[
F^{rs,lt}\Omega_{rs,x^p,x^1} - F^{aa}\Omega_{aa,x^1,x^1} = h_{x^p,x^1} + \partial_t(\varphi_{x^1,x^1}).
\]

by this and the concavity of \(F\)

\[
F^{aa}\sum_{p=0}^{3}\tilde{\lambda}_{1,x^p,x^p} - 4\partial_t\tilde{\lambda}_1 \geq \sum_{p=0}^{3} \left(F^{aa}\Omega_{aa,x^p,x^1} - \partial_t(\varphi_{x^1,x^1})\right) - C\mathcal{F} \geq -C\mathcal{F}.
\]

Substituting (4.6) into (4.5) we obtain the claimed inequality (4.4).

4.2. Bound for \(L(\alpha(|\nabla\varphi|^2))\).
First of all, since $|\nabla \varphi|^2 = \sum_r \varphi_r \varphi_r$ is real, we may compute
\[
L \left( \alpha(|\nabla \varphi|^2) \right) = \alpha'' F^{a\alpha} \sum_{p=0}^{3} \left( \sum_{r=1}^{n} (\varphi_{rx_p} \varphi_r + \varphi_r \varphi_{rx_p}) \right)^2
\]
\[
+ \alpha' F^{a\alpha} \sum_{p=0}^{3} \sum_{r=1}^{n} \left( \varphi_{rx_p} \varphi_r + 2|\varphi_{rx_p}|^2 + \varphi_r \varphi_{rx_p} \varphi_{rx_p} \right)
\]
\[
- \alpha' \sum_{r=1}^{n} \left( \partial_t (\varphi_r) \varphi_r + \varphi_r \partial_t (\varphi_r) \right)
\]
(4.7)
Differentiating the equation $\partial_t \varphi = F(A[\varphi]) - h$ yields
\[
\partial_t (\varphi_{x_p}) = F^{a\alpha} \Omega^{\varphi}_{a\alpha,x_p} - h_{x_p}, \quad \text{at } p_0.
\]
Together with Cauchy-Schwarz inequality and (4.1) this yields
\[
\alpha' F^{a\alpha} \sum_{r=1}^{n} (\varphi_{r\alpha} \varphi_r + \varphi_r \varphi_{r\alpha}) - \alpha' \sum_{r=1}^{n} \left( \partial_t (\varphi_r) \varphi_r + \varphi_r \partial_t (\varphi_r) \right)
\]
\[
= \alpha' \sum_{r=1}^{n} \left( (h_r - F^{a\alpha} \Omega_{a\alpha,r}) \varphi_r + \varphi_r (h_r - F^{a\alpha} \Omega_{a\alpha,r}) \right)
\]
\[
\geq -\frac{C}{N} (N^{1/2} + N^{1/2} \mathcal{F}) \geq -C \mathcal{F},
\]
Moreover, we have
\[
2\alpha' F^{a\alpha} \sum_{r=1}^{n} \sum_{p=0}^{3} |\varphi_{rx_p}|^2 \geq \frac{1}{2N} F^{a\alpha} \sum_{p=0}^{3} \varphi_{x_p x_p}^2 = \frac{8}{N} F^{a\alpha} \varphi_{a\alpha}^2
\]
\[
= \frac{8}{N} F^{a\alpha} (\lambda_a - \Omega_{a\alpha})^2 \geq \frac{4}{N} F^{a\alpha} \lambda_a^2 - C \mathcal{F}.
\]
Combining the last two inequalities with (4.7) we get
\[
L \left( \alpha(|\nabla \varphi|^2) \right) \geq \alpha'' F^{a\alpha} \sum_{p=0}^{3} \left( \sum_{r=1}^{n} (\varphi_{rx_p} \varphi_r + \varphi_r \varphi_{rx_p}) \right)^2 + \frac{4}{N} F^{a\alpha} \lambda_a^2 - C \mathcal{F}.
\]
(4.8)

4.3. Conclusion of the proof.

In view of (4.4) and (4.8), the main inequality (4.3) becomes
\[
0 \geq \alpha'' F^{a\alpha} \sum_{p=0}^{3} \left( \sum_{r=1}^{n} \text{Re}(\varphi_{rx_p} \varphi_r) \right)^2 - \frac{F^{a\alpha} |\Omega^\varphi_{11,a}|^2}{2\lambda_1 \sqrt{\lambda_1}}
\]
\[
+ \frac{4F^{a\alpha} \lambda_a^2}{N} + L (\beta(\tilde{v})) - C \mathcal{F}
\]
(4.9)
Since $p_0$ is a maximum point for $G$ we have
\[
0 = G_{x_p} = \frac{\Omega^\varphi_{11,x_p}}{\sqrt{\lambda_1}} + 2\alpha' \sum_{r=1}^{n} \text{Re}(\varphi_{rx_p} \varphi_r) + \beta' \tilde{v}_{x_p}.
\]
and therefore, by (4.1)
\[
\alpha'' F^{aa} \left( 2 \sum_{r=1}^{n} \text{Re}(\varphi_{\rho \rho} \varphi_{\rho}) \right)^2 = 2 F^{aa} \left( \frac{\Omega^{p}_{11, x_{p}^2}}{\sqrt{\lambda_1}} + \beta' \tilde{v}_{x_{p}^2} \right)^2 \geq 2 \varepsilon \frac{F^{aa} (\Omega^{p}_{11, x_{p}^2})^2}{\lambda_1} - \frac{2 \varepsilon}{1 - \varepsilon} (\beta')^2 F^{aa} \tilde{v}^2_{x_{p}^2},
\]
where we used the inequality \((a + b)^2 \geq \varepsilon a^2 - \frac{\varepsilon}{1 - \varepsilon} b^2\), which holds for \(\varepsilon \in (0, 1)\). Assuming without loss of generality that \(\sqrt{\lambda_1} > \frac{1}{4\varepsilon}\), we get
\[
\left( 4\varepsilon \sqrt{\lambda_1} - 1 \right) \frac{F^{aa} (\Omega^{p}_{11, x_{p}^2})^2}{2\lambda_1 \sqrt{\lambda_1}} \geq 0.
\]
Putting together (4.10), (4.11) and the calculation
\[
L (\beta(v)) = \beta'' F^{aa} |\tilde{v}_{a}|^2 + 4 \beta' F^{aa} \tilde{v}_{aa} - 4 \beta' \partial_t \tilde{v}
\]
(4.9) simplifies to
\[
0 \geq \frac{4 F^{aa} \lambda^2_a}{N} + \left( \beta'' - \frac{2 \varepsilon}{1 - \varepsilon} (\beta')^2 \right) F^{aa} |\tilde{v}_{a}|^2 + 4 \beta' (F^{aa} \tilde{v}_{aa} + \partial_t \tilde{v}) - C \mathcal{F}.
\]
If we choose \(\varepsilon = 1/(18S^2 + 1) < 1\), then (4.2) yields
\[
\beta'' - \frac{2 \varepsilon}{1 - \varepsilon} (\beta')^2 \geq 0,
\]
therefore we finally arrive at
\[
0 \geq \frac{4 F^{aa} \lambda^2_a}{N} + 4 \beta' (F^{aa} \tilde{v}_{aa} + \partial_t \tilde{v}) - C \mathcal{F}.
\]
Supposing \(\lambda_1 > R\) we have \(|\lambda(A[\varphi])| > R\) and we can then apply Lemma 11 according to which there exists \(\kappa > 0\) such that one of the following two cases occur:

- **Case 1:**
  \[
  \text{Re} F^{\rho \rho} (A[\varphi]) \left( A_{\rho \rho}[\varphi] - A_{\rho \rho}[\varphi] \right) - (\partial_t \varphi - \partial_t \varphi) > \kappa \sum_{r=1}^{n} F^{\rho \rho} (A[\varphi]),
  \]
  i.e. \(- F^{aa} \tilde{v}_{aa} + \partial_t \tilde{v} > \kappa \mathcal{F}\) at \(p_0\), where we recall that \(v = \varphi - \varphi_0\). This immediately gives
  \[
  F^{aa} \tilde{v}_{aa} - \partial_t \tilde{v} < -C_1 \mathcal{F}
  \]
  where \(C_1\) depends on \(||\partial_t v||_{C^{0}}\). Choosing \(S\) so large as to have \(\beta'(F^{aa} \tilde{v}_{aa} - \partial_t \tilde{v}) \geq C \mathcal{F}\) we deduce from (4.12) the inequality \(0 \geq \frac{4}{N} F^{aa} \lambda^2_a\) which is a contradiction, hence this case cannot occur.

- **Case 2:**
  \[
  F^{s s} (A[\varphi]) > \kappa \sum_{r=1}^{n} F^{\rho \rho} (A[\varphi]), \quad \text{for all } s = 1, \ldots, n,
  \]
  which in particular gives \(F^{11} > \kappa \mathcal{F}\) and thus \(F^{aa} \lambda^2_a \geq F^{11} \lambda^2_1 \geq \kappa \mathcal{F} \lambda^2_1\). We may assume \(F^{aa} \lambda_1 \leq F^{aa} \lambda^2_a/(6NS)\) because if this were not true we would have \(\kappa \mathcal{F} \lambda^2_1 < 6NS \mathcal{F} \lambda_1\)
  \[
  4 \beta' (F^{aa} \tilde{v}_{aa} - \partial_t \tilde{v}) \geq -12 SF^{aa} \varphi_{aa} - C \mathcal{F} \geq -\frac{2 F^{aa} \lambda^2_a}{N} - C \mathcal{F},
  \]
This last inequality and (4.12) finally give
\[ 0 \geq 2\kappa \frac{\lambda_1^2}{N} - C, \]
as was to be shown. The desired bound is valid at the maximum point \( x_0 \) of \( G \), and then also globally. \( \square \)

**Remark.** As in the elliptic case treated in [29], this is the only step of the proof of our main results that uses the assumption that the metric \( g \) is hyperkähler.

### 5. Gradient estimate

The bound find in the previous section is well-suited for a standard interpolation argument which allows us to obtain directly a gradient bound and consequently, also a Laplacian bound. We thank the anonymous referee for pointing out this proof to us, which simplifies our previous one.

**Proposition 13.** Suppose there is a uniform constant \( C \) such that
\[ \| \varphi \|_{C^0} \leq C, \quad \| \Delta_g \varphi \|_{C^0} \leq C (\| \nabla \varphi \|_{C^0} + 1), \]
then there is a uniform bound
\[ \| \varphi \|_{C^1} \leq C. \]

**Proof.** Interpolation theory (see [30, section 6.8]) reveals that for any \( \varepsilon > 0 \) and \( 0 < \alpha < 1 \) there is a constant \( C_\varepsilon > 0 \) such that
\[ \| \varphi \|_{C^1} \leq C_\varepsilon \| \varphi \|_{C^0} + \varepsilon \| \varphi \|_{C^{1,\alpha}} \leq C_\varepsilon C + \varepsilon \| \varphi \|_{C^{1,\alpha}}. \]
Choosing \( p = \frac{4n}{4 - \alpha} > 4n \) Morrey’s inequality and elliptic \( L^p \)-estimates for the Laplacian yield
\[ \| \varphi \|_{C^{1,\alpha}} \leq C' \| \varphi \|_{W^{2,p}} \leq C'' \left( \| \varphi \|_{L^p} + \| \Delta_g \varphi \|_{L^p} \right) \leq C'' \left( \| \varphi \|_{C^0} + \| \Delta_g \varphi \|_{C^0} \right) \]
for some constants \( C', C'' > 0 \) depending only on \( \alpha \).

Putting everything together, we obtain
\[ \| \varphi \|_{C^1} \leq C_\varepsilon C + \varepsilon C'' C (2 + \| \varphi \|_{C^1}), \]
from which we can conclude by choosing \( \varepsilon < (C'')^{-1} \). \( \square \)

### 6. Higher order estimates and long-time existence

Here we improve the Laplacian estimate to a Hölder estimate of the quaternionic Hessian of \( \varphi \). We do so by following an argument of Alesker [2] suitably adapted to our parabolic framework. By bootstrapping we then obtain estimates of any order on the solution of (1.1) and thus also long-time existence.

**Proposition 14.** For each \( k > 0 \), there exists a uniform constant \( C_k \) depending on the allowed data, \( k \), \( \| \nabla \varphi \|_{C^0} \) and an upper bound for \( \Delta_g \varphi \) such that
\[ \| \nabla^k \varphi \|_{C^0} \leq C_k, \quad (6.1) \]
where \( \nabla \) is the Levi-Civita connection with respect to \( g \). Moreover we have long-time existence for \( \varphi \), i.e. \( T = \infty \).
Proof. Assume (6.1) and suppose $T < \infty$. It follows form (3.1) that there exists a uniform constant $C$ such that

$$|\varphi| \leq T \sup_{M \times [0,T]} |\partial_t \varphi| \leq CT, \quad \text{on } M \times [0,T].$$

By this, (6.1) and short-time existence, one can extend the flow to $[0, T + \varepsilon_0)$ for some $\varepsilon_0 > 0$, which yields a contradiction. The interested reader can find more details about this standard discussion in the proof of [59, Theorem 3.1] (see also in [13, 67] and references therein).

We showed that it is enough to prove (6.1). And we claim that (6.1), follows once we have proved a Hölder bound for $\text{Hess}_g \varphi$ of the form

$$|\text{Hess}_g \varphi|_{C^{0,\alpha}(M \times [0,T])} \leq C_\varepsilon,$$

where $\varepsilon \in (0, T)$ and $C_\varepsilon$ is a uniform constant depending only on the initial data and $\varepsilon$. Indeed, given the Hölder bound (6.2) for the matrix $\text{Hess}_g \varphi$ and the second order estimate for $\varphi$, we can differentiate the flow (1.1) and then bootstrap using the Schauder estimates in order to obtain the uniform bound

$$|\nabla^k \varphi|_{C^{0,\alpha}(M \times [0,T])} \leq C_{\varepsilon,k}, \quad \text{for any } k > 0,$$

where $C_{\varepsilon,k}$ depends on $\varepsilon$ and $k$. But since by standard parabolic theory the solution $\varphi$ is uniquely determined by the initial and background data, we also have a uniform bound

$$|\nabla^k \varphi|_{C^{0,\alpha}(M \times [0,T])} \leq C_{\varepsilon,k}, \quad \text{for any } k > 0.$$

The estimate (6.2) is standard, we prove it as a separate Proposition below.

\[\square\]

**Proposition 15.** For each $\varepsilon \in (0, T)$ there exists $\alpha \in (0, 1)$ and a uniform constant $C_\varepsilon > 0$ depending only on the allowed data, $\varepsilon$, $\|\partial_t \varphi\|_{C^0}$, and an upper bound for $\Delta_g \varphi$ such that

$$|\text{Hess}_g \varphi|_{C^{0,\alpha}(M \times [0,T])} \leq C_\varepsilon.$$

**Proof.** The proof is classical in flavour and represents an adaptation of Alesker’s $C^{2,\alpha}$ estimate for the quaternionic Monge-Ampère equation obtained in [2] and inspired by the argument of Blocki [12].

Again the proof is local, since $M$ is locally flat. Let $\mathcal{O} \subset \mathbb{H}^n$ be an arbitrary open subset. For each $\alpha \in (0, 1)$, on $\mathcal{O}_T := \mathcal{O} \times [0,T)$, we define

$$[\varphi]_{\alpha, (x,t)} := \sup_{(y,s) \in \mathcal{O}_T \setminus (x,t)} \frac{|\varphi(y, s) - \varphi(x, t)|}{|y - x| + \sqrt{|s - t|}}^\alpha, \quad [\varphi]_{\alpha, \mathcal{O}_T} := \sup_{(x,t) \in \mathcal{O}_T} [\varphi]_{\alpha, (x,t)}.$$

The metric $g$ can be locally represented by a potential $w$ on $\mathcal{O}$, possibly shrinking $\mathcal{O}$ if necessary, in other words $g = \text{Hess}_g w$. Let us denote $u = w + \varphi$ and $U = \text{Hess}_g u$. By concavity of $F$, and the mean value theorem, for all $(x, t_1), (y, t_2) \in \mathcal{O} \times [0,T)$, we have

$$\text{Re} F^{rs}(y, t_2)(u_{rs}(x, t_1) - u_{rs}(y, t_2)) \geq \partial_t \varphi(x, t_1) - \partial_t \varphi(y, t_2) + h(x) - h(y) \geq \partial_t u(x, t_1) - \partial_t u(y, t_2) - C\|x - y\|,$$

for some constant $C$ depending on $\|h\|_{C^1}$.

At this point we recall the following algebraic lemma by Alesker [2, Lemma 4.9], which is analogous to [30] for the real case and [12, 46] in the complex setting.

**Lemma 16.** Let $\lambda, \Lambda \in \mathbb{R}$ satisfy $0 < \lambda < \Lambda < +\infty$. There exist a uniform constant $N$, unit vectors $\xi_1, \ldots, \xi_N \in \mathbb{H}^n$ and positive numbers $\lambda_1 < \Lambda_1 < +\infty$, depending only on $n, \lambda, \Lambda$ such
that any hyperhermitian matrix \( A \in \mathbb{H}^{n,n} \) with eigenvalues lying in the interval \([\lambda, \Lambda]\) can be written as
\[
A = \sum_{k=1}^{N} \beta_k (\xi^k)^* \otimes \xi^k, \quad \text{i.e.} \quad A_{rs} = \sum_{k=1}^{N} \beta_k \xi_r^k \xi_s^k,
\]
for some \( \beta_k \in [\lambda_*, \Lambda_*] \).

Applying the lemma to \( A = (F^{rs}(U)) \), immediately yields
\[
\text{Re} \, F^{rs}(U(y))(u_{rs}(y) - u_{rs}(x)) = \text{Re} \sum_{k=1}^{N} \beta_k(y) (\xi_k^r \xi_s^k)(u_{rs}(y) - u_{rs}(x)) \]
\[
= \sum_{k=1}^{N} \beta_k(y) (\Delta_{\xi^k}u(y) - \Delta_{\xi^k}u(x))
\]
for some functions \( \beta_k(y) \in [\lambda_*, \Lambda_*] \), where, for any unit vector \( \xi \in \mathbb{H}^n \), we denoted by \( \Delta_{\xi^k} \) the Laplacian on any translate of the quaternionic line spanned by \( \xi \), i.e.
\[
\text{Re} \, \text{tr}(\xi^* \otimes \xi)(u_{rs})) = \text{Re} \, \text{tr}(\xi^* u_{rs}) = \Delta_{\xi^k}u.
\]
Here we are using the well-known identity \( \text{Re} \, \text{tr}(B_1 B_2) = \text{Re} \, \text{tr}(B_2 B_1) \) valid for any two quaternionic matrices \( B_1, B_2 \) for which the product is defined.

For convenience, let us set \( \beta_0(y) \equiv 1 \) and \( \Delta_{\xi^0} = -\partial_t \). Then, from (6.3) we obtain
\[
\sum_{k=0}^{N} \beta_k \left( \Delta_{\xi^k}u(y, t_2) - \Delta_{\xi^k}u(x, t_1) \right) \leq C \| x - y \|.
\]

**Lemma 17.** For any \( k = 0, 1, \cdots, N \),
\[
\partial_t \Delta_{\xi^k}u \leq \text{Re} \, F^{rs}(\Delta_{\xi^k}u_{rs}) + \Delta_{\xi^k}h.
\]

**Proof.** For \( k = 0 \). Applying \( \partial_t \) to (1.1), we get
\[
\partial_t (\partial_t u) = \text{Re} \, F^{rs} \partial_t (u_{rs})
\]
and the lemma follows.

For other \( k \geq 1 \), write \( \xi^k = (\xi_1^k, \cdots, \xi_n^k) \). Differentiating (1.1) along \( \xi_p^k \) twice and taking sum over the index \( p \), gives
\[
\partial_t \Delta_{\xi^k}u = \text{Re} \, F^{rs}(\Delta_{\xi^k}u_{rs}) + \text{Re} \sum_{p=1}^{n} F^{rs,tl} u_{rs} \xi_p^k u_{rs} \xi_p^k - \Delta_{\xi^k}h
\]
\[
\leq \text{Re} \, F^{rs}(\Delta_{\xi^k}u_{rs}) - \Delta_{\xi^k}h,
\]
by the concavity of \( F \). Then the lemma follows. \( \Box \)

Fix \( \hat{t} \in [\varepsilon, T) \), and \( r \in (0, 1) \) such that \( 10r^2 \leq \hat{t} \). Define
\[
P_r = \{(x, t) \in \mathcal{O}_T : \|x\| \leq r, \hat{t} - 5r^2 \leq t \leq \hat{t} - 4r^2 \},
\]
\[
Q_r = \{(x, t) \in \mathcal{O}_T : \|x\| \leq r, \hat{t} - r^2 \leq t \leq \hat{t} \}.
\]
For every \( k = 0, 1, \cdots, N \), let us denote
\[
M_{k,r} = \sup_{Q_r} \Delta_{\xi^k}u, \quad m_{k,r} = \inf_{Q_r} \Delta_{\xi^k}u, \quad \eta(r) = \sum_{k=1}^{N} (M_{k,r} - m_{k,r}).
\]
To prove Proposition 15, it suffices to find a constant $C$ (depending only on $\varepsilon$), $r_0 > 0$ and $0 < \delta < 1$ such that
\[ \eta(r) \leq Cr^\delta, \quad \text{for all } r < r_0. \]

Let us define an operator $D = \frac{1}{4} \text{Re} F^{rs}(U) \partial_q \partial_p$. Let $(a_{ij}) \in \text{Sym}(4n, \mathbb{R})$ be the realization of $(F^{rs}(U))$. Then we can rewrite $D$ as
\[ D = \sum_{s,t=1}^{4n} a_{st} D_s D_t, \quad (6.5) \]

Since $F$ is uniformly elliptic on $\Gamma$, then $(a_{st}) \in \text{Sym}(4n, \mathbb{R})$ satisfies the uniform elliptic estimate $\lambda\|\xi\|^2 \leq \sum_{s,t} a_{st} \xi_s \xi_t \leq \Lambda\|\xi\|^2$ for some $0 < \lambda < \Lambda < \infty$ and any $\xi \in \mathbb{R}^{4n}$.

The following weak parabolic Harnack inequality is well-known.

**Lemma 18.** [43, Theorem 7.37] If $v \in W^{2,1}_{2n+1}$ is a nonnegative function and satisfies
\[ -\frac{\partial v}{\partial t} + \sum_{s,t} a_{st} D_s D_t v \leq h' \quad \text{on } Q_r, \]

where $h'$ is a bounded function and the matrix $(a_{st})$ is as in $(6.5)$. Then there exist positive constants $C, p$ depending on $n, \lambda, \Lambda$ such that
\[ \frac{1}{r^{4n+2}} \left( \int_{P_r} v^p \right)^{\frac{1}{p}} \leq C \left( \inf_{B_r} v + r^{\frac{4n}{4n+1}} \|h'\|_{L^{2n+1}_{2n+1}} \right). \quad (6.6) \]

For each $k = 0, 1, \cdots, N$, let us denote $v_k := M_{k,2r} - \Delta_{\xi_k} u$. Then $v_k \in W^{2,1}_{2n+1}$ is a non-negative function and since $\Delta_{\xi_k} u_{rs} = (\Delta_{\xi_k} u)_{rs}$ on $O_T$ it satisfies
\[ -\partial_t v_k + \text{Re} F^{rs}(v_k)_{rs} \leq h' \]
for a bounded function $h'$. Then by Lemmas 17 and 18,
\[ \frac{1}{r^{4n+2}} \left( \int_{P_r} (M_{k,2r} - \Delta_{\xi_k} u)^p \right)^{\frac{1}{p}} \leq C (M_{k,2r} - M_{k,r} + r^{\frac{4n}{4n+1}}), \quad (6.7) \]

On the other hand, let $(x, t_1), (y, t_2) \in Q_{2r}$, it then follows from $(6.4)$ that
\[ \beta_k \left( \Delta_{\xi_k} u(y, t_2) - \Delta_{\xi_k} u(x, t_1) \right) \leq Cr + \sum_{0 \leq \gamma \leq N, \gamma \neq k} \beta_{\gamma} \left( \Delta_{\xi_{\gamma}} u(x, t_1) - \Delta_{\xi_{\gamma}} u(y, t_2) \right). \]

For each $\varepsilon > 0$, pick a point $(x, t_1) \in Q_{2r}$ such that $m_{k,2r} \leq \Delta_{\xi_k} u(x, t_1) + \varepsilon$. As a consequence, after dividing the inequality above by $\beta_k$, we obtain
\[ \Delta_{\xi_k} u(y, t_2) - m_{k,2r} \leq Cr + C \sum_{0 \leq \gamma \leq N, \gamma \neq k} (M_{\gamma,2r} - \Delta_{\xi_{\gamma}} u(y, t_2)), \]
by arbitrariness of $\varepsilon$. Integrating for $(y,t_2)$ over $P_r$, and using the fundamental inequality $\|a+b\|_p \leq \|a\|_p + \|b\|_p$ for every $p > 1$, yields

$$\frac{1}{r^{4n+2}} \left( \int_{P_r} (\Delta_{\xi} u(y,t_2) - m_{k,2r})^p \right)^\frac{1}{p} \leq C \frac{1}{r^{4n+2}} \left( \int_{P_r} \left[ r + \sum_{0 \leq \gamma \leq N, \gamma \neq k} (M_{\gamma,2r} - \Delta_{\xi} u(y,t_2)) \right]^p \right)^\frac{1}{p} \leq C r + \frac{C}{r^{4n+2}} \sum_{0 \leq \gamma \leq N, \gamma \neq k} \left( \int_{P_r} [M_{\gamma,2r} - \Delta_{\xi} u(y,t_2)]^p \right)^\frac{1}{p}$$

(6.8)

where we have used the fact $0 < r < 1$ in the last inequality. In light of (6.7) and (6.8), and again the triangle inequality $\|a+b\|_p \leq \|a\|_p + \|b\|_p$, we obtain

$$M_{k,2r} - m_{k,2r} \leq C \frac{1}{r^{4n+2}} \left( \int_{P_r} (M_{k,2r} - \Delta_{\xi} u)^p \right)^\frac{1}{p} + \frac{C}{r^{4n+2}} \left( \int_{P_r} (\Delta_{\xi} u - m_{k,2r})^p \right)^\frac{1}{p} \leq C \sum_{\gamma=0}^{N} (M_{\gamma,2r} - M_{\gamma,r}) + C r^{\frac{4n}{4n+1}}.$$

Summing over $k$ we deduce

$$\eta(2r) \leq C \sum_{\gamma=0}^{N} (M_{\gamma,2r} - M_{\gamma,r}) + C r^{\frac{4n}{4n+1}}.$$

By definition, $m_{\cdot,s}$ is non-increasing in $s$, whence

$$\eta(2r) \leq C \sum_{\gamma=0}^{N} ((M_{\gamma,2r} - m_{\gamma,2r}) - M_{\gamma,r} + m_{\gamma,r}) + C r^{\frac{4n}{4n+1}} = C (\eta(2r) - \eta(r)) + C r^{\frac{4n}{4n+1}}.$$

Equivalently,

$$\eta(r) \leq \left(1 - \frac{1}{C}\right) \eta(2r) + C r^{\frac{4n}{4n+1}}.$$

Applying a standard iteration technique (see [30, Chapter 8] for more details), we finally infer that there exists a dimensional constant $\delta \in (0,1)$ such that $\eta(r) \leq C r^{\delta}$ as we wanted to show. This completes the proof of Proposition 15. \hfill \Box

7. Convergence of the flow and proof of Theorems 1 and 4

7.1. Li-Yau type inequality.

Now we consider the following Li-Yau [42] type equation

$$(\mathcal{L} - \partial_t)\psi = 0, \quad \psi > 0,$$

(7.1)
where \( L = \frac{1}{4} \text{Re} F^{ik} \partial_{q_k} \partial_{q_k} \).

If \( \Phi \) is a \( C^2 \) function and we let
\[
\Phi_k := \sum_{p=0}^{3} \Phi_{x^p} \bar{e}_p, \quad \Phi_k := \sum_{p=0}^{3} e_p \Phi_{x^p},
\]
where \( \Phi_{x^p} := \frac{\partial \Phi}{\partial x^p} \), and \( \bar{e}_p \) denotes the quaternionic conjugate of the quaternionic unit \( e_p \) for every \( p \), then we can rewrite \( L \) as
\[
L \Phi = \frac{1}{4} \text{Re} F^{ik} \partial_{q_k} \partial_{q_k} \Phi = F_{pq}^{ik} \Phi_{x^p x^q},
\]
where \( F_{pq}^{ik} := \frac{1}{4} \text{Re} \{ F^{ik} \bar{e}_p e_q \} \) for simplicity. This follows directly from the identity \( \text{Re}(ab) = \text{Re}(ba) \) valid for any pair of quaternions \( a, b \in \mathbb{H} \).

Let \( B \) be a constant so large that \( \psi = \partial_t \varphi + B \) is a solution to (7.1). We consider the quantity
\[
H = t(|\partial v|^2 - \alpha \partial_t v), \quad v = \log \psi,
\]
where \( \alpha \in (1, 2) \) is a constant and
\[
|\partial v|^2 = \frac{1}{4} \text{Re} F^{jl} v_j v_l = F_{rs}^{jl} v_{x^r} v_{x^s}.
\]

**Lemma 19.** There exists a constant \( C > 0 \) such that
\[
(L - \partial_t) H \geq \frac{t}{4n} \left( |\partial v|^2 - \partial_t v \right)^2 - 2 \langle \partial v, \partial H \rangle - (|\partial v|^2 - \alpha \partial_t v) - tC|\partial v|^2 - Ct, \quad (7.2)
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product defined by \( \langle \partial f, \partial g \rangle = \frac{1}{4} \text{Re} F^{ik} f_l g_k = F_{pq}^{ik} f_{x^p} g_{x^q} \) on real-valued \( C^1 \) functions.

**Proof.** The proof is local. For each \( z \in M \), we can find quaternionic coordinates \( q_1, \ldots, q_n \) on a local chart around \( z \). Plugging \( \psi = e^v \) into (7.1) we have
\[
L v - \partial_t v = -|\partial v|^2, \quad (7.3)
\]
giving
\[
H = -tL v - t(\alpha - 1) \partial_t v, \quad (7.4)
\]
and thus also
\[
t \partial_t (L v) = \frac{1}{t} H - \partial_t H - t(\alpha - 1) \partial_t^2 v. \quad (7.5)
\]
By a straightforward computation we get
\[
- \partial_t H = - (|\partial v|^2 - \alpha \partial_t v) - 2t \langle \partial v, \partial \partial_t v \rangle + t \alpha \partial_t^2 v - t \partial_t (F_{pq}^{ik} v_{x^p} v_{x^q}),
\]
\[
L H = tL(|\partial v|^2) - t \alpha L(\partial_t v). \quad (7.6)
\]
First we deal with the term \( L(|\partial v|^2) \). For convenience, let us define
\[
\mathcal{V} = F_{pq}^{ik} F_{rs}^{jl} v_{x^p x^r} v_{x^q x^s}, \quad \mathcal{W} = F_{pq}^{ik} F_{rs}^{jl} v_{x^p x^q} v_{x^r x^s}.
\]
By a direct calculation, we get
\[
L(|\partial v|^2) = \mathcal{V} + \mathcal{W} + L(F_{rs}^{jl} v_{x^r} v_{x^s} + F_{pq}^{ik} F_{rs}^{jl} v_{x^p} v_{x^q} v_{x^r} v_{x^s}),
\]
\[
+ F_{pq}^{ik} F_{rs}^{jl} v_{x^p} v_{x^q} v_{x^r} v_{x^s} + F_{pq}^{ik} F_{rs}^{jl} v_{x^p} v_{x^q} v_{x^r} v_{x^s} + F_{pq}^{ik} F_{rs}^{jl} v_{x^p} v_{x^q} v_{x^r} v_{x^s},
\]
\[
+ F_{pq}^{ik} F_{rs}^{jl} v_{x^p} v_{x^q} v_{x^r} v_{x^s} + F_{pq}^{ik} F_{rs}^{jl} v_{x^p} v_{x^q} v_{x^r} v_{x^s} + F_{pq}^{ik} F_{rs}^{jl} L(v_{x^p}) v_{x^q} v_{x^r} v_{x^s}.
\]
Note that $\varphi$ has uniformly bounded $C^k$ norms for every $k > 0$ by Proposition 14. Hence, analogously to the (almost) Hermitian case [19, 33], we deduce

$$|\mathcal{L}(F^{ij}_{rs}v_{x^i_{x^j}}v_{x^s})| \leq C|\partial v|^2. \quad (7.7)$$

For each $0 < \varepsilon < 1$, we have that

$$|F^{ik}_{pq}(F^{ij}_{rs}x^i_p v_{x^j_{x^i}}) + |F^{ik}_{pq}(F^{ij}_{rs}x^i_p v_{x^j_{x^i}})| + |F^{ik}_{pq}(F^{ij}_{rs}x^i_p v_{x^j_{x^i}})| \geq C|\partial v|^2 + 2\varepsilon W + 2\varepsilon V. \quad (7.8)$$

Observe that $(\mathcal{L}v)_{x^i} = (F^{ik}_{pq}v_{x^j_{x^i}}) - F^{ik}_{pq}v_{x^j_{x^i}} = (F^{ik}_{pq}v_{x^j_{x^i}})$. It follows that

$$F^{ij}_{rs}\mathcal{L}(v_{x^i_{x^j}}) + F^{ij}_{rs}\mathcal{L}(v_{x^i_{x^j}}) - 2(\partial v, \partial \mathcal{L}v)$$

$$= F^{ij}_{rs}v_{x^i_{x^j}}(\mathcal{L}(v_{x^i_{x^j}})) + F^{ij}_{rs}v_{x^i_{x^j}}(\mathcal{L}(v_{x^i_{x^j}}))$$

$$= -F^{ij}_{rs}v_{x^i_{x^j}}(F^{ik}_{pq}v_{x^j_{x^i}}) - F^{ij}_{rs}v_{x^i_{x^j}}(F^{ik}_{pq}v_{x^j_{x^i}})$$

$$\geq -C|\partial v|^2 - \varepsilon V - \varepsilon W. \quad (7.9)$$

On the other hand,

$$2t(\partial v, \partial \mathcal{L}v) \overset{(7.4)}{=} -2(\partial v, \partial H) - 2t(\alpha - 1)(\partial v, \partial t v)$$

$$\overset{(7.6)}{=} -2(\partial v, \partial H) - (\alpha - 1)\partial t H + \frac{\alpha - 1}{t} H - \alpha\partial t^2 v$$

$$-t(\alpha - 1)\partial t(F^{ik}_{pq}v_{x^j_{x^i}})$$

$$\geq -2(\partial v, \partial H) - (\alpha - 1)\partial t H + \frac{\alpha - 1}{t} H - \alpha\partial t^2 v - Ct|\partial v|^2. \quad (7.10)$$

It follows from (7.9) and (7.10) that

$$t\left(F^{ij}_{rs}\mathcal{L}(v_{x^i_{x^j}}) + F^{ij}_{rs}v_{x^i_{x^j}}\mathcal{L}(v_{x^i_{x^j}})\right)$$

$$\geq -2(\partial v, \partial H) - (\alpha - 1)\partial t H + \frac{\alpha - 1}{t} H - \alpha\partial t^2 v$$

$$- Ct|\partial v|^2 - \frac{Ct}{\varepsilon}|\partial v|^2 - t\varepsilon V - t\varepsilon W. \quad (7.11)$$

Now, we treat the second term of $\mathcal{L}H$ in (7.6). Using the Cauchy-Schwarz inequality, at $z$, we deduce

$$-t\alpha\mathcal{L}(\partial v) = -t\alpha\partial t(\mathcal{L}v) + t\alpha\partial t(F^{ik}_{pq}v_{x^j_{x^i}})$$

$$\overset{(7.5)}{=} -\frac{\alpha}{t} H + \alpha\partial t H + t\alpha(\alpha - 1)\partial t^2 v + t\alpha\partial t(F^{ik}_{pq}v_{x^j_{x^i}})$$

$$\geq -\frac{\alpha}{t} H + \alpha\partial t H + t\alpha(\alpha - 1)\partial t^2 v - \frac{Ct}{\varepsilon} |\partial v|^2 - t\varepsilon V,$$
Plugging (7.7), (7.8), (7.11) and (7.12) into (7.6), we get
\[
\mathcal{L}H \geq tW + t\mathcal{V} - Ct|\partial v|^2 - t\left(\frac{C}{\varepsilon}|\partial v|^2 + 2\varepsilon\mathcal{V} + 2\varepsilon W\right) - 2\langle \partial v, \partial H \rangle - (\alpha - 1)\partial_t H
\]
\[
+ \frac{\alpha - 1}{t}H - t\alpha(\alpha - 1)\partial_t^2 v - Ct|\partial v|^2 - \frac{Ct}{\varepsilon}|\partial v|^2 - t\varepsilon(\mathcal{V} + W)
\]
\[
- \frac{\alpha}{t}H + \alpha\partial_t H + t\alpha(\alpha - 1)\partial_t^2 v - \frac{Ct}{\varepsilon}|\partial v|^2 - t\varepsilon\mathcal{V}
\]
\[
\geq t(1 - 4\varepsilon)\mathcal{V} + t(1 - 3\varepsilon)W - \frac{4Ct}{\varepsilon}|\partial v|^2 + \partial_t H - \frac{1}{t}H - 2\langle \partial v, \partial H \rangle - \frac{Ct}{\varepsilon}.
\]
Thus, if we choose \(\frac{1}{16} \leq \varepsilon \leq \frac{1}{8}\),
\[
(\mathcal{L} - \partial_t)H \geq \frac{t}{2}\mathcal{V} - Ct|\partial v|^2 - (|\partial v|^2 - \alpha \partial_t v) - 2\langle \partial v, \partial H \rangle - Ct.
\] (7.13)
Applying the arithmetic-geometric mean inequality, and by (7.3),
\[
\mathcal{V} \geq \frac{1}{n}(\mathcal{L}v)^2 = \frac{1}{n}(\partial_t v - |\partial v|^2)^2.
\]
Plugging it into (7.13), we infer that
\[
(\mathcal{L} - \partial_t)H \geq \frac{t}{2n}(\partial_t v - |\partial v|^2)^2 - Ct|\partial v|^2 - (|\partial v|^2 - \alpha \partial_t v) - 2\langle \partial v, \partial H \rangle - Ct.
\]
By the arbitrariness of \(z\), this proves (7.2). \(\square\)

Using the parabolic maximum principle, we can prove the following lemma.

**Lemma 20.** On \(M \times (0, T)\), we have
\[
|\partial v|^2 - \alpha \partial_t v \leq \frac{8n\alpha^2}{t} + \sqrt{8n\alpha^2\left(C + \frac{nC^2\alpha^2}{2(\alpha - 1)^2}\right)}.
\]

**Proof.** Let us fix an arbitrary time \(t_0 \in (0, T)\). Suppose \(H(x, t)\) (as in (7.4)) achieves its maximum at the point \((\hat{q}, \hat{t}) \in M \times [0, t_0]\). We may assume \(\hat{t} > 0\), otherwise \(|\partial v|^2 - \alpha \partial_t v \leq 0\) on \(M \times [0, t_0]\) and we are done. It follows that
\[
H(\hat{q}, \hat{t}) \geq H(\hat{q}, 0) = 0.
\]
Using the maximum principle at \((\hat{q}, \hat{t})\), we deduce \((\mathcal{L} - \partial_t)H \leq 0\) and \(\partial H = 0\). Substituting this into (7.2) yields
\[
\frac{\hat{t}^2}{4n}(\partial_t v)^2 - C\hat{t}^2|\partial v|^2 - H \leq C\hat{t}^2.
\] (7.14)
Notice that at \((\hat{q}, \hat{t})\),
\[
\hat{t}^2(\partial_t v)^2 \geq \frac{\hat{t}^2}{\alpha^2}(\partial_t v)^2 - \alpha \partial_t v + (\alpha - 1)|\partial v|^2
\]
\[
= \frac{H^2}{\alpha^2} + \left(\frac{\alpha - 1}{\alpha}\right)^2\hat{t}^2|\partial v|^4 + \frac{2(\alpha - 1)\hat{t}H}{\alpha^2}|\partial v|^2
\]
\[
\geq \frac{H^2}{\alpha^2} + \left(\frac{\alpha - 1}{\alpha}\right)^2\hat{t}^2|\partial v|^4,
\] (7.15)
where we have used the fact that $H$ is nonnegative at $(\hat{q}, \hat{t})$. Using the elementary inequality $ax^2 + bx \geq -\frac{b^2}{4a}$, we get
\[
\frac{1}{4n} \left( \frac{\alpha - 1}{\alpha} \right)^2 \hat{t}^2 |\partial v|^4 - \hat{t}^2 C|\partial v|^2 \geq -\frac{nC^2\alpha^2}{2(\alpha - 1)^2} \hat{t}^2.
\]  
(7.16)

Plugging (7.15) and (7.16) into (7.14) gives
\[
\frac{H^2}{4n\alpha^2} \leq H + C\hat{t}^2 + \frac{nC^2\alpha^2}{2(\alpha - 1)^2} \hat{t}^2;
\]
from which we can deduce
\[
H(\hat{q}, \hat{t}) \leq 8n\alpha^2 + \sqrt{8n\alpha^2 \left( C + \frac{nC^2\alpha^2}{2(\alpha - 1)^2} \right)} \hat{t}.
\]

Hence, at each point $q \in M$,
\[
H(q, t_0) \leq H(\hat{q}, \hat{t}) \leq 8n\alpha^2 + \sqrt{8n\alpha^2 \left( C + \frac{nC^2\alpha^2}{2(\alpha - 1)^2} \right)} t_0.
\]

Consequently, at $(q, t_0)$,
\[
|\partial v|^2 - \alpha \partial_t v \leq \frac{8n\alpha^2}{t_0} + \sqrt{8n\alpha^2 \left( C + \frac{nC^2\alpha^2}{2(\alpha - 1)^2} \right)}.
\]

Then the lemma follows by arbitrariness of $t_0$. \hfill \Box

7.2. Parabolic Harnack inequality.

Let $\psi = \partial_t \varphi + B$ for a large constant $B$ such that $\psi > 0$ on $M$. By (3.3) we know
\[
\mathcal{L}\psi - \partial_t \psi = 0.
\]
(7.17)

With the results of the previous subsection we can prove the following useful parabolic Harnack inequality:

**Proposition 21.** Let $0 < t_1 < t_2 < T$. Then there exist constants $C_i$ $(i = 1, 2, 3)$ depending only on $(M, I, J, K)$, $\Omega$ and $f$ such that
\[
\sup_M \psi(\cdot, t_1) \leq \inf_M \psi(\cdot, t_2) \left( \frac{t_2}{t_1} \right)^{C_1} \exp \left( C_2 \frac{t_2 - t_1}{t_2 + C_3(t_2 - t_1)} \right).
\]  
(7.18)

**Proof.** With Lemmas 19 and 20, we can apply the procedure of [19, 33] verbatim. \hfill \Box

7.3. Convergence of the parabolic flow.

**Proposition 22.** Suppose $T = \infty$, $\text{osc}_M \varphi(\cdot, t) \leq C$ and $\|\nabla^k \varphi\|_{C^0} \leq C$ for any $k > 0$, where $C > 0$ is a uniform constant. Then the normalization $\tilde{\varphi}$ converges in $C^\infty$ topology to a smooth function $\tilde{\varphi}_\infty$ that satisfies
\[
F(A[\tilde{\varphi}_\infty]) = h + b,
\]
for some constant $b \in \mathbb{R}$.
Proof. Set $\psi = \partial_t \varphi + B$ for a large constant $B$ such that $\psi > 0$. For each $m \in \mathbb{N}$, we define

$$\psi_m(x, t) := \sup_M \psi(\cdot, m - 1) - \psi(x, m - 1 + t);$$

$$\tilde{\psi}_m(x, t) := \psi(x, m - 1 + t) - \inf_M \psi(\cdot, m - 1).$$

It is straightforward to verify that

$$(\partial_t - \mathcal{L}) \psi = (\partial_t - \mathcal{L}) \psi_m = (\partial_t - \mathcal{L}) \tilde{\psi}_m = 0.$$  

Applying the parabolic Harnack inequality (7.18), this yields

$$\sup_M \psi_m(\cdot, t_1) \leq C \inf_M \psi_m(\cdot, t_2), \quad \sup_M \tilde{\psi}_m(\cdot, t_1) \leq C \inf_M \tilde{\psi}_m(\cdot, t_2).$$

Choosing $t_1 = \frac{1}{2}, t_2 = 1$ we get

$$\sup_M \psi \left( \cdot, m - \frac{1}{2} \right) = \inf_M \psi(\cdot, m - 1) \leq C \left( \inf_M \psi(\cdot, m) - \inf_M \psi(\cdot, m - 1) \right),$$

$$\sup_M \psi(\cdot, m - 1) - \inf_M \psi \left( \cdot, m - \frac{1}{2} \right) \leq C \left( \sup_M \psi(\cdot, m - 1) - \sup_M \psi(\cdot, m) \right).$$

In light of (7.19), if we set

$$\vartheta(t) = \sup_M \psi(\cdot, t) - \inf_M \psi(\cdot, t)$$

for the oscillation, then we have

$$\vartheta(m - 1) + \vartheta \left( m - \frac{1}{2} \right) \leq C (\vartheta(m - 1) - \vartheta(m)), $$

which implies that $\vartheta(m) \leq e^{-\delta} \vartheta(m - 1)$, where $\delta := -\log(1 - \frac{1}{C}) > 0$, and by induction

$$\vartheta(t) \leq C e^{-\delta t}.$$  

Since we have $\int_M \partial_t \varphi = 0$, by the mean value theorem, there exists a point $x_t \in M$ such that $\partial_t \varphi(x_t, t) = 0$. Therefore,

$$\| \partial_t \varphi(x, t) \| = \| \partial_t \varphi(x, t) - \partial_t \varphi(x_t, t) \| \leq \text{osc}_M \partial_t \varphi(\cdot, t)$$

$$= \text{osc}_M \partial_t \varphi(\cdot, t) = \vartheta(t) \leq C e^{-\delta t},$$

which yields that $\varphi \leq C e^{-\delta t}$ (resp. $\varphi \leq C e^{-\delta t}$) is non-increasing (resp. non-decreasing) with respect to $t$. It then follows from the uniform bounds on $\varphi$ that $\bar{\varphi}$ is uniformly bounded in $C^\infty$ topology, therefore there is a sequence of times $t_j \to \infty$ such that $\bar{\varphi}(\cdot, t_j)$ converges smoothly to some smooth function $\bar{\varphi}_\infty$ and it is fairly standard to show that actually $\lim_{t \to \infty} \bar{\varphi} = \bar{\varphi}_\infty$ in the $C^\infty$ topology.

Finally, the limiting function $\bar{\varphi}_\infty$ satisfies

$$0 = \lim_{t \to \infty} \partial_t \bar{\varphi}(\cdot, t) = \lim_{t \to \infty} \left( F(A[\bar{\varphi}]) - h - \frac{\int_M \partial_t \varphi \Omega^n_0 \wedge \bar{\Omega}^n_0}{\int_M \Omega^n_0 \wedge \bar{\Omega}^n_0} \right) = F(A[\bar{\varphi}_\infty]) - h - b,$$

where we set

$$b = \lim_{t \to \infty} \frac{\int_M \partial_t \varphi \Omega^n_0 \wedge \bar{\Omega}^n_0}{\int_M \Omega^n_0 \wedge \bar{\Omega}^n_0}.$$  

□
8. Proof of Theorems 1–5

We are ready to complete the proofs of Theorems 1 and 4, from which we will infer Theorems 2, 3 and 5.

Proof of Theorem 1. Let \((M, I, J, K, g)\) be a compact flat hyperkähler manifold, \(\varphi, \tilde{\varphi}: M \to \mathbb{R}\) be the solution to (1.1) and its normalization (defined in (1.4)). The initial datum \(\varphi_0\) is assumed \(\Gamma\)-admissible and, since \(f\) is unbounded, every \(\Gamma\)-admissible function is automatically a parabolic \(C\)-subsolution. Hence we may apply Proposition 10 and deduce \(\text{osc}_M \varphi(\cdot, t) \leq C\) and \(\|\tilde{\varphi}\|_{C^0} \leq C\). This bounds allow to obtain from Propositions 12 and 13 a uniform constant \(C\) such that \(\Delta g \varphi \leq C\). Applying now Proposition 14 we infer long-time existence of \(\varphi\) and uniform bounds on its derivatives of any order. Finally, Proposition 22 yields smooth convergence of the normalization \(\tilde{\varphi}\) to some function \(\tilde{\varphi}_\infty\) which is a solution of (1.5), i.e.

\[
F(A[\tilde{\varphi}_\infty]) = h + b
\]

for a suitable constant \(b \in \mathbb{R}\).

\(\square\)

Proof of Theorem 4. The proof is quite similar to the one of Theorem 1. Indeed, suppose \(f\) is bounded on \(\Gamma\) and assume that it satisfies either one of the two conditions expressed in the statement of Theorem 4, we are still able to apply Proposition 10 and deduce \(\text{osc}_M \varphi(\cdot, t) \leq C\) and \(\|\tilde{\varphi}\|_{C^0} \leq C\). Now we can employ the arguments in the proof of Theorem 1 to complete the proof.

\(\square\)

Now we prove Theorem 2 and Theorem 3 as applications of Theorem 1.

Proof of Theorem 2. The result follows as a simple application of Theorem 1 once we choose \(f = \log \sigma_k\) defined over the cone

\[
\Gamma = \Gamma_k := \{ \lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \ldots, \sigma_k(\lambda) > 0 \},
\]

where \(\sigma_r\) is the \(r\)-th elementary symmetric function

\[
\sigma_r(\lambda) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}, \quad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.
\]

Indeed, on a locally flat hyperhermitian manifold a function \(u\) of class \(C^2\) lies in \(\text{QSH}_k(M, \Omega)\) if and only if it is \(\Gamma_k\)-admissible. The function \(f\) satisfies our structural assumptions C1–C3 (see e.g. [50]) and it is straightforward to check that it is unbounded over \(\Gamma_k\). Finally, with this setup, the quaternionic Hessian flow (1.6) becomes \(\partial_t \varphi = f(\lambda(A[\varphi])) - H\), as desired.

\(\square\)

Proof of Theorem 3. Define

\[
f = \log \sigma_n(T), \quad \Gamma = T^{-1}(\Gamma_n),
\]

where \(T: \mathbb{R}^n \to \mathbb{R}^n\) is the linear map defined by

\[
T(\lambda) = (T(\lambda)_1, \ldots, T(\lambda)_n), \quad T(\lambda)_k = \frac{1}{n - 1} \sum_{i \neq k} \lambda_i, \quad \text{for every } \lambda \in \mathbb{R}^n.
\]

An easy verification shows that assumptions C1–C3 are satisfied and that \(f\) is unbounded over \(\Gamma\). Setting

\[
\Omega := \text{Re} \left( g^{j_s}(\Omega_1) j_s \right) \Omega_0 - (n - 1)\Omega_1,
\]

one can easily see that \(u \in C^2(M, \mathbb{R})\) lies in \(\text{QPSH}_{n-1}(M, \Omega_1, \Omega_0)\) if and only if \(\lambda(A[u]) \in \Gamma\), where \(A[u] = g^{j_s}(\Omega_2 s + u j_s)\). We can then rewrite the \((n - 1)\)-quaternionic plurisubharmonic flow (1.8) as \(\partial_t \varphi = f(\lambda(A[\varphi])) - H\) and apply Theorem 1 to conclude.

\(\square\)
Finally, we conclude the paper with the proof of Theorem 5.

Proof of Theorem 5. Let $\varphi$ be an elliptic $C^\infty$-subsolution of the equation

$$F(A[\varphi]) = h,$$

which we have shown that can be seen as a time-independent parabolic $C^\infty$-subsolution of our flow (1.1). Consider flow (1.1) with a $\Gamma$-admissible initial datum $\varphi_0$, then condition (1.10) of Theorem 4 is trivially verified, and this concludes the proof. $\square$

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