Currents and correlations in Luttinger liquids and carbon nanotubes at finite temperature and size: a bosonization study

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We consider problems of one dimensional interacting fermions confined to a finite size, multichannel geometry. Concentrating on Luttinger liquids and carbon nanotubes, we use nontrivial boundary conditions to represent the effect of external leads, and apply our framework to transport problems in a Josephson junction setup. We present an exact computation of all correlation functions, including finite-size and temperature effects, for two sets of solvable boundary conditions. In all cases, we compute physical quantities like the Josephson current and the pairing order parameter profile.

I. INTRODUCTION

It is now common knowledge that one-dimensional interacting electronic systems possess metallic phases that are fundamentally different from those of their higher-dimensional brethren, which are for most practical purposes extremely well described by adaptations and refinements of Fermi liquid theory. The Luttinger liquid has become the paradigm for the former systems: the quasiparticle pole is no more, and instead one finds only collective spin and charge excitations living around two Fermi points. The most striking consequence of interactions are first that spin and charge excitations are not bound to one another anymore, and can thus travel at different velocities. Moreover, charge fractionalization can occur, whereby “fundamental” quantities like the electron’s charge can be split in many pieces (for a review, see e.g. [1]).

Experimental realizations of Luttinger liquids range from edge states in the fractional quantum Hall effect [2] (in this case, a chiral Luttinger liquid is found), quantum wires in semiconductor heterostructures [3], and perhaps most notably in single-walled carbon nanotubes (SWNT) [4]. The latter can be used to perform electrical transport experiments using different geometries, for example in junctions manipulated by mechanical means [5]. “Kinks” in the nanotubes or crossings of nanotubes can be produced by scanning tunneling microscopes (STM) tips, and these in turn provide realizations of backscattering impurities embedded in a Luttinger liquid or of connections through tunnel junctions. These junctions can act as rectifying diodes at room temperature, thus opening up a route towards the construction of nanoscale devices of all sorts. An example of recently proposed applications is that of an “entangler”, whereby two nanotubes coupled to a superconductor are expected to produce physically separated entangled pairs of excitations [6].

Single impurities embedded in Luttinger liquids have attracted a lot of theoretical interest since the early work of Kane and Fisher [7]. In particular, many approaches have been devised based on boundary conformal field theory (CFT) [8] and boundary integrability [9]. The trick here is to “fold” the system around the impurity, thereby replacing it with an effective boundary whose effects scale either to the weak- or strong-coupling regime depending on the nature of the bulk interactions in the system. For example, for a backscattering impurity in a Luttinger liquid, repulsive interactions enhance the backscattering amplitude, and therefore suppress the conductance.

Another extremely interesting situation to consider is that of a Luttinger liquid on a finite interval. This class of problems is a form of generalization of the previous single impurity ones, in the sense that we now have two different boundaries to deal with. One thus imagines electrons being trapped by either a direct physical cut of their support (maybe an ended nanotube), or prevented from propagating further by the presence of an excitation gap, maybe a superconducting gap, or voltage gates. We consider here the problem of a Josephson junction realized by bridging two superconductors with either quantum wires or carbon nanotubes. Experimental attempts at realizing this device can be found in [10]. Theoretically, this problem has been addressed in [11], where the Josephson current for “perfect” contacts was computed (here, by perfect, we mean a contact such that only Andreev reflection occurs; a poor contact would be one such that most of the reflection processes that occur would be normal reflection ones). A perturbative analysis of the problem was presented in [12], and an effective model amenable to boundary CFT was used in [13].
For certain cases, the problem is an integrable quantum field theory, and this was treated in [21]. However, only one particular (nontrivial) value of the interaction parameter can be fully solved, the others leading to rather tedious complications with the analytical properties of the thermodynamic Bethe Ansatz, which are yet to be fully resolved. One common factor of all these studies, however, is that they were devoted to single-channel Luttinger liquids.

In the case of nanotubes, the number of channels available for conduction is however by construction greater than one. Moreover, if one wants to describe interesting devices like the entangler mentioned above, or maybe even things like multiwall carbon nanotubes [22], it is important to have a theory for multichannel situations. It is very tempting to think that the one-dimensionality of the system opens the way to a full nonperturbative solution of transport problems of this class, as it can do for the single-channel case. However, from basic factorized scattering arguments, it is possible to show [23] that multichannel theories nontrivially coupled only at boundaries but not in the bulk, do not seem to have enough flexibility to be integrable (that is, only a trivial scattering matrix can satisfy the bootstrap equations). This holds, of course, if one considers having no boundary degrees of freedom. Relaxing this might lead to some other integrable theories, but not ones which can be directly applied to the problem at hand. One is therefore ultimately led straight back into the domain of perturbative computations as the only fully controllable approach, and most of what we will present here shall be based on a minimal use of it (that is, we solve all that we can exactly, using perturbation theory only when strictly needed).

Our plan is therefore to treat the general problem of a conducting multichannel system in a finite size domain, between two boundaries having nontrivial effects on the physics (but no independent degrees of freedom of their own). The noninteracting and interacting cases are treated in fundamentally different manners. The noninteracting problem is straightforward to solve thoroughly directly in the fermionic language, providing in some sense an adaptation of the well-known work of Blonder, Tinkham and Klapwijk [24] to two-boundary problems. The partition function can be computed exactly, which means that thermodynamic properties and correlation functions can be obtained exactly (for the case of correlation functions, we only solve the interacting case, since the noninteracting ones can be recovered easily by putting all interaction parameters to zero).

Including interactions, however, takes us to a different set of problems. First of all, the presence of appropriate Coulomb interactions renders the fermionic description untractable. Fortunately, however, we can use bosonization to perform our computations. This, as is well-known, allows one to treat interactions nonperturbatively. In the bulk, correlators are rather easily computed. Remember, however, that we are here dealing with two-boundary problems. This involves treating Luttinger liquids and carbon nanotubes in a finite size at finite temperatures, for which we present a computation of all multipoint correlation functions around simple fixed points represented by tractable boundary conditions. The scaling of the correlators then depends on whether the operators sit away from or near to the boundaries: the scaling near the boundaries differs from that in the bulk, and depends on the specifics of the boundary conditions that are in use. We venture to treat this problem in all generality at the level of bosonization. For our specific problems, we set up the perturbative formalism, and present computations for physical transport quantities like the Josephson current in different situations, as well as discussions about the pairing order parameter profile.

The plan of the paper is as follows. In section 2, we treat the noninteracting problem in fermionic language, and compute finite-temperature transport properties using canonical mode expansion methods. In section 3, we consider the interacting case. We consider separately the cases of boundary-coupled Luttinger liquids, and nanotubes. For each of these, we discuss separately the cases of normal and Andreev boundary conditions, representing “bad” and “good” coupling to the superconductors (again, in the sense described above in terms of normal versus Andreev reflection amplitudes; the microscopics of this is a rather tricky experimental issue). The physical results are listed in this part of the paper, and summarized in the conclusion. Important formulas for the bosonization, including computations of correlators at finite temperature in a finite-size geometry are collected in Appendix A. In Appendix B, we provide some definitions of (multivariable) \( \theta \)-functions appearing in our formulas for the correlation functions. Appendix C is devoted to some notes on the real-time, finite-temperature perturbative formalism we used.

## II. NONINTERACTING CASE

In this section, we concentrate on the case where interactions are absent in the bulk of the system. The main advantage of this is that the calculation of thermodynamic quantities and of correlators can be performed exactly using relatively straightforward methods. We choose to always work in a real-time formalism, even in the presence of finite temperatures. This allows an immediate adaptation of our framework to time-dependent perturbations (although we do not consider them here, they are in principle easy to address using our formulas, bearing the notes in Appendix C in mind). Moreover, the definitions and results obtained in this section, in particular the treatment of boundary phenomena, will prove to be a good starting point for other problems later on, when we will consider other physical cases with interactions in the bulk. Our intention is to provide a flexible framework for building theories of
e.g. transport through one-dimensional channels, in many different types of setups. Recent advances in experimental realizations of nanostructures provide good motivation for such types of theories.

Our starting point is a theory for \( N \) decoupled spinful channels of fermions living within a one-dimensional channel of finite size, \( x \in [0, R] \), which we often call the fundamental domain. The spin-1/2 Fermi fields are labeled by channel \( i = 1, \ldots, N \) and spin indices, \( \Psi_{\sigma}(x) \). The canonical equal-time anticommutation relations for the fermions are (for \( x, x' \in [0, R] \))

\[
\left\{ \Psi_{\sigma}^{ij}(x, t), \Psi_{\sigma'}^{ij}(x', t) \right\} = \delta_{\sigma\sigma'}\delta^{ij}\delta(x - x').
\]

In order to write proper mode expansions for our fermions, we have to be careful with what happens in the presence of the two boundaries. The crucial thing to preserve is the independence of \( \Psi_{\sigma}(0) \) at both boundaries: we do not want to identify fields at \( x = 0 \) and fields at \( x = R \), as one would normally do using a finite-size setup on a ring. We here have to start by unfolding the fields, and then requiring some form of periodicity. The way we do this is by extending the definitions of the fields to the whole real axis by reflecting the original field at both ends of the system, i.e. by taking

\[
\Psi_{\sigma}^{ij}(-x) = \Psi_{\sigma}^{ij}(x), \quad \Psi_{\sigma}^{ij}(R - x) = \Psi_{\sigma}^{ij}(R + x).
\]

This immediately shows that the proper periodicity of the fields is \( 2R \), and not \( R \) as we would have obtained in a closed ring of circumference \( R \).

Each channel has a value of the Fermi wavevector \( k_F^j \) and Fermi velocity \( v^j \) which we take as external parameters. This gives us the flexibility for example to treat the cases where the different channels are put at different external voltages \( V^j \). In the simplest case, the relationship between the external voltage \( V^j \) applied to channel \( j \) and the Fermi parameters is \( k_F^j = k_F^0 + \frac{2eV^j}{h} \), where \( k_F^0 \) is the Fermi wavevector at half-filling. This step in not crucial at this stage, as most of our formulas will comprise boundary backscattering potentials representing the voltages of leads, so we can simply keep the voltages explicitly instead of reabsorbing them. For transport, only voltage differences matter. The Fermi fields are then linearized as usual in terms of chiral left- and right-movers as

\[
\Psi_{\sigma}^{ij}(x) = e^{ik_F^j x}\Psi_{R\sigma}^{ij}(x) + e^{-ik_F^j x}\Psi_{L\sigma}^{ij}(x).
\]

with canonical equal-time anticommutation relations

\[
\left\{ \Psi_{L\sigma}^{ij}(x, t), \Psi_{L\sigma'}^{ij}(x', t) \right\} = \frac{1}{2}\delta_{\sigma\sigma'}\delta^{ij}\delta(x - x').
\]

and a similar equation for the right movers (the left and right fields anticommute in the fundamental domain; note the factor of a half, which is usual for chiral fields \( \Psi_{\sigma}^{ij} \)). In terms of left- and right-movers, the extensions \( \Psi_{L\sigma}^{ij} \) imply the analytical continuations of chiral fields

\[
\Psi_{L\sigma}^{ij}(-x) = \Psi_{R\sigma}^{ij}(x), \quad \Psi_{L\sigma}^{ij}(R - x) = e^{2ik_F^j x}\Psi_{R\sigma}^{ij}(R + x),
\]

which represent a full mapping between the left- and right-movers in the theory (that is, we can use one or the other, as long as we choose the right periodicity for getting the proper mode expansions). The appropriate periodicity of the fields then becomes

\[
\Psi_{L\sigma}^{ij}(2R + x) = e^{2ik_F^j x}\Psi_{L\sigma}^{ij}(x), \quad \Psi_{R\sigma}^{ij}(2R + x) = e^{-2ik_F^j x}\Psi_{R\sigma}^{ij}(x).
\]

The above formulas in turn imply what we term the normal boundary conditions at the left and right ends of the system, representing the fact that excitations hitting the boundary are simply normal-reflected back into the bulk of the system:

\[
\Psi_{L\sigma}^{ij}(0) = \Psi_{R\sigma}^{ij}(0), \quad \Psi_{L\sigma}^{ij}(R) = e^{2ik_F^j R}\Psi_{R\sigma}^{ij}(R).
\]

The important point to realize is that these boundary conditions represent a link between the fields of the theory: using them, all the fields of the theory can be described either by left- and right-movers in the fundamental domain, or using only e.g. left-movers, defined over an interval twice as big as the fundamental domain. This set of boundary conditions is used throughout (half) of the paper, and represents a basis from which bosonization can be done exactly. They should be kept closely in mind by the reader.
The action describing the left- and right-movers with normal boundary conditions is composed of a noninteracting bulk part and a specific boundary part which is chosen such as to canonically impose the normal boundary conditions:

\[ S_{\text{norm}} = S_0^{(\text{bulk})} + S_{\text{norm}}^{(\text{bdry})}. \]  

The noninteracting bulk action is most conveniently written in terms of a symmetric form of chiral left- and right-movers:

\[ S_0^{(\text{bulk})} = \int dt \int_{\sigma=\uparrow,\downarrow} \sum_{j} \left[ \frac{1}{2} \Psi_{L \sigma}^j i(\partial_t - v^j \partial_x) \Psi_{L \sigma}^j + \frac{1}{2} \Psi_{R \sigma}^j i(\partial_t - v^j \partial_x) \Psi_{R \sigma}^j + \right. \]

\[ + \left. \frac{1}{2} \Psi_{L \sigma}^j i(\partial_t + v^j \partial_x) \Psi_{L \sigma}^j + \frac{1}{2} \Psi_{R \sigma}^j i(\partial_t + v^j \partial_x) \Psi_{R \sigma}^j \right] + \sum_{\sigma} \left[ \Psi_{L \sigma} (0) \Psi_{R \sigma} (0) \right]. \]

The boundary action imposing normal boundary conditions is

\[ S_{\text{norm}}^{(\text{bdry})} = \int dt \sum_{j} \sum_{\sigma} \frac{v^j}{2} \left[ \Psi_{L \sigma}^j (0) \Psi_{R \sigma}^j (0) - \Psi_{R \sigma}^j (0) \Psi_{L \sigma}^j (0) - \right. \]

\[ \left. - \epsilon^{2ik^j R} \Psi_{L \sigma}^j (R) \Psi_{R \sigma}^j (R) + e^{-2ik^j R} \Psi_{R \sigma}^j (R) \Psi_{L \sigma}^j (R) \right]. \]

This theory is very simple: its partition function and correlators are straightforward to obtain, even at finite temperature. One can easily verify that the equations of motion at the boundaries coincide with the boundary conditions.

What really interests us, however, is to couple the system to external leads. One of the most interesting configurations that we can consider is the one where the system is in a Josephson junction geometry, whereby the left and right ends are in contact with two superconductors having different order parameters. On general grounds, one then expects currents to flow through the system even in the absence of voltage bias, part of the phenomena called the Josephson effects.

We therefore consider coupling our system to two external superconductors with bulk gaps well above the excitation energies existing within the conducting channel. In particular, this means that quasiparticle penetration within the superconductors from the channel are basically non-existent, and that the excitations are either normal- or Andreev-reflected immediately back into the channel by the contacts with the superconductors. An effective theory is therefore obtained by integrating out the superconductors themselves, leaving behind effective boundary actions taking the form of BCS-like couplings at the ends of the system. Therefore, we perturb the free action by the additional boundary actions

\[ S_{L}^{(\text{bdry})} = -\frac{1}{2} \int dt \sum_{i,j=1}^N \left[ \Delta_{L}^{ij} \Psi_{\sigma}^{\dagger i}(0) \Psi_{\sigma}^{ij}(0) + \Delta_{L}^{ij} \Psi_{\sigma}^{ij}(0) \Psi_{\sigma}^{\dagger i}(0) + V_{L}^{ij} \sum_{\sigma} \Psi_{\sigma}^{\dagger i}(0) \Psi_{\sigma}^{ij}(0) \right], \]

\[ S_{R}^{(\text{bdry})} = -\frac{1}{2} \int dt \sum_{i,j=1}^N \left[ \Delta_{R}^{ij} \Psi_{\sigma}^{\dagger i}(R) \Psi_{\sigma}^{ij}(R) + \Delta_{R}^{ij} \Psi_{\sigma}^{ij}(R) \Psi_{\sigma}^{\dagger i}(R) + V_{R}^{ij} \sum_{\sigma} \Psi_{\sigma}^{\dagger i}(R) \Psi_{\sigma}^{ij}(R) \right] \]  

The boundary BCS couplings \( \Delta_{L,R} \) are effective couplings coming from the integrating out procedure detailed above (we refer the reader to the discussion in paper [23], where this is presented in all necessary details). It is very important not to confuse them with the bulk couplings of the original superconductors: the boundary pairings contain information about the quality of the contact, the density of states of the superconductor at the junction, etc. We take them as external parameters. Their phase, however, can be reasonably supposed to be that of the original superconductors at low energies.

The one advantage of this form of the boundary perturbation is that it preserves the bilinear form of the action as far as the fermions are concerned. The theory can therefore again be solved exactly. Varying the action yields the
interpolating boundary conditions

\[
\Psi_{L\sigma}^j(0) - \frac{i}{v^i} \sum_j \sum_{\sigma'} \left[ \Delta_{L}^{ij} \epsilon_{\sigma\sigma'} \Psi_{L\sigma'}^j(0) + V_{L}^{ij} \delta_{\sigma\sigma'} \Psi_{L\sigma'}^j(0) \right] = \\
\Psi_{R\sigma}^i(0) + \frac{i}{v^i} \sum_j \sum_{\sigma'} \left[ \Delta_{R}^{ij} \epsilon_{\sigma\sigma'} \Psi_{R\sigma'}^i(0) + V_{R}^{ij} \delta_{\sigma\sigma'} \Psi_{R\sigma'}^i(0) \right],
\]

\[
e^{-ik^j_R} \Psi_{L\sigma}^j(R) + \frac{i}{v^i} \sum_j \sum_{\sigma'} \left[ \Delta_{R}^{ij} \epsilon_{\sigma\sigma'} e^{ik^j_R} \Psi_{L\sigma'}^j(R) + V_{R}^{ij} \delta_{\sigma\sigma'} e^{ik^j_R} \Psi_{L\sigma'}^j(R) \right] = \\
e^{ik^j_R} \Psi_{R\sigma}^i(R) - \frac{i}{v^i} \sum_j \sum_{\sigma'} \left[ \Delta_{R}^{ij} \epsilon_{\sigma\sigma'} e^{-ik^j_R} \Psi_{R\sigma'}^i(R) + V_{R}^{ij} \delta_{\sigma\sigma'} e^{-ik^j_R} \Psi_{R\sigma'}^i(R) \right].
\]  

(12)

where \(\epsilon^\uparrow_\downarrow = \epsilon^\downarrow_\uparrow = 1\). These boundary conditions allow us to move continuously between various extreme cases, where things simplify considerably. Besides the normal boundary conditions above, we also can tune our boundary parameters to achieve what we will call Andreev boundary conditions: taking

\[
V_{\downarrow\uparrow} = 0, \quad \Delta_{L}^{ij} = v^i \delta_{ij}, \quad \Delta_{R}^{ij} = v^i e^{i\chi} \delta_{ij},
\]

the Andreev boundary conditions read explicitly

\[
\Psi_{L\sigma}^j(0) = i\epsilon_{\sigma\sigma'} \Psi_{R\sigma'}^j(0), \quad \Psi_{R\sigma}^i(R) = -i e^{i\chi} \epsilon_{\sigma\sigma'} \Psi_{R\sigma'}^i(R).
\]  

(14)

The two cases of normal and Andreev boundary conditions are those for which one can compute thermodynamic quantities exactly in the interacting case, and we urge the reader to keep them in mind in what follows. Away from these, one has to rely on some form of perturbation theory.

In the absence of interactions, however, the exact solution of the theory is obtained by simple considerations. Since the fermions are free chiral fields in the bulk, we can write the mode expansions

\[
\Psi_{L\sigma}^j(x, t) = \int \frac{dk}{2\pi} e^{-ik(x+vt)} \Psi_{L\sigma}^j(k), \quad \Psi_{R\sigma}^i(x, t) = \int \frac{dk}{2\pi} e^{ik(x-vt)} \Psi_{R\sigma}^i(k).
\]  

(15)

Substituting (13) in (12) and defining the Nambu spinors

\[
\Psi_{L,R}(\omega) = \begin{pmatrix} \Psi_{L,R}^1(\frac{\omega}{v}) \\ \Psi_{L,R}^i(\frac{-\omega}{v}) \\ \vdots \end{pmatrix}
\]

(16)

we get the continuity equations

\[
[1 - iv^{-1}M_{L}] \Psi_{L}(\omega) = [1 + iv^{-1}M_{L}] \Psi_{R}(\omega),
\]

\[
[1 + iv^{-1}M_{R}] e^{-i\sigma_z \otimes kFR-1\otimes v^{-1}\omega R} \Psi_{L}(\omega) = [1 - iv^{-1}M_{R}] e^{i\sigma_z \otimes kFR+1\otimes v^{-1}\omega R} \Psi_{R}(\omega),
\]

(17)

where \(v = \text{diag}(v^i)\), and the boundary matrices are given by

\[
M_{L,R} = \begin{pmatrix} V_{L,R} & \Delta_{L,R} \\ \Delta_{L,R}^* & -V_{L,R} \end{pmatrix}.
\]

(18)

(note: we use here a Bogoliubov-de Gennes notation, with each entry of the 2x2 matrices above being an NxN matrix, where N is the number of channels. We order the tensor products as (BdG space) \(\otimes\) (channels space)). The quantization condition is then obtained by the requirement that a given eigenstate energy \(\omega\) fulfills both continuity equations. This implies

\[
\det[1 + F(\omega)] = 0,
\]

(19)

with matrix

\[
F(\omega) = -[1 - iv^{-1}M_{L}]^{-1} [1 + iv^{-1}M_{L}] e^{-i\sigma_z \otimes kFR-1\otimes v^{-1}\omega R} \times \\
\times [1 - iv^{-1}M_{R}]^{-1} [1 + iv^{-1}M_{R}] e^{-i\sigma_z \otimes kFR+1\otimes v^{-1}\omega R}.
\]

(20)
Noting that we can write
\[ [1 - i\mathbf{v}^{-1}\mathbf{M}]^{-1} [1 + i\mathbf{v}^{-1}\mathbf{M}] = e^{2i\arctan(\mathbf{v}^{-1}\mathbf{M})}, \] 
we see that \( F^\dagger = F^{-1} \), so all solutions to the quantization equation are such that \( \omega \in \mathbb{R} \).

The ground-state energy of the system is then given simply by the sum over a Fermi sea, which we can write as a contour integral:
\[ E_0 = \sum_{\omega < 0} \omega = \frac{1}{2\pi i} \int_C d\omega \frac{\partial \ln \det[1 + F(\omega)]}{\partial \omega} = -\frac{1}{2\pi i} \int_C d\omega \ln \det[1 + F(\omega)] \] 
where the contour \( C \) circles all the poles of \( \det[1 + F(\omega)]^{-1} \) on the negative real axis, in a clockwise direction. Using complex plane manipulations, we can write
\[ E_0 = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \ln \det[1 + F(\omega)] = -\frac{1}{2\pi} \int_0^\infty d\eta \ln \det \left([1 + F(-i\eta)][1 + F(i\eta)]\right) = \] 
\[ = -\frac{1}{2\pi} \int_0^\infty d\eta \ln \det \left([1 + F(-i\eta)][1 + F^{-1}(i\eta)]\right) + G_L + G_R \] 
where the first term is now manifestly convergent, and the additional contributions are the boundary intensive energies, which we ignore from now on since they have no influence on the physics that we are after.

We can now introduce finite temperatures by inserting temperature-dependent occupation numbers for all states, and recalculating the sum. We find, for the equilibrium distribution,
\[ \ln Z = \int_{-\infty}^{\infty} d\omega D(\omega) \ln \left[1 + e^{-\beta|\omega|}\right] - \beta E_0 \] 
where the thermal density of states is given by the expression
\[ D(\omega) = \frac{1}{\pi} \lim_{\delta \to 0} \Re \frac{\partial}{\partial \omega} \ln \det \left[1 + F(\omega - i\delta)\right] \] 
where \( \Re \) denotes the imaginary part. A more useful expression for numerical purposes is obtained by closing the contour around the poles of the Fermi distribution. We then get
\[ \ln Z = \sum_{n=0}^{\infty} \text{Tr} \ln \left[[1 + F(-2\pi(n + 1/2)i/\beta)][1 + F^{-1}(2\pi(n + 1/2)i/\beta)]\right]. \] 
Although the sum cannot be taken in closed form for general boundary couplings, higher-order terms become exponentially suppressed at a given finite temperature. For zero temperature, the sum becomes an integral reproducing the formula above.

The crucial point to bear in mind here is that we now have an expression for the partition function, for finite size and finite temperatures, for an arbitrary set of boundary parameters. These, hidden in the matrices above (which contain the boundary matrices), allow one in principle to compute many thermodynamic quantities in the system by using simple parameter derivatives. We will in what follows mostly concentrate on a particular one, but the reader in urgent to remember that the formulas above are more adaptable than we shall make out.

Let us now turn to the computation of interesting observables, the most important of which is the total current through the system. This is given by the expectation value of the operator (we put the electron’s charge to one)
\[ I_T(x) = i \sum_{j} v_{\sigma}^j \left[\Psi_{L\sigma}^\dagger(x)\Psi_{L\sigma}(x) - \Psi_{R\sigma}^\dagger(x)\Psi_{R\sigma}(x)\right]. \] 
in terms of chiral components. Now suppose that for all \( ij \), we have \( \Delta_{ij}^L = |\Delta_{ij}^L|e^{-ix} \), \( \Delta_{ij}^R = |\Delta_{ij}^R|e^{-ix} \). The physical interpretation of this is clear: the coherence length of the original superconductor is large enough to subject all channels to the same pairing phase (i.e., we have seen that the bulk pairing phase gets transferred more or less directly to the phase of the boundary pairing. If all the channels are coupled to the superconductor within a domain of typical size smaller than the coherence length, then the induced boundary pairing pairing phase is the same for all channels).
FIG. 1: The Josephson current at zero temperature for different values of the boundary parameters as a function of the phase $\chi$. This is for the 2-channel case, with no interactions, and with equal pairing amplitudes (that is, all the pairing matrix elements are equal). The critical current at zero temperature in this figure and the one below has been put to 1.

The modulus of the boundary pairings, however, can still vary from one channel to the other: microscopic details then have influence over them. In that case, it is straightforward to show the well-known fact that

$$I(\chi) = \langle I_T \rangle = -\frac{2}{\beta} \frac{d}{d\chi} \ln Z$$

(28)

using sources and an $x$-dependent gauge transformation. Therefore, a knowledge of the phase dependence of the partition function allows one to compute the current without further work, simply by using this derivative trick. This derivation is valid even in the presence of interactions, as long as these can be expressed as current-current interactions.

At zero temperature, we obtain the simple formula

$$I_0(\chi) = \frac{d}{d\chi} E_0$$

(29)

involving the ground-state energy above. This is plotted in figure [3] for the case of boundary pairing with equal entries, i.e. $\Delta^U = \Delta$. This essentially reproduces the plot in [21], done for a single channel, the only difference being that the perfect current occurs for $\Delta = 1/2$ instead of one. One can clearly see how the shape of the current-phase relationship changes from a $\sin \chi$ behaviour for small boundary pairing (so with normal reflection amplitude $\gg$ Andreev reflection amplitude) to the sawtooth function $\chi \pmod{2\pi}$ for optimal boundary pairing (where Andreev reflection amplitude $\gg$ normal reflection amplitude).

At finite temperatures, we can obtain a closed form expression by computing the $\chi$-derivative by hand. Defining the functions

$$\tilde{F}(-i\eta) = e^{-i\sigma_z \otimes k_f R - 1 \otimes \eta R - i} \sigma_z, e^{2i \arctan(v^{-1} M_L)} e^{-i\sigma_z \otimes k_f R - 1 \otimes v^{-1} \eta R}$$

(30)

we can then write

$$I(\chi) = -\beta^{-1} \sum_{n=0}^{\infty} \text{Tr} \left[ 1 + F(-i\eta_n) \right]^{-1} \tilde{F}(-i\eta_n) + \text{h.c.}$$

(31)
where \( \eta_n = \frac{2\pi}{\beta}(n + 1/2) \). Such a current is plotted in figure (2) for different temperatures and for a few values of the boundary parameters.

The appearance of the commutator in equation (30) is easily interpreted: the terms in (21) that do not commute with the \( \sigma_z \) matrix are precisely the boundary pairing terms driving Andreev reflection, and therefore the Josephson current.

This completes our discussion of the noninteracting case. From our formulas, it is easy to compute other transport properties, or to adapt to different systems with, for example, boundary terms breaking the spin symmetry. We leave these for now, and turn instead to the effects of interactions in the bulk.

### III. THE INTERACTING CASE

In this section, we study two general classes of interacting one-dimensional systems at finite size, coupled to the boundary conditions that simulate the superconducting leads to which the real system is coupled. We consider first the case of a multichannel Luttinger liquid in which the different channels only interact through the boundaries. Then we study the case of a carbon nanotube which can be modelled as a two channel Luttinger liquid with a particular choice of interchannel coupling. In both cases, we consider as the starting point the system coupled to either normal or Andreev boundary conditions, and introduce more general boundary conditions perturbatively. These lead to different types of behaviour, which we detail by presenting computations of the Josephson current (either perturbatively or exactly depending on the situation), and of the superconducting order parameter profile in the bulk of the system induced by the presence of the superconducting leads (the “proximity effect”).

#### A. Multichannel Luttinger liquid

It has long been appreciated theoretically that the interacting Luttinger liquid provides a one-dimensional metallic counterpart to the higher-dimensional Fermi-liquid state. However, there are no fermionic quasi-particles in a Luttinger liquid and the elementary excitations are rather bosonic collective charge and spin fluctuations dispersing with different
velocities. One needs to take interactions into account right from the beginning by using the bosonization technique. The power of this technique resides in the fact that it allows to express an interacting fermionic Hamiltonian in terms of a noninteracting bosonic one. Correlation functions of fermion operators become correlation functions of (combinations of) bosonic vertex operators, which can be computed exactly.

The bosonization procedure has been extensively reviewed in the literature \[25,28\]. Here we will only discuss in detail the particularities due to finite size and finite temperature, and how they depend on the boundary conditions. We start by noting that the fermion operators can be written in terms of bosons as

\[
\Psi_j^\sigma = \frac{n_j^\sigma}{\sqrt{4\pi}} e^{-i\sqrt{2\pi}[\cosh \xi_j^\sigma \phi_{jL}^\sigma + \sinh \xi_j^\sigma \phi_{jR}^\sigma + \sigma \cosh \xi_j^\sigma \phi_{jL}^\sigma + \sinh \xi_j^\sigma \phi_{jR}^\sigma]},
\]

\[
\Psi_j^{-\sigma} = \frac{n_j^{-\sigma}}{\sqrt{4\pi}} e^{-i\sqrt{2\pi}[\sinh \xi_j^\sigma \phi_{jL}^{-\sigma} + \cosh \xi_j^\sigma \phi_{jR}^{-\sigma} + \sigma \sinh \xi_j^\sigma \phi_{jL}^{-\sigma} + \cosh \xi_j^\sigma \phi_{jR}^{-\sigma}]} \tag{32}
\]

where the \(\eta\)'s are Klein factors obeying a diagonal Clifford algebra, ensuring anticommutation of different fermionic species. The index \(j = 1, \ldots, N\) labels the channel, and \(\sigma = \pm\) labels the spin. The interaction parameters \(\xi_j^\sigma\) vanish in the noninteracting case, and are related to other conventional notations by

\[
e^{2\xi_j^\sigma} = K_j^\sigma. \tag{33}\]

Repulsive interactions correspond to taking \(\xi < 0\) \((K < 1)\), whereas attractive interactions correspond to taking \(\xi > 0\) \((K > 1)\). In the following, we shall make use of both of these notations interchangeably.

The Hamiltonian of the bulk interacting theory is that of free chiral left- and right-moving bosons for the charge and spin sectors:

\[
H = \sum_j \sum_{\alpha = c,s} H_j^\alpha, \quad H_j^\alpha = \int_0^R dx \sum_{\sigma = \pm} \left[ (\partial_x \phi_j^\alpha)^2 + (\partial_x \phi_j^\alpha)^2 \right]. \tag{34}\]

The canonical commutation relations in the interval \([0, R]\) are

\[
\left[ \phi_j^\alpha(x, t), \partial_x \phi_j^\beta(x', t) \right] = \frac{i}{2} \delta^\alpha_\beta \delta(x - x'), \quad \left[ \phi_j^\alpha(x, t), \partial_x \phi_j^\beta(x', t) \right] = -\frac{i}{2} \delta^\alpha_\beta \delta(x - x'). \tag{35}\]

When working in the Hamiltonian formalism, our first objective is to provide adequate mode expansions for our fields. In a finite-size setup such as the one we are considering, the boundary conditions influence the particulars of this mode expansion: boundary conditions are in fact constraints on the fields that have to be properly taken into account. Unfortunately, in the interacting case, we cannot simply write down some interpolating boundary conditions like we have done for the fermionic fields in the noninteracting case. Instead, one has to impose some tractable boundary conditions, and use a perturbative setup around them. Two cases are detailed below, which we have already introduced in the noninteracting case: normal (eq. [2]) and Andreev (eq. [4]) boundary conditions.

1. **Normal boundary conditions**

The philosophy that we adopt for now is to start by imposing normal boundary conditions on the fermions \(\tilde{\phi}\). In terms of the charge and spin bosons, this means that we fix (being careful with the zero modes)

\[
\phi_j^\sigma(0, t) = \phi_j^\sigma(0, t) + \phi_j^\sigma(0, t) = \sqrt{\frac{z}{2K_j^\sigma}} [n_j^\sigma + n_j^\sigma] = \alpha_j^\sigma, \quad \phi_j^\sigma(R, t) = \phi_j^\sigma(R, t) + \phi_j^\sigma(R, t) = \sqrt{\frac{z}{2K_j^\sigma}} [n_j^\sigma - n_j^\sigma] = \alpha_j^\sigma,
\]

\[
\phi_j^\sigma(R, t) = \sqrt{\frac{z}{2K_j^\sigma}} [m_j^\sigma + m_j^\sigma + 1 - 2 \frac{K_j^\sigma R}{z}] = \beta_j^\sigma, \quad \phi_j^\sigma(R, t) = \sqrt{\frac{z}{2K_j^\sigma}} [m_j^\sigma - m_j^\sigma] = \beta_j^\sigma, \tag{36}\]

with \(n, m \in \mathbb{Z}\). The mode expansions satisfying these boundary conditions can be explicitly written down as

\[
\phi_{jL}(x, t) = \frac{\phi_{jL}^\sigma}{\sqrt{2}} + \tilde{\Pi}_j^\sigma \frac{x + v_j^\sigma t}{2R} + \zeta_{jL}^\sigma(x, t), \quad \phi_{jR}(x, t) = \frac{\phi_{jR}^\sigma}{\sqrt{2}} + \tilde{\Pi}_j^\sigma \frac{x - v_j^\sigma t}{2R} + \zeta_{jR}^\sigma(x, t), \tag{37}\]
with dynamical parts

\[ \zeta^j_{aL}(x, t) = \frac{i}{\sqrt{4\pi}} \sum_{n \in \mathbb{Z}} \frac{1}{n} \beta^j_{an} e^{-i\pi n(x-vjt)/R} = -\zeta^j_{aR}(-x, t) \]  

(38)

and commutation rules

\[ [\hat{\phi}^j_{aL}, \hat{\Pi}^k_{a'}] = -[\hat{\phi}^j_{aR}, \hat{\Pi}^k_{a'}] = i\delta^{jk}\delta_{aa'}, \quad [b^j_{an}, b^k_{m}] = n\delta_{n+m,0}\delta^{jk}\delta_{aa'}, \]  

(39)

with the left and right zero modes commuting.

The eigenvalues of the zero modes are set by the boundary conditions to relative values given by the topological number combinations \( [20] \):

\[ \hat{\phi}^j_{aL} + \hat{\phi}^j_{aR} = 2\alpha^j_a, \quad \hat{\Pi}^j_a = \beta^j_a - \alpha^j_a. \]  

(40)

In terms of operators, the charge and spin Hamiltonians become

\[ H^j_a = \frac{v^j_a}{2R} \hat{\Pi}^j_a + \frac{\pi v^j_a}{2R} \sum_{n \neq 0} b^j_{an} b^j_{a,n} = \frac{v^j_a}{2R} \hat{\Pi}^j_a + \frac{\pi v^j_a}{2R} \sum_{n > 0} b^j_{an} b^j_{a,n} - \frac{\pi v^j_a}{24R} \]  

(41)

Correlation functions can be obtained exactly in the finite-size geometry we are considering, even in the presence of a finite temperature. Our particular choice of fermionic boundary conditions yields a simple variation on the so-called open boundary conditions used in \([30]\), where the computation of the two-point correlator is made. In the Appendix \( A \) we have extended this result to the general case of the 2n-point correlation function. One thing to keep in mind is that we have computed canonical correlators, not (imaginary) time-ordered ones. The latter, however, can be obtained as combinations of the correlators we have derived.

The first thing that we are interested in computing is the correction to the partition function coming from the boundary couplings representing the effect of the superconductors. We perform this computation starting once again from a real-time formalism. There are some subtleties when deriving a Matsubara-like formalism in the particular finite-size geometry that we are using: we devote Appendix \( B \) to this issue.

The perturbing Hamiltonian, after making use of the boundary conditions, becomes

\[ H_1 = H_{1L} + H_{1R}, \]

\[ H_{1L} = 2 \sum_{i,j=1}^N \Delta_{ij}^L \left[ \Psi_{L_1}^j(0)\Psi_{L_1}^j(0) + h.c. \right], \]

\[ H_{1R} = 2 \sum_{i,j=1}^N \Delta_{ij}^R \left[ e^{i\chi+i(k^j_F+k^j_{F'})R} \Psi_{L_1}^j(R)\Psi_{L_1}^j(R) + h.c. \right]. \]  

(42)

The correction to the partition function comes from the usual Matsubara term. The first order vanishes; after a bit of algebra, we are left with the second order \( \chi \)-dependent contribution

\[ \langle \rho_{\mu\nu}^{(2)} \rangle = 16 \cos \chi \sum_{i,j=1}^N \Delta_{ij}^L \Delta_{ij}^R \frac{\theta}{\sqrt{\pi}} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \left[ (1 - \delta_{i,j})G_1^{(2)}(-i(\tau_1 - \tau_2), R)G_1^{(2)}(-i(\tau_1 - \tau_2), R) + \delta_{i,j}G_1^{(4)}(-i(\tau_1 - \tau_2), R) \right] \]  

(43)

where \( \Re \) denotes the real part, and the correlation functions can be computed using the formulas in Appendix \( A \). After a bit of algebra, and some simplifications, we obtain for the two-point contributions the expressions

\[ G_1^{(2)}(-i\tau, R) = N^{-1}_1 \tilde{Q}_1(-i\tau, R) \prod_{a=c,s} P_{a,i} \left[ F_a^0(-iv_{a,\tau}^0 + R) \right] - \frac{ze_{aL}}{2}, \]  

(44)

where the zero-mode part is

\[ \tilde{Q}_1(-i\tau, R) = -e^{\frac{k_{F,F'}\alpha_{c,s}^0}{4}} \frac{\theta_1(\tau^0\frac{\tau}{\sqrt{\pi}} + k^0_{F'}R|\tau^0)\theta_4(\tau^0\frac{\tau}{\sqrt{\pi}} + k^0_{F'}R|\tau^0) + \theta_4(\tau^0\frac{\tau}{\sqrt{\pi}} + k^0_{F'}R|\tau^0)\theta_1(\tau^0\frac{\tau}{\sqrt{\pi}} + k^0_{F'}R|\tau^0)}{\theta_2(\tau^0k^0_{F'}R|\tau^0)\theta_3(0|\tau^0) + \theta_3(\tau^0k^0_{F'}R|\tau^0)\theta_2(0|\tau^0)} . \]  

(45)
The reader unfamiliar with $\theta$-functions is referred to Appendix B. In order to make the bulk of our paper less clogged up, we have defined all functions and parameters in Appendix A. Suffice it to say here that the dynamical functions $F$ are given by

$$F_a(z) = \frac{\theta_1(\frac{z + \omega_a}{2R})}{\theta_1(\frac{z}{2R})}$$

(46)

in which $\omega_a$ is an implicit cutoff, i.e. $z \to z - i\alpha$. The periods of the theta functions are

$$\omega_a = i\frac{v_a\beta}{2R}.$$  

(47)

The phases are $P_{a,i} = e^{i\frac{z(1 + e^{-2\xi_i})}{4}}$. For the four-point function, we get

$$G_i^{(4)}(-i\tau_i, R) = N_2^{-1} \hat{Q}_i^{(4)} P_{c,i}^4 \left[F_c(-iv^c_\beta + R)\right]^{-2e^{-2\xi_i}}.$$

(48)

where the zero-mode part is

$$\hat{Q}_i^{(4)}(-i\tau_i, R) = -e^{i\sum_k N_{c,i} R} \frac{\theta_2(\tau_i | e^{-\frac{k_F R}{} + k^j F_i R} | \tau)^{\tau_i(0)} - \theta_3(\tau_i | e^{-\frac{k_F R}{} + k^j F_i R} | \tau)^{\tau_i(0)}}{\theta_2(\tau_i | e^{-\frac{k_F R}{} + k^j F_i R} | \tau)^{\tau_i(0)} + \theta_3(\tau_i | e^{-\frac{k_F R}{} + k^j F_i R} | \tau)^{\tau_i(0)}}.$$  

(49)

A few comments might be in order here. First of all, the definitions of the functions used here are to be found in Appendix A. For a quick match with the usual correlation function behaviours, however, one just has to remember that the functions $F_a(z)$ behave like $\propto z$ for zero temperature and infinite system size. Therefore, the scaling exponent of correlation functions is just (minus) the exponent of $F$ in the expression encountered. This gives the usual bulk power-law correlation functions so typical in this kind of business. In particular, at the level of the dynamical contributions through the $F$ functions, the charge and spin sector are completely separated, a phenomenon which is usually called spin-charge separation. The two types of excitations, spin and charge modes, move at different velocities, seemingly splitting the electron in different parts. Notice the crucial fact, however, that the zero-mode part above does not separate at all in distinct charge and spin sectors. Via the zero modes, these two sectors remain completely entangled, unless we neglect all finite-size effects. In the limit of an infinite size, the zero-mode parts simplify to trivial phases. We are, however, interested in finite-size effects, so we keep these contributions throughout our treatments. This will turn out to be true of all correlators we shall compute in all cases: the dynamical sector readily factorizes into charge and spin sectors, but the zero-modes entangle the two in a complicated manner.

The expression for the Josephson current to this order in perturbation theory is easily obtained by applying the derivative trick:

$$I(\chi) = 32\sin \chi \sum_{i,j=1}^N \Delta_{ij} \Delta_{ij} R^\beta \int_0^\beta d\tau_1 \int_0^\tau_1 d\tau_2 \left[(1 - \delta_{i,j})G_i^{(2)}(-i(\tau_1 - \tau_2), R)G_j^{(2)}(-i(\tau_1 - \tau_2), R) + \delta_{i,j} G_i^{(4)}(-i(\tau_1 - \tau_2), R)\right].$$

(50)

One thus obtains a $\sin \chi$ behaviour to this order in perturbation theory, with a normalization that is bilinear in the boundary pairings, and contains a rather nontrivial function of the system size, temperature, velocities and interaction parameters. This reproduces the expected form of the current near the normal fixed point. Higher-order terms in perturbation theory could now be taken into account, which would start showing deviations from the pure $\sin \chi$ behaviour. Moreover, each term in the perturbation theory would be composed of some nontrivial function of $\chi$ multiplied by a complicated integral over correlation functions, in other words a rather daunting function of system size. These would in turn conspire to make the current some scaling function of appropriate powers of the boundary pairings and the system size. We have not included those computations here, although they can in principle be performed using our formulas, given enough patience.

One of the features of equation (49) is that the current contains two different types of scaling functions of system size, with different exponents associated to them. In the limit of large system size and zero temperature, we can extract the leading behaviours of these, and write symbolically

$$I(\chi) \sim \sum_i I_i^0 R^{1 - 2\xi_i} + \sum_{ij} I_{ij}^0 R^{-1} [e^{-2\xi_i} + e^{-2\xi_j} + i \rightarrow j].$$

(51)
The first type of scaling comes from Cooper pair tunnelling purely into one of the channels, whereas the second type comes from Cooper pair tunnelling into two different channels (i.e., the Cooper pair splits up in two single-electron components penetrating channels $i$ and $j$). The particular form of these power laws depends crucially on the assumption that the system sits close to normal boundary conditions: if it is close to Andreev boundary conditions, the scaling is different, as will be seen later. One immediate thing to notice from these scaling functions is the expected fact that the current gets suppressed by increasing the repulsiveness of the interactions (that is, making $\xi_c$ smaller), whereas throwing in more attractive interactions enhances the current. This is very natural: we are dealing with a supercurrent, produced by pairing, and pairing works hand in hand with attractiveness.

Another object of interest is the pairing operator, defined as

$$ P^{ij}(x, t) = \Psi^\dagger_i(x, t)\Psi^\dagger_j(x, t) $$

within the bulk of the system. The fact that the expectation value of this operator can be nonvanishing in the bulk of the system represents the effects of the superconducting boundaries: these produce the well-known “proximity effect”, of which this is one manifestation. Using the normal boundary conditions, we can rewrite this pairing operator purely in terms of left-movers, and compute its expectation value to first order in perturbation theory. The two boundaries are independent to this order, so we take into account only the left one for simplicity. The result is

$$ \langle P^{ij}(x, 0) \rangle = 4\Delta_L^j \int_0^\beta d\tau \left[ (1 - \delta_{i,j})2\Re \left\{ G^{(2)}_i(-i\tau, x) \right\} \Re \left\{ G^{(2)}_j(-i\tau, x) \right\} - \delta_{ij} \Re \left\{ G^{(4)}_+(-i\tau, x) \right\} + G^{(4)}_-(i\tau, x) \right] \right) \right] \tag{53} $$

The two-point functions in either channel are here given by slightly more complicated functions (derived again from the formulas in Appendix A):

$$ G^{(2)}_i(-i\tau, x) = N^{-1}_1 \tilde{Q}_i(-i\tau, x) \prod_{a = c, s} \left[ F^\dagger_a(-iv_a\tau + x) \right]^{-\frac{1}{2}(1-e^{-2\xi_c})} \left| F_a(-iv_a\tau + x) \right|^{-\frac{1}{2}(1-e^{-2\xi_c})} |F_a(2x)|^{-\frac{1}{2}\sinh 2\xi_c} \tag{54} $$

where $|F|$ represents the modulus of $F$, and the zero mode parts are again given in terms of $\theta$-functions as

$$ \tilde{Q}_i(-i\tau, x) = e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_R^\tau|}} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_L^\tau|}} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_L^\tau|}} = \frac{\theta_2(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0) + \theta_3(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0)}{\theta_2(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0) + \theta_3(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0)}. \tag{55} $$

The four-point functions are of two types. This comes from the fact that, when writing the full pairing operator in terms of left-movers only, one obtains terms like $\Psi^\dagger_L^\tau(x)\Psi^\dagger_L^\tau(x)$ and $\Psi^\dagger_L^\tau(-x)\Psi^\dagger_L^\tau(x)$. These turn out to have slightly different $x$-dependencies. We label the first type $+$, and the second $-$.

The first type of correlator is

$$ G^{(4)}_{+; i}(-i\tau, x) = N^{-1}_2 \tilde{Q}^{(4)}_{+; i}(-i\tau, R) |F_c(-iv_c\tau + x)|^{-\frac{1}{2}(1-e^{-2\xi_c})} |F_c(2x)|^{-\frac{1}{2}\sinh 2\xi_c} \tag{56} $$

with

$$ \tilde{Q}^{(4)}_{+; i}(-i\tau, x) = e^{2k_F R x} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_R^\tau|}} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_L^\tau|}} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_L^\tau|}} = \frac{\theta_2(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0) + \theta_3(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0)}{\theta_2(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0) + \theta_3(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0)}. \tag{57} $$

The second type is

$$ G^{(4)}_{-; i}(-i\tau, x) = \tilde{Q}^{(4)}_{-; i}(-i\tau, R) |F_c(-iv_c\tau + x)|^{-\frac{1}{2}(1-e^{-2\xi_c})} |F_c(2x)|^{-\frac{1}{2}\sinh 2\xi_c} \tag{58} $$

with

$$ \tilde{Q}^{(4)}_{-; i}(-i\tau, x) = e^{2k_F R x} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_R^\tau|}} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_L^\tau|}} e^{\frac{\tau_2(\tau^2_0 + k_F R) + \frac{\pi}{2}\tau_1^2}{|F_L^\tau|}} = \frac{\theta_2(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0) + \theta_3(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0)}{\theta_2(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0) + \theta_3(\tau^2_0 + k_F R)\theta_2(0|\tau^2_0)}. \tag{59} $$

One therefore sees that the three terms in (53) have different scaling behaviour when interactions are added to the bulk of the system. In particular, the behaviour of the pairing expectation value depends on whether we are looking at channel-diagonal or crossed-channel pairing. The results can be summarized by the scaling laws

$$ \langle P^{ij}(x) \rangle \sim \delta_{ij} P_{+; i} x^{-1/2\xi_c} - x^{-1/2\xi_c} + x^{-3/2\xi_c} + x^{-1/2\xi_c} i + x^{-1/2\xi_c} j. \tag{60} $$

In these, $x$ represents the distance perpendicularly away from the superconducting boundary. The cross-channel term has a decay law that depends on the interaction parameters in both channels, whereas the channel diagonal term has two different behaviours, with different exponents depending on the nature of the operator involved. These results are collected in simplified form and discussed further in section [x].
2. Andreev boundary conditions

The other case that we consider, which leads to tractable calculations, requires extremely good coherence between the original bulk superconductors and the one-dimensional channel. This does not mean a perfect contact in the usual sense of the word, however. We invite the reader to consult reference \[21\], where basic considerations on the level of a simple lattice model point to the fact that there exists an optimization phenomenon, by which it should be possible to achieve perfect Andreev reflection at the boundaries by tuning available parameters.

We therefore want to reproduce the above results in this case. Since the specifics of the boundary conditions are crucial to the physics of the system, physical quantities like the Josephson current and the superconducting order parameter profile will be different from the previous results computed around the normal fixed point. We therefore start by imposing Andreev boundary conditions on the fermions, eq. (14). The right-moving fields can then be described in terms of left-moving ones via

\[
\psi_{L\sigma}(x) = i\epsilon_{\sigma\sigma}' \psi_{R\sigma}'(-x)
\]

with periodicity conditions

\[
\psi_{L\sigma}(x + 2R) = -e^{i\chi} \psi_{L\sigma}(x).
\]

In terms of the charge and spin bosons, this means that we fix

\[
\hat{\phi}_c^j(0, t) = \hat{\phi}_{cL}^j(0, t) - \hat{\phi}_{cR}^j(0, t) = \sqrt{\frac{\pi K^j_c}{2}} [m^j_c + n^j_c] \equiv \tilde{\alpha}^j_c, \quad \phi_s^j(0, t) = \sqrt{\frac{\pi}{2K^j_s}} [m^j_s - n^j_s - \frac{1}{2}] \equiv \alpha^j_s,
\]

\[
\hat{\phi}_c^j(R, t) = \sqrt{\frac{\pi K^j_c}{2}} [m^j_c + m^j_c + 1 - \frac{\chi^j}{\pi}] \equiv \tilde{\beta}^j_c, \quad \phi_s^j(R, t) = \sqrt{\frac{\pi}{2K^j_s}} [m^j_s - m^j_s + \frac{1}{2}] \equiv \beta^j_s,
\]

with \(n, m \in \mathbb{Z}\). The mode expansions satisfying these boundary conditions are the same as for normal boundary conditions for the spin channel. For the charge channel we have instead

\[
\hat{\phi}_{cL}^j(x, t) = \frac{\hat{\phi}_c^j}{2} + \hat{\Pi}_c^j x + \frac{v^j_c t}{2R} + \tilde{\zeta}_{cL}^j(x, t),
\]

\[
\hat{\phi}_{cR}^j(x, t) = \frac{\hat{\phi}_c^j}{2} + \hat{\Pi}_c^j x - \frac{v^j_c t}{2R} + \tilde{\zeta}_{cR}^j(x, t),
\]

with dynamical parts

\[
\tilde{\zeta}_{cL}^j(x, t) = \frac{i}{\sqrt{4\pi}} \sum_{n \in \mathbb{Z}} \frac{1}{n} b^j_{cm} e^{-i\pi n (x + v^j_c t)/R} = \tilde{\zeta}_{cR}^j(-x, t)
\]

and commutation rules

\[
[\hat{\phi}_{cL}^j, \hat{\Pi}_c^k] = [\hat{\phi}_{cR}^j, \hat{\Pi}_c^k] = i\delta^{jk}, \quad [b^j_{cm}, b^k_{cm}] = n\delta_{n+m,0} \delta^{jk},
\]

with the left and right zero modes, and charge and spin modes commuting.

The eigenvalues of the zero modes are the same as before for the spin modes, and for the charge sector they become

\[
\hat{\phi}_{cL}^j - \hat{\phi}_{cR}^j = 2\tilde{\alpha}_c^j, \quad \hat{\Pi}_c^j = \tilde{\beta}_c^j - \tilde{\alpha}_c^j.
\]

The fundamental difference between normal and Andreev boundary conditions is that for the Andreev case, there exists a Josephson current and a nonzero pairing operator to zeroth order in perturbation theory. That is, we can first of all compute the partition function in the Andreev case, which carries a phase dependence giving nonzero \(I(\chi)\) by using the derivative trick.

The partition function is given by \(Z = \text{Tr}[e^{-\beta \sum_{j} (H^j_{ch} + H^j_{ch})}]\) where \(H^j_{ch}\) is the charge or spin Hamiltonian for a given channel (eq. 14). Since the Andreev boundary conditions as well as the interactions are channel diagonal, the total partition function is a product of the partition function for each channel. Therefore the total current will be the sum of the contributions of each channel. In other words, we need to compute simply the partition function for one LL coupled to Andreev boundary conditions. This computation was already done by Maslov et al. in reference \[18\]. Here we only sketch the derivation. Since the Hamiltonian can be written in terms of a zero mode part plus a dynamical
part that commute with each other, the partition function becomes a product \( Z = Z_{zm}(\chi) \times Z_{dyn} \), where only the zero modes part depends on \( \chi \). Using the explicit expressions for \( \Pi_i \) and \( \Pi_x \) given in eqs. (14) and (15) respectively, and following the same steps described in Appendix A for the computation of the correlation functions, we obtain

\[
Z_{zm} = e^{ \frac{i\beta v_F^2}{R} \left[ \vartheta_2(\bar{\tau}_c \frac{\chi}{2}) \vartheta_2(0 | \tau_s) + \vartheta_3(\bar{\tau}_c \frac{\chi}{2}) \vartheta_3(0 | \tau_s) \right] } \tag{68}
\]

where \( \bar{\tau}_c = i \beta \frac{2\pi c}{R} \). We have dropped the channel index for simplicity. The current can be readily obtained by taking the derivative with respect to \( \chi \) of the above expression for each channel, and adding over the channels. Since the general form of the current for arbitrary temperature and length, involves derivatives of the \( \vartheta_{2,3} \) functions, it cannot be written in a closed form. The asymptotic values of the current for a given channel are:

\[
P^j(\chi^j) = e^{\frac{\beta v_F^2}{R} \chi^j} \tag{69}
\]

at low temperatures \( \left( \frac{\beta v_F^2}{R} >> 1 \right) \), and

\[
P^j(\chi^j) = 8eT e^{ -\frac{\beta v_F^2}{R} \left( \frac{2\pi c}{v_F} + \frac{2\pi c}{v_F} \right) } \sin(\chi^j) \tag{70}
\]

at high temperature \( \left( \frac{\beta v_F^2}{R} << 1 \right) \). This expression reflects the fact that at high temperatures, repulsive interactions \( (K^2 = e^{2\pi c} < 1) \) suppress the Josephson current \( [18] \).

Here as well, the channel diagonal component of the pairing operator (eq. (12)) can be computed exactly, without the need for perturbation theory. The general expression for \( P^{ii}(x,t) \) can be obtained from the expressions in the Appendix A by taking the coordinates of both fermions at the same point. Here we only write the asymptotic values of the order parameter at high and low temperatures.

At low temperatures we obtain

\[
< P^{ii}(x) > \approx e^{i \frac{\pi c}{2} x} \left( \sin \left( \frac{\pi x}{R} \right) \right)^{-\frac{1}{2}} (e^{-2\pi c} + e^{2\pi c}) \tag{71}
\]

In the limit of infinite length this correlator behaves like \( < P^{ii}(x) > \approx \frac{1}{2} \) with \( \gamma = (e^{-2\pi c} + e^{2\pi c}) / 2 \). This result was already derived by Maslov et al. in reference [18].

At high temperatures, the order parameter becomes

\[
< P^{ii}(x) > \approx e^{-\frac{\pi c}{4} x} (e^{-2\pi c} + e^{2\pi c}) \tag{72}
\]

We emphasize again that the two results above are completely nonperturbative, since the partition function and correlators can be calculated exactly at finite temperature in this finite-size geometry for what we have termed Andreev boundary conditions (eq. (14)). It is however possible that in the physical system described by the multichannel Luttinger liquid, the coupling to the superconducting leads is represented by non diagonal matrices \( \Delta_{R,L} \). This is certainly the case if the distance between the channels in the Luttinger liquid is smaller than the coherence length in the superconducting lead. In this case, it is necessary to take into account the off diagonal contributions of the boundary action, i.e., those terms corresponding to \( \Delta_{ij}^{R,L} \neq 0 \) for \( i \neq j \). This can only be done perturbatively, in an analogous way to the one described in the previous subsection. We do not include explicitly the calculations here, but the contributions of these perturbations to the scaling behaviour of the superconducting current and the pairing operator are summarized in section [14].

### B. Carbon nanotubes

Carbon nanotubes (CN), long, thin cylinders of graphite discovered in 1991 by S. Iijima [31], are large molecules that are unique for their size, shape, and remarkable physical properties. In fact, they can be metallic depending on the wrapping of the tube [3, 4, 32]. In the following we will be interested in those CN which are metallic and are known as “Single walled nanotubes” (SWNT). They exhibit unique quantum wire properties that derive from the tubes’ nanometer scale diameters in combination with the special electronic structure of graphite. Because of the quantization of circumferential modes, the tube’s electronic states do not form one wide electronic energy band but instead split into one-dimensional subbands with band onsets at different energies. These subbands are widely separated in energy, on the scale of 1 eV (much larger than the room-temperature thermal energy \( k_B T \), about 0.025
which we can now use to represent our fermions with the bosonization formulas

\[ \phi^T_{\mu a} = \frac{1}{\sqrt{2}} (\phi^1_{\mu a} + \tau \phi^2_{\mu a}). \]  

with \( \tau = \pm, \mu = c, s, \) and \( a = L, R. \) In terms of these operators, the Hamiltonian becomes (after taking the interactions into account nonperturbatively) that of a set of free bosons,

\[ H = \sum_{\mu=c,s} \sum_{\tau = \pm} \int_0^R dx v_\mu \left[ (\partial_x \phi^T_{\mu L})^2 + (\partial_x \phi^T_{\mu R})^2 \right] \]  

which we can now use to represent our fermions with the bosonization formulas

\[ \Psi^{1,2}_{L,\sigma} = \frac{1}{\sqrt{4\pi}} e^{i\sqrt{4\pi}[\cosh \xi_c \phi^+_{cL} + \sinh \xi_c \phi^+_{cR} \pm \phi^-_{cL} + \sigma(\phi^+_{cL} \pm \phi^+_{cR})]}, \]

\[ \Psi^{1,2}_{R,\sigma} = \frac{1}{\sqrt{4\pi}} e^{i\sqrt{4\pi}[\sinh \xi_c \phi^+_{cL} + \cosh \xi_c \phi^+_{cR} \pm \phi^-_{cL} + \sigma(\phi^+_{cL} \pm \phi^+_{cR})]} \]  

In the above, \( \xi_c \) is the interaction parameter representing the Coulomb interaction that involves the total charge of the system (in both channels of the nanotube).

1. Normal boundary conditions

As was the case with the Luttinger liquid, in order to set up our computations correctly in the finite size geometry that we are considering, we have to specify tractable boundary conditions. We start with normal boundary conditions for the fermions, which correspond in the bosonic language to taking

\[ \phi^\pm_c(0) = \frac{1}{2} \sqrt{\frac{\pi}{K_c^+}} [n^\pm_n + n^\mp_n] \equiv \alpha^\pm_c, \hspace{0.5cm} \phi^\pm_s(0) = \sqrt{\frac{\pi}{2}} [n^\pm_n - n^\pm_s] \equiv \alpha^\pm_s, \]

\[ \phi^\pm_c(R) = \frac{1}{2} \sqrt{\frac{\pi}{K_c^-}} [m^\pm_m + m^\mp_m + 1 \pm \frac{2k^\pm_c \pi}{\pi}] \equiv \beta^\pm_c, \hspace{0.5cm} \phi^\pm_s(R) = \sqrt{\frac{\pi}{2}} [m^\pm_m - m^\pm_s] \equiv \beta^\pm_s. \]  

where \( K^+_c = K_c, K^-_c = 1, \) and \( n^\pm_\sigma = n^\pm_\alpha \pm n^\pm_\beta \) with \( n^\pm_\alpha \) independent integers, and \( k^\pm_c = k^1_c \pm k^2_c. \) The mode expansions for charge and spin operators coincide with the ones given in eq. (77).

The second-order correction to the partition function can be calculated in a similar manner as for the multichannel Luttinger liquid. After a bit of algebra, and again making use of the correlators in the Appendix A, we find the \( \chi \)-dependent contribution

\[ \langle \rho^{(2)}_\chi \rangle = 16 \cos \chi \sum_{i,j=1,2} \Delta^i \Delta^j_R \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \Re \left[ (1 - \delta_{ij}) C^{(4)}_\parallel (-i(\tau_1 - \tau_2), R) + \delta_{ij} C^{(4)}_\perp (-i(\tau_1 - \tau_2), R) \right], \]

with correlators

\[ G^{(4)}_\parallel (-i\tau, R) = P_+ Q(u_\parallel) D_\parallel (-i\tau, R), \]

\[ G^{(4)}_\perp (-i\tau, R) = P_+ Q(u_\parallel) D_\perp (-i\tau, R), \]  

where we have defined \( P_+ = e^{i\sqrt{2}[1 + e^{-2\xi_c}]} \) (we remind the reader that \( K_c = e^{2\xi_c} \) is the only nonvanishing \( \xi \) for the nanotubes). The dynamical parts are for both correlators equal to

\[ D_\parallel (-i\tau, R) = [F_\parallel(-i\tau v_c + R)]^{-e^{-2\xi_c}} [F_\parallel(-i\tau v_s + R)]^{-1}, \]

\[ D_\perp (-i\tau, R) = [F_\perp(-i\tau v_c + R)]^{-e^{-2\xi_c}} [F_\perp(-i\tau v_s + R)]^{-1}, \]
independently of the channel index, and the zero-mode part is given in terms of the four-dimensional $\theta$-function (defined in the second appendix)

$$Q(u) = e^{i(2k_FR-\pi)\tau_\gamma/\beta} \frac{\theta(u + \frac{2\pi}{\Omega}(2k_FR-\pi)\hat{m}\Omega)}{\theta(\frac{2\pi}{\Omega}(2k_FR-\pi)\hat{m}\Omega)}$$

with vectors

$$u_\perp = \frac{\pi}{4R} \begin{pmatrix} i\tau[e^{-2\xi}v_c + v_s] + 2R \\ i\tau[e^{-2\xi}v_c - v_s] \\ i\tau[e^{-2\xi}v_c - v_s] \\ i\tau[e^{-2\xi}v_c + v_s] + 2R \end{pmatrix}, \quad u_\parallel = \frac{\pi}{4R} \begin{pmatrix} i\tau v_c[e^{-2\xi} + 1] + 2R \\ i\tau v_c[e^{-2\xi} + 1] + 2R \\ i\tau v_c[e^{-2\xi} - 1] \\ i\tau v_c[e^{-2\xi} - 1] \end{pmatrix},$$

and $\hat{m} = (1, 1, 1, 1)$. The interaction-dependent period matrix $\Omega$ is defined in equation (A27). We also remind the reader that $\tau_c = \frac{i\beta v_c e^{-2\xi}}{\Omega}$. To obtain the above results we have used that $G^{(4)|2}_{||\perp} = G^{(4)|2}_{\perp||}$ due to the symmetry properties of the matrix $\Omega$.

For the Josephson current, this then implies that, to second-order in perturbation theory, we find the expected $\sin \chi$ behaviour. More precisely, we get

$$I(\chi) = F(\beta, R) \sin \chi,$$

with a normalization amplitude function that depends on all the parameters of the system, and is given in terms of the integral expression

$$F(\beta, R) = 32 \sum_{i,j} \Delta_{ij}^L \Delta_{ij}^R \frac{1}{\beta} \int_0^\beta \int_0^{\tau_1} [1 - \delta_{ij}] G^{(4)}_{||\perp}(-i(\tau_1 - \tau_2), R) + \delta_{ij} G^{(4)}_{\perp||}(-i(\tau_1 - \tau_2), R).$$

As was the case for the multichannel Luttinger liquid, the scaling behaviour of the current splits up into two different types, depending on the way the Cooper pair penetrates into the conducting channel. Here, for the case of normal boundary conditions, the scaling behaviour is the same for channel-diagonal and cross-channel pair penetration (i.e., doesn’t depend on whether the pair penetrates into one channel only, or splits between the two):

$$I(\chi) \sim (I_\parallel + I_\perp)R e^{-2\xi\tau_c}.$$  (84)

The physical interpretation of this is easy: as Coulomb interactions involve the total charge density in both channels together, it doesn’t matter how the pair tunnels in, as the interaction cost is the same in both cases.

For the pairing operator defined in equation (42), a first-order perturbative computation similar to that in the Luttinger liquid case yields

$$\langle \Phi^{ij}(x, 0) \rangle = -4\Delta^L \int_0^\beta d\tau \Re \left[(1 - \delta_{ij}) \left(G^{(4)}_{||\perp}(-i\tau, x) + G^{(4)}_{\perp||}(-i\tau, x)\right) G^{(4)}_{||\perp}(-i\tau, x) + G^{(4)}_{\perp||}(-i\tau, x) \right],$$

with correlators splitting up into two different classes (channel-diagonal or cross-channel) and again two subclasses ($\pm$, depending on the particular operator combination; we refer the reader to the note at the end of the computation of the pairing for the multichannel Luttinger liquid with normal boundary conditions):

$$G^{(4)}_{\perp\perp}(-i\tau, x) = Q(u_{\perp\perp}) e^{(1\pm1)x^\pm/2} F_{\perp\perp}(-i\tau, x),$$

$$G^{(4)}_{||\parallel}(-i\tau, x) = Q(u_{||||}) e^{(1\pm1)x^\pm/2} F_{||\parallel}(-i\tau, x),$$

with dynamical parts given explicitly in terms of powers of $F$-functions (note in particular that if charge and spin velocity coincide, then the subsets $\pm$ become identical)

$$F_{\perp\perp}(-i\tau, x) = \left[F_c(-i\tau v_c - x)\right]^{-\frac{1}{4}(1 + e^{-2\xi})} \left[F_c(-i\tau v_c + x)\right]^{\frac{1}{2}(1 - e^{-2\xi})} \left[F_c(2x)\right]^{-\frac{\sinh 2\xi}{2}}\frac{\sinh 2\xi}{\left[F_c(-i\tau v_s - x)\right]},$$

$$F_{||\parallel}(-i\tau, x) = \left[F_c(-i\tau v_c + x)\right]^{-\frac{1}{2}(1 + e^{-2\xi})} \left[F_c(-i\tau v_s + x)\right]^{-\frac{1}{2}(1 - e^{-2\xi})} \left[F_c(2x)\right]^{-\frac{\sinh 2\xi}{2}}\frac{\sinh 2\xi}{\left[F_c(-i\tau v_c - x)\right]},$$

$$F_{||\parallel}(-i\tau, x) = \left[F_c(-i\tau v_c - x)\right]^{-\frac{1}{4}(1 + e^{-2\xi})} \left[F_c(-i\tau v_s + x)\right]^{\frac{1}{2}(1 - e^{-2\xi})} \left[F_c(2x)\right]^{-\frac{\sinh 2\xi}{2}}\frac{\sinh 2\xi}{\left[F_c(-i\tau v_c + x)\right]},$$

$$F_{||\parallel}(-i\tau, x) = \left[F_c(-i\tau v_c + x)\right]^{-\frac{1}{2}(1 + e^{-2\xi})} \left[F_c(-i\tau v_s + x)\right]^{-\frac{1}{2}(1 - e^{-2\xi})} \left[F_c(2x)\right]^{-\frac{\sinh 2\xi}{2}}\frac{\sinh 2\xi}{\left[F_c(-i\tau v_c + x)\right]},$$

(83)
The zero-mode parts are here given by equation (80), and the distance vectors appearing in the multivariable \( \theta \)-functions representing the contributions of the zero modes are defined by

\[
\mathbf{u}_{\perp \pm} = \frac{\pi}{4R} \begin{pmatrix}
  i\tau[e^{-2\xi c}\nu_c + \nu_s] \pm 2x \\
  i\tau[e^{-2\xi c}\nu_c - \nu_s] \\
  i\tau[e^{-2\xi c}\nu_c - \nu_s] \\
  i\tau[e^{-2\xi c}\nu_c + \nu_s] + 2x
\end{pmatrix},
\]

\[
\mathbf{u}_{\parallel \pm} = \frac{\pi}{4R} \begin{pmatrix}
  i\tau\nu_c[1 + e^{-2\xi c}] \pm 2x \\
  i\tau\nu_c[1 + e^{-2\xi c}] + 2x \\
  i\tau\nu_c[-1 + e^{-2\xi c}] \\
  i\tau\nu_c[-1 + e^{-2\xi c}]
\end{pmatrix}.
\] (88)

We therefore again find three different scaling behaviours for the pairing operator, as in the multichannel Luttinger liquid case. The scaling of the pairing operator expectation value again splits up in two different types of behaviour, which we can summarize as

\[
\langle P^{ij}(x) \rangle \sim ((1 - \delta_{ij})P_{\perp +} + \delta_{ij}P_{\parallel +})x^{-\frac{1}{4}[1 + e^{-2\xi c}]} + ((1 - \delta_{ij})P_{\perp -} + \delta_{ij}P_{\parallel -})x^{-\frac{1}{4}[1 + 3e^{-2\xi c}]}.
\] (89)

First of all, we note again that the scaling of the \( \pm \) operators coincide, as does the scaling of the \( \mp \) operators. That is, it’s not important (as far as scaling is concerned) how the Cooper pair penetrates the nanotube. These results are summarized and discussed in section II.

2. Andreev boundary conditions

For the relative bosons suitable to describe carbon nanotubes (73), Andreev boundary conditions can be written as

\[
\tilde{\phi}_c^\pm(0) = \frac{\sqrt{\pi}K^\pm}{2}[n^\pm_1 + n^\pm_2] \equiv \alpha_c^\pm, \quad \phi^\pm_c(0) = \frac{\sqrt{\pi}}{2}[n^\pm_1 - n^\pm_2 - \frac{1}{2} \pm \frac{1}{2}] \equiv \alpha_c^\pm,
\]

\[
\tilde{\phi}_c^\pm(R) = \frac{\sqrt{\pi}K^\pm}{2}[m^\pm_1 + m^\pm_2 - 1 \mp 1 - \frac{\chi^\pm}{\pi}] \equiv \beta_c^\pm, \quad \phi^\pm_c(R) = \frac{\sqrt{\pi}}{2}[m^\pm_1 - m^\pm_2 + \frac{1}{2} \pm \frac{1}{2}] \equiv \beta_c^\pm,
\] (90)

where \( n^\pm_1 = n^\pm_2 \) with \( n^\pm_1 \) independent integers, and \( \chi^\pm = \chi^1 \pm \chi^2 \). The mode expansion for charge and spin operators coincide with the one given in eqs. (84) and (87) respectively.

In order to compute the superconducting current, we consider the partition function. Once again, the only part of the partition function which is \( \chi \) dependent is the zero mode part \( Z_{zm} \). Since the two channels constituting the nanotube are coupled to the superconductor leads at the boundary in the same way, the two phases must satisfy \( \chi^1 = \chi^2 = \chi \), and the current is determined by \( Z_{zm}(\chi) \)

\[
Z_{zm}(\chi) = e^{\frac{i\pi\chi^2}{2\pi}}\theta\left(\frac{\tilde{\tau}_c\chi^2}{4}\tilde{m}\tilde{\Omega}\right)
\] (91)

where \( \tilde{\tau}_c = i\beta_c e^{2\xi c}/R \), and \( \tilde{m} = (1, 1, 1, 1) \). The multivariable theta function \( \theta \) and the matrix period \( \tilde{\Omega} \), defined in Appendix B and A respectively, appear naturally when computing correlation functions for carbon nanotubes, where the electronic interactions are not channel diagonal.

The total current is obtained by deriving \( \ln Z_{zm} \) with respect to the phase \( \chi \). Its asymptotic values are in this case given by the expressions

\[
I(\chi) = e^{2e\xi c\nu_c\frac{\chi}{\pi}}
\] (92)

at low temperature \( (\frac{\beta_c}{R} \gg 1) \), and

\[
I(\chi) = 16\pi e^{\frac{\beta_c}{R}(e^{-\xi c\nu_c/\pi} + \frac{1}{\pi})}\sin \chi
\] (93)

at high temperature \( (\frac{\beta_c}{R} \ll 1) \).

A physical realization of the coupling between a nanotube and a superconductor would however produce boundary pairing amplitudes \( \Delta^{ij} \) which should be approximately equal for all the channels (that is, even for the off-diagonal components). This is a fundamental difference between the present case and what one could expect from a two-channel Luttinger liquid: in the latter case, the off-diagonal pairing components could be taken to vanish if the contacts between the original superconductor and the two Luttinger liquids were separated by a finite distance. The diagonal components of the boundary pairing are easy to treat nonperturbatively as seen above. In view of the last
vanish unless we take into account off diagonal terms in the boundary couplings.

The perturbing Hamiltonian is given by

$$H_1 = H_{1L} + H_{1R},$$

where

$$H_{1L} = \frac{1}{2} \left[ \Delta_L^{12} \Psi_1^{11}(0) \Psi_1^{21}(0) + \Delta_L^{21} \Psi_1^{21}(0) \Psi_1^{11}(0) \right] + h.c.,$$

$$H_{1R} = \frac{1}{2} \left[ \Delta_R^{12} \Psi_1^{11}(R) \Psi_1^{21}(R) + \Delta_R^{21} \Psi_1^{21}(R) \Psi_1^{11}(R) \right] + h.c..$$

The first non vanishing correction is the second order one, which is given by

$$\langle \rho^{(2)}_{CN} \rangle = 2 \Delta_L \Delta_R \mathcal{R} \int_{0}^{\beta} dt_{1} \int_{0}^{\tau_{1}} d\tau_{e} e^{i2\epsilon_{e}R-i\tau_{e} \chi} \left( \theta(u_{\uparrow} + \frac{\pi}{4} x \hat{m} \Omega) + \theta(u_{\downarrow} + \frac{\pi}{4} x \hat{m} \Omega) \right) F_{e}(-i\epsilon_{e} \tau + R) - \frac{2\epsilon_{e}}{\beta} [F_{e}(-i\epsilon_{e} \tau - R)]^{-\frac{2\epsilon_{e}}{\beta}} [F_{e}(-i\epsilon_{e} \tau + R)]^{-\frac{2\epsilon_{e}}{\beta}}$$

where $\mathcal{R}$ denotes the real part, the vectors $u_{\uparrow}$ and $u_{\downarrow}$ are defined by

$$u_{\uparrow} = \frac{\pi}{4R} \left( -i(-\epsilon_{e} e^{2\epsilon_{e}} + v_s) \tau, -2R + i(-\epsilon_{e} e^{2\epsilon_{e}} + v_s) \tau, -2R + i(-\epsilon_{e} e^{2\epsilon_{e}} + v_s) \tau \right),$$

$$u_{\downarrow} = \frac{\pi}{4R} \left( -2R + i(-\epsilon_{e} e^{2\epsilon_{e}} + v_s) \tau, -i(-\epsilon_{e} e^{2\epsilon_{e}} + v_s) \tau, -i(-\epsilon_{e} e^{2\epsilon_{e}} + v_s) \tau \right).$$

respectively, and the correlation functions were computed using the formulas in Appendix A. The correction to the superconducting current will be given by $I^{(2)}(\chi) = -\frac{\partial}{\partial \chi} \rho^{(2)}_{CN}$. Even though we can not give a closed expression for this correction, we can study some simple limits. In particular, we can show that in the limit of low temperatures $I^{(2)}(\chi)$ will depend on the length of the system as $\frac{1}{\kappa L^2}$ times a complicated function of $\chi$, representing deviations from the sawtooth form of the perfect current. Thus, the off diagonal contribution to the superconducting current introduces a dependence on the length of the system which is interaction dependent. Due to the general structure of the correlation functions given in Appendix A, it is simple to see that higher order terms in the perturbative expansion will contribute with higher order powers of $\frac{1}{\kappa L^2}$ to the superconducting current at zero temperature.

The order parameter can be computed exactly if we only consider channel diagonal Andreev boundary conditions. In this case, using the boundary conditions (eq. [12]) we can express the pairing operator (eq. [22]) as

$$\langle P^{ii}(x) \rangle = -i \left[ \langle \Psi_{L_{1}}^{ii}(x,t) \Psi_{L_{1}}^{ii}(x,t) \rangle + \langle \Psi_{L_{1}}^{12}(x,t) \Psi_{L_{1}}^{12}(x,t) \rangle \right]$$

$$= -i \left( \frac{\theta(u_{\uparrow}) + \frac{\pi}{4} x \hat{m} \Omega}{\theta(\frac{\pi}{4} x \hat{m} \Omega)} \right) F_{e}(-2x) \left[ \left( \frac{-2\epsilon_{e}}{\beta} \right) \left( \frac{-2\epsilon_{e}}{\beta} \right) \right]$$

where $u_{\uparrow} = \frac{\pi}{2R}(1 + \sigma, 1 - \sigma, 0, 0)$ and $u_{\downarrow} = \frac{\pi}{2R}(0, 0, 1 + \sigma, 1 - \sigma)$. The contributions to $\langle P^{ii}(x) \rangle$ of the form $\langle \Psi_{L_{1}}^{12}(x,t) \Psi_{L_{1}}^{12}(x,t) \rangle$ as well as the off-diagonal components of the pairing operator ($\langle P^{12}(x) \rangle$, $\langle P^{21}(x) \rangle$) vanish unless we take into account off diagonal terms in the boundary couplings.

At low temperatures we obtain

$$\langle P(x)^{ii} \rangle \approx e^{i\frac{\pi}{4} x} (\sin \frac{\pi x}{R}) \frac{1}{1 + e^{-2\epsilon_{e} + 3}}$$

In the limit of infinite length this correlator behaves like $\langle P(x)^{ii} \rangle \approx \frac{1}{x^\gamma}$ with $\gamma = (e^{-2\epsilon_{e}} + 3)/4$.

At high temperatures, the order parameter becomes

$$\langle P(x)^{ii} \rangle \approx e^{-\frac{\pi}{4} x} \frac{e^{-2\epsilon_{e} + 3}}{1 + e^{-2\epsilon_{e}} + 3}$$

However, this is not the physical case since the LL constituting the CN couple to the boundary in the same way, introducing Andreev reflections between the two channels. Thus, we need to include perturbative corrections around
where the distance vectors are defined by example, the exact correlators that we have computed would have to be reproduced by any other method claiming good starting point for seeking a more precise exact solution to the type of problem that we have considered. For which are challenging to interpret straightforwardly. We have done this because it is feasible, and represents a very size or temperature. This results in rather complex expressions for correlators and their related physical quantities do not contribute to the expectation value of the off diagonal pairing operator to this order, due to the particular to this order the two boundaries remain uncoupled, for simplicity we only give the correction due to the left boundary

\[ (P_{12}(x,0))^{(1)} = -i \Delta_L \int_0^\beta d\tau \left[ e^{i2k_F R} G^{(4)}_{\downarrow \downarrow}(-x, i\tau) + e^{-i2k_F R} G^{(4)}_{\uparrow \uparrow}(-x, -i\tau) \right] \]

with correlators

\[ G^{(4)}_{\sigma\tau}(x, i\tau) = e^{i\pi(x + v_c e^{2\xi_c} \tau)} \theta(\frac{\xi_c + \frac{\pi}{4} \tilde{m}(\Omega))}{\theta(\frac{\pi}{4} \tilde{m}(\Omega))} \left[ F_c(i\tau v_c - x) \right]^{-\frac{2\xi_{c+1}}{2}} \left[ F_c(i\tau v_c + x) \right]^{-\frac{2\xi_{c-1}}{2}} \]

where the distance vectors are defined by

\[ u_j = \frac{\pi}{4R} \begin{pmatrix} -i\tau[e^{2\xi_c} v_c - v_s] \\ -i\tau[e^{2\xi_c} v_c + v_s] + 2x \\ -i\tau[e^{2\xi_c} v_c + v_s] + 2x \\ -i\tau[e^{2\xi_c} v_c - v_s] \end{pmatrix} \quad u_\uparrow = \frac{\pi}{4R} \begin{pmatrix} i\tau[e^{2\xi_c} v_c + v_s] + 2x \\ i\tau[e^{2\xi_c} v_c - v_s] \\ i\tau[e^{2\xi_c} v_c - v_s] \\ i\tau[e^{2\xi_c} v_c + v_s] + 2x \end{pmatrix}. \]

Thus, there is only one type of non vanishing contributions to this operator (to first order in perturbation theory), namely, what we have called \( G^{(4)}_{\downarrow \downarrow} \). The other terms present in the pairing operator of the form \( (\Psi_{L\uparrow}^i(x, t)\Psi_{L\uparrow}^{i\dagger}(x, t)) \) do not contribute to the expectation value of the off diagonal pairing operator to this order, due to the particular form of the boundary perturbation.

In general, this expectation value will be a complicated function of the distance to the superconductor \( x \). However, in the limit of zero temperature, and for very large systems, we can show that \( (P_{12}(x,0))^{(1)} \approx \frac{1}{x^2} \) with \( \gamma_\perp = (e^{-2\xi_c} + 3e^{2\xi_c})/4 \).

IV. SUMMARY AND DISCUSSION

Up to this point, we have presented most computations without making any approximations with respect to system size or temperature. This results in rather complex expressions for correlators and their related physical quantities which are challenging to interpret straightforwardly. We have done this because it is feasible, and represents a very good starting point for seeking a more precise exact solution to the type of problem that we have considered. For example, the exact correlators that we have computed would have to be reproduced by any other method claiming to be capable of dealing with similar systems (we are thinking here of for example massless limits of form factors for the sine-Gordon model).

In order to make the important results more accessible, and to paint a more transparent picture of the systems we considered, we provide here a short summary of our results. We remind the reader that the two systems that we have dealt with, namely multichannel Luttinger liquids and carbon nanotubes, have different bulk interaction structures. We have considered charge and spin interactions which are channel-diagonal for the Luttinger liquids, whereas the interaction for the nanotubes is of the form of a Coulomb term involving the total charge of the system. This has an influence on the specifics of the bosonization procedure, that was discussed in the bulk of our paper.

We have identified two different fixed points for our theory: one which we call the normal fixed point, and the other the Andreev fixed point. The normal fixed point is associated to the normal boundary conditions \([6]\), and similarly for the Andreev fixed point \([13]\). At each of these fixed points, it was shown that it is possible to do the bosonization exactly in finite size. All correlation functions are then readily computable, and we listed all correlators for all cases in Appendix A.

We have presented explicit computations for physical quantities like the Josephson current, and the pairing order parameter. We summarize here these results, and perform a stability analysis for the fixed points we have identified.

First, we define \( I_\parallel \) and \( I_\perp \) as the contributions to the Josephson current from channel diagonal and channel off-diagonal correlators respectively. Then we define the set of operators whose expectations values were already computed, and whose scaling needs to be assessed. The channel-diagonal and channel non-diagonal pairing operators are given by

\[ P_{\parallel \parallel} = \Psi_{L\uparrow}^i(x)\Psi_{L\uparrow}^{i\dagger}(x), \quad P_{\parallel \perp} = \Psi_{L\uparrow}^i(x)\Psi_{R\uparrow}^{i\dagger}(x), \quad P_{\perp \parallel} = \Psi_{L\uparrow}^i(x)\Psi_{L\downarrow}^{i\dagger}(x), \quad P_{\perp \perp} = \Psi_{L\uparrow}^i(x)\Psi_{R\downarrow}^{i\dagger}(x), \quad (i \neq j) \]

(104)
both of which occur in the physical pairing operator involving the full fermion. For completeness, we also define the back scattering bilinears

\[ V_{||+} = \Psi_{\sigma}^\dag(x)\Psi_{\sigma}(x), \quad V_{||-} = \Psi_{\sigma}^\dag(x)\Psi_{\sigma}(x), \]

\[ V_{\perp+} = \Psi_{\sigma}^\dag(x)\Psi_{\sigma}(x), \quad V_{\perp-} = \Psi_{\sigma}^\dag(x)\Psi_{\sigma}(x), \ (i \neq j). \] (105)

In Tables I and II we summarize the results obtained for the Josephson current and pairing operator in the limits of zero temperature and infinite size. For the diagonal and off-diagonal components of the Josephson current we list the necessary parameter.

**TABLE I: Scaling exponents (Luttinger liquids).**

| Operator | Normal Boundary Conditions | Andreev Boundary Conditions |
|----------|-----------------------------|-----------------------------|
| \( I_{||} \) | \( 2e^{-2\xi_c} - 1 \) | 1 |
| \( I_{\perp} \) | \[ \sum_{\alpha=c,s} \left( \frac{1}{4} e^{-2\xi_c} + e^{-2\xi_c} \right) - 1 \] | \[ \frac{1}{2} \left( e^{2\xi_c} + e^{-2\xi_c} + (i \to j) \right) - 1 \] |
| \( P_{||,+} \) | \[ \frac{1}{2} \left( e^{2\xi_c} + 3e^{-2\xi_c} \right) - 1 \] | \[ \frac{1}{2} \left( e^{2\xi_c} + e^{2\xi_c} \right) \] |
| \( P_{||,-} \) | \[ \frac{1}{2} \left( e^{2\xi_c} + e^{2\xi_c} \right) - 1 \] | \[ \frac{1}{2} \left( e^{2\xi_c} + e^{2\xi_c} \right) \] |
| \( P_{\perp} \) | \[ \frac{1}{2} \left( e^{2\xi_c} + 3e^{-2\xi_c} + (i \to j) \right) - 1 \] | \[ \frac{1}{2} \left( e^{2\xi_c} + e^{2\xi_c} + 3e^{-2\xi_c} + (i \to j) \right) - 1 \] |

**TABLE II: Scaling exponents (nanotube).**

| Operator | Normal Boundary Conditions | Andreev Boundary Conditions |
|----------|-----------------------------|-----------------------------|
| \( I_{||} \) | \( e^{-2\xi_c} \) | 1 |
| \( I_{\perp} \) | \( e^{-2\xi_c} \) | \( e^{2\xi_c} \) |
| \( P_{||,+} \) | \[ \frac{1}{4} \left( e^{2\xi_c} + 3e^{-2\xi_c} \right) \] | \[ \frac{1}{4} \left( e^{2\xi_c} + e^{2\xi_c} \right) \] |
| \( P_{||,-} \) | \[ \frac{1}{4} \left( e^{2\xi_c} + e^{2\xi_c} \right) \] | \[ \frac{1}{4} \left( e^{2\xi_c} + e^{2\xi_c} \right) \] |
| \( P_{\perp,+} \) | \[ \frac{1}{4} \left( e^{2\xi_c} + 3e^{-2\xi_c} \right) \] | \[ \frac{1}{4} \left( e^{2\xi_c} + e^{2\xi_c} + 3e^{-2\xi_c} \right) \] |
| \( P_{\perp,-} \) | \[ \frac{1}{4} \left( e^{2\xi_c} + 3e^{-2\xi_c} \right) \] | \[ \frac{1}{4} \left( e^{2\xi_c} + e^{2\xi_c} + 3e^{-2\xi_c} \right) \] |

As it is clear from the previous section, in the case of Andreev boundary conditions, the results listed for \( I_{||} \) and \( P_{||,-} \) are exact. \( I_{\perp} \) was obtained to second order in perturbation theory, and \( P_{||,+} \) as well as \( P_{\perp,+} \) were obtained to first order in perturbation theory. In all these cases the perturbative Hamiltonian corresponds to Andreev backscattering only. For carbon nanotubes and Andreev boundary conditions, we included in the bulk of the paper only the off-diagonal terms of the boundary perturbation. In this case \(< P_{\perp,+} > = 0\). However the result listed in Table II shows that this operator acquires a finite expectation value if the boundary perturbation includes diagonal terms. Finally, it is simple to see that the second order contribution to \( P_{\perp,-} \) vanishes due the particular form of the boundary perturbation. Therefore, although we include its naive exponent here, it should be remembered that in our setup, this vanishes. The scaling would then be determined by the first descendant of this operator.

These tables are simply a convenient way of viewing the various dependencies on the interaction parameters. The full functional expression for the operators’ expectation values, however, is much more complicated, containing nontrivial functions of the temperature, size of the system, velocities and interaction parameters. The results are too bulky to summarize here more precisely than the table above, and we refer the reader to the previous sections for the exact formulae.

What we have to do next is check the stability of our fixed points by computing the effective dimension of the sets of operators given by eqs. (104) and (105), when bulk interactions are introduced. We provide four tables giving the scaling dimensions of the pairing and backscattering operators around both the normal and Andreev fixed points for the (multichannel) Luttinger liquids and nanotube. These exponents are all simplified data coming from our exact correlators, and they are valid for zero temperature in a large system. In the Luttinger liquid case, the interaction parameters carry a channel index, whereas we have considered here only one nanotube, with one charge interaction parameter.
Three exponents are given, corresponding to the various regimes of the correlators that can be encountered. $\Delta$ is the usual bulk scaling exponent, valid away from the boundaries. $\Delta_\perp$ is the scaling exponent close to one boundary, with argument separation running perpendicular to the boundary. $\Delta_\parallel$ is the scaling exponent for correlators where operators both live close to the boundary, but with a finite time difference (the reader unfamiliar with these various exponents is invited to see e.g. [2] for a clear explanation).

To examine criteria of relevance in the RG sense in our theory, the exponents to look at are in the $\Delta_\parallel$ columns. That is the case in our framework, since our boundary perturbations include only fields sitting on the boundaries, integrated over an infinite time interval; in principle, we should perform a complete RG analysis, including even terms which represent the appearance of superconducting order within the one-dimensional channel, away from the boundary. We don’t to this here, as these contributions turn out to be much smaller. It is readily seen that, for normal boundary conditions, the pairing perturbations are irrelevant ($\Delta_\parallel < 1$) for repulsive interactions for both the multichannel Luttinger liquid and the carbon nanotubes. On the other hand, the diagonal backscattering term is marginal (the cross-channel backscattering is either irrelevant (Luttinger liquids) or marginal (carbon nanotubes)). Putting attractive interactions in the bulk, however, destabilizes the normal fixed point: in that case, the operator that drives the system away is the channel-diagonal pairing operator. One therefore expects, on general grounds, in the case of attractive bulk interactions (and supposing that our parameters are initially tuned to put our system near the normal fixed point), to witness a flow of our theory towards Andreev boundary conditions, corresponding to an optimization of channel-diagonal Andreev reflection at the boundary.

At the Andreev fixed point, the operator scaling dimensions are different, and are listed in the last two tables. We stress this fact again: the particulars of the boundary conditions influence in a nontrivial way the scaling of the system near the boundaries, and near the Andreev fixed point, the dimensions of the operators differ from those to be found near the normal fixed point. The situation is now reversed as compared to the scaling around the normal fixed point: most operators are now irrelevant for attractive interactions, with a few of them being marginal. If we were to turn the interactions in the bulk to the repulsive regime, we would see the system flowing back to the normal fixed point. We therefore recover a similar situation to the one in [20], in which the normal fixed point was the fixed point associated to repulsive bulk interactions, and the Andreev fixed point was associated to attractive bulk interactions. Our analysis here is rather short and simplistic: we could ask e.g. what other possibilities there might be for flows in the system, in particular by playing around with the interactions in the spin channels. A treatment similar to that in [14] would then be obtained. We leave this for future considerations.

The perturbative setup that we have presented here lends itself ideally to further extensions and refinements. First of all, the formulas we have presented for the noninteracting case would be appropriate to investigate the effects of voltages simultaneously present with various backscattering potentials, or could easily be modified for the addition of ferromagnetic couplings. Other possibilities more directly related to what we have done in the interacting case would be, for example, to take into account perturbatively the effects of quasiparticle penetration within the original superconductors. We have neglected those according to the logic of [20], and this should be correct at extremely low temperatures. However, finite temperatures make quasiparticle penetration possible if the energy scale starts approaching the (original, bulk) superconducting gap. This could again be done within our framework, by replacing the boundary action with some other action containing retarded effects. It is also important to realize that the class of problems that we have dealt with here forms a good basis from which to study other geometries. In view of the fact that experimental work is addressing these issues on a frequent basis, we think that there exist many opportunities for simple adaptations of our method. One could consider things like nanotube crossings of various kinds, tunnel junctions, etc. We leave such issues for later work.

| Operator | $\Delta$ | $\Delta_\perp$ | $\Delta_\parallel$ |
|----------|----------|----------------|------------------|
| $P_{\parallel,+}$ | $\cosh(2\xi_i^c)$ | $\frac{1}{4}[e^{2\xi_i^c} + 3e^{-2\xi_i^c}]$ | $e^{-2\xi_i^c}$ |
| $P_{\parallel,-}$ | $\frac{1}{2}[e^{-2\xi_i^c} + e^{2\xi_i^c}]$ | $\frac{1}{4}[3e^{-2\xi_i^c} + e^{2\xi_i^c}]$ | $e^{2\xi_i^c}$ |
| $P_{\perp,+}$ | $\frac{1}{4}[\cosh(2\xi_i^c) + \cosh(2\xi_j^c) + (i \to j)]$ | $\frac{1}{4}[e^{2\xi_i^c} + 3e^{-2\xi_i^c} + e^{2\xi_j^c} + 3e^{-2\xi_j^c} + (i \to j)]$ | $\frac{1}{4}[e^{-2\xi_i^c} + e^{-2\xi_j^c} + (i \to j)]$ |
| $V_{\parallel,+}$ | $1$ | $1$ | $1$ |
| $V_{\parallel,-}$ | $\frac{1}{4}[e^{2\xi_i^c} + e^{2\xi_j^c}]$ | $\frac{1}{4}[e^{2\xi_i^c} + e^{2\xi_j^c} + 2]$ | $1$ |
| $V_{\perp,+}$ | $\frac{1}{4}[\cosh(2\xi_i^c) + \cosh(2\xi_j^c) + (i \to j)]$ | $\frac{1}{4}[e^{2\xi_i^c} + 3e^{-2\xi_i^c} + e^{2\xi_j^c} + 3e^{-2\xi_j^c} + (i \to j)]$ | $\frac{1}{4}[e^{-2\xi_i^c} + e^{-2\xi_j^c} + (i \to j)]$ |
we compute multipoint correlation functions, in finite size and at finite temperature, for all the cases of interest in
the literature, we wish here to present formulas for any multipoint correlator. Systems are already known in finite size
and at finite temperatures, and multipoint correlators in the bulk are also in

| Operator | ∆ | ∆⊥ | ∆∥ |
|----------|---|----|----|
| $P_{||,+}$ | $\cosh(2\xi^i_c)$ | $1/4[3e^{2\xi^i_c} + e^{-2\xi^i_c}]$ | $e^{2\xi^i_c}$ |
| $P_{||,-}$ | $1/2[e^{-2\xi^i_c} + e^{2\xi^i_c}]$ | $1/4[e^{-2\xi^i_c} + e^{2\xi^i_c} + 2]$ | 1 |
| $P_{||,+}$ | $1/4[\cosh(2\xi^i_c) + \cosh(2\xi^i_j) + (i \rightarrow j)]$ | $1/16[3e^{2\xi^i_j} + e^{-2\xi^i_j} + e^{2\xi^i_j} + e^{-2\xi^i_j} + (i \rightarrow j)]$ | $1/4[e^{2\xi^i_j} + e^{-2\xi^i_j} + (i \rightarrow j)]$ |
| $V_{||,+}$ | 1 | 1 | 1 |
| $V_{||,-}$ | $1/2[e^{2\xi^i_c} + e^{2\xi^i_c}]$ | $1/4[e^{2\xi^i_c} + e^{2\xi^i_c}]$ | $e^{2\xi^i_c}$ |
| $V_{||,+}$ | $1/4[\cosh(2\xi^i_c) + \cosh(2\xi^i_j) + (i \rightarrow j)]$ | $1/16[3e^{2\xi^i_j} + e^{-2\xi^i_j} + e^{2\xi^i_j} + e^{-2\xi^i_j} + (i \rightarrow j)]$ | $1/4[e^{2\xi^i_j} + e^{-2\xi^i_j} + (i \rightarrow j)]$ |

| Operator | ∆ | ∆⊥ | ∆∥ |
|----------|---|----|----|
| $P_{||,+}$ | $1/2[\cosh(2\xi^i_c) + 1]$ | $1/4[e^{2\xi^i_c} + 3e^{-2\xi^i_c} + 4]$ | $1/4[e^{-2\xi^i_c} + 1]$ |
| $P_{||,-}$ | $1/2[e^{-2\xi^i_c} + 3]$ | $1/4[3e^{-2\xi^i_c} + 5]$ | $1/4[e^{-2\xi^i_c} + 1]$ |
| $P_{||,+}$ | $1/2[\cosh(2\xi^i_c) + 1]$ | $1/4[e^{2\xi^i_c} + 3e^{-2\xi^i_c} + 4]$ | $1/4[e^{-2\xi^i_c} + 1]$ |
| $P_{||,-}$ | $1/2[e^{-2\xi^i_c} + 3]$ | $1/4[3e^{-2\xi^i_c} + 5]$ | $1/4[e^{-2\xi^i_c} + 1]$ |
| $V_{||,+}$ | 1 | 1 | 1 |
| $V_{||,-}$ | $1/2[e^{2\xi^i_c} + 3]$ | $1/4[e^{2\xi^i_c} + 7]$ | 1 |

| Operator | ∆ | ∆⊥ | ∆∥ |
|----------|---|----|----|
| $P_{||,+}$ | $1/2[\cosh(2\xi^i_c) + 1]$ | $1/4[3e^{2\xi^i_c} + e^{-2\xi^i_c} + 4]$ | $1/4[e^{2\xi^i_c} + 1]$ |
| $P_{||,-}$ | $1/2[e^{-2\xi^i_c} + 3]$ | $1/4[e^{-2\xi^i_c} + 7]$ | 1 |
| $P_{||,+}$ | $1/2[\cosh(2\xi^i_c) + 1]$ | $1/4[3e^{2\xi^i_c} + e^{-2\xi^i_c} + 4]$ | $1/4[e^{2\xi^i_c} + 1]$ |
| $P_{||,-}$ | $1/2[e^{-2\xi^i_c} + 3]$ | $1/4[e^{-2\xi^i_c} + 7]$ | 1 |
| $V_{||,+}$ | 1 | 1 | 1 |
| $V_{||,-}$ | $1/2[e^{2\xi^i_c} + 3]$ | $1/4[3e^{2\xi^i_c} + 5]$ | $1/4[e^{2\xi^i_c} + 1]$ |

**APPENDIX A: CORRELATION FUNCTIONS**

The usefulness of bosonization really manifests itself in the computation of correlation functions. In this appendix,
we compute multipoint correlation functions, in finite size and at finite temperature, for all the cases of interest in
this work.

The first requirement when dealing with a finite size is to specify the boundary conditions used in the computation.
We consider here normal and Andreev boundary conditions introduced in the bulk of the paper. We present computations both for the Luttinger liquid, and for the carbon nanotubes. Although two-point correlators for such
systems are already known in finite size and at finite temperatures, and multipoint correlators in the bulk are also in
the literature, we wish here to present formulas for any multipoint correlator.
1. Luttinger liquid, normal boundary conditions

Let us consider the problem of computing correlation functions in finite size, at finite temperature, for normal boundary conditions. The general correlator that we want to compute can be taken to be

$$G^{(2n)}_{\{\sigma, \tau\}}(\{x, t; y, u\}) \equiv \prod_{i=1}^{n} \Psi^\dagger_{L_{\sigma_i}}(x_i, t_i) \prod_{j=1}^{n} \Psi_{L_{\tau_j}}(y_j, u_j) \langle N \rangle$$

(A1)

since all others can be obtained by simple manipulations (here, we mean in particular correlators with a different ordering of the $\Psi, \Psi^\dagger$ operators, which are easy to recover from the formulas we present; see later for a better explanation). Correlators involving right movers can be recovered by rewriting the right movers as left movers using the appropriate (here, normal) boundary conditions. Here, the angular brackets denote an average using the Hamiltonian \([41]\), and the subindex $N$ denotes normal boundary conditions.

For simplicity, we have suppressed the channel index. We remark that when dealing with multichannel LL, all the correlators needed to compute various physical quantities can be reduced to the above one for a given channel, or products of them evaluated in different channels.

The first step in the computation is to substitute the bosonization formulas \([42]\) for the fermions, and to make use of the normal boundary conditions, mode expansions and Hamiltonian (equations \([36]\) to \([41]\)) to reexpress the correlator as a product of zero-mode and dynamical contributions, averaged over a free theory. We write this as

$$G^{(2n)}_{\{\sigma, \tau\}}(\{x, t; y, u\}) = \delta_{\sum \sigma, \tau} J_{\{\sigma, \tau\}}(\{x, t; y, u\}) K_{\{\sigma, \tau\}}(\{x, t; y, u\}).$$

(A2)

Let us first consider the zero-mode part. Using the Campbell-Baker-Hausdorff formula to combine exponentials, we can reexpress this contribution as

$$J_{\{\sigma, \tau\}}(\{x, t; y, u\}) = P_c(\{x, t; y, u\}) P_{s\{\sigma, \tau\}}(\{x, t; y, u\}) Q_{\{\sigma, \tau\}}(\{x, t; y, u\})$$

(A3)

with phase functions

$$\ln P_c = -\frac{i\pi}{4R} \left( n \sum_{i=1}^{n} [v_c e^{-2\xi\pi(t_i - u_i)} + x_i - y_i] - \sum_{i < j = 1}^{n} [v_c e^{-2\xi\pi(t_i - t_j + u_i - u_j)} + x_i - x_j + y_i - y_j] \right),$$

$$\ln P_{s\{\sigma, \tau\}}(\{x, t; y, u\}) = -\frac{i\pi}{4R} \left( \sum_{i=1}^{n} [\sigma_i (v_s e^{-2\xi\pi t_i + x_i}) - \tau_i (v_s e^{-2\xi\pi u_i + y_i})] - \sum_{i < j = 1}^{n} [\sigma_i \sigma_j (v_s e^{-2\xi\pi(t_i - t_j) + x_i - x_j}) + \tau_i \tau_j (v_s e^{-2\xi\pi(u_i - u_j) + y_i - y_j})] \right)$$

(A4)

and zero-mode expectation value

$$Q_{\{\sigma, \tau\}}(\{x, t; y, u\}) = \langle e^{i\sqrt{2\pi} \sum_{i=1}^{n} [v_c e^{-\xi\pi(t_i - u_i)} + e^{\xi\pi}(x_i - y_i)]} e^{i\sqrt{2\pi} \sum_{i=1}^{n} [v_s e^{-\xi\pi(\sigma_i t_i - \tau_i u_i)} + e^{\xi\pi}(\sigma_i x_i - \tau_i y_i)]} \rangle.$$

(A5)

This average is readily computed in terms of $\theta$-functions (see next appendix for definitions and useful formulas). The final result is given below in equation \((A16)\). Note the crucial fact that the spin and charge sectors remain entangled in this contribution.

The dynamical part, on the other hand, readily factorizes in charge and spin sectors:

$$K_{\{\sigma, \tau\}}(\{x, t; y, u\}) = K_c(\{x, t; y, u\}) K_{s\{\sigma, \tau\}}(\{x, t; y, u\}).$$

(A6)

For clarity, we will here present details of the computation of the spin part only (the charge part doesn’t depend on the $\sigma, \tau$ subindices and is simple to recover from the formulas below). As an expectation value, this is

$$K_{s\{\sigma, \tau\}}(\{x, t; y, u\}) = \prod_{i=1}^{n} e^{i\sqrt{2\pi} \sigma_i [\cosh \xi_c (c_x(t_i, t_j) - \sinh \xi_c (c_x(t_i, t_j))] \prod_{i=1}^{n} e^{-i\sqrt{2\pi} \tau_i [\cosh \xi_s (c_y(y_i, u_j) - \sinh \xi_s (c_y(y_i, u_j))]}.$$  

(A7)

Since the average is over a free bosonic field, we can make use of the identity

$$\langle \prod_i e^{A_i} \rangle = e^{\langle \sum_i A_i A_i + \frac{1}{2} \sum_i A_i^2 \rangle}.$$  

(A8)
to rewrite the above expectation value as

$$\ln K_{s(\sigma,\tau)}(\{x, t; y, u\}) =$$

$$= 2\pi \cosh^2 \xi_\sigma \sum_{i<j} \sigma_i \tau_j B_s(x_i, t_i; y_j, u_j) - \sum_i \sigma_i \sigma_j B_s(x_i, t_i; x_j, t_j) - \sum_{i<j} \tau_i \tau_j B_s(y_i, u_i; y_j, u_j) +$$

$$+ 2\pi \sinh^2 \xi_\sigma \left[ \sum_i \sigma_i \tau_j B_s(-x_i, t_i; -y_j, u_j) - \sum_i \sigma_i B_s(-x_i, t_i; -x_j, t_j) - \sum_i \tau_i \tau_j B_s(-y_i, u_i; -y_j, u_j) \right] -$$

$$- \pi \sinh 2\xi_\sigma \left[ \sum_i \sigma_i \tau_j B_s(x_i, t_i; -y_j, u_j) + B_s(-x_i, t_i; y_j, u_j) \right] -$$

$$- \sum_i \sigma_i \sigma_j (B_s(x_i, t_i; -x_j, t_j) + B_s(-x_i, t_i; x_j, t_j)) - \sum_i \tau_i \tau_j (B_s(y_i, u_i; -y_j, u_j) + B_s(-y_i, u_i; y_j, u_j)) -$$

$$- \frac{1}{2} \sum_i (B_s(x_i, t_i; -x_i, t_i) + B_s(-x_i, t_i; x_i, t_i) + B_s(y_i, u_i; -y_i, u_i) + B_s(-y_i, u_i; y_i, u_i)) \tag{A9}$$

where

$$B_s(x, t; y, u) = \langle \zeta_{aL}(x, t) \zeta_{aL}(y, u) \rangle - \frac{1}{2} \langle \zeta_{aL}(x, t) \zeta_{aL}(x, t) \rangle - \frac{1}{2} \langle \zeta_{aL}(y, u) \zeta_{aL}(y, u) \rangle. \tag{A10}$$

The computation of this free correlator can also be done in terms of \(\theta\)-functions, with the result

$$B_s(x, t; y, u) = \frac{i}{4} \frac{v_a(t - u) + x - y}{2R} - \frac{1}{4\pi} \ln F_a(v_a(t - u) + x - y) \tag{A11}$$

where

$$F_a(z) = \frac{\vartheta_1(z/2|\omega_a)}{\vartheta_1(-i\pi/2R|\omega_a)} \tag{A12}$$

in which \(\alpha\) is an implicit cutoff, i.e. \(z \to z - \alpha\). The periods of the theta functions are

$$\omega_a = \frac{i v_a \beta}{2R}. \tag{A13}$$

This whole collection of formulas can now be assembled to yield our original correlator. In doing this, phases simplify somewhat; in the end, we obtain the general formula

$$G_{s(\sigma,\tau)}^{(2n)}(\{x, t; y, u\}) = \mathcal{N}_a^{-1} \delta_{\sum_\sigma, \sum_\tau, 2} Q_{\{\sigma,\tau\}}(\{x, t; y, u\}) \times$$

$$\times \prod_{a=c,s} \left[ P_{a-} \right]^{-\cosh^2 \xi_a} \left[ P_{a-} \right]^{-\sinh^2 \xi_a} \left[ P_{a+} \right]^{-\sinh \xi_a} \prod_{i=1}^n \left| F_a(2x_i) F_a(2y_i) \right|^{-\sinh 2\xi_a} \tag{A14}$$

where for \(\eta_i = \pm\) we have defined the functions

$$P^\pm_{c_1, c_2} = \frac{\prod_{i,j=1}^n F_c(v_c(t_i - t_j) + \eta_1 x_i + \eta_2 y_j)}{\prod_{i<j}^n F_c(v_c(t_i - t_j) + \eta_1 x_i + \eta_2 y_j)} \tag{A15}$$

and where the zero mode part is

$$Q_{\{\sigma,\tau\}}(\{x, t; y, u\}) = \frac{\vartheta_2(d^c_\tau + \tau_c k_F R |\tau_c) \vartheta_3(d^c_\tau |\tau_c) + \vartheta_3(d^c_\tau + \tau_c k_F R |\tau_c) \vartheta_2(d^c_\tau |\tau_c)}{\vartheta_2(\tau_c k_F R |\tau_c) \vartheta_3(0 |\tau_c) + \vartheta_3(\tau_c k_F R |\tau_c) \vartheta_2(0 |\tau_c)} e^{2d^c_\tau k_F R / \pi} \tag{A16}$$

in which we have defined \(\tau_a = i \frac{\omega_a v_a - 2\pi \alpha}{R}\) and

$$d^c_\tau = - \frac{\pi}{2R} \sum_{i=1}^n \left[ e^{-2\xi_c} v_c(t_i - u_i) + x_i - y_i \right],$$

$$d^s_\tau = - \frac{\pi}{2R} \sum_{i=1}^n \left[ e^{-2\xi_c} v_c(\sigma_i t_i - \tau_i u_i) + \sigma_i x_i - \tau_i y_i \right]. \tag{A17}$$
One can convince oneself that the correlators obey the correct (anti-)periodicity by shifting imaginary time like in the Matsubara formalism, and that moreover they obey the spatial periodicity by shifting distance arguments by twice the system size. This will of course be true of all correlators we present, in all differing boundary conditions cases.

Other correlators, for example of the type where the order of the \( \Psi, \Psi^\dagger \) differs, can be easily obtained from the above formulas: the only differences with the formulas that we have presented are some simple signs, in front of the coordinates and exponents. We have left these out here to keep the formulas more compact, and since these modifications are trivial to work out.

2. Luttinger liquid, Andreev boundary conditions

The general correlator is here again defined as

\[
\tilde{G}^{(2n)}_{\{\sigma,\tau\}}(\{x, t; y, u\}) = \prod_{i=1}^{n} \Psi_{L\sigma_i}^\dagger(x_i, t_i) \prod_{j=1}^{n} \Psi_{L\tau_j}(y_j, u_j) \]  

(A18)

where we now deal with Andreev boundary conditions, specified by the subindex \( A \). The computation is rather similar to the one above; in fact, one can convince oneself rather easily that the general formula becomes a simple variation of the one for normal boundary conditions, i.e.

\[
\times \left[ \prod_{\alpha=c,s} [P_{\alpha++} - \frac{\sinh2\xi_{\alpha}}{2}] [P_{\alpha--} + \frac{\sinh2\xi_{\alpha}}{2}] \right] [P_{c++}P_{c--} - \frac{\sinh2\xi_c}{4}] [P_{s++}P_{s--} - \frac{\sinh2\xi_s}{4}] \prod_{i=1}^{n} \frac{|F_c(2x_i)F_c(2y_i)|}{|F_s(2x_i)F_s(2y_i)|} \right]^{-\frac{\sinh2\xi_c}{8}}
\]

(A19)

where the \( P \) functions are as for normal boundary conditions, and where the zero mode part is

\[
\tilde{Q}_{\{\sigma,\tau\}}(\{x, t; y, u\}) = \frac{\theta_2(d_c^t + \hat{\tau}_c \chi/2|\tau_c)\theta_2(d_3^t|\tau_3) + \theta_3(d_c^t + \hat{\tau}_c \chi/2|\tau_c)\theta_3(d_3^t|\tau_3)}{\theta_2(\tau_c \chi/2|\tau_c)\theta_2(0|\tau_3) + \theta_3(\tau_c \chi/2|\tau_c)\theta_3(0|\tau_3)} e^{i\hat{d}_c^t \chi/\pi}
\]

(A20)

in which we have defined \( \hat{\tau}_c = i\frac{2\nu_c e^{2\xi_c}}{\pi} \) and

\[
\hat{d}_c^t = -\frac{\pi}{2R} \sum_{i=1}^{n} e^{2\xi_c v_c(t_i - u_i) + x_i - y_i}.
\]

(A21)

The other parameters are defined as in the previous section.

3. Carbon nanotube, normal boundary conditions

The general correlator for carbon nanotubes in the presence of normal boundary conditions can be written in terms of

\[
G^{(s,r)}_{\{\sigma,\tau\}}(\{x, t; y, u\}) = \prod_{i=1}^{n} \Psi_{L\sigma_i}^\dagger(x_i, t_i) \prod_{j=1}^{n} \Psi_{L\tau_j}(y_j, u_j) \]  

(A22)

where \( s_i, r_j = 1, 2 \) refer to the nanotube channel index. The computation of this general correlator is very similar to the computation presented above for a Luttinger liquid with normal boundary conditions. The final formula is

\[
\times \left[ P_{c++}^+ - \frac{\sinh2\xi_c}{2} \right] [P_{c--}^- + \frac{\sinh2\xi_c}{2}] [P_{c++}^- + \frac{\sinh2\xi_c}{2}] \left[ P_{c--}^+ P_{s++}^- \right]  \left[ P_{c++}^- P_{s+-}^- \right] \prod_{i=1}^{n} |F_c(2x_i)F_c(2y_i)|^{-\frac{\sinh2\xi_c}{8}}
\]

(A23)

In the above, \( \delta_{Klein} \) is a shorthand for the Kronecker deltas coming from the Klein factors, meaning that the set of (pairs of) indices associated to the \( \Psi^\dagger \) operators has to be a permutation of the set associated to the \( \Psi \) operators.
For later convenience, we also define $s(s_i)$ as $+1$ for the 1 channel, $-1$ for the 2 channel (and similarly for $s(r_j)$). We have defined the dynamical contributions

$$
P_{σ1;σ2} = \frac{\prod_{i,j=1}^{n}[F_{σ}(v_{σ}^+(t_i-u_j)+η_1x_i+η_2y_j)]\gamma_σ^+(i)\gamma_σ^+(j)}{\prod_{i<j=1}^{n}[F_{σ}(v_{σ}^+(t_i-t_j)+η_1x_i+η_2y_j)]\gamma_σ^+(i)\gamma_σ^+(j)},$$

(A24)

with indices

$$e^+_c(i) = 1, \quad e^-_c(i) = s(s_i), \quad e^+_s(i) = σ_i, \quad e^-_s(i) = s(s_i)σ_i,$$

$$γ^+_c(j) = 1, \quad γ^-_c(j) = s(r_j), \quad γ^+_s(j) = τ_j, \quad γ^-_s(j) = s(r_j)τ_j,$$

(A25)

and $v_a^+ = v_a^- = v_a$, with $a = c, s$. The zero mode part becomes here

$$Q^{s,r}(x, t; y, u) = \frac{θ(u + \frac{2}{π}(k_F-R(t))|Ω)}{θ(\frac{2}{π}(k_F-R(t))|Ω)} e^{-i d_c^+(1-2\text{e}^{-2ξ_c R})},$$

(A26)

where $Ω$ is a four-dimensional $θ$-function (such multivariable functions are defined in the next appendix). The four-dimensional period matrix is ($σ^2$ is the Pauli matrix)

$$Ω = i\frac{β}{4R} \left[ 1 \otimes (1 + 1) \right] \left( 1 + e^{-2ξ_c} - σ^1 \otimes (1 + σ^1) \right) \left( 1 - e^{-2ξ_c} + 1 \otimes (1 - σ^1) \right) v_s$$

(A27)

and $τ_c = i\frac{β_v e^{-2ξ_c}}{R}$. The vector $u$ is

$$u = \frac{1}{4} \left( \begin{array}{c} d_c^+ + d_c^- + d_s^+ + d_s^- \\ d_c^+ + d_c^- - d_s^+ + d_s^- \\ d_c^+ - d_c^- + d_s^+ - d_s^- \\ d_c^+ - d_c^- - d_s^+ + d_s^- \end{array} \right),$$

(A28)

in which the distances are given by

$$d_a^+ = -\frac{π}{2R} \sum_{i=1}^{n} \left[ e^{-2ξ_a^s} v_a (c_a^+ t_i - γ_a^+ u_i) + c_a^+ x_i - γ_a^+ y_i \right].$$

(A29)

Also, we put $ξ^- = ξ^+_s = 0$, $ξ^- = ξ_c$ and $k_F = k_F^2 \equiv k_F$.

4. Carbon nanotube, Andreev boundary conditions

For Andreev boundary conditions, we define

$$\tilde{G}_{σ,τ}^{s,r}(x, t; y, u) = \left( \prod_{i=1}^{n} Ψ_{σσ_i}^{r+}(x_i, t_i) \right) \left( \prod_{j=1}^{n} Ψ_{ττ_j}^{r+}(y_j, u_j) \right) A,$$

(A30)

where $s_i, r_j = 1, 2$ refer to the nanotube channel index. The computation is very similar to those above, and yields

$$\tilde{G}_{σ,τ}^{s,r}(x, t; y, u) = N^{-1} K_{\text{Klein}} Q_{σ,τ}^{s,r}(x, t; y, u) \times$$

$$\left[ P_{s+, c+}^{+} \right]^{-\frac{\text{cosh}^2}{4} t_F} \left[ P_{c+, c}^{+} \right]^{-\frac{\text{sinh}^2}{4} t_F} \left[ P_{c+, s+}^{+} \right]^{-\frac{\text{cosh}^2}{4} t_F} \left[ P_{s+, s+}^{+} \right]^{-\frac{\text{sinh}^2}{4} t_F} \left[ P_{c+, c-}^{+} P_{c+, s-}^{+} P_{s+, s-}^{+} \right]^{-1/4} \prod_{i=1}^{n} |F_c(2x_i) F_c(2y_i)|^{\frac{\text{sinh}^2}{4} t_F},$$

(A31)

The zero mode part becomes here

$$\tilde{Q}_{σ,τ}^{s,r}(x, t; y, u) = \frac{θ(u + \frac{2}{π} x|Ω)}{θ(\frac{2}{π} x|Ω)} e^{i d_c^+ x/π}$$

(A32)

with period matrix

$$Ω = i\frac{β}{4R} \left[ 1 \otimes (1 + 1) \right] \left( 1 + e^{-2ξ_c} - σ^1 \otimes (1 + σ^1) \right) \left( 1 - e^{-2ξ_c} + 1 \otimes (1 - σ^1) \right) v_s$$

(A33)
and \( \bar{\tau}_c = i \frac{\beta_c e^{2 \pi c}}{\alpha} \). The vector \( \bar{u} \) is

\[
\bar{u} = \frac{1}{4} \begin{pmatrix}
\hat{d}^+_c + d^-_c + d^+_s + d^-_s \\
\hat{d}^+_c + d^-_c - d^+_s - d^-_s \\
\hat{d}^+_c - d^-_c + d^+_s - d^-_s \\
\hat{d}^+_c - d^-_c - d^+_s + d^-_s
\end{pmatrix}, \tag{A34}
\]

The newly defined distances (only \( \hat{d}^+_c \) is actually needed, as in this particular setup the others coincide with the previous ones) are given by

\[
\hat{d}^+_a = -\frac{\pi}{2 R} \sum_{i=1}^{n} \left[ e^{2 i \xi^2} v_a (\epsilon^+_a t_i - (\gamma^+_a u_i) + (\epsilon^+_a x_i - (\gamma^+_a y_i)) \right]. \tag{A35}
\]

We have also put \( \chi_1 = \chi_2 \equiv \chi \).

APPENDIX B: \( \theta \)-FUNCTIONS, AND LIMITS OF CORRELATION FUNCTIONS

1. Traditional \( \theta \)-functions

Theta functions are defined with the following series:

\[
\theta_1(u|\tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n e^{i \pi \tau (n + 1/2)^2 + i (2n + 1) u}, \quad \theta_2(u|\tau) = \sum_{n \in \mathbb{Z}} e^{i \pi \tau (n + 1/2)^2 + i (2n + 1) u},
\]

\[
\theta_3(u|\tau) = \sum_{n \in \mathbb{Z}} e^{i \pi n^2 + i 2n u}, \quad \theta_4(u|\tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{i \pi n^2 + i 2n u}. \tag{B1}
\]

The most important property of these functions is their quasi-double-periodicity in the complex plane. We refer the reader to the standard literature for a list of all the properties obeyed by these remarkable functions [33].

In taking different limits like that of finite size and/or zero temperature, we need to consider various limits of theta functions with respect to their arguments and periods. We here collect a few useful formulas:

\[
\theta_1(u|\alpha) \to \alpha \to \infty 2 e^{-\pi \alpha/4} \sin(u) + ...
\]

\[
\theta_1(u|\alpha) \to \alpha \to 0 2 \alpha^{-1/2} e^{-\pi \alpha/4} \sinh \left( \frac{u}{\alpha} \right) + ...
\]

\[
\theta_2(u|\alpha) \to \alpha \to \infty 2 e^{-\pi \alpha/4} \cos(u) + ...
\]

\[
\theta_2(u|\alpha) \to \alpha \to 0 \alpha^{-1/2} e^{-\pi \alpha/4} \left[ 1 - 2 e^{-\pi \alpha} \cos \left( \frac{2u}{\alpha} \right) \right] + ...
\]

\[
\theta_3(u|\alpha) \to \alpha \to \infty 1 + 2 e^{-\pi \alpha} \cos(2u) + ...
\]

\[
\theta_3(u|\alpha) \to \alpha \to 0 \alpha^{-1/2} e^{-\pi \alpha/4} \left[ 1 + 2 e^{-\pi \alpha} \cos \left( \frac{2u}{\alpha} \right) \right] + ...
\]

\[
\theta_4(u|\alpha) \to \alpha \to \infty 1 - 2 e^{-\pi \alpha} \cos(2u) + ...
\]

\[
\theta_4(u|\alpha) \to \alpha \to 0 \alpha^{-1/2} e^{-\pi \alpha/4} \left[ 2 e^{-\pi/4 \alpha} \cos \left( \frac{3u}{\alpha} \right) \right] + ...
\]

where ... denote subdominant terms.

2. Multivariable theta functions

For the nanotube, the above \( \theta \)-functions are not sufficient. It is rather more convenient to use multivariable functions [33], defined as

\[
\theta(z|\Omega) = \sum_{n \in \mathbb{Z}} e^{i \pi n' \Omega n + 2i n' z}. \tag{B3}
\]
where $g$ is an integer (the “dimensionality” of the function), $\Omega$ is a $g \times g$ matrix with positive imaginary parts, and the sum is taken over all $g$-dimensional real vectors with integer coefficients. As for the usual $\theta$-functions, the multivariable version obeys two quasiperiodicity properties in the complex plane. These are, for any given $m \in \mathbb{Z}^g$,

$$\theta(z + \pi m|\Omega) = \theta(z|\Omega),$$

$$\theta(z + \pi \Omega m|\Omega) = e^{-i\pi m^T \Omega m - 2im^T z} \theta(z|\Omega).$$  \hspace{1cm} (B4)

The limits of low or high temperature are obtained by taking $\det \Omega \to \infty$ or $\det \Omega \to 0$ respectively. In the first case

$$\theta(z|\Omega) \to 1 + \sum_{i=1}^{g} e^{i\pi \Omega_{ii}} 2 \cos(2z_i)$$  \hspace{1cm} (B5)

For the high temperature limit we need to use a generalization of the Poisson summation formula to $\mathbb{Z}^g$ given by

$$\theta(z|\Omega) = \frac{1}{\sqrt{\det(-i\Omega)}} \sum_{n \in \mathbb{Z}^g} e^{-i\frac{1}{\Omega} (n + z\Omega)^{-1} (n + z)}.$$  \hspace{1cm} (B6)

The limit is then easily taken.

**APPENDIX C: NOTES ON THE REAL-TIME PERTURBATIVE FORMALISM IN FINITE-SIZE SYSTEMS**

In order to compute thermodynamic physical quantities for a given system at finite temperature, one usually uses the Matsubara formalism using imaginary time. If one desires to obtain finite-temperature time-dependent correlation functions, one then has to perform an analytical continuation back to real times, an operation which, done correctly, is often inconvenient.

We wish here to discuss some details of the derivation of the Matsubara formalism from first principles, and show that in our geometry of interest, i.e. a one-dimensional finite-size system, there are subtleties one has to bear in mind.

We adopt the following logic, using a real-time formalism to start with. Let us suppose that at $t = -\infty$, the Hamiltonian of the system is described by a known operatorial expression $H_0$. We then adiabatically turn on an additional piece $H_1$ with a modulation function $f(t)$. That is, $H(t) = H_0 + f(t)H_1$. The fundamental object that we need to consider is the density matrix $\rho(t)$, which obeys the time-evolution equation

$$i \frac{d}{dt} \rho(t) = [H(t), \rho(t)].$$  \hspace{1cm} (C1)

Defining $\rho(t) = \rho_0 + \rho_1(t)$ where $\rho_0 = e^{-\beta H_0}/Z_0$, we can obtain a self-consistency equation for $\rho_1(t)$:

$$\rho_1(t) = -i \int_{-\infty}^{t} dt_1 f(t_1) e^{-i H_0 (t - t_1)} [H_1, \rho_0 + \rho_1(t_1)] e^{i H_0 (t - t_1)}$$  \hspace{1cm} (C2)

which we can solve as a series $\rho_1(t) = \sum_{n=1}^{\infty} \rho_1^{(n)}(t)/Z_0$, each successive term containing one additional time integral and commutator with the unperturbed density operator.

Let us consider the first correction. For simplicity, we consider a modulation function with periodicity $f(t - i\beta) = f(t)$, although this is not a fundamental requirement. Using the notation $H_1(t) = e^{i H_0 t} H_1 e^{-i H_0 t}$, we obtain after changing integration variables and using the periodicity of $f(t)$

$$\rho_1^{(1)}(t) = -ie^{-\beta H_0} \left[ \int_{-\infty}^{-i\beta} dt_1 - \int_{-\infty}^{0} dt_1 \right] f(t_1 + t)H_1(t_1).$$  \hspace{1cm} (C3)

We can then combine the two integrals into one closed contour (using $f(-\infty) = 0$ by construction) and a leftover Matsubara piece:

$$\rho_1^{(1)}(t) = 2\pi e^{-\beta H_0} \sum_{s \in C(0)} \text{Res}(f(t_1 + t)H_1(t_1) - ie^{-\beta H_0} \int_{0}^{-i\beta} dt_1 f(t_1 + t)H_1(t_1)).$$  \hspace{1cm} (C4)
Let us suppose that we are in an infinite (and therefore aperiodic) system. If we choose to turn on our interaction $H_1$ at a very remote point in the past as compared to any object we want to calculate (say $t = 0$), then $f(t)$ will have poles only extremely far away, and the first correction will yield expectation values with extremely large time difference, which one can safely take to vanish. Moreover, near $t = 0$, we suppose that $f(t)$ has reached its asymptotic behaviour $f(t) \approx 1$. The second term then becomes

$$\rho^{(1)}_{1,\text{Mats}}(t) = -e^{-\beta H_0} \int_0^\beta d\tau H_1(-i\tau)$$

which is just the Matsubara contribution.

This logic, however, fails for a finite-size system described by an initial $H_0$ containing no dissipation. To see this, one simply has to realize that in such a finite-size system, correlation functions are by construction periodic in time, with a return time of the length of the system divided by the appropriate velocity. For example, one can take as $f(t)$ a step function triggering the perturbation $H_1$ at time $t_0$. In that case, the computation above gives

$$\rho^{(1)}_1(t) = e^{-\beta H_0} \int_0^\beta d\tau [H_1(t_0 - i\tau) - H_1(-i\tau)]$$

and no matter how far we put $t_0$ in the past, the first term never vanishes, since all correlations in the system are periodic. In all that we have done, we neglect the contribution from the first term, which we presume “die off” for $t_0$ enough in the past. For the second-order contribution, similar arguments yield

$$\rho^{(2)}_1(t) = \frac{e^{-\beta H_0}}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 f(t - i\tau_1) f(t - i\tau_2) T_\tau [H_1(-i\tau_1) H_1(-i\tau_2)],$$

where $T_\tau$ is the imaginary-time ordering operator which we use in the text with the same simplification as used in the first-order term, i.e. replace $f(t - i\tau) \approx 1$ in the form above.

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