Coherent Acceleration of Material Wavepackets

Farhan Saif
Department of Electronics, Quaid-i-Azam University, Islamabad 45320, Pakistan.
saif@fulbrightweb.org

Pierre Meystre
Department of Physics, The University of Arizona, Tucson, AZ 85721, USA.

Received Day Month Year
Revised Day Month Year
Communicated by Managing Editor

We study the quantum dynamics of a material wavepacket bouncing off a modulated atomic mirror in the presence of a gravitational field. We find the occurrence of coherent accelerated dynamics for atoms. The acceleration takes place for certain initial phase space data and within specific windows of modulation strengths. The realization of the proposed acceleration scheme is within the range of present day experimental possibilities.

Keywords: Matter waves; acceleration; coherence.

1. Introduction

Accelerating particles using oscillating potentials is an area of extensive research\(^1\), first triggered by the ideas of Fermi on the origin of cosmic rays. In his seminal paper 'On the origin of cosmic rays', he stated that “cosmic rays are originated and accelerated primarily in the interstellar space of the galaxy by collisions against moving magnetic fields”\(^2\). This understanding lead to the development of two major models: The Fermi-Ulam accelerator, which deals with the bouncing of a particle off an oscillating surface in the presence of another fixed surface parallel to it; and the Fermi-Pustyl’nikov accelerator, where the particle bounces off an oscillating surface in the presence of gravity. In the case of the Fermi-Ulam accelerator \(^3,4\) it was shown that the energy of the particle remains bounded and the unlimited acceleration proposed by Fermi is absent \(^5\). In the Fermi-Pustyl’nikov accelerator, by contrast, there exists a set of initial data within specific domains of phase space that result in trajectories speeding up to infinity.

In recent years the acceleration of laser-cooled atoms has become a topic of great interest for applications such as atom interferometry and the development of matter-wave based inertial sensors. Possible schemes of matter-wave acceleration have been proposed and studied. For example, a Bose Einstein condensate in a
frequency-chirped optical lattice and an atom in an amplitude modulated optical lattice in the presence of a gravitational field display acceleration. The $\delta$-kicked accelerator in the latter case operates for certain sets of initial data that originate in stable islands of phase space.

Here, we discuss an experimentally realizable technique to accelerate a material wavepacket in a coherent fashion. It consists of an atom optics version of the Fermi-Pustyl’nikov accelerator\(^9\) where a cloud of ultracold atoms falling in a gravitational field bounces off a spatially modulated atomic mirror. This scheme is different from previous accelerator schemes in the following ways: (i) The regions of phase space that support acceleration are located in the mixed phase space rather than in the islands of stability (or nonlinear resonances); (ii) The acceleration of the wavepacket is coherent; and (iii) It occurs only for certain windows of oscillation strengths.

### 2. The Model

We consider a cloud of laser-cooled atoms that move along the vertical $\tilde{z}$-direction under the influence of gravity and bounce back off an atomic mirror\(^{10}\). This mirror is formed by a laser beam incident on a glass prism and undergoing total internal reflection, thereby creating an optical evanescent wave of intensity $I(\tilde{z}) = I_0 \exp(-2k\tilde{z})$ and characteristic decay length $k^{-1}$ outside of the prism.

The laser intensity is modulated by an acousto-optic modulator as\(^{11}\) $I(\tilde{z}, \tilde{t}) = I_0 \exp(-2k\tilde{z} + \epsilon \sin \omega \tilde{t})$, where $\omega$ is the frequency and $\epsilon$ the amplitude of modulation. The laser frequency is tuned far from any atomic transition, so that there is no significant upper-state atomic population. The excited atomic level(s) can then be adiabatically eliminated, and the atoms behave for all practical purposes as scalar particles of mass $m$ whose center-of-mass motion is governed by the one-dimensional Hamiltonian

$$
\tilde{H} = \frac{\tilde{p}^2}{2m} + mg\tilde{z} + \frac{\hbar \Omega_{\text{eff}}}{4} e^{-2k\tilde{z} + \epsilon \sin \omega \tilde{t}},
$$

where $\tilde{p}$ is the atomic momentum along $\tilde{z}$ and $g$ is the acceleration of gravity.

We proceed by introducing the dimensionless position and momentum coordinates $\tilde{z} \equiv \tilde{z} \omega^2/g$ and $\tilde{p} \equiv \tilde{p} \omega/(mg)$, the scaled time $t \equiv \omega \tilde{t}$, the dimensionless intensity $V_0 \equiv \hbar \omega^2 \Omega_{\text{eff}}/(4mg^2)$, the steepness $\kappa \equiv 2kg/\omega^2$, and the modulation strength $\lambda \equiv \omega^2 \epsilon/(2kg)$ of the evanescent wave field.

When extended to an ensemble of non-interacting particles, the classical dynamics obeys the condition of incompressibility of the flow\(^3\), and the phase space distribution function $P(\tilde{z}, \tilde{p}, t)$ satisfies the Liouville equation. In the absence of mirror modulation, the atomic dynamics is integrable. For very weak modulations the incommensurate motion almost follows the integrable evolution and remains rigorously stable, as prescribed by the KAM theorem. As the modulation increases, though, the classical system becomes chaotic.

In the quantum regime, the atomic evolution obeys the corresponding Schrödinger equation. The commutation relation, $[\tilde{z}, \tilde{p}] = i(\omega^3/mg^2)\hbar \equiv ik$, natu-
rally leads to the introduction of the dimensionless Planck constant, \( \kappa \equiv \hbar \omega^3/(mg^2) \). It can easily be varied by changing for instance \( \omega \), thereby permitting to study the transition from the semiclassical to the purely quantum dynamics of the atoms.

3. Accelerated Dynamics

The classical version of the is characterized by the existence of a set of initial conditions resulting in trajectories that accelerate without bound\(^5\). More precisely, the classical evolution of the Fermi accelerator displays the onset of global diffusion above a critical modulation strength \( \lambda_1 = 0.24 \), while the quantum evolution remains localized until a larger value \( \lambda_u \) of the modulation\(^9,12,13\). Above that point both the classical and the quantum dynamics are diffusive. However, for specific sets of initial conditions that lie within phase space disks of radius \( \rho \), accelerating modes appear for values of the modulation strength \( \lambda \) within the windows \(^6\)

\[
s\pi \leq \lambda < \sqrt{1 + (s\pi)^2},
\]

where \( s \) can take integer and half-integer values for the sinusoidal modulation of the reflecting surface considered here.

We found numerically that for a modulation strength outside the windows of Eq. (2) the dynamics is dominantly diffusive. However, as the fundamental requirement for the acceleration is met by choosing a modulation strength within the windows the ensemble displays a nondispersive and coherent acceleration, Fig. 1. A small diffusive background results from a small part of the initial distribution which
F. Saei

Fig. 2. Mirror images of the classical and quantum mechanical momentum distributions, $P(p)$, plotted for (a) $\lambda = 1.7$ within acceleration window, and (b) $\lambda = 2.4$ outside the acceleration window. The spikes in the momentum distribution for $\lambda = 1.7$ are a signature of coherent accelerated dynamics. The initial width of the momentum distribution is $\Delta p = 0.5$ and the probability distributions are recorded after a scaled propagation time $t = 500$. The initial probability distributions have variance $\Delta p = 0.5$, which fulfills the minimum uncertainty relation.

is residing outside the area of phase space supporting acceleration. This coherent acceleration restricts the momentum space variance $\Delta p$ which then remains very small indicating the absence of diffusive dynamics. In the quantum case the Heisenberg Uncertainty Principle imposes a limit on the smallest size of the initial wavepacket. Thus in order to form an initial wavepacket that resides entirely within regions of phase space leading to coherent dispersionless acceleration, an appropriate value of the effective Planck constant must be chosen, for example, by controlling the frequency $\omega$. For a broad wavepacket, the coherent acceleration manifests itself as regular spikes in the marginal probability distributions $P(p, t) = \int dx P(x, p, t)$ and $P(x, t) = \int dp P(x, p, t)$.

This is illustrated in Fig. 2 which shows the marginal probability distribution $P(p, t)$ for (a) $\lambda = 1.7$ and (b) $\lambda = 2.4$, both in the classical and the quantum domains. In this example, the initial area of the particle phase-space distribution is taken to be large compared to the size of the phase-space regions leading to purely unbounded dispersionless acceleration. The sharp spikes in, $P(p, t)$ appear when the modulation strength satisfies the condition of Eq. (2), and gradually disappear as it exits these windows. These spikes are therefore a signature of the coherent...
accelerated dynamics. In contrast, the portions of the initial probability distribution originating from the regions of the phase space that do not support accelerated dynamics undergo diffusive dynamics.

From the numerical results of Fig. 2, we conjecture that the spikes are well described by a sequence of gaussian distributions separated by a distance $\pi$, both in momentum space and coordinate space. We can therefore express the complete time-evolved wavepacket composed of a series of sharply-peaked gaussian distributions superposed to a broad background due to diffusive dynamics, such that

$$P(p) = N e^{-p^2 / 4\Delta p^2} \sum_{n=-\infty}^{\infty} e^{-(p-n\pi)^2 / 4\epsilon^2},$$

where $\epsilon << \Delta p$, and $N$ is a normalization constant.

Further insight in the quantum acceleration of the atomic wavepacket is obtained by studying its temporal evolution. We find that within the window of acceleration the atomic wave packet displays a linear growth in the square of the momentum variance and in the coordinate space variance. Figure 3 illustrates that for modulation strengths within the acceleration window, the growth in square of the momentum variance displays oscillations of increasing periodicity whereas the variance in coordinate space follows with a phase difference of 180°. The out-of-phase oscillatory evolutions of $\Delta p^2$ and $\Delta z$ indicate a breathing of the wavepacket and is a signature of the coherence in accelerated dynamics. As a final point we note that outside of the acceleration window the linear growth in the square of the momentum variance, a consequence of normal diffusion, translates into a $t^\alpha$ law, with $\alpha < 1$ which is a consequence of anomalous diffusion.

4. Summary

We have investigated the classical and quantum evolution of atoms in a Fermi accelerator beyond the regime of dynamical localization where diffusive behavior occurs.
both in the classical and the quantum domains. We have identified the conditions leading to the coherent acceleration of the atoms, and found signatures of the behavior both for an ensemble of classical particles and for a quantum wavepacket. A quantum wavepacket with a broad initial variance, restricted by the Heisenberg Uncertainty Principle results in the coherent acceleration occurring on top of a diffusive background.

Acknowledgment

This work is supported in part by the US Office of Naval Research, the National Science Foundation, the US Army Research Office, the Joint Services Optics Program, the National Aeronautics and Space Administration, and J. William Fulbright foundation.

References

1. F. Saif, Phys. Rep. 419, 207 (2005); ibid. Phys. Rep. 425, 369 (2006).
2. E. Fermi, Phys. Rev. 75, 1169 (1949).
3. A. J. Lichtenberg and M. A. Lieberman, Regular and Stochastic Motion, (Springer, Berlin, 1983); Lichtenberg A. J. and M. A. Lieberman, Regular and Chaotic Dynamics (Springer, New York, 1992).
4. Reichl L. E., The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations (Springer-Verlag, Berlin, 1992).
5. Zaslavskii G. M., and B. Chirikov, Dokl. Akad. Nauk SSSR 159, 306 (1964) [Sov. Phys. Dokl. 9, 989 (1965) 989]; Pustilnikov L. D., Teor. Mat. Fiz. 57, 128 (1983); Pustilnikov L. D., Dokl. Akad. Nauk SSSR 292, 549 (1987) [Sov. Math. Dokl. 35, 88 (1987)].
6. Pustil’nikov L. D., Trudy Moskov. Mat. Obšč. Tom 34(2) (1977) 1 [Trans. Moscow Math. Soc. 2 (1978) 1].
7. S. Pöting, M. Cramer, C. H. Schwalb, H. Pu, and P. Meystre Phys. Rev. A 64, 023604 (2001).
8. A. Buchleitner et al., e-print physics/0501146; Z.-Y. Ma et al., Phys. Rev. A 73 013401 (2006) and references therein;
9. F. Saif et al., Phys. Rev. A 58, 4779 (1998); F. Saif, Phys. Lett. A 274, 98 (2000).
10. C. G. Aminoff et al., Phys. Rev. Lett. 71, 3083 (1993).
11. A. Steane et al., Phys. Rev. Lett. 74, 4972 (1995) have explored a regime of parameters that would lead to the observation of these effects in the long-time regime.
12. F. Benvenuto et al., Z. Phys. B 84, 159 (1991).
13. C. R. de Oliveira, I. Guarneri and G. Casati, Europhys. Lett. 27, 187 (1994).
14. F. Saif, and P. Meystre, arXiv:quant-ph/0607131.
15. F. Saif, and P. Meystre, in preparation.