Systems of PDEs obtained from factorization in loop groups

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**ABSTRACT**

We propose a generalization of a Drinfeld-Sokolov scheme of attaching integrable systems of PDEs to affine Kac-Moody algebras. With every affine Kac-Moody algebra \( \mathfrak{g} \) and a parabolic subalgebra \( \mathfrak{p} \), we associate two hierarchies of PDEs. One, called positive, is a generalization of the KdV hierarchy, the other, called negative, generalizes the Toda hierarchy. We prove a coordinatization theorem, which establishes that the number of functions needed to express all PDEs of the total hierarchy equals the rank of \( \mathfrak{g} \). The choice of functions, however, is shown to depend in a noncanonical way on \( \mathfrak{p} \). We employ a version of the Birkhoff decomposition and a “2-loop” formulation which allows us to incorporate geometrically meaningful solutions to those hierarchies. We illustrate our formalism for positive hierarchies with a generalization of the Boussinesq system and for the negative hierarchies with the stationary Bogoyavlenskii equation.

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References
Introduction

In recent years loop groups have been successfully used in the investigation of geometric objects, like surfaces of constant mean curvature in $\mathbb{R}^3$, harmonic maps into compact symmetric spaces, isometric immersions from space forms into space forms. At the heart of all these uses is the construction of solutions to certain nonlinear partial differential equations as compatibility conditions of a system of matrix equations. In this context it is an outgrowth of soliton theory. However, the geometric applications also yield some new features. While the classical uses of loop groups for finding solutions to certain nonlinear partial differential equations only use “positive flows”, the geometric applications require also “negative flows”. An application of this yields an extension of the potential KDV hierarchy by an equation investigated by Bogoyavlenskii [1] and also by Hirota-Satsuma [2]. It is therefore natural to extend to negative flows what is classical and well established for positive flows. In this paper we present such an extension.

The general setting is as follows: Let $G$ be a Banach Lie loop group, i.e. a certain group of maps from the unit circle $S^1$ into $\mathfrak{gl}(n, \mathbb{C})$. We also consider two subgroups $G^+$ and $G^-$ of $G$ and assume that $G^- G^+$ is open and dense in $G$. Finally we consider an abelian subgroup of $G$ given by $\exp\{... tE_{-1} + x E_1 ...\}$. For the groups considered in this paper we prove that for $h$ in $G$ and all sufficiently small $x \neq 0$ we have $x.h = \exp\{x E_1\} h$ is in $G^+ G^-$. If we denote by $h_-$ the part of $x.h$ which is in $G^-$ then we can split $t.h_- = \exp\{t E_{-1}\} h_- = g^- g^+$. Differentiating both sides one obtains a system of matrix equations for $g^-$ and $g^+$. The compatibility conditions for this system of equations then yield the (system) of scalar nonlinear partial differential equations we are primarily interested in. This scheme is well known to practitioners of soliton theory. Yet, by admitting a large class of choices of the subgroups $G^+$ and $G^-$, and a large abelian group we arrive at a framework unifying many of the known formulations into one theory.

This paper contains a reformulation of the theory due mainly to Drinfeld and Sokolov [3] (see also Wilson’s paper [4]), relating affine Kac-Moody algebras to certain classes of non-linear PDEs. The starting point for our reformulation is the idea of factorization outlined above. This point of view is not originally discussed in [3], even though it figures prominently in places where representation theory of affine Kac-Moody algebras is actually used to generate solutions to soliton equations. Using this reformulation we propose an extension of the formalism to include new hierarchies of equations. The basic result in [3] is that, given the Dynkin diagram of an affine algebra, one can associate to it a hierarchy of non-linear partial differential equations we are primarily interested in. This scheme is well known to practitioners of soliton theory. Yet, by admitting a large class of choices of the subgroups $G^+$ and $G^-$, and a large abelian group we arrive at a framework unifying many of the known formulations into one theory.

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We have extended the original picture of [3] in the following way. We consider a pair $(S, p)$ consisting of a Dynkin diagram $S$ of an affine Kac-Moody Lie algebra and an arbitrary parabolic $p$ related to the same graph. With $(S, p)$ we associate two hierarchies of
differential equations in two different matrix variables, called $\Omega_1$ and $\Omega_{-1}$. They generate what we call the positive and the negative hierarchies respectively. Each $\Omega$ appearing in the sequel generates a one parameter flow, and all these flows commute. The assignment of flow variables is that $t_{-1} \in \mathbb{C}$ is assigned to $\Omega_{-1}$, $x \in \mathbb{C}$ is assigned to $\Omega_1$, in general, $t_j \in \mathbb{C}$ is the flow variable for $\Omega_j$, $j \in \mathbb{Z}$. $\Omega_1$ and $\Omega_{-1}$ parametrize in the sense explained in the course of the paper all other flows. These two form a natural generalization of the pair giving the modified KdV and the sine-Gordon equation. The new feature here is that in the case when at least one point is removed from the Dynkin diagram one can in fact relate the two in a differential fashion, that is, $\Omega_{-1}$ can be expressed in terms of derivatives of elements of $\Omega_1$. This rather surprising fact we prove for all nontwisted affine algebras and all choices of points of their Dynkin diagrams except for those cases specified in Proposition (7.3.2) and Theorem (7.3.3). This result has interesting ramifications even for the well known case of the (potential) KdV equation, which comes from $A^{(1)}_1$ and a maximal parabolic subalgebra (one point removed, as the diagram of $A^{(1)}_1$ has only two nodes). It turns out that as a result of the differential dependence of $\Omega_{-1}$ on $\Omega_1$ the KdV variable $v = v(t_{-1}, x, t_3)$, which parametrizes $\Omega_1$, satisfies another differential equation with two independent variables, namely the stationary Bogoyavlenskii equation [1]. This equation is not an evolution equation, in complete analogy to the case of the modified KdV and the sine-Gordon equation.

We would like to point out that in our approach we consistently use group factorizations to both generate equations as well as to provide, in principle, solutions to these equations. This should be contrasted with the approach of Drinfeld and Sokolov who use formal pseudodifferential operators to formulate the Gelfand-Dikii hierarchy and its modified version, the modified Gelfand-Dikii hierarchy. Our approach is therefore somewhat more in a spirit of classical Lie theory. It is not transparent at all, however, how to bring out a hamiltonian aspect of theory, something quite prominent in [3]. On the other hand, we arrive naturally at numerous "cousins" of the Gelfand-Dikii hierarchy.

Our approach originates from the use of Kac-Moody algebras and groups and Grassmann like manifolds for the description of solutions to certain nonlinear partial differential equations as it was pioneered by Sato [5] and Segal-Wilson [6]. It is quite natural to work in this context with "full" affine Kac-Moody algebras, i.e. central extensions of loop algebras, additionally augmented by a degree derivation. In a completely algebraic context, the corresponding groups have been defined and investigated by Garland [7], Peterson-Kac [8] and Tits [9]. However, for a description of solutions to differential equations a "completion" of these groups and Lie algebras is more than just a matter of aesthetics. Indeed, it is well known by now that for those algebraic groups one obtains a sector of rational solutions only. The finite gap solutions on the other hand require at least a type of completion we are considering. This happens despite the fact that, with some extra work (see [10, chap14]), one can include in the algebraic approach based on those "thin groups" special finite gap solutions, namely solitons. Since the transition from problems involving PDEs, even if they are geometric in character, is not sufficiently refined yet to tell us the most convenient topology in which to work, we have chosen to work with a Banach topology. We have therefore been interested in Banach structures on Kac-Moody Lie algebras and the corresponding groups. Here one considers first loop algebras, i.e. on considers the derived algebra of an affine Kac-Moody algebra modulo its center. Following Goodman-Wallach [11] we obtain a large class of Banach structures that allow us to complete loop algebras and to obtain this way Banach Lie algebras. It is not difficult to find the associated Banach Lie groups. For our purposes we need a few additional specific features of the groups used. To prove those we rely mostly on the work of Goodman-Wallach [11] and Pressley-Segal [12]. Therefore we restrict our attention throughout this paper to non-twisted affine Kac-Moody algebras. We would hope that eventually the results of this paper will be extended to twisted affine Kac-Moody algebras, perhaps to even more general Kac-Moody
algebras. But it would be equally interesting to carry out the investigations of this paper with different Banach structures: the Banach structures used in this paper are all related with solution spaces containing only meromorphic solutions. It was shown, however, in [13] that by considering completely different function spaces, like the Fourier transforms of functions from $L^1(\mathbb{R})$, one can obtain solution spaces consisting of $L^1$ functions only.

In addition to using positive as well as negative flows and to using Banach structures we extend the setting of [3] in a third aspect: it turns out that one needs to use not only Kac-Moody algebras and groups, but one also needs to consider double loop algebras and groups. This was first noticed in [14]. There, an effort was made to describe the standard ”completely integrable” nonlinear partial differential equations in terms of the loop group/Grassmannian picture. It turned out that for the sine-Gordon equation one is naturally led to consider double loop groups and natural ”positive” and ”negative” subgroups. This was enhanced by [15]: while investigating constant mean curvature tori in $\mathbb{R}^3$, i.e. special (real) solutions to the sinh-Gordon equation, it was observed that not only should one consider double loop groups $G \times G$, but one should even consider $G^r \times G^R$, where $0 < r < 1 < R < \infty$ and $G^r$ and $G^R$ are defined on a circle of radius $r$ and $R$ respectively. The reason for this is that it is impossible to construct the solutions describing constant mean curvature tori in the ”1-loop” setting by the procedure above [16, Theorem 2.7]. Yet, it is possible to construct such solutions in the ”2-loop” setting. For more details on this see [15] and [17]. We should perhaps add that the term used there is an r-loop approach rather than a ”2-loop” approach used in the present paper.

Interestingly enough the hierarchies of PDEs we are considering can be used to obtain solutions to the self dual Yang-Mills equations [18]. From that perspective the theory we are putting forward comprises a part of the theory of the self dual Yang-Mills equations.

Here is a short description of the paper. In §1 and 2 we present some elementary facts about affine Kac-Moody algebras, their completions as well as basic properties of parabolic algebras and groups. The latter topic is further developed in §3. In particular in that chapter we prove Theorem (3.5.1), Corollary (3.5.3) and Theorem (3.6.1), crucial for the whole paper. In §4 we define our “2-loop” group setting. §5 contains a description of the Zero Curvature formulation of the systems of PDEs corresponding to our “2-loop” formulation. This is further developed in §6, where in particular we show, that all systems of PDEs appearing in the positive hierarchy depend on dim $g_0$ basic functions. This result is proven in Theorem (6.5.1). §6 ends with examples illustrating this part of the theory. The negative hierarchy is studied in §7. The main result here is Proposition (7.3.2). We give an example of the first flow in the negative hierarchy. This turns out to be an equation studied in [1] by Bogoyavlenskii. The appendices contain omitted proofs of two propositions from §3 and some results, used in the paper, regarding the map $adE$.

There are some very interesting and important questions that have not been addressed in this paper. One is the question of hamiltonicity of the solution spaces and to what extent the solution spaces, as Banach manifolds, are completely integrable in some rigorous sense. We feel that it is possibly advantageous to restrict the above questions to the dressing orbits on the solution spaces. This way one is perhaps able to make contact with the AKS (Adler-Kostant-Symes) method used by many authors for the construction of solutions to completely integrable nonlinear partial differential equations. Another question is to what extent one can find Miura like transformations from the solution space relative to one parabolic $p_1$ to the solution space of another parabolic $p_2$. We would like to point out that such Miura maps on the level of our solution spaces may not exist. As an example, in [19], it has been shown that there does not exist a Miura map from the solution space to the MKdV equation to the solution space to the potential KdV equation. Related with this is the question of how to define and to describe Bäcklund transformations for the equations considered in this paper. From their origin, Bäcklund transformations should be diffeomorphisms of the solutions spaces or at least maps from solution spaces to solution spaces. The actual use of Bäcklund transformations, however, seems to be different. It
will be very interesting to pursue the above questions in detail.

Finally, the referee has kindly pointed out to us that there already exists a generalization of the Drinfeld-Sokolov systems [20]. The Hamiltonian theory is discussed in [21] and [22]. The generalization we propose for positive flows is formally included in the other generalization. However, our way of parametrizing the occurring potential $\Omega_1$ is quite different from that of other authors. We illustrate the difference on the example of the potential KdV in §6. We also believe that the concept of the negative hierarchies for cases other than the minimal parabolic case (the Toda equations) is a key new element of our perspective, and this sets apart our approach from that of [20].
§1. Banach loop groups and algebras

In this section we will define the main objects of the entire paper: loop groups and algebras. Most of the definitions and results are taken from [11], but they are listed here for the convenience of the reader, and to fix notation.

1.1 A function \( w : \mathbb{Z} \to (0, \infty) \) is called a weight, if \( w(k+l) \leq w(k)w(l) \) for all \( k, l \in \mathbb{Z} \). A weight \( w \) is called symmetric, if \( w(-k) = w(k) \) for all \( k \in \mathbb{Z} \). Two classes of examples are:

- symmetric exponential-polynomial weights:
  \[ w_{a,t}(k) = (1 + |k|)^a \cdot e^{t|k|} \text{ for } a, t \geq 0. \]

- “Gevrey class”
  \[ w_{t,s}(k) = \exp(t \cdot |k|^s) \text{ for } t > 0, 0 < s < 1. \]

For a symmetric weight \( w \) define

\[
(1.1.1) \quad A_w := \{ f : S^1 \to \mathbb{C}, \lambda \mapsto \sum_{n \in \mathbb{Z}} a_n \lambda^n, \| f \|_w < \infty \}
\]

where

\[ \| f \|_w = \sum_{n \in \mathbb{Z}} |a_n| \cdot w(n). \]

One easily verifies that \( \| \cdot \|_w \) is a norm, thus \( A_w \) is a commutative Banach \(*\)-algebra, with pointwise multiplication, the \(*\)-operation being complex conjugation.

We call \( A_w \) the weighted Wiener algebra associated with the weight \( w \). In this paper we will use exclusively symmetric weights of non-analytic type, i.e. satisfying \( \lim_{n \to \infty} w(n) \frac{1}{n} = 1 \). We note that all Gevrey class weights are of non-analytic type, while in the first example only weights with \( t = 0 \) are non-analytic.

1.2 Now let \( \mathfrak{g} \) be a simple finite-dimensional complex Lie algebra of type \( X_l \) (i.e. \( X \in \{ A, B, C, D, E, F, G \} \)). Let \( \psi \) be an automorphism of the Dynkin diagram of order \( k \). Then, \( \psi \) can be extended to an automorphism of \( \mathfrak{g} \). Its order is \( k \), too. We will also call it \( \psi \).

Define

\[
(1.2.1) \quad \mathfrak{g}^{fin} := \left\{ x : \lambda \mapsto \sum_{-m \leq j \leq n} \lambda^j A_j : m, n \in \mathbb{N}_0, A_j \in \mathfrak{g} \right\}.
\]

such that \( x(\lambda \cdot e^{2\pi i/k}) = \psi(\lambda) \) for all \( \lambda \in S^1 \). This Lie algebra \( \mathfrak{g}^{fin} \) is called the affine Lie algebra of type \( X^{(k)}_l \). It differs from the affine Kac-Moody algebra of type \( X^k_l \) by a one-dimensional center. Nevertheless we will also use
the latter name for \( g^\text{fin} \). Now let \( \tilde{G} \) be the connected and simply-connected Lie group such that \( \text{Lie} \tilde{G} = \tilde{g} \). We may assume that \( \tilde{G} \) is a subgroup of a suitable \( SL_n(\mathbb{C}) \), \( \tilde{g} \) a subalgebra of \( sl_n(\mathbb{C}) \). Now recall from general Lie group theory that any Lie algebra automorphism can be “exponentiated” to a simply-connected Lie group \( \tilde{G} \) in a unique way. Therefore, there is a unique automorphism \( \phi \) of \( \tilde{G} \) such that \( d\phi(e) = \psi \). Obviously, \( \phi \) is of order \( k \), too.

Thus we may define:

\[
G_w := \left\{ g \in SL_n(A_w) : g(\lambda) \in \tilde{G} \quad \text{and} \quad g(\lambda e^{2\pi i/k}) = \phi(g(\lambda)) \quad \text{for all} \quad \lambda \in S^1 \right\}
\]

\[
\mathfrak{g}_w := \left\{ x \in s\ell_n(A_w) : x(\lambda) \in \tilde{g} \quad \text{and} \quad x(\lambda e^{2\pi i/k}) = \psi(x(\lambda)) \quad \text{for all} \quad \lambda \in S^1 \right\}.
\]

For these objects, Goodman and Wallach prove (essentially in [11; 5.1] and [12; 5.5, 6.8, 6.9]):

1. \( G_w \) is a complex Lie subgroup of \( SL_n(A_w) \), \( g_w \) is complex Lie subalgebra of \( s\ell_n(A_w) \).
2. \( \text{Lie} G_w = g_w \).
3. \( g_w \) is a Banach Lie algebra, the completion of \( g^\text{fin} \) w.r.t. to the norm defined by the symmetric weight \( w \).
4. \( G_w \) is connected and simply-connected (Lemma 5.5, [11]).

Therefore we will call \( G_w \) resp. \( g_w \) the Banach loop group resp. loop algebra of type \( X_l^{(k)} \) (w.r.t. the weight \( w \)).

Remark:

1. In [11], \( G_w \) is denoted by \( \widetilde{G}_w \) (\( g_w \) by \( \widetilde{g}_w \)).
2. Goodman and Wallach discuss in detail the case where \( \psi \) is the identity, i.e. the type \( X_l^{(n)} \) (also called “non-twisted case”, cf. [10]). However in the last two sections of chapter 6 they indicate how to generalize the results cited above to arbitrary affine algebras.
3. In the following we will write \( G \) resp. \( g \) instead of \( g_w \) and \( G_w \) when there is no ambiguity.

1.3 In this section we collect some properties of the Kac-Moody Lie algebra \( g^\text{fin} \) [10].

1. There exists a system of “Chevalley generators” \( \{e_i, f_i, h_i : i = 0, \cdots l\} \), i.e.
   - \( e_i, f_i, h_i \) generate \( g^\text{fin} \) as a Lie algebra
• \([h_i, h_j] = 0\)
• \([e_i, f_j] = \delta_{ij} h_i\)
• \([h_i, e_j] = a_{ij} e_j\)
• \([h_i, f_j] = -a_{ij} f_j\)

Moreover, for all \(i, j = 0 \cdots l, i \neq j\) we have
\((ad e_i)^{1-a_{ij}} e_j = 0, \ (ad f_i)^{1-a_{ij}} f_i = 0.\)

where \(A = (a_{ij})\) is a generalized Cartan matrix, i.e.
• \(a_{ii} = 2\) for all \(i = 0, \ldots, l\)
• \(a_{ij} \leq 0\) for all \(i \neq j\)
• \(a_{ij} = 0 \iff a_{ji} = 0.\)

For \(g^{fin}\), it is known (see e.g. [10, § 6.1]) that \(A\) is symmetrizable, that is, there exists an invertible diagonal matrix \(D\) such that \(DA\) is a symmetric matrix.

2. Let \(g^{fin}_-\) resp. \(g^{fin}_0\) resp. \(g^{fin}_+\) be the subalgebras generated by the \(f\)'s resp. \(h\)'s resp. \(e\)'s.

Then there is a natural triangular decomposition
\[
g^{fin} = g^{fin}_- \oplus g^{fin}_0 \oplus g^{fin}_+.\]

3. Let \(\Delta = \Delta_- \cup \Delta_+\) be a root system of \(g^{fin}\), \(\Delta_+\) denoting the set of positive resp. negative roots associated with \(g^{fin}_0\) and \(g^{fin}_\pm\). Then \(g^{fin}_\pm = \bigoplus_{\alpha \in \Delta_\pm} g^{fin}_\alpha\).

4. Denote by \(\Pi = \{\alpha_0, \cdots, \alpha_l\}\) a set of simple roots corresponding to \(\Delta\).

5. Finally by \(G^{fin}\) we denote the group generated by \(\{\exp x_\alpha\}\), where \(x_\alpha \in g^{fin}_\alpha\), \(\alpha \in \Delta^{re}\). The subset of real roots \(\Delta^{re}\) is defined and studied in [10, §5].
\section*{2. Borel and parabolic subgroups and subalgebras}

This section contains a collection of the results about Borel and - more generally - parabolic subgroups and algebras used later in the paper. For more details on the basic theory of these subalgebras see \cite{23, Ch. 8, Section 3.4}.

\subsection*{2.1 We start with}

**Theorem (2.1.1).** ([8, Theorem 3]). Every Borel subalgebra of $\mathfrak{g}^{\text{fin}}$ is $\text{Ad}(G^{\text{fin}})$-conjugate to $\mathfrak{g}_0^{\text{fin}} \oplus \mathfrak{g}_+^{\text{fin}}$ or $\mathfrak{g}_0^{\text{fin}} \oplus \mathfrak{g}_-^{\text{fin}}$.

**Remark:**

(a) A subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is called a Borel subalgebra if it is maximal completely solvable.

(b) A subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is called completely solvable in $\mathfrak{g}$ if there is a flag $\cdots \supset \mathfrak{b}_{-1} \supset \mathfrak{b}_0 \supset \mathfrak{b}_{+1} \supset \mathfrak{b}_2 \supset \cdots$ of $\text{ad}(\mathfrak{b})$-invariant subspaces of $\mathfrak{g}$ such that

\begin{itemize}
  \item $\mathfrak{g} = \bigcup_{i \in \mathbb{Z}} \mathfrak{b}_i$
  \item $\bigcap_{i \in \mathbb{Z}} \mathfrak{b}_i = \{0\}$
  \item $\mathfrak{b} = \mathfrak{b}_0$
  \item $\text{dim} \frac{\mathfrak{b}}{\mathfrak{b}_{i+1}} \leq 1$
\end{itemize}

In view of the Theorem above we may restrict ourselves to the case of the standard Borel subalgebra $\mathfrak{b}^{\text{fin}} := \mathfrak{g}_0^{\text{fin}} \oplus \mathfrak{g}_+^{\text{fin}}$, respectively, the standard Borel subgroup $B^{\text{fin}}$, the corresponding subgroup of $G^{\text{fin}}$. We define the standard Borel subalgebra $\mathfrak{b}_w$ of the Banach loop algebra $\mathfrak{g}_w$ as the completion of $\mathfrak{b}^{\text{fin}}$ in $\mathfrak{g}_w$. To obtain a similar statement on the group level we note that $\mathfrak{b}_w$ has a closed complement in $\mathfrak{g}_w$ (namely $(\mathfrak{g}_w)^-$). Therefore, by ([23], Ch. 3, §6, Theorem 2) there exists a connected Banach Lie group $B_w$ such that $\text{Lie} B_w = \mathfrak{b}_w$ and so that $B_w \subseteq G_w$ is an integral subgroup of $G_w$.

\subsection*{2.2 In general, a parabolic subalgebra of a Lie algebra is a subalgebra containing a Borel subalgebra. By virtue of Section 2.1 we may as well restrict ourselves to the following class:}

**Definition (2.2.1).** A subalgebra $\mathfrak{p}^{\text{fin}} \subseteq \mathfrak{g}^{\text{fin}}$ is called a standard-parabolic subalgebra (spsa) if $\mathfrak{g}_0^{\text{fin}} \oplus \mathfrak{g}_+^{\text{fin}} \subseteq \mathfrak{p}^{\text{fin}}$. For a subset $X \subseteq \Pi$ (the set of simple roots) let $\mathfrak{p}_X^{\text{fin}}$ be the smallest spsa containing all root spaces $\mathfrak{g}_-^{\text{fin}}\alpha$ for $\alpha \in X$.

**Remark:** Similarly, we define $\mathfrak{p}_w = \mathfrak{p}_X^{\text{fin}}$.

The following is well known [23, Ch. 8, Section 3.4, Ch. 4, Sect.2.6]:

**Proposition (2.2.2).**

(a) Let $\mathfrak{p}^{\text{fin}}$ be a spsa of $\mathfrak{g}^{\text{fin}}$. Then there is a subset $X \subseteq \Pi$ such that $\mathfrak{p}^{\text{fin}} = \mathfrak{p}_X^{\text{fin}}$.

(b) Let $X \subseteq \Pi$ and $\tilde{\Delta}_+ = \{\sum k_i \alpha_i \in \Delta_+ : \alpha_i \in X\}$. Then $\mathfrak{p}_X^{\text{fin}} = \bigoplus_{\alpha \in \tilde{\Delta}_+} \mathfrak{g}_-^{\text{fin}}\alpha \oplus \mathfrak{g}_0^{\text{fin}} \oplus \mathfrak{g}_+^{\text{fin}}$.  

10
(c) Let $q^\text{fin}_X := \bigoplus_{\alpha \in \Delta_+ \setminus \bar{\Delta}_+} g^\text{fin}_\alpha$. Then $q^\text{fin}_X$ is a subalgebra of $g^\text{fin}$, and $g^\text{fin} = q^\text{fin}_X \oplus p^\text{fin}_X$.

We call $q^\text{fin}_X$ the natural complement of $p^\text{fin}_X$. We define $q_w = \overline{q}^\text{fin}$. Then $q_w = \overline{q}^\text{fin}$. We therefore have a 1-1 correspondence between $\text{spsa}$'s and subsets of the set of simple roots. Let us now fix the subset $X$ and define $q^w = q^\text{fin}_X$. Then $q^w$ is a natural complement of $p^w$.

Now we turn to formulating the corresponding facts on the group level.

**Corollary (2.2.3).**

(a) For every spsa $p^\text{fin}_X \subset g^\text{fin}$ and its natural complement $q^\text{fin}_X$ there are unique connected subgroups $P^\text{fin}_X$ and $Q^\text{fin}_X \subset G^\text{fin}$ where $P^\text{fin}_X$ is generated by $\exp(p^\text{fin}_\alpha), p^\text{fin}_\alpha \subset p_w$ and $Q^\text{fin}_X$ by $\exp(q^\text{fin}_\alpha), q^\text{fin}_\alpha \subset q_w, \alpha \in \Delta^{re}$.

(b) For every spsa $p^w \subset g^w$ and its natural complement $q^w$ there are unique connected Banach Lie groups $P^w$ and $Q^w \subset G^w$ such that $\text{Lie } P^w = p_w$, $\text{Lie } Q^w = q_w$ and $P^w$ and $Q^w$ are integral subgroups of $G^w$.

**Proof.** (a) holds automatically by construction, (b) again follows from the fact that $p_w$ and $g^w$ are closed complements of each other. ■

2.3. For the sake of simplicity we drop the superscript $(\ )^\text{fin}$ in this section.

Let $g$ be an affine Kac-Moody algebra and $\{e_i, f_i, h_i : i = 0, \cdots, n\}$ be a set of canonical generators. The assignment

\[(2.3.1)\quad cdeg e_i := 1, \quad cdeg f_i := -1\]

for all $i$ defines a grading on $g$; we refer to it as the canonical grading. For $x \in g$ we denote the homogeneous component of $x$ of canonical degree $k$ by $x_k$ and write: $cdeg x_k = k$. Let $g_k$ be the subspace of all homogeneous elements of canonical degree $k$. Thus:

\[(2.3.2)\quad g = \bigoplus_{k \in \mathbb{Z}} g_k.\]

Let $g_- := \bigoplus_{k < 0} g_k, g_+ := \bigoplus_{k > 0} g_k$, then:

\[(2.3.3)\quad g = g_- \oplus g_0 \oplus g_+.\]

Note that $g_0$ is generated by $\{h_i : i = 0, \cdots, n\}$ and thus abelian.

For a given standard parabolic subalgebra $p$ of $g$ we define the $p$-grading of $g$ by assigning

\[(2.3.4)\quad pdeg f_i := \begin{cases} 0, & \text{if } f_i \in p \\ -1, & \text{otherwise} \end{cases}\]

\[(2.3.5)\quad pdeg e_i := -pdeg f_i \quad \text{for all } i.\]

For $x \in g$ let $x^{(k)}$ be the homogeneous component of $x$ of $p$-degree $k$ and $g^{(k)}$ the subspace of all such elements. In a similar manner as above:

\[(2.3.6)\quad g = \bigoplus_{k \in \mathbb{Z}} g^{(k)}.\]
We define:
\[
\begin{align*}
\mathfrak{g}^{(-)} := & \bigoplus_{k<0} \mathfrak{g}^{(k)}, \\
\mathfrak{g}^{(+)} := & \bigoplus_{k>0} \mathfrak{g}^{(k)},
\end{align*}
\]

\[2.3.6\]

\[
\mathfrak{g} = \mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+)}.
\]

Note that the canonical grading and the \(p\)-grading coincide iff \(p\) is a minimal parabolic subalgebra, i.e. the Borel subalgebra \(\mathfrak{g}_0 \oplus \mathfrak{g}_+\).

In general, \(\mathfrak{g}^{(0)}\) is no longer abelian. To understand better the structure of \(\mathfrak{g}^{(0)}\) we observe that \(\mathfrak{g}^{(0)}\) is generated by \({e_i, f_i : \alpha_i \in X}\) \(\cup \{h_i : i = 0, \cdots, n\}\). The following lemma is well known in the finite dimensional setting.

**Lemma (2.3.1).** Let \(\mathfrak{p} \neq \mathfrak{g}\) be a parabolic subalgebra of \(\mathfrak{g}\). Then \(\mathfrak{g}^{(0)}\) is a finite-dimensional reductive subalgebra of \(\mathfrak{g}\).

**Proof.** Let \(X \subset \Pi\) be such that \(\mathfrak{p} = \mathfrak{p}_X\). Since \(\mathfrak{p} \neq \mathfrak{g}\), \(X \neq \Pi\). To see that \(\mathfrak{g}^{(0)}\) is finite-dimensional, denote by \(\tilde{\mathfrak{g}}\) the subalgebra of \(\mathfrak{g}\) generated by the set \({e_i, f_i : \alpha_i \in X}\). Thus, the algebra \(\tilde{\mathfrak{g}}\) corresponds to a certain subdiagram of the Dynkin diagram of \(\mathfrak{g}\). By Lemma 4.4 in [10, Ch. 4] the Dynkin diagram of \(\tilde{\mathfrak{g}}\) is a disjoint union of diagrams corresponding to simple finite-dimensional Lie algebras. Thus \(\tilde{\mathfrak{g}}\) is finite-dimensional. Moreover, \(\mathfrak{g}^{(0)}/\tilde{\mathfrak{g}}\) is finite dimensional by the remark above. Thus \(\mathfrak{g}^{(0)}\) is finite dimensional. Since \([\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] = \tilde{\mathfrak{g}}, \mathfrak{g}^{(0)}\) is reductive (cf. [23, Ch. 1, §6.4]).

Although not obvious, the vector spaces \(\mathfrak{g}^{(k)}\) are finite dimensional for all \(k \in \mathbb{Z}\).

**Proposition (2.3.2).** Let \(\mathfrak{g}\) be graded with respect to a spsa: \(\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{(k)}\).

Then \(\dim \mathfrak{g}^{(k)} < \infty\) for all \(k \in \mathbb{Z}\).

**Proof.** See Corollary to Proposition 5.11 in [3].

2.4. The entire theory is based on the following facts.

**Theorem (2.4.1).**
(a) \(Q_w P_w\) is open in \(G_w\)
(b) \(P_w \cap Q_w\) is trivial (i.e. the identity).

**Proof.**
(a) It is easy to see that the map
\[
\psi : \begin{cases} 
q_w + p_w & \to G_w \\
q + p & \mapsto (\exp q)(\exp p)
\end{cases}
\]
is a local diffeomorphism at the identity. From this it follows that \(Q_w P_w\) is open in \(G_w\).

(b) Let \(g \in Q_w \cap P_w\). For every \(x \in \mathfrak{g}\) of \(p\)deg \(= k\) we have: \(\max(p\deg(\text{Ad}(g)x-x)) \leq k-1\) and \(\min(p\deg(\text{Ad}(g)x-x)) \geq k\) thus \(\text{Ad}(g) = I\). We claim however that \(\text{Ad}|Q_w\) is injective. To see that, we consider \(g \in Q_w\) such that \(\text{Ad}(g) = I\). Then we write \(g = (\exp(a))\hat{g}\) where \(p\deg(a) = m\) and \(\hat{g}\) is generated by \(\exp\) with \(\max(p\deg(y)) \leq m-1\) for some negative \(m\). Now applying \(\text{Ad}(g)\) to an arbitrary \(p\)-homogeneous element \(x\) implies that \([x,a] = 0\). Thus \([g,a] = 0\) and \(a = 0\) follows. This proves the claim and part (b).
§3. Birkhoff and Bruhat Decompositions

3.1 We retain the notation of §2. In particular, let \( G_w = G \) and \( g_w = g \) be as before and let \( P = P_w \) be a standard parabolic subgroup \( G \) with complement \( Q \) and \( p = \text{Lie } P, \ q = \text{Lie } Q \). From Corollary (2.2.3) we know that \( P \) and \( Q \) are integral subgroups of the connected Banach Lie group \( G \). By \( g^{fin}, p^{fin}, q^{fin} \) etc. we denote the finite linear combinations of the canonical generators, i.e. the Lie algebra in the sense of [10] or [8]. Recall from §1 and §2 that in our Banach topology we have

\[
(3.1.1) \quad g = g^{fin}, \ p = p^{fin}, \ q = q^{fin}.
\]

We also recall from 1.3 that by \( G^{fin}, P^{fin}, Q^{fin} \) etc. we denote the group generated by \( \{ \exp x_\alpha \} \), where \( \alpha \in \Delta^e \) and \( x_\alpha \in g_\alpha, \ g_\alpha \subset g^{fin}, p^{fin} \) and \( q^{fin} \) respectively. Then

\[
(3.1.2) \quad G = G^{fin}, \ P = P^{fin}, \ Q = Q^{fin}.
\]

From [23; Ch. 4, no.2.6] we know \( P^{fin} = P_X^{fin} \) for some \( X \subset \Pi = \{ \alpha_i \} \), where \( \Pi \) is a basis for the root system \( \Delta \) of \( g^{fin} \). For details and notation we refer to [23; Ch. 4] and [8].

We set \( U_\alpha = \exp g_\alpha, \alpha \in \Delta^e \). Then \( U_\alpha \subset G^{fin} \) and \( U_\alpha \) is closed in \( G \). Indeed, if

\[
\exp x_n \to A \in G
\]

then \( \exp x_n \) is a Cauchy sequence and thus

\[
\exp x_n(\exp x_m)^{-1} = \exp(x_n - x_m) \to I.
\]

We would like to conclude that \( x_n \) is a Cauchy sequence in \( g_\alpha \), which is closed. To this end we consider \( r_{mn} = \text{Ad}\exp(x_n - x_m)(h) = \exp\text{ad}(x_n - x_m)(h) = h + [x_n - x_m, h] + \epsilon \) where \( h \in \mathfrak{h} \). We know that \( r_{nm} \to h \). Moreover, \( \epsilon \) when expanded in terms of canonical degree has no components of degree zero or \( \text{deg}(x_n) = \text{deg}(x_m) \). This implies \( [x_n - x_m, h] \to 0 \). Consequently, \( x_n - x_m \to 0 \). Thus for some \( x \in g_\alpha, x_n \to x \) implying that \( \exp x_n \to \exp x = A \).

As in [8] we consider the subgroup \( U^{fin}_+ \) of \( G^{fin} \), the group generated by \( U_\alpha, \alpha \in \Delta^e_+ \), and similarly we define the subgroup \( U^{fin}_- \).

**Lemma (3.1.1).** \( U_+ = U^{fin}_+ \) and \( U_- = U^{fin}_- \) are connected closed Banach Lie groups, which are integral subgroups of \( G \) with Lie algebras \( g_+ \) and \( g_- \) respectively. Here \( g^{fin}_+ \) is generated by \( g_\alpha, \alpha \in \Delta^e_+ \) and \( g^{fin}_- \) is generated by \( g_\alpha, \alpha \in \Delta^e_- \).

**Proof.** Since \( e_i, i = 0, \cdots, l \) belongs to a real root, we know that the vector spaces \( g_\alpha, \alpha \in \Delta^e_+ \), generate \( g^{fin}_+ \). Hence the closure generates \( g_+ \). Similarly we obtain \( g_- \). From 1.3, we know that \( g_+ \) and \( g_- \) are closed complemented subalgebras of \( g \). Therefore, by [23; Ch. 3, §6, Theorem 2], there are connected integral subgroups \( \hat{U}_\pm \) such that \( \text{Lie } \hat{U}_\pm = g^{fin}_\pm \). Next we show that \( \hat{U}_\pm \) is closed in \( G \). Let \( u_m \in \hat{U}_+, \ u_m \to a \in G \). Then we know that \( u^{-1}_m a \) is in an arbitrary small neighborhood of \( I \), provided \( m \) is sufficiently large. It is easy to see that \( \text{Ad}(u_m) - I \) is an operator on \( g \) which maps \( r_k = \bigoplus \mathfrak{g}_j \) into \( r_{k+1} \). Therefore \( \text{Ad}(a) - I \) has the same property. But for \( u^{-1}_m a \) in a sufficiently small neighborhood of \( I \),
we know \(u_m^{-1}a = \exp y_m\). Now the degree shifting property mentioned above applied to 
\(Ad(\exp y_m) = \exp ady_m\) implies \(y_m \in \mathfrak{g}_+\). Therefore, \(a = u_m \exp y_m \in \hat{U}_+\). This shows 
\(\hat{U}_+\) is closed. Similarly one sees that \(\hat{U}_-\) is closed. This shows also \(\hat{U}_\pm = U_{\pm}^{fin} = U_\pm\). \]

A proof similar to the one above works for

**Theorem (3.1.2).** The groups \(B_w, P_w,\) and \(Q_w\) are closed in \(G_w\).

As a consequence we see that \(B_w, P_w,\) and \(Q_w\) are closed integral subgroups of \(G_w\). We show that those groups actually are Banach Lie subgroups in the sense of [23; Ch. 3, §1.3]. First we prove more generally

**Theorem (3.1.3).** Let \(a\) be a closed Lie subalgebra of \(\mathfrak{g}_w\) and \(b\) a closed subspace of \(\mathfrak{g}_w\) such that \(a + b = \mathfrak{g}_w\), \(a \cap b = 0\). Denote by \(A\) the integral subgroup associated with \(a\). Assume \(A \cap \exp V^0 = \{I\}\) for some open neighborhood \(V^0\) of \(0\) in \(b\). Then \(A\) is a Banach Lie subgroup of \(G_w\).

**Proof.** Let \(U\) denote an open neighborhood of \(0\) in \(a\), and \(V \subset V^0\) an open neighborhood of \(0\) in \(b\). We can assume that \(U \times V \mapsto \exp \hat{U} \exp V = R\) is a diffeomorphism. Consider now \(a \in A \cap R\). Then \(a = \exp \hat{a} \exp b, \hat{a} \in U, \hat{b} \in V\); therefore \(\exp(-\hat{a})a = \exp \hat{b} \in A \cap \exp(b)\), whence \(\hat{b} = 0, a = \exp \hat{a}\). This shows \(A \cap R = \exp U\). Now apply [23; Ch. 3, §1, Proposition 6] and obtain the claim.

**Corollary (3.1.4).** Let \(a\) and \(b\) be closed Lie subalgebras of \(\mathfrak{g}_w\) such that \(a + b = \mathfrak{g}\) and \(a \cap b = 0\). Assume also that for the associated integral subgroups \(A\) and \(B\) we have \(A \cap \exp V^0 = \{I\}\) and \(B \cap \exp U^0 = \{I\}\) where \(U^0\) and \(V^0\) are some open neighborhoods in \(a\) and \(b\) respectively. Then \(A\) and \(B\) are Banach Lie subgroups of \(G_w\). If \(A \cap B = \{I\}\), then \(AB = \{uv : u \in A, v \in B\} \cong A \times B\).

**Proof.** The first statement follows from the Theorem above. For the second statement we consider the map \(A \times B \mapsto G, (u, v) \mapsto uv\). Clearly, this map is analytic, and since \(a + b = \mathfrak{g}_w\) we see that \(AB\) is open in \(G\). Using \(U\) and \(V\) as in the proof of the Theorem above we see that the multiplication map is locally, around \((I, I)\) and \(I, I\), a diffeomorphism. Using translations we see that this true for any \((u, v)\) and \(uv\). Therefore it suffices to prove that the multiplication map is injective. It suffices to show that \(uv = I\) implies \(u = I = v\); but this follows from \(A \cap B = \{I\}\).

The above result and Theorem (2.4.1) imply that \(B_w, P_w,\) and \(Q_w\) are Banach Lie subgroups of \(G_w\).

**3.2** Next we consider the group \(H_{fin}\) generated by \(\exp h, h \in \mathfrak{h} = \mathfrak{g}_0^{fin}\). Since \(\mathfrak{h}\) is complemented in \(\mathfrak{g}\), \(H_{fin}\) is a finite dimensional, connected integral subgroup of \(G\) with Lie algebra \(\mathfrak{g}_0^{fin}\). However \(H_{fin}\) is closed in \(G\) and thus \(H_{fin}\) is a Lie subgroup of \(G\). This can be seen by a proof analogous to the one given for \(U_{\pm}\), or by Theorem (3.1.3). We thus set \(H = H_{fin}\). We denote by \(N_{fin}\) the normalizer of \(H\) in \(G_{fin}\). Subsequently we define \(W = N_{fin}/H\) and call it the Weyl group of \(G_{fin}\). We will show below that this is truly the Weyl group defined in terms of reflections acting on the affine Kac-Moody root system. In our setting, [8, Corollary 5] states:

**Theorem (3.2.1).**

(a) \(G_{fin} = U_{fin}^{fin}N_{fin}U_{fin}^{fin}\).

(b) \(G_{fin} = U_{fin}^{fin}N_{fin}U_{fin}^{fin}\).

(c) If \(g = uu', u, u' \in U_{fin}^{fin}, n \in N_{fin}\) then we can assume \(u \in nU_{fin}U_{fin}^{fin}\) and this decomposition is unique.
Remark.

(1) As mentioned in the introduction, we are primarily interested in loop algebras and loop groups. However, for the proof of many properties of the Banach Lie algebras and Lie groups used in this paper we will be using results on Kac-Moody Lie algebras and the associated groups. We follow [8, 24] and do not include the degree derivation in our Lie algebras. Instead, we use the root space decomposition and the principal grading induced from the full Kac-Moody algebra. For our purposes it would be therefore most convenient to use a Banach structure on the derived algebra of a Kac-Moody algebra, i.e. a central extension of a loop algebra. It is indeed possible to find such Banach structures and associated Banach Lie groups [11]. We find it therefore more appropriate for this paper to use only algebraic results for the central extension, to transport them via projection to our loop algebras and loop groups, and to extend them to our Banach loop algebras and groups. The only somewhat delicate point here is the definition of the projection map \( \pi \) on the group level. If one denotes by \( G' \) the group considered in [8] as opposed to \( G^\text{fin} \) considered in this paper, then

\[
\begin{array}{c}
G' \\
\downarrow \pi \\
G^\text{fin}
\end{array}
\]

\[
\rho' \quad G'/Z' \cong G^\text{fin}/Z^\text{fin} \subset G\ell(\mathfrak{g}^\text{fin} \oplus \mathbb{C}c)
\]

where \( Z^\text{fin} \) and \( Z' \) denote the centers of \( G \) and \( G' \) respectively, \( \rho' \) is the adjoint representation of \( G' \), and \( \rho \) denotes the “extended” adjoint representation [12; Proposition (4.3.3)].

Since we consider \( G \) simply connected, we need to lift results from \( G^\text{fin}/Z^\text{fin} \) to \( G^\text{fin} \). This will be straightforward in all the cases considered in this paper.

(2) Part(a) describes the Birkhoff decomposition of \( G^\text{fin} \), whereas part (b) describes the Bruhat decomposition of \( G^\text{fin} \). All the decompositions appearing in Theorem (3.2.1) are obtained by taking the projection of those of [8].

Lemma (3.2.2). \( W \cong \text{Weyl group of the corresponding full Kac-Moody algebra.} \)

Proof. We will use the superscript PK to distinguish between objects in our set up and those in [8]. Let us denote by \( \Pi \) the projection \( G^\text{PK} \to G^\text{fin} \). Then we have \( \Pi(N^\text{PK}) = N^\text{fin} \) and \( \Pi(H^\text{PK}) = H \) and the induced map \( W^\text{PK} \to W \) is surjective. To see that it is also injective we observe that \( \Pi^{-1}(H) = H^\text{PK} \).

We recall the definition of \( B^\text{fin} \) and \( B_w = \overline{B^\text{fin}} \) from §2.1. In what follows we will use \( B = B_+ = B_w \) and define \( B_- \) analogously. We note \( B_+ = HU_+ \) and \( B_- = HU_- \). We will also use parabolic algebras and groups \( \mathfrak{p}^\text{fin}, \mathfrak{p} = \overline{\mathfrak{p}^\text{fin}} \) and \( P^\text{fin}, P = \overline{P^\text{fin}} \) respectively. In view of 2.2 we know \( \mathfrak{p}^\text{fin} = \mathfrak{p}_X^\text{fin} \). Similarly we will use \( P_X^\text{fin}, \mathfrak{p}_X \) and \( P_X \).

Corollary (3.2.3). \( G^\text{fin} = \bigcup_{w \in W/W_X} U_-^\text{fin} w P_X^\text{fin} \), where \( W_X \) is the subgroup of \( W \) generated by \( \{r_\alpha; \alpha \in X\} \). Moreover, the union above is disjoint.

Proof. Let \( W' \subset W \) denote a set of representatives of \( W/W_X \). Then \( W = W'W_X \). From the Theorem above we know \( G^\text{fin} = \bigcup_{w \in W'} U_-^\text{fin} w' w U_+^\text{fin} \). Moreover, \( P_X^\text{fin} = \)}
Proposition (3.3.1) holds. 

\[ B^+_\text{fin} W X B^+_\text{fin} \] by [23, Ch. 4; §2.5]. Hence \( G^{\text{fin}} = \bigcup_{w' \in W'} U^\pm_{\text{fin}} w' P^\text{fin} \). To see that this union is disjoint, we note that for \( B^\pm_{\text{fin}} = H U^\pm_{\text{fin}} \) we have \( B^\pm_{\text{fin}} w B^\pm_{\text{fin}} r_i \subset B^\pm_{\text{fin}} w B^\pm_{\text{fin}} \cup B^\pm_{\text{fin}} w_r r_i B^\pm_{\text{fin}} \), where \( r_i = \exp j \exp(-e_i) \exp f_i \), as mentioned in [8]. With this, one obtains mutatis mutandis, [23, Ch.4, §2, Lemma 1] and then [23, Ch.4, §2.5, Remark 2], proving the claim.

**Remark.** We note that for \( w = 1 \) we obtain \( U^\pm_{\text{fin}} \cdot P^\text{fin} = Q^\text{fin} P^\text{fin} \).

### 3.3

The following technical result will be useful. We define \( \hat{G}_X \) as the connected subgroup of \( P \) generated by \( x_{\pm \alpha} \in \mathfrak{g}^\text{fin}_{\pm \alpha}, \alpha \in X \), with Lie algebra \( \hat{\mathfrak{g}}_X \). We denote by \( Q^+_{\hat{X}} \) the connected Banach subgroup of \( P = P_X \) with Lie algebra \( \mathfrak{q}^+_X \), where \( \mathfrak{q}^+_X \) is the natural complement of \( \hat{\mathfrak{g}}_X \) in \( \mathfrak{p} \).

Clearly, we have

\[ \mathfrak{q}^+_X = \mathfrak{a}_Q + \mathfrak{q}^{++}_X, \]

where \( \mathfrak{a}_Q = \mathfrak{h} \cap \mathfrak{q}^+_X \) and \( \mathfrak{q}^{++}_X = \mathfrak{g}^{(+)}. \) Then \( \hat{\mathfrak{g}}_X + \mathfrak{a}_Q = \mathfrak{g}^{(0)} \) and \( \mathfrak{q}^{++}_X \). Moreover, \( [\hat{\mathfrak{g}}_X, \mathfrak{a}_Q] = 0 \) holds.

**Proposition (3.3.1).** Let \( P = P_X \) be a parabolic subgroup of \( G \) and \( w \in W \). Then

(a) \( P = P_X = \hat{G}_X Q^+_{\hat{X}} = Q^+_{\hat{X}} \hat{G}_X \cong \hat{G}_X \times Q^+_{\hat{X}}. \)

(b) The stabilizer \( U^w_\alpha \) in \( U_\alpha \) of \( wP \in G/P \) is \( U^w_\alpha = U_\alpha \cap w P^{-1}. \) Moreover, \( \dim U^w_\alpha < \infty. \)

(c) There exists a closed subgroup \( V^w_\alpha \) of \( U_\alpha \) such that group multiplication induces a diffeomorphism \( U_\alpha \cong U^w_\alpha \times V^w_\alpha. \)

We relegate the proof of this proposition to Appendix A.

### 3.4

In this section we state the Birkhoff decomposition of \( G \). We refer the reader to [12, Ch.8.6] for more details.

**Theorem (3.4.1).**

(a) Let \( g \in G \). Then the orbit \( B_- g B_+ \) contains a unique \( w \in W \).

(b) \( G = \bigcup_{w \in W} B_- w B_+ \) is a disjoint union.

### 3.5

We consider the sets \( U_- w P, \ w \in W/W_X \). In general, these sets are not closed in \( G \). To investigate their closure we use the Bruhat order “\(<\)” on \( W \), i.e. the partial order generated by \( r_{i_1} \cdots r_{i_{s-1}} r_{i_{s+1}} \cdots r_{i_k} \prec w, 1 < s \leq k \), where \( w = r_{i_1} \cdots r_{i_k} \) is a reduced expression.

We will also need the notion of an \( X \)-reduced element [23 Ch.4, §1, exercise 3]: \( w \in W \) is called \( X \)-reduced if it has minimal length in the coset \( w W_X \). In particular, \( w \) is \( X \)-reduced iff \( w r_i \) is larger than \( w \) for all \( r_i \in X \). Moreover, any \( w \in W \) can be written in a unique way in the form \( w = w' q \), where \( w' \) is \( X \)-reduced and \( q \in W_X \). Hence a set \( W' \) of representatives of \( W/W_X \) can be chosen as the set of \( X \)-reduced elements in \( W \).

With this notation we can prove for \( P = P_X \)

**Theorem (3.5.1).**

\[
G = \bigcup_{w \in W \text{ is } X\text{-reduced}} B_- w P = \bigcup_{w \in W/W_X} B_- w P.
\]
Proof. Using Corollary 3.2.3 we have
\[ G = \bar{G}^{\text{fin}} = \bigcup_{w \in W} B^w_{\text{fin}} P^w_{\text{fin}} \supset \bigcup_{w \in W} B_{-w} P \supset \bigcup_{w \in W} B_{-w} B_+ = \bar{G} = G, \]
where we have also used Theorem 3.4.1. This established \( G = \bigcup B_{-w} P \). Since \( W_X \in P \) we get the claim.

Next we want to describe the closure of \( C_X(w) = B_{-w} P_X \). For this we recall the definition of \( U_{\alpha} \) from 3.1. Moreover, we set \( h_i = [f_i, e_i] \) and \( H_i = \exp C h_i \). We would also like to note that the Proposition below is most likely true for any norm given by a weight \( w \). However, at this point we are only able to prove it for the weights \( w \) satisfying our usual assumptions and, in addition,
\[ \sum_{n \in \mathbb{Z}} (1 + |n|) |f_n|^2 \leq C \| f \|_w, \quad f \in A_w, \]
for some \( C > 0 \).

**Proposition (3.5.2).** For every \( w \in W/W_X \) we have
\[ \overline{C_X(w)} = \bigcup_{M} C_X(w') \]
where \( M = \{ w' \in W/W_X, \ w' \succeq w \} \).

We relegate the proof of this proposition to Appendix A.

**Corollary (3.5.3).**
(a) \(QP\) is dense in \( G\).
(b) If \( w \in W, w \neq I \), then \( B_{-w} P \) has nonzero finite codimension in \( G\).

**Proof.** (a) It is easy to see that \( B_{-} = \bar{Q} P \) holds. Indeed, \( B_{-} = \bar{Q} \hat{B}_{-} \) as in Proposition (3.3.1). But \( \hat{B}_{-} = P \). Then, the Proposition above shows for \( w = 1 \)
\[ \overline{Q P} = \overline{B_{-} P} = \bigcup_{w' \geq 1} B_{-w'} B_+ , \]
and this is all of \( G \) by Theorem (3.4.1).
(b) Follows from Proposition (3.3.1).

3.6 The following result will be used to see that certain functions involved in our set-up are meromorphic in the flow variable. We recall (see [10]) that the cyclic element \( E \) of \( g \) is the sum of all canonical generators of (canonical) degree 1. We will review more facts about \( E \) in Sect.5.1. We continue to use the notation of 3.5 and the norm restriction used in Theorem (3.5.2).

**Theorem (3.6.1).** For every \( g \in G \) there exists some \( \varepsilon > 0 \) such that for all \( 0 < |t| < \varepsilon \) we have
\[ e^{tE} g \in Q P . \]
Proof. Clearly, the statement is trivial if \( g \not\in QP \). Hence we will assume \( g \not\in QP \). Then from Theorem (3.4.1) we know \( g = b_- wb_+ \) where \( b_\pm \in B_\pm \) and \( w \not\in I \). For our purpose we can assume \( b_+ = I \). Moreover we can assume \( b_- \in U_- \) and even \( b_- \in V^- \) by Proposition (3.3.1). Next we consider \( e^{tE}b_- \). Since \( b_- \in B_- B_+ \), for sufficiently small \( t \) we have \( e^{tE}b_- = u_-(t)u_+(t) \). We consider the closed subalgebra \( a = w^{-1}b_+w \) of \( g \). Then it is easy to see that

\[
a = a_- + a_+ ,
\]

where \( a_- = a \cap g_- \) and \( a_+ = a \cap b_+ \). Then \( a_\pm \) are closed subalgebras of \( a \) satisfying \( a_- \cap a_+ = 0 \). Moreover, for the corresponding integral subgroups \( A_\pm \) we have \( A_- \cap A_+ \subset U_- \cap B_+ = \{ I \} \). Therefore by Corollary (3.1.4) \( A_- \times A_+ \cong w^{-1}B_+w \). Thus

\[
e^{tE}b_-w = u_-(t)u_+(t)w = u_-(t)v_+(t)w q_+(t) ,
\]

where \( v_+(t) \in wU_-w^{-1} \cap U_+ \) and \( q_+(t) \in B_+ \). Using \([8, \text{Corollary 5}]\) and the fact that the factors of the factorization are analytic in \( t \) as well as \( u_-(0) = b_- \), we obtain \( v_+(0) = I \) and \( q_+(0) = I \). Hence we will assume \( g \not\in QP \). Then from Theorem (3.4.1) we know \( g = b_- wb_+ \) where \( b_\pm \in B_\pm \) and \( w \not\in I \). For our purpose we can assume \( b_+ = I \). Moreover we can assume \( b_- \in U_- \) and even \( b_- \in V^- \) by Proposition (3.3.1). Next we consider \( e^{tE}b_- \). Since \( b_- \in B_- B_+ \), for sufficiently small \( t \) we have \( e^{tE}b_- = u_-(t)u_+(t) \). We consider the closed subalgebra \( a = w^{-1}b_+w \) of \( g \). Then it is easy to see that

\[
a = a_- + a_+ ,
\]

where \( a_- = a \cap g_- \) and \( a_+ = a \cap b_+ \). Then \( a_\pm \) are closed subalgebras of \( a \) satisfying \( a_- \cap a_+ = 0 \). Moreover, for the corresponding integral subgroups \( A_\pm \) we have \( A_- \cap A_+ \subset U_- \cap B_+ = \{ I \} \). Therefore by Corollary (3.1.4) \( A_- \times A_+ \cong w^{-1}B_+w \). Thus

\[
e^{tE}b_-w = u_-(t)u_+(t)w = u_-(t)v_+(t)w q_+(t) ,
\]

where \( v_+(t) \in wU_-w^{-1} \cap U_+ \) and \( q_+(t) \in B_+ \). Using \([8, \text{Corollary 5}]\) and the fact that the factors of the factorization are analytic in \( t \) as well as \( u_-(0) = b_- \), we obtain \( v_+(0) = I \) and \( q_+(0) = I \). We claim that there exists some \( \varepsilon > 0 \) such that for all \( t \), \( 0 < |t| < \varepsilon \), we have

(1) \( v_+(t) \neq I \),
(2) \( v_+(t)w \in B_-B_+ . \)

To prove (1) we recall that \( v_+(t) \) is analytic in \( t \). Therefore, if (1) were wrong, then

(3) \( e^{tE}b_-w = u_-(t)w q_+(t) \) for all sufficiently small \( t \). This implies

\[
w^{-1}b_-^{-1}e^{tE}b_-w = w^{-1}b_-^{-1}u_-(t)w q_+(t) ,
\]

whence

(4) \( w^{-1}r_-(t)w q_+(t) = \exp\{ t \ Ad(w^{-1}b_-^{-1})E \} \)

where \( r_-(t) = b_-^{-1}u_-(t) \). In particular, we have \( r_-(0) = I = q_+(0) \).

Differentiating at \( t = 0 \) we obtain \( w^{-1}r_-(0)w + \dot{q}_+(0) = Ad(w^{-1}b_-^{-1})E \). Conjugating with \( w \) yields

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(5) \[ \hat{r}_-(0) + w\hat{q}_+(0)w^{-1} = b_-^{-1}Eb_- . \]

We note that \( b_-^{-1}Eb_- = E + S, \) where \( c \deg S \leq 0. \)

Next we decompose \( q_+(0) = h + u_+ , \) where \( h \in h \) and \( u_+ \in g_+ . \) Then (5) implies \( wu_+w^{-1} = E + T, \) where \( c \deg T \leq 0. \)

Now we write \( u_+ = \sum_{\alpha \in \Delta_+} v_\alpha, \) then \( wu_+w^{-1} = \sum_{\alpha \in \Delta_+} wv_\alpha w^{-1} = E + T . \) We know \( wv_\alpha w^{-1} \in g_{w(\alpha)}. \) We set \( A = \{ \alpha \in \Delta_+: w(\alpha) = \alpha_i \text{ for some } i \} \) and \( B = \Delta_+ \setminus A. \) Then \( wu_+w^{-1} = \sum c_\alpha e_{\alpha_i} + \sum_{\alpha \in B} wv_\alpha w^{-1}, \) and all \( c_\alpha \neq 0. \) In fact \( c_\alpha = 1. \) Therefore \( u_+ = \sum_{\alpha \in A} c_\alpha e_{w^{-1}(\alpha_i)} + \sum_{\alpha \in B} v_\alpha. \) This implies \( w^{-1}(\alpha_i) \in \Delta_+ \) for all \( i. \) But now [10, Lemma 3.11] shows \( w = 1, \) a contradiction. This proves (1).

To prove (2) note that from Theorem (3.4.1) we know that for every \( t \) we have \( v_+(t)wq_+(t) \in B_-w'B_+ \) for some \( w'. \) But then obviously \( (B_+wB_+) \cap (B_-w'B_+) \neq \emptyset, \) whence \( w \succeq w', \) (see e.g. [12; Theorem (8.4.6)]). We claim that \( w \neq w', \) if \( v_+(t) \neq 1. \) Suppose \( w = w', \) then \( v_+(t)wq_+(t) \in (B_+wB_+) \cap (B_-w'B_-) = wB_+, \) where the last equality follows from [12; Theorem 8.4.5]. In particular \( w^{-1}v_+(t)w \in B_+. \) But we had chosen \( v_+ \) above so that \( w^{-1}v_+(t)w \in U_. \) This contradicts \( v_+(t) \neq 1. \) As a consequence we have for all \( 0 < |t| \leq \varepsilon \) that the corresponding \( w' \) satisfies \( w \succeq w' \) and \( w \neq w'. \) From the definition of \( \succeq \) it is clear that there are only finitely many \( w' \in W, \) \( w \neq w'. \) Therefore, there exists a sequence \( t_j \to 0, \)

\[ b_-^{-1}e^{tE}b_-w = s_-(t)w's_+(t), \text{ for all } t = t_j, j = 1, 2, \ldots \text{ with } w'. \]

If \( w' \neq I, \) then we multiply from the left by \( b_-^{-1}e^{tE}b_- \) and \( s_-(t)^{-1}. \) We obtain \( s_-(t_j)^{-1}b_-^{-1}e^{tE}b_-s_-(t_j)w's_+(t_j) = l(r). \)

Now we can apply the above argument to \( b_- = b_-s_-(t_j), \) and obtain a sequence \( r_{jk} \to 0, \) \( w' \succeq w'' \), such that \( l(r_{jk}) \in B_-w''B_+ \) and \( w' \neq w'' \). But then \( s_-(t_j)^{-1}b_-^{-1}e^{t_{jk}E}e^{t_jE}b_-w \in B_-w''B_+. \)

Hence for every \( g \in G \) there exists a sequence \( z_j \to 0 \) such that \( e^{z_jE}g \in B_-B_+. \) We use again the standard representation of \( G \) in \( GL_{res}. \) Then for \( e^{tE}g \) to be in \( B_-B_+ \) it is necessary and sufficient that \( \chi(t) \equiv \sigma(exp(tE))g\mathcal{H} \neq 0, \) see [6, §5] for notation. From what we have shown above, the holomorphic function \( \chi \) vanishes for \( t = 0, \) but does not vanish identically. This proves (2). Finally, similar to Proposition (3.3.1) one can show \( B_- = Q(G\chi)_-. \) Therefore \( e^{tE}g \in B_-B_+ = QP \) for all \( 0 < |t| < \varepsilon. \)

3.7 We present two theorems for later use. Denote by \( \Gamma \) the connected Banach Lie group with Lie algebra \( Ker ad E \) (this is a closed complemented subalgebra, therefore \( \Gamma \) exists by [23, Ch.3, §6, Thm.2]). \( \Gamma \) is generated by \( exp T, \) \( T \in \text{Lie } \Gamma = Ker ad E. \) Thus \( (Ad(exp T))E = exp(ad T)E = E, \) i.e. \( (Ad g)E = E \) for all \( g \in \Gamma. \)

Lemma (3.7.1). \( \Gamma \) is closed.
Proof. Assume $g_n \to g$, $g_n \in \Gamma$. Then $Ad(g)E = \lim Ad(g_n)E = E$. Moreover, since $g^{-1}_n g \to I$ we know $g^{-1}_n g = \exp t_n$ for $n \geq n_0$, $n_0$ sufficiently large. As a consequence, $Ad(\exp(t_n))E = \exp(\text{ad}(t_n))E = E$.

Now consider $\exp(tE)\exp(t_n)\exp(-tE) = \exp(tE)(\exp(t_n)\exp(-tE)\exp(-t_n))\exp(t_n) = \exp(tE)\exp(-t(Ad(\exp(t_n)))E)\exp(t_n) = \exp(tE)\exp(-tE)\exp(t_n) = \exp(t_n)$. Therefore $\exp(t_n) = \exp(Ad(\exp(tE))t_n) = \exp((\exp \text{ad} tE)t_n)$. Thus for $t$ sufficiently small, since $\exp$ is bijective in a small neighborhood of $I$ we obtain $t_n = \exp(\text{ad} tE)t_n$. This implies $[E, t_n] = 0$, whence $t_n \in \text{Ker ad} E$. As a consequence, $g = g_n \exp t_n \in \Gamma$. 

Finally, one defines $\Gamma_{\pm}$ for $(\text{Ker ad} E)_{\pm}$. Note that by the above argument $\Gamma$, $\Gamma_+$, and $\Gamma_-$ are closed in $G$. Also note, these groups are abelian and $\Gamma = \Gamma - \Gamma_+$. 

Before we state and prove the next two theorems we formulate another decomposition of $\mathfrak{g}$. Namely: $\mathfrak{g} = \mathfrak{s} + \mathfrak{u}$, where $\mathfrak{s}$ is the principal Heisenberg algebra and $\mathfrak{u}$ is its orthogonal, graded complement. We also fix from now on a parabolic $p$ and write $q$ for its natural complement.

Next we set $F = \sum_i f_i$, the sum of all generators with $cdegree = 1$. We also note that $\text{ad} E$ and $\text{ad} F$ are invertible on $\mathfrak{u}$.

**Theorem (3.7.2).** Let $g \in Q$ and assume $e^{tE}ge^{-tE} \in U_-$ for all $t$ in some neighborhood of $t = 0$. Then $g \in \Gamma_-$. 

**Proof.** We will use for our algebras and groups the coordinates as in §14.3 of [10]. We would like to remark also that the proof of semisimplicity of $E$ in [10, Prop.14.2] carries over to the extension of $ad E$ to the loop algebra $gl(n)$. The nice thing about that particular choice of coordinates is that the canonical generators $e_i, f_i, h_i$ become homogeneous functions of $\lambda$ of degree 1, in particular $E = \lambda \hat{E}$. Consequently, expanding in terms of powers of $\lambda$ we can write $g = a_0 + a_{-1}/\lambda + a_{-2}/\lambda^2 \ldots$. The hypothesis implies that $(ad(E))^m g$ is analytic in $1/\lambda$ for all $m \geq 0$. In particular, we get $ad(E)a_0 = 0, (ad(E))^2a_{-1} = 0, \ldots, (ad(E))^{(k+1)}a_{-k} = 0, \ldots$. We claim that this implies that $a_{-k} \in \text{Ker ad}(E)$ for $k \geq 2$. We write $ad(E)^{k+1}a_{-k} = 0$ as $(ad(E))(ad(E))^{k}a_{-k} = 0$. Thus $ad(E)^{k}a_{-k} \in \text{Im ad}(E) \cap \text{Ker ad}(E) = \{0\}$ hence by the remark above $(ad(E))^{k}a_{-k} = 0$. Iterating this procedure we obtain that $ad(E)a_{-k} = 0$ for any positive integer $k$.

Next we prove similar to the last theorem:

**Theorem (3.7.3).** Let $g \in P$ and assume $e^{tF}ge^{-tF} \in P$ for all $t$ in some neighborhood of $t = 0$. Then $g \in \Gamma \cap P$. 

**Proof.** We use the same coordinates as in the previous theorem. In particular, $F = \lambda^{-1}\hat{F}$, where $\hat{F}$ is $\lambda$ independent. Furthermore, there exists a positive integer $k$ such that for every $g \in P$, $\lambda^kg$ is analytic in $\lambda$. Thus the hypothesis of the present theorem can be restated as $\lambda^k(e^{tF}ge^{-tF}) = e^{tF}(\lambda^kg)e^{-tF}$ is analytic in $\lambda$. The rest of the argument is the same as in the previous theorem with $\lambda$ replacing $1/\lambda$ and $F$ replacing $E$.

**§ 4. More Banach Lie groups and subgroups**

In this Section we want to introduce the “ingredients” which we will use later for factorization.

4.1 Let $G$ be a Banach loop group with Lie algebra $\mathfrak{g}$. In particular, $G$ and $\mathfrak{g}$ consist of functions defined on the unit circle $S^1$ with values in $\mathbb{C}^{n \times n}$ for some $n$.

For $R > 0$ we set

\[(4.1.1) \quad S^R = \{ z \in \mathbb{C} : |z| = R \} \]
and define

\[(4.1.2) \quad G^R = \{g^R : S^R \to \hat{G}, \lambda \mapsto g^R(R\lambda) \in G \quad \text{for} \quad \lambda \in S^1\} \]

Similarly we define $G^r$, $g^R$ and $g^r$. We will always assume $0 < r \leq 1 \leq R < \infty$. It is easy to see that $G^R \to G$, $g^R \mapsto g(\lambda) = g^R(R\lambda)$, $\lambda \in S^1$, is an isomorphism of groups. This way $G^R$ inherits naturally a Banach Lie group structure from $G$. The need for using two distinct circles is demonstrated clearly in Proposition 5.2 and Theorem 5.3.1 in [15].

Let $p$ be a standard parabolic subalgebra of $\mathfrak{g}$ and $q$ its natural complement. By $P$ and $Q$ we denote the corresponding connected Banach Lie subgroups. Via (4.1.2) we thus obtain Banach Lie groups $P^R, P^r, Q^R$ and $Q^r$ as well.

For this paper the following Banach Lie groups are particularly important:

\[(4.1.3) \quad \mathcal{H} = G^R \times G^r, \]

\[(4.1.4) \quad \mathcal{H}_- = Q^R \times P^r, \]

\[(4.1.5) \quad \mathcal{H}_+ = \left\{(g_1, g_2) \in \mathcal{H} : g_1 \in G^R \text{and} g_2 \in G^r \text{have the same holomorphic extension in the annulus} \quad r < |z| < R \right\}. \]

In particular, if $r = R$, then $\mathcal{H}_+$ is just the diagonal in $\mathcal{H} = G \times G$. The corresponding Lie algebras will be denoted by $\mathfrak{h}, \mathfrak{h}_-$ and $\mathfrak{h}_+$ respectively. On Fig.1 below we indicate the regions of the complex plane $z$ where the respective groups live.
In what follows we will frequently use the facts listed below.

**Theorem (4.2.1).** $H_+$ and $H_-$ are connected Banach Lie subgroups of $H$. Moreover, $H_+$ and $H_-$ have the following properties:

(a) $H_- H_+$ is open and dense in $H$.

(b) $H_- H_+$ is analytically diffeomorphic with $H_- \times H_+$.

(c) $H_+$ and $H_-$ are closed in $H$.

**Proof.** The first part of the theorem and Item (c) follow from Corollary (3.7.4) and the lemma below.

To prove (a) and (b) we observe that $h_- + h_+ = h$ and $h_- \cap h_+ = \{0\}$ holds [15]. From this it follows that the map

$$\begin{cases} h_- + h_+ & \rightarrow H \\ (h_-, h_+) & \mapsto \exp h_- \exp h_+ \end{cases}$$

is a local diffeomorphism at 0; therefore $H_- H_+$ is open in $H$ and, locally at the identity, analytically diffeomorphic with $H_- \times H_+$.

For general $h_- h_+ \in H_- H_+$, consider a neighborhood $U$ of $h_- h_+$ such that $h_-^{-1} U h_+^{-1}$ can be mapped diffeomorphically into $H_- \times H_+$. From this, (b) follows.

To see that $H_- H_+$ is dense in $H$, we pick an arbitrary $g = (g^R, g^r) \in H$. Since $QP$ is open and dense in $G$, in every neighborhood of $g$ we can find a $\tilde{g}$ such that

$$(\tilde{g}^r)^{-1} \in Q^r P^r.$$ 

Hence we may assume that $g^r \in P^r Q^r$. Write $g^r = p^r q^r$. Then

$$g = (g^R(\mu), g^r(\lambda)) = (g^R(\mu) q^r(\mu)^{-1}, p^r(\lambda)) (q^r(\mu), q^r(\lambda)).$$
If necessary, replace \( g^R \) by \( \hat{g}^R \) arbitrarily close to \( g^R \) such that
\[
\hat{g}^R(\mu)q^r(\mu)^{-1} = \hat{q}^R(\mu)\hat{p}^R(\mu) \in P^R P^R .
\]
Then we obtain
\[
(\hat{g}^R(\mu)q^r(\mu)^{-1}, p^r(\lambda)) = (\hat{q}^R(\mu)\hat{p}^R(\mu), p^r(\lambda))
\]
\[
= (\hat{q}^R(\mu), p^r(\lambda)\hat{p}^R(\lambda)^{-1})(\hat{p}^R(\mu), \hat{p}^r(\lambda)) ,
\]
where we used that \( P^R \) naturally embeds in \( P^r \). Thus \( (\hat{g}^R(\mu), g^r(\lambda)) \in H_+ \) and the proof is complete. 

**Lemma (4.2.2).** \( H_- \cap H_+ = \{ I \} \).

**Proof.** This is a coordinate dependent proof. We use the coordinates introduced earlier following §14.3 of [10]. Let \( (g_1, g_2) \in H_- \cap H_+ \). Since \( g_1 \in Q^R \), the Laurent expansion of \( g_1(\lambda) \) about \( \lambda = 0 \) contains only nonpositive exponents. Therefore, there exists some \( h = \sum h_n z^n \) such that \( \sum h_n R^n \lambda^n = g_1 \in Q \) and \( \sum h_n r^n \lambda^n = g_2 \in P \). Since \( h_n = 0 \) for \( n \geq 0 \), \( \sum h_n z^n \) converges for \( |z| \geq r \). Moreover, \( \sum h_n R^n \lambda^n \in P \cap Q = \{ I \} \).

**Remark.** The above section generalizes the setup in [15] to general loop groups as well as more general subgroups.

§5. Factorization

5.1 In this section we recall some facts about the cyclic element of \( g \), all of which can be found in [10]. By the cyclic element we mean \( E := \sum e_i \in g \), the sum of all canonical generators of canonical degree 1.

Its centralizer, \( s := \text{Cent}_g E = \text{Ker ad } E = \{ F \in g : [E, F] = 0 \} \), is an abelian subalgebra of \( g \), which is graded with respect to the canonical grading:

\[
(5.1.1) \quad s = \bigoplus_{k \in \mathbb{Z}} s_k .
\]

The integers \( k \), for which \( s_k \neq \{0\} \), are called the exponents of \( g \). If \( k \) is an exponent of \( g \), then \( \dim \mathfrak{s}_k \) is called the multiplicity of \( k \).

It turns out that the multiplicity is always one, with the exception of \( D_{2m}^{(1)} \), \( m \geq 2 \) [10, Ch.14]. Nevertheless the exponent 1 has always multiplicity one and the space \( s_1 \) is spanned by \( E \). (Remark 14.2 and table \( E_0 \) in [10]).

An important feature of \( E \) is that \( \text{ad } E : g \to g \) is bijective on its image, or equivalently,

\[
(5.1.2) \quad g = \text{Im ad } E \oplus \text{Ker ad } E .
\]

An easy proof of this can be deduced by extending the argument given by Kac in [10, Prop. 14.2] for the case of \( g \) simple and finite dimensional. Finally, we remark that \( \text{Im ad } E \) is graded with respect to the canonical grading.

5.2 For every integer \( k \) we choose a nonzero element \( E_k \in s_k \), if possible, and let \( E_k = 0 \) otherwise. We will always assume \( E_1 = E \). Then

\[
(5.2.1) \quad \{ \cdots, E_{-2}, E_{-1}, E_0, E_1, E_2, \cdots \}
\]
spans $s$ in case $g$ is not of type $D^{(1)}_\ell$, $\ell \geq 4$, $\ell$ even. In that case we choose two linearly independent elements $E_{\ell-1}, E'_{\ell-1}$. Note that $0$ is never an exponent; thus $E_0 = 0$. Nevertheless we include it for the sake of simplicity of notation.

We set

\[(5.2.2) \quad \underline{t} := (\cdots, t_{-2}, t_{-1}, t_0, t_1, t_2, \cdots)\]

for $t_k \in \mathbb{R}$ and $k \in \mathbb{Z}$,

\[(5.2.3) \quad \mathbb{R}^\mathbb{Z} = \{\underline{t} : \sum_{k \in \mathbb{Z}} t_k E_k \in g\}.

As above, we will double the coefficient $t_s$ if $g$ is of type $D^{(1)}_\ell$ and the exponent $s$ is of multiplicity two.

Define $E^R_k \in g^R$ as usual via

\[(5.2.4) \quad E^R_k := E_k(\mu) \quad \text{for} \quad \mu \in S^R,

Similarly we define $E^r_k \in g^r$.

For this paper the following action of $\mathbb{R}^\mathbb{Z}$ on $\mathcal{H}$ will be of particular importance.

\[(5.2.5) \left\{ \begin{array}{l}
\mathbb{R}^\mathbb{Z} \times \mathcal{H} \rightarrow \mathcal{H} \\
(t, (h_1, h_2)) \mapsto \underline{t}(h_1, h_2),
\end{array} \right.

\[(5.2.6) \quad \underline{t}(h_1, h_2) = \left( \exp \left( \sum_{k \in \mathbb{Z}} t_k E^R_k \right) h_1, \exp \left( \sum_{k \in \mathbb{Z}} t_k E^r_k \right) h_2 \right).

Note that $E^R_k$ and $h_1$ depend on $\mu \in S^R$, whereas $E^r_k$ and $h_r$ depend on $\lambda \in S^r$.

5.3 We recall from Theorem(4.2.1) that $\mathcal{H}_{-} \mathcal{H}_{+}$ is open and dense in $\mathcal{H}$. Therefore, if $(h_1, h_2) \in \mathcal{H}_{-} \mathcal{H}_{+}$, then also $\underline{t}(h_1, h_2) \in \mathcal{H}_{-} \mathcal{H}_{+}$ for all $\underline{t}$ in an open neighborhood of $\underline{t} = 0$. The proof of the following Proposition is in immediate adaptation of that in [15, Proposition 3.6].

**Proposition (5.3.1)** If $(h_1, h_2) \not\in \mathcal{H}_{-} \mathcal{H}_{+}$, then $\underline{t}(h_1, h_2) \in \mathcal{H}_{-} \mathcal{H}_{+}$ if $\underline{t}$ is contained in some open subset of $\{\underline{t} : t_1 \neq 0\}$.

So from now on we assume that $(h_1, h_2) \in \mathcal{H}_{-} \mathcal{H}_{+}$. Keeping this in mind we consider the Riemann-Hilbert splitting

\[(5.3.1) \quad \underline{t}(h_1, h_2) = \left( g^-(\underline{t}, \mu)^{-1}, g^+(\underline{t}, \lambda)^{-1} \right) \left( b_R(\underline{t}, \mu), b_r(\underline{t}, \lambda) \right),

such that the first factor lies in $\mathcal{H}_{-}$, the second one in $\mathcal{H}_{+}$. Note that $b_R(\underline{t}, \mu)$ and $b_r(\underline{t}, \lambda)$ are analytic in $\underline{t}$, since $\underline{t}(h_1, h_2)$ is analytic and the map $\mathcal{H}_{-} \mathcal{H}_{+} \rightarrow \mathcal{H}_{-} \times \mathcal{H}_{+}$ is an analytic diffeomorphism.

Let $\partial_j$ denote $\frac{\partial}{\partial t_j}$ for any $j \in \mathbb{Z}$. 

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Differentiating (5.3.1) and using that $s$ is abelian, we get

\[ (\partial j b_R)b_R^{-1} = \Omega_j^R, \quad \text{where} \]

\[ \Omega_j^R = (\partial j g^-)(g^-)^{-1} + g^- E_j^R (g^-)^{-1} \quad \text{and} \]

\[ (\partial j b_r)b_r^{-1} = \Omega_j^r, \quad \text{where} \]

\[ \Omega_j^r = (\partial j g^+)(g^+)^{-1} + g^+ E_j^r (g^+)^{-1}. \]

We call $\Omega_j$ the $j^{th}$ potential and call it positive (or negative) iff $j > 0$ (or $j < 0$). The equality of the mixed derivatives of $b_R$, i.e. $\partial_{ij} b_R = \partial_{ji} b_R$, yields the equation

\[ \partial_i \Omega_j^R - \partial_j \Omega_i^R = [\Omega_i^R, \Omega_j^R], \]

\[ \partial_i \Omega_j^r - \partial_j \Omega_i^r = [\Omega_i^r, \Omega_j^r]. \]

The conditions (5.3.6) and (5.3.7) are usually called Zero-Curvature Conditions (ZCC).

Recall from Proposition (2.2.2) that every $p$ is defined by choosing a subset $X$ of simple roots which in turn defines a subset of the positive root system called $\tilde{\Delta}_+$. Let us define $s$ to be the maximal height of the roots in $\tilde{\Delta}_+$. Since $\tilde{\Delta}_+$ is finite, $s$ is a finite, positive integer. We set $s = 0$ if $X$ is empty. With this notation we can describe another important feature of the $\Omega_j^*, \,* = R$ or $r$:

**Proposition (5.3.2).**

(a) If $j > 0$, then there exists a Laurent polynomial $\Omega_j$ in $\lambda$ such that $\Omega_j|S^r = \Omega_j^r$ and $\Omega_j|S^R = \Omega_j^R$. Moreover, $\Omega_j \in p$ and $\Omega_j$ contains only components of the $p$-degree between 0 and $j$, and the canonical degree between $-s$ and $j$.

(b) If $j < 0$, then there exists a polynomial $\Omega_j$ in $\lambda^{-1}$ such that $\Omega_j|S^r = \Omega_j^r$ and $\Omega_j|S^R = \Omega_j^R$. Moreover, $\Omega_j \in q$ and $\Omega_j$ contains only components of the $p$-degree between $j$ and $-1$, and the canonical degree between $j - s$ and $-1$.

**Proof.** (a) From (5.3.5) we see that $\Omega_j^r$ is in $p^r$ for $r > 0$. From (5.3.4) we know that $\Omega_j^r$ can be extended to the region between $S^r$ and $S^R$. Now (5.3.3) and (5.3.2) show that only finitely many $p$-degrees (resp. canonical degrees) can occur in $\Omega_j^r$. The rest is a matter of comparing $\Omega_j^r$ with $\Omega_j^R$. For example, in order to count the canonical degrees we count the powers of $\lambda$ appearing simultaneously in $\Omega_j^r$ and $\Omega_j^R$. For example, for $j > 0$, $\Omega_j^R$ has degrees $\leq j$ whereas $\Omega_j^r$ has degrees $\geq -s$ on account of the first term in (5.3.5). All other cases are dealt with analogously. 


§6. Systems of Partial Differential Equations obtained from Factorization

6.1 In this section we derive systems of PDE’s from the zero-curvature conditions (5.3.6) and (5.3.7). We treat here the case \( j > 0 \) for which we show that all potentials \( \Omega_j, j > 0 \), can be expressed in terms of a certain number of functions parametrizing \( \Omega_1 \). We give an example of the generalized Drinfeld-Sokolov system generalizing the results of Drinfeld and Sokolov [3] for maximal parabolic subalgebras. We would like to add that the possibility of this extension is already mentioned in [3, p.2014] where it is referred to as the “partially modified” generalized KdV equations.

In this chapter we will always assume \((h_1, h_2) \in \mathcal{H}_- \mathcal{H}_+\) in the splitting equation (5.3.1). In view of Proposition (5.3.1) this is a very mild restriction.

The assumption above implies that there exists some \((c_R, c_r) \in \mathcal{H}_+\) such that

\[
(6.1.1) \quad h_1 = h^- c_R \quad \text{for some} \quad h^- \in Q^R
\]

\[
(6.1.2) \quad h_2 = h^+ c_r \quad \text{for some} \quad h^+ \in P^r.
\]

Then (5.3.1) shows

\[
(6.1.3) \quad g^-(0, \mu) = h^-(\mu) \quad \text{and} \quad g^+(0, \lambda) = h^+(\lambda).
\]

Therefore, for sufficiently small \(t\), there exist

\[
q(t, \mu) \in q^R \quad \text{and} \quad p(t, \lambda) \in p^r
\]

such that

\[
(6.1.4) \quad g^-(t, \mu) = h^-(\mu) \exp (q(t, \mu)),
\]

\[
(6.1.5) \quad g^+(t, \lambda) = h^+(\lambda) \exp (p(t, \lambda)).
\]

Using this, the potentials defined in (5.3.3) and (5.3.5) can be expressed as

\[
(6.1.6) \quad \Omega_j^R = Ad(h^-)((\partial_j e^q)e^{-q} + e^q E_j^R e^{-q}),
\]

and

\[
(6.1.7) \quad \Omega_j^r = Ad(h^+)((\partial_j e^p)e^{-p} + e^p E_j^r e^{-p}),
\]

6.2 As outlined above, we are trying to express all potentials \( \Omega_j^R \) resp. \( \Omega_j^r, j > 0 \) in terms of a set of basic, “independent”, functions. To achieve this, we will make use of the following.

**Lemma (6.2.1).** Let \( h(t) \in \mathfrak{g} \). Then

\[
(6.2.1) \quad (\partial_j e^h)e^{-h} = \sum_{n \geq 1} \frac{1}{n!} (ad h)^{n-1} \partial_j h.
\]
In other words, \((\partial_j e^h)e^{-h} = \psi(\text{ad} h)\partial_j h\), where \(\psi : \mathbb{C} \to \mathbb{C}\) is the entire function defined by
\[
\psi(z) = \begin{cases} \frac{e^z - 1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}.
\]

**Remark.** For matrix groups, (6.2.1) can be proved by a straightforward induction. For a general proof, see [25, chapter II, Theorem 1.7] or [23, Ch. 3, §6, Proposition 12]. Note that for any Lie group, \((\partial_j e^h)e^{-h}\) is contained in its Lie algebra. In particular we have
\[
(6.2.2) \quad (\partial_j e^q)e^{-q} \in \mathfrak{q} \quad \text{and} \quad (\partial_j e^p)e^{-p} \in \mathfrak{p}.
\]

6.3 In this section we will show how every positive potential \(\Omega_j\) can be expressed as a differential polynomial in the components of \(\Omega_1\). To accomplish this, we need the following:

**Lemma (6.3.1).** For every sufficiently small \(q \in \mathfrak{q}\) there exist uniquely determined

\[ q_I \in (\text{Imad } E)_- = (\text{Imad } E) \cap \mathfrak{g}_- \quad \text{and} \quad q_K \in (\text{Kerad } E)_- = (\text{Kerad } E) \cap \mathfrak{g}_- \]

such that
\[
(6.3.1) \quad e^q = e^{q_I}e^{q_K}
\]

and \(q_I + q_K \in \mathfrak{q}\).

**Proof.** Recall (5.1.2): \( \mathfrak{g} = \text{Imad } E \oplus \text{Kerad } E\). Since \(\text{Imad } E\) and \(\text{Kerad } E\) are canonically graded, we have \(\mathfrak{g}_- = (\text{Imad } E)_- \oplus (\text{Kerad } E)_-\). On the other hand \(\mathfrak{q} \subset \mathfrak{g}_-\), therefore the map
\[
(6.3.2) \quad \begin{cases} (\text{Imad } E)_- + (\text{Kerad } E)_- & \to U_- \\ (q_I, q_K) & \mapsto e^{q_I}e^{q_K} \end{cases}
\]

is a local diffeomorphism. The last statement of the theorem is obtained by considering the curve \(e^{q_I}\) and using the Baker-Hausdorff formula for small \(t\).

We are now ready to prove

**Theorem (6.3.2).** For \(j > 0\) the potential \(\Omega_j\) is a universal \(\partial_1\)-differential polynomial in \(\Omega_1\) with rational coefficients.

**Remarks.** By \(y\) is a \(\partial_k\)-differential polynomial in \(x\)" we shall mean that each component of \(y\) is a polynomial in the components of \(x\) and its derivatives with respect to \(x\).

The word “universal polynomial” means that the coefficients of the polynomial are independent of \(\Omega_1\).

**Proof.** (a) We choose any \(h^- \in Q\) such that \(g^-(t) = h^- \exp(q)\) and \(\exp(q) = \exp(q_I)\exp(q_K)\). We then substitute \(g^-(t) = h^- \exp(q)\) into (6.1.4) and set \(j = 1\) in (6.1.6) to get
\[
(6.3.3) \quad \Omega_1^R = \text{Ad}(h^-)\{(\partial_1 e^{q_I})e^{-q_I} + e^{q_I}(\partial_1 e^{q_K})e^{-q_K} + e^{q_I}E^R e^{-q_I}\}.
\]

Since \(\text{Kerad } E^R\) is abelian, formula (6.2.1) shows
\[
(6.3.4) \quad (\partial_1 e^{q_K})e^{-q_K} = \partial_1 q_K.
\]
Using this and (6.1.7) we get

\[(6.3.5) \quad h^- \cdot \Omega_1^R = (\partial_1 e^{qI}) e^{-qI} + e^{qI} (\partial_1 q_K + E^R) e^{-qI}\]

where \(h^- \cdot \Omega_1^R = Ad^{-1}(h^-) \Omega_1^R\), and where we have used that \(q_K\) and \(E^R\) commute.

(b) We will show that \(q_I\) and \(\partial_1 q_K\) are determined by \(h^- \cdot \Omega_1\). To see this, we rewrite (6.3.5) as

\[(6.3.6) \quad (\partial_1 e^{-qI}) e^{qI} + e^{-qI} h^- \cdot \Omega_1^R e^{qI} = \partial_1 q_K + E^R.\]

In the next step we compare terms of the same canonical degree. For \(q_I, q_K \in g_-\) we may write:

\[(6.3.7) \quad q_I = q_{I,-1} + q_{I,-2} + \ldots,\]

\[(6.3.8) \quad q_K = q_{K,-1} + q_{K,-2} + \ldots.\]

Similarly, by Proposition (5.3.2), we have

\[(6.3.9) \quad h^- \cdot \Omega_1^R = (h^- \cdot \Omega_1^R)_1 + (h^- \cdot \Omega_1^R)_0 + (h^- \cdot \Omega_1^R)_{-1} + \ldots\]

We will prove by induction on the canonical degree \(m\) that \(q_I\) and \(\partial_1 q_K\) are determined by \(h^- \cdot \Omega_1^R\). For \(m = 1\) we obtain from (6.3.6) that \((h^- \cdot \Omega_1^R)_1 = E^R\) holds. For \(m = 0\) we get

\[[-q_{I,-1}, E^R] + (h^- \cdot \Omega_1^R)_0 = 0\]

Since \((h^- \cdot \Omega_1^R)_1 = E^R\) and \(ad E^R\) is bijective on its image, \(q_{I,-1}\) is uniquely determined by \(h^- \cdot \Omega_1^R\). To illustrate the procedure we consider the case \(m = -1\) separately. Here we have

\[\partial_1 q_{I,-1} + [-q_{I,-2}, E^R] + \frac{1}{2} [q_{I,-1}, [q_{I,-1}, E^R]]\]

\[= [q_{I,-1}, (h^- \cdot \Omega_1^R)_0] + (h^- \cdot \Omega_1^R)_{-1} = \partial_1 q_{K,-1}.\]

The sum

\[[q_{I,-2}, E^R] + \partial_1 q_{K,-1}\]

is therefore some \(\partial_1\)-differential polynomial in \((h^- \cdot \Omega_1^R)_0\) and \((h^- \cdot \Omega_1^R)_{-1}\). By projecting on \(Im ad E^R\) along \(Ker ad E^R\) and vice versa, we conclude that each term in the sum is a \(\partial_1\)-differential polynomial in \((h^- \cdot \Omega_1^R)_0\) and \((h^- \cdot \Omega_1^R)_{-1}\). Thus \(\partial_1 q_{K,-1}\) is a \(\partial_1\)-differential polynomial in \((h^- \cdot \Omega_1^R)_0\) and \((h^- \cdot \Omega_1^R)_{-1}\). Since \(ad E^R\) is bijective when restricted to its image, \(q_{I,-2}\) is a \(\partial_1\)-differential polynomial in \((h^- \cdot \Omega_1^R)_0\) and \((h^- \cdot \Omega_1^R)_{-1}\).

By the same token, for \(m \leq -1\), we obtain from (6.3.6) that

\[[q_{I,-i-1}, E^R] + \partial_1 q_{K,-i}\]
is some $\partial_1$-differential polynomial in $(h^- . \Omega^R_1)_0, (h^- . \Omega^R_1)_1, \cdots (h^- . \Omega^R_1)_{-i}$, and by repeating the argument given for $m = -1$ we see that

$$q_{l,-i-1} \quad \text{and} \quad \partial_1 q_{K,-i}$$

are $\partial_1$-differential polynomials in $(h^- . \Omega^R_1)_0, (h^- . \Omega^R_1)_1, \cdots (h^- . \Omega^R_1)_{-i}$. From the definition of $h^- . \Omega^R_1$ we immediately see that we can separate off the explicit dependence on $h^-$. As a result of this operation $q_{l,-i-1}$ and $\partial_1 q_{K,-i}$ become $\partial_1$-differential polynomials in $(\Omega^R_1)_0, (\Omega^R_1)_1, \cdots (\Omega^R_1)_{-i}$ and polynomials in the entries of $h$. Thus we can write $q_{l,-i}(h^-, \Omega^R_1)$ etc.

(c) After these preparations we are able to prove the theorem. From Proposition (5.3.2) we know that for $j > 0$, $\Omega_j$ has only components of nonnegative $p$-degree. Thus

$$(6.3.10) \quad \Omega^R_j = ((h^-) e^{q_j(h^-, \Omega^R_1)} E^R_j e^{q_j(h^-, \Omega^R_1)} (h^-)^{-1})^{(+)} ,$$

since $q_K$ and $E^R_j$ commute. Note that $\Omega^R_j$ is uniquely determined by $q_I$, which in turn is uniquely determined by $\Omega^R_1$. To see that $\Omega^R_j$ depends polynomially on $\Omega^R_1$, we use the fact that $\mathfrak{g}^1 \oplus \mathfrak{g}^0$ is a finite dimensional vector space and therefore any $x \in \mathfrak{g}_-$, such that $cdeg(x)$ is sufficiently negative, must be in $\mathfrak{q}$. Now, let us apply this remark to (6.3.10). We obtain that

$$\Omega^R_j = ((h^-) \sum_{i=0}^{N} \text{ad}^i(q_I(h^-, \Omega^R_1))(E^R_j)(h^-)^{-1})^{(+)}$$

$$= ((h^-) \sum_{i=0}^{N} \text{ad}^i(\sum_{l=1}^{N} q_I(h^-, \Omega^R_1)_{-l})(E^R_j)(h^-)^{-1})^{(+)},$$

for sufficiently large $N$. Thus $\Omega^R_j$ depends on finitely many $q_I(h^-, \Omega^R_1)_{-l}$, all of which are differential polynomials in $\Omega^R_1$ and polynomials in the entries of $h^-$. However, by repeating the whole argument for $h^-$ in a neighborhood of the identity we conclude that $q_I(\Omega^R_1), \partial_1 q_K(\Omega^R_1)$, in other words, both are constants as functions of $h^-$. Since the dependence of $q_{l,-i}$ on $h^-$ is polynomial, thus analytic, we obtain that $q_{l,-i}$ is a differential polynomial in $\Omega^R_1$, with no explicit dependence on $h^-$. Moreover, we used only linear operations (projections, commutators) to determine $q_I$, consequently, the coefficients in $q_I(\Omega^R_1)_{-l}$ are all rational numbers independent of $\Omega^R_1$. This completes the proof.

Below we will not use the superscripts $R$ or $r$ to distinguish the two circles appearing in our discussion. The context will clearly tell the reader if one needs to make a distinction. In particular we will not attach a superscript to $E$.

**Corollary (6.3.3).** The ZCC

$$(6.3.12) \quad \partial_j \Omega_i - \partial_i \Omega_j = [\Omega_j, \Omega_i], \quad i, j > 0 ,$$

is a system of equations for the scalar components of $\Omega_1$. In particular, for $i = 1$ we obtain a system of evolution equations for the scalar components of $\Omega_1$.  

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We will discuss (6.3.12) for \( i = 1 \) in more detail later. There, it will be important to know a priori how many equations we expect to obtain. To this end we consider

(6.3.13) \[ \mathfrak{U} = [q^{(-1)}, E^{(1)}], \]

where the superscript denotes as usual the \( p \)-degree. We note that

\[ (h^- E(h^-)^{-1})^{(1)} = E^{(1)} \]

since \( h^- \in \mathcal{Q} \). Thus, by (6.1.6), \( \Omega_1 - E \in \mathfrak{U} \), so essentially \( \mathfrak{U} \) is the space where \( \Omega_1^{(1)} \) naturally lives.

Proposition (6.3.4). Let \( q \) be any smooth function of \( t \) with values in \( q \) and set \( \Omega_1 = \text{Ad}(h^-)((\partial_1 e^q)e^{-q} + e^q E e^{-q}) \) and \( \Omega_j = (\text{Ad}(h^- e^q)E_j)^{(+)} \), \( j > 0 \). Then

\[ \partial_j \Omega_1 - \partial_1 \Omega_j - [\Omega_j, \Omega_1] \in \mathfrak{U}. \]

Proof. We know that \( \Omega_1 - E \) and all its partial derivatives are contained in \( \mathfrak{U} \). Next we note

\[ \partial_1 (\text{Ad}(h^- e^q)E_j) = \text{Ad}(h^-)((\partial_1 e^q)e^{-q}, \text{Ad}(e^q)E_j)) = [\Omega_1, \text{Ad}(h^- e^q)E_j]. \]

Hence

\[
\begin{align*}
\partial_j \Omega_1 - \partial_1 \Omega_j - [\Omega_j, \Omega_1] &= \partial_j \Omega_1 - [\Omega_1, \text{Ad}(h^- e^q)E_j]^{(+)} + [\Omega_1, \Omega_j] \\
&= \partial_j \Omega_1 - [\Omega_1, \Omega_j + (\text{Ad}(h^- e^q)E_j)^{(-)}]^{(+)}) + [\Omega_1, \Omega_j] \\
&= \partial_j \Omega_1 - [\Omega_1, (\text{Ad}(h^- e^q)E_j)^{(-)}]^{(+)} \\
&= \partial_j \Omega_1 - [E^{(1)}, (\text{Ad}(h^- e^q)E_j)^{(-1)}] \in \mathfrak{U},
\end{align*}
\]

where we used that \( \Omega_1^{(1)} = E^{(1)} \).

6.4 In the next section we intend to show that the entries of \( \Omega_1 \), relative to a certain basis, are all \( \partial_1 \)-polynomials in \( \dim \mathfrak{g}_0 \) “basic functions.” The present section collects all necessary algebraic facts needed in the proof of the forthcoming Theorem (6.5.1). The main idea of this theorem is to exploit (6.3.3) by projecting both sides of that equation onto a subspace of \( \mathfrak{U} \) given by the kernel of certain operator \( (\text{Ker}(B)) \) in the notation appearing in the proof of Theorem (6.5.1)). The dual of this subspace is \( \mathfrak{W}/\tilde{\mathfrak{W}} \) in the notation of this section. It is exactly the relation between \( \mathfrak{U}, \mathfrak{W} \) and \( \tilde{\mathfrak{W}} \) that we study in this section.

The discussion in this section deals exclusively with \( \mathfrak{g}^{\text{fin}} \). Therefore we omit the superscript “fin” in this section. First, we introduce the following notation. Let \( (\cdot, \cdot) \) denote the canonical nondegenerate symmetric bilinear form on \( \mathfrak{g} \) as in [10, Theorem 2.2]. We furthermore define

(6.4.1) \[ \mathfrak{b} = \mathfrak{g}^{(0)} \cap (\mathfrak{g}^- \oplus \mathfrak{g}_0) = \mathfrak{b}_0 \oplus \mathfrak{b}_{-1} \oplus \ldots \oplus \mathfrak{b}_{-s}, \]

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where \( s \) is the maximal integer such that \( b_s \neq \{0\} \).

\[
\tag{6.4.2} b^* = g^{(0)} \cap (g_0 \oplus g_+) = b_0 \oplus b_1 \oplus \ldots \oplus b_s ,
\]

\[
\tag{6.4.3} \mathfrak{M}_{-k} = [g_{-k-1}^{(-1)}, E^{(1)}] , \quad 0 \leq k \leq s
\]

\[
\tag{6.4.4} \mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \oplus \ldots \oplus \mathfrak{M}_s ,
\]

\[
\tag{6.4.5} \mathfrak{W}_k = \{ A_k \in g_k : ([q_{-k-1}, E]A_k) = 0 \} ,
\]

\[
\tag{6.4.6} \mathfrak{W} = \mathfrak{W}_0 \oplus \mathfrak{W}_1 \oplus \ldots \oplus \mathfrak{W}_s ,
\]

\[
\tag{6.4.7} \widetilde{\mathfrak{W}}_k = \{ A_k \in \mathfrak{W}_k : (A_k|\mathfrak{M}_{-k}) = 0 \} ,
\]

\[
\tag{6.4.8} \widetilde{\mathfrak{W}} = \widetilde{\mathfrak{W}}_0 \oplus \ldots \oplus \widetilde{\mathfrak{W}}_s .
\]

We also define

\[
\tag{6.4.9a} r_{-k} := (\text{Ker ad} E) \cap q_{-k} \quad \text{for} \quad k = 1, \ldots, s,
\]

\[
\tag{6.4.9b} r_k := (\text{Ker ad} E) \cap q_{-k}^{*} \quad \text{for} \quad k = 1 \ldots, s,
\]

where \( q_{-k}^{*} \) is the natural dual of \( q_{-k} \) with respect to \((.,.)\). Note that \( \dim r_{-k} = \dim r_k \) for \( k = 1, \ldots, s \).

We collect some information about the dimensions of the vector spaces \( \mathfrak{M}, \mathfrak{W} \) and \( \widetilde{\mathfrak{W}} \).

We will use the following notation. For \( \alpha \in \Delta, \alpha = \sum k_i \alpha_i \) we set \( ht(\alpha) = \sum k_i \). More generally, for a fixed \( X \subset \Pi \) we set \( pht(\check{\alpha}_i) = 0 \) if \( i \in \Pi \), 1 otherwise.

**Theorem (6.4.1).**

(a) \[
\tag{6.4.31} \mathfrak{W}_k = \{ A_k \in g_k : [E, A_k] \in b_{k+1} \} \quad \text{for} \quad k = 0, \ldots, s .
\]

(b) \[
\dim \mathfrak{W} = \dim b - \dim g_0 + \sum_{k=1}^{s} \dim r_{-k}
\]

(c) \[
\dim \mathfrak{M} + \dim \widetilde{\mathfrak{W}} = \dim b + \sum_{k=1}^{s} \dim r_k .
\]
Proof. (a) We know that

\[ \mathfrak{W}_k = \left\{ A_k \in \mathfrak{g}_k : (\mathfrak{g}_{-k-1} \cap [E, A_k]) = 0 \right\} \]
\[ = \left\{ A_k \in \mathfrak{g}_k : [E, A_k] \in \mathfrak{g}^{(0)} \cap \mathfrak{g}_{k+1} = \mathfrak{b}_{k+1} \right\}. \]

The latter equality holds because, for \( 1 \leq k \leq s \), \( (\mathfrak{g}_{-k})^\perp \cap \mathfrak{g}_k = \mathfrak{g}^{(0)} \cap \mathfrak{g}_k \). Indeed, if \( \alpha \in (\mathfrak{g}_{-k})^\perp \cap \mathfrak{g}_k \) then the height of \( \alpha \), \( ht(\alpha) \), equals \( k \). On the other hand the corresponding \( ph(\alpha) \), when only roots from \( X \) are counted, satisfies \( ph(\alpha) \geq 0 \). We want to show that \( ph(\alpha) = 0 \). Assume therefore that \( ph(\alpha) > 0 \). Then \( ph(-\alpha) < 0 \) and \( x_\alpha \in \mathfrak{g}_{-k} \). Since \( (x_\alpha, x_\alpha) \neq 0 \) we get a contradiction as \( x_\alpha \in (\mathfrak{g}_{-k})^\perp \). This proves that \( ph(\alpha) = 0 \). For \( k = s+1 \), on the other hand, \( \mathfrak{g}_{-s} \cap \mathfrak{g}_{s+1} = 0 \equiv \mathfrak{b}_{s+1} \).

(b) Since \((.,.)\) is nondegenerate on \( \mathfrak{g}_{-k} \times \mathfrak{g}_k \), we get from (6.4.5) the relation

\[ \dim \mathfrak{W}_k = \dim \mathfrak{g}_k - \dim [\mathfrak{g}_{-k-1}, E]. \]

Furthermore

(6.4.10) \[ \dim \mathfrak{W}_k = \dim \mathfrak{g}_k - \dim \mathfrak{g}_{k-1} + \dim \mathfrak{r}_{-k-1} \quad \text{for} \quad k = 0, 1, \ldots, s-1. \]

For \( \mathfrak{W}_s \) we note that \( \mathfrak{g}_{-s-1} = \mathfrak{g}_{-s-1} \), thus:

(6.4.11) \[ \mathfrak{W}_s = \left\{ A_s \in \mathfrak{g}_s : ([\mathfrak{g}_{-s-1}, E] \mid A_s) = 0 \right\} \]
\[ = (\text{Ker ad} E)_s. \]

From (6.4.10) and (6.4.11) , we get

\[ \dim \mathfrak{W} = \dim \mathfrak{g}_0 + \sum_{k=1}^{s-1} (\dim \mathfrak{g}_{-k} - \dim \mathfrak{q}_{-k}) - \dim \mathfrak{q}_s \]
\[ + \dim (\text{Ker ad} E)_s + \sum_{k=1}^{s} \dim \mathfrak{r}_{-k}. \]

Since \( \mathfrak{g}_{-k} = \mathfrak{q}_{-k} \oplus \mathfrak{b}_{-k} \), \( \dim \mathfrak{b}_{-k} = \dim \mathfrak{g}_{-k} - \dim \mathfrak{q}_{-k}, k = 1 \ldots s \). Therefore

\[ \dim \mathfrak{W} = \dim \mathfrak{g}_0 + \left( \sum_{k=1}^{s} \dim \mathfrak{b}_{-k} \right) - (\dim \mathfrak{g}_s - \dim (\text{Ker ad} E)_s) + \sum_{k=1}^{s} \dim \mathfrak{r}_{-k}. \]

Since \( \dim \mathfrak{g}_0 = \dim \mathfrak{b}_0 \), we obtain that

(6.4.12) \[ \dim \mathfrak{W} = \dim \mathfrak{b} - (\dim \mathfrak{g}_s - \dim (\text{Ker ad} E)_s) + \sum_{k=1}^{s} \dim \mathfrak{r}_{-k}. \]

By virtue of [10, Proposition 14.3a],

(6.4.13) \[ \dim \mathfrak{g}_s - \dim (\text{Ker ad} E)_s = \dim \mathfrak{g}_0, \]

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thus

\[(6.4.14) \quad \dim \mathfrak{W} = \dim b - \dim g_0 + \sum_{k=1}^{s} \dim r_{-k} .\]

(c) Let \( \mathfrak{V}_k = (\mathfrak{U}_{-k})^\perp \cap b_k .\) Then the space \( \mathfrak{V}_k \subseteq g_k^{(0)} \) satisfies for \( k = 0, 1 \ldots, s \)

\[(6.4.15) \quad \mathfrak{V}_k \subseteq b_k ,
\dim \mathfrak{U}_{-k} + \dim \mathfrak{V}_k = \dim b_k \quad \text{and}
(\mathfrak{U}_{-k} | \mathfrak{V}_k) = 0 , .\]

We claim that \( \mathfrak{W}_k = \mathfrak{V}_k \oplus r_k \) holds.

To verify \( \mathfrak{V}_k \subseteq \mathfrak{W}_k \), we decompose \( E = E^{(0)} + E^{(1)} \) and derive from the definitions of \( \mathfrak{V}_k \) and \( \mathfrak{U}_k \) immediately that

\[(6.4.16) \quad [\mathfrak{V}_k, E^{(1)}] = 0 \]

Moreover, since \( E^{(0)} \in g_1^{(0)} \), we have

\[(6.4.17) \quad [\mathfrak{V}_k, E^{(0)}] \subseteq b_{k+1} . \]

This implies

\[(6.4.18) \quad [\mathfrak{V}_k, E] \subseteq b_{k+1} . \]

From (a) we now obtain \( \mathfrak{V}_k \subseteq \mathfrak{W}_k \). This together with the definition of \( \mathfrak{V}_k \) and (6.4.8) shows

\[(6.4.19) \quad \mathfrak{V}_k \subseteq \mathfrak{W}_k . \]

Since \( r_k \perp p \) we obtain that \( r_k \subseteq \mathfrak{W}_k \). Since \( \mathfrak{V}_k \subseteq g^{(0)} \) and \( r_k \in g^{(+)} \), we have \( \mathfrak{V}_k + r_k = \mathfrak{V}_k \oplus r_k \).

Now let \( A_k \in \mathfrak{W}_k \). Note that

\[(6.4.20) \quad b_k \oplus (q_{-k})^* = g_k . \]

Let \( B_k \in b_k , C_k \in (q_{-k})^* \subseteq g^{(+)} \) such that \( A_k = B_k + C_k \). Then

\[(6.4.21) \quad (A_k | \mathfrak{U}_{-k}) = (B_k | \mathfrak{U}_{-k}) + (C_k | \mathfrak{U}_{-k}) . \]

Here the second term vanishes, since \( C_k \in g^{(+)} \) and \( \mathfrak{U}_{-k} \subseteq g^{(0)} \). Therefore, also \( B_k \in \mathfrak{V}_k \), whence \( B_k \in \mathfrak{W}_k \) by (6.4.19). This in turn means that \( C_k \in \mathfrak{W}_k \) by (6.4.8), and in particular \( C_k \in \mathfrak{W}_k \). Hence, by virtue of (a),

\[(6.4.22) \quad [C_k, E] \in b_{k+1} . \]
Since \( C_k \in g^{(1)} \) and \( E = E^{(0)} + E^{(1)} \),

(6.4.23) \[ [C_k, E] \in b_{k+1} \cap g^{(+)} = \{0\} \]

Thus

(6.4.24) \[ C_k \in (Ker ad E) \cap (q_{-k})^* , \]

whence \( \mathfrak{M}_k \subseteq \mathfrak{M}_k + r_k \).

This proves that \( \mathfrak{M}_k = \mathfrak{M}_k \oplus r_k \).

Now we have

(6.4.25) \[ \dim \mathfrak{M}_k = \dim \mathfrak{M}_k + \dim r_k \]

Hence (6.4.15) implies

(6.4.26) \[ \dim \mathfrak{M}_{-k} + \dim \mathfrak{M}_k = \dim \mathfrak{M}_{-k} + \dim \mathfrak{M}_k + \dim r_k \]
\[ = \dim b_k + \dim r_k \]

Summation over \( k = 0, \ldots, s \) completes the proof.

6.5. In this section we will roughly show that among the scalar components of \( \Omega_1 \), a subset of cardinality \( \text{rank } g = \dim g_0 \) can be chosen, such that the remaining functions are \( \partial_1 \)-differential polynomials in those.

More precisely, we show

**Theorem (6.5.1).** There exists a basis \( c_1, \ldots, c_m \) of \( \mathfrak{M} \), such that in the expansion

\[ \Omega_1 = E^{(1)} + \sum_{k=1}^m u_k c_k \]

the coefficient functions \( u_{l+1}, \ldots, u_m \) are \( \partial_1 \)-differential polynomials in \( u_1, \ldots, u_\ell \), where \( \ell = \text{rank } g = \dim g_0 \).

**Proof.** From Proposition (5.3.2) we know that

(6.5.1) \[ \Omega_1 = \Omega_1^{(0)} + \Omega_1^{(1)} , \]

where the superscript denotes as usual the \( p \)-grading. From (6.1.6) we obtain

(6.5.2) \[ \Omega_1^{(1)} = E^{(1)} \]

and we write \( \Omega_1 = E + U_0 + \ldots + U_{-s} \), where \( U_{-k} \) is homogeneous of canonical degree \(-k\).

Next we write (6.3.3) in the form

(6.5.3) \[ \Omega_1 = Ad(h^-)((\partial_1 e^{q_1}) e^{-q_1} + e^{q_1} (\partial_1 q_L + E) e^{-q_1}). \]

Decomposition into components of canonical degree \(-j\) yields:

(6.5.4a) \[ U_0 = [q_{L,-1}, E] + R_0(h^-), \]

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where \( R_0(h^-) = [h^-_1, E] \) for some \( h^-_1 \in g \),

\[
U_{-j} = [q_{I,\cdot,(j+1)}, E] + \partial_1 q_K_{-j}
\]
(6.5.4b)

\[+ R_j(q_{I,-1}, \ldots, q_{I,-j}, q_K_{-1}, \ldots, q_K_{-(j-1)}, h^-), \quad 1 \leq j \leq s,
\]

where \( R_j \) is a differential polynomial in its arguments. From Theorem A.1.2, it follows
that all dependence on \( q_K \) in (6.5.4b) drops out, thus (6.5.4b) simplifies to

\[
(6.5.4c) \quad U_{-j} = [q_{I,\cdot,(j+1)}, E] + R_j(q_{I,-1}, \ldots, q_{I,-j}, h^-).
\]

We can, furthermore, simplify (6.5.4c) by eliminating \( q_{I,-1}, \ldots q_{I,-j} \). Indeed, using (6.5.4a)
and solving (6.5.4c) for \( q_{I,-k} \), \( 0 \leq k \leq j \), we can write (6.5.4c) as:

\[
U_{-j} = [q_{I,\cdot,(j+1)}, E] + R_j(U_0, \ldots, U_{-j+1}, h^-), \quad 1 \leq j \leq s.
\]

Note, however, that in the neighborhood of the identity, \( \tilde{R}_j \) is independent of \( h^- \). Since \( \tilde{R}_j \)
is analytic in \( h^- \) we get that \( \tilde{R}_j = \tilde{R}_j(U_0, \ldots, U_{-j+1}) \). Relation (6.5.5) cut out a subset \( S \)
of \( \Omega \otimes \mathcal{R} \), where \( \mathcal{R} \) is the germ of holomorphic maps at \( x = 0 \). We proceed now to describe \( S \). We apply \((\cdot|A_j)\) to (6.5.5), where \( A_j \in \mathfrak{M}_j/\widetilde{\mathfrak{M}}_j \).

By (6.4.6), (6.4.8) and (6.5.4a) we get

\[
(6.5.6) \quad (U_{-j}|A_j) = (\tilde{R}_j(U_0, \ldots, U_{-j+1})|A_j), \quad 0 \leq j \leq s.
\]

Thus we get \( \dim \mathfrak{M}_j - \dim \widetilde{\mathfrak{M}}_j \) relations for the components of \((\Omega^{1})_{-j} = U_{-j} \). Let us now choose a basis \( b_1, \ldots, b_m \) of \( \Omega \), which is consistent with the canonical grading, i.e.

\[
\Omega_0 = \sum_{0 < j \leq l_0} u_j b_j, \quad \Omega_{-1} = \sum_{l_0 < j \leq l_1} u_j b_j, \ldots, \quad \Omega_{-s} = \sum_{l_{s-1} < j \leq l_s = m} u_j b_j
\]

and the \( u \)'s are in \( \mathcal{R} \). Then the system (6.5.6) takes the form

\[
(6.5.7) \quad \begin{pmatrix}
B_0 & 0 \\
0 & B_1 \\
\vdots & \ddots & \ddots \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots \\
\hat{u}_s
\end{pmatrix} = 
\begin{pmatrix}
\hat{v}_0 \\
\hat{v}_1 \\
\vdots \\
\hat{v}_s
\end{pmatrix}
\]

where \( B_j \) is a \((\dim \mathfrak{M}_j - \dim \widetilde{\mathfrak{M}}_j) \times \dim \Omega_{-j} \) matrix with entries in \( \mathbb{C} \), and

\[
\hat{u}_j = \begin{pmatrix}
u_{j-1+1} \\
\vdots \\
u_j
\end{pmatrix}.
\]

Note that by (6.5.4) \( \hat{v}_0 = 0 \). Moreover, in \( \hat{v}_j, j > 0 \), only terms which are differential polynomials in \( \hat{u}_0, \ldots, \hat{u}_{j-1} \) can occur. By construction rank \( B_j = \dim \mathfrak{M}_j - \dim \widetilde{\mathfrak{M}}_j \equiv d_j \).
Thus, in order to solve (6.5.7) we first solve $B_0 \hat{u}_0 = 0$ whose solution $\hat{u}_0$ is an arbitrary element in $\text{Ker} B_0$. Clearly, 

$$B_1 \hat{u}_1 = v_1(\hat{u}_0)$$

and 

$$\hat{u}_1 = \hat{u}_0 + B_1^{-1} v_1(\hat{u}_0), \hat{u}_1 \in \text{Ker} B_1.$$ 

An easy proof by induction shows that 

(6.5.8) 

$$\hat{u}_j = \hat{u}_0 + B_j^{-1} v_j(\hat{u}_0, \hat{u}_1, \hat{u}_{j-1}), \hat{u}_j \in \text{Ker} B_j.$$ 

Let us denote by $B$ the linear map $\mathfrak{U} \to \mathfrak{U}$, whose matrix of coefficients is given by $B_0, B_1, \ldots, B_s$ as in (6.5.7). 

Then we consider the map $\psi : \text{Ker} B \otimes \mathbb{R} \to \mathfrak{U} \otimes \mathbb{R},$ 

(6.5.9) 

$$\hat{u} = \sum_{i=0}^{s} \hat{u}_i \overset{\psi}{\mapsto} \sum_{i=0}^{s} \hat{u}_j$$

where $\hat{u}_j$ is given by (6.5.8). 

Note that $\dim \text{Ker} B = \dim \mathfrak{U} - (\dim \mathfrak{W} - \dim \tilde{\mathfrak{W}}) = \dim g_0$, the latter following from Theorem (6.4.1). 

The claim of the theorem is proven by choosing a basis 

$$\{c_1, \ldots, c_{\ell}\} \text{ of } \text{Ker} B \text{ and } \{c_{\ell+1}, \ldots, c_m\} \text{ of } \mathfrak{U}/\text{Ker} B.$$ 

6.6 In this section we illustrate how Theorem (6.5.1) works on three examples. First we analyze the well known case of the KdV equation or rather the potential KdV equation. The appropriate Kac-Moody algebra is $g = A^{(1)}_1$. We choose the Chevalley generators to be: 

$$e_0 = \lambda E_{21}, e_1 = E_{12},$$

$$f_0 = \lambda^{-1} E_{12}, f_1 = E_{21}$$

$$h_i := [e_i, f_i] \text{ for } i = 0, 1.$$ 

The parabolic algebra $\mathfrak{p}$ is generated by $\{e_0, e_1, f_1\}$. Our goal is to determine the form of $\Omega_1$. First, we find that 

$$q^{(-1)} = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) : \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}$$

and 

$$E^{(1)} = \left( \begin{array}{cc} 0 & 0 \\ \lambda & 0 \end{array} \right).$$

By (6.3.13) we get 

$$\mathfrak{U} = \left\{ \left( \begin{array}{cc} \alpha & 0 \\ \beta & -\alpha \end{array} \right) : \alpha, \beta \in \mathbb{C} \right\}.$$
Consequently,
\[ \mathfrak{U}_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \alpha \in \mathbb{C} \right\} \]
and
\[ \mathfrak{U}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} : \beta \in \mathbb{C} \right\}. \]

Thus \( \Omega_1 \) is of the form:
\[ \Omega_1 = \begin{pmatrix} \alpha & 1 \\ \lambda + \beta & -\alpha \end{pmatrix}. \]

Since the rank of \( \mathfrak{g} \) is 1, we expect to have only one function parametrizing \( \Omega_1 \). In order to see how Theorem (6.5.1) works in this case we essentially go through the main steps of the proof of that theorem. First we have to setup equations (6.5.4a) and (6.5.4b). To this end we observe that
\[ U_0 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \]
and
\[ U_{-1} = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}. \]

We are allowed to set \( h^- = I \) in computations. Thus, instead of (6.5.3), we use
\[ \Omega_1 = (\partial_1 e^{q_1}) e^{-q_1} + e^{q_1} (\partial_1 q_K + E^R) e^{-q_1} \]
where, as in Lemma (6.3.1), \( q_I + q_K \in \mathfrak{q} \). We get
\[ E + U_0 + U_{-1} = \partial_1 (q_I, -1 + q_K) + E + [q_I, -1, E] + [q_I, -2, E] + \frac{1}{2} [q_I, -1, [q_I, -1, E]] \]
from which we get
\[ U_0 = [q_I, -1, E], \]
\[ U_{-1} = \partial_1 (q_I, -1 + q_K) + [q_I, -2, E] + \frac{1}{2} [q_I, -1, [q_I, -1, E]]. \]

Now we solve the first equation for \( q_I, -1 \). The result is:
\[ q_I, -1 = \frac{U_0}{2} E^{-1}. \]

To get \( q_K, -1 \) we use \( q_I + q_K \in \mathfrak{q} \) which implies:
\[ q_K, -1 = \frac{[U_0]_{11}}{2} E^{-1}, \quad [U_0]_{11} = \alpha. \]

This brings us to (6.5.5), where all terms except for the very first one, which depends on \( q_I, -2 \), depend on \( U_{j}, j = 0, 1 \). Now we would like to recover (6.5.6). To this end we apply \( (\cdot, A_1), A_1 \in \mathfrak{m}_1 / \mathfrak{m}_1 \) to both sides of the last equation above. The symmetric, invariant,
bilinear form $(\cdot, \cdot)$ is defined as follows: $(x^\lambda^n, y^\mu^m) = \delta_{m,-n} \text{trace}(xy), x, y \in \mathfrak{g}$. One directly checks that $\mathfrak{W}_1/\mathfrak{W}_1 = \mathbb{C}E$ so that the result of applying $(\cdot, E)$ to the equation for $U_{-1}$ is:

$$\beta = \partial_1 \alpha - \alpha^2,$$

and the final result for $\Omega_1$ is:

$$\Omega_1 = E + \begin{pmatrix} \alpha \\ \partial_1 \alpha - \alpha^2 \\ 0 \end{pmatrix}.$$

This result should be contrasted with the situation appearing in other approach to a generalization of the Drinfeld-Sokolov systems [20] in which $U_1 - E$ is in a fixed complementary subspace to $[E^{(0)}, \mathfrak{g}^{(0)} \cap \mathfrak{g}_-]$ in $\mathfrak{b}$. This is not the case with us. To finish this example we note that the resulting hierarchy of equations on $\alpha$ is the potential KdV hierarchy.

So far we have looked at an example of a $2 \times 2$ system. Below we present two examples of systems of PDEs which can be obtained by using the results of Theorems (6.3.2) and (6.5.1), but which are related to $3 \times 3$ systems. The resulting equations are in some sense generalizations of the Boussinesq system studied usually in connection with the spectral problem [26] for the operator:

$$L = D^3 + pD + q, \quad \text{where} \quad D = \frac{d}{dx}.$$

Then the first Boussinesq flow is given by the following equations:

(6.6.1a)

$$p_t = 2q_x - p_{xx}$$

and

(6.6.1b)

$$q_t = q_{xx} - \frac{2}{3} p_{xxx} - \frac{1}{3} (p^2)_{xx},$$

or, after one differentiation of the former,

(6.6.2)

$$p_{tt} = \frac{1}{3} p_{xxx} - \frac{2}{3} (p^2)_{xx}.$$
and

\[ q^{(-1)} = \begin{pmatrix} 0 & 0 & \frac{\alpha}{\lambda} \\ 0 & 0 & \frac{\alpha^2}{\lambda} \\ \alpha & \alpha & 0 \end{pmatrix} \]

respectively. Thus (6.3.13) implies that \( \Omega_1 \) is of the form:

\[ \Omega_1 = \begin{pmatrix} u & 1 & 0 \\ \alpha & -v & 1 \\ \lambda & 0 & v - u \end{pmatrix} \]

where \( u, v, \alpha \) are parameters. Moreover, since the rank of \( \tilde{g} \) is 2, we will use \( u \) and \( v \) to parametrize \( \Omega_1 \) and consequently all other (at least all positive) potentials. Thus in the first place we have to see how \( \alpha \) is expressed in terms of \( u \) and \( v \). We observe that \( \alpha E_{21} \) carries the canonical degree -1 so we have to know \( q_{I, -1} \) and \( q_{I, -2} \). We proceed by implementing the proof of Theorem (6.5.1), in particular, (6.5.4a) and (6.5.4b). In the subsequent computations we take \( h = I \). The results are:

\[ q_{I, -1} = \begin{pmatrix} 0 & 0 & \frac{1}{3}u - v \\ \frac{1}{3}v - \frac{1}{3}u & 0 & \frac{1}{3}u - v \\ 0 & \frac{1}{3}v + 2u & 0 \end{pmatrix} \]

and

\[ q_{I, -2} = \begin{pmatrix} 0 & \frac{1}{6} - \frac{2u_x + 2v_x + u^2 - v^2}{\lambda} & 0 \\ 0 & 0 & \frac{1}{6} - \frac{2v_x + 4u_x + 4uv - 3u^2 - 2v^2}{\lambda} \\ -\frac{1}{3}u_x - \frac{2}{3}v_x - \frac{2}{3}uv + \frac{1}{3}u^2 + \frac{1}{3}v^2 & 0 & 0 \end{pmatrix} \]

For \( q_K \) part we get:

\[ q_{K, -1} = \begin{pmatrix} 0 & 0 & \frac{1}{3}(u_x + v_x) \\ \frac{1}{3}(u_x + v_x) & 0 & \frac{1}{3}(u_x + v_x) \\ 0 & \frac{1}{3}(u_x + v_x) & 0 \end{pmatrix} \]

Now, it is easy to compute \( \Omega_1 \) or rather \( \alpha \) in terms of \( u \) and \( v \). The final result is:

(6.6.3) \[ \Omega_1 = \begin{pmatrix} u & 1 & 0 \\ \alpha & -v & 1 \\ \lambda & 0 & v - u \end{pmatrix} \] where \( \alpha = u_x + v_x + uv - u^2 - v^2 \).

In a similar manner we can compute potentials corresponding to the higher flows. For example the second flow, i.e. \( j = 2 \), gives rise to

(6.6.4) \[ \Omega_2 = \begin{pmatrix} v_x + uv - v^2 & u - v & 1 \\ \alpha & u + uv - u^2 & -u \\ \lambda v & \lambda & -u_x - v_x - 2uv + u^2 + v^2 \end{pmatrix} \]
where
\[ \alpha = uu_x - vv_x + \frac{1}{3}v^3 - \frac{1}{3}u^3 - \frac{1}{3}u_{xx} + \frac{1}{3}v_{xx}. \]

The ZCC yields:

(6.6.5a) \[ u_t = \frac{2}{3}v^3 - \frac{2}{3}u^3 + 2u^2v - 2uv^2 + 2uv_x + \frac{2}{3}v_{xx} + \frac{1}{3}u_{xx} - 2v_{xx}, \]

(6.6.5b) \[ v_t = \frac{2}{3}v^3 - \frac{2}{3}u^3 + 2u^2v - 2uv^2 - 2vu_x - \frac{2}{3}u_{xx} - \frac{1}{3}v_{xx} + 2uu_x. \]

These equations can be thought of as a version of 6.6.1a and 6.6.1b. This assertion is further supported by the fact that after one differentiation with respect to \( t \) we get two copies of the potential Boussinesq equation 6.6.2, which is to say that \( u \) and \( v \) satisfy:

(6.6.6) \[ z_{tt} = -\frac{1}{3}z_{xxxx} + 4z_xz_{xx}. \]

For the sake of comparison we would like to quote the formulas for a maximal parabolic. In that case the only generator which has a nonzero \( p \)-degree is \( e_0 \). The remainder of computations is almost identical, except for the choice of the parametrization of \( \Omega_1 \). It is not sufficient to choose functions from the diagonal, as there is only one function there while we need two functions. We make the following choice of functions following [28]:

(6.6.7) \[ u = (\Omega_1)_{11}, \quad v = (\Omega_1)_{32} - (\Omega_1)_{21}. \]

Thus we arrive at the formulas:

\[ \Omega_1 = \begin{pmatrix} -\frac{1}{2}u^2 + \frac{1}{2}u_x - \frac{1}{2}v & 1 & 0 \\ uv - v_x + \lambda & -\frac{1}{2}u^2 + \frac{1}{2}u_x + \frac{1}{2}v & -u \end{pmatrix}, \]

\[ \Omega_2 = \begin{pmatrix} \frac{1}{2}u^2 - \frac{1}{2}u_x - \frac{1}{2}v & u & 1 \\ \alpha & -u^2 + u_x & -u \\ \beta & \gamma & -\frac{1}{2}u^2 - \frac{1}{2}u_x + \frac{1}{2}v \end{pmatrix}, \]

where

\[ \alpha = \frac{3}{2}uu_x + \frac{1}{2}uv - \frac{1}{2}u^3 - \frac{1}{2}u_{xx} - \frac{1}{2}v_x + \lambda, \]

\[ \beta = -\frac{1}{4}v^2 - \frac{3}{2}u^2u_x + \frac{1}{4}u^4 - \frac{1}{3}u_{xxx} + \frac{5}{4}(u_x)^2 + uu_{xx}, \]

and

\[ \gamma = -\frac{3}{2}uu_x + \frac{1}{2}uv + \frac{1}{2}u^3 + \frac{1}{2}u_{xx} - \frac{1}{2}v_x + \lambda. \]

As the result of the ZCC we obtain:

(6.6.8a) \[ u_t = -v_x, \]
These results should be compared with (6.6.1a), (6.6.1b) and their counterparts (6.6.5a) and (6.6.5b). After one more differentiation with respect to $t$ we obtain that $u$ satisfies the potential Boussinesq equation 6.6.6. On the other hand $v$ satisfies:

\[(6.6.9) \quad v_{tt} = \frac{1}{3} v_{xxxx} + 4u_xv_{xx}.\]

§7. Negative Potentials

In this section we will investigate the negative potentials, $\Omega_j$, $j < 0$. First, we outline the main results. We prove that every potential (both positive and negative ones) can be expressed in terms of the component of $p$-degree zero of $g^+(\ell, \lambda)$; cf. Section 5.3. The Zero-Curvature Condition (ZCC) gives systems of PDEs that include the so called two-dimensional Toda lattice (e.g. the Sinh-Gordon equation) in the case where $p$ is the minimal standard parabolic subalgebra, i.e. the standard Borel subalgebra.

Our second result is that if $p$ is not minimal, and some additional condition are satisfied (see Proposition (7.3.2)) then $\Omega_1$ uniquely determines every potential $\Omega_j$, $j \in \mathbb{Z}$.

7.1 Let $a \in G$ be the component of $p$-degree 0 of $g^+$, i.e.

\[(7.1.1) \quad g^+ = a \tilde{g}^+, \]

where $\tilde{g}^+ \in Q_X^+$ (see Proposition 3.3.1). Applying this to (5.3.5), we get

\[(7.1.2) \quad \Omega_j^R = (\partial_j a)a^{-1} + a((\partial_j \tilde{g}^+)(\tilde{g}^+)^{-1} + \tilde{g}^+ E_j^R(\tilde{g}^+)^{-1})a^{-1}. \]

First we look at $j = 1$; (5.3.3) reveals:

\[(7.1.3) \quad \Omega_1^R = (E_1^R)^{(1)} + \text{terms of nonpositive } p\text{-degree}, \]

whereas from (5.3.5) we derive

\[(7.1.4) \quad \Omega_1^R = (\partial_1 a)a^{-1} + a(E_1^R)^{(0)}a^{-1} + \text{terms of positive } p\text{-degree}, \]

thus by Proposition (5.3.1)

\[(7.1.5) \quad \Omega_1 = E_1^{(1)} + (\partial_1 a)a^{-1} + aE_1^{(0)}a^{-1}. \]

Similarly one gets

\[(7.1.6) \quad \Omega_{-1} = E_{-1}^{(0)} + aE_{-1}^{(-1)}a^{-1}. \]

From Theorem (6.3.2) we know that $\Omega_1$ determines every positive potential; we see therefore that $a$ determines every positive potential. Moreover, since $det(a) = 1$, both $\Omega_1$ and $\Omega_{-1}$ are $\partial_1$- differential polynomials in the entries of $a$. To see whether $\Omega_{-1}$ determines all negative potentials, we proceed as follows. Since $a \in G^{(0)}$, $a$ is a Laurent polynomial in
\( \lambda \in S^r \). Therefore it can be holomorphically extended to \( \mu \in S^R \). We use this to “swap” the original splitting (5.3.1). Let

\[
(7.1.7) \quad \tilde{g}^- = a^{-1}g^- ,
\]

where \( a \) is the same as in (7.1.1). Then we get

\[
\begin{align*}
\ell(h_1, h_2) &= ((g^-)^{-1}, (g^+)^{-1})(b_R, b_r) \\
&= ((\tilde{g}^-)^{-1}a^{-1}, (\tilde{g}^+)^{-1}a^{-1})(b_R, b_r) \\
&= ((\tilde{g}^-)^{-1}(\tilde{g}^+)^{-1}(a^{-1}b_R, a^{-1}b_r)
\end{align*}
\]

The second factor is still in \( H_+ \), whereas the first one now is in \( \tilde{H}_- := \mathcal{P}^R_{opp} \times Q^r_{opp} \), meaning the “opposite” standard parabolic subgroup and its complementary subgroup. In terms of Lie algebras this means:

\[
(7.1.9) \quad \tilde{H}_- := \mathcal{P}^R_{opp} \times Q^r_{opp} ,
\]

Roughly speaking, everything we did for positive potentials remains true if replace “(non) negative” by “(non) positive” and vice versa. In particular, we see that the negative potentials \( \tilde{\Omega}_{-k} \) (corresponding to the splitting into \( H_+ \) and \( \tilde{H}_- \)) are \( \partial_{-1} \)-differential polynomials in \( \tilde{\Omega}_{-1} \). From (5.3.5) we know how to express them:

\[
(7.1.12) \quad \tilde{\Omega}_j = (\partial_j \tilde{g}^+)(\tilde{g}^+)^{-1} + \tilde{g}^+E^r_j(\tilde{g}^+)^{-1}
\]

\[
= (\partial_j(a^{-1}))a + a^{-1}(\partial_j g^+)(g^+)^{-1} + g^+E^r_j(g^+)^{-1})a ,
\]

or

\[
(7.1.13) \quad \Omega_j = (\partial_j a)a^{-1} + a\tilde{\Omega}_j a^{-1}.
\]

Now we have the following chain:

\[
a \rightarrow \Omega_{-1} \rightarrow \tilde{\Omega}_{-1} \rightarrow \tilde{\Omega}_{-k} \rightarrow \Omega_{-k} , \quad k \geq 1 ,
\]

where “\( \rightarrow \)” stands for “determines”.

We summarize the results:

**Theorem (7.1.1).** Every positive potential \( \Omega_k \) is a \( \partial_1 \) differential polynomial in \( a \). Every negative potential \( \tilde{\Omega}_{-k} \) is a \( \partial_{-1} \)–\( \partial_{-k} \)- differential polynomial in \( a \). More precisely, \( \tilde{\Omega}_{-k} \) is gauge equivalent to \( \Omega_{-k} \) as in (7.1.13), where \( \tilde{\Omega}_{-k} \) is a \( \partial_{-1} \)- differential polynomial in \( a \).
Example In the case where \( \mathfrak{p} \) is the minimal parabolic subalgebra, there is a natural choice of functions parametrizing \( a \). In this case, \( G^{(0)} \) is an abelian subgroup and there is a \( w \in \mathfrak{g}_0 \) such that \( a = e^w \). Then (7.1.5) and (7.1.6) yield:

\[
(7.1.14) \quad \Omega_1 = E + \partial_1 w
\]

\[
(7.1.15) \quad \Omega_{-1} = e^w E_{-1} e^{-w},
\]

and the ZCC is equivalent to

\[
(7.1.16) \quad \partial_{1,-1} w = [e^w E_{-1} e^{-w}, E],
\]

the “two-dimensional Toda lattice.”

(for \( \mathfrak{g} = A_1^{(1)} \), this gives the Sinh-Gordon equation \( \partial_{1,-1} w = 2sinh(2w). \)

7.2 Before we attack the main question as to which extent \( \Omega_1 \) determines \( \Omega_{-1} \) in general, we prove the following partial result:

Lemma (7.2.1.) \( \partial_1 \Omega_{-j} \) is a universal \( \partial_1 - \partial_{-j} \)-differential polynomial in \( \Omega_1 \), for \( j \geq 1 \).

Proof. Along the lines of the proof of Theorem 6.3.2., we obtain

\[
(7.2.1) \quad h^- \Omega_{-j} = (\partial_{-j} e^{q_I}) e^{-q_I} + e^{q_I} ((\partial_{-j} q_K) + E_{-j}) e^{-q_I},
\]

where

\[
h^- \Omega_{-j} = \text{Ad} (h^-)^{-1} \Omega_{-j} \text{ and } h^- \in Q.
\]

We single out the part of the above expression which lies in \( \text{Ker ad} E \), namely,

\[
(7.2.2) \quad (\partial_{-j} e^{-q_I}) e^{q_I} + e^{-q_I} h^- \Omega_{-j} e^{q_I} = \partial_{-j} q_K + E_{-j}.
\]

Then we decompose (7.2.2) with respect to the canonical grading, using

\[
(7.2.3) \quad h^- \Omega_{-j} = (h^- \Omega_{-j})_{-1} + (h^- \Omega_{-j})_{-2} + \cdots.
\]

and \( (h^- \Omega_{-j})_{-1} = \Omega_{-j,-1} \), to obtain for \( \text{cdeg} = -1 \) :

\[
-\partial_{-j} q_{K_{-1}} + \Omega_{-j,-1} = \partial_{-j} q_{K_{-1}} + (E_{-j})_{-1}.
\]

From the proof of Theorem (6.3.2) we know, that \( q_I \) and \( \partial_1 q_K \) are \( \partial_1 \)- differential polynomials in \( h^- \Omega_1 \), therefore \( \partial_1 (h^- \Omega_{-j,-1}) \) is a \( \partial_1 - \partial_{-j} \)- differential polynomial in \( h^- \Omega_1 \).

Now we inspect the component of (7.1.2) of degree \(-i\):

\[
(h^- \Omega_{-j})_{-i} = \partial_{-j} q_{K_{-i}} + (F_{-j})_{-i} (q_I, (h^- \Omega_{-j})_{-1}, \cdots (h^- \Omega_{-j})_{-i+1})_{-i}
\]

where \( (F_{-j})_{-i} \) is a polynomial in its arguments. Inductively, we conclude, that \( \partial_1 ((h^- \Omega_{-j})_{-i}) \) is a differential polynomial in \( h^- \Omega_1 \) of the desired form. In fact, \( F_{-j} \) is also a polynomial in \( \Omega_1 \) with coefficients depending analytically on \( h^- \). However, in the neighborhood of the identity, there is no explicit dependence on \( h^- \) and thus \( F_{-i} \) is \( h^- \) independent.
7.3 We want to investigate more closely to which extent $\Omega_1$ determines the quantities involved in the splitting procedure.

To this end, let $h$ and $k \in \mathcal{H}$ be such that the associated potentials $\Omega^h_1$ and $\Omega^k_1$ are equal. Denote by $g^+_h$ and $g^+_k$ the splitting components associated with $h$ and $k$ respectively. With this notation we can show

**Proposition (7.3.1).**

(a) 

$$\Omega^h_1 = \Omega^k_1 \text{ iff }$$

(7.3.1) 

$$g^-_h = g^-_k \gamma^-, \gamma^- \in \Gamma^R \cap Q^R$$

and

(7.3.2) 

$$g^+_h = g^+_k e^{t_1 E^r} m^+ e^{-t_1 E^r}, \ m^+ \in P^r \text{ and } \partial_1 \gamma^- = \partial_1 m^+ = 0.$$ 

(b) If $\Omega^h_1 = \Omega^k_1$ then $\Omega^h_j = \Omega^k_j$, for all $j > 0$.

(c) $\Omega^h_{-1} = \Omega^k_{-1}$ iff

(7.3.3) 

$$g^-_h = g^-_k e^{-t_{-1} E^r} m^- e^{t_{-1} E^r}, \ m^- \in Q^R$$

and

(7.3.4) 

$$g^+_h = g^+_k \gamma^+, \gamma^+ \in \Gamma^r \cap P^r \text{ and } \partial_{-1} \gamma^+ = \partial_{-1} m^- = 0.$$ 

(d) If $\Omega^h_{-1} = \Omega^k_{-1}$ then $\Omega^h_j = \Omega^k_j$ for all $j < 0$.

(e) If $\Omega^h_1 = \Omega^k_1$ and $\Omega^h_{-1} = \Omega^k_{-1}$ then $\Omega^h_j = \Omega^k_j$ for all $j \in \mathbb{Z}$.

(f) If $\Omega^h_{-1} = \Omega^k_{-1}$ then $\Omega^h_1 = \Omega^k_1$

(g) If $\Omega^h_{-1} = \Omega^k_{-1}$ then $\Omega^h_j = \Omega^k_j$ for all $j \in \mathbb{Z}$.

(h) If $\Omega^h_{-1} = \Omega^k_{-1}$ and $\Omega^h_1 = \Omega^k_1$ then $g^-_h = g^-_k \gamma^-$ and $g^+_h = g^+_k \gamma^+$ with $\gamma^+$ and $\gamma^-$ independent of $t_1$ and $t_{-1}$.

**Proof.**

(a) From §5.3 we know

(7.3.5) 

$$(\Omega^h_1)^R = (\partial_1 g^-_k)(g^-_h)^{-1} + g^-_h E^R (g^-_h)^{-1} = \partial_1(g^+_h e^{t_1 E^r})(g^-_h e^{t_1 E^r})^{-1},$$

(7.3.6) 

$$(\Omega^h_1)^r = \partial_1(g^+_h e^{t_1 E^r})(g^+_h e^{t_1 E^r})^{-1}.$$ 

It is straightforward to show that

(7.3.7) 

$$(\partial_j g) g^{-1} = (\partial_j \tilde{g}) \tilde{g}^{-1} \text{ if and only if } g = \tilde{g} \text{ and } \partial_j \tilde{g} = 0.$$
Thus:

\begin{equation}
(7.3.8) \\
g_h^+ = g_k^+ e^{t_1 E^r} \gamma^- e^{-t_1 E^r}
\end{equation}

\begin{equation}
(7.3.9) \\
g_h^- = g_k^- e^{t_1 E^r} m^+ e^{-t_1 E^r}
\end{equation}

for some $\gamma^- \in G^R$, $m^+ \in G^r$ independent of $t_1$. Setting $t_1 = 0$ shows $\gamma^- \in Q^R$ and $m^+ \in P^r$. Moreover,

\begin{equation}
(7.3.10) \\
(g_k^-)^{-1} g_h^- = e^{t_1 E^R} \gamma^- e^{-t_1 E^R} \in Q^R,
\end{equation}

thus $\gamma^- \in \Gamma^r$ by Theorem (3.7.2). This settles the “only if” part.

The “if” part is straightforward.

\begin{equation}
(7.3.11) \\
(\Omega_1^h)^R = \partial_1 (g_k^- \gamma^- e^{t_1 E^R}) (g_k^- \gamma^- e^{t_1 E^R})^{-1} \\
= \partial_1 g_k^- (g_k^-)^{-1} + g_k^- (\partial_1 \gamma^-) (\gamma^-)^{-1} (g_k^-)^{-1} \\
+ g_k^- \gamma^- E^r (\gamma^-)^{-1} (g_k^-)^{-1} \\
= (\Omega_1^k)^R, \text{ since } \partial_1 \gamma = 0 \text{ and } [\gamma^-, E^R] = 0.
\end{equation}

\begin{equation}
(7.3.12) \\
(\Omega_1^h)^R = \partial_1 (g_k^+ e^{t_1 E^r} m^+) (g_k^+ e^{t_1 E^r} m^+)^{-1} \\
= (\partial_1 g_k^+)(g_k^+)^{-1} + g_k^+ E^r (g_k^+)^{-1} = (\Omega_1^k)^r,
\end{equation}

since $\partial_1 m^+ = 0$.

(b) Again, from 5.3 and (a) we know

\begin{equation}
(7.3.13) \\
(\Omega_j^h)^R = \partial_j (g_h^- e^{t_j E_j^R}) (g_h^- e^{t_j E_j^R})^{-1} \\
= \partial_j (g_h^- \gamma^- e^{t_j E_j^R}) (g_h^- \gamma^- e^{t_j E_j^R})^{-1}, \\
= (\Omega_j^k)^R + g_k^- (\partial_j \gamma^-) (\gamma^-)^{-1} (g_k^-)^{-1}.
\end{equation}

This shows

\begin{equation}
(7.3.14) \\
(\Omega_j^h)^R - (\Omega_j^k)^R \in \mathfrak{q}^R.
\end{equation}

On the other hand, we know from Proposition (5.3.1) that $(\Omega_j^s)^R \in \mathfrak{p}^R$, for $j > 0$. Whence

\begin{equation}
(\Omega_j^h)^R - (\Omega_j^k)^R \in \mathfrak{p}^R.
\end{equation}

This shows $(\Omega_j^h)^R = (\Omega_j^k)^R$. Note that the latter condition defines $\Omega_j$ uniquely, since two Laurent polynomials that coincide on the circle are equal. Thus: $\Omega_j^h = \Omega_j^k$.

(c) As in (a) one concludes

\begin{equation}
(7.3.15) \\
g_h^- = g_k^- e^{t_{-1} E_{-1}^R} m^- e^{-t_{-1} E_{-1}^R}.
\end{equation}
\[ g_h^+ = g_k^+ e^{t_1 E_{1-}} \gamma^+ e^{-t_1 E_{1-}} \]

for some \( m^- \in Q^r, \gamma^+ \in P^r \), such that \( \partial_{-1} \gamma^+ = \partial_{-1} m^- = 0 \).

Moreover,

\[ (g_k^+)^{-1} g_h^+ = e^{t_1 E_{1-}} \gamma^+ e^{-t_1 E_{1-}} \in P^r, \]

thus \( \gamma^+ \in \Gamma^r \cap P^r \) by Theorem (3.7.3). This settles the “only if” part. The “if” part is straightforward. (d) As in (b) one computes

\[ (\Omega_j^h)^r - (\Omega_j^k)^r \in p^r. \]

On the other hand we see from (5.3.3) that

\[ \Omega_j^R - (E_j^R)^{(0)} \in q^R, \]

thus

\[ (\Omega_j^h)^R - (\Omega_j^k)^R \in q^R. \]

This shows that \( \Omega_j^h = \Omega_j^k \).

(e) follows from (b) and (d)

(f) Using (c) we derive

\[ (\Omega_1^h)^r = \partial_1 (g_h^+ e^{t_1 E_r}) (g_k^+ e^{t_1 E_r})^{-1} \]

\[ = \partial_1 (g_k^+ \gamma^+ e^{t_1 E_r}) (g_k^+ \gamma^+ e^{t_1 E_r})^{-1} \]

\[ = (\Omega_1^k)^r + g_k^+ (\partial_1 \gamma^+) (\gamma^+)^{-1} (g_k^+)^{-1}. \]

We need to show that the second term vanishes. From (5.3.3) and (7.3.13) we know that

\[ \Omega_1^R - E^R \in (\mathfrak{g}^{(0)} \oplus \mathfrak{g}_0)^R \]

thus

\[ (\Omega_1^h)^r - (\Omega_1^k)^r = \text{Ad}(g_k^+) ((\partial_1 \gamma^+) (\gamma^+)^{-1}) \in (\mathfrak{g}^{(0)} + \mathfrak{g}_0)^r. \]

Now \( (\partial_1 \gamma^+) (\gamma^+)^{-1} \in (\text{Ker \ ad } E \cap p)^r \).

At this point we apply Theorem (B.1.2) from the Appendix B stating that \( \text{Ker \ ad } E \cap \mathfrak{g}^{(0)} = 0 \). Thus \( \min \text{ pdeg}(\text{Ad}(g_k^+) ((\partial_1 \gamma^+) (\gamma^+)^{-1}) \geq 1 \) which implies the claim.
(g) obvious from (b), (d), (f),
(h) follows from (a) and (c).

**Remark.** The results of Theorems (7.1.1) and (7.3.1) give the following picture (where “→” means “determines uniquely”):

\[ \Omega_1 \rightarrow \Omega_j \text{ (} j > 0 \text{)} \]
\[ a \uparrow \]
\[ \Omega_{-1} \rightarrow \Omega_j \text{ (} j < 0 \text{)} \]

**Proposition 7.3.2.** If \( \text{Ker} \, \text{ad} \, E \cap g^{(-1)} = \{0\} \), then \( \Omega^h_1 = \Omega^k_1 \) implies \( \Omega^h_{-1} = \Omega^k_{-1} \).

**Proof.** Assume \( \Omega^h_1 = \Omega^k_1 \). Then by Lemma (7.2.1) we have \( \partial_1 \Omega^h_{-1} = \partial_1 \Omega^k_{-1} \), hence

(7.3.8) \[ \Omega^h_{-1} = \Omega^k_{-1} + C_{-1}, \]

where \( C_{-1} \) depends on \( t_{-1} \) only. (We ignore variables different from \( t_1 \) and \( t_{-1} \) here.)

By Proposition (7.3.1 a)

(7.3.9) \[ g^h_\gamma = g^k_\gamma (t_{-1}) \quad \text{and} \quad \gamma^-(t_{-1}) \in \Gamma_\cap Q. \]

This implies (setting \( g^h_\gamma(0,0) = (h^-)^{-1}, g^h_\gamma(0,0) = (k^-)^{-1} \))

(7.3.10) \[ \gamma^-(0) = k^- (h^-)^{-1} \in \Gamma_. \]

Next we evaluate the \((−1)\) - potentials, using the formulas of §5.3:

\[
(\Omega^h_{-1})^R = (\partial_{-1} g^h_\gamma)(g^h_\gamma)^{-1} + g^h_\gamma (\partial_{-1} \gamma^-)(\gamma^-)^{-1}(g^h_\gamma)^{-1} + g^h_\gamma E^R_{-1}(g^h_\gamma)^{-1} = (\Omega^k_{-1})^R + g^h_\gamma (\partial_{-1} \gamma^-)(\gamma^-)^{-1}(g^h_\gamma)^{-1}
\]

From (7.3.8) we conclude

(7.3.11) \[ (\partial_{-1} \gamma^-)(\gamma^-)^{-1} = (g_k^-)^{-1} C_{-1} g^k_\gamma. \]

From (7.1.6) we see that

(7.3.12) \[ C_{-1} \in q^{(-1)}. \]

Upon decomposing \( C_{-1} \) with respect to the canonical grading we get

(7.3.13) \[ C_{-1} = c_{-1} + \cdots + c_{-m}, \quad \text{for some finite } m. \]

Let

(7.3.14) \[ (g_k^-)^{-1} = k^- \exp(r_{-1} + r_{-2} + \cdots), \]

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where \( k^- \in Q \) is \( t \) and \( x \) independent. Thus (7.3.11) turns into

\[
(\partial_1 \gamma^-)(\gamma^-)^{-1} = k^-(\exp(ad(r_1 + r_2 + s))C_{-1})(k^-)^{-1}.
\]

Let

\[
(\partial_1 \gamma^-)(\gamma^-)^{-1} = s_{-1} + s_{-2} + \cdots.
\]

Now compare in (7.3.15) and (7.3.16) components of the same canonical degree. Note that \( k^- \in Q \).

First, we consider \( cdeg = -1 \): we obtain \( s_{-1} \). Now (7.3.12) and (7.3.9) imply

\[
s_{-1} = c_{-1} \in Ker ad E \cap g^{(-1)} = \{0\}, \text{thus } s_{-1} = c_{-1} = 0.
\]

Next, we consider \( cdeg = -2 \): Since \( s_{-1} = 0 \) and \( c_{-1} = 0 \), we obtain \( s_{-2} = c_{-2} \). As above we conclude \( s_{-2} = c_{-2} = 0 \).

The remainder of the proof goes by induction on the canonical degree, thus yielding \( C = 0 \).

To see that the condition of the previous Proposition is nontrivial we briefly discuss the following two examples:

**Example 1:** In the case of minimal parabolics the \( p \)-grading coincides with the canonical grading. Thus the condition \( Ker ad E \cap q^{-1} = \{0\} \) does not hold and \( \Omega_{-1} \) is not uniquely determined by \( \Omega_1 \). To see that one only needs to examine (7.1.14) and (7.1.15). However, one has a natural parametrization of \( \Omega_1 \) in terms of \( w \in g_0 \), namely \( \Omega_1 = E + \partial_1 w \), which gives \( \Omega_{-1} = e^w E_{-1} e^{-w} \). It remains an interesting open question whether one can find a natural parametrization in terms of elements of \( g_0 \) in each case \( Ker ad E \cap q^{(-1)} \neq \{0\} \).

**Example 2:** Let \( g^{ln} \) be of type \( A^{(1)}_3 \), and \( g = \mathfrak{sl}_4(A_w) \) where \( A_w \) is the Wiener Algebra w.r.t. some weight \( w \) (cf. 1.1). Denote by \( E_{ij} \) the \( 4 \times 4 \)-matrix with 1 in the \((i,j)\)-position and 0’s elsewhere. A set of Chevalley generators is given by

\[
e_0 = \lambda E_{41}, \ e_1 = E_{12}, \ e_2 = E_{23}, \ e_3 = E_{34},
\]

\[
f_0 = \lambda^{-1} E_{14}, \ f_1 = E_{21}, \ f_2 = E_{32}, \ f_3 = E_{43},
\]

\[
h_i := [e_i, f_i] \quad \text{for } i = 0, 1, 2, 3.
\]

Let \( \mathfrak{p} \) be the standard parabolic subalgebra generated by \( g_0 \oplus g_+, f_1 \) and \( f_3 \), i.e.

\[
\mathfrak{p} = \left\{ \left( \sum_{n \geq 0} \lambda^n A_n \middle| \sum_{n \geq 0} \lambda^n B_n \right), \left( \sum_{n \geq 0} \lambda^n C_n \middle| \sum_{n \geq 0} \lambda^n D_n \right) : A_n, D_n \in \mathfrak{sl}_2(\mathbb{C}), \ B_n, C_n \in \mathfrak{gl}_2(\mathbb{C}) \right\}.
\]

It is easy to see that

\[
q^{(-1)} = \left\{ \left( \begin{array}{cc} 0 & \lambda^{-1} A \\ B & 0 \end{array} \right) : A, B \in \mathfrak{gl}_2(\mathbb{C}) \right\}.
\]
Thus, $q^{(-1)} \cap \text{Ker} \text{ad} E \neq \{0\}$, since

$$E_{-2} = E^{-2} = \begin{pmatrix} 0 & \lambda^{-1}I_2 \\ I_2 & 0 \end{pmatrix} \in q^{(-1)} \cap \text{Ker} \text{ad} E,$$

where $I$ denotes the $2 \times 2$ identity matrix.

For “nontwisted” $\mathfrak{g}$, i.e. $\mathfrak{g}$ of type $X_\ell^{(1)}$ we are able to determine all the cases where the condition of Proposition 7.3.2. is violated, i.e. where

$$\text{Ker} \text{ad} E \cap \mathfrak{g}^{(-1)} \neq \{0\}.$$

The result is as follows [27].

**Theorem (7.3.3).** Let $\mathfrak{g}$ be of type $X_\ell^{(1)}$, and $h$ the Coxeter number of $\mathfrak{g}$, where $\mathfrak{g}$ is finite dimensional and simple of type $X_\ell$. Let $k$ be a positive exponent of $\mathfrak{g}$.

The following are equivalent:

(a) $k$ divides $h$

(b) There is a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that

$$(\text{Ker} \text{ad} E)^{-k} \cap \mathfrak{g}^{(-1)} \neq \{0\}.$$

**Remark** The relations between $h, k$ and $\mathfrak{p}$ for which $(\text{Ker} \text{ad} E)^{-k} \cap \mathfrak{g}^{-1} \neq 0$ have been investigated in detail in [27]. The following table summarizes the main result of that paper.

| $X_\ell$ | $h$ | exponents $k$ | $k|h?$ |
|----------|-----|---------------|--------|
| $\ell$   | $\ell + 1$ | 1, 2, ..., $\ell$ | depends on $\ell$ |
| $B_\ell$ | $2\ell$ | 1, 3, ..., $2\ell - 1$ | depends on $\ell$ |
| $C_\ell$ | $2\ell$ | 1, 3, ..., $2\ell - 1$ | depends on $\ell$ |
| $D_\ell$ | $2\ell - 2$ | 1, 3, ..., $2\ell - 3$, $\ell - 1$ | depends on $\ell$ |
| $E_6$    | 12   | 1, 4, 5, 7, 8, 11 | $k = 4$ |
| $E_7$    | 18   | 1, 5, 7, 9, 11, 13, 17 | $k = 9$ |
| $E_8$    | 30   | 1, 7, 11, 13, 17, 19, 23, 29 | none |
| $F_4$    | 12   | 1, 5, 7, 11 | none |
| $G_2$    | 6    | 1, 5 | none |

This shows in particular that the situation is fairly easy for the exceptional Lie algebras. We will not pursue this any further in this paper.
7.4 We present below an example of a “negative potential” and the associated differential equation. We use the following notation: the derivatives with respect to \( x \) and \( t \) are denoted either by \( \partial_{x,t} \) or by using \( x \) or \( t \) as a subscript.

We will follow the procedure described in the last two chapters. The most important point we would like to emphasize is that by results of Section 6 it suffices to work in the neighborhood of the identity of the group \( G \). This allows one to carry out computations on a Lie algebraic level. In this example, we want to closely investigate to which extent \( \Omega_1 \) determines the quantities involved in the splitting procedure. We consider the case of \( A_1^{(1)} \). Let \( \mathfrak{g}^{fin} = sl_2(A_w) \) where \( A_w \) is the Wiener Algebra w.r.t. some weight \( w \) (cf. 1.1). Denote by \( E_{ij} \) the \( 2 \times 2 \)-matrix with 1 in the \((i,j)\)-position and 0’s elsewhere. A set of Chevalley generators is given by

\[
\begin{align*}
e_0 &= \lambda E_{21}, \quad e_1 = E_{12}, \\
f_0 &= \lambda^{-1} E_{12}, \quad f_1 = E_{21}, \\
\alpha_i^\vee &:= [e_i, f_i] \quad \text{for} \quad i = 0, 1.
\end{align*}
\]

Let \( \mathfrak{p} \) be the standard parabolic subalgebra generated by \( \mathfrak{g}_0 \oplus \mathfrak{g}_{+}, f_1 \), i.e.

\[
\mathfrak{p} = \left\{ \begin{pmatrix} \sum_{n \geq 0} \lambda^n a_n & \sum_{n \geq 0} \lambda^n b_n \\ \sum_{n \geq 0} \lambda^n c_n & -\sum_{n \geq 0} \lambda^n a_n \end{pmatrix} : a_n, b_n, c_n \in \mathbb{C} \right\}.
\]

The \( p \)-grading in this case agrees with the grading according to the powers of \( \lambda \), i.e. \( p \)-deg\((E_{ij} \lambda^k) = k \). To further simplify the computation we observe that by Theorem (6.3.2) we can use the formula

\[
(7.4.1) \quad \Omega_j^R = (\partial_j e^{q_l}) e^{-q_l} + e^{q_l} (\partial_j q_K + E_j^R) e^{-q_l}.
\]

In fact we can ignore the size of the contour and set \( R = 1 \). Our goal is to compute \( \Omega_{-1} \). To this end we will have to find the relevant contributions from both \( q_l \) and \( q_K \). We first decompose these two with respect to the canonical grading. This for the case at hand amounts to setting \( c\)-deg\((E_{ij} \lambda^k) = j - i + 2k \). This makes \( E \) homogeneous of degree 1, consequently \( E_j \) has degree \( j \). We write

\[
q_l = q_{l,-1} + q_{l,-2} + \ldots,
\]

similarly

\[
q_K = q_{K,-1} + q_{K,-2} + \ldots.
\]

By (7.1.6) the essential part of \( \Omega_{-1} \) lives in \( \mathfrak{g}^{(-1)} \). In the present case this gives

\[
(7.4.2) \quad \mathfrak{g}^{(-1)} = \lambda^{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a, b, c \in \mathbb{C}.
\]

We observe that the minimal canonical degree appearing in \( \Omega_{-1} \) is \(-3 \). We will therefore use the following variables \( \{ q_{l,-i}, q_{K,-i}, i = 1..3 \} \). In fact \( q_{K,-2} = 0 \) as \( E^{-2} \notin \mathfrak{g} \). Furthermore, it is useful to observe that \( v \in \text{Im ad}(E) \cap \mathfrak{g} \), if and only if there exists a diagonal matrix \( a = \text{diag}(\alpha, -\alpha), \alpha \in \mathbb{C} \), such that \( v = aE^i \). A similar result for \( \text{ker ad}(E) \) states that \( a \) has
to be a multiple of the identity. Thus we can parametrize \( \{q_1, -1, q_1, -2q_1, -3, q_\mathcal{K}, -1, q_\mathcal{K}, -3\} \) in terms of 5 scalars \( \{h_1, h_2, h_3, w_1, w_3\} \) by forming the appropriate diagonal matrices. Using (7.1.6) we can now express \( \Omega_{-1} \) in terms of these variables. This is a purely Lie algebraic computation and the result is:

\[
\Omega_{-1} = \begin{pmatrix}
\frac{2h_1 + \partial_t h_2 + 2\partial_t w_1 h_1}{\lambda}
& \frac{1 + \partial_t h_1 + \partial_t w_1}{\lambda}

(1 - \partial_t h_1 + \partial_t w_1) + a
& -\frac{2h_1 + \partial_t h_2 + 2\partial_t w_1 h_1}{\lambda}
\end{pmatrix}
\]

where

\[a = -\lambda^{-1}(2h_1^2 + h_1 \partial_t h_2 - h_2 \partial_t h_1 + 2\partial_t w_1 h_1^2 + 2\partial_t w_1 h_2 + 2h_2 - \partial_t w_3 + \partial_t h_3)\]

We parametrize the potential \( \Omega_1 \) as follows

\[
\Omega_1 = \begin{pmatrix} u \\
\lambda + u_x - u^2 \\
1 \end{pmatrix}
\]

Now we use (7.4.1) to express \( \{h_1, h_2, h_3\} \) in terms of \( u \) and its \( x \) derivatives. The result is:

\[
h_1 = \frac{1}{2} u, \ h_2 = -\frac{1}{2} u_x + \frac{1}{4} u^2, \ h_3 = \frac{1}{4} u_{xx} - \frac{1}{2} u u_x + \frac{1}{12} u^3.
\]

From the proof of Theorem (6.3.2) we know that \( \partial_x q_\mathcal{K} \) is a differential polynomial in \( \Omega_1 \), in the case of the present example we obtain:

\[
w_{1,x} = \frac{1}{2} u_x, \ w_{3,x} = -\frac{1}{4} (u_x)^2 - \frac{1}{8} u^2 u_x + \frac{1}{4} u u_{xx}.
\]

We would like to point out that in fact \( w_1 \) is uniquely determined. This is clear from the fact that if \( \exp(q_1) \exp(q_\mathcal{K}) \) and \( \exp(q_1) \exp(q_\mathcal{K}) \exp(cE_{-1}) \) satisfy (7.4.1) and both are in \( Q \) then this implies \( \exp(cE_{-1}) \in Q \) for some \( c \in \mathbb{C} \). Note, however, that in the present example \( g^{(-1)} \cap \ker ad(E) = 0 \), in particular \( E_{-1} \notin \mathfrak{q} \). Moreover the group \( Q = \overline{Q}^{fin} \) has the property that \( Q - I \) consists of functions analytic around \( \infty \) in \( \lambda \) and being zero there. This implies \( w_1 = \frac{1}{2} u \). Substituting (7.4.5) and \( w_1 \) back into \( \Omega_{-1} \) yields:

\[
\Omega_{-1} = \begin{pmatrix}
-\frac{u_{x,t} - 2uu_x - 2u}{2\lambda}
& \frac{1 + uu_x}{2\lambda}

\frac{1 + uu_x}{2\lambda} - 2uu_x - 2u
\end{pmatrix}
\]

where

\[b = -\lambda^{-1}\frac{1}{8}(-8u_x + 8u^2 + 7u^2 u_t + 2u_{xxt} - 6u_x u_t - 6uu_{xt} - 8w_{3,t}).\]

In the next step we use the ZCC to express \( \partial_t w_3 \) in terms of \( \partial_x - \partial_t \) derivatives of \( u \). The ZCC reads:

\[
\partial_t \Omega_1 - \partial_x \Omega_{-1} + [\Omega_1, \Omega_{-1}] = 0.
\]

The final formula for \( \Omega_{-1} \) takes now a simple form:
\[ \Omega_{-1} = \left( 1 - \frac{u_{xt} - 2uu_t - 2u}{2\lambda} \frac{2\lambda}{2\lambda} - \frac{u_{xx} + 2u^2 + 2u^2u_t + uu_{xt} - 4uu_{xt} - 2uu_t}{2\lambda} \right). \]

What now remains to do is to use the latter form of \( \Omega_{-1} \) to compute the ZCC. A straightforward computation leads to a simple ZCC:

\[
\frac{1}{\lambda} \begin{pmatrix}
0 & 0 \\
u_{xxx} - 4u_{xx} - 8u_{xt}u_t - 4u_tu_{xx} & 0 \\
0 & 0
\end{pmatrix} = 0.
\]

Summarizing, the \( \Omega_{-1} \) and \( \Omega_1 \)-generated flows give rise to the following partial differential equation:

\[ u_{xxx} = 4u_{xx} + 8u_{xt}u_t + 4u_tu_{xx}. \]

With the help of the simple substitution \( z(x, t) = u(4x, t) + t \) we can reduce this equation to

\[ z_{xxx} = z_{xx}z_t + 2z_xz_{xt}. \]

We would like to add that from the point of view of the KdV theory, to which one passes by setting \( u = z_x \), equation (7.4.10) is an integro-differential equation. Thus, if one thinks of it as a member of the KdV hierarchy, then one is confronted with the question of its utility for the KdV theory. This point remains to be clarified. Another remark is in order here. Equation (7.4.10) is a special case of the Bogoyavlenskii equation [1], namely,

\[ z_{xv} = z_{xxx} - z_{xx}z_t - 2z_xz_{xt}, \]

where \( v \) is the third independent variable. The factorization problem for this equation was formulated in [21].

Equation (7.4.10) is one of many reductions of (7.4.11), namely the one corresponding to the symmetry generator \( \partial_v \) (\( z \) is \( v \) independent). Another reduction which has been studied is the one corresponding to the symmetry generator \( \partial_t - \partial_v \). The resulting equation was studied by Hirota and Satsuma [2] as a model for shallow water waves. In particular the aforementioned authors found its soliton solutions.

We would like to conclude the discussion of this section with the remark that, since the time the first draft of this paper was written, a theory of ‘negative’ potentials has been further advanced by G. Haak [29].
Appendix A: Proofs of Propositions (3.3.1) and (3.5.2)

Proposition (3.3.1). Let $P = P_X$ be a parabolic subgroup of $G$ and $w \in W$. Then

(a) $P = P_X = G_X Q_X^+ = Q_X^+ G_X \cong G_X \times Q_X^+$.

(b) The stabilizer $U_w$ in $U_-$ of $wP \in G/P$ is $U_w = U_\cap wPw^{-1}$. Moreover, $\dim U_w < \infty$.

(c) There exists a closed subgroup $V_w$ of $U_-$ such that group multiplication induces a diffeomorphism $U_- \cong U_w \times V_w$.

Proof. (a) First we note that $g^{(0)}$ and $g^{(+)}$ are closed complementary subalgebras of $\mathfrak{p}$. Let $G^{(0)}$ and $G^{(+)}$ denote the corresponding integral subgroups of $P$. Arguing with the gradings as before, we see $Ad(G^{(0)} \cap G^{(+)}) = \{I\}$. But this implies $G^{(0)} \cap G^{(+)} = \{I\}$ as in the proof of Theorem 2.4.1. Therefore $P = P_X = G^{(0)} G^{(+)} \cong G^{(0)} \times G^{(+)}$ by Corollary 3.1.4.

To prove $P = P_X = G_X Q_X^+ = Q_X^+ G_X \cong G_X \times Q_X^+$ it now suffices to show $G^{(0)} \cong G_X \times A_Q$, where $A_Q = H \cap Q_X^+$. We want to apply Corollary (3.1.4). To this end we note that $\hat{\mathfrak{g}}$ and $\mathfrak{a}_Q$ are closed complementary subalgebras of $\mathfrak{g}^{(0)}$. Therefore, it suffices to prove $\hat{G}_X \cap A_Q = \{I\}$. In view of the Iwasawa decomposition [11, §5] of $G \cong K \times A \times U_+$ and $\hat{G}_X \cong K_X \times \hat{A} \times (\hat{G}_X)_+$ we see that it suffices to show $A_Q \cap \hat{A} = \{I\}$. We know that $[\hat{\mathfrak{g}}_X, \mathfrak{a}_Q] = 0$, so $Ad A_Q|\hat{G}_X = I$. Therefore for $g \in A_Q \cap \hat{A}$ we have $Ad g|\hat{G}_X = I$. Since $g \in \hat{A} \subseteq \hat{G}_X$, $g = I$ follows.

(b) The first statement is easy to verify. To see that $\dim U_w < \infty$ it suffices to note $l(w) = \#\{\alpha \in \Delta_+; \; w(\alpha) < 0\}$.

(c) The following proof is an adaptation of [12;8.6]. Set $u_w^\pm = u_- \cap wPw^{-1}$. In the overlying Kac-Moody algebra we see that $u_w^\pm$ is invariant under the action of the Cartan algebra. Therefore $u_w^\pm$ is a direct sum of root spaces. Moreover, $u_w^\pm = u_- \cap w p^{opp} w^{-1}$, $p^{opp}$ being the opposite parabolic, is a closed subalgebra of $u_-$ such that $u_- = u_w^\pm \oplus \hat{u}_w^\pm$. We note that $p^{opp}$ can be obtained by applying the Chevalley involution $\omega$ of $\mathfrak{g}$ to $\mathfrak{p}$ [10;§1.3]. Therefore, by [23, Ch.3, §6, Theorem 2], there exists a connected subgroup $V_w$ of $U_-$ with Lie algebra $\mathfrak{v}_w$. More precisely, we have the following closed, connected subgroups

$$U_w^\pm = U_- \cap w Pw^{-1} \quad \text{and} \quad V_w = U_- \cap w P^{opp} w^{-1}$$

with Lie algebra $u_w^\pm$ and $\hat{u}_w^\pm$ respectively. Let $u_-^{(k)}$ denote all elements in $u_-$ of sufficient high degree $\geq k$. Then $u_-^{(k)} \subset u_w^\pm$ and $u_w^\pm$ is contained in the natural complement of $u_-^{(k)}$ in $u_-$. Moreover, $u_\mathfrak{c}^{(k)}$ is an ideal in $u_-$, whence $\hat{u}_- = u_- \mod u_-^{(k)}$ is a finite dimensional, nilpotent, Lie algebra, with complementary subalgebras $\hat{u}_w^\pm$ and $\hat{\hat{u}}_w^\pm$. It is not difficult to see that for the corresponding groups $\hat{U}_w$, $\hat{U}_\mathfrak{c}^\pm$ and $\hat{V}_w$ we have $\hat{U}_- = U_w^\pm \hat{V}_w$. Hence $U_- = U_w^\pm V_w$ and the multiplication map $m : U_w^\pm \times V_w \to U_-$ is surjective. It is well known that $m$ is everywhere regular. Therefore it suffices now to show that $m$ is injective. This is equivalent with $U_w^\pm \cap V_w = \{I\}$. Let $g \in U_w^\pm \cap V_w$. Then we map $g$ to $\hat{g} \in \hat{U}_w \cap \hat{V}_w$. Since $\hat{U}_-$ is finite dimensional we know $\hat{g} = \exp(\hat{u}) = \exp(\hat{v})$, $u \in u_w^\pm$, $v \in \hat{u}_w^\pm$. Moreover, $\hat{u}_-$ has a faithful representation consisting of nilpotent matrices. Hence $\hat{u} = \hat{v}$; i.e. $u = v + \mathfrak{q}$, $\mathfrak{q} \in u_-^{(k)}$. But this implies $u = 0$, whence $g \in U_-^{(k)}$. Since we can choose $k$ arbitrary large, $g = I$ follows.
Proposition (3.5.2). For every \( w \in W/W \) we have
\[
C_X(w) = \bigcup_M C_X(w')
\]
where \( M = \{w' \in W/W, \ w' \succeq w\} \).

Proof. At first we follow [24]. Let \( w' \) be \( X \)-reduced and \( w' \succeq w \). Then for some \( i \)
\[
w' = w_1r_iw_2, \text{ where } w_1(\alpha_i) > 0 \text{ and } w_2^{-1}(\alpha_i) > 0 \text{ and } w = w_1w_2.
\]
Then
\[
B_-w_1w_2P = B_-w_1U_{-\alpha_i}w_2(\alpha_i)w_2P = B_-w_1U_{-\alpha_i}H_iU_{\alpha_i}w_2P.
\]
We know that the closure of \( U_{-\alpha_i}H_iU_{\alpha_i} \), where \( H_i = \exp C h_i \), is a subgroup \( G_i \) of \( G \) which is isomorphic with \( SL(2,\mathbb{C}) \). In particular, \( r_i \in G_i \), whence
\[
B_-w'P = B_-w_1r_iw_2P \subset B_-w_1U_{-\alpha_i}H_iU_{\alpha_i}w_2P \subset B_-w_1U_{-\alpha_i}H_iU_{\alpha_i}w_2P = B_-w_1w_2P.
\]
This proves \( C_X(w') \subset \overline{C_X(w)} \). Conversely, \( \overline{C_X(w)} = \bigcup_{w' \in J} C_X(w') \), where \( J \) is some subset of the set \( W' \) of \( X \)-reduced elements. Assume first that \( X = \emptyset \), i.e. \( B = P_X \). Then we consider the representation of \( \pi \) of \( \mathfrak{g} \) in \( gl_{res} \) described in [12;Ch.6]. Our assumptions on the weights \( w \) imply that \( \pi \) is continuous. In particular,
\[
\pi(C_\emptyset) \subset \overline{\pi(C_\emptyset)}.
\]
From [12,Proposition (7.3.3)] we can now conclude (see also [12, chap.8, in particular Theorem (8.7.2)]) that \( w' \succeq w \) for all \( w' \in J \). Consider now the general case. We note that
\[
C_X(w) = \bigcup_{\tilde{w} \in W_X} C_\emptyset(\tilde{w}w).
\]
It is important to note that here \( W_X \) is finite. Therefore, \( \overline{C_X(w)} = \bigcup_{\tilde{w} \in W_X} \overline{C_\emptyset(\tilde{w}w)} \). This implies
\[
\overline{C_X(w)} = \bigcup_{\tilde{w} \in W_X} \bigcup_{w' \succeq \tilde{w}w} C_\emptyset(w') = \bigcup_{w' \succeq w, X \text{-reduced}} C_X(w').
\]
Appendix B: Injectivity Problems

In the following we prove two theorems concerning the map \(ad E\) and its restriction to \(\mathfrak{g}^{(0)}\).

**Lemma B.1.1.**\(\text{Ker} \, ad \, E = \text{Ker} \, ad \, E_{-1}\) for any \(E_{-1} \in (\text{Ker} \, ad \, E)_{-1} \setminus \{0\}\).

**Proof.** Using the Chevalley involution \(\omega\), we get that \(\omega(\text{Ker} \, ad \, E) \subset \text{Ker} \, ad \, E_{-1}\). Moreover, by Proposition 14.3.a in [10] we get that \(\omega(\text{Ker} \, ad \, E) = \text{Ker} \, ad \, E\). So \(\text{Ker} \, ad \, E \subset \text{Ker} \, ad \, E_{-1}\). By the same argument we get \(\text{Ker} \, ad \, E_{-1} \subset \text{Ker} \, ad \, E\).

**Theorem B.1.2.** Let \(\mathfrak{g}\) be an affine Kac-Moody algebra and \(\mathfrak{p} \neq \mathfrak{g}\) a standard parabolic subalgebra of \(\mathfrak{g}\). Then: \(\text{Ker} \, ad \, E \cap \mathfrak{g}^{(0)} = \{0\}\).

**Proof.** The proof is broken down into nine subclaims:

**Claim 1:** \(\text{Ker} \, ad \, E \cap \mathfrak{g}_0 = \{0\}\).

Indeed the integer 0 is never an exponent [6; §14].

**Claim 2:** \(\text{Ker} \, ad \, E \cap \mathfrak{g}^{(0)} \subseteq \text{Ker} \, ad \, E^{(0)} \cap \mathfrak{g}^{(0)}\).

Let \(S \in \mathfrak{g}^{(0)}\) such that \([S, E] = 0\). Since \(E = E^{(0)} + E^{(1)}\), \([S, E^{(0)}] + [S, E^{(1)}] = 0\). The first term is in \(\mathfrak{g}^{(0)}\), whereas the second in \(\mathfrak{g}^{(0)}\), thus \([S, E^{(0)}] = 0\).

**Claim 3:** The map \(ad E^{(0)} : \mathfrak{g}^{(0)} \to \mathfrak{g}^{(0)}\) is nilpotent.

Clear, since \(cdeg \, E^{(0)} = 1\) and \(\mathfrak{g}^{(0)}\) is finite dimensional.

**Claim 4:** There are elements \(F^{(0)}, H^{(0)} \in \mathfrak{g}^{(0)}\) such that \(\mathfrak{Ω} := \mathbb{C}F^{(0)} \mathbb{C} \oplus H^{(0)} \oplus \mathbb{C}E^{(0)}\) is isomorphic to \(sl(2, \mathbb{C})\), i.e. \(H^{(0)} = [E^{(0)}, F^{(0)}], [H^{(0)}, E^{(0)}] = 2E^{(0)}, [H^{(0)}, F^{(0)}] = -2F^{(0)}\).

Since \(ad E^{(0)}\) is nilpotent, this follows from the Theorem of Jacobson-Morozov. Here, however, we can define the elements \(F^{(0)}\) and \(H^{(0)}\) explicitly. Let \(F^{(0)} := \sum c_i f_i\), then \(H^{(0)} = \sum c_i [e_i f_i] = \sum c_i h_i\), \([H^{(0)}, E^{(0)}] = \sum c_i [h_i, e_j] = \sum c_i a_{ij} e_j\) and \([H^{(0)}, F^{(0)}] = \sum c_i c_j [h_i, f_j] = -\sum c_j \left(\sum c_i a_{ij}\right) f_j\),

where \((a_{ij}) =: A\) is the Cartan matrix of \(\mathfrak{g}^{(0)} := [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}]\), the semisimple part of \(\mathfrak{g}^{(0)}\). Since \(\mathfrak{g}^{(0)}\) is finite dimensional, \(A\) is an invertible matrix, whence the system of equations

\[\sum_j \left(\sum_i (c_i a_{ij})\right) e_j = \sum_j 2e_j\]

has a unique solution. It is easy to verify that this gives \(F^{(0)}\) and \(H^{(0)}\) as required.

**Claim 5:** \(\alpha(H^{(0)}) = 2(ht \alpha)\) for every root \(\alpha\) of \(\mathfrak{g}^{(0)}\).

This statement is a direct consequence of \([H^{(0)}, e_i] = 2e_i, [H^{(0)}, f_i] = -2f_i\).
Claim 6: \( \text{Ker}\ ad\ E^{(0)} \cap g^{(0)} = \{0\} \).

Since \( g^{(0)} \) is an \( \mathfrak{a} \)-module, by Weyl’s Theorem it is the direct sum of a finite number of irreducible \( \mathfrak{a} \)-modules:
\[
g^{(0)} = V_1 \oplus \cdots \oplus V_r.
\]

Consider one of these irreducible \( \mathfrak{a} \)-module \( V = V_s \). Since \( \mathfrak{a} \cong s\ell(2, \mathbb{C}) \), there is a basis of \( V \) consisting of eigenvectors of \( \text{ad} \ H^{(0)} : v_{-m}, v_{-m+2}, \cdots, v_{m-2}, v_m \) such that \( [H^{(0)}, v_j] = j \cdot v_j \).

From (5) we see that \( m \) is always an even number, thus \( \dim V \) is odd. Since also \( [E^{(0)}, v_j] = v_{j+2} \), the kernel of \( \text{ad} E^{(0)} \) in \( V \) is spanned by the highest-weight vector \( v_m \). Since, by (5), this vector has a nonnegative canonical degree we conclude that \( \text{Ker}\ ad\ E^{(0)} \cap g^{(0)} = \{0\} \).

Claim 7: \( \text{Ker}\ ad\ E \cap g^{(0)} = \{0\} \).

This is a direct consequence of (2) and (6).

Claim 8: \( \text{Ker}\ ad\ E_{-1} \cap g^{(0)} = \{0\} \).

Copy the proof of (7), interchanging \( e_i \) and \( f_i \).

Claim 9: \( \text{Ker}\ ad\ E \cap g^{(0)} = \{0\} \).

This claim follows from (8) and Lemma 6.4.1.

Altogether, (1), (7) and (9) yield the claim.
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