Tensor space representations of Temperley–Lieb algebra via orthogonal projections of rank $r \geq 1$

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Abstract

Unitary representations of the Temperley–Lieb algebra $TL_N(Q)$ on the tensor space $(\mathbb{C}^n)^\otimes N$ are considered. Two criteria are given for determining when an orthogonal projection matrix $P$ of a rank $r$ gives rise to such a representation. The first of them is the equality of traces of certain matrices and the second is the unitary condition for a certain partitioned matrix. Some estimates are obtained on the lower bound of $Q$ for a given dimension $n$ and rank $r$. It is also shown that if $4r > n^2$, then $Q$ can take only a discrete set of values determined by the value of $n^2/r$. In particular, the only allowed value of $Q$ for $n = r = 2$ is $Q = \sqrt{2}$. Finally, properties of the Clebsch–Gordan coefficients of the quantum Hopf algebra $U_q(su_2)$ are used in order to find all $r = 1$ and $r = 2$ unitary tensor space representations of $TL_N(Q)$ such that $Q$ depends continuously on $q$ and $P$ is the projection in the tensor square of a simple $U_q(su_2)$ module on the subspace spanned by one or two joint eigenvectors of the Casimir operator $C$ and the generator $K$ of the Cartan subalgebra.

1 Introduction

The Temperley–Lieb (TL) algebras play an important role in the theory of subfactors, knot theory, and studies of discrete models in low dimensional physics. The Temperley–Lieb algebra of the type $A_{N-1}$ was introduced in [15]. Recall its definition.

Definition 1. Given $Q \in \mathbb{C}$ and an integer $N \geq 2$, the Temperley–Lieb algebra $TL_N(Q)$ is the unital algebra over $\mathbb{C}$ with generators $T_1, \ldots, T_{N-1}$ and the following defining relations:

\begin{align*}
T_k T_k &= Q T_k, & \text{for all } k, \quad (1) \\
T_k T_m &= T_m T_k & \text{for } |k - m| \geq 2, \quad (2) \\
T_k T_m T_k &= T_k, & \text{for } |k - m| = 1. \quad (3)
\end{align*}

The Temperley–Lieb algebra has a natural linear anti–involution:

$$T_k^* = T_k$$

for all $k$. (4)

In the present article, we will consider a particular class of representations of $TL_N(Q)$ on the tensor product space $(\mathbb{C}^n)^\otimes N$. We will denote by $M_n$ the ring of $n \times n$ complex matrices, by $I_n$ the $n \times n$ identity matrix, and by $\otimes$ the Kronecker product. Given $X \in M_n$, $X^*$ will stand for its conjugate transpose.
Definition 2. Given $Q > 0$ and an integer $n \geq 2$, a homomorphism $\tau : TL_N(Q) \to M_{nN}$ is a unitary tensor space representation of $TL_N(Q)$ if

$$
\tau(T_k) = I_n^\otimes(k-1) \otimes T \otimes I_n^\otimes(N-k-1), \quad k = 1, \ldots, N-1,
$$

and matrix $T \in M_{n^2}$ satisfies the following relations:

\begin{align}
(T1) & \quad T^* = T, \\
(T2) & \quad TT = QT, \\
(T3) & \quad T_{12} T_{23} T_{12} = T_{12}, \\
(T4) & \quad T_{23} T_{12} T_{23} = T_{23},
\end{align}

where $T_{12} \equiv T \otimes I_n$ and $T_{23} \equiv I_n \otimes T$.

Given $T \in M_{n^2}$ satisfying (T2)–(T4) with $Q > 0$, set $R = q I_n - T$, where $q$ is a root of the equation $q + q^{-1} = Q$. Note that $R$ is invertible, $R^{-1} = q^{-1} I_n - T$. Consider the following map $\mathcal{R} : \mathbb{C} \to M_{n^2}$:

$$
\mathcal{R}(u) = \begin{cases} 
  u R - R^{-1} & \text{if } Q \neq 2, \\
  u R + I_{n^2}, & \text{if } Q = 2.
\end{cases}
$$

One of the main motivations to study tensor space representations of the TL algebra is the following well–known fact: the map (6) provides a non–trivial example of an $R$–matrix, i.e. a solution to the Yang–Baxter equation:

$$
\mathcal{R}_{12}(u) \mathcal{R}_{23}(u \bullet v) \mathcal{R}_{12}(v) = \mathcal{R}_{23}(v) \mathcal{R}_{12}(u \bullet v) \mathcal{R}_{23}(u),
$$

where $\bullet$ stands for summation if $Q = 2$ and for multiplication otherwise.

In its turn, an $R$–matrix is the cornerstone for building quantum integrable models known as spin chains, see, e.g. [7]. From this perspective, the most interesting tensor space representations of $TL_N(Q)$ are those with varying $Q$, i.e. such representations where $T$ depends on some parameters and $Q$ varies within a certain range when the parameters change. Indeed, such a representation allows us to construct a parametric family of $R$–matrices and, therefore, of integrable models.

Example 1. The most known example of such a type is provided by

$$
T(q; \zeta) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & q & 0 & 0 \\
0 & 0 & \zeta & 0 \\
0 & 0 & 0 & q^{-1}
\end{pmatrix}, \quad Q = q + q^{-1}.
$$

For $q > 0$ and $|\zeta| = 1$, this $T(q; \zeta)$ defines a unitary tensor space representation of $TL_N(Q)$. For $q \neq 1$, the spin chain corresponding to $T(q; \pm 1)$ is called the XXZ model and, for $q = 1$, it is the Heisenberg spin chain.
Remark 1. Let a family of unitary tensor space representation of $TL_N(Q)$ be defined by $T(q)$ which satisfies (T1)–(T4) when $q$ varies continuously within a subset $S \subseteq \mathbb{C}$. If entries of $T(q)$ are rational functions in $q$ with poles contained in a subset $\tilde{S}$, then equations (T2)–(T4) imply that certain functions rational in $q$ vanish on $S$. But then these functions must vanish identically. Therefore, for $q \in \mathbb{C} \setminus (S \cup \tilde{S})$, $T(q)$ will not be Hermitian but it will satisfy (T2)–(T4) (where $Q$ may not be real) and thus will define a family of non–unitary tensor space representation of $TL_N(Q)$.

Example 2. For $q \in \mathbb{C} \setminus \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \{0\}$, $T(q; \zeta)$ given by (8) is not Hermitian but it satisfies relations (T2)–(T4).

Remark 2. Unitary tensor space representations of $TL_N(Q)$ that for some values of parameters extend to non–unitary ones can be used to construct non–Hermitian operators with real spectrum. For instance, if $T(q; \zeta)$ is given by (8), then $H = T_{12}(q; \zeta) - T_{23}(q; \zeta)$ is Hermitian only for real $q$ but its spectrum remains real also for $q = e^{i\gamma}$, $\gamma \in \mathbb{R}$ provided that $4 \cos^2 \gamma \geq 1$. See [6] for further examples of such a type.

The three important characteristics of a tensor space representation of $TL_N(Q)$ are the value of $Q$, the dimension $n$ which determines the size of $T$, and the rank of $T$, $r \equiv \text{rank}(T)$. In what follows, somewhat abusing the terminology, we will refer to $r$ as simply the rank of a representation.

In the rank one case, properties of spin chains based on the TL R–matrices (6), in particular, spectra of the TL Hamiltonians $H = T_{12} + T_{23} + \ldots$ have been studied by a large number of authors, see e.g. [11, 13, 16, 18, 14]. These studies used mainly the representation determined by $T(q, -1)$ or its higher spin analogue (cf. Section 4). This representation enjoys the great popularity because on the one hand it is a representation with varying $Q$ and thus it can be used to study parametric families of Hamiltonians and on the other hand it is related to the quantum Hopf algebra $U_q(sl_2)$ (cf. Section 4).

In the higher rank case, $r \geq 2$, some tensor space representations of $TL_N(Q)$ were constructed in [12, 16] for $r = n \geq 2$ but they correspond only to a specific value of $Q$, namely $Q = \sqrt{n}$. And, to the best of author’s knowledge, spin chains based on higher rank tensor space representations of $TL_N(Q)$ have not yet been studied.

The goal of the present article is to consider certain problems related to construction of unitary tensor space representations of $TL_N(Q)$ of an arbitrary rank $r$.

The paper is organized as follows. In Section 2, we first comment on a certain redundancy in equations (T1)–(T4). Then we give two criteria for determining when an orthogonal projection matrix $P$ gives rise to a unitary tensor space representation of $TL_N(Q)$ (matrix $T$ in (T1)–(T4) is always a scalar multiple of a projection matrix). The first of them is the equality of traces of certain matrices and the second is the unitary condition for a certain partitioned matrix. In Section 3, we give some estimates on the lower bound of $Q$ if the dimension $n$ and the rank $r$ are given. In particular, we show that $Q \geq n/r$ and that this yields the sharp lower bound if $r = 1$. Using the Jones–Wenzl projector, we show that if $r > n^2/4$, then $Q$ can take only a discrete set of values determined by the value of $n^2/r$. It follows, in particular, that the only allowed value of $Q$ for $n = r = 2$ is $Q = \sqrt{2}$. At the end of the section, the estimates on $Q$ are sharpened for some special
cases when $P$ is the projection on a subspace containing vectors whose matrices of coefficients have special properties. In Section 4, we use the results of Section 2 as well as some properties of the Clebsch–Gordan coefficients of the quantum Hopf algebra $U_q(su_2)$ (for a generic positive $q$ and for $q = 1$) in order to find all varying $Q$ unitary tensor space representations of $TL_N(Q)$ of rank one and rank two which are built from orthogonal projections in the tensor product of two spin $S$ representations of $U_q(su_2)$ on a subspace spanned by, respectively, one or two joint eigenvectors of $C$ and $K$ (the Casimir operator and the generator of the Cartan subalgebra). The proofs of all statements are given in the Appendix.

2 Criteria for an orthogonal projection

2.1 Remarks on equations (T1)–(T4)

We commence with the following simple remark. If $T, T' \in M_{n^2}$ are solutions to (T1)–(T4) corresponding to the same value of $Q$ and $\text{rank}(T) = \text{rank}(T')$, then, by the spectral theorem, these matrices are unitarily similar, $T' = GTG^*$, where $G$ is unitary. But the converse is not true: if $T \in M_{n^2}$ is a solution to (T1)–(T4) and $G \in U(n^2)$, then $T' = GTG^*$ is not in general a solution to (T3)–(T4). However, if $G = g \otimes g$, $g \in U(n)$, then the unitary similarity transformation

$$T' = (g \otimes g) T (g^* \otimes g^*) ,$$

does send a solution $T$ to equations (T1)–(T4) to another solution, $T'$, to these equations. Clearly, $T$ and $T'$ related as in (9) have equal ranks and correspond to the same value of $Q$. It is thus natural to study solutions to (T1)–(T4) up to the unitary equivalence (9).

Next, let us remark that relations (T1)–(T4) in the definition of a tensor space representation are somewhat redundant.

Here and below $\text{tr}$ denotes the standard matrix trace.

**Proposition 1.** a) If $T \in M_{n^2}$ satisfies relations (T3) and (T4), then $T^2 = QT$, where $Q \in \mathbb{C} \setminus \{0\}$ if $\text{tr} T \neq 0$ and $Q = 0$ if $\text{tr} T = 0$.

b) If $T \in M_{n^2}$ satisfies relation (T1) and any two of the three relations (T2)–(T4), then $T$ satisfies all the relations (T1)–(T4).

**Example 3.** For $T(q; \zeta)$ given by (3), we have $\text{tr} T(q; \zeta) = (q + q^{-1})$. $T(q; \zeta)$ is a scalar multiple of a rank one projection if $q^2 \neq -1$ and it is a nilpotent of order two if $q^2 = -1$. (The spin chain corresponding to the nilpotent case is known as the XX0 or XX model.)

2.2 Trace conditions

Every $T \in M_{n^2}$ that satisfies relations (T1) and (T2) is a scalar multiple of an orthogonal projection, i.e. $T = QP$, where $P \in M_{n^2}$, $P = P^2 = P^*$. Without a loss of generality, we always assume that $Q > 0$ (because a negative $Q$ can be made positive by the trivial transformation $T \rightarrow -T$).
If the rank of $P$ is $r$, then $\text{tr} \ T = Q \ r$. Furthermore,
\begin{equation}
\text{tr}_{123} (T_{12}) = \text{tr}_{123} (T_{23}) = Q \ r \ n,
\end{equation}
where $\text{tr}_{123}$ stands for the matrix trace in $M_{n^3}$.

The problem of constructing unitary tensor space representations of $TL_N(Q)$, that is, finding solutions $T$ to equations (T1)–(T4), amounts to finding suitable orthogonal projections in $(\mathbb{C}^n)^{\otimes 2}$. Remarkably, such projections can be characterised by just a single scalar condition.

**Theorem 1.** Let $P \in M_{n^2}$ be an orthogonal projection of rank $r \geq 1$ and suppose that $P_{12}P_{23} \neq 0$. Then a solution to (T3)–(T4) of the form $T = Q \ P$, where $Q > 0$, exists if and only if the following equality holds:
\begin{equation}
(\text{tr}_{123} (P_{12}P_{23}))^2 = n \ r \ \text{tr}_{123} (P_{12}P_{23})^2.
\end{equation}
If equality (11) holds, then relations (T1)–(T4) are satisfied for $T = Q \ P$, where
\begin{equation}
Q^2 = \frac{n \ r}{\text{tr}_{123} (P_{12}P_{23})}.
\end{equation}

As a consequence, matrix equations (T3)–(T4) in the definition of a unitary tensor space representation of $TL_N(Q)$ can be replaced by scalar equations as follows.

**Proposition 2.** Suppose that $T \in M_{n^2}$ has rank $r$ and satisfies relations (T1) and (T2) with $Q > 0$. Then the following statements are equivalent:

a) $T$ satisfies relations (T3)–(T4).

b) $\text{tr}_{123} (T_{12}T_{23}) \neq 0$ and the following equality holds:
\begin{equation}
(\text{tr}_{123} (T_{12}T_{23}))^2 = n \ r \ \text{tr}_{123} (T_{12}T_{23})^2.
\end{equation}

c) The following equalities hold:
\begin{equation}
\text{tr}_{123} (T_{12}T_{23}) = n \ r, \quad \text{tr}_{123} (T_{12}T_{23})^2 = n \ r.
\end{equation}

Theorem 1 can be used, in particular, in order to search for solutions to (T1)–(T4) numerically. For this purpose, we have to choose some orthonormal basis $\{y_a\}_{a=1}^{n^2}$ of $(\mathbb{C}^n)^{\otimes 2}$ and then test condition (11) for all projections of the form $P = \sum_{a=1}^{n^2} \varepsilon_a P_a$, where $\varepsilon_a$ is 0 or 1 and $P_a$ is the projection on the one-dimensional subspace spanned by $y_a$. Moreover, if we are interested in representations with varying $Q$, it suffices to check only the cases where $\sum_{a=1}^{n^2} \varepsilon_a \leq n^2/4$ (see Section 3).

### 2.3 Unitarity condition

Another way to characterize an orthogonal projection $P$ which gives rise to a unitary tensor space representation of $TL_N(Q)$ is to find a condition on the subspace on which $P$ projects.
Let \( \langle \cdot, \cdot \rangle \) denote the standard inner product on \( \mathbb{C}^n \) and let \( \mathcal{E} = \{e_a\}_{a=1}^n \) be a basis of \( \mathbb{C}^n \) orthonormal w.r.t. \( \langle \cdot, \cdot \rangle \). Then a vector \( v \in \mathbb{C}^n \otimes \mathbb{C}^n \) is determined by the matrix \( V \) of its coefficients, \( v = \sum_{a,b=1}^n V_{ab} e_a \otimes e_b \). Under a unitary change of the basis, \( e_a = \sum_{b=1}^n g_{ab} e'_b \), \( g \in U(n) \), the matrix of coefficients transforms as follows:

\[
V' = g^t V g.
\] (15)

Here and below we use the following notations for matrix operations: \( \bar{X} \), \( X^t \), and \( X^* \) stand, respectively, for the complex conjugate, the transpose, and the conjugate transpose of a matrix \( X \).

Given an \( r \)-dimensional vector subspace \( T \subset \mathbb{C}^n \otimes \mathbb{C}^n \), we will write \( T \sim \{V_1, \ldots, V_r\} \) if \( V_1, \ldots, V_r \) are the matrices of coefficients of an orthonormal set of vectors \( v_1, \ldots, v_r \) which is a spanning set of \( T \). The orthonormality condition implies that

\[
\langle v_s, v_m \rangle = \operatorname{tr}(V_s^* V_m) = \delta_{sm}.
\] (16)

The orthogonal projection onto \( T \) is given by \( P_T = \sum_{s=1}^r v_s \langle v_s, \cdot \rangle \). In the basis \( \mathcal{E} \), the operator \( e_a \langle e_b, \cdot \rangle \) is represented by the matrix \( E_{ab} \in M_n \) such that \( (E_{ab})_{ij} = \delta_{ai} \delta_{bj} \).

Therefore, the projection \( P_T \) is represented by the following matrix:

\[
P_T = \sum_{s=1}^r \sum_{a,b,c,d=1}^n (V_s)_{ab} (\bar{V}_s)_{cd} E_{ac} \otimes E_{bd},
\] (17)

where \( \otimes \) stands for the Kronecker product.

Each \( T \in M_{nr} \) which satisfies (T1)–(T2) and has rank \( r \) is determined by a set of matrices \( \{V_1, \ldots, V_r\} \) such that \( T = Q P_T \), where \( T \sim \{V_1, \ldots, V_r\} \).

**Example 4.** For \( T(q; \zeta) \) given by (2), we have \( T(q; \zeta) = (q + q^{-1}) P_T \) with \( T \sim \{V\} \), where

\[
V = \frac{1}{\sqrt{q^2 + 1}} \begin{pmatrix} 0 & \zeta & q \\ 1 & 0 & 0 \end{pmatrix}, \quad q > 0, \quad |\zeta| = 1.
\] (18)

Given an \( r \)-dimensional subspace \( T \subset \mathbb{C}^n \otimes \mathbb{C}^n \), \( T \sim \{V_1, \ldots, V_r\} \), let us introduce the following partitioned matrix consisting of \( r^2 \) blocks of the size \( n \times n \):

\[
W_T = \begin{pmatrix} V_1 \bar{V}_1 & V_2 \bar{V}_1 & \cdots \\ V_1 \bar{V}_2 & V_2 \bar{V}_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \sum_{s,m=1}^r E_{sm} \otimes V_m \bar{V}_s.
\] (19)

By Theorem [I] finding a solution to (T1)–(T4) amounts to finding a subspace \( T \) such that the corresponding orthogonal projection \( P_T \) satisfies relation (II). Let us reformulate relation (III) as a condition on the matrix \( W_T \).

**Theorem 2.** Let \( P_T \) be the orthogonal projection onto an \( r \)-dimensional subspace \( T \subset \mathbb{C}^n \otimes \mathbb{C}^n \), \( T \sim \{V_1, \ldots, V_r\} \) and let \( W_T \) be the corresponding matrix defined in (19). Then \( T = Q P_T \), \( Q > 0 \) is a solution to (T1)–(T4) if and only if \( Q W_T \) is a unitary matrix,

\[
Q W_T \in U(nr).
\] (20)
Since unitarity of \( Q W T \) implies unitarity of \( Q W T^t \), \( Q W T^* \), and \( Q W T^\ast \), we deduce the following.

**Corollary 1.** If \( T = Q P T, T \sim \{V_1, \ldots, V_r\} \) is a solution to (T1)-(T4), then so are \( T' = Q P T', T'' = Q P T'', \) and \( T''' = Q P T''' \), where \( T' \sim \{\bar{V}_1, \ldots, \bar{V}_r\} \), \( T'' \sim \{V'_1, \ldots, V'_r\} \), and \( T''' \sim \{V^*_1, \ldots, V^*_r\} \).

**Remark 3.** The validity of condition (20) depends neither on a particular choice of the orthonormal spanning set of \( T \) nor on a particular choice of the orthonormal basis \( E \) of \( \mathbb{C}^n \). Indeed, for two different orthonormal spanning sets, \( \{v_s\}_{s=1}^{r} \) and \( \{v'_s\}_{s=1}^{r} \), where \( v'_s = \sum_{k=1}^{r} h_{sk} v_k, \) \( h \in U(r) \), the corresponding \( W \) matrices are related by a unitary transformation, namely \( W'_T = (\bar{h} \otimes I_n) W_T (h^t \otimes I_n) \). For two different orthonormal bases of \( \mathbb{C}^n \), the matrices of coefficients are related as in (15) and so the \( W \) matrices corresponding to the same subspace \( T \) are also related by a unitary transformation, namely \( W'_T = (I_r \otimes g^t) W_T (I_r \otimes \bar{g}) \). In either case, the unitarity of \( W_T \) implies the unitarity of \( W'_T \).

**Remark 4.** Condition (20) admits also the following formulation. Let \( J \) be the unitary involutive automorphism of \( \mathbb{C}^n \otimes \mathbb{C}^n \) which maps a vector \( v \) with the coefficient matrix \( V \) into a vector \( J(v) \) with the coefficient matrix \( V' \). Note that, by (15), the map \( v \rightarrow J(v) \) is independent of a choice of the basis \( E \) of \( \mathbb{C}^n \). Observe that \( (W_T)^* = W_{J(T)} \). Therefore, condition (20) is equivalent to the requirement that \( W_{J(T)} \) is a scalar multiple of the inverse to \( W_T \).

### 3 On the range of \( Q \)

An interesting problem is to determine the range of possible values of \( Q \) in (T2) for solutions to (T1)–(T4) if the rank \( r \) and the dimension \( n \) of the underlying space \( \mathbb{C}^n \) are given.

**Example 5.** For \( T(q; \zeta) \) given by (8) with \( q > 0 \), we have \( Q = q + q^{-1} \in [2, +\infty) \). As we will see below, \( Q = 2 \) is the sharp lower bound in the \( n = 2, r = 1 \) case.

#### 3.1 Rank one case

For \( r = 1 \), the normalization condition (16) and the unitarity condition (20) acquire the following form:

\[
\text{tr}(V V^*) = 1 \quad \quad V \bar{V} V^t V^* = Q^{-2} I_n .
\]  

(21)

Clearly, \( V \) must be nonsingular. Taking this into account, we derive from (21) the following expressions for \( Q \):

\[
Q^2 = |\det V|^{-\frac{4}{n}} , \quad \quad Q^2 = \text{tr}((V^*V)^{-1}) .
\]  

(22)

They, in turn, allow us to find the lower bound for \( Q \) in the rank one case.
Proposition 3. Suppose that $V \in M_n$ satisfies relations \([21]\). Then

a) The following inequality holds:
\[ Q^2 \geq n^2. \]  \hspace{1cm} (23)

b) The equality in \([23]\) is achieved if and only if $V$ is a scalar multiple of a unitary matrix, that is
\[ V = \frac{1}{\sqrt{n}} G, \quad G \in U(n). \]  \hspace{1cm} (24)

Thus, in the rank one case, $Q = n$ is the sharp lower bound. Moreover, for every $n \geq 2$, a unitary tensor space representation of $TL_N(Q)$ of rank one exists for every $Q$ in the range $[n, +\infty)$ (see Theorem 5 in Section 4.2).

3.2 Higher rank case

Let us now establish some estimates on the lower bound for $Q$ in the higher rank case.

Theorem 3. If $T \in M_{n^2}$ has rank $r \geq 1$ and satisfies relations (T1)–(T4) with $Q > 0$, then the following inequalities hold:
\[ Q^4 \geq \frac{2n^2}{n^2 + r}, \]  \hspace{1cm} (25)
\[ Q \geq \frac{n}{r}. \]  \hspace{1cm} (26)

Inequality \([25]\) implies the following.

Corollary 2. $Q = 1$ is possible only for $r = n^2$ that is in the trivial case $T = I_{n^2}$.

Next, we will find certain restrictions on the possible values of $Q$ using the Jones–Wenzl orthogonal projector \([11, 17]\). Recall that, for a generic value of $Q$, the algebra $TL_N(Q)$ with the anti–involution \([4]\) possesses a unique non–zero element $P_N$ such that
\[ P_N P_N = P_N, \quad P_N^* = P_N, \quad T_k P_N = P_N T_k = 0, \quad \text{for } k = 1, \ldots, N - 1. \]  \hspace{1cm} (28)

For the first three values of $N$, these projectors are given by
\[ P_1 = 1, \quad P_2 = 1 - \frac{1}{Q} T_1, \quad P_3 = 1 - \frac{Q (T_1 + T_2)}{Q^2 - 1} + \frac{(T_1 T_2 + T_2 T_1)}{Q^2 - 1}. \]  \hspace{1cm} (29)

Let $\tau_{n,r}$ be the unitary tensor space representation of $TL_N(Q)$ determined by a matrix $T \in M_{n^2}$ which has rank $r \geq 1$ and satisfies (T1)–(T4). Denote $P_{n,r,N} = \tau_{n,r}(P_N)$ and $d_N(n,r) = \text{tr}_{1,\ldots,N}(P_{n,r,N})$, where $\text{tr}_{1,\ldots,N}$ is the matrix trace in $M_{nN}$.

Example 6. For the projectors given in \([29]\), we have (cf. \([10]\) and \([13]\))
\[ d_1(n,r) = n, \quad d_2(n,r) = n^2 - r, \quad d_3(n,r) = n^3 - 2rn. \]  \hspace{1cm} (30)
Note that relations (27) imply that $P_{n,r,N}$ is a positive semi–definite matrix. Therefore, $d_N(n,r)$ must be non–negative. But we see from (30) that $d_3(n,r) < 0$ for $r > n^2/2$. This implies that every representation $\tau_{n,r}$ of a rank $r > n^2/2$ can correspond only to the value $Q = 1$ (for which $P_3$ is not defined). By a similar analysis of values of $d_N(n,r)$ for $N \geq 3$, we establish the following statement.

**Theorem 4.** Suppose that $T \in M_{n^2}$ has rank $r > n^2/4$ and satisfies relations (T1)–(T4) with $Q > 0$. Then $Q$ in (T2) belongs to the following discrete sets of values:

$$
\text{if } 4\cos^2\left(\frac{\pi}{m+2}\right) \leq \frac{n^2}{r} < 4\cos^2\left(\frac{\pi}{m+3}\right), \quad m \in \mathbb{N},
$$

then 

$$
Q \in J_m \equiv \left\{2\cos\left(\frac{\pi}{k+2}\right), \quad k = 1, \ldots, m\right\}. 
$$

**Remark 5.** In the theory of von Neumann algebras, it is know [11, 17] that the algebra $TL_\infty(Q)$ with the anti–involution (4) admits a normalizable trace only if $Q \in J_\infty \cup [2, +\infty)$. The situation with unitary tensor space representations of $TL_N(Q)$ is somewhat different because the range of allowed values of $Q$ depends on the value of the parameter $n^2/r$. In particular, if $r \leq n^2/4$, then the positive definiteness of the Jones–Wenzl projector imposes no restrictions on $Q$.

Theorem 4 along with Corollary 2 imply, in particular, the following.

**Corollary 3.** a) There exists no unitary tensor space representation of $TL_N(Q)$ of rank $r \in \left(\frac{1}{2}n^2, n^2\right)$.

b) Each unitary tensor space representation of $TL_N(Q)$ of rank $r \in \left(\frac{1}{2}(3 - \sqrt{5})n^2, \frac{1}{2}n^2\right]$ corresponds to $Q = \sqrt{2}$.

c) Each unitary tensor space representation of $TL_N(Q)$ of rank $r \in \left(\frac{1}{2}n^2, \frac{1}{2}(3 - \sqrt{5})n^2\right]$ corresponds to either $Q = \sqrt{2}$ or $Q = \frac{1}{2}(1 + \sqrt{5})$.

**Example 7.** For $n = r = 2$, by Corollary 3 the only allowed value of $Q$ is $Q = \sqrt{2}$. In this case, a particular solution to (T1)–(T4) is given by

$$
T(\zeta) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & i\zeta \\
0 & 1 & i & 0 \\
0 & -i & 1 & 0 \\
-i\zeta^{-1} & 0 & 0 & 1
\end{pmatrix}, \quad |\zeta| = 1.
$$

The corresponding matrix $R$ appearing in (6) was listed as $R_{H0.2}$ in [10] among other constant solutions to the Yang–Baxter equation in the $n = 2$ case.

**Remark 6.** By Theorem 4 for $n = r = 3$, the only allowed values of $Q$ are $\sqrt{2}$, $\frac{1}{2}(1 + \sqrt{5})$, and $\sqrt{3}$. A solution corresponding to $Q = \sqrt{3}$ was constructed in [16].

### 3.3 Higher rank case, special cases

First, let us remark that every unitary tensor space representation of $TL_N(Q)$ of rank $r$ can be used to construct an infinite tower of representations of the same rank for underlying spaces of higher dimensions.
Proposition 4. Suppose that \( T = Q P_T, T \sim \{V_1, \ldots, V_r\} \) is a solution to (T1)–(T4). Given \( m \in \mathbb{N} \), define \( \tilde{T} \sim \{\tilde{V}_1, \ldots, \tilde{V}_r\} \), where \( \tilde{V}_k = \frac{1}{\sqrt{m}} I_m \otimes V_k \) for all \( k \) (or, alternatively, \( \tilde{V}_k = \frac{1}{\sqrt{m}} V_k \otimes I_m \) for all \( k \)). Then \( \tilde{T} = m Q P_{\tilde{T}} \) is a solution to (T1)–(T4).

Remark 7. Let us stress that \( \tilde{T} \) does not coincide with the Kronecker product of \( T \) and the identity matrix. Indeed, \( \tilde{T} \) has the same rank as \( T \). Even if \( T \) is the trivial solution, \( \tilde{T} \) is non–trivial for \( m > 1 \).

Next, we will refine the estimates on the value of \( Q \) for representations where the spanning vectors of the subspace \( T \) have certain specific properties.

Given an orthonormal basis \( \{e_a\}_{a=1}^n \) of \( \mathbb{C}^n \), we will write \( v \sim V \) if \( V \in M_n \) is the matrix of coefficients of a vector \( v \in \mathbb{C}^n \otimes \mathbb{C}^n \), i.e. \( v = \sum_{a,b=1}^n V_{ab} e_a \otimes e_b \). Relation (15) implies that the following characteristics of a vector in \( \mathbb{C}^n \otimes \mathbb{C}^n \) are independent of the choice of a basis of \( \mathbb{C}^n \):

a) \( v \sim V \) such that \( V \) is a symmetric or antisymmetric matrix;
b) \( v \sim V \) such that \( V \) is a scalar multiple of a unitary matrix.

Proposition 5. Suppose that \( T = Q P_T \in M_{n^2} \) has rank \( r \) and satisfies (T1)–(T4) and \( T \) contains a non–zero vector \( v \sim V \) such that matrix \( V \) is symmetric or antisymmetric. Then

\[
Q^2 \leq n^2. \tag{34}
\]

This statement along with Proposition 3 implies, in particular, the following.

Corollary 4. If \( T = Q P_T \in M_{n^2} \) satisfies (T1)–(T4) and \( T \sim \{V\} \), where matrix \( V \) is symmetric or antisymmetric, then \( Q = n \).

Example 8. See solutions listed in part a) of Theorem 5. The corresponding matrices \( V \) are (anti)symmetric by the symmetry (87) of the Clebsch–Gordan coefficients of the algebra \( U(su_2) \).

Proposition 6. Suppose that \( T = Q P_T \in M_{n^2} \) has rank \( r \) and satisfies (T1)–(T4) and \( T \) contains a non–zero vector \( v \sim V \) such that \( V \) is a scalar multiple of a unitary matrix.

a) Then

\[
Q^2 = \frac{n^2}{r}. \tag{35}
\]

b) If, in addition, \( \frac{n^2}{r} < r < n^2 \), then either \( r = \frac{n^2}{r} \) and \( Q = \sqrt{3} \) or \( r = \frac{n^2}{r} \) and \( Q = \sqrt{2} \).

Example 9. For \( T(\zeta) \) given by (33), we have \( T(\zeta) = \sqrt{2} P_T, T \sim \{V_1, V_2\} \), where

\[
V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \zeta & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \quad |\zeta| = 1. \tag{36}
\]

Both \( V_1 \) and \( V_2 \) are scalar multiples of unitary matrices. So, \( Q = \sqrt{2} \) as required by Proposition 6. Applying the recipe of Proposition 4, we can use \( V_1 \) and \( V_2 \) in order to construct a unitary tensor space representation of \( TL_N(Q) \) of rank two corresponding to \( Q = n/\sqrt{2} \) for any even \( n \).
Proposition 7. Suppose that $T = Q P T \in M_{n^2}$ has rank $r$ and satisfies (T1)–(T4) and $T \sim \{V_1, \ldots, V_r\}$, where matrix $V_1$ is nonsingular and either $V_k = V_1 g_k$ for $k = 2, \ldots, r$ or $V_k = g_k V_1$ for $k = 2, \ldots, r$, where, in both cases, all $g_k$ are unitary. Then the following inequality holds:

$$Q^2 \geq \frac{n^2}{r}. \quad (37)$$

Remark 8. In the rank one case, by Proposition 3, we have $Q \geq n$ and the lower bound is achieved only when the matrix of coefficients $V$ is a scalar multiple of a unitary matrix. Therefore, in view of Proposition 6, one could conjecture that $Q \geq n/\sqrt{r}$ if $r \leq n^2/4$. However, below (see Theorem 6) we will construct a family of rank two solutions to (T1)–(T4) for $n = 3$ for which $Q \in [2, +\infty)$. This example refutes the conjecture since $2 < 3/\sqrt{2}$.

Thus, in the case $r \leq n^2/4$, it remains an open problem to sharpen the estimate $Q \geq n/r$ established in Theorem 3.

4 Representations of rank one and two via $U_q(su_2)$

4.1 $U_q(su_2)$ and Clebsch–Gordan decomposition

Recall the definition of the universal enveloping Lie algebra $U(su_2)$ and its quantum deformation $U_q(su_2)$.

Definition 3. a) $U(su_2)$ is the unital $*$-algebra over $\mathbb{C}$ with generators $X^+, X^-, H$ and the following defining relations:

$$HX^\pm - X^\pm H = \pm X^\pm, \quad X^+X^- - X^-X^+ = 2H, \quad (38)$$

$$H^* = H, \quad (X^\pm)^* = X^{\mp}. \quad (39)$$

b) $U_q(su_2)$, $q > 0$, $q \neq 1$, is the unital $*$-algebra over $\mathbb{C}$ with generators $X^+, X^-, K, K^{-1}$ and the following defining relations:

$$KX^\pm = q^{\mp1}X^\pm K, \quad X^+X^- - X^-X^+ = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad (40)$$

$$KK^{-1} = K^{-1}K = 1, \quad (K^{\pm1})^* = K^{\mp1}, \quad (X^\pm)^* = X^{\mp}. \quad (41)$$

For both algebras, the center is generated by the corresponding Casimir element:

$$U(su_2): \quad C_1 = X^-X^+ + H(H + 1), \quad (42)$$

$$U_q(su_2): \quad C_q = X^-X^+ + \frac{(K - K^{-1})(qK - q^{-1}K^{-1})}{(q - q^{-1})^2}. \quad (43)$$

Both algebras become bialgebras if the comultiplication is defined as follows:

$$U(su_2): \quad \Delta(X^\pm) = X^\pm \otimes 1 + 1 \otimes X^\pm, \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \quad (44)$$

$$U_q(su_2): \quad \Delta(X^\pm) = X^\pm \otimes K + K^{-1} \otimes X^\pm, \quad \Delta(K^{\pm1}) = K^{\pm1} \otimes K^{\mp1}. \quad (45)$$
Remark 9. Setting formally $K^{±1} = q^{±H}$ and considering the limit $q \to 1$, one recovers from the defining relations and comultiplication of $U_q(su_2)$ those of $U(su_2)$. Furthermore, the q-number, i.e. a function $\mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}$ defined as follows: $[t]_q = t$ and $[t]_q = \frac{q^t - q^{-t}}{q - q^{-1}}$ for $q \neq 1$

(46)

is continuous at $q = 1$. For these reasons, one can regard $U(su_2)$ as the limit of $U_q(su_2)$ as $q \to 1$. In particular, the Clebsch–Gordan coefficients of $U_q(su_2)$ are continuous functions at $q = 1$ and their limit as $q \to 1$ yields the Clebsch–Gordan coefficients of $U(su_2)$.

An irreducible finite dimensional representation of $U_q(su_2)$ is characterized by its highest weight $\Lambda$ which is a non–negative integer. Following the terminology used in physics, we will refer to $S = \frac{1}{2} \Lambda$ as spin. We denote by $\mathcal{H}_S^q$ the irreducible $U_q(su_2)$–module of dimension $n = 2S + 1$. On $\mathcal{H}_S^q$, the Casimir element $C_q$ takes the value $\frac{1}{2}S(S+1)q$.

The tensor square $\mathcal{H}_S^q \otimes \mathcal{H}_S^q$ decomposes into a direct sum of irreducible modules, $\mathcal{H}_S^q \otimes \mathcal{H}_S^q = \bigoplus_{j=0}^{2S} \mathcal{H}_j^q$. Let $|m\rangle \in \mathcal{H}_S^q$ denote the eigenvector of $K$ (or $H$ if $q = 1$) such that $K|m\rangle = q^m|m\rangle$ (respectively, $H|m\rangle = m|m\rangle$). Let $|J,m\rangle \in \mathcal{H}_j^q \subset \mathcal{H}_S^q \otimes \mathcal{H}_S^q$ denote the joint eigenvector of $\Delta(K)$ and $\Delta(C_q)$ (or $\Delta(H)$ and $\Delta(C_1)$ if $q = 1$), i.e.

$q \neq 1: \quad \Delta(K)|J,m\rangle = q^m|J,m\rangle, \quad \Delta(C_q)|J,m\rangle = [J]_q[J+1]|J,m\rangle,

q = 1: \quad \Delta(H)|J,m\rangle = J(J+1)|J,m\rangle, \quad \Delta(C_1)|J,m\rangle = J(J+1)|J,m\rangle.$

(47)

The sets of vectors $\{|J,m\rangle\}_{m=-j,-j+1,...,j}$ and $\{|m\rangle \otimes |m\rangle\}_{m_1,m_2=-S,-S+1,...,S}$ provide two orthonormal bases for $\mathcal{H}_S^q \otimes \mathcal{H}_S^q$ related to each other as follows:

$|J,m\rangle = \sum_{m_1,m_2=-S}^{S} \{S,S,m_1,m_2|J,m\}\_q |m_1\rangle \otimes |m_2\rangle,

(48)$

where $\{S,S,m_1,m_2|J,m\}_q$ stands for the Clebsch–Gordan coefficient (see formulae (55)–(57) in the Appendix).

Let us identify the vector $e_a$ of the canonical basis $\{e_a\}_{a=1}^{2S+1}$ of $\mathbb{C}^{2S+1}$ with the vector $|S+1-a\rangle$ of $\mathcal{H}_S^q$. Then, by (18), the vector $|J,m\rangle \in \mathcal{H}_S^q \otimes \mathcal{H}_S^q$ can be associated with the matrix $V \in M_{2S+1}$ such that:

$V_{ab} = \delta_{a+b+m,2S+2} \{S,S,S+1-a,S+1-b|J,m\}_q.

(49)$

Example 10. For $S = 1/2$, matrices

$V_1 = \frac{1}{\sqrt{q^2+1}} \begin{pmatrix} 0 & q \\ -1 & 0 \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{q^2+1}} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}

(50)$

correspond, respectively to the vectors $|0,0\rangle_q$ and $|1,0\rangle_q$.

Observe that $V_1$ and $V_2$ in (50) coincide with (18) for $\zeta = -1$ and $\zeta = 1$ (up to a sign and transposition, respectively). Therefore they define unitary tensor space representations of $TL_N(Q)$. Motivated by this example, we will look for other unitary tensor space representations of $TL_N(Q)$ determined in the same sense by one or two joint eigenvectors of $\Delta(C_q)$ and $\Delta(K)$ (or their counterparts $\Delta(C_1)$ and $\Delta(H)$ if $q = 1$).
4.2 TL vectors and TL pairs

Below, \(\frac{1}{2}\mathbb{Z}_{\geq 0}\) stands for the set \(\{0, 1/2, 1, 3/2, \ldots\}\).

**Definition 4.**

a) Given \(S \in \frac{1}{2}\mathbb{Z}_{\geq 0}\) and \(q > 0\), a vector \(|J, m\rangle_q \in \mathcal{H}_S^q \otimes \mathcal{H}_S^q\) is called a TL vector if equations (T1)–(T4) admit a solution of the form \(T = QP_q\), where \(Q > 0\) and \(P_q\) is the projection in \(\mathcal{H}_S^q \otimes \mathcal{H}_S^q\) on the one dimensional subspace spanned by \(|J, m\rangle_q\).

b) Given \(S \in \frac{1}{2}\mathbb{Z}_{\geq 0}\) and \(q > 0\), a pair of orthogonal vectors \(|J_1, m_1\rangle_q, |J_2, m_2\rangle_q \in \mathcal{H}_S^q \otimes \mathcal{H}_S^q\) is called a TL pair if equations (T1)–(T4) admit a solution of the form \(T = QP_q\), where \(Q > 0\) and \(P_q\) is the projection in \(\mathcal{H}_S^q \otimes \mathcal{H}_S^q\) onto the two dimensional subspace spanned by these vectors.

**Proposition 8.** Given \(S \in \frac{1}{2}\mathbb{Z}_{\geq 0}\), a vector \(|J, m\rangle_q\) (respectively, a pair of orthogonal vectors \(|J_1, m_1\rangle_q, |J_2, m_2\rangle_q\)) is a TL vector (respectively, a TL pair) either for all \(q > 0\) or only for a finite (possibly empty) set of values of \(q\).

Recall that we are predominantly interested in solutions to (T1)–(T4) with varying \(Q\), i.e. solutions that depend on a parameter \(q\) in such a way that \(Q = Q(q)\) is a non–constant function of \(q\). Proposition 8 simplifies considerably the task of finding all such solutions if \(P_q\) is the projection onto a subspace spanned by one or two joint eigenvectors of \(\Delta(C_q)\) and \(\Delta(K)\). Indeed, Proposition 8 implies that it suffices to restrict consideration to the case \(q = 1\) and then verify which of the found solutions remain solutions to (T1)–(T4) for all \(q > 0\). Such a strategy allows us to establish the following.

**Theorem 5.**

a) For \(q = 1\), the exhaustive list of TL vectors \(|J, m\rangle_{q=1}\) is given by:

\[
\begin{align*}
|0,0\rangle_{q=1} & \quad \text{for all } S \in \frac{1}{2}\mathbb{Z}_{\geq 0}; \\
|1,0\rangle_{q=1} & \quad \text{for } S = \frac{1}{2}; \\
|2,0\rangle_{q=1} & \quad \text{for } S = \frac{3}{2}.
\end{align*}
\]

In all the three cases, the corresponding value of \(Q\) is \(Q = 2S + 1\).

b) The exhaustive list of vectors \(|J, m\rangle_q\) which are TL vectors for all \(q > 0\) is given by:

\[
\begin{align*}
|0,0\rangle & \quad \text{for all } S \in \frac{1}{2}\mathbb{Z}_{\geq 0}; \\
|1,0\rangle & \quad \text{for } S = \frac{1}{2}.
\end{align*}
\]

In both cases, the corresponding value of \(Q\) is \(Q = [2S + 1]_q\).

**Remark 10.** It was observed long ago in the physical literature that (51) and (54) are TL vectors, see [3] and [4], respectively.
Example 11. For $S = \frac{3}{2}$, vector $|2,0\rangle_q$ corresponds to the following matrix:

$$V = \frac{1}{\sqrt{q^{10} + q^6 + q^2 + 1}} \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & q^4 - q^2 - 1 & 0 \\ 0 & q^5 - q^3 - q & 0 & 0 \\ -q^4 & 0 & 0 & 0 \end{pmatrix}. \quad (56)$$

It is interesting to remark that, for $S = 3/2$, vector $|2,0\rangle_q$ is a TL vector not only at $q = 1$ but also at the points where $(q^4 - 1)^2 = 2q^4$, i.e. at $q = (2 \pm \sqrt{3})^\frac{1}{4}$. These points correspond to $Q^2 = 12 + 18\sqrt{6}$.

Remark 11. The fact that a vector $|J,m\rangle_q$ is not a TL vector at $q = 1$ does not exclude the possibility that it becomes a TL vector at some other value of $q$ (by Proposition 8 there can be only finite number of such values). Not aiming at finding all such cases, we give a particular example below.

Example 12. For $S = 1$, vector $|1,0\rangle_q$ corresponds to the following matrix:

$$V = \frac{1}{\sqrt{q^4 + 1}} \begin{pmatrix} 0 & 0 & q \\ 0 & q^2 - 1 & 0 \\ -q & 0 & 0 \end{pmatrix}. \quad (57)$$

Clearly, the corresponding vector is not a TL vector at $q = 1$ since $V$ is degenerate at this point. However it becomes a TL vector at the points where $q^2 - 1 = \pm q$, i.e. at $q = (\sqrt{5} \pm 1)/2$. These points correspond to $Q = 3$.

Theorem 6. a) For $q = 1$, the exhaustive list of TL pairs $|J_1,m_1\rangle_q$, $|J_2,m_2\rangle_q$ such that $J_1 \geq J_2$ is given by:

i) $|1,m\rangle_q$, $|1,-m\rangle_q$; \hspace{1cm} ii) $|1,m\rangle_q$, $|1,0\rangle_q$; \hspace{1cm} iii) $|2,m\rangle_q$, $|1,-m\rangle_q$; \hspace{1cm} iv) $|2,m\rangle_q$, $|1,0\rangle_q$; \hspace{1cm} v) $|2,m\rangle_q$, $|2,-m\rangle_q$.

In all of these cases, $S = 1$, $m = \pm 1$, and the corresponding value of $Q$ is $Q = 2$.

b) The exhaustive list of pairs of vectors $|J_1,m_1\rangle_q$, $|J_2,m_2\rangle_q$ such that $J_1 \geq J_2$ which are TL vectors for all $q > 0$ is given by:

i) $|2,1\rangle_q$, $|1,-1\rangle_q$; \hspace{1cm} ii) $|2,1\rangle_q$, $|1,1\rangle_q$.

In both cases, $S = 1$ and the corresponding value of $Q$ is $Q = q^2 + q^{-2} \equiv |2\rangle_q^2$.

Example 13. For $m = 1$, the TL pair ii) in (58) corresponds to

$$V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (62)$$

The first TL pair in (61) corresponds to

$$V_1 = \frac{1}{\sqrt{q^4 + 1}} \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{q^4 + 1}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q^2 \\ 0 & -1 & 0 \end{pmatrix}. \quad (63)$$

14
A Appendix

Proof of Proposition 1. Part a). The case $T = 0$ is trivial, so we assume that $T \neq 0$. Take $N = 4$ and consider $T_{12} = T \otimes I_n \otimes I_n$, $T_{23} = I_n \otimes T \otimes I_n$, and $T_{34} = I_n \otimes I_n \otimes T$. Using relations (T3)–(T4) and taking into account that $T_{12}$ commutes with $T_{34}$, we obtain

$$
T^2 \otimes T^m \equiv T_{12}^2 T_{34}^m = T_{12} T_{34} T_{12} T_{34}^{m-1} \equiv T_{12} T_{34} T_{23} T_{34} T_{12} T_{34}^{m-1}
$$

for every positive integer $m$. Whence, by taking the partial trace $\text{tr}_{34}$, we obtain

$$
T^2 \text{ tr} (T^m) = T \text{ tr} (T^{m+1}).
$$

Since $T \neq 0$, we infer from (65) that $\text{tr}(T^m) = 0$ implies $\text{tr}(T^{m+1}) = 0$. Therefore, if $\text{tr} T = 0$, then $\text{tr}(T^m) = 0$ for every positive integer $m$. Thus, $T$ is a nilpotent of order at most two. Suppose $T$ is a nilpotent of order $k$. The r.h.s. of (64) vanishes for $m = k - 1$ but the l.h.s. does not vanish (since $A \otimes B \neq 0$ if $A, B \neq 0$). Thus, $T$ is a nilpotent of order two.

Part b). By the part a), we know that relations (T3)–(T4) imply relation (T2) irrespective of whether or not $T$ is Hermitian. Let us show that (T1)–(T3) imply (T4). Consider the following Hermitian matrix: $H = T_{23} T_{12} T_{23} - T_{23}$. Then we have

$$
\text{tr}_{123} (H^2) \equiv \text{tr}_{123} (T_{23} T_{12} T_{23} - 2 T_{23} + T_{23}) \equiv \text{tr}_{123} (T_{23} - T_{23} T_{12} T_{23}) = \text{tr}(T_{23} - T_{12} T_{23})
$$

But, since $H$ is Hermitian, $\text{tr}(H^2) = 0$ implies that $H = 0$, that is relation (T4) holds.

Proof of Theorem 1. Given an orthogonal projection $P \in M_n$ of rank $r$, consider the following family of Hermitian matrices: $H(\alpha) = P_{12} P_{23} - \alpha P_{12}$, $\alpha \in \mathbb{R}$. If there exists $\alpha_0 > 0$ such that $H(\alpha_0) = 0$, then $T = P/\sqrt{\alpha_0}$ satisfies relations (T1)–(T3) and hence, by Proposition 1 relation (T4) as well. We have

$$
f(\alpha) \equiv \text{tr}_{123} (H^2(\alpha)) = \text{tr}_{123} (P_{12} P_{23} - 2 \alpha P_{12} + \alpha^2 P_{12}) = \alpha^2 nr - 2 \alpha \text{ tr}_{123} (P_{12} P_{23}) + \text{tr}_{123} (P_{12} P_{23})^2.
$$

Since $H(\alpha)$ is Hermitian, $H^2(\alpha)$ is positive semi-definite and therefore $f(\alpha) \geq 0$. This means that the discriminant of the quadratic polynomial $f(\alpha)$ is non-positive, i.e.

$$
(\text{tr}_{123} (P_{12} P_{23}))^2 \leq n r \text{ tr}_{123} (P_{12} P_{23})^2
$$

(66)
for every projection $P$. Hence a necessary and sufficient condition for equation $f(\alpha) = 0$ to have a solution is the condition that the discriminant of $f(\alpha)$ vanishes. Thus, $H(\alpha)$ vanishes for some $\alpha = \alpha_0$ if and only if the inequality (66) for a given $P$ becomes an equality. For such $P$, $f(\alpha)$ acquires the following form: $f(\alpha) = (\alpha n r - \text{tr} 123 (P_{12} P_{23}))^2 / (n r)$. Hence $f(\alpha_0) = 0$ for $\alpha_0 = \text{tr} 123 (P_{12} P_{23}) / (n r)$.

It remains to note that the condition $P_{12} P_{23} \neq 0$ guarantees that $\alpha_0 > 0$. Indeed, $\text{tr} 123 (P_{12} P_{23}) = \text{tr} 123 (P_{12}^2 P_{23}^2) = \text{tr} 123 ((P_{12} P_{23})(P_{12} P_{23})^*) \geq 0$, where the equality occurs only if $P_{12} P_{23} = 0$.

**Proof of Proposition 2.** Relations (T1)–(T2) imply that $T = Q P$, where $P$ is an orthogonal projection and $Q > 0$. Therefore, the hypotheses listed in b) imply the same hypotheses in terms of $P$, i.e. the hypotheses of Theorem 1 (as noted at the end of the proof of Theorem 1) the condition $\text{tr} 123 (P_{12} P_{23}) \neq 0$ is equivalent to $P_{12} P_{23} \neq 0$. Hence b) implies a).

Verification that a) implies c) is the following:

$$\text{tr} 123 (T_{12} T_{23})^2 = \text{tr} 123 (T_{12} T_{23} T_{12} T_{23}) \overset{(\ref{63})}{=} \text{tr} 123 (T_{12} T_{23}),$$

$$Q \text{tr} 123 (T_{12} T_{23}) \overset{(\ref{62})}{=} \text{tr} 123 (T_{12}^2 T_{23}) = \text{tr} 123 (T_{12} T_{23} T_{12}) \overset{(\ref{63})}{=} \text{tr} 123 (T_{12}) = Q r n.$$  \hspace{1cm} (68)

Finally, it is clear that c) implies b). \hspace{1cm} □

**Proof of Theorem 2.** Using that $E_{ab} E_{cd} = \delta_{bc} E_{ad}$ and $\text{tr}(E_{ab}) = \delta_{ab}$, it is straightforward to compute for $P_T$ given by (17) the following traces:

$$\text{tr} 123 ((P_T)_{12} (P_T)_{23}) = \sum_{s,m=1}^r \text{tr} \left(V_s V_m^t V_m^t V_s^* \right),$$

$$\text{tr} 123 ((P_T)_{12} (P_T)_{23})^2 = \sum_{s,s',m,m'=1}^r \text{tr} \left(V_s V_m^t V_m^t V_s^* V_s V_m^t V_m^t V_s^* \right).$$

Note that $\text{tr} 123 ((P_T)_{12} (P_T)_{23}) = \text{tr} W_T^* W_T$. Therefore $(P_T)_{12} (P_T)_{23} \neq 0$ iff $W_T \neq 0$ (cf. the proof of Theorem 1).

Consider the following partitioned matrix containing $r^2$ blocks of the size $n \times n$:

$$A_T = W_T W_T^* = \sum_{s,m=1}^r E_{sm} \otimes \left( \sum_{k=1}^r V_k V_k^t V_k^t V_k^* \right).$$

Observe that (69) and (70) coincide with $\text{tr} A_T$ and $\text{tr}(A_T^2)$, respectively. Therefore condition (11) for $P_T$ acquires the following form:

$$\text{tr} (A_T)^2 = nr \text{ tr}(A_T^2).$$

Suppose that $T = Q P_T$ is a solution to (T1)–(T4). Then, by Theorem 1 equality (72) holds and $W_T \neq 0$ so that $\beta^2 = \text{tr}(A_T^2)/nr > 0$. Since $A_T$ is Hermitian, matrix $H = (A_T - \beta^2 I_{nr})^2$ is positive semi–definite. But equality (72) implies that $\text{tr} H = 0$.  

16
Whence $H = 0$ and thus $A_T = \beta^2 I_{nr}$. (Another way to establish this result is to invoke the Cauchy inequality for the eigenvalues of $A_T$.) Therefore $QW_T$ is unitary for $Q = 1/\beta$. The converse implication is obvious: if $QW_T$ is unitary, then $A_T = Q^{-2} I_{nr}$ and therefore equality (72) holds. Hence, by Theorem 1, $T = QP_T$ is a solution to (T1)–(T4).

Proof of Proposition 3 a) Let $\lambda_1, \ldots, \lambda_n > 0$ be the singular values of $V$. The first relation in (21) means that $\sum_{a=1}^n \lambda_a^2 = 1$. Since $|\det V|^2 = |\det V*V| = \prod_{a=1}^n \lambda_a^2$, the arithmetic–geometric mean inequality implies that $|\det V|^2 \leq n^{-n}$. Substituting this inequality in the first formula in (22), we obtain the estimate (23). Let us remark that the same estimate follows from the second formula in (22) if we apply the Cauchy inequality: $|\det (VV*)^{-1}| = \sum_{a=1}^n \lambda_a^{-2} = (\sum_{a=1}^n \lambda_a^2)(\sum_{a=1}^n \lambda_a^{-2}) \geq (\sum_{a=1}^n \lambda_a \lambda_a^{-1})^2 = n^2$.

Proof of Theorem 3 Consider a family of Hermitian matrices: $J(\alpha) = T_{12} + T_{34} + \alpha T_{23}$, $\alpha \in \mathbb{R}$. For the trace in $M_{4n}$, we have (cf. (10) and (14)) $\text{tr}_{1234}(T_{12}) = Qn^2r$, $\text{tr}_{1234}(T_{12T_{23}}) = n^2r$, etc. Using these equalities, we find

$$f(\alpha) \equiv \text{tr}_{1234}(J(\alpha)^2) = rn^2Q^2\alpha^2 + 4rn^2\alpha + 2rQ^2(n^2 + r).$$

Since $J^2(\alpha)$ is positive semi–definite, the quadratic polynomial $f(\alpha)$ must be non–negative. The condition that the discriminant of $f(\alpha)$ is non–positive yields the first estimate in (26).

In order to prove the second estimate in (26), recall that if $\{\sigma_a\}_{a=1}^n$ are the singular values of a matrix $A \in M_n$, then $||A||_p = (\sum_{a=1}^n \sigma_a^p)^{1/p}$, $p \geq 1$, defines the Schatten $p$–norm of $A$. We will need the following properties of these norms (see, e.g. [5], Proposition 9.3.3 and Proposition 9.3.6):

$$||A + B||_p \leq ||A||_p + ||B||_p, \quad ||AB||_p \leq ||A||_{2p} ||B||_{2p}. \quad (73)$$

Since $T$ is a solution to (T1)–(T4), it has the form $T = QP_T$. Let $W_T$ be the corresponding $rn \times rn$ matrix given by (19). Using inequalities (73), we obtain

$$||W_T||_1 = \left|\left| \sum_{k,m=1}^r E_{km} \otimes V_m \tilde{V}_k \right|\right|_1 \leq \sum_{k,m=1}^r \left|\left| E_{km} \otimes V_m \tilde{V}_k \right|\right|_1 = \sum_{k,m=1}^r ||V_m \tilde{V}_k||_1 \leq \sum_{k,m=1}^r ||V_m||_2 ||V_k||_2 = r^2,$$

where we have taken into account that $||V_k||_2 = 1$ for all $k$ by the normalization condition (16). Thus, $||W_T||_1 \leq r^2$. On the other hand, by Theorem 2 $QW_T$ is a unitary matrix. Therefore, $Q||W_T||_1 = rn$. Hence the estimate (26) follows. \qed

17
Before proving Theorem 4, we will prove an auxiliary statement.

**Lemma 1.** For $d_N(n,r) = \text{tr}_{1,...,N}(P_{n,r,N})$, the following recurrence relation holds:

$$d_{N+1}(n,r) = n d_N(n,r) - r d_{N-1}(n,r).$$

(74)

For the initial values $d_0(n,r) = 1$ and $d_1(n,r) = n$, the solution to (74) is given by:

if $r \neq \frac{n^2}{4}$:  
$$d_N(n,r) = r^n \frac{\xi^{N+1} - \xi^{-N-1}}{\xi - \xi^{-1}}, \quad \text{where} \quad \xi + \xi^{-1} = \frac{n}{\sqrt{r}},$$

(75)

if $r = \frac{n^2}{4}$:  
$$d_N(n,r) = r^{n/2} (N + 1).$$

(76)

Note that $n^{-N} d_N(n,r)$ is a polynomial in $r/n^2$ of degree $\lfloor N/2 \rfloor$.

**Proof of Lemma 1.** It is well known [17] that the idempotent $P_N$ defined by relations (27) and (28) satisfies the following recursion relation:

$$P_{N+1} = P_N - \rho_N P_N T_N P_N,$$

(77)

where the scalar factor $\rho_N$ in turn satisfies a recursion relation, namely

$$\rho_{N+1} = (Q - \rho_N)^{-1}, \quad \rho_0 = 0.$$  

(78)

Now, taking into account that $T_N$ commutes with $P_{N-1}$, we verify relation (74):

$$n d_N(n,r) - d_{N+1}(n,r) = n \text{tr}_{1,...,N} P_{n,r,N} - \text{tr}_{1,...,N+1} P_{n,r,N+1}$$

$$\overset{(77)}{=} \rho_N \text{tr}_{1,...,N+1} (\tau_{n,r}(P_N T_N P_N)) = \rho_N \text{tr}_{1,...,N+1} (\tau_{n,r}(P_N T_N))$$

$$\overset{(74)}{=} \rho_N \text{tr}_{1,...,N+1} (\tau_{n,r}(P_{N+1} T_N - \rho_{N-1} P_{N+1} T_N - \rho_{N-1} T_N P_{N+1} P_{N-1}))$$

$$\overset{(72)}{=} \rho_N \text{tr}_{1,...,N+1} (\tau_{n,r}(P_{N+1} T_N - Q^{-1} \rho_{N-1} T_N P_{N+1} T_N P_{N-1}))$$

$$\overset{(78)}{=} (1 - Q^{-1} \rho_{N-1}) \rho_N \text{tr}_{1,...,N+1} (\tau_{n,r}(P_{N-1} T_N))$$

$$\overset{(78)}{=} Q^{-1} \text{tr}_{1,...,N+1} (P_{n,r,N-1} \otimes T) = r d_{N-1}(n,r).$$

The difference equation (74) can be rewritten in the form $v_N = B \cdot v_{N-1}$, where

$$B = \begin{pmatrix} n & -r \\ 1 & 0 \end{pmatrix}, \quad v_N = \begin{pmatrix} d_{N+1}(n,r) \\ d_N(n,r) \end{pmatrix}, \quad v_0 = \begin{pmatrix} n \\ 1 \end{pmatrix}.$$  

(79)

Computing the $N$-th power of the matrix $B$, we obtain solution (75)–(76). □

**Proof of Theorem 4.** First, note that, if $r < n^2/4$, then $\xi$ in (73) is positive and so $d_N(n,r)$ is also positive. For $r = n^2/4$, $d_N(n,r)$ is obviously positive as well.
The solution to the recursion relation (78) is given by

\[
\text{if } Q \neq 2 : \quad \rho_N = \frac{q^N - q^{-N}}{q^{N+1} - q^{-N-1}}, \quad \text{where } q + q^{-1} = Q \geq 1, \quad (80)
\]

\[
\text{if } Q = 2 : \quad \rho_N = \frac{N}{N + 1}, \quad (81)
\]

If \( q = e^{\pm \frac{i\pi}{2^r}} \), where \( k \in \mathbb{N} \), then \( \rho_m \) is finite for all \( m \leq k \) but \( \rho_{k+1} = \infty \). By (77), this implies that, for \( Q = 2 \cos\left(\frac{\pi}{2^r}\right) \), \( k \in \mathbb{N} \), the sequence of Jones–Wenzl projectors \( P_N \) is defined only up to \( N = k + 1 \).

For \( n^2 \geq r > n^2/4 \), we have \( |\xi| = 1 \) in (79) and therefore \( r^{N} d_N(r, n) = \frac{\sin[(N+1)\gamma]}{\sin \gamma} \), where \( 1 \leq 2 \cos \gamma = n/\sqrt{r} < 2 \). Therefore, for \( \gamma \in \left(\frac{\pi}{m+3}, \frac{\pi}{m+2}\right), m \in \mathbb{N} \), we have \( d_N(r, n) \geq 0 \) for all \( N \leq m + 1 \) and \( d_{m+2}(r, n) < 0 \). The latter inequality contradicts the positive semi-definiteness of \( P_{n,r,N} \) and therefore it requires the sequence of Jones–Wenzl projectors to terminate at some \( N \) not exceeding \( m + 1 \). Which, by the preceding consideration, restricts the allowed values of \( Q \) to the set \( \{2 \cos\left(\frac{\pi}{2^r}\right), 1 \leq k \leq m\} \). \( \square \)

**Proof of Proposition 4.** If \( \hat{V}_k = \frac{1}{\sqrt{m}} I_m \otimes V_k \), then \( W_\mathcal{T} = \frac{1}{m} I_m \otimes W_\mathcal{T} \). Therefore, if \( Q W_\mathcal{T} \) is unitary, so is \( m Q W_\mathcal{T} \). It remains to note that \( \text{tr}(\hat{V}_k \hat{V}_m^*) = \frac{1}{m} \text{tr}(I_m \otimes V_k V_m^*) = \frac{1}{m} \text{tr}(I_m) \text{tr}(V_k V_m^*) = \delta_{kp} \) as required. The case \( \hat{V}_k = \frac{1}{\sqrt{m}} V_k \otimes I_m \) is analogous. \( \square \)

**Proof of Proposition 5.** Without a loss of generality, we can assume that \( v \) is of length one. Taking Remark 3 into account, we can choose an orthonormal spanning set for \( \mathcal{T} \) in such a way that \( v \) is its first basis vector. That is, \( \mathcal{T} \sim \{V_1, \ldots, V_r\} \), where \( V_1 \) is (anti)symmetric. In this case, the upper–left block of \( Q W_\mathcal{T} \) is \( U \equiv Q V_1 V_1^* = \pm Q V_1 V_1^* \). Whence, by the normalization condition (10), we have \( \text{tr} U = Q \). By Theorem 2, \( Q W_\mathcal{T} \) is unitary. So, \( U \) is a principal submatrix of a unitary matrix and hence a contraction. Since \( U \) (or \( -U \)) is positive semi–definite, it implies that all its eigenvalues lie between 0 and 1. Therefore, \( |\text{tr} U| \leq n \), which imposes the restriction on \( Q \). \( \square \)

**Proof of Proposition 6.** a) Without a loss of generality, we can assume that \( v \) is of length one. Taking Remark 3 into account, we can choose an orthonormal spanning set for \( \mathcal{T} \) in such a way that \( v \) is its first basis vector. That is, \( \mathcal{T} \sim \{V_1, \ldots, V_r\} \), where \( V_1 \) is a scalar multiple of a unitary matrix. By Theorem 2, \( Q W_\mathcal{T} \) is unitary. Therefore the upper–left block of \( Q^2 W_\mathcal{T} W_\mathcal{T}^* \) equals to \( I_n \), that is

\[
Q^2 \sum_{k=1}^{r} V_k \tilde{V}_1 V_k^* = I_n. \quad (82)
\]

Since \( V_1 \) is a scalar multiple of a unitary matrix, we have \( \tilde{V}_1 V_1^* = \frac{1}{n} I_n \), where \( 1/n \) factor is due to the normalization (14). Substituting the latter relation in (82), taking trace, and using again the condition (16), we obtain that \( Q^2 = n^2/r \).
b) If \( n^2/4 < r < n^2 \), then Theorem 4 applies. Taking into account that \( Q^2 = n^2/r \), we conclude from (31) and (32) that

\[
\frac{n^2}{r} = 4 \cos^2 \left( \frac{\pi}{m + 2} \right), \quad m \in \mathbb{N}.
\] (83)

Note that the r.h.s. of (84) equals to \( e^{\frac{2\pi i}{m+2}} + e^{-\frac{2\pi i}{m+2}} \), which is a sum of three algebraic integers. Therefore, the r.h.s. of (83) is itself an algebraic integer. But the l.h.s. of (83) is a rational number. It is well known (see, e.g. Theorem 206 in [8]) that the only rational algebraic integers are ordinary integers. Thus, \( n^2/r \) is an integer from the interval (1,4). Hence \( n^2/r = 2 \) or \( n^2/r = 3 \).

\[ \Box \]

**Proof of Proposition 7.** First, consider the case \( V_k = V_1 g_k \). By Theorem 2, \( QW_T \) is unitary. Therefore equality (32) holds. Substituting \( V_k = V_1 g_k \) in (32) and taking into account that \( V_1 \) is invertible, we can rewrite (32) as follows:

\[
V_1 V_1^* + \sum_{k=2}^{r} g_k V_1 V_1^* g_k = Q^{-2}(V_1^* V_1)^{-1}.
\] (84)

Since \( g_k \) are unitary and \( V_1 \) satisfies (16), the trace of the l.h.s. of (84) is equal to \( r \). By the Cauchy inequality (cf. Proof of Proposition 3), the trace of the r.h.s. of (84) is greater or equal to \( n^2/Q^2 \). Whence follows inequality (37).

Now consider the case \( V_k = g_k V_1 \). Since \( T = QP_T \) is a solution to (T1)–(T4), so is \( T'' = QP_{T''} \) (cf. Corollary 1). But \( V_1'' = V_1^* \) and \( V_k'' = V_k^* g_k^* \). Therefore the preceding considerations apply to \( T'' \) yielding the same inequality on \( Q \).

\[ \Box \]

**Clebsch–Gordan coefficients for \( U_q(su_2) \)**

The Clebsch–Gordan coefficients appearing in (18) are given by [82]:

\[
\{S, S, k_1, k_2 | J, m\}_q = \delta_{k_1+k_2,m} q^{\frac{1}{2}(2S-J)(2S+J+1)+S(k_2-k_1)} [J]! \left( \frac{[2J+1]}{[2S+J+1]} \right)^{\frac{1}{2}}
\times \left( [2S-J] [S+k_1] [S-k_1] [S+k_2] [S-k_2] [J+k_1+k_2] [J-k_1-k_2] \right)^{\frac{1}{2}}
\times \sum_{l \geq 0} \frac{(-1)^{l} q^{-l(2S+J+1)}}{[l]! [2S-J-l] [S-k_1-l] [S+k_2-l] [J-S+k_1+l] [J-S-k_2+l]}.
\] (85)

Here \([l]! = \prod_{p=1}^{l} [p]_q \) if \( l \) is a positive integer, \([0]! = 1 \), and \([l]! = \infty \) if \( l \) is a negative integer. Due to the latter property, the sum in (85) always terminates. For \( q = 1 \), expression (85) recovers the Clebsch–Gordan coefficients for \( U_q(su_2) \).

The Clebsch–Gordan coefficients satisfy the orthogonality relation:

\[
\sum_{k_1, k_2=-S}^{S} \{S, S, k_1, k_2 | J_1, m_1\}_q \{S, S, k_1, k_2 | J_2, m_2\}_q = \delta_{J_1, J_2} \delta_{m_1, m_2}.
\] (86)

They also possess a number of symmetries including the following one:

\[
\{S, S, m_2, m_1 | J, m\}_q = (-1)^{2S-J} \{S, S, m_1, m_2 | J, m\}_{q^{-1}}.
\] (87)
Let \( V \) be the matrix associated by (49) to a vector \(|J, m\rangle_q = \mathcal{H}_S^{q=1} \otimes \mathcal{H}_S^{q=1} \), \( m \geq 0 \). Every row and column of \( V \) has at most one non–zero entry. Below we will need explicit expressions for the non–zero entries of the first few rows of \( V \).

**Lemma 2.** Let \( p \in \{0, 1, 2\} \). For all \( S \in \frac{1}{2} \mathbb{Z}_{\geq 1} \) and \( 0 \leq m \leq \min(J, 2S – p) \) we have

\[
\{ S, S, S-p, m+p-S | J, m \}_{q=1} = f_p(J, m) \left( \frac{(2J + 1)(2S - p)!}{p!(m + p)!} \right)^{\frac{1}{2}},
\]

where

\[
f_0(J, m) = 1, \quad f_1(J, m) = J(J + 1) - 2S(m + 1), \quad (89)
\]

\[
f_2(J, m) = f_1^2(J, m) + 2(m + 1 - 2S) f_1(J, m) + 2S(m + 1)(m - 2S).
\]

**Proof.** A direct verification with the help of formula (85) for \( q = 1 \). \( \square \)

**Remark 12.** The Clebsch–Gordan coefficient on the l.h.s. of (88) vanishes if \( p = 1 \) and \( m = 2S \) and also if \( p = 2 \) and \( m = 2S – 1 \) or \( m = 2S \). In these cases, we have \( f_1 = 0 \) and \( f_2 = 0 \), respectively. Indeed, \( m = 2S \) implies \( J = 2S \) and hence \( f_1 = f_2 = 0 \). Similarly, \( m = 2S – 1 \) implies either \( J = 2S \) or \( J = 2S – 1 \). In both cases, \(|f_1| = 2S \) and \( f_2 = 0 \).

**Proof of Proposition 8.** Let \( P_q \in M_{(2S+1)^2} \) stand for the orthogonal projection in \( \mathcal{H}_S^q \otimes \mathcal{H}_S^q \) on the one (or two) dimensional subspace spanned by the vector (respectively, the pair of vectors) under consideration. Consider the following function of \( q \):

\[
f(q) = \left( \text{tr}_{123} \left( (P_q)_{12}(P_q)_{23} \right) \right)^2 - (2S + 1) r \text{tr}_{123} \left( (P_q)_{12}(P_q)_{23} \right)^2,
\]

where \( r = \text{rank}(P_q) \). Our proof of the proposition will be based on the following lemma:

**Lemma 3.** Function \( f(q) \) is rational in \( q \).

An immediate consequence of Lemma 3 is that \( f(q) \) either vanishes identically for all \( q > 0 \) or it has only a finite (possibly empty) set of zeros on the semi–axis \( q > 0 \). But, by Theorem 1 a solution to (T1)–(T4) of the form \( T = QP_q \) exists only for such values of \( q \) that \( f(q) = 0 \). \( \square \)

**Proof of Lemma 3.** Let \( V_1, \ldots, V_r \) be the matrices associated by formula (48) to some set of vectors \(|J_1, m_1\rangle_q, \ldots, |J_r, m_r\rangle_q \) in \( \mathcal{H}_S^q \otimes \mathcal{H}_S^q \). Let \( P_q \) be the projection onto the subspace spanned by these vectors. Define the corresponding function \( f(q) \) by (91). Using (69) and (70) and taking into account that all entries of each \( V_k \) are real, we get

\[
f_1(q) \equiv \text{tr}_{123} \left( (P_q)_{12}(P_q)_{23} \right) = \sum_{k_1, k_2=1}^r \text{tr} (V_{k_1} V^t_{k_1} V^t_{k_2} V_{k_2}),
\]

\[
f_2(q) \equiv \text{tr}_{123} \left( (P_q)_{12}(P_q)_{23} \right)^2 = \sum_{k_1, k_2, k_3, k_4=1}^r \text{tr} (V_{k_1} V^t_{k_2} V^t_{k_3} V_{k_4} V^t_{k_1} V^t_{k_2} V_{k_3} V^t_{k_4} V_{k_4}).
\]

21
By (49), matrices $V_kV_k$, $V_kV_k^t$ and $V_k^tV_k$ are diagonal for every $k$. Observe that formula (55) along with the symmetry (57) imply that all non-zero entries of these matrices contain no square roots of $q$–factorials. Therefore, all non–zero entries of these matrices are rational functions in $q$. Whence it is evident that $f_1(q)$ is a rational function for all $r$ and $f_2(q)$ is a rational function for $r = 1$. Now, consider $f_2(q)$ for $r = 2$. In (93), the contributions from the terms with $k_1 = k_2$ or $k_3 = k_4$ are rational functions because the matrices in (93) can be multiplied in the following way: $\text{tr}((V_{k_1}V_{k_2}^t)(V_{k_3}^tV_{k_4}))$ and $\text{tr}((V_{k_1}V_{k_2}^tV_{k_3}^tV_{k_4}))$, respectively. If $k_1 \neq k_2$ and $k_3 \neq k_4$, then either $k_1 = k_4$ and $k_2 = k_3$ or $k_1 = k_3$ and $k_2 = k_4$. The contributions from such terms also yield rational functions because, in these cases, the matrices in (93) can be multiplied in the following way: $\text{tr}((V_{k_1}V_{k_2}V_{k_3}^tV_{k_4}))$ and $\text{tr}((V_{k_3}V_{k_1}V_{k_2}^tV_{k_4}))$, respectively. Thus, we conclude that both $f_1(q)$ and $f_2(q)$ are rational functions in $q$ if $r = 1, 2$. Hence the same holds for $f(q) = f_1(q) - (2S + 1)r f_2(q)$. \hfill \Box

**Proof of Theorem 5.** Let $V$ be the matrix associated to a vector $|J, m\rangle$ by formula (49). Note that $VV^*$ is diagonal and it is degenerate for all $m \neq 0$. Thus, we have to restrict consideration to the case $m = 0$.

For $J = m = 0$, the sum in (85) contains only one term $(l = S - k_1 = S + k_2)$ and, therefore, by (49), we have

$$V_{ab} = \delta_{a+b,2S+2} \frac{(-1)^{a-1}q^{S+1-a}}{\sqrt{[2S+1]_q}}, \quad a = 1, \ldots, 2S + 1.$$  \hfill (94)

Thus, $[2S + 1]_q^V V = (-1)^{2S} I_{2S+1}$. Hence, by Theorem 2, vector $|0, 0\rangle_q$ is a TL vector for all $q > 0$ and all $S \in \frac{1}{2} \mathbb{Z}_{>0}$.

For $q > 0$, $S = 1/2$, $J = 1$, $m = 0$, the corresponding matrix $V$ was given in (50). It is easy to see, invoking Theorem 2, that it corresponds to a TL vector for all $q > 0$.

In order to complete the proof, we have to consider vectors $|J, 0\rangle$ for $S \geq 1$ and $J \geq 1$.

First, setting $q = 1$, we will prove that, in this case, the list of TL vectors given in the part a) is exhaustive. To this end, we will invoke Theorem 2 again. For $q = 1$, matrix $V$ is either symmetric or anti-symmetric due to (57). Therefore, the diagonal matrix $VV^* = \pm VV^*$ can be a scalar multiple of a unitary matrix only if all the non–zero entries of $V$ have equal moduli. In view of formula (19), the latter condition is equivalent to the following requirement:

$$\{S, S, S - p, p - S | J, 0\}^2_{q=1} = \frac{1}{2S+1},$$  \hfill (95)

for $p = 0, 1, \ldots, 2S$. The value on the r.h.s. of (95) is due to the normalization condition, $\text{tr}(VV^*) = 1$.

For $S \geq 1$, equality of the expressions on the l.h.s. of (95) for $p = 0, 1, 2$ is equivalent by Lemma 2 to the following equalities:

$$(f_0(J, 0)(2S)!)^2 = (f_1(J, 0)(2S - 1)!)^2 = \frac{1}{2}f_2(J, 0)(2S - 2)!^2.$$  \hfill (96)
For $J > 0$, formulae (89) imply that the first equality in (96) holds iff

$$J(J + 1) = 4S.$$  \hfill (97)

Remarkably, under this condition, we have, by (89) and (90), $f_1(J, 0) = 2S$ and $f_2(J, 0) = 4S(1 - 2S)$ so that the second equality in (96) holds identically thus imposing no further restrictions on $S$ and $J$.

Equality (97) provides a necessary condition for $|J, 0⟩_{q=1}$ to be a TL vector. In order to obtain another necessary condition, we rewrite equality (95) for $p = 0$ with the help of (88) and (89) in the following form:

$$\frac{(2S - J)!(2S + J + 1)!}{(2S)!(2S + 1)!} = 2J + 1.$$ \hfill (98)

For $J > 2$, define $F(J) \equiv \left(\frac{J}{J - 2}\right)^J$. Observe that if $J$ and $S$ are related as in (97) and $J > 2$, then we have the following estimate for the l.h.s. of (98):

$$\frac{(2S - J)!(2S + J + 1)!}{(2S)!(2S + 1)!} = \prod_{k=1}^{J} \frac{2S + 1 + k}{2S - J + k} = \prod_{k=1}^{J} \left(1 + \frac{J + 1}{2S - J + k}\right) < \left(1 + \frac{J + 1}{2S - J - 1}\right)^J \equiv F(J).$$ \hfill (99)

Note that $F(J)$ decreases monotonically as $J$ grows and we have $F(6) = (3/2)^6 \approx 11.4 < 2 \cdot 6 + 1$. Therefore, for all $J \geq 6$, the l.h.s. of (98) is smaller then the r.h.s. So, it remains to check whether (98) holds for $1 \leq J \leq 5$ provided that $J$ and $S$ are related as in (97). A direct inspection shows that equalities (97) and (98) are not compatible for $J = 3, 4, 5$ but they hold for $J = 1$ and $J = 2$ if the corresponding values of $S$ are $S = 1/2$ and $S = 3/2$, respectively. The first of these cases, $J = 1$, $S = 1/2$ was already considered above. In the second case, it is easy to check that $|2, 0⟩_{q=1}$ is indeed a TL vector if $S = 3/2$ (cf. Example 11). This completes the proof of the part a) of the theorem.

What the part b) of the theorem is concerned, it was already explained at the beginning of this proof why the vectors listed in the part b) are TL vectors. This list is exhaustive because the third case found for $q = 1$, i.e. $|2, 0⟩_{q=1}$ for $S = 3/2$ is not a TL vector except for $q = 1$ and two other values of $q$ (cf. Example 11). □

**Proof of Theorem 6** Part a). Let $V_1$, $V_2$ be the matrices associated by formula (49) to a pair of orthogonal vectors $|J_1, m_1⟩_{q=1}$, $|J_2, m_2⟩_{q=1}$. Taking into account that $V_1$, $V_2$ are real, the unitarity condition (20) is equivalent to the following set of equations:

$$V_1 V_1^t V_1^t + V_2 V_1^t V_1^t = Q^{-2} I_n, \quad V_1 V_2 V_2^t V_1^t + V_2 V_2^t V_2^t = Q^{-2} I_n, \quad (99)$$

$$V_1 V_1^t V_2^t + V_2 V_1^t V_2^t = 0, \quad (100)$$

where $n = 2S + 1$ and $Q > 0$.

Denote $A \equiv V_2 V_1^t$, $B \equiv V_1 V_1^t$, $C \equiv V_2 V_2^t$. Recall that, by the symmetry (87), we have $V_1^t = (-1)^{2S - J_1} V_1$ and $V_1^t = (-1)^{2S - J_2} V_1$. Therefore, equations (99)–(100) can be
rewritten in the following form:

\[
B^2 = Q^{-2}I_n - A A^t, \quad C^2 = Q^{-2}I_n - A^t A, \quad (101)
\]
\[
B A + A C = 0. \quad (102)
\]

Let the singular value decomposition of \( A \) be given by \( A = O_1 \Sigma O_2 \), where \( O_1 \) and \( O_2 \) are orthogonal and \( \Sigma \geq 0 \) is diagonal (note that \( \Sigma \leq Q^{-1} I_n \)). Then, we infer from (101) that

\[
B^2 = O_1 (Q^{-2}I_n - \Sigma^2) O_1^t, \quad C^2 = O_2^t (Q^{-2}I_n - \Sigma^2) O_2. \quad (103)
\]

These equations determine the singular values of \( B^2 \) and \( C^2 \). Taking into account that \( B \) and \( C \) are diagonal and positive semi–definite, we conclude that

\[
B = O_1 (Q^{-2}I_n - \Sigma^2)^{\frac{1}{2}} O_1^t, \quad C = O_2^t (Q^{-2}I_n - \Sigma^2)^{\frac{1}{2}} O_2. \quad (104)
\]

where \((\ldots)^{\frac{1}{2}} \geq 0 \) and \( O_1, O_2 \) are orthogonal. Relations (103) and (104) are consistent if \((Q^{-2}I_n - \Sigma^2)\) commutes with \( O_1 O_1^t \) and \( O_2 O_2^t \). Therefore, we get

\[
B A = O_1 (Q^{-2}I_n - \Sigma^2)^{\frac{1}{2}} O_1^t O_1 \Sigma O_2 = O_1 O_1^t O_1 \Sigma (Q^{-2}I_n - \Sigma^2)^{\frac{1}{2}} O_2
\]
\[
= O_1 \Sigma (Q^{-2}I_n - \Sigma^2)^{\frac{1}{2}} O_2 = O_1 \Sigma (Q^{-2}I_n - \Sigma^2)^{\frac{1}{2}} O_2^t O_2 = A C. \quad (105)
\]

But then (102) holds only if

\[
B A = A C = 0. \quad (106)
\]

Equations (105) and (106) imply that \( \Sigma^2 (Q^{-2}I_n - \Sigma^2) = 0 \). Hence the only non–zero eigenvalue of \( \Sigma \) is \( Q^{-1} \). Recall that \( B \) and \( C \) are diagonal and positive semi–definite. Therefore, in view of (104), we have established the following.

**Lemma 4.** Diagonal matrices \( B \equiv V_1 V_1^t \) and \( C \equiv V_2 V_2^t \) are degenerate, have equal ranks, and all of their non–zero entries are equal to \( Q^{-1} \).

**Remark 13.** If \( B \) and \( C \) were non–degenerate then so would be \( V_1, V_2 \) and hence also \( A \). But then equalities (106) could not hold.

Recall that, in the rank one case (see the proof of Theorem 5), all the entries of the diagonal matrix \( V V^t \) are to be equal to \( Q^{-1} \). Lemma 4 shows that, in the rank two case, the situation is similar but somewhat more complicated because some of the diagonal entries of \( B \) and \( C \) can be equal to zero.

Note that if \( |J_1, m_1\rangle_{q=1}, |J_2, m_2\rangle_{q=1} \) is a TL pair, we must have \( m_1 m_2 \leq 0 \). Indeed, suppose this is not so, for instances, \( m_1 < 0 \) and \( m_2 < 0 \). Then the first row of \( V_1 \) and \( V_2 \) contains only zeroes and hence \( (B^2)_{11} = (A A^t)_{11} = 0 \) which contradicts relation (101). Thus, without a loss of generality, we can assume that \( m_1 \geq 0 \) and \( m_2 \leq 0 \). In this case,

\[
B_{ii} = 0 \quad \text{for all} \quad i > 2S + 1 - m_1. \quad (107)
\]
By Lemma 3, the remaining diagonal entries of $B$ are equ to either zero or $Q^{-1}$. In particular, the symmetry (37) implies that $B_{2S+1-m_1,2S+1-m_1} = B_{11} = Q^{-1}$.

Let us show that, for $S \geq 9/2$, there exist no $B$ and $C$ compatible with the requirements imposed by Lemma 3. First, we note that, by (107), $B$ has at most $2S + 1 - m_1$ non–zero entries and hence, by Lemma 3 and the normalization condition $\text{tr} B = 1$, the corresponding $Q$ must be an integer in the interval
\[
S + \frac{1}{2} \leq Q \leq 2S + 1 - m_1.
\] (108)

(The lower bound is imposed by Theorem 3). Inequalities (108) imply that $m_1 \leq S + 1/2$ and hence $2S + 1 - m_1 \geq S + 1/2$. Thus, for $S \geq 9/2$, matrix $B$ has at least five entries which are not a priori zero. Moreover, the first three of them, $B_{11}, B_{22}$, and $B_{33}$, are not related to each other by the symmetry (37).

By Lemma 2, $B_{11}$ is always non–zero. Therefore, by Lemma 4, we have one of the following cases: i) $B_{11} = Q^{-1}, B_{22} = B_{33} = 0$; ii) $B_{11} = B_{33} = Q^{-1}, B_{22} = 0$; iii) $B_{11} = B_{22} = Q^{-1}, B_{33} = 0$; iv) $B_{11} = B_{22} = B_{33} = Q^{-1}$.

Let now $f_1$ and $f_2$ stand for $f_1(J_1, m_1)$ and $f_2(J_1, m_1)$ defined in (39) and (40), respectively. (Note that, for $S \geq 9/2$ and $m_1 \leq S + 1/2$, we have $2S - m_1 - p > 0$ for $p = 0, 1, 2$ so that the r.h.s. of (39) is well defined.) The case i) requires that $f_1 = f_2 = 0$, which, by (39), requires that $m_1 = 2S$. This is impossible since we have $m_1 \leq S + 1/2$.

The case ii) requires that $B_{11} = B_{33}$ which, by (39), holds iff
\[
f_2^2 = 4S(2S - 1)(m_1 + 1)(m_1 + 2)(2S - m_1 - 1)(2S - m_1).
\]

On the other hand, by (39), equality $B_{22} = 0$, i.e. $f_1 = 0$, holds iff
\[
J_1(J_1 + 1) = 2S(m_1 + 1).
\]

In this case, (39) acquires the form $f_2 = 2S(m_1 + 1)(m_1 - 2S)$. Substituting the latter in (109), we infer that the case ii) holds only if $J_1 = 2S - 1$ and $m_1 = 2S - 2$. But the latter condition contradicts, for $S \geq 9/2$, the restriction $m_1 \leq S + 1/2$.

In the cases iii) and iv), equality $B_{11} = B_{22}$ holds, as seen from (39), only if
\[
f_1^2 = 2S(m_1 + 1)(2S - m_1).
\]

Therefore, $f_2$ given by (39) acquires the following form:
\[
f_2 = 2(m_1 + 1 - 2S)f_1.
\]

The case iii) requires that $f_2 = 0$ that is $m_1 = 2S - 1$. But this again contradicts the restriction $m_1 \leq S + 1/2$.

Finally, substituting (111) and (112) into (109) and taking into account that $m_1 \neq 2S - 1, 2S$, we infer that, for $S \geq 9/2$, the case iv) holds only if $m_1 = 0$. In this case, relations (39) and (111) imply that either $J_1 = 0$ or $J_1(J_1 + 1) = 4S$. Since $m_1 = 0$, we can assume that $m_2 \geq 0$ and, repeating the same analysis for matrix $C$, we draw the conclusion that $m_2 = 0$ and either $J_2 = 0$ or $J_2(J_2 + 1) = 4S$. But the vectors determined
by the matrices $V_1$ and $V_2$ must be orthogonal. Therefore, either $V_1$ or $V_2$ corresponds to $|0,0\rangle_{q=1}$ and so it is a non–degenerate matrix, cf. equation (94). However, this is in contradiction with Lemma 4 which asserts that both $B$ and $C$ are degenerate.

Thus, we have proved that there exist no TL pairs $|J_1, m_1\rangle_{q=1}, |J_2, m_2\rangle_{q=1}$ for all $S \geq 9/2$. A direct inspection shows that the only TL pairs for $S \leq 4$ are those listed in (55)–(60). This completes the proof of the part a).

Using explicit formulae (55) for the Clebsch–Gordan coefficients of $U_q(su_2)$, it is straightforward to check which of the pairs of vectors found in the part a) for $q = 1$ remain TL pairs for other values of $q$. □

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