In the stable marriage problem $N$ men and $N$ women have to be matched by pairs under the constraint that the resulting matching is stable. We study the statistical properties of stable matchings in the large $N$ limit using both numerical and analytical methods. Generalizations of the model including singles and unequal numbers of men and women are also investigated.

**Keywords**: Stable marriage, optimization

**I. INTRODUCTION**

The study of complex systems that are built up of many individuals interacting according to relatively simple laws has recently attracted a lot of interest among physicists. Concepts and methods developed in statistical physics have successfully been applied to describe collective and nonlinear phenomena arising in such systems. Studies on traffic flow, stock markets, voting, flocking birds, evolution etc. are only a few examples to mention here.

In the stable marriage problem the elements of two sets have to be matched by pairs. As possible applications one could think of job seekers and employers, lodgers and landlords, or simply men and women who want to get married. For the sake of clearness and simplicity we will exclusively refer to the paradigm of marriage in this paper. The purpose of most studies of the stable marriage problem has so far been to unravel the underlying mathematical structure and to develop new or improve existing algorithms for finding stable marriages. Only recently it has been recognized that the stable marriage problem has many features in common with classical disordered systems in statistical physics like spin glasses, in particular the element of frustration and the existence of many competing states. Applying methods developed in statistical mechanics, Nieuwenhuizen solved several variants of the stable marriage problem, including bachelors and polygamy. In his approach, Nieuwenhuizen focussed on the properties of globally optimal matchings which are advantageous for the society as a whole, but not necessarily for all individuals. Here we consider the opposite limit where all individuals pursue only their own egoistic objectives regardless of the consequences for others. Under these conditions, only matchings that are stable with respect to the actions of all individuals are relevant. In the framework of game theory this kind of stability is termed Nash equilibrium.

Our paper is organized as follows. In Section 2 we introduce the stable marriage model and set up the notation. The Gale-Shapley algorithm and the properties of the stable matchings obtained with it are presented in Section 3. In Section 4 we analyze the set of all stable matchings. Generalizations of the model, introducing an acceptance threshold and unequal number of men and women are discussed in Sections 5 and 6, respectively. Finally, we summarize our results in Section 7.

**II. THE STABLE MARRIAGE MODEL**

We consider a society consisting of $N$ men and $N$ women who have to be matched by pairs. The cost for man $i$ to be married with woman $j$ is $x(i,j)$ and the cost for woman $j$ to be married with man $i$ is $y(j,i)$. In order to investigate statistical properties of the model we assume that all cost functions are independent random variables uniformly distributed between 0 and 1.

Since all individuals simultaneously attempt to optimize their own benefit by being married with a partner of lowest possible cost, conflicting wishes arise, and in general it is impossible to find a matching where everybody is perfectly satisfied. Let us assume that a matching is formed where man $i$ is married with woman $j$. We define $x_i \equiv x(i,j)$ and $y_j \equiv y(j,i)$, i.e. $x_i$ is the cost for man $i$ in this particular matching and $y_j$ the corresponding cost for woman $j$. Instead of cost we will frequently use the expression energy in order to emphasize the analogy with physical systems where the energy is the quantity to be minimized in the groundstate. For later convenience, we introduce the total energies of men and women by

$$X = \sum_{i=1}^{N} x_i \quad Y = \sum_{j=1}^{N} y_j$$

(1)

In search of the rules according to which marriages can be formed or broken, essentially two points of view have

\[^{1}\text{In Ref. [1] a different probability distribution has been considered, namely } \rho(x) \propto x^r \exp(-x). \text{ The case } r = 0 \text{ corresponds essentially to our choice.}\]
emerged: In the first one, the goal is to find the globally best matching, i.e. the one where the total energy $U = X + Y$ is minimal. This problem has recently been studied by Nieuwenhuizen [4] using techniques borrowed from statistical physics of disordered systems, such as the replica trick. The second point of view, that we will adopt, emphasizes the role of individuals as decision-makers. As long as there is no supervising body like church or government that dictates who has to be married with whom no individual is obliged to stay in an unsatisfactory relationship if he or she can find a better one. This leads us to the notion of stability. A matching is called stable if there is no man and no woman who prefer each other to their actual partners. In other words, a matching is unstable, if there exists a man $i$ and a woman $j$, who are not married with each other, such that

$$x_i > x(i,j) \text{ and } y_j > y(j,i) \quad (2)$$

The particular significance of stable matchings lies in the fact that they cannot be broken, neither by the action of a single pair nor by any coalition of men and women. In contrast to the globally best matching which can only be found with the help of computers for very small numbers of men and women, there exist powerful algorithms to search for stable matchings, even for systems of several thousands of individuals. The simplest of these algorithms, introduced by Gale and Shapley some time ago [3], will be presented in the next section.

III. THE GALE-SHAPLEY ALGORITHM

In the man-oriented Gale-Shapley (GS) algorithm only men make proposals while women accept or reject. In the beginning every man makes up a list where he places the women according to his preferences. The smaller the cost $x(i,j)$ the higher the position of woman $j$ on the list of man $i$. Then a series of proposals starts. Whenever a man is not married he makes a proposal to the woman with the highest rank among those to whom he has not yet proposed. If the woman is not engaged she accepts the proposal and they get married. If the woman is already married, she accepts if she prefers the suitor to her husband. In this case she divorces and gets married with the proposer. The algorithm stops when the last woman has received an offer and gets married. The number of proposals $r$ that a man has to make on the average during the execution of the GS algorithm has been calculated in [3]. The result is

$$r \simeq \log N + C \quad (3)$$

where $C = 0.5772...$ is Euler’s constant and corrections are negative and of order $(\log N)^2/N$. Amazingly, the knowledge of $r$ is already sufficient to calculate the total energy of men ($X$) and women ($Y$) in the GS matching. With every proposal the energy of one man increases by $\simeq 1/N$ since he has to move forward one step on his preference list. Consequently, the total energy of men is on the average the same as the number of proposals that every man makes, i.e. $X = r$.

Each woman also receives on the average $r$ proposals. Keeping only the best offer i.e. the smallest out of $r$ random numbers between 0 and 1 yields a best value of $y = 1/(r + 1)$. Taking into account that the number of proposals that a woman receives is not fixed, but distributed according to a binomial distribution yields $y = (1-\exp(-r))/r \approx 1/r$ (see Appendix B). Using Eq. (3) we obtain the following estimates for the energies of men and women in the man-oriented GS matching

$$X \simeq \log N + C \quad Y \simeq \frac{N}{\log N + C} \quad (4)$$

which implies the relation $XY = N$.

FIG. 1. Average energy of men and women in the man-oriented Gale-Shapley algorithm as a function of $N$.

Fig. 1 shows the average energies $X$ and $Y$ of men and women obtained with the man-oriented GS algorithm as function of $N$. Each data point represents an average over several thousand realizations. A logarithmic scale is used for the energy axis in order to highlight the good agreement with the analytical result of Eq.(4) even for small values of $N$.

Of course, everything that has been said about the man-oriented GS algorithm is also valid in the woman-oriented version where the roles of men and women are interchanged, i.e. women propose and men judge. It has been shown [3] that the stable matching obtained from the man–oriented execution of the GS algorithm is man–optimal in the sense that no man can have lower energy in any other stable matching. Similarly, it is the worst possible stable matching for women. Therefore the energies obtained by performing the GS algorithm provide an upper and a lower bound for the possible energy range of all stable matchings. Furthermore, if the man and the woman–optimal matching happen to be identical we for sure that in this case there exists only one single stable matching.
IV. THE SET OF ALL STABLE MATCHINGS

If \( N \) is large, there are generally many other stable matchings besides the man and the woman-optimal matchings that can be obtained with the GS algorithm. Although the maximum number of stable matchings in a system of size \( N \) is not known, there exist lower bounds \(^2\); e.g. for \( N = 32 \) the maximum number of stable matchings is larger than \( 10^{11} \) (1).

In order to calculate the average number of stable matchings we consider an arbitrary matching of \( N \) men and \( N \) women. This matching is unstable, if there exists a man \( i \) and a woman \( j \) who are not married with each other, and whose mutual costs \( x(i,j) \) and \( y(j,i) \) are both lower than the ones in the existing matching, \( x_i \) and \( y_j \), respectively. The probability that man \( i \) and woman \( j \) prefer to stay with their respective partners is therefore \( p_{ij} = 1 - x_i, y_j \) and the whole matching is stable with probability

\[
P = \int_0^1 d^N x \int_0^1 d^N y \prod_{i \neq j} (1 - x_i, y_j) \tag{5}
\]

Although this \( 2N \)-fold integral, originally derived by Knuth \(^3\), looks quite simple, it cannot be done exactly unless \( N \) is very small. Using rather involved probabilistic arguments, Pittel \(^6\) has derived an asymptotic formula,

\[
P \approx \frac{\log N}{e \Gamma(N)} \tag{6}
\]

valid in the limit \( N \to \infty \). However, comparing Eq. (6) with numerical simulations we found rather serious discrepancies (see Fig. 2). In order to remove these discrepancies and to provide a more transparent approach we replace the factors \( 1 - x_i, y_j \) in Eq. (6) by \( \exp(-x_i, y_j) \), which is justified since the dominant contribution to the integral comes from the region where the products \( x_i, y_j \) are small. Dropping the constraint \( i \neq j \) we arrive at the approximation

\[
\hat{P} = \int_0^1 d^N x \int_0^1 d^N y \exp(-XY) \tag{7}
\]

where the variables \( X = \sum_{i=1}^N x_i \) and \( Y = \sum_{j=1}^N y_j \) are the energies of men and women, respectively. Eq. (7) can also be viewed as the partition sum of two species of particles confined to the interval \([0, 1]\) whose interaction is the product of their center of mass coordinates. It is convenient to replace the integral over the variables \( x_i \) and \( y_j \) by an integral over \( X \) and \( Y \)

\[
\hat{P} = \int_0^N dX \int_0^N dY \rho(X) \rho(Y) \exp(-XY) \tag{8}
\]

where the probability distribution of \( X \) is asymptotic to

\[
\rho(X) \simeq \frac{X^{N-1}}{\Gamma(N)} \left(1 - \exp\left(-\frac{N}{X}\right)\right)^N \tag{9}
\]

for small \( X \) (see Appendix A). Inserting Eq. (9) in Eq. (8) and introducing \( t = XY \) as new integration variable we obtain

\[
\hat{P} = \Gamma(N)^{-2} \int_0^{\infty} \frac{N}{X} dX X^{-1} \int_0^{\infty} \frac{N}{Y} dt t^{-1} \exp(-t) \infty = \Gamma(N)^{-1} \int_0^{\infty} \log \frac{N}{\Gamma(N)/N} \tag{10}
\]

where we have assumed that the factor \( \left(1 - \exp\left(-\frac{N}{X}\right)\right)^N \) imposes a cut-off \( \propto N/\log N \) on the \( X \)-integration and the same holds for \( Y \). The integral over \( t \) yields \( \Gamma(N) \), provided that \( X > \log N \). Comparison with Eq. (4) shows that the ratio \( \hat{P}/P \) approaches \( e = \exp(1) \) in the limit \( N \to \infty \) which we also checked numerically. Since there are in total \( N! = \Gamma(N + 1) \) matchings the average number of stable matchings is given by

\[
S \simeq \frac{N}{e} \left(\log N - 2 \log(\log N)\right) \tag{11}
\]

\[\text{FIG. 2.} \] Average number of stable matchings \( S \) as function of \( N \). Numerical simulations (circles) in comparison with the analytic expression of ref. [6] (solid curve) and our result of Eq. (11) (dashed curve).

\[S \text{ is plotted as function of } N \text{ in Fig. 2.} \] The circles are data obtained from numerical simulations using the algorithm of Gusfield and Irving \(^2\) that allows the determination of all stable matchings. Each data point represents an average over 200 random realizations of the preference lists. The dashed curve is the result of Eq. (11) and the solid curve is the asymptotic formula derived by Pittel \(^6\). Note the large error made by neglecting the \( \log(\log N) \) term.

In the following we investigate how the energies of men and women are correlated in stable matchings. From Eq.
we see that \( \rho(X,Y) = \rho(X)\rho(Y) \exp(-XY) \) can be considered as the probability density of finding a stable matching with energies \( X \) and \( Y \). Due to the factor \( \left(1 - \exp\left(-\frac{N}{N}\right)\right)^N \) that appears in \( \rho(X) \) the occurrence of stable matchings with \( X > N/\log N \) is very unlikely, and by symmetry, the same cutoff exists for \( Y \). This observation is in accordance with our analysis of the GS algorithm, where we found that the energy of women in the man-optimal stable matching (i.e. the worst possible for women) is \( \approx N/\log N \), and vice versa. In the region \( \log N < X,Y < N/\log N \) the probability distribution simplifies to \( \rho(X,Y) \propto (XY)^{N-1} \exp(-XY) \). It depends only on the product of men’s and women’s energies and is sharply peaked around the curve \( XY = N \) for large values of \( N \). Mean-value and standard deviation of the product \( XY \) are given by

\[
\langle XY \rangle = N, \quad \sigma(XY) = \sqrt{N}
\]

where \( \langle \ldots \rangle \) denotes the average calculated with \( \rho(X,Y) \).

At the symmetric point \( X = Y = \sqrt{N} \) the width of the distribution is \( O(1) \) which means that relative fluctuations go to zero for \( N \to \infty \).

![FIG. 3. Probability distribution of \( \log S \) for \( N = 50 \). Numerical simulations (circles) fitted by a Gaussian.](image)

Finally, we investigate the probability distribution of the number of stable matchings using numerical simulations. While \( S \) has a very asymmetric distribution which decays quite slowly for large values of \( S \), the distribution of \( \log S \) can be nicely fitted with a Gaussian, as shown in Fig. 3 for \( N = 50 \). Remarkably, the width of the distribution, i.e. the standard deviation \( \sigma(\log S) \) approaches a constant value \( \sigma_\infty \approx 0.48 \) for large \( N \) as shown in Fig. 4. We have no argument to explain this universal behavior.

V. ACCEPTANCE THRESHOLD

Up to now we have assumed that every person has to get married. In the following we soften this condition by introducing an acceptance threshold \( \Delta < 1 \) which is the energy of individuals who remain single. In principle, \( \Delta \) could be chosen at random for all individuals, or at least be different for men and women, but for the sake of simplicity we consider only the case where \( \Delta \) is the same for all men and all women.

Let us consider a situation where \( \Delta \) is sufficiently large that singles do not occur. The GS algorithm is readily generalized to the case with acceptance threshold. In the man-oriented version the only difference is that now women – even when unmarried – refuse proposals from men whose energy is higher than \( \Delta \). This increases the average number of proposals that are necessary to arrive at a stable matching by a factor of \( 1/\Delta \). The generalization of Eq. (11) to the case \( \Delta \neq 1 \) reads

\[
X(\Delta) \simeq \frac{\log N + C}{\Delta} \quad Y(\Delta) \simeq \Delta \frac{N}{\log N + C}
\]

Both energies coincide for \( \Delta = \Delta_c \), and we conclude that for \( \Delta < \Delta_c \) there exist only few or one single stable matching. This fact can be used to generate approximately sex-fair stable matchings – which is difficult to achieve by other means – using the GS algorithm and setting \( \Delta = \Delta_c \).

In the limit when \( \Delta \) is sufficiently small it becomes favorable for a certain fraction of men and women to remain single. In this case it is hard to calculate the average number of proposals \( r \) explicitly since the series of proposals is not only terminated when all men are married but also if no acceptable women are available. On the other hand the energies of men and women coincide in this regime since there exists only one stable matching. Let us assume that a woman has received \( r \) proposals out of which she keeps only the best one. If none of the proposals are better than the threshold \( \Delta \) she remains
single at the cost of $\Delta$. Analyzing the probability distribution of the best offer (see Appendix B) yields the average energy of women

$$Y = \frac{N}{X} (1 - \exp(-X\Delta))$$  \hspace{1cm} (15)$$

From the condition $X = Y$ we obtain

$$X^2 = N (1 - \exp(-X\Delta))$$  \hspace{1cm} (16)$$

In general this equation can only be solved numerically. The limiting cases are

$$X(\Delta) \simeq \left\{ \begin{array}{ll}
\frac{N\Delta}{\sqrt{N}} & \text{for } \Delta \ll \frac{1}{\sqrt{N}} \\
\frac{N\Delta}{\sqrt{N}} & \text{for } \Delta \gg \frac{1}{\sqrt{N}}
\end{array} \right.$$  \hspace{1cm} (17)$$

which means that for sufficiently small $\Delta$ it is advantageous for almost everybody to remain single (at the energy $\Delta$) while for $\Delta \gtrsim 1/\sqrt{N}$ the energies in sex-fair matchings are $X = Y = \sqrt{N}$, in agreement with our previous considerations.

In Fig. 5 we show the energy of men and women in the man-optimal stable matching as function of $\Delta$ for $N = 200$. Each data point represents the average over 1000 executions of the GS algorithm. The agreement between numerical and analytical results is very good with the exception of the cross-over region around $\Delta_c$ where neither of the conditions that were made to treat the problem (absence of singles in the case $\Delta > \Delta_c$, existence of only one stable matching in the case $\Delta < \Delta_c$) is strictly fulfilled.

In the following we investigate how the number of singles depends on the acceptance threshold $\Delta$. The probability that a woman who receives $r$ proposals rejects all of them and remains single is $p_s = (1 - \Delta)^r$. Taking into account that the number of proposals addressed to a particular woman is distributed according to a binomial distribution the average number of singles $N_s$ is related to the energy $X$ via

$$N_s(\Delta) = N \exp(-X\Delta)$$  \hspace{1cm} (18)$$

(see Appendix C) and can be calculated explicitly from the solution of Eq. (16). The limiting cases are

$$N_s(\Delta) \simeq \left\{ \begin{array}{ll}
N \exp(-\Delta^2 N) & \text{for } \Delta \ll \frac{1}{\sqrt{N}} \\
N \exp(-\Delta\sqrt{N}) & \text{for } \Delta \gg \frac{1}{\sqrt{N}}
\end{array} \right.$$  \hspace{1cm} (19)$$

When $\Delta$ is decreased, the first single appears when $N_s(\Delta) = 1$. This is the case for $\Delta = \log N/\sqrt{N} \simeq \Delta_c$ in accordance with our assumption that there are no singles for $\Delta > \Delta_c$.

Finally, we generalize our results concerning the average number of stable matchings to the case with threshold $\Delta < 1$. The only difference compared to the calculation for $\Delta = 1$ is that the probability distribution of $X$ (see Appendix A) has to be replaced by

$$\rho(X) \simeq \frac{X^{N-1}}{\Gamma(N)} \left(1 - \exp\left(-\frac{N\Delta}{X}\right)\right)^N$$  \hspace{1cm} (20)$$

which takes into account that all energies $x_i$ have to be smaller than $\Delta$. Making the same approximations as before we obtain for the average number of stable matchings

$$S(\Delta) \simeq \frac{N}{e} \left(\log N - 2 \log(\log N) + 2 \log \Delta\right)$$  \hspace{1cm} (21)$$

$S(\Delta)$ goes to zero at $\Delta = \log N/\sqrt{N} \simeq \Delta_c$ in agreement with the fact that man and woman-optimal stable matchings coincide for $\Delta < \Delta_c$.

VI. UNEQUAL NUMBER OF MEN AND WOMEN

Let us now consider the more general case where the number of men and women is not the same. To be specific, we assume that there are $N + 1$ men and $N$ women. It is obvious that in this case at least one man has to remain single. As before, the GS algorithm can be used to find the man-optimal and the woman-optimal stable matching, respectively, depending on who proposes and who judges. There is now a fundamental asymmetry between the man and the woman-oriented version: when women propose the algorithm stops as usual when the last woman has found a husband; in the man-oriented version, however, the series of proposals stops only when the first man has reached the bottom of his list, i.e. when he has been rejected by all women and must stay single. Therefore, the members of the minority group (the women in our case) are substantially favored.
When an acceptance threshold $\Delta$ is introduced, the number of stable matchings shrinks and singles occur. Three different regimes can be identified: For $\Delta > \Delta_c \approx \log N/\sqrt{N}$ there are many stable matchings whose energies extend over a wide range and no singles occur. For $1/\sqrt{N} < \Delta < \Delta_c$, there are very few or only one stable matching. In this regime the energy of men and women is nearly independent of $\Delta$ although the number of singles increases exponentially with decreasing $\Delta$. For $\Delta < 1/\sqrt{N}$ there are many singles and only a small fraction of married individuals. When the number of men and women is not the same, the number of stable matchings is also substantially reduced. Furthermore, the members of the minority group are strongly favored compared to the majority group, even if there is only one additional member.

The general ideas discussed in this paper could become important for investigations on the labour market since nowadays information about forthcoming job opportunities, on the one hand, and qualifications of applicants, on the other hand, can be made available to everybody via the internet. Preliminary results on applications of the stable marriage problem in the context of the labour market can be found in [6]. Further work on this subject is in progress.

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APPENDIX A: CALCULATION OF $\rho(X)$

In this appendix we calculate the distribution function $\rho(X)$ of the variable $X = \sum_i x_i$ which is formally given by

$$\rho(X) = \int_0^1 d^N x \delta(X - \sum_i x_i) \quad (A1)$$

According to the central limit theorem $\rho(X)$ converges to a normal distribution

$$\rho(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X - \bar{X})^2}{2\sigma^2}\right) \quad (A2)$$

with $\bar{X} = N/2$ and $\sigma^2 = N/12$ for large $N$. This is however only true for the central part of the distribution whereas the tails of $\rho(X)$ look very different. Let us introduce new variables $q_i$ such that $x_i = q_i - q_{i-1}$ for $i = 1, N$ and $q_0 = 0, q_N = X$. Disregarding the constraint $x_i < 1$ for the moment we obtain

$$\rho(X) = \frac{1}{\Gamma(N)} \int_0^X dq_1 \cdots \int_0^X dq_{N-1} = \frac{X^{N-1}}{\Gamma(N)} \quad (A3)$$
where the factor $\Gamma(N)$ occurs due to the fact that the $q_i$ have to be arranged in ascending order to satisfy the condition $x_i \geq 0$. The condition $x_i < 1$ can now be implemented as follows: Considering the $q_i$ as independent random variables uniformly distributed in the interval $[0, X]$ we obtain a Poisson distribution $p(x) = \lambda \exp(-\lambda x)$ for the distance $x$ between two consecutive numbers, where $\lambda = N/X$. The probability that all $x_i < 1$ is therefore given by

$$P(\text{all } x_i < 1) = \left( \int_0^x dx \ p(x) \right)^N = \left( 1 - \exp\left(-\frac{N}{X}\right) \right)^N \tag{A4}$$

neglecting the weak correlations among the $x_i$ due to the fact that their sum is fixed. Combining Eqs. (A3) and (A4) we obtain the final result

$$\rho(X) = \frac{X^{N-1}}{\Gamma(N)} \left( 1 - \exp\left(-\frac{N}{X}\right) \right)^N \tag{A5}$$

valid for $X < N/\log N$.

**APPENDIX B: AVERAGE ENERGY OF WOMEN**

Let us assume that men propose in the GS algorithm and that a woman receives $r$ proposals out of which she keeps only the best one. If none of the offers is better than $\Delta$ she remains single at the energy $\Delta$. The average energy thus obtained is

$$y = \frac{1}{r+1} \left( 1 - (1 - \Delta)^{r+1} \right) \tag{B1}$$

We have to take into account that the number $r$ of proposals that a woman receives is not fixed but distributed according to the binomial distribution

$$b(r) = \binom{R}{r} p^r (1-p)^{R-r} \tag{B2}$$

where $R$ is the total number of proposals that men make during the execution of the GS algorithm and $p = 1/N$ is the probability that such an offer is addressed to a given woman. Averaging Eq. (B3) with respect to $b(r)$ yields

$$\langle \frac{1}{r+1} \rangle = \sum_{r=0}^{\infty} \binom{R}{r} p^r (1-p)^{R-r} \frac{1}{r+1} = \frac{N}{R+1} \left( 1 - \left(1 - \frac{1}{N}\right)^{R+1} \right) \tag{B3}$$

and similarly

$$\langle \frac{(1-\Delta)^{r+1}}{r+1} \rangle = \frac{N}{R+1} \left( (1 - \frac{\Delta}{N})^{R+1} - \left(1 - \frac{1}{N}\right)^{R+1} \right) \tag{B4}$$

Combining Eqs. (B1), (B3) and (B4) we obtain

$$\langle y \rangle = \frac{N}{R+1} \left( 1 - \left(1 - \frac{\Delta}{N}\right)^{R+1} \right) \approx \frac{N}{R} \left( 1 - \exp\left(-\frac{R}{N}\Delta\right) \right) \tag{B5}$$

The average energy of women $Y = N\langle y \rangle$ is therefore related to the energy of men $X = R/N$ via

$$Y = \frac{N}{X} \left( 1 - \exp\left(-X\Delta\right) \right) \tag{B6}$$

**APPENDIX C: NUMBER OF SINGLES**

The probability for a woman to remain single after receiving $r$ proposals is $p_s = (1 - \Delta)^r$. Averaging with respect to the binomial distribution (B2) yields

$$\langle p_s \rangle = \sum_{r=0}^{R} \binom{R}{r} p^r (1-p)^{R-r} (1-\Delta)^r = \left( 1 - \frac{\Delta}{N} \right)^R \tag{C1}$$

where $p = 1/N$ and $R$ is the total number of proposals made. Using $X = R/N$ we obtain

$$\langle p_s \rangle = \exp(-X\Delta) \tag{C2}$$

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