Generalized bit cumulants for chaotic systems: Numerical results

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We propose generalized bit cumulants for chaotic systems, within nonextensive thermodynamic approach. In this work, we apply the first and second generalized cumulants to one dimensional logistic and logistic-like family of maps.

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I. INTRODUCTION

Bit cumulants offer a convenient characterization of the fluctuating bit numbers of probability distributions generated by chaotic systems [1]. Especially, the second bit cumulant which measures variance of bit number, is equivalent to heat capacity in the thermodynamic analogy. This quantity is also helpful to discuss sensitivity to correlations among subsystems [2].

Our purpose here is to generalize the bit cumulants within nonextensive thermostatistics of Tsallis [3]. The latter formalism is based on a non-logarithmic entropy

\[ S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}, \]

where \(|1 - q|\) is a measure of nonextensivity of the entropy i.e. its feature of non-additivity with respect to entropies of statistically independent subsystems. Based on the idea of q-deformed bit numbers [4,5], Tsallis entropy can be written in two equivalent forms

\[ S_q = - \sum_{i=1}^{W} [a_i] p_i = \sum_{i=1}^{W} [b_i] p_i^q, \]

where \(b_i = -a_i\) is the bit number and \([x] = \frac{q^x - 1}{q - 1}\). As \(q \to 1\), \([x] \to x\) and \(S_q \to -\sum p_i \ln p_i\), the Shannon entropy. Thermostatistics based on this formalism obeys the Legendre Transform structure of the standard formalism [6]. Apart from this, various (in)equalities of thermodynamics are properly generalized or are left invariant with respect to \(q\).

Tsallis formalism has found a number of significant applications, such as stellar polytropes [7], two-dimensional pure electron plasma turbulence [8], solar neutrinos [9], anomalous diffusion [10], dynamical response theory [11], to name only a few. These systems are characterized by one of the following: long range interactions, long term memory effects or multifractal-like phase space. Particularly, Tsallis formalism has yielded important insights into low-dimensional dissipative systems at the onset of chaos or at bifurcation points [12,13]. Recently, a nonextensive thermostatistics based on multifractal formalism was developed by the authors which related degree of nonextensivity \((1 - q)\) to the precision of a calculation [14,15]. Thus there is relevance to discuss the alternative tool of bit cumulants within nonextensive approach.

This paper is organized as follows: in section II we briefly discuss the standard bit cumulants. In section III, we present generalized version of bit cumulants and apply the first and second cumulant to logistic-like family of maps. Section IV concludes the study.

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II. STANDARD BIT CUMULANTS

Bit cumulants $\Gamma_k$ of order $k$ are defined via a generating function

$$G(\sigma) = \ln \left( \sum_i p_i \exp(-\sigma a_i) \right) = \sum_{k=0}^{\infty} \left( \frac{\sigma^k}{k!} \right) \Gamma_k.$$  \hfill (3)

Alternatively, we can write

$$\Gamma_k = \frac{\partial^k}{\partial \sigma^k} G(\sigma) \bigg|_{\sigma=0}.$$  \hfill (4)

The zeroth cumulant is zero. The first cumulant $\Gamma_1$ is Shannon entropy $-<a_i>$, of the distribution $\{p_i\}$. The second cumulant is variance of the bit number i.e., $\Gamma_2 = <a_i^2> - <a_i>^2$. From thermodynamics point of view, $\Gamma_2$ is of major importance. An important property of $\Gamma_k$ is that it is additive with respect to statistically independent systems or subsystems. Thus for a composite system $(I + II)$ whose joint probabilities factorize as:

$$p_{ij}^{(I+II)} = p_i^{(I)} p_j^{(II)},$$

we have $\Gamma_k^{(I+II)} = \Gamma_k^{(I)} + \Gamma_k^{(II)}$. This may be taken as extensive feature of the standard bit cumulants.

III. GENERALIZED BIT CUMULANTS

We have seen (Eq. (2)) that within Tsallis thermostatistics, the generalized bit-number is given by $-[a_i]$. Thus we define the new generating function of bit cumulants as

$$G^{(q)}(\sigma) = \ln\left( \sum_i p_i \exp(-\sigma [a_i]) \right) = \sum_{k=0}^{\infty} \left( \frac{\sigma^k}{k!} \right) \Gamma_k^{(q)}.$$  \hfill (5)

The generalized bit cumulant may be defined as

$$\Gamma_k^{(q)} = \left. \frac{\partial^k}{\partial \sigma^k} G^{(q)}(\sigma) \right|_{\sigma=0}.$$  \hfill (6)

It is easy to see that the first cumulant is Tsallis entropy $-<[a_i]>$. The second cumulant is the variance of the generalized bit number $-[a_i]$, and is given by

$$\Gamma_2^{(q)} = <[a_i]^2> - <[a_i]>^2.$$  \hfill (7)

Alternately, in terms of the bit number $[b_i]$ (Eq. (6)), one can write $\Gamma_2^{(q)} = \sum_i [b_i]^2 p_i^{(2q-1)} - \left( \sum_i [b_i] p_i^q \right)^2$. Note that it is also possible to obtain the second cumulant, by defining a generalized free energy $\Psi_q$ \cite{14} and using the relation

$$\Gamma_2^{(q)} = \left. -\frac{\partial^2 \Psi_q}{\partial \beta^2} \right|_{\beta=1}$$

such that

$$\Gamma_2^{(q)} = q \left\{ \sum_i [b_i]^2 p_i^{(2q-1)} - \left( \sum_i [b_i] p_i^q \right)^2 \right\}.$$  \hfill (8)

Thus the two cumulants differ only by factor of $q$. In the following, we will apply Eq. (8), as it is related to the general thermodynamic framework as established in \cite{14}.

An important distinctive feature of the new cumulants is that they are non-additive (nonextensive) with respect to independent subsystems. In this paper, we apply the first and second generalized cumulants to the study of chaotic systems, such as logistic map and logistic-like family of maps.
A. First cumulant

As said above, the first generalized bit cumulant is Tsallis entropy $S_q$ itself. For an ergodic map, we write $S_q = \frac{<\rho_1^{q-1}>}{1-q}$ where $\rho$ is the natural invariant density of the map. Consider the standard logistic map $x_{n+1} = r x_n (1-x_n)$, $x_n = [0,1]$, which is chaotic above $r = r_c = 3.569945 \ldots$. As Fig. 1 shows, for $q < 1$ with decrease in box size $\epsilon$, Tsallis entropy shows a corresponding increase. This behaviour is comparable to that shown in Fig. 2 by Shannon entropy $S_1 = -<\ln \rho>$, although at a given parameter value $r$, and for $q < 1$, $S_q > S_1$, at the same box size. This latter feature is already known for an equiprobability distribution [1], but here it is shown for a nonuniform distribution such as generated by logistic map. Moreover, Tsallis entropy provides notable variation with respect to $q$ (Fig. 3). For $q < 1$, as $(1-q)$ increases, $S_q$ also shows an increase. Thus results of Fig. 1 and Fig. 3 suggest an interesting possibility. Tsallis entropy evaluated at smaller box size and small $(1-q)$ can be matched by the value of entropy at large box size and large $(1-q)$ value. In other words, for a given value of Tsallis entropy, there can be a range of $(1-q)$ and box size values $\epsilon$ and it is interesting to see the relation between the two. For concreteness, we chose a fixed $r$ value and evaluate $S_q$ at some box size and given $q$ value. Then we keep $S_q$ fixed to within good approximation and plot the corresponding $1-q$ and $1/V = -1/\ln \epsilon$ values in Fig. 4. Note that $V$ is the volume parameter in thermodynamic analogy [2]. Thus thermodynamic limit $V \to \infty$ is equivalent to $\epsilon \to 0$. The direct proportionality between $(1-q)$ and $1/V$ plays important role in the nonextensive formalism for chaotic systems [3].

B. Second cumulant

In the canonical framework, second cumulant is equivalent to heat capacity. In the following, we make a detailed study of generalized second cumulants. In terms of probabilities $p_i$, we can write Eq. (8) as

$$\Gamma_2^{(q)} = \frac{q}{(q-1)^2} \left\{ \sum_i p_i^{2q-1} - (\sum p_i^q)^2 \right\}. \quad (9)$$

$\Gamma_2^{(q)}$ goes to second bit cumulant, $\Gamma_2 = <(\ln p_i)^2> - <\ln p_i>^2$ as $q \to 1$. To discuss non-additive property of $\Gamma_2^{(q)}$, consider again a composite system (I+II), for which $p_{ij}^{(I+II)} = p_i^{(I)} p_j^{(II)}$. Then

$$\Gamma_2^{(q)}(I+II) = \Gamma_2^{(q)}(I) + \Gamma_2^{(q)}(II)$$

$$- 2(1-q) \left\{ \Gamma_2^{(q)}(I) < [a_j] >_{II} + \Gamma_2^{(q)}(II) < [a_i] >_I \right\}$$

$$+ q(1-q) \left\{ <[a_i]^2> > I < [a_j]^2 >_{II} - <[a_i]^2 > < [a_j]^2 >_{II} \right\}.$$ 

The non-additive feature also indicates correlations among subsystems I and II when $q \neq 1$.

Now for ergodic maps, based on Eq. (3), we propose the generalized bit variance density or heat capacity given by

$$C_2^{(q)} = \frac{q}{(q-1)^2} <\rho^{2(q-1)}> - <\rho^q>^2. \quad (10)$$ 

For $q \to 1$, we have $C_2 = <(\ln \rho)^2> - <\ln \rho>^2$. We make a study of Eq. (10) for logistic-like family of maps. These are given by $x_{n+1} = 1 - a|x_n|^z$, $z > 1$, $0 < a < 2$ and $-1 \geq x \leq 1$. Especially for $z = 2$, we have standard logistic map in its centered representation. Fig. 5 shows both $C_2$ and $C_2^{(q)}$ vs. $a$ for $z = 2$. It appears there is a kind of scaling factor between $C_2$ and $C_2^{(q)}$. To check this, we plot $C_2$ vs. $C_2^{(q)}$ in Fig. 6 and note that most of the points can be fitted to a straight line.

One can ask how this relation between $C_2$ and $C_2^{(q)}$ depends on the nature of map. In Fig. 7, we show results for different $z$ values. The scaling factor between $C_2^{(q)}$ and $C_2$ which is measured by the slope of straight line fits to the graphs such as Fig. 6, shows a monotonic decrease with increasing $z$ value (Fig. 8). In other words, the deviation of the slope from unity decreases with increase in $z$ value, i.e. the function $C_2^{(q)}$ is less sensitive to $q$ for higher values of $z$.

Alternatively, for a given map (fixed $z$ value), one can enquire how the above mentioned slope changes with $q$. Naturally for $q \to 1$, $C_2^{(q)} \to C_2$ and the slope tends to unity. These results are shown in Fig. 9.
IV. CONCLUSION

We have generalized the bit cumulants within nonextensive approach. In this paper, we have concentrated on properties of first and second bit cumulants. We have seen how keeping Tsallis entropy (first cumulant) constant, we get a connection between box size (which represents precision of a calculation) and degree of nonextensivity \(1 - q\). Secondly, we have done detailed study on second bit cumulant applying it to Logistic-like family of maps. We note that for large \(z\), \(C_2^{(q)} \to C_2\). In the light of this, we would like to point out a feature seen in studies on sensitivity to initial conditions in similar systems [13]. There as \(z \to \infty\), the nonextensivity index \(q \to 1\). Further work elucidating this connection will be welcome.

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[1] C. Beck and F. Schlögl, *Thermodynamics of chaotic systems: an introduction* (Cambridge University Press, Cambridge, 1993).
[2] F. Schlögl, Z. Physik B 52, 51 (1983).
[3] C. Tsallis, J. Stat Phys. 52, 479 (1988).
[4] R. S. Johal, Phys. Rev. E 58, 4147 (1998).
[5] R. S. Johal, Phys. Lett. A 253, 47 (1999).
[6] E. M. F. Curado and C. Tsallis, J. Phys. A: Math. Gen. 24, L69 (1991); 24, 3187E (1991); 25, 1019E (1992).
[7] A.R. Plastino and A. Plastino, Phys. Lett. A 174, 384 (1993).
[8] B. M. Boghosian, Phys. Rev. E 53, 4754 (1996); C. Anteneodo and C. Tsallis, J. Mol. Liq. 71, 255 (1997).
[9] D.C. Clayton, Nature 249, 131 (1974); P. Quarati et. al., Nucl. Phys. A 621, 345c (1997).
[10] M. F. Shlesinger, G.M. Zaslavsky and U. Frisch, *Lévy Flights and Related Topics in Physics* (Springer, Berlin, 1995); D.H. Zanette and P.A. Alemany, Phys. Rev. Lett. 75, 366 (1995); C. Tsallis, S.V.F. Lévy, A.M.C. de Souza, and R. Maynard, Phys. Rev. Lett. 75, 3589 (1995); Phys. Rev. Lett. 77, 5442(E) (1996).
[11] A. K. Rajagopal, Phys. Rev. Lett. 76, 3469 (1996).
[12] M. L. Lyra and C. Tsallis, Phys. Rev. Lett. 80, 53 (1998).
[13] C. Tsallis, A. R. Plastino and W.-M. Zhang, Chaos, Solitons and Fractals 8, 885 (1997); U. M. S. Costa, M. L. lyra, A. R. Plastino and C. Tsallis, Phys. Rev. E 56, 245 (1997).
[14] R. S. Johal and R. Rai, *Nonextensive thermodynamic formalism for chaotic dynamical systems* - LANL preprint cond-mat/9909218.
FIG. 1. Tsallis entropy for logistic map vs. nonlinearity parameter \( r \) of the map for three different partitionings of unit interval: 1024, 2048 and 4096, number of boxes respectively. Curve with symbols is for 1024 boxes, \( q \) is set at 0.9. There is overall increase in Tsallis entropy in chaotic regions, as no. of boxes increase (box size decreases).
FIG. 2. Shannon entropy vs. $r$ for logistic map and same partitioning as in Fig. 1a. Curve with symbols is for 1024 boxes. We see Shannon entropy also increases as box size decreases, although its value remains lower than Tsallis entropy ($q < 1$) for the same box size.
FIG. 3. Tsallis entropy for logistic map vs. $r$, for different $q$ values: 0.9, 0.93, 0.96 respectively. Number of boxes is fixed at 1024. As $q$ increases, entropy decreases.
FIG. 4. Relation between $(1 - q)$ and $1/V$ using logistic map with $r = 3.81$ and keeping Tsallis entropy fixed, which is taken to be $9.15833$ at $q = 0.9$. Number of boxes is 1024.
FIG. 5. Heat capacity equivalent to bit variance is plotted against control parameter $a$ of logistic map ($z = 2$). Solid curve represents $q = 1$ case. Dashed curve is obtained by setting $q = 0.97$ and using Eq. (10). Higher values of variance in nonextensive ($q \neq 1$) case can be interpreted as due to (negative) nonextensive correlations.
FIG. 6. Scaling factor between $C_2^{(q)}$ and $C_2$ of Fig. 5, as given by the slope of the straight line fit to the data points.
FIG. 7. $C_2$ vs. $C_2^{(q)}$ plots for logistic-like family of maps. (a) $z = 1.5$, (b) $z = 3$, (c) $z = 4$, and (d) $z = 5$. $q$ is fixed at 0.97. Again the slope is a measure of scaling factor between $C_2$ and $C_2^{(q)}$. See also Fig. 8.
$z = 3$
$z = 4$
FIG. 8. Behaviour of slope as obtained from Fig. 7, against nature of map (\(z\) value).
FIG. 9. For a given logistic-like map (fixed $z$ value) slope vs. $q$ value. As $q$ tends unity, slope monotonically decreases to unity.