A Novel Symmetry in Sigma models

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Abstract

A class of non-linear sigma models possessing a new symmetry is identified. The same symmetry is also present in Chern-Simons theories. This hints at a possible topological origin for this class of sigma models. The non-linear sigma models obtained by non-Abelian duality are a particular case in this class.
1. Introduction

Non-linear sigma models in two dimensions possess remarkable features due to their rich symmetries. The symmetry properties a sigma model can have depend very much on the form of the metric and the torsion tensors. The most studied non-linear sigma model is the Wess-Zumino-Novikov-Witten (WZNW) model. This model enjoys an extra symmetry, generating a current algebra \[1\], precisely when the metric and the torsion tensors take the very specific forms

\[
G_{ij} = a_i^a e_j^b \eta_{ab} = 
\]

\[
H = -\frac{1}{3} \text{tr}(e \wedge e \wedge e) = \frac{1}{3} \text{tr}(\bar{e} \wedge \bar{e} \wedge \bar{e}),
\]

where we have introduced the differential forms on the group manifold \[2\]

\[
e = T_a e^a_i d\phi^i = g^{-1} dg
\]

\[
\bar{e} = T_a \bar{e}^a_i d\bar{\phi}^i = -dgg^{-1}.
\]

The torsion tensor \( H = \frac{1}{3} H_{ijk} d\phi^i \wedge d\phi^j \wedge d\phi^k \) is related to the antisymmetric tensor field \( B = \frac{1}{2} B_{ij} d\phi^i \wedge d\phi^j \) by the usual relation \( H = \frac{1}{2} dB \). Here \( \eta_{ab} \) is the invariant bi-linear form of the Lie algebra generated by \( T_a \) and \( g \) is a group element parametrised by \( \phi^i \).

In fact one can ask whether there exist other forms for \( G_{ij} \) and \( H_{ijk} \) which would lead to other forms of current algebras. This question was answered in refs.\[3, 4, 5, 6, 7\] where a generalisation of the above expressions for the metric and the torsion were found. The WZNW model is then just a particular case of this generalisation.

We explore, in this paper, the possibility of finding other non-linear sigma models that might have further symmetries depending on the forms of \( G_{ij} \) and \( B_{ij} \). Indeed, we identify a class of sigma models which have a very interesting symmetry. Remarkably, the same symmetry appears in Chern-Simons theories in three dimensions. This hints at a possible connection between the two theories.

2. The new symmetry

Consider the action for a general bosonic two-dimensional non-linear sigma model

\[
S(\varphi) = \int d^2x \sqrt{\gamma} \left( \gamma^\mu \gamma^\nu G_{ij}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j + \bar{\varphi}^\mu \partial_\mu \varphi^j \varphi^j \right).
\]
In this equation $\gamma_{\mu\nu}$ is the metric on the two-dimensional world sheet, $\gamma$ is its determinant and $\hat{\epsilon}^{\mu\nu} = \frac{1}{\sqrt{\gamma}} \epsilon^{\mu\nu}$ is the alternating tensor. This action can be written as

$$S(\varphi) = \int d^2 x \sqrt{\gamma} \left( \hat{\epsilon}^{\mu\nu} \eta_{ij} A^i_{\mu} \partial_\nu \varphi^j \right),$$

where we have introduced the gauge field-like quantity $A^i_{\mu}$

$$A^i_{\mu} = R^i_{\mu\nu} \eta_{jk} \hat{\epsilon}^\rho \partial_\rho \varphi^k,$$

$$R^i_{\mu\nu} = \eta^{jk} \eta^{il} (\gamma_{\mu\nu} G_{kl} + \hat{\epsilon}_{\mu\nu} B_{kl}).$$

with $\eta_{ij}$ a symmetric field-independent metric whose inverse is $\eta^{ij}$. Suppose now that $\eta_{ij}$ is the invariant bi-linear form of a Lie algebra whose structure constants we denote by $f^i_{jk}$ (which means that $\eta_{ij} f^j_{kl} + \eta_{kj} f^i_{il} = 0$).

We would like to investigate under which conditions the action (4) has a symmetry of the form

$$\delta \varphi^i = f^i_{jk} \xi^j F^{k}_{\mu\nu} \hat{\epsilon}^{\mu\nu},$$

where $\xi^j(x)$ is the infinitesimal gauge parameter and $F^{i}_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + f^i_{jk} A^j_\mu A^k_\nu$ is the field strength of the gauge field $A^i_\mu$ as given by (3). The transformation is suggested by the form of the action (4). The same kind of symmetry was identified in the context of non-Abelian duality in sigma models [8] and in non-Abelian gauge theories [1].

We found that the action is invariant, up to a total derivative, provided that the metric $G_{ij}$ and the antisymmetric tensor $B_{ij}$ satisfy

$$\partial_k R^{ij}_{\mu\nu} = \eta_{kl} f^j_{mn} R^{km}_{\mu\rho} R^{in}_{\nu\beta} \hat{\epsilon}^{\rho\beta}. $$

(7)

This condition can of course be expressed explicitly as two separate conditions on $G_{ij}$ and $B_{ij}$

$$\partial_k G_{ij} = f^l_{km} \eta^{mn} (G_{li} B_{nj} - G_{nj} B_{li}),$$

$$\partial_k B_{ij} = -f^l_{km} \eta^{mn} (G_{li} G_{nj} + B_{li} B_{nj})$$

(8)

This shows the special geometry of this class of sigma models. In particular, the Riemann tensor and the torsion will be given in closed forms as products of $G_{ij}$, $G^{ij}$ and $B_{ij}$.

Under these conditions, the equations of motion of the non-linear sigma model lead to

$$\hat{\epsilon}^{\mu\nu} F^{i}_{\mu\nu} = 0.$$ 

(9)

Therefore the above transformation vanishes on-shell. As seen later, this equation can also be thought of as deriving from a Chern-Simons theory.
It is straightforward to find the unique solution to the symmetry invariance condition in (7). In order to do this, we denote by \( \tilde{R}_{ij}^{\mu\nu} \) the inverse of \( R_{ij}^{\mu\nu} \) (that is, \( R_{ij}^{\mu\nu} \tilde{R}_{jk}^{\nu\alpha} = \delta^\alpha_i \delta^\beta_j \)). Equation (7) is then cast into the first order differential equation

\[
\partial_k \tilde{R}_{ij}^{\mu\nu} = -\eta_{kl} f_l^{ij} \tilde{R}_{ij}^{\mu\nu} \tag{10}
\]

whose general solution is given by

\[
\tilde{R}_{ij}^{\mu\nu} = - \left[ N_{ij}^{\mu\nu} + \tilde{\epsilon}_{ij}^{\mu\nu} \eta_{kl} f_l^{ij} \varphi^k \right] , \tag{11}
\]

where \( N_{ij}^{\mu\nu} = N_{ji}^{\nu\mu} \) is any field-independent matrix, and in general \( N_{ij}^{\mu\nu} = \gamma_{ij}^{\mu\nu} A + \tilde{\epsilon}_{ij}^{\nu\mu} C \).

The action can be cast into a form which is familiar in the context of non-Abelian duality [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. By extracting \( \partial_\mu \varphi^i \) from (5) and eliminating it in (4), one finds, after some straightforward manipulations, the following action

\[
S(\varphi, A) = \int d^2 x \sqrt{\gamma} \left( N_{ij}^{\mu\nu} A^i_\mu A^j_\nu + \tilde{\epsilon}_{ij}^{\mu\nu} \eta_{ij} F_{\mu\nu} \varphi^i \right) , \tag{12}
\]

where we have ignored a total derivative. If one treats \( A^i_\mu \) and \( \varphi^i \) as independent variables then the equations of motion for \( A^i_\mu \) are precisely those in (5). Indeed, this action is obtained (when \( A^i_\mu \) and \( \varphi^i \) are independent) by performing a non-Abelian duality transformation on the following action

\[
S(g) = \int d^2 x \sqrt{\gamma} N_{ij}^{\mu\nu} \eta^{ik} \eta^{jl} \text{tr} \left( T_k g^{-1} \partial_\mu g \right) \text{tr} \left( T_l g^{-1} \partial_\nu g \right) , \tag{13}
\]

where \( T_i \) are the generators of the Lie algebra \([T_i, T_j] = f^{k}_{ij} T_k\), \( g \) is a Lie group element and \( \text{tr} \) is the invariant bi-linear form \( \text{tr} (XY) = \eta_{ij} X^i Y^j \). This action is invariant under the global transformation \( g \rightarrow hg \). The non-Abelian dual theory is obtained by gauging this symmetry and at the same time restricting the gauge field strength to vanish [20]. We therefore obtain the action

\[
S(g, \varphi, A) = \int d^2 x \sqrt{\gamma} \left[ N_{ij}^{\mu\nu} \eta^{ik} \eta^{jl} \text{tr} \left( T_k g^{-1} D_\mu g \right) \text{tr} \left( T_l g^{-1} D_\nu g \right) + \tilde{\epsilon}_{ij}^{\mu\nu} \text{tr} (\varphi F_{\mu\nu}) \right] . \tag{14}
\]

The covariant derivative is \( D_\mu g = \partial_\mu g + A_\mu g \) with \( A_\mu \rightarrow h^{-1} A_\mu h - \partial_\mu hh^{-1} \), the gauge field is \( A_\mu = T_i A^i_\mu \) and the Lagrange multiplier is \( \varphi = T_i \varphi^i \) with \( \varphi \rightarrow h \varphi h^{-1} \). The gauge invariance allows us to choose a gauge such that \( g = 1 \). In this gauge \( S(g, \varphi, A) \) reproduces precisely \( S(g, A) \) as given by (12).

The dual of the principal chiral model is a special case of this construction [21, 22, 23]. The dual of the chiral model is obtained when \( A_{ij} = \eta_{ij} \) and \( C_{ij} = 0 \).
3. Generalisation

Another important case is obtained by splitting the field $\varphi^i$ of the previous non-linear sigma model as $\varphi^i = (X^a, Y^A)$ and restricting the transformation to the fields $X^a$ only. In this case, the non-linear sigma model action takes the form

$$S(X, Y) = \int d^2x \sqrt{\gamma} \left( \bar{\epsilon}^{\mu \nu} \eta_{ab} A^a_{\mu} \partial_\nu X^b + \bar{\epsilon}^{\mu \nu} \varepsilon^{a'b} M^{aA}_{\mu} \eta_{ab} L_{AB} \partial_\nu Y^B \partial_\nu X^b + \bar{R}^{\mu \nu}_{AB} \partial_\mu Y^A \partial_\nu Y^B \right),$$

(15)

The gauge field $A^a_{\mu}$ involves both $X^a$ and $Y^A$ through

$$A^a_{\mu} = R^{ab}_{\mu \nu} \bar{\epsilon}^{\nu \alpha} \eta_{bc} \partial_\alpha X^c + R^{aA}_{\mu \nu} \bar{\epsilon}^{\nu \alpha} L_{AB} \partial_\alpha Y^B .$$

(16)

The different quantities introduced here can depend on both $X^a$ and $Y^A$ and are such that

$$R^{ab}_{\mu \nu} = \eta^{ac} \eta^{bd} [\gamma^{\mu \nu} G_{cd} + \bar{\epsilon}^{\mu \nu} B_{cd}],$$

$$R^{ab}_{AB} = [\gamma^{\mu \nu} G_{AB} + \bar{\epsilon}^{\mu \nu} B_{AB}],$$

$$R^{aA}_{\mu \nu} + M^{aA}_{\mu \nu} = 2 \eta^{ab} L_{AB} [\gamma^{\mu \nu} G_{bB} + \bar{\epsilon}^{\mu \nu} B_{bB}] .$$

(17)

The symmetric matrices $\eta_{ab}$ and $L_{AB}$ are field-independent and their inverses are, respectively, $\eta^{ab}$ and $L^{AB}$. We then suppose that $\eta_{ab}$ is a bi-linear invariant form of a Lie algebra with structure constants $f^{a}_{bc}$.

Let us find the conditions under which the sigma model (15) is invariant under

$$\delta X^a = f^a_{bc} \bar{\epsilon}^{b \nu} F^{c \mu}_{\nu} \bar{\epsilon}^{\mu \nu}, \quad \delta Y^A = 0 .$$

(18)

We find that the action remains invariant, up to a total derivative, when the following conditions are fulfilled

$$M^{aA}_{\mu \nu} = R^{aA}_{\mu \nu},$$

$$\partial_\alpha R^{aA}_{\mu \nu} = \eta_{ab} f^{b}_{\nu \sigma} R^{ae}_{\mu \sigma} \varepsilon^{d \tau},$$

$$\partial_\alpha R^{aA}_{\mu \nu} = \eta_{ab} f^{b}_{\nu \sigma} R^{ae}_{\mu \sigma} \varepsilon^{a \tau},$$

$$\partial_\alpha R^{aA}_{AB} = \eta_{ab} f^{b}_{\nu \sigma} L_{AE} L_{BD} R^{ae}_{\mu \sigma} R^{eD}_{\tau \beta} \varepsilon^{\beta \nu} \varepsilon^{\beta \alpha} .$$

(19)

Notice that this set of equations cannot be obtained from (7) by simply splitting the field $\varphi^i$ as $(X^a, Y^A)$. The second equation of this set has the unique solution for the inverse of $R^{ab}_{\mu \nu}$, namely $\tilde{R}^{\mu \nu}_{ab}$, given by

$$\tilde{R}^{\mu \nu}_{ab} = - \left[ N^{\mu \nu}_{ab} (Y) + \bar{\epsilon}^{\mu \nu} \eta_{cd} f^{c}_{ab} X^d \right] .$$

(20)
The general solution of the remaining last two equations is provided by

\[
\begin{align*}
R^a_{\mu\nu} &= R^{ab}_{\mu\nu} W^{aA}_{\nu b} (Y) \\
R^a_{AB} &= L_{AE} L_{BD} R^{ab}_{\mu\nu} W^{\tau E}_{\sigma\alpha} W^{\rho D}_{\beta\beta} \hat{\epsilon}^{\tau\rho\sigma\beta} + T^\mu_{AB} (Y)
\end{align*}
\]  

(21)

with \(N^\mu_{ab}, W^a_{\mu b}\) and \(T^\mu_{ab}\) any arbitrary functions which depend on the field \(Y^A\) only.

Subject to these conditions, the variation with respect to \(X^a\) of our action leads to the equations of motion \(\hat{\epsilon}^{\mu\nu} F^a_{\mu\nu} = 0\), where \(F^a_{\mu\nu}\) is constructed from \(A^a_{\mu}\) in (16).

Similarly, by extracting \(\partial_\mu X^a\) from (16) and substituting in (15), we find (up to a total derivative)

\[
\begin{align*}
S (Y, X, A) &= \int d^2 x \sqrt{\gamma} \left[ T^\mu_{AB} (Y) \partial_\mu Y^A \partial_\nu Y^B + N^\mu_{ab} (Y) A^a_{\mu} A^b_{\nu} \right. \\
&\quad + \left. 2 W^a_{\mu b} (Y) L_{AB} \hat{\epsilon}^{\alpha\nu} A^a_{\mu} \partial_\nu Y^B + \hat{\epsilon}^{\mu\nu} \eta_{ab} X^a \gamma^b_{\mu\nu} \right].
\end{align*}
\]  

(22)

Again, the equations of motion for \(A^a_{\mu}\), if considered as an independent field, are precisely those in (16). The symmetry \(\delta X^a = f^a_{bc} \hat{\epsilon}^{bc} F^a_{\mu\nu} \hat{\epsilon}^{\mu\nu}\) is transparent in this case.

The same procedure can be applied here to find the non-Abelian dual theory. Consider now the action

\[
\begin{align*}
S (Y, g, A) &= \int d^2 x \sqrt{\gamma} \left[ T^\mu_{AB} (Y) \partial_\mu Y^A \partial_\nu Y^B + N^\mu_{ab} (Y) \eta_{ac} \eta_{bd} \left( T^b g^{-1} \partial_\mu g \right) \left( T^b g^{-1} \partial_\nu g \right) \right. \\
&\quad + \left. 2 W^a_{\mu b} (Y) L_{AB} \eta_{bc} \hat{\epsilon}^{\alpha\nu} \left( T^b g^{-1} \partial_\mu g \right) \partial_\nu Y^B \right]
\end{align*}
\]  

(23)

which is invariant under the left symmetry \(g \rightarrow h g\). This symmetry can be gauged by the replacement \(\partial_\mu g \rightarrow D_\mu g = \partial_\mu g + A_\mu g\). The dual theory is obtained when the Lagrange multiplier term \(\int d^2 x \sqrt{\gamma} \hat{\epsilon}^{\mu\nu} \text{tr} (X F^a_{\mu\nu})\) is added. Choosing then a gauge such that \(g = 1\) yields the action in (22).

Notice that in the above model (22) we have not assumed any transformation for the fields \(Y^A\). In fact these fields could transform when \(T^\mu_{AB}, N^\mu_{ab}\) and \(W^a_{\mu b}\) are restricted to satisfy certain conditions as shown below. It is found that the theory in (22), when \(A^a_{\mu}\) is treated as an independent field, has the infinitesimal local gauge symmetry

\[
\begin{align*}
\delta Y^A &= \lambda^a K^a_A (Y) \\
\delta A^a_{\mu} &= - \partial_\nu \lambda^a + f^a_{bc} \lambda^b A^c_{\mu}
\end{align*}
\]  

(24)

provided that the two quantities \(T^\mu_{AB}\) and \(K^a_A\) satisfy

\[
\begin{align*}
\partial_E T^\mu_{AB} K^E_a + T^\mu_{EB} \partial_A K^E_a + T^\mu_{AE} \partial_B K^E_a &= \hat{\epsilon}^{\mu\nu} \left( \partial_A V_{B\mu} - \partial_B V_{A\mu} \right) \\
K^A_a \partial_A K^B_b - K^A_b \partial_A K^B_a &= - f^c_{ab} K^B_c
\end{align*}
\]  

(25)
The second equation merely expresses the fact that the differential operators \(K_a = -K^A_a \frac{\partial}{\partial Y^A}\) form a representation of the Lie algebra defined by \(\eta_{ab}\) and \(f^c_{ab}\). The first equation defines the new quantity \(V_{Aa}\) which is required to satisfy

\[
\partial_D V_{Ab} K^D_c + \partial_A V_{Dc} K^D_b - \partial_D V_{Ac} K^D_b + V_{Db} \partial_A K^D_c = -f^d_{cb} V_{Ad} = 0
\]  

(26)

The remaining two quantities \(N^{\mu\nu}_{ab}\) and \(W^{\mu A}_{\nu a}\) are then given by

\[
N^{\mu\nu}_{ab} = T^{\mu\nu}_{AB} K^A_a K^B_b + \tilde{\epsilon}^{\mu\nu} V_{Ab} K^A_a \\
W^{\mu A}_{\nu a} = -L^{AE} \left( \tilde{\epsilon}_{\alpha\nu} T^{\alpha\mu}_{EB} K^B_a + \delta^\mu_\nu V_{Ea} \right)
\]  

(27)

By writing \(T^{\mu\nu}_{AB} = \gamma^{\mu\nu} G_{AB}(Y) + \tilde{\epsilon}^{\mu\nu} B_{AB}(Y)\), the equations (24)–(27) are precisely the equations needed to gauge the isometries of a general sigma model with metric \(G_{AB}\) and antisymmetric tensor \(B_{AB}\) \cite{24, 25}. Hence the non-linear sigma model obtained through a non-Abelian duality procedure is a particular case of this general construction.

4. Conclusions

It is worth mentioning that a symmetry similar to the one identified for the sigma model exists in Chern-Simons theory. To see this, consider a Chern-Simons theory for some gauge group \(\mathcal{G}\)

\[
I(A) = \int d^3 x \epsilon^{ijk} \left[ \text{tr} (A_i F_{jk}) - \frac{2}{3} \text{tr} (A_i A_j A_k) \right]
\]

(28)

where \(i, j, \ldots = 1, 2, 3\) and \(\epsilon^{123} = 1\). Let also \(\mu, \nu, \ldots = 1, 2\) and \(\epsilon^{\mu\nu}\) the corresponding alternating tensor, with \(\epsilon^{12} = 1\). By splitting the three-dimensional indices, the Chern-Simons action can be written as

\[
I(A) = 2 \int d^3 x \epsilon^{\mu\nu} \left[ \text{tr} (A_3 F_{\mu\nu}) - \text{tr} (A_\mu \partial_\nu A_3) - \text{tr} (\partial_\mu (A_\nu A_3)) \right]
\]

(29)

It is then clear, if we drop the total divergence term, that the Chern-Simons theory has a further symmetry given by

\[
A_3 \rightarrow A_3 + \epsilon^{\mu\nu} [\xi, F_{\mu\nu}]
\]

(30)

where \(\xi\) is a local Lie algebra-valued function. This symmetry is of the form (6). Furthermore, varying the Chern-Simons action with respect to \(A_3\) leads to an equation similar to (9). This hints at a deep connection between Chern-Simons theory and the class of sigma models we identified as having the new symmetry. We speculate that a non-trivial compactification to two dimensions of the Chern-Simons theory would lead to our sigma models.
As mentioned earlier, the new symmetry vanishes on-shell. Therefore, at the classical level this symmetry has no effects. We expect, however, that this symmetry would play a crucial role at the quantum level. We will report elsewhere on the work in progress regarding the quantisation of these models. The methods designed for the quantisation of Chern-Simons theories are essential to this investigation [26].

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