Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations

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Abstract
In this manuscript, we examine the existence and the Ulam stability of solutions for a class of boundary value problems for nonlinear implicit fractional differential equations with instantaneous impulses in Banach spaces. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. We provide some examples to indicate the applicability of our results.

Keywords: Boundary value problem; Generalized Hilfer fractional derivative; Fractional integral; Impulses; Fixed point; Measure of noncompactness; Implicit fractional differential equations; Ulam–Hyers–Rassias stability

1 Introduction
As it is known very well, the roots of fixed point theory go to the method of successive approximations (or Picard’s iterative method) that is used to solve certain differential equations. Roughly speaking, Banach derived the fixed point theorem from the method of successive approximations. In the last decades fixed point theory has been enormously and independently from the differential equations. But, recently, fixed point results turn to be the tools for the solutions of the differential equation. In this paper, we shall involve two interesting fixed point theorems (Darbo’s fixed point theorem and Mönch’s fixed point theorem) in the setting of “measure of noncompactness” to solve the boundary value problem for nonlinear implicit fractional differential equations with instantaneous impulses.

Differential equations of fractional order have been recently proved to be a powerful tool to study many phenomena in various fields of science and engineering such as electrochemistry, electromagnetics, viscoelasticity, finance, and so on. In the literature, it is very common to propose a solution for fractional differential equations by involving different kinds of fractional derivatives, see e.g. [1–10, 12, 13, 21, 22, 36]. On the other hand, there a few results that deal with the boundary value problems for fractional differential equations. The aim of the present paper is to underline the importance of the theory of impulsive differential equations. Further, by the help of these observations, we aim to un-
derstand several phenomena that are not clarified by the non-impulsive equations (see e.g. [15, 16, 18, 32]).

In 1940, Ulam [34, 35] raised the following problem of the stability of the functional equation (of group homomorphisms): “Under what conditions does it exist an additive mapping near an approximately additive mapping?”

In what follows, we pay attention to the weighted spaces of continuous functions

where

- \( D^{\alpha,\beta}_{u_k} \) is the generalized Hilfer fractional derivative of order \( \alpha \in (0,1) \) and type \( \beta \in [0,1] \) and \( \rho > 0 \);
- \( J^{1-\gamma}_{u_k} \) is the generalized Hilfer fractional integral of order \( 1-\gamma, (\gamma = \alpha + \beta - \alpha\beta) \);
- \( c_1, c_2, c_3 \) are reals with \( c_1 + c_2 \neq 0 \), \( J_k := (t_k, t_{k+1}) ; k = 0, \ldots, m \),
- \( a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b < \infty \);
- \( \lim_{t \to b^-} u(t_k + \epsilon) \) and \( \lim_{t \to a^+} u(t_k + \epsilon) \) represent the right-hand and left-hand limits of \( u(t) \) at \( t = t_k \), \( c_2 \in E \);
- \( f : (a, b) \times E \times E \rightarrow E \) is a given function over a Banach space \( (E, \| \cdot \|) \);
- \( \omega_k : E \rightarrow E; k = 1, \ldots, m \), are given continuous functions.

2 Preliminaries

In this section, we recall and recollect the basic notion, notations together with some fundamental results that will be necessary in the main results. Throughout the paper, \((E, \| \cdot \|)\) represents a Banach space. Set \( J = [a, b] \) where \( 0 < a < b \). The letter \( C \) is reserved to represent the Banach space which consists of all continuous functions \( u : J \rightarrow E \) where the norm is

In what follows, we pay attention to the weighted spaces of continuous functions

\[
C_{\gamma, \rho}(J) = \left\{ u : (a, b) \rightarrow E : \left( \frac{t^\alpha - a^\alpha}{\rho} \right)^{1-\gamma} u(t) \in C(J, \mathbb{R}) \right\},
\]
where $0 \leq \gamma < 1$,

$$C^n_{\gamma, \rho}(J) = \{ u \in C^{n-1}: u^{(n)} \in C_{\gamma, \rho}(J) \}, \quad n \in \mathbb{N},$$

$$C^0_{\gamma, \rho}(J) = C_{\gamma, \rho}(J),$$

with the norms

$$\| u \|_{C_{\gamma, \rho}} = \sup_{t \in J} \left\| \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} u(t) \right\|$$

and

$$\| u \|_{C^n_{\gamma, \rho}} = \sum_{k=0}^{n-1} \| u^{(k)} \|_{C_{\gamma, \rho}} + \| u^{(n)} \|_{C_{\gamma, \rho}}.$$

Consider the Banach space

$$PC(J) = \{ u: (a, b] \to E : u(t) \in C(J_k) \forall k = 0, \ldots, m, \text{ and there exist } u(t_k^-) \} \text{ and } ({}^{\rho}J_t^{1-\gamma} u)(t_k^-) \forall k = 0, \ldots, m, \text{ with } u(t_k^-) = u(t_k), \quad 0 \leq \gamma < 1.$$

Also, we consider the weighted space

$$PC_{\gamma, \rho}(J) = \{ u(t): \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} u(t) \in PC(J) \}, \quad 0 \leq \gamma < 1,$$

and

$$PC^n_{\gamma, \rho}(J) = \{ u \in PC^{n-1}: u^{(n)} \in PC_{\gamma, \rho}(J) \}, \quad n \in \mathbb{N},$$

$$PC^0_{\gamma, \rho}(J) = PC_{\gamma, \rho}(J),$$

equipped with the norm

$$\| u \|_{PC_{\gamma, \rho}} = \sup_{t \in J} \left\| \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} u(t) \right\|.$$

The letter $L^1(J)$ indicates the space of Bochner-integrable functions $f: J \to E$ with the norm

$$\| f \|_1 = \int_a^b \| f(t) \| \, dt.$$

**Definition 2.1** ([23]) Let $\alpha \in \mathbb{R}_+$ and $g \in L^1(J)$. The generalized Hilfer fractional integral of order $\alpha$ is

$$\left( {}^{\rho}J_t^a g \right)(t) = \int_a^t s^{a-1} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} g(s) \frac{ds}{\Gamma(\alpha)}, \quad \rho > 0, t > a,$$

with $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt$, $\alpha > 0$ (the Euler gamma function).
Definition 2.2 ([23] Generalized Hilfer fractional derivative) Set $\rho > 0$ and $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. The generalized Hilfer fractional derivative $^\rho D_a^\alpha$ of order $\alpha$ is defined by

$$
(^\rho D_a^\alpha g)(t) = \delta_\rho^n (^\rho \mathcal{J}^\alpha_{a^+} g)(t) = \left( (1 - \rho) \frac{d}{dt} \right)^n \mathcal{J}^\alpha_{a^+} (t^\rho - s^\rho)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds, \quad t > a, \rho > 0,
$$

where $n = [\alpha] + 1$ and $\delta_\rho^n = (1 - \rho) \frac{d}{dt} \frac{(t^\rho - s^\rho)^n}{\Gamma(n-\alpha)}$.

Theorem 2.3 ([23]) Let $\alpha > 0$, $\beta > 0$, $1 \leq \rho \leq \infty$, $0 < a < b < \infty$. Then, for $g \in L^1(J)$, we have

$$
(^\rho \mathcal{J}^\alpha_{a^+} g)(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_a^t (t^\rho - s^\rho)^{\alpha + \beta - 1} \frac{g(s)}{\Gamma(\alpha)} ds, \quad t > a, \rho > 0.
$$

Lemma 2.4 ([23, 27]) Let $\alpha > 0$ and $0 \leq \gamma < 1$. Then $^\rho \mathcal{J}^\alpha_{a^+}$ lies between $PC_{\gamma, \rho}(J)$ and $PC_{\gamma, \rho}(J)$.

Lemma 2.5 ([27]) Suppose that $0 \leq \gamma < 1$, $0 < a < b < \infty$, $\alpha > 0$, and $u \in PC_{\gamma, \rho}(J)$. If $\alpha > 1 - \gamma$, then $^\rho \mathcal{J}^\alpha_{a^+} u$ is continuous on $J$ and

$$
(^\rho \mathcal{J}^\alpha_{a^+} u)(a) = \lim_{t \to a^+} (^\rho \mathcal{J}^\alpha_{a^+} u)(t) = 0.
$$

Lemma 2.6 ([11]) Let $t > a$. Then, for $\alpha > 0$ and $\beta > 0$, we have

$$
\left[ ^\rho \mathcal{J}^\alpha_{a^+} \left( \frac{s^\rho - a^\rho}{\rho} \right)^{\beta - 1} \right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha + \beta - 1},
$$

$$
\left[ ^\rho D_a^\alpha \left( \frac{s^\rho - a^\rho}{\rho} \right)^{\alpha - 1} \right](t) = 0 \quad (\text{for } 0 < \alpha < 1).
$$

Lemma 2.7 ([27]) Suppose that $g \in PC_{\gamma, [a, b]}$ with $0 \leq \gamma < 1$ and $\alpha > 0$. Then we have

$$
(^\rho D_a^\alpha \mathcal{J}^\alpha_{a^+} g)(t) = g(t) \quad \text{for all } t \in (a, b).
$$

Lemma 2.8 ([27]) Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $g \in PC_{\gamma, [a, b]}$ and $^\rho \mathcal{J}^{1-\alpha}_{a^+} g \in PC_{\gamma, [a, b]}$, then

$$
(^\rho \mathcal{J}^\alpha_{a^+} D_a^\alpha g)(t) = g(t) - \frac{(^\rho \mathcal{J}^{1-\alpha}_{a^+} g)(a)}{\Gamma(\alpha)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha - 1} \quad \text{for all } t \in (a, b).
$$

Definition 2.9 ([27]) For a function $g \in PC_{\gamma, [a, b]}$ with $\rho > 0$, the generalized Hilfer-type fractional derivative is defined by

$$
(^\rho D_a^{\beta, \alpha} g)(t) = \left[ ^\rho \mathcal{J}^{\beta(n-\alpha)}_{a^+} \left( \frac{d}{dt} \right)^n \mathcal{J}^{1-\beta(n-\alpha)}_{a^+} g \right](t) = \left[ ^\rho \mathcal{J}^{\beta(n-\alpha)}_{a^+} \left( \delta_\rho^n \mathcal{J}^{1-\beta(n-\alpha)}_{a^+} g \right) \right](t),
$$

where type $\beta$ with $0 \leq \beta \leq 1$, and order $\alpha$ with $n - 1 < \alpha < n$ for $n \in \mathbb{N}$. 
Since \(0 < \alpha < 1\), we shall focus only on the case \(n = 1\).

**Property 2.10** ([27]) The operator \(\rho D^{\alpha,\beta}_{a+}\) can be written as

\[
\rho D^{\alpha,\beta}_{a+} = \rho J^{\beta(1-\alpha)}_{a+} \rho J^{1-\gamma}_{a+} = \rho J^{\beta(1-\alpha)}_{a+} \rho D^{\gamma}_{a+}, \quad \gamma = \alpha + \beta - \alpha \beta.
\]

**Property 2.11** ([27]) The fractional derivative \(\rho D^{\alpha,\beta}_{a+}\) is an interpolator of the following fractional derivatives: Hilfer (\(\rho \to 1\)), Hilfer–Hadamard (\(\rho \to 0^+\)), generalized (\(\beta = 0\)), Caputo-type (\(\beta = 1\)), Riemann–Liouville (\(\beta = 0, \rho \to 0^+\)), Caputo (\(\beta = 1, \rho \to 1\)), Caputo–Hadamard (\(\beta = 1, \rho \to 0^+\)), Liouville (\(\beta = 0, \rho \to 1, a = 0\)), and Weyl (\(\beta = 0, \rho \to 1, a = -\infty\)).

Let \(\gamma = \alpha + \beta - \alpha \beta\), where \(0 < \alpha, \beta, \gamma < 1\). Construct the space

\[
PC^{\alpha,\beta,\gamma}_{\rho}(J) = \left\{ u \in PC^{\gamma}_{\rho}(J), \rho D^{\alpha,\beta}_{a+} u \in PC^{\gamma}_{\rho}(J) \right\}
\]

and

\[
PC^{\beta}_{\gamma,\rho}(J) = \left\{ u \in PC^{\gamma}_{\rho}(J), \rho D^{\gamma}_{a+} u \in PC^{\gamma}_{\rho}(J) \right\},
\]

where \(k = 0, \ldots, m\).

Since \(\rho D^{\alpha,\beta}_{a+} u = \rho J^{\gamma(1-\alpha)}_{a+} \rho D^{\gamma}_{a+} u\), it follows from Lemma 2.4 that

\[
PC^{\beta}_{\gamma,\rho}(J) \subset PC^{\alpha,\beta,\gamma}_{\rho}(J) \subset PC^{\gamma}_{\rho}(J).
\]

**Lemma 2.12** ([27]) Let \(\gamma = \alpha + \beta - \alpha \beta\) with \(0 < \alpha < 1, 0 \leq \beta \leq 1\). If \(u \in PC^{\beta}_{\gamma,\rho}(J)\), then

\[
\rho J^{\gamma}_{a+} \rho D^{\alpha,\beta}_{a+} u = \rho J^{\alpha}_{a+} \rho D^{\beta,\gamma}_{a+} u
\]

and

\[
\rho D^{\gamma}_{a+} \rho J^{\alpha}_{a+} u = \rho D^{\beta(1-\alpha)}_{a+} u.
\]

**Definition 2.13** ([14]) Let \(X\) be a Banach space, and let \(\Omega_X\) be the family of bounded subsets of \(X\). The Kuratowski measure of noncompactness is the map \(\mu : \Omega_X \rightarrow [0, \infty)\) defined by

\[
\mu(M) = \inf \left\{ \epsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, \text{diam}(M_j) \leq \epsilon \right\},
\]

where \(M \in \Omega_X\).

For all \(M, M_1, M_2 \in \Omega_X\), the map \(\mu\) satisfies the following properties:

- \(M\) is relatively compact (\(\mu(M) = 0 \Leftrightarrow M\) is compact).
- \(\mu(M) = \mu(\overline{M})\).
- \(M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)\).
- \(\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)\).
• $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
• $\mu(\text{conv}M) = \mu(M)$.

**Lemma 2.14** ([19]) Let $D \subset PC_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then
(i) the function $t \rightarrow \mu(D(t))$ is continuous on $(a, b)$, and

$$\mu_{PC_{\gamma,\rho}}(D) = \sup_{t \in J} \mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} D(t)\right);$$

(ii) $\mu\left(\int_a^b u(s) \, ds : u \in D\right) \leq \int_a^b \mu(D(s)) \, ds$, where

$$D(t) = \{u(t) : u \in D\}, \quad t \in (a, b].$$

**Lemma 2.15** (Theorem 4.1, [27]) Let $f : (a, b] \times E \to E$ be a function such that $f(\cdot, u(_{\cdot}), \rho D_{t_k}^{\alpha,\beta} u(\cdot)) \in C_{\gamma,\rho}(J)$ for any $u \in PC_{\gamma,\rho}(J)$. Then $u \in PC_{\gamma,\rho}(J)$ is a solution of the differential equation

$$\left(\rho D_{t_k}^{\alpha,\beta} u\right)(t) = f(t, u(t), \rho D_{t_k}^{\alpha,\beta} u(t)), \quad \text{for each } t \in J_k, k = 0, \ldots, m, 0 < \alpha, \beta < 1,$

if and only if $u$ satisfies the following Volterra integral equation:

$$u(t) = \left(\rho D_{t_k}^{1-\gamma} u\right)(t_k^+) \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} + \frac{1}{\Gamma(\gamma)} \int_{t_k}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{1-\gamma} s^{\alpha-1} f(s, u(s), \rho D_{s_k}^{\alpha,\beta} u(s)) \, ds,$

where $\gamma = \alpha + \beta - \alpha \beta$.

**Theorem 2.16** (Mönch’s theorem [26]) Suppose that $D$ is a closed, bounded, and convex subset of a Banach space $X$ such that $0 \in D$. Let $T$ be a continuous mapping of $D$ into itself. If the implication

$$V = \text{conv}T(V), \quad \text{or} \quad V = T(V) \cup \{0\} \Rightarrow \mu(V) = 0$$

holds for every subset $V$ of $D$, then $T$ has a fixed point.

**Theorem 2.17** (Darbo’s theorem [17]) Suppose that $D$ is a nonempty, closed, bounded, and convex subset of a Banach space $X$. Let $T$ be a continuous mapping of $D$ into itself such that, for any nonempty subset $C$ of $D$,

$$\mu(T(C)) \leq k \mu(C),$$

where $0 \leq k < 1$, and $\mu$ is the Kuratowski measure of noncompactness. Then $T$ has a fixed point in $D$.

### 3 The existence of solutions

We start this section by stating the following linear fractional differential equation:

$$\left(\rho D_{t_k}^{\alpha,\beta} u\right)(t) = \psi(t), \quad t \in J_k, k = 0, \ldots, m,$$
where \(0 < \alpha < 1\), \(0 \leq \beta \leq 1\), \(\rho > 0\), with the conditions

\[
\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(t_k^+) = \left(\frac{\partial}{\partial t_{k-1}^\gamma}\right)u(t_k^-) + \sigma_k(u(t_k^-)), \quad k = 1, \ldots, m, \tag{7}
\]

and

\[
c_1\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(a^+) + c_2\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(b) = c_3, \tag{8}
\]

where \(\gamma = \alpha + \beta - \alpha\beta\), \(c_1, c_2 \in E\), \(c_1, c_2 \in \mathbb{R}\) with \(c_1 + c_2 \neq 0\) and \(\xi_1 = \frac{c_1}{c_1 + c_2}\), \(\xi_2 = \frac{c_2}{c_1 + c_2}\). The following theorem shows that problem (6)–(8) has a unique solution given by

\[
u(t) = \begin{cases} 
\frac{1}{\Gamma(\gamma)} \left(\frac{t^\alpha - a^\alpha}{\rho}\right)^{\gamma - 1} & \text{if } t \in I_0, \\
\frac{1}{\Gamma(\gamma)} \left(\frac{t^\alpha - a^\alpha}{\rho}\right)^{\gamma - 1} \int_{a}^{t} \left(\frac{s^\alpha - s^\alpha}{\rho}\right)^{\gamma - 1} s^{\alpha - 1} \psi(s) \, ds & \text{if } t \in I_1, \\
\frac{1}{\Gamma(\gamma)} \left(\frac{t^\alpha - a^\alpha}{\rho}\right)^{\gamma - 1} \int_{a}^{t} \left(\frac{s^\alpha - s^\alpha}{\rho}\right)^{\gamma - 1} s^{\alpha - 1} \psi(s) \, ds & \text{if } t \in I_2, k = 1, \ldots, m. \tag{9}
\end{cases}
\]

**Theorem 3.1** Let \(\gamma = \alpha + \beta - \alpha\beta\), where \(0 < \alpha < 1\) and \(0 \leq \beta \leq 1\). If \(\psi : (a, b] \to E\) is a function such that \(\psi(\cdot) \in C_{\gamma, \rho}(I)\), then \(u \in PC_{\gamma, \rho}(I)\) satisfies problem (6)–(8) if and only if it satisfies (9).

**Proof** Assume that \(u\) satisfies (6)–(8). If \(t \in I_0\), then

\[
\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(t) = \psi(t). \tag{10}
\]

Lemma 2.15 implies we have the solution that can be written as

\[
u(t) = \frac{\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(a^+)}{\Gamma(\gamma)} \left(\frac{t^\alpha - a^\alpha}{\rho}\right)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(\frac{s^\alpha - s^\alpha}{\rho}\right)^{\alpha - 1} s^{\alpha - 1} \psi(s) \, ds. \tag{10}
\]

If \(t \in I_1\), then Lemma 2.15 implies

\[
u(t) = \frac{\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(t_k^+)}{\Gamma(\gamma)} \left(\frac{t^\alpha - t_k^\alpha}{\rho}\right)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\frac{s^\alpha - s^\alpha}{\rho}\right)^{\alpha - 1} s^{\alpha - 1} \psi(s) \, ds
\]

\[
= \frac{\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(t_k^+)}{\Gamma(\gamma)} + \sigma_k(u(t_k^-)) \left(\frac{t^\alpha - t_k^\alpha}{\rho}\right)^{\gamma - 1} + \left(\frac{\partial}{\partial t_{k}^\gamma}\right)\psi(t)
\]

\[
= \left(\frac{t^\alpha - t_k^\alpha}{\rho}\right)^{\gamma - 1} \left[\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(a^+) + \sigma_k(u(t_k^-)) + \left(\frac{\partial}{\partial t_{k}^\gamma}\right)\psi(t_k^-)\right] + \left(\frac{\partial}{\partial t_{k}^\gamma}\right)\psi(t).
\]

If \(t \in I_2\), then Lemma 2.15 implies

\[
u(t) = \frac{\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(t_k^+)}{\Gamma(\gamma)} \left(\frac{t^\alpha - t_k^\alpha}{\rho}\right)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\frac{s^\alpha - s^\alpha}{\rho}\right)^{\alpha - 1} s^{\alpha - 1} \psi(s) \, ds
\]

\[
= \frac{\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(t_k^+)}{\Gamma(\gamma)} + \sigma_k(u(t_k^-)) \left(\frac{t^\alpha - t_k^\alpha}{\rho}\right)^{\gamma - 1} + \left(\frac{\partial}{\partial t_{k}^\gamma}\right)\psi(t)
\]

\[
= \left(\frac{t^\alpha - t_k^\alpha}{\rho}\right)^{\gamma - 1} \left[\left(\frac{\partial}{\partial t_{k}^\gamma}\right)u(a^+) + \sigma_k(u(t_k^-)) + \left(\frac{\partial}{\partial t_{k}^\gamma}\right)\psi(t_k^-)\right] + \left(\frac{\partial}{\partial t_{k}^\gamma}\right)\psi(t).
\]
which implies that

\[
(\mathcal{C}_a^{1,\gamma} u)(a^*) = \xi_2 - \xi_1 \sum_{i=1}^m \sigma_i(u(t_i)) - \xi_1 \sum_{i=1}^m (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(t_i)
- \xi_1 (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(b) \text{ if } t \in J_0,
\]

Substituting (13) into (11) and (10), we obtain (9).

Reciprocally, applying $\mathcal{C}_a^{1,\gamma}$ on both sides of (9) and using Lemma 2.6 and Theorem 2.3, we get

\[
(\mathcal{C}_a^{1,\gamma} u)(t) =
\begin{cases}
\xi_2 - \xi_1 \sum_{i=1}^m \sigma_i(u(t_i)) - \xi_1 \sum_{i=1}^m (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(t_i) - \xi_1 (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(b) & \text{if } t \notin J_0, \\
\xi_2 - \xi_1 \sum_{i=1}^m \sigma_i(u(t_i)) - \xi_1 \sum_{i=1}^m (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(t_i) - \xi_1 (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(b) + \sum_{i=1}^k (\mathcal{J}_{(t_{(i-1)}^t)}^{1-\gamma} \psi)(t_i) & \text{if } t \in J_k, k = 1, \ldots, m.
\end{cases}
\]
Next, taking the limit $t \to a^*$ of (14) and using Lemma 2.5, with $1 - \gamma < 1 - \gamma + \alpha$, we obtain

$$
(p \mathcal{J}^{1 - \gamma}_{a^*} u)(a^*) = \xi_2 - \xi_1 \sum_{i=1}^{m} \sigma_i(u(t_i^*)) - \xi_1 \sum_{i=1}^{m} \left( p \mathcal{J}^{1 - \gamma + \alpha}_{(t_i^*)}\psi \right)(t_i) - \xi_1 \left( p \mathcal{J}^{1 - \gamma + \alpha}_{t_i}\psi \right)(b).
$$

(15)

Now, taking $t = b$ in (14), we get

$$
(p \mathcal{J}^{1 - \gamma}_{t_m} u)(b) = \xi_2 + (1 - \xi_1) \left( \sum_{i=1}^{m} \sigma_i(u(t_i^*)) + \sum_{i=1}^{m} \left( p \mathcal{J}^{1 - \gamma + \alpha}_{(t_i^*)}\psi \right)(t_i) + \left( p \mathcal{J}^{1 - \gamma + \alpha}_{t_i}\psi \right)(b) \right).
$$

(16)

From (15) and (16), we find that

$$
c_1(p \mathcal{J}^{1 - \gamma}_{a^*} u)(a^*) + c_2(p \mathcal{J}^{1 - \gamma}_{t_m} u)(b) = c_3,
$$

which shows that the boundary condition $c_1(p \mathcal{J}^{1 - \gamma}_{a^*} u)(a^*) + c_2(p \mathcal{J}^{1 - \gamma}_{t_m} u)(b) = c_3$ is satisfied. Next, we apply operator $p \mathcal{D}^{\beta}_{t_k}$ on both sides of (9), where $k = 0, \ldots, m$. Then, from Lemma 2.6 and Lemma 2.12, we obtain

$$
(p \mathcal{D}^{\beta}_{t_k} u)(t) = (p \mathcal{D}^{\beta(1-\alpha)}_{t_k}\psi)(t).
$$

(17)

Since $u \in PC_{\gamma,\rho}(J)$ and by definition of $PC_{\gamma,\rho}(f)$, we have $p \mathcal{D}^{\beta}_{t_k} u \in PC_{\gamma,\rho}(J)$, then (17) implies that

$$
(p \mathcal{D}^{\beta}_{t_k} u)(t) = \left( \delta_{\rho} \mathcal{J}^{1 - \beta(1-\alpha)} \psi \right)(t) = (p \mathcal{D}^{\beta(1-\alpha)}_{t_k}\psi)(t) \in PC_{\gamma,\rho}(J).
$$

(18)

As $\psi(t) \in C_{\gamma,\rho}(J)$ and from Lemma 2.4, it follows

$$
(p \mathcal{J}^{1 - \beta(1-\alpha)}_{t_k}\psi) \in PC_{\gamma,\rho}(J).
$$

(19)

Regarding the definition of the space $PC_{\gamma,\rho}(f)$, together with (18), (19), we derive that

$$
(p \mathcal{J}^{1 - \beta(1-\alpha)}_{t_k}\psi) \in PC_{\gamma,\rho}(J).
$$

(20)

Applying operator $p \mathcal{J}^{\beta(1-\alpha)}_{t_k}$ on both sides of (17) and using Lemma 2.8, Lemma 2.5, and Property 2.10, we have

$$
(p p \mathcal{D}^{\beta}_{t_k} u)(t) = p \mathcal{J}^{\beta(1-\alpha)}_{t_k}(p \mathcal{D}^{\beta}_{t_k} u)(t) = \psi(t) - \frac{(p \mathcal{J}^{1 - \beta(1-\alpha)}\psi)(t_k)}{\Gamma(\beta(1-\alpha))} \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{\beta(1-\alpha) - 1} \psi(t),
$$

that is, (6) holds.
Lemma 3.2 Let $\gamma = \alpha + \beta - \alpha \beta$, where $0 \leq \beta \leq 1$ and $0 < \alpha < 1$. Suppose that $f : (a, b] \times E \times E \to E$ is a function so that $f(\cdot, u(\cdot), w(\cdot)) \in C_{\gamma, \rho}(J)$ for each $w, u \in PC_{\gamma, \rho}(J)$.

If $u \in PC_{\gamma, \rho}(J)$, then $u$ satisfies problem (1)–(3) iff $u$ is the fixed point of $\Psi : PC_{\gamma, \rho}(J) \to PC_{\gamma, \rho}(J)$ defined by

$$
\Psi u(t) = \frac{1}{\Gamma(\gamma)} \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{\gamma-1} \sum_{i=1}^m \xi_i \sigma_i(u(t_i)) - \sum_{i=1}^m \xi_i \left( \int_{t_{i-1}}^{t_i} f^1(t, u(t)) dt \right) - \sum_{a < t_k \leq t} \sigma_k(u(t_k)) + \sum_{a < t_k \leq t} \left( \int_{t_{i-1}}^{t_i} f^1(t, u(t)) dt \right) \right)
$$

(20)

where $h : (a, b] \to \mathbb{R}$ is defined as

$$
h(t) = f(t, u(t), h(t)).
$$

Evidently, $h \in C_{\gamma, \rho}(J)$. Also, by Lemma 2.4, $\Psi u \in PC_{\gamma, \rho}(J)$.

In the sequel, we shall use the following hypotheses efficiently:

(Ax1) The mapping $t \mapsto f(t, u, w)$ is measurable on $(a, b]$ for each $u, w \in E$, and the functions $u \mapsto f(t, u, w)$ and $w \mapsto f(t, u, w)$ are continuous on $E$ for a.e. $t \in (a, b]$, and $f(\cdot, u(\cdot), w(\cdot)) \in PC_{\gamma, \rho}^{\delta(1-\omega)}$ for any $u, w \in PC_{\gamma, \rho}(J)$.

(Ax2) There exists a continuous function $p : J \to (0, \infty)$ such that

$$
\|f(t, u, w)\| \leq p(t) \quad \text{for a.e. } t \in (a, b] \text{ and for each } u, w \in E.
$$

(Ax3) For each bounded set $B \subset E$ and for each $t \in (a, b]$, we have

$$
\mu(f(t, B, \{t^\rho D^\rho u\})) \leq \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} p(t) \mu(B),
$$

with $t^\rho D^\rho u = \{t^\rho D^\rho w : w \in B\}$ and $k = 1, \ldots, m$.

(Ax4) The functions $\sigma_k : E \to E$ are continuous and there exists $\eta^* > 0$ such that

$$
\|\sigma_k(u)\| \leq \eta^* \|u\| \quad \text{for each } u \in E, k = 1, \ldots, m.
$$
(Ax5) For any bounded set $B \subseteq E$ and for any $t \in (a, b]$, we find

$$\mu(B) = \eta^* \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} \mu(B), \quad k = 1, \ldots, m.$$ 

Set $p^* = \sup_{t \in J} p(t)$.

We are now in a position to investigate the existence result for problem (1)–(3) based on the fixed point theorem of Mönch.

**Theorem 3.3** Assume that (Ax1)–(Ax5) hold. If

$$\Omega := \frac{m \eta^*}{\Gamma(\gamma)} + p^* \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{m}{\Gamma(\gamma)\Gamma(2 - \gamma + \alpha)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma + \alpha} < 1,$$  

then problem (1)–(3) has at least one solution in $PC_{\gamma, \rho}(J) \subseteq PC_{\gamma, \rho}(J)$.

**Proof** Consider the operator $\Psi : PC_{\gamma, \rho}(J) \to PC_{\gamma, \rho}(J)$ defined in (20) and the ball $B_R := B(0, R) = \{w \in PC_{\gamma, \rho}(J) : \|w\|_{PC_{\gamma, \rho}} \leq R\}$.

For any $u \in B_R$ and any $t \in (a, b]$, we find

$$\left\| \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} (\Psi u)(t) \right\| \leq \frac{1}{\Gamma(\gamma)} \left[ \|s_2\| + \|s_1\| \sum_{i=1}^{m} \|\sigma_i(u(t_i))\| + \|s_1\| \sum_{i=1}^{m} \left( \frac{p^*}{\Gamma(\gamma)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma + \alpha} \|h(s)\| \right]$$

$$\leq \left[ \|s_2\| + \|s_1\| \left( \frac{m \eta^*}{\Gamma(\gamma)} + \frac{m \rho^*}{\Gamma(\gamma)\Gamma(2 - \gamma + \alpha)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma + \alpha} \|h(s)\| \right] \left( \frac{p^*}{\Gamma(\gamma)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma + \alpha} (\Psi u)(t) \right\|$$

By Lemma 2.6, we have

$$\left\| \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} (\Psi u)(t) \right\| \leq \left[ \|s_2\| + \|s_1\| \left( \frac{m \eta^*}{\Gamma(\gamma)} + \frac{m \rho^*}{\Gamma(\gamma)\Gamma(2 - \gamma + \alpha)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma + \alpha} \|h(s)\| \right] \left( \frac{p^*}{\Gamma(\gamma)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma + \alpha} (\Psi u)(t) \right\|$$
Hence, for all \( u \in PC_{\gamma,\rho}(f) \) and each \( t \in (a, b) \), we get

\[
\| (\Psi u)(t) - (\Psi u)(t) \| \leq \frac{\| \xi_1 \| + \| \xi_1 \| + 1}{\Gamma(\gamma)} \left( m \gamma R + \frac{m p^*}{\Gamma(2 - \gamma + \alpha)} \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha} \right) \\
+ \left( \frac{|\xi_1|^p}{\Gamma(\gamma)\Gamma(2 - \gamma + \alpha)} + \frac{\rho^*}{\Gamma(\alpha + 1)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha} \\
\leq R.
\]

Hence, we conclude that \( \Psi \) transforms the ball \( B_R \) into itself. Now, we prove that the operator \( \Psi : B_R \to B_R \) fulfills all conditions of Theorem 2.16. For the sake of transparency, we shall divide the proof in four steps.

**Step 1:** We shall prove that the operator \( \Psi : B_R \to B_R \) is continuous.

Suppose that the sequence \( \{u_n\} \) converges to \( u \) in \( PC_{\gamma,\rho}(f) \).

Then we get

\[
\left\| \left( (\Psi u_n)(t) - (\Psi u)(t) \right) \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1 - \gamma} \right\| \\
\leq \frac{1}{\Gamma(\gamma)} \left[ |\xi_1| \sum_{i=1}^{m} \left( \sum_{\delta \in t_i} (b_i)(s) \right) \right] + |\xi_1| \left( \frac{\rho}{\Gamma(\gamma)} \right) \left( 1 - \gamma \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha} \\
+ \left( \frac{|\xi_1|^p}{\Gamma(\gamma)} \right) \left( \frac{\rho^*}{\Gamma(\alpha + 1)} \right) \left( 1 - \gamma \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha} \\
+ \left( \frac{\rho}{\Gamma(\gamma)} \right) \left( \frac{\rho^*}{\Gamma(\alpha + 1)} \right) \left( 1 - \gamma \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha} \\
\leq R.
\]

for each \( t \in (a, b) \), where \( h_n, h \in C_{\gamma,\rho}(f) \) in a way that

\[
h(t) = f(t, u(t), h(t)),
\]

\[
h_n(t) = f(t, u_n(t), h_n(t)).
\]

Since \( u_n \to u \), then we get \( h_n(t) \to h(t) \) as \( n \to \infty \) for each \( t \in (a, b) \). So we find that

\[
\| \Psi u_n - \Psi u \|_{PC_{\gamma,\rho}} \to 0 \quad \text{as} \quad n \to \infty
\]

by the Lebesgue dominated convergence theorem.

**Step 2:** We shall indicate that \( \Psi(B_R) \) is equicontinuous and bounded.

Indeed, \( \Psi(B_R) \) is bounded since \( \Psi(B_R) \subseteq B_R \) and \( B_R \) is bounded. Next, let \( \epsilon_1, \epsilon_2 \in \mathbb{F} \), \( \epsilon_1 < \epsilon_2 \), and let \( u \in B_R \). Then

\[
\left\| \left( \frac{\epsilon_1^\rho - t_k^\rho}{\rho} \right)^{1 - \gamma} (\Psi u)(\epsilon_1) - \left( \frac{\epsilon_2^\rho - t_k^\rho}{\rho} \right)^{1 - \gamma} (\Psi u)(\epsilon_2) \right\| \\
\leq \frac{1}{\Gamma(\gamma)} \left[ \sum_{\epsilon_1 < t_k < \epsilon_2} \left( \sum_{\delta \in t_k} \| \sigma_k(t_k) \| \right) + \sum_{\epsilon_1 < t_k < \epsilon_2} \left( \frac{\rho^*}{\Gamma(\gamma)} \right) \left( 1 - \gamma \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha} \right]
\]
Thus from (21), we get

\[
\frac{p^*}{\Gamma(\alpha + 1)} \left| \left( \frac{\epsilon_1^0 - \epsilon_k^0}{\rho} \right) 1-\gamma + \left( \frac{\epsilon_2^0 - \epsilon_k^0}{\rho} \right) 1-\gamma \right| \rightarrow 0 \quad \text{as } \epsilon_1 \rightarrow \epsilon_2.
\]

Consequently, we conclude that $\Psi(B_R)$ is bounded and equicontinuous.

**Step 3:** The implication (4) of Theorem 2.16 holds.

Now let $D$ be an equicontinuous subset of $B_R$ such that $D \subset \bar{\Psi}(D) \cup \{0\}$, therefore the function $t \rightarrow d(t) = \mu(D(t))$ is continuous on $I$. Regarding the properties of the measure $\mu$, together with (Ax3) and (Ax5), for each $t \in (a, b]$, we find

\[
\left( \frac{t^\rho - t_k^\rho}{\rho} \right) 1-\gamma d(t) \leq \mu \left( \left. \left( \frac{t^\rho - t_k^\rho}{\rho} \right) 1-\gamma \right| (\Psi D(t) \cup \{0\}) \right) 
\]

\[
\leq \mu \left( \left. \left( \frac{t^\rho - t_k^\rho}{\rho} \right) 1-\gamma \right| (\Psi D(t)) \right) 
\]

\[
\leq \frac{1}{\Gamma(\gamma)} \left[ \sum_{a < t < c} \eta^* \left( \frac{t^\rho - t_k^\rho}{\rho} \right) 1-\gamma \mu(D(t)) \right. 
\]

\[
+ \sum_{a < t < c} \left( \frac{\rho \mathcal{J} 1-\gamma \alpha}{(1-\gamma)\rho} \left( \frac{s^\rho - t_k^\rho}{\rho} \right) 1-\gamma p(s) \mu(D(s)) \right) (t_k) 
\]

\[
+ \left( \frac{t^\rho - t_k^\rho}{\rho} \right) 1-\gamma \left( \frac{\rho \mathcal{J} 1-\gamma \alpha}{(1-\gamma)\rho} \left( \frac{s^\rho - t_k^\rho}{\rho} \right) 1-\gamma p(s) \mu(D(s)) \right) (t) 
\]

\[
\leq \frac{m n^*}{\Gamma(\gamma)} \|d\|_{PC_{\gamma, \rho}} + \frac{p^*}{\Gamma(\gamma + \alpha)} \left( \frac{b^\rho - a^\rho}{\rho} \right) \left( \frac{\rho \mathcal{J} 1-\gamma \alpha}{(1-\gamma)\rho} \left( \frac{s^\rho - t_k^\rho}{\rho} \right) 1-\gamma d(s) \right) (t) 
\]

\[
\leq \frac{m n^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\gamma + \alpha)} \left( \frac{b^\rho - a^\rho}{\rho} \right) \left( \frac{\rho \mathcal{J} 1-\gamma \alpha}{(1-\gamma)\rho} \left( \frac{s^\rho - t_k^\rho}{\rho} \right) 1-\gamma d(s) \right) (t) 
\]

Thus

\[
\|d\|_{PC_{\gamma, \rho}} \leq \mathcal{L} \|d\|_{PC_{\gamma, \rho}}.
\]

From (21), we get $\|d\|_{PC_{\gamma, \rho}} = 0$, that is, $d(t) = \mu(D(t)) = 0$, for each $t \in J_k$, $k = 0, \ldots, m$, and then $D(t)$ is relatively compact in $E$. In view of the Ascoli–Arzela theorem, $D$ is relatively compact in $B_R$. Applying now Theorem 2.16, we conclude that $\Psi$ has a fixed point $u^* \in PC_{\gamma, \rho}(J)$, which is a solution of problem (1)–(3).

**Step 4:** We show that such a fixed point $u^* \in PC_{\gamma, \rho}(J)$ lies in $PC_{\gamma, \rho}(J)$.
Regarding the fact that \( u^* \) is the only fixed point of the mapping \( \Psi \) at \( PC_{\gamma,\rho}(J) \), for any \( t \in J_k \), with \( k = 0, \ldots, m \), we have

\[
\begin{align*}
    u^*(t) &= \frac{1}{\Gamma(\gamma)} \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{\gamma-1} \left[ \xi_2 - \xi_1 \sum_{i=1}^{m} \sigma_i(u(t_i)) - \xi_1 \sum_{i=1}^{m} (\rho J_{(t_{i-1})}^{1-\gamma,\rho} h)(t_i) 
    \right. \\
    & \quad - \xi_1 (\rho J_{(t_k)}^{1-\gamma,\rho} h)(b) + \sum_{a \in \mathcal{C}_k} \sigma_h(u(t_k)) + \sum_{a \in \mathcal{C}_k} (\rho J_{(t_{k-1})}^{1-\gamma,\rho} h)(t_k) \\
    & \quad + \left. (\rho J_{(t_k)}^{1-\gamma,\rho} h)(t) \right],
\end{align*}
\]

where \( h \in C_{\gamma,\rho}(J) \) in a way that

\[
h(t) = f(t, u^*(t), h(t)).
\]

Applying \( \rho D_{t_k}^{\gamma} \) to both sides and by Lemma 2.6 and Lemma 2.12, we have

\[
\begin{align*}
    \rho D_{t_k}^{\gamma} u^*(t) &= (\rho D_{t_k}^{\gamma} J_{t_k}^{\rho} f(s, u^*(s), h(s)))(t) \\
    &= (\rho D_{t_k}^{\rho(1-\gamma)} f(s, u^*(s), h(s)))(t).
\end{align*}
\]

On account of \( \gamma \geq \alpha \), and (Ax1), it lies in \( PC_{\gamma,\rho}(J) \). Consequently, \( \rho D_{t_k}^{\gamma} u^* \in PC_{\gamma,\rho}(J) \), which implies that \( u^* \in PC_{\gamma,\rho}(J) \). Regarding Theorem 3.3, as an outcome of Step 1 to Step 4, we deduce that problem (1)–(3) has at least one solution in \( PC_{\gamma,\rho}(J) \). \( \square \)

Our second existence result for problem (1)–(3) is based on Darbo’s fixed point theorem.

**Theorem 3.4** Assume that (Ax1)–(Ax5) and (21) hold. Then problem (1)–(3) has at least one solution in \( PC_{\gamma,\rho}(J) \). \( \subset PC_{\gamma,\rho}(J) \).

**Proof** Consider the operator \( \Psi \) defined in (20). We know that \( \Psi : B_R \rightarrow B_R \) is bounded and continuous and that \( \Psi(B_R) \) is equicontinuous, we need to prove that the operator \( \Psi \) is a \( \mathcal{L} \)-contraction.

Let \( D \subset B_R \) and \( t \in J \). Then we have

\[
\begin{align*}
    \mu \left( \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} (\Psi D)(t) \right) \\
    = \mu \left( \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} (\Psi u)(t) : u \in D \right) \\
    \leq \frac{1}{\Gamma(\gamma)} \left[ \sum_{a \in \mathcal{C}_k} \eta^a \mu \left( \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} u(t) : u \in D \right) ight] \\
    + \sum_{a \in \mathcal{C}_k} \left\{ (\rho J_{(t_{k-1})}^{1-\gamma,\rho} P^a) \mu \left( \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} u(s) \right)(t_k) : u \in D \right\} \\
    + (b^\rho - a^\rho)^{1-\gamma} \left\{ (\rho J_{(t_k)}^{1-\gamma,\rho} P^a) \mu \left( \left( \frac{t^\rho - t_k^\rho}{\rho} \right)^{1-\gamma} u(s) \right)(t_k) : u \in D \right\}.
\end{align*}
\]
By Lemma, we have
\[
\mu_{PC_{\gamma,\rho}}(\Psi D) \leq \left[ \frac{m \eta^\gamma}{\Gamma(\gamma)} + \left( \frac{\rho^\gamma}{\Gamma(\alpha + 1)} + \frac{m \eta^\gamma}{\Gamma(2 - \gamma + \alpha)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1-\gamma+\alpha} \right] \mu_{PC_{\gamma,\rho}}(D).
\]

Therefore
\[
\mu_{PC_{\gamma,\rho}}(\Psi D) \leq L \mu_{PC_{\gamma,\rho}}(D).
\]

So, by \(21\), the operator \(\Psi\) is a \(L\)-contraction. □

As a consequence of Theorem 2.17 and using Step 4 of the last result, we deduce that \(\Psi\) possesses a fixed point. In particular, this fixed point forms a solution for problem (1)–(3).

**Example 3.5** Consider the Banach space
\[ E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\} \]
with the norm
\[ \|u\| = \sum_{n=1}^{\infty} |u_n|. \]

We examine the following impulsive boundary value problem of the generalized Hilfer FDE:
\[
\begin{align*}
\left( ^1D_{t_k}^{\frac{1}{2},0} u_n \right)(t) &= \frac{3t^2 - 20}{213e^{-t^3}(1 + |u_n(0)| + |^1D_{t_k}^{\frac{1}{2},0} u_n(t)|)}, \quad t \in J_k, k = 0, \ldots, 9, \quad (22) \\
\left( ^1J_{t_k}^{\frac{1}{2},0} u_n \right)(t_k^+) - \left( ^1J_{t_{k-1}}^{\frac{1}{2},0} u_n \right)(t_k^-) &= \frac{|u_n(t_k^-)|}{10(k + 3) + |u_n(t_k^-)|}, \quad k = 1, \ldots, 9, \quad (23) \\
\left( ^1J_{t_k}^{\frac{1}{2},0} u_n \right)(1^+) + 2\left( ^1J_{t_k}^{\frac{1}{2},0} u_n \right)(3) &= 0, \quad (24)
\end{align*}
\]
where \(J_k = (t_k, t_{k+1}], t_k = 1 + \frac{k}{9}\) for \(k = 0, \ldots, 9\), \(m = 9\), \(a = t_0 = 1\), and \(b = t_{10} = 3\).

Set
\[ f(t, u, w) = \frac{3t^2 - 20}{213e^{-t^3}(1 + \|u\| + \|w\|)}, \quad t \in (1, 3], u, w \in E. \]

We have
\[ PC_{\gamma,\rho}^{(1-\alpha)}([1, 3]) = PC_{\frac{1}{2},1}^0 ([1, 3]) = \{ g : (1, 3] \to \mathbb{R} : (\sqrt{t-t_k})g \in PC([1, 3]) \}, \]
where \(\alpha = \gamma = \frac{1}{2}, \beta = 0, \rho = 1\), and \(k = 0, \ldots, 9\).

Clearly, the continuous function \(f \in PC_{\frac{1}{2},1}^0 ([1, 2])\).

As a result, condition \((Ax I)\) is fulfilled.
For each \( u, w \in E \) and \( t \in (1, 3) \),
\[
\|f(t, u, w)\| \leq \frac{3t^2 - 20}{213e^{-t^3}}.
\]
Hence condition (Ax2) is satisfied with \( p^* = \frac{7}{213} \).
And let
\[
\sigma_k(u) = \frac{\|u\|}{10(k + 3) + \|u\|}, \quad k = 1, \ldots, 9, u \in E.
\]
Let \( u \in E \). Then we have
\[
\|\sigma_k(u)\| \leq \frac{1}{40}\|u\|, \quad k = 1, \ldots, 9,
\]
and so condition (Ax4) is satisfied with \( \eta^* = \frac{1}{40} \).
The condition (21) of Theorem 3.3 is satisfied for
\[
\Omega := \frac{m\eta^*}{\Gamma(\gamma)} + \left( \frac{p^*}{\Gamma(\alpha + 1)} + \frac{mp^*}{\Gamma(\gamma)(2 - \gamma + \alpha)} \right) \left( \frac{b^\rho - a^\rho}{\rho} \right)^{1 - \gamma + \alpha}
\]
\[
= \frac{9}{40\sqrt{\pi}} + 2\left( \frac{17}{213\sqrt{\pi}} + \frac{63}{213\Gamma(2)\sqrt{\pi}} \right)
\]
\[
\approx 0.55074703829 < 1.
\]
Then problem (22)–(24) has at least one solution in \( PC^{\frac{1}{2}}_{\frac{1}{2}}([1, 3]) \subset PC^{\frac{1}{2}}_{\frac{1}{2}}([1, 3]) \).

4 Ulam-type stability

Now, we consider the Ulam stability for problem (1)–(3). Let \( u \in PC_{\gamma, \rho}(f), \epsilon > 0, \tau > 0, \) and \( \vartheta : (a, b) \rightarrow [0, \infty) \) be a continuous function. We consider the following inequality:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\| (\epsilon D_{t_0}^{\alpha, \beta} u)(t) - f(t, u(t), (\epsilon D_{t_0}^{\alpha, \beta} u)(t)) \| \leq \epsilon \vartheta(t), & t \in J_k, k = 0, \ldots, m, \\
\| (\epsilon J_{t_0}^{1 - \gamma} u)(t_k) - (\epsilon J_{t_0}^{1 - \gamma} u)(t_k^*) - \sigma_k(u(t_k^*)) \| \leq \epsilon \tau, & k = 1, \ldots, m.
\end{array} \right.
\end{aligned}
\]

Definition 4.1 Problem (1)–(3) is Ulam–Hyers–Rassias (U-H-R) stable with respect to \( (\vartheta, \tau) \) if there exists a real number \( a_{f, m, \vartheta} > 0 \) such that, for each \( \epsilon > 0 \) and for each solution \( u \in PC_{\gamma, \rho}(f) \) of inequality (25), there exists a solution \( w \in PC_{\gamma, \rho}(f) \) of (1)–(3) with
\[
\| u(t) - w(t) \| \leq \epsilon a_{f, m, \vartheta} (\vartheta(t) + \tau), \quad t \in (a, b].
\]

Remark 4.2 A function \( u \in PC_{\gamma, \rho}(f) \) is a solution of inequality (25) if and only if there exist \( \sigma \in PC_{\gamma, \rho}(f) \) and a sequence \( \sigma_k, k = 0, \ldots, m, \) such that
\begin{enumerate}
\item \( \| \sigma(t) \| \leq \epsilon \vartheta(t) \) and \( \| \sigma_k \| \leq \epsilon \tau, \) \( t \in J_k, k = 1, \ldots, m; \)
\item \( (\epsilon D_{t_0}^{\alpha, \beta} u)(t) = f(t, u(t), (\epsilon D_{t_0}^{\alpha, \beta} u)(t)) + \sigma(t), \) \( t \in J_k, k = 0, \ldots, m; \)
\item \( (\epsilon J_{t_0}^{1 - \gamma} u)(t_k) = (\epsilon J_{t_0}^{1 - \gamma} u)(t_k^*) + \sigma_k(u(t_k^*)) + \sigma_k, k = 1, \ldots, m. \)
\end{enumerate}
Now we are concerned with the Ulam–Hyers–Rassias stability of our problem (1)–(3).

**Theorem 4.3** Assume that in addition to (A1)–(A5) and (21), the following hypotheses hold:

(Ax6) There exist a nondecreasing function $\vartheta \in PC_{\gamma,\rho}(J)$ and $\lambda_{\vartheta} > 0$ such that, for each $t \in (a, b]$, we have

$$
(\rho D_{a+}^\alpha \vartheta)(t) \leq \lambda_{\vartheta} \vartheta(t).
$$

(Ax7) There exists a continuous function $\chi : [a, b] \rightarrow [0, \infty)$ such that, for each $t \in J_k; k = 0, \ldots, m$, we have

$$
p(t) \leq \chi(t) \vartheta(t).
$$

Then equation (1) is U-H-R stable with respect to $(\vartheta, \tau)$.

Set $\chi^* = \sup_{t \in [a, b]} \chi(t)$.

**Proof** Consider the operator $\Psi$ defined in (20). Let $u \in PC_{\gamma,\rho}(J)$ be a solution of inequality (25), and let us assume that $w$ is the unique solution of the problem

$$
\begin{align*}
(\rho D_{a+}^\alpha w)(t) &= f(t, w(t), (\rho D_{a+}^\alpha w)(t)); \quad t \in J_k, k = 0, \ldots, m, \\
(\rho J_{a+}^{1-\gamma} w)(t_k) &= (\rho J_{a+}^{1-\gamma} w)(t_k) + \sigma_k(w(t_k)); \quad k = 1, \ldots, m, \\
c_1(\rho J_{a+}^{1-\gamma} u)(a^*) + c_2(\rho J_{a+}^{1-\gamma} w)(b) &= c_3, \\
(\rho J_{a+}^{1-\gamma} w)(t_k^*) &= (\rho J_{a+}^{1-\gamma} u)(t_k^*); \quad k = 0, \ldots, m.
\end{align*}
$$

By Theorem 3.1 and Lemma 3.2, we obtain for each $t \in (a, b]$

$$
w(t) = \frac{(\rho J_{a+}^{1-\gamma} w)(t_k^*)}{\Gamma(\gamma)} \left( \frac{t^\rho - t_k^{\rho}}{\rho} \right)^{\gamma-1} + (\rho J_{a+}^{\alpha} h)(t), \quad t \in J_k, k = 0, \ldots, m,
$$

where $h : (a, b] \rightarrow E$ is a function satisfying the functional equation

$$
h(t) = f(t, w(t), h(t)).
$$

Since $u$ is a solution of inequality (25), by Remark 4.2, we have

$$
\begin{align*}
(\rho D_{a+}^\alpha u)(t) &= f(t, u(t), (\rho D_{a+}^\alpha u)(t)) + \sigma(t); \quad t \in J_k, k = 0, \ldots, m; \\
(\rho J_{a+}^{1-\gamma} u)(t_k^*) &= (\rho J_{a+}^{1-\gamma} u)(t_k^*) + \sigma_k(u(t_k^*)) + \sigma_k, \quad k = 1, \ldots, m.
\end{align*}
$$

Clearly, the solution of (26) is given by

$$
u(t) = \frac{1}{\Gamma(\gamma)} \left( \frac{t^\rho - t_k^{\rho}}{\rho} \right)^{\gamma-1} \left[ (\rho J_{a+}^{1-\gamma} u)(a^*) + \sum_{a < t_k \leq t} \sigma_k(u(t_k^*)) + \sum_{a < t_k \leq t} \sigma_k \\
+ \sum_{a < t_k \leq t} (\rho J_{a+}^{1-\gamma} g)(t_k) + \sum_{a < t_k \leq t} (\rho J_{a+}^{1-\gamma} \sigma)(t_k) \right].
$$
\[ \rho J_{\alpha t}^\gamma g(t) + (\rho J_{\alpha t}^\gamma \sigma)(t) + k g(t), \quad t \in J_k, k = 0, \ldots, m, \]

where \( g : (a, b] \to E \) is a function satisfying the functional equation

\[ g(t) = f(t, u(t), g(t)). \]

We have, for each \( t \in J_k, k = 0, \ldots, m, \)

\[ \rho J_{\alpha t}^1 - \gamma t + k u(t) = \rho J_{\alpha t}^1 - \gamma a + u(a) + \sum_{a < t < b} (\rho J_{\alpha t}^1 - \gamma \sigma)(t_k) + \sum_{a < t < b} (\rho J_{\alpha t}^1 - \gamma \sigma)(t_k). \]

Hence, for each \( t \in (a, b], \) we have

\[ \| u(t) - w(t) \| \leq \rho J_{\alpha t}^1 \| g(s) - h(s) \| (t) + \rho J_{\alpha t}^1 \| \sigma(s) \| (t). \]

Thus,

\[ \| u(t) - w(t) \| \leq \rho J_{\alpha t}^1 \| g(s) - h(s) \| (t) + \rho J_{\alpha t}^1 \| \sigma(s) \| (t) \]

\[ \leq \epsilon \lambda_0 \theta(t) + \int_a^t \rho s^{\alpha - 1} \left( \frac{t^\alpha - s^\alpha}{\rho} \right) \frac{2 \chi^*(s)}{1} d\theta(s) \]

\[ \leq \epsilon \lambda_0 \theta(t) + 2 \chi^* \rho \theta(t) \theta(t) \]

\[ \leq \left( 1 + \frac{2 \chi^*}{\epsilon} \right) \lambda_0 \epsilon (\tau + \theta(t)) \]

\[ \leq a_0 \epsilon (\tau + \theta(t)), \]

where \( a_0 = (1 + \frac{2 \chi^*}{\epsilon}) \lambda_0 \). Hence, equation (1) is U-H-R stable with respect to \((\theta, \tau)\). \( \square \)

**Example 4.4** Consider the Banach space

\[ E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\} \]

with the norm

\[ \| u \| = \sum_{n=1}^{\infty} |u_n|, \]

and let the following impulsive anti-periodic boundary value problem hold:

\[ (\frac{1}{2} D_{\alpha t}^{1/0} u_n)(t) = \frac{(3t^3 + 5e^{-3})|u_n(t)|}{144e^{-t^3}(1 + \| u(t) \| + \| \frac{1}{2} D_{\alpha t}^{1/0} u(t) \|)} \quad \text{for each } t \in J_0 \cup J_1, \tag{27} \]

\[ (\frac{1}{2} J_{\alpha t}^{1/2} u_n)(2^+) - (\frac{1}{2} J_{\alpha t}^{1/2} u_n)(2^-) = \frac{|u_n(2^-)|}{\sqrt{7} e^{-t^4} + 2} \tag{28} \]
\[
\left( \frac{1}{2} J_{1}^{1} u \right)'(1^+) = -\left( \frac{1}{2} J_{1}^{1} u \right)(e),
\]
(29)

where \( f_0 = (1, 2], f_1 = (2, e], t_1 = 2, m = 1, a = t_0 = 1 \), and \( b = t_2 = e \).

Set
\[
f(t, u, w) = \frac{(3t^3 + 5e^{-3}) \| u \|}{144e^{-t+e}(1 + \| u \| + \| w \|)}, \quad t \in (1,e], u, w \in E.
\]

We have
\[
P_{C_{\gamma, \rho}^{(1, \alpha)}}(1, 2) = P_{C_{\frac{1}{2}, \frac{1}{2}}}^{0}([1, e]) = \{ g : (1, e) \rightarrow E : \sqrt{2}(\sqrt{1} - \sqrt{2t})\frac{1}{2} g \in C([1, e]) \},
\]
with \( \gamma = \alpha = \frac{1}{2}, \rho = \frac{1}{2}, \beta = 0 \), and \( k \in \{0, 1\} \). Clearly, the continuous function \( f \in P_{C_{\frac{1}{2}, \frac{1}{2}}}^{0}([1, e]) \).

Hence condition (Ax1) is satisfied.

For each \( u, w \in E \) and \( t \in (1,e] \),
\[
\| f(t, u, w) \| \leq \frac{(3t^3 + 5e^{-3})}{144e^{-t+e}}.
\]

Hence condition (Ax2) is satisfied with
\[
p(t) = \frac{(3t^3 + 5e^{-3})}{144e^{-t+e}}
\]
and
\[
p^* = \frac{(3e^3 + 5e^{-3})}{144}.
\]

And let
\[
\sigma_1(u) = \frac{\| u \|}{77e^{-t+4} + 2}, \quad u \in E.
\]

Let \( u \in E \). Then we have
\[
\| \sigma_2(u) \| \leq \frac{1}{77e^{-t+4} + 2} \| u \|,
\]
and so condition (Ax4) is satisfied with \( \eta^* = \frac{1}{77e^{-t+4} + 2} \).

The condition (21) of Theorem 3.3 is satisfied for
\[
\varphi := \frac{m \eta^*}{\Gamma(\gamma)} + \left( \frac{p^*}{\Gamma(\alpha + 1)} + \frac{mp^*}{\Gamma(\gamma) \Gamma(2 - \gamma + \alpha)} \right) \left( \frac{b^\alpha - a^\alpha}{\rho} \right)^{1 - \gamma + \alpha}
\]
\[
= \frac{1}{(77e^{-t+4} + 2)\sqrt{\pi}} + \frac{(2\sqrt{e} - 2)}{\sqrt{\pi}} \left( \frac{6e^3 + 10e^{-3}}{144\sqrt{\pi}} + \frac{3e^3 + 5e^{-3}}{144\sqrt{\pi} \Gamma(2)} \right)
\]
\[
\approx 0.92473323802 < 1.
\]

Then problem (27)–(29) has at least one solution in \( P_{C_{\frac{1}{2}, \frac{1}{2}}}^{1/2}([1, e]) \subset P_{C_{\frac{1}{2}, \frac{1}{2}}}^{1/2}([1, e]) \). Also, hypothesis (Ax6) is satisfied with \( \tau = 1, \varphi(t) = e^3 \), and \( \lambda, \beta = 3 \). Indeed, for each \( t \in (1, e] \), we
get
\[(\frac{1}{2}J^{1/2}_\lambda \vartheta)(t) \leq \frac{2e^3}{\Gamma(\frac{3}{2})} \leq \lambda \vartheta(t)\].

Let the function \( \chi : [1, e] \rightarrow [0, \infty) \) be defined by
\[\chi(t) = \frac{(3e^{-3}t^3 + 5e^{-6})}{144e^{-t+e}},\]
then, for each \( t \in (1, e] \), we have
\[p(t) = \chi(t) \vartheta(t)\]
with \( \chi^* = p^* e^{-3}. \) Hence, condition \((Ax7)\) is satisfied. Consequently, Theorem 4.3 implies that equation \((27)\) is U-H-R stable.

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