Anti-continuum approach on odd solitons in binary discrete media with cubic-quintic nonlinearity

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Abstract. The existence and stability of bright odd strongly localized modes in discrete binary media with cubic and quintic nonlinearities are investigated mainly by applying anti-continuum method. There are three different types of bright localized modes being studied; symmetric, antisymmetric and shifted odd modes. However, certain analytical analysis is carried out first applying the anti-continuum limit assumption to estimate initial excitations in the media. It can be deduced that, the quintic nonlinearity is a significant factor for stability in antisymmetric and shifted odd modes.

1. Introduction
The subject of nonlinear systems, in particular of the discrete nonlinear Schrödinger (DNLS) equation has been extensively studied since a decade ago, due to the fact of its application in the field of nonlinear optics and Bose-Einstein condensates (BEC). The DNLS realization in nonlinear optics began with the study of fabricated AlGaAs waveguide arrays [10]. This study has yielded numerous understanding regarding the interplay of discrete lattice dynamics with the effects of nonlinearity, for instance discrete solitons, modulational instabilities (MI), gap solitons, diffraction and diffraction management [2, 12, 11, 7, 18]. On the other hand, the application of DNLS equation in the context of matter waves began to flourish due to the study of the BEC dynamics in periodic potentials. Here the Gross-Pitaevskii equation with a periodic potential used as the model can be reduced to the DNLS equation for the BEC “droplets” [4, 20, 14].

The behaviour of discrete solitons in one-, two- and three-dimensional DNLS equations in arrays with cubic (Kerr) nonlinearity have been studied theoretically and experimentally [16]. More sophisticated nonlinearities in its integrable models were studied and such model attracted unique interests as they may exhibit some real life application. The dielectric response of a self-defocusing quintic correction to the self-focusing Kerr effect in glasses and organic optical media corresponds to the cubic-quintic (CQ) nonlinearity. On this particular model, multistability of discrete solitons have been studied and homoclinic solutions to the respective stationary discrete equation have been reported in [6]. Also, different conditions of the existence of MI in the CQ DNLS equation have been found in [1], where the regions of self-trapping and stability of localized modes are drastically affected by the presence of quintic nonlinear term.

Multicomponent system intrigues special attention because the additional interaction that arise strongly affecting the existence conditions of solitons. Vectorial solitons produce a family
of composite solutions, consists of two or more components that mutually self-trap in a nonlinear medium. Essentially, these structures could not exist if some of these components are not present. These bodies have been studied in various models, such as in carbon disulfide and photorefractive crystals. In optics application, cubic nonlinearity in DNLS model provides a nonlinear coupling between two components that copropagate in a waveguide, resulting the cross-phase modulation (XPM) and energy exchange. Strongly localized vectorial modes of coupled system have been found in for bright [8] and dark [17] types. In addition to known symmetric structure of odd and even modes from other DNLS models, fascinating structures of antisymmetric and shifted are also identified. For the case of coupled CQ DNLS, the phenomena of MI on the plane-wave solutions have been found [3], where the quintic nonlinearity is found to exhibit stronger effect on MI compared to that of the cubic nonlinearity.

The aim of this paper is to investigate methodically strongly localized vectorial solutions of the CQ DNLS equations. We derive approximate analytical expressions for strongly localized modes (SLM’s) of different topologies, analyze their stability by the mean of linearization, and validate the results obtained by numerical simulations.

The paper is organized as follows: In Sec. 2 the basic two-component DNLS equation is formulated. In Sec. 3, analytic expressions for vectorial bright odd SLM’s are derived and strong localization conditions are emphasized. In Sec. 4, numerical methods used in analyzing the existence and stability of SLM’s are introduced, and numerical simulations of the evolution of SLM’s are discussed. Finally in Sec. 5, we summarize the results.

2. Binary Discrete Media with Cubic-Quintic Nonlinearity

Binary discrete media with cubic-quintic nonlinearity can be described by two-component DNLS equations with cubic and quintic nonlinearities:

\[ i \frac{dA_n}{dz} = -c_a(A_{n+1} + A_{n-1}) - \lambda(|A_n|^2 + \beta|B_n|^2)A_n - \gamma(|A_n|^4 + 2\alpha|A_n|^2|B_n|^2 + \alpha|B_n|^4)A_n, \]

\[ i \frac{dB_n}{dz} = -c_b(B_{n+1} + B_{n-1}) - \lambda(|B_n|^2 + \beta|A_n|^2)B_n - \gamma(|B_n|^4 + 2\alpha|A_n|^2|B_n|^2 + \alpha|A_n|^4)A_n, \]

where \( A_n \equiv A_n(z) \) and \( B_n \equiv B_n(z) \) are the complex-valued wave function at site \( n \in \mathbb{Z} \); \( c_a, c_b \) denote the coupling constant between two adjacent sites of respective channel; \( \lambda \) and \( \gamma \) represent the cubic and quintic nonlinear coefficients respectively; \( \beta \) and \( \alpha \) are the cubic and quintic XPM coefficients respectively. Here, the self-phase modulation (SPM) coefficients are re-scaled to one. Eqs. (1)-(2) were studied for the first time in [3] to study the phenomena of MI on two plane waves. If we set \( \gamma = 0 \) and \( \beta = 1/\lambda \), Eqs. (1)-(2) become the two-component cubic DNLS system studied in [8]. Setting \( \beta = \alpha = 0 \) decouples Eqs. (1)-(2) and separates them into two scalar DNLS equations with CQ nonlinearity, which was recently investigated for the MI phenomenon [1, 21]. The binary DNLS system (1)-(2) was used in [19, 9] to study the interactions between optical solitons in bimodal cubic-quintic media.

The coupled CQ DNLS equations can be derived from the Hamiltonian, by assuming lattice of infinite sites, or periodic boundary conditions:

\[ H = H_a + H_b + H_{int}, \]
where

\[ H_a = \sum_n \left[ c_n(A_n^* A_{n+1} + A_n A_{n+1}^*) + \frac{\lambda}{2} |A_n|^4 + \frac{\gamma}{3} |A_n|^6 \right], \]

\[ H_b = \sum_n \left[ c_b(B_n^* B_{n+1} + B_n B_{n+1}^*) + \frac{\lambda}{2} |B_n|^4 + \frac{\gamma}{3} |B_n|^6 \right], \]

\[ H_{\text{int}} = \sum_n \left[ \beta \lambda |A_n|^2 |B_n|^2 + \alpha \gamma (|A_n|^2 |B_n|^2 + |A_n|^4 |B_n|^2) \right], \]

through

\[ i \frac{dA_n}{dz} = -\frac{\delta H}{\delta A_n^*}, \quad i \frac{dB_n}{dz} = -\frac{\delta H}{\delta B_n^*}. \]

The two-component CQDNLS equations has two conserved quantities, i.e. the Hamiltonian \( H \) and the total excitation norm \( N = \sum_n (|A_n|^2 + |B_n|^2) \).

3. Bright Odd SLM’s of the CQ DNLS Equations

Strongly localized solutions for Eqs. (1)-(2) can be found analytically by using the approach used in [8]. We consider only resting solutions, thus can assume the ansatz took the forms of \( A_n = \alpha_n \exp(iknz) \) and \( B_n = \beta_n \exp(ikbz) \). The stationary amplitudes in each channel \( \alpha_n = Aa_n \) and \( \beta_n = Bb_n \) are normalized by the maximum values of amplitudes \( A \) and \( B \), respectively. Without loss of generality, these maximum amplitudes can take either sign and are assumed to be real. Inserting \( A_n \) and \( B_n \) into Eqs. (1)-(2) we arrive at the following system of algebraic equations:

\[ k_a \alpha_n = c_a(\alpha_{n+1} + \alpha_{n-1}) + \lambda(\alpha_n^2 + \beta_n^2)\alpha_n + \gamma(\alpha_n^4 + 2\alpha_n^2\beta_n^2 + \alpha_n^4)\alpha_n, \]

\[ k_b \beta_n = c_b(\beta_{n+1} + \beta_{n-1}) + \lambda(\beta_n^2 + \alpha_n^2)\beta_n + \gamma(\beta_n^4 + 2\alpha_n^2\beta_n^2 + \alpha_n^4)\beta_n. \]

The solutions of these equations yield the stationary excitations in each waveguide of an array, or on each site of a lattice equivalently that essentially constitute SLM’s of different topologies. A particular topology of solutions will be discussed in detail, i.e. the odd SLM’s. Odd SLM’s are defined as the modes where the solution centered on site and exhibit different symmetries. As introduced in [8], symmetric, antisymmetric and shifted odd modes are examined.

3.1. Odd symmetric mode

Odd symmetric modes represent what is known as on-site mode in literature. We insert \( \alpha_n = A(\ldots, 0, a_1, 1, a_1, 0, \ldots) \) and \( \beta_n = B(\ldots, 0, b_1, 1, b_1, 0, \ldots) \) (refer Figure 1) into Eqs. (8)-(9) and, after straightforward derivation from equations for \( a_0 \) and \( b_0 \) and applying the condition that we are considering SLM’s in anti-continuum limit, we obtain the approximated analytic expression for the corresponding wave vectors \( k_a \) and \( k_b \) of the following forms:

\[ k_a = \lambda(A^2 + \beta B^2) + \gamma(A^4 + 2\alpha A^2 B^2 + \alpha B^4), \]

\[ k_b = \lambda(B^2 + \beta A^2) + \gamma(B^4 + 2\alpha A^2 B^2 + \alpha A^4). \]

Next, we use the strong localization conditions in anti-continuum limit, i.e. the secondary excitations \( a_1, b_1 \ll 1 \). From the equations for \( a_1 \) and \( b_1 \) we derive the approximate expression for \( a_1 \) and \( b_1 \) in the following forms:

\[ a_1 \approx \frac{c_a}{k_a}, \quad b_1 \approx \frac{c_b}{k_b}. \]

As has been done in [8], we limit ourselves to calculate the secondary excitations to the first-order approximations with respect to the small parameter \( a_1, b_1 \ll 1 \) and ignore the higher-order terms. These give us an adequate accuracy to understand the basic SLM properties.
3.2. Odd antisymmetric mode
Antisymmetric odd mode represents another kind of on-site solution for SLM that can be defined by a zero amplitude at the center of the lattice site. Here, we consider the antisymmetric odd mode took the forms $\alpha_n \approx A(\ldots, 0, a, 1, 0, s_a a, 0, \ldots)$ and $\beta_n \approx B(\ldots, 0, b, 1, 0, s_b b, 0, \ldots)$ (refer Figure 2) where $s_a$ and $s_b$ are the scaling factors which embrace the possibility of the change of the phase by $\pi$ across the mode. When substituting $\alpha_n$ and $\beta_n$ into Eqs. (8)-(9), it is immediately known that $s_a = s_b = -1$ revealing the antisymmetric nature of the solution. The wave vector $k_a, k_b$ and secondary excitations $a, b$ of this SLM coincide with those given by Eqs. (10) and (11). It is worth to note that this type of mode can be seen as a superposition of coupled symmetric odd SLM’s with a phase difference $\pi$.

3.3. Shifted mode
In the case of shifted mode, it is described as the centers of the component solitons are shifted with respect to each other. This mode can be represented by the form $\alpha_n \approx A(\ldots, 0, a_{-1}, 1, a_1, 0, 0, \ldots)$ and $\beta_n \approx B(\ldots, 0, 0, b_0, 1, b_2, 0, \ldots)$ (refer Figure 3). Substitution of these expression into Eqs. (8)-(9) and performing straightforward derivations we obtain the expressions for this SLM as

$$k_a = \lambda A^2 + \gamma A^4, \quad k_b = \lambda B^2 + \gamma B^4$$

(12)

for the corresponding wave vectors, and

$$a_{-1} \approx \frac{c_a}{\lambda A^2 + \gamma A^4}, \quad a_1 \approx \frac{c_a}{\lambda(A^2 - \beta B^2) + \gamma(A^4 - \alpha B^4)},$$

$$b_0 \approx \frac{c_b}{\lambda(B^2 - \beta A^2) + \gamma(B^4 - \alpha A^4)}, \quad b_2 \approx \frac{c_b}{\lambda B^2 + \gamma B^4}$$

(13)

for the secondary excitations. The existence of this SLM is due to its vectorial properties of coupling interaction. The corresponding lattice excitations $a_n$ and $b_n$ may exhibit the asymmetric structure.

However, it is worth to mention that the expressions (13) are valid in strong localization conditions, i.e. for $|a_{-1,1}| \ll 1$ and $|b_{0,2}| \ll 1$. As a consequence, the evolution of this mode is mainly determined by the ratio of the peak amplitudes $A/B$, as what has been found in [8].
Consequently at $O(\delta)$ and $O(\epsilon)$ the eigenvalue problem can be written as

$$\omega \begin{pmatrix} c_n \\ d_n^* \\ f_n \\ g_n^* \end{pmatrix} = J \begin{pmatrix} c_n \\ d_n^* \\ f_n \\ g_n^* \end{pmatrix}$$

where $J$ is the linear stability matrix of the form

$$J = \begin{pmatrix} \partial F_{a,i}/\partial \alpha_j & \partial F_{a,i}/\partial \alpha_j^* & \partial F_{a,i}/\partial \beta_j & \partial F_{a,i}/\partial \beta_j^* \\ -\partial F_{a,i}^*/\partial \alpha_j & -\partial F_{a,i}^*/\partial \alpha_j^* & -\partial F_{a,i}^*/\partial \beta_j & -\partial F_{a,i}^*/\partial \beta_j^* \\ \partial F_{b,i}/\partial \alpha_j & \partial F_{b,i}/\partial \alpha_j^* & \partial F_{b,i}/\partial \beta_j & \partial F_{b,i}/\partial \beta_j^* \\ -\partial F_{b,i}^*/\partial \alpha_j & -\partial F_{b,i}^*/\partial \alpha_j^* & -\partial F_{b,i}^*/\partial \beta_j & -\partial F_{b,i}^*/\partial \beta_j^* \end{pmatrix}$$

where

$$F_{a,i} = -k_\alpha a_n + c_a(\alpha_{n+1} + \alpha_{n-1}) + \lambda(\alpha_n \alpha_n^* + \beta_n \beta_n^*) \alpha_n + \gamma(\alpha_n^2 \alpha_n^2 + 2\alpha_\alpha_\alpha \alpha_n^\alpha_n^\beta_n \beta_n^\alpha_n \beta_n^\alpha_n^\beta_n^\beta_n^\beta_n) \alpha_n,$$

$$F_{b,i} = -k_\beta \beta_n + c_\beta(\beta_{n+1} + \beta_{n-1}) + \lambda(\beta_n \beta_n^* + \alpha_n \alpha_n^*) \beta_n + \gamma(\beta_n^2 \beta_n^2 + 2\alpha_\alpha_\beta \alpha_n \alpha_n^\beta_n \beta_n^\alpha_n \beta_n^\beta_n^\alpha_n \beta_n) \beta_n.$$

The instability of the solution is denoted by the presence of the imaginary part of the eigenvalues $\omega$.

To solve the ODE's in $z$ we employ the Runge-Kutta Order 4 method [5]. Here, we impose the free boundary condition. The initial conditions for the ODE's is taken from the solutions obtained from the Newton’s method with additional perturbation of $10^{-4}$. The conserved quantities of the problems, i.e. the Hamiltonian $H$ and the norm $I$ are monitored to precision $\sim 10^{-6}$ to ensure the accuracy of the computation.

In this paper, we explore the effect of weight contribution of cubic and quintic terms ($\lambda \neq 0$, $\gamma \neq 0$), only cubic ($\gamma = 0$), or only quintic ($\lambda = 0$) terms to overall SLM's, as has been considered in [3]. It is also interesting to consider the effect of difference in values of cubic, $\beta$ and quintic, $\alpha$ XPM coefficients.
4.1. Odd symmetric mode
For odd symmetric mode, we found that this mode is always stable (due to the absence of imaginary part of eigenvalues $\omega$). The stability of this mode is not factored either by the nonlinearity term or the XPM coefficients. Figure 4 shows the stability analysis and space-time evolution of an odd symmetric SLM for only quintic nonlinear terms (here, we set $\lambda = 0$, $\gamma = 1.0$) when $\alpha < \beta$. Furthermore, the simulations also show that the anti-continuum approximation for relatively large values of $c_a$ and $c_b$ is can be used to study the existence and stability of odd symmetric mode.

![Figure 4](image)

Figure 4: (a) Solution profiles and linear stability analysis; and (b) space-time evolution of an odd symmetric mode for $\lambda = \gamma = 1$, $c_a = c_b = 0.1$, $A = B = 1.0$, $\beta = 1.1$ and $\alpha = 1.0$.

4.2. Odd antisymmetric mode
The solutions in antisymmetric mode are found to be unstable for $\alpha < \beta$ when considering only cubic nonlinearity. In Figure 5, the presence of the imaginary part of the eigenvalue $\omega$ (Figure 5a) triggers the instability of the solution (Figure 5b). Similar result occurs in weighted distribution among the nonlinearities. From the simulations, it can be inferred that the absence of cubic term instigated the instability of the solution faster compared to the weighted distributed terms.

![Figure 5](image)

Figure 5: (a) Solution profiles and linear stability analysis; and (b) space-time evolution of an odd antisymmetric mode for $\lambda = 1.0$, $\gamma = 0$, $c_a = c_b = 0.1$, $A = B = 1.0$, $\beta = 1.1$ and $\alpha = 1.0$. 
Surprisingly, the solutions remain stable in the case of only quintic term (Figure 6). From the simulations also, since the difference between $\beta$ and $\alpha$ did not initiate the instability of the solution, these results supported the claim that the presence of cubic term triggers the instability of the solution for odd antisymmetric SLM.

Figure 6: (a) Solution profiles and linear stability analysis; and (b) space-time evolution of an odd antisymmetric mode for $\lambda = 0, \gamma = 1.0, c_a = c_b = 0.1, A = B = 1.0, \beta = 1.1$ and $\alpha = 1.0$.

4.3. Odd shifted mode

Figure 7 shows the dynamical evolution of Eqs. (1)-(2) for $\lambda = \beta = 1, A = 1.1, B = 1.0,$ and $c_a = c_b = 0.2$. For $\gamma = \alpha = 0$ Eqs. (1)-(2) has become a two-component cubic DNLS considered in [8]. The formation and stability of shifted SLM in this case is determined by $|A^2 - B^2|$. Simulations with different value of $A$ that the solutions are stable with very small oscillations of the peak intensities when $A \leq 0.9$, but unstable when $A \geq 1.1$ (Figure 7a).

Meanwhile, the dynamical evolution of Eqs. (1)-(2) for $\gamma = \alpha = 1$ is shown in Figure 7b. Compared to Figure 7a, the solutions are very stable. From expressions (13), in the case of equal coupling and nonlinear coefficients for both components, the shifted mode is formed and propagates stable if $|A^4 - B^4|$ exceeds some critical value.

Figure 7: Space-time evolution of odd shifted modes for $\lambda = \beta = 1, A = 1.1, B = 1.0,$ and $c_a = c_b = 0.2$. 
5. Conclusion
The conditions for the existence of bright odd SLM’s in coupled CQ DNLS equations are revealed. Approximate analytical expressions for different topologies of SLM’s are obtained. The stability of odd SLM’s in binary discrete media are analysed by the means of numerical computations and simulations. For the odd mode, the quintic nonlinearity found to be the key of the stability of antisymmetric and shifted odd modes. Conversely, the stability of symmetric odd mode does not determined by any nonlinearities.

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References
[1] Abdullaev F. Kh., Bouketir A, Messikh A and Umarov B A 2007 Physica D 232 54-61
[2] Aubry S 1997 Physica D 103 201-250
[3] Baizakov B B, Bouketir A, Messikh A and Umarov B A 2009 Phys. Rev. E 79 046605
[4] Brazhnyi V and Konotop V 2004 Mod. Phys. Lett. B 18 627-651
[5] Burden R L and Faires J D 2011 Numerical Analysis 9th Edition (USA: Brooks/Cole)
[6] Carretero-González R, Talley J D, Chong C and Malomed B A 2006 Physica D 216 77-89
[7] Christodoulides D N, Lederer F and Silberberg Y 2003 Nature 424 817-823
[8] Darmanyan S, Kobyakov A, Schmidt E and Lederer F 1998 Phys. Rev. E 57 3520-3530
[9] Desyatnikov A S, Mihalache D, Mazilu D, Malomed B A, Denz C and Lederer F 2005 Phys. Rev. E 71 026615
[10] Eisenberg H, Silberberg Y, Morandotti R, Boyd A R and Aitchison J S 1998 Phys. Rev. Lett. 81 3383-3386
[11] Flach S and Gorbach A 2008 Phys. Rep. 467 1-116
[12] Flach S and Wallis C R 1998 Phys. Rep. 195 181-264
[13] Kevrekidis P G 2009 The Discrete Nonlinear Schrödinger Equation: Mathematical Analysis, Numerical Computations and Physical Perspectives (Heidelberg: Springer-Verlag)
[14] Kevrekidis P, Frantzeskakis D and Carretero-González R 2008 Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment. (Heidelberg: Springer-Verlag)
[15] Kevrekidis P G, Kivshar Y S and Kovalev A S 2003 Phys. Rev. E 67 046604
[16] Kivshar Y S and Peyrard M 1992 Phys. Rev. A 46 3198-3205
[17] Kobyakov A, Darmanyan S, Lederer F and Schmidt E 1998 Opt. Quan. Elec. 30 795-808
[18] Lederer F, Stegeman G, Christodoulides D, Assanto G, Segev M and Silberberg Y 2008 Phys. Rep. 463 1-126
[19] Maimistov A, Malomed B and Desyatnikov A 1999 Phys. Lett. A 254 179-184
[20] Morsch O and Oberthaler M 2006 Rev. Mod. Phys. 78 179-215
[21] Zhang A X and Xue J K 2008 Phys. Lett. A 372 1147-1154