ESNault-Levine-Wittenberg Indices

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These are notes of a lecture about the papers [ELW12, Wit13] and related results of [Mer00, Ros01, CTM04, Zai10, Han13]. The papers do not define Esnault-Levine-Wittenberg indices, they focus on the two extreme cases $\text{elw}_0(X)$ (traditionally called the index) and $\text{elw}_{\dim X}(X)$. In retrospect, the method of [CTM04, Sec.5] is equivalent to the computation of $\text{elw}_1$ for Del Pezzo surfaces.

The basic properties are at least implicitly in the above papers, with the possible exception of the birational invariance (9) and the degree formula (11). However, the definition and systematic use of the ELW-indices streamlines several of the arguments.

It is interesting that the proof of the birational invariance does not rely on resolution. Instead, it uses what I call the Nishimura–Szabó lemmas (17–18).

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1. Definition and basic properties

Definition 1. Let $X$ be a proper scheme defined over a field $k$. For $0 \leq i \leq \dim X$ the Esnault-Levine-Wittenberg index $\text{elw}_i(X)$ is defined as

$$\text{elw}_i(X) := \langle \chi(X, F) : \dim F \leq i \rangle \subset \mathbb{Z},$$

where $F$ runs through all coherent sheaves of dimension $\dim F := \dim \text{Supp } F \leq i$ over $X$. I will think of $\text{elw}_i(X)$ as an ideal in $\mathbb{Z}$; it can be identified with its positive generator. It is convenient to set $\text{elw}_{-1}(X) = (0)$ and $\text{elw}(X) := \text{elw}_{\dim X}(X)$. It is clear that

$$\text{elw}_0(X) \subset \text{elw}_1(X) \subset \cdots \subset \text{elw}_{\dim X}(X).$$

If $X$ has a $k$-point $p \in X(k)$ then $\chi(X, k(p)) = 1$, hence $\text{elw}_0(X) = \text{elw}_1(X) = \cdots = \text{elw}_{\dim X}(X) = \mathbb{Z}$. Thus these notions are interesting only if $X$ does not have (or is not known to have) a $k$-point.

Usually $\text{ind}(X) := \text{elw}_0(X)$ is called the index of $X$. Its generator is the smallest positive degree of a (not necessarily effective) 0-cycle on $X$.

We see in (11) that if $X$ is integral then

$$\text{elw}(X) = \langle \text{elw}_{\dim X-1}(X), \chi(X, \mathcal{O}_X) \rangle.$$

This implies that if $\ell$ is a prime such that $\text{ord}_\ell \text{elw}(X) < \text{ord}_\ell \text{elw}_{\dim X-1}(X)$ then

$$\text{ord}_\ell \text{elw}(X) = \text{ord}_\ell \chi(X, \mathcal{O}_X).$$

This is why the results of [ELW12] involving $\chi(X, \mathcal{O}_X)$ are equivalent to statements about $\text{elw}(X)$. 

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Remark 2. The above definition makes sense for a proper scheme $X$ defined over a local Artin ring $A$. As a consequence of (4.1) we see that $\text{elw}_i(X) = \text{elw}_i(\text{red } X)$ and red $X$ can be viewed as a scheme over the residue field of $A$. Once the basic results are established over Artin rings, we concentrate on schemes over fields afterwards.

The following is useful in computations.

Lemma 3. Let $X$ be a proper scheme over a local Artin ring $A$ and $F$ a coherent sheaf on $X$. Let $Z_i \subset \text{Supp } F$ be the maximal dimensional irreducible components with generic points $\eta_i$. Then

$$
\chi(X, F) - \sum_i \text{length}_{\eta_i} F \cdot \chi(Z_i, \mathcal{O}_{Z_i}) \in \text{elw}_{\dim F - 1}(X). \tag{3.1}
$$

Proof. The $K$-group of coherent sheaves of dimension $\leq r$ is generated by the $\mathcal{O}_Z$ where $Z$ runs through all closed, integral subschemes of dimension $\leq r$. On this group

$$
F \mapsto \chi(X, F) - \sum_i \text{length}_{\eta_i}(F) \cdot \chi(Z_i, \mathcal{O}_{Z_i}) \in \mathbb{Z} / \text{elw}_{r-1}(X)
$$

is linear. Thus it is enough to check that it vanishes on the generators. If $\dim Z = r$ then

$$
\sum_i \text{length}_{\eta_i}(\mathcal{O}_Z) \cdot \chi(Z_i, \mathcal{O}_{Z_i}) = \chi(X, \mathcal{O}_Z).
$$

If $\dim Z < r$ then $\sum_i \text{length}_{\eta_i}(F) \cdot \chi(Z_i, \mathcal{O}_{Z_i}) = 0$ and $\chi(X, F) \in \text{elw}_{r-1}(X)$. \hfill \Box

Proposition 4 (Dévissage). Let $X$ be a proper scheme defined over a local Artin ring $A$. The ELW-indices can also be computed the following ways.

1. $\text{elw}_i(X) = (\chi(Z, \mathcal{O}_Z))$ where $Z$ runs through all integral subvarieties of dimension $\leq i$.
2. $\text{elw}_i(X) = (\chi(\bar{Z}, \mathcal{O}_Z))$ where $Z$ runs through all integral subvarieties of dimension $\leq i$ and $\bar{Z} \to Z$ denotes the normalization.
3. $\text{elw}_i(X) = (\chi(Z', \mathcal{O}_{Z'}))$ where $Z$ runs through all integral subvarieties of dimension $\leq i$ and $Z' \to Z$ is any proper birational morphism.
4. If $X$ is regular then $\text{elw}_i(X) = (\chi(X, E))$ where $E$ runs through all locally free sheaves on $X$.

Proof. The $K$-group of coherent sheaves of dimension $\leq i$ is generated by the $\mathcal{O}_Z$ where $Z$ runs through all integral subvarieties of dimension $\leq i$. This implies (1). If $X$ is regular then locally free sheaves also generate the $K$-group, giving (4). Finally (2–3) follow from this, (5.3) and induction on $i$. \hfill \Box

Lemma 5. Let $f : X \to Y$ be a morphism of proper $k$-schemes and $F$ a coherent sheaf on $X$. Then

1. $\chi(X, F) \in \text{elw}_r(Y)$ where $r = \dim f(\text{Supp } F)$.
2. If $Y$ is integral and $f$ is generically finite then $\chi(X, \mathcal{O}_X) - \deg(X/Y) \cdot \chi(Y, \mathcal{O}_Y) \in \text{elw}_{\dim Y - 1}(Y)$.
3. If $f$ is birational then $\chi(X, \mathcal{O}_X) - \chi(Y, \mathcal{O}_Y) \in \text{elw}_{\dim Y - 1}(Y)$.
4. If $X, Y$ are normal and $f$ is birational then $\chi(X, \mathcal{O}_X) - \chi(Y, \mathcal{O}_Y) \in \text{elw}_{\dim Y - 2}(Y)$.

Proof. By the Leray spectral sequence $\chi(X, F) = \sum (-1)^i \chi(Y, R^i f_* F)$ and the latter is in $\text{elw}_r(Y)$ where $r = \dim f(\text{Supp } F)$, showing (1).

If $X, Y$ are normal and $f$ is birational then $\dim \text{Supp } R^i f_* \mathcal{O}_X \leq \dim Y - 2$ for $i > 0$, thus $\sum_{i \geq 1} (-1)^i \chi(Y, R^i f_* \mathcal{O}_X) \in \text{elw}_{\dim Y - 2}(Y)$, giving (4).
If $Y$ is integral and $f$ is generically finite then the generic rank of $f_*\mathcal{O}_X$ equals the degree of $f$, thus (2) follows from (3) and (3) is a special case. \hfill $\square$

Applying (5.1) to all subvarieties we get the following.

Corollary 6. Let $f : X \to Y$ be a morphism of proper $k$-schemes. Then $\text{elw}_i(X) \subset \text{elw}_i(Y)$ for every $i$. \hfill $\square$

Corollary 7. Let $f : X \to Y$ be a morphism of proper $k$-schemes and $W \subset Y$ a closed subscheme such that $f$ is an isomorphism over $Y \setminus W$. Then

$$
\text{elw}_i(Y) = \text{elw}_i(X) + \text{elw}_i(W).
$$

Proof. It is clear that $\text{elw}_i(Y) \supset \text{elw}_i(X) + \text{elw}_i(W)$. Conversely, let $Z \subset Y$ be an integral subvariety. If $Z \subset W$ then set $Z' = Z$ and note that $\chi(Z', \mathcal{O}_{Z'}) \in \text{elw}_i(W)$.

If $Z \not\subset W$ then let $Z'$ be the birational transform of $Z$ in $X$. Thus $\chi(Z', \mathcal{O}_{Z'}) \in \text{elw}_i(X)$. Using (3) we see that $\text{elw}_i(Y) \subset \text{elw}_i(X) + \text{elw}_i(W)$. \hfill $\square$

A more interesting variant of (6) is the following.

Proposition 8. Let $f : X \dashrightarrow Y$ be a rational map of proper $k$-schemes. Assume that $X$ is regular. Then $\text{elw}_i(X) \subset \text{elw}_i(Y)$ for every $i$.

Proof. By induction on $i$. Let $Z \subset X$ be an integral subvariety of dimension $i$. By (7) we have a birational morphism $Z' \to Z$ and a morphism $Z' \to Y$. Thus $\chi(Z', \mathcal{O}_{Z'}) \in \text{elw}_i(Y)$ and $\chi(Z', \mathcal{O}_{Z'}) - \chi(Z, \mathcal{O}_Z) \in \text{elw}_{i-1}(X) \subset \text{elw}_{i-1}(Y)$ by induction. Thus $\chi(Z, \mathcal{O}_Z) \in \text{elw}_i(Y)$. \hfill $\square$

Corollary 9. For every $i$, the ELW-index $X \to \text{elw}_i(X)$ is a birational invariant of proper, regular $k$-schemes.

An immediate consequence of (5.1) and (8) is the following quite useful result.

Corollary 10. Let $X$ be a proper, regular $k$-scheme and $g : X \dashrightarrow Y$ a map. If $\chi(X, \mathcal{O}_X) \notin \text{elw}_r(Y)$ then $\dim g(X) > r$. \hfill $\square$

For $i = \dim X$, the following degree formula is in Mer00, Zai10, Hau13. (See [5] for its precise relationship to the versions given in Zai10, Hau13.)

Proposition 11. Let $f : X \dashrightarrow Y$ be a generically finite, rational map of proper $k$-schemes of the same dimension. Assume that $Y$ is integral and regular. Then

$$
\deg(X/Y) \cdot \text{elw}_i(Y) \subset \text{elw}_i(X) + \text{elw}_{i-1}(Y).
$$

Proof. Let $Z \subset Y$ be an integral subvariety of dimension $i$. By (18) we have a generically finite morphism $Z' \to Z$ of degree $d$ and a morphism $Z' \to X$. Thus $\chi(Z', \mathcal{O}_{Z'}) \in \text{elw}_i(X)$ by (5.1) and $\chi(Z', \mathcal{O}_{Z'}) - \deg(X/Y) \cdot \chi(Z, \mathcal{O}_Z) \in \text{elw}_{i-1}(Y) \subset \text{elw}_{i-1}(Y)$ by (5.2). Thus $\deg(X/Y) \cdot \chi(Z, \mathcal{O}_Z) \in \text{elw}_i(X) + \text{elw}_{i-1}(Y)$. \hfill $\square$

Corollary 12. Notation and assumptions as in (11). Fix a prime $\ell$ such that $\text{ord}_\ell \text{elw}_{i-1}(Y) > \text{ord}_\ell \text{elw}_i(Y)$ and $\ell \nmid \deg(X/Y)$.

Then $\text{ord}_\ell \text{elw}_i(X) = \text{ord}_\ell \text{elw}_i(Y)$. \hfill $\square$
Cycle class map.

The ELW-indices determine how far the Euler characteristic is from being linear on the Chow groups.

**Definition 13.** For an algebraic cycle \( Z = \sum_i a_i Z_i \subset X \) set
\[
\chi(Z) := \sum_i a_i \chi(Z_i, \mathcal{O}_{Z_i}) \quad \text{and} \quad \chi(\bar{Z}) := \sum_i a_i \chi(\bar{Z_i}, \mathcal{O}_{\bar{Z}_i}),
\]
where \( \bar{Z}_i \to Z_i \) denotes the normalization.

If \( W \subset X \) is an integral subvariety of dimension \( r \) then \( \chi(W, \mathcal{O}_W) - \chi(W, \mathcal{O}_W) \in \mathrm{elw}_{r-1}(X) \) by \([3]\).

Thus
\[
\chi(\bar{Z}) - \chi(Z) \in \mathrm{elw}_{\dim Z-1}(X).
\]

**Proposition 14.** Let \( B_r(X) \) denote the group of \( r \) dimensional cycles in \( X \) modulo algebraic equivalence. Then \( Z \mapsto \chi(Z) \) and \( Z \mapsto \chi(\bar{Z}) \) define the same well-defined linear map
\[
\chi : B_r(X) \to \mathrm{elw}_r(X)/\mathrm{elw}_{r-1}(X).
\]

**Proof.** Two \( r \)-cycles \( Z^1, Z^2 \) are algebraically equivalent if there are
1. an irreducible, nonsingular curve \( C \) with two points \( p_1, p_2 \in C(k) \),
2. a flat and proper morphism \( g : W \to C \),
3. a morphism \( \pi : W \to X \) and
4. an effective \( r \)-cycle \( Z^r \) such that \( \pi_* [g^{-1}(p_i)] = Z^i + Z^r \) for \( i = 1, 2 \).

Set \( W^i := g^{-1}(p_i) \) and let \( W^i \subset W \) be the irreducible components with multiplicities \( m_j^i \). By \([31]\)
\[
\chi(W^i, \mathcal{O}_{W^i}) - \sum_j m_j^i \cdot \chi(W_{j}^i, \mathcal{O}_{W_{j}^i}) \in \mathrm{elw}_{r-1}(W)
\]
for \( i = 1, 2 \). Furthermore, by \([32]\) \( \chi(\pi_* [W^i_j]) - \chi(W^i_{j}) \in \mathrm{elw}_{r-1}(X) \). Therefore
\[
\chi(Z^i) + \chi(Z^r) - \sum_j m_j^i \cdot \chi(W_{j}^i, \mathcal{O}_{W_{j}^i}) \in \mathrm{elw}_{r-1}(X).
\]
Finally \( \chi(W^i, \mathcal{O}_{W^i}) = \chi(W^2, \mathcal{O}_{W^2}) \) since \( g \) is flat. \( \square \)

**Example 15.** Let \( Q^n \subset \mathbb{P}^{n+1} \) be the (empty) quadric \((x_0^2 + \cdots + x_n^2 = 0)\) over \( \mathbb{R} \). Note that the quadric \( Q^2 \) is isomorphic to the product \( Q^1 \times Q^1 \). Let \( C_2, C_4 \subset Q^1 \times Q^1 \) be rational curves of bidegrees \((1,1)\) (resp. \((1,3)\)).

Both of these can be viewed as curves in \( Q^3 \) since \( Q^2 \subset Q^3 \). The cycles \( C_4 \) and \( 2 \cdot C_2 \) have the same degree, hence they are algebraically equivalent over \( \mathbb{C} \).

However, they are not algebraically equivalent over \( \mathbb{R} \) since \( \chi(C_4) = 1 \) but \( 2 \chi(C_2) = 2 \).

**Remark 16 (ML).** This shows that \( F \mapsto \chi(X, F) \in \mathbb{Z}/\mathrm{elw}_{\dim F-1}(X) \) can be viewed as the image of the push-forward map from the connective algebraic K-theory of \( X \) to the base-field. This is given by taking \( CK^q(X) \) to be the image of \( K_0(M^q(X)) \) in \( K_0(M^{q-1}(X)) \), where \( M^q(X) \) is the category of coherent sheaves on \( X \) with support in codim at least \( q \). This theory was first defined by \([\text{Ca08}]\). \([\text{DL12}]\) shows that \( CK^* \) is the universal theory with formal group law \( u + v - buv \in \mathbb{Z}[b][[u, v]] \).
2. The Nishimura–Szabó lemmas on rational correspondences

[Nis55] proved that if a regular $k$-scheme has a $k$-point then any proper $k$-scheme birational to it also has a $k$-point. Around 1992 Endre Szabó found a new short argument. (The proof is reproduced in [KS00, Prop.A.6] and [KSC04, p.183].) Another application of the method was also used in [KS00]; its generalization [18] is needed to prove the degree formula [11].

Lemma 17. Let $f : X \rightarrow Y$ be a rational map of proper $k$-schemes. Let $Z \subset X$ be a closed, integral subscheme that is not contained in Sing $X$. Then there is a birational morphism $Z' \rightarrow Z$ such that there exists a morphism $f_Z' : Z' \rightarrow Y$.

We make no claim about $f_Z'$ beyond its existence. In particular, it can be a constant map. Thus (11) is interesting only if $Y$ is not known to have a $k$-point.

Proof. By induction on dim $X$. If dim $X \leq$ dim $Z + 1$ then $f$ is defined at the generic point of $Z$, hence we can take $Z'$ to be the closure of the graph of $f|_Z$.

If dim $X >$ dim $Z + 1$, take the blow-up $B_Z X \rightarrow X$ and let $E_Z \subset B_Z X$ be the unique irreducible component of the exceptional divisor that dominates $Z$. Note that $E_Z \rightarrow Z$ is generically a projective space bundle, hence it has a rational section $Z_1 \subset E_Z$ that maps birationally to $Z$. Note that $f$ gives a rational map $f_E : E_Z \rightarrow Y$. Induction gives a birational morphism $Z_1' \rightarrow Z_1 \rightarrow Z$ and a morphism $Z_1' \rightarrow Y$.

The following is a variant of [KS00].

Lemma 18. Let $f : X \rightarrow Y$ be a generically finite, rational map of proper $k$-schemes of the same dimension. Assume that $Y$ is integral. Let $Z \subset Y$ be a closed, integral subscheme that is not contained in Sing $Y$. Then there are

1. a reduced, proper $k$-scheme $Z'$,
2. a generically finite morphism $Z' \rightarrow Z$ such that $\deg(Z'/Z) = \deg(X/Y)$

and

3. a morphism $Z' \rightarrow X$.

Proof. By induction on dim $Y$.

Replacing $X$ by the normalization of the closure of the graph of $f$, we may assume that $X$ is normal and $f : X \rightarrow Y$ is a morphism.

If dim $Y \leq$ dim $Z + 1$ then $f$ is finite over the generic point of $Z$. Let $Z_i \subset X$ be the irreducible components of $f^{-1}(Z)$ that dominate $Z$ and $e_i$ the ramification index of $f$ along $Z_i$. Then $\deg(X/Y) = \sum e_i \deg(Z_i/Z)$, thus we can take $Z'$ to be the disjoint union of $e_i$ copies of $Z_i$ for every $i$.

If dim $Y >$ dim $Z + 1$, take the blow-up $B_Z Y \rightarrow Y$ and let $E_Z \subset B_Z Y$ be the unique irreducible component of the exceptional divisor that dominates $Z$. Note that $E_Z \rightarrow Z$ is generically a projective space bundle, hence it has a rational section $Z_1 \subset E_Z$ that maps birationally to $Z$.

Replace $X$ by the the normalization of the closure of the graph of $X \rightarrow B_Z Y$. Let $F_i \subset X$ be the irreducible components of $f^{-1}(E_Z)$ that dominate $E_Z$ and $e_i$ the ramification index of $f$ along $F_i$. By induction, there are $Z_{1i}' \rightarrow Z_1$ and morphisms $Z_{1i}' \rightarrow F_i \rightarrow X$. Thus we can take $Z'$ to be the disjoint union of $e_i$ copies of $Z_{1i}'$ for every $i$. □
Remark 19. The computations of [Wit13, Lem.5.3] show that the above results also hold if the ambient variety has quotient singularities at the generic point of $Z$ and $k(Z)$ is perfect. (For $\dim Z > 0$ this restricts us to characteristic 0.)

Here quotient singularity is understood in the strong sense: it should be Zariski locally a quotient of a regular scheme. The analogous assertion is not true for singularities that are quotients only étale locally. For instance, take $X = (x^2 + y^2 + z^2 = 0) \subset \mathbb{A}^3_k$ and $Z = (0, 0, 0)$. After blowing up the origin, we get a surface with no real points. On the other hand, over $\mathbb{C}$ the singularity is isomorphic to $\mathbb{C}^2/(u, v) \sim (-u, -v)$.

3. Examples

Proposition [14] makes it relatively easy to compute the ELW-indices when generators of the groups $B_r(X)$ are known.

Example 20 (ML). Let $p$ be a prime and $X$ a nontrivial Severi–Brauer variety of dimension $p - 1$. Then

$$\text{elw}_0(X) = \cdots = \text{elw}_{p-2}(X) = p \quad \text{and} \quad \text{elw}_{p-1}(X) = 1.$$  

To see this note that a Severi–Brauer variety of dimension $n - 1$ has an (effective) 0-cycle of degree $n$ and it has a 0-cycle of degree 1 iff it is trivial. Thus $\text{elw}_0(X) = p$. At the other end, $\chi(X, \mathcal{O}_X) = 1$ shows that $\text{elw}_{p-1}(X) = 1$. Thus the only question is when $\text{elw}_i$ drops from $p$ to 1. Assume that $\text{elw}_{i-1}(X) = p$.

Let $K/k$ be a splitting field of degree $p$ and $L_K \subset X_K$ a linear subspace of dimension $i$. Let $L_1, \ldots, L_p$ be its conjugates over $k$. Then $Z := L_1 + \cdots + L_p$ is defined over $k$ and it generates the Chow group $A_i(X)$ if $i < \dim X$. $Z$ could be singular, but its normalization $\bar{Z}$ has Euler characteristic $\chi(\bar{Z}, \mathcal{O}_{\bar{Z}}) = p\chi(L_K, \mathcal{O}_{L_K}) = p$.

By [14], $Z \mapsto \chi(\bar{Z}, \mathcal{O}_{\bar{Z}}) \in \mathbb{Z}/p$ is a well defined linear map on $A_i(X)$ that vanishes on the generator of $A_i(X)$. Thus $\text{elw}_i(X) = p$.

(OW) notes that this is also a direct consequence of [34].

Next we compute the ELW-indices for certain products of general curves.

Proposition 21. Let $k$ be a field and $C_1, \ldots, C_n$ smooth, irreducible, projective curves over $k$. Assume that

1. $\text{elw}_0(C_i) \subset (m)$ for every $i$ for some $m \geq 1$ and
2. $\text{Pic}(C_1 \times \cdots \times C_n) = \pi_1^* \text{Pic}(C_1) + \cdots + \pi_n^* \text{Pic}(C_n)$ where $\pi_i$ denotes the $i$-th coordinate projection.

Then $\text{elw}_i(C_1 \times \cdots \times C_n) \subset (m)$ for $i < n$.

Proof. Set $X_r := C_1 \times \cdots \times C_r$. The proof is by induction on $r$.

Let $F$ be a coherent sheaf on $X_n$. Consider the coordinate projection $\Pi_n : X_n \to X_{n-1}$. If $\dim F \leq n - 2$ then $\dim \Pi_n(\text{Supp } F) \leq n - 2$, hence, by (5.1) and induction, $\chi(X_n, F) \in \text{elw}_{n-2}(X_{n-1}) \subset (m)$.

We are left with the case when $\dim F = n - 1$. By (3.1) it is enough to show that $\chi(D, \mathcal{O}_D) \in (m)$ for every effective divisor $D \subset X_n$. Using the exact sequence

$$0 \to \mathcal{O}_{X_n}(-D) \to \mathcal{O}_{X_n} \to \mathcal{O}_D \to 0$$

we are reduced to proving that

$$\chi(X_n, L) \equiv \chi(X_n, \mathcal{O}_{X_n}) \mod m$$
for every line bundle $L$ on $X_n$. By assumption (2), there are line bundles $L_i$ on $C_i$ such that $L \cong \otimes_i \pi_i^* L_i$. Therefore
\[
\chi(X_n, L) = \prod_i \chi(C_i, L_i) = \prod_i (\chi(C_i, \mathcal{O}_{C_i}) + \deg L_i).
\]
By assumption (1), $\deg L_i \in (m)$ for every $i$, thus
\[
\chi(X_n, L) \equiv \prod_i \chi(C_i, \mathcal{O}_{C_i}) = \chi(X_n, \mathcal{O}_{X_n}) \mod m. \quad \Box
\]

In applications the tricky part is to check the condition (21.2). Let us start over algebraically closed fields.

**Lemma 22.** Let $Y_i$ be normal, irreducible, proper varieties over an algebraically closed field $k$. The following are equivalent.

1. $\mathrm{Pic}(Y_1 \times \cdots \times Y_n) = \pi_1^* \mathrm{Pic}(Y_1) + \cdots + \pi_n^* \mathrm{Pic}(Y_n)$ where $\pi_i$ denotes the $i$th coordinate projection.
2. For $i \neq j$, every morphism $Y_i \to \mathrm{Pic}(Y_j)$ is constant.
3. For $i \neq j$, every morphism $\mathrm{Pic}^0(Y_i) \to \mathrm{Pic}^0(Y_j)$ is constant. \quad \Box

The above conditions hold if the $\mathrm{Pic}^0(Y_i)$ are sufficiently general and independent, but they may be hard to check in concrete situations. Consider the case when $Y_i = C_i$ are smooth curves. For very general curves $\mathrm{Pic}^0(C_i)$ is a simple Abelian variety; see [Mor76, Mor77, Zar00, Zar04] for explicit examples over $\mathbb{Q}$. Hence if, in addition, the $C_i$ all have different genera, then (22.3) holds. The same for sufficiently general and independent curves of the same genus $> 0$.

Over arbitrary fields, an extra complication comes from the Brauer group.

**23 (Map to the Brauer group).** Let $Y$ be normal, irreducible, proper variety over a perfect field $k$ with algebraic closure $\bar{k}$. Let $L$ be a line bundle on $Y_\bar{k}$ that is isomorphic to its Galois conjugates. Equivalently, $[L] \in \mathrm{Pic}(Y_\bar{k})(k)$. If the linear system $|L|$ is nonempty, then, as a subscheme of $\mathrm{Hilb}(Y)$; it is defined over $k$ and is isomorphic to a projective space over $\bar{k}$. Thus $[L]$ defines an element of the Brauer group $\mathrm{Br}(k)$. This gives an exact sequence
\[
\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_\bar{k})(k) \to \mathrm{Br}(k).
\]
(See [BLR90, Chap.8] for a more conceptual construction.) It is clear that if $[L_i] \in \mathrm{Pic}((Y_i)_\bar{k})(k)$ then
\[
\mathrm{br}_{\prod Y_i} (\otimes_i \pi_i^* L_i) = \sum_i \mathrm{br}_{Y_i}(L_i).
\]

**Lemma 24.** Let $Y_1, \ldots, Y_n$ be normal, irreducible, proper varieties over a perfect field $k$. Assume that they satisfy the equivalent conditions (22.1–3) over $\bar{k}$. The following are equivalent.

1. $\mathrm{Pic}(Y_1 \times \cdots \times Y_n) = \pi_1^* \mathrm{Pic}(Y_1) + \cdots + \pi_n^* \mathrm{Pic}(Y_n)$.
2. The subgroups $\mathrm{im} \left[ \mathrm{br}_{Y_i} : \mathrm{Pic}(Y_i)_\bar{k}(k) \to \mathrm{Br}(k) \right]$ are independent in $\mathrm{Br}(k)$.
3. (Subgroups $A_i$ of an Abelian group $A$ are independent if $a_i \in A_i$ and $\sum a_i = 0$ implies that $a_i = 0$ for every $i$.) \quad \Box

**Example 25 (Products of conics).** Let $C_i \subset \mathbb{P}^2$ be plane conics over a perfect field $k$. Set $Y_n = C_1 \times \cdots \times C_n$. The image of $\mathrm{br}_{C_i}$ equals the subgroup generated by $C_i$ in $\mathrm{Br}(k)$. Thus we see that the following are equivalent.

1. $\mathrm{ew}_{n-1}(Y_n) \subset (2)$ and
2. the classes $[C_i] \in \mathrm{Br}(k)$ are nonzero and independent.
If we are over \( \mathbb{Q} \) then any finite collection of conics has a point in a quadratic extension. Thus we obtain the following.

Set \( C_i := (x^2 + y^2 = p_i z^2) \) where the \( p_i \) are distinct primes congruent to 3 modulo 4 and \( Y_n := C_1 \times \cdots \times C_n. \) Then

\[
\text{elw}_0(Y_n) = \cdots = \text{elw}_{n-1}(Y_n) = (2) \quad \text{and} \quad \text{elw}_n(Y_n) = (1).
\]

**Example 26** (Products of hyperelliptic curves). We work over the field \( k = \mathbb{C}(t) \).

A hyperelliptic curve over \( k \) is given by an equation

\[
C := (z^2 = f(x, y)) \subset \mathbb{P}^2(1, 1, m)
\]

where \( f(x, y) \in \mathbb{C}(t)[x, y] \) is homogeneous of degree \( 2m \). We will look at it as a cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) where \( f^2 \) (with coordinates \( t, x/y \)) is an affine chart. Thus we have a double cover \( \pi : S \to \mathbb{P}^1 \times \mathbb{P}^1 \) with branch curve \( B \subset \mathbb{P}^1 \times \mathbb{P}^1 \). For very general \( S \),

\[
\pi^* \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \to \text{Pic}(S) \quad \text{is an isomorphism;}
\]

see [Bui83, RS09]. This implies that, for very general \( f \), \( \text{Pic}(C) = \mathbb{Z}[\pi^*O_{\mathbb{P}^1}(1)] \). In particular, if \( m \) is odd then \( \text{elw}_0(C) = 2 \) and \( \text{elw}_0(C) = 1 \).

For stronger examples over \( \mathbb{Q} \), see [Zar06, Zar10].

Note that \( \text{Pic}(C) \) can be viewed as the generic fiber of the family of the Picard varieties of the fibers of \( p_1 \circ \pi : S \to \mathbb{P}^1 \) where \( p_1 \) is the first coordinate projection of \( \mathbb{P}^1 \times \mathbb{P}^1 \). The singular fibers correspond to the branch points of \( p_1 : B \to \mathbb{P}^1 \).

Let us now choose our curves \( C_i \) such that the branch points of the corresponding \( p_1 : B_i \to \mathbb{P}^1 \) are all different. Then the different \( \text{Pic}^0(C_i) \) satisfy the condition (223). The Brauer group of \( \mathbb{C}(t) \) is trivial. Thus if \( Y_n := C_1 \times \cdots \times C_n \) is the product of such generic curves then

\[
\text{elw}_0(Y_n) = \cdots = \text{elw}_{n-1}(Y_n) = (2) \quad \text{and} \quad \text{elw}_n(Y_n) = (1).
\]

**Example 27** (Products of real curves). Let \( C \) be a geometrically irreducible smooth real curve. The interesting case is when \( C \) has no real points and even genus. Equivalently, when \( \text{elw}_0(C) = (2) \) and \( \text{elw}_1(C) = (1) \). Note that \( \text{Br}(\mathbb{R}) = \mathbb{Z}/2 \) and the Brauer map \( \text{br}_C : \text{Pic}(C)_{\mathbb{R}} \to \text{Br}(\mathbb{R}) = \mathbb{Z}/2 \) is surjective, essentially by a result that goes back to [Wit34] (see also [DK76]). Thus we see that for any product \( Y_n \) of such curves we have

\[
\text{elw}_0(Y_n) = (2) \quad \text{and} \quad \text{elw}_1(Y_n) = \cdots = \text{elw}_n(Y_n) = (1).
\]

4. **The sequence of ELW-indices**

There are several general questions about the ELW-indices that should be explored. Esnault asks about all possible sequences \( \text{elw}_0(X), \ldots, \text{elw}_{\dim X}(X) \).

**Definition 28.** Let \( \mu(Td_n) \) denote the denominator appearing in the Todd class in dimension \( n \). By [Hir50, 1.7.3], it is

\[
\mu(Td_n) = \prod_p p^{n/[p - 1]} \quad \text{where} \quad p \leq n + 1 \quad \text{is a prime.}
\]

(25.1)

The sequence starts as \( \mu(Td_1) = 2, \mu(Td_2) = 12, \mu(Td_3) = 24, \mu(Td_4) = 720. \)

Note that \( \mu(Td_n) \) is very close to \( n! \). Indeed, we can rewrite the formula as

\[
\mu(Td_n) = \prod_p p^{[n/p] + [n/p^2] + \cdots} \quad \text{while} \quad n! = \prod_p p^{[n/p] + [n/p^2] + \cdots}.
\]

Each \( \mu(Td_n) \) divides any later \( \mu(Td_m) \). There is also the more delicate relation

\[
n! \cdot \mu(Td_m) \div \mu(Td_{n+m-1}).
\]

(25.2)
Without much evidence, let me propose the following.

**Problem 29.** Let $e_0, \ldots, e_n$ be a sequence of natural numbers. Then there is a field $k$ and a $k$-scheme (or smooth $k$-variety) $X$ of dimension $n$ such that $elw_r(X) = (e_r)$ for every $r$ if the following hold.

1. $e_{r+1}|e_r$ for every $r$ and
2. $e_0|\mu(Td_r) \cdot e_r$ for every $r$.

Next we discuss some evidence supporting the above formulation of (29). The necessity of $e_{r+1}|e_r$ follows from (12). The divisibility $e_0|\mu(Td_r) \cdot e_r$ is more subtle and I do not know a complete proof.

**Lemma 30.** Let $X$ be a proper $k$-scheme over a field of characteristic 0. Then

$$\mu(\text{Td}_r) \cdot elw_r(X) \subset elw_0(X). \quad (30.1)$$

Proof. If $W$ is smooth, proper and of dimension $r$ then Riemann–Roch says that $\mu(\text{Td}_r) \cdot \chi(W, \mathcal{O}_W)$ is a polynomial in the Chern classes. In particular, $W$ has a 0-cycle of degree $\mu(\text{Td}_r) \cdot \chi(W, \mathcal{O}_W)$.

By (3), $elw_r(X)$ is generated by the $\chi(Z', \mathcal{O}_{Z'})$ where $Z$ runs through all integral subvarieties of dimension $\leq r$ and $Z' \rightarrow Z$ is any resolution. □

The only missing ingredient in positive characteristic is resolution of singularities. The method of [ELW12] shows that one can use de Jong’s alterations [dJ96, dJ97] (in the stronger form proved by Gabber, cf. [IL012, Exp.IX]) to prove that (30.1) holds in $\mathbb{Z}[\rho^{-1}]$.

A more interesting part of (29) is the claim that there are no additional relations between the $elw_i$. I do not even have a plausible argument, but the following example suggests that, for $n = \dim X$, there should not be any relations between $elw_n(X)$ and $elw_{n-1}(X)$ in general.

**Example 31.** Let $X$ be a smooth projective variety of dimension $n$ such that $\text{Pic}(X) = \mathbb{Z}[H]$ and the intersection number $(C \cdot H)$ is divisible by $m \cdot \mu(\text{Td}_{n-1})$ for some fixed number $m$ for every curve $C \subset X$. Such examples are K3 surfaces (where $m = (H^2)/2$ can take any value) or hypersurfaces of very high degree in $\mathbb{P}^4$ [Ko92].

By Riemann–Roch, $\chi(X, L) - \chi(X, \mathcal{O}_X)$ is divisible by $m$ for any line bundle $L$, thus $m | \chi(D, \mathcal{O}_D)$ for every divisor $D \subset X$. Thus (32) implies that

$$elw_{n-1}(X) \subset (elw_{n-2}(X), m).$$

Although I do not know if there are further relations between $elw_n(X)$ and $elw_{n-2}(X)$, the above computations suggest that once $elw_n(X)$ and $elw_{n-2}(X)$ are set, $elw_{n-1}(X)$ could be any ideal satisfying $elw_n(X) \supset elw_{n-1}(X) \supset elw_{n-2}(X)$.

An extreme case of (29) would be the following.

**Problem 32.** Find $n$-dimensional smooth, projective varieties such that $elw(X) = 1$ yet $elw_i(X) = (\mu(\text{Td}_i))$ for $0 \leq i < n$.

The higher dimensional examples of [Ko92] are not convincing for the current purposes, but the following should be possible to prove.

**Problem 33.** Given $e_0, e_1, e_2$, there is K3 surface $S$ over any field (or even over $\mathbb{Q}$) such that $elw_r(S) = (e_r)$ for $r = 0, 1, 2$ if
Another case when (30.1) is known connects our index formula (11) with the version in [Hau13]. The proof given in [Hau13] relies on [Mer02]; the claim also follows from properties of the numbers $\mu(Td_r)$.

Lemma 34. Let $X$ be a proper, nonsingular variety of dimension $n$. Then

$$\mu(Td_{n-1}) \cdot \text{elw}_{n-1}(X) \subset \text{elw}_0(X).$$

Proof. Let $F$ be a coherent sheaf of dimension $\leq n-1$ on $X$. Since $X$ is nonsingular, there are two vector bundles $E_1, E_2$ such that $[F] = [E_1] - [E_2]$ in $K_0(X)$. Thus, by Riemann–Roch,

$$\chi(X, F) = \int_X \text{ch}(E_1) \cdot \text{td}(T_X) - \int_X \text{ch}(E_2) \cdot \text{td}(T_X).$$

Since rank $E_1 = \text{rank } E_2$, the terms rank $E_1 \cdot \text{td}_n(T_X)$ and rank $E_2 \cdot \text{td}_n(T_X)$ cancel each other.

All other terms have denominators dividing $(n-r-1)! \cdot \mu(Td_r)$ for some $r \leq n-1$. By (28.2) these all divide $\mu(Td_{n-1})$. □

5. ELW-indices for special fields

We consider various fields for which the sequences $\text{elw}_0(X), \ldots, \text{elw}_{\dim X}(X)$ are special for all schemes, or at least for certain interesting classes of varieties.

Finite fields.

The following is proved in [Wit13].

Proposition 35. Let $X$ be a proper scheme over a finite field $k$. Then

$$\text{elw}_0(X) = \cdots = \text{elw}_{\dim X}(X) = (h^0(Z, O_Z) : Z \subset X \text{ and } Z \text{ is integral}).$$

The proof is a combination of two lemmas.

Lemma 36. Let $X$ be a proper $k$-scheme. Then

$$\text{elw}(X) \subset (h^0(Z, O_Z) : Z \subset X \text{ and } Z \text{ is integral}),$$

where $Z$ runs through all closed, integral subschemes of $X$ and $\bar{Z} \rightarrow Z$ denotes the normalization.

Proof. Let $W$ be a proper, integral scheme and $F$ a coherent sheaf over $W$. Then $H^i(W, F)$ is a vector space over the field $H^0(W, O_W)$, hence its dimension is divisible by $h^0(W, O_W)$. Thus $\chi(W, F)$ is divisible by $h^0(W, O_W)$. Applying this to $W = \bar{Z}$ we get that each $\chi(\bar{Z}, O_Z)$ is contained in $(h^0(Z, O_Z) : Z \subset X)$. By (31.2) this implies our claim. □

Lemma 37. Let $W$ be a proper, normal, integral variety over a finite field $k$. Then $\text{elw}_0(W) = h^0(W, O_W)$.

Proof. Assume first that $h^0(W, O_W) = k$. Then $W$ is geometrically integral. If $\dim W = 1$ then by Weil there are points in any large enough field extension; take two whose degrees are relatively prime.
If \( \dim W > 1 \) then use Bertini to get a geometrically integral hyperplane section. More precisely, such hyperplane sections exists over any large enough field extension; take two whose degrees are relatively prime.

In general, set \( K = H^0(W, \mathcal{O}_W) \). After base change to \( K \), the irreducible components \( W_i \subset W_K \) are geometrically integral. Thus there is a 0-cycle \( Z_1 \subset W_1 \) of degree 1. The sum of its conjugates gives a 0-cycle of degree \( \dim_k K \) on \( W \). \( \square \)

**Henselian fields with algebraically closed residue fields.**

One of the main questions proposed and investigated in [ELW12] is the following.

**Conjecture 38.** Let \( K \) be the quotient field of an excellent, Henselian DVR with algebraically closed residue field \( k \). Let \( X \) be a proper \( K \)-scheme. Then

\[
\text{elw}_0(X) = \text{elw}_1(X) = \cdots = \text{elw}_{\dim X}(X).
\]

(35) 1

The conjecture is almost proved in [ELW12].

**Theorem 39.** Let \( K \) be the quotient field of an excellent, Henselian DVR with algebraically closed residue field \( k \). Let \( X \) be a proper \( K \)-scheme. Then

1. If \( \text{char } k = 0 \) then \( \text{elw}_0(X) = \cdots = \text{elw}_{\dim X}(X) \).
2. If \( \text{char } k = p > 0 \) then \( \text{elw}_0(X) = \cdots = \text{elw}_{\dim X}(X) \) holds in \( \mathbb{Z}[p^{-1}] \).

The key step is the following.

**Lemma 40.** Let \( R \) be a Henselian DVR with quotient field \( K \) and algebraically closed residue field \( k \). Let \( X_K \) be a proper, regular \( R \)-scheme with generic fiber \( X_K \). Then \( \chi(X_K, \mathcal{O}_{X_K}) \in \text{elw}_0(X_K) \).

Proof. Write \( X_0 = \sum_{i \in I} m_i X_0^i \). If \( r|m_i \) for every \( i \) then set \( Z_r := \sum (m_i/r)X_0^i \). Note that

\[
\mathcal{O}_X(-iz_r)|_{Z_r} \cong \mathcal{O}_X(-iZ_r)|_{Z_r} \cong \mathcal{O}_{Z_r}.
\]

Thus \( \mathcal{O}_X(-iz_r)|_{Z_r} \) is numerically trivial, hence \( \chi(Z_r, \mathcal{O}_X(-iz_r)|_{Z_r}) = \chi(Z_r, \mathcal{O}_{Z_r}) \).

Therefore

\[
\chi(X_K, \mathcal{O}_{X_K}) = \chi(X_0, \mathcal{O}_{X_0}) = \sum_{i=1}^{r}\chi(Z_r, \mathcal{O}_X(-iz_r)|_{Z_r}) = r\chi(Z_r, \mathcal{O}_{Z_r}).
\]

This implies that \( \chi(X_K, \mathcal{O}_{X_K}) \in (m_i : i \in I) \). We conclude by noting that through a general point of \( X_0 \) there is a multi-section of degree \( m_i \). \( \square \)

**Real closed fields.**

For any scheme \( X \) over \( \mathbb{R} \), \( \text{elw}_0(X) = 1 \) iff \( X(\mathbb{R}) \neq \emptyset \). Otherwise \( \text{elw}_0(X) = 2 \).

Thus the only question is when the sequence \( \text{elw}_i \) drops form 2 to 1.

**Example 41.** Let \( \pi : S \to \mathbb{P}^2 \) be a double cover ramified along a curve of degree 2d. Let \( H \) denote the pull-back of a line in \( \mathbb{P}^2 \). Then \( \pi_* \mathcal{O}_S \cong \mathcal{O}_{\mathbb{P}^2} + \mathcal{O}_{\mathbb{P}^2}(-d) \), thus

\[
\chi(S, \mathcal{O}_S) = 1 + \frac{(d-1)(d-2)}{2} \quad \text{and} \quad K_S \sim (d-3)H.
\]

If \( C \sim rH \) is a curve in \( S \) then

\[
\chi(C, \mathcal{O}_C) = r(r + d - 3)\frac{(d^2)}{2} = r(r + d - 3).
\]

Thus if \( S(\mathbb{R}) = \emptyset \), \( d \equiv 2 \mod 4 \) and \( \text{Pic}(S) = \mathbb{Z}[H] \) then

\[
\text{elw}_0(S) = 2, \ \text{elw}_1(S) = 2, \ \text{elw}_2(S) = 1.
\]

Such surfaces can be obtained as small perturbations of

\[
S_0 := (x^{12} + y^{12} + z^{12} + w^2 = 0) \subset \mathbb{P}^3(1, 1, 1, 6).
\]
Probably there are similar higher dimensional examples. It is, however, quite difficult to understand all subvarieties of codimension $\geq 2$ of a given variety. I do not know how to compute the ELW-indices for higher dimensional hypersurfaces.

**Question 42 (OW).** Let $X$ be a smooth rationally connected variety over $\mathbb{R}$. Is $\text{elw}_1(X) = 1$?

It is known that if $X$ is rationally connected and $X(\mathbb{R}) \neq \emptyset$ then it contains a rational curve [Kol99, 1.7]. It is conjectured that $X$ contains a geometrically rational curve even if $X(\mathbb{R}) = \emptyset$; see [AK03, Rem.20]. (12) is a weaker variant of this. It is closely related to the conjecture that the function field of the empty real conic is $\mathbb{C}_1$; see [Lan53, p.379].

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