The basic problem posed by free will (FW) for physics appears to be not the physical one of whether it is compatible with the laws of physics, but the logical one of how to consistently define it, since it incorporates the contrary notions of freedom, which suggests indeterminism, as well as control, which bespeaks determinism. We argue that it must be a fundamentally new causal primitive, in addition to determinism and indeterminism. In particular, we identify FW in a physical theory with dynamics that is uncomputable, and hence effectively indeterministic within the theory. On the other hand, it would be deterministic in a higher order theory. An important consequence for artificial intelligence (AI) is that the FW aspect of cognitive systems may be fundamentally unsimulable. An implication for neuroscience is that FW will in general be experimentally undemonstrable. Apparently, it can only be subjectively experienced.

Keywords: Uncomputability, undecidability, free will, Turing machines

I. INTRODUCTION

We informally understand free will (FW) as the power to choose an alternative from many others. However for over two thousand years, philosophers, scientists and theologians have debated on what FW is and whether it exists [1–4]. A basic difficulty has been that FW incorporates two opposing notions: freedom and control. On the one hand, freedom suggests unpredictability and indeterminism, whereas on the other hand, control suggests intent and determinism. From this perspective, FW is an oxymoron.

Two broad philosophical positions on FW are compatibilism, according to which determinism is compatible with FW, and incompatibilism, according to which the two are incompatible. According to a compatibilist, a person may choose freely and yet an omniscient being may possess foreknowledge of that choice. An incompatibilist may reject FW, in which case he is a hard determinist, or reject determinism, in which case he is a libertarian.

FW has figured in some recent discussion in the foundations of quantum mechanics (QM), where it is treated simply as a form of unpredictability, or independence from all past information (with ‘past’ defined in an appropriately relativistically invariant way). Some works in physics that have taken a closer look at the existence or non-existence of FW include Refs. [5–11]. The important contribution of quantum mechanics to this debate is in introducing a concrete instance of fundamental indeterminism via the $|\psi|^2$ Born rule.

In this article, we hope to characterize libertarian FW. The article is structured as follows. Section II studies the problem of accommodating FW in physical laws. Section III examines the logical problem of reconciling unpredictability and control. Section IV develops a model of FW, in which this reconciliation is achieved by modelling FW as a new causal primitive, or fundamental kind of causation, which is not computable and hence deterministic within the theory. The model is applied to practical situations to elucidate its resolution of the paradox posed by FW. Finally, in Section VI we look at implications for AI, brain science and neuroscience.

II. FREE WILL AND PHYSICAL LAWS

If libertarian FW exists, then determinism is ruled out as universal. Is the lot of indeterminism any better? Intuitively, it seems so because the indeterminacy gives ‘elbow room’ for FW to act. But neither does indeterminism (as governed by some fixed probability rule, $P$) leave enough room for FW to act.
Consider the sample mean $X_n$ over $n$ trials

$$\lim_{n \to \infty} \Pr (|X_n - \mu| > \epsilon) = 0,$$

by the Weak Law of Large Numbers. This implies that there is a ‘probability pressure’ not to choose atypical sequences, which would cause deviations from the sample mean. Thus, there is a kind of long-run determinism, and hence a restriction on FW as we intuitively understand it.

Therefore, if we accept libertarian FW, the free-willed choice will potentially interfere with the underlying physical dynamics $D$. This interference will take the form of (a) overriding causality, if $D$ is deterministic; or (b) causing deviations from the relevant probability rule $P$, if $D$ is indeterministic.

Clearly, it is immaterial whether the underlying physics is classical or quantum. FW itself can’t be part of the dynamics $D$, for in that case it could not produce the required deviations. Therefore, we require a Cartesian dualism with a physical and an ‘extra-physical’ component making up a free-willed agent. Physical dynamics $D$ governs the former while FW comes from the latter. It is important to stress that this extraphysical agency must be something that is qualitatively different.

III. THE FUNDAMENTAL PROBLEM OF FREE WILL

The basic problem of FW is how to reconcile the notion of freedom, which suggests unpredictability and indeterminism, with the notion of control, which suggests the opposite. The following argument highlights this difficulty: (I) Given a set $X = \{x\}$ of possibilities, suppose the agent $A$ chooses $y = A(X)$, where $A$ is an algorithm representing $A$’s mechanism of choosing; (II) The principle of excluded middle implies that $A$ is deterministic or it is not; (III) In the former case, there is no genuine situation involving choice, and hence no FW. (IV) In the latter case, there is only randomness, and hence no control, and no FW.

We thus find again that the agent’s choice is deterministic or random, with no apparent room for (libertarian) FW. To us, this suggests the following model as a way out, in which the basic insight is that FW is not about unpredictability, but the power to make a choice against certain odds or compulsions.

Given the state $\rho$ of an agent, he is subject to two influences:

**Constraints:** coming from Nature, in the form of desires, instinctive drives and emotional tendencies. This corresponds to a computable function, described by Turing machine (TM) $C$ acting on an encoding of state $\rho$ (informally written $C(\rho)$).

**Guidance:** coming from a rational capacity to model the world, and to understand the (ethical, social, financial, etc.) implications of each choice. It is described by a computable function, computed by TM $G$, and has the action $G(\rho)$.

Here $\rho$ may be thought of as an integer, say the Gödel or Turing number, that encodes the physical state. In the simplest case, the dilemma facing the agent is whether to go with $C(\rho)$ or with $G(\rho)$.

Suppose there is a situation containing alternative possibilities, labelled $x \in X$. If there is no such thing as FW, then the combined action of $C$ and $G$ fully determine the outcome $x$. This may be expressed by saying that

$$\rho' = \phi(C(\rho), G(\rho)),$$

(1)

where $\phi$ is a computable function, that may be linear, nonlinear, deterministic or probabilistic. In this case, the output state can computed, and the dynamics represented by Eq. (1) understood. Such a model is suitable for a compatibilistic or hard deterministic world view. One can propose a more involved function that Eq. (1), such as:

$$\rho' = \chi(C, U)(\rho),$$

(2)

but the former suffices for our purpose.

If we accept the libertarian position, then a third element is required, which expresses the idea of empowering the agent to make a choice under the (possibly opposing) influences of $C(\rho)$ or $G(\rho)$. This is the faculty of volition, or:

**Freedom of Will.** The freedom to orient or align the choice in line with the Understanding $G(\rho)$, by overcoming, if necessary, the Constraint $C(\rho)$. We represent it by a process $A$.

For FW in this sense to be tenable, the process $A$ itself should be extra-physical, as noted earlier. In particular, in Eq. (1), function $\phi$ should be uncomputable in the Turing model. This can be shown in analogy with the proof of the uncomputability of the halting problem for TMs [12].
IV. UNCOMPUTABILITY AND FW

Suppose prediction \( Y \) is made about agent \( S \). If \( S \) believes \( Y \), \( S \) can falsify it by deliberately acting contrary to \( Y \). It may be supposed that a more detailed algorithm to arrive at an updated prediction \( Y' \) should be able to take into account \( S \)’s reaction. Yet \( S \) may simply choose to falsify that. Only if \( S \) disbelieves \( Y' \) will \( Y' \) hold with certainty. However, this would make \( S \) a sort of inconsistent agent, for disbelieving a truth. Thus, if \( S \) is consistent, then \( S \) must believe \( Y \). To avoid inconsistency, we conclude that in general there is no algorithm that can predict how an agent will behave. The similarity of this argument to Gödel’s celebrated incompleteness theorem \[13\] becomes apparent when we replace the notion of provability there with that of belief. An agent’s behavior may appear random because it cannot always be predicted, but is not (in this model) ontologically random. It is this ability to act in an unpredictable, yet non-random, way that we deem FW.

The similarity to the halting problem for TMs is also clear \[12\]. Consider the choice problem \( H_A \), mentioned above, of predicting the choice of an agent. The undecidability of \( H_A \) follows from the reduction of Turing’s Halting problem \( H \) to \( H_A \). If we had a decider \( A \) for \( H_A \) (one that can answer all instances of \( H_A \) in finite time), then one, denoted \( \mathcal{H} \), for \( H \) can be constructed as follows: \( \mathcal{H} \) constructs a TM \( N \) that outputs “0” if a TM \( M \) halts on input \( w \), and outputs “1” otherwise. To decide, \( \mathcal{H} \) can now evaluate \( A(N) \) and determine that \( M \) halts (does not halt) on \( w \) if the output is 0 (1). A non-reductive proof is also readily obtained via the usual diagonal argument.

Suppose an algorithm \( A \) exists that, knowing the state \( \rho \) of an agent \( S \), can predict in finite time whether \( S \) will choose “0” or “1” on input \( i \). That is,

\[
A(S; i) = \begin{cases} 
0 & \iff S(i) = 0 \\
1 & \iff S(i) = 1
\end{cases}
\]  
(3)

Then one can construct the program \( \mathcal{R} \), which represents an agent who reacts to a prediction about herself by falsifying it, as follows:

\[
\mathcal{R}(p) = \begin{cases} 
1 & \iff A(p; p) = 0 \\
0 & \iff A(p; p) = 1
\end{cases}
\]  
(4)

Applying \( A \) to \( \mathcal{R} \) we have from Eqs. (3) and (4):

\[
A(\mathcal{R}; p) = \begin{cases} 
1 & \iff A(p; p) = 0 \\
0 & \iff A(p; p) = 1
\end{cases}
\]  
(5)

Here we have assumed that \( \mathcal{R}, p \), etc. as arguments, stand for integer labels for the corresponding quantity. Eq. (5) leads to a contradiction when we set \( p = \mathcal{R} \), i.e., we choose \( p \) to be the Turing number of \( \mathcal{R} \). We conclude that to avoid it, \( A(\mathcal{R}; \mathcal{R}) \) loops infinitely.

In particular, the performance of \( A \) fails in cases like \( \mathcal{R}(\mathcal{R}) \), in that its truth value cannot be consistently computed by the agent. We can avert the inconsistency by stipulating that \( A(\mathcal{R}; \mathcal{R}) \) never halts, so that \( \mathcal{R}(\mathcal{R}) \) is not decidable, which is the Gödel sentence \[13\] for the formal system encompassing the agent, making it syntactically incomplete. Gödel undecidability thus is implied by Turing uncomputability \[12\]. Detangling the argument of Eq. (4), we find that a general predictive algorithm is logically impossible because it would help create an algorithm that is so powerful that, knowing its own future, it could act contradictory to its own prediction. Consistency dictates that arbitrarily powerful self-predicting programs are impossible. The question of self-reference in a FW-endowed entity was first considered by MacKay \[13, 16\], though he denied its connection to Gödel incompleteness or Turing non-computability.

If we assume that all properties of an agent that are physically directly accessible are computable, then the component of the agent (which we may call “mind”) responsible for process \( A \) must be something that is physically not directly accessible, and can be considered extra-physical in that sense. Something about the brain allows the mind to interface with matter in a way to allow FW.

FW as defined above is the new causal primitive or principle of causation, apart from physical determinism and indeterminism. Because the full dynamics is not accessible, the map which represents the act of choosing an element from a set, the physically accessible free-willed behavior, will appear seem probabilistic. Causality is no longer closed under physics.

A simple way to model this is as follows. Let \( |X| = d \). We represent \( \rho \) by a stochastic column vector \( \hat{\rho} \), and \( C \) and \( G \) be right stochastic matrices \( \hat{C} \) and \( \hat{G} \). The evolution of the state is given by:

\[
\hat{\rho}' = \left( \alpha \hat{G} + (1 - \alpha) \hat{C} \right) \hat{\rho} \equiv \hat{U} \hat{\rho},
\]  
(6)
where $0 \leq \alpha \leq 1$. The stochastic matrix $\hat{U}$ represents the free-willed action and encompasses the combined influence of guidance and constraints of nature. The number $\alpha$ and hence $\hat{U}$ are in general uncomputable [14], but can in principle be estimated experimentally.

The larger the ‘FW parameter’ $\alpha$, the more is the agent’s choice geared according to guidance $G$, and less the compulsion of nature. As a simple example, given a dichotomic choice space $\{0, 1\}$, let $G$ recommend choosing 0, while $C$ recommend 1. For any state $\hat{\rho} = (p, 1-p)^T$, we find

$$\hat{\rho}' = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix} \rho' + (1 - \alpha) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \hat{\rho}$$

(7)

V. APPLICATIONS

We will now see how this model resolves the logical paradox mentioned above. We define a Saint as an agent whose Understanding, and hence guidance $G$, is ethical and whose FW parameter $\alpha$ is near maximal. Consider the proposition: “Presented with the choice between the good and the evil, the Saint freely chooses the good.” In Eq. (7), setting 0 as “the good”, 1 as “evil”, and $\alpha = 1$, we find $\rho' = (1, 0)^T$. The Saint, by dint of high FW, can if necessary override any Constraints imposed by his human nature, to always choose the good. Thus, the deterministic behavior and predictability of the Saint does not preclude his free will, contrary to the incompatibilist implication (III) above. On the other end, predictable behavior can arise also because of low degree of FW, which obtains when we set $\alpha = 0$ in Eq. (7).

Suppose that an agent’s FW is not maximal. Then his choice fluctuates randomly between choosing according to the dictates of his nature and the recommendation of his Understanding. This is illustrated in the following proposition, whose intuitively apparent truth is validated by our definition of FW: “Presented with the choice between the good and the evil, the Conscientious Criminal vacillates” This criminal, being conscientious, has a clear Understanding of the virtue of ethical behavior, but, owing to lack of sufficiently high FW, cannot always overcome the compulsion of his criminal nature. In Eq. (7), setting $\alpha = 0.5$, $\rho' = (0.5, 0.5)^T$. His choice is random, being good sometimes and being evil at other times. Thus the randomness of his choice does not imply lack of free will. It only implies low free will. This explains why the pessimist implication (IV) fails.

VI. IMPLICATIONS FOR AI AND NEUROSCIENCE

Our analysis suggests that if we accept the proposition of libertarian FW, then something extra-physical or mind-like, and in particular, uncomputable, happens in the act of volition, at the instant where a new intent is generated in the brain. Any AI attempt to algorithmize consciousness must then fail because true FW cannot be simulated, at least in machines that implement first order logic. FW is then a facet of conscious beings that seems to set them apart from automatons.

Darwinists may argue that FW is a strategy necessary to prevent an organism or specie from settling down in local minima in the energy landscape appropriate to biological thermodynamics, and thus to ensure long-term self-preservation. A patient who submits to a short-term painful treatment for good health in the long run, or an altruist who starves to feed the needy, are, in this view, strategies reached through FW that could otherwise have been missed. In this view (that is freely held by one of the authors–CMN!), FW originated evolutionarily to enable deviation from the local gradient and thereby do better than locally optimize in an organism’s struggle for self-preservation. For example, in a mountaineous terrain, the higher the oxygen concentration the better, and the lower the geographical altitude, the higher the oxygen concentration. Organisms without FW would follow the local gradient and settle down in the local minimum valley. The intervention of FW will let an organism to climb over a mountain peak in order to reach a deeper valley. The evolution of FW also brought along “by-products” like altruism or self-harm.

The subject of FW rightly receives much attention in brain science and neuroscience [17][19]. For neuroscience, the implication of our findings is that any attempt to trace back a particular action of an organism to its first cause in a neuron (such as pyramidal neurons identified as the seat of origin of mouse whisker twichings [20]), or in a sub-neuronal structure [21], is bound to fail. It may be preserved to a great depth as one traces back down the signal pathway, but there will be a final ‘flicker’ that is irreducible to other observable causes, and thus visibly random, and yet quite rational when the bigger picture is taken into account.

But even to get there, an experimental test would require isolating systems of interest to rule out extraneous stimuli. Otherwise, one could not rule out a conventional, physical explanation, based on classical indeterminism [22] or even quantum indeterminism as being responsible for observed fluctuations in behavior.
In this sense, FW will fundamentally be experimentally inaccessible, and thus not scientifically falsifiable. One can only subjectively experience it, and also rule out an action as being free-willed (when it is manifestly explained by visible causes), but never point out FW directly. One can at best look for circumstantial evidence.

Our resolution of the problem of FW is to identify it with a kind of uncomputable determinism. It will be computable if one can access oracle machines, which are TMs equipped with a black box that is able to answer in one step any instance of the halting problem. But a halting problem, or equivalently the FW problem, exists for these oracle machines, when they are applied to machines equivalent to themselves. One can thus define a second-order FW. Extending the notion of oracle machines, one obtains the arithmetic hierarchy, where each higher stage brings a more powerful oracle machine and an even harder halting problem. It is thus an interesting question whether absolute FW exists!

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