NOTE ON THE PRIME NUMBER THEOREM

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ABSTRACT. We survey the classical results on the prime number theorem.

In this chapter, we are very interested in the asymptotic behavior of a single number theoretic function $\pi(n)$ which counts all prime numbers between 1 and $n$, or $\pi(x)$ which is extended to $\mathbb{R}$ and defined by

$$\pi(x) = \sum_{p \leq x} 1.$$ 

It is well-known that Euclid showed that

$$\lim_{x \to \infty} \pi(x) = \infty;$$

that is, there exist infinitely many prime numbers.

**Proposition 5.1.** There exists a constant $c > 0$ such that

$$\pi(x) \geq c \cdot \ln \ln x.$$ 

**Proof.** First of all, we prove that if $p_n$ is the $n$th prime number then we have that

$$p_n \leq 2^{2^n - 1}.$$ 

Since there must be some $p_{n+1}$ dividing the number $p_1 p_2 \cdots p_n - 1$ and not exceeding it, it follows from the induction step that

$$p_{n+1} \leq 2^{2^n} \cdot 2^{2^{n+1}} \cdot \cdots \cdot 2^{2^n} = 2^{2^n + 2^{n+1} + \cdots + 2^n} \leq 2^{2^n}.$$ 

If $x \geq 2$ is some real number, then we select the largest natural number $n$ satisfying $2^{2^n - 1} \leq x$, so that we have that $2^{2^n} > x$. Hence we conclude that

$$\pi(x) \geq n \geq \frac{1}{\ln 2} \cdot \ln \left( \frac{\ln x}{\ln 2} \right) \geq \frac{1}{\ln 2} \cdot \ln \ln x. \quad \square$$
Proposition 5.2. There exists a constant $c > 0$ such that

$$\pi(x) \geq c \cdot \ln x$$

for all sufficiently large $x$.

Proof. Since each square-free integer $n \leq x$ can be only be divided by $p_1, p_2, \cdots, p_{\pi(x)}$, $n$ can be written uniquely as

$$n = \prod_{k=1}^{\pi(x)} p_k^{\alpha_k}$$

where $\alpha_k$ takes only the values 0 or 1. Thus there are at most $2^{\pi(x)}$ square-free integers $n \leq x$. From Corollary 4.2.21, we see that the density of the square-free integers tends to $6/\pi^2$; that is, the number of square-free numbers $n \leq x$ grows asymptotically to $6x/\pi^2$. This implies that there is some constant $c_0 < 6/\pi^2$ such that

$$c_0 \cdot x \leq 2^{\pi(x)}$$

for all sufficiently large $x$. Hence we complete the proof. □

Neither of Proposition 5.1 and Proposition 5.2 describes the asymptotic behavior of $\pi(x)$ quite well. Long time ago, Legendre and Gauss conjectured that

$$\pi(x) \sim x \ln x.$$ 

The truth of this assertion is the core of the prime number theorem. For more delicate description of $\pi(x)$, we consider the integral logarithm function $\text{li} x$ defined as the Cauchy principal value integral

$$\text{li} x = \int_0^x \frac{1}{\ln t} dt = \lim_{\varepsilon \to 0} \left( \int_0^{1-\varepsilon} \frac{1}{\ln t} dt + \int_{1+\varepsilon}^x \frac{1}{\ln t} dt \right).$$

It follows from de l’Hospital’s rule that

$$\lim_{x \to \infty} \frac{\text{li} x}{\ln x} = \lim_{x \to \infty} \frac{1}{\ln x - \frac{1}{\ln^2 x}} = 1.$$ 

Thus we obtain the asymptotic behavior of $\text{li} x$ as follows:

$$\text{li} x \sim x / \ln x.$$ 

Hence the asymptotic relation $\pi(x) \sim \text{li} x$ is called the prime number theorem. In fact, Gauss conjectured that $\text{li} x$ describes $\pi(x)$ even better than $x/\ln x$.

Lemma 5.3. (a) $\sum_{n \leq x} \Lambda(n) \left\lfloor \frac{x}{n} \right\rfloor = x \ln x - x + O(\ln x)$.

(b) $\sum_{n \leq x} \Lambda(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x}{2n} \right\rfloor \right) = x \ln 2 + O(\ln x)$.

Proof. (a) By the definition of the Mangoldt function, we have that

$$\sum_{n \leq x} \ln n = \sum_{n \leq x} \sum_{m \mid n} \Lambda(m) = \sum_{m \leq x} \Lambda(m) \sum_{n \leq x, m \mid n} 1 = \sum_{m \leq x} \Lambda(m) \left\lfloor \frac{x}{m} \right\rfloor.$$
Thus it follows from Proposition 4.2.3 [the Euler’s sum formula] that
\[ \sum_{n \leq x} \Lambda(n) \left[ \frac{x}{n} \right] = \sum_{n \leq x} \ln n = \int_1^x \ln t \, dt + O(\ln x) = x \ln x - x + O(\ln x). \]

(b) By applying (a) and the fact that \( \sum_{x/2 < n \leq x} \Lambda(n) \left[ \frac{x}{2n} \right] = 0 \), we obtain that
\[ \sum_{n \leq x} \Lambda(n) \left( \left[ \frac{x}{n} \right] - 2 \left[ \frac{x}{2n} \right] \right) = \sum_{n \leq x} \Lambda(n) \left[ \frac{x}{n} \right] - 2 \sum_{n \leq x/2} \Lambda(n) \left[ \frac{x}{2n} \right] - 2 \sum_{x/2 < n \leq x} \Lambda(n) \left[ \frac{x}{n} \right] \]
\[ = x \ln x - x - 2 \left( x \ln \frac{x}{2} - \frac{x}{2} \right) + O(\ln x) \]
\[ = x \ln 2 + O(\ln x). \]

Hence we complete the proof. \( \square \)

**Theorem 5.4 [Chebyshev’s Theorem].** There exist two constants \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[ c_1 \cdot \frac{x}{\ln x} \leq \pi(x) \leq c_2 \cdot \frac{x}{\ln x} \]
for all sufficiently large \( x \).

*Proof.* Since \( \alpha - 2 \left[ \frac{\alpha}{2} \right] \) is always an integer and satisfies the following inequality
\[ -1 = \alpha - 1 - 2 \frac{\alpha}{2} < \alpha - 2 \left[ \frac{\alpha}{2} \right] < \alpha - 2 \left( \frac{\alpha}{2} - 1 \right) = 2, \]
we see that
\[ 0 \leq \alpha - 2 \left[ \frac{\alpha}{2} \right] \leq 1. \]

Thus by (5.1) and (b) of Lemma 5.3 we have that
\[ x \ln 2 + O(\ln x) = \sum_{n \leq x} \Lambda(n) \left( \left[ \frac{x}{n} \right] - 2 \left[ \frac{x}{2n} \right] \right) \]
\[ \leq \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \left[ \frac{\ln x}{\ln p} \right] \ln p \]
\[ \leq \ln x \sum_{p \leq x} 1 = \pi(x) \ln x, \]
and so we can get the first inequality by dividing by $\ln x$. For the second inequality, we observe that

$$
\pi(x) \ln x - \pi \left( \frac{x}{2} \right) \ln \frac{x}{2} = \ln \frac{x}{2} \left( \pi(x) - \pi \left( \frac{x}{2} \right) \right) + \pi(x) \ln 2
$$

$$
= \ln \frac{x}{2} \left( \pi(x) - \pi \left( \frac{x}{2} \right) \right) + O(x)
$$

$$
= O \left( \sum_{x/2 < p \leq x} \ln p + x \right)
$$

$$
= O \left( \sum_{x/2 < n \leq x} \Lambda(n) \cdot (1 - 0) + x \right)
$$

$$
= O \left( \sum_{x/2 < n \leq x} \Lambda(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x}{2n} \right\rfloor + x \right) \right)
$$

$$
= O \left( \sum_{n \leq x} \Lambda(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x}{2n} \right\rfloor + x \right) \right)
$$

$$
= O(x).
$$

From this, we have more generally the following estimate

$$
\pi \left( \frac{x}{2^k} \right) \ln \frac{x}{2^k} - \pi \left( \frac{x}{2^{k+1}} \right) \ln \frac{x}{2^{k+1}} = O \left( \frac{x}{2^k} \right), \quad k \in \mathbb{N}.
$$

Thus for any $K \in \mathbb{N}$ we obtain that

$$
\pi(x) \ln x - \pi \left( \frac{x}{2^K+1} \right) \ln \frac{x}{2^K+1} = \sum_{k=0}^{K} \pi \left( \frac{x}{2^k} \right) \ln \frac{x}{2^k} - \pi \left( \frac{x}{2^{k+1}} \right) \ln \frac{x}{2^{k+1}}
$$

$$
= O \left( \sum_{k=0}^{K} \frac{x}{2^k} \right) = O(x).
$$

This implies that $\pi(x) = O \left( \frac{x}{\ln x} \right)$. \hfill \Box

**Proposition 5.5.** The following asymptotic equation

$$
\pi(x) \sim \frac{x}{\ln x}
$$

is equivalent to the asymptotic equation $\psi(x) \sim x$ where the $\psi$-function is defined by

$$
\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p, \nu \geq 1, p^\nu \leq x} \ln p.
$$

(Here the function $\psi$ is introduced by Chebyshev.)

**Proof.** From the definition of the function $\psi$, we have that

$$
\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p \leq \ln x \sum_{p \leq x} 1 = \pi(x) \ln x.
$$

(5.2)
On the other hand, we note that for any \(y\) with \(1 < y < x\),
\[
\pi(x) = \pi(y) + \sum_{1 \leq \pi(y) + \sum_{y < p \leq x} \frac{\ln p}{\ln y}} \leq c_2 \cdot \frac{y}{\ln y} + \frac{\psi(x)}{\ln y}.
\]
Thus, multiplying by the factor \(\ln x/x\), the above inequality becomes
\[
\pi(x) \cdot \frac{\ln x}{x} \leq c_2 \cdot \frac{\ln x}{x \ln y} + \frac{\psi(x)}{x \ln y},
\]
If we set \(y = x/\ln x\) in (5.3), then we have that
\[
\pi(x) \cdot \frac{\ln x}{x} \leq \frac{c_2 \psi(x)}{\ln x - \ln \ln x} + \frac{1}{1 - \frac{\ln \ln x}{\ln x}}.
\]
Hence we complete the proof from (5.2) and (5.4). 

**Theorem 5.6[Mertens’ Theorem].** If \(p\) runs through all prime numbers, then we have the following asymptotic approximations;
\[
(a) \sum_{p \leq x} \frac{\ln p}{p} = \ln x + O(1), \quad (b) \sum_{p \leq x} \frac{1}{p} = \ln \ln x + c_3 + O \left( \frac{1}{\ln x} \right),
\]
\[
(c) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = \frac{c_4}{\ln x} \left( 1 + O \left( \frac{1}{\ln x} \right) \right),
\]
where \(c_3 > 0\) and \(c_4 > 0\) are some constants.

**Proof.** (a) From (a) of Lemma 5.3 and Theorem 5.4, we have that
\[
x \ln x - x + O(\ln x) = \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)
\]
\[
= \sum_{p \leq x} \left( \frac{x}{p} \right) \ln p + \sum_{p \leq \sqrt{x}, \nu \geq 2: p^\nu \leq x} \left( \frac{x}{p^\nu} \right) \ln p
\]
\[
= \sum_{p \leq x} \frac{\ln p}{p} \cdot x - \sum_{p \leq \sqrt{x}} \left( \frac{x}{p} \right) \ln p + O \left( \sum_{p \leq \sqrt{x}} \sum_{2 \leq \nu \leq \ln x/p} \frac{x}{p^\nu} \ln p \right)
\]
\[
= x \sum_{p \leq x} \frac{\ln p}{p} + O \left( \sum_{p \leq x} \ln p \right) + O \left( x \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \right)
\]
\[
= x \sum_{p \leq x} \frac{\ln p}{p} + O \left( \ln x \cdot c_2 \cdot \frac{x}{\ln x} \right) + O \left( x \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \right)
\]
\[
= x \sum_{p \leq x} \frac{\ln p}{p} + O(x).
\]
This implies the first one.
(b) It follows from Proposition 4.2.2”[Abel transformation] that
\[
\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{\ln p}{p} \cdot \frac{1}{\ln p} \\
= \frac{1}{\ln x} \sum_{p \leq x} \frac{\ln p}{p} + \int_2^x \sum_{p \leq t} \frac{\ln p}{p} \cdot \frac{1}{t \ln^2 t} dt \\
= 1 + O\left(\frac{1}{\ln x}\right) + \int_2^x \frac{1}{t \ln t} dt + \int_2^x \left(\sum_{p \leq t} \frac{\ln p}{p} - \ln t\right) \frac{1}{t \ln^2 t} dt.
\]

Since \(a(t) = \sum_{p \leq t} \frac{\ln p}{p} - \ln t\) is bounded by (a), the following integral
\[
\int_2^\infty \frac{a(t)}{t \ln^2 t} dt
\]
converges, and moreover we have that
\[
\int_2^\infty \frac{1}{t \ln t} dt = \ln \ln t - \ln 2.
\]

Therefore we conclude that
\[
\sum_{p \leq x} \frac{1}{p} = \ln \ln x + \left(1 - \ln 2 + \int_2^\infty \frac{a(t)}{t \ln^2 t} dt\right) + O\left(\frac{1}{\ln x} + \int_x^\infty \frac{|a(t)|}{t \ln^2 t} dt\right)
\]
\[
= \ln \ln x + c_3 + O\left(\frac{1}{\ln x}\right).
\]

(c) If we define the constant \(c_5\) by
\[
c_5 = \sum_{n=2}^\infty \frac{1}{n} \sum_{p \leq x} \frac{1}{p^n},
\]
then it follows from simple calculation that
\[
\ln \left(\prod_{p \leq x} \left(1 - \frac{1}{p}\right)\right) = \sum_{p \leq x} \ln \left(1 - \frac{1}{p}\right) = -\sum_{p \leq x} \sum_{n=1}^\infty \frac{p^{-n}}{n}
\]
\[
= -\sum_{p \leq x} \frac{1}{p} - \sum_{n=2}^\infty \frac{1}{n} \sum_{p \leq x} \frac{1}{p^n}
\]
\[
= -\sum_{p \leq x} \frac{1}{p} - c_5 + O\left(\sum_{n=2}^\infty \frac{1}{n} \sum_{p \leq x} \frac{1}{p^n}\right)
\]
\[
= -\sum_{p \leq x} \frac{1}{p} - c_5 + O\left(\sum_{n=2}^\infty \sum_{m > x} \frac{1}{m^n}\right)
\]
\[
= -\sum_{p \leq x} \frac{1}{p} - c_5 + O\left(\sum_{n=2}^\infty \frac{1}{n} \cdot \frac{1}{(n-1)x^{n-1}}\right)
\]
\[
= -\sum_{p \leq x} \frac{1}{p} - c_5 + O\left(\frac{1}{x}\right).
\]

Hence this implies the required result. \(\square\)
Lemma 5.7 [Tauberian Theorem of Ingham and Newman].
Let $F(t)$ be a bounded complex-valued function defined on $(0, \infty)$ and integrable over every compact subset of $(0, \infty)$, and let $G(z)$ be an analytic function defined on a domain containing the closed half-plane $\Pi = \{ z \in \mathbb{C} : \text{Re}(z) \geq 0 \}$. If $G(z)$ agrees with the Laplace transformation of $F(t)$ for all $z \in \Pi$, i.e.
\[
G(z) = \int_0^\infty F(t) e^{-zt} \, dt, \quad \text{Re}(z) > 0,
\]
then the improper integral
\[
\int_0^\infty F(t) \, dt
\]
converges.

Proof. Without loss of generality, we may assume that $|F(t)| \leq 1$ for all $t > 0$. For $\lambda > 0$, we set
\[
G_\lambda(z) = \int_0^\lambda F(t) e^{-zt} \, dt.
\]
Then we see that $G_\lambda(z)$ is analytic on $\mathbb{C}$. Thus it suffices to show that
\[
\lim_{\lambda \to \infty} G_\lambda(0) = \lim_{\lambda \to \infty} \int_0^\lambda F(t) \, dt = G(0).
\]
Fix $\varepsilon > 0$. Then there are $\delta = \delta(\varepsilon) > 0$ and $R > 0$ such that $1/R < \varepsilon/3$ and $G(z)$ is analytic on the compact region
\[
\Omega_{\delta,R} = \{ z \in \mathbb{C} : \text{Re}(z) \geq \delta, |z| \leq R \}
\]
with boundary $\partial\Omega_{\delta,R} = \gamma$ which is a simple closed contour oriented counterclockwise. By Cauchy integral formula, we have that
\[
G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_\gamma \frac{G(z) - G_\lambda(z)}{z} \, dz.
\]
We observe that for $x = \text{Re}(z) > 0$,
\[
|G(z) - G_\lambda(z)| = \left| \int_\lambda^\infty F(t) e^{-zt} \, dt \right| \leq \int_\lambda^\infty e^{-xt} \, dt = \frac{e^{-\lambda x}}{x},
\]
and for $x = \text{Re}(z) < 0$,
\[
|G_\lambda(z)| = \int_0^\lambda F(t) e^{-zt} \, dt \leq \int_0^\lambda e^{-xt} \, dt = \frac{e^{-\lambda x}}{|x|}.
\]
With technical reasons given in (5.6) and (5.7), the relation (5.5) can be written again as
\[
G(0) - G_\lambda(0) = \frac{1}{2\pi i} \int_\gamma |G(z) - G_\lambda(z)| e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) \, dz.
\]
If we denote by $\gamma_+$ the part of $\gamma$ lying in $\text{Re}(z) > 0$, then we see that
\[
\frac{1}{z} + \frac{z}{R^2} = \frac{2x}{R^2}
\]
on \( \gamma_+ \), and thus it follows from (5.6) and (5.8) that

\[
|G(0) - G_\lambda(0)| \leq \frac{1}{2\pi} \int_{\gamma_+} \left| [G(z) - G_\lambda(z)] e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) \right| dz \leq \frac{1}{2\pi} e^{-\lambda x} \cdot e^{\lambda x} \cdot \frac{2x}{R^2} \cdot \pi R = \frac{1}{R} < \frac{\varepsilon}{3}. \tag{5.9}
\]

If we denote by \( \gamma_- \) the part of \( \gamma \) lying in \( \text{Re}(z) < 0 \), then we have that

\[
\frac{1}{2\pi i} \int_{\gamma_-} G_\lambda(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz = \frac{1}{2\pi i} \int_{|z|=R} G_\lambda(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz
\]

since \( G_\lambda(z) \) is analytic on \( \mathbb{C} \). Thus similarly to (5.9) we obtain that

\[
\left| \frac{1}{2\pi i} \int_{\gamma_-} G_\lambda(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| \leq \frac{1}{R} < \frac{\varepsilon}{3}. \tag{5.10}
\]

Since the function \( G(z) \left( \frac{1}{z} + \frac{z}{R^2} \right) \) is analytic on \( \gamma_- \), there is a constant \( M = M(\delta, R) = M(\varepsilon) > 0 \) such that

\[
\left| G(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) \right| \leq M e^{\lambda \text{Re}(z)}
\]

for each \( z \in \gamma_- \). Since \( \text{Re}(z) < 0 \) for \( z \in \gamma_- \), the integral

\[
\frac{1}{2\pi i} \int_{\gamma_-} G(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz
\]

tends to zero as \( \lambda \to \infty \), and so there is a constant \( N > 0 \) such that

\[
\left| \frac{1}{2\pi i} \int_{\gamma_-} G(z) e^{\lambda z} \left( \frac{1}{z} + \frac{z}{R^2} \right) dz \right| < \frac{\varepsilon}{3}
\]

whenever \( \lambda > N \). Thus if \( \lambda > N \), then it follows from (5.9), (5.10), and (5.11) that

\[
|G(0) - G_\lambda(0)| < \varepsilon.
\]

Therefore we are done. \( \square \)

**Corollary 5.8** [Simplified Version of the Theorem of Weiner and Ikehara].

Let \( f(x) \) be a monotone nondecreasing function defined for \( x \geq 1 \) with \( f(x) = O(x) \). Suppose that \( g(z) \) is analytic in some region containing the closed half-plane \( \text{Re}(z) \geq 1 \) except for a simple pole at \( z = 1 \) with residue \( \alpha \) and, for any \( z \) with \( \text{Re}(z) > 1 \), \( g(z) \) coincides with the Mellin transform of \( f(x) \), i.e.

\[
g(z) = z \int_1^\infty f(x) x^{-z-1} dx, \ \text{Re}(z) > 1.
\]

Then we have that \( f(x) \sim \alpha x \).

**Proof.** We note that the function \( F(t) \) defined by

\[
F(t) = e^{-t} f(e^t) - \alpha
\]
is bounded on \((0, \infty)\) and integrable on each compact subset of \((0, \infty)\). Also its Laplace transform

\[
G(z) = \int_0^\infty [e^{-t} f(e^t) - \alpha] e^{-zt} \, dt = \int_1^\infty f(x) x^{-z-2} \, dx - \frac{\alpha}{z} = \frac{1}{z+1} g(z+1) - \frac{\alpha}{z}
\]

is well-defined in \(\text{Re}(z) > 0\). By the assumption, the right-hand side of (5.12) is analytic in some region containing the closed half-plane \(\text{Re}(z) \geq 0\). Thus it follows from Lemma 5.7 [Tauberian Theorem of Ingham and Newman] that the improper integral

\[
\int_0^\infty [e^{-t} f(e^t) - \alpha] \, dt = \int_1^\infty \frac{f(x) - \alpha x}{x^2} \, dx
\]

converges. Now we shall prove that \(f(x) \sim \alpha x\) by using the nondecreasing monotonicity of \(f\).

If \(\limsup_{x \to \infty} \frac{f(x)}{x} > \alpha\), then there exists some \(\delta > 0\) so that \(f(y) > (\alpha + 2\delta)y\) for infinitely many and arbitrarily large \(y\). Thus \(f(x) > (\alpha + 2\delta)y > (\alpha + \delta)x\) for all \(x\) with \(y < x < \left(\frac{\alpha + 2\delta}{\alpha + \delta}\right) y\), and

\[
\int_y^{\left(\frac{\alpha + 2\delta}{\alpha + \delta}\right) y} f(x) - \alpha x \, dx > \int_y^{\left(\frac{\alpha + 2\delta}{\alpha + \delta}\right) y} \frac{\delta}{x} \, dx = \delta \cdot \ln \left(\frac{\alpha + 2\delta}{\alpha + \delta}\right) > 0.
\]

This gives a contradiction. So we conclude that

\[
\limsup_{x \to \infty} \frac{f(x)}{x} \leq \alpha.
\]

If \(\liminf_{x \to \infty} \frac{f(x)}{x} < \alpha\), then there exists some \(\delta > 0\) with \(\delta < \alpha/2\) so that \(f(y) < (\alpha - 2\delta)y\) for infinitely many and arbitrarily large \(y\). Thus \(f(x) < (\alpha - 2\delta)y < (\alpha - \delta)x\) for all \(x\) with \(y < x < \left(\frac{\alpha - 2\delta}{\alpha - \delta}\right) y\), and

\[
\int_y^{\left(\frac{\alpha - 2\delta}{\alpha - \delta}\right) y} f(x) - \alpha x \, dx < \int_y^{\left(\frac{\alpha - 2\delta}{\alpha - \delta}\right) y} \frac{-\delta}{x} \, dx = -\delta \cdot \ln \left(\frac{\alpha - \delta}{\alpha - 2\delta}\right) < 0.
\]

This gives a contradiction. So we conclude that

\[
\liminf_{x \to \infty} \frac{f(x)}{x} \geq \alpha.
\]

Therefore we complete the proof from (5.13) and (5.14). □

**Lemma 5.9 [Mertens].** \(\zeta(z) \neq 0\) for any \(z\) with \(\text{Re}(z) = 1\) and \(z \neq 1\).

**Proof.** We observe that \(3 + 4 \cos \theta + \cos(2\theta) = 2(1 + \cos \theta)^2 \geq 0\) for any \(\theta \in \mathbb{R}\). If \(\zeta(1 + it) = 0\) for some \(t \neq 0\), then the equation

\[
\Theta(s) = \zeta(s)^3 \cdot \zeta(s + it)^4 \cdot \zeta(s + 2it)
\]

has a zero at \(s = 1\). Thus we have that

\[
\lim_{s \to 1} \ln |\Theta(s)| = -\infty.
\]
Now it follows from Theorem 4.3.11 that for any $s = \sigma > 1$,

\[
\ln |\zeta(\sigma + it)| = -\text{Re} \left( \sum_p \ln(1 - p^{-\sigma - it}) \right)
\]

\[
= \text{Re} \left( \sum_p \left( p^{-\sigma - it} + \frac{1}{2}(p^2)^{-\sigma - it} + \frac{1}{3}(p^3)^{-\sigma - it} + \cdots \right) \right)
\]

\[
= \text{Re} \left( \sum_{n=1}^{\infty} b_n n^{-\sigma - it} \right)
\]

where $b_n$’s are certain nonnegative constants. This leads to the following inequalities

\[
\ln |\Theta(\sigma)| = \text{Re} \left( \sum_{n=1}^{\infty} b_n n^{-\sigma}(3 + 4n^{-it} + n^{-2it}) \right)
\]

\[
= \sum_{n=1}^{\infty} b_n n^{-\sigma}(3 + 4\cos(t \ln n) + \cos(2t \ln n)) \geq 0,
\]

which contradict to (5.15). Hence we complete the proof. □

**Theorem 5.10 [Prime Number Theorem].**

If $\pi(x)$ denotes the number of prime numbers $p \leq x$, then we have that $\pi(x) \sim \frac{x}{\ln x}$.

**Proof.** First of all, by Theorem 5.4 [Chebyshev’s Theorem] we observe that

\[
\psi(x) = \sum_{p \leq x} \left[ \frac{\ln x}{\ln p} \right] \ln p \leq \ln x \sum_{p \leq x} 1
\]

\[
= \pi(x) \ln x = \mathcal{O}(x).
\]

By Proposition 5.5, it suffices to show that

\[
\psi(x) \sim x.
\]

By Theorem 4.3.18, the Mellin transform of $\psi(x)$ is

\[
-\frac{\zeta'(z)}{\zeta(z)} = z \int_{1}^{\infty} \frac{\psi(x)}{x^{z+1}} dx, \quad \text{Re}(z) > 1.
\]

In order to apply Corollary 5.8, we shall show that the function

\[
-\frac{\zeta'(z)}{\zeta(z)} \frac{1}{z-1}
\]

is analytic in some region containing the closed half-plane $\text{Re}(z) \geq 1$. By Proposition 4.3.16, there is some $\delta > 0$ so that

\[
\zeta(z) = \frac{1}{z-1}(1 + h(z))
\]

where $h(z)$ is analytic in $B(1; \delta)$ and $|h(z)| < 1$ there. Thus this implies that the function

\[
-\frac{\zeta'(z)}{\zeta(z)} \frac{1}{z-1} = -\frac{h'(z)}{1 + h(z)}
\]

is analytic at $z = 1$. Finally, it follows from Proposition 4.3.16 and Lemma 5.9 that the function

\[
-\frac{\zeta'(z)}{\zeta(z)} \frac{1}{z-1}
\]

is analytic at any other points $z$ with $\text{Re}(z) = 1$. Hence are are done. □
Corollary 5.11. Let $f(x)$ be a number theoretic function with nonnegative values and with
\[ \sum_{n \leq x} f(n) = O(x), \]
and let the Dirichlet series
\[ F(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \]
be analytic in $\text{Re}(z) > 1$ in the sense that the function
\[ F(z) - \frac{\alpha}{z - 1} \quad (\alpha \text{ is some fixed constant}) \]
is analytic in some region containing the closed half-plane $\text{Re}(z) \geq 1$. Then we have that
\[ \sum_{n \leq x} f(n) \sim \alpha x. \]

Proof. It easily follows from Corollary 5.8 and the following integral representation
\[ F(z) = z \int_{1}^{\infty} \left( \sum_{n \leq x} f(n) \right) x^{-z-1} \, dx. \]

Corollary 5.12. Let $f(n)$ and $g(n)$ be two number theoretic functions satisfying that $f(n) \geq 0$, $g(n) = O(f(n))$, and $\sum_{n \leq x} f(n) = O(x)$. If two Dirichlet series
\[ F(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} \frac{g(n)}{n^z} \]
are analytic in $\text{Re}(z) > 1$ in the sense that the functions
\[ F(z) - \frac{\alpha}{z - 1}, \quad G(z) - \frac{\beta}{z - 1} \quad (\alpha \text{ and } \beta \text{ are some fixed constants}) \]
are analytic in some region containing the closed half-plane $\text{Re}(z) \geq 1$, then we have that
\[ \sum_{n \leq x} g(n) \sim \gamma x. \]

Proof. First, we assume that $g(n)$ is real-valued. Let us choose some constant $K > 0$ so large that $|g(n)| \leq Kf(n)$ for all $n \in \mathbb{N}$. We now apply Corollary 5.11 to the Dirichlet series generated by the number theoretic function $h(n) = Kf(n) + g(n)$, given by
\[ H(z) = \sum_{n=1}^{\infty} \frac{h(n)}{n^z} = KF(z) + G(z). \]

By Corollary 5.11, we have that
\[ \sum_{n \leq x} h(n) = K \sum_{n \leq x} f(n) + \sum_{n \leq x} g(n) \sim K\alpha x + \sum_{n \leq x} g(n) \]
\[ \sum_{n \leq x} h(n) \sim K\alpha x + \beta x. \]

This implies the conclusion.

If \( g(n) \) is complex-valued, then we set \( G^*(z) = \overline{G(z)} \) and we consider

\[ G_1(z) \coloneqq \frac{1}{2} |G(z) + G^*(z)| = \sum_{n=1}^{\infty} \frac{\text{Re}(g(n))}{n^z} \]

and

\[ G_2(z) \coloneqq \frac{1}{2i} |G(z) - G^*(z)| = \sum_{n=1}^{\infty} \frac{\text{Im}(g(n))}{n^z}. \]

Hence we complete the proof by applying the above argument to \( G_1(z) \) and \( G_2(z) \). \( \square \)

In what follows, we furnish three examples as a foretaste of importance of Corollary 5.12.

**Corollary 5.13.** If \( \mu(n) \) is the Möbius function and \( \lambda(n) \) is the Liouville function, then we have that

\[ \sum_{n \leq x} \mu(n) = o(x) \quad \text{and} \quad \sum_{n \leq x} \lambda(n) = o(x). \]

**Proof.** By Proposition 4.3.15, we apply Corollary 5.12 to the associated Dirichlet series \( G(z) = \frac{1}{\zeta(z)} \) and \( G(z) = \frac{\zeta(2z)}{\zeta(z)} \) which are analytic in some region containing the closed half-plane \( \text{Re}(z) \geq 1 \). Since they have no singularity at \( z = 1 \), we conclude that \( \beta = 0 \). \( \square \)

As a third example, we consider the Dirichlet series

\[ \zeta_i(z) = \sum_{n=1}^{\infty} \frac{r(n)}{n^z} \]

generated by the number theoretic function \( r(n) \) which counts the number of the representations of \( n \) as the sum of two squares. By Proposition 3.25 in Chapter 3, \( r(n) \) can be considered as the number of representations \( n = \omega \overline{\omega} \) where \( \omega \) runs through the ring \( \mathbb{Z}(i) \). Thus we obtain that

\[ \zeta_i(z) = \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{1}{|\omega|^2} = \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{1}{|\omega|^2}, \]

which is called the \( \zeta \)-function for the number theory on the ring \( \mathbb{Z}(i) \). In order to keep track of the arguments of \( \omega \in \mathbb{Z}(i) \setminus \{0\} \), Hecke originated the following Dirichlet series

\[ \Xi(h, z) = \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{1}{|\omega|^2} \cdot e^{4i h \arg(\omega)}, \quad h \in \mathbb{Z}. \]

Then it is clear that \( \Xi(0, z) = \zeta_i(z) \) and

\[ \Xi(h, z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \left( \sum_{|\omega|^2 = n} e^{4i h \arg(\omega)} \right), \quad \text{Re}(z) > 1. \]

Its convergence for \( \text{Re}(z) > 1 \) follows from the convergence of \( \zeta_i(z) \) for \( \text{Re}(z) > 1 \); which can be derived from the estimate

\[ \sum_{x \leq n \leq y} \frac{r(n)}{n^z} = \frac{1}{y^z} \mathcal{O}(y-x) + z \int_{x}^{y} \mathcal{O}(t-x) t^{-z-1} dt = \mathcal{O} \left( \frac{1}{x^{z-1}} \right) \]

which is obtained by applying Proposition 4.2.2[Abel Transformation] and Proposition 4.2.8. The argument function \( \arg(\omega) \) in \( \Xi(h, z) \) is uniquely defined in \(-\pi < \arg(\omega) \leq \pi\).
Definition 5.14. Let $f$ be a complex-valued function defined on $\mathbb{Z}(i)$. Then $f$ is said to be multiplicative if $f \neq 0$ and

\begin{equation}
(5.16) \quad f(mn) = f(m)f(n)
\end{equation}

for any pair $(m,n) \in \mathbb{Z}(i) \times \mathbb{Z}(i)$ with no common prime factor. If (5.16) holds for any pair $(m,n) \in \mathbb{Z}(i) \times \mathbb{Z}(i)$, then we say that $f$ is completely multiplicative.

For instance, for $h \in \mathbb{Z}$ we consider the function $f(\omega) = e^{4h \arg(\omega)}$. Then it is certainly completely multiplicative and satisfies that $f(u) = 1$ for unit elements $u = 1, i, -1, -i$. This is the reason why the factor 4 in the exponent was taken in $\Xi(h, z)$.

Proposition 5.15. Let $f$ be a complex-valued function defined on $\mathbb{Z}(i)$ satisfying that $f(u) = 1$ for all units $u \in \mathbb{Z}(i)$. Suppose that the infinite series

$$F(z) = \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} f(\omega) \frac{1}{|\omega|^2z}$$

converges absolutely for $\Re(z) > \tau_0$.

(a) If $f$ is multiplicative, then we have that for all $z$ with $\Re(z) > 1$,

$$F(z) = 4 \prod_{p \in \mathbb{Z}_0^+(i)} \left( \sum_{\mu=1}^{\infty} \frac{f(p^\mu)}{|p|^{2\mu z}} \right)$$

where $\mathbb{Z}_0^+(i)$ is the set of all prime elements $p$ of $\mathbb{Z}(i)$ with $0 \leq \arg(p) < \pi/2$.

(b) If $f$ is completely multiplicative, then we have that for all $z$ with $\Re(z) > 1$,

$$F(z) = 4 \prod_{p \in \mathbb{Z}_0^+(i)} \frac{1}{1 - \frac{f(p)}{|p|^{2z}}}.$$

(c) For $h \in \mathbb{Z}$, we have that

$$\Xi(h, z) = 4 \prod_{p \in \mathbb{Z}_0^+(i)} \frac{1}{1 - \frac{e^{4h \arg(p)}}{|p|^{2z}}}, \quad \Re(z) > 1.$$

Proof. It easily follows from the modification of Proposition 4.3.13. \qed

Definition 5.16. We consider the function $\Lambda_i$ defined on $\mathbb{Z}(i)$ given by

$$\Lambda_i(\omega) = \begin{cases} 
\ln |p|, & \text{if } \omega = up^\nu \text{ for a unit } u \text{ and a prime } p \\
0, & \text{if } \omega \text{ is not such a prime power},
\end{cases}$$

which is called the generalized Mangoldt function.

In Chapter 4, we saw the relation between the Mangoldt function and the quotient $\zeta'(z)/\zeta(z)$. Similarly, in what follows we study the connection between the generalized Mangoldt function and the quotient

$$\frac{\Xi'(h, z)}{\Xi(h, z)},$$

in particular, this quotient will play an important role in the Mellin transform of the function

\begin{equation}
(5.17) \quad \psi_i(x) = \sum_{\omega \in \mathbb{B}_x(i)} \Lambda_i(\omega)
\end{equation}

where $\mathbb{B}_x(i) = \{\omega \in \mathbb{Z}(i) : |\omega|^2 \leq x\}$. 
Lemma 5.17. For Re\(z\) > 1 and \(h \in \mathbb{Z}\), we have that
\[
-\frac{\Xi'(h, z)}{\Xi(h, z)} = \frac{1}{2} \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{\Lambda_i(\omega)}{|\omega|^{2z}} e^{4ih\arg(\omega)}.
\]

Proof. Since \(\log(1 - e^{4ih\arg(p) \cdot |p|^{-2z}}) = O(|p|^{-2\text{Re}(z)})\), the series
\[
H(z) \approx \log 4 - \sum_{p \in \mathbb{Z}_p^+(i)} \log \left(1 - \frac{e^{4ih\arg(p) \cdot |p|^{-2z}}}{|p|^{-2z}}\right)
\]
converges uniformly in every compact subsets inside the half-plane Re\((z) > 1\), and so \(H(z)\) is analytic in Re\((z) > 1\). We also have the relation
\[
e^{H(z)} = \Xi(h, z).
\]
Thus we obtain that
\[
H'(z) \cdot \Xi(h, z) = \Xi'(h, z).
\]
Therefore we complete the proof by calculating \(H'(z)\) as follows;
\[
H'(z) = \sum_{p \in \mathbb{Z}_p^+(i)} \frac{1}{1 - \frac{e^{4ih\arg(p) \cdot |p|^{-2z}}}{|p|^{-2z}}} \cdot \frac{e^{4ih\arg(p) \cdot |p|^{-2z}}}{|p|^{-2z}} \cdot \log |p| \cdot \sum_{\mu=0}^{\infty} \frac{e^{4ih\arg(p^\mu)}}{|p^\mu|^{-2z}}
\]
\[
= \sum_{u \in \mathfrak{U}} \sum_{p \in \mathbb{Z}_p^+(i)} \sum_{\mu=1}^{\infty} \frac{\log |up| \cdot e^{4ih\arg((up)^\mu)}}{|(up)^\mu|^{-2z}}
\]
\[
= \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{\Lambda_i(\omega)}{|\omega|^{2z}} e^{4ih\arg(\omega)},
\]
where \(\mathfrak{U}\) denotes the set of all unit elements \(u\) of \(\mathbb{Z}(i)\).

Lemma 5.18. For all \(z\) with Re\((z) > 1\), we have the integral representation
\[
-\frac{\zeta'_i(z)}{\zeta_i(z)} = \frac{z}{2} \int_1^\infty \frac{\psi_i(x)}{x^{z+1}} \, dx
\]
where \(\psi_i\) is a function defined by \(\psi_i(x) = \sum_{\omega \in \mathbb{B}_z(i)} \Lambda_i(\omega)\).

Proof. By Lemma 5.17, we have that
\[
-\frac{\zeta'_i(z)}{\zeta_i(z)} = \frac{1}{2} \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{\Lambda_i(\omega)}{|\omega|^{2z}}.
\]
It also follows from Proposition 4.2.2 [Abel Transformation] that

\[(5.18) \quad \sum_{\omega \in \mathbb{B}_x(i) \setminus \{0\}} \frac{\Lambda_i(\omega)}{|\omega|^{2\pi}} = \frac{1}{x^2} \cdot \psi_i(x) - \int_1^x \psi_i(y) \cdot \frac{-\psi_i(y)}{y^2+1} \, dy.\]

From Proposition 3.24, we observe that

\[(5.19) \quad \sum_{p \in \mathbb{Z}_p(i), |p|^2 \leq x} 1 \sim \pi(x)\]

where \(\mathbb{Z}_p(i)\) denotes the set of all prime elements of \(\mathbb{Z}(i)\). Thus by the definition of \(\psi_i(x)\) and Theorem 5.4 [Chebyshev’s theorem] we obtain that

\[(5.20) \quad \psi_i(x) = \sum_{\omega \in \mathbb{B}_x(i)} \Lambda_i(\omega) = 4 \sum_{p \in \mathbb{Z}_p(i), |p|^2 \leq x} \left[ \frac{\ln x}{2 \ln |\omega|} \right] \ln |\omega| = \mathcal{O} \left( \sum_{p \in \mathbb{Z}_p(i), |p|^2 \leq x} \ln x \right) = \mathcal{O} \left( \sum_{p \in \mathbb{Z}_p(i), |p|^2 \leq x} 1 \right) = \mathcal{O}(x).\]

Taking the limit \(x \to \infty\) in (5.18), we can complete the proof. \(\square\)

**Lemma 5.19.** For \(h \in \mathbb{Z} \setminus \{0\}\), we have that

\[\sum_{\omega \in \mathbb{B}_x(i) \setminus \{0\}} e^{4ih \arg(\omega)} = \mathcal{O}(\sqrt{x} \ln x).\]

**Proof.** We write \(\omega = a + ib\) for \(a, b \in \mathbb{Z}\). Observing that \(\arg(a + ib) = \pi/2 - \arg(b + ia)\) for \(a, b \in \mathbb{N}\) and considering only the sum over non-associated elements, we have that

\[\sum_{\omega \in \mathbb{B}_x(i) \setminus \{0\}} e^{4ih \arg(\omega)} = 4 \sum_{a > 0} \sum_{b \geq \sqrt{a^2 + b^2} \leq x} e^{4ih \arg(a + ib)} \cos(4h \arg(a + ib)) + \mathcal{O}(\sqrt{x})\]

\[= 8 \sum_{a > 0} \sum_{b \geq \sqrt{a^2 + b^2} \leq x} \cos(4h \arg(a + ib)) + \mathcal{O}(\sqrt{x}).\]

Since \(\tan^{-1}\left(\frac{\sqrt{x^2 - a^2}}{a}\right) - \tan^{-1}1 = \mathcal{O}(1)\), it follows from Proposition 4.2.3 [The Euler Sum Formula] that

\[\sum_{\omega \in \mathbb{B}_x(i) \setminus \{0\}} e^{4ih \arg(\omega)}\]

\[= 8 \sum_{1 \leq a \leq \sqrt{x}} \left( \int_a^{\sqrt{x-a^2}} \cos\left(4h \tan^{-1}\left(\frac{y}{a}\right)\right) dy + \mathcal{O}\left(1 + \int_a^{\sqrt{x-a^2}} \frac{1}{a \left(1 + \frac{y^2}{a^2}\right)} \, dy\right)\right) + \mathcal{O}(\sqrt{x})\]

\[= 8 \sum_{1 \leq a \leq \sqrt{x}} \int_a^{\sqrt{x-a^2}} \cos\left(4h \tan^{-1}\left(\frac{y}{a}\right)\right) dy + \mathcal{O}(\sqrt{x})\]

\[= 8 \int_1^{\sqrt{x}} \int_t^{\sqrt{x-t^2}} \cos\left(4h \tan^{-1}\left(\frac{y}{t}\right)\right) dy \, dt\]

\[+ \mathcal{O}\left(\sqrt{x} + \int_1^{\sqrt{x}} \left|\frac{d}{dt} \int_t^{\sqrt{x-t^2}} \cos\left(4h \tan^{-1}\left(\frac{y}{t}\right)\right) dy\right| \, dt\right) + \mathcal{O}(\sqrt{x}).\]
We observe that
\[ \int_0^1 \int_{\sqrt{x-t^2}} \cos \left( 4h \tan^{-1} \left( \frac{y}{t} \right) \right) \, dy \, dt = O(\sqrt{x}) \]
and
\[ \frac{d}{dt} \int_{\sqrt{x-t^2}} \cos \left( 4h \tan^{-1} \left( \frac{y}{t} \right) \right) \, dy = \frac{1}{t^2} \int_{\sqrt{x-t^2}} \frac{4hy \sin(\tan^{-1}(\frac{y}{t}))}{1 + \frac{y^2}{t^2}} \, dy \]
\[ - \frac{t}{\sqrt{x-t^2}} \cos \left( 4h \tan^{-1} \left( \frac{\sqrt{x-t^2}}{t} \right) \right) - \cos(h\pi) \]
\[ = O \left( \int_{\sqrt{x-t^2}} \frac{y}{t^2 + y^2} \, dy + \frac{t}{\sqrt{x-t^2}} \right) \]
\[ = O \left( \ln \left( \frac{x}{2t^2} \right) + 1 \right). \]

Thus by applying polar coordinates \( t = r \cos \theta \) and \( y = r \sin \theta \) with \( 0 < r \leq \sqrt{x} \) and \( \pi/4 \leq \theta \leq \pi/2 \), we obtain that
\[ \sum_{\omega \in \mathbb{B}_r(i) \setminus \{0\}} e^{4ih \arg(\omega)} = 8 \int_0^\sqrt{x} \int_{\sqrt{x-t^2}} \cos \left( 4h \tan^{-1} \left( \frac{y}{t} \right) \right) \, dy \, dt + O(\sqrt{x} \ln x) \]
\[ = 8 \int_0^\sqrt{x} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(4h\theta) \, d\theta \, dr + O(\sqrt{x} \ln x) = O(\sqrt{x} \ln x), \]
because the last integral vanishes for \( h \in \mathbb{Z} \setminus \{0\} \). Therefore we complete the proof. \( \square \)

**Lemma 5.20.** Let \( f(n) \) be a number theoretic function satisfying
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N f(n) = \alpha. \]

For \( \text{Re}(z) > 1 \), we have the following formula
\[ \sum_{n=1}^\infty \frac{f(n)}{n^z} = \alpha \cdot \zeta(z) + \sum_{n=1}^\infty \left( \frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \left( \sum_{m=1}^n \frac{f(m)}{m} - n\alpha \right). \]

**Proof.** Applying Lemma 4.2.1 [Abel Transformation], we have that
\[ \sum_{n=1}^N \left( \frac{1}{(n+1)^z} - \frac{1}{n^z} \right) \left( \sum_{m=1}^n \frac{f(m)}{m} - n\alpha \right) \]
\[ = \sum_{n=1}^N \frac{1}{(n+1)^z} \left( \sum_{m=1}^{n+1} \frac{f(m)}{m} - \alpha(n+1) \right) - \left( \sum_{m=1}^n \frac{f(m)}{m} - n\alpha \right) \]
\[ - \frac{1}{(N+1)^z} \left( \sum_{m=1}^{N+1} \frac{f(m)}{m} - \alpha(N+1) \right) + (f(1) - \alpha) \]
\[ = \sum_{n=1}^{N+1} \frac{f(n) - \alpha}{n^z} - \frac{1}{(N+1)^z} \left( \alpha - \frac{1}{N+1} \sum_{m=1}^{N+1} f(m) \right) \]
\[ = \sum_{n=1}^{N+1} \frac{f(n)}{n^z} - \alpha \sum_{n=1}^{N+1} \frac{1}{n^z} + O \left( \frac{1}{(N+1)^{\text{Re}(z)-1}} \right). \]
Since \((N+1)^{-(\Re(z)-1)}\) tends to zero as \(N \to \infty\) for \(\Re(z) > 1\), and also
\[
\left| \sum_{n=1}^{N} \left( \frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \left( \sum_{m=1}^{n} f(m) - \alpha n \right) \right| = \left| \sum_{n=1}^{N} \left( \int_{n}^{n+1} \frac{1}{x^{z+1}} \, dx \right) \cdot n \left( \alpha - \frac{1}{n} \sum_{m=1}^{n} f(m) \right) \right|
= O \left( \sum_{n=1}^{N} \frac{|z|}{n^{\Re(z)}} \right)
\]
converges for \(\Re(z) > 1\), we can complete the proof by taking \(N \to \infty\). □

**Lemma 5.21.** For \(h \in \mathbb{Z} \setminus \{0\}\), \(\Xi(h, z)\) has an analytic continuation into the half-plane \(\Re(z) > 1/2\).
Similarly, the function
\[
\zeta_i(z) - \frac{\pi}{z - 1}
\]
has an analytic continuation into the half-plane \(\Re(z) > 1/2\) in the sense that \(\zeta_i(z)\) is analytic on \(\Re(z) > 1/2\) except for a simple pole at \(z = 1\) with residue \(\pi\).

**Proof.** If we set \(f(n) = \sum_{|\omega|^2=n} e^{4ih \arg(\omega)}\) for \(h \in \mathbb{Z} \setminus \{0\}\), then it follows from Lemma 5.19 that
\[
\alpha = \lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) = 0.
\]
By Lemma 5.20, we have that for \(h \in \mathbb{Z} \setminus \{0\}\),
\[
\Xi(h, z) = \sum_{n=1}^{\infty} \left( \frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \sum_{\omega \in \mathbb{B}_n(i) \setminus \{0\}} e^{4ih \arg(\omega)}.
\]
Thus it follows from Lemma 5.19 that the following sequence
\[
\sum_{n=M}^{N} \left( \frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \sum_{\omega \in \mathbb{B}_n(i) \setminus \{0\}} e^{4ih \arg(\omega)} = O \left( \frac{|z|}{\sqrt{n}} \sum_{n=M}^{N} \ln n \left| \int_{n}^{n+1} \frac{1}{x^{z+1}} \, dx \right| \right)
= O \left( \frac{|z|}{\sqrt{n}} \sum_{n=M}^{N} \frac{\ln n}{n^{\Re(z)+\frac{1}{2}}} \right)
\]
converges uniformly to zero as \(M \to \infty\) in every compact subsets of the half-plane \(\Re(z) > 1/2\). Hence this implies the analytic continuation of \(\Xi(h, z)\).

Similarly to the above, it follows from Lemma 5.20 that
\[
\zeta_i(z) = \sum_{n=1}^{\infty} \frac{r(n)}{n^z}
= \pi \cdot \zeta(z) + \sum_{n=1}^{\infty} \left( \frac{1}{n^z} - \frac{1}{(n+1)^z} \right) \left( \sum_{m=1}^{n} r(m) - n\pi \right).
\]
From Proposition 4.2.8, we see that
\[
\sum_{m=1}^{n} r(m) - n\pi = O(\sqrt{n}).
\]
Therefore we complete the proof by applying the above argument once again. □
Lemma 5.22. For $h \in \mathbb{Z} \setminus \{0\}$, $\Xi(h, z) \neq 0$ for any $z$ with $\text{Re}(z) = 1$.

Proof. It is trivial for the case $h = 0$ and $z = 1$, because $\zeta_i(z)$ has a pole at $z = 1$. For the other cases, we use a modified version of Lemma 5.9[Mertens].

Fix $h \in \mathbb{Z} \setminus \{0\}$. If $\Xi(h, 1 + it) = 0$ for some $t \neq 0$, then the equation

$$\Theta(s) = \zeta_i(z)^3 \cdot \Xi(h, s + it)^4 \cdot \Xi(2h, s + 2it)$$

has a zero at $s = 1$. Thus this implies that

$$\lim_{s \to 1} \ln |\Theta(s)| = -\infty.$$ 

Now it follows from Proposition 5.15, (c) that for any $s = \sigma > 1$,

$$\ln |\Xi(h, \sigma + it)| = \ln 4 - \sum_{p \in \mathbb{Z}_p^+(i)} \ln \left| 1 - e^{4ih \arg(p)} \right|_{|p|^{2\sigma + 2\sigma}} = \ln 4 + \sum_{p \in \mathbb{Z}_p^+(i)} \sum_{n=1}^{\infty} \frac{\cos n(4h \arg(p) - 2t \ln |p|)}{|p|^{2n\sigma}}.$$ 

This leads to the following inequalities

$$\ln |\Xi(s)| = 8 \ln 4 + \sum_{p \in \mathbb{Z}_p^+(i)} \sum_{n=1}^{\infty} 3 + 4 \cos n(4h \arg(p) - 2t \ln |p|) + \cos n(8h \arg(p) - 4t \ln |p|) \geq 0,$$ 

which contradicts to (5.21). Hence we complete the proof. \(\Box\)

Proposition 5.23. \(\psi_i(x) \equiv \sum_{\omega \in \mathbb{B}_r(i)} \Lambda_i(\omega) \sim 2x.\)

Proof. It is trivial that $\psi_i(x)$ is a monotone non-decreasing function on $[0, \infty)$. By (5.20), we have $\psi_i(x) = O(x)$. Thus it follows from Lemma 5.18 and Lemma 5.21 that the function $-\zeta_i(z)/\zeta_i(z)$ given by

$$-\frac{\zeta_i'(z)}{\zeta_i(z)} = z \int_1^\infty \frac{1}{2} \psi_i(x) \frac{1}{x^{z+1}} dx$$

is analytic in $\text{Re}(z) > 1$ and the function

$$-\frac{\zeta_i'(z)}{\zeta_i(z)} = \frac{1}{z - 1}$$

has an analytic continuation into some region containing the closed half-plane $\text{Re}(z) \geq 1$. Therefore Corollary 5.11 implies the conclusion. \(\Box\)

Proposition 5.24. \(\sum_{\omega \in \mathbb{B}_r(i)} e^{4ih \arg(\omega)} \Lambda_i(\omega) = o(x) \text{ for } h \in \mathbb{Z} \setminus \{0\}.\)

Proof. We observe that $e^{4ih \arg(\omega)} \Lambda_i(\omega) = O(\Lambda_i(\omega))$ for $h \in \mathbb{Z} \setminus \{0\}$ and $\omega \in \mathbb{Z}(i) \setminus \{0\}$. From Lemma 5.17 and Lemma 5.22, two Dirichlet series

$$-\frac{\zeta_i'(z)}{\zeta_i(z)} = \frac{1}{2} \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{\Lambda_i(\omega)}{|\omega|^{2z}} \quad \text{and} \quad -\frac{\Xi'(h, z)}{\Xi(h, z)} = \frac{1}{2} \sum_{\omega \in \mathbb{Z}(i) \setminus \{0\}} \frac{\Lambda_i(\omega) e^{4ih \arg(\omega)}}{|\omega|^{2z}}, \quad h \in \mathbb{Z} \setminus \{0\},$$

are analytic in $\text{Re}(z) > 1$ and have an analytic continuation with no singularity at $z = 1$ into some region containing the closed half-plane $\text{Re}(z) \geq 1$. Therefore Corollary 5.12 and Proposition 5.23 imply the required one. \(\Box\)
Theorem 5.25 [Hecke's Prime Number Theorem for the ring \( \mathbb{Z}(i) \)].

(a) If \( \pi_i(x) \) denotes the number of all non-associated prime elements \( p \) with \( |p|^2 \leq x \), i.e. the number of all prime elements in \( \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i) \), then we have that

\[
\pi_i(x) \sim \frac{x}{\ln x}.
\]

(b) If \( \pi_i(x; \alpha, \beta) \) denotes the number of all prime elements \( p \in \mathbb{Z}_p(i) \cap \mathbb{B}_x(i) \) with \( \alpha \leq \arg(p) < \beta \) for \( 0 \leq \alpha < \beta \leq 2\pi \), then we have that

\[
\pi_i(x; \alpha, \beta) \sim \frac{2}{\pi} (\beta - \alpha) \frac{x}{\ln x}.
\]

Proof. We observe the following estimate

\[
\sum_{k \geq 2} \sum_{p \in \mathbb{Z}_p^+(i): |p|^{2k} \leq x} e^{i k \arg (p)} \ln |p| = O \left( \ln x \sum_{k \geq 2} \sum_{p \in \mathbb{Z}_p^+(i): |p|^{2k} \leq x} 1 \right)
\]

\[
= O \left( \ln x \sum_{2 \leq k \leq \ln x / \ln 2} \frac{\sqrt{x}}{\ln \sqrt{x}} \right)
\]

\[
= O(\sqrt{x \ln x}).
\]

This implies that

\[
4 \sum_{p \in \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i)} e^{i \arg (p)} \ln |p| = 4 \sum_{k \geq 1} \sum_{p \in \mathbb{Z}_p^+(i): |p|^{2k} \leq x} e^{i k \arg (p)} \ln |p| + O(\sqrt{x \ln x})
\]

\[
= \sum_{\omega \in \mathbb{B}_x(i)} e^{i \arg (\omega)} \Lambda_i(\omega) + O(\sqrt{x \ln x})
\]

\[
= \begin{cases} 2x + o(x), & h = 0, \\ o(x), & h \neq 0. \end{cases}
\]

Thus it follows from the above estimate and Proposition 4.2.2 [Abel Transformation] that

\[
\sum_{p \in \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i)} e^{i \arg (p)} = \sum_{p \in \mathbb{Z}_p^+(i): 2 \leq |p|^2 \leq x} e^{i \arg (p)} \ln |p|^2 \cdot \frac{1}{\ln |p|^2}
\]

\[
= \frac{1}{\ln x} \sum_{p \in \mathbb{Z}_p^+(i): 2 \leq |p|^2 \leq x} e^{i \arg (p)} \ln |p|^2
\]

\[
- \int_2^x \frac{1}{t \ln^2 t} dt
\]

\[
= \frac{2}{\ln x} \sum_{p \in \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i)} e^{i \arg (p)} \ln |p| + O \left( \int_2^x \frac{1}{t \ln^2 t} dt \right)
\]

\[
= \begin{cases} \frac{x}{\ln x} + O (\frac{x}{\ln x}), & h = 0, \\ o (\frac{x}{\ln x}), & h \neq 0. \end{cases}
\]
(a) By (5.22) on $h = 0$, we have that
\[ \pi_i(x) = \sum_{p \in \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i)} e^{4ih \arg(p)} = \frac{x}{\ln x} + o\left(\frac{x}{\ln x}\right). \]

(b) It easily follows from (5.22) on $h \neq 0$ that
\[ \lim_{x \to \infty} \frac{1}{4\pi_i(x)} \sum_{p \in \mathbb{Z}_p(i) \cap \mathbb{B}_x(i)} e^{2\pi ih \left(\frac{x}{2\pi} \arg(p)\right)} = \lim_{x \to \infty} \frac{1}{\pi_i(x)} \sum_{p \in \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i)} e^{2\pi ih \left(\frac{x}{2\pi} \arg(p)\right)} \]
\[ = \lim_{x \to \infty} \frac{\ln x}{x} \sum_{p \in \mathbb{Z}_p^+(i) \cap \mathbb{B}_x(i)} e^{4ih \arg(p)} = 0. \]

Thus by Theorem 2.13 [Weyl’s Criterion] we see that the sequence
\[ \{\theta_{p,x} = \frac{2}{\pi} \arg(p) : p \in \mathbb{Z}_p(i) \cap \mathbb{B}_x(i), x \in \mathbb{R}_+\} \]
is uniformly distributed modulo $2\pi$. Hence by Proposition 2.12 we have that
\[ (5.23) \lim_{x \to \infty} \frac{1}{4\pi_i(x)} \sum_{p \in \mathbb{Z}_p(i) \cap \mathbb{B}_x(i)} f(\arg(p)) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \]
for any real-valued Riemann integrable function $f(\theta)$ on $[0, 2\pi)$. If we take $f(\theta) = \chi_{[\alpha,\beta]}(\theta)$ in (5.23), we obtain that
\[ \lim_{x \to \infty} \frac{\pi_i(x; \alpha, \beta)}{4\pi_i(x)} = \lim_{x \to \infty} \frac{1}{4\pi_i(x)} \sum_{p \in \mathbb{Z}_p(i) \cap \mathbb{B}_x(i)} f(\arg(p)) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = \frac{1}{2\pi} (\beta - \alpha). \]

Therefore this implies the required result. □

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