Lie symmetries are discussed for the Wheeler-DeWitt equation in Bianchi Class A cosmologies. In particular, we consider General Relativity, minimally coupled scalar field gravity and Hybrid Gravity as paradigmatic examples of the approach. Several invariant solutions are determined and classified according to the form of the scalar field potential. The approach gives rise to a suitable method to select classical solutions and it is based on the first principle of the existence of symmetries.

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1. INTRODUCTION

Nowadays astronomical observations have shown that if we consider our Universe on a large scale, its visible structure is accelerating, homogeneous and isotropic and, essentially, filled with pressureless dust. The simplest cosmic model which describes a Universe with the above properties is the Friedmann-Lemaître-Robertson-Walker (FLRW) model [1]. The evolution of the Universe from the radiation dominant epoch till the present cosmic acceleration can be well-explained by the FLRW model with a cosmological constant (the so-called ΛCDM model). However, it fails if one tries to address the whole early and late history of the Universe starting from the origin and the inflation epoch where quantum effects should be taken into account.

Anisotropies observed in the cosmic microwave background (CMB) are small enough to suggest that anisotropic models of spacetimes become isotropic ones by evolving in time [2–4]. One can expect that pre-inflationary anisotropies played an important role (for example they can be responsible for the coupling between the gravitational field and the inflaton field) so if inflationary models are considered, one should understand the dynamics of anisotropies. Models describing anisotropic but homogeneous universes are the so-called Bianchi cosmologies. They can be considered in standard General Relativity and in its extensions containing scalar fields.

In this paper we will consider the Lie symmetries of the Wheeler-DeWitt equation (WDW) in General Relativity and in scalar field cosmology assuming Bianchi spatially homogeneous spacetimes. We will use the Lie symmetries in order to define the unknown potential and derive exact solutions for the WDW equation and for the field equations. Symmetries are considered to play a central role in physical problems because they provide first integrals which can be utilized in order to simplify a given system of differential equations and thus to determine the integrability of the system. Indeed, in [5, 7] it has been shown that the Lie symmetries of a dynamical system are related to the geometry of the underlying space where dynamics occurs.

The application of symmetries in gravitational theories is an important tool which could lead to exact solutions of the field equations. In particular, the symmetries which can be used are the Noether symmetries of the Lagrangian of the field equations and they have been applied in several models such as scalar tensor cosmology [8–13], $f(R)$ gravity and higher order theories of gravity [14–21], spherically symmetric spacetimes [22–23] and many others. [24–28]. The application of the Noether symmetries in Bianchi spatially homogeneous spacetimes can be found in [2]. According to this results one can deal with the so-called Noether Symmetry Approach.

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In this work we will not apply the Noether symmetries of the field equations but the Lie symmetries of the WDW equation. It has been proved in [7] that the Lie symmetries of the WDW equation could form a greater Lie algebra than the Noether symmetries of the Lagrangian of the field equations. Hence, it is possible to determine new cases where the field equations are integrable. This method has been applied to some axisymmetric quantum cosmologies with scalar fields [33, 34]. Recently a similar method has been applied to some asymetric quantum cosmologies with scalar fields [35].

The layout of the paper is the following. In Sec. 2 we give the basic definition of the Bianchi classification while in Sec. 3 we study the Lie symmetries of the WDW equation of class A Bianchi spacetimes in the case of General Relativity. In Sec. 4 the previous results are applied in order to reduce the WDW equation by using the Lie invariants and determine invariant solutions. Secs. 5 and 6 are devoted to the same analysis in the case of a minimally coupled scalar tensor gravity and we use the Lie symmetries in order to determine the field potential by using the Lie symmetries of the WDW equation as a geometric criterion. We show that, in scalar tensor cosmology, there exist invariant solutions of the WDW equation: in Bianchi I spacetime, for constant potential and for exponential potential and, in Bianchi II spacetime, for a kination scalar field. For convenience of the reader, we present the Lie symmetry classification for each model in tables. Furthermore in Sec. 7 we show how these results are related with the so-called Hybrid Gravity and conformal transformations. Finally, in Sec. 8 we discuss our results and draw conclusions. Appendix A completes our presentation. Here the basic theory of Lie symmetries is briefly discussed.

2. THE CLASS A OF BIANCHI SPACETIMES

The class of Bianchi spatially homogeneous cosmologies contains several important cosmological models which have been used for the discussion of anisotropies of primordial Universe and for its evolution towards the observed isotropy of the present epoch [2, 36–39]. In these models, the spacetime manifold is foliated along the time axis, with three dimensional homogeneous hypersurfaces which admit a group of motions $G_3$. Bianchi classified all three dimensional real Lie algebras and has shown that there are nine possible $G_3$ groups. This results in nine types of Bianchi spatially homogeneous spacetimes. The importance of Bianchi cosmological models is that, in these models, the physical variables depend on the time only, reducing the Einstein and other governing equations to ordinary differential equations.

In the $(3 + 1)$ decomposition of spacetime manifold (Arnowitt-Deser-Misner (ADM) formalism), the line element of the Bianchi models can be written in the following form [40, 41]

$$ds^2 = -\frac{1}{N(t)^2}dt^2 + \bar{g}_{ij}(t)\omega^i \otimes \omega^j,$$

(1)

where $N(t)$ is the lapse function and $\{\omega^i\}$ denotes the canonical basis 1-forms satisfying the Lie algebra

$$d\omega^i = C^i_{jk} \omega^j \wedge \omega^k,$$

(2)

where $C^i_{jk}$ are the structure constants of the algebra. The spatial metric $\bar{g}_{ij}$ is diagonal (following the notation of [5, 30] and references therein) and can be factorized as follows

$$\bar{g}_{ij}(t) = e^{2\lambda(t)} e^{-2\beta_{ij}(t)},$$

(3)

where $e^{\lambda(t)}$ is the scale factor of the universe and the matrix $\beta_{ij}$ is diagonal and traceless. The matrix $\beta_{ij}$ depends on two independent quantities $\beta_1, \beta_2$ which are called the anisotropy parameters. The matrix $\beta_{ij}$ can be selected as

$$\beta_{ij} = diag \left( \beta_1, -\frac{1}{2} \beta_1 + \sqrt{3} \beta_2, -\frac{1}{2} \beta_1 - \sqrt{3} \beta_2 \right).$$

(4)

and, in these variables, it is $\sqrt{\bar{g}} = e^{3\lambda}$. The structure constants of the Lie algebra $G_3$ can be expressed in terms of a three dimensional vector field $a^i$ and a symmetric $3 \times 3$ tensor $m^{ij}$ as follow [42]

$$C^i_{jk} = \epsilon_{ikl} m^{l} + \delta^i_k a_j - \delta^i_j a_k,$$

(5)

1 These are the Misner variables [41].
TABLE I: The Ricci scalar of the 3d hypersurfaces of the class A Bianchi spacetimes.

| Model          | $R^* (\lambda, \beta_1, \beta_2)$ |
|----------------|-----------------------------------|
| Bianchi I      | 0                                 |
| Bianchi II     | $-\frac{1}{e^{2\lambda}} e^{4\beta_1} (e^{2\beta_1} + e^{-2(\beta_1 + \sqrt{3}\beta_2)}) + 2e^{\beta_1 + \sqrt{3}\beta_2}$ |
| Bianchi VII/VI | $-\frac{1}{2} e^{-2\lambda} \left( e^{4\beta_1} + e^{-2\beta_1} \left( e^{\sqrt{3}\beta_2} + e^{-\sqrt{3}\beta_2} \right)^2 \right)$ |
| Bianchi VIII   | $-\frac{1}{2} e^{-2\lambda} \left( e^{4\beta_1} + e^{-2\beta_1} \left( e^{\sqrt{3}\beta_2} + e^{-\sqrt{3}\beta_2} \right)^2 \right)$ + 1 |
| Bianchi IX     | $-\frac{1}{2} e^{-2\lambda} \left( e^{4\beta_1} + e^{-2\beta_1} \left( e^{\sqrt{3}\beta_2} + e^{-\sqrt{3}\beta_2} \right)^2 \right)$ + 1 |

and the Bianchi models are grouped in two classes: class A for $a^i = 0$ and Class B for $a^i \neq 0$. Each class is divided into several types according to the rank and the signature of the tensor $m^{ij}$. Specifically, the Bianchi models are divided into two subclasses A ($a_i = 0$) and B ($a_i \neq 0$) containing Bianchi types corresponding to the form of the metric $m^{ij}$. In this paper, we are interested in the class A models for which there exists a Lagrangian of the field equations.

For the line element (1) with the definitions (3) and (4), the Ricci scalar of the Bianchi class A spacetimes can be written as

$$ R = R_{(4)} + R^* $$

where

$$ R_{(4)} = \frac{3}{2} N \left( 4N\dot{\lambda} + 4\dot{N}\lambda + 8N\ddot{\lambda}^2 + N\dot{\beta}_1^2 + N\dot{\beta}_2^2 \right) $$

and $R^* = R^* (\lambda, \beta_1, \beta_2)$ is the component of the three dimensional hypersurface which depends on the structure constants of the algebra $N_{1-3}$ of the Killing algebra of the Bianchi algebras [43]. The general form of $R^*$ is

$$ R^* = -\frac{1}{2} e^{-2\lambda} \left[ N_1^2 e^{4\beta_1} + e^{-2\beta_2} \left( N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2} \right)^2 \right] \right] + \frac{1}{2} N_1 N_2 N_3 (1 + N_1 N_2 N_3) $$

and the special forms for the class A spacetimes can be found in Table I.

In the case of General Relativity, when the action of the field equations is the Einstein-Hilbert action (we consider that the spacetime is empty)

$$ S_{GR} = \int dx^4 \sqrt{-g} R $$

the field equations for the Bianchi class A spacetimes follow from the Lagrangian

$$ L (N, \lambda, \beta_1, \beta_2, \dot{\lambda}, \dot{\beta}_1, \dot{\beta}_2) = N(t) e^{3\lambda} \left( 6\dot{\lambda}^2 - \frac{3}{2} \left( \dot{\beta}_1^2 + \dot{\beta}_2^2 \right) \right) + \frac{e^{3\lambda}}{N(t)} R^* $$

and the corresponding field equations are the Euler-Lagrange equations with respect to the variables $\{N, \lambda, \beta_1, \beta_2\}$. The Euler-Lagrange equations for the variables $\beta_1, \beta_2$ are:

$$ \ddot{\beta}_1(1,2) + \frac{\dot{N}}{N} \dot{\beta}_1(1,2) + 3\dot{\lambda} \dot{\beta}_1(1,2) + \frac{1}{3N^2} R^* (1,2) = 0 $$

for the variable $\lambda$:

$$ 4\ddot{\lambda} + \left( 6\dot{\lambda}^2 + \frac{3}{2} \left( \dot{\beta}_1^2 + \dot{\beta}_2^2 \right) \right) + \frac{\dot{N}}{N} \dot{\lambda} - \frac{1}{N^2} \left( R^* + \frac{1}{3} \frac{\partial R^*}{\partial \lambda} \right) = 0 $$
and for the variable $N$ we have the $G_0^0 = 0$ Einstein equation

$$N(t) e^{3\lambda} \left( 6\lambda^2 - \frac{3}{2} \left( \frac{\dot{\lambda}^2}{\dot{\lambda}_1^2} + \frac{\dot{\lambda}^2}{\dot{\lambda}_2^2} \right) \right) - \frac{e^{3\lambda}}{N(t)} R^* = 0. \tag{13}$$

Lagrangian (10) is singular since $\frac{\partial L}{\partial \dot{N}} = 0$, however, if we consider that $N(t) = N_0$ or $N(t) = N(\lambda(t), \beta_1(t), \beta_2(t))$ then Lagrangian (10) becomes a regular time independent Lagrangian which admits always, as a Noether integral, the Hamiltonian constant. Hence, equation (13) can be seen as the energy constrain of the field equations.

In the following we will quantize equation (13) in order to write the Wheeler-DeWitt (WDW) equation and to perform a symmetry analysis using the Lie point symmetries in the case of General Relativity and minimally coupled scalar tensor cosmology.

3. SYMMETRIES OF THE WDW EQUATION IN GENERAL RELATIVITY

In order to simplify equation (13), we consider $N(t) = \bar{N}(t) e^{-3\lambda}$ in the line element (1). Furthermore, we consider the following change of the variables $(\lambda, \beta_1, \beta_2) = (\sqrt[3]{6x}, \sqrt[3]{3y}, \sqrt[3]{3z})$, then the Lagrangian (10) becomes

$$L = \frac{1}{2} \bar{N} \left( x^2 - y^2 - z^2 \right) + \frac{1}{N} e^{\sqrt[3]{3x}} R^* . \tag{14}$$

Therefore, we define the momentum $p(x,y,z) = \frac{\partial L}{\partial \dot{x}}$ and equation (13) has now the form

$$\frac{1}{2} \left( p_x^2 - p_y^2 - p_z^2 \right) - e^{\sqrt[3]{3x}} R^* = 0. \tag{15}$$

Equation (15) can be seen as the Hamiltonian of a particle moving in the space $M^3$ with potential $V(x,y,z) = -e^{\sqrt[3]{3x}} R^*$. Furthermore, the field equations are the Hamiltonian constraint (15) and the Hamilton equations

$$\dot{x} = \frac{1}{N} p_x , \quad \dot{y} = -\frac{1}{N} p_y , \quad \dot{z} = -\frac{1}{N} p_z \tag{16}$$

$$\dot{p}_x = -\sqrt[3]{3} \left(1 + (\ln R^*)_x \right) e^{\sqrt[3]{3x}} R^* \tag{17}$$

$$\dot{p}_y = e^{\sqrt[3]{3x}} R^*_y , \quad \dot{p}_z = e^{\sqrt[3]{3x}} R^*_z \tag{18}$$

Since the minisuperspace is flat; that is the Ricci scalar vanishes, the WDW equation can be achieved by a standard quantization, assuming the conjugate momenta $p_i = \frac{\partial L}{\partial \dot{x}_i}$. Hence from (15), we have the WDW equation of the form

$$\Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - 2 e^{\sqrt[3]{3x}} R^* \Psi = 0. \tag{19}$$

which is nothing else but the Klein Gordon equation in the $M^3$ space.

In order to determine the Lie symmetries of (19), we will use the geometric results in Ref. [7]. The $M^3$ spacetime admits a ten dimensional conformal algebra. In particular, it admits a six dimensional Killing algebra which is the $T^3 \cup SO(3)$ with elements:

$$K_{(x)} = \partial_x , \quad K_{(y)} = \partial_y , \quad K_{(z)} = \partial_z$$

$$R_{(xy)} = y \partial_x + x \partial_y , \quad R_{(xz)} = z \partial_x + x \partial_z , \quad R_{(yz)} = z \partial_y - y \partial_z$$

one gradient homothetic Killing vector (HV)

$$H = x \partial_x + y \partial_y + z \partial_z , \quad \psi_H = 1$$
However, we want to give a concrete example of application of the Lie symmetries. The reduction of the (1+2) wave equation and the invariant solution $s$ have been studied in [44] and recently in [45].

It is possible to find invariant solutions for the WDW equation for Bianchi I and II spacetimes. WDW equation (19) and, if possible, to determine invariant solutions. From the results of Table II, we can see that for the Bianchi II model, five Lie symmetries, $\Psi \partial \psi$, $K_{(x)}$, $K_{(y)} - \frac{i}{2}K_{(y)}$, $R_{(x)} = -\frac{i}{2}R_{(y)}$, $\frac{i}{2}K_{(y)}$, for the Bianchi I model, eleven Lie symmetries, $\Psi \partial \psi$, $K_{(x)}$, $K_{(y)}$, $K_{(z)}$, $R_{(x)}$, $R_{(y)}$, $H$, and three special conformal Killing vectors (CKVs) which are

\[
\begin{align*}
C_{(x)} &= \frac{1}{2} (x^2 + y^2 + z^2) \partial_x + xy \partial_y + xz \partial_z, \quad \psi_{(x)} = x \\
C_{(y)} &= xy \partial_x + \frac{1}{2} (x^2 + y^2 - z^2) \partial_y + yz \partial_z, \quad \psi_{(y)} = y \\
C_{(z)} &= xz \partial_x + yz \partial_y + \frac{1}{2} (x^2 + z^2 - y^2) \partial_z, \quad \psi_{(z)} = z.
\end{align*}
\]

See Ref. [7] for details. Furthermore, by applying the results in [7], we find that the WDW equation (19) admits: i) for the Bianchi I model, eleven Lie symmetries, ii) for the Bianchi II model, five Lie symmetries, iii) two Lie symmetries for the models VIII/IX and iv) one Lie symmetry, the linear one, for the models VIII and IX. In Table II we give the corresponding Lie symmetries of the WDW equation (19) for each Bianchi model.

### 4. Invariant Solutions of the WDW Equation in General Relativity

In this section, we will apply the zeroth order invariants of the Lie point symmetries to reduce the order of the WDW equation (19) and, if possible, to determine invariant solutions. From the results of Table II, we can see that it is possible to find invariant solutions for the WDW equation for Bianchi I and II spacetimes.

#### 4.1. Bianchi I Cosmology

Since for Bianchi I spacetimes hold the property that $R^I = 0$, the WDW equation (19) is the (1+2) wave equation. The reduction of the (1+2) wave equation and the invariant solutions have been studied in [44] and recently in [45]. However, we want to give a concrete example of application of the Lie symmetries.

Let us consider the Lie algebra $\{X^I_{(x)}, X^I_{(y)}\}$ with commutators $[X^I_{(x)}, X^I_{(y)}] = 0$ where

\[
X^I_{(x)} = K_{(x)} + \mu_i \Psi \partial \psi
\]

hence the solution of equation (19) is

\[
\Psi_I(x, y, z) = \exp(\mu_1 x + \mu_2 y) \left[ c_1 \exp \left( \frac{\sqrt{\mu_1^2 - \mu_2^2}}{z} \right) + c_2 \exp \left( -\sqrt{\mu_1^2 - \mu_2^2} z \right) \right].
\]

One can also consider the Lie algebra $\{X^I_{(x)}, X^I_{(yz)} = R_{(yz)} + \mu \Psi \partial \psi\}$ for which the invariant solution is

\[
\Psi_I(x, y, z) = \left[ c_1 I_{i\mu} \left( \mu_1 \sqrt{y^2 + z^2} \right) + c_2 K_{i\mu} \left( \left( \mu_1 \sqrt{y^2 + z^2} \right) \right) \right] \exp \left( \mu \arctan \frac{y}{z} + \mu_1 x \right)
\]

where $I_{i\mu}, K_{i\mu}$ are the modified Bessel functions of the first and the second kinds. Similarly, for the others Lie subalgebras, we can construct invariant solutions. We want also to recall that the WDW equation is a linear second order partial differential equation and any linear combination of the solutions is also a solution.
4.1.1. The WKB approximation and the classical solution

Adopting the WKB approximation to the WDW equation, equation (19) becomes

\[
\left( \frac{\partial S}{\partial x} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 = 0
\] (23)

which is the null Hamilton-Jacobi equation of the Hamiltonian system (15)-(18), where \( p_i = \frac{\partial S}{\partial x^i} \). The solution of (23) is

\[
S(x, y, z) = s_1 y + s_2 z + \pm \sqrt{s_1^2 + s_2^2} x + s_0, \text{ with } \varepsilon = \pm 1.
\] (24)

Hence the Hamilton equations (16)-(18) reduce to the following system (where \( \bar{N} = 1 \))

\[
\dot{x} = \varepsilon \sqrt{s_1^2 + s_2^2}, \quad \dot{y} = -s_1, \quad \dot{z} = -s_2
\] (25)

and therefore we have the solutions

\[
(\lambda(t), \beta_1(t), \beta_2(t)) = \left( \frac{\sqrt{3}}{6} \varepsilon \sqrt{s_1^2 + s_2^2 t}, -\frac{\sqrt{3}}{3} s_1 t, -\frac{\sqrt{3}}{3} s_2 t \right)
\] (26)

and, by the coordinate transformation \( d\tau = e^{3\lambda}dt \) in the line element (1), we obtain a Kasner spacetime.

4.2. Bianchi II cosmology

For the Bianchi II spacetime, using Table II, we have that if we use the Lie symmetries \( \{ X_{(z)}, X_{(x)} - \frac{1}{2} X_{(y)} \} \) with zero commutator, the invariant solution is

\[
\Psi_{1I}^2(x, y, z) = \exp \left( \frac{2\mu_{12}}{3} (y + 2x) + \mu_3 z \right) (c_1 I_{c_0} (u(x, y)) + c_2 K_{c_0} (u(x, y)))
\] (27)

where \( \mu_{12} = \mu_1 - \mu_2, \ c_0 = \sqrt{\frac{12\mu_{12}^2 - 9\mu_3^2}{3}}, \ u(x, y) = \exp \left( \frac{\sqrt{3}(2y+x)}{3} \right) \) and \( I_{c_0}, K_{c_0} \) are the modified Bessel functions of the first and the second kinds, respectively. For the Lie algebra

\[
K_{(x)}^I - \frac{1}{2} K_{(y)}, R_{(xz)} - \frac{1}{2} R_{(yz)}
\]

we have the invariant solution

\[
\Psi_{1I}^2(x, y, z) = (c_1 I_0 (u(x, y)) + c_2 K_0 (u(x, y))).
\] (28)

The last solution \( \Psi_{1I}^2 \) is included in solution \( \Psi_{1I}^1 \) for \( \mu_{12} = \mu_3 = 0 \).

4.2.1. The WKB approximation and the classical solution

One can also apply the WKB approximation for the Bianchi II model to the equation (19):

\[
\left( \frac{\partial S}{\partial x} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 + e^{\frac{2\sqrt{3}}{3}(2y+x)} = 0
\] (29)

Let us adopt the coordinate transformation \( y = w - \frac{x}{2} \) in the Hamiltonian system. Hence, the new Hamilton-Jacobi equation (29) becomes

\[
\left( \frac{\partial S}{\partial x} \right)^2 + \left( \frac{\partial S}{\partial x} \right) \left( \frac{\partial S}{\partial w} \right) - \frac{3}{4} \left( \frac{\partial S}{\partial w} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 + e^{\frac{2\sqrt{3}}{3}w} = 0
\] (30)
with the solution
\[
S(x, w, z) = S_1(x) + S_2(w) + S_3(z),
\]
where the functions \(S_1, S_2, S_3\) follow from the system
\[
S_1(x) = c_1 x, \quad S_3(z) = c_2 z
\]
(32)
\[
\left(\frac{dS_2}{dw}\right)^2 = \frac{4}{3} \left[ c_1 \left( c_1 + \frac{dS_2}{dw} \right) + c_2^2 - 2e^{\frac{4\sqrt{3}}{3} w} \right] = 0.
\]
(33)
The Hamilton function \(S(x, w, z)\) is
\[
S(x, w, z) = c_1 x + c_2 z + \frac{2}{3} c_1 w - \varepsilon \frac{\sqrt{3}}{3} \left( M(w) - \sqrt{c_{12}} \arctan \left( \frac{M(w)}{\sqrt{c_{12}}} \right) \right), \quad \text{with} \quad \varepsilon = \pm 1
\]
(34)
where \(c_{12} = 3c_2^2 - 4c_1^2\) and \(M(w) = \sqrt{3e^{\frac{4\sqrt{3}}{3} w} - c_{12}}\). Therefore the reduced Hamiltonian system is (recall that \(\bar{N} = 1\))
\[
\dot{z} = -2c_2, \quad \dot{w} = \varepsilon M(w)
\]
(35)
\[
\dot{x} = \frac{8}{3} c_1 - \varepsilon \frac{2}{3} M(w)
\]
(36)
Hence the solution of the system (35), (36) is
\[
z(t) = -2c_2 t
\]
(37)
\[
w(t) = -\frac{\sqrt{3}}{4} \ln \left( \cos^2 \left( \frac{2\sqrt{3}}{3} \varepsilon \sqrt{c_{12}} (t + w_0) \right) \right)
\]
(38)
\[
x(t) = \frac{\sqrt{3}}{6} \ln \left[ \cos^2 \left( \frac{2\sqrt{3}}{3} \varepsilon \sqrt{c_{12}} (t + w_0) \right) \right] + \frac{8}{3} c_1 t + x_0.
\]
(39)
which is the solution of the empty Bianchi II spacetime in General Relativity.

In the following section we will apply the same procedure in the case of minimally coupled scalar-tensor cosmology. Furthermore, we will use the Lie symmetries of the WDW equation in order to determine the unknown potential of the scalar field.

5. SYMMETRIES OF THE WDW EQUATION IN MINIMALLY COUPLED SCALAR-TENSOR COSMOLOGY

Let us continue the Lie symmetry analysis of the WDW equation for cosmological containing a minimally coupled scalar field in the gravitational action. The Noether symmetry classification of the field equations has been studied in [30–32]. Noether symmetries have been adopted in the jet space in [3]. In Ref. [6], a detailed study of integrable cosmological models with non-minimally coupled scalar fields is presented.

Let us now take into account the following action:
\[
S = \int dx^4 \sqrt{-g} \left( R - \frac{1}{2} g^{\mu \nu} \phi,_{\mu} \phi,_{\nu} + V(\phi) \right).
\]
(40)
From the line element (1) and equations (9), (11) we find the Lagrangian
\[
L \left( N, \lambda, \beta_1, \beta_2, \phi, \dot{\lambda}, \dot{\beta}_1, \dot{\beta}_2, \dot{\phi} \right) = N(t)e^{3\lambda} \left( 6\dot{\lambda}^2 - \frac{3}{2} \left( \dot{\beta}_1^2 + \dot{\beta}_2^2 \right) - \frac{1}{2} \dot{\phi}^2 \right) + \frac{e^{3\lambda}}{N(t)} \left( V(\phi) + R^* \right).
\]
(41)
Hence, by applying the Euler-Lagrange vector in (41), we find four field equations which are the two equations (11) and

\[ 4\dot{\lambda} + \left( 6\dot{\lambda}^2 + \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2) + \frac{1}{2}\dot{\beta}_3^2 \right) + \frac{N}{N^2}\dot{\lambda} - \frac{1}{N^2} \left( V + R^* + \frac{1}{3}\frac{\partial R^*}{\partial \lambda} \right) = 0 \]  

(42)

\[ Ne^{3\lambda} \left( 6\dot{\lambda}^2 - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2) - \frac{1}{2}\dot{\beta}_3^2 \right) - \frac{e^{3\lambda}}{N} (V + R^*) = 0. \]  

(43)

The last corresponds to the the \( G_0^0 = T_0^0 \) Einstein equation. Furthermore, from the Euler-Lagrange equation \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} \), we obtain the field equation for the scalar field

\[ \ddot{\phi} + 3\lambda \dot{\phi} + \frac{N}{N^2} \dot{\phi} + \frac{1}{N^2} \frac{\partial V}{\partial \phi} = 0. \]  

(44)

which can also be derived by the Bianchi identity \( T^{\mu\nu} = 0 \), where \( T^{\mu\nu} \) is the energy-momentum tensor for scalar field. As in the case of General Relativity, the coordinate transformations \( (\lambda, \beta_1, \beta_2) = \left( \frac{\sqrt{\pi}}{6} x, \frac{\sqrt{\pi}}{3} y, \frac{\sqrt{\pi}}{3} z \right) \) and \( N(t) = \bar{N}(t) e^{-3\lambda} \), can be adopted. Hence, equation (43) becomes

\[ \frac{1}{2} \bar{N} \left( \dot{x}^2 - \dot{y}^2 - \dot{z}^2 - \dot{\phi}^2 \right) - \frac{1}{N} e^{\sqrt{\pi}x} (V(\phi) + R^*) = 0. \]  

(45)

Moreover, by using the momentum \( p_{(x,y,z,\phi)} = \frac{\partial L}{\partial x^i} \), equation (45) becomes

\[ \frac{1}{2} \left( p_x^2 - p_y^2 - p_z^2 - p_\phi^2 \right) - e^{\sqrt{\pi}x} (V(\phi) + R^*) = 0 \]  

(46)

and hence, from (46), we have the following WDW equation

\[ \Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Psi_{,\phi\phi} - 2e^{\sqrt{\pi}x} (V(\phi) + R^*) \Psi = 0 \]  

(47)

since the minisuperspace of equation (46) is the flat space \( M^4 \). We note that equation (47) is the Klein Gordon equation in \( M^4 \).

Since we want to use the geometric approach in Ref. [7], we need the conformal algebra of the \( M^4 \) spacetime. The four dimensional flat space \( M^4 \) admits ten Killing vectors (KVs) which are

\[ K(x), K(y), K(z), K(\phi) = \partial_\phi, \]

\[ R(xy), R(xz), R(yz), R(x\phi) = \phi \partial_x + x \partial_\phi, \]

\[ R(y\phi) = \phi \partial_y - y \partial_\phi, \quad R(z\phi) = \phi \partial_z - z \partial_\phi, \]

one gradient HV

\[ \bar{H} = x \partial_x + y \partial_y + z \partial_z + \phi \partial_\phi, \]

and four special CKVs

\[ \bar{C}(x) = C(x) + \frac{1}{2} \phi^2 \partial_x + \phi x \partial_\phi, \quad \psi(x) = x \]

\[ \bar{C}(y) = C(y) - \frac{1}{2} \phi^2 \partial_y + \phi y \partial_\phi, \quad \psi(y) = y \]

\[ \bar{C}(z) = C(z) - \frac{1}{2} \phi^2 \partial_z + \phi z \partial_\phi, \quad \psi(z) = z \]
TABLE III: Lie symmetries of the WDW equation of the Class A Bianchi models in scalar field cosmology for $V(\phi) = 0$.

| Model | $V(\phi) = 0$ | Lie Symmetries |
|-------|---------------|----------------|
| Bianchi I | 16 | $\psi \partial \psi, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, R_{(x)}, R_{(y)}, R_{(y)}$ |
| Bianchi II | 7 | $\psi \partial \psi, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, R_{(x)}, R_{(y)}, R_{(y)}$ |
| Bianchi VIII/IX | 2 | $\psi \partial \psi, K_{(z)}, K_{(z)}$ |

TABLE IV: Lie symmetries of the WDW equation of the Class A Bianchi models in scalar field cosmology for non-zero potentials.

| Model | $V(\phi) = V_0 \exp^{\mu \phi}$ | Lie Symmetries |
|-------|-----------------|----------------|
| Bianchi I | 7 | $\psi \partial \psi, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, R_{(x)}, R_{(y)}, R_{(y)}$ |
| Bianchi II | 4 | $\psi \partial \psi, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, K_{(z)}, R_{(x)}, R_{(y)}, R_{(y)}$ |
| Bianchi VIII/IX | 1 | $\psi \partial \psi$ |

$\bar{C}(\phi) = x\phi \partial x + y\phi \partial y + z\phi \partial z + \frac{1}{2} (x^2 + \phi^2 - y^2 - z^2) \partial \phi, \psi(\phi) = \phi$

One can find that the WDW equation (47) admits Lie symmetries for an arbitrary potential $V(\phi)$; in the case of Bianchi I, spacetime equation (47) admits four Lie symmetries; for Bianchi II, one has two Lie symmetries and one Lie symmetry for the rest of the Bianchi models. Therefore, for special forms of potentials $V(\phi)$, it is possible that the WDW equation (47) admits extra symmetries. The special forms of the potentials that we found are: (a) $V(\phi) = 0$, in which the scalar field behaves as stiff matter; (b) $V(\phi) = V_0$ with $V_0 \neq 0$ and (c) $V(\phi) = V_0 \exp^{\mu \phi}$. The results of symmetry analysis are collected in Tables III and IV.

We will continue our analysis using the results of Tables III and IV in order to determine invariant solutions of the WDW equation (47) for cases where it is possible.

6. IN Variant Solutions of the WDW Equation in Scalar Field Cosmology

From the symmetries in Tables III and IV, we observe that invariant solutions for the WDW equation can be determined for the Bianchi type I model for (a) $V(\phi) = 0$, (b) $V(\phi) = V_0$, (c) $V(\phi) = V_0 \exp^{\mu \phi}$; and, for the Bianchi II model, for zero potential. It is worth noticing that, in the following, we will choose $\bar{N} (t) = 1.$
6.1. Bianchi I cosmology

For the Bianchi I spacetime, the WDW equation becomes

\[ \Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Psi_{,\phi\phi} - 2e^{\sqrt{3}\phi}V(\phi)\Psi = 0. \] (48)

If the potential \( V(\phi) = 0 \), then (48) becomes the (1+3) wave equation in \( E^3 \), hence we will omit this case. When \( V(\phi) = V_0 \), \( V_0 \neq 0 \), the field equations are equivalent to the case of General Relativity with stiff matter and a cosmological constant. In this case, we use zero order invariants of the Lie symmetries

\[ \bar{X}_y (A) = K_{(A)} + \mu_{(A)} \Psi \partial_\Psi, \ A = y, z, \phi \] (49)

which form a closed Lie algebra. In this case, equation (48) reduces to the linear second order ordinary differential equation

\[ \Phi'' - \left( \mu_y + \mu_z + \mu_\phi \right) \Phi = 0 \] (50)

where \( \Phi = \Phi(x) \exp \left( \mu_y y + \mu_z z + \mu_\phi \phi \right) \) and \( \Phi' = \frac{d\Phi(x)}{dx} \). Therefore, the solution of equation (50) is

\[ \Phi(x) = \Phi_1 J_\nu \left( \frac{2\sqrt{6}V_0}{3} e^{\sqrt{3}\phi} \right) + \Phi_2 Y_\nu \left( \frac{2\sqrt{6}V_0}{3} e^{\sqrt{3}\phi} \right) \] (51)

where \( J_\nu, Y_\nu \) are the Bessel functions of the first and second kind and the constant is \( c = \frac{2\sqrt{3}}{3} \left( \mu_y^2 + \mu_z^2 + \mu_\phi^2 \right) \).

For the exponential potential, we apply the zero order invariants of the Lie symmetries

\[ \bar{X}_{(y)}, \bar{X}_{(z)}, \sqrt{\frac{3}{2}} \mu \bar{K}_{(x)} - K_{(\phi)} + \nu \Psi \partial_\Psi \] (52)

and the WDW equation becomes

\[ (3 - \mu^2) \Phi''(w) + 6\nu \Phi' - \left( \left( \mu_y^2 + \mu_z^2 \right) \mu^2 - 3\nu^2 + 2V_0 \nu^2 e^{\mu w} \right) \Phi = 0 \] (53)

where \( \Psi(x, y, z, \phi) = \Phi(w) \exp \left( \sqrt{3}w + \mu_y y + \mu_z z \right) \), \( \Phi' = \frac{d\Phi(w)}{dw} \) and \( w = \sqrt{3}x + \phi \). Hence, for various values of the constant \( \mu \) from (53), we have that

\[ \Phi(w) = \exp \left( \frac{3\nu w}{\mu^2 - 3} \right) \left[ \Phi_1 J_\nu \left( 2 \sqrt{\frac{2V_0}{\mu^2 - 3}} e^{\frac{\phi}{\nu}} \right) + \Phi_2 Y_\nu \left( 2 \sqrt{\frac{2V_0}{\mu^2 - 3}} e^{\frac{\phi}{\nu}} \right) \right], \text{ with } \mu \neq \sqrt{3} \] (54)

where \( \tilde{c} = \sqrt{ \frac{2}{|\mu^2 - 3|} \left( 3\nu^2 - (\mu^2 - 3) \left( \mu_y^2 + \mu_z^2 \right) \right) } \) and

\[ \Phi(w) = \Phi_0 \exp \left[ \frac{1}{2\nu} \left( \mu_y^2 + \mu_z^2 \right) - \frac{\nu}{2} w + \frac{\sqrt{3}}{3} \frac{V_0}{\nu} e^{\sqrt{3}w} \right], \text{ for } |\mu| = \sqrt{3}, \nu \neq 0. \] (55)

6.1.1. The WKB approximation and the classical solutions

In the WKB approximation the WDW equation (48) becomes the Hamilton-Jacobi equation

\[ \frac{1}{2} \left[ \left( \frac{\partial S}{\partial x} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 - \left( \frac{\partial S}{\partial \phi} \right)^2 \right] - e^{\sqrt{3}\phi}V(\phi) = 0 \] (56)

where \( S = S(x, y, z, \phi) \) describes a motion of a particle in the \( M^4 \) space. The solution of the Hamilton Jacobi function leads us to the following reduced Hamiltonian system

\[ \dot{x} = \frac{\partial S}{\partial x}, \quad \dot{y} = \frac{\partial S}{\partial y}, \quad \dot{z} = \frac{\partial S}{\partial z}, \quad \dot{\phi} = \frac{\partial S}{\partial \phi} \] (57)
For $V(\phi) = 0$, from equation (50) we have that

$$S_0(x, y, z, \phi) = c_1 y + c_2 z + c_3 \phi + \varepsilon \sqrt{c_1^2 + c_2^2 + c_3^2} x, \quad \varepsilon = \pm 1.$$  (58)

Then from (57) and (58) we have

$$x(\tau) = \varepsilon \sqrt{c_1^2 + c_2^2 + c_3^2} t + x_0.$$  (59)

$$y(t) = -c_1 t + y_0, \quad z(t) = -c_2 t + z_0, \quad \phi(t) = -c_3 t + \phi_0.$$  (60)

Similarly, for constant potential, i.e. $V(\phi) = V_0$, we have

$$S_{V_0}(x, y, z, \phi) = c_1 y + c_2 z + c_3 \phi + \varepsilon \frac{2\sqrt{3}}{3} \left( L(x) + \sqrt{c_{1-3}} \arctan \frac{L(x)}{\sqrt{c_{1-3}}} \right)$$  (61)

where $L(x) = \sqrt{c_1^2 + c_2^2 + c_3^2 + 2V_0 e^{\sqrt{3}x}}$ and $c_{1-3} = c_1^2 + c_2^2 + c_3^2$. Hence the reduced Hamilton equations (57) are

$$\dot{x} = L(x), \quad \dot{y} = -c_1, \quad \dot{z} = -c_2, \quad \phi = -c_3$$  (62)

and the exact solution of the field equations is

$$x(t) = \frac{1}{3} \ln \left[ \frac{c_{1-3}}{2V_0} \left( \tanh \left( \frac{\sqrt{3}}{2} \sqrt{c_{1-3}} (t + x_0) \right) - 1 \right) \right]$$  (63)

$$y(t) = c_1 t + y_0, \quad z(t) = c_2 t + z_0, \quad \phi(t) = c_3 t + \phi_0.$$  (64)

For the exponential potential, $V(\phi) = V_0 e^{\mu \phi}$, as we saw previously, there exist different solutions of the WDW equation for different values of the constant $\mu$. Hence, the solution of the Hamilton-Jacobi equation (50) is determined by the various values of $\mu$.

Let us set $\mu = -\sqrt{3}$. Applying the coordinate transformation $\phi = \psi + x$ in the Hamiltonian system, the new Hamilton-Jacobi equation is

$$\frac{1}{2} \left[ \left( \frac{\partial S}{\partial x} \right)^2 - 2 \left( \frac{\partial S}{\partial y} \right) \left( \frac{\partial S}{\partial \psi} \right) - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 \right] - V_0 e^{-\sqrt{3} \psi} = 0$$  (65)

and the reduced Hamiltonian system is

$$\dot{x} = \left( \frac{\partial S}{\partial x} \right), \quad \dot{y} = -\frac{\partial S}{\partial y}, \quad \dot{z} = -\frac{\partial S}{\partial z}, \quad \dot{\psi} = -\frac{\partial S}{\partial x}$$  (66)

Therefore from (65) the Hamilton action is

$$S(x, y, z, \psi) = c_1 x + c_2 y + c_3 z + \frac{(c_2^2 + c_3^2 - c_1^2)}{2c_1} \psi - \frac{\sqrt{3}V_0}{6c_1} e^{-\sqrt{3} \psi}$$  (67)

and the field equations reduce to the system

$$\dot{x} = \frac{(c_2^2 - c_3^2 - c_1^2)}{2c_1} - \frac{\sqrt{3}V_0}{6c_1} e^{\sqrt{3} \psi}, \quad \dot{y} = -c_2, \quad \dot{z} = -c_3, \quad \dot{\psi} = -c_1$$  (68)

with the solution

$$y(t) = -c_2 t + y_0, \quad z(t) = -c_3 t + z_0, \quad \psi(t) = -c_1 t + \psi_0$$  (69)

and

$$x(t) = \frac{3}{2} c_1 t - \frac{(c_2^2 + c_3^2)}{2c_1} t + \frac{\sqrt{3}V_0}{6c_1} e^{-\sqrt{3} \psi_0} e^{\sqrt{3} c_1 t} + x_0.$$  (70)
Furthermore, for \( |\mu| \neq \sqrt{3} \), we apply the coordinate transformation \( \phi = \psi - \frac{\sqrt{3}}{\mu} x \),

\[
\frac{1}{2} \left[ \left( \frac{\partial S}{\partial x} \right)^2 - \frac{2\sqrt{3}}{\mu} \left( \frac{\partial S}{\partial x} \right) \left( \frac{\partial S}{\partial \psi} \right) - \left( 1 - \frac{3}{\mu^2} \right) \left( \frac{\partial S}{\partial \psi} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2 - \left( \frac{\partial S}{\partial z} \right)^2 \right] - V_0 e^{\mu \psi} = 0
\]

(71)

hence from the Hamilton-Jacobi equation (56) we have

\[
S(x, y, z, \psi) = c_1 x + c_2 y + c_3 z + \gamma(\psi)
\]

(72)

where

\[
\left( \frac{d\gamma}{d\psi} \right)^2 = \frac{\mu^2}{\mu^2 - 3} \left( \frac{2\sqrt{3}}{\mu} \frac{d\gamma}{d\psi} - \frac{V_0 e^{\mu \psi}}{c_1 - c_2 - c_3} \right)
\]

(73)

The reduced Hamiltonian system (57) in the new coordinates becomes

\[
\dot{y} = -\frac{\partial S}{\partial y}, \quad \dot{z} = -\frac{\partial S}{\partial z}
\]

(74)

\[
\dot{x} = \left( \frac{\partial S}{\partial x} \right) - \frac{\sqrt{3}}{\mu} \left( \frac{\partial S}{\partial \psi} \right), \quad \dot{\psi} = -\frac{\sqrt{3}}{\mu} \left( \frac{\partial S}{\partial x} \right) - \left( 1 - \frac{3}{\mu^2} \right) \left( \frac{\partial S}{\partial \psi} \right)
\]

(75)

and therefore from (72) and (73) the last becomes

\[
\dot{y} = -c_2, \quad \dot{z} = -c_3
\]

(76)

\[
\dot{x} = c_1 - \frac{3 - \sqrt{3} \sqrt{(3 - \mu^2) c_1 - 3} V_0 e^{\mu \psi}}{\mu^2 - 3}
\]

(77)

\[
\dot{y} = -\frac{2\sqrt{3}}{\mu} c_1 - \frac{1}{\mu} \sqrt{(3 - \mu^2) c_1 - 3} V_0 e^{\mu \psi}
\]

(78)

However, if one wants to write an analytical solution of this system, we have to perform another transformation that is \( dt \to d\tau \). The exact solution of the exponential potential in Bianchi I scalar field cosmology was found in [5] so we will omit the derivation it in this work.

### 6.2. Bianchi II cosmology

One can observe from Tables [III] and [IV] that, for Bianchi type II spacetimes, we can determine invariant solution of the WDW equation only for zero potential, i.e. the scalar field is a kination fluid acting as stiff matter \( p_\phi = \rho_\phi \). In this case, the WDW equation (47) becomes

\[
\Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Psi_{,\phi \phi} + e^{2\sqrt{3} \left(2y+x\right)} \Psi = 0.
\]

(79)

By applying the zero order invariants of the following three dimensional closed Lie algebra with vanishing commutators,

\[
K_{(x)} - \frac{1}{2} K_{(y)} + \nu \Psi \partial_\psi, \quad K_{(z)} + \mu(z) \Psi \partial_\psi, \quad K_{(\phi)} + \mu(\phi) \Psi \partial_\psi
\]

(80)

we find the invariant solution

\[
\Psi_1 (x, y, z, \phi) = \exp \left( \frac{2\nu}{3} (y + 2x) + \mu(z) z + \mu(\phi) \phi \right) \left( \Psi_1 (u(x, y)) + \Psi_2 K_{(x)} (u(x, y)) \right)
\]

(81)

where \( \lambda = \frac{1}{3} \sqrt{12\nu^2 - 9 \left( \mu^2(z) + \mu^2(\phi) \right)} \) and \( u(x, y) = \exp \left( \frac{2\sqrt{3}}{3} (2y + x) \right) \).
One can also consider the Lie algebra \( \{ K(z), K(x) - \frac{1}{2} K(y), R(xz) - \frac{1}{2} R(yz) \} \) for which we have the invariant solution
\[
\Psi_2 (x, y, z, \phi) = (\Psi_3 e^{c_1 \phi} + \Psi_4 e^{-c_1 \phi}) (\Psi_1 I_{ic_1} (u (x, y)) + \Psi_2 K_{ic_1} (u (x, y)))
\] (82)
whereas, for the Lie algebra \( \{ K(\phi), K(x) - \frac{1}{2} K(y), R(x\phi) - \frac{1}{2} R(y\phi) \} \), the invariant solution is
\[
\Psi_3 (x, y, z, \phi) = (\Psi_3 e^{c_1 \phi} + \Psi_4 e^{-c_1 \phi}) (\Psi_1 I_{ic_1} (u (x, y)) + \Psi_2 K_{ic_1} (u (x, y))).
\] (83)
The Lie algebras \( \{ R(z\phi), K(x) - \frac{1}{2} K(y), R(x\phi) - \frac{1}{2} R(y\phi) \} \) and \( \{ R(z\phi), K(x) - \frac{1}{2} K(y), R(x\phi) - \frac{1}{2} R(y\phi) \} \) provide the solution of the form
\[
\Psi_4 (x, y, z, \phi) = \Psi_1 I_0 (u (x, y)) + \Psi_2 K_0 (u (x, y)).
\] (84)

In WKB approximation, where the WDW equation reduces to the Hamilton-Jacobi equation, we apply the same approach as for the case of General Relativity, Sec. 4.2.1, hence we will omit it. However, we would like to note that the solution for the kination scalar field is \( \phi (t) = c_0 t \), where \( c_0 \) is a constant. In the following section, we discuss how these solutions can be applied, under a conformal transformation, in the case of \( f (\mathcal{R}) \) Hybrid Gravity. This means that the approach can be easily extended to higher-order gravity and non-minimally coupled cases.

7. HYBRID GRAVITY IN BIANCHI COSMOLOGY

In this section we consider the action of the Hybrid metric-Palatini Gravity with the action of the form [46–48]
\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + f(\mathcal{R})],
\] (85)
where \( R \) is the metric Ricci curvature scalar and \( f(\mathcal{R}) \) is a function of the Palatini curvature scalar which is constructed by an independent connection \( \hat{\Gamma} \). A variation of the above action with respect to the metric gives the gravitational field equations
\[
G_{\mu\nu} + f'(\mathcal{R}) R_{\mu\nu} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = 0
\] (86)
where \( G_{\mu\nu} \) is the Einstein tensor for metric \( g_{\mu\nu} \) while \( R_{\mu\nu} \) is the Ricci tensor constructed by the conformally related metric \( h_{\mu\nu} = f'(\mathcal{R}) g_{\mu\nu} \). It is well known that Hybrid Gravity is equivalent to a non-minimally coupled scalar tensor theory [48]. In particular, if we consider a new scalar field \( \psi = f'(\mathcal{R}) \) by using a Lagrange multiplier and the relation between the two Ricci scalars \( R \) and \( \mathcal{R} \), action (85) can be written in the following form
\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + \frac{3}{2\psi} \partial^\mu \psi \partial_\mu \psi - V(\psi)].
\] (87)
where
\[
V(\psi) = \mathcal{R} f'(\mathcal{R}) - f(\mathcal{R}).
\] (88)
is a Clairaut equation with the singular solution
\[
\frac{\partial V(f'(\mathcal{R}))}{\partial f'(\mathcal{R})} - \mathcal{R} = 0.
\] (89)
Furthermore, from the action [87] and for the Bianchi spacetimes [11], we have that the Lagrangian of the field equations is:
\[
L_{f(\mathcal{R})} = N(t) e^{3\lambda} \left[ (1 + \psi) \left( 6\lambda^2 - \frac{3}{2} \left( \dot{\beta}_1^2 + \dot{\beta}_2^2 \right) \right) + 6\dot{\lambda} \dot{\psi} + \frac{3}{2\psi} \dot{\psi}^2 \right] + e^{3\lambda} N(t) \left( V(\psi) + (1 + \psi) R^* \right).
\] (90)
As it is shown in [29], the action [87] of the Hybrid Gravity is equivalent to a phantom minimally coupled scalar field under the conformal transformation \( \bar{g}_{ij} = (1 + \psi) g_{ij} \), and action [87] becomes
\[
S = \int d^4x \sqrt{-\bar{g}} \left[ \bar{R} + \frac{3}{2} \left( \frac{2\phi + 3}{2(1 + \phi) \phi} \right) g^{ij} \dot{\psi}_i \dot{\psi}_j - \frac{1}{(1 + \psi)^2} V(\psi) \right]
\] (91)
where $\bar{R}$ is the Ricci scalar of the conformal metric $\bar{g}_{\mu \nu}$; therefore under the transformation 
\[ d\phi = \frac{1}{\sqrt{2(1+\psi)}} \sqrt{V(\psi)} d\psi \]
we have the action of the form of (10). From the transformation $\psi \to \phi$, we find that the only potential which admits Lie point symmetries has the following form
\[ V(\psi) = V_0 (1 + \psi)^2 \exp\left(-\kappa \arctan \sqrt{\psi}\right) \]  
which is the exponential potential in the case of minimally coupled scalar field cosmology with $\kappa = \kappa(\mu)$ and $\kappa(0) = 0$, i.e. $V(\psi) = V_0 (1 + \psi)^2$. This potential transforms to the constant potential in the case of minimally coupled scalar field [54].

8. DISCUSSION AND CONCLUSIONS

In this work we studied the Lie symmetries of the WDW equation in the Bianchi Class A spacetimes for General Relativity and scalar field cosmologies, considering minimally coupled scalar tensor gravity and non-minimally coupled gravity coming from Hybrid Gravity. We applied the Lie invariants in order to construct solutions of the WDW equation. In the case of General Relativity, we found exact solutions of the WDW equation for the Bianchi I and Bianchi II spacetimes. In scalar field cosmology, we applied the Lie symmetries as a criterion for the selection of the unknown potential of the scalar field and we were able to construct exact solutions for the Bianchi I spacetime for zero potential $V(\phi) = 0$, constant potential $V(\phi) = V_0$, and exponential potential $V(\phi) = e^{\mu \phi}$. For the Bianchi II spacetime we obtained solutions for the zero potential case. In each case, we show that when the WDW equation is invariant under the action of the three dimensional Lie algebra with zero commutators, the Hamilton-Jacobi equation of the Hamiltonian system which was defined by the field equations, can be solved by the method of separation of variables; that means that the field equations are Liouville integrable. It is important to note that, in the case of FLRW scalar cosmology, we have more potentials where the WDW admits Lie symmetries. However, since the Lie symmetries are connected to the conformal algebra of the minisuperspace, in the case of FLRW scalar field cosmology, the dimension of the minisuperspace is two, which means that the last admits an infinite number of conformal killing vectors, whereas, for the Bianchi models, the minisuperspace has dimension four and admits a fifteen dimensional conformal algebra, i.e. less possible generators for the Lie symmetries of the WDW equation.

Finally, we studied the case of the Hybrid Gravity in the Bianchi Class A spacetimes. Since the Hybrid Gravity is equivalent to a scalar tensor theory, we were able to related all the potentials we found in the case of minimally coupled scalar field to that of Hybrid Gravity.

This analysis is important in the sense that can be used in order to construct solutions of the wave function of the Universe and, at the same time, conservation laws, and classical solutions for the field equations. Following the discussion in [54], the presence of symmetries gives rise to a straightforward interpretation of the Hartle criterion: the symmetries generates oscillatory behaviors in the wave function of the Universe and then allow correlations among physical variables. This give rise to classically observable cosmological solutions. Here we generalized this result considering Bianchi models. On the other hand, other general selection rules can be identified in Quantum cosmology, as discussed in [56]. This will be the topic of forthcoming papers.

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Appendix A: Symmetries of differential equations and invariant functions

In this appendix, we briefly discuss the basic properties and definitions of the symmetries of differential equations (DE). Furthermore we discuss the application of the Lie symmetries of the WDW equation showing that if the WDW equation admits Lie symmetries, which form a Lie algebra with zero commutators, then the WDW equation admits oscillatory terms in the solution as many as the dimension of the minisuperspace and, at the same time, the Hamiltonian system which defined by the field equations is Liouville integrable; that is, the field equations can be solved by quadratures. In such a case, classical cosmological solutions can be derived.
1. Lie point symmetries and invariant functions

A second order DE is a function $H = H(x^i, u^A, u^A_{,i}, u^A_{,ij})$ in the jet space $B_M$, where $x^i$ are the independent variables and $u^A$ are the dependent ones. Let

$$X = \xi^i(x^k, u^B) \partial_{x^i} + \eta^A(x^k, u^B) \partial_{u^A}. \tag{A1}$$

be the generator of the infinitesimal point transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i(x^k, u^B) \tag{A2}$$

$$\bar{u}^A = \bar{u}^A + \varepsilon \eta^A(x^k, u^B). \tag{A3}$$

The function $H = 0$, is invariant under the action of the infinitesimal point transformation (A2), (A3) if there exists a function $\lambda$ such that

$$X[2](H) = \lambda H \tag{A4}$$

The vector field $X$ is called Lie point symmetry of the function $H$ and $X[n]$ is the second prolongation of $X$ in the jet space $B_M$

$$X[2] = X + \eta^A_i \partial_{x^i} + \eta^A_{ij} \partial_{u^A_{,ij}}, \tag{A5}$$

where

$$\eta^A_i = \eta^A_{i} + u^B_{,i} u^A_{,j} - \xi^i_{,i} u^A_{,j} - u^A_{,i} u^B_{,j} \xi^i_{,j}, \tag{A6}$$

and

$$\eta^A_{ij} = \eta^A_{ij} + 2u^A_{,i} u^B_{,j} - \xi^k_{,ij} u^A_{,k} + \eta^A_{BC} u^B_{,i} u^C_{,j} - 2\xi^k_{,(i)B} u^B_{,(j)k} u^A_{,k}$$

$$- \xi^k_{,BC} u^B_{,i} u^C_{,j} u^A_{,k} + \eta^A_{ji} u^B_{,i} - 2\xi^k_{,(i)j} u^A_{,k} - \xi^k_{,B} \left(u^A_{,i} + 2u^B_{,(j)} u^A_{,i} \right). \tag{A7}$$

The importance of Lie point symmetries is that the last one can be used in order to reduce an order of a differential equation. When a reduction is possible, one can determine invariant solutions or transform them to another ones [51]. From condition (A4) one defines the Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du}{\eta} = \frac{du_i}{\eta_{ij}} = \frac{du_{ij}}{\eta_{ij}}$$

whose solution provides the characteristic functions

$$Z[0] \left(x^k, u \right), \ Z[1] \left(x^k, u, u_i \right), \ Z[2] \left(x^k, u, u_i, u_{ij} \right).$$

The solution $Z[k]$ is called the $k$th order invariant of the Lie symmetry vector (A1). By writing the DE in terms of the invariants $Z[k]$, we can reduce the order of the DE, for details see for instance [51, 52]. Below we discuss the application of the Lie symmetries and of the Lie invariants for the WDW equation.

2. Reduction and invariant solutions of the WDW equation by Lie point symmetries

In order to determine the Lie symmetries of the WDW equation we apply a geometric method which is established by Paliathanasis & Tsamparlis [5]. The method relates the Lie symmetries of the Klein Gordon equation to the conformal algebra of the underlying geometry. Hence, in the following we will not present the construction of the Lie symmetries of the WDW equation but we will give the results.

In particular, the general form of a Lie symmetry of the WDW equation is

$$X = \xi^i(x^k) \partial_i + \left(\frac{2-n}{2} \psi \psi + a_0 \psi \right) \partial_{\psi}, \tag{A8}$$
where $\xi^i (x^k)$ is a CKV of the metric which defines the conformal Laplace operator, (in our consideration the min-superspace) and $\psi (x^k)$ is the conformal factor of the CKV, recall that since $\xi^i$ is a CKV of $g_{ij}$, it means that, $L_{\xi} g_{ij} = 2 \psi g_{ij}$.

Furthermore, it is possible to consider a coordinate transformation $x^i \to \bar{x}^i$ so that $\xi^i (x^k) \partial_i \to \partial_J$ (these are called normal coordinates). In the normal coordinates the symmetry vector takes the following simple form,

$$X = \partial_J + \left( \frac{2-n}{2} \psi \Psi + a_0 \Psi \right) \partial_a$$

where now either with the method of Lie invariants, or with the method of linear differential operators (see [33] for details) we find the following expression for the solution of $\Psi$,

$$\Psi (\bar{x}^b, \bar{x}^I) = \Phi (\bar{x}^b) \exp \left[ \int \left( \frac{2-n}{2} \psi - Q_0 \right) d\bar{x}^J \right]$$

from which follows again that the coordinate $x^J$ is factored out from the solution $\Psi$.

Furthermore, in [7] it was also shown that the symmetries of the WDW equation can be used in order to find Noether symmetries for classical particles\(^2\). The exact relation among the Lie symmetries of the WDW equation, the Noetherian conservation laws of the field equations and the interpretability conditions are given in [33].

\[^2\] A similar methodology by using the variational symmetries has been established recently in [53].
