Myriad non-linearity for GNSS robust signal processing

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Abstract: The robustness of standard correlation-based Global Navigation Satellite System (GNSS) signal processing can be significantly improved by pre-processing the input samples with a zero-memory non-linearity (ZMNL). A paradigm for the design of ZMNLs is provided by the M-estimator framework where heavy-tailed probability density functions (pdfs) are used to model the impairments affecting the input samples. The myriad non-linearity, obtained considering a Cauchy pdf, is analysed for the acquisition and tracking of GNSS signals in the presence of pulsed interference. The impact of the myriad non-linearity is theoretically characterised and Monte Carlo simulations are used to support theoretical findings. Finally, real GNSS signals collected in the presence of jamming are processed: the myriad non-linearity provides a significant performance improvement with respect to standard GNSS signal processing which is unable to acquire and track the samples affected by interference.

1 Introduction

The robustness of Global Navigation Satellite System (GNSS) signal processing can be significantly improved by introducing a zero-memory non-linearity (ZMNL) for pre-processing the samples [1, 2] provided by a GNSS receiver front-end. The ZMNL is used to weight the amplitude of the input samples: if a sample is an outlier or is affected by interference, its amplitude is significantly reduced by the ZMNL. In this way, the impact of outliers and interference is effectively mitigated.

Standard GNSS signal processing blocks, such as acquisition and tracking, are designed to solve a least squares (LS) problem where GNSS signal parameters are estimated by minimising a quadratic cost function [3]. The LS approach is obtained by assuming that the digital samples provided by the receiver front-end are affected by Gaussian noise. More general problems are obtained by relaxing the Gaussian hypothesis: this type of problems has been considered by several authors in the literature and has led to development of robust statistical approaches and robust signal processing schemes [4–7]. Robust signal processing is effectively able to deal with model uncertainties and model mismatches when the Gaussian hypothesis is not valid.

Robust signal processing techniques have been recently considered for the computation of the user position using raw GNSS observables [8] and have the potential to significantly improve the performance of GNSS receivers in the presence of jamming and pulsed interference. In particular, robustness can be obtained by introducing a ZMNL for pre-processing the input samples [1, 2]. In this way, a robust version of the cross-ambiguity function (CAF), the fundamental function computed by standard GNSS signal processing blocks, is obtained. The robust CAF can be interpreted as a form of locally optimum (LO) detector/generalised correlator [2, 7] and can enable receiver operations even in the presence of significant levels of interference.

A paradigm for the design of ZMNLs is provided by the M-estimator framework [4, 5] where heavy-tailed probability density functions (pdfs) are used to model the impairments affecting the input samples. In particular, the ZMNL is directly related to the pdf of the input samples.

This paper analyses the case where the ZMNL is designed assuming samples affected by complex Cauchy noise. In this way, the myriad non-linearity is obtained. The term ‘myriad’ is from the robust signal processing literature and, in particular, from the myriad filter [5, 9] which has been derived considering Cauchy pdfs. It is noted that, although strictly related, the myriad filter and the myriad ZMNL implement different processing schemes.

The Cauchy distribution has been proven to lead to the design of robust estimators which are effective not only in the presence of Cauchy-distributed noise [4, 5]. In particular, algorithms based on the Cauchy assumption are generally able to work effectively in the presence of contamination models [4] and of pulsed interference. This is because the Cauchy distribution is heavy-tailed and thus it better describes the behaviour of samples contaminated by pulsed interference than the standard Gaussian model. This fact is empirically demonstrated in Section 5.

The myriad non-linearity depends on a linearity parameter, K, which allows different compromises between robustness and loss of efficiency in the case of input Gaussian noise. In this respect, acquisition and tracking have to be interpreted as estimators which provide noisy estimates. In the presence of Gaussian noise only, non-linearities lead to an increase of variance in the estimates, i.e. the efficiency of the estimators is reduced. The linearity parameter, K, determines the loss of efficiency and allows different levels of robustness. The myriad non-linearity is theoretically analysed and a closed-form expression is obtained for the loss of efficiency in the presence of Gaussian noise only. Theoretical findings are supported by Monte Carlo simulations. In particular, a good agreement between theoretical and simulation results is obtained.

Finally, the benefits of the myriad non-linearity are empirically investigated using the data collected in [10] where GNSS samples affected by jamming were recorded. The front-end used for the data collection is narrow-band and the jamming signal periodically enters and exits the receiver bandwidth. In this way, it is perceived as pulsed interference. The data collected and the tests performed show the effectiveness of the myriad non-linearity in mitigating the impact of pulsed jamming.

It is noted that the focus of this paper is on pulsed interference. In this way, only a certain percentage of the input samples is affected by interference. The myriad non-linearity is not designed to operate in the presence of other types of interference such as continuous wave (CW) signals. When a CW signal is present, all the samples are affected by interference and the myriad non-linearity is not able to distinguish the CW from the useful signal components. The extension of the robust approaches considered in this paper to other types of interference is left for future work.
The remaining of this paper is organised as follows: Section 2 briefly introduces standard and robust GNSS signal processing schemes. The myriad non-linearity is discussed in Section 3 and theoretically analysed in Section 4. Experimental results are presented in Section 5 and conclusions are finally drawn in Section 6.

2 GNSS signals and GNSS signal processing

The signal at the antenna of a GNSS receiver can be modelled as [3]

\[ y(t) = \sqrt{2} \tilde{C}d(t - t_0)c(t - t_0)\cos(2\pi (f_{RF} + f_\gamma)t + \phi_\eta) + \eta(t) \]  

(1)

where \( C \) is the received signal power; \( d(\cdot) \) is the navigation message; \( c(\cdot) \) is a pseudo-random sequence extracted from a family of quasi-orthogonal codes modulated using rectangular pulses. \( c(\cdot) \) spreads the signal spectrum and can be made of several components including a subcarrier [11] which further shapes the signal power spectral density (PSD); \( t_0, f_\gamma \) and \( \phi_\eta \) are the delay, Doppler frequency and phase introduced by the communication channel, respectively; \( f_{RF} \) is the centre frequency of the GNSS signal; and \( \eta(t) \) is a zero-mean noise process.

Although a GNSS receiver recovers several signals from different satellites, a single component is considered in (1). This model assumption is based on the fact that each signal is characterised by a specific code and that the receiver is able to process each component independently.

The receiver front-end filters, down-converts and digitises signal (1). In the following, sampling and quantisation effects are neglected.

After analogue-to-digital conversion, (1) becomes

\[ y[n] = \sqrt{2} \tilde{C}d(nT_s - t_0)c(nT_s - t_0)\cos(2\pi (f_{RF} + f_\gamma)nT_s + \phi_\eta) + \eta_{\text{AD}}[n] \]  

(2)

where the notation ‘\( y[n] \)’ is used to denote a discrete-time sequence sampled at the frequency \( f_s = (1/T_s) \). The subscript ‘BB’ denotes a base-band signal and symbol \( \tilde{\cdot} \) is used to indicate that the useful signal component may be filtered by the front end.

\( \eta_{\text{AD}}[n] \) is a noise term which is obtained from the processing of \( \eta(t) \) in (1) and accounts for possible front-end disturbances such as quantisation noise. \( \eta_{\text{AD}}[n] \) is assumed to be a circularly symmetric white noise process with independent and identically distributed (i.i.d) real and imaginary parts. In the classical framework, \( \eta_{\text{AD}}[n] \) is assumed to be an additive white Gaussian noise (AWGN) with real and imaginary parts each with variance, \( \sigma^2 \). This assumption is removed to obtain robust GNSS signal processing schemes.

Under the Gaussian assumption, variance \( \sigma^2 \) is usually modelled as

\[ \sigma^2 = N_0 B_{\text{RF}} \]  

(3)

where \( B_{\text{RF}} \) is the front-end one-sided bandwidth and \( N_0 \) is the PSD of the input noise, \( \eta(t) \). The ratio between the carrier power, \( C \) and the noise power spectral density, \( N_0 \) defines the carrier-to-noise power spectral density ratio \( C/N_0 \), one of the main signal quality indicators used in GNSS.

Although, front-end filtering may introduce correlation among noise samples, the front-end bandwidth, \( B_{\text{RF}} \) is generally much wider than that of the correlator blocks used to process the input signal. Thus, the i.i.d. hypothesis is adopted here. More complex signal models accounting for correlation among the noise samples are discussed in [12, 13].

2.1 Standard GNSS signal processing

A GNSS receiver estimates the signal parameters, \( t_0, f_\gamma \) and \( \phi_\eta \) from the input samples, \( y[n] \). In a classical framework, i.e. under the assumption of Gaussian input noise, the estimation process is based on the minimisation of the non-linear LS cost function

\[ J(\tau, f_\gamma, \phi) = \sum_{n=0}^{N-1} |y[n] - Ac(nT_s - \tau)e^{j2\pi f_\gamma nT_s + \phi}|^2 \]  

(4)

where \( N \) is the number of samples used during the estimation process and \( A \) accounts for the unknown signal amplitude. The product, \( T_s = NT_c \) defines the coherent integration time.

The signal parameters are obtained as

\[ \{\hat{\tau}, \hat{f}_\gamma, \hat{\phi}\} = \arg \min_{\tau, f_\gamma, \phi} J(\tau, f_\gamma, \phi) \]  

(5)

where \( \hat{\tau}, \hat{f}_\gamma, \hat{\phi} \) denote the final signal estimates.

Minimisation (5) is implemented in a two-stage process involving signal acquisition and tracking. Acquisition determines the signal presence and provides raw estimates of the code delay and Doppler frequency; signal tracking completes the minimisation process using a gradient descent/ascent approach. As a result, tracking refines the estimates provided by the acquisition stage, determines the carrier phase and tracks variations in the signal parameters. Under the hypothesis of Gaussian input noise, the estimates obtained according to (5) are optimal in the Gaussian maximum-likelihood (ML) sense [14].

Minimisation (5) is equivalent to the maximisation problem

\[ \{\hat{\tau}, \hat{f}_\gamma, \hat{\phi}\} = \arg \max_{\tau, f_\gamma, \phi} \sum_{n=0}^{N-1} \Re\{y[n]c(nT_s - \tau)e^{-j2\pi f_\gamma nT_s - j\phi}\} \]  

(6)

where \( \Re(\cdot) \) is the real part operator. Problem (6) does not depend on \( A \) and is found by expanding the squares in (4) and discarding the terms that do not depend on the signal parameters to be estimated. Moreover, this maximisation process can be simplified using the following result:

\[ \sum_{n=0}^{N-1} \Re\{y[n]c(nT_s - \tau)e^{-j2\pi f_\gamma nT_s - j\phi}\} = \Re\{C(\tau, f_\gamma)\} \]  

(7)

where

\[ C(\tau, f_\gamma) = \sum_{n=0}^{N-1} y[n]c(nT_s - \tau)e^{-j2\pi f_\gamma nT_s} \]  

(8)

is the CAF [3]. From (7), it is possible to note that the cost function to be maximised can be factored in two terms. The first is the absolute value of the CAF and depends only on \( \tau \) and \( f_\gamma \). The second term is a cosine which also depends on \( \phi \). The cosine can be maximised by setting

\[ \phi = \angle C(\hat{\tau}, \hat{f}_\gamma) \]  

(9)

In this way, problem (6) can be restated as

\[ \{\hat{\tau}, \hat{f}_\gamma\} = \arg \max_{\tau, f_\gamma} |C(\tau, f_\gamma)| \]  

\[ \hat{\phi} = \angle C(\hat{\tau}, \hat{f}_\gamma) \]  

(10)

Result (10) is from the literature [3, 13, 15] and it is not further discussed here.

Standard acquisition and tracking algorithms are designed to maximise the absolute value of the CAF and estimate the signal phase using (10). The CAF is computed using a bank of correlators [3, 15].

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Fig. 1 Schematic representation of the processing performed by the myriad non-linearity: the samples are at first pre-processed using (23)

2.2 Robust CAF

The M-estimator framework introduced by Huber [4, 16] allows one to obtain robust versions of the LS problem discussed in the previous section. In particular, (4) can be generalised by replacing the squares with less rapidly increasing functions of the residuals [4, 17] which are defined as

\[ r[n] = y[n] - Ac(nT_s - \tau)e^{jz Alternative signal amplitude, \( z \) is used to indicate the argument of \( \rho \). The properties of \( \rho(\cdot) \) are discussed in [4, 16].

To obtain a robust version of CAF (8), it is necessary to expand \( \rho(z) \) in (12) in Taylor series. The Taylor series expansion is justified by the fact that received GNSS signals are weak and the signal amplitude, \( A \), can be assumed to be small. Function \( \rho(z) \) can be regarded as a real function of two real variables, the real and imaginary parts of \( z \):

\[ \rho(z) = \rho(z_r, z_i). \] (13)

Thus, for a small increment, \( \Delta z \approx \Delta z_r + j\Delta z_i \)

\[ \rho(z + \Delta z) \approx \rho(z) + \frac{\partial \rho(z)}{\partial z} \Delta z + j \frac{\partial \rho(z)}{\partial z} \Delta z \]

\[ = \rho(z) + \rho_1(z) \Delta z, \] (14)

where \( * \) denotes complex conjugate and

\[ \rho_1(z) = \rho(z) + j\rho_1(z) = \frac{\partial \rho(z)}{\partial z} + j \frac{\partial \rho(z)}{\partial z_i}. \] (15)

Using (14), it is possible to approximate (12) as (see (16)) where the second summation in the first line of (16) defines a robust version of the CAF

\[ C_\rho(\tau, f_\delta) = \sum_{n=0}^{N-1} \rho(y[n])c(nT_s - \tau)e^{jz z f_\delta + j \rho_1(\tau, f_\delta)}. \] (17)

In this case, the input samples, \( y[n] \), are pre-processed with the ZMNL, \( \rho_1(\cdot) \), before correlation with the local signal replicas.

Approximate cost function (16) is minimised when the robust CAF is maximised. Thus, a robust equivalent of (10) is obtained

\[ \{ \hat{\tau}, \hat{f}_\delta \} = \arg \max \{ |C_\rho(\tau, f_\delta)| \} \]

\[ \hat{\phi} = \arg C_\rho(\hat{\tau}, \hat{f}_\delta). \] (18)

The M-estimator framework [4, 16] also provides an effective way for the selection of \( \rho(z) \) and \( \rho_1(z) \). In particular, it is suggested to select \( \rho(z) \) as

\[ \rho(z) = - \log f(z) \] (19)

where \( f(z) \) is the pdf of a possibly heavily-tailed complex random variable. The usage of heavily-tailed pdf introduces robustness in the estimation process. Problem (5) is found when \( f(z) \) is the pdf of a complex Gaussian variable. In this paper, a complex Cauchy distribution is assumed and the myriad non-linearity is considered.

3 Myriad non-linearity

The myriad non-linearity is obtained considering the Cauchy distribution, which, for complex random variables is defined as [18]

\[ f(z) = \frac{\sqrt{K}}{2\pi (K + 1|z|^2)} \] (20)

where \( z \) is a complex variable and \( K \) is the so-called linearity parameter [5]. The meaning of \( K \) will be better clarified in the following.

Using the M-estimator approach [4], it is possible obtain \( \rho(z) \) from (20)

\[ \rho(z) = - \log(f(z)) = \frac{3}{2} \log(K + 1|z|^2) + \frac{1}{2} \log \left( \frac{4\pi^2}{K} \right). \] (21)

Finally, the complex non-linearity used to pre-process the input samples is obtained from (21) using (15)

\[ \rho_1(z) = \frac{3z}{K + 1|z|^2}. \] (22)

Note that any scaled version of (22) provides equivalent results with respect to the maximisation process defined by (18). In this respect, it is convenient to adopt the normalised ZMNL

\[ \rho_1(z) = \frac{Kz}{K + 1|z|^2}. \] (23)

In this way, \( \rho_1(z) \to z \) for large \( K \). This justifies the term, ‘linearity parameter’, used to denote \( K \): as \( K \) increases \( \rho_1(z) \) becomes more and more linear. In the limit case, \( K \to \infty, \rho_1(z) \) is the identity and the system becomes linear.

The processing defined by (23) is denoted as myriad non-linearity in analogy to the myriad filter [5, 9] which is also derived under the hypothesis of input Cauchy noise.

Myriad non-linearity, (23), performs a scaling on the input samples: when an outlier is present, its amplitude is significantly reduced. The amplitude of each sample is reduced depending on the tunable parameter, \( K \), which allows one to select the most
appropriate compromise between robustness and efficiency. The attenuation provided by the myriad non-linearity is analysed in Fig. 2 as a function of the amplitude of the input samples and for different linearity parameters.

For small values of \( K \), samples with large amplitudes are strongly attenuated. The attenuation becomes less abrupt as \( K \) increases and, in the limit case, the attenuation is almost constant. Fig. 2 also provides the attenuation profile of a pulse blanker [10, 19]. In pulse blanking (PB), the absolute value of the input samples is compared with a blanking threshold. If the absolute value is greater than the threshold then the corresponding sample is set to zero. In Fig. 2, a blanking threshold equal to 20 was arbitrarily selected for comparison purposes. The attenuation provided by the blanker is equal to 0 dB for the samples which have an amplitude lower than the blanking threshold. When this threshold is passed, the attenuation goes to infinity, i.e. the signal is blanked. For the myriad non-linearity, the attenuation profile is continuous and \( K \) plays a role comparable to that of the blanking threshold for PB.

### 4 Coherent output signal-to-noise ratio (SNR)

This section analyses the loss of efficiency introduced by the myriad non-linearity. This loss is leveraged by the robustness with respect to outliers introduced by the non-linearity.

Estimators produce noisy estimates the variance of which depends on the noise contaminating the input samples and on the processing adopted. An estimator is more efficient than another if it depends on the noise contaminating the input samples and on the myriad non-linearity. This loss is leveraged by the robustness with respect to outliers introduced by the non-linearity.

The coherent output SNR is defined as [12, 20]

\[
\text{SNR}_{\text{out}} = \max_{\tau, f_d} \frac{\mathbb{E}[|C(\tau, f_d)|^2]}{(1/2)\text{Var}(C(\tau, f_d))} \tag{24}
\]

and requires the evaluation of the first two statistical moments of the CAF which is considered as a random variable.

When standard acquisition and tracking are used, it is possible to show [12, 20] that

\[
\text{SNR}_{\text{out}} = \frac{N^2 C}{N \sigma^2} = N \frac{C}{N_B R_x} = N \frac{C}{N (f_d/2)} = \frac{2 C}{N_b} T_c \tag{25}
\]

where the following assumption has been used

\[
B_{R_x} \approx \frac{f_d}{2} \tag{26}
\]

The first moment of the CAF obtained using the samples preprocessed with the myriad non-linearity can be computed as follows:

\[
\mathbb{E}[C(\tau, f_d)] = \sum_{n=-N_f}^{N_f} \mathbb{E}\left[ \frac{K y[n]}{K + |y[n]|^2} \right] e^{j 2 \pi f_d n T_c}. \tag{27}
\]

The expected value, \( \mathbb{E}[\mathbb{E}[|K y[n]|/(K + |y[n]|^2)] \), can be computed using the results presented in the Appendix. In particular, it is shown that, for weak signal conditions

\[
\mathbb{E}\left[ \frac{K y[n]}{K + |y[n]|^2} \right] \approx \mathbb{E}[y[n]] \frac{K}{2 \sigma^2} \left( 1 - \frac{K}{2 \sigma^2} e^{K / \sigma^2} \right) \tag{28}\]

where \( \mathbb{E}[\cdot] \) is the exponential integral ([21], page 228).

\[
E(x) = \int_{-\infty}^{e^x} e^{-t} \frac{dt}{t}. \tag{29}\]

Condition (28) shows that the expected value of the samples processed with the myriad non-linearity is proportional to the first moment of the original samples, \( y[n] \). This implies that the myriad non-linearity does not introduce biases in the estimation process. Since the signal parameters are estimated through the maximisation of the CAF, scaling does not change the results of the maximisation process. This fact has been empirically verified when processing real data: no biases were observed in the estimates provided by acquisition and tracking.

The second important remark from (28) is that the average scaling introduced by the myriad non-linearity only depends on the ratio

\[
K \rightarrow \frac{K}{2 \sigma^2} \tag{30}\]

that is the linearity parameter scaled by the total variance of the input samples.

The variance of (17) can be computed as

\[
\text{Var}\{C(\tau, f_d)\} = N \text{Var}\left[ \frac{K y[n]}{K + |y[n]|^2} \right] = N \left( \mathbb{E}\left[ \frac{K y[n]}{K + |y[n]|^2} \right] - \mathbb{E}\left[ \frac{K y[n]}{K + |y[n]|^2} \right]^2 \right) \tag{31}\]

where the independence of the samples, \( y[n] \), and the fact that the myriad non-linearity does not introduce memory have been exploited.

The second moment of the samples processed with the myriad non-linearity is evaluated in the Appendix for weak signal conditions. In particular, the second moment can be approximated as

\[
\mathbb{E}\left[ \frac{K y[n]}{K + |y[n]|^2} \right] \approx \frac{K^2}{2 \sigma^2} \left( 1 + \frac{K}{2 \sigma^2} e^{K / \sigma^2} \right), \tag{32}\]

In approximation (32), the second moment of the pre-processed samples does not depend on the signal amplitude and, for weak signal conditions, is significantly greater than the square of the first moment. For this reason, the variance of the pre-processed samples can be approximated with second moment (32). This approximation corresponds to neglecting the second term in the last part of (31).

Using these results, it is finally possible to evaluate the coherent output SNR in the myriad non-linearity case

\[
\text{SNR}_{\text{out}, r} = \max_{\tau, f_d} \frac{\sum_{n=-N_f}^{N_f} \mathbb{E}[y[n]] e^{j 2 \pi f_d n T_c}}{N} \tag{33}\]

where

\[
L_{\text{out}}\left( \frac{K}{2 \sigma^2} \right) = \left( 1 - \frac{K}{2 \sigma^2} e^{K / \sigma^2} \right) \tag{34}\]

is the loss introduced by the myriad non-linearity.
This was expected as for large values of $K$ the myriad non-linearity converges to the identity. For $K > 3\sigma^2$, the efficiency loss is lower than 0.5 dB. These loss levels can be considered acceptable for the processing of GNSS signals.

### 4.1 Simulation results

Monte Carlo simulations have been used to support the theoretical results obtained above. In particular, noisy GPS L1 Coarse Acquisition (C/A) signals have been generated using the parameters reported in Table 1. The samples have been processed using the myriad non-linearity and correlated with local signal replicas aligned in time and frequency. In this way, it was possible to evaluate the coherent output SNR, which is obtained when the parameters of the signal local replicas match those of the input samples. Simulations were performed as a function of the input $C/N_0$ and for different values of $K$. For each $C/N_0$ and for each $K$ value, $10^5$ correlator outputs were computed and used to evaluate the expected value and the variance in (24). Finally, the coherent output SNR was determined.

Simulation results are presented in Fig. 3 which provides the loss of efficiency caused by the myriad non-linearity in the presence of input Gaussian noise. The analysis is performed for $C/N_0$ values in the [20 – 50] dB-Hz interval. These $C/N_0$ levels are representative of the values typically observed during normal GNSS receiver operations. As predicted by the theory developed above, the loss caused by the myriad non-linearity is constant with respect to the $C/N_0$: signals with a $C/N_0$ in the [20 – 50] dB-Hz range are sufficiently weak to justify the simplifications adopted in the Appendix for the computation of the first two moments of the samples pre-processed with the myriad non-linearity.

A good agreement between theoretical and simulation results clearly emerges from Fig. 3. A significant loss is experienced for low values of $K$. Moreover, the loss decreases with increasing $K$. This was expected as for large $K$, the myriad non-linearity converges to the identity. For $K > 3\sigma^2$, the efficiency loss is lower than 0.5 dB. These loss levels can be considered acceptable for the processing of GNSS signals.

### 5 Experimental analysis

The robustness provided by the myriad non-linearity is experimentally evaluated in this section. In particular, GNSS signals affected by jamming and collected using a narrow-band front-end are analysed. The front-end is a Realtek RTL2832u device and its characteristics are summarised in Table 2. The jammer used in the experiment is a cigarette lighter device [22] with a transmitted power of about 9 dBm. The jamming signal spans a frequency range of about 16.7 MHz with a sweep period of about 9 $\mu$s. The data collected are the same used in [10] to analyse the impact of PB on the processing of GPS signals affected by jamming.

The Realtek RTL2838u is sufficiently narrow-band such that the jamming signal periodically enters and exits the receiver bandwidth. In this way, the jamming signal is perceived as a form of pulsed interference.

The front-end was placed on a car moving at different speeds. The car performed several loops passing in front of the jammer which was placed in a static position on the side of the road selected for the experiment. In this way, the car was periodically approaching and getting away from the jammer. The minimum distance from the jammer was about 5 m. The received jamming power varied as a function of the distance between the jammer and the front-end.

To provide an indication of the level of interference recorded during the experiment, the jammer-to-noise power ratio ($J/N$) estimated during one of the passages of the car in front of the jammer is depicted in Fig. 4. The $J/N$ was estimated by dividing the dataset in small data blocks. For each data block, the total received power was estimated as the variance of the collected samples. The noise power was determined using samples at the beginning of the dataset when the car was far from the jammer and only noise was recorded (GPS signals have a negligible impact on the total received power). Finally, the jamming power was obtained as the difference between the total and the noise power. The time interval considered in Fig. 4 corresponds to the same interval used in the following to analyse tracking. From Fig. 4, it emerges that the car experiment considered here allows one to test algorithms under different $J/N$ conditions ranging from about $-10$ dB to about 25 dB.

In the close proximity of the jammer, with a $J/N$ of about 25 dB, the jamming signal is so strong that standard processing is
unable to acquire GPS signals. This fact clearly emerges from the left part of Fig. 5 which shows the CAF obtained using standard processing when the front-end is at the minimum distance from the jammer. The useful signal is hidden by the jamming component and no clear peak emerges from the CAF. In this way, the receiver is unable to acquire the useful signal component.

The same samples have been pre-processed using the myriad non-linearity in (23) and \( K = 6\sigma^2 \). The pre-processed samples were then correlated with the locally generated replicas, obtaining the CAF shown in the right part of Fig. 5. In this case, the useful signal peak clearly emerges from the CAF. Using the myriad non-linearity it is possible to acquire the useful GPS signal even in the close proximity of the jammer. The two CAFs in Fig. 5 were obtained using a coherent integration time equal to 1 ms and a single non-coherent integration: the advantages brought by the myriad non-linearity clearly emerges from the comparison of the two CAFs.

The linearity parameter, \( K = 6\sigma^2 \), was selected according to the CAF peak SNR estimated as

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**Fig. 3** Loss of efficiency caused by the myriad non-linearity as a function of the \( C/N_0 \) and for different values of \( K \). Comparison between Monte Carlo simulations and theoretical results

**Table 2** Characteristics of the Realtek RTL2832u front-end

| Parameter       | Value            |
|-----------------|------------------|
| sampling frequency | \( f_s = 2.046 \text{ MHz} \) |
| sampling type    | complex I/Q      |
| no. of bits      | 8                |

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**Fig. 4** \( J/N \) estimated during the car experiment. A maximum \( J/N \) value of about 25 dB is observed when the jammer is in the close proximity of the victim receiver.
Signal-to-noise ratio (SNR) is given by

\[
\text{SNR}_p(K) = \frac{\max_{\tau, f_d} \{ C(\tau, f_d) \}}{\text{median} \{ C(\tau, f_d) \}}. \tag{35}
\]

Metric (35) is the ratio between the maximum value in the CAF and the noise floor estimated using the median operator. The median is used to obtain an estimate of the noise floor only marginally influenced by the signal presence. Quantity (35) provides an indication of the quality of the CAF and is a function of the linearity parameter, \(K\). The CAF peak SNR is provided in Fig. 6 as a function of \(K\). The CAF used for the computation of (35) was obtained from the samples affected by jamming considered in Fig. 5.

The myriad non-linearity allows the acquisition of GPS signals even for small values of \(K\). However, for small values of \(K\), a significant SNR loss is observed. The peak SNR progressively increases with the linearity parameter, \(K\), up to a maximum value of about 17 dB. Then, an almost flat region for \(K\) in the interval \([5\sigma^2, 15\sigma^2]\) is observed. These considerations have been used for the selection of \(K = 6\sigma^2\). A value of \(K\) close to the minimum value in the interval \([5\sigma^2, 15\sigma^2]\) was selected to obtain more robustness to pulsed interference. The definition of a more systematic approach for the selection of the optimum value of \(K\) represents an independent research topic and it is thus out of the scope of this paper.

For large value of \(K\), the myriad non-linearity does not provide sufficient robustness to jamming and does not allow the acquisition of GPS signals. This fact is clearly shown in Fig. 6: when \(K\) is too large (greater than a value close to \(10^3 \sigma^2\) in Fig. 6), the acquisition block fails to detect the useful signal component. The decision threshold for acquisition was set in order to obtain a system false alarm probability [23] equal to 0.1.

The benefits brought by the myriad non-linearity are further analysed in Fig. 7 which compares the effective \(C/N_0\) values [12, 20] estimated during the jamming experiment. The effective \(C/N_0\) is continuously estimated by the receiver and it can be interpreted as a scaled version of coherent output SNR. Different processing strategies are considered in Fig. 7: standard processing without interference mitigation, PB with blanking threshold equal to 12 and two different myriad non-linearities, for \(K = 3\sigma^2\) and \(K = 6\sigma^2\). PB
has been included as comparison term. The blanking threshold has been selected according to the criteria analysed in [10].

At the beginning of the experiment considered in Fig. 7, the car is far from the jammer and all the processing strategies lead to similar $C/N_0$ values.

When the car approaches the jammer, the jamming signal causes a significant loss of $C/N_0$ when no mitigation is applied. PB and the myriad non-linearity allow receiver operations even in the close proximity of the jammer. The processing strategy employing the myriad non-linearity with $K = 6\sigma^2$ leads to the best performance.

The performance of the myriad non-linearity is better analysed in Fig. 8 which compares the $C/N_0$ values between standard and myriad processing in the absence of jamming, i.e. in the first portion of the dataset, and between PB and myriad processing during the part of the experiment most affected by interference. The $C/N_0$ difference with respect to standard processing is shown in the upper part of Fig. 8: this part of the experiment is considered to analyse the loss introduced by myriad processing in the absence of interference. Since a large value of $K$ is used, the averaged loss observed is about 0.1 dB. This loss is lower than that theoretically predicted due front-end filtering and front-end imperfections which are not considered in Section 4. The loss observed is very small and it is largely justified by the robustness introduced by the myriad non-linearity. The upper part of Fig. 8 also provides the $C/N_0$ difference between standard processing and PB. The two approaches have, in practice, the same performance.

The $C/N_0$ difference between PB and myriad processing is provided in the bottom part of Fig. 8: in this case, the myriad non-linearity with $K = 6\sigma^2$ provides the best performance and a
performance gain is observed. This fact is reflected in the bottom part of Fig. 8 by a negative loss. The myriad non-linearity outperforms PB and achieves an average effective $C/N_0$ gain of about 0.9 dB.

The results provided in this section show the benefits of the myriad non-linearity which is a low-complexity mitigation technique enabling receiver operations even in the presence of significant levels of jamming. The myriad non-linearity and PB have similar computational complexities and the experimental analysis provided above shows that the myriad non-linearity is a valid alternative to PB.

### 6 Conclusions

This paper considered the case of complex Cauchy noise for the design of the myriad non-linearity which has been adopted to introduce robustness in GNSS signal processing. The myriad non-linearity leads to a robust CAF where the input samples are first pre-processed in order to reduce the impact of outliers. The myriad non-linearity depends on a linearity parameter which allows one to select a compromise between loss of efficiency, when only Gaussian noise is present, and robustness against outliers.

The myriad non-linearity has been theoretically characterised and a closed-form expression for the loss of efficiency as a function of the linearity parameter has been derived. Theoretical results have been supported by Monte Carlo simulations. Finally, the benefits brought by the myriad non-linearity were empirically analysed considering the case of GPS signals corrupted by jamming. Myriad processing significantly improves the performance of a GNSS receiver enabling receiver operations even in the close proximity of a jammer. In the case considered, myriad processing slightly outperforms competing interference mitigation techniques such as PB. In this respect, myriad processing is an effective low-complexity interference mitigation technique.

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### 8 Appendix

#### 8.1 Properties of the myriad non-linearity

In this appendix, the first two moments of the samples pre-processed using the myriad non-linearity are derived under the assumption of input Gaussian noise and under weak signal conditions.

The first moment can be computed as

$$E\left[ \left| K + y \right| \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2} \left( \frac{y - Kx}{\sigma} \right)^2} A e^{i \theta_0} \, dy \, dx$$

where $A$ and $\theta_0$ are the signal amplitude and phase. The integration in (36) is performed over the whole complex plane. To evaluate (36), the transformation to circular coordinates is used and

$$E\left[ \left| K + y \right| \right] = \int_{0}^{2\pi} \int_{0}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2} \left( \frac{r^2}{\sigma^2} \right)} K \, r \, dr \, d\theta$$

where the last passage is justified by the fact that $e^{i \theta_0 \cos \theta - \theta_0} \sigma$ is an even function in $(\theta - \theta_0)$ and the multiplication by the sine component leads to a term with a zero integral over the $(-\pi, \pi)$ interval.

The last integral in (37) is directly connected to the integral definition of a modified Bessel function of first kind and order 1 [21]. In particular, using Eq. 9.6.19 of [21] (page 376), (37) becomes

$$E\left[ \left| K + y \right| \right] = \int_{0}^{\infty} \int_{0}^{2\pi} \frac{1}{2\pi} e^{-\frac{1}{2} \left( \frac{r^2}{\sigma^2} \right)} K \, r \, dr \, d\theta$$

The term within square brackets is the pdf of a Rician (non-central $\chi$) random variable with four degrees of freedom ([24], Eq. 2.21).
The term $Ae^{j\theta}$ is the mean of the input sample, $y[n]$, and can be replaced with $E[y[n]]$.

Integral (38) seems not to admit a simple closed-form solution. An approximation can be obtained by considering weak signal conditions, i.e.

$$\text{SNR} = \frac{A^2}{\sigma^2} \ll 1. \quad (39)$$

The quantity defined in (39) is the input SNR and it is a property of the input samples before any type of processing. It should not be confused with the coherent output SNR analysed in Section 4.

Under low SNR conditions, it is possible to approximate the pdf of a non-central $\chi$-squared distribution with that of a central random variable. In particular

$$E\left[ \frac{K^2}{K + |y[n]|^2} \right] \simeq \frac{K^2}{2\sigma^2} \frac{1}{K + x} e^{-\frac{x}{2\sigma^2}} dx. \quad (40)$$

The last integral can be finally computed using the change of variable $t = 1 + x/K$ and

$$\int_{0}^{\infty} \frac{1}{2\sigma^2} \frac{x}{K + x} e^{-\frac{x}{2\sigma^2}} dx = 1 - \frac{K^2}{2\sigma^2} e^{K^2/2\sigma^2} E_1\left( \frac{K^2}{2\sigma^2} \right). \quad (41)$$

where $E_1(\cdot)$ denotes the exponential integral ([21], page 228). Result (28) follows from (41):

$$E\left[ \frac{K^2}{K + |y[n]|^2} \right] \simeq E[y[n]] \frac{K^2}{2\sigma^2} \left[ 1 - \frac{K^2}{2\sigma^2} e^{K^2/2\sigma^2} E_1\left( \frac{K^2}{2\sigma^2} \right) \right]. \quad (42)$$

This approximation has been derived for low SNR conditions and its validity is investigated at the end of the Appendix.

The second moment of the pre-processed samples can be computed using a similar approach. In particular

$$E\left[ \frac{K^2}{K + |y[n]|^2} \right] \simeq E[y[n]] \frac{K^2}{2\sigma^2} \left[ 1 - \frac{K^2}{2\sigma^2} e^{K^2/2\sigma^2} E_1\left( \frac{K^2}{2\sigma^2} \right) \right]. \quad (43)$$

where $x = |y[n]|^2$. Moment (43) only depends on $x$ which is a non-central $\chi$-squared random variable with two degrees of freedom. An approximation for weak signal conditions is obtained using the pdf of central $\chi$-squared distribution with two degrees of freedom.

In this way

$$E\left[ \frac{K^2}{K + |y[n]|^2} \right] \simeq \frac{K^2}{2\sigma^2} \left[ 1 + \frac{K^2}{2\sigma^2} e^{K^2/2\sigma^2} E_1\left( \frac{K^2}{2\sigma^2} \right) \right]. \quad (44)$$

Results (44) have been obtained using the properties of the generalised exponential integral [21].

The validity of (42) and (44) is investigated in Fig. 9 which compares the approximations obtained with numerical results. In particular, trapezoidal integration has been used to compute the integrals in (38) and (43) where the exact pdf of non-central $\chi$-squared random variables have been used.

The first moment (denoted as ‘M1’) is analysed in the upper part of the figure, whereas the second moment (denoted as ‘M2’) is considered in the bottom part. From the figure, it emerges that the approximations derived are effective for input SNR values lower than −5 dB. Since GNSS signals are generally characterised by SNR values significantly lower than −5 dB, the approximations derived effectively describe the statistical properties of the pre-processed input samples. The results presented in Fig. 9 also support the effectiveness of the approximations derived for the different values of $K$. 

Fig. 9  First two moments of the samples pre-processed using the myriad non-linearity: comparison between numerical results and analytical approximations.