Effects of Poynting–Robertson drag on the circular restricted Three-Body Problem with variable masses

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ABSTRACT
This paper presents an investigation on the dynamical effect of Poynting–Robertson drag on the circular restricted three-body problem by considering all the masses are variable (primaries as well as infinitesimal body). After determining the equations of motion, the rate of change of the Jacobi constant with time has been evaluated. Further, in the numerical analysis section, the equilibrium points, Poincaré surface of section and Newton–Raphson basins of attraction have been plotted with the effect of Poynting–Robertson drag by Mathematica software. More exactly, we have noted that the equilibrium points are towards the origin, Newton–Raphson basins of attraction shrink and the Poincaré surface of sections, in two planes shrink and in one plane they expand. All these phenomena appear when we increase the effect of Poynting–Robertson drag. Finally, it is shown that the equilibrium points are unstable.

1. Introduction
The study of the motion of particles is a very interesting topic in the solar system. Many scientists and researchers are emphasizing this study by a very common model known as restricted Three-Body Problem. They have studied this model by considering one of the three bodies (infinitesimal body) is moving under the effect of the other two bodies (the primaries) but not affecting them. They have also considered many perturbations, such as shapes of the primaries, variable masses, radiation pressure, drags, resonance and magnetic fields. Many scientists have also studied the so-called Four-Body Problem with various perturbations. Poynting [1] studied the radiation in the solar system that affects temperature and small bodies. Robertson [2] studied the dynamical effects of drag in the solar system and derived the expression for the times of fall from circular orbits. Stanley [3] discussed the Poynting–Robertson effects in the solar system and found that it is due to the general radiation of a galaxy and is entirely too small to appreciably affect the dynamic of interstellar dust. Chernikov [4] studied the restricted Three-Body Problem (Sun–Planet–Particle) with the effects of solar radiation pressure and investigated the stability of six libration points by Lyapunov’s Method. Schuerman [5] explored the Roche potential with the effect of radiation pressure due to one component of a binary system and found that the energy considerations of the modified Roche potential have greater tendency to form rings. Ishwar [6] examined the linear stability of triangular equilibrium points in the generalized photogravitational restricted Three-Body Problem with Poynting–Robertson drag by considering one of the smaller primaries as an oblate spheroid. They found that all the equilibrium points are unstable. Kushvah [7–9] performed a normalization of the Hamiltonian in the generalized photogravitational restricted Three-Body Problem with Poynting–Robertson drag. They used the Whittaker method to transform the second order part of the Hamiltonian into a normal form. They found that the triangular points are stable in the non-linear sense except three critical mass ratios for which KAM (Kolmogorov, Arnold and Moser) theorem fails. Singh [10] investigated the stability of triangular equilibrium points in the restricted Three-body Problem with Poynting–Robertson drag and a smaller triaxial primary. In their study, they observed that these points are unstable. Lhotka [11] examined the stability of triangular equilibrium points in the elliptic restricted Three-Body Problem with the radial component of Poynting–Robertson drag and found that the temporary stability of particles displayed a tadpole motion in the $1:1$ resonance. Jain and others [12,13] studied the stability of the equilibrium points in the circular restricted Three-Body Problem with the drags and found that all the equilibrium points are unstable.

Many researchers have studied the variable masses in the restricted Three-Body Problem. Jeans [13] studied the two body problem with a variable mass. Meshcher-skii [14,15] worked on the mechanics of bodies with a variable mass. Ishwar [16] studied in his thesis about the restricted problem of three bodies with a variable mass. Singh [17–21] investigated the effect of perturbations...
on the locations and stability of triangular equilibrium points in the restricted Three-body problem with variable mass. Zhang [22] investigated the stability of the triangular equilibrium points in the restricted Three-body problem with a variable mass by using space-time transformation of Meshcherskii. Abouelmagd [23] studied the location of the out of plane equilibrium points and shown the forbidden movement regions in the restricted three-body problem with a variable mass. On the other hand, many other researchers have studied the basins of attraction in the restricted problem as Douskos [24], Alhussain [25], Ansari [26–28], Zotos [29–31], etc.

Taking in consideration all these contributions to the topic, we have investigated the effects of Poynting–Robertson drag on the circular restricted Three-Body Problem where the masses of the bodies vary. So the present study is organized as follow. In the second section, we have derived the equations of motion and the time variation of Jacobi constant under the effect of Poynting–Robertson drag. In the third section, we have done all the numerical analysis (equilibrium points, Poincaré surface of sections and Newton–Raphson basins of attraction). In the fourth section, we have examined the stability of the equilibrium points. Finally, in the fifth section, we have extracted the principal results of our problem.

Our principal motivation is that we believe this problem has many applications in this space age and can serve for further investigations. More precisely, as a possible physical application of our problem is on the celestial bodies. For example, we can consider any two celestial bodies with variable masses (because every celestial bodies are varying their masses according to Jean’s Law) and the third infinitesimal body as satellite which is also varying its mass. Although all the variations are very small. We also have taken the effect of P–R drag which is affecting to the motion of the satellite. We have shown this effect in our study.

Finally, we have considered two perturbations; variable masses and P–R drag. And since the main result of our problem is equation (3), if we reduce in this equation, the above perturbations (i.e. k = 1, α1 = 0 and q1 = 1) the obtained results coincide with the classical and well known results.

2. Equations of motion

Let \( m_1, m_2 \) and \( m \) be the masses of the primaries and infinitesimal body respectively and assume that they vary with time. The two primaries are revolving in circular orbits around their centre of mass which is considered as the origin in the xy-plane. The two primaries are placed at the x-axis. We have considered the Poynting–Robertson drag on \( m \) due to \( m_1 \) in the direction of \( m_1 \) to \( m \). Let us consider the synodic coordinate system, initially coincident with the inertial coordinate system, with angular velocity \( \omega \) about the z-axis (Figure 1). Using the procedure used in Abouelmagd [23] and Ansari [32], we can write the equations of motion of the infinitesimal variable mass \( m \) when the variation of mass is non-isotropic and originates from one point with the effect of Poynting–Robertson drag as

\[
\begin{align*}
\frac{m}{m}(\dot{x} - \alpha y) + (\dot{x} - \alpha y - 2\omega \dot{y} - \omega^2 x) &= W_x, \\
\frac{m}{m}(\dot{y} + \alpha x) + (\dot{y} + \alpha x + 2\omega \dot{x} - \omega^2 y) &= W_y, \\
\frac{m}{m}\dot{z} + \ddot{z} &= W_z,
\end{align*}
\]

where

\[
\begin{align*}
W_x &= -\frac{\mu_1(x - x_1)q_1}{r_1^3} - \frac{\mu_2(x - x_2)q_1}{r_2^3} - \frac{\mu_1(1 - q_1)}{C_{dr}^2} \left(\frac{(x - x_1)x + yy + zz}{r_1}\right) + \dot{x} - \alpha y, \\
W_y &= -\frac{\mu_1'yq_1}{r_1^3} - \frac{\mu_2' y}{r_2^3} - \frac{\mu_1(1 - q_1)}{C_{dr}^2} \times \left[\frac{y}{r_1}((x - x_1)x + yy + zz) + \dot{y} + \omega(x - x_1)\right], \\
W_z &= -\frac{\mu_1'zq_1}{r_1^3} - \frac{\mu_2' z}{r_2^3} - \frac{\mu_1(1 - q_1)}{C_{dr}^2} \times \left[\frac{z}{r_1}((x - x_1)x + yy + zz) + \dot{z}\right],
\end{align*}
\]

\( r_i^2 = (x - x_i)^2 + y^2 + z^2, \) (\( i = 1, 2 \)) are the distances from the primaries to the infinitesimal body, \( \mu_i = m_i/(m_1 + m_2) \), (\( i = 1, 2 \)) are the masses of the primaries, \( q_1 \) is the mass reduction factor and \( C_{dr} \) is the dimensionless velocity of light.
Using the Meshcherskii transformation (Meshcherskii [14])

\[ x = \xi R(t), \quad y = \eta R(t), \quad z = \zeta R(t), \quad \frac{dR}{dt} = R^2(t), \]

\[ r_i = \rho_i R(t), \quad (i = 1, 2), \]

the particular solutions of the Gylden–Meshcherskii problem are given by,

\[ \omega(t) = \frac{\omega_0}{R^2(t)}, \quad x_1 = \xi_1 R(t), \quad x_2 = \xi_2 R(t), \]

and the unified Meshcherskii law

\[
\begin{align*}
\mu(t) &= \frac{\mu_0}{R(t)}, \quad \mu_1(t) = \frac{\mu_{10}}{R(t)}, \quad \mu_2(t) = \frac{\mu_{20}}{R(t)}, \\
\mu(t) &= \mu_1(t) + \mu_2(t), \\
m &= \frac{m_0}{R(t)}, \quad R(t) = \sqrt{at^2 + 2bt + c},
\end{align*}
\]

where \(a, b, c, \mu_0, \mu_{10}, \mu_{20}, m_0\) are constants. We then get

\[
\begin{align*}
\dot{\xi} &= \xi' + (a + b^2)\xi, \\
\dot{\eta} &= \eta' + (a c - b^2)\eta, \\
\dot{\zeta} &= \zeta' + (a + b^2)\zeta, \\
\Delta &= a c - b^2, \xi_1 = -\frac{\mu_{20}}{\mu_0} \rho_1, \xi_2 = \frac{\mu_{10}}{\mu_0} \rho_1.
\end{align*}
\]

Taking the unit of mass, distance and time at initial time to such that \(\mu_0 = G, \rho_1 = 1, \omega_0 = 1, a_0 + b = a_1\) (constant).

So that \(\Delta = 1 - k\), where \(k\) is constant of a particular integral of Gylden–Meshcherskii problem and consequently, \(G = k\), the mass parameter \(\nu\) is given by

\[
\begin{align*}
\frac{\mu_{10}}{\mu_0} &= 1 - \nu, \\
\frac{\mu_{20}}{\mu_0} &= \nu, \quad 0 < \nu \leq \frac{1}{2}.
\end{align*}
\]

After applying all the transformations, the equations of motion (1) become

\[
\begin{align*}
\xi'' &= W_\xi, \\
\eta'' &= W_\eta, \\
\zeta'' &= W_\zeta,
\end{align*}
\]

where Dash (') is the differentiation w.r.t. \(t\) and

\[
W_\xi = (\alpha_1^2 + k)\xi + \alpha_1 \xi' - \alpha_1 \eta - \frac{(1 - \nu)(\xi + \nu)kq_1}{\rho_1^3}
\]

\[
- \frac{\nu k(\xi + \nu - 1)}{\rho_2^3}
\]

\[
- \frac{(1 - \nu)(1 - q_1)}{C_d \rho_1^2} \left[ \frac{\xi + \nu}{\rho_1^3} \right]^{(\xi + \nu)} \times (\xi' + \alpha_1 \xi) + \eta (\eta' + \alpha_1 \eta) + \xi (\xi' + \alpha_1 \xi)
\]

\[
+ (\xi' + \alpha_1 \xi) - \eta \right],
\]

\[ W_\eta = (\alpha_1^2 + k)\eta + \alpha_1 \eta' - \alpha_1 \xi - \frac{(1 - \nu)\eta kq_1}{\rho_1^3} \]

\[
- \frac{\nu k \eta}{\rho_2^3}
\]

\[
- \frac{(1 - \nu)(1 - q_1)}{C_d \rho_1^2} \left[ \frac{\eta}{\rho_1^3} \right]^{(\xi + \nu)} (\xi' + \alpha_1 \xi)
\]

\[
+ \eta (\eta' + \alpha_1 \eta) + \xi (\xi' + \alpha_1 \xi)
\]

\[
+ (\eta' + \alpha_1 \eta) + (\xi + \nu) \right],
\]

\[ W_\zeta = (\alpha_1^2 + k - 1) \xi + \alpha_1 \zeta' - \frac{(1 - \nu)\xi k q_1}{\rho_1^3} - \frac{\nu k \zeta}{\rho_2^3}
\]

\[
- \frac{(1 - \nu)(1 - q_1)}{C_d \rho_1^2} \left[ \frac{\xi}{\rho_1^3} \right]^{(\xi + \nu)} (\xi' + \alpha_1 \xi)
\]

\[
+ \eta (\eta' + \alpha_1 \eta) + \xi (\xi' + \alpha_1 \xi) \right] + (\xi' + \alpha_1 \xi)
\]

\[
\rho_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2, \quad (i = 1, 2), \xi_1 = -\nu, \xi_2 = 1 - \nu.
\]

When there are no dissipative forces, there is a constant of motion (Liou [33]), the Jacobi constant, defined by

\[
C = 2W - (\xi^2 + \eta^2 + \zeta^2). \quad (4)
\]

From Equations (3) and (4), we can write the time variation of the Jacobi constant due to Poynting–Robertson drag as

\[
C'(t) = -2(1 - \nu)(1 - q_1) \frac{1}{C_d \rho_1^2} \left[ \frac{\xi + \nu}{\rho_1^3} \right]^{(\xi + \nu)} \times (\xi' + \alpha_1 \xi) + \eta (\eta' + \alpha_1 \eta)
\]

\[
+ \alpha_1 (\xi' + \eta \eta' + \zeta') + (\xi^2 + \eta^2 + \zeta^2)
\]

\[
+ \nu \eta + \xi \xi' - \eta \eta' \right],
\]

\[
- 2\alpha_1 (\xi^2 + \eta^2 + \zeta^2).
\]

3. Numerical analysis

In this section, we have studied numerically equilibrium points, Poincaré surfaces of section and basins of attraction. We plotted them by using Mathematica software.
3.1. Equilibrium Points

By considering all derivatives are equal to zero in Equation (3), we obtain

\[
(\alpha_1^2 + k)\xi - \alpha_1 \eta - \frac{(1 - \upsilon)(\xi + \upsilon)kq_1}{\rho_1^3} - \frac{\upsilon k(\xi + \upsilon - 1)}{\rho_2^3} - \frac{(1 - \upsilon)k(1 - q_1)}{C_d\rho_1^2} \left[ \frac{\alpha_1(\xi + \upsilon)}{\rho_1^2} (\xi(\xi + \upsilon) + \eta^2 + \zeta^2) \right] + \alpha_1 \xi + \eta = 0, \tag{6}
\]

\[
(\alpha_1^2 + k)\eta - \alpha_1 \xi - \frac{(1 - \upsilon)\eta kq_1}{\rho_1^3} - \frac{\upsilon k\eta}{\rho_2^3} - \frac{(1 - \upsilon)k(1 - q_1)}{C_d\rho_1^2} \left[ \frac{\alpha_1\eta}{\rho_1^2} (\xi(\xi + \upsilon) + \eta^2 + \zeta^2) \right] + \alpha_1 \eta + \xi + \upsilon = 0, \tag{7}
\]

\[
(\alpha_1^2 + k - 1)\zeta - \frac{(1 - \upsilon)\zeta kq_1}{\rho_1^3} - \frac{\upsilon k\zeta}{\rho_2^3} - \frac{(1 - \upsilon)k(1 - q_1)}{C_d\rho_1^2} \left[ \frac{\alpha_1\zeta}{\rho_1^2} (\xi(\xi + \upsilon)) \right] + \eta^2 + \zeta^2 + \alpha_1 \zeta = 0. \tag{8}
\]

In this subsection, we have plotted the equilibrium points graphically by solving equations (6), (7) and (8) through Mathematica Software and found that there are at most five equilibrium points.

I. \((\xi, \eta)\)-plane \(\alpha_1 = 0.2, k = 0.4, C_d = 299,792,458, \upsilon = 0.019, \zeta = 0\).

In this plane, we found five equilibrium points with the effects of Poynting–Robertson drag. (Figure 2).

II. \((\eta, \zeta)\)-plane \(\alpha_1 = 0.2, k = 0.4, C_d = 299,792,458, \upsilon = 0.019, \eta = 0\).

In this plane, we got three equilibrium points with the effects of Poynting–Robertson drag. (Figure 3).

III. \((\xi, \zeta)\)-plane. \(\alpha_1 = 0.2, k = 0.4, C_d = 299,792,458, \upsilon = 0.019, \xi = 0\).

In this plane, we found three equilibrium points with the effects of Poynting–Robertson drag. (Figure 4).

In all three planes, we observe that increasing the effects of Poynting–Robertson drag, the equilibrium points are towards the origin.

3.2. Poincaré Surface of Sections

We have plotted the Poincaré surface of sections in three phase spaces \((\xi, \xi')\), \((\eta, \eta')\) and \((\zeta, \zeta')\) and observed that by changing the effect of Poynting–Robertson drag, the surface of sections are shrinking in the cases \((\xi, \xi')\) (Figure 5(a)), and \((\eta, \eta')\) (Figure 5(b)) and expanding in \((\zeta, \zeta')\)-case (Figure 5(c)).
3.3. Newton–Raphson basins of attraction

We have plotted the basins of attraction by using the Newton–Raphson iterative method. The iterative algorithm of our problem is given by the system

\[
\begin{align*}
\xi_n &= \xi_{n-1} - \left( \frac{W_{\xi \xi} W_{\eta \eta} - W_{\xi \eta} W_{\eta \xi}}{W_{\xi \xi} W_{\eta \eta} - W_{\xi \eta} W_{\eta \xi}} \right) \xi_{n-1}, \\
\eta_n &= \eta_{n-1} - \left( \frac{W_{\eta \xi} W_{\xi \xi} - W_{\eta \eta} W_{\xi \eta}}{W_{\xi \xi} W_{\eta \eta} - W_{\xi \eta} W_{\eta \xi}} \right) \eta_{n-1},
\end{align*}
\]

where \( \xi_{n-1}, \eta_{n-1} \) are the values of the \( \xi \) and \( \eta \) coordinates of the \( (n-1)_{th} \) step of the Newton–Raphson iterative process. The initial point \( (\xi, \eta) \) is a member of the basin of attraction of the root if this point converges rapidly to one of the equilibrium points. This process stops when the successive approximation converges to an attractor with some predefined accuracy. In this process the successive approximation points create a crooked path line (Figure 9). For the classification of the equilibrium points on the \( (\xi, \eta) \)-plane, a colour code will be used. In each figure, it appears different colours for different equilibrium points. In Figure 6(a), \( L_1 \) represents a purple colour region, \( L_2 \) represents a yellow colour region, \( L_3 \) represents a magenta colour region and \( L_4 \) represents a red colour region. The basins of attraction corresponding to the libration points of \( L_1, L_2, L_3, L_4 \) regions extend to infinity, but the basin of attraction covers a finite area corresponding to the libration point in the \( L_5 \) cyan colour region, which can be clearly visible in zoomed Figure 7(b) on the \( L_5 \) region of Figure 7(a). Furthermore, in Figure 8(a), \( L_1 \) represents a blue colour region, \( L_2 \) represents a yellow colour region, \( L_3 \) represents a purple colour region and \( L_4 \) represents a red colour region. The basins of attraction corresponding to

\[ L_3 \text{ represents a magenta colour region and } L_4 \text{ represents a red colour region.} \]
the libration points of $L_1$, $L_2$, $L_3$, $L_4$ regions extend to infinity, but the basin of attraction covers finite area corresponding to the libration point in the $L_5$ light green colour region, which can be clearly visible in zoomed Figure 8 (b) on the $L_5$ region of Figure 8(a). In this way, we get a complete view of the basins structure created by the attractors. We have plotted the basins of attraction with the effects of Poynting–Robertson drag ($q_1 = 1$, $q_1 = 0.74$, $q_1 = 0.50$) by taking 100 iterations and found that the basins of attraction are shrinking.

4. Stability of the equilibrium points

Following procedure given by Mishra [34], we examine the stability of the equilibrium points. When $\xi = \xi_0 + \alpha$, $\eta = \eta_0 + \beta$, $\zeta = \zeta_0 + \gamma$, Equation (3) becomes

$$
\begin{align*}
\alpha'' - 2\beta' &= W_\xi^0 + \alpha W_\xi^0 \xi + \beta W_\xi^0 \eta + \gamma W_\xi^0 \zeta + \alpha' W_\xi^0 \xi' + \\
&\quad + \beta' W_\xi^0 \eta' + \gamma' W_\xi^0 \zeta', \\
\beta'' + 2\alpha' &= W_\eta^0 + \alpha W_\eta^0 \xi + \beta W_\eta^0 \eta + \gamma W_\eta^0 \zeta + \alpha' W_\eta^0 \xi' + \\
&\quad + \beta' W_\eta^0 \eta' + \gamma' W_\eta^0 \zeta', \\
\zeta'' &= W_\zeta^0 + \alpha W_\zeta^0 \xi + \beta W_\zeta^0 \eta + \gamma W_\zeta^0 \zeta + \alpha' W_\zeta^0 \xi' + \\
&\quad + \beta' W_\zeta^0 \eta' + \gamma' W_\zeta^0 \zeta',
\end{align*}
$$

(10)

where $\alpha$, $\beta$ and $\gamma$ are the small displacements of the infinitesimal body from the equilibrium point. Superscript “zero” denotes the value at the equilibrium point. To solve Equations (10), let $\alpha = Ae^{\lambda t}$, $\beta = Be^{\lambda t}$, $\gamma = Ce^{\lambda t}$, where $A$, $B$ and $C$ are parameters. Substituting
these values in Equation (10) and rearranging, we get

\[
A(\lambda^2 - \lambda W_{\xi \xi}^0 - W_{\xi}^0) - B(2\lambda + \lambda W_{\eta \eta}^0 + W_{\xi}^0) = 0, \\
A(2\lambda - \lambda W_{\eta \eta}^0 - W_{\eta}^0) + B(\lambda^2 - \lambda W_{\eta \eta}^0 - W_{\eta}^0) = 0, \\
C(\lambda^2 - \lambda W_{\xi \xi}^0 - W_{\xi}^0) = 0.
\]

(11)

From the third equation of the system (11), we get

\[
\lambda = \frac{W_{\xi}^0 \pm \sqrt{(W_{\xi}^0)^2 + 4W_{\xi \xi}^0}}{2}.
\]

(12)

The first two equations of the system (11) will have a non-trivial solution for \( A \) and \( B \) if

\[
\begin{vmatrix}
\lambda^2 - \lambda W_{\xi \xi}^0 - W_{\xi}^0 & 2\lambda - \lambda W_{\eta \eta}^0 - W_{\xi}^0 \\
2\lambda - \lambda W_{\eta \eta}^0 - W_{\eta}^0 & \lambda^2 - \lambda W_{\eta \eta}^0 - W_{\eta}^0
\end{vmatrix} = 0.
\]

i.e.

\[
\lambda^4 + a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0,
\]

(13)

where

\[
a_0 = -W_{\xi \xi}^0 - W_{\eta}^0, \\
a_1 = 4 - W_{\xi}^0 + 2W_{\eta \eta}^0 - W_{\eta}^0 - 2W_{\eta \eta}^0, \\
& - W_{\eta \eta}^0 W_{\xi}^0 + W_{\xi}^0 W_{\eta}^0, \\
a_2 = 2W_{\xi}^0 W_{\eta}^0 - 2W_{\eta \eta}^0 W_{\eta \eta}^0 + W_{\xi}^0 W_{\eta \eta}^0, \\
& - W_{\xi}^0 W_{\xi}^0 W_{\eta}^0 + W_{\xi}^0 W_{\eta}^0 W_{\eta}^0, \\
a_3 = W_{\xi}^0 W_{\eta}^0 W_{\eta}^0.
\]

After finding all the values of \( \lambda \)'s at \((\xi_0, \eta_0, \xi' = 0, \eta' = 0, \zeta' = 0)\), we observed that in all cases, there exists at least one positive real characteristic root (the values of \( \lambda \) from Equations (12) and (13)). Hence, it can be concluded that all the equilibrium points are unstable.

5. Conclusion

In this paper, we have investigated the dynamical effect of Poynting–Robertson drag on the circular restricted three-body problem by considering all masses to be variable (primaries as well as infinitesimal body). We have determined the equations of motion which differ from the equations of motion found with only infinitesimal variable mass by the factors \( q_1, \alpha_1 \). We also have evaluated the rate of change of the Jacobi constant with time. In the numerical analysis section, we have plotted the equilibrium points with the effect of Poynting–Robertson drag in the \((\xi, \eta)\)-plane, we found that there are at most five equilibrium points (Figure 2). In the \((\xi, \zeta)\)-plane, we found that there are at most three equilibrium points (Figure 3) and in the \((\eta, \zeta)\)-plane, we found that there are at most three equilibrium points (Figure 4) and observed that in all cases the equilibrium points are towards the origin. The Poincaré surface of sections in three phase spaces \((\xi, \xi')\), \((\eta, \eta')\) and \((\zeta, \zeta')\) have been drawn and we observed that with the variation of the Poynting–Robertson drag, these surfaces are shrinking in the cases \((\xi, \xi')\) (Figure 5(a)), and \((\eta, \eta')\) (Figure 5(b)), and expanding in \((\xi, \zeta')\)-plane (Figure 5(c)). Finally, using Newton–Raphson iterative process, the Newton–Raphson basins of attraction have been determined. In this process, the successive approximation points create a crooked path line (Figure 9). For the classification of the equilibrium points on the \((\xi, \eta)\)-plane, a colour code will be used. In each figure, it appears different colours for different equilibrium points. In Figure 6(a), \( L_1 \) represents a purple colour region, \( L_2 \) represents a yellow colour region, \( L_3 \) represents a magenta colour region and \( L_4 \) represents a red colour region. The basins of attraction corresponding to the libration points of \( L_1, L_2, L_3, L_4 \) regions extend to infinity, but the basin of attraction covers
finite area corresponding to the libration point in the L5 cyan colour region, which can be clearly visible in zoomed Figure 6(b) on the L5 region of Figure 6(a). Further in Figure 7(a), L1 represents a blue colour region, L2 represents a yellow colour region, L3 represents a magenta colour region and L4 represents a red colour region. The basins of attraction corresponding to the libration points of L1, L2, L3, L4 regions extend to infinity, but the basin of attraction covers finite area corresponding to the libration point in the L5 cyan colour region, which can be clearly visible in zoomed Figure 7(b) on the L5 region of Figure 7(a). Furthermore, in Figure 8(a), L1 represents a blue colour region, L2 represents a yellow colour region, L3 represents a purple colour region and L4 represents a red colour region. The basins of attraction corresponding to the libration points of L1, L2, L3, L4 regions extend to infinity, but the basin of attraction covers finite area corresponding to the libration point in the L5 light green colour region, which can be clearly visible in zoomed Figure 8(b) on the L5 region of Figure 8(a). In this way we get a complete view of the basins structure created by the attractors. We have plotted the basins of attraction with the effects of Poynting–Robertson drag \( q_1 = 1, q_1 = 0.74, q_1 = 0.50 \) by taking 100 iterations and found that the basins of attraction are shrinking. At the end, we have examined the stability for each equilibrium point and have found that all the equilibrium points are unstable.

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