SEMI–FREDHOLM SINGULAR INTEGRAL OPERATORS
WITH PIECEWISE CONTINUOUS COEFFICIENTS ON
WEIGHTED VARIABLE LEBESGUE SPACES ARE FREDHOLM

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Abstract. Suppose \( \Gamma \) is a Carleson Jordan curve with logarithmic whirl points, \( \varrho \) is a Khvedelidze weight, \( p : \Gamma \to (1, \infty) \) is a continuous function satisfying \(|p(\tau) - p(t)| \leq -\text{const}/\log |\tau - t|\) for \(|\tau - t| \leq 1/2\), and \( L^p(\Gamma, \varrho) \) is a weighted generalized Lebesgue space with variable exponent. We prove that all semi-Fredholm operators in the algebra of singular integral operators with \( N \times N \) matrix piecewise continuous coefficients are Fredholm on \( L^p(\Gamma, \varrho) \).

1. Introduction

Let \( X \) be a Banach space and \( B(X) \) be the Banach algebra of all bounded linear operators on \( X \). An operator \( A \in B(X) \) is said to be \( n \)-normal (resp. \( d \)-normal) if its image \( \text{Im} A \) is closed in \( X \) and the defect number \( n(A; X) := \dim \ker A \) (resp. \( d(A; X) := \dim \ker A^* \)) is finite. An operator \( A \) is said to be semi-Fredholm on \( X \) if it is \( n \)-normal or \( d \)-normal. Finally, \( A \) is said to be Fredholm if it is simultaneously \( n \)-normal and \( d \)-normal. Let \( N \) be a positive integer. We denote by \( X_N \) the direct sum of \( N \) copies of \( X \) with the norm \( \|f\| = \|(f_1, \ldots, f_N)\| := (\|f_1\|^2 + \cdots + \|f_N\|^2)^{1/2} \).

Let \( \Gamma \) be a Jordan curve, that is, a curve that is homeomorphic to a circle. We suppose that \( \Gamma \) is rectifiable. We equip \( \Gamma \) with Lebesgue length measure \( |d\tau| \) and the counter-clockwise orientation. The \textit{Cauchy singular integral} of \( f \in L^1(\Gamma) \) is defined by

\[
(Sf)(t) := \lim_{R \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t, R)} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma),
\]

where \( \Gamma(t, R) := \{ \tau \in \Gamma : |\tau - t| < R \} \) for \( R > 0 \). David [7] (see also [3, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator.
$S$ on the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$, if and only if $\Gamma$ is a Carleson (Ahlfors-David regular) curve, that is,
\[
\sup_{r \in \Gamma} \sup_{R > 0} \frac{\left| \Gamma(t,R) \right|}{R} < \infty,
\]
where $|\Omega|$ denotes the measure of a measurable set $\Omega \subset \Gamma$. We can write $\tau - t = |\tau - t| e^{i \arg(\tau - t)}$ for $\tau \in \Gamma \setminus \{t\}$, and the argument can be chosen so that it is continuous on $\Gamma \setminus \{t\}$. It is known [3, Theorem 1.10] that for an arbitrary Carleson curve the estimate
\[
\arg(\tau - t) = O(- \log |\tau - t|) \quad (\tau \to t)
\]
holds for every $t \in \Gamma$. One says that a Carleson curve $\Gamma$ satisfies the *logarithmic whirl condition* at $t \in \Gamma$ if
\[
\arg(\tau - t) = - \delta(t) \log |\tau - t| + O(1) \quad (\tau \to t)
\]
with some $\delta(t) \in \mathbb{R}$. Notice that all piecewise smooth curves satisfy this condition at each point and, moreover, $\delta(t) \equiv 0$. For more information along these lines, see [2], [3, Chap. 1], [4].

Let $t_1, \ldots, t_m \in \Gamma$ be pairwise distinct points. Consider the Khvedelidze weight
\[
g(t) := \prod_{k=1}^{m} |t - t_k|^{\lambda_k} \quad (\lambda_1, \ldots, \lambda_m \in \mathbb{R}).
\]
Suppose $p : \Gamma \to (1, \infty)$ is a continuous function. Denote by $L^{p(\cdot)}(\Gamma, g)$ the set of all measurable complex-valued functions $f$ on $\Gamma$ such that
\[
\int_{\Gamma} |f(\tau)g(\tau)/|\lambda|^{p(\tau)}|d\tau| < \infty
\]
for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm
\[
\|f\|_{p(\cdot), g} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(\tau)g(\tau)/|\lambda|^{p(\tau)}|d\tau| \leq 1 \right\}.
\]
If $p$ is constant, then $L^{p(\cdot)}(\Gamma, g)$ is nothing else than the weighted Lebesgue space. Therefore, it is natural to refer to $L^{p(\cdot)}(\Gamma, g)$ as a *weighted generalized Lebesgue space with variable exponent* or simply as weighted variable Lebesgue spaces. This is a special case of Musielak-Orlicz spaces [24]. Nakano [25] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma, g)$ are referred to as weighted Nakano spaces.

If $S$ is bounded on $L^{p(\cdot)}(\Gamma, g)$, then from [13, Theorem 6.1] it follows that $\Gamma$ is a Carleson curve. The following result is announced in [16, Theorem 7.1] and in [18, Theorem D]. Its full proof is published in [20].

**Theorem 1.1.** Let $\Gamma$ be a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ be a continuous function satisfying
\[
|p(\tau) - p(t)| \leq -A\Gamma/\log |\tau - t| \quad \text{whenever} \quad |\tau - t| \leq 1/2,
\]
where $A$ is a constant.
where $A_\Gamma$ is a positive constant depending only on $\Gamma$. The Cauchy singular integral operator $S$ is bounded on $L^{p(.)}(\Gamma, \varrho)$ if and only if

$$0 < 1/p(t_k) + \lambda_k < 1 \quad \text{for all} \quad k \in \{1, \ldots, m\}. \quad (3)$$

We define by $PC(\Gamma)$ as the set of all $a \in L^\infty(\Gamma)$ for which the one-sided limits

$$a(t \pm 0) := \lim_{\tau \to t \pm 0} a(\tau)$$

exist and finite at each point $t \in \Gamma$; here $\tau \to t - 0$ means that $\tau$ approaches $t$ following the orientation of $\Gamma$, while $\tau \to t + 0$ means that $\tau$ goes to $t$ in the opposite direction. Functions in $PC(\Gamma)$ are called piecewise continuous functions.

The operator $S$ is defined on $L^{p(.)}_N(\Gamma, \varrho)$ elementwise. We let stand $PC_{N \times N}(\Gamma)$ for the algebra of all $N \times N$ matrix functions with entries in $PC(\Gamma)$. Writing the elements of $L^{p(.)}_N(\Gamma, \varrho)$ as columns, we can define the multiplication operator $aI$ for $a \in PC_{N \times N}(\Gamma)$ as multiplication by the matrix function $a$. Let $\text{alg} (S, PC; L^{p(.)}_N(\Gamma, \varrho))$ denote the smallest closed subalgebra of $B(L^{p(.)}_N(\Gamma, \varrho))$ containing the operator $S$ and the set $\{aI : a \in PC_{N \times N}(\Gamma)\}$.

For the case of piecewise Lyapunov curves $\Gamma$ and constant exponent $p$, a Fredholm criterion for an arbitrary operator $A \in \text{alg} (S, PC; L^{p(.)}_N(\Gamma, \varrho))$ was obtained by Gohberg and Krupnik [10] (see also [11] and [22]). Spitkovsky [29] established a Fredholm criterion for the operator $aP + Q$, where $a \in PC_{N \times N}(\Gamma)$ and

$$P := (I + S)/2, \quad Q := (I - S)/2,$$

on the space $L^{p(.)}_N(\Gamma, w)$, where $\Gamma$ is a smooth curve and $w$ is an arbitrary Muckenhoupt weight. He also proved that if $aP + Q$ is semi-Fredholm on $L^{p(.)}_N(\Gamma, w)$, then it is automatically Fredholm on $L^{p(.)}_N(\Gamma, w)$. These results were extended to the case of an arbitrary operator $A \in \text{alg} (S, PC; L^{p(.)}_N(\Gamma, w))$ in [12]. The Fredholm theory for singular integral operators with piecewise continuous coefficients on Lebesgue spaces with arbitrary Muckenhoupt weights on arbitrary Carleson curves was accomplished in a series of papers by Böttcher and Yu. Karlovich. It is presented in their monograph [3] (see also the nice survey [4]).

The study of singular integral operators with discontinuous coefficients on generalized Lebesgue spaces with variable exponent was started in [17, 19]. The results of [3] are partially extended to the case of weighted generalized Lebesgue spaces with variable exponent in [13, 14, 15]. Suppose $\Gamma$ is a Carleson curve satisfying the logarithmic whirl condition (1) at each point $t \in \Gamma$, $\varrho$ is a Khvedelidze weight, and $p$ is a variable exponent as in Theorem 1.1. Under these assumptions, a Fredholm criterion for an arbitrary operator $A$ in the algebra $\text{alg} (S, PC; L^{p(.)}_N(\Gamma, \varrho))$ is obtained in [14, Theorem 5.1] by using the Allan-Douglas local principle [5, Section 1.35] and the two projections theorem [9]. However, this approach does not allow us to get additional information about semi-Fredholm and Fredholm operators in this algebra. For instance, to obtain an index formula for Fredholm operators in this algebra, we need other means (see, e.g., [15, Section 6]). Following the ideas of [10, 29, 12], in this paper we present a self-contained proof of the following result.
THEOREM 1.2. Let $\Gamma$ be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point $t \in \Gamma$, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). If an operator in the algebra $\mathcal{A}(S, PC; L^p_n(\Gamma, \varrho))$ is semi-Fredholm, then it is Fredholm.

The paper is organized as follows. Section 2 contains general results on semi-Fredholm operators. Some auxiliary results on singular integral operators acting on $L^p_n(\Gamma, \varrho)$ are collected in Section 3. In Section 4, we prove a criterion guaranteeing that $aP + bQ$, where $a \in PC(\Gamma)$, has closed image in $L^p_n(\Gamma, \varrho)$. This criterion is intimately related with a Fredholm criterion for $aP + Q$ proved in [14]. Notice that we are able to prove both results for Carleson Jordan curves which satisfy the additional condition (1). Section 5 contains the proof of the fact that if the operator $aP + bQ$ is semi-Fredholm on $L^p_n(\Gamma, \varrho)$, then the coefficients $a$ and $b$ are invertible in the algebra $L^\infty_{\mathbb{N}}(\Gamma)$. In Section 6, we prove that the semi-Fredholmness and Fredholmness of $aP + bQ$ on $L^p_n(\Gamma, \varrho)$, where $a$ and $b$ are piecewise continuous matrix functions, are equivalent. In Section 7, we extend this result to the sums of products of operators of the form $aP + bQ$ by using the procedure of linear dilation. Since these sums are dense in $\mathcal{A}(S, PC; L^p_n(\Gamma, \varrho))$, Theorem 1.2 follows from stability properties of semi-Fredholm operators.

2. General results on semi-Fredholm and Fredholm operators

2.1. The Atkinson and Yood theorems

For a Banach space $X$, let $\Phi(X)$ be the set of all Fredholm operators on $X$ and let $\Phi_+(X)$ (resp. $\Phi_+(X)$) denote the set of all $n$-normal (resp. $d$-normal) operators $A \in \mathcal{B}(X)$ such that $d(A; X) = +\infty$ (resp. $n(A; X) = +\infty$).

THEOREM 2.1. Let $X$ be a Banach space and $K$ be a compact operator on $X$.

(a) If $A, B \in \Phi(X)$, then $AB \in \Phi(X)$ and $A + K \in \Phi(X)$.
(b) If $A, B \in \Phi_+(X)$, then $AB \in \Phi_+(X)$ and $A + K \in \Phi_+(X)$.
(c) If $A \in \Phi(X)$ and $B \in \Phi_+(X)$, then $AB \in \Phi_+(X)$ and $BA \in \Phi_+(X)$.

Part (a) is due to Atkinson, parts (b) and (c) were obtained by Yood. For a proof, see e.g. [11, Chap. 4, Sections 6 and 15].

THEOREM 2.2. (see e.g. [11], Chap. 4, Theorem 7.1) Let $X$ be a Banach space. An operator $A \in \mathcal{B}(X)$ is Fredholm if and only if there exists an operator $R \in \mathcal{B}(X)$ such that $AR - I$ and $RA - I$ are compact.

2.2. Stability of semi-Fredholm operators

THEOREM 2.3. (see e.g. [11], Chap. 4, Theorems 6.4, 15.4) Let $X$ be a Banach space.

(a) If $A \in \Phi(X)$, then there exists an $\varepsilon = \varepsilon(A) > 0$ such that $A + D \in \Phi(X)$ whenever $\|D\|_{\mathcal{B}(X)} < \varepsilon$. 
(b) If $A \in \Phi_\pm(X)$, then there exists an $\varepsilon = \varepsilon(A) > 0$ such that $A + D \in \Phi_\pm(X)$ whenever $\|D\|_{B(X)} < \varepsilon$.

**Lemma 2.4.** Let $X$ be a Banach space. Suppose $A$ is a semi-Fredholm operator on $X$ and $\|A_n - A\|_{B(X)} \to 0$ as $n \to \infty$. If the operators $A_n$ are Fredholm on $X$ for all sufficiently large $n$, then $A$ is Fredholm, too.

**Proof.** Assume $A$ is semi-Fredholm, but not Fredholm. Then either $A \in \Phi_-(X)$ or $A \in \Phi_+(X)$. By Theorem 2.3(b), either $A_n \in \Phi_-(X)$ or $A_n \in \Phi_+(X)$ for all sufficiently large $n$. That is, $A_n$ are not Fredholm. This contradicts the hypothesis. □

We refer to the monograph by Gohberg and Krupnik [11] for a detailed presentation of the theory of semi-Fredholm operators on Banach spaces.

### 2.3. Semi-Fredholmness of block operators

Let a Banach space $X$ be represented as the direct sum of its subspaces $X = X_1 + X_2$. Then every operator $A \in B(X)$ can be written in the form of an operator matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{ij} \in B(X_j, X_i)$ and $i, j = 1, 2$. The following result is stated without proof in [27]. Its proof is given in [28] (see also [23, Theorem 1.12]).

**Theorem 2.5.**

(a) Suppose $A_{21}$ is compact. If $A$ is $n$-normal ($d$-normal), then $A_{11}$ (resp. $A_{22}$) is $n$-normal (resp. $d$-normal).

(b) Suppose $A_{12}$ or $A_{21}$ is compact. If $A_{11}$ (resp. $A_{22}$) is Fredholm, then $A_{22}$ (resp. $A_{11}$) is $n$-normal, $d$-normal, Fredholm if and only if $A$ has the corresponding property.

### 3. Singular integrals on weighted variable Lebesgue spaces

#### 3.1. Duality of weighted variable Lebesgue spaces

Suppose $\Gamma$ is a rectifiable Jordan curve and $p : \Gamma \to (1, \infty)$ is a continuous function. Since $\Gamma$ is compact, we have

$$1 < \frac{1}{p} := \min_{t \in \Gamma} p(t), \quad \overline{p} := \max_{t \in \Gamma} p(t) < \infty.$$ 

Define the conjugate exponent $p^*$ for the exponent $p$ by

$$p^*(t) := \frac{p(t)}{p(t) - 1} \quad (t \in \Gamma).$$ 

Suppose $\varrho$ is a Khvedelidze weight. If $\varrho \equiv 1$, then we will write $L^{p(-)}(\Gamma)$ and $\| \cdot \|_{p(-)}$ instead of $L^{p(\cdot)}(\Gamma, 1)$ and $\| \cdot \|_{p(\cdot), 1}$, respectively.
THEOREM 3.1. (see [21], Theorem 2.1) If \( f \in L^p(\Gamma) \) and \( g \in L^{p^*}(\Gamma) \), then \( fg \in L^1(\Gamma) \) and
\[
\|fg\|_1 \leq (1 + 1/p - 1/p^*) \|f\|_{p(\cdot)} \|g\|_{p^*(\cdot)}.
\]
The above Hölder type inequality in the more general setting of Musielak-Orlicz spaces is contained in [24, Theorem 3.13].

THEOREM 3.2. The general form of a linear functional on \( L^p(\Gamma, \varrho) \) is given by
\[
G(f) = \int_{\Gamma} f(\tau) g(\tau) |d\tau| \quad (f \in L^p(\Gamma, \varrho)),
\]
where \( g \in L^{p^*}(\Gamma, \varrho^{-1}) \). The norms in the dual space \( [L^p(\Gamma, \varrho)]^* \) and in the space \( L^{p^*}(\Gamma, \varrho^{-1}) \) are equivalent.

The above result can be extracted from [24, Corollary 13.14]. For the case \( \varrho = 1 \), see also [21, Corollary 2.7].

### 3.2. Smirnov classes and Hardy type subspaces

Let \( \Gamma \) be a rectifiable Jordan curve in the complex plane \( \mathbb{C} \). We denote by \( D_+ \) and \( D_- \) the bounded and unbounded components of \( \mathbb{C} \setminus \Gamma \), respectively. We orient \( \Gamma \) counter-clockwise. Without loss of generality we assume that \( 0 \in D_+ \). A function \( f \) analytic in \( D_+ \) is said to be in the Smirnov class \( E^q(D_+) \) \((0 < q < \infty)\) if there exists a sequence of rectifiable Jordan curves \( \Gamma_n \) in \( D_+ \) tending to the boundary \( \Gamma \) in the sense that \( \Gamma_n \) eventually surrounds each compact subset of \( D_+ \) such that
\[
\sup_{n \geq 1} \int_{\Gamma_n} |f(z)|^q |dz| < \infty. \tag{4}
\]
The Smirnov class \( E^q(D_-) \) is the set of all analytic functions in \( D_- \cup \{\infty\} \) for which (4) holds with some sequence of curves \( \Gamma_n \) tending to the boundary in the sense that every compact subset of \( D_- \cup \{\infty\} \) eventually lies outside \( \Gamma_n \). We denote by \( E^q(D_-) \) the set of functions in \( E^q(D_-) \) which vanish at infinity. The functions in \( E^q(D_+) \) have nontangential boundary values almost everywhere on \( \Gamma \) (see, e.g. [8, Theorem 10.3]). We will identify functions in \( E^q(D_+) \) with their nontangential boundary values. The next result is a consequence of the Hölder inequality.

**Lemma 3.3.** Let \( \Gamma \) be a rectifiable Jordan curve. Suppose \( 0 < q_1, \ldots, q_r < \infty \) and \( f_j \in E^{q_j}(D_+) \) for all \( j \in \{1, 2, \ldots, r\} \). Then \( f_1 f_2 \ldots f_r \in E^q(D_+) \), where
\[
\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_r}.
\]

Let \( \mathcal{R} \) denote the set of all rational functions without poles on \( \Gamma \).

**Theorem 3.4.** Let \( \Gamma \) be a rectifiable Jordan curve and \( 0 < q < \infty \). If \( f \) belongs to \( E^q(D_+) + \mathcal{R} \) and its nontangential boundary values vanish on a subset \( \gamma \subset \Gamma \) of positive measure, then \( f \) vanishes identically in \( D_+ \).
This result follows from the Lusin-Privalov theorem for meromorphic functions (see, e.g. [26, p. 292]).

We refer to the monographs by Duren [8] and Privalov [26] for a detailed exposition of the theory of Smirnov classes over domains with rectifiable boundary.

**Lemma 3.5.** Let $\Gamma$ be a Carleson Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). Then $P^2 = P$ and $Q^2 = Q$ on $L^{p^*}(\Gamma, \varrho)$.

This result follows from Theorem 1.1 and [13, Lemma 6.4].

In view of Lemma 3.5, the Hardy type subspaces $PL^p(\Gamma, \varrho)$, $QL^p(\Gamma, \varrho)$, and $QL^{p^*}(\Gamma, \varrho) + \mathbb{C}$ of $L^p(\Gamma, \varrho)$ are well defined. Combining Theorem 1.1 and [13, Lemma 6.9] we obtain the following.

**Lemma 3.6.** Let $\Gamma$ be a Carleson Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). Then

$$E_1^1(D_+) \cap L^p(\Gamma, \varrho) = PL^p(\Gamma, \varrho),$$
$$E_0^1(D_-) \cap L^p(\Gamma, \varrho) = QL^p(\Gamma, \varrho),$$
$$E_1^1(D_-) \cap L^{p^*}(\Gamma, \varrho) = QL^{p^*}(\Gamma, \varrho) + \mathbb{C}.$$

### 3.3. Singular integral operators on the dual space

For a rectifiable Jordan curve $\Gamma$ we have $d\tau = e^{i\Theta(\tau)}|d\tau|$ where $\Theta_{\Gamma}(\tau)$ is the angle between the positively oriented real axis and the naturally oriented tangent of $\Gamma$ at $\tau$ (which exists almost everywhere). Let the operator $H_\Gamma$ be defined by $(H_\Gamma \varphi)(t) = e^{-i\Theta_{\Gamma}(t)}\overline{\varphi(t)}$ for $t \in \Gamma$. Note that $H_\Gamma$ is additive but $H_\Gamma(\alpha \varphi) = \overline{\alpha} H_\Gamma \varphi$ for $\alpha \in \mathbb{C}$. Evidently, $H_\Gamma^2 = I$.

From Theorem 1.1 and [13, Lemma 6.6] we get the following.

**Lemma 3.7.** Let $\Gamma$ be a Carleson Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). The adjoint operator of $S \in B(L^p(\Gamma, \varrho))$ is the operator $-H_\Gamma S H_\Gamma \in B(L^{p^*}(\Gamma, \varrho^{-1}))$.

**Lemma 3.8.** Let $\Gamma$ be a Carleson Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). Suppose $a \in L^\infty(\Gamma)$ and $a^{-1} \in L^\infty(\Gamma)$.

(a) The operator $aP + Q$ is $n$-normal on $L^{p^*}(\Gamma, \varrho)$ if and only if the operator $a^{-1}P + Q$ is $d$-normal on $L^{p^*}(\Gamma, \varrho^{-1})$. In this case

$$n(aP + Q; L^{p^*}(\Gamma, \varrho)) = d(a^{-1}P + Q; L^{p^*}(\Gamma, \varrho^{-1})).$$

(b) The operator $aP + Q$ is $d$-normal on $L^{p^*}(\Gamma, \varrho)$ if and only if the operator $a^{-1}P + Q$ is $n$-normal on $L^{p^*}(\Gamma, \varrho^{-1})$. In this case

$$d(aP + Q; L^{p^*}(\Gamma, \varrho)) = n(a^{-1}P + Q; L^{p^*}(\Gamma, \varrho^{-1})).$$
Proof. By Theorem 3.2, the space $L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})$ may be identified with the dual space $[L^p(\cdot)_{\Gamma}(\Gamma, \varrho)]^*$. Let us prove part (a). The operator $aP + Q$ is $n$-normal on $L^p(\cdot)_{\Gamma}(\Gamma, \varrho)$ if and only if its adjoint $(aP + Q)^*$ is $d$-normal on the dual space $L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})$ and

$$n(aP + Q; L^p(\cdot)_{\Gamma}(\Gamma, \varrho)) = d((aP + Q)^*; L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})).$$

(6)

From Theorem 3.2 it follows that

$$(aI)^* = H_\Gamma aH_\Gamma.$$  

(7)

Combining Lemma 3.7 and (7), we get

$$(aP + Q)^* = H_\Gamma(P + QaH)H_\Gamma.$$  

(8)

On the other hand, taking into account Lemma 3.5, it is easy to check that

$$P + QaI = (I + Pa^{-1}Q)(a^{-1}P + Q)(I - Qa^{-1}P)aI,$$  

(9)

where $I + Pa^{-1}Q$, $I - Qa^{-1}P$, and $aI$ are invertible operators on $L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})$. From (8) and (9) it follows that $(aP + Q)^*$ and $a^{-1}P + Q$ are $d$-normal on the space $L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})$ only simultaneously and

$$d((aP + Q)^*; L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})) = d(a^{-1}P + Q; L^p(\cdot)_{\Gamma}^*(\Gamma, \varrho^{-1})).$$  

(10)

Combining (6) and (10), we arrive at (5). Part (a) is proved. The proof of part (b) is analogous. □

Denote by $L^\infty_{N \times N}(\Gamma)$ the algebra of all $N \times N$ matrix functions with entries in the space $L^\infty(\Gamma)$.

**Lemma 3.9.** Let $\Gamma$ be a Carleson Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). Suppose $a \in L^\infty_{N \times N}(\Gamma)$ and $a^T$ is the transposed matrix of $a$. Then the operator $P + aQ$ is $n$-normal (resp. $d$-normal) on $L^p(\cdot)_{N}(\Gamma, \varrho)$ if and only if the operator $a^TP + Q$ is $d$-normal (resp. $n$-normal) on $L^p(\cdot)_{N}^*(\Gamma, \varrho^{-1})$.

**Proof.** In view of Theorem 3.2, the space $L^p(\cdot)_{N}^*(\Gamma, \varrho^{-1})$ may be identified with the dual space $[L^p(\cdot)_{N}(\Gamma, \varrho)]^*$, and the general form of a linear functional on $L^p(\cdot)_{N}(\Gamma, \varrho)$ is given by

$$G(f) = \sum_{j=1}^N \int_\Gamma f_j(\tau) \overline{g_j(\tau)} \, |d\tau|,$$

where $f = (f_1, \ldots, f_N) \in L^p(\cdot)_{N}(\Gamma, \varrho)$ and $g = (g_1, \ldots, g_N) \in L^p(\cdot)_{N}^*(\Gamma, \varrho^{-1})$, and the norms in $[L^p(\cdot)_{N}(\Gamma, \varrho)]^*$ and in $L^p(\cdot)_{N}^*(\Gamma, \varrho^{-1})$ are equivalent. It is easy to see that $(aI)^* = H_\Gamma a^T H_\Gamma$, where $H_\Gamma$ is defined on $L^p(\cdot)_{N}^*(\Gamma, \varrho^{-1})$ elementwise.
From Lemma 3.7 it follows that $P^* = H_T Q H_T$ and $Q^* = H_T P H_T$ on $L_N^{p^*}(\Gamma, g^{-1})$. Then

$$(P + aQ)^* = H_T (Pa^T I + Q) H_T.$$  \hspace{1cm} (11)

On the other hand, it is easy to see that

$$Pa^T I + Q = (I + Pa^T Q)(a^T P + Q)(I - Qa^T P),$$  \hspace{1cm} (12)

where the operators $I + Pa^T Q$ and $I - Qa^T P$ are invertible on $L_N^{p^*}(\Gamma, g^{-1})$. From (11) and (12) it follows that $(P + aQ)^*$ and $a^T P + Q$ are $n$-normal (resp. $d$-normal) on $L_N^{p^*}(\Gamma, g^{-1})$ only simultaneously. This implies the desired statement. \hfill $\Box$

4. Closedness of the image of $aP + Q$ in the scalar case

4.1. Functions in $L^p(\Gamma, g)$ are better than integrable if $S$ is bounded

**Lemma 4.1.** Suppose $\Gamma$ is a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ is a continuous function satisfying (2). If $g$ is a Khvedelidze weight satisfying (3), then there exists an $\varepsilon > 0$ such that $L^p(\Gamma, g)$ is continuously embedded in $L^{1+\varepsilon}(\Gamma)$.

**Proof.** If (3) holds, then there exists a number $\varepsilon > 0$ such that

$$0 < (1/p(t_k) + \lambda_k)(1 + \varepsilon) < 1 \quad \text{for all} \quad k \in \{1, \ldots, m\}.$$  

Hence, by Theorem 1.1, the operator $S$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma, g^{1+\varepsilon})$. In that case the operator $g^{1+\varepsilon} S g^{-1-\varepsilon} I$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$. Obviously, the operator $V$ defined by $(Vg)(t) = tg(t)$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$, and

$$((AV - VA)g)(t) = \frac{g^{1+\varepsilon}(t)}{\pi i} \int_{\Gamma} \frac{g(\tau)}{g^{1+\varepsilon}(\tau)} d\tau.$$  

Since $AV - VA$ is bounded on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$, there exists a constant $C > 0$ such that

$$\left| \int_{\Gamma} \frac{g(\tau)}{g^{1+\varepsilon}(\tau)} d\tau \right| \|g^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} = \left\| g^{1+\varepsilon} \int_{\Gamma} \frac{g(\tau)}{g^{1+\varepsilon}(\tau)} d\tau \right\|_{p(\cdot)/(1+\varepsilon)} \leq C \|g\|_{p(\cdot)/(1+\varepsilon)}$$

for all $g \in L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$. Since $g(\tau) > 0$ a.e. on $\Gamma$, we have $\|g^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} > 0$. Hence

$$\Lambda(g) = \int_{\Gamma} \frac{g(\tau)}{g^{1+\varepsilon}(\tau)} e^{i\theta(\tau)} |d\tau|$$

is a bounded linear functional on $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$. From Theorem 3.2 it follows that $g^{-1-\varepsilon} \in L^{p(\cdot)/(1+\varepsilon)*}(\Gamma)$, where

$$\left( \frac{p(t)}{1 + \varepsilon} \right)^* = \frac{p(t)}{p(t) - (1 + \varepsilon)}$$

is the conjugate exponent for $p(\cdot)/(1 + \varepsilon)$. By Theorem 3.1,

$$\int_{\Gamma} |f(\tau)|^{1+\varepsilon} |d\tau| \leq C_{p(\cdot), \varepsilon} \|f\|^{1+\varepsilon} g^{1+\varepsilon} \|g^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} \|g^{-1-\varepsilon}\|_{p(\cdot)/(1+\varepsilon)*}. \hspace{1cm} (13)$$
It is easy to see that
\[ \| |f|^{1+\varepsilon} g^{1+\varepsilon} \|_{\ell^{p(\cdot)/(1+\varepsilon)}} = \| f \|_{\ell^{p(\cdot), \varrho}}^{1+\varepsilon} = \| f \|_{\ell^{p(\cdot), \varrho}}^{1+\varepsilon}. \] (14)

From (13) and (14) it follows that \( \| f \|_{1+\varepsilon} \leq C_{p(\cdot), \varrho} \| f \|_{p(\cdot), \varrho} \) for all \( f \in L^{p(\cdot)}(\Gamma, \varrho) \), where \( C_{p(\cdot), \varrho} := (C_{p(\cdot), \varrho} \| f \|_{p(\cdot)/(1+\varepsilon)})^{1/(1+\varepsilon)} < \infty \). \( \square \)

4.2. Criterion for Fredholmness of \( aP + Q \) in the scalar case

THEOREM 4.2. (see [14], Theorem 3.3) Let \( \Gamma \) be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point \( t \in \Gamma \), let \( p : \Gamma \rightarrow (1, \infty) \) be a continuous function satisfying (2), and let \( \varrho \) be a Khvedelidze weight satisfying (3). Suppose \( a \in PC(\Gamma) \). The operator \( aP + Q \) is Fredholm on \( L^{p(\cdot)}(\Gamma, \varrho) \) if and only if \( a(t \pm 0) \neq 0 \) and
\[
-\frac{1}{2\pi} \arg \frac{a(t)-0}{a(t)+0} + \frac{\delta(t)}{2\pi} \log \frac{|a(t)-0|}{a(t)+0} + \frac{1}{p(t)} + \lambda(t) \notin \mathbb{Z} \quad (15)
\]
for all \( t \in \Gamma \), where
\[
\lambda(t) := \begin{cases} 
\lambda_{k}, & t = t_{k}, \quad k \in \{1, \ldots, m\}, \\
0, & t \notin \Gamma \setminus \{t_{1}, \ldots, t_{m}\}.
\end{cases}
\]

The necessity portion of this result was obtained in [13, Theorem 8.1] for spaces with variable exponents satisfying (2) under the assumption that \( S \) is bounded on \( L^{p(\cdot)}(\Gamma, w) \), where \( \Gamma \) is an arbitrary rectifiable Jordan curve and \( w \) is an arbitrary weight (not necessarily power). The sufficiency portion follows from [13, Lemma 7.1] and Theorem 1.1 (see [14] for details). The restriction (1) comes up in the proof of the sufficiency portion because under this condition one can guarantee the boundedness of the weighted operator \( wS w^{-1}I \), where \( w(\tau) = |(t-\tau)^{\gamma}| \) and \( \gamma \in \mathbb{C} \). If \( \Gamma \) does not satisfy (1), then the weight \( w \) is not equivalent to a Khvedelidze weight and Theorem 1.1 is not applicable to the operator \( wS w^{-1}I \), that is, a more general result than Theorem 1.1 is needed to treat the case of arbitrary Carleson curves. As far as we know, such a result is not known in the case of variable exponents. For a constant exponent \( p \), the result of Theorem 4.2 (for arbitrary Muckenhoupt weights) is proved in [2] (see also [3, Proposition 7.3] for the case of arbitrary Muckenhoupt weights and arbitrary Carleson curves).

4.3. Criterion for the closedness of the image of \( aP + Q \)

THEOREM 4.3. Let \( \Gamma \) be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point \( t \in \Gamma \), let \( p : \Gamma \rightarrow (1, \infty) \) be a continuous function satisfying (2), and let \( \varrho \) be a Khvedelidze weight satisfying (3). Suppose \( a \in PC(\Gamma) \) has finitely many jumps and \( a(t \pm 0) \neq 0 \) for all \( t \in \Gamma \). Then the image of \( aP + Q \) is closed in \( L^{p(\cdot)}(\Gamma, \varrho) \) if and only if (15) holds for all \( t \in \Gamma \).
Proof. The idea of the proof is borrowed from [3, Proposition 7.16]. The sufficiency part follows from Theorem 4.2. Let us prove the necessity part. Assume that \(a(t \pm 0) \neq 0\) for all \(t \in \Gamma\). Since the number of jumps, that is, the points \(t \in \Gamma\) at which \(a(t - 0) \neq a(t + 0)\), is finite, it is clear that

\[
- \frac{1}{2\pi} \arg \frac{a(t - 0)}{a(t + 0)} + \frac{\delta(t)}{2\pi} \log \frac{|a(t - 0)|}{|a(t + 0)|} + \frac{1}{1 + \epsilon} \notin \mathbb{Z},
\]

for all \(t \in \Gamma\) and all sufficiently small \(\epsilon > 0\). By Theorem 4.2, the operators \(aP + Q\) and \(a^{-1}P + Q\) are Fredholm on the Lebesgue space \(L^{1+\epsilon}(\Gamma)\) whenever \(\epsilon > 0\) is sufficiently small. From Lemma 4.1 it follows that we can pick \(\epsilon_0 > 0\) such that

\[
L^{p(-)}(\Gamma, \varrho) \subset L^{1+\epsilon_0}(\Gamma), \quad L^{p(+)}(\Gamma, \varrho^{-1}) \subset L^{1+\epsilon_0}(\Gamma)
\]

and \(aP + Q\), \(a^{-1}P + Q\) are Fredholm on \(L^{1+\epsilon_0}(\Gamma)\). Then

\[
n(aP + Q; L^{p(-)}(\Gamma, \varrho)) \leq n(aP + Q; L^{1+\epsilon_0}(\Gamma)) < \infty, \tag{16}
\]

and taking into account Lemma 3.8(b),

\[
d(aP + Q; L^{p(-)}(\Gamma, \varrho)) = n(a^{-1}P + Q; L^{p(+)}(\Gamma, \varrho^{-1})) \leq n(a^{-1}P + Q; L^{1+\epsilon_0}(\Gamma)) < \infty. \tag{17}
\]

If (15) does not hold, then \(aP + Q\) is not Fredholm on \(L^{p(-)}(\Gamma, \varrho)\) in view of Theorem 4.2. From this fact and (16)–(17) we conclude that the image of \(aP + Q\) is not closed in \(L^{p(-)}(\Gamma, \varrho)\), which contradicts the hypothesis. \(\square\)

5. Necessary condition for semi-Fredholms of \(aP + bQ\). The matrix case

5.1. Two lemmas on approximation of measurable matrix functions

Let the algebra \(L^\infty_{N \times N}(\Gamma)\) be equipped with the norm

\[
\|a\|_{L^\infty_{N \times N}(\Gamma)} := N \max_{1 \leq i,j \leq N} \|a_{ij}\|_{L^\infty(\Gamma)}.
\]

Lemma 5.1. (see [23], Lemma 3.4) Let \(\Gamma\) be a rectifiable Jordan curve. Suppose \(a\) is a measurable \(N \times N\) matrix function on \(\Gamma\) such that \(a^{-1} \notin L^\infty_{N \times N}(\Gamma)\). Then for every \(\epsilon > 0\) there exists a matrix function \(a_\epsilon \in L^\infty_{N \times N}(\Gamma)\) such that \(\|a_\epsilon\|_{L^\infty_{N \times N}(\Gamma)} < \epsilon\) and the matrix function \(a - a_\epsilon\) degenerates on a subset \(\gamma \subset \Gamma\) of positive measure.

Lemma 5.2. (see [23], Lemma 3.6) Let \(\Gamma\) be a rectifiable Jordan curve. If \(a\) belongs to \(L^\infty_{N \times N}(\Gamma)\), then for every \(\epsilon > 0\) there exists an \(a_\epsilon \in L^\infty_{N \times N}(\Gamma)\) such that \(\|a - a_\epsilon\|_{L^\infty_{N \times N}(\Gamma)} < \epsilon\) and \(a_\epsilon^{-1} \in L^\infty_{N \times N}(\Gamma)\).
5.2. Necessary condition for $d$-normality of $aP + Q$ and $P + aQ$

**Lemma 5.3.** Suppose $\Gamma$ is a Carleson Jordan curve, $p : \Gamma \to (1, \infty)$ is a continuous function satisfying (2), and $\varrho$ is a Khvedelidze weight satisfying (3). If $a \in L_{N \times N}^\infty(\Gamma)$ and at least one of the operators $aP + Q$ or $P + aQ$ is $d$-normal on $L_{N}^p(\Gamma, \varrho)$, then $a^{-1} \in L_{N \times N}^\infty(\Gamma)$.

**Proof.** This lemma is proved by analogy with [23, Theorem 3.13]. For definiteness, let us consider the operator $P + aQ$. Assume that $a^{-1} \notin L_{N \times N}^\infty(\Gamma)$. By Lemma 5.1, for every $\varepsilon > 0$ there exists an $a_\varepsilon \in L_{N \times N}^\infty(\Gamma)$ such that $\|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} < \varepsilon$ and $a_\varepsilon$ degenerates on a subset $\gamma \subset \Gamma$ of positive measure. We have

$$
\|(P + aQ) - (P + a_\varepsilon Q)\|_{\mathcal{B}(L_N^p(\Gamma, \varrho))} \leq \|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} \|Q\|_{\mathcal{B}(L_N^p(\Gamma, \varrho))} = O(\varepsilon)
$$

as $\varepsilon \to 0$. Hence there is an $\varepsilon > 0$ such that $P + a_\varepsilon Q$ is $d$-normal together with $P + aQ$ due to Theorem 2.3. Since the image of the operator $P + a_\varepsilon Q$ is a subspace of finite codimension in $L_N^p(\Gamma, \varrho)$, it has a nontrivial intersection with any infinite-dimensional linear manifold contained in $L_N^p(\Gamma, \varrho)$. In particular, the image of $P + a_\varepsilon Q$ has a nontrivial intersection with linear manifolds $M_j$, $j \in \{1, \ldots, N\}$, of those vector-functions, the $j$-th component of which is a polynomial of $1/z$ vanishing at infinity and all the remaining components are identically zero. That is, there exist

$$
\psi_j^+ \in PL_N^p(\Gamma, \varrho), \quad \psi_j^- \in QL_N^p(\Gamma, \varrho), \quad h_j \in M_j, \quad h_j \neq 0
$$

such that $\psi_j^+ + a_\varepsilon \psi_j^- = h_j$ for all $j \in \{1, \ldots, N\}$. Consider the $N \times N$ matrix functions

$$
\Psi_+ := [\psi_1^+, \psi_2^+, \ldots, \psi_N^+], \quad \Psi_- := [\psi_1^-, \psi_2^-, \ldots, \psi_N^-], \quad H := [h_1, h_2, \ldots, h_N],
$$

where $\psi_j^+$, $\psi_j^-$, and $h_j$ are taken as columns. Then $H - \Psi_+ = a_\varepsilon \Psi_-$. Therefore,

$$
\det(H - \Psi_+) = \det a_\varepsilon \det \Psi_- \quad \text{a.e. on} \quad \Gamma.
$$

The left-hand side of this equality is a meromorphic function having a pole at zero of at least $N$-th order. Thus, it is not identically zero in $D_+$. On the other hand, each entry of $H - \Psi_+$ belongs to $PL_N^p(\Gamma, \varrho) + \mathcal{R} \subset E^1(D_+) + \mathcal{R}$ (see Lemma 3.6). Hence, by Lemma 3.3, the function $\det(H - \Psi_+) \in E^1/N(D_+) + \mathcal{R}$ and $\det(H - \Psi_+)$ degenerates on $\gamma$ because $a_\varepsilon$ degenerates on $\gamma$. In view of Theorem 3.4, $\det(H - \Psi_+)$ vanishes identically in $D_+$. This is a contradiction. Thus, $a^{-1}$ belongs to $L_{N \times N}^\infty(\Gamma)$. $\square$
5.3. Necessary condition for semi-Fredholmness of $aP + bQ$

THEOREM 5.4. Let $\Gamma$ be a Carleson Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (2), and let $\varrho$ be a Khvedelidze weight satisfying (3). If the coefficients $a$ and $b$ belong to $L^\infty_{N \times N}(\Gamma)$ and the operator $aP + bQ$ is semi-Fredholm on $L^p_{N}(\Gamma, \varrho)$, then $a^{-1}, b^{-1} \in L^\infty_{N \times N}(\Gamma)$.

Proof. The proof is analogous to the proof of [23, Theorem 3.18]. Suppose $aP + bQ$ is $d$-normal on $L^p_{N}(\Gamma, \varrho)$. By Lemma 5.2, for every $\varepsilon > 0$ there exist $a_\varepsilon \in L^\infty_{N \times N}(\Gamma)$ such that $a_\varepsilon^{-1} \in L^\infty_{N \times N}(\Gamma)$ and $\|a - a_\varepsilon\|_{L^\infty_{N \times N}(\Gamma)} < \varepsilon$. Since

$$
\|(aP + bQ) - (a_\varepsilon P + bQ)\|_{B(t^p_{N}(\Gamma, \varrho))} \leq \|a - a_\varepsilon\|_{L^\infty_{N \times N}(\Gamma)} \|P\|_{B(t^p_{N}(\Gamma, \varrho))} = O(\varepsilon)
$$

as $\varepsilon \to 0$, from Theorem 2.3 it follows that $\varepsilon > 0$ can be chosen so small that $a_\varepsilon P + bQ$ is $d$-normal on $L^p_{N}(\Gamma, \varrho)$, too. Since $a_\varepsilon^{-1} \in L^\infty_{N \times N}(\Gamma)$, the operator $a_\varepsilon I$ is invertible on $L^p_{N}(\Gamma, \varrho)$. From Theorem 2.1 it follows that the operator $P + a_\varepsilon^{-1}bQ = a_\varepsilon^{-1}(a_\varepsilon P + bQ)$ is $d$-normal. By Lemma 5.3, $b^{-1}a_\varepsilon$ belongs to $L^\infty_{N \times N}(\Gamma)$. Hence $b^{-1} = b^{-1}a_\varepsilon a^{-1} \in L^\infty_{N \times N}(\Gamma)$.

Furthermore, $b^{-1}aP + Q = b^{-1}(aP + bQ)$ and the operator $b^{-1}aP + Q$ is $d$-normal on $L^p_{N}(\Gamma, \varrho)$. By Lemma 5.3, $a^{-1} \in L^\infty_{N \times N}(\Gamma)$, then $a^{-1} = a^{-1}bb^{-1}$ belongs to $L^\infty_{N \times N}(\Gamma)$. That is, we have shown that if $aP + bQ$ is $d$-normal on $L^p_{N}(\Gamma, \varrho)$, then $a^{-1}, b^{-1} \in L^\infty_{N \times N}(\Gamma)$.

If $aP + bQ$ is $n$-normal on $L^p_{N}(\Gamma, \varrho)$, then arguing as above, we conclude that the operator $P + a_\varepsilon^{-1}bQ$ is $n$-normal on $L^p_{N}(\Gamma, \varrho)$. By Lemma 3.9, the operator $(a_\varepsilon^{-1}b)^TP + Q$ is $d$-normal on $L^p_{N}(\Gamma, \varrho^{-1})$. From Lemma 5.3 it follows that $[(a_\varepsilon^{-1}b)^T]^{-1} \in L^\infty_{N \times N}(\Gamma)$. Therefore, $b^{-1} = (a_\varepsilon^{-1})^{-1}a^{-1} \in L^\infty_{N \times N}(\Gamma)$. Furthermore, $b^{-1}aP + Q = b^{-1}(aP + bQ)$ and the operator $b^{-1}aP + Q = b^{-1}(aP + bQ)$ is $n$-normal on $L^p_{N}(\Gamma, \varrho)$. From Lemma 3.9 we get that the operator $P + (b^{-1}a)^TQ$ is $d$-normal on $L^p_{N}(\Gamma, \varrho^{-1})$. Applying Lemma 5.3 to the operator $P + (b^{-1}a)^TQ$ acting on $L^p_{N}(\Gamma, \varrho^{-1})$, we obtain $a^{-1}b \in L^\infty_{N \times N}(\Gamma)$. Thus $a^{-1} = a^{-1}bb^{-1} \in L^\infty_{N \times N}(\Gamma)$. □

6. Semi-Fredholmness and Fredholmness of $aP + bQ$ are equivalent

6.1. Decomposition of piecewise continuous matrix functions

Denote by $PC^0(\Gamma)$ the set of all piecewise continuous functions $a$ which have only a finite number of jumps and satisfy $a(t - 0) = a(t)$ for all $t \in \Gamma$. Let $C_{N \times N}(\Gamma)$ and $PC^0_{N \times N}(\Gamma)$ denote the sets of $N \times N$ matrix functions with continuous entries and with entries in $PC^0(\Gamma)$, respectively. A matrix function $a \in PC_{N \times N}(\Gamma)$ is said to be nonsingular if $\det a(t \pm 0) \neq 0$ for all $t \in \Gamma$.

LEMMA 6.1. (see [6], Chap. VII, Lemma 2.2) Suppose $\Gamma$ is a rectifiable Jordan curve. If a matrix function $f \in PC^0_{N \times N}(\Gamma)$ is nonsingular, then there exist an upper-triangular nonsingular matrix function $g \in PC^0_{N \times N}(\Gamma)$ and nonsingular matrix functions $c_1, c_2 \in C_{N \times N}(\Gamma)$ such that $f = c_1gc_2$. 

6.2. Compactness of commutators

**Lemma 6.2.** Let \( \Gamma \) be a Carleson Jordan curve, let \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying (2), and let \( \varrho \) be a Khvedelidze weight satisfying (3). If \( c \) belongs to \( C_{N\times N}(\Gamma) \), then the commutators \( cP - PcI \) and \( cQ - QcI \) are compact on \( L^p_N(\Gamma, \varrho) \).

This statement follows from Theorem 1.1 and [13, Lemma 6.5].

6.3. Equivalence of semi-Fredholmness and Fredholmness of \( aP + bQ \)

**Theorem 6.3.** Let \( \Gamma \) be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point \( t \in \Gamma \), let \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying (2), and let \( \varrho \) be a Khvedelidze weight satisfying (3). If \( a, b \in PC_{N\times N}^0(\Gamma) \), then \( aP + bQ \) is semi-Fredholm on \( L^p_N(\Gamma, \varrho) \) if and only if it is Fredholm on \( L^p_N(\Gamma, \varrho) \).

**Proof.** The idea of the proof is borrowed from [29, Theorem 3.1]. Only the necessity portion of the theorem is nontrivial. If \( aP + bQ \) is semi-Fredholm, then \( a \) and \( b \) are nonsingular by Theorem 5.4. Hence \( b^{-1}a \) is nonsingular. In view of Lemma 6.1, there exist an upper-triangular nonsingular matrix function \( g \in PC_{N\times N}^0(\Gamma) \) and continuous nonsingular matrix functions \( c_1, c_2 \) such that \( b^{-1}a = c_1gc_2 \). It is easy to see that

\[
aP + bQ = bc_1[(gP + Q)(PcI + Qc_1^{-1}I) + g(c_2P - PcI) + (c_1^{-1}Q - Qc_1^{-1}I)] .
\]

(18)

From Lemma 6.2 it follows that the operators \( c_2P - PcI \) and \( c_1^{-1}Q - Qc_1^{-1}I \) are compact on \( L^p_N(\Gamma, \varrho) \) and

\[
(PcI + Qc_1^{-1}I)(c_2^{-1}P + c_1Q) = I + K_1 , \quad (c_2^{-1}P + c_1Q)(PcI + Qc_1^{-1}I) = I + K_2 ,
\]

where \( K_1 \) and \( K_2 \) are compact operators on \( L^p_N(\Gamma, \varrho) \). In view of these equalities, by Theorem 2.2, the operator \( PcI + Qc_1^{-1}I \) is Fredholm on \( L^p_N(\Gamma, \varrho) \). Obviously, the operator \( bc_1I \) is invertible because \( bc_1 \) is nonsingular. From (18) and Theorem 2.1 it follows that \( aP + bQ \) is \( n \)-normal, \( d \)-normal, Fredholm if and only if \( gP + Q \) has the corresponding property.

Let \( g_j, j \in \{1, \ldots, N\} \), be the elements of the main diagonal of the upper-triangular matrix function \( g \). Since \( g \) is nonsingular, all \( g_j \) are nonsingular, too. Assume for definiteness that \( gP + Q \) is \( n \)-normal on \( L^p_N(\Gamma, \varrho) \). By Theorem 2.5 (a), the operator \( g_1P + Q \) is \( n \)-normal on \( L^p(\Gamma, \varrho) \). Hence the image of \( g_1P + Q \) is closed. From Theorem 4.3 it follows that (15) is fulfilled with \( g_1 \) in place of \( a \). Therefore, the operator \( g_1P + Q \) is Fredholm on \( L^p_N(\Gamma, \varrho) \) due to Theorem 4.2. Applying Theorem 2.5(b), we deduce that the operator \( g^{(1)}P + Q \) is \( n \)-normal on \( L^p_{N-1}(\Gamma, \varrho) \), where \( g^{(1)} \) is the \( (N - 1) \times (N - 1) \) upper-triangular nonsingular matrix function obtained from \( g \) by deleting the first column and the first row. Arguing as before with \( g^{(1)} \) in place of \( g \), we conclude that \( g_2P + Q \) is Fredholm on \( L^p_{N-1}(\Gamma, \varrho) \) and \( g^{(2)}P + Q \) is \( n \)-normal on \( L^p_{N-2}(\Gamma, \varrho) \), where \( g^{(2)} \) is the \( (N - 2) \times (N - 2) \) upper-triangular
nonsingular matrix function obtained from $g^{(1)}$ by deleting the first column and the first row. Repeating this procedure $N$ times, we can show that all operators $g_jP + Q$, $j \in \{1, \ldots, N\}$, are Fredholm on $L^{p(\cdot)}(\Gamma, \varrho)$.

If the operator $gP + Q$ is $d$-normal, then we can prove in a similar fashion that all operators $g_jP + Q$, $j \in \{1, \ldots, N\}$, are Fredholm on $L^{p(\cdot)}(\Gamma, \varrho)$. In this case we start with $g_N$ and delete the last column and the last row of the matrix $g^{(j-1)}$ on the $j$-th step (we assume that $g^{(0)} = g$).

Since all operators $g_jP + Q$ are Fredholm on $L^{p(\cdot)}(\Gamma, \varrho)$, from Theorem 2.5(b) we obtain that the operator $gP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, \varrho)$. Hence $aP + bQ$ is Fredholm on $L^{p(\cdot)}(\Gamma, \varrho)$, too. □

7. Semi-Fredholmness and Fredholmness are equivalent for arbitrary operators in $\text{alg}(S, PC, L^{p(\cdot)}_N(\Gamma, \varrho))$

7.1. Linear dilation

The following statement shows that the semi-Fredholmness of an operator in a dense subalgebra of $\text{alg}(S, PC, L^{p(\cdot)}_N(\Gamma, \varrho))$ is equivalent to the semi-Fredholmness of a simpler operator $aP + bQ$ with coefficients of $a, b$ of larger size.

**Lemma 7.1.** Suppose $\Gamma$ is a Carleson Jordan curve, $p : \Gamma \to (1, \infty)$ is a continuous function satisfying (2), and $\varrho$ is a Khvedelidze weight satisfying (3). Let

$$A = \sum_{i=1}^{k} A_{ii} A_{i2} \ldots A_{ir},$$

where $A_{ii} = a_{ii}P + b_{ii}Q$ and all $a_{ii}, b_{ii}$ belong to $PC_{0}(\Gamma)$. Then there exist functions $a, b \in PC_{0}(\Gamma)$, where $D := N(k(r+1)+1)$, such that $A$ is $n$-normal ($d$-normal, Fredholm) on $L^{p(\cdot)}_N(\Gamma, \varrho)$ if and only if $aP + bQ$ is $n$-normal ($d$-normal, Fredholm) on $L^{p(\cdot)}_D(\Gamma, \varrho)$.

**Proof.** The idea of the proof is borrowed from [10] (see also [1, Theorem 12.15]). Denote by $O_s$ and $I_s$ the $s \times s$ zero and identity matrix, respectively. For $\ell = 1, \ldots, r$, let $B_{\ell}$ be the $kN \times kN$ matrix

$$B_{\ell} = \text{diag}(A_{1\ell}, A_{2\ell}, \ldots, A_{k\ell}),$$

then define the $kN(r+1) \times kN(r+1)$ matrix $Z$ by

$$Z = \begin{bmatrix}
I_{kN} & B_1 & O_{kN} & \ldots & O_{kN} \\
O_{kN} & I_{kN} & B_2 & \ldots & O_{kN} \\
& \vdots & \vdots & \ddots & \vdots \\
O_{kN} & O_{kN} & O_{kN} & \ldots & B_r \\
O_{kN} & O_{kN} & O_{kN} & \ldots & I_{kN}
\end{bmatrix}.$$
Put
\[ X := \text{column}(O_N, \ldots, O_N, -I_N, \ldots, -I_N), \quad Y := (I_N, \ldots, I_N, O_N, \ldots, O_N). \]

Define also \( M_0 = (I_N, \ldots, I_N) \) and for \( \ell \in \{1, \ldots, r\} \), let
\[ M_\ell := (A_{11}A_{12} \ldots A_{1\ell}, A_{21}A_{22} \ldots A_{2\ell}, \ldots, A_{k_1}A_{k_2} \ldots A_{k_\ell}). \]

Finally, put
\[ W := (M_0, M_1, \ldots, M_r). \]

It can be verified straightforwardly that
\[ \begin{bmatrix} I_{kN(r+1)} & O \\ W & I_N \end{bmatrix} \begin{bmatrix} I_{kN(r+1)} & O \\ O & A \end{bmatrix} = \begin{bmatrix} Z & X \\ O & I_N \end{bmatrix} \begin{bmatrix} Z & X \\ Y & O_N \end{bmatrix}. \quad (19) \]

It is clear that the outer terms on the left-hand side of (19) are invertible. Hence the middle factor of (19) and the right-hand side of (19) are \( n \)-normal \((d\text{-normal, Fredholm})\) only simultaneously in view of Theorem 2.1. By Theorem 2.5(b), the operator \( A \) is \( n \)-normal \((d\text{-normal, Fredholm})\) if and only if the middle factor of (19) has the corresponding property. Finally, note that the left-hand side of (19) has the form \( aP + bQ \), where \( a, b \in PC_0^{D \times D}(\Gamma) \). □

### 7.2. Proof of Theorem 1.2

Obviously, for every \( f \in PC(\Gamma) \) there exists a sequence \( f_n \in PC_0(\Gamma) \) such that
\[ \| f - f_n \|_{L^\infty(\Gamma)} \to 0 \text{ as } n \to \infty. \]
Therefore, for each operator \( \alpha P + \beta Q \), where \( \alpha = (\alpha_{rs})_{r,s=1}^N, \beta = (\beta_{rs})_{r,s=1}^N \) and \( \alpha_{rs}, \beta_{rs} \in PC(\Gamma) \) for all \( r,s \in \{1, \ldots, N\} \), there exist sequences \( \alpha^{(n)} = (\alpha_{rs}^{(n)})_{r,s=1}^N, \beta^{(n)} = (\beta_{rs}^{(n)})_{r,s=1}^N \) with \( \alpha_{rs}^{(n)}, \beta_{rs}^{(n)} \in PC_0(\Gamma) \) for all \( r,s \in \{1, \ldots, N\} \) such that

\[ \|(\alpha P + \beta Q) - (\alpha^{(n)} P + \beta^{(n)} Q)\|_{B(\mathcal{L}_N^{\infty}(\Gamma, \mathfrak{g}))} \]
\[ \leq N \max_{1 \leq r,s \leq N} \|\alpha_{rs} - \alpha_{rs}^{(n)}\|_{L^\infty(\Gamma)} \|P\|_{B(\mathcal{L}_N^{\infty}(\Gamma, \mathfrak{g}))} \]
\[ + N \max_{1 \leq r,s \leq N} \|\beta_{rs} - \beta_{rs}^{(n)}\|_{L^\infty(\Gamma)} \|Q\|_{B(\mathcal{L}_N^{\infty}(\Gamma, \mathfrak{g}))} = o(1) \]

as \( n \to \infty \).

Let \( A \in \text{alg}(S, PC; L_N^{p_1}(\Gamma, \mathfrak{g})) \). Then there exists a sequence of operators \( A^{(n)} \) of the form \( \sum_{i=1}^k A_{i1}^{(n)} A_{i2}^{(n)} \ldots A_{ir}^{(n)} \), where \( A_{ij}^{(n)} = a_{ij}^{(n)} P + b_{ij}^{(n)} Q \) and \( a_{ij}^{(n)}, b_{ij}^{(n)} \) belong to \( PC_{N \times N}(\Gamma) \), such that \( \| A - A^{(n)} \|_{B(\mathcal{L}_N^{p_1}(\Gamma, \mathfrak{g}))} \to 0 \) as \( n \to \infty \). In view of what has been said above, without loss of generality, we can assume that all matrix functions \( a_{ij}^{(n)}, b_{ij}^{(n)} \) belong to \( PC_0^{N \times N}(\Gamma) \).
If $A$ is semi-Fredholm, then for all sufficiently large $n$, the operators $A^{(n)}$ are semi-Fredholm by Theorem 2.3. From Lemma 7.1 it follows that for every semi-Fredholm operator $\sum_{i=1}^{k} A_{i1}^{(n)} A_{i2} \ldots A_{ir}^{(n)}$ there exist $a^{(n)}, b^{(n)} \in PC_{D \times D}(\Gamma)$, where $D := N(k(r+1) + 1)$, such that $a^{(n)} P + b^{(n)} Q$ is semi-Fredholm on $L_{D}^{-1}(\Gamma, \varrho)$. By Theorem 6.3, $a^{(n)} P + b^{(n)} Q$ is Fredholm on $L_{D}^{-1}(\Gamma, \varrho)$. Applying Lemma 7.1 again, we conclude that $\sum_{i=1}^{k} A_{i1}^{(n)} A_{i2} \ldots A_{ir}^{(n)}$ is Fredholm on $L_{N}^{-1}(\Gamma, \varrho)$. Thus, for all sufficiently large $n$, the operators $A^{(n)}$ are Fredholm. Lemma 2.4 yields that $A$ is Fredholm. □

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