Warped products and FRW cosmology

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We study the Friedmann-Robertson-Walker cosmological model to investigate non-smooth curvatures associated with multiple discontinuities involved in the evolution of the universe, by exploiting Lorentzian warped product scheme. Introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions, we also discuss non-smooth features of the spatially flat Friedmann-Robertson-Walker universe.

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I. INTRODUCTION

A warped product spacetime was introduced by Bishop and O’Neill long ago [1]. The warped product scheme was later applied to general relativity [2] and semi-Riemannian geometry [3]. Recently, this warped products were extended to multiply warped products with non-smooth metric [4], and the Banados-Teitelboim-Zanelli (BTZ) black hole [5] and Reissner-Nordström-anti-de Sitter (RN-AdS) black hole [6]. The concept of the warped products was used in higher dimensional theories. For instance, the warped products were exploited in the Randall-Sundrum model in five-dimensions [7] and in the Kaluza-Klein supergravity theory in seven-dimensions [8].

On the other hand, the standard big bang cosmological model based on the Friedmann-Robertson-Walker (FRW) spacetimes has led to the inflationary cosmology [9] and nowadays to the M-theory cosmology with bouncing universes [10]. These spacetimes are foliated by a special set of spacelike hypersurfaces such that each hypersurface corresponds to an instant of time.

In this paper, as a cosmological model we will exploit the FRW spacetimes, which can be treated as a warped product manifold possessing warping function (or scale factor) with time dependence, to investigate the non-smooth curvature originated from the multiple discontinuities involved in the evolution of the universe. In particular, we will analyze the spatially flat FRW universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in the astrophysical phenomenology.

II. WARPED PRODUCTS WITH SINGLE DISCONTINUITY

In differential geometry, a multiply warped product manifold \((M = B \times f_1 \times \ldots \times f_n \times F_n, g)\) is defined to consist of the Riemannian base manifold \((B, g_B)\) and fibers \((F_i, g_i)\) \(i = 1, \ldots, n\) associated with the Lorentzian metric,

\[
g = \pi_B^*g_B + \sum_{i=1}^{n} (f_i \circ \pi_B)^2 \pi_i^*g_i
\]

where \(\pi_B, \pi_i\) are the natural projections of \(B \times F_1 \times \ldots \times F_n\) onto \(B\) and \(F_i\), respectively, and \(f_i\) are positive warping functions. For the specific case of \((B = R, g_B = -d\mu^2)\), the Lorentzian metric is rewritten as

\[
g = -d\mu^2 + \sum_{i=1}^{n} f_i^2 g_i.
\]

It is well-known that the Randall-Sundrum model is described by the metric of the form

\[
g = -N^2(t, y) dt^2 + A^2(t, y) dx^2 + dy^2,
\]

which can be thus regarded as the Lorentzian metric for the (4+1) higher dimensional warped product manifold.
Next, let $M = B \times f_1 \times \ldots \times f_n$, $F_n$ be a multiply warped products with Riemannian curvature tensor $R$ and flow vector field $U = \partial_t$. If $X, Y \in \mathcal{V}(B)$, $U_i, V_i \in \mathcal{V}(F_i)$ ($n = 1, 2, \ldots, n$), $d_i = \dim F_i$, $f_i \in C^0(S)$ at a single point $p \in B$, and $S = \{p\} \times f_1 \times \cdots \times f_n$, $F_n$, then we can obtain the Ricci components as follows

\[
\begin{align*}
\text{Ric}(X, Y) &= -\sum_{i=1}^n d_i X^1 Y^1 f_i''(t) + \delta(t - p) \left( f_i^+ - f_i^- \right), \\
\text{Ric}(X, U_i) &= 0, \\
\text{Ric}(U_i, V_i) &= f_i \text{Ric}(U_i, V_i) + \langle U_i, V_i \rangle f_i''(t) + \delta(t - p) \left( f_i^+ - f_i^- \right) \\
&\quad + \langle U_i, V_i \rangle \left[ (d_i - 1) f_i^+ - f_i^- \right] \\
&\quad + \sum_{j \neq i} d_j \left( f_i^+ - f_j^+ - f_i^- - f_j^- \right), \\
\text{Ric}(U_i, U_j) &= 0, \quad \text{for } i \neq j,
\end{align*}
\]

where $X = X^1 \partial/\partial_t$ and $Y = Y^1 \partial/\partial_t$ and $\delta(t - p)$ is the delta function.

### III. WARPED PRODUCTS WITH MULTIPLE DISCONTINUITIES

Now, in order to study multiple discontinuities in the warped products, we consider the FRW metric of the form

\[
g = -dt^2 + f^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right),
\]

where $f$ is a scale factor and $k$ is a parameter denoting the spatially flat ($k = 0$), 3-sphere ($k = 1$) and hyperboloidal ($k = -1$) universes. Treating $f$ as a warping function associated with the Lorentzian metric $\text{FRW}$, the FRW spacetime can be regarded as the warped product manifold $M = B \times f F$ where the base manifold $B$ is an open interval of $R$ with usual metric $g_B = -dt^2$, the fiber is a 3-dimensional Riemannian manifold $(F, g_F)$ and the warping function $f$ is any positive function on $B$. The Lorentzian metric $\text{FRW}$ for the Friedman-Robertson-Walker spacetime is then rewritten as $g = -dt^2 + f^2(t)g_F$. Here the warping function $f$ is a function of time alone and it measures how physical separations change with time. The dynamics of the expanding universe only appears implicitly in the time dependence of the warping function $f$.

Given a warped product manifold $M = B \times f F$, we assume $f$ be a positive smooth function on $B = (t_0, t_\infty)$ such as $f \in C^\infty$ for $t \neq t_i$ and $f \in C^0$ at $t = t_i$ ($i = 1, 2, \ldots, n$). When $f \in C^1$ at points $t_i \in (t_0, t_\infty)$ and $S = \{t_i\} \times f F$, we define $f \in C^0(S)$ as a collection of functions $\{f^{(i)}\}$ with $f^{(i)}$ piecewisely defined on the intervals $t_i \leq t \leq t_{i+1}$ ($i = 0, 1, 2, \ldots, n$) with $t_{n+1} = t_\infty$. Since $f \in C^0(S)$, we have $f^{(i-1)} = f^{(i)}$, but $f^{(i+1)} = f^{(i)}$. We will use the unit step function $\mu$ of discontinuity of $f^{(i)}$ at $t = t_i$.

In order to evaluate the Ricci curvatures, we derive $f'$ in terms of the collection of functions $\{f^{(i)}\}$ with $f^{(i)}$ piecewisely defined on the intervals $t_i \leq t \leq t_{i+1}$. For a single discontinuity $n = 1$ case, $f'$ is trivially given by

\[
f' = f^{(1)'} \mu(t - t_1) + f^{(0)'} \mu(t_1 - t),
\]

with $\mu(t - t_i)$ being the unit step function. For double discontinuities $n = 2$ case, $f'$ is similarly given by

\[
f' = \left( f^{(2)'} - \frac{1}{2} f^{(1)'} + \frac{1}{2} f^{(0)'} \right) \mu(t - t_2) + \left( \frac{1}{2} f^{(1)'} - \frac{1}{2} f^{(0)'} \right) \mu(t - t_1) \\
+ \left( \frac{1}{2} f^{(1)'} + \frac{1}{2} f^{(0)'} \right) \mu(t_2 - t) + \left( -\frac{1}{2} f^{(1)'} + \frac{1}{2} f^{(0)'} \right) \mu(t_1 - t).
\]

By using iteration method, one can obtain for an arbitrary $n$ case

\[
f' = \left( f^{(n)'} - f^{(n-1)'} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)'} \right) \mu(t - t_n) + \sum_{l=1}^{n-1} \left( -f^{(l-1)'} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)'} \right) \mu(t_l - t) \\
+ \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)'} \mu(t_n - t) + \sum_{l=1}^{n-1} \left( -f^{(l)'} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)'} \right) \mu(t_l - t).
\]
To proceed to calculate \( f'' \), we use the derivative of the unit step function \( \mu(t - t_i) \). For all \( t \neq t_i \) this is well-defined, \( \mu'(t - t_i) = 0 \). However, at \( t = t_i \) there exists a jump discontinuity so that we cannot define classical derivative and thus we use the \( \delta \)-function, \( \mu'(t - t_i) = \delta(t - t_i) \) to yield

\[
f'' = \left( f^{(n)n} - f^{(n-1)n} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)n} \right) \mu(t - t_n) + \sum_{i=1}^{n-1} \left( -f^{(i-1)n} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)n} \right) \mu(t - t_i)
\]

\[
+ \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)n} \mu(t_n - t) + \sum_{i=1}^{n-1} \left( -f^{(i)n} + \frac{1}{n} \sum_{k=0}^{n-1} f^{(k)n} \right) \mu(t_i - t)
\]

\[
+ \left( f^{(n)n} - f^{(n-1)n} \right) \delta(t - t_n) + \sum_{i=1}^{n-1} \left( f^{(i)n} - f^{(i-1)n} \right) \delta(t - t_i).
\]  

(3.5)

We can then obtain the Ricci components

\[
\text{Ric}(U, U) = -\frac{3f''}{f}
\]

\[
\text{Ric}(U, X) = 0
\]

\[
\text{Ric}(X, Y) = \left( \frac{2(f'^{2} + k)}{f^{2}} + \frac{f''}{f} \right) \langle X, Y \rangle, \text{ if } X, Y \perp U,
\]  

(3.6)

and the Einstein scalar curvature

\[
R = 6 \left( \frac{f'^{2}}{f^{2}} + \frac{f''}{f} + \frac{k}{f^{2}} \right),
\]  

(3.7)

where \( f' \) and \( f'' \) are given by (3.4) and (3.5).

IV. FRW UNIVERSE AS WARPED PRODUCTS

In the spatially flat FRW cosmology with \( k = 0 \), the early universe was radiation dominated, the adolescent universe was matter dominated, and the present universe is now entering into lambda-dominant phase in the absence of vacuum energy. If the universe underwent inflation, there was a very early period when the stress-energy was dominated by vacuum energy. The Friedmann equation may be integrated to give the age of the universe in terms of present cosmological parameters. We have the scale factor \( f \) as a function of time \( t \) which scales as \( f(t) \propto t^{1/2} \) for a radiation-dominated (RD) universe, and scales as \( f(t) \propto t^{2/3} \) for a matter-dominated (MD) universe, and scales as \( f(t) \propto e^{Kt} \) for a lambda-dominated (LD) universe. Note that the transition from the radiation-dominated phase to the matter-dominated is not an abrupt one; neither is the later transition from the matter-dominated phase to the exponentially growing lambda-dominated phase. With the above astrophysical phenomenology in mind, we consider the spatially flat FRW spacetime \((M, g)\) with Lorentzian metric \( g = -dt^{2} + f^{2}(t)g_{F} \) in the form of warped products as in (3.1) with \( k = 0 \). Here \( f \) is a smooth function on \( B = (t_0, \infty) \) except at \( t \neq t_i \) \((i = 1, 2)\), that is \( f \in C^{\infty}(S) \) (where \( S = \{ t_i \} \times F \)) for \( t \neq t_i \) and \( f \in C^{0}(S) \) at \( t = t_i \) to yield

\[
f = \begin{cases} 
  f^{(0)} = c_{0}t^{1/2}, & \text{for } t < t_1 \\
  f^{(1)} = c_{1}t^{2/3}, & \text{for } t_1 \leq t \leq t_2 \\
  f^{(2)} = c_{2}e^{Kt}, & \text{for } t > t_2
\end{cases}
\]  

(4.1)

with the boundary conditions

\[
c_{0}t_{1}^{1/2} = c_{1}t_{1}^{2/3}, \quad c_{1}t_{2}^{2/3} = c_{2}e^{Kt_{2}}.
\]  

(4.2)

Experimental values for \( t_1 \) and \( t_2 \) are given by \( t_1 = 4.7 \times 10^{4} \) yr and \( t_2 = 9.8 \) Gyr [11]. Moreover \( c_1 \) and \( c_2 \) are given in terms of \( c_0, t_1 \) and \( t_2 \) as follows

\[
c_1 = c_0 t_1^{-1/6}, \quad c_2 = c_0 t_1^{-1/6} t_2^{2/3} e^{-Kt_2}.
\]

Note that in the spatially flat FRW model, \( f \in C^{0}(S) \) since if we assume \( f \in C^{1}(S) \) one could have the boundary conditions \( \frac{1}{2}c_{0}t_{1}^{-1/2} = 2c_{1}t_{1}^{-1/3} \) and \( 3c_{1}t_{2}^{-1/3} = Ke^{Kt_{2}} \), which cannot satisfy the above boundary conditions (4.2) simultaneously.
Substituting $f$ in (4.1) into (3.4) and (3.5), one can readily obtain

$$f' = \left(\frac{1}{4}c_0 t^{-1/2} - \frac{1}{3}c_1 t^{-1/3} + c_2 e^{Kt}\right)\mu(t-t_2) + \left(-\frac{1}{4}c_0 t^{-1/2} + \frac{1}{3}c_1 t^{-1/3}\right)\mu(t-t_1)$$

$$+ \left(\frac{1}{4}c_0 t^{-1/2} + \frac{1}{3}c_1 t^{-1/3}\right)\mu(t_2-t) + \left(-\frac{1}{4}c_0 t^{-1/2} - \frac{1}{3}c_1 t^{-1/3}\right)\mu(t_1-t)$$

(4.3)

$$f'' = \left(\frac{1}{8}c_0 t^{-3/2} + \frac{1}{9}c_1 t^{-4/3} + c_2 e^{Kt}\right)\mu(t-t_2) + \left(-\frac{1}{8}c_0 t^{-3/2} - \frac{1}{9}c_1 t^{-4/3}\right)\mu(t_2-t)$$

$$+ \left(-\frac{1}{8}c_0 t^{-3/2} - \frac{1}{9}c_1 t^{-4/3}\right)\mu(t_1-t)$$

(4.4)

$$+ \left(\frac{2}{3}c_1 t^{-1/3} + c_2 e^{Kt}\right)\delta(t-t_2) + \left(-\frac{1}{2}c_0 t^{-1/2} + \frac{2}{3}c_1 t^{-1/3}\right)\delta(t-t_1),$$

with $\mu(t-t_i)$ and $\delta(t-t_i)$ being the unit step function and the delta function, respectively.

For the spatially flat FRW spacetime $M = B \times_f F$ with Riemannian curvature $R$, flow vector field $U = \partial_t$ and warping function $f \in C^0(S)$, we can then obtain the Ricci curvature for vector fields $X, Y, Z \in V(F)$,

$$\text{Ric}(U, U) = -\frac{3f''}{f},$$

$$\text{Ric}(U, X) = 0,$$

$$\text{Ric}(X, Y) = \left(\frac{2f'^2}{f^2} + \frac{f''}{f}\right)\langle X, Y \rangle, \quad \text{if } X, Y \perp U,$$

and the Einstein scalar curvature

$$R = 6 \left(\frac{f'^2}{f^2} + \frac{f''}{f}\right),$$

(4.6)

where $f$, $f'$ and $f''$ are given by (4.1), (4.3) and (4.4), respectively.

V. CONCLUSIONS

We have considered the FRW cosmological model in the warped product scheme to investigate the non-smooth curvature originated from the multiple discontinuities associated with the evolution of the universe. In particular we have analyzed the non-smooth features of the spatially flat FRW universe phenomenologically by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions.

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