Cabibbo-Kobayashi-Maskawa matrix: rephasing invariants and parameterizations

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Abstract. In this work we study two topics: the first one considers the general phase invariant monomials built out of the CKM matrix elements and their conjugates. We show, that there exist 30 fundamental phase invariant monomials and 18 of them are a product of 4 CKM matrix elements and 12 are a product of 6 CKM matrix elements. In our main result we show that all rephasing invariant monomials can be expressed as a product of at most 5 factors with positive powers. Next, we propose a general method of a recursive construction of the CKM matrix for any number of generations. This allows to construct a parameterization with desired properties. As an application we generalize the Wolfenstein parameterization to the case of 4 generations and obtain restrictions on the CKM suppression of the fourth generation.

1. Introduction

It is well known that the quark fields are defined up to a phase, therefore it is possible to introduce the states $u_\alpha \to e^{i\varphi_\alpha}u_\alpha$ and $d_i \to e^{i\phi_i}d_i$ so that the elements of the CKM matrix become

$$V_{\alpha i} \to e^{i(\phi_i - \varphi_\alpha)}V_{\alpha i}.$$  

With this redefinition one can easily check that the two simplest rephasing invariant (RI) functions of the CKM matrix are

$$W_{\alpha i} \equiv |V_{\alpha i}|^2,$$  

$$Q_{\alpha i\beta j} \equiv V_{\alpha i}V_{\beta j}V_{\alpha j}^*V_{\beta i}^*, \quad \alpha \neq \beta \quad i \neq j.$$  

In order to avoid $Q_{\alpha i\beta j} = W_{\alpha i}W_{\beta j}$ the conditions $\alpha \neq \beta$ and $i \neq j$ must be satisfied. The CKM RIs have been studied in detail by many authors [1, 2, 3].

Higher order invariants can be generated, for example,

$$V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^*,$$  

$$V_{11}V_{12}^2V_{21}^2V_{22}V_{32}^2V_{33}^2(V_{12}V_{23}^3V_{31}^3)^*.$$  

In [4] is demonstrated the possibility of expressing all higher rephasing invariants in terms of the RIs in Eqs. (1) and (2), but not all the powers of these RIs are positive, for example

$$V_{11}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^* = W_{12}^{-1}V_{12}V_{22}V_{33}V_{13}^*V_{21}^*V_{32}^* = W_{12}^{-1}Q_{1122}Q_{3312}.$$  

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We will prove in section one that all higher order rephasing invariants of the CKM matrix can be expressed as product of fundamental rephasing invariants taken to positive powers and the product of the squares of the absolute values of the CKM matrix elements also taken to positive powers. This result is independent of the unitary of the CKM matrix. The discussion of the RIs depends on the number of generations and we will discuss these invariants for 3 generations.

In section two we outline how to construct and parameterize the \((n \times n)\) CKM matrix \(V^{(n-1)}\). Our algorithm allows to set up a parameterization with desired properties. As an application we generalize the Wolfenstein parameterization to the case of 4 generations and obtain restrictions on the CKM suppression of the fourth generation.

2. A new form to construct rephasing invariants
Let us denote by \(P(m,n)\) the most general monomial constructed from the CKM matrix elements and its conjugates

\[
P(m,n) = A \prod_{\alpha i} (V_{\alpha i})^{m_{\alpha i}} \prod_{\beta j} (V^*_{\beta j})^{n_{\beta j}},
\]

here \(m\) and \(n\) are \(3 \times 3\) matrices with integer non negative matrix elements \([m]_{\alpha i} = m_{\alpha i}\) and \([n]_{\beta j} = n_{\beta j}\). We can assume, without loss of generality \(A = 1\). The condition that the elements of the matrices \(m\) and \(n\) are integers may be relaxed, but the CKM observables are monomials that contain only integer powers.

The monomials \(P(m,n)\) fulfills the following properties

a. \(P(m_1,n_1) \cdot P(m_2,n_2) = P(m_1 + m_2, n_1 + n_2)\).

b. \(P(m,n)^* = P(n,m)\).

c. \(P(m,m) = \prod_{\alpha i} |(V_{\alpha i})^2|^{m_{\alpha i}}\).

The monomial \(P(m,n)\) in general is not rephasing invariant. Suppose that we make the following phase transformation of the CKM matrix

\[
V_{\text{CKM}} \rightarrow \text{diag}(e^{i\phi_1}, 0, 0)V_{\text{CKM}},
\]

then the monomial \(P(m,n)\) is transformed in the following way

\[
P(m,n) \rightarrow e^{i\phi_1(m_{11}+m_{12}+m_{13}+n_{11}+n_{12}+n_{13})} P(m,n),
\]

so we see that \(P(m,n)\) is invariant under the transformation in Eq. (7) only if the sum of the elements of the first row of the matrices \(m\) and \(n\) are equal. From this one obtains theorem 1.

**Theorem 1** The monomial \(P(m,n)\) is rephasing invariant if the sums of the elements of the corresponding rows and columns of the matrices \(m\) and \(n\) are equal. It means that for the rephasing invariant monomial \(P(m,n)\) the matrices \(m\) and \(n\) fulfill the following conditions

\[
\sum_{i=1}^{3} m_{\alpha i} = \sum_{i=1}^{3} n_{\alpha i}, \quad \sum_{i=1}^{3} m_{i\alpha} = \sum_{i=1}^{3} n_{i\alpha}, \quad \alpha = 1, 2, 3.
\]

**Definition 1** The rephasing invariant monomial of the CKM matrix which cannot be factored out into the product of the absolute values of the elements of the CKM matrix and other invariant is called the pure rephasing invariant (PRI).

**Example 1**- The rephasing invariant \(V_{11}V_{22}V_{33}V_{12}V_{21}V_{33}^* = |V_{33}|^2 V_{11}V_{22}V_{12}^*V_{21}\) is not a PRI because it has the factor \(|V_{33}|^2\). The remaining part \(V_{11}V_{22}V_{12}^*V_{21}\) is a PRI.
Table 1. Fundamental rephasing invariants.

| 4-th order FRIs \((J_1, J_2, \ldots, I_{18})\) | 6-th order FRIs \((I_1, I_2, \ldots, I_{12})\) |
|---------------------------------------------|---------------------------------------------|
| \(J_1 = V_{11}V_{22}V_{12}V_{21}\)       | \(I_1 = V_{11}V_{22}V_{33}V_{13}V_{21}V_{32}\) |
| \(J_5 = V_{11}V_{33}V_{23}V_{12}V_{31}^*\) | \(I_4 = V_{11}V_{33}V_{23}V_{12}V_{31}^*\) |
| \(J_3 = V_{11}V_{23}V_{13}V_{21}V_{31}^*\) | \(I_2 = V_{11}V_{23}V_{32}V_{12}V_{31}^*\) |
| \(J_6 = V_{12}V_{33}V_{13}V_{23}V_{32}^*\) | \(I_5 = V_{12}V_{33}V_{13}V_{23}V_{32}^*\) |
| \(J_7 = V_{21}V_{32}V_{12}V_{31}^*\)     | \(I_3 = V_{11}V_{23}V_{32}V_{12}V_{31}^*\) |
| \(J_4 = V_{11}V_{32}V_{12}V_{31}^*\)     | \(I_6 = V_{12}V_{21}V_{33}V_{12}V_{31}^*\) |
| \(J_8 = V_{21}V_{33}V_{23}V_{12}V_{31}^*\) |                                |
| \(J_9 = V_{22}V_{33}V_{23}V_{12}V_{32}^*\) |                                |
| \(J_{9+k} = (J_k)^\ast, \ k = 1, \ldots, 9\) |                                |

The PRIs can be represented by two matrices \(m\) and \(n\) but it can also be represented by

\[
B(p) = \prod_{p_{\alpha i} \geq 0} (V_{\alpha i})^{p_{\alpha i}} \prod_{p_{\beta j} \leq 0} (V_{\beta j}^*)^{-p_{\beta j}},
\]

where \(p\) is a 3 \(\times\) 3 matrix with the following properties:
1. The matrix elements of \(p\) are integers (positive, negative or 0).
2. The sum of the elements of \(p\) in each row and column is equal to 0.
3. A permutation of the rows and columns of the \(p\) matrix is reversible and the resulting matrix is also the \(p\) matrix of pure rephasing invariant.

**Example 2.** The matrix \(p\) for the PRI, defined in Eq. (4) is

\[
p = \begin{pmatrix}
1 & -3 & 2 \\
2 & 1 & -3 \\
-3 & 2 & 1
\end{pmatrix}.
\]

The matrix \(p\) in the example 2 can be decomposed into the sum of matrices, all of them with the same properties that \(p\), for example

\[
p = \begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 0 \\
-1 & 1 & 0
\end{pmatrix} + \begin{pmatrix}
0 & -1 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & -1 \\
-1 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix}.
\]

If one carefully analyses the decomposition in the previous example Eq. (11), we can see that exist a subset within the PRIs from which all rephasing invariants can be constructed. It leads us to the following definition.

**Definition 2** The fundamental rephasing invariant (FRI) is such a pure rephasing invariant monomial that is the product of 4 or 6 CKM matrix elements and its complex conjugates.

There are 30 fundamental rephasing invariants Table 1, 18 of them are the products of 4 CKM matrix elements and the others 12 are the products of 6 CKM matrix elements. For each FRI in there corresponds a \(p\) matrix, e.g.,

\[
J_1 \rightarrow p_{J_1} = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}\quad I_1 \rightarrow p_{I_1} = \begin{pmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}, \quad \text{etc.}
\]
All the matrices \( p_{I_k} \) and \( p_{I_1} \) corresponding to the invariants in Table 1 can be obtained by the permutations of the rows and columns of the matrices \( p_{I_k} \) and \( p_{I_1} \) that are given in Eq. (12).

**Theorem 2** Any pure rephasing invariant can be expressed in a unique way as the product of positive powers of at most 4 fundamental rephasing invariants. Not more than one of these invariants can be from the 6-th order FRIs and the remaining are from the 4-th order FRIs.

To prove the theorem we analyze the matrices \( p_{I_k} \) and \( p_{I_1} \) and we show that the matrix corresponding to a PRI can be decomposed in a unique way as a linear combination with positive coefficients of at most 4 \( p \) matrices corresponding to the fundamental rephasing invariants defined in table 1. The inverse theorem is not true, the product of two or more FRIs is rephasing invariant, but it does not have to be the PRI.

From the unitarity of the CKM matrix it follows that the 6-th order FRIs in table 1 can be expressed by the 4-th order FRIs and the squares of the CKM matrix elements, e.g.,

\[
I_1 = V_{11} V_{22} V_{33} V_{13}^{*} V_{21}^{*} V_{32}^{*} = |V_{22}|^2 V_{12} V_{33} V_{13}^{*} V_{21}^{*} V_{32}^{*} - |V_{13}|^2 V_{22} V_{33} V_{24}^{*} V_{32}^{*} = |V_{22}|^2 J_6 - |V_{13}|^2 J_9, \quad (13)
\]

and there are analogous formulas for the remaining \( I_j \)'s. It should be emphasized that without the unitarity of the CKM matrix there are no simple relations between the invariants of the 4-th and 6-th order. Thus relation in Eq. (13) is also a test of the unitarity of the CKM matrix.

### 3. Recursive construction of the CKM matrix

The presented algorithm does not impose any conditions on the parameterization of the matrix \( V^{(n-1)} \) and assume that the explicit form of the matrix \( V^{(n-1)} \) is known. We write the CKM matrix \( V^{(n)} \) in terms of column vectors

\[
V^{(n)} = \left( v_1^{(n)}, v_2^{(n)}, \ldots, v_n^{(n)} \right) \quad \text{with} \quad v_k^{(n)} = \left( \begin{array}{c} V_{1k}^{(n)} \\ V_{2k}^{(n)} \\ \vdots \\ V_{nk}^{(n)} \end{array} \right), \quad k = 1, 2, \ldots, n., \quad (14)
\]

The CKM matrix \( V^{(n)} \) is built from \( V^{(n-1)} \) in two steps:

a. We construct \( n \) real column vectors \( e_1, \ldots, e_n \), normalized to 1 and orthogonal \( e_i \cdot e_j = \delta_{ij} \), with \( n \) rows that depend on \( (n - 1) \) independent parameters.

b. The columns of the CKM matrix \( V^{(n)} \) are obtained from the vectors \( e_k \) and the elements of the matrix \( V_{ij}^{(n-1)} \) in the following way

\[
v_k^{(n)} = V_{1k}^{(n-1)} e_1 + \sum_{\ell=2}^{n-1} V_{\ell k}^{(n-1)} e^{-i \delta_{\ell-1,n}} e_\ell, \quad k = 1, \ldots, n - 1, \quad n \geq 2, \quad (15)
\]

The matrix \( V^{(n)} \) constructed in such a way is unitary and it depends on parameters of the matrix \( V^{(n-1)}, (n - 1) \) parameters of the vectors \( e_1, \ldots, e_n \) and on \( (n - 2) \) phase factors \( e^{-i \delta_{\ell,n}} \) from Eq. (15).

### 4. Wolfenstien parameterization for 4 quark generations

We shall generalized the Wolfenstein parameterization [5] to the case of 4 quark generations using the method outlined earlier.

\[
V^{(3)} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^2(\rho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^2(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix}. \quad (16)
\]
The Wolfenstein parameterization given in Eq. (16) is approximate. It can be given an exact meaning by assuming a one to one correspondence between the Wolfenstein parameters $A$, $\lambda$, $\rho$ and $\eta$ and the parameters of the standard parameterization $s_{12} \equiv \lambda$, $s_{23} \equiv A \lambda^2$ and $s_{13} e^{i \delta_{13}} \equiv A \lambda^3 (\rho + i \eta)$ [6, 7].

First we have to construct the vectors $e_i$, which are expressed in the spirit of the Wolfenstein parameterization in terms of the powers of $\lambda$. The vectors $e_i$ are real and are chosen in the following way

$$
e_1 = N_1 \begin{pmatrix} 1 + z_1^2 & 0 \\ -z_1 z_2 & -z_1 \end{pmatrix}, \quad e_2 = N_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_3 = N_3 \begin{pmatrix} 0 \\ -z_3 \end{pmatrix}, \quad e_4 = N_4 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \tag{17}
$$

Here $z_i = A_i^{(4)} \lambda^{k_i}$, $A_i^{(4)} \sim 1$, $k_i \geq 1$ are integers and $N_i$ are suitable normalization factors. The powers $k_i$ are considered to be constants (suppression factors) and $A_i^{(4)}$, $i = 1, 2, 3$ are free parameters. It is easy to verify that the vectors $e_i$ are normalized to 1 and orthogonal.

The vectors $e_i$ were constructed to fulfill the following two additional properties

$$(e_i)_j |_{A_i=0} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \quad V^{(4)}|_{\delta_{1,4}=0, \delta_{2,4}=0} = \begin{pmatrix} V^{(3)} & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3. \tag{18}
$$

The columns of the CKM matrix for 4 generations are then equal

$$
v^{(4)}_1 = V^{(3)}_{11} e_1 + e^{-i \delta_{1,4}} V^{(3)}_{21} e_2 + e^{-i \delta_{2,4}} V^{(3)}_{31} e_3, \quad v^{(4)}_2 = V^{(3)}_{12} e_1 + e^{-i \delta_{1,4}} V^{(3)}_{22} e_2 + e^{-i \delta_{2,4}} V^{(3)}_{32} e_3, \quad v^{(4)}_3 = V^{(3)}_{13} e_1 + e^{-i \delta_{1,4}} V^{(3)}_{23} e_2 + e^{-i \delta_{2,4}} V^{(3)}_{33} e_3, \quad v^{(4)}_4 = e_4, \tag{19}
$$

and the CKM matrix is

$$V^{(4)} = \begin{pmatrix} v^{(4)}_1 & v^{(4)}_2 & v^{(4)}_3 & v^{(4)}_4 \end{pmatrix}. \tag{20}
$$

The matrix $V^{(4)}$ in Eq. (20) is described by 9 parameters: $\lambda$, $A$, $\rho$, $\eta$ of the matrix Eq. (16), introduced by Wolfenstein, $A_1^{(4)}$, $A_2^{(4)}$, $A_3^{(4)}$ of the vectors in Eqs. (17) and $\delta_{1,4}$, $\delta_{2,4}$ from Eq. (15). Not all these parameters can be determined from the experimental data. On the other hand we can derive some restrictions on the powers $k_i$ of the suppression factors for the 4-th generation from the experimental information for the CKM matrix for 3 generations. We have the following information

$$|V_{12}| \sim \lambda, \quad |V_{21}| \sim \lambda, \quad |V_{23}| \sim \lambda^2, \quad |V_{32}| \sim \lambda^2, \quad |V_{13}| \sim \lambda^3, \quad |V_{31}| \sim \lambda^3, \quad |(V_{12} - |V_{21}|) | \sim \lambda^3, \quad |(V_{23} - |V_{32}|) | \sim \lambda^4. \tag{21}
$$

Now, using the information in Eq. (21) with the explicit representation of the $4 \times 4$ CKM matrix in Eqs. (19) and (20) we obtain the following restrictions on the powers $k_i$

$$k_1 \geq 1, \quad k_1 + k_2 \geq 3, \quad k_2 + k_3 \geq 4, \quad k_1 + k_3 \geq 3, \tag{22}
$$

which can be resolved and give $k_1 \geq 1, \quad k_2 \geq 2, \quad k_3 \geq 2$ and the vector $v_4$ for the minimal values of $k_i$ has the following form

$$v_4 = N_4 \begin{pmatrix} A_1^{(4)} \lambda \\ A_2^{(4)} \lambda^2 \\ A_3^{(4)} \lambda^2 \\ 1 \end{pmatrix}. \tag{23}$$
In the case of 3 generations the suppression has totally different structure. However, we would like to note that the real suppression may be different, because we have only obtained the lower limits of the suppression powers.

5. Conclusions
From theorem 2 follows that any rephasing invariant monomial of the CKM matrix for 3 generations is the product of no more than 5 factors: 4 fundamental rephasing invariants taken to positive powers and the product of the squares of the absolute values of the CKM matrix elements also taken to positive powers. Another important conclusion that can be obtained from theorem 2 and the unitarity of the CKM matrix is that the imaginary part of any rephasing invariant monomial is proportional to the Jarlskog invariant or equal to 0. Let us note that general rephasing invariant monomials constructed from the CKM matrix elements appear at higher orders of the renormalization group equations for the CKM matrix elements. The systematic analysis of such equations based on the results obtained in this paper will be published elsewhere.

From the theoretical point of view all exact parameterizations of the CKM matrix are equivalent. From the practical point of view the situation is less obvious, because certain experimental facts can be presented in a more transparent way in one parameterization than in the other. The scheme of the construction of the parameterization of the CKM matrix presented in this work allows to adjust properties of the parameterization according to the needs. Such an approach has not been discussed before and it can facilitate the discussion of the properties of the CKM in the Standard Model or its extensions.

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