On Hausdorff dimension of some Cantor attractors

G. Levin
Dept. of Math., Hebrew Univ.
Jerusalem 91904, Israel
levin@math.huji.ac.il

F. Przytycki
Inst. of Math. of PAN
Warsaw 00-950, Poland
F.Przytycki@impan.gov.pl

Abstract
We study what happens with the dimension of Feigenbaum-like attractors of smooth unimodal maps as the order of the critical point grows

1 Introduction

Let $f$ be a smooth unimodal map of an interval. We assume that $f$ is infinitely-renormalizable with stationary combinatorics. Then $f$ has an attractor $C(f)$ both in metric and topological senses, which is a Cantor set and which is the $\omega$-limit set of the critical point of $f$. In this note we consider the following question motivated by [1], [15], and [8]: what happens with the Hausdorff dimension of $C(f)$ as the order $\ell$ of the critical point grows to infinity? We show that it must grow to at least $2/3$. In the orientation reversing case (which includes the classical Feigenbaum’s one) we also prove that the Hausdorff dimension has a limit as $\ell$ tends to infinity, this limit is less than 1, and it is equal to the Hausdorff dimension of an attractor of some limit unimodal dynamics defined in [8].

Denote by $HD(E)$ the Hausdorff dimension of a set $E$ in $\mathbb{R}^n$.

It is well-known [9] (and follows from convergence of renormalizations), that the Hausdorff dimension $HD(C(f))$ of the attractor $C(f)$ of $f$ depends actually only on the stationary combinatorics $\mathcal{N}$ of the map $f$ and the criticality order $\ell$ of its critical point provided that $\ell$ is an even integer. It allows us to write $D(\mathcal{N}, \ell) = HD(C(f))$ for all smooth $f$ with fixed $\mathcal{N}$ and $\ell$.

(Note here that once the convergence of renormalizations is established for all real big enough criticalities $\ell$ all results and proofs of the paper hold true for such $\ell$.)

We have a priori:

$$0 < HD(\mathcal{N}, \ell) < 1. \quad (1)$$
Comment 1 (1) If $\ell = 2$, then the upper bound in (1) can be strengthened [3]: there is a number $\sigma < 1$, such that $HD(\mathbb{N}, 2) \leq \sigma$ for all combinatorics $\mathbb{N}$.

(2) Feigenbaum’s case $|\mathbb{N}| = 2$ with the quadratic critical point ($\ell = 2$) has been studied intensively, see [10], [14], particularly in the framework of Feigenbaum’s universality [3], [4]. Numerically, $D(\mathbb{N}, 2) = 0.538...$, see [16].

(3) Although $HD(\mathbb{N}, \ell)$ is always positive, it is not difficult to construct a sequence of stationary combinatorics $\mathbb{N}_n$, such that, for every $\ell$, $HD(\mathbb{N}_n, \ell) \to 0$ as $n \to \infty$. For instance, $\mathbb{N}_n$ can be defined by the following first $n - 1$ itineraries of the critical value: $n - 2$ times ”plus” and one time ”minus”. Then bounds (real or complex) imply that if $f_n(z) = z^\ell + c_n$ is infinitely-renormalizable with the stationary combinatorics $\mathbb{N}_n$, then $HD(C(f_n)) \to 0$ as $n \to \infty$.

Note that the number $D(\mathbb{N}, 2)$ ($|\mathbb{N}| = 2$) as well as the numbers $HD(\mathbb{N}_n, \ell)$ (with fixed $\ell$ and big $n$) are less than $2/3$.

Theorem 1 For every $\mathbb{N}$,

$$\liminf D(\mathbb{N}, \ell) > \frac{2}{3}$$

as $\ell$ tends to infinity along the even integers.

To state our result about the upper bound, we need to introduce some notions.

Non-symmetry. For a unimodal map $f$ with a single critical point at $c$, denote by $I_f$ the involution map defined in a neighborhood of $c$ by $I_f : x \mapsto \hat{x}$, where $I_f(c) = c$, and otherwise $I_f(x)$ is the unique $\hat{x} \neq x$, such that $f(x) = f(\hat{x})$. If $f$ is of the form $|E(x)|^\ell$, where $\ell > 1$ and $E$ is a $C^2$-diffeomorphism, then $I_f$ is also $C^2$, and $I_f'(c) = -1$. The non-symmetry $N(f)$ of $f$ is said to be the number $N(f) = |I_f''(c)/2|$. It is easy to check that $N(f) = |E''(c)/E'(c)|$.

Orientation reversing combinatorics of an infinitely-renormalizable unimodal map $f$ is such stationary combinatorics $\mathbb{N}$, that the rescaling factor of the renormalization is negative. In other words, the maps $f$ and $f^{[\mathbb{N}]}$ have at the critical point of $f$ different type of extrema (maximum and minimum). Examples: $|\mathbb{N}| = 2, 3$; more generally, $\mathbb{N}_n$ ($n \geq 1$) defined in Comment 1(3).

For a combinatorial type $\mathbb{N}$ and an even integer $\ell$, denote by $H_{\mathbb{N}, \ell}$ the unique universal unimodal map normalized so that $H_{\mathbb{N}, \ell} : [0, 1] \to [0, 1]$ and $H_{\mathbb{N}, \ell}(0) = 1$ (see next Section for complete definition). It is shown in [9], that the sequence $\{H_{\mathbb{N}, \ell}\}_{\ell}$ converges uniformly to a unimodal map $H_{\mathbb{N}} : [0, 1] \to [0, 1]$.

We prove in Lemma 4.3 that if the combinatorial type $\mathbb{N}$ reverses orientation, then the sequence of non-symmetries $N(H_{\mathbb{N}, \ell})$, $\ell = 2, 4, ...$, is uniformly bounded.

Theorem 2 For a given combinatorial type $\mathbb{N}$, assume that the sequence of non-symmetries $N(H_{\mathbb{N}, \ell})$, $\ell = 2, 4, ...$, is uniformly bounded. Then the Hausdorff dimension of the attractor is continuous at $\ell = \infty$: there exists

$$\lim_{\ell \to \infty} D(\mathbb{N}, \ell) = HD(C(H_{\mathbb{N}})) < 1.$$
Consequently, (3) holds when \( \aleph \) reverses orientation.

**Comment 2** It is not clear if the non-symmetry \( N(H_{\aleph,\ell}) \) is uniformly bounded in \( \ell \) for any type \( \aleph \).

The proof of Theorems \([1, 2]\) is based on recent results of \([8]\): see next Sect. where we reduce the statements to Theorem \([4]\).

(Note however that in the proof of the lower 2/3-bound we use only a part of the main result of \([8]\), namely, the compactness (Theorem 4 in \([8]\)).)

In turn, to prove Theorem \([4]\) we use some results of \([10, 13]\), see Sect. \([3]\).

From now on, we fix the type \( \aleph \). Denote \( p = |\aleph| \).

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# 2 Reduction to fixed-point maps

## 2.1 Universal maps

For every real number \( \ell > 1 \), we consider a unimodal map \( g_\ell : [-1, 1] \to [-1, 1] \) with the critical point at 0 of order \( \ell \). More precisely, \( g_\ell \) is assumed to be in the following form: \( g_\ell(x) = E_\ell(|x|^{\ell}) \), where \( E_\ell : [0, 1] \to \mathbb{R} \) is a \( C^2 \)-diffeomorphism onto its image. The map \( g = g_\ell \) is normalized so that \( g_\ell(0) = 1 \). It is further assumed to be infinitely renormalizable with the fixed combinatorial order type \( \aleph \) and to satisfy the fixed point equation:

\[
\alpha g^{[\aleph]}(x) = g(\alpha x) .
\] (4)

with \( |\alpha| > 1 \). By renormalization theory, see \([14]\), a fixed point \( g_\ell \) for any \( \ell > 1 \) can be represented as \( E_\ell(|x^{\ell}|) \) with \( E_\ell \) which is a diffeomorphism in Epstein class (i.e. a diffeomorphism \( E \) of a real interval \( T' \) onto another real interval \( T \) such that the inverse map \( E^{-1} : T \to T' \) extends to a univalent map \( E^{-1} : (\mathbb{C} \setminus \mathbb{R}) \cup T \to (\mathbb{C} \setminus \mathbb{R}) \cup T' \)).

It will be useful to deal with another unimodal map \( H_\ell \), which is related to \( g_\ell \) as follows: \( H_\ell(x) = |g_\ell(x^{1/\ell})|^{\ell} = |E_\ell(x)|^{\ell}, \) \( 0 \leq x \leq 1 \). Then \( H_\ell \) is a unimodal map of \([0, 1]\) into itself, with a strict minimum attained at some \( x_\ell \in (0, 1) \). It also satisfies the equation:

\[
\tau H^{[\aleph]}(x) = H(\tau x) .
\] (5)

with \( \tau = |\alpha|^{\ell} \).

We denote by \( C(g_\ell) \) and \( C(H_\ell) \) the attracting Cantor sets of the maps \( g_\ell : [-1, 1] \to [-1, 1] \) and \( H_\ell : [0, 1] \to [0, 1] \) respectively. Clearly, \( HD(C(g_\ell)) = \)
\(\text{HD}(C(H_\ell))\). Indeed, \(E\) conjugates \(H = H_\ell\) to \(g\) restricted to \([g(1), 1]\), therefore it maps \(C(H)\) to \(C(g)\) and is a diffeomorphism between neighbourhoods of these sets.

Assume now that the order \(\ell\) is an even integer. Then the equation (14) with the normalization as above does have a unique solution, for every fixed \(\ell\) and \(\aleph\), see [14], [11]. Consequently, \(H_\ell = |g_\ell(x^{1/\ell})|^\ell\) is the unique solution of (15) with the normalization as above.

In what follows, \(\ell\) is an even integer, and \(H_\ell\) denotes this unique solution of (5), with its own scaling constant \(\tau_\ell > 1\). (Remind that the type \(\aleph\) is fixed.)

### 2.2 Limit dynamics

The following result is proved in [8] (even for real \(\ell\)), see Theorems 1-2 and Proposition 3 there:

**Theorem 3** The sequence of maps \(H_\ell\) converges as \(\ell \to \infty\), uniformly on \([0, 1]\), to a unimodal function \(H = H_\infty\), which satisfies the following properties:

1. \(\lim_{\ell \to \infty} \tau_\ell = \tau > 1\) exists. and \(H, \tau\) satisfy the fixed point equation \(\tau H^p(x) = H(\tau x)\) for every \(0 \leq x \leq \tau^{-1}\). Here (as always) \(p = \aleph\).

2. \(H\) has analytic continuation to the union of two topological disks \(U_-\) and \(U_+\) and this analytic continuation will also be denoted by \(H\).

3. For some \(R > 1\), \(H\) restricted to either \(U_+\) or \(U_-\) is a covering (unbranched) of the punctured disk \(V := D(0, R) \setminus \{0\}\) and \(U_+ \cup U_- \subset D(0, R)\).

4. \(U_\pm\) are both symmetric with respect to the real axis and their closures intersect exactly at \(x_0\); \([0, x_0) \subset U_-\), \((x_0, 1] \subset U_+\).

5. Each \(H_\ell\) extends to complex-analytic map defined in \(U_- \cup U_+\); this sequence of analytic extensions converges to \(H\), as \(\ell_m \to \infty\), uniformly on every compact subset of \(U_- \cup U_+\).

6. For any two open intervals \(I, J\) of the real axis, if \(0 \notin J\) and \(H : I \to J\) is one-to-one, then the branch \(H^{-1} : J \to I\) extends to a univalent map to the slit complex plane \((\mathbb{C} \setminus \mathbb{R}) \cup J\) (this follows from the same property for \(H_\ell\) with \(\ell\) finite).

7. The mapping \(G_\infty(x) := H^p - (\tau^{-1} x)\) fixes \(x_0\) and \(G_\infty^2\) has the following power series expansion at \(x_0\):

\[
G_\infty^2(x) = x - a(x - x_0)^3 + O(|x - x_0|^4)
\]

with \(a > 0\).

8. For each \(\ell\), the mapping \(G_\ell := H_\ell^p(\tau_\ell^{-1} x)\) fixes the critical point \(x_\ell\) of \(H_\ell\), \(G_\ell'(x_\ell) = \pm 1/\tau_\ell^{-1/\ell}\), and \(G_\ell\) converge to \(G_\infty\) uniformly in a (complex) neighborhood of \(x_0\).
9. The unimodal map map $H : [0,1] \to [0,1]$ has a unique attractor $C(H)$, which (as for finite $\ell$) is the closure of iterates of the critical point.

2.3 The reduction

Since we know already that $HD(C(f))$ depends merely on $\aleph$ and $\ell$, Theorems 1-2 are covered by the following statement

**Theorem 4** The following holds.

(a) \[ \liminf_{\ell \to \infty} HD(C(H_\ell)) \geq HD(C(H_\infty)). \] (6)

(b) \[ \frac{2}{3} < HD(C(H_\infty)) < 1; \] (7)

(c) if the non-symmetries $N(H_\ell)$ are uniformly bounded as $\ell \to \infty$, then the Hausdorff dimension is continuous at infinity:

\[ \lim_{\ell \to \infty} HD(C(H_\ell)) = HD(C(H_\infty)). \] (8)

The rest of the paper is devoted to the proof of this statement.

3 Background in dynamics

We prove Theorem 4 by reducing it finally to known statements about infinite conformal iterated function systems (c.i.f.s.) [10] and asymptotics near parabolic maps [13], which are given here.

3.1 C.I.F.S.

We follow [10] restricting ourself to dimension one. Let $X$ be a closed real interval, and $\sigma$ be a positive continuous function on $X$, which defines a new metric $d\rho = \sigma dx$ on $X$. Let $I$ be a countable index set, $|I| > 1$, and let $S = \{ \phi_i : X \to X, i \in I \}$ be a collection of injective uniform contractions w.r.t. the metric $\rho$: there is $\lambda < 1$, such that $\rho(\phi_i(x), \phi_i(y)) \leq \lambda \rho(x, y)$ for all $i$ and all $x, y$. For every finite word $w = w_1...w_n$, denote $\phi_w = \phi_{w_1} \circ ... \circ \phi_{w_n}$. (Note that the metric $\rho$ can be replaced by the Euclidean one by replacing $\phi_i$ by $\phi_w$, where $w$ runs over all finite words of some fixed length $n$, s.t. $\lambda^n ||\sigma|| < 1$.) For any infinite word of symbols $w = w_1w_2...w_j...$, $w_j \in I$, denote $w|n = w_1w_2...w_n$. The limit set $L$ of $S$ is $L = \cup_{w \in I^\infty} \cap_{n=1}^\infty \phi_w|n(X)$. The system $S$ is said to be conformal if:

(a) $\phi_i(Int(X)) \subset Int(X)$ and $\phi_i(Int(X)) \cap \phi_j(Int(X)) = \emptyset$ for all indexes $i \neq j$. 

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(b) There is an open set $Y \supset X$, such that all maps $\phi_i$ extend to $C^{1+\epsilon}$ diffeomorphisms of $V$ into $V$.

(c) There is $K \geq 1$, such that $|D\phi_w(y)| \leq K|D\phi_w(x)|$ for every finite word $w$ and all $x, y \in Y$, where $D\phi_w(x)$ means the derivative w.r.t. the metric $\rho$.

The main object of our interest is the Hausdorff dimension of the limit set $L$. Note that it is the same w.r.t. the metric $\rho$ as w.r.t. the standard Euclidean metric.

For every integer $n \geq 1$ and every $t \geq 0$ define $p_n(t) = \sum_w \|D\phi_w\|^t$ where $w$ runs over all words of length $n$, and $\|\cdot\|$ means the sup-norm. Consequently, $P(t) = \lim_{n \to \infty} \frac{1}{n} \log p_n(t)$ is called the pressure of $S$ at $t$. The parameter $\theta = \theta_S$ of the system is defined as $\inf \{t : p_1(t) < \infty\}$.

**Theorem 5**

1. (see [10], Prop. 3.3) $P(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$.

2. (see [10], Thm. 3.15) $HD(L) = \sup \{HD(L_F) : F \subset I \ \text{is finite}\} = \inf \{t : P(t) \leq 0\}$; if $P(t) = 0$ then $t = HD(L)$.

3. If the series $p_1(\theta)$ diverges, then $P(HD(L)) = 0$ and $\theta < HD(L)$.

(Note that 3 follows directly from 1-2.)

The system with $P(t) = 0$ is called regular. The system is regular if and only if there is a $t$-conformal measure, i.e. a probability measure $m$ such that $m(L) = 1$ and for every Borel set $A \subset X$ and every $i \in I$ $m(\phi_i(A)) = \int_A |D\phi_i|^t dm$ and $m(\phi_i(X) \cap \phi_j(X)) = 0$ for all $i \neq j$ from $I$.

### 3.2 Dominant convergence and forward Poincaré series

Here we follow [13] adapting the statements slightly for our applications.

Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic maps which converges uniformly in a topological disk $U$ of the plane to a holomorphic map $f : U \to \mathbb{C}$. Assume that $c_n \to c \in U$, and the following expansions hold: $f_n(z) = c_n + \lambda_n(z - c_n) + b_n(z - c_n)^2 - a_n(z - c_n)^3 + \ldots$, where $0 < \lambda_n < 1$, $b_n, a_n \in \mathbb{R}$, and $f(z) = z - a(z - c)^3 + \ldots$, where $a > 0$, i.e. $f$ is parabolic with two ("real") attracting petals at $c$. (In particular, $b_n \to 0$ and $a_n \to a$.) Then $f_n$ is said to converge to $f$ dominantly, if there is $M > 0$ such that $|b_n| \leq M|\lambda_n - 1|$ for all $n$.

For every $g = f_n$ and $t > 0$ define the (forward) Poincare series $P_t(g, x) = \sum_{i \geq 0} \|g^i(x)\|^t$, and, for any open set $V \subset U$, define $P_t(g, V, x) = \sum_{g^i(x) \in V} \|g^i(x)\|^t$. We say the Poincare series for $(f_n, t_n)$ converge uniformly, if, for any compact set $K$ ($c \notin K$) in an attracting petal of $f$, and any $\epsilon > 0$ there exists a neighborhood $V$ of $c$, such that $P_{t_n}(f_n, V, x) < \epsilon$ for all $n$ large enough and all $x \in K$. We will need
Theorem 6  Let \( f_n, f \) be as above, and \( t_n \to t > 2/3 \). If \( f_n \to f \) dominantly, then
the Poincare series for \((f_n, t_n)\) converge uniformly.

This is a particular case of Theorem 10.2 proven in [13]. For completeness, we give a short proof of Theorem 6, see Appendix.

4  Proof of Theorem 4

4.1  Presentation system for the Cantor attractor

We repeat (with modifications) a construction from [8] (cf. [7], [2]), which is crucial for our proof. Let \( H \) be either one of \( H_\ell \) or the limit map \( H_\infty \). Consequently, let \( G \) be either the corresponding \( G_\ell \) or \( G_\infty \). We construct the presentation system for the attractor \( C(H) \), which is an infinite iterated function system \( \Pi \) on an interval \( I \) so that \( C(H) \cap I \) is (up to a countable set) the limit set of \( \Pi \). Moreover, this picture converges, as \( \ell \to \infty \), to the corresponding picture of the limit map.

Denote \( c_j = H^{j-1}(0) \), \( j \geq 0 \), the \( j \)-iterate of the critical point \( c_0 \) of \( H \) (i.e., \( c_0 = x_\ell \) for \( H = H_\ell \) and \( c_0 = x_0 \) for \( H = H_\infty \)). Let \( I = [c_p, c_{2p}] \). Then we define a sequence of maps \( \psi_{k,m} : I \to I \), \( k = 1, 2, ..., m = 1, 2, ..., p - 1 \), as follows. Let \( H^{-(p-m)} : [c_p, c_{2p}] \to [c_m, c_{p+m}] \) denote corresponding one-to-one branch of \( H^{-(p-m)} \). Then set
\[
\psi_{k,m} = G^k \circ H^{-(p-m)}. \tag{9}
\]

Lemma 4.1  (a)
\[
I_{k,m} := \psi_{k,m}(I) = [c_{pk+m}, c_{p(k+(p+m))}] \subset I.
\]

The intervals \( I_{k,m} \) are pairwise disjoint.

(b) Let \( L \) be the limit set of the system \( \{\psi_{k,m}\} \) (in other words, \( L \) is the set of non-escaping points of the inverse maps \( \psi_{k,m}^{-1} : I_{k,m} \to I \)). Then the closure \( \overline{L} = L \cup P \), where \( P \) is a subset of pre-images of the critical point \( c_0 \), and
\[
\overline{L} = C(H) \cap I.
\]

Proof. From the functional equation for \( H \), \( G(c_j) = c_{pj} \), \( j \in \mathbb{Z} \), where \( c_j \), for \( j < 0 \) is an \( H^j \)-preimage of \( c_0 \). The rest follows.

\[\square\]

Denote by \( \Pi_\ell = (\psi^{(\ell)}_{k,m})_{k,m} \), resp. \( \Pi_\infty = (\psi^{(\infty)}_{k,m})_{k,m} \), the presentation system of \( H_\ell \), resp. \( H_\infty \).

The notation \( B(E) \) stands for the round disk which is based on an interval \( E \subset \mathbb{R} \) as a diameter.
Lemma 4.2 Let $\Pi = \{\psi_{k,m} : I \to I_{k,m}\}_{k,m}$ be either $\Pi_\ell$ or $\Pi_\infty$.

(1) There exists a fixed open interval $J$, which contains $I$ for all $\ell$ large enough (including $\ell = \infty$), such that each $\psi_{k,m}$ extends to a univalent map $\psi_{k,m} : B(J) \to B(J_{k,m})$, where $J_{k,m} = \psi_{k,m}(J)$ are pairwise disjoint intervals properly contained in $J$.

Therefore, there is $\lambda < 1$ (dependent only on the type $\mathfrak{K}$), such that $\|D\psi_{k,m}\|_\rho < \lambda$, for all $k, m$, and $\ell \leq \infty$ large enough, where $\|D\psi_{k,m}\|_\rho$ denotes the supremum on the interval $I$ of the derivative of $\psi_{k,m}$ in the hyperbolic metric $\rho$ of $B(J)$.

(2) $\Pi$ (with the metric $\rho$ restricted to the closed subinterval $I$ of $J$) is an infinite conformal iterated function system, such that:

(a) $\theta_{\Pi_\ell} = 0$ for $\ell < \infty$;
(b) $\theta_{\Pi_\infty} = 2/3$, $P(\theta_{\Pi_\infty}) = \infty$;
(c) $\Pi_{\ell}, \ell \leq \infty$, is regular.

Proof. (1) follows from Theorem 3 (7), and from another representation of the maps of the system: $\psi_{k,m} = H^{-1} \circ \tau^{-k} \circ H^{-((p-m)-1)}$ which is a consequence of the eq. $H \circ G = \tau^{-1} \circ H$. (2a) is immediate because $c_0$ is the attracting fixed point of $G$ for finite $\ell$.

(2b)-(2c): since $G = G_\infty$ has a neutral fixed point with two attracting petals, and $\psi_{k,m}'(x) = (G^k)'(H^{-((p-m)-1)}(x))(H^{-((p-m)-1)}(x))$, we obtain the following asymptotics, as $k \to \infty$, for the presentation system: $|\psi_{k,m}'(x)|_{k^{-3/2}} \to a_m(x)$ where, for fixed $m = 1, ..., p - 1$, the function $a_m(x)$ is continuous and positive on $I$. It follows from here that the critical exponent $\theta$ of the system is $\theta = 2/3$. Thus, $p_1(\theta) = \infty$ for all $\ell \leq \infty$. Hence, by Theorem 3, the system $\{\psi_{k,m}\}$ is regular.

4.2 Hausdorff dimension for the limit map

As a corollary, we obtain Theorem 4 (a)-(b):

Corollary 4.1 (1) $2/3 < HD(C(H_\infty)) < 1$,

(2) $\liminf_{\ell \to \infty} HD(C(H_\ell)) \geq HD(C(H_\infty)) > \frac{2}{3}$.

Proof. Denote $H = H_\infty$. Since $H$ is regular and $P(2/3) = \infty$, then $HD(C(H)) > 2/3$. On the other hand, the Lebesgue measure of $I \setminus \cup_{k,m} I_{k,m}$ is positive. Therefore (10), $HD(C(H)) = HD(C(H) \cap I) = HD(D) < 1$.

(2) follows from Theorem 6 for every $\delta > 0$, there is a finite subsystem $F_{\infty}$ of $\Pi_\infty$ with the Hausdorff dimension of its limit set at least $HD(C(H_\infty)) - \delta$. Since corresponding finite subsystem $F_\ell$ converges to $F_{\infty}$ as $\ell \to \infty$, then the Hausdorff dimension of the limit set of $F_\ell$ is at least $HD(C(H_\infty)) - 2\delta$, for all $\ell$ large enough. The result follows.
4.3 Non-symmetry and dominant convergence

It remains to prove Theorem 4(c).

Denote $\epsilon = 1$ or $2$ depending on whether $G'_\infty(x_0) = 1$ or $-1$.

Lemma 4.3

1. The sequence $G'_\ell$ converges to $G'^\infty$ dominantly if and only if the sequence of non-symmetries $N(H_\ell)$ is bounded.

2. If the combinatorics reverses orientation, then $G'^\ell_\ell$ converges dominantly to $G'^2_\ell$, and the non-symmetries $N(H_\ell)$ are uniformly bounded.

Proof. Let $H = H_\ell$ and $G = G_\ell$, $\tau = \tau_\ell$, and $I = I_H$. We have: $H(G(I(x))) = \tau^{-1}H(I(x)) = \tau^{-1}H(x) = H(G(x))$, i.e. $I \circ G = G \circ I$. The latter equation gives us: $|(G'\ell''(x_\ell)| = N(H)\lambda(1 - \lambda)$, where $\lambda = \lambda_\ell = (G'\ell''(x_\ell) \in (0, 1)$. This implies 1.

To prove 2, notice that the combinatorics reverses orientation if and only if $G'_\infty(x_0) = -1$. Then we get the dominant convergence, because $|(G''\ell''(x_\ell)| = |G''(x_\ell)||\lambda(1 - |\lambda|| and $G''(x_\ell) = G''_\ell(x_\ell)$ converges to the number $G''_\infty(x_0)$, as $\ell \to \infty$. (One can also refer formally to [13], Proposition 7.3.)

4.4 Conformal measures of the presentation systems

Remind that $\Pi_\ell = (\psi_{k,m}^{(\ell)} : I^\ell \to I^\ell_{k,m})_{k,m}$, resp. $\Pi_\infty = (\psi_{k,m}^{(\infty)} : I^\infty \to I^\infty_{k,m})_{k,m}$, the presentation system of $H_\ell$, resp. $H_\infty$. We know that $\Pi_\ell, \Pi_\infty$ are regular. Denote by $\mu_\ell, \mu_\infty$, the unique probability $h_\ell$-conformal, resp. $h_\infty$-conformal, measure of $\Pi_\ell$, resp. $\Pi_\infty$, where $h_\ell = HD(C(H_\ell) \cap I^\ell) = HD(C(H_\infty))$, $h_\infty = HD(C(H_\infty) \cap I^\infty) = HD(C(H_\infty))$. (Notice that the measures have nothing to do with conformal measures of $H_\ell, H_\infty$, because the dynamics are completely different.) Since any regular system has a unique conformal measure, to prove that $h_\ell \to h_\infty$, it is enough to prove that a weak limit $\nu$ of a subsequence of $\mu_\ell$ is a conformal measure of $\Pi_\infty$. For this to be true, it is enough to check that the support of $\nu$ is contained in the limit set $L_\infty$ of $\Pi_\infty$. Note that by Lemma 4.1(b), the set $\overline{L_\ell} \setminus L_\infty$ is countable. Therefore, it is enough to prove that $\nu$ has no atoms. Thus Theorem 4(c) follows from

Lemma 4.4 If the non-symmetries $N(H_\ell)$ are uniformly bounded, then the measure $\nu$ has no atoms.

Proof. Let the point $a \in supp(\nu) = \overline{L_\infty}$, where $L_\infty$ is the limit set of $\Pi_\infty$, be an atom of $\nu$. Then there is $\sigma > 0$ such that for all $r > 0$ small enough $\mu_\ell(B(a, r)) > \sigma$ along a subsequence of $\ell$‘s. Since $\psi_{k,m}$ are uniform contractiones and the measures are probabilities, one sees that $a \in \overline{L_\ell} \setminus L_\infty$, i.e., afterall, one can assume that $a = x_0$. Now $\mu_\ell(B(x_0, r)) \leq \sum_{\ell k,m \cap B(x_0, r) \neq \emptyset} \int_{I^\ell} |D\psi_{k,m}^{(\ell)}|^h d\mu_\ell \leq C \sum |(G_{\ell k}^{(\ell)}(y_{\ell, m})|^h_\ell$, where $h_\ell = HD(C(H_\ell) \cap I^\ell)$ and $h_\ell = HD(C(H_\infty) \cap I^\infty) = HD(C(H_\infty))$. (Notice that the measures have nothing to do with conformal measures of $H_\ell, H_\infty$, because the dynamics are completely different.) Since any regular system has a unique conformal measure, to prove that $h_\ell \to h_\infty$, it is enough to prove that a weak limit $\nu$ of a subsequence of $\mu_\ell$ is a conformal measure of $\Pi_\infty$. For this to be true, it is enough to check that the support of $\nu$ is contained in the limit set $L_\infty$ of $\Pi_\infty$. Note that by Lemma 4.1(b), the set $\overline{L_\ell} \setminus L_\infty$ is countable. Therefore, it is enough to prove that $\nu$ has no atoms. Thus Theorem 4(c) follows from

Lemma 4.4 If the non-symmetries $N(H_\ell)$ are uniformly bounded, then the measure $\nu$ has no atoms.
for some fixed $C > 0$, some points $y_{\ell,m}$ from a fixed compact set $K$, $x_0 \notin K$ (if $\ell$ is big enough), and the latter sum runs over such $k$ that $G_{\ell}^k(y_{\ell,m}) \in B(x_0, r')$, where $r' \to 0$ as $r \to 0$. Then a contradiction follows directly from Lemma 13 and Theorem 6 (note that $t > 2/3$ by Corollary 14). 

\[ \square \]

5 Appendix: proof of Theorem 6

1. If $h_n \to h$ is a sequence of injective holomorphic maps in a fixed neighborhood of $c$, which converges to an injective $h$ uniformly, then the Poincaré series for $(f_n, t_n)$ converge uniformly iff the Poincaré series for $(h_n \circ f_n \circ h_n^{-1}, t_n)$ converge uniformly. In particular, one can assume that $c_n = c = 0$.

2. (see Theorem 7.2 of [13]). Let $h_n(z) = z - B_n z^2$, where $B_n = b_n/(\lambda_n(\lambda_n - 1))$. Since $|b_n| \leq M|\lambda_n - 1|$ for all $n$, there is a subsequence of $h_n$ as in Step 1. On the other hand, $h_n \circ f_n \circ h_n^{-1}(z) = \lambda_n z + O(z^3)$. It means one can assume that $f_n(z) = \lambda_n - a_n z^2 + \ldots$ where $a_n \to a > 0$, $0 < \lambda_n < 1$ and $\lambda_n \to 1$.

3. For $f_n$, make a change $z = \hat{h}_n(w) = d_n w^{-1/2}$, where $w \in F = \{w : Re(w) > R_0\}$ and $d_n = (\lambda_n^3/(2a_n))^{1/2}$. For $g_n = \hat{h}_n^{-1} \circ f_n \circ \hat{h}_n$, it holds $g_n(w) = \sigma_n w + 1 + \alpha_n(w)$, where $\sigma_n = \lambda_n^2 > 1$ and $\alpha_n \to 1$, $\alpha_n$ converge uniformly in $F$ to the corresponding $\alpha$ for $g = \hat{h}^{-1} \circ f \circ \hat{h}$, $\hat{h} = \lim \hat{h}_n$, and $\alpha_n(w) = O(|w|^{-1/2})$, $\alpha(w) = O(|w|^{-1/2})$.

To deal with $g_n$, we prove the following simple Claim. This is weaker than Theorems 8.1-8.3 of [13], but still enough for our needs.

**Claim 1:** For every $\delta > 0$ there is $R_\delta > R_0$ and, for every $n$, there is $1 + \delta$ quasi-conformal map $\phi_n$ of the plane that fixes 0, 1, and $\infty$, such that $\phi_n^{-1} \circ g_n \circ \phi_n = T_n$, where $T_n(w) = \sigma_n w + 1$, for $Re(w) > R_\delta$. Passing to a subsequence, one can assume that $\phi_n \to \phi$, so that $\phi^{-1} \circ g \circ \phi = T$, $T(w) = w + 1$.

**Proof.** Fix $\delta > 0$. Denote $\Pi(R_1, R_2) = \{w : R_1 < Re(w) < R_2\}$. Then $|\alpha_n(w)|$ and $|\alpha'_n(w)| \leq \sup(\{|\alpha_n(t)| : |t - w| < 1\})$ are uniformly arbitrary small as $w \in L := \{Re(w) = R_\delta\}$ and $R_\delta \to \infty$. Therefore, all $\sigma_n w$ can be joined to $z(w) := \sigma_n w + 1 + \alpha_n(w)$ by disjoint intervals $I(w)$ in the strip between $\sigma_n L$ and $z(L)$. The mapping $\phi_n$, which is affine on each interval $[\sigma_n w, \sigma_n w + 1]$ onto $I(w)$ together with the identity on $\Pi(R_\delta, \sigma_n R_\delta)$, is $1 + \delta$ quasi-conformal on $\Pi(R_\delta, \sigma_n R_\delta + 1)$. Then we extend $\phi_n$ to $Re(w) > \sigma_n R_\delta + 1$ by the (conformal) dynamics of $g_n$, $T_n$, and define it identity on the rest of the plane.

**Claim 2.** For every real $p > 1$, there is $M$ such that $\frac{|T_n^i y(w)|}{|T_n^i(w)|^p} \leq M i^{-p}$ for all $i, n$, and all $w > 1$.

Indeed, denote $C(i, n) = \sigma_n^i$. Consider any subsequence $(i_j, n_j), j \to \infty$. If $C(i, n)$ is bounded from above along this subsequence, then applying as in [12], Sect.6, the inequality between arithmetic and geometric means, we can write

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$T^i_n(w) = \sigma^i_n w + (1 + \sigma_n + \ldots + \sigma_n^{i-1}) \geq (i + 1) w^{1/(i+1)} \sigma_n^{i/2} \geq C(i,n)^{1/2} i$, so that
\[
\left| \frac{|f_n^i(w)|}{|T^i_n(w)|^p} \right| \leq C(i,n)^{1-p/2} 2^{-p} = O(i^{-p}) \text{ along the subsequence. If now } C(i,n) \to \infty \text{ along } (i,j,n) \text{ (and } \sigma_n \to 1), \text{ then } \left| \frac{|T^i_n(w)|}{|f_n^i(w)|^p} \right| = \left| \frac{|\sigma_n^i|}{|\sigma_n^i + (\sigma_n - 1)/(\sigma_n - 1)|^p} \right| \sim C(i,n)|\sigma_n - 1|^p/C(i,n)^p \sim (\log C(i,n))^p/C(i,n)^{p-1} 2^{-p} = o(i^{-p}).
\]

4. From Steps 1-2, Claim 1, and Koebe distortion theorem, it follows that it is enough to prove the theorem assuming that the compact $K$ is a point $x$, which moreover lies on an attracting direction of $f$, and small neighborhood $V$ can be replaced by big indexes. We have: $|(f_n^i)'(x)| = K |(g_n^i)'(w)|/|g_n^i(w)|^{3/2}$, where $K > 0$ and $w > R$ depend only on $x > 0$. Thus we need to show that, if $t_n \to t > 2/3$, for a given $w > 0$ close enough to $+\infty$, for any $\epsilon > 0$ there exists an index $i_0$, such that $S(g_n, i_0, t_n) := \sum_{i \geq i_0} |(g_n^i)'(w)|/|g_n^i(w)|^{3/2} t_n < \epsilon$ for all $n$ large enough. Claim 2 (with $p = 3/2$) implies immediately that this is true for $g_n = T_n$.

To handle $S(g_n, i_0, t_n)$ in general, we compare it with $S(T_n, i_0, t_n)$ and proceed similar to [13], Sect.10. Due to Koebe distortion theorem, one can replace the derivative by the ratio of diameters. By Claim 1, the change of the diameters when passing from $g_n$ to $T_n$ is Hölder with the exponent arbitrary close to 1. Then we apply Claim 2 with $p$ arbitrary close to 3/2.

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