Three-dimensional solutions of M-theory on $S^1/Z_2 \times T_7$

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Abstract: We review various three-dimensional solutions in the low-energy description of M-theory on $S^1/Z_2 \times T_7$. These solutions have an eleven dimensional interpretation in terms of intersecting M-branes.

1 Introduction

Eleven dimensional supergravity, the low-energy effective theory of M-theory, has four 1/2-supersymmetric solutions: the M-2-brane, the M-5-brane, the M-wave and (in the $S^3$ Kaluza-Klein vacuum) the M-monopole. We will refer to them collectively as M-branes. Intersections of these four basic configurations give rise to solutions preserving a smaller amount of supersymmetry (for a review see for instance [1, 2] and references therein). In the following, we will review various three-dimensional solutions in the low-energy description of M-theory on $S^1/Z_2 \times T_7$. These solutions have an eleven dimensional interpretation in terms of orthogonally intersecting M-branes.

The eleven-dimensional space-time line element describing a supersymmetric configuration of orthogonally intersecting M-branes is given in terms of harmonic functions which depend on some of the overall transverse directions [3, 4, 5]. If these overall transverse directions do not include the eleven-th dimension $x_{11}$, then its compactification on $S^1/Z_2 \times T_7$ down to three dimensions is also a solution of the low-energy effective field theory of ten-dimensional heterotic string theory compactified on a seven-torus. Such a solution is determined in terms of harmonic functions $H(z, \bar{z}) = f(z) + \bar{f}(\bar{z})$, where $z$ and $\bar{z}$ denote the spatial dimensions. The static solutions presented in [3, 4, 5] are of this type. If, on the other hand, the overall transverse directions include the eleven-th dimension $x_{11}$, then compactifying the eleven-dimensional configuration on $S^1/Z_2 \times T_7$ gives rise to a solution which is determined in terms of harmonic functions $H(x_{11}, z, \bar{z})$. An example of such a supergravity solution has recently been given [6] in the context of M-theory compactified...
on $T_k$. There, it was also shown that this solution corresponds to a BPS state with mass that goes like $1/g^3$. Evidence for the existence of BPS states with masses that go like $1/g^3$ or higher inverse powers of the string coupling constant $g$ in M-theory compactifications down to three (and lower) dimensions has been emerging in studies of U-duality multiplets of M-theory on tori [3, 8].

If the harmonic functions $H$ are taken to be independent of the eleven-th dimension, then the resulting three-dimensional solutions can be turned into finite energy solutions [3, 7] by utilizing a mechanism first discussed in the context of four-dimensional stringy cosmic string solutions [10]. For instance, the solutions presented in [7] have, at spatial infinity ($z \to \infty$), an asymptotic behaviour corresponding to $f(z) \propto \ln z$. At finite distance these asymptotic solutions become ill-defined and so need to be modified. The associated corrections are all encoded in $f(z)$. They can be determined by requiring the solutions to have finite energy. This requirement, together with the appropriate asymptotic behaviour, determines $f(z)$ to be given by $f(z) \propto j^{-1}(z)$. In section 2 we will review some of the three-dimensional solutions constructed in [7], namely those which have a ten-dimensional interpretation in terms of a fundamental string, a wave and up to three orthogonally intersecting NS 5-branes as well as up to three Kaluza–Klein monopoles. These solutions are labelled by an integer $n$ with $n = 1, 2, 3, 4$. The energy $E$ carried by these (one-center) solutions is given by $E = 2n \pi^6$ (in units where $8\pi G_N = 1$).

Another class of solutions constructed in [7] consists of solutions carrying energies $E = n \pi^6$, where $n = 1, 2, 3, 4$. An example of a solution with $E = \pi^6$ is given by a wrapped M-monopole. In section 3 we briefly review this solution. We then compare it to the solution recently discussed in [8], which can be constructed by considering a periodic array of M-monopoles along a transverse direction and by identifying this transverse direction with the eleven-th dimension. This latter solution is thus specified by a harmonic function $H(x_{11}, z, \bar{z})$ on $S^1/Z_2 \times \mathbb{R}^2$.

## 2 A class of finite energy solutions

The effective low-energy field theory of the ten-dimensional heterotic string compactified on a seven-dimensional torus is obtained from reducing the ten-dimensional $N = 1$ supergravity theory coupled to $U(1)^{16}$ super Yang–Mills multiplets (at a generic point in the moduli space). The massless ten-dimensional bosonic fields are the metric $G_{MN}^{(10)}$, the antisymmetric tensor field $B_{MN}^{(10)}$, the $U(1)$ gauge fields $A_M^{(10)}$ and the scalar dilaton $\Phi^{(10)}$ with $(0 \leq M, N \leq 9, \ 1 \leq I \leq 16)$. The field strengths are $F_{MN}^{(10)} = \partial_M A_N^{(10)} - \partial_N A_M^{(10)}$ and $H_{MNP}^{(10)} = (\partial_M B_{NP}^{(10)} - \frac{1}{2} A_M^{(10)} F_{NP}^{(10)}) +$ cyclic permutations of $M, N, P$.

The bosonic part of the ten-dimensional action is

$$ S \propto \int d^{10} x \sqrt{-G^{(10)}} e^{-\Phi^{(10)}} [R^{(10)} + G^{(10)MN} \partial_M \Phi^{(10)} \partial_N \Phi^{(10)} - \frac{1}{12} H_{MNP}^{(10)} H^{(10)MNP} - \frac{1}{4} F_{MN}^{(10)} F^{(10)IMN}]. \quad (1) $$
The reduction to three dimensions (see [3] and references therein) introduces
the graviton $g_{\mu\nu}$, the dilaton $\phi \equiv \Phi^{(10)} - \ln \sqrt{\det G_{mn}}$, with $G_{mn}$ the internal 7D metric,
30 $U(1)$ gauge fields $A_{\mu}^{(a)} \equiv (A_{\mu}^{(1)m}, A_{\mu}^{(2)}, A_{\mu}^{(3)I})$ ($a = 1, \ldots, 30$, $m = 1, \ldots, 7$, $I = 1, \ldots, 16$), where $A_{\mu}^{(1)m}$ are the 7 Kaluza-Klein gauge fields coming from the
reduction of $G_{MN}^{(10)}$, $A_{\mu}^{(2)} \equiv B_{\mu mn} + B_{nm}A_{\mu}^{(3)n} + \frac{1}{2} a_{m}^{I} A_{\mu}^{(3)I}$ are the 7 gauge fields coming from the
reduction of $B_{MN}^{(10)}$, and $A_{\mu}^{(3)I} \equiv A_{\mu}^{I} - a_{m}^{I} A_{\mu}^{(1)m}$ are the 16 gauge fields from $A_{M}^{(10)I}$. The field strengths $F_{\mu\nu}^{(a)}$ are given by $F_{\mu\nu}^{(a)} = \partial_{\mu} A_{\nu}^{(a)} - \partial_{\nu} A_{\mu}^{(a)}$. Finally, $B_{MN}^{(10)}$
induces the two-form field $B_{\mu\nu}$ with field strength $H_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} - \frac{1}{2} A_{\mu}^{(b)} L_{ab} F_{\nu\rho}^{(b)} +$ cyclic permutations.

The bosonic part of the three-dimensional action in the Einstein frame is then

$$
S = \frac{1}{4} \int d^{3}x \sqrt{-g} \left( R - g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{12} e^{-2\phi} g^{\mu\nu\rho} g^{\mu\nu\rho} H_{\mu\nu\rho} H_{\mu\nu\rho} + \frac{1}{4} e^{-2\phi} g^{\mu\nu\rho} F_{\mu\nu}^{(a)} (L_{ab} F_{\mu\nu}^{(b)}) + \frac{1}{8} g^{\mu\nu\rho} \text{Tr} (\partial_{\mu} ML \partial_{\nu} ML) \right), \tag{2}
$$

where $a = 1, \ldots, 30$. Here $M$ denotes a matrix comprising the scalar fields $G_{mn}$, $a_{m}$ and $B_{mn}$ [1], [2].

We now construct static solutions by setting the associated three-dimensional
Killing spinor equations to zero. In doing so, we restrict ourselves to backgrounds
with $H_{\mu\nu\rho} = 0$ and $a_{m} = 0$. It can be checked that the resulting solutions to the
Killing spinor equations satisfy the equations of motion derived from (2). The
associated three-dimensional Killing spinor equations in the Einstein frame are [3]

$$
\delta \chi^{I} = \frac{1}{2} e^{-2\phi} F_{\mu\nu}^{(3)I} \gamma^{\mu\nu} \varepsilon, \\
\delta \lambda = -\frac{1}{2} e^{-\phi} \partial_{\mu} \{ \phi + \ln \det e_{m}^{a} \} \gamma^{\mu} \otimes I_{8} \varepsilon + \frac{1}{4} e^{-2\phi} [-B_{mn} F_{\mu\nu}^{(1)n} + F_{\mu\nu}^{(2)}] \gamma^{\mu\nu} \gamma^{4} \otimes \Sigma^{m} \varepsilon + \frac{1}{4} e^{-2\phi} \partial_{\mu} B_{mn} \gamma^{\mu} \otimes \Sigma^{mn} \varepsilon, \\
\delta \psi_{\mu} = \partial_{\mu} \varepsilon + \frac{1}{4} \omega_{\alpha\beta} \gamma^{\alpha\beta} \varepsilon + \frac{1}{4} (e_{\alpha\beta} e_{\gamma\delta} - e_{\alpha\gamma} e_{\beta\delta}) \partial_{\mu} \phi \gamma^{\alpha\beta} \varepsilon + \frac{1}{8} (e_{a}^{\alpha} \partial_{\mu} e_{\alpha}^{\beta} - e_{\alpha}^{\alpha} \partial_{\mu} e_{\beta}^{\alpha}) I_{4} \otimes \Sigma^{ab} \varepsilon + \frac{1}{4} e^{-\phi} [e_{a}^{m} F_{\mu\nu}^{(2)} - e_{ma} F_{\mu\nu}^{(1)m}] \gamma^{\mu\nu} \gamma^{4} \otimes \Sigma^{a} \varepsilon + \frac{1}{8} \partial_{\mu} B_{mn} I_{4} \otimes \Sigma^{mn} \varepsilon + \frac{1}{4} e^{-2\phi} B_{mn} F_{\mu\nu}^{(1)n} \gamma^{\mu\nu} \gamma^{4} \otimes \Sigma^{m} \varepsilon, \\
\delta \psi_{d} = -\frac{1}{4} e^{-\phi} (e_{d}^{m} \partial_{\mu} e_{ma} + e_{ma} \partial_{\mu} e_{md}) \gamma^{\mu\nu} \gamma^{4} \otimes \Sigma^{a} \varepsilon + \frac{1}{4} e^{-2\phi} e_{d}^{m} B_{mn} F_{\mu\nu}^{(1)n} \gamma^{\mu\nu} \varepsilon + \frac{1}{4} e^{-2\phi} e_{d}^{m} e_{ma} \partial_{\mu} B_{mn} \gamma^{\mu\nu} \gamma^{4} \otimes \Sigma^{a} \varepsilon - \frac{1}{8} e^{-2\phi} [e_{md} F_{\mu\nu}^{(1)m} + e_{md} F_{\mu\nu}^{(2)}] \gamma^{\mu\nu} \varepsilon, \tag{3}
$$

where $\delta \psi_{d} \equiv e_{d}^{m} \delta \psi_{m}$ denotes the variation of the internal gravitini. Here we have performed a $3 + 7$ split of the ten-dimensional gamma matrices [3].

In the following, we review a particular class of static solutions to the Killing
spinor equations [3] constructed in [3]. The solutions in this class are labelled by
an integer $n$ ($n = 1, 2, 3, 4$). The associated space-time line element is given by

$$
ds^{2} = -dt^{2} + H^{2n} d\omega d\bar{\omega}, \quad H = f(\omega) + \bar{f}(\bar{\omega}), \tag{4}
$$
where $\omega = a \ln z = a \ln r + i\theta = \hat{r} + i\hat{\theta}$. Here, $a = \frac{n+1}{2\pi} \sqrt{|\alpha_i \alpha_{i+7}|}$, where $\alpha_i$ and $\alpha_{i+7}$ denote two electric charges carried by each of the solutions in this class. The associated field strengths of the three-dimensional gauge fields are 

$$F_{t\beta}^{(1)i} = \eta_{\alpha_i} \sqrt{G_{ii}} \frac{\partial_{\beta} H}{H^2} \quad , \quad F_{t\beta}^{(2)i} = -\eta_{\alpha_i} \sqrt{G_{ii}} \frac{\partial_{\beta} H}{H^2} \quad , \quad \beta = \hat{r}, \hat{\theta} \quad , \quad \eta_{\alpha_i} = \pm .$$

The three-dimensional dilaton field is given by $e^{-\phi} = H$. The i-th component of the internal metric $G_{mn}$ reads $G_{ii} = |\alpha_i \alpha_{i+7}|$. In addition, there are (depending on the integer $n$) various additional non-constant background fields $G_{mn}$ and $B_{mn}$, which are also determined in terms of $f(\omega)$.

Solving the Killing spinor equations (3) does not determine the form of $f(\omega)$. Its form can be determined by demanding the solution to behave as $f(\omega) \approx \omega^2 (n+1)$ at spatial infinity and by requiring the solution to have finite energy, as follows. The energy carried by any of the solutions in this class is computed to be (in units where $8\pi G_N = 1$)

$$E = i 2n \int d\omega d\bar{\omega} \frac{\partial_\omega f \partial_{\bar{\omega}} \bar{f}}{(f + \bar{f})^2} = i 2n \int dz \bar{z} \frac{\partial_z \hat{f} \partial_{\bar{z}} \bar{\hat{f}}}{(\hat{f} + \bar{\hat{f}})^2} ,$$

where we have introduced $\hat{f} = \frac{n+1}{\pi} f$ for later convenience. There is an elegant mechanism for rendering the integral finite. Let us take $z$ to be the coordinate of a complex plane. Then there is a one-to-one map from a certain domain $F$ on the $\hat{f}$-plane (the so called ‘fundamental’ domain) to the $z$-plane. This map is known as the j-function, $j(\hat{f}) = z$. By means of this map, the integral (6) can be pulled back from the $z$-plane to the domain $F$ (the $z$-plane covers $F$ exactly once). Then, by using integration by parts, this integral can be related to a line integral over the boundary of $F$, which is evaluated to be

$$E = 2n \frac{2\pi}{12} = 2n \frac{\pi}{6}$$

and, hence, is finite. By expanding $j(\hat{f}) = e^{2\pi \hat{f}} + 744 + O(e^{-2\pi \hat{f}}) = z$ we recover $f(\omega) \approx \frac{\omega}{2(n+1)}$ at spatial infinity.

We note that the solutions discussed above represent one-center solutions. They can be generalised to multi-center solutions via $j(\hat{f}(z)) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials in $z$ with no common factors. These are the analogue of the multi-string configurations discussed in [10].

It can be checked that the curvature scalar $\mathcal{R} \propto \partial_\omega f \partial_{\bar{\omega}} \bar{f}$ blows up at the special point $\hat{f} = 1$ (at this point, the $j$-function and its derivatives are given by $j = 1728, j' = 0$), whereas it is well behaved at the point $\hat{f} = e^{i\pi/6}$ (at this point, the $j$-function and its derivatives are given by $j = j' = j'' = 0$). It would be interesting to understand the physics at this special point in moduli space further.

As mentioned in the introduction the three-dimensional solutions reviewed in this section can be given an eleven dimensional interpretation in terms of orthogonally intersecting M-branes. The two gauge fields (5), in particular, arise from a wave and from a M-2-brane in eleven dimensions, respectively.
3 Wrapped M-monopoles

Another class of solutions constructed in [3] is the class of solutions carrying energy $E = n \frac{\pi}{6}$, where $n = 1, 2, 3, 4$. An example of a solution carrying energy $E = \frac{\pi}{6}$ is obtained by compactifying the M-monopole in the following way. The M-monopole in eleven dimensions is given by the metric [12]

$$ds_{11}^2 = -dt^2 + H dy_i^2 + H^{-1}(d\psi + A_i dy_i)^2 + dx_1^2 + \ldots + dx_6^2, \quad i = 1, 2, 3,$$  

(8)

with $H = H(y_i)$, $F_{ij} = \partial_i A_j - \partial_j A_i = c \varepsilon_{ijk} \partial_k H$, $c = \pm$. Here, $\psi$ denotes a periodic variable. Identifying one of the $x_i$ with the eleven-th coordinate and compactifying the remaining $x_i$ on a five-torus yields the five-dimensional line element

$$ds_5^2 = -dt^2 + H dy_i^2 + H^{-1}(d\psi + A_i dy_i)^2.$$

(9)

Now, consider the case that $H$ only depends on two of the coordinates $y_i$, that is $H = H(z, \bar{z})$ with $z = y_2 + iy_3$. Then we can set $A_2 = A_3 = 0$, and $\partial_2 A_1 = -c \partial_3 H$, $\partial_3 A_1 = c \partial_2 H$. The metric is now

$$ds_5^2 = -dt^2 + H dz d\bar{z} + H dy_i^2 + H^{-1}(d\psi + A_i dy_i)^2$$

$$= g_{\mu\nu} dx^\mu dx^\nu + G_{mn} dx^m dx^n,$$

(10)

where the off-diagonal internal metric is given by

$$G_{mn} = \begin{pmatrix} H + A_2^2 H^{-1} & A_1 H^{-1} \\ A_1 H^{-1} & H^{-1} \end{pmatrix}.$$  

(11)

The resulting three-dimensional solution can be turned into a finite energy solution with energy $E = \frac{\pi}{6}$ by using the mechanism described in the previous section.

Let us now compare this solution to the one obtained by identifying the eleven-th dimension not with one of the $x_i$, but rather with one of the $y_i$ [3]. Compactifying the eleven-dimensional line element (8) on a six-torus yields again the five-dimensional line element (9). Compactifying over $\psi$ yields the four-dimensional line element

$$ds_4^2 = -dt^2 + H dy_i^2$$

(12)

with internal metric component $G_{\psi\psi} = H^{-1}$ and magnetic gauge field $F_{ij} = c \varepsilon_{ijk} \partial_k H$.

The Bianchi identity of $F$ yields $\Delta H = 0$. Now, identifying one of the transverse coordinates $y_i$ with the eleven-th coordinate (say $y_1 = x_{11}$) and setting $z = y_2 + i y_3$ yields the Laplacian as $\Delta = \partial_{x_{11}}^2 + 4 \partial_z \partial_{\bar{z}}$. Thus, $H$ is now a harmonic function on $S^1/Z_2 \times \mathbb{R}^2$. In [13] a solution to the Laplace equation on $S^1 \times \mathbb{R}^2$ was constructed. It can be readily adapted to the case at hand. We take the range of $x_{11}$ to be $x_{11} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, with a peridic identification $x_{11} \sim x_{11} + 1$ of the endpoints. The $Z_2$ symmetry acts as $x_{11} \rightarrow -x_{11}$. A $Z_2$ invariant solution to the Laplace equation on $S^1/Z_2 \times \mathbb{R}^2$ is then given by

$$H(x_{11}, z, \bar{z}) = \sum_{n \in \mathbb{Z}} \left(\frac{1}{|n - \frac{1}{2}|^2} - \frac{1}{\sqrt{(x_{11} - (n - \frac{1}{2}))^2 + z \bar{z}}}\right).$$  

(13)
The constant term in (13) is such that \( H \) is regular at \( z = 0 \) on the orbifold plane \( x_{11} = 0 \). For \( |z| \to \infty \), on the other hand, \( H \) reduces to \( H \approx \log z \bar{z} \) on the plane \( x_{11} = 0 \).

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