A VARIETY THAT CANNOT BE DOMINATED BY ONE THAT LIFTS

REMY VAN DOBBEN DE BRUYN

ABSTRACT

We prove a precise version of a theorem of Siu and Beauville on morphisms to higher genus curves, and use it to show that if a variety $X$ in characteristic $p$ lifts to characteristic 0, then any morphism $X \to C$ to a curve of genus $g \geq 2$ can be lifted along. We use this to construct, for every prime $p$, a smooth projective surface $X$ over $\mathbb{F}_p$ that cannot be rationally dominated by a smooth proper variety $Y$ that lifts to characteristic 0.

INTRODUCTION

Given a smooth proper variety $X$ over a field $k$ of characteristic $p > 0$, a lift of $X$ to characteristic 0 consists of a DVR $^1 R$ of characteristic 0 with residue field $k$ and a flat proper $R$-scheme $\mathcal{X}$ whose special fibre $\mathcal{X}_0$ is isomorphic to $X$.

Varieties that lift enjoy some of the properties of varieties in characteristic 0. For example, minimal surfaces of general type that lift to characteristic 0 satisfy the Bogomolov–Miyaoka–Yao inequality [Lie13, Ex. 11.5]; and for varieties of dimension $d \leq p$ that lift over the Witt ring $W(k)$, or even its characteristic $p^2$ quotient $W_2(k)$, the Hodge–de Rham spectral sequence degenerates and ample line bundles satisfy Kodaira vanishing [DI87]. Liftability of K3 surfaces [Del81] plays an important role in the recent proofs of the Tate conjecture for K3 surfaces over finitely generated fields [Cha13], [Mad15], [KM16, Appendix A].

However, Serre showed [Ser61] that not every smooth projective variety can be lifted to characteristic 0. Serre’s example is constructed as a quotient $Y \to X$ of a liftable variety by a finite group action. The following well-known open problem arises naturally from this construction:

**Question 1.** Given a smooth proper variety $X$ over $\mathbb{F}_p$, does there exist a smooth proper variety $Y$ and a surjection $Y \twoheadrightarrow X$ such that $Y$ lifts to characteristic 0?

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1One can also define lifts over a more general base, but this reduces to the case of a DVR at the expense of enlarging the residue field (see e.g. [vDdB18a, Lem. 6.1.3]).
In other words, can one “resolve characteristic $p$ pathologies”, in much the same way one resolves the singularities of a variety? The main result of this paper is a negative answer to this question:

**Theorem 1.** Let $C$ be a supersingular curve over $\overline{\mathbb{F}}_p$ of genus $g \geq 2$, and let $X \subseteq C \times C \times C$ be a sufficiently general divisor. If $Y$ is a smooth proper variety admitting a dominant rational map $Y \dashrightarrow X$, then $Y$ cannot be lifted to characteristic 0.

Here, a curve $C$ is **supersingular** if its Jacobian is a supersingular abelian variety, and by a *sufficiently general divisor* we mean that there is a Zariski open $U \subseteq \text{NS}(C^3) \otimes \mathbb{Q}$ such that for every very ample line bundle $L$ whose Néron–Severi class lands in $U$, a general member $X \in |L^\otimes n|$ for $n \gg 0$ satisfies the conclusion of the theorem. See Theorem 6.3 and Remark 6.5.

Another motivation for Question 1 comes from motives. A positive answer would give a strategy for deducing cohomological statements in positive characteristic from characteristic 0 situations. Indeed, if $f: Y \to X$ is a surjective morphism of smooth proper varieties, then the pullback $f^*: H^*(X) \to H^*(Y)$ for any Weil cohomology theory $H$ is injective [Kle68, Prop. 1.2.4]. If we can find such $Y$ that lifts to characteristic 0, then one can try to deduce properties of $H^*(X)$ from characteristic 0 analogues.

For example, if $X$ is a smooth projective variety over $\overline{\mathbb{F}}_p$ and $\alpha \in \text{CH}^i(X)_{\mathbb{Q}}$ is an algebraic cycle, then it is expected$^2$ that the vanishing or nonvanishing of $\text{cl}(\alpha) \in H^2_{\acute{e}t}(X, \mathbb{Q}_\ell)$ does not depend on the prime $\ell$. A strategy for this problem would be to dominate the pair $(X, \alpha)$ by a pair $(Y, \beta)$ that can be lifted. The present paper shows that this is not even possible in absence of the cycle $\alpha$.

If the ground field is $\overline{\mathbb{F}}_p$, then there is no cohomological obstruction to the main question, since Honda showed [Hon68] that the $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$ appear inside the cohomology of abelian varieties admitting CM lifts.

**Outline of the proof**

Like in Serre’s example, we have no direct obstruction to liftability of $X$ or $Y$. Rather, we prove that additional geometric structure can be lifted along, and then set up our example to obtain a contradiction.

In Serre’s argument, the additional structure that lifts is a finite étale Galois cover $X' \to X$. This structure lives above $X$, so we have no way to use it on $Y$. Instead, we lift structure below $X$:

**Theorem 2.** Let $X$ be a variety in characteristic $p$, and let $\mathcal{X} \to \text{Spec } R$ be a lift over a DVR $R$. Let $\phi: X \to C$ be a morphism to a smooth projective curve of genus $g \geq 2$ such that $\phi_*\mathcal{O}_X = \mathcal{O}_C$. Then $\phi$ can be lifted to a morphism $\tilde{\phi}: \mathcal{X} \to \mathcal{C}$, up to an extension of $R$ and a Frobenius twist of $C$.

$^2$See also [vdDdB18b] for the equivalence of this conjecture to other classical conjectures on independence of $\ell$ of étale cohomology of varieties over finite fields.
A precise version is given in Theorem 2.1. The proof relies on a classification of morphisms \( X \to C \) to higher genus curves depending only on the fundamental group of \( X \). For this, we need the following precise version of Siu–Beauville’s theorem [Siu87, Thm. 4.7], [Cat91, Appendix].

**Theorem 3.** Let \( X \) be a smooth proper variety over an algebraically closed field \( k \) of characteristic 0, let \( \ell \) be a prime, and let \( g_0 \geq 2 \). Then the association \( \phi \mapsto \phi_* \) induces a bijection

\[
\left\{ \phi: X \to C \mid g(C) \geq g_0 \right\} \sim \to \left\{ \rho: \pi_1^{\text{et}}(X) \to \Gamma_g^\ell \text{ open} \mid g \geq g_0 \right\}
\]
on equivalence classes for naturally defined equivalence relations.

Here, \( \Gamma^\ell_g \) denotes the pro-\( \ell \) fundamental group of a genus \( g \) smooth projective curve. On the left hand side, two pairs \((C_1, \phi_1), (C_2, \phi_2)\) are equivalent if they both factor through a third pair \((C, \phi)\) (Definition 1.2). On the right hand side, two open maps \( \rho_1, \rho_2 \) are equivalent if their abelianisations both factor through the abelianisation of a third map \( \rho \) (Definition 1.5).

The classical statement of Siu–Beauville is recalled in Theorem 1.1, and our version is Theorem 1.6. The proof is a refinement of Beauville’s argument [Cat91, Appendix].

To deduce Theorem 2 from Theorem 3, we use the specialisation isomorphism

\[ sp: \pi_1^{\text{et}}(X_K) \sim \to \pi_1^{\text{et}}(X_k), \]

where \( k \) is the residue field and \( K \) the fraction field of \( R \). The map \( \phi: X \to C \)
gives rise to a map \( \phi_*: \pi_1^{\text{et}}(X_K) \cong \pi_1^{\text{et}}(X_k) \to \Gamma_g^\ell \), which in characteristic 0 comes from some morphism \( \phi': X_K \to C' \) to a higher genus curve. The proof is carried out by relating \((\phi', C')\) to the pair \((\phi, C)\) we started with.

With Theorem 2 in place, we want to study varieties admitting many morphisms to higher genus curves. We will work on a product \( \prod_{i=1}^r C_i \) of curves of genera \( g_i \geq 2 \), and we define for each \( i \in \{1, \ldots, r\} \) an obstruction \( E_i(\mathcal{L}) \) for a line bundle on \( \prod_i C_i \) to lift to \( \prod_i C_i \) for lifts \( C_i \) of the \( C_i \). The definition and main properties of \( E_i(\mathcal{L}) \) are given in Section 3. The isomorphism

\[ \text{Pic} \left( \prod_{i=1}^r C_i \right) \cong \prod_{i=1}^r \text{Pic}(C_i) \times \prod_{i<j} \text{Hom}_k(J_i, J_j) \]
suggests that we should look at supersingular curves \( C_i \), because the supersingular abelian varieties \( J_i = \text{Jac} C_i \) have more automorphisms than is possible in characteristic 0. In Section 4 we construct a line bundle \( \mathcal{L} \) on a power \( C^3 \) of a supersingular curve \( C \) of genus \( g \geq 2 \) such that no multiple \( \mathcal{L} \otimes n \) for \( n > 0 \) can be lifted to \( \prod_i C_i \) for any choice of lifts \( C_i \) of \( C_i \); see Lemma 4.3.

The proof of Theorem 1 then roughly goes as follows. Choose a line bundle \( \mathcal{L} \) on \( C^3 = \prod_i C_i \) as above, and let \( X \in |\mathcal{L}| \) be a general member. If \( Y \to X \) is a surjective morphism (for simplicity), then consider the projections \( \phi_i: Y \to C_i \),
By Theorem 2, if $Y$ is a lift of $Y$, then the $\phi_i$ can be lifted to maps $\tilde{\phi}_i: Y \to C_i$ (for simplicity we ignore Stein factorisation and Frobenius twists). Then the image of the product map

$$\tilde{\phi}: Y \to \prod_{i=1}^{3} C_i$$

is a divisor whose special fibre is a multiple of the reduced divisor $X$. But then a power of $O_{\prod C_i}(X) = \mathcal{L}$ lifts to $\prod C_i$, contradicting the choice of $\mathcal{L}$.

There are some additional technical difficulties one runs into, coming from the fact that the morphisms $Y \to C_i$ do not lift on the nose. Rather, one has to take their Stein factorisation $Y \to C'_i \to C_i$ first, and then the morphisms $Y \to C'_i$ only lift up to a power of Frobenius $F: C'_i \to C''_i$ (see Theorem 2).

One therefore has to devise an argument that is flexible with respect to finite covers $C'_i \to C_i$. We facilitate this as follows:

- We show that the obstruction $E_i(\mathcal{L})$ to the liftability of $\mathcal{L}$ to $\prod C_i$ is well-behaved with respect to pullback under finite morphisms (Lemma 3.10). This is the reason we use this intermediate obstruction, rather than working directly with nonliftable line bundles.
- At the end of the argument, we take the scheme-theoretic image. This is only well-behaved with respect to pushforward, not pullback. Pullback and pushforward can be interchanged as long as the inverse image of $X$ under $\prod C'_i \to \prod C_i$ is still irreducible (Lemma 6.2).
- But we have to define $X$ before we know what the finite covers $C'_i \to C_i$ are. We call a divisor $X \subseteq \prod C_i$ stably irreducible if its inverse image in $\prod C'_i$ is irreducible, regardless of the covers $C'_i \to C_i$. A Bertini theorem proves that a general member of $|\mathcal{L}^\otimes n|$ for $n \gg 0$ satisfies this property (Proposition 5.3).

**Structure of the paper**

The paper is divided into three (roughly) equal parts, each spanning two sections:

- In Section 1 we prove Theorem 3, which we then use in Section 2 to prove Theorem 2. This is the geometric part of the argument.
- In Section 3, we study line bundles $\mathcal{L}$ on a product $\prod C_i$ and define an obstruction $E_i(\mathcal{L})$ for $\mathcal{L}$ to lift. We use this in Section 4 to construct a line bundle on the third power $C^3$ of a supersingular curve that cannot be lifted. This is the cohomological part of the argument.
- In Section 5, we construct stably irreducible divisors in $|\mathcal{L}^\otimes n|$ for $n \gg 0$. This gives the variety $X$ of Theorem 1. In Section 6, we carry out the construction and proof.

This paper presents the main result of the author’s dissertation [vDdB18a]. The statement and proof of Theorem 3 are new; in [vDdB18a] we use a more complicated argument relying on results from nonabelian Hodge theory [Sim91, Thm. 10], [CS08] to deduce Theorem 2. The dissertation further contains proofs of well-known results for which no detailed account in the literature was known to the author; we occasionally refer the reader there for extended discussion.
**NOTATION**

A *variety* over a field $k$ will mean a separated scheme of finite type over $k$ that is geometrically integral. When we say *curve*, *surface*, *thefold*, etc., this is always understood to be a variety.

A smooth proper variety $X$ over a perfect field $k$ of characteristic $p > 0$ is *supersingular* if for all $i$, all Frobenius slopes on $H^i_{\text{crys}}(X/W(k))[1/p]$ equal $\frac{i}{2}$. If $X$ is an abelian variety, this reduces to $i = 1$, hence it recovers the usual notion. If $X$ is a curve, then it is supersingular if and only if its Jacobian is.

For two $S$-schemes $X$ and $Y$, we will write $\text{Mor}_S(X,Y)$ for the set of morphisms of $S$-schemes $X \to Y$, and $\text{Mor}_S(X,Y)$ for the functor $\text{Sch}_S \to \text{Set}$ mapping $T \to S$ to $\text{Mor}_T(X_T,Y_T)$ (or the scheme representing this functor, if it exists). Similarly, if $A$ and $B$ are abelian schemes over $S$, then $\text{Hom}_S(A,B)$ will denote the group of homomorphisms $A \to B$ of abelian schemes, and $\text{Hom}_S(A,B)$ will denote the group scheme of homomorphisms (see e.g. [vDdB18a, Cor. 4.2.4] for representability). The group $\text{Hom}_S(A,B) \otimes \mathbb{Q}$ will be denoted $\text{Hom}_S^\vee(A,B)$.

We will write $\pi^\text{et}_1(X)$ for the étale fundamental group of a scheme $X$, $\pi^\text{et,\ell}_1(X)$ for its maximal pro-$\ell$ quotient, and $\pi^\text{top}_1(X)$ for the topological fundamental group of a $\mathbb{C}$-variety $X$. All maps between profinite groups are assumed continuous. We will write $\Gamma_g$, $\hat{\Gamma}_g$, and $\Gamma^\text{pro}_g$ for the topological, étale, and pro-$\ell$ fundamental groups of a smooth projective genus $g$ curve over $\mathbb{C}$ respectively. We consistently ignore the choice of base point, because it does not affect the arguments.

If $\mathcal{P}$ is a property of schemes and $X \to S$ is a morphism of schemes, then $\mathcal{P}$ holds for a general fibre $X_s$ if there exists a dense open $U \subseteq S$ such that $\mathcal{P}(X_s)$ holds for all $s \in U$. We will sometimes omit mention of $\mathcal{P}$ and say that $X_s$ with $s \in U$ is a general member of the family. If $S$ is a variety over a finite field $k$, then there need not exist a general member that is defined over $k$.

A rng is a ring without unit, and a $\mathbb{Q}$-rng is a rng which is also a $\mathbb{Q}$-vector space.

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1. A precise version of Siu–Beauville

The following result was obtained independently by Siu [Siu87, Thm. 4.7] and Beauville [Cat91, Appendix].

**Theorem 1.1** (Siu–Beauville). Let $X$ be a compact Kähler manifold, and let $g_0 \geq 2$. Then $X$ admits a surjection $\phi: X \to C$ to a compact Riemann surface $C$ of genus $g(C) \geq g_0$ if and only if there exists a surjection $\rho: \pi^\top_{1}(X) \to \Gamma_{g_0}$.

We will upgrade this in Theorem 1.6 to a bijection between suitable sets of surjections $X \to C$ and maps $\pi_{1}(X) \to \Gamma_{g}$ with finite index image. For our mixed characteristic application, it is convenient to work with the profinite fundamental group (if $k = \mathbb{C}$) or the profinite fundamental group.

**Definition 1.2.** Write $\operatorname{Mor}(X, \geq g_0)$ for the set of pairs $(C, \phi)$ where $C$ is a smooth projective curve of genus $\geq g_0$ and $\phi: X \to C$ is a nonconstant morphism, up to isomorphism (as schemes under $X$). By de Franchis’s theorem, this is a finite set if $g_0 \geq 2$ [dFr13] (see e.g. [Mar88] for a modern proof).

For $(C_1, \phi_1), (C_2, \phi_2) \in \operatorname{Mor}(X, \geq g_0)$, write $(C_1, \phi_1) \sim (C_2, \phi_2)$ if there exists $(C, \phi) \in \operatorname{Mor}(X, \geq g_0)$ such that both $\phi_i$ factor through $\phi$. This is equivalent to the statement that in the Stein factorisations

$$X \overset{\phi_i}{\to} C_i \overset{f_i}{\to} C,$$

of the $\phi_i$, the pairs $(C_i', \phi_i')$ agree in $\operatorname{Mor}(X, \geq g_0)$.

**Remark 1.3.** If $f: X \to Y$ is a dominant morphism of normal varieties, then $f_*: \pi_1^\top(X) \to \pi_1^\top(Y)$ is an open homomorphism [Kol03, Lem. 11] (see also [vDdB18a, Lem. 5.3.1] for a shorter and more general proof), which is surjective if $f$ has geometrically connected fibres. Recall that a continuous homomorphism of profinite groups is open if and only if its image has finite index.

**Remark 1.4.** The group $H^1(X, \mathbb{Z}_{\ell})$ can be identified with the (continuous) group cohomology $H^1(\pi^\et_1(X), \mathbb{Z}_{\ell}) = \operatorname{Hom}_{cts}(\pi^\et_1(X), \mathbb{Z}_{\ell})$, so any morphism $\rho: \pi^\et_1(X) \to \Gamma_{g}^\ell$ induces a pullback

$$\rho^*: H^1(\Gamma_{g}^\ell, \mathbb{Z}_{\ell}) \to H^1(X, \mathbb{Z}_{\ell}).$$

**Definition 1.5.** Write $\operatorname{Hom}^\circ(\pi^\et_1(X), \Gamma_{g \geq g_0}^\ell)$ for the set of open homomorphisms $\rho: \pi^\et_1(X) \to \Gamma_{g}^\ell$ for $g \geq g_0$, up to isomorphism (as groups under $\pi^\et_1(X)$). For elements $\rho_1, \rho_2 \in \operatorname{Hom}^\circ(\pi^\et_1(X), \Gamma_{g \geq g_0}^\ell)$, we write $\rho_1 \sim \rho_2$ if there exists $\rho \in \operatorname{Hom}^\circ(\pi^\et_1(X), \Gamma_{g \geq g_0}^\ell)$ such that the abelianisations $\rho^{\ab}_1: \pi^\et_1(X)^\ab \to \Gamma_{g_i}^\ell$ for $i \in \{1, 2\}$ both factor through $\rho^{\ab}$. Equivalently, the pullbacks in $H^1(X, \mathbb{Z}_{\ell})$ satisfy

$$\rho_i^* H^1(\Gamma_{g_i}^\ell, \mathbb{Z}_{\ell}) \subseteq \rho^* H^1(\Gamma_{g}^\ell, \mathbb{Z}_{\ell}),$$

as subgroups of $H^1(X, \mathbb{Z}_{\ell})$, for $i \in \{1, 2\}$. 

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Theorem 1.6. Let $X$ be a smooth proper variety over an algebraically closed field $k$ of characteristic $0$, let $\ell$ be a prime, and let $g_0 \geq 2$. Then the association $\phi \mapsto \phi^*$ induces a bijection

$$\alpha: \text{Mor}(X, g_0) \sim \rightarrow \text{Hom}^\circ\left(\pi_1^{\text{ét}}(X), \Gamma_{g_0}\right)$$

on equivalence classes for the relations of Definition 1.2 and Definition 1.5.

The proof will be given after Corollary 1.13. By Corollary 1.9 below, the classical Siu–Beauville theorem amounts to the statement that one side is nonempty if and only if the other is.

Remark 1.7. In $\text{Mor}(X, g_0)$, every equivalence class $C$ for $\sim$ has a canonical representative given by the unique $(C, \phi) \in C$ such that $\phi^* O_X = O_C$. A priori, equivalence classes in $\text{Hom}^\circ\left(\pi_1^{\text{ét}}(X), \Gamma_{g_0}\right)$ do not have a preferred representative (see also Remark 1.14). In fact, it is not even clear that $\sim$ defines an equivalence relation on $\text{Hom}^\circ\left(\pi_1^{\text{ét}}(X), \Gamma_{g_0}\right)$; this will follow from the proof.

We first discuss some general properties of the groups $\Gamma_g^\ell$. The following result is an unstated consequence of [And74].

Lemma 1.8. Let $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ be an exact sequence of finitely generated groups. If $H$ is an $\ell$-group and $N^\ell$ has trivial centre, then the pro-$\ell$ completion

$$1 \rightarrow N^\ell \rightarrow G^\ell \rightarrow H^\ell \rightarrow 1$$

is exact.

Proof. By [And74, Prop. 3 and Cor. 7], it suffices to show that the image of the conjugation action $G \rightarrow \text{Aut}(N^\text{ab}/\ell)$ is an $\ell$-group. But this map is trivial on $N$ since conjugation of $N$ acts trivially on $N^\text{ab}$. Hence, it factors through $G/N = H$, which is an $\ell$-group. □

Corollary 1.9. Let $g \geq 2$, and let $U \subseteq \Gamma_g^\ell$ be an open subgroup of index $\ell^n$. Then $U \cong \Gamma_g^\ell(\ell^{\ell^n}(g-1)+1)$. The corresponding statement for $\Gamma_g$ (resp. $\hat{\Gamma}_g$) follows from topology (resp. algebraic geometry). The difficulty is that pro-$\ell$ completion (resp. maximal pro-$\ell$ quotient) is in general only right exact.

Proof. By solubility of $\ell$-groups, we may reduce to the case that $U$ is normal (or even $n = 1$). If $H = \Gamma_g^\ell/U$, then consider the surjection $\Gamma_g \rightarrow H$. Its kernel is $\Gamma_{g/(g-1)+1}$, whose pro-$\ell$ completion has trivial centre [And74, Prop. 18]. Therefore, Lemma 1.8 implies that $U \cong \Gamma_{g/(g-1)+1}$. □

The following lemma addresses injectivity of the map of Theorem 1.6. It also holds in positive characteristic, which will be used in the proof of Theorem 2.1.

Lemma 1.10. Let $X$ be a smooth proper variety over an algebraically closed field $k$, and let $\ell$ be a prime invertible in $k$. Let $(C_i, \phi_i) \in \text{Mor}(X, g_2)$ for $i \in \{1, 2\}$, and consider the following statements.
(1) The $\phi_i$ satisfy $(C_1, \phi_1) \sim (C_2, \phi_2)$;
(2) The product morphism $\phi: X \to C_1 \times C_2$ is not dominant;
(3) The pullbacks $\phi^*_i H^1_{\et}(C_1, \mathbb{Z}_\ell) \subseteq H^1_{\et}(X, \mathbb{Z}_\ell)$ have nonzero intersection.

Then the implications $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ hold. If $\phi_{1,*} \mathcal{O}_X = \mathcal{O}_{C_1}$ for $i \in \{1, 2\}$, then the reverse implication $(2) \Rightarrow (3)$ holds as well.

**Proof.** Note that $\phi$ is not dominant if and only if its image is contained in a (possibly singular) curve in $C_1 \times C_2$, showing $(1) \Leftrightarrow (2)$. If $\phi: X \to C_1 \times C_2$ is dominant, then the pullback

$$
\phi^*: H^1_{\et}(C_1 \times C_2, \mathbb{Z}_\ell) \to H^1_{\et}(X, \mathbb{Z}_\ell)
$$

is injective [Kle68, Prop. 1.2.4]. Therefore, the pullbacks $\phi^* H^1_{\et}(C_i, \mathbb{Z}_\ell)$ are linearly disjoint, which proves $(3) \Rightarrow (2)$. Finally, if $\phi_{1,*} \mathcal{O}_X = \mathcal{O}_{C_1}$, then $(C_1, \phi_1) \sim (C_2, \phi_2)$ implies $(C_1, \phi_1) \cong (C_2, \phi_2)$, so that $(1) \Rightarrow (3)$ is clear. \qed

The implication $(1) \Rightarrow (3)$ is false in general: if $X = C \subseteq C_1 \times C_2$ is a smooth very ample divisor, then $H^1_{\et}(X, \mathbb{Z}_\ell) \cong H^1_{\et}(C_1, \mathbb{Z}_\ell) \oplus H^1_{\et}(C_2, \mathbb{Z}_\ell)$, so that the parts coming from $C_1$ and $C_2$ are linearly disjoint.

**Definition 1.11.** Let $X$ be a variety over an algebraically closed field $k$, and let $\ell$ be a prime number that is invertible in $k$. For a class $\eta \in H^1_{\et}(X, \mathbb{Z}_\ell)$ and $n \in \mathbb{Z}_{> 0}$, we write $\mathcal{L}_n(\eta)$ for the $\ell^n$-torsion line bundle given by

$$
\eta \pmod{\ell^n} \in H^1_{\et}(X, \mathbb{Z}/\ell^n) \cong H^1_{\et}(X, \mu_{\ell^n}),
$$

and we write $\langle \eta \rangle \subseteq \text{Pic}^0(X)[\ell^\infty]$ for the group generated by $\mathcal{L}_n(\eta)$ for $n \in \mathbb{Z}_{> 0}$. The set of $\eta \in H^1_{\et}(X, \mathbb{Z}_\ell)$ such that $H^1(X, \mathcal{L}) \neq 0$ for all $\mathcal{L} \in \langle \eta \rangle$ is denoted by $S_1^1$ (in analogy with the Green–Lazarsfeld loci $S^i \subseteq \text{Pic}^0_X$ [GLS75]).

The key input of the proof of **Theorem 1.6** is the following $\ell$-adic version of Beauville’s corollary [Bea88, Thm. 1] of the Green–Lazarsfeld generic vanishing theorem [GLS75]. The proof relies on Beauville’s result [loc. cit.].

**Proposition 1.12.** Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $0$, and let $\eta \in H^1_{\et}(X, \mathbb{Z}_\ell)$ be a nonzero element. Then the following are equivalent:

(1) There exists $(C, \phi) \in \text{Mor}(X, \geq 2)$ such that $\eta \in \phi^* H^1_{\et}(C, \mathbb{Z}_\ell)$;
(2) There exists $\rho \in \text{Hom}^\vee(\pi_{1, \ell}^\et(X), \Gamma_{\geq 2})$ such that $\eta \in \rho^* H^1(\Gamma_{\eta}, \mathbb{Z}_\ell)$;
(3) For all $\mathcal{L} \in \langle \eta \rangle$, we have $H^1(X, \mathcal{L}) \neq 0$;
(4) For infinitely many $\mathcal{L} \in \langle \eta \rangle$, we have $H^1(X, \mathcal{L}) \neq 0$.

**Proof.** If $\eta \in \phi^* H^1_{\et}(C, \mathbb{Z}_\ell)$ for some surjection $\phi: X \to C$, then the easy direction of [Bea88, Thm. 1] gives $H^1(X, \mathcal{L}) \neq 0$ for all $\mathcal{L} \in \langle \eta \rangle$. This shows $(1) \Rightarrow (3)$. Implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$ are obvious.

If $H^1(X, \mathcal{L}) \neq 0$ for infinitely many $\mathcal{L} \in \langle \eta \rangle$, then by [Bea88, Thm. 1] there exist morphisms $\phi_i: X \to C_i$ with $g(C_i) \geq 2$ such that $\mathcal{L}_i \in \phi_i^* \text{Pic}^0(C_i)$. Since there are only finitely many possible $C_i$, one of them must occur infinitely many times, which forces $\langle \eta \rangle \subseteq \phi_i^* \text{Pic}^0(C_i)$ since $\phi_i^*$ is a group homomorphism. This immediately implies $\eta \in \phi_i^* H^1_{\et}(C_i, \mathbb{Z}_\ell)$, proving $(4) \Rightarrow (1)$. 

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Finally, assume \( \eta \in \rho^*H^1(\Gamma_g^f, \mathbb{Z}_\ell) \) for some \( \rho \in \text{Hom}^\circ(\pi_1^\dagger, \Gamma_g) \) with \( g \geq 2 \). By Corollary 1.9, we may assume \( \rho \) is surjective (increasing \( g \) if necessary). Let \( \tau: \Gamma_g^f \to \mathbb{Z}_\ell \) be the homomorphism such that \( \eta = \rho^*(\tau) \in H^1(X, \mathbb{Z}_\ell) \). If \( \tau = 0 \), then clearly \( \eta \in S \), so we may assume \( \tau \) is surjective. Then the surjection \( \pi_1^\dagger(X) \to \mathbb{Z}_\ell^\dagger \to \mathbb{Z}/\ell^n \) corresponds to the cyclic \( \mathbb{Z}/\ell^n \)-cover

\[
\pi_n: X_n = \text{Spec}_X \left( \bigoplus_{i=0}^{\ell^n-1} \mathcal{L}_n(\eta)^{\otimes i} \right) \to X,
\]

where \( \mathcal{L}_n(\eta) \) is as in Definition 1.11. In particular, we find

\[
H^1(X_n, \mathcal{O}_{X_n}) = H^1(X, \pi_n^*\mathcal{O}_{X_n}) \cong \bigoplus_{i=0}^{\ell^n-1} H^1(X, \mathcal{L}_n(\eta)^{\otimes i}). \tag{1.1}
\]

But \( \pi_1^\dagger(X_n) \) surjects onto \( \ker(\Gamma_g^f \to \mathbb{Z}/\ell^n) \), which is isomorphic to \( \Gamma_g^f \) by Corollary 1.9, so Hodge theory gives

\[
h^1(X_n, \mathcal{O}_{X_n}) \geq \ell^n(g-1)+1.
\]

Thus, by (1.1) there are infinitely many \( n \) such that \( H^1(X_g, \mathcal{L}_n(\eta)^{\otimes i}) \neq 0 \) for some \( i \in (\mathbb{Z}/\ell^n)^\times \), showing that the final implication \((2) \Rightarrow (4)\). \( \square \)

**Corollary 1.13.** Let \( X \) be a smooth projective variety over an algebraically closed field \( k \) of characteristic 0. Then the locus \( S \subseteq H^1_\dagger(X, \mathcal{O}_X) \) of Definition 1.11 is the finite union

\[
S = \bigvee_{\substack{\phi: X \to C \\phi_*\mathcal{O}_X = \mathcal{O}_C}} \phi^*H^1_\dagger(C, \mathbb{Z}_\ell).
\]

of the linearly disjoint saturated \( \mathbb{Z}_\ell \)-submodules \( \phi^*H^1_\dagger(C, \mathbb{Z}_\ell) \subseteq H^1_\dagger(X, \mathbb{Z}_\ell) \) for \( \phi: X \to C \) a morphism to a curve \( C \) of genus \( g \geq 2 \) satisfying \( \phi_*\mathcal{O}_X = \mathcal{O}_C \).

**Proof.** By property (1) of Proposition 1.12, every element \( \eta \in S \) is contained in \( \phi^*H^1_\dagger(C, \mathbb{Z}_\ell) \) for some surjection \( \phi: X \to C \) to a smooth projective curve \( C \) of genus \( g \geq 2 \). Taking Stein factorisation, we may assume \( \phi_*\mathcal{O}_X = \mathcal{O}_C \). For different \( C \), these spaces are pairwise linearly disjoint by Lemma 1.10. The saturatedness statement follows since property (4) of Proposition 1.12 for any \( \eta \in (\mathbb{Z}/\ell^n)^\times \) implies the same for \( \eta \). \( \square \)

**Proof of Theorem 1.6.** It is well-known that rational maps to curves of genus \( \geq 1 \) extend (see e.g. [vDdB18a, Cor. 4.1.4]). Moreover, \( \pi_1 \) is a birational invariant [SGA1, Exp. X, Cor. 3.4], hence both sides of the statement only depend on the birational isomorphism class of \( X \). Then Chow’s lemma [EGA2, Thm. 5.6.1] and resolution of singularities [Hir64] reduce us to the smooth projective case.

By Proposition 1.12 and Corollary 1.13, the union of the pullbacks \( \rho^*H^1(\Gamma_g^f, \mathbb{Z}_\ell) \) for \( \rho \in \text{Hom}^\circ(\pi_1^\dagger, \Gamma_g) \) is the wedge sum

\[
S = \bigvee_{\substack{\phi: X \to C \\phi_*\mathcal{O}_X = \mathcal{O}_C}} \phi^*H^1_\dagger(C, \mathbb{Z}_\ell). \tag{1.2}
\]
This defines a map

$$\beta: \text{Hom}^\circ \left( \pi_1^\et, \ell_1(X), \Gamma_\ell \geq g_0 \right) \longrightarrow \text{Mor}(X, \geq g_0) \sim \sim,$$

taking $\rho$ to the unique $(C, \phi)$ with $\phi_* \mathcal{O}_X = \mathcal{O}_C$ corresponding to the component of the wedge sum (1.2) in which $\rho^* H^1(\Gamma_\ell, \mathbb{Z}_\ell)$ lands. Moreover, the fibres of $\beta$ are exactly the equivalence classes of $\sim$, showing that $\sim$ is an equivalence relation. Then $\beta$ descends to a two-sided inverse of $\alpha$.

**Remark 1.14.** The proof shows that the surjections $\rho: \pi_1^\et, \ell_1(X) \twoheadrightarrow \Gamma_\ell \geq g_0$ for which $\rho^* H^1(\Gamma_\ell, \mathbb{Z}_\ell)$ is inclusionwise maximal correspond to the maximal linear subspaces of the set $S$ of Proposition 1.12. One can use this to state Theorem 1.6 in terms of maximal elements instead of equivalence classes.

However, there may be multiple surjections $\rho: \pi_1^\et, \ell_1(X) \twoheadrightarrow \Gamma_\ell \geq g_0$ for which the pullbacks $\rho^* H^1(\Gamma_\ell, \mathbb{Z}_\ell)$ form the same maximal subspace of $S$, so they are only maximal in a weak sense. For this reason, we chose to state Theorem 1.6 in terms of equivalence classes.

**Corollary 1.15.** Let $X$ be a smooth proper variety over an algebraically closed field $k$ of characteristic 0, and let $g \geq 2$. Then the set of (isomorphism classes of) surjections $\phi: X \rightarrow C$ with $\phi_* \mathcal{O}_X = \mathcal{O}_C$ to a smooth projective curve $C$ of genus $g$ only depends on $\pi_1^\et, \ell_1(X)$. In particular, if $k = \mathbb{C}$, it is a homotopy invariant of $X$.

**Proof.** Apply Theorem 1.6 to $g_0 = g$ and $g_0 = g + 1$.

In contrast, the original Siu–Beauville theorem (Theorem 1.1) only addresses whether or not there exists a morphism $X \rightarrow C$ to a curve of some fixed genus $g \geq 2$, not how many there are.

**Remark 1.16.** A similar result was obtained by Catanese [Cat91, Thm. 2.25], which deals more generally with *Albanese general type fibrations* $\phi: X \rightarrow Y$, i.e. maps to Kähler manifolds $Y$ with $q(Y) > \dim(Y)$ such that the image of $\text{alb}: Y \rightarrow \text{Alb}(Y)$ has dimension $\dim(Y)$. Catanese’s characterisation uses certain real subspaces of $H^1(X, \mathbb{C})$ instead of the fundamental group. As such, it does not generalise well to other algebraically closed fields of characteristic 0.

Catanese also shows that if $\phi: X \rightarrow C$ is a morphism from a smooth proper scheme $X$ to a smooth projective curve $C$ of genus $g \geq 2$ over a field $k$ of characteristic 0, then the forgetful transformation

$$\text{Def}(\phi: X \rightarrow C) \longrightarrow \text{Def}_C$$

is an isomorphism of deformation functors [Cat91, Rmk. 4.10]. The proof was rediscovered by the present author [vDdB18a, Thm. 5.5.1]. It relies on a Kodaira type vanishing theorem of Kollár [Kol86, Thm. 2.1]; there are various reasons the required vanishing theorem fails in positive characteristic [Cat91, Rmk. 4.12], [vDdB18a, Ex. 5.5.4]. See also Question 2.4 below.
2. LIFTING MORPHISMS TO CURVES

We apply Theorem 1.6 to prove that a morphism \( \phi: X \to C \) of smooth proper varieties to a curve \( C \) of genus \( g \geq 2 \) lifts along with any lift of \( X \).

**Theorem 2.1.** Let \( R \) be a DVR of characteristic 0 with fraction field \( K \) and algebraically closed residue field \( k \). Let \( X \to \text{Spec} R \) be a smooth proper morphism, let \( C \) be a smooth projective curve over \( k \) of genus \( g \geq 2 \), and let \( \phi: \mathcal{X}_0 \to C \) be a morphism with \( \phi_* \mathcal{O}_{\mathcal{X}_0} = \mathcal{O}_C \). Then there exists a generically finite extension \( R \to R' \) of DVRs, a smooth projective curve \( C' \) over \( R' \), a morphism \( \phi': \mathcal{X}_R \to C' \), and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}_0 & \xrightarrow{\phi} & C \\
\downarrow{\phi_0} & & \downarrow{F} \\
C' & \xrightarrow{\phi'} & C_0',
\end{array}
\]

where \( F \) is purely inseparable. In particular, \( F \) is a power of the relative Frobenius if \( \text{char } k = p > 0 \), and \( F \) is an isomorphism if \( \text{char } k = 0 \).

**Remark 2.2.** That is, if \( X \) can be lifted, then so can any morphism \( \phi: X \to C \) with \( \phi_* \mathcal{O}_X = \mathcal{O}_C \) to a curve \( C \) of genus \( g \geq 2 \), up to a generically finite extension \( R \to R' \) and a purely inseparable morphism \( C \to C_0' \) with \( \mathcal{O}_C = C^{(p^{-e})} \).

**Remark 2.3.** If \( \text{char } k = 0 \), i.e. \( R \) is a DVR of equicharacteristic 0, then we may in fact choose \( R = R' \) by Catanese’s deformation theoretic result (Remark 1.16). We do not know if the extension \( R \to R' \) and the purely inseparable morphism \( C \to C_0' \) are actually needed in mixed characteristic, nor whether a variant of the result is true in pure characteristic \( p > 0 \). In fact, as far as we know the following is still open (see also [Cat91, Rmk. 4.12]):

**Question 2.4.** Let \( X \) and \( X' \) be deformation equivalent smooth proper varieties over an algebraically closed field \( k \) of characteristic \( p > 0 \). If \( X \) admits a dominant morphism to a curve of genus \( g \geq 2 \), then does \( X' \) as well?

**Proof of Theorem 2.1.** The morphism \( \phi: \mathcal{X}_0 \to C \) induces a surjection

\[ \rho: \pi^\text{ét,\ell}_1(\mathcal{X}_0) \to \Gamma_\ell' \]

By [SGA1, Exp. X, Cor. 3.9], we have an isomorphism \( \pi^\text{ét,\ell}_1(\mathcal{X}_0) \cong \pi^\text{ét,\ell}_1(\mathcal{X}_K) \), hence we may view \( \rho \) as a map \( \pi^\text{ét,\ell}_1(\mathcal{X}_K) \to \Gamma_\ell' \). By Proposition 1.12 and Corollary 1.13 there exists a unique morphism \( \phi': \mathcal{X}_K \to C' \) with \( \phi'_* \mathcal{O}_{\mathcal{X}_K} = \mathcal{O}_C \) to a smooth projective curve \( C' \) over \( K \) of genus \( g(C') \geq 2 \) such that

\[ \rho^* H^1(\Gamma_\ell', \mathbb{Z}_{et}) \subseteq \phi'^* H^1_\text{ét}(C', \mathbb{Z}_\ell). \tag{2.1} \]

There exists a finite extension \( K' \) of \( K \) and a smooth projective curve \( C' \) over \( K' \) such that \( C'_K \cong C' \). Extending \( K' \) if necessary, we may assume that \( C' \) has a rational point and that the morphism \( \phi': \mathcal{X}_K \to C' \) is defined over \( K' \). We then have a \( \text{Gal}(\bar{K}/K') \)-equivariant surjection

\[ \phi'_*: \pi^\text{ét,\ell}_1(\mathcal{X}_K) \to \pi^\text{ét,\ell}_1(\bar{C}'). \tag{2.2} \]
Let $R'$ be the localisation of the integral closure of $R$ in $K'$ at any prime above $m$. Then the $\Gal(K/K')$-action on $\pi_1^{\text{ét, f}}(X_K)$ is unramified since $X_K$, has good reduction, hence by the surjection (2.2) the same is true for the $\Gal(K/K')$-action on $\pi_1^{\text{ét, f}}(C')$. By Takayuki Oda’s “Néron–Ogg–Shafarevich for curves” [Oda95, Thm. 3.2]3, this implies that $C'$ has good reduction. Thus, there exists a smooth proper curve $C' \to \Spec R'$ with generic fibre $C'$.

Since $g(C') \geq 1$, the Néron mapping property [BLR, Cor. 4.4.4] for the abelian scheme $\text{Alb}_{C'/R}$ implies that the morphism $\phi': X_{K'} \to C'$ extends uniquely to a morphism $\phi': X_R \to C'$. Since formation of $\phi'_*$ commutes with flat base change, we get $\phi'_*\mathcal{O}_{X_{K'}} = \mathcal{O}_{C'}$, hence $\phi'$ has geometrically connected fibres [EGA3, Cor. 4.3.2]. Now consider the Stein factorisation of its special fibre $\phi'_0$:

$$X_0 \xrightarrow{f} \tilde{C}' \xrightarrow{F} C'_0.$$ 

Since $\phi'_0$ has geometrically connected fibres, the finite part $F$ is radicial. Since $C'_0$ and $\tilde{C}'$ are smooth projective curves over an algebraically closed field, this implies $F: C'_0^{(F)} \to C'_0$ is a power $\text{Frob}^\alpha$ of the relative Frobenius. Finally, the specialisation isomorphism [SGA1, Exp. X, Cor. 3.9] and topological invariance of the étale site [SGA1, Exp. IX, Thm. 4.10] give an isomorphism

$$\pi_1^{\text{ét, f}}(C'_0) \cong \pi_1^{\text{ét, f}}(\tilde{C}') .$$

Under this isomorphism and the comparison $\pi_1^{\text{ét, f}}(X_K) \cong \pi_1^{\text{ét, f}}(X_0)$, the maps $f_*: \pi_1^{\text{ét, f}}(X_0) \to \pi_1^{\text{ét, f}}(\tilde{C}')$ and $\phi'_*: \pi_1^{\text{ét, f}}(X_K) \to \pi_1^{\text{ét, f}}(C'_0)$ agree. Translating (2.1) to this notation gives

$$\phi^*H^1_{\text{ét}}(C, \mathbb{Z}_\ell) \subseteq f^*H^1_{\text{ét}}(\tilde{C}', \mathbb{Z}_\ell) ,$$

so Lemma 1.10 (3) $\Rightarrow$ (1) forces $(\phi, C) \cong (f, \tilde{C}')$. 

\[ \square \]

### 3. Line bundles on products of curves

We will give a criterion for a line bundle on a product $\prod C_i$ of curves in characteristic $p > 0$ that implies it cannot be lifted to $\prod C_i$ for any lifts $C_i$ of the curves $C_i$; see Proposition 3.14 below. We will give an example of this obstruction in Section 4. The main definitions are given in Definition 3.6 and Definition 3.9. It is based on the following lemma.

**Lemma 3.1.** Let $S$ be a scheme, and let $X_i \to S$ for $i \in \{1, \ldots, r\}$ be flat proper morphisms of finite presentation for which the Picard functor $\text{Pic}_{X_i/S}$ and the Albanese $\text{Alb}_{X_i/S}$ are representable (as scheme or algebraic space). Write $X \to S$ for the fibre product $X_1 \times_S \cdots \times_S X_r$. Then any choice of sections $\sigma_i$ of $f_i$ induces an isomorphism

$$\text{Pic}_{X/S} \cong \prod_{i=1}^r \text{Pic}_{X_i/S} \times \prod_{i<j} \text{Hom}_S(\text{Alb}_{X_i/S}, \text{Pic}^0_{X_j/S}) .$$

---

3Oda’s paper only states the result over a number field, but the methods work over any DVR. See e.g. [Tam97, Thm. 0.8] for a proof over an arbitrary DVR.
See [FGA, TDTE VI, Thm 3.3[iii]] for the definition and main existence theorem of the Albanese. We include a sketch of the proof of this well-known lemma because we want to refer to the argument later. A detailed discussion can be found in [vDdB18a, §4.4].

**Proof of Lemma (sketch).** When \( r = 2 \) the sections \( \sigma_i \) induce a section

\[
\text{Pic}_{X/S} \to \text{Pic}_{X_1/S} \times \text{Pic}_{X_2/S}
\]

\[
\mathcal{L} \mapsto (\sigma_2^* \mathcal{L}, \sigma_1^* \mathcal{L})
\]

to the natural external tensor product map. The kernel consists of line bundles \( \mathcal{L} \) on \( X \) trivial along the coordinate axes \( \sigma_1 \times X_2 \) and \( X_1 \times \sigma_2 \). The trivialisation along \( X_1 \times \sigma_2 \) (viewed as a rigidificator for the Picard functor \( \text{Pic}_{X_2/S} \)) shows that this data corresponds to a morphism

\[
\phi: X_1 \to \text{Pic}_{X_2/S}.
\]

The trivialisation along \( \sigma_1 \times X_2 \) shows that \( \phi(\sigma_1) = 0 \), hence \( \phi \) lands inside \( \text{Pic}_{X_2/S}^0 \). The Albanese property shows that \( \phi \) factors uniquely through

\[
\text{Alb}_{X_1/S} \to \text{Pic}_{X_2/S}^0.
\]

This proves the result for \( r = 2 \), and the general case follows by induction. \( \square \)

**Remark 3.2.** The choice to use \( \text{Hom}(\text{Alb}_i, \text{Pic}_j^0) = \text{Hom}(\text{Alb}_{X_i/S}, \text{Pic}_{X_j/S}^0) \) instead of the version \( \text{Hom}(\text{Alb}_{ij}, \text{Pic}_0^i) \) with \( i \) and \( j \) swapped is arbitrary. If we use the same sections \( \sigma_i \), then replacing \( \text{Hom}(\text{Alb}_i, \text{Pic}_j^0) \) by \( \text{Hom}(\text{Alb}_j, \text{Pic}_i^0) \) takes the map \( \phi: \text{Alb}_i \to \text{Pic}_j^0 \) to its transpose \( \phi^\top: \text{Alb}_j \to \text{Pic}_i^0 \).

Indeed, by Lemma 3.1 applied to both \( X_i \times S X_j \) and \( \text{Alb}_i \times S \text{Alb}_j \), the Albanese map \( X_i \times S X_j \to \text{Alb}_i \times S \text{Alb}_j \) induces an isomorphism on the \( \text{Hom} \) factor of the lemma (line bundles trivialised along a coordinate cross). Hence, we may reduce to the case of abelian schemes, where it follows from the definition of the transpose.

**Remark 3.3.** The choice of sections \( \sigma_i \) of \( X_i \to S \) does not affect the projection \( \text{Pic}_{X_i/S} \to \prod_{i < j} \text{Hom}(\text{Alb}_{X_i/S}, \text{Pic}_{X_j/S}^0) \). Indeed, we may reduce to the case \( r = 2 \). Then the map \( X_i \to \text{Pic}_{X_j/S}^0 \) is given by \( x_i \mapsto \mathcal{L}_{x_i \times S X_j} \otimes \mathcal{L}_{\sigma_j^{-1} x_i \times S X_j}^{-1} \), which visibly does not depend on \( \sigma_j \).

For dependence on \( \sigma_i \), use Remark 3.2 to swap the roles of \( i \) and \( j \). We can also argue directly: changing \( \sigma_i \) gives maps \( \text{Alb}_{X_i/S} \to \text{Pic}_{X_j/S}^0 \) that differ by at most a translation, so they have to agree since they are morphisms of abelian varieties. (See also [vDdB18a, Lem. 4.4.5] for an alternative point of view and additional details.)

For the rest of this section, we will work in the following setup.

**Setup 3.4.** Let \( k \) be a field, let \( r \in \mathbb{Z}_{>0} \), and let \( C_1, \ldots, C_r \) be smooth projective curves over \( k \) with \( C_i(k) \neq \emptyset \). Let \( X = \prod_i C_i \) be their product. The principal polarisation from the theta divisor induces an isomorphism \( \text{Pic}_{C_i/k}^0 \cong \text{Alb}_{C_i/k} \), and we will denote both by \( J_i \).
Corollary 3.5. The choice of rational points \( c_i \in C_i(k) \) induces an isomorphism

\[
\text{Pic}(X) \cong \prod_{i=1}^{r} \text{Pic}(C_i) \times \prod_{i<j} \text{Hom}_k(J_i, J_j).
\]

The projection \( \text{Pic}(X) \to \prod_{i<j} \text{Hom}_k(J_i, J_j) \) does not depend on the choice of rational points \( c_i \in C_i(k) \).

Proof. Immediate from Lemma 3.1 and Remark 3.3. \( \square \)

Definition 3.6. Given a line bundle \( \mathcal{L} \in \text{Pic}(X) \), we write

\[
\mathcal{L} = \left( \left( \mathcal{L}_i \right)_i, \left( \phi_{ji} \right)_{i<j} \right) \in \prod_{i=1}^{r} \text{Pic}(C_i) \times \prod_{i<j} \text{Hom}(J_i, J_j).
\]

For each \( i \in \{1, \ldots, r\} \), write \( E_i(\mathcal{L}) \subseteq \text{End}^0(J_i) \) for the \( \mathbb{Q} \)-subrng (equivalently, \( \mathbb{Q} \)-subspace) generated by the compositions

\[
\phi_{i_1 \ldots i_m} = \phi_{i_1 i_2} \circ \cdots \circ \phi_{i_{m-1} i_m}
\]

for any \( m \geq 2 \) and \( i_1, \ldots, i_m \in \{1, \ldots, r\} \) with \( i_1 = i_m = i \). Here we write \( \phi_{ij} = \phi_{ji}^{-1} \) if \( i < j \); see Remark 3.2. By Remark 3.3, the \( E_i(\mathcal{L}) \) do not depend on the choice of rational points \( c_i \in C_i(k) \) (but the \( \mathcal{L}_i \) do).

Remark 3.7. The reason we only adjoin \( \phi_{i_1 \ldots i_m} \) for \( m \geq 2 \) and do not include the empty composition \( \phi_{\emptyset} = 1 \) is that \( 1 \) is not preserved under pullback by finite morphisms \( C'_i \to C_i \) (Lemma 3.10), as well as with respect to specialisation (Lemma 3.12). In particular, the \( E_i(\mathcal{L}) \) provide an obstruction for a line bundle \( \mathcal{L} \) to lift to \( \prod C_i \) for any lifts \( C_i \) of \( C_i \) (Proposition 3.14).

Definition 3.9. Let \( C_i \) and \( X \) be as in Setup 3.4, and let \( \mathcal{L} \in \text{Pic}(X) \). Then \( \mathcal{L} \) corresponds to an isogeny factor \( A \) of \( J_i \) if there exists an isogeny factor \( J_i \to C_i \) such that

\[
E_i(\mathcal{L}) = \iota \text{End}^0(A) \pi.
\]

Here, \( \pi \) is a surjective homomorphism and \( \iota \) is an element of \( \text{Hom}^0(A, J_i) \) such that \( \pi \iota = \text{id} \).
Equivalently, $E_i(\mathcal{L}) = p \text{End}^g(J_i)p$ for some idempotent $p \in \text{End}^g(J_i)$. Indeed, isogeny factors as in (3.1) correspond to idempotents $p \in \text{End}^g(J_i)$ by setting $p = \iota \pi$, and under this correspondence we have

$$\iota \text{End}^g(A)\pi = p \text{End}^g(J_i)p.$$ 

If $E_i(\mathcal{L}) = \text{End}^g(J_i)$, then we say that $\mathcal{L}$ generates all endomorphisms of $J_i$. This is a special case of the above, where we take $A = J_i$, or equivalently $p = \text{id}$.

**Lemma 3.10.** Let $C_i$ and $X$ be as in Setup 3.4, and let $C'_i$ and $X'$ satisfy the same assumptions. For each $i$, let $f_i : C'_i \to C_i$ be a finite morphism, and denote their product by $f : X' \to X$. Let $\mathcal{L} \in \text{Pic}(X)$, and let $\mathcal{L}' = f^* \mathcal{L}$. Then for all $i$, we have

$$E_i(\mathcal{L}') = f_i^* E_i(\mathcal{L}) f_{i,*}.$$ 

If $\mathcal{L}$ corresponds to an isogeny factor $A$ of $J_i$, then $\mathcal{L}'$ corresponds to the isogeny factor $A$ of $J'_i$. The converse holds if $g(C'_i) = g(C_i)$.

**Proof.** Let $c'_i \in C'_i(k)$ be rational points, and let $c_i \in C_i(k)$ be their images. Let $\mathcal{L}_i(\mathcal{L}')$ and $\phi_{j,i}(\phi'_{j,i})$ denote the components of $\mathcal{L}(\mathcal{L}')$ as in Definition 3.6, with respect to the sections $c_i$ and $c'_i$. The map $C'_i \to J'_j$ of the proof of Lemma 3.1 factors as $C'_i \to C_i \to J_i \to J'_j$, so on the Albanese we get

$$\phi_{j,i} = f_{j,i}^* \phi_{j,i} f_{i,*}.$$ 

Since $f_{i,*}f_i^* : J_i \to J_i$ is multiplication by $\text{deg}(f_i)$, we deduce that

$$\phi'_{j,i} = \text{deg}(f_i) \cdot f_{i,*} \phi_{j,i} = f_{i,*} \phi_{j,i}.$$ 

Taking $\mathbb{Q}$-vector spaces spanned by these elements proves the first statement.

For the final statements, note that the pair $(\iota, \pi) = (\frac{1}{\text{deg}(J_i)} f_i^*, f_{i,*})$ as in (3.1) realises $J_i$ as an isogeny factor of $J'_i$, and this is an isogeny if $g(C'_i) = g(C_i)$. \hfill $\square$

Next, we look at how the $E_i(\mathcal{L})$ interact with specialisation of endomorphisms.

**Definition 3.11.** If $R$ is a DVR with fraction field $K$ and residue field $k$, $S = \text{Spec} R$ is its spectrum, and $T$ is an $S$-scheme satisfying the valuative criterion of properness, then we get a specialisation map

$$\text{sp} : T(K) \cong T(R) \to T(k).$$

In particular, we may apply this to $T = \text{Pic}(X/S)$ for $X \to S$ a smooth proper $S$-scheme with geometrically integral fibres, or to $T = \text{Hom}_S(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are abelian schemes over $S$. In the latter case, the specialisation map is an injective group homomorphism (see e.g. [vDdB18a, Cor. 4.3.4]).

**Lemma 3.12.** Let $R$ be a DVR, and let $C_i$ be smooth projective geometrically integral curves over $\text{Spec} R$ with sections $\sigma_i$. Let $X$ be their fibre product. Let $\mathcal{L}_K \in \text{Pic}(C_i, K)$, and let $\mathcal{L}_0 \in \text{Pic}(C_i, k)$ be its specialisation. Then for all $i$, we have

$$\text{sp}(E_i(\mathcal{L}_K)) = E_i(\mathcal{L}_0).$$

If $\mathcal{L}_0$ corresponds to an isogeny factor $A_0$ of $\text{Jac} C_{i,n}$, then $\mathcal{L}_K$ corresponds to the isogeny factor $A_K$ of $\text{Jac} C_{i,K}$ for an abelian scheme $\mathcal{A}$ over $R$ whose special fibre $A_0$ is isogenous to $A_0$. 

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Proof. Let \( \mathcal{L}_{i,K} (\mathcal{L}_0) \) and \( \phi_{ji,K} (\phi_{ji,0}) \) denote the components of \( \mathcal{L}_K (\mathcal{L}_0) \) as in Definition 3.6. Since specialisation acts componentwise on the right hand side of Lemma 3.1, we get \( \text{sp}(\phi_{ji,K}) = \phi_{ji,0} \). We deduce that

\[
\text{sp}(\phi_{i_1 \cdots i_m,K}) = \phi_{i_1 \cdots i_m,0}.
\]

Taking \( \mathbb{Q} \)-vector spaces spanned by these elements proves the first statement.

For the final statement, if \( E_i(\mathcal{L}_0) = p \End^\mathbb{Q}(\text{Jac}_{C_{i,o}})p \) for some idempotent \( p \), then \( p = \text{sp}(q) \) for some \( q \in \End^\mathbb{Q}(\text{Jac}_{C_{i,K}}) \). Since specialisation is injective, we conclude that such \( q \) is unique, and that \( q \) is an idempotent as well. Similarly, \( \psi_K \in \End^\mathbb{Q}(\text{Jac}_{C_{i,K}}) \) satisfies \( q\psi_K = \psi_K = \psi_K q \) if and only if \( \psi_0 = \text{sp}(\psi_K) \) satisfies \( p\psi_0 = \psi_0 = \psi_0 p \). Thus, we conclude from the first statement that \( E_i(\mathcal{L}_K) = q \End^\mathbb{Q}(\text{Jac}_{C_{i,K}})q \).

Let \( A_K \) be the isogeny factor corresponding to \( q \). Then \( A_K \) has good reduction by Néron–Ogg–Shafarevich [ST68, Thm. 1], since \( \text{Jac}_{C_{i,K}} \) does. Let \( A \) be the Néron model over \( \text{Spec} \, R \). Let \( (i, \pi) \) correspond to the idempotent \( q \) as in Definition 3.9. By the Néron property of abelian schemes, \( \pi \) extends uniquely to a morphism \( \pi: \text{Pic}_{C_i/R} \to A \). Similarly, if \( n \) is such that \( n\pi \in \Hom(A_K, \text{Jac}_{C_{i,K}}) \), then \( n\pi \) extends uniquely to a morphism \( A \to \text{Pic}_{C_i/R} \), which we also denote \( n\pi \). The uniqueness statement implies that \( \pi_{0\pi 0} = \id \) and \( \pi_0 \pi_0 = p \). Therefore \( p \) corresponds to the reduction \( \mathcal{A}_0 \) of \( A_K \), hence \( \mathcal{A}_0 \) is isogenous to \( \mathcal{A}_0 \).

Remark 3.13. Unlike Lemma 3.10, there is no converse to the final statement of Lemma 3.12. For example, if \( \End^\mathbb{Q}(\text{Jac}_{C_{i,o}}) \) is larger than \( \End^\mathbb{Q}(\text{Jac}_{C_{i,K}}) \) and \( \mathcal{L}_{i,K} \) generates all endomorphisms of \( \text{Jac}_{C_{i,K}} \), then \( \mathcal{L}_{i,0} \) does not generate all endomorphisms of \( \text{Jac}_{C_{i,o}} \) and in fact does not correspond to any isogeny factor.

Proposition 3.14. Let \( C_1, \ldots, C_r \) be smooth projective curves over a field \( k \) of characteristic \( p > 0 \) such that all endomorphisms of \( J_1, \ldots, J_r \) are defined over \( k \). Let \( \mathcal{L} \) be a line bundle on \( X = \prod \, C_i \) that corresponds to a nonzero supersingular isogeny factor \( A \) of \( \End^\mathbb{Q}(J_i) \) for some \( i \) (see Definition 3.9). Then for any DVR \( R \) with residue field \( k \) and any lifts \( C_i \to \text{Spec} \, R \) of \( C_i \), no multiple \( \mathcal{L}^{\otimes m} \) for \( m > 0 \) can be lifted to \( X \).

Proof. Note that \( E_i(\mathcal{L}^{\otimes m}) = E_i(\mathcal{L}) \), so we may take \( m = 1 \). Suppose \( C_i \) are lifts of the \( C_i \) and \( \mathcal{L} \) is a lift of \( \mathcal{L} \). By Lemma 3.12, \( \mathcal{L}_K \) corresponds to a lift \( A_K \) of \( A \) (up to isogeny). From the equality \( \text{sp}(E_i(\mathcal{L}_K)) = E_i(\mathcal{L}_0) \), it follows that specialisation \( \End^\mathbb{Q}(A_K) \to \End^\mathbb{Q}(A) \) is an isomorphism. But \( A \) is supersingular, so by a dimension count it is impossible to lift all its endomorphisms simultaneously (see e.g. [vDdB18a, Cor. 4.3.9]).

4. Generation by Rosati dual elements

In Lemma 4.3 below, we give an example of the situation of Proposition 3.14. The following slightly more technical result is needed to make an example of minimal dimension in Theorem 1. The reader who does not care about such matters may skip the proof; see Remark 4.5.
Theorem 4.1. Let \((A, \phi)\) be a polarised supersingular abelian variety of dimension \(g \geq 2\) over a field \(k\) containing \(\mathbb{F}_p\). Then there exists an element \(x \in \text{End}^g(A)\) such that \(x\) and \(x^\dagger = \phi^{-1} x^\top \phi\) generate \(\text{End}^g(A)\) as \(\mathbb{Q}\)-ring.

Proof. Any supersingular abelian variety over a field containing \(\mathbb{F}_p\) is isogenous to \(E^g\), where \(E\) is a supersingular elliptic curve. Then \(D = \text{End}^g(E)\) is the quaternion algebra over \(\mathbb{Q}\) ramified only at \(p\) and \(\infty\), and \(\text{End}^g(A) \cong M_g(D)\). Moreover, when \(A\) is supersingular, the Rosati involution on \(\text{End}^g(A)\) does not depend on the rational polarisation used [Eke87, Prop. 1.4.2], so we may assume that \(\phi\) is the product polarisation. Then the Rosati involution on \(M_g(D)\) is given by

\[
(-)^\dagger : M_g(D) \to M_g(D)
\]

\[
(a_{ij}) \mapsto (a_{ji}^\dagger),
\]

where \(a^\dagger = \text{Trd}(a) - a\) is the Rosati involution on \(D = \text{End}^g(E)\).

Write \(\mathcal{A}(M_g(D))\) for \(M_g(D)\) viewed as affine space over \(\mathbb{Q}\), and note that the ring operations are given by morphisms of \(\mathbb{Q}\)-varieties. Then the set \(U\) of elements \(x \in \mathcal{A}(M_g(D))\) such that \(x\) and \(x^\dagger\) generate \(M_g(D)\) as \(\mathbb{Q}\)-ring is Zariski open. Indeed, for every subset \(W \subseteq \mathbb{Z}^{2g} \setminus \{e\}\) of size \(4g^2\) of nontrivial words, the locus in \(\mathcal{A}(M_g(D))\) where \(\{w(x, x^\dagger) \mid w \in W\}\) generates \(M_g(D)\) as \(\mathbb{Q}\)-vector space is given by the nonvanishing of a certain \(4g^2 \times 4g^2\) determinant whose coefficients depend on \(x\) through the structure coefficients for multiplication and involution. For each \(W\) this gives an open set where the \(w(x, x^\dagger)\) generate, and \(U\) is the union of these open sets over all sets \(W \subseteq \mathbb{Z}^{2g} \setminus \{e\}\) of size \(4g^2\).

But an open subset \(U \subseteq \mathcal{A} \mathcal{Q}\) has a \(\mathbb{Q}\)-point if and only if it is nonempty, i.e. if and only if it has a \(\mathbb{Q}\)-point. Thus, it suffices to study \(\text{End}^g(A) \otimes \mathbb{Q} \mathbb{Q}\). The algebra \(\text{End}^g(A) \otimes \mathbb{Q} \mathbb{Q}\) is isomorphic to \(M_{2g}(\mathbb{Q})\), with involution \((-)^\dagger\) given by

\[
\begin{pmatrix}
(a_{11} & b_{11} & \cdots & a_{1g} & b_{1g} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
(d_{11} & -b_{11} & \cdots & d_{1g} & -b_{1g} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
(c_{11} & d_{11} & \cdots & c_{1g} & d_{1g} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
(-c_{11} & a_{11} & \cdots & -c_{1g} & a_{1g} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
(a_{g1} & b_{g1} & \cdots & a_{gg} & b_{gg} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
(d_{g1} & -b_{g1} & \cdots & d_{gg} & -b_{gg} \\
\end{pmatrix}
\]

Now consider the matrix

\[
x = \begin{pmatrix}
0 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots \\
1 & 0 & \cdots & 1
\end{pmatrix}.
\]

We want to show that the \(\mathbb{Q}\)-subrng \(B \subseteq M_{2g}(\mathbb{Q})\) generated by \(x\) and \(x^\dagger\) is \(M_{2g}(\mathbb{Q})\). One easily computes

\[
x^{2g-1} = e_{1.2g},
\]

\[
x^{2g-3} = e_{1.2g-2} + e_{2.2g-1} + e_{3.2g},
\]

\[
(x^\dagger)^{2g-3} = (x^{2g-3})^\dagger = - (e_{2g-3.2} + e_{2g.1} + e_{2g-1.4}).
\]
Write \( a = x^{2g-1} \) and \( b = (x^1)^{2g-3} \), which makes sense because \( g \geq 2 \). Then \( ab = -e_{11} \), hence \( bab = -e_{2g,1} \). Thus \( x - bab \) is the rotation matrix \( \rho \) given by \( e_i \mapsto e_{i-1} \) for \( i > 1 \) and \( e_1 \mapsto e_{2g} \). Now the matrices \( \rho^i e_{11} \rho^j \) for various \( a \) and \( b \) give all standard basis vectors \( e_{ij} \), hence the matrix algebra \( M_{2g}(\mathbb{Q}) \) is generated (as \( \mathbb{Q}\text{-rng} \)) by \( x - bab \) and \( ab \). Thus, \( B = M_{2g}(\mathbb{Q}) \). \( \square \)

**Remark 4.2.** The theorem is false for \( g = 1 \). Indeed, for any \( x \in D \), we have \( x^\dagger = \text{Trd}(x) - x \), so in particular \( x \) and \( x^\dagger \) commute. Therefore, the non-commutative algebra \( D \) can never be generated by an element and its Rosati transpose.

Using the theorem, we can construct an example of the situation of Proposition 3.14.

**Lemma 4.3.** Let \( k \) be an extension of \( \mathbb{F}_p \), and let \( C \) be a supersingular curve over \( k \) of genus \( g \geq 2 \). Let \( r \geq 3 \), and set \( C_i = C \) for all \( i \in \{1, \ldots, r\} \). Then there exists a very ample line bundle \( \mathcal{L} \) on \( \prod_i C_i \) such that \( E_1(\mathcal{L}) = \text{End}^g(J_1) \).

**Remark 4.4.** That is, \( \mathcal{L} \) generates all endomorphisms of \( J_1 \) (in the sense of Definition 3.9), hence in particular corresponds to a nonzero supersingular isogeny factor. Thus, Proposition 3.14 implies that no multiple \( \mathcal{L}^{\otimes m} \) for \( m > 0 \) can be lifted to \( \prod C_i \), for any lifts \( C_i \) of \( C \).

**Proof of Lemma.** By Theorem 4.1, there exists \( x \in \text{End}^g(J_1) \) such that \( x \) and \( x^\dagger = x^\top \) generate \( \text{End}^g(J_1) \) as \( \mathbb{Q}\text{-rng} \). Now set \( \phi_{21} = x \), and \( \phi_{31} = \phi_{32} = 1 \). Then the maps

\[
\phi_{121} = \phi_{31}^\top \phi_{32} \phi_{21} : J_1 \rightarrow J_1 \\
\phi_{123} = \phi_{21}^\top \phi_{32} \phi_{31} : J_1 \rightarrow J_1
\]

are given by \( x \) and \( x^\top \) respectively. In Picture 3.8, this corresponds to going around the following loops (where all unmarked arrows are the identity):

\[\xymatrix{\ast & \ast \ar[l] \ar[r] & \ast}
\]

If \( P \in C(k) \) is a rational point, then the line bundle \( \mathcal{O}(P)^{\otimes r} \) is very ample. Hence, for \( d \gg 0 \), the line bundle

\[
\mathcal{L} = \left( (\mathcal{L}_i(dP))_{i < j} \right)
\]

is very ample and satisfies \( E_1(\mathcal{L}) = \text{End}^g(J_1) \). \( \square \)

**Remark 4.5.** For \( r \geq 4 \) we do not need to use Theorem 4.1. Indeed, by Albert’s theorem on generation of separable algebras [Alb44] there exist \( x, y \in \text{End}^g(J_1) \) that generate it as \( \mathbb{Q}\text{-alg} \) (see also [vDdB18a, Thm. 7.2.1] for an elementary geometric proof analogous to our proof of Theorem 4.1 above). Hence the elements \( 1, x, \) and \( y \) generate \( \text{End}^g(J_1) \) as \( \mathbb{Q}\text{-rng} \).

Then we can run the argument of Lemma 4.3 using \( \phi_{42} = x, \phi_{43} = y \), and all other \( \phi_{ij} \) equal to 1. The loops \( \phi_{1421} = x, \phi_{1431} = y \), and \( \phi_{1321} = 1 \) then show \( E_1(\mathcal{L}) = \text{End}^g(J_1) \).
This corresponds to going around the following loops in Picture 3.8 (where again all unmarked arrows are the identity):

\[
\begin{array}{ccc}
  & \bullet & \\
  \downarrow^x & & \downarrow^y \\
  \bullet & & \bullet
\end{array}
\]

5. STABLY IRREDUCIBLE DIVISORS

We introduce the following property that plays a role in the proof of Theorem 1, as explained at the end of the Introduction.

**Definition 5.1.** Let \( k \) be a field, and let \( C_1, \ldots, C_r \) be smooth projective curves over \( k \). Then an effective divisor \( D \subseteq \prod_{i=1}^r C_i \) is stably irreducible if for all finite coverings \( f_i : C'_i \rightarrow C_i \) of the \( C_i \) by smooth projective curves \( C'_i \), the inverse image \( D' \subseteq \prod_{i=1}^r C'_i \) of \( D \) under \( f : \prod_{i=1}^r C'_i \rightarrow \prod_{i=1}^r C_i \) is geometrically irreducible. In particular, \( D \) itself is geometrically irreducible.

We will show that a sufficiently general divisor \( D \) satisfies this property; see Proposition 5.3. The local computation we use is the following.

**Lemma 5.2.** Let \( r \geq 2 \), let \( C_1, \ldots, C_r \) be smooth projective curves over a field \( k \), and let \( D \subseteq \prod_{i=1}^r C_i \) be an ample effective divisor. Assume that all of the following hold:

1. \( D \) is geometrically normal;
2. \( D \cap \pi_i^{-1}(x_i) \) is generically smooth for all \( i \) and all \( x_i \in C_i \);
3. \( D \) does not contain \( \pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j) \) for any closed points \( x_i \in C_i \) and \( x_j \in C_j \) for \( i \neq j \).

Then \( D \) is stably irreducible.

**Proof.** Since all statements are geometric, we may assume \( k \) is algebraically closed. Let \( f_i : C'_i \rightarrow C_i \) be finite coverings by smooth projective curves. If \( f_i \) is purely inseparable, then it is a universal homeomorphism. This does not affect irreducibility, so we only have to treat the case that the \( f_i \) are separable, i.e. generically étale.

The inverse image \( D' = f^{-1}(D) \) is ample since \( D \) is [EGA2, Cor. 6.6.3], hence \( D' \) is connected since \( r \geq 2 \) [Har77, Cor. III.7.9]. Since \( D' \) is a divisor in a regular scheme, it is Cohen–Macaulay [Stacks, Tag 02JN]. We will show that the assumptions on \( D \) imply that \( D' \) is regular in codimension 1. Then Serre’s criterion implies that \( D' \) is normal [EGA4II, Thm. 5.8.6]. Then \( D' \) is integral, since it is normal and connected [EGA4II, 5.13.5].

Now let \( x' \in D' \) be a point of codimension 1, and consider the image \( x'_i \) of \( x' \) in \( C'_i \). Let \( \eta'_i \) be the generic point of \( C'_i \). Let \( x, x_i \), and \( \eta_i \) be the images of \( x' \) in \( X \), of \( x'_i \) in \( C_i \), and of \( \eta'_i \) in \( C_i \) respectively. Consider the set

\[
I = \{ i \in \{1, \ldots, r \} \mid x'_i \neq \eta'_i \} = \{ i \mid x_i \neq \eta_i \}
\]
of \( i \in \{1, \ldots, r \} \) such that \( x' \) does not dominate the factor \( C'_i \), i.e. \( x' \) lies in a closed fibre of the projection \( X' \to C'_i \).

If \(|I| > 2\), then \( \mathfrak{p}^t(x) \in \pi^{-1}(x_i) \cap \pi_j^{-1}(x_j) \cap \pi_k^{-1}(x_k) \) for some \( i, j, k \in \{1, \ldots, r\} \) pairwise distinct, contradicting the fact that \( x' \) has codimension 1 in \( D' \). If \(|I| = 2\), then \( x' \) is the generic point of \( \pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j) \) for \( i \neq j \), hence \( D \) contains \( \pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j) \), contradicting assumption (3). Hence, \(|I| \leq 1\).

If \(|I| = 0\), then \( x \) maps to \( \eta_i \) for each \( i \), hence \( \mathcal{O}_{D,x} \) contains the fields \( \kappa(\eta_i) \) for all \( i \). Since \( f_i \) is separable, the field extension \( \mathcal{O}_{C_i,\eta_i} \to \mathcal{O}_{C'_i,\eta'_i} \) is étale. Hence, \( x \) is in the étale locus of \( D' \to D \). But \( \mathcal{O}_{D,x} \) is regular by assumption (1), so the same goes for \( \mathcal{O}_{D',x'} \) [EGA4IV, Prop. 17.5.8].

Finally, if \(|I| = 1\), then \( x \) is the generic point of a component of \( D \cap \pi_i^{-1}(x_i) \) for some \( i \), and similarly for \( x' \). As in the case \(|I| = 0\), the extensions \( C'_i \to C_j \) for \( j \neq i \) do not affect normality at \( x \), so we may assume that \( C'_i = C_j \) for \( j \neq i \).

Then the natural map \( D' \cap \pi_i^{-1}(x'_i) \to D \cap \pi_i^{-1}(x_i) \) is an isomorphism, since \( \pi_i^{-1}(x'_i) = \prod_{j \neq i} C'_j \to \prod_{j \neq i} C_j = \pi_i^{-1}(x_i) \).

Consider the local homomorphism \( \mathcal{O}_{C'_i,x'_i} \to \mathcal{O}_{D',x'} \). It is flat, since (3) implies that every irreducible component of \( D' \) dominates \( C'_i \). Moreover, the fibre \( \mathcal{O}_{D',x'}/m_{x'}\mathcal{O}_{D',x'} \) is a field, since \( D' \cap \pi_i^{-1}(x'_i) = D \cap \pi_i^{-1}(x_i) \) is generically smooth by assumption (2). Since \( \mathcal{O}_{C'_i,x'_i} \) is regular and \( \mathcal{O}_{C'_i,x'_i} \to \mathcal{O}_{D',x'} \) is flat and local, we conclude that \( \mathcal{O}_{D',x'} \) is regular [EGA4II, Prop. 6.5.1(ii)].

**Proposition 5.3.** Let \( r \geq 3 \), and let \( C_1, \ldots, C_r \) be smooth projective curves over \( k \). Let \( H \) be an ample divisor on \( \prod_{i=1}^r C_i \). Then there exists \( n_0 \in \mathbb{Z}_{>0} \) such that for all \( n \geq n_0 \), a general divisor \( D \in [nH] \) is stably irreducible.

**Proof.** There exists \( n_0 \) such that for all \( n \geq n_0 \), the divisor \( nH \) is very ample. By the usual Bertini smoothness theorem, a general \( D \in [nH] \) is smooth, so in particular geometrically normal. Increasing \( n_0 \) if necessary, for a general \( D \) all fibres \( D \cap \pi_i^{-1}(x_i) \) are generically smooth (see e.g. [vDdB18a, Lem. 3.1.2]). Similarly, we may avoid any finite type family of positive-dimensional subvarieties (see e.g. [vDdB18a, Lem. 3.1.3]), so a general \( D \) does not contain any double fibre \( \pi_i^{-1}(x_i) \cap \pi_j^{-1}(x_j) \) (these are positive-dimensional since \( r \geq 3 \)). Then Lemma 5.2 shows that these \( D \) are stably irreducible.

**Remark 5.4.** On the other hand, for \( r \leq 2 \) no effective divisor \( D \subseteq \prod C_i \) is stably irreducible. This is obvious if \( r \leq 1 \) and for \( r = 2 \) if \( D \) is pulled back from either curve. For ‘diagonal’ divisors \( D \subseteq C_1 \times C_2 \), we can first apply a cover to \( C_1 \) to make its degree in \( C_1 \times \text{Spec} \ K(C_2) \) larger than 1. Then it picks up an \( L \)-rational point after a finite extension \( L = K(C'_2) \) of \( K(C_2) \), hence it becomes reducible in \( C'_1 \times C'_2 \).

**Example 5.5.** The conclusion of Proposition 5.3 is not true for all smooth divisors \( D \in [nH] \). For example, let \( r = 3 \), \( C_i = \mathbb{P}^1 \) with coordinates \( [x_i : y_i] \), and let \( D \) be given by \( x_1x_2x_3 - y_1y_2y_3 \in H^0((\mathbb{P}^1)^3, \mathcal{O}(1)\mathbb{P}^1) \). Consider the affine charts associated with inverting one of \( \{x_i, y_i\} \) for each \( i \). Then the local equations are \( xy - z = 1 \) and \( xy - z = 1 \), both of which define a smooth surface.
However, if we take the covers given by $C'_i = \mathbb{P}^1$ with map $C'_i \to C_i$ given by $[x_i : y_i] \mapsto [x_i^2 : y_i^2]$, then $D_i^*$ splits as $V(x_1 x_2 x_3 - y_1 y_2 y_3) \cup V(x_1 x_2 x_3 + y_1 y_2 y_3)$.

So even when $D$ is smooth (in arbitrary characteristic), it is not always stably irreducible. This $D$ violates assumption (3) of Lemma 5.2 because it contains $\pi^{-1}_1([0 : 1]) \cap \pi^{-1}_2([1 : 0])$.

6. Main construction

For every prime $p$, we construct a smooth projective surface $X$ over $\overline{\mathbb{F}}_p$ with the property that no smooth proper variety $Y$ dominating $X$ can be lifted to characteristic 0; see Construction 6.1 and Theorem 6.3 below.

Construction 6.1. Let $p$ be a prime, let $r \geq 3$ be an integer, and let $k$ be an algebraically closed field of characteristic $p$. Let $C_1 = \ldots = C_r = C$ be a supersingular curve over $k$ of genus $g \geq 2$. For example, the Fermat curve $x^{q+1} + y^{q+1} + z^{q+1} = 0$ is supersingular if $q$ is a power of $p$ [SK79, Lem. 2.9]. Alternatively, a smooth member of Moret-Bailly’s family [Mor81; Mor79] is a supersingular of curve genus 2. Both examples are defined over $\overline{\mathbb{F}}_p$.

By Lemma 4.3 (or Remark 4.5 if $r \geq 4$), there exists a very ample line bundle $\mathcal{L}$ on $\prod_i C_i$ that generates all endomorphisms of $J_1$ in the sense of Definition 3.9. Finally, we define $X \subseteq \prod_i C_i$ as a smooth divisor in $|n\mathcal{L}|$ for $n \gg 0$ that is stably irreducible (see Definition 5.1). Such a divisor exists by Proposition 5.3 and the usual Bertini smoothness theorem.

Lemma 6.2. Let $f : X \to Y$ be a finite flat morphism of finite type $k$-schemes. Let $V \subseteq X$ be an integral subscheme, and let $W = f(V)$ be its image. If $f^{-1}(W)$ is irreducible, then $f^* [W] = d \cdot [V]$ for some $d \in \mathbb{Z}_{>0}$.

Proof. Note that $W$ is irreducible since $V$ is. Since specialisations lift along finite morphisms, we have $\dim(V) = \dim(W) = \dim(f^{-1}(W))$. Hence, $V$ is a component of $f^{-1}(W)$. Since $f^{-1}(W)$ is irreducible, we conclude that $V = f^{-1}(W)$ holds set-theoretically. Therefore, $f^*[W]$ is a multiple of $[V]$. \hfill $\square$

Theorem 6.3. Let $X$ be as in Construction 6.1. If $k \subseteq k'$ is a field extension and $Y$ is a smooth proper $k'$-variety with a dominant rational map $Y \dashrightarrow X \times_k k'$, then $Y$ cannot be lifted to characteristic 0.

Remark 6.4. Since $X$ is a divisor in a product of $r \geq 3$ curves, we get examples in every dimension $\geq 2$. Of course, if $X$ is an example of dimension $d$ and $Z$ is any $m$-dimensional smooth projective variety, then $X \times Z$ is an example of dimension $d + m$.

Since curves are unobstructed, the result in dimension 2 is the best possible. If $p \geq 5$, then the Bombieri–Mumford classification [Mum69b; BM77; BM76] together with existing liftability results in the literature [Del81], [Mum69a], [NO80], [Sei88] imply that every smooth projective surface $X$ of Kodaira dimension $\kappa(X) \leq 1$ can be dominated by a liftable surface [vDdB18a, Thm. 6.3.1]. Therefore, our surface of general type is the ‘easiest’ example possible.
Proof of Theorem. We may replace \( k' \) by \( \overline{\mathbb{F}} \) and then replace \( k \) by \( k' \). This does not change the supersingularity of the \( C_i \), the generation of all endomorphisms of \( J_1 \), or the stable irreducibility of \( X \). This reduces us to the case \( k = k' \).

Any rational map \( Y \to C \) to a curve of genus \( \geq 1 \) can be extended to a morphism \( Y \to C_i \) (see e.g. [vDdB18a, Cor. 4.1.4]). In particular, the rational maps \( \phi_i : Y \to X \to C_i \) extend, hence so does the map

\[
\phi : Y \to X \subseteq \prod_{i=1}^r C_i.
\]

Now assume \( R \) is a DVR with residue field \( k \) and \( Y \) is a lift of \( Y \) over Spec \( R \). Consider the Stein factorisation

\[
Y \xrightarrow{\phi_i} C^i \xrightarrow{f_i} C_i \tag{6.1}
\]

of the maps \( \phi_i : Y \to C_i \). By Theorem 2.1, after possibly extending \( R \) there exist smooth proper curves \( C^i \) over \( R \), morphisms \( \phi^i : Y \to C^i \), and commutative diagrams

\[
\begin{array}{ccc}
\phi_i' & \xrightarrow{Y_0} & \phi^i_0 \\
\downarrow & & \downarrow \\
C^i & \xrightarrow{F_i} & C^i_0,
\end{array}
\tag{6.2}
\]

where \( F_i \) is purely inseparable (hence a power of Frobenius). Write \( Z = \prod C_i \), \( Z' = \prod C^i \), and \( Z'' = \prod C^i_0 \). The product over all \( i \) of (6.1) and (6.2) gives the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\phi} & Z' \\
\xrightarrow{f} & & \xrightarrow{F} \\
Z & \xrightarrow{\phi'} & Z''
\end{array}
\]

Then the image \( X' = \phi'(Y) \subseteq Z' \) satisfies \( f(X') = X \subseteq Z \). The preimage \( f^{-1}(X) \subseteq Z' \) is irreducible since \( X \) is stably irreducible by Construction 6.1. Thus, Lemma 6.2 implies that

\[
f^*[X] = a \cdot [X'] \tag{6.3}
\]

for some \( a \in \mathbb{Z}_{>0} \). Since \( F : Z' \to Z'' \) is radicial, another application of Lemma 6.2 shows that the image \( X'' = \phi^i_0(Y) \subseteq Z'' \) satisfies

\[
F^*[X''] = b \cdot [X'] \tag{6.4}
\]

for some \( b \in \mathbb{Z}_{>0} \). Finally, let \( X''_0 = \phi^i_0(Y) \subseteq Z'' \) be the scheme-theoretic image of \( \phi^i_0 : Y \to Z'' \). Then \( X''_0 \) is flat over \( R \) since the image factorisation \( \mathcal{O}_{Z''} \to \mathcal{O}_{X''} \to \phi^i_0 \mathcal{O}_Y \) realises \( \mathcal{O}_{X''} \) as a subsheaf of the \( R \)-torsion-free sheaf \( \phi^i_0 \mathcal{O}_Y \). Hence, \( X''_0 \) is a lift of its special fibre \( X''_0 \) as a divisor. Since \( X''_0 \) agrees set-theoretically with the reduced divisor \( X'' = \phi^i_0(Y) \), we conclude that

\[
[X''_0] = c \cdot [X''] \tag{6.5}
\]

for some \( c \in \mathbb{Z}_{>0} \).
Combining (6.3), (6.4), and (6.5), we conclude that
\[ bc \cdot f^* [X] = a \cdot F^* [X''_0]. \] (6.6)

But \([X] = [L]\) is given by a line bundle \(L\) that generates all endomorphisms of the supersingular abelian variety \(J_1\), by Construction 6.1. Hence, \(f^* [L]\) corresponds to the supersingular isogeny factor \(J_1\) of \(J'_1\) by Lemma 3.10. Hence the same holds for \([X''_0]\) by (6.6) and Lemma 3.10. Finally, Proposition 3.14 then shows that \(O_{Z''}(X''_0)\) does not lift to a line bundle on \(Z''\). This contradicts the fact that \(X''\) is a lift of \(X''\) as a divisor. \(\square\)

**Remark 6.5.** The proofs of Theorem 4.1 and Lemma 4.3 show that the set of \(L \in \text{Pic}(\prod C_i)\) that generate all endomorphisms of \(J_1\) (as in Definition 3.9) form the integral points of the intersection of the ample cone with a Zariski open subset of \(\text{NS}(\prod C_i) \otimes \mathbb{Q}\). Similarly, the set of stably irreducible divisors \(X \in |L \otimes n|\) contains a Zariski open (which is nonempty for \(n \gg 0\)) by the proof of Proposition 5.3. This shows that ‘most’ divisors in \(\prod C_i\) give counterexamples to the main question.

**Remark 6.6.** Our methods do not address the weaker question of dominating \(X\) by a smooth proper variety \(Y\) that admits a formal lift to characteristic \(0\). Similarly, our methods do not answer Bhatt’s question [Bha10, Rmk. 5.5.5] whether every smooth projective variety \(X\) can be dominated by a smooth proper variety \(Y\) that admits a lift to the length 2 Witt vectors \(W_2(k)\).

**References**

[Alb44] A. A. Albert, *Two element generation of a separable algebra*. Bull. Amer. Math. Soc. 50, p. 786–788 (1944).

[And74] M. P. Anderson, *Exactness properties of profinite completion functors*. Topology 13, p. 229–239 (1974).

[Bea88] A. Beauville, *Annulation du \(H^1\) et systèmes paracanoniques sur les surfaces*. J. Reine Angew. Math. 388, p. 149–157 (1988).

[Bha10] B. Bhatt, *Derived direct summands*. PhD thesis, 2010.

[BM76] E. Bombieri and D. Mumford, *Enriques’ classification of surfaces in char. \(p\). III*. Invent. Math. 35, p. 197–232 (1976).

[BM77] E. Bombieri and D. Mumford, *Enriques’ classification of surfaces in char. \(p\). II*. Complex analysis and algebraic geometry, p. 23–42, 1977.

[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 21, 1990.

[Cat91] F. Catanese, *Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations*. Invent. Math. 104.2, p. 119–145 (1991).

[Cha13] F. Charles, *The Tate conjecture for \(K3\) surfaces over finite fields*. Invent. Math. 194.1, p. 119–145 (2013).

[CS08] K. Corlette and C. Simpson, *On the classification of rank-two representations of quasiprojective fundamental groups*. Compos. Math. 144.5, p. 1271–1331 (2008).

[Del81] P. Deligne, *Relèvement des surfaces \(K3\) en caractéristique nulle*. Surfaces algébriques (Orsay, 1976–78), p. 58–79. LNM 868, 1981.

[DI87] P. Deligne and L. Illusie, *Relèvements modulo \(p^2\) et décomposition du complexe de de Rham*. Invent. Math. 89.2, p. 247–270 (1987).

[vDdB18a] R. van Dobben de Bruyn, *Dominating varieties by liftable ones*. PhD thesis, 2018.

[vDdB18b] R. van Dobben de Bruyn, *The equivalence of several conjectures on independence of \(\ell\)*. 2018. arXiv: 1808.00119.
[EGA2] A. Grothendieck, Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. Inst. Hautes Études Sci. Publ. Math. 8 (1961).

[EGA3] A. Grothendieck, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math. 11 (1961).

[EGA4] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. Inst. Hautes Études Sci. Publ. Math. 24 (1965).

[Ega87] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. IV. Inst. Hautes Études Sci. Publ. Math. 32 (1967).

[Eke87] T. Ekedahl, On supersingular curves and abelian varieties. Math. Scand. 60, p. 151–178 (1987).

[FGA] A. Grothendieck, Fonsdements de la géométrie algébrique. Extrait des Séminaire Bourbaki 1957–1962, 1962.

[GL87] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. 90, p. 389–407 (1987).

[Hart77] R. Hartshorne, Algebraic geometry. GTM 52, 1977.

[Hir64] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I. Ann. of Math. (2) 79, p. 109–203 (1964).

[Kol03] J. Kollár, Higher direct images of dualizing sheaves. I. Ann. of Math. (2) 123, p. 11–42 (1986).

[Kol13] J. Kollár, Rationally connected varieties and fundamental groups. Higher dimensional varieties and rational points (Budapest, 2001), p. 69–92. Bolyai Soc. Math. Stud. 12, 2003.

[Lie13] C. Liedtke, Algebraic surfaces in positive characteristic. Birational geometry, rational curves, and arithmetic, p. 229–292. Simons Symp. 2013.

[Mad15] K. Madapusi Pera, The Tate conjecture for K3 surfaces in odd characteristic. Invent. Math. 210, p. 625–668 (2015).

[Mart68] H. H. Martens, Observations on morphisms of closed Riemann surfaces. II. Bull. London Math. Soc. 20, p. 233–254 (1988).

[Mor79] L. Moret-Bailly, Polarisations de degré 4 sur les surfaces abéliennes. C. R. Acad. Sci. Paris Sér. A-B 289, p. A869–A870 (1979).

[Mor81] L. Moret-Bailly, Familles de courbes et de variétés abéliennes sur \( \mathbb{P}^1 \). Astérisque 86, p. 109–140 (1981).

[Mum69a] D. Mumford, Bi-extensions of formal groups. Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), p. 307–322, 1969.

[Mum69b] D. Mumford, Ensembles de variétés projectives, Problèmes de classement de surfaces char. p. I. Global Analysis (Papers in Honor of K. Kodaira), p. 325–339, 1969.

[Nor80] P. Norman and F. Oort, Moduli of abelian varieties. Ann. of Math. (2) 112, p. 413–439 (1980).

[Oda95] T. Oda, A note on ramification of the Galois representation on the fundamental group of an algebraic curve. II. J. Number Theory 53, p. 342–355 (1995).

[Sei88] W. K. Seiler, Deformations of Weierstrass elliptic surfaces. Math. Ann. 281, p. 263–278 (1988).

[Ser61] J.-P. Serre, Ensembles de variétés projectives en caractéristique p non relevables en caractéristique zéro. Proc. Nat. Acad. Sci. U.S.A. 47, p. 108–109 (1961).

[ST68] J.-P. Serre and J. Tate, Good reduction of abelian varieties. Ann. of Math. (2) 88, p. 492–517 (1968).

[SGA1] A. Grothendieck, Séminaire de Géométrie Algébrique du Bois Marie 1960–1966 : Revêtements étalés et groupe fondamental (SGA 1). LNM 224, 1971.

[SK79] T. Shioda and T. Katsura, On Fermat varieties. Tôhoku Math. J. (2) 31, p. 97–115 (1979).
[Sim91] C. T. Simpson, *The ubiquity of variations of Hodge structure*. Complex geometry and Lie theory (Sundance, UT, 1989), p. 329–348. Proc. Sympos. Pure Math. 53, 1991.

[Siu87] Y. T. Siu, *Strong rigidity for Kähler manifolds and the construction of bounded holomorphic functions*. Discrete groups in geometry and analysis, p. 124–151. Progr. Math. 67, 1987.

[Stacks] The Stacks Project Authors, *The Stacks project*, 2005–2019.

[Tam97] A. Tamagawa, *The Grothendieck conjecture for affine curves*. Compos. Math. 109.2, p. 135–194 (1997).