SOME RESULTS ON INHOMOGENEOUS DISCRIMINANTS

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Abstract. We study generalized Horn-Kapranov rational parametrizations of inhomogeneous sparse discriminants from both a theoretical and an algorithmic perspective. We show that all these discriminantal parametrizations are birational, we prove some results on the corresponding implicit equations, and we propose a combinatorial algorithm to compute their degree in the uniform case of (co)dimension 3.

1. Introduction

Given a configuration of \( n \) lattice points \( A \) in \( \mathbb{Z}^{d-1} \), let \( F_A = \sum_{a \in A} x_a t^a \) denote the generic polynomial in \( d-1 \) variables \((t_1, \ldots, t_{d-1})\) with exponents in \( A \). Under certain general conditions, Gelfand, Kapranov and Zelevinsky [15] showed that there exists an irreducible polynomial with integer coefficients \( D_A = D_A(x) \) in the vector of coefficients \( x = (x_a : a \in A) \) (defined up to sign), called the \( A \)-discriminant, which vanishes for each choice of coefficients \( c \) for which \( F_A \) and all its partial derivatives have a common root in the torus \((\mathbb{C}^*)^{d-1}\). The \( A \)-discriminant is an affine invariant, in the sense that any configuration of points affinely isomorphic to \( A \) has the same discriminant.

Given \( A \), we form the \( d \times n \) integer matrix (also called \( A \)) whose first row consists of ones, and whose columns are given by the points \((1, a)\) for all \( a \in A \). The kernel of this matrix expresses the affine dependencies among the given configuration of points. Let \( B = (b_{ij}) \in \mathbb{Z}^{n \times (n-d)} \) be a matrix whose column vectors are a basis of the integer kernel of the matrix \( A \), i.e. a Gale dual of \( A \). \( B \) is of full rank and its maximal minors have g.c.d. \( g_B = 1 \). Since the first row of \( A \) is the vector \((1, \ldots, 1) \in \mathbb{Z}^n \), the row vectors of \( B \) must always add up to 0. We say that \( B \) is regular when this condition is satisfied.

Set \( m = n - d \). The \( A \)-discriminant \( D_A \) is \( A \)-homogeneous, i.e it is quasi-homogenous relative to the weight defined by any vector in the row span of \( A \). Therefore, “taking out these homogeneities” we get a polynomial \( \Delta_B \) in \( m \) variables \( y_1, \ldots, y_m \) which is in fact the implicit equation of (the closure of) the image \( S_B \) of the rational Horn-Kapranov parametrization [15] [20] [12].
defined by

\[ y_j = \prod_{i=1}^{n} (b_{i1}u_1 + \cdots + b_{im}u_m)^{b_{ij}}, \quad j = 1, \ldots, m. \]

Extracting the homogeneities means the following. As we said, there exists \( v \in \mathbb{Z}^d \) such that all monomials \( c^v \) that occur in \( D_A = \sum_{\nu} d_{\nu} x^\nu \) satisfy \( A \cdot \nu = v \). Therefore, for any \( \nu_0 \) such that \( A \cdot \nu_0 = v \),

\[ D_A(x) = x^{\nu_0} \sum_{\nu} d_{\nu} x^{\nu-\nu_0}, \]

where \( d_{\nu} \in \mathbb{Z} \setminus \{0\} \) and \( \nu - \nu_0 \in \ker_{\mathbb{Z}}(A) \). Write each \( \nu - \nu_0 \) as a linear combination of the columns \( v^{(1)}, \ldots, v^{(m)} \) of \( B \). Then, there is a Laurent polynomial \( \Delta_B(y) \) in \( m \) variables such that up to a monomial, \( \Delta_B(x^{v^{(1)}}, \ldots, x^{v^{(m)}}) \) equals \( D_A(x) \). In particular, \( \Delta_B \) has the same number of monomials and the same coefficients as \( D_A \).

Via the well known Cayley trick, the computation of mixed sparse resultants may be reduced to the computation of sparse discriminants [15, 3]. The computation of sparse discriminants \( D_A \) (including all sparse resultants), is then equivalent to the implicitization problem for the parametric varieties given by (1). These implicitization problems are thus interesting and very hard, since they involve the whole of sparse elimination.

It is easy to see that when the matrix \( B \in \mathbb{Z}^{n \times m} \) defining the parametrization (1) is not of maximal rank \( m \), then \( S_B \) is certainly not a hypersurface. This is the case in which the corresponding homogenized discriminant \( D_A \) is just the constant polynomial 1. However, the image of the parametrization may fail to be a hypersurface even if \( B \) is of maximal rank (c.f. Example 2.4). The classification of such defective configurations is a hard combinatorial problem [8, 12, 15]. On the other side, the condition of non defectiveness can be checked algorithmically.

When trying to compute discriminants, natural reductions as in [3] lead to integer matrices \( B \) which are still regular and of maximal rank, but whose columns do not necessarily generate a saturated lattice \( \mathbb{Z}B \), i.e. such that the gcd of its maximal minors takes any value \( g_B \in \mathbb{Z} \setminus \{0\} \). On the other side, sparse discriminants describe the singular locus of \( A \)-hypergeometric systems of PDE's, while the inhomogeneous discriminants \( \Delta_B \) describe the singular locus of classical Horn hypergeometric differential equations [17]. In this setting, it is again natural and necessary to consider matrices \( B \) with arbitrary gcd \( g_B \) [10]. Another interest on studying this case stems in the precise version we give of a result of Kapranov [20], which asserts that the rational hypersurfaces they define have an interesting geometric characterization.

In this article we focus then on rational parametrizations of the form (11) for regular non defective integer matrices \( B \in \mathbb{Z}^{n \times m} \) of maximal rank \( m \) and any value of gcd \( g_B \neq 0 \). To emphasize the fact that the lattice generated
by the columns of the matrix is not necessarily saturated (equivalently, the lattice generated by the rows is not necessarily \( \mathbb{Z}^m \)), we will call our matrix \( C \) and we will keep the name \( B \) in case \( g_B = 1 \). The irreducible equation of the closure of the image \( S_C \), denoted by \( \Delta_C \in \mathbb{Z}[y_1, \ldots, y_m] \), will be referred to as a generalized inhomogeneous discriminant.

We prove in Section 2 that, up to multiplication by an element \( \lambda \in (\mathbb{C}^*)^m \), the parametrizations à la Horn-Kapranov associated to such matrices \( C \) correspond precisely to those rational hypersurfaces \( S_C \) for which the logarithmic Gauss map is birational. In fact, we follow line by line Kapranov’s proof in [20] and we correct slightly his statement in the sense that the condition \( g_C = 1 \) is not necessary. In particular, we show that all such parametrizations are birational, thus recovering and explaining the results in [22, Section 3.3].

We then study in Section 3 the precise relation between the generalized inhomogeneous discriminant \( \Delta_C \) which gives the equation of \( S_C \) and the equation \( \Delta_B \) of \( S_B \), where the columns of \( B \) are a \( \mathbb{Z} \)-basis of the saturated of the lattice \( \mathbb{Z}C \) spanned by the columns of \( C \). We also give other tips to simplify the search for the equation \( \Delta_C \).

Finally, in Section 4 we focus on the case \( m = 3 \). The case of codimension \( m = 2 \) has been studied in detail in [11]. We give a combinatorial algorithm for the computation of the degree of \( S_C \) in case \( C \) is uniform, based on the intersection formula (15) and the algorithm for the computation of local multiplicites at the base points given in [9, Prop. 1.5]. This degree could be also obtained via Theorem 2.2 in [12] which should be made explicit to give a dehomogenized version of Theorem 1.3 in that paper. It would be interesting to develop intersection formulas which do not require to take a common denominator in order to work in projective space, to avoid the new base points coming from this approach. We also give several examples which show the difficulty of computing the local multiplicities in the general case.

2. Generalized inhomogeneous discriminants and birational Gauss maps

We start by setting some notations and first consequences of the definition of the Horn-Kapranov rational parametrizations \( \psi_C \) associated to \( C \)-matrices and their closed images \( S_C \). We then show that these parametrizations are proper (i.e. \( \deg(\psi_C) = 1 \)) and that (torus translates of) varieties of the form \( S_C \) give all rational varieties with birational logarithmic Gauss map.

2.1. The setting. Given a matrix \( C \in \mathbb{Z}^{n \times m} \), \( n \geq m \), we denote by \( C_1, \ldots, C_n \in \mathbb{Z}^m \) the row vectors of \( C \). Each \( C_k \) defines a linear form

\[
l_k(u_1, \ldots, u_m) := \langle C_k, (u_1, \ldots, u_m) \rangle.
\]

Throughout this paper, we will always assume that \( C \) is regular (i.e. its columns sum up to zero), and has no zero rows. We associate to the matrix
C an algebraic variety $S_C$ in $\mathbb{C}^m$ which is the closure of the image of the rational parametrization:

$$\psi_C : \mathbb{C}^m \to \mathbb{C}^m \quad (u_1, \ldots, u_m) \mapsto (y_1, \ldots, y_m),$$

where

$$y_k = \prod_{i=1}^{n} l_i(u_1, \ldots, u_m)^{c_{i,k}} \forall k = 1, \ldots, m. \quad (3)$$

Call $f_0 = \prod_{i=1}^{n} l_i^{\min \{0, c_{i,k} : k = 1, \ldots, m\}}$ the least common denominator of all the $y_k$’s, and write

$$y_k = \frac{f_k}{f_0}, \quad k = 1, \ldots m. \quad (4)$$

Since the coordinates of the parametrization are given as a product of linear forms whose exponents sum up to 0, all $f_i$ have the same degree $d_C$, where

$$d_C = -\sum_{i=1}^{n} \min \{0, c_{i,k} : k = 1, \ldots, m\}. \quad (5)$$

We can think of the mapping $\psi_C$ as a rational function between projective spaces

$$\psi_C : \mathbb{P}^{m-1} \to \mathbb{P}^m, \quad (6)$$

where $\psi_C = (f_0 : f_1 : \cdots : f_m)$ is defined outside the base point locus $Z = V(f_0, \ldots, f_m)$. We again denote by $S_C$ the projective variety defined by the closure of the image of this map.

The linear forms $l_k$ define a hyperplane arrangement in $\mathbb{C}^m$ and in $\mathbb{P}^{m-1}$. Let $F$ be a flat in this arrangement, i.e. a linear space defined by the vanishing of a subset $\{l_{i_1}, \ldots, l_{i_r}\}$ of the given linear forms. Denote by $\mathcal{L}(F)$ all linear forms $l_j$ vanishing on $F$, i.e. all linear forms $l_j$ which lie in the linear span of $\{l_{i_1}, \ldots, l_{i_r}\}$. Note that a given $f_k$ vanishes on $F$ if and only if it contains a linear factor from $\mathcal{L}(F)$. A given flat $F$ of the arrangement will be called basic if all of $f_0, \ldots, f_m$ vanish on $F$. We then have

**Lemma 2.1.** The base point locus $Z$ equals the union of all basic flats.

**Remark 2.2.** By taking out any common factor from $f_0, \ldots, f_m$, we can always assume that $\text{codim}(Z) \geq 2$. Then, in the case of codimension 3 that we will study in Section 4, we can assume without loss of generality that the number of base points is finite. Note however that their local structure can be very complicated. When $m > 3$, the base point locus variety has in general positive dimension.

Recall that $C$ is called defective if $\text{codim}(S_C) > 1$. We easily have

**Lemma 2.3.** Let $C \in \mathbb{Z}^{n \times m}$ be a regular matrix of rank $r < m$. Then $\text{codim}(S_C) > m - r$, so $C$ is defective.
Proof. Assume $C_1, \ldots, C_r$ are linearly independent and write any other $C_j = \sum_{i=1}^r \lambda_{j,k}^i C_k$, $j = r + 1, \ldots, n$. It follows that $y_1, \ldots, y_m$ can be written as homogeneous rational functions of $l_1, \ldots, l_r$ of degree 0, and so they can be written as rational functions of the $r - 1$ variables $l_1/l_r, \ldots, l_{r-1}/l_r$. Therefore, the codimension of $S_C$ is at least $m - (r - 1) = m - r + 1$, as claimed. \hfill $\square$

Example 2.4. As we have remarked in the Introduction, the converse to Lemma 2.3 is not true and the classification of defective matrices involves subtle combinatorial questions. As a simple example of a full rank defective matrix, let $n = 2n'$ be even and $m \leq n'$ arbitrary. Pick any set of integer vectors $C_1, \ldots C_n'$ whose $\mathbb{Q}$-linear span is $\mathbb{Q}^m$ and take $C_{n'+k} = -C_k$ for all $k = 1, \ldots, n'$. Then, it is straightforward to check that $\psi_C$ is a constant map.

We will assume from now on that $C$ has full rank $m$, so that the gcd of its maximal minors $g_C$ is non zero.

2.2. Birationality of $\psi_C$. Given an algebraic group $G$ with Lie algebra $\mathcal{G}$, let $l_g : h \mapsto gh$ denote the left translation by an element $g \in G$. Let $S \subset G$ be an irreducible algebraic hypersurface. The left Gauss map of the hypersurface $S$ is the rational map $\gamma_S : S \rightarrow \mathbb{P}(\mathcal{G}^*)$, taking a smooth point $y \in S$ to the hyperplane $d(l_y^{-1})(T_y S) \subset T_y(G) = \mathcal{G}$. The case which we are interested in is $G = (\mathbb{C}^*)^m$. So $\mathcal{G} = \mathbb{C}^m$, and left translation is the usual coordinatwise multiplication map by an element in the torus. If $S$ is a hypersurface defined by a minimal equation ($\Delta = 0$), then the Gauss map is simply

$$
(7) \quad \gamma(y) = (y_1 \frac{\partial \Delta}{\partial y_1}(y) : \ldots : y_m \frac{\partial \Delta}{\partial y_m}(y)),
$$

mapping a regular point $y \in S$ to a projective point in $\mathbb{P}^{m-1}$. Assume $S^* = S \cap (\mathbb{C}^*)^m$ is non empty. Geometrically, the map $\gamma_S$ corresponds to looking at the image of $S^*$ via the map $\log(y) = (\log(y_1), \ldots, \log(y_m))$ and then considering the Gauss map of $\log(S^*)$.

The following theorem is essentially due to Kapranov [20]. Our contribution lies in making the statement precise by removing the incorrect hypothesis about the gcd of the matrix defining the parametrization. We will show in Section 3 why a quick first thought may lead to conjecture that this gcd should equal 1 (cf. Remark 3.2).

**Theorem 2.5.** Let $S \subset \mathbb{C}^m$ be an algebraic irreducible hypersurface.

The Gauss map $\gamma_S : S \rightarrow \mathbb{P}^{m-1}$ is birational if and only if there exist a non defective and regular integer matrix $C \in \mathbb{Z}^{n \times m}$ of full rank, and a constant $\lambda \in (\mathbb{C}^*)^m$ such that $S = \lambda \cdot S_C$, i.e. $S$ is a torus translate by $\lambda$ of a generalized inhomogeneous discriminant hypersurface.

Moreover, in this case, $\lambda \cdot \psi_C$ is birational and the logarithmic Gauss map $\gamma_S$ is its inverse.
As we said in the Introduction, the proof is exactly Kapranov’s original proof, except that one can check that the hypothesis $g_C = 1$ is not used in the “if” direction and that in the “only if” direction his last statement about the fact that $g_C$ needs to be equal to 1 for $\psi_C$ to be of degree 1, is not true. For the convenience of the reader, we sketch only the “if” direction, i.e. the proof that for any $C$ as in the statement, the logarithmic Gauss map $\gamma_{S_C}$ is indeed a birational inverse to $\psi_C$.

We first give a simple example.

**Example 2.6.** [The discriminant of a generic univariate cubic polynomial]

The $A$-discriminant associated to the $2 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$  

i.e. the discriminant $D_A(x_0, x_1, x_2, x_3)$ of the generic polynomial $F_A = x_0 + x_1t + x_2t^2 + x_3t^3$ equals

$$D_A(x) = -27x_3^2x_0^2 + 18x_3x_0x_2x_1 + x_2^2x_1^2 - 4x_2^3x_0 - 4x_3x_1^3.$$  

Equivalently, $(D_A = 0)$ is the dual variety of the projectively embedded toric variety parametrized by monomials with exponents in $A$, i.e., the well known twisted cubic.

Consider the following choice of matrix $B$:

$$B = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

Note that $g_B = 1$, so that the columns of $B$ are a basis of the integer kernel of $A$. Calling $l_0(u, v) := u + 2v$, $l_1(u, v) := -2u - 3v$, $l_2(u, v) := u$ and $l_3(u, v) := v$, the parametrization $\psi_B$ equals

$$\begin{cases} y_1 := \frac{l_0 l_1}{l_1^2} \\ y_2 := \frac{l_0 l_3}{l_1} \end{cases}.$$  

Its closed image is the hypersurface $S_B = \{\Delta_B = 0\}$, where

$$\Delta_B(y_1, y_2) = -4y_2 - 27y_2^2 + y_1^2 + 18y_2y_1 - 4y_1^3,$$  

which can be computed in MAPLE as

$$-\text{normal}(\text{resultant}(l_1^2 \ast y_1 - l_0 \ast l_2, l_1^3 \ast y_2 - l_0^2 \ast l_3, u) / v^6)$$  

(the factor $v^6$ appears because MAPLE computes an affine resultant instead of an homogeneous one with respect to $(u : v)$). Equivalently, it could be computed as the dehomogenization

$$\Delta_B(y_1, y_2) = D_A(1, 1, y_1, y_2).$$
Conversely, up to a monomial $D_A$ equals $\Delta_B(x_0 x_2/x_1^2, x_0^2 x_3/x_1^3)$. The associated logarithmic Gauss map equals $\gamma_B = (\gamma_1, \gamma_2)$, with

$$\gamma_1 := 12y_1^3 - 2y_1^2 - 18y_1y_2, \quad \gamma_2 := -4y_2 - 54y_2^2 + 18y_1y_2.$$ 

If we compute $(z_1(u, v), z_2(u, v)) := (\gamma_B \circ \psi_B)(u, v)$, we obtain:

$$z_1 = 4(u + 2v)^2 u(u^3 + 9u^2v + 27uv^2 + 27v^3)/(2u + 3v)^6,$$

$$z_2 = 4(u + 2v)^2 v(u^3 + 9u^2v + 27uv^2 + 27v^3)/(2u + 3v)^6.$$ 

Note that we do not recover $(u, v)$ but it only holds that $z_1/z_2 = u/v$, or $(z_1 : z_2) = (u : v)$.

Consider now the following matrix $C$:

$$C = \begin{pmatrix} 1 & 2 \\ 0 & -3 \\ -3 & 0 \\ 2 & 1 \end{pmatrix}.$$ 

Note that $A \cdot C = 0$ but that $g_C = 3$. In fact, $C = B \cdot M$, where $M$ is the square matrix:

$$M = \begin{pmatrix} -3 & 0 \\ 2 & 1 \end{pmatrix}.$$ 

Calling $L_0(u, v) := u + 2v$, $L_1(u, v) := -3v$, $L_2(u, v) := -3u$ and $L_3(u, v) := 2u + v$, the parametrization $\psi_C$ equals

$$\begin{cases} 
  y_1 := \frac{L_0 L_2}{L_3} \\
  y_2 := \frac{L_2 L_3}{L_1} 
\end{cases}.$$ 

Its closed image is the hypersurface $S_C = \{ \Delta_C = 0 \}$, where

$$\Delta_C(y_1, y_2) = -1296y_2y_1^3 - 8748y_2^2y_1^3 - 19683y_2^3y_1^3 + y_2y_1 + 4698y_2^2y_1^2 - 64y_1^3 - 64y_2^3 + 24y_2y_1 + 24y_2y_1^2 - 1296y_2y_1 - 8748y_2y_1^2.$$ 

The corresponding logarithmic Gauss map $\gamma_C(y)$ composed with $\psi_C(u, v)$ gives back $(r(u, v) \cdot u, r(u, v) \cdot v)$, where $r$ denotes the rational function $r(u, v) = -4/729(u + 2v)^3(2u + v)^3(16u^4 + 84uv^3 + 143u^2v^2 + 84v^4 + 16u^4)(v + u)^4/(u^6 v^6)$. We will study in Section 4 the precise relation between the irreducible polynomials $\Delta_B$ and $\Delta_C$.

**Proof of the “if” part in Theorem 2.5** Let $C$ be a regular non defective $n \times m$ integer matrix, a point $\lambda \in (\mathbb{C}^*)^m$ in the torus, and consider the map $\psi'_C := \lambda \psi_C$. We need to show that the logarithmic Gauss map is its birational inverse. Denote by $\Delta$ an irreducible equation of its closed image. The principal observation is that the Jacobian matrix of $\log(\psi'_C)$ is symmetric since

$$\frac{\partial}{\partial u_k} \log((\psi'_C)_{ij}) = \sum_{i=1}^n c_{i,k} c_{i,j}/l_i(u).$$
Moveover, a straightforward computation shows that for any point \( u \) in the preimage of the torus, the Jacobian matrices \( J(\psi'_C) \) and \( J(\log(\psi'_C)) \) have the same rank since \( J(\log(\psi'_C)) = J(\psi'_C) \cdot D \), where \( D \) is the diagonal matrix with diagonal entries the multiplicative inverses of the coordinates of \( \psi'_C \).

This rank is equal to \( m - 1 \) by our hypothesis that \( C \) is non defective. Now, on one side, implicit partial differentiation of the equality \( \Delta(\psi'_C(u)) = 0 \) implies that the vector \( \gamma_C(y) \) lies in the kernel of the transposed Jacobian matrix \( J(\log(\psi'_C))^t \) for any \( y \) in the image of \( \psi'_C \). On the other side, since the coordinates of \( \psi'_C \) are homogeneous forms of degree 0, it follows from Euler’s formula applied to the coordinates of \( \log(\psi'_C) \) that any point \( u \) in the preimage of the torus lies in the kernel of \( J(\log(\psi'_C))(u) \). Then, \( u \) is proportional to \( \gamma_C(\psi'_C(u)) \), when this vector is non zero.

\[ \square \]

3. Monomial changes of coordinates and factorizations

Given any matrix \( C \in \mathbb{Z}^{n \times m} \) with \( g_C \neq 0 \), we proved that the parametrization \( \psi_C \) is birational if \( C \) is non defective (i.e. when \( S_C \) has codimension 1). Indeed, the converse is trivially true. The defectiveness condition can be checked by computing the generic rank of the Jacobian matrix \( J(\psi_C)(u) \). The \((m - 1) \times (m - 1)\) minors of this matrix are rational functions, and with probability 1 at least one of them will be non zero at a generic point \( u \) when \( C \) is non defective. The tropical approach in [12] provides another algorithm to check this property.

It is obvious that given a regular non defective matrix \( C \in \mathbb{Z}^{n \times m} \), we can replace all row vectors in \( C \) lying in the same one-dimensional flat \( F \) by their sum, without essentially changing the coordinates of the parametrization \( \psi_C \) except for constants (if the sum gives the zero vector, we keep the constants but we don’t keep a zero row). So, we can always start simplifying the problem by making this replacement. Note however that we could change the gcd, so this operation could lead from a Gale dual matrix with gcd equal to 1 to a matrix with arbitrary (non zero) gcd [8].

Assume \( C \) and \( B \) are non-defective \( n \times m \) regular integer matrices of full rank \( m \) such that the lattice \( \mathbb{Z}B \) generated by the columns of \( B \) is the saturated of the lattice \( \mathbb{Z}C \). Then, there exists a square matrix \( M \in \mathbb{Z}^{m \times m} \) with determinant equal to \( \pm g_C \) such that \( C = B \cdot M \), as in Example 2.6. The lattice ideal \( I(\mathbb{Z}C) \) (in \( n \) variables) is radical with \( |g_C| \) primary components, which correspond to torus translates of the toric variety defined by the lattice ideal \( I(\mathbb{Z}B) \) [13]. We will see in Theorem 3.4 how this is reflected in the precise relation between the irreducible \( m \)-variate polynomials \( \Delta_B \) and \( \Delta_C \).

We begin by recalling Lemma 6.3.1 in [7], which shows the relation between the Horn-Kapranov parametrizations associated to two regular integer \( n \times m \) matrices \( C_1, C_2 \) with \( C_1 = C_2 \cdot M \) for a square matrix \( M \).
We associate to \( M = (M_{ij}) \in \mathbb{Z}^{m \times m} \) a linear map \( \Lambda_M : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1} \):

\[
\Lambda_M(u) = (\sum_{j=1}^{m} M_{1,j} u_j : \ldots : \sum_{j=1}^{m} M_{m,j} u_j) = M \cdot u^t ,
\]

and, denoting by \( M^{(1)}, \ldots, M^{(m)} \) the column vectors of \( M \), the (multiplicative) monomial map \( \alpha_M : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m \):

\[
\alpha_M(y) = (\prod_{i=1}^{m} y^{M_{i,1}}, \ldots, \prod_{i=1}^{m} y^{M_{i,m}}) = (y^{M^{(1)}}, \ldots, y^{M^{(m)}}).
\]

For any given \( m \times m \) matrices \( M_1, M_2 \), it clearly holds that \( \Lambda_{M_1} \cdot M_2 = \Lambda_{M_1} \circ \Lambda_{M_2} \) and

\[
\alpha_{M_1} \cdot M_2 = \alpha_{M_2} \circ \alpha_{M_1} .
\]

**Lemma 3.1.** The following diagram

\[
\begin{array}{ccc}
\mathbb{P}^{m-1} & \xrightarrow{\Lambda_M} & \mathbb{P}^{m-1} \\
\psi_{C_1} \downarrow & & \downarrow \psi_{C_2} \\
(\mathbb{C}^*)^m & \xleftarrow{\alpha_M} & (\mathbb{C}^*)^m
\end{array}
\]

is commutative.

The proof of Lemma 3.1 is a straightforward verification.

**Remark 3.2.** Assume \( C_2 \) has \( \text{gcd} \) equal to 1, and call \( C = C_1, B = C_2 \). Then, \( |\text{det}(M)| = g_C \). Suppose that we didn't know Theorem 2.5, but instead we suspected (or proved) that \( \psi_B \) is birational. From the equality \( \psi_C = \alpha_M \circ \psi_B \circ \Lambda_M \), where \( \Lambda_M \) is birational and \( \alpha_M \) is a \( g_C \) to 1 mapping, one is tempted to deduce that \( \psi_C \) is also a \( g_C \) to 1 mapping. But indeed, we have already proved that it is birational. The explanation is given in the next lemma.

**Lemma 3.3.** With definitions and notations as in (8) and (9), the restriction of the map \( \alpha_M \) to the zero set of \( \Delta_{C_2} \) defines by corestriction a birational map

\[
\tilde{\alpha}_M = \alpha_M|_{(\Delta_{C_2} = 0)} : (\Delta_{C_2} = 0) \dashrightarrow (\Delta_{C_1} = 0),
\]

where as before \( C_1 = C_2, M \) and \( C_1, C_2 \) are non defective regular integer matrices.

**Proof.** The image of \( \psi_{C_2} \) is dense in \( (\Delta_{C_2}) = 0 \), and so the image of \( \psi_{C_2} \circ \Lambda_M \) is also dense. For any point \( y \) of the form \( y = \psi_{C_2}(\Lambda_M(u)) \), its image by \( \alpha_M \) equals the point \( \psi_{C_1}(u) \). Therefore, it lies in \( (\Delta_{C_1} = 0) \). Thus, we have a rational map \( \tilde{\alpha}_M : (\Delta_{C_2} = 0) \dashrightarrow (\Delta_{C_1} = 0) \) which has to be 1–1 by Theorem 2.5 applied to both \( \psi_{C_2} \) and \( \psi_{C_1} \).
We now present the relation between \( \Delta_{C_1} \) and \( \Delta_{C_2} \). Note that since the corresponding Horn-Kapranov parametrizations are given by rational forms with rational coefficients, we can assume that these polynomials have integer coefficients and content 1. They are thus defined up to sign, but we will usually omit this sign in the notation.

Given an integer square matrix \( M \) of size \( m \), denote by \( G_M \) the multiplicative group
\[
G_M := \{ \varepsilon \in (\mathbb{C}^*)^m : \alpha_M(\varepsilon) = (1, \ldots, 1) \}
\]
with the induced coordinatewise multiplication. We then have:

**Theorem 3.4.** Let \( C_1, C_2 \) are non defective \( n \times m \) regular integer matrices such that \( C_1 = C_2 M \). There exists \( v \) in the lattice \( \mathbb{Z} M \) spanned by the columns of \( M \) (or equivalently, verifying \( \varepsilon^v = 1 \) for all \( \varepsilon \in G_M \)) such that
\[
\Delta_{C_1} \circ \alpha_M(y) = y^v \prod_{\varepsilon \in G_M} \Delta_{C_2}(\varepsilon \cdot y).
\]

Before giving the proof, we revisit Example 2.6.

**Example 3.5.** [Example 2.6 cont.] With the notations of Example 2.6, the group \( G_M \) consists of those \( (\varepsilon_1, \varepsilon_2) \in (\mathbb{C}^*)^2 \) such that \( \varepsilon_1^3 = 1, \varepsilon_2 = 1 \) and so we have
\[
\Delta_C \left( \frac{y_2^3}{y_1^3}, y_2 \right) = \frac{y_2^3}{y_1^3} \prod_{\varepsilon^3 = 1} \Delta_B(\varepsilon y_1, y_2).
\]

Of course, we can move the factor \( y_1^3 \) to the left hand side to produce an equality in the polynomial ring \( \mathbb{C}[y_1, y_2] \) but this is really an equality over the Laurent polynomial ring \( \mathbb{C}[y_1^\pm 1, y_2^{\pm 1}] \). The exponent vector \( v = (-9, 3) \) equals 3 times the difference of the columns of \( M \).

**Proof.** Given any point \( y \in (\mathbb{C}^*)^m \) in the image of \( \psi_{C_2} \) (thus, in a dense subset of \( (\Delta_{C_2} = 0) \)), using that \( \Lambda_M \) is an isomorphism and Lemma 3.1, we can write \( y = \psi_{C_2}(\Lambda_M(u)) \) for some \( u \in \mathbb{P}^{m-1} \) and so \( \alpha_M(y) \in (\Delta_{C_1} = 0) \). For any \( \varepsilon \in G_M \) we have that \( \alpha_M(y) = \alpha_M(\varepsilon \cdot y) \) and therefore it also holds that \( \Delta_{C_1}(\varepsilon \cdot y) = 0 \). Reciprocally, pick any point \( y \in (\mathbb{C}^*)^m \) such that \( \alpha_M(y) \) lies in the image of \( \psi_{C_1} \). Then, there exists \( u \in \mathbb{P}^{m-1} \) such that \( \alpha_M(\psi_{C_2}(\Lambda_M(u))) = \psi_{C_1}(u) = \alpha_M(y) \). Therefore, there exists \( \varepsilon \in G_M \) such that \( y = \varepsilon^{-1} \psi_{C_2}(\Lambda_M(u)) \), and then \( \Delta_{C_2}(\varepsilon \cdot y) = 0 \). By density and properness arguments, we deduce that that
\[
(\Delta_{C_1} \circ \alpha_M(y) = 0) \cap (\mathbb{C}^*)^m = \bigcup_{\varepsilon \in G_M} (\Delta_{C_2}(\varepsilon \cdot y) = 0) \cap (\mathbb{C}^*)^m
\]

Observe now that, as a consequence of Lemma 3.3, the irreducible polynomials \( \Delta_{C_2}(\varepsilon \cdot y) \) with \( \varepsilon \) varying in \( G_M \), are pairwise coprime. In fact, if \( \Delta_{C_2}(\varepsilon \cdot y) \) is proportional to \( \Delta_{C_2}(\varepsilon' \cdot y) \), with \( \varepsilon, \varepsilon' \in G_M \), then writing \( \delta = \varepsilon' \cdot \varepsilon^{-1} \), we have that a point \( y \in (\Delta_{C_2} = 0) \cap (\mathbb{C}^*)^m \) if and only if the point \( \delta \cdot y \in (\Delta_{C_2} = 0) \cap (\mathbb{C}^*)^m \). Since \( \alpha_M \) is birational, we deduce that
δ is the unit element in \( G_M \), i.e. that \( ε = ε' \). Then, we deduce from the Nullstellensatz that there exist positive integers \( n_ε \), and a unit \( qy^v \) (where \( q \) is a constant and \( v \in \mathbb{Z}^m \)) in the Laurent polynomial ring \( \mathbb{C}[y_1^{±1}, \ldots, y_m^{±1}] \) such that

\[
\Delta_{C_1}(α_M(y)) = qy^v \prod_{ε \in G_M} \Delta_{C_2}(ε \cdot y)^{n_ε}.
\]

Substituting \( y \mapsto δ \cdot y \) for any \( δ \in G_M \) in the above factorization we get

\[
\Delta_{C_1}(α_M(y)) = \Delta_{C_1}(α_M(δ \cdot y)) = qδ^vy^v \prod_{ε \in G_M} \Delta_{C_2}(ε \cdot y)^{n_εδ^{-1}}.
\]

By uniqueness of the irreducible factorization, it follows that all \( n_ε \) are equal to some \( N \in \mathbb{N} \) and that moreover \( δ^v = (1, \ldots, 1) \) for all \( δ \in G_M \).

It is clear that this last property holds whenever \( v \) lies in \( \mathbb{Z}^m \). To prove the converse, assume that \( δ^v = 1 \) for all \( δ \in G_M \). To see that \( v \in \mathbb{Z}^m \) we write \( M \) in its Smith normal form:

\[
M = U \cdot \begin{pmatrix} d_1 & \cdots & 0 \\ 0 & \cdots & d_m \\ \end{pmatrix} \cdot V,
\]

where \( U, V \in \mathbb{Z}^{m \times m} \) are invertible over \( \mathbb{Z} \) and \( d_1 | d_2 | \ldots | d_m \) in \( \mathbb{Z} \). By the composition formula (10), it is enough by formula (10) to prove the result for the factors \( U, V, D \).

Assume that all \( n_ε = N > 1 \) and differentiate both sides of equation (13). Since the Jacobian of \( α_M \) is invertible at any point of the torus, we deduce from the Chain rule and the fact that \( ̂α_M \) is birational, that the Jacobian of \( \Delta_{C_1} \) vanishes along \( (\Delta_{C_1} = 0) \). But this contradicts the irreducibility of \( \Delta_{C_1} \), so \( N \) must equal 1.

Finally, we show that \( q = ±1 \). As \( Δ_{C_1} \) has content 1, we need to show that the product

\[
P(y) := \prod_{ε \in G_M} Δ_{C_2}(ε \cdot y)
\]

has integer coefficients and content 1 too. First, note that if we write \( M \) in its Smith Normal Form (14), it is enough by formula (10) to prove the result for the factors \( U, V, D \). This is obvious for \( U \) and \( V \). We can further decompose \( D \) as a product of diagonal matrices, all of whose diagonal entries are equal to \( 1 \) except for a single entry which is a prime \( p \). So, assume that \( D = (d_{ij}) \) is the diagonal matrix with \( d_{ii} = 1 \) for all \( i = 1, \ldots, m - 1 \), and \( d_{mm} = p \). But then, the coefficients of \( P \) are symmetric polynomials with coefficients in \( \mathbb{Z}[s_1, \ldots, s_p] \), where \( s_1, \ldots, s_p \) are the elementary symmetric functions on the \( p \)-th roots of unity. Since all \( s_i \) equal either \( 1 \) or \( -1 \), we
deduce that $P \in \mathbb{Z}[y_1, \ldots, y_{m-1}][y_m]$. In fact, there exist a polynomial $Q$ with the same coefficients, such that $P(y_1, \ldots, y_m) = Q(y_1, \ldots, y_{m-1}, y_m^p)$. Moreover, for every fixed values of the first $m-1$ coordinates, the roots of $Q$ are the $p$-th powers of the roots of $\Delta_{C_2}$ in the last variable $y_m$, so that one can trace recursively the relation between the coefficients of $Q$ and the coefficients of $\Delta_{C_2}$ (which has content 1) to deduce that the gcd of the coefficients of $Q$ is also equal to 1.

A different argument to prove that $q = \pm 1$ is the following. Assume again that we have a diagonal matrix with all diagonal entries equal to 1, except for a single entry which equals a prime number $p$. Fix a primitive $p$-th root of unity $w$ and take any monomial degree ordering $\prec$. Call $b\gamma$ the leading term of $\Delta_{C_2}(\gamma)$. Then, the leading term of $P$ is $b\gamma w^\gamma [\sum_{i-1}^1 y_i^{q^\gamma}] = \pm 1 b\gamma y^{q\gamma}$. Since $\Delta_{C_2} \in \mathbb{Z}[\gamma]$, then $q \in \mathbb{Q}$. But as all the coefficients of $P$ lie in $\mathbb{Z}(w)$, and $q \in \mathbb{Q}$ we deduce that $P \in \mathbb{Z}[\gamma]$. In fact, $q = 1/s$ where $s = \text{cont}(P)$. Assume $s \neq \pm 1$ and let $a$ be a prime dividing $s$. Suppose first that $a \neq p$. Since $a$ divides the content of $P(\gamma) \in \mathbb{Z}[\gamma]$ we have that $P = 0$ in the extension field $\mathbb{Z}_a(w)$. Therefore, one of the factors of $P$ must be zero. But given that the content of $\Delta_{C_2}$ is $\pm 1$, this cannot happen. If $a = p$, first reduce $\Delta_{C_2}$ mod $p$ and then look at its leading coefficient, which we call $b\gamma$. Then, the coefficient of the monomial $y^{q\gamma}$ in $P$ is not divisible by $p$, a contradiction. \hfill \square

The moral of the factorization provided by Theorem 5.3 is that when trying to compute a generalized discriminant $\Delta_C$ it is possible to compute a “simpler” discriminant $\Delta_B$ where $\mathbb{Z}C \subseteq \mathbb{Z}B$ and then reconstruct $\Delta_C$ from $\Delta_B$. Even if the monomial $y^w$ is in principle unknown, its function is to clear denominators without introducing extra monomial factors. The group $G_M$ can be computed via the Smith Normal Form (14) of $M$. One easier way to recover $\Delta_{C_2}$ from $\Delta_{C_2}$ is the following. Substitute $\gamma = \alpha_{\text{Adj}(M)}(z)$ in (14) and denote $g := g_{C_2}$. The key point is that $\alpha_M \circ \alpha_{\text{Adj}(M)}(z) = z^g$. Furthermore, since we know that the exponent $v$ lies on the lattice spanned by the columns of $M$, we deduce that the specialized product on the left hand side must be a Laurent polynomial in the variables $z_1^2, \ldots, z_m^2$. So, we only need to divide all exponents in the resulting polynomial by $g$ to recover $\Delta_{C_2}(z)$.

In fact, this is also useful when dealing with saturated lattices $\mathbb{Z}B$, i.e. when trying to compute $A$-discriminants. Given a regular $n \times m$ non defective matrix $B$, we can first look for a reduced basis of $\mathbb{Z}B$ using the LLL-algorithm (18), available at any Computer Algebra System, and then put these reduced generators as the columns of a matrix $B'$. Write $B = B'M$ with $\det(M) = 1$. Then $G_M$ consists of the single element $(1, \ldots, 1)$ and so (12) reduces to the equality

$$\Delta_B(y) = y^w \Delta_{B'}(\alpha_{M^{-1}}(y)),$$
which can in fact be easily proved since the homogenizations of both discriminants give the same discriminant $D_A$ (Here, $B$ and $B'$ are Gale duals of $A$). Since the coefficients in the parametrization $\psi_{B'}$ are smaller, the implicit equation $\Delta_{B'}$ can be obtained using standard elimination techniques in cases in which the systems would crash when trying to compute $\Delta_B$ directly.

Here is a simple example.

**Example 3.6.** [Example 2.6 cont.] Recall that for the matrix $B$ we proposed in Example 2.6, the inhomogeneous discriminant had degree 3. Consider now another choice $B'$ of a Gale dual of $A$ whose entries are integers with larger absolute value:

$$B' = \begin{pmatrix} -5 & -3 \\ 13 & 8 \\ -11 & -7 \\ 3 & 2 \end{pmatrix}.$$

If we try to compute the inhomogeneous discriminant $\Delta_{B'}$ using the resultant formula, we get a common factor in the coefficients equal to $3^{88}$. After dividing by this quantity, we recover the following polynomial of degree 16:

$$\Delta_{B'}(y) = -27y_2^{16} + 18y_2^8y_1^5 - 4y_2^5y_1^7 - 4y_2^3y_1^8 + y_1^{10}.$$

Note that in general, the degree of a dehomogenization of a sparse discriminant $D_A$ can be as large as wanted.

**4. The degree of $\Delta_C$ and the computation of local multiplicities**

Let $C \in \mathbb{Z}^{n \times m}$ be, as before, a regular non defective integer matrix with no zero rows. We set $m = 3$ and we assume w.l.o.g. that the variety of base points is finite. Base points $p_F \in \mathbb{Z}$ are indexed by basic flats $F$ as in Section 2. In this section, we concentrate on the algorithmic computation of the degree of the generalized homogeneous variety $S_C$, based on the following well known intersection theory formula \[14\]

\[d_C^2 = \deg(\psi_C) \deg(S_C) + \sum_{F \text{ basic}} e_F,\]

where $e_F$ denotes the Hilbert-Samuel multiplicity of $p_F$ \[14\] \[2\] \[5\].

Since we have an easy formula for $d_C$ in (5) and we know that $\deg(\psi_C) = 1$ by Theorem 2.5, we would need to compute the Hilbert-Samuel multiplicities. In fact, this is a delicate notion and there is no efficient deterministic algorithm for the general case (cf. \[16\] \[21\] for a Gröbner/Standard bases approach). We will start by recalling the definition of multiplicity, together with some known properties. In particular, there is a probabilistic algorithm, which reduces the problem to the tractable case of local complete intersection. We will present however several examples, which show that in general one cannot expect that the base points that occur in discriminant parametrizations are local complete intersections, or even almost local complete intersections with one more generator. It follows that the known
algorithms to find the implicit equation do not work in principle in these cases [1, 4]. One way out would be to compute the Newton polytope of \( \Delta_C \) using the results in [12], and then compute its coefficients via an efficient interpolation.

We show in Proposition 4.3 that when \( C \) is uniform, i.e. when all its maximal minors are non zero, one can use the combinatorial algorithm from [9] to compute the Hilbert-Samuel multiplicities. We then turn things upside-down in Corollary 4.8 to compute the dimension of the local vector space at the origin of \( d \) sparse polynomials in \( d \) variables with generic coefficients.

There are several ways to define the algebraic multiplicity of a base point. Our definition follows [19]. We will state the results for the case of dimension two, but they hold with the obvious changes in any dimension. We refer to [6] for more details on the definitions and examples in this section.

Let \( p = p_F \in \mathbb{Z} \). Consider the Noetherian local ring \( A_p := \mathcal{O}_{\mathbb{P}^2, p} \) and the localized base point locus ideal \( I_p := \langle f_0, f_1, f_2, f_3 \rangle_{A_p} \). The Samuel function of \( A_p \) with respecto to \( I_p \) is defined as:

\[
\chi_{I_p}^{A_p}(r) = l(A_p/I_p^{r+1}) \quad \text{for all } r \in \mathbb{N},
\]

where \( l(\cdot) \) is the length function of \( I_p \) as an \( A_p \)-module, that is, the length of a composition series of the module. Since we are working over the algebraically closed field \( \mathbb{C} \), this length coincides with the vector space dimension \( \dim_{\mathbb{C}}(A_p/I_p^{r+1}) \). The Samuel function is polynomial for large values of \( r \), that is, there is a polynomial \( \text{PS}_{I_p}^{A_p}(X) \) in \( \mathbb{Q}[X] \) (which takes integer values over \( \mathbb{Z} \)) such that we have \( \text{PS}_{I_p}^{A_p}(r) = \chi_{I_p}^{A_p}(r) \) for \( r >> 0 \). Moreover, this polynomial has degree 2 and its leading coefficient is \( e/2! \) with \( e \in \mathbb{N}_0 \).

Then, the local multiplicity \( e_F \) of the base point locus at \( p_F \) is defined as \( e_F := e \), i.e. \( 2! \) times the leading coefficient of the polynomial \( \text{PS}_{I_p}^{A_p} \).

Therefore,

\[
e_F = \lim_{r \to \infty} \frac{\dim_{\mathbb{C}}(A_p/I_p^{r+1}) \cdot 2!}{r^2}.
\]

When the base point \( p = p_F \) is a local complete intersection, i.e., when the ideal \( I_p \) admits two generators, then \( e_F \) is just the vector space dimension \( A_p/I_p \) of the local quotient. This dimension can thus be computed algorithmically via a standard basis computation using a local order \( \prec \), and counting the number of monomials not in \( \text{in}_{\prec}(I_p) \). Even if this is not the case, we always have the following probabilistic approach to compute the local multiplicity.

Consider the ideal \( J_p \) generated by 2 generic linear combinations of the 4 generators:

\[
J_p := 
\langle \sum_{i=0}^{1} v_i^0 f_0 + \sum_{i=0}^{1} v_i^0 f_1 + \sum_{i=0}^{1} v_i^0 f_2 + \sum_{i=0}^{1} v_i^0 f_3 \rangle,
\]

with \( v_i^j \in \mathbb{C} \). Then \( J_p \) is generically a complete intersection inside \( I_p \) (and a reduction ideal of \( I_p \)). Thus, with probability 1, we can compute \( e_F = \dim_{\mathbb{C}}(A_p/J_p) \).
As a corollary, we always have the inequality $e_F \geq \dim_C(A_p/J_p)$, so that in any case
\[
\deg(\Delta_C) \leq d_C^2 - \sum_{F \text{ basic}} \dim_C(A_{p_F}/I_{p_F}).
\]

On the other side when $I$ is a monomial ideal, there exists a combinatorial way of computing this multiplicity, as stated in [9]. We reproduce this result below.

If $p = (1 : 0 : 0)$ is a base point (which we can assume after a translation) and the localized ideal $I_p$ is monomial, we have the following algorithm to compute the Hilbert-Samuel multiplicity $e_p$ at $p$:

**Algorithm 4.1. Computation of Hilbert-Samuel Multiplicities for the monomial case and $m = 3$.**

- Set $x_0 = 1$ and let $\tilde{I}_p$ be the specialization of the ideal $I_p$.
- Compute the convex hull $C$ of the exponents of the bivariate monomials in $\tilde{I}_p$.
- Then: $e_p = 2! \cdot \text{Vol}(\mathbb{N}_0^2 \setminus C)$ equals the normalized volume of the complement $K$ of $C$ in the first orthant.

Note that in the very simple case in which the ideal is both monomial and a complete intersection, generated by $\{x_1^{m_1}, x_2^{m_2}\}$, the local multiplicity $m_1 \times m_2$ equals both the normalized volume of the triangle $K$ with vertices $(0,0), (m_1,0), (0,m_2)$, which is the complement in $\mathbb{N}_0^2$ of the convex hull of the staircase of the ideal, and the number of standard monomials $\{x_1^{k_1}x_2^{k_2} / 0 \leq k_i < m_i, i = 1,2\}$, which is the dimension of the quotient by the ideal.

**Remark 4.2.** The Volume Algorithm 4.1 does not work for general ideals. In fact, it might seem reasonable to expect the same algebraic multiplicity for $I_p$ and for any initial (monomial) ideal in $\prec(I_p)$ with respect to a local order, but this is not in general the case. We illustrate this issue in Example 4.5.

**Proposition 4.3.** Assume $C \in \mathbb{Z}^{n \times 3}$ is uniform. Then $C$ is non defective and each local ideal $I_p$ becomes a monomial ideal modulo a linear change of coordinates. So, the degree of the generalized discriminant surface $S_C$ can be combinatorially computed using formula (15) and Algorithm 4.1.

**Proof.** The fact that $C$ is non defective follows from [12], and more precisely from [8, Section 5]. Since any flag is generated by only two linear forms $l_i, l_j$, after dehomogenizing and localizing, the four generators of each local ring $I_p$ are products of powers of $l_i, l_j$, i.e. they are monomials in two independent variables $l_i, l_j$. □
Example 4.4. We first give a very simple example to illustrate Proposition 4.3. Let $C$ be the uniform matrix:

$$
C = \begin{pmatrix}
2 & 1 & 3 \\
-2 & -1 & -2 \\
1 & 1 & 0 \\
-1 & -1 & -1
\end{pmatrix}.
$$

We read from the parametrization $\psi_C$ that

$$f_1 = l_1^2 l_3, \quad f_2 = l_1 l_2 l_3, \quad f_3 = l_1^3, \quad f_0 = l_2^3 l_4.$$

There are two base points: $p = p_{\{1,4\}} = (-2 : 1 : 1)$ and $p' = p_{\{1,2\}} = (1 : -2 : 0)$. The localized ideals are the following monomial ideals $I_p = \langle l_1^2, l_1^3, l_4 \rangle = \langle l_1, l_4 \rangle$ and $I_{p'} = \langle l_1^2, l_1 l_2, l_1^3, l_2^2 \rangle = \langle l_1^2, l_1 l_2, l_2^2 \rangle$ (in variables $l_1, l_4$ and $l_1, l_2$ and not in the $u$ variables). The first ideal is moreover a complete intersection. It is straightforward in this case to compute the multiplicities: $e_p = 1$ and $e_{p'} = 4$, while the dimension of the local ring at $p'$ equals 3. Thus, $\deg(S_C) = 3^2 - 1 - 4 = 4$. Indeed, in this case $g_C = 1$, so that $C$ is a Gale dual of the matrix $A \in \mathbb{Z}^{1 \times 4}$ with all four entries equal to 1. So, the homogeneous $A$-discriminant $D_A(x) = x_1 + x_2 + x_3 + x_4$. Then, we can obtain $\Delta_B$ by dehomogenizing $D_A$. From the equations $y_1 = \frac{x_1^2 x_3}{x_2 x_4}$, $y_2 = \frac{x_2 + x_3}{x_2 x_4}$, $y_3 = \frac{x_3}{x_2 x_4}$, we get that

$$D_A(x) = x_1 \left(1 + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_4}{x_1}\right) = x_1 \left(1 + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_2 y_3}{y_1^2}\right).$$

Clearing denominators, we get the equation

$$\Delta_C(y) = y_1^2 y_3 + y_1 y_2 + y_1^3 + y_2^2 y_3^2,$$

of degree 4, as predicted. In fact, this same procedure holds for any matrix $B$ of size $(m + 1) \times m$ and $g_B = 1$. The associated discriminant $\Delta_B$ has $(m + 1)$ monomials and all coefficients are equal to 1.

We address now a more complicated example, which will help us illustrate several features.

Example 4.5. Consider the matrix

$$
C = \begin{pmatrix}
1 & -1 & 0 \\
1 & -1 & 1 \\
1 & -1 & 0 \\
-1 & 2 & 0 \\
-1 & 1 & -2 \\
-1 & 0 & 1
\end{pmatrix}.
$$

Observe that the first and the third rows of $C$ are identical. We have:

$$f_0 = l_1^4 l_5, \quad f_1 = l_1^3 l_6^2, \quad f_2 = l_1^2 l_2 l_4^2, \quad f_3 := l_1^2 l_2 l_4 l_6^2 l_6. $$

There are seven base points: $p_{\{1,3,4\}} = (0 : 0 : 1)$, $p_{\{1,2,3,5\}} = (1 : 1 : 0)$, $p_{\{1,6\}} = (1 : 1 : 1)$,
Figure 1. Region corresponding to the ideal $I_p$

Figure 2. Region corresponding to the ideal $\text{in}_\prec(I_p)$

Let’s focus on $p = p\{1,3,4\}$. The local ideal equals $I_p = \langle l_1^4; l_1^3; l_1^2l_4 \rangle$, or changing the name of the variables, $I_p = \langle x_1^4; x_2^3; x_1^2x_2 \rangle$. By the volume formula in Algorithm 4.1 we get $e_p = 10$ (see Figure 1). The dimension of the local quotient by $I_p$ equals 8 < 10.

On the other side, if we write the linear forms $l_i$ in the affine coordinates $(u_1, u_2)$, we look at the generators of $I_p$ in the polynomial ring $\mathbb{C}[u_1, u_2]$. If we consider the local order $\prec = ds$ (with $u_2 \prec u_1$) in SINGULAR, we get the following initial ideal:

\[ \text{in}_\prec(I_p) = \langle u_1^3; u_1^2u_2; u_1u_2^3; u_2^4 \rangle, \]

which, by the same volume algorithm as above, has multiplicity 11 > 10. See Figure 2.

The implicit equation $\Delta_C(y)$ can in this case be easily computed with SINGULAR by Gröbner bases methods and we get (up to sign) the following
polynomial of degree 13:

\[-8y_1^4y_2^2y_3^2 + 39y_1^2y_2y_3^2 + 16y_3^3 + 1000y_1^3y_2^2y_3^2 + 3y_1y_2y_3 + y_1^3
\]
\[y_2^3y_3 + y_3 + 3125y_1^4y_2^2y_3^2 + 27y_1^2y_2 + 16y_1^5y_2y_3^2 - 225y_1^3y_2y_3 -
\]
\[-225y_1^3y_3^2y_2 + 500y_1^3y_2^2y_3 + 160y_1^3y_2^2 + 800y_1y_2y_3^3 + 2y_3^2 - 8y_3 + 80y_3y_1y_2.\]

**Example 4.6.** One could try to reduce the computation of multiplicities to the case of monomial ideals in the following way. Consider for instance the matrix

\[
C := \begin{pmatrix}
1 & 1 & 2 \\
-1 & 0 & 1 \\
0 & 1 & 3 \\
0 & -1 & -2 \\
0 & -1 & -4
\end{pmatrix},
\]

and the base point \( p = p_{(2,3,4)}. \) Calling \( x_1 = l_4(1, u_2, u_2), x_2 = l_5(1, u_1, u_2) \) we have that \( 2l_3 = -(l_4 + l_5), \) so the local ideal \( I_p \) is generated by

\[I_p = \langle x_1 + x_2 \rangle x_1 x_2^3, (x_1 + x_2)^3, x_1^2 x_2^2 \rangle.\]

This is not a monomial ideal in these coordinates, but after the linear change \( x_1 = u + 3v, x_2 = -u + v, \) it becomes a monomial ideal with the following generators obtained after easy algebraic manipulations: \( \langle u^4v, u^6, v^3 \rangle, \) and we can compute its multiplicity \( e_p = 18 \) by means of Algorithm [14]. However, this case is very special and there doesn’t seem to be a general pattern about when such a change of coordinates is possible.

For instance, it is possible to prove that the local ideal at the base point \( p_{\{1,4,5\}} \) of the parametrization \( \psi_C \) associated to the matrix

\[
C = \begin{pmatrix}
1 & 1 & 3 \\
1 & 0 & 2 \\
0 & 1 & 1 \\
-2 & -2 & 0 \\
0 & 0 & -6
\end{pmatrix},
\]

cannot be transformed into a monomial ideal by a linear change of coordinates.

**Example 4.7.** Consider the matrix \( C: \)

\[
C = \begin{pmatrix}
1 & -7 & -6 \\
-1 & 4 & 3 \\
1 & 0 & 4 \\
0 & 1 & -1 \\
-1 & 2 & 0
\end{pmatrix}
\]

and the base point \( p = p_{\{1,2\}} = (-1 : -1 : 1). \) Calling \( x = l_1, y = l_2, \) the local ideal at \( p \) equals \( I_p = \langle x^8, y^5, xy^2, x^7y \rangle, \) which is not an almost complete intersection.
We end with an application to unmixed sparse polynomial systems. Fix an exponent set $A = \{\alpha_1, \ldots, \alpha_r\} \subseteq \mathbb{N}_0^d$, with $r \geq d$, and consider $d$ generic polynomials $F_1, \ldots, F_d$ with exponents in $A$ and coefficients in $\mathbb{C}$:

$$F_i(x) = \sum_{j=1}^r c^j_i x^{\alpha_j}, \quad i = 1, \ldots, d; \quad x = (x_1, \ldots, x_d)$$

with $(c^j_1)_j, \ldots, (c^j_d)_j \in \mathbb{C}^r$ generic. Then, by Bernstein’s theorem, the total number of common roots in the torus $(\mathbb{C}^*)^d$ equals the normalized volume of $A$. Assume moreover that $\alpha_i = \lambda_i e_i$ for all $i = 1, \ldots, d$, i.e. that a monomial which is a pure positive power of each of the variable occurs in the polynomials $F_1, \ldots, F_d$. Using the previous results, we are able to compute geometrically their multiplicity at the origin (see also [15, Chapter 5, § 2.E] for a general version of this result).

**Corollary 4.8.** Let $A = \{\alpha_1, \ldots, \alpha_r\} \subseteq \mathbb{N}_0^d$ such that $\alpha_i = \lambda_i e_i$, $i = 1, \ldots, d$, where $\lambda_i \in \mathbb{N}$. Given generic sparse polynomials $F_1, \ldots, F_d$ with exponents in $A$, their multiplicity at the origin

$$e_0 = \dim_{\mathbb{C}} \left( \mathbb{C}[x_1, \ldots, x_d] \langle F_1, \ldots, F_d \rangle_0 \right)$$

coincides with the normalized volume of the complement $K_A$ in the first orthant of the convex hull of $\{\alpha_1 + \mathbb{R}_{\geq 0}^d\} \cup \ldots \cup \{\alpha_r + \mathbb{R}_{\geq 0}^d\}$.

**Proof.** By our hypothesis about $A$, it follows that the monomial ideal $I$ generated by $\{x^{\alpha_1}, \ldots, x^{\alpha_r}\}$ is supported at the origin $0 \in \mathbb{C}^d$. We identify it with its localization $I_0 = I \mathbb{C}[x_1, \ldots, x_d]_0$. Note that the localized ideal $J_0 := \langle F_1, \ldots, F_d \rangle_0$ is a generic complete intersection inside the zero dimensional ideal $I_0$. By the probabilistic algorithm (in dimension $d$) for the computation of the local multiplicity $e$ of $I_0$, we know that $e$ equals the local multiplicity of the reduction ideal $J_0$ of $I_0$, and then $e = e_0$. Finally, $e$ coincides with the normalized volume of $K_A$ by the corresponding version of Algorithm 4.1 in dimension $d$. \hfill $\blacksquare$

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