AdS Lovelock thermodynamics with pressure

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Abstract

In this work are studied, for Lovelock gravities, the consequences of including the variation of the AdS radius $l$ as a parameter in the space of solutions. The result corresponds to an extended thermodynamics space that includes the pair (volume-pressure), with the pressure determined, universally, by $p \sim l^{-2}$. Some known solutions are studied in this context.

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I. INTRODUCTION

The discovery that accretion processes of a black hole solution can be understood as thermodynamic processes was certainly one of the greatest breakthroughs in theoretical physics. The original derivation by Carter [1], Bardeen [2], Bekenstein [3], Hawking [4], and many others, was based, roughly speaking, on the idea that in-falling matter, in pseudo-adiabatic processes, introduces small perturbations on the black hole that can be expressed as infinitesimal changes in the space of parameters that characterize the black hole solution. Given that the black hole must evolve into another black hole solution, those changes are physically constrained in the form of the first law of the black hole thermodynamics. For instance, in four dimensions, where there are only three parameters to consider, mass $M$, angular momentum $J$ and electric charge $Q$, the first law of the black hole thermodynamics takes the form

$$\delta M = T \delta S + \Omega_H \delta J + \Phi_H \delta Q.$$  \hspace{1cm} (1)

Here $T$ is the Hawking temperature, $S$ the entropy, $\Omega_H$ is the angular velocity and $\Phi_H$ the electric potential. In the original derivation $M$ was considered the energy of the system.

Undoubtedly the most remarkable result of the derivation of the first law of black hole thermodynamics was the recognition that the black entropy is proportional to the black hole area, $\sim r_+^2$ in four dimensions. Although heretofore there is no agreement or clarity about the microstates that give rise to this entropy, in many ways this can be considered the starting point upon later the holographic principle was proposed by t’Hooft and Susskind [5, 6].

After the original derivation was obtained a large number of alternatives derivation arose. Although, these derivations are expected to be connected, in one way or another, but there is not certainty that every known derivation is connected with the rest. See [7] for a discussion. In the case of asymptotically locally AdS one can refer to [8] where different approaches to define conserved charges are discussed.

In the original derivation $M$ was identified with the energy of the black hole. However, one can notice that if this is the case then the work term, $-P \delta v$ is absent in Eq. (1). Obviously to incorporate such a term would require a definition for pressure and volume of a black hole, but that is not all. In [9, 10] was proposed that $M$ should be identified with the enthalpy of the system, $H$, instead of the energy, $E$. See [11, 12] for different discussions on this. Let
us recall that the the first law of \textit{black hole} thermodynamics in terms of the enthalpy would adopt the form

\[ \delta H = \delta M = T \delta S + \Omega h \delta J + \Phi \delta Q + v \delta P, \]

To connect this law with the first law of the black thermodynamics, \( v \) is identified with the \textit{three dimensional volume} surrounded by the horizon \( v = (4/3)\pi r_+^3 \). The pressure of the system, on the other hand, is defined as \( P \sim -\Lambda \), with \( \Lambda \) the cosmological constant. One can notice that the difference between Eqs.\([11,12]\) is irrelevant if \( \Lambda \) is \textit{constant}. The reason for the conceptual shift \( M = E \rightarrow M = H \) is due to both entropy and volume are functions of the horizon radius \( r_+ \), and therefore the directions along \( dv \) and \( ds \) are not independent in a law of the form \( dM = Tds + pdV + \ldots \). Conversely, with the identification of \( M \) with the enthalpy, \( H \), the expression of the first law of thermodynamics in Eq.\([2]\), the since \( dP \) and \( dS \) define independent directions, is perfectly well defined.

Now rest to clarify the physical meaning a change in the cosmological constant. In four dimensional Einstein gravity a modification of the cosmological constant can be interpreted as a change of scale, or constant conformal transformation. To see that, let us consider \( \Lambda = -3l^{-2} \) and the transformation \( l \rightarrow (1 + \sigma)l = l + \delta l \). On can think of \( |\sigma| \ll 1 \) for simplicity. In this case,

\[ -2\Lambda \sqrt{g} d^4x = \frac{6}{l^2} \sqrt{\tilde{g}} d^4x \rightarrow \frac{6}{l^2} (1 - 2\sigma) \sqrt{g} d^4x = \frac{6}{l^2} \sqrt{\tilde{g}} d^4x \]

where \( \tilde{g}_{\mu\nu} = (1 - \sigma)g_{\mu\nu} \), a rigid conformal transformation.

However, it is worth to recall that for an asymptotically (locally) AdS spaces a scale transformation, as Eq.\([3]\), cannot affect the structure of the conformal infinity. In fact, any scale transformation becomes essentially a (global) conformal transformation of the conformal infinity, and therefore can be reabsorbed. This, in the context of the AdS/CFT conjecture, translates into a IR reparametrization of the RG (on the bulk). In layman terms, these considerations imply that the asymptotic charges cannot be altered by a change of scale for locally asymptotically AdS space \( (\Lambda < 0) \). In higher odd dimensions, says \( d \geq 5 \), the presence of a potential conformal anomaly in the \( d-1 \) dimensional asymptotic (conformal) boundary determines that this argument must be revisited. Although no connection has been established, this case coincides with the arise of a vacuum energy for odd \( d \) ALAdS spaces \([13]\).
The situation at the horizon is quite different since this indeed is affected by a change of scale. This implies that not only charges of the horizon are affected but also that an additional term must arise, at the horizon, to compensate the transformation of scale. This becomes manifest in the context of Wald’s approach \[14\] to the black hole thermodynamics as will be shown in the next sections but the same argument is valid for a Hamiltonian approach \[15\]. The additional terms for Einstein gravity in four dimensions \[10\] is given by

\[ v\delta P = \frac{4}{3}\pi r^3_+\delta\left(\frac{1}{L^2}\right). \]

In the next sections this problem will be studied in general terms for asymptotically locally AdS solutions of Lovelock gravity. To do this it will be constructed an extension of the formalism developed in \[16\] to incorporate changes along distance scales. Although for simplicity the computations will be carried out in first order formalism of gravity \[17\], the translation to second order formalism is straightforward.

II. PHASE SPACE AND CHARGES

A. Noether Charges

Let us start by rephrasing the construction of the Noether currents associated with a symmetry. In general the most general infinitesimal transformation of a field \(\phi(x)\) is given

\[ x \to x' = x + \xi \quad \text{and} \quad \phi(x) \to \phi'(x'). \]

Now, the infinitesimal transformation, defined as \(\delta\phi = \phi'(x') - \phi(x)\), can be split into \(\delta\phi = \phi'(x') - \phi(x') + \phi'(x') - \phi(x)\). Here one can recognize the usual function variation \(\delta_0\phi = \phi'(x') - \phi(x')\) and the Lie derivative, \(\phi(x') - \phi(x) = \mathcal{L}_\xi\phi\), along the diffeomorphism defined by \(\xi(x)\).

Any transformation defines a symmetry of an action principle, says

\[ I = \int_{M_d} L(\phi) \]

where \(L\) is \(d\)–form Lagrangian, provided \(L(\phi)\) and \(L(\phi + \delta\phi)\) have the same equations of motion (EOM). This can written formally in terms of the transformations as

\[ \delta L(\phi) = \delta_0 L(\phi) + \mathcal{L}_\xi L(\phi) = d\Psi. \]
where
\[ \delta_0 L(\phi) = \text{EOM}_\phi \delta_0 \phi + d\Theta(\delta_0 \phi, \phi). \]

where EOM\(_\phi\) stands for the equations of motion associated with \( \phi \). Here \( \Theta(\delta_0 \phi, \phi) \) is called the *boundary term* and it is worth to stress that in order to have a proper action principle \( \Theta(\delta_0 \phi, \phi) \) must vanish on the boundary conditions. Finally, it is worth to recall that \( L_\xi L = dI_\xi L \) since \( dL \equiv 0 \). Therefore, for any symmetry transformation is possible to define
\[
d(\Theta + I_\xi L(\phi) - \Psi) = -\text{EOM}_\phi \delta_0 \phi.
\]

(6)

With this in mind one can define the \( n-1 \)-form current,
\[
^*J = \Theta + I_\xi L(\phi) - \Psi
\]

(7)

whose divergence vanishes on shell, *i.e.* \( d^*J \big|_{\text{On Shell}} = 0 \). This is called the Noether current. This implies that at least locally \( ^*J = dQ \). In the next section the exact form of this current will be discussed for Lovelock gravity [18].

The definition of Noether charge \( ^*J \) is just a first step to define a conserved charge. In fact, to compute from Eq. (7) a conserved charge is necessary to impose at least two additional conditions. First, the manifold \( M_d \) must have at least an asymptotic time-like Killing symmetry. For simplicity one can consider a stationary space \( M_d = \mathbb{R} \otimes \Sigma_{d-1} \), where \( \mathbb{R} \) stands for a time direction, but in general it is only necessary that \( \mathbb{R} \otimes \partial \Sigma_\infty \subset \partial M_d \). The second condition is that the transformation of \( \phi \) must be defined by a (Killing) symmetry in the space of solutions, *i.e.*, by a transformation that maps solutions into solutions. In the case of diffeomorphism, where \( \delta \phi = 0 = \delta_0 \phi + L_\xi \phi \), this last condition merely implies that \( \xi \) must be a Killing vector of \( M_d \).

One aspect to consider, in addition, is that it is necessary to have a proper action principle, meaning finite evaluated on any solution that satisfies the boundary conditions, to have proper conserved charges. For asymptotically AdS spaces this implies the need to implement a regularization on the action principle. See for instance [19–22] for different discussions on this.
B. The presymplectic form and charges

In Hamiltonian formalism the generator of diffeomorphisms associated to $x \rightarrow x + \xi$ is given by

$$G(\xi) = \int_{\Sigma} H_{\mu} \xi^\mu + \int_{\partial \Sigma} g(\xi).$$

(8)

$g(\xi)$ is an $n-2$-form whose presence is necessary to construct a proper generator on the phase space of the theory [15]. The Hamiltonian charges come from this definition as the value on-shell for the Killing vector, i.e.,

$$G(\xi)_{\text{on-shell}} = \int_{\Sigma} H_{\mu} \xi^\mu + \int_{\partial \Sigma} g(\xi)_{\text{on-shell}}.$$

In this way, a definition of a (conserved) charge at each boundary of $\partial \Sigma$ arises in terms of $g(\xi)$.

In principle, for a given Killing vector the corresponding Nöther and Hamiltonian charges can differ. Fortunately, it is possible to connect them. To do that here the phase space method, developed in [16], will be used. Let us define the $d-1$-form

$$\Xi = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi),$$

where $\delta_1$ and $\delta_2$ stand for transformations of the form Eq.(4). As mentioned above, for simplicity it will be considered only the stationary case $M_d = \mathbb{R} \times \Sigma_{d-1}$. In addition it must be imposed that $\partial \Sigma_{d-1} = \partial \Sigma_{\infty} \oplus \partial \Sigma_{\mathcal{H}}$ where $\partial \Sigma_{\mathcal{H}}$ is to be connected with existence of a Killing horizon in the manifold. Under these conditions [16],

$$\int_{\Sigma} \Xi = 0,$$

(9)

provided either $\delta_1$, or $\delta_2$, are transformations along the space of solutions, as mentioned above. From Eq.(9) is direct to construct the thermodynamics of a black hole [14]. The case where one of the transformation is a diffeomorphisms with $\delta_0 \phi = \delta_\xi \phi = -L_\xi \phi$, following [14, 16], one obtains

$$\Omega(\phi, \delta_\xi \phi, \delta_\phi) = d \left( \delta(Q) + I_\xi \Theta(\phi, \delta_\phi) \right),$$

(10)

and therefore Eq.(9) implies that

$$\int_{\partial \Sigma_{\infty}} \delta(Q) + I_\xi \Theta(\phi, \delta_\phi) = \int_{\partial \Sigma_{\mathcal{H}}} \delta(Q) + I_\xi \Theta(\phi, \delta_\phi).$$

(11)
Remarkably, by using the result in [16], variation \( \delta \) at any of the boundaries of the generator \( G(\xi) \) in the form

\[
\delta G(\xi)|_{\partial \Sigma} = \int_{\partial \Sigma} \delta g(\xi) = \int_{\partial \Sigma} \delta (Q) + I_\xi \Theta(\phi, \delta \phi), \tag{12}
\]

where \( \partial \Sigma \) stands either for both the asymptotic region or the horizon. In this way, in principle, one can compute (conserved) Hamiltonian charges \( g(\xi) \) by direct integration of Eq.\( (12) \). This must be done such that the boundary conditions on each boundary hold.

In general, one is concerned only in the case where \( \xi \) stands for either time translation, rotations or a linear combination of them may define a Killing horizon on the space. It is direct to notice, see [23] for instance, that for rotations the second term of Eq.\( (12) \) vanish identically. This implies that Nöther charge associated with a rotational symmetry and the Hamiltonian one, that can be cast as the angular momentum, are one and the same.

For the time translation, for stationary spaces, the second term of Eq.\( (12) \) may contribute in general, and therefore the Nöther and Hamiltonian charges may differ in general. This will be discussed in detail in the next section for gravity.

### III. LOVELOCK ACTION PRINCIPLE

In the previous section both Hamiltonian and Noether charges have been identified and related. In this section these results will be applied on Lovelock gravity one of the simplest generalization of General Relativity in higher dimensions \( d > 4 \). The Lovelock Lagrangian is the addition, with arbitrary coefficients \( \{ \tilde{\alpha}_p \} \) of the lower dimensional Euler densities \([17, 24]\). The Lagrangian can be written in first order formalism as

\[
L = \sum_{p=0}^{[(d-1)/2]} \tilde{\alpha}_p R^p e^{d-2p} \tag{13}
\]

where \( [(d-1)/2] \) is the integer part of \( (d-1)/2 \), and

\[
R^p e^{d-2p} = R^{a_1 a_2} \wedge \ldots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \ldots \wedge e^{a_d} \epsilon_{a_1 \ldots a_d}.
\]

The variation of this action principle is given by

\[
\delta_0 L = \sum_{p=0}^{[(d-1)/2]} p \tilde{\alpha}_p d(\delta_0 \omega R^{-1} e^{2n-2p}) + \text{EOM}_e \delta_0 e + \text{EOM}_\omega \delta_0 \omega, \tag{14}
\]
where \[ EOM_e = \sum_{p=0}^{\lceil (d-1)/2 \rceil} \frac{(d-2p)}{2} \tilde{\alpha}_p R^p \varepsilon^{d-2p-1} = 0. \quad (15) \]

On the other hand, \( EOM_{\omega} = 0 \) is satisfied, in general, if \( \omega^{ab} = \omega_0^{ab} \), the Levi-Civita connection, i.e., if \( T^a = de^a + \omega_0^a b \wedge e^b = 0 \) is satisfied. In this work the Chern-Simons case is ignored, see \[ 17 \], where a Levi-Civita connection, though a solution, is not the most general solution to \( EOM_{\omega} \). From now on, the \( \wedge \)-product will be omitted as its presence is self-explanatory on the equations.

**A. Noether Currents and Symplectic Form**

After a straightforward computation the Noether current associated with the diffeomorphisms, defined by \( x \rightarrow x + \xi \), is given by \[ 19 \]

\[
*J = -d \left( I_\xi \omega^{ab} \frac{\partial L}{\partial R^{ab}} \right)
\]

where

\[
\frac{\partial L}{\partial R^{ab}} = \sum_{p=0}^{\lceil (d-1)/2 \rceil} p\tilde{\alpha}_p \varepsilon_{abc_1...c_{d-2}} R^{c_1c_2} \ldots R^{c_{2p-3c_{2p-2}} e_{c_{2p-1}} \ldots e_{c_{d-2}}}.
\]

In the same fashion, the variation of the Hamiltonian generator, Eq.\[ 12 \], is given by

\[
\delta G(\xi)_{\partial \Sigma} = \int_{\partial \Sigma} \delta \left( I_\xi \omega^{ab} \frac{\partial L}{\partial R^{ab}} \right) + I_\xi \left( \delta (\omega^{ab}) \frac{\partial L}{\partial R^{ab}} \right). \quad (17)
\]

**B. The ground states and regularization**

To analyze the asymptotic structure of Lovelock gravity solutions let us consider that \( \tilde{\alpha}_p = 0 \) for \( p > I \) with \( \lceil (n-1)/2 \rceil \geq I \geq 1 \). Now, one can notice that the equations of motion can be written as

\[
G_{a_d} = (R^{a_1a_2} + \kappa_1 e^{a_1} e^{a_2}) \ldots (R^{a_{2l-1}a_{2l}} + \kappa_{2l} e^{a_{2l-1}} e^{a_{2l}}) e^{a_{2l+1}} \ldots e^{a_{d-1}} e_{a_1...a_d} = 0, \quad (18)
\]

where \( \kappa_i \) are a set of constant to be determined from the set \( \tilde{\alpha}_p \). Now, it is straightforward to notice that any space of constant curvature \( \kappa_i \) is a solution. These will be identified as the
ground states of the theory. Now, by introducing the constant curvature ansatz \( R^{ab} = x e^a e^b \), Eq. (18) becomes \( G_{a d} = P_1(x) e^{a_1} \ldots e^{a_{d-1}} \varepsilon_{a_1 \ldots a_d} \) where

\[
P_1(x) = \sum_{p=0}^{l} \tilde{\alpha}_p x^p = (x + \kappa_1) \ldots (x + \kappa_1).
\] (19)

Now, one can notice that, even though \( \forall \tilde{\alpha}_p \in \mathbb{R} \), the coefficients \( \kappa_i \) can be complex numbers. This restricts the potential ground states defined by the coefficients \( \{ \tilde{\alpha}_p \} \). This is a dynamical selection of the ground states. Following this, one can assert that for positive or null \( \kappa_i \), the possible asymptotic behaviors, which must match one of the ground states, are restricted as well. The case of a \( \kappa_i \) negative, which would correspond to a dS ground state, stands apart since there are no asymptotic regions in this case. It is worth mentioning that, as noticed in [25], in certain cases the definition of a ground state can be extended to non-constant curvature spaces.

Now, let us consider the case where \( \kappa_1 = \ldots = \kappa_k = -l^2 \), with \( l \in \mathbb{R} \) and \( \kappa_i \neq -l^2 \) for \( i > k \). This corresponds to the existence of a \( k \)-fold degenerate solution of \( P_1(x) = 0 \). By construction its associate family of solutions satisfies \( \lim_{x \to \partial \Sigma\infty} R^{ab} = -l^2 e^a e^b \). For this implies that Eq. (18) can be written as

\[
\left( R + \frac{e^2}{l^2} \right)^k \left[ \sum_{q=0}^{[(d-1)/2]-k} \tilde{\beta}_q R^q e^{d-2k-2q-1} \right] = 0,
\] (20)

where the \( \tilde{\beta}_q \) are arbitrary coefficients. The original \( \tilde{\alpha}_p \) coefficients, mentioned above, can be written as

\[
\tilde{\alpha}_p = \frac{1}{d-2p} \sum_{j=0}^{[(d-1)/2]-k} l^{2(j-k)} \left[ \begin{array}{c} k \\ p-j \end{array} \right] \tilde{\beta}_j.
\] (21)

It is direct to confirm that in this case the action in principle Eq. (13) diverges as asymptotically becomes proportional to the element of volume of the space. Unfortunately, there are two additional problems presented. The Noether’s charges also diverge. This can be confirmed by direct evaluation of the Eq. (16) or by noticing that the charge becomes proportional to the spatial volume element, \( e^{d-2} |_{\Sigma} \), that diverges as \( x \to \partial \Sigma\infty \), the asymptotic region. On top of that, the boundary conditions are not well defined either. One can notice that the boundary term,

\[
\Theta(\omega, e, \delta \omega) = \delta \omega^{ab} \frac{\partial L}{\partial R^{ab}} = \sum_{p=0}^{[(d-1)/2]} \tilde{p} \tilde{\alpha}_p \delta \omega R^{p-1} e^{d-2p},
\]
asymptotically satisfies that \( \lim_{x \to \partial \Sigma} \delta \omega R^{p-1} e^{d-2p} \approx \delta \omega e^{d-2} \). As previously \( \lim_{x \to \partial \Sigma} e^{d-2} \) diverges, making not obvious the boundary conditions to be imposed. It must be stressed that these three problems are to be solved to attain a holographic dual interpretation.

Those three problems can be solved by the introduction of a regulator in the action principle. This is the underlying principle of holographic renormalization [20], for instance. In [26] for even dimensions, and later extended and improved for odd dimensions in [21, 22], was introduced a different method based on the addition of topological densities that solve the three problems, mentioned above, simultaneously, regularizing the action principle almost univocally. The method in even dimensions is reviewed in appendix A but essentially corresponds to the completion of the Lovelock series by the addition of the corresponding \( 2n \) Euler density with a coupling constant, says \( \alpha_n \), determined from the previous \( n - 1 \) \( \alpha_p \) coefficients.

In the next section the boundary conditions for an internal, meaning the event horizon of a black hole \( \partial \Sigma_H \), will be discussed. In particular, see below, at the horizon is natural to fix \( \omega^{ab} \) as no divergences might come from \( \partial L / \partial R^{ab} \). This condition is equivalent to fix the temperature [27] of the black hole as the surface gravity is defined by the second fundamental form of the horizon, which is the projection of \( \omega \) onto the horizon, see [28]. In the next sections this will be discussed for a static geometry where the relation between fixing \( \omega^{ab} \) and fixing the temperature of the horizon becomes manifest.

**IV. SCALE TRANSFORMATIONS AND A NEW PRESYMPLECTIC FORM**

In this section will be introduced a generalization of the pre-symplectic form that includes global scale transformations of the form

\[
e \rightarrow (1 + \sigma)^{-1} e.
\]

One can notice, by recognizing the presence of quotient \( e/l \) in \( R + (e/l)^2 = 0 \), the ground state considered, and in the Lovelock action, that this transformation is dual to a change background’s radius considered, namely

\[
l \rightarrow l' = (1 + \sigma)l.
\]

Now, one can recall that global rigid dilatation transformations, namely scale transformations, cannot affect the conformal structure of the asymptotic region. Roughly speaking,
$\mathbb{R} \times \Sigma_{\infty}$ must be considered merely a representative of a family of conformal manifolds. This is merely the same idea discussed in four dimensions, mentioned above, but in this case the power that connect radius $-l^2$ and the transformation of the cosmological constant for $d \neq 4$ may be fractional.

Now, by dimensional analysis, one can notice that the coefficients $\{\tilde{\alpha}_p\}$ not only depend on $l$ but they do it with different conformal weights. Therefore, in order to variate along $l \to l + \delta l$, is convenient to make explicit their dependency on $l$ by defining a new set of coefficients $\{\alpha_p\}$ functionally independent of $l$. This can be done as follows. Let $L$ be the unit of length, after fixing $c = \hbar = \kappa_b = 1$. Notice that with these definitions the units of [energy (enthalpy)] = $L^{-1}$, [entropy] = $L^0$, [temperature] = $L^{-1}$ and [force] = $L^{-2}$ in any dimension. The definition of units also yields [volume] = $L^{d-1}$ and [pressure = force/area] = $L^{-d}$ as expected. From this, it is direct to notice that $\tilde{\alpha}_1$, which corresponds to the $d$-dimensional gravitational constant, must have units of $L^{d-2}$ to match the force units. On the other hand, $\tilde{\alpha}_1$ must be independent of $l$, or otherwise the gravitational strength would depend on the scale/radius $l$ of the space. By the same token, one can take $\alpha_1 = \tilde{\alpha}_1$. Now, in order to comply with the units $\hat{\delta} l$, which corresponds to the $d$-dimensional gravitational constant, must have units of $L^{d-2}$ to match the force units. On the other hand, $\alpha_1$ must be independent of $l$, or otherwise the gravitational strength would depend on the scale/length $l$ of the space. By the same token, one can take $\alpha_1 = \tilde{\alpha}_1$. Now, in order to comply with the units $\alpha_p = l^{2p-2} \alpha_p$ where $\alpha_p$ are functionally independent of $l$ and $[\alpha_p] = L^{2-d} \forall p$. With this in mind, the action principle can be written as,

$$L = \frac{[(d-1)/2]}{d-2} \sum_{p=0}^{[(d-1)/2]} l^{2p-2} \alpha_p R^p e^{d-2p} = \frac{[(d-1)/2]}{d-2} \sum_{p=0}^{[(d-1)/2]} \alpha_p \left( \frac{e}{L^0} \right)^{d-2p} . \tag{22}$$

After this analysis of the dependency on $l$ of the coefficients one can construct a pre-symplectic form that incorporates the $l \to l + \delta l$ transformation. This was proposed in [29] for the four dimensional Einstein Hilbert action in similar terms.

Now we can consider the $\hat{\delta}$-variation of the Lovelock action in Eq.(22) including the variation a long $l$. Therefore, the variation on-shell can written as

$$\hat{\delta} L = \sum_p l^{2p-2} p \alpha_p d \left( \hat{\delta} (\omega) R^{p-1} e^{d-2p} \right) + (2p - 2) \alpha_p \hat{\delta} l \left( R^p e^{d-2p} \right) . \tag{23}$$

Now, for notation, let us assume that last term can be written as a total derivative, i.e., $(2p - 2) \alpha_p \hat{\delta} l \left( R^p e^{d-2p} \right) = d \theta_p$ . This is direct to prove, see below, for a static space. Therefore,

$$\hat{\delta} L = \sum_p l^{2p-2} p \alpha_p d \left( \hat{\delta} (\omega) R^{p-1} e^{d-2p} \right) + d \theta_p .$$

Now, using this definition, the improved presymplectic form has the form
\[
\hat{\delta}\Omega(\phi, \delta\phi) - \delta\Omega(\phi, \hat{\delta}\phi) = \int_{\partial\Sigma} \sum_p \left( \hat{\delta}(Q_p) + I_\xi \left( \ell^{2p-2}\hat{\delta}(\omega)R^{p-1}e^{d-2p} + \theta_p \right) \right)
\]  

(24)

where
\[
Q_p = -\ell^{2p-2}p\alpha_p (I_\xi\omega)R^{p-1}e^{d-2p}
\]

is the \(p\)-term of the Noether charge defined in Eq. (16) above.

V. STATIC SOLUTION

Any static solution in Schwarzschild coordinates is given by the vielbein
\[
e^0 = f(r)dt, \quad e^1 = f(r)^{-1}dr \quad \text{and} \quad e^i = r\tilde{e}^i
\]  

(25)

where \(\tilde{e}^i\) is the intrinsic vielbein for a constant curvature transverse section, \(\Omega\), which must be finite and closed. Therefore the intrinsic curvature of the transverse section satisfies \(\tilde{R}^{ij} = \gamma \tilde{e}^i \tilde{e}^j\) with \(\gamma\) a constant. Without loss of generality one can take \(\gamma = \pm 1, 0\).

One can notice, see Eq. (25), that the vielbein has been written such as \(r \to \infty\) defines the asymptotic region \(\mathbb{R} \times \partial\Sigma_\infty\). Conversely, \(f(r)^2 = 0\) defines an event horizon. The spin connections are given by
\[
\omega^{01} = \frac{1}{2} \frac{df}{dr}f(r)^2 dt, \quad \omega^{1i} = f(r)\tilde{e}^i \quad \text{and} \quad \omega^{ij} = \tilde{\omega}^{ij}
\]  

(26)

where \(\tilde{\omega}^{ij}\) is the intrinsic Levi-Civita spin connection defined from \(\tilde{e}^i\). The curvatures are
\[
R^{01} = -\frac{1}{2} \frac{d^2}{dr^2}f(r)^2 dt \wedge dr, \quad R^{0i} = -\frac{1}{2} \frac{df}{dr}f(r)^2 dt \wedge \tilde{e}^i
\]
\[
R^{1i} = -\frac{1}{2} \frac{df}{dr}f(r)^2 f^{-1} dr \tilde{e}^i \quad \text{and} \quad R^{ij} = (\gamma - f(r)^2)\tilde{e}^i \wedge \tilde{e}^j.
\]  

(27)

By using the ansatz in Eq. (25) together with the time like Killing vector \(\xi = \partial_t\), one can show that
\[
R^p e^{2n-2p} = \frac{d^2}{dr^2} \left( (\gamma - f(r)^2)\ell^{p-2n}r^{2n-2p} \right) dt \wedge dr \wedge d\Omega
\]
\[
= -d \left( \frac{df}{dr} \left( (\gamma - f(r)^2)\ell^{p-2n}r^{2n-2p} \right) dt \wedge d\Omega \right)
\]
\[
I_\xi\omega R^{p-1}e^{2n-2p} = \frac{df(r)^2}{dr} \left( \gamma - f(r)^2 \right) \ell^{p-2n}r^{2n-2p} d\Omega
\]  

(28)
\[
\hat{\delta}\omega R^{p-1}e^{2n-2p} = \hat{\delta} \left( \frac{df(r)^2}{dr} \left( \gamma - f(r)^2 \right) \ell^{p-2n}r^{2n-2p} \right) dt \wedge d\Omega
\]
\[
+ (2n - 2p)\hat{\delta}(f(r)^2) \left( (\gamma - f(r)^2)\ell^{p-1}r^{2n-2p} \right) dt \wedge d\Omega
\]
where \( d\Omega = \varepsilon_{i_1 \ldots i_{d-2}} \tilde{e}^{i_1} \wedge \ldots \wedge \tilde{e}^{i_{d-2}} \). Let us define for simplicity \( \Omega = \int d\Omega \) as well.

With these results in mind one can evaluate Eq. (24). First one can notice that for \( \xi = \partial_t \), \( \theta_p \) is given by

\[
I_\xi \theta_p = -(2p - 2) \alpha_p l^{2p-3} \delta l \left[ \frac{d}{dr} \left( \gamma - f(r)^2 \right)^{p-1} r^{2n-2p} \right] d\Omega
\]

As noticed previously the presymplectic form can be separated into two contributions from \( \partial \Sigma_\infty \) and \( \partial \Sigma_H \) that cancel each other. In both surfaces it is satisfied that

\[
\Xi = \int_{\partial \Sigma} \sum_p \alpha_p \left[ -p(2p - 2) \delta l l^{2-3} \left( \frac{d}{dr} f(r)^2 \right) \left( \gamma - f(r)^2 \right)^{p-1} r^{2n-2p} \right. \\
- \left. l^{2p-2} \delta l \left( \frac{d}{dr} f(r)^2 \right) \left( \gamma - f(r)^2 \right)^{p-1} r^{2n-2p} \right] d\Omega.
\]

It is direct to notice that some formal simplifications occur, however the explicit form depends on the boundary considered and its corresponding boundary conditions. Because of that, those simplifications will be carried out only after the boundary conditions are discussed in the next sections.

A. Asymptotic Behavior

Let us go back to the case discussed above where the equations of motion have \( k \)-degenerated ground state of constant curvature \(-l^2\), \( i.e., \) when EOM have the form

\[
\frac{\partial L}{\partial e} = \frac{\partial L_R}{\partial e} = l^{2n-3} \left( R + \frac{e^2}{l^2} \right)^k \left[ \sum_{q=0}^{[(d-1)/2]-k} \beta_q R^q \left( \frac{e}{l} \right)^{d-2k-2q-1} \right] = 0
\]

where the \( \beta_q = l^{2n-2} \beta_q \) are arbitrary coefficients. One can notice that the EOM behaves asymptotically in the branch \( \lim_{x \to \partial \Sigma_\infty} R^{ab} = -l^2 e^{a} e^{b} \) as

\[
\lim_{x \to \partial \Sigma_\infty} \frac{\partial L}{\partial e} \propto l^{2n-3} \left( R + \frac{e^2}{l^2} \right)^k \left( \frac{e}{l} \right)^{d-2k-1}.
\]

This implies that the solutions of this branch must behave asymptotically as

\[
\lim_{r \to \infty} f(r)^2 \sim \gamma + \frac{r^2}{l^2} - \left( \frac{C}{r^{d-2k-1}} \right)^{1/k}
\]
where $C$ is a constant to be determined from the exact solution. Remarkably, and in general, knowing this asymptotic behavior is enough to compute the variation of the asymptotic Nöther charges, Eq. (16). However, as mentioned previously, one still has to concern about regularization of the action principle to obtain the proper Nöther charges.

**B. Even Dimensions**

In even dimensions $d = 2n$ the process of regularization is straightforward, see appendix A. In the case hand, the Killing vector $\xi = \partial_t$ defines the Killing horizon and defines the mass parameter.

$$Q^{2n} (\partial_t) = \int_{\partial \Sigma_\infty} I_\omega \omega^{ab} \frac{\partial L_R}{\partial R^{ab}} = C l^{2k-2} \left[ \sum_{q=0}^{n-1-k} \beta_q (-1)^q \right] \Omega. \quad (34)$$

By identifying $M = Q (\partial_t)$ one can fix $C$ such that

$$M = C l^{2k-2} \left[ \sum_{q=0}^{n-1-k} \beta_q (-1)^q \right] \Omega \leftrightarrow C = l^{2-2k} \frac{M}{\Omega} \left[ \sum_{q=0}^{n-1-k} \beta_q (-1)^q \right]^{-1}. \quad (35)$$

It is direct to check that $[M] = L^{-1}$, as expected, while $[C] = L^{d-2k-1}$.

**C. Odd Dimensions**

The process of regularization in odd dimensions requires to consider boundary terms that cannot be expressed in a closed form, in terms of $R^{ab}$ and $e$. In fact, this requires of the extrinsic curvature contained in the second fundamental form of the boundary. For a discussion see [21, 22]. A review of how the Noether charges are obtained in this case will be skipped and only the final result will be used. The relevant difference of odd dimensions case, namely $d = 2n + 1$, is the presence of an additional term in the Nöther charge corresponding to the vacuum energy. Unfortunately that term is not independent of the (Lovelock) gravitational theory considered. For the static case it can be shown that

$$Q^{2n+1} (\partial_t) = M + E_0 \text{ with } E_0 \sim l^{2(n-1)} \Lambda P, \quad (36)$$

corresponding to the vacuum energy of any (L)AdS [13]. Here $\Lambda P$ stands for a constant, with units $[\Lambda P] = L^{1-2n}$, depending on the theory considered but independent of the AdS
radius $l$. This determines that

$$C = l^{2-2k} \frac{M}{\Omega} \left[ \sum_{q=0}^{n-k} \beta_q (-1)^q \right]^{-1} + \Gamma l^{2n-2k}, \quad (37)$$

with $\Gamma$ a dimensionless constant, $[\Gamma] = L^0$, independent of $l$.

D. Variation along the space of solutions

First, one must stress that the constant $C$ is merely a function of the integration constants and therefore $C$ lacks of any physical meaning by itself. Conversely, the Noether and Hamiltonian charges are the physical meaningful quantities. In this way, $C$ must be defined in terms of $M$ and $l$ to acquire a physical meaning.

One can notice that for the construction of the presymplectic form is necessary to consider the variation along $M$ and $l$, and thus necessary to construct the variation of the conserved charges. In general, for the variation along $M$ the presence of $E_0$ is irrelevant. Conversely, the presence of $E_0$ for the variation along $l$ is quite relevant.

The existence of Eqs. (35,36) is not necessary to compute the variation of the conserved charges, not even $M$ which in this context can be understood as the integral of $\delta M$. However, since Eqs. (35,36) actually fix the dependency of $C$ on $l$ they can be considered shortcuts to compute $\delta f(r)^2$.

E. Hamiltonian Variation

The variation of the Hamiltonian charges Eq. (17) at the asymptotic region is given by

$$\delta g(\partial_t) = \lim_{r \to \infty} \sum_{p=0}^{[(d-1)/2]} \alpha_p \left( l^{2p-2} r^{\delta f(r)^2 (\gamma + f(r)^2)^{p-1} (2n - 2p)} - (2p - 1)(2n - 2p) \right) d\Omega. \quad (38)$$

It is straightforward to notice that the form above can be casted as

$$\delta g(\partial_t) = \frac{\partial g}{\partial M} \delta M + \frac{\partial g}{\partial l} \delta l.$$

To compute each of the contribution one needs to separate the variation of $f(r)^2$ as

$$\delta f(r)^2 = \frac{\partial f(r)^2}{\partial M} \delta M + \frac{\partial f(r)^2}{\partial l} \delta l.$$
where \( f(r)^2 \) is given by Eq.(33). It direct to compute the variation along \( \hat{\delta}M \)

\[
\frac{\partial g}{\partial M} \bigg|_{\partial \Sigma_{\infty}} = \lim_{r \to \infty} \sum_{p=0}^{[(d-1)/2]} \alpha_p \left( l^{2p-2} \frac{p}{2} f^2(r)(\gamma + f(r)^2)^{p-1}(2n - 2p)r^{2n-2p-1} \right) d\Omega
\]

The variation along \( \hat{\delta}l \) requires a careful discussion

\[
\frac{\partial g}{\partial l} \bigg|_{\partial \Sigma_{\infty}} = \lim_{r \to \infty} \sum_{p=0}^{[(d-1)/2]} \alpha_p \left( l^{2p-2} \frac{p}{2} f^2(r)(\gamma + f(r)^2)^{p-1}(2n - 2p)r^{2n-2p-1} \right) - (2p-1)(2n-2p)(\gamma - f(r)^2)^{p-1} \right) d\Omega
\]

After a cumbersome computation, the final result is given by

\[
\frac{\partial g}{\partial l} \bigg|_{\partial \Sigma_{\infty}} = \begin{cases} 
0 & \text{for } d = 2n \\
0 & \text{for } d = 2n + 1 \text{ and } n - k > 2 \\
-2 kl^{2n-1} \Gamma \left[ \sum_{q=0}^{n-k} \beta_q (-1)^q \right] & \text{otherwise}
\end{cases} (39)
\]

Finally this yields

\[
\hat{\delta}G(\partial_t) \bigg|_{\partial \Sigma_{\infty}} = \hat{\delta}M
\]

as expected for \( d = 2n \) or \( d = 2n + 1 \) and \( n - k < 2 \). On the other hand, for the non vanishing case

\[
\hat{\delta}G(\partial_t) \bigg|_{\partial \Sigma_{\infty}} = \hat{\delta}M + kl^{2n} \Gamma \left[ \sum_{q=0}^{n-k} \beta_q (-1)^q \right] \Omega \hat{\delta} \left( \frac{1}{l^2} \right) \hat{\delta}p (41)
\]

The presence of an effective \( v - p \) term from infinity represents a novelty. One can notice that the effective volume

\[
v_\infty \sim l^{2n}
\]

is proportional to the volume defined by the AdS radius \( l \).

\[ \text{F. The Horizon} \]

To address the boundary conditions at the horizon one must reanalyze Eq.(14). Unlike the asymptotic region, at the horizon, given that \( \partial L / \partial R^{ab} \) is finite, the simplest solution is
fixing $\delta \omega |_{\partial \Sigma_H} = 0$ in order to the boundary term $\Theta$ vanish. See Eq. (14). Now, considering the variation along the parameter of the solution in Eq. (25), this is given by
\[
\hat{\delta} \omega^{ab} = \frac{1}{2} \delta^{ab}_{01} \left( \frac{d}{dr} f(r)^2 \right) dt - \delta^{ab}_{0i} \hat{\delta} f(r) e^i = 0 \tag{42}
\]
and thus both $f(r)^2$ and its derivative must be fixed along any trajectory in the space of parameters of solutions. Now, both have different interpretations. To fix the derivative of $f(r)^2$ corresponds to fix the temperature in this case. On the other hand, $\hat{\delta} f^2(r) = 0$ is to be understood as relation between the variations of the horizon radius and the rest the parameters of the solution such that $f(r)^2 = 0$ at the new horizon be preserved. For this to happen one has to notice that $f^2(r)$ evaluated at $r = r_+$ is actually promoting $f(r_+)^2 \to f^2(r_+, M, l, \ldots)$. In this way,
\[
\hat{\delta} f^2(r) = 0 = \frac{\partial}{\partial r_+} f^2(r) \hat{\delta} r_+ + \frac{\partial}{\partial M} f^2(r) \hat{\delta} M + \frac{\partial}{\partial l} f^2(r) \hat{\delta} l + \ldots = 0
\]
This relation must be equivalent to the first law of the black hole thermodynamics defined by Eq. (11). Otherwise, the thermodynamic evolution of the system would be inconsistent by having two different tangent vectors at each point. In fact, $\hat{\delta} f^2(r) = 0$ is the layman approach to obtain the first law of thermodynamics in this case.

Now, Eq. (30) can be evaluated considering $f(r)^2 = 0$ is fixed as well as its derivative has a fixed value. This yields
\[
\hat{\delta} g(\partial_i) \bigg|_{\partial \Sigma_H} = \sum_{p=0}^{[d-1]/2} \alpha_p \left( -l^{2p-2} p \left( \frac{d}{dr} f(r_+)^2 \right) \gamma^p \delta(r_+^{2n-2p}) \right.
+ \left. \left( (2p-2)(2n-2p)(\gamma)^p r_+^{2n-2p-1} l^{2p-3} \delta l \right) d\Omega \right) \tag{43}
\]
In Eq. (43) is straightforward to notice that first part, the component along $dr_+$, can be identified with $T \hat{\delta} S$, where $T = 1/(4\pi) (df(r)^2/dr)_+$, as was introduced in [14] and given, after a cumbersome computation, in this case
\[
T \hat{\delta} S = T \left( \gamma + \frac{r_+^2}{l^2} \right)^{k-1} \left[ \sum_{i=0}^{d-2k-1} \zeta_i \gamma^p r_+^{2(n-k-i-1)} \right] \hat{\delta} r_+ \tag{44}
\]
where $\zeta_i$ are proportional to $\beta_i$ mentioned above in Eq. (32). It is direct to show that in even dimensions this is equivalent to the usual expression in [14, 30]
\[
T \hat{\delta} S = T \hat{\delta} \left( 2\pi \int_{\partial \Sigma_H} \frac{\partial L}{\partial R^{01}} \right) . \tag{45}
\]
In odd dimensions the situation is similar, however the boundary terms, see [21], depends also on the intrinsic curvature and therefore the expression can not be shaped as a pure derivative with respect to the Riemann two-form.

The second term in Eq. (43) corresponds to the generalization of the \( V \hat{\delta} P \) term mentioned above. In this case, however, the connection with the cosmological constant and the volume of the black hole is not direct as for GR in four dimensions. For simplicity this term will be called

\[
\hat{w} \hat{\delta} l = \sum_{p=0}^{[(d-1)/2]} \alpha_p \left( (2p - 2)(2n - 2p)(\gamma)^p r_+^{2n-2p-1} \right) l^{2p-3} \hat{\delta} l
\]

and computed for some relevant case in the next section.

**VI. RELEVANT CASES**

In this section some relevant cases will be discussed.

**A. Einstein in \( d \) dimensions**

Probably the simplest example of the previous construction is GR in \( d > 3 \) dimensions. In this case \( \alpha_0 \) and \( \alpha_1 \) are the only two non null coefficients and they are fixed such that the EOM are given by

\[
\frac{\partial L}{\partial e} = \beta_0 \left( R + \frac{e^2}{l^2} \right) \frac{e^{d-2}}{l^{d-1}} = 0.
\]

The static solution is defined by

\[
f(r)^2 = \gamma + \frac{r^2}{l^2} - \frac{m}{r^{d-3}}
\]

where \( m = 2M(\Omega \beta_0)^{-1} \) with \( \hat{\delta} M \) the variation of the enthalpy. By using this solution is direct to compute the different parts of the first law of thermodynamics above, \( \hat{\delta} M = T \hat{\delta} S + \hat{w} \hat{\delta} l \). In this case,

\[
\hat{\delta} S = \beta_0 \hat{r}_+^{d-2} \Omega
\]

and

\[
\hat{w} \hat{\delta} l = \hat{v} \hat{\delta} P = \beta_0 (r_+^{d-1} \Omega) \hat{\delta} \left( \frac{1}{l^2} \right)
\]

In this relation one can notice that the dependence of pressure on the scale is not altered by the dimension, \( p \sim l^{-2} \). The dependence of the volume is also not altered, namely \( v \sim r_+^{d-1} \).
The only difference is given by numerical factors that can be fixed by the definition of gravitational constant in $d$ dimensions, namely by $\beta_0$.

B. Five dimensional Gauss-Bonnet gravity

In [31] was restudied the static solution of the five dimensional Lovelock gravity equations of motion

$$l^2 \left( 5\alpha_0 \frac{e^4}{l^4} + 3\alpha_1 R \frac{e^2}{l^2} + \alpha_3 R^2 \right) = l^2 \left( R + \frac{e^2}{l^2} \right) \left( \beta_1 R + \beta_0 \frac{e^2}{l^2} \right) = 0. \quad (47)$$

This solution was originally found in [32]. In Schwarzschild coordinates (see Eq.(25)) this solution is defined by [31]

$$f(r)^2 = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha} \sqrt{1 + \frac{16\alpha m}{r^4} + 4\frac{\alpha \Lambda}{3}}, \quad (48)$$

where the coefficient are given by

$$\alpha = \frac{l^2 \beta_1}{2(\beta_0 + \beta_1)}, \; \Lambda = \frac{6\beta_0}{l^2(\beta_0 + \beta_1)} \quad \text{and} \quad m = \frac{M}{2(\beta_0 + \beta_1)\Omega}, \quad (49)$$

with $\delta M$ is to be considered the variation of both the mass [13] but also of the enthalpy of the solution. In this case the vacuum energy is given by [13]

$$E_0 = \frac{1}{8} l^2 \beta_0 \Omega$$

and therefore $p - v$ term is given by

$$\omega_\infty \delta l = \beta_0 \frac{l^4}{8} \Omega \delta \left( \frac{1}{l^2} \right)$$

See Eq.41. Here one can identify, as previously $\delta P \sim \delta(l^{-2})$ and

$$v_\infty = \beta_0 \frac{l^4}{8} \Omega.$$

In this case the first law, $\delta M + v_\infty dp = T \delta S + w \delta l$, can be separated as

$$T \delta S = T \left( \beta_1 r_+^2 + 2\beta_1 l^2 + \beta_0 r_+^2 \right) \Omega \delta r_+,$$

which coincides the usual Wald’s expression in Eq.45. The last term is given by

$$w \delta l = - \left( (\beta_0 r_+^4 + \beta_1 l^4) \Omega \right) \frac{2}{l^3} \delta l,$$

$$= \left( (\beta_0 r_+^4 + \beta_1 l^4) \Omega \right) \delta \left( \frac{1}{l^2} \right). \quad (50)$$
In this case one write down the first law of thermodynamics \( \delta M = T \delta S + v \delta p \) where the *volume* is given by

\[
v = \left( \beta_0 r_+^4 + \left( \beta_1 - \frac{\beta_0}{8} \right) l^4 \right) \Omega.
\]

One can notice an additional term due to the presence of the Gauss Bonnet term, \( \beta_1 \neq 0 \) in Eq.(47) and a correction due to vacuum energy assoicate to \( \beta_0 \). These terms can be understood as Van der Waals corrections to the volume. The pressure term, \( p \sim l^{-2} \), has the same dependency as for GR. This will be a general feature in the rest of the examples, see below.

### C. Born-Infeld

In this case \( d = 2n \) and the Lagrangian, once the regulator added, has the form of a perfect binomial

\[
L_R = \beta_0 l^{2n-3} \left( R + \frac{e^2}{l^2} \right)^n,
\]

and the EOM are

\[
l^{2n-5} \beta_0 \left( R + \frac{e^2}{l^2} \right)^{n-1} e = 0
\]

The solution in this case is defined by

\[
f(r)^2 = \gamma + \frac{r^2}{l^2} - \left( \frac{m}{r} \right) \frac{1}{r}
\]

and the mass/enthalpy is given by

\[
M = 4 \beta_0 l^{2n-4} m \Omega.
\]

From this it is direct to check explicitly that \( T \delta S \) coincides with the definition in Eq.(45). The \( \dot{w} \delta l \) is given in this case by

\[
\dot{w} \delta l = \beta_0 l^{2n-3} \left( \gamma + \frac{r_+^2}{l^2} \right)^{n-2} \left( (n-2) \gamma - \frac{r_+^2}{l^2} \right) \frac{1}{l^2}
\]

Once again in this case one can take Eq.(55) in the form \( V \delta P \) with \( P \sim l^{-2} \) by defining the effective volume,

\[
V_{\text{effective}} = \beta_0 l^{2n-3} \left( \gamma + \frac{r_+^2}{l^2} \right)^{n-2} \left( (n-2) \gamma - \frac{r_+^2}{l^2} \right).
\]

In this case the effective volume becomes proportional to volume only for \( r_+ \gg l \) which is not an adequate limit in this case.
D. Pure Lovelock

Pure Lovelock theory corresponds to just considering a single term in the Lovelock series plus the term associated with $\alpha_0$ in Eq. (13). The EOM in this case can be cast in the form,

$$l^{2k-3}\gamma_k \left( R^k \pm \left( \frac{\epsilon}{l} \right)^{2k} \right) e^{d-2k-1} = 0. \quad (56)$$

where $\alpha_k = (d-2k)^{-1}\gamma_k$ and $\alpha_0 = d^{-1}\gamma_k$.

The double sign $\pm$ in Eq. (56) comes from the fact that the cases $k$ even and odd differ. For even $k$ there are no constant curvature ground states with $+ \left( \frac{\epsilon}{l} \right)^{2k}$, but only for $- \left( \frac{\epsilon}{l} \right)^{2k}$ which correspond to $R \pm e^2/l^2 = 0$. However, only $R + e^2/l^2 = 0$ is of interest for this work since dS spaces has no asymptotic regions. Conversely, for odd $k$, $+ \left( \frac{\epsilon}{l} \right)^{2k}$ defines a single constant curvature ground state satisfying $R + e^2/l^2 = 0$. $- \left( \frac{\epsilon}{l} \right)^{2k}$, on the other hand, defines a single dS ground state which is not relevant for this work.

Following the form of Eq. (20) the equation of motion can be rewritten as

$$l^{d-3}\gamma_k \left( R + \frac{\epsilon^2}{l^2} \right) \left( \sum_{j=0}^{k-1} R^j \left( -\frac{e^2}{l^2} \right)^{k-j} \right) \left( \frac{\epsilon}{l} \right)^{d-2k-1} = 0, \quad (57)$$

In this case the $\beta_i$ constant mentioned above in Eq. (31) can be written as

$$\beta_i = \begin{cases} 
\frac{l^{2d-3}\gamma_k (-1)^{k-1-i}}{i \leq k-1} \\
0 \quad i > k-1
\end{cases}$$

It is straightforward to check that

$$\sum_i \beta_i = l^{d-3}\gamma_k.$$ 

In this case the static solution [33] is defined by

$$f(r)^2 = 1 + \left( \frac{r^{2k}}{l^{2k}} - \frac{m}{r^{d-2k-1}} \right)^k. \quad (58)$$

It is direct to check that this solution asymptotically satisfies

$$\lim_{r \to \infty} f(r)^2 \sim 1 + \frac{r^2}{l^2} - \frac{m}{r^{d-3}}$$

as expected from the analysis above. From this last expression is direct to compute the mass/enthalpy, by integrating Eq. (40), is given by

$$m = \frac{l^2 M}{\Omega \beta_0}.$$
Analogously, the term \( w \hat{\delta}l \) is given by

\[
w \hat{\delta}l = \alpha_0 ((k - 1)(d - 2k)r_+^{d-2k-1}) l^{2k} \Omega \left( \frac{1}{l^2} \right)
\] (59)

where one can recognize the volume as

\[
V = \alpha_0 ((k - 1)(d - 2k)r_+^{d-2k-1}) l^{2k} \Omega.
\] (60)

In this case this effective volume differs from the volume \( \sim r_+^{d-2} \). In fact, even though the effective volume in this case is a monotonically increasing function of \( r_+ \), there is no way the effective volume can reach a dependence as \( \sim r_+^{d-2} \).

VII. CONCLUSION AND PROSPECTS

In this work was studied the consequences of incorporating scale changes as thermodynamic variables for asymptotically (locally) AdS static black hole solutions of Lovelock gravity. The result is consistent with considering the mass term as the enthalpy of the black hole, instead of the energy. The final result is the arise of \( vdp \) term in the first law of black hole thermodynamics where pressure \( p \) can be cast universally as \( p \sim l^{-2} \) while the volume \( v \) changes according to the particular case considered. For instance, for Einstein gravity in \( d \) dimensions \( v \) corresponds to volume of a \( d - 1 \)-ball of radius \( r_+ \), the horizon radius.

In general the asymptotically locally AdS structure of the solution determine that change of scales does not alter the conserved charges structure of the solutions. Conversely the horizon structure is modified by the scale change which determines the arise of the \( vdp \) term mentioned above.

It must be stressed that from only studying the thermodynamics, without having the (regularized) conserved charges at hand, is not possible to determine the existence of the vacuum energy of the odd dimensional AdS spaces for \( (n - k) > 2 \). However, its existence becomes manifest for \( (n - k) \leq 2 \) since it incorporates an additional term in the first law of the thermodynamics. This new term actually implies a modification of the effective volume \( v \), but not of the condition \( p \sim l^{-2} \). In the context of the AdS/CFT conjecture this seems to imply that only for certain CFT theories, those dual to the gravities satisfying \( (n - k) < 2 \), the conformal anomaly can contribute to the CFT thermodynamics. This is a very interesting topic and will be studied in a future work. Another aspect to be addressed
in future works is the analysis of the phase transitions, such as a generalization of the Hawking-Page transitions, that should be presented.

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Appendix A: Regulation in even dimensions

For simplicity, in this work only the even dimensional case will be discussed. In this case the term to be added is the Euler density in \( d = 2n \) dimensions, which can be considered as merely the addition of the last term in the Lovelock Lagrangian. Notice that \( R^n \) is a topological density and thus it does not alter the EOM of \( L \). In this way,

\[
L \rightarrow L_R = L + \tilde{\alpha}_n R^n
\]  

where \( \tilde{\alpha}_n \) is to be fixed by any of the three conditions mentioned above. For instance, considering the improved action principle therefore,

\[
\delta L_R = \sum_{p=0}^{[(d-1)/2]} p\tilde{\alpha}_p d(\delta_0 \omega R^{p-1} e^{2n-2p}) + n\tilde{\alpha}_n (\delta_0 \omega R^n) + \text{EOM}_\omega \delta_0 e + \text{EOM}_e \delta_0 \omega,
\]  

and thus formally the boundary term can be written as

\[
\Theta_R = \delta_0 \omega \left( \sum_{p=0}^{[(d-1)/2]} p\tilde{\alpha}_p R^{p-1} e^{2n-2p} + n\tilde{\alpha}_n R^{n-1} \right).
\]  

However, for an asymptotically (locally) AdS space of radius \(-l^2\) as \( x \rightarrow \mathbb{R} \times \partial \Sigma_\infty \) is satisfied that \( e^2 \rightarrow -l^2 R \). Therefore,

\[
\lim_{x \rightarrow \mathbb{R} \times \partial \Sigma_\infty} \Theta_R = \delta_0 \omega \left( \sum_{p=0}^{[(d-1)/2]} p\tilde{\alpha}_p (-l^2)^{n-p} + n\tilde{\alpha}_n \right) R^{n-1},
\]  

which implies that, provided \( \delta_0 \omega \) is finite, the boundary term vanishes if

\[
\tilde{\alpha}_n = -\frac{1}{n} \sum_{p=0}^{[(d-1)/2]} p\tilde{\alpha}_p (-l^2)^{n-p},
\]  

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and therefore a proper action principle is at hand. In this way it is obtained a new boundary term, defined by
\[ \Theta_R = \delta_0 \omega^{ab} \frac{\partial L_R}{\partial R^{ab}}, \] (A6)
which vanishes identically at \( \mathbb{R} \times \partial \Sigma_\infty \) provided \( \delta \omega \) is arbitrary but finite. In the same fashion, one can show that action principle is also regularized by the introduction \( \tilde{\alpha}_n R^n \).

Roughly speaking, the Lagrangian
\[ \lim_{x \to \mathbb{R} \times \partial \Sigma_\infty} L_R \approx \left[ \sum_p \tilde{\alpha}_p (-l^2)^{n-p} \left( 1 - \frac{2}{n} \right) R^n \right] \]
which vanishes for \( \tilde{\alpha}_p \) defined by Eq.(21). Therefore, by the addition of \( \tilde{\alpha}_n R^n \) the divergences from the asymptotic AdS region has been removed from the action principle and the new one is finite.

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