AN ELEMENTARY PROOF OF THE GENERALIZATION OF THE BINET FORMULA FOR \( k \)-BONACCI NUMBERS

HAROLD R. PARKS AND DEAN C. WILLS

Abstract. We present an elementary proof of the generalization of the \( k \)-bonacci Binet formula, a closed form calculation of the \( k \)-bonacci numbers using the roots of the characteristic polynomial of the \( k \)-bonacci recursion.

1. Introduction

The Binet formula for the Fibonacci numbers is:

\[
F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]

Noting that \( \phi_1 = (1 + \sqrt{5})/2 \) and \( \phi_2 = (1 - \sqrt{5})/2 \) are the roots of \( x^2 - x - 1 = 0 \), the characteristic equation of the Fibonacci recursion, we can rewrite (1.1) as

\[
F_n = \frac{\phi_1^n - \phi_2^n}{\phi_1 - \phi_2},
\]

which in turn can be written in the symmetric form

\[
F_n = \frac{\phi_1^n}{\phi_1 - \phi_2} + \frac{\phi_2^n}{\phi_2 - \phi_1}.
\]

This leads one to conjecture that (1.2) ought to generalize in a natural way to the \( k \)-bonacci numbers, \( k > 2 \). Indeed, such a natural generalization, given in (1.3) below, is true.

Certainly, this result is known. The subject has been explored in various forms in all the papers, [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11], listed as references.

We call special attention to [9] and [10]. In [9], a more general result is developed using the theory of symmetric functions. The particular application to the generalized \( k \)-bonacci sequence is Example 1 on page 291. In [10], the Binet formula for the \( k \)-bonacci sequence is equation (2)" on page 749. In that paper, the result is obtained via the general theory of difference equations.

Despite the available references, the authors feel that the proof given below, relying only on Vandermonde determinants, is simpler and more succinct than any available elsewhere in the literature.

Theorem 1.1. For \( k \geq 2 \), let \( \phi_i \), \( i = 1, 2, \ldots, k \), be the solutions of the characteristic equation, \( x^k - x^{k-1} - \cdots - 1 = 0 \), of the \( k \)-bonacci recursion. If the \( k \)-bonacci numbers, \( F_n^{(k)} \), are defined with initial values of \( F_n^{(k)} = 0 \), for \( 0 \leq n < k - 1 \), and

\[2010 \ Mathematics \ Subject \ Classification. \ Primary \ 15A15; \ Secondary \ 11B39.\]
Similarly, define the minor

\[ \text{Lemma 2.2.} \]

\[ F_{k-1}^{(k)} = 1, \text{ then} \]

\[ (1.3) \]

\[ F_n^{(k)} = \sum_{i=1}^{k} \frac{\phi_i^n}{(\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_i - \phi_{i+1}) \cdots (\phi_i - \phi_k)} = \sum_{i=1}^{k} \frac{\phi_i^n}{\prod_{j \neq i}(\phi_i - \phi_j)} \]

The expressions in the center and right-hand side of (1.3) simply use different notations for the same denominator. It is important to know that the \( k \) solutions of the characteristic equation are all distinct, so that no division by zero occurs. To see this, note that if the characteristic polynomial had a multiple root, then that same multiple root would be a multiple root of \( (x-1)(x^k-x^{k-1} \cdots -1) = x^{n+1} - 2x^n + 1 \) and thus a root of \( d(x^n - 2x^n + 1)/dx = (n+1)x^n - 2nx^{n-1} \). This last polynomial has only rational roots, and the Rational Root Theorem tells us that they are not roots of \( x^k - x^{k-1} \cdots - 1 \).

2. Proofs

The proof of the next lemma and the theorem will involve various Vandermonde determinants, so we make the following definition.

**Definition 2.1.** Set

\[ V(\phi, k) = \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & \phi_1^{k-1} \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & \phi_2^{k-1} \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & \phi_3^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & \phi_k^{k-1} \end{vmatrix} = \prod_{1 \leq m < n \leq k} (\phi_n - \phi_m). \]

Similarly, define the minor

\[ V_1(\phi, k) = \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{i-1} & \phi_{i-1}^2 & \cdots & \phi_{i-1}^{k-2} \\ 1 & \phi_i & \phi_i^2 & \cdots & \phi_i^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_{k-1} & \phi_{k-1}^2 & \cdots & \phi_{k-1}^{k-2} \end{vmatrix} = \prod_{1 \leq m < n \leq k} (\phi_n - \phi_m). \]

**Lemma 2.2.** Let \( K \) be a field, and \( \phi_i, f_i \in K \), for \( 1 \leq i \leq k \). If the \( \phi_i \) are all distinct, then

\[ (2.1) \]

\[ \sum_{i=1}^{k} \frac{f_i}{\prod_{j \neq i}(\phi_i - \phi_j)} = \frac{1}{V(\phi, k)} \begin{vmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & f_1 \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & f_2 \\ 1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & f_k \end{vmatrix} \]
Proof. Expanding the determinant on the right-hand side of (2.1) along the last column, we see that
\[
\begin{vmatrix}
1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{k-2} & f_1 \\
1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{k-2} & f_2 \\
1 & \phi_3 & \phi_3^2 & \cdots & \phi_3^{k-2} & f_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \phi_k & \phi_k^2 & \cdots & \phi_k^{k-2} & f_k \\
\end{vmatrix}
= \sum_{i=1}^{k} (-1)^{k-i} f_i \cdot \mathcal{V}_i (\phi, k)
\]

For each \( i \), observe that
\[
\prod_{1 \leq m<n \leq k} (\phi_n - \phi_m) = (-1)^{k-i} \prod_{\substack{1 \leq m<n \leq k \\text{if} \, m,n \neq i}} (\phi_n - \phi_m) \prod_{j \neq i} (\phi_i - \phi_j)
\]
so
\[
\sum_{i=1}^{k} \frac{f_i}{\prod_{j \neq i} (\phi_i - \phi_j)} = \sum_{i=1}^{k} (-1)^{k-i} f_i \cdot \mathcal{V}_i (\phi, k) \mathcal{V} (\phi, k)
\]
holds, and the result follows. \(\square\)

Proof of Theorem 1.1. For \( n = 0, 1, \ldots, k - 2 \), Lemma 2.2 tells us that the right-hand side of (1.3) equals \( \mathcal{V} (\phi, k)^{-1} \times \) the value of a determinant with a repeated column, thus the right-hand side of (1.3) equals \( 0 \). When \( n = k - 1 \), Lemma 2.2 tells us that the right-hand side of (1.3) equals \( \mathcal{V} (\phi, k)^{-1} \times \) the value of a Vandermonde determinant and that Vandermonde determinant clearly equals \( \mathcal{V} (\phi, k) \).

Now suppose that \( n \geq k - 1 \). Arguing inductively and noting that \( \phi^k = \sum_{m=0}^{k-1} \phi^m \), we have
\[
F_{n+1}^{(k)} = \sum_{m=0}^{k-1} F_{(n+1-k)+m}^{(k)} = \sum_{m=0}^{k-1} \sum_{i=1}^{k} \phi_i^{(n+1-k)+m} \prod_{j \neq i} (\phi_i - \phi_j) = \sum_{i=1}^{k} \phi_i^{n+1-k} \prod_{j \neq i} (\phi_i - \phi_j) \]
\[
= \sum_{i=1}^{k} \frac{\phi_i^{n+1-k} \cdot \mathcal{V}_i (\phi, k)}{\prod_{j \neq i} (\phi_i - \phi_j)}.
\]
\(\square\)

REFERENCES

[1] Gregory P. Dresden and Zhaohui Du. A simplified binet formula for \( k \)-generalized fibonacci numbers. J. Integer Seq., 17:14.4.7, 2014.
[2] David E. Ferguson. An expression for generalized fibonacci numbers. Fibonacci Quarterly, 4(3):270–272, October 1966.
[3] Ivan Flores. Direct calculation of \( k \)-generalized fibonacci numbers. Fibonacci Quarterly, 5(3):259–266, October 1967.
[4] Hyman Gabai. Generalized fibonacci \( k \)-sequences. Fibonacci Quarterly, 8(1):31–38, February 1970.
[5] Gautam S. Hathiwalla and Devbhadra V. Shah. Binet – type formula for the sequence of tetranacci numbers by alternate methods. Mathematical Journal of Interdisciplinary Sciences, 6(1):37–48, May 2019.
[6] Dan Kalman. Generalized fibonacci numbers by matrix methods. Fibonacci Quarterly, 20(1):73–76, February 1982.
[7] David Kessler and Jeremy Schiff. A combinatoric proof and generalization of Ferguson’s formula for $k$-generalized Fibonacci numbers. *Fibonacci Quarterly*, 42(3):266–273, August 2004.

[8] Gwang-Yeon Lee, Sang-Gu Lee, Jin-Soo Kim, and Hang-Kyun Shin. The Binet formula and representations of $k$-generalized Fibonacci numbers. *Fibonacci Quarterly*, 39(2):158–164, May 2001.

[9] Claude Levesque. On $m$-th order linear recurrences. *Fibonacci Quarterly*, 23(4):290–293, November 1985.

[10] E. P. Miles. Generalized fibonacci numbers and associated matrices. *The American Mathematical Monthly*, 67(8):745–752, 1960.

[11] W. R. Spickerman. Binets’s formula for the Tribonacci sequence. *Fibonacci Quarterly*, 20(2):118–120, May 1982.

Department of Mathematics, Oregon State University, Corvallis, Oregon 97331 USA

Email address: hal.parks@oregonstate.edu

AppDynamics, San Francisco, California 94107 USA

Email address: dean@lifetime.oregonstate.edu