Stochastic Characteristic Analysis of Lid-Driven Cavity Flow

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Abstract. The stochastic characteristic of lid-driven cavity flow under the condition of Re=100 is investigated in this paper by a non-intrusive polynomial chaos method. The lid-driven velocity and viscosity coefficient are assumed to be two independent stochastic input variables with Gaussian distributions. The velocities at every grid points are the response of interest and no longer deterministic but stochastic. Their stochastic characteristics are represented by Hermite polynomials with interaction effects between two input variables considered. The statistical results, including mean value, variance and the relative contribution of each input variable to the variance of output are analysed. It is found that the variance of velocity shows similar structures with its magnitude of mean value and the correlation coefficient is larger than 0.96. The nearly linear correlation coefficient shows that the lid-driven velocity is the main factor affecting the velocity distribution and the influence of viscosity coefficient on most regions is unimportant. The analysis of stochastic flow field helps understand the inherent mechanism in the lid-driven cavity flow.

1. Introduction
The Polynomial Chaos (PC) method originated from the isotropic chaos theory proposed by Wiener [1] in the study of the spectral space expansion of stochastic variables in turbulence problems. Ghanem [2] applied it to finite element calculations, Xiu and Karniadakis [3] extended Hermite PC suitable for Gaussian stochastic processes to Wiener-Askey PC.

Most of the existing researches on lid-driven cavity flow are deterministic CFD simulations. Wang Xiaodong and Kang Shun [4] use PC method to simulate the propagation of uncertainty in the flow field by considering the uncertainty of the lid-driven velocity uw and the viscosity coefficient v, respectively. Liu Zhiyi [5] et al. studied the uncertainty of the flow structure caused by assuming uw and v satisfying Gaussian distribution, and found that the coupled effect of two input uncertain variables weakens the effect of uncertainty on the result. This paper use Sobol’s indices [6] to quantify the output uncertainty due to the uncertainty in the input variables.

2. Polynomial Chaos expansion
Assume a stochastic input variable ξ that satisfies a determined probability density distribution. For any stochastic output variable y can be expanded in the spectral space formed by the orthogonal basis functions series of the input variable, ie:

\[ y = \sum_{j-o}^{p} \alpha_{j} \psi_{j}(\xi) \]  

(1)
Where $\alpha_j$ is the deterministic component, also known as freedom, $\psi_j$ is the basis function of the stochastic variable. The inner product of two basis functions is defined by:

$$ \langle \psi_i, \psi_j \rangle = \int \psi_i(\xi) \psi_j(\xi) f(\xi) d\xi = \delta_{ij} \langle \psi_i, \psi_j \rangle $$  \hspace{1cm} (2)

Where $\{\psi_j, j \geq 0\}$ is the sequence of orthogonal polynomials with a weight function of $f(\xi)$. For a uniformly distributed stochastic variable, the optimal basis function sequence is a Legendre polynomial; and for a normally distributed stochastic variable, the optimal basis function sequence is Hermite polynomial. The optimization mentioned here refers to the convergence rate of the statistical information output of the system with the increase of the order of polynomials consistent with the theory. In this paper, Non-Intrusive Polynomial Chaos (NIPC) method is used to solve the degree of freedom in the expansion by regression method. The response values $\{y_1, y_2, \ldots, y_N\}$ of the samples $\{\xi_1, \xi_2, K, \xi_N\}$ are obtained by deterministic CFD method. The number of samples $N$ is related to the number of the degree of freedom in PC,

$$ N = n p (P + 1) = n p \frac{(n + p)!}{n! p!} $$  \hspace{1cm} (3)

Where $n$ is the dimension of the input stochastic variable, $p$ is the order of the PC, and $n_p$ is defined as the oversampling rate, set to 2. An overdetermined equation is formed,

$$ \Psi^T \alpha = Y $$  \hspace{1cm} (4)

$\Psi$ is the response of samples matrix, $\psi_i(\xi)$, $\alpha$ can be solved by the least squares method. Then, the output statistics are,

$$ \mu(y) = \alpha_0 $$

$$ \sigma^2(y) = \sum_{j=1}^p \alpha_j^2 \langle \psi_j, \psi_j \rangle $$  \hspace{1cm} (5)

The Sobol’s indices are used in global sensitivity analysis as a tool for ranking the contribution of input stochastic variables.

$$ \frac{\alpha_j^2 \langle \psi_j, \psi_j \rangle}{\sigma^2} $$  \hspace{1cm} (6)

3. Numerical method
The calculation domain of the lid-driven cavity flow is $[0, 1] \times [0, 1]$, the lid-driven velocity $uw$ and viscosity coefficient $\nu$ satisfy the Gaussian distribution with mean values of 1 and 0.01. The standard deviation is 10% of the mean value. The boundary condition is isothermal solid wall. The structure grid 100×100 is used and 2-D incompressible Navier-Stokes equation is solved by the SIMPLE algorithm with the second-order upwind scheme. The results are consistent with Ghia’s [7].

4. Uncertainty quantification
In this section, two input stochastic variables are used to analyze uncertainty. First, the two input variables are transformed.

$$ \xi = \frac{u_w - \mu(u_w)}{\sigma(u_w)}, \eta = \frac{\nu - \mu(\nu)}{\sigma(\nu)} $$  \hspace{1cm} (7)

Where $\mu$ and $\sigma$ represent the mean and standard deviation of the stochastic variables, respectively. The fifth-order PC expansion with Hermite basis function is adopted. The general process is: determine
the number of samples needed according to formula (3), generate samples by Latin hypercube sampling method, get the deterministic output. The deterministic component in the expansion is obtained by the regression method. In this paper, the output variable of interest is the velocities U and V at every grid points. The mean and standard deviation of the velocities can be obtained according to the formula (5). In Figure 1, it shows the streamline obtained from the mean velocities, and in Figure 2, it shows the mean velocity patterns of two sections and the deterministic calculation results are in good agreement with the mean.

![Figure 1. Streamline of the mean velocities](image1)

![Figure 2. Comparison between mean velocity and deterministic calculation results of two sections](image2)

The standard deviation and mean contour of the velocity U are given in the Figure 3 and Figure 4. The two figures have similar structure and the correlation coefficient between the mean and standard deviation of the velocity U is 0.9721. The standard deviation and mean contour of the velocity V are given in the Figure 5 and Figure 6. They also have similar structure and the correlation coefficient is 0.9625.

![Figure 3. Standard deviation contour of velocity U](image3)

![Figure 4. Amplitude contour of mean velocity U](image4)
Figure 5. Standard deviation contour of velocity V

Figure 6. Amplitude contour of mean velocity V

According to formula (6), Sobol’s indices are used to measure the contribution of each input variable, including linear terms, higher order terms, and cross terms. Figure 7 show the contribution of the linear terms (stochastic variables $u_w$ and $v$), quadratic terms (stochastic variables $u_w$ and $v$), and quadratic cross terms in the velocity U-expansion. Figure 8 show the contribution of each term in the velocity V-expansion. Both figures show the linear term of two stochastic variables plays a major role, and the influence of quadratic cross term only on a few grid points cannot be ignored. Therefore, it can be concluded that in most areas of the flow field, only the first three terms are important in the expansion, namely the mean term and the two linear terms. In Figure 9 and Figure 10, we check the correlation between the velocity mean and the coefficient of the linear term. The correlation coefficients of the two are 0.9768 and 0.9804 respectively, which is higher than the amplitude of the aforementioned. So it can be said that the degree of freedom of the linear term in PC expansion is approximately proportional to the mean value of the output.
Figure 8. The contribution of linear term, quadratic term and cross term in velocity $V$-expansion

Figure 9. The relationship between linear term and mean of velocity $U$

Figure 10. The relationship between the linear term order and mean of velocity $V$

Four spatial points are selected to examine the probability density distribution of the output by the kernel smoothing method, as shown in Table 1 and Figure 9. The distribution of points 2 and 3 approximates the Gaussian distribution, while the point 1 deviates slightly because $H_z(\xi)$ and $H_z(\eta)$ of the PC expansion have a non-negligible contribution. The probability density function of point 4 deviates farther from the Gaussian distribution because the contribution of $H_z(\xi)H_z(\eta)$, $H_z(\xi)$ and $H_z(\eta)$ these three terms is dominant.
Table 1. The coordinates of the four points

| Point number | Pt 1   | Pt 2   | Pt 3   | Pt 4   |
|--------------|--------|--------|--------|--------|
| Coordinates  | (0.44,0.59) | (0.74,0.29) | (0.06,0.77) | (0.22,0.86) |

Figure 11. velocity U probability distribution at spatial point
(top left: point 1; top right: point 2; bottom left: point 3; bottom right: point 4)

Finally, we study the distribution of linear term in PC expansion of velocity U and V, as shown in Figures 11 and 12, to analyze the effects of the variation of u_w and ν on flow and its mechanism. From the distribution of \( H_1(\xi) \) term in Figure 4, for the velocity U, the coefficient of the upper half of the cavity is positive, and the middle and lower part is negative. For the velocity V, the coefficient of the left half of the cavity is positive, and the right half is negative. Therefore, when the u_w increases, the amplitudes of the velocity U and V increase, and the driven flow in the cavity is strengthened, which is consistent with the intuition. The effect of variation of ν on flow is more complicated. When ν increases, the velocity U will increase due to the viscous effect near the top cover, and the fluid in the upper left part will also accelerate. However, in the vicinity of other walls, the velocity amplitude will decrease due to the viscous hindrance, and the angular vortices on both sides of the bottom will decrease at the same time, which is consistent with the law of the influence of the driven Reynolds number on the angular vortices.
Figure 12. Linear term distribution in velocity U-expansion

Figure 13. Linear term distribution in velocity V-expansion

5. Conclusion
In this paper, the PC method is used to study the propagation and effect of the uncertainty caused by the lid-driven velocity and viscosity coefficient in the cavity flow. The response model established by the 5th-order PC method has high prediction accuracy. The contour of velocity mean and standard deviation distribution shows that they have a good linear relationship. The Sobol’s indices are accurate descriptor of the sensitivity, the first-order and total sensitivity indices are usually used. Several special locations are examined to analyze the distribution of response results.

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