Thermodynamic geometry of novel 4-D Gauss Bonnet AdS Black Hole

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Abstract: In this paper, we employ the new thermodynamic geometry formalism (NTG) [1] to investigate phase transitions and critical behaviors of the four dimension Gauss Bonnet black hole, proposed in [2]. In this regard, the extrinsic curvature of a certain hypersurface in thermodynamic manifold contains useful information about stability/instability of the heat capacities. In extended phase space, through sitting on the $Q$-zero hypersurface, we calculate the intrinsic curvature for 4D neutral Gauss Bonnet black hole. More importantly, there is a positive intrinsic curvature near the region $0 < v < 2$, which indicates a repulsive interaction among the microstructure of the black hole system. This is in contrast to the five dimensional (uncharged) case where the attractive interaction is dominant in the microstructure of the black hole system.
1 Introduction

The study of thermodynamic properties of black holes in anti-de Sitter (AdS) space have gained significant importance due to the gauge/gravity duality [3]. This duality establishes a relationship between gravitational theories on the bulk of AdS spacetime and a quantum field theory which lives on the boundary of that AdS spacetime. In addition, the critical behaviors of charged AdS black holes in the extended phase space, where the cosmological constant treats as the thermodynamic pressure and its conjugate quantity as a thermodynamic volume of the black hole, and its similarity with the Van der Waals (VdW) liquid-gas phase transition have been studied in [4–6].

Over the past years, some attempts have also been made to investigate thermodynamic systems by means of geometric methods [7–15]. In particular, the Riemann scalar curvature associated with these geometries can provide useful information about black hole phase transitions. The history of the study of thermodynamic geometry began with the seminal paper of Weinhold [7], who introduced, in the seventies, the phase thermodynamic space and had developed a geometric description of the equilibrium space of thermodynamic system. Motivated by this Ruppeiner [8, 9], considering the fluctuation theory of equilibrium states, proposed a different metric structure for the equilibrium space. More precisely, Weinhold’s metric components are those of the Hessian of the internal energy function, whereas Ruppeiner’s metric elements are defined by the Hessian matrix of the entropy. It turns out that the line elements of the Weinhold and Ruppeiner geometries are conformally related to each other. These geometries had done by using the curvature of the Riemannian manifold, representing the equilibrium space of thermodynamic system, to find a direct correspondence between divergences of the curvature and phase transitions. But, in some
contradictory examples [12, 16], these metrics fail to explain this relation. Recently, we proposed a new formulation of the Ruppeiner’s metric, developed from considerations about the thermodynamic potentials related to the mass (instead of the entropy) by Legendre transformations [1, 10, 18, 19]. This new formalism of thermodynamic geometry (NTG) represents a one-to-one correspondence between the divergences of the heat capacities and curvature singularities. In Ref. [1] we have shown that the geometrothermodynamics (GTD) [15] can be constructed by an explicit conformal transformation, which is singular at unphysical points were generated in GTD metric, from NTG geometry.

In Refs. [20, 21] authors have also introduced a general Ruppeiner geometry, starting the Boltzmann entropy formula, to study the AdS black hole microstructure. In this formalism, the fluctuation coordinates are taken as the temperature and volume in extended phase space. When this metric is applied to the van der Waals fluid only a dominant attractive interaction was observed, while for the RN AdS black hole, in a small parameter range, the repulsive interaction is also found in addition to the dominant attractive interaction between the black hole molecules. This approach has been extended to other black hole systems [22–28]. It is interesting that this formalism can be reproduced in NTG geometry by a choice of the free energy potential in the coordinate of the temperature and the volume.

Recently, Glavan and Lin have proposed a novel theory of the Einstein Gauss-Bonnet gravity in 4- dimensional spacetime which bypasses the conclusions of Lovelock’s theorem and avoids Ostrogradsky instability [2]. In $D$- dimensions, by rescaling the Gauss-Bonnet coupling $\alpha$ by a factor of $1/(D - 4)$, and taking the limit $D \to 4$, the Gauss-Bonnet term gives rise to non-trivial contributions to gravitational dynamics in four dimensions [2]. The stability and shadow of this black hole and quasinormal modes of a scalar, electromagnetic, and gravitational perturbations have been studied in [29]. The solutions of charged black hole [30] and a rotating analogy of this black hole using Newman-Janis algorithm [31] have been investigated. In Refs. [31, 32] one can find the possible range of GB coupling parameter by modelling the $M87^{*}$ as the rotating 4D GB black hole. Moreover, the thermodynamics of asymptotically AdS black hole in the four dimensional Einstein-Gauss-Bonnet theory has been reported in [33]. From isotherms in $P - V$ diagram, Gibbs free energy and specific heat plots, the authors found that both charged and neutral cases of the 4D GB black hole exhibits a phase transition similar to that of van der Waals system [33]. Moreover, thermodynamics and $P - V$ criticality of Bardeen-AdS Black Hole in 4D Einstein-Gauss-Bonnet Gravity has been obtained in [34]. So far, thermodynamic geometry have not been studied in 4D Einstein Gauss-Bonnet models, which is the aim of this paper.

The organization of this paper is as follows. In Section 2, we review the new formalism of thermodynamic geometry (NTG). We further investigate NGT metrics on all thermodynamical potentials generated by Legendre transformations and investigate correspondence between curvature singularities and phase transitions. In Section 3 we study the phase transitions of 4D charged GB black hole by making use of the NTG formalism on certain hypersurface. In Section 4, we first briefly discuss about some thermodynamical properties for this novel Black hole and then apply the NTG formalism to the extended phase space. This allow us to study the critical behaviours and microstructures of the black hole via extrinsic and intrinsic curvatures. Final remarks are presented in section 5.
2 The New formalism of Thermodynamic geometry

In thermodynamic geometry, it is important to construct an appropriate metric which explains the one-to-one correspondence between phase transitions and singularities of the scalar curvature. In Ref. [1] we have introduced a new formalism of the thermodynamic geometry (NTG) which confirms this correspondence. The NTG geometry is defined by

\[ dl^2_{NTG} = \frac{1}{T} \left( \eta^j_i \frac{\partial^2 \Xi}{\partial X^j \partial X^l} dX^i dX^l \right) \] (2.1)

where \( \eta^j_i = \text{diag}(-1,1,...,1) \) and \( \Xi \) is the thermodynamic potential and \( X^i \) can be intensive and extensive variables [1]. In two dimensional equilibrium state space with first law of thermodynamics, \( dM = TdS + \Phi dQ \), it is straightforward to verify that, by taking \( \Xi = M(S,Q) \) and \( X^i = (S,Q) \), the curvature singularities correspond precisely to the phase transitions of \( C_Q \). Moreover, as one selects the thermodynamical potential by Legendre transformation \( \Xi = H(S,\Phi) = M - \Phi Q \), the totality of curvature singularities of the NTG metric is exactly the same as the totality of phase transitions for \( C_\Phi \). It seems natural to consider NTG metrics generated by all thermodynamical potentials produced by Legendre transformations. By using Legendre transformations, we can introduce a set of new additional thermodynamic potentials that depend on different combinations of extensive and intensive variables. Therefore, all thermodynamic potentials can be written as

\[ H(S,\Phi) = M(S,Q(S,\Phi)) - \Phi Q(S,\Phi) \] (2.2)
\[ F(T,Q) = M(S(T,Q),Q) - TS(T,Q) \] (2.3)
\[ G(T,\Phi) = M(S(T,\Phi),Q(T,\Phi)) - TS(T,\Phi) - \Phi Q(T,\Phi) \] (2.4)

where \( H \), \( F \), and \( G \) are enthalpy, free energy, Gibbs energy, respectively. In following, we shall demonstrate that the NTG metrics coming from free energy, \( F \) and enthalpy, \( H \), have the same singularity as the one associated with the divergence of \( C_\Phi \). By choosing \( \Xi = F(T,Q) \) with \( X^i = (T,Q) \), the NTG metric yields

\[ g^F_{NTG} = \frac{1}{T} \left( \begin{array}{cc} -\frac{\partial^2 F}{\partial T^2} & 0 \\ 0 & \frac{\partial^2 F}{\partial Q^2} \end{array} \right) = \frac{1}{T} \left( \begin{array}{cc} \frac{\partial S}{\partial T} & 0 \\ 0 & \frac{\partial \Phi}{\partial Q} \end{array} \right) \] (2.5)

here we have used the first law of thermodynamics for free energy, \( dF = -SdT + \Phi dQ \). One can express metric elements in \( (S,\Phi) \) (it is coordinate of NTG metric associated with enthalpy potential.) coordinate as follows.

\[ g_{TT} = \frac{1}{T} \left( \frac{\partial S}{\partial T} \right)_Q = \frac{1}{T} \{S,Q\}_{S,\Phi} \] (2.6)
\[ g_{QQ} = \frac{1}{T} \left( \frac{\partial \Phi}{\partial Q} \right)_T = \frac{1}{T} \{\Phi,T\}_{S,\Phi} = -\frac{1}{T} \{\Phi,T\}_{T,\Phi} \] (2.7)
Appx. A can be consulted for a brief introduction to the bracket notation. In \((S, \Phi)\) coordinate, the metric elements must be changed by
\[
\hat{g} = J^T g^N \hat{T} J
\]
where \(J^T\) is the transpose of the Jacobian matrix \(J\) which defined as
\[
J = \frac{\partial (T, Q)}{\partial (S, \Phi)} = \begin{pmatrix}
\frac{\partial T}{\partial S} & \frac{\partial T}{\partial \Phi} \\
\frac{\partial Q}{\partial S} & \frac{\partial Q}{\partial \Phi}
\end{pmatrix}
\]
(2.8)

Under varying coordinates and using Maxwell relation, \(\frac{\partial T}{\partial \Phi} \bigg|_S = -\frac{\partial Q}{\partial S} \bigg|_\Phi\), the metric form (2.5) takes the following form
\[
\hat{g} = \frac{1}{T} \begin{pmatrix}
\frac{\partial T}{\partial S} & 0 \\
0 & -\frac{\partial Q}{\partial \Phi} \bigg|_S
\end{pmatrix}
\]
(2.9)

In the last part, we have used the first law of thermodynamic for entalpy potential, i.e. \(dH = TdS - Qd\Phi\). Clearly, their associated metrics are negative of each other, i.e.
\[
g^N_H = -J^T g^N_F J \quad \text{or} \quad dl^2(H) = -dl^2(F)
\]
(2.10)

Therefore, the singularity of both \(R_F\) and \(R_H\) correspondences to the divergence of \(C_\Phi\). It worth mentioning that this result is true for any conjugate potential pairs \((\Xi, \bar{\Xi})\) which satisfies the following relation [17].
\[
\Xi + \bar{\Xi} = 2M - TS - \sum_i \Phi_i dQ_i
\]
(2.12)

Therefore, for conjugate pair \((M, G)\), one can show that \(g^N_M = -N^T g^N_G N^T\) by Jacobian matrix \(N = \frac{\partial (T, \Phi)}{\partial (S, \Omega)}\). In Table 1, one can see the relation between curvature singularities and heat capacity phase transition in three dimensional thermodynamic space with the first law, \(dM = TdS + \Phi dQ + \Omega dJ\). In the next section, we first apply the NTG metrics to study correspondences between curvature singularities and phase transitions for

| Thermodynamic potentials \((\Xi, \bar{\Xi})\) | Jacobian matrix \(g^N_E = -J^T g^N_F J\) | Heat Capacities \(C\) |
|---|---|---|
| \((M, M - TS - Q\Phi - \Omega J)\) | \(\frac{\partial (T, \Phi, \Omega)}{\partial (S, \Omega, \Phi)}\) | \(C_{Q, J}\) |
| \((M - Q\Phi, M - TS - \Omega J)\) | \(\frac{\partial (T, \Phi, J)}{\partial (S, \Omega, \Phi)}\) | \(C_{\Phi, J}\) |
| \((M - \Omega J, M - TS - Q\Phi)\) | \(\frac{\partial (T, \Phi, J)}{\partial (S, \Omega, \Phi)}\) | \(C_{Q, \Omega}\) |
| \((M - Q\Phi - \Omega J, M - TS)\) | \(\frac{\partial (T, \Phi, J)}{\partial (S, \Omega, \Phi)}\) | \(C_{\Phi, \Omega}\) |

**Table 1**: The relationship between curvature singularities and heat capacity divergences.
the charged 4D Gauss Bonnet black holes [30]. Then, by taking the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume, we reconsider NTG metrics in order to see the critical behaviour of both neutral and charged 4D GB-AdS black hole.

3 Intrinsic and extrinsic curvature singularities and phase transition signals

The action for the Einstein-Maxwell-Gauss-Bonnet gravity theory in 4D reads [30].

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R + \frac{3}{l^2} + \frac{\alpha}{D-4} \mathcal{G} - F_{\mu\nu}F^{\mu\nu} \right] \quad (3.1) \]

where \( l \) is AdS radius, \( \alpha \) is a dimensionless coupling constant and \( \mathcal{G} \) is the Gauss Bonnet invariant which defined by

\[ \mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \quad (3.2) \]

and the Maxwell field strength is defined by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) with \( A_\mu \) as the vector potential. The spherically symmetric solution to its equations of motion is

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \quad (3.3) \]

\[ f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + 4\alpha \left( -\frac{1}{l^2} + \frac{2M}{r^3} - \frac{Q^2}{r^4} \right)} \right) \]

in which \( Q \) and \( M \) are charge and mass of the black hole [30]. The explicit form for \( M \) is obtained by using the condition \( f(r_+) = 0 \) as

\[ M = \frac{r_+^3}{2l^2} + \frac{Q^2}{2r_+} + \frac{\alpha}{2r_+} + \frac{r_+}{2} \quad (3.4) \]

Entropy for this black hole is also defined by

\[ S = \pi r_+^2 + 4\pi \log(r_+) \quad (3.5) \]

Note that that the entropy has a logarithmic correction term in comparison with RN-AdS case. Using the first law of thermodynamics, \( dM = TdS + \Phi dQ \), the Hawking temperature, \( T \), the electric potential, \( \Phi \), and the specific heat capacity at fixed electric charge, \( C_Q \), are given by

\[ T = \left( \frac{\partial M}{\partial S} \right)_Q = \frac{\{M, Q\}_{r_+, Q}}{\{S, Q\}_{r_+, Q}} = \frac{3r_+^4 - l^2(Q^2 - r_+^2 + \alpha)}{4l^2\pi r_+(r_+^2 + 2\alpha)} \quad (3.6) \]

\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_S = \frac{\{M, S\}_{r_+, Q}}{\{Q, S\}_{r_+, Q}} = \frac{Q}{r_+} \quad (3.7) \]

\[ C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = T \left( \frac{\{S, Q\}_{r_+, Q}}{\{T, Q\}_{r_+, Q}} \right) = \frac{2\pi (r_+^2 + 2\alpha)^2(3r_+^4 - l^2(Q^2 - r_+^2 + \alpha))}{3(r_+^6 + 6r_+^4\alpha + l^2(-r_+^4 + 5r_+^2\alpha + 2\alpha^2 + Q^2(3r_+^4 + 2\alpha)))} \quad (3.8) \]
Figure 1: Left: Graph of the phase transition of $C_Q$ (dashed green curve) and the scalar curvature $R^{NTG}(S, Q)$ (solid blue curve) with respect to entropy, $S$, with $Q = 0.5$ for charged GB-AdS black hole. Right: Graph of the phase transition of $C_{\Phi}$ (dashed green curve) and the scalar curvature $R^{NTG}(S, \Phi)$ (solid blue curve) with respect to entropy, $S$, with $\Phi = 0.5$ for charged GB-AdS black hole. In both diagrams, we have considered $l = 6$, and $\alpha = 0.2$.

Allow us to apply the NTG method to study the critical behaviors of heat capacity $C_Q$. To do this, one needs to plug the thermodynamic potential $\Xi = M(S, Q)$ with $X^i = (S, Q)$ into Eq. (2.1), i.e.,

$$\left(\frac{d{l^{NTG}}}{2}\right) = \frac{1}{T} \left( -\frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial Q^2} dQ^2 \right)$$

and the denominator of $R^{NTG}$ reads

$$D(R^{NTG}) = \pi \left( (r_+^2 + 2\alpha)^2(3r_+^4 - l^2(Q^2 - r_+^2 - \alpha)) \right) \times \left( 3(r_+^6 + 6r_+^4\alpha) + l^2(-r_+^4 + 5r_+^2\alpha + 2\alpha^2 + Q^2(3r_+^2 + 2\alpha) \right)^2$$

(3.10)

It is obvious that the leading term in the denominator is zero only at the extremal limit ($T = 0$) which is forbidden by the third law of thermodynamics, while the roots of the second part give us all the phase transitions of $C_Q$. This result has been illustrated in the left hand side of Fig. (1). It should be noted that positive regions of heat capacity diagram correspond to a stable system whereas a negative regions indicate the instability of the system. As a consequence of the NTG method, the curvature singularities occur exactly at the phase transitions with no other additional roots. Furthermore, by using Eq. (3.7) for $Q$, one can define the heat capacity at fixed electric potential as

$$C_{\Phi} = \frac{2\pi(r_+^2 + 2\alpha)^2(3r_+^4 - l^2(\alpha + r_+^2(-1 + \Phi^2))))}{3(r_+^6 + 6r_+^4\alpha) + l^2(2\alpha^2 + r_+^2\alpha(5 - 2\Phi^2) + r_+^4(-1 + \Phi^2))}$$

(3.11)

Let us construct the NTG metric in this case. Starting from NTG metric (2.1), and considering $\Xi = H(S, \Phi) = M(S, Q(S, \Phi)) - Q(S, \Phi)\Phi$ and $X^i = (S, \Phi)$, we arrive at

$$d{l^{NTG}}^2 = \frac{1}{T} \left( -\frac{\partial^2 H}{\partial S^2} dS^2 + \frac{\partial^2 H}{\partial \Phi^2} d\Phi^2 \right)$$

(3.12)
Thus the denominator of the scalar curvature is

\[
D(R_{NTG}) = \pi r_+^2 (3r_+^6 + 6r_+^4 \alpha + \ell^2(2\alpha^2 + r_+^2(5 - 2\Phi^2) + r_+^4(-1 + \Phi^2)))
\times \left(3(r_+^6 + 6r_+^4 \alpha + \ell^2(2\alpha^2 + r_+^2(5 - 2\Phi^2) + r_+^4(-1 + \Phi^2)))\right)^2
\]  
(3.13)

It is interesting that the curvature singularities give us the phase transition points of \(C_{\Phi}\). (See the right hand side of Fig. (1)). In summary, the NTG geometry can provide a powerful tool to achieve a one-to-one correspondence between singularities and phase transitions.

Although the thermodynamic geometry curvature determines the phase transition points, it is not able to explain thermal stability of a thermodynamic system. In Ref. [19] we have shown that the extrinsic curvature of a certain hypersurface in thermodynamic space contains useful information about stability of a thermodynamic system. Strictly speaking, the extrinsic curvature of a cretin hypersurface has the same sign as the heat capacity around the phase transition points while the intrinsic curvature of this hypersurface is divergent at the critical points but has no information about the sign of the heat capacity.

Let us briefly review the basic concept of the extrinsic curvature in thermodynamic manifold. For an \(D\)-dimensional thermodynamic manifold \(M\) with coordinate \(X^i\), a special hypersurface \(\Sigma\) embedded in \(M\) is defined by surface equation \(P(X^i) = 0\) and the orthogonal normal vector [35],

\[
n_{\mu} = \frac{\partial_\mu P}{\sqrt{|\partial_\mu P \partial^\mu P|}}
\]  
(3.14)

Therefore, the extrinsic curvature tensor on this hypersurface is written as [19]

\[
K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} n^\mu\right)
\]  
(3.15)

This curvature opens an interesting and impressive avenue to study of thermal phase transition behavior. In order to see the behavior of \(C_Q\) around phase transitions, we must restrict ourselves to live on a constant \(Q\) hypersurface with the normal vector \(n_Q = \frac{1}{\sqrt{g_{NTG}^{\Phi Q}}}\). From Eq.3.15, we have

\[
K_{NTG}^{\Phi} = \sqrt{r_+^2 + 2\alpha (lQr_+^2 (l^2 + 6r_+^2)) \left(\pi (3r_+^6 - l^2(Q^2 - r_+^2 + \alpha))\right)^{-\frac{1}{2}}}
\]  
(3.16)

Clearly, the denominator term indicates the phase transition points and the second term in numerator is only zero at \(T = 0\). From the left hand side of Fig. 2, one can see that the extrinsic curvature has the same sign as heat capacity does, while the left hand side of Fig. 1 shows that the scalar curvature does not have the same sign as the heat capacity around the phase transition points. It means that the extrinsic curvature manifests more information such as the stability/instability of heat capacity than the Ricci scalar curvature does. It is surprising that the same result obtains for \(C_{\Phi}\) when one sits on a constant \(\Phi\) hypersurface with unit normal vector \(n_\Phi = \frac{1}{\sqrt{g_{NTG}^{\Phi \Phi}}}\). From Eq. (3.15), the extrinsic curvature is calculated by

\[
K_{NTG}^{\Phi} = \frac{lQr_+^2 + 2\alpha (3r_+^4 + \ell^2 \alpha) \Phi \left(\pi (3r_+^6 - l^2(\alpha + r_+^2(\Phi^2 - 1)))\right)^{-\frac{1}{2}}}{3(r_+^6 + 6r_+^4 \alpha + \ell^2(2\alpha^2 + r_+^2(5 - 2\Phi^2) + r_+^4(\Phi^2 - 1))}
\]  
(3.17)
Figure 2: Left: Graph of the phase transition of $C_Q$ (dashed green curve) and the scalar curvature $K_{NTG}(S,Q)$ (solid blue curve) with respect to entropy, $S$, with $Q = 0.5$ for charged GB-AdS black hole. Right: Graph of the phase transition of $C_\Phi$ (dashed green curve) and the scalar curvature $K_{NTG}(S,\Phi)$ (solid blue curve) with respect to entropy, $S$, with $\Phi = 0.5$ for charged GB-AdS black hole. In both diagrams, we have considered $l = 6$, and $\alpha = 0.2$.

Clearly, the extrinsic curvature diverges at phase transition points and exhibit a similar behavior around the transition points as shown in the right side of Fig. (2).

4 NTG geometry, critical behaviors, and phase transitions in the extend phase space

In this section, we investigate our formalism of thermodynamic geometry for a charged GB-AdS black hole in extended phase space. In extended phase space, the cosmological constant (or AdS radius) is treated as thermodynamic variable pressure using the relation $P = \frac{3}{8\pi l^2}$ [6]. Rewriting the AdS radius $l$ in terms of the pressure $P$, the first law for the black hole is

$$dM = TdS + VdP + \Phi dQ + A d\alpha$$  \hspace{1cm} (4.1)

where $V$ and $A$ are the thermodynamic quantities conjugating to pressure $P$, Gauss Bonnet coupling coefficient $\alpha$, respectively. According to Eq. (4.1), it is clear that the black hole mass $M$ should be treated as the enthalpy, i.e., $M \equiv H$ rather than the internal energy $E$ of the system [6]. By making use of Eq. (3.4) and the first law (4.1), the thermodynamic volume is obtained by following relation.

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,\alpha} = \frac{4}{3} \pi v^3 = \frac{\pi}{6} v^3$$  \hspace{1cm} (4.2)

where $v = 2r_+$ denotes the specific volume. Now, one can obtain free energy as follows.

$$F = E - TS = M - PV - TS = \frac{4Q^2 + v^2 - \pi T v^3 + 4\alpha - 16\alpha T v \ln(v/\bar{v})}{4v}$$  \hspace{1cm} (4.3)
According to the differential form for free energy, \( dF = -SdT - PdV + \Phi dQ + \alpha d\alpha \), we have

\[
S = -\left( \frac{\partial F}{\partial T} \right)_{V,Q,\alpha} = \frac{\pi}{4} \left( v^2 + 16\alpha \ln \left( \frac{v}{2} \right) \right) \tag{4.4}
\]

\[
P = -\left( \frac{\partial F}{\partial V} \right)_{T,Q,\alpha} = -\frac{2}{\pi v^2} \left( \frac{\partial F}{\partial v} \right)_{T,Q,\alpha} = \frac{2Q^2}{\pi v^4} - \frac{1}{2\pi v^2} + \frac{T}{v} + \frac{2\alpha}{\pi v^4} + \frac{8T\alpha}{v^3} \tag{4.5}
\]

\[
\Phi = \left( \frac{\partial F}{\partial Q} \right)_{T,V,\alpha} = 2Q \tag{4.6}
\]

\[
A = \left( \frac{\partial F}{\partial \alpha} \right)_{T,V,Q} = \frac{1}{v} - 4\pi T \ln \left( \frac{v}{2} \right) \tag{4.7}
\]

Note that the form of Eq. (4.5) is reminiscent of the state equation for the Van der Waals gas. In Ref. [33], it has been shown that there exists a small-large black hole phase transition of VdW type for 4D GB black hole case via isotherms in \( P-V \) diagram. The critical point can be obtained by solving \((\partial_v P)_T = (\partial_{v,v} P)_T = 0\), which gives [33]

\[
T_c = \frac{8\alpha + 3Q^2 - \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2}}{48\pi \alpha^2} \sqrt{6\alpha + 3Q^2 + \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2}} \tag{4.8}
\]

\[
v_c = 2 \left( 6\alpha + 3Q^2 + \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2} \right)^{1/2}. \tag{4.9}
\]

The signal of a phase transition typically arises when a specific heat capacity changes its sign, which indicates a change of stability. More precisely, a positive heat capacity implies the stability of a thermal system whereas a negative heat capacity shows the instability of the system under small perturbation. Using above equations, the specific heat at constant pressure is given by

\[
C_{P,Q,\alpha} = T \left( \frac{\partial S}{\partial T} \right)_{P,Q,\alpha} = T \left\{ \frac{\partial}{\partial v} \left( S_{P,Q,\alpha} \right) \right\}_{T,Q,\alpha} = \frac{\pi^2 T v (v^2 + 8\alpha)^2}{2 \left( 8Q^2 + v^2 (\pi T v - 1) + 8\alpha (1 + 3\pi T v) \right)} \tag{4.10}
\]

where stability requires \( C_{P,Q,\alpha} > 0 \). Clearly, the specific heat \( C_{P,Q,\alpha} \) becomes singular exactly at the critical point. We here use our formalism of thermodynamic geometry to analysis phase transition behaviors of \( C_{P,Q,\alpha} \). By setting the thermodynamic potential by \( \Xi = H = M = E + PV \) \(^1\) with \( X^i = (S, P, Q, \alpha) \) in Eq. (2.1), the NTG metric is given by

\[
g_{\mu\nu}^{NTG} = \frac{1}{T} \begin{pmatrix}
-H_{SS} & 0 & 0 & 0 \\
0 & H_{PP} & H_{PQ} & H_{P\alpha} \\
0 & H_{QP} & H_{QQ} & H_{Q\alpha} \\
0 & H_{\alpha P} & H_{\alpha Q} & H_{\alpha\alpha}
\end{pmatrix} \tag{4.11}
\]

Since all thermodynamic parameters are written as a function of \( (T, v, Q, \alpha) \), therefore it is convenient to rewrite metric elements from coordinate \( X^i = (S, P, Q, \alpha) \) to favorite

\(^1\)The conjugate potential, \( \Xi = E - TS - \Phi Q - \alpha A \) leads to the same result as the one obtained using enthalpy potential.
coordinate \((T, v, Q, \alpha)\). For this purpose, we first need to redefine metric elements as follows,

\[
H_{SS} = \left(\frac{\partial T}{\partial S}\right)_{P,Q,\alpha} = \frac{\{T, P, Q, \alpha\}_{T,v,Q,\alpha}}{\{S, P, Q, \alpha\}_{T,v,Q,\alpha}} = \frac{-16Q^2 - 2v^2 + 2\pi Tv^3 + 16\alpha + 48\pi Tv\alpha}{\pi^2 v(v^2 + 8\alpha)^2}
\]

\[
H_{PP} = \left(\frac{\partial V}{\partial P}\right)_{S,Q,\alpha} = \frac{\{V, S, P, \alpha\}_{T,v,Q,\alpha}}{\{P, S, Q, \alpha\}_{T,v,Q,\alpha}} = 0
\]

\[
H_{QQ} = \left(\frac{\partial \Phi}{\partial Q}\right)_{S,P,\alpha} = \frac{\{\Phi, S, P, \alpha\}_{T,v,Q,\alpha}}{\{Q, S, P, \alpha\}_{T,v,Q,\alpha}} = \frac{-2}{v}
\]

\[
H_{\alpha \alpha} = \left(\frac{\partial A}{\partial \alpha}\right)_{S,P,Q} = \frac{\{A, S, P, Q\}_{T,v,Q,\alpha}}{\{\alpha, S, P, Q\}_{T,v,Q,\alpha}} = \frac{16}{v(8Q^2 + v^2(\pi Tv - 1) + 8\alpha(1 + \pi Tv)^2)}
\]

\[
H_{PQ} = H_{QP} = \left(\frac{\partial V}{\partial Q}\right)_{S,P,\alpha} = \frac{\{V, S, P, \alpha\}_{T,v,Q,\alpha}}{\{Q, S, P, \alpha\}_{T,v,Q,\alpha}} = 0
\]

\[
H_{P\alpha} = H_{\alpha P} = \left(\frac{\partial A}{\partial P}\right)_{S,Q,\alpha} = \frac{\{A, S, Q, \alpha\}_{T,v,Q,\alpha}}{\{P, S, Q, \alpha\}_{T,v,Q,\alpha}} = \frac{-4\pi v^3 \ln \left(\frac{v}{2}\right)}{v^2 + 8\alpha}
\]

\[
H_{Q\alpha} = H_{\alpha Q} = \left(\frac{\partial A}{\partial Q}\right)_{S,P,\alpha} = \frac{\{A, S, P, \alpha\}_{T,v,Q,\alpha}}{\{Q, S, P, \alpha\}_{T,v,Q,\alpha}} = \frac{-4\pi v^3 \ln \left(\frac{v}{2}\right)}{v^2 + 8\alpha}
\]

Then transferring from the coordinate \((S, P, Q, \alpha)\) to \((T, v, Q, \alpha)\) by below Jacobian matrix,

\[
J = \frac{\partial (S, P, Q, \alpha)}{\partial (T, v, Q, \alpha)} = \begin{pmatrix}
0 & \pi \left(\frac{v}{2} + \frac{4\alpha}{\pi}\right) & 0 & 4\pi \ln \left(\frac{v}{2}\right) \\
\frac{v^2 + 8\alpha}{v^2} & -8Q^2 + v^2 - \pi Tv^3 - 8\alpha(1 + \pi Tv) & 4Q & \frac{4\pi v^2}{\pi^2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

the metric elements convert to

\[
\hat{g} = J^T g^{NTG}_i J = \begin{pmatrix}
0 & 0 & 0 & -\frac{4\pi}{T} \ln \left(\frac{v}{2}\right) \\
0 & -8Q^2 + v^2 (\pi Tv - 1) + 8\alpha(1 + \pi Tv) & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{4\pi}{T} \ln \left(\frac{v}{2}\right) & 0 & 0 & 0
\end{pmatrix}
\]

Thus, the denominator of the scalar curvature is given by

\[
D(R^{NTG}) = \pi \left(8Q^2 + v^2 (\pi Tv - 1) + 8\alpha(1 + \pi Tv)^2\right)^2 \ln \left(\frac{v}{2}\right)^2
\]

The scalar curvature and heat capacity \(C_{P,Q,\alpha}\) is plotted against the specific volume in Fig. 3. From it, we can see that for \(T < T_c\) there exists two divergent points for \(C_{P,Q,\alpha}\). The stable phases with positive specific heat happen in the lower volume and higher volume regions, while the intermediate phase with negative \(C_{P,Q,\alpha}\) value is unstable phase. Therefore when the temperature is smaller than critical value there are three phases possible, i.e., small black hole (SBH), intermediate black hole (IBH) and large black hole (LBH).
Figure 3: Diagram of specific heat ($\times 10^6$) and scalar curvature ($\times 10^4$) versus specific volume $v$ for 4D GB- AdS black hole. From left to right we consider $T = \{0.0444, 0.0474, 0.0494\}$ where $T_c = 0.0474$ for fixed value of electric charge $Q = 0.5$ and Gauss-Bonnet coupling $\alpha = 0.2$.

By increasing temperature to $T = T_c$, these two divergent points get close and coincide at $v = v_c = 4.079$ to form a single divergence where the unstable region disappears. For $T > T_c$ the heat capacity is always positive and the divergent point vanishes. It means that the black hole is stable and there is no phase transition. Moreover, in all diagrams shown in Fig. 3, the scalar curvature is positive in $0 < v < 2$ which implies a repulsive interaction between the microscopic molecules. In continue, we shall discuss about this region.

Allow us now to analyze the nature of the phase transition through the concept of thermodynamic hypersurface in lower dimensions. To do this, we force ourselves to sit down on the constant $Q$ hypersurface with the orthogonal normal vector,

$$n_Q = \frac{1}{\sqrt{|g^{QQ}|}} = \sqrt{\frac{2}{Tv}}.$$  \hspace{1cm} (4.16)

Therefore, the extrinsic curvature is given by

$$K_{NTG} = \frac{4Q\sqrt{2Tv}}{8Q^2 + v^2(\pi Tv - 1) + 8\alpha(1 + 3\pi Tv)}$$  \hspace{1cm} (4.17)

It is obvious that this curvature diverges at the phase transition point and exhibit a similar sign behavior around the transition points as illustrated in Fig. 4. Interestingly, for a
neutral GB-AdS black hole case, we need to live on the $Q$-zero hypersurface. It should be noted that, within the hypersurface framework, setting $Q$ to zero is equivalent to living on the constant $Q$ hypersurface ($Q$-zero hypersurface). By using metric elements in Eq. (4.14), the metric elements induced on this hypersurface are obtained by

$$g^{in} = \begin{pmatrix} 0 & 0 & -\frac{4\pi}{T} \ln \left( \frac{v}{2} \right) \\ 0 & -\frac{v^2(\pi Tv - 1) + 8\alpha(1 + 3\pi Tv)}{2Tv^3} & 0 \\ -\frac{4\pi}{T} \ln \left( \frac{v}{2} \right) & 0 & 0 \end{pmatrix}$$  \tag{4.18}

Therefore, the intrinsic curvature $R^{in}$ of $Q$-zero hypersurface reads

$$R^{in} = \frac{\pi T v^2 - \pi T v^3 - 8\alpha - 24\pi T v \alpha - 2(v^2(-1 + 2\pi T v(2 + \pi T v) + 8\alpha(1 + 2\pi T v))) \ln \left( \frac{v}{2} \right)}{\pi \left( v^2(\pi T v) - 1 + 8\alpha(1 + 3\pi T v)^2 \ln \left( \frac{v}{2} \right)^2 \right)}$$  \tag{4.19}

Note that the Ricci scalar in four-dimensions is related to the above curvature in three dimensions via Gauss-Codazzi relation. Before discussing about the case study, it seems
useful to identify the critical point for neutral GB-AdS black hole. With the help of the equation of state Eq. (4.5), the critical point in neutral GB-AdS is given by

\[ T_c = \frac{2\sqrt{3} - 3}{6\pi \sqrt{2\alpha}} \quad v_c = 2\sqrt{2\alpha} \sqrt{3 + 2\sqrt{3}} \quad P_c = \frac{15 - 8\sqrt{3}}{288\pi \alpha} \] (4.20)

and the ratio,

\[ \frac{P_c v_c}{T_c} = \frac{1}{12} \left( 6 - \sqrt{3} \right) \] (4.21)

is slightly smaller than Van-der Waals ratio $3/8$ [6]. In the reduced parameter space, the equation of state (4.5) has the following form

\[ \hat{P} = \frac{9 - 4\sqrt{3}}{11\hat{v}^4} + 4(3\sqrt{3} - 4) \frac{T}{11\hat{v}^3} - 6(1 + 2\sqrt{3}) \frac{1}{11\hat{v}^2} + 4(6 + \sqrt{3}) \frac{T}{11\hat{v}} \] (4.22)

where the reduced pressure, temperature, and specific volume are defined by $\hat{P} = \frac{P}{P_c}$, $\hat{T} = \frac{T}{T_c}$, and $\hat{v} = \frac{v}{v_c}$. More interestingly, this reduced state equation does not depend on $\alpha$ parameter. Now one can identify the spinodal curve which satisfies $(\partial_v \hat{P})_{\hat{T}} = 0$. Indeed, this curve separates the metastable phase from the unstable phase, and the heat capacity and the scalar curvature diverge along this curve. Solving this condition, the spinodal curve has a below compact form

\[ T_{sp} = \frac{3(2 + \sqrt{3})\hat{v}^2 - \sqrt{3}}{3\hat{v} + (3 + 2\sqrt{3})\hat{v}^3} \] (4.23)

where $\frac{\sqrt{2} - \sqrt{3}}{\sqrt{3}} < \hat{v} < 1$ is for the small black hole spinodal curve, while $\hat{v} > 1$ is for the large black hole spinodal curve (see left had side of Fig. 5) \(^2\).

\[ \text{Figure 5: Left: Spinodal curve of the four-dimensional neutral GB-AdS black hole, Right: The sign-changing curve where the scalar curvature } R^{\text{in}} \text{ changes its sign.} \]

In Fig. 6, we have drawn the behavior of $R^{\text{in}}$ and $C_{P,\alpha} = C_{P,Q,\alpha}(Q = 0)$ for $T > T_c$, $T = T_c$ and $T < T_c$, respectively. When $T > T_c$, we observe three negative divergent

\(^2\) Because of the logarithmic correction term appeared in entropy, it may be hard to obtain an analytical expression for the coexistence curve for 4D Gauss-Bonnet AdS black holes, therefore we leave it for readers to verify.
points where the fist happens at $v = 2$ generated from $\ln(v/2)$ in denominator of curvature. Moreover, the other two points get closer with the increase of $T$. At the critical temperature $T = T_c = 0.0571$, these two divergent points merge to form a single divergence at $v = v_c = 3.215$. More importantly, in all figures, we observe a positive $R^{in}$ near the region $0 < v < 2$, which indicates a repulsive interaction among the microstructure of the black hole system. This is in contrast to the five dimensional (uncharged) case where the attractive interaction is dominant in the microstructure of the black hole system [22]. In the right hand side of Fig. 5, we also plot sign-changing curve of $R^{in}$. Below the sign-changing curve, $R^{in}$ is positive, whereas above it, $R^{in}$ will be negative. It is clear that by increasing the temperature value, this positive region shown in Fig. 6, disappears and the scalar curvature takes the negative values.

5 Conclusions

In this paper, we first have generalized the NTG formalism by considering other thermodynamical potentials via Legendre transformations. This formulation yields a one-to-one correspondence between heat capacity phase transitions and curvature singularities. Next
we applied the NTG geometry to study the critical behaviors of heat capacities when \(\alpha\) and \(\Lambda\) (or \(l\)) don’t have any variations in first law of thermodynamics. We found that the critical behavior of a heat capacities on an explicit hypersurface can be explained by using intrinsic and extrinsic curvatures of this hypersurface. By including the cosmological constant as a thermodynamic pressure in the first law of black hole thermodynamics, we studied the thermodynamic phase transition and NTG geometry of the four dimensional charged GB-AdS black hole. By setting on \(Q\)-zero hypersurface, we studied phase transitions and the microstructure of a neutral 4D GB black hole. More importantly, there is a positive intrinsic curvature, \(R^{in}\), near the region \(0 < v < 2\), which indicates a repulsive interaction among the microstructure of the black hole system. At high temperature, this region disappears and scalar curvature takes the negative values, which indicates that only attractive interaction exists among the microstructures.

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**A  Bracket notation and Partial derivative**

The partial derivative of functions with explicit form of \(n + 1\) variables can be defined by [18],

\[
\left(\frac{\partial f}{\partial g}\right)_{h_1,\ldots,h_n} = \frac{\{f, h_1,\ldots,h_n\}_{q_1,q_2,\ldots,q_{n+1}}}{\{g, h_1,\ldots,h_n\}_{q_1,q_2,\ldots,q_{n+1}}} \tag{A.1}
\]

where all of \(f, g,\) and \(h_n\) \((n = 1, 2, 3, \ldots)\) are functions of \(q_i, i = 1,\ldots,n + 1\) variables and \{\ldots\} is Nambu brackets which is defined as,

\[
\{f, h_1,\ldots,h_n\}_{q_1,q_2,\ldots,q_{n+1}} = \sum_{ijk\ldots l=1}^{n+1} \varepsilon_{ijk\ldots l} \frac{\partial f}{\partial q_i} \frac{\partial h_1}{\partial q_j} \frac{\partial h_2}{\partial q_k} \ldots \frac{\partial h_n}{\partial q_l} \tag{A.2}
\]

In a simple case, when we consider only \(f, g,\) and \(h\) as explicit functions of \((a, b)\), the above formula reduces to

\[
\left(\frac{\partial f}{\partial g}\right)_h = \{f, h\}_{a,b} \tag{A.3}
\]

where \{\ldots\} is Poisson bracket defined by

\[
\{f, h\}_{a,b} = \left(\frac{\partial f}{\partial a}\right)_b \left(\frac{\partial h}{\partial b}\right)_a - \left(\frac{\partial f}{\partial b}\right)_a \left(\frac{\partial h}{\partial a}\right)_b \tag{A.4}
\]

**B  Some of the other heat capacities**

In this appendix, we can verify phase transition signals of some specific heats like,

\[
C_{V,Q,\alpha} = T \left(\frac{\partial S}{\partial T}\right)_{V,Q,\alpha} = T \frac{\{S,V,Q,\alpha\}_{T,V,Q,\alpha}}{\{T,V,Q,\alpha\}_{T,V,Q,\alpha}} = 0 \tag{B.1}
\]
and
\[ C_{P,\Phi,A} = T \left( \frac{\partial S}{\partial T} \right)_{P,\Phi,A} = T \left\{ S, P, \Phi, A \right\}_{T,v,Q,\alpha} \]
\[ = -\frac{4\pi^2 T \ln \left( \frac{v}{2} \right) \left[ (1 + 4\pi T v)(v^2 + 8\alpha) + 2(4Q^2 + \pi^2 T v - 1) + 8\alpha(1 + 3\pi T v) \right] \ln \left( \frac{v}{2} \right)}{(1 + 4\pi T v)^2} \]  

We find that there is no phase transition for the above heat capacities. This result can also be satisfied by using NTG geometry. To do this, let us select \( \Xi = F = E - TS = M - PV - TS \) with coordinate \( X_i = (T, V, Q, \alpha) \) in Eq. (2.1) which yields
\[ g^\text{NTG}_F = \frac{1}{T} \begin{pmatrix} -F_{TT} & 0 & 0 & 0 \\ 0 & F_{VV} & F_{VQ} & F_{V\alpha} \\ 0 & F_{QV} & F_{QQ} & F_{Q\alpha} \\ 0 & F_{\alpha V} & F_{\alpha Q} & F_{\alpha\alpha} \end{pmatrix} = \begin{pmatrix} \frac{C_{V,Q,\alpha} T^4}{\pi^2 T v} & 0 & 0 & 0 \\ 0 & \frac{2(3Q^2 + v^2(\pi T v - 1) + 8\alpha(1 + 3\pi T v))}{\pi^2 T v} & -\frac{4Q}{\pi T v^3} & -\frac{2 + 8\pi T v}{\pi T v^3} \\ 0 & -\frac{4Q}{\pi T v^3} & \frac{2}{T v} & 0 \\ 0 & -\frac{2 + 8\pi T v}{\pi T v^3} & 0 & 0 \end{pmatrix} \]  

in which we find that its vanishing heat capacity \( C_{V,Q,\alpha} \) leads to \( g_{TT} \to 0 \) or \( g_{TT} \to \infty \) and so the metric is not invertible. As a simple trick to evaluate scalar curvature, we will treat \( C_{V,Q,\alpha} \) as a constant with its value infinitely close to zero [20]. Therefore, we can obtain the normalize the scalar curvature \( R_N \) as
\[ R_N^\text{NTG} = C_{V,Q,\alpha} R^\text{NTG} = -\frac{3 + 8\pi T v(2 + \pi T v)}{(1 + 4\pi T v)^2} \]  

Analogous with the charged AdS black hole, this normalized scalar curvature \( R_N \) is independent of \( \alpha \). Clearly, we observe \( R \) is always negative, which implies a attractive interaction among the microstructure of the black hole system. Moreover, for constant \( T \) hypersurface with normal vector \( n_T = \sqrt{\frac{C_{V,Q,\alpha} T}{T^4}} \), the extrinsic curvature is
\[ K_N^\text{NTG} = \sqrt{C_{V,Q,\alpha} K^\text{NTG}} = -\frac{3 + 4\pi T v}{(1 + 4\pi T v)^2} \]  

which manifests that there is no phase transition.

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