ON THE VERTICAL EQUILIBRIUM OF THE LOCAL GALACTIC DISK AND THE SEARCH FOR DISK DARK MATTER

F. J. Sánchez-Salcedo¹, Chris Flynn²,³, and A. M. Hidalgo-Gámez⁴

¹ Instituto de Astronomía, Universidad Nacional Autónoma de México, Ciudad Universitaria, Apt. Postal 70 264, C.P. 04510, Mexico City, Mexico; jsanchez@astro.unam.mx
² Department of Physics and Astronomy, University of Sydney, NSW 2006, Australia
³ Finnish Centre for Astronomy with ESO, University of Turku, FI-21500 Piikkiö, Finland
⁴ Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, U.P. Adolfo López Mateos, C.P. 07738, Mexico City, Mexico

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ABSTRACT

Estimates of the dynamical surface mass density at the solar Galactocentric distance are commonly derived assuming that the disk is in vertical equilibrium with the Galactic potential. This assumption has recently been called into question, based on the claim that the ratio between the kinetic and the gravitational energy in such solutions is a factor of three larger than required if virial equilibrium is to hold. Here we show that this ratio between energies was overestimated and that the disk solutions are likely to be in virial equilibrium after all. We additionally demonstrate, using one-dimensional numerical simulations, that the disks are indeed in equilibrium. Hence, given the uncertainties, we find no reason to cast doubt on the steady-state solutions which are traditionally used to measure the matter density of the disk.

Key words: Galaxy: disk – Galaxy: kinematics and dynamics – Galaxy: structure

1. INTRODUCTION

Studies on the amount of unseen mass associated with the Galactic disk have been carried out since Oort (1932) and up to the present day (e.g., Kuijken & Gilmore 1991; Bahcall et al. 1992; Crézé et al. 1998; Siebert et al. 2003; Holmberg & Flynn 2004; Kalberla et al. 2007; Moni Bidin et al. 2010). All recent studies indicate that there is no dynamically significant amount of disk dark matter at the solar position, although a small fraction (<10%) cannot be ruled out. In order to derive the dynamical surface mass density, the usual way to proceed is to consider a tracer of the potential, such as K giant stars or H I gas. In principle, ideal tracers are old enough (>1 Gyr) that they are expected to be well mixed into the Galactic potential, so that they satisfy the virial condition. In a recent paper, Garrido Pestaña & Eckhardt (2010, hereafter GPE) compare the kinetic energy $T$ of stellar and gaseous components in the local Galactic disk, to their gravitational energy $W$, and conclude that the vertical configuration is not in a state of equilibrium because $|2T/W| \approx 3$ and, therefore, the classical Poisson–Boltzmann approach used to interpret the data is an inappropriate and futile exercise. If the energy in vertical random motions is a factor of three larger than in virial equilibrium, the local portion of the disk should be in a phase of violent relaxation. Under such conditions, the thickness of the disk would inflate by a factor of 5/3 on average (see Section 2) after just a few vertical crossing times ($\approx 10^8$ yr). It is unlikely that we would be observing the disk at a time just prior to such a violent event, and if we were, the classical view that the disk thickening is due to slow dynamical heating due to encounters with massive molecular clouds, spiral arms, and gravitational perturbations caused by minor mergers should change dramatically. In this Letter, we show, however, that there is no reason for abandoning the assumption that the vertical configuration is in a steady state.

2. STEADY-STATE CONFIGURATION

Consider a rotationally symmetric disk stratified in the $z$-direction, where $z = 0$ is the disk plane of symmetry, and composed of $N$ components in equilibrium. Each component, either if it is collisionless (stars and dark matter) or collisional (gas), satisfies a Boltzmann equation. The vertical Jeans equations can be combined to obtain

$$\sum_{i=1}^{N} \frac{\partial (\rho_i \sigma_i^2)}{\partial z} = -\rho \frac{\partial \Phi}{\partial z},$$

where $\sigma_i(R, z)$ is the vertical velocity dispersion of the component $i$, $\Phi(R, z)$ is the gravitational potential, and $\rho(R, z) \equiv \sum \rho_i(R, z)$. For gaseous components, $\sigma_i$ is the effective velocity dispersion, which includes the pressure support by turbulent motions, magnetic fields, and cosmic rays. In a real galaxy, gradients in the radial and azimuthal directions generate nondiagonal terms of the velocity dispersion tensor, i.e., $\sigma_{Rz}$ and $\sigma_{\theta z}$, in the vertical Jeans equation. At $z \lesssim 1$ kpc, the fractional corrections due to the cross terms are less than 1% and are usually ignored except when studying stars at $z > 2$ kpc (e.g., Bahcall 1984; Olling 1995; Becquaert & Combes 1997; Kalberla 2003; Moni Bidin et al. 2010).

Following GPE, we start by considering an infinite plane-parallel layer of total column density $\Sigma_\infty$ in equilibrium with its own gravity. By choosing the zero of the potential at $z = 0$ and multiplying the above equation by $z$ and integrating over $z$, one obtains the virial theorem, linking the kinetic and gravitational energies with surface terms. If the surface terms are null at $z \to \pm \infty$, the relationship is $2T = W$, where $W$ and $T$ are the gravitational and kinetic energies (due to vertical motions), respectively, per surface unit in the disk. GPE show that the virial theorem can be rewritten as

$$\tilde{K} = \frac{\rho(0)(\sigma^2)}{\pi G \Sigma_\infty},$$

and

$$Q = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a \rho(z) p(z + a) dz da,$$

Note that $W > 0$ in this case. GPE use a different sign convention.
where \( \rho(z) = \rho(z)/\Sigma_{\infty} \). Here, \( (\sigma_i^2) \) is the mass-weighted mean-square velocity dispersion, i.e., \( (\sigma_i^2) \equiv 2T/\Sigma_{\infty} \).

For the particular case of having a single self-gravitating isothermal component \((N = 1)\), it holds that \( Q \) is exactly \( 1/2 \), whereas it is slightly different from \( 1/2 \) for a Gaussian density profile in equilibrium \((Q = 0.45)\). GPE argue that \( Q \simeq 1/2 \) and evaluate the left-hand side of Equation (2) for the Galactic disk at the solar position and find that it has the value \( 1.6 \). Since this is a factor of \( \sim 3 \) larger than required if the virial equilibrium is to hold, they conclude that the disk at the solar position is not in a steady-state vertical configuration.\(^6\)

In the following, we will check the robustness of the assumption that \( Q \simeq 1/2 \). To do so, we start by integrating Equation (1) from the midplane to a certain distance \( z \) above the midplane and obtain

\[
\sum_{i=1}^{N} \rho_i(z)\sigma_i^2(z) = \sum_{i=1}^{N} \rho_i(0)\sigma_i^2(0) - \int_{0}^{z} \rho(z')\frac{\partial \Phi}{\partial z'} dz'.
\]

For a system such that the volume density of kinetic energy decays to zero at large \( z \), Equation (4) becomes

\[
\sum_{i=1}^{N} \rho_i(0)\sigma_i^2(0) = \int_{0}^{\infty} \rho(z')\frac{\partial \Phi}{\partial z'} dz',
\]

and tells us that the total pressure at the midplane must be in balance with the weight of all the material above \( z = 0 \).

In the particular case of a stratified plane-parallel layer, all the variables depend only on \( z \). Integration of the Poisson equation allows us to find the vertical acceleration. If the layer is confined by its own gravity, the vertical acceleration for \( z > 0 \) reads

\[
\frac{d\Phi}{dz} = 2\pi G \Sigma(z),
\]

where \( \Sigma(z) \) is defined for \( z > 0 \) as

\[
\Sigma(z) = \int_{-z}^{z} \rho(z')dz'.
\]

Combining Equations (5) and (6), and using the relation

\[
\rho(z) = \frac{1}{2} \frac{d\Sigma}{dz},
\]

we find

\[
\sum_{i=1}^{N} \rho_i(0)\sigma_i^2(0) = \frac{\pi G}{2} \int_{0}^{\infty} \Sigma(z')\frac{d\Sigma}{dz'} dz' = \frac{1}{2} \pi G \Sigma_{\infty}^2,
\]

which can be expressed as

\[
\frac{\sum_{i=1}^{N} \rho_i(0)\sigma_i^2(0)}{\pi G \Sigma_{\infty}^2} = \frac{1}{2}.
\]

Note that the above equation is exact for a plane-parallel layer in equilibrium. Therefore, the total pressure at \( z = 0 \) (not \( \rho(0)/(\pi G \Sigma_{\infty}^2) \) as stated in GPE) divided by \( \pi G \Sigma_{\infty}^2 \) must be \( 1/2 \) in a steady-state plane-parallel layer. GPE analysis applied to a plane-parallel layer is strictly correct only if \( \rho(0)/(\pi G \Sigma_{\infty}^2) \) is a factor of \( 2 \) the dark matter halo, respectively, \( Q \equiv R^{-1}\partial\Phi/\partial R \), and \( G \equiv R(\partial\Omega/\partial R) \) (e.g., Kuijken & Gilmore 1989; Ferréria 1998).

At \( R > 5 \) kpc and within a few kpc from the disk plane, \( \Omega + G \simeq 0 \) (e.g., Kuijken & Gilmore 1989) and thus

\[
\frac{\partial \Phi}{\partial z} = 2\pi G \Sigma(z) + \frac{v_z^2}{R} \frac{z}{\sqrt{R^2 + z^2}},
\]

where \( v_z(R) \) is the contribution of the dark halo to the rotation curve: \( v_z \simeq 140 \) km s\(^{-1}\) at the solar position (e.g., Flynn et al. 1996; Klypin et al. 2002). In the particular case of a massless disk embedded in the spherical potential of the isothermal dark halo, we recover the classical result that the vertical epicyclic frequency coincides with the orbital frequency at the midplane, \( \beta^2 \Phi/\partial z^2 |_{z=0} = \Omega^2 \). Substituting Equation (12) into Equation (5) and for \( z \ll R \), we obtain

\[
\sum_{i=1}^{N} \rho_i(0)\sigma_i^2(0) \simeq \frac{1}{2} + (\ln 2) \left( \frac{v_z^2}{2\pi G \Sigma_{\infty}} \right) \sum_{i=1}^{N} \left( \frac{\Sigma_i}{\Sigma_{\infty}} \right),
\]

where the index \( i \) refers to the disk components, \( \Sigma_i \) is the vertical scale height of population \( i \), and \( \Sigma_{\infty} = \Sigma_{\infty}(z = \infty) \).

Evaluation of \( \sum \rho_i(0)\sigma_i^2 \) (not including the stellar halo) in the updated disk model of Holmberg & Flynn (2004) reported in Table 2 of Flynn et al. (2006) gives 20.2 \( M_{\odot} \) pc\(^{-2}\) km\(^2\) s\(^{-2}\). Taking the total surface density in disk components of \( \Sigma_{\infty} = 48.7 \) \( M_{\odot} \) pc\(^{-2}\), the left-hand side of Equation (13) is 0.61. We then compute the right-hand side of Equation (13) at \( R = R_0 \approx 8 \) kpc and find that it is \( \simeq 0.56 \), which is only 9% smaller than the left-hand side. This simple estimate indicates that the disk model of Flynn et al. (2006) is very close to satisfying the virial theorem. In order to assess whether this difference of 10% is significant, we have calculated \( \rho_i(z) \) in the model of Flynn et al. (2006) and computed the right-hand side of Equation (13) exactly. We confirmed that the disk model fulfills the virial condition.

Why did GPE obtain that the disk models by Bahcall et al. (1992) and Flynn et al. (2006) do not comply with the virial theorem? It was a consequence of the combination of three factors: (1) they adopted \( Q = 1/2 \), which is in error by a factor of two (see below), (2) the stellar halo was treated as a disk component, which introduces an additional artificial factor of 1.25, and (3) the contribution of the dark halo in the vertical confinement of disk stars was not taken into account. As a consequence, the ratio between kinetic and gravitational energies was overestimated by a factor of three.

For a stratified plane-parallel self-gravitating layer, it is true that \( R = Q \) in virial equilibrium. However, \( Q \) is not necessarily \( 1/2 \) in a generic disk. In fact, the form of the left-hand side of
the equilibrium condition used in GPE (Equation (2)) can be preserved by introducing a “correction” factor $\epsilon$ on the right-hand side:

$$\frac{\rho(0)(\sigma^2)}{\pi G \Sigma^2} = \frac{\epsilon}{2},$$

(14)

with

$$\epsilon = \frac{\rho(0)(\sigma^2)}{\sum \rho_i(0)\sigma_i^2}. \quad (15)$$

The value of $\epsilon$ depends on the vertical structure of the disk. For a single Gaussian (non-isothermal) density profile in $z$, $\epsilon = (\sigma^2)/\sigma^2(0) = 2\sqrt{2}/\pi = 0.9$, and hence $Q = 0.45$, as reported in GPE. However, for the 14th components of the updated disk model of Holmberg & Flynn (2004), the $\epsilon$ value in the virial theorem criterion for a state of equilibrium is 2 and thus $Q \approx 1$. To be more precise, we adopted the values of $\rho(0)$ and $\sigma_i$ for the 14th components of Flynn et al. (2006) and solved the Poisson–Boltzmann equation for such a disk with no dark halo, in order to compare with the GPE analysis on a common ground. We obtain $\rho(0) = 0.091 M_\odot $ pc$^{-3}$, $\Sigma_\infty = 55 M_\odot $ pc$^{-2}$, $(\sigma^2)^{0.5} = 22.25 km$ s$^{-1}$, and $\bar{R} = Q = 1.1$. Therefore, GPE analysis applied to the infinite plane-parallel multicomponent layer overestimates the $T/W$ ratio by a factor of 2.2 because

$$\frac{\rho(0)(\sigma^2)}{\sum \rho_i(0)\sigma_i^2} = 2.2. \quad (16)$$

3. SIMULATIONS

In the last section, we have shown that the virial theorem is satisfied for the Galactic disk models under consideration. However, the virial theorem is a relation between global quantities and is not a guarantee that the system is in a steady-state equilibrium. In order to dispel any doubt, we have tested the temporal stability of the type of disks considered here using $N$-body simulations, in which stars are initially set up in a one-dimensional density and velocity distribution, based on solutions to the Poisson–Boltzmann equation, after which we calculate their vertical motions under the gravitational force generated by the overall density distribution of the disk.

At each time step in the simulation, the downward acceleration at height $z$ above the disk is found by integrating numerically the density profile of all the stars (Equation (6)) to absolute height $|z|$. A leap-frog integrator is then used to compute new positions and velocities for the particles, and the process repeated. We keep track of the velocity dispersion, scale height, and other quantities for the ensemble to check if it is in vertical equilibrium or not.

Three disk types were considered. First, we looked at an “isothermal disk,” in which the initial conditions are a $\text{sech}^2(z/h)$ distribution in density (with a scale height $h$) and a (constant) velocity dispersion $\sigma(z)$, given by $\sigma = h\sqrt{2\pi G \rho(0)}$, where $\rho(0)$ is the disk central density. The density distribution in such a disk is given by $\rho(z) = \rho(0)\text{sech}^2(z/2h)$. We adopt $\rho(0) = 0.1 M_\odot $ pc$^{-3}$ and $h = 250 $ pc, implying a total disk surface density of $50 M_\odot $ pc$^{-2}$ and a velocity dispersion of $\sigma = 13.0 km$ s$^{-1}$. These are similar to the measured values for the local Galactic disk.

Second, we looked at an “exponential disk,” in which the density distribution is given by $\rho(z) = \rho(0) \exp(-|z|/h)$, and the velocity dispersion is a function of $z$, being smaller close to the midplane and rising asymptotically to a constant value at large height. The velocity dispersion $\sigma(z)$ is given by

$$\sigma(z) = 2h\sqrt{\pi G \rho(0)}(1 - 0.5 \exp(-|z|/h)). \quad (17)$$

Adopting a scale height of $h = 250 $ pc and a central density of $0.1 M_\odot $ pc$^{-3}$, this disk has a surface density of $50 M_\odot $ pc$^{-2}$, like the isothermal disk. The central velocity dispersion for this disk rises from $13.0 km$ s$^{-1}$ at $z = 0$ to $18.4 km$ s$^{-1}$ at large $z$.

The simulations of these two disks were run with 10 million particles, for 5 Gyr, over 1 Myr time steps. This time step was found to be a good compromise between the length of the runs and the accuracy of the orbits. Note that the oscillation time for stars in the tested potentials is typically 30–80 Myr. Both disks are completely self-gravitating, with the gravitational field provided by the density distribution of the particles alone.

Over 5 Gyr, the central density, mean velocity dispersion, and scale heights of the simulated disks remain within a few percent of their initial values. Thus, the exponential disk and the isothermal disk are found to be stationary in the simulations. Furthermore, this is what we expect, since GPE’s analysis shows that, for both simulated disks, the virial equilibrium condition is met very precisely. Thus, no long-term secular evolution is expected for either disk, and this we fully confirm with the simulations.

We then simulated the much more complex system, consisting of all components, stellar and gaseous, of the disk in Flynn et al. (2006). This disk consists of 14 isothermal components, with a range of central densities and (isothermal) velocity distributions. The stellar halo, which is the 15th component of the Flynn et al. (2006) disk model, is ignored (including it in the solutions is found to have no effect on the disk stability, but we focus in the following on disk components only). The gaseous components are modeled as collisionless particles, just like the stars, since the aim is to test the Poisson–Boltzmann solution, which implicitly makes this assumption. We also include the effect of the halo dark matter in these simulations, not by simulating it with particles, but by adding it into the computation of the vertical force as an extra density term (as a function of $z$).

The simulation is run with 10 million particles for a period of 5 Gyr with time steps of 1 Myr. Energy conservation in the simulation is better than 0.1 percent. We find that this complex disk is also stationary. The density and velocity distributions of the particles after 5 Gyr are very similar to their initial distributions, and so no large scale secular evolution takes place. If this rather complex disk were substantially out of equilibrium initially, we would expect rapid evolution in these quantities. Figure 1 shows the results. Panel (a) shows the evolution of kinetic energy (dotted line), potential energy (dashed line), and total energy (solid line) over the 5 Gyr of the simulation. Panel (b) shows the total velocity dispersion as a function of time. In panels (c) and (d), we show the initial (circles) and final (crosses) density and velocity distributions of the particles as a function of $z$ height. The initial and final distributions are essentially indistinguishable. Panel (e) shows the time evolution of the central density, and panel (f) the scale height $h$, of an exponential fit to the density profile, measured from the midplane to $z \approx 750 $ pc. None of the plots show any significant evolution in these quantities with time. We conclude from the simulations that the Flynn et al. (2006) model disk is indeed in equilibrium, as we concluded from theoretical analysis in the previous section, and contrary to the claims in GPE.

In order to illustrate the role of the dark halo in confining the disk, we rerun the simulation, but the dark halo contribution to the force was turned off so that the disk is not initially in
Figure 1. Simulation of the secular evolution of the Flynn et al. (2006) model disk. Panel (a) shows the total energy (solid line), the kinetic energy (dotted), and the potential energy (dashed) as functions of time, from 0 to 5 Gyr. Panel (b) shows the total velocity dispersion of all the particles as a function of time. The central panels, (c) and (d), show the initial (circles) and final (crosses) density and velocity dispersion distributions of the particles as functions of height above the disk $z$; the distributions are close to identical, indicating that the disk solutions are stationary and that there is no long-term evolution in its structure. Panels (e) and (f) show the evolution of central density and scale height as functions of time. All indicators demonstrate that the disk is in a stationary state in the long term.

Figure 2. Simulation of the secular evolution of the Flynn et al. (2006) model disk, as in Figure 1, but with the special case that the dark matter halo has been removed from the simulation. All panels are as for Figure 1, except panel (f), which shows the quantities $\tilde{R}$ (dashed line) and $Q$ (solid line). Note that the time axis shows 0–1 Gyr, to highlight the rapid response of the disk. The disk is initially out of virial equilibrium, but adjusts itself to equilibrium within a few dynamical times, arriving at the state $\tilde{R} = Q$. Since the gravity of the dark halo is lacking, which would help to confine the disk, the response is of the potential energy to increase which kinetic energy decreases, i.e., the disk thickens (panel e) and becomes kinematically cooler (panel d).

4. DISCUSSION AND CONCLUSIONS

We have examined the claim by GPE that solutions for the density distribution of matter in the local Galactic disk are seriously in error because such solutions are not in virial equilibrium. GPE claim that the ratio between the kinetic and the gravitational energy in such disks is a factor of three larger than required for virial equilibrium to hold. We have shown that this ratio between energies was overestimated by GPE because (1) they assumed that $Q = 1/2$, (2) the stellar halo was treated as a disk component, and (3) the vertical confinement provided by the dark halo was ignored.

Taking into account these corrections, the local disk model of Flynn et al. (2006) is likely to be in virial equilibrium after all. Our technique could easily be applied to any such model (e.g., Bahcall et al. 1992; Just & Jahreiss 2010) and yields a similar result. Simulations of the oscillation of stars in model disks show that the initial distributions of stars in density and velocity are stationary, backing up our theoretical analysis.

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