The classification of finite-dimensional triangular Hopf algebras over an algebraically closed field of characteristic 0

Pavel Etingof
Massachusetts Institute of Technology
Department of Mathematics
Cambridge, MA 02139, USA
email: etingof@math.mit.edu

Shlomo Gelaki
Technion-Israel Institute of Technology
Department of Mathematics
Haifa 32000, Israel
email: gelaki@math.technion.ac.il

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1 Introduction

The first step towards the classification of finite-dimensional triangular Hopf algebras $H$ over an algebraically closed field $k$ of characteristic 0 was taken in [EG1, Theorem 2.1] where it was proved that if $H$ is semisimple then it is obtained from the group algebra $k[G]$ of a finite group $G$ by twisting its comultiplication in the sense of Drinfeld [Dr]. The proof of this theorem relies in an essential way on a theorem of Deligne on Tannakian categories [De1] which characterizes symmetric tensor rigid categories over $k$ which are equivalent to representation categories of affine proalgebraic groups intrinsically as those categories in which the categorical dimensions of objects are non-negative integers. One can apply Deligne Theorem [De1] since the representation category $\text{Rep}(H)$ of $H$ has this property (maybe after modifying its commutativity constraint). Later on we used [EG1, Theorem 2.1] and the theory of Movshev on twisting in finite groups [M] to completely classify semisimple triangular Hopf
algebras over \( k \) in terms of certain quadruples \( (G, A, V, u) \) of group-theoretical data \([EG2]\), and obtained a similar classification in positive characteristic \([EG2]\).

However, for non-semisimple finite-dimensional triangular Hopf algebras \( H \) over \( k \) it is no longer true that the categorical dimensions of objects in \( \text{Rep}(H) \) are non-negative integers; so Deligne Theorem \([De1]\) cannot be applied. Nevertheless, in \([AEG]\) it was realized that all known examples of finite-dimensional triangular Hopf algebras \( H \) over \( k \) have the Chevalley property; namely, the semisimple part of \( H \) is itself a Hopf algebra. Then in \([AEG, \text{Theorem 5.1.1}]\), using \([EG1, \text{Theorem 2.1}]\) and hence Deligne Theorem \([De1]\), it was proved that \( H \) has the Chevalley property if and only if it is obtained from a certain modification of the supergroup algebra \( k[G] \) of a finite supergroup \( G \) by twisting its comultiplication. Later on we used \([AEG, \text{Theorem 5.1.1}]\) to completely classify finite-dimensional triangular Hopf algebras over \( k \) with the Chevalley property in terms of certain septuples \( (G, W, A, Y, B, V, u) \) of group-theoretical data \([EG3]\). Nevertheless, we could not show that this list contains all possible finite-dimensional triangular Hopf algebras over \( k \).

Very recently, Deligne has generalized his theorem on Tannakian categories \([De1]\) to the super-case \([De2]\). The purpose of this note is to explain how this remarkable generalized theorem, combined with the results from \([AEG,EG1,EG2,EG3]\), lead to the complete and explicit classification of finite-dimensional triangular Hopf algebras over an algebraically closed field \( k \) of characteristic 0, and to answer some questions from \([AEG,G2]\) about triangular Hopf algebras and symmetric rigid tensor categories over \( k \), positively. We also use Deligne Theorem and the lifting theorem of Etingof-Nikshych-Ostrik \([ENO]\) to prove that a symmetric fusion category over a field of characteristic \( p > 0 \), whose global dimension is nonzero, is equivalent to a representation category of a unique finite group whose order is not divisible by \( p \) (see Theorem 5.4 below). We emphasize that although the results of this note are useful for Hopf algebra theory, they (with the exception of Theorem 5.4) follow from \([De2]\) and \([AEG,EG1,EG2,EG3]\) in a fairly straightforward manner.

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2 Preliminaries

Throughout this note, unless otherwise stated, \( k \) will denote an algebraically closed field of characteristic 0.

Recall that an affine algebraic supergroup is the spectrum of a finitely generated supercommutative Hopf superalgebra over \( k \) (see \([De2]\)). In other words, it is a functor \( G \) from the category of supercommutative algebras to the category of groups defined by \( A \mapsto G(A) := \text{Hom}(H, A) \), where \( \text{Hom}(H, A) \) is the set of algebra maps \( H \to A \) preserving the parity (so called functor of points). An inverse limit of affine algebraic supergroups
is called an \textit{affine proalgebraic supergroup}. A \textit{finite supergroup} is an affine algebraic supergroup, whose function algebra is finite-dimensional. In this case, the dual of this function algebra (called the \textit{supergroup algebra}) is a supercocommutative Hopf superalgebra of the form \( k[G \ltimes V] = k[G] \ltimes AV \), where \( G \) is a finite group and \( V \) is a finite-dimensional \( k \)-representation of \( G \) (see e.g. [AEG] for more details).

Let \( G \) be an affine proalgebraic supergroup over \( k \) and let \( p : \mathbb{Z}_2 \to G \) be a morphism such that \( \text{Ad}(p(-1)) \) is the parity automorphism of \( G \). Let \( \text{Rep}(G, p) \) denote the category of all finite-dimensional algebraic representations of \( G \) over \( k \) in which \( p(-1) \) acts as the parity operator. Then \( \text{Rep}(G, p) \) is a \( k \)-linear abelian symmetric rigid tensor category with \( \text{End}(1) = k \), where \( 1 \) denotes the unit object of \( \text{Rep}(G, p) \) (see [DM]).

\textbf{Definition 2.1} Let \( \mathcal{C} \) be a \( k \)-linear (abelian) symmetric tensor category which is equivalent to \( \text{Rep}(G, p) \) for some \( G, p \). Then \( \mathcal{C} \) is said to be of supergroup type. If in addition \( G \) is a finite supergroup then \( \mathcal{C} \) is said to be of finite supergroup type.

\section{Deligne Theorem}

Let \( \mathcal{C} \) be a \( k \)-linear abelian symmetric rigid tensor category with \( \text{End}(1) = k \). Recall that for any \( X \) in \( \mathcal{C} \) its length, denoted by \( \text{length}(X) \), is defined to be the maximal possible length of a strictly increasing filtration of \( X \). We will always assume that all objects in \( \mathcal{C} \) have finite length. We are now ready to state Deligne Theorem.

\begin{theorem}[see [De2], Proposition 0.5, Theorem 0.6, and the sentence after Theorem 0.6] Suppose that for any object \( X \) in \( \mathcal{C} \) there exists a constant \( d(X) > 0 \) such that \( \text{length}(X \otimes^n) \) is dominated by \( d(X)^n \). Then \( \mathcal{C} \) is of supergroup type.
\end{theorem}

\begin{corollary}[[De2], Corollaries 0.7 and 0.8] If \( \mathcal{C} \) has finitely many classes of simple objects, then \( \mathcal{C} \) is of supergroup type. In particular, if in addition \( \mathcal{C} \) is semisimple then \( \mathcal{C} \) is equivalent to \( \text{Rep}(G) \) where \( G \) is a finite group, possibly with a modified symmetric structure.
\end{corollary}

If \( \mathcal{C} \) is equivalent, as a \( k \)-linear abelian category, to \( \text{Rep}(A) \), where \( A \) is a finite-dimensional algebra, then \( \mathcal{C} \) is said to be \textit{finite}. It is known that this condition is equivalent to the condition that \( \mathcal{C} \) has finitely many isomorphism classes of simple objects, and any simple object has a projective cover. It follows from Corollary 3.2 that if \( \mathcal{C} \) is finite then it is of finite supergroup type.
4 Applications to Hopf algebras

Corollary 3.2 implies Theorem 2.1 of [EG1] on the classification of triangular semisimple Hopf algebras.

Theorem 4.1 [EG1, Theorem 2.1] Let \((H, R)\) be a semisimple triangular Hopf algebra over \(k\), with Drinfeld element \(u\). Set \(R_u := 1/2(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)\) and \(\tilde{R} := RR_u\). Then there exist a finite group \(G\) and a twist \(J \in k[G] \otimes k[G]\) such that \((H, \tilde{R})\) and \((k[G]^J, J_{21}^{-1}J)\) are isomorphic as triangular Hopf algebras.

Proof: Let \(\text{Rep}(H)\) denote the \(k\)-linear abelian symmetric tensor rigid category of all finite-dimensional \(k\)-representations of \(H\). Clearly, \(\text{End}(1) = k\). By Corollary 3.2 there exists a finite group \(G\) such that \(\text{Rep}(H)\) is equivalent to \(\text{Rep}(G)\), possibly with modified symmetric structure. The rest is as in [EG1] (see also [G1]).

Remark 4.2 The proof we gave in [EG1] relied on the weaker version of Deligne Theorem [De1] and hence required some Hopf algebra theory e.g. Larson-Radford Theorem that the antipode of a semisimple Hopf algebra over \(k\) is an involution [LR]. We stress that Hopf algebra theory is no longer needed for the proof of Theorem 4.1.

Recall [AEG] that a finite-dimensional triangular Hopf algebra \((H, R)\) is called a modified supergroup algebra if its \(R\)-matrix \(R\) is of rank \(\leq 2\). The reason for this terminology can be found in Corollaries 2.3.5 and 3.3.3 in [AEG] where it is proved that such finite-dimensional triangular Hopf algebras correspond to (finite) supergroup algebras. The correspondence respects the tensor categories of representations [AEG, Theorem 3.1.1] and the twisting procedure [AEG, Proposition 3.2.1]. Recall also that \(H\) is said to have the Chevalley property if its quotient by its radical is a Hopf algebra itself [AEG]. In [AEG, Theorem 5.1.1] it is proved that \(H\) is twist equivalent to a modified supergroup algebra (by twisting of comultiplication) if and only if \(H\) has the Chevalley property. In Question 5.5.1 in [AEG] we asked if any finite-dimensional triangular Hopf algebra over \(k\) has the Chevalley property. We now have

Theorem 4.3 Let \(H\) be a finite-dimensional triangular Hopf algebra over \(k\). Then \(H\) is twist equivalent to a modified supergroup algebra. In particular, \(H\) has the Chevalley property.

Proof: This follows from Corollary 3.2 and the preceding remarks.

Recall from [EG3] that a triangular septuple is a septuple \((G, W, A, Y, B, V, u)\) where \(G\) is a finite group, \(W\) is a finite-dimensional \(k\)-representation of \(G\), \(A\) is a subgroup of \(G\), \(Y\) is an \(A\)-invariant subspace of \(W\), \(B\) is an \(A\)-invariant nondegenerate element in \(S^2Y\), \(V\) is an irreducible projective \(k\)-representation of \(A\) of dimension \(|A|^{1/2}\), and \(u \in G\) is a central element of order \(\leq 2\) acting by \(-1\) on \(W\). In [EG3, Section 2] it is explained how to assign a
finite-dimensional triangular Hopf algebra with the Chevalley property $H(G, W, A, Y, B, V, u)$ over $k$ to any triangular septuple. Therefore, Theorem 4.3 and [EG3, Theorem 2.2] imply now the following explicit classification of finite-dimensional triangular Hopf algebras over $k$.

**Theorem 4.4** The assignment $(G, W, A, Y, B, V, u) \rightarrow H(G, W, A, Y, B, V, u)$ is a bijection between:

1. isomorphism classes of triangular septuples, and
2. isomorphism classes of finite-dimensional triangular Hopf algebras over $k$.

**Remark 4.5** In [EG2, Theorem 5.2] we proved that for $W = Y = B = 0$ the bijection given in Theorem 4.4 reduces to a bijection between quadruples $(G, A, V, u)$ and semisimple triangular Hopf algebras over $k$.

As a result of Theorem 4.3 and [AEG, Remark 5.5.2] we can now answer Question 3.6 from [G2] positively.

**Theorem 4.6** Let $H$ be a finite-dimensional triangular Hopf algebra over $k$ and let $u$ be its Drinfeld element. Then $u^2 = 1$ and consequently $S^4 = id$. Moreover, if $\dim(H)$ is odd then $u = 1$ and $H$ is semisimple.

Finally, let us explain how Deligne Theorem can be applied to cotriangular Hopf algebras $H$ over $k$. Let $\text{Corep}(H)$ denote the $k$–linear abelian symmetric tensor rigid category of all finite-dimensional $k$–corepresentations of $H$. Clearly, $\text{End}(1) = k$.

**Theorem 4.7** Let $H$ be a cotriangular Hopf algebra over $k$. Then the category $\text{Corep}(H)$ is equivalent to the category $\text{Rep}(G, p)$ for a unique affine proalgebraic supergroup $G$ and morphism $p$.

**Proof:** Set $\mathcal{C} := \text{Corep}(H)$. If $X$ is in $\mathcal{C}$ then it is clear that $\text{length}(X^{\otimes n})$ is at most $\dim(X)^n$, where $\dim(X)$ is the usual linear algebraic dimension of $X$. Thus, by Theorem 3.1 $\mathcal{C}$ is of the form $\text{Rep}(G, p)$. The uniqueness of $(G, p)$ follows from the uniqueness of the super fiber functor, which follows from Sections 3 and 4 of [De2].

**Important remark.** In [EG4, Theorem 3.3] we showed that any *pseudoinvolutive* cotriangular Hopf algebra over $k$ (i.e. such that $\text{tr}(S^2_C) = \dim(C)$ on any finite-dimensional subcoalgebra $C$) is twist equivalent (by twisting of multiplication) to $\mathcal{O}(G)$ for an affine proalgebraic group $G$, and vice versa. One might expect that this correspondence would
extend to supergroups, if one drops the pseudoinvolutivity condition. As we saw, this is
definitely true in the finite-dimensional case. Nevertheless, in the infinite-dimensional case,
such a generalization fails, and the situation is much more nontrivial. Namely, Theorem 4.7
implies that the coalgebras $H$ and $O(G)$ are Morita equivalent, but it does not imply that
they are isomorphic, since the equivalence of Theorem 4.7 need not preserve linear algebraic
dimensions (as, unlike in [EG4], they need not be equal to the categorical dimensions). In
fact, even for $G = SL(2)$, for any integer $N > 2$ there exist cotriangular Hopf algebras $H$
with $\text{Comod}(H) = \text{Rep}(G)$, such that the 2-dimensional vector representation of $G$
corresponds to an $N$-dimensional object in $\text{Comod}(H)$. (For examples of such Hopf algebras, see
[GM,B]). This shows that the theory of (infinite-dimensional) cotriangular Hopf algebras is
much richer than the theory of triangular Hopf algebras.

5 Applications to tensor categories

We first use Deligne Theorem to answer Question 5.5.5 in [AEG]. Recall from [AEG] that
a tensor category $\mathcal{C}$ is said to have the Chevalley property if the tensor product of any two
simple objects in $\mathcal{C}$ is semisimple.

**Theorem 5.1** Let $\mathcal{C}$ be a finite symmetric tensor rigid category with $\text{End}(1) = k$. Then $\mathcal{C}$
has the Chevalley property.

**Proof:** This follows from Corollary 3.2 since $\text{Rep}(G, p)$ has the Chevalley property when
$G$ is a finite supergroup. ■

**Theorem 5.2** Let $\mathcal{C}$ be a finite symmetric tensor rigid category with $\text{End}(1) = k$. Then $\mathcal{C}$
is equivalent to a category $\text{Rep}(H)$ where $H$ is a finite-dimensional triangular Hopf algebra
with $R$–matrix of rank $\leq 2$.

**Proof:** By Corollary 3.2, $\mathcal{C}$ is equivalent to the category of representations of a finite
supergroup $G \ltimes V$ on supervector spaces, in which a fixed element of order 2 acting by parity
on $G \ltimes V$ is represented by the parity operator. Thus, $\mathcal{C}$ is equivalent to the category of
representations of the cocommutative triangular Hopf superalgebra $k[G \ltimes V]$. Modifying it
into a finite-dimensional triangular Hopf algebra $H$ with $R$–matrix of rank $\leq 2$ (as in [AEG,
Corollary 3.3.3]), we obtain the theorem. ■

**Remark 5.3** In Questions 5.5.5 and 5.5.6 in [AEG] we accidentally omitted the assumption
that the category $\mathcal{C}$ has enough projectives (i.e. each simple object has a projective cover).
Without this condition one may take for example the category $\mathcal{C} := \text{Rep}(G_a)$ of algebraic
representations of the additive group; it is definitely not a representation category of a
finite-dimensional triangular Hopf algebra.
We now apply Deligne Theorem to symmetric fusion categories over fields with positive characteristics. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Recall that if \( \mathcal{C} \) is a fusion category (= semisimple rigid tensor category with finitely many simple objects and \( \text{End}(1) = k \)), the global dimension \( \text{dim}(\mathcal{C}) \) of \( \mathcal{C} \) is defined to be the sum of squares of dimensions of its simple objects (see e.g. [ENO]).

**Theorem 5.4** Let \( \mathcal{C} \) be a symmetric fusion category over a field \( k \) of positive characteristic \( p \). Assume that \( \text{dim}(\mathcal{C}) \) is nonzero. Then \( \mathcal{C} \) is equivalent to \( \text{Rep}_k(G) \) for a unique (up to isomorphism) finite group \( G \) whose order is not divisible by \( p \).

**Proof:** According to [ENO, Theorem 9.3], the category \( \mathcal{C} \) admits a unique lifting to a category \( \mathcal{C}_{W(k)} \) over the ring of Witt vectors \( W(k) \). Let \( K \) be the algebraic closure of the field of quotients \( Q \) of \( W(k) \). Let \( \mathcal{C}_Q, \mathcal{C}_K \) be the categories obtained from \( \mathcal{C}_{W(k)} \) by extension of scalars to \( Q, K \) respectively. By Corollary [3.2] there exists a finite group \( G \) such that \( \mathcal{C}_K = \text{Rep}_K(G) \). Thus there exists a finite extension \( Q' \) of \( Q \) such that \( \mathcal{C}_{Q'} = \text{Rep}_{Q'}(G) \).

The order of the group \( G \) is the global dimension of \( \mathcal{C}_K \), hence it is equal to the global dimension of \( \mathcal{C} \) modulo \( p \). Thus, \( p \) does not divide \( |G| \).

Let \( W' \) be the ring of integers in \( Q' \). It is a local ring. Let \( I \) be the maximal ideal in \( W' \). Then the residue field \( W'/I \) is equal to \( k \).

Consider the tensor categories \( \mathcal{C}_{W'} \) and \( \text{Rep}_{W'}(G) \). The reductions modulo \( I \) of these categories (namely, \( \mathcal{C} \) and \( \text{Rep}_k(G) \), respectively) have nonzero global dimensions. The localizations \( \mathcal{C}_{Q'}, \text{Rep}_{W'}(G) \) are equivalent. By [ENO, Theorem 9.6(ii)], this implies that the reductions \( \mathcal{C} \) and \( \text{Rep}_k(G) \) are equivalent as well, as symmetric categories.

The uniqueness of \( G \) is well known (see [DM]). The theorem is proved. 

**Remark 5.5** if \( \text{dim}(\mathcal{C}) = 0 \), the theorem is false, and much more interesting categories than \( \text{Rep}_k(G) \) can occur. A counterexample is the reduction mod \( p \) of the fusion category of representations of \( U_q(sl_2) \), \( q = e^{\pi i/p} \). This category is symmetric, but is not equivalent to the category of representations of a finite group, since the Frobenius-Perron dimensions of its objects are not integers. (see [ENO], Remark 9.5).

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