On the coupling of tensors to gauge fields:
$D = 5, \, N = 2$ supergravity revisited

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Abstract: A general free differential algebra encoding the anti-Higgs mechanism among two-index antisymmetric tensors and gauge vectors is analyzed at the full group theoretical level. $N = 2$ supergravity in five dimensions coupled to tensor, vector and hyper multiplets with all possible couplings included is reconsidered from this point of view. Within our approach, we find that some of the constraints on the couplings usually considered are too stringent and may in fact be relaxed. This generalization also affects the scalar potential.

Keywords: Supergravity models, Flux compactifications.
1. Introduction

The role of tensor multiplets in supergravity has seen in the last years a revived interest, in connection with the study of flux compactifications in superstring or M-theory.

Two-index antisymmetric tensors are 2-form gauge fields whose field-strengths are invariant under the (tensor)-gauge transformation $B \to B + d\Lambda$, $\Lambda$ being any 1-form. A physical pattern to introduce massive tensor fields is the anti-Higgs mechanism, where the dynamics allows the tensor to take a mass by a suitable coupling to some vector field. The mass term plays the role of magnetic charge in the theory. The investigation of the role of massive tensor fields was particularly fruitful for the $N = 2$ theory in 4 dimensions, where the study of the coupling of tensor-scalar multiplets (obtained by Hodge-dualizing scalars
covered by derivatives in the hypermultiplet sector) to \(N = 2\) supergravity was considered, both as a CY compactification [1] and at a purely four dimensional supergravity level [2, 3]. When this model was extended, in [4, 5, 6], to include the coupling to gauge multiplets, it allowed to construct new gaugings containing also magnetic charges, and to find the electric/magnetic duality completion of the \(N = 2\) scalar potential.

However, a general formulation of \(N = 2\), \(D = 4\) supergravity coupled to tensor-vector multiplets (obtained by Hodge-dualizing scalars in the vector multiplet sector) is still missing, even if important steps in that direction appeared quite recently [7, 8]. On the other hand, the situation appears more promising in five dimensional supergravity. There, 2-index antisymmetric tensors appear in the gauge sector, since the field-strengths of massless two-index tensors are Hodge-dual to vector field-strengths, and they naturally appear in the compactification of higher dimensional theories \(^1\). Various approaches to construct a general coupling to tensor multiplets in the \(N = 2\) theory have been given [12, 13, 14, 15, 16, 17, 18, 19, 8]. Towards a general understanding of the four dimensional case, we adopted the strategy of first looking at the five dimensional theory in a framework as general as possible. In particular, an ingredient generally used for the construction of the couplings is the “self-duality in odd dimensions” [20] that allows to work with massive, self-dual tensors from the very beginning. However, in this way much of the algebraic structure underlying the theory is not manifest. To find the most general theory in five dimensions in a way which can give insight into the algebraic structure also for the four dimensional case, we found useful to examine first at the bosonic level and in full generality the algebraic structure which any theory coupled to tensors and gauge vectors is based on. This requires the extension of the notion of gauge algebra to that of free differential algebra (FDA in the following) that naturally accommodates in a general algebraic structure the presence of \(p\)-forms \((p > 1)\). We have then devoted the first part of the paper, section 2, to the study of the gauge properties of a general FDA involving gauge vectors (1-forms) and two-index antisymmetric tensors (2-forms). The discussion will be completely general, and will not rely on the dimensions of space-time (apart from the obvious request \(D \geq 4\), in order to have dynamical 2-forms) nor on supersymmetry. Our procedure allows the FDA structure to be further generalized, for \(D \geq 5\), by including also couplings to higher order forms, as is the case, in general, for flux compactifications. This is left to a future investigation.

When applying our results to the case of \(D = 5\), \(N = 2\) supergravity, in section 3, we find some possible generalizations with respect to the current literature in the subject [12, 13, 14, 15, 16, 17, 18, 19, 8]. Besides the fact, already pointed out in [19] and [21], that it is possible to include in the 3-form field-strength a coupling of the type \(d_{\Lambda \Sigma \Pi} F^\Lambda \wedge A^\Sigma\) (where \(\Lambda\) enumerates gauge fields and \(M\) tensor fields), we find that the mass matrix for the tensor fields, which in five dimensional supergravity has to be antisymmetric \((m^{MN} = -m^{NM})\), is however not necessarily proportional to the symplectic metric \(\Omega^{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\), which is the case to which the literature on the subject usually refers to. On the contrary, any general antisymmetric matrix may be considered. This may be understood, for example, \(^1\)For example, the \(N = 8\) gauged theory requires that a subset of the gauge vectors be dualized to tensors [9, 10, 11].
by looking at the \( D = 5 \) \( N = 2 \) theory obtained by Scherk–Schwarz dimensional reduction from six dimensions \([22]\). In this case, indeed, the tensor mass-matrix is the Scherk–Schwarz phase and has in general different eigenvalues. Therefore for a general five dimensional \( N = 2 \) theory the generators of the gauge algebra are not necessarily in a symplectic representation, and constraints from supersymmetry give milder constraints on the gauging than the ones usually considered. As a consequence of these generalizations, the scalar potential of the theory has some differences with respect to the previous investigations.

The FDA approach allows to interpret the resulting structure in a general group-theoretical way which is not evident with other approaches. Our starting point is a general gauge algebra, which is represented via generators with indices in the adjoint representation of the gauge group. The building blocks of the FDA are then p-form potentials (with, for our case, \( p = 1, 2 \), that is \( A = A_\mu dx^\mu \) and \( B = B_{\mu \nu} dx^\mu \wedge dx^\nu \)) and their field-strengths. Let us emphasize that in this way the fields are subject to gauge constraints, and are therefore massless. The mechanism for which the 2-forms become massive is left to the dynamics of the Lagrangian (or alternatively, in the supersymmetric case, also of the supersymmetric Bianchi identities). At the bosonic level, this is implemented via the anti-Higgs mechanism, that is by fixing the gauge invariance of the system \((A, B)\):

\[
\begin{cases}
\delta B &= d\Lambda \\
\delta A &= d\Theta - m\Lambda \quad ,
\end{cases}
\]

with field-strengths

\[
\begin{cases}
H &= dB \\
F &= dA + mB \quad ,
\end{cases}
\]

via the tensor-gauge fixing \( \bar{\Lambda} = \frac{1}{m} A \).

Since our analysis does not rely on the space-time dimension, we expect to retrieve in particular, with our approach, also the results already known for the \( D = 5, N = 2 \) theory. However, there is a subtle point here, because it appears not evident how to reconcile the anti-Higgs mechanism with the fact that supersymmetry constrains massive tensors in \( D = 5 \) supergravity to obey the self-duality condition:

\[
m \partial_\mu B_{\nu \rho |\lambda} \propto \epsilon_{\mu \nu \rho \sigma \lambda} B^{\sigma \lambda} , \quad \mu, \nu, \cdots = 0, 1, \ldots, 4. \]

The way out from this puzzle may be found by looking again at the subclass of models obtained by Scherk–Schwarz dimensional reduction from six dimensions. Indeed, the six-dimensional Lorentz algebra admits as irreducible representations self-dual tensors, satisfying

\[
\partial_{[\hat{\mu} \hat{\nu} \hat{\rho}]} B_{\hat{\lambda} \hat{\sigma} M} = \frac{1}{6} \epsilon_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\lambda} \hat{\sigma}} \partial^{\hat{\theta}} B_{\hat{\lambda} \hat{\sigma} M}^{\hat{\theta}}, \quad \hat{\mu}, \hat{\nu}, \cdots = 0, 1, \ldots, 5.
\]

Since \( N = 2 \) matter-coupled supergravity in six dimensions contains one antiself-dual and \( n_T \) self-dual tensors in the vector representation of \( SO(1, n_T) \), one can use the \( SO(n_T) \subset SO(1, n_T) \) global symmetry of the model to dimensionally reduce the theory on a circle down to five dimensions à la Scherk–Schwarz \([22]\), with S-S phase \( m^{MN} = -m^{NM} \in \)
\[ B_{\mu \bar{\nu} M}(x, y_5) = \left( \exp[my_5] \right)_M^N \sum_n B_{\mu \bar{\nu} N}^{(n)}(x) \exp \left[ \frac{i \pi n_5}{2} \right]. \] (1.5)

Applying (1.5) to the self-duality relation (1.4), we find

\[ \partial_{[\mu} B_{\nu \rho] M} = \frac{1}{6} \epsilon_{\mu \nu \rho \sigma \lambda_5} \left( m_M^N B_{\sigma \lambda}^N + 2 F_{\lambda_5}^N \right), \quad \mu = 0, 1, \ldots, 4 \] (1.6)

where \( F_{\lambda_5}^N \equiv \partial_{[\sigma} B_{\lambda]5,N} \). Eq. (1.6) expresses the self-duality obeyed by the tensors in five dimensional supergravity. However, it also shows that the field-strengths of the vectors \( B_{\mu 5,N} \), that give mass to the tensors \( B_{\mu \nu M} \) via the anti-Higgs mechanism, are in fact the Hodge-dual of the tensors \( B_{\mu \nu M} \) themselves. From our analysis applied to \( N = 2 \) supergravity in five dimensions, we find this to be a general fact, not necessarily related to theories admitting a six dimensional uplift: in each case, the massive tensor fields belong to short representations of supersymmetry, and the dynamical interpretation of the mechanism giving mass to the tensors requires the coupling of the massless tensors to gauge vectors which are the Hodge-dual of the tensors themselves.

The paper is organized as follows: In section 2 we study the general FDA describing the coupling of two-index antisymmetric tensor fields to non-abelian gauge vectors and show in detail, for the general case, how the anti-Higgs mechanism takes place. In section 3, we apply the formalism to the case of \( N = 2 \) five dimensional supergravity, using the geometric approach to find the Lagrangian, supersymmetry transformations rules and constraints on the scalar geometry and gauging. Our results are summarized in the concluding section, while we left to the appendices some technical details and the comparison of our notations with the ones of [13] and of [18].

2. A general bosonic theory with massive tensors and non-abelian vectors

In this section we are going to study the gauge structure of a general theory with two-index antisymmetric tensor fields coupled to gauge vectors. The discussion here will be general, with no need to make reference to any particular dimension of space-time nor to any possible supersymmetric extension of the model. Later, in section 3, we will consider the supersymmetrization of the model, specifying the discussion to the case of \( N = 2 \) five dimensional supergravity coupled to vector, tensor and hyper multiplets. The corresponding four dimensional case of \( N = 2 \) supergravity coupled to vector-tensor multiplets is under investigation, and is left to a future publication.

2.1 FDA and the anti-Higgs mechanism

2.1.1 Abelian case

The simplest case of a FDA including 1-form and 2-form potentials \(^2\) is described by a set of abelian gauge vectors \( A^M \) and of massless tensor two-forms \( B_M \) \((M = 1, \ldots, n_T.)\)

\(^2\)0-forms will also be included in section 3, when considering a supersymmetric version of the theory
interacting by a coupling \( m^{MN} \). The field-strengths are:

\[
\begin{align*}
F^M &= dA^M + m^{MN}B_N \\
H_M &= dB_M
\end{align*}
\]

(2.1)

and are invariant under the gauge transformations:

\[
\begin{align*}
\delta A^M &= d\Theta^M - m^{MN}\Lambda_N \\
\delta B_M &= d\Lambda_M
\end{align*}
\]

(2.2)

with \( \Theta^M \) parameters of infinitesimal U(1) gauge transformations and \( \Lambda_M \) one-form parameters of infinitesimal tensor-gauge transformations of the two-forms \( B_M \). In this case the system undergoes the anti-Higgs mechanism, and it is possible to fix the tensor-gauge so that:

\[
\begin{align*}
A^M &\rightarrow A'^M = -m^{MN}\bar{\Lambda}_N \\
B_M &\rightarrow B'_M = B_M + d\bar{\Lambda}_M;
\end{align*}
\]

(2.3)

In this way the gauge vectors \( A^M \) disappear from the spectrum providing the degrees of freedom necessary for the tensors to acquire a mass, since:

\[
\begin{align*}
F'^M &= m^{MN}B_N \\
H'_M &= dB_M.
\end{align*}
\]

(2.4)

### 2.1.2 Coupling to a non-abelian algebra

The model outlined above may be generalized by including the coupling of this system to \( n_V \) gauge vectors \( A^\Lambda (\Lambda = 1, \ldots n_V) \), with gauge algebra \( G_0 \) (not necessarily semisimple), if the index \( M \) of the tensors \( B_M \) and of the abelian vectors \( A^M \) runs over a representation of \( G_0 \). In this case the FDA becomes:

\[
\begin{align*}
F^\Lambda &= dA^\Lambda + \frac{i}{2} f_{\Sigma\Gamma}^\Lambda A^\Sigma \wedge A^{\Gamma} \\
F^M &= dA^M - T_{\Lambda^M} A^\Lambda \wedge A^N + m^{MN}B_N \\
&\equiv DA^M + m^{MN}B_N \\
H_M &= dB_M + T_{\Lambda^M} A^\Lambda \wedge B_N + d_{\Lambda NM} F^A \wedge A^N \\
&\equiv DB_M + d_{\Lambda NM} F^A \wedge A^N
\end{align*}
\]

(2.5)

Here \( f_{\Sigma\Gamma}^\Lambda \) are the structure constants of the gauge algebra \( G_0 \) and \( T_{\Lambda^M} \), \( d_{\Lambda MN} \) suitable couplings. The closure of the FDA \( (d^2 A^\Lambda = d^2 A^M = d^2 B_M = 0) \) gives the following constraints:

\[
f_{[\Lambda^\Sigma^\Gamma]^{A}}^{\Omega} = 0
\]

(2.6)

\[
T_{[\Lambda^M]P} T_{\Sigma^P}^{N} = \frac{1}{2} f_{\Sigma\Gamma}^{\Lambda} T_{\Gamma^{M^N}}
\]

(2.7)

\[
T_{\Lambda^M} = -d_{\Lambda^M}^P m^{NP} = d_{\Lambda^P M} m^{PN}
\]

(2.8)

\[
T_{\Lambda^M} m^{NP} = -T_{\Lambda^P} m^{MN}
\]

(2.9)

\[
T_{\Sigma^M} d_{\Gamma^{NP}} + T_{\Sigma^P} d_{\Gamma^{NM}} - f_{\Sigma^A} d_{\Lambda^M} = 0.
\]

(2.10)

\(^3\)We will generally assume, here and in the following, that the tensor mass-matrix \( m^{MN} \) is invertible. In case it has some 0-eigenvalues, we will restrict to the submatrix with non-vanishing rank. This is not a restrictive assumption, because any tensor corresponding to a zero-eigenvalue of \( m \) may be dualized to a gauge vector and so included in the set of \( \{A^\Lambda\} \).
Eqs (2.6), (2.7) show in particular that the structure constants $f_{\Lambda \Sigma \Gamma}$ do indeed close the algebra $G_0$ and that $T_{\Lambda M}^N$ are generators of $G_0$ in the representation spanned by the tensor fields. Eqs (2.8) and (2.9) imply:

$$m^{MN} = \mp m^{NM},$$

$$d_{\Lambda M N} = \pm d_{\Lambda N M},$$

and (2.10) is a consistency condition that, when multiplied by $m^{PQ}$, is equivalent to (2.7) (upon use of (2.9)).

When (2.6) - (2.10) are satisfied, the Bianchi identities read:

$$\begin{align*}
\delta A^A &= dA^A + f_{\Sigma \Gamma}^{\Lambda} A^{\Sigma} \wedge F^{\Gamma} \\
\delta A^M &= \delta \Theta^M - T_{\Lambda N} M A^\Lambda \Theta^N + T_{\Lambda N} M A^\Lambda m^{MN} A_N \\
\delta B_M &= \delta \Lambda_M + T_{\Lambda N} M A^\Lambda \wedge A_N - d_{\Lambda M N} A^\Lambda \wedge A_N - T_{\Lambda N} A_M H_N \\
&= D\Lambda_M - d_{\Lambda M N} A^\Lambda \wedge \delta \Theta^N - T_{\Lambda N} B_M \epsilon^\Lambda,
\end{align*}$$

with:

$$\begin{align*}
\delta F^A &= f_{\Sigma \Gamma}^{\Lambda} F^{\Sigma} \epsilon^{\Gamma} \\
\delta F^M &= T_{\Lambda N} M F^N \epsilon^\Lambda \\
\delta H_M &= -T_{\Lambda M} H_N \epsilon^\Lambda.
\end{align*}$$

Fixing the gauge of the tensor-gauge transformation as:

$$\begin{align*}
A^A &\rightarrow A'^A = A^A \\
A^M &\rightarrow A'^M = -m^{MN} A_N \\
B_M &\rightarrow B'_M = B_M + D\Lambda_M,
\end{align*}$$

we find:

$$\begin{align*}
F'^A &= F^A \\
F'^M &= m^{MN} B_N \\
H'_M &= DB_M
\end{align*}$$

When the tensor-gauge is fixed as in (2.15),(2.16), the vectors $A^M$ disappear from the spectrum while the tensors $B_M$ acquire a mass. As anticipated in the introduction, this is in particular the starting point of the formulation adopted in the literature to describe $D = 5, N = 2$ supergravity coupled to massive tensor multiplets [17, 18, 19].

However, let us observe that in this more general case the abelian gauge vectors $A^M$, providing the degrees of freedom needed to give a mass to the tensors via the anti-Higgs
mechanism, are charged under the gauge algebra $G_0$. It is not possible to make the gauge transformation of the vectors $A^M$ compatible with that of the $A^\Lambda$ unless all together the vectors $\{A^\Lambda, A^M\} \equiv A^I$ form the co-adjoint representation of some larger non semisimple gauge algebra $G \supset G_0$.

The relations so far obtained may then be written with the collective index $\tilde{I} = (\Lambda, M)$, in terms of structure constants $f_{\tilde{J}\tilde{K}}{}^I$ restricted to the following non vanishing entries:

$$ f_{\tilde{J}\tilde{K}}{}^I = (f_{\Lambda^\Sigma^\Gamma}, f_{\Lambda^M}{}^N = -T_{\Lambda M}{}^N), $$  \hspace{1cm} (2.17)

and of the couplings:

$$ m_{\tilde{I}M} \equiv \delta_{\tilde{I}N}^I m_{N M}, \quad d_{\tilde{I}JM} \equiv \delta_{\tilde{I}J}^A \delta_{N}^A d_{\Lambda N M}. $$  \hspace{1cm} (2.18)

In terms of the tilded quantities the FDA (2.5) reads:

$$ \begin{cases} 
F^I & \equiv dA^I + \frac{1}{2} f_{\tilde{J}\tilde{K}}{}^I A^{\tilde{J}} \wedge A^{\tilde{K}} + m_{\tilde{I}M} B_M \\
H_M & \equiv dB_M + T_{\tilde{I}M}{}^N A^I B_N + d_{\tilde{I}JM} F^I \wedge A^J 
\end{cases} \hspace{1cm} (2.19) $$

with Bianchi identities:

$$ \begin{cases} 
dF^I + \left( f_{\tilde{J}\tilde{K}}{}^I + m_{\tilde{I}M} d_{\tilde{K}JM} \right) A^J F^K & = m_{\tilde{I}M} H_M \\
dH_M + \left( T_{\tilde{I}M}{}^N + m_{\tilde{J}N} d_{\tilde{J}IM} \right) A^I H_N & = d_{\tilde{I}JM} F^I F^J 
\end{cases} \hspace{1cm} (2.20) $$

provided the following relations, equivalent to (2.6) - (2.10), hold:

$$ \begin{align*}
f_{\tilde{J}\tilde{K}}{}^L f_{\tilde{K}\tilde{L}}{}^M & = 0 \\
[T_{\tilde{I}}, T_{\tilde{J}}] & = f_{\tilde{I}\tilde{J}}{}^K T_{\tilde{K}} \\
T_{\tilde{I}M}{}^{(N, m_{\tilde{I}P})} & = 0 \\
m_{\tilde{I}M} T_{\tilde{J}N}{}^{M} & = f_{\tilde{I}\tilde{K}}{}^I m_{\tilde{K}M} \\
T_{\tilde{I}M}{}^N & = d_{\tilde{I}JM} m_{\tilde{J}N} \\
T_{\tilde{I}M}{}^{N} d_{\tilde{K}J}{}^{[N} & = \left( f_{\tilde{I}\tilde{K}}{}^L + m_{\tilde{I}N} d_{\tilde{K}J}{}^{[N} \right) d_{\tilde{L}J}{}^{M]} - \frac{1}{2} f_{\tilde{I}\tilde{J}}{}^L d_{\tilde{K}LM} = 0.
\end{align*} \hspace{1cm} (2.21) $$

Subject to the constraints (2.21), the system is covariant under the gauge transformations:

$$ \begin{cases} 
\delta A^I & = d\epsilon^I + f_{\tilde{I}\tilde{K}}{}^I A^{\tilde{J}} \epsilon^{\tilde{K}} - m_{\tilde{I}M} \Lambda_M \\
\delta B_M & = d\Lambda_M + T_{\tilde{I}M}{}^N A^I \Lambda_N - d_{\tilde{I}JM} F^I e^J - T_{\tilde{I}M}{}^N e^I B_N
\end{cases} \hspace{1cm} (2.22) $$

implying the gauge transformation of the field strengths:

$$ \begin{cases} 
\delta F^I & = -\left( f_{\tilde{I}\tilde{K}}{}^I + m_{\tilde{I}M} d_{\tilde{K}JM} \right) e^J F^K \\
\delta H_M & = -\left( T_{\tilde{I}M}{}^N + m_{\tilde{J}N} d_{\tilde{J}IM} \right) e^I H_N
\end{cases} \hspace{1cm} (2.23) $$

2.1.3 A general FDA

We now observe that the restrictions on the couplings (2.17) and (2.18) have been set to exactly reproduce eqs. (2.5) while exhibiting the fact that $A^I$ collectively belong to the coadjoint of some algebra $G \supset G_0$. Actually eqs. (2.5) and (2.21) allow in fact a more general gauge structure than the one declared in (2.17), (2.18). Let $T_{\tilde{I}} \in \text{Adj} G$ be the
gauge generators dual to $A^I$. For the case of (2.17), $G$ has the semisimple structure $G = G_0 \ltimes \mathbb{R}^{n^T}$, and the generators $T_\Lambda \in G_0$ may be realized in a block-diagonal way (with entries $T_{\Lambda \Sigma}^\Gamma = f_{\Lambda \Sigma}^\Gamma$, $T_{\Lambda M}^N = -f_{\Lambda M}^N$) while the $T_M$ are off-diagonal (with entries $T_{\Lambda M}^N = f_{\Lambda M}^N$). However, any gauge algebra $G$ with structure constants $f_{\tilde{I} \tilde{J} \tilde{K}}$ may in principle be considered, provided it satisfies the constraints (2.21). In the general case, to match (2.21) one must also relax the restrictions on the couplings (2.17), (2.18), and allow for more general $f_{\tilde{I} \tilde{J} \tilde{K}}$ and $d_{IJM}$. This includes in particular the case

$$f_{\Lambda \Sigma M} \neq 0, \quad d_{\Lambda \Sigma M} \neq 0$$

which was considered in [19] and [21]. In this case, $G$ cannot be semisimple, and $G_0$ is not a subalgebra of $G$. This implies that the vectors $A^M$ do not decouple anymore at the level of gauge algebra, and this, at first sight, would be an obstruction to implement the anti-Higgs mechanism. However, this apparent obstruction may be simply overcome in the FDA framework, due to the freedom of redefining the tensor fields as [23]:

$$B_M \to B_M + k_{\tilde{I} \tilde{J} \tilde{M}} A^{\tilde{I}} \wedge A^{\tilde{J}},$$

for any $k_{\tilde{I} \tilde{J} \tilde{M}}$ antisymmetric in $\tilde{I}, \tilde{J}$. It is then possible to implement the anti-Higgs mechanism with the tensor-gauge fixing (which includes a field redefinition as in (2.25)):

$$\begin{cases}
    A^\Lambda \to A'^\Lambda = A^\Lambda \\
    A^M \to A'^M = -m^{MN} \bar{\Lambda}_N \\
    B_M \to B'_M = B_M - \frac{1}{2} d_{\Lambda \Sigma M} A^\Lambda \wedge A^\Sigma + D \bar{\Lambda}_M
  \end{cases}$$

This still gives:

$$\begin{cases}
    F'^\Lambda = F^\Lambda \\
    F'^M = m^{MN} B_N \\
    H'_M = D B_M
  \end{cases}$$

provided that:

$$m^{MN} d_{[\Lambda \Sigma]N} = f_{\Lambda \Sigma M}.$$  

With this observation, we may now analyze in full generality which non trivial structure constants may be turned on in (2.19) in a way compatible with the anti-Higgs mechanism.

First of all, it is immediate to see that if:

$$f_{\tilde{I} \tilde{M} \Sigma} \neq 0,$$

it is impossible to implement the anti-Higgs mechanism, because they introduce a coupling to the gauge vectors $A^M$ in the field-strengths $F^\Lambda$ which is not possible to reabsorb by any field-redefinition.

Considering then the case:

$$f_{MN P} \neq 0, \quad d_{MNP} \neq 0.$$  

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We acknowledge an enlightening discussion with Maria A. Lledó on this point.
we see that $f_{MN}^P$ would introduce a non-abelian interactions among the vectors $A^M$ and in particular, for the case $\Lambda = 0$, this would imply that the $A^M$ close a non-abelian gauge algebra. This case may be treated in a way quite similar to the case (2.24), since again we may use the freedom in (2.25) to absorb the non-abelian contribution to $F^M$ in a redefinition of $B_M$. The anti-Higgs mechanism may then be implemented via the tensor-gauge fixing:

\[
\begin{align*}
A^\Lambda & \to A^\Lambda = A^\Lambda \\
A^M & \to A^M = -m^{MN} \bar{\Lambda}_N \\
B_M & \to B'_M = B_M - \frac{1}{2} d_{NP} m^N A^N \wedge A^P + D \bar{\Lambda}_M
\end{align*}
\] (2.31)

giving, as before:

\[
\begin{align*}
F^\Lambda & = F^\Lambda \\
F^M & = m^{MN} B_N \\
H'_M & = D B_M
\end{align*}
\] (2.32)

provided that:

\[
m^{MQ} d_{[NP]} = f_{NP}^M.
\] (2.33)

This shows that also non-abelian gauge vectors $A^M$ may be considered, and still may decouple from the gauge-fixed theory by giving mass to the tensors $B_M$. For this case, however, the constraints (2.21), together with (2.33), give the following conditions on the couplings:

\[
\begin{align*}
d_{MNP} & = d_{[MNP]} \\
m^{MN} & = +m^{NM}
\end{align*}
\] (2.34)

As we are going to discuss in the next section, for the $D = 5$, $N = 2$ theory the matrix $m^{MN}$ has to be antisymmetric, and this then implies, for this theory, $f_{MN}^P = 0$. We conclude that even if the algebra (2.19) can have non trivial extensions with new couplings, this is not the case for the $D = 5$, $N = 2$ theory we shall be concerned with in section 3, so that the couplings $f_{MN}^P$ and $d_{MNP}$ will be set to zero.

2.2 General properties of the FDA

A further observation concerns eq.s (2.20) and (2.23). In these equations, as in all the relations involving the physical field strengths $F^I$ and $H_M$, the following objects appear:

\[
\begin{align*}
\hat{f}_{JK}^I & = f_{JK}^I + m^I d_{KJ}^M \\
\hat{T}_{IMN} & = T_{IMN} + m^N d_{IJ}^M = 2 d_{(ij)M} m^N.
\end{align*}
\] (2.35)

The generalized couplings $\hat{f}_{JK}^I$ belong to a representation of the gauge algebra $G$ which is not the adjoint, since they are not antisymmetric in the lower indices. In particular we find:

\[
\begin{align*}
\hat{f}_{IJ}^K m^JM & = \hat{T}_{IN}^M m^JN \\
\hat{f}_{IJ}^K m^IM & = 0.
\end{align*}
\] (2.36)
However, the $\hat{f}_{J^K}^I$ and $\hat{T}_{I^N}^M$ can be understood as representations of generators $\hat{f}_I$ and $\hat{T}_I$ that still generate the gauge algebra $G$. Indeed the following relations hold (subject to the constraints (2.21)):

\[
\begin{align*}
\{ \hat{f}_I, \hat{f}_J \} &= -\hat{f}_{J^K}^I \hat{f}_K, \\
\{ \hat{T}_I, \hat{T}_J \} &= \hat{f}_{J^K}^I \hat{T}_K.
\end{align*}
\] (2.37)

The generalized couplings $\hat{f}$ and $\hat{T}$ express the deformation of the gauge structure due to the presence of the tensor fields. In particular, only the structure constants of $G_0$ are unchanged, corresponding to the fact that this is the algebra realized exactly in the interacting theory (2.19) after the anti-Higgs mechanism has taken place. The rest of the gauge algebra $G$ is instead spontaneously broken by the anti-Higgs mechanism (which requires, if $f_{A\Sigma}^M \neq 0$, also a tensor redefinition, as explained in (2.26)). However, the entire algebra $G$ is still realized, even if in a more subtle way, as eqs (2.37) show. From a physical point of view, this is expected by a counting of degrees of freedom, since the degrees of freedom required to make a two-index tensor massive are the ones of a gauge vector connection\(^5\), so that also the vectors $A^M$, besides the $A^A$, are expected to be massless gauge vectors. This algebra indeed closes provided the Jacobi identities $f_{[I^J}^L f_{K^L]_M} = 0$ are satisfied. We find indeed, using (2.37):

\[
\begin{align*}
\left[ \{ \hat{f}_I, \hat{f}_J \}, \hat{f}_{K} \right]_L & = \hat{f}_{[I^J}^N f_{K^N]}^M \hat{f}_{M^L}^P = 0 \\
\left[ \{ \hat{T}_I, \hat{T}_J \}, \hat{T}_{K} \right]_M & = \hat{f}_{[I^J}^M f_{K^M]}^L \hat{T}_{L^M}^N = 0
\end{align*}
\] (2.38)

The hatted generators $\hat{f}, \hat{T}$ play the role of physical couplings when the gauge structure is extended to include charged tensors. They have then to be considered as the appropriate generators of the free differential structure. It may be useful to recast the theory in terms of all the couplings appearing in the Bianchi identities (2.20), that is the hatted generators and the symmetric part $d_{(i^j)M}$ of the Chern–Simons-like coupling $d_{i^jM}$. This is done by the field redefinition:

\[
B_M \rightarrow \tilde{B}_M = B_M + \frac{1}{2} d_{[i^j]M} A^{i} \wedge A^{j}
\] (2.39)

so that the FDA takes the form:

\[
\begin{align*}
F^I & \equiv dA^I + \frac{1}{2} f_{J^K}^I A^{j} \wedge A^{K} + m^I M \tilde{B}_M, \\
H_M & \equiv dB^M + \frac{1}{2} T_{IM}^N A^{i} \tilde{B}_N + d_{i^jM} F^{i} \wedge A^{j} + K_{M^I^J^K^N} A^{i} \wedge A^{j} \wedge A^{K}
\end{align*}
\] (2.40)

\(^5\)Indeed, the on-shell degrees of freedom of a massless (2-index) tensor and of a vector in $D$ dimensions are $(D-2)(D-3)/2$ and $(D-2)$ respectively, while the ones of a massive tensor are $(D-1)(D-2)/2 = (D-2)(D-3)/2 + (D-2)$.\]
and the constraints (2.21) in the new formulation read, after introducing \( \tilde{f}_{ij}^K \equiv \hat{f}_{ij}^K \):

\[
\begin{align*}
\hat{f}_{ij}^M \hat{f}_{K}\hat{L} & = 2m_{LM}K_{M[jj}\hat{K}] \\
\hat{T}_{ij}^{MN}\hat{T}_{j}^{NP} & = \hat{f}_{ij}^K\hat{T}_{K} + 12K_{M[ij}\hat{K}m_{K]} \\
\hat{T}_{ij}^{MN}m_{IP} & = 0 \\
\frac{1}{2}m_{IN}\hat{T}_{j}^{MN} & = \hat{f}_{jK}\hat{m}_{K} \\
\hat{T}_{ij}^{IN} & = 2d_{(ij)M}m_{IN} \\
\hat{T}_{i[m}^{NN}\hat{d}_{(j]}^{L}N & = 2\hat{f}_{iK}\hat{L}d_{(j]}^{L}M - \hat{f}_{ij}^{L}d_{(K}^{L}M = -6K_{M[ij}\hat{K} \\
\hat{K}_{N[jK}\hat{L}\hat{T}_{i]}^{MN} & = 3K_{MIP[ij}\hat{P}^{L]} = 0.
\end{align*}
\]  

In eqs (2.40) and (2.41) we have introduced the definition:

\[
K_{M[ij}\hat{K}] = \frac{1}{2}\hat{f}_{ij}^{L}d_{(K}^{L}M + \frac{2}{3}d_{(j]}^{L}M\hat{f}_{iK}\hat{L}.
\]  

that could also be found by directly studying the closure of the FDA (2.40) without referring to its derivation from (2.19).

Eq. (2.40), which is expressed in terms of the physical couplings only, is completely equivalent to (2.19). This is in fact the formulation used in [21], for the study of \( N = 8 \) supergravity in 5 dimensions. However, as eqs (2.41) shows, in the formulation (2.40) the gauge structure is not completely manifest, because for the “structure constants” \( \hat{f}_{ij}^K \) the Jacobi identities fail to close.

Equation (2.19) (or, equivalently, (2.40)) is the most general FDA involving vectors and 2-index antisymmetric tensors. Any other possible deformation of (2.19) is indeed trivial (unless the system is also coupled to higher order forms) as we will show in detail in Appendix A.

As a final remark, let us observe that, given the definitions (2.19), the FDA still enjoys a scale invariance under the transformation, with parameter \( \alpha \):

\[
\begin{align*}
m^{MN} & \rightarrow \alpha m^{MN} \\
B_{M} & \rightarrow \frac{1}{\alpha}B_{M} \\
d_{ijM} & \rightarrow \frac{1}{\alpha}d_{ijM}
\end{align*}
\]  

As we will see in the following, for the \( N = 2 \) theory in five dimensions this freedom corresponds to the possibility of choosing an overall normalization for the tensor contributions to the Chern–Simons Lagrangian.

3. \( D = 5, N = 2 \) supergravity revisited

3.1 Generalities and differences from previous approaches

In this section we are going to apply the general analysis of section 2 to the case of \( N = 2 \) supergravity theory in five dimensions coupled to vector- and tensor-multiplets.

The field content of the theory, in the absence of couplings, is
the gravity supermultiplet

\[ (V^a_\mu, \psi^A_\mu, A^0_\mu), \quad a = 0, 1, \ldots 4, \quad \mu = 0, 1, \ldots 4, \quad A = 1, 2 \]

where \( V^a_\mu \) is the space-time vielbein (with a tangent-space indices and \( \mu \) world-indices), \( \psi^A_\mu \) the gravitino, with R-symmetry index in the fundamental representation of \( Sp(2, \mathbb{R}) \), and \( A^0 \) the graviphoton;

- \( n_V \) gauge multiplets

\[ (A^i_\mu, \lambda^i_A, \varphi^i), \quad i = 1, \ldots n_V \]

with \( \varphi^i, \lambda^i_A \) the scalar partners of the gauge vectors \( A^i \) and the \( Sp(2, \mathbb{R}) \)-valued gaugini respectively. Since the gauge vectors mix in the interacting theory, in the following we will introduce the index \( \Lambda = (0, i) = 0, 1, \ldots n_V \) running over all the gauge-vector indices, that is: \( A^\Lambda \equiv (A^0, A^i) \);

- \( n_T \) massless tensor multiplets

\[ (B^i_M|\mu\nu, \lambda^M_A, \varphi^M), \quad i = 1, \ldots n_T \]

with \( \varphi^M, \lambda^M_A \) the scalar and spinor partners respectively of the tensors \( B_M \);

- \( n_H \) hypermultiplets

\[ (q^u, \zeta^\alpha), \quad u = (A\alpha) = 1, \ldots 4n_H; \quad \alpha = 1, \ldots 2n_H \]

where the scalars \( q^u \) span a quaternionic manifold of quaternionic dimension \( n_H \) and their spin-1/2 partners \( \zeta^\alpha \) are labeled with an index in the fundamental representation of \( Sp(2n_H, \mathbb{R}) \).

Before entering in the explicit construction of the theory, let us emphasize the differences of our approach with respect to the existing literature on \( D = 5, \; N = 2 \) supergravity. Inspired by the analysis of the previous section, we are interested in exploiting all the rich gauge structure underlying the bosonic sector of the model, so we want to retrieve and possibly to extend the results in the existing literature by starting with massless tensors and letting them take mass via the anti-Higgs mechanism. Let us discuss this point in some more detail than what has already done in the introduction.

While the anti–Higgs mechanism is very well understood at the bosonic level, to implement it within a supersymmetric theory is a non trivial task. This is due to the fact that the supersymmetry constraints require the vectors \( A^M \) giving mass to the tensors (in the notations of section 2) to be related to the tensors themselves in a non local way, involving Hodge-duality. This relation is codified in the so-called “self-duality-in-odd-dimensions” condition to which all the tensor fields in odd-dimensional supergravity theories have to comply [20]:

\[ m^{MN} H_{N|abc} \propto \epsilon_{abcde} F^{M|de}. \]  

(3.1)

In particular, for the five dimensional case the tensors are further required to be complex.
In fact, in the approach currently adopted in the literature [12, 13, 14, 15, 16, 17, 18, 19, 8], the tensors $B_M$ in the tensor multiplets are taken to be massive (and constrained to satisfy (3.1)) from the very beginning, without any tensor-gauge freedom.

Naively, to implement the anti-Higgs mechanism at the supersymmetric level one could think of directly supersymmetrizing the FDA (2.19), and try to give mass to the whole tensor multiplets by coupling them to $n_T$ extra abelian vector multiplets added to the theory:

$$(A^M, \phi^M, \chi^M),$$

where the vectors $A^M$ and the tensors $B_M$ admit the couplings and gauge invariance as in (2.19) and (2.22). If this would be the case, in the interacting theory the fields in the extra vector multiplets would couple to the tensor multiplets and one would end up with $n_T$ long massive multiplets. We found, however, from explicit calculation that this is not the case, since supersymmetry transformations never relate the tensors $B_M$ to the spinors $\chi^M$ nor to the scalars $\phi^M$ in (3.2). Then the only way compatible with supersymmetry to couple $N = 2$ supergravity with $n_T$ massive tensors involves short BPS tensor multiplets

$$(B_M|_{\mu\nu}, \lambda^M, \varphi^M)$$

where the massive tensors $B_M$ (that are complex because of CPT invariance of the BPS multiplet) have to satisfy (3.1) (see eq. (3.48)). This is evident for the models having a six dimensional uplift, as discussed in the introduction, since for these cases the mass of the tensors is the BPS central charge gauged by the graviphoton $g_{\mu 5}$. Then, in order to understand the $N = 2$ supergravity theory in five dimensions coupled to tensor and vector multiplets as a supersymmetrization of the FDA discussed in section 2, we will adopt the following strategy: we start from the massless theory with field content as outlined at the beginning of this section, but we also introduce $n_T$ extra auxiliary abelian vectors $A^M$ coupled to the system. The closure of the supersymmetry algebra will then fix their field-strengths, on-shell, to be the Hodge-dual of the field-strengths of the tensors $B_M$. When the theory also includes non-abelian gauge multiplets gauging some algebra $G_0$, and the tensor multiplets are charged under some representation of $G_0$, then the spin-one part $(B_M, A^M, A^A)$ of the bosonic sector is coupled as in (2.19). In this case the closure of the supersymmetry algebra also involves the non abelian field-strengths and give the set of constraints (3.45) - (3.54) below.

According to the discussion in section 2, to simplify the notation we will generally use the index $\tilde{I} = (\Lambda, M)$, valued in a representation of a group $G \supset G_0$ in the notations of section 2, that runs over all the vectors (including the auxiliary ones)

$$A^{\tilde{I}} \equiv (A^\Lambda, A^M)$$

and over the scalar sections

$$X^{\tilde{I}}(\varphi^x) \equiv (X^\Lambda, X^M)$$

($G$-valued functions of the scalar fields $\varphi^x \equiv (\varphi^i, \varphi^M)$) which appear in the supersymmetry transformations of the vector and tensor fields. The world-index $x = 1, \ldots, n_V + n_T$ will
collectively enumerate the scalar fields \( \varphi^x \) and the spinors \( \lambda^x A \) both in the tensor and vector multiplets.

With respect to the analysis of [17], our discussion will be a bit more general as we will include a non-zero Chern–Simons coupling \( d_{\Lambda \Sigma M} \), as in [19].

Another important point, not considered so far in this general context, concerns the couplings \( m^{MN} \). The closure of the FDA (2.19) demands (recalling (2.21)) the generators \( T_{IM}^N \) to be related to the couplings \( d_{IJM} \) and to the structure constants \( f_{JK}^I \) respectively by

\[
T_{IN}^M = m_{ji} d_{ijN}, \quad m_{ijN} T_{jn}^M = f_{jk}^i m_{KM}.
\]

(3.3)

Setting in the second relation \( \tilde{I} = P \) and \( \tilde{J} = \Lambda \), it is immediate to obtain the following:

\[
d_{\Lambda MN} m^{MQ} m^{NP} = -d_{\Lambda NM} m_{PN} m^{MQ}.
\]

(3.4)

Eq. (3.4) in principle admits two different solutions: either \( d_{\Lambda MN} \) is symmetric and \( m^{MN} \) antisymmetric or the opposite. But since these couplings enter the Lagrangian of five dimensional supergravity respectively in the kinetic term for the tensors \( m^{MN} B_M d_B N \) (which for \( m^{MN} \) symmetric is a total derivative) and in the Chern–Simons term \( d_{IM} A^I F^M F^N \) (which is zero if \( d_{IM} \) is antisymmetric in \( M \) and \( N \)), we are forced to consider only the former solution\(^6\). Furthermore it should be noted that this same choice forbids the presence of \( d_{MNP} \) couplings, due to eq. (2.34).

We want to stress, however, that this constraint leaves the freedom for the tensor mass-matrix \( m = -m^T \) to have \( n_T \) different eigenvalues \( \pm i m_\ell \) \( (\ell = 1, \ldots n_T/2) \). As anticipated in the introduction, this is the case, for example, of the five dimensional theory obtained by Scherk–Schwarz generalized dimensional reduction [24, 25] from the (2,0) theory in six dimensions [22]. In this theory the mass matrix \( m^{MN} \) is in fact the S-S phase, in the Cartan subalgebra of the global symmetry \( SO(n_T) \subset SO(1,n_T) \), which is the isometry group of the scalar sector of the tensor multiplets in the \( D = 6 \) parent theory. The five dimensional theory one obtains in this way is a gauged theory with flat group given by the semidirect product \( U(1) \times R_V \), where \( R_V \) is an \( n_V \)-dimensional representation of \( SO(1,n_T) \), and the \( U(1) \) group is gauged by the vector coming from the metric in six dimensions. As remarked in [22], such a situation was not considered in previous classifications.

On the other hand, if we take all the eigenvalues of the matrix \( m^{MN} \) equal, which is the case generally considered in the literature [12, 13, 14, 15, 16, 17, 18, 19, 8], then \( m^{MN} \) may be set in the form \( m^{MN} = m \Omega^{MN} \) where \( m \) is one constant real parameter and \( \Omega \) the symplectic metric. In this case the constraints (3.4) require the generators \( T_{AI}^J \) to belong to a symplectic representation of the gauge group \( T_{AI}^J \cdot \Omega + \Omega \cdot T_{AI} = 0 \).

\(^6\)This is not necessarily true for other cases, like the four dimensional theories, where the equation (3.4) also seems to allow the alternative solution

\[
d_{IM} = -d_{IM} \quad m^{MN} \rightleftharpoons m^{NM}.
\]

(3.5)

\(^7\)For \( n_T \) odd there is one extra zero-eigenvalue. However, this case is excluded when the theory is embedded in \( N = 2 \) supergravity, since in this case, as we already anticipated, the closure of the superalgebra requires a self-duality condition [20] which needs an even number of tensors.
3.2 The construction of the theory

Since our approach involves some generalizations with respect to those in the existing literature, as discussed above, we have rederived right from the beginning the theory in its full generality. We have used the superspace geometric approach as far as the solution of Bianchi identities is concerned (from which the supersymmetry transformation laws of the fields follow) and the superspace rheonomic Lagrangian for the derivation of the Lagrangian on space-time. We also tried to use, as much as possible, the notations existing in the previous literature; however in some cases we found useful to adopt different normalizations for the fields and couplings with respect to the seminal papers on the subject [13, 17, 18]. A dictionary between our normalizations and those adopted in the previous papers is given in appendix E.

Our starting point, for the construction of the theory, is the generalization of the bosonic FDA of section 2 to a super-FDA in superspace. Consequently, we introduce the supergravity one-forms $\Omega^a_b$, $V^a$ and $\Psi^A$ denoting respectively the spin-connection, the vielbein and the gravitino in superspace ($V^a$ and $\Psi^A$ spanning a basis on superspace), together with their “supercurvatures” two-forms $R^{ab}$, $T^a$ and $\rho^A$. We further introduce zero-forms for the scalars $\phi^x$, $q^u$ and spin 1/2 fields $\lambda^{xA}$, $\zeta^\alpha$ and their curvatures (covariant derivatives).

The $D = 5$, $N = 2$ super-FDA is:

$$\mathcal{R}^a_b = d\Omega^a_b - \Omega^a_c \wedge \Omega^c_b \quad (3.6)$$

$$T^a = dV^a - \Omega^a_b V^b - \frac{i}{2} \nabla A^a \Psi^A \quad (3.7)$$

$$F^I = dA^I + \frac{1}{2} f_{ijk} A^j \wedge A^k + m^{IM} B_M + i X^I \Psi_A \Psi^A \quad (3.8)$$

$$H_M = dB_M + T^N I A^I \wedge B_N + d_{IJM} \left( F^I - i X^I \Psi_A \Psi^A \right) A^J + + i X_M \Psi_A \Gamma^a \Psi^A \quad (3.9)$$

$$D\phi^x = d\phi^x + k^x_I A^I \quad (3.10)$$

$$Dq^u = dq^u + k^u_I A^I \quad (3.11)$$

$$\rho^A = d\Psi^A - \frac{1}{4} \Omega_{ab} \Gamma^{ab} \Psi^A + \tilde{\omega}^A_B \Psi^B \quad (3.12)$$

$$\nabla \lambda^{xA} = d\lambda^{xA} - \frac{1}{4} \Omega_{ab} \Gamma^{ab} \lambda^{xA} + \tilde{\Gamma}^x \lambda^{yA} + \tilde{\omega}^A_B \lambda^{xB} \quad (3.13)$$

$$\nabla \zeta^\alpha = d\zeta^\alpha - \frac{1}{4} \Omega_{ab} \Gamma^{ab} \zeta^\alpha + \tilde{\Delta}^\alpha_\beta \zeta^\beta \quad (3.14)$$

In (3.12) - (3.14) the gauged connections on the scalar $\sigma$-models $\mathcal{M}(\phi)$ and $\mathcal{M}_H(q)$ appear, where $\mathcal{M}(\phi)$ is parametrized by the scalars in the vector and tensor multiplets while $\mathcal{M}_H(q)$ is parametrized by the scalars of the hypermultiplets (the quaternionic sector is unaffected by the presence of tensor multiplets). They are defined as:

$$\tilde{\Gamma}^x = \Gamma^x + A^I \partial_k k^x_I \quad (3.15)$$

$$\tilde{\omega}^{AB} = \omega^{AB} + \frac{3}{2} A^I P^I_{AB} \quad SU(2) \text{ connection}$$

$$\tilde{\Delta}^\alpha_\beta = \Delta^\alpha_\beta + A^I \partial_u k^u_I U^\alpha U^\beta \quad Sp(2n_H) \text{ connection}. $$
Here $\Gamma^x_{y} (\varphi)$ is the Christoffel connection one-form of $\mathcal{M}(\varphi)$, while $\omega^{AB}(q)$ and $\Delta^\alpha_\beta(q)$ are respectively the $Sp(2, \mathbb{R})$ R-symmetry and the $Sp(2n_H, \mathbb{R})$ connections on $\mathcal{M}_H(q)$. Furthermore $k_\lambda^x (\varphi)$ and $k_\beta^x(q)$ denote the Killing vectors on the two $\sigma$-models. In eq. (3.15) also appear the geometric quantities $\mathcal{U}_a A^a$ and $\mathcal{P}^A_{IB}$. They are the vielbein ($\mathcal{U}^{Aa} = \mathcal{U}^{a}_a dq^a$) and prepotential on $\mathcal{M}_H$. For their definition and geometric properties, we refer the reader to the standard literature, in particular [26, 27] where the same notations are used.

We adopted the following conventions for raising and lowering $Sp(2, \mathbb{R})$ and $Sp(2n_H)$ indices:

$$\xi_A = \epsilon_{AB} \xi^B; \quad \xi^A = - \epsilon^{AB} \xi_B; \quad \xi_\alpha = \xi_{\alpha\beta} \xi^\beta; \quad \xi^\alpha = - \xi^{\alpha\beta} \xi_\beta$$

(3.16)

while the flat space-time indices $a, b$ are raised or lowered with the metric

$$\eta_{ab} = \text{diag}(+, -, -, -).$$

(3.17)

With these definitions, the explicit construction proceeds by first solving the super-Bianchi’s following from (3.6) - (3.14):

$$R^a_{\ b} V^b = i \overline{\Psi} a \Gamma^a \rho^A$$

(3.18)

$$D R^a_{\ b} = 0$$

(3.19)

$$D F^I = m I M (H_M - i X_M \overline{\Psi} A \Gamma_a \Psi^a V^a) + i D X I \overline{\Psi} A \Psi^A - 2 i X I \overline{\Psi} A \rho^A$$

(3.20)

$$D H_M = d I J M \left( F^I - i X I \overline{\Psi} A \Psi^A \right) \wedge \left( F^J - i X J \overline{\Psi} B \Psi^B \right) +$$

$$+ i D X_M \overline{\Psi} A \Gamma_a \Psi^a V^a - 2 i X_M \overline{\Psi} A \rho^A V^a - \frac{1}{2} X_M \overline{\Psi} A \Psi^A \overline{\Psi} B \Gamma^a \Psi^B$$

(3.21)

$$D^2 \varphi^x = k^x_I \left( F^I - i X I \overline{\Psi} A \Psi^A \right)$$

(3.22)

$$D^2 q^u = k^u_I \left( F^I - i X I \overline{\Psi} A \Psi^A \right)$$

(3.23)

$$\nabla \rho^A = - \frac{1}{4} R_{ab} \Gamma^{ab} \Psi^A - \hat{\mathcal{R}}^A B \Psi^B$$

(3.24)

$$\nabla^2 \lambda^x A = - \frac{1}{4} R_{ab} \Gamma^{ab} \lambda^x A - \hat{\mathcal{R}}^x y \lambda^y B - \hat{\mathcal{R}}^A B \lambda^x B$$

(3.25)

$$\nabla^2 \zeta^a = - \frac{1}{4} R_{ab} \Gamma^{ab} \zeta^a - \hat{\mathcal{R}}^\alpha_\beta \zeta^\beta$$

(3.26)

where we have defined:

$$D F^I \equiv dF^I + \hat{f}_{j K} I A J F^K$$

(3.27)

$$D X^I \equiv dX^I + \hat{f}_{j K} I A J X^K$$

(3.28)

$$D H_M \equiv dH_M + \hat{T}^N_{I M} A I H_N$$

(3.29)

$$D X_M \equiv dX_M + \hat{T}^N_{I M} A I X_N$$

(3.30)

with the $\hat{f}_{j K}, \hat{T}^N_{I M}$ introduced in (2.35), and

$$\hat{\mathcal{R}}^x y \equiv d\tilde{\mathcal{R}}^x y + \tilde{\mathcal{R}}^x y \tilde{\Gamma}^y_x$$

(3.31)

$$\hat{\mathcal{R}}^A B \equiv d\tilde{\mathcal{R}}^A B + \tilde{\mathcal{R}}^A C \tilde{\mathcal{R}}^C B$$

(3.32)

$$\hat{\mathcal{R}}^\alpha_\beta \equiv d\tilde{\mathcal{R}}^\alpha_\beta + \tilde{\mathcal{R}}^\alpha_\gamma \tilde{\mathcal{R}}^\gamma_\beta.$$

(3.33)
Eq.s (3.18) - (3.26) are solved by parametrizing the supercurvatures on superspace (from which the supersymmetry transformation laws follow) as:

\[ T^a = 0 \]  
\[ \mathcal{R}_{ab} = \mathcal{R}_{abcd}V^cV^d - i\bar{\Psi}_A\Gamma_{[a}P^A_{b]c} V^c - \frac{i}{8} X_I \check{F}_{[abcd}\epsilon_{abcde}\bar{\Psi}_A\Gamma^c\Psi^A - \frac{i}{2} X_I \check{F}_{ab}\bar{\Psi}_A\Psi^A + \]  
\[ + \frac{1}{4} g_{xy} \left( \chi A \Gamma^c \psi B \right) - \chi A \Gamma^c \psi B - \frac{1}{2} \chi A \Gamma^c \psi B \right) + \]  
\[ + i S^{AB}\bar{\Psi}_A \Gamma^{abc} \Psi_B - \frac{1}{4} \zeta_{abc} \bar{\Psi}_A \Gamma^c \Psi^A \]  
\[ F^I = \hat{F}_{ab} V^a V^b - 2 f^I_{x} \bar{\Psi}_A \Gamma_{[a} \lambda^x V^a \]  
\[ H_M = \hat{H}_{M[abc} V^a V^b V^c - h_{MX} \bar{\Psi}_A \Gamma_{[a} \lambda^x V^a V^b \]  
\[ D_{\phi}^x = \hat{D}_a \phi^x V^a + \bar{\Psi}_A \lambda^x \]  
\[ D_{\phi}^a = \hat{D}_a \phi^x V^a + U^{\alpha} A_{\alpha} \bar{\Psi}_A \lambda^x \]  
\[ \rho^A = \rho^A_{ab} V^a V^b - \frac{1}{8} X_I \check{F}_{[abc} \left( \Gamma_{abc} - 4 \eta_{abc} \Gamma_{[d]} \right) \Psi^A V^a + S^{AB} \Gamma_{a} \Psi_B V^a + \]  
\[ + \frac{1}{4} g_{xy} \left[ \chi A \Gamma^b \psi B \right) \psi B + \frac{1}{4} \chi A \Gamma^{abc} \psi B \right] V^c + \]  
\[ - \frac{i}{8} \zeta_{abc} \psi B \]  
\[ \nabla^x = \bar{\nabla}_a \lambda^x V^a + \frac{i}{2} \hat{D}_a \phi^x \Gamma^a \Psi^A + \frac{i}{4} g_{xy} \hat{F}_{ab} \Gamma^{abc} \Psi^A + i W^x AB \psi_B + \]  
\[ + \frac{1}{4} T^x_{yz} \left( -3 \chi A \lambda^x \psi B \psi_B + \chi A \lambda^x \psi_B + \frac{1}{2} \chi A \psi_B \right) \]  
\[ \nabla^\alpha = \bar{\nabla}_a \psi B \]  
\[ f^I_x \]  
\[ h_{Mx} \]  
\[ g_{xy} \]  
\[ g^x_I \]  
\[ T^x_{yz} \]  

in terms of a set of scalar-dependent quantities:

\[ f^I_x, \ h_{Mx}, \ g_{xy}, \ g^x_I, \ T^x_{yz} \]

and of the fermion-shifts due to the gauging \( S^{AB}, W^x AB, N^\alpha_A \). The ‘hat’ on the field-strengths and covariant derivatives denotes the supercovariant part. Eq.s (3.18) - (3.26) give a set of constraints among the quantities appearing in the parametrizations (3.35) - (3.42). Part of them are reported below:

\[ f^I_x = D_x X^I \]  
\[ h_{Mx} = - D_x X_M \]  
\[ D_{(y f^I_x)} = T^x_{xy} f^I_x + X^I g_{xy} \]  
\[ T^x_{[xy]} = 0 \]  
\[ X_M = -2 d_{ij} M X^I X^J \]  
\[ \check{H}_{M[abc} = - \frac{1}{6} \epsilon_{M[ab} \epsilon_{cde} \check{F}_{1de} \]  
\[ X^I X_J + f^I_x g_{xy} = \delta^I_J \]  
\[ f^I_x W^x [AB] = \frac{1}{2} m^I M X_M \epsilon^{AB} \]
\[ S^{AB} = X^{\bar{I}} \mathcal{P}_{\bar{I}}^{AB} \; ; \quad S^{[AB]} = 0 \]  
(3.51)

\[ 2X_M S^{(AB)} = \hbar_{Mx} W^{x(AB)} \]  
(3.52)

\[ \mathcal{P}_{\bar{I}}^{AB} m^{\bar{I}M} = k^{\bar{I}}_m m^{\bar{I}M} = 0 \]  
(3.53)

\[ \mathcal{P}_{\bar{I}}^{AC} \mathcal{P}_{\bar{J}}^{CB} = \frac{1}{3} \epsilon_{\bar{I} \bar{J}} \mathcal{P}^{AB}. \]  
(3.54)

In eq. (3.48) we have introduced the matrix

\[ a_{\bar{I} \bar{J}} = X_{\bar{I}} X_{\bar{J}} + \hbar_{\bar{I} \bar{J}} g^x \]  
(3.55)

which will appear in the Lagrangian as kinetic matrix for the vector field-strengths. Eq. (3.48) expresses, at the supersymmetric level, the duality relation among the B-field-strengths and the vector field-strengths.

Since the analysis has been done only at 2-fermion level, these are not the totality of the algebraic and geometric constraints of the theory. Further constraints are more easily evaluated from the equations of motion in superspace of the rheonomic Lagrangian given in appendix B and will be reported in the next subsection.

### 3.3 The Lagrangian

Writing the action as:

\[ S = \int \sqrt{-g} d^5 x \mathcal{L}, \]  
(3.56)

the Lagrangian of the theory is:

\[ \mathcal{L} = \mathcal{L}_{\text{Grav}} + \mathcal{L}_{\text{Kin}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_4 \]  
(3.57)

with:

\[ \mathcal{L}_{\text{Grav}} = R + \frac{i}{\sqrt{-g}} \bar{\Psi}_{A \mu} \Gamma^{\mu \nu \rho} \bar{\psi}_\nu \]  
(3.58)

\[ \mathcal{L}_{\text{Kin}} = -\frac{3}{8} a_{\bar{I} \bar{J}} X_{\bar{I}} X_{\bar{J}} + \frac{3}{16} \sqrt{-g} \epsilon^{\mu \nu \rho \lambda} m^{MN} B_{M \mu \rho} B_{N \lambda} + \]  
(3.59)

\[ \mathcal{L}_{\text{Pauli}} = -\frac{3}{8} i X_{\bar{I}} \bar{X}_{\bar{J}} \bar{\Psi}_{A \mu} \bar{\Psi}_{A \rho} \Gamma^{\mu \nu} \Gamma^{\nu} \bar{\Psi}_A + \frac{3}{4} \epsilon_{\bar{I} \bar{J}} X_{\bar{I}} X_{\bar{J}} \bar{\Psi}_{A \mu} \bar{\Psi}_{A \rho} \Gamma^{\mu \nu} \Gamma^{\nu} \bar{\Psi}_A + \]  
(3.60)

\[ \mathcal{L}_{\text{gauge}} = -i S^{AB} \bar{\Psi}_A \Gamma_{\mu} \bar{\psi}_B - 3 g_{x y} W^{x A B} \bar{\chi}^{\mu} \Gamma_{\mu} \bar{\psi}_B + 2 N_{\alpha} \bar{\Psi}_{A \mu} \Gamma^{\mu} \zeta_{\alpha} + \]  
(3.61)

\[ \mathcal{L}_{\text{CS}} = \frac{3}{16} 2 m^{MN} B_{M \mu \nu} d_{\bar{I} J N} \left( \bar{F}_{\bar{I} \bar{J}}^{\rho \sigma} - i X_{\bar{I}} \bar{\Psi}_{A \rho} \bar{\Psi}_A \right) A_{\bar{J}}^\rho \]  

$$+\frac{1}{3}t_{1JK}A_\mu^I\partial_\nu A_\rho^J\partial_\sigma A_\tau^K + \frac{1}{4} \left( t_{1LM}f_{JK}^L + 4d_{1JM}m^{MN}d_{MKN} \right) A_\mu^I A_\nu^J A_\rho^K \partial_\sigma A_\tau^M + \frac{1}{20} \left( t_{1LM}f_{JK}^L + 4d_{1JM}m^{MN}d_{MKN} \right) f_{NP}^M A_\mu^I A_\nu^J A_\rho^K A_\sigma^\tilde{\nu} A_\tau^\tilde{\rho} \right] \epsilon^{\mu
u\rho\sigma}\right) \right) \right)$$

(3.62)

where:

$$M_{xy|AB} = (g_{yM}k^y_j f_j - \frac{1}{2} h_{Mx}m^{MN}h_{Ny}) \epsilon^{AB} - 2f_2 T^x_{xy} P^{AB}_I$$

(3.63)

$$M^\alpha_\beta = \frac{1}{2} U^a_{\alpha} u_{\beta} D_{[\mu} k^\nu_{,\nu} X^I$$

(3.64)

$$M_{\alpha_\alpha} = -2U^a_{\alpha} k^u_j f_j$$

(3.65)

and:

$$\left\{ \begin{align*}
S^{AB} &= X^I P^I_{AB} \\
W^{xAB} &= g^{xy}(\frac{1}{2} h_{Iy} m^{\tilde{I} M} X_M^{AB} - 2f_2 P^{AB}_I) \\
\mathcal{N}^{\alpha_\alpha} &= 2U^a_{\alpha} k^u_j X^I
\end{align*} \right.$$  

(3.66)

Finally, $t_{1JK}$ introduced in (3.62) is a covariantly constant tensor.

In (3.62), the freedom under rescaling (2.43) has been used to fix the overall normalization. More details on the calculation are given in appendix B. The 4-fermions contributions to the Lagrangian, from [13], is reported in appendix C.

The scalar potential is

$$V = -12S^{AB} S_{AB} + \frac{3}{2} g_{xy} W^{xAB} W_{AB}^{xy} + N_{AB} A_{\alpha} A_{\alpha}$$

$$= 6P^{AB} P_{JAB} \left( f_2 X^I X^J - \frac{1}{2} X^I X^J \right) + \frac{3}{4} X_M X_N M^{MP} M^{NL} h_{x^y} h_{y^x} +$$

$$+ 4g_{uv} k^y_j k^u_j X^I X^J.$$  

(3.67)

The following Ward-identity on the gauging holds

$$V^{B}_{\alpha} = -24 S^{BC} S^{\alpha} + 3g_{xy} W^{xCB} W_{CA}^{xy} + 2N_{AB} A_{\alpha}.$$  

(3.68)

Eq. (3.68) is identically satisfied, given eqs (3.66), for any $SU(2)$-valued $P^{AB}_I = P^{\alpha_\alpha}_I$. The Lagrangian (3.57) is left invariant by the supersymmetry transformation rules (with supersymmetry parameter $\epsilon^A$):

$$\delta V^{\alpha}_\mu = -i\overline{\psi}_{A \mu} \Gamma^a \epsilon^a$$

$$\delta A^I_\mu = 2i X^I \overline{\psi}_{A \mu} \epsilon^A - 2f_2 \Gamma_{A \mu} \lambda^x A$$

$$\delta B_{\mu \rho} = 2id_{1JM} X^I A^I_{[\mu} \overline{\psi}_{\rho]} \epsilon^A + 2i X_M \overline{\psi}_{[\mu} \Gamma_{\rho]} \epsilon^A - h_{Mx} \overline{\psi}_{A \mu} \Gamma_{\rho} \lambda^x A$$

$$\delta \varphi^x = \overline{\lambda}_A \lambda^x A$$

$$\delta q^u = U^a_{\alpha} \overline{\psi}_{A \alpha} \epsilon^A$$

$$\delta \Phi^A = D_{\mu} \overline{\psi}_{A \mu} \epsilon^A$$

$$\delta \lambda^x A = D_{\mu} \overline{\psi}_{A \mu} \epsilon^A - \omega_{x \mu} \overline{\psi}_{A \mu} \epsilon^A$$

$$\delta \Delta^{\alpha}_u = D_{\mu} \overline{\psi}_{A \mu} \epsilon^A$$

(3.69)

$$\delta \xi^a = -i\overline{\psi}_{A \mu} \Gamma^a \epsilon^A$$

$$\delta \zeta^\alpha = -i\overline{\psi}_{A \mu} \Gamma^\alpha \epsilon^A$$
For our calculations, we used the geometrical (rheonomic) approach which, as is well known, provides not only the space-time Lagrangian and the superspace equations of motion, but also the value of the generalized curvatures in superspace thus providing constraints on the physical fields of the theory. This is of course equivalent to require space-time supersymmetry.

The extra constraints we find besides those already given by closure of the Bianchi identities (3.45)-(3.54) are:

\[ t_{i j K} X^i X^j X^K = 1 \]  
(3.70)

\[ X^i = t_{i j K} X^j X^K = a_{i j} X^j \]  
(3.71)

\[ f_x^i = D_x X^i \]  
(3.72)

\[ g_{x y} = -2t_{i j K} X^K f_x^i f_y^j = a_{i j} f_x^i f_y^j \]  
(3.73)

\[ T_{x y}^z = t_{i j K} g^{x w} f_w^i f_y^j f_y^j \]  
(3.74)

\[ h_{i x} = -D_x X_i = a_{i j} f_x^j = g_{x y} g_i^y \]  
(3.75)

In particular, (3.70) defines the equation of the surface generally characterizing the scalar geometry of \( D = 5, N = 2 \) tensor and vector multiplet sector. Furthermore, the above relations also imply the constraints on the curvature of \( M(\varphi) \) characterizing its geometry:

\[ R_{x y z t} = \left( \delta_{x [y}^{x} g_{z]}^{y} + T_{w [z}^{x} T_{w z]}^{x} \right) \]  
(3.82)

and a relation between the constant \( t_{i j K} \) defining the surface and the scalar-dependent couplings:

\[ t_{i j K} = \frac{1}{2} \left( 5X_i X_j X_K - 3a_{i j K} X_K + 2T_{x y z} g^{i} g^{y} g^{z} \right) \]  
(3.83)

Since the geometrical properties of the \( \sigma \)-model \( M(\varphi) \) have been discussed thoroughly in the original paper [13], we omit further comments on this point.

### 3.4 Comments on the scalar potential

The scalar potential that we find in eq. (3.67):

\[ V = 6P_i^{A B} P_{j A B} \left( f_x^i f_y^j - 2X^i X^j \right) + \frac{3}{4} X_M X_N M^P M^N L^L h_{x z} g^{i} g^{y} h_{L y} + 4 g_{x i} k_i^u k_j^v X^i X^j \]  
(3.84)
is formally the same as the one found in the literature \[18\] (a rescaling of the fields is required for a precise comparison; a map is given in Appendix E). However, since we are considering more general couplings and a non-trivial \( m^{MN} \) matrix, a few comments are in order.

First of all, it is already known that the presence of the tensor multiplets allows a non-zero \( W_{\alpha \beta} = \frac{1}{2} g^{\alpha \beta} h_{ij} m^{ij} X_M \epsilon^{AB} \) but that the contribution to the scalar potential coming from the tensors is always positive, so that Anti de Sitter solutions may only be accounted for a non trivial (possibly constant) \( P_i^{AB} \), giving a mass to the gravitino, while, in the case \( P_i^{AB} = 0 \), only Minkowski vacua are attainable, for

\[
h_{MX} m^{MN} X_N = 0. \quad (3.85)
\]

This is in particular the case when one considers as \( N = 2 \) model the Scherk–Schwarz generalized dimensional reduction of a six dimensional theory, as discussed in [22]. For the case of the S-S dimensionally reduced theory, to have a non negative scalar potential a cancellation is needed between the gaugino and gravitino contributions, proportional to the prepotential \( P_i^{AB} \). This does not appear instead to be necessary in more general, purely five dimensional, cases, still allowing, however, a general antisymmetric matrix \( m^{MN} \). Then, when \( m^{MN} \) has general skew-eigenvalues, eq. (3.85) may have solutions more general than the “symplectic-orthogonality” condition between \( h_{MX} \) and \( X_N \).

Let us now see the implications of having \( d_{\Lambda \Sigma M} \neq 0 \). Eq. (3.85) has a solution for:

\[
X_M = t_{ijM} X^i X^j = -4 d_{\Lambda \Sigma M} X^\Lambda X^N - 2 d_{\Lambda \Sigma M} X^\Lambda X^\Sigma = 0 \quad (3.86)
\]

where we used the relation (3.80) Eq. (3.86) must be solved together with the defining equation of the scalar geometry

\[
t_{ij\hat{k}} X^i X^j X^{\hat{k}} = 1 \quad (3.87)
\]

that is:

\[
X^\Lambda \left( t_{\Lambda \Sigma \Gamma} X^\Sigma X^\Gamma + 2 t_{\Lambda \Sigma M} X^\Sigma X^M + 2 t_{\Lambda M N} X^M X^N \right) = 1. \quad (3.88)
\]

In (3.88) we used the fact that, for \( m^{MN} \) invertible, \( t_{MNP} = 0 \), as explicitly shown in appendix B, eq. (B.17). Eq. (3.88) requires \( X^\Lambda \neq 0 \) for at least one value of \( \Lambda \) (e.g. \( X^\Lambda |_{vac} \propto \delta_0^\Lambda \)), and it then implies that the v.e.v. of the scalars \( X^M \) are now shifted from zero, since eq. (3.86) is solved for

\[
d_{\Lambda M N} X^N |_{vac} = -\frac{1}{2} d_{\Lambda M \Sigma} X^\Sigma |_{vac} \neq 0. \quad (3.89)
\]

4. Conclusions and outlook

In the present paper we have studied the \( D = 5, N = 2 \) theory coupled to vector, tensor and hyper multiplets by including all possible couplings compatible with gauge symmetry and supersymmetry. We paid particular attention in analyzing the algebraic structure of the
FDA which underlies the theory. This allowed to relax some constraints on the couplings usually considered, and correspondingly to write-down a scalar potential a bit more general than usually considered. It would be interesting to analyze in detail the critical points of models exhibiting the features described here, as in particular a magnetic coupling $m^{MN}$ with arbitrary skew-eigenvalues. Models of this kind (an example of which is found by Scherk–Schwarz compactification from six dimensions [22]) should appear in general flux compactifications from superstring or M-theory.

Our investigation may now be extended in various directions. At a group-theoretical point of view, it would be interesting to extend the FDA to include also higher order forms, as is the case, in general, in theories corresponding to compactifications from superstrings or M-theory. We would also like to perform an analysis, on the same lines of the one presented here, for the $D = 4 N = 2$ theory coupled to vector-tensor multiplets. These developments are under investigation and will be discussed elsewhere.

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References

[1] J. Louis, A. Micu, Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes, Nucl. Phys. B635 (2002) 395, [hep-th/0202168]

[2] B. de Wit, R. Philippe, A. Van Proeyen, THE IMPROVED TENSOR MULTIPLET IN N=2 SUPERGRAVITY, Nucl. Phys. B219 (1983) 143

[3] U. Theis, S. Vandoren, $N = 2$ supersymmetric scalar-tensor couplings, JHEP 04 (2003) 042, [hep-th/0303048]

[4] R. D’Auria, L. Sommovigo and S. Vaulà, “$N = 2$ supergravity Lagrangian coupled to tensor multiplets with electric and magnetic fluxes,” JHEP 0411 (2004) 028 [arXiv:hep-th/0409097].

[5] G. Dall’Agata, R. D’Auria, L. Sommovigo, S. Vaulà, $D = 4$, $N = 2$ gauged supergravity in the presence of tensor multiplets, Nucl. Phys. B682 (2004) 243, [hep-th/0312210]

[6] L. Sommovigo, S. Vaulà, $D = 4$, $N = 2$ supergravity with Abelian electric and magnetic charge, Phys. Lett. B602 (2004) 130, [hep-th/0407205]

[7] M. Gunaydin, S. McReynolds, M. Zagermann, Unified $N = 2$ Maxwell-Einstein and Yang-Mills-Einstein supergravity theories in four dimensions, JHEP 09 (2005) 026, hep-th/0507227

[8] M. Gunaydin, S. McReynolds, M. Zagermann, The R-map and the coupling of $N = 2$ tensor multiplets in 5 and 4 dimensions, JHEP 01 (2006) 168, [hep-th/0511025]
[9] M. Gunaydin, L. J. Romans, N. P. Warner, *Gauged N=8 supergravity in five-dimensions*, Phys. Lett. B154 (1985) 268

[10] M. Gunaydin, L. J. Romans, N. P. Warner, *Compact and noncompact gauged supergravity theories in five-dimensions*, Nucl. Phys. B272 (1986) 598

[11] M. Pernici, K. Pilch, P. van Nieuwenhuizen, *Gauged N=8 D = 5 supergravity*, Nucl. Phys. B259 (1985) 460

[12] M. Gunaydin, G. Sierra, P. K. Townsend, *Exceptional Supergravity theories and the magic square*, Phys. Lett. B133 (1983) 72

[13] M. Gunaydin, G. Sierra, P. K. Townsend, *The geometry of N=2 Maxwell-Einstein Supergravity and Jordan algebras*, Nucl. Phys. B242 (1984) 244

[14] M. Gunaydin, G. Sierra, P. K. Townsend, *Vanishing potentials in gauged N=2 Supergravity: an application of Jordan algebras*, Phys. Lett. B144 (1984) 41. M. Gunaydin, G. Sierra, P. K. Townsend, *Gauging the d = 5 Maxwell–Einstein Supergravity Theories: More on Jordan Algebras*, Nucl. Phys. B253 (1985) 573. M. Gunaydin, G. Sierra, P. K. Townsend, *Quantization of the gauge coupling constant in a five- dimensional Yang-Mills / Einstein Supergravity theory*, Phys. Rev. Lett. 53 (1984) 322.

[15] G. Sierra, *N=2 Maxwell matter Einstein Supergravities in D = 5, D = 4 and D = 3*, Phys. Lett. B157 (1985) 379

[16] A. Lukas, B. A. Ovrut, K. S. Stelle, D. Waldram, *The universe as a domain wall*, Phys. Rev. D59 (1999) 086001, [hep-th/9803235]. A. Lukas, B. A. Ovrut, K. S. Stelle, D. Waldram, *Heterotic M-theory in five dimensions*, Nucl. Phys. B552 (1999) 246, [hep-th/9806051]

[17] M. Gunaydin, M. Zagermann, *The gauging of five-dimensional, N = 2 Maxwell-Einstein supergravity theories coupled to tensor multiplets*, Nucl. Phys. B572 (2000) 131, [hep-th/9912027]. M. Gunaydin, M. Zagermann, *The vacua of 5d, N = 2 gauged Yang-Mills/Einstein/tensor supergravity: Abelian case*, Phys. Rev. D62 (2000) 044028, [hep-th/0002228]. M. Gunaydin, M. Zagermann, *Gauging the full R-symmetry group in five-dimensional, N = 2 Yang-Mills/Einstein/tensor supergravity*, Phys. Rev. D63 (2001) 064023, [hep-th/0004117]. M. Gunaydin, M. Zagermann, *Unified Maxwell-Einstein and Yang-Mills-Einstein supergravity theories in five dimensions*, JHEP 07 (2003) 023, [hep-th/0304109].

[18] A. Ceresole, G. Dall’Agata, *General matter coupled N = 2, D = 5 gauged supergravity*, Nucl. Phys. B585 (2000) 143, [hep-th/0004111]

[19] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren, A. Van Proeyen, *N = 2 supergravity in five dimensions revisited*, Class. Quant. Grav. 21 (2004) 3015, [hep-th/0403045]

[20] P. K. Townsend, K. Pilch, P. van Nieuwenhuizen, *Selfduality in odd dimensions*, Phys. Lett. 136B (1984) 38

[21] B. de Wit, H. Samtleben, M. Trigiante, *The maximal D = 5 supergravities*, Nucl. Phys. B716 (2005) 215, [hep-th/0412173]

[22] L. Andrianopoli, S. Ferrara, M. A. Lledo, *No-scale D = 5 supergravity from Scherk-Schwarz reduction of D = 6 theories*, JHEP 06 (2004) 018, [hep-th/0406018]

[23] G. Dall’Agata, R. D’Auria, S. Ferrara, *Compactifications on twisted tori with fluxes and free differential algebras*, Phys. Lett. B619 (2005) 149, hep-th/0503122
A. A trivial deformation of the FDA

In this appendix we show that a further possible deformation of the FDA (2.19) via an extra 3-vector contribution in the field strengths $H_M$ can be always reabsorbed by a field redefinition provided we do not couple the system to higher order forms. It is in fact possible to deform the FDA (2.19) as follows

$$\begin{align*}
F^I & \equiv dA^I + \frac{1}{2} f_{JK}^I A^J \wedge A^K + m^{IM} B_M \\
H_M & \equiv dB_M + T_{IM}^N A^I B_N + d_{IJM} F^I \wedge A^J + e_{MIJK} A^K \wedge A^J \wedge A^K
\end{align*}$$

(A.1)

with the constant $e_{MIJK} = e_{M[IK]}$ completely antisymmetric in the last 3 indices. This is a deformation of the FDA structure, which leaves unchanged the Bianchi identities (2.20), but modifies the constraints in the following way:

$$\begin{align*}
[f_{[ij}^L f_{K]}^M] & = 2 e_{MIJK} m^{iM} \\
[T_{IJ}]^P_M & = f_{[ij}^K T^K_{KM}^P + 6 e_{MIJK} m^{KP} \\
T_{IM}^{(N} m_{i]}^P & = 0 \\
m^{IN} T_{iJ}^M & = f_{jK}^i m^{KM} \\
T_{IM}^N & = d_{IJM} m^{iN} \\
T_{[i}^N d_K^J ]^N & = \frac{1}{2} f_{[i}^{JK} d_{[i]}^L d_{]_j]}^N - \frac{1}{7} f_{[i}^{JK} d_{]_K}^L + 3 e_{MIJK} = 0 \\
e_{N[J]KL} T_{iM}^N & = \frac{1}{2} e_{MP} \tilde{\epsilon} \tilde{\epsilon} = 0.
\end{align*}$$

(A.2)

The last equation of (A.2) means that $e_{MIJK}$ is a cocycle of the Lie algebra $G$.

For non-zero $e_{MIJK}$, the gauge transformations of the system are deformed into

$$\begin{align*}
\delta A^I & = de^I + f_{JK}^I A^J \epsilon^K - m^{iM} A_M \\
\delta B_M & = d\Lambda_M + T_{IM}^N A^I \Lambda_N - d_{IJM} F^I \epsilon^J - T_{IM}^N \epsilon^I B_N - 3 e_{MIJK} A^J \wedge A^K \epsilon^K
\end{align*}$$

(A.3)

but give, for the field strengths, the same gauge transformation of the undeformed theory:

$$\begin{align*}
\delta F^I & = -f_{JK}^N \epsilon^J F^K \\
\delta H_M & = -T_{IM}^N \epsilon^J H_N.
\end{align*}$$

(A.4)

As we see from (A.2), in this case the Jacobi identities fail to close and the $T_{IM}^N$ do not generate anymore the algebra $G$, which is explicitly broken. However, for any general value
of $e_{MIJK}$ subject to (A.2), the entire algebra $G$ is still generated by the hatted generators $\hat{f}, \hat{T}$, that still satisfy eqs (2.37). The consistency of the extended theory is guaranteed since, from (2.37) we have, for any $e_{MIJK}$:

$$\begin{align}
\left[ \left[ \hat{f}_{[j}, \hat{f}_{j]}, \hat{f}_{K} \right], \hat{f}_{K} \right]_{L}^{P} &= -f_{[i} \hat{f}_{j]N} \hat{f}_{K}^{N} \hat{f}_{ML}^{P} = -2e_{MIJK} \hat{f}_{M}^{N} \hat{f}_{K}^{P} = 0 \\
\left[ \hat{T}_{[j}, \hat{T}_{j]}, \hat{T}_{K} \right]_{M}^{N} &= -f_{[i} \hat{f}_{j]M} \hat{f}_{K}^{M} \hat{f}_{LM}^{N} = -2e_{PIJK} \hat{f}_{P}^{M} \hat{f}_{K}^{N} = 0
\end{align}$$

(A.5)

due to (A.2) and in particular to

$$m^{IN} T_{jN}^{M} - f_{jK} \hat{f}_{M}^{K} = 0 \quad \Rightarrow \quad \hat{f}_{K} \hat{f}_{M}^{K} = 0$$

$$T_{IM}^{(N} m_{J)P} = 0 \quad \Rightarrow \quad T_{IM}^{N} m_{P} = 0.$$  

(A.6)

Therefore we can state that the gauge algebra $G$, even if not anymore realized in an abstract way, still closes when acting on the physical generators $\hat{f}, \hat{T}$ appearing in the Bianchi identities. To complete the proof that the extension of the FDA (2.19) to include the $e_{MIJK}$ is trivial, we are now going to show that, when expressed only in terms of the physical couplings, the structure of the FDA is not affected by any possible contribution in $e_{MIJK}$. To do so, let us recast the theory in terms of the physical couplings appearing in the Bianchi identities (2.20), as we did in section 2.2 for the FDA (2.19). As shown in section 2.2, this is done by the field redefinition 2.39. Then the FDA (A.7) takes the form:

$$\begin{align}
P^{I} &= dA^{I} + \frac{1}{2} f_{JK} \hat{f}_{I}^{J} A^{K} + m^{IM} \tilde{B}_{M} \\
H_{M} &= dB_{M} + \frac{1}{2} T_{IM}^{N} A^{J} \hat{B}_{N} + d_{I(j)M} F^{I} A^{J} + K_{MIJK} A^{I} A^{J} A^{K}
\end{align}$$

(A.7)

where:

$$K_{MIJK} = e_{MIJK} - \frac{1}{2} T_{IM}^{N} d_{jK}^{j} - \frac{1}{4} f_{ij} \hat{f}_{K}^{L} d_{kL}^{j} + \frac{1}{4} d_{LMK}^{j} f_{ij}^{L},$$

(A.8)

and:

$$\begin{align}
\hat{f}_{j}^{M} \hat{f}_{K}^{N} &= 2m^{L} K_{M(jK]} \\
\hat{T}_{IM}^{N} \hat{T}_{jN}^{P} &= \hat{f}_{j}^{K} \hat{T}_{K}^{P} + 12K_{MIJK} m^{KP} \\
\hat{T}_{IM}^{N} m_{IP} &= 0 \\
\frac{1}{2} m^{IN} \hat{T}_{jN}^{M} &= \hat{f}_{j}^{M} \hat{f}_{M}^{K} \\
\hat{T}_{IM}^{N} &= 2d_{I(j)M} m^{IP} \\
\hat{T}_{jM}^{N} d_{(j)N}^{k} &= 2\hat{f}_{j}^{K} \hat{d}_{(j)N}^{k} \hat{d}_{kL}^{j} - \hat{f}_{j}^{L} \hat{d}_{(j)N}^{k} \hat{d}_{kL}^{j} = -6K_{MIJK} \\
K_{N(jK} \hat{T}_{l)}^{N} m_{M} &= -3K_{MP} \hat{f}_{j}^{K} \hat{T}_{l}^{P} = 0.
\end{align}$$

(A.9)

Note that, by substituting the value of $e_{MIJK}$ given by (A.2), eq. (A.8) may be rewritten as

$$K_{MIJK} = \frac{1}{2} f_{ij} \hat{d}_{(j)N}^{k} \hat{d}_{kL}^{j} + \frac{2}{3} d_{(j)M} m^{IP} \hat{f}_{j}^{K} \hat{L},$$

(A.10)

which is identical to (2.42). This shows that, when the FDA is expressed only in terms of the physical couplings $\hat{f}_{i}, \hat{T}_{j}$ and $d_{(j)M}$, it does not depend on any possible contribution from $e_{MIJK}$ in (A.1). We conclude that the couplings $e_{MIJK}$ give a trivial deformation of the FDA (2.19). This means that the cocycle $e_{MIJK}$ is in fact a coboundary. According to the general construction of the free differential algebras, one can however expect that, if one enlarges the FDA by introducing 3-form potentials and the associated curvatures, a 4-form associated to $e_{MIJK}$ could play a role. This possibility will not be pursued here, but left for a future publication.
B. The D=5 Rheonomic Lagrangian

We write down explicitly here the rheonomic lagrangian, up to 4-fermion terms. We recall that, in the rheonomic approach, the action is written as

\[ S = \int_{\mathcal{M}_5} \mathcal{L} \]  

(B.1)

where \( \mathcal{M}_5 \) is a generic bosonic surface embedded in superspace. \( \mathcal{L} \) is a 5-form written in terms of superfields without use of the Hodge-duality operator. It is then first-order in the kinetic terms, that is auxiliary fields \( F_{ab}^I, X_I^a, Q_a^\alpha \) are introduced and are then fixed in terms of the physical field-strengths by solving their field equations. The space-time lagrangian is retrieved by restricting the rheonomic lagrangian along the space-time differentials \( dx^\mu \), at zero fermionic coordinates \( \Theta^A = d\Theta^A = 0 \).

With this approach, the field equations are valid all over superspace. The equations of motion on space-time are given by the field equations along the bosonic vielbein \( V^a \) of superspace, while the field equations with at least one fermionic direction \( \Psi^A \) yield the constraints on the supercurvatures and couplings.

The Lagrangian density up to 4-fermion terms can be written as:

\[ \mathcal{L} = \mathcal{L}_{\text{Grav}} + \mathcal{L}_{\text{Kin}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{Tors}} + \mathcal{L}_{\text{gauge}} \]  

(B.2)

where

\[ \mathcal{L}_{\text{Grav}} = R^{ab} V^c V^d V^e \epsilon_{abcde} - 6i \bar{\Psi}_A \Gamma_{ab} \phi^A V^a V^b \]  

(B.3)

\[ \mathcal{L}_{\text{Kin}} = -\frac{3}{4} a_{ij} F_{ab}^I \left( F_{IJ} + 2f_{ij}^I \bar{\Psi}_A \Gamma_{\ell} \lambda^A \lambda^\ell \right) \epsilon_{abcdef} V^c V^d V^e + \right. \]
\[ \left. + \frac{3}{4} g_{xy} X^x_a \left( D_{\varphi^y} - \bar{\Psi}_A \lambda^A \lambda^y \right) + g_{uv} Q^\mu_a \left( D_{\varphi^v} - \bar{\Psi}_A \lambda^A \lambda^v \right) \right] \epsilon_{abcde} V^a V^b V^c V^d V^e + \right. \]
\[ \left. - \frac{1}{10} \left[ \frac{3}{4} g_{xy} X^x_a \left( D_{\varphi^y} - \bar{\Psi}_A \lambda^A \lambda^y \right) + g_{uv} Q^\mu_a \left( D_{\varphi^v} - \bar{\Psi}_A \lambda^A \lambda^v \right) \right] \epsilon_{abcde} V^a V^b V^c V^d V^e + \right. \]
\[ \left. + \frac{3}{4} g_{xy} \bar{\lambda}_A \Gamma_a \lambda^A \lambda^y + \frac{1}{2} \bar{\epsilon}_{\alpha} \Gamma_{a} \lambda^A \lambda^y \right] \epsilon_{abcde} V^a V^b V^c V^d V^e \]  

(B.4)

\[ \mathcal{L}_{\text{Pauli}} = \mathcal{F}^I \left[ -\frac{9}{2} J_I \bar{\Psi}_A \Gamma_a \Psi^A V^a - \frac{9}{2} b_{Ia} \bar{\Psi}_A \Gamma_{ab} \lambda^x V^a V^b + \right. \]
\[ \left. - \frac{1}{2} \left( \Phi_{Ij} \bar{\lambda}_A \Gamma_{abc} \lambda^b \lambda^c + 3 X_{j} \bar{\epsilon}_{\alpha} \Gamma_{abc} \lambda^b \lambda^c \right) \right] + \right. \]
\[ \left. (3 g_{xy} D_{\varphi^y} \bar{\lambda}_A \Gamma_{abc} \lambda^b \lambda^c + 4 U_a \phi^A D_{\varphi^y} \bar{\Psi}_A \Gamma_{abc} \lambda^b \lambda^c \right) \]  

(B.5)

\[ \mathcal{L}_{\text{Tors}} = -3i \bar{\Gamma}_a \Psi^A \left( \bar{\Psi}_A \Gamma^a + \frac{3}{4} g_{xy} \bar{\lambda}_A \Gamma_{bc} \lambda^b \lambda^c + \frac{1}{2} \bar{\epsilon}_{\alpha} \Gamma_{bc} \lambda^b \lambda^c \right) \]  

(B.6)

\[ \mathcal{L}_{\text{gauge}} = \left( \frac{3}{2} g_{xy} W^y A^B \bar{\Psi}_B + N_{\alpha} \bar{\Psi}_A \Gamma^a \lambda^A \lambda^a \right) \epsilon_{abcde} V^a V^b V^c V^d V^e + \right. \]
\[ \left. + 6i S A^B \bar{\Psi}_A \Gamma_{abc} \Psi_B V^a V^b V^c - \frac{1}{10} V(\phi) \epsilon_{abcde} V^a V^b V^c V^d V^e + \right. \]
\[ \left. - \frac{1}{10} \left( 3i \bar{M}_{xy} A^B \bar{\Psi}_B \lambda^A + 4i M^{x} \bar{\epsilon}_{\alpha} \lambda^A \lambda^x + 2i M^{x} \bar{\epsilon}_{\alpha} \lambda^A \right) \epsilon_{abcde} V^a V^b V^c V^d V^e \right) \]  

(B.7)
The Chern-Simons Lagrangian can be written down in terms of $A^\tilde{I}$ and $B_M$:

$$\mathcal{L}_{CS} = \alpha m^{MN} B_M d_B + s_i^{MN} B_M B_N A^{\tilde{I}} + s^i_{ij} B_M A^{\tilde{I}} d A^{\tilde{J}} + s^i_{ijk} B_M A^{\tilde{I}} A^{\tilde{J}} A^{\tilde{K}} + \frac{3}{4} t_{ij\tilde{K}} A^{\tilde{I}} d A^{\tilde{J}} d A^{\tilde{K}} + r_{ij\tilde{K}}[L A^{\tilde{I}} A^{\tilde{J}} A^{\tilde{K}} d A^{\tilde{L}} + r_{ij\tilde{K}L\tilde{M}} A^{\tilde{I}} A^{\tilde{J}} A^{\tilde{K}} A^{\tilde{L}} A^{\tilde{M}} \tag{B.8}$$

where the gauge invariance of $\mathcal{L}_{CS}$ implies:

$$s_i^{MN} = \alpha m^{MP} T_{iP}^N \tag{B.9}$$

$$s_{ij}^M = 2a d_{i[i} m^{MN} \tag{B.10}$$

$$s_{ij\tilde{K}} = \alpha m^{MN} d_{L[i]N} \tilde{f}_{\tilde{K}]L} \tag{B.11}$$

$$s_{ij} = \frac{1}{4} t_{ij\tilde{K}} m^{\tilde{K}M} \tag{B.12}$$

$$r_{ij\tilde{K}}[L = \frac{1}{16} t_{\tilde{K}LM} \tilde{f}_{ij} \tilde{M} + \frac{i}{2} d_{\tilde{K}[i} m^{MN} m^{\tilde{L]}j} N \tag{B.13}$$

$$r_{ij\tilde{K}L\tilde{M}} = \frac{1}{80} i N Q f_{ij} \tilde{K} \tilde{M} \tilde{Q} + \frac{\alpha}{5} d_{i[j} m^{MN} d_{\tilde{K}L] M} \tilde{f}_{\tilde{K}LM} \tilde{N} \tag{B.14}$$

Conditions (B.9)-(B.14), required for gauge invariance of the action, in particular imply that $t_{MNP} = 0$. To show this, let us consider eq. (B.12) and multiply it by $m^N m^p$:

$$\frac{1}{4} t_{ij\tilde{K}} m^{MP} m^{QN} = s_{ij} \tag{B.15}$$

But due to eq. (B.10) this is related to the physical coupling $\hat{T}$:

$$\frac{1}{4} t_{ij\tilde{K}} m^{MP} m^{QN} = 2a d_{i[j} Q m^{PQ} m^{QN} = \alpha \hat{T}_{iQ} N m^{IM} m^{PQ} \tag{B.16}$$

This last term vanishes due to eq. (2.41), so that, since we generally take $m^{MN}$ invertible, it gives:

$$t_{MNP} = 0. \tag{B.17}$$

As a final remark, let us observe that, given (B.9) - (B.14), all the Chern-Simons Lagrangian (3.4) contains a free multiplicative parameter, $\alpha$. However, recalling the discussion at the end of section 2, the theory still has the scale invariance (2.43), which may be used to fix the parameter $\alpha$ at our wish. We set $\alpha = \frac{2}{7}$. This finally gives eq. (3.62).

C. The four-fermions Lagrangian

The 4-fermions contributions to the Lagrangian, from [13], with our notations reads:

$$\mathcal{L}_{4f} = \left\{ -\frac{1}{16} D_{\mu\nu} T_{xy} \bar{\lambda}_A^{\mu} \lambda^b A_{\lambda B}^{\nu} \lambda^{wB} + \frac{1}{2} R_{xyzw} \left( \bar{\lambda}_A^{\mu \nu} A_{\lambda B}^{\nu} A_{\lambda B}^{\mu} + \bar{\lambda}_A^{\mu} A_{\lambda B}^{\nu} \lambda^{wB} \lambda^{wB} \right) + \frac{3}{4} g_{xy} g_{zw} \left( \bar{\lambda}_A^{\mu \nu} A_{\lambda B}^{\nu} A_{\lambda B}^{\mu} - \bar{\lambda}_A^{\mu} A_{\lambda B}^{\nu} \lambda^{wB} \lambda^{wB} \right) - \frac{3}{16} \bar{\lambda}_A^{\mu} A_{\lambda B}^{\nu} \lambda^{wB} \lambda^{wB} \right] + \left\{ 2 i T_{xyz} \left( \bar{\lambda}_A^{\mu} \psi_{\lambda B} \lambda^b A_{\lambda B}^{\nu} - \frac{1}{2} \bar{\lambda}_A^{\mu} \psi_{\lambda B} \lambda^{wB} \lambda^{wB} \right) \right\} \tag{3.42}$$

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\[ \left\{ \frac{3}{16} g_{xy} \left[ \bar{\Psi}_{\mu A} \Psi_{\nu B} \bar{\lambda}^{x A} \left( 3g^{\mu \nu} + \frac{1}{2} \Gamma^{\mu \nu} \right) \lambda^{y B} - \bar{\Psi}_{\mu A} \Gamma_{\rho \sigma} \Psi_{\nu B} \bar{\lambda}^{x A} \left( 3g^{\mu \nu} \Gamma_{\rho \sigma} + 2g^{\mu \rho} \Gamma^{\nu \sigma} + \frac{1}{2} \Gamma_{\mu \nu \rho \sigma} \right) \lambda^{y B} + \right. \]

\[ \left. - \frac{1}{2} \bar{\Psi}_{\mu A} \Gamma_{\rho \sigma} \Psi_{\nu B} \bar{\lambda}^{x A} \left( 2g^{\mu \rho} \Gamma_{\nu \sigma} - 4g^{\mu \rho} \Gamma_{\nu \sigma} - 2g^{\mu \nu} \Gamma^{\rho \sigma} + \frac{1}{2} \Gamma_{\mu \nu \rho \sigma} \right) \lambda^{y B} \right] \right\} f_{2 \lambda} + \]

\[ + \left\{ \frac{3}{16} \bar{\Psi}_{\mu A} \mu \nu B \Psi_{\nu B} - \frac{1}{8} \bar{\Psi}_{\mu A} \Gamma^{\nu \rho \sigma} \Psi_{\nu B} \Psi_{\sigma} + \frac{1}{2} \bar{\Psi}_{\mu A} \Gamma^{\nu B} \Psi_{\nu B} \Psi_{y B} + \right. \]

\[ \left. + \frac{1}{2} \Psi_{\mu A} \Gamma_{\nu \rho} \Psi_{\nu B} \Psi_{y B} - \frac{1}{2} \Psi_{\mu A} \Gamma_{\nu \rho} \Psi_{\nu B} \Psi_{y B} \right\} f_{2 \Psi} + \]

\[ + \bar{\Psi}_{\nu B} \zeta_{\alpha} \bar{\Psi}_{\gamma} \left( \Gamma^{\mu \nu} + \Gamma^{\mu \nu} \right) \zeta_{\alpha} + \frac{1}{16} \bar{\zeta}_{\alpha} \Gamma^{\mu \nu} \zeta_{\beta} \Gamma^{\mu \nu} \zeta_{\beta} + \]

\[ \left. \frac{3}{16} g_{xy} \bar{\zeta}_{\alpha} \Gamma^{\mu \nu} \lambda^{y B} - \frac{1}{4} \Omega^{\alpha \beta \gamma \delta} (5 \bar{\zeta}_{\alpha} \zeta_{\beta} \zeta_{\gamma} \zeta_{\delta} - \bar{\zeta}_{\alpha} \Gamma_{\mu \sigma} \zeta_{\gamma} \Gamma^{\mu \delta}) \right) \]

where \( \Omega^{\alpha \beta \gamma \delta} = \mathcal{R}^{\alpha \beta}_{\mu \nu} \mathcal{U}^{\gamma \alpha}_{\nu \rho} \mathcal{U}^{\delta \rho}_{\nu \sigma} \epsilon_{\alpha \beta} \).

**D. Useful relations with \( \Gamma \)-matrices and Fierz identities**

\[ \epsilon^{a_{1} \ldots a_{p} b_{1} \ldots b_{q}} \epsilon_{a_{1} \ldots a_{p} c_{1} \ldots c_{q}} = p! q! \delta^{b_{1} \ldots b_{q}}_{c_{1} \ldots c_{q}}, \quad (p + q = 5) \]  \hfill (D.1)

\[ \Gamma^{abcd} = \epsilon^{a b c d e} \Gamma_{e} \]  \hfill (D.2)

\[ \Gamma^{a b c} = - \frac{1}{2} \epsilon^{a b c d e} \Gamma_{d e} \]  \hfill (D.3)

\[ \Gamma_{a b c} = \Gamma_{a b c} + 2 \delta_{[a}^{b[c} \Gamma_{d]c]} \]  \hfill (D.4)

\[ \Gamma_{b c a} = \Gamma_{a b c} - 2 \delta_{[a}^{b[c} \Gamma_{d]c]} \]  \hfill (D.5)

\[ \Gamma^{a b c} \Gamma_{c d} = \Gamma^{a b d} - 4 \delta^{[a}_{[c} \Gamma^{b]}_{d]} - 2 \delta_{a}^{b c} \]  \hfill (D.6)

\[ \Gamma_{[a} \Gamma_{c d] \Gamma_{b]} = \Gamma_{a b c d e} + 4 \delta_{a}^{b c} \]  \hfill (D.7)

\[ \Gamma_{a b c_{1} \ldots c_{p}} = (5 - p) \Gamma_{b_{1} \ldots b_{p}}, \quad 0 \leq p \leq 4 \]  \hfill (D.8)

\[ \Gamma^{a b c_{1} \ldots c_{p}} = -(5 - p) (4 - p) \Gamma_{c_{1} \ldots c_{p}}, \quad 0 \leq p \leq 3 \]  \hfill (D.9)

\[ \Gamma^{a b} \Gamma_{c_{1} \ldots c_{p}} = 2 \Gamma_{bc} - 4 \delta_{b c} \]  \hfill (D.10)

\[ \Gamma^{a b} \Gamma_{c d} = \Gamma_{c d} \]  \hfill (D.11)

\[ \Gamma^{a b} \Gamma_{c} = -4 \eta_{[a}^{b c] \Gamma_{d]} = \Gamma_{a b c} \Gamma^{a} \]  \hfill (D.12)

\[ \Gamma^{a b} \Gamma_{c} = -3 \Gamma_{c} \]  \hfill (D.13)

\[ \Gamma^{a b} \Gamma_{c d} = 4 \Gamma_{c d} \]  \hfill (D.14)

Recalling that

\[ \Psi_{A} \equiv \epsilon_{A B} \Psi_{B}; \quad \Psi^{A} = - \epsilon^{A B} \Psi_{B} \]  \hfill (D.16)

and that the currents of spinor one-forms have the symmetry properties

\[ \bar{\Psi}_{A} \Psi_{B} = - \bar{\Psi}_{B} \Psi_{A} \]  \hfill (D.17)

\[ \bar{\Psi}_{A} \Gamma^{a} \Psi_{B} = - \bar{\Psi}_{B} \Gamma^{a} \Psi_{A} \]  \hfill (D.18)

\[ \bar{\Psi}_{A} \Gamma^{a b} \Psi_{B} = \bar{\Psi}_{B} \Gamma^{a b} \Psi_{A} \]  \hfill (D.19)
the following Fierz-identities follow:

\[
\Psi_A \wedge \Psi_B = \frac{1}{4} (\Psi_B \Psi_A + \Gamma_a \Psi_B \Gamma^a \Psi_A) - \frac{1}{8} \Gamma_{ab} \Psi_B \Gamma^{ab} \Psi_A \quad (D.20)
\]

\[
\Psi_A \wedge \Psi_B \wedge \Psi_C \equiv -\frac{1}{2} \epsilon_{BC} \Xi_A \quad (D.21)
\]

\[
\Psi_A \wedge \Psi_B \wedge \Gamma^a \Psi_C = -\frac{1}{2} \epsilon_{BC} \left( \Xi^a_A + \frac{1}{5} \Gamma^a \Xi_A \right), \quad \Gamma_a \Xi_A = 0 \quad (D.22)
\]

\[
\Psi_A \wedge \Psi_B \wedge \Gamma^{ab} \Psi_C = \Xi^{ab}_{(ABC)} - \frac{2}{3} \epsilon_{AB} \Gamma^{[a \Xi^b]}_{(C)} + \frac{1}{5} \epsilon_{A(B} \Gamma^{ab} \Xi^c_{(C)}), \quad \Gamma_a \Xi_{(ABC)} = 0 \quad (D.23)
\]

so that

\[
\Gamma^a \Psi_A \wedge \Psi_B \wedge \Gamma_a \Psi_B = \Psi_A \wedge \Psi_B \wedge \Psi_B \quad (D.24)
\]

\[
\Gamma^{ab} \Psi_A \wedge \Psi_B \wedge \Gamma_{ab} \Psi_C = -4 \delta^A_{(B} \Psi_{C)} \wedge \Psi_L \wedge \Psi_L \quad (D.25)
\]

E. Matching of notations

We have collected in the first table the differences in notation between [13] (GST) and the present paper (ADS) for the vector and tensor multiplet sector, and in the second the differences with [18] (CD) for the hypermultiplet sector.

| GST  | \( \eta_{ab} \) | \( \Gamma^a \) | \( \Gamma_a \) | \( \epsilon^{AB} \) | \( h^I \) | \( h_i \) | \( F_{ab}^I \) | \( \phi^x \) | \( \psi^A \) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| ADS  | \(-\eta_{ab}\)  | \(i\Gamma^a\)   | \(-i\Gamma_a\)  | \(-\epsilon^{AB}\) | \(\frac{4}{\sqrt{6}} X^I\) | \(\frac{\sqrt{6}}{2} X_i\) | \(2F_{ab}^I\) | \(\phi^x\)   | \(\sqrt{2} \psi^A\) |
| GST  | \(h^I_x\)       | \(h_i^x\)       | \(\hat{a}_{ij}\) | \(g_{xy}\)       | \(T^{xy}_{yz}\)   | \(\phi_{i\bar{x}}\)   | \(C_{ij\bar{k}}\) | \(K^x_{yz}\)  | \(\lambda^{xA}\)  |
| ADS  | \(-2f^I_x\)     | \(-\frac{2}{\sqrt{6}} g^x_i\) | \(\frac{3}{2} a_{ij}\) | \(\frac{3}{2} g_{xy}\) | \(-\sqrt{\frac{3}{2}} T_{yz}^x\) | \(\frac{1}{4} \phi_{i\bar{x}}\) | \(\sqrt{\frac{27}{8}} t_{ij\bar{k}}\) | \(R^x_{yz}\)   | \(-\frac{i}{\sqrt{2}} \lambda^{xA}\) |

| CD   | \(C_{\alpha\beta}\) | \(q^x\) | \(\zeta^\alpha\) | \(U_{uA}^\alpha\) | \(g_{uv}\) | \(k^u_i\) | \(N^{A\alpha}\) | \(M^{A\alpha}_\beta\) |
|------|---------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| ADS  | \(-C_{\alpha\beta}\) | \(q^x\) | \(-i\zeta^\alpha\) | \(-\sqrt{2}U_{uA}^\alpha\) | \(2g_{uv}\) | \(k^u_i\) | \(-\frac{1}{\sqrt{2}} N^{A\alpha}\) | \(-\frac{1}{\sqrt{2}} M^{A\alpha}_\beta\) |