GOHBERG LEMMA, COMPACTNESS, AND ESSENTIAL SPECTRUM OF OPERATORS ON COMPACT LIE GROUPS

APARAJITA DASGUPTA AND MICHAEL RUZHANSKY

Abstract. In this paper we prove a version of the Gohberg lemma on compact Lie groups giving an estimate from below for the distance from a given operator to the set of compact operators on compact Lie groups. As a consequence, we prove several results on bounds for the essential spectrum and a criterion for an operator to be compact. The conditions are given in terms of the matrix-valued symbols of operators.

1. Introduction

In this paper we establish a version of the Gohberg lemma in the setting of compact Lie groups and apply it to study the compactness of pseudo-differential operators and give bounds for their essential spectrum. The original Gohberg lemma has been obtained by Gohberg [Goh60] in the investigation of integral operators, and its version on the unit circle $T^1$ has been recently obtained by [MW10], with application to the spectral properties of operators, see [Mol11, Pir11]. In this paper we establish the Gohberg lemma on general compact Lie groups using the matrix quantization of operators developed in [RT10, RT12]. In particular, we give estimates for the distance from a given operator to the set of compact operators as well as for the essential spectrum of the operator in terms of some quantities associated to the matrix symbols. The results contain the corresponding results obtained in [Mol11, Pir11] on the unit circle. The matrix-valued symbols have been quite useful in other studies of compactness of operators in cases when conditions on the kernel are less effective, for example by providing criteria for operators to belong to Schatten classes, see [DR13b], and criteria for nuclearity in $L^p$-spaces, see [DR13a].

In Section 2 we briefly recall the necessary notions of the Fourier analysis on compact Lie groups and of the matrix quantization of operators. In Section 3 we state our results. In Section 4 we prove the Gohberg lemma given in Theorem 3.1, and in Section 5 we prove an application of the Gohberg lemma given in Theorem 3.2.

2. Fourier analysis and matrix symbols on compact Lie groups

Let $G$ be a compact Lie group with the unit element $e$, and let $\hat{G}$ be its unitary dual, consisting of the equivalence classes $[\xi]$ of the continuous irreducible unitary
representations \( \xi : G \to \mathbb{C}^{d_\xi \times d_\xi} \) of dimension \( d_\xi \). For a function \( f \in C^\infty(G) \) we can define its Fourier coefficient at \( \xi \) by
\[
\hat{f}(\xi) := \int_G f(x) \xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi},
\]
where the integral is (always) taken with respect to the Haar measure on \( G \). The Fourier series becomes
\[
f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} \left( \xi(x) \hat{f}(\xi) \right),
\]
with the Plancherel’s identity taking the form
\[
\|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \hat{G}} d_\xi \|\hat{f}(\xi)\|_{2\mathbb{H}}^2 \right)^{1/2} =: \|\hat{f}\|_{L^2(\hat{G})},
\]
which we take as the definition of the norm on the Hilbert space \( L^2(\hat{G}) \), and where \( \|\hat{f}(\xi)\|_{2\mathbb{H}} = \text{Tr}(\hat{f}(\xi)\hat{f}(\xi)^*) \) is the Hilbert–Schmidt norm of the matrix \( \hat{f}(\xi) \).

Given an operator \( T : C^\infty(G) \to C^\infty(G) \) (or even \( T : C^\infty(G) \to D'(G) \)), we define its matrix symbol by \( \sigma_T(x, \xi) := \xi(x)^* (T\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi} \), where \( T\xi \) means that we apply \( T \) to the matrix components of \( \xi(x) \). In this case we can prove that
\[
Tf(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} \left( \xi(x)\sigma_T(x, \xi) \hat{f}(\xi) \right).
\]
The correspondence between operators and symbols is one-to-one, and we will write \( T_\sigma \) for the operator given by (2.2) corresponding to the symbol \( \sigma(x, \xi) \). The quantization (2.2) has been extensively studied in [RT10, RT12], to which we refer for its properties and for the corresponding symbolic calculus.

We note that the matrix components of \( \xi(x) \) are the eigenfunctions of the Laplacian (Casimir element) \( \mathcal{L} \) on \( G \) corresponding to one eigenvalue which we denote by \( \lambda_\xi^2 \), i.e. we have \( \mathcal{L}\xi(x)_{ij} = -\lambda_\xi^2 \xi(x)_{ij} \) for all \( 1 \leq i, j \leq d_\xi \). We denote \( \langle \xi \rangle := (1 + \lambda_\xi^2)^{1/2} \).

We now briefly describe the class \( \Psi^0(G) \) of Hörmander’s pseudo-differential operators on \( G \) in terms of the matrix symbols. Here, \( \Psi^0(G) \) stands for the usual class of operators that have symbols in Hörmander’s class \( S^0_{1,0}(\mathbb{R}^n) \) in every local coordinate system.

It was proved in [RTW10] that \( T \in \Psi^0(G) \) is equivalent to the condition that its matrix-valued symbol \( \sigma \) satisfies
\[
\|\partial_\xi^\beta \Delta_\xi^\alpha \sigma(x, \xi)\|_{op} \leq C_{\alpha, \beta} \langle \xi \rangle^{-|\alpha|}
\]
for all \( x \in G \) and \([\xi] \in \hat{G} \), and for all \( \alpha, \beta \), where \( \| \cdot \|_{op} \) stands for the operator norm of the matrix multiplication. The difference operators \( \Delta_\xi^\gamma \) in (2.3) are defined as follows. Let \( q_1, \ldots, q_m \in C^\infty(G) \) be such that \( q_j(e) = 0, \nabla q_j(e) \neq 0 \), for all \( 1 \leq j \leq m \), the unit element \( e \) is the only common zero of the family \( \{q_j\}_{j=1}^m \), and such that \( \text{rank}\{\nabla q_1(e), \ldots, \nabla q_m(e)\} = \dim G \). We call such a collection strongly admissible. Then we set \( \Delta_\xi f(\xi) := \hat{q}_j f(\xi) \) and \( \Delta_\xi := \Delta_{q_1}^{\alpha_1} \cdots \Delta_{q_m}^{\alpha_m} \). We refer to [RT10] and especially to [RTW10] for the analysis of such difference operators.
It was shown in [RTW10] that the operator \( T \in \Psi^0(G) \) is elliptic if and only if its matrix symbol \( \sigma(x, \xi) \) is invertible for all but finitely many \( [\xi] \in \widehat{G} \), and for all such \( \xi \) we have
\[
\| \sigma(x, \xi)^{-1} \|_{op} \leq C
\]
for all \( x \in G \).

3. Gohberg lemma and applications

We define \( \| \sigma(x, \xi) \sigma(x, \xi)^* \|_{\min} \) to be the smallest eigenvalue of the positive matrix \( \sigma(x, \xi) \sigma(x, \xi)^* \), that is, if \( \lambda_1(x, \xi), \lambda_2(x, \xi), \ldots, \lambda_d(x, \xi) \geq 0 \) are the eigenvalues of \( \sigma(x, \xi) \sigma(x, \xi)^* \) then we set
\[
\| \sigma(x, \xi) \sigma(x, \xi)^* \|_{\min} := \min_{1 \leq i \leq d} \lambda_i(x, \xi).
\]

We formulate a version of the Gohberg Lemma first for operators in the Hörmander class \( \Psi^0 \) to relate it with the well-known theory and to be used in the application in Theorem 3.2, but later, in Remark 4.2, we note that the result remains valid for a much more general class of operators.

**Theorem 3.1** (Gohberg Lemma). Let \( \sigma(x, \xi) \) be the matrix symbol of \( T_\sigma \in \Psi^0(G) \). Then for all compact operators \( K \) on \( L^2(G) \), we have
\[
\| T_\sigma - K \|_{L(L^2(G))} \geq d_{\min},
\]
where
\[
d_{\min} := \limsup_{\langle \xi \rangle \to \infty} \left\{ \sup_{x \in G} \frac{\| \sigma(x, \xi) \sigma(x, \xi)^* \|_{\min}}{\| \sigma(x, \xi) \|_{op}} \right\}.
\]

We note that \( d_{\min} \) is well-defined. Indeed, we have
\[
\| \sigma(x, \xi) \sigma(x, \xi)^* \|_{\min} \leq \| \sigma(x, \xi) \sigma(x, \xi)^* \|_{op} \leq \| \sigma(x, \xi) \|_{op}^2,
\]
implying that
\[
d_{\min} \leq \limsup_{\langle \xi \rangle \to \infty} \sup_{x \in G} \| \sigma(x, \xi) \|_{op} < \infty
\]
in view of (2.3) with \( \alpha = \beta = 0 \).

We note again that the condition \( T_\sigma \in \Psi^0(G) \) in Theorem 3.1 can be substantially relaxed, see Remark 4.2.

To formulate an application of the Gohberg lemma, let us first introduce some notation. Let \( A : X \to X \) be a closed linear operator with dense domain \( D(A) \) in the complex Banach space \( X \). Then the spectrum \( \Sigma(A) \) of \( A \) is denoted by \( \Sigma(A) := \mathbb{C} \setminus \Phi(A) \), where \( \Phi(A) \) is the resolvent set of \( A \) given by
\[
\Phi(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is bijective} \}.
\]
The essential spectrum \( \Sigma_{\text{ess}}(A) \) of \( A \) is defined by \( \Sigma_{\text{ess}}(A) := \mathbb{C} \setminus \Phi_{\text{ess}}(A) \), where
\[
\Phi_{\text{ess}}(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm and } i(A - \lambda I) = 0 \}.
\]
Theorem 3.2. Let \( \sigma \) be the matrix symbol of a pseudo-differential operator \( T_\sigma \in \Psi^0(\mathbb{G}) \). Let
\[
d_{\text{max}} := \limsup_{\langle \xi \rangle \to \infty} \left\{ \sup_{x \in G} \| \sigma(x, \xi) \|_{\text{op}} \right\}.
\]
Then for \( T_\sigma \) on \( L^2(\mathbb{G}) \) we have
\[
\Sigma_{\text{ess}}(T_\sigma) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq d_{\text{max}} \}.
\]
Moreover, if \( d_{\text{max}} = 0 \), then \( T_\sigma \) is a compact operator on \( L^2(\mathbb{G}) \).

We observe that it follows from Theorem 3.1 that if \( d_{\text{min}} \neq 0 \), then \( T_\sigma \) is not compact, since otherwise we could take \( K = T_\sigma \). In other words, if \( T_\sigma \) is compact, then \( d_{\text{min}} = 0 \). From this point of view, the converse to this is given by the last statement of Theorem 3.2.

We note that we always have \( d_{\text{min}} \leq d_{\text{max}} \) in view of (3.2). If \( \mathbb{G} = \mathbb{T}^n \) is the torus, we have \( d_{\text{min}} = d_{\text{max}} \).

4. Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1.

First we observe that by (3.2), \( d_{\text{min}} \) is well-defined, and hence for every \( \xi \in \hat{\mathbb{G}} \) there exists \( x_\xi \in G \) such that
\[
\frac{\| \sigma(x_\xi, \xi_n)^* \sigma(x_\xi, \xi_n) \|_{\text{min}}}{\| \sigma(x_\xi, \xi_n) \|_{\text{op}}} = \sup_{x \in G} \frac{\| \sigma(x, \xi_n)^* \sigma(x, \xi_n) \|_{\text{min}}}{\| \sigma(x, \xi_n) \|_{\text{op}}}.
\]
From the definition of \( d_{\text{min}} \) there exists a sequence \( (x_\xi, \xi_n) \) such that \( \langle \xi_n \rangle \to \infty \) and \( \frac{\| \sigma(x_\xi, \xi_n)^* \sigma(x_\xi, \xi_n) \|_{\text{min}}}{\| \sigma(x_\xi, \xi_n) \|_{\text{op}}} \to d_{\text{min}} \). Let \( u \in L^2(\mathbb{G}) \) be sufficiently smooth. We define \( u_{\xi_n} \in C_{d_\xi \times d_\xi} \) by
\[
u_{\xi_n}(x) := d_\xi^{-1/2} \xi_n(x) u(x \cdot x_\xi^{-1}).
\]
For a matrix-valued function \( w = w(x) \in C_{d_\xi \times d_\xi} \), we define its \( L^2 \)-matrix norm by
\[
\| w \|_{L^2(\mathbb{G})} := \left( \int_{\mathbb{G}} \| w(x) \|_{\text{HS}}^2 \, dx \right)^{1/2}.
\]
Then we have
\[
\| u_{\xi_n} \|_{L^2(\mathbb{G})}^2 = \int_{\mathbb{G}} \| u_{\xi_n}(x) \|_{\text{HS}}^2 \, dx
\]
\[
= \int_{\mathbb{G}} d_\xi^{-1} \| \xi_n(x) u(x \cdot x_\xi^{-1}) \|_{\text{HS}}^2 \, dx
\]
\[
= d_\xi^{-1} \| \xi_n \|_{\text{HS}}^2 \| u \|_{L^2(\mathbb{G})}^2
\]
\[
= \| u \|_{L^2(\mathbb{G})}^2.
\]
Therefore, we have the equality
\[
\| u_{\xi_n} \|_{L^2(\mathbb{G})} = \| u \|_{L^2(\mathbb{G})} = \| \hat{u} \|_{L^2(\hat{\mathbb{G}})}.
\]
Let now $\phi \in C^\infty(G)$. Then, with $y = x \cdot x_{\xi_n}^{-1}$, we have

$$
\int_G u_{\xi_n}(x)\phi(x)dx = d_{\xi_n}^{-1/2} \int_G \xi_n(y)u(y)\phi(y \cdot x_{\xi_n})\xi_n(x_{\xi_n})dy
$$

so that

$$
\int_G u_{\xi_n}(x)\phi(x)dx = d_{\xi_n}^{-1/2}u\phi(x_{\xi_n})(\xi_n^*\xi_n(x_{\xi_n}),
$$

where $\xi_n^*(x) = \xi_n(x)^*$. Since

$$
\|u\phi(x_{\xi_n})\|^2_{L^2} \leq C\|u\|^2_{L^2}
$$

with a constant $C$ independent of $x_{\xi_n}$, we have $u\phi(x_{\xi_n}) \in \ell^2(G)$ uniformly in $x_{\xi_n}$. Hence, from (4.2) and (2.1), it follows that $d_{\xi_n}\|u\phi(x_{\xi_n})(\xi_n^*)\|^2_{\text{HS}} \to 0$ as $\langle \xi_n^* \rangle \to \infty$.

This implies

$$
\int_G u_{\xi_n}(x)\phi(x)dx\|_{\text{HS}} = \|d_{\xi_n}^{-1/2}u\phi(x_{\xi_n})(\xi_n^*)\xi_n(x_{\xi_n})\|_{\text{HS}} \leq \|u\phi(x_{\xi_n})(\xi_n^*)\|_{\text{HS}},
$$

so that

$$
u_{\xi_n} \to 0 \text{ as } \langle \xi_n \rangle \to \infty
$$

weakly. Hence for a compact operator $K$ we have

$$
\|Ku_{\xi_n}\|_{L^2(G)} \to 0
$$

as $\langle \xi_n \rangle \to \infty$. Then for any $\epsilon > 0$ and sufficiently large $n$ we have by compactness

$$
\|Ku_{\xi_n}\|_{L^2(G)} \leq \epsilon\|u_{\xi_n}\|_{L^2(G)} = \epsilon\|u\|_{L^2(G)},
$$

where $u$ is fixed and $\|Ku_{\xi_n}\|_{L^2(G)} = (\int_G \|Ku_{\xi_n}(x)\|^2_{\text{HS}}dx)^{1/2}$. We now define

$$
T_{\sigma} u_{\xi_n} := (T_{\sigma}(u_{\xi_n})_{ij})_{1 \leq i,j \leq d_{\xi_n}} \in \mathbb{C}^{d_{\xi_n} \times d_{\xi_n}}
$$

by $T_{\sigma}$ acting on the components of the matrix-valued function $u_{\xi_n}$.

**Lemma 4.1.** We have $\|u_{\xi_n}\sigma(\cdot, \xi_n) - T_{\sigma} u_{\xi_n}\|_{L^2(G)} \to 0$ as $\langle \xi_n \rangle \to \infty$.

We postpone the proof of Lemma 4.1 and continue with the proof of Theorem 3.1

Let us fix $u \in C^\infty(G)$ such that $u \neq 0$. Then for any $\epsilon > 0$ there exists $N(u)$ such that for any $n \geq N(u)$ we have

$$
\|u_{\xi_n}\sigma(\cdot, \xi_n)\|_{L^2(G)} - \|T_{\sigma} u_{\xi_n}\|_{L^2(G)} \leq \epsilon\|u\|_{L^2(G)}
$$

for sufficiently large $\langle \xi_n \rangle$. Now since $\sigma$ satisfies (2.3) with $\alpha = 0$, its $x$-derivatives are uniformly bounded, and hence for $\epsilon > 0$ there exists an open neighbourhood $V$ of the unit $e$ of the group such that for all $x \cdot x_{\xi_n}^{-1} \in V \subseteq G$ we have

$$
\|\sigma(x, \xi_n) - \sigma(x_{\xi_n}, \xi_n)\|_{\text{op}} < \epsilon.
$$
Let now \( u \in C^\infty(G) \) be such that \( u(x) = 0 \) for all \( x \notin V \). Then \( u_{\xi_n}(x) = 0 \) for all \( x \notin x_{\xi_n}V \), i.e. for \( x \cdot x_{\xi_n}^{-1} \notin V \). Then

\[
\begin{aligned}
\|u_{\xi_n}\sigma(x_{\xi_n}, \xi_n)\sigma(x_{\xi_n}, \xi_n)^*\|_{L^2(G)} - \|u_{\xi_n}\sigma(\cdot, \xi_n)\|_{L^2(G)} &
\leq \|u_{\xi_n}\sigma(x_{\xi_n}, \xi_n)\|_{L^2(G)} - \|u_{\xi_n}\sigma(\cdot, \xi_n)\|_{L^2(G)} \\
&\leq \|u_{\xi_n}\sigma(x_{\xi_n}, \xi_n) - u_{\xi_n}\sigma(\cdot, \xi_n)\|_{L^2(G)} \\
&\leq \left( \int_{x_{\xi_n}V} \|\sigma(x, \xi_n) - \sigma(x_{\xi_n}, \xi_n)\|_{op}^2 \|u_{\xi_n}(x)\|_{HS}^2 dx \right)^{1/2} \\
\leq & \epsilon \left( \int_{x_{\xi_n}V} \|u_{\xi_n}(x)\|_{HS}^2 \right)^{1/2} = \epsilon \|u_{\xi_n}\|_{L^2(G)} = \epsilon \|u\|_{L^2(G)},
\end{aligned}
\]

the last inequality following from (4.2). Therefore,

\[
\begin{aligned}
\|u\|_{L^2(G)} \|T_\sigma - K\|_{L^2(G)} &= \|u_{\xi_n}\|_{L^2(G)} \|T_\sigma - K\|_{L^2(G)} \\
&\geq \|T_\sigma - K\|u_{\xi_n}\|_{L^2(G)} \\
&\geq \|T_\sigma u_{\xi_n}\|_{L^2(G)} - \|K u_{\xi_n}\|_{L^2(G)} \\
&\geq \|u_{\xi_n}\sigma(\cdot, \xi_n)\|_{L^2(G)} - 2\epsilon \|u\|_{L^2(G)} \\
&\geq \frac{\|u_{\xi_n}\sigma(x_{\xi_n}, \xi_n)\sigma(x_{\xi_n}, \xi_n)^*\|_{L^2(G)}}{\|\sigma(x_{\xi_n}, \xi_n)\|_{op}} - 3\epsilon \|u\|_{L^2(G)},
\end{aligned}
\]

using (4.5) and (4.6) for the last inequalities. Since \( \sigma(x_{\xi_n}, \xi_n)\sigma(x_{\xi_n}, \xi_n)^* \) is normal there exist a unitary matrix \( U \) such that

\[
\sigma(x_{\xi_n}, \xi_n)\sigma(x_{\xi_n}, \xi_n)^* = U \Lambda U^*,
\]

where

\[
\Lambda = \begin{pmatrix}
\lambda_{11} & 0 & \ldots & 0 \\
0 & \lambda_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_{d_{\xi_n}d_{\xi_n}}
\end{pmatrix},
\]

with \( \lambda_{ii}(x_{\xi_n}, \xi_n) \) being the eigenvalues of \( \sigma(x_{\xi_n}, \xi_n)\sigma(x_{\xi_n}, \xi_n)^* \). Now let

\[
\lambda_{mm}(x_{\xi_n}, \xi_n) = \min_{1 \leq i \leq d_\xi} \lambda_{ii}(x_{\xi_n}, \xi_n).
\]

We want to show that

\[
\|u_{\xi_n}(x)U \Lambda U^*\|_{HS}^2 \geq \lambda_{mm}^2 \|u_{\xi_n}(x)\|_{HS}^2.
\]

Let \( M = U \Lambda U^* \). Then \( M \) is symmetric and \( \lambda_{mm}(x_{\xi_n}, \xi_n) \) is the minimum eigenvalue of the matrix \( M > 0 \). Let \( w_{\xi_n} := u_{\xi_n}(x)M \). To prove (4.8) it is enough to show that

\[
\|w_{\xi_n}\|_{HS} \geq \lambda_{mm}(x_{\xi_n}, \xi_n)\|w_{\xi_n}M^{-1}\|_{HS},
\]

where \( M^{-1} = U \Lambda^{-1} U^* \). This is true since \( \lambda_{mm}^{-1}(x_{\xi_n}, \xi_n) \) is the maximum eigenvalue of \( M^{-1} \), that is, \( \|M^{-1}\|_{op} = \lambda_{mm}^{-1}(x_{\xi_n}, \xi_n) \). This proves (4.8).
Using (4.7) and (4.8), we can estimate
\[
\| u \|_{L^2(G)} \| T_\sigma - K \|_{\mathcal{L}(L^2(G))} \geq \left( \int_G \| u_{x_n, \xi_n}(x) \|_{L^2(G)}^2 \right)^{1/2} - 3\epsilon \| u \|_{L^2(G)}
\]
\[
\geq \frac{\left( \lambda_{\text{min}}^2 \int_G \| u_{x_n, \xi_n}(x) \|_{L^2(G)}^2 \right)^{1/2}}{\| \sigma(x_{x_n, \xi_n}) \|_{\text{op}}} - 3\epsilon \| u \|_{L^2(G)}
\]
\[
= \frac{\lambda_{\text{min}} \| u_{x_n} \|_{L^2(G)}}{\| \sigma(x_{x_n, \xi_n}) \|_{\text{op}}} - 3\epsilon \| u \|_{L^2(G)}
\]
\[
= \left( \frac{\lambda_{\text{min}}(x_{x_n, \xi_n})}{\| \sigma(x_{x_n, \xi_n}) \|_{\text{op}}} - 3\epsilon \right) \| u \|_{L^2(G)}.
\]
(4.9)

Now, as \( \langle \xi_n \rangle \to \infty \), we have
\[
\| u \|_{L^2(G)} \| T_\sigma - K \|_{\ast} \geq (d_{\text{min}} - 3\epsilon) \| u \|_{L^2(G)},
\]
that is, for any \( \epsilon > 0 \),
\[
\| T_\sigma - K \|_{\ast} \geq d_{\text{min}} - 3\epsilon.
\]

Now, using the fact that \( \epsilon \) is an arbitrary positive number, we have
\[
\| T_\sigma - K \|_{\ast} \geq d_{\text{min}}.
\]

This completes the proof of Theorem 3.1.

**Proof of Lemma 4.1.** Let \( x, z \in G \). Let us define
\[
R(x, z) := \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} (\sigma(x, \xi)\xi(z)).
\]

Now we can write
\[
T_\sigma u(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} (\xi(x)\sigma(x, \xi)\widehat{u}(\xi))
\]
\[
= \int_G \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} (\xi(x)\sigma(x, \xi)\xi^*(y)) u(y) \, dy
\]
\[
= \int_G \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} (\sigma(x, \xi)(y^{-1}x)) u(y) \, dy
\]
\[
= \int_G R(x, y^{-1}x) u(y) \, dy
\]
\[
= \int_G R(x, z) u(xz^{-1}) \, dz,
\]
(4.10)
where \( z = y^{-1}x \). Then from the definition of \( u_{\xi_n}(x) = d_{\xi_n}^{-1/2}z_n(x)u(x_{\xi_n}^{-1}) \) and \((4.10)\) we obtain

\[
T_{\sigma}u_{\xi_n}(x) = d_{\xi_n}^{-1/2} \int_G R(x, z) \xi_n(xz^{-1})u(xz^{-1}x_{\xi_n}^{-1})dz
\]

\[
(4.11)
\]

Let us denote

\[
v_{\xi_n}^{z^{-1}} := v_{\xi_n}(xz^{-1}) := d_{\xi_n}^{-1/2}z_n(x)u(xz^{-1}x_{\xi_n}^{-1}),
\]

so that we have

\[
(4.13)
\]

For a given collection of \( m \) strongly admissible difference operators \( \Delta_1, \ldots, \Delta_m \) with the corresponding functions \( q_1, \ldots, q_m \in C^\infty(G) \) with \( \Delta_j \tilde{f}(\xi) = q_j f(\xi) \) we have the Taylor expansion formula, see \([RT10, RTW10]\),

\[
u(x) = u(e) + \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} q^\alpha(x^{-1}) \partial^{(\alpha)} u(e) + O(h(x)^N),
\]

where \( h(x) \to 0 \), \( h(x) \) is the geodesic distance from \( x \) and \( e \), and \( \partial^{(\alpha)} \) are some left-invariant differential operators on \( G \), and \( q^\alpha(x) = q_1(x)^{a_1} \cdots q_m(x)^{a_m} \). Assuming that \( u \) is sufficiently smooth, from the Taylor expansion formula we have

\[
v_{\xi_n}^{z^{-1}} = v_{\xi_n}^e(e) + \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} q^\alpha(z) \partial^{(\alpha)} v_{\xi_n}(e) + O(h(z)^N).
\]

Then by the left-invariance of \( \partial^{(\alpha)} \) we obtain

\[
(4.14)
\]

Using \((4.13)\) and \((4.14)\), we can now write

\[
T_{\sigma}u_{\xi_n}(x) = \int_G R(x, z)v_{\xi_n}(x)\xi_n^*(z)dz + \int_G R(x, z) \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} q^\alpha(z) \partial^{(\alpha)} v_{\xi_n}(x)\xi_n^*(z)dz
\]

\[
+ \int_G R(x, z)O(h(z)^N)\xi_n^*(z)dz.
\]

We denote

\[
I_1 := \int_G R(x, z)v_{\xi_n}(x)\xi_n^*(z)dz,
\]

\[
I_2 := \int_G R(x, z) \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} q^\alpha(z) \partial^{(\alpha)} v_{\xi_n}(x)\xi_n^*(z)dz,
\]

and

\[
I_3 := \int_G R(x, z)O(h(z)^N)\xi_n^*(z)dz.
\]
Then we have

\[
I_1 = \int_G R(x, z) v_{\xi_n}(x) \xi_n^*(z) dz \\
= v_{\xi_n}(x) \sigma(x, \xi_n) \\
= d_{\xi_n}^{-1/2} \xi_n(x) u(x^{-1} \xi_n) \sigma(x, \xi_n) = u_{\xi_n}(x) \sigma(x, \xi_n).
\]

(4.15)

Calculating \(I_2\), we have

\[
I_2 = \int_G R(x, z) \partial(\alpha) v_{\xi_n}(x) \xi_n^*(z) \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} q^\alpha(z) dz \\
= \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} \partial(\alpha) v_{\xi_n}(x) \int_G R(x, z) \xi_n^*(z) q^\alpha(z) dz \\
= \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} \partial(\alpha) v_{\xi_n}(x) (\mathcal{F} q^\alpha \mathcal{F}^{-1} \sigma) (\xi_n) \\
= \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} \partial(\alpha) v_{\xi_n}(x) \Delta q^\alpha \sigma(x, \xi_n).
\]

(4.16)

And calculating \(I_3\), we have

\[
I_3 = \int_G R(x, z) \mathcal{O}(h(z)^N) \xi_n^*(z) dz \\
= \int_G R(x, z) q^N(z) \xi_n^*(z) dz \\
= \Delta q^N \sigma(x, \xi_n),
\]

(4.17)

where we can denote \(q^N := \mathcal{O}(h(x)^N)\) so that \(q^N\) vanishes at \(e\) of order \(N\), but keep in mind that it is matrix-valued. Then we have

\[
T_\sigma u_{\xi_n}(x) - u_{\xi_n}(x) \sigma(x, \xi_n) = \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} \partial(\alpha) v_{\xi_n}(x) \Delta q^\alpha \sigma(x, \xi_n) \\
+ \Delta q^N \sigma(x, \xi_n).
\]

(4.18)

We denote

\[
T_1^N := \sum_{|\alpha|=1}^{N-1} \frac{1}{\alpha!} \partial(\alpha) v_{\xi_n}(x) \Delta q^\alpha \sigma(x, \xi_n)
\]

and

\[
T_2^N := \Delta q^N \sigma(x, \xi_n).
\]

We can estimate

\[
\|T_1^N(x)\|_{HS} \leq \|\Delta q^\alpha \sigma(x, \xi_n)\|_{op} \sum_{0<|\alpha|\leq N} \frac{1}{\alpha!} \|\partial(\alpha) v_{\xi_n}(x)\|_{HS}
\]

where \(z \in G\). So, using (4.12) and (2.3), for some operator \(\tilde{\partial}^{(\alpha)}\) we have
\[
\|T_N^1(x)\|_{HS} \leq C \left( \sum_{0<|\alpha| \leq N} |\hat{\partial}_z^{(\alpha)} u(x \cdot x_{\xi_n})| \right) \frac{d\xi_n^{-1/2}}{\xi_n} \|\xi_n(x)\|_{HS} \langle \xi_n \rangle^{-|\alpha|} \\
\leq C \sum_{0<|\alpha| \leq N} \|u\|_{H^{|\alpha|}} \langle \xi_n \rangle^{-|\alpha|} \\
(4.19) \\
\leq C' \langle \xi_n \rangle^{-1},
\]
where \(1 \leq |\alpha| \leq N\), \(N\) is fixed and \(\| \cdot \|_{H^{|\alpha|}}\) is the Sobolev norm. Similarly,
\[
\|T_N^2\|_{HS} \leq C' \langle \xi_n \rangle^{-N}.
\]
Therefore, as \(\langle \xi_n \rangle \to \infty\) we have \(\|T_N^1(x)\|_{HS} \to 0\) and \(\|T_N^2(x)\|_{HS} \to 0\) for all \(x \in G\) which gives
\[
\|T_N^1\|_{L^2(G)}^2 = \int_G \|T_N^1(x)\|^2_{HS} dx \to 0
\]
as \(\langle \xi_n \rangle \to \infty\) and, similarly, \(\|T_N^2\|_{L^2(G)} \to 0\). This implies
\[
\|T_n u_{\xi_n} - u_{\xi_n} \sigma(\cdot, \xi_n)\|_{L^2(G)} \to 0
\]
as \(\langle \xi_n \rangle \to \infty\), and we note that it is sufficient to take \(N = 1\) in the above argument.

\(\square\)

**Remark 4.2.** Looking at what we have used in the proof, we note that we have (with the same proof and \(N = 1\)) the following extension of the Gohberg lemma without making an assumption that the operator belongs to \(\Psi(G)\), namely:

Let \(T_\sigma : L^2(G) \to L^2(G)\) be a bounded operator with the matrix symbol \(\sigma(x, \xi)\) satisfying, for some \(\rho > 0\),
\[
\|\sigma(x, \xi)\|_{op} \leq C, \quad \|\partial_x \sigma(x, \xi)\|_{op} \leq C, \quad \|\Delta \sigma(x, \xi)\|_{op} \leq C \langle \xi \rangle^{-\rho}
\]
for all \(q \in C^\infty(G)\) with \(q(e) = 0\), and all \(x \in G\) and \(|\xi| \in \tilde{G}\). Then the conclusion of the Gohberg lemma in Theorem 5.1 remains true, namely, the estimate (3.11) holds for all compact operators \(K\) on \(L^2(G)\).

5. **Proof of Theorem 3.2**

We first recall the following theorem which is known as the Atkinson theorem which gives another equivalent definition of Fredholm operators.

**Theorem 5.1.** Let \(A\) be a closed linear operator from a complex Banach space \(X\) into a complex Banach space \(Y\) with a dense domain \(D(A)\). Then \(A\) is Fredholm if and only if we can find a bounded linear operator \(B : Y \to X\), a compact operator \(K_1 : X \to X\) and a compact operator \(K_2 : Y \to Y\) such that \(BA = I + K_1\) on \(D(A)\) and \(AB = I + K_2\) on \(Y\).

We recall that the Wolf spectrum \(\Sigma_w(A)\) of \(A\) is defined by \(\Sigma_w(A) := \mathbb{C} \setminus \Phi_w(A)\), where
\[
\Phi_w(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm} \}.
\]
Clearly, we have \(\Sigma_w(A) \subseteq \Sigma_{ess}(A) \subseteq \Sigma(A)\).
Proof of Theorem 3.2 Let \( \lambda \in \mathbb{C} \) be such that \( |\lambda| > d_{\text{max}} \). Then there exists \( \epsilon > 0 \) such that
\[
|\lambda| > d_{\text{max}} + \epsilon.
\]
Now, by the definition of \( d_{\text{max}} \) in Theorem 3.2 we have for some \( R > 0 \) and for all \( \langle \xi \rangle \geq R \) that
\[
\sup_{\langle \xi \rangle \geq R} \sup_{x \in G} \|\sigma(x, \xi)\|_{\text{op}} \leq (d_{\text{max}} + \epsilon/2).
\]
Then for \( \langle \xi \rangle \geq R \), we can estimate
\[
(5.1) \quad \| (\sigma(x, \xi) - \lambda I)^{-1} \|_{\text{op}} \leq \sum_{k=1}^{\infty} \frac{\lambda^{-k} - 1}{|\lambda|^{k+1}} \|\sigma(x, \xi)^{k}\|_{\text{op}}
\]
\[
\leq \sum_{k=1}^{\infty} \frac{(d_{\text{max}} + \epsilon/2)^{k}}{|\lambda|^{k+1}} \leq \sum_{k=1}^{\infty} \frac{(d_{\text{max}} + \epsilon/2)^{k}}{(d_{\text{max}} + \epsilon)^{k+1}} \leq C \sum_{k=1}^{\infty} \frac{(d_{\text{max}} + \epsilon/2)^{k}}{(d_{\text{max}} + \epsilon)^{k}} < \infty.
\]
Hence from (2.4) it follows that the operator \( T_{\sigma} - \lambda I \) is elliptic and hence it is a Fredholm operator from \( L^{2}(G) \rightarrow L^{2}(G) \), see e.g. [Hör07, Section 19.5].

Thus
\[
\{ \lambda \in \mathbb{C} : |\lambda| > d_{\text{max}} \} \subseteq \Phi_{w}(T_{\sigma}),
\]
which implies that
\[
\Sigma_{w}(T_{\sigma}) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq d_{\text{max}} \}.
\]
Since \( \{ \lambda \in \mathbb{C} : |\lambda| > d_{\text{max}} \} \) is a connected component of \( \Phi_{w}(T_{\sigma}) \) it follows that
\[
i(T_{\sigma} - \lambda I) \text{ is constant for all } \lambda \in \{ \lambda \in \mathbb{C} : |\lambda| > d_{\text{max}} \}.
\]
Now again
\[
\Phi(T_{\sigma}) \cap \{ \lambda \in \mathbb{C} : |\lambda| > d_{\text{max}} \} \neq \emptyset.
\]
Therefore \( i(T_{\sigma} - \lambda I) = 0 \) for all \( \{ \lambda \in \mathbb{C} : |\lambda| > d_{\text{max}} \} \). This implies
\[
\Sigma_{\text{ess}}(T_{\sigma}) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq d_{\text{max}} \},
\]
completing the proof of 3.3.

To prove the last part of Theorem 3.2 we start by recalling the definition of the Calkin algebra. Let \( \mathcal{L}(L^{2}(G)) \) and \( \mathcal{K}(L^{2}(G)) \) be respectively the \( C^{*} \) algebra of bounded linear operators on \( L^{2}(G) \) and the ideal of compact operators on \( L^{2}(G) \). The Calkin algebra, \( \mathcal{L}(L^{2}(G))/\mathcal{K}(L^{2}(G)) \), is a \( \ast \)-algebra. The product and the involution are defined here as
\[
[A][B] := [AB]
\]
and
\[
[A]^{\ast} := [A^{\ast}],
\]
for all \( A \) and \( B \) in \( \mathcal{L}(L^{2}(G)) \). Let \( [A] \) and \( [B] \) be members of \( \mathcal{L}(L^{2}(G))/\mathcal{K}(L^{2}(G)) \). Then
\[
[A] = [B] \iff A - B \in \mathcal{K}(L^{2}(G)).
\]
The norm \( \| \cdot \|_{\mathcal{C}} \) in \( \mathcal{L}(L^{2}(G))/\mathcal{K}(L^{2}(G)) \) is given by
\[
\|[A]\|_{\mathcal{C}} := \inf_{K \in \mathcal{K}(L^{2}(G))} \| A - K \|_{\mathcal{L}(L^{2}(G))}, \quad [A] \in \mathcal{L}(L^{2}(G))/\mathcal{K}(L^{2}(G)).
\]
By using the Calkin algebra the Gohberg lemma Theorem 3.1 can be reformulated as the inequality
\[
\|[T_{\sigma}]\|_{\mathcal{C}} \geq d_{\min}.
\]
We now prove the last part of Theorem 3.2. We assume that $d_{\text{max}} = 0$ and observe that $T_{\sigma}$ is compact if and only if $[T_{\sigma}] = 0$ in the Calkin algebra $\mathcal{L}(L^2(G))/\mathcal{K}(L^2(G))$. We also observe that the operator $T_{\sigma}$ is essentially normal on $L^2(G)$, i.e. $T_{\sigma}T_{\sigma}^* - T_{\sigma}^*T_{\sigma}$ is compact. Indeed, this is the operator of order $-1$ so the compactness follows from the compactness of the embedding $H^1 \hookrightarrow L^2$. Consequently, $[T_{\sigma}]$ is normal in $\mathcal{L}(L^2(G))/\mathcal{K}(L^2(G))$, and, therefore,

$$r(T_{\sigma}) = \|[T_{\sigma}]\|_C,$$

where $r(T_{\sigma})$ is the spectral radius of $[T_{\sigma}]$. On the other hand we know that $\Sigma_{\text{ess}}(T_{\sigma}) \subset \{0\}$ by the first part of Theorem 3.2 in (3.3). This implies that $T_{\sigma} - \lambda I$ is Fredholm for $\lambda \neq 0$. So using Atkinson’s Theorem 5.1 this implies that there exists a bounded operator $B$ such that

$$(T_{\sigma} - \lambda I)B = I + K,$$

where $K$ is a compact operator. That is, for $\lambda \neq 0$, $[(T_{\sigma} - \lambda I)]$ is invertible, which implies that for $\lambda \neq 0$, we have $\lambda \notin \Sigma([T_{\sigma}])$, the spectrum of $[T_{\sigma}]$. So $\Sigma([T_{\sigma}]) \subseteq \{0\}$. Consequently,

$$\|[T_{\sigma}]\|_C = r(T_{\sigma}) = 0.$$

Therefore $[T_{\sigma}] = 0$, and hence $T_{\sigma}$ is compact, completing the proof.

\[ \square \]

References

[DR13a] J. Delgado and M. Ruzhansky. Lp-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups. arXiv:1303.4792, 2013.

[DR13b] J. Delgado and M. Ruzhansky. Schatten classes and traces on compact Lie groups. arXiv:1303.3914, 2013.

[Goh60] I. C. Gohberg. On the theory of multidimensional singular integral equations. Soviet Math. Dokl., 1:960–963, 1960.

[Hör07] L. Hörmander. The analysis of linear partial differential operators. III. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[Mol11] S. Molahajloo. A characterization of compact pseudo-differential operators on $S^1$. In Pseudo-differential operators: analysis, applications and computations, volume 213 of Oper. Theory Adv. Appl., pages 25–29. Birkhäuser/Springer Basel AG, Basel, 2011.

[MW10] S. Molahajloo and M. W. Wong. Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $S^1$. J. Pseudo-Differ. Oper. Appl., 1(2):183–205, 2010.

[Pir11] M. Pirhavati. Spectral theory of pseudo-differential operators on $S^1$. In Pseudo-differential operators: analysis, applications and computations, volume 213 of Oper. Theory Adv. Appl., pages 15–23. Birkhäuser/Springer Basel AG, Basel, 2011.

[RT10] M. Ruzhansky and V. Turunen. Pseudo-differential operators and symmetries. Background analysis and advanced topics, volume 2 of Pseudo-Differential Operators. Theory and Applications. Birkhäuser Verlag, Basel, 2010.

[RT12] M. Ruzhansky and V. Turunen. Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces. Int Math Res Notices IMRN, doi: 10.1093/imrn/rns122, 2012.

[RTW10] M. Ruzhansky, V. Turunen, and J. Wirth. Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity. arXiv:1004.4396, 2010.

Aparajita Dasgupta:
Department of Mathematics
Imperial College London
180 Queen’s Gate, London SW7 2AZ
United Kingdom
E-mail address a.dasgupta@imperial.ac.uk

Michael Ruzhansky:
Department of Mathematics
Imperial College London
180 Queen’s Gate, London SW7 2AZ
United Kingdom
E-mail address m.ruzhansky@imperial.ac.uk