Quantum Hall Transition in an Array of Quantum Dots

K. Ziegler

Max-Planck-Institut für Physik Komplexer Systeme
Außenstelle Stuttgart, Postfach 800665, D-70506 Stuttgart, Germany

Abstract:
A two-dimensional array of quantum dots in a magnetic field is considered. The electrons in the quantum dots are described as unitary random matrix ensembles. The strength of the magnetic field is such that there is half a flux quantum per plaquette. This model exhibits the Integer Quantum Hall Effect. For \( N \) electronic states per quantum dot the limit \( N \to \infty \) can be solved by a saddle point integration of a supersymmetric field theory. The effect of level statistics on the density of states and the Hall conductivity is compared with the effect of temperature fluctuations.

PACS Nos.: 71.55Jv, 73.20Dx, 73.40Hm

We consider a two-dimensional array of quantum dots in a homogeneous magnetic field perpendicular to the array. A quantum dot in an array is a complex finite system of electrons, subject to strong Coulomb interaction and a confining potential. Even if the number of electrons is small there is a large number of electronic states in a given energy interval. Therefore, we are forced to use a statistical description of the quantum dot. A typical feature of such a complicated non-integrable system is level repulsion. The latter, also found in other complex many-particle systems like atomic nuclei [1], atoms [2] or metallic particles [3], can be conveniently described by random matrix ensembles [4]. Since the magnetic field breaks the time–reversal invariance in the dot, an appropriate model is the Gaussian unitary ensemble (GUE). Electrons can travel in the array of quantum dots due to tunneling between neighboring dots. On the square–array, which will be considered in this article, the tunneling rates are \( t \) and \( t' \) for nearest and next nearest neighbors, respectively (cf. Fig.1). The coupling between the individual quantum dots due to these tunneling processes is weak. This allows us to assume that the statistical occupation of the electronic states in each dot is uncorrelated between different dots. Thus the quantum dots
can be represented by independent random matrix ensembles. Moreover, we also assume for simplicity that the tunneling processes are independent, i.e., the tunneling electrons do not interact with each other.

For very weak tunneling rates the array should behave like an insulator because of the fluctuations of the energy levels. One would expect for increasing tunneling rates that a metallic regime can be reached where the array becomes conducting. However, due to the statistical fluctuations of the energy levels in the dots the effect of Anderson localization must play a crucial role in the array. Anderson localization prevents a two-dimensional system to become metallic, at least if no or only a weak magnetic field is present [5]. On the other hand, in the two-dimensional electron gas in a homogeneous magnetic field quantum Hall transitions (QHT) have been observed which are accompanied by delocalized electronic states [6]. A QHT occurs if a gap opens in a band of electronic states. This phenomenon is known, for instance, from electrons which are subject to a homogeneous magnetic field and a periodic potential [7]. Depending on the magnetic field the electrons form several subbands where each subband contributes $e^2/h$ to the Hall conductivity [8,9,10]. As an approximation of the periodic potential one can use a tight binding model where the lattice constant is given by the period of the potential. In this article we will study the effect of the statistics of energy levels and the effect of thermal fluctuations on the QHT.

There are two different approaches to the transport in quantum dots. One is based on the S-matrix, the other one on the Hamiltonian. The former is very useful for numerical simulations because it describes directly the reflection and transmission of the electrons through the quantum dots [11,12]. The latter, however, requires the application of linear response theory to get a conductivity via Kubo’s formula. In this article the Hamiltonian representation will be used. The effective Hamiltonian of an array of quantum dots reads as a quadratic form $\sum \hat{H}_{r,r'}^{\alpha,\alpha'} c_{r,\alpha} c_{r',\alpha'}^{\dagger}$ in the fermion creation and annihilation operators $c, c^{\dagger}$ with the matrix elements

$$\hat{H}_{r,r'}^{\alpha,\alpha'} = H_{r,\alpha}^{\alpha'} \delta_{r,r'} + H_{r,r'}' \delta_{\alpha,\alpha'} + V_r \delta_{r,r'} \delta_{\alpha,\alpha'},$$  \hspace{1cm} (1)

where $\alpha, \alpha' = 1, ..., N$ label the $N$ electronic states in the quantum dots and $r$ and $r'$ label positions of the quantum dots in the two-dimensional array. In general, tunneling between all $N$ states should be allowed with some probability, depending exponentially on the energy of the states $\alpha$ and $\alpha'$. To include this would require a detailed knowledge of these states. Therefore, we choose as a simplifying approximation the assumption that there is tunneling only between states with the same $\alpha$ at nearest or next nearest neighbor dots with fixed tunneling rates. The distance between neighboring dots is measured in units of $(\phi_0/2B)^{1/2}$. Typical distances are $a = 100...500nm$ [13]. The magnetic field for the creation of one flux quantum per plaquette is $B = \phi_0/a^2 \approx 0.016...0.4T$. This regime is accessible in natural crystals ($a \approx 0.5nm$) only with astronomical magnetic fields.
The electron can occupy statistically states inside the quantum dot which are represented by the matrix elements $H_{r,r'}^{\alpha,\alpha'}$. $H$ is the $N \times N$ Hermitean Hamiltonian ($H^\dagger = H$) of a quantum dot with $N^2$ statistically independent matrix elements. They are Gaussian distributed with zero mean and $\langle H_{r,r'}^{\alpha,\alpha'} H_{r,r''}^{\alpha'',\alpha'''} \rangle = (g/N) \delta^{\alpha\alpha''} \delta^{\alpha'\alpha'''} \delta_{r,r''}$. $g$ is the strength of the level fluctuations. This depends on the strength of the interaction between the electrons inside the dot. Therefore, $g$ increases with the number of electrons per dot and with increasing confinement.

The tunneling is represented by the Hamiltonian $H'$. This reads in Landau gauge (with $r = (x,y)$) for flux $\phi$ per plaquette

$$H'_{r,r'} = te^{2i\pi y\phi/\phi_0} \delta_{r',r+e_x} + t\delta_{r',r+e_y} \pm it'e^{2i\pi y\phi/\phi_0} \delta_{r',r+e_x \pm e_y} + h.c. \quad (2)$$

For the special case of half a flux quantum per plaquette ($\phi = \phi_0/2$) the phase factor in (2) is real and changes only sign between nearest neighbors in $y$-direction. Finally, the potential term $V_r$ represents an additional external (e.g. electric) field. Here we regard a staggered chemical potential $V_r = (-1)^{x+y}\mu$ which opens a gap $2\mu$ in the spectrum of the electrons, as will be explained below [14]. It is probably difficult to implement a staggered potential in a real sample of quantum dots. However, the parameter $\mu$ plays only the role of a gap which could be created also by other means.

We choose for the tunneling rate $t = 1$. Therefore, $\mu$, $t'$ are measured in units of $t$ and $g$ is measured in units of $t^2$. If we identify fermions with the four corners of the unit cell (Fig.1) the tunneling matrix $H'$ can be diagonalized by a Fourier transformation. This gives a $4 \times 4$ matrix in Fourier space. $H$, the Hamiltonian of a dot, is a diagonal matrix with respect to the four corners in the sublattice representation $H = (H_1^{\alpha,\alpha'}, H_2^{\alpha,\alpha'}, H_3^{\alpha,\alpha'}, H_4^{\alpha,\alpha'})$. A similar model with correlated randomness $H_1^{\alpha,\alpha'} = H_3^{\alpha,\alpha'} = -H_2^{\alpha,\alpha'} = -H_4^{\alpha,\alpha'}$ was considered in Ref. [15].

We begin the discussion of the model with the analysis of an array where the interaction of the electrons inside the quantum dots are neglected. It can be understood as a tight-binding model for non-interacting electrons in a metal with some electronic bands in a magnetic field [14,16]. The Fourier components of $H'$ can be expanded around the four nodes $k = (\pm \pi, \pm \pi)$ for $k = (\pm \pi, \pm \pi) + ap$ with small $p$ vectors. After a global orthogonal transformation the Hamiltonian reads

$$H''(p) = 2 \begin{pmatrix}
\begin{array}{cccc}
\mu - t' & ip_x - p_y & -2t'(p_x + p_y) & 0 \\
-ip_x - p_y & -\mu + t' & 0 & -2t'(p_x - p_y)
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
\mu - t' \\
-ip_x - p_y \\
-2t'(p_x + p_y) \\
0
\end{array}
\end{pmatrix}
\equiv \begin{pmatrix}
H''_{11} & H''_{12} \\
H''_{21} & H''_{22}
\end{pmatrix} \quad (3)$$
The last equation combines the $4 \times 4$–structure to a $2 \times 2$–structure with $2 \times 2$ block matrices $H''_{ij}$. Neglecting terms $O(p^2)$ the Green’s function $(\hat{H} + i\omega)^{-1}$ decays into a diagonal block structure

$$
\hat{G}(i\omega) = \begin{pmatrix} (H''_{11} 1_N + h_1 + i\omega)^{-1} & 0 \\ 0 & (H''_{22} 1_N + h_1 + i\omega)^{-1} \end{pmatrix}
$$

(4)

with the diagonal matrix $h_1 = (H_1 + H_3, H_2 + H_4)$. Thus the diagonal elements are statistically independent. $1_N$ is the $N \times N$–unit matrix. It is interesting to notice that the matrices $H''_{jj} = m_j\sigma_z + i\nabla_x\sigma_x \mp i\nabla_y\sigma_y$ represent two independent two-dimensional Dirac Hamiltonians with masses $m_1 = \mu - t'$ and $m_2 = \mu + t'$, respectively.

The current density in a Dirac model can be calculated from the response to an external vector potential $q_y$ [17]. The introduction of this vector potential is equivalent to a change of the boundary conditions, a concept extensively used in numerical investigations of Anderson localization [18]. The response to the vector potential leads to the Hall conductivity $\sigma_{xy}$ in terms of Green’s functions. We obtain for $q_y \sim 0$ the expression [14,15]

$$
\sigma_{xy} \approx i \sum' \int Tr[\sigma_x \hat{G}_{r'r'}(E - i\omega)\hat{G}_{r''r''}(E - i\omega)\sigma_y \hat{G}_{r''r'}(E - i\omega)] \frac{d\omega}{2\pi}.
$$

(5)

Here $\sum'$ is the sum normalized with the number of quantum dots in the array and the number of energy levels $N$. If there is only one electron per dot the energy spectrum has discrete levels which are well–separated. For instance, with a harmonic oscillator potential for the dot we have $E_n = \hbar\omega_p(n + 1/2)$. The separation of the energy levels in the single electron case allows us to neglect all levels with $n > 0$. Consequently, there is no statistics of energy levels and we can write $h_1 = 0$. For the Hall conductivity in units of $e^2/h$ we find

$$
\sigma_{xy} = (1/2)[\text{sign}(m_1)\Theta(|m_1| - |E|) + \text{sign}(m_2)\Theta(|m_2| - |E|)],
$$

(6)

where $\Theta$ is the Heaviside step function. This result reflects correctly the qualitative behavior of the Hall conductivity at the QHT: The Hall conductivity of the original lattice fermion problem is the sum of the Hall conductivities from the light Dirac mass ($m_1$) and the heavy Dirac mass ($m_2$), such that the total $\sigma_{xy}$ has a jump from 0 to 1 if the light mass changes the sign (i.e., exchange of particles and holes in the Dirac model). Thus the Dirac fermions, together with the Hall conductivity of Eq. (5), represent a simple picture for a Hall transition. Special cases are $\mu = 0$ which gives $\sigma_{xy} = 0$ and the (unrealistic) case $t' = 0$ with $\sigma_{xy} = (\text{sign}(\mu)/2\pi)\Theta(|m_1| - |E|)$. The sharp step–like QHT is only possible in an ideal systems of non-interacting lattice electrons at zero temperature. In order to compare with real systems we have to include the statistical fluctuations of the energy levels as well as thermal fluctuations. The latter are taken into account by replacing the
integral over $\omega$ in (5) by a summation over discrete Matsubara frequencies $\omega_n = (2n+1)\pi T$ ($n = 0, \pm 1, \pm 2, \ldots$). This leads to a thermal broadening of the step–like behavior of $\sigma_{xy}$.

The effect of the level fluctuations is evaluated by averaging $\sigma_{xy}$ over the random matrix elements of $h_1$. In order to derive a simple expression for the limit of infinitely many energy levels per dot ($N \to \infty$) it is convenient to write the product of Green’s functions $G = (H_0 + h_1 + z\sigma_0)^{-1}$ ($H_0$ is either $H_{11}^0$ or $H_{22}^0$) in the expression of the Hall conductivity formally as a functional integral of a supersymmetric model [15,19,20]

$$G^{\alpha\alpha'}_{\mu'\mu}, G^{\beta\beta'}_{\nu'\nu} = \langle \bar{\Psi}_{\alpha}^{\prime} \Psi_{\alpha}^{\prime} \chi_{\beta}^{\prime} \chi_{\beta}^{\prime} \bar{\Psi}_{\gamma}^{\prime} \Psi_{\gamma}^{\prime} \rangle_S - \langle \chi_{\gamma} \chi_{\gamma} \bar{\Psi}_{\gamma} \Psi_{\gamma} S \rangle$$

with $\langle \ldots \rangle_S = \int \ldots \exp(-S_1) \prod_r d\Phi_r d\bar{\Phi}_r$ and with the supersymmetric action (sum convention for $\alpha$)

$$S_1 = -is_z \sum_{r, r', \mu, j, j'} \Phi_{r, \mu, j} (H_0 + z\sigma_0)_{r, j, r', j'} \bar{\Phi}_{r', \mu, j'} - is_z \sum_{r, \mu, j} (\Phi_{r, \mu, j} h_{r, \mu, j} \Phi_{r, \mu, j})$$

where $s_z = \text{sign}(\text{Im} z)$ and the field $\Phi_{r, j} = (\Psi_{r, \alpha}^{\prime}, \chi_{r, \gamma}^{\prime})$. The first component is Grassmann and the second complex. $\mu = 1, 2$ labels the complex and the Grassmann components, and $j = 1, 2$ labels the two components of the Dirac model. This choice guarantees a normalized functional. Consequently, the averaging with respect to the Gaussian distributed matrix elements of $h_1$ can be performed in the functional integral as $\langle \exp(-S_1) \rangle_{h_1} = \exp(-S_2)$ with the effective action $S_2$. The latter is obtained from $S_1$ by replacing the second term with $(g/N) \sum_{r, \mu, j} (\Phi_{r, \mu, j} \Phi_{r, \mu, j})^2$. Thus we have derived an effective field theory for $\Phi$ which serves as a generating functional for the average product of Green’s functions. It is important to notice that not only $h_1$ creates the interaction in $S_2$ but also other types of random terms in $S_1$. For instance, the interaction can also be created by a term $(N/g)(Q_{r, \mu, j})^2 - 2is_z Q_{r, \mu, j} \Phi_{r, \mu, j} \Phi_{r, \mu, j}$ as the second term in $S_1$, followed by an integration over the matrix field $Q$. This field, in contrast to the random matrix $h_1$, does not depend on the index $\alpha$ of the electronic states inside the quantum dot. This means that the distribution $h_1$ can be transformed into another distribution with a new ‘random variable’ $Q$ (which does not have a probability measure but some generalized distribution including Grassmann variables). In other words, we can write, after integrating out the field $\Phi$,

$$\langle [(H_0 + h_1 + z\sigma_0)^{-1}]^{\alpha \alpha \ldots} \rangle_{h_1} = \langle [(H_0 + 2Q + z\sigma_0)^{-1}]^{\alpha \alpha \ldots} \rangle_Q$$

The distribution which belongs to $\langle \ldots \rangle_Q$ was investigated in detail in [20]. Here we need only the result for leading order in $N$: $\langle \ldots \rangle_Q = \int \ldots \exp(-NS(Q, P)) \prod_r dP_r dQ_r$ with diagonal matrix fields $Q_r, P_r$ and

$$S(Q, P) = \frac{1}{g} \sum_r [Tr(Q_r^2) + Tr(P_r^2)]$$

5
\[ + \log \det(H_0 + 2Q + z\sigma_0) - \log \det(H_0 - 2iP + z\sigma_0) \]  

The number of levels \( N \) appears in front of the action. Thus the effect of the statistics of the energy levels can be evaluated for \( N \to \infty \) in saddle point (SP) approximation. The SP equation reads

\[
\frac{\delta}{\delta Q} \left[ g \frac{1}{2} \text{Tr}(Q^2) + \log \det(H_0 + 2Q + z\sigma_0) \right] = 0. \tag{11}
\]

A second SP equation appears from the variation of \( P \) by replacing \( Q \to -iP \). As an ansatz we take a uniform SP solution \( Q_0 = -iP_0 = (1/2)[i\eta\sigma_0 + M_s\sigma_3] \). Then (11) leads to the conditions \( \eta = (\eta + \omega - iE)gI, M_s = -m_1 gI/(1 + gI) \) with the integral \( I = \int [(m_1 + M_s)^2 + (\eta + \omega - iE)^2 + k^2]^{-1}d^2k/2\pi^2 \). This result means that disorder shifts the frequency \( \omega \to \omega + \eta \) and the Dirac mass \( m_1 \to M' = m_1 + M_s \), where \( \eta(m_1, \omega) \) and \( M_s(m_1, \omega) \) are solutions of the SP equation. For instance, with \( \omega = 0 \) we have \( \eta^2 = (1/4)(M_c^2 - m_1^2)\Theta(M_c^2 - m_1^2) \) where \( M_c = 2e^{-\pi/g} \). The sign of \( \eta \) is fixed by the condition that \( \eta \) must be analytic in \( \omega \). This implies \( \text{sign}(\eta) = \text{sign}(\omega) \). The average density of states (DOS) is proportional to \( \eta \) in the \( N \to \infty \)-limit. Thus we have a narrow DOS for the array of quantum dots of width \( 2M_c \) in contrast to the isolated dot which has a semicircular density of width \( 2\sqrt{g} \). The DOS vanishes for \( E = 0 \) in the absence of level fluctuations. The creation of a non–zero DOS due to level fluctuations is a non–perturbative effect.

At \( T = 0 \) and \( E = 0 \) the Hall conductivity per fermion level reads in the limit \( N \to \infty \) and with the approximation that \( M' \) and \( \eta \) do not depend on \( \omega \)

\[
\sigma_{xy} \approx 1/2 + \text{sign}(m_1) \left[ 1/2 - (1/\pi)\arctan(\sqrt{M_c^2/m_1^2 - 1})\Theta(M_c^2 - m_1^2) \right]. \tag{12}
\]

The Hall conductivities are plotted in Fig.2 for \( T = 0.1 \) with and without level fluctuations. It is remarkable that the Hall conductivity is enhanced by the level fluctuations for \( \sigma_{xy} < 1/2 \) whereas it is suppressed for \( \sigma_{xy} > 1/2 \). The effect of these fluctuations is strictly constrained to the interval \( 2M_c \).

**Conclusions** In a square–array of quantum dots with \( N \) electronic states per dot we have investigated the DOS and the Hall conductivity. Both quantities are significantly affected by the statistical fluctuations of the energy levels. In particular, the Hall conductivity, which is step–like at the QHT in the absence of fluctuations, has a more complicated behavior in the presence of level fluctuations. Thermal fluctuations have a different effect on the Hall conductivity; they lead to a simple broadening of the step–like behavior.

Only the average quantities have been considered. However, it is possible within the same method described in this article to study also higher moments of these quantities.
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Figure Captions

Fig.1: Schematic picture of an array of quantum dots with nearest \((t)\) and next nearest neighbor \((t')\) tunneling. The square denotes the unit cell of the translational invariant array with magnetic flux \(\Phi = \Phi_0/2\).

Fig.2: Hall conductivity \(\sigma_{xy}\) in units of \(e^2/h\) as a function of the effective chemical potential \(m = \mu - t'\) at temperature \(T = 0.1\). The circles are without level fluctuations and the full curve is with level fluctuations with variance \(g = 1.36\).
