INVARIANTS OF UNIPOTENT TRANSFORMATIONS
ACTING ON NOETHERIAN RELATIVELY FREE ALGEBRAS

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Abstract. The classical theorem of Weitzenböck states that the algebra of invariants $K[X]^g$ of a single unipotent transformation $g \in GL_m(K)$ acting on the polynomial algebra $K[X] = K[x_1, \ldots, x_m]$ over a field $K$ of characteristic 0 is finitely generated. This algebra coincides with the algebra of constants $K[X]^{\delta}$ of a linear locally nilpotent derivation $\delta$ of $K[X]$. Recently the author and C. K. Gupta have started the study of the algebra of invariants $F_m(V)^g$ where $F_m(V)$ is the relatively free algebra of rank $m$ in a variety $V$ of associative algebras. They have shown that $F_m(V)^g$ is not finitely generated if $V$ contains the algebra $UT_2(K)$ of $2 \times 2$ upper triangular matrices. The main result of the present paper is that the algebra $F_m(V)^g$ is finitely generated if and only if the variety $V$ does not contain the algebra $UT_2(K)$. As a by-product of the proof we have established also the finite generation of the algebra of invariants $T_{nm}^g$ where $T_{nm}$ is the mixed trace algebra generated by $m$ generic $n \times n$ matrices and the traces of their products.

Introduction

Let $K$ be any field of characteristic 0 and let $X = \{x_1, \ldots, x_m\}$, where $m > 1$. Let $g \in GL_m = GL_m(K)$ be a unipotent linear operator acting on the vector space $KX = Kx_1 \oplus \cdots \oplus Kx_m$. By the classical theorem of Weitzenböck, the algebra of invariants $K[X]^g = \{f \in K[X] \mid f(g(x_1), \ldots, g(x_m)) = f(x_1, \ldots, x_m)\}$ is finitely generated. A proof in modern language was given by Seshadri. An elementary proof based on the ideas of was presented by Tyc. Since $g - 1$ is a nilpotent linear operator of $KX$, we may consider the linear locally nilpotent derivation

$$
\delta = \log g = \sum_{i \geq 1} (-1)^{i-1} \frac{(g - 1)^i}{i}
$$

called a Weitzenböck derivation. (The $K$-linear operator $\delta$ acting on an algebra $A$ is called a derivation if $\delta(\mu v) = \delta(\mu) v + \mu \delta(v)$ for all $\mu, v \in A$.) The algebra of invariants $\mathbb{C}[X]^\delta$ coincides with the algebra of constants $\mathbb{C}[X]^{\delta} (= \ker(\delta))$. See the book by Nowicki for a background on the properties of the algebras of constants of Weitzenböck derivations.

Looking for noncommutative generalizations of invariant theory, see e.g. the survey by Formanek for $K(X) = K\langle x_1, \ldots, x_m \rangle$ be the free unitary associative

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algebra freely generated by $X$. The action of $GL_m$ is extended diagonally on $K\langle X \rangle$ by the rule

$$h(x_{j_1} \cdots x_{j_n}) = h(x_{j_1}) \cdots h(x_{j_n}), \ h \in GL_m, \ x_{j_1}, \ldots, x_{j_n} \in X.$$  

For any PI-algebra $R$, let $T(R) \subset K\langle X \rangle$ be the T-ideal of all polynomial identities in $m$ variables satisfied by $R$. The class $\mathfrak{U} = \var(R)$ of all algebras satisfying the identities of $R$ is called the variety of algebras generated by $R$ (or determined by the polynomial identities of $R$). The factor algebra $F_m(\mathfrak{U}) = K\langle X \rangle/T(R)$ is called the relatively free algebra of rank $m$ in $\mathfrak{U}$. We shall use the same symbols $x_j$ and $X$ for the generators of $F_m(\mathfrak{U})$. Since $T(R)$ is $GL_m$-invariant, the action of $GL_m$ on $K\langle X \rangle$ is inherited by $F_m(\mathfrak{U})$ and one can consider the algebra of invariants $F_m(\mathfrak{U})^G$ for any linear group $G$. As in the commutative case, if $g \in GL_m$ is unipotent, then $F_m(\mathfrak{U})^g$ coincides with the algebra $F_m(\mathfrak{U})^{\mathfrak{g}}$ of the constants of the derivation $\delta = \log g$.

Till the end of the paper we fix the integer $m > 1$, the variety $\mathfrak{U}$, the unipotent linear operator $g \in GL_m$ and the derivation $\delta = \log g$.

The author and C. K. Gupta have started the study of the algebra of invariants $F_m(\mathfrak{U})^g$. They have shown that if $\mathfrak{U}$ contains the algebra $UT_2(K)$ of $2 \times 2$ upper triangular matrices and $g$ is different from the identity of $GL_m$, then $F_m(\mathfrak{U})^g$ is not finitely generated for any $m > 1$. They have also established that, if $UT_2(K)$ does not belong to $\mathfrak{U}$, then, for $m = 2$, the algebra $F_2(\mathfrak{U})^g$ is finitely generated.

In the present paper we close the problem for which varieties $\mathfrak{U}$ and which $m$ the algebra $F_m(\mathfrak{U})^g$ is finitely generated. Our main result is that this holds, and for all $m > 1$, if and only if the variety $\mathfrak{U}$ does not contain the algebra $UT_2(K)$.

It is natural to expect such a result by two reasons. First, it follows from the proof of Tyc, see also the earlier paper by Onoda, that the algebra $K[X]^g$ is isomorphic to the algebra of invariants of certain $SL_2$-action on the polynomial algebra in $m + 2$ variables. One can prove a similar fact for $F_m(\mathfrak{U})^g$ and $(K[y_1, y_2] \otimes_K F_m(\mathfrak{U}))^{SL_2}$. Second, the results of Vonessen, Domokos and the author give that $F_m(\mathfrak{U})^G$ is finitely generated for all reductive $G$ if and only if the finitely generated algebras in $\mathfrak{U}$ are one-side noetherian. For unitary algebras this means that $\mathfrak{U}$ does not contain $UT_2(K)$ or, equivalently, $\mathfrak{U}$ satisfies the Engel identity $[x_2, x_1, \ldots, x_1] = 0$. In our proof we use the so called proper polynomial identities introduced by Specht, the fact that the Engel identity implies that the vector space of proper polynomials in $F_m(\mathfrak{U})$ is finite dimensional and hence $F_m(\mathfrak{U})$ has a series of ideals such that the factors are finitely generated $K[X]$-modules. As a by-product of the proof we have established also the finite generation of the algebra of invariants $T_{nm}^g$, where $T_{nm}$ is the mixed trace algebra generated by $m$ generic $n \times n$ matrices $x_1, \ldots, x_m$ and and the traces of their products $\text{tr}(x_{i_1} \cdots x_{i_k})$, $k \geq 1$.

1. Preliminaries

We fix two finite dimensional vector spaces $U$ and $V$, $\dim U = p$, $\dim V = q$, and representations of the infinite cyclic group $G = \langle g \rangle$:

$$\rho_U : G \to GL(U) = GL_p, \ \rho_V : G \to GL(V) = GL_q,$$

where $\rho_U(g)$ and $\rho_V(g)$ are unipotent linear operators. Fixing bases $Y = \{y_1, \ldots, y_p\}$ and $Z = \{z_1, \ldots, z_q\}$ of $U$ and $V$, respectively, we consider the free left $K[Y]$-module
$M(Y, Z)$ with basis $Z$. Then $g$ acts diagonally on $M(Y, Z)$ by the rule

$$g : \sum_{j=1}^{q} f_j(y_1, \ldots, y_p)z_j \rightarrow \sum_{j=1}^{q} f_j(g(y_1), \ldots, g(y_p))g(z_j), \quad f_j \in K[Y],$$

where, by definition, $g(y_i) = \rho(g)(y_i)$ and $g(z_j) = \rho(g)(z_j)$. Let $M(Y, Z)^\delta$ be the set of fixed points in $M(Y, Z)$ under the action of $g$. Since $\rho(g)$ and $\rho(g)$ are unipotent operators, the operators $\delta_U = \log \rho(g)$ and $\delta_V = \log \rho(g)$ are well defined. Denote by $\delta$ the induced derivation of $K[Y]$. We extend $\delta$ to a derivation of $M(Y, Z)$, denoted also by $\delta$, i.e. $\delta$ is the linear operator of $M(Y, Z)$ defined by

$$\delta : \sum_{j=1}^{q} f_j(Y)z_j \rightarrow \sum_{j=1}^{q} \delta(f_j(Y))z_j + \sum_{j=1}^{q} f_j(Y)\delta(z_j).$$

It is easy to see that $\delta = \log g$ on $M(Y, Z)$ and $M(Y, Z)^\delta$ coincides with the kernel of $\delta$, i.e. the set of constants $M(Y, Z)^\delta$. Changing the bases of $U$ and $V$, we may assume that $\delta_U$ and $\delta_V$ have the form

$$\delta_U = \begin{pmatrix} J_{p_1} & 0 & \cdots & 0 & 0 \\ \vdots & J_{p_2} & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{p_{r-1}} & 0 \\ 0 & 0 & \cdots & 0 & J_{p_r} \end{pmatrix}, \quad \delta_V = \begin{pmatrix} J_{q_1} & 0 & \cdots & 0 & 0 \\ \vdots & J_{q_2} & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{q_{i-1}} & 0 \\ 0 & 0 & \cdots & 0 & J_{q_i} \end{pmatrix},$$

where $J_r$ is the $(r+1) \times (r+1)$ Jordan cell

$$J_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with zero diagonal.

We denote by $W_r$ the irreducible $(r+1)$-dimensional $SL_2$-module. It is isomorphic to the $SL_2$-module of the forms of degree $r$ in two variables $x, y$. This is the unique structure of an $SL_2$-module on the $(r+1)$-dimensional vector space which agrees with the action of $\delta$ (and hence of $g$) as the Jordan cell (1). We can think of $\delta$ as the derivation of $K[x, y]$ defined by $\delta(x) = 0$, $\delta(y) = x$. We fix the “canonical” basis of $W_r$

$$u^{(0)} = x^r, u^{(1)} = \frac{x^{r-1}y}{1!}, u^{(2)} = \frac{x^{r-2}y^2}{2!}, \ldots, u^{(r-1)} = \frac{xy^{r-1}}{(r-1)!}, u^{(r)} = \frac{y^r}{r!}.$$  

We give $U$ and $V$ the structure of $SL_2$-modules

$$U = W_{p_1} \oplus \cdots \oplus W_{p_k}, \quad V = W_{q_1} \oplus \cdots \oplus W_{q_i},$$

and extend it on $K[Y]$ and $M(Y, Z)$ via the diagonal action of $SL_2$. Again, this agrees with the action of $g$ and $\delta$. Then $K[U]$ and $M(Y, Z)$ are direct sums of irreducible $SL_2$-modules $U_{r_i} \subset K[Y]$ and $W_{r_j} \subset M(Y, Z)$ isomorphic to $W_r$, $i, j = 1, 2, \ldots, r = 0, 1, 2, \ldots,$ with canonical bases $\{u_{r_i}^{(0)}, u_{r_i}^{(1)}, \ldots, u_{r_i}^{(r)}\}$ and $\{w_{r_j}^{(0)}, w_{r_j}^{(1)}, \ldots, w_{r_j}^{(r)}\},$ respectively.
Lemma 1. The elements \( u \in K[Y] \) and \( w \in M(Y, Z) \) belong to \( K[Y]^\delta \) and \( M(Y, Z)^\delta \), respectively, if and only if they have the form

\[
(4) \quad u = \sum_{r,i} \alpha_{ri} u_r^{(0)}, \quad w = \sum_{r,j} \beta_{rj} u_r^{(0)}, \quad \alpha_{ri}, \beta_{rj} \in K.
\]

Proof. Almkvist, Dicks and Formanek [11] translated in the language of \( g \)-invariants results of De Concini, Eisenbud and Procesi [2] and proved that, in our notation, \( g(u) = u \) and \( g(w) = w \) if and only if \( u \) and \( w \) have the form \( Y, Z \). Since \( g(u) = u \) if and only if \( \delta(u) = 0 \), and similarly for \( w \), we obtain that \( \delta \) holds if and only if \( u \) and \( w \) are \( \delta \)-constants. (The same fact is contained in the paper by Tyc [14] but in the language of representations of the Lie algebra \( sl_2(K) \).

In each component \( W_r \) of \( U \) in [2], using the basis [2], we define a linear operator \( d \) by

\[
d(u^{(k)}) = (k + 1)(r - k)u^{(k+1)}, \quad k = 0, 1, 2, \ldots, r,
\]

i.e., up to multiplicative constants, \( d \) acts by \( u^{(0)} \to u^{(1)} \to u^{(2)} \to \cdots \to u^{(r)} \to 0 \). We extend \( d \) to a derivation of \( K[Y] \). As in the case of \( \delta \), again we can think of \( d \) as the derivation of \( K[x, y] \) defined by \( d(x) = y, d(y) = 0 \).

Lemma 2. (i) The derivation \( d \) acts on each irreducible component \( U_{ri} \) of \( K[Y] \) by

\[
d(u_{ri}^{(k)}) = (k + 1)(r - k)u_{ri}^{(k+1)}, \quad k = 0, 1, \ldots, r.
\]

(ii) If \( f = f(Y) \in K[Y] \), then \( \delta^{s+1}(f) = 0 \) if and only if \( f \) belongs to the vector space

\[
(5) \quad K[Y]_s = \sum_{t=0}^s d^t(K[Y]^\delta).
\]

Proof. Part (i) follows from the fact that the \( SL_2 \)-action on \( U \) is the only action which agrees with the action of \( \delta \) as well as with the action of \( d \) (as the derivations of \( K[x, y] \) defined by \( \delta(x) = 0, \delta(y) = x \) and \( d(x) = y, d(y) = 0 \), respectively), and the extension of this \( SL_2 \)-action to \( K[U] \) also agrees with the action of \( \delta \) and \( d \) on \( K[U] \).

(ii) Since the irreducible \( SL_2 \)-submodules of \( K[Y] \) are \( \delta \)- and \( d \)-invariant, it is sufficient to prove the statement only for \( f \in W_r \subset K[Y] \). Considering the basis [2] of \( W_r \), we have that \( \delta^{s+1}(f) = 0 \) if and only if

\[
f = \alpha_0 u^{(0)} + \alpha_1 u^{(1)} + \cdots + \alpha_s u^{(s)}, \quad \alpha_k \in K.
\]

Since \( W_r^\delta = Ku^{(0)} \) and \( d^s(u^{(0)}) \in Ku^{(0)} \), we obtain that \( W_r \cap K[Y]_s \) is spanned by \( u^{(0)}, u^{(1)}, \ldots, u^{(s)} \) and coincides with the kernel of \( \delta^{s+1} \) in \( W_r \).

In principle, the proof of the following proposition can be obtained following the main steps of the proof of Tyc [14] of the Weitzenböck theorem. The proof of the three main lemmas in [14] uses only the fact that the ideals of the algebra \( K[Y] \) are finitely generated \( K[Y]^\delta \)-modules. Instead, we shall give a direct proof, using the idea of the proof of Lemma 3 in [14].

Proposition 3. The set of constants \( M(Y, Z) \) is a finitely generated \( K[Y]^\delta \)-module.
Proof. Let $N_i$ be the $K[Y]$-submodule of $M(Y,Z)$ generated by the basis elements $z_j$ of $V = K z_1 \oplus \cdots \oplus K z_q$ corresponding to the $i$-th Jordan cell $J_i$. Since $M(Y,Z) = N_1 \oplus \cdots \oplus N_q$ and $M(Y,Z) = K[Y] \oplus \cdots \oplus K[Y]$, it is sufficient to show that each $N_i$ is a finitely generated $K[Y]$-module. Hence, without loss of generality we may assume that $q = r + 1$ and $\delta(z_0) = 0, \delta(z_j) = z_{j-1}, j = 1, 2, \ldots, r$. Let

$$f = f_0(Y) z_0 + f_1(Y) z_1 + \cdots + f_r(Y) z_r \in M(Y,Z), \quad f_j(Y) \in K[Y].$$

Then

$$\delta(f) = (\delta(f_0) + f_1) z_0 + (\delta(f_1) + f_2) z_1 + \cdots + (\delta(f_{r-1}) + f_r) z_{r-1} + \delta(f_r) z_r$$

and this implies that

$$\delta(f_j) = -f_{j+1}, \quad j = 0, 1, \ldots, r - 1,$$

$$\delta(f_r) = \delta^2(f_{r-1}) = \cdots = \delta^r(f_1) = \delta^{r+1}(f_0) = 0.$$

Hence, fixing any element $f_0(Y)$ from $K[Y]^r$, we determine all the coefficients $f_1, \ldots, f_r$ from (6). By Lemma 2 it is sufficient to show that the $K[Y]$-module generated by $d^r(K[Y]^r)$ is finitely generated. By the theorem of Weitzenböck, $K[Y]^r$ is a finitely generated algebra. Let $\{h_1, \ldots, h_n\}$ be a generating set of $K[Y]^r$. Then $d^r(K[Y]^r)$ is spanned by the elements $d^r(h_1^{t_1} \cdots h_n^{t_n})$. Since $d$ is a derivation, $d^r(K[Y]^r)$ is spanned by elements of the form

$$h_1^{t_1} \cdots h_n^{t_n} \left( \prod d^{t_1}(h_1) \right) \cdots \left( \prod d^{t_n}(h_n) \right), \quad \sum t_1 + \cdots + \sum t_n = t.$$

There is only a finite number of possibilities for $t_1, \ldots, t_n$, and we obtain that $d^r(K[Y]^r)$ generates a finitely generated $K[Y]^r$-module. \qed

Corollary 4. Let, in the notation of this section, $U$ and $V$ be polynomial $GL_m$-modules, let $g \in GL_m$ be a unipotent matrix and let $M(Y,Z)$ be equipped with the diagonal action of $GL_m$. Then, for every $GL_m$-submodule $M_0$ of $M(Y,Z)$, the natural homomorphism $M(Y,Z) \to M(Y,Z)/M_0$ induces an epimorphism $M(Y,Z)^g \to (M(Y,Z)/M_0)^g$, i.e. we can lift the $g$-invariants of $M(Y,Z)/M_0$ to $g$-invariants of $M(Y,Z)$.

Proof. The lifting of the constants was established in [3] in the case of relatively free algebras and the same proof works in our situation. Since $U$ and $V$ are polynomial $GL_m$-modules, the module $M(Y,Z)$ is completely reducible. Hence $M(Y,Z) = M_0 \oplus M'$ for some $GL_m$-submodule $M'$ of $M(Y,Z)$ and $M(Y,Z)/M_0 \cong M'$. If $w + M_0 = \bar{w} \in (M(Y,Z)/M_0)^g$, then $w = w_0 + w'$, $w_0 \in M_0$, $w' \in M'$, and $g(w) = g(w_0) + g(w')$. Since $g(\bar{w}) = \bar{w}$, we obtain that $g(w') = w'$ and the $g$-invariant $\bar{w}$ is lifted to the $g$-invariant $w'$. \qed

Remark 5. The proof of Proposition 4 gives also an algorithm to find the generators of $M(Y,Z)^g$ in terms of the generators of $K[Y]^g$. The latter problem is solved by van den Essen [7] and his algorithm uses Gröbner bases techniques.

2. The Main Results

The following theorem is the main result of our paper. For $m = 2$ it was established in [3] using the description of the $g$-invariants of $K(x,y)$.

Theorem 6. For any variety $\mathcal{V}$ of associative algebras which does not contain the algebra $UT_2(K)$ of $2 \times 2$ upper triangular matrices, the algebra of invariants $F_m(\mathcal{V})^g$ of any unipotent $g \in GL_m$ is finitely generated.
Proof. We shall work with the linear locally nilpotent derivation \( \delta = \log g \) instead with \( g \).

It is well known that any variety \( \mathfrak{V} \) which does not contain \( UT_2(K) \) satisfies some Engel identity \([x_2, x_1, \ldots, x_1] = 0\). By the theorem of Zelmanov \( \mathbb{L} \) any Lie algebra over a field of characteristic zero satisfying the Engel identity is nilpotent. Hence we may assume that \( \mathfrak{V} \) satisfies the polynomial identity of Lie nilpotency \([x_1, \ldots, x_{c+1}] = 0\). (Actually, this follows from much easier and much earlier results on PI-algebras.)

Let us consider the vector space \( B_m(\mathfrak{V}) \) of so called proper polynomials in \( F_m(\mathfrak{V}) \). It is spanned by all products \([x_{i_1}, \ldots, x_{i_k}] \cdots [x_{j_1}, \ldots, x_{j_l}]\) of commutators of length \( \geq 2 \). One of the main results of the paper by the author \( \mathbb{1} \) states that if \( \{f_1, f_2, \ldots\} \) is a basis of \( B_m(\mathfrak{V}) \), then \( F_m(\mathfrak{V}) \) has a basis
\[
\{x_{i_1}^{p_1} \cdots x_{i_k}^{p_k} f_i \mid p_j \geq 0, i = 1, 2, \ldots\}.
\]

Let \( B_m^{(k)}(\mathfrak{V}) \) be the homogeneous component of degree \( k \) of \( B_m(\mathfrak{V}) \). It follows from the proof of Theorem 5.5 in \( \mathbb{1} \), that for any Lie nilpotent variety \( \mathfrak{V} \) and for a fixed positive integer \( m \), the vector space \( B_m(\mathfrak{V}) \) is finite dimensional. Hence \( B_m^{(n)}(\mathfrak{V}) = 0 \) for \( n \) sufficiently large, e. g. for \( n > n_0 \). Let \( I_k \) be the ideal of \( F_m(\mathfrak{V}) \) generated by \( B_m^{(k+1)}(\mathfrak{V}) + B_m^{(k+2)}(\mathfrak{V}) + \cdots + B_m^{(m)}(\mathfrak{V}) \). Since \( w x_i = x_i w + [w, x_i], w \in F_m(\mathfrak{V}) \), applying Lemma 2.4 \( \mathbb{1} \), we obtain that \( I_k/I_k+1 \) is a free left \( K[X] \)-module with any basis of the vector space \( B_m^{(k)}(\mathfrak{V}) \) as a set of free generators. Since \( \delta \) is a nilpotent linear operator of \( U = K X = K x_1 \oplus \cdots \oplus K x_m \), it acts also as a nilpotent linear operator of \( V_k = B_m^{(k)}(\mathfrak{V}) \). Proposition \( \mathbb{8} \) gives that \( (I_k/I_k+1)\delta \) is a finitely generated \( K[X]^\delta \)-module. Clearly, \( B_m^{(0)}(\mathfrak{V}) = K, B_m^{(1)}(\mathfrak{V}) = 0, B_m^{(2)}(\mathfrak{V}) \) is spanned by the commutators \([x_{i_1}, x_{i_2}]\), etc. Hence \( I_0/I_1 \cong K[X] \) and by the theorem of Weitzenböck \( (I_0/I_1)^\delta \) is a finitely generated algebra. We fix a system of generators \( f_1, \ldots, f_n \) of the algebra \((I_0/I_1)^\delta \) and finite sets of generators \( \{f_{k_1}, \ldots, f_{k_{nk}}\} \) of the \( K[X]^\delta \)-modules \((I_k/I_k+1)^\delta \), \( k = 2, 3, \ldots, n_0 \). The vector space \( U \) is a \( GL_m \)-module and its \( GL_m \)-action makes \( V_k \) a polynomial \( GL_m \)-module. We apply Corollary \( \mathbb{4} \) and lift all \( \tilde{f}_i \) and \( \tilde{f}_{kj} \) to some \( \delta \)-constants \( f_i, f_{kj} \in F_m(\mathfrak{V})^\delta \). The algebra \( S \) generated by \( f_1, \ldots, f_n \) maps onto \((I_0/I_1)^\delta \) and hence \((I_k/I_k+1)^\delta \) is a left \( S \)-module generated by \( f_{k_1}, \ldots, f_{k_{nk}} \). The condition \( I_{n_0+1} = 0 \) together with Corollary \( \mathbb{4} \) gives that the \( f_i \) and \( f_{kj} \) generate \( F_m(\mathfrak{V})^\delta \).

Together with the results of \( \mathbb{9} \) Theorem \( \mathbb{6} \) gives immediately:

**Corollary 7.** For \( m \geq 2 \) and for any fixed unipotent operator \( g \in GL_m \), \( g \neq 1 \), the algebra of \( g \)-invariants \( F_m(\mathfrak{V})^g \) is finitely generated if and only if \( \mathfrak{V} \) does not contain the algebra \( UT_2(K) \).

We refer to the books \( \mathbb{1} \) and \( \mathbb{2} \) for a background on the theory of matrix invariants. We fix an integer \( n > 1 \) and consider the generic \( n \times n \) matrices \( x_1, \ldots, x_m \). Let \( C_{nm} \) be the pure trace algebra, i. e. the algebra generated by the traces of products \( \text{tr}(x_{i_1} \cdots x_{i_k}), k = 1, 2, \ldots \), and let \( T_{nm} \) be the mixed trace algebra generated by \( x_1, \ldots, x_m \) and \( C_{nm} \). It is well known that \( C_{nm} \) is finitely generated. (The Nagata-Higman theorem states that the nil polynomial identity \( x^n = 0 \) implies the identity of nilpotency \( x_1 \cdots x_d = 0 \). If \( d(n) \) is the minimal \( d \) with this property, one may take as generators \( \text{tr}(x_{i_1} \cdots x_{i_k}) \) with \( k \leq d(n) \).) Also, \( T_{nm} \) is a finitely generated \( C_{nm} \)-module.
Theorem 8. For any unipotent operator $g \in GL_m$, the algebra $T^g_{nm}$ is finitely generated.

Proof. Consider the vector space $U$ of all formal traces $y_i = \text{tr}(x_{i1} \cdots x_{ik})$, $i_j = 1, \ldots, m$, $1 \leq k \leq d(n)$. Let $Y$ be the set of all $y_i$. It has a natural structure of a $GL_m$-module and hence $\delta = \log g$ acts as a nilpotent linear operator on $U$. Also, consider a finite system of generators $f_1, \ldots, f_a$ of the $C_{nm}$-module $T_{nm}$. We may assume that the $f_j$ do not depend on the traces and fix some elements $h_j \in K\langle X \rangle$ such that $h_j \mapsto f_j$ under the natural homomorphism $K\langle X \rangle \to T_{nm}$ extending the mapping $x_i \mapsto x_i$, $i = 1, \ldots, m$. Let $V$ be the $GL_m$-submodule of $K\langle X \rangle$ generated by the $h_j$. Again, $\delta$ acts as a nilpotent linear operator on $V$. We fix a basis $Z = \{z_1, \ldots, z_q\}$ of $V$. Consider the free $K[Y]$-module $M(Y, Z)$ with basis $Z$. Proposition 3 gives that $M(Y, Z)^\delta$ is a finitely generated $K[Y]^\delta$-module and the theorem of Weitzenböck implies that $K[Y]^\delta$ is a finitely generated algebra. Since the algebra $C_{nm}$ and the $C_{nm}$-module $T_{nm}$ are homomorphic images of the algebra $K[Y]$ and the $K[Y]$-module $M(Y, Z)$, Corollary 4 gives that $K[Y]^\delta$ and $M(Y, Z)^\delta$ map on $C_{nm}^\delta$ and $T_{nm}^\delta$, respectively. Hence $T_{nm}$ is a finitely generated module of the finitely generated algebra $C_{nm}^\delta$ and, therefore, the algebra $T_{nm}^\delta$ is finitely generated. \hfill $\Box$

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