A CORRECTION IN THE COSMOLOGICAL PARALLAX–DISTANCE FORMULA

ASHOK K. SINGAL
ASTRONOMY & ASTROPHYSICS DIVISION, PHYSICAL RESEARCH LABORATORY,
NAVRANGPURA, AHMEDABAD - 380 009, INDIA; asingal@prl.res.in

Received _______________; accepted _______________
ABSTRACT

It is shown that the standard cosmological parallax-distance formula, as found in the literature, including text-books on cosmology, requires a correction. This correction arises from the fact that any chosen baseline in a gravitationally bound system does not partake in the cosmological expansion and therefore two ends of the baseline used by the observer for parallax measurements cannot form a set of co-moving co-ordinates, contrary to what seems to have been implicitly assumed in the standard text-book derivation of the parallax distance formula. At large redshifts, the correction in parallax distance could be as large as a factor of three or more, in the currently favoured cosmologies (viz. $\Omega_\Lambda = 0.73, k = 0$). Even otherwise, irrespective of the amount of corrections involved, it is necessary to have formulae bereft of any shortcomings. We further show that the parallax distance does not increase indefinitely with redshift and that even the farthest observable point (i.e., at redshift approaching infinity) will have a finite parallax value, a factor that needs to be carefully taken into account when using distant objects as the background field against which the parallax of a foreground object is to be measured.

*Subject headings:* cosmological parameters — cosmology: miscellaneous — cosmology: observations — cosmology: theory — distance scale
1. INTRODUCTION

Cosmological distance formulae are standard formulae for different world models in use for more than half a century and are available in review articles and textbooks on cosmology. The two most commonly used formulae for cosmological distant sources are the luminosity distance formula and the angular diameter distance formula. A use of either of these in conjunction with the measured values of flux densities or angular sizes of a suitable sample, for ascertaining the geometry of the universe, depends upon the assumption of the existence of a standard candle/rod, and a successful interpretation is very much dependent on the cosmological evolution of the intrinsic luminosities/physical sizes of the parent population of the sources of interest. Not only the large spread in their intrinsic values makes it very uncertain to define a standard candle/rod, in fact in most cases the effects of the evolutionary changes in source properties are overwhelmingly larger than those expected due to differences in geometry between different world models. The other two formulae, which presently are being used only for nearby objects, are the proper motion distance formula and the parallax distance formula.

For testing different world models, application of the parallax distance in particular, unlike other distance measures, does not depend upon any assumption about the intrinsic properties of the observed sources. Thus it is also independent of the chosen frequency band since no source property is involved, as long as the source is detectable in that band. At a given redshift, the observed parallax depends only on the chosen baseline of the observer and the world–model geometry.

Weinberg (1970) showed that even otherwise, the measurement of redshift and luminosity (or angular diameter) distance cannot in principle determine the sign and magnitude of the spatial curvature unless supplemented with a dynamical model. However, this ambiguity can be resolved by parallax measurements at cosmological distances. With
the achieved angular resolutions already in the micro-arcsec domain (Fomalont & Kobayashi 2006), it could with the advancement of technology become important rather sooner than expected.

The cosmological parallax–distance formulation is available in text-books on cosmology (Weinberg 1972; Peacock 1999), review articles (von Hoerner 1974) or reference books (Lang 1980), with expressions for the different world models given there. Here we show that a subtle correction is required in the standard formulation found in the contemporary literature. The correction stems from the fact that the physical dimensions of a gravitationally bound system like that of the solar system (or for that matter even of larger systems like that of a galaxy), do not change with the cosmological expansion of the universe. Therefore two ends of a baseline, used by the observer for parallax measurements, do not partake in the free-fall-like motion of the cosmic fluid and thus cannot be considered to form a set of co-moving co-ordinates, contrary to what seems to have been implicitly assumed in the standard text-book derivations of the parallax distance formula. The correction can be very large (a factor of three or even more) at large redshifts. In any case, irrespective of the amount of corrections involved, it is imperative to have formulae bereft of any shortcomings. In the next section we shall briefly review the formulation given in the literature and in the section following that, we spell out the required corrections.

2. FORMULATION

In a homogeneous and isotropic universe, the line element can be expressed in the Robertson-Walker metric form,

$$ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{(1 - kr^2)^{1/2}} + r^2 (d\theta^2 + \sin^2 \theta \ d\phi^2) \right],$$
where $R(t)$, a function of time $t$, is known as the cosmic scale factor, $k$ is the curvature index that can take one of the three possible values $+1, 0$ or $-1$ and $(r, \theta, \phi)$ are the time-independent co-moving co-ordinates.

From Einstein’s field equations, one can relate the curvature index $k$ and the present values of the cosmic scale factor $R_0$ to the Hubble constant $H_0$, the matter energy density $\Omega_0$ and the vacuum energy (dark energy) density $\Omega_\Lambda$ as (Peacock 1999),

$$\frac{k c^2}{H_0^2 R_0^2} = \Omega_0 + \Omega_\Lambda - 1 .$$  \hspace{1cm} (1)

The space is flat ($k = 0$) if $\Omega_0 + \Omega_\Lambda = 1$.

As shown by Weinberg (1972), an observer using a baseline $b$ to make measurements of a source at radial co-ordinate $r$ will infer a parallax angle,

$$\psi = \frac{b (1 - k r^2)^{1/2}}{R_0 r} .$$  \hspace{1cm} (2)

Defining a parallax distance, as in Euclidean geometry, by $d_p = b/\psi$ we can write,

$$d_p = \frac{R_0 r}{(1 - k r^2)^{1/2}} .$$  \hspace{1cm} (3)

In general it is not possible to express $d_p$ in terms of the cosmological redshift $z$ of the source in a close-form analytical expression and one may have to evaluate it numerically. For example, in the $\Omega_0 + \Omega_\Lambda = 1, \Omega_\Lambda \neq 0$ world-models, $r$ is given by,

$$r = \frac{c}{H_0 R_0} \int_1^{1+z} \frac{dz}{(\Omega_\Lambda + \Omega_0 z^3)^{1/2}} ,$$  \hspace{1cm} (4)

and since $k = 0$, one can write,

$$d_p = R_0 r = \frac{c}{H_0} \int_1^{1+z} \frac{dz}{(\Omega_\Lambda + \Omega_0 z^3)^{1/2}} .$$  \hspace{1cm} (5)

For a given $\Omega_\Lambda$, one can evaluate $d_p$ from (5) by a numerical integration.
However for $\Omega_\Lambda = 0$ cosmologies, where the deceleration parameter $q_0 = \Omega_0/2$, it is possible to express $r$ in an analytical form (Mattig 1959),

$$r = \frac{c}{H_0 R_0} \frac{q_0 z + (q_0 - 1) (-1 + \sqrt{1 + 2q_0 z})}{q_0^2 (1 + z)}.$$  \hspace{1cm} (6)

Also Equation (1) now becomes,

$$\frac{k c^2}{H_0^2 R_0^2} = 2q_0 - 1.$$  \hspace{1cm} (7)

Then it is straightforward to get,

$$d_p = \frac{c}{H_0} \frac{q_0 z + (q_0 - 1) (-1 + \sqrt{1 + 2q_0 z})}{\left[q_0^4 (1 + z)^2 - (2q_0 - 1) \{q_0 z + (q_0 - 1) (-1 + \sqrt{1 + 2q_0 z}) \}^2 \right]^{1/2}},$$  \hspace{1cm} (8)

which is the expression derived by Weinberg (1972), quoted in Lang (1980) and also used by von Hoerner (1974). One can get rid of the square-root in the denominator (Peacock 1999) to write Equation (8) in an alternative form,

$$d_p = \frac{c}{H_0 (q_0 - 1) (q_0 - 1 - q_0 z) + (2q_0 - 1) \sqrt{1 + 2q_0 z}}.$$  \hspace{1cm} (9)

3. THE CORRECTION

In the derivation of the expression for the parallax angle (Weinberg 1972), while tracing the light path from the source to the observer, the baseline ends were defined by co-moving co-ordinates. But the two ends of any rod or baseline, be it the sun-earth line or some larger baseline in the solar system (or still larger ones but as long as one is confined within a gravitationally bound system like our galaxy), cannot be freely falling with the expanding cosmic fluid as the distance between the two ends of the rod is taken to be fixed. One can consider one end of the baseline to be at rest with respect to the underlying cosmic fluid, but then the other end, at a fixed proper distance $b$, will have a velocity $v = -H_0 b$ with respect to the underlying co-moving sub-stratum. That means the second end of the
baseline, because of its motion with respect to the co-moving sub-stratum, will have an aberration, \( \delta \approx bH_0/c \). This will add to the value of the parallax angle as given by (2) and the actually measured parallax angle will be,

\[
\Psi = \psi + \delta = b \left[ \frac{(1 - kr^2)^{1/2}}{R_0 r} + \frac{H_0}{C} \right].
\]

(10)

Then for the parallax distance, defined by \( D_p = b/\Psi \), we can write,

\[
D_p = \frac{R_0 r}{(1 - kr^2)^{1/2} + R_0 rH_0/C}.
\]

(11)

In the \( \Omega_\Lambda = 0 \) cosmologies, the modified formula for parallax distance now becomes,

\[
D_p = \frac{c}{H_0} \left[ \frac{q_0 z + (q_0 - 1) \left( -1 + \sqrt{1 + 2q_0 z} \right)}{(q_0 - 2)(q_0 - 1 - q_0 z) + (3q_0 - 2) \sqrt{1 + 2q_0 z}} \right].
\]

(12)

Actually the relations (6), (8), (9) and (12) are clumsy when to be used for small \( q_0 \) and \( z \) as then one has to evaluate these expressions in the limit. Terrell (1977) provided a much simpler form for \( r \) as an alternative to Mattig’s expression,

\[
r = \frac{c}{H_0 R_0 (1 + z)} \left[ \frac{1 + z + \sqrt{1 + 2q_0 z}}{1 + q_0 z + \sqrt{1 + 2q_0 z}} \right].
\]

(13)

Using Equation (3) and (13), after some algebraic manipulations we get \( d_p \) as,

\[
d_p = \frac{c}{H_0} \frac{z \left[ 1 + z + \sqrt{1 + 2q_0 z} \right]}{\left[ (1 + z + z^2 + q_0 z - q_0 z^2) + (1 + z) \sqrt{1 + 2q_0 z} \right]}.
\]

(14)

Similarly one gets a much simpler form for \( D_p \) also,

\[
D_p = \frac{c}{H_0} \frac{z \left[ 1 + z + \sqrt{1 + 2q_0 z} \right]}{\left[ (1 + 2z + 2z^2 + q_0 z - q_0 z^2) + (1 + 2z) \sqrt{1 + 2q_0 z} \right]}.
\]

(15)

The expressions (14) and (15) are much simpler to use, especially when evaluating \( D_p \) for small \( q_0 \) or \( z \) values as one can avoid going through the process of taking a limit.

As both the parallax and the correction for the parallax angle are proportional to the baseline \( b \), the relative correction to the parallax distance is independent of the length of
the baseline and could become appreciable at large redshifts. Figure (1) shows a plot of the
text-book expression $d_p$ as well as the corrected expression $D_p$, for different world models
$(q_0 = 0, 0.5, 1)$ in the $\Omega_\Lambda = 0$ cosmologies. We see that at large redshifts the corrected
values for the parallax distance are at least a factor of two or three lower. Also the parallax
distance does not seem to increase indefinitely with redshift. From the expressions (14)
for $d_p$ we find that as $z \to \infty$, $d_p$ reduces to $c/[H_0(1 - q_0)]$ while $D_p$ from Equation (15)
 reduces to $c/[H_0(2 - q_0)]$. Thus the corrected parallax distance values are smaller by a
factor $(2 - q_0)/(1 - q_0)$ at large redshifts, which for $q_0 = 0$ case (Milne’s world model) is a
reduction factor of 2 and in the flat space ($q_0 = 0.5$) it is a factor of 3, the factor is still
larger for positive curvature spaces with $q_0 > 0.5$.

Recent observations indicate that $\Lambda > 0$ and the space may be flat with $k = 0$. In such
a case, $D_p$ has to be evaluated numerically from,

$$D_p = \frac{c}{H_0} \frac{I}{1 + I},$$

(16)

where

$$I = \int_1^{1+z} \frac{dz}{(\Omega_\Lambda + \Omega_0 z^3)^{1/2}}.$$

Figure (2) shows a plot of parallax distance with redshift for the flat space for different $\Omega_\Lambda$
values, including the most likely value $\Omega_\Lambda = 0.73$ as inferred from the WMAP observations
(Hinshaw et al. 2009). We see that at large redshifts, the parallax distances calculated
from the modified expressions could be lower than from the uncorrected ones by as much
as a factor of three or more in the currently favoured cosmologies (viz. $\Omega_\Lambda = 0.73, k = 0$).
REFERENCES

Fomalont, E. B., & Kobayashi, H. 2006, Radio Sc. Bull., 318, 13

Hinshaw, G., et al. 2009, ApJS, 180, 225

Lang, K. R. 1980, Astrophysical Formulae: A Compendium for the Physicist and Astrophysicist (Berlin: Springer)

Mattig, V. W. 1958, Astronomische Nachrichten, 284, 109

Peacock, J. A. 1999, Cosmological Physics (Cambridge: Cambridge)

Terrell, J. 1977, Am. J. Phys., 45, 869

von Hoerner S. 1974, ‘Cosmology’ in Galactic and Extra-galactic Radio Astronomy, ed. G. L. Verschuur, & K. I. Kellermann (Berlin: Springer), 353

Weinberg, S. 1970, ApJ, 161, L233

Weinberg, S. 1972, Gravitation and Cosmology : Principles and Applications of the General Theory of Relativity (New York: Wiley)

This manuscript was prepared with the AAS \LaTeX\ macros v5.2.
Fig. 1.— A plot of the parallax distance with redshift for different world models in Λ = 0 cosmologies. The upper three curves (broken lines) are for $d_p$, calculated from the existing expressions in the literature, while the lower plots (continuous lines) for the same three world models are for $D_p$, calculated using the modified formulae. The bold lines are for the flat-space ($q_0 = 0.5$).
Fig. 2.— A plot of the parallax distance with redshift for different world models in $k = 0$ (flat space) cosmologies. The upper five curves (broken lines) are for $d_p$ calculated from the existing expressions in the literature for different $\Omega_\Lambda$ values, while the lower plots (continuous lines) are for the same five world models for $D_p$, calculated using the modified formulae. The bold lines for the presently considered to be the most–likely value of $\Omega_\Lambda = 0.73$. 