Universal and Generalized Cartan Calculus on Hopf Algebras

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Abstract

We extend the universal differential calculus on an arbitrary Hopf algebra to a “universal Cartan calculus”. This is accomplished by introducing inner derivations and Lie derivatives which act on the elements of the universal differential envelope. A new algebra is formulated by incorporating these new objects into the universal differential calculus together with consistent commutation relations. We also explain how to include nontrivial commutation relations into this formulation to obtain the “generalized Cartan calculus”.

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1 Introduction

The question of how to endow a quantum group with a differential geometry has been studied extensively [1]-[6]. Most of these approaches, however, are rather specific: many papers dealing with the subject consider the quantum group in question as defined by its R-matrix, and others limit themselves to particular cases. In this paper, we attempt a more abstract and general formulation, where we consider an arbitrary Hopf algebra rather than a quantum group as the basis for our differential geometry.

The approach we take starts with a Hopf algebra \( A \) and its associated universal differential calculus \( (\Omega(A), \delta) \) [7]. By introducing another Hopf algebra \( U \) which is dually paired with \( A \), we then construct a larger class of generalized derivations (given in terms of elements of \( U \)) which act on \( \Omega(A) \); these play the roles of Lie derivatives and inner derivations. It is then possible to extend \( \Omega(A) \) to a larger algebra by finding the commutation relations (rather than merely actions) of these new objects between themselves and elements of \( \Omega(A) \). We call this new algebra the “universal Cartan calculus” associated with \( A \).

However, just as the universal differential calculus approach does not assume any explicit commutation relations between elements of the algebra \( A \), and therefore does not coincide with the “textbook” differential calculus, neither does our universal Cartan calculus reduce to the “textbook” version, since we take only the basic properties of a Hopf algebra as given. Additional structure on \( \Omega(A) \) (e.g. commutation relations) may be incorporated into our formulation [8]; in Section 3 we review the conditions under which this is possible, and elaborate on how it is accomplished.

We begin by presenting reviews of the basics of the universal differential calculus and Hopf algebras.

1.1 Universal Differential Calculus

(See [7, 9] for a more detailed discussion of the material in this subsection.)

Let \( A \) be a unital associative algebra over a field \( k \), and \( \Gamma(A) \) an \( A \)-bimodule such that there exists a linear map \( \delta : A \rightarrow \Gamma(A) \) which satisfies the following:

\[
\delta(1_A) = 0,
\]
\[ \delta(ab) = \delta(a)b + a\delta(b), \]  

(1)

where \(1_A\) is the unit in \(A\), and \(a, b \in A\). Note that the latter of these conditions implies that \(\Gamma(A)\) is the span of elements of the form \(a\delta(b)\).

As a concrete example, we take \(\Gamma(A)\) to be equal to \(\ker m\) as a vector space \((m : A \otimes A \to A\) is the algebra multiplication, which we will usually suppress), \(i.e.\) the span of elements of the form \(\sum_i a_i \otimes b_i\) where \(\sum_i a_i b_i = 0\). \(\Gamma(A)\) is made into an \(A\)-bimodule by defining the left and right actions of \(A\) to be \(c(\sum_i a_i \otimes b_i) = \sum_i (ca_i) \otimes b_i\) and \((\sum_i a_i \otimes b_i)c = \sum_i a_i \otimes (b_i c)\), \(c \in A\). The \(\delta\) which satisfies all the needed conditions is given by \(\delta(a) := 1_A \otimes a - a \otimes 1_A\).

We now introduce \(\Omega(A)\), the differential envelope associated with \(A\); it is the algebra which is spanned by elements of \(A\), together with formal products of elements of \(\Gamma(A)\) modulo the relations (1), namely, elements of the form \(a_0 \delta(a_1) \delta(a_2) \ldots \delta(a_p)\). Such elements are called \(p\)-forms \((e.g.\) \(0\)-forms are elements of \(\Omega(A)\), \(1\)-forms elements of \(\Gamma(A)\), etc.\)). \(\Omega(A)\) is easily seen to be associative and unital (with unit \(1 = 1_A\)); furthermore, \(\delta\) can be extended to a linear map \(\delta : \Omega(A) \to \Omega(A)\) by requiring

\[
\begin{align*}
\delta(1) &= 0, \\
\delta^2(\alpha) &= 0, \\
\delta(\alpha\beta) &= \delta(\alpha)\beta + (-1)^p \alpha\delta(\beta),
\end{align*}
\]

(2)

where \(\alpha, \beta \in \Omega(A)\), \(\alpha\) a \(p\)-form. Thus, \(\delta\) maps \(p\)-forms to \((p + 1)\)-forms. \(\delta\) is the exterior derivative on \(\Omega(A)\), and we call \((\Omega(A), \delta)\) the universal differential calculus \((UDC)\) associated with \(A\).

**Note:** throughout the remainder of the paper, we shall use the terms “0-form” and “function” interchangably to refer to any element of \(A\).

### 1.2 Hopf Algebras

(See [10]-[13] for more information about Hopf algebras.)

A Hopf algebra \(A\) is an associative unital algebra (with multiplication \(m\)) over a field \(k\), equipped with a coproduct \(\Delta : A \to A \otimes A\), an antipode \(S : A \to A\) (in this paper, we assume the inverse \(S^{-1}\) also exists, although this

\(^4\)To be precise, we should actually say that \(e.g.\) \(0\)-forms are elements of \(\iota(A)\), where \(\iota : A \to \Omega(A)\) is the inclusion map, but we will be glib and suppress this notation throughout this paper.

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is not necessarily true for an arbitrary Hopf algebra), and a counit \( \epsilon : \mathcal{A} \to k \); these maps satisfy the usual consistency conditions:

\[
(\Delta \otimes \text{id})\Delta(a) = (\text{id} \otimes \Delta)\Delta(a),
\]

\[
(\epsilon \otimes \text{id})\Delta(a) = (\text{id} \otimes \epsilon)\Delta(a) = a,
\]

\[
m(S \otimes \text{id})\Delta(a) = m(\text{id} \otimes S)\Delta(a) = 1_A\epsilon(a),
\]

\[
\Delta(ab) = \Delta(a)\Delta(b), \quad \epsilon(ab) = \epsilon(a)\epsilon(b),
\]

\[
\Delta(1_A) = 1_A \otimes 1_A, \quad \epsilon(1_A) = 1_k,
\]

for all \( a, b \in \mathcal{A} \). (We will often use Sweedler’s \([12]\) notation for the coproduct:

\[
\Delta(a) \equiv a(1) \otimes a(2),
\]

\[
(\Delta \otimes \text{id})\Delta(a) \equiv a(1) \otimes a(2) \otimes a(3),
\]

and so forth, where summation is understood.)

We call two Hopf algebras \( \mathcal{U} \) and \( \mathcal{A} \) \textit{dually paired} if there exists a nondegenerate inner product \( \langle \cdot, \cdot \rangle : \mathcal{U} \otimes \mathcal{A} \to k \) such that

\[
\langle xy, a \rangle = \langle x \otimes y, \Delta(a) \rangle \equiv \langle x, a(1) \rangle \langle y, a(2) \rangle,
\]

\[
\langle x, ab \rangle = \langle \Delta(x), a \otimes b \rangle \equiv \langle x(1), a \rangle \langle x(2), b \rangle,
\]

\[
\langle S(x), a \rangle = \langle x, S(a) \rangle,
\]

\[
\langle x, 1_A \rangle = \epsilon(x), \quad \langle 1_U, a \rangle = \epsilon(a),
\]

for all \( x, y \in \mathcal{U} \) and \( a, b \in \mathcal{A} \).

## 2 Universal Cartan Calculus

The purpose of this section is to generalize the “classical” case, namely the familiar situation where \( \Omega(\mathcal{A}) \) is graded-commutative and the Cartan calculus contains Lie derivatives and inner derivations which act on \( \Omega(\mathcal{A}) \). Our “deformed” version presented here assumes not only (possible) noncommutativity of \( \Omega(\mathcal{A}) \), but also a Hopf algebraic structure on \( \mathcal{A} \). However, just as in the classical case, we need specify only how the derivations act on and commute with 0- and 1-forms; the extension to arbitrary \( p \)-forms in \( \Omega(\mathcal{A}) \) follows immediately.
We begin with two dually paired Hopf algebras $\mathcal{A}$ and $\mathcal{U}$, and take $(\Omega(\mathcal{A}), \delta)$ be the UDC associated with $\mathcal{A}$. Recall that $\mathcal{U}$ can be interpreted as an algebra of left-invariant generalized derivations which act on elements of $\mathcal{A}$ via the action
\begin{equation}
 x \triangleright a = a_{(1)} \langle x, a_{(2)} \rangle,
\end{equation}
where $x \in \mathcal{U}$ and $a \in \mathcal{A}$. The action of $x$ on a product of functions $a, b \in \mathcal{A}$ is given in terms of the coproduct of $x$:
\begin{equation}
 x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b).
\end{equation}
This motivates the introduction of a product structure on the “cross product” algebra $\mathcal{A} \times \mathcal{U}$ \cite{14-16} via the commutation relation
\begin{equation}
 xa = a_{(1)} \langle x_{(1)}, a_{(2)} \rangle x_{(2)}.
\end{equation}

We now introduce for each $x \in \mathcal{U}$ a new object, the Lie derivative $\mathcal{L}_x$; it is linear in $x$, and is a linear map taking $\Omega(\mathcal{A})$ into itself such that $p$-forms map to $p$-forms. Furthermore, we require that
\begin{equation}
 \mathcal{L}_x \delta = \delta \mathcal{L}_x.
\end{equation}
This relation allows us to uniquely recover the action of $\mathcal{L}_x$ on all of $\Omega(\mathcal{A})$ from its action on $\mathcal{A}$, i.e. 0-forms. Just as in the classical case, the action of the Lie derivative on $a \in \mathcal{A}$ is defined to be the same as that of the corresponding differential operator, i.e.
\begin{equation}
 \mathcal{L}_x (a) = x \triangleright a = a_{(1)} \langle x, a_{(2)} \rangle,
\end{equation}
and likewise for its commutation relations with 0-forms:
\begin{equation}
 \mathcal{L}_x a = a_{(1)} \langle x_{(1)}, a_{(2)} \rangle \mathcal{L}_{x_{(2)}} = \mathcal{L}_{x_{(1)}} (a) \mathcal{L}_{x_{(2)}}.
\end{equation}
From (12) and (14) we can find the action on and commutation relation with a 1-form\footnote{We use parentheses to delimit operations like $\delta$, $i_x$ and $\mathcal{L}_x$, e.g. $\delta a = \delta(a) + a \delta$. However, if the limit of the operation is clear from the context, we will suppress the parentheses, e.g. $\delta(\mathcal{L}_x \delta a) \equiv \delta(\mathcal{L}_x (\delta(a)))$.}:
\begin{align}
 \mathcal{L}_x (\delta a) &= \delta(a_{(1)}) \langle x, a_{(2)} \rangle \\
 \mathcal{L}_x \delta (a) &= \delta(a_{(1)}) \langle x_{(1)}, a_{(2)} \rangle \mathcal{L}_{x_{(2)}} = \mathcal{L}_{x_{(1)}} (\delta a) \mathcal{L}_{x_{(2)}}.
\end{align}
At this point we introduce for each \( x \in U \) the corresponding inner derivation \( i_x \). The guideline for this generalization of the classical case will be the Cartan identity\(^6\)

\[
\mathcal{L}_x = i_x \delta + \delta i_x
\]  

(16)

(so \( i_x \) is linear in \( x \)). To find the action of \( i_x \) on \( \Omega(\mathcal{A}) \) we can now attempt to use (16) in the identity \( \mathcal{L}_x(a) = i_x(\delta a) + \delta(i_x a) \). We take as an assumption that the action of \( i_x \) on 0-forms like \( a \) vanishes; therefore, we obtain

\[
i_x(\delta a) = a(1) \langle x, a(2) \rangle.
\]  

(17)

However, this cannot be true for any \( x \in U \) because, by assumption, \( \delta(1) = 0 \). From (17), \( i_x(\delta 1) = 1 \epsilon(x) \), which is not necessarily zero. We see that the trouble arises when dealing with \( x \in U \) with \( \epsilon(x) \neq 0 \). Noting that \( \epsilon(x - 1_U \epsilon(x)) = 0 \), we modify equation (17) to read

\[
i_x(\delta a) = a(1) \langle x - 1_U \epsilon(x), a(2) \rangle,
\]  

(18)

so that \( i_x(\delta 1) \) does indeed vanish for all \( x \). Also note that this requires the consistency condition

\[
i_{1_U} \equiv 0.
\]  

(19)

To allow for all \( x \in U \) with nonzero counit, we also need to modify equation (19) to

\[
\mathcal{L}_{x - 1_U \epsilon(x)} = i_x \delta + \delta i_x,
\]  

(20)

or, in view of (14), identifying \( \mathcal{L}_{1_U} \equiv \text{id} \) and using the linearity of the Lie derivative,

\[
\mathcal{L}_x = i_x \delta + \epsilon(x) \text{id} + \delta i_x
\]  

(21)

(here \( \text{id} \) is the identity map on \( \Omega(\mathcal{A}) \), and therefore the unit in the algebra of generalized derivations). We call this the universal Cartan identity.

To find the complete commutation relations of \( i_x \) with elements of \( \Omega(\mathcal{A}) \) rather than just its action on them, we need only find out how \( i_x \) moves through 0- and 1-forms. Both of these can be found by commuting \( \mathcal{L}_x \) through a function \( a \in \mathcal{A} \), using (14) and (21): the left-hand side of the former equation gives (using the Leibniz rule)

\[
\mathcal{L}_x a = i_x(\delta a) + i_x a \delta + \epsilon(x) a + \delta i_x a
\]  

(22)

\(^6\)The idea is to use this identity as long as it is consistent and modify it when needed.
and the right-hand side gives

\[ a(1) \langle x(1), a(2) \rangle \mathcal{L}_{x(2)} = \]
\[ a(1) \langle x(1), a(2) \rangle \delta i_{x(2)} + a(1) \langle x, a(2) \rangle + a(1) \langle x(1), a(2) \rangle i_{x(2)} \delta. \] (23)

Equating the two and using (2), (12), (18) and \( i_x(a) = 0 \), we obtain

\[ i_x \delta(a) - i_x(\delta a) + \mathcal{L}_{x(1)}(\delta a)i_{x(2)} = \left\{ -i_x a + i_x(a) + \mathcal{L}_{x(1)}(a) i_{x(2)}, \delta \right\}. \] (24)

Therefore, we propose the commutation relation

\[ i_x \alpha = i_x(\alpha) + (-1)^p \mathcal{L}_{x(1)}(\alpha) i_{x(2)} \] (25)

for any 0- or 1-form \( \alpha \), so that both sides of (24) vanish.

Missing in our list are commutation relations of Lie derivatives with themselves and inner derivations. To find the \( \mathcal{L}-\mathcal{L} \) relations, we note that it follows from the Hopf algebra axioms that the product in \( \mathcal{U} \) can be expressed as

\[ x \cdot y \equiv (x^{ad} \triangleright y)_{x(2)}, \] (26)

where \( x^{ad} \triangleright y := (x^{(1)} y S(x^{(2)})) \) is the adjoint action on \( \mathcal{U} \). As before, we extend the properties of the elements of \( \mathcal{U} \) to those of the corresponding Lie derivatives to find

\[ \mathcal{L}_x \mathcal{L}_y = \mathcal{L}_{(x^{ad} \triangleright y)_{x(2)}}, \] (27)

and therefore, using (21),

\[ \mathcal{L}_x i_y = i_{(x^{ad} \triangleright y)_{x(2)}}, \] (28)

(It would seem that (21) could also give the relation

\[ i_x \mathcal{L}_y = \mathcal{L}_{(x^{ad} \triangleright y)_{x(2)}}(i_x + i_{x-1\epsilon(x)}^{ad} \triangleright y), \]

but this is inconsistent with the commutation relation (25).)
To recap our results of this section, we present a summary of the actions of the Lie derivatives and inner derivations with 0- and 1-forms:

\[
\mathcal{L}_x(a) = a(1) \left< x, a(2) \right>,
\]

(29)

\[
\mathcal{L}_x(\delta a) = \delta(a(1)) \left< x, a(2) \right>,
\]

(30)

\[
i_x(a) = 0,
\]

(31)

\[
i_x(\delta a) = a(1) \left< x - 1_U \epsilon(x), a(2) \right>,
\]

(32)

where, as usual, \( x \in U, a \in A \). The commutation relations are therefore

\[
\mathcal{L}_x \alpha = \mathcal{L}_{x(1)}(\alpha) \mathcal{L}_{x(2)},
\]

(33)

\[
i_x \alpha = i_x(\alpha) + (-1)^p \mathcal{L}_{x(1)}(\alpha) i_{x(2)},
\]

(34)

where \( \alpha \in \Omega(A) \) is a \( p \)-form. (The actions and commutation relations for \( \delta \) were already given when the UDC was introduced.) Finally, here are the relations between the derivations themselves:

\[
\{ \delta, \delta \} = 0,
\]

(35)

\[
[\delta, \mathcal{L}_x] = 0,
\]

(36)

\[
\{ \delta, i_x \} = \mathcal{L}_x - \epsilon(x)i\text{id},
\]

(37)

\[
\mathcal{L}_x \mathcal{L}_y = \mathcal{L}_{(x(1) \triangleright y)} \mathcal{L}_{x(2)}
\]

(38)

\[
\mathcal{L}_x i_y = i_{(x(1) \triangleright y)} \mathcal{L}_{x(2)}
\]

(39)

Note that at this point we do not have \( i - i \) commutation relations. This is not a problem; an expression like \( i_x i_y \) is simply an element of the calculus whose action on and commutation relations with \( p \)-forms are perfectly well-defined. This is much like the fact that \( \delta(a) \delta(b) \) and \( \delta(b) \delta(a) \) are simply elements of \( \Omega(A) \); the UDC does not a priori impose relations such as \( \delta(a) \delta(b) + \delta(b) \delta(a) \equiv 0 \) (unlike the “classical” case). However, we will see in Section 3 that such restrictions between elements of \( \Omega(A) \) are possible in some cases, and we will comment on the possibility of \( i - i \) commutation relations.

### 2.1 Cartan-Maurer Forms

The most general left-invariant 1-form can be written

\[
\omega_b := S(b(1)) \delta(b(2)) = -\delta(Sb(1))b(2),
\]

(40)
corresponding to a function \( b \in \mathcal{A} \). (To connect with the classical case, if \( \mathcal{A} \) is an \( m \times m \) matrix representation of some Lie group with \( \Delta(g^i_j) = g^i_k \otimes g^k_j \), \( S(g^i_j) = (g^{-1})^i_j \) and \( \epsilon(g^i_j) = \delta^i_j \) for \( g \in \mathcal{A} \), then \( \omega_g = g^{-1}\delta(g) \), i.e. \( \omega_g \) is the well-known left-invariant Cartan-Maurer form.) Here is a nice formula for the exterior derivative of \( \omega_b \):

\[
\delta(\omega_b) = \delta(Sb(1))\delta(b(2)) = \delta(Sb(1))b(2)S(b(3))\delta(b(4)) = -\omega_{b(1)}\omega_{b(2)}. \tag{41}
\]

The Lie derivative on \( \omega_b \) is

\[
\mathcal{L}_x(\omega_b) = \mathcal{L}_{x(1)}(Sb(1))\mathcal{L}_{x(2)}(\delta b(2)) = \left\langle x(1), S(b(1)) \right\rangle S(b(2))\delta(b(3)) \left\langle x(2), b(4) \right\rangle = \omega_{b(2)} \left\langle x, S(b(1))b(3) \right\rangle. \tag{42}
\]

The contraction of left-invariant forms with \( i_x \) gives a number in the field \( k \), rather than a function in \( \mathcal{A} \) (as was the case for \( \delta(a) \)):

\[
i_x(\omega_b) = i_x(-\delta(Sb(1))b(2)) = -i_x(\delta Sb(1))b(2) = -\left\langle x - 1_\mathcal{U}\epsilon(x), S(b(1)) \right\rangle S(b(2))b(3) = -\left\langle x, S(b) \right\rangle + \epsilon(x)\epsilon(b). \tag{43}
\]

(This result is a consequence of the fact that \( \mathcal{U} \) was interpreted as an algebra of left-invariant differential operators, so \( i_x(\omega_b) \) must be a left-invariant 0-form, i.e. proportional to 1.)

As an exercise, as well as a demonstration of the consistency of our results, we will compute the same expression in a different way:

\[
i_x(\omega_b) = i_x(S(b(1))\delta(b(2))) = \left\langle x(1), S(b(1)) \right\rangle S(b(2))i_x(\delta b(2)) = \left\langle x(1), S(b(1)) \right\rangle S(b(2))b(3) \left\langle x(2) - 1_\mathcal{U}\epsilon(x(2)), b(4) \right\rangle = \left\langle x(1), S(b(1)) \right\rangle \left\langle x(2) - 1_\mathcal{U}\epsilon(x(2)), b(2) \right\rangle = \epsilon(x)\epsilon(b) - \left\langle x, S(b) \right\rangle. \tag{44}
\]
As a final observation, if \( \{e_i\} \) and \( \{f^i\} \) are, respectively, (countable) bases of \( \mathcal{U} \) and \( \mathcal{A} \) with \( \langle e_i, f^j \rangle = \delta^j_i \), the action of \( \delta \) on functions \( a \in \mathcal{A} \) may be expressed as

\[
\delta(a) = \mathcal{L}_{e_i}(a) \omega_{f^i} = -\omega_{S^{-1}(f^i)} \mathcal{L}_{e_i}(a);
\]

so that the Cartan-Maurer forms form a left-invariant basis for \( \Gamma(\mathcal{A}) \).

## 3 Further Commutation Relations

### 3.1 Generalized Differential Calculus

Recall that, so far, the only commutation relations we have in \( \Omega(\mathcal{A}) \) are those which follow from the Leibniz rule (2); we assume nothing else. Here we review the standard method of introducing nontrivial commutation relations into the differential envelope which maintains the covariance properties we have chosen (e.g. left-invariance of the Cartan-Maurer forms).

Let \( \mathcal{K} \equiv \ker \epsilon \subset \mathcal{A} \), and suppose there exists a subalgebra \( \mathcal{R} \subset \mathcal{A} \) which satisfies

1. \( \mathcal{R} \subseteq \mathcal{K} \),
2. \( \mathcal{R} \mathcal{A} \subseteq \mathcal{R} \),
3. \( \Delta^{\text{Ad}}(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{A} \)

(where \( \Delta^{\text{Ad}}(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)} \) is the adjoint coaction on \( \mathcal{A} \)). We define the submodule \( \mathcal{N}_\mathcal{R} \subseteq \Gamma(\mathcal{A}) \) as the space spanned by 1-forms of the form \( a \omega_r \), where \( a \in \mathcal{A} \) and \( r \in \mathcal{R} \). The above properties of \( \mathcal{R} \) imply properties of \( \mathcal{N}_\mathcal{R} \): (1) and (2) give \( \mathcal{N}_\mathcal{R} \mathcal{A} \subseteq \mathcal{N}_\mathcal{R} \), and (3) gives \( \Delta_A(\mathcal{N}_\mathcal{R}) \subseteq \mathcal{N}_\mathcal{R} \otimes \mathcal{A} \). Such an \( \mathcal{R} \) always exists; \( \{0\} \) and \( \mathcal{K} \) both satisfy all three conditions.

With \( \mathcal{R} \) as above, we can construct the \( \mathcal{A} \)-module \( \Gamma_\mathcal{R} := \Gamma(\mathcal{A})/\mathcal{N}_\mathcal{R} \). When \( \mathcal{R} = \{0\} \), and therefore \( \mathcal{N}_\mathcal{R} = \{0\} \), the only commutation relations between elements of \( \mathcal{A} \) and \( \Gamma_\mathcal{R} \) are those allowed by the Leibniz rule, and we recover the UDC; when \( \mathcal{R} = \mathcal{K} \), \( \mathcal{N}_\mathcal{R} = \Gamma(\mathcal{A}) \), so \( \Gamma_\mathcal{R} = \{0\} \), and we end up with a trivial differential calculus. However, if there exists an \( \mathcal{R} \) in between these two extreme cases, then there are additional commutation relations between elements of \( \Gamma_\mathcal{R} \), namely those given by \( \omega_r \simeq 0 \) for \( r \in \mathcal{R} \) (\( \simeq \) being the equivalence relation in \( \Gamma_\mathcal{R} \)). Furthermore, we find explicit commutation
relations between elements of \( \Gamma_R \) by using (41) and the properties of \( R \), i.e. \( \omega_{r(1)} \omega_{r(2)} \simeq 0 \). Therefore, we no longer have a UDC, but rather a differential envelope with nontrivial commutation relations which is constructed using \( A \) and \( \Gamma_R \); we refer to this envelope as \( \Omega_R \), and the pair \((\Omega_R, \delta)\) is referred to as the generalized differential calculus (GDC) associated with \( A \) and \( R \).

### 3.2 Generalized Cartan Calculus

How do we incorporate our Cartan calculus into this scheme? We start by defining a subspace \( \mathcal{T}_R \subset U \), given by

\[
\mathcal{T}_R := \{ x \in U | \varepsilon(x) = 0; \langle x, S(r) \rangle = 0, \ r \in R \}.
\]  

(46)

It is easily seen that the defining properties for \( R \) imply, respectively,

1. \( 1_U \notin \mathcal{T}_R \),
2. \( \Delta(\mathcal{T}_R) \subseteq U \otimes (\mathcal{T}_R \oplus 1_U) \),
3. \( U^{ad} \subseteq \mathcal{T}_R \).

Note that for \( x \in \mathcal{T}_R \) and \( a \in A \),

\[
\iota_x(\omega_a) = -\langle x, S(a) \rangle.
\]  

(47)

Suppose this vanishes; then either \( x = 0 \), \( a = 1_A \), or \( a \in R \). Therefore, if we restrict \( a \) to be in \( K/R \), then the vanishing of (47) implies that \( x = 0 \) or \( a = 0 \), i.e. the inner product \( \langle \ , \ \rangle : \mathcal{T}_R \otimes K/R \rightarrow k \) defined by

\[
\langle \langle x, a \rangle \rangle := -\langle x, S(a) \rangle
\]  

(48)

is nondegenerate. Hence, \( \mathcal{T}_R \) and \( K/R \) are dual to one another. The nondegeneracy of (48) guarantees that the map from \( K/R \rightarrow \mathcal{T}_R^* \) given by \( a \mapsto \omega_a \) is bijective, insuring that \( \Gamma_R \) is the space of all 1-forms over \( A \). Therefore, to consistently define our Cartan calculus on all of \( \Omega_R \), we must restrict the arguments of the Lie derivative and inner derivation from \( U \) to \( \mathcal{T}_R \), and the argument of \( \omega \) from \( A \) to \( K/R \).
(As an example of how this works, note that for $x \in T_R$ and $a\omega_r \in N_R$,

$$\mathcal{L}_x a\omega_r = a(1)\omega_{r(2)} \left< x(1), a(2)S(r(1))r(3) \right> \mathcal{L}_{x(2)},$$

$$i_x a\omega_r = -a(1)\omega_{r(2)} \left< x(1), a(2)S(r(1))r(3) \right> i_{x(2)},$$

(49)

Property (3) of $\mathcal{R}$ guarantees that $a(1)\omega_{r(2)} \left< x, a(2)S(r(1))r(3) \right> \in N_R$ for all $x \in U$, so both sides of the two preceding equations are $\simeq 0$ in $\Gamma_R$.

A specific example where a nontrivial $\mathcal{R}$ exists is the case of the quantum group $GL_q(N)$. It was shown in [17] that nontrivial commutation relations between 0- and 1-forms can be expressed using the $GL_q(N)$ quantum matrix $A$ and R-matrix $R$; these are obtained using the above prescription, with $\mathcal{R}$ being the subalgebra of $GL_q(N)$ generated by the $N^4$ elements

$$r^{ij}_{k\ell} := (A_1A_2 - A_2 - R^{-1}A_1R^{-1}_{21} + R^{-1}R^{-1}_{21})^{ij}_{k\ell}.$$  

(50)

The consistency of the resultant relation is entirely dependent upon the fact that the R-matrix for $GL_q(N)$ satisfies a quadratic characteristic equation.

When the Cartan calculus for this case was found in [3], consistent $i-i$ commutation relations, written in terms of the R-matrix, were given. These relations were also dependent upon the form of the $GL_q(N)$ R-matrix characteristic equation. However, we have not yet found a method for explicitly expressing such $i-i$ relations in a form depending manifestly on $\mathcal{R}$, i.e. in the flavor of $\omega_{r(1)}\omega_{r(2)} \simeq 0$.

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