Extension theorems for Hamming varieties over finite fields

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Abstract. We study the finite field extension estimates for Hamming varieties $H_j$, $j \in \mathbb{F}_q^*$, defined by $H_j = \{ x \in \mathbb{F}_q^d : \prod_{k=1}^d x_k = j \}$, where $\mathbb{F}_q^d$ denotes the $d$-dimensional vector space over a finite field $\mathbb{F}_q$ with $q$ elements. We show that although the maximal Fourier decay bound on $H_j$ away from the origin is not good, the Stein-Tomas $L^2 \to L^r$ extension estimate for $H_j$ holds.

1. Introduction

The extension or restriction problem is one of central open questions in Euclidean harmonic analysis. In 2002, Mockenhaupt and Tao \[16\] initially studied this problem for algebraic varieties in the finite field setting. Let $\mathbb{F}_q^d$ be the $d$-dimensional vector space over a finite field with $q$ elements. Throughout this paper, we assume that $q$ is an odd prime power. Given complex-valued functions $f, g$ on $\mathbb{F}_q^d$ and $1 \leq s < \infty$, we define

$$
\|g\|_{L^s(\mathbb{F}_q^d)} := \left( \sum_{m \in \mathbb{F}_q^d} |g(m)|^s \right)^{1/s} \quad \text{and} \quad \|f\|_{L^s(\mathbb{F}_q^d)} := \left( q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^s \right)^{1/s}.
$$

In addition, it is defined that $\|g\|_{L^\infty(\mathbb{F}_q^d)} := \max_{m \in \mathbb{F}_q^d} |g(m)|$ and $\|f\|_{L^\infty(\mathbb{F}_q^d)} := \max_{x \in \mathbb{F}_q^d} |f(x)|$. The notation $\|g\|_{L^s(\mathbb{F}_q^d)}$ indicates that the function $g$ is defined on the space $\mathbb{F}_q^d$ with counting measure. On the other hand, the notation $\|f\|_{L^s(\mathbb{F}_q^d)}$ tells us that the function $f$ is defined on the space $\mathbb{F}_q^d$ with normalized counting measure. Let $V$ be an algebraic variety in $\mathbb{F}_q^d$. We endow $V$ with a normalized surface measure $d\sigma$ which means that the mass of each point of $V$ is $1/|V|$, where $|V|$ denotes the cardinality of the set $V$. For a function $f : V \to \mathbb{C}$ and $1 \leq s < \infty$, we define

$$
\|f\|_{L^s(V,d\sigma)} := \left( \frac{1}{|V|} \sum_{x \in V} |f(x)|^s \right)^{1/s}.
$$

We also define $\|f\|_{L^\infty(V,d\sigma)} := \max_{x \in V} |f(x)|$.

The Fourier transform of $g$, denoted by $\hat{g}$, is defined by

$$
\hat{g}(x) = \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m),
$$

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where $\chi$ denotes the canonical additive character of $\mathbb{F}_q$, and $m \cdot x$ is the usual dot-product of $m$ and $x$. We recall that the orthogonality of $\chi$ states that

$$\sum_{\alpha \in \mathbb{F}_q^d} \chi(n \cdot \alpha) = \begin{cases} 0 & \text{if } n \neq (0, \ldots, 0) \\ q^d & \text{if } n = (0, \ldots, 0). \end{cases}$$

The inverse Fourier transform of $f$, denoted by $f^\vee$, is defined by

$$f^\vee(m) := q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x)f(x).$$

Furthermore, the inverse Fourier transform of the measure $f \sigma$ is given by

$$(f \sigma)^\vee(m) := \frac{1}{|V|} \sum_{x \in V} \chi(m \cdot x)f(x).$$

We denote by $R^*(p \to r)$ the smallest constant such that the following extension estimate

$$\|(f \sigma)^\vee\|_{L^p(V, \sigma)} \leq R^*(p \to r)\|f\|_{L^p(V, \sigma)}$$

holds for all functions $f$ on $V$. Note that $R^*_V(p \to r)$ may depend on $q$, the size of the underlying finite field $\mathbb{F}_q$. The extension problem for the variety $V$ is to determine all exponents $1 \leq p, r \leq \infty$ such that $R^*_V(p \to r)$ is independent of $q$. For $A, B > 0$, we will write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$ independent of $q$. We will also use $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. By a well-known duality, the inequality (1.1) is the same as the following restriction estimate:

$$\|g\|_{L^{r'}(V, \sigma)} \leq R^*_V(p \to r)\|g\|_{L^r(V, \sigma)},$$

where $p', r'$ denote the dual exponents of $p, r$, respectively (i.e. $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$).

When $|V| \sim q^{d-1}$, necessary conditions for $R^*_V(p \to r)$ bound can be obtained from the size of a maximal affine subspace lying on $V$. Indeed, Mockenhaupt and Tao [16] showed that if $|V| \sim q^{d-1}$ and $V$ contains an affine subspace $H$ with $|H| = q^k$, then necessary conditions for $R^*_V(p \to r)$ bound are given by

$$r \geq \frac{2d}{d-1} \quad \text{and} \quad r \geq \frac{p(d-k)}{(p-1)(d-1-k)}.$$

In dimension two, the extension problem for algebraic curves $V$ was completely solved by Shen and the second listed author [10] who showed that the above necessary conditions are also sufficient conditions for $R^*_V(p \to r)$ bound. For this reason, we will restrict ourselves to the case when $d \geq 3$. In particular, we have the following conjecture for $R^*_V(2 \to r)$ bound.

**Conjecture 1.1.** Let $V$ be an algebraic variety in $\mathbb{F}_q^d$. Suppose that $|V| \sim q^{d-1}$ and $V$ contains an affine subspace $H$ with $|H| = q^k$. Then we have

$$R^*_V(2 \to r) \lesssim 1 \quad \text{if} \quad \frac{2(d-k)}{d-k-1} \leq r \leq \infty.$$

By the norm nesting property (see Section 2), one can check that if $1 \leq r_1 \leq r_2 \leq \infty$, then $R^*_V(2 \to r_2) \leq R^*_V(2 \to r_1)$. This implies that a smaller exponent gives a better result on the restriction problem. Thus if we want to establish the sharp $R^*_V(2 \to r)$ bound, then we only needs to find the smallest exponent $r$ such that $R^*_V(2 \to r) \lesssim 1$. Namely, to confirm Conjecture 1.1 it suffices to prove that

$$R^*_V \left(2 \to \frac{2(d-k)}{d-k-1} \right) \lesssim 1.$$
The finite field extension problem has been studied only for few algebraic varieties with relatively simple structures such as spheres, paraboloids, or cones. For example, Mockenhaupt and Tao [16] addressed results on the problem for paraboloids and cones, and their work for those varieties has been recently improved by other researchers (see [11, 4, 14, 12, 8, 6, 17, 9, 13]). For spheres, Iosevich and the second listed author [5] obtained nontrivial results which have been improved in the papers [7, 9]. While several new methods have been used in studying the Euclidean extension problem, there are only few known skills to deduce the results on the finite field extension problem. Among other things, the Stein-Tomas argument can be applied in the finite field case to deduce $R^*_V(2 \to r)$ bound. Indeed, Mockenhaupt and Tao [16] introduced the finite field Stein-Tomas argument. In particular, we have the following lemma which is a special case of Lemma 6.1 in [16].

**Lemma 1.2 (The finite field Stein-Tomas argument).** Let $d\sigma$ be the normalized surface measure on an algebraic variety $V$ in $\mathbb{F}_q^d$. Suppose that

\begin{equation}
|V| \sim q^{d-1}
\end{equation}

and

\begin{equation}
\max_{m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}} |(d\sigma)^\vee(m)| \lesssim q^{-\frac{4}{d}}
\end{equation}

for some $\alpha > 0$. Then we have

\begin{equation}
R^*_V \left(2 \to \frac{2(\alpha + 2)}{\alpha} \right) \lesssim 1.
\end{equation}

For a general version of Lemma 1.2 we refer readers to [1]. To apply Lemma 1.2 one needs to compute the maximal Fourier decay bound on the measure $d\sigma$ away from the origin. For example, when $V$ is a sphere or a paraboloid, it is well-known that one can take $\alpha = (d-1)/2$ in (1.3), and thus $R^*_V(2 \to (2d+2)/(d-1)) \lesssim 1$ (see [16, 5]). This result is called as the Stein-Tomas result which gives the optimal $R^*_V(2 \to r)$ bound in general. Now, we pose an interesting question.

**Question 1.3.** Does there exist a variety $V$ such that the condition (1.3) does not hold with $\alpha = (d-1)/2$ but we still have the Stein-Tomas result for $V$?

There exist several varieties $V$ in $\mathbb{F}_q^d$ such that the condition (1.3) does not hold with $\alpha = (d-1)/2$ and the Stein-Tomas result can not be obtained. For example, if $d$ is even and $d\sigma$ is the normalized surface measure on the variety $V := \{x \in \mathbb{F}_q^d : x_1^2 - x_2^2 + \cdots + x_{d-1}^2 - x_d^2 = 0\}$, then

\begin{equation}
\max_{m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}} |(d\sigma)^\vee(m)| \sim q^{-\frac{(d-2)}{2}}
\end{equation}

and the optimal $L^2 \to L^r$ extension estimate for $V$ is that $R^*_V(2 \to 2d/(d-2)) \lesssim 1$, which is much weaker than the Stein-Tomas result (see Theorem 2.1 and Lemma 4.1 in [11]).

Our main purpose of this paper is to provide a concrete variety which gives a positive answer to the above question.

For each $j \in \mathbb{F}_q^*$, the Hamming variety $H_j$ in $\mathbb{F}_q^d$ is defined by

\begin{equation}
H_j = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d : \prod_{k=1}^d x_k = j \right\}.
\end{equation}

Since $j \neq 0$, it is not hard to see $|H_j| = (q-1)^{d-1} \sim q^{d-1}$. Our main result is as follows.
Theorem 1.4. Let $d\sigma_j$ denote the normalized surface measure on the Hamming variety $H_j$ in $\mathbb{F}_q^d$ defined as in (1.4). Then, for every $j \neq 0$ and $d \geq 3$, we have

$$R^*_H(2 \to \frac{2d + 2}{d - 1}) \lesssim 1.$$  

It is not hard to see that the Hamming variety $H_j$ with $j \neq 0$ does not contain any line. Taking $k = 0$ in Conjecture [1,1] we may conjecture that

$$R^*_H(2 \to r) \lesssim 1 \text{ if } \frac{2d}{d - 1} \leq r \leq \infty.$$  

Theorem [1,4] is much weaker than this conjecture, but for $d \geq 4$ it can not be obtained by simply applying Lemma [1,2]. To see this, notice from Corollary [3,3] in Section 3 that

$$\max_{m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}} |(d\sigma_j)^{\vee}(m)| \sim q^{-1} \text{ for } d \geq 4.$$  

Combining this with Lemma [1,2] we only see that $R^*_H(2 \to 4) \lesssim 1$ which is much weaker than Theorem [1,4] for $d \geq 4$. To prove Theorem [1,4] we decompose the surface measure on $H_j$ into $(d + 1)$ surface measures and each of them will be analyzed.

Remark 1.5. In even dimensions, much progress on extension problems for the paraboloid $P$ in $\mathbb{F}_q^d$ has been made by improving the additive energy estimate for subsets of the paraboloid (see, for example, [12, 6, 17]). Recall that for a set $E$ in $\mathbb{F}_q^d$, the additive energy of the set $E$, denoted by $\Lambda(E)$, is defined by

$$\Lambda(E) = \sum_{x, y, z, w \in E, x + y = z + w} 1.$$  

When a set $E$ lies on the Hamming variety $H_j$, it seems that it is a challenging problem to obtain a good upper bound of $\Lambda(E)$.

2. Discrete Fourier analysis

In this section, we review the discrete Fourier analysis which will be our main tool in proving our main result. The proofs of all statements in this section can be found in Been Green’s lecture note [3]. In the finite field setting, the norm nesting properties hold: for $1 \leq p_1 \leq p_2 \leq \infty$,

$$\|g\|_{\ell^2(\mathbb{F}_q^d)} \leq \|g\|_{\ell^1(\mathbb{F}_q^d)}$$  

and

$$\|f\|_{L^{p_1}(\mathbb{F}_q^d)} \leq \|f\|_{L^{p_2}(\mathbb{F}_q^d)}, \quad \|f\|_{L^{p_1}(V, d\sigma)} \leq \|f\|_{L^{p_2}(V, d\sigma)}.$$  

The Plancherel theorem states that

$$\|\hat{g}\|_{L^2(\mathbb{F}_q^d)} = \|g\|_{\ell^2(\mathbb{F}_q^d)} \text{ or } \|f^{\vee}\|_{\ell^2(\mathbb{F}_q^d)} = \|f\|_{L^2(\mathbb{F}_q^d)}$$  

which can be easily deduced by the orthogonality of $\chi$. We also note that $(\hat{f}^{\vee}) = f$. Given functions $g_1, g_2 : \mathbb{F}_q^d \to \mathbb{C}$, the convolution function of $g_1$ and $g_2$, denoted by $g_1 * g_2$, is defined by

$$g_1 * g_2(m) = \sum_{n \in \mathbb{F}_q^d} g_1(m - n)g_2(n).$$  

One can easily check that $\hat{g_1} * \hat{g_2} = \hat{g_1g_2}$. We recall that Young’s inequality for convolutions states that if $1 \leq a, b, r \leq \infty$ satisfy $1/r = 1/a + 1/b - 1$, then

$$\|g_1 * g_2\|_{\ell^r(\mathbb{F}_q^d)} \leq \|g_1\|_{\ell^a(\mathbb{F}_q^d)}\|g_2\|_{\ell^b(\mathbb{F}_q^d)}.$$  

We will invoke the following well-known interpolation theorem.
Theorem 2.1 (Riesz-Thorin). Let $1 \leq p_0, p_1, r_0, r_1 \leq \infty$ with $p_0 \leq p_1$ and $r_0 \leq r_1$. Suppose that $T$ is a linear operator and the following two estimates hold for all functions $g$ on $\mathbb{F}_q^d$:

$$\|Tg\|_{L^p_0(\mathbb{F}_q^d)} \leq M_0\|g\|_{L^p_0(\mathbb{F}_q^d)} \quad \text{and} \quad \|Tg\|_{L^p_1(\mathbb{F}_q^d)} \leq M_1\|g\|_{L^p_1(\mathbb{F}_q^d)}.$$ 

Then we have

$$\|Tg\|_{L^p(\mathbb{F}_q^d)} \leq M_0^{1-\theta}M_1^\theta\|g\|_{L^p(\mathbb{F}_q^d)}$$

for any $0 \leq \theta \leq 1$ with

$$\frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1}{r} \quad \text{and} \quad \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}.$$

3. Fourier decay on Hamming varieties

Recall that $d\sigma_j$ denotes the normalized surface measure on the Hamming variety $H_j$ in $\mathbb{F}_q^d$. In this section, we introduce an explicit form of $(d\sigma_j)^\vee$ which makes a crucial role in proving Theorem 1.4.

Lemma 3.1. For each $j \in \mathbb{F}_q^*$, let $d\sigma_j$ be the normalized surface measure on the Hamming variety $H_j$ in $\mathbb{F}_q^d$. For each $m \in \mathbb{F}_q^d$, denote by $\ell_m$ the number of zero components of $m$. Then we have

$$(d\sigma_j)^\vee(m) = (-1)^{(d-\ell_m)}(q-1)^{-(d-\ell_m)} \quad \text{if} \quad 1 \leq \ell_m \leq d.$$

In addition, if $\ell_m = 0$, then $|(d\sigma_j)^\vee(m)| \leq q^{-\frac{(d-1)}{2}}$.

Proof. Since $j \neq 0$, we see that $|H_j| = (q-1)^{d-1} \sim q^{d-1}$ and all components of any element in the Hamming variety $H_j$ are not zero. By definition, it follows

$$(d\sigma_j)^\vee(m) = \frac{1}{|H_j|} \sum_{x \in H_j} \chi(m \cdot x) = \frac{1}{|H_j|} \sum_{x_1, x_2, \ldots, x_d \in \mathbb{F}_q^*: x_1x_2\cdots x_d = j} \chi(m \cdot x).$$

Case 1. Assume that $\ell_m = d$. Then $m = (0, \ldots, 0)$ and so $(d\sigma)^\vee(m) = 1$.

Case 2. Assume that $\ell_m = d - k$ for some $k = 1, 2, \ldots, (d-1)$. Without loss of generality, we may assume that $m_1 = m_2 = \cdots = m_k = 0$, and $m_i \neq 0$ for $k+1 \leq i \leq d$. It follows that

$$(d\sigma_j)^\vee(m) = \frac{1}{|H_j|} \sum_{x_1, x_2, \ldots, x_d \in \mathbb{F}_q^*} \chi(m_1 x_1 + m_2 x_2 + \cdots + m_d x_d - 1) \chi\left(\frac{jm_d}{x_1 x_2 \cdots x_d - 1}\right)
\quad = \frac{1}{|H_j|} \sum_{x_1, x_2, \ldots, x_d \in \mathbb{F}_q^*} \chi(m_1 x_{k+1} + \cdots + m_d x_d - 1) \chi\left(\frac{jm_d}{x_1 x_2 \cdots x_d - 1}\right),$$

where we assume that if $k = d - 1$, then $\chi(m_1 x_{k+1} + \cdots + m_d x_d - 1) = 1$. Since $j, m_d \neq 0$, we see from the orthogonality of $\chi$ that for each $x_2, \ldots, x_{d-1} \in \mathbb{F}_q^*$,

$$\sum_{x_1 \in \mathbb{F}_q^*} \chi\left(\frac{jm_d}{x_1 x_2 \cdots x_d - 1}\right) = -1.$$

Therefore we have

$$(d\sigma_j)^\vee(m) = -\frac{1}{|H_j|} \sum_{x_1, x_2, \ldots, x_d \in \mathbb{F}_q^*} \chi(m_1 x_{k+1} + \cdots + m_d x_d - 1) \left(\sum_{x_2, \ldots, x_k \in \mathbb{F}_q^*} 1\right).$$

Since $|H_j| = (q-1)^{d-1}$ and $m_{k+1}, \ldots, m_{d-1} \neq 0$, we conclude from the orthogonality of $\chi$ that

$$(d\sigma_j)^\vee(m) = (-1)^{d-k}(q-1)^{-(d-k)} \quad \text{if} \quad 1 \leq \ell_m = k \leq d - 1,$$
which completes the proof in the case when $1 \leq \ell_m \leq (d - 1)$.

**Case 3.** Assume that $\ell_m = 0$. Then all components of $m$ are not zero. As in Case 2, we can write

$$(d\sigma_j)^\vee (m) = \frac{1}{|H_j|} \sum_{x_1, x_2, \ldots, x_{d-1} \in \mathbb{F}_q} \chi(m_1 x_1 + m_2 x_2 + \cdots + m_{{d-1}} x_{{d-1}}) \chi \left( \frac{j m_d}{x_1 x_2 \cdots x_{{d-1}}} \right).$$

Since $|H_j| \sim q^{d-1}$, the last part of the theorem is a direct consequence from the following theorem due to Deligne [2]:

**Theorem 3.2 (Multiple Kloosterman sums).** For $a_1, a_2, \ldots, a_s, b \in \mathbb{F}_q^*$, we have

$$\left| \sum_{x_1, x_2, \ldots, x_s \in \mathbb{F}_q^*} \chi(a_1 x_1 + \cdots + a_s x_s + bx_1^{-1} x_2^{-1} \cdots x_s^{-1}) \right| \leq (s + 1) q^{\frac{3}{2}}.$$

To find further references for Multiple Kloosterman sums, we refer readers to [P.254, 15]. □

The following result follows immediately from Lemma 3.1.

**Corollary 3.3.** For each $j \in \mathbb{F}_q^*$, let $d\sigma_j$ denote the normalized surface measure on $H_j$ in $\mathbb{F}_q^d$. Then we have

$$\max_{m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}} |(d\sigma)^\vee (m)| \sim q^{-1} \quad \text{for} \quad d \geq 3.$$

### 4. Proof of Theorem 1.4

We aim to prove that the extension estimate

$$\|(f d\sigma_j)^\vee\|_{L^2(H_j, d\sigma_j)} \lesssim \|f\|_{L^2(H_j, d\sigma_j)}$$

holds for all complex-valued functions $f$ on $H_j$. By duality, it suffices to prove that the restriction estimate

$$\|\hat{g}\|_{L^2(H_j, d\sigma_j)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}$$

holds for all complex-valued functions $g$ on $\mathbb{F}_q^d$. By the $RR^*$ method (see [3]), we see that

$$\|\hat{g}\|^2_{L^2(H_j, d\sigma_j)} = \langle g, g * (d\sigma_j)^\vee \rangle_{\ell^2(\mathbb{F}_q^d)}.$$

Here, we recall that if $g, h : \mathbb{F}_q^d \to \mathbb{C}$, then

$$\langle g, h \rangle_{\ell^2(\mathbb{F}_q^d)} := \sum_{m \in \mathbb{F}_q^d} g(m) \overline{h(m)}.$$

For each $k = 0, 1, \ldots, d$, define

$$N_k = \{m \in \mathbb{F}_q^d : k \text{ components of } m \text{ are exactly zero}\}.$$

We decompose $(d\sigma_j)^\vee$ as

$$(d\sigma_j)^\vee (m) = ((d\sigma_j)^\vee 1_{N_0}) (m) + \sum_{k=1}^d ((d\sigma_j)^\vee 1_{N_k}) (m).$$
It follows that
\[ \|g\|_{L^2(H_j, d\sigma_j)}^2 = \langle g, g \cdot ((d\sigma_j)^{1/2}N_0) \rangle_{L^2(\mathbb{F}_q^d)} + \left\langle g, g \cdot \sum_{k=1}^d (d\sigma_j)^{1/2}N_k \right\rangle_{L^2(\mathbb{F}_q^d)} \]
\[ = \langle g, g \cdot ((d\sigma_j)^{1/2}N_0) \rangle_{L^2(\mathbb{F}_q^d)} + \sum_{k=1}^d \langle g, g \cdot ((d\sigma_j)^{1/2}N_k) \rangle_{L^2(\mathbb{F}_q^d)} . \]

Hence, to complete the proof, it will be enough to show that the following two inequalities hold for all functions \( g : \mathbb{F}_q^d \to \mathbb{C} \) and for all \( k = 1, 2, \ldots, d \):
\[ \langle g, g \cdot ((d\sigma_j)^{1/2}N_0) \rangle_{L^2(\mathbb{F}_q^d)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}^{2d} \]
\[ \text{and} \]
\[ \langle g, g \cdot ((d\sigma_j)^{1/2}N_k) \rangle_{L^2(\mathbb{F}_q^d)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}^{d} . \]

In the following subsections, we will give the proofs of inequalities (4.1) and (4.2), which completes the proof of Theorem 1.4.

### 4.1. Proof of inequality (4.1)

By Hölder’s inequality, we have
\[ \langle g, g \cdot ((d\sigma_j)^{1/2}N_0) \rangle_{L^2(\mathbb{F}_q^d)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}^{2d} \|g \cdot ((d\sigma_j)^{1/2}N_0)\|_{L^2(\mathbb{F}_q^d)} . \]

Thus, in order to prove the inequality (4.1), it is enough to prove the following:
\[ \|g \cdot ((d\sigma_j)^{1/2}N_0)\|_{L^2(\mathbb{F}_q^d)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}^{d-1} \|g\|_{L^2(\mathbb{F}_q^d)} . \]

By the Riesz-Thorin interpolation theorem (Theorem 2.1), it suffices to prove the following two inequalities:
\[ \|g \cdot ((d\sigma_j)^{1/2}N_0)\|_{L^\infty(\mathbb{F}_q^d)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}^{d-1} \]
\[ \text{and} \]
\[ \|g \cdot ((d\sigma_j)^{1/2}N_0)\|_{L^2(\mathbb{F}_q^d)} \lesssim \|g\|_{L^2(\mathbb{F}_q^d)}^{d} . \]

For the first inequality (4.3), using the Plancherel theorem gives us the following estimate:
\[ \|g \cdot ((d\sigma_j)^{1/2}N_0)\|_{L^2(\mathbb{F}_q^d)} = \|\hat{g} \cdot ((d\sigma_j)^{1/2}N_0)\|_{L^2(\mathbb{F}_q^d)} \]
\[ \leq \max_{x \in \mathbb{F}_q^d} \left| ((d\sigma_j)^{1/2}N_0)(x) \right| \left\|g\right\|_{L^2(\mathbb{F}_q^d)} . \]

Now, for each fixed \( x \in \mathbb{F}_q^d \), we have
\[ \left| ((d\sigma_j)^{1/2}N_0)(x) \right| = \sum_{m \in N_0} \chi(-m \cdot x) \frac{1}{H_j} \sum_{y \in H_j} \chi(m \cdot y) = \frac{1}{|H_j|} \sum_{y \in H_j} \sum_{m \in N_0} \chi(m \cdot (y - x)) . \]

Using a change of variable by letting \( z = y - x \), and the triangle inequality,
\[ \left| ((d\sigma_j)^{1/2}N_0)(x) \right| \leq \frac{1}{|H_j|} \sum_{z \in H_j - x} \left| \sum_{m \in N_0} \chi(m \cdot z) \right| \lesssim \frac{1}{|H_j|} \sum_{z \in \mathbb{F}_q^d} \left| \sum_{m \in N_0} \chi(m \cdot z) \right| . \]
We decompose $\sum_{z \in \mathbb{F}_q^d}$ into $\sum_{k=0}^d \sum_{z \in \mathbb{N}_k}$, and use the orthogonality of $\chi$. Then we obtain

$$\left| \left( (d \sigma_j)^{\mathcal{V}} 1_{N_0} \right)(x) \right| \lesssim \frac{1}{q^{d-1}} \sum_{k=0}^d \sum_{m \in \mathbb{N}_k} \sum_{m_1, \ldots, m_d \in \mathbb{F}_q^d} \chi(m \cdot z)$$

$$= \frac{1}{q^{d-1}} \sum_{k=0}^d (q-1)^k |N_k| \lesssim q,$$

where we also used the fact that $|N_k| = \binom{d}{k} q^{d-k} \sim q^{d-k}$. Thus the inequality (4.3) holds.

The second inequality (4.4) follows by using Young’s inequality for convolutions and the second part of Lemma 3.1. More precisely, we have

$$\|g * ((d \sigma_j)^{\mathcal{V}} 1_{N_0})\|_\ell^\infty(\mathbb{F}_q^d) \leq \|((d \sigma_j)^{\mathcal{V}} 1_{N_0})\|_\ell^\infty(\mathbb{F}_q^d) \|g\|_{\ell^1(\mathbb{F}_q^d)} \lesssim q^{-\frac{d-1}{2}} \|g\|_{\ell^1(\mathbb{F}_q^d)}.$$

Hence, the proof of the inequality (4.4) is complete.

4.2. Proof of inequality (4.2). We will prove much better inequality than the inequality (4.2). Notice from the norm nesting property (2.1) that

$$\|g\|_{\ell^2(\mathbb{F}_q^d)} \leq \|g\|_{\ell^2(\mathbb{F}_q^d)}.$$ 

To complete the proof of the inequality (4.2), it will be enough to show that for each $k = 1, 2, \ldots, d$,

$$\langle g, g * ((d \sigma_j)^{\mathcal{V}} 1_{N_k}) \rangle_{\ell^2(\mathbb{F}_q^d)} \lesssim \|g\|_{\ell^2(\mathbb{F}_q^d)}^2 \|g * ((d \sigma_j)^{\mathcal{V}} 1_{N_k})\|_{\ell^2(\mathbb{F}_q^d)}.$$

By Hölder’s inequality, we have

$$\langle g, g * ((d \sigma_j)^{\mathcal{V}} 1_{N_k}) \rangle_{\ell^2(\mathbb{F}_q^d)} \leq \|g\|_{\ell^2(\mathbb{F}_q^d)} \|g * ((d \sigma_j)^{\mathcal{V}} 1_{N_k})\|_{\ell^2(\mathbb{F}_q^d)}.$$

As seen in (4.5), it suffices by the Plancherel theorem to prove that

$$\left[ \max_{x \in \mathbb{F}_q^d} \left( (d \sigma_j)^{\mathcal{V}} 1_{N_k}(x) \right) \right] \lesssim 1.$$

Fix $x \in \mathbb{F}_q^d$ and $k \in \{1, 2, \ldots, d\}$. From (3.1) of Lemma 3.1, we see that for each $m \in \mathbb{F}_q^d$,

$$(d \sigma_j)^{\mathcal{V}} 1_{N_k}(m) = (-1)^{d-k} (q-1)^{-d+k} \chi(m \cdot x).$$

Therefore, we have

$$\left| ((d \sigma_j)^{\mathcal{V}} 1_{N_k}(x) \right| = (q-1)^{-d+k} \left| \hat{N_k}(x) \right|.$$

Since $|\hat{N_k}(x)| = |\sum_{m \in \mathbb{N}_k} \chi(-m \cdot x)| \leq |N_k| = \binom{d}{k} q^{d-k} \sim q^{d-k}$, we conclude that

$$\left| ((d \sigma_j)^{\mathcal{V}} 1_{N_k}(x) \right| \lesssim 1.$$

Thus, the inequality (4.7) holds, which completes the proof of the inequality (4.2).

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