On the Hardy-Sobolev-Maz’ya inequality and its generalizations

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Abstract

The paper deals with natural generalizations of the Hardy-Sobolev-Maz’ya inequality and some related questions, such as the optimality and stability of such inequalities, the existence of minimizers of the associated variational problem, and the natural energy space associated with the given functional.

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1 Introduction

The term “inequalities of Hardy-Sobolev type” refers, somewhat vaguely, to families of inequalities that in some way interpolate the Hardy inequality

\[ \int_{\Omega} |\nabla u(x)|^p \, dx \geq C(N, p, K, \Omega) \int_{\Omega} \frac{|u(x)|^p}{\text{dist}(x, K)^p} \, dx \quad u \in C_0^\infty(\Omega \setminus K), \quad (1.1) \]
where $\Omega \subset \mathbb{R}^N$ is an open domain and $K \subset \bar{\Omega}$ is a nonempty closed set, and the Sobolev inequality

$$
\int_{\Omega} |\nabla u(x)|^p \, dx \geq C \left( \int_{\Omega} |u(x)|^{p^*} \, dx \right)^{p/p^*} \quad u \in C^\infty_0(\Omega),
$$

(1.2)

where $C > 0$, $1 < p < N$, and $p^* \overset{\text{def}}{=} pN/(N-p)$ is the corresponding Sobolev exponent. Throughout the paper we repeatedly consider the following particular case.

**Example 1.1.** Let $\Omega = \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, where $1 \leq m < N$, and let $K = \mathbb{R}^n \times \{0\}$. We denote the variables of $\mathbb{R}^n$ and $\mathbb{R}^m$ as $z$ and $y$ respectively, and set $\mathbb{R}^N_0 \overset{\text{def}}{=} \mathbb{R}^n \times (\mathbb{R}^m \setminus \{0\})$. It is well known that the Hardy inequality (1.1) holds with the best constant

$$
C(N,p,\mathbb{R}^n \times \{0\},\mathbb{R}^N) = \frac{|m-p|}{p}.
$$

(1.3)

An elementary family of Hardy-Sobolev inequalities can be obtained by Hölder interpolation between the Hardy and the Sobolev inequalities. More significant inequalities of Hardy-Sobolev type with the best constant in the Hardy term can be derived as consequences of Caffarelli-Kohn-Nirenberg inequality ([7, 15]) that provides estimates in terms of the weighted gradient norm $\int |\xi|^\alpha |\nabla u|^p d\xi$. The substitution $u = |y|^\beta v$ into the Caffarelli-Kohn-Nirenberg inequality can be used to produce inequalities that combine terms with the critical exponent and with the Hardy potential. Such inequalities are known as Hardy-Sobolev-Maz’ya (or HSM for brevity) inequalities. In particular, in [18, Section 2.1.6, Corollary 3] Maz’ya proved the HSM inequality

$$
\int_{\mathbb{R}^N_0} |\nabla u|^2 \, dy \, dz - \left( \frac{m-2}{2} \right)^2 \int_{\mathbb{R}^N_0} \frac{|u|^2}{|y|^2} \, dy \, dz \geq C \left( \int_{\mathbb{R}^N_0} |u|^{2^*} \, dy \, dz \right)^{2/2^*} \quad u \in C^\infty_0(\mathbb{R}^N_0),
$$

(1.4)

where $C > 0$, $N > 2$, and $1 \leq m < N$. This HSM inequality is false for $m = N$ and reduces to the Sobolev inequality for $m = 2$. Since the left-hand
side of (1.4) induces a Hilbert norm, the inequality holds on $D^{1,2}(\mathbb{R}^N_0)$, the completion of $C^\infty_0(\mathbb{R}^N_0)$ in the gradient norm, which coincides with $D^{1,2}(\mathbb{R}^N)$ for all $m > 1$, in particular, $C^\infty_0(\mathbb{R}^N_0)$ may be replaced by $C^\infty_0(\mathbb{R}^N)$ unless $m = 1$.

A joint paper of Filippas, Maz’ya and Tertikas [10] gives the following generalization of the HSM inequality (1.4).

**Example 1.2.** Let $2 \leq p < N$, $p \neq m < N$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $K$ be a compact $C^2$-manifold without boundary embedded in $\mathbb{R}^N$, of codimension $m$ such that $K \Subset \Omega$ for $1 < m < N$ (i.e., $K$ is compact in $\Omega$), or $K = \partial \Omega$ for $m = 1$. Assume further that

$$-\Delta_p \left[ \text{dist} (\cdot, K)^{(p-m)/(p-1)} \right] \geq 0 \quad \text{in } \Omega \setminus K, \quad (1.5)$$

where $\Delta_p(u) \overset{\text{def}}{=} \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian. Then for all $u \in C_0^\infty(\Omega \setminus K)$ we have

$$\int_\Omega |\nabla u(x)|^p \, dx - \frac{m-p}{p} \int_\Omega \frac{|u(x)|^p}{\text{dist} (x, K)^p} \, dx \geq C \left( \int_\Omega |u(x)|^{p^*} \, dx \right)^{p/p^*}. \quad (1.6)$$

For $N = 3$ Benguria, Frank and Loss [4] have shown recently that the best constant $C$ in (1.4) is the Sobolev constant $S_3$. Mancini and Sandeep [16] have studied the analog of HSM on the hyperbolic space and its close connection to the original HSM inequality.

In the present paper we consider a nonnegative functional $Q$ of the form

$$Q(u) \overset{\text{def}}{=} \int_\Omega (|\nabla u|^p + V|u|^p) \, dx \quad u \in C_0^\infty(\Omega), \quad (1.7)$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $V \in L^\infty_{\text{loc}}(\Omega)$, and $1 < p < \infty$. We study several questions related to extensions of inequalities (1.4) and (1.6). In Section 2, we deal with generalizations of these HSM inequalities for the functional $Q$. It turns out, that in the subcritical case a weighted HSM inequality holds true, where the weight appears in the Sobolev term. In the critical case, one needs to add a Poincaré-type term (a one-dimensional $p$-homogeneous functional), and we call it Hardy-Sobolev-Maz’ya-Poincaré (or HSMP for brevity) inequality. We show that under “small” perturbations such HSM-type inequalities are preserved (with the original Sobolev weight). We also
address the question concerning the optimal weight in the generalized HSM inequality.

In Section 3 we study a natural energy space $\mathcal{D}^{1,2}_V(\Omega)$ for nonnegative singular Schrödinger operators, and discuss the existence of minimizers for the HSM inequality in this space, that is, minimizers of the equivalent Caffarelli-Kohn-Nirenberg inequality. Finally, in Section 4 we prove that a related functional $\hat{Q}$ which satisfies $C^{-1}Q \leq \hat{Q} \leq CQ$ for some $C > 0$ induces a norm on the cone of nonnegative $C^0_\infty(\Omega)$-functions. For $p = 2$, this norm coincides (on the above cone) with the $\mathcal{D}^{1,2}_V(\Omega)$-norm defined in [20]. It is our hope that this approach paves the way to circumvent the general lack of convexity of the nonnegative functional $Q$ for $p \neq 2$.

## 2 Generalization of HSM inequality

We need the following definition.

**Definition 2.1.** Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $V \in L^\infty_{\text{loc}}(\Omega)$, and $1 < p < \infty$. Assume that the functional

$$Q(u) = \int_\Omega (|\nabla u|^p + V |u|^p) \, dx$$

is nonnegative on $C^\infty_0(\Omega)$. A function $\varphi \in C^1(\Omega)$ is a *ground state* for the functional $Q$ if $\varphi$ is an $L^p_{\text{loc}}$-limit of a nonnegative sequence $\{\varphi_k\} \subset C^\infty_0(\Omega)$ satisfying

$$Q(\varphi_k) \to 0, \quad \text{and} \quad \int_B |\varphi_k|^p \, dx = 1,$$

for some fixed $B \Subset \Omega$ (such a sequence $\{\varphi_k\}$ is called a *null sequence*). The functional (1.7) is called *critical* if $Q$ admits a ground state and *subcritical* or *weakly coercive* if it does not.

The following statement (see [22]) is a generalization of HSM inequality. Inequality (2.4) might be called Hardy-Sobolev-Maz’ya-Poincaré (HSMP)-type inequality.

**Theorem 2.2.** Let $Q$ be a nonnegative functional on $C^\infty_0(\Omega)$ of the form (1.7), and let $1 < p < N$. 
(i) The functional $Q$ does not admit a ground state if and only if there exists a positive continuous function $W$ such that

$$Q(u) \geq \left( \int_{\Omega} W|u|^p \, dx \right)^{p/p^*} \quad u \in C_0^\infty(\Omega). \quad (2.2)$$

(ii) If $Q$ admits a ground state $\varphi$, then $\varphi$ is the unique global positive (super)-solution of the Euler-Lagrange equation

$$Q'(u) \overset{\text{def}}{=} -\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega. \quad (2.3)$$

Moreover, there exists a positive continuous function $W$ such that for every function $\psi \in C_0^\infty(\Omega)$ with $\int_{\Omega} \psi \varphi \, dx \neq 0$, the following inequality holds

$$Q(u) + C \left| \int_{\Omega} \psi u \, dx \right|^p \geq \left( \int_{\Omega} W|u|^{p^*} \, dx \right)^{p/p^*} \quad u \in C_0^\infty(\Omega) \quad (2.4)$$

with some suitable constant $C > 0$.

**Remark 2.3.** For the relationships between the criticality of $Q$ in $\Omega$ and the $p$-capacity (with respect to the functional $Q$) of closed balls see [22, Theorem 4.5] and [26, 27].

Theorem 2.2 applies to the case of $\Omega = \mathbb{R}_0^N$ and the Hardy potential (see Example 1.1 and in particular (1.4)), but it does not specify that the weight $W$ in the Sobolev term is the constant function. We note that Example 1.2 provides another Hardy-type functional satisfying the HSM inequality with the weight $W = \text{constant}$.

On the other hand, let $\Omega = \mathbb{R}_0^N$, with $m = N$, then the corresponding Hardy functional admits a ground state $\varphi(x) = |x|^{(p-N)/p}$, and therefore the HSM inequality does not hold with any weight. Moreover, the HSMP inequality (2.4), which by Theorem 2.2 holds with some weight $W$, is false with the weight $W = \text{constant}$ ([11] and Example 2.5).

Let us present few other examples which illustrate further the question of the admissible weights in the HSM and HSMP inequalities. The first two examples are elementary but general. In the first one the HSM inequality (2.2) holds with the constant weight function, while in the second example (Example 2.5) such an inequality is false.
Example 2.4. Consider a nonnegative functional $Q$ of the form (1.7), where $V \in L^\infty_{\text{loc}}(\Omega)$ is nonzero function, and $1 < p < N$. For $\lambda \in \mathbb{R}$ we denote

$$Q_\lambda(u) \overset{\text{def}}{=} \int_\Omega (|\nabla u|^p + \lambda V |u|^p) \, dx.$$  

Then for every $\lambda \in (0, 1)$ there exists $C > 0$ such that

$$Q_\lambda(u) \geq C \|u\|_{p^*}^p, \quad u \in C_0^\infty(\Omega),$$

where $C = C(N, p, \lambda) > 0$. This HSM inequality follows from

$$Q_\lambda(u) = (1 - \lambda) \int_\Omega |\nabla u|^p \, dx + \lambda Q(u) \geq (1 - \lambda) \int_\Omega |\nabla u|^p \, dx,$$

and the Sobolev inequality.

Example 2.5. Let $Q \geq 0$ be as in (1.7), where $1 < p < N$. Suppose that $Q$ admits ground state $\varphi \notin L^{p^*}(\Omega)$, and let $\{\varphi_k\}$ be a null sequence (see Definition 2.1) such that $\varphi_k \rightharpoonup \varphi$ locally uniformly in $\Omega$ (for the existence of a locally uniform convergence null sequence, see [22, Theorem 4.2]). Let $V_1 \in L^\infty(\Omega)$ be a nonzero nonnegative function with a compact support. Then

$$Q(\varphi_k) + \int_\Omega V_1 |\varphi_k|^p \, dx \to \int_\Omega V_1 |\varphi|^p \, dx < \infty,$$

while Fatou’s lemma implies that $\|\varphi_k\|_{p^*} \to \infty$. Therefore, the subcritical functional

$$Q_{V_1}(u) \overset{\text{def}}{=} Q(u) + \int_\Omega V_1 |u|^p \, dx$$

does not satisfy the HSM inequality (2.2) with the constant weight. Similar argument shows that the critical functional $Q$ does not satisfy the HSMP inequality with the constant weight.

Remark 2.6. Example 2.5 can be slightly generalized by replacing the assumption $\varphi \notin L^{p^*}(\Omega)$ with $\varphi \notin L^{p^*}(\Omega, W \, dx)$, where $W$ is a continuous positive weight function. Under this assumption it follows that the functional $Q_{V_1}$ and $Q$ do not satisfy HSM and respectively HSMP inequality with the weight $W$. 

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Example 2.7. In [12, Theorem C], Filippas, Tertikas and Tidblom proved that a nonnegative functional $Q$ of the form (1.7) with $p = 2$ satisfies the HSM inequality in a smooth domain $\Omega$ with $W = \text{constant}$ if the equation $Q'(u) = 0$ has a positive $C^2$-solution $\varphi$ such that the following $L^1$-Hardy-type inequality

$$
\int_{\Omega} \varphi^{2(N-1)/(N-2)} |\nabla u| \, dx \geq C \int_{\Omega} \varphi^{N/(N-2)} |\nabla \varphi| \, |u| \, dx \quad u \in C_0^\infty (\Omega).
$$

holds true.

Example 2.8. Consider the function

$$X(r) \overset{\text{def}}{=} (\log |r|)^{-1} \quad r > 0.$$

Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be a bounded domain and let $D > \sup_{x \in \Omega} |x|$. The following inequality is due to Filippas and Tertikas [11, Theorem A, and the corresponding Corrigendum], see also [1].

$$
\int_{\Omega} |\nabla u|^2 \, dx - \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx \geq
C \left( \int_{\Omega} |u|^{2^*} X(|x|/D)^{1+N/(N-2)} \, dx \right)^{2/2^*} \quad u \in C_0^\infty (\Omega). \quad (2.6)
$$

In this case the HSM inequality does not hold with $W = \text{constant}$ (cf. Example 2.5 and Remark 2.6).

We now consider the question whether the weight $W$ in the HSM inequality (2.2) is preserved (up to a constant multiple) under small perturbations.

**Theorem 2.9.** Let $\Omega$ be a domain in $\mathbb{R}^N$, $N > 2$, and let $V \in L_{\text{loc}}^\infty (\Omega)$. Assume that the following functional $Q$ satisfies the HSM inequality

$$
Q(u) \overset{\text{def}}{=} \int_{\Omega} (|\nabla u|^2 + V|u|^2) \, dx \geq \left( \int_{\Omega} W|u|^{2^*} \, dx \right)^{2/2^*} \quad u \in C_0^\infty (\Omega)
$$

with some positive continuous function $W$. Let $\tilde{V} \in L_{\text{loc}}^\infty (\Omega)$ be a nonzero potential satisfying

$$
|\tilde{V}|^{N/2} W^{2-N)/2} \in L^1(\Omega), \quad (2.8)
$$

with some positive continuous function $W$. Let $\tilde{V} \in L_{\text{loc}}^\infty (\Omega)$ be a nonzero potential satisfying

$$
|\tilde{V}|^{N/2} W^{2-N)/2} \in L^1(\Omega).
$$
and consider the one-parameter family of functionals $\tilde{Q}_\lambda$ defined by

$$\tilde{Q}_\lambda(u) \overset{\text{def}}{=} Q(u) + \lambda \int_\Omega \tilde{V}|u|^2 \, dx,$$

where $\lambda \in \mathbb{R}$.

(i) If $\tilde{Q}_\lambda$ is nonnegative on $C^\infty_0(\Omega)$ and does not admit a ground state, then

$$\tilde{Q}_\lambda(u) \geq C \left( \int_\Omega W|u|^{2^*} \, dx \right)^{2/2^*} \quad u \in C^\infty_0(\Omega), \quad (2.9)$$

where $C$ is a positive constant.

(ii) If $\tilde{Q}_\lambda$ is nonnegative on $C^\infty_0(\Omega)$ and admits a ground state $v$, then for every $\psi \in C^\infty_0(\Omega)$ such that $\int_\Omega \psi v \, dx \neq 0$ we have

$$\tilde{Q}_\lambda(u) + C_1 \left( \int_\Omega \psi u \, dx \right)^2 \geq C \left( \int_\Omega W|u|^{2^*} \, dx \right)^{2/2^*} \quad u \in C^\infty_0(\Omega) \quad (2.10)$$

with suitable positive constants $C, C_1 > 0$.

(iii) The set

$$S \overset{\text{def}}{=} \{ \lambda \in \mathbb{R} \mid \tilde{Q}_\lambda \geq 0 \text{ on } C^\infty_0(\Omega) \}$$

is a closed interval with a nonempty interior which is bounded if and only if $\tilde{V}$ changes its sign on a set of a positive measure in $\Omega$. Moreover, $\lambda \in \partial S$ if and only if $\tilde{Q}_\lambda$ is critical in $\Omega$.

Proof. (i)–(ii) Let $D^{1,2}_{\lambda \tilde{V}}(\Omega)$ denote the completion of $C^\infty_0(\Omega)$ with respect to the norm defined by the square root of the left-hand side of (2.9) if $\tilde{Q}_\lambda$ does not admit a ground state, and by the square root of the left-hand side of (2.10) if $\tilde{Q}_\lambda$ admits a ground state (see [20]). Similarly, we denote by $D^{1,2}_V(\Omega)$ the completion of $C^\infty_0(\Omega)$ with respect to the norm defined by the square root of the left-hand side of (2.7). We denote the norms on $D^{1,2}_{\lambda \tilde{V}}(\Omega)$ and $D^{1,2}_V(\Omega)$ by $\| \cdot \|_{D^{1,2}_{\lambda \tilde{V}}}$ and $\| \cdot \|_{D^{1,2}_V}$ respectively.

Assume that (2.9) (respect. (2.10)) does not hold. Then there exists a sequence $\{u_k\} \subset C^\infty_0(\Omega)$ such that

$$\|u_k\|_{D^{1,2}_{\lambda \tilde{V}}} \to 0, \quad \text{and} \quad \int_\Omega W|u_k|^{2^*} \, dx = 1. \quad (2.11)$$

By [20, Proposition 3.1], the space $D^{1,2}_{\lambda \tilde{V}}(\Omega)$ is continuously imbedded into $W^{1,2}_{\text{loc}}(\Omega)$ and therefore, $u_k \to 0$ in $W^{1,2}_{\text{loc}}(\Omega)$. Consequently, for any $K \subseteq \Omega$ we
have
\[
\lim_{k \to \infty} \int_K |\tilde{V}| |u_k|^2 \, dx = 0. \tag{2.12}
\]

On the other hand, (2.8) and Hölder inequality imply that for any \( \varepsilon > 0 \) there exists \( K_\varepsilon \subset \Omega \) such that
\[
\left| \int_{\Omega \setminus K_\varepsilon} |\tilde{V}| |u_k|^2 \, dx \right| \leq \left( \int_{\Omega \setminus K_\varepsilon} |\tilde{V}|^{N/2} W^{(2-N)/2} \, dx \right)^{2/N} \left( \int_{\Omega} W |u_k|^{2^*} \, dx \right)^{2/2^*} < \varepsilon.
\]

(2.13)

Since
\[
|u_k|_{D_1,2} \leq |u_k|_{D_1,2} + \left| \int_{\Omega} \lambda \tilde{V} |u_k|^2 \, dx \right|^{1/2},
\]

it follows from (2.11)–(2.13) that the sequence \( u_k \to 0 \) in \( D_1,2^* (\Omega) \). Therefore, (2.7) implies that \( \int_{\Omega} W |u_k|^{2^*} \, dx \to 0 \) which contradicts the assumption \( \int_{\Omega} W |u_k|^{2^*} \, dx = 1 \). Consequently, (2.9) (resp. (2.10)) holds true.

(iii) It follows from [21, Proposition 4.3] that \( S \) is an interval, and that \( \lambda \in \text{int} \, S \) implies that \( Q_\lambda \) is subcritical in \( \Omega \). The claim on the boundedness of \( S \) is trivial and left to the reader.

On the other hand, suppose that for some \( \lambda \in \mathbb{R} \) the functional \( \tilde{Q}_\lambda \) is subcritical. By part (i), \( \tilde{Q}_\lambda \) satisfies the HSM inequality with weight \( W \). Therefore, (2.13) (with \( K_\varepsilon = \emptyset \)) implies that
\[
\tilde{Q}_\lambda (u) \geq C \left( \int_{\Omega} W |u|^{2^*} \, dx \right)^{2/2^*} \geq C_1 \left| \int_{\Omega} \tilde{V} |u|^2 \, dx \right| \quad u \in C_0^\infty (\Omega). \tag{2.14}
\]

Therefore, \( \lambda \in \text{int} \, S \). Consequently, \( \lambda \in \partial S \) implies that \( \tilde{Q}_\lambda \) is critical in \( \Omega \). In particular, \( 0 \in \text{int} \, S \).

\[\square\]

**Example 2.10.** Let \( \Omega = \mathbb{R}^N \), where \( N \geq 3 \), and let \( V \in L^{N/2} (\mathbb{R}^N) \) such that \( V \not\equiv 0 \) (so, \( V \) is a short range potential). Fix \( \mu < (N-2)^2/4 \). Then the classical Hardy inequality together with Example 2.4 and Theorem 2.9 imply that there exists \( \lambda^* > 0 \) such that for \( \lambda < \lambda^* \), we have the following HSM inequality
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx + \lambda \int_{\mathbb{R}^N} V(x) |u|^2 \, dx \geq C_\lambda \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{2/2^*} \quad u \in C_0^\infty (\mathbb{R}^N). \tag{2.15}
\]
On the other hand, if $\lambda = \lambda^*$, then the associated functional is critical and satisfies the corresponding HSMP inequality with the weight function $W = \text{constant}$. Recall that the HSM and HSMP inequalities for $\mu = (N - 2)^2/4$ are false with the weight $W = \text{constant}$ (see, Example 2.5 and [11]).

**Example 2.11.** Consider again Example 1.2 with $p = 2 < N$, and $2 \neq m < N$. By [10, Theorem 1.1], there exists $M \leq 0$ such that the following HSM inequality holds true

$$Q(u) \overset{\text{def}}{=} \int_\Omega |\nabla u|^2 \, dx - \left(\frac{m-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{\text{dist} (x,K)^2} \, dx - M \int_\Omega |u|^2 \, dx \geq C \left(\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*} u \in C_0^\infty(\Omega \setminus K). \quad (2.16)$$

We note that if (1.5) is satisfied, then (2.16) holds with $M = 0$.

Let $V \in L^1_{\text{loc}}(\Omega) \cap L^{N/2}(\Omega)$ be a nonzero function, and consider the one-parameter family of functionals $Q_\lambda$ defined by

$$Q_\lambda(u) \overset{\text{def}}{=} Q(u) + \lambda \int_\Omega V|u|^2 \, dx,$$

where $\lambda \in \mathbb{R}$. By Theorem 2.9, the set $S$ of all $\lambda$ such that $Q_\lambda$ is nonnegative on $C_0^\infty(\Omega)$ is a nonempty closed interval with a nonempty interior. Moreover, for $\lambda \in \text{int } S$ there exists a positive constant $c_\lambda$ such that

$$Q_\lambda(u) \geq c_\lambda \left(\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*} u \in C_0^\infty(\Omega \setminus K). \quad (2.17)$$

On the other hand, if $\lambda \in \partial S$, then $Q_\lambda$ admits a ground state $v$. Therefore, Theorem 2.9 implies that for every $\psi \in C_0^\infty(\Omega \setminus K)$ satisfying $\int_\Omega \psi v \, dx \neq 0$ there exist constants $C, C_1 > 0$ such that

$$Q_\lambda(u) + C \left(\int_\Omega \psi^2 \, dx\right)^2 \geq C_1 \left(\int_\Omega |u|^{2^*} \, dx\right)^{2/2^*} u \in C_0^\infty(\Omega \setminus K).$$

We note that if $K = \partial \Omega$ is smooth (that is, $m = 1$) and $V = 1$, one actually deals with the case considered by Brezis and Marcus in [5, Theorem 1.1]. In particular, let $\lambda^*$ be the supremum of all $\lambda \in \mathbb{R}$ such that the inequality

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{|u|^2}{\text{dist} (x,\partial \Omega)^2} \, dx - \lambda \int_\Omega |u|^2 \, dx \geq 0 \quad u \in C_0^\infty(\Omega) \quad (2.18)$$

\text{10}
holds true ($\lambda^* > -\infty$ and is attained by [5, Theorem 1.1]). Then Theorem 2.9 implies that for each $\lambda < \lambda^*$ there exists $C_\lambda > 0$ such that

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{|u|^2}{\text{dist} (x, \partial \Omega)^2} \, dx - \lambda \int_\Omega |u|^2 \, dx \geq C_\lambda \left( \int_\Omega |u|^2 \, dx \right)^{2/2^*} \quad u \in C_0^\infty (\Omega). \quad (2.19)$$

Moreover, Theorem 2.9 implies that for $\lambda = \lambda^*$, the functional defined by the left-hand side of (2.19) is critical, and satisfies the HSMP inequality with weight $W = \text{constant}$. In particular, the corresponding Euler-Lagrange equation $Q_{\lambda^*}(u) = 0$ in $\Omega$ admits a unique positive (super)-solution.

Theorem 1.1 of [5] has been extended by Marcus and Shafrir in [17, Theorem 1.2] to the case $1 < p < \infty$ and a perturbation $0 < V(x) = O(\text{dist} (x, \partial \Omega) \gamma)$, where $\gamma > -p$ (cf. our assumption (2.8), where $p = 2$). Following [17], let $\lambda^*$ be the supremum of all $\lambda \in \mathbb{R}$ such that the inequality

$$\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{|u|^2}{\text{dist} (x, \partial \Omega)^2} \, dx - \lambda \int_\Omega V(x)|u|^2 \, dx \geq 0 \quad u \in C_0^\infty (\Omega). \quad (2.20)$$

holds true. It follows that Theorem 2.9 with the constant weight applies also to this functional if in addition $V \in L^\infty_{\text{loc}} (\Omega) \cap L^{N/2} (\Omega)$.

**Remark 2.12.** We note that even under the less restricted assumptions of [17, Theorem 1.2], with $p = 2$ and $\lambda = \lambda^*$, one can show that the positive solution $u_*$ of Equation (1.14) in [17] is actually a ground state. Therefore, $u_*$ is the unique (up to a multiplicative constant) global positive supersolution of that equation, and the corresponding functional is critical.

Indeed, Lemma 5.1 of [17] implies that any positive supersolution of [17, Equation (1.14)] satisfies

$$C u(x) \geq \text{dist} (x, \partial \Omega)^{1/2} \quad x \in \Omega. \quad (2.21)$$

On the other hand, [17, Theorem 1.2] implies that the positive solution $u_*$ satisfies

$$u_*(x) \approx \text{dist} (x, \partial \Omega)^{1/2} \quad x \in \Omega, \quad (2.22)$$

where $f \approx g$ means that there exists a positive constant $C$ such that $C^{-1} \leq f/g \leq C$ in $\Omega$. Now, take a positive supersolution $u$, and let $\varepsilon$ be the maximal
positive number such that \( u - \varepsilon u^* \geq 0 \) in \( \Omega \). Note that by (2.21) and (2.22), \( \varepsilon \) is well defined. By the strong maximum principle it follows that either \( u = \varepsilon u^* \), or \( u - \varepsilon u^* > 0 \). Consequently, (2.21) and (2.22) imply that there exists a positive constant \( C_1 \) such that

\[
    u - \varepsilon u^* \geq C \operatorname{dist}(x, \partial \Omega)^{1/2} \geq C_1 u^* \quad \text{in } \Omega,
\]

which is a contradiction to the definition of \( \varepsilon \).

3 The space \( D^{1,2}_V(\Omega) \) and minimizers for the HSM inequality

Consider again the HSM inequality (1.4). This inequality clearly extends to \( D^{1,2}(\mathbb{R}^N) \) for \( m > 2 \) and to \( D^{1,2}(\mathbb{R}^N_0) \) for \( m = 1 \), but since the quadratic form \( Q(u) \) in the left-hand side of (1.4) induces a scalar product on \( C_0^{\infty}(\mathbb{R}^N_0) \), the natural domain of \( Q \) is the completion of \( C_0^{\infty}(\mathbb{R}^N) \) with respect to the norm \( Q(\cdot)^{1/2} \). Recall [20] that given a general subcritical functional \( Q \) of the form (1.7) (with \( p = 2 \)), we denote such a completion by \( D^{1,2}_V(\Omega) \). Similarly to the standard definition of \( D^{1,2}(\mathbb{R}^N) \) for \( N = 1, 2 \), when \( Q \) admits a ground state, one appends to \( Q(u) \) a correction term of the form \( \int \psi u^2 \, dx \). Hence, by (2.2) and (2.4) the space \( D^{1,2}_V(\Omega) \) is continuously imbedded into a weighted \( L^2 \)-space.

In the particular case (1.4), \( V \) is the Hardy potential \( [(m - 2)/2]|y|^{-2} \). By (1.4), the space \( D^{1,2}_V(\mathbb{R}^N_0) \) is continuously imbedded into \( L^{2^*}(\mathbb{R}^N_0) \), thus its elements can be identified as measurable functions. The substitution \( u = |y|^{(2-m)/2}v \) transforms HSM inequality (1.4) into an inequality of Caffarelli-Kohn-Nirenberg type:

\[
    \int_{\mathbb{R}^N} |y|^{2-m}|
abla v|^2 \, dy \, dz \\
    \geq C \left( \int_{\mathbb{R}^N} |y|^{2-m}2^{2^*} |v|^{2^*} \, dy \, dz \right)^{2/2^*} \quad v \in D^{1,2}(\mathbb{R}^N_0 ; |y|^{2-m} \, dy \, dz). \quad (3.1)
\]

The left-hand side of (3.1) defines a Hilbert space isometric to \( D^{1,2}_V(\mathbb{R}^N_0) \). However, the Lagrange density

\[
    |\nabla u|^2 - \left( \frac{m - 2}{2} \right)^2 \frac{|u|^2}{|y|^2}
\]

(3.2)
is no longer integrable for an arbitrary \( u \in D^{1,2}_V(\mathbb{R}^N_0) \). The integrable Lagrange density of (3.1), \( |y|^{2-m} |\nabla (u|y|^{(m-2)/2})|^2 \) can be equated to (3.2) by partial integration when \( u \in C^\infty_0(\mathbb{R}^N_0) \), but this connection does not extend to the whole of \( D^{1,2}_V(\mathbb{R}^N_0) \) as the terms that mutually cancel in the partial integration on \( C^\infty_0(\mathbb{R}^N_0) \) might become infinite. In particular, it should not be expected a priori that the minimizer for HSM inequality in \( D^{1,2}_V(\mathbb{R}^N_0) \) would have a finite gradient in \( L^2(\mathbb{R}^N_0, dx) \).

Existence of minimizers for the variational problem associated with (3.1) is proved in [25] for all codimensions \( 0 < m < N \), where \( N > 3 \). The existence proof is based on concentration compactness argument that utilizes invariance properties of the problem. Similarly to other problems where lack of compactness stems from a noncompact equivariant group of transformations, some general domains and potentials admit minimizers and some do not, and analogy with similar elliptic problems in \( D^{1,2}(\mathbb{R}^N) \) provides useful insights (see for example [23]).

4 Convexity properties of \( Q \) for \( p > 2 \)

The definition of \( D^{1,2}_V(\Omega) \) cannot be applied to other values of \( p \), since for \( p \neq 2 \) the positivity of the functional \( Q \) on \( C^\infty_0(\Omega) \) does not necessarily imply its convexity, and thus it does not give rise to a norm. For the lack of convexity when \( p > 2 \), see an elementary one-dimensional counterexample at the end of [8], and also the proof of Theorem 7 in [14]. For \( p < 2 \), see [13, Example 2].

On the other hand, by [21, Theorem 2.3], the functional \( Q \) is nonnegative on \( C^\infty_0(\Omega) \) if and only if the equation \( Q'(u) = 0 \) in \( \Omega \) admits a positive global solution \( v \). With the help of such a solution \( v \), one has the identity [9, 2, 3]:

\[
Q(u) = \int_\Omega L_v(w) \, dx \quad u \in C^\infty_0(\Omega),
\]

where \( w \overset{\text{def}}{=} u/v \), the Lagrangian \( L_v(w) \) is defined by

\[
L_v(w) \overset{\text{def}}{=} |v \nabla w + w \nabla v|^p - |w|^p |\nabla v|^p - pw^{p-1} v |\nabla v|^{p-2} \nabla v \cdot \nabla w \geq 0 \quad w \in C^\infty_{0+}(\Omega),
\]

and \( C^\infty_{0+}(\Omega) \) denotes the cone of all nonnegative functions in \( C^\infty_0(\Omega) \).

The following proposition claims that the nonnegative Lagrangian \( L_v(w) \), which contains indefinite terms, is bounded from above and from below by multiples of a simpler Lagrangian.
Proposition 4.1 ([19] Lemma 2.2]). Let $v$ be a positive solution of the equation $Q'(u) = 0$ in $\Omega$. Then

$$L_v(w) \asymp v^2 |\nabla w|^2 (w|\nabla v| + v|\nabla w|)^{p-2} \quad \forall w \in C_{0+}^{\infty}(\Omega). \quad (4.2)$$

In particular, for $p \geq 2$, we have

$$L_v(w) \asymp \hat{L}_v(w) \overset{\text{def}}{=} v^p |\nabla w|^p + v^2 |\nabla v|^{p-2} w^{p-2} |\nabla w|^2 \quad \forall w \in C_{0+}^{\infty}(\Omega). \quad (4.3)$$

Define the simplified energy $\hat{Q}$ by

$$\hat{Q}(u) \overset{\text{def}}{=} \int_{\Omega} \hat{L}_v(w) \, dx \quad w = u/v \in C_{0+}^{\infty}(\Omega). \quad (4.4)$$

It is shown in [19] that for $p > 2$ neither of the terms in the simplified energy $\hat{Q}$ is dominated by the other.

It follows from Proposition 4.1 that

$$Q(u) = Q(|u|) \asymp \hat{Q}(|u|) \quad u \in C_{0}^{\infty}(\Omega).$$

In [24], the solvability of equation $Q'(u) = f$ is proved in the class of functions $u$ satisfying $Q^{**}(u) < \infty$, where $Q^{**} \leq Q$ is the second convex conjugate (in the sense of Legendre transformation) of $Q$. If the inequality $Q \leq CQ^{**}$ is true, then $Q^{**(1/p)}(u)$ would define a norm, and $Q$ would extend to a Banach space, which should be regarded as the natural energy space for the functional $Q$.

On the other hand, if $p > 2$, it is not clear whether the functional $\hat{Q}$ is convex due to the second term in (4.3). It has, however, the following convexity property.

Proposition 4.2. Assume that $p \geq 2$, and let $v \in C_{1}^{1, \text{loc}}(\Omega)$ be a fixed positive function. Consider the functional

$$Q(\psi) \overset{\text{def}}{=} \hat{Q}(v \psi^{2/p}) \quad \psi \in C_{0+}^{\infty}(\Omega),$$

where $\hat{Q}$ is defined by (4.3) and (4.4). Then the functional $Q$ is convex on $C_{0+}^{\infty}(\Omega)$. 

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Proof. We first split each of the functionals \( \hat{Q} \) and \( Q \) into a sum of two functionals:

\[
\hat{Q}_1(u) \overset{\text{def}}{=} \int_\Omega v^p|\nabla w|^p \, dx, \quad \hat{Q}_2(u) \overset{\text{def}}{=} \int_\Omega v^2|\nabla v|^{p-2} |\nabla w|^2 \, dx \quad w = u/v \in C_0^\infty(\Omega),
\]

\[
Q_1(\psi) \overset{\text{def}}{=} \hat{Q}_1(v\psi^2/p) = \int_\Omega v^p|\nabla (\psi^{2/p})|^p \, dx \quad \psi \in C_0^\infty(\Omega),
\]

\[
Q_2(\psi) \overset{\text{def}}{=} \hat{Q}_2(v\psi^2/p) = \int_\Omega v^2|\nabla v|^{p-2} \psi^{2(p-2)/p}|\nabla (\psi^{2/p})|^2 \, dx \quad \psi \in C_0^\infty(\Omega).
\]

Thus, \( \hat{Q} = \hat{Q}_1 + \hat{Q}_2 \), and \( Q = Q_1 + Q_2 \).

For \( t \in [0, 1] \) and \( w_0, w_1 \in C_0^\infty(\Omega) \), let

\[
w_t \overset{\text{def}}{=} \left[ (1 - t)w_0^{p/2} + tw_1^{p/2} \right]^{2/p}.
\]

Then

\[
\nabla w_t = \frac{(1 - t)w_0^{p/2-1}\nabla w_0 + tw_1^{p/2-1}\nabla w_1}{\left[ (1 - t)w_0^{p/2} + tw_1^{p/2} \right]^{1-2/p}}.
\]

Therefore,

\[
|\nabla w_t| \leq \frac{\left[ (1 - t)^{2/p}w_0 \right]^{p/2-1} + \frac{(t^{2/p}w_1)^{p/2-1}t^{2/p}}{\left[ (1 - t)w_0^{p/2} + tw_1^{p/2} \right]^{1-2/p}}}{\left[ (1 - t)w_0^{p/2} + tw_1^{p/2} \right]^{1-2/p}}. \quad (4.5)
\]

Applying Hölder inequality to the sum in the numerator of (4.5) (with the terms \((1 - t)^{2/p}w_0\) and \(t^{2/p}\nabla w_1\) raised to the power \(p/2\)) and taking into account that the conjugate of \(p/2\) is reciprocal to \(1 - 2/p\), we have

\[
|\nabla w_t|^{p/2} \leq (1 - t)|\nabla w_0|^{p/2} + t|\nabla w_1|^{p/2}. \quad (4.6)
\]

From (4.6) it follows easily that

\[
|\nabla w_t|^p \leq (1 - t)|\nabla w_0|^p + t|\nabla w_1|^p.
\]

Setting \( \psi_t \overset{\text{def}}{=} w_t^{p/2}, \) \( t \in [0, 1] \), we immediately conclude that \( Q_1 \) is convex as a function of \( \psi \). The same conclusion extends to \( Q_2 \) once we note that

\[
w^{p-2}|\nabla w|^2 = (2/p)^2|\nabla w^{p/2}|^2,
\]

and use (4.6) for \( p = 4 \). \( \square \)
Let
\[ N(\psi) \overset{\text{def}}{=} \left[ Q(\psi) \right]^{1/2} = \left[ \hat{Q}(\nu \psi^{2/p}) \right]^{1/2} \]
\[ \psi \in C^\infty_0(\Omega). \tag{4.7} \]

It is immediate that \( N(\psi) > 0 \) for \( \psi \in C^\infty_0(\Omega) \), unless \( \psi = 0 \), and that \( N(\lambda \psi) = \lambda N(\psi) \) for \( \lambda \geq 0 \). Due to Proposition 4.2, the functional \( N(\cdot) \) satisfies the triangle inequality
\[ N(\psi_1 + \psi_2) \leq N(\psi_1) + N(\psi_2) \quad \psi_1, \psi_2 \in C^\infty_0(\Omega). \]

Thus, we have equipped the cone \( C^\infty_0(\Omega) \) with a norm. For \( p = 2 \) the functional \( Q = \hat{Q} \) is a positive quadratic form, and thus convex. Consequently, in the subcritical case, \( Q^{1/2} \) extends the functional \( N \) to a norm on the whole \( C^\infty_0(\Omega) \), and then by completion, to the Hilbert space \( D^{1,2}_V(\Omega) \). It would be interesting to introduce \( D^{1,p}_V(\Omega) \) for \( p > 2 \) once one finds an extension of \( N \) to \( C^\infty_0(\Omega) \).

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