Quantization of the Schrödinger-Virasoro Lie algebra *

Yucai SU, Lamei YUAN

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China

© Higher Education Press and Springer-Verlag Berlin Heidelberg 2010

Abstract In this paper, we use the general quantization method by Drinfel’d twists to quantize the Schrödinger-Virasoro Lie algebra whose Lie bialgebra structures were recently discovered by Han-Li-Su. We give two different kinds of Drinfel’d twists, which are then used to construct the corresponding Hopf algebraic structures. Our results extend the class of examples of noncommutative and noncocommutative Hopf algebras.

Keywords Lie bialgebras, quantization, Schrödinger-Virasoro Lie algebra

MSC 17B05, 17B37, 17B62, 17B68

1 Introduction

In Hopf algebra or quantum group theory, two standard methods to yield new bialgebras from old ones are by twisting the product by a 2-cocycle but keeping the coproduct unchanged, and by twisting the coproduct by a Drinfel’d twist but preserving the product. Constructing quantizations of Lie bialgebras is an important approach to producing new quantum groups (see [1, 2, 4] and references therein). Drinfel’d in [3] formulated a number of problems in quantum group theory, including the existence of a quantization for Lie bialgebras. In the paper [5] Etingof and Kazhdan gave a positive answer to some of Drinfel’d’s questions. In particular, they showed the existence of quantizations for Lie bialgebras, namely, any classical Yang-Baxter algebra can be quantized. Since then the interests in quantizations of Lie bialgebras have been growing in the mathematical literatures (e.g., [6, 7, 8, 18, 22]).

The Schrödinger-Virasoro Lie algebra considered in this paper was introduced in the context of non-equilibrium statistical physics during the process of investigating the free Schrödinger equations (see [11, 12]). This Lie algebra
2 Yucai SU, Lamei YUAN

is closely related to Schrödinger algebra and Virasoro algebra, both of which play important roles in many areas of mathematics and physics (e.g., statistical physics). The Schrödinger-Virasoro Lie algebra, denoted by $\mathcal{L}$, is an infinite-dimensional vector space with basis $\{L_n, Y_p, M_n \mid n \in \mathbb{Z}, p \in \frac{1}{2} + \mathbb{Z}\}$ and the following non-vanishing Lie brackets

\[
\begin{align*}
[L_m, L_n] &= (n-m)L_{n+m}, \\
[L_m, M_n] &= nM_{n+m}, \\
[L_n, Y_p] &= (p-n^2)Y_{p+n}, \\
[Y_p, Y_q] &= (q-p)M_{p+q},
\end{align*}
\]

for all $m, n \in \mathbb{Z}$ and $p, q \in \mathbb{Z} + 1/2$. This kind of Lie algebras has been investigated in a number of papers. Some of these investigations [9, 14, 15, 16] focus on its structure theory including derivations, central extension and automorphism groups, others [13, 17, 19, 20] on its representations. Recently, the Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra $\mathcal{L}$ were discussed in [21], which turned out to be not all coboundary triangular (for definition, see p.28, [4]). In the present paper, we use the general quantization method by Drinfel’d twists (cf. [1]) to quantize explicitly the newly determined triangular Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra $\mathcal{L}$. Actually, this process completely depends on the construction of Drinfel’d twists determined by the $r$-matrix (namely, the triangular Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra). Our results extend the class of examples of noncommutative and noncocommutative Hopf algebras.

The main results of this paper are listed as follows:

**Theorem 1.** With the choice of two distinguished elements $h := \frac{1}{n_0}L_0$ and $e := M_{n_0}$ ($n_0 \neq 0$) such that $[h, e] = e$ in $\mathcal{L}$, there exists a structure of noncommutative and noncocommutative Hopf algebra $(U(\mathcal{L})[[t]], m, \imath, \Delta, S, \epsilon)$ on $U(\mathcal{L})[[t]]$ over $\mathbb{F}[[t]]$ with $U(\mathcal{L})[[t]]/tU(\mathcal{L})[[t]] \cong U(\mathcal{L})$, which leaves the product and counit of $U(\mathcal{L})[[t]]$ undeformed but with the deformed comultiplication and antipode defined by:

\[
\begin{align*}
\Delta(L_n) &= 1 \otimes L_n + L_n \otimes (1 - et)^{-n_0} + n_0 h \otimes (1 - et)^{-1}M_{n+n_0} t, \\
\Delta(M_k) &= 1 \otimes M_k + M_k \otimes (1 - et)^{-k_0}, \\
\Delta(Y_p) &= 1 \otimes Y_p + Y_p \otimes (1 - et)^{-p_0}, \\
S(L_n) &= -(1 - et)^{-\frac{n_0}{2}}(L_n - n_0 M_{n+n_0} h_1 t), \\
S(M_k) &= -(1 - et)^{-\frac{k_0}{2}} \cdot M_k, \\
S(Y_p) &= -(1 - et)^{-\frac{p_0}{2}} \cdot Y_p.
\end{align*}
\]

**Theorem 2.** With the choice of two distinguished elements $h := \frac{1}{n_0}L_0$ and
product and counit of $U(\mathcal{L})[[t]]$ undeformed but with the deformed comultiplication and antipode defined by:

\[
\Delta(L_n) := 1 \otimes L_n + L_n \otimes (1 - et)^{\frac{n_0}{n}} + \frac{n(n - n_0)}{4} h^{(2)} \otimes (1 - et)^{-2} M_{n+n_0} t^2,
\]

\[
\Delta(M_k) := 1 \otimes M_k + M_k \otimes (1 - et)^{\frac{k_0}{k}},
\]

\[
\Delta(Y_p) := 1 \otimes Y_p + Y_p \otimes (1 - et)^{\frac{p_0}{p}} - (p - \frac{n_0}{2}) h \otimes (1 - et)^{-1} M_{p+n_0} t,
\]

\[
S(L_n) := -(1 - et)^{-\frac{n_0}{n}} (L_n + \frac{n-n_0}{2} Y_n + \frac{n_0}{4} h_1 h_2 t^2),
\]

\[
S(Y_p) := -(1 - et)^{-\frac{p_0}{p}} (Y_p + (p - \frac{n_0}{2}) M_{p+n_0} h_1 t),
\]

\[
S(M_k) := -(1 - et)^{-\frac{k_0}{k}} M_k.
\]

Throughout this paper $\mathbb{F}$ denotes a field of characteristic zero. All vector spaces and algebras are assumed to be over $\mathbb{F}$. $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}^*$ stand for the sets of integers, nonnegative and nonzero integers respectively.

## 2 Preliminaries

In this section, we summarize some basic definitions and results concerning Lie bialgebra structures which will be used in the following discussions. For a detailed discussion of this subject we refer the reader to the literatures (e.g. \[18, 21\] and references therein).

Let $\mathcal{L}$ be the Schrödinger-Virasoro Lie algebra defined in (1) and $U(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$. Then $U(\mathcal{L})$ is equiped with a natural Hopf algebraic structure $(U(\mathcal{L}), m, \iota, \Delta_0, S_0, \epsilon)$, i.e.,

\[
\Delta_0(X) = X \otimes 1 + 1 \otimes X, \quad S_0(X) = -X, \quad \epsilon(X) = 0 \quad \text{for } X \in \mathcal{L}.
\]

where $\Delta_0$ is a comultiplication, $\epsilon$ is a counit and $S_0$ is an antipode. In particular,

\[
\Delta_0(1) = 1 \otimes 1 \quad \text{and} \quad \epsilon(1) = S_0(1) = 1.
\]

In order to search for the solutions of the Yang-Baxter quantum equation, Drinfel’d in [1] introduced the notion of Lie bialgebras in 1983. Since then, a great deal of attention has been paid to the study of the quantization of Lie bialgebras as well as Lie bialgebra structures of some Lie algebras (e.g.,
define a linear map
\[ \Delta_r(x) = x \cdot r = [x, a] \otimes b - b \otimes [x, a] + a \otimes [x, b] - [x, b] \otimes a, \quad \text{for } x \in L. \] (3)

Then \( \Delta_r \) equips \( L \) with a structure of triangular coboundary Lie bialgebra.

Equation (3) implies that \( \Delta_r \) is an inner derivation of \( L \). For the Schrödinger-Virasoro Lie algebra \( L \) defined in (1), it is shown in [21] that a Lie bialgebra \( (L, [\cdot, \cdot], \Delta) \) is triangular coboundary if and only if \( \Delta \) is an inner derivation, which is determined by the classical Yang-Baxter \( r \)-matrix \( r \). From the above proposition, we notice that the classical Yang-Baxter \( r \)-matrix is uniquely expressed as the antisymmetric tensor of two distinguished elements \( a, b \) up to nonzero scalars satisfying \( [a, b] = kb \) \((k \neq 0)\). In fact, for a given \( r \)-matrix, we may take two distinguished elements of the form \( h := k^{-1}a \) and \( e := kb \) such that \( [h, e] = e \) with \( 0 \neq k \in \mathbb{F} \).

**Definition 1.** Let \((H, m, \iota, \Delta_0, S_0, \epsilon)\) be a Hopf algebra over a commutative ring \( R \). A Drinfel’d twist \( F \) on \( H \) is an invertible element of \( H \otimes H \) such that
\[
(F \otimes 1)(\Delta_0 \otimes \text{Id})(F) = (1 \otimes F)(\text{Id} \otimes \Delta_0)(F),
\]
\[
(\epsilon \otimes \text{Id})(F) = 1 \otimes 1 = (\text{Id} \otimes \epsilon)(F).
\]
The following result is well known (see [1, 4], etc.).

**Lemma 1.** Let \((H, m, \iota, \Delta_0, S_0, \epsilon)\) be a Hopf algebra over a commutative ring and \( F \) a Drinfel’d twist on \( H \), then \( w = m(\text{Id} \otimes S_0)(F) \) is invertible in \( H \) with \( w^{-1} = m(S_0 \otimes \text{Id})(F^{-1}) \). Moreover, if we define \( \Delta : H \rightarrow H \otimes H \) and \( S : H \rightarrow H \) by
\[
\Delta(x) = F \Delta_0(x) F^{-1}, \quad S = wS_0(x)w^{-1}, \quad \text{for all } x \in H.
\]
Then \((H, m, \iota, \Delta, S, \epsilon)\) is a new Hopf algebra, which is called the twisting of \( H \) by the Drinfel’d twist \( F \).

Let \( F[[t]] \) be a ring of formal power series. Assume that \( L \) is a triangular Lie bialgebra with a classical Yang-Baxter \( r \)-matrix \( r \) (see [1, 4]). Denote by \( U(L) \) the universal enveloping algebra of \( L \), with the standard Hopf algebra structure \((U(L), m, \iota, \Delta_0, S_0, \epsilon)\). Now let us consider the topologically free \( F[[t]] \)-algebra \( U(L)[[t]] \) (for definition, see p.4, [1]), which can be viewed as an associative \( F \)-algebra of formal power series with coefficients in \( U(L) \). Naturally, \( U(L)[[t]] \) is equipped with an induced Hopf algebra structure arising from that on \( U(L) \). By abuse of notation, we denote it by \((U(L)[[t]], m, \iota, \Delta_0, S_0, \epsilon)\).

**Definition 2.** (See Definition 1.4, [22]) For a triangular Lie bialgebra \( L \), the classical Yang-Baxter \( r \)-matrix is uniquely expressed as the antisymmetric tensor of two distinguished elements \( a, b \) up to nonzero scalars satisfying \( [a, b] = kb \) \((k \neq 0)\). In fact, for a given \( r \)-matrix, we may take two distinguished elements of the form \( h := k^{-1}a \) and \( e := kb \) such that \( [h, e] = e \) with \( 0 \neq k \in \mathbb{F} \).
An algebra $A$ equipped with a classical Yang-Baxter $r$-matrix $r$ is called a classical Yang-Baxter algebra. It is showed in [5] that any classical Yang-Baxter algebra can be quantized.

For any element $x$ of a unital $R$-algebra ($R$ a ring) and $a \in R$, we set (see, e.g., [10])

$$x^{(n)} := (x + a)(x + a + 1) \cdots (x + a + n - 1)$$
$$x^{[n]} := (x + a)(x + a - 1) \cdots (x + a - n + 1)$$

and $x^{(0)} := x^{(n)}$, $x^{[0]} := x^{[n]}$.

**Lemma 2.** (See [10, 8]) For any element $x$ of a unital $F$-algebra, $a, b \in F$, and $r, s, t \in \mathbb{Z}$, one has

$$x^{(s+t)} = x^{(s)} x^{(t)} = x^{[s]} x^{[t]} = x^{(s+t+1)},$$

$$\sum_{s+t=r} \frac{(-1)^t}{s! t!} x^{[s]} x^{[t]} = (a - b) \frac{(a - b - r + 1)}{r!},$$

$$\sum_{s+t=r} \frac{(-1)^t}{s! t!} x^{[s]} x^{[t]} = (a - b + r - 1) \frac{(a - b) \cdots (a - b + r - 1)}{r!}.$$  

**Remark 1.** One can see that the right-hand sides of the last two equations do not depend on $x$ from the proof process (see e.g., Lemma 3, [8]).

The following popular result will be frequently used in the third part of this paper.

**Lemma 3.** (see e.g., Proposition 1.3(4), [23]) For any elements $x, y$ of an associative algebra $A$, and $m \in \mathbb{Z}_+$, one has

$$xy^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} y^{m-k} (\text{ad } y)^k(x).$$

**3 Proof of the main results**

To describe quantizations of $U(\mathcal{L})$, we need to construct explicitly Drinfel’d twists according to Lemma 1. Fix $n_0 \in \mathbb{Z}^*$. Set

$$h := \frac{1}{n_0} L_0, \quad e := M_{n_0}, \quad \text{or} \quad h := \frac{2}{n_0} L_0, \quad e := Y_{\frac{n_0}{2}} \quad \text{(only when } n_0 \text{ is odd)}. $$

Clearly, one has $[h, e] = e$ by equation 1. Then from Proposition 1 it follows that the Lie bialgebra $(\mathcal{L}, [\cdot, \cdot], \Delta_r)$ with $r = h \otimes e - e \otimes h$ is triangular. We shall quantize this triangular Lie bialgebra structure in this section. To do this, we need some necessary calculations, which are useful to the construction
(i) If \( h = \frac{1}{n_0} L_0 \), and \( e = M_{n_0} \), then
\[
L_n h_a^{(i)} = h_a^{(i)} \frac{-n}{n_0} L_n, \quad L_n h_a^{[i]} = h_a^{[i]} \frac{-n}{n_0} L_n, \\
M_n h_a^{(i)} = h_a^{(i)} \frac{-n}{n_0} M_n, \quad M_n h_a^{[i]} = h_a^{[i]} \frac{-n}{n_0} M_n, \\
Y_p h_a^{(i)} = h_a^{(i)} \frac{-n}{n_0} Y_p, \quad Y_p h_a^{[i]} = h_a^{[i]} \frac{-n}{n_0} Y_p. \tag{8}
\]

(ii) If \( h = \frac{2}{n_0} L_0 \), and \( e = Y_{\frac{n}{n_0}} \), then
\[
L_n h_a^{(i)} = h_a^{(i)} \frac{-2n}{n_0} L_n, \quad L_n h_a^{[i]} = h_a^{[i]} \frac{-2n}{n_0} L_n, \\
M_n h_a^{(i)} = h_a^{(i)} \frac{-2n}{n_0} M_n, \quad M_n h_a^{[i]} = h_a^{[i]} \frac{-2n}{n_0} M_n, \\
Y_p h_a^{(i)} = h_a^{(i)} \frac{-2n}{n_0} Y_p, \quad Y_p h_a^{[i]} = h_a^{[i]} \frac{-2n}{n_0} Y_p. 
\]

(iii) In both cases,
\[
e^n h_a^{(i)} = h_a^{(i)} e^n, \quad e^n h_a^{[i]} = h_a^{[i]} e^n. \tag{9}
\]

**Proof.** We only prove the first equation of (9) (the others can be obtained similarly). We have \([L_n, h] = \frac{n}{n_0} L_n\) and \(L_n h = (h - \frac{n}{n_0}) L_n\), i.e., it holds for \( i = 1 \). Suppose that it holds for \( i \), then we have
\[
L_n h_a^{(i+1)} = L_n h_a^{(i)} (h + a + i) = h_a^{(i)} \frac{-n}{n_0} L_n (h + a + i) \\
= h_a^{(i)} \frac{-n}{n_0} (h - \frac{n}{n_0} + a + i) L_n = h_a^{(i+1)} \frac{-n}{n_0} L_n. 
\]

For any \( a \in \mathbb{F} \), we set
\[
\mathcal{F}_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} \otimes e^r t^r, \quad F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r, \tag{10}
\]
\[
u_a = m \cdot (S_0 \otimes \text{Id})(F_a), \quad v_a = m \cdot (\text{Id} \otimes S_0)(\mathcal{F}_a). 
\]

Write \( \mathcal{F} = \mathcal{F}_0, F = F_0, u = u_0, v = v_0 \). Since \( S_0(h_a^{(r)}) = (-1)^r h_{-a}^{[r]} \) and \( S_0(e^r) = (-1)^r e^r \), we have
\[
u_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a}^{[r]} e^r t^r, \quad v_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} e^r t^r. \tag{11}
\]

**Lemma 5.** For any \( a, b \in \mathbb{F} \), one has
\[
\mathcal{F}_a F_b = 1 \otimes (1 - et)^{a-b}, \quad v_a u_b = (1 - et)^{-(a+b)}. 
\]

**Proof.** By equations (5) and (10), we have
\[
\mathcal{F}_a F_b = \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r! s!} h_a^{[r]} h_b^{[s]} \otimes e^r e^s t^r t^s \\
= \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r! s!} \cdot (1 - et)^{a+b}. 
\]
From (6), (9) and (11), we obtain that
\[ v_a u_b = \sum_{r,s=0}^{\infty} \frac{(-1)^s}{r!s!} h_{-b}^r e^s h_{-b}^r e^{r+s} \]
\[ = \sum_{m=0}^{\infty} \sum_{r+s=m} \frac{(-1)^s}{r!s!} h_{-b}^r h_{-b}^s e^m t^m \]
\[ = \sum_{m=0}^{\infty} \left( a + b + m - 1 \right) e^m t^m = (1 - et)^{-1}. \]
□

**Corollary 1.** For any \( a \in \mathbb{F} \), the elements \( F_a \) and \( u_a \) are invertible with \( F_a^{-1} = \mathcal{F} \), \( u_a^{-1} = v_{-a} \). In particular, \( F_a^{-1} = \mathcal{F} \), \( u_a^{-1} = v \).

**Lemma 6.** For any \( a \in \mathbb{F} \) and \( r \in \mathbb{Z}_+ \), one has \( \Delta_0(h^r) = \sum_{i=0}^{r} \binom{r}{i} \Delta_0(h) \otimes h^{r-i} \).

In particular, one has \( \Delta_0(h^r) = \sum_{i=0}^{r} \binom{r}{i} h^i \otimes h^{r-i} \).

**Proof.** It can be proved by induction on \( r \). □

**Lemma 7.** The element \( \mathcal{F} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^r \otimes e^r t^r \) is a Drinfel’d twist on \( U(L)[[t]] \).

**Proof.** It can be proved directly by using the similar arguments as those presented in the proof of [22, Proposition 2.5]. □

Now we can perform the process of twisting the standard Hopf structure \((U(L), m, \iota, \Delta_0, S_0, \varepsilon)\) defined in (2) by the Drinfel’d twist \( \mathcal{F} \) constructed above. The following lemmas are very useful to our main results.

**Lemma 8.** For \( p, q \in \frac{1}{2} + \mathbb{Z} \) and \( s \in \mathbb{Z}_+ \), one has
\[ Y_p Y_q^s = Y_q^s Y_p - s(p - q) Y_q^{s-1} M_{p+q}. \]

**Proof.** It follows from Lemma 3 that
\[ Y_p Y_q^s = \sum_{i=0}^{s} (-1)^i \binom{s}{i} Y_q^{s-i} (\text{ad} Y_q)^i Y_p. \] (12)

By (11), we have
\[ (\text{ad} Y_q)^i Y_p = \begin{cases} Y_p, \quad i = 0, \\ (p - q) M_{p+q}, \quad i = 1, \\ 0, \quad \text{others} \end{cases} \] (13)
Lemma 9. For $a \in \mathbb{F}$, $n, k \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $h = \frac{1}{n_0}L_0$ and $e = M_{n_0}$, we have

\[
(L_n \otimes 1)F_a = F_a - \frac{n}{n_0} (L_n \otimes 1),
\]

\[
(M_k \otimes 1)F_a = F_a - \frac{k}{n_0} (M_k \otimes 1),
\]

\[
(Y_p \otimes 1)F_a = F_a - \frac{p}{n_0} (Y_p \otimes 1),
\]

\[
(1 \otimes M_k)F_a = F_a(1 \otimes M_k), (1 \otimes Y_p)F_a = F_a(1 \otimes Y_p),
\]

\[
(1 \otimes L_n)F_a = F_a(1 \otimes L_n) + n_0 F_{a+1}(h_a^{(1)} \otimes M_{n+n_0} t).
\]

Proof. The former three equations can be directly obtained by the definition of $F_a$ and Lemma 4(ii). Since both $Y_p$ and $M_k$ commute with $e$, the next two become obvious. It is left to verify the last one. From equations (4), (10) and Lemma 3 one has

\[(1 \otimes L_n)F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes L_n e^t t^r = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes (\sum_{i=0}^{r} (-1)^i \binom{r}{i} e^{r-i}(ad e)^i L_n) t^r
\]

\[= \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^i}{(r+i)!} \binom{r+i}{i} h_a^{(r+i)} \otimes e^t (ad e)^i L_n t^{r+i}
\]

\[= \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^i}{r!} h_a^{(r)} h_a^{(i)} \otimes e^t (ad e)^i L_n t^{r+i}
\]

\[= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} F_{a+i}(h_a^{(i)} \otimes (ad e)^i L_n t^i)
\]

\[= F_a(1 \otimes L_n) + n_0 F_{a+1}(h_a^{(1)} \otimes M_{n+n_0} t). \]

Lemma 10. For $a \in \mathbb{F}$, $n, k \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $h = \frac{2}{n_0}L_0$ and $e = Y_{2n_0}$, we have

\[(L_n \otimes 1)F_a = F_a - \frac{n}{n_0} (L_n \otimes 1), (M_k \otimes 1)F_a = F_a - \frac{k}{n_0} (M_k \otimes 1),
\]

\[(Y_p \otimes 1)F_a = F_a - \frac{p}{n_0} (Y_p \otimes 1), (1 \otimes M_k)F_a = F_a(1 \otimes M_k),
\]

\[(1 \otimes L_n)F_a = F_a(1 \otimes L_n) - \frac{n}{2} n_0 F_{a+1}(h_a^{(1)} \otimes Y_{n+n_0}) t
\]

\[+ \frac{n(n-n_0)}{4} F_{a+2}(h_a^{(2)} \otimes M_{n+n_0}) t^2,
\]

\[(1 \otimes Y_p)F_a = F_a(1 \otimes Y_p) - (p-n_0) F_{a+1}(h_a^{(1)} \otimes M_{p+n_0}) t.
\]

Proof. It only needs to verify the last two formulas since the other four are obvious because of Lemma 4(ii). By (4), (10) and Lemma 3 one has

\[(1 \otimes L_n)F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes L_n e^t t^r
\]

\[+ \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes \prod_{s=0}^{r} \left( Y_{n_0} - \frac{n_0}{2} s \right)^2.
\]
Again by (4), (10) and Lemma 8, one has

\[ F_a(1 \otimes L_n) - \frac{n - n_0}{2} F_{a+1}(h_a^{(1)} \otimes Y_{n+\frac{n_0}{2}}) t + \frac{n(n - n_0)}{4} F_{a+2}(h_a^{(2)} \otimes M_{n+n_0}) t^2. \]

Again by (4), (10) and Lemma 8 one has

\[
(1 \otimes Y_p) F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes \left( e^r Y_p - r(p - \frac{n_0}{2}) e^{r-1} M_{p+\frac{n_0}{2}} \right) t^r
\]

\[
= F_a(1 \otimes Y_p) - (p - \frac{n_0}{2}) \sum_{r=0}^{\infty} \frac{1}{(r-1)!} h_a^{(r)} \otimes e^{r-1} M_{p+\frac{n_0}{2}} t^r
\]

\[
= F_a(1 \otimes Y_p) - (p - \frac{n_0}{2}) \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r+1)} \otimes e^r M_{p+\frac{n_0}{2}} t^{r+1}
\]

\[
= F_a(1 \otimes Y_p) - (p - \frac{n_0}{2}) \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} h_a^{(1)} \otimes e^r M_{p+\frac{n_0}{2}} t^{r+1}
\]

\[
= F_a(1 \otimes Y_p) - (p - \frac{n_0}{2}) F_{a+1}(h_a^{(1)} \otimes M_{p+\frac{n_0}{2}}) t. \tag*{\blacksquare}
\]

**Lemma 11.** For \( a \in \mathbb{F}, \ n, k \in \mathbb{Z}, \ p \in \frac{1}{2} + \mathbb{Z}, \ h = \frac{1}{n_0} L_0 \) and \( e = M_{n_0}, \) we have

\[
L_n u_a = u_{a + \frac{n}{n_0}} (L_n - n_0 M_{n+n_0} h_1^{[1]} t), \tag{14}
\]

\[
M_k u_a = u_{a + \frac{n}{n_0}} M_k, \quad Y_p u_a = u_{a + \frac{p}{n_0}} Y_p. \tag{15}
\]

**Proof.** From Lemma 11), and since both \( Y_p \) and \( M_k \) commute with \( e, \) one can easily obtain the last two equations of (15). By (4), (11) and Lemma 8 we have

\[
L_n u_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} L_n e^r t^r
\]

\[
= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} e^r L_n + n_0 e^{r-1} M_{n+n_0} t^r
\]

\[
= u_{a + \frac{n}{n_0}} L_n + n_0 \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!} h_a^{[r]} e^{r-1} M_{n+n_0} t^r
\]

\[
= u_{a + \frac{n}{n_0}} L_n - \frac{n_0}{r!} h_a^{[r+1]} e^r M_{n+n_0} t^{r+1}
\]

\[
= u_{a + \frac{n}{n_0}} L_n - n_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_a^{[r]} e^{r} M_{n+n_0} t^{r+1}
\]
Lemma 12. For $a \in \mathbb{F}$, $n, k \in \mathbb{Z}$, $p \in \frac{1}{2} + \mathbb{Z}$, $h = \frac{2a}{n_0} L_0$ and $e = Y_{n_0}$, we have

\[
M_k u_a = u_a + \frac{2a}{n_0} M_k, \quad Y_p u_a = u_a + \frac{2a}{n_0} \left( Y_p + \left( p - \frac{n_0}{2} \right) M_p + \frac{n_0}{2} h^{[1]} \right), \quad (16)
\]

\[
L_n u_a = u_a + \frac{2a}{n_0} \left( L_n + \frac{n-n_0}{2} Y_n + \frac{n_0}{4} h^{[1]}_1 - a t + \frac{n(n-n_0)}{4} M_{n+n_0} h^{[2]}_2 - a t^2 \right). \quad (17)
\]

Proof. The first equation of (16) is obvious by Lemma 4(ii) and since $e^{10} Yucai SU, Lamei YUAN

\[
\sum_{i=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} L_n e^{-i} L_n t^r
\]

\[
= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} \left( \sum_{i=0}^{r} (-1)^i \binom{r}{i} e^{ir} (ad e)^i L_n t^r \right)
\]

\[
= \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} \left( \binom{r+i}{i} e^{ir} (ad e)^i L_n t^{r+i} \right)
\]

\[
= \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} \left( \binom{r+i}{i} e^{ir} (ad e)^i L_n t^{r+i} \right)
\]

\[
= \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} \left( \binom{r+i}{i} e^{ir} (ad e)^i L_n t^{r+i} \right)
\]

\[
= u_a + \frac{2a}{n_0} \left( \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h^{[i]}_{-a - \frac{2a}{n_0}} (ad e)^i L_n t^i \right)
\]

\[
= u_a + \frac{2a}{n_0} \left( L_n + \frac{n-n_0}{2} Y_n + \frac{n_0}{4} h^{[1]}_1 + \frac{n(n-n_0)}{4} M_{n+n_0} h^{[2]}_2 - a t^2 \right)
\]

In addition, by Lemma 8, one has

\[
Y_p u_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} Y_p e^{ir} t^r
\]

\[
= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} \left( e^{r} Y_p - r \left( p - \frac{n_0}{2} \right) e^{-r-1} M_p + \frac{n_0}{2} \right) t^r
\]

\[
= u_a + \frac{2a}{n_0} \left( Y_p - \left( p - \frac{n_0}{2} \right) \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} e^{-r-1} M_p + \frac{n_0}{2} \right) t^r
\]

\[
= u_a + \frac{2a}{n_0} \left( Y_p - \left( p - \frac{n_0}{2} \right) \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} e^{r} M_p + \frac{n_0}{2} \right) t^{r+1}
\]

\[
= u_a + \frac{2a}{n_0} \left( Y_p - \left( p - \frac{n_0}{2} \right) \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} h^{[r]}_{-a - \frac{2a}{n_0}} e^{r} M_p + \frac{n_0}{2} \right) t^{r+1}
\]
Now we have enough in hand to prove our main results in this paper.

**Proof of Theorem 1**  By Lemmas 1, 5, Corollary 1, Lemmas 7 and 10, we have
\[
\Delta(L_n) = \mathcal{F} \cdot \Delta_0(L_n) \cdot \mathcal{F}^{-1} = \mathcal{F} \cdot (L_n \otimes 1) \cdot F + \mathcal{F} \cdot (1 \otimes L_n) \cdot F \\
= \mathcal{F} \cdot F_n \frac{\delta}{n_0} \cdot L_n \otimes 1 + \mathcal{F} \cdot (F \cdot 1 \otimes L_n + n_0 F_1 \cdot h^{(1)} \otimes M_{n_0 + n} t) \\
= 1 \otimes (1 - et) \frac{\delta}{n_0} \cdot L_n \otimes 1 + 1 \otimes L_n + n_0 \otimes (1 - et)^{-1} \cdot (h^{(1)} \otimes M_{n_0 + n} t) \\
= 1 \otimes L_n + L_n \otimes (1 - et) \frac{\delta}{n_0} + n_0 h \otimes (1 - et)^{-1} M_{n_0 + n} t.
\]
\[
\Delta(M_k) = \mathcal{F} \cdot (M_k \otimes 1) \cdot F + \mathcal{F} \cdot (1 \otimes M_k) \cdot F \\
= \mathcal{F} \cdot F_n \frac{\delta}{n_0} \cdot (M_k \otimes 1) + \mathcal{F} \cdot F(1 \otimes M_k) \\
= 1 \otimes (1 - et) \frac{\delta}{n_0} \cdot (M_k \otimes 1) + 1 \otimes M_k = M_k \otimes (1 - et) \frac{\delta}{n_0} + 1 \otimes M_k.
\]
\[
\Delta(Y_p) = \mathcal{F} \Delta_0(Y_p) \mathcal{F}^{-1} = \mathcal{F} \cdot Y_p \otimes 1 \cdot F + \mathcal{F} \cdot 1 \otimes Y_p \cdot F \\
= \mathcal{F} \cdot F_n \frac{\delta}{n_0} \cdot Y_p \otimes 1 + \mathcal{F} \cdot F \cdot 1 \otimes Y_p \\
= 1 \otimes (1 - et) \frac{\delta}{n_0} \cdot Y_p \otimes 1 + 1 \otimes Y_p = Y_p \otimes (1 - et) \frac{\delta}{n_0} + 1 \otimes Y_p.
\]

Again by Lemmas 1, 5, Corollary 1 and Lemma 11 we have
\[
S(L_n) = -v L_n u = -v \cdot u_n \frac{\delta}{n_0} (L_n - n_0 M_{n_0 + n_0} h^{[1]} t) \\
= - (1 - et)^{-1} \frac{\delta}{n_0} (L_n - n_0 M_{n_0 + n_0} h^{[1]} t).
\]
\[
S(M_k) = u^{-1} S_0(M_k) u = -v \cdot M_k \cdot u = -v u_n \frac{\delta}{n_0} M_k = -(1 - et)^{-1} \frac{\delta}{n_0} \cdot M_k.
\]
\[
S(Y_p) = u^{-1} S_0(Y_p) u = -v \cdot Y_p \cdot u = -v \cdot u_n \frac{\delta}{n_0} \cdot Y_p = -(1 - et)^{-1} \frac{\delta}{n_0} \cdot Y_p.
\]

Hence, we get the results. \( \square \)

**Proof of Theorem 2**  By Lemmas 1, 5, Corollary 1, Lemmas 7 and 10, we have
\[
\Delta(L_n) = \mathcal{F} \Delta_0(L_n) \mathcal{F}^{-1} = \mathcal{F} \cdot L_n \otimes 1 \cdot F + \mathcal{F} \cdot 1 \otimes L_n \cdot F \\
= \mathcal{F} \cdot F_n \frac{\delta}{n_0} \cdot L_n \otimes 1 + \mathcal{F} \cdot (F \cdot 1 \otimes L_n + \frac{n_0 - n}{2} F_1 \cdot h \otimes Y_n + \frac{n_0}{2} t) \\
+ \frac{n(n - n_0)}{4} F_2 \cdot h^{(2)} \otimes M_{n_0 + n_0} t^2 \\
= 1 \otimes (1 - et) \frac{2n_0}{n_0} L_n \otimes 1 + 1 \otimes L_n + \frac{n_0 - n}{2} \otimes (1 - et)^{-1} h \otimes Y_n + \frac{n_0}{2} t \\
+ \frac{n(n - n_0)}{4} \otimes (1 - et)^{-2} \cdot h^{(2)} \otimes M_{n_0 + n_0} t^2 \\
= L_n \otimes (1 - et) \frac{2n_0}{n_0} + 1 \otimes L_n + \frac{n_0 - n}{2} h \otimes (1 - et)^{-1} Y_n + \frac{n_0}{2} t \\
+ \frac{n(n - n_0)}{4} h^{(2)} \otimes (1 - et)^{-2} M_{n_0 + n_0} t^2.
\]
\[ \Delta(M_k) = \mathcal{F} \cdot M_k \otimes 1 \cdot F + \mathcal{F} \cdot 1 \otimes M_k \cdot F = \mathcal{F} \cdot F \cdot M_k \mathcal{F} \cdot 1 + \mathcal{F} \cdot F \cdot 1 \otimes M_k \]
\[ = 1 \otimes (1 - et) \frac{2k}{n} \cdot M_k \otimes 1 + 1 \otimes M_k = M_k \otimes (1 - et) \frac{2k}{n} + 1 \otimes M_k. \]

\[ \Delta(Y_p) = \mathcal{F} \Delta_0(L_n) \mathcal{F}^{-1} = \mathcal{F} \cdot Y_p \otimes 1 \cdot F + \mathcal{F} \cdot 1 \otimes Y_p \cdot F = \mathcal{F} F \frac{2k}{n_0} \cdot Y_p \otimes 1 + \mathcal{F} \cdot (F(1 \otimes Y_p) - (p - \frac{n_0}{2})F_1(h \otimes M_p + \frac{n_0}{2})t) \]
\[ = 1 \otimes (1 - et) \frac{2k}{n} \cdot Y_p \otimes 1 + 1 \otimes Y_p - (p - \frac{n_0}{2}) \otimes (1 - et)^{-1} \cdot h \otimes M_p + \frac{n_0}{2}t \]
\[ = 1 \otimes Y_p + Y_p \otimes (1 - et) \frac{2k}{n} - (p - \frac{n_0}{2})h \otimes (1 - et)^{-1} M_p + \frac{n_0}{2}t. \]

Again by Lemmas 11, 13, Corollary 1 and Lemma 12, we have

\[ S(L_n) = u^{-1} S_0(L_n)u = -vL_nu \]
\[ = -vu \frac{2n}{n_0} (L_n + \frac{n - n_0}{2} Y_{n + \frac{n_0}{2}} h_1^{[1]} t + \frac{n(n - n_0)}{4} M_{n + n_0} h_2^{[2]} t^2) \]
\[ = -(1 - et) \frac{2n}{n_0} (L_n + \frac{n - n_0}{2} Y_{n + \frac{n_0}{2}} h_1^{[1]} t + \frac{n(n - n_0)}{4} M_{n + n_0} h_2^{[2]} t^2). \]

\[ S(Y_p) = u^{-1} S_0(Y_p)u = -v Y_p u = -v \cdot u \frac{2n}{n_0} \cdot (Y_p + (p - \frac{n_0}{2})M_{p + \frac{n_0}{2}} h_1^{[1]} t) \]
\[ = -(1 - et) \frac{2n}{n_0} (Y_p + (p - \frac{n_0}{2})M_{p + \frac{n_0}{2}} h_1^{[1]} t). \]

\[ S(M_k) = u^{-1} S_0(M_k)u = -v \cdot M_k u = -v \cdot u \frac{2k}{n_0} M_k = -(1 - et) \frac{2k}{n_0} M_k. \]

So the proof is complete! \( \square \)

**Remark 2.** In this paper, we have presented two kinds of Hopf algebraic structures on \( U(\mathcal{L})[[t]] \) using the Drinfel’d twists. During the process of constructing the Drinfel’d twists, we see that any of them is definitely determined by some classical Yang-Baxter \( r \)-matrix \( r \) (namely, the Lie bialgebra structures of \( \mathcal{L} \)). So any two different elements \( h, c \in \mathcal{L} \) such that \( [h, c] = e \) can determine a Drinfel’d twist and thus a Hopf algebraic structure. This is one of the reasons why it is difficult to determine all Hopf algebraic structures on \( U(\mathcal{L})[[t]] \). It is sure that there exist other Hopf algebraic structures different from that given in our paper. One clear example is to take \( h = \frac{L_n}{n_0} \) and \( e = L_{n_0} \) for a fixed nonzero integer \( n_0 \). It is easy to see \( [h, e] = e \). Thus one can get another Hopf algebraic structure using the similar arguments as above.
References

1. Drinfel’d V G. Quantum groups. Proceeding of the International Congress of Mathematicians, Vol. 1, 2, Berkeley, California, 1986, American Mathematical Society, 1987, 798–820
2. Drinfel’d V G. Constant quasiclassical solutions of the Yang-Baxter quantum equation. Soviet Mathematics Doklady, 1983, 28(3): 667–671
3. Drinfel’d V G. On some unsolved problems in quantum group theory. Lecture Notes in Mathematics, 1992, 1510: 1–8
4. Etingof P, Schiffmann O. Lectures on Quantum groups, 2nd ed. International Press, USA, 2002
5. Etingof P, Kazhdan D. Quantization of Lie bialgebras I. Selecta Mathematica (New Series), 1996, 2: 1–41
6. Enriquez B, Halbout G. Quantization of Γ-Lie bialgebras. Journal of Algebra, 2008, 319: 3752–3769
7. Etingof P, Kazhdan D. Quantization of Lie bialgebras, part VI: Quantization of generalized Kac-Moody algebras. Transformation Groups, 2008, 13: 527–539
8. Grunspan C. Quantizations of the Witt algebra and of simple Lie algebras in characteristic $p$. Journal of Algebra, 2004, 280: 145–161
9. Gao S L, Jiang C B, Pei Y F. Structure of the extended Schrödinger-Virasoro Lie algebra. Algebra Colloquium, in press, 2008
10. Giaquinto A, Zhang J. Bialgebra action, twists and universal deformation formulas. Journal of Pure and Applied Algebra, 1998, 128(2): 133–151
11. Henkel M. Schrödinger invariance and strongly anisotropic critical systems. Journal of Statistical Physics, 1994, 75: 1023–1029
12. Henkel M, Unterberger J. Schrödinger invariance and space-time symmetries. Nuclear Physics B, 2003, 660: 407–412
13. Li J B, Su Y C. Representations of the Schrödinger-Virasoro algebras. Journal of Mathematical Physics, 2008, 49: 053512
14. Li J B, Su Y C. The derivation algebra and automorphism group of the twisted Schrödinger-Virasoro algebra. arXiv:0801.2207v1, 2008
15. Li J B, Su Y C. Leibniz central extension on centerless twisted Schrödinger-Virasoro algebras. Frontiers of Mathematics in China, 2008, 3(3): 337–344
16. Li J B, Su Y C, Zhu L S. 2-cocycles of original deformative Schrödinger-Virasoro algebras. Science in China Series A: Mathematics, 2008, 51: 1989–1999
17. Roger C, Unterberger J. The Schrödinger-Virasoro Lie group and algebra: representation theory and cohomological study. Annales Henri Poincaré, 2006, 7: 1477–1529
18. Song G A, Su Y C. Lie bialgebras of generalized-Witt type. Science in China Series A: Mathematics, 2006, 49(4): 533–544
19. Tan S B, Zhang X F. Automorphisms and Verma modules for Generalized Schrödinger-Virasoro algebras. arXiv:0804.1610v2, 2008
20. Unterberger J. On vertex algebra representations of the Schrödinger-Virasoro algebra. arXiv:cond-mat/0703214v2, 2007
21. Han J Z, Li J B, Su Y C. Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra. Journal of Mathematical Physics, 2009, 50: 083504, 12 pp
22. Hu N H, Wang X L. Quantizations of generalized-Witt algebra and of Jacobson-Witt algebra in the modular case. Journal of Algebra, 2007, 312: 902–929
23. Strade H, Farnsteiner R. Modular Lie Algebras and Their Representations, Monographs. Textbooks, Pure and Applied Mathematics, vol.116, Marcel Dekker, 1988