DM particles: how warm they can be?

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Abstract. One of important questions concerning particles which compose the Dark Matter (DM) is their average speed. We consider the model of relativistic weakly interacting massive particles and try to impose an upper bound on their actual and past warmness through the analysis of density perturbations and comparison with the LSS data. It is assumed that the DM can be described by the recently invented model of reduced relativistic gas (RRG). The equation of state of the RRG model is closely reproducing the one of the Maxwell distribution, while being much simpler. This advantage of the RRG model makes our analysis very efficient. As a result we arrive at the rigid and model-independent bound for the DM warmness without using the standard (much more sophisticated) approach based on the Einstein-Boltzmann system of equations.

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1 Introduction

The Sun is shining bright in Brazil. However, independently on geography, cosmologists say the Universe is dominated by a darkness. Namely, the energy balance of the present-day Universe shows that the relative energy densities of the Dark Energy and Dark Matter (DM) are close to $\Omega_\Lambda^0 = 0.7$ and $\Omega_{DM}^0 = 0.25$, respectively, while the visible (more precise, baryonic) matter is represented by a modest less than 5% of the total energy density \[1\].

Despite the existing variety of the models for Dark Energy, the Cosmological Constant (CC) $\Lambda$ is the most natural candidate. The presence of a $\Lambda$-term is dictated by the requirement of consistency of quantum field theory in curved space. At the same time the enormous fine-tuning which is necessary for adjusting the value of $\Lambda$ to the astronomical observations creates a longstanding CC problem (see, e.g., \[2,3\] for discussion). However, in this paper we will concentrate on the second dark component, which is equally mysterious. The main candidate to be DM is the gas of weakly interactive massive particles (WIMPs) which could be part of a multiplet composition of some extension of the Standard Model of elementary particle physics. For example, those can be superpartners of observable particles in MSSM or in some supergravity model. One can find an extensive discussion of the DM issue in the books \[4–7\] or in the recent reviews \[8–10\].

In simplest terms one can describe the DM problem as follows. The astronomical observations show that the stars and interstellar gas clouds in the spiral galaxies have the rotation curves different from the ones produced by gravitational field of the visible matter. The typical spiral galaxy has an almost flat structure, while the gravitational field is apparently produced by some almost spherical distribution of mass, total amount of it should be a few times greater than the one of the visible part. The hidden mass presumably forms a halo and is called DM. The main question is from what the DM is made. Obviously, the constituents of the DM should have properties distinct from the ones of the baryonic matter, for otherwise the two kinds of matter would be distributed in the same way. Furthermore, in the cosmological setting, DM is necessary for the cosmic structure formation.

One can distinguish the three main kinds of DM. The first one is cold DM, e.g., formed by WIMPs. Another one is hot DM, which can be represented, e.g., by massive neutrinos (with a mass of some eV). Hot dark matter leads to the so-called up-bottom scenario, where structures of clusters of galaxies are formed first, while cold dark matter implies the bottom-up scenarios, forming first small objects of scales smaller than a galaxy. Even if the cold dark matter scenario seems more favored, each scenario has its own problems, with
suppression (hot DM) or excess (cold DM) of power at small scales \([11]\). The intermediate scenario of warm DM has been invoked to solve this problem \([12–14, 16, 17]\). Warm DM may be composed, e.g., by relatively heavy sterile neutrino with the \(keV\)-scale mass (see, e.g., \([15, 16, 18, 19]\)) or by some other particles such as light gravitinos \([20]\).\(^1\) The structure formation in these models has been explored using fluid description \([17]\) (see also \([16]\) and references therein) and also \(N\)-body simulations methods \([9, 21]\).

The concepts of hot and warm DM imply that the DM constituents are relativistic in the early Universe and then cool down when the Universe expands. Qualitatively similar evolution takes place also for the baryonic matter, however the last emits radiation and therefore cools down faster. Due to the growing amount of the available experimental data, the requirements for the cosmological models are becoming stronger and in particular it would be desirable to have more precise description of the expansion of the Universe. In both cases of baryonic and DM one needs to have a model which could take into account, in a natural way, the continuous evolution of the equation of state of the matter content. In other words, this should be a model with the radiation-like behavior in the early Universe, becoming more like a dust or like a DM in the later epoch.

In the present paper we start the detailed analysis of the new, relatively simple model of reduced relativistic gas (RRG). This model is reproducing the equation of state of the Maxwell distribution with high precision \([22]\). The short technical introduction to the RRG model is postponed for the next section, but some general discussion is in order here. The RRG model enables one to consider the matter content which is hot in the early Universe and continuously cools down when the Universe expands. Let us note that the same purpose can be achieved by taking, e.g., relativistic distributions for the ideal gas case, such as the Maxwell, Fermi-Dirac or Bose-Einstein ones. However, in all these cases the relation between the energy density and the pressure have rather complicated form. In particular, for the most simple Maxwell distribution this relation has the form of the ratio of the two modified Bessel functions \([23]\) (see also, e.g., \([24]\)). Needless to say that such expressions are difficult to deal with in the cosmological framework. It is important to remember that the equation of state is the unique relevant element of the matter content, as far as we are interested only in the behavior of the conformal factor of the metric (zero-order cosmology). In this respect the RRG model \([22]\) provides a real advantage. Due to its simplicity, one can easily integrate the Friedmann equation and develop the zero-order cosmological model in the economic and analytic form. From the other side, as we have already mentioned above, the RRG model provides the equation of state which is very close to the one corresponding to the Maxwell distribution. Therefore the properties of the RRG-based cosmological model can be safely attributed to the model of the Universe filled by an ideal gas of massive relativistic particles.

In the framework of the RRG model one arrives at the single-fluid cosmology which interpolates, in a natural way, between the radiation-like and dust-like regimes. Therefore, at the level of zero-order cosmology the RRG model is proved to be a useful tool. Even more important, one can generalize the RRG model by implementing the possibility of the energy exchange between RRG gas and other fluids by introducing viscosity (see, e.g., \([25]\) and references therein). In this way we may expect to obtain the most precise zero-order cosmology which can be hopefully given by an analytical solution and be very useful. In future, we may have a chance to see the impact of different interaction types of the DM particles, e.g., on the cosmic perturbations spectrum.

\(^1\)See, e.g., \([10]\) for further references.
However, before the RRG model can be considered as a tool for creating the precise cosmological model, it has to be submitted to another important test. The present-day cosmological model does not deserve confidence if it produces good results only at the zero-order level. So, the next step in the development of the RRG model is to check whether it can produce acceptable results for the cosmic perturbations. If the RRG model can pass this test, it is worthwhile to use it as a building block for constructing the realistic cosmological models in a way described above. For this reason, in the present paper we address the issue of density perturbations spectrum in the simple RRG model [22]. The output of the analysis of the cosmic perturbations in the RRG model can be compared to existing detailed description of the perturbations in the warm and hot DM models (see, e.g., [12, 26] and also [5, 6, 8, 9] for the review). Indeed, our purpose is not to compete with the standard results, which are based on the numerical solution of the much more detailed description in the framework of the complicated Einstein-Boltzmann system. Instead of this, we want to see whether the results derived within the RRG model are compatible with the standard ones and, in this way, to try our model.

Let us remember that the RRG model is reproducing the ideal relativistic gas, which is isotropic and, therefore, can not provide full information on the motion of the DM particles. In the framework of such simplified approach we have a restricted choice of relevant physical observables, the most obvious is an average speed of the DM particles. Hence, our immediate purpose here is to establish an upper bound for the velocities of the DM constituents, both in the present and earlier epochs of the Universe. The upper bound for the DM velocities comes from the fit with the LSS data [27]. In this way, we can test to which extent the dark matter can be hot or at least warm. It is well known that the standard way to impose the bound on the warmness of the DM particles is through the analysis of cosmic perturbations in the Einstein-Boltzmann coupled system [26]. As we shall see in what follows, the use of the relatively simple RRG model enables one to achieve similar restrictions in a much more economic way. Thus, using this approach, we circumvent the technical difficulties related to the analysis of the Einstein-Boltzmann system without losing the essential features.

The paper is organized as follows. In the next section we present a very brief introduction to the RRG model. The reader can find further details in [22]. In section 3 the equations for density perturbations and their numerical analysis are considered and in section 4 we present some discussions and draw our conclusions.

2 Reduced model for relativistic gas

The equation of state for the ideal relativistic gas of identical massive particles has been derived in [23]. This equation involves a ratio of two modified Bessel functions and is rather difficult to apply for the cosmological purposes. One can simplify things considerably if assuming that, instead of the Maxwell law, all particles have equal kinetic energies. An elementary consideration leads to the following relation between pressure $P$ and energy density $\rho = n\varepsilon$:

$$P = \frac{\rho}{3} \left[ 1 - \left( \frac{mc^2}{\varepsilon} \right)^2 \right], \quad (2.1)$$

where $\varepsilon = mc^2/\sqrt{1 - \beta^2}$, $n$ is a number of particles per unit of volume and $\beta = v/c$. One can introduce the new notation for the density of the rest energy

$$\rho_d = nmc^2, \quad (2.2)$$
where \( n \) is the number of particles for a unit of 3d volume. This density depends on the scale factor in the usual way

\[
\rho_d = \frac{a^3_0}{a^3} \rho_d^0,
\]

where \( \rho_d^0 \) is the rest energy density at present, when \( a = a_0 \). Using this quantity, one can rewrite eq. (2.1) in the form

\[
P = \frac{\rho}{3} \cdot \left[ 1 - \frac{\rho_d^2}{\rho^2} \right].
\]

Both forms (2.1) and (2.3) will be useful for us in what follows. An important observation is that the expression (2.3) reproduce the Maxwell-based equation of state with very good precision. According to the plot obtained in [22], the maximal difference between the equations of state \( \rho = \rho(P) \) in the two cases is just 2.5% of the absolute value of \( \rho \) and, moreover, this discrepancy goes to zero pretty fast in the UV, when \( \rho \gg \rho_d \).

Let us emphasize that the difference between the equation of state which follows from relativistic Maxwell distribution and the equation of state in our simplified model is so small that it can be seen as negligible, when we use this equation of state, e.g., in the Friedmann equation. Therefore, the cosmological model which we are going to develop on this background, will be based on the following two assumptions: 1) that the massive particles (e.g. the ones of DM or baryonic matter) go from one equilibrium state to another in a sufficiently smooth way, such that the fluid composed by these particles can be described by the equation of state instead of the Boltzmann equation (if compared to the standard approach [26]). 2) That the interaction between these particles is negligible. Indeed, the main advantage of our model is that it enables one to introduce interactions between the particles in a very elegant way. We shall treat this issue in a separate paper and now concentrate on the ideal gas case.

Since the RRG model is really close to the Maxwell distribution, in what follows we shall refer to the velocity of the particles in the RRG as to “average speed”. This term will help us to keep in mind that the results of our calculations provide the reliable information not only about the proper RRG model, but also about the Maxwell-distributed relativistic gas.

Using the conservation law

\[
-\frac{d\rho}{\rho + P} = \frac{3da}{a},
\]

and the equation of state (2.3), one can easily arrive at the RRG density scaling rule

\[
\rho = \rho(z) = \rho_c^0 \frac{\Omega_M^0}{\sqrt{1 + b^2}} (1 + z)^3 \sqrt{1 + b^2 (1 + z)^2},
\]

where \( \Omega_M^0 = \Omega_{DM}^0 + \Omega_{BM}^0 \) is a total relative present-day matter energy density, \( \rho_c^0 \) is the present day critical density and \( z = -1 + a_0/a \) is the red-shift parameter. The dimensionless parameter \( b \) shows whether the velocity of the RRG particles is large or small or, in other words, whether the matter is “cold”, or “warm”, or “hot”. In order to better understand the physical sense of this parameter, let us express it in two different (albeit equivalent) forms as follows:

\[
b = \frac{\rho_d^0}{\rho^0} = \frac{\beta}{\sqrt{1 - \beta^2}}.
\]
Indeed, $b \approx 0$ means that the particles are nonrelativistic and that the RRG is nothing but the dust. Furthermore, for small velocities one can just set $b = \beta$. The main purpose of this paper is to establish the upper bound for the parameter $b$ from the analysis of cosmic perturbations.

It is easy to see that the expressions (2.1), (2.3) and (2.6) interpolate between the dust (the limit $b = 0$) and radiation ($b \to \infty$) extreme cases. It is important to note that the expression (2.5) does not represent a simple sum of the pressureless and radiation components. Instead this is a formula for the ideal relativistic gas of massive particles which undergoes an adiabatic expansion. At high red shift $z \to \infty$ the gas is compressed and its temperature is high. In this case the expression above looks as ultrarelativistic one. When $z \to -1$, the gas becomes almost pressureless and the above expression is close to the one for the dust. Of course, we are interested in the finite time intervals and, for this reason, can not separate the RRG equation of state and scale dependence (2.5) to the radiation and dust parts. Due to this feature the expression above looks as a useful tool for various problems of cosmology. In particular, here we will use the RRG model as a new tool for testing the warmness of the DM today and in the early Universe.

Let us consider the cosmological solution in the RRG model [22]. The Friedmann-Lemaître equation has the form

$$H^2(z) = \frac{8\pi G}{3}[\rho(z) + \rho_\Lambda] + H_0^2\Omega_k^0(1 + z)^2,$$  

(2.7)

where $\rho(z)$ is given by (2.5) and $\rho_\Lambda = \Lambda/8\pi G$. This equation can be presented in the explicit form

$$H^2 = H_0^2\left[\Omega_k^0(1 + z)^2 + \frac{\Omega_M^0}{\sqrt{1 + b^2}}(1 + z)^3 \sqrt{1 + b^2(1 + z)^2} + \Omega_\Lambda^0\right].$$  

(2.8)

Let us introduce the following two useful notations:

$$g(z) = \frac{(1 + z)H}{3[H^2 - \Omega_k^0H_0^2(1 + z)^2]}, \quad f_1(z) = \frac{\rho(z)}{\rho_t(z)} = \frac{(1 + z)(H^2)' - 2\Omega_k^0H_0^2(1 + z)^2}{[H^2 - \Omega_k^0H_0^2(1 + z)^2](4 - r)},$$  

(2.9)

where the prime means derivative $d/dz$. In the last formulas we denoted the ratio of the square of the rest energy density (2.2) and the energy density $\rho$ as

$$r = r(z) = \frac{\rho_t^2(z)}{\rho_t^2(z)},$$

and also applied the usual sum rule $\Omega_M^0 + \Omega_\Lambda^0 + \Omega_k^0 = 1$.

The cosmological model based on RRG with the presence of the cosmological constant admits an analytic solution [22]. This solution, as one should expect, does interpolate between the ones for the radiation and the dust cases. In the very early Universe, when the temperature is high, the evolution of the Universe is close to the one in the radiation-dominated case. At the other end of the energy scale, in the late Universe, the solution becomes close to the one for the pressureless matter case. Such interpolation between the two regimes is qualitatively similar to the more conventional case where the matter content is composed by a sum of the dust and radiation. In the conventional case, also, radiation dominates at high $z$ and dust dominates at low $z$. However, in the RRG case we observe this property in a cosmological model with a single fluid. This property makes RRG model a useful tool
for modeling the behaviour of a DM particles in the expanding Universe. The analytic zero-
order solution can be, in principle, extended for the combination of the relativistic gas of
massive particles and radiation, or for the combination of several distinct RRG-like fluids
(for example, this can be done by using the method of [28]).

3 Perturbations spectrum

Consider the cosmic perturbations in the RRG model described above. We shall follow the
scheme elaborated for the exploration of another model which is motivated by quantum
corrections [31] and consider simultaneous perturbations of metric, energy density and the
4-velocity in the co-moving coordinates

\[ U^\alpha \to U^\alpha + \delta U^\alpha, \quad \rho \to \rho (1 + \delta), \quad g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \]  

(3.1)

In the synchronous coordinates we have \( h_{0\mu} = 0 \). The perturbation of the pressure should
be derived from the eq. (2.1), such that

\[ \delta P = \frac{\delta \rho (1 - r)}{3}. \]

In this way we arrive at the following 00-component of the Einstein equation

\[ h' - \frac{2h}{(1 + z)} = -\frac{f_1 (2 - r)}{g} \delta, \]  

(3.2)

where \( \hat{h} = \partial_t \left( h_{kk}/a^2 \right) \). Other equations follow from the variation of the conservation law
\( \delta (\nabla_\mu T^\mu) = 0 \) and have the form

\[ \delta' - \frac{1}{(1 + z)} \left[ 4 - r - \frac{(1 + z)\rho'}{\rho} \right] \delta + \frac{4 - r}{3H(1 + z)} \left( \frac{\hat{h}}{2} - \frac{v}{f_1} \right) = 0, \]  

(3.3)

and

\[ v' + \left( \frac{\rho'}{\rho} - \frac{r'}{4 - r} - \frac{5}{1 + z} - \frac{f_1'}{f_1} \right) v + \frac{k^2 (1 + z) f_1}{H} \frac{1 - r}{4 - r} \delta = 0, \]  

(3.4)

where \( v = f_1 (\nabla_k \delta U^k) \) and we used Fourier expansions for \( \delta(z) \) and \( v(z) \)

\[ \delta(x, z) = \int \frac{d^3k}{(2\pi)^3} \delta(k, z) e^{ik \cdot x}, \quad v(x, z) = \int \frac{d^3k}{(2\pi)^3} v(k, z) e^{ik \cdot x}, \quad k = |k|. \]

In order to explore the equations (3.2), (3.3) and (3.4) one has to choose the initial
conditions, related to the choice of the transfer function. We have performed the numerical
analysis using two kinds of these functions. The more complicated one was introduced in [29]
and was explained in details, e.g., in [30, 31]. The second option is a more simple transfer
function from the book [7]. Both transfer functions assume a scale invariant primordial
spectrum, and determine the spectrum today considering the Universe with the cosmological
constant and filled by DM. Using the transfer functions we can fix the initial conditions at
a redshift after the recombination epoch. It is remarkable that the results for the power
spectrum obtained through these transfer functions actually coincide. The most relevant
Figure 1: Power spectrum for the RRG-Λ model, for fixed $\Omega_0^B = 0.04$, $\Omega_0^{DM} = 0.21$ and $\Omega_0^\Lambda = 0.75$ (flat Universe), with the values $b = 10^{-5}$, $b = 10^{-4}$, $b = 2 \times 10^{-4}$ and $b = 10^{-3}$. The theoretic plots are presented together with the LSS data from the 2dFGRS [27]. The ordinate axis represents the log of $P(k) = |\delta_m(k)|^2$ at $z = 0$. In the abscissa we have the log of the wave number $k$ given in $h\text{ Mpc}^{-1}$ units.

The relevant quantity to be compared with the observational data is the power spectrum parameter defined by

$$P_k = \delta_k^2,$$

where $\delta_k$ is the component of the Fourier transform of the density contrast $\delta(t)$, which is computed by integrating the equations for the cosmic perturbations (3.2), (3.3) and (3.4) for a given value of $k$ and with a given initial conditions (as indicated above).

In the present case, since the upper bound for the possible values of $b$ has great physical relevance, it proves useful to establish this bound with more certainty. For this end we performed calculation for a set of different values of $b$ and then applied the statistical method to compare the result to the power spectrum data of the 2dFGRS observational program. The quality of the fit between the theoretical estimate and the observational data can be
characterized by the quantity

\[ \chi^2 = \sum_i \left( \frac{P_{oi}^{k_i} - P_{ti}^{k_i}}{\sigma_i} \right)^2 , \]

(3.6)

where \( P_{oi}^{k_i} \) is the observational data for the power spectrum for a given wavenumber \( k_i \), \( P_{ti}^{k_i} \) is the corresponding theoretical result obtained by the numerical integration of the equations for the perturbations (3.2), (3.3), (3.4) and \( \sigma_i \) are the observational error bars related to the measurement. As smaller is the parameter \( \chi^2 \), better is the fit. Of course, since our theoretical model depends on some input parameters such as \( b, \Omega_0^M \) and \( \Omega_0^\Lambda \), the value of \( \chi^2 \) depends also on these parameters. At a first step, in order to obtain estimations for the relevant parameters, we reduce the three-dimensional probability distribution to the one-dimensional one by choosing the values of the present day cosmological parameters

\[ \Omega_0^M = \Omega_B^0 + \Omega_{DM}^0 = 0.04 + 0.21 = 0.25 \quad \text{and} \quad \Omega_0^\Lambda = 0.75 , \]

which correspond to the flat space section of the space-time manifold.

Using the quantity \( \chi^2 \), the probability distribution is given by

\[ P = A e^{-\chi^2/2} , \]

(3.7)

where \( A \) is a normalization constant. The final result for the one-dimensional probability distribution function (PDF) for the parameter \( b \) is displayed in figure 2. Using this plot we can see that this probability distribution goes sharply to zero for the values above \( b = 5 \times 10^{-5} \).

We can conclude that there is an upper bound for the “warmness” parameter \( b \), that means \( b \leq 3 \times 10^{-5} \sim 4 \times 10^{-5} \).

Before starting to discuss physical interpretations of our results, let us present some comments concerning the validity and restrictions of the bound on the parameter \( b \) obtained above. The power spectrum, which is defined in the space of the \( k' \)’s, is the Fourier transformation of the two-point correlation function, which is defined in the real space, \( x \). Hence,
the power spectrum implies a Fourier decomposition. The different Fourier modes are independent only when the linear approximation is valid; non-linearity leads to the mixing of the different Fourier modes, which do not remain independent anymore. The non-independence of Fourier modes must be taken into account when the statistical analysis in the $k$ space is performed. This is done through the use of the covariance matrix, $C_{ij}$. According to this method, the likelihood function for a given set of data $\Delta_i$ is given by

$$L \propto \exp\left\{ -\frac{1}{2} \Delta_i (C^{-1})_{ij} \Delta_j \right\},$$

(3.8)

where $(C^{-1})_{ij}$ is the inverse of the covariance matrix. If the conditions of linearity are satisfied, the covariance matrix becomes diagonal, and the likelihood function reduces to the usual expression encoded in the $\chi^2$ parameter.

We will use the 2dFGRS data such that $0 < k h^{-1} < 0.185 \, Mpc^{-1}$ [27]. This interval lies outside the non-linear regime, generally fixed as $k h^{-1} \gtrsim 0.8 \, Mpc^{-1}$, corresponding to a scale greater than $8 \, Mpc$. If the linear approximation is valid, the covariance matrix is diagonal, and the use of a $\chi^2$ statistic is justified. However, the limits on the wavelength where the linear regime can be safely applied is not very well established. In reference [33], it has been argued that the confidence on the linear approximation for the 2dFGRS data restricts the scales to $k h^{-1} \leq 0.15 \, Mpc^{-1}$. On the other hand, the cosmic variance leads to a restriction such that $k h^{-1} \geq 0.02 \, Mpc^{-1}$. If we use data ranging in the interval $0.02 \, Mpc^{-1} \leq k h^{-1} \leq 0.15 \, Mpc^{-1}$ the results does not change substantially, as it will be discussed latter.

To obtain a better estimate for the velocities of the DM particles in halo of galaxies, we must enter deeply into the non-linear regime. All considerations presented in this work are based on the applicability of the linear analysis and therefore we assume it to be valid. Using the matter power spectra data, we can not avoid in this case the use of the full covariance matrix, since the different modes are not independent anymore (see for example reference [34] for a beautiful analysis of the non-linear effects and the consequent use of the covariance matrix in the context of the baryonic acoustic oscillations). Hence, our previous estimations must be seen as a lower bound, since the process of contraction in the formation of the galaxy must increase the velocities of the particles.

Furthermore, in the previous analysis, we have fixed the quantity of dark matter and dark energy, based on the 5-years results of Wmap. In order to verify how this restriction may influence the obtained bounds for the parameter $b$, we now consider a two-dimensional parameter space, varying at the same time $b$ and the dark matter quantity $\Omega_{dm0}$ (or equivalently the dark energy density $\Omega_{\Lambda 0}$, since we are restricted to a flat spatial section). The results are shown in figure 4 using the data $0 \leq kh^{-1} \leq 0.185 \, Mpc^{-1}$, and in figure 3 using $0.2 \, Mpc^{-1} \leq kh^{-01} \leq 0.150 \, Mpc^{-1}$. According to these plots, the bounds for the value of $b$ remain essentially the same when we perform these additional variations of other cosmic parameters. Moreover, they are also in a good agreement with the case when we used only the best fit value for $\Omega_{\Lambda 0}$ and $\Omega_{dm0}$ given by Wmap.

Now we are in a position to discuss physical significance of the bounds obtained from cosmic perturbations analysis. The restriction for the parameter $b$ which we derived from the numerical analysis of cosmic perturbations, can be easily translated into the bound for the average velocity (or warmness) of the DM constituents. For this purpose we have to note that $b$ is necessarily small and therefore the relation (2.6) converts into $b = \beta = \nu/c$. Then we arrive at the bound for the average speed of the massive relativistic particles of the DM in the
Figure 3: The two-dimensional probability distribution, for the flat spatial section, when both $b$ and $\Omega_{\Lambda 0}$ are varied (left), considering the modes $0 \leq k h^{-1} \leq 0.185 Mpc^{-1}$. Higher probabilities are indicated by the brighter regions. The one-dimensional probability distribution for $b$, after marginalizing on $\Omega_{\Lambda 0}$ is shown at the plot on the right.

Figure 4: The two-dimensional probability distribution, for the flat spatial section, when both $b$ and $\Omega_{\Lambda 0}$ are varied (left), but restricting the modes to $0.2 Mpc^{-1} \leq k h^{-01} \leq 0.150 Mpc^{-1}$. Higher probabilities are indicated by the brighter regions. The one-dimensional probability distribution for $b$, after marginalizing on $\Omega_{\Lambda 0}$ is shown at right.

present-day Universe $v \leq v_0$, $v_0 \approx 10 - 12 \, \text{km/s}$. This bound agrees with the standard evaluations (see, e.g., [4, 6]) obtained from the numerical simulations of the structure formation in the neutrono-dominated Universe and also from the model-dependent considerations.
It looks tentative to compare this bound for the speeds of the DM particles with some astronomic observable, as it is usually done in the framework of standard approaches [26]. One can, for instance, try to compare this bound to the known one for the spiral galaxies, which is about\(^2\) 240 km/s. Obviously, there is no correspondence between the two numbers. However, let us note that the galaxy is an object which definitely lies out of the linear perturbation regime which we deal with here. In general, any comparison of the results obtained from the linear approximation to the cosmological perturbations, including the one for the upper limit of the velocity of DM particles with the dynamics of virialized systems in the Universe (galaxies, clusters of galaxies, etc.) must be performed with great caution, because those virialized objects are in the deep non-linear regime, with typical densities hundreds of times larger than the critical density.

In general, the estimations of velocity of dark matter particles strongly depend on the nature of the DM candidate under consideration and especially on the reference frame where the velocity is evaluated. For example, in the paper [35] were obtained a dispersion of primordial velocities (which is not affected by non-linear effects) with respect to the Hubble flow. The result is of the order of magnitude about \(10^{-12}\) in unities of the velocity of light for the WIMPS with a typical mass of a few GeV’s and about \(10^{-17}\) in the same units for the axion, even if the typical mass of the axions is some \(10^{16}\) orders smaller than the typical mass of the WIMPs. This case shows how the nature of the dark matter candidate influences the estimate for its primordial velocity. In reference [36], the possibility that the warm dark matter can be described by sterile neutrinos has been analysed. The possible mass range goes from 1 keV to tenths of keV. Now, using the results for the WIMPS quoted above, and the usual non-relativistic relation for the ratio of velocities of particles having the same energy but different masses, we find an upper limit for the velocity of the warm dark matter particle of the order of \(10^{-9}\) in unities of \(c\). This is far below our upper limit near the maximum of probability distribution. On the other hand, the two estimates correspond to distinct reference frames and therefore these results do not contradict our analysis.

We could try to use, for instance, the data of weak lensing investigations concerning the dark halo of galaxies. But, the results obtained in this way still concern a very non-linear regime, see for example [37]. In order to chose the appropriate object, we note that among the galaxies, the dwarf spheroidal elliptical ones are those with the large proportion of dark matter to baryonic matter, with a mass/luminosity ratio that can be as large as 500, in solar units, while ordinary galaxies have a ratio of the order of some 10 – 50 [38]. Even if these objects are deep in the non-linear regime it is remarkable that the dispersion velocity becomes essentially constant far from the center with a typical value of the order of \(v \sim 10 \text{ km/s}\) [39], comparable with the velocity bounds we have obtained for the dark matter particle.

Since the dwarf spheroidal galaxies represent the extreme case of virialized system dominated by dark matter, such agreement of the typical velocities with our bound is perhaps not meaningless, even if a clear determination of the extension of the dark halo would be necessary to put this comparison into more solid grounds. For the dwarf irregular or spiral galaxies the typical radial speed of the stars is evaluated to be about 10 – 12 km/s [40] and we meet a nice correspondence with our bound. It is amusing that we arrived at this correspondence by using a very simple RRG model [22] and not the complicated approach based on the Einstein-Boltzmann system [26].

\(^2\)This approach is close to the one of [32], which is based on the Maxwellian dark matter velocity distribution for spherically symmetric and isotropic halo.
The last problem which we can easily address within the RRG model is the dynamics of the DM average speeds in the expanding Universe. In other words, it would be interesting to calculate how this speed depends on the red-shift parameter $z$, or on the temperature $T_{\text{CMB}}$ of the cosmic background radiation (CMB). In order to address this issue one has to use the relations (2.6) and (2.5). The unique role of the parameter $b$ is to define the “warmness” of the DM in the last of these relations, so it is obvious that at the higher $z$ we have $b(z) = b(1 + z)$, where $b$ is the modern value. Furthermore, since the upper bound for $b$ nowadays is much less than $10^{-4}$, for the potentially relevant $z \leq 1000$ we can, according to (2.6), safely use the formula $v = bc$. In this way we arrive at the following relation for the average speed of the DM particles

$$v(z) = cb(z) = cb(1 + z) = v \times \frac{T_{\text{CMB}}}{T_{\text{CMB}}^{(0)}},$$

(3.9)

where we used $T_{\text{CMB}} \sim (1 + z)$ and denoted $T_{\text{CMB}}^{(0)}$, $b$ and $v$ the corresponding quantities for $z = 0$. So, in the framework of the RRG model we note that the average kinetic energy of the DM particles is proportional to the square of the CMB temperature, the result which is familiar from the conventional (but more complicated) considerations (see, e.g., [9]).

4 Discussions and conclusions

We have considered the structure formation in the model where the DM is described by the ideal relativistic gas of identical massive particles. Instead of using Maxwell distribution, we have employed the RRG model [22] which is closely reproducing Maxwell distribution and, at the same time, is rather simple. As a result we arrive at the strong limit on the parameter $b$, which should satisfy the upper bound $b \leq 3 - 4 \times 10^{-5}$. According to the relation (2.6), this is equivalent to the upper bound on the velocities of the DM particles $v \leq v_0 = 3 - 4 \times 10^{-5}c = 10 - 12 \text{km/s}$. This restriction is much more severe than, e.g., the one discussed earlier in [41–44] on the basis of the nonrelativistic Maxwell distribution [42] and is essentially smaller than the typical velocities of the stars in the spiral galaxies. Also, it is about two order of magnitude smaller than the escape velocities for the spiral galaxies.

Does our result mean that the actual velocities of DM particles can not be greater than the mentioned bound $v_0$? An obvious answer is no. Let us remember that both DM and baryonic matter can acquire an extra kinetic energy after the galaxy starts to form and the linear regime of the cosmological perturbations can not be applied. One can see the corresponding process as a kind of the usual transformation of the potential gravitational energy into the kinetic one. This process has nothing to do with the linear perturbations we have studies here. However taking smaller astrophysical objects such as dwarf galaxies, we arrive at the surprisingly nice correspondence between the observed average speeds of the stars in such galaxies and our upper bound $v_0$. This correspondence shows that the RRG model is, perhaps, the simplest way to arrive at the reasonable estimates concerning not only the behavior of the conformal factor, but also the linear cosmological perturbations.

The result described above is universal in the sense there is no dependence on the origin and properties of the WDM constituents. In particular, the restriction on velocities does not interfere with the one for the masses of the WDM particles, which can be even macroscopic ones. We were just treating them at a component of ideal relativistic gas and derive restrictions on their velocities. Hence, these restrictions apply equally to all known models of WDM, see, e.g., refs. [18, 20, 21, 45, 46]. For example, they apply to the models of
DM particles which do not interact with other matter and with themselves, except gravitationally [8, 47, 48]. According to our results, even this simplified model can produce good predictions for the spectrum of linear perturbations, if we assume the ideal gas equation of state for these weakly interacted particles. Another interesting point is that our results show that the anisotropy of the DM distribution does not play a critical role in the definition of the perturbations spectrum and that the equilibrium distributions such as the Maxwell one (which is closely reproduced by RRG) is, in principle, sufficient to arrive at the reasonable estimates for the speed of the DM particles.

Last, but not least. The RRG model may be successfully applied for the investigation of more complicated situations, including two distinct non-interacting ideal gases [49] and also may be useful in describing the interaction between these gases, e.g. through the use of viscosity (see, e.g., [25]). We expect to explore these issues in the near future. In general, our model proved useful in exploring relativistic properties of the ideal gas of massive particles, it can be applied for solving various problems of gravitational physics.

In order to verify to which extent the model described above may correct the excess of power of the ΛCDM at small scales, the non-linear regime may be explored. As a result one may hope to achieve a more detailed description of the structure formation. At the moment it is unclear whether the RRG model can be a useful tool in such case. We hope to explore this issue at the consequent stage of our work.

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