THE KATOK’S ENTROPY FORMULA FOR AMENABLE GROUP ACTIONS

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Abstract. In this paper we generalize Katok’s entropy formula to a large class of infinite countably amenable group actions.

1. Introduction. Topological entropy was introduced in [1] by Adler, Konheim and McAndrew who formulated it in terms of open covers, in analogy with the Kolmogorov-Sinai measure entropy picture. Here too topological entropy turned out to be a very basic invariant with many applications. Topological entropy is related to measure entropy by the variational principle which asserts that for a continuous map on a compact metric space the topological entropy equals the supremum of the measure entropy taken over all the invariant probability measures. For other related work, see [2, 4, 5, 7, 8, 9, 12, 15, 16].

Definitions of topological entropy based on separated and spanning sets with respect to a metric were given independently by Bowen [3] and Dinaburg [6]. In [10], for Z-action dynamical system, Katok introduce an entropy with respect to Borel probability T-invariant ergodic measure µ by a similar manner. The metric entropy turns out to be the asymptotic value of the same kind with some subsets of positive

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measure instead of the whole space $X$. The Katok’s entropy has the advantage to simplify some computations and has many applications in diffeomorphic dynamical systems.

In order to state Katok’s result we proceed to define the relevant quantities. Let $(X, d)$ be a compact metric space and $T : X \to X$ a homeomorphism of $X$, and $d_n$ an increasing system of metrics on $X$ defined by:

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y).$$

Denote by $M_T(X)$ the space of $T$-invariant probability measures on $X$ and $\mathcal{B}(X)$ the set of all Borel subsets of $X$. Let $\epsilon > 0$. A set $Z \subseteq X$ is called $(d_n, \epsilon)$-spanning if for any $x \in X$ there exists some $z \in Z$ with $d_n(x, z) < \epsilon$. Denote by $\text{span}(X, d_n, \epsilon)$ the minimal cardinality of $(d_n, \epsilon)$-spanning subsets of $X$.

For $\mu \in M_T(X)$, $\epsilon > 0$ and $0 < \delta < 1$, Katok defined $N_\mu(n, \epsilon, \delta)$ as the minimum number of $\epsilon$-balls in the $d_n$-metric which cover a set of measure strictly bigger than $1 - \delta$ i.e.,

$$N_\mu(n, \epsilon, \delta) = \inf_{\mu(X') > 1 - \delta} \text{span}(X', d_n, \epsilon).$$

In [10] Katok obtained an entropy formula as follows:

**Theorem 1.1.** Let $\mu$ be an ergodic $T$-invariant probability measure. For every $1 > \delta > 0$,

$$h_\mu(T) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(n, \epsilon, \delta)}{n} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log N_\mu(n, \epsilon, \delta)}{n}.$$

In this paper, we shall discuss extensions of Katok’s entropy formula to the class of infinitely countable discrete amenable groups. The class of amenable groups includes all finite groups, solvable groups and compact groups, and actions of these groups on a compact metric space are a natural extension of the $\mathbb{Z}$-actions considered in Katok’s theorem: the foundations of the theory of amenable group actions were laid by Ornstein and Weiss in their pioneering paper [14].

Lindenstrauss [13] established the Shannon-McMillan-Breiman theorem for countably amenable group actions along tempered Følner sequences (with some mild growth conditions). Based on Lindenstrauss’s result, we obtain the extension of Katok’s entropy formula to a dynamic system of a countable discrete amenable group actions.

**Main Theorem.** Let $G$ be a countably infinite amenable group. Let $G \curvearrowright X$ be a continuous action on the compact metric space $(X, d)$ and $\mu$ be an ergodic and $G$-invariant Borel probability measure. For every $0 < \delta < 1$, one has

$$h_\mu(X, G) = \lim_{\epsilon \to 0} \liminf_{F} \frac{\log N_\mu(F, \epsilon, \delta)}{|F|} = \lim_{\epsilon \to 0} \limsup_{F} \frac{\log N_\mu(F, \epsilon, \delta)}{|F|}.$$

For the definitions of limsup and liminf in Theorem 1.1, please refer to (1) and (2) in Section 2. For the definition of the number $N_\mu(F, \epsilon, \delta)$, please refer to (6) in Section 3.

2. **Amenable group.** In this section, we give some basic properties of countably amenable group.

Let $G$ be a countable discrete group. Denote by $\mathcal{F}(G)$ the set of all nonempty finite subsets of $G$. For $K \in \mathcal{F}(G)$ and $\delta > 0$ write $\mathcal{B}(K, \delta)$ for the collection of all
$F \in \mathcal{F}(G)$ satisfying $|KF\setminus F| < \delta |F|$. The group $G$ is called amenable if $\mathcal{B}(K, \delta)$ is nonempty for every $(K, \delta)$.

The collection of pairs $(K, \delta)$ forms a net $\Lambda$ where $(K', \delta') \succ (K, \delta)$ means $K' \supseteq K$ and $\delta' \leq \delta$. For an $\mathbb{R}$-valued function $\varphi$ defined on $\mathcal{F}(G)$, we define

$$\limsup_{F} \varphi(F) := \lim_{(K, \delta) \in \Lambda} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F)$$

and

$$\liminf_{F} \varphi(F) := \lim_{(K, \delta) \in \Lambda} \inf_{F \in \mathcal{B}(K, \delta)} \varphi(F).$$

From the definition of the partial order $\succ$ it is clear that $\mathcal{B}(K', \delta') \subseteq \mathcal{B}(K, \delta)$ if $(K', \delta') \succ (K, \delta)$. Thus it follows that

$$\limsup_{F} \varphi(F) = \inf_{(K, \delta) \in \Lambda} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F),$$

$$\liminf_{F} \varphi(F) = \sup_{(K, \delta) \in \Lambda} \inf_{F \in \mathcal{B}(K, \delta)} \varphi(F).$$

Furthermore, we state the Følner property of the infinitely countable amenable group which is one of fundamental characterizations of amenability. The Følner property is useful for exhibiting amenable groups which are not locally finite.

**Definition 2.1.** A sequence $\{F_n\}$ of nonempty finite subsets of $G$ is called a (left) Følner sequence if $|sF_n \Delta F_n|/|F_n| \to 0$ as $n \to \infty$ for every $s \in G$.

A key point in proving pointwise convergence results for amenable group action is working with appropriate Følner sequences. Now we introduce the following condition introduced by A. Shulman.

**Definition 2.2.** A sequence $\{F_n\}$ of nonempty finite subsets of $G$ is said to be tempered if there is a $b > 0$ such that $|\bigcup_{k=1}^{n-1} F_k^{-1} F_n| \leq b|F_n|$ for every $n > 1$.

In this paper, we need the following proposition which is stated in [13].

**Proposition 1.** Every Følner sequence $\{F_n\}$ has a tempered subsequence. In particular, every amenable group has a tempered Følner sequence.

Lindenstrauss [13] established the Shannon-McMillan-Breiman theorem for countably amenable group actions along tempered Følner sequences $\{F_n\}$ with $|F_n|/\log n \to \infty$. Such tempered Følner sequences with that mild growth condition always exist, as one can start with any Følner sequence and recursively construct a subsequence with these properties. For completeness, we will illustrate the existence of the tempered Følner sequences $\{F_n\}$ with $|F_n|/\log n \to \infty$ for an infinitely countable amenable group $G$ in the next few paragraphs.

**Definition 2.3.** Let $K, F$ be nonempty finite subsets of $G$ and $\delta > 0$. We say that $F$ is $(K, \delta)$-invariant if $|\{s \in F : Ks \subseteq F\}| \geq (1 - \delta)|F|$.

**Remark 1.** The above definition is a less intuitive formulation of approximate invariance than the one implicitly expressed by Definition 2.1. More precisely, the sequence $\{F_n\}$ is Følner if and only if for every finite set $K \subseteq G$ and $\delta > 0$ there is an $N \in \mathbb{N}$ such that $F_n$ is $(K, \delta)$-invariant for all $n \geq N$ (see [11] p.92-93).

With above definitions and arguments, we have the following results.

**Fact 2.4.** Let $G$ be an infinitely countable amenable group and $\{F_n\}$ be a Følner sequence of $G$. Then $\lim_{n \to \infty} |F_n| = +\infty$. 
Proof. Otherwise, there is a constant $M > 0$ such that $|F_n| \leq M$ for all $n \in \mathbb{N}$. Since $G$ is infinite, we can choose a nonempty finite subset $K$ of $G$ with $|K| > M + 1$. Thus, for any $n \in \mathbb{N}$, one has

$$\{ s \in F_n : Ks \subseteq F_n \} = \emptyset.$$ 

Therefore, for every $n \in \mathbb{N}$, $F_n$ is NOT $(K, 1/2)$-invariant which contradicts the conclusion in Remark 1.

Hence Fact 2.4 is obtained. \hfill \Box

**Proposition 2.** Let $G$ be an infinitely countable amenable group $G$. Then every Følner sequence of $G$ has a tempered Følner subsequence $\{ F_n \}$ with $|F_n|/\log n \to \infty$.

**Proof.** Let $\{ F'_n \}$ be a Følner sequence of $G$. From Proposition 1, we may assume that the Følner sequence $\{ F'_n \}$ is tempered. By Fact 2.4, we know that $\lim_{n \to \infty} |F'_n| = +\infty$. Thus, for each $n \in \mathbb{N}$, there exist positive integer $m_n$ with $m_n > m_{n-1}$ ($m_0 = 0$) such that $|F_{m_n}|/\log n > n$. Set $F_n = F'_{m_n}$. Then the sequence $\{ F_n \}$ is our desired. \hfill \Box

Let $G$ be a countably infinite amenable group. Thus we can find a family of finite subsets $\{ G_n \}_{n=1}^\infty$ of $G$ which satisfies that $e \in G_1 \subseteq G_2 \subseteq \cdots$ and $G = \bigcup_{n=1}^\infty G_n$.

For the sake of our following proofs, we present some simple facts and give the proofs for completeness here.

**Fact 2.5.** Let $\{ G_n \}_{n=1}^\infty$ be a family of finite subsets of $G$ which satisfies that $e \in G_1 \subseteq G_2 \subseteq \cdots$ and $G = \bigcup_{n=1}^\infty G_n$. For an $\mathbb{R}$-valued function $\varphi$ defined on $\mathcal{F}(G)$, one has

$$\limsup_{F} \varphi(F) = \lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F),$$

$$\liminf_{F} \varphi(F) = \lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F).$$

**Proof.** We only to prove the first equation. The proof of the second equation is similar.

From (3) in this section, it follows that

$$\limsup_{F} \varphi(F) \leq \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F) \quad \text{for each} \quad n \in \mathbb{N}.$$ 

Thus, one has

$$\limsup_{F} \varphi(F) \leq \lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F).$$

Let $K \in \mathcal{F}(G)$ and $\delta > 0$. Then there exists $N = N(K, \delta) \in \mathbb{N}$ such that $K \subseteq G_n$ and $1/n < \delta$ for all $n \geq N$ which means that $\mathcal{B}(G_n, 1/n) \subseteq \mathcal{B}(K, \delta)$ ($n \geq N$). Thus, we have

$$\sup_{F \in \mathcal{B}(K, \delta)} \varphi(F) \geq \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F) \quad \text{for all} \quad n \geq N,$$

which implies

$$\sup_{F \in \mathcal{B}(K, \delta)} \varphi(F) \geq \lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F).$$

From the arbitrariness of $(K, \delta)$ we get

$$\inf_{(K, \delta) \in \mathcal{A}} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F) \geq \lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F).$$

Hence the first equation is proved. \hfill \Box
Fact 2.6. Let $G$ be a countably infinite amenable group and $\varphi$ an $\mathbb{R}$-valued function defined on $\mathcal{F}(G)$. Then there exist two Følner sequences $\{F_m\}$ and $\{F_m'\}$ such that
\[
\limsup_F \varphi(F) = \lim_{m \to \infty} \varphi(F_m),
\]
\[
\liminf_F \varphi(F) = \lim_{m \to \infty} \varphi(F'_m).
\]

Proof. We only to prove the first equation. By Fact 2.5, we have
\[
\limsup_F \varphi(F) = \lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F).
\]
We may assume that $\limsup_F \varphi(F)$ is finite. For any $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that
\[
\limsup_F \varphi(F) - \frac{1}{m} < \sup_{F \in \mathcal{B}(G_{n_m}, 1/n_m)} \varphi(F) < \limsup_F \varphi(F) + \frac{1}{m}.
\]
Thus there is a finite subset $F_m \in \mathcal{B}(G_{n_m}, 1/n_m)$ such that
\[
\limsup_F \varphi(F) - \frac{1}{m} < \varphi(F_m) < \limsup_F \varphi(F) + \frac{1}{m}.
\]
Since $\bigcup_{m=1}^{\infty} G_{n_m} = G$ and $\lim 1/n_m = 0$, it is easy to see that $\{F_m\}$ is a Følner sequence of $G$ and
\[
\lim \varphi(F_m) = \limsup_F \varphi(F).
\]
Hence the fact is proved. \qed

3. The Shannon entropy for the amenable group action. Let $(X,d)$ be a compact metric space and $G$ be a countably amenable group with the identity element $e$. Throughout $G \curvearrowright X$ is a continuous action on a compact metric space $(X,d)$.

We write $M(X)$ for the set of Borel probability measures on $X$, which is a weak* compact subset of the dual space $C(X)^*$. We write $M_G(X)$ for the weak* closed set of $G$-invariant measures in $M(X)$.

Let $\mu \in M(X)$ be a probability measure and $\mathcal{P} = \{A_1, \cdots, A_m\}$ a finite partition of $X$. The information function of $\mathcal{P}$ is defined as
\[
I_{\mathcal{P}} = -\sum_{i=1}^{m} 1_{A_i} \log \mu(A_i).
\]
The (Shannon) entropy of $\mathcal{P}$ is then defined as
\[
H(\mathcal{P}) = \int_X I_{\mathcal{P}} \, d\mu = -\sum_{i=1}^{m} \mu(A_i) \log \mu(A_i).
\]
We also give a condition version of Shannon entropy with respect to a second finite partition $\mathcal{Q} = \{B_1, \cdots, B_l\}$. The conditional information function is defined as
\[
I_{\mathcal{P}, \mathcal{Q}} = -\sum_{j=1}^{l} \sum_{i=1}^{m} 1_{A_i \cap B_j} \log \frac{\mu(A_i \cap B_j)}{\mu(B_j)}.
\]
The Conditional (Shannon) entropy of $\mathcal{P}$ given $\mathcal{Q}$ is then defined by
\[
H(\mathcal{P} | \mathcal{Q}) = \int_X I_{\mathcal{P}, \mathcal{Q}} \, d\mu = \sum_{j=1}^{l} \sum_{i=1}^{m} -\mu(A_i \cap B_j) \log \frac{\mu(A_i \cap B_j)}{\mu(B_j)}.
\]
Proposition 3. Let $\mathcal{P}, \mathcal{Q}$ be finite partitions of $X$, and let $\mu \in M(X)$ be a probability measure. Then

- if $\mathcal{P} \leq \mathcal{Q}$, then $H(\mathcal{P}) \leq H(\mathcal{Q})$.
- $H(\mathcal{P} \vee \mathcal{Q}) = H(\mathcal{P}) + H(\mathcal{Q} | \mathcal{P})$.

Notation. For a finite partition $\mathcal{P}$ of $X$ and a empty finite set $F \subseteq G$, we write $\mathcal{P}^F$ for the join $\bigvee_{s \in F} s^{-1} \mathcal{P}$, unless $F$ is empty in which case we interpret this as the trivial partition $\{X\}$.

Let $\mu \in M_G(X)$. For each finite partition $\mathcal{P}$ of $X$ the function $F \to H(\mathcal{P}^F)$ on the collection of finite subsets of $G$ satisfies two conditions: $G$-invariant and Shearer’s inequality. Thus

$$\lim_{n \to \infty} \frac{1}{|F_n|} H(\mathcal{P}^{F_n}) = \inf_{F} \frac{1}{|F|} H(\mathcal{P}^F)$$

where $F$ ranges over nonempty finite subsets of $G$ and $\{F_n\}$ is a Følner sequence of $G$.

Definition 3.1. Let $\mu \in M_G(X)$. For a finite partition $\mathcal{P}$ of $X$ we define $h_\mu(\mathcal{P})$ to be the above limit. The entropy of the action $G \acts (X, \mu)$ is then defined as

$$h_\mu(X, G) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}).$$

Let $\mathcal{P} = \{A_1, \cdots, A_m\}$ a finite partition of $X$. We denote by

$$\text{diam}(\mathcal{P}) = \max_{1 \leq i \leq m} \text{diam}(A_i) \quad \text{and} \quad \partial(\mathcal{P}) = \bigcup_{i=1}^{m} \partial(A_i)$$

where $\text{diam}(A_i)$ denotes the diameter of the set $A_i$ and $\partial(A_i)$ denote the boundary of the set $A_i$.

In order to prove our result, we need the following lemmas. For the detailed proofs of these lemmas, please refer to [11].

Lemma 3.2. Let $(X, d)$ be a compact metric space, $\mu \in M(X)$ and $\kappa > 0$. Then there is a finite partition of $X$ whose member all have diameter less than $\kappa$ and $\mu$-measure zero.

Lemma 3.3. Let $\mathcal{P}$ be a finite partition of $X$ and let $\epsilon > 0$. Then there exists a $\eta > 0$ such that, for every finite partition $\mathcal{Q}$ of $X$ with the property that for all $A \in \mathcal{P}$ there is a set $B$ in the $\sigma$-algebra generated by $\mathcal{Q}$ satisfying $\mu(A \Delta B) < \eta$, one has

$$H(\mathcal{P} | \mathcal{Q}) < \epsilon.$$

Lemma 3.4 (The combinations bound). If $\binom{n}{k}$ denotes the number of combinations of $n$ objects taken $k$ at a time and $\delta < 1/2$ then

$$\sum_{k \leq n \delta} \binom{n}{k} \leq 2^{nH(\delta)}$$

where $H(\delta) = -\delta \log \delta - (1-\delta) \log(1-\delta)$.

The conclusion of the following lemma is well known, we give a proof here for completeness.
Lemma 3.5. Let \((X,d)\) be a compact metric space and \(\mu \in M(X)\). Suppose that \(\mu\) is \(G\)-invariant and \(h_\mu(X,G) < \infty\). Then, for every \(\epsilon > 0\), there exists a finite partition \(\mathcal{P}\) of \(X\) such that \(\text{diam}(\mathcal{P}) < \epsilon\), \(\mu(\partial(\mathcal{P})) = 0\) and
\[
h_\mu(X,G) < h_\mu(\mathcal{P}) + \epsilon.
\]

Proof. Let \(\epsilon > 0\). By the definition of \(h_\mu(X,G)\), there is a finite partition \(\mathcal{Q} = \{A_1, \cdots, A_r\}\) of \(X\) such that \(\mu(A_i) > 0 (i = 1, \cdots, r)\) and
\[
h_\mu(X,G) < h_\mu(\mathcal{Q}) + \epsilon/2.
\]

Let \(0 < \eta = \eta(\epsilon/2) < \epsilon\) be as in Lemma 3.3. Since every probability measure on the compact metric space is regular, there exist the compact sets \(K_1 \subseteq A_i\) \((i = 1, \cdots, r)\) such that
\[
\mu(A_i \setminus K_i) < \frac{\eta}{r+1}.
\]

Denote by
\[
\kappa = \min \left\{ \eta, \min_{i \neq j} \text{dist}(K_i, K_j) \right\}.
\]

By Lemma 3.2, there is a finite partition \(\mathcal{P}\) of \(X\) such that \(\text{diam}(\mathcal{P}) < \kappa/2\) and \(\mu(\partial(\mathcal{P})) = 0\). For \(1 \leq i < r\), we define
\[
B_i = \bigcup_{p \in \mathcal{P}, B \neq \emptyset} p.
\]

and
\[
B_r = X \setminus \bigcup_{i=1}^{r-1} B_i.
\]

Thus, we get a finite partition \(\mathcal{P}^* = \{B_1, \cdots, B_r\}\). It is clear that \(\mathcal{P}^* \leq \mathcal{P}\).

Claim. For \(1 \leq i \leq r\), one has \(B_i \setminus A_i \subseteq X \setminus \bigcup_{j=1}^{r} K_j\).

Let \(x \in B_i \setminus A_i\) for \(1 \leq i \leq r\).

Suppose that \(1 \leq i < r\). Due to \(K_i \subseteq A_i\), we have \(x \not\in K_i\). Assume that \(x \in p\) for some \(p \in \mathcal{P}\) with \(p \cap K_i \neq \emptyset\). If \(x \in K_j\) for some \(j \neq i\), \(p \cap K_j \neq \emptyset\). Thus one has \(\text{dist}(K_i, K_j) \leq \text{diam}(p) < \kappa/2\) which contradicts that \(\text{dist}(K_i, K_j) \geq \kappa\). Hence \(x \in X \setminus \bigcup_{i=1}^{r} K_i\).

Suppose that \(x \in B_r \setminus A_r\). Since \(K_r \subseteq A_r\), it follows that \(x \not\in K_r\). Combing with the assumption \(x \in B_r\) and \(B_r = X \setminus \bigcup_{i=1}^{r-1} B_i\), we deduce that \(x \not\in B_i\) for \(1 \leq i \leq r - 1\). Noting that \(K_i \subseteq B_i\), we know that \(x \not\in K_i\) for \(1 \leq i \leq r - 1\). Thus \(x \in X \setminus \bigcup_{i=1}^{r} K_i\).

Hence the claim is obtained.

For \(1 \leq i \leq r\), noting \(K_i \subseteq B_i\) and Claim, it is not hard to see that
\[
\mu(B_i \Delta A_i) \leq \mu(B_i \setminus A_i) + \mu(A_i \setminus B_i)
\]
\[
\leq \mu(X \setminus \bigcup_{j=1}^{r} K_j) + \mu(A_i \setminus K_i)
\]
\[
< \frac{r \eta}{r+1} + \frac{\kappa}{r+1}
\]
\[
\leq \eta.
\]

By Lemma 3.3, we have
\[
H(\mathcal{Q} | \mathcal{P}^*) < \frac{\epsilon}{2}.
\]
Thus, it follows that
\[ h_\mu(X, G) - \frac{\epsilon}{2} < h_\mu(\mathcal{Q}) \leq h_\mu(\mathcal{R}^*) + H_\mu(\mathcal{Q}|\mathcal{R}^*) \]
\[ < h_\mu(\mathcal{P}) + \frac{\epsilon}{2}. \]
Hence we get
\[ h_\mu(X, G) < h_\mu(\mathcal{P}) + \epsilon. \]

**Definition 3.6.** Let \( \mathcal{P} \) be a finite partition of \( X \) and \( F \) be a nonempty finite subset of \( G \). We define the Hamming pseudometric \( H_{\mathcal{P}, F}(\cdot, \cdot) \) on \( X \) as follows:

For any \( x, y \in X \), we denote by
\[ F_{x,y} = \{ s \in F : \text{the points of } sx \text{ and } sy \text{ are NOT in the same member of } \mathcal{P} \}. \]

The Hamming pseudometric \( H_{\mathcal{P}, F}(x, y) \) is then defined as
\[ H_{\mathcal{P}, F}(x, y) = \frac{|F_{x,y}|}{|F|}. \]

**Notation.** Let \( \mathcal{P} \) be a finite partition of \( X \). For \( x \in X \) we write \( \mathcal{P}(x) \) for the member of \( \mathcal{P} \) which contains \( x \).

Actually, we have an alternative view on the quality \( H_{\mathcal{P}, F} \). Let \( x, y \in X \).

Suppose that
\[ \mathcal{P}_F(x) = \bigcap_{s \in F} s^{-1}\varphi(s) \quad \text{and} \quad \mathcal{P}_F(y) = \bigcap_{s \in F} s^{-1}\psi(s) \]
where \( \varphi, \psi : F \to \mathcal{P} \) are two maps from the finite set \( F \) into the finite set \( \mathcal{P} \). Thus it follows that
\[ \mathcal{F}_{x,y} = \{ s \in F : \varphi(s) \neq \psi(s) \}. \]

Denote by \( B_{H_{\mathcal{P}, F}}(x, \epsilon) \) the ball of radius \( \epsilon \) in the pseudometric \( H_{\mathcal{P}, F} \) around \( x \).

It is not hard to see that \( B_{H_{\mathcal{P}, F}}(x, \epsilon) \) is the union of some members of \( \mathcal{P}_F \).

Denote by
\[ \mathcal{P}_{x,\epsilon} = \{ c \in \mathcal{P}_F : c \subseteq B_{H_{\mathcal{P}, F}}(x, \epsilon) \}. \]

**Lemma 3.7.** For \( x \in X \) and \( 0 < \epsilon < 1/2 \), one has
\[ |\mathcal{P}^F_{x,\epsilon}| \leq |\mathcal{P}|^e |F|^{eH(\epsilon)} \]
where \( H(\epsilon) = -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon) \).

**Proof.** Suppose \( \mathcal{P}^F(x) = \bigcap_{s \in F} s^{-1}\varphi(s) \) where \( \varphi : F \to \mathcal{P} \). Let \( c \in \mathcal{P}^F_{x,\epsilon} \) be a member of \( \mathcal{P}_F \). Suppose \( c = \bigcap_{s \in F} s^{-1}\psi(s) \) where \( \psi : F \to \mathcal{P} \). Denote by
\[ \mathcal{F}_c = \{ s \in F : \psi(s) \neq \varphi(s) \}. \]

The assumption \( c \in \mathcal{P}^F_{x,\epsilon} \) implies that \( H_{\mathcal{P}, F}(x, y) < \epsilon \) for any \( y \in c \), i.e.
\[ |\mathcal{F}_c| \leq \epsilon |F|. \]

We define a set \( \text{Map}^e_{\mathcal{P}, F} \) which consists of some maps from \( F \) into \( \mathcal{P} \) as follows:
\[ \text{Map}^e_{\mathcal{P}, F} = \{ \psi_c : \psi_c : F \to \mathcal{P} \text{ and } c \in \mathcal{P}^F_{x,\epsilon} \}. \]

It is clear that
\[ |\text{Map}^e_{\mathcal{P}, F}| = |\mathcal{P}^F_{x,\epsilon}|. \]
Let \( \mathcal{L} \) be the collection of subsets of \( F \) with the cardinality at most \( \epsilon|F| \), that is,
\[
\mathcal{L} = \{ D \subseteq F : |D| \leq \epsilon|F| \}.
\]
It is easy to see that
\[
|\mathcal{L}| \leq \sum_{k \leq \epsilon|F|} \binom{|F|}{k}.
\] (5)
For each \( D \in \mathcal{L} \), we write \( \text{Map}_D \) for the collection of some maps from \( F \) into \( \mathcal{P} \) as follows:
\[
\text{Map}_D = \{ \psi : F \to \mathcal{P} \mid \psi|_{F \setminus D} = \varphi|_{F \setminus D} \quad \text{and} \quad \psi(s) \neq \varphi(s) \quad \text{for all} \quad s \in D \}.
\]
Thus, the inequality (4) implies that
\[
|\text{Map}_{\mathcal{P},F}| \subseteq \bigcup_{D \in \mathcal{L}} \text{Map}_D.
\]
Hence, by a simple computation, we have
\[
|\text{Map}_{\mathcal{P},F}| \leq \sum_{D \in \mathcal{L}} |\text{Map}_D| \leq \sum_{D \in \mathcal{L}} (|\mathcal{P}|-1)^{|D|}
\leq |\mathcal{P}|^{|F|} |\mathcal{L}| \leq |\mathcal{P}|^{|F|} \sum_{k \leq \epsilon|F|} \binom{|F|}{k}
\leq |\mathcal{P}|^{|F|} 2^{|F| H(\epsilon)}.
\]
Hence, the lemma is proved. \( \square \)

The following theorem is the classical Shannon-McMillan-Breiman theorem for amenable groups. For the details, please refer to [13].

**Theorem 3.8 (Shannon-McMillan-Breiman Theorem).** Let \( G \) be an amenable group acting ergodically on a measure space \((X, \mathcal{B}, \mu)\). Let \( \mathcal{P} \) be a finite partition of \( X \), and \( \{F_n\} \) be a tempered Følner sequence for \( G \) with \( |F_n|/\log n \to \infty \).
Then
\[
\lim_{n \to \infty} -\frac{1}{|F_n|} \log \mu \left( \mathcal{P}^{F_n}(x) \right) = h_\mu(\mathcal{P})
\]
pointwise a.e. and in \( L^1 \).

**Definition 3.9.** Let \( F \) be a nonempty finite subset of \( G \). Define on \( X \) the metric 
\[
d_F(x, y) = \max_{s \in F} d(sx, sy).
\]
For every \( \epsilon > 0 \) we denote by \( B_{d_F}(x, \epsilon) \) the open Bowen ball of radius \( \epsilon \) in the metric \( d_F \) around \( x \), i.e.,
\[
B_{d_F}(x, \epsilon) = \{ y \in X : d_F(x, y) < \epsilon \}.
\]

**Definition 3.10.** Let \( F \) be a nonempty finite subset of \( G \), \( \epsilon > 0 \) and \( S \subseteq X \). A set \( D \subseteq S \) is said to be \((d, S, F, \epsilon)\)-separated if \( d_F(x, y) \geq \epsilon \) for all distinct \( x, y \in D \), and \((d, S, F, \epsilon)\)-spanning if for every \( x \in S \) there is \( y \in D \) such that \( d_F(x, y) < \epsilon \).

**Definition 3.11.** Let \( F \) be a nonempty finite subset of \( G \), \( \epsilon > 0 \) and \( S \subseteq X \). We write \( \text{sep}(d, S, F, \epsilon) \) for the maximum cardinality of a \((d, S, F, \epsilon)\)-separated subset of \( S \), and \( \text{span}(d, S, F, \epsilon) \) for the minimum cardinality of a \((d, S, F, \epsilon)\)-spanning subset of \( S \).
Let $0 < \delta < 1$, $\epsilon > 0$ and $\mu \in M(X)$. We define the quantity $N_\mu(F, \epsilon, \delta)$ as follows:

$$N_\mu(F, \epsilon, \delta) = \inf_{S \in B(X)} \mu(S) > 1 - \delta \text{span}(d, S, F, \epsilon).$$

(6)

Therefore, according to the notations of Section 2, we have the following two quantities:

$$\limsup_F \frac{\log N_\mu(F, \epsilon, \delta)}{|F|} \text{ and } \liminf_F \frac{\log N_\mu(F, \epsilon, \delta)}{|F|}.$$ 

Now we prove the following theorem which implies our main result.

**Theorem 3.12.** Let $G$ be a countably infinite amenable group and $\{F_n\}$ be a tempered Følner sequence for $G$ with $|F_n|/\log n \to \infty$. Let $G \curvearrowright X$ be a continuous action on the compact metric space $(X, d)$ and $\mu$ be an ergodic and $G$-invariant Borel probability measure. For every $0 < \delta < 1$, one has

$$h_\mu(X, G) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(F_n, \epsilon, \delta)}{|F_n|} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log N_\mu(F_n, \epsilon, \delta)}{|F_n|}.$$ 

Proof. We divide our proof into two steps.

**Step 1.** We show that the quantity in the right part of the formula does not exceed $h_\mu(X, G)$.

To prove this it is enough to show that:

$$\limsup_{n \to \infty} \frac{\log N_\mu(F_n, \epsilon, \delta)}{|F_n|} \leq h_\mu(X, G)$$

for every $\epsilon > 0$ and $1 > \delta > 0$.

Let $0 < \delta < 1$ and $\epsilon > 0$. Let $\mathcal{P}$ be a finite partition of $X$ with $\text{diam}(\mathcal{P}) < \epsilon/2$. By SMB Theorem and Egorov Theorem, we can deduce that there is a Borel set $S_0 \subseteq X$ such that $\mu(S_0) > 1 - \delta$ and

$$-\frac{1}{|F_n|} \log \mu(\mathcal{P}^{F_n}(x)) \Rightarrow h_\mu(\mathcal{P}) \text{ uniformly on } S_0.$$ 

So there is $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$ and all $x \in S_0$, one has

$$\mu(\mathcal{P}^{F_n}(x)) > \exp(-|F_n|(h_\mu(\mathcal{P}) + \epsilon)).$$

For each $n \geq N_0$, we denote that

$$\mathcal{L}_n = \{c \in \mathcal{P}^{F_n} : c \cap S_0 \neq \emptyset\} \quad \text{and} \quad A_n = \bigcup_{c \in \mathcal{L}_n} c.$$ 

Thus the set $A_n$ is the union of some members of $\mathcal{P}^{F_n}$ and

$$\mu(c) > \exp(-|F_n|(h_\mu(\mathcal{P}) + \epsilon)) \quad \text{for each } c \in \mathcal{L}_n.$$ 

Hence, it follows that

$$1 \geq \mu(A_n) = \sum_{c \in \mathcal{L}_n} \mu(c) > |\mathcal{L}_n| \exp(-|F_n|(h_\mu(\mathcal{P}) + \epsilon)).$$

So we get

$$|\mathcal{L}_n| \leq \exp(|F_n|(h_\mu(\mathcal{P}) + \epsilon)).$$

(7)
For each member \( p \in \mathcal{P}^{F_n} \), we choose a point \( x_p \in p \). Since \( \text{diam}(\mathcal{P}) < \epsilon/2 \), by a simple computation, it is not hard to see that \( p \) is contained in a \( d_{F_n} \)-ball center at \( x_p \) and radius \( \epsilon \), i.e.,

\[
p \subseteq B_{d_{F_n}}(x_p, \epsilon).
\]

Thus the set \( A_n \) (or the set \( S_0 \)) can be covered by at most \( |Z_n| \) Bowen balls with radius \( \epsilon \) in the metric \( d_{F_n} \). Hence,

\[
N_\mu(F_n, \epsilon, \delta) \leq \text{span}(d, S_0, F_n, \epsilon) \leq |Z_n| \leq \exp(|F_n|)(h_\mu(\mathcal{P}) + \epsilon)
\]

which implies that

\[
\limsup_{n \to \infty} \frac{\log N_\mu(F_n, \epsilon, \delta)}{|F_n|} \leq h_\mu(\mathcal{P}) + \epsilon.
\]

Since \( \epsilon \) can be taken arbitrarily small and \( h_\mu(\mathcal{P}) \leq h_\mu(X, G) \), we obtain the conclusion of Step 1.

**Step 2.** We prove that the second half of the theorem, i.e.,

\[
h_\mu(X, G) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(F_n, \epsilon, \delta)}{|F_n|}.
\]

Let \( 0 < \epsilon < 1 - \delta \). By Lemma 3.5, there exists a finite partition \( \mathcal{P} \) of \( X \) such that

\[
h_\mu(X, G) \leq h_\mu(\mathcal{P}) + \epsilon \quad \text{and} \quad \mu(\partial \mathcal{P}) = 0. \tag{8}
\]

Suppose that \( \mathcal{P} = \{p_1, \ldots, p_m\} \). Let \( \gamma > 0 \) and \( p_i \in \mathcal{P} \) for \( i \in \{1, \ldots, m\} \). Denote by

\[
U_\gamma(p_i) = \{x \in X : \text{dist}(x, X \setminus p_i) < \gamma\}
\]

and

\[
U_\gamma(\mathcal{P}) = \bigcap_{i=1}^m U_\gamma(p_i).
\]

**Claim 1.** \( \bigcap_{\gamma > 0} U_\gamma(\mathcal{P}) = \partial \mathcal{P} \).

Let \( x \in \partial \mathcal{P} \). Thus \( x \in \partial p_{i_0} \) for some \( i_0 \in \{1, \ldots, m\} \).

Let \( \gamma > 0 \). So there are two points \( y_1 \in p_{i_0} \) and \( y_2 \in X \setminus p_{i_0} \) such that \( d(x, y_j) < \gamma \) for \( j = 1, 2 \). Since \( \mathcal{P} \) is a partition, we have \( y_1 \in X \setminus p_j \) for all \( j \neq i_0 \). Combing with \( d(x, y_1) < \gamma \), we get

\[
x \in U_\gamma(p_j) \quad \text{for all} \quad j \neq i_0.
\]

The assumptions \( y_2 \in X \setminus p_{i_0} \) and \( d(x, y_2) < \gamma \) imply that

\[
x \in U_\gamma(p_{i_0}).
\]

So we get \( x \in \bigcap_{\gamma > 0} U_\gamma(\mathcal{P}) \). Hence we have

\[
\partial \mathcal{P} \subseteq \bigcap_{\gamma > 0} U_\gamma(\mathcal{P}).
\]

Let \( y \in \bigcap_{\gamma > 0} U_\gamma(\mathcal{P}) \). Suppose that \( y \notin \partial \mathcal{P} \).

Since \( \mathcal{P} \) is a partition, we know that \( y \in p_{j_0} \) for some \( j_0 \in \{1, \ldots, m\} \). Due to \( y \notin \partial p_{j_0} \), it follows that \( y \) belongs to the interior of the set \( p_{j_0} \), that is, there exists \( \gamma_0 > 0 \) such that

\[
B_d(y, \gamma_0) \subseteq p_{j_0}.
\]

Thus we have that \( \text{dist}(y, X \setminus p_{j_0}) \geq \gamma_0 \) which contradict the assumption \( y \in \bigcap_{\gamma > 0} U_\gamma(\mathcal{P}) \). So Claim 1 is proved.
Hence, by Claim 1 and \( \mu(\partial \mathcal{P}) = 0 \), there exists \( 0 < \gamma_0 < \epsilon \) such that, for all \( 0 < \gamma' < \gamma_0 \),

\[
\mu(U_{\gamma'}(\mathcal{P})) < \frac{\epsilon^2}{4}.
\]

Let us denote for brevity the characteristic function of the set \( U_{\gamma'}(\mathcal{P}) \) by \( \chi_{\gamma'} \) and let

\[
D_{F_n, \epsilon} = \left\{ x \in X : \sum_{s \in F_n} \chi_{\gamma'}(sx) < \frac{|F_n|}{2} \right\}.
\]

**Claim 2.** Suppose that \( x, y \in X \) and \( d_{F_n}(x, y) < \gamma' \). Then, for any \( s \in F_n \), either \( sx \) and \( sy \) belong to the same member of \( \mathcal{P} \) or both of them belong to the set \( U_{\gamma'}(\mathcal{P}) \).

If the pair points \( s'x, s'y \in X \) are in the deferent members of \( \mathcal{P} \) for some \( s' \in F_n \), then we claim that the two point are both in \( U_{\gamma'}(\mathcal{P}) \).

Suppose that the pair points \( s'x \) and \( s'y \) are NOT in the same member of \( \mathcal{P} \), that is, \( s'x \in P \) and \( s'y \in X \setminus P \) for some \( P \in \mathcal{P} \).

Note that the assumption \( d_{F_n}(x, y) \) implies that \( d(s'x, s'y) \) which implies that \( d(s'x, X \setminus P) \geq \gamma' \). Noting \( s'y \) implies that \( d(s'x, s'y) \geq \gamma' \) which contradicts that \( d(s'x, s'y) < \gamma' \). So \( s'x \in U_{\gamma'}(\mathcal{P}) \). With the same argument we can get \( s'y \in U_{\gamma'}(\mathcal{P}) \). So both of the points \( s'x \) and \( s'y \) belong to the set \( U_{\gamma'}(\mathcal{P}) \). Hence Claim 2 is obtained.

Since \( \mu \in M_\mathcal{G}(X) \) is a \( G \)-invariant measure, we have

\[
\int_X \chi(x)d\mu = \int_X \chi(gx)d\mu \quad \text{for any} \quad g \in G.
\]

Thus one has

\[
\frac{|F_n|\epsilon^2}{4} \geq |F_n| \int_X \chi_{\gamma'}(x)d\mu = \sum_{s \in F_n} \int_X \chi_{\gamma'}(sx)d\mu
\]

\[
= \int_X \sum_{s \in F_n} \chi_{\gamma'}(sx)d\mu \geq \int_{X \setminus D_{F_n, \epsilon}} \sum_{s \in F_n} \chi_{\gamma'}(sx)d\mu
\]

\[
\geq \int_{X \setminus D_{F_n, \epsilon}} \frac{\epsilon|F_n|}{2} d\mu \quad \text{(since (9))}
\]

\[
= \frac{\epsilon|F_n|}{2} \mu(X \setminus D_{F_n, \epsilon}).
\]

So we get \( \mu(X \setminus D_{F_n, \epsilon}) < \epsilon/2 \), i.e.,

\[
\mu(D_{F_n, \epsilon}) \geq 1 - \frac{\epsilon}{2}.
\]

**Claim 3.** Let \( 0 < \gamma' < \epsilon \). For every \( x \in D_{F_n, \epsilon} \), one has

\[
B_{d_{F_n}}(x, \gamma') \subseteq B_{H_{\mathcal{P}, F_n}}(x, \frac{\epsilon}{2})
\]

where \( H_{\mathcal{P}, F_n} \) is the Hamming pseudometric.

Denote by

\[
W_1^n = \{ s \in F_n : sx \in U_{\gamma'}(\mathcal{P}) \} \quad \text{and} \quad W_2^n = \{ s \in F_n : sx \notin U_{\gamma'}(\mathcal{P}) \}.
\]
It is clear that $F_n = W^n_1 \sqcup W^n_2$. Since $x \in D_{F_n, \epsilon}$, it follows that
\[
\sum_{s \in F_n} \chi_{U_{x'}(\mathcal{P})}(sx) < \frac{\epsilon|F_n|}{2},
\]
which implies that
\[
|W^n_1| < \frac{\epsilon|F_n|}{2}.
\]
(11)

Let $y \in B_{d_{F_n}}(x, \gamma')$ and $g \in F_n$. By Claim 2, we know that $gx$ and $gy$ belong to the same member of $\mathcal{P}$ or both of them belong to the set $U_{x'}(\mathcal{P})$. If $g \notin W_2^n$ (i.e. $gx$ is not in $U_{x'}(\mathcal{P})$), the above conclusion implies that both of the points $gx, gy$ are in the same member of $\mathcal{P}$. Thus we get
\[
|\{s \in F_n : sx \text{ and } sy \text{ are NOT in the same member of } \mathcal{P}\}| \leq |W^n_1|.
\]
It follows that
\[
H_{\mathcal{P}, F_n}(x, y) = \frac{1}{|F_n|} |\{s \in F_n : sx \text{ and } sy \text{ are NOT in the same member of } \mathcal{P}\}| \leq \frac{|W^n_1|}{|F_n|} < \frac{\epsilon}{2}.
\]
Hence Claim 3 is proved.

Recall that
\[
N_\mu(F_n, \gamma', \delta) = \inf_{\mathcal{B} \in \mathcal{B}(X)} \frac{\mu(x)}{\mu(x) > 1 - \delta} \text{ span}(d, S, F_n, \gamma').
\]
Since $N_\mu(F_n, \gamma', \delta)$ is a finite integer, there exists a Borel set $S_n \subseteq X$ such that $\mu(S_n) > 1 - \delta$ and
\[
N_\mu(F_n, \gamma', \delta) = \text{span}(d, S_n, F_n, \gamma').
\]
Since $\mu(D_{F_n, \gamma'}) \geq 1 - \frac{\epsilon}{2}$ and $\gamma' < \epsilon < 1 - \delta$, we have
\[
\mu(D_{F_n, \gamma'} \cap S_n) > \frac{1 - \delta}{2}.
\]

Let $E_n$ be a $(d, F_n, \gamma')$-spinning subset of $D_{F_n, \gamma'} \cap S_n$ of minimum cardinality. It is clear that
\[
|E_n| \leq \text{span}(d, S_n, F_n, \gamma') = N_\mu(F_n, \gamma', \delta).
\]
(12)
From the definition of the spanning set we know that the balls $\{B_{d_{F_n}}(y, \gamma')\}_{y \in E_n}$ can cover the set $D_{F_n, \gamma'} \cap S_n$. By Claim 3, we know that the Hamming metric balls
\[
\{B_{H_{\mathcal{P}, F_n}}(y, \epsilon)\}_{y \in E_n}
\]
cover the set $D_{F_n, \gamma'} \cap S_n$.

Denote by
\[
\mathcal{W}_n = \{c \in \mathcal{P}^F_n : c \subseteq B_{H_{\mathcal{P}, F_n}}(y, \epsilon) \text{ for some } y \in E_n\}.
\]
Thus one has
\[
D_{F_n, \gamma'} \cap S_n \subseteq \bigcup_{c \in \mathcal{W}_n} c.
\]
(13)
Furthermore, by Lemma 3.7 we have
\[
|\mathcal{W}_n| \leq |E_n| |\mathcal{P}| |F_n| |F_n| |H(\epsilon)\| \leq N_\mu(F_n, \gamma', \delta) |\mathcal{P}| |F_n| |F_n| |H(\epsilon)|.
\]
(14)
By SMB Theorem and Egorov Theorem, we know that there is a Borel set $T \subseteq X$ such that

$$\mu(T) > \frac{1+3\delta}{4}$$

and

$$\frac{1}{|F_n|} \log \mu(P_{F_n}(x)) \Rightarrow h_\mu(\mathcal{P}) \quad \text{uniformly on } T.$$

So there is $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$ and all $x \in T$, one has

$$\mu(P_{F_n}(x)) < \exp(-|F_n|(h_\mu(\mathcal{P}) - \epsilon)).$$

Since $\mu(T) > \frac{1+3\delta}{4}$ and $\mu(D_{F_n,\gamma'} \cap S_n) > \frac{1-\delta}{2}$, it follows that

$$\mu(D_{F_n,\gamma'} \cap S_n \cap T) \geq \frac{1-\delta}{4}. \quad (16)$$

Let $c \in \mathcal{U}_n$ be a member of $\mathcal{P}_{F_n}$. If $c \cap T \neq \emptyset$, owing to (15), then one has

$$\mu(c) < \exp(-|F_n|(h_\mu(\mathcal{P}) - \epsilon)). \quad (17)$$

Denote by

$$\mathcal{U}_n^* = \{ c \in \mathcal{U}_n : \mu(c) < \exp(-|F_n|(h_\mu(\mathcal{P}) - \epsilon)) \}.$$

Due to (13) and (17), we know that

$$D_{F_n,\gamma'} \cap S_n \cap T \subseteq \bigcup_{c \in \mathcal{U}_n^*} c.$$

Define

$$D_n^* = \bigcup_{c \in \mathcal{U}_n^*} c.$$

Combing (16), we have $\mu(D_n^*) \geq (1-\delta)/4$. Moreover,

$$\frac{1-\delta}{4} \leq \mu(D_n^*) = \sum_{c \in \mathcal{U}_n^*} \mu(c) \leq |\mathcal{U}_n^*| \exp(-|F_n|(h_\mu(\mathcal{P}) - \epsilon)) \leq |\mathcal{U}_n| \exp(-|F_n|(h_\mu(\mathcal{P}) - \epsilon)) \leq N_\mu(F_n,\gamma',\delta) N_{\mathcal{P}} e^{2|F_n|H(\epsilon)} \exp(-|F_n|(h_\mu(\mathcal{P}) - \epsilon)) \quad \text{(since (14))}$$

where $H(\epsilon) = -\log \epsilon - (1-\epsilon) \log(1-\epsilon)$. So, we get

$$h_\mu(\mathcal{P}) < \frac{\log N_\mu(F_n,\gamma',\delta)}{|F_n|} + H(\epsilon) \log 2 + \epsilon(1 + \log |\mathcal{P}|) - \frac{\log((1-\delta)/4)}{|F_n|}.$$

Combining (8), one has $h(X,G) < h_\mu(\mathcal{P}) + \epsilon$. Thus

$$h(X,G) \leq \lim_{\gamma' \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(F_n,\gamma',\delta)}{|F_n|} + H(\epsilon) \log 2 + \epsilon(2 + \log |\mathcal{P}|).$$

Since $\epsilon$ can be taken arbitrarily small and $\lim_{\epsilon \to 0} H(\epsilon) = 0$, we obtain

$$h(X,G) \leq \lim_{\gamma' \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(F_n,\gamma',\delta)}{|F_n|}.$$

\[\square\]
By the above theorem, we can easily obtain our main result.

**Proof of Main Theorem.** From Fact 2.6, there exist two Følner sequences \( \{F_m\} \) of \( G \) such that
\[
\limsup_{F} \frac{\log N_{\mu}(F, \epsilon, \delta)}{|F|} = \lim_{m \to \infty} \frac{\log N_{\mu}(F_m, \epsilon, \delta)}{|F_m|};
\]
\[
\liminf_{F} \frac{\log N_{\mu}(F, \epsilon, \delta)}{|F|} = \lim_{m \to \infty} \frac{\log N_{\mu}(F_m, \epsilon, \delta)}{|F_m|}.
\]
By Proposition 2, we can get two subsequences \( \{F_n\} \) of \( \{F_m\} \) and \( \{F^*_n\} \) of \( \{F^*_m\} \) which are tempered and satisfy \( |F_n|/\log n \to \infty \) and \( |F^*_n|/\log n \to \infty \). At the same time, we have
\[
\limsup_{F} \frac{\log N_{\mu}(F, \epsilon, \delta)}{|F|} = \lim_{n \to \infty} \frac{\log N_{\mu}(F_n, \epsilon, \delta)}{|F_n|};
\]
\[
\liminf_{F} \frac{\log N_{\mu}(F, \epsilon, \delta)}{|F|} = \lim_{n \to \infty} \frac{\log N_{\mu}(F^*_n, \epsilon, \delta)}{|F^*_n|}.
\]
By Theorem 3.12, we get
\[
h_{\mu}(X, G) = \lim_{\epsilon \to 0} \liminf_{F} \frac{\log N_{\mu}(F, \epsilon, \delta)}{|F|} = \lim_{\epsilon \to 0} \limsup_{F} \frac{\log N_{\mu}(F, \epsilon, \delta)}{|F|}.
\]
Hence Main Theorem is proved.

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