Error-disturbance uncertainty relations in Faraday measurements

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We examine error-disturbance relations in the quantum measurement of spin systems using an atom-light interface scheme. We model a single spin-1/2 system that interacts with a polarized light meter via a Faraday interaction. We formulate the error and disturbance of the model and examine the uncertainty relations. We found that for the coherent light meter in pure polarization, both the error and disturbance behave the cyclic oscillations due to the Faraday rotation in both the light and spin polarizations. We also examine a class of polarization squeezed light meter, where we apply the phase-space approximation and characterize the role of squeezing. We derive error-disturbance relations for these cases and find that the Heisenberg-Arthurs-Kelly uncertainty is violated while the tight Branciard-Ozawa uncertainty always holds. We note that, in the limit of weak interaction strength, the error and disturbance become to obey the unbiasedness condition and hence the Heisenberg-Arthurs-Kelly relation holds. The work would contribute to our understanding of quantum measurement of spin systems under the atom-light interface framework, and may hold potential applications in quantum metrology, quantum state estimation and control.

I. INTRODUCTION

Quantum measurements play a crucial role in the characterization of physical systems, which elucidate hidden quantum properties to the classical world [1]. Moreover, many measurements come with more than just one observables that do not commute, and thus have an enormous impact from the fundamental verification such as Bell non-locality and entanglement [2–4], quantum steering [5], quantum metrology [6, 7] to quantum information technologies including quantum key distribution [8], quantum dense coding [9–12], quantum cryptography [9, 13], and non-local quantum measurement [14–16].

An important intrinsic property of quantum measurements is the uncertainty relation in which it is infeasible to measure incompatible observables with arbitrary precision. This is the fundamental restriction in the attainable precision of quantum measurements. In the early stage of quantum mechanics, Heisenberg [17] was first formulated such an uncertainty relation between the position measurement and the disturbance of the momentum that satisfies \( \epsilon_0 \eta_p \approx \hbar/2 \), where \( \epsilon_0 \) and \( \eta_p \) represent the root-mean-square error and root-mean-square disturbance, respectively. The study then was paraphrased under the form of the standard deviations by Kennard [18], Weyl [19], and later Robertson [20], for a general pair of operators \( A \) and \( B \), reads \( \sigma_A \sigma_B \geq C_{A,B} \), where \( C_{A,B} = |\langle \psi | [A,B] | \psi \rangle | / 2 \), and \( \sigma_A = \sqrt{\langle \Lambda^2 \rangle - \langle \Lambda \rangle^2} \) represents the standard deviation of \( \Lambda \), with \( \langle \Lambda \rangle = \langle \psi | \Lambda | \psi \rangle \) is the expectation value for a pure quantum state \( | \psi \rangle \), and \( \Lambda \equiv A \) or \( B \). However, this mathematical relation in the form of standard deviation has no direct connection to the limitation on measurements, and thus could not cover Heisenberg’s interpretation uncertainty. Preferably, Arthurs and Kelly [21] provided an error-disturbance relation and then was generalized to [22, 23]

\[
\epsilon_A \eta_B \geq C_{A,B},
\]

which states that if the measurement of an observable \( A \) with an error \( \epsilon_A \), then it also disturbs an observable \( B \) with a disturbance \( \eta_B \) satisfying such a relation. So far, it is known that this relation is not universally valid (see, for example, Ref. [24].) Hereafter, we call (1) the Heisenberg-Arthurs-Kelly uncertainty.

Ozawa has theoretically derived a universal error-disturbance relation [25, 26] through an indirect measurement following the von Neumann paradigms [27]. The measurement consists of an interaction between a quantum system and a meter. A measurement of \( A \) in the system was done indirectly via a measurement of \( M \) in the meter. At the same time, this process affects back to the system, and thus it disturbs the subsequent measurement of observable \( B \) in the system. According to Ozawa, the error-disturbance relation for any input state \( | \psi \rangle \) is expressed by [25, 26]

\[
\epsilon_A \eta_B + \epsilon_A \sigma_B + \eta_B \sigma_A \geq C_{A,B} \,
\]

This relation has been experimentally confirmed recently by using a state-preparation method [28–32], weak probe method [33–36], continuous-variable entangled states [37, 38], and others [39, 40].

Subsequently, Branciard [41, 42], and Ozawa [43] have considered a rigorous relation reads

\[
\epsilon_A^2 \sigma_B^2 + \sigma_A^2 \eta_B^2 + 2 \epsilon_A \eta_B \sqrt{\sigma_A^2 \sigma_B^2 - C_{A,B}^2} \geq C_{A,B}^2,
\]

that claimed tighter than relation (2) and has been experimentally verified [31, 36–38]. Hereafter, we call (3) the Branciard-Ozawa uncertainty. Recently, numerous alternative approaches have been used to revisit the uncertainty relation theoretically and experimentally [39, 44–67].
Recently, the Faraday measurements of spin based on an atom-light interface framework have been studied actively [68–78]. It has contributed to our understanding of quantum measurement and has various applications in quantum metrology of atomic ensemble [79], quantum information processing [80], strongly correlated systems [81], and many-body systems [82]. The Faraday effect causes the rotation of the polarized light via the interaction with the spin system and thus allows indirect measurement of the spin system through the polarized light meter. Such a measurement contains fundamental limits in the sensitivity caused by the quantum nature of light. Likewise, the back-action of the polarized light meter perturbs the spin state, which causes disturbance on the subsequent measurements of the spin system.

Recently, uncertainty relation in the Faraday measurement has been studied by examining the relation between preparation (prediction) and postselection (retrodiction) [75], where the authors consider the Weyl-Robertson relation for the approximate canonical position and momentum of the spin of atoms. However, the obtained relation cannot be considered as the error-disturbance relation. Also, the approximate canonical observables used there are only applicable in the case of weak interaction and unbiased measurements. Thus, more precise and appropriate analysis of the error-disturbance uncertainty relation in Faraday measurement is necessary.

In this paper, we formulate an atom-light interface scheme in the Faraday measurement and evaluate the error, disturbance and their uncertainty relations. We consider an atom as a single spin-1/2 particle interacting with a polarized light meter. We first consider a classical coherent polarized light as the light meter. We then consider an atom as a single spin-1/2 particle interacting with a polarized light meter. With the help of the canonical phase-space approximation for classical coherent polarized light as the light meter, we derive the error and disturbance under the atom-light interface framework. In Sec. II, we derive the error-disturbance interface framework for classical coherent light meter and polarization squeezed light meter. The error-disturbance relations are provided in Sec. IV. We give a brief summary and outlook in Sec. V.

## II. MEASUREMENT PROCESS

We consider a measurement model in which a spin-1/2 system interacts with a polarized light meter based on the Faraday interaction under the standard von Neumann paradigm [27]. The spin system is a single particle characterized by Pauli matrices $\sigma_i$, with $i = x, y, z$, while the polarized light meter is given by the Stokes operators $S_i$ [83]. For light propagating along the z-direction, we explicitly have

\begin{align}
S_0 &= a_H^\dagger a_H + a_V^\dagger a_V = n_H + n_V, \\
S_x &= a_H^\dagger a_H - a_V^\dagger a_V = n_H - n_V, \\
S_y &= a_H^\dagger a_V + a_V^\dagger a_H, \\
S_z &= -i(a_H^\dagger a_V - a_V^\dagger a_H),
\end{align}

where $H$ and $V$ stand for the light modes of horizontal and vertical linear polarizations, respectively, $a_{H,V} (a_{H,V}^\dagger)$ are the annihilation (creation) operators in the corresponding polarization modes, and $n = a^\dagger a$ the photon number operator. The Stokes operators obey the angular momentum commutation relation $[S_z, S_y] = 2iS_z$, and cyclic permutations.

The unitary evolution of the Faraday interaction is given by

$$U_T = e^{-igA @ S_z},$$

where $g = \int_0^T g(t) \, dt$ is the interaction strength over the time interval $T$. Here $A$ is the being measured observable in the system. Under such an atom-light interface, the polarization state of the light meter rotates through the Faraday effect by an amount proportional to $A$, and thus allows the indirect measurement of $A$. Likewise, under the back-action effect, the system state is rotated around the $z$-axis by an amount proportional to $S_z$, and thus disturbs the system.

Assume that the spin system is prepared in state $|\psi\rangle$ and the light meter state is $|\xi\rangle$. They are initially uncorrelated, so that $|\Psi\rangle = |\psi\rangle \otimes |\xi\rangle$. The unitary operator $U_T$ in Eq. (8) describes the time evolution of the joint system-meter during the interaction time. After the interaction, the joint state is given by $|\Psi'\rangle = U_T|\Psi\rangle$, and the measuring expectation value of an observable $M$ in the meter will be

\begin{align}
(\langle I \otimes M \rangle) &= \langle \Psi' | (I \otimes M) | \Psi' \rangle \\
&= \langle \Psi | U_T^\dagger (I \otimes M) U_T | \Psi \rangle. 
\end{align}

Let us choose $M = S_y$, and in the Heisenberg picture, we consider $(I \otimes S_y)_T = U_T^\dagger (I \otimes S_y) U_T$ the time-dependent operator after the interaction. Particularly, for $A^2 = I$, as in Pauli operators, using the Baker-Campbell-Hausdorff (BCH) formula [84], we obtain (see Appendix A)

$$\langle I \otimes S_y \rangle_T = (I \otimes S_y)_0 \cos(2g) + (A \otimes S_z)_0 \sin(2g).$$

\begin{align}
S_0 &= a_H^\dagger a_H + a_V^\dagger a_V = n_H + n_V, \\
S_x &= a_H^\dagger a_H - a_V^\dagger a_V = n_H - n_V, \\
S_y &= a_H^\dagger a_V + a_V^\dagger a_H, \\
S_z &= -i(a_H^\dagger a_V - a_V^\dagger a_H),
\end{align}
Here and hereafter, the bra-ket symbol \langle \ldots \rangle means \langle \Psi | \ldots | \Psi \rangle \equiv |\psi\rangle\langle\psi| \equiv \langle\psi|\psi\rangle, whereas \langle \ldots \varphi \rangle stands for \langle\varphi|\ldots|\varphi\rangle \equiv \langle\varphi|\varphi\rangle. We omit the subscript 0 in the R.H.S without confusion. Here, the mean value of the meter’s observable \langle S_y \rangle is determined by the sign of \langle A \rangle: note that the eigenvalues of \langle A \rangle are ±1, since \langle A^2 \rangle = I. Then, we measure the expectation value of the meter \langle (A \otimes I) \rangle_T that provides the information of an indirect measurement performed on the system. In our model, the expectation value gives
\begin{align}
\langle (I \otimes S_y)_T \rangle = \langle S_y \rangle \cos(2g) + \langle A \rangle \langle S_x \rangle \sin(2g). \quad (11)
\end{align}

Here and hereafter, the bra-ket symbol \langle \ldots \rangle means \langle \Psi | \ldots | \Psi \rangle whereas \langle \ldots \varphi \rangle stands for \langle\varphi|\ldots|\varphi\rangle. We omit the subscript 0 in the R.H.S without confusion. Here, the mean value of the meter’s observable will shift from the initial value by an amount proportional to the mean value of the system’s observable \langle A \rangle. Without loss of generality, we can choose the initial mean of the meter is zero, i.e., \langle S_y \rangle = 0. Then, we thus can indirectly measure the value of the system operator \langle A \rangle via a calibrated meter operator \langle (A \otimes I) \rangle_T = (\langle I \otimes S_y \rangle_T) / \langle S_x \rangle \sin(2g). The calibration is designed so that \langle M_T \rangle is unbiased, i.e., \langle M_T - (A \otimes I)_0 \rangle = 0 irrespective of \langle A \rangle, given that \langle S_y \rangle = \langle B \rangle = 0. The calibration factor \langle (I \otimes S_y) \rangle \sin(2g) can be determined independently in practical experiments.

In this scenario, to measure \langle A \rangle of the system before the interaction, we measure \langle S_y \rangle of the meter after the interaction. If these two observables are perfectly correlated in any given system state \langle \psi \rangle, the measurement is said to be accurate [33, 85]. However, in general, they would not be perfectly correlated and thus become inaccurate because of possible noise and error in the measurement process. Moreover, when another observable \langle B \rangle in the system is measured after the measurement of \langle A \rangle, it would be disturbed by the back-action effect caused by the prior interaction in the \langle A \rangle measurement. In the following, we will consider the error and disturbance in our measurement model.

### III. ERROR AND DISTURBANCE

#### A. Exact solution for classical coherent light meter

In the following, we will consider the measurements of \langle A = \sigma_z \rangle and \langle B = \sigma_x \rangle in a single spin system. In the joint space, we denote
\begin{align}
A_0 = (\sigma_z \otimes I)_0, \quad B_0 = (\sigma_x \otimes I)_0. \quad (12)
\end{align}
for the operators at the time 0. We denote the measurement operators at time T as
\begin{align}
M_T = \frac{(I \otimes S_y)_T}{\langle S_x \rangle \sin(2g)}, \quad B_T = (\sigma_x \otimes I)_T. \quad (13)
\end{align}

We get [See Eqs. (B.1, B.8) in Appendix B]
\begin{align}
M_T &= \frac{(I \otimes S_y)_0 \cot(2g)}{(S_x)_{\xi \sin(2g)}}, \quad (14)
\end{align}
\begin{align}
B_T &= (\sigma_x \otimes \cos(2g S_z))_0 - (\sigma_y \otimes \sin(2g S_z))_0. \quad (15)
\end{align}

The error can be evaluated by the error operator \langle N_{\sigma_z} \rangle, and the disturbance is defined through the disturbance operator \langle D_{\sigma_z} \rangle, as follows
\begin{align}
N_{\sigma_z} = M_T - A_0, \quad \text{and} \quad D_{\sigma_z} = B_T - B_0. \quad (16)
\end{align}

Then, the square error and the square disturbance are given by [25, 26, 86],
\begin{align}
\xi_{\sigma_z}^2 = \langle N_{\sigma_z}^2 \rangle, \quad \text{and} \quad \eta_{\sigma_z}^2 = \langle D_{\sigma_z}^2 \rangle. \quad (17)
\end{align}

In the following, we choose the initial system state \langle \psi \rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), an eigenstate of \langle A \rangle that maximize the R.H.S. of the error-disturbance relations, Eqs. (1), (2) and (3). We also choose the light meter state to be a coherent state in the horizontal linear polarization:
\begin{align}
\langle \xi \rangle \equiv |\alpha \rangle_H |0\rangle_V = \exp(\alpha a_H^+ - \alpha^* a_H) |0\rangle_H |0\rangle_V, \quad (18)
\end{align}
where \langle |\alpha \rangle \rangle is the coherent state with the coherent amplitude \alpha, and \langle 0 \rangle is the vacuum state of light. For this meter state, \langle S_z \rangle = |\alpha|^2 \eta and \langle S_y \rangle = \langle \xi \rangle = \langle S_z \rangle \eta = 0.\langle \eta \rangle. We readily find \langle N_{\sigma_z} \rangle = \langle S_z \rangle = 0 and thus \langle N_{\sigma_z} \rangle = 0 irrespective of \langle \psi \rangle, i.e., \langle M_T \rangle is unbiased as mentioned earlier. For the disturbance, however, \langle D_{\sigma_z} \rangle = \langle S_z \rangle \eta = 0 in general, since \langle \cos(2g S_z) \rangle \eta \neq I even though \langle S_z \rangle = 0 and \langle \sin(2g S_z) \rangle = 0. The non-zero mean disturbance comes from the noise (fluctuation) in \langle S_z \rangle, which randomly rotates the spin system about the z-axis and effectively reduces the \langle x \rangle-\langle \text{component of the spin. This behavior is the imprint of the back-action effect on the spin system caused by the light meter, which disturbs (rotates) the spin system on its Bloch sphere.}

Then, the square error and disturbance read (see detailed calculation in Appendix B)
\begin{align}
\xi_{\sigma_z}^2 &= \frac{1}{|\alpha|^2 \sin^2(2g)}, \quad (19)
\end{align}
\begin{align}
\eta_{\sigma_z}^2 &= 2(1 - e^{-2|\alpha|^2 \sin^2 2g}). \quad (20)
\end{align}
Note that, for the polarized coherent state of light, the root-mean-square noise in \langle S_y \rangle and also in \langle S_z \rangle is \langle |\alpha \rangle \rangle. This noise is imprinted in \langle M_T \rangle as |\alpha| / \langle S_z \rangle \eta \langle 2g \rangle = 1 / |\alpha| \langle \sin(2g) \rangle, and thus results in the square error \xi_{\sigma_z}^2 in Eq. (19). Also, as mentioned above, the noise in \langle S_z \rangle contributes to the disturbance in \langle \sigma_z \rangle with a bias and thus results in the square disturbance \eta_{\sigma_z}^2 given in Eq. (20).

In Fig. 1, we show the square error \xi_{\sigma_z}^2 and square disturbance \eta_{\sigma_z}^2 as functions of the interaction strength g for several coherent amplitudes \langle |\alpha \rangle \rangle. When n = \pi/2 where n is an integer number, \xi_{\sigma_z}^2 diverges because no shift of \langle S_y \rangle is expected in the meter. With increasing g,
due to the rotation of the light polarization that causes a certain amount of shift of $S_y$ in the meter depending on $\sigma_z$ in the system, the square error gradually decreases as $\epsilon_{\sigma_z}^2 \approx g^{-2}$. When $g = \pi/4 + n\pi/2$, $\epsilon_{\sigma_z}^2$ reaches its minimum value $1/|\alpha|^2$, i.e., the minimum square error that can be achieved by the coherent light meter. Likewise, the square disturbance $\eta_{\sigma_z}^2$ exhibits periodic behavior as a function of $g$. When $g = n\pi$, the square disturbance $\eta_{\sigma_z}^2$ vanishes because the spin system is rotated by integer multiples of $2\pi$ for any integer values of $S_z$ and thus returns to its original state. This phenomenon can be regarded as a kind of quantum revival, which essentially reflects the discrete nature of the observable, i.e., $S_z$. When $g = \pi/2 + n\pi$, $\eta_{\sigma_z}^2$ reaches its maximum $2(1 - e^{-2|\alpha|^2}) \sim 2$ for large $|\alpha|$. In this case, the spin system is rotated about the $z$-axis by $0$ or $\pi$ at almost even probabilities depending on the even or odd number of $S_z$, so that the square disturbance becomes approximately $(2^2 + 0^2)/2 = 2$. This analysis provides us a complete and accurate insight on the quantum measurement of spin systems via the Faraday interaction.

B. Phase-space approximation (canonical approximation) for polarization squeezed light meter

To further investigate the error and disturbance in various light meter states, we apply the phase-space approximation (PSA) for the light system. We introduce two canonical operators as $q = S_y/\sqrt{|S_z|}$ and $p = S_z/\sqrt{|S_x|}$ for a finite $|S_z|$. These operators approximately obey the canonical commutator relation $[q, p] = 2iS_z/|S_z| \approx 2\pi$. This approximation is valid when $S_z$ can be regarded as a classical positive constant that does not change during the measurement process. Practically, under the PSA the evolution of $q$ in the BCH formula (Eq. (A.2) in Appendix A) is approximated up to its first order (first and second terms).

Here, we discuss the error and disturbance using the impact of a class of the polarization squeezed state in the light meter space, which is given by

$$|\xi\rangle = \left(\frac{1}{2\sigma^2}\right)^{1/4} \int e^{-\frac{q^2}{\sigma^2}} |q\rangle \, dq,$$

where $\sigma$ represents the squeezing parameter. For $\sigma = 1$, it is a coherent state, the cases $\sigma < 1$ and $\sigma > 1$ correspond to an amplitude-squeezed state and a phase-squeezed state, respectively [87] (See App. C for detailed.) Here, $q$ and $|q\rangle$ are the eigenvalue and eigenstate of the position operator $q$, such that $q|q\rangle = q|q\rangle$. We illustrate such a polarization squeezed state in a Poincaré sphere in Fig. 2.

The interaction evolution (8) is recast as

$$U_T = e^{-ig\sigma_z \otimes p},$$

where we have set $\sqrt{|S_z|} = |\alpha|$. Under the PSA as mentioned above and using the BCH formula, we have

$$(I \otimes q) T = (I \otimes q)_0 + 2g|\alpha| \langle \sigma_z \otimes I \rangle_0.$$

Using the calibrated meter operator $(I \otimes q) T / 2g |\alpha|$, we obtain the corresponding information of $(\sigma_z \otimes I)_0$ in the system. Thus, the error operator is given by

$$N_{\sigma_z} = (I \otimes q)_0 / 2g|\alpha| - (\sigma_z \otimes I)_0 = (I \otimes q)_0 / 2g|\alpha|.$$

As a result, the square error is appropriate to the variance of the meter, i.e., $(q^2)_{\xi}/4g^2|\alpha|^2$ when $(q)_{\xi} = 0$. Similarly, the square disturbance operator is given by $2(1 - \cos(2g|\alpha| p))_{\xi}$ (see App. C). Straightforward calculating gives

$$\epsilon_{\sigma_z}^2 = \frac{1}{4\chi^2}, \quad \eta_{\sigma_z}^2 = 2(1 - e^{-2\chi^2}),$$

FIG. 1. (Color online) The plot of the square error and square disturbance as functions of interaction strength $g$ for some values of amplitude $|\alpha|^2$ as shown in the figure. The square error is large for small $g$ and reaches the minimum when $g = \pi/4$ and increases again for $g$ increases to $\pi/2$. The procedure is repeated when continuously increasing $g$. Similarly, the square disturbance increases along with $g$ and reaches the maximum of two, then it reduces to zero as $g$ increasing to $\pi$.

FIG. 2. (Color online) Illustration of the class of polarization squeezed light in the Poincaré sphere.
where $\chi = g|\alpha|/\sigma$ represents the measurement strength.

In Fig. 3, we show the square error (solid curve) and disturbance (short-dashed curve) for PSA as functions of $\chi$. When $g|\alpha|$ is fixed (not require to be small), hence, the squeezing parameter $\sigma$ plays the role of the measurement strength: for large $\sigma$ the measurement is weak, likewise, for small $\sigma$ the measurement is strong. Correspondingly, small (large) $\chi$ implies weak (strong) measurement. Weak interaction approximation (WIA): the plot of square error (solid curve) and square disturbance (long-dashed curve) in the WIA as functions of the measurement strength $\chi$.

In this subsection, to investigate the impact of the error and disturbance in such a far off-resonant region, we consider a weak interaction approximation (WIA), i.e., $\chi \ll 1$. In this approximation, we have

$$
\epsilon_{\sigma}^2 \approx \frac{1}{4\chi^2}, \text{ and } \eta_{\sigma}^2 \approx 4\chi^2, \tag{26}
$$

where $\chi = g|\alpha|$ (see, App. D). Noting that in the coherent-state case, $\chi = 1$. We show Fig. 3 for the square error (solid curve) and square disturbance (long-dashed curve), denoted by ‘WIA’. While the square error is the same as in the PSA case, the square disturbance gradually increasing from zero when increasing $\chi$. This square disturbance is different from that of the PSA case because the WIA is applied to both the spin system and the light meter (we can neglect the higher-order terms of $g$ in both the spin system and the light meter.) It is thus provides us the impact of weak Faraday interaction on the error and disturbance in spin measurements. We also confirm that the joint unbiasedness condition is satisfied, i.e., $\langle N_{\sigma} \rangle = \langle D_{\sigma} \rangle = 0$ irrespective of the initial system state $|\psi\rangle$ (see App. D), which is sufficient for holding the Heisenberg-Arthurs-Kelly uncertainty [25], i.e., $\epsilon_{\sigma}^2, \eta_{\sigma}^2 = 1$.

### IV. ERROR-DISTURBANCE RELATIONS

This section examines the error-disturbance relations for the measurement of a single spin system with two cases of the meter state: exact solution of the classical coherent light [Eq. (18)] and phase-space approximation (PSA) of the polarization squeezed light [Eq. (21)].

We consider the Heisenberg-Arthurs-Kelly relation [the L.H.S of Eq. (1)] and the Brainciard-Ozawa relation [the L.H.S. of Eq. (2)], which are denoted as HAK and BO, respectively. With our choice of the spin system, we have $\Delta_{\sigma} = 1, \Delta_{\sigma} = 1$, and $\mathcal{C}_{\sigma, \sigma} = 1$. We straightforwardly rewrite these relations as

$$
\text{HAK} = \epsilon_{\sigma}^2 \eta_{\sigma}^2 \geq 1, \tag{27}
$$

$$
\text{BO} = \epsilon_{\sigma}^2 + \eta_{\sigma}^2 \geq 1. \tag{28}
$$

We also consider a tighter Brainciard-Ozawa relation, where the condition of $B^2 = I$ is satisfied, here $B = \sigma_{x}$. Following Refs. [41], we replace $\eta_{\sigma}$ by $\eta_{\sigma} \sqrt{1 - \frac{\eta_{\sigma}^2}{4}}$ in Eq. (28) and recast it as

$$
\text{BOt} = \epsilon_{\sigma}^2 + \frac{\eta_{\sigma}^2}{4} \left(1 - \frac{\eta_{\sigma}^2}{4}\right) \geq 1. \tag{29}
$$

We examine an error-disturbance tradeoff, which shows the dependence of the disturbance on the error or vice versa. The result is shown in Fig. 4. It can be seen that the error-disturbance tradeoff in the exact case behaves the dependence on the Faraday and spin rotations: for small $g$, the error is large and the disturbance is small, then increasing $g$ results in the reducing of the error and
FIG. 4. (Color online) The error-disturbance tradeoffs. The short-dashed curve is the Heisenberg-Arthurs-Kelly bound given in Eq. (27) and denoted by HAK. The left region is the forbidden region where the HAK relation is violated. Similarly, the long-dashed curve is the tight Branciard-Ozawa bound given in Eq. (29) and denoted by BOt. The solid curves show the error-disturbance tradeoff obtained from the atom-light interface in this work for two cases of exact solution and phase space approximation (PSA). For the weak interaction approximation (WIA) it follows the HAK bound.

increasing of the disturbance. After the error reaches the minimum, the disturbance toward two, while the error gradually increases, results in a straight line in the tradeoff. Here, we show the result for $|\alpha|^2 = 6$. For large $|\alpha|^2$, the tradeoff asymptotically reaches that of the PSA. Moreover, the tradeoff can reach the BOt relation in the PSA, while the HAK relation is violated. Concretely, it can be seen that for large square error (small $\chi$), the error-disturbance tradeoff reaches the HAK bound, while for small square error (large $\chi$), the error-disturbance tradeoff reaches the maximum of two, the BOt bound.

V. CONCLUSION

We have discussed the error, disturbance and their uncertainly relations in Faraday measurements. For a single spin interacting with coherent polarization of the light meter, we derived the exact behaviors of error and disturbance without approximation. Under the Faraday rotation of the coherent light polarization and its back-action to the spin system, the error and disturbance behave cyclic oscillations. In the case of polarization squeezed light meter, to which we apply the canonical phase-space approximation, the squeezing parameter acts as a factor that modifies the measurement strength. In this approximation, the square error monotonically decreases to 0 while the square disturbance monotonically increases and approaches to 2 with increasing measurement strength. In the cases above, Heisenberg-Arthurs-Kelly uncertainty is violated while the tight Branciard-Ozawa uncertainty always holds. It is worth mentioning that, under the weak interaction approximation, the Heisenberg-Arthurs-Kelly uncertainty holds because the error and disturbance both become to satisfy the unbiasedness.

Our analysis would contribute to deeper understanding of error, disturbance and uncertainty relations in quantum measurements under the atom-light interface, and provide an insight into quantum metrology [88, 89], quantum sensing [90], and quantum state estimation [91]. This analytical work would also be a testbed for further experimental studies.

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Appendix A: Heisenberg equation of motions

1. Heisenberg equation for $S_y$

In the Heisenberg picture, the meter’s operator $S_y$ evolves with time according to

$$(I \otimes S_y)_T = U_T^\dagger (I \otimes S_y)_0 U_T, \quad (A.1)$$

where $U_T = e^{-igA \otimes S_y}$.

Using the Baker–Campbell–Hausdorff formula [84]

$$e^{EF}e^{-E} = F + [E,F] + \frac{1}{2!}[E,[E,F]] + \cdots, \quad (A.2)$$

for $E = igA \otimes S_z$ and $F = I \otimes S_y$, we have

$$F = I \otimes S_y,$$

$$[E,F] = ig[A \otimes S_z, I \otimes S_y]$$

$$= igA \otimes [S_z, S_y]$$

$$= 2gA \otimes S_x$$

$$\frac{1}{2!}[E,[E,F]] = -\frac{(2g)^2}{2!} A^2 \otimes S_y,$$

$$\frac{1}{3!}[E,[E,[E,F]]] = -\frac{(2g)^3}{3!} A^3 \otimes S_x,$$

$$\cdots.$$
Then Eq. (A.2) is recast as
\[ e^EFe^{-E} = (\cos(2gA) \otimes S_y)_0 + (\sin(2gA) \otimes S_x)_0. \]
(A.3)

Submitting Eq. (A.3) into the R.H.S. of Eq. (A.1), we have
\[ (I \otimes S_y)_T = (\cos(2gA) \otimes S_y)_0 + (\sin(2gA) \otimes S_x)_0. \]
(A.4)

In the case of \( A^2 = I \), as in Pauli operators, we obtain
\[ (I \otimes S_y)_T = (I \otimes S_y)_0 \cos(2g) + (A \otimes S_x)_0 \sin(2g). \]
(A.5)

which is given in Eq. (10) in the main text.

2. Heisenberg equation for \( \sigma_x \)

Next, we consider the particular case where \( A = \sigma_z \) and \( B = \sigma_x \), and calculate the Heisenberg equation of motion for \( \sigma_x \) in the spin system. We consider
\[ (\sigma_x \otimes I)_T = U_T^\dagger (\sigma_x \otimes I)_0 U_T. \]
(A.6)

where \( U_T = e^{-ig\sigma_z \otimes S_z} \). Using the BCH formula for \( E = ig\sigma_z \otimes S_z \) and \( F = \sigma_x \otimes I \), we have
\[ F = \sigma_x \otimes I, \]
\[ [E, F] = ig[\sigma_z \otimes S_z, \sigma_x \otimes I] = -2g\sigma_y \otimes S_z, \]
\[ \frac{1}{2!} [E, [E, F]] = -\frac{(2g)^2}{2!}\sigma_x \otimes S_z^2, \]
\[ \ldots. \]

Then Eq. (A.6) gives
\[ (\sigma_x \otimes I)_T = (\sigma_x \otimes \cos(2gS_z))_0 - (\sigma_y \otimes \sin(2gS_z))_0. \]
(A.7)

Appendix B: Error and disturbance

In this section, we provide detailed calculation of root mean-square (rms) error \( \epsilon_{\sigma_z} \) and rms disturbance \( \eta_{\sigma_z} \).

1. The error

We first consider \( M_T \) operator:
\[ M_T = \frac{1}{(S_x)_0 \sin(2g)} (I \otimes S_y)_T \]
\[ = \frac{(I \otimes S_y)_0 \cot(2g)}{(S_x)_0} + \frac{(\sigma_z \otimes S_z)_0}{(S_x)_0}, \]
(B.1)

where we have used \( A = \sigma_z \) in Eq. (A.5). Then, the noise operator is given by
\[ N_{\sigma_z} = \frac{(I \otimes S_y)_0 \cot(2g)}{(S_x)_0} + \frac{(\sigma_z \otimes S_z)_0}{(S_x)_0} - (\sigma_z \otimes I)_0. \]
(B.2)

Then, we obtain
\[ N_{\sigma_z}^2 = \left[ \frac{(I \otimes S_y)_0 \cot(2g)}{(S_x)_0} + \frac{(\sigma_z \otimes S_z)_0}{(S_x)_0} - (\sigma_z \otimes I)_0 \right]^2. \]
(B.3)

Now, we calculate the average \( \langle N_{\sigma_z}^2 \rangle \) over the initial joint state \( |\psi\rangle \otimes |\xi\rangle \). We have
\[ \langle Y^2 \rangle = \frac{(S_y)^2}{(S_x)^2}; \langle X^2 \rangle = \frac{(S_x)^2}{(S_x)^2}; \langle Z^2 \rangle = 1, \]
\[ \langle YX \rangle = \langle XY \rangle = 0, \langle YZ \rangle = \langle ZY \rangle = 0, \]
\[ \langle XZ \rangle = \langle ZX \rangle = 1. \]

Explicitly, we express the meter coherent state \( |\xi\rangle \) into two modes as \( |\xi\rangle = |\alpha_H, 0\rangle \). We have
\[ \langle S_x \rangle_\xi = \langle \alpha_H, 0 | (a_H^\dagger a_H - a_V^\dagger a_V) | \alpha_H, 0 \rangle = |\alpha|^2, \]
(B.4)

where \( \langle S_x \rangle_\xi = \langle \alpha_H, 0 | (a_H^\dagger a_V + a_V^\dagger a_H) | \alpha_H, 0 \rangle = 1 \)
\[ = \langle \alpha_H, 0 | (a_H^\dagger a_V a_H^\dagger a_V + a_H^\dagger a_V a_H^\dagger a_V) | \alpha_H, 0 \rangle \]
\[ = \langle \alpha_H, 0 | (a_H^\dagger a_V + a_V^\dagger a_H) | \alpha_H, 0 \rangle = |\alpha|^2, \]
(B.5)

and
\[ \langle S_y \rangle_\xi = \langle \alpha_H, 0 | (a_H^\dagger a_x - a_V^\dagger a_y) | \alpha_H, 0 \rangle \]
\[ = |\alpha|^2 + |\alpha|^4. \]
(B.6)

Then, we obtain the square error:
\[ \epsilon_{\sigma_z}^2 = \langle N_{\sigma_z}^2 \rangle = \frac{1}{|\alpha|^2} \left( \cot^2(2g) + 1 \right) \]
\[ = \frac{1}{|\alpha|^2 \sin^2(2g)}. \]
(B.7)

2. The disturbance

Next, we calculate the rms disturbance, starting from \( B_T \) operator in Eq. (A.7),
\[ B_T \equiv (\sigma_x \otimes I)_T \]
\[ = (\sigma_x \otimes \cos(2gS_z))_0 - (\sigma_y \otimes \sin(2gS_z))_0. \]
(B.8)
The disturbance operator reads

\[ D_{\sigma_x} = (\sigma_x \otimes I)_T - (\sigma_x \otimes I)_0. \]  

(B.9)

Substituting Eq. (B.8) into Eq. (B.9) and taking the square of both sides, we have

\[ D_{\sigma_x}^2 = \left( \left( \sigma_x \otimes \left[ \cos(2gS_z) - I \right] \right)_{0Y} - \left( \sigma_y \otimes \sin(2gS_z) \right)_{0X} \right)^2. \]  

(B.10)

Now, we calculate the average \( D_{\sigma_x}^2 \) over the initial joint state \( |\psi\rangle \otimes |\zeta\rangle \). We have

\[ \langle X^2 \rangle = \langle (\cos(2gS_z) - I)^2 \rangle_{\xi}, \]
\[ \langle Y^2 \rangle = \langle \sin^2(2gS_z) \rangle_{\xi}, \]
\[ \langle XY \rangle = \langle YX \rangle = 0. \]

Finally, we have

\[ \langle D_{\sigma_x}^2 \rangle = 2(1 - \langle \cos(2gS_z) \rangle_{\xi}). \]  

(B.11)

For \( |\xi\rangle = |\alpha_H, 0\nu\rangle \) and using the operator ordering relation [92–94], such as \( e^{e^{i}a^*a} = e^{(e^{i}1)aa^*} \), we have

\[ \langle \cos(2gS_z) \rangle_{\xi} = e^{-2|\alpha|^2 \sin^2 \theta}. \]  

(B.12)

Finally, we obtain the square disturbance:

\[ \eta_{\sigma_x} = \langle D_{\sigma_x}^2 \rangle = 2(1 - e^{-2|\alpha|^2 \sin^2 \theta}), \]  

(B.13)

as shown in Eq. (20) in the main text.

Appendix C: Phase-space approximation and polarization-squeezed light meter

In this Appendix, we examine a class of the polarization squeezed coherent state states using phase-space approximation. Let us consider a class of the squeezed coherent state of the two polarization modes as

\[ |\xi\rangle = |\alpha, z\rangle = D(\alpha)S(z)|0_H\rangle|0_V\rangle, \]  

(C.1)

where \( D(\alpha) = \exp[\alpha a_H^* - \alpha^* a_H] \) is the displacement operator with \( \alpha = |\alpha|e^{i\phi} \) in the polar form, and the two-mode squeezing operator is chosen to be \( S(z) = \exp[z a_L a_R^* - a_L^* a_R] \) with \( z = re^{i\phi} \), and \( a_{L(R)} = (a_H \pm ia_V)/\sqrt{2} \). This squeezing operator is equivalent to \( S(z) = S_H(z)S_V(z) \) where \( S_H(z) = \exp[\frac{1}{2} z^* a_H^2 - \frac{1}{2} \frac{z}{2} (a_H^* a_H)^2] \) and \( S_V(z) = \exp[\frac{1}{2} z^* a_V^2 - \frac{1}{2} \frac{z}{2} (a_V^* a_V)^2] \).

It is known that when \( r > 0, \phi - \theta/2 = 0 \) results in the amplitude-squeezed coherent state, while the phase-squeezed coherent state will happen when \( \phi - \theta/2 = \pm \pi/2 \) [95]. In the following, we choose \( \phi - \theta/2 = 0 \).

The mean photon number in the two modes are

\[ \langle n_H \rangle = \langle a_H^\dagger a_H \rangle = |\alpha|^2 + \sinh^2 r, \]
\[ \langle n_V \rangle = \langle a_V^\dagger a_V \rangle = \sinh^2 r, \]

(C.2)

(C.3)

and the variances are

\[ \sigma_{n_H}^2 = |\alpha|^2 e^{-2r} + 2 \cosh^2 r \sinh^2 r \]
\[ \sigma_{n_V}^2 = |\alpha|^2 e^{-2r} + \frac{1}{2} \sinh^2 2r, \]

(C.4)

(C.5)

The expectation values of the Stokes operators give

\[ \langle S_0 \rangle = |\alpha|^2 + 2 \sinh^2 r, \]
\[ \langle S_x \rangle = |\alpha|^2, \] and \( \langle S_y \rangle = \langle S_z \rangle = 0 \).

(C.6)

(C.7)

The variances give

\[ \sigma_{S_0}^2 = \sigma_{S_x}^2 = \sigma_{S_y}^2 = |\alpha|^2 e^{-2r} + \sinh^2 2r \]
\[ \sigma_{S_z}^2 = |\alpha|^2 e^{-2r}. \]

(C.8)

(C.9)

For \( |\alpha| \gg e^{3r} \), we can ignore the term \( \sinh^2 2r \) and read

\[ \sigma_{S_0}^2 = \sigma_{S_x}^2 = \sigma_{S_y}^2 = |\alpha|^2 e^{-2r}, \]
\[ \sigma_{S_z}^2 = |\alpha|^2 e^{-2r}. \]

(C.10)

(C.11)

Thus, for \( r > 0 \), we observe squeezing in \( \sigma_{S_0}^2, \sigma_{S_x}^2 \) and \( \sigma_{S_z}^2 \), while anti-squeezing appears in \( \sigma_{S_y}^2 \).

We introduce the canonical operators

\[ q = \frac{S_y}{\sqrt{|\langle S_z \rangle|}}, \text{ and } p = \frac{S_z}{\sqrt{|\langle S_z \rangle|}} \]

(C.12)

which proportional to the polarized Stokes operators \( S_y \) and \( S_z \), respectively. The commutation relation \( [q, p] = 2iS_x/|\langle S_z \rangle| \approx 2i \) indicates that \( q \) and \( p \) can be regarded as a pair of canonical operators when \( S_z \) can be approximated by a classical positive constant as \( |\langle S_z \rangle| \) so that \( |S_x, S_y| \approx 0 \) and \( |S_x, S_z| \approx 0 \). It means that the third and higher terms in (A.2) can be ignored as in the case where \( g \ll 1 \), and that the Stokes operators, which essentially hold the discrete nature of the photon number, are replaced by the canonical continuous variable operators. Again, this approximation is valid only when \( g \ll 1 \) so that the change in \( S_x/|\langle S_z \rangle| \), which is in the order of \( g^2 \), is sufficiently small.

Then, the variances in \( q \) and \( p \) read

\[ \sigma_q^2 = e^{-2r}; \sigma_p^2 = e^{2r}. \]

(C.13)

We can define the wave function for the polarization squeezed coherent state as

\[ \psi(q) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} e^{-\frac{q^2}{2\sigma^2}}. \]  

(C.14)
where $\sigma = \sqrt{\sigma_q^2} = e^{-r}$ represents a squeezing parameter:

\[
\left\{ \begin{array}{ll}
\sigma < 1 & \Rightarrow \text{amplitude-squeezed} \\
\sigma = 1 & \Rightarrow \text{no squeezed} \\
\sigma > 1 & \Rightarrow \text{phase-squeezed}
\end{array} \right.
\]

Then, the meter light state $|\xi\rangle$ can be defined as

\[
|\xi\rangle = \int \psi(q)|q\rangle dq = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} \int e^{-\frac{q^2}{4\sigma^2}}|q\rangle dq,
\]

which we name as polarization squeezed coherent state.

1. The error

The interaction evolution is defined by

\[
U_T = e^{-i\sigma_\alpha|\sigma_\xi\otimes p},
\]

(C.16)

Using the BCH formula, we have

\[
\langle I \otimes q \rangle = e^{i\sigma_\alpha|\sigma_\xi\otimes p} \langle I \otimes q \rangle_0 e^{-i\sigma_\alpha|\sigma_\xi\otimes p} = \langle I \otimes q \rangle_0 + 2\sigma_\alpha(|\sigma_\xi \otimes I|_0).
\]

(C.17)

Therefore, if we measure the calibrated meter operator $(I \otimes q)/2\sigma_\alpha$, we will obtain the corresponding information of $(\sigma_\xi \otimes I)_0$ in the system.

We first calculate the error $\epsilon_{\sigma_\xi}^2 = \langle N_{\sigma_\xi}^2 \rangle_\xi = \langle q^2 \rangle_\xi/4\sigma_\alpha^2$. Particularly, for $|\xi\rangle = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} \int e^{-\frac{q^2}{4\sigma^2}}|q\rangle dq$, we have

\[
\langle q^2 \rangle_\xi = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} \int e^{-\frac{q^2}{4\sigma^2}}q|q\rangle \cdot \int q_1^*|q_1\rangle |q_1\rangle dq_1.
\]

\[
= \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} \int e^{-\frac{q^2}{4\sigma^2}}|q_2\rangle dq_2
\]

\[
= \left( \frac{1}{2\pi\sigma^2} \right)^{1/2} q^2 e^{-\frac{q^2}{2\sigma^2}} dq_2
\]

\[
= \left( \frac{1}{2\pi\sigma^2} \right)^{1/2} 2\sqrt{8\pi\sigma^6}
\]

\[
= \sigma^2.
\]

(C.18)

Then, we obtain the square error

\[
\epsilon_{\sigma_\xi}^2 = \langle q^2 \rangle_\xi/4\sigma_\alpha^2 = 1/4\chi^2,
\]

(C.19)

where in the last equally, we have set $\chi = g|\alpha|/\sigma$.

2. The disturbance

Doing similar to App. B, we obtain

\[
\langle D_{\sigma_\xi}^2 \rangle_\xi = 2(1 - \langle \cos(2g|\alpha|p) \rangle_\xi).
\]

(C.20)

Using the Fourier transformation, we recast the meter state $|\xi\rangle$ in the momentum representation as $|\xi\rangle = \left( \frac{2\sigma^2}{\pi} \right)^{1/4} \int e^{-\frac{p^2}{4\sigma^2}}|p\rangle dp$. Then, we obtain

\[
\eta_{\sigma_\xi}^2 = \langle D_{\sigma_\xi}^2 \rangle_\xi = 2(1 - e^{-2\chi^2}),
\]

(C.21)

as shown in Eq. (25) in the main text.

### Appendix D: Weak Interaction Approximation (WIA)

Under the WIA, we assume $g \ll 1$ and $|\alpha| \ll 1$. From Eqs. (B.7, B.13), we obtain

\[
\epsilon_{\sigma_\xi}^2 \approx \frac{1}{4g^2|\alpha|^2}, \quad \text{and} \quad \eta_{\sigma_\xi}^2 \approx 4g^2|\alpha|^2.
\]

(D.1)

From Eqs. (C.19, C.21), we obtain for $\chi \ll 1$,

\[
\epsilon_{\sigma_\xi}^2 \approx \frac{1}{4\chi^2}, \quad \text{and} \quad \eta_{\sigma_\xi}^2 \approx 4\chi^2,
\]

(D.2)

which are equivalent to Eqs. (D.1) when $\sigma = 1$. Note that, in Eq. (C.19), we have already assumed that $g \ll 1$ in the phase-space approximation. In Eqs. (D.1) and (D.2), we observe $\epsilon_{\sigma_\xi}^2, \eta_{\sigma_\xi}^2 = 1$ and thus Heisenberg-Arthurs-Kelly uncertainty is valid with minimal uncertainty.

We also show that, under the WIA, both the noise and disturbance are unbiased. As described in the main text, the noise operator (B.2) is already unbiased, i.e., $\langle N_{\sigma_\xi} \rangle_\xi = 0$ and thus $\langle N_{\sigma_\xi} \rangle = 0$ irrespective of $|\psi\rangle$, provided that $\langle S_\psi \rangle = 0$. Obviously, it is also true in the case of WIA. For the disturbance operator, from Eq. (B.8) and (B.9), we get

\[
D_{\sigma_\xi} = (\sigma_\xi \otimes \cos(2gS_z) - I)_0 - (\sigma_y \otimes \sin(2gS_z))_0
\]

(D.3)

and

\[
\langle D_{\sigma_\xi} \rangle_\xi = \sigma_x \left[ \langle \cos(2gS_z) \rangle_\xi - 1 \right] - \sigma_y \langle \sin(2gS_z) \rangle_\xi
\]

\[
= \sigma_x (e^{-2|\alpha|^2\sin^2 g} - 1).
\]

(D.4)

Here, we use Eq. (B.12) and $\langle \sin(2gS_z) \rangle_\xi = 0$. Thus, $\langle D_{\sigma_\xi} \rangle_\xi \approx 0$ when $g \ll 1$ and $|\alpha| \ll 1$. Consequently, for our initial meter state under the WIA, both the noise and disturbance operators are unbiased, i.e., $\langle N_{\sigma_\xi} \rangle_\xi = \langle D_{\sigma_\xi} \rangle = 0$ irrespective of the initial system state $|\psi\rangle$.

This joint-unbiasedness condition is sufficient for holding the Heisenberg-Arthurs-Kelly uncertainty [25].
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