MINIMAX GAMES, SPIN GLASSES AND THE
POLYNOMIAL-TIME HIERARCHY OF COMPLEXITY CLASSES

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Abstract. We use the negative replica method, which was originally de
developed for the study of overfrustration in disordered system, to investigate the
statistical behaviour of the cost function of minimax games. These games
are treated as hierarchical statistical mechanical systems, in which one of the
components is at negative temperature.

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1. Introduction

The theory of spin glasses has found interesting applications in several branches
of science [1]. In the theory of combinatorial optimization it inspired the invention
of the so called simulated annealing heuristic search technique [2]. With the help
of the replica method, several authors [3, 4, 5] managed to obtain analytical insight
about optimal solutions of some hard optimization problems. The interest in these
studies was driven by the fact that many of these problems were members of the NP
complexity class, which means that to check their solutions requires only polynomial
time, but to find them is presumably much harder.

NP is among the first few members of a hierarchy of complexity classes of in-
creasing difficulty, the polynomial-time hierarchy PH of Meyer and Stockmeyer
[6]. Examples of problems from this hierarchy are adversary games, where the first
player tries to minimize the objective function while the second one tries to maxi-
mize it. In this paper we treat the case in which one of the players has control over
the spins of a spin glass, while the other controls the external magnetic field, and
the objective function is the energy of the spin configuration.

The standard machinery of statistical mechanics provides information on the
ground state (minimum of energy), as the temperature approaches zero. To study
the maximum, we need to approach zero from negative direction. Fortunately, this
step can be incorporated into the replica method by allowing the number of replicas
to be negative. The method of negative replicas was invented by Dotenko, Franz
and Mezard to study partial annealing and overfrustration in disordered systems [7].
(Some related works are [8, 9, 10, 11].) We use this framework for the investigation
of minimax games.

In Sec. II we give a short, nontechnical description of the polynomial-time hi-
erarchy of complexity classes. In Sec. III we apply this extension of the replica
method for three simple models. Sec. IV contains an extension of the negative
replica method for multi-move games.

Key words and phrases. spin glass, minimax games, optimization, complexity.
2. The polynomial-time hierarchy

In this section we closely follow the exposition of Stockmeyer [6]. To formulate a rough definition of the complexity classes it is easier to use decision problems than optimization ones. We define the complexity class \( P \) as those problems which are solvable by a deterministic (and sequential) computer in time bounded by some polynomial of the size of the problem. Of course, one should spell out in a little more detail the kind of computers used (usually a Turing or Random Access Machine), however, the class \( P \) is remarkably stable with respect to changes of the computational model.

The definition of the class \( NP \) is similar, but in this case the use of nondeterministic computers is allowed. The nondeterministic model of computation is more powerful than the deterministic one. Let us take for example the most representative problem of \( NP \), the satisfiability of an arbitrary Boolean expression. If there is an assignment of truth values to the variables of the expression such that the expression evaluates to ‘true’, then a nondeterministic computer is able to verify that in polynomial time. In the first few steps it correctly guesses that assignment, and then by a deterministic algorithm it verifies that the assignment indeed satisfies the Boolean expression. These steps take only polynomial time. So \( NP \) contains those problems, whose solutions, if exist, can be checked in polynomial time. A basic conjecture of computer science is that the inclusion \( P \subset NP \) is proper, i.e. there are problems easy to check but hard to solve.

In the case of spin glasses the decision problem is that given a \( J_{ij} \) coupling constants matrix and a number \( K \), is there any spin configuration \( s_i \) such that \( E_J(s_i) = \sum_{i,j} J_{ij} s_is_j \leq K \). (To keep the size of the problem under control, \( J_{ij} \) should take only discrete (maybe \( \pm 1 \)) values). A closely related problem class is \( \text{co-NP} \), the complement of \( NP \). Here the task is to recognize those problems which has no solution. For example in the spin glass case one needs to prove that there is no spin configuration with energy less than a given constant. It is unlikely that such proof of polynomial length exists for a random \( J_{ij} \) matrix, so it is believed that \( NP \neq \text{co-NP} \).

Optimization problems requires the ability to solve both \( NP \) and \( \text{co-NP} \) problems. To prove that \( U_0 \) is the minima of \( E_J(s) \), one should find first \( s \) such that \( U_0 = E_J(s) \), then solve a \( \text{co-NP} \) problem proving that there is no such \( s \) that \( E_J(s) < U_0 \).

Several ways exist to obtain problems harder than \( NP \). The most obvious is to allow more (say exponential) time for the computation. A more subtle way to increase the power of the computational model is the use of oracle machines. They have an additional instruction ‘Call-Oracle’. When the machine executes this instruction, it presents the oracle a problem from the oracle’s problem class for which the oracle gives returns the solution or gives ‘no’ answer in a single step. The power of an oracle computer depends on the oracle’s problem class \( C \). Since the oracle recognizes non-membership in \( C \), too, the oracles \( C \) and \( \text{co-C} \) have the same computational power. In this manner, \( NP(C) \) (resp. \( P(C) \)) is defined as the decision problem class, which satisfiability can be decided by a nondeterministic (resp. deterministic) computer with oracle \( C \) in polynomial time.

By denoting \( P = \Sigma_0^p \), the polynomial-time hierarchy is defined as

\[
\Sigma_k^p = \text{NP}(\Sigma_{k-1}^p), \quad \Delta_k^p = \text{P}(\Sigma_{k-1}^p), \quad \Pi_k^p = \text{co-}\Sigma_k^p.
\]
Members of this hierarchy occurs in problems involving the alternation of existential and universal quantifiers. The satisfiability of the Boolean formula $f(x)$ (i.e. $f \in \text{NP}$) means $\exists x f(x)$, while its non-satisfiability (i.e. $f \in \text{co-NP}$) is the same as $\forall x \neg f(x)$. Boolean formulas $\exists x_1 \forall x_2 \ldots \exists x_{l+1} f(x_1, x_2, \ldots)$ or $\exists x_1 \forall x_2 \ldots \forall x_{k} f(x_1, x_2, \ldots)$ with $(k-1)$-fold alternation of existential and universal quantifiers provides natural examples for problems from $\Sigma^P_k$. The determination of the satisfiability of such formulas can be described as a game between two adversary players. The first player’s objective is to satisfy the formula, while the second one tries to set the variables $x_2, x_4, \ldots$ so that the formula is not satisfied.

An optimization problem from the polynomial-time hierarchy is the determination of the outcome

$$M = \max_{x_1} \min_{x_2} \ldots c(x_1, x_2, \ldots)$$

of a minimax game. For many functions $c(x_1, x_2, \ldots)$, the computation of $M$ is a $\Delta^P_k$ type problem if there are $k-1$ alternation of the the $\min$ and $\max$ operators.

In the next section we treat the case where $x_1$ and $x_2$ represent sets of discrete spin variables and $c(x_1, x_2)$ is the energy function of spin configurations.

3. Spin games

In this section we study two-move minimax games. The objective function is denoted by $H(u, v)$, where $u$ and $v$ are two sets of variables. The first player (the minimizer) controls the $u$ variables, while the second one (the maximizer) controls the $v$ variables. If both players play optimally, then the outcome of the game is

$$M = \inf_u \left( \sup_v H(u, v) \right).$$

To apply the methods of statistical mechanics, $\inf_u h(u)$ (resp. $\sup_v H(u, v)$) is replaced by the free energy of a system with Hamiltonian $h$ (resp. $H$) at low positive (resp. negative) temperature. For that purpose we introduce

$$M(\beta_u, \beta_v) = -\frac{1}{\beta_u} \ln \sum_{\{a\}} \exp -\beta_u \left( \frac{1}{\beta_v} \ln \sum_{\{v\}} \exp \beta_v H(u, v) \right) \frac{1}{\beta_u} \ln \sum_{\{a\}} \exp \beta_u H(u, v).$$

$$= -\frac{1}{\beta_u} \ln \sum_{\{a\}} \left( \sum_{\{v\}} \exp \beta_v H(u, v) \right)^{-\beta_u/\beta_v}$$

$$= -\frac{1}{\beta_u} \lim_{n \to 0} \frac{1}{n} \left[ \left( \sum_{\{u\}} \exp \beta_u \sum_{\{a, \alpha\}} H(u, v) \right)^{-1} - 1 \right].$$

There are $n$ replicas of $u$ and $nk = -n \beta_u / \beta_v$ copies of $v$. If $\beta_u, \beta_v \to \infty$, then $M(\beta_u, \beta_v) \to M$. (At least if the zero temperature entropy vanishes, which is true even in the mean field theory of spin glasses.)

To gain some experience with the method of negative replicas, we apply it first for the non-random Hamiltonian

$$H(u, v) = \frac{2}{N} \left( \sum_i u_i \right) \left( \sum_i v_i \right) + g \sum_i u_i + h \sum_i v_i, \quad i = 1..N.$$
In this case the application of the $n \to 0$ limit is not necessary, so the $u$ spins are not replicated. The partition function is

$$Z = \sum_{u,v} \exp \beta_v \sum_{\alpha} H(u,v^\alpha) =$$

$$= \int \frac{N \beta_u dx dy}{4i\pi} \exp \left\{ -N \left[ \frac{\beta_v}{2} xy - \log (2 \cosh [\beta_v k (g + x/k)]) - k \log (2 \cosh [\beta_v (h + y)]) \right] \right\}.$$

The large $\beta$ saddle point equations are

$$\frac{y_0}{2} = \tanh \left[ -\beta_u (g + \frac{x_0}{k}) \right] \approx \text{sign} \left( -g + \frac{x_0}{k} \right),$$

$$\frac{x_0}{2k} = \tanh \left[ \beta_v (h + y_0) \right] \approx \text{sign} \left( h + y_0 \right).$$

Using $\log (2 \cosh \beta z) \approx \beta |z|$ for $\beta \gg 1$, one can check that

$$\lim_{\beta_u, \beta_v \to \infty} \frac{-1}{N\beta_u} \log Z = \min_{u \in \{-1,1\}} \left[ 2u \text{sign} (2u + h) + gu + h \text{sign} (2u + h) \right],$$

where the last expression is the outcome of the game if both players play optimally, since at optimal play $v = \text{sign} (2u + h)$.

Expected outcome in constant magnetic field

In the next example random magnetic field has been added to the model:

$$H(u, v) = \frac{2}{N} \left( \sum_i u_i \right) \left( \sum_i v_i \right) + \sum_i (g + g_i) u_i + \sum_i (h + h_i) v_i,$$
where \( \overline{h}_i = \overline{g}_i = 0 \) and \( \overline{h}_i^2 = \overline{g}_i^2 = 1 \). After some tedious but standard calculations, we obtain that \( M(\beta_u, \beta_v) \) is equal to the saddle-point value of

\[
\begin{align*}
(7) & \quad -\frac{1}{\beta_u} \left\{ \frac{\beta_u}{2} pq + \int \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \log \left( 2 \cosh \left( \beta_u \sqrt{J} p + g + z \right) \right) \right\} \\
(8) & \quad \frac{\beta_u}{\beta_v} \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} \log \left( 2 \cosh \left( \beta_v \sqrt{J} q + h + w \right) \right)
\end{align*}
\]

with respect to \( p \) and \( q \). The saddle point equations are

\[
\begin{align*}
(9) & \quad p_0 = 2\sqrt{J} \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} \tanh \left( \beta_v \sqrt{J} q_0 + h + w \right) \\
(10) & \quad q_0 = 2\sqrt{J} \int \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tanh \left( -\beta_u \sqrt{J} p_0 + g + z \right)
\end{align*}
\]

The numerical solution of these equations is presented on the following graph:

![Graph showing expected outcome in random magnetic field](image)

Expected outcome in random magnetic field

Since the Hamiltonian (6) is fairly simple, the expression (7) can be derived without the use of replicas, too. For that purpose, we assume that \( M(\beta_u, \beta_v) \) receives its dominant contribution from spin configurations where the \( u \) spins' average
magnetisation is $\bar{u}$. Then
\[
M(\beta_u, \beta_v) = \frac{-1}{\beta_u N} \int \prod_i \frac{dg_i}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_i g_i^2} \log \left\{ \int d\lambda \sum_{\{u_i\}} e^{-i\lambda \left( \sum_i u_i - N\bar{u} \right)} \exp \left( -\beta_u \left( \sum_i (g_i + u_i) + Nf_v(\bar{u}) \right) \right) \right\},
\]
where $f_v(\bar{u})$ is the free-energy of the $v_i$ spins in the external field of the $u_i$ spin variables:
\[
f_v(\bar{u}) = \frac{1}{\beta_v} \int \frac{dw}{\sqrt{2\pi}} e^{-\frac{1}{2} w^2} \log \left( 2 \cosh \left[ \frac{-\beta_u (g_i + z) + i\lambda}{\beta_v (2J\bar{u} + h + w)} \right] \right).
\]
This formula should be computed at its saddle-point value with respect to $\lambda$ and its minimum with respect to $\bar{u} = [ -1, 1 ]$. After the change of variables $\bar{u} = q/(2\sqrt{J})$ and $i\lambda = \beta_u \sqrt{J}p$, the expressions (7) and (12) coincide. Since we managed to evaluate $M(\beta_u, \beta_v)$ without the use of replicas, too, this example is certainly not the most impressive application of the negative replica method. Nevertheless, this model provides an example where one can analytically prove that the replica method works.

Finally, we attempt to treat the case of a spin-glass type objective function
\[
H_J(s_i, h_i) = \sum_{1 \leq i \leq j \leq N} J_{ij} s_i s_j + g \sum_{1 \leq i \leq N} h_i s_i, \quad h_i = \pm 1, \quad s_i = \pm 1,
\]
where $J_{ij}$ is random variable with Gaussian distribution
\[
d\mu(J_{ij}) = \sqrt{N/2\pi} \exp \left( -\frac{1}{2} J_{ij}^2 / N \right) dJ_{ij}.
\]
The minimizer makes the first move and controls the $h_i$ variables, while the maximizer makes the second move and controls the $s_i$ spins. The partition function of this system is
\[
Z_{n,k} = \int \prod_{i < k} d\mu(J_{ik}) \sum_{\{h_i^a, s_i^a\}} \exp \beta_v \sum_{a \alpha} H_J(h_i, s_i) \exp \left\{ -nk \frac{\beta_u^2}{4} + \frac{\beta_v^2}{2} \sum_{a \alpha < b \beta} Q_{aab\beta}^2 - \log \sum_{\{S_{a\alpha}, H_a\}} \exp \beta_v \left[ \sum_{a \alpha < b \beta} Q_{aab\beta} S_{a\alpha}^{a\alpha} S_{b\beta}^{b\beta} - g \sum_a H_a \sum_{\alpha} S_{a\alpha}^{a\alpha} \right] \right\},
\]
where $k = -\beta_u / \beta_v$. In the replica symmetric approximation
\[
Q_{aaoa} = p, \quad Q_{aab\beta} = q \quad \text{for} \quad a \neq b,
\]
The expression for the partition function $Z_{n,k}$ is:

$$Z_{n,k} = \int \prod_{a \neq b} \left( \sqrt{\frac{N \beta}{2\pi}} dQ_{a \alpha b \beta} \right) \exp -N \left\{ \beta_v^2 \left( \frac{nk}{4} + \frac{n(n-1)k^2}{4}q^2 + \frac{nk(k-1)}{4}p^2 + \frac{nk}{2}p \right) - \log \int dx e^{-\frac{1}{2}x^2} \right\}$$

From this equation one obtains the expected outcome of the game:

$$m = \frac{1}{N} \min_{(s_i)} \left( \max_{(h_i)} H_f(s_i, h_i) \right) = \lim_{n \to 0, \beta_v \to \infty, k \to 0} \frac{1}{k \beta_v n N} (Z_{n,k} - 1) = \frac{\beta_v}{4} (1 + kq^2 + (1 - k)p^2 - 2p) +$$

$$+ \frac{1}{k \beta_v} \int dx e^{-\frac{1}{2}x^2} \log \int dy e^{-\frac{1}{2}y^2} \left\{ [2 \cosh (\sqrt{q}x + \sqrt{p - q}y + g)]^k + [2 \cosh (\sqrt{q}x + \sqrt{p - q}y - g)]^k \right\}$$

where the last expression should be evaluated at its saddle point. This expression is very similar to the free energy of a spin glass at the one stage replica symmetry breaking approximation [12]. Indeed, $Q_{a \alpha b \beta}$ might be regarded as an $nk \times nk$ matrix broken into blocks of size $k \times k$. However, here $k$ is a fixed negative number. $m(p, q)$ has a minimum at $p = q = 1$ on the line $p = q$. In this approximation

$$m_{\text{minimax}}(g) = \int dx e^{-\frac{1}{2}x^2} \min |x + g|, |x - g|).$$

A better approximation is achieved if we search for the saddle point on the $(p, q)$ plane. Since the first term of $m$ scales as $O(\beta)$ as $\beta \to \infty$, while the second has finite limit, the saddle point should be on the curve $1 + kq^2 + (1 - k)p^2 - 2p$. We evaluated numerically $m$ as the function of $g$. We plot the function $m_{\text{minimax}}(g)$ (solid line on the figure).
The exact value of $m_{\text{minimax}}(g)$ is smaller than $m_{\text{spinglass}}(g)$, (where $m_{\text{spinglass}}(g)$ is the maximal value of $\sum J_{ij}s_is_j + g\sum s_i$), since the minimizer tries to set $h_i$ into directions least favorable for the maximizer, while the constant magnetic field is equivalent to a randomly chosen $h_i$ configuration. However, $m_{\text{minimax}}(g) \geq m_{\text{spinglass}}(0)$, since one of $\pm \sum h_is_i$ is always nonnegative. $m_{\text{minimax}}(g) \geq (g - m_{\text{spinglass}}(0))$ also holds, since if the spins are set to the same direction as $h_i$, then the contribution of $\sum J_{ij}s_is_j$ cannot be less than $-m_{\text{spinglass}}(0)$ by the symmetry of the couplings $J_{ij}$. We also expect that $m_{\text{minimax}}(g)$ converges to $g - m_{\text{spinglass}}(0)$ as $g \to \infty$. These considerations provide upper and lower bounds for $m_{\text{minimax}}(g)$ (dotted lines on the figure). Unfortunately, the lower bound is violated for small $g$, while its assymptotics is correctly reproduced. It would be interesting to know if a better, replica symmetry breaking solution would cure this problem.

4. Multi-move games

Up to this point only two-moves games were treated. The extension for multi-move games is straightforward. For example, the outcome of the four-move game

$$M = \inf_u \left\{ \sup_v \left[ \inf_w \left( \sup_z H(u, v, w, z) \right) \right] \right\}$$

is

$$\lim_{\beta_u, \beta_v, \beta_w, \beta_z \to \infty} -\frac{1}{\beta_u n} \lim_{n \to 0} \left[ \sum_{u^a, v^{a\alpha}, w^{a\alpha\beta}, z^{a\alpha\beta\gamma}} \exp \beta_z \sum_{a\alpha\beta\gamma} H(u^a, v^{a\alpha}, w^{a\alpha\beta}, z^{a\alpha\beta\gamma}) - 1 \right]$$

where the ranges of the indices are $|a| = n$, $|\alpha| = -\beta_u/\beta_v$, $|\beta| = -\beta_v/\beta_w$, $|\gamma| = -\beta_w/\beta_z$. 
The limit $\beta_{u,v,w,z} \to \infty$ corresponds to the optimal strategies of the players. Finite $\beta$ simulates non-exact optimization, i.e. players with bounded computational capabilities. An interesting case is when one player’s temperature is infinite, so the other ones play against random moves. Such games are called 'games against Nature' \[13\].

5. Discussion

In the previous sections we used the method of negative replicas to examine optimization problems arising in minimax games. Such games provide examples of very difficult combinatorial problems. In principle our method is able to estimate the expected outcome of some adversary games. Unfortunately, due to the complexity of the calculations emerging in problems of spin glass type, we manage to treat only fairly simple optimization problems. Nevertheless, the method of negative replicas provides a natural framework to treat game theoretical problems with the machinery of statistical mechanics.
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