A Dataset-Level Geometric Framework for Ensemble Classifiers

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Abstract

Ensemble classifiers have been investigated by many in the artificial intelligence and machine learning community. Majority voting and weighted majority voting are two commonly used combination schemes in ensemble learning. However, understanding of them is incomplete at best, with some properties even misunderstood. In this paper, we present a group of properties of these two schemes formally under a dataset-level geometric framework. Two key factors, every component base classifier’s performance and dissimilarity between each pair of component classifiers are evaluated by the same metric - the Euclidean distance. Consequently, ensembling becomes a deterministic problem and the performance of an ensemble can be calculated directly by a formula. We prove several theorems of interest and explain their implications for ensembles. In particular, we compare and contrast the effect of the number of component classifiers on these two types of ensemble schemes. Empirical investigation is also conducted to verify the theoretical results when other metrics such as accuracy are used. We believe that the results from this paper are very useful for us to understand the fundamental properties of these two combination schemes and the principles of ensemble classifiers in general. The results are also helpful for us to investigate some issues in ensemble classifiers, such as ensemble performance prediction, selecting a small number of base classifiers to obtain efficient and effective ensembles.

Keywords: Ensemble learning; Geometric framework; Classification; Majority voting; Weighted majority voting

1 Introduction

In the last three decades, ensemble learning has been investigated by many researchers. This technique has seen use in diverse tasks such as classification,
regression, and clustering among others. Research has been conducted at multiple levels: from feature selection, component classifier generation and selection, to the ensemble model (Jurek et al., 2014; Dong et al., 2020). Some of the ensemble approaches have been very successful in international machine learning competitions such as Kaggle, KDD-Cups, etc., and the technologies have been used in various application areas (Oza and Tumer, 2008).

Although some ensemble models (such as stacking, bagging, random forest, AdaBoost, gradient boosting machines, deep neural network-based models, and others) are more complicated, two relatively simple combination schemes, majority voting and weighted majority voting, have been used widely for the ensemble model (Sagi and Rokach, 2018). Even in those more complicated models, majority voting and weighted majority voting are still used frequently at intermediate or final combination stages.

How many component classifiers to use and how to select a subset from a large group for an ensemble are two related questions. Zhou et al. (2002) investigated the impact of the number of component classifiers on ensemble performance. It is found that the number of component classifiers is not always a positive factor for improving performance when majority voting is used for combination. Later their finding is often referred to as the “many-could-be-better-than-all” theorem.

However, this finding is not echoed by many others. More empirical evidence indicates that the size of an ensemble has a positive impact on performance (Hernández-Lobato et al., 2013). To balance ensemble performance and efficiency, many papers investigate how to achieve best possible performance by combining a fixed or a small number of classifiers selected from a large number of candidates (Latitine et al., 2001; Oshiro et al., 2012; Xiao et al., 2014; Dias and Windeatt, 2014; Bhardwaj et al., 2016; Ykhlef and Bouchaffra, 2017; Zhu et al., 2019).

Possibly inspired by Zhou et al. (2002)’s work, Bonab and Can (2019) asserted that for weighted majority voting: the ideal condition for the ensemble to achieve maximum accuracy is for the number of component classifiers to equal the number of output classes. However, their theoretical analysis is limited by the strength of assumptions used, which experimental results do not always support.

Diversity among component classifiers is a factor that may influence ensemble performance (Bi, 2012; Jain et al., 2020; Zhang et al., 2020). However, there is no generally accepted definition of diversity (Kuncheva and Whitaker, 2003). Many measures, such as Yule’s Q statistics, the correlation coefficient, the disagreement measure, F-score, the double fault measure, Kohavi-Wolpert’s variance, Kuncheva’s entropy, etc., have been proposed and investigated (Tang et al., 2006; Visentini et al., 2016). Moreover, many diversity measures such as the F-score, the disagreement measure, the double fault measure, etc. are or related to performance measures. This explains why the effect of diversity on ensemble performance is unclear (Tang et al., 2006; Bi, 2012).

In short, although ensemble learning has garnered considerable attention from researchers, much of the literature comprise methodological and empirical
studies, while the theory is underrepresented. Many fundamental questions remain unclear. We list some of these below:

1. What is the difference between majority voting and weighted majority voting?
2. When should we use majority voting rather than weighted majority voting, or vice versa?
3. There are a lot of weighting assignment methods for weighted majority voting, which one is the best?
4. How does the number of component classifiers affect ensemble performance?
5. How does each component classifier affect ensemble performance?
6. How does performance of component classifiers affect ensemble performance?
7. How does diversity of component classifiers affect ensemble performance?

The goal of this paper is to set up a geometric framework for ensemble classifiers, thereby enabling us to clearly answer the above questions. In this framework, the result from each base classifier is represented as a point in a multi-dimensional space. We can use the same measure - the Euclidean distance - for both performance and diversity. Ensemble learning becomes a deterministic problem. Many interesting theorems about the relation of component classifiers and ensemble classifiers can be proven. Armed with this framework, we discover and present some useful features of majority voting and weighted majority voting.

The rest of this paper is organized as follows: Section 2 presents some related work. In Section 3, we present the geometric framework for ensemble classifiers. Some characteristics of majority voting and weighted majority voting are presented and compared. Discussion regarding the questions raised previously is given in Section 4, supported by theory. In Section 5, we present some empirical investigation results to confirm our findings in Section 3. Finally Section 6 is the conclusion.

2 Related work

Majority voting and weighted majority voting are commonly used in many ensemble models. Their features and some related tasks have been investigated by many researchers. Zhou et al. (2002) investigated majority voting theoretically and empirically through an ensemble of a group of neural networks. It is found that selecting a subset may be able to achieve better performance than combining all available
classifiers. Note that in both their theoretical and empirical study, weighted majority voting is mentioned, but not considered.

Fumera et al. (2008) set up a probabilistic framework to analyse the bagging misclassification rate when the ensemble size increases. Majority voting is used for combination. Both their theoretical analysis and empirical investigation show that bagging misclassification rate decreases when more and more component classifiers are combined. This is somewhat inconsistent with the finding of Zhou et al. (2002).

Latíne et al. (2001) used the McNemar test to decide if two sets of component classifiers are significantly different. More and more classifiers are added to the pool and the McNemar test is carried out repeatedly, the process stops if no significant difference can be observed.

Oshiro et al. (2012) raised the problem of how many component classifiers (trees) should be used for an ensemble (random forest). Experimenting with 29 real datasets in the biomedical domain, they observed that higher accuracy is achievable when more trees are combined by majority voting. However, when the number of trees is relatively large (say, over 32 or 64), the improvement is no longer significant. Therefore, a good choice is to consider both ensemble accuracy and computing cost. This issue is also addressed in Hernández-Lobato et al. (2013); Adnan and Islam (2014); Probst and Boulesteix (2017).

Many researchers find that if more base classifiers are combined, then the ensemble is able to achieve better performance. Although it is possible to achieve better performance by fusing more base classifiers, it costs more. Quite a few papers investigated the ensemble pruning problem that aims at increasing efficiency by reducing the number of base classifiers, without losing much performance at the same time. Various kinds of methods have been proposed. See Xiao et al. (2010); Dias and Windeatt (2014); Bhardwaj et al. (2016); Ykhlef and Bouchaffra (2017); Zhu et al. (2019) for some of them.

How to assign proper weights for weighted majority vote has been investigated by quite a few researchers. Some methods may consider different aspects of base classifiers for the weights: performance-based in Opitz and Shavlik (1994); Wozniak (2008), a model probability-based in Duan et al. (2007), prediction confidence-based in Schapire and Singer (1999), and Matthews correlation coefficient-based in Haque et al. (2016). Some methods may have different goals for weight optimisation: minimizing classification error in Kuncheva and Diez (2014); Mao et al. (2015); Bashir et al. (2017), reducing variance while preserving given accuracy in Derbeko et al. (2002), minimizing Euclidean distance in Bonab and Can (2018). Different search methods are also used: A linear programming-based method is presented in Zhang and Zhou (2011); Georgion et al. (2004) applied a game theory to weight assignment, Liu et al. (2014) applied the genetic algorithm to weight assignment. Dynamic adjustment of weights is investigated in Valdovinos and Sánchez (2009).

Wu and Crestani (2015) proposed a geometric framework for data fusion in information retrieval. In this query-level framework, for a certain query each component retrieval system assigns scores to all the documents in the collection,
which indicates their relevance to the query. Also predefined values are given to documents that are relevant or irrelevant to the query, which are ideal scores for the documents involved. Therefore, the scores from each component retrieval system can be regarded as a point in a multiple dimensional space, and the ideal scores form an ideal point. Under this framework, performance and dissimilarity can be measured by the same metric - the Euclidean distance. Both majority voting and weighted majority voting can be explained and calculated in the geometric space. However, the framework is set up at the query level, which is equivalent to the instance level in ensemble classifiers.

Bonab and Can (2018, 2019) adapted the model in Wu and Crestani (2015) to ensemble classifiers. Some properties of majority voting and weighted majority voting are presented. However, as in Wu and Crestani (2015), their framework is at the instance level. This means that the theorems hold for each individual instance, but it is unclear if they remain true for multiple instances collectively. The latter is a more important and realistic situation we should consider. As we know, a training dataset or a test dataset usually comprises a group of instances. It is desirable to know the collective properties of an ensemble classifier over all the instances, rather than that of any individual instance. This generalization is the major goal of this paper.

To do this, we first define a dataset-level framework and then go about proving a number of useful theorems.

3 The geometric framework

In this section, we introduce the dataset-level geometric framework.

Suppose for a machine learning problem we have \( p \) classes \( CL = \{cl_1, cl_2, \cdots, cl_p\} \), the ensemble has \( m \) component base classifiers \( CF = \{cf_1, cf_2, \cdots, cf_m\} \), and the dataset \( DT \) has \( n \) instances \( T = \{t_1, t_2, \cdots, t_n\} \). For every instance \( t_i \) and every class \( cl_j \), each base classifier \( cf_k \) provides a score \( s_{kj} \), which indicates the estimated probability score that \( t_i \) is an instance of class \( cl_k \) given by \( cf_j \). \( s_{kj} \in [0, 1] \). Each instance \( t_i \) has a real label for each class \( cl_k \), which is 0 or 1. We may set up a \( n \times p \)-dimensional space for the above problem. There are \( m \) points \( \{S^1, S^2, \cdots, S^m\} \), each represents the scores given by a specific classifier to all the instances in the dataset for all the classes. Point \( S^k \) is:

\[
S^k = \{(s_{11}^k, s_{12}^k, \cdots, s_{1p}^k), (s_{21}^k, s_{22}^k, \cdots, s_{2p}^k), \cdots, (s_{n1}^k, s_{n2}^k, \cdots, s_{np}^k)\} = \{s_{11}^k, s_{21}^k, \cdots, s_{np}^k\}
\]

As such, we can always organize all the scores first by instances and then by classes. Thus a two-dimensional array with \( n \) elements in one dimension and \( p \) in the other is transformed to a list of \( n \times p \) elements: \( s_{ij}^k \) becomes \( s_{ij}^k \). The ideal point is also represented in the same style.
Table 1: Notation used in this paper

| Symbols | Meaning |
|---------|---------|
| $C$     | Centroid of a group of points in $X$ |
| $CL$    | A set of classes |
| $cl_j$  | One of the classes in $CL$ |
| $CF$    | A set of component classifiers |
| $cf_k$  | One of the component classifiers in $CF$ |
| $DT$    | A dataset has components $CF$, $CL$, $S$, $T$, and $O$ |
| $ed(S^i, S^j)$ | the Euclidean distance between two points $(S^i, S^j)$ in $X$ |
| $F$     | Fused point by linear combination of a group of points in $X$ |
| $m$     | Number of component classifiers in $CF$ |
| $n$     | Number of instances in $T$ |
| $O$     | All the real labels for $T$ constitute the ideal point in $X$ |
| $o_{ij}$ | One of the elements of $O$ |
| $o_l$   | Another form of $o_{ij}$, in which $l = (i - 1) * n + j$ |
| $p$     | Number of classes in $CL$ |
| $S$     | Scores given by $CF$ members relating to $CL$, a set of points in $X$ |
| $S^k$   | Scores given by $cf_k$ for $T$ relating to $CL$, a point in $X$ |
| $s_{ij}$ | A score given by $cf_k$ for $t_i$ relating to $c_j$ |
| $s_{ij}^l$ | Another form of $s_{ij}^k$, in which $l = (i - 1) * n + j$ |
| $T$     | A set of instances |
| $t_i$   | One of the instances in $T$ |
| $w^k$   | Weight assigned to component classifier $cf_k$ |
| $X$     | a $n * p$-dimensional space relating to $DT$ |
which indicates the real labels of every instance in the dataset to each of the classes involved: 1 for a true label and 0 for a false label. The notation used in this paper is summarized in Table 1.

This framework is a generalization of the one presented in Bonab and Can (2018, 2019). If the dataset only has one instance, then the above framework is the same as the one in Bonab and Can (2018, 2019). This framework is suitable for soft voting (Cao et al., 2015), in which each component classifier provides probability scores for every instance relating to each class. If no such probability scores are provided (hard voting), then it is still possible to apply it to the geometric framework if we transform estimated class labels into proper scores. The geometric framework is applicable to both single-label and multi-label classification problems.

**Example 1.** A dataset includes two instances \( t_1 \) and \( t_2 \), and three classes \( cl_1 \), \( cl_2 \), and \( cl_3 \). \( t_1 \) is an instance of \( cl_2 \) but not the other two, and \( t_2 \) is an instance of both class \( cl_1 \) and \( cl_3 \) but not \( cl_2 \). The scores got from the three base component classifiers \( cf_1 \), \( cf_2 \), and \( cf_3 \) are as follows:

| Instance | Classifier | Class \( cl_1 \) | Class \( cl_2 \) | Class \( cl_3 \) |
|----------|------------|-----------------|-----------------|-----------------|
| \( t_1 \) | \( cf_1 \)  | \( s_{11} = 0.5 \) | \( s_{12} = 0.6 \) | \( s_{13} = 0.3 \) |
|          | \( cf_2 \)  | \( s_{21} = 0.4 \) | \( s_{22} = 0.7 \) | \( s_{23} = 0.2 \) |
|          | \( cf_3 \)  | \( s_{31} = 0.6 \) | \( s_{32} = 0.8 \) | \( s_{33} = 0.4 \) |
| Real label | \( o_{11} = 0 \) | \( o_{12} = 1 \) | \( o_{13} = 0 \) |
| \( t_2 \) | \( cf_1 \)  | \( s_{21} = 0.7 \) | \( s_{22} = 0.3 \) | \( s_{23} = 0.9 \) |
|          | \( cf_2 \)  | \( s_{31} = 0.3 \) | \( s_{32} = 0.6 \) | \( s_{33} = 0.7 \) |
|          | \( cf_3 \)  | \( s_{41} = 0.2 \) | \( s_{42} = 0.6 \) | \( s_{43} = 0.8 \) |
| Real label | \( o_{21} = 1 \) | \( o_{22} = 0 \) | \( o_{23} = 1 \) |

In a 6-dimensional geometric space, we may set up four points to represent this scenario: \( S^1 = \{0.5, 0.6, 0.3, 0.7, 0.3, 0.9\} \), \( S^2 = \{0.4, 0.7, 0.2, 0.3, 0.6, 0.7\} \), \( S^3 = \{0.6, 0.8, 0.4, 0.2, 0.6, 0.8\} \), and \( O = \{0, 1, 0, 1, 0, 1\} \).

In the following we use point and component result interchangeably if no confusion will be caused. We can calculate the Euclidean distance of two points \( S^u \) and \( S^v \)

\[
ed(S^u, S^v) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{p} (s^u_{ij} - s^v_{ij})^2} = \sqrt{\sum_{i=1}^{n \cdot p} (s^u_{i} - s^v_{i})^2} \tag{1}
\]

We may use the Euclidean distance to evaluate the performance of classifier \( cf_u \) over \( n \) instances and \( p \) classes.
\[ ed(S^n, O) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{p} (s^n_{ij} - o_{ij})^2} = \sqrt{\sum_{l=1}^{n+p} (s^n_l - o_l)^2} \]  

(2)

It is an advantage of the geometric framework to evaluate both performance of a component result and dissimilarity of two component results by using the same metric. They will be referred to as performance distance and dissimilarity distance later in this paper.

**Definition 1 (Performance distribution).** In a geometric space \( X \), there are \( m \) points \( S = \{S^1, S^2, \ldots, S^m\} \) \((m \geq 1)\). \( O \) is the ideal point. Performance distribution of these \( m \) points in \( S \) is defined as \( ed(S_i, O) \) for \((1 \leq i \leq m)\).

**Definition 2 (Dissimilarity distribution).** In a geometric space \( X \), there are \( m \) points \( S = \{S^1, S^2, \ldots, S^m\} \) \((m \geq 1)\). Dissimilarity distribution of these \( m \) points in \( S \) is \( ed(S_i, S_j) \) for \((1 \leq i \leq m, 1 \leq j \leq m, i \neq j)\).

From their definitions, we can see that performance distribution and dissimilarity distribution are two completely different aspects of \( S \), and not related to each other. Their independence is a good thing for us to investigate the properties of ensembles, especially the effect of each on ensemble performance.

### 3.1 Majority voting

In a \( n \times p \)-dimensional space, there are \( m \) points \( S = \{S^1, S^2, \ldots, S^m\} \) \((m \geq 2)\). Combining them by majority voting can be understood to be finding the centroid of these \( m \) points. It is referred to as the centroid-based fusion method in Wu and Crestani (2015).

**Theorem 1.** In a geometric space \( X \), there are \( m \) points \( S = \{S^1, S^2, \ldots, S^m\} \) \((m \geq 2)\). \( C \) is the centroid of these \( m \) points and \( O \) is the ideal point. The distance between \( C \) and \( O \) is no longer than the average distance between each of the \( m \) points and \( O \):

\[ m \sqrt{\sum_{i=1}^{n+p} \left( \frac{1}{m} \sum_{k=1}^{m} s^k_i - o_i \right)^2} \leq \frac{1}{m} \sum_{k=1}^{m} \sum_{l=1}^{n+p} (s^k_l - o_l)^2 \]  

(3)

Proof: Replace \( C \) with its definition \( c_l = \frac{1}{m} \sum_{k=1}^{m} s^k_l \) for \((1 \leq l \leq n \times p)\) in Equation 3 and move \( \frac{1}{m} \) on the right side to the left as \( m \), we get

\[ m \sqrt{\sum_{i=1}^{n+p} \left( \frac{1}{m} \sum_{k=1}^{m} s^k_i - o_i \right)^2} \leq \sum_{k=1}^{m} \sum_{l=1}^{n+p} (s^k_l - o_l)^2 \]  

(4)

The Minkowski Sum Inequality (Minkowski 2024) is

\[ \left( \sum_{i=1}^{n+p} (a_i + b_i)^q \right)^{\frac{1}{q}} \leq \sum_{i=1}^{n+p} a_i^q + \sum_{i=1}^{n+p} b_i^q \]

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Consider the points in Example 1. Example 2. No researchers that the results from the ensemble are stable and good. This theorem confirms the observation by Euclidean distance as the performance metric, then the performance of the

Then we have:

\[
\sum_{k=1}^{m} \sqrt{\sum_{l=1}^{n*p} (s^k_l - o_l)^2} = \sqrt{\sum_{l=1}^{m} (s^l)^2}
\]

\[
\geq \sqrt{\sum_{l=1}^{n*p} (a_1 + a_2)^2 + \sum_{k=3}^{m} \sum_{l=1}^{n*p} (a_l)^2}
\]

\[
\geq ... \geq \sqrt{\sum_{l=1}^{n*p} (a_1 + a_2 + ... + a_m)^2} = \sqrt{\sum_{l=1}^{n*p} \sum_{k=1}^{m} a_k)^2}
\]

Now notice the left side of Inequation \[ \]

\[
m \sqrt{\sum_{i=1}^{m} \left( \frac{1}{m} \sum_{k=1}^{m} s^k_l - o_l \right)^2} = \sqrt{\sum_{l=1}^{n*p} \sum_{k=1}^{m} (s^k_l - m * o_l)^2} = \sqrt{\sum_{l=1}^{n*p} \sum_{k=1}^{m} a_k)^2} \]

This theorem tells us, if we take all the instances together and use the Euclidean distance as the performance metric, then the performance of the ensemble by majority voting is at least as good as the average performance of all component classifiers involved. This theorem confirms the observation by many researchers that the results from the ensemble are stable and good.

**Example 2.** Consider the points in Example 1. \( C = \{0.5, 0.7, 0.3, 0.4, 0.5, 0.8\} \), \( ed(S^1, O) = 0.83, ed(S^2, O) = 1.11, ed(S^3, O) = 1.26, ed(C, O) = 1.04, (ed(S^1, O) + ed(S^2, O) + ed(S^3, O))/3 = 1.06. ed(C, O) \) is slightly smaller than the average of \( ed(S^1, O), ed(S^2, O), \) and \( ed(S^3, O) \). \[ \]

**Theorem 2.** In a space \( X \), suppose that \( S = \{S^1, S^2, \ldots, S^m\} \) and \( O \) are known points. \( C \) is the centroid of \( S^1, S^2, \ldots, S^m \). The distance between \( C \) and \( O \) can be represented as

\[
ed(C, O) = \frac{1}{m} \sqrt{m \sum_{i=1}^{m} ed(S^i, O)^2 - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} ed(S^i, S^j)^2} \] (5)

Proof. Assume that \( O = (0, \ldots, 0) \), which can always be done by coordinate transformation. According to its definition \( C = (\frac{1}{m} \sum_{i=1}^{m} s^1_i, \ldots, \frac{1}{m} \sum_{i=1}^{m} s^{n*p}_i) \), we have

\[
ed(C, O) = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (s^1_i)^2 + \ldots + \frac{1}{m} \sum_{i=1}^{m} (s^{n*p}_i)^2}
\]

\[
= \frac{1}{m} \sqrt{\sum_{i=1}^{m} \sum_{k=1}^{n*p} (s^i_k)^2 + 2 \sum_{k=1}^{n*p} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} s^i_k \cdot s^j_k}
\]

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Note that the distance between \( S^i \) and \( S^j \) is 
\[
ed(S^i, S^j) = \sqrt{\sum_{k=1}^{n+p} (s^i_k - s^j_k)^2}
\]
or 
\[
ed(S^i, S^j)^2 = \sum_{k=1}^{n+p} (s^i_k)^2 + \sum_{k=1}^{n+p} (s^j_k)^2 - 2 \sum_{k=1}^{n+p} s^i_k \cdot s^j_k
\]
Because 
\[
ed(S^i, O)^2 = \sum_{k=1}^{n+p} (s^i_k)^2 \quad \text{and} \quad \ned(S^j, O)^2 = \sum_{k=1}^{n+p} (s^j_k)^2
\]
we get
\[
2 \sum_{k=1}^{n+p} s^i_k \cdot s^j_k = \ned(S^i, O)^2 + \ned(S^j, O)^2 - \ned(S^i, S^j)^2
\]
Considering all possible pairs of points \( S^i \) and \( S^j \) and we get
\[
2 \sum_{\substack{k=1 \atop k \neq i, j}}^{n+p} s^i_k \cdot s^j_k = (m - 1) \sum_{i=1}^{m} \ned(S^i, O)^2 - \sum_{i=1}^{m} \sum_{j=i+1}^{m} \ned(S^i, S^j)^2
\]
In Equation [8] we use the right side of the above equation to replace
\[
2 \sum_{\substack{k=1 \atop k \neq i, j}}^{n+p} s^i_k \cdot s^j_k
\]
Also note that 
\[
\sum_{i=1}^{m} \sum_{j=i+1}^{m} (s^i_k)^2 = \sum_{i=1}^{m} \ned(S^i, O)^2
\]
we obtain
\[
ed(C, O) = \frac{1}{m} \sqrt{m \sum_{i=1}^{m} \ned(S^i, O)^2 - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \ned(S^i, S^j)^2} \quad \square
\]
This theorem tells us that the ensemble performance is completely decided by \( \ned(S^i, O) \) (for \( 1 \leq i \leq m \)) and \( \ned(S^j, S^j) \) (for \( 1 \leq i \leq m, 1 \leq j \leq m, i \neq j \)). The impact of both performance of component classifiers and dissimilarity of all pairs of component classifiers on ensemble performance can be seen clearly. According to Equation 2, in order to minimize \( \ned(C, O) \), we need to minimize \( \sum_{i=1}^{m} \ned(S^i, O)^2 \) and maximize \( \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \ned(S^i, S^j)^2 \) at the same time. Therefore, for performance distribution, both average and variance affect ensemble performance. Lower average and lower variance lead to better performance. For dissimilarity distribution, both average and variance also affect ensemble performance. Higher average and higher variance lead to better performance.

**Example 3.** Assume that \( S^1 = \{S^1, S^2\}, S^2 = \{S^3, S^4\} \), \( \ned(S^1, S^2) = \ned(S^3, S^4) = \ned(S^3, O) = 0.5, \ned(S^2, O) = 0.5, \ned(S^3, O) = 0.4, \ned(S^4, O) = 0.6 \). The centroid of \( S^1 \) and \( S^2 \) is \( C^1 \), and the centroid of \( S^3 \) and \( S^4 \) is \( C^2 \). We have \( \ned(C^1, O) < \ned(C^2, O) \).

In this example, because dissimilarity distance is the same for both \( S^1 \) and \( S^2 \), we only need to consider performance distribution. The average of \( \ned(S^1, O) \) and \( \ned(S^2, O) \) is 0.5, and the average of \( \ned(S^3, O) \) and \( \ned(S^4, O) \) is also 0.5. The variance of \( S^1 \) is smaller than that of \( S^2 \). \( \ned(S^1, O)^2 + \ned(S^2, O)^2 = 0.50, \ned(S^1, O)^2 + \ned(S^4, O)^2 = 0.52 \). Because \( \ned(S^1, S^2) = \ned(S^3, S^4) \), we get \( \ned(C^1, O) \) < \( \ned(C^2, O) \). \[\square\]
Example 4. In the figure below, there are six points $S_i$ ($1 \leq i \leq 6$). Among these points, $S^1$ is the closest to $O$. It is followed by $S^2$ and $S^3$, which are equally distant to $O$. Finally, $S^4$, $S^5$ and $S^6$ are equally distant to $O$ and they are all further from $O$ than the other three points. Now we try to work out a subset of these to maximize performance.

![Diagram of points](image)

Now, if we select one point only, then $S^1$ is the best option; if we select two points, then combining $S^2$ and $S^3$ is the best option, in this instance the centroid would be $C^2$; if we select three points, then combining $S^4$, $S^5$ and $S^6$ is the best option, this gives the centroid $O$. Fusing all six points is also a good option, but it may not be as good as fusing $S^4$, $S^5$ and $S^6$. The “many-could-be-better-than-all” theorem seems reasonable in this case. Because the centroid of a group of points is decided by the positions of all the points collectively, each of which has an equal weight, removing or adding even a single point may change the position of the centroid of a group of points significantly. It also indicates that it is not an easy task to find the best subset from a large group of base classifiers. As a matter of fact, it is an NP-hard problem, as our next theorem proves.

**Theorem 3.** $S = \{S_1, S_2, \cdots, S_m\}$ for ($m \geq 3$) is a group of points and $O$ is an ideal point. For a given number $m'$ ($2 \leq m' < m$), the problem is to find a subset of $m'$ points from $S$ to minimize the distance of the centroid of these $m'$ points to the ideal point $O$. The above question is NP-hard. 

Proof: First let us have a look at the maximum diversity problem (MDP), which is a known NP-hard problem [Wang et al., 2014]. The MDP is to identify a subset $E'$ of $m'$ elements from a set $E$ of $m$ elements, such that the sum of the pairwise distance between the elements in the subset is maximized. More precisely, let $E = \{e_1, e_2, \cdots, e_m\}$ be a group of elements, and $d_{ij}$ be the distance between elements $e_i$ and $e_j$. The objective of the MDP can be formulated as:

Maximize

$$ f(x) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij} x_i x_j $$

subject to $\sum_{i=1}^{m} x_i = m', x_i \in \{0, 1\}, i = 1, \cdots, m$
where each \( x_i \) is a binary (zero-one) variable indicating whether an element \( e_i \in E \) is selected to be a member of \( E' \).

On the other hand, according to Theorem 2, our problem may be written as minimizing \( ed(C, O)^2 \). Here \( C \) is the centroid of \( m' \) selected points.

\[
ed(C, O)^2 = \frac{1}{m'} \sum_{i=1}^{m} ed(S^i, O)^2 * x_i - \frac{1}{m' * m} \sum_{i=1}^{m} \sum_{j=1}^{m} ed(S^i, S^j)^2 * x_i * x_j
\]

subject to \( \sum_{i=1}^{m} x_i = m' \), \( x_i \in \{0, 1\}, i = 1, \cdots, m \)

where each \( x_i \) is also a binary (zero-one) variable indicating whether a point \( S^i \in S \) is selected to be a member of \( S' \).

If we assume that \( ed(S^i, O) \) equals to each other for all \( i = 1, \cdots, m \), then \( \frac{1}{m'} \sum_{i=1}^{m} ed(S^i, O)^2 * x_i \) becomes a constant and minimizing \( ed(C, O)^2 \) equals to maximizing \( div(C, O)^2 \)

\[
div(C, O)^2 = \frac{1}{m' * m'} \sum_{i=1}^{m} \sum_{j=1}^{m} ed(S^i, S^j)^2 * x_i * x_j
\]

Let \( g(x) = div(C, O)^2 \) and \( d_{ij} = \frac{2}{m' * m} ed(S^i, S^j)^2 \), then the above equation can be rewritten as

\[
g(x) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij} * x_i * x_j \quad (8)
\]

Comparing Equation \( 11 \) and \( 8 \), we can see that a simplified version of our problem is an MDP problem. Therefore, our problem is an NP-hard problem. □

Theorem 3 tells us: for a given number of classifiers, choosing a subset for best ensemble performance by majority voting is an NP-hard problem.

Although the “many-could-be-better-than-all” theorem may be applicable to some cases, it does not tell the whole story. Now let us have a look at how the size of the ensemble impacts its performance. Because the performance of an ensemble is affected by a few different factors, we need to find a way of separating this from other factors.

**Theorem 4.** In a space \( X \), \( S = \{ S^1, S^2, \cdots, S^m \} \) and \( O \) are known points. \( C \) is the centroid of \( S^1, S^2, \cdots, S^m \), \( C^1 \) is the centroid of \( m-1 \) points \( S^2, S^3, \cdots, S^m \), \( C^2 \) is the centroid of \( m-1 \) points \( S^1, S^3, \cdots, S^m \), \( C^m \) is the centroid of \( m-1 \) points \( S^1, S^2, \cdots, S^{m-1} \). We have

\[
ed(C, O) \leq \frac{1}{m} \sum_{i=1}^{m} ed(C^i, O) \quad (9)
\]

Proof: According to the definition of \( C^1, C^2, \cdots, C^m \), \( C \) is the centroid of these \( m \) points. The theorem can be proven by applying Theorem 1. □
Theorem 4 can be used repeatedly to prove more general situations in which a subset includes \( m - 2, m - 3, \ldots, \) or 2 points. This demonstrates that the number of component results has a positive effect on ensemble performance.

**Example 5.** In a space \( X, \{S^1, S^2, S^3, S^4\} \) and \( O \) are known points. There are four different combinations of three points and six combinations of two points. We use \( C^{1234} \) to represent the centroid of all 4 points. Similarly, \( C^{23} \) represents the centroid of \( S^2 \) and \( S^3 \), and so on. Applying Theorem 4 repeatedly, we have

\[
ed(C^{1234}, O) \leq \frac{1}{4}[\ned(C^{123}, O) + \ned(C^{124}, O) + \ned(C^{134}, O) + \ned(C^{234}, O)]
\leq \frac{1}{6}[\ned(C^{12}, O) + \ned(C^{13}, O) + \ned(C^{14}, O) + \ned(C^{23}, O) + \ned(C^{24}, O) + \ned(C^{34}, O)]
\leq \frac{1}{4}[\ned(S^1, O) + \ned(S^2, O) + \ned(S^3, O) + \ned(S^4, O)]
\]

Although Theorem 4 shows that the number of component classifiers has a positive effect on ensemble performance, it is not clear how significant the effect is. Next let us look at this matter quantitatively. In order to focus on the number of component classifiers, we make a few simplifying assumptions. Suppose that \( S^1, S^2, \ldots, S^m \) and \( O \) are known points in space \( X \). \( \ned(S^i, O) = c_p \) for any \( 1 \leq i \leq m \), \( \ned(S^i, S^j) = c_d \) for any \( 1 \leq i \leq m, 1 \leq j \leq m, i \neq j \), and \( c_d = \theta * c_p \). According to Theorem 2 and the above assumptions, we have

\[
ed(C, O)^2 = \frac{1}{m^2}[m \sum_{i=1}^{m} \ned(S^i, O)^2 - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \ned(S^i, S^j)^2]
= c_p^2 - \frac{m - 1}{2m} c_d^2
= (1 - \frac{m - 1}{2m} \theta^2) c_p^2
\]

Therefore,

\[
ed(C, O) = \sqrt{(1 - \frac{m - 1}{2m} \theta^2) * c_p} \quad (10)
\]

By definition, \( \ned(C, O) \) cannot be negative and \( (1 - \frac{m - 1}{2m} \theta^2) \geq 0 \) should hold. Therefore, for a given \( m \), \( \theta \) must have a maximal limit. If \( m = 2 \) and \( \theta = 2 \), then \( \ned(C, O) = 0 \). \( \theta = 2 \) must be the maximal value in this case. Likewise, when \( m = 3 \), the maximal value for \( \theta \) is \( \sqrt{3} \).

Fig. 1 shows the values of \( \ned(C, O) \) in \( c_p \) unit for \( \theta = 0.25, 0.5, 0.75, 1 \) and \( m = 2, 3, \ldots, 100 \). From Fig. 1 we can see that in all four cases, \( \ned(C, O) \) decreases with \( m \). However, as \( m \) becomes larger and larger, the rate of decrease becomes smaller and smaller. When \( m \) tends to infinity, \( \ned(C, O) \) approaches
Fusion performance in $c_p$ units. They are 0.984, 0.935, 0.848, and 0.707 when $\theta$ equals to 0.25, 0.5, 0.75, and 1, respectively. However, adding more component results does not help much if there is already a large number. For example, when $\theta = 1.0$ and $m = 27$, $ed(C, O)$ is 0.720 $c_p$ units. It is close to the limit of 0.707 $c_p$ units. It suggests that fusing 30 or more component results may not be very useful for further improving ensemble performance. This has been observed in some empirical studies before, such as in Oshiro et al. (2012), and others.

Theorem 1 tells us that the ensemble performance is at least as good as the average performance of all the component classifiers involved. This may not be positive enough for many applications of the technique. Theorem 3 indicates that it may take too much time to choose a subset from a large group of candidates for good ensemble performance. This problem may be solved in other ways. In particular, if we want the ensemble performance to be better than the best component classifier, more favourable conditions are required for those component classifiers. It means that we need to apply some restrictions to all the component classifiers involved. Theorem 5 can be useful for this.

**Theorem 5.** In a space $X$, $S = \{S^1, S^2, \cdots, S^m\}$ and $O$ are known points. At least one of the points in $S$ is different from the others. $C$ is the centroid of $S^1, S^2, \cdots, S^m$. If $ed(S^1, O) = ed(S^2, O) = \cdots = ed(S^m, O)$, then $ed(C, O) < ed(S^1, O)$ must hold.

Proof: According to Theorem 2, we have
Therefore, we obtain \( ed(C, O) < ed(S^1, O) \).

Theorem 5 tells us if all the component classifiers are equally effective, then majority voting is able to do a better job than Theorem 1’s guarantee. In practice, this has been implemented in various situations. For example, if using bagging with random forest or neural networks (Oshiro et al., 2012; Yang et al., 2013), then we can generate a large number of almost equally-effective component classifiers. Good performance is achievable by fusing such classifiers. See Section 4 for more discussions.

3.2 Weighted majority voting

Weighted majority voting is a generalization of majority voting. It is more flexible than its counterpart because different weighting schemes can be defined. It might be believed that both are similar, and this is indeed true for the cases when weights across component classifiers are largely similar. However, for the weighted majority voting, we have little interest in the universe of all possible weighting schemes, but are rather focused on the optimum weighting scheme. We delve into the following two questions especially.

1. How to find the optimum weights for a group of component classifiers?
2. What are the properties of weighted majority voting with the optimum weights?

Let us begin with the first question. In a space \( X \), \( S = \{S^1, S^2, \ldots, S^m\} \) and \( O \) are known points. Let \( F \) be the fused point for linear combination of \( m \) points in \( S \) with weighting \( w^1, w^2, \ldots, w^m \).

\[
ed(F, O)^2 = \sum_{i=1}^{n} \sum_{j=1}^{p} \left( \sum_{k=1}^{m} (w^k * s^k_{ij}) - o_{ij} \right)^2
\]  

Our goal is to minimize \( ed(F, O)^2 \). Assuming \( f(w^1, w^2, \ldots, w^m) = ed(F, O)^2 \), we have

\[
\frac{\partial f}{\partial w^q} = \sum_{i=1}^{n} \sum_{j=1}^{p} 2 \left( \sum_{k=1}^{m} (w^k * s^k_{ij}) - o_{ij} \right) s^q_{ij} 
\]

Let \( \frac{\partial f}{\partial w^q} = 0 \) for \( q=1,2, \ldots, m \), then

\[
\sum_{k=1}^{m} w^k \sum_{i=1}^{n} \sum_{j=1}^{p} (s^k_{ij} s^q_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{p} o_{ij} s^q_{ij}
\]
Let \( a_{qk} = \sum_{i=1}^{n} \sum_{j=1}^{p} s_{ij}^{q} \) for \( 1 \leq q \leq m \) and \( 1 \leq k \leq m \), and \( b_q = \sum_{i=1}^{n} \sum_{j=1}^{p} o_{ij} \delta_{ij} \) for \( 1 \leq q \leq m \). Thus we obtain the following \( m \) linear equations with \( m \) variables \( w^1, w^2, \cdots, w^m \):

\[
\begin{pmatrix}
a_{11}w^1 + a_{12}w^2 + \cdots + a_{1m}w^m = b_1 \\
a_{21}w^1 + a_{22}w^2 + \cdots + a_{2m}w^m = b_2 \\
\vdots \\
a_{m1}w^1 + a_{m2}w^2 + \cdots + a_{mm}w^m = b_m
\end{pmatrix}
\] (12)

The optimum weights can be calculated by finding the solution to these \( m \) linear equations. Note that minimizing \( ed(F, O)^2 \) and minimizing \( ed(F, O) \) is equivalent for us to find the optimum weights because \( ed(F, O) \) can not be negative.

**Theorem 6.** In a \( n \times p \) dimensional space \( X, S=\{S^1, S^2, \cdots, S^m\} \) and \( O \) are known points. If every point in \( S \) is linearly independent from the others, then the above process and Equation 12 can find the unique solution to the problem.

Proof: The independency of each point in \( S \) indicates that \( m \leq n \times p \) holds. For the same reason, any point that can be represented linearly by these \( m \) points has a unique representation. We may write \( ed(F, O) \) as \( f(w^1, w^2, \cdots, w^m) \), which is a continuous function. In the whole space, there is only one minima and no other saddle points or maxima. The point at which all partial derivatives of the function to all variables equal to zero must be the minimum point. The \( m \) equations set up in Equation 12 are able to find the point with a unique representation of weights.

**Intuitively,** in a \( n \times p \) dimensional space \( X, m \) points in \( S=\{S^1, S^2, \cdots, S^m\} \) comprise a subspace \( X' \) in \( X \). For any point \( O \), there exists one and only one point in \( X' \) that has the shortest distance to \( O \). This point can be linearly represented by \( S^1, S^2, \cdots, S^m \).

Theorem 6 can be explained as follows. For a (training) dataset with \( n \) instances, \( p \) classes, and \( m \) classifiers, each of the classifiers gives a score for each instance and each class. For each instance, we also have real labels relating to all the classes. Then we are able to find a group of weights \( w^1, w^2, \cdots, w^m \) for \( S^1, S^2, \cdots, S^m \) to achieve the best ensemble performance by weighted majority voting.

The following Theorems 7 and 8 answer the second question.

**Theorem 7.** In a \( n \times p \) dimensional space \( X, S = \{S_1, S_2, \cdots, S_m\} \) is a group of points, \( m < n \times p \) and all \( m \) points are independent of each other. \( O \) is an ideal point. For a given number \( m' \) \( (2 \leq m' < m) \), the problem is to find a subset of \( m' \) points from \( S \) so as to let the fused point of the chosen \( m' \) points by weighed majority voting to the ideal point \( O \) as close as possible. This problem is NP-hard.

Proof. As shown in Theorem 6, for each group of \( m' \) points, their optimum weights can be calculated by least-squares with a time complexity of \( O(m'^2 \times n \times p) \) [Boyd and Vandenberghe 2004]. To choose \( m' \) points from a total number of \( m \) points, there are
combinations. When $m'$ approaches $m/2$, the number of combinations grows exponentially with $m$. Note that because each group of points is different from the other groups, their weights need to be calculated by least-squares separately. Therefore, the problem is an NP-hard problem.

**Theorem 8.** In a $n \times p$ dimensional space $X$, $S^1 = \{S_1, S_2, \ldots, S^m\}$, $S^2 = \{S_1, S_2, \ldots, S^m, S^{m+1}\}$, and $O$ is an ideal point. If the optimum weights are used for both $S^1$ and $S^2$, then the performance of Group $S^2$ is at least as effective as that of Group $S^1$.

**Proof.** Assume that $w_1, w_2, \ldots, w_m$ are optimum weights for $S_1, S_2, \ldots, S^m$ of $S^1$ to obtain the best performance. For $S^2$, if using the same weights $w_1, w_2, \ldots, w_m$ for $S_1, S_2, \ldots, S^m$, and 0 for $S^{m+1}$, then the ensemble performance of $S^2$ will be the same as that of $S^1$. Note that the above weighting scheme is by no means the best for $S^2$ and it is also possible to find more profitable weights for $S^2$.

Theorem 8 can be explained as follows: for a given dataset, consider two groups of classifiers. Group 1 has $n$ classifiers: $c_{f1}, c_{f2}, \ldots, c_{fm}$, and Group 2 has $m+1$ classifiers, $c_{f1}, c_{f2}, \ldots, c_{fm}, c_{f_{m+1}}$. $m$ classifiers in both groups are the same. If using weighted majority voting with the optimum weights, then the ensemble performance of Group 2 is at least as effective as that of Group 1.

**Corollary 8.1.** In a $n \times p$ dimensional space $X$, assume that weighted majority voting is applied with the optimum weights. When more and more points are added, the ensemble performance is monotonically non-decreasing.

**Proof.** It can be proven by applying Theorem 8 repeatedly.

Intuitively, when more and more points are added, the subspace becomes bigger and bigger. During this process, it is possible to find new points that has the shortest distance to the ideal point.

Corollary 8.1 can be explained in this way. Assume for a given dataset, an ideal ensemble is implemented by weighted majority voting with the optimum weights. When more and more classifiers are added into such an ensemble, its performance is non-decreasing monotonically.

Theorem 2 tells us about how individual classifier performance ($ed(S^i, O)$) and dissimilarity between classifiers ($ed(S^i, S^j)$) affect ensemble performance with the majority voting scheme. We may regard weighted majority voting as a variation of majority voting. Before applying Theorem 2, weighted majority voting changes the positions of all component results’ position by a linear weighting scheme, thus Equation 5 becomes

\[
ed(C, O) = \frac{1}{m} \sqrt{m \sum_{i=1}^{m} ed(w^i \cdot S^i, O)^2 - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} ed(w^i \cdot S^i, w^j \cdot S^j)^2}
\]

where $w^i \cdot S^i = \sum_{j=1}^{n} \sum_{k=1}^{p} w^i \cdot s^i_{jk}$. After that, both can be treated in the same way.
4 Discussion

In Section 3 we have set up a geometric framework and presented the properties of majority voting and weighted majority voting. Now we are in a good position to compare the two different levels of geometric frameworks and answer the questions raised in the first section of this paper.

4.1 Dataset-level vs. instance-level frameworks

A major objective of the ensemble problem is to try to provide a solution for all the instances in a dataset. A framework at different levels has certain impact on the way we can deal with the problem. For an instance-level framework, we need an approach to expand it to cover all the instances in the whole dataset, while the dataset-level framework does not need it.

In the dataset-level framework, all the instances in the whole dataset are concatenated to form a super-instance. Thus, all the properties stand in the instance-level framework also stand in the dataset-level framework.

However, a few differences need to be noted. One is the dimensionality of the geometric space involved. For a classification problem with $p$ classes and a dataset with $n$ instances, the dimensionality of the instance-level geometric space is $p$, while that of the dataset-level geometric space is $p \times n$.

How to calculate optimal weights for weighted majority voting is another place where we may have different solutions. The solution given in Section 3 of this paper is to minimize the Euclidean distance between the linear combination of all the component points and the ideal point (refer to Equation 11). Recall that all the instances in the dataset is transformed to a single super-instance. However, for the instance-level framework, we still need to consider multiple instances together. One possible way is to minimize the sum of the distance over all instances, or

$$
\sum_{i=1}^{n} ed(F, O) = \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{p} \sum_{k=1}^{m} (w^k \ast s^k_{ij} - o_{ij})^2}
$$

(14)

It is tricky to optimise Equation 14 directly. To simplify, we may optimise $\sum_{i=1}^{n} ed(F, O)^2$ instead (Wu and Crestani, 2015; Bonab and Carl, 2018). In this way, it is the same as Equation 11. It demonstrates that there are connections between the two levels of frameworks. For the instance-level framework, optimising $\sum_{i=1}^{n} ed(F, O)^2$ approximates optimising $\sum_{i=1}^{n} ed(F, O)$. On the other hand, for the dataset-level framework, optimising $ed(F, O)^2$ is the same as optimising $ed(F, O)$. It means that the weights obtained are optimum.

4.2 The size of ensemble

As discussed in Section 3, we proved that the number of component classifiers has positive impact on ensemble performance for both majority voting and weighted majority voting. However, this contradicts the assertion in
Bonab and Can (2019): for a multi-classification problem with \( k \) classes, \( k \) is the ideal number of base classifiers to constitute an optimum ensemble by weighted majority voting. Let us analyse this further.

**Example 6.** Consider a classification problem with two classes and three base classifiers. One instance is shown in the figure below. Weighted majority voting is used for combination.

In the figure above, it shows a two-dimensional space with three points \( S^1 \), \( S^2 \), \( S^3 \) to be a combination. The ideal point is \( O \). We can see that combining three points can lead to the optimal results of zero distance than fusing any two. Therefore, in this example the assertion in Bonab and Can (2019) even does not hold at the instance level. However, as shown in this example, for a \( n \) dimensional space, \( n + 1 \) independent points are enough. □

We may add some restrictions to the points involved. For example, for a binary classification problem, we let all the points to be on the line segment of \([0,1] \) and \([1,0] \). The ideal point is either \([1,0] \) or \([0,1] \). In this way, a maximum of two points are needed for the optimal fusion results. In the figure below, any two of the three points \( S^1 \), \( S^2 \), and \( S^3 \) are competent for this task.

Anyhow, the assertion at the instance level is not very useful. To consider the problem in a more realistic way, we need to look at it at the dataset level. A dataset usually comprises at least a good number of instances. If we consider three instances with a binary classification problem, then the dimensionality of the dataset-level geometric framework is up to \( 2^3 = 6 \), and not \( 2 \) any more. If the dataset has more instances, then we may include even more independent classifiers. With an increased number of base classifiers, we will likely get better ensemble performance (Corollary 8.1).

On the other hand, we can obtain the same conclusion as Corollary 8.1 even under the instance-level framework. Assume that the whole dataset has \( n \) instances. \( F_i \) and \( O_i \) are the fused point and the ideal point for instance \( t_i \), respectively. The optimal weighting for \( m \) base classifiers are \( w^1, w^2, \ldots, w^m \). Then we have
Now one more base classifier is added. We can set a new weighting scheme as \( w_1^{\text{new}} = w_1, w_2^{\text{new}} = w_2, \ldots, w_m^{\text{new}} = w_m, w_{m+1}^{\text{new}} = 0. \)

\[
\sum_{i=1}^{n} ed(F_i, O_i) = \sum_{i=1}^{n} \left[ \sum_{j=1}^{p} \left( \sum_{k=1}^{m} (w_k \ast s_{ij}^k) - o_{ij} \right)^2 \right]
\]

(15)

Then \( \sum_{i=1}^{n} ed(F_i, O_i) = \sum_{i=1}^{n} ed_{\text{new}}(F_i, O_i) \), \( w_1, w_2, \ldots, w_m \) is the optimal weighting scheme for \( m \) base classifiers, while \( w_1^{\text{new}}, w_2^{\text{new}}, \ldots, w_m^{\text{new}}, w_{m+1}^{\text{new}} \) may not be optimal for \( m+1 \) base classifiers. Therefore, if the optimal weighting scheme is used, then fusing \( m+1 \) base classifiers can achieve at least the same performance as fusing \( m \) base classifiers. This is exactly what Corollary 8.1 tells us.

4.3 Answer to some questions

In Section 1, we listed some outstanding questions. Now let us discuss them one by one.

Question 1: What is the difference between majority voting and weighted majority voting?

In a sense, both weighting schemes can potentially enhance ensemble performance. However, there are certain aspects, including their abilities, to consider. Weighted majority voting can be better than the best component classifier if optimum weights are used, while majority voting can be better than the average of all component classifiers. Majority voting is a “mild” method because all the component results are treated equally and the centroid is the solution, while weighted majority voting is an “extreme” method because it does not treat all component results equally and it takes the most effective solution from all possible ones.

Question 2: When should we use majority voting rather than weighted majority voting, or vice versa?

A general answer is: in cases majority voting does not work well, then weighted majority voting should be used. Now a further question is: when is majority voting a good method? Performance of all component results, dissimilarity of all pairs of component results, number of component results have positive impact on ensemble performance. A judicious decision should consider these factors thoroughly. More specifically, Theorem 2 answers this question quantitatively. One easy noticeable situation is that when all the component classifiers are of equal or very close performance, then majority voting may be able to achieve better ensemble performance than the best component classifier (Theorem 5).
Question 3: There are a lot of weighting assignment methods for weighted majority voting, which one is the best?

The least squares is the best weighting assignment method for the measure of Euclidean distance. Compared with many others, it is efficient and effective at the same time. Almost all other weighting assignment methods are either heuristic or optimisation methods. For the former, its effectiveness is not guaranteed; for the latter, it is timing-consuming.

Question 4: How does the number of component classifiers affect ensemble performance?

The number of component classifiers has a positive effect on ensemble performance. The situation is straightforward for weighted majority voting. When more and more component results are added to an ensemble, its performance becomes better and better. However, the situation for majority voting is more complicated. Adding more component classifiers into an ensemble cannot always improve performance. When the number is small, then its impact on ensemble performance is large. When the number increases, its impact becomes smaller.

Question 5: How does each component classifier affect ensemble performance?

It is related to the first question. For majority voting, each contributes equally; for weighted majority voting, each contributes differently in order to get the optimal results for the whole data set. If a very good component classifier is added, then weighted majority voting can take advantage of it. On the other hand, for majority voting, if many component classifiers are poor, then a few good ones will not be able to improve performance very much.

Question 6: How does performance of component classifiers affect ensemble performance?

Performance of component classifiers is the most important aspect that affects ensemble performance. For majority voting, a high-performance point is able to move the centroid of the group closer to the ideal point. For weighted majority voting, a high-performance point very likely enables the subspace to expand with some points closer to the ideal point.

Question 7: How does diversity of component classifiers affect ensemble performance?

For diversity, there are many different types of definitions before. In this paper, we define it as the dissimilarity distribution of all pairs of component results. Apart from performance, diversity is another aspect that impacts ensemble performance significantly. For majority voting, a comparative investigation about it and performance has been done in Subsection 3.1. Based on a simplified situation, the importance ratio between diversity and performance is calculated to be in the range of (0.25,0.5], varying with the number of component classifiers. For weighted majority voting, high diversity among component results will make the subspace bigger, thus it is more likely to find closer points to the ideal point in such a space.

One final comment about the framework is: all the theorems in the geometric framework hold when the Euclidean distance is used for measuring performance. When other metrics are used, the conclusions we obtain may hold for many of the
instances, but not every single instance. However, there is strong correlations between any other meaningful performance metrics and the Euclidean distance. If enough instances are observed, we may expect consistent conclusions.

5 Empirical investigation

In this section we are going to investigate how theoretical conclusions presented in Section 3 can be confirmed for practical use. Within the geometric framework, all the theorems hold perfectly when Euclidean distance is used as performance metric on the same dataset. We would like to see how they behave when the conditions are partially satisfied. Specifically, two points are considered:

- Usually classification accuracy or some other metrics, rather than Euclidean distance, is used for performance evaluation, although Euclidean distance and all those commonly used metrics are strongly correlated;
- When the component classifiers and ensemble models are trained using some training data, they need to be used and tested in the test dataset, which may be somewhat different from the training dataset.

For the above purpose, we carried out the empirical investigation by using the WEKA machine learning suite\(^1\) and 20 datasets downloaded from the UCI Machine Learning Repository\(^2\). The main statistics of the these 20 datasets are listed in Table 2.

We set up ensembles (random forests) by using up to 30 random trees as base classifiers. Both classical random forest and weighted random forest are tested. The weighting scheme for weighted random forest is to optimise the Euclidean distance. For a given dataset, all the instances are divided into two disjoint partitions after stratification, in which 80% is taken as the training partition and the remaining 20% as the test partition. During the training process, sampling with replacement is used to extract data for training classifiers. 30 component random trees are generated with \(K\) randomly chosen features, where \(K\) is the square root of the total number of features in that dataset. First we use all 30 trees for an ensemble. Then one of them is chosen randomly and removed, and the remaining 29 are ensembled. The above process is repeated until there are two, which is the minimal number for an ensemble.

Both random forest and weighted random forest are tested. Apart from these two combination schemes, the best component classifier and average of all component classifiers are also calculated for comparison. Two metrics including Euclidean distance and classification accuracy are used for performance evaluation. Tables 3 and 4 show the results for the training partition and test partition, respectively. Each figure in these two tables is the average of 30 ensembles.

For the training partition, we can see from Table 3 that the weighted random forest is always better than the classical random forest when Euclidean distance

\(^1\)http://www.cs.waikato.ac.nz/ml/weka/
\(^2\)https://archive.ics.uci.edu/ml/index.php
Table 2: Statistics of two datasets used in the study

| Data set               | No. of instances | No. of attributes | No. of classes |
|------------------------|------------------|-------------------|----------------|
| Anneal                 | 898              | 39                | 6              |
| Credit-g               | 1000             | 21                | 2              |
| Data-cortex-nuclear    | 1080             | 81                | 8              |
| Germancredit           | 1000             | 21                | 2              |
| Hypothyroid            | 3772             | 30                | 4              |
| Kr-vs-kp               | 3196             | 37                | 2              |
| Mfeat-factors          | 2000             | 217               | 10             |
| Mfeat-fourier          | 2000             | 77                | 10             |
| Mfeat-karhunen         | 2000             | 65                | 10             |
| Mfeat-zernike          | 2000             | 48                | 10             |
| Optdigits              | 5620             | 65                | 10             |
| Pendigits              | 10992            | 17                | 10             |
| Secom                  | 1567             | 591               | 2              |
| Segment                | 2310             | 20                | 7              |
| Semeion                | 1593             | 257               | 10             |
| Sick                   | 3772             | 30                | 2              |
| Soybean                | 683              | 36                | 19             |
| Spambase               | 4601             | 58                | 2              |
| Splice                 | 3190             | 61                | 3              |
| Vowel                  | 990              | 14                | 11             |
Table 3: Classification performance of classical random forest and its weighted counterpart in the training partition; a pair of figures in bold indicates that random forest performs better than its weighted counterpart for the given measure, and a pair of underlined figures indicates there is a tie between the two methods.

| Dataset             | Random forest    | Weighted random forest |
|---------------------|------------------|------------------------|
|                     | Accuracy(%)      | Distance               | Accuracy(%) | Distance               |
| Anneal              | 99.93            | 1.74                   | 99.94       | 1.42                   |
| Credit-g            | 98.74            | 7.03                   | 98.74       | 6.63                   |
| Data-cortex-nuclear | 99.72            | 3.90                   | 99.73       | 3.35                   |
| Germancredit        | 98.72            | 7.05                   | 98.73       | 6.65                   |
| Hypothyroid         | **99.91**        | 3.64                   | **99.88**   | 2.88                   |
| Kr-vs-kp            | 99.86            | 4.25                   | 99.87       | 3.89                   |
| Mfeat-factors       | 99.58            | 6.39                   | 99.60       | 5.97                   |
| Mfeat-fourier       | 98.88            | 9.83                   | 98.96       | 8.96                   |
| Mfeat-karhunen      | 99.23            | 8.67                   | 99.30       | 7.75                   |
| Mfeat-zernike       | 98.38            | 10.73                  | 98.50       | 10.00                  |
| Optdigits           | 99.66            | 9.54                   | 99.68       | 9.01                   |
| Pendigits           | 99.87            | 7.90                   | 99.88       | 7.72                   |
| Secom               | 99.13            | 5.46                   | 99.16       | 5.30                   |
| Segment             | 99.84            | 3.98                   | 99.86       | 3.75                   |
| Semeion             | 99.18            | 7.37                   | 99.19       | 7.37                   |
| Sick                | **99.89**        | 4.15                   | **99.89**   | 3.87                   |
| Soybean             | 99.47            | 4.47                   | 99.52       | 4.25                   |
| Spambase            | 99.60            | 7.90                   | 99.61       | 7.66                   |
| Splice              | 99.39            | 10.17                  | 99.41       | 9.14                   |
| Vowel               | 99.39            | 5.51                   | 99.42       | 4.95                   |
Table 4: Classification performance of classical random forest and its weighted counterpart in the test partition; a pair of figures in bold indicates that random forest performs better than its weighted counterpart for the given measure, and a pair of underlined figures indicates there is a tie between the two methods.

| Dataset            | Random forest | Weighted random forest |
|--------------------|---------------|------------------------|
|                    | Accuracy(%)   | Distance               | Accuracy(%)   | Distance               |
| Anneal             | 98.96         | 2.15                   | 98.95         | 2.08                   |
| Credit-g           | 72.39         | **8.56**               | 72.73         | **8.63**               |
| Data-cortex-nuclear| 98.95         | 3.72                   | 98.97         | 3.29                   |
| Germancredit       | 73.19         | 8.49                   | 73.36         | 8.56                   |
| Hypothyroid        | 98.92         | 3.90                   | 99.04         | 3.47                   |
| Kr-vs-kp           | 98.74         | 4.34                   | 98.90         | 4.10                   |
| Mfeat-factors      | 95.73         | 6.38                   | 95.84         | 6.18                   |
| Mfeat-foursier     | **80.88**     | 11.27                  | **80.66**     | 11.10                  |
| Mfeat-karhunen     | 92.74         | 8.88                   | 92.78         | 8.43                   |
| Mfeat-zernike      | 75.68         | **11.92**              | 75.71         | **11.94**              |
| Optdigits          | 96.03         | 10.66                  | 96.09         | 10.37                  |
| Pendigits          | 98.68         | 8.16                   | 98.68         | 8.09                   |
| Secom              | 92.82         | **6.41**               | 93.02         | **6.45**               |
| Segment            | **97.33**     | 4.60                   | **97.29**     | 4.57                   |
| Semeion            | 85.06         | 9.47                   | 85.76         | 9.17                   |
| Sick               | 98.63         | 4.22                   | 98.65         | 4.04                   |
| Soybean            | 93.05         | 4.47                   | 93.34         | 4.25                   |
| Spambase           | 94.27         | **8.96**               | 94.30         | **9.00**               |
| Splice             | 91.57         | 11.36                  | 92.02         | 10.70                  |
| Vowel              | 94.59         | 5.67                   | 94.60         | 5.42                   |

Table 5: Statistics of paired T test and correlation test for the results of classical random forest and weighted random forest

| Partition | Metric  | Correlations | Compare means |
|-----------|---------|--------------|---------------|
|           | Correlation | Significance | Mean  | Significance |
| Training  | Accuracy  | .998         | .000  | -.025        | .003   |
| Training  | Distance  | .995         | .000  | .458         | .000   |
| Testing   | Accuracy  | 1.000        | .000  | -.124        | .012   |
| Testing   | Distance  | .998         | .000  | .188         | .001   |
Figure 2: Performance comparison of two combination schemes with different number of base classifiers for dataset Soybean ("Best" denotes the best base classifier and "Average" denotes the average of all base classifiers involved.)
is used as the metric for evaluation. The only exception is Semeion, in which there is a tie. This is trivial because Euclidean distance is the optimisation goal. When accuracy is used as the metric, the classical random forest is better than the weighted counterpart on one dataset and there are ties on two datasets, while the weighted random forest is better than the classical on other 17 datasets. As Table 4 shows that the situation is very similar for the test partition, although the classical random forest is a little better than it is in the training partition. Considering all the datasets, it is better than the weighted on three and four datasets, for accuracy and Euclidean distance, respectively. Besides, there is a tie on one dataset when measured by accuracy. However, for the majority datasets, the weighted random forest performs better than the classical.

Although the weighted random forest performs better than the classical one on both training and test partitions, it is noticeable that the difference between them is very small. This is because all generated base classifiers are very close in performance and it is a very good condition for the classical random forest to do good work. To further compare the difference between them, we carried out paired $t$ test plus correlation test for them. Table 5 shows the result. From Table 5 we can see that the difference between classical random forest and weighted random forest is significant in both partitions, either accuracy or Euclidean distance is used as the metric. Among them, the lowest significance level, .012, is still very high and happens in the test partition with accuracy as the metric. On the other hand, in all four cases, the correlation between two groups of results are always very strong. All four correlation coefficients are very close to 1.

Finally, we take a look at the effect of number of base classifiers on ensemble performance. Figure 2 shows the performance of different ensemble methods with the Soybean dataset. Performances of random forest, weighted random forest, the best base classifier, and the average of all base classifiers are presented. Note that the curves for all other datasets are very similar in shapes. It is clear that performance increases for both random forest and weighted random forest when more base classifiers are involved. From Figure 2, we have a few other observations. First, for the metric of accuracy, performance of both random forest and weighted random forest increases very rapidly with the number of base classifiers when the number is small. When the number is bigger, it slows down very quickly. When the number is 10 or more, the increase rate becomes very small. On the other hand, such a phenomenon is not so prominent for the Euclidean distance. Second, the difference between random forest and weighted random forest is more noticeable for the Euclidean distance than for accuracy. It seems that only a small percentage of the deduction in distance has been transferred to the increase in accuracy.

From the above experimental results, it demonstrates that the Euclidean distance is a very useful measure for performance evaluation and especially for optimisation. Therefore, we conclude that in general, the theorems we obtain from the geometric framework still make sense even when other metrics such as accuracy is used for performance evaluation.
6 Conclusions

In this paper, we have presented a dataset-level geometric framework for ensemble classifiers. The most important advantage of the framework is it makes ensemble learning a deterministic problem. Both performance and dissimilarity can be measured by the same metric - the Euclidean distance, thus it is a good platform for us to understand the fundamental properties of ensembles clearly and investigate many issues in ensemble classifiers, such as the impact of multiple aspects on ensemble performance, predicting ensemble performance, selecting a small number of base classifiers to get efficient and effective ensembles, etc. Otherwise, it is very challenging to grasp even an incomplete picture. This is why up to now some of the properties of majority voting and weighted majority voting have not been fully understood.

Compared with the instance-level framework in [Wu and Crestani (2015); Bonab and Can (2019)], the dataset-level framework presented in this paper is a step forward. It maps the ensemble classifier problem for a whole dataset into one multi-dimensional space, thus it is more convenient for us to investigate the properties of ensembles. Otherwise, we have to deal with multiple spaces at the same time, each for one instance. To find out the collective properties in those spaces is more complicated. Based on the dataset-level framework, we have deduced some useful theorems which had not been found before.

An empirical investigation has also been conducted to see how those theorems in the geometric framework hold when accuracy rather than the Euclidean distance is used for performance evaluation. The experimental results show that the theorems are still meaningful for other metrics.

In this paper, the setting for the proposed framework is traditionally with a batch of training data. In recent years, data stream classification has attracted some attention [Gomes et al. (2017)]. How to adapt the geometric framework for this is worth further research. Especially, incorporating dynamic updates is a key point. Another research topic is multi-model data fusion [Gao et al. (2020)]. Again how to adapt the framework to support multimodal data fusion is an interesting research issue. One possible solution is to use a separate framework for each mode and then to combine them. These research issues remain to be our future work.

References

Md Nasim Adnan and Md Zahidul Islam. Optimizing the number of trees in a decision forest to discover a subforest with high ensemble accuracy using a genetic algorithm. *Knowl. Based Syst.*, 110:86–97, 2016.

Saba Bashir, Usman Qamar, and Farhan Hassan Khan. Heterogeneous classifiers fusion for dynamic breast cancer diagnosis using weighted vote based ensemble. *Quality & Quantity*, 49:2061–2076, 2015.
Manju Bhardwaj, Vasudha Bhatnagar, and Kapil Sharma. Cost-effectiveness of classification ensembles. *Pattern Recognit.*, 57:84–96, 2016.

Yaxin Bi. The impact of diversity on the accuracy of evidential classifier ensembles. *Int. J. Approx. Reason.*, 53(4):584–607, 2012.

Hamed R. Bonab and Fazli Can. GOOWE: geometrically optimum and online-weighted ensemble classifier for evolving data streams. *ACM Trans. Knowl. Discov. Data*, 12(2):25:1–25:33, 2018.

Hamed R. Bonab and Fazli Can. Less is more: A comprehensive framework for the number of components of ensemble classifiers. *IEEE Trans. Neural Networks Learn. Syst.*, 30(9):2735–2745, 2019.

Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

Jingjing Cao, Sam Kwong, Ran Wang, Xiaodong Li, Ke Li, and Xiangfei Kong. Class-specific soft voting based multiple extreme learning machines ensemble. *Neurocomputing*, 149:275–284, 2015.

Philip Derbeko, Ran El-Yaniv, and Ron Meir. Variance optimized bagging. In Tapio Elomaa, Heikki Mannila, and Hannu Toivonen, editors, *Machine Learning: ECML 2002*, 13th European Conference on Machine Learning, Helsinki, Finland, August 19-23, 2002, Proceedings, volume 2430 of *Lecture Notes in Computer Science*, pages 60–71. Springer, 2002.

Kaushala Dias and Terry Windeatt. Dynamic ensemble selection and instantaneous pruning for regression used in signal calibration. In Stefan Wermter, Cornelius Weber, Wlodzislaw Duch, Timo Honkela, Petia D. Koprinkova-Hristova, Sven Magg, Günther Palm, and Alessandro E. P. Villa, editors, *Artificial Neural Networks and Machine Learning - ICANN 2014 - 24th International Conference on Artificial Neural Networks, Hamburg, Germany, September 15-19, 2014. Proceedings*, volume 8681 of *Lecture Notes in Computer Science*, pages 475–482. Springer, 2014.

Xibin Dong, Zhiwen Yu, Wenming Cao, Yifan Shi, and Qianli Ma. A survey on ensemble learning. *Frontiers Comput. Sci.*, 14(2):241–258, 2020.

Qingyun Duan, Newsha K. Ajami, Xiaogang Gao, and Soroosh Sorooshian. Multi-model ensemble hydrologic prediction using bayesian model averaging. *Advances in Water Resources*, 30(5):1371–1386, 2007.

Giorgio Fumera, Fabio Roli, and Alessandra Serrau. A theoretical analysis of bagging as a linear combination of classifiers. *IEEE Trans. Pattern Anal. Mach. Intell.*, 30(7):1293–1299, 2008.

Jing Gao, Peng Li, Zhikui Chen, and Jianing Zhang. A survey on deep learning for multimodal data fusion. *Neural Computation*, 32(5):829–864, 2020.
Harris V. Georgiou, Michael E. Mavroforakis, and Sergios Theodoridis. A game-theoretic approach to weighted majority voting for combining SVM classifiers. In Stefanos D. Kollias, Andreas Stafyllopatis, Wlodzislaw Duch, and Erkki Oja, editors, Artificial Neural Networks - ICANN 2006, 16th International Conference, Athens, Greece, September 10-14, 2006. Proceedings, Part I, volume 4131 of Lecture Notes in Computer Science, pages 284–292. Springer, 2006.

Heitor Murilo Gomes, Jean Paul Barddal, Fabrício Enembreck, and Albert Bifet. A survey on ensemble learning for data stream classification. ACM Comput. Surv., 50(2):23:1–23:36, 2017.

Mohammad Nazmul Haque, Nasimul Noman, Regina Berretta, and Pablo Moscato. Optimising weights for heterogeneous ensemble of classifiers with differential evolution. In IEEE Congress on Evolutionary Computation, CEC 2016, Vancouver, BC, Canada, July 24-29, 2016, pages 233–240. IEEE, 2016.

Daniel Hernández-Lobato, Gonzalo Martínez-Muñoz, and Alberto Suárez. How large should ensembles of classifiers be? Pattern Recognit., 46(5):1323–1336, 2013.

Siddhartha Jain, Ge Liu, Jonas Mueller, and David Gifford. Maximizing overall diversity for improved uncertainty estimates in deep ensembles. In The Thirty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2020, The Thirty-Second Innovative Applications of Artificial Intelligence Conference, IAAI 2020, The Tenth AAAI Symposium on Educational Advances in Artificial Intelligence, EAAI 2020, New York, NY, USA, February 7-12, 2020, pages 4264–4271. AAAI Press, 2020.

Anna Jurek, Yaxin Bi, Shengli Wu, and Chris D. Nugent. A survey of commonly used ensemble-based classification techniques. Knowledge Eng. Review, 29(5):551–581, 2014.

Ludmila I. Kuncheva and Juan José Rodríguez Diez. A weighted voting framework for classifiers ensembles. Knowl. Inf. Syst., 38(2):259–275, 2014.

Ludmila I. Kuncheva and Christopher J. Whitaker. Measures of diversity in classifier ensembles and their relationship with the ensemble accuracy. Mach. Learn., 51(2):181–207, 2003.

Patrice Latinne, Olivier Debeir, and Christine Decaestecker. Limiting the number of trees in random forests. In Josef Kittler and Fabio Roli, editors, Multiple Classifier Systems, Second International Workshop, MCS 2001 Cambridge, UK, July 2-4, 2001, Proceedings, volume 2096 of Lecture Notes in Computer Science, pages 178–187. Springer, 2001.

Nan Liu, Jiujuan Cao, Zhiping Lin, Pin Pin Pek, Zhi Xiong Koh, and Marcus Eng Hock Ong. Evolutionary voting-based extreme learning machines. Mathematical Problems in Engineering, 2014, 2014.
Shasha Mao, Licheng Jiao, Lin Xiong, Shuiping Gou, Bo Chen, and Sai-Kit Yeung. Weighted classifier ensemble based on quadratic form. *Pattern Recognit.*, 48(5):1688–1706, 2015.

Minkowski. [http://mathworld.wolfram.com/minkowskisinequalities.html](http://mathworld.wolfram.com/minkowskisinequalities.html), 2020.

David W. Opitz and Jude W. Shavlik. Actively searching for an effective neural network ensemble. *Connect. Sci.*, 8(3):337–354, 1996.

Thais Mayumi Oshiro, Pedro Santoro Perez, and José Augusto Baranauskas. How many trees in a random forest? In Petra Perner, editor, *Machine Learning and Data Mining in Pattern Recognition - 8th International Conference, MLDM 2012, Berlin, Germany, July 13-20, 2012. Proceedings*, volume 7376 of *Lecture Notes in Computer Science*, pages 154–168. Springer, 2012.

Nikunj C. Oza and Kagan Tumer. Classifier ensembles: Select real-world applications. *Inf. Fusion*, 9(1):4–20, 2008.

Philipp Probst and Anne-Laure Boulesteix. To tune or not to tune the number of trees in random forest. *J. Mach. Learn. Res.*, 18:181:1–181:18, 2017.

Omer Sagi and Lior Rokach. Ensemble learning: A survey. *Wiley Interdiscip. Rev. Data Min. Knowl. Discov.*, 8(4), 2018.

Robert E. Schapire and Yoram Singer. Improved boosting algorithms using confidence-rated predictions. *Mach. Learn.*, 37(3):297–336, 1999.

E. Ke Tang, Ponnuthurai N. Suganthan, and Xin Yao. An analysis of diversity measures. *Mach. Learn.*, 65(1):247–271, 2006.

Rosa Maria Valdovinos and José Salvador Sánchez. Combining multiple classifiers with dynamic weighted voting. In Emilio Corchado, Xindong Wu, Erkki Oja, Álvaro Herrero, and Bruno Baruque, editors, *Hybrid Artificial Intelligence Systems, 4th International Conference, HAIS 2009, Salamanca, Spain, June 10-12, 2009. Proceedings*, volume 5572 of *Lecture Notes in Computer Science*, pages 510–516. Springer, 2009.

Ingrid Visentini, Lauro Snidaro, and Gian Luca Foresti. Diversity-aware classifier ensemble selection via f-score. *Inf. Fusion*, 28:24–43, 2016.

Yang Wang, Jin-Kao Hao, Fred W. Glover, and Zhipeng Lü. A tabu search based memetic algorithm for the maximum diversity problem. *Eng. Appl. Artif. Intell.*, 27:103–114, 2014.

Michal Wozniak. Classifier fusion based on weighted voting - analytical and experimental results. In Jeng-Shyang Pan, Ajith Abraham, and Chin-Chen Chang, editors, *Eighth International Conference on Intelligent Systems Design and Applications, ISDA 2008, 26-28 November 2008, Kaohsiung, Taiwan, 3 Volumes*, pages 687–692. IEEE Computer Society, 2008.
Shengli Wu and Fabio Crestani. A geometric framework for data fusion in information retrieval. *Information Systems*, 50:20–35, 2015.

Jin Xiao, Changzheng He, Xiaoyi Jiang, and Dunhu Liu. A dynamic classifier ensemble selection approach for noise data. *Inf. Sci.*, 180(18):3402–3421, 2010.

Jing Yang, Xiaoqin Zeng, Shuiming Zhong, and Shengli Wu. Effective neural network ensemble approach for improving generalization performance. *IEEE Trans. Neural Networks Learn. Syst.*, 24(6):878–887, 2013.

Hadjer Ykhlef and Djamel Bouchaffra. An efficient ensemble pruning approach based on simple coalitional games. *Inf. Fusion*, 34:28–42, 2017.

Li Zhang and Weida Zhou. Sparse ensembles using weighted combination methods based on linear programming. *Pattern Recognit.*, 44(1):97–106, 2011.

Wentao Zhang, Jiawei Jiang, Yingxia Shao, and Bin Cui. Efficient diversity-driven ensemble for deep neural networks. In *36th IEEE International Conference on Data Engineering, ICDE 2020, Dallas, TX, USA, April 20-24, 2020*, pages 73–84. IEEE, 2020.

Zhihua Zhou, Jianxin Wu, and Wei Tang. Ensembling neural networks: Many could be better than all. *Artificial Intelligence*, 137(1-2):239–263, 2002.

Xuhui Zhu, Zhiwei Ni, Liping Ni, Feifei Jin, Meiyong Cheng, and Jingming Li. Improved discrete artificial fish swarm algorithm combined with margin distance minimization for ensemble pruning. *Comput. Ind. Eng.*, 128:32–46, 2019.