Universal instability for wavelengths below the ion Larmor scale

Matt Landreman, Thomas M. Antonsen, Jr, and William Dorland
Institute for Research in Electronics and Applied Physics,
University of Maryland, College Park, MD, 20742, USA
(Dated: November 10, 2014)

We demonstrate that the universal mode driven by the density gradient in a plasma slab can be absolutely unstable even in the presence of reasonable magnetic shear. Previous studies from the 1970s that reached the opposite conclusion used an eigenmode equation limited to \( L_x \gg \rho_i \), where \( L_x \) is the scale length of the mode in the radial direction, and \( \rho_i \) is the ion Larmor radius. Here we instead use a gyrokinetic approach which does not have this same limitation. Instability is found for perpendicular wavenumbers \( k_y \) in the range \( 0.7 \lesssim k_y \rho_i \lesssim 100 \), and for magnetic shear \( L_s/L_n \geq 17 \), where \( L_s \) and \( L_n \) are the scale lengths of magnetic shear and density. Thus, the plasma drift wave in a sheared magnetic field may be unstable even with no temperature gradients, no trapped particles, and no magnetic curvature.

INTRODUCTION

Following a series of papers in 1978 [1–3], it has been widely accepted that drift waves in a plasma in a sheared magnetic field, in the absence of parallel current, temperature gradients, and magnetic curvature, are absolutely stable. In other words, the “universal mode” driven by the density gradient in slab geometry [4–6] becomes absolutely stable if any amount of magnetic shear is included. This conclusion is important for the edge of magnetic confinement plasma experiments, in which the slab-like drive from radial gradients (\( \sim 1/L_s \) for scale length \( L_s \)) greatly exceeds the curvature drive (\( \sim 1/R \) for major radius \( R \)), and in which the temperature gradient may tend to be relatively weaker than the density gradient since the former is more strongly equilibrated over ion orbits [6]. Knowing the stability of the plasma slab is also important as a basic problem of plasma physics, and for understanding the physical mechanisms underlying instabilities in more complicated configurations. However, in contrast to this previous work, here we demonstrate numerically that the universal mode is in fact unstable for a range of perpendicular wavenumbers \( k_y \) satisfying \( k_y \rho_i \gtrsim 0.7 \), where \( \rho_i \) is the ion gyroradius. We reach a different conclusion to the 1978 work because these earlier calculations relied on an eigenmode equation derived under the assumption \( \rho_i \ll L_x \), where \( L_x \) is the radial scale length in untwisted coordinates. In contrast, we use a gyrokinetic approach which does not share this limitation. Using this approach, we find unstable modes with \( \rho_i \gg L_x \), outside the domain of validity of the 1978 work. Our work also contradicts several later analytic studies of drift wave stability [7,8].

GYROKINETIC MODEL

We consider slab geometry with magnetic shear, so the magnetic field in Cartesian coordinates \((\hat{z}, \hat{y}, \hat{z})\) is \( B = [\hat{e}_z + (\hat{x}/L_s)\hat{e}_y]B \) for some constant \( B \) and shear length \( L_s \). A field-aligned coordinate system is introduced: \( x = \hat{x}, \ y = y - \hat{x} \hat{z}/L_s, \ z = \hat{z} \), so \( \mathbf{B} \cdot \nabla x = \mathbf{B} \cdot \nabla y = 0 \) and \( \mathbf{B} \cdot \nabla z = B \). In this geometry, we consider the linear electrostatic gyrokinetic equation [9–11], dropping toroidal effects, collisions, and temperature gradients. For perturbations varying in \( y \) as \( \exp(ik_y y) \) and independent of \( x \), the gyrokinetic equation is

\[
\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} = q_s \frac{f_{Ms}}{T_s} J_{0s} \frac{\partial \Phi}{\partial t} - \frac{i k_y f_{Ms}}{B L_{en}} J_{0s} \Phi
\]  

and the quasineutrality condition is

\[
\sum_s \left[ - \frac{q_s^2 n_s \Phi}{T_s} + q_s \int d^3v J_{0s} h_s \right] = 0. \tag{2}
\]

Here \( s \in \{i, e\} \) denotes species, \( \Phi(t, z) \) is the electrostatic potential, \( h_s(t, z, \mathcal{E}, \mu) \) is the anisotropic distribution function, \( \mathcal{E} = v^2/2, \mu = v_z^2/(2B) \), and \( f_{Ms} \) is the leading-order Maxwellian. The scale length of the equilibrium electron density \( n = n_e \) is \( L_n = -n/(dn/dx) \), \( q_e = e = -q_i \) is the proton charge, \( T_i \) is the species temperature, \( J_{0s}(z) = J_0(k_z v_{\perp}/\Omega_s) \), \( k_\perp(z) = k_y \sqrt{1 + (z/L_s)^2} \), \( \Omega_s = q_i B/m_s \), and \( J_0 \) is a Bessel function.

For modes with time dependence \( \propto \exp(-i\omega t) \), a single dispersion equation can be obtained by introducing the Fourier transformed potential \( \Phi(x) \) satisfying

\[
\Phi(z) = \int dx \hat{\Phi}(x) \exp \left( \frac{i k_y x z}{L_s} \right), \tag{3}
\]

with an analogous transform for \( h_s \). We use \( x = \hat{x} \) as the transform variable because \( \hat{\Phi}(x) \) represents the radial mode structure in the untwisted coordinates. Using [3], the \( \partial/\partial z \) operator in [1] becomes the algebraic factor \( k_\perp = i k_y x/L_s \), so [4] may be solved for \( h_s \). Using [2] and approximating \( J_{0e} \approx 1 \), we obtain the integral equation...
\[ 0 = \left( \frac{T_e}{T_i} + 1 \right) \Phi(x) + \frac{1 - \omega_s}{\omega} \left[ \frac{L_s}{k_y x} \right] v_e Z \left( \left[ \frac{L_s}{k_y x} \right] \omega \right) + \left( \frac{T_e}{T_i} + \frac{\omega_s}{\omega} \right) \left( \frac{k_y}{2\pi L_s} \right)^2 \int dx' \int dz' \int dz \int dz' \times \Gamma_0(z, z') \Phi'(z) \exp \left( i \frac{k_y}{L_s} [x' - x' - x + x'' z'] \right) \times \frac{L_s}{k_y x'} \left[ \frac{L_s}{k_y x'} \right] \omega \frac{v_i}{v_i} \frac{\omega}{\nu_i} \frac{\nu_i}{\nu_i} \right] (4) \]

where

\[ \Gamma_0(z, z') = I_0 \left( \frac{k_y \rho_i^2}{2} \sqrt{1 + \frac{z^2}{L_z^2}} \sqrt{1 + \frac{z'^2}{L_z^2}} \right) \times \exp \left( - \frac{k_y \rho_i^2}{4} \left[ 2 + z^2 + z'^2 \right] \right) \] (5)

Here, \( v_s = \sqrt{2T_s/m_s}, \rho_i = v_i/\Omega_i, I_0 \) is a modified Bessel function, \( Z \) is the plasma dispersion function, and \( \omega_s = k_y T_e/(eBL_i) \) is the drift frequency. The multiple Fourier transforms in (4) arise because the \( Z \) function arguments are naturally written in terms of \( k \parallel \propto x \), whereas the Bessel function arguments are naturally written in terms of \( z \).

The integral equation (4) is manifestly different from the differential eigenmode equations solved in Refs. [1–3]. Then as eigenmodes of (4) are equivalent to eigenmodes of the gyrokinetic system (1), the 1978 conclusions regarding stability of the universal mode will not necessarily agree with gyrokinetic calculations.

**NUMERICAL DEMONSTRATION OF INSTABILITY**

We solve the system (1)–(2) using two independent codes, gs2 [12] and gene [13,15]. A demonstration of linear instability is given in figure 1 for the case \( L_s/L_n = 32 \). In this figure and hereafter, \( T_i = T_n \). We have taken great care to verify that this instability is robust and not a numerical artifact. To this end, we demonstrate in figure 1 that there is precise agreement between the two codes, which use different velocity-space coordinates and different numerical algorithms. The figure shows calculations from gs2 run as initial-value simulations, and from gene run as a direct eigenmode solver using the SLEPc library [10]. For each code, results are plotted for many different resolutions to demonstrate convergence. (For gs2, results are overlaid for 8 sets of numerical parameters: a base resolution, \( 2 \times \) number of parallel grid points, \( 2 \times \) parallel box size, \( 2 \times \) number of energy grid points, \( 2 \times \) maximum energy, \( 2 \times \) number of pitch angle grid points, 1/2 timestep, and \( 2 \times \) maximum time. For gene, results are again overlaid for 8 sets of numerical parameters: a base resolution, \( 2 \times \) number of parallel grid points, \( 2 \times \) parallel box size, \( 2 \times \) number of energy grid points, \( 2 \times \) maximum energy, \( 2 \times \) number of pitch angle grid points, 1/2 timestep, and \( 2 \times \) maximum time.)

![Figure 1](image-url)  
**FIG. 1.** (Color online) Demonstration that the universal instability is numerically robust. The positive growth rate and real frequency are plotted for the unstable range of \( k_y \) using calculations from two independent codes – gs2 run as an initial-value simulation and gene run as a direct eigenvalue solver – with results agreeing to high precision. For each code, results from many different numerical resolutions are overlaid in the figure. A reduced mass ratio \( m_i/m_e = 25 \) is used here to further minimize the possibility of numerical artifacts.
MODE PROPERTIES

Figure 2 displays the dependence of the frequency and growth rate on perpendicular wavenumber and magnetic shear, this time using the true deuterium-electron mass ratio. Across this range of parameters shown, the phase velocity is in the same direction as in the absence of shear (Re(ω/ωi) > 0, appearing in the figures as Re(ω) < 0 due to the codes’ sign convention), and the mode satisfies |ωLs/vi| ≪ 1 and |ω/ωi| ≪ 1. The growth rate is smaller in magnitude than the real frequency.

Instability is found for k_yawρi > 0.7, extending beyond k_yawρi > 100 for the highest values of Ls/Ln. Instability is found for at least some values of k_yaw whenever Ls/Ln exceeds a critical value of ∼ 17. Interestingly, the growth rate does not increase monotonically with Ls/Ln, as one can see along the slice k_yawρi = 7. Another surprising feature is a region of stability around k_yawρi ≈ 2, surrounded by instability at higher and lower k_yaw. Unfortunately, an understanding of these mode characteristics would require analytical solution of the integral equation (4), a daunting task.

A typical unstable eigenmode is shown in figure 3. The example shown is obtained using gs2 at Ls/Ln = 50 and k_yawρi = 10, near the maximum growth rate in figure 2. In the field-aligned parallel coordinate z, the mode has a parallel extent ≈ L_s. Fourier transforming, we find the extent of the radial eigenmode is ≲ ρ_e. Most of the mode structure occurs where the ion Z function has an argument of order 1, |x| ∼ |ωLs/(k_yawv_i)|, so the electron Z function has small argument. The eigenmode amplitude is small in the inner electron region where |x| < |ωLs/(k_yawv_e)|.

Dependence of the instability on temperature gradients and collisionality is shown in figure 4 for the case L_s/L_n = 50 and m_i/m_e = 3600. The growth rate generally decreases with increasing temperature gradients, although around k_yawρi ∼ 2 the opposite trend can be found. Collisions (using the operator in [17]) reduce the growth rate, particularly at the highest k_yaw values. For sufficient collisionality, ν_ee ∼ 0.05v_i/L_n or more, the mode is stabilized at all k_yaw (for this L_s/L_n).

RELATIONSHIP TO PREVIOUS WORK

Let us now examine in greater detail why we find instability for the sheared slab with weak temperature gradient, whereas previous authors found stability. First, consider Ref. [3], which gave an analytical proof of stability in the limit T_i ≪ T_e. We find no contradiction using gyrokinetic simulations, obtaining instability only when T_i/T_e is not small. (Recall T_i = T_e for all figures here.)

Next, consider Refs. [1] and [2]. These authors considered a differential eigenvalue equation, not equivalent to (4). This differential eigenvalue equation was derived assuming

\[ \rho_i^2 |\hat{\Phi}^{-1} \partial^2 \hat{\Phi} / \partial x^2| \ll 1 \]  (6)

(see e.g. [18].) No such assumption was made in deriving (4), so this integral equation is therefore more general. Considering the typical unstable eigenmode in figure 3b, there is structure on scales smaller than ρ_i, violating (6). Therefore, the approach used in Refs. [1] and [2] is inapplicable for the modes that are unstable.

Ref. [2] claims to give a proof of stability which does not rely on (6). However, this reference uses an integral eigenmode equation which differs from the general gyrokinetic result (4), and so it is not surprising that we reach different conclusions. The eigenmode equation in [2] is effectively a WKB approximation, the zero-shear dispersion relation but with k_z replaced by −i d/dx. This WKB approximation is not justified for modes such as the one in figure 3, in which the wavelengths in x are comparable to or much longer than the scale of variation.
in the ion and electron Z functions. The problems with the integral equation in [8] are discussed in the introduction of [8] and references therein.

Ref [8] appears to give a proof of stability using the full gyrokinetic integral eigenmode equation (4) without further approximation, in direct opposition to our numerical results. Although we retain \( J_{\text{Be}} \) for all numerical results shown here while \( J_{\text{Be}} \) is set to 1 in Ref. [8], this difference is not the cause of the disagreement: we find little change in the instability at low \( k_y \) if \( J_{\text{Be}} \) is set to 1 in gs2. The proof in [8] is accomplished in two steps. The electron nonadiabatic term in (4) (the term proportional to \( 1 - \omega_s/\omega \)) is multiplied by a parameter \( \lambda \), so that \( \lambda = 1 \) corresponds to the true eigenmode equation and \( \lambda = 0 \) corresponds to the adiabatic electron approximation. The authors first prove that there is no instability when \( \lambda = 0 \), using a bounding of quadratic forms. We do not disagree with the first part of this proof, as we find no instability in the numerical gyrokinetic calculations when the electrons are taken to be adiabatic. The authors then argue that no instability is introduced if \( \lambda \) is continuously increased from 0 to 1. We do disagree with this second part of the proof, and in particular with the argument that there can be no marginally stable modes for \( 0 < \lambda < 1 \), i.e. that no solutions for \( \omega \) can cross the real axis as \( \lambda \) is increased from 0 to 1. Including the \( \lambda \) parameter in gs2, we find that modes do continuously transform from damped to unstable as \( \lambda \) is increased from 0 to 1. For the case \( L_s/L_n = 50 \) and \( k_y\rho_i \) shown in figure 3, marginal stability occurs when \( \lambda = 0.032 \). Therefore, the problematic step in [8] is item (iii) on p750. Indeed, when \( \lambda > 0 \), a nonadiabatic electron term must be included in (38)-(55), and this term can have opposite sign to the other terms in (55), so (56) no longer follows.

CONCLUSIONS

In summary, we find that the slab with weak or moderate magnetic shear (\( L_s/L_n > 17 \)) is generally unstable at low collisionality, even when temperature gradients are weak, in contrast to previous work. The instability is seen robustly in multiple gyrokinetic codes, and occurs for short perpendicular wavelengths, \( k_y\rho_i \gtrsim 1 \). The instability may be relevant to the edge of magnetically confined plasma experiments, where the density gradient drive is strong, particularly in future experiments with low edge collisionality. Previous work that apparently showed stability of the universal mode [1][3] assumed either \( T_i \approx T_e \) or \( T_i \ll T_e \), whereas we allow \( T_i \sim T_e \) and use a full gyrokinetic description that does not require [0].

FIG. 3. (Color online) Eigenmodes computed with gs2 for \( L_s/L_n = 50 \), \( m_i/m_e = 3600 \), \( k_y\rho_i = 10 \) (near the maximum growth rate in figure 2). In (b), noting \( k_y = k_y x/L_s \), vertical lines indicate the values of \( x \) at which the arguments of the electron and ion Z functions in (4) have unit magnitude.

FIG. 4. (Color online) Growth rate and real frequency vs. wavenumber for \( L_s/L_n = 50 \) and \( m_i/m_e = 3600 \), showing trends with \( \eta = (d\ln T/dx)/(d\ln n/dx) \) and collisionality.
Ref. [7] employed a less accurate dispersion relation, and [8] appears to make a mathematical error. As the universal mode can indeed be absolutely unstable in the presence of magnetic shear, it should be considered alongside temperature-gradient-driven modes, trapped particle instabilities, ballooning modes, and tearing modes as one of the fundamental plasma microinstabilities.

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Science, under Award Numbers DEFG0293ER54197 and DEFC0208ER54964. Computations were performed on the Edison system at the National Energy Research Scientific Computing Center, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231. We wish to thank Tobias Görler for providing the gene code. We also acknowledge conversations about this work with Peter Catto, Greg Hammett, Adil Hassam, Edmund Highcock, Wrick Sengupta, Jason TenBarge, and George Wilkie.

* mattland@umd.edu

[1] D. W. Ross and S. M. Mahajan, Phys. Rev. Lett. 40, 324 (1978).