A SHIFT MAP WITH A DISCONTINUOUS ENTROPY FUNCTION

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ABSTRACT. Let $f: X \rightarrow X$ be a continuous map on a compact metric space with finite topological entropy. Further, we assume that the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. It is well-known that this implies the continuity of the localized entropy function of a given continuous potential $\phi : X \rightarrow \mathbb{R}$. In this note we show that this result does not carry over to the case of higher-dimensional potentials $\Phi : X \rightarrow \mathbb{R}^m$. Namely, we construct for a shift map $f$ a 2-dimensional Lipschitz continuous potential $\Phi$ with a discontinuous localized entropy function.

1. Introduction. Let $f : X \rightarrow X$ be a continuous map on a compact metric space with finite topological entropy, and let $\mathcal{M}$ denote the set of all $f$-invariant Borel probability measures on $X$ endowed with the weak* topology. This makes $\mathcal{M}$ a compact convex metrizable topological space. For a continuous $m$-dimensional potential $\Phi = (\phi_1, \ldots, \phi_m) : X \rightarrow \mathbb{R}^m$ we define

$$\mathcal{R}(\Phi) = \{rv(\mu) : \mu \in \mathcal{M}\},$$

where $rv(\mu) = (\int \phi_1 \, d\mu, \ldots, \int \phi_m \, d\mu)$. It follows that $\mathcal{R}(\Phi)$ is a compact and convex subset of $\mathbb{R}^m$. The set $\mathcal{R}(\Phi)$ is frequently referred to as the rotation set of $\Phi$ (see e.g. [3, 9, 10, 12, 13, 15, 20]), while in the context of multifractal analysis it is often referred to as the spectrum of (Birkhoff) ergodic averages (see e.g. [1, 2, 4, 6]). The localized entropy function of $\Phi$ on $\mathcal{R}(\Phi)$ is defined by

$$\mathcal{H}(w) = \mathcal{H}_\Phi(w) = \sup \{h_\mu(f) : rv(\mu) = w\},$$

where $h_\mu(f)$ denotes the measure-theoretic entropy of $\mu$. We note that for various systems and potentials the localized entropy function coincides with the entropy of certain multifractal level sets (e.g. [2, 4]). Recall that the measure-theoretic entropy is an affine function on $\mathcal{M}$. This shows that $w \mapsto \mathcal{H}(w)$ is concave which implies its continuity on the interior of $\mathcal{R}(\Phi)$, see e.g. [18]. If $\mathcal{R}(\Phi)$ has empty interior we still obtain the continuity of $\mathcal{H}$ on the relative interior of $\mathcal{R}(\Phi)$, i.e., the interior of $\mathcal{R}(\Phi)$ considered as a subset of the affine hull of $\mathcal{R}(\Phi)$. Another frequently considered
condition is the upper semi-continuity of the entropy map $\mu \mapsto h_\mu(f)$, which holds for example when $f$ is expansive [19], when $f$ is a $C^\infty$-map on a compact smooth Riemannian manifold [16] or when $f$ satisfies entropy-expansiveness (as for example certain partial hyperbolic systems [5]). The upper semi-continuity of the entropy map immediately implies that the supremum in (2) is actually a maximum and more importantly that $w \mapsto \mathcal{H}(w)$ is upper semi-continuous. One might suspect that the latter even guarantees the continuity of the localized entropy function for all dimensions $m$. Indeed, it was stated by Jenkinson [12, p. 3723] that the upper semi-continuity of the entropy map implies the continuity of the localized entropy. This claim was restated by Kucherenko and Wolf in [13, 14, 15]. However, it turns out that the argument in [12] is incomplete. While the continuity of every upper semi-continuous concave function with domain in $\mathbb{R}$ is immediate, the situation in higher dimensions is more delicate. Indeed, a striking result by Dale, Klee and Rockafellar [8] shows that for a compact convex set $D \subset \mathbb{R}^m$ the property that every concave upper semi-continuous function on $D$ is continuous is equivalent to $D$ being a polyhedron. We point out that $\mathcal{R}(\Phi)$ being a polyhedron actually occurs in relevant situations, e.g. for subshifts of finite type (SFT) and locally constant potentials in [12, 20], and for certain non-locally constant potentials in [12, 13]. On the other hand, the results in [8] do not imply that $w \mapsto \mathcal{H}(w)$ can be discontinuous. After all $\mathcal{H}$ is a rather special upper semi-continuous concave function. In this note we show that the continuity of the localized entropy function can even fail in the case of shift maps and Lipschitz continuous potentials. More precisely, we have the following result (see Example 1 and Theorem 3.2 in the text).

**Main Theorem.** Let $f : X \to X$ be a shift map on a one-sided full shift with 3 symbols. Then there exists a Lipschitz continuous potential $\Phi : X \to \mathbb{R}^2$ with the following properties:

(i) The set $\mathcal{R}(\Phi)$ has non-empty interior and countably many extreme points of which all but one are isolated;

(ii) The localized entropy function $w \mapsto \mathcal{H}(w)$ is discontinuous at the non-isolated extreme point.

Further, one can show that the localized entropy function in the Main Theorem is analytic on the interior of $\mathcal{R}(\Phi)$. This follows from a more general analyticity result for so-called STP-maps (including SFT’s, uniformly hyperbolic systems and expansive homeomorphisms with specification) and for Hölder continuous potentials, see [2, 11, 13]. We note that the reason for formulating the Main Theorem for one-sided shift maps on a shift space with 3 symbols is for the ease of presentation. Our techniques can be applied to obtain similar discontinuity results for more general SFT’s in the one-sided and two-sided case.

Next we mention some related work. Fan, Feng and Wu consider in [6] the dimension spectrum for $m$-dimensional continuous potentials $\Phi$ and show that it is concave and upper semi-continuous and hence continuous on the interior of $\mathcal{R}(\Phi)$. They ask the question (see [6, p. 242/243]) if the continuity of the dimension spectrum extends to the boundary of $\mathcal{R}(\Phi)$. We note that our Main Theorem also

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1We note that the theorems in [12, 13, 14, 15] do not rely on the continuity of the localized entropy function. The only exception is Theorem A in [15] whose proof uses the continuity of $\mathcal{H}$ restricted to a line segment, i.e., $m = 1$. As noted above, for $m = 1$ the localized entropy is always continuous.

2We note that the results in [8] are formulated in terms of lower semi-continuous convex functions.
displays the possibility of a discontinuous dimension spectrum and consequently gives a negative answer to the question of Fan, Feng and Wu in [7]. They consider the V-statistics for real-valued potentials defined on \( X \). Among other results they construct examples with discontinuous entropy spectra for the full-shift and \( r = 3 \). We point out that the discontinuity results in [7] stem from the fact that in general the entropy spectrum of the V-statistics is not concave whereas the discontinuity result in our Main Theorem is despite the localized entropy function being concave. We refer to [7] for more details.

We end the introduction with the discussion of a simple example of a upper semi-continuous concave function that fails to be continuous. Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous concave function that fails to be continuous. Let \( \mu \) be a Borel probability measure endowed with the weak* topology, and let \( \mathcal{M}_E \subset \mathcal{M} \) denote the subset of ergodic measures. Recall that \( \mathcal{M} \) is a compact convex metrizable topological space. Given \( \mu \in \mathcal{M} \) we denote by \( h_\mu(f) \) the measure-theoretic entropy of \( \mu \), see [19] for the definition and details. Clearly \( f \) is expansive and consequently the entropy map \( \mu \mapsto h_\mu(f) \) is upper semi-continuous. We say \( t = t_1 t_2 \cdots t_k \in \mathcal{A}^k \) is a block of length \( k \) and write \( |t| = k \). Further, \( \epsilon \) denotes the empty block. Moreover, we say \( s = s_1 s_2 \cdots s_l \) is a subblock of \( t \) if there exists \( 1 \leq i \leq k \) with \( i + l - 1 \leq i \leq k \) such that \( s_1 = t_i, s_2 = t_{i+1}, \ldots, s_l = t_{i+l-1} \). Given \( \xi \in X \), we write \( \pi_k(\xi) = \xi_1 \cdots \xi_k \in \mathcal{A}^k \). For \( \xi \in \mathcal{A} \) and \( k \in \mathbb{N} \) we write \( \xi_k = \xi_1 \cdots \xi_k \in \mathcal{A}^k \) and define the concatenation of blocks \( t \) and \( s \) by \( ts = t_1 \cdots t_k s_1 \cdots s_l \). Moreover, we denote by \( t^k \) the \( k \)-times concatenation of the block \( t \). We denote the cylinder of length \( k \) generated by \( t \) by \( \mathcal{C}_k(t) = \{ \xi \in X : \xi_1 = t_1, \ldots, \xi_k = t_k \} \). Given \( \xi \in X \) and \( k \in \mathbb{N} \),
we call \( C_k(\xi) = C(\pi_k(\xi)) \) the cylinder of length \( k \) generated by \( \xi \). Further, we call \( O(t) = t_1 \cdots t_k t_1 \cdots t_k \cdots \in X \) the periodic point with period \( k \) generated by \( t \). We denote by \( \text{Per}_n(f) \) the set of periodic points of \( f \) with prime period \( n \) and by \( \text{Per}(f) \) and the set of periodic points of \( f \). Let \( x \in \text{Per}_n(f) \). We call \( \tau_x = x_1 \cdots x_n \) the generating segment of \( x \), that is \( x = O(\tau_x) \). For \( x \in \text{Per}_n(f) \), the unique invariant measure supported on the orbit of \( x \) is given by

\[
\mu_x = \frac{1}{n} (\delta_x + \cdots + \delta_{f^{n-1}(x)}),
\]

where \( \delta_y \) denotes the Dirac measure on \( y \). We also call \( \mu_x \) the periodic point measure of \( x \). Obviously, \( \mu_x = \mu_{f(x)} \) for all \( l \in \mathbb{N} \). We write \( \mathcal{M}_{\text{Per}} = \{ \mu_x : x \in \text{Per}(f) \} \) and observe that \( \mathcal{M}_{\text{Per}} \subset \mathcal{M}_E \).

3. Construction of the example. In this section we give an example of a shift map and a 2-dimensional Lipschitz continuous potential that exhibits a discontinuous localized entropy function. For convenience we consider here a one-sided shift map on a shift space with 3 symbols. We note that our construction can be modified to obtain discontinuous localized entropy functions for other shift maps with positive entropy. We begin by constructing a certain compact convex subset of \( \mathbb{R}^2 \) that will become \( \mathcal{R}(\Phi) \) in our example.

Fix \( a, b > 0 \) and fix \( \lambda \in \mathbb{N} \) with \( \lambda \geq 3 \). Fix \( \theta \in (0, 1) \). We consider a continuous function \( h : [0, a] \to \mathbb{R} \) which is strictly increasing and strictly concave. Further assume \( h(0) = 0 \) and \( h(a) = b \). Let \( (x_k)_{k \in \mathbb{N}} \) be a strictly decreasing sequence with \( x_k \in (0, a) \) for all \( k \geq 1 \) such that \( v_k \overset{\text{def}}{=} (x_k, h(x_k)) \) satisfies

\[
||v_k|| < C \theta^k
\]

for all \( k \in \mathbb{N} \) and some \( C > 0 \). The existence of such a sequence \( (x_k) \) follows from the continuity of \( h \) at 0. Since \( ||(x_k, 0)|| \leq ||v_k|| \) for all \( k \in \mathbb{N} \), equation (5) also holds for all \( (x_k, 0) \) instead of \( v_k \). Let \( w_\infty = (0, 0) \) and \( w_0 = (a, 0) \). Further, for \( k \geq 1 \) we define

\[
w_k = \frac{1}{k + \lambda} \left( \lambda w_0 + \sum_{j=1}^{k} v_j \right).
\]

Define \( \mathcal{V} = \{ w_k : k \geq 0 \} \cup \{ w_\infty \} \). Further, let \( \mathcal{R} = \text{conv}(\mathcal{V}) \) denote the convex hull of \( \mathcal{V} \). For \( k \geq 1 \) let \( m_k \) denote the slope of the line segment joining \( w_k \) and \( w_{k-1} \). Since \( (x_k)_k \) is strictly decreasing it follows that the x-coordinates of \( (w_k)_k \) are strictly decreasing. Thus, \( m_k \in \mathbb{R} \) for all \( k \geq 1 \). We refer to Figure 1 for an illustration.

**Proposition 1.** The set \( \mathcal{R} \) has the following properties:

(i) \( \lim_{k \to \infty} w_k = w_\infty \) and \( \mathcal{R} \) is compact;

(ii) The sequence \( (m_k)_{k \geq 2} \) is strictly decreasing;

(iii) The boundary of \( \mathcal{R} \) is an infinite polygon with extreme point set \( \mathcal{V} \).

**Proof.**

(i) That \( \lim_{k \to \infty} w_k = w_\infty \) follows from (5) and (6). Hence, \( w_\infty \) is the only accumulation point of \( \mathcal{V} \). We conclude that \( \mathcal{V} \) is compact which implies the compactness of its convex hull \( \mathcal{R} \).

(ii) By (6),

\[
w_{k+1} = \frac{k + \lambda}{k + 1 + \lambda} w_k + \frac{1}{k + 1 + \lambda} v_{k+1}.
\]

**Definition:** Temps
It now follows from an elementary induction argument that the points $w_k$ lie strictly below the graph of $h$. Therefore, the statement that $m_k$ is strictly decreasing follows from $h$ being strictly increasing.

(iii) First notice that $w_0$ and $w_\infty$ are extreme points of $\mathcal{R}$. This holds since $\mathcal{R}$ has empty intersection with $\{(x,y) : x < 0\}$, $\{(x,y) : x > a\}$ and $\{(x,y) : y < 0\}$. Finally, for $k \geq 1$ that $w_k$ is an extreme point of $\mathcal{R}$ follows from statement (ii). 

Example 1. Let $f : X \to X$ be the one-sided full shift with alphabet $\{0,1,2\}$ endowed with the $\theta$-metric where $\theta$ is as in (5). We construct a potential $\Phi$ as follows: First, we define several subsets of $X$. Let $S = \{0,1\}$. We define

\[ X(l) = \{ \xi \in X : \xi_1, \ldots, \xi_{l-1} \in S, \xi_l = 2 \}, \]
\[ X_0(\lambda) = \bigcup_{l=1}^\lambda X(l), \]
\[ X(\infty) = \{ \xi \in X : \xi_l \in S \text{ for all } l \in \mathbb{N} \} = S^\mathbb{N}. \]

Note that $X(1) = C_1(2) = \{ \xi \in X : \xi_1 = 2 \}$. We define a potential $\Phi : X \to \mathbb{R}^2$ by

\[
\Phi(\xi) = \begin{cases} 
w_0 & \text{if } \xi \in X_0(\lambda) \\
(x_{l-\lambda}, 0) & \text{if } \xi \in X(l) \setminus C_{l-1}(1^{l-1}), \ l > \lambda \\
v_{l-\lambda} & \text{if } \xi \in X(l) \cap C_{l-1}(1^{l-1}), \ l > \lambda \\
w_\infty & \text{if } \xi \in X(\infty) 
\end{cases} \tag{8}
\]

Throughout the remainder of this paper we study the potential $\Phi$ defined in the Example 1.

Proposition 2. The potential $\Phi$ defined in (8) is Lipschitz continuous and $\mathcal{R}(\Phi) = \mathcal{R}$. 
Proof. Let \( \xi, \eta \in X \) with \( \Phi(\xi) \neq \Phi(\eta) \). First we assume \( \xi_k \neq \eta_k \) for some \( k \leq \lambda \). Let \( C_1 = \sup \{ \|u - v\| : u, v \in \Phi(X) \} \). Then

\[
\|\Phi(\xi) - \Phi(\eta)\| \leq C_1 \leq C_1 \theta^{-\lambda} d(\xi, \eta). \tag{9}
\]

Next we consider the case \( l = \min\{j : \xi_j \neq \eta_j\} > \lambda \). It follows from the definition of \( \Phi \) that neither \( \Phi(\xi) \) nor \( \Phi(\eta) \) belong to \( \{(x_j, 0), v_j : j = 1, \cdots, \lambda - 1\} \cup \{u_0\} \) since otherwise \( \Phi(\xi) = \Phi(\eta) \). Therefore, it is sufficient to consider the case \( \Phi(\xi), \Phi(\eta) \in \{(x_j, 0), v_j : j \geq \lambda - 1\} \cup \{w_\infty\} \). Applying (5) yields

\[
\|\Phi(\xi) - \Phi(\eta)\| \leq 2C \theta^{-\lambda} = 2C \theta^{-\lambda} d(\xi, \eta). \tag{10}
\]

By combining (9) and (10) we conclude that \( \Phi \) is Lipschitz continuous with Lipschitz constant \( \max\{C_1 \theta^{-\lambda}, 2C \theta^{-\lambda}\} \).

Next we prove \( R(\Phi) = R \). Recall that \( R(\Phi) \) is convex. Therefore, in order to prove \( R \subset R(\Phi) \) it suffices to show that each extreme point of \( R \) (i.e. each point in \( \gamma \)) coincides with the rotation vector of some invariant measure. For \( k \in \mathbb{N} \) let \( \xi^k = O(1^{k+\lambda-1}, 2) \), that is \( \xi^k \) is the periodic point whose generating segment \( \tau_{\xi^k} \) is given by \( k + \lambda - 1 \)'s followed by a 2. Hence \( \xi^k \in \text{Per}_{k+\lambda}(f) \). It follows from equations (4), (6) and the definition of \( \Phi \) (see (8)) that \( \text{rv}(\mu_{\xi^k}) = w_k \). Further, we clearly have \( \text{rv}(\mu_{O(02)}) = w_0 \) and \( \text{rv}(\mu_{O(0)}) = w_\infty \). Hence \( \mathbb{V} \subset \{ \text{rv}(\mu_x) : x \in \text{Per}(f) \} \) which implies \( R \subset R(\Phi) \).

Finally, we prove \( R(\Phi) \subset R \). It is well-known that the periodic point measures \( \mathbb{N}_{\text{Per}} \) are weak* dense in \( \mathbb{N} \), see [17]. Thus, by compactness of \( R \) it suffices to show that \( \{ \text{rv}(\mu_x) : x \in \text{Per}(f) \} \subset R \). Let \( x \in \text{Per}_n(f) \) for some \( n \in \mathbb{N} \). Recall that \( \tau_x = x_1 \cdots x_n \) denotes the generating segment of \( x \). If \( x = O(1) \) then \( \text{rv}(\mu_x) = w_\infty \). Assume now that \( x \neq O(1) \). Thus, at least one of the \( x_i \)'s in \( \tau_x \) is not equal to 1. Taking a different point in the (finite) orbit of \( x \) if necessary, we may assume \( x_n \neq 1 \). It follows from (8) that the \( y \)-coordinate of \( \Phi(\xi) \) is positive if and only if

\[
\xi \in \bigcup_{k \in \mathbb{N}} C_{k+\lambda}(\xi^k). \tag{11}
\]

It follows that if \( \tau_x \) does not contain a subblock in \( \{\tau_k : k \in \mathbb{N}\} \) then \( \text{rv}(\mu_x) \in [0, a] \times \{0\} \subset R \). It remains to consider the case when \( \tau_x \) contains at least one block in \( \{\tau_k : k \in \mathbb{N}\} \). By replacing \( x \) with a point in the orbit of \( x \) if necessary, we can write \( \tau_x \) as a finite concatenation of blocks of the form

\[
\tau_x = \eta_1 \tau_{k_1} \cdots \eta_l \tau_{k_l}, \tag{12}
\]

where the \( \eta_i \)'s are blocks that do not have a subblock contained in \( \{\tau_k : k \in \mathbb{N}\} \) and whose last symbol is either 0 or 2. The latter ensures that the blocks \( \tau_k \) are of maximal length. We note that some of the \( \eta_i \)'s in (12) may be the empty block. Let \( n_i \) denote the length of \( \eta_i \). For each \( i \) there exists a \( m_i \) such that \( f^{m_i}(x) = \eta_i \tau_{k_i} \eta_{i+1} \tau_{k_{i+1}} \cdots \). We define \( u_i = \frac{1}{n_i} \sum_{k=0}^{n_i-1} \Phi(f^{m_i+k}(x)) \). It follows from the construction that \( u_i \in [0, a] \times \{0\} \). We conclude that

\[
\text{rv}(\mu_x) = \frac{1}{n} \sum_{k=0}^{n-1} \Phi(f^k(x)) = \frac{n_1}{n} u_1 + \frac{k_1 + \lambda}{n} w_{k_1} + \cdots + \frac{n_l}{n} u_l + \frac{k_l + \lambda}{n} w_{k_l}. \tag{13}
\]

Notice that \( n = \lambda + \sum_{i=1}^{l} n_i + k_l \). Therefore, (13) shows that \( \text{rv}(\mu_x) \) is a convex combination of points in \( R \) which implies that \( \text{rv}(\mu_x) \in R \).

Corollary 1. Let \( k \in \mathbb{N} \) and let \( x \in \text{Per}(f) \). Then \( \text{rv}(\mu_x) = w_k \) if and only if \( \mu_x = \mu_{\xi^k} \).
Proof. The statement follows from (13), $u_1, \ldots, u_t \in [0, a] \times \{0\}$ and the fact that $w_k$ is an extreme point of $R(\Phi)$. \hfill \square

We will make use of the following trivial facts that hold for all measure-preserving transformations.

**Lemma 3.1.** Let $\mu \in M$, $A \subset X$ and $B \subset f^{-1}(A)$ then $\mu(B) \leq \mu(A)$, in particular $\mu(B) \leq \mu(f(B))$. Moreover, if $B \subset f^{-1}(A)$ then $\mu(B) = \mu(A)$ if and only if $\mu(f^{-1}(A) \setminus B) = 0$.

We continue to use the notation from Proposition 2. Recall that $\xi^k = O(1^{k+\lambda-1})$ and $\nu(\xi^k) = w_k$.

**Proposition 3.** Let $k \in \mathbb{N}$ and $p = \frac{1}{\lambda + 1}$. Let $\mu \in M$ with $\mu(\Phi^{-1}(w_0)) = \lambda p$ and $\mu(C_{t+\lambda}(\xi^l)) = p$ for $l = 1, \ldots, k$. Then $\mu = \mu_{\xi^k}$.

**Proof.** We first notice that since the cylinders $C_{t+\lambda}(\xi^l), l = 1, \ldots, k$ are pairwise disjoint, the assumptions of the proposition imply

$$\mu \left( \Phi^{-1} \left( \{w_\infty \} \cup \bigcup_{l>k} \{v_l\} \cup \bigcup_{l \geq 1} \{ (x_l, 0) \} \right) \right) = 0. \tag{14}$$

Recall that $\tau_1 = 1^{t+\lambda-1}$ denotes the generating segment of $\xi^l$. We define cylinders

$C^0 = C_{1+k+\lambda}(2\tau_k)$ and $C^1 = C_{t+1+k+\lambda}(1^{\lambda}2\tau_k)$ for $1 \leq l \leq \lambda - 1$. It follows from the construction that the $C^i \cap C^j = \emptyset$ for all $0 \leq i, j \leq \lambda - 1$ with $i \neq j$. Further, by Lemma 3.1,

$$p = \mu(C_{t+\lambda}(\tau_k)) \geq \mu(C^0) \geq \cdots \geq \mu(C^{\lambda-1}). \tag{15}$$

First, we prove the following.

**Claim 1.** $\mu(C^l) = p$ for all $0 \leq l \leq \lambda - 1$.

For the case $l = 0$ we note that $\mu(C_{k+\lambda}(\xi^k)) = \mu(C_{k+\lambda}(\tau_k)) = p$. Moreover,

$$f^{-1}(C_{k+\lambda}(\tau_k)) = C^0 \cup \bigcup_{i=0}^{1} C_{1+k+\lambda}(i\tau_k). \tag{16}$$

Since $\Phi(\bigcup_{i=0}^{1} C_{1+k+\lambda}(i\tau_k)) = \{(x_{k+1}, 0), v_{k+1}\}$ we may conclude from equation (14) that $\mu(\bigcup_{i=0}^{1} C_{1+k+\lambda}(i\tau_k)) = 0$. Therefore, the case $l = 0$ follows from (16) and Lemma 3.1. Clearly,

$$C_{k+\lambda+1}(2\tau_k), f^{-1}(C_{k+\lambda+1}(2\tau_k)), \ldots, f^{-(\lambda-1)}(C_{k+\lambda+1}(2\tau_k)) \tag{17}$$

are pairwise disjoint sets with

$$\mu \left( \bigcup_{r=0}^{\lambda-1} f^{-r}(C_{k+\lambda+1}(2\tau_k)) \right) = w_0 \text{ that satisfy } \mu \left( f^{-r}(C_{k+\lambda+1}(2\tau_k)) \right) = p \tag{18}$$

for $r = 0, \ldots, \lambda - 1$. Hence

$$\mu \left( \bigcup_{r=0}^{\lambda-1} f^{-r}(C_{k+\lambda+1}(2\tau_k)) \right) = \lambda p. \tag{19}$$

Assume that the claim is false. Then, it follows from (15) that $\mu(C^{\lambda-1}) < p$. Since $\mu(f^{-(\lambda-1)}(C_{k+\lambda+1}(2\tau_k))) = p$, there exists $\eta = \eta_1 \cdots \eta_{\lambda-2}$ such that $C_{k+2\lambda}(\eta\tau_k) \not\subset C^{\lambda-1}$ with $\mu(C_{k+2\lambda}(\eta\tau_k)) > 0$. Here $C_{k+2\lambda}(\eta\tau_k) \not\subset C^{\lambda-1}$ means that $\eta_{i} \neq 1$ for some $i = 1, \ldots, \lambda - 1$. We conclude that there must exist a cylinder $C = C(\eta)$ of length $k + 2\lambda + 1$ contained in $f^{-1}(C_{k+2\lambda}(\eta\tau_k))$ with $\mu(C) > 0$. Since $\eta_{i} \neq 1$, $v_1 \notin \Phi(C)$. 
Moreover, since \( \mu(\Phi^{-1}((x_1, 0))) = 0 \) we conclude that \( \Phi(C) \neq \{(x_1, 0)\} \). Hence \( \Phi(C) = \{u_0\} \). On the other hand, \( C \cap \bigcup_{r=0}^{\lambda-1} f^{-r}(C_{k+\lambda+1}(2 \tau_k)) = \emptyset \). Therefore (19) implies \( \mu(\Phi^{-1}(u_0)) > \lambda p \) with is a contraction. This proves Claim 1.

Next we define cylinder \( \tilde{C}^0 = C_1(2) \) and \( \tilde{C}^l = C_{l+1}(1^l 2) \) for \( 1 \leq l \leq \lambda - 1 \).

**Claim 2.** \( \mu(\tilde{C}^l) = p \) for all \( l = 0, \cdots, \lambda - 1 \).

Obviously, \( \tilde{C}^l \subset \tilde{C}^l \) for all \( 0 \leq l \leq \lambda - 1 \). Thus, by Claim 1, \( \mu(\tilde{C}^l) \geq p \). On the hand hand, we observe that \( \tilde{C}^0, \cdots, \tilde{C}^{\lambda - 1} \) are pairwise disjoint cylinders with \( \Phi(\tilde{C}^l) = w_0 \) for all \( 0 \leq l \leq \lambda - 1 \). Hence, \( \sum_{l=0}^{\lambda-1} \mu(\tilde{C}^l) \leq \lambda p \). Putting these facts together proves the claim.

**Claim 3.** \( \mu(\{\xi^k\}) = p \).

The statement \( \mu(\{\xi^k\}) \leq p \) follows from \( \mu(C_{k+\lambda}(\xi^k)) = p \) since \( \xi^k \in C_{k+\lambda}(\xi^k) \). Since \( \mu(C_{j+\lambda}(\xi^k))_{j \geq 1} \) is a non-increasing sequence with limit \( \mu(\{\xi^k\}) \), it suffices to show that \( \mu(C_{j(k+\lambda)}(\xi^k)) \geq p \) for all \( j \geq 1 \). Note that the case \( j = 1 \) is part of the assumption. Suppose on the contrary that there exists \( j > 1 \) such that \( \mu(C_{j(k+\lambda)}(\xi^k)) < p \). Further, suppose \( j \) is the smallest integer with this property. Recall that \( \tau_{j-1} \) denotes the \( (j - 1) \)-times concatenation of the block \( \tau_k \). It follows that there exists a block \( \eta = \eta_1 \cdots \eta_{k+\lambda} \) with \( \eta \neq \tau_k \) such \( \mu(C_{j(k+\lambda)}(\tau_{j-1}^{-1} \cdot \eta)) > 0 \). We conclude from Lemma 3.1 that

\[
0 < \mu(C_{j(k+\lambda)}(\tau_{j-1}^{-1} \cdot \eta)) = \mu \left( f^{(j-1)(k+\lambda) - 1}(C_{j(k+\lambda)}(\tau_{j-1}^{-1} \cdot \eta))) \right) = \mu(C_{1+k+\lambda}(2 \eta)).
\]

Note that \( C_{1+k+\lambda}(2 \eta) \) and \( C_{1+k+\lambda}(2 \tau_k) \) are disjoint cylinders contained in \( C_1(2) = \tilde{C}^0 \). Since \( \mu(C_{1+k+\lambda}(2 \tau_k)) = \mu(\tilde{C}^l) \) we are able to deduce from (20) that \( \mu(C_1(2)) > p \) which is a contradiction to Claim 2 with \( l = 0 \). This completes the proof of the Claim 3.

To complete the proof of the proposition it remains to show that Claim 3 holds for all points in the orbit of \( \xi^k \). Let \( \xi^k = f^l(\xi^k) \) for some \( l = 1, \cdots, k + \lambda - 1 \). Since \( f^{k+\lambda-l}(\xi^k) = \xi^k \) we conclude from Lemma 3.1 that \( \mu(\{\xi^k\}) \leq p \). A similar argument shows \( \mu(\{\xi^k\}) \leq \mu(\{\xi^k\}) \). Hence \( \mu(\{\xi^k\}) = \mu(\{\xi^k\}) = p \), and the proof of the proposition is complete.

**Theorem 3.2.** Let \( \Phi \) be the potential defined in (8). Then \( \mathcal{H}(w_k) = 0 \) for all \( k \in \mathbb{N} \) and \( \mathcal{H}(w_\infty) = \log 2 \). In particular, \( w \mapsto \mathcal{H}(w) \) is discontinuous at \( w_\infty \).

**Proof.** Since \( w_\infty \) is an extreme point of \( \mathcal{R}(\Phi) \) as well as an extreme point of \( \Phi(X) \), we obtain that \( \text{rv}(\mu) = w_\infty \) if and only \( \mu(\Phi^{-1}(w_\infty)) = 1 \). Note that \( f_{|\Phi^{-1}(w_\infty)} = f_{|\{(0,1)\}} \) has topological entropy equal to \( \log 2 \). Therefore, \( \mathcal{H}(w_\infty) = \log 2 \) is consequence of the variational principle for the entropy, see e.g. [19].

Fix \( k \in \mathbb{N} \) and let \( \mu \in \mathcal{M} \) with \( \text{rv}(\mu) = w_k \). Our goal is to show that \( \mu = \mu_{\xi^k} \) which obviously suffices to prove the theorem.

Recall from (8) that \( \Phi^{-1}(v_1) = C_{1+\lambda}(1^{l+\lambda-1} 2) \) which implies

\[
f(\Phi^{-1}(v_{l+1})) = \Phi^{-1}(v_1)
\]
for all \( l \in \mathbb{N} \). Therefore, by Lemma 3.1,

\[
\cdots \leq \mu(\Phi^{-1}(v_{l+1})) \leq \mu(\Phi^{-1}(v_1)) \leq \cdots \leq \mu(\Phi^{-1}(v_2)) \leq \mu(\Phi^{-1}(v_1)).
\]

Next, we define several sets. We define \( V_1(1) = \Phi^{-1}(v_1) = C_{1+\lambda}(1^{l+2}) \) and \( p = p_1(1) \triangleq \mu(V_1(1)) \). Since \( \text{rv}(\mu) = w_k \), (8) and (22) imply \( p_1(1) > 0 \). For \( l \geq 2 \) and
\[ i = 0, 1, 2 \text{ we define } V_i(i) = C_{i+\lambda}(i1^{i+\lambda-2}) \text{ and } p_2(i) = \mu(V_i(i)). \]

Since

\[ f^{-1}(V_l(1)) = V_{l+1}(0) \cup V_{l+1}(1) \cup V_{l+1}(2) \]

we have \( p_l(1) = p_{l+1}(0) + p_{l+1}(1) + p_{l+1}(2) \) for all \( l \geq 1 \). Moreover, \( \Phi|_{V_l(0)} = (x_l, 0) \), \( \Phi|_{V_l(1)} = v_l \) and \( \Phi|_{V_l(2)} = w_0 \). Next we consider the pre-images of \( V_l(0) \). Fix \( l \geq 2 \) and let \( j \in \mathbb{N} \). We define

\[ U^j_l = \{ \xi \in f^{-j}(V_l(0)) : \xi_1 = 2 \text{ and } \xi_2, \cdots, \xi_j \in \{0, 1\} \}, \]

that is

\[ U^j_l = \bigcup_{i_2, \cdots, i_j \in \{0, 1\}} C_{j+l+\lambda}(2i_2 \cdots i_j 01^{l+\lambda-2}) \subset f^{-j}(V_l(0)). \]

We note that \( U^j_l \) represents \( U^j_l \) as a pairwise disjoint union of sets. Moreover, \( U^j_l \cap U^{j'}_l = \emptyset \) for all \( l, l' \in \mathbb{N} \) and all \( j, j' \geq 1 \) whenever \( l \neq l' \) or \( j \neq j' \) or both. We define \( U_l = \bigcup_{j \geq 1} U^j_l \) and claim the following.

**Claim 1.** \( \mu(U_l) = \mu(V_l(0)) = p_l(0) \).

To prove the claim we consider sets

\[ \tilde{U}_l(j) = \bigcup_{i_1, \cdots, i_j \in \{0, 1\}} C_{j+l+\lambda}(i_1 \cdots i_j 01^{l+\lambda-2}) \subset f^{-j}(V_l(0)). \]

It follows that \( (\tilde{U}_l(j))_{j \geq 1} \) is a sequence of pairwise disjoint sets. Hence,

\[ \lim_{j \to \infty} \mu(\tilde{U}_l(j)) = 0. \]

Further, by construction,

\[ \mu(V_l(0)) - \mu(\tilde{U}(j)) = \mu(f^{-j}(V_l(0))) - \mu(\tilde{U}(j)) = \sum_{i=1}^{j} \mu(U_i^j). \]

Therefore, Claim 1 follows from \( \mu(U_l) = \sum_{i=1}^{\infty} \mu(U_i^j) \).

**Claim 2.** \( \mu(\Phi^{-1}(w_0)) \geq \lambda p \).

To prove the claim we first construct \( \tilde{Y}_1 = \tilde{Y}_1(\mu) \subset X \) with \( \mu(\tilde{Y}_1) = p \) such that for all \( \xi \in \tilde{Y}_1 \) we have \( \xi_1 = 2 \) and \( \xi_2, \cdots, \xi_\lambda \neq 2 \). By construction, \( (V_l(1))_{l \geq 1} \) is a sequence of pairwise disjoint sets. Thus, \( \lim_{l \to \infty} \mu(V_l(1)) = 0 \). By applying that the sets

\[ V_2(0), V_2(2), V_3(0), V_3(2), \cdots, V_l(0), V_l(2), \cdots \]

are pairwise disjoint, a similar argument as in the proof of Claim 1 shows

\[ \mu \left( \bigcup_{i=2}^\infty V_i(0) \cup V_i(2) \right) = \sum_{i=2}^\infty \mu(V_l(0)) + \mu(V_l(2)) = V_1(1) = p. \]

Evidently, \( U_l \cap U_{l'}(2) = \emptyset \) for all \( l, l' \geq 2 \). Therefore, we may conclude from (29) and Claim 1 that

\[ \mu \left( \bigcup_{l \geq 2} U_l \cup V_l(2) \right) = p. \]

We define \( \tilde{Y}_1 = \bigcup_{l \geq 2} U_l \cup V_l(2) \). By construction, if \( \xi \in \tilde{Y}_1 \) then \( \xi_1 = 2 \) and \( \xi_2, \cdots, \xi_\lambda \neq 2 \). Define \( \tilde{Y}_2 = f^{-1}(\tilde{Y}_1), \cdots, \tilde{Y}_\lambda = f^{-\lambda-1}(\tilde{Y}_1) \). It follows that \( \tilde{Y}_1, \cdots, \tilde{Y}_\lambda \) are pairwise disjoint sets with \( \mu(\tilde{Y}_1) = \cdots = \mu(\tilde{Y}_\lambda) = p \). We define

\[ Y_0(\lambda) = \tilde{Y}_1 \cup \cdots \cup \tilde{Y}_\lambda. \]
Hence, $\mu(Y_0(\lambda)) = \lambda p$. We note that $\Phi(\xi) = w_0$ for all $\xi \in Y_0(\lambda)$ which completes the proof of Claim 2.

Next we compute $\text{rv}(\mu)$ by integrating $\Phi$ over various subsets of $X$. Define

$$Y_k = \bigcup_{l=1}^{k} V_l(1) = \Phi^{-1}(\{v_1, \ldots, v_k\}) \quad \text{and} \quad Y_{gk} = \Phi^{-1}\left(\bigcup_{l>k} \{v_l\}\right)$$

(32)

and

$$Y_0 = \Phi^{-1}([0, a] \times \{0\}) \setminus Y_0(\lambda).$$

(33)

Moreover, define $p_{gk} = \mu(Y_{gk})$ and $p_0 = \mu(Y_0)$. Thus $X = Y_0(\lambda) \cup Y_k \cup Y_{gk} \cup Y_0$ is a union of pairwise disjoint sets. To compute $\int_{Y_0(\lambda) \cup Y_k} \Phi \, d\mu$ we define $p_k = p_k(1)$ and $p_l = p_l(1) - p_{l+1}(1)$ for $l = k-1, \ldots, 1$. Hence $p = \sum_{l=1}^{k} p_l$. By making a telescope sum argument and applying (6) we obtain

$$\int_{Y_0(\lambda) \cup Y_k} \Phi \, d\mu = \lambda p w_0 + \sum_{l=1}^{k} p_l(1) v_l$$

$$= \sum_{l=1}^{k} p_l \left(\lambda w_0 + \sum_{i=1}^{l} v_i\right)$$

(34)

$$= \sum_{l=1}^{k} p_l (l + \lambda) w_l.$$

Evidently we have $p_l \leq 1/(l + \lambda)$ for $l = 1, \ldots, k$. If $\mu(Y_{gk}) \neq 0$ we define

$$\int_{Y_{gk}} \Phi \, d\mu = \sum_{l=k+1}^{\infty} p_l(1) v_l = \mu(Y_{gk}) \sum_{l=k+1}^{\infty} \frac{p_l(1)}{p_{gk}} v_l \overset{\text{def}}{=} \mu(Y_{gk}) v_\infty,$$

(35)

otherwise we set $v_\infty = 0$. Similarly, if $\mu(Y_0) \neq 0$, we define

$$\int_{Y_0} \Phi \, d\mu = \mu(Y_0) \left(\frac{1}{\mu(Y_0)} \int_{Y_0} \Phi \, d\mu\right) \overset{\text{def}}{=} \mu(Y_0) \omega_0,$$

(36)

otherwise we set $\omega_0 = 0$. It follows from the definition of $Y_0$ that $\omega_0 \in [0, a] \times \{0\}$. Let $\ell_1$ denote the line through $w_k$ and $w_{k-1}$, and let $\ell_2$ denote the line through $v_{k+1}$ and $w_k$. Let $\mathcal{G}$ denote the intersection of the two closed half-spaces of points on and below the lines $\ell_1$ and $\ell_2$ respectively. Clearly $\mathcal{G}$ is convex. By definition, the closed line segment $[w_k, w_{k-1}]$ is contained in $\ell_1$. Moreover, by (6) the closed line segment $[w_{k+1}, w_k]$ is contained in $\ell_2$. We conclude that $\ell_i$ is the supporting hyperplane of the face $[w_{k+i-1}, w_{k+i-2}]$ of $\mathcal{R}(\Phi)$. This shows that $\mathcal{R}(\Phi) \subset \mathcal{G}$. It follows from Proposition 1 (ii) that $w_k$ is an extreme point of $\mathcal{G}$. Moreover, $\{v_l : l \geq k + 1\} \subset \mathcal{G}$ which together with (35) implies that $v_\infty \in \mathcal{G}$. It follows from

$$\text{rv}(\mu) = \int_{Y_0(\lambda) \cup Y_k} \Phi \, d\mu + \int_{Y_{gk}} \Phi \, d\mu + \int_{Y_0} \Phi \, d\mu$$

(37)

and equations (34),(35) and (36) that $\text{rv}(\mu)$ is a convex combination of the points $w_1, \ldots, w_k, v_\infty, \omega_0$ all of which belong to $\mathcal{G}$. By using that $w_k = \text{rv}(\mu)$ is an extreme point of $\mathcal{G}$ we may conclude that this convex combination must coincide with $w_k$ itself. Hence $p_1 = \cdots = p_{k-1} = 0$, $\mu(Y_0) = \mu(Y_{gk}) = 0$ and $p = p_1(1) = p_2(1) = \cdots = p_k(1) = p_k = \frac{1}{k+1}$. This shows that $\mu$ satisfies the assumptions of Proposition 3. Thus, by Proposition 3 we have $\mu = \mu_{X_k}$ which completes the proof of the theorem. \qed
A SHIFT MAP WITH A DISCONTINUOUS ENTROPY FUNCTION

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