Parity-odd anomalies and correlation functions on conical defects

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Abstract.

We analyse parity-odd (“P-type”) surface anomalies (“Graham-Witten anomalies”) of energy-momentum correlators in conformal field theories defined in $d$-dimensional spacetime supplemented with a conical defect, with an emphasis on $d = 4$ and $d = 3$ cases. In $d = 4$ we show that the trace anomaly will receive such surface contribution if the bulk trace anomaly contains P-type anomaly given by Pontryagin (pseudo)tensor, and as a consequence 2-point correlation function of energy-momentum tensor in flat spacetime will be nonvanishing as it receives corresponding surface contributions. In the process, we construct the most general P-type surface trace anomaly on singular 2-dimensional surface in 4-dimensional spacetime by performing consistency analysis. We show that there are two independent terms, one is the outer curvature (pseudo)scalar and the other is quadratic in the traceless part of the second fundamental tensor. For the special case of conical singularity we calculate the coefficient of the first term. Though we were unable to directly calculate the coefficient of the second term from methods used in this paper, we present an argument in favour of its vanishing. In $d = 3$ dimensions we calculate a surface contribution to the expectation value of an energy-momentum tensor in flat spacetime with conical singularity and show that it is also nonvanishing.
\section{Introduction}

When defined on a curved spacetime conformal field theories (CFT) may develop quantum anomalies. A well-known example are the trace (also called Weyl) anomalies in $d = 2k$, i.e. in even-dimensional spacetimes \cite{1,2,3,4}:

\begin{equation}
\langle T(x) \rangle_g = A(x), \quad T(x) \equiv T_{\mu \nu}^g(x)
\end{equation}

where $T_{\mu \nu}$ is the energy-momentum tensor of the theory, and $A$ is the (local expression for) trace anomaly which is generally some combination of monomials of $k$-th order in Riemann tensor. The terms in the anomaly $A$ are local covariant expressions constructed from the background metric $g_{\mu \nu}$. Riemann tensor and covariant derivatives $A$ may come in three types: type A which is the $k$-th Euler invariant, type B which are contracted tensor products of Weyl tensor, and type P consisting of exterior products of the Riemann curvature two-forms. The types A and B are parity-even, while type P is parity-odd and, though allowed by consistency conditions, is usually neglected in the literature.

There has been a renewed interest in gravitational mechanisms of CP violation, see e.g., \cite{7,8,9,10,11,12,13,14,15}. This includes analysis of appearance of parity-odd type P anomalies \cite{3,4,16,17,18,19,20,21,22}. Indeed, in \cite{20,21} an old result of \cite{22} was rediscovered and it was shown that in 4-dimensional quantum field theories with chiral fermions in which the numbers of left and right chiralities are not the same, type P anomalies are indeed present. This paper aims to fill some gaps in understanding of parity-odd sector of gravity-induced anomalies.

Now, accepting a standard definition for correlation functions of energy momentum tensor in flat space, an immediate consequence of (1) is that nonvanishing trace anomaly induces terms in correlation functions

\begin{equation}
\langle T_{\mu_1 \nu_1}(x_1) \cdots T_{\mu_n \nu_n}(x_n) T(x) \rangle = (-1)^n \frac{2}{\sqrt{g(x_1)}} \cdots \frac{2}{\sqrt{g(x_n)}} \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} \cdots \frac{\delta}{\delta g^{\mu_n \nu_n}(x_n)} A(x) \bigg|_{g_{\mu \nu} = \delta_{\mu \nu}}
\end{equation}

with $n \geq k$. However, correlation functions for $n < k$ vanish in the regular flat spacetime.

It is of interest to study non-regular spacetimes containing singular submanifolds. Orbifolds and branes provide important examples. In some instances, possibly after analytical extensions, these singular surfaces are equivalent to conical defects. One notable example is provided by the replica method \cite{23} for calculation of entanglement entropy (for reviews see \cite{24,25}), in which one effectively (by analytical continuation) introduces conical singularity on the entangling surface (which is codimension 2 submanifold) with a deficit parameter $\alpha$ ($\alpha = 1$ means no defect) and calculates a linear term in expansion in $(1 - \alpha)$ which carries relevant information.

It was observed in \cite{26} that when one introduces a conical defect on a codimension 2 surface $\Sigma$ in otherwise flat Euclidean spacetime, in general one should expect that correlation functions \cite{27} for $n = k - 1$ which were zero in flat case now receive nonvanishing contributions localized on $\Sigma$, so called surface anomalies.\footnote{Our conventions are as follows. Riemann tensor is $R^{\alpha \mu \nu \beta} = \partial_\nu \Gamma^\alpha_{\mu \beta} - \partial_\beta \Gamma^\alpha_{\mu \nu} + \Gamma^\alpha_{\nu \gamma} \Gamma^\gamma_{\mu \beta}$ and Ricci tensor is $R_{\mu \nu} = R^{\alpha \mu \nu \beta}$. The energy-momentum tensor is defined by $T_{\mu \nu} = 2g^{-1/2} \delta S / \delta g^{\mu \nu}$, where $S$ is the action, and so $\langle T_{\mu \nu} \rangle = 2g^{-1/2} \delta W / \delta g^{\mu \nu}$ where the functional $W[g]$ is $W[g] = -\ln Z[g] = -\ln \int D\phi \exp(-S)$. All calculations in this paper are performed in the Euclidean time.} One can infer this by using two methods: by straightforwardly using Eqs. (1) and (2) (Method 1), or by using a more general formula proposed in \cite{28} which connects the $n$-point correlation functions in a regular spacetime with $(n - 1)$-point correlation functions in spacetime with conical singularity (Method 2). In view of Method 1, these surface contributions follow from the fact that trace anomaly, beside the bulk contribution, now contains also contributions localized on the $(d - 2)$-dimensional surface $\Sigma$. Now, Method 1 uses the assumption that the trace anomaly in a spacetime with the conical defect keeps the same geometric form as in the regular spacetime, and the surface anomalies are then calculated in a geometric fashion. If the result for correlation function with $n = k - 1$ obtained by Method 2 exactly matches the result obtained by Method 1, this confirms the assumption of Method 1, at least in the linearised level in $(1 - \alpha)$ expansion around flat metric. The results offer interesting information, and the lowering of the rank of nonvanishing correlation functions may have some practical

\begin{equation}
\langle A(x) \rangle_g = \text{const}
\end{equation}

\begin{equation}
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advantages. Indeed, in [29] this program was successfully applied to parity-even sector, containing type A and B anomalies, and corresponding contributions to correlation functions of the type (2) with $n = k - 1$ were calculated.

In this paper we extend the analysis of [29] to parity-odd sector containing type P trace anomaly. We analyse in detail the case of $d = 4$ dimensions and show that the results for parity-odd part of correlation function (2) with $n = 1$ obtained by Methods 1 and 2 exactly match in linear order in $(1 - \alpha)$. In addition, we perform a complete general cohomological analysis of the parity-odd surface trace anomalies and show that there are only two linearly independent nontrivial terms – one is given by the extrinsic curvature (pseudo)scalar on $\Sigma$, while the other is quadratic in the second fundamental tensor on $\Sigma$ and as such cannot contribute to the mentioned correlation function with $n = 1$. For a particular case of surface trace anomaly on a conical defect, by using the result of Method 2 we were able to fix the coefficient of the first term. The value of the second coefficient, however, is left undetermined, though we presented an argument in favour of its vanishing. At the end we briefly discuss higher-dimensional case $k > 2$.

Finally, we study parity-odd contributions to correlation functions (2) in flat odd-dimensional spacetimes with conical singularity induced by the presence of gravitational Chern-Simons Lagrangian terms. These contributions are not connected with the trace anomaly (as there are no such in odd-dimensional spacetimes), though they may be generated by quantum loop contributions in QFT’s. For $d = 3$ case we perform calculations for $n = 0$ correlation function with both methods and show that they give the same result.

2 Conical defects

We assume that in otherwise regular $d$-dimensional spacetime with metric $g_{\mu \nu}$ a conical defect is introduced in a standard fashion such that there is a $(d - 2)$-dimensional singular surface $\Sigma$. Following [29] we assume that there are coordinates $x^\mu$, $\mu = 1, \ldots, d$ in which the conical defect is described by having an angle deficit in $x_1$-$x_2$ plane equal to $2\pi (1 - \alpha)$, with “the tip of the cone” being $(d - 2)$-dimensional surface $\Sigma$ defined by $x_1 = x_2 = 0$. We shall be interested only in lowest-order correction in expansion over $(1 - \alpha)$. In [30] it was argued that the sole effect of an introduction of a conical singularity in otherwise regular spacetime on contracted products of Riemann tensors, such as those appearing in the trace anomaly $\mathcal{A}(x)$ in (1), can be taken into account by using

$$ (R^{\mu \nu \rho \sigma})_{\Sigma} = R^{\mu \nu \rho \sigma} + 2\pi (1 - \alpha) \left( n^\mu_{(1)} n^\nu_{(2)} - n^\mu_{(2)} n^\nu_{(1)} \right) \delta_\Sigma + O((1 - \alpha)^2) $$  (3)

where the quantities on the right hand side refer to a regular spacetime without conical singularity. Here $\delta_\Sigma$ is a covariant $\delta$-function with a support on the conical singularity plane $\Sigma$, while

$$ n^{\mu \nu} \equiv \sum_{a=1}^{2} n^{\mu}_{(a)} n^{\nu}_{(a)}. $$  (4)

where $n^{\mu}_{(a)}$, $a = 1, 2$, are two orthonormal vectors orthogonal to $\Sigma$. It was later shown in [31,32] that this is a valid procedure only when the second fundamental tensor (extrinsic curvatures) of $\Sigma$ is vanishing, and generally there are extra terms not encoded by (3). However, these extra terms are of too high power in $\partial_\mu g_{\nu \rho}$ to be able to contribute to the flat space correlation functions we are interested in. For example, these extra contributions to (3) are quadratic in the second fundamental form, which means quadratic in $\partial_\mu g_{\nu \rho}$ and we shall see below that such terms cannot contribute to our results. In conclusion, for the purposes of this paper we are allowed to use (3). In Sec. 3.3 we shall undertake a brief excursion into curved backgrounds and analyse the form of this extra terms relevant to type P trace anomaly in $d = 4$ in more general settings.

More precisely, one should refer to integrals like integrated trace anomaly $\mathcal{A}_\omega \equiv \int d^d x \sqrt{g} \omega(x) \mathcal{A}(x)$. After using (3) to evaluate $\mathcal{A}(x)$ it is assumed that integral is performed over the regular spacetime, i.e., with $\alpha = 1$. 

3
3 Conical trace anomaly in $d = 4$

3.1 Correlation functions in flat spacetime: Method 1

When defined in curved 4-dimensional spacetime CFT’s develop trace anomaly $\mathcal{A}$ where

$$\mathcal{A} = -\frac{a}{64} E_2 + \frac{c}{64} (W_{\mu\nu\rho\sigma})^2 + p P_2$$

where $a$, $c$ and $p$ are constants depending on CFT in question, $E_2$ is the second Euler tensor (Gauss-Bonnet tensor), $W_{\mu\nu\rho\sigma}$ is the Weyl tensor, and $P_2$ is the Pontryagin (pseudo)tensor

$$P_2 = \frac{1}{2\sqrt{g}} \varepsilon^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu} R^{\alpha\beta}_{\rho\sigma}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol (equal 0 or $\pm 1$). First two terms in (5) correspond to type A and type B anomalies, respectively, and are parity-even, while the third term is parity-odd type $P$ anomaly, which is the focus of our study here.

The starting assumption of Method 1 is that expression (5) remains valid when the conical defect is introduced in the spacetime as described in Sec. 2. The fact that the same results will be obtained by different method shows that this assumption is valid at least in linear order in $h_{\mu\nu}$.

Using (3) in (2) one gets that in a flat regular spacetime correlation functions $T(x') \langle T(x) \rangle_c$ in (1) become

$$\langle T(x') \rangle_c = -\frac{a\pi}{4} (1 - \alpha) \left( \gamma_{\mu\nu} - \gamma_{\mu\nu} \gamma^{\alpha\beta} \delta_{\Sigma} \partial_{\alpha} \partial_{\beta} \delta^{(4)}(x - x') \right)$$

where $\gamma_{\mu\nu} = \delta_{\mu\nu} - n_{\mu} n_{\nu}$ is the induced metric on $\Sigma$.

Here we calculate the parity-odd contribution to the same correlation function. Assuming that expression for anomaly in (5) applies also in the presence of a conical singularity (which is denoted by an index $c$), from (2) follows

$$\langle (T(x') \langle T(x) \rangle_c)_{\text{p-odd}} \rangle = -2p \frac{\delta P_2(x') \langle C \rangle}{\delta g^{\mu\nu}(x)} \bigg|_{g_{\mu\nu} = \delta_{\mu\nu}}$$

By using (3), (6) and the fact that in the flat space $R_{\mu\nu\rho\sigma} = 0$, we obtain

$$\langle (T_{\mu\nu}(x) T(x')) \rangle_{c_{\text{p-odd}}} = -4\pi p (1 - \alpha) \delta_{\Sigma} \left( n^\alpha n^\beta - n^\nu n^\rho \right) \varepsilon^{\gamma\kappa\rho\sigma} \frac{\delta R_{\alpha\beta\gamma\kappa}(x')}{\delta g^{\mu\nu}(x)} \bigg|_{g_{\mu\nu} = \delta_{\mu\nu}} + O((1 - \alpha)^2)$$

In our conventions one has

$$\frac{\delta R_{\alpha\beta\gamma\kappa}(x')}{\delta g^{\mu\nu}(x)} \bigg|_{g_{\mu\nu} = \delta_{\mu\nu}} = -\frac{1}{4} (\delta_{\alpha\mu} \delta_{\nu\nu} \delta_{\gamma\beta} + \delta_{\beta\mu} \delta_{\gamma\nu} \delta_{\alpha\nu} ) \delta^{(4)}(x - x') + (\mu \leftrightarrow \nu) - (\gamma \leftrightarrow \kappa)$$

Using this in (10) we obtain

$$\langle (T_{\mu\nu}(x) T(x')) \rangle_{c_{\text{p-odd}}} = -8\pi (1 - \alpha) p \varepsilon_{\mu\nu\rho\sigma} n^\rho n^\sigma \delta_{\Sigma} \partial_\gamma \partial_\delta \delta^{(4)}(x - x') + (\mu \leftrightarrow \nu) + O((1 - \alpha)^2)$$

Now we choose the coordinates such that $\Sigma$ is defined by $x_1 = x_2 = 0$. Then, $\delta_{\Sigma} = \delta(x_1) \delta(x_2)$ and we can take $n^\mu = (1, 0, 0, 0)$ and $n^{\nu(2)} = (0, 1, 0, 0)$. Using all this (11) becomes

$$\langle (T_{\mu\nu}(x) T(x')) \rangle_{c_{\text{p-odd}}} = -8\pi (1 - \alpha) p \varepsilon_{\mu\nu} \delta(x_1) \delta(x_2) \partial_\mu \partial_\nu \delta^{(4)}(x - x') + (\mu \leftrightarrow \nu) + O((1 - \alpha)^2)$$
where \( \hat{a} = 1, 2 \) denotes directions normal to \( \Sigma \), while \( a = 3, 4 \) denotes directions tangential to \( \Sigma \), and \( \hat{\varepsilon} \) and \( \varepsilon \) are 2-dimensional Levi-Civita symbols living on normal and tangential space, respectively (binormal and volume form on \( \Sigma \), respectively)

\[
\hat{\varepsilon}_{12} = 1 \, , \quad \varepsilon_{\mu a} = 0 \, , \quad \hat{\varepsilon}_{34} = 1 \, , \quad \hat{\varepsilon}_{\rho \sigma} = 0 \tag{13}
\]

From the expression (12) it is obvious that the correlation function is nonvanishing only if one of the indices \( \mu \) or \( \nu \) is normal while the other one is tangential to \( \Sigma \). One of the consequences is that the trace of (12) vanishes

\[
\langle \langle T(x) T(x') \rangle \rangle_{\text{P-odd}} = 0 \tag{14}
\]

As it was shown in [29] that the same is true for the type B anomaly, the only contribution to the trace

\[
\langle \langle T(x) T(x') \rangle \rangle_{\text{A-type}} = -\frac{a \pi}{4} \{1 - (1 - \alpha) \delta(x_1) \delta(x_2) \partial_\sigma \partial^\sigma \delta^{(4)}(x - x') \}
\]

where again \( a = 3, 4 \) are tangential coordinates on \( \Sigma \) and (12) was used in the second equality.

### 3.2 Correlation functions in flat spacetime: Method 2

As argued in [28], there is a correspondence between the correlation functions on flat space with and without the conical defect given by

\[
\mathcal{P} \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \rangle_{\text{C}_a} = \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \rangle_{K_0} \tag{16}
\]

where

\[
\mathcal{P} = -\lim_{\alpha \to 1} \frac{\partial}{\partial \alpha}
\]

and operator \( K_0 \) is

\[
K_0 = -2\pi \int d^{d-2}y \int_0^\infty dz_1 z_1 T_{22}(z_1, z_2 = 0, \bar{y}) \tag{17}
\]

where directions 1 and 2 are normal while directions 3, \ldots, \( d \) are tangential to the conical defect surface \( \Sigma \), and \( \bar{y} = (z_3, \ldots, z_d) \).

Applying this to the particular case of parity-odd contribution to correlation functions of energy-momentum tensor in \( d = 4 \) enables us to check the result from the last subsection by independent method. From (16) follows

\[
\mathcal{P} \langle \langle T(x) T_{\mu \nu}(y) \rangle \rangle_{\text{P-odd}} = -2\pi \int dz_1 \int dz_4 \int_0^\infty dz_3 \langle \langle T(x) T_{\mu \nu}(y) T_{22}(z_1, z_2 = 0, z_3, z_4) \rangle \rangle \tag{18}
\]

Now, from (15) it can be shown that in the regular flat spacetime one has

\[
\langle \langle T(x) T_{\mu \nu}(y) T_{\rho \sigma}(z) \rangle \rangle_{\text{P-odd}} = 2p \varepsilon_{\mu \nu \rho \sigma} (\partial_\sigma \partial_\nu - \delta_\sigma_\nu \partial^2) [\partial^\rho \delta^{(4)}(x - y) \delta^{(4)}(x - z)] + (\rho \leftrightarrow \sigma) + (\mu \leftrightarrow \nu) \tag{19}
\]

where the differential operator inside round brackets is explicitly given by

\[
\partial_\sigma \partial_\nu - \delta_\sigma_\nu \partial^2 = \frac{\partial^2}{\partial y_\sigma \partial z_\nu} - \delta_\sigma_\nu \frac{\partial}{\partial y_\kappa} \frac{\partial}{\partial z_\kappa}
\]

Plugging (20) into (18) we obtain

\[
\mathcal{P} \langle \langle T(x) T_{\mu \nu}(y) \rangle \rangle_{\text{P-odd}} = -8\pi p \varepsilon_{\mu \nu \rho \sigma} \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_4 \int_0^\infty dz_1 \times \left( \partial_2 \partial_\nu - \delta_2_\nu \partial_\sigma \partial_\sigma \right) \left[ \partial_\rho \delta^{(4)}(x - y) \partial_\sigma \delta^{(4)}(x - z) \right] + (\mu \leftrightarrow \nu) \tag{21}
\]
anomalies which in the case of trace anomaly (5) is defined by calculate the conical surface trace anomaly for general spacetime metric. For the moment we shall ignore the possibility of these extra terms and apply the logic of Method 1 to anomalies corresponding to 2-dimensional singular surfaces in 4-dimensional spacetime. conical surface trace anomaly, let us find out what is the most general form of parity-odd surface trace which makes a presence. Before coming back to the question of exact expression for the function in the flat spacetime analysed in this paper obtained by combining Eqs. (3) and (5), modulo terms which do not affect particular 2-point correlation function in the flat spacetime analysed in this paper. 4.1 it follows that integration over gives in the lowest order in (1 − α) the following result (|T(x) Tµν(y)|)P−odd = −8π p∂2∂x2δ(4)(x − y) + (µ ↔ ν) (25) where we used 2-dimensional Levi-Civita symbols defined in (13). As expected, the result is the same as the corresponding one obtained by the Method 1 in Sec. 3.1 (see Eq. (12)).

3.3 Type P surface trace anomalies in d = 4

3.3.1 Type P conical surface trace anomaly: take 1

In Sec. 3.1 we have assumed that the trace anomaly in a curved spacetime with the conical defect can be obtained by combining Eqs. and , modulo terms which do not affect particular 2-point correlation function in the flat spacetime analysed in this paper. Let us now analyse this issue more carefully. For the moment we shall ignore the possibility of these extra terms and apply the logic of Method 1 to calculate the conical surface trace anomaly for general spacetime metric.

Now, in the analyses of anomalies, instead of local anomalies it is more fruitful to consider integrated anomalies which in the case of trace anomaly is defined by

\[ A_\omega = \int d^4x \sqrt{g(x)} \omega(x)A(x) \] (26)

where A is defined in and ω is a parameter for infinitesimal Weyl transformation on which metric transforms as

\[ \delta_\omega g_{\mu\nu}(x) = \omega(x) g_{\mu\nu}(x) \] (27)

As a first take, let us calculate the integrated conical surface trace anomaly in linear order in (1 − α) by using in type P part of . We obtain

\[ A_\omega^{(P)} = 4\pi p(1 - \alpha) \int d^4x \sqrt{\gamma} \epsilon_{\mu\nu\lambda\rho} \gamma^{\mu\sigma} \gamma^{\nu\tau} n^{\lambda\alpha} n^{\rho\beta} R_{\sigma\tau\alpha\beta} \delta_\Sigma \]

\[ = 4\pi p(1 - \alpha) \int d^2y \sqrt{\gamma} \epsilon_{\mu\nu\lambda\rho} \gamma^{\mu\sigma} \gamma^{\nu\tau} n^{\lambda\alpha} n^{\rho\beta} R_{\sigma\tau\alpha\beta} \] (28)

where \( \epsilon_{\mu\nu\rho\sigma} \) is the Levi-Civita tensor and \( \gamma_{\mu\nu} \) is the induced metric (first fundamental tensor) of \( \Sigma \). More details on mathematical background and terminology can be found in, e.g., [33]. However, experience with parity-even sector shows that extra terms we mentioned above, which are not encoded in the rule and which contain second fundamental tensor (extrinsic curvatures), cannot be a priori dismissed. In fact, both for type A and B anomalies it was explicitly shown that there is an extra term which makes a presence. Before coming back to the question of exact expression for the conical surface trace anomaly, let us find out what is the most general form of parity-odd surface trace anomalies corresponding to 2-dimensional singular surfaces in 4-dimensional spacetime.

\footnote{An explicit example of such terms is given in the second term of Eq. [52], see also note 1 in Section 3.3.3.}
3.3.2 Parity-odd surface trace anomaly: general cohomology analysis

Viable candidates for the trace anomaly must satisfy Wess-Zumino consistency conditions. These consistency conditions combine trace (Weyl) and diffeomorphism anomalies in pairs, \( A_\omega \) and \( A_\xi \), and read

\[
(\delta_\omega + \delta_\xi)(A_\omega + A_\xi) = 0
\]

which splits into

\[
\delta_\omega A_\omega = 0, \quad \delta_\omega A_\xi + \delta_\xi A_\omega = 0, \quad \delta_\xi A_\xi = 0 \tag{29}
\]

Here \( \xi_\mu(x) \) parametrizes infinitesimal difft-transformations which act on the metric as

\[
\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{30}
\]

where \( \nabla_\mu \) is the covariant derivative. The transformation parameters \( \omega(x) \) and \( \xi(x) \) are anticommuting Grassmann fields. If the pair \( A_\omega \) and \( A_\xi \) is such that there exists a local term \( C \) satisfying

\[
A_\omega = \delta_\omega C \quad \text{and} \quad A_\xi = \delta_\xi C \tag{31}
\]

then such anomaly pair is considered to be trivial as it can be cancelled by adding the local term \( C \) to the quantum action. In this way the search for possible anomalies has been turned into solving cohomology problem.

So, for (28) to be a viable candidate for the surface trace anomaly, we must check that it satisfies consistency conditions (29). As (28) is manifestly covariant it is obvious that it satisfies \( \delta_\xi A_\omega^{(P)} = 0 \) so trace and diff- anomalies separate. We can then take \( A_\xi^{(P)} = 0 \) which means that cohomology conditions (29) turn into

\[
\delta_\omega A_\omega^{(P)} = 0 \tag{32}
\]

To check this we need to know how objects appearing in (28) transform under Weyl rescaling (27). One obtains:

\[
\delta_\omega \gamma_{\mu\nu} = \omega \gamma_{\mu\nu}, \quad \delta_\omega n_{\mu\nu} = \omega n_{\mu\nu}, \quad \delta_\omega \delta_\Sigma = -\omega \delta_\Sigma, \quad \delta_\omega \sqrt{g} = 2\omega \sqrt{g}
\]

Using this it is straightforward to show not only that (28) satisfies (32), but that its density satisfies a stronger condition of Weyl-invariance

\[
\delta_\omega \left( \sqrt{g(x)} A^{(P)}(x) \right) = 0 \tag{33}
\]

We see that (28) is indeed a viable candidate for the parity-odd surface trace anomaly. The question now is: Does it give the most generic expression? As we now show, the answer is no.

We shall prove this by finding the most general expression for parity-odd surface anomalies by performing Wess-Zumino consistency analysis. The idea is to complete the analysis from Sec. 3.2 of [34], where parity-even sector was discussed. We have found two more parity-odd terms satisfying the consistency conditions (29) (and Weyl invariance of the corresponding densities):

\[
\int_\Sigma \sqrt{\Omega} \Gamma, \quad \int_\Sigma \omega e^{\mu\nu\rho\sigma} K_{\mu\tau\rho} K_{\nu\tau\sigma} \tag{34}
\]

where \( K_{\mu\nu\alpha} \equiv \gamma_\mu^\rho \gamma_\nu^\sigma \nabla_\rho \gamma_\sigma\alpha \) is the second fundamental tensor, while \( \Omega \) is the outer curvature (pseudo) scalar on \( \Sigma \) obtained from the outer curvature tensor \( \Omega_{\mu\nu\alpha\beta} \) through

\[
\Omega \equiv \frac{1}{2} e^{\mu\nu\alpha\beta} \Omega_{\mu\nu\alpha\beta} \tag{35}
\]

Integrated anomalies are variations of a quantum effective action \( W[g] \) on corresponding infinitesimal transformations, in our case \( A_\omega = \delta_\omega W[g] \) and \( A_\xi = \delta_\xi W[g] \).

*Note that \( \Omega_{\mu\nu\alpha\beta} \) has only one independent component, so \( \Omega \) contains the complete information on the outer curvature tensor.
By using Ricci equation the outer curvature tensor can be expressed as

\[ \Omega_{\mu \nu \alpha \beta} = \gamma_{\mu}^{\rho} \gamma_{\nu}^{\sigma} n_{\alpha}^{\delta} n_{\beta}^{\epsilon} R_{\rho \delta \sigma \epsilon} + K_{\mu \rho \beta} K_{\nu}^{\rho} - K_{\nu \rho \beta} K_{\mu}^{\rho} \]  

(36)

Consistency of terms in (34) follows from the following transformation properties under Weyl rescaling

\[ \delta \omega \Omega = -\omega \Omega, \quad \delta \omega K_{\mu \nu} = -\frac{1}{2} \gamma_{\mu \nu} n^{\alpha} n^{\beta} \partial_{\alpha} \omega \]  

(37)

Both of the terms in (34) are cohomologically nontrivial, as is (28). To our knowledge this exhausts the possibilities, i.e., there are no other linearly independent terms of the required kind\(^7\). However, the terms (28) and (34) are not linearly independent as the Ricci equation (36) implies

\[ \epsilon_{\mu \nu \rho \sigma} \gamma_{\lambda}^{\alpha} n_{\alpha}^{\delta} n_{\nu}^{\beta} R_{\rho \sigma \alpha \lambda} = 2 \Omega + 2 \epsilon_{\mu \nu \rho \sigma} K_{\mu \tau \rho} K_{\nu}^{\tau} \]  

(38)

It follows that by purely using cohomology analysis we infer that the surface type P integrated trace anomaly must have the following form

\[ A_{\omega}^{(P)} = 4\pi p (1 - \alpha) \int_{\Sigma} \omega \epsilon_{\mu \nu \rho \sigma} (k_1 \gamma_{\rho}^{\lambda} \gamma_{\sigma}^{\epsilon} n_{\mu}^{\alpha} n_{\nu}^{\beta} R_{\sigma \tau \alpha \lambda} + 2(k_2 - k_1) K_{\mu \tau \rho} K_{\nu}^{\tau}) \] 

\[ = 8\pi p (1 - \alpha) \int_{\Sigma} \omega [k_1 \Omega + k_2 \epsilon_{\mu \nu \rho \sigma} K_{\mu \tau \rho} K_{\nu}^{\tau}] \]  

(39)

where \( p \) is the constant appearing in (5) which depends on particulars of a quantum field theory in question, while \( k_1 \) and \( k_2 \) are two numbers left undetermined. We emphasize that this result is generic, valid for all parity-odd surface trace anomalies on singular 2-dimensional surfaces in 4-dimensional spacetime.

3.3.3 Type P conical surface trace anomaly: take 2

Now we can use the results from the previous two subsections to constrain the values of \( k_1 \) when singular surface is of conical defect type. For this purpose we note the following:

1. Because the second term in (34) does not contain linear terms in \( h_{\mu \nu} \) when metric is expanded around flat space, \( g_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu} \), it follows that this term cannot contribute to 2-point correlation functions calculated in the previous two subsections. In other words, our calculations of correlation functions are blind to \( k_2 \) and cannot be used to fix its value.

2. Because of the agreement between the results for these correlation functions obtained with Method 1 and 2, we can conclude that \( k_1 = 1 \), because with this choice (together with \( k_2 = 0 \)) (39) is equal to (28), which was used in Method 1.

3. Though we cannot fix the value of \( k_2 \), in principle we could use again Method 2 to claim that it must have a universal value, equal for all QFT’s. This conclusion would follow immediately from applying Eq. (16) to the calculation of higher-point \((n > 1)\) correlation functions of the type (2) in a flat spacetime with conical singularity, because it is obvious that no model dependent parameters can creep in aside overall factor \( p \).

4. It is interesting to note that the outer curvature scalar \( \Omega \) can be written as the total 2-dimensional gradient, and is in some sense an outer analogue of Euler term in \( d = 2 \) (which is an intrinsic Ricci scalar)\(^8\). As this “topological” nature of \( \Omega \) on \( \Sigma \) is compatible with the topological nature of

\(^7\)Note that for the surface anomalies, given as integrals over two-dimensional surface \( \Sigma \), one has to take into account expressions containing covariant derivatives of \( \omega \). In this case covariant derivatives cannot be moved off \( \omega \) by partial integrations, as is possible for bulk anomalies. However, we have not found parity-odd terms of this kind satisfying consistency conditions.

\(^8\)Let us note that Penrose showed that a spinor approach leads naturally to the construction of a single complex curvature invariant on 2-dimensional submanifolds whose real part is Euler term (intrinsic Ricci scalar) and imaginary part is the outer curvature scalar \( \Omega \).
Pontryagin term $P_2$ in the bulk, it is tempting to speculate that the value of $k_2$ in (39) is $k_2 = 0$, as in this case surface type $P$ local trace anomaly in $d = 4$ would be given purely by $\Omega$. We leave this question to our future research.

To summarize, we have shown that the surface type $P$ trace anomaly on conical defect in $d = 4$ is of the form

$$A_{\alpha}^{(P)} = 8\pi p (1 - \alpha) \int_{\Sigma} \omega \left[ \Omega + k_2 \epsilon^{\mu\nu\sigma\rho} K_{\mu\tau\rho} K_{\nu\tau\sigma} \right]$$

(40)

where $p$ is the model dependent constant appearing also in the bulk trace anomaly, while $k_2$ is some universal constant which we conjecture to be vanishing.

4 Conical singularity and correlation functions in $d = 3$

4.1 Method 1

When 3-dimensional QFT’s are defined in a curved spacetime, expectation value of the energy-momentum tensor may develop parity-odd contribution of the form

$$\langle T_{\mu\nu}(x) \rangle_{P-\text{odd}} = \frac{iw}{48\pi} \epsilon_{\alpha\beta(\mu} \nabla^\beta R^{\alpha)}_{\nu} = -\frac{iw}{48\pi} C_{\mu\nu}$$

(41)

where $C_{\mu\nu}$ is known as Cotton-York tensor. As the integer part of the coefficient $w$ can be removed by adding to the classical action a local counterterm, which is well-known gravitational Chern-Simons term, it is sometimes stated that $w$ is defined modulo 1 [17].

It is known that in regular spacetimes Cotton-York tensor is traceless and covariantly conserved

$$C_{\mu\mu} = 0, \quad \nabla_{\mu} C^{\mu\nu} = 0$$

(42)

so as a consequence (11) does not contribute to the trace anomaly, which is expected from the general theorem stating that there are no trace anomalies in CFT’s defined in odd-dimensional spacetimes.

Now we add into the spacetime a conical defect with deficit angle $2\pi (1 - \alpha)$ in the same manner as before. By assuming that (11) is valid also when conical defect is present, and using

$$(R_{\mu\nu})_{C_{\alpha}} = R_{\mu\nu} + 2\pi (1 - \alpha) n_{\mu\nu} \delta_{\Sigma}$$

(43)

which follows from (3), we obtain that for the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$

$$\langle (T_{\mu\nu}(x))_{C_{\alpha}} \rangle_{P-\text{odd}} = \frac{iw}{24} (1 - \alpha) \epsilon_{\alpha\beta(\mu} \partial^{\beta} \left( n_{\nu)} \delta_{\Sigma} \right) = \frac{iw}{48} (1 - \alpha) \hat{\epsilon}_{\nu\alpha\beta} \partial^{\beta} \left[ \delta(x_1) \delta(x_2) \right] + (\mu \leftrightarrow \nu)$$

(44)

where in the second equality we used (13). Again, this expression is nonvanishing only if one of the indices is in normal direction (1 or 2) while the other one is in tangential direction (3). It is easy to see that (14) is traceless and covariantly conserved. This means that, as expected, it will not contribute to the trace anomaly.

4.2 Method 2

For a CFT defined in a regular flat spacetime a consequence of (11) is

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle_{P-\text{odd}} = \frac{iw}{192\pi} \epsilon_{\mu\alpha\sigma} (\partial_{\nu} \partial_{\beta} - \delta_{\nu\beta} \partial^2) \partial^\sigma \delta^{(3)}(x - y) + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta)$$

(45)

9Arguments based on path integral quantization suggest that (i) coupling constant of the purely gravitational Chern-Simons Lagrangian term in all odd spacetime dimensions is imaginary in Euclidean regime, and (ii) the value of the coupling is quantized [11]. The parametrization used in (11) is such that the contribution to $w$ from the Lagrangian gravitational CS term must be an integer if the only restriction on the spacetime is that it is a spin manifold.
Now we can use the correspondence \(^{(16)}\) to independently calculate expectation value of the energy momentum tensor in a flat space with a conical singularity. In this way we obtain

\[
\mathcal{P} \langle T_{\mu\nu}(x) \rangle_{\mathcal{C}_n} = \langle T_{\mu\nu}(x) K_0 \rangle = -2\pi \int_{-\infty}^{\infty} dy_3 \int_{0}^{\infty} dy_1 y_1 \langle T_{\mu\nu}(x) T_{22}(y_1, y_2 = 0, y_3) \rangle \tag{46}
\]

By using \(^{(45)}\) we obtain

\[
\langle T_{\mu\nu}(x) K_0 \rangle_{\text{P-odd}} = \frac{i w}{48} \epsilon_{\mu\nu\alpha} (\partial_\alpha \partial_2 - \delta_{\alpha 2} \partial^2) \left[ \delta(x_2) \int_{0}^{\infty} dy_1 y_1 \partial^1 \delta(x_1 - y_1) \right] + (\mu \leftrightarrow \nu) \tag{47}
\]

Now, \(\sigma \neq 3\) and \(\nu \neq 3\) because integration over \(y_3\) would be vanishing. From this follows that \(\sigma = 1\) and one gets

\[
\langle T_{\mu\nu}(x) K_0 \rangle_{\text{P-odd}} = \frac{i w}{48} \delta_{\mu3} \delta_{\nu3} \partial^\alpha [\delta(x_1) \delta(x_2)] + (\mu \leftrightarrow \nu) = \frac{i w}{48} \epsilon_{\mu\alpha\beta} n_\alpha \partial^\beta \delta_\Sigma + (\mu \leftrightarrow \nu) \tag{48}
\]

where \(\alpha = 1, 2\) and \(\delta_\Sigma = \delta(x_1) \delta(x_2)\) in the particular Cartesian coordinates we use here. Plugging this into \(^{(10)}\), using \(^{(17)}\) and integrating over \(\alpha\) we obtain that in the leading order in \((1 - \alpha)\) the final result is

\[
\langle T_{\mu\nu}(x) \rangle_{\mathcal{C}_n} = \frac{i w}{48} (1 - \alpha) \epsilon_{\alpha\beta\mu} n_\mu \partial^\beta \delta_\Sigma + (\mu \leftrightarrow \nu) = \frac{i w}{24} (1 - \alpha) \epsilon_{\alpha\beta\mu} \partial^\beta \left( n_\mu \delta_\Sigma \right) \tag{49}
\]

We see that the final result is the same as the one obtained by Method 1, which is \(^{(44)}\).

5 Generalisations to \(d > 4\)

The analyses from the previous sections may be rather straightforwardly extended to \(d = 4k\) and \(d = 4k - 1\) dimensions, respectively. However, as formulae are rather cumbersome, we restrict ourselves here to some general comments.

The trace anomaly for CFTs in \(d = 4k\) dimensions may contain parity-odd P-type terms, which are generalizations of Pontryagin pseudoscalar, in the form of invariant polynomials of Riemann tensor. A difference from \(d = 4\) case is that for \(k > 1\) beside irreducible monomial \(\text{tr} (R^{2k})\) (where \(R\) is a Riemann 2-form and a product is the wedge-product) there are also several reducible monomials. For example, in \(d = 8\) beside irreducible term \(\text{tr} (R^4)\) there is one reducible term \((\text{tr} (R^2))^2\). If CFT is defined on a regular flat space, then correlation functions \(^{(2)}\) with \(n \geq 2k - 2\) are non-vanishing. If one introduces a conical defect then this is so for \(n \geq 2k - 2\). Again, explicit expressions in general case are rather cumbersome and we shall not write them here. They can be straightforwardly constructed by following procedures from Sec. \(^{3}\) with the result that the general properties are similar as in \(d = 4\).

In \(d = 4k - 1\) dimensions with \(k > 1\) expectation value of energy-momentum tensor may contain generalized Cotton tensors \(^{(9)}\). They are a consequence of the presence of gravitational Chern-Simons terms in the quantum effective action. For \(k > 1\) the number of such terms is larger then one, as reducible terms are also present. In fact, these gravitational Chern-Simons terms can be obtained from above mentioned invariant polynomials through transgressions. If CFT is defined on a regular flat space, then correlation functions \(^{(2)}\) with \(n \geq 2k - 1\) are non-vanishing. If one introduces a conical defect then this is so for \(n \geq 2k - 2\). Again, explicit expressions are somewhat cumbersome but again can be constructed by following procedures from Sec. \(^{4}\).
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References

[1] D. M. Capper and M. J. Duff, “Trace anomalies in dimensional regularization,” Nuovo Cim. A 23 (1974) 173.
[2] S. Deser, M. J. Duff and C. J. Isham, “Nonlocal Conformal Anomalies,” Nucl. Phys. B 111 (1976) 45.
[3] L. Bonora, P. Pasti and M. Bregola, “Weyl Cocycles,” Class. Quant. Grav. 3 (1986) 635.
[4] S. Deser and A. Schwimmer. “Geometric classification of conformal anomalies in arbitrary dimensions,” Phys. Lett. B 309 (1993) 279 [hep-th/9302047].
[5] Y. Nakayama, “CP-violating CFT and trace anomaly,” Nucl. Phys. B 859 (2012) 288 [arXiv:1201.3428 [hep-th]].
[6] Y. Nakayama, “Scale invariance vs conformal invariance,” Phys. Rept. 569 (2015) 1 [arXiv:1302.0884 [hep-th]].
[7] S. Alexander and N. Yunes, “Chern-Simons Modified General Relativity,” Phys. Rept. 480 (2009) 1 [arXiv:0907.2562 [hep-th]].
[8] L. Bonora, M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, “Gravitational Chern-Simons Lagrangians and black hole entropy,” JHEP 1107 (2011) 085 [arXiv:1104.2523 [hep-th]].
[9] L. Bonora, M. Cvitan, P. D. Prester, S. Pallua and I. Smolić, “Gravitational Chern-Simons Lagrangian terms and spherically symmetric spacetimes,” Class. Quant. Grav. 28 (2011) 195009 [arXiv:1105.4792 [hep-th]].
[10] L. Bonora, M. Cvitan, P. D. Prester, S. Pallua and I. Smolić, “Gravitational Chern-Simons terms and black hole entropy. Global aspects,” JHEP 1210 (2012) 077 [arXiv:1207.6969 [hep-th]].
[11] L. Bonora, M. Cvitan, P. D. Prester, S. Pallua and I. Smolić, “Stationary rotating black holes in theories with gravitational Chern-Simons Lagrangian term,” Phys. Rev. D 87 (2013) 024047 [arXiv:1210.4035 [hep-th]].
[12] L. Bonora, M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, “Symmetries and gravitational Chern-Simons Lagrangian terms,” Phys. Lett. B 725 (2013) 468 [arXiv:1305.0432 [hep-th]].
[13] S. Alexander, S. Cormack, A. Marcian and N. Yunes, “Gravitational-Wave Mediated Preheating,” Phys. Lett. B 743 (2015) 82 [arXiv:1405.4288 [gr-qc]].
[14] T. Azeyanagi, R. Loganayagam, G. S. Ng and M. J. Rodriguez, “Covariant Noether Charge for Higher Dimensional Chern-Simons Terms,” arXiv:1407.6364 [hep-th].
[15] S. Mauro and I. L. Shapiro, “Anomaly-induced effective action and Chern-Simons modification of general relativity,” arXiv:1412.5002 [gr-qc].
[16] S. N. Solodukhin, “Holographic description of gravitational anomalies,” JHEP 0607 (2006) 003 [hep-th/0512216].
[17] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski and N. Seiberg, “Comments on Chern-Simons Contact Terms in Three Dimensions,” JHEP 1209 (2012) 091 [arXiv:1206.5218 [hep-th]].

[18] N. Banerjee, S. Dutta, S. Jain, R. Loganayagam and T. Sharma, “Constraints on Anomalous Fluid in Arbitrary Dimensions,” JHEP 1303 (2013) 048 [arXiv:1206.6499 [hep-th]].

[19] R. Loganayagam, “Anomalies and the Helicity of the Thermal State,” JHEP 1311 (2013) 205 [arXiv:1211.3850 [hep-th]].

[20] L. Bonora, S. Giacchini and B. Lima de Souza, “Trace anomalies in chiral theories revisited,” JHEP 1407, 117 (2014) [arXiv:1403.2606 [hep-th]].

[21] L. Bonora, A. D. Pereira and B. L. de Souza, “Regularization of energy-momentum tensor correlators and parity-odd terms,” [arXiv:1503.03326 [hep-th]].

[22] S. M. Christensen and M. J. Duff, ”Axial and conformal anomalies for arbitrary spin in gravity and supergravity”, Phys. Lett. 76B (1978) 571.

[23] C. G. Callan, Jr. and F. Wilczek, “On geometric entropy,” Phys. Lett. B 333 (1994) 55 [hep-th/9401072].

[24] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 0608 (2006) 045 [hep-th/0605073].

[25] S. N. Solodukhin, “Entanglement entropy of black holes,” Living Rev. Rel. 14 (2011) 8 [arXiv:1104.3712 [hep-th]].

[26] S. N. Solodukhin, “The a-theorem and entanglement entropy,” [arXiv:1304.4411 [hep-th]].

[27] C. R. Graham and E. Witten, “Conformal anomaly of submanifold observables in AdS / CFT correspondence,” Nucl. Phys. B 546 (1999) 52 [hep-th/9901021].

[28] M. Smolkin and S. N. Solodukhin, “Correlation functions on conical defects,” Phys. Rev. D 91 (2015) 4, 044008 [arXiv:1406.2512 [hep-th]].

[29] S. N. Solodukhin, “Conformal a-charge, correlation functions and conical defects,” Phys. Lett. B 736 (2014) 283 [arXiv:1406.5368 [hep-th]].

[30] D. V. Fursaev and S. N. Solodukhin, “On the description of the Riemannian geometry in the presence of conical defects,” Phys. Rev. D 52 (1995) 2133 [hep-th/9501127].

[31] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, “Distributional Geometry of Squashed Cones,” Phys. Rev. D 88 (2013) 4, 044054 [arXiv:1306.4001 [hep-th]].

[32] S. N. Solodukhin, “Entanglement entropy, conformal invariance and extrinsic geometry,” Phys. Lett. B 665 (2008) 305 [arXiv:0802.3117 [hep-th]].

[33] B. Carter, “Essentials of classical brane dynamics,” Int. J. Theor. Phys. 40 (2001) 2099 [gr-qc/0012036].

[34] A. Schwimmer and S. Theisen, “Entanglement Entropy, Trace Anomalies and Holography,” Nucl. Phys. B 801 (2008) 1 [arXiv:0802.1017 [hep-th]].

[35] R. Penrose and W. Rindler, “Spinors and Space-time: Volume 1, Two-Spinor Calculus and Relativistic Fields,” Cambridge, Uk: Univ. Pr. (1984) (Cambridge Monographs On Mathematical Physics)

[36] E. Witten, “Three-Dimensional Gravity Revisited,” [arXiv:0706.3359 [hep-th]].