Chaotifying continuous-time systems by symmetry

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Abstract

In this letter we present a method of constructing dynamical systems with any preassigned number of equilibria by adding symmetry to another system with at least one equilibrium point. If the resulting system is chaotic, we call this procedure chaotification by symmetry since the resulting system is chaotic with many symmetrical equilibrium points.

Keywords: Chaotification, symmetry, equilibrium points, chaos, continuous-time systems.

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1 Introduction

There are several well-known situations involving computation of equilibria, such as Nash equilibria and market equilibria. For these two cases, the equilibrium is the result of individual agents acting on their own in a non-competitive but selfish manner. See [10] for more details. On the other hand, the existence of many equilibria in a system makes its dynamics more complex and shows some special structure. Examples include the well-known
multi-scroll attractors [1-2-4-5-6-7 and references therein], chaotic attractors with multiple-merged basins of attraction [3], scroll grid attractors [8], and 2n-wing and n × m-wing Lorenz-like attractors [12-13].

In [14] a methodology was presented for constructing some simple chaotic systems with any preassigned number of equilibria by adding symmetry to a simple 3-D quadratic system with only one stable equilibrium.

In this letter, we generalize the above method to any m-dimensional system by introducing a singular transformation to generate a system with an explicit formula and n equilibrium points. The importance of this result is that systems with many equilibria are generally more complex and display chaotic attractors with special structure. See for example [9].

2 Generating m-dimensional systems with n equilibrium points

In this section, we present our method to generate an m-dimensional system with exactly n equilibrium points. Consider the following m-dimensional system:

\[
\begin{align*}
    x'_1 &= f_1(x_1, x_2, \ldots, x_m) \\
    x'_2 &= f_2(x_1, x_2, \ldots, x_m) \\
    \vdots \\
    x'_m &= f_m(x_1, x_2, \ldots, x_m)
\end{align*}
\]

(1)

and consider the following coordinate transformation

\[
\begin{align*}
    u_1 &= \left(\sqrt{x_1^2 + x_m^2}\right)^{\frac{1}{n}} \cos \left(\frac{1}{n} \arccos \left(\frac{x_1}{\sqrt{x_1^2 + x_m^2}}\right)\right) \\
    u_j &= x_j, \text{ for } j = 2, 3, \ldots, m-1 \\
    u_m &= \left(\sqrt{x_1^2 + x_m^2}\right)^{\frac{1}{n}} \sin \left(\frac{1}{n} \arccos \left(\frac{x_1}{\sqrt{x_1^2 + x_m^2}}\right)\right)
\end{align*}
\]

(2)

where n is the number of desired equilibria. Transformation (2) can add a \( \{x_j, j = 2, 3, \ldots, m-1\} \)-axis rotation symmetry, \( \mathbb{R}_{(x_j, j = 2, 3, \ldots, m-1)}(\frac{2}{n}\pi) \), to the original system.

The first and the last relations of the transformation (2) are deduced from the relation \( x_1 + ix_m = (u_1 + i u_m)^n \) in the complex plane, where \( u_1 + i u_m \) is the unknown variable. Let \( x_1 + ix_m = r (\cos \theta + i \sin \theta) \), where \( r = \sqrt{x_1^2 + x_m^2} \), \( \cos \theta = \frac{x_1}{\sqrt{x_1^2 + x_m^2}} \), and \( \sin \theta = \frac{x_m}{\sqrt{x_1^2 + x_m^2}} \). Let \( u_1 + i u_m = s (\cos \alpha + i \sin \alpha) \), where

\[s = \left(\sqrt{x_1^2 + x_m^2}\right)^{\frac{1}{n}} \]
\[ s = \sqrt{u_1^2 + u_m^2}, \cos \alpha = \frac{u_1}{\sqrt{u_1^2 + u_m^2}}, \sin \alpha = \frac{u_m}{\sqrt{u_1^2 + u_m^2}}. \] By comparison, we have \( r = s^n \) and \( \alpha = \frac{\theta + 2k\pi}{n}, k = 0, 1, \ldots, n-1. \) Here we can choose \( \theta = \arccos \left( \frac{x_1}{\sqrt{x_1^2 + x_m^2}} \right) (k = 0). \) The other cases are similar, but are complex in their analysis. Finally, we can obtain the formulas for \( u_1 \) and \( u_m \) as in (2).

On the other hand, we have \( x_1 = s^n \cos n\alpha \) and \( x_m = s^n \sin n\alpha. \) Hence we have

\[
\begin{align*}
x_1 &= \left( \sqrt{u_1^2 + u_m^2} \right)^n \cos \left( n \arccos \left( \frac{u_1}{\sqrt{u_1^2 + u_m^2}} \right) \right) = \varphi_1 (u_1, u_m) \\
x_j &= u_j, \text{ for } j = 2, 3, \ldots, m - 1 \\
x_m &= \left( \sqrt{u_1^2 + u_m^2} \right)^n \sin \left( n \arccos \left( \frac{u_1}{\sqrt{u_1^2 + u_m^2}} \right) \right) = \varphi_2 (u_1, u_m)
\end{align*}
\]

By differentiating this equation with respect to \( t \), we have

\[
\begin{align*}
u_1' &= \left( \frac{\partial \varphi_1}{\partial u_1} \frac{\partial \varphi_1}{\partial u_m} - \frac{\partial \varphi_2}{\partial u_1} \frac{\partial \varphi_1}{\partial u_m} \right) f_1 - \frac{\partial \varphi_1}{\partial u_1} f_m - \frac{\partial \varphi_2}{\partial u_m} f_1 \\
u_j' &= x_j' = f_j (\varphi_1 (u_1, u_m), u_2, u_3, \ldots, \varphi_2 (u_1, u_m)), \text{ for } j = 2, 3, \ldots, m - 1 \\
u_m' &= \frac{\partial \varphi_2}{\partial u_1} f_m - \frac{\partial \varphi_2}{\partial u_m} f_1
\end{align*}
\]

where \( f_k = f_k (\varphi_1 (u_1, u_m), u_2, u_3, \ldots, \varphi_2 (u_1, u_m)) \) for \( k \in \{1, m\} \). Here we use the facts that \( (\varphi_1 (u_1, u_m))' = \frac{\partial \varphi_1}{\partial u_1} u_1' + \frac{\partial \varphi_1}{\partial u_m} u_m' \) and Transformation (2) is singular and not defined at the points \((0, x_2, x_3, \ldots, 0)\). Hence the two systems (1) and (4) are not globally but only locally topologically equivalent.

Equation (4) is well defined if and only if \( \frac{\partial \varphi_1}{\partial u_1} \neq 0 \) and \( \frac{\partial \varphi_1}{\partial u_3} - \frac{\partial \varphi_2}{\partial u_1} \neq 0. \) But these relations are not true for all variables \( u_1 \) and \( u_m \) since, for example, \( \frac{\partial \varphi_1 (u_1, u_m)}{\partial u_1} \) can vanish for the set of points satisfying \( \tan \left( n \arccos \frac{u_1}{\sqrt{u_1^2 + u_m^2}} \right) = \frac{-u_1}{u_m} \).

The equilibrium points of the new system (4) are the real solutions of the algebraic equations \( u_j' = 0 \) for all \( j = 1, 2, \ldots, m. \) Thus we have

\[
f_j (\varphi_1 (u_1, u_m), u_2, u_3, \ldots, \varphi_2 (u_1, u_m)) = 0
\]

for all \( j = 1, 2, \ldots, m. \) Assume that the original system (1) has at least one equilibrium point \((a_1, a_2, \ldots, a_m)\). Then let \((b_1, b_2, \ldots, b_m)\) be an equilibrium
point of the new system (4). From (3) we have

\[
\begin{align*}
    a_1 &= \left( \sqrt{b_1^2 + b_m^2} \right)^n \cos \left( n \arccos \left( \frac{b_1}{\sqrt{b_1^2 + b_m^2}} \right) \right) \\
    b_j &= a_j, \text{ for } j = 2, 3, \ldots, m - 1 \\
    a_m &= \left( \sqrt{b_1^2 + b_m^2} \right)^n \sin \left( n \arccos \left( \frac{b_1}{\sqrt{b_1^2 + b_m^2}} \right) \right)
\end{align*}
\]  

(6)

Firstly, from the first and the last equation of (6), we have

\[
\sqrt{b_1^2 + b_m^2} = \left( \sqrt{a_1^2 + a_m^2} \right)^{\frac{1}{n}}.
\]

Secondly, in order to obtain \( n \) values for \( b_1 \), we can assume that \( a_1 = 0 \), (but \( a_m \neq 0 \), otherwise the transformation is not defined) that is

\[
\cos \left( n \arccos \left( \frac{b_1}{\left( |a_m| \right)^{\frac{1}{n}}} \right) \right) = T_n \left( \frac{b_1}{\left( |a_m| \right)^{\frac{1}{n}}} \right) = 0,
\]

where \( T_n \) is the Chebyshev polynomial of the first kind (well defined here since \( \frac{b_1}{\sqrt{b_1^2 + b_m^2}} \in (-1, 1) \) which has \( n \) simple roots (multiplicity here is zero) of the form \( \rho_k = \frac{b_1^{(k)}}{\left( |a_m| \right)^{\frac{1}{n}}} = \cos \left( \frac{(2k-1)\pi}{2n} \right) \), for all \( k = 1, \ldots, n \), in the interval \((-1, 1)\). Hence \( b_1 \) has \( n \) different values of the form \( b_1^{(k)} = \left( |a_m| \right)^{\frac{1}{n}} \cos \left( \frac{(2k-1)\pi}{2n} \right) \), for all \( k = 1, \ldots, n \). The last equation of (6) and the fact that \( \theta = \arccos \left( \frac{x_1}{\sqrt{x_1^2 + x_m^2}} \right) = \arcsin \left( \frac{x_m}{\sqrt{x_1^2 + x_m^2}} \right) \)

implies that \( a_m = |a_m| \sin \left( n \arcsin \left( \frac{b_m}{\left( |a_m| \right)^{\frac{1}{n}}} \right) \right) \). Without loss of generality, we can assume that \( a_m > 0 \). Hence \( b_m = a_m^\frac{1}{n} \sin \frac{\pi}{2n} \). Thus the \( n \) equilibrium points of the new system (4) are of the form

\[
\begin{align*}
    b_1^{(k)} &= a_m^\frac{1}{n} \cos \left( \frac{(2k-1)\pi}{2n} \right), \text{ for } k = 1, \ldots, n \\
    b_j &= a_j, \text{ for } j = 2, 3, \ldots, m - 1 \\
    b_m &= a_m^\frac{1}{n} \sin \frac{\pi}{2n}
\end{align*}
\]  

(7)

The above procedure can generate \( m \)-dimensional systems with \( n \) known equilibrium points. From equations (7), we remark that if the original system (1) has \( q \) equilibrium points, then the new system (4) has \( nq \) equilibrium points. If the resulting system is chaotic, then we call this procedure *chaotification by symmetry* since the resulting system is chaotic with many symmetrical equilibrium points as shown in [14] for some elementary examples.
3 Conclusion

In this letter a method was presented for constructing dynamical systems with any preassigned number of equilibria by adding symmetry to another $m$-dimensional system with at least one equilibrium point. The importance of this result is that systems with many equilibria are generally more complex and display chaotic attractors with special structures as described in the current literature.

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