Means, quasi-scalar product and covariance of fuzzy numbers

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Abstract. In this paper, we discuss extreme properties of means of fuzzy numbers. The proposed new definition quasi-scalar product and covariance between fuzzy numbers, and their properties are considered. As an application of the introduced concepts, the problem of optimal linear approximation of a fuzzy number by a given system of fuzzy numbers is considered.

1. Extreme properties of means of fuzzy numbers
By a fuzzy number, we assume a fuzzy subset of the universal set of real numbers that has a compact support and a normal, convex, and semi-continuous membership function (see, for example, [1]).

We will use the interval representation of fuzzy numbers.

As known, the α-level intervals of a fuzzy number \( \tilde{z} \) with the membership function \( \mu_{\tilde{z}}(x) \) are defined by the relations

\[
Z_\alpha = \{ x | \mu_{\tilde{z}}(x) \geq \alpha \} \quad (\alpha \in (0, 1]), Z_0 = supp(A).
\]

According to the accepted assumptions, all α-levels of a fuzzy number are closed and bounded intervals of the real axis. Denote the left border of the interval by \( z^-(\alpha) \), and the right border by \( z^+(\alpha) \), i.e. \( Z_\alpha = [z^-(\alpha), z^+(\alpha)] \). Sometimes \( z^-\) and \( z^+ \) are called the left and right indices of a fuzzy number, respectively.

We will assume below that the indices of the fuzzy numbers under consideration are quadratically summable by \([0, 1]\). The set of such fuzzy numbers is denoted by \( J_2 \).

Let \( f : [0, 1] \to \mathbb{R} \) be a non-negative quadratically summable weight function satisfying the condition \( \int_0^1 f(\alpha)d\alpha = 1 \), and not equals 0 almost everywhere on \([0, 1]\).

As known, the weighted average value of a fuzzy number \( \tilde{z} \) can be determined using the interval representation in the following way

\[
m_f(\tilde{z}) = \frac{1}{2} \int_0^1 (z^-(\alpha) + z^+(\alpha)) f(\alpha)d\alpha.
\]  

Note that the mean (1) is linear.
Example 1. Consider a fuzzy triangular number ˜z characterized by a triple (a, b, c) with 
\( a < b < c \) defining the membership function
\[
\mu_{\tilde{z}}(x) = \begin{cases}
\frac{x-a}{b-a}, & \text{if } x \in [a, b]; \\
\frac{x-c}{b-c}, & \text{if } x \in [b, c]; \\
0, & \text{otherwise}. 
\end{cases}
\]

As known, in this case, the lower and, respectively, the upper bound of the \( \alpha \)-interval has the form
\[
z^-(\alpha) = (b - a)\alpha + a, \quad z^+(\alpha) = -(c - b)\alpha + c.
\]

It is not difficult to calculate that the average (1) for a fuzzy triangular number in the case 
of \( f(\alpha) \equiv 1 \) is \( m_1(\tilde{z}) = \frac{1}{2}(a + 2b + c) \).

On a set of fuzzy numbers, we can put definitions of distances between them in different ways. 
The interval approach often uses Hausdorff distances between sets of fuzzy numbers (see, 
for example, [2]). However, we would like to consider such a definition of distance that the mean 
(1) minimizes some error in the corresponding metric. As such a distance, it is convenient to 
consider the following.

Let \( \tilde{z} \) and \( \tilde{u} \) be two fuzzy numbers. Set the weighted distance \( \rho_f(\tilde{z}, \tilde{u}) \) between them by the formula
\[
\rho_f(\tilde{z}, \tilde{u}) = \left( \int_0^1 ((z^-(\alpha) - u^-(\alpha))^2 + (z^+(\alpha) - u^+(\alpha))^2)f(\alpha)d\alpha \right)^{1/2}.
\]

Here \([z^-(\alpha), z^+(\alpha)]\) and \([u^-(\alpha), u^+(\alpha)]\) – intervals of \( \alpha \)-levels of fuzzy numbers \( \tilde{z} \) and \( \tilde{u} \), respectively.

This definition of the distance between fuzzy numbers for \( f \equiv 1 \) is used, for example, in [3].

Denote by \( \hat{y} \) a singleton corresponding to the number \( y \in R \), i.e. a fuzzy number, the 
membership function \( \mu_{\hat{y}}(x) \) which is equal to 1 if \( x = y \) and 0 otherwise. Note that the left and 
right indexes of \( \hat{y} \) for all \( \alpha \in [0, 1] \) coincide with \( y \).

Statement 1. The mean value (1) of a fuzzy number \( \tilde{z} \) minimizes the value
\[
\delta(y) = \rho^2_f(\tilde{z}, \hat{y}) = \int_0^1 ((z^-(\alpha) - y)^2 + (z^+(\alpha) - y)^2)f(\alpha)d\alpha \quad \forall y \in R.
\]

It is the only solution of this extreme problem.

The proof follows from the minimum condition for the function \( \delta(y) \), since \( \delta'(y) = 4(y - m(\tilde{z})) \), and 
the second derivative \( \delta''(y) = 4 \quad (\forall y \in R) \). Uniqueness implies a strong convexity of the 
function \( \delta(y) \).

In many papers, including [4], the following definition of the distance between fuzzy numbers 
was used
\[
d_f(\tilde{z}, \tilde{u}) = \left( \int_0^1 f(\alpha) \int_0^1 (\lambda(z^+(\alpha) - u^+(\alpha)) + (1 - \lambda)(z^-(\alpha) - u^-(\alpha)))^2d\lambda d\alpha \right)^{1/2},
\]

where the parameter \( \lambda \in [0, 1] \) An interesting fact is that the mean (1) also has an extreme 
property relative to the distance (3).

Statement 2. Mean (1) is the solution to the optimal problem
\[
d_f(\tilde{z}, \hat{y}) \to \min \quad (y \in R).
\]

The proof follows from the minimum condition for the function \( \delta_1(y) = d_f(\tilde{z}, \hat{y}) \).

Note that even though many papers devoted to the means of fuzzy numbers (see, for example., 
[5-7]) extreme properties of the type of statements 1 and 2 were not previously noted.
2. Quasi-scalar product and the covariance

Let the fuzzy number $\tilde{z}$ correspond to $\alpha$-levels $Z_\alpha = [z^- (\alpha), z^+ (\alpha)]$. Let’s put as it is accepted in interval analysis

$\text{mid } Z_\alpha = \frac{1}{2} (z^+ (\alpha) + z^- (\alpha)), \quad \text{rad } Z_\alpha = \frac{1}{2} (z^+ (\alpha) - z^- (\alpha)).$

Here $\text{mid } Z_\alpha$ characterizes the mean, for each $\alpha \in [0, 1]$, and $\text{rad } Z_\alpha$ is the range.

For the fuzzy numbers $\tilde{z}$ and $\tilde{u}$ from $J_2$, we define the weighted quasi-scalar product

$$\langle \tilde{z}, \tilde{u} \rangle_f = \int_0^1 (\text{mid } Z_\alpha \text{ mid } U_\alpha + \text{rad } Z_\alpha \text{ rad } U_\alpha)f(\alpha)d\alpha =$$

$$0.5 \int_0^1 (z^+ (\alpha)u^+ (\alpha) + z^- (\alpha)u^- (\alpha))f(\alpha)d\alpha. \tag{4}$$

The weighted quasinorm $\tilde{z}$ is $\langle \tilde{z}, \tilde{z} \rangle_f^{1/2}$.

Example 2. Consider two triangular numbers $\hat{z}_1$ and $\hat{z}_2$, characterized by triples $a_i, b_i, c_i$ for $a_i < b_i < c_i \ (i=1,2)$. By determining their right and left indices (see example 1) and according to (4), the quasi-scalar product $\langle \hat{z}_1, \hat{z}_2 \rangle_f$ for $f \equiv 1$ is calculated by the formula

$$\langle \hat{z}_1, \hat{z}_2 \rangle_1 = \frac{2}{3} b_1 b_2 + \frac{1}{3} (a_1 a_2 + c_1 c_2) + \frac{1}{6} (a_1 b_2 + b_1 a_2 + b_1 c_2 + c_1 b_2).$$

The quasinorm $\hat{z}_i$ is defined by the equality $||\hat{z}_i||^2 = \frac{1}{4} (2b_i^2 + a_i^2 + c_i^2 + a_1 b_1 + b_1 c_1)$.

Below, the sum of fuzzy numbers with indexes $z^- (\alpha), z^+ (\alpha)$ and $u^- (\alpha), u^+ (\alpha)$ is understood as a fuzzy number with intervals $\alpha$-level $[z^- (\alpha) + u^- (\alpha), z^+ (\alpha) + u^+ (\alpha)]$. Multiplication by a positive number $c$ is characterized by the intervals $\alpha$-level of $[cz^- (\alpha), cz^+ (\alpha)]$, and multiplication by a negative number $c$ – intervals $\alpha$-level of $[cz^+ (\alpha), cz^- (\alpha)]$.

**Theorem 1.** The following properties of the quasi-scalar product (4) are satisfied.

1) $\langle \tilde{z}, \tilde{u} \rangle_f = \langle \tilde{u}, \tilde{z} \rangle_f \forall \tilde{u}, \tilde{z} \in J_2$; 2) $\langle c_1 \tilde{z}, c_2 \tilde{u} \rangle_f = c_1 c_2 \langle \tilde{z}, \tilde{u} \rangle_f$, provided for the product of the numbers $c_1 c_2 > 0$; 3) $\langle \tilde{z}_1 + \tilde{z}_2, \tilde{u} \rangle_f = \langle \tilde{z}_1, \tilde{u} \rangle_f + \langle \tilde{z}_2, \tilde{u} \rangle_f \forall \tilde{z}_1, \tilde{z}_2 \in J_2$; 4) $\langle \tilde{z}, \tilde{z} \rangle_f \geq 0$, and the condition $\langle \tilde{z}, \tilde{z} \rangle_f = 0$ is equivalent to zero left and right indexes of $\tilde{z}$.

5) Generalized Cauchy-Bunyakovsky inequality $|\langle \tilde{z}, \tilde{u} \rangle_f| \leq \langle \tilde{z}, \tilde{z} \rangle_f^{1/2} \langle \tilde{u}, \tilde{u} \rangle_f^{1/2} \forall \tilde{u}, \tilde{z} \in J_2$.

A fuzzy number $\tilde{z}$ with $\alpha$-level intervals $[z^- (\alpha), z^+ (\alpha)]$ match the vector function $\tilde{z}(\alpha) = (z^+ (\alpha), z^- (\alpha))^T$. For vector functions $\tilde{z}, \tilde{u}$ formula

$$\langle \tilde{z}, \tilde{u} \rangle_1 = 0.5 \int_0^1 (z^+ (\alpha)u^+ (\alpha) + z^- (\alpha)u^- (\alpha))d\alpha$$

is an ordinary scalar product, and $||\tilde{z}||_1 = \langle \tilde{z}, \tilde{z} \rangle_f^{1/2}$ is the norm. Note that the distance $p_\tilde{z}^2(\tilde{z}, \tilde{u}) = 2||\tilde{z} - \tilde{u}||_1^2$.

The semi-scalar product of fuzzy numbers similar to (4) is used in [8]. It has the form

$$\langle \tilde{z}, \tilde{u} \rangle = 0.25 \int_0^1 (z^+ (\alpha) + z^- (\alpha))(u^+ (\alpha) + u^- (\alpha))d\alpha = \int_0^1 \text{mid } Z_\alpha \text{ mid } U_\alpha d\alpha.$$

However, this expression does not take into account the range of fuzzy numbers. Herewith, the equality $\langle \tilde{z}, \tilde{z} \rangle_f^{1/2} = 0$ does not guarantee that the left and right indexes of $\tilde{z}$ are equal to zero.
Other definitions of the scalar product between fuzzy numbers are also considered in the literature (see, for example, [9]). Our definition is convenient because it is related to the distance (2) and the covariance between fuzzy numbers introduced below.

Consider the problem of nonlinear regression of a fuzzy number \( \tilde{z} \) with arbitrary Borel functions of a fuzzy number \( \tilde{y} \). Let’s call the conditional \( \tilde{z} \) by \( \tilde{y} \) and denote \( m[\tilde{z}/\tilde{y}] \) is a fuzzy number for which \( \langle \tilde{z} - m[\tilde{z}/\tilde{y}], \phi(\tilde{y}) \rangle_1 = 0 \) for all Borel functions \( \phi \). Here the function of a fuzzy number is understood in the sense of the generalization principle Of L. Zadeh.

**Statement 3.** The conditional average of \( \tilde{z} \) by \( \tilde{y} \) is the solution to the following optimization problem

\[
\rho_1(\tilde{z}, \phi(\tilde{y})) \rightarrow \min,
\]

for all Borel functions \( \phi \).

The following representation is used for the proof

\[
\rho^2(\tilde{z}, \phi(\tilde{y})) = \int_0^1 ((\langle z^+ - (\phi(\tilde{y}))^+ \rangle + (z^- - (\phi(\tilde{y}))^-))^2) d\alpha =
\]

\[
= ||z^+ - m[\tilde{z}/\tilde{y}]||^2 + ||m[\tilde{z}/\tilde{y}]^+ - \phi(\tilde{y})||^2 + 2 \langle \tilde{z} - m[\tilde{z}/\tilde{y}], m[\tilde{z}/\tilde{y}] - \phi(\tilde{y}) \rangle.
\]

This statement is similar to the corresponding statement for conditional mathematical expectations of random variables (see, for example, [10] ch. 1, §6).

For fuzzy numbers \( \tilde{z}_1, \tilde{z}_2 \) with average values \( m_1 \) and \( m_2 \), we define their covariance by the formula

\[
cov_{1}[\tilde{z}_1, \tilde{z}_2]_f = \langle \tilde{z}_1 - m_1, \tilde{z}_2 - m_2 \rangle =
\]

\[
0.5 \int_0^1 ((z^+_1 - m_1)(z^+_2 - m_2) + (z^-_1 - m_1)(z^-_2 - m_2)) f(\alpha) d\alpha.
\]

Denote the variance \( D_f(\tilde{z}) = \text{cov}_{1}[\tilde{z}, \tilde{z}]_f \) and \( \sigma_f(\tilde{z}) = \sqrt{D_f(\tilde{z})} \) — mean square deviation.

**Theorem 2.** The following covariance properties hold (5):

1. \( \text{cov}_{f}[\tilde{z}_1 + \tilde{z}_2, \tilde{u}] = \text{cov}_{f}[\tilde{z}_1, \tilde{u}] + \text{cov}_{f}[\tilde{z}_2, \tilde{u}] \) (\( \forall \tilde{z}_1, \tilde{z}_2, \tilde{u} \in J_2 \));
2. \( \text{cov}_{f}[c_1 \tilde{z}, c_2 \tilde{u}] = c_1 c_2 \text{cov}_{f}[\tilde{z}, \tilde{u}] \) (\( \forall c_1, c_2 \in J_2 \)) for any real \( c_1, c_2 \) such that \( c_1 c_2 > 0 \);
3. a specific covariance property

\[
\text{cov}_{f}[\tilde{z}_1, \tilde{z}_2] = \langle \tilde{z}_1, \tilde{z}_2 \rangle - m_1 m_2, \quad (\forall \tilde{z}_1, \tilde{z}_2 \in J_2) \text{ where } m_1 \text{ and } m_2 \text{ are weighted averages of the fuzzy numbers } \tilde{z}_1 \text{ and } \tilde{z}_2.
\]

**Theorem 3.** The usual properties of variance hold:

1. \( D_f(c \tilde{z}) = c^2 D_f(\tilde{z}) \) for any real number \( c \);
2. \( D_f(\tilde{z} + \tilde{u}) = D_f(\tilde{z}) + D_f(\tilde{u}) + 2c \text{cov}_{f}[\tilde{z}, \tilde{u}] \) for \( \forall \tilde{z}, \tilde{u} \in J_2 \);
3. \( c \text{ov}_{f}[\tilde{z}, \tilde{z}] = D_f(\tilde{z}) = \frac{1}{2} \rho^2_f(\tilde{z}, \tilde{m}) \) (\( \forall \tilde{z} \in J_2 \)), where \( m \) is weighted average value of a fuzzy number \( \tilde{z} \).

We emphasize that property 3 characterizes the relationship between the considered distance (2) on the set of fuzzy numbers and the variance we introduced.

Let’s define the correlation coefficient between the fuzzy numbers \( \tilde{z}_1 \) and \( \tilde{z}_2 \) by the formula

\[
k_f(\tilde{z}_1, \tilde{z}_2) = \frac{\text{cov}_{f}[\tilde{z}_1, \tilde{z}_2]}{\sigma_f(\tilde{z}_1) \sigma_f(\tilde{z}_2)}.
\]

**Theorem 4.** The following properties of the correlation coefficient hold (6):

1. \( |k_f| \leq 1 \);
2) the correlation coefficient \( k_f = 1 \) if and only if there is \( \lambda > 0 \) for which \( (\hat{z}_1 - \hat{m}_1) = \lambda (\hat{z}_2 - \hat{m}_2) \), where \( m_i \) are weighted averages of fuzzy numbers \( \tilde{z}_i \) \( (i = 1, 2) \);

3) correlation coefficient \( k_f = -1 \) if and only if for almost all \( \alpha \in [0, 1] \)

\[
\begin{align*}
z_1^+(\alpha) - m_1 &= -\frac{\sigma(\hat{z}_1)}{\sigma(\hat{z}_2)}(z_2^+(\alpha) - m_2), \\
z_1^-(\alpha) - m_1 &= -\frac{\sigma(\hat{z}_1)}{\sigma(\hat{z}_2)}(z_2^-(\alpha) - m_2).
\end{align*}
\]

We explain that property 1) follows from the generalized Cauchy-Bunyakovsky inequality. Properties 2) and 3) are provided by the condition of equality in the Cauchy-Bunyakovsky inequality. Note that property 3) does not match the equality \( (\tilde{z}_1 - \hat{m}_1) = -\lambda (\tilde{z}_2 - \hat{m}_2) \) (for \( \lambda = \sigma(\hat{z}_1)/\sigma(\hat{z}_2) \)), which is equivalent to the aggregate \( z_1^+(\alpha) - m_1 = -\lambda(z_2^+(\alpha) - m_2) \), \( z_1^-(\alpha) - m_1 = -\lambda(z_2^-(\alpha) - m_2) \). So the analogy with random variables is not preserved in this case.

In a number of works (see, for example, [6, 11]) the following expression is considered as a weighted covariance of the fuzzy numbers \( \tilde{z}_1, \tilde{z}_2 \)

\[
cov_f[\tilde{z}_1, \tilde{z}_2] = \int_0^1 \left( \frac{z_1^+(\alpha) - z_1^-(\alpha)}{2} \right) \left( \frac{z_2^+(\alpha) - z_2^-(\alpha)}{2} \right) f(\alpha) d\alpha.
\]

Accordingly, the variance

\[ D_f(\tilde{z}_1) = \text{cov}_f[\tilde{z}_1, \tilde{z}_2]. \]

With this definition, \( \text{cov}_f[\tilde{z}_1, \tilde{z}_2] \) is always non-negative, which does not correspond to the standard covariance properties (for random variables). Therefore, definition (7) does not provide an opportunity to introduce a correlation coefficient that has standard properties.

A number of papers (see, for example, [12]) consider the concept of covariance between fuzzy random variables and study the corresponding properties. However, this is a qualitatively different object of research compared to fuzzy numbers.

The definition of the quasi-scalar product and the covariance of fuzzy numbers introduced above can be useful, for example, in the problem of optimal approximating fuzzy numbers. Note that even in the case of the weight function \( f(\alpha) \equiv 1 \), the mentioned definitions seem new to us.

3. Problem of optimal approximation of fuzzy numbers

Consider the problem of approximating a fuzzy number \( \tilde{z} \) by a linear combination of the fuzzy number system \( \tilde{z}_1, ..., \tilde{z}_n \).

In this point, we consider the case when the weight function \( f(\alpha) \equiv 1 \). In this case, the index 1 in the metric, quasisquare work and covariances will be omitted.

Let us first study an extreme problem with non-negative coefficients \( \beta_i \geq 0 \) \( (i = 1, ..., n) \)

\[
\rho(\hat{z}, \sum_{i=1}^n \beta_i \tilde{z}_i) \rightarrow \min \quad (\forall \beta_i \geq 0).
\]

Take place

**Lemma 1.** Let the numbers \( \tilde{z}_i \) be quasi-orthogonal for \( i \neq j \), and their quasinorms \( \kappa_j := \langle \tilde{z}_j, \tilde{z}_j \rangle^{1/2} \neq 0 \) \( (j = 1, ..., n) \). Let, in addition, the non-negativity condition be satisfied

\[
(\tilde{z}, \tilde{z}_j) \geq 0 \quad (j = 1, ..., n).
\]

Then problem (8) has a non-negative solution, and the only one. It has the form \( \beta_i^* = \frac{1}{\kappa_i^2} \langle \hat{z}, \tilde{z}_i \rangle \), \( (i = 1, ..., n) \).
Indeed, let’s put
\[ F(\beta_1, ..., \beta_n) = \rho^2(\sum_{i=1}^{n} \beta_i \tilde{z}_i). \]

Due to the assumption that the coefficients \( \beta_i \) are non-negative for each \( \alpha \in [0, 1] \), the left index \( \sum_{i=1}^{n} \beta_i \tilde{z}_i \) is \( \sum_{i=1}^{n} \beta_i z_i^{-} (\alpha) \), and the right index \( \sum_{i=1}^{n} \beta_i z_i^{+} (\alpha) \). Then, omitting the \( \alpha \) argument, we can write
\[ F(\beta_1, ..., \beta_n) = \int_{0}^{1} \left( (z^{+} - \sum_{i=1}^{n} \beta_i z_i^{+})^2 + (z^{-} - \sum_{i=1}^{n} \beta_i z_i^{-})^2 \right) d\alpha. \]

This is a quadratic form in \( \beta_1, ..., \beta_n \).

Differentiating \( F \) by \( \beta_j \) and equating the derivative to zero, we get
\[ \sum_{i=1}^{n} \beta_i \langle \tilde{z}_i, \tilde{z}_j \rangle = \langle \tilde{z}, \tilde{z}_j \rangle \ (j = 1, ..., n). \]  

We introduce the following notation. Vector \( B \) with coefficients \( b_i = \langle \tilde{z}, \tilde{z}_i \rangle \), matrix \( A \) with coefficients \( a_{ij} = \langle \tilde{z}_i, \tilde{z}_j \rangle \), vector \( \beta \) with coefficients \( \beta_i \).

In vector form, system (10) has the form\[ A \beta = B. \]

Matrix \( A \) due to the quasi-orthogonality of the system \( \tilde{z}_i \) has a diagonal form, with positive numbers \( z_i^{\pm} \ (i = 1, ..., n) \) on the diagonal. Then \( \beta^* = A^{-1} B \), i.e. \( \beta_i^* = \frac{b_i}{z_i^{\pm}} = \frac{1}{z_i^+} \langle \tilde{z}, \tilde{z}_i \rangle \ (i = 1, ..., n). \)

The nonnegativity of the obtained coefficients \( \beta_i \) is provided by condition (9).

To make sure that \( \beta^* = A^{-1} B \) is the minimum point, consider the second derivatives of
\[ \frac{\partial^2 F}{\partial \beta_j \partial \beta_k} = 2 \int_{0}^{1} (z_i^{+} z_j^{+} + z_i^{+} z_j^{-}) d\alpha = \langle \tilde{z}_i, \tilde{z}_j \rangle. \]

A sufficient sign of the minimum is the positive definiteness of the Hesse matrix \( \{ \frac{\partial^2 F}{\partial \beta_j \partial \beta_k} \} \).

And this is provided by the quasi-orthogonality of the system \( \{ \tilde{z}_i \} \).

Note 1. The condition \( \langle \tilde{z}, \tilde{z}_i \rangle > 0 \) means that there is an acute angle between the fuzzy numbers \( \tilde{z} \) and \( \tilde{z}_i \). In other words, the fuzzy numbers \( \tilde{z} \) and \( \tilde{z}_i \) increase in this sense (or decrease) at the same time.

Remark 2. In the conditions of Lemma 1, we can reject the requirement of pairwise orthogonality of fuzzy numbers \( \tilde{z}_1, ..., \tilde{z}_n \). It is sufficient to require positive invertibility of their Gram matrix \( A \) with coefficients \( a_{ij} = \langle \tilde{z}_i, \tilde{z}_j \rangle \).

In addition, if instead of pairwise quasi-orthogonality of the elements of the system \( \{ \tilde{z}_i \} \), we consider their pairwise uncorrelability, then based on property 3 of the covariance, the question is reduced to the positive invertibility of the Gram matrix for the system \( \{ m(\tilde{z}_i) \} \) of the corresponding means.

Consider the question of the best approximation of a given fuzzy number \( \tilde{z} \) by convex combinations of fuzzy numbers \( \tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_n \).

Statement 4. Let the fuzzy numbers \( \tilde{z}_j \) be quasi-orthonormal, i.e. they are pairwise quasi-orthogonal and their quasinorms \( \langle \tilde{z}_j, \tilde{z}_j \rangle^{1/2} = 1 \ (j = 1, ..., n) \). Let, in addition, the nonnegativity condition (9) be satisfied. Then the problem
\[ \rho(\tilde{z}, \sum_{i=1}^{n} \beta_i \tilde{z}_i) \to \min \ (\forall \beta_i \geq 0, \sum_{i=1}^{n} \beta_i = 1) \]  

(11)
has a solution defined by the equalities

\[ \beta_j = \frac{1}{n} (1 - \sum_{j=1}^{n} \langle \hat{z}, \tilde{z}_j \rangle) + \langle \hat{z}, \tilde{z}_j \rangle \quad (j = 1, \ldots, n). \]  

(12)

The proof is performed using the necessary extremum condition for the Lagrange function

\[ L(\beta, \lambda) = \rho^2(\tilde{z}, \sum_{i=1}^{n} \beta_i \tilde{z}_i) - \lambda (\sum_{i=1}^{n} \beta_i - 1). \]

It can be shown that for the coefficients (12) a sufficient minimum condition is also satisfied. The positivity of the right-hand expression in (12) by assumption (9) is provided, for example, by the proximity condition of all \( \langle \hat{z}, \tilde{z}_j \rangle \) (difference of order \( \frac{1}{n} \)).

We now reject the nonnegativity of the coefficients \( \beta_i \) in the optimal approximation problem (8).

Define the constant \( c_* = \max_{j=1,\ldots,n} \{ \frac{||z||}{||z_j||} \} \).

**Theorem 5.** Let the fuzzy numbers \( \tilde{z}_i \) be quasi-orthogonal for \( i \neq j \), and all their quasinorms \( \kappa_i \neq 0 \) (\( i = 1, \ldots, n \)). Then the problem is

\[ \rho(\tilde{z}, \sum_{i=1}^{n} \beta_i \tilde{z}_i) \to \min \quad (\beta_i \in [-c_*, \infty)) \]

(13)

has a solution, and the only one. It has the form \( \beta_i^* = \frac{1}{c_*} \rho(\tilde{z}, \tilde{z}_i) \) (\( i = 1, \ldots, n \)).

We emphasize that although problem (13) does not assume that the coefficients \( \beta_i \) are positive (and conditions (9)), the formula for the coefficients \( \beta_i \) has the same form as in Lemma 1.

In the proof of theorem 5, the following special property of the distance (2) between fuzzy numbers will be used.

**Lemma 2.** For any fuzzy numbers \( \tilde{z}, \tilde{u}, \text{ and } \tilde{w}, J_2 \) has the equality

\[ \rho(\tilde{z} + \tilde{w}, \tilde{u} + \tilde{w}) = \rho(\tilde{z}, \tilde{u}). \]

In fact, this is true, since taking into account the rules of interval addition for left indexes, we have

\[ (\tilde{z} + \tilde{w})^- = z^- + w^-, \quad (\tilde{u} + \tilde{w})^- = u^- + w^-, \]

and similarly for right-hand indexes.

After substituting the corresponding expressions in (2), we obtain the required equality.

**Proof of theorem 5.** Let the condition \( \langle \hat{\tilde{z}} \rangle > 0 \) not be satisfied for at least one \( j \). Consider a fuzzy number \( \tilde{y} = \tilde{z} + c_* \sum_{i=1}^{n} \tilde{z}_i \). According to the definition of \( c_* \) we have \( \langle \hat{\tilde{y}}, \tilde{z}_j \rangle > 0 \) (\( j = 1, \ldots, n \)).

Consider for \( \tilde{y} \) problem (8). Let \( \gamma_i \geq 0 \) be the optimal coefficients of the linear combination \( \sum_{i=1}^{n} \gamma_i \tilde{z}_i \) for \( \tilde{y} \) obtained by solving problem (8). They are defined by the formulas \( \gamma = A^{-1} f \), for \( f_i = \langle \hat{\tilde{y}}, \tilde{z}_i \rangle \).

We show that the coefficients \( \gamma_i - c_* \) are optimal for linear approximation of a fuzzy number \( \tilde{z} \) by the system \( \{ \tilde{z}_i \} \).

Consider the distance \( \rho(\tilde{z}, \sum_{i=1}^{n} (\gamma_i - c_*) \tilde{z}_i) \). By Lemma 2 and taking into account the definition of \( \tilde{y} \), we have

\[ \rho(\tilde{z}, \sum_{i=1}^{n} (\gamma_i - c_*) \tilde{z}_i) = \rho(\tilde{z} + \sum_{i=1}^{n} c_* \tilde{z}_i, \sum_{i=1}^{n} \gamma_i \tilde{z}_i) = \rho(\tilde{y}, \sum_{i=1}^{n} \gamma_i \tilde{z}_i). \]  

(14)
Then (14) implies the inequality
\[
\rho(\tilde{y}, \sum_{i=1}^{n} \gamma_i \tilde{z}_i) \leq \rho(\tilde{y}, \sum_{i=1}^{n} \xi_i \tilde{z}_i) = \rho(\tilde{z} + \sum_{i=1}^{n} c_* \tilde{z}_i, \sum_{i=1}^{n} (\xi_i - c_*) \tilde{z}_i + \sum_{i=1}^{n} c_\ast \tilde{z}_i).
\]
Using Lemma 2 again, we get
\[
\rho(\tilde{y}, \sum_{i=1}^{n} \gamma_i \tilde{z}_i) \leq \rho(\tilde{z}, \sum_{i=1}^{n} (\xi_i - c_*) \tilde{z}_i).
\]
Then (14) implies the inequality
\[
\rho(\tilde{z}, \sum_{i=1}^{n} (\xi_i - c_*) \tilde{z}_i) \leq \rho(\tilde{z}, \sum_{i=1}^{n} c_\ast \tilde{z}_i).
\]
Since here \((\xi_i - c_*)\) are arbitrary coefficients from \([-c_\ast, \infty)\), then \((\gamma_i - c_\ast)\) are optimal coefficients.

Note that by definition \(\tilde{y}\) and according to Lemma 2
\[
\gamma_j = \frac{1}{x_j^2} \langle \tilde{y}, \tilde{z}_j \rangle = \frac{1}{x_j^2} \langle \tilde{z} + \sum_{i=1}^{n} c_\ast \tilde{z}_i, \tilde{z}_j \rangle.
\]
Then, taking into account the quasi-orthogonality of the system \(\{\tilde{z}_i\}\), we get
\[
\gamma_j = \frac{1}{x_j^2} \langle \tilde{z} \rangle + c_\ast \langle \tilde{z}_j \rangle = \frac{\langle \tilde{z}, \tilde{z}_j \rangle}{x_j^2} + c_\ast.
\]
Hence, the optimal coefficients for \(\tilde{z}_j\) in linear approximation, \(\tilde{z}\), which have the form \((\gamma_j - c_\ast)\), are defined by the equality \(\beta_j = \frac{1}{x_j^2} \langle \tilde{z}, \tilde{z}_j \rangle\) \((j = 1, \ldots, n)\). This is what the statement implies.

Note that the explicit form of the formula for the distance \(\rho(\tilde{z}, \sum_{i=1}^{n} \beta_i \tilde{z}_i)\) in the case of coefficients \(\beta_i\) of arbitrary sign is inconvenient for research, since in this case the product of the number \(\beta\) on the interval \([z^-, z^+]\) has a cumbersome form
\[
\beta[z^-, z^+] = [\min\{\beta z^-, \beta z^+\}, \max\{\beta z^-, \beta z^+\}].
\]
Let us now consider the situation when the fuzzy number \(\tilde{z}\) is approximated by the fuzzy numbers \(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n\), which are pairwise uncorrelated, and the variances \(D\tilde{z}_i = \sigma_i^2\) are non-zero. In this situation, a statement similar to theorem 5 takes place. Denote by \(\hat{m}\) and \(\hat{m}_i\) the singletons corresponding to the average \(m(\tilde{z})\) and \(m(\tilde{z}_i)\) and let \(r_\ast = \max_{i=1,\ldots,n} \frac{\sigma_i(z)}{\sigma_i(\tilde{z}_i)}\).

**Theorem 6.** Let the fuzzy numbers \(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n\) be pairwise uncorrelated and \(D\tilde{z}_i = \sigma_i^2 \neq 0\) \((i = 1, \ldots, n)\). Then the problem is
\[
\rho(\hat{w}, \hat{m} + \sum_{i=1}^{n} \beta_i (\tilde{z}_i - \hat{m}_i)) \to \min\ (\beta_i \in [-r_\ast, \infty])
\]
has a single solution. It has the form
\[
\beta_i^\ast = \frac{1}{\sigma_i^2} \frac{\text{cov}[\tilde{z}, \tilde{z}_i]}{m(\tilde{z}_i)} \quad (i = 1, \ldots, n).
\]
Indeed, consider the fuzzy numbers \( \tilde{w}_i = \hat{z}_i - \hat{m}_i \). According to the definitions of the quasiscalar product \((4)\) and covariance \((5)\) \( \text{cov}[\tilde{z}_i, \tilde{z}_j] = \langle \tilde{w}_i, \tilde{w}_j \rangle \). So if \( \hat{z}_i \) and \( \hat{z}_j \) are uncorrelated, then \( \tilde{w}_i \) and \( \tilde{w}_j \) are quasi-orthogonal. In this case, \( D(\tilde{z}_i) = \langle \tilde{w}_i, \tilde{w}_i \rangle = \sigma_{\tilde{w}}^2 \neq 0 \). Note that the optimal coefficients \((16)\) in the new notation have the form \( \beta_i^* = 1/\sigma_{\tilde{w}}^2 \). Note that the optimal estimate \((17)\) has the maximum correlation coefficient with \( \tilde{z} \) among all estimates of the form \( \hat{z} \). According to the definitions of the covariance

\[
\text{cov}[\tilde{z}, \hat{z}] = \sum_{i=1}^{n} \beta_i^* \sigma_i^2 = \sum_{i=1}^{n} \beta_i^* \langle \tilde{w}, \tilde{w}_i \rangle = \sum_{i=1}^{n} \beta_i^* \sigma_i^2.
\]

In addition, by virtue of the pairwise quasi-orthogonality \( \tilde{w}_i \), we have

\[
D(\hat{z}) = \sum_{i=1}^{n} \beta_i^2 \sigma_i^2 = \sum_{i=1}^{n} \beta_i^2 \sigma_i^2.
\]

Then the coefficient of correlation defined by formula \((6)\) equal to

\[
k[\tilde{z}, \hat{z}] = \frac{\sum_{i=1}^{n} (\beta_i^2 \sigma_i^2)}{\sigma(\tilde{z})(\sum_{i=1}^{n} (\beta_i^2 \sigma_i^2))^{1/2}} = \frac{1}{\sigma(\tilde{z})} \left( \sum_{i=1}^{n} (\beta_i^2 \sigma_i^2) \right)^{1/2}.
\]

Besides,

\[
\text{cov}[\hat{z}, \hat{z}] = \sum_{i=1}^{n} \beta_i \langle \tilde{w}, \tilde{w}_i \rangle = \sum_{i=1}^{n} \beta_i \sigma_i^2 \beta_i^*.
\]

Herewith

\[
D(\hat{z}) = \sum_{i=1}^{n} \beta_i \sigma_i^2 = \sum_{i=1}^{n} \beta_i \sigma_i^2.
\]
Then the correlation coefficient

\[ k[\tilde{z}, \hat{m} + \sum_{i=1}^{n} \beta_i \tilde{w}_i] = \frac{1}{\sigma(\tilde{z})} \left( \frac{\sum_{i=1}^{n} (\beta_i^* \sigma_i)(\beta_i \sigma_i)}{\left( \sum_{i=1}^{n} (\beta_i^2 \sigma_i^2) \right)^{1/2}} \right). \]

Hence, by the Cauchy-Bunyakovsky inequality

\[ |k[\tilde{z}, \hat{m} + \sum_{i=1}^{n} \beta_i \tilde{w}_i]| \leq \frac{1}{\sigma(\tilde{z})} \left( \frac{\sum_{i=1}^{n} (\beta_i^*)^2 \sigma_i^2}{\left( \sum_{i=1}^{n} (\beta_i^2 \sigma_i^2) \right)^{1/2}} \right) = \frac{1}{\sigma(\tilde{z})} \left( \frac{\sum_{i=1}^{n} (\beta_i^2 \sigma_i^2)}{\left( \sum_{i=1}^{n} (\beta_i^2 \sigma_i^2) \right)^{1/2}} \right) = k[\tilde{z}, \hat{m} + \sum_{i=1}^{n} \beta_i^* \tilde{w}_i]. \]

Q.E.D.

Note that theorem 7 resembles a well-known result for random variables about the maximum correlation of the optimal linear estimate with the predicted random variable (see, for example, [13], Chapter 5, §5.6).

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