Analytical stellar models of neutron stars in teleparallel gravity

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Abstract: In this paper, we use the theory of teleparallel gravity to develop three analytical models and obtain a new class of solutions describing compact stellar structures. We investigate the anisotropic nature of stellar configurations in general and solve teleparallel gravity equations. To thoroughly analyze the various parameters of the stars, we develop three models by choosing various physically acceptable forms of metric potential $e^{\beta(r)}$ and radial pressure $p_r(r)$. We also analyze the impact of teleparallel gravity’s parameters $\beta$ and $\beta_1$ on the description of the stellar structures. We calculated model parameters so that models describing various observed neutron stars obey all physical conditions necessary to be potentially stable and causal. By analyzing the impact of various parameters of teleparallel gravity on the description of anisotropic stellar structures, we show that these three models can describe anisotropic neutron stars ranging from low density to high density. Finally, we obtain a quadratic equation of state for each model describing various neutron stars, which can be utilized to find compositions of the stellar structures. It is extremely useful to find models that can exhibit quadratic EOS, because various authors have discovered that the material compositions of real neutron stars and strange stars exhibit quadratic EOS. Since a nonlinear $f(T)$ model gives a high deviation of EOS from the quadratic behavior, we work with a linear $f(T)$ function by using a diagonal tetrad to model realistic compact stars.

Keywords: Teleparallel gravity; Stellar structures; Neutron stars; EOS

1. Introduction

It is well understood that general relativity describes gravity as an effect of space-time curvature. Thus, general relativity is a geometrical theory of gravitation. It has been very successful in describing various astronomical and astrophysical phenomena with great precision. However, recent observations of galactic dynamics cannot be understood in the tenets of general relativity. As a result, various new concepts such as dark matter and dark energy have been introduced to better understand galactic dynamics and the expansion of the universe [1–4]. However, another way to account for observations of galactic dynamics like phenomena is to modify general relativity [5–12]. The Lagrangian formulation of the general relativity is constructed from the Einstein–Hilbert action written in terms of the Ricci scalar and the metric tensor [5]. Different modified gravity theories can be constructed by modifying the Einstein-Hilbert action. Modified field equations can be obtained by varying the modified Einstein–Hilbert action and implementing the Euler–Lagrangian equation. In recent years, there has been a huge effort to modify gravity to solve the issue of non-renormalizability of general relativity [13, 14].

The general relativity is constructed from its dynamical variable - the metric tensor, defined on a pseudo-Riemannian manifold. In general relativity, connections are considered to be torsion-less. However, in order to account for torsion, Einstein introduced the Teleparallel Equivalent of General Relativity known as TEGR [15, 16]. In this theory, Einstein introduced absolute parallelism and the tangent space as a frame of reference to understand the effect of torsion. In TEGR, tetrad fields are used instead of a metric tensor to construct the theory [16]. Since a tangent space frame of reference is used, the curvature at each point becomes zero, and the only effect that remains is that of torsion [17]. Thus, TEGR describes the gravity as due to the effect of torsion of space-time. TEGR also can be constructed by the Lagrangian formulation by considering a tetrad field as dynamical variables instead of the metric...
tensor. Thus, we can also modify this action to construct modified gravity in terms of the torsion mechanism. This modified theory of gravity is known as $f(T)$ gravity [18–20]. A recent detailed review on the topic of teleparallel gravity and modified $f(T)$ gravity [21] can be considered by the readers for more discussion on the subject.

In recent years, teleparallel gravity has been found to be very successful in describing various astronomical and astrophysical phenomena. $f(T)$ gravity is also a widely used theory besides $f(R)$ gravity to explain the expansion of the universe and dark matter through the modification of general relativity [22–38]. The compact astrophysical objects such as black holes and compact stars have been modeled usingTEGR [39–46] as well as other theories of modified gravity [47–51]. Recently, Wang et al [52] studied a spherically symmetric static solution in $f(T)$ gravity models and showed that only a limited class of $f(T)$ gravity models can be solved in this frame. C. Deliduman and B. Yaprak investigated neutron stars under modified $f(T)$ gravity [53] and found that the relativistic neutron star solution in $f(T)$ gravity models is possible only if $f(T)$ is a linear function of the torsion scalar $T$ for diagonal tetrad. A black hole with a cosmological constant in the context of $f(T)$ gravity has been studied in [54]. The violation of the first law of black hole thermodynamics in $f(T)$ gravity due to the lack of local Lorentz invariance is studied in [55]. Many authors have also studied compact star structures in other modified theories of gravity. Capozziello et al have studied compact stars in $f(R)$ gravity [56]. The authors have also investigated gravitational waves generated by the compact objects in $f(T, B)$ gravity [57]. Astashenok et al have studied maximal neutron star mass in modified gravity [58] and developed stable neutron star models from $f(R)$ gravity [59]. Pretel et al have recently investigated neutron stars in $f(R, T)$ gravity with conserved energy-momentum tensor [60]. Xu et al have studied neutron stars in massive scalar-Gauss-Bonnet gravity [61].

Here, we choose the linear function of $f(T)$ as $f(T) = \beta T + \beta_1$. This is essentially known as teleparallel gravity with the extra factors $\beta$ and $\beta_1$. The effect of this linear $f(T)$ on the modeling of anisotropic neutron stars has been the subject of various studies. However, many of the studies are performed by setting the parameter $\beta_1$ equal to zero to describe and analyze graphs of various parameters of neutron stars [43, 44, 46]. In this work, we have developed three models to understand the effect of various parameters like metric potential, radial pressure, $\beta$, $\beta_1$, etc., on the properties of the neutron stars. We considered different physically acceptable forms of the metric potential $\mathcal{e}^{\mu(r)}$ and the radial pressure $p_r(r)$ to solve equations of teleparallel gravity and, to generate models that can describe various types of neutron stars ranging from low density to high density. In the first two models, we considered $\beta$ to be a constant of value $\beta = 2$. Then we compute the physically acceptable value of $\beta_1$ along with other parameters of the models. Thus, in the first two models, we analyze the effect of the parameter $\beta_1$ on the properties of observed neutron stars. In the third model, we considered $\beta_1 = 0$, and observed the effect of the parameter $\beta$ on the properties of observed neutron stars. We found that these three models can describe anisotropic neutron stars ranging from low density to high density.

We have organized this paper as follows: In Sect. 2, we formulate the field equations of teleparallel gravity. In that section, we briefly review the idea of the tetrad field and write down the elements of the manifold in terms of the tetrad field. By varying the action and considering the Weitzenböck connection, we write down equations of teleparallel gravity for an anisotropic matter distribution. In Sects. 3, 4, and 5, we develop the first, second, and third model describing anisotropic neutron stars in teleparallel gravity, respectively. In each of those sections, we develop our model by considering physically acceptable forms of the metric potential $\mathcal{e}^{\mu(r)}$ and the radial pressure $p_r(r)$. After that, we impose physical conditions so that the model becomes physically acceptable, followed by detailed calculations of various model parameters for describing observed neutron stars. We calculate model parameters and neutron star’s physical parameters for each neutron star mentioned in this paper. We also plot various parameters of neutron stars to study the general behavior of each model on the description of the stellar structures. Our analysis shows that we obtain a quadratic equation of state for each model describing various neutron stars, which can be utilized to find compositions of the stellar structures. In Sect. 6, we analyze adiabatic stability of neutron stars described by three models. Finally, we summarize the results obtained through the discussion Sect. (7).

2. Formulation of the field equations in the teleparallel gravity

The dynamical variable of general relativity is the metric tensor, which is defined on a pseudo-Riemannian manifold. In teleparallel gravity, on the other hand, a tetrad field is used as a dynamical variable, forming an orthonormal basis in tangent space at each point. Now let us denote Latin indices $(i, j, \text{etc.})$ for coordinates of the tangent space and Greek indices $(\mu, \nu, \text{etc.})$ for space-time coordinates. Thus, basis vectors and covectors for the tangent space are given by $e_i = \partial_i$ and $\mathcal{e}^i = dx^i$. Basis vectors for space-time coordinates are given by $e_\mu = \partial_\mu$ and $\mathcal{e}^\mu = dx^\mu$. The vectors
and covectors can be transformed from one base to another as follows

\[ \partial_i = e_{i\mu} \partial_{\mu} \quad \text{and} \quad dx^i = e_{i\mu} dx^\mu \]  

(1)

and conversely

\[ \partial_{\mu} = e^\mu_i \partial_i \quad \text{and} \quad dx^\mu = e^\mu_i dx^i \]  

(2)

where \( e^i_\mu \) is known as tetrad field and \( e^{\mu}_i \) is the inverse of it, that satisfy the relations \( e^i_\mu e^{j}_\nu = \delta^i_j \) and \( e^{\mu}_i e^{\nu}_i = \delta^\nu^\mu \). The relation between the metric of space-time and the tetrad field is given by

\[ g_{\mu\nu} = \eta_{ij} e^{i}_\mu e^{j}_\nu \]  

(3)

where \( \eta_{ij} = \text{diag}[1, -1, -1, -1] \). From Eq. (1), the line element of the manifold is given in terms of the tetrad field as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij} e^{i}_\mu e^{j}_\nu dx^\mu dx^\nu \]  

(4)

Now in teleparallel gravity, the curvature-less connection known as Weitzenböck connection is given by

\[ \Gamma^\nu_{\mu\nu} = e_i^\nu \partial_{\nu} e^i_\mu \]  

(5)

The torsion tensor in terms of the Weitzenböck connection is given by

\[ T^\nu_{\mu\nu} = \Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\mu\nu} \]  

(6)

Con-torsion and superpotential tensors have been introduced to calculate the torsion scalar. The con-torsion and superpotential tensors are, respectively,

\[ K^{\mu\nu} = -\frac{1}{2} (T^{\mu\nu} - T^{\nu\mu} - T^\mu_{\nu\nu}) \]  

(7)

\[ S^{\mu\nu} = \frac{1}{2} (K^{\mu\nu} + \delta^\mu_\nu T^{\mu\nu} - \delta^\nu_\mu T^{\mu\nu}) \]  

(8)

The torsion scalar can be defined as

\[ T = S^{\mu\nu} T^\nu_{\mu\nu} \]  

(9)

In f(T) gravity, the modified gravitational action is written in terms of the torsion scalar in the units of \( G = c = 1 \) as

\[ S = \int d^4x \left[ \frac{1}{16\pi} f(T) + \mathcal{L}_m \right] e \]  

(10)

where \( f(T) \) is a function of the torsion scalar and \( \mathcal{L}_m \) is the Lagrangian density for the matter field. Also

\[ e = \sqrt{-g} = \text{det}[e^i_\mu]. \]

The variation in action (10) gives the field equations of f(T) gravity [35, 44, 62]

\[ S^\mu_\nu \partial_\tau T_{\tau\nu} + e^{-1} e'_\mu \partial_\nu (e e'_i S^i_\nu) f_T + T^3 e^{\nu}_{\tau\nu} f_T + \frac{1}{4} \delta_\nu^\rho f = 4\pi T^\nu_{\mu} \]  

(11)

where \( T^\nu_{\mu} \) is the energy-momentum tensor of matter and \( f_T \) and \( f_{TT} \) are the first and second derivative of \( f(T) \) with respect to \( T \), respectively. Now in the case of anisotropic matter distribution, the energy-momentum tensor is given by

\[ T^\nu_{\mu} = (\rho + p_r) u^\mu u^\nu - p_t \delta^\nu_\mu + (p_r - p_t) v^\mu v^\nu \]  

(12)

where \( \rho, p_r, p_t \) are the energy density, the radial pressure and the tangential pressure, respectively. \( u^\mu \) and \( v^\mu \) are the four velocity and radial four vectors, respectively. The anisotropy of the compact stellar structure is defined as \( S = (p_r - p_t)/\sqrt{3} \).

3. First anisotropic model of neutron star in teleparallel gravity

We consider the static and spherically symmetric model of the star. Consequently, the space-time metric that describes the star’s interior is given by

\[ ds^2 = e^{2\alpha} dt^2 - e^{2\beta} r^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(13)

For the metric (13), we calculate the tetrad matrix as

\[ e^i_\mu = \text{diag}[e^{\alpha}, e^{\beta}, r, r\sin(\theta)] \]  

(14)

From equation (14) we obtain,

\[ e = \text{det}[e^i_\mu] = e^{(\alpha\beta)} r^2 \sin(\theta) \]  

(15)

Now we determine the torsion scalar as

\[ T(r) = \frac{2e^{-d(r)}}{r} \left( c' + \frac{1}{r} \right) \]  

(16)

Now from Eq. (11), we write the equations of f(T) gravity for the metric (13) as [35, 44, 62]

\[ 4\pi \rho = \frac{f}{4} - \frac{f_T}{2} \left( T - \frac{e^{-d}}{r} (c' + d') - \frac{1}{r^2} \right) \]  

(17)

\[ 4\pi p_r = \frac{f_T}{2} \left( T - \frac{1}{r^2} \right) - \frac{1}{4} \]  

(18)

\[ 4\pi p_t = \left[ \frac{T}{2} + e^{-d} \left( \frac{c''}{2} + \left( \frac{c' + 1}{2r} \right) (c' - d') \right) \right] \frac{f_T}{2} - \frac{f}{4} \]  

(19)

\[ \cot(\theta) \frac{2}{r^2} T f_{TT} = 0 \]  

(20)

where single prime and double prime denote the first and
second derivatives with respect to $r$. Now, Eq. (20) leads to the following linear form of $f(T)$

$$f(T) = \beta T + \beta_1$$

(21)

where $\beta$ and $\beta_1$ are constants. Thus, the theory becomes comparable to teleparallel gravity with the extra factors of $\beta$ and $\beta_1$. We will study the effects of these parameters by generating three different models. Now in order to solve Eqs. (17) to (19), we choose the following form of the metric function,

$$e^{\varphi(r)} = 1 - \frac{ar^2}{R^2}$$

(22)

where $a$ is a constant and $R$ is the radius of the neutron star. We consider $a < 1$, so that the metric does not become singular at any point for $r < R$. We have chosen the particular form of the metric potential because it is very useful in describing various astrophysical systems. A similar kind of metric has been used to study the compact anisotropic star in general theory of relativity [63]. A similar form of metric function has also been used to study anisotropic compact objects in $f(T)$ gravity with Finch–Skea geometry [64]. Thus, we will use the simplest yet most useful form of the metric function described by Eq. (22) to develop the first model of anisotropic neutron star in the teleparallel gravity. In this model we consider $\beta = 2$ in Eq. (21). By substituting Eq. (22) in (17), we obtain

$$4\pi \rho = \frac{p_0}{R^2} \left(\frac{3 - a^2}{1 - a^2}\right) + \frac{\beta_1}{4}$$

(23)

From Eq. (18), we write

$$c'(r) = \frac{(4\pi \rho_r)e^{\varphi(r)} - \frac{e^{\varphi(r)} - 1}{r} + \frac{\beta_1}{4}re^{\varphi(r)}}{4}$$

(24)

In order to solve Eq. (24), we consider the following form of $4\pi \rho_r$,

$$4\pi \rho_r = \frac{p_0}{R^2} \left(1 - \frac{r^2}{R^2}\right)$$

(25)

where $p_0$ is another constant, which is the parameter of the model such that $\frac{p_0}{R^2}$ represents the central pressure of the neutron star. The condition on the $p_0$ is that it must be greater than 1, ensuring that the radial pressure is positive for $0 < r < R$. The particular form of $p_r$ is physically reasonable because it is a monotonically decreasing function of $r$ and satisfies the condition of vanishing radial pressure at the surface of the neutron star $r = R$. By substituting values of Eqs. (22 and 25) into (24), we obtain

$$c'(r) = r \left(\frac{p_0 - a}{R^2} + \frac{\beta_1}{4}\right) - r^2 \left(\frac{p_0 + \beta_1 a}{4R^2}\right)$$

(26)

by integrating the above equation, we obtain the metric function as

$$e^{\varphi(r)} = Ke^{\frac{\varphi_0}{R^2} \left(\frac{r^2}{R^2} - 1\right)}$$

(27)

where $K$ is the integration constant. Hence, the interior space-time of the neutron star in this model is given by

$$ds^2 = \left(Ke^{\frac{\varphi_0}{R^2} \left(\frac{r^2}{R^2} - 1\right)}\right) dr^2 - \left(1 - \frac{ar^2}{R^2}\right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(28)

3.1. Constraining parameters for a physically well-behaved model

3.1.1. Matching exterior space-time with interior space-time

The space-time metric should be continuous at the boundary of the star. Thus, the interior metric (28) of the star should be matched to the exterior Schwarzschild space-time metric at $r = R$ [44, 46].

$$ds^2 = \left(1 - \frac{2M}{r}\right) dr^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(29)

The interior metric (28) will be matched to the exterior Schwarzschild metric at the surface of the star if

$$1 - a = \left(1 - \frac{2M}{r}\right)^{-1}$$

(30)

and

$$Ke^{\left(\frac{\varphi_0}{R^2} \left(\frac{r^2}{R^2} - 1\right)\right)} = 1 - \frac{2M}{R}$$

(31)

From Eqs. (30 and 31), we obtain the value of $K$ as

$$K = \frac{1}{1 - a} e^{-\left(\frac{\varphi_0}{R^2} \left(\frac{R^2}{R^2} - 1\right)\right)}$$

(32)

For a given value of $a$, $p_0$, $\beta_1$, and $R$, the value of constant $K$ in Eq. (28) can be calculated from Eq. (32).

3.1.2. Imposing physical conditions to the model

Since compact stars are physical objects that exist in nature, their physical parameters, such as energy density and pressure, must obey the fundamental laws of physics incorporated in physical conditions. Thus, physically
acceptable model of the compact star should obey the following physical conditions for $0 < r < R$ [44, 46]

(i) $\rho, p_r, p_t \geq 0$
(ii) $\frac{dp}{dr}, \frac{\rho}{p_r}, \frac{dp_r}{dr} \leq 0$
(iii) $0 \leq \frac{dp_r}{dr}, \frac{dp_t}{dr} \leq 1$
(iv) $\frac{dp_r}{dr} - \frac{dp_t}{dr} < 0$

where condition (i) is known as weak energy condition and condition (ii) implies that the density and pressure of the star must be a decreasing function of radius. The condition (iii) is known as causality condition, which implies that the radial and the transverse speed of sound represented by $v_r = \frac{\sqrt{\rho}}{\sqrt{\rho_r}}$ and $v_t = \frac{\sqrt{\rho_t}}{\sqrt{\rho_r}}$, respectively, must not exceed the speed of light. The condition (iv) is known as the stability condition, which implies that the radial speed of sound must be greater than the transverse speed of sound for the star to be potentially stable for $0 < r < R$.

We have already chosen parameters such that $a < 1$ and $p_0 > 0$ for $\rho, p_r \geq 0$. We can get the $p_t \geq 0$ if the following condition is satisfied

$$- 64p_0 + 64ap_0 + R^2 \beta_1^2 + 3a^2(4 + R^2 \beta_1)^2$$
$$- a^3(4 + R^2 \beta_1)^2 - aR^2 \beta_1(16 + 3R^2 \beta_1) \geq 0$$

(33)

Now in this model we already have $\frac{dp}{dr}, \frac{\rho}{p_r}, \frac{dp_r}{dr} \leq 0$. In order to get $\frac{dp_r}{dr} \leq 0$, we impose the following condition

$$- R^2 \beta_1^2 - 2a^2R^2 \beta_1(4 + R^2 \beta_1) - a^3(4 + 8p_0 - 7R^2 \beta_1)(4 + R^2 \beta_1) + 8p_0(16 + 5R^2 \beta_1) + a(48 - 32R^2 \beta_1 - 9R^4 \beta_1^2 + 24p_0(4 + R^2 \beta_1)) + a(-24p_0(8 + R^2 \beta_1) + R^2 \beta_1(16 + 5R^2 \beta_1)) \geq 0$$

(34)

The conditions (iii) and (iv) are satisfied if the model parameters satisfy following conditions, which we deduced by calculating $\frac{dp_r}{dr}, \frac{dp_t}{dr}$ and $\frac{dp}{dr} - \frac{dp_t}{dr}$ at $r = R$ and $r = r$.

$$0 \leq \frac{(1 - a)p_0}{5a^2} \leq 1 \quad \text{and} \quad 0 \leq \frac{(1 - a)^2p_0}{(5 - a)a^2} \leq 1$$

(35)

$$0 \leq \frac{1}{64(-5 + a)a^2}(R^4 \beta_1^2 + 2a^2R^2 \beta_1(4 + R^2 \beta_1)) + a(4 + 8p_0 - 7R^2 \beta_1)(4 + R^2 \beta_1) - 8p_0(16 + R^2 \beta_1) + a(48 + 32R^2 \beta_1 + 9R^4 \beta_1^2 - 24p_0(4 + R^2 \beta_1)) + a(-24p_0(8 + R^2 \beta_1) - R^2 \beta_1(16 + 5R^2 \beta_1)) \leq 1$$

(36)

$$48a^2 + 16p_0^2 - 16aR^2 \beta_1 + R^4 \beta_1^2 + 8p_0(-8 + R^2 \beta_1) > 0$$

(38)

$$R^4 \beta_1^2 + 2a^2R^2 \beta_1(4 + R^2 \beta_1) + a^3(4 + 8p_0 - 7R^2 \beta_1)(4 + R^2 \beta_1) - 8p_0(8 + R^2 \beta_1) + a^2(48 + 32R^2 \beta_1 + 9R^4 \beta_1^2 - 8p_0(4 + 3R^2 \beta_1)) + a(8p_0(8 + 3R^2 \beta_1) - R^2 \beta_1(16 + 5R^2 \beta_1)) > 0$$

(39)

3.2. Physical analysis

In this section, we examine the behavior of physical parameters such as energy density and pressure at the interior of the star for five observed neutron stars. We calculated parameters for each star such that the first model describing five neutron stars mentioned in Table 1 satisfies all bound conditions from (i) to (iv). We obtained the mass ($M$) and the radius ($R$) of the neutron stars from references [46, 65, 66] and computed the central density ($\rho_c$), the surface density ($\rho_s$), the central pressure ($p_c(r = 0) = p_c(r = R) = p_c$) and the surface tangential pressure ($p_t(r = R) = p_t$) of each star, shown in Table 2. We have plotted various parameters of the star shown in Figs. 1 and 2 of a neutron star SAX J1808.4-3658(SS2) shown in Table 1 described by the first model of the neutron stars in teleparallel gravity. In Fig. 1 (c), $v_t^2$ is the speed of sound having two components given by $v_t^2 = \frac{dp}{dr}$ and $v_t^2 = \frac{dp_r}{dr}$. The figures indicate that all the physical parameters of the star are well-behaved and follow all the conditions (i) to (iv) at every interior points of the star.

3.3. Equation of state

We have derived a physically acceptable model of the neutron star. To predict the material composition of the stellar configuration, we need to obtain an equation of state (EOS) of the stellar configuration. By obtaining an EOS governed by the physical laws of the system, one can parametrically relate the energy density and the radial

| Star             | R (km) | M (M₆) | a   | $\beta_1 \times 10^4$ | $p_0$ (km⁻²) |
|------------------|--------|--------|-----|----------------------|--------------|
| HER X-1          | 7.7    | 0.88   | -0.512 | -0.843              | 0.09         |
| RX J1856-3754    | 12.7   | 1.4    | -0.486 | -0.248              | 0.08         |
| SMC X-1          | 8.301  | 1.04   | -0.592 | -1.161              | 0.11         |
| RX J1856.5-3754  | 12     | 1.4    | -0.529 | -0.382              | 0.095        |
| SAX J1808.4-3658(SS2) | 7.951 | 0.9    | -0.506 | -0.759              | 0.089        |
pressure, which is useful in predicting the composition of the stellar configuration. By using Eqs. (23 and 25), we have plotted the variation of the radial pressure against the energy density, as shown in Fig. 1(h). From the plot, we find an EOS of the form \( p_r = l \rho^2 + m \rho + n \), where \( l \), \( m \) and \( n \) are constants. By approximating EOS in Fig. 1(h), we obtain \( l = -20 \), \( m = 0.18 \) and \( n = -0.00015 \).

4. Second anisotropic model of neutron star in teleparallel gravity

In this section we develop a second model to describe the neutron stars. We consider the same metric potential as given by Eq. (22). Thus, we get the same energy density as in Eq. (23).

In this new model, we consider a different form of radial pressure as follows

\[
4 \pi p_r = \frac{\rho_0}{\rho} \left( 1 - \frac{r^2}{R^2} \right) \frac{1}{1 - \frac{a^2}{R^2}}
\] (40)

where \( \rho_0 \) is also another constant, that is the parameter of this model such that \( \frac{\rho_0}{\rho} \) denotes the central pressure of the neutron star. Again, the constraint on the \( \rho_0 \) is that \( \rho_0 > 1 \), so that the radial pressure remains positive for \( 0 < r < R \). The particular form of \( p_r \) is also physically reasonable because it is a monotonically decreasing function of \( r \) and satisfies the condition of vanishing radial pressure at the surface of the neutron star \( r = R \). By substituting values of Eqs. (22 and 40) into (24), we obtain

\[
c'(r) = \frac{\rho_0 c}{\rho R} \left( 1 - \frac{r^2}{R^2} \right) + r \left( \frac{\beta_1}{4} - \frac{a}{R^2} \right) - r^3 \beta_1 a
\] (41)

By integrating the above equation, we obtain the metric function as

\[
e^{\chi(r)} = K \left( 1 - \frac{r^2}{R^2} \right)^{\frac{\rho_0 c}{\rho R}} e^{\frac{\rho_0 c}{\rho R} \left( \frac{\beta_1}{4} \right) \left( 1 + \frac{a}{R^2} \right)} \frac{\rho_0 c}{\rho R}
\] (42)

where \( K \) is the integration constant. The interior space-time of the neutron star in this model is given by

\[
d^2s = K \left( 1 - \frac{a^2}{R^2} \right)^{\frac{\rho_0 c}{\rho R}} e^{\frac{\rho_0 c}{\rho R} \left( \frac{\beta_1}{4} \right) \left( 1 + \frac{a}{R^2} \right)} \frac{\rho_0 c}{\rho R} \left( r^2 + \sin^2 \theta d\phi^2 \right) - r^2 (dt^2 - \frac{dr^2}{1 - \frac{a^2}{R^2}})
\]

(43)

4.1. Constraining parameters for a physically well-behaved model

4.1.1. Matching exterior space-time with interior space-time

The space-time metric should be continuous at the boundary of the star. Thus, the star’s interior metric (43) should be matched to the exterior Schwarzschild space-time metric (29) at \( r = R \) for this second model. The interior metric (43) will be matched to the exterior Schwarzschild metric at the surface of the star if

\[
1 - a = \left( 1 - \frac{2M}{R} \right)^{-1}
\] (44)

and

\[
K (1 - a)^{\frac{\rho_0 c}{\rho R} \left( \frac{\beta_1}{4} \right) \left( 1 + \frac{a}{R^2} \right)} = 1 - \frac{2M}{R}
\] (45)

From Eqs. (44 and 45), we obtain the value of \( K \) for the second model as

\[
K = \frac{1}{(1 - a)^{\frac{\rho_0 c}{\rho R} \left( \frac{\beta_1}{4} \right) \left( 1 + \frac{a}{R^2} \right)} - 1}
\] (46)

4.1.2. Imposing physical conditions to the second model

Physically acceptable model of the compact star should obey physical conditions (i) to (iv) as described in section (3.1.2). We impose the condition (i) by enforcing the following inequality computed from \( p_i \geq 0 \) in the second model.

\[
-64p_0 + R^4 \beta_1^2 + 3a^2 (4 + R^2 \beta_1)^2 - a^3 (4 + R^2 \beta_1)^2 \geq 0
\]

(47)

Now, in the second model we already have \( \frac{dp}{dr} \leq 0 \). In

Table 2: Values of the physical parameters of the five observed neutron stars described by the first model

| Star               | R (km) | M (M_\odot) | \( p_r \times 10^3 \) (km^{-2}) | \( p_i \times 10^3 \) (km^{-2}) | \( p_t \times 10^4 \) (km^{-2}) | \( p_e \times 10^4 \) (km^{-2}) |
|--------------------|--------|-------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| HER X-1            | 7.7    | 0.88        | 2.06                          | 1.05                          | 1.21                          | 0.55                          |
| RX J1856-3754      | 12.7   | 1.4         | 0.72                          | 0.38                          | 0.39                          | 0.19                          |
| SMC X-1            | 8.301  | 1.04        | 2.05                          | 0.97                          | 1.27                          | 0.63                          |
| RX J1856.5-3754    | 12     | 1.4         | 0.88                          | 0.44                          | 0.53                          | 0.23                          |
| SAX J1808.4-3658(SS2) | 7.951  | 0.9         | 1.91                          | 0.98                          | 1.12                          | 0.49                          |
Fig. 1 (a) Density (b) pressure curves (c) anisotropy curve (d) $\frac{dp}{dr}$ curves (e) $v_c^2$ curves (f) $v_t^2 - v_l^2$ curve for the first model describing neutron star SAX J1808.4-3658(SS2) mentioned in Table 1
order to get \( \frac{dp}{dr} \leq 0 \), we impose the following condition to the second model

\[
\begin{align*}
- R^4 \beta_1^2 + 2a^2 R^4 \beta_1(4 + R^2 \beta_1) + 8p_0(16 + R^2 \beta_1) + a^2(-16 + 24R^2 \beta_1 + 7R^4 \beta_1^2) + a^2(-48 - 32R^2 \beta_1 - 9R^4 \beta_1^2 + 8p_0(4 + R^2 \beta_1))
+ a(-16p_0(-4 + R^2 \beta_1) + R^2 \beta_1 * (16 + 5R^2 \beta_1)) & \geq 0
\end{align*}
\]

Conditions (iii) and (iv) are satisfied if the parameters of the second model satisfy the following conditions, which we obtained by calculating \( \frac{dp}{dr} \) and \( \frac{dp}{dr} \) at \( r = 0 \) and \( r = R \).

\[
0 \leq \frac{p_0(1 - 2a)}{5a^2} \leq 1 \quad \text{and} \quad 0 \leq \frac{p_0(1 - a)}{a^2(a - 5)} \leq 1
\]

\[
0 \leq \frac{48a^2 - 128p_0 + 192a \rho_0 + 16\rho_0^2 - 16aR^2 \beta_1 + 8p_0R^2 \beta_1 + R^4 \beta_1^2)}{320a^2} \leq 1
\]

\[
0 \leq \frac{1}{64(-5 + a)a^2}((R^4 \beta_1^2 + 2a^2 R^4 \beta_1(4 + R^2 \beta_1) - 8p_0(16 + R^2 \beta_1)
+ a^2(16 - 24R^2 \beta_1 - 7R^4 \beta_1^2) + a^2(48 + 32R^2 \beta_1 + 9R^4 \beta_1^2)
- 8p_0(4 + R^2 \beta_1)) + a(16p_0(-4 + R^2 \beta_1) - R^2 \beta_1(16 + 5R^2 \beta_1))) \leq 1
\]

\[
48a^2 - 64p_0 + 64a \rho_0 + 16\rho_0^2 - 16aR^2 \beta_1
+ 8p_0R^2 \beta_1 + R^4 \beta_1^2 > 0
\]

\[
R^4 \beta_1^2 + 2a^2 R^4 \beta_1(4 + R^2 \beta_1) - 8p_0(8 + R^2 \beta_1)
+ a^2(16 - 24R^2 \beta_1 - 7R^4 \beta_1^2)
+ a^2(48 + 32R^2 \beta_1 + 9R^4 \beta_1^2 - 8p_0(4 + R^2 \beta_1))
+ a(16p_0(-8 + R^2 \beta_1) - R^2 \beta_1(16 + 5R^2 \beta_1)) > 0
\]

### 4.2. Physical analysis

In this section we analyze the behavior of physical parameters such as energy density and pressure at the interior of the star for the second model describing four observed neutron stars. We calculated parameters for each star such that the second model describing four neutron stars mentioned in Table 3, satisfies all bound conditions from (i) to (iv). We obtained the mass (M) and the radius (R) of the neutron stars from references [67, 68] and calculated the central density (\( \rho_c \)), the surface density (\( \rho_s \)), the central pressure (\( p_c(r = 0) = p_i(r = 0) = p_s \)), and the surface tangential pressure (\( p_t(r = R) = p_s \)) of each star, shown in Table 4. It can be observed that this second model describes potentially stable neutron stars of higher central density and higher central pressure than neutron stars described by the first model. The modification in the radial pressure for the second model leads to a description of stable neutron stars of comparable higher density and higher pressure profile. We have plotted various parameters of the star shown in Figs. 3 and 4 of a neutron star RX J 1856-37 (1) shown in Table 3, described by the second model of the neutron stars in teleparallel gravity. The figures indicate that all the star’s physical parameters are

### Table 3 Values of the model parameters of the second model describing four observed neutron stars with \( \beta = 2 \)

| Star      | R (km) | M (M\(_{\odot}\)) | a     | \( \beta_1 \times 10^4 \) | \( p_0 \) (km\(^{-2}\)) |
|-----------|--------|-----------------|-------|---------------------------|--------------------------|
| RX J 1856-37 | 6      | 0.9031          | -0.805 | -119.44                   | 0.21                     |
| Cen X-3   | 9.51   | 1.49            | -0.869 | -49.757                   | 0.2                      |
| EXO 1785-248 | 8.99 | 1.3             | -0.751 | -37.119                   | 0.2                      |
| PSR J1614-2230 | 13 | 1.908          | -0.77  | -18.343                   | 0.2                      |
well-behaved and follow all the conditions (i) to (iv) at all interior points of the star.

4.3. Equation of state

We have derived a second physically acceptable model of the neutron star. To predict the material composition of the stellar configuration, we need to generate an equation of state of the stellar configuration for this model. By using Eqs. (23 and 40), we have plotted the variation of the radial pressure against the energy density for the second model, as shown in Fig. 2(h). From the plot, we find an EOS of the form

\[ p_r = l \rho^2 + m \rho + n, \]

where \( l, m, \) and \( n \) are constants.

By approximating the EOS in Fig. 2(h), we obtain

\[ l = 9.6, \quad m = 0.077 \text{ and } n = -18 \times 10^{-5}. \]

5. Third anisotropic model of neutron star in teleparallel gravity

Now, we develop the third model for the neutron star. We consider the following form of the metric potential:

\[ \epsilon^{(r)} = \left(1 - \frac{a \rho^2}{R^2}\right)^2 \]

where \( a \) is constant and \( R \) is the radius of the neutron star. We consider \( a < 1 \), so that the metric does not become singular at any point for \( r < R \). By substituting Eqs. (54) in (17), we obtain

\[ 4\pi\rho = \frac{\beta_1}{4} + \frac{\beta}{2} \left(\frac{\alpha \rho^2 - 2a}{1 - a \rho^2}\right)^2 - \frac{2a\beta}{(1 - a \rho^2)^3} \]

Now from Eq. (18), we write

\[ c'(r) = \left(4\pi p_r\right)\frac{\rho (4\pi p_r)}{\rho} + \frac{\epsilon^{(r)} - 1}{r} + \frac{\beta_1 e^{(r)}}{2\beta} \]

In order to solve the equation, we consider the following form of \( 4\pi p_r \),

\[ 4\pi p_r = \frac{\rho_0}{1 - \frac{r^2}{R^2}} \]

where \( \rho_0 \) is constant in this model also, which is the parameter of the model such that \( \rho_0 \) denotes the central pressure of the neutron star. The constraint on the \( \rho_0 \) is that \( \rho_0 > 1 \), so that the radial pressure remains positive for

Table 4 Values of the physical parameters of the four observed neutron stars described by the second model

| Star          | R (km) | M (\( M_\odot \)) | \( \rho_c \times 10^3 \) (km\(^{-2}\)) | \( \rho_s \times 10^3 \) (km\(^{-2}\)) | \( \rho_c \times 10^3 \) (km\(^{-2}\)) | \( \rho_s \times 10^3 \) (km\(^{-2}\)) |
|---------------|-------|------------------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|
| RX J 1856-37  | 6     | 0.9031           | 5.32                                   | 1.84                                   | 4.64                                   | 0.261                                  |
| Cen X-3       | 9.51  | 1.49             | 2.19                                   | 0.75                                   | 1.76                                   | 0.73                                   |
| EXO 1785-248  | 8.99  | 1.3              | 2.15                                   | 0.83                                   | 1.97                                   | 0.65                                   |
| PSR J1614-2230| 13    | 1.908            | 1.05                                   | 0.40                                   | 0.98                                   | 0.32                                   |

Fig. 3 (a) Density curve and (b) pressure curves for the second model describing neutron star RX J 1856-37 mentioned in Table 3
0 < r < R. The particular form of \( p_r \) is physically reasonable because it is a monotonically decreasing function of \( r \) and satisfies the condition of vanishing radial pressure at the surface of the neutron star \( r = R \). By substituting values of Eqs. (54 and 57) into (56), we obtain...
By integrating the above equation, we obtain the metric function as

\[ c'(r) = Ke^{\frac{\beta_0 r}{\beta R^2}} \left( 1 - \frac{r^2}{R^2} \right) \left( 1 - \frac{a r^2}{R^2} \right) \left( 1 - \frac{2r}{R} \right)^{\frac{\beta}{\beta_0}} \]  

(58)

By integrating the above equation, we obtain the metric function as

\[ c'(r) = Ke^{\frac{\beta_0 r}{\beta R^2}} \left( 1 - \frac{r^2}{R^2} \right) \left( 1 - \frac{a r^2}{R^2} \right) \left( 1 - \frac{2r}{R} \right)^{\frac{\beta}{\beta_0}} \]  

(59)

Now the interior space-time of the neutron star in this model is given by

\[ ds^2 = \left( Ke^{\frac{\beta_0 r}{\beta R^2}} \left( 1 - \frac{r^2}{R^2} \right) \left( 1 - \frac{a r^2}{R^2} \right) \left( 1 - \frac{2r}{R} \right)^{\frac{\beta}{\beta_0}} \right) dr^2 \]

\[ - \left( 1 - \frac{a r^2}{R^2} \right) 2 dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(60)

5.1. Constraining parameters for a physically well-behaved model

5.1.1. Matching exterior space-time with interior space-time

The space-time metric should be continuous at the boundary of the star. Thus, the star’s interior metric (60) should be matched to the exterior Schwarzschild space-time metric (29) at \( r = R \) for the third model. Now, the interior metric (60) would be matched to the exterior Schwarzschild metric at the surface of the star if

\[ (1 - a)^2 = \left( 1 - \frac{2M}{R} \right)^{-1} \]  

(61)

and

\[ Ke^{\frac{\beta_0 R^2}{\beta R^2}} \left( a^2 + 3a + 3 \right) \left( 1 + \frac{a}{R} \right)^{\frac{\beta}{\beta_0}} = 1 - \frac{2M}{R} \]  

(62)

From Eqs. (61) and (62) we obtain the value of \( K \) for the third model as

\[ K = 1 \left( 1 - \frac{a}{R} \right)^{-2} e^{-\frac{\beta_0 R^2}{\beta R^2}} \left( a^2 + 3a + 3 \right) \left( 1 + \frac{a}{R} \right)^{\frac{\beta}{\beta_0}} \]  

(63)

For a given value of \( a, p_0, \beta, \beta_1 \) and \( R \), the value of constant \( K \) in Eq. (60) can be calculated from Eq. (63).

5.1.2. Imposing physical conditions to the third model

As physically acceptable model of the compact star must obey physical conditions (i) to (iv) as described in section (3.1.2). We impose the condition (i) by enforcing the following inequality computed from \( p_r \geq 0 \) in the third model.

\[ 32\beta p_0 - \beta_1 R^4 + 5a\beta_1 R^2 + 5a^2(2\beta + \beta_1 R)^2 + a^2(2\beta + \beta_1 R)^2 \]

\[ + 16\beta \left(-p_0 + \beta_1 R^2 \right) + 2a^3(24\beta^2 + 22\beta_1 R^2 + 5\beta_1^2 R^4) \]

\[ - 2a^2(24\beta^2 - 16\beta_1 R + 2\beta_1^2 R^2) \leq 0 \]  

(64)

Now, in the third model we already have \( \frac{dp_r}{dr} \leq 0 \). In order to get \( \frac{dp_r}{dr} \leq 0 \) we impose the following condition to the third model

\[ - 64\beta p_0 + \beta_1 R^2(-8p_0 + \beta_1 R^2) \]

\[ - 8a^3(2\beta + \beta_1 R^2)(\beta - p_0 + 2\beta_1 R^2) \]

\[ + 8a^3(2p_0 - \beta_1 R^2)(11\beta + 5\beta_1 R^2) + 8a(24\beta p_0 - 2\beta_1 R^2) \]

\[ + 5\beta_1 p_0 R^2 - \beta_1^2 R^4 \]

\[ + a^4(4\beta^2 + 8\beta_1 R + 3\beta_1^2 R^4) + a^2(48\beta^2 - 240\beta p_0 \]

\[ + 56\beta_1 \beta_1 R^2 - 80\beta_1 p_0 R^2 \]

\[ + 25\beta_1^2 R^4 \]  

\[ + a^4(12\beta^2 - 80(p_0 - \beta_1 R^2) \]

\[ + 5\beta_1 R^2(-8p_0 + 7\beta_1 R^2)) \geq 0 \]  

(65)

Now, conditions (iii) and (iv) would be satisfied if the

| Star | \( R \) (km) | \( M \) (\( M_{\odot} \)) | \( \rho_c \times 10^3 \) (km\(^{-3} \)) | \( \rho_v \times 10^3 \) (km\(^{-3} \)) | \( p_c \times 10^4 \) (km\(^{-2} \)) | \( p_v \times 10^4 \) (km\(^{-2} \)) |
|-------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| HER X-1 | 10.7655 | 0.85 | 1.24 | 0.79 | 0.70 | 0.30 |
| PSR J0737-3039A | 14.68 | 1.4 | 0.64 | 0.42 | 0.33 | 0.14 |
| PSR J0437-4715 | 13.8 | 1.4 | 0.73 | 0.47 | 0.40 | 0.17 |

Table 5: Values of the model parameters of the third model describing three neutron stars with \( \beta_1 = 0 \)

Table 6: Values of the three observed neutron stars described by the third model
Fig. 5  (a) Density curve and (b) pressure curves (c) anisotropy curve (d) $\frac{dp}{dr}$ curves (e) $v_s^2$ curves (f) $v_t^2$ curve for the third model describing neutron star J0737-3039A mentioned in Table 5.
parameters of the third model satisfy the following conditions, which we deduced by calculating \( \frac{dp}{dr} \) and \( \frac{dp}{d\rho} \) at \( r = 0 \) and \( r = R \).

\[
0 \leq -\frac{2\rho_0(a-1)}{15\alpha a^2} \leq 1 \quad \text{and} \quad 0 \leq -\frac{2\rho_0(a-1)^3}{\beta a^2(a^2 - 4a + 15)} \leq 1 \quad (66)
\]

\[
0 \leq -\frac{48a^2\beta^2 - 16a\beta\beta_1R^2 + 16p_0 + \beta_1^2R^4 + 8p_0(-8\beta + \beta_1R^2)}{240a^2\beta^2} \leq 1 \quad (67)
\]

\[
0 \leq \frac{1}{16a^2(15 - 4a + \alpha^2)}(64\beta p_0 + \beta_1 R^2(8 p_0 - \beta_1 R^2)) + 8a\beta(2\beta + \beta_1 R^2)
\]

\[
\times (\beta - p_0 + 2\beta_1 R^2) - 8\alpha^3(2p_0 - \beta_1 R^2)(11\beta + 5\beta_1 R^2) - 8a(24p_0 - 2\beta_1 R^2 + 5p_0 R^2 - \beta_1^2 R^4 - \alpha^6(4\beta^2 + 8\beta\beta_1 R^2 + 3\beta_1^2 R^4)
\]

\[
+ a^2(-48\beta^2 + 8\beta(30\beta_0 - 7\beta_1 R^2 + 5\beta_1 R^2(16 p_0 - 5\beta_1 R^2))
\]

\[
+ a^2(-12\beta^2 + 5\beta_1 R^2(8p_0 - 7\beta_1 R^2) + 80\beta(p - \beta_1 R^2))) \leq 1 \quad (68)
\]

\[
48a^2\beta^2 - 32\beta p - 16a\beta(2p + \beta_1 R^2) + (4p + \beta_1 R^2)^2 < 0 \quad (69)
\]

\[
\frac{1}{16a^2(\alpha^2 - 4a + 15)}(32\beta p + \beta_1 R^2(8p - \beta_1 R^2))
\]

\[
+ 8a^3(2\beta + \beta_1 R^2)
\]

\[
\times (\beta - p + 2\beta_1 R^2) - 8\alpha^3(18\beta p - 11\beta_1 R^2)
\]

\[
+ 10\beta_1 p R^2 - 5\beta_1^2 R^4) \quad (70)
\]

\[
- 8a(12\beta p - 2\beta_1 R^2 + 5\beta_1 p R^2 - \beta_1^2 R^4)
\]

\[
- a^6(4\beta^2 + 8\beta\beta_1 R^2 + 3\beta_1^2 R^4)
\]

\[
+ a^2(-48\beta^2 + 8\beta(18p - 7\beta_1 R^2 + 5\beta_1 R^2(16p - 5\beta_1 R^2))
\]

\[
+ a^2(-12\beta^2 + 5\beta_1 R^2(8p - 7\beta_1 R^2) + 80\beta(p - \beta_1 R^2))) > 0
\]

5.2. Physical analysis

This section analyzes the behavior of physical parameters of the third model describing three observed neutron stars. We calculate parameters for each star such that the third model describing three neutron stars mentioned in Table 5. satisfies all bound conditions from (i) to (iv). We obtained the mass (M) and the radius (R) of the neutron stars from references [69, 70] and calculated the central density(\( \rho_c \)), surface density(\( \rho_s \)), central pressure(\( p_c(r = 0) = p_c(r = 0) = p_c \)), and the surface tangential pressure(\( p_t(r = R) = p_t \)) of each star, shown in Table 6. From Table 6, we can observe that this third model describes stable neutron stars of low mass and higher outer radius comparable to neutron stars described by the first two models. Neutron stars described by the third model have low density and low pressure profile comparable to neutron stars described by the first two models. The particular modification in the metric potential and the chosen model parameters for the third model lead to a description of stable neutron stars of lower density and lower pressure profile comparable to both the first two models. We have plotted various parameters of the star shown in the Figs. 5 and 6 of a neutron star PSR J0737-3039A (2) shown in Table 5, described by this third model of the neutron stars in teleparallel gravity. The figures indicate that all the star’s physical parameters are well-behaved and follow all the conditions (i) to (iv) at all interior points of the star.

5.3. Equation of state

We have derived a third physically acceptable model of the neutron star. To predict the material composition of the stellar configuration, we need to generate an equation of state of the stellar configuration for this model. By using Eqs. (55 and 57), we have plotted the variation of the radial
pressure against the energy density for the third model, as shown in Fig. 3(h). From the plot, we find an EOS of the form $p_r = l_2 q^2 + m_2 q + n$, where $l_2, m_2$ and $n_2$ are constants.

By approximating the EOS in Fig. 3(h), we obtain $l_2 = -99$, $m_2 = 0.26$ and $n_2 = -9.3 \times 10^{-5}$.

6. Adiabatic stability of neutron stars described by three models

In the previous sections, we imposed energy conditions as well as the requirement that the star’s density and pressure be monotonically decreasing functions of radial distance. We also investigated the causality and stability of neutron stars described by the models in terms of the speed of sound. In addition, we showed that the three models describing anisotropic neutron stars satisfy all conditions (i) to (iv) successfully. Apart from these conditions, it is also beneficial to investigate the adiabatic stability of compact stars. The adiabatic index ($\Lambda$), related to the thermodynamical quantity, serves this purpose quite well. Using the variational method, the condition for dynamical stability of the star was investigated by Chandrasekhar [71]. The dynamical stability conditions have also been investigated for shearing viscous objects as well as anisotropic stars [72, 73]. According to these studies, the adiabatic index $\Lambda$ can be written as

$$\Lambda = \frac{\rho + \rho_r \frac{d\rho_r}{d\rho}}{\rho_r}$$  \hspace{1cm} (71)

For adiabatic stability of compact stars, the adiabatic index $\Lambda$ should be greater than $4/3$, i.e., $\Gamma > 4/3$, where $\Gamma$ is given by Eq. (71). We have plotted the adiabatic index $\Gamma$ for the three models developed in this paper in Fig. 7. The plots show that for all three models, $\Gamma > 4/3$ throughout the interior of the stars. It is also a monotonically increasing function of the radial distance. Hence, the condition of adiabatic stability is satisfied.
7. Conclusions

The idea of teleparallel gravity has been used to provide a novel class of solutions for three analytical models that describe anisotropic stellar structures for different observed neutron stars. The concept of the tetrad field and the construction of the teleparallel gravity field equations have been briefly examined. Then, we solved the teleparallel gravity equations for a spherically symmetric anisotropic matter distribution using some physically reliable metric potentials and radial pressures. Thus, we developed three analytical models describing the anisotropic stellar structures of neutron stars. To make the models physically plausible, we determined various model parameters by applying physical bound constraints to them. We calculated model parameters and the star’s physical parameters for various observed neutron stars described by the three models developed in this paper. The physical analysis of each model revealed that specific changes in the metric potential and radial pressure result in a description of stable neutron stars ranging from low density and pressure to high density and pressure. We demonstrated that models with very good approximations admit quadratic equations of state, which is very useful in determining the compositions of stellar structures of specific neutron stars as well as strange stars, as well as in modelling such realistic neutron stars. It is also worth noting that, the linear form of $f(T)$ is responsible for the quadratic behavior of the EOS. The non-linear forms of $f(T)$ lead to a high deviation of the EOS from the quadratic behavior. Thus, the authors chose to work with the linear EOS. We have also plotted graphs of various parameters of observed neutron stars described by each model developed in this paper and studied their adiabatic stability. The graph shows that each model is physically reliable and realistic in all aspects. The research work presented in this paper provides an understanding of how a modification in the model parameters can lead to a description of various neutron stars described by teleparallel gravity. This work can also be utilized to investigate the reasons for anisotropy in neutron stars due to pion condensation, hyperon interaction and electromagnetic forces in teleparallel gravity theory.

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