Generalized $R^p$-attractor Cosmology in the Jordan and Einstein Frames: New Type of Attractors and Revisiting Standard Jordan Frame $R^p$ Inflation

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In this work we shall study a new class of attractor models which we shall call generalized $R^p$-attractor models. This class of models is based on a generalization of the Einstein frame potential of $R^p f(R)$ gravity models in the Jordan frame. We present the attractor properties of the corresponding non-minimally coupled Jordan frame theory, and we calculate the observational indices of inflation in the Einstein frame. As we show, there is a large class of non-minimally coupled scalar theories, with an arbitrary non-minimal coupling which satisfies certain conditions, that yield the same Einstein frame potential, this is why these models are characterized attractors. As we demonstrate, the generalized $R^p$-attractor models are viable and well fitted within the Planck constraints. This includes the subclass of the generalized $R^p$-attractor models, namely the Einstein frame potential of $R^p$ inflation in the Jordan frame, a feature also known in the literature. We also highlight an important issue related to the $R^p$ inflation in the Jordan frame, which is known to be non-viable. By conformal invariance, the $R^p$ inflation model should also be viable in the Jordan frame, which is not. We pinpoint the source of the problem using two different approaches in the $f(R)$ gravity Jordan frame, and as we demonstrate, the problem arises in the literature due to some standard simplifications made for the sake of analyticity. We demonstrate the correct way to analyze $R^p$ inflation in the Jordan frame, using solely the slow-roll conditions.

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I. INTRODUCTION

The physics of the primordial Universe is currently in the focus of both theoretical and observational cosmologists. The next fifteen years will be sensational for physicists since several experiments which will probe the early Universe, will commence their operation, like the stage 4 Cosmic Microwave Background (CMB) experiments \cite{1, 2} and the gravitational wave future interferometers like LISA \cite{3, 4}, DECIGO \cite{5, 6}, BBO \cite{7, 8} and other experiments \cite{9–11}. The main aim of these experiments is to probe the early Universe in two ways: firstly to observe whether a relic stochastic gravitational wave background which was generated primordially exists in the Universe, and secondly whether any tensor modes are imprinted in the CMB, the so called B-modes (curl modes). Any observation of the aforementioned two, will verify the existence of the inflationary era. Inflation \cite{12–15} is one of the most appealing candidate scenarios for describing the early Universe, with alternatives being bouncing cosmologies \cite{16–22}. The inflationary scenario will be put into test the next fifteen years, so it is of profound important to study many aspects of inflationary dynamics. Inflationary scenarios can arise from many theoretical frameworks, with the two mainstreams descriptions being scalar field theory \cite{12, 14, 22} and modified gravity \cite{24, 26}.

One appealing property of the scalar field models is that these can be classified to attractor models \cite{27}, and several works already appear in the literature studying the attractor properties of scalar models, see for example Refs. \cite{27, 65} and references therein. In this paper the focus is on a new type of attractor models which we shall call generalized $R^p$-attractor models. These models are based on a scalar potential which is a direct generalization of the Einstein frame potential corresponding to the Jordan frame $R^p$ gravity of the form $f(R) = R + \beta R^p$. We shall study the general features of the generalized $R^p$-attractor models and specifically we shall present the features of the non-minimally coupled scalar field theory which corresponds to the Einstein frame $R^p$-attractor models scalar potential. As we shall show, the generalized $R^p$-attractor models in the Einstein frame generate a viable phenomenology, including the Einstein frame potential corresponding to the Jordan frame $R^p$ theory. This latter feature has also been pointed out in the literature \cite{66, 67}. However, it is known from the literature that power-law $R^p$ inflation in the Jordan frame is not a viable scenario. This is in contrast to the general rule that dictates that conformal invariant quantities of conformally related theories in the Jordan and Einstein frames, should be the same in the two frames \cite{68, 70}. In order to pinpoint the source of the problem, we present two formalisms of Jordan frame $R^p$ inflation. As we demonstrate, the two approaches lead to different results, and as we show, the problem with the standard approach of $R^p$ inflation in the literature is probably a simplification made for the sake of analyticity for Jordan frame $R^p$ inflation. As a concluding remark, our work indicates that although generalized $R^p$ inflation can be difficult to study in the Jordan frame, the Einstein frame potential is rather easy to work with and the conformal invariant quantities should...
be identical, thus the Einstein frame study suffices to describe the generalized $R^p$ inflation. Our study could have potentially interesting implications in neutron stars, since inflationary potentials and inflationary modified gravity theories are frequently being used in these contexts, see for example [70–74].

This work is organized as follows: In section II we present the standard $R^p$ inflation scenario in the Einstein frame and we introduce the generalized $R^p$-attractor potential in the Einstein frame. We describe the attractor properties of the generalized $R^p$-attractor models in the Jordan frame in terms of the non-minimally coupled scalar theory. We also investigate the case which leads to the standard $R^p$ inflation scenario in the Einstein frame. We study in detail the inflationary properties of generalized $R^p$-attractor models and we show that the predictions are very well fitted within the Planck constraints. In section III we discuss the problem of $R^p$ inflation in the Jordan and Einstein frames and we pinpoint the probable reason which leads to the conflict between the two frames. We also discuss the difficulties of studying the generalized $R^p$-attractor model in the Jordan frame.

Before we proceed to the core of this study, let us mention that we shall assume that the spacetime is described by a flat Friedmann-Robertson-Walker metric, with the line element being,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,$$

where $a(t)$ denotes as usual the scale factor. Furthermore, we assume that the metric connection is a metric compatible, symmetric, and torsion-less affine connection, the Levi-Civita connection. For the FRW metric, the Ricci scalar becomes,

$$R = 6(2H^2 + \dot{H}) ,$$

where $H$ denotes the Hubble rate $H = \dot{a}/a$. Also we use the natural units physical system.

II. GENERALIZED $R^p$-ATTRACTORS AND $F(R)$ GRAVITY DESCRIPTION

In this section we shall discuss the essential features of the $R^p$ inflation in the Einstein frame and we shall introduce a generalized $\alpha$-attractor-like potential which we shall call for simplicity generalized $R^p$-attractor potential. This Einstein frame theory corresponds to a large class of scalar theories in the Jordan frame, and this justifies the terminology attractor for the $\alpha$-attractor like potential which we called generalized $R^p$-attractor potential.

A. $R^p$ Inflation in the Einstein Frame

Before we get into the core of our study, let us recall the essential features of a Jordan frame vacuum $F(R)$ gravity, for details see the reviews [24–26]. The Jordan frame action for vacuum $F(R)$ gravity is,

$$S = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} F(R) ,$$

with $\hat{g}_{\mu\nu}$ being the Jordan frame metric tensor. Upon introducing the auxiliary scalar field $A$ for the Jordan frame action, the gravitational action takes the following form,

$$S = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} (F'(A)(R - A) + F(A)) .$$

Upon variation of the action with respect to the scalar field $A$, one obtains the solution $A = R$, which proves the mathematical equivalence of the gravitational actions and .

In order to obtain the Einstein frame scalar field potential corresponding to the $F(R)$ gravity, we perform a conformal transformation,

$$g_{\mu\nu} = e^{-\varphi} \hat{g}_{\mu\nu} ,$$

with $g_{\mu\nu}$ being the Einstein frame metric. The Jordan and Einstein frames are connected via the canonical transformation,

$$\varphi = \sqrt{\frac{3}{2}} \ln(F'(A)) ,$$
where $\varphi$ denotes the Einstein frame canonical scalar field. The conformally transformed action is,

$$
\hat{S} = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} \left( \frac{F''(A)}{F'(A)} \right)^2 g^{\mu\nu} \partial_\mu A \partial_\nu A - \left( \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \right) \right) 
$$

$$
= \int d^4x \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right). 
$$

(7)

The corresponding canonical scalar field potential $V(\varphi)$ in the Einstein frame is,

$$
V(\varphi) = \frac{1}{2} \left( \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \right) = \frac{1}{2} \left( e^{-\sqrt{2/3}r} R \left( e^{\sqrt{2/3}r} \right) - e^{-2\sqrt{2/3}r} F \left( R \left( e^{\sqrt{2/3}r} \right) \right) \right). 
$$

(8)

Also the Ricci scalar in terms of the canonical scalar field can easily be obtained by solving Eq. (6) with respect to $F(R)$ which is well studied in the recent literature [66, 67]. The scalar field $R$ in terms of the canonical scalar field and also in Eq. (10), one may obtain the actual $F(R)$ gravity which can realize a specific scalar potential. This can be done by differentiating both sides of Eq. (8) with respect to the Ricci scalar and also by using $\frac{d\varphi}{dR} = \sqrt{\frac{3}{2} \frac{F''(R)}{F'(R)}}$. The resulting equation is the following,

$$
RF_R = 2 \sqrt{\frac{3}{2}} \frac{d}{d\varphi} \left( \frac{V(\varphi)}{e^{-2\left(\sqrt{2/3}\varphi\right)}} \right) 
$$

(9)

where $F_R$ stands for $F_R = \frac{dF(R)}{dR}$. The above differential equation in conjunction with Eq. (9) basically yields the vacuum $F(R)$ gravity in the Jordan frame which generates the canonical scalar field potential $V(\varphi)$ in the Einstein frame. In the same vain, Eq. (8) can yield the scalar potential that corresponds to a specific $F(R)$ gravity. Now let us use the above relations and let us discuss the $R^p$ inflation theory in the Einstein frame.

B. Generalized $R^p$ Attractor Potential and the Attractor Property in the Scalar-Jordan Frame Theory

For attractor models, the terminology “attractors” indicate that these models belong to a class of models which produce the same inflationary indices, see for example [27, 33, 35, 36]. Many well known inflationary models belong to some attractor potential category, like the $R^2$ [75, 76], and the Higgs models [77]. All the attractor models are basically Einstein frame models in vacuum described of course by a minimally coupled scalar field. In addition, these models have a Jordan frame counterpart theory described by a non-minimally coupled scalar field theory, but all the models also have an $F(R)$ gravity description. So in order not to confuse the Jordan frame $F(R)$ gravity description and the Jordan frame non-minimally coupled scalar theories, we shall refer to the latter as “$\phi$-Jordan frame” and to the Jordan frame $F(R)$ gravity as “Jordan frame $F(R)$ gravity theory”.

Here we shall consider generalizations of the following Einstein frame scalar potential,

$$
V(\varphi) = V_0 e^{-2\sqrt{\frac{3}{2}}\kappa \varphi} \left( e^{\sqrt{\frac{3}{2}}\kappa \varphi} - 1 \right)^{\frac{p}{2}}, 
$$

(10)

which is well studied in the recent literature [66, 67]. The scalar field $\varphi$ describes an Einstein frame minimally coupled canonical scalar field, and also in Eq. (10), $p$ is an arbitrary number for the Einstein frame theory and recall $\kappa = 1/M_p$ where $M_p$ is the reduced Planck mass. The Einstein frame potential above corresponds to the following Jordan frame $F(R)$ gravity theory,

$$
F(R) = R + \beta R^p, 
$$

(11)

where $\beta$ is a free parameter with mass dimensions $[\beta] = [m]^{-2-2p}$. As we will show explicitly in a later section, in order for the model (11) to describe an inflationary model in the Jordan $F(R)$ theory, the parameter $p$ has to be chosen in the range $\frac{1}{2} < p < 2$, however in the Einstein frame such a constraint does not apply.

The generalization of the potential which we shall consider in this work has the following form,

$$
V(\varphi) = V_0 e^{-2\sqrt{\frac{3}{2}}\kappa \varphi} \left( e^{\sqrt{\frac{3}{2}}\kappa \varphi} - 1 \right)^{\frac{p}{2}}, 
$$

(12)
which for \( \alpha = 1 \) becomes identical with the \( R^p \) model in the Einstein frame. Now let us demonstrate to which attractor class does the potential (12) belong to, and to this end, consider the following \( \phi \)-Jordan frame action,

\[
S_J = \int d^4x \left( \frac{\Omega(\phi)}{2\kappa^2} R - \frac{\omega(\phi)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V_J(\phi) \right),
\]

(13)

where the scalar field describes the non-canonical Jordan frame scalar field and \( \Omega(\phi) = 1 + \xi f(\phi) \) with \( \xi \) being an arbitrary parameter and \( f(\phi) \) and analytic function of \( \phi \). The particular choice for the function \( \Omega(\phi) = 1 + \xi f(\phi) \) is motivated by many non-minimally coupled theories of gravity having this form, like the Higgs scalar tensor theory (see Eq. (52) of [78] and Eq. (31) of [79]).

We shall call the attractor class of the Einstein frame potential (12) as “\( R^p \)-attractors”, and these are realized when the scalar potential in the \( \phi \)-Jordan frame is chosen to be of the following form,

\[
V_J(\phi) = V_0 (\Omega(\phi) - 1)^{\frac{p}{p-1}},
\]

(14)

and also the kinetic term function \( \omega(\phi) \) is chosen to be,

\[
\omega(\phi) = \frac{1}{4\xi} \left( \frac{d\Omega(\phi)}{d\phi} \right)^2.
\]

(15)

Thus basically, the \( R^p \)-attractors correspond the choices (14) and (15). Performing the following conformal transformation on the Jordan frame metric \( g_{\mu\nu} \),

\[
\tilde{g}_{\mu\nu} = \Omega(\phi) g_{\mu\nu},
\]

(16)

where \( \tilde{g}_{\mu\nu} \) is the Einstein frame metric tensor, we obtain the following Einstein frame action,

\[
S_E = \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^2} - \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),
\]

(17)

where the “tilde” denotes Einstein frame quantities, and the Einstein frame potential \( V(\phi) \) is related with the Jordan frame potential \( V_J(\phi) \) in the following way,

\[
V(\phi) = \Omega^{-2}(\phi) V_J(\phi).
\]

(18)

The general relation between the Jordan frame scalar field \( \phi \) and the canonical Einstein frame scalar field \( \varphi \) is [69],

\[
\left( \frac{d\varphi}{d\phi} \right)^2 = \frac{3}{2} \left( \frac{d\Omega(\phi)}{d\phi} \right)^2 + \frac{\omega(\phi)}{\Omega(\phi)},
\]

(19)

so for the case of the generalized \( R^p \)-attractors for which the kinetic term function \( \omega(\phi) \) is given by Eq. (15), by integrating the relation between the Jordan frame scalar field \( \phi \) and the canonical Einstein frame scalar field \( \varphi \), namely Eq. (19), we get,

\[
\Omega(\phi) = e^{\sqrt{\frac{3p}{6\xi}} \varphi},
\]

(20)

where the parameter \( \alpha \) is defined to be,

\[
\alpha = 1 + \frac{1}{6\xi}.
\]

(21)

It is apparent that by substituting Eq. (20) in Eq. (18) one obtains the generalized \( R^p \)-attractor potential of Eq. (12).

Before we proceed, let us discuss several important issues related to the generalized \( R^p \)-attractor. Firstly, in the Einstein frame there is no constraint on the parameter \( p \), in contrast to the Jordan frame \( F(R) \) description of \( R^p \) inflation, in which case in order for having an inflationary description, the parameter \( p \) is constrained to be \( 1 + \frac{\sqrt{3}}{2} < p < 2 \). Secondly, the \( R^p \) theory is not a viable inflationary theory in the \( F(R) \) Jordan frame, in contrast to its Einstein frame counterpart theory with potential (10), as it can be seen in Refs. [66, 67]. Furthermore, we
abviously used the terminology $R^p$-attractors to characterize the class of attractor potentials \cite{12}, since the Einstein frame potential \cite{12} does not originate from an $R^p$ theory in the Jordan $F(R)$ frame. In fact it is impossible to find the $F(R)$ Jordan frame theory for a general $p$ and can be found for specific values of $\alpha$ and $p$, without these values guaranteeing the viability of the inflationary theory (see the Appendix A for an explicit example for which the $F(\tilde{R})$ gravity can be evaluated). Moreover, the case $\alpha = 1$ can be realized when $\xi \to \infty$, or equivalently when $\Omega(\phi) \ll \frac{3}{2} \left( \frac{\alpha(\phi)}{\omega(\phi)} \right)^2$ (see for example Ref. \cite{39}). The $\phi$-Jordan frame counterpart theory to the potential \cite{12} will yield the same inflationary phenomenology with the Einstein frame theory with the potential \cite{12} for a general $\alpha$. However for $\alpha = 1$, the same does not apply to the $F(R)$-Jordan frame theory, which as we show in a later section is not a viable inflationary theory for any value of the parameter $p$. This is a peculiar result, since by the conformal equivalence the $\alpha = 1$ theory with potential \cite{10} and the $R^p$ theory should be equivalent. This however is not true, and in a later section we shall show that this is due to the peculiarity of the actual $R^p$ inflation scenario in the Jordan frame.

Let us proceed to the phenomenology of the generalized $R^p$-attractor potential \cite{12} in some detail. The results should coincide with the phenomenology of the model \cite{10} studied in Refs. \cite{66, 67} for $\alpha = 1$, which in the case of the generalized $R^p$-attractors is obtained in the limit $\xi \to \infty$. To start with, let us calculate the spectral indices for the generalized $R^p$-attractor potential, and since the theory is a minimally coupled scalar field theory, the slow-roll indices are given by (see for example the review \cite{24}),

$$\epsilon = \frac{1}{2\kappa^2} \left( \frac{V'}{V} \right)^2,$$  \hspace{1cm} (22)

$$\eta = \frac{1}{\kappa^2} \frac{V''}{V},$$  \hspace{1cm} (23)

while the spectral index of the primordial scalar perturbations has the form,

$$n_s = 1 - 6\epsilon + 2\eta,$$  \hspace{1cm} (24)

and the tensor-to-scalar ratio is,

$$r = 16\epsilon.$$  \hspace{1cm} (25)

The $e$-foldings number can be expressed in terms of the scalar field potential as follows,

$$N = \kappa^2 \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi,$$  \hspace{1cm} (26)

where $\phi_i$ is the value of the scalar field when the corresponding scalar perturbation mode exits the horizon at the beginning of the inflationary era and $\phi_f$ is the value of the scalar field at the end of the inflationary era. Let us proceed to calculate the inflationary parameters and observational indices in detail for the generalized $R^p$-attractor potential \cite{12}. Firstly let us quote the expressions for the slow-roll indices and these are,

$$\epsilon = \frac{(p - 2)e\sqrt{\frac{p}{3}} \sqrt{\frac{\alpha}{\omega}} - 2p + 2)}{3\alpha(p - 1)^2 \left( e\sqrt{\frac{p}{3}} \sqrt{\frac{\alpha}{\omega}} - 1 \right)^2},$$  \hspace{1cm} (27)

$$\eta = \frac{2 \left( -5p^2 + 13p - 8 \right) e\sqrt{\frac{p}{3}} \sqrt{\frac{\alpha}{\omega}} + (p - 2)^2e^2\sqrt{\frac{p}{3}} \sqrt{\frac{\alpha}{\omega}} + 4(p - 1)^2)}{3\alpha(p - 1)^2 \left( e\sqrt{\frac{p}{3}} \sqrt{\frac{\alpha}{\omega}} - 1 \right)^2}. \hspace{1cm} (28)$$

Having these available the final value of the scalar field at the end of inflation $\phi_f$ is found by solving $\epsilon(\phi) = O(1)$, and we get,

$$\phi_f = \frac{\sqrt{\frac{2}{\alpha}} \ln \left( \frac{3\alpha(3\alpha - 2)p^2 + \sqrt{\frac{2}{\alpha}} \sqrt{\alpha(p - 1)^2p^2 - 6(\alpha - 1)p - 4}}{3\alpha(3\alpha - 1)p^2 + (4 - 6\alpha)p - 4} \right)}{\sqrt{\frac{1}{\alpha}}},$$  \hspace{1cm} (29)
and by using Eq. (20) we can easily find the value of the scalar field at the first horizon crossing at the beginning of inflation \( \varphi_i \) which is,

\[
\varphi_i = \sqrt{\frac{s}{2}} \ln \left( \frac{e^{-\frac{3(H - f)}{\alpha \kappa \varphi}}(p - 2)e^{\frac{3}{2}(p - 2)}(p - 2)e^{\frac{3}{2}H} + 2 - 2p + 2)}{p - 2} \right). 
\]

Accordingly, the spectral index as a function of the scalar field is,

\[
n_s = \frac{3\alpha(p - 1)^2(e^{\frac{3}{2}(p - 2)} - 1)^2}{(3\alpha + (3\alpha - 2)p^2 + (8 - 6\alpha)p - 8)e^{\frac{3}{2}H}(p - 1)(-3\alpha + (3\alpha - 2)p + 8)e^{\frac{3}{2}H}(3\alpha - 8)(p - 1)^2},
\]

and the tensor-to-scalar ratio as a function of the scalar field is,

\[
r = \frac{16(p - 2)e^{\frac{3}{2}H}(p - 2)e^{\frac{3}{2}H} - 2p + 2)^2}{3\alpha(p - 1)^2(e^{\frac{3}{2}H}(p - 2)e^{\frac{3}{2}H} - 1)^2}.
\]

Hence having these analytic relations available we can now proceed with the confrontation of the theory with the Planck 2018 data [80] which constrain the spectral index and the tensor-to-scalar ratio as follows,

\[
n_s = 0.962514 \pm 0.00406408, \quad r < 0.064.
\]

Firstly, let us give several values on the free parameters \( \alpha \) and \( p \) in order to see how the phenomenology behaves depending on the values of these free parameters. For both the indices, the values of \( p \) and \( \alpha \) that yield a viable phenomenology are in the ranges \( 1.9 < p < 2.01 \) and \( 1 < \alpha < 8 \). Also small values of the parameter \( \alpha \leq 1 \) significantly affect the tensor-to-scalar ratio which render it small, however most of these values strongly affect the spectral index making it to have values non compatible with the Planck data. If however, \( p \) takes values quite close to \( p \approx 2 \), for example \( p = 1.9999 \) and \( 0.1 \leq \alpha \leq 0.9 \) then the tensor-to-scalar becomes quite small and the spectral index is compatible with the Planck 2018 data. The results of our analysis where we confront the model for various parameter values can be found in Fig. 1, where we present the model’s predictions compared to the Planck likelihood curves. In the upper left plot of Fig. 1 we used \( 1.9 < p < 2.01 \) and \( 1 < \alpha \leq 8 \) for \( N = 60 \), in the upper right plot we used \( 1.9 < p \leq 1.9999 \) and \( 0.1 \leq \alpha \leq 0.9 \) for \( N = 50 \), whereas in the bottom plot we used \( 1.9 \leq p \leq 1.9999 \) and \( \alpha = 1 \) for \( N = [50, 60] \). As it can be seen in all cases, the model is quite well fitted within the Planck likelihood curves, and in some cases the model is found at the sweet spot of the Planck likelihood curves.

III. \( R^p \) Gravity in the Jordan Frame: A Difference on Inflationary Dynamics

In this section we shall reveal an important feature of the \( R^p \) inflation in the \( f(R) \) Jordan frame. In the previous section we demonstrated that the \( R^p \) inflation is a viable theory in the Einstein frame, for various values of the parameter \( \alpha \), including the value \( \alpha = 1 \), and the model was found to be compatible with the Planck data. The same applies for the \( \phi \)-Jordan frame theory described by the non-minimally coupled generalized \( R^p \)-attractors. However, as we now demonstrate shortly, the \( R^p \) inflation in the \( f(R) \) Jordan frame is not a viable inflationary theory, at least when studied in the formal way appearing in the literature. The focus in this section is investigating the reason why this peculiarity occurs, since the scalar spectral index and the tensor-to-scalar ratio are conformal invariant quantities, thus one would expect that these should be the same in all conformally related frames. As we will see, the incompatibility between the two frames is possibly due to a simplification made in the \( f(R) \) Jordan frame for the sake of analyticity. A similar discussion for the complexity of the Jordan frame counterpart theories, can be found in the Appendix B where a detailed study of the \( \phi \)-Jordan frame theory is presented in detail. Consider the \( R^p \) inflationary theory in the \( f(R) \) Jordan frame,

\[
f(R) = R + \beta R^p,
\]

with \( p \neq 2 \). The Friedman equation of \( f(R) \) gravity is,

\[
3H^2F = \frac{RF - f}{2} - 3HF,
\]
with \( F = \frac{\partial f}{\partial R} \). During the inflationary era, we have approximately \( F \sim p^3 R^{p-1} \), at leading order, so the Friedman equation (35) reads,

\[
3H^2 p^3 R^{p-1} = \frac{\beta(p-1)R^{p-1}}{2} - 3p(p-1)\beta H R^{p-2} \dot{R}. \tag{36}
\]

Also \( R = 12H^2 + 6\dot{H} \), which during the slow-roll inflationary era becomes at leading order \( R \sim 12H^2 \) and \( \dot{R} \sim 24H\dot{H} \), since the slow-roll assumptions in the \( f(R) \) Jordan frame are,

\[
\dot{H} \ll H^2, \quad H \ll H\dot{H}, \tag{37}
\]

and therefore the Friedman equation takes the following form at leading order,

\[
3H^2 p^3 \simeq 6\beta(p-1)H^2 - 6p\beta(p-1)\dot{H} + 3\beta(p-1)\ddot{H}, \tag{38}
\]

which by solving it, yields the following solution for the Hubble rate,

\[
H(t) = \frac{-2p^2 + 3p - 1}{(p - 2)t}. \tag{39}
\]

The evolution in Eq. (39) can describe an inflationary evolution in the \( f(R) \) Jordan frame only if the parameter \( p \) takes values in the range \( \frac{2}{3}\sqrt{2} < p < 2 \). This is in contrast to the Einstein frame theory in which case the parameter \( p \) is unconstrained. Using the evolution of Eq. (39), we can obtain the slow-roll indices for the power law \( f(R) \) gravity model (34), and these have the following form,

\[
c_1 = \frac{p - 2}{1 - 3p + 2p^2}, \quad c_2 \simeq 0, \quad c_3 = (p - 1)c_1, \quad c_4 = \frac{p - 2}{p - 1}. \tag{40}
\]
The spectral index of scalar perturbations and the tensor-to-scalar ratio for a vacuum $f(R)$ gravity in the Jordan frame have the following form \[24, 25\],

$$n_s = 1 - 6\epsilon_1 - 2\epsilon_4, \quad r = 48\epsilon_1^2. \quad (41)$$

By performing a simple analysis, it is easy to show that the only value of the parameter $p$ that renders the scalar spectral index compatible with the Planck data is $p = 1.81$, however the model is not viable since for that value of $p$ the tensor-to-scalar ratio takes the value $r = 0.13$. This is in contrast to the Einstein frame theory in which case the model was found viable in the previous section, for $\alpha = 1$. We think that the reason for this inconsistency is the following: it has to do with the approximation we made $F \sim p\beta R^{p-1}$, which we applied it in order to obtain analytic results.

Now let us use another formalism in order to study the $R^p$ inflation in the $f(R)$ Jordan frame, this will also provide useful insights regarding the inconsistency in the two frames. As we now show, using the formalism we shall now present, the $R^p$ inflation in the $f(R)$ Jordan frame is a viable theory compatible with the Planck data. This result seems to be correct and perfectly aligned with the principle which dictates that conformally related frames should yield identical results for conformal invariant quantities. Let us quote for convenience again the field equations, which for the FRW metric read,

$$0 = -\frac{f(R)}{2} + 3\left(H^2 + \dot{H}\right)F_R(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)F_{RR}(R), \quad (42)$$

$$0 = \frac{f(R)}{2} - \left(\dot{H} + 3H^2\right)F_R(R) + 6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + 2H\dot{R}\right)F_{RR}(R) + 36\left(4H\dot{H} + \ddot{H}\right)^2F_{RRR}, \quad (43)$$

where $F_{RR} = \frac{d^2f}{dR^2}$, and $F_{RRR} = \frac{d^3f}{dR^3}$. Also the Ricci scalar as a function of the Hubble rate for the FRW metric is $R = 12H^2 + 6H$. We shall now derive the $n_s - r$ relation for $f(R)$ gravity and we shall apply the formalism for the power-law $f(R)$ gravity of Eq. (44). Let us recall the general form of the slow-roll indices relevant for studying $f(R)$ gravity inflation in vacuum, namely $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, which are \[24, 81\],

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = 0, \quad \epsilon_3 = \frac{\dot{F}_R}{2HF_R}, \quad \epsilon_4 = \frac{\ddot{F}_R}{HF_R}. \quad (44)$$

Now, let us assume that $\epsilon_i \ll 1$, $i = 1, 3, 4$, and the scalar spectral index and the tensor-to-scalar ratio are \[24, 81, 82\],

$$n_s = 1 - \frac{4\epsilon_1 - 2\epsilon_3 + 2\epsilon_4}{1 - \epsilon_1}, \quad r = 48\frac{\epsilon_1^2}{(1 + \epsilon_3)^2}. \quad (45)$$

Now, the Raychaudhuri equation dictates that,

$$\epsilon_1 = -\epsilon_3(1 - \epsilon_4), \quad (46)$$

and due to the slow-roll assumption for the slow-roll indices, we have,

$$n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4. \quad (47)$$

Furthermore regarding the tensor-to-scalar ratio, we have $r \simeq 48\epsilon_1^2$, and due to the fact that $\epsilon_1 \simeq -\epsilon_3$, we finally have,

$$r \simeq 48\epsilon_1^2. \quad (48)$$

The analytic calculation of the slow-roll index $\epsilon_4 = \frac{\dot{F}_R}{HF_R}$ is of profound importance in order to obtain the $n_s - r$ relation, so let us express it in terms of the slow-roll index $\epsilon_1$, and we find,

$$\epsilon_4 = \frac{\dot{F}_R}{HF_R} = \frac{\frac{d}{dt}\left(F_{RR}\dot{R}\right)}{HF_{RRR}} = \frac{F_{RRR}\ddot{R}^2 + F_{RRR} \frac{d(R)}{dt}}{HF_{RRR}\dot{R}}. \quad (49)$$

We need to find $\dot{R}$, so by using the slow-roll assumption $\dot{H} \ll H\ddot{H}$, we have,

$$\dot{R} = 24\dot{H}H + 6\ddot{H} \simeq 24H\dot{H} = -24H^3\epsilon_1. \quad (50)$$
Now upon combining Eqs. (50) and (49), after some algebra we obtain,
\[
\epsilon_4 \simeq -\frac{24F_{RRR}H^2}{F_{RR}}\epsilon_1 - 3\epsilon_1 + \frac{\dot{\epsilon}_1}{H\epsilon_1},
\] (51)
and due to the fact that the term \(\dot{\epsilon}_1\) is,
\[
\dot{\epsilon}_1 = -\frac{\ddot{H}H^2 - 2\dot{H}^2H}{H^3} = -\frac{\ddot{H}}{H^2} + \frac{2\dot{H}^2}{H^3} \simeq 2H\epsilon_1^2,
\] (52)
the slow-roll index \(\epsilon_4\) reads,
\[
\epsilon_4 \simeq -\frac{24F_{RRR}H^2}{F_{RR}}\epsilon_1 - \epsilon_1.
\] (53)

Now we shall introduce the following dimensionless parameter \(x\),
\[
x = \frac{48F_{RRR}H^2}{F_{RR}},
\] (54)
so the slow-roll index \(\epsilon_4\) can be written in terms of \(x\) as follows,
\[
\epsilon_4 \simeq -\frac{x}{2}\epsilon_1 - \epsilon_1.
\] (55)

Now, by combining Eqs. (55) and (47), we can directly express the spectral index as a function of the first slow-roll index \(\epsilon_1\) as follows,
\[
n_s - 1 = -4\epsilon_1 + x\epsilon_1.
\] (56)
By eliminating \(\epsilon_1\) and combining Eqs. (55) and (48), we finally obtain,
\[
r \simeq \frac{48(1-n_s)^2}{(4-x)^2},
\] (57)
which is a quite valuable relation for vacuum \(f(R)\) gravity, and it is derived solely on the slow-roll assumptions. Now, let us consider the \(f(R)\) gravity of relation (34) and let us calculate the \(n_s - r\) relation for this \(f(R)\) gravity. The crucial parameter to be calculated is the dimensionless parameter \(x\), which for the particular \(f(R)\) gravity under study reads,
\[
x = \frac{48(p - 2)H^2}{R}.
\] (58)
Recall that \(R = 12H^2 + 6\dot{H}\) for a flat FRW metric, so due to the slow-roll assumption, \(R \sim 12H^2\), hence the parameter \(x\) is constant for the \(f(R)\) gravity at hand and it is equal to \(x = 4(p - 2)\). Thus, the \(n_s - r\) relation of Eq. (57) becomes,
\[
r = 3\frac{(1-n_s)^2}{(3-p)^2}.
\] (59)
The above relation can generate a viable phenomenology for several values of \(p\) if \(n_s\) is within the acceptable parameter values ranges of the Planck data. For example, if \(n_s = 0.961\) for \(p = 1.81\) we get \(r = 0.00322223\), which is well fitted with the Planck observational data. This result is in more agreement with the results of the previous section, since the \(R^p\) theory in the \(f(R)\) Jordan frame is viable. In general, according to our approach, if the spectral index is compatible with the observational data, then the tensor-to-scalar ratio is also compatible with the observations, a feature absent in the other approach presented earlier. However, this result should be discussed in the context of both formalisms presented in this section, in order to pinpoint the source of the inconsistency. If for example, the first formalism is assumed to hold true, then the second formalism cannot be used since \(\epsilon_1 =\text{const.}\). However, we believe that the second formalism is the correct approach and thus the \(R^p\) theory in the \(f(R)\) Jordan frame is viable, a result which is also supported by the viability of the Einstein frame counterpart theory. The problem with the first formalism which we presented in the first part of this section, is mainly the assumption \(F_R \sim \beta nR^{p-1}\) which we did for the sake of analyticity and only for that. It seems therefore that the \(R^p\) theory in the \(f(R)\) Jordan frame
is a viable inflationary theory, contrary to what was believed to date. This result is further supported by the fact that the counterpart theory in the Einstein frame is viable, as it should be due to the conformal relation between the two frames. It can be checked that the Einstein frame counterpart scalar theory yields almost identical results with the Jordan frame theory \(^{(30)}\). Also another fact that further supports the second formalism we presented for the \(R^p\) theory in the \(f(R)\) Jordan frame is the irrelevance of the values of the parameter \(p\) in order to obtain a viable inflationary theory, in contrast to the first formalism, in which case an inflationary theory occurs for \(\frac{1+\sqrt{3}}{2} < p < 2\). In the Einstein frame theory inflation depends on the parameter \(p\) but the values of \(p\) are not directly constrained.

As a final comment, the differences in the observational indices in conformally related theories, are mainly due to the different approximations used for the sake of lack of analyticity. In the Appendix B we shall present the \(\phi\)-Jordan frame inflationary scalar theory and we shall show that it is quite hard to be studied analytically, without making several assumptions. These assumptions might affect the final forms of the observational indices.

### IV. CONCLUSIONS

In this paper we studied a new class of cosmological inflationary attractors, which we named generalized \(R^p\)-attractors, since the Einstein frame scalar potential is a generalization of the Einstein frame scalar potential corresponding to the Jordan frame \(R^p\) gravity. We presented several features of the corresponding non-minimally coupled scalar field theory that yield an attractor type property, since a large class of models with an arbitrary non-minimal coupling having specific properties, yields the same Einstein frame scalar potential and consequently the same inflationary phenomenology. We studied the inflationary phenomenology of the resulting Einstein frame scalar field theory and as we showed, the viability of the generalized \(R^p\)-attractors is guaranteed for a large range of the two parameters that characterize the model. Also we discussed an important issue related to the \(R^p\) inflation in the Jordan and Einstein frame. The \(R^p\) inflation in the Einstein frame is a viable theory and this is in contrast to the well-known non-viability of the standard power-law \(R^p\) model in the Jordan frame. Following the general principle that conformal invariant quantities between conformally related frames, should be identical when calculated in the two frames, the result of \(R^p\) inflation between the two frames is rather unappealing. We pinpoint the reason why \(R^p\) inflation in the Jordan frame is rendered non-viable, and it seems that it has to do with an approximation made for the sake of analyticity. We presented a formalism that may alleviate this problem, and as we showed, by using the slow-roll conditions solely, the Jordan frame \(R^p\) theory can be viable. To be more accurate, if the spectral index of the \(R^p\) theory is compatible with the Planck data, then the tensor-to-scalar ratio is also viable, based on our approach. The whole problem arises to our opinion, due to the lack of analyticity and the complexity of the Jordan frame theory. We also present the inflationary phenomenology of the non-minimally coupled theory in the Jordan frame, which further shows how difficult it is to obtain analytic results in the Jordan frame. However, by conformal invariance of the spectral index and of the tensor-to-scalar ratio, the Einstein frame study suffices for the inflationary study. Finally, the same consideration made in this article can be extended to Einstein-Gauss-Bonnet inflationary theories, a task for which work is in progress.

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### Appendix A: An Explicit Example for which the \(F(R)\) Gravity Can be Obtained

In this Appendix we shall evaluate the \(f(R)\) gravity which can generate the generalized \(R^p\)-attractor potential \(^{(12)}\) for specific values of the parameters \(\alpha\) and \(p\). Apart from the value \(\alpha = 1\), one case for which we can extract analytical results is the case \(\alpha = 4\) and \(p = 3/2\) in which case the equation \(^{(9)}\) takes the form,

\[
frR - 4 \left( \sqrt{frR} - 1 \right)^2 \left( \frac{5\sqrt{frR}}{4} - \frac{1}{2} \right) frV_0 = 0 ,
\]  

\[
(60)
\]
where recall that $f_R = \frac{df}{dR}$, which when solved it yields,

$$f_R = \frac{18}{25} - \frac{\sqrt{2} \left( -1800RV_0^3 - 441V_0^3 \right)}{225V_0^2 \sqrt{625R^2V_0^4 + 25\sqrt{625R^2V_0^4 - 24600R^3V_0^3 + 244020R^2V_0^6 - 38416RV_0^{10} + 13300RV_0^5 - 686V_0^6}}$$

$$+ \frac{\sqrt{2}}{25\sqrt{2}V_0^2 \left( 625R^2V_0^4 + 25\sqrt{625R^2V_0^4 - 24600R^3V_0^3 + 244020R^2V_0^6 - 38416RV_0^{10} + 13300RV_0^5 - 686V_0^6 \right)},$$

and this is the only thing that can analytically be evaluated. Thus in conclusion, the $f(R)$ Jordan frame evaluation and study of inflation is particularly difficult to be performed analytically, so one must rely on the Einstein frame study and the conformal equivalence of the two frames in order to study the inflationary dynamics of generalized $R^p$ inflation.

**Appendix B: Inflationary Dynamics in the φ-Jordan Frame**

In this section we shall present the field equations and the formalism of inflationary dynamics in the φ-Jordan frame for the generalized $R^p$ inflation. As we shall demonstrate, as in the case presented in the Appendix A, the study of generalized $R^p$ inflation in the φ-Jordan frame is particularly difficult to study analytically, thus one must rely on the Einstein frame calculation and the conformal equivalence of the frames in order to study generalized $R^p$ inflation analytically. To start with, consider the gravitational action of a the non-minimally coupled scalar field in the φ-Jordan frame,

$$S_J = \int d^4x \left( \frac{\Omega(\phi)}{2\kappa^2} R - \frac{\omega(\phi)}{2} g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi - V_J(\phi) \right), \quad (62)$$

and the slow-roll indices are defined as follows, $[24, 81]$,

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\ddot{\Omega}(\phi)}{2H\dot{\Omega}(\phi)}, \quad \epsilon_4 = \frac{\ddot{E}(\phi)}{H\dot{E}(\phi)}, \quad (63)$$

where the function $E(\phi)$ is defined as follows,

$$E(\phi) = \frac{\Omega(\phi)}{\kappa^2 \phi^2} \left( \omega(\phi)\dot{\phi}^2 + \frac{3\Omega(\phi)^2}{2\Omega(\phi)\kappa^2} \right). \quad (64)$$

In the present case, the field equations read,

$$3\Omega(\phi) = \frac{1}{2} \omega(\phi)\dot{\phi}^2 + V(\phi) - \frac{3H\ddot{\Omega}(\phi)}{\kappa^2}, \quad (65)$$

$$- \frac{2\ddot{\Omega}(\phi)\dot{H}}{\kappa^2} = \omega(\phi)\dot{\phi}^2 - \frac{\dot{H}\ddot{\phi}}{\kappa^2} + \frac{\ddot{\Omega}(\phi)}{\kappa^2} + \frac{\ddot{\omega}(\phi)}{\kappa^2} + \frac{\dot{\omega}(\phi)\dot{\phi}}{\kappa^2}, \quad (66)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{\omega(\phi)} \left( V(\phi) - \frac{R\ddot{\phi}}{2\kappa^2} \right), \quad (67)$$

where the “prime” indicates differentiation with respect to the scalar field. Apparently, the simultaneous presence of the functions $\omega(\phi)$, $\Omega(\phi)$ and $V(\phi)$ makes simply impossible to analytically study the inflationary dynamics in the φ-Jordan frame for the generalized $R^p$ inflation. Thus one must rely on the Einstein frame calculation and the conformal equivalence of the frames in order to study analytically the the generalized $R^p$ inflationary theory.

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