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The Construction of Shapley Value in Cooperative Game and its Application on Enterprise Alliance

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Abstract

On the base of the nature of Shapely Value, the construction of Shapely Value in cooperative game is analyzed in this paper, and it is concluded that Shapely Value is made up of two parts, one part depends on individual itself, and the other part depends on the state of individual in alliance. Then the construction of Shapely Value is used to discuss the enterprise alliance and the rent of enterprise alliance is put forward, finally some interesting conclusions are presented in the paper.

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Keywords-Shapely Value; cooperative game; rent of enterprise alliance; efficiency of system

1. Introduction

The cooperative game is a competitive decision-making model in which the individual player cooperate with some player to get result as most as possible. [1] In real life, the complementarity and two-win are both the result of cooperative game. In cooperative game, the players are both in competition and in cooperation with others for more benefit and what they do are helpful to the development of society. [2]

In cooperative game, the key and core of game is the profit allocation of players. Shapely Value is one of the most important methods, and is widely used in the study of dynamic enterprise alliance. In paper [3], LIU Lang discussed application of the Shapely Value in game analysis of benefit distribution of agile virtual enterprise. ZHENG Liqun studied the residual Claim assignment mechanism of Stock Corporation and its axiomatic analysis in [4]. ZHANG Daowu put up a method to synthesize the performance of alliance members of cooperation innovation in [5]. Paper [6] showed a strategy for cost sharing among
partners under VMI based on the Shapley Value. DAI Jian-hua discussed the strategy of profit allocation among partners in dynamic alliance based on the Shapley Value in paper [7]. In the literature of Shapley Value, there are mainly applications of Shapley Value, and lack of the construction analysis. The paper will discuss the construction of Shapley Value and its economy implications, on which the rent of alliance and the benefit of alliance system is studied. The rest of this paper is organized as following. In section II, based on the model of Shapley Value and its solution, the construction of Shapley Value is discussed and its generality is proved. We propose the rent of enterprise alliance and individual, also the benefit of alliance system in section III. Based on the theory and new analysis of this problem, we present the example of enterprise alliance to illustrate the economy significance in section IV, and we draw conclusions in section V.

2. The construction of shapley value

2.1 The model and solution of shapley Value

Example-1 Assume that three main enterprises, $A$, $B$ and $C$, will compose one enterprise alliance. This is a typical tri-cooperative game. It is estimated that the profit allocation of different alliance system is shown in Table I.

In Table I, $v_A$, $v_B$ and $v_C$ are respectively the benefit of players when they operate without cooperation.

| alliance system | $A$ | $B$ | $C$ |
|----------------|-----|-----|-----|
| Operation independence | $v_A$ | $v_B$ | $v_C$ |
| Alliance of two of three enterprises, the left independence | $A \cup B$ | Part of $v_{AB}$ | $v'_C$ |
| | $B \cup C$ | $v'_A$ | Part of $v_{BC}$ |
| | $A \cup C$ | Part of $v_{AC}$ | $v'_B$ | Part of $v_{AC}$ |
| Alliance of three enterprises | $A \cup B \cup C$ | Part of $v_{ABC}$ |

**Definition 1** (independence profit of player) In cooperative game, the return of the player when it operates itself without cooperation is called independence profit of player, marked as $v$. The independence profit of player-$A$ is noted as $v_A$.

Then according to the Formula (1) of Shapley Value,

$$\Phi_{v_{A}}(v) = \sum_{S \subseteq N} \frac{(|S|-1)!(n-|S|)!v(S) - v(S-\{A\})}{n!}$$

(1)

Here, $|S|$ means the number of the member of alliance, $w_{|S|} = \frac{(|S|-1)!(n-|S|)!}{n!}$.

The count process of $v_A$ is presented by Table II.
So that, the profit allocation of player- $A$ is

$$\Phi_A(v) = \frac{1}{3}v_A + \frac{1}{6}(v_{AB} - v_A) + \frac{1}{6}(v_{AC} - v_A) + \frac{1}{3}(v_{ABC} - v_A)$$

$$= \frac{1}{6}(v_{AB} + v_{AC} - v_A) + \frac{1}{3}(v_B - v_{AC} + v_{ABC}) \quad (2)$$

### TABLE II. The count process of $V_A$

| $S \setminus A$ | $A \cup B$ | $A \cup C$ | $A \cup B \cup C$ |
|-----------------|-------------|-------------|-----------------|
| $v(S)$          | $V_A$       | $V_{AB}$    | $V_{AC}$        |
| $S \setminus A$ | 0           | $V_B$       | $V_C$           | $V_{BC}$       |
| $\nu(S) - \nu(S \setminus A)$ | $V_A$     | $v_{AB} - v_A$ | $v_{AC} - v_A$ | $v_{ABC} - v_A$ |
| $|S|$ | 1 | 2 | 2 | 3 |
| $\nu|S|$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{3}$ |
| $\phi_A$ | $\frac{1}{3}v_A$ | $\frac{1}{6}(v_{AB} - v_A)$ | $\frac{1}{6}(v_{AC} - v_A)$ | $\frac{1}{3}(v_{ABC} - v_A)$ |

Similarly,

$$\Phi_B(v) = \frac{1}{6}(v_{AB} + v_{BC} - v_A - v_C) + \frac{1}{3}(v_B - v_{AC} + v_{ABC}) \quad (3)$$

$$\Phi_C(v) = \frac{1}{6}(v_{AC} + v_{BC} - v_A - v_B) + \frac{1}{3}(v_C - v_{AB} + v_{ABC}) \quad (4)$$

The result of mode is $\Phi = [\Phi_A, \Phi_B, \Phi_C]^T$.

### 2.2 The construction of Shapley Value

According to the axiom of Shapley Value [2], there exists the nature of super additivity: $v(S) + \sum_{i \in S} v(i)$ if $R, S \subseteq N$, and $R \cap S = \emptyset$, then $v(R \cup S) \geq v(R) + v(S)$.

So assume $V_{AB} \geq V_A + V_B$, let $V_{AB} = V_A + V_B + a_{AB}$, in which,

$$a_{AB} \geq 0 \quad (5)$$

Similarly, assume
$v_{BC} \geq v_B + v_C$, let $v_{BC} = v_B + v_C + a_{BC}$, of which,
\[ a_{BC} \geq 0, \quad (6) \]

$v_{AC} \geq v_A + v_C$, let $v_{AC} = v_A + v_C + a_{AC}$, of which,
\[ a_{AC} \geq 0, \quad (7) \]

$v_{ABC} \geq v_A + v_B + v_C$, let $v_{ABC} = v_A + v_B + v_C + a_{ABC}$, of which,
\[ a_{ABC} \geq 0. \quad (8) \]

Take the value of Formula (5)-(8) into Formula (2)-(4), there are
\[ \Phi_A(v) = v_A + \frac{1}{6}(a_{AB} + a_{AC}) + \frac{1}{3}(a_{ABC} - a_{BC}) \quad (9) \]
\[ \Phi_B(v) = v_B + \frac{1}{6}(a_{AB} + a_{BC}) + \frac{1}{3}(a_{ABC} - a_{AC}) \quad (10) \]
\[ \Phi_C(v) = v_C + \frac{1}{6}(a_{BC} + a_{AC}) + \frac{1}{3}(a_{ABC} - a_{AB}) \quad (11) \]

It is obvious that the profit allocation of player in tri-cooperative game is made up two parts.

One part is $v_A, v_B$ and $v_C$, which respectively is the benefit of player when it operates independence, which depends on the capability of itself.

The other part is the exceed return which comes from the cooperation system. It depends on four coefficients, ($a_{AB}, a_{BC}, a_{AC}$ and $a_{ABC}$). The coefficients are respectively the above benefit of alliance of subscript. And the relationship of coefficients represents the important and the contribution of player to alliance.

**Theorem 1** In a cooperative game with $n$ players, the profit of one player by Shapely Value composes two parts, one is the independence profit of player, the other is the exceed return from the contribution of player to alliance.

**Prove.** In a cooperative game with $n$ players, independence profit of player $1, 2, \cdots, n$ are respectively $v_1, v_2, \cdots, v_n$, and $v_{i,j}$ ($i, j = 1, 2, \cdots, n, i \neq j$) is the benefit of alliance of subscript.

Let $v_{i,j} = v_i + v_j + a_{i,j}$, $a_{i,j} \geq 0$, similarly, the profit of alliance of random $r$ players is $t_1, t_2, \cdots, t_r = 1, 2, \cdots, n$, $t_1 \neq t_2 \neq \cdots \neq t_r$, and let $v_{t_1, t_2, \cdots, t_r} = v_{t_1} + v_{t_2} + \cdots + v_{t_r} + a_{t_1, t_2, \cdots, t_r}$.

According to Formula (1), the profit allocation of player-$k$ ($k = 1, 2, \cdots, n$) is discussed.

When $|s| = 1$,
\[
\sum_{s \in N} \frac{(|s| - 1)! (n - |s|)! [v(S) - v(S - \{A\})]}{n!} = \frac{(n - 1)!}{n!} (v_i - v_s) = \frac{1}{n} v_i,
\]

When $|s| = 2$,
\[
\sum_{s \in N} \frac{(s - 1)! (n - s)! [v(S) - v(S - \{k\})]}{n!} = \frac{1}{n} v_i + \frac{(n - 2)!}{n!} \sum_{i \neq k} a_{i,k},
\]
When $|s| = 3$,
\[
\sum_{S \subseteq N} \frac{|s|-1!(n-|s|)!v(S) - v(S \{A\})}{n!} = \frac{v_k}{n} + \frac{2!(n-3)!}{n!} \sum_{i \neq j,k} (a_{i,j} - a_{i,j}) + \ldots .
\]

When $|s| = r$,
\[
\sum_{S \subseteq N} \frac{|s|-1!(n-|s|)!v(S) - v(S \{A\})}{n!} = \frac{v_k}{n} + \frac{(r-1)!(n-r)!}{n!} \sum_{k \neq i,j} (a_{k,i} - a_{k,i}) + \ldots .
\]

When $|s| = n$,
\[
\sum_{S \subseteq N} \frac{|s|-1!(n-|s|)!v(S) - v(S \{A\})}{n!} = \frac{v_k}{n} + \frac{(n-1)!}{n!} (a_{1,2,\ldots,n}) + \ldots .
\]

Then $\Phi_k(v) = v_k + \frac{(n-2)!}{n!} \sum_{i \neq k} a_{i,k} + \ldots + \frac{2!(n-3)!}{n!} \sum_{i \neq j,k} (a_{i,j} - a_{i,j}) + \ldots + \frac{(n-1)!}{n!} (a_{1,2,\ldots,n} - a_{1,2,\ldots,n})$

\[= v_k + \sum_{S \subseteq N, |S| = 1} \frac{(s-1)!}{n!} (a_S - a_{S-\{k\}})\]

$\triangleq Part I + Part II$

So that “Part I” is the independence profit of player-$k$, and “Part II” behalves the contribution of player-$k$ in alliance. (Proved)

3. The rent of Enterprise Alliance

Base on the construction of Shapley Value, the rent of enterprise alliance and individual will be discussed.

**Definition 2** (the rent of enterprise alliance) In cooperative game with $n$ players, the exceed benefit of enterprise alliance $N$, which is more than the sum of independence profit of all players, is called the rent of enterprise alliance, marked $RA$, that is,
\[
RA(N) = v(N) - \sum_{i \in N} v(i) = \sum_{k \in S} \sum_{S \subseteq N} \frac{(s-1)!}{n!} (a_S - a_{S-\{k\}}).
\]

**Definition 3** (the individual rent of enterprise alliance) In cooperative game with $n$ players, the exceed benefit of some one player in enterprise alliance $N$, which is more than the independence profit of it, is called the individual rent of enterprise alliance, marked $SRA$, that is, the individual rent of enterprise alliance of player-$k$ is
\[SR{A}_k(N) = \Phi_k(v) - v_k = \sum_{S \subseteq N} \frac{(|S| - 1)! (n - |S|)! (a_S - a_{S \setminus \{k\}})}{n!}, \quad k = 1, 2, \ldots, n\] (13)

**Theorem 2** In cooperative game with \(n\) players, the rent of enterprise alliance \(N\) is the sum of the individual rent of enterprise alliance of all player members of enterprise alliance \(N\), that is to say,

\[RA(N) = \sum_{k \in S} SRA_k(N)\] (14)

Prove. From profit allocation and Formula (12) and (13),

\[RA(N) = v(N) - \sum_{i \in N} v(i) = \sum_{k \in N} \Phi_k(v) - \sum_{i \in N} v(i)\]

\[= \sum_{k \in N} (\Phi_k(v) - v_k) = \sum_{k \in N} SRA_k(N)\]

(Proved.)

**Definition 4** (the efficiency of alliance system) the ratio of the rent of enterprise alliance to the cost of alliance establishment is called the efficiency of alliance system, noted as \(EA(N)\). And \(EA(N) = RA/CA(S)\), here \(CA(S)\) is the cost of alliance \(N\) establishment.

**Definition 5** (the efficiency of individual in alliance) the ratio of the individual rent of enterprise alliance to the individual cost of alliance establishment is called the efficiency of individual in alliance in alliance \(N\), noted as \(SEA(N)\). And \(SEA(N) = SRA/SCA\), here \(SCA\) is the cost of individual for alliance \(N\) establishment.

It is obvious that the efficiency of alliance system is important index which decides the stable station of alliance. The more the efficiency of alliance system, the more likely the player goes into alliance, and the more stable of the alliance. One the hand, the stability of individuals of alliance depends on the efficiency of individual in alliance.

**Proposition 1** In cooperative game with \(n\) players, the member, who controls the most resource of alliance, will get the most individual rent of enterprise alliance.

It will be proved in the condition of three members. Assume that Player-A, B, and C would establish an alliance. And the most competitive resource is controlled by member-B. So the alliance of A& bait will get a small rent of enterprise alliance, and presume \(a_{AC}\) has a small value, and then, \(a_{AC} = \delta \ll a_{AB}, a_{AC} = \delta \ll a_{BC}\), and \(a_{AC} = \delta \ll a_{ABC}\).

In this condition, there will be \(SRA_b(N) \succ SRA_A(N)\) and \(SRA_b(N) \succ SRA_C(N)\).

**Prove.** By Formula (9) and (13),

\[SRA_A(N) = \Phi_A(v) - v_A\]

\[= \frac{1}{6} (a_{AB} + 2a_{ABC} - 2a_{BC} + a_{AC})\] (15)

By Formula (10), (11) and (13), then

\[SRA_B(N) = \Phi_B(v) - v_B\]

\[= \frac{1}{6} (a_{AB} + 2a_{ABC} - 2a_{AC} + a_{BC})\] (16)
\[ SRA_C(N) = \Phi_C(v) - v_C \]

\[ = \frac{1}{6} (a_{BC} + 2a_{ABC} - 2a_{AB} + a_{AC}) \quad (17) \]

By (15)-(17) and the situation above mentioned,

\[ SRA_B(N) - SRA_A(N) = \frac{1}{6} (3a_{BC} - 3a_{AC}) = \frac{1}{2} (a_{BC} - a_{AC}) > 0 \]

\[ SRA_A(N) - SRA_C(N) = \frac{1}{6} (3a_{AB} - 3a_{AC}) = \frac{1}{2} (a_{AB} - a_{AC}) > 0 \]

Therefore, \( SRA_B(N) > SRA_A(N) \) and \( SRA_B(N) > SRA_C(N) \), that is to say, the member which controls key resource archives the most individual rent of enterprise alliance.

This conclusion is general in the real life. Take the sale of production as an example, some super-market with great scale controls the channels of sale which is the most important channel between enterprise and customers. Hence this super-market will get the most individual rent of enterprise alliance if the customers are always faithful.

4. Example

Example-2 There are three different schools (A, B, and C) which will cooperate in a training project in an area. Before cooperation, they have their resource and stable income. Now assume that only A has the credential of training, B and C have not. Assume that the return function of alliance is shown in Table III, and the cost of individual of A, B, and C is respectively 700, 400, and 500 thousands RMB.

| Table III the income function of alliances of three schools |
|----------------------------------------------------------|
| \( v(S) \) | \( v_A \) | \( v_B \) | \( v_C \) | \( v_{AB} \) | \( v_{AC} \) | \( v_{BC} \) | \( v_{ABC} \) |
|----------------|-----|-----|-----|------|------|------|------|
| Ten thousands (RMB) | 200 | 100 | 150 | 450  | 550  | 300  | 200  |

By the Formula (1), (12), (13), we could get the rent of enterprise alliance and the individual rent of enterprise alliance. And by the Definition 4 and 5, the result is presented in Table IV.

It is concluded from Table IV that, school-A will get the most individual rent of enterprise alliance (141.7), and also the most efficiency of individual in alliance. Why the situation happens is that school-A has the most import resource of alliance.

5. Conclusion

The paper analyzed the construction of Shapley Value based on which the rent and the efficiency of enterprise alliance are discussed. And the conclusion prove that the member of alliance with competitive resource will gain the most individual rent of enterprise alliance.
Table IV the result of Example-1(Ten thousands RMB)

| Symbol | $A$  | $B$  | $C$  |
|--------|------|------|------|
| Shapley Value | $\Phi_i (v)$ | 341.7 | 166.7 | 241.7 |
| the cost of individual for alliance establishment | $SCA$ | 70 | 40 | 50 |
| the individual rent of enterprise alliance | $SRA$ | 141.7 | 66.7 | 91.7 |
| the efficiency of individual in alliance | $SEAN$ | 2.02 | 1.67 | 1.83 |
| the rent of enterprise alliance | $RA$ | 300 | |
| the cost of alliance establishment | $CA(S)$ | 160 | |
| the efficiency of alliance system | $EA(N)$ | | 1.88 |

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