CERTAIN TRANSFORMATIONS PRESERVING FAMILIES OF UNIVALENT ANALYTIC FUNCTIONS

MILUTIN OBRADOVIĆ, SAMINATHAN PONNUSAMY, AND KARL-JOACHIM WIRTHS

Abstract. The article deals with the family \( U(\lambda) \) of all functions \( f \) analytic and univalent in the unit disk \(|z| < 1\) with the Taylor series \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) such that \( |(z/f(z))^2 f'(z) - 1| < \lambda \) for \(|z| < 1\) and for some \( 0 < \lambda \leq 1 \). First we show that the family \( U(\lambda) \) is preserved under rotation, conjugation, dilation and omitted value transformations. We show by an example that this family is not preserved under the \( n \)-th root transformation for each \( n \geq 2 \). This is a basic here which helps to generate a number of new theorems and in particular provides a way for constructions of functions from the family \( U(\lambda) \).

1. Introduction

Let \( \mathbb{D} \) denote the open the unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{A} \) the family of all functions \( f \) analytic in \( \mathbb{D} \) with the Taylor series \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \). Let \( \mathcal{S} \) denote the subset of \( \mathcal{A} \) consisting of functions that are univalent in \( \mathbb{D} \). See [3, 5] for the general theory of univalent functions. Let \( \mathcal{U} = \{ f \in \mathcal{A} : |U_f(z)| < 1 \text{ in } \mathbb{D} \} \), where \( U_f(z) = (z/f(z))^2 f'(z) - 1 \). In the recent years, a number of properties of \( \mathcal{U} \) and its various generalizations are investigated for example in [4, 6, 8, 11]. Because \( f'(z)(z/f(z))^2 (f \in \mathcal{U}) \) is bounded, it follows that \( (z/f(z))^2 f'(z) \neq 0 \) in \( \mathbb{D} \) and thus, each \( f \in \mathcal{U} \) is non-vanishing in \( \mathbb{D}\setminus\{0\} \) and can be written as

\[
\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k, \quad z \in \mathbb{D}.
\]

Moreover, it is known [1] that \( \mathcal{U} \not\subseteq \mathcal{S} \). One of the sufficient conditions for functions \( f \) of this form to belong to the class \( \mathcal{U} \) is that (see [6, 8])

\[
\sum_{n=2}^{\infty} (n-1)|b_n| \leq 1.
\]

It follows [4, 6, 11] that neither \( \mathcal{U} \) is included in \( \mathcal{S}^\ast \) nor includes \( \mathcal{S}^\ast \). When we say that \( f \in \mathcal{U} \) in \(|z| < r\) it means that the inequality \( |U_f(z)| < 1 \) holds in the subdisk \(|z| < r \) of \( \mathbb{D} \), which is indeed same as saying that \( r^{-1}f(rz) \) belongs to the

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class $\mathcal{U}$. Before we proceed further, we may let $0 < \lambda \leq 1$ and consider one of the
generalizations of the class $\mathcal{U}$, namely,
$$U(\lambda) = \{ f \in A : |U_f(z)| < \lambda \text{ for } z \in \mathbb{D} \}$$
so that $\mathcal{U} := U(1)$. Moreover, every $f \in U(\lambda)$ can be expressed as (cf. [8])
\begin{equation}
\frac{z}{f(z)} = 1 - az - \lambda \int_0^z \frac{\omega(t)}{t^2} dt, \quad a_2 = \frac{f''(0)}{2},
\end{equation}
for some $\omega \in B_1$, where $B_1$ denotes the class of functions $\omega$ analytic in $\mathbb{D}$ such
that $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. More recently, Vasudevarao
and Yanagihara [12] discussed the class $U(\lambda)$ in geometric perspectives. See also [2]
where some related studies are initiated.

It is well-known that the class $\mathcal{S}$ is preserved under a number of elementary trans-
formations, eg. conjugation, rotation, dilation, disk automorphisms (i.e. Koebe
transformations), range, omitted-value and square-root transformations to say few.
We show that $\mathcal{U}$ as a subset of $\mathcal{S}$ preserves some of these properties and as a conse-
quence we derive few applications.

2. Main Results

Lemma 1. The class $\mathcal{U}$ is preserved under rotation, conjugation, dilation and
omitted-value transformations.

Proof. Let $f \in \mathcal{U}$ and define $g(z) = e^{-i\theta}f(ze^{i\theta})$, $h(z) = \overline{f(z)}$ and $\psi(z) = r^{-1}f(rz)$. Then we see that $g'(z) = f'(ze^{i\theta})$, $h'(z) = f'(z)$, $\psi'(z) = f'(rz)$,
\begin{align*}
\left(\frac{z}{g(z)}\right)^2 g'(z) - 1 &= \left(\frac{ze^{i\theta}}{f(ze^{i\theta})}\right)^2 f'(ze^{i\theta}) - 1, \\
\left(\frac{z}{h(z)}\right)^2 h'(z) - 1 &= \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1, \quad \text{and} \\
\left(\frac{z}{\psi(z)}\right)^2 \psi'(z) - 1 &= \left(\frac{rz}{f(rz)}\right)^2 f'(rz) - 1.
\end{align*}
It follows that $g$, $h$ and $\psi$ belong to $\mathcal{U}$, since $f \in \mathcal{U}$.

Finally, if $f \in \mathcal{U}$ and $f(z) \neq c$ for some $c \neq 0$, then the function $F$ defined by
$$F(z) = \frac{cf(z)}{c - f(z)}$$
obviously belongs to $\mathcal{S}$. Thus, $z/F(z)$ is non-vanishing in $\mathbb{D}$, and it is a simple
exercise to see that
\begin{equation}
U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = z \frac{z}{f(z)} - z \frac{z}{f(z)} - 1, \quad z \in \mathbb{D}.
\end{equation}
Using [3], it is easy to see that $U_F(z) = U_f(z)$ for $z \in \mathbb{D}$. Consequently, $F \in \mathcal{U}$.
The proof is complete. $\square$
From the proof of the lemma it follows easily that the class $U(\lambda)$ is preserved under rotation, conjugation, dilation and omitted value transformations. On the other hand, the class $U$ (and hence, $U(\lambda)$) is not preserved under the square-root transformation. For example, we consider the function

$$f_1(z) = \frac{z}{1 + (1/2)z + (1/3)z^3}.$$ 

Then we see that $z/f_1(z)$ is non-vanishing in $D$, and it is a simple exercise to see that $U_{f_1}(z) = -(2/3)z^3$ showing that $f_1 \in U$. In particular, $f_1$ is univalent in $D$. 

On the other hand if we define $g_1$ by

$$g_1(z) = \sqrt{f_1(z^2)} = z\sqrt{\frac{f_1(z^2)}{z^2}},$$

then, because $S$ is preserved under the square-root transformation, it follows that $g_1$ is univalent in $D$ whereas

$$\left(\frac{z}{g_1(z)}\right)^2 g_1'(z) - 1 = \left(\frac{z}{f_1(z)}\right)^{3/2} f_1'(z) - 1 = \frac{1 - (2/3)z^6}{\sqrt{1 + (1/2)z^2 + (1/3)z^6}} - 1,$$

which approaches the value $\frac{2\sqrt{5}-3}{3} > 1$ as $z \to i$. This means that $U_{g_1}(\mathbb{D})$ cannot be a subset of the unit disk $\mathbb{D}$ and hence, the square-root transformation $g_1$ of $f_1$ does not belong to $U$. 

More generally if we consider

$$f(z) = \frac{z}{1 + (1/n)z + (-1)^n(1/(n+1))z^{n+1}}$$

then a computation shows that $f \in U$ whereas the $n$-th root transformation $g$ of $f$, given by

$$g(z) = \sqrt[n]{f(z^n)} = z\sqrt[n]{\frac{f(z^n)}{z^n}},$$

does not belong to the class $U$ for each $n \geq 2$. Thus, for any $n \geq 2$, $U$ is not preserved under the $n$-th root transformation unlike the class $S$.

**Theorem 1.** Let $f \in A$ and

$$\frac{z}{f(z)} = 1 + b_1z + \sum_{n=2}^{\infty} (-1)^n b_n z^n,$$

where $b_n \geq 0$ for $n \geq 2$. Then $f \in S$ if and only if $\sum_{n=2}^{\infty} (n-1)b_n \leq 1$.

**Proof.** For $f \in S$, by Lemma [1] we have that $g(z) = -f(-z) \in S$. Since

$$\frac{z}{-f(-z)} = 1 - b_1z + \sum_{n=2}^{\infty} b_n z^n,$$

then by the characterization given in [2] (see also the survey article [3]), $g \in U$ if and only if $\sum_{n=2}^{\infty} (n-1)b_n \leq 1$ if and only if $g \in S$. The desired conclusion follows. \qed
Problem 1. It will be interesting to find necessary and/or sufficient conditions (as in [9]) for the function $f \in \mathcal{A}$ of the following form to be univalent in $D$:

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} (-1)^{n-1} b_n z^n \quad \text{or} \quad \frac{z}{f(z)} = 1 + b_1 z - \sum_{n=2}^{\infty} b_n z^n,$$

where $b_n \geq 0$ for $n \geq 2$.

A function $f$ analytic in $D$ is called $n$-fold symmetric ($n = 1, 2, \ldots$) if

$$f(e^{i2\pi/n}z) = e^{i2\pi/n}f(z) \quad \text{for} \quad z \in D.$$ 

In particular, every $f \in \mathcal{A}$ is 1-fold symmetric and every odd $f$ is 2-fold symmetric.

Every $n$-fold symmetric function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ can be written as

$$f(z) = z + a_{n+1}z^{n+1} + a_{2n+1}z^{2n+1} + \cdots.$$ 

Properties of various geometric subclasses of $n$-fold symmetric functions from $S$ have been investigated by many authors. We now investigate certain analogous problems associated with the class $\mathcal{U}$.

Theorem 2. Let $f \in \mathcal{U}$ be given by (1). Then for each $n \geq 2$, the function $f_n(z)$ defined by

$$\frac{z}{f_n(z)} = 1 + \sum_{k=1}^{\infty} b_{nk} z^{nk}$$

also belongs to the class $\mathcal{U}$, whenever $f_n(z) \neq 0$ in $D$. More generally, if $f \in \mathcal{U}(\lambda)$ is given by (1), then $f_n \in \mathcal{U}(\lambda)$ whenever it is non-vanishing in $D$.

Proof. Let $f \in \mathcal{U}$ with $\phi(z) = z/f(z)$. Then $\phi(z)$ is nonvanishing and analytic in $D$ and has the form

$$\frac{z}{f(z)} = \phi(z) = 1 + \sum_{k=1}^{\infty} b_k z^k.$$ 

Now, we define $\Phi_n$ by $\Phi_n(z) = z/f_n(z)$ and $\omega = e^{i2\pi/n}$. Then, $\{\omega^k : k = 1, 2, \ldots, n\}$ is the set of all $n$ $n$-th roots of unity. It is a simple exercise to see that

$$\Phi_n(z) := \frac{1}{n} \sum_{k=1}^{n} \phi(\omega^k z) = \frac{1}{n} \sum_{k=1}^{n} \frac{z}{\omega^{-k}f(\omega^k z)} = 1 + \sum_{k=1}^{\infty} b_{nk} z^{nk}.$$ 

Since $f \in \mathcal{U}$, by Lemma 1 for each $k$, the function $F_k(z)$ defined by $F_k(z) = \omega^{-k}f(\omega^k z)$ clearly belongs to the class $\mathcal{U}$. By calculation and the relation (3), it follows that

$$U_{f_n}(z) = \frac{1}{n} \sum_{k=1}^{n} U_{F_k}(z) = \frac{1}{n} \sum_{k=1}^{n} \left[ \left( \frac{\omega^k z}{f(\omega^k z)} \right)^2 f'(\omega^k z) - 1 \right]$$

and thus, $|U_{f_n}(z)| < 1$ in $D$ for each $n \geq 2$. The proof is complete. \qed

From the proof of the following corollary, we see that the non-vanishing condition $f_n(z) \neq 0$ in $D$ in the above theorem can be dropped for the case $n = 2$. 

\[\text{Corollary.} \quad \text{If } f \in \mathcal{U}(\lambda) \text{ is given by (1), then } f_n \in \mathcal{U}(\lambda) \text{ whenever it is non-vanishing in } D.\]
Corollary 1. If \( f \in \mathcal{U} \), then the odd function \( f_2 \) defined by
\[
\frac{z}{f_2(z)} = \frac{1}{2} \left( \frac{z}{f(z)} + \frac{z}{-f(-z)} \right)
\]
also belongs to the class \( \mathcal{U} \). More generally, if \( f \in \mathcal{U}(\lambda) \), then \( f_2 \in \mathcal{U}(\lambda) \).

Proof. Let \( f \in \mathcal{U} \). Then, by Lemma 1, \( F \) defined by \( F(z) = -f(-z) \) belongs to \( \mathcal{U} \). Moreover, the condition \( f(z) - f(-z) \neq 0 \) for \( z \in \mathbb{D} \setminus \{0\} \) is satisfied, because if \( f(z) = f(-z) \) for some \( z \in \mathbb{D} \setminus \{0\} \), then, since \( f \) is univalent, we have \( z = -z \), i.e. \( z = 0 \), which is a contradiction. Consequently,
\[
\frac{z}{f_2(z)} = \frac{z^2}{f(z)f(-z)} \left( \frac{f(z) - f(-z)}{2} \right)
\]
is non-vanishing in \( \mathbb{D} \). Moreover, a calculation gives that if \( f \in \mathcal{U} \) is given by (1), then \( f_2 \) takes the form
\[
\frac{z}{f_2(z)} = 1 + \sum_{k=1}^{\infty} b_{2k} z^{2k}
\]
and thus, by Theorem 2, \( f_2 \in \mathcal{U} \). \( \square \)

From the proof of Theorem 2, the following general result could be proved easily and so, we omit its details.

Theorem 3. Let \( g_k \in \mathcal{U}(\lambda_k) \) for \( k = 1, 2, \ldots, n \) and \( \mu_k, \lambda_k \in [0, 1] \) for \( k = 1, 2, \ldots, n \) such that \( \mu_1 \lambda_1 + \cdots + \mu_n \lambda_n = 1 \). If \( \Phi \) defined by
\[
\Phi(z) = \sum_{k=1}^{n} \mu_k \frac{z}{g_k(z)} = \frac{z}{\Psi(z)}
\]
is non-vanishing in \( \mathbb{D} \), then the function \( \Psi(z) = \frac{1}{\Phi(z)} \) belongs to the class \( \mathcal{U} \).

Proof. It suffices to observe that
\[
U_{\Psi}(z) = \sum_{k=1}^{n} \mu_k U_{g_k}(z)
\]
and the rest follows by taking absolutely sign on both sides and use the triangle inequality. \( \square \)

Corollary 2. Let \( f \in \mathcal{U} \) be given by (1). For \( \theta \in [0, 2\pi) \), the functions \( f_3 \) and \( f_4 \) defined by
\[
\frac{z}{f_3(z)} = 1 + \sum_{n=1}^{\infty} b_n \cos(n\theta) z^n \quad \text{and} \quad \frac{z}{f_4(z)} = 1 + \sum_{n=1}^{\infty} b_n \sin(n\theta) z^n
\]
also belong to the class \( \mathcal{U} \) (whenever \( f_3 \) and \( f_4 \) are non-vanishing in \( \mathbb{D} \)).
Proof. Lemma 1 shows that the functions 
\[ g_1(z) = e^{-i\theta}f(ze^{i\theta}) \text{ and } g_2(z) = e^{i\theta}f(ze^{-i\theta}) \]
belong to the class \( U \) and so does its convex combination (by Theorem 3 with 
\( \mu_1 = \mu_2 = 1/2 \) and \( \lambda_1 = \lambda_2 = 1 \)). Moreover, it follows from the power series 
representation of \( z/f(z) \) that 
\[
\frac{z}{f_3(z)} = \frac{1}{2} \left( \frac{z}{e^{-i\theta}f(ze^{i\theta})} + \frac{z}{e^{i\theta}f(ze^{-i\theta})} \right) = 1 + \sum_{n=1}^{\infty} b_n \cos(n\theta)z^n
\]
from which we conclude that \( f_3 \in U \), by Theorem 3.

In order to prove that \( f_4 \) belongs to \( U \), we first observe that 
\[
\frac{z}{f_4(z)} = 1 + \frac{1}{2i} \left( \frac{ze^{i\theta}}{f(ze^{i\theta})} - \frac{ze^{-i\theta}}{f(ze^{-i\theta})} \right) = 1 + \sum_{n=1}^{\infty} b_n \sin(n\theta)z^n,
\]
and, by a computation, we have 
\[
|U_{f_4}(z)| = \left| \frac{1}{2i} \left( U_{f}(ze^{i\theta}) - U_{f}(ze^{-i\theta}) \right) \right| \leq \frac{1}{2} \left( |U_{f}(ze^{i\theta})| + |U_{f}(ze^{-i\theta})| \right) < 1,
\]
showing that \( f_4 \in U \). \( \square \)

In particular, if we set \( \theta = \pi/2 \), then \( f_3(z) \) and \( f_4(z) \) takes the forms 
\[
\frac{z}{f_3(z)} = 1 - b_2z^2 + b_4z^4 - \cdots \quad \text{and} \quad \frac{z}{f_4(z)} = 1 + b_1z - b_3z^3 + \cdots
\]
and thus, the above corollary provides us with new functions from \( U \).

**Theorem 4.** Let \( f \in U \) be given by (1). Then the function \( g_2 \) defined by 
\[
\frac{z}{g_2(z)} = 1 + \sum_{k=1}^{\infty} \Re \{b_k\} z^k,
\]
with \( z/g_2(z) \neq 0 \) in \( \mathbb{D} \), also belongs to the class \( U \). More generally, if \( f \in U(\lambda) \), then \( g_2 \in U(\lambda) \).

**Proof.** Let \( f \in U \). Then, by Lemma 1 \( h(z) = \overline{f(z)} \) belongs to \( U \). Now, we observe that 
\[
\frac{z}{g_2(z)} = \frac{1}{2} \left[ \left( 1 + \sum_{k=1}^{\infty} b_k z^k \right) + \overline{ \left( 1 + \sum_{k=1}^{\infty} b_k z^k \right) } \right] = \frac{1}{2} \left( \frac{z}{f(z)} + \frac{z}{h(z)} \right)
\]
and thus, we easily have 
\[
U_{g_2}(z) = \frac{z}{g_2(z)} - z \left( \frac{z}{g_2(z)} \right)' - 1 = \frac{U_{f}(z) + U_{h}(z)}{2}
\]
Clearly, the last relation implies that \( g_2 \in U \). \( \square \)
Theorem 5. Let \( f \in \mathcal{U} \) be given by (1). Then the function \( F \) defined by
\[
\frac{z}{F(z)} = 1 + \sum_{n=1}^{\infty} b_{2n} z^n
\]
belongs to the class \( \mathcal{U} \). More generally, if \( f \in \mathcal{U}(\lambda) \) is given by (1), then \( F \in \mathcal{U}(\lambda) \).

Proof. If \( f \in \mathcal{U} \), then we have the representation (see (2))
\[
\frac{z}{f(z)} = 1 + b_1 z - z \int_0^z \frac{\omega(t)}{t^2} \, dt, \quad b_1 = -a_2,
\]
where \( \omega \in \mathcal{B}_1 \). If we put
\[
\omega_1(z) = \int_0^z \frac{\omega(t)}{t^2} \, dt,
\]
then \( \omega_1 \) is analytic in \( \mathbb{D} \), \( \omega_1(0) = 0 \) and \( |\omega_1(z)| \leq |z| \). Moreover, \( |\omega'_1(z)| = |\omega(z)/z^2| \leq 1 \) for every \( z \in \mathbb{D} \). Consequently, for \( f \in \mathcal{U} \) one has
\[
\frac{z}{f(z)} = 1 + b_1 z - z \omega_1(z).
\]
and thus, the function \( \Psi \) defined by
\[
\Psi(z) = 1 + \sum_{n=1}^{\infty} b_{2n} z^{2n}
\]
and observe that \( F \) defined by
\[
\frac{z}{F(z)} = \Psi(\sqrt{z}) = 1 - z W(z) := 1 - \frac{z}{2} \left( \frac{\omega_1(\sqrt{z})}{\sqrt{z}} - \frac{\omega_1(-\sqrt{z})}{\sqrt{z}} \right)
\]
is analytic in \( \mathbb{D} \), where \( W \) is analytic in \( \mathbb{D} \). Next, we observe that
\[
U_F(z) = \frac{z}{F(z)} - z \left( \frac{z}{F(z)} \right)' = 1 - z^2 W'(z)
\]
and, in view of the fact that \( |\omega(z)| \leq |z|^2 \) and \( |\omega'(z)| = |\omega(z)/z^2| \leq 1 \), we can easily see that \( |z^2 W'(z)| < 1 \) in \( \mathbb{D} \), which means that \( F \in \mathcal{U} \).

\( \square \)

Theorem 6. Let \( f \in \mathcal{S} \) and \( f \) be given by (1). Then the function \( F \) defined by
\[
\frac{z}{F(z)} = 1 + \sum_{n=1}^{\infty} b_{2n} z^n
\]
belongs to the class \( \mathcal{U} \) at least in the disk \( |z| < r_0 = 0.778387 \) (implying \( F \) is univalent in \( |z| < r_0 \)), where \( r_0 \in (0, 1) \) is the root of the equation
\[
\frac{r(1-r^2)^2}{2} \log \left( \frac{1+r}{1-r} \right) - (4 + r^4 - 7r^2) = 0.
\]
Proof. Assume that $f \in S$ and is given by (1). In order to show that $F \in U$ in the disk $|z| < r_0$, we need to prove that the function $G$ defined by $G(z) = r^{-1}F(rz)$ belongs to $U$ in $D$ for each $0 < r \leq r_0$. Thus, we begin to consider the function $G$ defined by

$$G(z) = r - 1 F(rz),$$

where $0 < r \leq 1$. To prove $G \in U$, it suffices to show that

$$S =: \sum_{n=2}^{\infty} (n - 1)|b_{2n}|r^n \leq 1$$

for $0 < r \leq r_0$. To do this, we need to recall first the following inequality, namely, for $f \in S$, the necessary coefficient inequality ([5, Theorem 11 on p.193 of Vol. 2])

$$\sum_{n=2}^{\infty} (n - 1)|b_n|^2 \leq 1.$$

This in particular gives that $\sum_{n=2}^{\infty} (2n - 1)|b_{2n}|^2 \leq 1$. Now, we find that

$$S = \sum_{n=2}^{\infty} \sqrt{2n - 1}|b_{2n}|\frac{(n - 1)}{\sqrt{2n - 1}}r^n \leq \left( \sum_{n=2}^{\infty} (2n - 1)|b_{2n}|^2 \right)^{\frac{1}{2}} \left( \sum_{n=2}^{\infty} \frac{(n - 1)^2}{2n - 1}r^{2n} \right)^{\frac{1}{2}} \leq \left( \sum_{n=2}^{\infty} \frac{(n - 1)^2}{2n - 1}r^{2n} \right)^{\frac{1}{2}}.$$

By a computation we see that

$$\sum_{n=2}^{\infty} \frac{(n - 1)^2}{2n - 1}r^{2n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{n - 3}{2} + \frac{1}{2(2n - 1)} r^{2n} = \frac{1}{2} \left( \frac{r^2}{(1 - r^2)^2} - r^2 \right) - \frac{3r^4}{4(1 - r^2)} - \frac{r^2}{4} + \frac{r}{8} \log \left( \frac{1 + r}{1 - r} \right) = \frac{r^2(3r^2 - 1)}{4(1 - r^2)^2} + \frac{r}{8} \log \left( \frac{1 + r}{1 - r} \right)$$

and thus, $S \leq 1$ holds provided

$$\frac{r^2(3r^2 - 1)}{4(1 - r^2)^2} + \frac{r}{8} \log \left( \frac{1 + r}{1 - r} \right) \leq 1,$$

i.e. if $0 < r \leq r_0 = 0.778387$, where $r_0$ is the root of the equation ([7]). It means that $F$ is in the class $U$ in the disc $|z| < r_0$.

In [7], as a corollary to a general result, it has been shown that $|z| < 1/\sqrt{2}$ is the largest disk centered at the origin such that every function in $S$ is included in $U$. \qed
More precisely,

$$\sup \{ r > 0 : r^{-1}f(rz) \in \mathcal{U} \text{ for every } f \in \mathcal{S} \} = \frac{1}{\sqrt{2}}.$$

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M. Obradović, Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia.
E-mail address: obrad@grf.bg.ac.rs

S. Ponnusamy, Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India.
E-mail address: samy@isichennai.res.in, samy@iitm.ac.in

K.-J. Wirths, Institut für Analysis und Algebra, TU Braunschweig, 38106 Braunschweig, Germany.
E-mail address: kjwirths@tu-bs.de