Abstract. In this paper, we consider the second order semilinear impulsive differential equations with state-dependent delay. First, we consider a linear second order system and establish the approximate controllability result by using a feedback control. Then, we obtain sufficient conditions for the approximate controllability of the considered system in a separable, reflexive Banach space via properties of the resolvent operator and Schauder’s fixed point theorem. Finally, we apply our results to investigate the approximate controllability of the impulsive wave equation with state-dependent delay.

1. Introduction. Second order differential equations emerge in many areas of science and engineering. One aspect of studying second order systems is through an equivalent formulation of first order equations, but this transformation may lack important information about the original evolution systems. Therefore, it is more advantageous to study a second order system directly. For the basic theory of second order differential equations, the interested readers are referred to see [12, 21, 49], etc. We observe that the strongly continuous cosine family is an important tool in studying the second order evolution equations. The theory of strongly continuous cosine family was introduced by Fattorini in [19, 20]. Later, Travis and Webb have made essential additions to the theory of strongly continuous cosine family (cf. [49, 50]).

There are several realistic evolutionary processes subject to abrupt changes in state occur at certain negligible time instant. These processes are mathematically
modelled by impulsive differential equations (IDE). Impulsive differential equations have found a variety of applications in various fields of engineering (cf. [22, 47, 48], etc). On the other hand, delay differential equations also present in many evolutionary processes such as heat conduction in materials with fading memory, inferred grinding models and ecological models. In these phenomena, the current state of a system is influenced by the previous states. In the study of delay differential equations, many authors considered the time delay as constant under the state variable to make the study easier. However, state-dependent delays are more prevalent and adequate in applications, some nice examples of state-dependent delay models are given in [1, 15] and the references therein.

Controllability plays a vital role in designing and analyzing control systems. The theory of controllability of the first and second order deterministic or stochastic systems in infinite dimensional spaces is well developed, and has been studied extensively, see for instance [6, 7, 8, 10, 13, 16, 28, 31, 42], etc. Controllability means that a dynamical system can be steered to the desired final state constrained to an admissible class of controls. For the infinite dimensional systems, two famous notions of controllability, namely exact controllability and approximate controllability have been studied extensively by many researchers account to its wide range of applications. Exact controllability means that a system can achieve a desired final state in finite time period, while the approximate controllability implies that a system can steer into an arbitrary small neighborhood of the final state. In the infinite dimensional setting, it has been observed that exact controllability rarely holds (see [52]). However, the approximate controllability is extensive and more appropriate in applications. Therefore, the approximate controllability of nonlinear evolution systems is still seeking attention by many researchers and has emerged as an important area of investigation (cf. [4, 5, 14, 24, 30, 32, 33, 37, 43, 46], etc and the references therein).

Recently, many works reported on the approximate controllability of the impulsive dynamical control system with delay via fixed point methods, see for example, [4, 5, 32, 38, 46], etc. In [4], a set of sufficient conditions is established for the approximate controllability of the semilinear impulsive functional differential system with nonlocal initial conditions by invoking Schauder’s fixed point theorem. Urvashi et.al. in [5], investigated the approximate controllability of second order semilinear stochastic systems in Hilbert spaces with variable delay in control. Li and Huang in [32], considered second order impulsive stochastic differential equations with state-dependent delay in Hilbert spaces and examined the approximate controllability of the system. Mahumudov [38], discussed the approximate controllability of a class of second order evolution differential inclusions, using Bohnenblust-Karlin’s fixed point theorem.

It is observed that the approximate controllability of the impulsive functional control systems in Banach spaces has not got much attention in the literature. In [37], Mahumudov examined the approximate controllability of the semilinear deterministic control systems in Banach spaces. Sumit et.al. [2, 3], established sufficient conditions for the approximate controllability of impulsive evolution systems with delay in separable reflexive Banach spaces. There are a few articles, namely [30, 33, 44], etc investigated the approximate controllability of the second order impulsive functional differential equations in Banach spaces. We observe that the resolvent operator defined in these articles is not well defined in general Banach spaces. We also point out that if the state space is a Hilbert space (identified with
its own dual), then the resolvent operator defined in the works [30, 33, 44], etc are well defined (see (2.1) below for the resolvent operator definition on general Banach spaces). This fact motivates us to consider the second order semilinear impulsive differential equations with state-dependent delay in Banach spaces and examine the approximate controllability of such systems via resolvent operator condition and Schauder’s fixed point theorem.

Let $X$ be a separable reflexive Banach space (having a strictly convex dual $X^*$) and $H$ be a separable Hilbert space. We consider the following second order impulsive system with state-dependent delay:

$$\begin{align*}
\ddot{x}(t) &= Ax(t) + f(t, x_{\rho(t,x)}) + Bu(t), \quad t \in J = [0, T], \quad t \neq \tau_k, \quad k = 1, \ldots, m, \\
x_0 &= \phi \in \mathcal{P}, \quad \dot{x}(0) = \zeta_0 \in X, \\
\Delta x |_{=\tau_k} &= I_k(x(\tau_k)), \quad k = 1, \ldots, m, \\
\Delta \dot{x} |_{=\tau_k} &= g_k(x(\tau_k)), \quad k = 1, \ldots, m, 
\end{align*}$$

where $A$ is a linear operator on $X$, $B$ is a bounded linear operator from $H$ into $X$ and the control function $u \in L^2(J; H)$. Also, $f : J \times \mathcal{P} \to X$ is a nonlinear function, where $\mathcal{P}$ is a phase space, which will be specified later. The impulsive functions $I_k, g_k : X \to X$, for $k = 1, \ldots, m$, and $\Delta x |_{=\tau_k} = x(\tau_k^+) - x(\tau_k^-)$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = T$, for $k = 1, \ldots, m$. The function $x_{\rho} : (-\infty, 0) \to X$, $x_{\rho}(t) = x(t + \theta)$, belong to the phase space $\mathcal{P}$ and the function $\rho : J \times \mathcal{P} \to (-\infty, T]$, is continuous.

This article is structured as follows: The next section presents the basic definitions and results required to develop the approximate controllability of the second order system (1.1). In section 3, the approximate controllability of the linear control system corresponding to the semilinear system (1.1) is discussed. Section 4 is devoted for establishing the approximate controllability of the system (1.1), by using the theory of strongly continuous cosine family and resolvent operators. In the final section, we discuss a concrete example to validate the theory developed in sections 3 and 4.

2. Preliminaries. In this section, we introduce some basic notations, assumptions and definitions to be used in succeeding sections to prove the approximate controllability results for the evolution system (1.1). The norms in the state space $X$, its dual space $X^*$ and control space $H$ are denoted by $\| \cdot \|_X$, $\| \cdot \|_{X^*}$ and $\| \cdot \|_H$, respectively. The inner product in $H$ is represented by $\langle \cdot, \cdot \rangle$ and the duality pairing between $X$ and its topological dual $X^*$ is denoted by $\langle \cdot, \cdot \rangle$. The space of all bounded linear operators from $H$ to $X$ is denoted by $\mathcal{L}(H; X)$ endowed with norm $\| \cdot \|_{\mathcal{L}(H; X)}$. The notation $\mathcal{L}(X)$, represents the space of all bounded linear operators on $X$ endowed with the norm $\| \cdot \|_{\mathcal{L}(X)}$. A function $x : [\mu, \sigma] \to X$ is called the normalised piecewise continuous function on the interval $[\mu, \sigma]$, if it is piecewise continuous on $[\mu, \sigma]$ and left continuous on $(\mu, \sigma]$. The space of all normalised piecewise continuous functions from $[\mu, \sigma]$ to $X$ is represented by $\mathcal{PC}([\mu, \sigma]; X)$ endowed with norm $\|x\|_{\mathcal{PC}} = \sup_{s \in [\mu, \sigma]} \|x(s)\|_X$. For convenience of notations, we use $\mathcal{PC}$ instead of $\mathcal{PC}(J; X)$ throughout the article.

2.1. The strongly continuous cosine family. In this subsection, we first construct a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in $X$, generated by the
linear operator $A$, and then discuss some special properties of the strongly continuous cosine family. Let us impose the following assumptions on the linear operator $A : D(A) \to X$.

**Assumption 2.1.** The operator $A$ satisfies the following:

(R1) The linear operator $A$ is closed and the domain $D(A)$ is dense in $X$.

(R2) For real $\lambda$, $\lambda > \omega$, $\lambda^2$ is in the resolvent set $\rho(A)$ of $A$. The resolvent $R(\lambda^2; A)$ exists, strongly infinitely differentiable, and satisfies

$$\left\| \left( \frac{d}{d\lambda} \right)^k (\lambda R(\lambda^2; A)) \right\|_{L(X)} \leq \frac{Mk!}{(\lambda - \omega)^{k+1}},$$

for $k = 0, 1, 2, \ldots$

(R3) $R(\lambda^2; A)$ is compact for some $\lambda$ with $\lambda^2 \in \rho(A)$.

**Definition 2.1** ([50]). A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space $X$ is called a strongly continuous cosine family if

1. $C(s + t) + C(s - t) = 2C(s)C(t)$, for $s, t \in \mathbb{R}$,
2. $C(0) = 1$, where $I$ denotes the identity operator,
3. $C(t)x$ is strongly continuous in $t$ on $\mathbb{R}$, for each fixed $x \in X$.

**Proposition 2.1** (Proposition 2.7, [51]). Let us assume that (R1)-(R2) hold. Then $A$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, of type $(M, \omega)$.

**Remark 2.1.** A strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is said to be of type $(M, \omega)$, if there exists a constant $M$ such that $\|C(t)\|_{L(X)} \leq Me^{\omega|t|}$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ associated with the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in $X$ is defined as

$$S(t)x = \int_0^t C(s)xds, \quad x \in X, \quad t \in \mathbb{R}.$$ 

Let us assume that, there exists $M, M', \omega \geq 0$ such that $\|C(t)\|_{L(X)} \leq Me^{\omega t}$ and $\|S(t)\|_{L(X)} \leq M'e^{\omega t}$, for every $t \in J$.

Note that if $C(t)$ is a strongly continuous cosine family of type $(M, \omega)$ with the infinitesimal generator $A$, then by the Theorem 2.1, Chapter 2, [21], for any $f \in X$ and $\lambda > \omega$, we have

$$\left( \frac{d}{d\lambda} \right)^k (\lambda R(\lambda^2; A))f = (-1)^k \int_0^\infty t^k e^{-\lambda t} C(t)f dt, \quad k = 0, 1, 2, \ldots$$

In this paper, we consider the second order problems only, for higher order abstract Cauchy problems, we refer the interested readers to [11].

**Proposition 2.2** (Proposition 5.2, [51]). Let $\{C(t) : t \in \mathbb{R}\}$, be a strongly continuous cosine family in $X$, with the infinitesimal generator $A$ and associated sine family $\{S(t) : t \in \mathbb{R}\}$. The following are equivalent:

(i) The operator $S(t)$ is compact for every $t \in \mathbb{R}$.

(ii) $R(\lambda^2; A)$ is compact for some $\lambda$ with $\lambda^2 \in \rho(A)$.

The infinitesimal generator $A$ of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2}C(t)x \bigg|_{t=0}, \quad x \in D(A),$$
where

\[ D(A) = \{ x \in \mathbb{X} : C(t)x \text{ is twice continuously differentiable function in } t \}, \]

endowed with the norm

\[ \|x\|_{D(A)} = \|x\|_\mathbb{X} + \|Ax\|_\mathbb{X}, \ x \in D(A). \]

We also define the set

\[ E = \{ x : C(t)x \text{ is once continuously differentiable function of } t \}, \]

endowed with the norm

\[ \|x\|_1 = \|x\|_\mathbb{X} + \sup_{0 \leq t \leq 1} \|AS(t)x\|_\mathbb{X}, \ x \in E, \]

forms a Banach space, see [29]. The operator valued function

\[ \mathcal{H}(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}, \]

is a strongly continuous group of bounded linear operators on the space \( E \times \mathbb{X} \), generated by the operator \( A = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \) defined on \( D(A) \times E \) (see Proposition 2.6, [50]). From this, it follows that \( AS(t) : E \to \mathbb{X} \) is a bounded linear operator and that \( AS(t)x \to 0 \) as \( t \to 0 \), for each \( x \in E \).

### 2.2. Phase space

We now define an axiomatic definition of the phase space \( \mathcal{P} \), introduced by Hino, Murakami and Naito in [26] and suitably modify to treat the impulsive evolution equations (cf. [40]). Specifically \( \mathcal{P} \), is a linear space of all functions from \((-\infty, 0]\) into \( \mathbb{X} \) endowed with the seminorm \( \| \cdot \|_P \) and satisfying the following axioms:

(A1) If \( x : (-\infty, \mu + \sigma) \to \mathbb{X}, \ \sigma > 0, \) such that \( x_\mu \in \mathcal{P} \) and \( x|_{[\mu, \mu + \sigma]} \in \mathcal{PC}([\mu, \mu + \sigma]; \mathbb{X}) \), then for every \( t \in [\mu, \mu + \sigma] \), the following conditions hold:

(i) \( x_t \) is in \( \mathcal{P} \).

(ii) \( \|x_t\|_P \leq \Lambda(t - \mu) \sup\{\|x(s)\|_\mathbb{X} : \mu \leq s \leq t\} + \Upsilon(t - \mu) \|x_\mu\|_P \), where \( \Lambda, \ U : [0, \infty) \to [0, \infty) \) such that \( \Lambda \) is continuous, \( \Upsilon \) is locally bounded, and both \( \Lambda, \ U \) are independent of \( x(.) \).

(A2) The space \( \mathcal{P} \) is complete.

For any \( \phi \in \mathcal{P} \), the function \( \phi_t, \ t \leq 0, \) defined as \( \phi_t(\theta) = \phi(t + \theta), \ \theta \in (-\infty, 0], \) Therefore, if the function \( x(.) \) satisfies the axiom (A1) with \( x_0 = \phi \) then we may extend the mapping \( t \mapsto x_t \) by setting \( x_t = \phi_t, \ t \leq 0, \) to the whole interval \((-\infty, T] \).

Moreover, let us introduce a set

\[ \mathcal{Z}(\rho^-) = \{ \rho(s, \varphi) : \rho(s, \varphi) \leq 0, \ \text{for} \ (s, \varphi) \in J \times \mathcal{P} \}, \]

where the function \( \rho \) is same as the one defined in section 1. We give the following hypothesis on \( \phi_t \): the function \( t \mapsto \phi_t \), defined from \( \mathcal{Z}(\rho^-) \) into \( \mathcal{P} \) is continuous, and there exists a continuous and bounded function \( \Theta^\phi : \mathcal{Z}(\rho^-) \to (0, \infty) \) such that

\[ \|\phi_t\|_P \leq \Theta^\phi(t) \|\phi\|_P. \]

**Lemma 2.1** (Lemma 2.3, [44]). Let \( x : (-\infty, T) \to \mathbb{X} \), be a function such that \( x_0 = \phi \) and \( x|_J \in \mathcal{PC} \). Then

\[ \|x_s\|_P \leq H_1 \|\phi\|_P + H_2 \sup\{\|x(\theta)\|_\mathbb{X} : \theta \in [0, \max\{0, s\}]\}, \ s \in \mathcal{Z}(\rho^-) \cup J, \]
where
\[ H_1 = \sup_{t \in \mathbb{Z}^{(\rho-)}} \Theta^\phi(t) + \sup_{t \in J} \Upsilon(t), \quad H_2 = \sup_{t \in J} \Lambda(t). \]

2.3. Resolvent operator and mild solution. In this subsection, we recall the duality mapping and resolvent operator in Banach spaces, which is introduced in [37], and then define the mild solution of the system (1.1).

Definition 2.2. A duality mapping \( \mathcal{J} : X \to 2^{X^*} \) is defined as (see [9])
\[ \mathcal{J}[x] = \{ x^* \in X^* : \langle x, x^* \rangle = \| x \|_X^2 = \| x^* \|_{X^*}^2 \}, \text{ for all } x \in X. \]

Since the space \( X \) is reflexive, \( X \) can be renormed such that \( X \) and \( X^* \) becomes strictly convex (Theorem 1.1, Chapter 1, [9]). From the strict convexity of \( X^* \), we obtain that the duality mapping \( \mathcal{J} : X \to X^* \) is single valued and demicontinuous (Theorem 1.2, Chapter 1, [9]), that is,
\[ x_k \to x \text{ in } X \text{ implies } \mathcal{J}[x_k] \xrightarrow{w} \mathcal{J}[x] \text{ in } X^*. \]

Let us define the operators
\[
\begin{align*}
L_Tu &:= \int_0^T S(T-t)Bu(t)dt, \\
\Psi_T^0 &:= \int_0^T S(T-t)BB^*S(T-t)dt = L_T(L_T^*), \\
R(\lambda, \Psi_T^0) &:= (\lambda I + \Psi_T^0)^{-1}, \lambda > 0.
\end{align*}
\]

Whenever, \( X \) is a separable Hilbert space, which can be identified by its own dual (the duality mapping \( \mathcal{J} \) becomes \( I \), the identity operator), then one can define the resolvent operator as \( R(\lambda, \Psi_T^0) := (\lambda I + \Psi_T^0)^{-1}, \lambda > 0 \). It is clear from the second expression in (2.1) that the operator \( \Psi_T^0 : X^* \to X \) is a nonnegative symmetric operator.

Definition 2.3. A function \( x(\cdot, \phi, \zeta_0, u) : (-\infty, T] \to X \) is called a mild solution of (1.1), if it satisfies the following:
(i) \( x(t) = \phi(t) \in P, \quad t \in (-\infty, 0], \quad x'(0) = \zeta_0 \in X, \)
(ii) \( \Delta x|_{t=\tau_k} = I_k(x(\tau_k)), \quad k = 1, \ldots, m, \)
(iii) \( \Delta x|_{t=\tau_k} = g_k(x(\tau_k)), \quad k = 1, \ldots, m, \)
(iv) \( x(\cdot)|_J \in \mathcal{PC} \) and the following integral equation is verified:
\[
x(t) = C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)f(s, x_p(s, x_s))ds + \int_0^t S(t-s)Bu(s)ds \\
+ \sum_{0<\tau_k<t} C(t-\tau_k)I_k(x(\tau_k)) + \sum_{0<\tau_k<t} S(t-\tau_k)g_k(x(\tau_k)), \quad t \in J.
\]

Definition 2.4. The system (1.1) is said to be approximately controllable on \( J \), for any initial function \( \phi \in P \), if the closure of reachable set is the whole space \( X \), where the reachable set is defined as
\[ \mathcal{R}(T, \phi, \zeta_0) := \{ x(T; \phi, \zeta_0, u) : u(\cdot) \in L^2(J; \mathbb{H}) \}. \]

In order to establish the approximate controllability results for the system (1.1), we impose the following assumptions.
Assumption 2.2. \((H0)\) For every \(h \in X\), \(z_\lambda(h) = \lambda R(\lambda, \Psi_0^T)(h) \to 0\) as \(\lambda \downarrow 0\) in the strong topology, where \(z_\lambda(h)\) is a solution of the equation
\[
\lambda z_\lambda + \Psi_0^T J[z_\lambda] = \lambda h. \tag{2.3}
\]

\((H1)\) \(S(t), t \in \mathbb{R}\) is compact.

\((H2)\) The linear operator \(B : \mathbb{H} \to \mathbb{X}\) is bounded with \(\|B\|_{\mathbb{L}(\mathbb{H}; \mathbb{X})} = M_B\).

\((H3)\) \((i)\) Let \(x : (-\infty, T] \to \mathbb{X}\) be such that \(x_0 = \phi\) and \(x_j \in PC\). The function \(t \mapsto f(t, x_{\rho(t, x)})\) is strongly measurable on \(J\) and \(t \mapsto f(s, x_t)\) is continuous on \(Z(\rho^{-}) \cup J\), for every \(s \in J\). Also, for each \(t \in J\), the function \(f(t, \cdot) : \mathcal{P} \to \mathbb{X}\) is continuous.

\((ii)\) For each positive integer \(r\), there exist a function \(\gamma_r(\cdot) \in L^1(J; \mathbb{R}^+\) such that
\[
\sup_{\|\phi\|_\mathcal{P} \leq r} \|f(t, \phi)\|_\mathbb{X} \leq \gamma_r(t), \text{ for a.e. } t \in J \text{ and } \phi \in \mathcal{P},
\]
with
\[
\lim_{r \to \infty} \inf \int_0^T \frac{\gamma_r(t)}{r} dt = \delta < \infty.
\]

\((H4)\) For \(k = 1, \ldots, m\),

\((i)\) the impulses \(I_k : \mathbb{X} \to \mathbb{X}\) are completely continuous,

\((ii)\) the impulses \(g_k : \mathbb{X} \to \mathbb{X}\) are continuous.

Moreover, there exist constants \(c_k\)’s and \(d_k\)’s such that
\[
\|I_k(x)\|_\mathbb{X} \leq c_k \text{ and } \|g_k(x)\|_\mathbb{X} \leq d_k,
\]
for all \(x \in \mathbb{X}\), \(k = 1, \ldots, m\).

Remark 2.2. Since \(\Psi_0^T\) is the nonnegative symmetric operator and \(\mathbb{X}\) is a separable reflexive Banach space, then by using the Lemma 2.2 [37], we obtain that for every \(h \in \mathbb{X}\) and \(\lambda > 0\), the equation (2.3) has a unique solution \(z_\lambda(h) = \lambda(\lambda I + \Psi_0^T J)^{-1}(h) = \lambda R(\lambda, \Psi_0^T)(h)\) and
\[
\|z_\lambda(h)\|_\mathbb{X} = \|J[z_\lambda(h)]\|_\mathbb{X} \leq \|h\|_\mathbb{X}. \tag{2.4}
\]

3. Linear control problem. In this section, we consider the linear problem corresponding to the system (1.1), we formulate an optimal control problem and then discuss about its connection to the approximate controllability of the linear control system. First, we consider the following system:
\[
\begin{aligned}
\dot{z}(t) &= Az(t) + h(t), \quad 0 \leq s, t \leq T, \\
z(s) &= \zeta_0, \quad \dot{z}(s) = \zeta_1.
\end{aligned} \tag{3.1}
\]

If the function \(h : J \to \mathbb{X}\) is integrable, then a continuous function \(z : [0, T] \to \mathbb{X}\) is said to be a mild solution of (3.1), if the equation
\[
z(t) = C(t - s)\zeta_0 + S(t - s)\zeta_1 + \int_s^t S(t - \xi)h(\xi)d\xi, \quad t \in J, \tag{3.2}
\]
is satisfied. Moreover, when \(\zeta_0 \in E\), the function \(z(\cdot)\) is continuously differentiable and
\[
\dot{z}(t) = AS(t - s)\zeta_0 + C(t - s)\zeta_1 + \int_s^t C(t - \xi)h(\xi)d\xi, \quad t \in J.
\]
Such a solution (3.2) is called a strong solution. Furthermore, if \( \zeta_0 \in D(A) \), \( \zeta_1 \in E \) and \( h \) is a continuously differentiable function, then the function \( z(\cdot) \) become a classical solution of the system (3.1). The existence of mild as well as strong solutions for the above system has been discussed in [49].

3.1. **Optimal control problem for the linear system.** In this subsection, first we prove the existence of an optimal pair, which minimizes a cost functional consisting in the linear-quadratic regulator problem. The cost functional is given by

\[
\mathcal{F}(x, u) = \|x(T) - x^n\|_X^2 + \lambda \int_0^T \|u(t)\|_U^2 \, dt,
\]

where \( \lambda > 0 \), \( x_T \in X \) and \( x(\cdot) \) is the mild solution of the linear system:

\[
\begin{cases}
    \dot{x}(t) = Ax(t) + Bu(t), & t \in J, \\
    x(0) = \phi(0), \quad \dot{x}(0) = \zeta_0, \quad \phi(0) \in \mathcal{P}, \quad \zeta_0 \in \mathcal{X},
\end{cases}
\]

with the control \( u \). We take the admissible control class as \( \mathcal{U}_{ad} = L^2(J; H) \), consisting of the controls \( u \). Since \( Bu \in L^1(J; X) \), there exists a unique mild solution \( x(\cdot) \) of the system (3.4) satisfying (see [49])

\[
x(t) = C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)Bu(s)ds, \quad t \in J.
\]

**Definition 3.1** (Admissible class). The admissible class \( \mathcal{A}_{ad} \) of pairs \((x, u)\) is defined as the set of states \( x(\cdot) \) solving the system (3.4) with the control \( u \in \mathcal{U}_{ad} \). That is,

\[
\mathcal{A}_{ad} := \{ (x, u) : x \text{ is a unique mild solution of (3.4) with the control } u \in \mathcal{U}_{ad} \}.
\]

Since there exists a unique mild solution of the system (3.4) for any \( u \in \mathcal{U}_{ad} \), it follows that the admissible class \( \mathcal{A}_{ad} \) is nonempty. By using the above definition of \( \mathcal{F}(\cdot, \cdot) \), we formulate the optimal control problem as:

\[
\min_{(x, u) \in \mathcal{A}_{ad}} \mathcal{F}(x, u).
\]

A solution to the problem (3.6) is called an optimal solution. The optimal pair will be denoted by \((x^0, u^0)\). The control \( u^0 \) is called an optimal control.

**Theorem 3.1** (Existence of an optimal pair). Let \( \phi(0), \zeta_0 \in \mathcal{X} \) be given. Then there exists at least one pair \((x^0, u^0) \in \mathcal{A}_{ad}\) such that the functional \( \mathcal{F}(x, u) \) attains its minimum at \((x^0, u^0)\), where \( x^0 \) is the unique mild solution of the system (3.4) with the control \( u^0 \).

**Proof.** Let us first define

\[
\mathcal{F} := \inf_{u \in \mathcal{U}_{ad}} \mathcal{F}(x, u).
\]

Since, \( 0 \leq \mathcal{F} < +\infty \), there exists a minimizing sequence \( \{u^n\}_{n=1}^{\infty} \in \mathcal{U}_{ad} \) such that

\[
\lim_{n \to \infty} \mathcal{F}(x^n, u^n) = \mathcal{F},
\]

where \( x^n(\cdot) \) is the unique mild solution of the system (3.4), with the control \( u^n \) and the initial data \( x^n(0) = \phi(0) \) and \( \dot{x}^n(0) = \zeta_0 \). Note that \( x^n(\cdot) \) satisfies

\[
x^n(t) = C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)Bu^n(s)ds, \quad \text{if } t \in J.
\]
Since $0 \in \mathcal{U}_0$, without loss of generality, we may assume that $F(x^n, u^n) \leq F(x, 0)$, where $(x, 0) \in \mathcal{U}_0$, Using the definition of $F(\cdot, \cdot)$, we easily get
\[
\|x^n(T) - xT\|_X^2 + \lambda \int_0^T \|u^n(t)\|_{\mathcal{H}}^2 dt \leq \|x(T) - xT\|_X^2 \leq 2\left(\|x(T)\|_X^2 + \|xT\|_X^2\right) < +\infty.
\] (3.8)

From the above relation, it is clear that, there exist an $R > 0$, large enough such that
\[
0 \leq F(x^n, u^n) \leq R < +\infty.
\]
In particular, there exists a large $C > 0$, such that
\[
\int_0^T \|u^n(t)\|_{\mathcal{H}}^2 dt \leq C < +\infty.
\] (3.9)
Moreover, from (3.7), we have
\[
\|x^n(t)\|_X \leq \|C(t)\phi(0)\|_X + \|S(t)\zeta_0\|_X + \int_0^t \|S(t - s)Bu^n(s)\|_X ds
\leq \|C(t)\|_L(\mathcal{X})\|\phi(0)\|_X + \|S(t)\|_L(\mathcal{X})\|\zeta_0\|_X
+ \int_0^t \|S(t - s)\|_L(\mathcal{X})\|B\|_L(\mathcal{H}, \mathcal{X})\|u^n(s)\|_\mathcal{H} ds
\leq Me^{\omega t}\|\phi(0)\|_X + M'e^{\omega t}\|\zeta_0\|_X + M'B\left(\int_0^t |e^{\omega(t-s)}| ds\right)^{1/2}\left(\int_0^t \|u^n(s)\|_{\mathcal{H}}^2 ds\right)^{1/2}
\leq Me^{\omega t}\|\phi(0)\|_X + M'e^{\omega t}\|\zeta_0\|_X + M'B\bar{e}^{\omega t}t^{1/2} < +\infty,
\] (3.10)
for all $t \in J$. Since $L^2(J; X)$ is reflexive, an application of the Banach-Alaoglu theorem yields the existence of a subsequence $\{x^{n_k}\}_{k=1}^\infty$ of $\{x^n\}_{n=1}^\infty$ such that
\[
x^{n_k} \rightharpoonup x^0 \text{ in } L^2(J; X), \quad \text{as } k \to \infty.
\] (3.11)
From (3.9), we also infer that the sequence $\{u^n\}_{n=1}^\infty$ is uniformly bounded in the space $L^2(J; \mathcal{H})$. Since $L^2(J; \mathcal{H})$ is a separable Hilbert space (in fact reflexive), using the Banach-Alaoglu theorem, we can find a subsequence $\{u^{n_k}\}_{k=1}^\infty$ of $\{u^n\}_{n=1}^\infty$ such that
\[
u^{n_k} \rightharpoonup u^0 \quad \text{in } L^2(J; \mathcal{H}) = \mathcal{U}_0, \quad \text{as } k \to \infty.
\]
Since $B$ is a bounded linear operator from $\mathcal{H}$ to $X$, the above convergence also implies
\[
Bu^{n_k} \rightharpoonup Bu^0 \quad \text{in } L^2(J; X).
\] (3.12)
Note that
\[
\left\|\int_0^t S(t - s)Bu^{n_k}(s) ds - \int_0^t S(t - s)Bu^0(s) ds\right\|_X \to 0, \quad \text{as } k \to \infty,
\] (3.13)
for all $t \in J$. Here, we used the weak convergence given in (3.12) and the compactness of the operator $(Qf)(\cdot) = \int_0^\cdot S(\cdot - s)f(s) ds : L^2(J; X) \to C(J; X)$. Using the above convergence, we estimate
\[
\|x^{n_k}(t) - x^*(t)\|_X \leq \left\|\int_0^t S(t - s)Bu^{n_k}(s) ds - \int_0^t S(t - s)Bu^0(s) ds\right\|_X
\to 0, \quad \text{as } k \to \infty, \quad \text{for all } t \in J,
\] (3.14)
Lemma 3.1. Assume that the cost functional (3.6) is convex, the constraint (3.4) is linear and the class \( \mathcal{U}_{ad} = L^2(J; \mathbb{H}) \) is convex, then the optimal control obtained in Theorem 3.1 is unique. The explicit expression of the optimal control \( u^* \) is given by the following lemma:

**Lemma 3.1.** Assume that \( u \) is the optimal control satisfying (3.4) and minimizing the cost functional (3.6). Then \( u \) is given by

\[
    u_\lambda(t) = B^*S^*(T - t)J[R(\lambda, \Psi_0^T)p(x(\cdot))], \quad t \in J,
\]

with

\[
    p(x(\cdot)) = x_T - (C(T)\phi(0) + S(T)\zeta_0).
\]

A proof of the above Lemma can be obtained by proceeding similarly as in the proof of Lemma 3.1, [43].

3.2. Approximate controllability of the linear system. Here, we discuss about the approximate controllability of the second order linear control system (3.4), by proving the following Theorem:

**Theorem 3.2.** The following statements are equivalent:

(i) The linear control system (3.4) is approximately controllable on \( J \).

(ii) The operator \( \Psi_0^T \) is positive, that is, \( \langle x^*, \Psi_0^T x^* \rangle > 0 \), for all nonzero \( x^* \in \mathbb{X}^* \).

(iii) The Assumption \((H0)\) holds.
Proof. Claim 1. (ii)$\iff$(iii). Since the operator $\Psi_0^T$ is symmetric, more precisely $\langle x_1^*, \Psi_0^T x_2^* \rangle = \langle x_2^*, \Psi_0^T x_1^* \rangle$, for all $x_1^*, x_2^* \in \mathbb{X}$, using Theorem 2.3 [37], we know that the operator $\Psi_0^T$ is positive if and only if the Assumption (H0) holds, which proves our claim.

Claim 2. (i)$\implies$(iii). Let us first assume that the Assumption (H0) holds true. We know that for every $\lambda > 0$ and $x_T \in \mathbb{X}$, the mild solution $x_\lambda \in C(J; \mathbb{X})$ for the system (3.4) can be written as

$$x_\lambda(t) = C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)Bu_\lambda(s)ds, \ t \in J$$

with $u_\lambda(t) = B^*S^*(T-t)J[R(\lambda, \Psi_0^T)p(x(\cdot))]$, and $p(x(\cdot)) = x_T - (C(T)\phi(0) + S(T)\zeta_0)$.

Using (3.16), it can be easily seen that

$$x_\lambda(T) = C(T)\phi(0) + S(T)\zeta_0 + \int_0^T S(T-s)Bu_\lambda(s)ds$$

$$= C(T)\phi(0) + S(T)\zeta_0 + \Psi_0^T[\lambda R(\lambda, \Psi_0^T)p(x(\cdot))]$$

$$= x_T - p(x(\cdot)) + \Psi_0^T[\lambda R(\lambda, \Psi_0^T)p(x(\cdot))]$$

$$= x_T - (\lambda T + \Psi_0^T[\lambda R(\lambda, \Psi_0^T)p(x(\cdot))])$$

$$= x_T - \lambda T + \Psi_0^T[\lambda R(\lambda, \Psi_0^T)p(x(\cdot))],$$

and since $\|C(T)\phi(0)\|_X \leq Me^\omega T\|\phi(0)\|_X$, $\|S(T)\zeta_0\|_X \leq M'e^\omega T\|\zeta_0\|_X$ and $x_T \in \mathbb{X}$, we have

$$\|x_\lambda(T) - x_T\|_X \leq \|\lambda T + \Psi_0^T[\lambda R(\lambda, \Psi_0^T)p(x(\cdot))]\|_X$$

$$\to 0, \ \text{as} \ \lambda \downarrow 0.$$ 

Thus, it follows that the system (3.4) is approximately controllable, and hence (iii)$\Rightarrow$(i).

Conversely, we assume that the linear control system (3.4) is approximately controllable on $J$. Then, for arbitrary $x_T \in \mathbb{X}$, there exist a sequence $\{u^n\}_{n=1}^\infty$ in $\mathcal{U}_{ad} = L^2(J; \mathbb{H})$ such that

$$\|x^n(T) - x_T\|_X \to 0 \ \text{as} \ n \to \infty,$$

where $x^n(\cdot)$ is the unique mild solution of the system (3.4) with the control $u^n \in \mathcal{U}_{ad}$. For all $n \geq 1$, we have

$$\|x^n(T) - x_T\|^2_X \leq \|x^n(T) - x_T\|^2_X + \lambda \int_0^T \|u^n(t)\|^2_\mathcal{H} dt$$

$$\leq \|x^n(T) - x_T\|^2_X + \lambda \int_0^T \|u^n(t)\|^2_\mathcal{H} dt,$$

where $(x^0, u^0) \in \mathcal{A}_{ad}$ is the optimal pair at which the functional (3.3) takes its minimum value. For given $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$\|x^n(T) - x_T\|_X < \frac{\varepsilon}{\sqrt{2}}, \ \text{for all} \ n \geq n_0.$$

Now we can choose a $\delta > 0$ sufficiently small such that

$$\lambda \int_0^T \|u^{n_0}(t)\|^2_\mathcal{H} dt < \frac{\varepsilon^2}{2}.$$
for all $0 < \lambda < \delta$. Taking $n = n_0$ in (3.19), we get
\[ \|x^0(T) - x_T\|_X^2 \leq \|x^{n_0}(T) - x_T\|_X^2 + \lambda \int_0^T \|u^{n_0}(t)\|_H^2 dt < \varepsilon, \tag{3.20} \]
for all $0 < \lambda < \delta$. Invoking Lemma 3.1, we know that
\[ u^0(t) = B^*S^*(T - t)J[R(\lambda, \Psi_0^T)p(x(\cdot))], \quad t \in J, \]
with
\[ p(x(\cdot)) = x_T - (C(T)\phi(0) + S(T)\zeta_0). \]
Proceeding in a similar way as in the proof of the equality (3.17), we obtain
\[ x^0(T) = x_T - \lambda R(\lambda, \Psi_0^T)p(x(\cdot)). \tag{3.21} \]
Since $x_T \in X$ is arbitrary, using the inequality (3.20) together with the expression (3.21), we deduce that the Assumption (H0) follows.

\begin{remark}
Note that for $x^* \in X^*$ and $u \in L^2(J; H)$, we have
\[ ((L_T)^*x^*, u)_{L^2(J; H)} = \langle x^*, L_Tu \rangle = \left\langle x^*, \int_0^T S(T - t)Bu(t)dt \right\rangle \]
\[ = \int_0^T \langle x^*, S(T - t)Bu(t) \rangle dt = \int_0^T (B^*S^*(T - t)x^*, u(t))dt \]
\[ = (B^*S^*(T - t)x^*, u)_{L^2(J; H)}, \tag{3.22} \]
therefore, it is immediate that $(L_T)^* = B^*S^*(T - t)$. Note that $B^*S^*(T - t)x^* = 0$ on $J$ implies $x^* = 0$, and then by the above fact, we easily obtain that the operator $\Psi_0^T$ is positive and vice versa.
\end{remark}

4. Approximate controllability of the nonlinear system. In the present section, we investigate the approximate controllability of the system (1.1). In order to do this, we first show that, for every $\lambda > 0$ and $x_T \in X$, the system (1.1) has at least one mild solution with the control
\[ u_\lambda(t) = B^*S^*(T - t)J[R(\lambda, \Psi_0^T)g(x(\cdot))], \tag{4.1} \]
where
\[ g(x(\cdot)) = x_T - C(T)\phi(0) - S(T)\zeta_0 - \int_0^T S(T - s)f(s, \tilde{x}_\rho(s, \tilde{x}))ds \]
\[ - \sum_{k=1}^m C(T - \tau_k)I_{k}(\tilde{x}(\tau_k)) - \sum_{k=1}^m S(T - \tau_k)g_k(\tilde{x}(\tau_k)), \tag{4.2} \]
with $\tilde{x} : (-\infty, T] \to X$ such that $\tilde{x}(t) = \phi(t)$ on $t \in (-\infty, 0)$ and $\tilde{x} = x$ on $J$.

**Theorem 4.1.** Let the Assumptions (R1)-(R3) and (H1)-(H4) hold true. Then for every $\lambda > 0$ and for fixed $x_T \in X$, the system (1.1) with the control (4.1) has at least one mild solution on $J$, provided that
\[ M'e^{\omega T} \left\{ 1 + \frac{1}{\lambda} TM^2M_B^2 e^{2\omega T} \right\} H_2 \delta < 1. \tag{4.3} \]
Proof. Let $Z = \{x \in \mathcal{PC} : x(0) = \phi(0)\}$ be the space endowed with the norm $\| \cdot \|_{\mathcal{PC}}$. We consider a set

$$Q_r = \{x \in Z : \|x\|_{\mathcal{PC}} \leq r\}.$$  

For $\lambda > 0$, we define an operator $\Phi_\lambda : Z \to Z$ as

$$(\Phi_\lambda x)(t) = z(t),$$

where

$$z(t) = C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)\left[ f(s, \dot{x}, \ddot{x}) \right] ds + Bu_\lambda(s) \right| ds$$

$$+ \sum_{0 < \tau_k < t} C(t - \tau_k)I_k(\dot{x}(\tau_k)) + \sum_{0 < \tau_k < t} S(t - \tau_k)g_k(\dot{x}(\tau_k)), \ t \in J. \quad (4.4)$$

It is clear from the definition $\Phi_\lambda$, for $\lambda > 0$, that the fixed point of $\Phi_\lambda$ is a mild solution of system (1.1). We prove that the operator $\Phi_\lambda$ has a fixed point in the following steps.

Step 1. $\Phi_\lambda(Q_r) \subset Q_r$, for some $r$. Indeed, suppose that our assertion is false. Then for any $\lambda > 0$ and for all $r > 0$, there exists $x^r(\cdot) \in Q_r$, such that $\| (\Phi_\lambda x^r)(t) \|_X > r$, for some $t \in J$, where $t$ may depend upon $r$. First, by using the expressions defined in (4.1), (2.4), and Assumption 2.2, we evaluate

$$\| u_\lambda(t) \|_E = \| B^* S^*(T-t) \mathcal{J} [R(\lambda, \Psi^\tau) g(x(\cdot)) \| \|_E$$

$$\leq \frac{1}{\lambda} \| B^* \|_\mathcal{L}(X^*, E) \| S^*(T-t) \mathcal{J} [R(\lambda, \Psi^\tau) g(x(\cdot)) \|_X$$

$$\leq \frac{M_B M' e^{\omega T}}{\lambda} \left( \| xT \|_X + M e^{\omega T} \| \phi(0) \|_X + M' e^{\omega T} \| \zeta_0 \|_X$$

$$+ M' e^{\omega T} \int_0^T \| f(s, \dot{x}, \ddot{x}) \|_X ds + M e^{\omega T} \sum_{k=1}^m \| I_k(\dot{x}(\tau_k)) \|_X$$

$$+ M' e^{\omega T} \sum_{k=1}^m \| g_k(\dot{x}(\tau_k)) \|_X \right) \leq \frac{M_B M' e^{\omega T}}{\lambda} \left( \| xT \|_X + M e^{\omega T} \| \phi(0) \|_X + M' e^{\omega T} \| \zeta_0 \|_X$$

$$+ M' e^{\omega T} \int_0^T \| f(s, \dot{x}, \ddot{x}) \|_X ds + M e^{\omega T} \sum_{k=1}^m c_k + M' e^{\omega T} \sum_{k=1}^m d_k \right). \quad (4.5)$$

Inequality (4.5) ensures that $\| u_\lambda(t) \|_E$ is bounded for all $t \in J$. Let us define $\| u_\lambda \|_\infty := \sup_{t \in J} \| u_\lambda(t) \|_E$. Now, we estimate

$$r < \| (\Phi_\lambda x^r)(t) \|_X$$

$$= \| C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)\left[ f(s, \dot{x}, \ddot{x}) \right] ds + Bu_\lambda(s) \right| ds$$

$$+ \sum_{0 < \tau_k < t} C(t - \tau_k)I_k(\dot{x}(\tau_k)) + \sum_{0 < \tau_k < t} S(t - \tau_k)g_k(\dot{x}(\tau_k)) \|_X$$
\[ \leq \|C(t)\phi(0)\|_X + \|S(t)\zeta_0\|_X + \left\| \int_0^t S(t-s)[f(s, x^T_{\rho(s,x^*_s)}) + Bu_s(s)]ds \right\|_X \]
\[ + \left\| \sum_{0<\tau_k<t} C(t-\tau_k)I_k(x^T_{\tau_k}) \right\|_X + \left\| \sum_{0<\tau_k<t} S(t-\tau_k)g_k(x^T_{\tau_k}) \right\|_X. \quad (4.6) \]

For any \( x \in Q_r \), by applying Lemma 2.1, we compute
\[ \left\| x^T_{\rho(s,x^*_s)} \right\|_p \leq H_1\|\phi\|_p + H_2r = r'. \]

Using (4.5) and (4.6), and the Assumption 2.2, we obtain
\[ r < \| (\Phi x^T)(t) \|_X \]
\[ \leq M e^{\omega t}\|\phi(0)\|_X + M' e^{\omega t}\|\zeta_0\|_X + M' e^{\omega t}\left[ M_B T\|u_\lambda\|_\infty + \int_0^T \gamma_r(s)ds \right] + M e^{\omega t}\sum_{k=1}^m c_k + M' e^{\omega t}\sum_{k=1}^m d_k, \quad (4.7) \]

where \( \tilde{N} = \|x_T\|_X + M e^{\omega T}\|\phi(0)\|_X + M' e^{\omega T}\|\zeta_0\|_X + M e^{\omega T}\sum_{k=1}^m c_k + M' e^{\omega T}\sum_{k=1}^m d_k. \)

From the Assumption 2.2 (H3), it is immediate that
\[ \lim_{r \to \infty} \int_0^T \frac{\gamma_r(t)}{r} dt = \lim_{r \to \infty} \left( \int_0^T \frac{\gamma_r(t)}{r'} dt \cdot \frac{r'}{r} \right) = H_2\delta. \]

Dividing by \( r \) in the inequality (4.7) and then passing \( r \to \infty \), we get
\[ M' e^{\omega T}\left\{ 1 + \frac{1}{\lambda} TM_B^2 e^{2\omega T} \right\} H_2\delta > 1, \]
which is a contradiction to (4.3).

**Step 2.** Our next aim is to show that the operator \( \Phi_\lambda \) is a continuous operator. In order to do this, we consider a sequence \( \{x^n\}_{n=1}^\infty \subseteq Q_r \) such that \( x^n \to x \) in \( Q_r \), that is,
\[ \lim_{n \to \infty} \|x^n - x\|_{PC} = 0. \]

Applying Lemma 2.1, we have
\[ \|x^n_s - \tilde{x}_s\|_p \leq H_2 \sup_{\theta \in [0,T]} \|x^n(\theta) - \tilde{x}(\theta)\|_X = H_2\|x^n - x\|_{PC} \to 0 \quad \text{as} \quad n \to \infty, \]
for all \( s \in Z(\rho^-) \cup J \). Since \( \rho(s, x^n_s) \in Z(\rho^-) \cup J \), for all \( k \in \mathbb{N} \), then we conclude that
\[ \|x^n_{\rho(s,x^n_s)} - \tilde{x}_{\rho(s,x^n_s)}\|_p \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad s \in J \quad \text{and} \quad k \in \mathbb{N}. \]

In particular, we take \( k = n \) and using the above convergence together with the Assumption 2.2 (H3) and evaluate
Next, we estimate \(H\)orem and Assumption 2.2 (where we used the convergence (4.8) together with the dominated convergence theorem for each \(t\)). Since the operator \(S(t)\) is compact for each \(t\), we know that the duality mapping \(\Psi_R^*\) is compact for each \(t\). Thus, by using (4.11) and compactness of the operator \(S(t)\), we easily get

\[
\| R(\lambda, \Psi_R^*)g(x^n(\cdot)) - R(\lambda, \Psi_R^*)g(x(\cdot)) \|_X \leq \frac{1}{\lambda} \| \lambda R(\lambda, \Psi_R^*)g(x(\cdot)) - g(x(\cdot)) \|_X \\
\leq \frac{1}{\lambda} \| g(x^n(\cdot)) - g(x(\cdot)) \|_X \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]  

(4.10)

We know that the duality mapping \(\mathcal{J}\) is demicontinuous, and thus we have

\[
\mathcal{J} [R(\lambda, \Psi_R^*)g(x^n(\cdot))] \xrightarrow{\text{w}} \mathcal{J} [R(\lambda, \Psi_R^*)g(x(\cdot))] \text{ as } n \rightarrow \infty \text{ in } X^*.
\]  

(4.11)

Since the operator \(S(t)\) is compact for each \(t \in \mathbb{R}\), implies that the operator \(S^*(t)\) is also compact for each \(t \in \mathbb{R}\). Thus, by using (4.11) and compactness of the operator \(S^*(\cdot)\), we obtain

\[
\| u_s^*(t) - u_s(\cdot) \|_\mathcal{H} \\
= \| B^* S^*(t - t) (\mathcal{J} [R(\lambda, \Psi_R^*)g(x^n(\cdot))] - \mathcal{J} [R(\lambda, \Psi_R^*)g(x(\cdot))]) \|_\mathcal{H} \\
\leq \| B^* \|_{L_2(X, \mathcal{H})} \| S^*(t - t) (\mathcal{J} [R(\lambda, \Psi_R^*)g(x^n(\cdot))] - \mathcal{J} [R(\lambda, \Psi_R^*)g(x(\cdot))]) \|_X \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]  

(4.12)
\[ \lambda \text{ theorem. From the convergence (4.13), it follows that } \Phi_{H} \text{ is continuous.} \]

\[ \text{Step 3. } \Phi_{\lambda} \text{ is a compact operator. To prove this, first we show that } \Phi_{\lambda}(Q_r) \text{ is equicontinuous. For } 0 \leq t_1 \leq t_2 \leq T \text{ and } x \in Q_r, \text{ we compute} \]

\[ \| (\Phi_{\lambda} x)(t_2) - (\Phi_{\lambda} x)(t_1) \|_{\mathcal{X}} \]

\[ \leq \| C(t_2) \phi(0) - C(t_1) \phi(0) \|_{\mathcal{X}} + \| S(t_2) \zeta_0 - S(t_1) \zeta_0 \|_{\mathcal{X}} \]

\[ + \left\| \int_{t_1}^{t_2} S(t_2 - s) \left[ B u_{\lambda}(s) + f(s, \tilde{x}_{\rho(s, \tilde{x})}) \right] ds \right\|_{\mathcal{X}} \]

\[ + \left\| \int_{0}^{t_1} [S(t_2 - s) - S(t_1 - s)] [B u_{\lambda}(s) + f(s, \tilde{x}_{\rho(s, \tilde{x})})] ds \right\|_{\mathcal{X}} \]

\[ + \left\| \sum_{0 < \tau_k < t_1} [C(t_2 - \tau_k) - C(t_1 - \tau_k)] I_k(\tilde{x}(\tau_k)) \right\|_{\mathcal{X}} \]

\[ + \left\| \sum_{0 < \tau_k < t_1} S(t_2 - \tau_k) g_k(\tilde{x}(\tau_k)) \right\|_{\mathcal{X}} \]

\[ + \left\| \sum_{t_1 \leq \tau_k < t_2} S(t_2 - \tau_k) g_k(\tilde{x}(\tau_k)) \right\|_{\mathcal{X}} \]

\[ + \left\| \sum_{t_1 \leq \tau_k < t_2} C(t_2 - \tau_k) I_k(\tilde{x}(\tau_k)) \right\|_{\mathcal{X}} \]

\[ \leq \| C(t_2) \phi(0) - C(t_1) \phi(0) \|_{\mathcal{X}} + \| S(t_2) \zeta_0 - S(t_1) \zeta_0 \|_{\mathcal{X}} \]

\[ + \int_{t_1}^{t_2} \left\| S(t_2 - s) \right\|_{L(\mathcal{X})} \left\| B u_{\lambda}(s) + f(s, \tilde{x}_{\rho(s, \tilde{x})}) \right\|_{\mathcal{X}} ds \]

\[ + \int_{0}^{t_1} \left\| S(t_2 - s) - S(t_1 - s) \right\|_{L(\mathcal{X})} \left\| B u_{\lambda}(s) + f(s, \tilde{x}_{\rho(s, \tilde{x})}) \right\|_{\mathcal{X}} ds \]

\[ + \sum_{0 < \tau_k < t_1} \left\| C(t_2 - \tau_k) I_k(\tilde{x}(\tau_k)) - C(t_1 - \tau_k) I_k(\tilde{x}(\tau_k)) \right\|_{\mathcal{X}} \]

\[ + \sum_{0 < \tau_k < t_1} \left\| S(t_2 - \tau_k) - S(t_1 - \tau_k) \right\|_{L(\mathcal{X})} \left\| g_k(\tilde{x}(\tau_k)) \right\|_{\mathcal{X}} \]

where we used (4.8), (4.12), Assumption 2.2 (H4), and the dominated convergence theorem. From the convergence (4.13), it follows that \( \Phi_{\lambda} \) is continuous.

\[ \text{(4.13)} \]
Assumption 4.1. The following assumption is imposed on $f$ \begin{align*}
\end{align*}

Theorem 4.2. By using the Theorem 4.1, for every $x \in \mathcal{X}$, uniformly for $t_1 \leq t_2$, the compactness of the operator $Q(t)$ follows from the uniform operator topology. Therefore, $\Phi_{\lambda}$ and $S(t)$ is precompact in $\mathcal{X}$. In order to do this, we prove that for every $t \in J$, the operator $S(t)$ is uniformly continuous on $J$ and the operator $S(t)$ is continuous in the uniform operator topology. Therefore, $\Phi_{\lambda}(Q_r)$ is equicontinuous.

Next, we show that for each $\lambda > 0$, the operator $\Phi_{\lambda}$ maps $Q_r$ into a relative compact subset of $Q_r$. In order to do this, we prove that for every $t \in J$, the set $V(t) = \{ (\Phi_{\lambda}x)(t) : x \in Q_r \}$ is precompact in $\mathcal{X}$. By the Assumption 2.2 (H1) and (H2), we know that the operators $S(t)$, for $t \in J$ and $I_k$, for $k = 1, \ldots, m$, are compact. Thus, the precompactness of $V(t)$, for each $t \in J$, follows from the compactness of the operators $S(t)$, for $t \in J$ and $I_k$'s, for $k = 1, \ldots, m$, and also the compactness of the operator $(Qf)(\cdot) = \int_0^t S(t-s)f(s)ds : L^2(J; \mathcal{X}) \to C(J; \mathcal{X})$ (Lemma 3.2, Corollary 3.3, Chapter 3, [35]). By the Arzelá-Ascoli Theorem, it follows that the operator $\Phi_{\lambda}$ is compact. Thus, invoking Schauder’s fixed point theorem, $\Phi_{\lambda}$ has a fixed point in $Q_r$.

In order to establish the approximate controllability of the system (1.1), the following assumption is imposed on $f(\cdot, \cdot)$.

Assumption 4.1. (H5) The function $f : J \times P \to \mathcal{X}$ satisfies the Assumption H3.\((i)\) and uniformly bounded, that is, there exists a constant $\hat{N} > 0$ such that

\[ ||f(t, \phi)||_{\mathcal{X}} \leq \hat{N}, \quad \text{for all } (t, \phi) \in J \times P. \]

Theorem 4.2. Suppose that the Assumptions (H0)-(H2), (H4)-(H5) and the conditions of the Theorem 4.1 hold. Then the system (1.1) is approximately controllable on $J$.

**Proof.** By using the Theorem 4.1, for every $\lambda > 0$ and $x_T \in \mathcal{X}$, there exists a mild solution $x^\lambda(\cdot)$ such that

\begin{align*}
  x^\lambda(t) &= C(t)\phi(0) + S(t)\zeta_0 + \int_0^t S(t-s)f\left(s, x^\lambda_\rho(s, x_T)\right)ds + \int_0^t S(t-s)Bu^\lambda(s)ds \\
  &+ \sum_{0 < \tau_k < t} C(t-\tau_k)I_k(x^\lambda(\tau_k)) + \sum_{0 < \tau_k < t} S(t-\tau_k)g_k(x^\lambda(\tau_k)), \quad t \in J.
\end{align*}
with the control 
\[ u^\lambda(t) = B^* S^*(T - t) \mathcal{J} [R(\lambda, \Psi^T_0) g(x^\lambda(\cdot))], \] 
where 
\[ g(x^\lambda(\cdot)) = x_T - C(T) \phi(0) - S(T) \zeta_0 - \int_0^T S(T - s) f(s, x^\lambda_{\rho(s, x^\lambda)}) \, ds \]
\[ - \sum_{k=1}^m C(T - \tau_k) I_k (x^\lambda(\tau_k)) - \sum_{k=1}^m S(T - \tau_k) g_k(x^\lambda(\tau_k)). \]

Using the control given in (4.15), it is easy to verify that
\[ x^\lambda(T) = x_T - \lambda R(\lambda, \Psi^T_0) g(x^\lambda(\cdot)). \]

By the Assumption 4.1 (H5), we have
\[ \int_0^T \left\| f(s, \tilde{x}^\lambda_{\rho(s, x^\lambda)}) \right\|_X^2 \, ds \leq \tilde{N}^2 T. \]
That is, the sequence \( \{ f(s, \tilde{x}^\lambda_{\rho(s, x^\lambda)}) : \lambda > 0 \} \) is a bounded sequence in \( L^2(J; X) \).

Then, by the Banach-Alaoglu theorem, there exists a subsequence, relabeled as \( \{ f(s, \tilde{x}^\lambda_{\rho(s, x^\lambda)}) : \lambda > 0 \} \), which is weakly convergent to, say \( f \in L^2(J; X) \). Furthermore, using the Assumption 2.2 (H4), we obtain
\[ \left\| I_k (x^\lambda(\tau_k)) \right\|_X \leq c_k \quad \text{and} \quad \left\| g_k(x^\lambda(\tau_k)) \right\|_X \leq d_k, \]
for each \( k = 1, \ldots, m \). Thus, the sequences \( \{ I_k (x^\lambda(\tau_k)) : \lambda > 0 \} \) and \( \{ g_k(x^\lambda(\tau_k)) : \lambda > 0 \} \) are bounded in \( X \). Once again using the Banach-Alaoglu theorem, we can find weakly convergent subsequences relabeled as \( \{ I_k (x^\lambda(\tau_k)) : \lambda > 0 \} \) and \( \{ g_k(x^\lambda(\tau_k)) : \lambda > 0 \} \), with pointwise weak limits \( \mu_k \) and \( \nu_k \) respectively, for each \( k = 1, \ldots, m \). We now estimate
\[ \left\| g(x^\lambda(\cdot)) - \eta \right\|_X \leq \left\| \int_0^T S(T - s) \left[ f(s, x^\lambda_{\rho(s, x^\lambda)}) - f(s) \right] \, ds \right\|_X \]
\[ + \sum_{k=1}^m \left\| C(T - \tau_k) \left( I_k (x^\lambda(\tau_k)) - \mu_k \right) \right\|_X \]
\[ + \sum_{k=1}^m \left\| S(T - \tau_k) \left( g_k(x^\lambda(\tau_k)) - \nu_k \right) \right\|_X, \]
where
\[ \eta := x_T - S(T) \phi(0) - C(T) \zeta_0 - \int_0^T S(T - s) f(s) \, ds - \sum_{k=1}^m C(T - \tau_k) \mu_k \]
\[ - \sum_{k=1}^m S(T - \tau_k) \nu_k. \]

Further, using the Assumption 2.2 and bounds of \( C(t), S(t) \), for all \( t \in J \), we have
\[ \left\| C(T - \tau_k) (I_k (x^\lambda(\tau_k)) - \mu_k) \right\|_X \leq M e^{\omega T} \left( \left\| I_k (x^\lambda(\tau_k)) \right\|_X + \| \mu_k \|_X \right) \]
\[ \leq M e^{\omega T} (c_k + \| \mu_k \|_X). \]
Next, we evaluate
\[
\| \lambda R(\lambda, \Psi_0^T)(g(x^\lambda(\cdot)) - \eta) \|_X
\]
\[
\leq \| \lambda R(\lambda, \Psi_0^T) \int_0^T S(T - s) \left[ f \left( s, \tilde{x}^\lambda_{\rho(s), x^\lambda_s} \right) - f(s) \right] ds \|_X
\]
\[
+ \sum_{k=1}^m \| \lambda R(\lambda, \Psi_0^T) C(T - \tau_k)(I_k(\tilde{x}^\lambda_{\tau_k})) - \mu_k \|_X
\]
\[
+ \sum_{k=1}^m \| \lambda R(\lambda, \Psi_0^T) S(T - \tau_k)(g_k(\tilde{x}^\lambda_{\tau_k})) - \nu_k \|_X
\]
\[
\leq \| \lambda R(\lambda, \Psi_0^T) \|_{L(X)} \left\| \int_0^T S(T - s) \left[ f \left( s, \tilde{x}^\lambda_{\rho(s), x^\lambda_s} \right) - f(s) \right] ds \right\|_X
\]
\[
+ \sum_{k=1}^m \| \lambda R(\lambda, \Psi_0^T) C(T - \tau_k)(I_k(\tilde{x}^\lambda_{\tau_k})) - \mu_k \|_X
\]
\[
+ \sum_{k=1}^m \| \lambda R(\lambda, \Psi_0^T) S(T - \tau_k)(g_k(\tilde{x}^\lambda_{\tau_k})) - \nu_k \|_X
\]
\[
\to 0 \text{ as } \lambda \to 0. \tag{4.17}
\]
In the right hand side of (4.17), the first term goes to zero using the compactness of the operator \((Qf)(\cdot) = \int_0^T S(-t)f(s)ds : L^2(J; X) \to C(J; X)\) (see Lemma 3.2, Chapter 3, [35]), the second term tends to zero making the use of the Hypothesis \((H0)\) and the final term tends to zero by using the compactness of the sine family \(S(\cdot)\). Finally, by using the result (4.17) and Assumption 2.2 \((H0)\), we obtain
\[
\| x^\lambda(T) - x_T \|_X = \| \lambda R(\lambda, \Psi_0^T) g(x^\lambda(\cdot)) \|_X
\]
\[
\leq \| \lambda R(\lambda, \Psi_0^T) \eta \|_X + \| \lambda R(\lambda, \Psi_0^T)(g(x^\lambda(\cdot)) - \eta) \|_X \to 0 \text{ as } \lambda \to 0,
\]
and hence, the control system (1.1) is approximately controllable on \(J\). \(\square\)

5. Application. The Cauchy problem for one dimensional wave equation has been studied extensively in the literature, see for instance [17, 23, 27, 34], etc. In this section, as an application of the theory developed in the previous sections, we investigate the approximate controllability of an impulsive wave equation with state-dependent delay.

Example 5.1. Let us consider the following wave equation:
\[
\begin{cases}
\frac{\partial^2}{\partial t^2} z(t, \varsigma) = \frac{\partial^2}{\partial \varsigma^2} z(t, \varsigma) + \mu(t, \varsigma) + \int_{-\infty}^t b(s - t)z(s - \rho_1(t)\rho_2(||z(t)||), \varsigma)ds,
\varsigma \in [0, \pi], \ t \in J = [0, T], \ t \neq \tau_k, \ k = 1, \ldots, m,
\end{cases}
\]
\[
z(t, 0) = 0 = z(t, \pi), \ t \in J,
\]
\[
z(\theta, \varsigma) = \varphi(\theta, \varsigma), \frac{\partial}{\partial \theta} z(0, \varsigma) = \zeta_0(\varsigma), \varsigma \in [0, \pi], \ \theta \leq 0,
\]
\[
\Delta z(\tau_k, \varsigma) = I_k(z(\tau_k, \varsigma)), \ k = 1, \ldots, m, \ \varsigma \in [0, \pi],
\]
\[
\Delta z' (\tau_k, \varsigma) = g_k(z(\tau_k, \varsigma)), \ k = 1, \ldots, m, \ \varsigma \in [0, \pi],
\]
where the functions \(\rho_i : [0, \infty) \to [0, \infty]\), for \(i = 1, 2\), and \(b : [0, \infty) \to \mathbb{R}\) are continuous. Also the function \(\mu : J \times [0, \pi] \to [0, \pi]\) is continuous in \(t\), and the
functions \( I_k \)'s, \( g_k \)'s and \( \varphi \) satisfy appropriate conditions which will be specified later.

**Step 1.** Resolvent operator and strongly continuous families. Let \( X = L^p([0, \pi]; \mathbb{R}) \), for \( p \in [2, \infty) \), and \( U = L^2([0, \pi]; \mathbb{R}) \). The operator \( A : D(A) \subset X \rightarrow X \) is defined as

\[
Af(\xi) = f''(\xi), \quad \text{where } D(A) = W^{2,p}([0, \pi]; \mathbb{R}) \cap W_0^{1,p}([0, \pi]; \mathbb{R}). \tag{5.2}
\]

Note that \( C_0^\infty([0, \pi]; \mathbb{R}) \subset D(A) \) and hence \( D(A) \) is dense in \( X \). Also, it is easy to verify that \( A \) is closed, and hence the condition \((R1)\) of Assumption 2.1 is fulfilled. The spectrum of the operator \( A \) defined in (5.2) is given by \( \sigma(A) = \{-n^2 : n \in \mathbb{N}\} \). Moreover, for every \( f \in D(A) \), the operator \( A \) can be written as

\[
Af = \sum_{n=1}^\infty -n^2 \langle f, w_n \rangle w_n, \quad \langle f, w_n \rangle := \int_0^\pi f(\xi)w_n(\xi)d\xi,
\]

where \( w_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi) \) are the normalized eigenfunctions of the operator \( A \) corresponding to the eigenvalues \(-n^2\), \( n \in \mathbb{N} \). Next, we show that the Assumption 2.1 \((R2)\) holds true. For any \( f \in X \), the resolvent operator of \( A \) can be written as

\[
R(\lambda^2; A)f = (\lambda^2 I - A)^{-1}f = \sum_{n=1}^\infty \frac{1}{\lambda^2 + n^2} \langle f, w_n \rangle w_n. \tag{5.3}
\]

Clearly, from the above expression, for any \( \lambda > 0 \), \( \lambda^2 \in \rho(A) = \mathbb{C}\setminus\sigma(A) \). Now, we prove that for any \( \lambda > 0 \), we have

\[
\left\| \frac{d^k}{d\lambda^k}(\lambda \mathbb{R}(\lambda^2; A)) \right\|_{L(X)} \leq \frac{Mk!}{(\lambda - \omega)^{k+1}}, \quad k = 0, 1, 2, \ldots, \tag{5.4}
\]

with \( M = 1 \) and \( \omega = 0 \). One can use the orthogonality of eigenfunctions to obtain the above result in Hilbert spaces (for example, in our case \( L^2([0, \pi]; \mathbb{R}) \)). But in Banach spaces (in our context \( L^p([0, \pi]; \mathbb{R}), 2 < p < \infty \)), establishing the estimate (5.4) is not easy. For completeness, we provide a proof for the first few values of \( k \). For \( k = 0 \), we consider the equation

\[
\begin{cases}
\lambda^2 g(\xi) - g''(\xi) = \lambda f(\xi), & 0 < \xi < \pi, \\
g(0) = 0, \quad g(\pi) = 0.
\end{cases} \tag{5.5}
\]

Taking product both side with \( |g(\xi)|^{p-2}g(\xi) \) in (5.5) and integrating between 0 to \( \pi \), we get

\[
\lambda^2 \int_0^\pi |g(\xi)|^{p-2}|g(\xi)|^2d\xi + (p - 1) \int_0^\pi |g(\xi)|^{p-2}|g'(\xi)|^2d\xi = \lambda \int_0^\pi f(\xi)|g(\xi)|^{p-2}g(\xi)d\xi.
\]

In the above equality, an application of H"older's inequality yields

\[
\lambda^2 \int_0^\pi |g(\xi)|^p d\xi \leq \lambda \left( \int_0^\pi |f(\xi)|^p d\xi \right)^{1/p} \left( \int_0^\pi |g(\xi)|^p d\xi \right)^{\frac{p-1}{p}}.
\]

Thus, from (5.5), we infer that

\[
\|\lambda \mathbb{R}(\lambda^2; A)f\|_X = \|g\|_X \leq \frac{1}{\lambda} \|f\|_X. \tag{5.6}
\]
For $k = 1$, we evaluate
\[
\frac{d}{d\lambda} (\lambda R(\lambda; A)f) = \frac{d}{d\lambda} \left( \sum_{n=1}^{\infty} \frac{\lambda}{\lambda^2 + n^2} (f, \omega_n) \omega_n \right)
= \sum_{n=1}^{\infty} \left( \frac{1}{\lambda^2 + n^2} - \frac{2\lambda^2}{(\lambda^2 + n^2)^2} \right) (f, \omega_n) \omega_n
= R(\lambda^2; A)f - 2\lambda^2 (R(\lambda^2; A))^2 f.
\]
Using the inequality (5.6), we estimate
\[
\left\| \frac{d}{d\lambda} (\lambda R(\lambda^2; A)f) \right\|_X \leq \left\| R(\lambda^2; A)(f - 2\lambda^2 (R(\lambda^2; A))^2 f) \right\|_X
\leq \frac{1}{\lambda^2} \left\| I - 2\lambda^2 R(\lambda^2; A) \right\|_{L(X)} \| f \|_X. \tag{5.7}
\]
Note that the eigenvalues of the operator $I - 2\lambda^2 R(\lambda^2; A)$ are given by $\frac{n^2 - \lambda^2}{\lambda^2 + n^2}$, $n \in \mathbb{N}$.
It can be easily seen that $\left| \frac{n^2 - \lambda^2}{\lambda^2 + n^2} \right| < 1$, for all $\lambda > 0$ and $n \in \mathbb{N}$, and the spectral radius of the operator is 1. Furthermore, the operator $I - 2\lambda^2 R(\lambda^2; A)$ is normal in the sense of Definition 1.1, [39] (for normal operators in Banach spaces, see [41] also). Since $I - 2\lambda^2 R(\lambda^2; A)$ is a bounded operator, invoking the Corollary 1.4, [39], we infer that
\[
\left\| I - 2\lambda^2 R(\lambda^2; A) \right\|_{L(X)} = 1. \tag{5.8}
\]
Thus, from (5.7), we obtain
\[
\left\| \frac{d}{d\lambda} (\lambda R(\lambda^2; A)f) \right\|_{L(X)} \leq \frac{1}{\lambda^2}. \tag{5.9}
\]
For $f \in X$, we estimate
\[
\left\| \frac{d^2}{d\lambda^2} R(\lambda^2; A)f \right\|_X = \left\| -2\lambda (R(\lambda^2; A))^2 f \right\|_X \leq \frac{2}{\lambda^3} \| f \|_X,
\]
where we used (5.6). Thus, it is immediate that $\left\| \frac{d}{d\lambda} R(\lambda^2; A) \right\|_{L(X)} \leq \frac{2}{\lambda^2}$. Let us now consider
\[
\left\| \frac{d^2}{d\lambda^2} (\lambda R(\lambda^2; A)f) \right\|_{L(X)} = \left\| \frac{d}{d\lambda} (R(\lambda^2; A)(I - 2\lambda^2 R(\lambda^2; A))) \right\|_{L(X)}
\leq \left\| \frac{d}{d\lambda} R(\lambda^2; A) \right\|_{L(X)} \left\| I - 2\lambda^2 R(\lambda^2; A) \right\|_{L(X)} \leq \frac{2}{\lambda^3}. \tag{5.10}
\]
Next, we consider
\[
\left\| \frac{d^2}{d\lambda^2} R(\lambda^2; A) \right\|_{L(X)} = \left\| \frac{d}{d\lambda} (-2\lambda R(\lambda^2; A)^2) \right\|_{L(X)}
= 2 \left\| R(\lambda^2; A)^2 (I - 4\lambda^2 R(\lambda^2; A)) \right\|_{L(X)}
\leq \frac{2}{\lambda^4} \left( 2 \left\| I - 2\lambda^2 R(\lambda^2; A) \right\|_{L(X)} + 1 \right) \leq \frac{3}{\lambda^4}, \tag{5.11}
\]
where we used (5.6) and (5.8). For $k = 3$, using (5.8) and (5.11), we find
\[
\left\| \frac{d^3}{d\lambda^3} (\lambda R(\lambda^2; A)) \right\|_{L(X)} = \left\| \frac{d^2}{d\lambda^2} (R(\lambda^2; A)(I - 2\lambda^2 R(\lambda^2; A))) \right\|_{L(X)}
\leq \frac{3}{\lambda^4}. \tag{5.12}
\]
In (5.15), we used the fact that

\[ \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} = 4, \]

using a calculation similar to (5.12) yields

\[ \left\| \frac{d^3}{d\lambda^3} R(\lambda^2; A) \right\|_{\mathcal{L}(X)} \leq \frac{4!}{\lambda^4}. \]  

(5.12)

Moreover, we have

\[ \left\| \frac{d^3}{d\lambda^3} R(\lambda^2; A) \right\|_{\mathcal{L}(X)} = \left\| 24\lambda(R(\lambda^2; A))^3(I - 2\lambda^2 R(\lambda^2; A)) \right\|_{\mathcal{L}(X)} \leq \frac{4!}{\lambda^4}. \]  

(5.13)

For the case \( k = 4 \), using a calculation similar to (5.12) yields

\[ \left\| \frac{d^4}{d\lambda^4} (\lambda R(\lambda^2; A)) \right\|_{\mathcal{L}(X)} \leq \frac{5!}{\lambda^5}. \]  

(5.14)

The estimate \( \left\| \frac{d^4}{d\lambda^4} (\lambda R(\lambda^2; A)) \right\|_{\mathcal{L}(X)} \leq \frac{5!}{\lambda^5} \), can be obtained from the following computation:

\[
\left\| \frac{d^4}{d\lambda^4} R(\lambda^2; A) \right\|_{\mathcal{L}(X)} = 24 \left\| R(\lambda^2; A)^3(I - 12\lambda^2 R(\lambda^2; A) + 16\lambda^4 R(\lambda^2; A)^2) \right\|_{\mathcal{L}(X)} \\
\leq \frac{24}{\lambda^6} \left\| 3(I - 2\lambda^2 R(\lambda^2; A))^2 - 2(I - 2\lambda^4 R(\lambda^2; A)^2) \right\|_{\mathcal{L}(X)} \\
\leq \frac{24}{\lambda^6} \left( 3\|I - 2\lambda^2 R(\lambda^2; A)\|^2_{\mathcal{L}(X)} + 2\|I - 2\lambda^4 R(\lambda^2; A)^2\|_{\mathcal{L}(X)} \right) \\
\leq \frac{5!}{\lambda^5}.
\]  

(5.15)

In (5.15), we used the fact that

\[ \|I - 2\lambda^4 R(\lambda^2; A)^2\|_{\mathcal{L}(X)} = 1. \]  

(5.16)

The above equality can be seen in the following way. The eigenvalues of the operator \( I - 2\lambda^4 R(\lambda^2; A)^2 \) are given by \( \frac{n^4 + 2n^2 \lambda^2 - 2\lambda^4}{n^4 + 2n^2 \lambda^2 + \lambda^4} \), \( n \in \mathbb{N} \). It can be easily seen that

\[ \left| \frac{n^4 + 2n^2 \lambda^2 - 2\lambda^4}{n^4 + 2n^2 \lambda^2 + \lambda^4} \right| < 1, \quad \text{for all } \lambda > 0 \text{ and } n \in \mathbb{N}, \]

and the spectral radius of the operator is 1. Applying Corollary 1.4, [39], we obtain (5.16). Note that the estimates similar to (5.8) and (5.16) yield

\[ \left\| \frac{d^{k-1}}{d\lambda^{k-1}} R(\lambda^2; A) \right\|_{\mathcal{L}(X)} \leq \frac{k!}{\lambda^{k+1}}, \quad \text{for } k \geq 1, \]

which easily implies:

\[ \left\| \frac{d^k}{d\lambda^k} (\lambda R(\lambda^2; A)) \right\|_{\mathcal{L}(X)} \leq \frac{k!}{\lambda^{k+1}}, \quad \text{for } k \geq 1. \]

Next, we show that the resolvent operator \( R(\lambda^2; A) \) is compact for some \( \lambda \) with \( \lambda^2 \in \rho(A) \) for \( p \in [2, \infty) \). Taking \( \lambda = 1 \), the compactness of the operator \( R(1; A) \) follows in a similar way as in [43]. Hence, the condition (R3) of the Assumption 2.1 holds. Thus, the existence of a strongly continuous cosine family \( C(t), \ t \in \mathbb{R} \) in \( \mathbb{X} \), follows by Proposition 2.1 and then using Proposition 2.2, we obtain that the associated strongly continuous sine family \( S(t), \ t \in \mathbb{R} \) is compact on \( \mathbb{X} \). Moreover, the explicit expression for the strongly continuous cosine \( \{ C(t) : t \in \mathbb{R} \} \) and sine \( \{ S(t) : t \in \mathbb{R} \} \) families are given by

\[ C(t)f = \sum_{n=1}^{\infty} \cos(nt) \langle f, w_n \rangle w_n, \quad f \in \mathbb{X}, \]

\[ S(t)f = \sum_{n=1}^{\infty} \frac{1}{n} \sin(nt) \langle f, w_n \rangle w_n, \quad f \in \mathbb{X}. \]
Since $M = 1$ and $\omega = 0$, the strongly continuous cosine family $C(t)$ is of type $(1,0)$.

**Step 2.** Phase space. (1) Let $g : (-\infty, 0] \to \mathbb{R}^+$ be a Lebesgue integrable function, which is locally bounded. Take $\mathcal{P} = \mathcal{PC}_r \times L^1_r(\mathbb{X})$ as the space of all functions $\phi : (-\infty, 0] \to \mathbb{X}$ such that $\phi|_{[r,0]} \in \mathcal{PC}([-r,0]; \mathbb{X})$, for some $r > 0$, $\phi$ is Lebesgue measurable on $(-\infty, -r)$, and $g\|\phi(\cdot)\|_\mathbb{X}$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\mathcal{P}$ is defined as

$$
\|\phi\|_\mathcal{P} := \int_{-r}^{0} \|\phi(\theta)\|_\mathbb{X} d\theta + \int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|_\mathbb{X} d\theta.
$$

Moreover, there exists a locally bounded function $G : (-\infty, 0] \to \mathbb{R}^+$ such that $g(t + \theta) \leq G(t)g(\theta)$, for all $t \leq 0$ and $\theta \in (-\infty, 0) \setminus N_t$ where $N_t \subseteq (-\infty, 0)$ is a set with Lebesgue measure zero. A simple example of $g$ is given by $g(\theta) = e^{t\theta}$, for some $\nu > 0$.

In order to verify that the space $\mathcal{P}$ is a phase space, first we show that it satisfies the axiom (A1). In the context of our example, we choose $\sigma = T$, $\mu = 0$. Let $x(\cdot)$ be a function with $x_0 \in \mathcal{P}$ and $x|_J \in \mathcal{PC}(J; \mathbb{X})$. For $t \in J$, we verify that $x_t$ is in $\mathcal{P}$.

We know that the function $x_t : (-\infty, 0] \to \mathbb{X}$ is defined as

$$x_t(\theta) = x(t + \theta), \quad \text{for each } t \in J.$$ 

One can easily see that for each $t \in J$, the function $x_t|_{[-r,0]} \in \mathcal{PC}([-r,0]; \mathbb{X})$, $r > 0$ and Lebesgue measurable on $(-\infty, -r)$. For $t \in [0,r]$, we compute

$$
\|x_t\|_{\mathcal{P}} = \int_{-r}^{0} \|x(t + \theta)\|_\mathbb{X} d\theta + \int_{-\infty}^{-r} g(\theta)\|x(t + \theta)\|_\mathbb{X} d\theta
$$

$$= \int_{-r}^{t} \|x(t + \theta)\|_\mathbb{X} d\theta + \int_{-t}^{0} \|x(t + \theta)\|_\mathbb{X} d\theta + \int_{-\infty}^{-r} g(\theta)\|x(t + \theta)\|_\mathbb{X} d\theta
$$

$$+ \int_{-t}^{-r} g(\theta)\|x(t + \theta)\|_\mathbb{X} d\theta
$$

$$\leq \int_{-r}^{0} \|x(\theta)\|_\mathbb{X} d\theta + \int_{0}^{t} \|x(\theta)\|_\mathbb{X} d\theta + \int_{-\infty}^{-r} g(\theta - t)\|x(\theta)\|_\mathbb{X} d\theta
$$

$$+ \int_{-r}^{0} g(\theta - t)\|x(\theta)\|_\mathbb{X} d\theta
$$

$$\leq \int_{-r}^{0} \|x(\theta)\|_\mathbb{X} d\theta + t \sup_{\theta \in [0,t]} \|x(\theta)\|_\mathbb{X} + G(-t)\int_{-\infty}^{-r} g(\theta)\|x(\theta)\|_\mathbb{X} d\theta
$$

$$+ G(-t)\int_{-r}^{0} g(\theta)\|x(\theta)\|_\mathbb{X} d\theta
$$

$$\leq \left(1 + G(-t) \sup_{\theta \in [-r,0]} g(\theta)\right)\int_{-r}^{0} \|x(\theta)\|_\mathbb{X} d\theta + t \sup_{\theta \in [0,t]} \|x(\theta)\|_\mathbb{X}
$$

$$+ G(-t)\int_{-\infty}^{-r} g(\theta)\|x(\theta)\|_\mathbb{X} d\theta.$$

(5.18)

Similarly for $t > r$, we obtain

$$\|x_t\|_{\mathcal{P}} \leq \left(1 + G(-t) \sup_{\theta \in [-r,0]} g(\theta)\right)\int_{-r}^{0} \|x(\theta)\|_\mathbb{X} d\theta$$

(5.19)
\[ + \left( t + \int_{-t}^{-r} g(\theta) d\theta \right) \sup_{\theta \in [0,t]} \|x(\theta)\|_X + G(-t) \int_{-\infty}^{-r} g(\theta) \|x(\theta)\|_X d\theta. \]  \tag{5.19}

It is clear from (5.18) and (5.19) that the axiom (A1) holds with
\[ \Lambda(t) = \begin{cases} t, & \text{for } 0 \leq t \leq r, \\ t + \int_{-t}^{-r} g(\theta) d\theta, & \text{for } r < t, \end{cases} \]
and
\[ \Upsilon(t) = \max \left\{ 1 + G(-t) \sup_{\theta \in [-r,0]} g(\theta), G(-t) \right\}. \]

Finally, the completeness of the space \( P \), under the norm \( \| \cdot \|_P \) given in (5.17) follows in a similar way as in Theorem 1.3.1 [26] and thus the condition (A2) holds. In particular, we take \( r = 0 \) and in this case, we consider the following condition:

(C1) The function \( \varphi \in PC_0 \times L^1_g(X) \) and \( K := \sup_{\theta \in (-\infty,0]} \frac{|b(-\theta)|}{g(\theta)}. \)

(2) Let us now take another example of phase space, \( P = PC_g(X) \), as the space of all functions \( \phi : (-\infty, 0] \to X \) such that \( \phi \) is left continuous, \( \phi|_{[-r,0]} \in PC([-r,0]; X) \), for \( r > 0 \), and \( \int_{-\infty}^{0} \frac{\|\phi(\theta)\|_X}{g(\theta)} d\theta < \infty \). The norm \( \| \cdot \|_P \) is defined as
\[ \|\phi\|_P := \int_{-\infty}^{0} \frac{\|\phi(\theta)\|_X}{g(\theta)} d\theta, \]  \tag{5.20}
where \( g : (-\infty, 0] \to [1, \infty) \) is a continuous function with \( g(0) = 1 \), which satisfies the following conditions:

1. \( \lim_{\theta \to -\infty} g(\theta) = \infty, \)
2. the function \( G(t) := \sup_{-\infty < \theta \leq -t} \frac{g(t+\theta)}{g(\theta)} \) is locally bounded for \( t \geq 0 \).

One can easily verify that \( P = PC_g(X) \), is a phase space which satisfies the axioms (A1) and (A2) with \( \Lambda(t) = t \) and \( \Upsilon(t) = G(t) \). In this case, we consider the following condition:

(C2) The function \( \varphi \in PC_g(X) \) and \( K' := \sup_{\theta \in (-\infty,0]} |b(-\theta)|g(\theta). \)

**Step 3.** Abstract formulation and approximate controllability. The PDE (5.1) can be modeled as the abstract impulsive second order functional differential equation (1.1) in the following way:
\[ x(t)(\zeta) := z(t,\zeta), \quad \text{for } t \in J \text{ and } \zeta \in [0, \pi], \]
and the bounded linear operator \( B : L^2([0, \pi]; \mathbb{R}) \to X \) as
\[ B(u(t))(\zeta) := u(t)(\zeta) = \mu(t,\zeta), \quad t \in J, \ \zeta \in [0, \pi]. \]
The function \( \phi : (-\infty, 0] \to X \), is defined as
\[ \phi(t)(\zeta) = \varphi(t,\zeta), \quad \zeta \in [0, \pi]. \]

Next, we define the functions \( f, \rho : J \times P \to X \) as
\[ f(t,\varphi)(\zeta) := \int_{-\infty}^{0} b(-\theta)\varphi(\theta,\zeta) d\theta, \]
\[ \rho(t,\varphi) = t - \rho_1(s)\rho_2(\|\varphi(0)\|_X), \]
for \( \varsigma \in [0, \pi] \). Clearly, \( f \) is continuous. Moreover, if \( \mathcal{P} = \mathcal{P}C_0 \times L_q(\mathbb{X}) \), then the function, \( f \) is bounded by \( K \), and if \( \mathcal{P} = \mathcal{P}C_0(\mathbb{X}) \), then the function \( f \) is bounded by \( K' \). These facts ensure the condition \((H3)\) of the Assumption 2.2 and also ensure the condition \((H5)\) of the Assumption 4.1. Next, we define the impulses \( I_k, g_k : \mathbb{X} \to \mathbb{X} \) as

\[
I_k(z(\tau_k, \varsigma)) := I_k(z(\tau_k, \varsigma)), \\
g_k(z(\tau_k, \varsigma)) := g_k(z(\tau_k, \varsigma)),
\]

for \( k = 1, \ldots, m \) and \( \varsigma \in [0, \pi] \). Let us choose

\[
I_k(z(\tau_k, \varsigma)) = \int_0^\pi \rho_k(\varsigma, \zeta) \cos^2(z(\tau_k, \zeta)) d\zeta, \\
g_k(z(\tau_k, \varsigma)) = z(\tau_k, \varsigma),
\]

where \( \rho_k \in C([0, \pi] \times [0, \pi]; \mathbb{R}) \), \( \varsigma \in [0, \pi] \). The impulses \( I_k, g_k \), for each \( k = 1, \ldots, m \) satisfy the condition \((H4)\) of the Assumption 2.2 (cf. [36, 45], etc).

Using the above substitution, the system (5.1) can be modeled as abstract form of (1.1), which satisfies the Assumption 2.1 \((R1)-(R3)\), Assumptions 2.2 \((H1)-(H4)\) and Assumption 4.1. Finally, we verify that the corresponding linear system of the equation (1.1) is approximately controllable. For this, we consider \( B^*S^*(T - s)x^* = 0 \), for any \( x^* \in \mathbb{X}^* \), since \( B = I \) in the system (5.1), then we have

\[
B^*S^*(T - s)x^* = 0 \Rightarrow S^*(T - s)x^* = 0 \Rightarrow x^* = 0.
\]

Thus, by Theorem 3.2, the linear system corresponding to (5.1) is approximately controllable, and the condition \((H0)\) of the Assumption 2.2 holds. Hence, by Theorem 4.2, the semilinear system (5.1) is approximately controllable.

Acknowledgments: S. Arora would like to thank Council of Scientific and Industrial Research, New Delhi, Government of India (File No. 09/143(0931)/2013 EMR-I), for financial support to carry out his research work and Department of Mathematics, Indian Institute of Technology Roorkee (IIT Roorkee), for providing stimulating scientific environment and resources. M. T. Mohan would like to thank the Department of Science and Technology (DST), Govt of India for Innovation in Science Pursuit for Inspired Research (INSPIRE) Faculty Award (IFA17-MA110).

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Received June 2020; revised September 2020.

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