A counterexample to a conjecture on the chromatic number of \( r \)-stable Kneser hypergraphs

Hamid Reza Daneshpajouh

School of Mathematical Sciences, University of Nottingham Ningbo China, Ningbo, China

Correspondence
Hamid Reza Daneshpajouh, School of Mathematical Sciences, University of Nottingham Ningbo China, 199 Taikang East Rd, 315100 Ningbo, China. Email: Hamid-Reza.Daneshpajouh@nottingham.edu.cn

Abstract
Let \( \mathcal{F} \) be a set of subsets of \([1, \ldots, n]\) and \( r \geq 2 \) be an integer. The \( r \)-colorability defect \( cd_r(\mathcal{F}) \) of \( \mathcal{F} \) is the least number of elements of \([1, \ldots, n]\) that need to be removed such that the remaining ground set can be \( r \)-colored without inducing monochromatic sets in \( \mathcal{F} \). The set \( \mathcal{F}_{r-stab} \) is a subset of \( \mathcal{F} \) containing all elements \( A \) of \( \mathcal{F} \) such that any two members of \( A \) are at least at distance \( r \) apart on the \( n \)-cycle. The main purpose of this note is to give a counterexample to a conjecture raised by Florian Frick, which says if \( \mathcal{F} \) is a set of subsets of \([1, \ldots, n]\) and \( r \geq 3 \), then the number of blocks needed to partition the elements of \( \mathcal{F}_{r-stab} \) such that no \( r \) pairwise disjoint sets lie in the same block is at least \( \frac{cd_r(\mathcal{F})}{r-1} \).

Keywords
chromatic number, colorability defect, kneser hypergraphs

1 | INTRODUCTION

Throughout this note, the symbol \([n]\) is used for the set \([1, \ldots, n]\), the set of all \( k \)-subsets of \([n]\) is denoted by \( \binom{[n]}{k} \), and the set of all subsets of a set \( X \) is denoted by \( 2^X \). A hypergraph or \( \mathcal{H} = (V, E) \) is a pair \((V, E)\) where \( V \) is a finite set of elements called vertices, and \( E \) is a set of nonempty subsets of \( V \) called edges. A hypergraph \( \mathcal{H} \) is called \( r \)-uniform hypergraph if every edge of \( \mathcal{H} \) has size \( r \). An \( m \)-coloring of a hypergraph \( \mathcal{H} \) is a map \( c : V(\mathcal{H}) \rightarrow \{1, \ldots, m\} \) with no monochromatic edge, that is, \( |c(e)| \geq 2 \) for all \( e \in \mathcal{H} \). We say a hypergraph is \( m \)-colorable if it admits an \( m \)-coloring. The chromatic number \( \chi(\mathcal{H}) \) of a hypergraph \( \mathcal{H} \) is the minimum integer
such that $\mathcal{H}$ is $m$-colorable. For an integer $r \geq 2$, the $r$-colorability defect $\text{cd}_r(\mathcal{H})$ of a hypergraph $\mathcal{H} = (V, E)$ is the minimum number of vertices that must be removed from the vertex set of $\mathcal{H}$ such that the induced hypergraph on the remaining vertices is $r$-colorable. For a set system $\mathcal{F}$ of subsets of a set and a nonnegative integer $r$, $\text{KG}^r(\mathcal{F})$ is an $r$-uniform hypergraph whose vertices are all elements of $\mathcal{F}$ and whose edges are all sets $\{A_1, \ldots, A_r\}$ of $r$ vertices where $A_i \cap A_j = \emptyset$ for $i \neq j$. A subset $\sigma \subseteq [n]$ is called $s$-stable if $s \leq |i - j|$ for $i, j \in \sigma$ distinct; similarly $\sigma \subseteq [n]$ is called almost $s$-stable if $s \leq |i - j|$ for $i, j \in \sigma$ distinct. For a set system $\mathcal{F}$ of the subsets of $[n]$, the set of all $s$-stable (almost $s$-stable) members of $\mathcal{F}$ is denoted by $\mathcal{F}_{s\text{-stab}}$.

In 1978, Lovász [3] proved that

$$\chi\left(\text{KG}^2\left(\left[\begin{array}{c} n \\ k \end{array}\right]_{2\text{-stab}}\right)\right) = n - 2(k - 1) \quad \text{for all } n \geq 2k$$

and shortly afterward, Schrijver [4] showed that even after deleting all non-$2$-stable vertices from $\left[\begin{array}{c} n \\ k \end{array}\right]_{2\text{-stab}}$ the chromatic number is not changed, that is,

$$\chi\left(\text{KG}^2\left(\left[\begin{array}{c} n \\ k \end{array}\right]_{2\text{-stab}}\right)\right) = n - 2(k - 1) \quad \text{for all } n \geq 2k.$$

Later, Alon–Frankl–Lovász [1] generalized Lovász's result as follows:

$$\chi\left(\text{KG}^r\left(\left[\begin{array}{c} n \\ k \end{array}\right]_{r\text{-stab}}\right)\right) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil \quad \text{for all } n \geq rk \text{ and } r \geq 2.$$

Ziegler [5] conjectured that the Alon–Frankl–Lovász result holds if even all the non-$r$-stable vertices are removed, that is,

**Conjecture 1.** If $r \geq 2$ and $n \geq rk$, then

$$\chi\left(\text{KG}^r\left(\left[\begin{array}{c} n \\ k \end{array}\right]_{r\text{-stab}}\right)\right) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

And finally, the following generalization of the Ziegler conjecture is raised in [2].

**Conjecture 2.** Let $r \geq 3$ and let $\mathcal{F}$ be a set system on the ground set $[n]$. Then

$$\chi(\text{KG}^r((\mathcal{F}_{r\text{-stab}}))) \geq \left\lceil \frac{\text{cd}_r(\mathcal{F})}{r - 1} \right\rceil.$$

The main aim of this note is to disprove Conjecture 2.

---

1 A set system $\mathcal{F}$ on a ground set $X$ is a family of nonempty subsets of $X$ which can be seen as the hypergraph $(X, \mathcal{F})$, and so these two words are used interchangeably throughout this note.
2 | MAIN RESULT

Let $\mathcal{F}(n, r)$ be the set system $\binom{\{n\}}{2} \setminus \binom{\{n\}}{2}_{r-stab}$ on ground set $[n]$.

**Proposition 3.** Let $r \geq 2$, and $n = kr + 1$ for some $k \geq 1$. Then, we have

$$cd_r(\mathcal{F}(n, r)) = 1.$$  

**Proof.** First of all it is easy to see that $cd_r(\mathcal{F}(n, r)) \leq 1$. Indeed, after removing the vertex $n$, the induced family on the remaining vertices is $r$-colorable; for each $1 \leq i \leq r$ assign the color $i$ to the vertices $i, i + r, i + 2r, \ldots, i + (k - 1)r$. Now, assume for a contradiction that $cd_r(\mathcal{F}(n, r)) = 0$. This means that there is a proper coloring of $\mathcal{F}(n, r)$ with $r$ colors, namely, $1, \ldots, r$. First note that each pair of distinct vertices among $\{1, \ldots, r\}$ must receive different colors as any two of them constitute an element of $\mathcal{F}(n, r)$. Without loss of generality, assume that each vertex $i$ receives the color $i$ for $1 \leq i \leq r$. Since $\{i, i + 1\} \in \mathcal{F}(n, r)$ for every $2 \leq i \leq r$, the vertex $r + 1$ must be colored by 1. This implies the vertex $r + 2$ must be colored by $2$ as $\{r + 1, r + 2\} \in \mathcal{F}(n, r)$ and $\{i, r + 2\} \in \mathcal{F}(n, r)$ for every $3 \leq i \leq r$. Inductively, a similar argument shows that each of the vertex of the form $lr + j$ must receive the color $j$ for each $1 \leq j \leq r$ and $0 \leq l \leq k - 1$. This implies that the vertex $n = kr + 1$ must be colored with 1. But this is a contradiction as $\mathcal{F}(n, r)$ must be $r$-colorable. Therefore, $cd_r(\mathcal{F}(n, r)) = 1$ and this finishes the proof. \(\square\)

But on the other hand, by definition, we have $\mathcal{F}(n, r)_{r-stab} = \emptyset$ for every $n$ which implies that $\chi(\text{KG}^r(\mathcal{F}(n, r)_{r-stab})) = 0$. Thus, the family $\mathcal{F}(n, r)$ when $n = kr + 1^2$ for some $k \geq 1$ gives a counterexample to Conjecture 2. It should be noted that Frick mentioned [2, see the text before Thm. 4.2] the statement of Conjecture 2 cannot be valid for $r = 2$ with the same reason (in our notation, his counterexample was $\mathcal{F}(5, 2)$), and hence he formulated the conjecture for $r \geq 3$. It is worth mentioning that while this counterexample produces a hypergraph without edges, the construction can be used (see Proposition 4) as a building block to produce counterexamples with larger chromatic number. So this conjecture is not just false for ‘trivial’ reasons.

**Proposition 4.** Let $r \geq 2$, $n = r(2r - 1)$ and set

$$\mathcal{F} = \mathcal{F}(n, r) \cup \{(1 + ir, 1 + (i + 1)r) : i = 0, \ldots, 2r - 3\} \cup \{(2r - 2)r + 1, 1\}.$$  

We have

$$cd_r(\mathcal{F}) \geq r \quad \& \quad \chi(\text{KG}^r(\mathcal{F}_{r-stab})) = 1.$$  

**Proof.** First note that

---

\(^2\)Actually, a similar argument shows that the family $\mathcal{F}(n, r)$ provides a counterexample for Conjecture 2 provided $n$ is not a multiple of $r$ and $n \geq r$. 

---
\( \mathcal{F}_{r-\text{stab}} = \{[1 + ir, 1 + (i + 1)r] : i = 0, ..., 2r - 3 \} \cup \{((2r - 2)r + 1, 1)\}, \)

which implies \( \chi(KG'(\mathcal{F}_{r-\text{stab}})) = 1 \) as \( \mathcal{F}_{r-\text{stab}} \neq \emptyset \) and there are no \( r \)-pairwise disjoint sets in \( \mathcal{F}_{r-\text{stab}} \). Indeed, the latter property is simply because the size of the set \( \bigcup \mathcal{F}_{r-\text{stab}} = \{1, 1 + r, ..., 1 + r(2r - 2)\} \) is \( 2r - 1 \), and hence any collection of 2-subsets of \( \mathcal{F}_{r-\text{stab}} \) of size \( r \) contains at least two sets with a nonempty intersection.

For the other part, suppose the contrary, that is, \( cd_r(\mathcal{F}) \leq r - 1 \). So, we can remove a set \( B \) of \( r - 1 \) elements of the ground set \([n]\) such that the induced family on the remaining elements is \( r \)-colorable. First note that, \( B \) must contain at least one element from each of the following sets:

\[ A_i = \{1 + ir, ..., 1 + (i + 1)r\} \quad \text{for} \quad i = 0, ..., 2r - 3. \]

This is because, if we do not delete an element from an \( A_i \) for some \( i \), then each element of this set must receive a different color as any 2-subset of \( A_i \) is in \( \mathcal{F} \). But, then this is impossible as the size of \( A_i \) is \( r + 1 \) and we have just \( r \) colors. Next, note that for \( i < j \)

\[ A_i \cap A_j \neq \emptyset \quad \text{if and only if} \quad j = i + 1, \]

and moreover \( A_i \cap A_{i+1} = \{1 + (i + 1)r\} \) for \( i = 0, ..., 2r - 3 \). These properties beside the fact that we have \( 2r - 2 \) sets \( A_i \) and the size of \( |B| = r - 1 \) uniquely determine \( B \), that is,

\[ B = \{1 + r, 1 + 3r, ..., 1 + (2r - 3)r\}. \]

So, in particular, \( B \) does not have any elements of the set

\[ C = \{1 + (2r - 2)r, 2 + (2r - 2)r, ..., (2r - 1)r, 1\}. \]

But, then again we need \( r + 1 \) colors to color this part as any 2-subset of \( C \) appears in \( \mathcal{F} \) and \( |C| = r + 1 \). This contradiction finishes the proof. \( \square \)

We believe the following weaker version of Conjecture 2 might be true.

**Conjecture 5.** Let \( r \geq 2 \) and let \( \mathcal{F} \) be a set system. Then

\[ \chi(KG'(\mathcal{F}_{r-\text{stab}})) \geq \left\lceil \frac{cd_r(\mathcal{F})}{r - 1} \right\rceil. \]

Conjecture 5 is known for \( r = 2 \) [2, Thm. 4.3], the conjectured lower bound applies to 2-stable sets for \( r > 2 \), and in general, it applies to \( s \)-stable sets if \( r > 6s - 6 \) is a prime power [2, Thm. 4.9].

**Remark 6.** If Conjecture 5 is true, then

\[ \left\lceil \frac{cd_r(\mathcal{F})}{r - 1} \right\rceil - \chi(KG'(\mathcal{F}_{r-\text{stab}})) \leq 1. \]
To confirm this claim it is enough to show that

\[ \chi(KG'(\mathcal{F}_{r-stab})) - \chi(KG'(\mathcal{F}_{r-stab})) \leq 1, \]

where \( \mathcal{F} \) is an arbitrary family on the ground set \([n]\). Let \( c \) be a proper coloring of \( KG'(\mathcal{F}_{r-stab}) \). We extend this coloring to a proper coloring \( c' \) for \( KG'(\mathcal{F}_{r-stab}) \) with one more new color, namely, \(*\). Define

\[ c'(F) = \begin{cases} c(F) & \text{if } F \in \mathcal{F}_{r-stab}, \\ * & \text{if } F \in \mathcal{F}_{r-stab} \setminus \mathcal{F}_{r-stab}. \end{cases} \]

It is easy to check that this map gives a proper coloring of \( KG'(\mathcal{F}_{r-stab}) \). Indeed, note that if

\[ F \in \mathcal{F}_{r-stab} \setminus \mathcal{F}_{r-stab} \]

then \( F \cap \{1, \ldots, r-1\} \neq \emptyset \). Thus, there are two sets with a nonempty intersection among any collection of \( r \) such sets, which verifies the claim.

ACKNOWLEDGMENTS
The author would like to thank the anonymous reviewers for constructive criticism of the manuscript.

DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

REFERENCES
1. N. Alon, P. Frankl, and L. Lovász, The chromatic number of Kneser hypergraphs, Trans. Amer. Math. Soc. 298 (1986), no. 1, 359–370.
2. F. Frick, Chromatic numbers of stable Kneser hypergraphs via topological Tverberg-type theorems, Int. Math. Res. Not. 2020 (2020), no. 13, 4037–4061.
3. L. Lovász, Kneser’s conjecture, chromatic number and homotopy, J. Combin. Theory Ser. A. 25 (1978), 319–324.
4. A. Schrijver, Vertex-critical subgraphs of Kneser graphs, Nieuw Archief voor Wiskunde. Derde Serie. 26 (1978), 454–461.
5. G. M. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, Invent. Math. 147 (2002), no. 3, 671–691.

How to cite this article: H. R. Daneshpajouh, A counterexample to a conjecture on the chromatic number of \( r \)-stable Kneser hypergraphs, J. Graph Theory. 2023;103:762–766. https://doi.org/10.1002/jgt.22945