Stability Analysis of Rotating-Disk Flows with Uniform Suction*

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Instability of the flow on a rotating disk is governed by linearized disturbance equations of the partial differential with respect to the radial distance from the rotation axis and the normal distance from the disk surface. Applying uniform suction from the surface brings a small parameter associated with displacement thickness of the circumferential velocity profile into a dimensionless form of the equation system. Two kinds of series solutions expanded by the powers of this parameter are obtained to describe the cross-flow and centrifugal instabilities of the flow having a twisted velocity profile. The leading terms of the series solutions are determined from two eigenvalue problems of slightly different ordinary differential equations, and the superposition of those equations leads to an eigenvalue problem applicable to multiple-instability characteristics of such three-dimensional boundary layers.

Key Words: Instability, Boundary Layer, Cross Flow, Streamline Curvature

1. Introduction

The instability of three-dimensional boundary layers and their transition to turbulence is a fundamental and most important subject of fluid mechanics, because understanding, prediction and control of the associated phenomena are increasingly required in the fields of aeronautical and mechanical engineering. The flow on a rotating disk is well known as a simple and typical example of these flows, and has therefore been designated as a basic flow for theoretical studies on instability and experimental ones on transition of three-dimensional boundary layers. The flow on an infinitely large rotating disk is described exactly by Kármán’s solution of Navier-Stokes equations written in cylindrical polar coordinates. This exact solution has been extended to some varieties including Hannah’s flows on a rotating disk placed in an axial flow, and Stuart’s flows on a rotating disk subject to uniform suction from the surface. Since they possess three-dimensional characteristics similar to Kármán’s flow with different emphasis, stability analysis of the extended flows may be a promising way to broaden our basic knowledge in this field, but little seems to have been done up to now.

Studies on the instability and transition of three-dimensional boundary layers substantially began with the experiment and theory by Gregory et al., where Gregory and Walker succeeded in clearly visualizing so-called cross-flow vortices on a rotating disk, and Stuart explained the related cross-flow instability with Rayleigh’s inviscid theory concerning inflectional velocity distribution of the flow. The subsequent research about cross-flow instability and its development to turbulence has been carried out and properly and conveniently reviewed by various authors, including a large literature cited. However, it is emphasized here that two kinds of instabilities can occur in three-dimensional boundary layers, and more remarkably in the flow on a rotating disk. Superposition of the instabilities due to inviscid profiles of the cross-flow perpendicular to external streamlines and due to the curvature of the streamlines makes stability characteristics of the flow so complicated that even the method of linear stability calculations has not been completely established yet.

The familiar Orr-Sommerfeld equation may be applied to the cross-flow instability (referred to as ‘‘C-F mode’’ herein-after) of the original inviscid inflection point, but cannot describe the streamline-curvature instability (S-C mode) because no curvature terms are included. Thus, various forms of stability equation system have been proposed to determine the complex dispersion relation for Kármán’s rotating-disk flow (i.e., Malik and others), though none of them seems to be commonly acceptable in terms of reliability and convenience. It may be natural to expect that different instabilities should be described by different stability equations, as those for Tollmien-Schlichting instability and Görtler instability of two-dimensional boundary layers along a weakly concave surface. Small disturbances superimposed on a rotating-disk flow are governed by partial differential equations, which may be rewritten into two kinds of dimensionless forms, one being for centrifugal instability and slightly different from the other for cross-flow instability. If the basic flow is subjected to uniform suction from the surface as investigated by Stuart, then each of the dimensionless equation systems involves a small parameter associated with the displacement thickness of the circumferential velocities. It follows justly that solutions of the partial differential equations can be expanded into powers of this parameter, resulting in a sequence of ordinary differential equations to form an eigenvalue problem and higher-order corrections.

The objective of this paper is to clarify the fundamental roles of centrifugal force in the instabilities of rotating-
disk flows with uniform suction, and then propose a simple method of stability analysis for application to multiple instabilities of three-dimensional boundary layers.

2. Basic Flows

We use the cylindrical polar coordinates (r, φ, z) held fixed on a disk rotating around the z-axis with angular velocity ω in the counterclockwise direction in rest fluid with density ρ. Uniform suction at the surface of the disk is denoted by the velocity $w_s = s \sqrt{\nu a}$, where $\nu$ is kinematic viscosity and the dimensionless quantity $s$ is called the suction parameter. The case of $s=0$ corresponds to Kármán’s flow without suction, while the limit $s \to \infty$ gives an asymptotic suction profile. It may be noted that increasing $s$ weakens cross-flow velocities in the radial direction and decreases the displacement thickness $\delta^* = s \sqrt{v/a}$ defined from the boundary-layer profile of circumferential velocities.

Stuart\(^3\) obtained a unique relation between suction parameter $s$ and displacement coefficient $e$, and suggested suitability of the similarity variable $\zeta = z/\delta^*$ for comparing velocity profiles. Following his lead, we adopt this variable and write the velocities $(v_r, v_\phi, v_z)$ and pressure $p_\phi$ of the undisturbed laminar flow in a dimensionless form

$$
\frac{v_r}{ra} = U(\zeta; \epsilon) = e^2 \hat{U}(\zeta; \epsilon), \quad \frac{v_\phi}{ra} = V(\zeta; \epsilon), \quad \frac{v_z}{Ra} = \frac{W(\zeta; \epsilon)}{R}, \quad \frac{p_\phi}{\rho(\epsilon)^2} = \frac{\epsilon^4 P(\zeta; \epsilon)}{R^2},
$$

where $\epsilon$ is used instead of $s$ as a parameter denoting the strength of the suction, two expressions, $U$ and $\hat{U}$, of the radial velocity are introduced for asymptotic analyses given later, and the local Reynolds number is defined as $R = ra \delta^*/\nu$. Substitution of this form into Navier-Stokes equations leads to the similarity equations and boundary conditions

$$U'' - WU' + (V + 1)^2 - \epsilon^4 \hat{U}^2 = 0,$n
$$V'' - WW' - 2\epsilon^2 \hat{U}(V + 1) = 0,$n
$$W' = -2\epsilon^2 \hat{U}, \quad P' = -2(\hat{U}' - W\hat{U}),$$

where the prime denotes differentiation with respect to $\zeta$.

At the limit of $\epsilon = 0$ while keeping $\epsilon \zeta = 1$, these equations can be easily solved to yield the asymptotic suction profiles

$$U(\zeta; 0) = e^{-\zeta} - e^{-2\zeta}/2, \quad V(\zeta; 0) = e^{-\zeta} - 1, \quad W(\zeta; 0) = -1,$$

showing that the peak of cross-flow velocities is given by $\hat{U}_p = 1/8$ at $\zeta_\phi = \ln 2$. Table 1 shows Stuart’s relation between $s$ and $\epsilon$, and the dependence of the asymptotic suction profile and other members of Stuart’s family on $s$ or $\epsilon$. Almost all of the peak values $\hat{U}_p$ and peak positions $\zeta_\phi$ agree very well with each other. Even the difference between the asymptotic suction profile and Kármán’s flow is within 11% and 5%, respectively.

Figure 1(a) shows three distributions of the radial velocity for $s=0$, 1 and $\infty$, where an almost complete similarity in shape is found except for slightly smaller values of Kármán’s flow than the others. A more complete similarity for the circumferential velocities of the asymptotic suction profile and of Kármán’s flow can be seen in Fig. 1(b), suggesting similar distributions of other members in Stuart’s family. These results indicate that, if necessary, the flow on a rotating disk with suction may be approximated with the asymptotic suction profile for almost the entire range of $0 \leq \epsilon \leq 1.2712$, and then parameter $\epsilon$ plays only the role as an index identifying each member of Stuart’s family.

3. Linearized Disturbance Equations

We consider small and wavy disturbances superimposed on the basic flows given in the previous section. The disturbances may be assumed to be periodic in the azimuthal coordinate $\phi$ with wave number $m$, because the basic flows are of rotational symmetry. They are also assumed to be an exponential function of time $t$ with a complex frequency $f$ whose real part correctly gives the angular frequency of wavy disturbances, while the imaginary part denotes temporal growth rate as an index of stability or instability of the steady basic flow; that is, only the temporally dependent problem of stability is discussed in this paper for simplicity. Moreover, the radial dependence of disturbances may be divided into a wavy variation with a constant or slowly varying wave number $h$ and an amplitude of the wavy disturbance concerned. Thus, disturbance velocities and pressure are assumed to be proportional to the exponential function defined by

$$E \equiv \exp\left(i \int hdr + im\phi - 2\pi if t\right),$$
and amplitudes of the wavy motion are regarded as functions of two spatial coordinates in the radial and axial directions, so that the governing equations should be partial differential.

In this section, the displacement thickness \( \delta^* = \varepsilon \sqrt{\nu / \dot{a}} \) is chosen as the reference length, but two reference velocities are used: one is \( \alpha a \) appropriate to the basic flows, and the other is a constant reference velocity, say \( \nu_0 \) for disturbance velocities so as to reflect radial variations properly on the amplitude functions. Therefore, in addition to Eq. (1), all other quantities are made dimensionless as

\[
\frac{r}{\delta^*} = \frac{R}{\varepsilon^2}, \quad \frac{z}{\delta^*} = \zeta, \quad \frac{\tilde{v}_r}{\nu_0 \varepsilon^2} = \frac{u(\zeta, R)}{\nu_0 \varepsilon^2} = v(\zeta, R),
\]

\[
\frac{\tilde{v}_\phi}{\nu_0 \varepsilon^2} = w(\zeta, R), \quad \frac{\tilde{p}}{\rho \nu_0 \varepsilon^2} = p(\zeta, R),
\]

\[
h\delta^* = \alpha, \quad m = \frac{\beta R}{\varepsilon^2} \quad \frac{2\pi f}{\alpha} = \frac{\omega R}{\varepsilon^2},
\]

where the tilde denotes disturbance velocities and pressure, while \( R, (\alpha, \beta) \) and \( \omega \) are local Reynolds number, wave numbers in the \((r, \phi)\) directions and frequency, respectively.

Substituting the sum of the basic flow and a small disturbance into the Navier-Stokes equations and extracting the terms linearly proportional to disturbance velocities and pressure, we have linear disturbance equations in a primitive form. After applying Eq. (1) to the basic flows and Eq. (4) to the wavy disturbances and introducing the symbols \( D = \nu_0 \varepsilon^2 / \partial \zeta / \partial \zeta \) for convenience, the disturbance equations in the dimensionless form become

\[
\left[ \frac{1}{R} \left( i\alpha + \frac{\varepsilon^2}{R} + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) \left( i\alpha + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) - \beta^2 + D^2 \right]
\]

\[
- \frac{\varepsilon^4}{R^3} + i(\omega - \alpha U - \beta V) - \frac{W}{R} \frac{D}{R} \frac{U}{R} \left( 1 + R \frac{\partial}{\partial R} \right) u
\]

\[
+ \frac{2\varepsilon^2}{R^3} \left( V + 1 - i\beta R \right) \cdot \left( U + 1 - i\beta R \right) \cdot \left( \left( i\alpha + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) \cdot \left( i\alpha + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) \right)
\]

\[
= 0,
\]

\[
\left[ \frac{1}{R} \left( i\alpha + \frac{\varepsilon^2}{R} + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) \left( i\alpha + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) - \beta^2 + D^2 \right]
\]

\[
- \frac{\varepsilon^4}{R^3} + i(\omega - \alpha U - \beta V) - \frac{W}{R} \frac{D}{R} \frac{U}{R} \left( 1 + R \frac{\partial}{\partial R} \right) v
\]

\[
- \frac{2\varepsilon^2}{R^3} \left( V + 1 - i\beta R \right) \cdot \left( U + 1 - i\beta R \right) \cdot \left( \left( i\alpha + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) \cdot \left( i\alpha + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) \right)
\]

\[
= 0,
\]

\[
\left( i\alpha + \frac{\varepsilon^2}{R} + \frac{\varepsilon^2}{R} \cdot R \frac{\partial}{\partial R} \right) u + i\beta U + Dw = 0.
\]

The above disturbance equations include an additional parameter \( \varepsilon \) that approaches zero in inverse proportion to the suction parameter \( s \) as \( s \rightarrow \infty \). This suggests that the solution may be expanded into a power series of this parameter, resulting in an infinite sequence of ordinary differential equations whose leading terms pose an eigenvalue problem, while higher-order terms give corrections of the eigenvalues.

4. Centrifugal Instability of Asymptotic Suction Profile

As well known, the flow on a rotating disk is susceptible to both cross-flow instability and streamline-curvature instability, the latter of which belongs to the centrifugal type in the same sense as Göltler’s instability of two-dimensional boundary layers along a concave wall. The main effect of centrifugal force on stability formulation in the present problem comes from the term \( 2\varepsilon^2 R^{-4}(V + 1)w \) in the first of Eq. (5), which becomes a smaller order as \( \varepsilon \) tends to zero, and can therefore play a fundamental role in the occurrence of centrifugal instability only if the other terms, such as \( \beta, \omega, U \) and viscous terms, are all of \( O(\varepsilon^2) \). Following Göltler’s transformations in the problem of concave walls \( U = \varepsilon^2 U \), we express these considerations in the form

\[
U = \varepsilon^2 U, \quad u = \varepsilon^2 u, \quad w = \varepsilon^2 w, \quad p = \varepsilon^4 p,
\]

\[
\omega = \varepsilon^2 \omega, \quad \beta = \varepsilon^2 \beta, \quad R = \varepsilon^2 R,
\]

where hatted quantities are assumed to be \( O(1) \) as \( \varepsilon \rightarrow 0 \). Substitution of these transformations into Eq. (5) gives the disturbance equations appropriate to centrifugal instability.

On the assumption that the independent parameters are \( \alpha, \beta, R \), it is found that solutions of the resultant equations can be expanded into powers of the small parameter \( \varepsilon^2 \) in the form

\[
\hat{v}(\zeta, \alpha, \beta, \hat{R}) = \sum_{n=0}^{\infty} v_n(\zeta, \alpha, \beta, \hat{R})\varepsilon^{2n},
\]

\[
\hat{\omega}(\alpha, \beta, \hat{R}) = \sum_{n=0}^{\infty} \omega_n(\alpha, \beta, \hat{R})\varepsilon^{4n},
\]

where vectors \( \hat{v} \equiv (u, v, w, \hat{p}) \) and \( \nu_n \equiv (\nu_n, v_n, w_n, p_n) \) are introduced for convenience. Substituting these into the governing equations and separating out each power of \( \varepsilon^2 \), we have an infinite sequence of ordinary differential equations

\[
\left[ D^2 - \alpha^2 + i\hat{R}(\omega_0 - \alpha \hat{U} - \beta V) - W \hat{D} \right] u_n + 2(V + 1)w_n
\]

\[
- \hat{R} \hat{U}' w_n - i\alpha \hat{R} p_n = \hat{R} f^{(1)}_n,
\]

\[
\left[ D^2 - \alpha^2 + i\hat{R}(\omega_0 - \alpha \hat{U} - \beta V) - W \hat{D} \right] v_n
\]

\[
= \hat{R} f^{(1)}_n,
\]

\[
\left[ D^2 - \alpha^2 + i\hat{R}(\omega_0 - \alpha \hat{U} - \beta V) - W \hat{D} \right] w_n
\]

\[
- \hat{R} \hat{D} p_n = \hat{R} f^{(3)}_n,
\]

\[
i\alpha u_n + i\beta v_n + D w_n = f^{(4)}_n,
\]

where \( n = 0, 1, 2, \ldots \), and the forcing terms for \( n = 0 \) are all zero, indicating that the leading terms of the series solutions are governed by homogeneous equations, while the
forcing terms of inhomogeneous equations for \( n \geq 1 \) are given by

\[
f_n^{(1)} = \frac{1}{R} \left\{ -i \frac{\alpha}{R} \left( 1 + 2R \frac{\partial}{\partial R} \right) + \beta^2 + \hat{U} \left( 1 + \hat{R} \frac{\partial}{\partial R} \right) \right\} u_{n-1}
\]

\[
- i \sum_{j=1}^{n} \omega_j u_{n-j} + \frac{2i\beta}{R} v_{n-1} + \frac{1}{R} \left( 1 + \hat{R} \frac{\partial}{\partial R} \right) p_{n-1}
\]

\[
+ \frac{1}{R} \left( 1 - \hat{R} \frac{\partial}{\partial R} - \hat{R}^2 \frac{\partial^2}{\partial R^2} \right) u_{n-2},
\]

\[
f_n^{(2)} = \frac{1}{R} \left\{ -i \frac{\alpha}{R} \left( 1 + 2R \frac{\partial}{\partial R} \right) + \beta^2 + \hat{U} \left( 1 + \hat{R} \frac{\partial}{\partial R} \right) \right\} v_{n-1}
\]

\[
- i \sum_{j=1}^{n} \omega_j v_{n-j} + \frac{2i\beta}{R} (V+1)u_{n-1} + \hat{\beta} p_{n-1}
\]

\[
+ \frac{1}{R} \left( 1 - \hat{R} \frac{\partial}{\partial R} - \hat{R}^2 \frac{\partial^2}{\partial R^2} \right) v_{n-2} - \frac{2i\hat{\beta}}{R} u_{n-2},
\]

\[
f_n^{(3)} = \frac{1}{R} \left\{ -i \frac{\alpha}{R} \left( 1 + 2R \frac{\partial}{\partial R} \right) + \beta^2 + \hat{U} \frac{\partial}{\partial R} \right\} w_{n-1}
\]

\[
- \sum_{j=1}^{n} \omega_j w_{n-j} - \frac{1}{R} \left( \hat{R} \frac{\partial}{\partial R} + \hat{R}^2 \frac{\partial^2}{\partial R^2} \right) w_{n-2},
\]

\[
f_n^{(4)} = -\frac{1}{R} \left( 1 + \hat{R} \frac{\partial}{\partial R} \right) u_{n-1},
\]

under the condition that terms with negative subscripts are conventionally ignored. Since the original equation system is linear and homogeneous, certain normalization is necessary for the solutions to be uniquely determined. In this paper, the series solutions, Eq. (7), are assumed to be subject to normalization of the form

\[
dv_{n}/d\xi = 1, \quad dv_{n}/d\xi = 0 \quad \text{for} \quad n = 1, 2, \ldots,
\]

at \( \zeta = 0 \) \hspace{1cm} (10)

which may be regarded as an extension of the so-called solvability condition for inhomogeneous equations with the same or almost same principal operators as the corresponding homogeneous equations.

The leading terms of the series solutions are governed by homogeneous equations, which may be written, after eliminating radial velocity and pressure, in the form

\[
(D^2 - \alpha^2 + i\omega \hat{R} - i\beta \hat{R} \hat{U} - i\beta \hat{R} V - WD) v - \hat{R} V \hat{\omega} = 0,
\]

\[
\left[ (D^2 - \alpha^2 + i\omega \hat{R} - i\beta \hat{R} \hat{U} - i\beta \hat{R} V - WD - W') (D^2 - \alpha^2) \right. 
\]

\[
\left. + i \hat{R} \left( \alpha \hat{U}'' + \beta V'' \right) \right] \hat{v} - 2i\alpha [(V+1)D + V^{'2}] v = 0,
\]

(11)

where the subscript 0 is neglected for simplicity. Since the transformation, Eq. (6), places special emphasis on the curvature terms, we call Eq. (11) “stability equations of curved-flow approximation.” Boundary conditions to be imposed are

\[
v = \hat{v} = \hat{w} = 0 \quad \text{at} \quad \zeta = 0,
\]

\[
v' - \rho_1 v = \hat{w}'' - (\rho_2 + \rho_3) \hat{v}' - \rho_2 \rho_3 \hat{w}'
\]

\[
= \hat{w}'' - (\rho_2 + \rho_3) \hat{v}' - \rho_2 \rho_3 \hat{w}'
\]

\[
= 0 \quad \text{at} \quad \zeta = \zeta_0,
\]

(12)

where

\[
\rho_1 = \rho_3 = W_c/2 - \sqrt{W_c^2/4 + \alpha^2 - i\omega \hat{R} - i\beta \hat{R}}, \quad \rho_2 = -\alpha,
\]

and the boundary-layer edge \( \zeta_0 \) is located far away from the wall, and the basic flow in its neighborhood is assumed as \( \hat{U} = 0, V = 1, W \approx W_c \equiv W(\zeta_0), W' = 0 \), so that external stability equations can be solved analytically, giving rise to the above relations to be satisfied at the edge.\(^{14}\)

The homogeneous equations, together with the homogeneous boundary conditions, pose an eigenvalue problem to determine the complex dispersion relation

\[
\hat{\omega} = \hat{\omega}(\alpha, \beta, \hat{R}),
\]

(13)

which furnishes significant information on the propagation and development of wavy disturbances, provided that higher-order terms in the series solutions, Eq. (7), are negligible. Imposition of the neutrally stable condition \( \text{Im}[\hat{\omega}(\alpha, \beta, \hat{R})] = 0 \) on the dispersion relation defines a neutral surface in the three-dimensional space with the rectangular coordinates \( (\alpha, \beta, \hat{R}) \). We can draw neutral curves enclosing the unstable regions on the \( (\alpha, \beta) \) plane for some fixed values of \( \hat{R} \), as shown by neglecting hats in Fig. 2, which presents neutral curves for streamwise-curvature instability of the asymptotic suction profile in the region of negative \( \beta \) consistently with current results, such as Fig. 4 of Faller\(^8\) and Fig. 2 of Itoh.\(^9\) It is also shown that a neutral curve for a large Reynolds number extends to the region of positive \( \beta \), where the curve seems to reflect the cross-flow mode in rather distorted appearances, because the present stability equations do not completely include Rayleigh’s inviscid equation.

Since the forcing terms, Eq. (9), include the differential operator with respect to Reynolds number \( \hat{R} \), the inhomoge-

![Fig. 2. Neutral curves obtained from the curved-flow approximation, Eq. (11), for the asymptotic suction profile.](image-url)
neous equations, Eq. (8), for \( n \geq 1 \) have to be solved together with their derivatives; that is, we are concerned with the simultaneous equations for \( u_n^{(k)}, v_n^{(k)}, w_n^{(k)} \) and \( p_n^{(k)} \) with \( k = 0, 1, \cdots, n \) at each step of the successive calculations, where the superscript denotes \( k \)-th derivatives with respect to \( \tilde{R} \) of the dependent variables. These inhomogeneous equations can be uniquely solved subject to the well-known solvability condition together with the normalization conditions, Eq. (10), which determines the unknown parameter \( \omega_n \) in the forcing terms, Eq. (9), at the \( n \)-th step of the series solutions. An example of computational results is shown in Fig. 3, where variations of the complex eigenvalue for \( \alpha = 0.2747 \) and \( \tilde{\beta} = -0.0305 \) corresponding to the maximum growth rate at \( \tilde{R} = 400 \) are plotted against \( \varepsilon \) for several values of the truncation number \( N \), as

\[
\tilde{\omega} = \omega_n + i\omega_{n} = \sum_{n=0}^{N} \omega_n \eta_n \quad \text{for} \quad N = 0, 1, 2, 3, 4. \tag{14}
\]

The parameter \( \varepsilon \) participating in this expansion has another important role of identifying each member in Stuart’s similarity family on a rotating disk, which ranges from the asymptotic flow at \( \varepsilon = 0 \) (\( s = \infty \)) to Kármán’s flow without suction at \( \varepsilon = 1.271 \) (\( s = 0 \)). The figure shows negligible effects of higher-order terms in Eq. (14) on the asymptotic flow in the range of \( 0 \leq \varepsilon \leq 1.271 \) may be estimated by eigen solutions of the homogeneous equations, Eq. (11).

5. Cross-Flow Instability of Stuart’s Flows

It is well known that so-called “cross-flow instability” is due to an inflection point on the velocity distribution of basic flow in the direction of the wave-number vector of small disturbances. This instability therefore requires that the velocities \( (U, V) \) and wave numbers \( (\alpha, \beta) \) all remain in the order of unity if certain small-parameter expansion is applied to the governing equations, Eq. (5), for the derivation of stability equations appropriate for the cross-flow mode. Thus, we consider a member at \( \varepsilon = \varepsilon_1 \) of Stuart’s family as the basic flow for stability analysis, and assume that the basic-flow terms in Eq. (5) are given by \( U(\zeta; \varepsilon_1), V(\zeta; \varepsilon_1) \) and \( W(\zeta; \varepsilon_1) \) of \( O(1) \). On the other hand, \( \varepsilon \) appearing explicitly in Eq. (5) is assumed to be free from \( \varepsilon_1 \) and acts as the parameter of expansion for series solutions. If the necessary number of coefficients in the series is determined, we substitute \( \varepsilon = \varepsilon_1 \) into the series solution to estimate the stability characteristics of the given basic flow.

Along this line, the solutions of Eq. (5) are written in the forms

\[
\begin{align*}
    v &= \sum_{n=0}^{\infty} v_n(\zeta; \alpha, \beta, R) \left( \frac{\varepsilon}{R} \right)^n, \\
    \omega &= \sum_{n=0}^{\infty} \omega_n(\alpha, \beta, R) \left( \frac{\varepsilon}{R} \right)^n, 
\end{align*}
\tag{15}
\]

where \( \varepsilon/R \) is chosen as the expansion parameter so as to avoid excessive dependence of the coefficients on the Reynolds number. Substituting these into Eq. (5) and equating the coefficients of each power of the expansion parameter to zero, we have ordinary differential equations

\[
\begin{align*}
    &\left[ D^2 - \alpha^2 - \beta^2 + iR(\omega_0 - \alpha U - \beta V) - WD \right] u_n \\
    &- RU' w_n - i\alpha R p_n = R f_n^{(1)}, \\
    &\left[ D^2 - \alpha^2 - \beta^2 + iR(\omega_0 - \alpha U - \beta V) - WD \right] v_n \\
    &- RV' w_n - i\beta R p_n = R f_n^{(2)}, \\
    &\left[ D^2 - \alpha^2 - \beta^2 + iR(\omega_0 - \alpha U - \beta V) - WD - W' \right] w_n \\
    &- RDP_n = R f_n^{(3)},
\end{align*}
\tag{16}
\]

for \( n = 0, 1, 2, \cdots \). The forcing terms on the right-hand side are zero for \( n = 0 \), but are given for \( n \geq 1 \) by

\[
\begin{align*}
    f_n^{(1)} &= \left\{ -\frac{i\alpha}{R} \left( 3 - 2n + 2R \frac{\partial}{\partial R} \right) + U \left( 2 - n + R \frac{\partial}{\partial R} \right) \right\} u_{n-1} \\
    &- \sum_{j=1}^{n} \omega_j u_{n-j} - 2 \left( V + 1 - i\beta \right) v_{n-1} \\
    &+ \left( 2 - n + R \frac{\partial}{\partial R} \right) p_{n-1} \\
    &+ \left( -3 + 4n - n^2 - R \frac{\partial}{\partial R} - R^2 \frac{\partial^2}{\partial R^2} \right) u_{n-2}, \\
    f_n^{(2)} &= \left\{ -\frac{i\alpha}{R} \left( 3 - 2n + 2R \frac{\partial}{\partial R} \right) + U \left( 2 - n + R \frac{\partial}{\partial R} \right) \right\} v_{n-1} \\
    &- \sum_{j=1}^{n} \omega_j v_{n-j} + 2 \left( V + 1 - i\beta \right) u_{n-1} \\
    &+ \left( -3 + 4n - n^2 - R \frac{\partial}{\partial R} - R^2 \frac{\partial^2}{\partial R^2} \right) v_{n-2}, \\
    f_n^{(3)} &= \left\{ -\frac{i\alpha}{R} \left( 3 - 2n + 2R \frac{\partial}{\partial R} \right) + U \left( 1 - n + R \frac{\partial}{\partial R} \right) \right\} w_{n-1}
\end{align*}
\]
while neglecting negative-subscript terms. Since the principal operators on the left-hand side include all terms composing Rayleigh’s inviscid stability equation, this formulation clearly describes the cross-flow instability, though it fails to express the streamline-curvature instability because there are no terms related to centrifugal force on the left-hand side.

The lowest-order equations for \( n = 0 \) include no curvature terms and may therefore be called “stability equations of straight-flow approximation,” which leads to an eigenvalue problem to define the complex dispersion relation

\[
\omega = \omega (\alpha, \beta, R),
\]

depending on the parameter \( \epsilon_1 \) through the basic flow included in the equations. The neutral curves drawn on the \((\alpha, \beta)\) plane for several values of \( R \) are shown in Fig. 4, where a rather large value of \( \epsilon_1 \) is chosen because the cross-flow component \( U(\zeta; \epsilon_1) \) of the basic flow is proportional to \( \epsilon_1^2 \), so that the critical Reynolds numbers become extremely high for smaller values of \( \epsilon_1 \). It is clear from a comparison with Fig. 2 that the dispersion relation, Eq. (18), is concerned only with the C-F mode. In other words, pure features of cross-flow instability have been revealed by eliminating the centrifugal terms from the lowest-order equations. It is also worth noting that neglecting the vertical velocity component \( W(\zeta; \epsilon_1) \) in Eq. (16) gives rise to the familiar stability equations of the parallel-flow theory, whose neutral curves are shown by fine dotted lines for the same Reynolds numbers in the figure (see also Itoh\(^1\)), indicating quite weak effects of the vertical velocity on cross-flow instability.

In order to determine higher-order terms in the series solutions, Eq. (15), we have to solve inhomogeneous equations for \( \varphi_n^k \) with \( k = 0, 1, \ldots, n \) at each step of \( n \) under suitable boundary, solvability and normalization conditions in the same way as the centrifugal case. Numerical computations have been carried out using a complicated but routine procedure similar to that given in the previous section, and an example of the results is presented in Fig. 5, where the effects of higher-order terms in the same truncated series as Eq. (14) on the eigenvalues at the maximum growth rate with \( \alpha = 0.4211, \beta = 0.0356 \) and \( \omega = (0.00135, 0.00202) \) for \( R = 1,000 \) are plotted against \( \epsilon \). The location of \( \epsilon_1 = 0.810 \) \((s = 1.0)\) is indicated by an arrow, where serial values of the different truncations seem to converge nearly to a point, which is slightly away from the lowest-order approximation because the very small frequency and growth rate of this example have a remarkably enlarged effect of the second terms on the whole series. It may be said that the lowest-order eigenvalue problem provides the first approximation to the series solution with reasonable accuracy in the sense that the corrections due to higher-order terms remain within the same range of small magnitude as the leading terms.

### 6. Eigenvalue Problems for Multiple Instabilities

As a typical example of three-dimensional boundary layers, Kármán’s rotating-disk flow without suction has been an important target of stability analysis for a long time. However no rational method for calculating stability has been established yet, because the flow is susceptible to two kinds of instabilities resulting in very complicated stability characteristics in the same way as Stuart’s flows. The asymptotic analyses given above indicate that the original disturbance equations, Eq. (5), admit different series solutions corresponding to the two instabilities and that the two eigenvalue problems associated with the leading terms are both found to give quite good approximations to those series solutions. It is therefore expected that superposition of the two eigenvalue problems may yield simple and reliable stability equations for rotating-disk flows with and without suction, and furthermore establish a practical method of calculating stability applicable to such complicated stability characteristics.

It is usual in the stability analysis of rotating-disk flows to use the characteristic length \( \delta = \sqrt{\nu/\omega_B} \) in the definition of...
the similarity variable $\tilde{\zeta} = z/\delta$. We adopt this length scale for common expression in this section, and substitute

$$
\alpha = \varepsilon^2 \tilde{\alpha}, \quad \beta = \varepsilon^2 \tilde{\beta} = \varepsilon \tilde{\beta}, \quad \omega = \varepsilon^2 \tilde{\omega} = \varepsilon \tilde{\omega},
$$

$$
R = \varepsilon^{-2} \tilde{R} = \varepsilon \tilde{R}, \quad \zeta = \tilde{\zeta}/\varepsilon, \quad U = \varepsilon^2 \tilde{U}, \quad V = \tilde{V},
$$

$$
W = \varepsilon \tilde{W}, \quad \tilde{u} = \varepsilon^{-2} u, \quad \tilde{w} = \varepsilon^{-2} w, \quad \tilde{p} = \varepsilon^{-4} p
$$

(19)

into the straight-flow and curved-flow approximations, and then superpose them for application to multiple instabilities of rotating-disk flows. Neglecting the upper bars denoting the dimensionless quantities scaled with the characteristic length $\delta$ for simplicity, the composite equations are

$$
\left[ D^2 - \alpha^2 - \beta^2 + i\alpha R - i\alpha RU - i\beta RV - WD \right] u + 2(V + 1)v - RU' w - i\alpha R p = 0,
$$

$$
\left[ D^2 - \alpha^2 - \beta^2 + i\alpha R - i\alpha RU - i\beta RV - WD \right] v - RV' w - i\beta R p = 0,
$$

$$
\left[ D^2 - \alpha^2 - \beta^2 + i\alpha R - i\alpha RU - i\beta RV - WD - W \right] w - R D p = 0,
$$

$$
\ iau + i\beta v + Dw = 0,
$$

(20)

together with the boundary conditions

$$
v = v = v' = 0 \text{ at } \zeta = 0,
$$

$$
v' - \rho_1 v - C_T \left( \rho_3 w' - \rho_1 w \right) = w'' - \left( \rho_2 + \rho_3 \right) w' - \rho_2 \rho_3 w
$$

$$
= w'' - \left( \rho_2 + \rho_3 \right) w'' - \rho_2 \rho_3 w' = 0 \text{ at } \zeta = \xi, \quad \text{ (21)}
$$

where

$$
\rho_1 = \rho_3 = W_c/2 - \sqrt{\frac{W_c^2}{4} + \alpha^2 + \beta^2 - i\alpha R - i\beta R},
$$

$$
\rho_2 = -\sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad C_T = i\beta/\left(\alpha^2 + \beta^2\right).
$$

It may be noteworthy that the above equations differ from the straight-flow approximation obtained by putting $n = 0$ in Eq. (16) only by the curvature terms $2(V + 1)v$.

Neutral curves obtained from Eq. (20) for Kármán’s flow are shown in Fig. 6, which clarifies two distinct areas of positive growth rate corresponding to the cross-flow instability on the upper side and the streamline-curvature instability downward, properly corresponding to the C-F mode in Fig. 4 and the S-C mode in Fig. 2, respectively. The fine dotted lines in Fig. 6 denote the neutral curves of slightly simpler stability equations that are derived from ignoring the vertical velocity $W(\zeta)$ in Eq. (20), consisting of the parallel-flow theory plus the curvature terms, as previously given by the author. The discrepancy between solid lines and fine lines in the figure seems to be so small that the effects of the vertical velocity component on the stability characteristics of the flow may be ignored, at least in this range of parameters. A comparison of the neutral curves of the parallel-flow theory and those in this figure confirms again that the existence of curvature terms in the stability equations makes it possible to describe the S-C mode belonging to centrifugal instability.

The critical Reynolds numbers of the C-F and S-C instabilities are found in Fig. 6 as 258 and 50.6, respectively, which are slightly lower than the values $R_c = 290$, 282 and 284 for C-F mode given by Lingwood,11) Itoh,12) and Pier,13) and $R_c = 69$, 64 and 63 for the S-C mode by Faller,14) Balakumar and Malik15) and Itoh,12) respectively, as well as the experimentally observed lower limits of C-F mode listed by Malik et al.17) However, the Orr-Sommerfeld equation gives $R_c = 177$ for the C-F mode,15) which is much lower than the above non-parallel estimations. Since the present study restricts non-parallel terms only to those of the most fundamental activity in the multiple instabilities, it is reasonable that the equations proposed give a critical value midway between the parallel approach and the non-parallel attempts.

7. Concluding Remarks

If the displacement thickness concerning circumferential velocities is used as the reference length of scaling, the flow on a rotating disk with uniform suction can be approximated by a perturbation from the asymptotic suction profile, as already shown by Stuart.3) Application of this length scale to stability analysis of the rotating-disk flows brings a small parameter inversely proportional to the surface suction into the partial differential equations governing small and wavy disturbances superimposed on the basic flows. Expanding the solutions into power series of this parameter decomposes the governing equations into an infinite sequence of ordinary differential equations, which forms an eigenvalue problem and higher-order corrections of the eigenvalue obtained. This procedure justifies the classical description of stability characteristics for rotating-disk flows with eigenvalue problems of ordinary differential equations.

Rotating-disk flows are susceptible to two kinds of instabilities; one is caused by the centrifugal force acting on the rotating fluids. Different treatment of the centrifugal terms in the original partial differential equations yields different kinds of series solutions. These coefficients are determined by solving two systems of ordinary differential equations; one for cross-flow...
instability and the other for streamline-curvature instability. The eigenvalue problem governing the leading terms of the series for cross-flow instability includes all of the terms corresponding to inviscid stability equations in the parallel-flow theory but no centrifugal term. However, the eigenvalue problem for streamline-curvature instability is strongly affected by the centrifugal terms included, as in Görtler instability of two-dimensional boundary layers along a concave wall. Computational results for higher-order correction terms in the two series show that the solutions may be approximated only by the leading terms nearly throughout the whole range of expansion parameters, from the asymptotic suction profile to Kármán’s flow without suction.

The present study has clarified, in a rational way, that rotating-disk flow suffers from two kinds of instabilities, which are approximately described by eigenvalue problems of slightly different ordinary differential equations. For simpler stability calculations meeting practical uses, a composite eigenvalue problem is proposed through the primitive method of superposing the two stability equations. The resultant equations are still very simple and seem to give sufficiently reasonable estimation for multiple instability characteristics of the flow. It is also pointed out that the composite equations compensate for the deficiency of the curved-flow stability equations that are applicable only to small values of the circumferential wave number.

The flow on a rotating disk is a typical example of three-dimensional boundary layers, but more general examples are found on the swept wings of aircraft. The practically important boundary layers also have curved streamlines of the external flow and cross-flow velocity components inside, confirming that the two instabilities mentioned above are possible there. The composite eigenvalue problem presented in this paper may therefore be applicable to stability analysis of the three-dimensional boundary layers, though some contrivances are needed to properly express the curvature terms.

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