Horseshoes and invariant tori in cosmological models with a coupled field and non-zero curvature

Leo T Butler

Department of Mathematics, University of Manitoba, Winnipeg, MB, R2J 2N2, Canada
E-mail: leo.butler@umanitoba.ca

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Abstract
This paper studies the dynamics of a family of Hamiltonian systems that originate from Friedman–Lemaître–Robertson–Walker space-times with a coupled field and non-zero curvature. In four distinct cases, previously considered by Maciejewski, Przybylska, Stachowiak and Szydowski, it is shown that there are homoclinic connections to invariant submanifolds and the connections split. These results imply the non-existence of a real-analytic integral independent of the Hamiltonian.

Keywords: FLRW space-time, coupled field, non-integrability, Hamiltonian mechanics

(Some figures may appear in colour only in the online journal)

1. Introduction
Almost one hundred years ago, in a pair of ground-breaking papers, Alexander Friedman introduced a simplified solution to Einstein’s equations of general relativity [22–25]. This solution implicitly determines the evolution of the radius of the Universe [22, equation (5)]. Lemaître independently rediscovered this equation [39, equation (2)]. Robertson [52–54] and Walker [61] separately considered spatially homogeneous space-times and their properties.

In [44], Maciejewski, Przybylska, Stachowiak and Szydowski investigate two families of Hamiltonians motivated by the Friedman–Lemaître–Robertson–Walker models in cosmology.
The first, the minimally-coupled field, has the Hamiltonian \[ H = -\frac{1}{2} \left( A^2 + 2ka^2 - 2\Lambda a^4 \right) + \frac{1}{2} \left( a^2 B^2 + 2(\omega/ab)^2 + 2(ma^2b)^2 \right), \tag{1} \]

where \( k, \Lambda \) and \( m \) are scalar parameters and \( \omega \) is angular momentum. As noted, Maciejewski, \textit{et al} consider only the case where angular momentum vanishes. They prove in theorem 5.i [44], that if \( H \) is integrable, then \( 9 - 4m^2/\Lambda \) is a perfect square. Conjecture 5.1.i conjectures that, in fact, the only integrable case is \( m = 0 \) [44]. The first result of the present paper is

**Theorem 1.1.** Assume \( k, \Lambda > 0 \) and \( \omega \) are fixed. For all \( \omega \) sufficiently small, if the Hamiltonian (1) has a second, independent real-analytic integral of motion, then \( m = 0 \).

This theorem is proven by demonstrating the existence of a horseshoe in the dynamics of a family of Hamiltonians that includes the unreduced variant of (1).

In the same paper, the authors consider a second Hamiltonian derived from a minimally-conformally-coupled field [44, equation (18)]

\[ H = -\frac{1}{2} \left( A^2 + ka^2 - \Lambda a^4 \right) + \frac{1}{2} \left( B^2 + kb^2 + (\omega/b)^2 + \frac{1}{2} \lambda b^4 + (mab)^2 \right) \tag{2} \]

where the parameters are the same as in (1) with the exception that a self-excitation term has been added with a strength of \( \lambda \). In theorem 7.i, it is proven that if the Hamiltonian is integrable and angular momentum vanishes, then either \( k = 0 \) or \( \lambda = \Lambda \) and \( m^2 = -\Lambda, -3\Lambda \).

**Theorem 1.2.** Assume that \( k, \Lambda > 0 \) and \( \lambda, \omega \) are fixed. For all \( \omega \) sufficiently small, if the Hamiltonian (2) has a second, independent real-analytic integral of motion, then \( m \) is imaginary.

Similar to theorem 1.1, this theorem is proven by demonstrating the existence of horseshoes in the dynamics; perhaps surprisingly, the proof also suggests the integrable case where \( m^2 = -\Lambda \), but makes no constraint on \( \lambda \).

The technique used to prove these theorems exploits a well-known mechanism: both Hamiltonians enjoy saddle-centre equilibria with connecting orbits. In a suitably re-scaled limit, the DE decouple and enjoy normally hyperbolic invariant manifolds which are foliated by invariant tori. Using essentially the same machinery as the variational DE employed in [44], one computes the Poincaré–Melnikov function \( M \) for the connecting orbits and shows that \( M \) has non-degenerate zeros. By well-known arguments, this proves the existence of transverse homo/hetero-clinic orbits, a suspended horseshoe and the absence of a second constant of motion that is independent of the Hamiltonian \( H \).

The case of zero curvature and angular momentum (\( k = 0, \omega \)) is studied in [42] by Mahdi, Llibre and Valls. They use the weighted homogeneity of the Hamiltonian to prove that there does not exist a second real-analytic constant of motion except in the known integrable cases (see below). Although their results are only stated for the case where the kinetic part of the Hamiltonian is positive definite, the arguments based on [42, proposition 2], extend to the indefinite case. It should be noted that this result is weaker than theorems 1.1 and 1.2: it remains an open question if there are horseshoes in the dynamics.

In the case of negative curvature (\( k = -1 \)), the present results are less definitive than the positive curvature case. It is proven that

**Theorem 1.3.** Assume that \( k, \Lambda < 0 \) and \( \omega \) are fixed. For all \( \omega \) sufficiently small, if the Hamiltonian (1) has a second, independent real-analytic integral of motion, then \( m = 0 \).
Theorem 1.4. Assume that $k, \Lambda, -\lambda < 0$ and $\omega$ are fixed. For all $\omega$ sufficiently small there is a countable set $E_\omega$ of real numbers such that, if the Hamiltonian (2) has a second, independent real-analytic integral of motion, then $m \in E_\omega$.

The last theorem is proven using similar tools to the first three theorems, but the details are rather different as is the result. The origin is a saddle critical point for the unreduced Hamiltonian (for all $m$ and $\omega = 0$). With the assumption that $\lambda > 0$, when $m = 0$ the saddle’s stable and unstable manifolds coincide. It is shown that for $m \neq 0$ sufficiently small, these manifolds split and create transverse homoclinic orbits. This implies that a family of nearby hyperbolic periodic orbits also have transverse homoclinic orbits and hence horseshoes in the dynamics. The homoclinic orbits where the splitting is detected exist for all $m$ and due to the real-analytic dependence of the stable and unstable manifolds of the saddle, the set $E_\omega$ of $m$ where those manifolds are not transverse along the homoclinic orbits, is a closed real-analytic subset of the reals with a non-empty complement. By real-analyticity in $\omega$, there is a similarly defined set $E_\omega$ for each $\omega$ sufficiently small. It is likely that $E_\omega$ is empty in all cases, but the present techniques cannot prove this. On the other hand, theorem 1.4 is the only theorem where we show that the horseshoe is on the zero energy level.

1.1. Implications

The results of the present paper suggest an investigation into the implications for quantum cosmology. De Oliveira and Soares [15] consider this in the context of model that is inspired by Hartle and Hawking’s work [30]. In a similar vein, the latter work shows that there is a positive probability of a quantum tunnelling between a bounded and unbounded Universe; the present work implies that a similar phenomenon can happen in the classical models. Numerical investigations might be illuminating in this regard.

1.2. Outline

The outline of the present note is: section 2 reviews related work; section 3 reviews the Lagrangian derivation of the Hamiltonians following the presentation in [44]; section 4 sets up the Poincaré–Melnikov integral for the saddle connections of (1); section 5 does likewise for (2); section 6 explains the computation of the Poincaré–Melnikov integral in 3 of the 4 cases (the remaining case is dealt with in section 5.2); section 7 proves the existence of KAM tori in the case of $k = -1$ with minimal conformal coupling; section 8 concludes; references follow.

2. Non-integrable and chaotic dynamics

The existence of non-integrable or chaotic dynamics in several cosmological models is well-known. Belinsky, Khalatnikov and Lifshitz [5] conjecture that the nature of singularities in space-time are dictated asymptotically by Bianchi IX space-time and that the transition between singularities is governed by the Gauss map. This work was later amplified in [34]. It should be noted that this characterization remains conjectural—Cushman and Śniatycki prove that the Hamiltonian flow is locally integrable but the proof is not constructive [14]. Indeed, to prove integrability, they construct a Lyapunov function (which by flow-box coordinates yields local integrability), but the flow has no recurrence. In [16], de Oliveira, Soares and Stuchi demonstrate chaotic dynamics in a reduction of the Bianchi IX model coupled with a scalar field.

In the case of Friedman–Lemaître–Robertson–Walker models, Calzetta and El Hasi study the minimal conformally coupled model with a real scalar field and $\Lambda = 0$, i.e. there is no
accelerating inflation nor self-excitation of the scalar field [11, equation (5)]. They demonstrate the existence of horseshoes on the zero energy level and provide numerical phase portraits as evidence of it, too. In a related vein, Bombelli and Calzetta show that a relativistic particle in motion around a Schwarzschild black-hole has hyperbolic periodic orbits with coincident homoclinic connections; for a generic periodic perturbation of the Schwarzschild metric, the connections split and create horseshoes [6]. Bombelli, Lombardo and Castagnino revisit the homoclinic connections; for a generic periodic perturbation of the Schwarzschild metric, the in motion around a Schwarzschild black-hole has hyperbolic periodic orbits with coincident evidence of it, too. In a related vein, Bombelli and Calzetta show that a relativistic particle the existence of horseshoes on the zero energy level and provide numerical phase portraits as evidence and offer a heuristic reason that in a neighbourhood of the saddle-centre equilibrium there is a family of hyperbolic periodic orbits with split homoclinic connections and therefore horseshoes. Other work connects this with a possible mechanism to explain inflation [16, 17].

Ziglin’s ground-breaking work on meromorphic integrability included his proof that the Yang–Mills Hamiltonian is non-integrable [64, 65]. The Yang–Mills Hamiltonian can be obtained from the Hamiltonian (2) by specializing \( \lambda = \Lambda = \omega = k = 0 \) and applying a complex rotation in the \((a, A)\) plane–such a change of variables destroys the real phase portrait of the Hamiltonian but it leaves invariant its integrability in the class of meromorphic integrals. Coelho, Skea and Stuchi [13] trod similar ground to Maciejewski, et al: they use differential Galois theory as developed by Morales-Ruiz and Ramis to demonstrate the non-integrability of the minimal conformally coupled Hamiltonian (2) when \( \omega = 0 \). They show that when \( \Lambda = \lambda = 0 \) and \( m \neq 0 \), then the Hamiltonian does not possess a second, independent meromorphic constant of motion and when \( k, \lambda, \Lambda \neq 0 \), the same is true except when \( \Lambda = \lambda = -m^2 \) or \(-m^2/3\) [13, theorems 3, 5]. It should be noted that when \( m \neq 0 \) is real, the first result of Coelho, et al is implied by the works of Calzetta and El Hasi and Bombelli, Lombardo and Castagnino. Although the latter works seem to imply otherwise (see [11, p 1828], [7, p 6048]), a straightforward rescaling shows that in this case all such Hamiltonian flows are conjugate up to a constant reparameterization of conformal time (see [33, equation (5)]). Helmi and Vucetich use Painlevé analysis to determine the possible integrable cases of (2) [31].

More recently, Shi and Li [58] examine the generalized Yang–Mills Hamiltonian

\[
H = \frac{1}{2} \left( a^3 + \alpha a^2 \right) + \frac{1}{2} \left( b^3 + \beta b^2 \right) + \frac{1}{4} a^4 + \frac{1}{2} \mu (ab)^3 + \frac{1}{4} \eta b^4,
\]

from the viewpoint of the theory of Morales-Ruiz and Ramis and the higher-order theory of Morales-Ruiz, Ramis and Simo [45, 47, 48]. There are several known integrable cases of (3):

(a) \( \alpha = \beta, \mu = \eta = 1 \): the rotationally-invariant case with \( F = aB - Ba \);

(b) \( \mu = 0 \): the separable case;

(c) \( \alpha = \beta, \mu = 3, \eta = 1 \) due to Bountis, Segur and Vivaldi [8]; as noted in [28, p 2293], this a special case of the previous case where \( \eta = 1 \) and the potential separates after a rotation by \( \pi/4 \);

(d) \( \beta = 4\alpha, \mu = 3, \eta = 8 \): Dorizzi, Grammaticos and Ramani discovered this case in their work on Darboux’s ‘direct method’ for finding integrable two-dimensional potentials. Under suitable simplifying assumptions, the potential satisfies a linear second-order PDE; the current case in the notation of [19, equation (18)] is \( \frac{1}{4} V_2 + \frac{1}{4} V_4 \) [21];

(e) \( \beta = 4\alpha, \mu = 6, \eta = 16 \): similar to the previous case [28, equation (4.6)], this is a superposition of two integrable potentials;
\( \beta \neq \alpha, \mu = \eta = 1 \): this is the two-dimensional Garnier system studied in [60]. The integral in [58, p 1646] is incorrect, as is that in [60, p 158], the correct integral appears on p. 168 of Vanhaecke’s paper.

Shi and Li demonstrate that when \( \alpha \neq \beta \), the generalized Yang–Mills Hamiltonian is not meromorphically integrable except for the above listed cases (c)–(e). Of the known integrable cases of (3), only the first two cases are relevant for the purposes of this paper.

In a sequence of papers, Llibre and Vidal [43], Lembarki and Llibre [40] and Jiménez-Lara and Llibre [33] use averaging theory to show the existence of a family of isolated periodic orbits that are parameterized by energy and have non-trivial Floquet multipliers to the origin in a Hamiltonian motivated by the Friedman–Lemaître–Robertson–Walker model (cf (2) and (9) below). The existence of such periodic orbits is taken as an indication that the Hamiltonians do not enjoy a second, independent \( C^1 \) first integral. However, two important qualifications need to be made: first, these arguments can only prove that the Hamiltonian vector field of any first integral must be co-linear along these orbits to the given vector field—to obtain stronger results, one needs a topological or metric characterization of the set of such periodic orbits (e.g., their closure forms a horseshoe); second, [43] considers only \( k = 1 \) and \( \lambda, \Lambda < 0 \) and inspection of [43, equation (10)] shows that the proof does not extend to the case where either \( \lambda > 0 \) or \( \Lambda > 0 \).

dos Santos and Vidal consider the stability of the origin for a 2 and 3 degree-of-freedom version of the Hamiltonian (2) for \( k = 1 \) [20, section 7]. They prove for the 2 degree-of-freedom case that when \( 3\Lambda + m^2 < 0 \) (resp. > 0) the origin is Lyapunov unstable (resp. formally stable); and a similar result is proven for 3 degrees of freedom with a sparse coupling.

In [51], Palacián, Vidal, Vidarte and Yanguas study a 3 degree-of-freedom version of the Hamiltonian (2) for \( k = 1 \) distinct from the one in the previous paragraph. Similarly, however, the paper focuses on the critical point at the origin and uses multi-scale KAM theory to prove the existence of invariant 3 tori near that critical point.

### 3. Friedman–Lemaître–Robertson–Walker space-time

Let us motivate the equations following the approach taken in [44]. The metric on space-time, modeled as \( \mathbb{R} \times M \), is postulated to be

\[
d s^2 = a(\eta)^2 \left(-d\eta^2 + g\right)
\]

where \( g \) is a metric on the space-like manifold \( M \). The time-like variable \( \eta \) is conformal time; the time measured by an external observer would be determined by \( dt = |a(\eta)| d\eta \). The selection principle for \( ds^2 \) is determined by the action functional

\[
I = \int_{\mathbb{R} \times M} \left[ Ric - 2\Lambda - \frac{1}{2} \left( \|\nabla \Psi\|^2 + V(\Psi) + \xi \text{Ric} |\Psi|^2 \right) \right] d\text{vol},
\]

where \( Ric \) is the Ricci (scalar) curvature of \( ds^2 \), \( \Lambda \) is the cosmological constant, \( \Psi : \mathbb{R} \times M \to \mathbb{R}^n \) is a field, \( |\Psi| \) is the Euclidean norm, \( \nabla \) is the gradient operator of the metric \( ds^2 \) which is extended component-wise for vector-valued functions, \( \|\nabla \Psi\|^2 \) is the \( ds^2 \)-inner product of \( \nabla \Psi \) with itself, also extended component-wise, \( V : \mathbb{R}^n \to \mathbb{R} \) is a potential function, \( \xi \) is a coupling constant, \( \rho \) is ‘fluid’ density and \( d\text{vol} \) is the volume form of \( ds^2 \).

Let us assume the following:

\( H1 \) (\( M, g \)) is a finite-volume homogeneous Riemannian manifold whose (constant) scalar curvature is \( 6k \neq 0 \);
H2 The volume of \((M, g)\) is unity;
H3 The field \(\Psi\) is spatially homogeneous, hence depends only on \(\eta\);
H4 The density \(\rho = ca^{-d}\) where \(d = 1 + \dim M\) and \(c\) is a constant;
H5 The potential \(V : \mathbb{R}^n \to \mathbb{R}\) is a polynomial of degree \(\leq d\) such that \(V\) decomposes into a sum \(V_2 + \cdots + V_d\) where \(V_1\) is homogeneous of degree \(k\) and \(V_2\) is positive definite;
H6 The dimension of space-time \(d = 4\);
H7 The cosmological constant \(\Lambda\) has \(k \Lambda > 0\).

Since these assumptions imply that the integrand of \(I\) (Lagrangian) is independent of the spatial variables, in the case \(d = 4\) scalar curvature reduces to \(6aa'' + 6ka^2\) and the action functional reduces to

\[
I = \int_{\mathbb{R}} \left[ 6(1 - \frac{1}{2} \xi |\Psi|^2)(aa'' + ka^2) + \frac{1}{2} |\Psi|^2 a^2 - a^4 V(\Psi) - 2\Lambda a^4 - c \right] d\eta. \tag{6}
\]

Integration by parts, combined with the assumption that \(a'a\) and \(a'|\Psi|^2\) are equal at \(\eta = \pm \infty\) yields the Lagrangian

\[
L = (-6 + 3\xi |\Psi|^2)(d')^2 + 6\xi (a\Psi, a'\Psi') + \frac{1}{2} a^2 |\Psi'|^2 - \frac{1}{2} a^4 V(\Psi) - 3\xi ka^2 |\Psi|^2
+ 6ka^2 - 2\Lambda a^4 - c, \tag{7}
\]

where \((,\)\) is the Euclidean inner product on \(\mathbb{R}^n\). The ‘kinetic’ part of the Lagrangian retains an indefinite character for all \(\xi\), but the off-diagonal part makes analysis difficult.

There are two straightforward routes to simplify the Lagrangian further:

H8A Minimal coupling: set \(\xi = 0\) to uncouple the field \(\Psi\) from the scalar curvature term;
H8B Minimal conformal coupling: set \(\Psi = \tau/a\) and \(\xi = 1/6\) to minimize the coupling of the rescaled field.

3.1. Minimal coupling

In the case of minimal coupling, the Lagrangian \(L\) produces a Hamiltonian \(H\), that after suitable rescaling becomes

\[
H = -\frac{1}{2} \left[ \frac{A^2 + ka^2}{\mu_{(1)}} - \frac{1}{2} \Lambda a^4 \right] + \frac{1}{2} \left[ a^{-2} |B|^2 + a^4 V(b) / \mu_{(2)} \right]. \tag{8}
\]

The Hamiltonian \(\mu^{(2)}\) has an apparent singularity at \(a = 0\); however, the singularity is not essential and part of the proof below involves removing the singularity.

3.2. Minimal conformal coupling

In the case of minimal conformal coupling, after the change of variables an off-diagonal term is left unless the coupling constant \(\xi = 1/6\). In that case the Lagrangian produces a Hamiltonian \(H\)

\[
H = -\frac{1}{2} \left[ \frac{A^2 + ka^2}{\mu_{(1)}} - \frac{1}{2} \Lambda a^4 \right] + \frac{1}{2} \left[ |B|^2 + k|b|^2 + a^4 V(b/a) / \mu_{(2)} \right]. \tag{9}
\]
In this case, the apparent singularity in $V$ is resolved by the assumption $\text{H5}$ on $V$.

**Remark 3.1.** I have largely adopted the terminology and notation of [44], so I should point out a number of differences. In [44, equation (2)], it is assumed that $\Psi$ is a complex-valued function (and ultimately real-valued for the minimally-coupled case), but this is not necessary for the mathematics. Similarly, the form of the potential $V(\Psi) = \frac{1}{2}m^2|\Psi|^2 + \frac{\lambda}{2\ell^4}|\Psi|^4$ (with $\lambda = 0$ in the minimally-coupled case) is used [44, equations (3) and (12)], but while this may make physical sense, it is not necessary for the mathematical results here. The density $\rho$ allows us to study integrability on an arbitrary energy level.

### 3.3. The phase space

It is useful to clarify the phase space of the Hamiltonians in question. It makes mathematical sense to choose the largest space on which the Hamiltonians can be defined while simultaneously preserving their algebraic character. On the other hand, the physical origins of the model indicate that the locus $\{a = 0\}$ is one with special meaning and the model ceases to be meaningful near this set. Belinsky, Khalatnikov and Lifshitz met such concerns by stating that general relativity is a purely gravitational theory and their studies were meant to clarify that theory. Similar comments are appropriate here. It is also important to note that the sign of $a$ has no intrinsic meaning in the model and that the correct phase space is the quotient of $\{(a, A, B)\}$ obtained by identifying points $(a, A, B)$ and $(-a, -A, B)$. As is so often the case, the behaviour of the Hamiltonian $H$ in a neighbourhood of the singular variety $\{(0, 0, B)\}$ in the reduced space contains a great deal of information and so we desingularize it to obtain that information. Or, in other words, we simplify matters by studying $H$ on a de-singularized phase space where the sign of $a$ is defined—but the conclusions must be independent of this latter fact.

### 4. Minimal coupling

Let us investigate the normal form for the minimally-coupled Hamiltonian $H$ (8). Recall that by assumption $\text{H5}$, $V = V_2 + V_3 + V_4$.

**Lemma 4.1.** The function $\nu = \nu(x, y, A, B) = xA + yB/x$ is a generating function of the symplectic transformation

$$a = x, \quad b = y/x, \quad A = X, \quad B = xY. \quad (10)$$

The Hamiltonian (8) is transformed to

$$H = -\frac{1}{2} \left[ \frac{X^2}{\mu^{(1)}} + kx^2 - \frac{1}{2} \Lambda x^4 \right] + \frac{1}{2} \left[ \frac{Y^2}{\mu^{(2)}} + V_4(y) + xV_3(y) + x^2V_2(y) \right]. \quad (11)$$

**Proof.** By definition, the change of variables is defined from the equations

$$X = \nu_A = A, \quad Y = \nu_B = B/x, \quad a = \nu_A = x, \quad b = \nu_B = y/x.$$  

This defines a symplectic change of variables; the remainder is clear. \qed

In the new coordinate system, courtesy of the assumption $\text{H5}$ on the potential $V$, the Hamiltonian $H$ has forgotten the singularity at $a = 0 (= x)$. Roughly speaking, the transformation has glued $[0, \infty) \times \mathbb{R}^n$ and $(-\infty, 0) \times \mathbb{R}^n$ along the singular variety $\{0\} \times \mathbb{R}^n$ to produce a copy of $\mathbb{R} \times \mathbb{R}^n$ where the potential of the system is regular.
Lemma 4.2. Assume \( k = \pm 1 \) (i.e. the scalar curvature of \( g \) is \( \pm 6 \)) and \( k\Lambda > 0 \). Let \( \alpha^2 = 1/k\Lambda, \varepsilon > 0 \), and
\[
\begin{align*}
x &= au, & X &= -\alpha U, & y &= \sqrt{\varepsilon}w, & Y &= \sqrt{\varepsilon}W.
\end{align*}
\] (12)

Then the Hamiltonian differential equations of \( H \) are transformed to
\[
\begin{align*}
U' &= u, & u'' &= -ku\left(1 - u^2\right) + \varepsilon uV_2(w) + \frac{1}{2}\varepsilon^2\alpha^{-1}V_3(w), \\
w' &= W, & w'' &= -\frac{1}{2}\left[\alpha^2u^2\nabla V_2(w) + \alpha\varepsilon u\nabla V_3(w) + \varepsilon\nabla V_4(w)\right].
\end{align*}
\] (13)

The proof of the lemma is a calculation. Note that without imposing the equality \( \varepsilon = \alpha^2 \), the DE are no longer in canonical form. That is a price worth paying in order to examine the system near \( y = Y = 0 \).

By hypothesis **H5**, the quadratic form \( V_2 \) is positive definite. Therefore, there is an orthonormal change of variables such that \( V_2 \) is transformed to a weighted sum of squares, weighted by its eigenvalues. Since the linear change of variables does not affect the structure of the DE (13), it can be assumed without loss of generality that

H9 The quadratic form \( V_2 \) equals
\[
V_2(w) = \frac{1}{2}k\Lambda \langle \varphi(w), \varphi(w) \rangle, \quad \text{where} \quad \varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} > 0. \] (14)

To make this a regular perturbation problem, one can assume either

H10A \( V_3 = O(\sqrt{\varepsilon}) \), i.e. \( V_3 = \sqrt{\varepsilon}V_3 \) for some homogeneous cubic \( \tilde{V}_3 \); or

H10B \( V_3 \equiv 0 \), i.e. the potential function is even.

And, in all cases, \( k\Lambda > 0 \) is fixed.

Lemma 4.3. Assume **H9** and either **H10A** or **H10B**. Then, the system of DE (13) is quadratic in the parameter \( \varepsilon \) and for \( \varepsilon = 0 \), the system is transformed to
\[
\begin{align*}
U' &= u, & u'' &= -ku(1 - u^2), \\
w' &= W, & w'' &= -\frac{1}{2}\left[\varphi^2 \left[1 - (1 - u^2)\right]\right] w.
\end{align*}
\] (15)

At this point, the treatment of the cases \( k = 1 \) and \( -1 \) diverge somewhat. The case of \( k = 1 \) is treated first.

4.1. \( k = 1 \)

The system (15) has a pair of saddle-centre critical points at \( (u = \pm 1, U = 0, w = W = 0) \). Moreover, for \( \sigma = \pm 1 \) the hyper-planes
\[
N_\sigma = \{(u = \sigma, U = 0, w, W) | w, \ W \in \mathbb{R}^n\}
\] (16)

are normally hyperbolic invariant manifolds that are foliated by invariant tori. Let \( W^+/-(N_\sigma) \) be the stable/unstable manifolds of \( N_\sigma \). A connected component of the stable manifold of \( N_\sigma \) less \( N_\sigma \), i.e. \( W^-(N_\sigma) - N_\sigma \), coincides with a connected component of the unstable manifold of
Figure 1. The contours of $H^{(1)}_k$ (17) with the saddle fixed points and the separatrixes in blue. The arrows indicate the direction of the flow lines.

$N_{-\sigma}$ less $N_{-\sigma}$, $W^+(N_{-\sigma}) - N_{-\sigma}$ (see figure 1). These invariant manifolds are contained in the zero set of the function $H^{(1)}_k = H^{(1)}_1$ where

$$H^{(1)}_k(u, U, w, W) = \frac{1}{2} U^2 - \frac{k}{4} (1 - u^2)^2,$$

which is an integral of motion of (15).

For $\varepsilon > 0$ sufficiently small, the local normally hyperbolic invariant manifolds $N^{\text{loc}}_{\sigma}$ will be perturbed to normally hyperbolic invariant manifolds $N^{\text{loc}}_{\sigma,\varepsilon}$ that are invariant for the flow of (13) and are graphs over $N^{\text{loc}}_{\sigma}$. The stable and unstable manifolds, $W^-(N^{\text{loc}}_{\sigma,\varepsilon})$ and $W^+(N^{\text{loc}}_{-\sigma,\varepsilon})$, will generally no longer ‘coincide’ as for $\varepsilon = 0$. Since they are codimension-1 submanifolds, the distance between them can be measured by $H^{(1)}$.$\varepsilon$. Specifically, the Poincaré–Melnikov function measures the $O(\varepsilon)$ separation between the two perturbed invariant manifolds. In the present case,

$$M(\varepsilon) = \int_{-\infty}^{\infty} U(t) u(t) V_2(w(t)) \, dt = \frac{1}{2} \sum_{j=1}^{n} \int_{-\infty}^{\infty} w_j(t) \frac{d}{dt} \left( u^2 - \sigma^2 \right) \, dt,$$

where $P = (u(0), U(0), w(0), W(0)) \in W^-(N_{\sigma}) \cap W^+(N_{-\sigma})$ and $P(t) = (u(t), U(t), w(t), W(t))$ is the solution to the unperturbed DE (15).

4.2. $k = -1$

In this case, the origin $(u = 0, U = 0, w = 0, W = 0)$ is a saddle-degenerate centre equilibrium of the DE (13). In this case, the remainder of the discussion in the previous subsection carries over with $\sigma = 0$ in place of $\sigma = \pm 1$ and $k = -1$ in place of $k = 1$. In particular, the Poincaré–Melnikov function (18) describes the $O(\varepsilon)$ separation of the local stable and unstable manifolds of the local normally hyperbolic invariant manifold $N^{\text{loc}}_{0,\varepsilon}$.

The computation of the Poincaré–Melnikov integral (18) is deferred to section 6.
5. Minimal conformal coupling

Lemma 5.1. Assume H5. Then the Hamiltonian of the minimal conformal coupling model (9) is transformed to the sum of the Hamiltonian of the minimal coupling model (11) and \( k \times \frac{1}{2} |y|^2 \) under the identity transformation \( x = a, X = A, y = b, Y = B \).

5.1. \( k = 1 \)

The lemma implies that for \( k = 1 \), the minimal coupling and minimal conformal coupling models are virtually identical—the sole change being that the equation for \( u'' \) in (15) has an additional term of \(-w\) on the right-hand side.

5.2. \( k = -1 \)

However, for \( k = -1 \), the treatment of the models diverges somewhat because the origin in the latter model is a non-degenerate saddle equilibrium. In this case, we are forced to make some additional assumptions about the potential \( V \). A minimal requirement is that \( V_2 \) be positive-definite so that \( H^{(1)} \) is proper and \( V_3 \equiv 0 \) so that \( \{ x = X = 0 \} \) is invariant. On the other hand, if \( H^{(2)} \) is non-integrable on the plane \( \{ x = X = 0 \} \), then there is nothing to be proven, so we make the assumption that \( H^{(2)} \) is integrable on this plane, too. Finally, as the discussion of the integrable cases of the generalized Yang–Mills Hamiltonian in the introduction makes clear, there are very few known integrable cases. The only two cases that are relevant to the particular problem here are the cases where \( V \) is either rotationally invariant (under the action of \( SO_n \)) or separable; likewise \( H \) should be separable.

H11 The potential \( V_\varepsilon = \varepsilon V_2 + \varepsilon^2 V_3 + V_4 \), \( V_4 \) is positive and either rotationally-invariant or separable while \( V_2 \) satisfies H9.

If hypothesis H11 is assumed, then for \( \varepsilon = 0 \), the Hamiltonian \( H^{(2)} \) has a saddle critical point at \( y = Y = 0 \). It follows that the Hamiltonian \( H_\varepsilon = -H^{(1)} + H^{(2)} \) has a saddle critical point \( s \) at \( x = X = 0, y = Y = 0 \) that persists for all \( \varepsilon \) (to be clear, the saddle point for \( H_\varepsilon \) is denoted by \( s \), below). By the hypothesis H11, the stable and unstable manifolds \( W^\pm(s) \) are coincident, lagrangian submanifolds for \( \varepsilon = 0 \). For \( \varepsilon \) non-zero and sufficiently small, the local manifolds are lagrangian graphs over the unperturbed local manifolds. Since the local stable manifold is contractible, the theory of lagrangian submanifolds implies that there are analytic functions \( \nu_\varepsilon^\pm : W^\pm \to \mathbb{R} \) such that \( W^\pm_{\loc} = \text{graph of } \nu_\varepsilon^\pm \) and \( \nu_\varepsilon = \nu_\varepsilon^+ - \nu_\varepsilon^- = \varepsilon \nu_0 + O(\varepsilon^2) \).

Definition 5.1. The function \( \nu_\varepsilon \) described in the previous paragraph is called the Poincaré–Melnikov splitting potential; the function \( \nu_0 \) is its lowest-order term.

A critical point of \( \nu_\varepsilon \) is a point of intersection of \( W^-_{\loc} \) and \( W^+_{\loc} \); since the Hamiltonian vector field \( X_\varepsilon \) is tangent to each manifold, such a critical point is not isolated but instead lies on a smooth curve of critical points. The maximal rank of the hessian \( \text{hess}\nu_\varepsilon \) at such a critical point is therefore \( n \) and at such points, the local manifolds intersect transversely as submanifolds of the common energy level (i.e. they are each \( n + 1 \) dimensional submanifolds of a \( 2n + 1 \) dimensional iso-energy manifold that intersect along a curve).

Proposition 5.1. Assume that \( \nu_0 \) has a critical point at \( P \in W^-_{\loc} \). If the hessian \( \text{hess} \nu_0 \) has rank \( n \) at \( P \), then the perturbed stable and unstable manifolds intersect transversely at a nearby point \( P_\varepsilon = P + O(\varepsilon) \).

As mentioned, transversality means the submanifolds intersect transversely in the energy level. The proof of this proposition may be reconstructed along the lines of [59, theorem 3.4];
see also [18] for an exposition. The lowest-order term $\nu_0$ can be computed from the decomposition of $H_\varepsilon = H_0 + \varepsilon H_1$ by

$$
\nu_0(P) = \int_{-\infty}^{\infty} H_1 \circ \phi'(P) \, dt,
$$

(19)

where $\phi'$ is the Hamiltonian flow of $H_0$ and $P \in W^\pm_{\text{loc}}$.

**Proposition 5.2.** Assume hypothesis H11. If

1. $V_4$ is rotationally invariant; or
2. $V_4$ is separable and the polynomial $\Delta$ (29) is non-zero,

then $W^\pm_{\text{loc}}$ and $W^\pm_{\text{loc}}$ do not coincide for all $\varepsilon \neq 0$ sufficiently small. If

1. $V_4$ is rotationally invariant and the eigenvalues of $\varphi$ are distinct; or
2. $V_4$ is separable and the polynomial $\Delta$ (29) does not vanish at 5,

then $W^\pm_{\text{loc}}$ and $W^\pm_{\text{loc}}$ intersect transversely for all $\varepsilon \neq 0$ sufficiently small.

**Proof.** In both cases, the lowest-order term $\nu_0$ of the Poincaré–Melnikov potential is

$$
\nu_0(P) = \frac{1}{2} L \times \int_{-\infty}^{\infty} x(t)^2 \left| \varphi(y(t)) \right|^2 \, dt
$$

(20)

where $P = (x(0), y(0), z(0), w(0)) \in W^\pm_{\text{loc}}$ and $P(t) = (x(t), y(t), z(t), w(t))$ is the solution to the Hamiltonian DEs for $H_0$ and $L = k\Lambda > 0$.

1. Assume that $V_4$ is rotationally invariant, so $V_4(y) = \frac{1}{2} \lambda |y|^2$ for some $\lambda > 0$. Let us change variables

$$
x = \alpha u, \quad y = \beta r \theta, \quad \text{where } \alpha^{-2} = L, \beta^{-2} = \lambda,
$$

(21)

and $r \in [0, \infty)$ and $\theta \in \mathbb{S}^{n-1}$. Because $H^{(2)}$ is rotationally invariant, the momentum map $\Psi = \frac{1}{2} (yY^T - Yy^T)$ is a first integral which vanishes identically on $W^\pm_{\text{loc}}$. This implies that the Hamiltonian DEs of $H_0$ are transformed, along the saddle, to

$$
\begin{align*}
\dot{u} &= u(1 - u^2), & \quad \dot{r} &= r(1 - r^2), & \quad \theta' &= 0.
\end{align*}
$$

(22)

Let $\mu(t)$ be an even solution to the DE for $u$; then the solutions $u = \pm \mu(t - t_0)$ and $r = \pm \mu(t - t_1)$ for some $t_0, t_1$. Finally, with $\tau = t_1 - t_0$,

$$
\nu_0(P) = \nu_0(\tau, \theta) = \frac{1}{2} \beta^2 |\varphi(\theta)|^2 \int_{-\infty}^{\infty} \mu(s)^2 \mu(s - \tau)^2 \, ds.
$$

(23)

The choice of the solution $\mu$ has induced coordinates $(t_0, \tau, \theta)$ on $W^\pm_{\text{loc}}$. The invariance of $\nu_0$ under the unperturbed flow means it depends only on $(\tau, \theta)$.

By even/odd symmetry, $\frac{\partial \nu_0}{\partial \theta} = 0$ at $\tau = 0$. Thus, if $\theta_0$ is a maximum point of $\theta \to |\varphi(\theta)|^2$, then $(\tau = 0, \theta = \theta_1)$ is a critical point of $\nu_0$ (and a simple argument using the Cauchy–Schwarz inequality shows it is a global maximum). On the other hand,

$$
\left| \frac{\partial^2 \nu_0}{\partial \tau^2} \right|_{\tau=0} = \frac{1}{2} \beta^2 |\varphi(\theta)|^2 \int_{-\infty}^{\infty} \mu(s)^4 \left( (2 - \mu(s)^2)/2 + (1 - \mu(s)^2) \right) \, ds
$$

$$
= -\frac{32}{15} \beta^2 |\varphi(\theta)|^2.
$$

(24)
This proves case (1), since the hessian of $\nu_0$ is non-trivial at $(\tau = 0, \theta = \theta_1)$ (hence $\nu_0 \neq$ constant) and it has rank $n$ if the eigenvalues of $\varphi$ are distinct.

(2) Assume that $V_4$ is separable, so $V_4(y) = \frac{1}{2} \sum_{\lambda=1}^{n} \lambda |y|^4$ for some positive scalars $\lambda_1, \ldots, \lambda_n$. Let us note that the separating coordinates are not necessarily the coordinates in which $\varphi$ is diagonal. Let us change variables

$$x = \alpha u, \quad y = \beta w, \quad \text{where } \alpha^{-2} = L, \ \beta^{-2} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}. \quad (25)$$

The change of coordinates implies that the Hamiltonian DEs of $H_0$ are transformed to

$$u'' = u(1 - u^2), \quad w'' = w(1 - w^2), \quad i = 1, \ldots, n. \quad (26)$$

Let $\Phi = (\varphi \beta)^2$ and $\mu$ be as above. Then, similar to case (1),

$$\nu_0(P) = \nu_0(\tau) = \frac{1}{2} \sum_{i=1}^{n} \Phi_{ij} \int_{-\infty}^{\infty} \mu(s)^2 \mu(s - \tau_i) \mu(s - \tau_j) \mathrm{ds}. \quad (27)$$

Similar to case (1), $\tau = 0$ is always a critical point. Let $\sigma(\Phi)$ be the diagonal matrix whose $i, i$ entry is the sum of the elements in the $i$th row of $\Phi$; define $\delta(\Phi, z) = -\Phi + z\sigma(\Phi)$. Calculations similar to case (1) show that

$$\text{hess } \nu_0|_{\tau=0} = -\frac{8}{5} \times \delta(\Phi, 5). \quad (28)$$

Hence, the critical point $\tau = 0$ is non-degenerate iff

$$\Delta(z) = \det \delta(\Phi, z) \quad (29)$$

is non-zero at $z = 5$. To investigate when hess$\nu_0$ is non-trivial, assume $\Delta(z)$ is a non-zero constant multiple of $(z - 5)^n$. Then, since $\Phi$ is positive definite, the characteristic polynomial of $\Phi$ is $(z - 1/5)^n$. Since $\Phi$ is symmetric, this forces $\Phi = 5\sigma(\Phi)$, so $\Phi$ is diagonal and therefore $\Phi$ is zero. But $\Phi$ is positive definite, a contradiction. This proves case (2). □

**Remark 5.1.** In the decoupled ($\varepsilon = 0$) limit the plane $\{y = Y = 0\}$ is a normally-hyperbolic invariant manifold and, aside from the saddle at the origin and its connections, the plane is fibred by periodic orbits (similarly, $\{x = X = 0\}$ is foliated by normally-hyperbolic invariant tori). A consequence of proposition 5.2 is that the homoclinic connections split for energy levels close to 0 and all $\varepsilon \neq 0$ sufficiently small. Hence, the Hamiltonian flow enjoys horseshoes on all energy levels near the zero level.

One can be more quantitative: the lowest-order term $\nu_0$ in the Poincaré–Melnikov splitting potential is an explicit integral involving the function $\mu$ and the Jacobi $\text{dn}$ (negative energy) or $\text{cn}$ function. In case (1), the expression is the same as (23) except the integral is changed to

$$\nu_0(\tau) = \int_{-\infty}^{\infty} u(s)^2 \mu(s - \tau)^2 \mathrm{ds} \quad (30)$$

where $u$ is a solution to the first DE in (22). Figure 2 plots the second derivative of $\nu_0(\tau)$ at the critical point $\tau = 0$.

**Proof of theorem 1.4.** Let $V_2(b) = \frac{1}{2}\lambda |b|^4$ and $V_2(b) = \varepsilon (ma |b|^2)^2$ for $b \in \mathbb{R}^2$ where $\lambda < 0$ and $m > 0$ are fixed. By proposition 5.2 the stable and unstable manifolds of the saddle fixed
Figure 2. The second derivative of $\nu_0$ with respect to $\tau$ at $\tau = 0$ as a function of $H^{(1)} := H^{(1)}(17)$. The hessian vanishes at the saddle-centre where $H^{(1)} = -1/4$; the indicated point at zero energy coincides with the value in (24). Inset: the contours of $H^{(1)}$ in the $(u, u')$ plane.

point at the origin intersect transversely modulo rotations for all $\varepsilon \neq 0$ sufficiently small. Therefore, the nearby hyperbolic periodic tori in the $\{a = A = 0\}$ plane have stable and unstable manifolds that intersect transversely modulo rotations, too. If the angular momentum $\omega$ is fixed and small enough, then the reduction of the Hamiltonian yields a Hamiltonian in the form of (2) and the normally hyperbolic tori are reduced to hyperbolic periodic orbits with transverse homoclinic points. This implies the theorem.

6. Computation of the Poincaré–Melnikov integral

Let us explain how the Poincaré–Melnikov integral (18) is computed. The integral is of the form

$$M(P) = \int_{-\infty}^{\infty} q'(t)w(t)^{2} \, dt \quad \text{subject to} \quad u'' + [\beta^{2} - q(t)] w = 0,$$

where $q(t)$ and $q'(t)$ both vanish at $t = \pm \infty$. To compute the integral $M$, let us make the simplifying assumption that $q$ is an even function. If $u_{0}$ (resp. $u_{1}$) is the unique solution such that $u_{0}(0) = 1$, $u'_{0}(0) = 0$ (resp. $u_{1}(0) = 0$, $u'_{1}(0) = 1$), then the general solution $w = c_{0}u_{0} + c_{1}u_{1}$ and $M(P) = 2c_{0}c_{1}m_{01}$ where $m_{01} = \int_{-\infty}^{\infty} q'(t)w_{0}(t)w_{1}(t) \, dt$. That is, in the coordinates $(t_{0}, c_{0}, c_{1})$, $M$ is either identically zero ($m_{01} = 0$) or it is an indefinite quadratic form and therefore its zero locus is $\{c_{0}c_{1} = 0\}$ and $dM \neq 0$ on the zero locus except at $c_{0} = c_{1} = 0$.

The computation diverges somewhat depending on the value of $k$.

6.1. $k = -1$

In this case, the minimal coupling model’s DE (15) and the Poincaré–Melnikov integral (18) translate to $q = -\frac{1}{2} \varphi^{2}u^{2}$ and $\beta = 0$ in equations (31) and (32). Since $\beta = 0$, the fundamental solutions are $w_{0} = 1$ and $w_{1} = t$. Then, integration by parts gives $m_{01} = -\frac{1}{2} \varphi^{2} \int_{-\infty}^{\infty} u^{2} \, dt = -2 \varphi^{2}$. 

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6.2. \(k = 1\)

To compute \(M\) explicitly in this case it is more convenient to complexify. Let \(w^\tau_\sigma(t)\) be solutions to the DE (32) such that \(w^\tau_\sigma(t)\) is asymptotic to \(\exp((-1)^\sigma i\beta t)\) at \(t = (1)^\infty\) for \(\tau, \sigma \in \{0, 1\}\). Then, there is a unique change of basis \(a = \{a_{ij}\}\) from \(\{w^\tau_0, w^\tau_1\}\) to \(\{w^\sigma_0, w^\sigma_1\}\) such that \(w^\sigma_i = a_{00}w^\tau_0 + a_{01}w^\tau_1\) for \(i = 0, 1\). Let \(w = c_0w^\tau_0 + c_1w^\tau_1\) be an expansion of the solution \(w\) in terms of the basis \(\{w^\tau_0, w^\tau_1\}\) of solutions. It is proven in [10, corollary 3.1] that the integral \(M\) equals \(m_{00}c_0^2 + 2m_{01}c_0c_1 + m_{11}c_1^2\) where the constants \(m_{ij}\) are the coefficients of the complexified Poincaré–Melnikov form \(M\) with respect to this basis. These coefficients are explicitly calculable in terms of the scattering matrix \(a\). Due to the fact that the real form of \(M\) is either zero or indefinite, it suffices to prove that the complexified \(M\) has a non-zero determinant (it will necessarily be positive).

To compute the determinant of \(M\) in the present situation, one rewrites the DE (15) with the variable \(u\) as an independent variable in place of \(t\) (since along the separatrix solution \(u\) is monotone). In that case the DE for \(w\) is transformed to a Legendre DE:

\[
(1 - u^2) \frac{d^2w}{du^2} - 2u \frac{dw}{du} + \left(\nu(\nu + 1) - \frac{\mu^2}{1 - u^2}\right)w = 0. \tag{33}
\]

One sees that

1. In the minimal coupling case (with \(\beta = \varphi\)), \(\nu = \frac{-1 + \sqrt{1 - 4\varphi^2}}{2}\) and \(\mu = \pm i\varphi\).
2. In the minimal conformal coupling case (with \(\beta = \sqrt{2 + \varphi^2}\)), \(\nu = \frac{-1 + \sqrt{1 - 4\varphi^2}}{2}\) and \(\mu = \pm \sqrt{2 + \varphi^2} = \pm i\beta\).

It can be demonstrated that the connection matrix for (33) is

\[
[a_{ij}] = \begin{bmatrix} B & 2\bar{\nu}A \\ 2^{1/2}A & B \end{bmatrix} \quad \text{where} \quad A = \frac{\Gamma(c)\Gamma(1 - c)}{\Gamma(a)\Gamma(b)}, \quad B = \frac{\Gamma(c)\Gamma(c - 1)}{\Gamma(c - a)\Gamma(c - b)}
\]

\[
a = -\nu, \quad b = 1 + \nu, \quad c = 1 - \mu. \tag{34}
\]

From this, the determinant of the complexified Poincaré–Melnikov quadratic form \(M\) equals

\[
det \; M = -4\beta^4 \left(1 - (|A| + |B|)^2\right) \left(1 - (|A| - |B|)^2\right) = 16\beta^4|A|^2, \tag{35}\]

where the fact that the connection matrix has unit determinant is used.

**Proposition 6.1.** The following holds when \(k = 1:\)

1. In the minimal coupling model, for all \(\varphi = \beta > 0\), \(\det M \neq 0\);
2. In the minimal conformal coupling model, for all \(\varphi > 0 \; (\beta > \sqrt{2})\), \(\det M \neq 0\).

**Proof.** Assume that \(\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} - \varphi^2}\) and \(\mu = i\beta\) are independent parameters.

First, assume that \(\varphi > \frac{1}{2}\). In this case, the proof is similar to that of [9, theorem 4.1]: let \(a = \frac{1}{2} + is\) where \(s = \sqrt{\varphi^2 - \frac{1}{4}}\) is real and positive. Equations (34) and (35) and [1, 6.1.28–30] imply that for \(s, \beta > 0\)

\[
|A|^2 = \frac{(\cosh \frac{\pi s}{\pi \beta})^2}{\sinh \frac{\pi s}{\pi \beta}}. \tag{36}
\]
Second, assume $0 < \varphi < \frac{1}{4}$. Then $a = \frac{1}{2} + s$ and $b = \frac{1}{2} - s$ where $s = \sqrt{\frac{1}{4} - \varphi^2} \in (0, \frac{1}{2})$ and so $a, b \in (0, 1)$. In this case, the reflection formula cited above along with $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$ for $z \in (0, 1) + i\mathbb{R}$ [1, 6.1.17] imply that for $\beta > 0$ and $0 < s < \frac{1}{2}$

$$|A|^2 = \left( \frac{\cos \pi s}{\sinh \pi \beta} \right)^2,$$

which does not vanish for $s \in (0, \frac{1}{2})$.

Finally, it is apparent that $\det M \neq 0$ when $s = 0 (\varphi = \frac{1}{4})$, too.

This proves case 1 where $\beta = \varphi > 0$. Case 2 follows since $\varphi > 0$ and $\beta = \sqrt{2 + \varphi^2}$. □

Remark 6.1. Figures 3 and 4 graph the re-scaled determinant of the complex Poincaré–Melnikov form versus $\beta$ for the minimal and minimal conformal coupling models. Note that although $\beta > \sqrt{2}$ in the latter, the graph is extended over the interval $[0, \sqrt{2}]$ where one sees that the integrable case of $m^2/\Lambda = -1$ is identified at $\beta = 0$.

The literature on saddle-centre equilibria in Hamiltonian systems is extensive. Lerman [41] and Lerman and Kol’tsova [36–38] study the general case of a 2-degree of freedom Hamiltonian with a saddle-centre equilibrium, and subsequently an $n + 1$-degree of freedom Hamiltonian with an equilibrium that decomposes as a saddle and $n$ centres. In the former case, they prove that for a generic Hamiltonian, there is family of nearby hyperbolic periodic orbits which enjoy a pair of transverse homoclinic orbits; and in the latter case, similar results hold. In the $n + 1$-degree of freedom setting, the generic case is that there is a Cantor family of normally hyperbolic invariant tori near the saddle-centre and these tori enjoy transverse homoclinic orbits [27, theorem 1]. Moreover, under generic conditions, one can demonstrate the existence of transition chains of tori and Arnol’d diffusion [4, 63]. Grotta-Ragazzo [29] gives an alternative, geometric, proof of the result of Lerman and Koltsvoa and derives several corollaries from that proof. The first corollary is that, in the notation here, the separatrixes of the homoclinic connection split if the connection matrix $[a_{ij}]$ is not diagonal (i.e. $A \neq 0$) [29, theorem 4] which is exactly the condition here that the complexified Poincaré–Melnikov form $M$ be non-degenerate (35). The results of [29] extend well beyond this, though. In section 4, the paper connects the non-triviality of $A$ with the non-triviality of squares in the monodromy group of the variational equation [29, theorem 8]. This point of view is elaborated in subsequent papers by Morales-Ruiz and Peris and Yagasaki [46, 62] where the differential galois group is brought in. In recent work, Giles, Lamb and Turaev [26] revisit the saddle-centre problem and derive a novel proof of the splitting result using Lyapunov–Schmidt reduction and the Poincaré–Melnikov potential. On the other hand, the classic work of Holmes and Marsden, although couched in slightly different terminology, proves the existence of horseshoes on all super-energy levels in a neighbourhood of a saddle-centre separatrix under the assumption that the unperturbed Hamiltonian is separable [32, theorem 3.2 and example 4.1].

7. KAM tori

This section investigates the near-integrability of certain Hamiltonians that originate from the minimal coupling model. The results here are intended to be illustrative rather than exhaustive. The paper by Palacián, et al [51] contains more expansive results in a more specialized context.

Let us revisit the minimal coupling Hamiltonian (11) with $k = -1$ and $L = k\Lambda > 0$. In addition, the following hypothesis is assumed

H12 The field is scalar, i.e. $n = 1$. 

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In this case, the homogeneous terms in the potential of the scalar field can be written as $V_i(y) = v_i y^i$ for $i = 1, 2, 3$. By hypothesis $H_9$, $v_2 = \frac{1}{4}L\varphi^2$.

With these hypotheses, it follows that there is an elliptic critical point of $H$ at $(x = \alpha = 1/\sqrt{L}, X = 0, y = 0, Y = 0)$.

**Lemma 7.1.** Assume $k = -1$ and $L = k\Lambda > 0$. Let $\alpha^2 = 1/L, \varepsilon > 0$, and define the canonical transformation by

$$
    x = \alpha + u, \quad X = U, \quad y = w/\sqrt{\varphi}, \quad Y = \sqrt{\varphi}W.
$$

Then the Hamiltonian $H = -H^{(1)} + H^{(2)} (11)$ is transformed to
\[ H^{(1)} = \frac{1}{2} \left[ U^2 + 2u^2 + 2\sqrt{L}u^3 + \frac{1}{2}Lu^4 \right] \]

\[ H^{(2)} = \frac{1}{2} \varphi \left[ W^2 + \frac{1}{2}w^2 + \varphi \sqrt{Lu}w^2 + \frac{1}{2}\varphi L(uw)^2 + v_3u^3/\sqrt{\varphi^3L} \right. \]

\[ \left. + v_3uw^3/\sqrt{\varphi^3} + v_3w^4/\varphi^2 \right] . \tag{39} \]

The quadratic terms in \( H \) can be used to put \( H \) into Birkhoff normal form:

**Lemma 7.2.** Let

\[ I_1 = \frac{1}{\sqrt{2}} \left[ \frac{1}{2} U^2 + u^2 \right], \quad I_2 = \frac{1}{\sqrt{2}} \left[ W^2 + \frac{1}{2} w^2 \right] . \tag{40} \]

If \( \varphi \notin 2 \{ 0, 1/3, 1/2, 1, 2, 3 \} \), then there is a canonical transformation \((\theta_1, I_1, \theta_2, I_2) \rightarrow (u, U, w, W)\) that transforms \( H \) to

\[ H = -\sqrt{2}I_1 + \frac{\varphi}{\sqrt{2}}I_2 + A_{11}I_1^2 + 2A_{12}I_1I_2 + A_{22}I_2^2 + O(5), \quad \text{where} \tag{41} \]

\[ A_{22} = \frac{L^2\varphi(2\varphi^2 - 3) + 24L\varphi^2v_3(\varphi^2 - 1) - 60v_3^2(\varphi^2 - 1)}{16L\varphi^4(\varphi^2 - 1)} \quad \text{and} \]

\[ A_{12} = -\frac{L\varphi}{8} \left( \frac{3\varphi^2 - 2}{\varphi^2 - 1} \right), \quad \text{A}_{11} = 3L/4. \]

Let \( H = H_2 + O(5) \), where \( H_2 \) is the second-order (in \( I_1, I_2 \)) Birkhoff invariant of \( H \) at the fixed point. The canonical variables \((\theta_1, I_1, \theta_2, I_2)\) constitute a system of (singular) angle-action variables for \( H_2 \). The theory of Kolmogorov, Arnol’d and Moser on the preservation of conditionally periodic motion in perturbations of integrable, real-analytic Hamiltonians implies that if either the hessian or bordered hessian of \( H_2 \) is non-singular at \( I_1 = I_2 = 0 \), then there is a positive-measure set of invariant tori for \( H \) whose density approaches 1 as \( I_1^2 + I_2^2 \rightarrow 0 \) \([2, 3, 35, 49, 50]\). Moreover, since an invariant 2-torus separates a three-dimensional iso-energy surface of \( H \), the equilibrium is stable.

**Theorem 7.1.** Assume the hypotheses of lemmas 7.1 and 7.2. Then, there exists a positive-measure set of invariant tori for \( H \) that accumulates on the elliptic critical point \((x = \alpha = 1/\sqrt{\varphi}, \ X = 0, \ y = 0, \ Y = 0)\).

**Proof.** By the remarks preceding the theorem, it suffices that for all \( \varphi \) in the non-resonant set, i.e. \( \varphi \notin 2 \{ 0, 1/3, 1/2, 1, 2, 3 \} \), either the hessian or bordered hessian of \( H_2 \) is non-degenerate at \( I_1 = I_2 = 0 \). By inspection, each respective determinant is a rational function of \( \varphi \). Let \( m \) and \( n \) be the numerators of the respective determinants, with factors of the form \( \varphi^k \) removed. One computes that

\[ \text{resultant}(m, n; \varphi) = \begin{cases} 2^{12} 3^6 5^6 L^{24} v_3^{12} & \text{if } v_3 \neq 0, \\
2^{12} 3^4 L^{10} v_4^4 & \text{if } v_3 = 0, v_4 \neq 0, \\
1 & \text{if } v_3 = v_4 = 0. \end{cases} \tag{42} \]

Since the resultant is never zero, the determinants do not vanish simultaneously at a non-zero value of \( \varphi \). \( \square \)
Remark 7.1. There is a second, less pedestrian and computation-free, proof of theorem 7.1 when \( \varphi \) satisfies a Diophantine condition. If the Birkhoff normal form of \( H \) is trivial at all orders and \( \varphi \) satisfies a Diophantine condition, then by a theorem of Rüssmann [55], \( H \) is conjugate to its linearization. This cannot happen since \( H \) also possesses a hyperbolic critical point at the origin. This implies that for Diophantine, and hence for almost all, \( \varphi \), the Birkhoff normal form of some order is not trivial and therefore the image of the frequency map does not lie in a line through the origin. A second work of Rüssman implies that the Hamiltonian is ‘non-degenerate enough’ that a positive measure set of invariant tori exist in a neighbourhood of the critical point [56, 57]. This idea was used by Churchill, Pecelli, Sacolick and Rod in their study of a Yang–Mills-type Hamiltonian [12].

8. Conclusion

This paper studies a Friedman–Lemaître–Robertson–Walker space-time with a coupled field. In positively-curved space-times, with either a minimal or minimal conformal coupled field, the coupling between the field and the radius of space splits a saddle connection and creates a family of horseshoes on a nearby energy levels. The \( C^1 \) structural stability of a horseshoe implies that for ‘nearby’ coupled models (i.e. for lagrangians \( L(7) \) with \( \xi \) close to 0 or 1/6), the horseshoes persist. This implies real-analytic non-integrability on a general energy. However, the situation on the important zero energy level still remains inaccessible using the current techniques.

The situation is similar and different for negatively-curved space-times. Similar, in that the \( C^1 \) structural stability extends the results to nearby, non-minimally coupled, models. Different, in the minimal conformal coupled model, because the existence of horseshoes is proven only for Hamiltonians that have a weak coupling and because the horseshoe is shown to exist on the zero energy level.

These results suggest many intriguing questions. I pose a few:

- **Q1** Can the splitting results be extended from a neighbourhood of the saddle-centre to prove the hyperbolic periodic orbits on \( \{ H = 0 \} \) also have split connections?

- **Q2** In the minimal conformal coupled model, does the splitting results extend from weakly coupled to all coupling strengths?

- **Q3** The proof of theorem 1.4 is the only place where the hypothesis that the field \( \Psi \) is \( \mathbb{R}^n \)-valued is really needed; does the theorem extend to the case when \( \Psi \) takes values in a smooth manifold?

- **Q4** With reference to either (1) or (2), what happens to the family of hyperbolic periodic orbits with large values of angular momentum \( \omega \)? Are there values where the connections do not split?

- **Q5** Beyond the remarks above, what can be proven for the non-minimally coupled models?

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ORCID iDs
Leo T Butler 🐘 https://orcid.org/0000-0002-0188-0607

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