INARIANT HOPF 2-COCYCLES FOR AFFINE ALGEBRAIC GROUPS

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Abstract. We generalize the theory of the second invariant cohomology group $H^2_{\text{inv}}(G)$ for finite groups $G$, developed in [Da2, Da3, GK], to the case of affine algebraic groups $G$, using the methods of [EG1, EG2, G1]. In particular, we show that for connected affine algebraic groups $G$ over an algebraically closed field of characteristic 0, the map $\Theta$ from [GK] is bijective (unlike for some finite groups, as shown in [GK]). This allows us to compute $H^2_{\text{inv}}(G)$ in this case, and in particular show that this group is commutative (while for finite groups it can be noncommutative, as shown in [GK]).

1. Introduction

An interesting invariant of a tensor category $\mathcal{C}$ is the group of tensor structures on the identity functor of $\mathcal{C}$ (i.e., the group of isomorphism classes of tensor autoequivalences of $\mathcal{C}$ which act trivially on the underlying abelian category) up to an isomorphism [Da1, Da4, BC, GK, PSV, Sc]. This group is called the second invariant (or lazy) cohomology group of $\mathcal{C}$ and denoted by $H^2_{\text{inv}}(\mathcal{C})$. In particular, if $\mathcal{C} := \text{Corep}(H)$ is the category of finite dimensional comodules over a Hopf algebra $H$, then $H^2_{\text{inv}}(H) := H^2_{\text{inv}}(\mathcal{C})$ is the group of invariant Hopf 2-cocycles for $H$ modulo the subgroup of coboundaries of invertible central elements of $H^*$.

In [Da2, GK] the authors study $H^2_{\text{inv}}(\mathcal{C})$ for $\mathcal{C} := \text{Corep}_k(\mathcal{O}(G)) = \text{Rep}_k(G)$, where $G$ is a finite group, and $\mathcal{O}(G)$ is the algebra of functions on $G$ with values in an algebraically closed field $k$. Namely, they define the set $\mathcal{B}(G)$ of pairs $(A, R)$, where $A$ is a normal commutative subgroup in $G$ and $R$ is a non-degenerate $G$-invariant class in $H^2(\hat{A}, k^*)$, and a map $\Theta : H^2_{\text{inv}}(G) := H^2_{\text{inv}}(\mathcal{C}) \to \mathcal{B}(G)$. This allows one to compute $H^2_{\text{inv}}(G)$ in many examples, although not always, as in general the map $\Theta$ is neither surjective nor injective. For example, it was proved
in [Da2] that if the order of $G$ is coprime to 6 then $H^2_{\text{inv}}(G)$ is the direct product of $\mathcal{B}(G)$ and the group $\text{Out}_{\text{cl}}(G)$ of class-preserving outer automorphisms of $G$.

The goal of this paper is to generalize the theory of [Da2, GK] to the case of affine algebraic groups, using the methods of [EG1, EG2, G1]. In particular, we show that for connected affine algebraic groups over a field $k$ of characteristic zero, the map $\Theta$ is, in fact, bijective (so the situation is simpler than for finite groups). This allows us to compute the second invariant cohomology group in this case.

More specifically, we show in Theorem 5.1 that if $k$ has characteristic 0 and $U$ is a unipotent algebraic group with $u := \text{Lie}(U)$, then $H^2_{\text{inv}}(U) \cong (\wedge^2 u)^u$.

Furthermore, let $G$ be any connected affine algebraic group over $k$ of characteristic 0. Let $G_u$ be the unipotent radical of $G$, and let $G_r := G/G_u$. Let $Z$ be the center of $G$, and let $Z_u, Z_r$ be the unipotent and reductive parts of $Z$. Let $g_u, j_u, j_r$ be the Lie algebras of $G_u, Z_u, Z_r$, respectively. Our main result Theorem 7.8 is that one has

$$H^2_{\text{inv}}(G) \cong \text{Hom}(\wedge^2 \hat{Z}_r, k^\times) \times (j_r \otimes j_u) \times (\wedge^2 g_u)^G,$$

where $\hat{Z}_r$ is the character group of $Z_r$. In particular, this group is commutative (while for finite groups it can be noncommutative, as shown in [GK]).

The organization of the paper is as follows. Section 2 is devoted to preliminaries. In particular, we recall the definition of the second invariant cohomology group of a Hopf algebra and its basic properties, and study the space $(\wedge^2 g)^0$ for a Lie algebra $g$.

In Section 3 we use [G1] to describe the structure of $H^2_{\text{inv}}(G)$, where $G$ is a commutative affine algebraic group over $k$ of characteristic 0 (see Theorem 3.4).

In Section 4 we use [G1] to study the support group $A$ of an invariant Hopf 2-cocycle $J$ for the function Hopf algebra $\mathcal{O}(G)$ of an arbitrary affine algebraic group over $k$. In particular, we show that $A$ is a closed normal commutative subgroup of $G$ (see Theorem 4.2).

In Section 5 we extend the definitions of the set $\mathcal{B}(G)$ and map $\Theta : H^2_{\text{inv}}(G) \to \mathcal{B}(G)$ from [GK] to affine algebraic groups $G$ over $k$, and study the basic properties of $\Theta$ in Theorem 5.1.

In Section 6 we focus on unipotent algebraic groups over $k$ of characteristic 0, and prove Theorem 6.1.

Finally, Section 7 is devoted to the proof of our main result Theorem 7.8 which is a generalization of Theorem 6.1 to arbitrary connected affine algebraic groups $G$ over $k$ of characteristic 0.
Acknowledgements. We are grateful to Vladimir Popov for communicating to us Lemma [7.2] We thank A. Davydov and M. Yakimov for suggesting the current formulation of Lemma [2.8] and A. Davydov for references. The work of P.E. was partially supported by the NSF grant DMS-1502244. S.G. thanks the University of Michigan and MIT for their hospitality.

2. Preliminaries

Throughout the paper, unless otherwise specified, we shall work over an algebraically closed field \( k \) of arbitrary characteristic.

2.1. Hopf 2-cocycles. Let \( H \) be a Hopf algebra over \( k \) with multiplication map \( m \). An invertible element \( J \in (H \otimes H)^* \) is called a (right) Hopf 2-cocycle for \( H \) if it satisfies the two conditions

\[
\sum J(a_1b_1,c)J(a_2,b_2) = \sum J(a,b_1c_1)J(b_2,c_2),
\]

\[
J(a,1) = \varepsilon(a) = J(1,a)
\]

for all \( a,b,c \in H \) (see, e.g., [Da]).

We have a natural action of the group of invertible elements \((H^*)^x\) on Hopf 2-cocycles for \( H \). Namely, if \( J \) is a Hopf 2-cocycle for \( H \) and \( x \in (H^*)^x \), then the linear map \( J^x : H \otimes H \to k \) defined by

\[
J^x(a,b) = \sum x(a_1b_1)J(a_2,b_2)x^{-1}(a_3)x^{-1}(b_3), \quad a,b \in H,
\]

is also a Hopf 2-cocycle for \( H \). We say that two Hopf 2-cocycles for \( H \) are gauge equivalent if they belong to the same \((H^*)^x\)-orbit, and that they are strongly gauge equivalent if they belong to the same orbit under the action of the group of invertible central elements of \( H^* \).

Given a Hopf 2-cocycle \( J \) for \( H \), one can construct a new Hopf algebra \( H^J \) as follows. As a coalgebra \( H^J = H \), and the new multiplication \( m' \) is given by \( m' = J^{-1} * m * J \), i.e.,

\[
m^J(a \otimes b) = \sum J^{-1}(a_1,b_1)a_2b_2J(a_3,b_3), \quad a,b \in H.
\]

Equivalently, every Hopf 2-cocycle \( J \) for \( H \) defines a tensor structure on the forgetful functor \( \text{Corep}_k(H) \to \text{Vec} \), where \( \text{Corep}_k(H) \) is the tensor category of finite dimensional comodules over \( H \). Recall that if \( J \) and \( J' \) are gauge equivalent then the Hopf algebras \( H^J \) and \( H^{J'} \) are isomorphic.

Note that if \( K \) is a Hopf subalgebra of \( H \) and \( J \) is a Hopf 2-cocycle for \( H \) then the restriction \( \text{res}(J) \) of \( J \) to \( K \) defines a Hopf 2-cocycle for \( K \). Also, if \( K \) is a Hopf algebra quotient of \( H \) and \( J \) is a Hopf 2-cocycle for \( K \) then the lifting \( \text{lift}(J) \) of \( J \) to \( H \) defines a Hopf 2-cocycle for \( H \).
2.2. Invariant Hopf 2-cocycles. Let $H$ be a Hopf algebra over $k$ with multiplication map $m$. A Hopf 2-cocycle $J$ for $H$ is called invariant (or lazy in, e.g., [GK, Section 1.4]) if $m^J = m$ (equivalently, if $J * m = m * J$). In particular, when $H$ is cocommutative, every Hopf 2-cocycle is invariant.

Note that if $K$ is a Hopf subalgebra of $H$ and $J$ is an invariant Hopf 2-cocycle for $H$ then res($J$) defines an invariant Hopf 2-cocycle for $K$. However, if $K$ is a Hopf algebra quotient of $H$ and $J$ is an invariant Hopf 2-cocycle for $K$ then lif($J$) is not necessarily an invariant Hopf 2-cocycle for $H$.

It is clear that invariant Hopf 2-cocycles for $H$ form a group under multiplication, which is denoted by $Z^2_{inv}(H)$ (=$Z^2_{l}(H)$ in [GK]). In fact, we make the following observation.

**Lemma 2.1.** Let $J$ be an invariant Hopf 2-cocycle for $H$, and let $F$ be a (not necessarily invariant) Hopf 2-cocycle for $H$. Then $J * F$ is a Hopf 2-cocycle for $H$.

**Proof.** By Eq. (1) for $J$ and $F$, and the invariance of $J$, we have

$$
\sum (J * F)(a_1b_1,c)(J * F)(a_2,b_2)
= \sum J(a_1b_1,c_1)F(a_2b_2,c_2)J(a_3,b_3)F(a_4,b_4)
= \sum (J(a_1b_1,c_1)J(a_2,b_2))(F(a_3b_3,c_2)F(a_4,b_4))
= \sum (J(a_1,b_1c_1)J(b_2,c_2))(F(a_2,b_3c_3)F(b_4,c_4))
= \sum J(a_1,b_1c_1)F(a_2,b_2c_2)J(b_3,c_3)F(b_4,c_4)
= \sum (J * F)(a,b_1c_1)(J * F)(b_2,c_2)
$$

for all $a, b, c \in H$, as desired. \qed

Furthermore, the group $Z^2_{inv}(H)$ contains a central subgroup $B^2_{inv}(H)$ (= $B^2_{l}(H)$ in [GK]) consisting of all the invariant Hopf 2-cocycles $\partial(x)$, where $x \in (H^*)^\times$ is central in $H^*$ and

$$\partial(x)(a,b) = \sum x(a_1b_1)x^{-1}(a_2)x^{-1}(b_2), \ a, b \in H.$$

Following [GK], we define the quotient group

$$H^2_{inv}(H) := Z^2_{inv}(H)/B^2_{inv}(H)$$

(= $H^2_{l}(H)$ in [GK]). For $J \in Z^2_{inv}(H)$, let $[J]$ denote its class in $H^2_{inv}(H)$. By definition, $[J] = [J]$ if and only if $J$ and $J$ are strongly gauge equivalent (see Subsection 2.1).
It is straightforward to verify that a Hopf 2-cocycle $J$ for $H$ is invariant if and only if it defines a tensor structure on the identity functor on $\mathcal{C} := \text{Corep}_k(H)$. Hence, $H^2_{\text{inv}}(H)$ is isomorphic to the group $\text{Aut}_0(\mathcal{C})$ of isomorphism classes of tensor structures on the identity functor of $\mathcal{C}$, i.e., of tensor autoequivalences of $\mathcal{C}$ which act trivially on the underlying abelian category. The second invariant cohomology $\text{Aut}_0(\mathcal{C})$ for an arbitrary tensor category $\mathcal{C}$ was introduced first by Davydov [Da1] and studied also in [PSV].

**Proposition 2.2.** [DEN, Corollary 3.16] $H^2_{\text{inv}}(H)$ is an affine proalgebraic group, and if $H$ is finitely presented, it is an affine algebraic group. □

**Example 2.3.** As pointed out in [Da1, Section 8] (see also [Sc]), when $H$ is cocommutative, the group $H^2_{\text{inv}}(H)$ coincides with Sweedler’s second cohomology group $H^2_{\text{Sw}}(H,k)$ of $H$ with coefficients in the algebra $k$ [Sw]. In particular, if $G$ is a group then $H^2_{\text{inv}}(k[G]) \cong H^2(G,k^\times)$ [Sw, Theorem 3.1], and if $\mathfrak{g}$ is a Lie algebra then $H^2_{\text{inv}}(U(\mathfrak{g})) \cong H^2(\mathfrak{g},k)$ [Sw, Theorem 4.3].

Recall that a Hopf algebra automorphism of $H$ of the form $\text{Ad}(x) := x \ast \text{id} \ast x^{-1}$ for some $x \in (H^*)^\times$ is called cointernal (see, e.g., [BC]). For example, if $x : H \to k$ is an algebra homomorphism (i.e., $x \in H^*_{\text{fin}}$ is a grouplike element) then $\text{Ad}(x)$ is cointernal and it is called coinner. The set of all cointernal Hopf algebra automorphisms of $H$ is denoted by $\text{CoInt}(H)$, and the set of all coinner Hopf algebra automorphisms of $H$ is denoted by $\text{CoInn}(H)$. It is easy to see that $\text{CoInn}(H) \subseteq \text{CoInt}(H)$ are normal subgroups of $\text{Aut}_{\text{Hopf}}(H)$ (see, e.g., [BC, Lemma 1.12]).

The following lemma is dual to [GK, Proposition 1.7(a)].

**Lemma 2.4.** Let $J \in Z^2_{\text{inv}}(H)$ and let $x \in (H^*)^\times$. Then $J^x \in Z^2_{\text{inv}}(H)$ if and only if $\text{Ad}(x) \in \text{CoInt}(H)$.

**Proof.** By definition, $J^x \in Z^2_{\text{inv}}(H)$ if and only if $J^x * m = m * J^x$, i.e., if and only if

$$(x \circ m) * J * (x^{-1} \otimes x^{-1}) * m = m * (x \circ m) * J * (x^{-1} \otimes x^{-1}).$$

Hence, since $J$ is invariant, it follows that $J^x \in Z^2_{\text{inv}}(H)$ if and only if

$$(x \circ m) * (x^{-1} \otimes x^{-1}) * m = m * (x \circ m) * (x^{-1} \otimes x^{-1}),$$

which is equivalent to

$$(x^{-1} \otimes x^{-1}) * m * (x \otimes x) = (x^{-1} \circ m) * m * (x \circ m).$$
But by definition, the latter is equivalent to $\text{Ad}(x)$ being a Hopf automorphism of $H$. We are done. \hfill \Box

Following [GK] Proposition 1.7, we have the following result.

**Proposition 2.5.** The quotient group $\text{CoInt}(H)/\text{CoInn}(H)$ acts freely on $H^2_{\text{inv}}(H)$, and the associated map

$$\text{(4)} \quad \text{CoInt}(H)/\text{CoInn}(H) \rightarrow H^2_{\text{inv}}(H), \quad \text{Ad}(x) \mapsto (\varepsilon \otimes \varepsilon)^x,$$

defines an embedding of $\text{CoInt}(H)/\text{CoInn}(H)$ as a subgroup of $H^2_{\text{inv}}(H)$.

**Proof.** By Lemma 2,4 we have an action of the group $\text{CoInt}(H)$ on $H^2_{\text{inv}}(H)$, given by $\text{Ad}(x) \cdot [J] = [J^x]$ for every $\text{Ad}(x) \in \text{CoInt}(H)$ and $[J] \in H^2_{\text{inv}}(H)$. This is a well-defined action since if $\text{Ad}(x) = \text{Ad}(y)$ then $x \ast y^{-1}$ is central in $H^*$, and hence $[J^x] = [J^y]$.

Now every invariant Hopf 2-cocycle $J$ for $H$ is fixed by every coinner Hopf automorphism $\text{Ad}(x)$ with $x \in G(H^*)$, since

$$J^x = (x \circ m) \ast J \ast (x^{-1} \otimes x^{-1}) = J \ast (x \circ m) \ast (x^{-1} \otimes x^{-1}) = J.$$

Hence, we get an action of the quotient group $\text{CoInt}(H)/\text{CoInn}(H)$ on $H^2_{\text{inv}}(H)$.

Fix $[J] \in H^2_{\text{inv}}(H)$, and suppose $\text{Ad}(x) \in \text{CoInt}(H)$ is such that $\text{Ad}(x) \cdot [J] = [J]$. Then $[J] = [J^x]$, hence $J$ and $J^x$ are strongly gauge equivalent, i.e.,

$$J = (z \circ m) \ast (x \circ m) \ast J \ast (x^{-1} \otimes x^{-1}) \ast (z^{-1} \otimes z^{-1})$$

for some central invertible element $z$ in $H^*$. Since $J$ is invariant,

$$(z \circ m) \ast (x \circ m) \ast J = J \ast (z \circ m) \ast (x \circ m),$$

hence we have $\varepsilon \otimes \varepsilon = ((z \ast x) \circ m) \ast ((z \ast x)^{-1} \otimes (z \ast x)^{-1})$. In other words, we have $z \ast x \in G(H^*)$. But $z$ is central, hence $\text{Ad}(x) = \text{Ad}(z \ast x)$ is in $\text{Inn}(H)$, which implies the freeness of the action.

Finally, it follows from the above that the map \ref{1} is injective, and since $(\varepsilon \otimes \varepsilon)^x$ is invariant for every $x \in (H^*)^\times$ such that $\text{Ad}(x) \in \text{CoInt}(H)$, it is straightforward to verify that $(\varepsilon \otimes \varepsilon)^{xy} = (\varepsilon \otimes \varepsilon)^x \ast (\varepsilon \otimes \varepsilon)^y$ for every $x, y \in (H^*)^\times$ such that $\text{Ad}(x), \text{Ad}(y) \in \text{CoInt}(H)$, hence \ref{1} is a group homomorphism.

**Remark 2.6.** Let $\mathcal{C} := \text{Corep}_k(H)$. It is straightforward to verify that $\text{CoInt}(H)/\text{CoInn}(H)$ is the stabilizer of the standard fiber functor of $\mathcal{C}$ in $\text{Aut}_0(\mathcal{C})$.\hfill \Box
2.3. Cotriangular Hopf algebras. Recall (see, e.g., [EGNO, Definition 8.3.19]) that \((H, R)\) is a cotriangular Hopf algebra if \(R \in (H \otimes H)^*\) is an invertible element such that

\begin{itemize}
  \item \(\sum R(h_1, g_1)R(g_2, h_2) = \varepsilon(g)\varepsilon(h)\) (i.e., \(R^{-1} = R_{21}\)),
  \item \(R(h, gl) = \sum R(h_1, g)R(h_2, l)\),
  \item \(R(hg, l) = \sum R(g, l_1)R(h, l_2)\), and
  \item \(\sum R(h_1, g_1)g_2 h_2 = \sum h_1 g_1 R(h_2, g_2)\)
\end{itemize}

for every \(h, g, l \in H\).

Recall that the Drinfeld element of \((H, R)\) is the grouplike element \(u\) in \(H^*\), given by \(u(h) = \sum R(S(h_2), h_1)\) for every \(h \in H\).

Given a Hopf 2-cocycle \(J\) for \(H\), \((H_J, R_J)\) is also cotriangular, where \(R_J := J^{-1} * R * J\).

Recall that if \(J\) and \(\bar{J}\) are gauge equivalent then \(\text{Ad}(x) : (H_J, R_J) \to (H^{\bar{J}}, R^{\bar{J}})\) is an isomorphism of cotriangular Hopf algebras.

**Proposition 2.7.** Let \((H, R)\) be a cotriangular Hopf algebra, and let \(I := \{a \in H \mid R(b, a) = 0, b \in H\}\) be the right radical of \(R\). Then \(I\) coincides with the left radical \(\{b \in H \mid R(b, a) = 0, a \in H\}\) of \(R\), and is a Hopf ideal of \(H\).

**Proof.** See [G1, Proposition 2.1]. \qed

A cotriangular Hopf algebra \((H, R)\) such that \(R\) is non-degenerate (i.e., \(I = 0\)) is called minimal. By Proposition 2.7 any cotriangular Hopf algebra \((H, R)\) has a unique minimal cotriangular Hopf algebra quotient \(H/I\), which we will denote by \((H_{\text{min}}, R_{\text{min}})\).

Recall that the finite dual Hopf algebra \(H^*_\text{fin}\) of \(H\) consists of all the elements in \(H^*\) that vanish on some finite codimensional ideal of \(H\).

Note that in the minimal case, the non-degenerate form \(R\) defines two injective Hopf algebra maps \(R_+, R_- : H \hookrightarrow H^*_\text{fin}\), given by \(R_+(h)(a) = R(h, a)\) and \(R_-(h)(a) = R(S(a), h)\) for every \(a, h \in H\).

2.4. Invariant solutions to the classical Yang-Baxter equation.

Recall that a Lie algebra \(\mathfrak{h}\) over \(k\) is called quasi-Frobenius with symplectic form \(\omega\) if \(\omega \in \wedge^2 \mathfrak{h}^*\) is a non-degenerate 2-cocycle, i.e., \(\omega : \mathfrak{h} \times \mathfrak{h} \to k\) is a non-degenerate skew-symmetric bilinear form satisfying

\[\omega([x, y], z) + \omega([z, x], y) + \omega([y, z], x) = 0\]

for every \(x, y, z \in \mathfrak{h}\).

Let \(\mathfrak{g}\) be a Lie algebra over \(k\). Recall that an element \(r \in \wedge^2 \mathfrak{g}\) is a solution to the classical Yang-Baxter equation if

\[\text{CYB}(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.\]
Lemma 2.8. Assume support the hgb. Proposition 2.9. x,y,z permutation of quasi-Frobenius Lie subalgebra with symplectic form ω. We will call h the support of r.

The following result appear in [Da3] Lemma 5.5.2.

Lemma 2.8. Assume k has characteristic ≠ 2. Then the components of any element r ∈ (∧²g)⁰ commute. Hence, any element in (∧²g)⁰ is a solution to the classical Yang-Baxter equation.

Proof. Let h be the span of the components of r. Since r is g-invariant, h is a Lie ideal of g. Moreover, h carries a g-invariant symplectic form ω = r⁻¹. Thus, ω([x,y], z) = −ω(z,[x,y]) = −ω([z,x],y) for every x,y,z ∈ h. I.e., ω([x,y], z) is anti-invariant under the cyclic permutation of x,y,z. Since this permutation has order 3, we get ω([x,y], z) = 0 for every x,y,z ∈ h. But since ω is non-degenerate, [x,y] = 0 for every x,y ∈ h, as desired.

Proposition 2.9. Assume k has characteristic ≠ 2,3. If r,s ∈ (∧²g)⁰ then

[r, s] = Δ(z) − z ⊗ 1 − 1 ⊗ z
for some central element z ∈ U(g) of degree 3. Moreover, we have [r, [r, s]] = [s, [r, s]] = 0.

Proof. Let a and b be the supports of r and s, respectively. By Lemma 2.8 a and b are abelian Lie ideals in g. Therefore, a + b is a Lie subalgebra (in fact, a Lie ideal) of g, which is nilpotent of index ≤ 2. Namely, let h := a∩b, and let a₀ and b₀ be some complements of h in a and b, respectively (as vector spaces). Then a + b = a₀ ⊕ b₀ ⊕ h. Since a, b are abelian Lie ideals in a + b, we have that h is central in a + b, and the only nontrivial component of the bracket is [ , ] : a₀ × b₀ → h.

We have r = r' + r'', where r' ∈ (a₀ ⊗ h) ⊕ ∧²h and r'' ∈ ∧²a₀. Let r'' = i x_i ⊗ x'_i be a shortest presentation of r'' in ∧²a₀. Let b ∈ b₀. Since [Δ(b), r] = 0, we have i ([b, x_i] ⊗ x'_i + x_i ⊗ [b, x'_i]) = 0. Hence, [b, x_i] = [b, x'_i] = 0 for all i. Thus, x_i, x'_i are central in a + b. So [r'', s] = 0, and it suffices to show that [r', s] = Δ(z) − z ⊗ 1 − 1 ⊗ z for some central element z ∈ U(g) of degree 3.

Similarly, we have a decomposition s = s' + s'', where the components of s'' are central in a + b. Therefore, [r', s''] = 0, and it suffices to show that [r', s'] = Δ(z) − z ⊗ 1 − 1 ⊗ z.

Let {h_i} be a basis of h. Then r' = i a_i ∧ h_i and s' = i b_i ∧ h_i, modulo ∧²h, where a_i ∈ a₀ and b_i ∈ b₀. Thus, [r', s'] = i,j ([a_i, b_j] ⊗ h_i h_j + h_i h_j ⊗ [a_i, b_j]).
Now write \([a_i, b_j] = \sum_k c_{ijk} h_k\). Since \(r'\) is \((a + b)\)-invariant, it follows that \(\sum_i[a_i, b_j] \wedge h_i = 0\) for every \(j\), i.e., \(\sum_i c_{ijk} h_k \wedge h_i = 0\) for every \(j\). Thus, \(c_{ijk}\) is symmetric in \(i, k\). Similarly, since \(s'\) is \((a + b)\)-invariant, \(c_{ijk}\) is symmetric in \(j, k\). Thus \(c_{ijk}\) is symmetric in all three indices.

Finally, we have

\([r, s] = [r', s'] = \sum_{i,j,k} c_{ijk} (h_k \otimes h_i h_j + h_i h_j \otimes h_k)\).

Thus we may take \(z := \frac{1}{3} \sum_{i,j,k} c_{ijk} h_i h_j h_k = \frac{1}{6} \text{mult}([r, s])\), where \(\text{mult}\) denotes multiplication of components, and obtain that

\([r, s] = [r', s'] = \Delta(z) - z \otimes 1 - 1 \otimes z\),

as desired. Note that \(z\) is central in \(U(g)\), since \(r\) and \(s\) are \(g\)-invariant, and \([r, [r, s]] = [s, [r, s]] = 0\) since \([r, h_i] = [s, h_i] = 0\) for every \(i\). □

**Example 2.10.** Let \(g\) be the Heisenberg Lie algebra with basis \(a, b, c\) such that \([a, b] = c, [a, c] = [b, c] = 0\), \(r := a \wedge c\) and \(s := b \wedge c\). Then \([r, s] = c \otimes c^2 + c^2 \otimes c\), so we may take \(z := c^3/3\). (See [Da3, Example 5.5.9] for more details.)

Finally, we let \(B(g)\) denote the set of pairs \((\mathfrak{h}, \omega)\) such that \(\mathfrak{h} \subseteq g\) is an abelian Lie ideal with a \(g\)-invariant symplectic form \(\omega\). As mentioned above, we have a bijection \(B(g) \xrightarrow{\cong} \wedge^2 g^\mathfrak{h}\), given by \((\mathfrak{h}, \omega) \mapsto \omega^{-1}\).

3. **Hopf 2-cocycles for commutative affine algebraic groups**

In this section we will assume that \(k\) has characteristic 0.

3.1. **Invariant Hopf 2-cocycles for affine algebraic groups.** Let \(G\) be a (possibly disconnected) affine algebraic group over \(k\), and let \(G_u\) be the unipotent radical of \(G\). Recall that we have a split projection \(G \rightarrow G/G_u\), i.e., \(G = G_r \ltimes G_u\) is a semidirect product for a (unique, up to conjugation) closed subgroup \(G_r \cong G/G_u\) (a “Levi subfactor”) such that \(G_r\) is reductive.

Equivalently, we have a split exact sequence of Hopf algebras

\[\mathcal{O}(G_r) \hookrightarrow \mathcal{O}(G) \twoheadrightarrow \mathcal{O}(G_u)\]

with the Hopf algebra section \(\mathcal{O}(G) \cong \mathcal{O}(G_r)\) corresponding to the inclusion \(G_r \hookrightarrow G\).

Set

\[Z^2_{inv}(G) := Z^2_{inv}(\mathcal{O}(G))\] and \(H^2_{inv}(G) := H^2_{inv}(\mathcal{O}(G))\).
Let us say that $J$ is a Hopf 2-cocycle for $G$ if $J$ is a Hopf 2-cocycle for $\mathcal{O}(G)$, and that $J$ is minimal if the cotriangular Hopf algebra $(\mathcal{O}(G)^{\hat*}, J_{21}^{-1} \ast J)$ is minimal (see Subsection 2.3).

For $J \in Z_{\text{inv}}^2(G)$, we will call $J_r := \text{lif}(\text{res}(J))$ (with respect to $\iota$ and $q$; see Subsection 2.1) the reductive part of $J$. It is clear that $J_r$ is a Hopf 2-cocycle for $G$.

**Lemma 3.1.** Let $J \in Z_{\text{inv}}^2(G)$ be an invariant Hopf 2-cocycle for $G$. Then $J_r^{-1}$ is a Hopf 2-cocycle for $G$.

**Proof.** Since $J$ is invariant, $J^{-1}$ is also a Hopf 2-cocycle for $G$. Hence, $J_r^{-1} = \text{lif}(\text{res}(J^{-1}))$ is a Hopf 2-cocycle for $G$ too.

By Lemmas 2.1 and 3.1 if $J \in Z_{\text{inv}}^2(G)$ then $J_u := J \ast J_r^{-1}$ is a Hopf 2-cocycle for $G$. Clearly the restriction of $J_u$ to $\mathcal{O}(G_\iota)$ is trivial (i.e., it defines a fiber functor on $\text{Rep}(G)$, which coincides with the standard one on the semisimple tensor subcategory $\text{Rep}(G/G_u)$). We will call $J_u$ the unipotent part of $J$. We thus obtain

**Proposition 3.2.** For every invariant Hopf 2-cocycle $J$ for $G$, we have $J = J_u \ast J_r$.

### 3.2. Hopf 2-cocycles for commutative affine algebraic groups.

Let $A$ be a commutative affine algebraic group over $k$. Recall that $A \cong A_r \times A_u$ is a direct product, and $A_r \cong T \times A_f$ is a direct product of an algebraic torus $T \cong \mathbb{G}_m^n$ and a finite abelian group $A_f$, and $A_u \cong \mathbb{G}_a^m$ is an additive group. The diagonalizable closed subgroup $A_r$ of $A$ is the reductive part of $A$, and the closed subgroup $A_u$ is the unipotent radical of $A$.

Let $\widehat{A} := \text{Hom}(A, k^\times)$ denote the character group of $A$. Recall that $\widehat{T} \cong \mathbb{Z}^n$. We have $\mathcal{O}(A_r) \cong k[\widehat{A}_r]$, the group algebra of $\widehat{A}_r \cong \widehat{T} \times \widehat{A_f}$ with its standard Hopf algebra structure, and $\widehat{A} = \widehat{A}_r$.

Recall also that $\mathcal{O}(A_u)$ is a polynomial algebra on $m$ variables, with its standard Hopf algebra structure, and we have

$$\mathcal{O}(A) \cong \mathcal{O}(A_r) \otimes \mathcal{O}(A_u) \cong k[\widehat{A}_r] \otimes \mathcal{O}(A_u),$$

as Hopf algebras.

**Corollary 3.3.** Let $A \cong A_r \times A_u$ be a commutative affine algebraic group as above, and let $a, a_r,$ and $a_u$ be the Lie algebras of $A, A_r$ and $A_u$, respectively. Let $J$ be a Hopf 2-cocycle for $A$. Then $J_r$ and $J_u$ are (invariant) Hopf 2-cocycles for $A$, and $J = J_r \ast J_u = J_u \ast J_r$. Furthermore, we have $[J_u] = [\exp(r/2)]$ for a unique element $r$ in $a \wedge a_u = a_r \otimes a_u \oplus a_u \otimes a_u$. In particular, the Drinfeld element of the cotriangular Hopf algebra $(\mathcal{O}(A), J_{21}^{-1} \ast J)$ is trivial.
Proof. The first assertion follows from Proposition 3.2. The second assertion is a special case of [EG2, Theorem 3.2], but here is a direct proof. Consider \( \rho := 2 \log(J_u) \) (it is well defined since \( J_u \) is unipotent). Then \( \rho \) is a Hochschild 2-cocycle of \( \mathcal{O}(A) \) with trivial coefficients. Since by the Hochschild-Kostant-Rosenberg theorem, \( HH^i(\mathcal{O}(A), k) \cong \wedge^i \mathfrak{a} \), this isomorphism maps the class of \( \rho \) to an element \( r \in \wedge^2 \mathfrak{a} \), which is moreover in \( \mathfrak{a} \wedge \mathfrak{a}_u \) since \( \rho \) is nilpotent. The claim follows. \( \square \)

**Theorem 3.4.** Let \( A \cong A_r \times A_u \) be a commutative affine algebraic group as above, and let \( \mathfrak{a} \) and \( \mathfrak{a}_u \) be the Lie algebras of \( A \) and \( A_u \), respectively. Then the following hold:

1. We have group isomorphisms
   \[ H^2_{inv}(A_r) \cong H^2(\widehat{A_r}, k^\times) \cong \text{Hom}(\wedge^2 \widehat{A_r}, k^\times). \]
   Furthermore, minimal Hopf 2-cocycles for \( A_r \) correspond under this isomorphism to non-degenerate alternating bicharacters on \( \widehat{A_r} \).

2. We have a vector space isomorphism
   \[ \wedge^2 \mathfrak{a}_u \cong H^2_{inv}(A_u), \quad r \mapsto [\exp(r/2)]. \]
   Furthermore, minimal Hopf 2-cocycles for \( A_u \) correspond under this isomorphism to non-degenerate elements of \( \wedge^2 \mathfrak{a}_u \). In particular, \( \mathcal{O}(A_u) \) has a minimal Hopf 2-cocycle if and only if \( m \) is even.

3. We have group isomorphisms
   \[ H^2_{inv}(A) \cong H^2(\widehat{A}, k^\times) \times (\mathfrak{a} \wedge \mathfrak{a}_u) \cong \text{Hom}(\wedge^2 \widehat{A}, k^\times) \times (\mathfrak{a}_r \otimes \mathfrak{a}_u \oplus \wedge^2 \mathfrak{a}_u). \]

Proof. (1) and (2) follow from the definitions in a straightforward manner (see [G1, Theorem 5.3 & Proposition 5.4]).

(3) follows from (1) and Corollary 3.3. \( \square \)

**Remark 3.5.** The group \( H^2_{inv}(A) \) plays the role of the group of skew-symmetric bicharacters of \( A \) when \( A_u \) is non-trivial.

**Remark 3.6.** The group isomorphism
   \[ H^2(\widehat{A}, k^\times) \cong \text{Hom}(\wedge^2 \widehat{A}, k^\times) \]
   is given by \( J \mapsto R^J := J_{21}^{-1} J \).

**Remark 3.7.** The dimension of an algebraic torus having a minimal Hopf 2-cocycle need not be even (see [G1, Remark 4.2]).

Moreover, if \( A \cong T \times A_f \) has a minimal Hopf 2-cocycle then the order of \( A_f \) does not have to be a perfect square (contrary to the case of finite groups). For example, let \( A = \mathbb{G}_m^2 \times \mathbb{Z}/n\mathbb{Z} \), let \( q \in k^\times \) be a
non-root of unity, $\zeta \in k$ a primitive $n$-th root of unity, and let $R$ be the bicharacter of $A$ given by $R((x, y, a), (x', y', a')) = q^{xy' - yx'} \zeta^{xa' - ax'}$. Then $R$ is non-degenerate, hence defines a minimal cotriangular structure on $\mathcal{O}(A)$. It corresponds to a minimal Hopf 2-cocycle $J$ for $A$ given by $J((x, y, a), (x', y', a')) = q^{xy' \zeta^{xa'}}$.

This raises a question which finite groups can arise as component groups $H/H^0$ of affine algebraic groups $H$ such that $\mathcal{O}(H)$ has a minimal Hopf 2-cocycle. Clearly, this can be the product of any group $\Gamma$ of central type with any finite abelian group, by taking $H$ to be the product of $\Gamma$ with a number of copies of the example above for various values of $n$.

**Example 3.8.** Let $A_r := \mathbb{G}_m$ and $A_u := \mathbb{G}_a$, with Lie algebra bases $x$ and $y$, respectively. Then $r := x \wedge y$ defines a minimal Hopf 2-cocycle $\exp(r/2)$ for $A_r \times A_u$.

**Example 3.9.** Let $A_r := \mathbb{G}_m^2$ and $A_u := \mathbb{G}_a$, with Lie algebra bases $\{x_1, x_2\}$ and $y$. Then for every irrational $a$, $r_a := (x_1 + ax_2) \wedge y$ defines a minimal Hopf 2-cocycle $\exp(r_a/2)$ for $A_r \times A_u$, with a 3-dimensional support. (See [G1, Example 4.13].)

**Remark 3.10.** Since a commutative affine algebraic group $A$ over $k$ has only 1-dimensional irreducible representations, all fiber functors on $\text{Rep}_k(A) = \text{Corep}_k(\mathcal{O}(A))$ preserve dimensions, so all of them arise from Hopf 2-cocycles for $A$.

Let us now consider commutative affine algebraic groups over a field $k$ of characteristic $p > 0$.

**Proposition 3.11.**
1. If $A$ is a finite $p$-group then $H^2_{\text{inv}}(A)$ is the trivial group.
2. If $A$ is a vector group, we have a vector space isomorphism $H^2_{\text{inv}}(A) \cong \wedge^2 \mathfrak{a}$, where $\mathfrak{a}$ is the Lie algebra of $A$.

**Proof.** (1) Follows from [G2, Corollary 6.9].
(2) Since in this case $\mathcal{O}(A) \cong U(\mathfrak{a}^*)$, we have $H^2_{\text{inv}}(A) = H^2_{\text{inv}}(U(\mathfrak{a}^*)) = H^2(\mathfrak{a}^*, k) = \wedge^2 \mathfrak{a}$ (see Example [23]).

4. **The Support of an Invariant Hopf 2-Cocycle for an Affine Algebraic Group**

Let $G$ be an affine algebraic group over $k$ and let $\mathcal{O}(G)$ be its function algebra. Then $(\mathcal{O}(G), \varepsilon \otimes \varepsilon)$ is a cotriangular commutative Hopf algebra.
**Theorem 4.1.** Let $J$ be a Hopf 2-cocycle for $G$. Then there exist a closed subgroup $H$ of $G$ (called the support of $J$), determined uniquely up to conjugation, and a minimal Hopf 2-cocycle $\bar{J}$ for $H$ such that $J$ is gauge equivalent to $\bar{J}$ (viewed as a Hopf 2-cocycle for $G$).

**Proof.** For the existence of $H$ and $\bar{J}$, see [G1] Theorem 3.1.

Now let $H, \bar{H} \subseteq G$ be two closed normal subgroups of $G$, and let $J \in (\mathcal{O}(H) \otimes \mathcal{O}(H))^*$ and $L \in (\mathcal{O}(\bar{H}) \otimes \mathcal{O}(\bar{H}))^*$ be two minimal Hopf 2-cocycles for $H$ and $\bar{H}$, respectively. Suppose $J, L$ are gauge equivalent as Hopf 2-cocycles for $G$. Then our job is to show that $H, \bar{H}$ are conjugate in $G$.

By definition, $L = J^x = (x \circ m) \ast J \ast (x^{-1} \otimes x^{-1})$ for some element $x \in (\mathcal{O}(G)^*)^\times$. Hence, we have $R^L = (x \otimes x) \ast R^J \ast (x^{-1} \otimes x^{-1})$. Then by the minimality of $R^L$ and $R^J$, we have

$$ (x \otimes x) \ast (\mathcal{O}(H) \otimes \mathcal{O}(H))^* \ast (x^{-1} \otimes x^{-1}) = (\mathcal{O}(\bar{H}) \otimes \mathcal{O}(\bar{H}))^* $$

inside the algebra $(\mathcal{O}(G) \otimes \mathcal{O}(G))^*$. It follows that

$$ (x \circ m) \ast (x^{-1} \otimes x^{-1}) = L \ast (x \otimes x) \ast J^{-1} \ast (x^{-1} \otimes x^{-1}) \in (\mathcal{O}(\bar{H}) \otimes \mathcal{O}(\bar{H}))^* $$

is a symmetric Hopf 2-cocycle for $\bar{H}$. Therefore,

$$ (x \circ m) \ast (x^{-1} \otimes x^{-1}) = (y \circ m) \ast (y^{-1} \otimes y^{-1}) $$

for some $y \in \mathcal{O}((\bar{H})^*)^\times$. Equivalently, $x \ast y^{-1} \in \mathcal{O}(G)^*$ is a grouplike element, so $x \ast y^{-1} = g$ for some $g \in G$. Hence, we have

$$ (g \otimes g) \ast (\mathcal{O}(H) \otimes \mathcal{O}(H))^* \ast (g^{-1} \otimes g^{-1}) = (\mathcal{O}(\bar{H}) \otimes \mathcal{O}(\bar{H}))^* $$

inside $(\mathcal{O}(G) \otimes \mathcal{O}(G))^*$, which is equivalent to $gHg^{-1} = \bar{H}$.

We will denote the conjugacy class of $H$ by $\text{Supp}(J)$, and sometimes write $\bar{H} = \text{Supp}(J)$ by abuse of notation.

The following result was proved for finite groups in [Da2 Theorem 2.4] (see also [GK Lemma 4.4(a)]).

**Theorem 4.2.** Let $J \in Z^2_{\text{inv}}(G)$ be an invariant Hopf 2-cocycle with support $A := \text{Supp}(J)$. Then $A$ is a closed normal commutative subgroup of $G$.

Note that since $A$ is normal, it is well defined as a closed subgroup of $G$, not just up to conjugation.

**Proof.** Since $J$ is invariant, it follows that $(\mathcal{O}(G), J_{21}^{-1} \ast J)$ is a commutative cotriangular Hopf algebra and $(\mathcal{O}(A), J_{21}^{-1} \ast J)$ is a minimal commutative cotriangular Hopf algebra. Hence, $\mathcal{O}(A)$ is isomorphic to a Hopf subalgebra of its finite dual Hopf algebra $\mathcal{O}(A)^*_{\text{fin}}$ (see Subsection 2.3). Since $\mathcal{O}(A)^*_{\text{fin}}$ is cocommutative, it follows that $\mathcal{O}(A)$ is cocommutative, which is equivalent to $A$ being commutative.
Take arbitrary \( g \in G \). By Lemma 2.4, \( J^g \) is also invariant, thus \((O(G), (J^g)^{-1} * J^g)\) is a commutative cotriangular Hopf algebra too. Now since \( J \) is invariant, we have

\[
\begin{align*}
(J^g)^{-1} * J^g &= ((g \otimes g) * J_{21}^{-1} * (g^{-1} \circ m)) * ((g \circ m) * J * (g^{-1} \otimes g^{-1})) \\
&= (g \otimes g) * J_{21}^{-1} * J * (g^{-1} \otimes g^{-1}) \\
&= J_{21}^{-1} * J,
\end{align*}
\]

which implies that \( gAg^{-1} = A \), as desired. \( \square \)

**Corollary 4.3.** The connected component of the identity of the support of an invariant Hopf 2-cocycle \( J \) for a reductive algebraic group \( G \) over \( k \) is a torus. In particular, if \( G \) is semisimple then the support of \( J \) is a finite group. \( \square \)

5. **The set \( \mathcal{B}(G) \)**

Let \( G \) be an affine algebraic group over \( k \). Following [Da2] (see also [GK]), we let \( \mathcal{B}(G) \) denote the set of pairs \((A, R)\) such that \( A \) is a commutative closed normal subgroup of \( G \), \((O(A), R)\) is a minimal cotriangular Hopf algebra with trivial Drinfeld element, and \( R \) is \( G \)-invariant (see Subsection 2.3). We set \( e := (\{1\}, \varepsilon \otimes \varepsilon) \).

By Theorem 4.2, every invariant Hopf 2-cocycle \( J \) for \( G \) gives rise to an element \((\text{Supp}(J), R^J) \in \mathcal{B}(G)\), where \( R^J := J_{21}^{-1} * J \).

Set \( \text{Int}(G) := \text{CoInt}(O(G)) \), and observe that since \( G \) is the group of grouplike elements of \( O(G)^* \), we have \( \text{CoInn}(O(G)) = \text{Inn}(G) \). Recall that by Proposition 2.5, \( \text{Int}(G) / \text{Inn}(G) \) is a subgroup of \( H^2_{\text{inv}}(G) \).

Following [Da2] Section 6 (see also [GK] Lemma 4.4(b) & Theorem 4.5), we have the following result.

**Theorem 5.1.** The following hold:

1. The assignment

\[
\Theta : H^2_{\text{inv}}(G) \to \mathcal{B}(G), \quad [J] \mapsto (\text{Supp}(J), R^J),
\]

is a well-defined map of sets.

2. The fibers of \( \Theta \) are the left cosets of \( \text{Int}(G) / \text{Inn}(G) \) inside \( H^2_{\text{inv}}(G) \).

In particular, \( \Theta \) is injective if and only if \( \text{Int}(G) = \text{Inn}(G) \).

3. \( \Theta^{-1}(e) \cong \text{Int}(G) / \text{Inn}(G) \) as groups.

**Proof.** (1) If \([J] = [\bar{J}]\) then by Theorem 4.1 and Theorem 4.2, we have \( \text{Supp}(J) = \text{Supp}(\bar{J}) \) and \( R^J = R^{\bar{J}} \). Thus, \( \Theta([J]) \) is independent of the choice of a representative from \([J]\).

(2) Suppose \( \Theta([J]) = \Theta([\bar{J}]) \). Then \( J, \bar{J} \) are gauge equivalent as Hopf 2-cocycles for \( G \), supported on the same commutative closed normal
subgroup $A$ of $G$, and $R^J = R^\bar{J}$. Since $J, \bar{J}$ are invariant, the equality between the $R$-matrices implies that $JJ^{-1}$ is a symmetric Hopf 2-cocycle for $G$, hence $\bar{J} = J^x$ for some element $x \in \mathcal{O}(G)^*$ such that $\text{Ad}(x) \in \text{Int}(G)$, which defines a unique element in $\text{Int}(G)/\text{Inn}(G)$.

(3) Follows from (2). □

**Corollary 5.2.** If $G$ is a commutative affine algebraic group then $\Theta$ is bijective and therefore induces a group structure on $\mathcal{B}(G)$.

**Proof.** Follows from Theorems 3.4, 5.1 and the fact $\text{Int}(G) = \text{Inn}(G)$ is trivial. □

Corollary 5.2 implies that for a fixed closed normal commutative subgroup $A$ of $G$, the set of elements $R$ such that $(A, R) \in \mathcal{B}(G)$ is in bijection with the set of non-degenerate elements in $H^2_{\text{inv}}(A)^G$, where $H^2_{\text{inv}}(A)$ is described in Theorem 3.4.

### 6. Unipotent algebraic groups

In this section we will assume that $k$ has characteristic 0.

**Theorem 6.1.** If $U$ is a unipotent algebraic group over $k$ then $\Theta$ is bijective. Moreover, $H^2_{\text{inv}}(U) \cong (\wedge^2 u)^u$ as affine algebraic groups, where $u := \text{Lie}(U)$, so in particular $H^2_{\text{inv}}(U)$ is commutative.

**Proof.** We first show that $\Theta$ is surjective. Let $(A, R) \in \mathcal{B}(U)$. By Theorem 3.4, we have $R = \exp(r)$ for some $r \in (\wedge^2 u)^u$ (as $A$ is commutative unipotent, i.e., a vector group). Let $J := \exp(r/2)$. Then $J$ is invariant since $R$ is, and we have $\Theta([J]) = (A, R)$.

Next we show that $\Theta$ is injective. By Theorem 5.1, we have to show that $\text{Int}(U) = \text{Inn}(U)$.

Let $m$ be the augmentation ideal of $\mathcal{O}(U)^*$. Then $\mathcal{O}(U)^*$ is a complete local ring with unique maximal ideal $m$. Thus, if $\alpha$ is a co-inner automorphism of $U$ then $\alpha$ is given by conjugation by an element $a$ in $1 + m$. Hence the element $\log(a)$ defines a derivation of $u$, given by $d(x) = \log(a, x)$, and it suffices to show that this derivation is inner: $d(x) = [u, x]$ for some $u \in u$. Then we may take $a = \exp(u) \in U$, and get that $\alpha$ is inner.

The derivation $d$ defines a class $[d] \in H^1(u, u)$, and its image in $H^1(u, \mathcal{O}(U)^*)$ is zero (as $d(x) = \log(a, x)$, where $\log(a) \in \mathcal{O}(U)^*$). Thus, it suffices to show that $u$ is a direct summand in $\mathcal{O}(U)^*$ as a $u$-module under the adjoint action. In other words, we need to show that the surjective morphism $b : \mathcal{O}(U) \rightarrow u^*$, given by $\beta \mapsto d\beta(1)$, splits as a morphism of $u$-modules, i.e., there exists a $u$-morphism $\phi : u^* \rightarrow \mathcal{O}(U)$ such that $b \circ \phi = \text{id}$. The morphism $\phi$ can be viewed as an element
of $O(U) \otimes u$, i.e., as a regular function $\phi : U \to u$. Thus it suffices to show that there exists a regular $u$-invariant function $\phi : U \to u$ such that $d\phi(1) = \text{id}$. But in the unipotent case we can simply take $\phi(X) = \log(X)$. The proof that $\Theta$ is injective is complete.

Finally, let us show that the bijection

$$\Theta^{-1} \circ \exp : (\wedge^2 u)^u \to H^2_{\text{inv}}(U), \ r \mapsto [J_r],$$

is a group homomorphism. By Proposition 2.9 $[r, [r, s]] = [s, [r, s]] = 0$ for any $r, s \in (\wedge^2 u)^u$. So we have $J_r \ast J_s = J_{r+s} \exp([r, s]/8)$, i.e., $J_r \ast J_s$ is gauge equivalent to $J_{r+s}$ by the gauge transformation given by the element $\exp(z/8)$, where $z$ is as in Proposition 2.9.

The proof of the theorem is complete. \qed

**Remark 6.2.** Theorem 6.1 is a special case of Theorem 7.8 but its proof is simpler and we decided to give it separately.

**Example 6.3.** Let $U$ be the 3-dimensional Heisenberg group. Then $u$ has a basis $a, b, c$ such that $c$ is central and $[a, b] = c$. It is straightforward to verify that $(\wedge^2 u)^u = \{(\alpha a + \beta b) \wedge c \mid \alpha, \beta \in k\}$, and by Corollary 6.1

$$H^2_{\text{inv}}(U) \cong \{\exp((\alpha a + \beta b) \wedge c) \mid \alpha, \beta \in k\} \cong \mathbb{C}^2_a.$$

7. Connected affine algebraic groups

In this section we will assume that $k$ has characteristic 0.

7.1. The bijectivity of $\Theta$.

**Proposition 7.1.** Let $G$ be a connected affine algebraic group over $k$ with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is a direct summand of $O(G)^*$ as a $\mathfrak{g}$-module (under the adjoint action).

**Proof.** We will show that the surjective morphism $b : O(G) \to \mathfrak{g}^*$, given by $\beta \mapsto d\beta(1)$, splits as a morphism of $\mathfrak{g}$-modules, i.e., there exists a $\mathfrak{g}$-morphism $\phi : \mathfrak{g}^* \to O(G)$ such that $b \circ \phi = \text{id}$. The morphism $\phi$ can be viewed as an element of $O(G) \otimes \mathfrak{g}$, i.e., as a regular function $\phi : G \to \mathfrak{g}$. Thus it suffices to prove the following lemma, communicated to us by Vladimir Popov.

**Lemma 7.2.** There exists a regular $G$-invariant function $\phi : G \to \mathfrak{g}$ such that $d\phi(1) = \text{id}$.

**Proof.** This almost follows from [LPR, Theorem 10.2 and Proposition 10.5]. More precisely, the map $\phi := \gamma_c \circ \theta \circ \gamma_c^{-1}$ in [LPR, 2nd line of p. 961] actually is defined at 1 and moreover is étale at 1 (i.e., its differential is an isomorphism). Indeed, $\theta$ is étale at $[1, 1]$ (because $\epsilon$ is
étale at 1 by [LPR, Lemma 10.3], and \( \tau \) is an isomorphism [LPR, 4.3]), and by [LPR (3.1)], \( \gamma_c \) and \( \gamma_C \) are étale at \([1, 0]\) and \([1, 1]\), respectively.

So by multiplying \( \phi \) by the inverse of \( d\phi(1) \), we can make \( d\phi(1) = \text{id} \). The remaining problem is that \( \phi \) is only a rational map. To make it regular, we use [LPR, Proposition 10.6 and Lemma 10.7] to construct a regular function \( f \) on \( G \) such that \( f(1) = 1 \), and \( f \circ \phi \) is regular. Then \( d(f \circ \phi)(1) = d\phi(1) = \text{id} \), so we may replace \( \phi \) with \( f \circ \phi \). We are done. \( \square \)

This completes the proof of the proposition. \( \square \)

**Proposition 7.3.** Let \( G \) be a connected affine algebraic group over \( k \). Then \( \text{Int}(G) = \text{Inn}(G) \).

**Proof.** Assume first that \( G \) is reductive. Then it follows from the well known description of the automorphism group of \( G \) that an automorphism of \( G \) is inner if and only if it does not permute the irreducible representations of \( G \). Hence, the claim follows in this case.

Consider now the general case. Suppose \( x \in (\mathcal{O}(G)^*)^\times \) is such that \( \alpha := \text{Ad}(x) \) is an automorphism of \( G \). Let \( G_u \) be the unipotent radical of \( G \), let \( G_r := G/G_u \), and let \( \psi : \mathcal{O}(G)^* \to \mathcal{O}(G_r)^* \) be the corresponding algebra surjection. Then \( \psi(x) \) implements an automorphism of \( G_r \), which is inner by the above argument. Hence, multiplying \( x \) by an element of \( G \), we are reduced to the situation when \( \psi(x) = 1 \). Let \( I \) be the kernel of \( \psi \), then \( x \in 1 + I \).

Thus we can define the element \( \log(x) := \sum (-1)^{n-1}(x-1)^n/n \in I \) (as any series \( \sum_n a_n \), where \( a_n \in I^n \), converges in the topology of \( \mathcal{O}(G)^* \)). We have \( [\log(x), y] = d(y) \), where \( d := \log(\alpha) \) is a derivation of \( g := \text{Lie}(G) \). Thus \( \log(x) \) defines a class in \( H^1(g, \mathcal{O}(G)^*) \) (where \( g \) acts on \( \mathcal{O}(G)^* \) by the adjoint action). Therefore, since by Proposition 7.1 \( g \) is a direct summand of \( \mathcal{O}(G)^* \) as a \( g \)-module, the class of \( \log(x) \) in \( H^1(g, \mathcal{g}) \) is zero, i.e., \( [\log(x), y] = [b, y] \) for all \( y \in g \) and some element \( b \in g \). Then it is easy to see that \( \text{Ad}(x)y = \text{Ad}(\exp(b))y \) for all \( y \in g \), hence \( xgx^{-1} = \exp(bg) \exp(b)^{-1} \) for all \( g \in G \). Thus, \( \text{Ad}(x) \) is an inner automorphism, as desired. \( \square \)

**Theorem 7.4.** Let \( G \) be a connected affine algebraic group over \( k \). Then \( \Theta \) is bijective.

**Proof.** By Theorem 5.1 and Proposition 7.3, \( \Theta \) is injective.

We now show that \( \Theta \) is surjective. Let \((A, R) \in \mathcal{B}(G)\). We have \( A = A_r \times A_u \), the product of the reductive and unipotent parts. Since \( \text{Aut}(A_r) \) is discrete, \( G \) acts on \( A_r \) trivially, i.e., \( A_r \) is central in \( G \). Let \( a := \text{Lie}(A) \), \( a_u := \text{Lie}(A_u) \). By Theorem 3.4 we have \( R = R_r * R_u \),
where \( R_u = \exp(s) \) for some \( s \in \mathfrak{a} \wedge \mathfrak{a}_u \). Let \( J_u := \exp(s/2) \), and let \( J_r \) be any Hopf 2-cocycle for \( A_r \) such that \((J_r)_{21}^{-1} \ast J_r = R_r\). Then \( J_r \) is invariant since \( A_r \) is central in \( G \). So setting \( J := J_r \ast J_u \), we see that \( J \) is \( G \)-invariant and \( \Theta([J]) = (A, R) \).

**Remark 7.5.** If \( G \) is a finite group of odd order, it follows from \([\text{Da}2\text{, Lemma 6.1}]\) that \( \Theta \) is surjective. (See also \([\text{GK}\text{, Corollary 4.6}]\).) However, there exist finite groups of even order for which \( \Theta \) is not surjective (see \([\text{GK}\text{, 7.3}]\)).

### 7.2. The Structure of \( H^2_{\text{inv}}(G) \) for Nilpotent \( G \) with a Commutative Reductive Part

Let \( G := G_r \times G_u \) be a (possibly disconnected) nilpotent affine algebraic group over \( k \), with a commutative reductive part. Let \( \mathfrak{g}_u \) be the Lie algebra of \( G_u \), and let \( \mathfrak{z}_u \) be the center of \( \mathfrak{g}_u \). Let \( \mathfrak{g}_r \) be the Lie algebra of \( G_r \). Then \( \mathfrak{g} := \mathfrak{g}_r \oplus \mathfrak{g}_u \) is the Lie algebra of \( G \). Note that \( \mathfrak{g}_r \) is central in \( \mathfrak{g} \).

**Lemma 7.6.** Let \( r \in \mathfrak{g} \wedge \mathfrak{g}_u \) be a \( \mathfrak{g} \)-invariant element. Write \( r = r' + r'' \), where \( r' \in \mathfrak{g}_r \otimes \mathfrak{g}_u \) and \( r'' \in \wedge^2 \mathfrak{g}_u \). Then \( r' \in \mathfrak{g}_r \otimes \mathfrak{z}_u \) and \( r'' \in (\wedge^2 \mathfrak{g}_u)^{\mathfrak{g}_u} \).

**Proof.** Let \( b \in \mathfrak{g}_u \). We have \([\Delta(b), r' + r''] = 0\), \([\Delta(b), r''] \in \wedge^2 \mathfrak{g}_u\), and \([\Delta(b), r'] = [1 \otimes b, r'] \in \mathfrak{g}_r \otimes \mathfrak{g}_u \). Thus \([\Delta(b), r''] = [\Delta(b), r'] = 0\), and it follows that \( r' \in \mathfrak{g}_r \otimes \mathfrak{z}_u \).

**Lemma 7.7.** Let \( G := G_r \times G_u \) be a nilpotent affine algebraic group with a commutative reductive part as above. The following hold:

1. \( \text{Int}(G) = \text{Inn}(G) \), so \( \Theta : H^2_{\text{inv}}(G) \to \mathcal{B}(G) \) is injective.
2. The group \( H^2_{\text{inv}}(G) \) is commutative and naturally isomorphic to \( \text{Hom}(\wedge^2 \widehat{G}_r, k^\times) \times (\mathfrak{g}_r \otimes \mathfrak{z}_u) \times (\wedge^2 \mathfrak{g}_u)^{\mathfrak{g}_u} \).

**Proof.** (1) Every \( \gamma \in \text{Int}(G) \) acts trivially on \( G_r \) (since \( G_r \) is central in \( G \)), and defines an internal automorphism of \( G_u \) by restriction. Hence, by Theorem 6.1, \( \gamma \) is inner.

(2) We have a natural group homomorphism

\[ \psi : \text{Hom}(\wedge^2 \widehat{G}_r, k^\times) \times (\mathfrak{g}_r \otimes \mathfrak{z}_u) \times (\wedge^2 \mathfrak{g}_u)^{\mathfrak{g}_u} \to H^2_{\text{inv}}(G), \]

given by \( \psi(K, r', r'') = [K \ast \exp(r/2)] \), where \( r := r' + r'' \) and \( K \) is viewed as an element in \( H^2_{\text{inv}}(G_r) = \text{Hom}(\wedge^2 \widehat{G}_r, k^\times) \). Thus, we have \((\Theta \circ \psi)(K, r', r'') = (A, R)\), where \( A \) is the corresponding commutative normal subgroup to \( R := K_{21}^{-1} \ast K \ast \exp(r) \). It is clear that \( \Theta \circ \psi \) is injective, hence \( \psi \) is injective as well.

Now let us show that \( \psi \) is surjective. By Part (1), for this it suffices to prove that \( \Theta \circ \psi \) is surjective. Let \((A, R) \in \mathcal{B}(G)\). Write \( A = A_r \times A_u \), and let \( \mathfrak{a}_r, \mathfrak{a}_u \) be the Lie algebras of \( A_r, A_u \). Then Theorem 3.4 implies that \( R = R_r \ast R_u \), where \( R_r = K_{21}^{-1} \ast K \) and \( R_u = \exp(r) \) for some
r \in \mathfrak{a}_r \otimes \mathfrak{a}_u \oplus \wedge^2 \mathfrak{a}_u$. Since $A_r \subset G_r$ is central in $G$, we have that $K$ and $r$ are $G$-invariant. Hence, by Lemma 7.6, $r = r' + r''$, where $r' \in \mathfrak{g}_r \otimes \mathfrak{z}_u$ and $r'' \in (\wedge^2 \mathfrak{g}_u)^{\mathfrak{g}_u}$. Thus, $(A, R) = (\Theta \circ \psi)(K, r', r'')$, i.e., $\psi$ is an isomorphism, as desired.

7.3. The structure of $H^2_{\text{inv}}(G)$ for connected $G$. Let $G$ be a connected affine algebraic group over $k$. Let $G_u$ be the unipotent radical of $G$, $\mathfrak{g}_u$ its Lie algebra, and $G_r := G/G_u$. Let $Z$ be the center of $G$, $Z_r$ its reductive part and $Z_u$ its unipotent part (so, $Z = Z_r Z_u$). Let $\mathfrak{z}_r$, $\mathfrak{z}_u$ and $\mathfrak{g}_u$ be the Lie algebras of $Z$, $Z_r$ and $Z_u$, respectively.

**Theorem 7.8.** Let $G$ be a connected affine algebraic group over $k$. Then the group $H^2_{\text{inv}}(G)$ is commutative and is naturally isomorphic to

$$\text{Hom}(\wedge^2 \mathfrak{z}_r, k^\times) \times (\mathfrak{z}_r \otimes \mathfrak{z}_u) \times (\wedge^2 \mathfrak{g}_u)^G$$

**Proof.** We have a natural group homomorphism

$$\psi : \text{Hom}(\wedge^2 \mathfrak{z}_r, k^\times) \times (\mathfrak{z}_r \otimes \mathfrak{z}_u) \times (\wedge^2 \mathfrak{g}_u)^G \to H^2_{\text{inv}}(G),$$

given by $\psi(K, r', r'') = [K \ast \exp(r/2)]$, where $r := r' + r''$ and $K$ is viewed as an element in $H^2_{\text{inv}}(Z_r) = \text{Hom}(\wedge^2 \mathfrak{z}_r, k^\times)$. Thus, we have $(\Theta \circ \psi)(K, r', r'') = (A, R)$, where $A$ is the corresponding commutative normal subgroup to $R := K_2 \ast K \ast \exp(r)$. It is clear that $\Theta \circ \psi$ is injective, hence $\psi$ is injective as well.

Let us now prove that $\psi$ is surjective. By Theorem 7.3, $\Theta$ is bijective, so it suffices to show that $\Theta \circ \psi$ is surjective. Let $(A, R) \in \mathcal{B}(G)$. Write $A = A_r \times A_u$. Then $A_r$, $A_u$, and $A_u$ are normal in $G$. Hence, $A_r$ is central in $G$ (since the group of automorphisms of $A_r$ is discrete, while $G$ is connected). Hence, $A \subset G' := Z_r G_u$, and $(A, R)$ is an element of $\mathcal{B}(G')$. Since $G'$ is nilpotent with a commutative reductive part, it follows from Lemma 7.7 that $(A, R) = (\Theta \circ \psi)(K, r', r'')$, where $K$ is a Hopf 2-cocycle for $Z_r$, $r' \in \mathfrak{z}_r \otimes \mathfrak{g}_u$, and $r'' \in (\wedge^2 \mathfrak{g}_u)^{\mathfrak{g}_u}$. Moreover, $R$ is $G$-invariant, hence so are $r'$, $r''$. Therefore, $r' \in \mathfrak{z}_r \otimes \mathfrak{z}_u$ and $r'' \in (\wedge^2 \mathfrak{g}_u)^G$, i.e., $\psi$ is an isomorphism, as desired.

**Remark 7.9.** If $G$ is a reductive algebraic group over $k$, Theorem 7.8 says that $H^2_{\text{inv}}(G) = H^2(\mathfrak{z}, k^\times) = \text{Hom}(\wedge^2 \mathfrak{z}, k^\times)$, which also follows for $k = \mathbb{C}$ from [NT] Theorem 7. In particular, if $G$ is semisimple then $H^2_{\text{inv}}(G)$ is a finite group, and if the center of $G$ is cyclic then $H^2_{\text{inv}}(G)$ is the trivial group.

**Remark 7.10.** The construction of Hopf 2-cocycles for affine algebraic groups over $k$ of characteristic 0 from solutions to the classical Yang-Baxter equation was introduced in [EG1] (see, e.g., [EG1] Theorem 5.5), following a method of Drinfeld. The map $r \mapsto J_r := \exp(r/2)$
used throughout this paper is a special case of this construction in the case when $r$ spans an abelian Lie algebra.

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