Quantum mechanics and permutation invariants of finite groups

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Abstract. We study quantum behavior from a constructive “finite” point of view, since the introduction of continuum or other actual infinities into physics poses serious conceptual and technical difficulties without any need for these concepts in physics as an empirical science. Taking this approach, we can show that the quantum-mechanical problems can be formulated in the invariant subspaces of permutation representations of finite groups, while the quantum interferences occur as phenomena that are observable in these subspaces. The scalar products in the invariant subspaces (which are needed for formulating the Born rule — the main postulate of quantum mechanics that links mathematical description with experiment) are linear combinations of independent bilinear invariant forms of the permutation representation. A complete set of such forms for any permutation group can be easily calculated by a simple algorithm. Slightly more sophisticated algorithms are required for expressing quantum observables in terms of these forms.

1. Introduction
Quantum behavior is a natural consequence of symmetries of physical systems. In fact, it is a manifestation of the fundamental impossibility to trace the identity of indistinguishable objects in the process of their evolution. The only that can be extracted from any physical observation is information about invariant relations and quantities which describe ensembles of such objects. Thus, the fundamental problem is to study invariants of groups of symmetries of physical systems. Invariant bilinear forms, i.e., scalar products are particularly important for quantum physics. These forms in accordance with the Born rule provide the link between mathematical description and experimental observations.

Physics, as an empirical science, is insensitive to whether finite or infinite concepts are used in formulating its laws. Moreover, effective modeling is possible only when the problem is formulated in finite terms. From the theory of quantum computing it is known that any unitary operator can be presented with arbitrary precision by a finite combination of matrix operators from a finite set of universal quantum gates. Such universal sets of gates generate an infinite group $G_\infty$ which is everywhere dense in the full unitary group. By the theorem of Mal’cev [1], the group $G_\infty$ — as being a finitely generated matrix group — is residually finite [2], i.e., there is a rich set of nontrivial homomorphisms from $G_\infty$ to finite groups. Combining these facts, we can replace the full unitary group by some finite group $G$ in any particular problem without distorting its physical content.
It is well known that any linear representation of a finite group is unitary. Any unitary representation of a finite group is a subrepresentation of some permutation representation. Let $U$ be a representation of the group $G$ in a $K$-dimensional Hilbert space $\mathcal{H}_K$. Then $U$ can be embedded into a permutation representation $P$ of $G$ in an $N$-dimensional Hilbert space $\mathcal{H}_N$, where $N \geq K$. The representation $P$ is equivalent to an action of $G$ on a set of things $\Omega = \{\omega_1, \ldots, \omega_N\}$ by permutations. In the proper case, i.e., when $N > K$, the embedding has the structure

$$T^{-1}PT = \begin{pmatrix} 1 & V \\ \mathcal{H}_{N-K} & U \end{pmatrix} \mathcal{H}_K, \quad \mathcal{H}_N = \mathcal{H}_{N-K} \oplus \mathcal{H}_K,$$

where $1$ is the trivial one-dimensional representation, mandatory for any permutation representation; $V$ is a complementary subrepresentation which may or may not be. $T$ is a matrix of transition from the basis of the permutation representation $P$ to the basis in which the permutation space $\mathcal{H}_N$ is split into the invariant subspaces $\mathcal{H}_{N-K}$ and $\mathcal{H}_K$. For brevity, we will refer to this basis as “quantum basis”. The data in the spaces $\mathcal{H}_K$ and $\mathcal{H}_{N-K}$ are independent since both spaces are invariant subspaces of the space $\mathcal{H}_N$. So we can consider the data in $\mathcal{H}_{N-K}$ as “hidden parameters” with respect to the data in $\mathcal{H}_K$.

A trivial approach would be to set arbitrary (e.g., zero) data in the complementary subspace $\mathcal{H}_{N-K}$. This approach is not interesting since it is not falsifiable by means of the standard quantum mechanics. In fact, it leads to the standard quantum mechanics modulo the empirically unobservable distinction between the “finite” and the “infinite”. The only difference is technical: we can replace the linear algebra in the $K$-dimensional space $\mathcal{H}_K$ by permutations of $N$ things.

A more promising approach requires some changes in the concept of quantum amplitudes. We assume [3, 4] that they are projections onto invariant subspaces of the vectors of multiplicities (“occupation numbers”) of elements of the set $\Omega$ on which the group $G$ acts by permutations. The vectors of multiplicities

$$|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix}$$

are elements of the module $\mathcal{H}_N = \mathbb{N}^N$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the semiring of natural numbers. We start with the natural permutation representation of $G$ in the module $\mathcal{H}_N$. In order to turn the module $\mathcal{H}_N$ into the Hilbert space $\mathcal{H}_N$, it is sufficient just to add roots of unity — an algebraic incarnation of periodicity — to the natural numbers. Linear combinations of $P$th roots of unity with natural coefficients form a semiring $\mathbb{N}_P$. The period $P$ (it is called also conductor) depends on the structure of the group $G$. Generally, it is a multiple of some divisor of the exponent of $G$, which is defined as the least common multiple of the orders of elements of $G$. We will always assume that $P \geq 2$. In this case, using the standard identities for the roots of unity, we can introduce the negative numbers, and thus, $\mathbb{N}_P$ becomes a ring of cyclotomic integers. To complete the conversion of the module $\mathcal{H}_N$ into the Hilbert space $\mathcal{H}_N$, we introduce the cyclotomic field $\mathbb{Q}_P$ as a field of fractions of the ring $\mathbb{N}_P$. If $P \geq 3$, then $\mathbb{Q}_P$ is a dense subfield of the field of complex numbers $\mathbb{C}$. In particular, this gives a trivial explanation of why the complex numbers are so important in quantum mechanics. In fact, the algebraic properties of elements of $\mathbb{Q}_P$ are quite sufficient for all our purposes, so we can forget the possibility to embed $\mathbb{Q}_P$ into $\mathbb{C}$.

The connection between mathematical description and observation is provided by the Born rule: the probability to register a particle described by the amplitude $|\psi\rangle$, by an apparatus tuned

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1 An additional advantage of the transition to finite groups is a quite natural explanation of the “unitarity” of quantum mechanics: it follows from this trivial property of finite groups.
to the amplitude $|\phi\rangle$, is

$$P(\phi, \psi) = \frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle}.$$  

In the “finite” background the only reasonable interpretation of probability is the frequency interpretation: the probability is the ratio of the number of “favorable” combinations to the total number of combinations. So we expect that $P(\phi, \psi)$ must be a rational number if everything is arranged correctly.

To calculate via the Born rule quantum interferences of projections of natural amplitudes, it is sufficient to know the expressions for scalar products in invariant subspaces of the permutation representation. These expressions can be obtained by straightforward computation using the matrix of transition from the “permutation” to “quantum” basis. Unfortunately, there is no satisfactory algorithm for splitting a module over an associative algebra into irreducible submodules — and, hence, for computing the corresponding transition matrix — over important to us number systems of zero characteristic. However, we could try to cope with these difficulties if we could get the expressions for the scalar products directly, i.e., without the use of transition matrices. We describe here one of the possible approaches. This approach can be used for splitting invariant forms of representations for which one can easily calculate the centralizer ring. Permutation representations belong just to this type of representations.

2. Basis of permutation invariant forms

For a start we construct the set of all invariant bilinear forms in the permutation basis. Suppose that the group $G \equiv G(\Omega)$, that acts on the set $\Omega = \{\omega_1, \ldots, \omega_N\}$, is generated by $K$ elements $g_1, \ldots, g_k, \ldots, g_K$. Clearly, it suffices to verify invariance of the form with respect to these elements. For convenience, the set $\Omega$ is identified canonically with the set of indices of its elements: $\Omega = \mathbb{N} = \{1, \ldots, N\}$. Denote matrices of permutation representation of the generators by symbols $P_k = P(g_k)$. Obviously, $P_k^T = P(g_k^{-1})$. The condition of invariance of the bilinear form $A = (a_{ij})$ under the group $G$ can be written as a system of matrix equations

$$A = P_k A P_k^T, \quad 1 \leq k \leq K.$$  

It is easy to check that in terms of components these equations have the form

$$a_{ij} = a_{i g_k j g_k}.$$  

(1)

Thus, the basis of all invariant bilinear forms is in one-to-one correspondence with the set of orbits $\Delta_1, \ldots, \Delta_R$ of the action of $G$ on the Cartesian product $\Omega \times \Omega$. Orbits of the group on the product $\Omega \times \Omega$, called orbitals, play an important role in the theory of permutation groups and their representations [6, 7]. If the group is transitive, then there is a single orbital, which consists of all pairs of the form $(i, i)$. Such an orbital is called trivial or diagonal. The number of orbitals $R$ is called the rank of the group $G$. Each orbital $\Delta_r$ can be associated with

(i) a directed graph, whose vertices are the elements of $\Omega$ and edges are pairs $(i, j) \in \Delta_r$;

(ii) an $N \times N$ matrix $A(\Delta_r)$, which is constructed according to the rule

$$A(\Delta_r)_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \Delta_r, \\ 0, & \text{if } (i, j) \notin \Delta_r. \end{cases}$$  

Properties of the graphs of orbitals reflect important properties of the groups. For example, a transitive permutation group is primitive if and only if the graphs of all its non-trivial orbitals are connected.

2 The most practical and popular algorithm called MeatAxe [5] is designed only for algebras over finite fields.
Matrices of orbitals form a basis of the **centralizer ring** of permutation representation of $G$. This ring plays an important role in the theory of group representations. We will denote it by $ZR \equiv ZR (G(\Omega))$. The multiplication table of basis elements of the ring $ZR$ has the form

$$A(\Delta_p) A(\Delta_q) = \sum_{r=1}^{R} a^p_q r A(\Delta_r),$$

where all the structure constants $a^p_q r$ are natural numbers.

Note that the centralizer ring can be obtained via slightly different approach [8] which is originated from the works of Schur. The approach is based on the study of orbits of stabilizers of points in the set $\Omega$. Recall that the stabilizer of the element $i \in \Omega$ is the maximal subgroup $G_i$ of the group $G$ that fixes this element, i.e., $g \in G_i \iff ig = i$. In the general case, the stabilizer $G_i$ of a transitive group $G$ acts on the set $\Omega$ intransitively, generating a set of orbits $\Lambda_1 = \{i\}, \Lambda_2, \ldots, \Lambda_R$. There is a trivial one-to-one correspondence between the orbits of the stabilizer (sometimes they are called suborbits) and the orbitals (see, e.g., [6, 7]). Therefore the number $R$ of orbits of the stabilizer $G_i$ coincides with the rank of the permutation group $G(\Omega)$. Via the orbits of stabilizers the basis matrices of the centralizer ring are constructed as follows. We consider the orbits of the stabilizer $G_1$. Each orbit $\Lambda$ of the stabilizer $G_1$ is associated with the matrix $A(\Lambda)$ by the rule

$$A(\Lambda)_{ij} = \begin{cases} 1, & \text{if there exist } g \in G \text{ and } l \in \Lambda \text{ such that } 1g = i \text{ and } lg = j, \\ 0, & \text{otherwise.} \end{cases}$$

Technical advantage of this approach is the use of the isomorphism between the orbits of stabilizers of different points: initially, we construct the first row of the matrix using the orbits of the stabilizer of the point 1, and then, the remaining rows are created via the group translations.

We will call the matrices of orbitals $A_r \equiv A(\Delta_r)$ **basic forms**, since any permutation invariant bilinear form can be presented by their linear combination.

Algorithm for computing the basic forms is reduced to construction of orbitals in accordance with (1). In a few words, it scans the elements of the set $\Omega \times \Omega$ in some, say, lexicographic, order and distributes these elements over the equivalence classes. The output of the algorithm is a complete basis $A_1, \ldots, A_R$ of permutation invariant bilinear forms. The algorithm is quite simple. Our implementation is just a few lines in C.

The following identity follows directly from the construction

$$A_1 + A_2 + \cdots + A_R = L^T L = J_N \equiv \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $L$ is a covector of the form $(1, 1, \ldots, 1)$, and $J_N$ is an $N \times N$ “matrix of ones”.

Let us illustrate the algorithm with the example of the group $\mathbb{Z}_3$ acting on the set $\Omega = \{1, 2, 3\}$. The group is generated by a single element, for example, $g_1 = (1, 2, 3)$. We need to distribute the set of pairs of indices

$$\begin{align*} (1, 1) & \quad (1, 2) \quad (1, 3) \\ (2, 1) & \quad (2, 2) \quad (2, 3) \\ (3, 1) & \quad (3, 2) \quad (3, 3) \end{align*}$$

\[ (2) \]
over the equivalence classes in accordance with (1). If we start from the top left corner of table (2) and will look for untreated pairs in the lexicographic order, the orbitals will be constructed in the following order:

$$\Delta_1 = \{(1,1), (2,2), (3,3)\}, \Delta_2 = \{(1,2), (2,3), (3,1)\}, \Delta_3 = \{(1,3), (2,1), (3,2)\}.$$ 

The corresponding complete basis of invariant forms reads

$$A_1 = \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad A_3 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.$$ 

Note that in general, if we start the algorithm with the pair $(1,1)$ and $G$ is transitive, then the first basis form will always be the identity matrix $A_1 = I_N$, corresponding to the trivial orbital.

3. Permutation invariant forms and decomposition into irreducible components

Despite the ease of obtaining the permutation invariant forms, they are sufficiently informative. Consider the decomposition of permutation representation into irreducible components using the transformation matrix $T$. This decomposition for transitive groups has the form

$$T^{-1}P(g)T = \begin{pmatrix} I_{m_1} \otimes U_1(g) \\ \vdots \\ I_{m_k} \otimes U_k(g) \\ \vdots \\ I_{m_{N_{\text{Irr}}} \otimes U_{N_{\text{Irr}}}(g)} \end{pmatrix}, \quad g \in G.$$ 

Here $N_{\text{Irr}}$ is the total number of different irreducible representations $U_k$ ($U_1 \equiv 1$) of the group $G$, which are contained in the permutation representation $P$; $m_k$ is the multiplicity of the subrepresentations $U_k$ in the representation $P$; $\otimes$ denotes the Kronecker product of matrices; $I_m$ is an $m \times m$ identity matrix.

The most general permutation invariant form is a linear combination of the basic forms

$$A = a_1A_1 + a_2A_2 + \cdots + a_R A_R,$$ 

where the coefficients $a_i$ are elements of some abelian number field\(^3\) $\mathcal{F}$ which is defined concretely in the computations to be described below.

It is easy to show (see [6, 8]) that in a basis, which splits the permutation representation into irreducible components, the form in (3) takes the form

$$T^{-1}AT = \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \otimes I_{d_2} \\ \vdots \\ \mathcal{B}_k \otimes I_{d_k} \\ \vdots \\ \mathcal{B}_{N_{\text{Irr}}} \otimes I_{d_{N_{\text{Irr}}}} \end{pmatrix}.$$ 

\(^3\) An abelian number field is an extension of the field $\mathbb{Q}$ with abelian Galois group. Due to the Kronecker–Weber theorem any such field is a subfield of some cyclotomic field $\mathbb{Q}_p$. 

Here $B_k$ is an $m_k \times m_k$ matrix, whose elements are linear combinations of the coefficients $a_i$ from (3), while $m_k$ is a multiplicity of the irreducible component $U_k$; $d_k$ is the dimension of $U_k$. The structure of matrix (4) implies that the rank $R$ of the group (i.e., the dimension of the centralizer ring) is equal to the sum of the squares of the multiplicities: $R = 1 + m_2^2 + \cdots + m_{N_{\text{irr}}}^2$.

Let us now consider the determinant $\det(A)$. In terms of the variables $a_1, \ldots, a_R$ this determinant is a homogeneous polynomial of degree $N$. Since the determinant of a form does not depend on the choice of basis, and the determinant of a block diagonal matrix is the product of determinants of its blocks, from decomposition (4) it follows

$$\det(A) = \det(B_1) \cdots \det(B_{N_{\text{irr}}}).$$

Here we have used the identity $\det(X \otimes Y) = (\det(X))^m (\det(Y))^n$ for the Kronecker product of an $n \times n$ matrix $X$ and an $m \times m$ matrix $Y$.

It is clear that $\det(B_k)$ is a homogeneous polynomial of degree $m_k$ in the variables $a_1, \ldots, a_R$: $\det(B_k) = E_k(a_1, \ldots, a_R)$. Thus, we have the following

**Proposition.** The determinant of linear combination (3) has the following decomposition into factors over a certain ring of cyclotomic integers:

$$\det \sum_{i=1}^{R} a_i A_i = \prod_{k=1}^{N_{\text{irr}}} E_k(a_1, \ldots, a_R)^{d_k}, \quad \deg E_k(a_1, \ldots, a_R) = m_k,$$

where $E_k$ is an irreducible polynomial corresponding to the irreducible component $U_k$ in the representation $P$. Recall that $N_{\text{irr}}$ is the number of different irreducible components in $P$; $d_k$ is dimension of $U_k$; $m_k$ is the multiplicity of $U_k$ in $P$.

From this proposition the idea of an algorithm to compute the invariant forms in irreducible subspaces of the permutation representation follows.

We must first calculate the polynomial $\det(A)$. This is a relatively simple task. In particular, algorithms based on the Gaussian elimination have a cubic complexity in the size of the matrix.

Then, the polynomial $\det(A)$ must be decomposed into a maximum number of irreducible factors. Note that advanced algorithms of polynomial factorization automatically determine an algebraic extension of rationals that guarantees maximal factorization. Such algorithms are called algorithms of “absolute factorization”. There are many practical implementations of polynomial factorization algorithms with different estimates of the complexity.\footnote{To improve efficiency we can try to write a specialized algorithm that takes into account the fact that in our case the polynomials are factored over the rings of cyclotomic integers, whose conductors are divisors of group exponents. However, the benefits of such work require a separate study.}

Thus, constructing decomposition (5) is an algorithmically realizable task. Solving it, we obtain the complete information about the dimensions and multiplicities of all irreducible subrepresentations.

The next natural step is to try to compute explicitly invariant forms $B_k$ in the irreducible subspaces of the permutation representation. To do this, we exclude from consideration the factor $E_k(a_1, \ldots, a_R)$, related to the component $B_k$, and equate to zero the other factors. That is, we write the system of equations

$$E_1 = \cdots = \hat{E}_k = \cdots = E_{N_{\text{irr}}} = 0.$$  

3.1. Example with irreducible components of multiplicity one.

If all the multiplicities $m_i = 1$ (under this condition $R = N_{\text{irr}}$ and the centralizer ring $ZR$ is commutative), then all the polynomials $E_i$ are linear. In this case, the computation of the scalar products in the invariant subspaces can easily be completed.
As an example, consider the group \( \text{SL}(2, 3) \) defined as a group of special linear transformations of two-dimensional space over the field of three elements \( \mathbb{F}_3 \). This group is used in particle physics where it is often denoted by \( T' \) or \( 2T \) since it is a double cover of the symmetry group \( T \cong A_4 \) of a tetrahedron. We will consider its faithful permutation action of degree 8, which can be generated, for example, by the following two permutations

\[
g_1 = (1, 5, 3, 2, 6, 4)(7, 8) \quad \text{and} \quad g_2 = (1, 3, 7, 2, 4, 8)(5, 6).\]

The 8-dimensional permutation representation will be denoted by \( 8 \).

The following four matrices — obtained via constructing orbitals — form a basis of the ring \( \mathbb{Z}R \equiv \mathbb{Z}R \left( \text{SL}(2, 3) \left( \mathbb{F}_3 \right) \right) \) of permutation invariant forms:

\[
A_1 = \begin{pmatrix}
1 & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & 1
\end{pmatrix},
A_2 = \begin{pmatrix}
1 & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & 1
\end{pmatrix},
A_3 = \begin{pmatrix}
. & 1 & 1 & 1 & 1 & . & . & . \\
. & 1 & 1 & 1 & 1 & . & . & . \\
. & 1 & 1 & 1 & 1 & . & . & . \\
. & 1 & 1 & 1 & 1 & . & . & . \\
1 & . & 1 & 1 & 1 & . & . & . \\
1 & 1 & . & 1 & 1 & . & . & . \\
1 & 1 & 1 & . & 1 & . & . & . \\
1 & 1 & 1 & 1 & . & . & . & .
\end{pmatrix},
A_4 = \begin{pmatrix}
. & 1 & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & 1 & 1 & 1 & . & . \\
. & 1 & 1 & 1 & 1 & 1 & . & . \\
1 & . & 1 & 1 & 1 & 1 & . & . \\
1 & 1 & . & 1 & 1 & 1 & . & . \\
1 & 1 & 1 & . & 1 & 1 & . & . \\
1 & 1 & 1 & 1 & . & 1 & . & .
\end{pmatrix}.
\]

The determinant of their linear combination \( A = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 \) is decomposed into linear factors over the ring of cyclotomic integers \( \mathbb{Z}_3 \):

\[
\det A = (a_1 + a_2 + 3a_3 + 3a_4) \\
\{ a_1 - a_2 + (1 + 2r) a_3 - (1 + 2r) a_4 \}^2 \\
\{ a_1 - a_2 - (1 + 2r) a_3 + (1 + 2r) a_4 \}^2 \\
(a_1 + a_2 - a_3 - a_4)^3,
\]

where \( r \) is a third primitive root of unity. From this formula the structure of decomposition of the representation \( 8 \) into irreducible components is seen immediately:

\[
8 = 1 \oplus 2 \oplus 2' \oplus 3.
\]

Excluding sequentially the linear factors corresponding to the representations \( k = 1, 2, 2', 3 \), and equating to zero the other factors, we obtain four systems of three linear equations.

Consider for example the subrepresentation \( 2 \). For this component, the system of equations (6) takes the form

\[
a_1 + a_2 + 3a_3 + 3a_4 = 0, \quad (7) \\
a_1 - a_2 - (1 + 2r) a_3 + (1 + 2r) a_4 = 0, \quad (8) \\
a_1 + a_2 - a_3 - a_4 = 0. \quad (9)
\]
The linear systems of this type consist of $R - 1$ equations, but contain $R$ variables. Since the bilinear form (3) describes a non-degenerate scalar product, the coefficient $a_1$ at the diagonal basis form cannot vanish. Thus, one can treat $a_1$ as a parameter and solve the system of equations for the remaining variables. In principle, the coefficient $a_1$ can be taken to be an arbitrary (nonzero) parameter, since Born’s probability is independent of its value. However, it is reasonable to choose the value

$$a_1 = \frac{d_k}{N}$$  \hspace{1cm} (10)$$

for each irreducible component. Under such a normalization the sum of scalar products in invariant subspaces will be equal to the standard scalar product in the space of permutation representation.

Solving linear system (7—9) and setting $a_1 = 2/8 = 1/4$, we obtain

$$a_2 = -\frac{1}{4}, \quad a_3 = -\frac{1 + 2r}{12}, \quad a_4 = \frac{1 + 2r}{12}.$$  

Applying the same procedure to all other irreducible components we come to the following set of forms defining scalar products in the invariant subspaces

$$B_1 = \frac{1}{8} (A_1 + A_2 + A_3 + A_4) \equiv \frac{1}{8} J_8,$$

$$B_2 = \frac{1}{4} \left( A_1 - A_2 - \frac{1 + 2r}{3} A_3 + \frac{1 + 2r}{3} A_4 \right),$$

$$B_{2'} = \frac{1}{4} \left( A_1 - A_2 + \frac{1 + 2r}{3} A_3 - \frac{1 + 2r}{3} A_4 \right),$$

$$B_3 = \frac{3}{8} \left( A_1 + A_2 - \frac{1}{3} A_3 - \frac{1}{3} A_4 \right).$$

It is easy to check that normalization (10) ensures the identity $B_1 + B_2 + B_{2'} + B_3 = A_1 \equiv I_8$. Recall that in all such tasks $B_1$ does not require calculation, since the inner product in the subspace of the trivial representation always has the form $\frac{1}{N} J_N$.

Let $|n\rangle = (n_1, \ldots, n_8)^T$ and $|m\rangle = (m_1, \ldots, m_8)^T$ be natural vectors in the permutation basis, and $|\Psi_k\rangle$ and $|\Phi_k\rangle$ their projections onto the invariant subspaces, where $k = 1, 2, 2', 3$. The scalar products of these projections in the invariant subspaces can be expressed in terms of the natural vectors as

$$\langle \Phi_k | \Psi_k \rangle = \langle m | B_k | n \rangle.$$  

For example, for the subrepresentation 1 we have

$$\langle \Phi_1 | \Psi_1 \rangle = \frac{1}{8} (m_1 + m_2 + \cdots + m_8) (n_1 + n_2 + \cdots + n_8).$$

It is clear that this expression never vanishes for natural vectors $|n\rangle$ and $|m\rangle$. In fact, in the general case the trivial one-dimensional subrepresentation contained in any permutation representation can be interpreted as the “counter of particles”, since the permutation invariant $\sum_{i=1}^N n_i$ corresponding to this subrepresentation is the total number of elements of the set $\Omega$ in the ensemble.

As for the other subrepresentations, we can observe non-trivial “quantum behavior” in their invariant subspaces. In particular, the equations $\langle m | B_k | n \rangle = 0$ — the conditions for destructive interference — have infinitely many solutions in natural vectors for $k = 2, 2', 3$.

Note also, that Born’s probabilities computed via the forms $B_2$ and $B_{2'}$ take irrational values for natural vectors. This contradicts the idea that any probability in the finite background must
be rational. The contradiction is resolved by the fact that the permutation action, that we consider, is imprimitive, i.e., it moves some nontrivial subsets of $\Omega$ as single units. Such subsets are called blocks. In our example, the block system (called also the system of imprimitivity) is the following $\{(1, 2), (3, 4), (5, 6), (7, 8)\}$. Thus, we can not take the subrepresentations $2$ and $2'$ separately and must consider their sum $2 \oplus 2'$ instead. The invariant form for this sum $B_{2 \oplus 2'} = \frac{1}{2} (A_1 - A_2)$ does not contain irrationals.

3.2. The case of multiple subrepresentations
In the case of multiple subrepresentations the situation becomes more complicated because of the nonlinearity of relations. The question of whether it is possible to develop a general algorithm for the case of multiple subrepresentations requires a deeper additional study. In many concrete examples, using analogy to the case of single subrepresentations and some ad hoc tricks, it is possible to carry out the complete decomposition of scalar products into irreducible components.

The source of nonlinearity is an excess in the number of parameters, the definite values of which are inessential for computing the Born probabilities. As one can see from the structure of decomposition (4), any block of multiple components $B_k \otimes I_d$ contains $m_k \times m_k$ such parameters. Our convention (10) allows to fix $m_k$ diagonal elements of the matrix $B_k$. So we need to fix somehow the remaining $m_k^2 - m_k$ parameters. In some cases, the examination of the structure of the centralizer ring helps to do this.

As an illustrative example, let us consider the Coxeter group $A_2$, which is also the Weyl group of the Lie group, say, $SU(3)$. We will consider the natural action of $A_2$ on its root system $\Omega = \{1, 2, \cdots , 6\}$. The vectors of this root system form a two-dimensional regular hexagon. The generators of the action are, for example, $g_1 = (1, 4)(2, 3)(5, 6)$ and $g_2 = (1, 3)(2, 5)(4, 6)$.

Computation of orbitals gives the following basis of the centralizer ring

\[
A_1 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & 1 & \cdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \cdots & 1 & \cdots \\ \cdots & 1 & \cdots \\ 1 & \cdots & 1 \end{pmatrix},
\]

\[
A_4 = \begin{pmatrix} \cdots & 1 \\ \cdots & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} \cdots & 1 \\ \cdots & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} \cdots & 1 \\ \cdots & 1 \end{pmatrix}.
\]

Applying the algorithm of absolute factorization to the generic linear combination

\[
A = a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 + a_5 A_5 + a_6 A_6,
\]

we obtain the following decomposition:

\[
\det A = (a_1 + a_2 - a_3 - a_4 - a_5 + a_6) \\
(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) \\
\{a_1^2 + a_2^2 - a_3^2 - a_4^2 - a_5^2 + a_6^2 - a_1 a_2 - a_1 a_6 - a_2 a_6 + a_3 a_4 + a_3 a_5 + a_4 a_5\}^2.
\]

The structure of the action of the Weyl group $A_2$ on its roots in terms of the permutation representation follows immediately from (11): $\overline{6} = 1 \oplus 1' \oplus 2 \oplus 2$. Here we have four irreducible
components and a six-dimensional centralizer ring. Thus, we need to eliminate the two extra
degrees of freedom. Basis matrices derived from the orbitals are \((0,1)\)-matrices with disjoint sets
of unit entries. A result of addition of such matrices belongs to the same type of matrices. So,
the natural idea is to sum up some of the matrices \(A_i\) in order to reduce their total number.
Recall that if the centralizer ring is commutative, then all the multiplicities are equal to one.
So we retain the commutative subset of matrices unchanged. In our case, such a subset consists
of the matrices \(A_1, A_2\) and \(A_6\). “Killing noncommutativity” by replacing the remaining non-
commuting matrices by their sum \(A_\Sigma = A_3 + A_4 + A_5\), we write the linear combination of four
matrices
\[
A' = a_1 A_1 + a_2 A_2 + a_6 A_6 + a_\Sigma A_\Sigma.
\]
Now the determinant is factorizable to the linear factors over the cyclotomic integer ring \(\mathbb{N}_3\):
\[
\det A' = (a_1 + a_2 + a_6 - 3 a_\Sigma)
\begin{align*}
& (a_1 + a_2 + a_6 + 3 a_\Sigma) \\
& \{ a_1 + r a_2 - (1 + r) a_6 \}^2 \\
& \{ a_1 - (1 + r) a_2 + r a_6 \}^2,
\end{align*}
\]
where \(r\) is a third root of unity. After the same manipulations as above, we come to the following
set of scalar product forms in the invariant subspaces
\[
B_1 = \frac{1}{6} (A_1 + A_2 + A_6 + A_\Sigma) \equiv \frac{1}{6} J_6, \\
B_1' = \frac{1}{6} (A_1 + A_2 + A_6 - A_\Sigma), \\
B_2 = \frac{1}{3} \{ A_1 - (1 + r) A_2 + r A_6 \}, \\
\tilde{B}_2 = \frac{1}{3} \{ A_1 + r A_2 - (1 + r) A_6 \}.
\]
Here \(B_2\) and \(\tilde{B}_2\) are different coordinate presentations of the same form, associated with the
irreducible representation \(2\).

4. Conclusions
Elimination of actual infinities from the description of physical reality removes many technical
difficulties. This allows to focus on the content aspects of physical problems. We have considered
some implications of the idea that quantum mechanics, as any reasonable physical theory, must
allow effective modeling by finite means.

Using mathematical arguments of general nature we can show that any quantum problem
can be reduced to permutations. If we assume also that the entities, which are subject to the
permutations, have a physical meaning, we come to a very simple and self-consistent picture
of the quantum behavior. To study the consequences of this assumption, we need to know the
inner products in the invariant subspaces of permutation representations.

In this paper, we have considered a possible algorithmic approach to calculating these
inner products. With this approach, we can for any permutation representation obtain the
full information about the structure of its decomposition into irreducible components. If
all irreducible components have unit multiplicities then the invariant inner products can be
easily computed. There are many observations which indicate that in the case of multiple
subrepresentations a reasonable algorithm is also possible.
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