Parameter Estimation of Two Classes of Nonlinear Systems with Non-separable Nonlinear Parameterizations

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Abstract: In this paper we address the challenging problem of designing globally convergent estimators for the parameters of nonlinear systems containing a non-separable exponential nonlinearity. This class of terms appears in many practical applications, and none of the existing parameter estimators is able to deal with them in an efficient way. The proposed estimation procedure is illustrated with two modern applications: fuel cells and human musculoskeletal dynamics. The procedure does not assume that the parameters live in known compact sets, that the nonlinearities satisfy some Lipschitzian properties, nor rely on injection of high-gain or the use of complex, computationally demanding methodologies. Instead, we propose to design a classical on-line estimator whose dynamics is described by an ordinary differential equation given in a compact precise form. A further contribution of the paper is the proof that parameter convergence is guaranteed with the extremely weak interval excitation requirement.

Keywords: Nonlinear systems, Observers, Estimation algorithms, Regression estimates, Excitation.

1. INTRODUCTION

To comply with the stringent monitoring and control requirements in modern applications an accurate model of the system is vital. It is well-known that nonlinear parameterizations (NLP) are inevitable in any realistic dynamic model of practical problems with complex dynamics. Constitutive relations and conservation equations used to characterize physical variables always involve NLP. Classical examples are friction dynamics (Armstrong-Hélouvry et al., 1994), biochemical processes (Dochain, 2003) and in more recent technological developments we can mention fuel cells (Pukrushpan et al., 2004), photovoltaic arrays (Masters, 2013), windmill generators (Heier, 2014) and biomechanics (Winter, 2009). However, one of the assumptions that pervades almost all results in adaptive estimation and control is linearity in the unknown parameters and there are very few results available for NLP systems. Quite often, in practical problems, there are only few physical parameters that are uncertain and occur nonlinearly in the underlying dynamic model. In some cases, it is possible to use suitable transformations so as to convert it into a problem where the unknown parameters occur linearly, usually involving overparameterizations. This procedure, however, suffers from serious drawbacks including the enlarging of dimension of the parameter space, with the subsequent increase in the excitation requirements needed to ensure parameter convergence. The reader is referred to (Ortega et al., 2020) for a thorough discussion on the drawbacks of overparameterization.

Some results for gradient estimators have been reported in the literature for convexly parameterized systems. It was first reported in (Fomin et al., 1981) (see also (Ortega, 1995)) that convexity is enough to ensure that the gradient search “goes in the right direction” in a certain region of the estimated parameter space. The idea is then to apply a standard adaptive scheme in this region, while in the “bad” region either the adaptation is frozen and a robust constant parameter controller is switched-on (Fradkov et al., 2001) or, as proposed in (Annaswamy et al., 1998), the adaptation is running all the time and stability is ensured with a high-gain mechanism which is suitably adjusted incorporating prior knowledge on the parameters. In (Netto et al., 2000) reparametrization to convexify an otherwise non-convexly parameterized system is proposed. See also (Tyukin et al., 2003, 2007) for some interesting results along these lines, where the controller and the estimator switch between over/underbounding convex/concave functions. Some calculations invoking computationally demanding set membership principles—similar...
to fuzzy systems—have recently been reported in (Adetola et al., 2014). Using the Immersion and Invariance adaptation laws proposed in (Astolfi et al., 2008), stronger results were obtained in (Liu et al., 2010, 2011) invoking the property of monotonicity, see also (Tyukin et al., 2003, 2007) for related results. The main advantage of using monotonicity, instead of convexity, is that in the former case the parameter search “goes in the right direction” in all regions of the estimated parameter space—this is in contrast to the convexity-based designs where, as pointed out above, this only happens in some regions of this space. See the recent work (Ortega et al., 2022) where these results relying on monotonicity have been significantly extended. The reader is referred to (Ortega et al., 2020, 2022) for recent reviews of the literature on parameter estimation and adaptive control of NLP systems. Unfortunately, the monotonicity property can be exploited only for the case of separable NLP. That is for the case where we can factor the parameter dependent terms as $h_i(u, y, \theta) = h_i(u, y)\psi_i(\theta)$, where $u$ and $y$ are measurable and $\theta_i$ is the unknown parameter. However, there are many practical application models where this factorization is not possible, we refer to this case as non-separable NLP. Two often encountered cases are cos($\theta_i h_i(u, y)$) or $e^{\theta_i h_i(u, y)}$. In particular, the last example appears in many physical processes including Arrenhius laws (Silberberg, 2006), biochemical reactors (Dochain, 2003), friction models (Armstrong-Hélouvy et al., 1994), windmill systems (Bobtsov et al., 2022b), fuel cell systems (Xing et al., 2022), photovoltaic arrays (Bobtsov et al., 2022a) and models of elastic moments (Schauer et al., 2015; Sharma et al., 2012; Yang and de Queiroz, 2010). This paper is devoted to the development of a systematic methodology for the parameter identification of systems containing this kind of exponential terms. More precisely, we consider systems of the form

$$\dot{x} = F_x(u, y, \theta), \quad y = H_x(u, y, \theta)$$

with $u$ and $y$ measurable and $\theta$ a vector of unknown parameters, with some of its elements entering into the functions $F_x$ and/or $H_x$ via exponential terms of the form $e^{\theta_i h_i(u, y)}$. The objective is to design an on-line estimator

$$\hat{x} = F_{\hat{x}}(\chi, u, y), \quad \dot{\theta} = H_{\hat{x}}(\chi, u, y)$$

with $\chi(t) \in \mathbb{R}^n$, such that we ensure global exponential convergence (GEC) of the estimated parameters. That is, for all $x(0) \in \mathbb{R}^n$, $\chi(0) \in \mathbb{R}^n$, and all continuous $u$ that generates a bounded state trajectory $x$ we ensure

$$\lim_{t \to \infty} |\dot{\theta}(t)| = 0, \quad \text{(exp)}$$

where $\hat{\theta} := \theta - \theta$ is the parameter estimation error, with all signals remaining bounded.

Notice that, in contrast with the existing approaches for non-separable NLP systems, we do not assume that the parameters live in known compact sets, that the nonlinearities satisfy some Lipschitzian properties, nor rely on injection of high-gain to dominate the nonlinearities or the use of complex, computationally demanding methodologies like min-max optimizations, parameter projections or set membership techniques. Instead, we propose to design a classical on-line estimator whose dynamics is described by an ordinary differential equation given in a compact precise form.

We identify in the paper two classes of systems for which the problem formulated above can be solved. The design procedure consists of the construction—from the non-separable NLP containing an exponential term—a new NLP regression equation (NLPRE) of the form $Y(u, y) = \phi^\top(u, y)G(\theta)$, where the functions $Y(u, y)$ and $\phi(u, y)$ are known and $G(\theta)$ is a nonlinear mapping. To estimate the parameters $\theta$ from the NLPRE we invoke the recent result of (Ortega et al., 2022), where a least-squares plus dynamic regression equation and mixing (Aranovskiy et al., 2017) (LS+DREM) estimator applicable for this kind of NLPRE is reported. A key feature of the LS+DREM estimator is that it ensures GEC imposing an extremely weak interval excitation (IE) assumption (Kreisselmeier and Rietz-Augst, 1990; Tao, 2003) of the regressor $\phi$. On the other hand, this estimator requires that the mapping of the NLPRE satisfies a rather weak monotonizability property—that is captured by the verifiability of a linear matrix inequality (LMI) imposed on $G(\theta)$. We give two practical examples of the application of the proposed estimation method and illustrate their performance with some simulations.

**Notation.** $I_n$ is the $n \times n$ identity matrix and $0_{s,r}$ is an $s \times r$ matrix of zeros. $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the positive real and integer numbers, respectively. For $q \in \mathbb{Z}_+$ we define the set $q := \{1, 2, \ldots, q\}$. For $a \in \mathbb{R}^n$, we denote $|a|^2 := a^\top a$, and for any matrix $A$ its induced norm is $\|A\|$. All functions and mappings are assumed smooth and all dynamical systems are assumed to be forward complete. Given a function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ we define its transposed gradient via the differential operator $\nabla(h(x, u) := \left[\frac{\partial h}{\partial (x, u)}\right]^\top)$. For a mapping $G : \mathbb{R}^n \to \mathbb{R}^m$ we denote its Jacobian by $\nabla G(\eta) := \left[\frac{\partial G}{\partial \eta}\right]$. To simplify the notation, the arguments of all functions and mappings are written only when they are first defined and are omitted in the sequel.

### 2. FIRST CLASS OF SYSTEMS

In this section we consider NLP systems of the form

$$\begin{align*}
\dot{x} &= f_1(x, u) + f_2(x, u) \nabla G(\eta) \\
y &= \left[\begin{array}{c}
   y_1 \\
   \vdots \\
   y_m
\end{array}\right] = \left[\begin{array}{c}
   h_1(x, u) + h_2(x, u) \theta_2 + h_3(x, u) e^{\lambda_1(y_1)} \\
   \vdots \\
   \vdots
\end{array}\right] \quad \text{(2a)}
\end{align*}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^{n+1}$ and $u(t) \in \mathbb{R}^m$ the systems states, output and control, respectively. The functions $f_1$, $i = 1, 2$, and $h_i$, $i = 1, \ldots, 4$, are known nonlinear functions, $G : \mathbb{R}^n \to \mathbb{R}^m$, $p_k > n_q$, is a known mapping of the unknown parameters $\eta \in \mathbb{R}^{n_q}$, and $\theta_i \in \mathbb{R}$, $i = 1, 2$ are also unknown parameters. Hence, the overall vector of unknown parameters, which needs to be estimated on-line, consists of $\theta := \text{col}(\theta_1, \theta_2, \eta) \in \mathbb{R}^{l_1}$, where $l_1 := 2 + n_q$.

We make the important observation that, in view of the presence of the exponential term in the signal $y_1$, the parameterization of the system is nonlinear and non-separable. As discussed in the Introduction none of the existing parameter estimators can deal with this difficulty—but often encountered in practice—scenario.

#### 2.1 Assumptions

We make the following assumptions on the system.
Lemma 1. Consider the system recently reported in (Ortega et al., 2022; Pyrkin et al., 2022). We could redefine a new vector of unknown parameters \( \eta \) we could redefine a new vector of unknown parameters \( \eta \).

\[ T_G \nabla G(\eta) + [\nabla G(\eta)]^T T_G^T \geq \rho_G I_{n_\eta}, \]

for some \( \rho_G > 0 \).

Discussion on the assumptions D1 In (Ortega et al., 2020, Proposition 1) it is shown that (7) ensures the mapping \( T_G \mathcal{G}(\eta) \) is strictly monotonic (Pavlov et al., 2004).

This is the fundamental property that is required by the LS+DREM estimator used in the next section.

D2 The assumption that the state trajectories of (2) are bounded is standard in parameter estimation theory (Ljung, 1987; Sastry and Bodson, 1989). Similarly, the assumption that the dimension \( n_\eta \) of the unknown parameters vector \( \eta \) is smaller than \( p_\eta \) is reasonable, otherwise we could redefine a new vector of unknown parameters \( \bar{\eta} := \mathcal{G}(\eta) \in \mathbb{R}^{n_\bar{\eta}} \) without overparameterization and get a LRE.

2.2 Regression Equation for Parameter Estimation

In this section derive the regression equation that will be used to estimate the unknown parameters \( \theta \). As expected, this regression equation is nonlinearly parameterized, which hampers the application of standard estimation techniques. Therefore, we are compelled to appeal—

In Section 4—to the LS+DREM parameter estimator recently reported in (Ortega et al., 2022; Pyrkin et al., 2022).

Lemma 1. Consider the system (2) verifying Assumptions A1, A2. There exists measurable, scalar signals \( Y_t(x,u,y) = \phi_t(x,u,y)W_1(\theta), \)

where we defined the mapping \( W_1 : \mathbb{R}^{l_1} \rightarrow \mathbb{R}^{p_1} \).

\[ W_1(\theta) := \left[ \theta_1 \theta_2 \theta_1 \theta_2 \theta_1 \theta_2 \theta_1 \theta_2 \right] \]

Discussion on the regression equation D3 It is possible to construct another NLPRE proceeding as follows. First, exploiting the monotonicity property of Assumption A2 and using the LS+DREM algorithm estimate the parameters \( \eta \) filtering (2a). Then, use this estimate in the (approximate) calculation of \( \hat{h}_4 \), yielding

\[ \hat{h}_4 = \nabla \mathcal{G}(\eta)[f_1 + f_2 \mathcal{G}(\eta)]. \]

Applying the certainty equivalent principle, and this expression in the chain of implications of the proof of Lemma 1 in Appendix A would then yield a simpler NLPRE where only the terms \( \theta_1, \theta_2, \theta_1 \theta_2 \) will appear. Of course, the drawback of this approach is that we rely on the fast convergence of \( \eta \approx \eta \) to zero.

D4 In the system (2) the function \( h_4 \) appearing in the exponential does not depend on \( u \). It is possible to adapt the result of Lemma 1 to consider that case in the following way. The expression for \( \hat{h}_4 \) given in (A.2) would need to be replaced by

\[ \hat{h}_4 = \nabla \mathcal{G}(\eta)[f_1 + f_2 \mathcal{G}(\eta)] + \nabla \mathcal{G}(\eta) u. \]

To construct the NLPRE as in Lemma 1 for this case it is clearly necessary to know \( u \). However, in many practical applications the control law contains an integral action—e.g., in PID control—therefore this signal is available for measurement.

2.3 Construction of a Strictly Monotonic Mapping

To estimate the parameters \( \theta \) from the NLPRE (5) we invoke the recent result of (Ortega et al., 2022), where the LS+DREM estimator proposed in (Pyrkin et al., 2022), which is applicable for linear regression equations, was extended to deal with NLPRE. However, this estimator requires that the mapping of the NLPRE satisfies a monotonicity property, which is not verified by \( W_1(\theta) \) given in (6). Therefore, in this section we construct a new mapping verifying the required monotonicity condition.

Lemma 2. Consider the mapping \( W(\theta) \) given in (6) with \( \mathcal{G}(\eta) \) verifying Assumption A2. There exists a constant \( \alpha_m > 0 \) such that for all \( \alpha \geq \alpha_m \) the mapping \( W_1(\theta) \) satisfies the LMI

\[ T_{W_1} \nabla W_1(\theta) + [\nabla W_1(\theta)]^T T_{W_1} \geq \rho_{W_1} I_{l_1}, \]

for some \( \rho_{W_1} > 0 \), with the matrix

\[ T_{W_1} := \begin{bmatrix} \alpha & 0 & 0 & 0_{1 \times p_n} & 0_{1 \times p_n} \\ 0 & \alpha & 0 & 0_{1 \times p_n} & 0_{1 \times p_n} \\ 0 & 0 & \text{sign}(\theta_1) T_G & 0_{n \times p_n} \end{bmatrix} \in \mathbb{R}^{l_1 \times s}. \]

Discussion on the mapping D5 Notice that the only prior knowledge needed to construct the matrix \( T_{W_1} \) is \( \text{sign}(\theta_1) \). On the other hand, to select the value of \( \alpha \) some prior knowledge on the parameters \( \theta \) is required. Specifically, as shown in the proof of Lemma 2 in Appendix A, it is necessary to know an upper bound on \( \| T_{W_1} \mathcal{G}(\eta) \| \).

3. SECOND CLASS OF SYSTEMS

In this section we consider second order systems of the form

\[ \dot{x} = f_1(x) + f_2(x, \dot{x}) \mathcal{G}(\eta) + h_3(x) e^{\theta_1 \dot{x}(x)} + u \]

(8a)

\[ y = x \]

(8b)

with \( x(t) \in \mathbb{R} \) and \( u(t) \in \mathbb{R} \). The functions \( f_i, i = 1, 2, \) and \( h_i, i = 1, 3, \) are known nonlinear functions, \( \mathcal{G} : \mathbb{R}^{n_\eta} \rightarrow \mathbb{R}^{p_\eta}, p_\eta > n_\eta, \) is a known mapping of the unknown parameters \( \eta \in \mathbb{R}^{n_\eta} \), and \( \theta_1 \in \mathbb{R} \) is also an unknown parameter. Hence, the overall vector of unknown parameters, which needs to be estimated on-line, consists of \( \theta := \text{col}(\theta_1, \eta) \in \mathbb{R}^{l_{II}}, \)

where \( l_{II} := 1 + n_\eta. \)

Notice that, in contrast to system (2), in this case the dynamics is second order and the nasty exponential term enters into the state question instead of the readout map. Moreover, note that the control signal is scalar and enters linearly in the state equation. In particular, observe that the function \( h_3 \) appearing in the exponential does not depend on \( u \) now.\footnote{To simplify the presentation, but with an obvious abuse of notation, we keep the same symbol for both functions.}


To simplify the calculations, in the model (8) we do not include unknown parameters multiplying the function $h_3$ or the control $u$. As explained in Discussion D7 below, this can be easily added redefining $h_3(x) := \theta_3 h_3(x)$ and $u := \theta_3 \tilde{u}$, where the functions $h_3$ and $\tilde{u}$ are known but $\theta_2$ and $\theta_3$ are unknown parameters.

### 3.1 Assumptions

We make on the system (8) Assumptions A1, A2 together with the following.

**A3** [Separability] The function $f_2(x, \dot{x})$ verifies
\[
\nabla_x f_2(x, \dot{x}) = \psi_a(x) \psi_b(\dot{x}),
\]
for some functions $\psi_a(x)$ and $\psi_b(\dot{x})$.

**Discussion on Assumption A3** D6 As shown in the proof of Lemma 3 given in Appendix A, Assumption A3 is needed to be able to generate—via LTI filtering—a measurable regressor in the NLPRE. We observe that the function $\nabla_x f_2 \in \mathbb{R}^{p_n}$ hence, for $p_n > 1$, this is a vector function. However, there is no restriction on the dimensions of the functions $\psi_a$ and $\psi_b$ as long as they comply with the dimensionality requirement $\psi_a \psi_b \in \mathbb{R}^{p_n}$. This degree of freedom relaxes the condition of the assumption.

### 3.2 Regression Equation for Parameter Estimation

As in Subsection 2.2 we derive here the NLPRE that will be used to estimate the unknown parameters $\theta$.

**Lemma 3.** Consider the system (8) verifying Assumptions A1-A3. There exists measurable, scalar signals $Y_{II}(x, u, y), \phi_{II}(x, u, y), i = 1, \ldots, s_{II}, s_{II} := 1 + 2p_n$, such that the following NLPRE holds:
\[
Y_{II}(x, u, y) = \phi_{II}(x, u, y) W_{II}(\theta), \quad (9)
\]
where we defined the mapping $W_{II} : \mathbb{R}^{s_{II}} \rightarrow \mathbb{R}^{s_{II}}$
\[
W_{II}(\theta) := [\theta_1 \nabla^T(\eta) \theta_3 \nabla^T(\eta)]^T. \quad (10)
\]

**Discussion on regression equation D7** To include an unknown multiplicative parameter in the function $h_3$ or the control $u$ we proceed as follows. Define $h_3(x) := \theta_2 h_3(x)$ and $u := \theta_3 \tilde{u}$, where the functions $h_3$ and $\tilde{u}$ are known but $\theta_2$ and $\theta_3$ are unknown parameters. Tracing back the proof of Lemma 3 given in Appendix A, in the first step where we divide the model equation by $h_3$ we divide instead by $h_3$. Then, the parameter $\theta_2$ appears multiplying the exponential in the term in parenthesis and it is removed in the next line. That is, the first three lines of the proof become
\[
\frac{1}{h_3} \dot{x} = \theta_2 e^{h_3 \theta_1} + f_3 + \tilde{f}_3^T \nabla(\eta) + \theta_3 \tilde{u} h_3,
\]
\[
\frac{1}{h_3} \dot{h}_3 + \frac{1}{h_3} \frac{d^3 x}{dt^3} = \theta_1 h_3 \left( e^{h_3 \theta_1} \right) + \tilde{f}_3 + \tilde{f}_3^T \nabla(\eta) - \theta_3 \left( \frac{\dot{h}_3}{h_3} \tilde{u} - \frac{\dot{u}}{h_3} \right),
\]
\[
\frac{1}{h_3} \dot{f}_3 + \frac{1}{h_3} \frac{d^3 x}{dt^3} = \theta_1 h_3 \left( \frac{1}{h_3} \dot{x} - f_3 - \tilde{f}_3^T \nabla(\eta) - \theta_3 \tilde{u} h_3 \right) + \tilde{f}_3 + \tilde{f}_3^T \nabla(\eta) - \theta_3 \tilde{u} h_3,
\]
with the new definitions
\[
\tilde{f}_3 := \frac{f_1}{h_3}, \quad \tilde{f}_3 := \frac{1}{h_3} f_2.
\]
The remaining part of the proof remains unchanged leading to a NLPRE similar to (9), with the new $(\cdot)$ terms and adding to the parameter vector $\theta_1$ and $\theta_2 \theta_3$. As proven in Proposition 1, from this NLPRE we can estimate exponentially fast $(\theta_1, \theta_2, \eta)$. Therefore, we can replace their estimates in the model (8) leading to the system
\[
\dot{z} = f_1(z) + \tilde{f}_3^T(\dot{z}) \nabla(\eta) + \theta_3 h_3(z)e^{h_3 \theta_1(z)} + \tilde{\theta}_3 \tilde{u},
\]
which is a classical linear parameterized system from which we can estimate $\theta_3$ with standard filtering plus gradient descent techniques.

### 3.3 Construction of a Strictly Monotonic Mapping

Similarly to the calculations presented in Subsection 2.3 we present here the matrix $T_{W_{II}} \in \mathbb{R}^{s_{II} \times s_{II}}$ that defines the new monotonic mapping. The proof of this lemma is trivial, therefore it is omitted for brevity.

**Lemma 1.** Consider the mapping $W_{II}(\theta)$ given in (10) with $\nabla(\eta)$ verifying Assumption A2. The mapping $W_{II}(\theta)$ satisfies the LMI
\[
T_{W_{II}} \nabla W_{II}(\theta) + [\nabla W_{II}(\theta)]^T T_{W_{II}} \geq \rho_{W_{II}} I_{s_{II}}, \quad (11)
\]
with the matrix
\[
T_{W_{II}} := \begin{bmatrix} 1 & 0_{1 \times p_n} & 0_{1 \times p_n} \\ 0_{n_n \times 1} & T_{\phi} & 0_{n_n \times p_n} \end{bmatrix} \in \mathbb{R}^{s_{II} \times s_{II}}.
\]

**Discussion on the mapping D8** Notice that, in contrast with the construction of Subsection 2.3, here there is no requirement of prior knowledge on the parameter $\theta_1$. These stems from the fact that, as seen in (10), the mapping $\nabla(\eta)$ appears once without multiplying this parameter—compare with (6). Therefore, Assumption A2 is sufficient to construct the new monotonic mapping.

### 4. A GLOBALLY EXPONENTIALLY CONVERGENT ESTIMATOR OF $\theta$

In this section we present the main result of the paper, that is, an estimator of the parameters $\theta$ that achieves GEC of the parameter error. We proceed from the NLPREs constructed in Lemmata 1 and 3 and, as explained in Subsection 2.3, we propose to use the LS+DREM estimator recently reported in (Ortega et al., 2022). Towards this end, we use the new mappings identified in Lemmata 2 and 4 that verify the monotonicity conditions required by the LS+DREM estimator. To simplify the notation we avoid the subindices $(\cdot)_I$ and $(\cdot)_II$ of the various terms appearing in previous sections and present a single proposition applicable to both classes of systems.

Therefore, we consider a general scalar NLPRE of the form
\[
Y(t) = \phi(t) W(\theta) \quad (12)
\]
with $W : \mathbb{R}^\ell \rightarrow \mathbb{R}^r$. The main feature of the LS+DREM estimator is that it ensures GEC imposing the following extremely weak IE assumption (Kreisselmeier and Rietz-August, 1990; Tao, 2003) of the regressor $\phi$. 


Some simple calculations give us terms $Y$ and $G$.

A detailed review of the literature on parameter estimation is referred to (Xing et al., 2022), where nonlinear parameterizations (Pukrushpan et al., 2004) of fuel cell systems are considered. However, an accurate description of the system described by equations (11) to (14) of (Yang and de Queiroz, 2018), see also (Schauer et al., 2005; Sharma et al., 2012) and concentrate our attention on the problem of estimating the parameters of a widely accepted mathematical model of this system. Namely, the system described by equations (11) to (14) of (Yang and de Queiroz, 2018), that we repeat here for ease of reference

$$J\ddot{x} + b_1 \dot{q} + b_2 \text{sign}(\dot{x}) + k_1 e^{-b_2 x}(x - q_0) + mg\ell \sin(x) = u,$$

where $(x, \dot{x})$ are assumed measurable and all the parameters are assumed unknown. The reader is referred to this reference for further details on the model, in particular, the physical interpretation of the different terms in the model, and the overall formulation of the neuromuscular electrical stimulation problem.

Verification of the conditions from the general result

The following clarifications regarding our formulation of the parameter estimation problem are in order.

C1 As indicated in (Yang and de Queiroz, 2018), the term $\text{sign}(\dot{x})$ of our model (13) is replaced in equation (12) of (Yang and de Queiroz, 2018) by the function $\tanh(b_2 \dot{x})$, with a large value for $b_2 > 0$, which is a smooth approximation of the sign function. This approximation is made for mathematical convenience of their calculations that rely on a smoothness assumption, but is not required in our approach that can deal with discontinuous nonlinearities.

C2 In this paper we assume that the term $q_0$, which is the constant resting knee angle, and the constant inertia $J$ are known. Therefore the uncertain parameters in our case are $\text{col}(b_1, b_2, k_1, \ell)$. The assumption of known $J$ is not to restrictive because the inertia can be predicted from the subject’s anthropometric data (Winter, 2009).

C3 In (Yang and de Queiroz, 2018) there is an additional, bounded, unstructured, additive term in (13) that is omitted here for brevity. As shown in Proposition 1 we achieve GEC of the parameter estimates, therefore this term could be easily accommodated in our analysis to ensure practical stability.

The dynamics (13) belongs to the second class of systems given by (8) with $n_\eta = p_\eta = 3$, and the following definitions for the functions

$$f_1(x) = 0, f_2(x, \dot{x}) = \frac{1}{\ell} \text{col}(-\dot{x}, -\text{sign}(\dot{x}), g \sin(x)), h_3(x) = k_1 \dot{h}_3(x) := k_1 \frac{1}{\ell} (x - q_0), h_4(x) = -x,$$

and the parameters

$$\theta_1 = k_2, \ G(\eta) = \eta = \text{col}(b_1, b_2, \ell), \ \theta := \text{col}(\theta_1, \eta^\top).$$

We bring to the readers attention the fact that the model (13) has a parameter $k_1$ multiplying the exponential term.
Therefore, it is necessary to invoke the two-stage certainty-equivalent based procedure described in Discussion D7. That is, we estimate with the NLPRE (9) the parameters \( (k_2, b_1, b_2, mℓ) \) and then estimate, e.g., with some filtering and a standard gradient, the remaining parameter \( k_1 \).

To comply with Assumption A1, we assume that \(|x - q_0| > 0.2\). Clearly, since \( g(η) = η \), Assumption A2 is satisfied with \( T_0 = \hat{M}_1 I_3 \), with any \( ρ_2 > 0 \). Finally Assumption A3 is satisfied with the functions

\[
ψ_a(x) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
ψ_b(\hat{x}) := \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{\theta} \\ \dot{\theta} \\ \dot{\theta} \\ \dot{\theta} \\ \dot{\theta} \end{bmatrix}.
\]

The mapping \( W_{II} : \mathbb{R}^4 \rightarrow \mathbb{R}^7 \) is given as

\[
W_{II}(\theta) := [k_2 \ b_1 \ b_2 \ mℓ \ k_2 b_1 \ k_2 b_2 \ k_2 mℓ]^\top.
\]

Some simple calculations give us terms \( Y_{II} \) and \( φ_{II}^\top \) for the NLPRE (9).

Finally, the matrix \( T_{W_{II}} \in \mathbb{R}^{4×7} \) of Lemma 4 is given as

\[
T_{W_{II}} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ρ_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ρ_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ρ_2 & 0 & 0 & 0 \end{bmatrix}.
\]

6. SIMULATION RESULTS

6.1 First class of systems

Consider the "synthetic" model of the first class of the systems in the form (2), where \( f_1 = -x + u, f_2 = u, h_1 = x^2, h_2 = x - 353, h_3 = 0.1 + x^2, h_4 = -1/x \).

Since \( G = θ_3 \) the mapping \( W_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^5 \) defined in (6) is simpler and given as

\[
W_1(\theta) := [θ_1 \ θ_2 \ θ_1 \ θ_2 \ θ_1 \ θ_2 \ θ_1]^\top.
\]

Some simple calculations show that the terms of the NLPRE (5) are given as

\[
Y_I = F(p) p(H - l_3) \quad \text{and} \quad φ_I^\top = F(p) \begin{bmatrix} \frac{h_2}{h_3} \ f_1(y - h_1) & \frac{f_2(y - h_1)}{x^2h_3} & \frac{f_1 h_2}{x^2h_3} - \frac{f_2 h_2}{x^2h_3} \end{bmatrix}.
\]

For simulations we used next parameter values: filters parameter \( λ = 600 \), \( \hat{W}(0) = [0 \ 0 \ 0 \ 0 \ 0] \), \( f_0 = 1 \),
\[
\dot{θ} = \text{col}[0 \ 0 \ 0], \quad Γ = 10^2, \quad \hat{W}_V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Fig. 1 ... Fig. 3 demonstrate error transients of parameter estimations. In simulations we switch on our observer on the fifth second. Figures demonstrate that error transients tends to the zero.

Adding a simple logic and a discontinuous function we can easily avoid the singularity points and replace this assumption by the knowledge of a set such that \( q_0 \in [q_0^0, q_0^h] \).

2 Adding a simple logic and a discontinuous function we can easily avoid the singularity points and replace this assumption by the knowledge of a set such that \( q_0 \in [q_0^0, q_0^h] \).

6.2 Second class of systems

Parameters of the human shank model were chosen as in (Yang and de Queiroz, 2018). For estimation of shank model parameters we used algorithm from proposition 1 with \( W(0) = [0 \ 0 \ 0 \ 0 \ 0 \ 0], f_0 = 10^{-3}, Γ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \ 0 & 1 & 0 & 0 & 0 \ 0 & 0 & 1 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 \ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \),
\[
T_I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \ 0 & 1 & 0 & 0 & 0 \ 0 & 0 & 1 & 0 & 0 \ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \hat{θ} = \text{col}[0 \ 0 \ 0 \ 0 \ 0] \quad \text{and} \quad λ = 1
\]
in the filters.

Consider computer simulations of human shank system with dynamic robust control from (Yang and de Queiroz, 2018)
\[ u(t) = (k_1 + 1)r(t) - (k_1 + 1)r(0) + \int_0^t [(k_1 + 1)k_2 r(\tau) + k_3 \text{sign}(r(\tau))] d\tau \]

where \( e = x - x_d\) and \( r = \dot{e} - \mu e \). For simulation we used

\[
\dot{x}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6000 & -1300 & -80 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \\
\begin{cases} 1000\pi/3, & 0 \leq t < 20s, \\ 1000\pi, & 10 \leq t < 40s, \\ 2500\pi, & t \geq 40s, \end{cases}
\]

\[ x_1 = [x_d \dot{x}_d \ddot{x}_d]^T. \]

For simulation we used \( \mu = 4, k_1 = 1, k_2 = 2 \) and \( k_3 = 40 \) for control algorithm (simulation results for Human Shank with control are shown on Fig. 4 and Fig. 5) and \( \lambda = 10, \gamma = 10^6, \Gamma = 10^5 I, f_0 = 0.001 \).

Fig. 4 demonstrates transient of the system output \( x \). Fig. 5 demonstrates transient of the error \( x - x_d \). Fig. 6...9 demonstrate transients of estimation errors.

If parameters \( \theta_2 \) and \( \eta \) are known then we can estimate parameter \( \theta_1 \) using for instance standard gradient observer. We found parameter \( \theta_1 \) with standard gradient observer using \( \hat{\theta}_2 \) and \( \hat{\eta} \) instead real values with adaptation gain \( 10^6 \) (see simulation result on the Fig. 10).

Simulation results demonstrate convergence estimates to real values.

7. CONCLUDING REMARKS

We have presented in this paper a constructive procedure to design GEC estimators for the parameters of two classes of nonlinear, NLP systems containing nonseparable nonlinearities of the form \( e^{\theta_1 h_i(u,y)} \). Although this class of nonlinearities seems to be very particular, as discussed in the Introduction, it appears in many practical applications, including the two thoroughly studied in the paper, and is not amenable for the application of the existing parameter estimation techniques. The design procedure consists of
with \( \theta \) unmeasurable.

We would like to bring to the readers attention that knowledge, only this estimator is capable of dealing with this kind of NLPREs. Moreover, the excitation requirement needed to ensure GEC is the very weak condition of IE defined in Assumption A4.

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Appendix A. PROOF OF LEMMAS

Proof of Lemma 1

We make the key observation that the function $h_4$ verifies

$$\dot{h}_4 = \nabla^T h_4 [f_1 + f_2 G(\eta)],$$

where we have used (2a).

To simplify the notation in the sequel we define

$$h_5 := \nabla^T h_4 f_1, \ h_6 := \frac{f_2^T \nabla h_4}{h_4}, \ h_7 := \frac{y_1 - h_5}{h_6}, \ h_8 := -\frac{h_2}{h_3}.$$  \hspace{1cm} (A.1)

We observe that using this notation we can write

$$\dot{h}_8 = h_5 + h_6 G(\eta).$$  \hspace{1cm} (A.2)

From the definition of $y_1$ in (2b) we get

$$e^{h_4 \phi_1} = h_7 + h_8 \phi_2$$

$$\quad \Rightarrow \theta_1 \dot{h}_4 e^{h_4 \phi_1} = \dot{h}_7 + \dot{h}_8 \phi_2$$

$$\Rightarrow \dot{h}_5 = h_6 \dot{h}_7 \theta_1 + h_8 \phi_2 - \dot{h}_8 \phi_2$$

$$\Rightarrow \dot{h}_7 = [h_5 + h_6 G(\eta)] \dot{h}_7 \theta_1 + [h_5 + h_6 G(\eta)] h_8 \phi_2$$

$$- \dot{h}_8 \phi_2$$

$$\Rightarrow \dot{h}_7 = \theta_1 h_5 \dot{h}_7 - \dot{\theta}_2 \dot{\phi}_2 h_8 + \dot{\theta}_1 \dot{\phi}_1 h_7 + \dot{\theta}_1 \dot{\phi}_2 G(\eta) h_7 - \dot{\theta}_2 \dot{\phi}_2 G(\eta) h_6$$

$$+ \theta_2 \dot{\phi}_2 G(\eta) h_6,$$

where we used (A.2) to get the fourth implication. To obtain from the last identity a measurable NLPRE we apply the standard filtering technique (Jing and Ioannou, 1996; Tao, 2003). Toward this end, we fix a constant $\lambda > 0$ and define the stable filter $\frac{\dot{\lambda}}{\lambda}$, where $\dot{\lambda} := \frac{d}{dt}$. Applying this filter to the last equation above, and recalling the definitions (A.1), we get appropriate vectors $Y_1$ and $\phi_1$ for the NLPRE (5) completing the proof.  \hspace{1cm} 3

Proof of Lemma 2

The Jacobian of $W(\theta)$ is given as

$$\nabla W(\theta) = \begin{bmatrix} 1 & 0 & 0_{1 \times n_\eta} \\ 0 & 1 & 0_{1 \times n_\eta} \\ \theta_2 & \theta_1 \theta_2 & \theta_1 \theta_2 \\ G(\eta) \theta_1 \theta_2 & \theta_1 \theta_2 \nabla G(\eta) \\ \theta_2 G(\eta) \theta_1 & \theta_1 \theta_2 \nabla G(\eta) \end{bmatrix}.$$  \hspace{1cm} 4

The symmetric part of the matrix $T_W \nabla W(\theta)$ takes the form

$$T_W \nabla W(\theta) + [\nabla W(\theta)]^T T_W$$

$$= \begin{bmatrix} 2\alpha_{12} \theta_1 & \cdots & \left\{ \sign(\theta_1) G(\eta) T_G^+ \right\} \\ \cdots & \cdots & \cdots \\ \left\{ \sign(\theta_1) T_G \nabla G(\eta) \theta_1 \theta_2 \nabla G(\eta) \right\} \end{bmatrix}.$$  \hspace{1cm} 5

Let us introduce the notation

$$B := \left\{ \sign(\theta_1) G(\eta) T_G^+ \right\},$$

$$C := \left\{ \theta_1 | (T_G \nabla G(\eta)) + (\nabla G(\eta)) T_G^+ \right\}.$$  \hspace{1cm} 6

\hspace{1cm} 3 Notice that the term $\phi_{2,2}$ can be computed without differentiation via the proper filtering $\frac{\dot{\lambda}_p}{\lambda_p^2} (\frac{h_2}{h_3}).$
A simple Schur complement calculation proves that the matrix $T_W \nabla W(\theta) + [\nabla W(\theta)]^T T_W^T$ is positive definite if and only if
\[
C > \frac{1}{2\alpha} B^T B. \tag{A.3}
\]
On the other hand, from Assumption A3 we have that $C > \|\theta_1\| \|\phi_{II}\| > 0$. From which we conclude that (A.3) holds for sufficiently large $\alpha$, concluding the proof.

**Proof of Lemma 3**

To simplify the notation in the sequel we define
\[
f_3 := \frac{f_1}{h_3}, f_4 := \frac{1}{h_3} f_2. \tag{A.4}
\]
We observe that using this notation and (8) we get the following chain of implications
\[
\begin{align*}
\frac{1}{h_3} \dot{x} &= e^{h_3 \theta_1} f_3 + f_4^T \dot{G}(\eta) + \frac{u}{h_3} \\
\Rightarrow \quad \frac{\dot{h}_3}{h_3} \dot{x} + \frac{1}{h_3} d^3 x &= \theta_1 h_4 \left( e^{h_3 \theta_1} \right) f_3 + f_4^T \dot{G}(\eta) \\
&= \dot{h}_3 f_3 + f_4^T \dot{G}(\eta) - \dot{h}_3 u + \dot{u} \\
\Rightarrow \quad \frac{\dot{h}_3}{h_3} \dot{x} + \frac{1}{h_3} d^3 x &= \theta_1 h_4 \left( \dot{x} - f_3 - f_4^T \dot{G}(\eta) - u \right) \\
&= \dot{h}_3 f_3 + f_4^T \dot{G}(\eta) - \dot{h}_3 u + \dot{u} \\
\quad \frac{\dot{h}_3}{h_3} \dot{x} + \frac{1}{h_3} d^3 x &= \theta_1 h_4 \left( \dot{x} - f_3 - f_4^T \dot{G}(\eta) - u \right) \\
\end{align*}
\]
where the last right hand term can be computed without differentiation. Clearly, the same procedure can be applied to the term $\frac{h_4}{h_3} \ddot{x}$, leading to
\[
\begin{align*}
\frac{\lambda}{p + \lambda} \left( \frac{h_3}{h_3} \ddot{x} \right) &= \frac{\lambda}{p + \lambda} \left( \frac{h_3}{h_3} \dot{G}(\eta) \right) = \frac{1}{2} \frac{\lambda}{p + \lambda} \left[ h_4 p(\dot{x}^2) \right] \\
&= \frac{1}{2} \left[ \frac{h_4}{p + \lambda} \left( \dot{x}^2 \right) - \frac{\lambda}{p + \lambda} \left( h_4 \dot{x} \dot{x} + \lambda(\dot{x}^2) \right) \right],
\end{align*}
\]
where, again, the last right hand term can be computed without differentiation.

Now, regarding the term $h_3 f_4$, from (A.4), we have that
\[
\begin{align*}
h_3 f_4 &= \ddot{f}_4 - \frac{h_3}{h_3} f_2 \\
&= \nabla_x f_2 \ddot{x} + \nabla_x f_2 \ddot{x} - \frac{h_3}{h_3} f_2 \\
&= \nabla_x f_2 \ddot{x} + \psi_a(x) \psi_a(\ddot{x}) - \frac{h_3}{h_3} f_2 \\
&= \nabla_x f_2 \ddot{x} + \psi_a(x) \psi_a(\ddot{x}) - \frac{h_3}{h_3} f_2,
\end{align*}
\]
where we used Assumption A3 in the third identity and defined the function $\psi_a := \int \psi_b(s) ds$. Applying the Swapping Lemma we can take care of the term $\lambda \psi_a \psi_a$. Let
\[
\lambda = \frac{p + \lambda}{\psi_a \psi_a} = \psi_a \psi_a - \frac{1}{p + \lambda} \left( \psi_a \psi_a \psi_a \right),
\]
which is clearly computable.

Applying the second order filter to (A.5) and invoking the calculations above we get, after lengthy but straightforward calculations, get appropriate vectors $Y_{II}$ and $\phi_{II}$ for the NLPRE (9) completing the proof.