Line crossing problem for biased monotonic random walks in the plane

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Abstract
In this paper, we study the problem of finding the probability that the two-dimensional (biased) monotonic random walk crosses the line \( y = \alpha x + d \), where \( \alpha, d \geq 0 \). A \( \beta \)-biased monotonic random walk moves from \((a, b)\) to \((a + 1, b)\) or \((a, b + 1)\) with probabilities \( \frac{1}{\beta + 1} \) and \( \frac{\beta}{\beta + 1} \), respectively. Among our results, we show that if \( \beta \geq \lceil \alpha \rceil \), then the \( \beta \)-biased monotonic random walk, starting from the origin, crosses the line \( y = \alpha x + d \) for all \( d \geq 0 \) with probability 1.

1 Introduction

It is well-known that the standard one-dimensional random walk meets every lattice point infinitely many times almost surely. In particular, Pólya’s random walk constant in dimension 1 is 1, where Pólya’s random walk constant \( d(n) \) is defined to be the probability that the (lattice) random walk in dimension \( n \) returns to the departure point; see [3]. Pólya himself showed that \( d(1) = d(2) = 1 \) and \( d(n) < 1 \) for \( n > 2 \). Equivalently, \( d(1) = 1 \) implies that the probability that all partial sums of an infinite series of \(-1\)'s and \(1\)'s are nonzero is 0. Chung and Fuchs [11] gave generalizations of Pólya’s random
walk problem to sums of identically distributed random variables. See also [2] for an asymptotic evaluation of the probability of return at step $n$ for a bounded lattice distribution.

Any one-dimensional lattice path is in one-to-one correspondence with a monotonic lattice path in the plane (modulo a choice of departure points). By a monotonic lattice path in the plane, we understand a sequence of points $P_0P_1\ldots P_l$ such that each $P_i$ has integer coordinates and $P_{i+1} - P_i \in \{(0,1), (1,0)\}$ for all $i \leq l-1$. Hence, the monotonic random walk in the plane is simply the one-dimensional random walk, and so for instance, $d(1) = 1$ implies that the monotonic random walk in the plane intersects the line $y = x$ infinitely many times almost surely. To illustrate the combinatorial features of the problems that we will discuss, we prefer to work in this two-dimensional setting of the one-dimensional random walk.

We say a random walk in the plane is $\beta$-biased monotonic, if it moves from $(a,b)$ to $(a+1,b)$ or $(a,b+1)$ with probabilities $1/(\beta + 1)$ and $\beta/(\beta + 1)$, respectively. Here the term ‘biased random walk’ is borrowed from cell biology. A bacteria’s trajectory in presence of food shows many characteristics close to those of a random walk. However, depending on the food supply gradient, the bacteria shows bias in choosing directions of movement. Throughout this paper, we use the notation $W_\beta$ to refer to the $\beta$-biased monotonic random walk in the plane starting from the origin.

As we will show in Theorem 2, $W_\beta$ returns to meet the line $y = x$ with probability $2/(\beta + 1)$. More generally, we are interested in calculating $\Phi(\alpha,d)$, the probability that $W_\beta$ crosses the line $y = \alpha x + d$, $\alpha, d \geq 0$. In this introduction section, we first describe the combinatorial aspect of the problem.

A $(p+1)$-good path is a monotonic lattice path with the property that no lattice point on the path is strictly above the line $y = px$. Let $M(p,n)$ denote the number of $(p+1)$-good paths from $(0,0)$ to $(n,pn)$. In [4], P. Hilton and J. Pedersen showed that

\[
M(p,n) = \frac{1}{pn+n+1} \binom{pn+n+1}{n}.
\]  

(1.1)

When $p = 1$, one obtains the Catalan numbers $M(1,n) = \binom{2n}{n}/(n+1)$. Catalan numbers have many combinatorial interpretations which naturally lead to the generalized Catalan numbers $M(p,n) = C(pn+n,n)/(pn+1)$; see [4] for a detailed discussion on three such combinatorial interpretations, namely the number of $(p+1)$-ary trees with $n$ source nodes, the number of
ways of associating \( n \) applications of a \((p + 1)\)-ary operation, and the number of ways of dividing a convex polygon into \( n \) disjoint \((p + 2)\)-gons by means of non-intersecting diagonals. Also see [8] for more than 60 combinatorial interpretations of Catalan numbers.

Let \( H_p \) be the generating function for the sequence \( M(p, n) \), \( n \geq 0 \). In other words,

\[
H_p(x) = \sum_{n=0}^{\infty} M(p, n)x^n ,
\]

(1.2)

where \( M(p, 0) = 1 \). It then follows that

\[
\Phi_\beta(p, 0) = \sum_{n=0}^{\infty} M(p, n)\frac{\beta^{pn+1}}{(\beta + 1)^{pn+n+1}} = \frac{\beta}{\beta + 1} H_p\left(\frac{\beta^p}{(\beta + 1)^{p+1}}\right) ,
\]

(1.3)

since the probability that a \( \beta \)-biased random walk meets the lattice point \((n, pn + 1)\) as its first lattice point above the line \( y = px \) is given by the \( n \)th term of the series above. In section 2, we will prove a functional identity which shows that \( H_p \) satisfies an implicit equation for which \( H_p(x) \) is the smallest positive root. We will establish the following theorems in section 3.

**Theorem 1.** Let \( \beta > 0 \) and \( p, d \) be nonnegative integers. Then \( \Phi_\beta(p, d) = \Phi_\beta(p, 0)^{d+1} \), and \( \Phi_\beta(p, 0) \) is the smallest positive root of the equation:

\[
y^{p+1} - (\beta + 1)y + \beta = 0 .
\]

(1.4)

In particular, if \( \beta \geq p \), then the \( \beta \)-biased monotonic random walk in the plane starting from the origin crosses the line \( y = px + d \) with probability 1, for all \( d \geq 0 \).

Next, let \( \Psi_\beta(p, d) \) denote the probability that \( W_\beta \) meets a lattice point on the line \( y = px + d \) after the departure.

**Theorem 2.** Let \( \beta > 0 \) and \( p, d \) be nonnegative integers. Then \( \Psi_\beta(p, d) = \Phi_\beta(p, 0)^d \) for \( d > 0 \), and

\[
\Psi_\beta(p, 0) = 2 \left( 1 - \frac{\beta}{(\beta + 1) \Phi_\beta(p, 0)} \right) .
\]

In particular, if \( \beta \geq p \), then \( \Psi_\beta(p, d) = 1 \) for \( d > 0 \), and \( \Psi_\beta(p, 0) = 2/(\beta + 1) \) for \( p > 0 \).
2 The Generating Function $H_p$.

We need the following proposition in calculating $H_p$.

**Proposition 3.** Let $\alpha \geq 0$ be fixed. Then for all $z \in [0, 1/(\alpha + 1))$, we have:

$$\sum_{n=0}^{\infty} \frac{1}{n\alpha + n + 1} \binom{n\alpha + n + 1}{n} z^n (1-z)^{n\alpha+1} = 1. \quad (2.1)$$

**Proof.** If $\alpha = 0$, then the assertion is clear. Thus, suppose $\alpha > 0$ and let $\lambda_n$ denote the $n$th term on the left hand side of (2.1). Then, by Stirling’s Approximation Theorem for the Gamma function [9], we conclude that for $z \in (0, 1)$,

$$\lambda_n^{1/n} \simeq \left( \frac{\Gamma(n\alpha + n + 1)}{\Gamma(n+1)\Gamma(n\alpha + 1)} \right)^{1/n} z(1-z)^\alpha \quad (2.2)$$

where by $f(n) \simeq g(n)$ we mean $f(n)/g(n) \to 1$ as $n \to \infty$. It follows from the root test for convergence [7] that for a fixed $\alpha > 0$, the left hand side of (2.1) is convergent for all $z$ with

$$|z(1-z)^\alpha| < \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}}. \quad (2.3)$$

A simple differentiation shows that $z(1-z)^\alpha$ attains its maximum on $[0, 1]$ at $1/(\alpha + 1)$. It follows that all $z \in [0, 1]$ with the exception of $z = 1/(\alpha + 1)$ satisfy (2.3) and the series (2.1) is absolute convergent on $[0, 1/(\alpha + 1))$ as a function of $z$ for a fixed $\alpha \geq 0$.

Next, recall that the Taylor expansion of $(1-z)^{n\alpha+1}$ is given by the Newton series

$$(1-z)^{n\alpha+1} = \sum_{l=0}^{\infty} (-1)^l \binom{n\alpha + 1}{l} z^l.$$

For $z \in [0, 1/(\alpha + 1))$, we can use this expansion in (2.1) and re-arrange the terms to obtain a power series in $z$. Such re-arrangements in working with
series are allowed as long as all of the series involved are absolute convergent. In our case, all of the series involved are absolute-convergent for all \( z \in [0, 1/(\alpha + 1)) \). The coefficient of \( z^k \) for \( k \geq 1 \) is then given by

\[
\sum_{n+l=k} (-1)^l \binom{n\alpha + n + 1}{n} \left( \binom{n\alpha + 1}{l} \right) = \sum_{n=0}^{k} \frac{(-1)^{k-n}}{k} \binom{k}{n} \left( \binom{n\alpha + n}{k-1} \right).
\]

It is left to show that these coefficients are all zero for \( k \geq 1 \), i.e.

\[
\sum_{n=0}^{k} (-1)^n \binom{k}{n} \binom{n\alpha + n}{k-1} = 0, \quad \forall \alpha \in \mathbb{R}, \quad \forall k \geq 1.
\]

(2.4)

Since the left hand side is a polynomial in \( \alpha \), it is sufficient to show the identity above is valid for all positive even integers \( \alpha = m \). Let

\[
J(y) = \sum_{n=0}^{k} \binom{k}{n} y^{nm+n} = (1 + y^{m+1})^k.
\]

By taking \( k - 1 \) derivatives of \( J(y) \), we get

\[
0 = J^{(k-1)}(y) = (k-1)! \sum_{n=0}^{k} \binom{k}{n} \left( \binom{nm+n}{k-1} y^{nm+n-k+1} \right).
\]

The equation (2.4) follows from the above equation by setting \( y = -1 \). This completes the proof of (2.1).

**Corollary 4.** Let \( p \in \mathbb{N} \). Then the power series \( H_p(x) \), defined by (1.2), is convergent for all \( x \) with

\[
|x| \leq \frac{p^p}{(p+1)p+1}.
\]

(2.5)

Moreover, for any fixed nonnegative \( x \) in the domain above, \( H_p(x) \) is given by the smallest positive root of the equation:

\[
xy^{p+1} = y - 1.
\]

(2.6)

**Proof.** For each nonnegative \( x \) satisfying (2.5), let \( z(x) \) denote the smallest positive root of \( x = z(1-z)^p \). The function \( z(1-z)^p \) is increasing on \( [0, 1/(p+1)] \) and so its inverse \( z(x) \) is continuous in \( x \).
with \( z = z(x) < 1/(p + 1) \) implies that \( H_p(x) = 1/(1 - z(x)) \) and so \( H_p(x) \) satisfies (2.6) in the case of \( x < x_0 = p^p/(p + 1)^{p+1} \). Next, we show that \( H_p(x_0) = 1/(1 - z(x_0)) \). It is sufficient to show that \( H_p(x_0) < \infty \), since as soon as \( H_p(x_0) \) exists, we have:

\[
H_p(x_0) = \lim_{x \to x_0^-} H_p(x) = \lim_{x \to x_0^-} \frac{1}{1 - z(x)} = \frac{1}{1 - z(x_0)},
\]

by Abel’s Theorem [7]. The identity (2.1) with \( \alpha = p \) and \( z = 1/(t + 1) \) with any \( t > p \) implies that:

\[
\sum_{n=0}^{\infty} M(p, n) \frac{t^{pn+1}}{(t + 1)^{pn+n+1}} = 1.
\]

It follows that for each \( N \),

\[
\sum_{n=0}^{N} M(p, n) \frac{p^{pn+1}}{(p + 1)^{pn+n+1}} = \lim_{t \to p^+} \sum_{n=0}^{N} M(p, n) \frac{t^{pn+1}}{(t + 1)^{pn+n+1}} \leq 1,
\]

and so \( H_p(x_0) < \infty \). If \( s > 0 \) is any other root of (2.6), then \( x = z'(1 - z')^p \) with \( z' = 1 - 1/s \). Since \( z(1 - z)^p \) is increasing (hence, one-to-one) on \([0, 1/(p + 1)]\), we conclude that \( s \geq H_p \). \( \square \)

Let \( N(p, n) \) denote the number of monotonic lattice paths from \((0, 0)\) to \((n, pn)\) that are strictly under the line \( y = px \) except for the first and the last points. Let \( G_p(x) \) denote the generating function for \( N(p, n), n \geq 0 \), i.e.

\[
G_p(x) = \sum_{n=0}^{\infty} N(p, n)x^n,
\]

where \( N(p, 0) = 0 \). In the remainder of this section, we compute \( G_p(x) \).

**Proposition 5.** For \( n, p \in \mathbb{N} \), we have:

\[
M(p, n) = \sum_{m=0}^{n} N(p, m)M(p, n - m). \tag{2.7}
\]

In particular,

\[
H_p(x) = 1 + G_p(x)H_p(x),
\]

and \( G_p(x) \) is the smallest nonnegative root of the equation

\[
y(1 - y)^p = x.
\]
Proof. Any monotonic lattice path from \((0,0)\) to \((n, pn)\) has to intersect the line \(y = px\) at \((m, pm)\) for some \(m\) with \(1 \leq m \leq n\). The number of monotonic lattice paths from \((0,0)\) to \((n, pn)\) that intersect the line \(y = px\) for the first time at \((m, pm)\) is given by \(N(p, m)M(p, n - m)\). Then (2.7) follows by summing over \(m\). The last statement follows from (2.6).

3 Probability of crossing the line \(y = \alpha x + d\).

Recall that \(\Phi_\beta(p, d)\) denotes the probability that \(W_\beta\) crosses the line \(y = px + d\), i.e. the probability that it meets any of the lattice points \((n, pm + d + 1)\), \(n \geq 0\). For a fixed \(p \in \mathbb{N}\) and every nonnegative integer \(d\), let \(S(n, d)\) be the number of monotonic lattice paths from \((0,0)\) to \((n, pn + d)\) that are weakly below the line \(y = px + d\). We let \(S(0, d) = 1\) for \(d \geq 0\). Also, define the generating function of the sequence \(S(n, d), n \geq 0\), by setting

\[
S_d(x) = \sum_{n=0}^{\infty} S(n, d)x^n.
\]

Lemma 6. Let \(p \in \mathbb{N}\). Then for all nonnegative integers \(d\), we have

\[
S_d(x) = S_0^{d+1}(x) = H_p^{d+1}(x).
\]  (3.1)

In particular,

\[
\Phi_\beta(p, d) = \Phi_\beta(p, 0)^{d+1}.
\]  (3.2)

Proof. Any lattice path from \((0,0)\) to \((n, pn + d + 1)\) has to meet the line \(y = px + d\) at some point. Let \((m_\gamma, pm_\gamma + d)\) denote the first lattice point on \(y = px + d\) that \(\gamma\) meets. Then:

\[
S(n, d + 1) = \sum_{i=0}^{n} |\{\gamma : m_\gamma = i\}| = \sum_{i=0}^{n} S(i, d)S(n - i, 0). \quad (3.3)
\]

We prove (3.1) by induction on \(d \geq 0\). The assertion is clearly true for \(d = 0\). Assuming (3.1) for \(d\), we conclude from (3.3) that \(S_{d+1} = S_dS_0 = S_0^{d+2}\) by the inductive hypothesis. Finally, we compute
\[ \Phi_\beta(p, d) = \sum_{n=0}^{\infty} S(n, d) \left( \frac{\beta^{pn + d + 1}}{(\beta + 1)^{pn + n + d + 1}} \right) = \left( \frac{\beta}{\beta + 1} \right)^{d + 1} S_d \left( \frac{\beta^p}{(\beta + 1)^{p+1}} \right) = (\Phi_\beta(p, 0)^{d+1}, \] by (1.3).

Now we are ready to present the proof of Theorem 1.

**Proof of Theorem 1.** It follows from equations (1.3) and (2.6) that \( \Phi_\beta(p, 0) \) is the smallest positive root of (1.4). It is left to examine the case \( \beta \geq p \). We show that in this case 1 is the smallest positive root of (1.4). Otherwise, if \( y < 1 \) was a smaller positive root, by noticing

\[ y^{p+1} - (\beta + 1)y + \beta = (y - 1)(y^p + y^{p-1} + \ldots + y - \beta), \]

we would have \( p > y^p + \ldots + y = \beta \), which contradicts \( \beta \geq p \). Theorem 1 implies that \( \Phi_\beta(p, 0) = 1 \). It then follows from (3.2) that \( \Phi_\beta(p, d) = 1 \) for all \( \beta \geq p \) and all \( d \geq 0 \).

Let \( p \) be a fixed positive integer and let \( T(n, d) \) denote the number of monotonic lattice paths from \((0, 0)\) to \((n, pn + d)\) that are strictly under the line \( y = px + d \) except for the first and the last points. As in Proposition 5, one shows that

\[ S(n, d) = \sum_{m=0}^{n} T(m, d) M(p, n - m), \]

where \( T(0, d) = 1 \) for \( d > 0 \) and \( T(0, 0) = 0 \). It follows that the generating function of the sequence \( T(n, d), n \geq 0 \), given by

\[ T_d = \sum_{n=0}^{\infty} T(n, d)x^n \]

satisfies the equation

\[ S_d(x) = T_d(x) H_p(x), \] for \( d > 0 \), and so \( T_d(x) = H_p(x)^d \) by (3.1).
Proof of Theorem 2. There are two cases:

Case i) $d > 0$. The probability that $W_\beta$ meets the line $y = px + d$ at $(n, pn + d)$ for the first time after its departure from the origin is given by $T_d(p, n)\beta^{pn+d}(\beta+1)^{pn-n-d}$. Hence, by combining equations (3.4), (3.1), and (1.3), we have

$$\Psi_\beta(p, d) = \sum_{n=0}^{\infty} T_d(p, n)\beta^{pn+d}/(\beta+1)^{pn+n+d} = \left(\frac{\beta}{\beta+1}\right)^d T_d\left(\frac{\beta^p}{(\beta+1)^{p+1}}\right)$$

which implies that $\Psi_\beta(p, d) = 1$ if $\beta \geq p$ and $d > 0$.

Case ii) $d = 0$. In this case, the number of monotonic lattice paths starting from $(0, 0)$ that meet the line $y = px$ at $(n, pn)$ for the first time after the departure is given by $2T(n, 0) = 2N(p, n)$. The reason for the factor 2 is that the path could initially move over the line $y = px$. More precisely, the lattice paths from $(0, 0)$ to $(n, pn)$ that are strictly under the line $y = px$ except for the endpoints are in one-to-one correspondence with the lattice paths from $(0, 0)$ to $(n, pn)$ that are strictly above the line $y = px$ except for the endpoints. Hence,

$$\Psi_\beta(p, 0) = 2 \sum_{n=0}^{\infty} N(p, n)\beta^{pn}/(\beta+1)^{pn+n} = 2G_p\left(\frac{\beta^p}{(\beta+1)^{p+1}}\right)$$

by Proposition 5. If $\beta \geq p > 0$, then $\Phi_\beta(p, 0) = 1$ and so $\Psi_\beta(p, 0) = 2/(\beta+1)$. This completes the proof of Theorem 2. □

4 Conclusion and further remarks

The function $\Phi_\beta(\alpha, d)$ for non-integer values of $\alpha$ and $d$ can be defined similarly, namely $\Phi_\beta(\alpha, d)$ is the probability that $W_\beta$ crosses the line $y = \alpha x + d$. It is straightforward to check that $\Phi_\beta(\alpha, 0)$ is non-increasing in $\alpha$, it is continuous at every irrational $\alpha$, and it is at least right-continuous at every rational number. For each $\beta > 0$, there exists a unique $\alpha_\beta$ such that $\Phi_\beta(\alpha, 0) = 1$, $\forall \alpha \in (0, \alpha_\beta)$.  

\[9\]
Theorem 1 implies that
\[ |\beta| \leq \alpha \beta \leq |\beta| + 1. \]

To see this, let \( p = |\beta|. \) Then \( \beta \geq p \) and so \( \Phi_\beta(p, 0) = 1. \) On the other hand, \( \Phi_\beta(p+1, 0) \) is the smallest positive root of
\[ f(y) = y^{p+1} + y^p + \ldots + y - \beta = 0, \]
which has a solution in \((0, 1)\) by the Intermediate-value Theorem, since \( f(0) < 0 \) and \( f(1) = p + 1 - \beta > 0. \) It follows that \( p \leq \alpha \beta \leq p + 1. \)

In the rest of this section, we analyze the asymptotic behavior of \( \Phi_\beta(p, 0). \) Equations (2.6) and (1.3) imply that \( H_p(x_1) - 1 = x_1 H_p(x_1)^{p+1} \), where \( x_1 = \beta^p / (\beta + 1)^{p+1}. \) Since \( \lim H_p(x_1) = 1 \) as \( p \to \infty, \) we conclude that there exists a constant \( c \in (0, 1) \) depending only on \( \beta \) such that for \( p \) large enough, we have \( H_p(x_1) \leq c \min\{\beta + 1, 1 + 1/\beta\}. \) It follows that
\[ H_p(x_1) - 1 = x_1 H_p(x_1)^{p+1} \leq c^{p+1}, \]
and so
\[ 0 \leq H_p(x_1) - 1 = x_1 (1 + H_p(x_1) - 1)^{p+1} \leq x_1 (1 + c^{p+1})^{p+1}. \]

Since the function \( (1 + c^{p+1})^{p+1} \) is decreasing to 1 as \( p \to \infty, \) we conclude that:
\[ \Phi_\beta(p, 0) = \frac{\beta}{\beta + 1} (1 + x_1) + x_1 o_p, \]
where \( o_p \to 0 \) as \( p \to \infty. \)

References

[1] K.L. Chung and W.H.J. Fuchs, *On the distribution of values of sums of random variables*, Mem. Amer. Math. Soc. (1951), no. 6.

[2] C. Domb, *On multiple returns in the random-walk problem*, Proc. Cambridge Philos. Soc. 50 (1954), 586-591.

[3] S.R. Finch, *Pólya’s Random Walk Constant*, §5.9 in Mathematical Constants. Cambridge, England: Cambridge University Press (2003), 322-331.
[4] P. Hilton and J. Pedersen, \textit{Catalan numbers, their generalization, and their uses}, Math. Intelligencer 13 (1991), no. 2, 64-75.

[5] D.A. Klarner, \textit{Correspondences between plane trees and binary sequences}, J Comb. Theory 9 (1970), 401-411.

[6] S.K. Lando, \textit{Lectures on Generating Functions}, AMS Student Mathematical Library, vol. 23, AMS, (2003).

[7] K.A. Ross, \textit{Elementary Analysis: The Theory of Calculus}, Springer, (1980).

[8] R.P. Stanley, \textit{Enumerative Combinatorics}, vol. 2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, (1999).

[9] G.N. Watson and E.T. Whittaker, \textit{A Course in Modern Analysis}, fourth edition, Cambridge University Press, (2002).