Representations of simple Lie algebras with vectors having a zero stationary subalgebra

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Abstract. The paper solves the problem of describing the representations $V$ of compact Lie groups $G$ and vectors $v \in V$ with single stationary subgroups. It builds the classification of the representation of simple Lie algebras with vectors having a zero stationary subalgebra.

If the codimension of the submanifold $M_0$ of dimension $m$ in an $N$-dimensional Euclidean space $R^N$ is less than the dimension and the group $G$ is simple, then all these cases are described in the corresponding tables in sections 3 and 4.

1. Introduction

Let $G$ be a compact Lie group acting linearly in an $n$-dimensional vector space $V$. For any point $v \in V$, the dimension of its orbit $G_v$ is equal to the dimension of the subspace $\hat{G}_v$, where $\hat{G}$ is the Lie algebra of the Lie group $G$.

Papers [1-3] consider $\hat{G}$-modules of $V$, for which the following condition holds true: $\dim V < \dim \hat{G}$.

Paper [4] considers compact Lie groups for which $\dim V \geq \dim \hat{G}$. To make possible representations $V$ of the Lie algebra $G$, the dimension of the $\hat{G}$-module of $V$ is limited by the condition

$$\dim \hat{G} \leq \dim V < 2 \dim \hat{G}. \tag{1}$$

Not only simple Lie algebras, but also reductive ones are taken as the algebra $\hat{G}$.

One can distinguish three groups of modules among the $\hat{G}$-modules of $V$ that satisfy the inequality (1):

1. Reducible $\hat{G}$-modules of $V$.
2. Irreducible $\hat{G}$-modules of $V$ with irreducible complexification $\{\hat{G}^C, V^C\}$.
3. Irreducible $\hat{G}$-modules of $V$ with reducible complexification $\{\hat{G}^C, V^C\}$.

Let a $\hat{G}$ module of $V$ be given. Let us move on to its complexification $\{\hat{G}^C, V^C\}$. Its components are pairs of the same dimension or admit a bilinear invariant symmetric form.

Note that a direct sum of one-dimensional submodules $V^1$ can be added to the $\hat{G}$-module of $V$; the algebra $\hat{G}$ acts trivially on them. The number of terms in the sum is limited by condition (1) on dimensions $\hat{G}$ and $V$. Therefore, from this point on, without loss of generality, we can assume that $V$ does not contain one-dimensional submodules.

If the $\hat{G}$-module of $V$ is irreducible, then the transition to the complexification $\{\hat{G}^C, V^C\}$ offers two possibilities:
1. The $\mathcal{G}^c$-module of $V^c$ is irreducible. Then the condition (1) for space dimensions will be written down as

$$\dim_{\mathcal{C}} \mathcal{G}^c \leq \dim_{\mathcal{C}} V^c < 2 \dim_{\mathcal{C}} \mathcal{G}^c,$$

(2)

2. The $\mathcal{G}^c$-module of $V^c$ is reducible. Then the space of $V$ is a complex space $\mathcal{G}^c$, acts on $V$, and the inequality (1) is written down as

$$\frac{1}{2} \dim_{\mathcal{C}} \mathcal{G}^c \leq \dim_{\mathcal{C}} V < \dim_{\mathcal{C}} \mathcal{G}^c,$$

(3)

where $\dim_{\mathcal{C}}$ means a complex dimension.

In [4], the problem of describing the $\mathcal{G}$ modules of $V$, satisfying the inequality (1), is solved completely for the simple Lie algebras $\mathcal{G}$. For semi-simple Lie algebras, only irreducible representations satisfying the inequality (1) were considered. For them, this problem was not fully solved, since the possible infinite number of these representations forced us, in a number of cases, to impose a more stringent condition on their dimensions,

$$\dim V = \dim \mathcal{G} + \delta, \delta = 0, 2,$$

under which all such possible irreducible $\mathcal{G}$ modules of $V$ are found.

The present article builds the classification of the representation of simple Lie algebras with vectors having a zero stationary subalgebra.

2. On the existence of a vector with a zero stationary subalgebra

Before we select such representations $V$ of Lie algebras $\mathcal{G}$ that have vectors with a zero stationary subalgebra from the known and obtained by the author, we prove the following statement.

Statement. Let a $\mathcal{G}$-module of $V$ is written down as $V = \mathcal{G} + W$, where $\mathcal{G}$ is a simple Lie algebra, $W$ is a standard representation of the Lie algebra $\mathcal{G}$. Then there is a vector $v \in V$, which has a zero stationary subalgebra $\mathcal{G}_v$.

Proof. Let $v \in V$. Then we represent the vector $v$ in the form of $v = v_1 + v_2$, where $v_1 \in \mathcal{G}$, $v_2 \in W$. The stationary subalgebra $\mathcal{G}_v$ of the vector $v$ is equal to the intersection of stationary subalgebras of the elements $v_1$ and $v_2$. If $v_1$ is a regular element, then $\mathcal{G}_v$ coincides with the Cartan subalgebra $H$ of the Lie algebra $\mathcal{G}$. Since the restriction of a standard representation for $H$ is diagonal, then there is an element $v_2 \in W$, the stationary subalgebra of which in $H$ is equal to zero.

It follows from the Statement that all the representations listed in table 1 (see [4, table 1], containing the adjoint representation as components and at least one standard representation, have a zero stationary subalgebra.

| $\mathcal{G}^c$ | $V^c$ | $n$ |
|-----------------|--------|-----|
| $E_6$           | $2(\mathcal{C}^{27} + \mathcal{C}^{2\mathcal{C}}) + n\mathcal{C}^1$ | $n = 0, 47$ |
|                 | $E_6 + \mathcal{C}^{27} + \mathcal{C}^{2\mathcal{C}} + n\mathcal{C}^1$ | $n = 0, 23$ |
| $E_7$           | $(\mathcal{C}^{56} + \mathcal{C}^{5\mathcal{C}}) + n\mathcal{C}^1$ | $n = 0, 41$ |
|                 | $\mathcal{C}^{56} + \mathcal{C}^{5\mathcal{C}} + \mathcal{C}_7 + n\mathcal{C}^1$ | $n = 0, 20$ |
| $G_2$           | $3\mathcal{C}_7 + n\mathcal{C}^1$ | $n = 0, 6$ |
|                 | $\mathcal{C}_7 + G_2 + n\mathcal{C}^1$ | $n = 0, 6$ |
| $F_4$           | $\mathcal{C}^{26} + F_4 + n\mathcal{C}^1$ | $n = 0, 25$ |
|                 | $3\mathcal{C}^{26} + n\mathcal{C}^1$ | $n = 0, 25$ |
3. A case of exceptional Lie algebras

Let us consider exceptional Lie algebras. Let $\mathcal{G} = G_2$, $V = 3C^7$. If $G_2$ acts on the standard representation of $3C^7$, then $\mathcal{G}_\nu = A_2$ (see [1, table 1]). The action of $A_2$ on $C^7 = 2C^3 + C^1$ gives $A_3$ as a semisimple component of the stationary subalgebra. The action of $A_1$ on $C^7$ gives a zero stationary subalgebra.

Similarly, it can be shown that:

a) if $\mathcal{G} = F_4, V = 3C^{26}$, then the restriction of a stationary subalgebra on $C^{26}$ gives $D_4$, and $\mathcal{G}_\nu = \text{so}(2)$;

b) if $\mathcal{G} = E_6, V = 2(C^{26} + C^{26*})$, then $\mathcal{G}_\nu = D_4$;

c) if $\mathcal{G} = F_7, V = 2(C^{56} + C^{56*})$, then $\mathcal{G}_\nu = D_4$.

All compact exceptional Lie algebras $\mathcal{G}$ and $\mathcal{G}$-modules of $V$ satisfying the inequality (1) and having vectors with a zero stationary subalgebra are finally listed in table 2.

| $\mathcal{G}^C$ | $V^C$ | Condition on $n$ |
|-----------------|-------|------------------|
| $E_6$           | $C^27 + E_6 + nC^1$ | $n = 0.50$ |
| $E_7$           | $C^{56} + C^{56*} + E_7 + nC^1$ | $n = 0.20$ |
| $G_2$           | $3C^7 + nC^1$ | $n = 0.6$ |
| $G_2$           | $C^7 + G_2 + nC^1$ | $n = 0.6$ |
| $F_4$           | $C^{26} + F_4 + nC^1$ | $n = 0.25$ |

4. A case of classic Lie algebras

If $\mathcal{G}$ is a classic algebra, $V$ is its reducible representation from table 3 (see [4, table 2]), then, according to Lemma 1, the representations containing adjoint and standard representations as their components have a vector with a zero stationary subalgebra.

| $\mathcal{G}$ | $V$ | Condition on coefficients |
|---------------|-----|-------------------------|
| $so(n), n \geq 2$ | $so(n) + lR^n + kR^1$ | $l < (n - 1)/2$ |
|               |     | $lR^n + kR^1$ | $l < n - 1$ |
| $sp(n), n \geq 1$ | $sp(n) + (\Lambda^2C^{2n} - C^3)$ | $n < (m - 1)/2$ |
|               |     | $sp(n) + lQ^n + mC^1$ | $l < (m + 1)/2$ |
|               |     | $(\Lambda^2C^{2n} - C^3) + lQ^n + mC^1$ | $m = 0.1$ |
|               |     | $2(\Lambda^2C^{2n} - C^3) + Q^n + mC^1$ | $l < 2n + 1$ |
|               |     | $lQ^n + mC^1$ | $l < 2n + 1$ |
| $su(n), n \geq 2$ | $su(n) + mC^1$ | $m < n^2 - 1$ |

Let us find stationary subalgebras of the rest of the representations from table 3. Let

a) $\mathcal{G} = so(n), V = IR^n + kR^1$. As $l < n - 1$, then $\mathcal{G}_\nu = so(2)$;

b) if $\mathcal{G} = su(n), V = su(n) + mC^1$, then $\mathcal{G}_\nu = H$;
c) if \( \mathcal{G} = sp(n), V = \Lambda^2 C^{2n} - C^1 + lQ^n \), the stationary subalgebra has a dimension no greater than unity for reasons of dimension, therefore, if \( l > 1 \), then \( \mathcal{G}_v = 0 \);

d) if \( \mathcal{G} = sp(n), V = \Lambda^2 C^{2n} - C^1 + sp(n) \), then the stationary subalgebra of the universal position vector in \( sp(n) \) is equal to the Cartan subalgebra. Then in the space \( \Lambda^2 C^{2n} - C^3 \) you can always choose such an external form \( \omega \) that \( \forall h \in H, h \neq 0 \), it follows that \( \mathcal{G}_v = 0 \);

e) if \( \mathcal{G} = sp(n), V = lQ^n, l = n \), then \( \mathcal{G}_v = 0 \);

f) if \( \mathcal{G} = su(n), V = su(n) + mC^1 \), then \( \mathcal{G}_v = H \neq 0 \).

All representations \( V \) of Lie algebras \( \mathcal{G} \), satisfying the inequality (1) and having vectors with a zero stationary subalgebra, are finally listed in table 4.

| \( \mathcal{G}^C \) | \( V^C \) | Condition on coefficients |
|-----------------|---------|-----------------------------|
| \( so(n) \)     | \( so(n) + lC^n + kC^1 \) | \( l < (n - 1)/2 \) |
| \( sp(n) \)     | \( sp(n) + lQ^n + mC^1 \) | \( l < (n + 1)/2 \) |
| \( \Lambda^2 C^{2n} - C^1 + lQ^n + mC^1 \) | \( l = 2(n + 2)/2 \) |
| \( 2(\Lambda^2 C^{2n} - C^1) + Q^n + mC^1 \) | \( m = 0,1 \) |
| \( lQ^n + mC^1 \) | \( l = n \) |
| \( \Lambda^2 C^{2n} - C^1 + sp(n) + mC^1 \) | \( m < 2n + 1 \) |

Now let the reducible representations of the Lie algebra \( so(n) \) contain a spinor representation as a component. From all admissible representations listed in table 3, we eliminate representations of the form \( V = W_1 + W_2 \), where \( W_1 \) is the standard representation of the corresponding algebra \( D_r \) \( (r \neq 8) \) or \( B_r \) \( (r \neq 7, 8) \). \( W_2 \) is the spinor representation, since the minimal stationary subalgebra \( \mathcal{G}^C \) is equal to \( so(2) \) for them. In case when \( \mathcal{G}_v = D_{16}, B_7 \) or \( B_8 \) the dimension of the spinor representation exceeds the dimension of the algebra, therefore, \( \mathcal{G}_v = 0 \).

All such representations \( V \) of the Lie algebra \( G \) are listed in table 5.

| \( \mathcal{G}^C \) | \( V^C \) | Conditions on coefficients |
|-----------------|---------|-----------------------------|
| \( so(8) \)     | \( kC^2 + so(8) + nC^8 + pC^1 \) | \( k = 1,2; 8k + 8n + p < 28 \) |
| \( so(16) \)    | \( 2C^2 + nC^{16} + pC^1 \) | \( n = 0,6; 16n + p < 112 \) |
| \( so(7) \)     | \( 2C^2 + so(7) + pC^1 \) | \( p = 0.4 \) |
|                 | \( C^2 + so(7) + C^7 + pC^1 \) | \( p = 0.5 \) |
| \( so(9) \)     | \( so(9) + C^2 + 2C^9 + pC^1 \) | \( p = 0.1 \) |
| \( so(9) \)     | \( so(9) + C^2 + C^9 + pC^1 \) | \( p = 0.10 \) |
| \( so(9) \)     | \( so(9) + C^2 + pC^1 \) | \( p = 0.19 \) |
| \( so(9) \)     | \( so(9) + 2C^2 + pC^1 \) | \( p = 0.3 \) |

Let us consider the case when the \( \mathcal{G} \)-module of \( V \) is irreducible. Work [4, table 4] indicates irreducible \( \mathcal{G} \)-modules of \( V \) with irreducible complexification \( \{ V^C, \mathcal{G}^C \} \) satisfying the inequality (2). It
is obvious that the stationary subalgebras $\hat{G}_v$ of these representations for the corresponding Lie algebras $\hat{G}$ are equal to zero, since $\dim V > \dim \hat{G}$ by hypothesis.

If the complexification of $\{V^c, \hat{G}^c\}$ is reducible, then such $\hat{G}^c$-modules of $V^c$, satisfying inequality (3), are presented in [4, table 5]. When passing on to the compact form of Lie algebras, we pass on from complex stationary subalgebras to compact ones. In the stationary subalgebra, one should take a maximal compact subalgebra, but since they are unsolvable, then the maximal compact subalgebra will be nonzero and, therefore, the stationary subalgebra $\hat{G}_v = 0$.

5. Conclusion

The work describes the representations $V$ of compact Lie groups $G$ and vectors $v \in V$ with single stationary subgroups. It builds the classification of the representation of simple Lie algebras with vectors having a zero stationary subalgebra. All cases are noted when the codimension of the submanifold $M_0$ of dimension $m$ in an $N$-dimensional Euclidean space $R^N$ is less than the dimension, and the group $G$ is simple.

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