Remarks on pure and simple shear

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In a 2012 article in the International Journal of Non-Linear Mechanics, Destrade et al. showed that for nonlinear elastic materials satisfying Truesdell’s so-called empirical inequalities, the deformation corresponding to a Cauchy pure shear stress is not a simple shear. Similar results can be found in a 2011 article of L. A. Mihai and A. Goriely. We confirm their results under weakened assumptions and consider the case of a shear load, i.e. a Biot pure shear stress. In addition, conditions under which Cauchy pure shear stresses correspond to (idealized) pure shear stretch tensors are stated and a new notion of idealized finite simple shear is introduced, showing that for certain classes of nonlinear materials, the results by Destrade et al. can be simplified considerably.

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1 Introduction

A (homogeneous) simple shear deformation is a mapping \( \varphi: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) of a unit cube of the form \( \varphi(x_1, x_2, x_3) = (x_1 + \gamma x_2, x_2, x_3) \) with the amount of shear \( \gamma \in \mathbb{R} \), which is often put in the wrong context with the notion of a pure shear stress \( T^{s} = s(e_1 \otimes e_2 + e_2 \otimes e_1) \) with the amount of shear stress \( s \in \mathbb{R} \).

\[
\nabla \varphi = F_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^{s} = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

In nonlinear elasticity, contrary to the linear case, it is known [1, 3, 2] that a non-trivial Cauchy pure shear stress \( \sigma = T^{s} \) never corresponds to a simple shear deformation \( F_\gamma \).

Guiding questions:

1. Independent of the particular elasticity law, which kind of deformations correspond to pure shear stress?
2. Which of these deformations are suitable to be called “shear”?
3. Which constitutive requirements ensure that only “shear” deformations correspond to pure shear Cauchy stress?

2 Pure shear stress

Starting with the first question, we use the fact that the left Cauchy-Green deformation tensor \( B = F F^T \) and the corresponding Cauchy stress tensor \( \tilde{\sigma}(B) \) commute for any isotropic stress response. Thus \( B \) and \( \tilde{\sigma}(B) \) are simultaneously diagonalizable [5].

If \( B = F F^T \) commutes with a Cauchy pure shear stress tensor \( \tilde{\sigma}(B) = T^{s} \), then \( F \) is uniquely determined up to an arbitrary rotation \( Q \in SO(3) \) by

\[
F = F_\gamma \text{diag}(a, b, c) Q \rightarrow \text{diag}(a, b, c) Q.
\]

Fig. 1: Left finite simple shear deformation with amount of shear \( \gamma = \tanh(2\alpha) \).
3 Finite simple shear deformation

In order to answer the second question, we introduce the notion of an idealized shear deformation which translates the characteristic infinitesimal properties of the simple shear in linear elasticity to the setting of finite elasticity:

We call \( F = VR \in GL^+(3) \) with \( V \in Sym^+(3) \) and \( R \in SO(3) \) an (idealized) finite shear deformation if the following requirements are satisfied:

i) The stretch \( V \) (or, equivalently, the deformation \( F \)) is volume preserving, i.e. \( \det V = 1 \).

ii) The stretch \( V \) is planar, i.e. \( V \) has the eigenvalue 1 to the eigenvector \( e_3 \).

iii) The rotation \( R \) is such that the deformation \( F \) is ground parallel, i.e. \( e_1, e_3 \) are eigenvectors of \( F \).

These considerations lead to deformations which exhibit the general form \( F = F_\gamma \text{ diag}(a, b, c) Q \) and are suitable to be called “shear”:

For \( \alpha \in \mathbb{R} \), we call \( F \in GL^+(3) \) an (idealized) left finite simple shear deformation gradient [4] and \( V \in Sym^+(3) \) a finite pure shear stretch if \( F = F_\alpha \) and \( V = V_\alpha \) have the form

\[
F_\alpha = \frac{1}{\cosh(2\alpha)} \begin{pmatrix} 1 & \sinh(2\alpha) & 0 \\ 0 & \cosh(2\alpha) & 0 \\ 0 & 0 & \sqrt{\cosh(2\alpha)} \end{pmatrix}, \quad V_\alpha = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}
\]

The definition of a finite shear deformation \( F_\alpha \) is a direct generalization of the infinitesimal behavior because the infinitesimal simple shear \( F_\gamma \) is planar, ground parallel and infinitesimally volume preserving (\( \text{tr} (\varepsilon_\gamma) = 0 \)). The transition mechanism of infinitesimal to finite stretch is the matrix exponential via the identification \( \gamma = 2\alpha \).

4 Constitutive conditions

For the third question, it is important to note that whether or not a deformation gradient \( F \) corresponding to a Cauchy pure shear stress is a finite shear deformation depends on the particular stress response function. Similarly, not every constitutive law ensures that every finite simple shear (2) induces a Cauchy pure shear stress tensor.

In particular, for a given stress response, a finite pure shear stretch always induces a pure shear stress if and only if for all \( \lambda \in \mathbb{R}_+ \), there exists \( s \in \mathbb{R} \) such that \( \lambda_1 = \frac{s}{2} \lambda = \lambda' \) and \( \lambda_3 = 1 \) imply \( \sigma_1 = -\sigma_2 = s \) and \( \sigma_3 = 0 \), where \( \sigma_i \) denotes the \( i \)-th eigenvalue of \( \hat{\sigma}(FF^T) \) (the principle stresses) for \( F = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \). It follows [4]:

Let \( W: GL^+(3) \to \mathbb{R} \) be an elastic energy of the form \( W(F) = W_{tc}(F) + f(\det F) \), where \( W_{tc}: GL^+(3) \to \mathbb{R} \) is a sufficiently smooth tension-compression symmetric energy function, i.e. \( W_{tc}(F^{-1}) = W_{tc}(F) \) for all \( F \in GL^+(3) \), and \( f: \mathbb{R}_+ \to \mathbb{R} \) with \( f'(1) = 0 \). Then \( \hat{\sigma}(B) \) is a pure shear stress for every finite pure shear stretch \( V = V_\alpha \).

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