A gradient system on the quantum information space realizing the averaged learning equation of Hebb type for the principal component analyzer

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Abstract

The averaged learning equation (ALEH) applicable to the principal component analyzer is studied from both quantum information geometry and dynamical system viewpoints. On the quantum information space (QIS), the space of regular density matrices endowed with the quantum SLD-Fisher metric, a gradient system is given as an extension of the ALEH; on the submanifold, consisting of the diagonal matrices, of the QIS, the gradient flow coincides with the ALEH up to a local diffeomorphism.

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1 Introduction

Quantum computing has been widely known to be a research area in rapid-rate progress. It has been recognized also as an interdisciplinary research area in which numbers of researchers with a variety of backgrounds are working. An ambition of breaking theoretical boundaries of binary computing has been a great driving force behind many investigations to discover quantum algorithms beyond the boundaries. As celebrated examples beyond the boundaries, Shor’s algorithms and Grover’s one are well-known for the discrete logarithms, the prime factorization and the data search, respectively [1, 2]. To trace the history of quantum computing, see [3] for example.
On turning to algorithms in conventional sense of computing, there exist various excellent algorithms developed in engineering and systems science. It might happen that some of them admit a similar mathematical structure: In 1990’s, Nakamura revealed integrability, Lax-type structure and gradient-system structure in a matrix-eigenvalue computing [4], the gradient system on the space of multinomial distributions [5], the Karmarkar flow for linear programming [6] and an averaged learning equation of Hebb type [7].

In a series of papers by the authors [8, 9, 10, 11], a counterpart to Nakamura’s gradient-system structure was successfully found in the quantum information space (QIS) for the gradient system on the space of multinomial distributions [5] and the Karmarkar flow for linear programming [6]. The QIS in this paper stands for the space of regular density matrices endowed with the quantum SLD-Fisher metric. The former system is realized as the gradient system on the QIS associated with the negative von Neumann entropy, the latter with a trace of the square of density matrix times a cost-coefficient matrix. As a continuation of those papers of the authors, the aim of the present paper is to construct a counterpart of the averaged learning equation of Hebb type for the principal component analyzer [7, 12], which takes the gradient-system form in the QIS. The present paper is placed in an interdisciplinary research area of dynamical systems on the QIS, algorithms and applied differential geometry. The result would be expected to be a clue to realize algorithms in the QIS in mathematical sense first, and in a more realistic physical sense in future. In what follows, the contents of the present paper are outlined.

Section 2 is the preliminaries for the averaged learning equation of Hebb type (ALEH). The ALEH is derived from a synaptic neuron model and its gradient-system form is introduced. Section 3 is devoted to geometric devices for realizing the ALEH on the QIS. In order to transfer the ALEH on a dense submanifold $S_m$ of the sphere to that on the submanifold $D_m$ of the QIS consisting of diagonal matrices, an immersion of $S_m$ to $D_m$ is introduced. On regarding the immersion as a local diffeomorphism, the ALEH is understood to be transferred to a multi-fold copy of the ALHE. It is worth pointing out that the geometric devices presented in the present paper are quite different from those in the papers [10, 11] for the Karmarkar flow. Section 4 is the core part of the present paper, where the gradient system on the QIS realizing the ALEH is given explicitly. Section 5 is for concluding remarks.
2 Preliminary: The ALEH

In this section, we review an averaged learning equation of Hebb-type (ALEH) following Nakamura [7] and Oja [12]. The gradient-system form derived by Nakamura [7] of the ALEH is also reviewed on the \((m - 1)\)-dimensional unit sphere.

2.1 Learning in the synaptic neuron model

Let us introduce a vector-valued variable \(X = (X_1, X_2, \cdots, X_m)^T \in \mathbb{R}^m\) to express the \(m\) presynaptic signals and a scalar-valued variable \(Y \in \mathbb{R}\) to express the postsynaptic signal; the values at the time \(s\) of \(X\) and \(Y\) are described as \(X(s)\) and \(Y(s)\), respectively. The synaptic neuron model dealt with in this paper starts with the following timewise linear relation

\[ Y(s) = W^T(s)X(s) \] (1)

between \(X\) and \(Y\), where \(W(s) = (W_1(s), W_2(s), \cdots, W_m(s))^T \in \mathbb{R}^m\) stands for the vector of coupling strengths of the neuron. Throughout this paper, the superscript \(^T\) indicates the transpose operation.

According to Hebb’s hypothesis [12][13], learning in the synaptic neuron models amounts to updating efficacies of extracting inputs with high probability: In the model (1), the vector of coupling strengths \(W(s)\) is understood to be updated through a recurrence relation along with repetitive inputs of signals into neuron. Following Oja [12], we consider the discrete-time recurrence relation

\[ W(s + 1) = \frac{W(s) + \eta Y(s)X(s)}{\|W(s) + \eta Y(s)X(s)\|} \] (2)

for the coupling strength \(W(s)\) in (1), where \(\eta\) is a positive constant indicating the learning rate and the symbol \(\|\cdot\|\) stands for the standard Euclidean norm of vectors in \(\mathbb{R}^m\). As a significant characteristic of the recurrence relation (2), the following norm-preserving property is worth pointed out;

\[ \|W(s)\| = 1 \quad \text{for} \quad \|W(0)\| = 1. \] (3)

We derive a differential equation providing an approximation of the learning system (1) with Oja’s recurrence rule (2) in what follows. Equation (2) takes the form

\[ W(s + 1) = \frac{W(s) + \eta X(s)X^T(s)W(s)}{\|W(s) + \eta X(s)X^T(s)W(s)\|} \] (4)
under the relation \( \Pi \), which admits the Maclaurin expansion

\[
W(s + 1) = W(s) + \eta \{ X(s)X^T(s)W(s) 
\quad - (W^T(s)X(s)X^T(s)W(s))W(s) \} + O(\eta^2)
\]

if the leaning rate \( \eta \) is sufficiently small. The \( O(\eta^2) \) in (5) denotes the second-order infinitesimal. Elimination of the term \( O(\eta^2) \) from the rhs of (5) therefore provides us with the equation

\[
W(s + 1) = W(s) + \eta \{ X(s)X^T(s)W(s) 
\quad - (W^T(s)X(s)X^T(s)W(s))W(s) \}.
\]

### 2.2 Averaging

We proceed an averaging of (6) in what follows. Let \( X(s) \) and \( W(s) \) be stochastic processes, which are statistically independent to each other. On taking the expectation of (6), we obtain

\[
E[W(s + 1)|W(s)] - E[W(s)] = \eta \{ E[X(s)X^T(s)]E[W(s)] 
\quad - (E[W(s)]^T E[X(s)X^T(s)] E[W(s)])E[W(s)] \},
\]

where the symbol \( E[\cdot] \) denotes expectation operation. On assuming the stochastic process \( X(s) \) to be stationally, the correlation matrix \( E[X(s)X^T(s)] \) is kept invariant along \( s \), which is thereby diagonalized by an orthogonal matrix \( G \) to

\[
C = \text{diag}(c_1, c_2, \cdots, c_m) = G^T E[X(s)X^T(s)]G.
\]

Note that the orthogonal matrix \( G \) do not depend on \( s \), so that all the eigenvalues \( c_j \) of \( E[X(s)X^T(s)] \) are kept invariant along \( s \), too. The change of variables

\[
w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^T = G^T W(t/\eta)
\]

with the time-scaling

\[
t = \eta s
\]

brings (7) into the form,

\[
w(t + \eta) - w(t) = \eta \left( Cw(t) - (w^T(t)Cw(t))w(t) \right).
\]
The differential equation
\[
\frac{dw}{dt} = Cw - (w^T Cw)w
\] (12)
thereby emerges from (11) as a continuous-time approximation of (4) \[7, 12\] in view of the stochastic approximation theory \[14\]. Throughout this paper, we will refer to (12) as the averaged learning equation of Hebb type (ALEH).

2.3 The gradient-system form

We start with showing the norm preserving property
\[
\|w(t)\| = 1 \quad \text{for} \quad \|w(0)\| = 1
\] (13)
of the ALEH (12), which is understood to be a counterpart to (3) of (1) with Oja’s rule (2). Indeed, for any solution \(w(t)\) of (12) with \(\|w(0)\| = 1\), the calculation below shows (13):
\[
\frac{d}{dt}\|w(t)\|^2 = 2w(t)^T \frac{dw}{dt}(t) = 2(w(t)^T C w(t))(1 - \|w(t)\|^2) = 0.
\] (14)

Putting (9), (10) and (14) together, we obtain
\[
\|W(s)\| = \|Gw(\eta s)\| = \|w(\eta s)\| = \|w(0)\| = 1
\] (15)
for \(\|W(0)\| = \|w(0)\| = 1\), as the counterpart to (3).

Owing to the norm preserving property (13), we can restrict the ALEH on the \((m - 1)\)-dimensional unit sphere
\[
S^{m-1} = \left\{ w \in \mathbb{R}^m \left| \|w\| = 1 \right. \right\}
\] (16)
in \(\mathbb{R}^m\). We endow \(S^{m-1}\) with the standard Riemannian metric
\[
\langle (u, u') \rangle_{w}^{Sph} = u^T u' \quad (u, u' \in T_w S^{m-1}, \ w \in S^{m-1}),
\] (17)
where \(T_w S^{m-1}\) denotes the tangent space of \(S^{m-1}\) at \(w\) defined to be
\[
T_w S^{m-1} = \left\{ u \in \mathbb{R}^m \left| w^T u = 0 \right. \right\} \quad (w \in S^{m-1}).
\] (18)

According to Nakamura \[7\], the ALEH on \(S^{m-1}\) admits the gradient-system form. Let the function,
\[
\Lambda(w) = -\frac{1}{2} w^T C w = -\frac{1}{2} \sum_{k=1}^{m} c_k w_k^2 \quad (w \in S^{m-1}),
\] (19)
on $S^{m-1}$ be taken as the potential for the gradient-system form. The gradient vector field $\text{grad}\Lambda$ for the gradient system $(S^{m-1}, (\cdot, \cdot)^{Sph}, \Lambda)$ is defined as follows: For a sufficiently small interval $[a, b]$ with $a < 0 < b$, let us associate a smooth curve $\gamma : [a, b] \rightarrow S^{m-1}$ with any $u \in T_wS^{m-1}$ in the manner

$$\tau \in [a, b] \mapsto \gamma(\tau) \in S^{m-1}, \quad \gamma(0) = w, \quad \frac{d\gamma}{d\tau}\bigg|_{\tau=0} = u \in T_wS^{m-1}. \quad (20)$$

Then the gradient vector field $\text{grad}\Lambda$ is defined to satisfy

$$((\text{grad}\Lambda(w), u))^{Sph}_w = \frac{d}{d\tau}\bigg|_{\tau=0} \Lambda(\gamma(\tau)) (u \in T_wS^{m-1}). \quad (21)$$

Accordingly, $\text{grad}\Lambda$ turns out to be

$$\text{grad}\Lambda(w) = -Cw + (w^TCw)w, \quad (22)$$

which coincides with the minus of the rhs of (12). The ALEH on $S^{m-1}$ is thus written in the gradient-system form.

3 Geometric devices

This section provides geometric devices for realizing the ALEH in the QIS.

3.1 The QIS

Following Uwano et al [9], we introduce the quantum information space (QIS), the space of regular density matrices endowed with the quantum SLD (symmetric logarithmic derivative) Fisher metric, in what follows.

Let us consider the space of $m \times m$ regular density matrices

$$\dot{P}_m = \{ \rho \in M(m, m) | \rho^\dagger = \rho, \quad \text{tr}\rho = 1, \quad \rho : \text{positive definite} \}, \quad (23)$$

where $M(m, m)$ denotes the set of $m \times m$ complex matrices. The $\dot{P}_m$ is endowed with the quantum SLD Fisher metric $(\cdot, \cdot)^{QF}$ as follows.

Let the tangent space of $\dot{P}_m$ at $\rho$ be defined by

$$T_\rho\dot{P}_m = \{ \Xi \in M(m, m) | \Xi^\dagger = \Xi, \quad \text{tr}\Xi = 0 \}. \quad (24)$$

The symmetric logarithmic derivative (SLD) to any tangent vector $\Xi \in T_\rho\dot{P}_m$ is defined to provide the Hermitean matrix $L_\rho(\Xi) \in M(m, m)$ subject to

$$\frac{1}{2} \{ \rho L_\rho(\Xi) + L_\rho(\Xi) \rho \} = \Xi \quad (\Xi \in T_\rho\dot{P}_m). \quad (25)$$
The quantum SLD Fisher metric, denoted by \( \langle (\cdot, \cdot) \rangle_{QF} \), is then defined to be
\[
\langle (\Xi, \Xi') \rangle_{QF}^\rho = \frac{1}{2} \text{tr} \left[ \rho (L_\rho(\Xi)L_\rho(\Xi') + L_\rho(\Xi')L_\rho(\Xi)) \right] \quad (\Xi, \Xi' \in T_\rho \dot{P}_m) \tag{26}
\]
(see also \cite{8, 9, 16, 17, 18, 19}).

We wish to present a more explicit expression of \( \langle (\cdot, \cdot) \rangle_{QF} \) in what follows. Let \( \rho \in \dot{P}_m \) be expressed as
\[
\rho = h \Theta h^\dagger, \quad h \in \text{U}(m)
\]
\[
\Theta = \text{diag}(\theta_1, \ldots, \theta_m) \quad \text{with} \quad \text{tr}\Theta = 1, \quad \theta_k > 0 \quad (k = 1, 2, \cdots, m),
\]
where \( \text{U}(m) \) denotes the group of \( m \times m \) unitary matrices. Expressing \( \Xi \in T_\rho \dot{P}_m \) as
\[
\Xi = h \chi h^\dagger
\]
with \( h \in \text{U}(m) \) in (27), we obtain an explicit expression,
\[
(h^\dagger L_\rho(\Xi)h)_{jk} = \frac{2}{\theta_j + \theta_k} \chi_{jk} \quad (j, k = 1, 2, \cdots, m),
\]
(29)
of the SLD to \( \Xi \in T_\rho \dot{P}_m \) \cite{9}. Putting (27)-(29) into (26), we have
\[
\langle (\Xi, \Xi') \rangle_{QF}^\rho = \sum_{j,k=1}^{m} \frac{\chi_{jk} \chi'_{jk}}{\theta_j + \theta_k}
\]
(30)
where \( \Xi' \in T_\rho \dot{P}_m \) is expressed as (9)
\[
\Xi' = h \chi' h^\dagger.
\]
(31)

The space of \( m \times m \) regular density matrices, \( \dot{P}_m \), endowed with the quantum SLD Fisher metric \( \langle (\cdot, \cdot) \rangle_{QF} \) defined above is what we are referring to as the quantum information space (QIS) in the present paper, which will be denoted also as the pair \( (\dot{P}_m, \langle (\cdot, \cdot) \rangle_{QF}) \) henceforth.

### 3.2 Metric preserving map

We start with the Riemannian submanifold
\[
\mathcal{S}_m = S^{m-1} \setminus \bigcup_{k=1}^{m} \mathcal{N}^{(k)}_m,
\]
(32)
with
\[
\mathcal{N}^{(k)}_m = \{ w \in S^{m-1} \mid w_k = 0 \} \quad (k = 1, 2, \cdots, m)
\]
(33)
of $S^{m-1}$. From (32) and (33), we immediately obtain the coincidence

$$T_wS_m = T_wS_m^{m-1} \quad (w \in S_m \subset S^{m-1}) \quad (34)$$

of the tangent spaces $T_wS_m$ and $T_wS_m^{m-1}$ (see also (18)). Then, the metric $(\langle \cdot, \cdot \rangle)^S$ of $S_m$ is defined by

$$\langle (u, u') \rangle_w^S = \langle (\iota^S_{*,w}(u), \iota^S_{*,w}(u')) \rangle_w^{S_{ph}} \quad (u, u' \in T_wS_m), \quad (35)$$

where $\iota^S_{*,w}$ denotes the differential of the inclusion map

$$\iota^S: w \in S_m \mapsto w \in S^{m-1} \quad (36)$$

at $w \in S_m$ (see Appendix A for the differential of maps). According to (36) and Appendix A, $\iota^S_{*,w}(u)$ turns out to be

$$\iota^S_{*,w}(u) = u \in T_w\hat{P}_m \quad (u \in T_wS_m). \quad (37)$$

We move to consider the Riemannian submanifold

$$D_m = \left\{ \Theta \in \hat{P}_m \left| \Theta = \operatorname{diag} (\theta_1, \ldots, \theta_m), \sum_{k=1}^m \theta_k = 1, \theta_k > 0 \ (k = 1, 2, \ldots, m) \right\} \quad (38)$$

of the QIS $\left(\hat{P}_m, (\langle \cdot, \cdot \rangle)^{QF}\right)$. The tangent space $T_{\Theta}D_m$ at $\Theta$ takes the form

$$T_{\Theta}D_m = \left\{ Z \in M(m, m) \left| Z = \operatorname{diag} (\zeta_1, \ldots, \zeta_m), \sum_{j=1}^m \zeta_j = 0 \right\} \subset T_{\Theta}\hat{P}_m. \quad (39)$$

The metric $(\langle \cdot, \cdot \rangle)^D$ of $D_m$ is defined by

$$\langle (Z, Z') \rangle^D_{\Theta} = \langle (\iota^D_{*,\Theta}(Z), \iota^D_{*,\Theta}(Z')) \rangle^D_{\Theta} \quad (Z, Z' \in T_{\Theta}D_m), \quad (40)$$

where $\iota^D_{*,\Theta}$ denotes the differential of the inclusion map

$$\iota^D: \Theta \in D_m \mapsto \Theta \in \hat{P}_m \quad (41)$$

at $\Theta \in D_m$ (see Appendix A). Like in the case of $\iota^S_{*,w}$ (see (37)), $\iota^D_{*,\Theta}(Z)$ turns out to be

$$\iota^D_{*,\Theta}(Z) = Z \in T_{\Theta}\hat{P}_m \quad (Z \in T_{\Theta}D). \quad (42)$$
Under the preparation above, we consider the smooth map $\mu$ of $S_m$ to $D_m$ of the form
\[
\mu : w = (w_1, w_2, \ldots, w_m)^T \in S_m \mapsto \text{diag}(w_1^2, w_2^2, \ldots, w_m^2) \in D_m,
\]
whose differential of the map $\mu_*,w$ at $w$ takes the form
\[
\mu_* w(u) = 2 \text{diag}(u_1, u_2, \ldots, u_m) (u \in T_{w}S_m).
\]
We have the following lemma for $\mu$.

**Lemma 3.1.** The map $\mu$ satisfies (i)-(v) in the following:

(i) The $\mu$ is surjective.

(ii) The $\mu$ is metric-preserving up to the constant multiple 4:
\[
4 (u, u')_w^S = ((\mu_*,w(u), \mu_*,w(u'))^D_{\mu(w)} (w \in S_m, u, u' \in T_w S_m)
\]
holds true, where $T_w S_m$ is defined by $34$ with $18$.

(iii) Under the $(Z_2)^m$-action on $S_m$ defined by
\[
\phi_\sigma : w = (w_1, w_2, \ldots, w_m)^T \in S_m \mapsto (\sigma_1 w_1, \sigma_2 w_2, \ldots, \sigma_m w_m)^T \in S_m
\]
with
\[
\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)^T \in (Z_2)^m \quad (Z_2 = \{-1, 1\}),
\]
the $\mu$ is invariant;
\[
\mu(\phi_\sigma(w)) = \mu(w) \quad (w \in S_m, \sigma \in (Z_2)^m).
\]

(iv) The coincidence $\mu(w) = \mu(w')$ holds true if and only if there exists a certain $\sigma \in (Z_2)^m$ subject to $w' = \phi_\sigma(w)$.

(v) The restrict of $\mu$ to each of
\[
S^\sigma = \{w \in S_m | \sigma_j w_j > 0 (j = 1, 2, \ldots, m)\} \quad (\sigma \in (Z_2)^m)
\]
is diffeomorphic (smooth, injective and surjective) of $S^\sigma$.

The items other than (ii) in Lemma 3.1 are proved easily by straightforward calculations. The proof of (ii) is consigned in Appendix B.

**Remark 1.** If we apply $34$ formally to any $w \in N^{(k)}_m$, $\mu(w)$ is out of $\tilde{P}_m$. This is the account for eliminating $N^{(k)}_m$'s from $S^{m-1}$ to consider $S_m$.

**Remark 2.** From $44$ and $34$ with $18$, we have rank $\mu_*,w = m - 1 = \dim D_m$. The map $\mu$ is hence called an immersion $15$.

**Remark 3.** In view of (i) and (iv) in Lemma 3.1, $D_m$ is understood as the quotient space $S_m/(Z_2)^m$ of $S_m$. Further, combining (v) with (i) and (iv), we see that $S^\sigma$ with every $\sigma \in (Z_2)^m$ is diffeomorphic to $D_m$. 


\[

\]
4 The ALEH on the QIS

We are now in a position to construct the gradient system on the QIS realizing an extension of the ALEH by making full use of the geometric devices developed in section 3.

4.1 Mapping the ALEH through $\mu^*, w$

In this subsection, the vector field for the ALEH (12) on $S_m$ is shown to be mapped to a vector field on $D_m$. Note that the fact above is not so trivial since the map $\mu$ is not injective (see (ii) of Lemma 3.1 and Remark 3).

In view of (i) of Lemma 3.1 and Remark 1, we start with the restrict of the ALEH (12) to $S_m$. Since $\text{grad}\Lambda$ satisfies

$$\text{grad}\Lambda(w) \in T_w N_m^{(k)} \quad (k = 1, 2, \ldots, m),$$

all trajectories with initial condition $w(0) \in N_m^{(k)}$ are confined in $N_m^{(k)}$. On the same account, all the trajectories with $w(0) /\not\in \bigcup_{k=1}^m N_m^{(k)}$ never intersect with $\bigcup_{k=1}^m N_m^{(k)}$. Accordingly, the ALEH (12) can be dealt with on $S_m = S_m - \bigcup_{k=1}^m N_m^{(k)}$.

The next thing to show is the $(\mathbb{Z}_2)^m$-invariance of $\text{grad}\Lambda$, which is equivalent for

$$\text{grad}\Lambda(\phi\sigma(w)) = (\phi\sigma)_s w(\text{grad}\Lambda(w)) \quad (w \in S_m, \sigma \in (\mathbb{Z}_2)^m)$$

(see (46) and (47) for $\phi\sigma$). Indeed, on taking the expression

$$(\phi\sigma)_s w(u) = (\sigma_1 u_1, \sigma_2 u_2, \ldots, \sigma_m u_m)^T \quad (u \in T_w S, \sigma \in (\mathbb{Z}_2)^m)$$

into account (see Appendix A for the differential of maps), Equation (51) is shown to hold true by a simple calculation.

The $(\mathbb{Z}_2)^m$-invariance thus shown is put together with (iv) of Lemma 3.1 to ensure the existence of the vector field $\mu^* \text{grad}\Lambda$ on $D_m$ subject to

$$\mu^* \text{grad}\Lambda(\mu(w)) = \mu^* w(\text{grad}\Lambda(w)) \quad (w \in S_m).$$

(53)

Note that $\mu^* \text{grad}\Lambda$ is well-defined since the equation

$$\mu^* w(\text{grad}\Lambda(w)) = (\phi\sigma)_s w(\text{grad}\Lambda(w))$$

holds true. The first equality follows from (51) and the second one from

$$(\phi\sigma)_s w \circ (\phi\sigma)_s w = (\mu \circ \phi\sigma)_s w = \mu^* w$$

(55)

with (48). Equation (55) follows from (44) with (52) immediately.
Remark 4. On closing this subsection, we give a naive description of the vector field $\mu_*\nabla\Lambda$ on $D_m$. Recalling Remark 3, we may understand that the image $D_m$ of the map $\mu$ is a $2^m$-fold copy of $S_m$. Therefore, in view of (54), the vector field $\mu_*\nabla\Lambda$ on $D_m$ can be regarded as a $2^m$-folded copy of the ALEH on $S_m$.

4.2 The gradient system on the QIS realizing the ALEH

We are now in a position to seek a gradient system on the QIS realizing the ALEH. Namely, what we are to seek is a gradient system that realizes

$$\text{grad}L(\Theta) = \mu_*\nabla\Lambda(\Theta) \quad (\Theta \in D_m),$$

where $L$ denotes the potential for the gradient system. We note here that the gradient vector field $\text{grad}L$ associated with the potential $L$ is defined on the QIS $(\dot{P}_m, (\cdot, \cdot)^{QF})$ in the following way (cf. (20) and (21)). For a sufficiently small interval $[a, b]$ with $a < 0 < b$, let us associate a smooth curve $r : [a, b] \to \dot{P}_m$ with any $\Xi \in T_{\dot{P}_m}$ in the manner

$$\tau \in [a, b] \mapsto r(\tau) \in \dot{P}_m, \quad r(0) = \rho, \quad \frac{dr}{d\tau}\bigg|_{\tau=0} = \Xi \in T_{\dot{P}_m}. \quad (57)$$

Then the gradient vector field $\text{grad}L$ is defined to satisfy

$$((\text{grad}L(\rho), \Xi)^{QF}_\rho = \left. \frac{d}{d\tau}\right|_{\tau=0} L(r(\tau)) \quad (\Xi \in T_{\dot{P}_m}). \quad (58)$$

Instead of (56), we are to consider a weaker condition than (56) below in order that we can fix a candidate easily for the potential. The weaker condition to be dealt with is

$$((\text{grad}L(\Theta), i_{\ast\Theta}^D Z)^{QF}_\Theta = \left. \text{(\mu_*\nabla\Lambda(\Theta), Z)}^D\right|_{\Theta} (\Theta \in D_m, Z \in T_{\Theta}D_m). \quad (59)$$

We show the following Lemma.

Lemma 4.1. If the potential $L$ for a gradient system $(\dot{P}_m, (\cdot, \cdot)^{QF}, L)$ satisfies

$$L(\mu(w)) = 4\Lambda(w) \quad (w \in S_m),$$

then (59) holds true.
Proof It follows from (43) that the unique \( w \in \mathcal{S}_m^{\sigma} \) with \( \sigma = (1, 1, \cdots, 1)^T \) subject to \( \mu(w) = \Theta \), where \( \mathcal{S}_m^{\sigma} \) is defined by (49). Further, recalling (v), we can take the unique \( u \in T_w \mathcal{S}_m \) subject to \( \mu\star w u = Z \). Then for a sufficiently small interval \([a, b]\) with \( a < 0 < b \), we consider a curve \( \gamma(\tau) \) subject to (20) and a curve \( r(t) = \mu(\gamma(t)) \). The setting above is put together with (21), (45), (53), (58) and the assumption (60) to show

\[
\left( \langle \nabla L(\Theta), \mu\star w \nabla \Lambda(w) \rangle \right)_{\Theta} = (\langle \nabla \Lambda(w), Z \rangle)_{\Theta} = \frac{d}{d\tau} \bigg|_{\tau=0} L(\mu(\gamma(\tau)))
\]

This completes the proof.

Owing to Lemma 4.1 we can choose \( L(\rho) = -2 \text{tr}(C\rho) \) (62) as a candidate for the potential realizing (56), where \( C \) is the diagonal matrix given in (8). Note that we can confirm (60) for \( L \) of (62) by the calculation,

\[
L(\mu(w)) = -2 \text{tr}(C\mu(w)) = -2 \sum_{j=1}^m c_j w_j^2 = 4\Lambda(w),
\]

made with (19) and (43). We move to draw the gradient equation for \( L \) of (62) along with the framework of gradient systems on the QIS developed by the authors [10, 11]. Following to [10, 11], we define the Hermitian matrix \( \mathcal{M}(L) \) for \( L \) to be

\[
(\mathcal{M}(L))_{jk} = \begin{cases} 
\frac{\partial L}{\partial \rho_{jk}} = \frac{\partial L}{\partial \rho_{jk}} & (1 \leq j < k \leq m) \\
\frac{\partial L}{\partial \rho_{jj}} & (j = k = 1, 2, \cdots, m),
\end{cases}
\]

where the partial differentiations, \( \partial/\partial \rho_{jk} \) and \( \partial/\partial \rho_{jk} \), stand for

\[
\frac{\partial}{\partial \rho_{jk}} = \frac{1}{2} \left( \frac{\partial}{\partial \xi_{jk}} - i \frac{\partial}{\partial \eta_{jk}} \right) \quad (1 \leq j < k \leq m)
\]

\[
\frac{\partial}{\partial \rho_{jk}} = \frac{1}{2} \left( \frac{\partial}{\partial \xi_{jk}} + i \frac{\partial}{\partial \eta_{jk}} \right) \quad (j = k = 1, 2, \cdots, m),
\]
with
\[ \xi_{jk} = \Re(\rho_{jk}), \quad \eta_{jk} = \Im(\rho_{jk}) \quad (1 \leq j < k \leq m). \]  
(66)
The symbols $\Re$ and $\Im$ indicate the real part and the imaginary one, respectively. In contrast, the $\rho_{jj}$s are thought of as real variables to give rise to $\partial/\partial \rho_{jj}$s in usual way. In terms of $\mathcal{M}(L)$, the gradient equation for $L$ is written as
\[ \text{grad}L(\rho) = \frac{1}{2} \left( \rho \mathcal{M}(L) + \mathcal{M}(L)\rho \right) - \left( \text{tr}(\rho \mathcal{M}(L)) \right)\rho \]  
(67)
(see [10, 11]). Since we have
\[ \mathcal{M}(L) = -2C \]  
(68)through a straightforward calculation of (64) with (62), the gradient vector field for $L$ is expressed as
\[ \text{grad}L(\rho) = -(\rho C + C \rho) + 2\left( \text{tr}(\rho C) \right)\rho \]  
(69)
with the diagonal matrix $C$ of (8). To summarize, we have the following.

**Lemma 4.2.** The gradient system $(\dot{P}_m, ((\cdot, \cdot))^{QF}, L)$ is governed by the differential equation of motion
\[ \frac{d\rho}{dt} = (\rho C + C \rho) - 2\left( \text{tr}(\rho C) \right)\rho, \]  
(70)
where $C$ is the diagonal matrix of (8).

We are at the final stage to show (56) for $L$ of (62). On combining (22) with (44), the rhs of (56) is calculated to be
\[ \mu_{*,w}(\text{grad} \Lambda(w)) = 2 \text{diag} \left( w_1(\text{grad} \Lambda(w))_1, w_2(\text{grad} \Lambda(w))_2, \cdots, w_m(\text{grad} \Lambda(w))_m \right) \]  
(71)
with
\[ (\text{grad} \Lambda(w))_j = -c_j w_j + \left( \sum_{k=1}^{m} c_k w_k^2 \right) w_j \quad (j = 1, 2, \cdots, m), \]  
(72)
while (69) is put together with (13) to show
\[ \text{grad}L(\mu(w)) = -2 \text{diag}(c_1 w_1^2, c_2 w_2^2, \cdots, c_m w_m^2) \]  
\[ + 2 \left( \sum_{k=1}^{m} c_k w_k^2 \right) \text{diag}(c_1 w_1^2, c_2 w_2^2, \cdots, c_m w_m^2) \]  
(73)
for $w \in \mathcal{S}_m$. Equations (71) - (73) therefore confirms (56) for $L$ of (62). Finally, we reach to the following theorem.
Theorem 4.3. The gradient system $(\dot{P}_m, \langle \cdot , \cdot \rangle^{QF}, L)$ governed by the equation of motion (70) realizes the ALEH in the QIS in the following sense: In the manner of (56), the gradient vector field $\text{grad} L$ realizes the vector field mapped through (53) from the gradient vector field describing the ALEH on $S_m$.

5 Concluding remarks

We have constructed successfully the gradient system on the QIS (GS-QIS) which realizes the ALEH on the submanifold $D_m$. The success is due to the geometric devices developed in Section 3: Especially, a key fact is that the immersion $\mu$ of $S_m$ to $D_m$ is put together with the $(\mathbb{Z}_2)^m$-symmetry of the ALEH to realize the $2^m$-folded copy of the ALEH on $D_m$.

The behavior of trajectories of the GS-QIS not on $D$ is open still, though the trajectories on $D$ are understood by Nakamura [7]: The GS-QIS is expected to have a ‘global convergence’ property.

Integrability of the GS-QIS is an open question, too. However, in the special case of $C = I$, integrability is allowed like in the cases of [8] and [10, 11] with $C = -2I$, since the $U(m)$ action,

$$\rho \in \dot{P}_m \mapsto h\rho h^\dagger \in \dot{P}_m \quad (h \in U(m)), \quad (74)$$

is allowed to be a symmetry, whose role is studied in [8].

Relation of the gradient systems obtained in [8, 9, 10, 11] to physical systems would be a big problem. The horizontal lift of gradient vectors to those on the space of ordered-tuples of multi-qubit states developed in [8, 9] could be a clue, and the work of Braunstein [16] another one. Further, the authors might say that the complementary property between gradient vector fields and the Hamiltonian ones in classical mechanics is worth taken together with the horizontal lift.

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A Differential of maps

With the aim of a brief and clear description, we restrict to our attention to smooth maps among spaces in $M(m, m)$, the space of $m \times m$ complex
matrices. Let $M_1$ and $M_2$ be manifolds in $M(m, m)$ and $\psi$ a map of $M_1$ to $M_2$:

$$\psi : M_1 \rightarrow M_2. \quad (75)$$

For a given point $p \in M_1$, the differential $\psi_{*,p}$ of $\psi$ at $p$ is defined as follows. Like (20), for a sufficiently small interval $[a, b]$ with $a < 0 < b$, let us associate a smooth curve $q : [a, b] \rightarrow M_1$

$$\tau \in [a, b] \mapsto q(\tau) \in M_1, \quad q(0) = w, \quad \frac{dq}{d\tau} \bigg|_{\tau=0} = w \in T_p M_1. \quad (76)$$

On using the smooth curve $q(\tau)$ introduced above, the differential $\psi_{*,p}$ of the map $\psi$ at $p$ is defined to be

$$\psi_{*,p}(v) = \left. \frac{d}{d\tau} \right|_{\tau=0} \psi(q(\tau)) \quad (v \in T_p M_1). \quad (77)$$

(see [15], for example). The differential maps, $\psi^*, w, \iota_D^*, \mu_*^*, w$ and $(\phi_\sigma)_*, w$ are defined by (76) and (77) with $(\mathcal{S}_m, \mathcal{S}_m^{-1}, w, \psi^S), (\mathcal{D}_m, \mathcal{D}_m, \Theta, \mu^D), (\mathcal{S}_m, \mathcal{D}_m, \mathcal{w}, \psi_\sigma)$ in place of $(M_1, M_2, p, \psi)$.

### B Proof of Lemma 3.1

We show (ii) of Lemma 3.1. On setting $h = id$, $\Xi = Z$ and $\Xi' = Z'$ in (28) and (31), Equation (30) is put together with (40) to show

$$\langle (Z, Z') \rangle^D = \sum_{k=1}^{m} \frac{\zeta_k \zeta_k'}{\theta_k}. \quad (78)$$

Further the substitutions $Z = \mu_*^*(w)$ and $Z' = \mu_*^*(w')$ with (44) bring us to have

$$\langle (Z, Z') \rangle^D = \sum_{k=1}^{m} \frac{\zeta_k \zeta_k'}{\theta_k} = \sum_{k=1}^{m} \frac{(2w_k u_k)(2w_k u'_k)}{w_k^2} = 4 \sum_{k=1}^{m} u_k u'_k = 4(\langle u, w' \rangle)_{w}^S. \quad (79)$$

This ends the proof.

### References

[1] P.W.Shor, Proceedings of 35th Annual Symposium on Foundations of Computer Science (Los Alamitos, IEEE Press), 124 (1994).
[2] L.K.Grover, *Proceedings of 28th Annual ACM Symposium on the Theory of Computing* (New York, ACM), 212 (1996).

[3] M.A.Nielsen and I.L.Chuang, *Quantum Computation and Quantum Information* (Cambridge: Cambridge UP), Chaps 1 and 2 (2000).

[4] Y.Nakamura, *Japan J. Indust. Appl. Math.* 9, 133 (1992).

[5] Y.Nakamura, *Japan J. Indust. Appl. Math.* 10, 179 (1993).

[6] Y.Nakamura, *Japan J. Indust. Appl. Math.* 11, 1 (1994).

[7] Y.Nakamura, *Japan J. Indust. Appl. Math.* 11, 11 (1994).

[8] Y.Uwano, *Czech. J. Phys.* 56, 1311 (2006).

[9] Y.Uwano, H.Hino and Y.Ishiwatari, *Phys. Atom. Nuclei* 70, 784 (2007).

[10] Y.Uwano and H.Yuya, arXive:0807.4053v1 [math.DS] (2008).

[11] Y.Uwano and H.Yuya, *A gradient system on the quantum information space realizing the Karmarkar flow for linear programming – a clue to effective algorithms – (a revised version of [10]), submitted to EJTP* (2009).

[12] E.Oja, *J. Math. Biology*, 15, 267 (1982).

[13] D.O.Hebb, *The Organization of Behavior* (New York, Wiley), (1949).

[14] H.J.Kushner and D.S.Clark, *Stochastic Approximation Methods for Constrained and Unconstrained Systems* (New York, Springer-Verlag), (1978).

[15] S.Kobayashi and K.Nomizu, *Foundations of Differential Geometry vol.2* (New York, John Wiley), 337 (1969).

[16] S.L.Braunstein, *Physics Letters A*, 219, 169 (1996).

[17] A.Fujiwara, *Geometry in Present Day Science*, eds O.E.Bandorff-Nielsen and E.B.V.Jensen (World Scientific, Singapore), 35 (1999).

[18] S.Amari and H.Nagaoka, *Methods of Information Geometry*, Translations of Mathematical Monographs vol.191 (Providence, AMS), Chap. 7.3 (2000).

[19] C.W.Helstrom, *Quantum Detection Theory* (New York, Academic Press), 117 (1976).