Abstract: We introduce the natural \((t, q)\)-deformation of the Q-system algebra in type A. The \(q\)-Whittaker limit \(t \to \infty\) gives the quantum Q-system algebra of Di Francesco and Kedem (Lett Math Phys 107(2):301–341, DFK17), a deformation of the Groethendieck ring of finite dimensional Yangian modules, compatible with graded tensor products (Hatayama et al. in: Recent Developments in Quantum Affine Algebras and Related Topics (Raleigh, NC, 1998), Volume 248 of Contemporary Mathematics, Amer. Math. Soc., Providence, HKO+99; Feigin and Loktev in: Differential Topology, Infinite-Dimensional Lie Algebras, and Applications, Volume 194 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, FL99; Di Francesco and Kedem in Int Math Res Not IMRN 10:2593–2642, DFK14). We show that the \((q, t)\)-deformed algebra is isomorphic to the spherical double affine Hecke algebra of type \(\mathfrak{gl}_N\). Moreover, we describe the kernel of the surjective homomorphism from the quantum toroidal algebra (Miki in J Math Phys 48(12):123520, Mik07) and the elliptic Hall algebra (Schiffmann and Vasserot in Compos Math 147(1):188–234, SV11) to this new algebra. It is generated by \((q, t)\)-determinants, new objects which are a deformation of the quantum determinant associated with the quantum Q-system. The functional representation of the algebra is generated by generalized Macdonald operators, obtained from the usual Macdonald operators by the \(SL_2(Z)\)-action on the spherical Double Affine Hecke Algebra. The generating function for generalized Macdonald operators acts by plethysms on the space of symmetric functions. We give the relation to the plethystic operators from Macdonald theory of Bergeron et al. (J Comb 7(4):671–714, BGLX16) in the limit \(N \to \infty\). Thus, the \((q, t)\)-deformation of the Q-system cluster algebra leads directly to Macdonald theory.
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1. Introduction and Main Results

1.1. Introduction. The classical Q-systems are recursion relations satisfied by characters of Kirillov–Reshetikhin (KR)-modules of current algebras, Yangians or quantum affine algebras. They first appeared in the context of state counting in the Bethe Ansatz solutions of generalized Heisenberg spin chains [KR87]. The connection between Q-systems and fermionic formulas for the multiplicities of irreducible g-modules in the tensor product of KR-modules is explained in [HKO+99,DFK08].

Quantum Q-systems are a natural non-commutative deformation of Q-systems by a parameter q, introduced in [DFK14] as a tool to prove the q-graded multiplicity formulas for the same tensor products [KR87,HKO+99]. Such formulas were first introduced in [KR87] in terms of the Bethe ansatz combinatorics. In the case of current algebras g[t], the definition of the grading on tensor products was given in [FL99]. The physical meaning of the grading [KKMM93] originates from the observation that, for sufficiently uniform Heisenberg spin chains, there is an effective conformal field theory which describes the model in the critical limit. The partition function of the chiral conformal field theory, given by characters of Virasoro modules, is given by the appropriate limit of the graded tensor product characters [Ked04].

The physical motivation for a further deformation, by an additional parameter t, is inspired by the AGT correspondence between (q, t)-deformed conformal blocks involving deformed Gaiotto vectors for the Verma modules of the deformed Virasoro algebra [AGT10,AFH+11] and Nekrasov instanton partition functions in five dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theories [Nek03]. The Fermionic character formulas are a degeneration of these in special cases, and in particular have a similar combinatorial structure.

The mathematical motivation for the t-deformation is suggested by the structure of the algebra \( A_{(q)} \) associated with the quantum Q-system [DFK17], originally defined [DFK14] using the natural quantization of the associated cluster algebra [BZ05]. In [DFK17], we explored the structure of the algebra for \( g = \mathfrak{g}l_N \) as a current algebra. It was shown to be isomorphic to an N-dependent quotient of a quantum affine algebra \( U_v(n[t, t^{-1}]) \subset U_v(\mathfrak{sl}_2) \), with \( v = \sqrt{q} \), by the ideal generated by polynomial relations given by a quantum determinant of generators [DFK11].

Moreover, the algebra \( A_{(q,t)} \) has a natural representation in terms of finite q-difference operators acting on the space of symmetric polynomials, which are spanned by the characters of graded tensor products of KR-modules. These difference operators, and the generators they represent, are more naturally understood as \( t \rightarrow \infty \) limits of generators of a new algebra, \( A_{(q,t)}^{(q)} \), which is less singular than \( A_{(q)}^{(q)} \).

We prove that \( A_{(q,t)}^{(q)} \) is isomorphic to the spherical double affine Hecke algebra (sDAHA) of type \( \mathfrak{g}l_N \). The difference operators which generate the functional representation of \( A_{(q,t)}^{(q)} \) are \( SL_2(\mathbb{Z}) \)-translated Macdonald operators. In the limit \( t \rightarrow \infty \), these correspond to operators acting on the graded characters by adding one additional KR-module to the tensor product.

In addition we show that, as in the quantum case, the algebra \( A_{(q,t)}^{(q)} \) can be understood as a quotient of the Ding–Iohara–Miki algebra, also known as quantum toroidal algebra of \( \mathfrak{g}l_N \) or deformed W-algebra [Mik07], which is a deformation of the quantum affine algebra. The quotient is again given by an ideal generated by polynomial relations, which we refer to as \( (q,t) \)-determinants.

The study of the polynomiality of the \( (q,t) \)-determinants is at the heart of this paper. That such relations should exist originates in the quantum cluster algebra formula-
tion of the quantum Q-system: the cluster structure ensures many (Laurent) polynomial properties that still hold after quantization. More such relations can be obtained in the $t$-deformed setting by using known correspondences between type $A_\infty$ sDAHA and Elliptic Hall Algebra (EHA) [SV11].

1.2. Quantum Q-systems and fusion products. Let us recall some relevant results about quantum Q-systems [DFK14] and the algebra $\mathcal{A}_{\mathfrak{gl}_N}(q)$ generated by the solutions of the quantum Q-system in type $A$. These are relations satisfied by the generators of a non-commutative algebra obtained from the natural quantization [BZ05] of the Q-system cluster algebras [Ked08,DFK11]. The classical cluster algebra is defined by interpreting the Q-systems as mutation or exchange relations for cluster variables, thus yielding the initial quiver.

In [DFK14], we defined this algebra for $\mathfrak{g}$ a simple, simply-laced Lie algebra of rank $r$.

Definition 1.1. The quantized algebra $\mathcal{N}_{\mathfrak{g}}(q)$ is a subalgebra of the quantum Q-system cluster algebra, generated by the non-commuting elements

\begin{equation}
Q_{\alpha,k}Q_{\beta,k+1} = q^{\lambda_{\alpha,\beta}}Q_{\beta,k+1}Q_{\alpha,k}, \quad \alpha, \beta \in [1, r], \ k \in \mathbb{Z},
\end{equation}

and the $q$-exchange relations (quantum mutations):

\begin{equation}
q^{\lambda_{\alpha\alpha}}Q_{\alpha,k+1}Q_{\alpha,k-1} = Q_{\alpha,k} - \prod_{\beta: C_{\alpha,\beta} = -1} Q_{\beta,k}, \quad \alpha \in [1, r], \ k \in \mathbb{Z}.
\end{equation}

Here, $C$ is the Cartan matrix of $\mathfrak{g}$ and $\lambda = |\det C|C^{-1}$.

As for any quantum cluster algebra, the cluster variables $Q_{\alpha,k}$ are polynomials in, for example, the finite set of elements of $S_0 := \{Q_{\alpha,0}, Q_{\alpha,1} : 1 \leq \alpha \leq r\}$ (the initial cluster), due to the Laurent property [BZ05] of quantum cluster algebras. Note that these elements belong to a localization of the algebra $\mathcal{N}_{\mathfrak{g}}(q)$: In a cluster algebra, all cluster variables are assumed to be invertible.

As in [DFK17], we focus on the case $\mathfrak{g} = \mathfrak{gl}_N$. In this case, it is more convenient to use renormalized generators $M_{\alpha,k}$, which satisfy the so-called M-system:

\begin{equation}
q^\alpha M_{\alpha,k+1} M_{\alpha,k-1} = M_{\alpha,k}^2 - M_{\alpha+1,k}M_{\alpha-1,k}, \quad \alpha \in [1, N],
\end{equation}

\begin{equation}
M_{\alpha,k}M_{\beta,k+1} = q^{\min(\alpha,\beta)} M_{\beta,k+1} M_{\alpha,k}, \quad \alpha, \beta \in [1, N], \ k \in \mathbb{Z},
\end{equation}

\begin{equation}
M_{0,k} = 1, \quad M_{N+1,k} = 0.
\end{equation}

The following result shows that in the case $\mathfrak{g} = \mathfrak{gl}_N$, the algebra $\mathcal{N}_{\mathfrak{gl}_N}(q)$ is generated by a finite subset of generators, without the use of inverses:

Theorem 1.2 ([DFK17, Theorem 2.11]). The algebra $\mathcal{N}_{\mathfrak{gl}_N}(q)$ is generated by any subset of $2N$ elements with consecutive indices of the set $S = \{M_{1,n} : n \in \mathbb{Z}\}$. Each generator $M_{\alpha,n}$ can be expressed as a homogeneous polynomial of degree $\alpha$ in the elements of $S$. This polynomial is the quantum determinant of the discrete
Wronskian matrix \((M_{1; n+i+j-\alpha-1})_{1 \leq i, j \leq \alpha})\), defined in terms of the generating function

\[ M(z) := \sum_{n \in \mathbb{Z}} z^n M_{1; n} \]

as follows:

\[ M_{\alpha; n} = Q \left( \frac{1}{(u_1 \cdots u_{\alpha})^n} \prod_{1 \leq i < j \leq \alpha} (1 - qu_j / u_i) \prod_{i=1}^{\alpha} M(u_i) \right), \]

where the notation \(Q\) for the constant term in the variables \(u_i\) in the expansion of the Laurent series.

The fact that only a finite subset of generators in \(S\) is needed is not immediate, but follows from the fact that the generators in \(S\) satisfy a linear recursion relation with constant coefficients, which we interpret as the conserved quantities of quantum Q-system viewed as a discrete time recursion relation.

Therefore, \(N^q_{\mathfrak{g}l_N}\) is the algebra generated by \(\{M_{1, n} : n \in \mathbb{Z}\}\) modulo the relations (1.1). These relations have a rather compact expression in terms of generating functions.

**Theorem 1.3.** [DFK17] The relations among the generators in the set \(S\) are given by the exchange relations

\[ (z - q w)M(z)M(w) + (w - qz)M(w)M(z) = 0, \quad (1.3) \]

and the finite rank condition amounts to the vanishing of the \((N + 1)\)st quantum determinants

\[ M_{N+1; n} = 0, \quad n \in \mathbb{Z}. \quad (1.4) \]

That is, the algebra \(N^q_{\mathfrak{g}l_N}\) is isomorphic to the quotient of \(U_{\sqrt{q}}(n[t, t^{-1}])\) by the ideal generated by all the order \(N + 1\) quantum Wronskian determinants. Here, \(n \subset \mathfrak{sl}_2\) is the nilpotent subalgebra.

For any simple Lie algebra \(\mathfrak{g}\), the quantum Q-system is the “right” deformation of the Grothendieck ring of the current algebra which accounts for the graded structure of the fusion product:

**Theorem 1.4** ([DFK14, Theorem 5.17]). For any simple Lie algebra, there is a linear functional \(\phi\) from the algebra \(N^q_{\mathfrak{g}l_N}\) to the set of Weyl-symmetric functions with coefficients in \(\mathbb{C}q = \mathbb{C}[q, q^{-1}]\), such that the image of the ordered product \(\prod_{j=1}^{k} \prod_{\alpha=1}^{r} Q^{n_{\alpha,j}}_{\alpha, j}\) is the \(q\)-graded character of the Feigin–Loktev fusion product [FL99] of the corresponding Kirillov–Reshetikhin modules.

The fusion product of KR-modules is specified by the multi-index \(n = \{n_{\alpha, k} : \alpha \in [1, r], k \in \mathbb{N}\}\), where \(n_{\alpha, k}\) is the number of KR-modules with \(\mathfrak{g}\)-highest weight \(k\omega_{\alpha}\) (denoted by \(\text{KR}_{\alpha, k}\)) in the product. The character \(\text{ch}_n(q, x)\) of such a fusion product is a Weyl-symmetric function of the variables \(x = (x_1, \ldots, x_r)\) with coefficients in \(\mathbb{Z}_+[q]\). To describe how adding one more factor to the fusion changes the character, one needs to specify an action on the space of symmetric functions.
Theorem 1.5. [DFK17] When \( \mathfrak{g} = \mathfrak{gl}_N \), adding the factor \( \mathfrak{KR}_{\alpha,k} \) to the fusion product specified by the multi-index \( \mathbf{n} \) corresponds to acting on the graded character \( \text{ch}_n(q^{-1}, \mathbf{x}) \) by the difference operator

\[
\mathcal{D}_{\alpha,k}^{(q)} = \sum_{I \subseteq [1,N]} \left( \prod_{i \in I} x_i \right)^k \left( \prod_{i \in I, j \notin I} \frac{x_i}{x_i - x_j} \right) \prod_{i \in I} \Gamma_i, \tag{1.5}
\]

where \( \Gamma_i \) acts as a multiplicative shift of the \( i \)-th variable \( x_i \) by \( q \):

\[
\Gamma_i f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) = f(x_1, \ldots, x_{i-1}, q x_i, x_{i+1}, \ldots, x_N). \tag{1.6}
\]

Here, it is assumed that \( \max \{m : n_{\alpha,m} > 0\} \leq k \).

One can understand these difference operators as “raising operators” on the space of symmetric functions spanned by the graded characters. The proof of the Theorem amounts to showing that the difference operators (1.5) obey the M-system (1.1).

In [DFK17], we also defined the algebra \( A_{\mathfrak{gl}_N}^{(q)} \), obtained, for finite \( N \), by adjoining to \( N_{\mathfrak{gl}_N}^{(q)} \) the element \( M_{N,0}^{-1} \), or equivalently, the invertible grading element \( \Delta^{-1} \). In this case one can write \( M_{N,k} = A^k \Delta \), where \( \Delta M_{\alpha,k} = q^{\alpha k} M_{\alpha,k} \Delta \). The invertible element \( A \) is identified as a coefficient in the cluster algebra. Equivalently,

Definition 1.6. The algebra \( A_{\mathfrak{gl}_N}^{(q)} \) is the algebra generated by the elements \( \{M_{\alpha,k}; \alpha \in [1,N], k \in \mathbb{Z}\} \) and the elements \( \{M_{\alpha,k}; \alpha \in [1,N], k \in \mathbb{Z}\} \).

If the generating function \( M(z) \) corresponds to Drinfeld currents \( x^+ \) of the quantum affine algebra of \( \mathfrak{sl}_2 \), the generating function \( \check{M}(z) \) corresponds to the currents \( x^- \). The resulting algebra satisfies relations in a quotient of a non-standard presentation of \( U_q(\widehat{\mathfrak{sl}}_2) \). In particular, in the limit \( N \to \infty \), the generators \( \check{M}_{1,k} \) vanish, and the limit is quite singular.

1.3. \( t \)-deformation: the main results. The exchange relations (1.3) and the difference operators (1.5) both suggest the natural \( t \)-deformation of the algebra \( A_{\mathfrak{gl}_N}^{(q,t)} \), which we call \( A_{\mathfrak{gl}_N}^{(q,t)} \). This algebra is, in several ways, more natural. In particular, the limit \( N \to \infty \) of the algebra \( A_{\mathfrak{gl}_N}^{(q,t)} \) retains all its generators, and it becomes a rather well-known algebra.

The first observation is that the difference operators (1.5) are a limit as \( t \to \infty \) of a generalization of the Macdonald difference operators for \( \mathfrak{g} = \mathfrak{gl}_N \) [Mac95]. In this limit, the Macdonald operators tend to the operators \( \mathcal{D}_{\alpha,0}^{(q)} \) of Eq. (1.5).

Definition 1.7. The generalized \( \mathfrak{gl}_N \) Macdonald operators are defined as

\[
\mathcal{D}_{\alpha,k}^{(q,t)} := \sum_{I \subseteq [1,N]} \left( \prod_{i \in I} x_i \right)^k \left( \prod_{i \in I, j \notin I} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \right) \prod_{i \in I} \Gamma_i, \quad \alpha \in [1,N], \; k \in \mathbb{Z}, \tag{1.7}
\]

with \( \theta := t^{1/2} \).
In the limit \( t \to \infty \), we recover the difference operators \((1.5)\), after scaling by a power of \( \theta \):

\[
\lim_{t \to \infty} \theta^{-\alpha(N-\alpha)} D_{\alpha;k}^{(q,t)} = D_{\alpha;k}^{(q)} .
\]

**Remark 1.8.** In the limit \( N \to \infty \), the operators \( D_{\alpha;n}^{(q,t)} \) with \( \alpha = 1 \) first appeared in the context of the deformed W-algebra in [Mik07].

**Definition 1.9.** The algebra \( \mathcal{A}_{gl_N}^{(q,t)} \) is generated by the operators \( D_{\alpha;n}^{(q,t)} \) \((1.7)\), together with \( \tilde{D}_{\alpha;n}^{(q,t)} := D_{\alpha;n}^{(-1,t^{-1})} \), \( n \in \mathbb{Z}, \alpha \in [1,N] \), in the functional (faithful) representation.

There are at least two ways of understanding the structure of the algebra \( \mathcal{A}_{gl_N}^{(q,t)} \). First, Macdonald theory can be understood via the Double Affine Hecke Algebra (DAHA) approach of Cherednik [Che05], and we shall identify the difference operators \((1.7)\) as generators of the spherical DAHA of type \( gl_N \) (Sect. 2) in the functional representation. The DAHA is generated by “position” \( X_i \) and “momentum” \( Y_i \) operators, \( i = 1, 2, \ldots, N \) as well as Hecke operators \( T_1, T_2, \ldots, T_{N-1} \) subject to relations (see Definition 2.1). The Macdonald operators \( D_{\alpha;0}^{(q,t)} \) correspond to the elementary symmetric functions of the \( Y \)'s, \( D_{\alpha;0} := e_\alpha(Y_1, \ldots, Y_N) \), in the functional representation of DAHA, when restricted to acting on symmetric functions.

In Sect. 2, it is shown that the natural \( SL_2(\mathbb{Z}) \) action on DAHA by automorphisms can be used to define generalized operators \( D_{\alpha;n} := q^{-an} e_\alpha(Y_1, n, \ldots, Y_N) \), where \( Y_{i,n} \) is the \( n \)-iterate action on \( Y_i \) of the automorphism \( \tau_q: Y_i \mapsto (X_1 \cdots X_{i-1})^{-1} Y_i (X_1 \cdots X_i) \). The main result in Sect. 2 is the following:

**Theorem 1.10.** Let \( \rho \) be the functional representation of the DAHA. The operators \( \rho(D_{\alpha;n}) \) leave the space of symmetric functions of the variables \( x_1, \ldots, x_N \) invariant, and are equal to

\[
\rho(D_{\alpha;n})|_{S_N} = D_{\alpha;n}^{(q,t)} . \tag{1.8}
\]

Theorem 2.14 then gives the isomorphism of algebras, \( \mathcal{A}_{gl_N}^{(q,t)} \simeq s\text{DAHA}(gl_N) \).

A second approach to understanding the algebra \( \mathcal{A}_{gl_N}^{(q,t)} \) is in terms of the quantum toroidal algebra of \( gl_1 \). There is an analogue of Theorem 1.2: The subalgebra \( \mathcal{N}_{gl_N}^{(q,t)} \), generated by \( \{ D_{\alpha;k}^{(q,t)} : k \in \mathbb{Z}, \alpha \in [1,N] \} \), is alternatively generated by the subset of elements represented by \( S = \{ D_{\alpha;1;n}^{(q,t)} \}_{n \in \mathbb{Z}} \), subject to quadratic exchange relations which are a \( r \)-deformation of \((1.3)\): the factor \((z - qw)(z - tw)(z - t^{-1} w)(z - q^{-1} t w) \). This factor, which appears in Definition 3.2, characterizes the quantum toroidal algebra [Mik07]. Moreover, the generators of the full algebra \( \mathcal{A}_{gl_N}^{(q,t)} \) satisfy all the relations of the quantum toroidal algebra at level \((0,0)\). Defining the generating functions \( e_1(z), f_1(z) \) by

**Definition 1.11.**

\[
e_1(z) := \frac{q^{\frac{1}{2}}}{1-q} \sum_{n \in \mathbb{Z}} q^{n/2} z^n D_{1;n}^{(q,t)} , \quad f_1(z) := e_1(z)|_{q \to q^{-1}, t \to t^{-1}} .
\]
Theorem 1.12. The fundamental Macdonald currents $\psi_1(z)$, $\psi_1(z)$, together with the series obtained from their commutator
\[
\psi^{\pm}(z) := \prod_{i=1}^{N} \frac{(1 - q^{1/2}t(zx_i)^{\pm 1})(1 - q^{1/2}t^{1}(zx_i)^{\pm 1})}{(1 - q^{-1/2}(zx_i)^{\pm 1})(1 - q^{-1/2}(zx_i)^{\pm 1})} \in \mathbb{C}[[z^{\pm 1}]] [x_1^{\pm 1}, \ldots, x_N^{\pm 1}] S_N,
\]
(1.9)
satisfy the level $(0, 0)$ quantum toroidal $\widehat{gl}_1$ algebra relations of Definition 3.2.

Therefore there is a homomorphism from the quantum toroidal algebra to $A_{\widehat{gl}_1}^{(q,t)}$. We claim that the kernel of the homomorphism is generated by $(q,t)$-determinants of degree $N + 1$. The $(q,t)$-determinant is defined by an equation analogous to (1.2), but with the function $(1 - qu_j/u_i)$ replaced by the function $(1 - q^{u_j}/u_i)^{(1-qt)/u_j}/u_i^t$, see Corollary 6.7 and Theorem 6.8. On the level of the functional representation, this is due to the fact that the number of variables $x_1, \ldots, x_N$ is finite.

This definition of the $(q,t)$-determinant does not give polynomial relations between the generators, due to the presence of the rational function in the definition. To see that it has a polynomial expression, we make use of a third formulation of the spherical DAHA in terms of the elliptic Hall algebra. To this end, we introduce a further generalization of the Macdonald operators:

Definition 1.13. For $\alpha \in [1, N]$, we consider the map $A_\alpha : \mathcal{F}_\alpha \to \text{Diff}_N$ from the space $\mathcal{F}_\alpha$ of symmetric rational functions of $\alpha$ variables to the space $\text{Diff}_N$ of difference operators acting on $\mathcal{F}_N$, defined as:
\[
A_\alpha(P) := \frac{1}{\alpha!(N-\alpha)!} \text{Sym} \left( P(x_1, x_2, \ldots, x_\alpha) \prod_{1 \leq i < j \leq N} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \Gamma_1 \Gamma_2 \cdots \Gamma_\alpha \right),
\]
(1.10)
where the symmetrization is over the $N$ variables $x_1, \ldots, x_N$.

The original Macdonald difference operators $D_{\alpha:0}^{(q,t)}$ correspond to $P = 1$ and the operators $D_{\alpha:k}^{(q,t)}$ in Eq. (1.7) corresponds to $P = (x_1 x_2 \cdots x_\alpha)^k$. In general, let $P = s_{a_1,\ldots,a_\alpha}(x_1, \ldots, x_\alpha) \in \mathcal{F}_\alpha$, $a_1, \ldots, a_\alpha \in \mathbb{Z}$ be the generalized Schur function
\[
s_{a_1,\ldots,a_\alpha}(x_1, x_2, \ldots, x_\alpha) := \frac{\det_{1 \leq i,j \leq \alpha} (x_i^{a_j + \alpha - i})}{\prod_{1 \leq i < j \leq \alpha} x_i - x_j} = \text{Sym} \left( \prod_{I=1}^{\alpha} x_i^{a_i + \alpha - i} \prod_{I \leq j} x_i - x_j \right).
\]
(1.11)

Definition 1.14. The operators $D_{a_1,\ldots,a_\alpha} = A(s_{a_1,\ldots,a_\alpha}(x_1, x_2, \ldots, x_\alpha))$ are difference operators corresponding to generalized Schur functions:
\[
D_{a_1, a_2, \ldots, a_\alpha}^{(q,t)} = \sum_{I \subseteq [1, N]} s_{a_1,\ldots,a_\alpha}(x_I) \prod_{i \in I} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \prod_{i \in I} \Gamma_i,
\]
(1.12)
where $x_I$ stands for the ordered collection of variables $(x_i)_{i \in I}$. 
As a particular case, we recover $\mathcal{D}_{\alpha;k}^{(q,t)} = \mathcal{D}_{k,k,\ldots,k}^{(q,t)}$, $k$ repeated $\alpha$ times, which when $k > 0$ corresponds to the usual Schur function $s_{k,k,\ldots,k}(x_1, \ldots, x_\alpha) = (x_1 x_2 \cdots x_\alpha)^k$.

We proceed by introducing yet another map $\mathcal{B}_\alpha : \mathcal{F}_\alpha \to \text{Diff}_N$:

**Definition 1.15.** To each symmetric rational function $P(x_1, x_2, \ldots, x_\alpha) \in \mathcal{F}_\alpha$ we associate the difference operator $\mathcal{B}_\alpha(P)$:

$$
\mathcal{B}_\alpha(P) := \frac{1}{\alpha!} \text{CT}_u \left( P(u_1^{-1}, u_2^{-1}, \ldots, u_\alpha^{-1}) \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - u_j)(u_i - qu_j)}{(u_i - tu_j)(u_i - qt^{-1}u_j)} \prod_{i=1}^\alpha \partial(u_i) \right)
$$

where the symbol CT stands for the constant term, the coefficient of $u_1^0 u_2^0 \cdots u_\alpha^0$ in the expansion, and where the generating function $\partial(u)$ is defined as:

$$
\partial(u) := \sum_{n \in \mathbb{Z}} z^n \mathcal{D}_{1;n}^{(q,t)} = \frac{1 - q}{q^t} \epsilon_1(q^{-\frac{1}{t}}z).
$$

We have the following remarkable result (see Sect. 7):

**Theorem 1.16.** For any symmetric rational function $P(x_1, \ldots, x_\alpha) \in \mathcal{F}_\alpha$, $1 \leq \alpha \leq N$, we have the identity:

$$
\mathcal{B}_\alpha(P) = A_\alpha(P)
$$

with $A_\alpha(P)$ as in Definition 1.13.

Using $\mathcal{D}_{a_1,\ldots,a_\alpha}^{(q,t)} = \mathcal{B}_\alpha(s_{a_1,\ldots,a_\alpha})$, this difference operator is expressed it in terms of a (possibly infinite) series involving degree $\alpha$ expressions of the generators $\mathcal{D}_{1;n}^{(q,t)}$, $n \in \mathbb{Z}$. The particular case $a_1 = a_2 = \cdots = a_\alpha = k$ is the $(q, t)$-determinant which generates the kernel of the homomorphism from the quantum toroidal algebra to $\mathcal{A}_\beta^{(q,t)}$. In general, we conjecture polynomiality of the $(q, t)$-determinant.

**Conjecture 1.17.** The generalized Macdonald operators $\mathcal{D}_{a_1,\ldots,a_\alpha}^{(q,t)}$ may be expressed as polynomials of finitely many $\mathcal{D}_{1;n}^{(q,t)}$’s. These polynomials are $t$-deformation of the quantum determinant expression (1.2).

We prove the conjecture for $a_1 = a_2 = \cdots = a_\alpha = n$ in Sect. 5, and for arbitrary $a$’s in the cases $\alpha = 2, 3$ in Sect. 7.

In Sect. 5, we use the known homomorphism between the EHA and the spherical DAHA in the limit $N \to \infty$ to derive an explicit expression for $\mathcal{D}_{\alpha;n}^{(q,t)}$ as a polynomial of the $\mathcal{D}_{1;n}^{(q,t)}$’s.

**Theorem 1.18.** The operator $\mathcal{D}_{a;n}^{(q,t)}$ is expressible as a homogeneous polynomial of degree $\alpha$ in the variables $\mathcal{D}_{1;n}^{(q,t)}, \mathcal{D}_{1;n \pm 1}^{(q,t)}$, with coefficients in $\mathbb{C}(q,t)$. 
This provides the proof of the polynomiality of the \((q, t)\)-determinant \(D_{\alpha,k}\). Moreover, we describe a method to systematically obtain explicit formulas for the corresponding polynomials. This completes our program of deriving the \((q, t)\)-determinants for \(\mathfrak{gl}_N\).

Finally we investigate the limit \(N \to \infty\) of the algebra \(A_{\mathfrak{gl}_N}\). In this limit, the algebra becomes the quantum toroidal algebra of \(\mathfrak{gl}_1\) with non-trivial horizontal central charge \((1,0)\). As part of the investigation of this limit, we clarify the relations with several structures encountered in recent work of Bergeron et al. \[BGLX16\]. There is a direct connection between plethysms and bosonization in the \(N \to \infty\) limit. We find an interpretation of the Nabla operator of \[BGLX16\] in terms of the \(SL_2(\mathbb{Z})\) action on the DAHA, and recover commutation relations of the \(D_k\) operators of \[BGLX16\] as those of fundamental currents in the level \((1,0)\) representation of the quantum toroidal algebra of \(\mathfrak{gl}_1\).

1.4. Outline of the paper. The paper is organized as follows. In Sect. 2 we find the generators of sDAHA whose functional representation is \(D_{\alpha,k}\). Moreover, we describe a method to systematically obtain explicit formulas for the corresponding polynomials. This completes our program of deriving the \((q, t)\)-determinants for \(\mathfrak{gl}_N\).

In Sect. 3, we show (Theorem 3.5) that the generating functions for the elements \(D_{1,n}^{(q,t)}\) and \(\tilde{D}_{1,n}^{(q,t)}\) are Drinfeld generators which satisfy the relations of the quantum toroidal algebra of \(\mathfrak{gl}_1\) at level-(0, 0). In Sect. 4, we give the plethystic formulation, or bosonization, of the generating currents, and investigate their behavior in the limit of an infinite number of variables \(N \to \infty\). In this limit, the functional representation of \(A_{\mathfrak{gl}_1}\) naturally acquires a non-trivial horizontal central charge \((1,0)\).

Section 5 presents a functional representation of the EHA in terms of the generalized Macdonald operators. The homomorphism from EHA to sDAHA(\(\mathfrak{gl}_\infty\)) \[SV11\] can be restricted to the quotient corresponding to finite \(N\) by the ideal generated by the \((q, t)\)-determinants. The formulation of the generators in terms of EHA generators gives new formulas for the operators \(D_{\alpha,n}^{(q,t)}\) of (1.7), and this can be used to prove the polynomiality of the corresponding \((q, t)\)-determinants, Conjecture 1.17.

In Sect. 6, we prove constant term identities relating the generalized Macdonald operators. In particular, we prove Theorem 1.16, the equivalence between the two maps \(A_\alpha(P)\) and \(B_\alpha(P)\) to generalized Macdonald difference operators. The second definition allows to write the multiple current generating function for the operators \(D_{\alpha_1,\ldots,\alpha_n}^{(q,t)}\) of (1.12) in terms solely of those of the \(D_{\alpha}^{(q,t)}\)'s (Theorem 6.5). We conjecture (Conjecture 1.17) that these may be reduced to polynomial expressions modulo the relations of the quantum toroidal algebra, and give the proof in the case \(\alpha = 2\) (Theorem 6.9) and \(\alpha = 3\) (Theorem 6.11).

In Sect. 7, we show that the definition of \(B_\alpha(P)\) is naturally compatible with a suitably defined non-commutative product \(* : \mathcal{F}_\alpha \times \mathcal{F}_\beta \to \mathcal{F}_{\alpha+\beta}\) (the shuffle product, see [FO98, Neg14] for related definitions), which satisfies the morphism property \(B_{\alpha+\beta}(P \ast P') = B_\alpha(P)B_\beta(P')\) (see Theorem 7.2). We may therefore translate some of the relations between the generalized Macdonald operators into shuffle product identities, which are sometimes easier to prove.

In Sect. 8, we explore the \(q\)-Whittaker limit \(t \to \infty\) of the constructions of this paper. In particular, we relate the finite \(t\) generalized Macdonald operators \(D_{1,n}^{(q,t)}\) to their \(t \to \infty\) limit \(D_{1,n}^{(q)}\). We also find an explicit formula for the \(t \to \infty\) limit of the operators \(D_{\alpha_1,\alpha_2,\ldots,\alpha_n}^{(q,t)}\) as a quantum determinant, which involves a summation over
2. Generalized Macdonald Operators and DAHA

In this section, we identify the elements in the spherical double affine Hecke algebra (sDAHA) whose functional representation are the operators $\mathcal{D}^{(q,t)}_{\alpha;n}$ of Eq. (1.7). We show that they are $\text{SL}_2(\mathbb{Z})$-translated Macdonald operators in the functional representation.

2.1. DAHA: definition. The standard definition and properties of the DAHA of type $\text{gl}_N$ can be found in [Che05]. Given indeterminates $q$ and $\theta = t^{\frac{1}{2}}$, this algebra generated over $C_{q,t} := C(q, \theta)$ by the elements $\{X_i^{\pm 1}, Y_i^{\pm 1}, T_j : i \in [1, N], j \in [1, N - 1]\}$, modulo the relations:

$$
\begin{align*}
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}; \\
(T_i - \theta)(T_i + \theta^{-1}) &= 0; \\
X_i &\cdots X_j = X_j X_i; \\
Y_i &\cdots Y_j = Y_j Y_i; \\
T_i X_i &= T_i X_i; \\
T_i Y_i &= Y_i T_i, \quad (1 \leq i \leq N - 1); \\
T_i X_j &= Y_j X_i; \\
T_i Y_j &= Y_j T_i, \quad (j \neq i, i + 1); \\
X_1 Y_2 &= Y_2 T_1^2 X_1; \\
Y_1 \cdots Y_N X_j &= q X_j Y_1 \cdots Y_N; \\
X_1 \cdots X_N Y_j &= q^{-1} Y_j X_1 \cdots X_N.
\end{align*}
$$

(2.1)

In Appendix A, we summarize various useful relations which can be derived from these. In particular, following Cherednik [Che05], we define an invertible element $\pi$ in Sect. 9.6, which allows to present the algebra solely in terms of the $T$’s, $X$’s and $\pi$, in terms of which $Y_i = T_i T_{i+1} \cdots T_{N-1} \pi^{-1} T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}$, whereas $\pi^{-1} X_{i+1} = X_i \pi^{-1}$, for $i > 1$, $\pi^{-1} X_1 = q X_N \pi^{-1}$ and $\pi^{-1} T_{i+1} = T_i \pi^{-1}$ for $i > 1$.

2.2. The functional representation of the DAHA. As the generators $X_1, \ldots, X_N$ commute among themselves, one can define the (faithful) functional representation $\rho$ of the DAHA acting on $V = C_{q,t}(x_1, \ldots, x_N)$, where $X_i$ acts as multiplication by the variable $x_i$. Let $s_i$ denote the elementary transposition of the variables $(x_i, x_{i+1})$. Then

$$
\begin{align*}
\rho(T_i) &= \theta s_i + \frac{\theta - \theta^{-1}}{x_i x_{i+1}^{-1}} (s_i - 1) = \frac{\theta x_i - \theta^{-1} x_{i+1}}{x_i - x_{i+1}} s_i - x_{i+1} \frac{\theta - \theta^{-1}}{x_i - x_{i+1}}; \\
\rho(T_i^{-1}) &= \rho(T_i) - \theta + \theta^{-1} = \frac{\theta x_i - \theta^{-1} x_{i+1}}{x_i - x_{i+1}} s_i - x_{i+1} \frac{\theta - \theta^{-1}}{x_i - x_{i+1}}; \\
\rho(\pi) f(x_1, x_2, \ldots, x_N) &= f(x_2, x_3, \ldots, x_N, q^{-1} x_1), \quad f \in V; \\
\rho(Y_i) &= \rho(T_i T_{i+1} \cdots T_{N-1} \pi^{-1} T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}) \quad (i = 1, 2, \ldots, N)
\end{align*}
$$
where the element $\pi$ is defined in Sect. 9.6.

In particular, the $q$-shift operators $\Gamma_i$ of (1.6) correspond to:

$$\Gamma_i = s_i s_{i+1} \cdots s_{N-2} s_{N-1} \rho(\pi^{-1}) s_1 s_2 \cdots s_{i-1}, \quad i = 1, 2, \ldots, N.$$  

(2.4)

In the following, we shall need the following.

**Definition 2.1.** We define the quantities

$$A := x_1 x_2 \cdots x_N, \quad \Delta := \Gamma_1 \Gamma_2 \cdots \Gamma_N,$$

corresponding respectively to the operators $X_1 X_2 \cdots X_N$ and $Y_1 Y_2 \cdots Y_N = \pi^{-N}$ in the functional representation.

2.3. Macdonald difference operators. The operators $Y_1, \ldots, Y_N$ commute among themselves. Therefore one can define the elementary symmetric functions $e_\alpha(Y_1, \ldots, Y_N)$ unambiguously.

**Definition 2.2.** The DAHA elements corresponding to Macdonald operators are

$$D_\alpha := e_\alpha(Y_1, \ldots, Y_N), \quad (\alpha = 0, 1, 2, \ldots, N).$$

(2.5)

It is well-known that the operators $\rho(D_\alpha)$ in the functional representation act on the space $\mathcal{F}_N$ of symmetric functions in the variables $x_1, \ldots, x_N$. The following is a standard result of Macdonald theory:

**Theorem 2.3.** The restriction of the operators $\rho(D_\alpha)$ to $\mathcal{F}_N$ is

$$\rho(D_\alpha)|_{\mathcal{F}_N} =: D^{(q,t)}_{\alpha;0}$$

where $D^{(q,t)}_{\alpha;0}$ are the Macdonald operators (1.7).

2.4. More commuting operators. In this section, we introduce families of commuting operators $\{Y_{i,n}\}_{i \in [1,N]}$ for each $n \in \mathbb{Z}$. These are related to Cherednik’s $SL_2(\mathbb{Z})$ action on $Y_i$ by $n$ iterations of the generator $\tau_+$ of $SL_2(\mathbb{Z})$.

2.4.1. Definition and commutation

**Definition 2.4.** For each $i \in \{1, \ldots, N\}$ and $n \in \mathbb{Z}$, define

$$Y_{i,n} := (X_1 X_2 \cdots X_{i-1})^{-n} Y_i (X_1 X_2 \cdots X_i)^n, \quad (i = 1, 2, \ldots, N; n \in \mathbb{Z}).$$

For example, $Y_{i,0} = Y_i$, $Y_{1,n} = Y_1 X_1^n$ and

$$Y_{N,n} = (X_1 \cdots X_{N-1})^{-n} Y_N (X_1 \cdots X_N)^n = q^n X_N^n Y_N.$$

**Lemma 2.5.** For fixed $n \in \mathbb{Z}$, the elements $\{Y_{i,n} : i \in [1, N]\}$ are mutually commuting:

$$Y_{i,n} Y_{j,n} = Y_{j,n} Y_{i,n} \quad \forall i, j \in [1, N].$$
Below, we will give another definition of the $Y_{i,n}$’s in terms of the adjoint action of an element $\gamma$ in a completion of the DAHA, which makes it clear that the elements are mutually commuting. However, for completeness, we provide the proof in the DAHA itself.

**Proof.** Writing $j = i + k, k > 0$, we have:

\[
(X_1 \cdots X_{i-1})^n Y_{i,n} Y_{i+k,n} (X_1 \cdots X_i)^{-n} \\
= Y_{i}(X_{i+1} \cdots X_{i+k-1})^n Y_{i+k} (X_{i+1} \cdots X_{i+k})^n \\
= Y_{i+k} Y_{i}(T_{i+k-1} \cdots T_{i+1})(X_{i+2} \cdots X_{i+k})^{-n} \\
(T_{i+1}^{-1} \cdots T_{i+k-1})^{-1} (X_{i+1} \cdots X_{i+k})^n \\
= Y_{i+k} (T_{i+k-1} \cdots T_{i+1})(Y_{i} X_{i+1}^n)(T_{i+1}^{-1} \cdots T_{i+k-1}^{-1}) \\
= Y_{i+k} (T_{i+k-1} \cdots T_{i}) X_i^n (T_{i}^{-1} \cdots T_{i+k-1}^{-1}) Y_i
\]

where we have first used the relation (A.1):

\[
(X_{i+1} \cdots X_{i+k-1})^{-n} Y_{i+k} = Y_{i+k} (T_{i+k-1} \cdots T_{i+1})(X_{i+2} \cdots X_{i+k})^{-n} (T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})
\]

then Lemma A.1:

\[
(T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})(X_{i+1} \cdots X_{i+k}) = (X_{i+1} \cdots X_{i+k})(T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})
\]

and finally $Y_{i} X_i^n = T_i X_i^n T_i^{-1} Y_i$ by iteration of (A.1). Likewise, we have:

\[
(X_1 \cdots X_{i-1})^n Y_{i+k,n} Y_{i,n} (X_1 \cdots X_i)^{-n} \\
= (X_i \cdots X_{i+k-1})^{-n} Y_{i+k} (X_i \cdots X_{i+k})^n Y_i \\
= Y_{i+k} (T_{i+k-1} \cdots T_{i})(X_{i+1} \cdots X_{i+k})^{-n} (T_{i+k-1}^{-1} \cdots T_{i+1}^{-1})(X_i \cdots X_{i+k})^n Y_i \\
= Y_{i+k} (T_{i+k-1} \cdots T_{i}) X_i^n (T_{i}^{-1} \cdots T_{i+k-1}^{-1}) Y_i
\]

by use of the relations

\[
(X_{i+1} \cdots X_{i+k-1})^{-n} Y_{i+k} = Y_{i+k} (T_{i+k-1} \cdots T_{i})(X_{i+1} \cdots X_{i+k})^{-n} (T_{i+k-1}^{-1} \cdots T_{i+1}^{-1}) \\
(T_{i}^{-1} \cdots T_{i+k-1}^{-1})(X_i \cdots X_{i+k}) = (X_i \cdots X_{i+k})(T_{i}^{-1} \cdots T_{i+k-1}^{-1})
\]

The lemma follows. □

2.4.2. **Expression in the functional representation** We introduce the following symmetric function of $X_1, \ldots, X_N$:

\[
\gamma := \exp \left\{ \sum_{i=1}^{N} \frac{\log(X_i)^2}{2\log(q)} \right\}.
\]  

(2.6)

This element does not belong to the DAHA, but to a suitable completion (see [Che05, Chapter 3, Sect. 3.7]). Nevertheless, it has some useful commutation relations with elements of the DAHA.
Lemma 2.6.

\[ \pi^{-1} \gamma = q^{\frac{1}{2}} X_N \gamma \pi^{-1} \]

Proof. Using \( \pi^{-1} X_i = X_{i-1} \pi^{-1} \), for \( i \in [2, N] \) and \( \pi^{-1} X_1 = q X_N \pi^{-1} \), we compute

\[ \pi^{-1} \left( \sum_{j=1}^{r+1} \frac{\log(X_j)^2}{2\log(q)} \right) = \left( \sum_{j=1}^{r+1} \frac{\log(X_j)^2}{2\log(q)} + \frac{X_{N+1} + \log(q)}{2} \right) \pi^{-1} \]

and the Lemma follows. \( \square \)

In the functional representation \( \rho \) of Sect. 2.2,

Lemma 2.7.

\[ \Gamma_i \rho(\gamma) = q^{\frac{1}{2}} x_i \rho(\gamma) \Gamma_i. \]

Proof. This follows from

\[ \Gamma_i \left( \sum_{j=1}^{r+1} \frac{\log(x_j)^2}{2\log(q)} \right) = \left( \sum_{j=1}^{r+1} \frac{\log(x_j)^2}{2\log(q)} + \frac{\log(q)}{2} \right) \Gamma_i. \]

\( \square \)

Theorem 2.8. For all \( n \in \mathbb{Z} \),

\[ Y_{i,n} = q^{\frac{n}{2}} \gamma^{-n} Y_i \gamma^n. \]

Proof. As \( \gamma \) is a symmetric function of the \( X_i \)’s, it commutes with \( s_i \), and with all the \( T_i \) in the functional representation. Then

\[ \gamma^{-n} Y_i \gamma^n = \gamma^{-n} T_i \ldots T_{N-1} \pi^{-1} T_1^{-1} \ldots T_{i-1}^{-1} \gamma^n \]

\[ = T_i \ldots T_{N-1} \gamma^{-n} \pi^{-1} \gamma^n T_1^{-1} \ldots T_{i-1}^{-1} \]

\[ = q^{\frac{n}{2}} T_i \ldots T_{N-1} X_N^n \pi^{-1} T_1^{-1} \ldots T_{i-1}^{-1} \]

\[ = q^{-\frac{n}{2}} T_i \ldots T_{N-1} (X_1 \ldots X_{i-1})^{-n} (X_1 \ldots X_{i-1})^n \pi^{-1} X_1^n T_{i-1}^{-1} \ldots T_{i-1}^{-1} \]

\[ = q^{-\frac{n}{2}} (X_1 \ldots X_{i-1})^{-n} T_i \ldots T_{N-1} \pi^{-1} (X_1 \ldots X_i)^n T_1^{-1} \ldots T_{i-1}^{-1} \]

\[ = q^{-\frac{n}{2}} (X_1 \ldots X_{i-1})^{-n} Y_i (X_1 \ldots X_i)^n = q^{-\frac{n}{2}} Y_{i,n}, \]

using Lemma 2.6 and the commutations relations between the \( X \)’s and the \( T \)’s. \( \square \)

Remark 2.9. Theorem 2.8 immediately implies the commutation of the elements \( \{Y_{i,n}, i \in [1, N]\} \) for any fixed \( n \). However, the element \( \gamma \) (2.6) only belongs to a completion of the DAHA. The direct proof of Lemma 2.5 bypasses this complication.
2.4.3. Comparison with the standard $SL(2, \mathbb{Z})$ action on DAHA According to Theorem 2.8, conjugation by $\gamma^{-1}$ is the action of the generator $\tau_+$ of the standard $SL(2, \mathbb{Z})$ action on DAHA [Che05]. Indeed, using the definition\(^1\):
\[
\tau_+(X_i) = X_i, \quad \tau_+(T_i) = T_i, \quad \tau_+(q) = q, \quad \tau_+(t) = t,
\]
\[
\tau_+(Y_1 Y_2 \cdots Y_i) = q^{-i/2} (Y_1 Y_2 \cdots Y_i)(X_1 X_2 \cdots X_i).
\]

Therefore $Y_{i,n} = q^{n/2} \tau^*_n(Y_i)$, from which follows:

**Lemma 2.10.** The generator $\tau_+$ of the standard $SL(2, \mathbb{Z})$ action on DAHA reads:
\[
\tau_+ = \text{ad}_{\gamma^{-1}}
\]

namely it acts by conjugation w.r.t. $\gamma^{-1}$ of (2.6).

The second generator $\tau_-$ of the standard $SL(2, \mathbb{Z})$ action on DAHA is obtained by use of the anti-involution $\epsilon$ of the DAHA acting on generators and parameters as:
\[
\epsilon : (X_i, Y_i, T_i, q, t) \mapsto (Y_i, X_i, T_i^{-1}, q^{-1}, t^{-1})
\]
and such that
\[
\tau_- = \epsilon \tau_+ \epsilon.
\]

This leads to the following:

**Lemma 2.11.** The action of the element $\tau_-$ on the generators of the DAHA is equal to the adjoint action of the element $\eta^{-1}$ on the generators, where:
\[
\eta := \exp \left\{ - \sum_{i=1}^{N} \frac{\log(Y_i)^2}{2\log(q)} \right\},
\]
\[
(2.7)
\]

That is,
\[
\tau_- = \text{ad}_{\eta^{-1}}.
\]

**Proof.** Apply the anti-involution $\epsilon$ to $\gamma$, and note that $\epsilon(\gamma) = \eta$. \(\Box\)

**Remark 2.12.** The operator $\eta^{-1}$ is very similar to the nabla operator $\nabla$ of [BG99], restricted to a finite number $N$ of variables. Denote $\eta^{-1} = \nabla^{(N)}$. The Macdonald polynomial $P_{\lambda}(x_1, \ldots, x_N)$ is an eigenvector of the functional representation of any symmetric function $f((Y_i)_{i=1}^{N})$, with eigenvalue $f((t^{-\frac{N+1-i}{2}}q^{\lambda_i})_{i=1}^{N})$ [Che05]. In particular, this holds for $\nabla^{(N)}$. That is,
\[
\nabla^{(N)} P_{\lambda} = \exp \left\{ \frac{1}{2\log(q)} \sum_{i=1}^{N} \left( \frac{N+1}{2} - i \right) \log(t) + \lambda_i \log(q) \right\} P_{\lambda}
\]
\[
= \left( C_N \prod_{i=1}^{N} q^{\frac{\lambda_i^2}{2}t^{\frac{N+1-i}{2}}\lambda_i} \right) P_{\lambda} = C_N u_{\lambda} P_{\lambda}
\]

\(^1\) This definition is compatible with the definitions of Chapter 1 of [Che05]. They are dual to those appearing in Chapter 3, Sect. 3.7 in which conventions are different.
where $\log(C_N) = \frac{N(N^2-1)}{24} \log(t)^2$ and $u_\lambda = t^{\frac{N-1}{2} |\lambda| - n(\lambda)} q^{\frac{1}{2} |\lambda| + n(\lambda')}$, where $n(\lambda) = \sum_i (i-1) \lambda_i$, and $\lambda'$ is the transposed Young diagram, so that $n(\lambda') = \sum_j \lambda_j (\lambda_i - 1)/2$. In [BG99], the $\nabla$ operator is defined to have eigenvalue $t^{n(\lambda)} q^{n(\lambda')}$ on the modified Macdonald polynomials $\tilde{H}_\lambda$, obtained from $P_\lambda$ by a certain transformation. Therefore, $\nabla(N)$ is an analogue of the operator $\nabla$, acting instead on the Macdonald polynomials $P_\lambda$. We complete this identification in Remark 4.8 of Sect. 4.2.2.

2.5. Generalized Macdonald difference operators.

Definition 2.13. In analogy with Definition 2.2, we define elements $D_{\alpha;n}$ of DAHA in terms of the elementary symmetric functions of \{\(Y_i,n : i \in [1, N]\)\} by:

$$D_{\alpha;n} := q^{-an} \sum_{1 \leq i_1 < i_2 < \cdots < i_\alpha \leq N} Y_{i_1,n} Y_{i_2,n} \cdots Y_{i_\alpha,n} \quad \alpha \in [0, 1, \ldots, N]. \quad (2.8)$$

It follows from Theorem 2.8 that in the functional representation of DAHA:

$$\rho(D_{\alpha;n}) = q^{-an} \rho(\gamma)^{-n} \rho(D_{\alpha;0}) \rho(\gamma)^n, \quad (2.9)$$

where $D_{\alpha;0} = D_\alpha$ are given by (2.5).

Theorem 1.10. The operators $\rho(D_{\alpha;n})$ leave the space $F_N$ of symmetric functions invariant. They are identified as follows:

$$\rho(D_{\alpha;n})|_{F_N} =: D^{(q,t)}_{\alpha;n} \quad (2.10)$$

where $D^{(q,t)}_{\alpha;n}$ are the generalized Macdonald operators (1.7).

Proof. We use Lemma 2.7 to write for any index subset $I$ of cardinality $\alpha$:

$$\Gamma_I \rho(\gamma) = (\prod_{i \in I} \Gamma_i) \rho(\gamma) = q^{\frac{a}{2} x_I} \rho(\gamma) \Gamma_I.$$ 

Starting with Eq. (2.9), act on the formula of Theorem 2.3 for $\rho(D_\alpha) = \rho(D_{\alpha;0})$ with the adjoint action of $\rho(\gamma)^{-n}$:

$$\rho(D_{\alpha;n}) = q^{-an} \rho(\gamma)^{-n} \rho(D_\alpha) \rho(\gamma)^n$$

$$= q^{-an} \sum_{|I| = \alpha, I \subset [1, N]} \prod_{i \in I} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \rho(\gamma)^{-n} \Gamma_I \rho(\gamma)^n$$

$$= \sum_{|I| = \alpha, I \subset [1, N]} (x_I)^n \prod_{i \in I, j \notin I} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \Gamma_I. $$

The Theorem follows. □
This can be rephrased into a simple relation between Macdonald and generalized Macdonald operators:

\[
D^{(q,t)}_{\alpha,n} = q^{-\frac{an}{2}} \gamma^n D^{(q,t)}_{\alpha,0} \gamma^n
\]  
(2.11)

where by a slight abuse of notation we write $\gamma$ for $\rho(\gamma)$, which amounts to replacing all $X_i \rightarrow x_i$ in the expression for the Gaussian $\gamma$.

Note that by construction, as $e_{N+1}(Y_1, Y_2, \ldots, Y_N) = 0$, we have the following values:

\[
D^{(q,t)}_{N+1,n} = 0 \quad (n \in \mathbb{Z}).
\]  
(2.12)

### 2.6. Spherical DAHA and $A^{(q,t)}_{\text{gl}_N}$

The spherical DAHA is obtained from DAHA by projection via the (unnormalized) idempotent:

\[
s = \sum_{\sigma \in S_N} \theta^{\ell(\sigma)} T_{\sigma},
\]

where $T_{\sigma} = T_1 T_2 \cdots T_{\ell}$ for any reduced decomposition of the permutation $\sigma$ into a product of $\ell = \ell(\sigma)$ elementary transpositions $\sigma = s_{i_1} \cdots s_{i_{\ell}}$. This is a projection onto the common eigenspace of all $T_i$ with eigenvalue $\theta$, a.s. $T_i s = \theta I$. Moreover, using the expression (A.2) for $Y_i$, we get:

\[
s Y_i s = \theta^{N+1-2i} s \pi - 1 s.
\]

As a result, sDAHA is generated by symmetric polynomials of respectively the $X_i$, the $X_i^{-1}$, the $Y_i$ and the $Y_i^{-1}$, in other words by the power sums

\[
p_k = \sum_{i=1}^{N} X_i^k \quad \text{and} \quad P_k = \sum_{i=1}^{N} Y_i^k \quad (k \in \mathbb{Z} \setminus \{0\}).
\]

The $SL_2(\mathbb{Z})$ action is inherited from that on DAHA.

The functional representation of Sect. 2.2 restricts to sDAHA, which preserves the space $\mathcal{F}_N$ of symmetric functions of the $x$’s. The generalized Macdonald operators are therefore in this representation of sDAHA, and correspond to elementary symmetric functions of the $Y_i$’s and their translates under $\tau_+$. In particular, the elements $(X_1 X_2 \cdots X_N)^{\pm 1}$ and $(Y_1 Y_2 \cdots Y_N)^{\pm 1}$ belong to sDAHA, and correspond to $A^{\pm 1}$ and $\Delta^{\pm 1}$ in the functional representation (c.f. Definition 2.1). In particular, we have

\[
\rho(D_{N;n}) = \rho(q^{-Nn} Y_1 \cdots Y_N (X_1 \cdots X_N)^n) = A^n \Delta = D^{(q,t)}_{N,n}.
\]

We are now ready for the main result of this section:

**Theorem 2.14.** The algebra $A^{(q,t)}_{\text{gl}_N}$ is isomorphic to the spherical DAHA for $\text{gl}_N$.

**Proof.** We work with the functional representation of sDAHA, which is known to be faithful. We have the mapping of generators:

\[
M_{\alpha;n} \mapsto D^{(q,t)}_{\alpha,n} \equiv D^{(q,t)}_{\alpha,0} = \rho(q^{-an/2} \tau_+^n (e_\alpha([Y_i])))
\]

\[
\tilde{M}_{\alpha;n} \mapsto \tilde{D}^{(q^{-1},t^{-1})}_{\alpha,n} \equiv D^{(q^{-1},t^{-1})}_{\alpha,0} = \rho(q^{an/2} \tau_-^n (e_\alpha([Y_i^{-1}])))
\]

\[
A \mapsto \rho(X_1 X_2 \cdots X_N) \quad \Delta \mapsto \rho(Y_1 Y_2 \cdots Y_N).
\]
First, noting that \( e_\alpha ([Y_i^{-1}]) = e_{N-\alpha} ([Y_i]) (Y_1 Y_2 \cdots Y_N)^{-1} \), we compute, using (2.8) for \( n = 0 \):

\[
\rho (e_\alpha ([Y_i^{-1}])) = \rho (D_{N-\alpha}; 0) \Delta^{-1} = D_{N-\alpha}; 0 \Delta^{-1}
\]

\[
= \sum_{J \subseteq [1, N]} \prod_{j \in J \atop |J| = \alpha} \frac{\theta^{-1} x_j - \theta x_i}{x_j - x_i} \Gamma_j^{-1} = D_{\alpha; 0} (q^{-1}, t^{-1})
\]

where we have rewritten the sum (1.7) for \( \alpha \to N - \alpha \) and \( n \to 0 \) over \( I \subseteq [1, N], |I| = N - \alpha \) in terms of its complement \( J = [1, N] \setminus I \), and used \( \Gamma_I \Delta^{-1} = \Gamma_j^{-1} \). The action of \( SL_2(\mathbb{Z}) \) follows immediately. \( \square \)

### 3. The Quantum Toroidal Algebra of \( \mathfrak{gl}_1 \) and \( \mathcal{A}^{(q, t)}_{\mathfrak{gl}_N} \)

In this section, we show that the generalized Macdonald operators \( \{ D^{(q, t)}_{n} : n \in \mathbb{Z} \} \) of Eq. (1.7) satisfy the relations of the quantum toroidal algebra of \( \mathfrak{gl}_1 \) [Mik07] at “level” \((0, 0)\). When formulated as a quotient of a so-called Ding–Iohara algebra, this algebra has a presentation in terms of Drinfeld type currents, which makes evident that it is a deformation of the quantum affine algebra of \( \widehat{sl}_2 \), \( U_{\sqrt{q}}(\widehat{sl}_2) \).

We call the generating functions of the generators \( \{ D^{(q, t)}_{n} : n \in \mathbb{Z} \} \) the fundamental currents: In Sect. 5 (Theorem 1.18) we show, by use of the elliptic Hall algebra, that the other generators of the algebra \( \mathcal{A}^{(q, t)}_{\mathfrak{gl}_N}, \mathcal{D}^{(q, t)}_{\alpha; n} \), are polynomials in the fundamental currents. Therefore, the algebra \( \mathcal{A}^{(q, t)}_{\mathfrak{gl}_N} \) is a quotient of the quantum toroidal algebra.

#### 3.1. The quantum toroidal algebra of \( \mathfrak{gl}_1 \)

For generic parameters \( q, t \in \mathbb{C}^* \), let

\[
g(z, w) := (z - qw)(z - t^{-1} w)(z - q^{-1} tw).
\]

**Definition 3.1.** The quantum toroidal algebra of \( \widehat{\mathfrak{gl}}_1 \) is defined in terms of generators and relations, where the generators are the modes of the series \( e(z) = \sum_{n \in \mathbb{Z}} e_n z^n \) and \( f(z) = \sum_{n \in \mathbb{Z}} f_n z^n \), subject to the relations\(^2\)

\[
\begin{align*}
g(z, w) e(z) e(w) + g(w, z) e(w) e(z) &= 0, \\
g(w, z) f(z) f(w) + g(z, w) f(w) f(z) &= 0, \\
[\epsilon(z), f(w)] &= \frac{1}{g(1, 1)} \left( \delta(z/w) \varphi^+(z) - \delta(\hat{z}^{-2} z/w) \varphi^-(z) \right),
\end{align*}
\]

\(^2\) This algebra is usually presented with the generators \( x^\pm \) and \( \varphi^\pm \) with the following dictionary:

\[
\begin{align*}
x^+(z) &= \frac{1 - q (1 - t^{-1})}{q^{1/2}} e(q^{-1/2} z), & x^-(z) &= \frac{1 - q^{-1} (1 - t)}{q^{-1/2}} f(q^{-1/2} \hat{z}^{-1} z), \\
\varphi^+(z) &= \psi^-(q^{-1/2} \hat{z}^{1/2} z), & \varphi^-(z) &= \psi^+(q^{-1/2} \hat{z}^{-1/2} z).
\end{align*}
\]
where $\psi^\pm(z)$ are a generating series expanded in $z^{\pm 1}$: $\psi^\pm(z) = \sum_{n \geq 0} \psi^\pm_n z^n$, with $\psi^\pm_0 = \hat{\delta}^{\pm 1}$, and $\hat{\delta}$ and $\hat{\gamma}$ are two central elements. Here, the formal delta function is

$$\delta(u) = \sum_{n \in \mathbb{Z}} u^n$$

(3.3)

with the property that $\delta(u)f(u) = \delta(u)f(1)$ for any function $f$ which is non-singular at $u = 1$. Further relations in the algebra are given by

$$[\psi^\pm(z), \psi^\pm(w)] = 0,$$

$$g(z, w)g(w, \hat{\gamma}^{-2}z)\psi^\pm(z)\psi^\pm(w) = g(w, z)g(\hat{\gamma}^{-2}z, w)\psi^\pm(w)\psi^\pm(z)$$

$$g(z, w)\psi^\pm(z)\epsilon(w) + g(w, z)\epsilon(w)\psi^\pm(z) = 0,$$

$$g(w, z)\psi^\pm(z)\epsilon(w) + g(z, w)\epsilon(w)\psi^\pm(z) = 0,$$

$$g(\hat{\gamma}^{\pm 2}w, z)\psi^\pm(z)\epsilon(w) + g(\hat{\gamma}^{\pm 2}w, z)\epsilon(w)\psi^\pm(z) = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} \left( \frac{z_2}{z_3} e(z_1), [e(z_2), e(z_3)] \right) = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} \left( \frac{z_2}{z_3} f(z_1), [f(z_2), f(z_3)] \right) = 0.$$  

(3.4)

(3.5)

The cubic relations (3.5) are Serre-type relations which distinguish the quantum toroidal algebra of Miki [Mik07] from the original Ding–Iohara algebra [DI97].

A particular class of representations [FHH+09] indexed by integers $(\ell_1, \ell_2) \in \mathbb{Z}_+^2$, referred to as horizontal and vertical levels, corresponding to action of the central elements $\hat{\gamma}$, $\hat{\delta}$ by the eigenvalues $\gamma^{\ell_1}$, $\gamma^{\ell_2}$, respectively, where $\gamma = (tq^{-1})^{1/2}$.

3.1.1. Level $(0, 0)$ quantum toroidal $\widehat{gl}_1$. Except when discussing the limit $N \to \infty$ in Sect. 4.2, the algebra relevant to $\mathcal{A}^{(q;l)}_{gl_N}$ has level-$(0, 0)$, corresponding to $\hat{\gamma} = \hat{\delta} = 1$. In this case the relations simplify to

**Definition 3.2.** The level $(0, 0)$ quantum toroidal $\widehat{gl}_1$ is the algebra generated by $\{e_n, f_n, \psi^\pm_n : n \in \mathbb{Z}, m \in \mathbb{Z}_+\}$ with relations:

$$g(z, w)\epsilon(z)\epsilon(w) + g(w, z)\epsilon(w)\epsilon(z) = 0,$$

$$g(w, z)f(w) + g(z, w)f(z) = 0,$$

$$g(z, w)\psi^\pm(z)\epsilon(w) + g(w, z)\epsilon(w)\psi^\pm(z) = 0,$$

$$g(w, z)\psi^\pm(z)f(w) + g(z, w)f(z)\psi^\pm(z) = 0,$$

$$\left[ \epsilon(z), f(w) \right] = \frac{\delta(z/w)}{g(1,1)} \left( \psi^+(z) - \psi^-(z) \right),$$

$$\text{Sym}_{z_1, z_2, z_3} \left( \frac{z_2}{z_3} e(z_1), [e(z_2), e(z_3)] \right) = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} \left( \frac{z_2}{z_3} f(z_1), [f(z_2), f(z_3)] \right) = 0,$$

with $\psi^\pm_0 = 1$ and $\{\psi^\pm_n\}$ mutually commuting for all $n \in \mathbb{Z}_+$.

**Remark 3.3.** In the limit $t \to \infty$, let $g_0(z, w) = \lim_{t \to \infty} t^{-1}g(z, w) = z - qw$. Then the first relations in Definition 3.2 become relations in the quantum affine algebra of $sl_2$ in the Drinfeld presentation (with a non-standard deformation parameter $\sqrt{q}$, as in the Hall algebra of [Kap97]).
3.2. Macdonald currents. Define the generating series

$$e_\alpha(z) := \frac{q^\frac{\alpha}{2}}{(1 - q)^{\alpha}} \sum_{n \in \mathbb{Z}} q^{n\alpha/2} z^n D_{\alpha; n}^{(q, t)}$$

$$f_\alpha(z) := \frac{q^{-\frac{\alpha}{2}}}{(1 - q^{-1})^{\alpha}} \sum_{n \in \mathbb{Z}} q^{-n\alpha/2} z^n D_{\alpha; n}^{(q^{-1}, t^{-1})},$$

(3.6)

for any $\alpha \in [1, N]$. Note that:

$$D_{\alpha; n}^{(q, t)} = \sum_{|I| = \alpha, I \subseteq [1, N]} (x_I)^n \prod_{i \in I, j \notin I} \frac{\theta^{-1} x_i - \theta x_j}{x_i - x_j} \Gamma_I^{-1}$$

(3.7)

where $\Gamma_I^{-1}$ acts on functions of $x_1, \ldots, x_N$ by multiplying the $i$-th variable $x_i$ by $q^{-1}$.

**Remark 3.4.** Let $S$ denote the involution acting on the space of functions in $(x_1, \ldots, x_N)$ by sending $x_i \mapsto x_i^{-1}$ for all $i$. Then $\Sigma \Gamma_I S = \Gamma_I^{-1}$ and $D_{\alpha; -n}^{(q^{-1}, t^{-1})} = S D_{\alpha; n}^{(q, t)} S$, yielding the identification

$$S e_\alpha(z) S = (-1)^\alpha f_\alpha(z^{-1}).$$

(3.8)

Note that we may also write: $D_{\alpha; -n}^{(q^{-1}, t^{-1})} = A^{-n} D_{n-\alpha; N}^{(q, t)} \Delta^{-1}$, with $A$ and $\Delta$ as in Definition 2.1, which implies:

$$f_\alpha(z) = (-1)^\alpha \frac{q^{\frac{\alpha-N}{2}}}{(1 - q)^{\frac{\alpha-N}{2}}} e_{N-\alpha} (q^{\frac{\alpha-N}{2}} A^{-1} z^{-1}) \Delta^{-1}.$$

Using equation (1.7) the currents (3.6) are expressed in terms of the formal $\delta$ function (3.3):

$$e_\alpha(z) = \frac{q^\frac{\alpha}{2}}{(1 - q)^{\alpha}} \sum_{|I| = \alpha, I \subseteq [1, N]} \delta(q^{\alpha/2} x_I) \prod_{i \in I, j \notin I} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \Gamma_I,$$

$$f_\alpha(z) = \frac{q^{-\frac{\alpha}{2}}}{(1 - q^{-1})^{\alpha}} \sum_{|I| = \alpha, I \subseteq [1, N]} \delta(q^{-\alpha/2} x_I) \prod_{i \in I, j \notin I} \frac{\theta^{-1} x_i - \theta x_j}{x_i - x_j} \Gamma_I^{-1}.$$

Due to the finite number $N$ of variables, there are vanishing relations for the $\alpha = N+1$ currents:

$$e_{N+1}(z) = 0 \quad \text{and} \quad f_{N+1}(z) = 0,$$

(3.9)

as a consequence of the expressions $D_{N+1, n}^{(q, t)} = 0$ (2.12).
3.3. Exchange relations satisfied by Macdonald currents. We claim that the fundamental currents \( e_1(z), f_1(z) \) satisfy the relations of Definition 3.2 of the quantum toroidal algebra at level \((0, 0)\). For finite \( N \), there are further relations given by the vanishing conditions (3.9), once we express the conditions (2.12) as \((q, t)\)-determinant polynomial relations between the \( D^{(q, t)}_{1:n} \) (see Theorem 5.2, in Sect. 5). This will be formulated as a constant term relation between the fundamental currents (see Theorem 6.8 in Sect. 6).

**Theorem 3.5.** The Macdonald currents \( e_1(z) \) and \( f_1(z) \) (3.6), together with the series:

\[
\psi^\pm(z) := \prod_{i=1}^N \frac{(1 - q^{-\frac{1}{2}} t(zx_i)^{\pm1})(1 - q^{\frac{1}{2}} t^{-1}(zx_i)^{\pm1})}{(1 - q^{-\frac{1}{2}} (zx_i)^{\pm1})(1 - q^{\frac{1}{2}} (zx_i)^{\pm1})} \tag{3.10}
\]

satisfy the level \((0, 0)\) quantum toroidal \( \hat{g}_1 \) algebra relations of Definition 3.2.

The proof follows from the following lemmas. We will prove that the currents satisfy the exchange relations in Lemmas 3.6–3.8. The Serre relations in Lemma 3.9 can be shown by a similar calculation, but the proof is more cleanly presented in the language of shuffle algebras, and follows from Lemma 7.4.

**Lemma 3.6.** The following relations hold for the difference operators (3.6) with \( \alpha = 1 \):

\[
\begin{align*}
&g(z, w)e_1(z)e_1(w) + g(w, z)e_1(w)e_1(z) = 0, \\
g(w, z)f_1(z)f_1(w) + g(z, w)f_1(w)f_1(z) = 0.
\end{align*}
\]

**Proof.** The second equation follows from the first by taking \((q, t) \mapsto (q^{-1}, t^{-1})\). Using the explicit expression,

\[
\begin{align*}
\frac{(1 - q)^2}{q} e_1(z)e_1(w) &= \sum_{i=1}^{r+1} \delta(q^{\frac{1}{2}}zx_i) \prod_{k \neq i} \frac{\theta x_i - \theta^{-1} x_k}{x_i - x_k} \Gamma_i \\
&\quad \times \sum_{j=1}^{r+1} \delta(q^{\frac{1}{2}}wx_j) \prod_{k \neq j} \frac{\theta x_j - \theta^{-1} x_k}{x_j - x_k} \Gamma_j \\
&= \sum_{1 \leq i \neq j \leq r+1} \delta(q^{\frac{1}{2}}zx_i) \delta(q^{\frac{1}{2}}wx_j) \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \frac{\theta x_j - \theta^{-1} x_i}{x_j - x_i} \\
&\quad \times \prod_{k \neq i, j} \frac{\theta x_i - \theta^{-1} x_k}{x_i - x_k} \frac{\theta x_j - \theta^{-1} x_k}{x_j - x_k} \Gamma_i \Gamma_j \\
&\quad + \sum_{i=1}^{r+1} \delta(q^{\frac{1}{2}}zx_i) \delta(q^{\frac{3}{2}}wx_i) \prod_{k \neq i} \frac{\theta x_i - \theta^{-1} x_k}{x_i - x_k} \frac{q \theta x_i - \theta^{-1} x_k}{q x_i - x_k} \Gamma_i^2.
\end{align*}
\]

The second term is proportional to \( \delta(z/(qw)) \). Since \((z - qw)\delta(z/(qw)) = 0\),

\[
\begin{align*}
\frac{(1 - q)^2}{q} (z - qw)e_1(z)e_1(w) &= \sum_{1 \leq i \neq j \leq r+1} \delta(q^{\frac{1}{2}}zx_i) \delta(q^{\frac{1}{2}}wx_j) \\
&\quad \times (z - qw)(\theta x_i - \theta^{-1} x_j)(\theta x_j - \theta q^{-1} x_i) \\
&\quad \times (x_i - x_j)(x_j - q x_i)
\end{align*}
\]
Using the definition, where
\[ \psi \]
This makes the quantity \((z - q_w)(z - t^{-1}w)(z - tq^{-1}w)\) manifestly skew-symmetric under the interchange \(z \leftrightarrow w\). □

**Lemma 3.7.**

\[ [\varepsilon_1(z), \hat{f}_1(w)] = \frac{\delta(z/w)}{g(1, 1)} (\psi^+(z) - \psi^-(z)) \]

where \(\psi^\pm(z)\) are the power series (3.10) in \(z^\pm 1\).

**Proof.** Using the definition,

\[
(1 - q)(1 - q^{-1})\varepsilon_1(z)\hat{f}_1(w)
\]

\[
= \sum_{i=1}^{r+1} \delta(q^{1/2}z_{x_i}) \prod_{k \neq i} \frac{\theta x_i - \theta^{-1}x_k}{x_i - x_k} \Gamma_i \sum_{j=1}^{r+1} \delta(q^{-1/2}w_{x_j}) \prod_{k \neq j} \frac{\theta^{-1}x_j - \theta x_k}{x_j - x_k} \Gamma_j^{-1}
\]

\[
= \sum_{1 \leq i \neq j \leq r+1} \delta(q^{1/2}z_{x_i}) \delta(q^{-1/2}w_{x_j}) \frac{\theta x_i - \theta^{-1}x_j}{x_i - x_j} \frac{\theta^{-1}x_j - \theta x_i}{x_j - q x_i}
\]

\[
\times \prod_{k \neq i, j} \frac{(\theta x_i - \theta^{-1}x_k)(\theta^{-1}x_j - \theta x_k)}{(x_i - x_k)(x_j - x_k)} \Gamma_i \Gamma_j^{-1}
\]

\[
+ \sum_{i=1}^{r+1} \delta(q^{1/2}z_{x_i}) \delta(q^{-1/2}w_{x_i}) \prod_{k \neq i} \frac{\theta x_i - \theta^{-1}x_k}{x_i - x_k} \frac{\theta^{-1}q x_i - \theta x_k}{q x_i - x_k}
\]

and

\[
(1 - q)(1 - q^{-1})\hat{f}_1(w)\varepsilon_1(z)
\]

\[
= \sum_{j=1}^{r+1} \delta(q^{-1/2}w_{x_j}) \prod_{k \neq j} \frac{\theta^{-1}x_j - \theta x_k}{x_j - x_k} \Gamma_j^{-1} \sum_{i=1}^{r+1} \delta(q^{1/2}z_{x_i}) \prod_{k \neq i} \frac{\theta x_i - \theta^{-1}x_k}{x_i - x_k} \Gamma_i
\]

\[
= \sum_{1 \leq i \neq j \leq r+1} \delta(q^{1/2}z_{x_i}) \delta(q^{-1/2}w_{x_j}) \frac{\theta^{-1}x_j - \theta x_i}{x_j - x_i} \frac{\theta x_i - q^{-1}\theta x_j}{x_i - q^{-1}x_j}
\]

\[
\times \prod_{k \neq i, j} \frac{(\theta x_i - \theta^{-1}x_k)(\theta^{-1}x_j - \theta x_k)}{(x_i - x_k)(x_j - x_k)} \Gamma_i \Gamma_j^{-1}
\]

\[
+ \sum_{i=1}^{r+1} \delta(q^{-1/2}w_{x_i}) \delta(q^{1/2}z_{x_i}) \prod_{k \neq i} \frac{\theta^{-1}q x_i - \theta^{-1}x_k}{q^{-1}x_i - x_k} \frac{\theta^{-1}x_i - \theta x_k}{x_i - x_k}
\]
In the commutator \([\epsilon_1(z), f_1(w)]\), the terms corresponding to \(i \neq j\) cancel out, while the remaining terms are proportional to \(\delta(z/w)\):

\[
[\epsilon_1(z), f_1(w)] = \frac{\delta(z/w)}{(1 - q)(1 - q^{-1})} \sum_{i=1}^{r+1} \left\{ \delta(q^{1/z}w_i) \prod_{k \neq i} \theta_{x_i - \theta^{-1}x_k} \frac{\theta^{-1}q_{x_i} - \theta x_k}{q x_i - x_k} \right\}. 
\]

Recalling that \(\delta(z) = \sum_{n \in \mathbb{Z}} z^n\), this is in agreement with the partial fraction decomposition of (3.10), which can be written as

\[
\psi^+(z) = 1 + \frac{g(1, 1)}{(1 - q)(1 - q^{-1})} \sum_{i=1}^{r+1} \left\{ \frac{1}{1 - q^{1/z}w_i} \prod_{k \neq i} \theta_{x_i - \theta^{-1}x_k} \frac{\theta^{-1}q_{x_i} - \theta x_k}{q x_i - x_k} \right\}, 
\]

\[
\psi^-(z) = 1 - \frac{g(1, 1)}{(1 - q)(1 - q^{-1})} \sum_{i=1}^{r+1} \left\{ \frac{1}{1 - q^{-1/z}w_i} \prod_{k \neq i} \theta_{x_i - \theta^{-1}x_k} \frac{\theta^{-1}q_{x_i} - \theta x_k}{q x_i - x_k} \right\}, 
\]

as power series of \(z\) and \(z^{-1}\), respectively. Here, \(g(1, 1)/(1 - q)(1 - q^{-1}) = (1 - t^{-1})(1 - t q^{-1})/(1 - q^{-1})\). □

**Lemma 3.8.** The following relations hold between the Macdonald currents and the series (3.10):

\[
\begin{align*}
g(z, w)\psi^\pm(z) \epsilon_1(w) + g(w, z)\epsilon_1(w) \psi^\pm(z) &= 0 \\
g(w, z)\psi^\pm(z) f_1(w) + g(z, w)f_1(w) \psi^\pm(z) &= 0.
\end{align*}
\]

**Proof.** The second relation again follows by changing \((q, t) \mapsto (q^{-1}, t^{-1})\). Using the definitions,

\[
\frac{1 - q}{q^{1/z}} \psi^+ (z) \epsilon_1 (w) = \prod_{j=1}^{r+1} (1 - q^{-1/z}w_j)(1 - q^{1/z}w_j) \\
\prod_{j=1}^{r+1} (1 - q^{-1/z}w_j)(1 - q^{1/z}w_j) \\
\times \delta(q^{1/z}w_i) \prod_{k \neq i} \frac{\theta_{x_i - \theta^{-1}x_k}}{x_i - x_k} \Gamma_i
\]
Lemma 7.4. is given below by use of the shuffle algebra, where it follows from the identity

\[ \frac{1 - q^{1/2}}{q^{1/2}} \epsilon_1(w) \psi^+(z) = \sum_{i=1}^{r+1} \delta(q^{1/2} w x_i) \prod_{k \neq i} (1 - q^{1/2} x_k z) \prod_{j=1}^{r+1} \frac{\theta x_i - \theta^{-1} x_k}{x_i - x_k} \Gamma_i, \]

Using

\[ g(z, w) \frac{(w - q^{-1} t z)(w - t^{-1} z)}{(w - q^{-1} z)(w - z)} = -g(w, z) \frac{(w - t z)(w - q t^{-1} z)}{(w - z)(w - q z)} \]

we see that \( g(z, w) \psi^+(z) \epsilon_1(w) + g(w, z) \epsilon_1(w) \psi^+(z) = 0 \). The exchange relation with \( \psi^- \) follows a similar calculation. \( \square \)

The final two relations are the cubic, Serre-type relations for the fundamental currents.

**Lemma 3.9.** The fundamental currents satisfy the cubic relations

\[
\text{Sym}_{z_1, z_2, z_3} \left( \sum_{z_3} \left[ \epsilon_1(z_1), [\epsilon_1(z_2), \epsilon_1(z_3)] \right] \right) = 0. \tag{3.14}
\]

\[
\text{Sym}_{z_1, z_2, z_3} \left( \sum_{z_3} \left[ \tilde{f}_1(z_1), [\tilde{f}_1(z_2), \tilde{f}_1(z_3)] \right] \right) = 0. \tag{3.15}
\]

The proof of (3.14) is again by straightforward but tedious calculation. A shorter proof is given below by use of the shuffle algebra, where it follows from the identity of Lemma 7.4.

Therefore, the currents \( \epsilon_1(z), \tilde{f}_1(z), \psi^\pm(z) \) satisfy all the relations of the level (0, 0) quantum toroidal algebra \( gl_1 \), and Theorem 3.5 follows.

4. Plethysms, Bosonization and the Limit \( N \to \infty \)

The action of the difference operators \( \epsilon_1(z), \tilde{f}_1(z) \) on the space of symmetric functions can be re-written in terms of plethysms, which act naturally on symmetric functions when they are expressed in terms of the power sum symmetric functions \( p_k, k \in \mathbb{Z} \setminus \{0\} \).

In the infinite rank limit \( N \to \infty \), the power sum symmetric functions are algebraically...
independent, and together with the derivatives with respect to these functions, they
generate an infinite-dimensional Heisenberg algebra. The plethystic expression becomes,
in this limit, a formula for the bosonization of the difference operators. This formulation
will be compared to recent work of Bergeron et al. [BGLX16] and the bosonization of
[FHH+09].

4.1. Action of power sums on difference operators as plethysms. Let \( \{p_k\}_{k \in \mathbb{Z} \setminus \{0\}} \) be
an infinite set of algebraically independent variables. Let \( \mathcal{P} \) be the space of functions
which can be expressed as formal power series of elements in this set. For any (finite
or infinite) collection \( X \) of variables, functions in \( \mathcal{P} \) can be evaluated by using the
substitution \( p_k \mapsto p_k[X] := \sum_{i=1}^{N} x_i^k \) for all \( k \in \mathbb{Z}^* \). Any \( F \in \mathcal{P} \) becomes a symmetric
function of the variables in \( X \), \( F[X] \), by taking \( p_k = p_k[X] \). If the number of variables
\( N \) is finite, the resulting space of functions is a quotient of \( \mathcal{P} \), because the functions
\( p_k[X] \) are not algebraically independent. By a slight abuse of notation we still denote
the resulting space of functions by \( \mathcal{P} \). We also define the space \( \mathcal{P}_+ \subset \mathcal{P} \) of formal power
series of the positive power sums \( \{p_k\}_{k \in \mathbb{Z}_{>0}} \).

The action of the difference operators \( \varepsilon_1(z) \), \( \varepsilon_1(z) \) on \( \mathcal{P} \) can be rewritten in terms of
plethysms, for any \( N \).

**Lemma 4.1.** For any \( N, i \leq N \), and \( X = (x_1, x_2, \ldots, x_N) \), the following commutation
relations hold:

\[
[p_k[X], \mathcal{D}^{(q,t)}_{1;n+k}] = (1 - q^k) \mathcal{D}^{(q,t)}_{1;n+k}, \quad \text{and} \quad [p_k[X], \mathcal{D}^{(q^{-1},t^{-1})}_{1;n+k}] = (1 - q^{-k}) \mathcal{D}^{(q^{-1},t^{-1})}_{1;n+k}.
\]

**Proof.** By direct computation, using \( [p_k[X], \Gamma_i] = (1 - q^k) x_i^k \Gamma_i \), with \( \Gamma_i \) as in (1.6). \( \Box \)

**Corollary 4.2.** For any \( N, k \in \mathbb{Z}^* \), and \( X \) as above,

\[
\varepsilon_1(z) p_k[X] = \left( p_k[X] + \frac{q^{k/2} - q^{-k/2}}{z^k} \right) \varepsilon_1(z),
\]

\[
\hat{f}_1(z) p_k[X] = \left( p_k[X] - \frac{q^{k/2} - q^{-k/2}}{z^k} \right) \hat{f}_1(z).
\]

Therefore, up to a scalar multiple, the difference operators \( \varepsilon_1(z) \), \( \hat{f}_1(z) \) act on \( \mathcal{P} \)
by plethysms (see the notes [Hai99] for a detailed exposition). In \( \lambda \)-ring notation, let
\( X = (x_1, x_2, \ldots) \) denote an alphabet, and \( p_k[X] = \sum_i x_i^k \). Given two alphabets \( X, Y \),
the sum \( X + Y \) refers to the concatenation of the two alphabets, so that \( p_k[X + Y] = p_k[X] + p_k[Y] \). For any scalar \( \lambda \), \( p_k[\lambda X] = \lambda^k p_k[X] \), and for a single variable alphabet
\( \mu \), \( p_k[X + \mu] = p_k[X] + \mu^k \).

In the plethystic notation, Corollary 4.2 can be written as

\[
\varepsilon_1(z) F[X] = F \left[ X + \frac{q^{1/2} - q^{-1/2}}{z} \right] \varepsilon_1(z),
\]

\[
\hat{f}_1(z) F[X] = F \left[ X - \frac{q^{1/2} - q^{-1/2}}{z} \right] \hat{f}_1(z) \quad (4.1)
\]

for any \( F \in \mathcal{P} \), where the alphabet \( X \) is extended by two variables, \( q^{1/2}/z \) and \( q^{-1/2}/z \).
Note that when acting on \( F \in \mathcal{P} \) the currents in (4.1) contain both positive and negative
obtained by acting on the constant \( F \). The plethystic part of the action is given in Eq. (4.1). The scalar factors are powers of \( 892 \). Di Francesco, R. Kedem, Theorem 4.3. The currents \( e_1(z) \) and \( f_1(z) \) act as the following plethysms on symmetric functions \( F[X] \in \mathcal{P} \):

\[
e_1(z) F[X] = \frac{(q/t)^{1/2}}{(1-q)(1-t^{-1})} \left( t^{N} \frac{C^+(q^{1/2}t^{-1}z)}{C^+(q^{1/2}z)} - t^{-N} \frac{C^-(q^{1/2}t^{-1}z)}{C^-(q^{1/2}z)} \right) F \left[ X + \frac{q^{1/2} - q^{-1/2}}{z} \right],
\]

\[
f_1(z) F[X] = \frac{(i/q)^{-1/2}}{(1-q^{-1})(1-t)} \left( t^{-N} \frac{C^+(q^{1/2}tz)}{C^+(q^{1/2}z)} - t^{N} \frac{C^-(q^{1/2}tz)}{C^-(q^{1/2}z)} \right) F \left[ X - \frac{q^{1/2} - q^{-1/2}}{z} \right],
\]

where the first term in the prefactors must be expanded as a series of \( z \), and the second as a series of \( z^{-1} \).

**Proof.** The plethystic part of the action is given in Eq. (4.1). The scalar factors are obtained by acting on the constant \( F[X] \equiv 1 \). For example,

\[
e_1(z) \cdot 1 = \frac{q^{1/2}}{1-q} \sum_{i=1}^{N} \delta(q^{1/2}z x_i) \prod_{j \neq i} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j}
= \frac{q^{1/2}}{1-q} \sum_{i=1}^{N} \left\{ \frac{t^{N-1}}{1-q^{1/2}z x_i} \prod_{j \neq i} \frac{x_i - t^{-1} x_j}{x_i - x_j} + \frac{t^{-N-1}}{1-q^{-1/2}z^{-1} x_i} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} - \prod_{j \neq i} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \right\}.
\]

Observe that

\[
\sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} = \frac{t^{N-1}}{1-t^{-1}} - \frac{t^{-N+1}}{1-t^{-1}} = - \left\{ \frac{t^{N+1}}{1-t} + \frac{t^{-N+1}}{1-t} \right\},
\]

because the left hand side is a symmetric rational function, with denominator the Vandermonde determinant \( \prod_{i<j} x_i - x_j \), and total degree 0. Therefore, it is a constant \( c_N \), which can be found by taking the limit \( x_1 \to \infty \):

\[
c_N = \theta^{N-1} + \theta^{-1} \sum_{i=2}^{N} \prod_{j \neq i, j > 1} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} = \theta^{N-1} + \theta^{-1} c_{N-1}.
\]
Since \( c_0 = 0 \), this implies \( c_N = \frac{\theta^N - \theta^{-N}}{\theta - \theta^{-1}} \), and (4.4) follows.

Moreover, by simple fraction decomposition,

\[
\frac{C'(q^{1/2} t^{-1} z)}{C'(q^{1/2} z)} = \prod_{i=1}^{N} \frac{1 - q^{1/2} t^{-1} z x_i}{1 - q^{1/2} z x_i} = t^{-N} + (1 - t^{-1})
\]

\[
\times \sum_{i=1}^{N} \frac{1}{1 - q^{1/2} z x_i} \prod_{j \neq i} \frac{x_i - t^{-1} x_j}{x_i - x_j}.
\]

Therefore

\[
\frac{t^{-N} - 1}{1 - t^{-1}} \frac{C'(q^{1/2} t^{-1} z)}{C'(q^{1/2} z)} = \frac{t^{-N} - 1}{1 - t^{-1}} + t^{-N} \sum_{i=1}^{N} \frac{1}{1 - q^{1/2} z x_i} \prod_{j \neq i} \frac{x_i - t^{-1} x_j}{x_i - x_j}.
\]

By using the change of variables \((q, t, z, x_i) \to (q^{-1}, t^{-1}, z^{-1}, x_i^{-1})\),

\[
\frac{t^{-N} - 1}{1 - t} \frac{C'(q^{-1/2} t^{-1} z^{-1})}{C'(q^{-1/2} z^{-1})} = \frac{t^{-N} - 1}{1 - t} + t^{-N} \sum_{i=1}^{N} \frac{1}{1 - q^{-1/2} t^{-1} z^{-1} x_i^{-1}} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j}.
\]

Summing the two above contributions yields the full action of \( \tilde{\epsilon}(z) \) on 1. Similarly, the action of \( \tilde{f}(z) \) on 1 follows immediately by sending \((q, t) \to (q^{-1}, t^{-1})\). \( \square \)

The Cartan currents act as scalars on symmetric functions of \( x_1, x_2, \ldots, x_N \):

\[
\psi^+(z) = \frac{C'(q^{-1/2} t z) C'(q^{1/2} t^{-1} z)}{C'(q^{-1/2} t z) C'(q^{1/2} t^{-1} z)}, \quad \psi^-(z) = \frac{C'(q^{-1/2} t z) C'(q^{1/2} t^{-1} z)}{C'(q^{-1/2} t z) C'(q^{1/2} t^{-1} z)},
\]

and are respectively series of \( z \) and \( z^{-1} \).

**Remark 4.4.** The plethystic formulas above for \( \tilde{\epsilon}_1 \) and \( -\tilde{f}_1 \), as well as \( \psi^+ \) and \( \psi^- \), are exchanged under the involution \( x_i \mapsto 1/x_i \) for all \( i \), and \( z \mapsto z^{-1} \), under which \( C'(z) \mapsto C'(z^{-1}) \), and the one-variable plethysm \([X + \mu] \mapsto [X + \mu^{-1}]\). This is in agreement with Remark 3.4 for \( \alpha = 1 \).

### 4.2. Bosonization formulas in the \( N \to \infty \) limit.

In the infinite rank \( N \to \infty \) limit, the power sum symmetric functions \( p_k \) become algebraically independent for all \( k \in \mathbb{Z} \setminus \{0\} \), and together with the derivatives with respect to these functions, form two Heisenberg algebras. However for the limit to make sense in view of the formulas of Theorem 4.3, we must choose a particular condition on \( t \), e.g. \(|t| > 1\). With this choice only the limit \( \hat{\epsilon}(z) \) of \( \tilde{\epsilon}_1(z) \) makes sense. Our strategy is to supplement this limit with another \( \hat{f}(z) \) current, via the substitution \((q, t) \to (q^{-1}, t^{-1})\) applied on \( \epsilon \). We stress that this is not the limit of the current \( \hat{f}_1(z) \) of Theorem 4.3. Moreover, it turns out that the new currents \( \hat{\epsilon}(z) \) and \( \hat{f}(z) \) act on \( \mathcal{P}_+ \).
4.2.1. Plethystic formulas for the $N \to \infty$ limit The collection $X = (x_1, x_2, \ldots)$ is now infinite. Define

$$\hat{\epsilon}(z) := \lim_{N \to \infty} \frac{1}{N} \epsilon_1(z),$$

$$\hat{f}(z) := \hat{\epsilon}(z)|_{q \to q^{-1}, t \to t^{-1}}$$

(4.5)

(under the assumption that $|t| > 1$). Similarly, let

$$\hat{C}^+(z) := \lim_{N \to \infty} C^+(z) = \prod_{i=1}^{\infty} (1 - zx_i), \quad \hat{C}^-(z) := \lim_{N \to \infty} C^-(z) = \prod_{i=1}^{\infty} (1 - zx_i^{-1}),$$

so that the limiting Cartan current is

$$\hat{\psi}^+(z) = \frac{\hat{C}^+(q^{-1/2}tz)\hat{C}^+(q^{1/2}t^{-1}z)}{\hat{C}^+(q^{-1/2}z)\hat{C}^+(q^{1/2}z)},$$

(4.7)

to be understood as a power series of $z$.

**Theorem 4.5.** The limiting currents $\hat{\epsilon}(z), \hat{f}(z)$ act on functions $F[X] \in \mathcal{P}_+$ as:

$$\hat{\epsilon}(z) F[X] = \frac{q^{1/2}}{(1 - q)(1 - t^{-1})} \hat{C}^+(q^{1/2}t^{-1}z) \hat{C}^+(q^{1/2}z) F \left[ X + \frac{q^{1/2} - q^{-1/2}}{z} \right],$$

$$\hat{f}(z) F[X] = \frac{q^{-1/2}}{(1 - q^{-1})(1 - t)} \hat{C}^+(q^{-1/2}tz) \hat{C}^+(q^{-1/2}z) F \left[ X - \frac{q^{1/2} - q^{-1/2}}{z} \right].$$

(4.8)

(Note that the plethystic parts give rise to series of $z^{-1}$, whereas the coefficients are series of $z$ in both cases). Moreover, the following relations hold:

$$[\hat{\epsilon}(z), \hat{f}(w)] = \frac{1}{g(1, 1)} \left\{ \delta(z/w)\hat{\psi}^+(z) - \delta(qt^{-1}z/w)\hat{\psi}^-(z) \right\}$$

$$\hat{\epsilon}(z) \hat{\epsilon}(w) = -\frac{g(w, z)}{g(z, w)} \hat{\epsilon}(w) \hat{\epsilon}(z), \quad \hat{f}(z) \hat{f}(w) = -\frac{g(z, w)}{g(w, z)} \hat{f}(w) \hat{f}(z)$$

$$\hat{\psi}^+(z) \hat{\epsilon}(w) = -\frac{g(w, z)}{g(z, w)} \hat{\psi}^+(z) \hat{\epsilon}(w), \quad \hat{\psi}^+(z) \hat{f}(w) = -\frac{g(z, w)}{g(w, z)} \hat{\psi}^+(z) \hat{f}(w)$$

$$\hat{\psi}^-(z) \hat{f}(w) = -\frac{g(qt^{-1}z, w)}{g(w, qt^{-1}z)} \hat{\psi}^-(z) \hat{f}(w), \quad \hat{\psi}^-(z) \hat{\psi}^+(w) = \frac{g(w, z)g(qt^{-1}z, w)}{g(z, w)g(w, qt^{-1}z)} \hat{\psi}^+(w) \hat{\psi}^-(z)$$

where $\hat{\psi}^+(z)$ is defined in (4.7), and $\hat{\psi}^-(z)$ is a series of $z^{-1}$ acting on symmetric functions $F[X] \in \mathcal{P}_+$ as:

$$\hat{\psi}^-(z) F[X] = F \left[ X + \frac{(q^{1/2} - q^{-1/2})(1 - tq^{-1})}{z} \right].$$

In addition, the “Serre relations” of Lemma 3.9 still hold with $(\epsilon_1, \overset{\theta}{f}_1) \to (\hat{\epsilon}, \hat{f})$. 

Proof. The expressions for \( \hat{\epsilon}(z) \) and \( \hat{\eta}(z) \) follow from Theorem 4.3 and the definition (4.5). To compute the commutator of these two generators, the plethysms must be commuted through the prefactors. Noting that

\[
\frac{\hat{\mathcal{C}}^+(q^{-1/2}tw)}{\hat{\mathcal{C}}^+(q^{-1/2}w)} \left[ X + \frac{q^{1/2} - q^{-1/2}}{z} \right] = \frac{1}{1 - t} \frac{(1 - tw/z)(1 - q^{-1}w/z)}{1 - w/z(1 - q^{-1}tw/z)} \frac{\hat{\mathcal{C}}^+(q^{1/2}tw)}{\hat{\mathcal{C}}^+(q^{-1/2}w)},
\]

\[
\frac{\hat{\mathcal{C}}^+(q^{1/2}t^{-1}z)}{\hat{\mathcal{C}}^+(q^{1/2}z)} \left[ X - \frac{q^{1/2} - q^{-1/2}}{w} \right] = \frac{1}{1 - t^{-1}} \frac{(1 - t^{-1}z/w)(1 - qz/w)}{1 - z/w(1 - qt^{-1}z/w)} \frac{\hat{\mathcal{C}}^+(q^{1/2}t^{-1}z)}{\hat{\mathcal{C}}^+(q^{1/2}z)},
\]

this gives

\[
[\hat{\epsilon}(z), \hat{\eta}(w)] = \frac{1}{(1 - q)(1 - q^{-1})(1 - t)(1 - t^{-1})} \frac{\hat{\mathcal{C}}^+(q^{1/2}t^{-1}z)\hat{\mathcal{C}}^+(q^{-1/2}tw)}{\hat{\mathcal{C}}^+(q^{1/2}z)\hat{\mathcal{C}}^+(q^{-1/2}w)} \times \left\{ \frac{(1 - tw/z)(1 - q^{-1}w/z)}{1 - w/z(1 - q^{-1}tw/z)} - \frac{(1 - t^{-1}z/w)(1 - qz/w)}{1 - z/w(1 - qt^{-1}z/w)} \right\} \times F \left[ X + (q^{1/2} - q^{-1/2}) \left( \frac{1}{z} - \frac{1}{w} \right) \right] = \frac{1}{g(1, 1)} (\delta(z/w) - \delta(qt^{-1}z/w)) \frac{\hat{\mathcal{C}}^+(q^{1/2}t^{-1}z)\hat{\mathcal{C}}^+(q^{-1/2}tw)}{\hat{\mathcal{C}}^+(q^{1/2}z)\hat{\mathcal{C}}^+(q^{-1/2}w)} \times F \left[ X + (q^{1/2} - q^{-1/2}) \left( \frac{1}{z} - \frac{1}{w} \right) \right].
\]

The other relations are obtained by similar computations. \( \square \)

Corollary 4.6. The currents \( \hat{\epsilon}(z) \), \( \hat{\eta}(z) \) and series \( \hat{\psi}^{\pm}(z) \in \mathbb{C}[[z^{\pm1}]] \) satisfy the relations of the quantum toroidal algebra of Definition 3.1 with non-trivial central charge, \((\ell_1, \ell_2) = (1, 0)\), i.e. \( \hat{\gamma} = \gamma = (tq^{-1})^{1/2} \) and \( \hat{\delta} = 1 \). That is, they act on the space \( \mathcal{P}_+ \) by the “horizontal representation”.

4.2.2. Comparison with the plethystic operators of Bergeron et al. In [BGLX16], the authors define the difference operators \( D_k, D_k^* \), \( k \in \mathbb{Z}_+ \), acting on the space of symmetric functions in infinitely many variables. Extending this definition to \( k \in \mathbb{Z} \), these generating functions

\[
D(z) = \sum_{k \in \mathbb{Z}} z^k D_k \quad \text{and} \quad D^*(z) = \sum_{k \in \mathbb{Z}} z^k D_k^*
\]

act as

\[
D(z) F[X] = \hat{\mathcal{C}}^+(z) F \left[ X + \frac{(1 - t)(1 - q)}{z} \right], \quad (4.9)
\]

\[
D^*(z) F[X] = \frac{1}{\hat{\mathcal{C}}^+(z)} F \left[ X - \frac{(1 - t^{-1})(1 - q^{-1})}{z} \right]. \quad (4.10)
\]
The commutation relations of [BGLX16], extended to $k \in \mathbb{Z}$, are

$$[D(z), D^*(\frac{w}{qt})]F[X] = \frac{(1-t)(1-q)}{qt-1} \left\{ \delta(\frac{z}{w}) \frac{\hat{C}^+(z)}{\hat{C}^+(\frac{z}{qt})} F[X] 
- \delta(\frac{qtz}{w}) F\left[X + \frac{(1-t)(1-q)(1-\frac{1}{qt})}{z}\right] \right\}, \quad (4.11)$$

or in terms of coefficients,

$$[D_a, D^*_b]F[X] = \frac{(1-t)(1-q)}{qt-1} \left\{ (qt)^b h_{a+b}[X(\frac{1}{qt} - 1)]F[X] 
- F\left[X + \frac{(1-t)(1-q)(1-\frac{1}{qt})}{z}\right] \bigg|_{z^{a+b}} \right\},$$

where the notation $F(z)|_{z^{a+b}}$ stands for the coefficient of $z^{a+b}$ in $F$, and $h_n[X]$ are the complete symmetric functions. These specialize to the commutator $[D_a, D^*_b]$ of [BGLX16] when $a, b > 0$.

The action on $\mathcal{P}_+$ of the difference operators $D_k, D^*_k$ of [BGLX16] is related to that of $\hat{\epsilon}, \hat{f}$ of Theorem 4.5 above as follows:

**Theorem 4.7.** Let $\Sigma$ be the operator acting by the plethysm $[X] \mapsto [X/(t-1)]$ (accordingly $\Sigma^{-1}$ acts by $[X] \mapsto [X(t-1)]$). When restricted to acting on $\mathcal{P}_+$,

$$\hat{\epsilon}(z) = \frac{q^{\frac{z}{2}}}{(1-q)(1-t^{-1})} \left( \Sigma^{-1} D(q^{\frac{z}{2}}) \Sigma \right) \bigg|_{t \to t^{-1}},$$

$$\hat{f}(z) = \frac{q^{-\frac{z}{2}}}{(1-q^{-1})(1-t)} \left( \Sigma^{-1} D^*(\frac{z}{qt}) \Sigma \right) \bigg|_{t \to t^{-1}}.$$

**Proof.** The bosonized expressions are matched using (4.9) and (4.10). Defining $F = \Sigma G$,

$$F\left[X \pm \frac{(1-t)(1-q)}{q^{\frac{1}{2}}u}\right] = G\left[\frac{X}{t-1} \pm \frac{q-1}{q^{\frac{1}{2}}u}\right],$$

while

$$\Sigma^{-1} \hat{C}^+(q^{\frac{1}{2}}u) \Sigma = \frac{\hat{C}^+(tq^{\frac{1}{2}}u)}{\hat{C}^+(q^{\frac{1}{2}}u)}, \quad \Sigma^{-1} \frac{1}{\hat{C}^+(\frac{u}{q^{2}t})} \Sigma = \frac{\hat{C}^+(t^{-1}q^{-\frac{1}{2}}u)}{\hat{C}^+(q^{-\frac{1}{2}}u)}.$$

Hence

$$\Sigma^{-1} D(q^{\frac{1}{2}}u) \Sigma G[X] = \Sigma^{-1} D(q^{\frac{1}{2}}u) F[X] = \frac{\hat{C}^+(tq^{\frac{1}{2}}u)}{\hat{C}^+(q^{\frac{1}{2}}u)} G\left[X + \frac{q-1}{q^{\frac{1}{2}}u}\right],$$
4.2.3 Bosonization formulas for ˆf functions in N X collection and all other commutators are 0. These relations still hold when evaluated on the infinite (Corollary 4.9.

The generating functions Theorem 4.5 can be rewritten as polynomials ˜latter as a function of [X]. The Theorem follows by taking t → t−1 in the above, and comparing with (4.8).

Remark 4.8. Theorem 4.7 allows a refinement of Remark 2.12. The modified Macdonald polynomials ˜Hλ are defined in terms of the Macdonald polynomial ˜X where 1/2 the action of ˜x, 1/2 1 2

\begin{align*}
\Sigma^{-1}D^*\left(\frac{\mu}{q^\frac{1}{2}t}\right)\Sigma G[X] &= \Sigma^{-1}D^*\left(\frac{\mu}{q^\frac{1}{2}t}\right)F[X] \\
&= \frac{\hat{C}^*(t^{-1}q^{-\frac{1}{2}}u)}{\hat{C}^*(q^{-\frac{1}{2}}u)} G\left[\frac{X - q - 1}{u}\right],
\end{align*}

where the action of \(\Sigma^{-1}\) on \(\hat{C}^*(q^{-\frac{1}{2}}u)^{\pm 1}G\left[\frac{X}{t-1} \pm \frac{q-1}{q^\frac{1}{2}u}\right]\) is expressed by considering the latter as a function of \([X]\). The Theorem follows by taking \(t \rightarrow t^{-1}\) in the above, and comparing with (4.8).

\(\eta\) defined in Eq. (2.7) acts when \(N\) is finite as

\[\eta^{-1} = \nabla^{(N)} = C_N \left(\frac{N-1}{2}\right) q^d \left(\Sigma^{-1} \nabla \Sigma\right)|_{t \rightarrow t^{-1}},\]

where \(d\) \(\lambda\), \(C_N\) as in Remark 2.12, and \(\nabla\) is restricted to act on symmetric functions in \(N\) variables.

4.2.3 Bosonization formulas for ˆf(z), ˆf(z), ˆψ±(z) Bosonization refers to expressing all the algebra generators in terms of Heisenberg algebra generators \(\{p_k, \frac{\partial}{\partial p_k} : k > 0\}\) with

\[\left[\frac{\partial}{\partial p_j}, p_k\right] = \delta_{jk}\]

and all other commutators are 0. These relations still hold when evaluated on the infinite collection \(X = (x_1, x_2, \ldots)\), as the \(p_k[X]\) remain independent variables.

As \(\{p_k[X], k > 0\}\) are independent, a plethysm which adds one additional variable to the alphabet \(X\) can be written as

\[F[X + \mu] = \exp\left\{\sum_{k>0} \mu_k \frac{\partial}{\partial p_k[X]}\right\} F[X].\]

Theorem 4.5 can be rewritten as

Corollary 4.9. The generating functions ˆf(z), ˆf(z) and ˆψ±(z), acting on the space \(\mathcal{P}_+\), can be expressed in terms of the Heisenberg algebra generators as follows:

\[
\hat{\epsilon}(z) = \frac{q^{1/2}}{(1 - q)(1 - t^{-1})} e^{\sum_{k>0} p_k[X] z^{k/2}(1 - t^{-k})} \frac{1}{z^k} e^{\sum_{k>0} \frac{q^{k/2} - q^{-k/2}}{k} \frac{a}{\eta_p[X],}}.
\]
\[ \hat{f}(z) = \frac{q^{-1/2}}{(1-q^{-1})(1-t^{-1})} e^{\sum_{k>0} p_k[X] \frac{r^{k/2}(q^{-1})^k}{k}} e^{-\sum_{k>0} \frac{q^{k/2-q^{-k/2}}}{q^{pk}X}}, \]
\[ \hat{\psi}^+(z) = e^{\sum_{k>0} p_k[X] \frac{r^{k/2-q^{-k/2}}(q^{-1})^{k/2}}{k}} \frac{\partial}{\partial p_k[X]}, \]
\[ \hat{\psi}^-(z) = e^{\sum_{k>0} \frac{q^{k/2}-q^{-k/2}}{k} (1-(q-1)^k)} \frac{\partial}{\partial p_k[X]}. \]

**Remark 4.10.** These expressions are identical to the bosonized expressions for the level \((1,0)\) representation of the quantum toroidal algebra [FHH+09], up to simple redefinitions of the generators and identification of the bosonic modes for \(k > 0\) as

\[ a_k = p_k[X], \quad a_{-k} = k \frac{1-q^k}{1-\ell^k} \frac{\partial}{\partial p_k[X]}. \]

Note that the \((0,0)\) representation is very different. It corresponds to taking two mutually commuting families of harmonic oscillators, namely \(a_k = p_k[X]\) for \(k > 0\), and \(a_k = p_k[X]\) for \(k < 0\), together with their respective adjoints \(\partial/\partial p_k[X]\) for \(k > 0\) and for \(k < 0\). This is related to the fact that if \(\ell_1 = 0\), the modes \(a_k\) and \(a_{-\ell}\) commute for all \(k, \ell > 0\), leading to commuting \(\psi^\pm\).

### 4.3. Plethystic formulas in the dual Whittaker limit

The plethystic expressions make sense in the limit \(t \to \infty\). We use the following definition for the limiting currents:

\[ \epsilon^{(q)}(u) := \lim_{t \to \infty} t^{-\frac{N-1}{2}} \epsilon_1(u), \quad f^{(q)}(u) := \lim_{t \to \infty} t^{-\frac{N-1}{2}} f_1(u), \]
\[ \psi^{\pm(q)}(z) := \lim_{t \to \infty} t^{-N} \psi^{\pm}(z), \quad \text{(4.13)} \]

and note the degeneration of the function \(g\): \(g^{(q)}(z, w) := \lim_{t \to \infty} -\frac{q^{-1}}{z^w} g(z, w) = z - q^w\). We have in particular:

\[ \psi^{\pm(q)}(z) = (-q^{-1/2} z^{\pm 1})^N \prod_{i=1}^{N} \frac{(x_i)^{\pm 1}}{(1-q^{-1/2} (x_i)^{\pm 1})(1-q^{1/2} (x_i)^{\pm 1})}, \quad \text{(4.14)} \]

whereas the other relations of Theorem 3.5 reduce to those of the Drinfeld presentation of \(U_{\sqrt{q}}(\hat{sl}_2)\) for \(\epsilon^{(q)}\), \(f^{(q)}\) and \(\psi^{\pm(q)}\), with zero central charge. However, the Cartan currents \(\psi^{\pm(q)}\) have a non-standard valuation of \(z^{\pm N}\) and as a consequence vanish when \(N \to \infty\), whereas in the standard Drinfeld presentation of the quantum affine algebra, the zero modes of the Cartan currents are invertible elements.

Define \(A := x_1 x_2 \cdots x_N\). The function \(\text{Log } A\) can be understood as a renormalized version of the power sum \(p_0[X]\):

\[ \text{Log } A = \lim_{\epsilon \to 0} \frac{p_\epsilon[X]}{\epsilon} - N. \]

By a slight abuse of notation, we will write \(A = e^{p_0}\). In the dual Whittaker limit,

\[ \psi^{+(q)}(z) = \frac{(-q^{-1/2} z)^N A}{C^+(q^{-1/2} z) C^+(q^{1/2} z)} = (-q^{-1/2} z)^N e^{p_0 + \sum_{k>0} p_k(q^{k/2}+q^{-k/2})^+}, \]
\[ \psi^{-(q)}(z) = \frac{(-q^{-1/2}z^{-1})^N A^{-1}}{C^{-}(q^{-1/2}z)C^{-}(q^{1/2}z)} = (-q^{-1/2}z^{-1})^N e^{-p_0 + \sum_{k>0} p_k (q^{k/2} + q^{-k/2}) \frac{z^k}{k}}. \]

The necessity of the introduction of the quantity \( A \) (or the Heisenberg mode \( p_0 \)) means it must be considered as an independent variable, so plethysms now act on the space \( \mathcal{P}_0 \) of functions that are power series of all \( p_k, k \in \mathbb{Z} \) (including \( k = 0 \)). There is also an extra commutation relation in addition to those of Lemma 4.1.

**Lemma 4.11.**

\[ \mathcal{D}^{(q,t)}_{1:n} p_0[X] = (p_0[X] + \log q) \mathcal{D}^{(q,t)}_{1:n}, \quad \text{and} \quad \mathcal{D}^{(q^{-1},t^{-1})}_{1:n} p_0[X] = (p_0[X] - \log q) \mathcal{D}^{(q^{-1},t^{-1})}_{1:n}. \quad (4.15) \]

**Proof.** This follows from the \( q \)-commutation relations

\[ \mathcal{D}^{(q,t)}_{1:n} A = q A \mathcal{D}^{(q,t)}_{1:n}, \quad \text{and} \quad \mathcal{D}^{(q^{-1},t^{-1})}_{1:n} A = q^{-1} A \mathcal{D}^{(q^{-1},t^{-1})}_{1:n}, \quad (4.16) \]

due to \( \Gamma_i^{\pm 1} A = q^{\pm 1} A \Gamma_i \). \( \square \)

Plethysms act on functions \( \{p_k : k \in \mathbb{Z}\} \), so that the case of addition of one variable,

\[ p_k[X + \mu] = \begin{cases} p_k[X] + \mu^k & \text{if } k \neq 0, \\ p_0[X] + \log \mu & \text{if } k = 0. \end{cases} \]

As a check,

\[ p_k \left[ X + \frac{q^{1/2} - q^{-1/2}}{z} \right] = \begin{cases} p_k[X] + \frac{2^{k/2} - q^{-k/2}}{z^k} & \text{if } k \neq 0, \\ p_0[X] + \log q & \text{if } k = 0. \end{cases} \]

With this notation, the action of the currents in the dual Whittaker limit on functions \( F[X] \in \mathcal{P}_0 \) is

\[ e^{(q)}(z) F[X] = \frac{q^{1/2}}{1 - q} \left\{ \frac{1}{C^+(q^{1/2}z)} \right\} F \left[ X + \frac{q^{1/2} - q^{-1/2}}{z} \right], \]

\[ f^{(q)}(z) F[X] = \frac{q^{-1/2}}{1 - q^{-1}} \left\{ \frac{1}{C^-(q^{-1/2}z)} \right\} F \left[ X - \frac{q^{1/2} - q^{-1/2}}{z} \right]. \quad (4.17) \]

### 5. The Elliptic Hall Algebra and \( \mathcal{A}^{(q,t)}_{d1_{\mathcal{N}}} \)

In this section, we use the homomorphism between the Elliptic Hall Algebra (EHA) and sDAHA [SV11] to obtain new relations between the generalized Macdonald operators. These are t-deformed versions of the quantum determinants. We derive an expression for the operators \( \mathcal{D}^{(q,t)}_{\alpha; n} \) as polynomials [“\( q, t \)-determinants”] of the fundamental operators \( \mathcal{D}^{(q,t)}_{1:n}, \mathcal{D}^{(q,t)}_{1:n \pm 1} \). For simplicity and whenever unambiguous, we shall drop the superscript \( (q, t) \) and simplify the notation for the fundamental generators, \( \mathcal{D}_n := \mathcal{D}^{(q,t)}_{1:n} \) and \( \tilde{\mathcal{D}}_n := \mathcal{D}^{(q^{-1},t^{-1})}_{1:n \pm 1} \).
5.1. Elliptic Hall algebra: definition and isomorphisms. In this section, we recall the relevant results from [BS12,Sch12] for the EHA. Let

\[ \alpha_k := \frac{1}{k} (1 - q^k)(1 - t^{-k})(1 - q^{-k}t^k), \quad (k \in \mathbb{Z}^+) \]  

(E1) \[ [u_{c,d}, u_{a,b}] = 0 \quad \text{if } (0,0), (a,b), (c,d) \text{ are aligned} \]

and

(E2) \[ [u_{c,d}, u_{a,b}] = \frac{\epsilon_{a,b;c,d}}{\alpha_1} \theta_{a+c,b+d}, \quad \text{if } \gcd(a, b) = 1 \quad \text{and} \quad \Delta_{a,b;c,d} = \emptyset, \]

where \( \Delta_{a,b;c,d} \) is the intersection with \( \mathbb{Z}^2 \) of the strict interior of the triangle \((0, 0), (a, b), (a + c, b + d)\), and \( \epsilon_{a,b;c,d} = \sgn(ad - bc) \). The generators \( \theta \) and \( u \) are related by

(E3) \[ 1 + \sum_{n=1}^{\infty} \theta_{n(a,b)} z^n = e^{\sum_{k=1}^{\infty} a_k u_{k(a,b)} z^k}, \quad (\gcd(a, b) = 1), \]

in particular \( \theta_{n,p} = \alpha_1 u_{n,p} \) whenever \( \gcd(n, p) = 1 \).

The EHA is isomorphic to the quantum toroidal algebra Definition 3.2 of Sect. 3 via the following assignments:

\[ e(z) = \sum_{n \in \mathbb{Z}} u_{1,n} z^n, \quad f(z) = \sum_{n \in \mathbb{Z}} u_{-1,n} z^n, \quad \psi_{\pm}(z) = 1 + \sum_{n \geq 1} \theta_{0,\pm n} z^{\pm n}. \] (5.2)

There is a natural action of \( SL(2, \mathbb{Z}) \) on the generators \( u_{a,b} \) given by:

\[ \left( \begin{array}{cc} a_0 & a_1 \\ a_2 & a_3 \end{array} \right) \cdot u_{a,b} = u_{a_0a+a_2b,a_1a+a_3b}. \]

The two generators

\[ T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \quad \text{and} \quad U = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \] (5.3)

act on the EHA generators by \( T u_{a,b} = u_{a,a+b} \) and \( U u_{a,b} = u_{a+b,b} \).

In [SV11], the authors constructed a surjective homomorphism between EHA and sDAHA in the limit \( N \to \infty \). With this homomorphism, the natural action of \( SL(2, \mathbb{Z}) \) on EHA maps onto the natural \( SL(2, \mathbb{Z}) \) action on the DAHA [Che05]:

\[ T \mapsto \tau_+ = \ad_{\gamma-1}, \quad \text{and} \quad U \mapsto \tau_- = \ad_{\eta-1} \]

with \( \gamma \) and \( \eta \) as in (2.6) and (2.7), and Lemmas 2.10 and 2.11 respectively.

The homomorphism of algebras maps the generators \( u_{0,k} \) and \( u_{k,0} \) to the power sums \( p_k = \sum_i (X_i)^k \) and \( P_k := \sum_i (Y_i)^k \) respectively, \( k \in \mathbb{Z}\setminus\{0\} \), in sDAHA. The main result of this paper is the explicit computation of the kernel of this homomorphism: we claim that it is precisely expressed by the vanishing of “\((q,t)\)-determinant” homogeneous polynomials of degree \( N + 1 \) expressing \( D^{(q,t)}_{N+1,n} \) in terms of the generators \( D_k \)'s.
5.2. EHA representation via generalized Macdonald operators. Comparing Eq. (5.2) with the representation of the Macdonald currents in Eq. 3.6, we deduce the following representation of the EHA, or rather a quotient thereof corresponding to the finite number \( N \) of variables:

\[
  u_{1,n} = \frac{q^{1/2}}{1-q} q^n D_n, \quad u_{-1,n} = \frac{q^{-1/2}}{1-q^{-1}} q^{-n} \tilde{D}_n,
\]

(5.4)

whereas the relation:

\[
  1 + \sum_{n \geq 1} z^{\pm n} \theta_{0,\pm n} = e^{\sum_{k \geq 1} \frac{z^{\pm k}}{k} p_{\pm k} q^{\frac{k}{2}} (1-t^{-1}) (1-t^{-1} q^{-k})}
\]

fixes the values:

\[
  u_{0,\pm k} = \frac{q^{\frac{k}{2}}}{(1-q^k)} p_{\pm k} \quad (k \in \mathbb{Z}_{>0}).
\]

(5.5)

Moreover, from the homomorphism to sDAHA, it is easy to identify the \( u_{k,0} \) from the sDAHA power sum operators, which in the functional representation are: \( P_k := \sum_{i=1}^{N} \rho(Y_i)^k |s_N \) and are related for \( k > 0 \) to the Macdonald operators \( D_{\alpha;0} \) via:

\[
  \rho(\prod_{i=1}^{N} (1-zY_i))|s_N = \sum_{\alpha=0}^{N} (-z)^\alpha D_{\alpha;0} = e^{-\sum_{k \geq 1} P_k \frac{k}{2}},
\]

(5.6)

while for \( k < 0 \) they are expressed in terms of the dual operators \( \tilde{D}_{\alpha;0} \equiv D_{(q^{-1},t^{-1})}^{\alpha;0} \) via:

\[
  \sum_{\alpha=0}^{N} (-z)^\alpha \tilde{D}_{\alpha;0} = e^{-\sum_{k \geq 1} P_{-k} \frac{k}{2}}.
\]

(5.7)

Finally from the action of the involution \( -\epsilon \) which maps \( u_{0,k} \) to \( u_{k,0} \), we have the identification:

\[
  u_{\pm k,0} = \frac{q^{\frac{k}{2}}}{(1-q^k)} P_{\pm k} \quad (k \in \mathbb{Z}_{>0}).
\]

(5.8)

5.3. EHA and relations between generalized Macdonald operators. The functional representation of the EHA provides an alternative way to derive between the generalized Macdonald operators, so that the difference operators \( D_{\alpha,n} \) can be expressed as an explicit polynomial of the \( D_m \)'s.

**Theorem 1.18.** The operator \( D_{\alpha;n} \) is expressible as a homogeneous polynomial of degree \( \alpha \) in the variables \( D_n, D_{n\pm 1} \), with coefficients in \( \mathbb{C}(q,t) \).

\(^3\) These assignments first appeared in [Mik07, Proposition 3.3], with different notations \((q, y, y_i) \mapsto (\theta = t^{1/2}, q, \theta, z_i)\).
Proof. **Step 1:** The coefficient of $z^\alpha$ Eq. (5.6) gives

$$\mathcal{D}_{\alpha,0} = \varphi_\alpha(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N) = \sum_{|n_i|} (-1)^{\alpha + \sum n_i} \prod_i \mathcal{P}_i^{n_i} n_i!,$$

(5.9)

where $\varphi_\alpha$ is a quasi-homogeneous polynomial of total degree $\alpha$, setting $\deg \mathcal{P}_i = i$.

**Step 2:** Use the definition of EHA to compute the representation of $\theta_{n,0}$ as an iterated commutator of $\mathcal{D}_i$ with $i = 0, \pm 1$. Note that

$$u_{n-1,1} = [u_{n-2,1}, u_{1,0}] = \cdots = [[\cdots [u_{1,1}, u_{1,0}], u_{1,0}], \cdots, u_{1,0}]$$

$$= \frac{q^{\frac{z}{2}}}{(1-q)^{n-1}} [[\cdots [\mathcal{D}_1, \mathcal{D}_0], \cdots, \mathcal{D}_0], \cdots, \mathcal{D}_0].$$

Moreover,

$$\theta_{n,0} = \alpha_1 [u_{n-1,1}, u_{1,1}] = \frac{\alpha_1 q^{\frac{1}{2}}}{(1-q)^{n}} [[\cdots [\mathcal{D}_1, \mathcal{D}_0], \cdots, \mathcal{D}_0], \cdots, \mathcal{D}_0] \cdot (n > 1)$$

(5.10)

$$\theta_{1,0} = \alpha_1 u_{1,0} = \frac{\alpha_1 q^{\frac{1}{2}}}{(1-q)} \mathcal{D}_0.$$  

(5.11)

This gives an expression of $\theta_{n,0}$ as a homogeneous polynomial of degree $n$ of the variables $\mathcal{D}_0, \mathcal{D}_{\pm 1}$.

**Step 3:** Use the relation (E3) of Definition 5.1 between $\theta_{n,0}$ and the $u_{k,0}$

$$1 + \sum_{n>0} \theta_{n,0} z^n = e^{\sum_{k>0} \alpha_k u_{k,0} z^k} = e^{\sum_{k>0} q^{k/2} (1-t^{-k})(1-t^{-k}) \mathcal{P}_k \mathcal{T}}.$$  

(5.12)

By solving a triangular system, this gives a way to rewrite each $\mathcal{P}_k$ first as a quasi-homogeneous polynomial $\phi_\alpha(\theta_{1,0}, \theta_{1,1}, \ldots, \theta_{0,0})$, and then by substituting the expressions (5.10–5.11), as a homogeneous polynomial of the variables $\mathcal{D}_0, \mathcal{D}_{\pm 1}$.

Finally, eliminating $\mathcal{P}_k$ between the expressions obtained in Step 1 and Step 3, we are left with an algebraic relation expressing $\mathcal{D}_{\alpha,0}$ as a homogeneous polynomial of degree $\alpha$ in the three variables $(\mathcal{D}_1, \mathcal{D}_0, \mathcal{D}_{-1})$, with coefficients in $\mathbb{C}(q, t)$. This proves the Theorem for $n = 0$. For arbitrary $n$, use the conjugation by $\gamma^{-n}$ and then (2.11) to conclude. The Theorem follows. \(\square\)

We are now ready for the main result of this paper, expressing that the finite number $N$ of variables is characterized by the vanishing of all elementary symmetric functions of $N + 1$ variables, which is equivalent to the vanishing relations (2.12), turned into polynomial relations in the generators via Theorem 1.18, and its analogue obtained by applying the transformation $(q, t) \rightarrow (q^{-1}, t^{-1})$.

**Theorem 5.2.** The algebra $\mathcal{A}_{\mathbb{C}[q, t]}^{(q, t)}$ is isomorphic to the quotient of the EHA by the ideal generated by the relations $\mathcal{D}_{N+1,n} = \mathcal{D}_{N+1,n} = 0, n \in \mathbb{Z}$, expressed as homogeneous polynomials of degree $N + 1$ in the generators $\mathcal{D}_n \propto u_{1,n}$ and $\mathcal{D}_n \propto u_{-1,n}$ respectively, via Theorem 1.18.

**Corollary 5.3.** The kernel of the Schiffman–Vasserot homomorphism from EHA to sDAHA is generated by the polynomial relations of Theorem 5.2.
Let us write the explicit polynomial relation of Theorem 1.18 above for \( \alpha = 1, 2, 3 \), by following the Steps in the proof.

**Step 1:** We express \( D_{\alpha:0} \)'s in terms of \( P_i \)'s via (5.9):

\[
D_{1:0} = D_0 = P_1, \quad D_{2:0} = \frac{P_1^2 - P_2}{2}, \quad D_{3:0} = \frac{P_3^3 - 3P_1P_2 + 2P_3}{6}.
\] (5.13)

**Step 2:** We express \( \theta_{n,0}, n = 1, 2, 3 \) in terms of the \( D_0, D_{\pm 1} \) via (5.11):

\[
\theta_{1,0} = \alpha_1 \frac{q^{1/2}}{1-q} D_0, \quad \theta_{2,0} = \alpha_1 \frac{q}{(1-q)^2} [D_1, D_{-1}],
\]

\[
\theta_{3,0} = \alpha_1 \frac{q^{3/2}}{(1-q)^3} [D_1, D_0, D_{-1}].
\]

**Step 3:** We express the \( P_i \)'s in terms of the \( \theta_{i,0} \)'s via (5.12), which gives:

\[
P_1 = \frac{\theta_{1,0}}{q^{1/2}(1-t^{-1})(1-q^{-1}t)}, \quad P_2 = \frac{2\theta_{2,0} - \theta_{1,0}^2}{q(1-t^{-2})(1-q^{-2}t^2)},
\]

\[
P_3 = \frac{3\theta_{3,0} - 3 \theta_{1,0} \theta_{2,0} + \theta_{1,0}^3}{q^{3/2}(1-t^{-3})(1-q^{-3}t^3)}.
\]

These are reexpressed in terms of \( D_1, D_0, D_{-1} \) by using the expressions of Step 2 as:

\[
P_1 = D_0,
\]

\[
P_2 = \frac{\alpha_1}{(1-q)^2(1-t^{-2})(1-q^{-2}t^2)} \left( 2[D_1, D_{-1}] - \alpha_1 D_0^2 \right),
\]

\[
P_3 = \frac{\alpha_1}{(1-q)^3(1-t^{-3})(1-q^{-3}t^3)} \left( 3[D_1, D_0, D_{-1}] - 3\alpha_1 D_0[D_1, D_{-1}] + \alpha_1^2 D_0^3 \right).
\]

Substituting these into (5.13) gives the polynomial relations \( D_{1:0} = D_0 \) and:

\[
D_{2:0} = \frac{t}{(1+t)(q+t)} \left\{ (1+q)(D_0)^2 - \frac{q}{1-q} [D_1, D_{-1}] \right\},
\]

\[
D_{3:0} = \frac{t^2}{(1+t)(q+t)(1+t+q^2)(q^2+q t+t^2)} \times \left\{ (q(q+t^2) + t(1+q+q^2+q^3))(D_0)^3 
+ \frac{q((1+q)(1+t)(q+t) + t(1-q+q^2))}{1-q} D_0[D_1, D_{-1}] 
+ \frac{q^2}{(1-q)^2} [D_1, D_0, D_{-1}] \right\}.
\]

Finally conjugating \( n \) times w.r.t. \( \gamma^{-1} \) gives:

\[
D_{2:n} = \frac{t}{(1+t)(q+t)} \left\{ (1+q)(D_n)^2 + \frac{q}{1-q} [D_{n-1}, D_{n+1}] \right\},
\] (5.14)
\[ \mathcal{D}_{3,n} = \frac{t^2}{(1 + t)(q + t)(1 + t + t^2)(q^2 + qt + t^2)} \times \left\{ \frac{q((1 + q)(1 + t)(q + t) + t(1 - q + q^2))}{1 - q} \mathcal{D}_n \mathcal{D}_{n+1}, \mathcal{D}_{n-1} \right\} \]

For instance, the \( \mathfrak{gl}_2 \) quotient of EHA for the \( N = 2 \)-variable case corresponds to the relations \( \mathcal{D}_{3,n} = 0 \) and \( \tilde{\mathcal{D}}_{3,n} = 0 \), with \( \mathcal{D}_{3,n} \) as in (5.15).

### 5.4. The functional representation of EHA and the operators of [BGLX16].

One can continue the comparison with the work of [BGLX16] started in Remarks 2.12 and 4.8, which makes a connection to the positive quadrant of the Elliptic Hall Algebra, with generators \( Q_{a,b} := u_{a,b} \) with \( a, b \geq 0 \), by setting

\[ Q_{1,k} = D_k \ (k \geq 0), \quad \text{and} \quad Q_{0,1} = -e_1 = -p_1, \]

where the \( D_k \)'s are defined via their generating current in (4.9). The identification is extended via the \( SL_2(\mathbb{Z}) \)-action on the EHA. By Theorem 4.7,

\[ \lim_{N \to \infty} t^{\frac{1-N}{2}} \mathcal{D}_n = \frac{1}{1 - t^{-1}} (\Sigma^{-1} D_n \Sigma) \big|_{t \to t^{-1}}, \]

\[ \lim_{N \to \infty} -p_1(x_1, x_2, \ldots, x_N) = -p_1(x_1, x_2, \ldots). \]

Comparing with the functional representation of EHA in the limit of infinite number of variables of \( x_1, x_2, \ldots, \)

\[ \lim_{N \to \infty} t^{\frac{1-N}{2}} u_{1,k} = \frac{q^{k+1}}{(1 - q)(1 - t^{-1})} (\Sigma^{-1} Q_{1,k} \Sigma) \big|_{t \to t^{-1}} \]

and

\[ \lim_{N \to \infty} u_{0,1} = \frac{q^1}{(1 - q)(1 - t^{-1})} (\Sigma^{-1} (-p_1) \Sigma) \big|_{t \to t^{-1}}. \]

Moreover, by using (4.12), and \( \nabla^{(N)} u_{a,b} \nabla^{(N)-1} = u_{a+b,b} \) we have:

\[ \lim_{N \to \infty} t^{\frac{(1-N)(k+1)}{2}} \nabla^{(N)} u_{1,k} \nabla^{(N)-1} = \lim_{N \to \infty} t^{\frac{(1-N)(k+1)}{2}} u_{k+1,k} \]

\[ = \lim_{N \to \infty} t^{\frac{(1-N)(k+1)}{2}} (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^d (\Sigma^{-1} \nabla \Sigma) \big|_{t \to t^{-1}} u_{1,k} \]

\[ \times (\Sigma^{-1} \nabla^{-1} \Sigma) \big|_{t \to t^{-1}} (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^{-d} \]

\[ = \frac{q^{k+1}}{(1 - q)(1 - t^{-1})} \lim_{N \to \infty} t^{\frac{(1-N)(k)}{2}} (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^d \]
\[
(\Sigma^{-1} \nabla Q_{1,k} \nabla^{-1} \Sigma)_{t \to t^{-1}} (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^{-d}.
\]

Note that the action of \(Q_{1,k}\) on \(\tilde{H}_\lambda\) is a linear combination of terms in which \(k\) boxes are added to \(\lambda\), and similarly for \(u_{1,k}\) acting on \(P_\lambda\). We deduce that the conjugation by \((t^{N-1} q^{\frac{1}{2}})^d\) amounts to a factor \((t^{N-1} q^{\frac{1}{2}})^k\), and using \(\nabla Q_a \nabla^{-1} = Q_a + b\) \(b\) (from [BGLX16]),

\[
\lim_{N \to \infty} t^{\frac{(1-N)(k+1)}{2}} u_{k+1,k} = \frac{q^{\frac{2k+1}{2}}}{(1-q)(1-t^{-1})} (\Sigma^{-1} Q_{k+1,k} \Sigma)_{t \to t^{-1}}.
\]

Repeating the inductive proof of [BGLX16], we arrive in general at:

\[
\lim_{N \to \infty} t^{\frac{(1-N)a}{2}} u_{a,b} = \frac{q^{a+b}}{(1-q)(1-t^{-1})} (\Sigma^{-1} Q_{a,b} \Sigma)_{t \to t^{-1}}
\]

for any coprime \((a, b)\).

6. Constant Term Identities and Macdonald (Multi-)currents

6.1. A constant term identity for generalized Macdonald operators. The relations between the generating functions of the previous sections can be reformulated as constant term identities. This gives an expression for the difference operator \(D^{(q,t)}_{\alpha_1, \alpha_2, \ldots, \alpha_\alpha}\) of Eq. (1.12) in terms of the fundamental generators \(D^{(q,t)}_{1;n}\). In this section, we drop the superscript \((q,t)\) and write \(D_n = D^{(q,t)}_{1;n}\) for the fundamental generators.

6.1.1. Generating currents  Recall the generating functions \(d(z) = \sum_{n \in \mathbb{Z}} z^n D_n\) of Eq. (1.14). By the homogeneity of \(g(u, v)\) (3.1), Theorem 3.6 implies

\[
g(u, v) d(u) \partial(v) + g(v, u) \partial(u) d(v) = 0.
\]

In terms of components, this is equivalent to

\[
\mu_{a,b} := qt D_{a-3} D_{b} - (t^2 + q^2 t + q) D_{a-2} D_{b-1} + (qt^2 + t + q^2) D_{a-1} D_{b-2} - qt D_{a} D_{b-3} = - \mu_{b,a}
\]

for all \(a, b \in \mathbb{Z}\).

6.1.2. Constant term identities

**Definition 6.1.** The constant term of any rational symmetric function \(f(u_1, \ldots, u_\alpha) \in \mathcal{F}_\alpha\) is

\[
CT_{u_1, \ldots, u_\alpha} (f(u_1, \ldots, u_\alpha)) := \prod_{i=1}^{\alpha} \oint \frac{du_i}{2i\pi u_i} f(u_1, \ldots, u_\alpha),
\]

where the contour integral picks up the residues at \(u_i = 0\).

In particular, we have \(CT_v (f(v) \delta(u/v)) = f(u)\). We now prove the following remarkable result:
Theorem 1.16. For any symmetric rational function \( P(x_1, \ldots, x_\alpha) \in \mathcal{F}_\alpha, 1 \leq \alpha \leq N \),

\[
\mathcal{B}_\alpha(P) = \mathcal{A}_\alpha(P)
\]  

(6.4)

where \( \mathcal{A}_\alpha(P) \) is defined in Eq. 1.13 and \( \mathcal{B}_\alpha(P) \) 1.15.

**Proof.** Let us compute:

\[
\prod_{i < j} \frac{u_i - u_j}{u_i - tu_j} \frac{u_i - qu_j}{u_i - qt^{-1}u_j} \prod_{i=1}^\alpha \vartheta(u_i) = \prod_{i < j} \frac{u_i - u_j}{u_i - tu_j} \frac{u_i - qu_j}{u_i - qt^{-1}u_j}
\]

\[
\times \sum_{i_1 \neq i_2 \neq \ldots \neq i_\alpha} \prod_{i=1}^\alpha \delta(u_j x_{i_j}) \prod_{k < \ell} \frac{\theta x_{i_k} - \theta^{-1} x_{i_\ell}}{x_{i_k} - x_{i_\ell}} \frac{\theta x_{i_\ell} - q \theta^{-1} x_{i_k}}{x_{i_\ell} - qx_{i_k}}
\]

\[
\times \prod_{j=1}^\alpha \prod_{i_1 \neq i_2 \neq \ldots \neq i_\alpha} \frac{\theta x_{i_j} - \theta^{-1} x_i}{x_{i_j} - x_i} \Gamma_{i_1} \cdots \Gamma_{i_\alpha}
\]

\[
= \sum_{i_1 \neq i_2 \neq \ldots \neq i_\alpha} \prod_{j=1}^\alpha \delta(u_j x_{i_j}) \prod_{i_1 \neq i_2 \neq \ldots \neq i_\alpha} \frac{\theta x_{i_k} - \theta^{-1} x_i}{x_{i_k} - x_i} \Gamma_{i_1} \cdots \Gamma_{i_\alpha}
\]

\[
= \alpha! \sum_{i_1 < i_2 < \ldots < i_\alpha} \prod_{j=1}^\alpha \delta(u_j x_{i_j}) \prod_{i_1 \neq i_2 \neq \ldots \neq i_\alpha} \frac{\theta x_{i_k} - \theta^{-1} x_i}{x_{i_k} - x_i} \Gamma_{i_1} \cdots \Gamma_{i_\alpha}
\]

\[
= \alpha! \frac{1}{\alpha! (N - \alpha)!} \text{Sym} \left( \prod_{k=1}^\alpha \delta(u_k x_k) \prod_{1 \leq i < j \leq N} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \Gamma_{i_1} \cdots \Gamma_{i_\alpha} \right).
\]

Here, we used the expression

\[
\vartheta(u) = \sum_{i=1}^N \delta(u x_i) \prod_{j=1}^N \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \Gamma_i.
\]

Noting that terms with any two identical \( i_k = i_\ell, k < \ell \), in the sum must vanish. This is due to the prefactor \((u_k - qu_\ell)\) which when multiplying the delta function \(\delta(u_k x_{i_k})\delta(u_\ell qx_{i_\ell})\) yields a zero result. Comparing with Eq. (1.10), the constant term (6.4) follows immediately. \( \square \)

As a by-product of the proof of Theorem 1.16 above, we note that if \( \alpha > N \), then there are no terms in which all \( i_k \) are distinct (as there are at most \( N \) of them), thus causing the result to vanish. We deduce the following:

**Corollary 6.2.** For any symmetric rational function \( P(x_1, \ldots, x_\alpha) \in \mathcal{F}_\alpha, \alpha > N \), we have:

\[
\mathcal{A}_\alpha(P) = \mathcal{B}_\alpha(P) = 0 \quad \forall \alpha > N.
\]  

(6.5)
This implies in particular that $D_{N+1; n} = 0$, in agreement with (2.12). Recalling the definition (1.12), Theorem 1.16 has also the following immediate application to $P = s_{a_1, \ldots, a_\alpha}(x_1, \ldots, x_\alpha)$:

**Corollary 6.3.** We have:

\[
D_{a_1, \ldots, a_\alpha} = \frac{1}{\alpha!} \text{CT}_u \left( s_{a_1, \ldots, a_\alpha}(u^{-1}) \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - u_j)(u_i - qu_j)}{(u_i - tu_j)(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \vartheta(u_i) \right). \tag{6.6}
\]

In particular, this implies:

\[
D_{\alpha; n} = \frac{1}{\alpha!} \text{CT}_u \left( (u_1 u_2 \cdots u_\alpha)^{-n} \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - u_j)(u_i - qu_j)}{(u_i - tu_j)(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \vartheta(u_i) \right), \tag{6.7}
\]

or equivalently, in terms of the currents $e_\alpha(z)$ of (3.6):

**Corollary 6.4.** We have the following constant term identities:

\[
e_\alpha(z) = \frac{1}{\alpha!} \text{CT}_u \left( \delta(u_1 u_2 \cdots u_\alpha/z) \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - u_j)(u_i - qu_j)}{(u_i - tu_j)(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \epsilon_1(u_i) \right). \tag{6.8}
\]

**Proof.** The equation results from combining the expression (3.6) for the generating function $e_\alpha(z)$ and the constant term identity (6.7), and the change of variables $u_i \mapsto q^{1/2} u_i$ for all $i$ and $z \mapsto q^{\alpha/2} z$. \qed

### 6.1.3. Generating multi-currents

The result of Corollary 6.3 may be rephrased in terms of generating (multi-)currents as follows. We consider the generating multi-current with argument $v = (v_1, v_2, \ldots, v_{\alpha})$:

\[
D_\alpha(v) := \sum_{a_1, \ldots, a_\alpha \in \mathbb{Z}} D_{a_1, \ldots, a_\alpha} v_1^{a_1} v_2^{a_2} \cdots v_{\alpha}^{a_{\alpha}}
\]

\[
= \frac{1}{\prod_{j=1}^{\alpha} v_j^{\alpha-j} A_{\alpha}} \left( \frac{\det \left( (\delta(x_i, v_j))_{1 \leq i, j \leq \alpha} \right)}{\prod_{1 \leq i < j \leq \alpha} (x_i - x_j)} \right). \tag{6.9}
\]

**Theorem 6.5.** The generating multi-current for the generalized Macdonald operators (1.12) is

\[
D_\alpha(v) = \prod_{1 \leq i < j \leq \alpha} \frac{(v_i - qv_j)}{(t - v_i v_j^{-1})(v_i - qt^{-1}v_j)} \prod_{i=1}^{\alpha} \vartheta(v_i). \tag{6.10}
\]
Proof. Using the identity (6.6), and the expression (1.11) for the generalized Schur function,

\[
\mathcal{D}_\alpha(v) = \frac{1}{\alpha!} \text{CT}_u \left( \sum_{a_1, \ldots, a_\alpha \in \mathbb{Z}} \det \left( u_i^{-a_j - \alpha + j} \right) v_1^{a_1} v_2^{a_2} \cdots v_\alpha^{a_\alpha} \right) \\
\times \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - qu_j)}{(tu_i^{-1} - u_j^{-1})(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \mathcal{D}(u_i) \\
= \frac{1}{\alpha!} \text{CT}_u \left( \det \left( \delta(v_j/u_i) v_j^{j-\alpha} \right) \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - qu_j)}{(tu_i^{-1} - u_j^{-1})(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \mathcal{D}(u_i) \right) \\
= \frac{1}{\alpha!} \text{CT}_u \left( \det \left( \delta(v_j/u_i) \right) \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - qu_j)}{(t - u_i u_j^{-1})(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \mathcal{D}(u_i) \right) \\
= \prod_{1 \leq i < j \leq \alpha} \frac{(v_i - qv_j)}{(t - v_i v_j^{-1})(v_i - vt^{-1}v_j)} \prod_{i=1}^{\alpha} \mathcal{D}(v_i),
\]

where in the last step we have used the skew-symmetry of both the determinant and the quantity next to it, to see that each of the \(\alpha!\) terms in the expansion of the determinant contributes the same as the diagonal term \(\prod \delta(v_i/u_i) u_i^{i-\alpha}\). \(\square\)

Corollary 6.6. The following is an alternative expression for the generalized Macdonald operators (1.12):

\[
\mathcal{D}_{a_1, \ldots, a_\alpha} = \text{CT}_u \left( \prod_{i=1}^{\alpha} u_i^{-a_i} \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - qu_j)}{(t - u_i u_j^{-1})(u_i - qt^{-1}u_j)} \prod_{i=1}^{\alpha} \mathcal{D}(u_i) \right). \tag{6.11}
\]

Proof. The constant term (6.11) picks up the coefficient of \(u_1^{a_1} u_2^{a_2} \cdots u_\alpha^{a_\alpha}\) in \(\mathcal{D}_\alpha(u)\). \(\square\)

We also have the corresponding current version for \(\mathcal{E}_\alpha\), when \(a_1 = a_2 = \cdots = a_\alpha = n\), and for \(\mathcal{F}_\alpha\) by the transformation \((q, t) \rightarrow (q^{-1}, t^{-1})\).

Corollary 6.7. The generating function \(\mathcal{E}_\alpha\), \(\mathcal{F}_\alpha\) is expressed in terms of the fundamental currents \(\mathcal{E}_1\) and \(\mathcal{F}_1\) respectively as follows:

\[
\mathcal{E}_\alpha(z) = \text{CT}_u \left( \det \left( \delta(u_1 u_2 \cdots u_\alpha/z) \right) \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - qu_j)}{(t - u_i u_j^{-1})(u_i - q/tu_j)} \prod_{i=1}^{\alpha} \mathcal{E}_1(u_i) \right), \tag{6.12}
\]

\[
\mathcal{F}_\alpha(z) = \text{CT}_u \left( \det \left( \delta(u_1 u_2 \cdots u_\alpha/z) \right) \prod_{1 \leq i < j \leq \alpha} \frac{(u_j - qu_i)}{(t - u_j u_i^{-1})(u_j - q/tu_i)} \prod_{i=1}^{\alpha} \mathcal{F}_1(u_i) \right). \tag{6.13}
\]

This allows to rephrase our main result of Theorem 5.2 as follows in terms of quantum toroida algebra.
Theorem 6.8. The algebra $\mathcal{A}_{\mathfrak{g}l_1}^{(q, t)}$ is isomorphic to the quotient of the level $(0, 0)$ quantum toroidal algebra of $\widehat{\mathfrak{g}}_1$ of Definition 3.2 by the relations:

\[
\begin{align*}
\text{CT}_u \left( \delta(u_1 u_2 \cdots u_{N+1}/z) \prod_{1 \leq i < j \leq N+1} \frac{(u_i - q u_j)}{(t - u_i u_j^{-1})(u_i - q/u_j)} \prod_{i=1}^{N+1} \varepsilon(u_i) \right) &= 0, \\
\text{CT}_u \left( \delta(u_1 u_2 \cdots u_{N+1}/z) \prod_{1 \leq i < j \leq N+1} \frac{(u_j - q u_i)}{(t - u_j u_i^{-1})(u_j - q/u_i)} \prod_{i=1}^{N+1} f(u_i) \right) &= 0.
\end{align*}
\]

6.2. Main conjecture: polynomiality of the $(q, t)$-determinant. In this section, we state the following:

Conjecture 1.17. The generalized Macdonald operators $D_{a_1}, \ldots, a_q$ may be expressed as homogeneous polynomials of degree $a_1 + \cdots + a_q$ in finitely many $D_{p_i}$'s. These polynomials are $t$-deformations of the quantum determinant expression (8.10) of Sect. 8.1.4.

Note that if they exist, such polynomials are not necessarily unique, as they can be modified by use of the exchange relations (6.2).

We already derived evidence for this conjecture in Sect. 5.3, via Theorem 1.18 which proves the conjecture for arbitrary $a_1 = a_2 = \cdots = a_\alpha = n \in \mathbb{Z}$.

We first give the proof of the conjecture in the case $\alpha = 2$ for arbitrary $D_{a,b}, a, b \in \mathbb{Z}$, by deriving an explicit polynomial expression (see Theorem 6.9), and the sketch of the proof in the case $\alpha = 3$ (Theorem 6.11).

Theorem 6.9. For all $a, b \in \mathbb{Z}$, the operator $D_{a,b}$ can be expressed as an explicit quadratic polynomial of the $D_{p_i}$'s, with coefficients in $\mathbb{C}(q, t)$. More precisely, we have:

\[
D_{a,b} = \frac{q(q + t^2)v_{a,b} - t(1 + q)v_{a+1,b-1} + (q + t^2)v_{b-1,a+1} - qt(1 + q)v_{b-2,a+2}}{(q - 1)(q^2 - t^2)(1 - t^2)},
\]

where $v_{a,b}$ stands for the following quantum determinant:

\[
v_{a,b} = \left| \begin{array}{c} D_a \\ D_{a+1} \\ D_b \\ D_{b-1} \end{array} \right|_q := D_a D_b - q D_{a+1} D_{b-1}.
\]

Proof. The current relation of Theorem 6.5 is

\[
\mathcal{D}_2(v_1, v_2) = \frac{(v_1 - q v_2)}{(t - v_1 v_2^{-1})(v_1 - q t^{-1} v_2)} \mathcal{D}(v_1) \mathcal{D}(v_2).
\]

The exchange relation (6.1) is equivalent to

\[
\frac{(v_1 - q v_2)}{(t v_2 - v_1)(v_1 - q t^{-1} v_2)} \mathcal{D}(v_1) \mathcal{D}(v_2) + \frac{(v_2 - q v_1)}{(t v_1 - v_2)(v_2 - q t^{-1} v_1)} \mathcal{D}(v_2) \mathcal{D}(v_1) = 0.
\]

As a consequence, the double current $\mathcal{N}_2(v_1, v_2) := \mathcal{D}_2(v_1, v_2)/v_2$ is skew-symmetric, and we may rewrite (6.16) as:

\[
\delta_{1,2} \mathcal{N}_2(v_1, v_2) = \frac{v_1 - q v_2}{v_1 v_2} \mathcal{D}(v_1) \mathcal{D}(v_2) =: \mu_2(v_1, v_2),
\]

(6.17)
with the notation

\[ \delta_{i,j} := \left( t - \frac{v_i}{v_j} \right) \left( 1 - q t^{-1} \frac{v_j}{v_i} \right). \]

Using the skew-symmetry of \( \mathcal{R}_2 \),

\[ \delta_{2,1} \mathcal{R}_2(v_1, v_2) = -\mu_2(v_2, v_1). \] (6.18)

Next, we eliminate the prefactors by using the “inversion” relation:

\[ \eta_{1,2} \delta_{1,2} - \theta_{1,2} \delta_{2,1} = 1, \quad \eta_{i,j} := t \frac{q(q + t^2) - t(1 + q) v_j}{(q - 1)(1 - t^2)(q^2 - t^2)} \]

\[ \theta_{i,j} := t \frac{(q + t^2) - t(1 + q) v_j}{(q - 1)(1 - t^2)(q^2 - t^2)}, \] (6.19)

derived by decomposing \( 1/(\delta_{1,2} \delta_{2,1}) \) into simple fractions. Therefore,

\[ \mathcal{D}_2(v_1, v_2) = v_2 \mathcal{R}_2(v_1, v_2) = v_2(\eta_{1,2} \mu_2(v_1, v_2) + \theta_{1,2} \mu_2(v_2, v_1)). \]

Defining \( v_2(v_1, v_2) := v_2 \mu_2(v_1, v_2) = \left( 1 - q \frac{v_2}{v_1} \right) \mathcal{D}(v_1) \mathcal{D}(v_2), \)

\[ \mathcal{D}_2(v_1, v_2) = \eta_{1,2} v_2(v_1, v_2) + \frac{v_2}{v_1} \theta_{1,2} v_2(v_2, v_1). \] (6.20)

Introducing the mode expansion: \( v_2(v_1, v_2) = \sum_{a,b \in \mathbb{Z}} v_{a,b} v^a q^b \), with \( v_{a,b} \) as in (6.15), the formula (6.14) follows from the mode expansion of (6.20). \( \square \)

Remark 6.10. Note that the expression (6.14) expresses \( \mathcal{D}_{a,b} \) in terms of more variables that just \( \mathcal{D}_a, \mathcal{D}_{a+1}, \mathcal{D}_{b-1}, \mathcal{D}_b \) which appear in the usual quantum determinant \( v_{a,b} \). In the limit when \( t \to \infty \), defining \( \mathcal{D}_{a,b}^{(q)} := \lim_{t \to \infty} t^{-(N-2)} \mathcal{D}_{a,b}^{(q,t)} \), \( \mathcal{D}_a^{(q)} = \lim_{t \to \infty} t^{-(N-1)} \mathcal{D}_{a}^{(q,t)} \), and \( n_{a,b} = \lim_{t \to \infty} t^{-(N-1)} v_{a,b} \), the quantum determinant is recovered,

\[ \mathcal{D}_{a,b}^{(q)} = \frac{q n_{a,b} + n_{b-1,a+1}}{q - 1} = n_{a,b} = \mathcal{D}_{a}^{(q)} \mathcal{D}_{b}^{(q)} - q \mathcal{D}_{a+1}^{(q)} \mathcal{D}_{b-1}^{(q)} = \begin{vmatrix} \mathcal{D}_a^{(q)} & \mathcal{D}_{a+1}^{(q)} \\ \mathcal{D}_b^{(q)} & \mathcal{D}_{b-1}^{(q)} \end{vmatrix}_{q}, \]

by use of the \( t \to \infty \) exchange relation: \( n_{b-1,a+1} + n_{a,b} = 0 \).

For finite \( t \) the expression (6.14) is not unique: it is unique only up to the exchange relation (6.2) for the \( \mathcal{D}_n \)'s. Here is a simple example. Revisiting the proof of the Theorem, we note that there is another inverting pair \( (\eta_{1,2}', \theta_{1,2}') = (-\theta_{2,1}, -\eta_{2,1}) \) obtained by acting with the transposition (12) on the inversion relation (6.19), namely we also have:

\[ \eta_{1,2}' \delta_{1,2} - \theta_{1,2}' \delta_{2,1} = 1. \]

This choice leads to alternative expressions:

\[ \mathcal{D}_2(v_1, v_2) = v_2 \mathcal{R}_2(v_1, v_2) = -v_2 \left( \theta_{2,1} \mu_2(v_1, v_2) + \eta_{2,1} \mu_2(v_2, v_1) \right), \]

\[ = -\theta_{2,1} v_2(v_1, v_2) - \frac{v_2}{v_1} \eta_{2,1} v_2(v_2, v_1) \]
\[
D_{a,b} = t \frac{qt(1 + q) v_{a-1,b+1} - (q + t^2) v_{a,b} + t(1 + q) v_{b,a} - q(q + t^2) v_{b-1,a+1}}{(q - 1)(q^2 - t^2)(1 - t^2)}.
\]

(6.21)

The difference between the two expressions (6.14) and (6.21) is proportional to:

\[
(q + t^2)(v_{a,b} + v_{b-1,a+1}) - t(v_{a+1,b-1} + v_{b,a}) - qt(v_{a-1,b+1} + v_{b-2,a+2}).
\]

(6.22)

Rewriting the exchange relation (6.2) as:

\[
\varphi_{a,b} := qt v_{a-3,b} - (q + t^2) v_{a-2,b-1} + t v_{a-1,b-2} = -\varphi_{b,a}.
\]

(6.23)

the quantity in Eq. (6.22) vanishes, as it reads

\[-\varphi_{a+2,b+1} - \varphi_{b+1,a+2} = 0.
\]

\[
\text{Theorem 6.11. The Conjecture 1.17 holds for } \alpha = 3.
\]

Proof. Sketch of the proof. Use simple fraction decomposition of the quantity $1/\prod_{1 \leq i < j \leq \alpha} \delta_{i,j}$ to obtain a relation of the form:

\[
\sum_{\sigma \in S_3} \text{sgn}(\sigma) A_\sigma(v_1, v_2, v_3) \delta_{\sigma(1),\sigma(2)} \delta_{\sigma(1),\sigma(3)} \delta_{\sigma(2),\sigma(3)} = 1
\]

with explicit Laurent polynomials $A_\sigma(v_1, \ldots, v_\alpha)$ (too cumbersome to be presented here). This allows to express the skew-symmetric current $\mathcal{N}_\alpha = D_\alpha/(v_2 v_3^2 \cdots v_{\alpha-1}^\alpha)$ as a sum over the symmetric group $S_3$ of Laurent polynomials of the $v$'s times permuted products of the fundamental currents $\vartheta(v_i)$'s. The polynomiality property of the coefficients follows.

More generally, one could try to generalize the above argument. Defining the skew-symmetric function $\mathcal{N}_\alpha = D_\alpha/(v_2 v_3^2 \cdots v_{\alpha-1}^\alpha)$, we wish to invert the relation

\[
\left( \prod_{1 \leq i < j \leq \alpha} \delta_{i,j} \right) \mathcal{N}_\alpha(v_1, \ldots, v_\alpha) = \prod_{1 \leq i < j \leq \alpha} \frac{v_i - q v_j}{v_i v_j} m(v_1) \cdots m(v_\alpha) =: \mu_\alpha(v_1, \ldots, v_\alpha).
\]

Acting with the permutation group of the $v$'s, we have accordingly for all $\sigma \in S_\alpha$:

\[
\prod_{1 \leq i < j \leq \alpha} \delta_{\sigma(i),\sigma(j)} \mathcal{N}_\alpha(v_1, \ldots, v_\alpha) = \text{sgn}(\sigma) \mu_\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(\alpha)}).
\]

Inverting the system could be done by looking for Laurent polynomials $A_\sigma(v_1, \ldots, v_\alpha)$ such that

\[
\sum_{\sigma \in S_\alpha} \text{sgn}(\sigma) A_\sigma(v_1, \ldots, v_\alpha) \prod_{1 \leq i < j \leq \alpha} \delta_{\sigma(i),\sigma(j)} = 1.
\]

If such $A_\sigma$'s existed, then we could write

\[
D_\alpha = v_2 v_3^2 \cdots v_{\alpha-1}^\alpha \sum_{\sigma \in S_\alpha} A_\sigma(v_1, \ldots, v_\alpha) \mu_\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(\alpha)})
\]

and polynomiality would follow.
6.3. Plethystic formulation. Using the plethystic formulas of Sect. 4, we derive a plethystic formula for the action of the higher currents \( \varepsilon_\alpha \) on symmetric polynomials.

**Theorem 6.12.** The current \( \varepsilon_\alpha(z) \) acts on functions \( F[X] \) as follows:

\[
\varepsilon_\alpha(z) \cdot F[X] = \left( \frac{q^{1/2}t^{-1/2}}{(1-q)(1-t^{1})} \right)^\alpha C_T \left( \prod_{1 \leq i < j \leq \alpha} \frac{(u_i - u_j)}{(u_i - tu_j)(u_i - t^{-1}u_j)} \right) \times \prod_{i=1}^\alpha \left\{ \frac{C^+(q^{1/2}t^{-1}u_i)}{C^+(q^{1/2}u_i)} - t^{-\frac{\alpha}{2}} \frac{C^-(q^{-1/2}tu_i^{-1})}{C^-(q^{-1/2}u_i^{-1})} \right\} \delta(u_1u_2\cdots u_\alpha/z) \cdot F \left[ X + (q^{1/2} - q^{-1/2}) \sum_{i=1}^\alpha \frac{1}{u_i} \right],
\]

with \( C^\pm(z) \) as in (4.2–4.3).

**Proof.** Using the plethystic formula for \( \varepsilon_1(z) \) of Theorem 4.3 and Eq. (6.12), one can compute the action of the plethysm \( [X] \mapsto [X + \frac{q^{1/2} - q^{-1/2}}{t}] \) on \( \varepsilon_1(w) \), it acts by

\[
\frac{C^+(q^{1/2}t^{-1}w)}{C^+(q^{1/2}w)} \mapsto \frac{(z - qt^{-1}w)(z-w)}{(z-t^{-1}w)(z-qw)} \frac{C^+(q^{1/2}t^{-1}w)}{C^+(q^{1/2}w)}
\]

\[
\frac{C^-(q^{-1/2}tw^{-1})}{C^-(q^{-1/2}w^{-1})} \mapsto \frac{(w - q^{-1}tw)(w-z)}{(w-tw)(w-q^{-1}z)} \frac{C^-(q^{-1/2}tw^{-1})}{C^-(q^{-1/2}w^{-1})}.
\]

The theorem follows by repeated applications until all \( u_i \)'s are exhausted. \( \square \)

Similarly, there is a plethystic action of the multi-current \( D_\alpha(v) \) of (6.9).

**Theorem 6.13.** The multi-current \( D_\alpha(v) \) for the generalized Macdonald operators acts on functions \( F[X] \) as follows.

\[
D_\alpha(v) \cdot F[X] = \left( \frac{t^{1/2}}{t-1} \right)^\alpha \prod_{1 \leq i < j \leq \alpha} \frac{(v_i - v_j)}{(t - v_i v_j^{-1})(t^{-1}v_i)} \times \prod_{i=1}^\alpha \left\{ \frac{C^+(t^{-1}v_i)}{C^+(v_i)} - t^{-\frac{\alpha}{2}} \frac{C^-(tv_i^{-1})}{C^-(v_i^{-1})} \right\} \cdot F \left[ X + (q - 1) \sum_{i=1}^\alpha \frac{1}{v_i} \right].
\]

**Proof.** Starting from (6.9), we substitute \( \delta(v_i) = q^{-1/2}(1 - q)\varepsilon_1(q^{-1/2}v_i) \), and then use the proof of Theorem 6.12 to rearrange the prefactors of the plethysms. \( \square \)

This immediately implies the following corollary for the limit of infinite alphabet \( N \to \infty \):

**Corollary 6.14.** Defining the limiting multi-current \( \hat{D}_\alpha(v) := \lim_{N \to \infty} t^{-\frac{\alpha N}{2}} D_\alpha(v) \), we have the following plethystic formula:

\[
\hat{D}_\alpha(v) F[X] = \left( \frac{t^{1/2}}{t-1} \right)^\alpha \prod_{1 \leq i < j \leq \alpha} \frac{(v_i - v_j)}{(t - v_i v_j^{-1})(t v_i - v_j)} \times \prod_{i=1}^\alpha \frac{\hat{C}^+(t^{-1}v_i)}{\hat{C}^+(v_i)} \cdot F \left[ X + (q - 1) \sum_{i=1}^\alpha \frac{1}{v_i} \right],
\]

with \( \hat{C}^+ \) as in (4.6).
7. Identities via the Shuffle Product

One can rephrase relations satisfied by difference operators in terms of the shuffle algebra [FO98, Neg14]. The shuffle product is an associative, non-commutative product \( \star : \mathcal{F}_\alpha \times \mathcal{F}_\beta \to \mathcal{F}_{\alpha+\beta} \). It can be used to give elegant proofs of relations in the algebra which involve only products of \( e(z) \) generators (or only \( f(z) \)).

For instance, we give alternative proofs of the exchange relation (6.1), the Serre relations of Lemma 3.9, and provide an expression for the multi-current Theorem 6.5.

The main observation which relates the shuffle product to the algebra \( \mathcal{A}(q,t)_{gl_N} \) is that the product of two difference operators \( \mathcal{A}_\alpha(P) \) and \( \mathcal{A}_\beta(P') \) can be written as a difference operator \( \mathcal{A}_{\alpha+\beta}(P \star P') \).

7.1. Shuffle product: definition and properties. Let

\[
\zeta(x) := \frac{1-tx}{1-x} \frac{1-qt^{-1}x}{1-qx}.
\]

**Definition 7.1.** The shuffle product \( P \star P' \in \mathcal{F}_{\alpha+\beta} \) of \( P \in \mathcal{F}_\alpha \) and \( P' \in \mathcal{F}_\beta \) is defined as the symmetrized expression:

\[
P \star P'(x_1, \ldots, x_{\alpha+\beta}) := \frac{1}{\alpha! \beta!} \text{Sym} \left( P(x_1, \ldots, x_\alpha) P'(x_{\alpha+1}, \ldots, x_{\alpha+\beta}) \prod_{1 \leq i \leq \alpha < j \leq \alpha+\beta} \zeta(u_j/u_i) \right)
\]

(7.1)

where the symmetrization \( \text{Sym} \) is over the group \( S_{\alpha+\beta} \) acting on the set \( \{x_1, x_2, \ldots, x_{\alpha+\beta}\} \).

The following theorem shows that definition of the shuffle product is compatible with the maps \( \mathcal{A}_\alpha \) and \( \mathcal{B}_\alpha \) of Equations (1.10) and (1.15).

**Theorem 7.2.** For any rational functions \( P \in \mathcal{F}_\alpha \) and \( P' \in \mathcal{F}_\beta \),

\[
\mathcal{A}_\alpha(P) \mathcal{A}_\beta(P') = \mathcal{A}_{\alpha+\beta}(P \star P'), \quad \mathcal{B}_\alpha(P) \mathcal{B}_\beta(P') = \mathcal{B}_{\alpha+\beta}(P \star P'),
\]

(7.2)

with \( \mathcal{A}_\alpha, \mathcal{B}_\alpha \) as in Definitions 1.13 and 1.15, and \( P \star P' \in \mathcal{F}_{\alpha+\beta} \) defined by (7.1).

**Proof.** Theorem 1.16 shows that the one equality implies the other. Let \( u = (u_1, u_2, \ldots, u_{\alpha+\beta}) \). Then

\[
\mathcal{B}_\alpha(P) \mathcal{B}_\beta(P')
\]

\[
= \text{CT}_u \left( \frac{P(u_1^{-1}, \ldots, u_\alpha^{-1}) P'(u_{\alpha+1}^{-1}, \ldots, u_{\alpha+\beta}^{-1})}{\alpha! \beta!} \right.
\]

\[
\times \prod_{i < j \in [1, \alpha] \text{ or } [\alpha+1, \alpha+\beta]} \zeta(u_j/u_i)^{-1} \prod_{k=1}^{\alpha+\beta} \delta(u_k)
\]

\]

(7.2)
\[= CT_u \left( \frac{P(u^{-1}_1, \ldots, u^{-1}_\alpha)}{\alpha! \beta!} \right) \]
\[= \frac{1}{(\alpha + \beta)!} CT_u \left( P \ast P'(u^{-1}_1, \ldots, u^{-1}_{\alpha+\beta}) \prod_{1 \leq i < j \leq \alpha+\beta} \zeta(u_j/u_i) \prod_{k=1}^{\alpha+\beta} \mathcal{D}(u_k) \right) \]
where we have used the symmetry of the last factor in the second line, and the fact that the constant term is preserved under symmetrization, that is, \(CT_{u_1, \ldots, u_m}(\text{Sym}(f(u_1, \ldots, u_m))) = m! CT_{u_1, \ldots, u_m}(f(u_1, \ldots, u_m))\). The Theorem follows. \(\Box\)

The following are a few applications of Theorem 7.2.

7.2. Application: Macdonald current relations. The generating function \(\mathcal{D}(v)\) of Eq. (1.14) can be written as
\[\mathcal{D}(v) = A_1(\delta(vx_1)).\]

Theorem 7.2 and the associativity of the shuffle product can be used iteratively to write
\[\mathcal{D}(v_1) \mathcal{D}(v_2) \cdots \mathcal{D}(v_\alpha) = A_\alpha(\delta(v_1x_1) \ast \delta(v_2x_1) \ast \cdots \ast \delta(v_\alpha x_1)).\]

We list a few instances of this relation below respectively for \(\alpha = 2, 3\) and arbitrary \(\alpha\).

The exchange relation Eq. (6.1) is equivalent to a shuffle product identity:

Lemma 7.3. The following relation holds in the shuffle algebra.
\[g(u, v) \delta(ux_1) \ast \delta(vx_1) + g(v, u) \delta(vx_1) \ast \delta(ux_1) = 0.\]

Proof. From Definition 7.1,
\[g(u, v) \delta(ux_1) \ast \delta(vx_1) = g(u, v)\text{Sym} \left( \frac{\delta(ux_1) \delta(vx_2)(x_2 - tx_1)(x_2 - qt^{-1}x_1)}{(x_2 - x_1)(x_2 - qx_1)} \right)\]
\[= g(u, v) \frac{(u - tv)(u - qt^{-1}v)}{(u - v)(u - qv)} \text{Sym} \delta(ux_1) \delta(vx_2)\]
\[= \frac{(v - tu)(u - tv)(v - qt^{-1}u)(u - qt^{-1}v)}{q(u - v)} (\delta(ux_1), \delta(vx_2) + \delta(vx_1) \delta(ux_2))\]
which is manifestly skew-symmetric in \((u, v)\). \(\square\)

We now provide a proof of the Serre relation Lemma 3.9 for \(\mathcal{D}\), which is the identity
\[\text{Sym}_{v_1, v_2, v_3} \left( \frac{v_2}{v_3} \left[ \mathcal{D}(v_1), [\mathcal{D}(v_2), \mathcal{D}(v_3)] \right] \right) = 0.\]
This is equivalent to the following shuffle identity:
Lemma 7.4. The following relations hold in the shuffle algebra:

\[
\text{Sym}_{v_1,v_2,v_3} \left\{ \frac{v_2}{v_3} \delta(v_1x_1) * \left( \delta(v_2x_1) * \delta(v_3x_1) - \delta(v_3x_1) * \delta(v_2x_1) \right) - \delta(v_2x_1) * \delta(v_3x_1) - \delta(v_3x_1) * \delta(v_2x_1) \right\} = 0.
\]

Proof. Using the proof of Lemma 7.5 for \(\alpha = 3\),

\[
\delta(v_1x_1) * \delta(v_2x_1) * \delta(v_3x_1) = \prod_{1 \leq i < j \leq 3} \frac{(tv_j - v_i)(v_i - qt^{-1}v_j)}{(v_i - qv_j)(v_j - v_i)} \text{Sym}_{x_1,x_2,x_3} (\delta(v_1x_1)\delta(v_2x_2)\delta(v_3x_3)).
\]

Noting that the symmetrized term is also symmetric in \((v_1, v_2, v_3)\). Therefore, the statement of the Lemma is equivalent to

\[
\text{Sym}_{v_1,v_2,v_3} \left( \frac{v_2}{v_3} (1 - (23) - (123) + (13)) \prod_{1 \leq i < j \leq 3} \frac{(tv_j - v_i)(v_i - qt^{-1}v_j)}{(v_i - qv_j)(v_j - v_i)} \right) = 0,
\]

where the permutations on the left act by permuting the \(v\)'s. This last identity is easily checked. \(\square\)

Finally Theorem 6.5 for the expression for the multi-current \(\mathcal{Q}_\alpha(v)\) is equivalent to following shuffle product identity:

Lemma 7.5. The following relations hold in the shuffle algebra:

\[
\det \left( \left( \frac{\delta(x_i v_j)}{1 \leq i, j \leq \alpha} \right) \right) = \prod_{i=1}^{\alpha} v_j^{\alpha-1} \prod_{1 \leq i < j \leq \alpha} \frac{v_i - qv_j}{(tv_j - v_i)(v_i - qt^{-1}v_j)} \delta(v_1x_1) * \delta(v_2x_1) * \cdots * \delta(v_\alpha x_1).
\]

Proof. Starting from the left hand side,

\[
\delta(v_1x_1) * \delta(v_2x_1) * \cdots * \delta(v_\alpha x_1)
\]

\[
= \text{Sym} \left( \delta(v_1x_1)\delta(v_2x_2)\cdots\delta(v_\alpha x_\alpha) \prod_{1 \leq i < j \leq \alpha} \frac{x_j - tx_i}{x_j - x_i} \frac{x_j - (qt^{-1}x_i)}{x_j - qx_i} \right)
\]

\[
= \frac{1}{(\prod_{i=1}^{\alpha} v_j)^{\alpha-1}} \prod_{1 \leq i < j \leq \alpha} \frac{(tv_j - v_i)(v_i - qt^{-1}v_j)}{(v_i - qv_j)} \left( \frac{\delta(v_1x_1)\delta(v_2x_2)\cdots\delta(v_\alpha x_\alpha)}{\prod_{1 \leq i < j \leq \alpha} (x_i - x_j)} \right)
\]

and the result follows, as the symmetrization produces the desired determinant.

We now turn to the proof of the following simplest \(t\)-deformed M-system relations.
Lemma 7.6. We have the following relations between generalized Macdonald operators, valid for all \( n \in \mathbb{Z} \):

\[
\begin{align*}
D_n D_{n+1} - q D_{n+1} D_n &= (1 - q) D_{n+1,n} \\
(D_n)^2 - q D_{n+1} D_{n-1} &= (qt^{-1} + 1 + t) D_{n,n} - q D_{n+1,n-1} \\
(D_n)^2 - q^{-1} D_{n-1} D_{n+1} &= (t^{-1} + 1 + q^{-1} t) D_{n,n} - q^{-1} D_{n+1,n-1}.
\end{align*}
\]

**Proof.** The equivalent shuffle identities in \( \mathcal{F}_2 \) read respectively:

\[
\begin{align*}
(x_1^n) \ast (x_1)^{n+1} - q (x_1)^{n+1} \ast (x_1)^n &= (1 - q) s_{n+1,n}(x_1, x_2) \\
(x_1^n) \ast (x_1)^n - q (x_1)^{n+1} \ast (x_1)^{n-1} &= (qt^{-1} + 1 + t)s_{n,n}(x_1, x_2) \\
q^{-1} (x_1^n) \ast (x_1)^{n-1} \ast (x_1)^{n+1} &= (t^{-1} + 1 + q^{-1} t)s_{n,n}(x_1, x_2) \\
q^{-1} s_{n+1,n-1}(x_1, x_2).
\end{align*}
\]

The first relation follows from:

\[
\begin{align*}
\operatorname{Sym} \left( \frac{(x_1^n x_2 + q x_1^{n+1} x_2^n)(x_2 - tx_1)(x_2 - qt^{-1} x_1)}{(x_2 - x_1)(x_2 - qx_1)} \right) \\
= (x_1 x_2)^n \operatorname{Sym} \left( \frac{(x_2 - tx_1)(x_2 - qt^{-1} x_1)}{(x_2 - x_1)} \right) = (1 - q)(x_1 x_2)^n(x_1 + x_2).
\end{align*}
\]

The last two from:

\[
\begin{align*}
\operatorname{Sym} \left( \frac{(x_1^n x_2 - q x_1^{n+1} x_2^{n-1})(x_2 - tx_1)(x_2 - qt^{-1} x_1)}{(x_2 - x_1)(x_2 - qx_1)} \right) \\
= (x_1 x_2)^{n-1} \operatorname{Sym} \left( \frac{x_1(x_2 - tx_1)(x_2 - qt^{-1} x_1)}{x_2 - x_1} \right) \\
= (x_1 x_2)^{n-1}((qt^{-1} + 1 + t)x_1 x_2 - q(x_1^2 + x_1 x_2 + x_2^2))
\end{align*}
\]

and

\[
\begin{align*}
\operatorname{Sym} \left( \frac{(x_1^n x_2 - q^{-1} x_1^{n-1} x_2^{n+1})(x_2 - tx_1)(x_2 - qt^{-1} x_1)}{(x_2 - x_1)(x_2 - qx_1)} \right) \\
= -q^{-1}(x_1 x_2)^{n-1} \operatorname{Sym} \left( \frac{x_2(x_2 - tx_1)(x_2 - qt^{-1} x_1)}{x_2 - x_1} \right) \\
= -q^{-1}(x_1 x_2)^{n-1}((x_1^2 + x_1 x_2 + x_2^2) - (qt^{-1} + q + t)x_1 x_2)
\end{align*}
\]

The final results follow from the values \( s_{n+1,n} = (x_1 x_2)^n(x_1 + x_2) \), \( s_{n,n} = (x_1 x_2)^n \) and \( s_{n+1,n-1} = (x_1 x_2)^{n-1}(x_1^2 + x_1 x_2 + x_2^2) \). \( \square \)

**Remark 7.7.** The three quantities

\[
\begin{align*}
D_{n+1,n} &= \frac{1}{1 - q}(D_n D_{n+1} - q D_{n+1} D_n) \\
D_{n,n} &= t(1 - q^2)(D_n)^2 + q [D_{n-1}, D_{n+1}] \\
&= \frac{t}{(1 - q)(1 + t)(q + t)}
\end{align*}
\]
\[
D_{n+1,n-1} = t \frac{(1-q)(q+t^2)(D_n)^2 + (1+t)(q+t) \left[ D_{n-1}, D_{n+1} \right]_q - tq \left[ D_{n-1}, D_{n+1} \right]}{(1-q)(1+t)(q+t)}
\]

are instances of \((q,t)\)-determinants which also follow from Theorem 6.9.

8. The Dual \(q\)-Whittaker Limit \(t \to \infty\)

We now compare the results of this paper to the limit \(t \to \infty\), the dual Whittaker limit, which was the subject [DFK17]. We consider in particular the degenerations of our generalized Macdonald operators and of the shuffle product.

8.1. The quantum \(Q\)-system and quantum determinants.

8.1.1. The operators \(D^{(q)}_n\) The generalized Macdonald operators \(D^{(q,t)}_{\alpha;n}\) (1.7) were introduced in [DFK17] as the natural \(t\)-deformation of the difference operators of Eq. (1.5), with the limit \(\lim_{t \to \infty} \theta^{-\alpha(N-\alpha)}D^{(q,t)}_{\alpha;n} = D^{(q)}_{\alpha;n}\). Conversely, the \(t\)-deformed operators \(D^{(q,t)}_{\alpha;n}\) can be expressed in terms of the limiting difference operators \(D^{(q)}_{\alpha,n}\) as follows.

**Lemma 8.1.**

\[
D^{(q,t)}_n = \frac{t^{N+1}}{t-1} \sum_{j=0}^{N} (-t^{-1})^j e_j(x_1, \ldots, x_N) D^{(q)}_{n-j},
\]

where \(e_i\) are the elementary symmetric functions. Equivalently, the generating functions \(\vartheta(z)\) and \(\vartheta^{(q)}(z) = \sum_{n \in \mathbb{Z}} z^n D^{(q)}_{\alpha;n}\) are related via:

\[
\vartheta(u) = \frac{t^{N+1}}{t-1} C^+(t^{-1}u) \vartheta^{(q)}(u),
\]

with \(C^+(u)\) as in (4.2).

**Proof.** Evaluating the relation

\[
\prod_{i=1}^{N} (tx - x_i) = \sum_{j=0}^{N} (-1)^j t^{N-j} x^{N-j} e_j(x_1, \ldots, x_N)
\]

at \(x = x_k\) gives

\[
\prod_{i \neq k} (tx_k - x_i) = \frac{t^N}{t-1} \sum_{j=0}^{N} (-t^{-1})^j x_k^{N-1-j} e_j(x_1, \ldots, x_N).
\]

Therefore,

\[
x_k^n \prod_{i \neq k} \theta x_k - \theta^{-1} x_i x_k - x_i = \frac{t^{N+1}}{t-1} \sum_{j=0}^{N} (-t^{-1})^j e_j(x_1, \ldots, x_N) x_k^{n-j} \prod_{i \neq k} (x_k - x_i).
\]

Substituting this into (1.7) and summing this over \(k\), Eq. (8.1) follows. Eq. (8.2) follows by computing the generating functions for both sides of (8.1), and using the fact that \(C^+(t^{-1}u) = \sum_{j=0}^{N} (-t^{-1}u)^j e_j(x_1, \ldots, x_N). \) \( \square \)
Remark 8.2. The relation (8.2) can also be obtained by comparing the bosonized expression for $\varepsilon_1(z)$ of Theorem 4.3 and that for the limiting current $\varepsilon^{(q)}(z)$ (4.17), by using the identity:

$$C^-(u) = (-u)^N A^{-1} C^+(u^{-1}).$$

Indeed, this identity allows to rewrite the second term of the scalar prefactor in the bosonized expression of $\varepsilon_1$:

$$t^{-\frac{N}{2}} \frac{C^-(q^{-1/2}z^{-1})}{C^-(q^{-1/2}z^{-1})} = (-t^{1/2}q^{-1/2}z^{-1})^N A^{-1} \frac{C^+(q^{1/2}t^{-1}z)}{C^-(q^{-1/2}z^{-1})}$$

which up to the factor $t^{N/2}$ is $C^+(q^{1/2}t^{-1}z)$ times that for the limiting current $\varepsilon^{(q)}$ of (4.17). This results in the relation $\varepsilon_1(z) = t^{(N+1)/2}(t-1) C^+(q^{1/2}t^{-1}z)\varepsilon^{(q)}(z)$ which boils down to (8.2) upon rescaling $z \to q^{-1/2}z$ and multiplying by $(1-q)/q^{1/2}$.

8.1.3. Symmetrization and constant terms

We define the limiting difference operators, $\mathcal{D}_{\alpha}^{(q)}$, together with the quantities $\Delta = \Gamma_1 \Gamma_2 \cdots \Gamma_N$ and $A = x_1 x_2 \cdots x_N$, satisfy the M-system relations (1.1) which have an added coefficient $A$. These relations are, for $\alpha, \beta \in [1, N]$ and $n \in \mathbb{Z}$:

$$\mathcal{D}_{\alpha; n+1}^{(q)} \mathcal{D}_{\alpha; n}^{(q)} = q^{\min(|\alpha - \beta|, |p-n|)} \mathcal{D}_{\beta; n}^{(q)} \mathcal{D}_{\alpha; n}^{(q)}, \quad |n-p| \leq |\alpha - \beta| + 1,$$

$$q^{\alpha} \mathcal{D}_{\alpha; n+1}^{(q)} \mathcal{D}_{\alpha; n-1}^{(q)} = (\mathcal{D}_{\alpha; n}^{(q)})^2 - \mathcal{D}_{\alpha+1; n}^{(q)} \mathcal{D}_{\alpha-1; n}^{(q)},$$

$$\mathcal{D}_{0; n}^{(q)} = 1, \quad \mathcal{D}_{N+1; n}^{(q)} = 0, \quad \mathcal{D}_{N,n}^{(q)} = A^n \Delta,$$

$$\Delta \mathcal{D}_{\alpha; n}^{(q)} = q^{\alpha n} \mathcal{D}_{\alpha; n}^{(q)} \Delta, \quad \mathcal{D}_{\alpha; n}^{(q)} A = q^\alpha A \mathcal{D}_{\alpha; n}^{(q)}, \quad \Delta A = q^N A \Delta.$$  

(8.6)

8.1.3. Symmetrization and constant terms

We define the limiting difference operators, for any $P \in \mathcal{F}_\alpha$,

$$A_{\alpha}(P) := \lim_{t \to \infty} t^{-\alpha(N-\alpha)} A_{\alpha}(P) = B_{\alpha}(P) := \lim_{t \to \infty} t^{-\alpha(N-\alpha)} B_{\alpha}(P),$$  

(8.7)

by

$$A_{\alpha}(P) = \frac{1}{\alpha!(N-\alpha)!} \text{Sym} \left( P(x_1, \ldots, x_\alpha) \prod_{1 \leq i \leq \alpha < j \leq N} \frac{x_i}{x_i - x_j} \Gamma_1 \cdots \Gamma_\alpha \right),$$

(8.8)

$$B_{\alpha}(P) = \frac{1}{\alpha!} \text{CT}_u \left( P(u_1^{-1}, \ldots, u_\alpha^{-1}) \prod_{1 \leq i < j \leq \alpha} (u_i^{-1} - u_j^{-1})(u_i - q u_j) \prod_{i=1}^{\alpha} \mathcal{D}_{(q)}^{(q)}(u_i) \right).$$

(8.9)

Moreover, define

$$\mathcal{D}_{a_1, \ldots, a_\alpha}^{(q)} := \lim_{t \to \infty} t^{-\alpha(N-\alpha)} \mathcal{D}_{a_1, \ldots, a_\alpha}^{(q)},$$

which is a polynomial of degree $\alpha$ in the $\mathcal{D}_{\ell}^{(q)}$'s, as a direct consequence of Eq. (8.9) and the fact that $s_{a_1, \ldots, a_\alpha}$ is a Laurent polynomial of $x_1, \ldots, x_\alpha$. 
8.1.4. Quantum determinants Taking the limit $t \to \infty$ of Eq. (6.11) of Corollary 6.6 gives the quantum determinant expression of Equation (2.25) in [DFK17],
\[
\mathcal{D}_{a_1, \ldots, a_\alpha}^{(q)} = CT_u \left( \prod_{i=1}^\alpha u_i^{-a_i} \left( \prod_{1 \leq i < j \leq \alpha} 1 - q \frac{u_j}{u_i} \right) \prod_{i=1}^{m(u_i)} \right) =: \left| \mathcal{D}_{a_j+i-j}^{(q)} \right|_{1 \leq i, j \leq \alpha, q}. \tag{8.10}
\]
In terms of the multi-current expression, this is equivalent to
\[
\mathcal{D}_{\alpha}^{(q)}(v_1, \ldots, v_\alpha) := \sum_{a_1, \ldots, a_\alpha \in \mathbb{Z}} \mathcal{D}_{a_1, \ldots, a_\alpha}^{(q)} v_1^{a_1} v_2^{a_2} \cdots v_\alpha^{a_\alpha} = \left( \prod_{1 \leq i < j \leq \alpha} 1 - q \frac{v_j}{v_i} \right) \prod_{i=1}^{\alpha} \mathcal{D}^{(q)}(v_i). \tag{8.11}
\]

8.1.5. The quantum determinant in terms of alternating sign matrices There is a very nice expression of the quantum determinant (8.10) as a sum over Alternating Sign Matrices (ASM). This is because the quantity $\prod_{i < j} (v_i + \lambda v_j)$ is the $\lambda$-determinant $\lambda \det(V_n)$ [RR86] of the Vandermonde matrix $V_n := (v_i^{n-j})_{1 \leq i, j \leq n}$.

Recall that an $n \times n$ ASM $A$ has elements $a_{i,j} \in \{0, 1, -1\}$ such that each row and column sum is 1, and the non-zero entries alternate in sign along each row and column. We denote by $ASM_n$ the set of all such matrices. The inversion number of an ASM is $I(A) = \sum_{i > k, j < \ell} A_{i,j} A_{k,\ell}$, and the total number of $-1$s in $A$ is denoted by $N(A)$. Let $v = (n - 1, n - 2, \ldots, 1, 0)^t$, and define $m_i(A) := (Av)_i$. Taking $\lambda = -q$ for the $\lambda$-determinant of the $\alpha \times \alpha$ Vandermonde matrix $V_q$,
\[
\prod_{1 \leq i < j \leq \alpha} (v_i - q v_j) = \sum_{A \in ASM_n} (-q)^{I(A) - N(A)} (1 - q)^{N(A)} \prod_{i=1}^{n} v_i^{m_i(A)}.
\]
Combining this with (8.10), we have:

**Theorem 8.3.** The quantum determinant of the matrix $\left( \mathcal{D}_{a_j+i-j}^{(q)} \right)_{1 \leq i, j \leq \alpha}$ is
\[
\left| \mathcal{D}_{a_j+i-j}^{(q)} \right|_{1 \leq i, j \leq \alpha, q} = \sum_{A \in ASM_n} (-q)^{I(A) - N(A)} (1 - q)^{N(A)} \prod_{i=1}^{\alpha} \mathcal{D}_{a_i+\alpha - i - m_{i}(A)}^{(q)}. \tag{8.12}
\]

**Example 8.4.** For $\alpha = 2$, there are two ASMs,
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
with, respectively, $I(A) = 0, 1$ and $N(A) = 0, 0$, and $(m_1(A), m_2(A)) = (1, 0), (0, 1)$. Eq. (8.12) becomes
\[
\mathcal{D}_{a_1, a_2}^{(q)} = \begin{vmatrix} \mathcal{D}_{a_1}^{(q)} & \mathcal{D}_{a_2}^{(q)} \\ \mathcal{D}_{a_1+1}^{(q)} & \mathcal{D}_{a_2+1}^{(q)} \end{vmatrix}_q := \mathcal{D}_{a_1}^{(q)} \mathcal{D}_{a_2}^{(q)} - q \mathcal{D}_{a_1+1}^{(q)} \mathcal{D}_{a_2+1}^{(q)}.
\]
For $\alpha = 3$, there are seven ASMs:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & -1 & 1
\end{pmatrix}.
\]

They have, respectively, $I(A) = 0, 1, 1, 3, 2, 2, 2$, $N(A) = 0, 0, 0, 0, 0, 1$, and the values of $(m_1(A), m_2(A), m_3(A)) = (2, 1, 0), (1, 2, 0), (2, 0, 1), (0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 1, 1)$. Eq. (8.12) becomes

\[
D_{a_1, a_2, a_3}^{(q)} = \begin{vmatrix}
D_{a_1}^{(q)} & D_{a_2}^{(q)} & D_{a_3}^{(q)} \\
D_{a_1+1}^{(q)} & D_{a_2}^{(q)} & D_{a_3}^{(q)} \\
D_{a_1+2}^{(q)} & D_{a_2+1}^{(q)} & D_{a_3}^{(q)}
\end{vmatrix}_q
\]

\[
:= D_{a_1}^{(q)} D_{a_2}^{(q)} D_{a_3}^{(q)} - q D_{a_1+1}^{(q)} D_{a_2}^{(q)} D_{a_3}^{(q)} - q D_{a_1}^{(q)} D_{a_2+1}^{(q)} D_{a_3}^{(q)}
- q^3 D_{a_1+2}^{(q)} D_{a_2}^{(q)} D_{a_3}^{(q)} + q^2 D_{a_1}^{(q)} D_{a_2}^{(q)} D_{a_3+1}^{(q)}
+ q^2 D_{a_1+1}^{(q)} D_{a_2+1}^{(q)} D_{a_3}^{(q)} - q (1 - q) D_{a_1+1}^{(q)} D_{a_2}^{(q)} D_{a_3}^{(q)}.
\]

8.2. The $t \to \infty$ limit of the shuffle product and the quantum Q-system. Define the limiting product $\star$ of the shuffle product $\ast$ is compatible with the limiting difference operators (8.7). It is defined as follows. Let $(P, P') \in \mathcal{F}_\alpha \times \mathcal{F}_\beta$. Then

\[
P \ast P'(x_1, \ldots, x_{\alpha+\beta}) := \frac{1}{\alpha! \beta!} \text{Sym} \left( \frac{P(x_1, \ldots, x_\alpha) P'(x_{\alpha+1}, \ldots, x_{\alpha+\beta})}{\prod_{1 \leq i \leq \alpha < j \leq \alpha+\beta} (x_j - x_i)(x_j - qx_i)} \right).
\]

This is the limit

\[
P \ast P'(x_1, \ldots, x_{\alpha+\beta}) = \lim_{t \to \infty} t^{-\alpha \beta} P \ast P'(x_1, \ldots, x_{\alpha+\beta}).
\]

The compatibility:

\[
D_\alpha(P) D_\beta(P') = D_{\alpha+\beta}(P \ast P')
\]

follows from

\[
\lim_{t \to \infty} t^{-\alpha(N-\alpha)} D_\alpha(P) t^{-\beta(N-\beta)} D_\beta(P') = \lim_{t \to \infty} t^{-(\alpha+\beta)(N-\alpha-\beta)} D_{\alpha+\beta}(t^{-2\alpha \beta} P \ast P').
\]

Recall that $D_{a, n}^{(q)} = A_a ((x_1 x_2 \cdots x_\alpha)^n)$. The renormalized quantum Q-system relations (8.6) are equivalent to the following relations in the $\star$ shuffle algebra:

**Lemma 8.5.** For $\alpha, \beta \in [1, N-1],$

\[
(x_1 x_2 \cdots x_\alpha)^n \ast (x_1 x_2 \cdots x_\beta)^p
= q^{\min(\alpha, \beta)(p-n)} (x_1 x_2 \cdots x_\beta)^p \ast (x_1 x_2 \cdots x_\alpha)^n,
\]

\[
(n, p \in \mathbb{Z}, |p-n| \leq |\alpha - \beta| + 1),
\]

\[
q^\alpha (x_1 x_2 \cdots x_\alpha)^{n+1} \ast (x_1 x_2 \cdots x_\alpha)^{n-1}
= (x_1 x_2 \cdots x_\alpha)^n \ast (x_1 x_2 \cdots x_\alpha)^n
- (x_1 x_2 \cdots x_{\alpha+1})^n \ast (x_1 x_2 \cdots x_{\alpha-1})^n, \quad (n \in \mathbb{Z}).
\]
Example 8.6. In $\mathcal{F}_2$ we have the shuffle identities:

$\ns_{n+k,n}(x_1, x_2) = x_1^{k+n} \star x_1^n - q x_1^{k+n+1} \star x_1^{n-1} \quad (k \in \mathbb{Z}).$

These follow from:

$\text{Sym} \left( \frac{x_1^{k+1}(x_1 x_2)^{n-1}}{(x_2^{-1} - x_1^{-1})} \right) = (x_1 x_2)^n \text{Sym} \left( \frac{x_1^{k+1}}{(x_1 - x_2)} \right) = (x_1 x_2)^n s_{k,0}(x_1, x_2) = s_{n+k,n}(x_1, x_2)$

where Sym denotes the symmetrization in $x_1, x_2$. The shuffle identities above are equivalent to the quantum determinant identities:

$\mathcal{D}_{n+k,n}^{(q)} = \mathcal{D}_{n+k}^{(q)} \mathcal{D}_n^{(q)} - q \mathcal{D}_{n+k+1,n}^{(q)} \mathcal{D}_{n-1}^{(q)} = \left| \begin{array}{cc} \mathcal{D}_{n+k}^{(q)} & \mathcal{D}_{n-1}^{(q)} \\ \mathcal{D}_{n+k+1}^{(q)} & \mathcal{D}_n^{(q)} \end{array} \right|_q.$

obtained by picking the coefficient of $v_1^{n+k} v_2^n$ in $\mathcal{D}_2^{(q)}(v_1, v_2)$ of Eq. (8.11). For respectively $k = -1$ and $k = 0$ the above shuffle relations are equivalent to the first two relations $\alpha = \beta = 1$ of the renormalized quantum $Q$-system, namely:

$\mathcal{D}_n^{(q)} \mathcal{D}_{n+1}^{(q)} = q \mathcal{D}_{n+1}^{(q)} \mathcal{D}_n^{(q)}, \quad (\mathcal{D}_n^{(q)})^2 - q \mathcal{D}_{n+1}^{(q)} \mathcal{D}_{n-1}^{(q)} = \mathcal{D}_{n,n}^{(q)}.$

9. Conclusion

9.1. EHA and $t$-deformed quantum cluster algebra. In this paper, we considered the representation of the EHA acting on the space of functions in finitely many variables $x_1, x_2, \ldots, x_N$. This representation is not faithful, and the kernel is generated by the $(q, t)$-determinant relations $\mathcal{D}_{N+1,n}^{(q,t)} = 0, n \in \mathbb{Z}$. These are polynomial relations in the generators $\mathcal{D}_n$ by Theorem 1.18.

To clarify the relations between the various algebras and their generators, we show in Fig. 1 the lattice of generators $u_{a,b}$ of the EHA. The three columns $a = -1, 0, 1$, emphasized with square vertices, correspond to the modes of the currents $f_1(z)$, $\psi^\pm(z)$ and $\epsilon_1(z)$, respectively, in the quantum toroidal algebra. The horizontal axis corresponds to the power sum versions $\mathcal{P}_\ell$ (5.6–5.7) of the Macdonald operators, that is, power sum symmetric functions in the $Y_i$’s.

Our quotient of EHA is a natural $t$-deformation of the renormalized quantum $A_{N-1} Q$-system algebra. The defining $M$-system relations (1.1) are indeed explicitly $t$-deformed. For instance, the $t$-deformation of the $q$-commutation relation $\mathcal{D}_n^{(q)} \mathcal{D}_{n+1}^{(q)} = q \mathcal{D}_{n+1}^{(q)} \mathcal{D}_n^{(q)}$ is $\mathcal{D}_n^{(q,t)} \mathcal{D}_{n+1}^{(q,t)} - q \mathcal{D}_{n+1}^{(q,t)} \mathcal{D}_n^{(q,t)} = (1 - q) \mathcal{D}_{n+1,n}^{(q,t)}$, obtained from Eq. (7.6) of Lemma 7.6. Noting that $\theta^{1-N} \mathcal{D}_n^{(q,t)}$ and $\theta^{2(2-N)} \mathcal{D}_{n,p}^{(q,t)}$ have finite limits, we have

$\left( \theta^{1-N} \mathcal{D}_n^{(q,t)} \right) \left( \theta^{1-N} \mathcal{D}_{n+1}^{(q,t)} \right) - q \left( \theta^{1-N} \mathcal{D}_{n+1}^{(q,t)} \right) \left( \theta^{1-N} \mathcal{D}_n^{(q,t)} \right) = t^{-1} (1 - q) \left( \theta^{2(2-N)} \mathcal{D}_{n+1,n}^{(q,t)} \right)$.
Fig. 1. The representation of EHA of Sect. 5. The generators correspond to the vertices of $\mathbb{Z}^2$. We have emphasized the three columns $a = -1, 0, 1$ corresponding respectively to the elements $u_{-1,k} = \frac{q^{k+1}}{1-q^{-1}}D_k$, $u_{0,\pm k} = \frac{q^{1/2}}{1-q}D_1$, and $u_{1,k} = \frac{q^{k+1}}{1-q}D_k$, as well as the horizontal axis corresponding to the power sum versions of the Macdonald operators, $u_{\pm \ell,0} = \frac{q^{\ell/2}}{1-q^{\ell}}P_{\pm \ell}$, which shows that the r.h.s. is subleading when $t \to \infty$. Similarly, the $t$-deformation of the relation $(D_n^{(q,t)})^2 - q D_n^{(q,t)} D_{n-1}^{(q,t)} = D_2^{(q)}$ is $(D_n^{(q,t)})^2 - q D_{n+1}^{(q,t)} D_{n-1}^{(q,t)} = (qt^{-1} + 1 + t)D_{2,n}^{(q,t)} - q D_{n+1,n-1}^{(q,t)}$ (see (7.4)). Indeed, repeating the scaling analysis, we may write

$$\theta^{2(1-N)}(D_n^{(q,t)})^2 - q \theta^{1-N}D_{n+1}^{(q,t)}\theta^{1-N}D_{n-1}^{(q,t)} = (1 + t^{-1} + qt^{-2})\theta^{2(2-N)}D_{2,n}^{(q,t)} - qt^{-1}\theta^{2(2-N)}D_{n+1,n-1}^{(q,t)}$$

from which we recover the correct $t \to \infty$ limit.

The renormalized quantum $Q$-system relations were shown in [DFK17] to be solved by the quantum determinant expressions (8.10) with $a_1 = a_2 = \cdots = a_q = n$ for $D_{\alpha;n}^{(q)}$ as polynomials of $(D_{\alpha;n}^{(q)})_{|\alpha|<\alpha}$, with the condition that $D_{\alpha;n+1}^{(q)} = 0$ for all $n \in \mathbb{Z}$. More generally, we may consider the quantum determinant expression (8.10) for $D_{a_1,a_2,\ldots,a_q}^{(q)}$. Rewriting the multi-current relation of Theorem 6.5 as
both sides have finite limits: \( \lim_{t \to -\infty} \theta^{(\alpha-N)}(\theta(v)) = \Delta_{\alpha}^{(q)}(v) \) and \( \lim_{t \to -\infty} \theta^{1-N}(\theta(v)) = \Delta_{\alpha}^{(q)}(v) \). The r.h.s. of (9.1) is simply the generating function for the quantum determinants \( (\theta^{(\alpha-N)}(\theta(v)))_{1 \leq i, j \leq \alpha|q} \) which tend to \(|(\theta^{1-N}(\theta(v)))_{1 \leq i, j \leq \alpha|q}| \) when \( t \to -\infty \).

Expanding the l.h.s. in powers of \( t \) at large \( t \), we see that the dominant term is \( \theta^{(\alpha-N)}(\theta(v)) \to \Delta_{\alpha}^{(q)}(v) \), and all other terms are of strictly smaller order. This displays explicitly in which sense this relation is a \( t \)-deformation of the quantum determinant relation (8.10).

9.2. \((q, t)\)-determinant. One of the goals of this paper is the construction of a \( t \)-deformation of the quantum determinant associated with the type A quantum Q-system. In particular, we conjecture that the \((q, t)\)-determinant \( \Delta_{a_1 a_2 \ldots a_\alpha}^{(q, t)} \) is a polynomial in the generators \( \{ \Delta_{a_i a_{i+1}}^{(t)} : n \in \mathbb{Z} \} \). We proved this in the case \( \alpha = 2, 3 \) in Theorems 6.9 and 6.11, and in the case when \( a_1 = a_2 = \cdots = a_\alpha = n \) in Theorem 1.18. The latter case uses explicitly the relations in the Elliptic Hall Algebra.

Unlike the quantum determinant (8.10), the polynomial expression for \( \Delta_{a_1 a_2 \ldots a_\alpha}^{(q, t)} \) depends, in general, on more than just the elements of the discrete Wronskian matrix \( \{ \Delta_{a_i a_{i+1}}^{(t)} : 1 \leq i, j \leq \alpha \} \). The polynomial is unique modulo the relations of the quantum toroidal algebra (6.1) and (3.14). The expressions (7.6), (7.7), and the alternative expression (6.21) for \( n = 2, b = n \), are all polynomials of only \( \{ \Delta_{a_i a_{i+1}}^{(t)} : 1 \leq i, j \leq \alpha \} \), since

\[
\Delta_{n+1, n}^{(q, t)} = \frac{\Delta_n^{(q, t)} \Delta_{n+1}^{(q, t)}}{\Delta_{n+1}^{(q, t)} \Delta_n^{(q, t)}} = \frac{1}{(1-q)} \left[ \frac{\Delta_{n+1}^{(q, t)} \Delta_n^{(q, t)}}{\Delta_n^{(q, t)} \Delta_{n+1}^{(q, t)}} \right]_q
\]

\[
\Delta_{n, n+1}^{(q, t)} = \frac{\Delta_n^{(q, t)} \Delta_{n+1}^{(q, t)}}{\Delta_{n+1}^{(q, t)} \Delta_n^{(q, t)}} = \frac{t}{(1-q)(1+t)(q+t)} \times \left( (q^{-1} - q)(\Delta_n^{(q, t)})^2 + \left[ \Delta_{n+1}^{(q, t)} \Delta_n^{(q, t)} \right]_q \right)
\]

\[
\Delta_{n+2, n+1}^{(q, t)} = \frac{\Delta_{n+2}^{(q, t)} \Delta_{n+1}^{(q, t)}}{\Delta_{n+1}^{(q, t)} \Delta_{n+2}^{(q, t)}} = \frac{t}{(q-1)(1-t^2)(q^2 - t^2)} \times \left[ (1+q)(\Delta_n^{(q, t)})^2 \right]_q^2 - (q+t^2) \left[ \Delta_{n+2}^{(q, t)} \Delta_n^{(q, t)} \right]_q^2
\]

The property still holds for \( \Delta_{n+3, n}^{(q, t)} \), but breaks down for \( \Delta_{n+4, n}^{(q, t)} \) which can at best be expressed as a polynomial of \( \Delta_{n+4}^{(q, t)}, \Delta_{n+5}^{(q, t)}, \Delta_{n+1}^{(q, t)}, \Delta_{n+3}^{(q, t)} \).
The \( (q, t) \)-determinant is therefore a subtle deformation of the quantum determinant. It is desirable to find an expression with good combinatorial properties, generalizing Eq. (8.12) of Theorem 8.3.

9.3. Relation to Macdonald raising/lowering operators. In [DFK17] it was noted that the difference operators \( D_{\alpha;n}^{(q)} \) for \( n = \pm 1 \) are the \( t \to \infty \) limits of the raising and lowering operators \( K_{\alpha}^{\pm} \) for Macdonald polynomials introduced by Kirillov and Noumi [KN99]. More precisely, the \( t \to \infty \) limits of the Macdonald polynomials are (dual) \( q \)-Whittaker functions \( \psi_{\lambda}(x) \), which obey:

\[
D_{\alpha;1}^{(q)} \psi_{\lambda} = c_{\alpha,\lambda}(q) \psi_{\lambda+\omega_\alpha}, \quad D_{\alpha;-1}^{(q)} \psi_{\lambda} = d_{\alpha,\lambda}(q) \psi_{\lambda-\omega_\alpha},
\]

where \( \lambda = \sum_{i=1}^{N} \lambda_i \epsilon_i, \omega_\alpha = \epsilon_1 + \cdots + \epsilon_\alpha \) in the canonical basis \( \{\epsilon_i\} \) of \( \mathbb{C}^N \), and \( c, d \) are scalar functions that vanish whenever \( \lambda \pm \omega_\alpha \) is not a partition.

At finite \( t \), the difference operators \( D_{\alpha;1}^{(q,t)} \) have an expression as a linear combination of the raising and lowering operators of [KN99]. These can be inverted into expressions for \( K_{\alpha}^{\pm} \) involving only “hook” difference operators of the form \( D_{k,1,1,...,1} \) and elementary symmetric functions. We will return to this question in future work.

9.4. Relation to graded characters. The difference operators \( D_{\alpha;n}^{(q)} \) were introduced in [DFK17] to generate graded characters of tensor products of Kirillov–Reshetikhin modules by iterated action on the constant function 1, namely:

\[
\text{ch}_n(q^{-1}, x) = q^{-\frac{1}{2}} \sum_{i,j,\alpha,\beta} n_{\alpha,i} \text{Min}(i,j) \text{Min}(\alpha,\beta) n_{\beta,j} \sum_{i,\alpha} i \epsilon n_{\alpha,i} \prod_{m=k}^{N} \prod_{\alpha=1}^{N-1} D_{\alpha;m}^{(q)} \cdot 1,
\]

where the graded characters are generating functions for \( q \)-graded multiplicities in the decomposition of tensor products of Kirillov–Reshetikhin modules \( KR_{\alpha;i} \) into irreducibles \( V_\lambda \) (with character \( s_\lambda(x) \)): \( \text{ch}_n(q, x) = \sum_\lambda \text{Mult}_q \left( \bigotimes_{i,\alpha} K R_{\alpha;i}^{\otimes n_{\alpha,i}}, V_\lambda \right) s_\lambda(x) \). A particular case is the restriction to “level 1,” where \( n_{\alpha,i} = 0 \) for all \( i > 1 \). In this case, the graded characters are the dual \( q \)-Whittaker functions \( \psi_\lambda(x) \) of Sect. 9.3 with Young diagrams \( \lambda \) made of \( n_{\alpha;1} \) consecutive columns of \( \alpha \) boxes, \( \alpha = N - 1, N - 2, \ldots, 1 \).

As a consequence, any expression of the form \( \prod_{\alpha=1}^{N-1} (D_{\alpha;i}^{(q)})^{n_{\alpha,i}} \cdot 1 \) for \( n_{\alpha,i} \in \mathbb{Z}_+ \) is Schur positive with coefficients in \( \mathbb{Z}_+[q^{\pm 1}] \). This is not the case for the \( t \)-deformed version. For example, \( D_{2,1,1}^{(q,t)} \cdot 1 = (q^{N-1}(s_2 - t^{-1}s_{1,1})). \) However, Schur positivity is restored upon acting with the plethysm \( \Sigma \) of Sect. 4. Further investigation of the significance of these expressions is underway.

9.5. Nil-DAHA and quantum \( Q \)-system. It is known that the DAHA admits degenerations called the nil-DAHA, corresponding to either the \( t \to 0 \) or \( t \to \infty \) limits [CF13]. An interesting problem concerns the algebraic formulation of the quantum \( Q \)-system algebra directly from the nil-DAHA generators. We expect an isomorphism between spherical nil-DAHA and our quantum algebra \( A_{N}^{(q)} \). We will address this question in a later publication.
9.6. Possible generalizations. The Q-system and DAHA discussed in this paper are those associated with \( g = \mathfrak{gl}_N \). Some of the constructions of this paper can be extended to other types. The DAHA is defined for other types [Che05] as are the quantum Q-systems. Our preliminary investigations indicate that the \( t \)-deformation of the quantum Q-system and its difference operator representation can be defined as compatible with the DAHA for all classical Lie algebras \( g \).

Another interesting direction, even in the \( A_{N-1} \) case, is to try to understand the meaning of the other cluster variables, not of the form \( \mathcal{D}^{(q)}_{\alpha, n} \) in the quantum cluster algebra of the \( Q \)-system. Preliminary investigations show that those other variables are also difference operators. Understanding these could be a step in the direction of fully comprehending the \( t \)-deformation of the quantum cluster algebra.

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Appendix A. Some Useful Relations in the DAHA

A.0.1. Other useful relations. We list here several useful relations among the generators of the DAHA which follow from the definition of Sect. 2.1. Some of them can be found in [Che05] [see Sect. Some relations on pp. 103–104, Eqs. (1.4.64–69)].

The following relations give a way to reorder \( X_i \) and \( Y_j \):

\[
X_i \ Y_{i+1} = Y_{i+1} X_i \ T_i^2 X_i = Y_{i+1} T_i X_{i+1} T_i^{-1} = T_i^{-1} Y_i T_i X_i, \quad (i = 1, 2, \ldots, N - 1);
\]

\[
Y_i \ X_{i+1} = X_{i+1} T_i^{-2} Y_i = X_{i+1} T_i^{-1} Y_i T_i^{-1} Y_i T_i, \quad (i = 1, 2, \ldots, N - 1).
\]

More generally, let \( 1 \leq i \leq j \leq N \). Then

\[
X_i \ Y_{j+1} = Y_{j+1} (T_i T_{j-1} \cdots T_{i+1} T_i T_{i+2} \cdots T_j) X_i = Y_{j+1} (T_i T_{j-1} \cdots T_i) X_{i+1} (T_i^{-1} T_{i+1}^{-1} \cdots T_j^{-1});
\]

\[
Y_i \ X_{j+1} = X_{j+1} (T_j^{-1} T_{j-1}^{-1} \cdots T_{i+1}^{-1} T_i^{-2} T_{i+1} T_{i+2} \cdots T_j) Y_i = X_{j+1} (T_j^{-1} T_{j-1}^{-1} \cdots T_i^{-1}) Y_{i+1} (T_i T_{i+1} \cdots T_j).
\]

Moreover,

\[
(X_j X_{j-1} \cdots X_i) \ Y_{j+1} = Y_{j+1} (T_j T_{j-1} \cdots T_{i+1} T_i T_{i+2} \cdots T_j) (X_j X_{j-1} \cdots X_i) = Y_{j+1} (T_j T_{j-1} \cdots T_j) (X_{j+1} X_j \cdots X_{i+1}) (T_i^{-1} T_{i+1}^{-1} \cdots T_j^{-1}).
\]

This equation can be iterated to obtain

\[
(X_j X_{j-1} \cdots X_i)^n Y_{j+1} = Y_{j+1} (T_j T_{j-1} \cdots T_j) (X_j X_j \cdots X_{i+1})^n (T_i^{-1} T_{i+1}^{-1} \cdots T_j^{-1}). \quad (A.1)
\]
Lemma A.1. For all $1 \leq i \leq j \leq N$,

$$(X_i X_{i+1} \cdots X_j)(T_{i+1}^{-1} T_{i+2}^{-1} \cdots T_{j-1}^{-1}) = (T_i^{-1} T_{i+1}^{-1} \cdots T_{j-1}^{-1})(X_i X_{i+1} \cdots X_j).$$

Proof. For all $1 \leq i \leq j \leq N$,

$$(X_i X_{i+1} \cdots X_j)(T_{i+1}^{-1} T_{i+2}^{-1} \cdots T_{j-1}^{-1}) = (X_i X_{i+1} \cdots X_{j-1})(T_{i+1}^{-1} T_{i+2}^{-1} \cdots T_{j-2}^{-1})T_{j-1} X_{j-1}$$

$$= X_i(T_i X_i \cdots T_{j-2} X_{j-2} T_{j-1} X_{j-1})$$

$$= T_i^{-1} X_{i+1} X_i(T_{i+1} X_{i+1} \cdots T_{j-1} X_{j-1})$$

$$= T_i^{-1} X_{i+1}(T_{i+1} X_{i+1} \cdots T_{j-1} X_{j-1}) X_i$$

$$= T_i^{-1} T_{i+1}^{-1} X_{i+2}(T_{i+2} X_{i+1} \cdots T_{j-1} X_{j-1}) X_i X_{i+1}$$

$$= (T_i^{-1} T_{i+1}^{-1} \cdots T_{j-1}^{-1})(X_i X_{i+1} \cdots X_j).$$

The lemma follows. ☐

A.0.2. The generator $\pi$. It is particularly helpful to define

$$\pi = Y_1^{-1} T_1 T_2 \cdots T_{N-1}.$$

Using $Y_{i+1} = T_i^{-1} Y_i T_i^{-1}$, we may express each of the $Y_i$s as:

$$Y_i = T_i T_{i+1} \cdots T_{N-1} \pi^{-1} T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1} \quad (i = 1, 2, \ldots, N). \quad (A.2)$$

In other words, we may express $\pi$ in $N$ different manners:

$$\pi = T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_{i+1} T_{i+2} \cdots T_{N-1} \quad (i = 1, 2, \ldots, N). \quad (A.3)$$

The following two Lemmas show that $\pi$ acts as a translation operator on $T_i$s and $X_i$s:

Lemma A.2.

$$\pi T_i = T_{i+1} \pi \quad (i = 1, 2, \ldots, N - 2).$$

Proof. Using the $i$th expression for $\pi$ (A.3),

$$\pi T_i = (T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}) Y_i^{-1} (T_i T_{i+1} T_i)(T_{i+2} \cdots T_{N-1})$$

$$= (T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}) Y_i^{-1} T_{i+1} (T_i T_{i+1} \cdots T_{N-1})$$

$$= T_{i+1}(T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}) Y_i^{-1} (T_i T_{i+1} \cdots T_{N-1}) = T_{i+1} \pi.$$

by using the braid relations (2.1) and the commutation relations (2.2). ☐

Lemma A.3.

$$\pi X_i = X_{i+1} \pi \quad (i = 1, 2, \ldots, N - 1) \quad \text{and} \quad \pi X_N = q^{-1} X_1 \pi.$$
Proof. Using $Y_iX_{i+1} = X_{i+1}T_i^{-2}Y_i$, which implies $Y_i^{-1}X_{i+1}T_i^{-2} = X_{i+1}Y_i^{-1}$, and the expression (A.3) for $\pi$,

$$\pi X_i = (T_1^{-1}T_2^{-1} \cdots T_{i-1}^{-1})Y_i^{-1}(T_i T_{i+1} \cdots T_{N-1})X_i$$

$$= (T_1^{-1}T_2^{-1} \cdots T_{i-1}^{-1})Y_i^{-1}X_i(T_{i+1} \cdots T_{N-1})$$

$$= (T_1^{-1}T_2^{-1} \cdots T_{i-1}^{-1})Y_i^{-1}X_i T_i^{-1}(T_{i+1} \cdots T_{N-1})$$

$$= X_{i+1}(T_1^{-1}T_2^{-1} \cdots T_{i-1}^{-1})Y_i^{-1} T_i^{-1}(T_{i+1} \cdots T_{N-1}) = X_{i+1} \pi.$$

The last relation is obtained by using

$$\pi X_1 \cdots X_N = q^{-1}X_1 \cdots X_N \pi,$$

a direct consequence of the relations (2.3). $\square$

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