Superconductivity in $2+1$ dimensions via Kosterlitz-Thouless Mechanism: Large-N and Finite-Temperature Analyses

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Abstract

We analyse a $2+1$ dimensional model with charged, relativistic fermions interacting through a four-Fermi term. Taking advantage of its large-$N$ renormalizability, the various phases of this model are studied at finite temperature and beyond the leading order in $1/N$. Although the vacuum expectation value (VEV) of a charged order parameter is zero at any non-zero temperature, the model nevertheless exhibits a rich phase structure in the strong coupling régime, because of the non-vanishing VEV of a neutral order parameter and due to the non-trivial dynamics of the vortex excitations on the plane. These are: a confined-vortex phase which is superconducting at low temperatures, an intermediate-temperature phase with deconfined vortices, and a high-temperature phase, where the neutral order parameter vanishes. The manifestation of superconductivity at low-temperatures and its disappearance above a critical temperature is explicitly shown to be due to the vortex confinement/deconfinement mechanism of Kosterlitz and Thouless. The ground state does not break parity or time-reversal symmetries and the ratio of the energy gap to $T_c$ is bigger than the conventional BCS value, for $N \lesssim 22$.

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I. INTRODUCTION

Of late, there has been considerable interest in planar field theoretical models due to their relevance to condensed matter systems.\textsuperscript{1,2} Furthermore, the recent observation\textsuperscript{3} of the renormalizability of four-Fermi couplings in 2 + 1-dimensions in large-$N$ perturbation theory,\textsuperscript{4,5} has given a tremendous boost to the systematic analysis of the phase structures of various interacting theories.\textsuperscript{3,6,7} The possibility of observing a superconducting ground state in some of these phases is obviously of great practical and theoretical interest. This is due to the fact that the conventional explanation of this phenomenon based on the spontaneous breaking of a $U(1)$ symmetry may not be applicable to a lower-dimensional world.

For quite some time, it has been known from the works of Hohenberg,\textsuperscript{8} Mermin and Wagner\textsuperscript{9} and Coleman\textsuperscript{10} that breaking of a continuous symmetry (SSB) does not take place in $1 + 1$ dimensions due to infrared divergences. This is strictly true in the absence of long-range interactions. Similar infrared singularities also arise in $2 + 1$ dimensions at finite temperature, making the vacuum expectation value (VEV) of a charged order parameter vanish for non-zero temperatures. It is not very difficult to identify the source of this problem. The low-energy field theory in the broken symmetry phase is governed by a massless Bose field $\phi$ (the Goldstone mode), which in the imaginary-time formalism\textsuperscript{11} can be expanded as $\phi(x, \tau) = \sum_n \phi_n(x)e^{i\omega_n \tau}$, where the energy takes on values $\omega_n = 2n\pi/\beta$, $\beta$ being the inverse temperature. The $n = 0$ mode is governed by $S[\phi_0] = \int_0^\beta d\tau \int d^2x L(\phi_0, \partial \phi_0) = \beta \int d^2x L(\phi_0, \partial \phi_0)$, a purely two-dimensional action. Hence, it is not surprising that the afflictions of $1 + 1$ dimensional theories will manifest themselves here. However, it is worth mentioning that a neutral order parameter (possibly breaking a discrete symmetry) can have a VEV in $2 + 1$ dimensions at finite temperature.\textsuperscript{3}

In light of the above constraint, to achieve planar superconductivity, one usually invokes a small interplanar coupling or takes recourse in unconventional mechanisms, \textit{e.g.}, anyon superconductivity.\textsuperscript{12,13,14,15,16,17,18,19} In this paper, an alternate model\textsuperscript{20} exhibiting superconductivity in the strong coupling phase due to the vortex confinement mechanism of Kosterlitz and Thouless (KT),\textsuperscript{21,22} is analysed in some detail via large-$N$ perturbation theory. The model under consideration is a 2+1 dimensional relativistic theory with a four-Fermi interaction (reminiscent of the BCS theory\textsuperscript{23}) in the presence of a weak and external electromagnetic field. In the previous work\textsuperscript{20}, the low-energy effective action up to two-derivatives has been explicitly computed to leading order in $1/N$ using two independent methods.\textsuperscript{24,25}

The primary goal of the present analysis is to go beyond the leading order in $1/N$ in order to establish the renormalizability of the model and to show the absence of any ultraviolet or infrared divergences that could destabilize the vacuum structure. The model exhibits three different phases as a function of temperature and chemical potential. The low-temperature phase exhibits perfect diamagnetism with a pole in the current-current correlator because the vortices appear in tightly bound pairs by virtue of a logarithmic attractive potential. It is worth mentioning that on a plane the phase degree of freedom of a complex order parameter is multivalued, thereby giving rise to vortex configurations.
At a temperature $T_{KT}$, the vortices are deconfined; it is explicitly shown that this is also the temperature where superconductivity vanishes. This result was anticipated, but not proven, in some recent works in the literature.\textsuperscript{26,27,28} At a still higher temperature, the neutral order parameter vanishes through a second order phase transition.

The motivation for analysing the above model lies in the appearance of relativistic fermions as the relevant low-energy degrees of freedom in the strong coupling limit of the Hubbard and related models,\textsuperscript{29,30} widely believed to be of relevance to the high-$T_c$ superconductors. Furthermore, four-Fermi couplings also arise naturally in the above-mentioned models in the presence of doping.\textsuperscript{30,31}  In the absence of precise knowledge regarding the form and strength of these couplings, we consider a four-Fermi term of the BCS type, preserving the relativistic invariance of the original Lagrangian. The method of analysis can be easily generalised to other types of couplings. Finally, although not conclusive, there is some experimental indication as to the KT nature of the phase transition in the high-$T_c$ materials,\textsuperscript{32} giving urgency to the analyses of various models exhibiting such behaviour.

The goal of the study is to capture the generic properties of this class of models using the large-$N$ renormalizability of the four-Fermi couplings in $2 + 1$ dimensions and find the similarities and dissimilarities with the standard BCS theory. Large-$N$ perturbation theory is inherently non-perturbative: a given order in this approximation scheme represents a class of diagrams with various powers of the conventional weak-coupling expansion parameter. The different orders in this perturbation theory also have nice geometrical interpretation in terms of Riemann surfaces; \textit{e.g.}, the leading order expression is the sum of all planar diagrams (genus zero surface).\textsuperscript{5} In the present context the large-$N$ expansion is ideal because of the fact that the weak-coupling perturbation is not renormalizable as can be easily seen from power counting. Interestingly, our analysis reveals that unlike the BCS case, superconductivity in the model under consideration occurs in the strong coupling phase.

The organization of the paper is as follows. In Sec. II, we give a brief review of the derivation of the Landau-Ginzburg effective action in the BCS theory and proceed to do the same in our model. We start with the analysis of the effective potential $V_{\text{eff}}$ and show that to leading order in the $1/N$ expansion a neutral order parameter can have a non-vanishing VEV up to a critical temperature $T_0$. Subsequently, the low-energy effective action is computed and used to show that the VEV of a charged order parameter at any non-zero temperature vanishes, because of the presence of a massless mode. In Sec. III, we discuss the renormalizability of the model after taking into account the next-to-leading order corrections. We also show that the VEV of the neutral order parameter is not destabilized by quantum fluctuations. Sec. IV is devoted to the study of the occurrence of superconductivity in this model, taking into account the non-trivial dynamics of the phase degrees of freedom. Finally, we conclude in Sec. V with some comments and future directions of work.
II. ONE-LOOP EFFECTIVE ACTION

In the discussions of BCS superconductivity by field theorists, a common approach is the following. One starts from the second-quantized BCS Lagrangian, including an effective electron-electron interaction due to phonon exchange:

\[ L_{\text{BCS}} = \psi_\uparrow^\dagger (i\partial_t - \epsilon(p)) \psi_\uparrow + \psi_\downarrow^\dagger (i\partial_t - \epsilon(p)) \psi_\downarrow - \lambda \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow. \]  

(1)

Here \( \epsilon(p) \) is the kinetic energy, including a chemical potential and coupling to electromagnetism, if desired. This has the obvious \( U(1) \) phase symmetry \( \psi \to e^{i\theta} \psi \). One then argues that the low-energy behaviour of the theory is best described in terms of a condensate field rather than the fermions. Physically this is connected to the fact that there is a gap in the fermion spectrum below the critical temperature, as a result of which long-wavelength excitations of the condensate are much cheaper to create than fermionic excitations.

In order to achieve an effective theory for the condensate, one performs a Hubbard-Stratonovich transformation, yielding

\[ L_1 = \psi_\uparrow^\dagger (i\partial_t - \epsilon(p)) \psi_\uparrow + \psi_\downarrow^\dagger (i\partial_t - \epsilon(p)) \psi_\downarrow + \sqrt{\lambda g} (\phi^* \psi_\downarrow \psi_\uparrow + \phi \psi_\uparrow^\dagger \psi_\downarrow^\dagger) + g \phi^* \phi. \]  

(2)

The advantage of (2) is that its bilinear nature in the fermion fields permits one to eliminate the fermion, at least formally. In the language of path integrals, the generating functional of (2) is

\[ Z = \int D\psi^\dagger D\psi D\phi^* D\phi e^{iS_1}, \]  

(3)

where \( S_1 = \int d^4x L_1 \). Notice that the functional integral over the scalar field can be done, yielding an “effective action” for the fermion which is none other than the integral of (1). Alternatively, one can perform the fermion functional integral:

\[ Z = \int D\phi^* D\phi e^{iS_{\text{eff}}[\phi^*, \phi]}, \]  

where

\[ e^{iS_{\text{eff}}[\phi^*, \phi]} = \int D\psi^\dagger D\psi e^{iS_1} \]  

(4a)

and

\[ S_{\text{eff}} = -i \text{Tr} \log \left( \frac{p_0 - \epsilon(p)}{\sqrt{\lambda g} \phi^*} \frac{-\sqrt{\lambda g} \phi}{p_0 + \epsilon(p)} \right) + \int d^4x g \phi^* \phi. \]  

(4b)

Here the matrix, if sandwiched between \( \psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger) \) and \( \psi = (\psi_\uparrow, \psi_\downarrow^\dagger)^T \), gives the fermion-dependent part of (2).

Next, the effective action can be evaluated in a gradient expansion. The lowest-order term, with no derivatives, is the effective potential for the condensate; at finite temperature, the result is

\[ V_{\text{eff}} \sim a \log \frac{T}{T_c} \phi^* \phi + b (\phi^* \phi)^2, \]  

(5)
where \( a, b \) are positive parameters, the details of which are not important to us. One sees from the effective potential that \( \phi \) attains a nonzero VEV below a certain critical temperature. When that happens, electromagnetism is broken and superconductivity results. The phase transition is of second order.

In this paper, we pursue a similar analysis in a relativistic setting taking into account the peculiarities of 2 + 1 dimensional space-time.

Probably the first situation in which a calculation similar to the above was done in a relativistic setting was the work of Nambu and Jona-Lasinio.\(^{37}\) This work, which is actually approximately concurrent with BCS, introduced the idea of dynamical symmetry breaking in the context of the pion-nucleon system. A four-Fermi interaction was shown to break chiral symmetry, leading to what are now known as Nambu-Goldstone bosons, which Nambu and Jona-Lasinio identified with the pions.

Another more recent field theory which mimics the BCS theory as discussed above is the Gross-Neveu model,\(^{38}\) a 1+1-dimensional model described by the following Lagrangian:

\[
\mathcal{L}_{\text{GN}} = \bar{\psi}_i i \frac{\partial}{\partial \psi_i} \psi_i + \frac{g^2}{2} (\bar{\psi}_i \psi_i)^2. \tag{6}
\]

Again one often introduces an auxiliary field to remove the four-Fermi interaction, replacing the last term in (6) with

\[
g\phi \bar{\psi}_i \psi_i - \frac{1}{2} \phi^2. \tag{7}
\]

It is well known that the discrete chiral transformation \( \psi \to \gamma_5 \psi \) is broken spontaneously. This symmetry breaking, together with the fact that the model is asymptotically free, makes it a good testing ground for QCD (although the fact that it is in 1+1 dimensions makes the connection rather tenuous).

One obvious difference between the Gross-Neveu model and the BCS theory is that the “condensate” \( \phi \) in (7) is a real (uncharged) scalar, while that in BCS is doubly charged. One can of course couple a charged scalar to fermions via an interaction of the form \( \bar{\psi} \psi |\phi|^2 \), but this is not terribly reminiscent of BCS. In order to find a closer analog we must find a doubly-charged bilinear in the fermion. This can be done using the charge-conjugate fermion field, \( \psi^c = C \bar{\psi}^T \), \( C \) being the charge conjugation matrix.

The appropriate relativistic generalization of (1), which we will analyze in detail, is thus

\[
\mathcal{L} = \bar{\psi}_\alpha (i \partial - eA) \psi_\alpha - \frac{1}{4\lambda N} \bar{\psi}_\alpha \psi^c_\alpha \bar{\psi}_\beta \psi_\beta. \tag{8}
\]

Here, \( \alpha, \beta \) range from 1 to \( N \), \( N \) being the parameter of the large-\( N \) expansion, and repeated indices imply summation. Furthermore, the coupling constant \( 1/\lambda \) has canonical dimension (mass)\(^{-1}\) and hence conventional perturbation theory is non-renormalizable. However, as mentioned earlier this type of four-Fermi coupling is renormalizable in large-\( N \) perturbation theory,\(^{3}\) making it an ideal tool to analyse the phase structure. Coupling to the photon field endows the model with a U(1) gauge symmetry. Like the BCS theory, it is assumed that the short range attractive force represented by the four-Fermi
term, originating from the underlying vibrations of the microscopic theory, dominates the Coulomb repulsion between the electrons at low-temperatures so that the net interaction is attractive.

In what follows, we will remove the four-Fermi interaction in favour of a Yukawa-type coupling to an auxiliary charged scalar field $\phi$ and then perform a derivative expansion of the effective theory for $\phi$. As a first step, the effective potential will be computed to leading order in $1/N$ to study the question of radiative breaking of the $U(1)$ symmetry à la Coleman-Weinberg.\(^{39}\) After answering this question in the affirmative, finite temperature effects will then be incorporated to show that the VEV of $\phi$ is zero when $T \neq 0$, although the neutral order parameter $|\phi|$ has a non-zero VEV. At a still higher temperature the neutral order parameter vanishes, the phase transition being of second order.

Were $\phi$ a true dynamical field, rather than an auxiliary field, we would argue that the two-derivative term induced by the fermion loop serves only to renormalize the already-present kinetic term for $\phi$. However, being an auxiliary field, it has no kinetic term and the induced two-derivative term is physically significant as it stands. In order for the expansion up to two derivatives to be sensible, therefore, the coefficient of this term must be positive. We check this, and find that it is.

We use a diagrammatic approach to compute the derivative expansion,\(^{24}\) making use of the interpretation of the effective action in terms of one-particle-irreducible Green’s functions. The approach is essentially a generalization of the method devised by Coleman and Weinberg\(^ {39}\) to compute the effective potential. An alternative approach using an expansion of the trace-log representation of the effective action\(^ {25}\) is employed in Sec. III as a check, and also to compute the next-to-leading order effects.

It is worth pointing out that the Lagrangian under consideration here, with additional scalar self-couplings, has been analysed before for the purpose of finding zero-modes of the Dirac equation in the presence of a vortex, either without\(^ {40,41}\) or with\(^ {42,43}\) a Chern-Simons term. Also, radiative symmetry breaking in various similar models in $2+1$ dimensions has been discussed in Ref. 44. Hence our investigation supplements these previous works.

Introducing auxiliary fields $\phi$ and $\phi^*$ to decouple the four-Fermi term in (8), we get:

$$\mathcal{L} = \bar{\psi}_{\alpha}(i\partial - eA)\psi_{\alpha} + \frac{1}{2} (\phi^* \bar{\psi}_{\alpha}^c \psi_{\alpha} - \phi \bar{\psi}_{\alpha} \psi_{\alpha}^c) - \lambda N \phi^* \phi. \quad (9)$$

To maintain gauge invariance of the Yukawa term in (9), the charge of the complex field is $2e$.

The effective action for the scalar and gauge fields is attained by integrating over the fermions:

$$e^{iS_{\text{eff}}[A,\phi,\phi^*]} = \int D\bar{\psi}D\psi e^{iS}. \quad (9)$$

In fact, the dependence of $S_{\text{eff}}$ on $A$ can be deduced from gauge invariance, so often in what follows we set $A$ to zero.
A nonlocal object, $S_{\text{eff}}$ cannot be computed exactly. However, if our goal is to describe the low-energy, long-wavelength excitations of the condensate, it is reasonable to attempt a gradient expansion of $S_{\text{eff}}$. Indeed, the equivalent of the trace-log in (4) can be computed as a derivative expansion directly.\textsuperscript{25,45} However, an approach which is perhaps more physical is to appeal to the interpretation of the effective action as an object which incorporates the effects of scalar field interactions mediated by internal fermion lines. Thus, $S_{\text{eff}}$ is the generating functional of one-particle irreducible vertex functions:

$$S_{\text{eff}}[\phi,\phi^*] = -i \sum_n \frac{1}{n!^2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \cdots \frac{d^3p_n}{(2\pi)^3} \frac{d^3q_n}{(2\pi)^3} (2\pi)^3 \delta^3 \left( \sum (p_i + q_i) \right) \Gamma^{(n,n)}(p_1,\cdots,p_n;q_1,\cdots,q_n) \tilde{\phi}(p_1) \tilde{\phi}^*(q_1) \cdots \tilde{\phi}(p_n) \tilde{\phi}^*(q_n).$$

(10)

Alternatively, simply by recognizing $S_{\text{eff}}$ as a functional of $\phi$, it can be expressed in terms of the gradients of $\phi$ and $\phi^*$:

$$S_{\text{eff}}[\phi,\phi^*] = \int d^3x \left( -V_{\text{eff}}(\phi) + Z_1(\phi) \partial_\mu \phi \partial^\mu \phi^* + Z_2(\phi) (\phi^* \partial_\mu \phi + \phi \partial_\mu \phi^*)^2 + \cdots \right).$$

(11)

The key feature of the first expansion is that each term is calculable: $\Gamma^{(n,n)}$ is the one-fermion-loop diagrams with $n$ external $\phi$’s and $\phi^*$’s. On the other hand, the second, a derivative expansion, is a convenient way of capturing the long-wavelength behaviour of the scalar field as induced by the fermion. In what follows we will compute the first two terms in the derivative expansion through judicious combinations of the 1PI Green’s functions. To leading order the massless fermion propagator is $i/p$; the two vertices coming from $\phi \overline{\psi} \psi^c$ couplings to fermion fields $\psi_i$ and $\psi_j^*$ are $iC^{-1} \delta_{ij}/2$ and $iC \delta_{ij}/2$ respectively.

We concentrate first on the effective potential. By rewriting (11) in momentum space, $V_{\text{eff}}$ can be calculated from the Green’s functions at zero momentum since the other terms vanish in this limit. It is not difficult to see that

$$V_{\text{eff}}(\phi) = V^0 + i \sum_{n=1}^{\infty} \left( \frac{\phi^* \phi}{n!^2} \right) \Gamma^{(n,n)}(0,\cdots;0,\cdots).$$

(12)

Here, $V^0$ is the classical potential, \textit{i.e.}, the last term in (9). The sum over all zero-momentum diagrams appears formidable, but as shown by Coleman and Weinberg, it is actually quite doable. The only unusual twist in the present case is actually in the Feynman rules: the $\phi \overline{\psi} \psi^c$ coupling, for example, gives a vertex with an incoming $\phi$ and two outgoing fermions. Formally, the effective potential is given by the classical potential plus the sum of diagrams depicted in Fig. 1, with external fields at zero momentum.

The above sum can be carried out, yielding

$$V_{\text{eff}} = V^{(0)} + iN \int \frac{d^3k}{(2\pi)^3} \log \left( 1 - \frac{\phi^* \phi}{k^2} \right).$$

(14)
$V_{\text{eff}}$ is linearly divergent; we thus impose a cutoff $\Lambda$. Although $V_{\text{eff}}$ itself is not very illuminating, the gap equation involving $v_0 \equiv \langle |\phi| \rangle$,

$$\frac{1}{N} \frac{\delta V_{\text{eff}}}{\delta v_0} = 0 = 2\lambda v_0 - \frac{v_0}{2\pi} \left( \sqrt{\Lambda^2 + v_0^2} - v_0 \right),$$

(15)
yields a non-trivial VEV for $v_0$ if $\lambda$ is less than a critical value $\lambda_c \equiv \Lambda/4\pi$. Notice that the four-Fermi coupling is $\sim 1/\lambda$ and hence $\lambda < \lambda_c$ is a strong coupling régime. This result has to be contrasted with the BCS model where the gap arises in the weak coupling phase. The origin of this difference lies in the behavior of the density of states. In case of nonrelativistic fermions the density of states is taken to be constant near the Fermi surface whereas in the relativistic model presented here the density of states $g(\epsilon) = \frac{\epsilon}{4\pi}$ goes to zero in the $\epsilon \to 0$ limit.

There are two interesting limits one can consider for the solutions of the gap equation. The first limit $\lambda < \lambda_c$ is more physical since the mean-field approach is generally sensible near the critical point. Here, $v_0 \simeq 4\pi(\lambda_c - \lambda)$; obtaining this solution we have assumed that $v_0 \ll \Lambda$, and the solution indeed shows that this is self-consistent; consequently, the mass gap in the fermion spectrum is also small. Here, $V_{\text{eff}}$ can also be written compactly as

$$V_{\text{eff}} = N (\lambda - \lambda_c) \phi^* \phi + \frac{N}{6\pi} (\phi^* \phi)^{3/2},$$

(16)
from which it is evident that $\lambda_c$ is indeed the critical value of $\lambda$

The second limit corresponds to the very strong coupling régime, $\lambda \ll \lambda_c$; one gets from the solution of the gap equation $v_0 \approx 2\pi \lambda_c^2/\lambda$. Here, the mass gap is large and the fermion mass plays the role of the cut-off.

In what follows, we will confine ourselves to the first régime in the spirit of the mean-field approach.

It is worth mentioning at this point that the presence of the linear ultraviolet divergence, although troubling at first sight, can be handled in large-$N$ perturbation theory, where these four-Fermi couplings are renormalizable. This cannot be done in standard perturbation theory as seen from a simple power counting argument. In the following section we will give the renormalization prescription once we have the two-loop results.

We now analyse the temperature corrections to the effective potential in order to determine the nature of the phase transition. In the imaginary-time formalism, finite temperature is achieved by making field configurations periodic or antiperiodic in time for bosons or fermions respectively, with period $\beta = 1/T$ (with the Boltzman constant put to one). In momentum space this amounts to replacing $-i \int d^3 k/(2\pi)^3$ by $\beta^{-1} \sum_n \int d^2 k/(2\pi)^2$, where $k^0 = 2\pi i \beta^{-1} (n + \frac{1}{2})$ for fermions. Finite density is taken into account by adding a chemical potential $\mu$ to $k^0$. The effective potential at finite temperature and zero chemical potential is then

$$\frac{1}{N} V_{\text{eff}}(\phi, \phi^*; T) = \lambda \phi^* \phi - \frac{1}{\beta} \sum_{-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \log \left( 1 + \frac{\phi^* \phi}{k^2 + \frac{4\pi^2}{\beta^2} (n + \frac{1}{2})^2} \right).$$

(17)
Like the zero temperature case, we analyse the gap equation for a non-trivial constant solution $v_T \equiv \langle |\phi| \rangle$:

$$
\lambda v_T - \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\sqrt{k^2 + v_T^2}} \frac{1}{\cosh(\beta \sqrt{k^2 + v_T^2})} = 0.
$$

(18)

After carrying out the sum over frequencies, we get

$$
v_T \left( \lambda - \int \frac{d^2 k}{(2\pi)^2} \frac{1}{\sqrt{k^2 + v_T^2}} \sinh(\beta \sqrt{k^2 + v_T^2}) \cosh(\beta \sqrt{k^2 + v_T^2}) + 1 \right) = 0.
$$

(19)

Introducing the cutoff $\Lambda$ to regulate the linear divergence as for the $T = 0$ case (finite temperature effects do not introduce any new ultraviolet divergence), the non-trivial solution $v_T \neq 0$ satisfies the equation

$$
\lambda - \frac{\Lambda}{4\pi} + \frac{1}{4\pi \beta} \log [2(\cosh(\beta v_T) + 1)] = 0.
$$

At $T = 0$, we get $v_0 = 4\pi(\lambda_c - \lambda)$, with $\lambda_c = \Lambda/4\pi$, which matches with our zero-temperature calculation. After simplification, the finite-temperature gap equation is

$$
\frac{1}{\beta} \log [2(\cosh(\beta v_T) + 1)] = v_0.
$$

(20)

The critical temperature $T_0$ is defined to be the point where $v_T = 0$, yielding

$$
T_0 = \frac{v_0}{2 \log 2}.
$$

(21)

In general, allowing for a finite chemical potential leads to a critical line in the $\mu - T$ plane; here the simplified gap equation generalizes to

$$
\frac{1}{\beta} \log [2(\cosh(\beta v_T) + \cosh(\beta \mu))] = v_0.
$$

(22)

The critical line in the $\mu - T$ plane is derived from (22) by putting $v_T = 0$, and subsequently solving for $\mu$ as a function of temperature. The resulting phase diagram is displayed in Fig. 2.

A physically relevant quantity is the ratio of the gap parameter (at zero temperature) to the critical temperature. From the above, we have $2v_0/T_0 = 4 \log 2 \simeq 2.77$, which is less than the BCS gap-to-transition temperature ratio of 3.52. However, as will be shown in Sec. IV, $T_0$ is not the superconducting transition temperature: that temperature will be lower than $T_0$ and hence the gap-to-transition temperature ratio will turn out to be larger than $2v_0/T_0$. Indeed, it will turn out to be larger than the BCS value for $N \lesssim 22$.

The neutral order parameter approaches zero smoothly, with critical exponent $\beta = 1/2$ (this is precisely the mean-field value), so this is a second-order phase transition.
We proceed to compute the kinetic term of the effective action. If we are to have a sensible Ginzburg-Landau theory for the condensate, the two-derivative term must have a positive coefficient. The computation is very similar to that of the effective potential. Here we present the zero-temperature case; at finite temperature, the calculation is rendered somewhat more complicated by the fact that Lorentz invariance is lost; this case is discussed in the Appendix.

For the kinetic term, it is clearly Green’s functions with two nonvanishing external momenta which contribute. In real space these terms can be arranged as in (11) to exhibit the charge conjugation symmetry of the original theory. (This arrangement also makes it easier to take into account gauge interactions by minimal substitution: notice that the last term is already gauge covariant.) To calculate the constant coefficients $Z_1$ and $Z_2$, we compute the following derivatives of the two-derivative part of (11):

$$\frac{\delta^2 S^{\text{kin}}}{\delta \phi(x) \delta \phi^*(0)} \bigg|_{\phi=\phi^*=v_0} = -(Z_1 + 2Z_2 v_0^2) \partial_\mu \partial^\mu \delta^3(x), \quad (23a)$$

and

$$\frac{\delta^2 S^{\text{kin}}}{\delta \phi(x) \delta \phi(0)} \bigg|_{\phi=\phi^*=v_0} = -2Z_2 v_0^2 \partial_\mu \partial^\mu \delta^3(x). \quad (23b)$$

Hence,

$$-Z_1 \partial_\mu \partial^\mu \delta(x) = \left( \frac{\delta^2 S^{\text{kin}}}{\delta \phi(x) \delta \phi^*(0)} - \frac{\delta^2 S^{\text{kin}}}{\delta \phi(x) \delta \phi(0)} \right),$$

$$-Z_2 \partial_\mu \partial^\mu \delta(x) = \frac{1}{2v_0^2} \frac{\delta^2 S}{\delta \phi(x) \delta \phi(0)}. \quad (24)$$

In momentum space, the above expressions become

$$Z_1 = \frac{1}{6} \frac{\partial^2}{\partial q^\mu \partial q_\mu} \left( D^{-1}_{\phi^*}(q) - D^{-1}_{\phi}(q) \right)_{q=0},$$

$$Z_2 = \frac{1}{12v_0^2} \frac{\partial^2}{\partial q^\mu \partial q_\mu} \left( D^{-1}_{\phi^*}(q) \right)_{q=0}. \quad (25)$$

Here, the $D_{i,j}$’s are the inverse propagators for the respective fields, which can be explicitly calculated. Consider first $D_{\phi^*}^{-1}$. It gets contributions from all one-loop diagrams where $\phi$ and $\phi^*$ carry momenta $p$ and $-p$, respectively; analytically, we have

$$\frac{N}{2} \tilde{\phi}^*(q) \tilde{\phi}(q) \sum_n v_0^{2n} \int \frac{d^3k}{(2\pi)^3} \sum_{j=0}^n \frac{1}{(k+q)^{2j+1}} \frac{1}{k^{2n+1-2j}}
= N \tilde{\phi}^*(q) \tilde{\phi}(q) \int \frac{d^3k}{(2\pi)^3} \frac{k(k-q)}{(k^2-v_0^2)((k-q)^2-v_0^2)}
= i\tilde{\phi}^*(q) \phi(-q)D_{\phi^*}^{-1}(q).$$
The above series is depicted in Fig. 3.

One finds
\[
\frac{1}{N} D^{-1}_{\phi^*}(p) = \frac{\Lambda}{4\pi} - \frac{v_0}{2\pi} - \frac{-p^2 + 2v_0^2}{4\pi \sqrt{-p^2}} \arctan \frac{\sqrt{-p^2}}{2v_0}.
\]

Hence,
\[
\frac{\partial^2}{\partial q_\mu \partial q_\mu} D^{-1}_{\phi^*}(q) \bigg|_{q=0} = \frac{5N}{16\pi v_0}
\]

A similar calculation yields
\[
\frac{\partial^2}{\partial q_\mu \partial q_\mu} D^{-1}_{\phi}(q) \bigg|_{q=0} = -\frac{N}{16\pi v_0};
\]

whence
\[
Z_1 = \frac{N}{16\pi v_0},
Z_2 = -\frac{N}{192\pi v_0^3}.
\]

Incorporation of electromagnetism is trivial; since we know \( \phi \) has charge \( 2e \), the kinetic term must be modified to an appropriate covariant derivative; furthermore, the last term in (11) is already gauge-covariant. Thus, we get the zero-temperature effective Lagrangian
\[
\mathcal{L}_{\text{eff}} = \frac{N}{16\pi v_0} |(\partial_\mu + 2ie A_\mu)\phi|^2 - \frac{N}{192\pi v_0^3}(\phi^* \partial_\mu \phi + \phi \partial_\mu \phi^*)(\phi^* \partial_\mu \phi + \phi \partial_\mu \phi^*) - V_{\text{eff}}.
\]

At this point it is worth asking if a Chern-Simons term can be present in our effective action, thereby leading to parity and time-reversal violating effects. This term could arise from one-loop diagrams with two external photons carrying momentum (in addition to any number of scalar fields at zero momentum). However it is easy to see that such diagrams can never generate a Chern-Simons term. This is essentially due to the fact that there must be an even number of external scalar legs (to preserve gauge invariance), and therefore an even number of fermion propagators. Since the fermion is massless there results a trace of an even number of propagators; however with two-component spinors a Chern-Simons term would only arise from the trace of three gamma matrices (proportional to \( \epsilon_{ijk} \)). Thus, there is no induced Chern-Simons term and no spontaneous parity or time-reversal violation, in contrast with the simplest anyon superconductivity models\(^{46}\) and in accord with experiments on high-temperature superconductors\(^{47,48}\).

Once we have the full effective action, the expectation value of the charged order parameter can be computed at finite temperatures. Writing \( \phi = \rho e^{i\theta} \), it will be sufficient to analyse the VEV of \( \phi \) when fluctuations of \( \rho \) can be neglected. Hence,
\[
\langle \phi(x) \rangle = \langle \rho e^{i\theta(x)} \rangle \simeq \nu_T e^{-\frac{1}{2} \langle \theta^2(x) \rangle}.
\]

11
Going over to momentum space and expanding \( \theta(x, \tau) = \sum_n \theta_n(x) e^{i\omega_n \tau} \), where \( \omega_n = 2\pi n/\beta \), we have

\[
\langle \theta^2(x) \rangle = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \int \frac{d^2q}{(2\pi)^2} \langle \theta_n(q) \theta_{-n}(-q) \rangle
= \frac{1}{\beta} \sum_n \int \frac{d^2q}{(2\pi)^2} D_\theta(q, n). \tag{29}
\]

The propagator, whose derivation can be found in the Appendix, is

\[
D_\theta(q, n) = \frac{8\pi}{N v_T \tanh(\beta v_T/2)} \left( q^2 + \left( 1 - \frac{\beta v_T}{\sinh(\beta v_T)} \right) (2n\pi/\beta)^2 \right). \tag{30}
\]

Substituting this into (29), the sum can be performed, yielding

\[
\langle \theta^2 \rangle = \frac{2}{N v_T \left( 1 - \frac{\beta v_T}{\sinh(\beta v_T)} \right)^{1/2} \tanh(\beta v_T/2)} \int_0^\infty dq \coth \left( \frac{\beta q}{2} \left( 1 - \frac{\beta v_T}{\sinh(\beta v_T)} \right)^{1/2} \right).
\]

The resulting integral over \( q = |q| \) diverges in both the ultraviolet and infrared; introducing respective cutoffs \( \Lambda \) and \( \eta \), we get

\[
\langle \theta^2 \rangle = \frac{2}{N v_T \left( 1 - \frac{\beta v_T}{\sinh(\beta v_T)} \right)^{1/2} \tanh(\beta v_T/2)} \left( \Lambda - 2 \frac{\beta}{\beta} \left( 1 - \frac{\beta v_T}{\sinh(\beta v_T)} \right)^{1/2} \log \beta \eta \right),
\]

up to finite parts.

We can eliminate the above ultraviolet divergence (a next to leading order effect) by suitably renormalizing \( \rho \), appearing in (28). To show how this happens, it is sufficient to work at zero temperature (since finite temperature effects do not introduce any new ultraviolet divergence). The VEV of \( \phi \), then, reduces to

\[
\langle \phi \rangle = v_0 e^{-\Lambda \frac{N}{v_0}}
\]

which up to order \( 1/N \) becomes

\[
\langle \phi \rangle \approx v_0 - \frac{\Lambda}{N}. \tag{31}
\]

Now recall that at the leading order \( v_0 = 4\pi(\lambda_c - \lambda) \) with \( \lambda_c = \frac{\Lambda}{4\pi} \). After integrating out the \( \theta \) degree of freedom, as we have done in (28), we expect the critical coupling to get shifted, the shift being of order \( 1/N \) in the next-to-leading order. As will be shown explicitly in the following section, the new critical coupling up to order \( 1/N \) is \( \lambda_c^{(1)} = (1 - \frac{1}{N}) \frac{\Lambda}{4\pi} \) and the new VEV is \( \langle \phi \rangle = 4\pi(\lambda_c^{(1)} - \lambda) \), which precisely agrees with the expression in (31). This justifies our assertion that the ultraviolet divergence can be soaked up in a redifined \( \rho \).

It is not possible, however, to get rid of the infrared divergence, the dependance of \( \langle \phi \rangle \) on the infrared cutoff \( \eta \) is:

\[
\langle \phi \rangle = \text{const}(\beta \eta)^\chi,
\]

with \( \chi = 2/N(\beta v_T \tanh(\beta v_T/2)) \). We see that in the limit \( \eta \to 0 \), \( \langle \phi \rangle \to 0 \), since \( \chi \) is positive. This proves that \( \theta \), the massless excitation associated with the Goldstone mode, is responsible for the vanishing of the VEV of the charged order parameter \( \phi \) at non-zero temperatures.
In this section, we compute the next-to-leading order corrections to ensure that the quantum fluctuations do not destabilize the vacuum, either due to ultraviolet or infrared effects. Furthermore, we show explicitly the renormalizability of the theory and give a renormalization group invariant expression for the gap. The approach used for this purpose is the same as the one advocated in Ref. 3, except that we have an additional massless field. Interestingly, the analysis of the three point vertex shows that it is not modified in the next-to-leading order. This is similar to the lack of radiative corrections to the electron-phonon vertex at low-energy, a result known as Migdal’s theorem in the literature.

For calculational convenience, we use a reducible basis of $\gamma$-matrices, and a Majorana basis for the spinors. A common convention for the $2 \times 2$ Dirac $\gamma$-matrices is

$$\gamma^0 = \sigma^2, \gamma^1 = i \sigma^3, \gamma^2 = i \sigma^1,$$

where $\sigma^i$'s are the Pauli matrices. We can write a complex spinor in terms of two real ones as

$$\psi = \chi_1 + i \chi_2.$$

With $\phi = \phi_2 + i \phi_1$ and $\psi^c \equiv C \psi^T = \chi_1 - i \chi_2$, the fermionic part of the linearized Lagrangian in equation (9) (restricting for simplicity to one flavour for the moment) can be written as:

$$L = \bar{\psi} \partial^\mu \psi - \phi_1 \bar{\psi} \psi - i \phi_2 \bar{\psi} \gamma_5 \psi - \lambda N (\phi_1^2 + \phi_2^2) - e \bar{\psi} A_5 \gamma_5 \psi.$$

Here, we have used a reducible basis involving $4 \times 4$ $\gamma$-matrices

$$\tilde{\gamma}_\mu = \gamma_\mu \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the four-component spinor $\Psi = (\chi_1 \chi_2)^T$. Furthermore, the $\gamma_5$ matrix has been defined as:

$$-i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this formulation, the method of Refs. 25,45 can conveniently be used to calculate the derivative expansion of the effective action. For instance, using the fact that $\{\gamma_5, \gamma_\mu\} = 0$, and that $\gamma_5^2 = 1$, we obtain, assuming that the $\phi^i$'s are constant,

$$V_{\text{eff}} = V^0 + \frac{iN}{2} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \log(p^2 + \phi_1 - i \phi_2 \gamma_5)$$

$$= N \lambda \phi^* \phi + iN \int \frac{d^3p}{(2\pi)^3} \log(p^2 - \phi^* \phi). \quad (34)$$

This expression is equivalent to the one derived in (14) modulo a $\phi$ independent term, but appears rather easily in this approach. We have checked for consistency that the full covariant action of the previous section follows from this alternative method.

To show renormalizability, the bare Lagrangian can be written (neglecting the external gauge field) with three substraction constants in the form:

$$L^{(\text{bare})} = Z_1 \bar{\Psi} i \phi \Psi - Z_2 \phi_1 \bar{\Psi} \Psi - iZ_2 \phi_2 \bar{\Psi} \gamma_5 \Psi - \lambda N Z_3 \left( \frac{Z_2}{Z_1} \right)^2 (\phi_1^2 + \phi_2^2). \quad (35)$$
Here, $Z_1$, $Z_2$ and $Z_3$ are overall normalizations of the fields and will drop out of the S-matrix.

Henceforth, we will work in Euclidean space for convenience; the Wick-rotated functional with the fermionic sources $\bar{\eta}$ and $\eta$ reads:

$$Z(\bar{\eta}, \eta) = \int D\bar{\Psi} D\Psi D\phi_1 D\phi_2 e^{-\int d^3 x (\mathcal{L}^{(E)} - \bar{\eta}\Psi - \Psi\eta)},$$

where

$$\mathcal{L}^{(E)} = Z_1 \bar{\Psi} i\gamma^0 (E) \Psi + Z_2 \phi_1 \bar{\Psi} \Psi + i Z_3 \phi_2 \bar{\Psi} \gamma_5 \Psi + \lambda N Z_3 \left( \frac{Z_2}{Z_1} \right)^2 (\phi_1^2 + \phi_2^2).$$

(36)

(37)

Without loss of generality, we can assume that $\phi_1$ takes a VEV and hence write

$$\phi_1 = \frac{Z_1}{Z_2} (M + \tilde{\phi}_1),$$

$$\phi_2 = \frac{Z_1}{Z_2} \tilde{\phi}_2;$$

(38)

where $M = M_0 + M_1/N$ and $Z_i = 1 + \hat{Z}_i/N$. Since we will be computing corrections up to order $1/N$, the parameters have been conveniently expanded in the same powers. The variables $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are the fluctuations in the respective fields.

The Feynman rules can be read off easily from the Lagrangian. Notice that to next-to-leading order we have the fermion mass insertion $-M_1/N$, given diagrammatically by $i-|j$ and the two tadpoles given by $-2\lambda N M_0$ and $-2\lambda \hat{Z}_3 (M_0 + M_1)$ arising respectively due to the second and last term of (37). For the proof of renormalizability, it is sufficient to show that the full VEV of $\phi_1$ and that the connected Greens functions $\langle \Psi(x)\bar{\Psi}(y) \rangle$, $\langle \phi_i(x)\phi_i(y) \rangle, i = 1, 2$, are finite.

In the momentum space, the inverse fermionic two point function is

$$\Gamma^{ij}(p) = Z_1 \delta^{ij} \Gamma(p),$$

where

$$\Gamma(p) = i\gamma^0 + M_0 + \frac{M_1}{N} - \frac{1}{N}$$

$$- \frac{1}{N}$$

(39)

The dashed and the dotted lines in the above expression correspond to the $\tilde{\phi}_1$ and $\tilde{\phi}_2$ propagators respectively.
The inverse mesonic two-point function is:

\[ = N \left( \frac{Z_2}{Z_1} \right)^2 D_1^{-1}(p) + 2\lambda \hat{Z}_3 + 2 \]

\[ + 2 \]

\[ + 2 \]  

(40)

The vertex function is:

\[ = -Z_2 \left( 1 + \frac{1}{N} \right) \]

\[ + \frac{1}{N} \quad + \text{finite diagrams} \]  

(41)

To leading order, the gap equation (9) can be easily derived by demanding that the sum of the tadpoles give zero contribution. To next-to-leading order, the tadpole contributions diagrammatically represented as:

gives

\[ \frac{M_1}{D_1(0)} = -2\lambda \hat{Z}_3 M_0 - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} D_\alpha(q; M) \frac{\partial D_\alpha^{-1}(q; M)}{\partial M} \bigg|_{M=M_0} . \]  

(42)

Here, \( D_1^{-1}(q; M) \) and \( D_2^{-1}(q; M) \) are the inverse propagators for the \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) fields respectively and are given by:

\[ D_1^{-1}(q, M) = 2\lambda + \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \frac{1}{(i\gamma_\mu + M)} \frac{1}{(i\gamma_\nu - iq + M)} \right] \]

\[ = -\frac{M_0}{\pi} + \frac{M}{\pi} + \frac{(q^2 + 4M^2)}{2q\pi} \arctan \frac{q}{2M} . \]  

(43)

and

\[ D_2^{-1}(q, M) = 2\lambda - \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \frac{1}{(i\gamma_\mu + M)\gamma_5} \frac{1}{(i\gamma_\nu - iq + M)\gamma_5} \right] \]

\[ = -\frac{M_0}{\pi} + \frac{M}{\pi} + \frac{q}{2\pi} \arctan \frac{q}{2M} . \]  

(44)
Note that the ultraviolet behaviour of these propagators is given by:

$$D_{1,2}(q)|_{\lim q \to \infty} \sim \frac{4}{q}.$$  \hfill (45)

A straightforward calculation then yields:

$$M_1 = -2\lambda \hat{Z}_3\pi - \frac{2\Lambda}{\pi} - \frac{4M_0}{\pi^2}\log(\Lambda) + \text{finite terms}.$$  \hfill (46)

Now we consider the fermion self-energy correction due to the $\tilde{\phi}_1$ exchange; using the large $q$ behaviour of the $\tilde{\phi}_1$ propagator the divergent part can be identified:

$$\equiv \Sigma_1(p) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{i\not{q} - i\not{q} + M} D_{1}(q)$$

$$\equiv a(0) + i\tilde{a}(0)p + \text{finite terms}.$$  \hfill (47)

Taking the trace yields

$$a(0) = M_0 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} \frac{4}{q} = \frac{2M_0}{\pi^2}\log(\Lambda).$$  \hfill (48)

Similarly, after some algebra,

$$\tilde{a}(0) = -\lim_{p \to 0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{p^2} \frac{p \cdot (p - q)}{(p - q)^2 + M_0^2}$$

$$= -\frac{2}{3\pi^2}\log(\Lambda).$$  \hfill (49)

Writing the contribution of the $\tilde{\phi}_2$ field to the fermion self-energy as $\Sigma_2(p) = b(0) + i\tilde{b}(0)p + \text{finite terms}$, we obtain $b(0) = -(2M_0/\pi^2)\log(\Lambda)$ and $\tilde{b}(0) = -(2/3\pi^2)\log(\Lambda)$. The divergent part of the $\phi_1$ and $\phi_2$ vertex corrections are respectively given by $c(0) = (2/\pi^2)\log(\Lambda)$ and $d(0) = -(2/\pi^2)\log(\Lambda)$. Hence, the renormalization condition reads

$$\hat{Z}_1 - \tilde{a}(0) - \tilde{b}(0) = 0,$$

$$\hat{Z}_1M_0 + M_1 - a(0) - b(0) = 0,$$

$$\hat{Z}_2 + c(0) + d(0) = 0.$$  \hfill (50)

These conditions give

$$\hat{Z}_1 = -\frac{4}{3\pi^2}\log(\Lambda),$$

$$\hat{Z}_2 = 0,$$

$$M_1 = \frac{4}{3\pi^2}M_0\log(\Lambda).$$  \hfill (51)
By using the gap equation for $M_1$, we then obtain

$$2 \lambda \hat{Z}_3 = -\frac{2 \Lambda}{\pi^2} - \frac{16 M_0}{3 \pi^3} \log(\Lambda)$$  \hspace{1cm} (52)

and hence

$$2 \lambda Z_3 = \frac{2 \Lambda}{\pi^2} (1 - \frac{1}{N}) - \frac{M_0}{\pi} - \frac{16 M_0}{3 \pi^3 N} \log(\Lambda).$$

At this point, with all the substraction constants properly defined, the consistency of the renormalization procedure can be checked by computing the inverse of the connected, mesonic two-point function and showing that it is finite, which we now do.

The zero-momentum inverse two-point function is

$$|_{p=0} = N \left( \frac{Z_2}{Z_3} \right)^2 \left( 2 \lambda + \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(i\psi + M_0)^2} \right) + 2 \lambda \hat{Z}_3 - 2 M_1 \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(i\psi + M_0)^3}$$

$$+ \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left[ \partial^\alpha D^\alpha - \frac{M_{\text{phys}}}{M} \partial M \frac{\partial D^\alpha}{\partial M} + D^\alpha \frac{\partial^2 D^\alpha}{\partial M^2} \right] |_{M=M_0}$$  \hspace{1cm} (53)

where $\alpha = 1, 2$.

A rather lengthy calculation yields

$$|_{p=0} = \frac{2 M_0}{\pi} (\hat{Z}_2 - \hat{Z}_1) + 2 \lambda \hat{Z}_3 + \frac{2 M_1}{\pi} + \frac{2 \lambda}{\pi^2} + \text{finite terms}.$$  \hspace{1cm} (54)

By using the previously calculated substraction constants $\hat{Z}_1$, $\hat{Z}_2$ and $\hat{Z}_3$ the divergent part of the above two-point function is zero leaving only the finite parts.

Introducing an arbitrary substraction point $\mu$, we get

$$Z_1 = 1 - \frac{4}{3 \pi^2 N} \log \left( \frac{\Lambda}{\mu} \right)$$

$$Z_2 = 1$$

$$2 \lambda Z_3 = \frac{2 \Lambda}{\pi^2} (1 - \frac{1}{N}) - \frac{M_0}{\pi} - \frac{16 M_0}{3 \pi^3 N} \log \left( \frac{\Lambda}{\mu} \right).$$  \hspace{1cm} (55)

By demanding that the theory not depend on the scale $\mu$, we are led to consider $M = M(\mu)$. In particular, since $2 \lambda Z_3$ should be independent of $\mu$, we get

$$\mu \frac{\partial M}{\partial \mu} + \frac{16}{3 \pi^2 N} \left( \mu \frac{\partial M}{\partial \mu} \log \left( \frac{\Lambda}{\mu} \right) - M \right) = 0,$$  \hspace{1cm} (56)

yielding $M(\mu) \sim \mu^{16/3 \pi^2 N}$. By substituting this expression in $2 \lambda Z_3$, we get,

$$\frac{2 \lambda Z_3}{A} = \frac{2}{\pi^2} (1 - \frac{1}{N}) - \frac{1}{\pi} \left( \frac{M_{\text{phys}}}{A} \right)^{1-16/3 \pi^2 N}.$$  \hspace{1cm} (57)

With $\tilde{\lambda} \equiv 2 \lambda Z_3/A$ and $\tilde{\lambda}_c \equiv \frac{2}{\pi^2} (1 - 1/N)$, the $\beta$-function can be easily computed:

$$\beta_{\tilde{\lambda}} \equiv -\frac{d \tilde{\lambda}}{d \log(A)} = (1 - \frac{16}{3 \pi^2 N}) (\tilde{\lambda} - \tilde{\lambda}_c).$$  \hspace{1cm} (58)
IV. SUPERCONDUCTIVITY

To analyse the occurrence of superconductivity in the model presented above, we study the low-energy effective action using a duality transformation.\textsuperscript{28,50} This is necessitated by the fact that, on a plane, the phase of a complex order parameter is multivalued and gives rise to so-called vortex excitations; a naive calculation fails to capture the contribution of the vortices. As will be shown below, the description in terms of the dual variables allows the effective separation of the single and the multivalued components of the complex field, and hence is quite useful in identifying the two phases: one where vortices and anti-vortices are tightly bound to one another, and the other where they are free. The phase transition between these two régimes is the celebrated Kosterlitz-Thouless transition.\textsuperscript{21}

In this section we will show that in the low-temperature régime, where the vortices occur in tightly bound pairs, the relevant contribution to the path integral comes only from the topologically trivial piece, leading to a ground state which exhibits superconductivity. The photon develops a mass in this phase and the current-current correlation function shows a massless pole, indicative of zero resistance. When the vortices are free, their contribution to the path integral will be seen to precisely negate the contribution of the topologically trivial part, destroying superconductivity. This then proves the earlier assertion that $T_{KT}$ is indeed the superconducting critical temperature.

We wish to analyse the long-wavelength London limit, neglecting fluctuations of $|\phi|$. We therefore write $\phi = \rho e^{i\theta}$, with $\rho$ fixed at $v_T$. As was shown in Refs. 28,50, a direct substitution of this change of variables into the Lagrangian (11) is inappropriate, since the correspondence between $\phi$ and $\theta$ is not one-to-one. Rather, to capture the physics of vortex excitations, we make the replacement $\partial_\mu \theta \rightarrow \partial_\mu \theta - i\phi^* \partial_\mu \varphi$ with $\phi^* \varphi = 1$. The new variable $\theta$ is now treated as a single-valued field and $\varphi$ describes the vortex dynamics. One can define a vortex current by

$$J_{\mu}^{\text{vort}} = (2\pi i)^{-1} \epsilon_{\mu\nu\lambda} \partial^\nu (\varphi^* \partial^\lambda \varphi).$$

This current is identically conserved, and the corresponding integrated charge gives the vortex number.

The dynamics is described by the finite-temperature version of the Lagrangian (27), where the finite-temperature effective potential determines the value of $v_T$ (see (20)), and the first term of (27) determines the Lagrangian for $\theta$, $\phi$. (As remarked earlier, the second term is independent of these variables.) The coefficient of the first term at finite temperature is computed in the Appendix. For static properties we can replace $Z_1$ by its spatial finite-temperature counterpart $Z_1^{(1)}$ given in (A.6b), so that the Lagrangian becomes

$$\mathcal{L}_{\text{eff}} = \frac{N \tanh(\beta v_T/2)}{16\pi v_T} \frac{e^2}{v_T^2} \left( \partial^\mu \theta - i\varphi^* \partial^\mu \varphi + 2eA^\mu \right)^2.$$

It is useful to rewrite this in terms of an auxiliary current $J^\mu = \partial^\mu \theta - i\varphi^* \partial^\mu \varphi + 2eA^\mu$:

$$\frac{1}{N} \mathcal{L}_{\text{eff}} = -\frac{1}{2K} J^\mu J_\mu - J_\mu \left( \partial^\mu \theta - i\varphi^* \partial^\mu \varphi + 2eA^\mu \right),$$

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where $\ K = v_T \tanh(\beta v_T/2)/8\pi$. Since $\theta$ is now single-valued, we can integrate it out, obtaining the constraint $\partial^\mu J_\mu = 0$; this is readily solved by

$$J_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu a^\lambda;$$  \hspace{1cm} (61)

here $a_\mu(x)$ is a vector field which is defined globally. In terms of this auxiliary gauge field $a_\mu$ the new Lagrangian, after an integration by parts, becomes:

$$\frac{1}{N} \mathcal{L}_{\text{eff}} = -\frac{1}{4K} f_{\mu\nu}^2 - 2e\epsilon_{\mu\nu\lambda} a^\mu \partial^\nu A^\lambda - 2\pi a^\mu J^{\text{vort}}_\mu,$$  \hspace{1cm} (62)

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Notice that the above description of $J_\mu$ in terms of the gauge field $a_\mu$ brings in an unphysical gauge degree of freedom, which must be removed by gauge fixing.

The integration over $a_\mu$ can now be performed, and the result, in the Landau gauge, is

$$\mathcal{L}_{\text{eff}} = -2\pi^2 NK J^{\text{vort}}_\mu(x) \frac{g_{\mu\nu}}{\partial^2} J^{\text{vort}}_\nu(x) - 4\pi e NK J^{\text{vort}}_\mu g_{\mu\nu} \epsilon_{\nu\lambda\rho} \partial^\lambda A^\rho - e^2 NK F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu}. $$  \hspace{1cm} (63)

The last term is the contribution of the topologically trivial piece and is relevant for superconductivity. Explicitly, neglecting for the moment the second term, which describes the interaction of the electromagnetic field with the vortex, the current-current correlation function is:

$$\delta^2 S_{\text{eff}} / \delta A_\mu(x) \delta A_\nu(y) = \langle j^\mu(x) j^\nu(y) \rangle = -4e^2 NK \left( \frac{\partial^\mu \partial^\nu}{\partial^2} - g^{\mu\nu} \right) \delta^3(x-y).$$  \hspace{1cm} (64)

This indicates a pole at zero momentum, and hence superconductivity. To see the Meissner effect, namely the presence of a pole in the photon propagator at nonzero momentum, we have to include the Maxwell term. With the addition of $\mathcal{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, the photon propagator does indeed have such a pole, corresponding to a photon mass of $M^2_v = \frac{e^2 N v_0}{2\pi}$. Since the scalar mass near the critical point is given by $M^2_s = 8v_0^2$, the ratio $M_s/M_v = \frac{1}{\pi} \sqrt{\frac{16\pi v_0}{N}}$ can give information about the nature of superconductivity (type I or type II). However, without knowledge of the dimensionful coupling constants $\lambda$ and $e$ (which in principle are determined by the underlying model), we are not able to draw any meaningful conclusion.

Now we proceed to analyse the crucial question: When can one neglect the vortex contribution? As we will now show, at low temperatures, when the vortices are confined, their contribution to the $F_{\mu\nu} \partial^2 F_{\mu\nu}$ term is zero and hence they do not affect superconductivity. Physically, this is perfectly reasonable: assuming that the interaction of the vortices with the electromagnetic field is small, in the confined phase a slowly varying magnetic field sees pairs of tightly bound vortices with opposite vortex charges, and hence the net contribution from each pair vanishes, leaving untouched the contribution of the single-valued part. In contrast, above the KT phase transition the vortex contribution to
this term is crucial: it exactly cancels the contribution from the single-valued part, thereby destroying superconductivity.

To see this more explicitly, consider for simplicity only the static vortex density configurations ($J_v^0 = 0$ and $\partial_0 J_v^0 = 0$), and let us compute the vortex contribution to the current-current correlation function. Knowing that the vortex current is conserved, we can write

$$J_v^0(x) \equiv \rho_v(x) = \sum_a m_a \delta(x - x_a(t))$$

where $m_a$ is the integer-valued vorticity and $x_a$ is the position of the $a^{th}$ vortex. And making use of the green function property in two dimensions

$$\frac{1}{\nabla^2} F(x) = \frac{1}{2\pi} \int dy \log|x - y|$$

the effective action reduces to

$$S_{\text{eff}}/N = \pi K_\beta \sum_{a,a'} m_a m_{a'} \log|x_a - x_{a'}| + 2eK_\beta \sum_a m_a \int d^2r B(x) \log|x - x_a|; \quad (66)$$

without the interaction term this is the action of the familiar XY model. In the static limit, the contribution of the vortices to the current-current correlation function is

$$\langle j^i(q)j^j(-q) \rangle_v = \frac{\delta^2 S_{\text{eff}}}{\delta A^i(q) \delta A^j(-q)} = \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right) q^2 \langle \rho_v(q) \rho_v(-q) \rangle.$$

In real space,

$$\rho_v(q) = \sum_a m_a e^{iq \cdot x_a}$$

and hence in the confined phase

$$\langle \rho_v(q) \rho_v(-q) \rangle = \sum_{a,a'} \langle e^{iq \cdot (x_a - x_{a'})} \rangle = -2\langle e^{iq \cdot R} \rangle + (m_a = 2, 3 \ldots). \quad (68)$$

The expectation value of $\langle e^{iq \cdot R} \rangle$ can be easily computed using the action for the XY model, yielding

$$\langle e^{iq \cdot R} \rangle \sim \frac{1}{q^{2-2P}}$$

and

$$\langle \rho_v(q) \rho_v(-q) \rangle \sim \frac{1}{q^{2-2P}}; \quad (69)$$

here $P = \pi NK\beta$. In the confined phase $P > 2$, $\langle \rho_v(q) \rho_v(-q) \rangle \sim q^{p-2}$, where $p > 4$. With (67), this implies that the zero momentum pole in $\langle j^i(q)j^j(-q) \rangle$ does not get a
contribution from the confined vortices. This then implies that in $\mathcal{L}_{\text{eff}}$ the $F^{\mu\nu} \partial^{-2} F_{\mu\nu}$ term and hence the Meissner effect is not affected by the vortex sector.

As an aside, it is perhaps worth pointing out that in position space

$$\langle \rho^{\text{vort}}(x) \rho^{\text{vort}}(y) \rangle \sim |x - y|^{-2P},$$

which is the well-known power law behaviour of the correlation functions in the confined phase.

In contrast, at temperatures above the vortex confinement-deconfinement transition, when the vortices are free and the interaction between them is screened as in a plasma, we expect a rather substantial contribution from that sector. In this phase directly integrating out the $J_\mu$ variable yields:

$$\mathcal{L}_{\text{eff\ vort}}^{\text{vort}} = N K e^2 F^{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu}$$  \hspace{1cm} (70)

which is precisely of opposite sign to that of the previously computed contribution of the single-valued sector (cf. (63)). This substantiates the physical argument given above that in the deconfined phase superconductivity is destroyed and $T_c = T_{KT}$. This is one of the main results of this paper.

We note that the same mechanism has been invoked in other models 26,27 to explain superconductivity. In these works the logarithmic confining potential was due to the exchange of massless gauge field, however the disappearance of the Meissner effect above $T_{KT}$ has not been shown explicitly. Our analysis may also be applicable to these models; this is currently under investigation.

The confining-deconfining phase transition in this model occurs at $\pi K N \beta = 2$, or,

$$\frac{\beta v_T}{2} \tanh \frac{\beta v_T}{2} = \frac{8}{N}.$$  \hspace{1cm} (71)

Here $v_T$, the expectation value of $|\phi|$ at finite temperature, is given by (20). Eqs. (20) and (71) can (in principle) be solved for the KT transition temperature $T_{KT}$, which as explained above is also the superconducting transition temperature. It is not difficult to see that for any $N$ there is a unique solution to (71); indeed, we display the left side of (71) as a function of inverse temperature in Fig. 4. Its intersection with the line $8/N$ gives the KT transition temperature. Clearly $T_{KT}$ increases with $N$; its minimum value is obviously $\beta_0$, the inverse of the critical temperature for the transition wherein $|\phi|$ obtains an expectation value.

We are now in a position to estimate the ratio of the gap parameter to the superconducting critical temperature, $2v_0/T_c = 2v_0/T_{KT}$, as a function of $N$. This ratio is a decreasing function of $N$, with minimum value $4 \log 2 \simeq 2.77$ as $N \to \infty$. For $N$ small (more precisely, for values of $N$ for which $8/N \gg 1$), the ratio tends to the asymptotic form $32/N$. (Since $N$ is integral, this asymptotic form is reasonable for $N \leq 3$.) In many physically interesting models 29 related to the Hubbard model, $N = 2$, in which case the above ratio is approximately 16, considerably larger than the BCS value of 3.52. In fact, it is found numerically that the ratio is larger than the BCS value for $N \lesssim 22$. 

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V. CONCLUSIONS

To conclude, we have presented a relativistic model in 2 + 1 dimensions which exhibits superconductivity up to a critical temperature $T_c$; unlike the anyonic theories, the ground state does not violate parity or time-reversal symmetries. Interestingly, the gap-to-$T_c$ ratio is higher than the conventional BCS value. It should be emphasized that relativistic field theories are not uncommon in condensed matter systems; quasi-particles satisfying linear dispersion relations are described by such theories. Prominent examples appear in magnetic systems, e.g., the $O(3)$ sigma model describes the low-energy modes of the antiferromagnetic spin system and Dirac fermions are the relevant excitations of the so-called flux-phase ground state. These models are widely believed to be relevant for high-$T_c$ materials because of their connection with the Hubbard model.

The KT mechanism of vortex binding played a crucial role in the occurrence of superconductivity in our model. These vortices arise due to the multivaluedness of the phase degree of freedom of the order parameter. We have explicitly demonstrated how the photon mass vanishes once the vortices unbind above some critical temperature. Concurrently, the pole in the current-current correlator disappears. We note that this analysis can also be applied to other models\cite{26,27,28,52} and to effective field theories describing the quantized Hall effect.\cite{50} Furthermore, we remark that the relevance of the KT transitions to quasi-two-dimensional systems has been discussed previously;\cite{53} such an extension to the model described here can also be carried out. Work along these lines is currently under progress and will be reported elsewhere.

Another crucial ingredient of our analysis is the large-$N$ perturbation theory; this technique has been extensively used in condensed matter physics. In fact it has been recently applied to the $O(3)$ sigma model leading to interesting results.\cite{54} It should be noted that interesting gap-to-$T_c$ ratio is reached for $N \lesssim 22$ in our model; it is amusing that multiplets of fermions appear in the low-energy dynamics of Hubbard and related models.\cite{29} Hence, the precise connection of the model elucidated here to that of the microscopic lattice Hamiltonians needs further scrutiny to establish the values of the parameters used and the relevant low-energy degrees of freedom.

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Here, we present the finite temperature propagator for \( \theta(x) \) when \(|\phi|\) takes a non-vanishing VEV. Since Lorentz invariance is lost when \( T \neq 0 \), the two derivative part of the effective action has to be separated into temporal and spatial parts; the coefficients are different at non-zero temperatures. The two derivative part can then be written as:

\[
\mathcal{L}^{\text{kin}} = Z_1^{(0)} \partial_0 \phi \partial_0 \phi^* - Z_1^{(1)} \nabla \phi \cdot \nabla \phi^* + Z_2^{(0)} (\phi^* \partial_0 \phi + \phi \partial_0 \phi^*)^2 - Z_2^{(1)} (\phi^* \nabla \phi + \phi \nabla \phi^*)^2 . \tag{A.1}
\]

For low-energy calculations we neglect the fluctuations in the magnitude of \( \phi \), writing \( \phi = (v_T) e^{i \theta} \). Then the last two terms drop out and the kinetic term becomes

\[
\mathcal{L}^{\text{kin}} = v_T^2 \left[Z_1^{(0)} (\partial_0 \theta)^2 - Z_1^{(1)} (\nabla \theta)^2 \right] . \tag{A.2}
\]

After performing an integration by parts:

\[
S(\theta) = \int d^3 x \frac{1}{2} \theta(x) D^{-1}_\theta(0) \theta(x) . \tag{A.3}
\]

Here, the inverse propagator for \( \theta(x) \) is given by:

\[
D^{-1}_\theta(q; \omega) = 2v_T^2 \left(Z_1^{(0)} \omega^2 + Z_1^{(1)} q^2 \right) . \tag{A.4}
\]

To calculate the \( Z_1^{(i)} \)'s and \( Z_2^{(i)} \)'s we proceed as for the zero temperature case. We notice that

\[
\frac{\delta^2 S}{\delta \phi(x) \delta \phi^*(0)} = \left(-Z_1^{(0)} \partial_0^2 + Z_1^{(1)} \nabla^2 - 2Z_2^{(0)} v_T^2 \partial_0^2 + 2Z_2^{(1)} v_T^2 \nabla^2 \right) \delta^3(x) . \tag{A.5a}
\]

Similarly,

\[
\frac{\delta^2 S}{\delta \phi(x) \delta \phi(0)} = \left(-2Z_2^{(0)} v_T^2 \partial_0^2 + 2Z_2^{(1)} v_T^2 \nabla^2 \right) \delta^3(x) . \tag{A.5b}
\]

Although we are in fact only interested in \( Z_1^{(i)} \), they are not isolated in (A.5a) and we must calculate \( Z_2^{(i)} \) as well.

A straightforward calculation gives

\[
Z_1^{(0)} = -N \frac{\beta v_T - \sinh(\beta v_T)}{32 \pi v_T \cosh^2(\beta v_T / 2)} = \frac{N}{16 \pi v_T} \left[ \tanh(\beta v_T / 2) - \frac{\beta v_T}{2 \cosh^2(\beta v_T / 2)} \right] , \tag{A.6a}
\]

and

\[
Z_1^{(1)} = \frac{N}{16 \pi v_T} \tanh(\beta v_T / 2) . \tag{A.6b}
\]
Similarly,

\[
Z_2^{(0)} = \frac{N}{192\pi v_T^3} \left( \frac{(\beta v_T)^2 \tanh(\beta v_T/2)}{\cosh^2(\beta v_T/2)} + \left( \frac{\beta v_T}{2} \frac{1}{\cosh^2(\beta v_T/2)} - \tanh(\beta v_T/2) \right) \right) \tag{A.7a}
\]

and

\[
Z_2^{(1)} = \frac{N}{192\pi v_T^3} \left[ \frac{\beta v_T}{2 \cosh^2(\beta v_T/2)} - \tanh(\beta v_T/2) \right]. \tag{A.7b}
\]

Using these expressions, we get

\[
D_\theta(q^2; \omega) = \frac{8\pi}{Nv_T \tanh(\beta v_T/2)} \frac{1}{\left( 1 - \frac{\beta v_T}{\sinh(\beta v_T)} \right) \omega^2 + q^2}. \tag{A.8}
\]

Identifying the frequency \( \omega \) with the Matsubara frequency \( \omega_n = 2\pi n/\beta \), we obtain (30), as desired.
FIGURE CAPTIONS

1. One-loop contribution to $V_{\text{eff}}$.
2. Critical line in the $\mu - T$ plane.
3. One-loop diagrams contributing to the two-derivative term in $S_{\text{eff}}$.
4. Graphical solution of Eq. (71).
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