We study in detail the evaluation of the gauge invariant overlap for analytic solutions constructed out of elements in the $K_{Bc}$ algebra in open string field theory. We compute this gauge invariant observable using analytical and numerical techniques based on the silver frame $L_0$ and traditional Virasoro $L_0$ level expansions of the solutions.
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1 Introduction

It is well-known that the analytic solutions for tachyon condensation [1, 2, 3, 4, 5, 6, 7] in open bosonic string field theory [8] can be formally given in terms of elements in the $KBc$ algebra [9, 10]. Once a solution $\Psi$ is given, the next step is to evaluate relevant physical gauge invariant quantities, like the energy and the gauge invariant overlap $\langle i|V(i)|\Psi\rangle$ discovered in [11, 12, 13]. As argued by Ellwood [14], the gauge invariant overlap represents the shift in the closed string tadpole of the solution relative to the perturbative vacuum. Moreover, using an appropriate zero momentum vertex operator $V$, defined in [15], it has been shown that the value of the energy can be obtained from the gauge invariant overlap.

The analytic computation of the gauge invariant overlap for Schnabl’s tachyon vacuum solution has been performed in [13]. Although the evaluation of this gauge invariant appears to be simpler than the energy, the computation presented in [13] was a bit cumbersome, and the reason for this subtlety was that the authors used a representation of the solution as given in Schnabl’s original work [1]. As we will see, the computation of the gauge invariant overlap can be enormously simplified if we express Schnabl’s solution in terms of elements in the $KBc$ algebra.

Concerning the numerical analysis of the gauge invariant overlap for analytic solutions within the $KBc$ algebra, in reference [13], using the traditional Virasoro $L_0$ level truncation scheme the authors have evaluated the gauge invariant overlap for the case of Schnabl’s original solution. Regarding the case of the so-called Erler-Schnabl’s solution, although the analytical computation of the gauge invariant overlap for this solution has been performed in [2], up to now, using the Virasoro $L_0$ level truncation scheme, the
analysis of the gauge invariant overlap for this type of solution was not performed. Moreover, the analysis of the gauge invariant overlap by means of the curly $L_0$ level truncation scheme has not been carried out neither for Schnabl nor for Erler-Schnabl’s solution. And in the case of the new real tachyon vacuum solution discovered in reference [6] (called as Jokel’s solution [7]), neither the numerical nor the analytical computation was presented for the gauge invariant overlap.

Motivated by the above results and open issues, in this work, using analytical and numerical techniques based on the curly $L_0$ and the traditional Virasoro $L_0$ level truncation schemes, we show a detailed and pedagogical way of computing the gauge invariant overlap for solutions constructed out of elements in the $KBC$ algebra. As explicit examples of our generic results, we present the analytical and numerical computation of the gauge invariant overlap for Schnabl’s, Erler-Schnabl’s and Jokel’s solutions.

By expanding the solutions $\Psi$ in the basis of curly $L_0$ eigenstates, we surprisingly discover that the result for the gauge invariant overlap $\langle I|\mathcal{V}(i)|\Psi \rangle$ turns out to be a finite series. This result is in contrast to the case of the energy, where the series has an infinite number of terms and diverges, though this divergent series can be resummed numerically by means of Padé approximants to give a good approximation to the expected value of the D-brane tension [1, 2, 16, 17].

Regarding the numerical result of the gauge invariant overlap for Erler-Schnabl’s and Jokel’s solution obtained by means of Virasoro $L_0$ level truncation computations, we would like to mention that the main reason for performing this numerical computation is to see whether or not higher level contributions yield to increasingly convergent results which approach the expected answer. We will show that the series that represents the gauge invariant overlap for these solutions turn out to be a divergent ones, therefore we will be required to use Padé approximants.

This paper is organized as follows. In section 2, we introduce the sliver frame and discuss some conventions and definitions that will be used in the rest of the paper. In section 3, we review the $KBC$ algebra. In sections 4, 5 and 6, we evaluate analytically and numerically the gauge invariant overlap for solutions expressed in terms of elements in the $KBC$ algebra. In section 7, a summary and further directions of exploration are given.

2 The sliver frame: conventions and definitions

Originally, the sliver frame has been defined as the $\tilde{z}$ coordinate obtained by the map [1]

$$\tilde{z} = \arctan z,$$

(2.1)

where $z$ is a point on the upper half-plane (UHP). It is known that the gluing prescription entering into the definition of the star product simplifies if one uses the $\tilde{z}$ coordinate.
Under the map (2.1), the UHP looks as a semi-infinite cylinder of circumference $\pi$ denoted by $C_\pi$.

There is another convention for the definition of the sliver frame which uses the following map

\[ \tilde{z} = \frac{2}{\pi} \arctan z. \]  

(2.2)

This map has been used in reference [2], and in this case, the UHP looks as a semi-infinite cylinder of circumference 2 denoted by $C_2$.

Since the expressions written in terms of elements in the $KBC$ algebra which are used in the construction of analytic solutions look different depending on the convention adopted for the $\tilde{z}$ coordinate, it is always useful to mention, from the beginning, what of those conventions will be chosen, i.e., the one given by (2.1) or (2.2).

In the literature some authors use the convention (2.1) and others (2.2), in this work we are going to use a rather generic definition which takes into account these two conventions. Let us define the $\tilde{z}$ coordinate by the map

\[ \tilde{z} = \frac{l}{\pi} \arctan z, \]  

(2.3)

so that the UHP looks as a semi-infinite cylinder of circumference $l$ denoted by $C_l$. Note that the case $l = \pi$ corresponds to the convention (2.1) while the case $l = 2$ corresponds to (2.2).

Let us define the operators $\hat{L}$, $\hat{B}$ and $\hat{c}_p$ which are very useful in the construction of elements in the $KBC$ algebra. These operators are related to the worldsheet energy-momentum tensor $T$, the $b$ and $c$ ghosts fields respectively. Using the map (2.3), we can write the explicit definition of the operators $\hat{L}$, $\hat{B}$ and $\hat{c}_p$

\[ \hat{L} \equiv L_0 + L_0^\dagger = \oint dz \frac{1}{2\pi i} (1 + z^2)(\arctan z + \text{arccot} z) T(z), \]  

(2.4)

\[ \hat{B} \equiv B_0 + B_0^\dagger = \oint dz \frac{1}{2\pi i} (1 + z^2)(\arctan z + \text{arccot} z) b(z), \]  

(2.5)

\[ \hat{c}_p = \left( \frac{l}{\pi} \right)^p \oint dz \frac{1}{2\pi i (1 + z^2)^{2-h}} (\arctan z)^{p-2} c(z). \]  

(2.6)

In general, if we have a primary field $\phi$ with conformal weight $h$, using the map (2.3), we obtain

\[ \tilde{\phi}_p \equiv \oint d\tilde{z} \frac{1}{2\pi i} \tilde{z}^{p+h-1} \tilde{\phi}(\tilde{z}) = \left( \frac{l}{\pi} \right)^p \oint dz \frac{1}{2\pi i (1 + z^2)^{1-h}} (\arctan z)^{p+h-1} \phi(z). \]  

(2.7)

Using this equation (2.7), let us define the operators $L_{-1}$ and $B_{-1}$ which are useful in the
computation of the star product of string fields involving the operators $\hat{L}$ and $\hat{B}$

\[
\mathcal{L}_{-1} \equiv \oint \frac{dz}{2\pi i} \hat{T}(\hat{z}) = \frac{\pi}{l} \oint \frac{dz}{2\pi i} (1 + z^2) T(z) = \frac{\pi}{l} [L_{-1} + L_1],
\]

\[
\mathcal{B}_{-1} \equiv \oint \frac{dz}{2\pi i} \hat{b}(\hat{z}) = \frac{\pi}{l} \oint \frac{dz}{2\pi i} (1 + z^2) b(z) = \frac{\pi}{l} [b_{-1} + b_1].
\]

To compute the star product of string fields involving the operators $\hat{L}$, $\hat{B}$ and $\hat{c}_p$, we will need to know the following commutator and anti-commutator relations

\[
[\mathcal{L}_{-1}, \hat{L}] = [\mathcal{L}_{-1}, \hat{B}] = 0, \quad [\mathcal{L}_{-1}, \hat{c}_p] = (2 - p) \hat{c}_{p-1}.
\]

\[
[\hat{B}, \hat{L}] = [\mathcal{B}_{-1}, \hat{L}] = [\mathcal{B}_{-1}, \hat{B}] = 0, \quad \{\mathcal{B}_{-1}, \hat{c}_p\} = \delta_{p-1,0}.
\]

To represent the elements in the $KBc$ algebra, we will require to know the operator $U^l_1 U^r$. This operator can be written in terms of the operator $\hat{L}$

\[
U^l_1 U^r = \exp \left[ \frac{2 - r}{2} \hat{L} \right].
\]

### 3 Star products and the $KBc$ algebra

Before defining the basic elements belonging to the $KBc$ algebra, we are going to write the star product of string fields containing the operators $\hat{L}$ and $\hat{B}$. Given two string fields $\phi_1$ and $\phi_2$, we can show that

\[
(\hat{B} \phi_1) * \phi_2 = \hat{B}(\phi_1 * \phi_2) + (-1)^{g_b(\phi_1)} \frac{l}{2} \phi_1 * \mathcal{B}_{-1} \phi_2,
\]

\[
\phi_1 * (\hat{B} \phi_2) = (-1)^{g_b(\phi_1)} \hat{B}(\phi_1 * \phi_2) - (-1)^{g_b(\phi_1)} \frac{l}{2} (\mathcal{B}_{-1} \phi_1) * \phi_2,
\]

\[
(\hat{B} \phi_1) * (\hat{B} \phi_2) = -(-1)^{g_b(\phi_1)} \frac{l}{2} \hat{B} \mathcal{B}_{-1} (\phi_1 * \phi_2) + (\frac{l}{2})^2 (\mathcal{B}_{-1} \phi_1) * (\mathcal{B}_{-1} \phi_2),
\]

\[
(\hat{L}^n \phi_1) * \phi_2 = \sum_{n' = 0}^{n} \left( \begin{array}{c} n \\ n' \end{array} \right) \left( \frac{l}{2} \right)^{n'} \hat{L}^{n-n'} (\phi_1 * \mathcal{L}_{-1}^{n'} \phi_2),
\]

\[
\phi_1 * (\hat{L}^n \phi_2) = \sum_{n' = 0}^{n} \left( \frac{n}{n'} \right) \left( \frac{l}{2} \right)^{n'} (-1)^{n'} \hat{L}^{n-n'} ((\mathcal{L}_{-1}^{n'} \phi_1) * \phi_2),
\]

\[
(\hat{L}^m \phi_1) * (\hat{L}^n \phi_2) = \sum_{m' = 0}^{m} \sum_{n' = 0}^{n} \left( \begin{array}{c} m \\ m' \end{array} \right) \left( \begin{array}{c} n \\ n' \end{array} \right) \left( \frac{l}{2} \right)^{m'+n'} (-1)^{m'} \hat{L}^{m+n-m'-n'} ((\mathcal{L}_{-1}^{n'} \phi_1) * (\mathcal{L}_{-1}^{n'} \phi_2))
\]

where $g_n(\phi)$ takes into account the Grassmannality of the string field $\phi$. If we set $l = \pi$, the above results match the results given in reference [1].
The action of the BRST, $\mathcal{L}_{-1}$, and $\mathcal{B}_{-1}$ operators on the star product of two string fields are given by

\[ Q(\phi_1 \ast \phi_2) = (Q\phi_1) \ast \phi_2 + (-1)^{gn(\phi_1)} \phi_1 \ast (Q\phi_2), \]

\[ \mathcal{L}_{-1}(\phi_1 \ast \phi_2) = (\mathcal{L}_{-1}\phi_1) \ast \phi_2 + \phi_1 \ast (\mathcal{L}_{-1}\phi_2), \]

\[ \mathcal{B}_{-1}(\phi_1 \ast \phi_2) = (\mathcal{B}_{-1}\phi_1) \ast \phi_2 + (-1)^{gn(\phi_1)} \phi_1 \ast (\mathcal{B}_{-1}\phi_2). \]

Given a operator $\tilde{\phi}(\tilde{z})$ defined in the $\tilde{z}$ coordinate, let us write the wedge state with insertion as follows

\[ U_r^\dagger U_r \tilde{\phi}(\tilde{z}) |0\rangle, \]

where $U_r = (2/r)^{\mathcal{L}_0}$ is the scaling operator in the $\tilde{z}$ coordinate. The star product of two states $U_r^\dagger U_r \tilde{\phi}(\tilde{x}) |0\rangle$ and $U_s^\dagger U_s \tilde{\psi}(\tilde{y}) |0\rangle$ can be derived using the usual gluing prescription

\[ U_r^\dagger U_r \tilde{\phi}(\tilde{x}) |0\rangle \ast U_s^\dagger U_s \tilde{\psi}(\tilde{y}) |0\rangle = U_r^\dagger U_s \tilde{\phi}(\tilde{x} + \frac{l}{4}(s - 1)) \ast \tilde{\psi}(\tilde{y} - \frac{l}{4}(r - 1)) |0\rangle, \]

where by $\tilde{\phi}(\tilde{z})$ we have denoted the local operator $\phi(z)$ expressed in the sliver frame. For instance, in the case of a primary field with conformal weight $h$, $\tilde{\phi}(\tilde{z})$ is given by

\[ \tilde{\phi}(\tilde{z}) = \left( \frac{dz}{d\tilde{z}} \right)^h \phi(z) = \left( \frac{\pi}{l} \right)^h \cos^{-2h} \left( \frac{\pi \tilde{z}}{l} \right) \phi \left( \tan \left( \frac{\pi \tilde{z}}{l} \right) \right). \]

The elements in the $K\mathcal{B}c$ algebra are constructed out of the basic string fields $K$, $\mathcal{B}$, and $c$. These fields can be represented in terms of operators acting on the identity string field $|I\rangle = U_r^\dagger U_r |0\rangle$

\[ K \equiv \frac{1}{l} \hat{\mathcal{L}} U_r^\dagger U_r |0\rangle, \]

\[ \mathcal{B} \equiv \frac{1}{l} \hat{\mathcal{B}} U_r^\dagger U_r |0\rangle, \]

\[ c \equiv U_r^\dagger U_r \tilde{c}(0) |0\rangle. \]

Let us derive the algebra associated to the set of operators defined by equations (3.13)-(3.15). As a pedagogical illustration, we explicitly compute $\{B, c\}$

\[ \{B, c\} \equiv Bc + cB = \frac{1}{l} \hat{\mathcal{B}} U_r^\dagger U_r |0\rangle \ast U_r^\dagger U_r \tilde{c}(0) |0\rangle + \frac{1}{l} U_r^\dagger U_r \tilde{c}(0) |0\rangle \ast \hat{\mathcal{B}} U_r^\dagger U_r |0\rangle, \]

using equations (3.1), (3.2) and the anti-commutator (2.11), we obtain

\[ \{B, c\} = U_r^\dagger U_r |0\rangle = |I\rangle, \]

therefore, we have that $\{B, c\} = 1$. 

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Following the same steps, using equations (3.1)-(3.6), the commutator and anti-commutator relations (2.10), (2.11), we can show that

\[ [K, B] = 0, \quad \{B, c\} = 1, \quad \partial c = [K, c], \quad B^2 = 0, \quad c^2 = 0, \quad (3.18) \]

where the expression \( \partial c \) has been defined as \( \partial c \equiv U_1^\dagger U_1 \partial c(0) |0\rangle \).

The action of the BRST operator \( Q \) on the basic string fields \( K, B, \) and \( c \) is given by

\[ QK = 0, \quad QB = K, \quad Qc = cKc. \quad (3.19) \]

Employing the elements in the \( KBc \) algebra, we can construct a rather generic solution

\[ \Psi = Fc \frac{KB}{1 - F^2} cF, \quad (3.20) \]

which formally satisfies the string field equation of motion \( Q\Psi + \Psi \Psi = 0 \). For this solution to be a well defined string field, the function \( F(K) \) must satisfy some holomorphicity conditions stated in reference [3]. From now, we will assume that \( \Psi \) belongs to the set of well defined string fields.

Let us list some solutions of the form (3.20). As a first example, consider the analytic solution for the tachyon vacuum [1], where \( F(K) = e^{-lK/4} \), actually Schnabl’s original solution corresponds to the case where \( l = \pi \). Recall that in this work, we are considering the map \( \tilde{z} = \frac{l}{\pi} \arctan z \), and therefore the Schnabl’s solution looks like

\[ \Psi_{Sch} = e^{-lK/4} c \frac{KB}{1 - e^{-lK/2}} c e^{-lK/4}. \quad (3.21) \]

There is a subtlety with this solution, as shown in references [1] [3] when one performs the expansion of \( K/(1 - e^{-lK/2}) \) as the sum \( \sum_n K e^{-lKn/2} \), the truncation of this sum produces a remnant which still contributes to certain observable [4]. This is the origin of the phantom term \( \Psi_N \). Taking into account the phantom term, the solution (3.21) can be written as

\[ \Psi_{Sch} = 2 l \lim_{N \to \infty} \left[ \psi_N - \sum_{n=0}^{N-1} \frac{d\psi_n}{dn} \right], \quad (3.22) \]

where

\[ \psi_n = e^{-lK/4} cBe^{-lKn/2} c e^{-lK/4}. \quad (3.23) \]

As a second example, let us consider the solution discovered by Erler and Schnabl, namely, the so-called simple tachyon vacuum solution [2]

\[ \Psi_{Er-Sch} = \frac{1}{\sqrt{1 + K}} cB(1 + K) c \frac{1}{\sqrt{1 + K}}. \quad (3.24) \]
Note that in this case, \( F(K) = 1/\sqrt{1+K} \), and as shown in references \[2, 3\] there is no need for a phantom like term. It is possible to provide an integral representation of the solution (3.24), this is given by writing the inverse square root of \( 1 + K \) as
\[
\frac{1}{\sqrt{1+K}} = \frac{1}{\sqrt{\pi}} \int_0^\infty dt \frac{1}{\sqrt{t}} e^{-t} \Omega^t, \tag{3.25}
\]
where \( \Omega^t \) is the wedge state which can be written as \[18, 19\]
\[
\Omega^t = e^{-K^t} = U^I_{\hat{t}+1} U^I_{\hat{t}+1} |0\rangle. \tag{3.26}
\]

And as a last example, we consider the so-called, real tachyon vacuum solution without square roots, or Jokel’s real solution for short \[6, 7\]. This solution takes the form
\[
\Psi_{\text{Jok}} = \frac{1}{4} \left( \frac{1}{1+K} c + c \frac{1}{1+K} + c \frac{B}{1+K} c + \frac{1}{1+K} \frac{1}{1+K} \right) + Q\text{-exact terms}, \tag{3.27}
\]
where the \( Q\)-exact terms are given by
\[
\frac{1}{2} \left[ Q(Bc) \frac{1}{1+K} + \frac{1}{1+K} Q(Bc) \right] + \frac{1}{4} \frac{1}{1+K} Q(Bc) \frac{1}{1+K}. \tag{3.28}
\]
Interestingly, the solution does not take the factorized form (3.20), and is both real and simple, namely without square roots and phantom terms. For this real solution, the corresponding energy has been computed and shown that the value is in agreement with the value predicted by Sen’s conjecture.

4 The gauge invariant overlap: analytical computations

In this section, we are going to study the analytic computation of the gauge invariant overlap for solutions given in terms of elements in the \( KBc \) algebra. This gauge invariant observable has been considered in references \[11, 12, 13, 15, 20, 21\]. For a given solution \( \Psi \) of the string field equations of motion, the gauge invariant overlap is defined as the evaluation of the quantity
\[
\langle \mathcal{V} | \Psi \rangle = \langle I | \mathcal{V}(i) | \Psi \rangle, \tag{4.1}
\]
where \(|I\rangle\) is the identity string field, and the operator \( \mathcal{V}(i) \) is an on-shell closed string vertex operator \( \mathcal{V}(i) = c \overline{c} V^m \) which is inserted at the midpoint of the string field \( \Psi \). As argued by Ellwood \[14\], the gauge invariant overlap represents the shift in the closed string tadpole of the solution relative to the perturbative vacuum.

\(^1V^m\) is a weight \((1,1)\) conformal matter primary field.
To evaluate the gauge invariant overlap for solutions given in terms of elements in the $KBC\mathbb{C}$ algebra, it will be useful the following results

\begin{align}
\langle V|^{\Omega_1 c^{\Omega_2}} &= (t_1 + t_2)C_V, \\
\langle V|^{\Omega_1 Bc^{\Omega_2} c^{\Omega_3}} &= t_2 C_V, \\
\langle V|^{\Omega_1 c^{\Omega_2} Bc^{\Omega_3}} &= (t_1 + t_3)C_V,
\end{align}

where the coefficient $C_V$ represents the correlator

$$C_V = \langle V(i\infty)c(0)\rangle_{C_1},$$

which is the closed string tadpole evaluate on a cylinder $C_1$ of unit circumference. The proofs of the above results (4.2)-(4.4) are based on usual scaling arguments and can be found in references [2, 22].

As an application of equations (4.2)-(4.4), we are going to compute the gauge invariant overlap for Schnabl’s tachyon vacuum solution. We would like to mention that in reference [13], after performing lengthy computations the authors have evaluated the gauge invariant overlap for Schnabl’s solution. However, as we will see, this computation can be performed in a few lines if one uses Schnabl’s solution expressed in terms of the basic string fields $K, B$ and $c$

$$\langle V|\Psi_{Sch} \rangle = \frac{2}{l} \lim_{N \to \infty} \left[ \langle V|\psi_N \rangle - \sum_{n=0}^{N-1} \frac{d \langle V|\psi_n \rangle}{dn} \right],$$

therefore, we need to compute $\langle V|\psi_n \rangle$. Using equation (3.23), we can write

$$\langle V|\psi_n \rangle = \langle V|e^{-lK/4} cBe^{-lKn/2} c e^{-lK/4} \rangle = \langle V|^{\Omega_1/4 c^{\Omega_2} B^{\Omega_3} c^{\Omega_4}}.$$ 

Employing equation (4.3), from equation (4.7), we get

$$\langle V|\psi_n \rangle = \frac{l}{2} C_V,$$

plugging this result (4.8) into equation (4.6), we obtain

$$\langle V|\Psi_{Sch} \rangle = C_V = \langle V(i\infty)c(0)\rangle_{C_1}.$$

This result coincides with the expected answer of closed string tadpole on the disk [14]. Note that the result (4.9) does not depend on the parameter $l$ which explicitly appears in the solution (3.22).

Next we would like to evaluate the gauge invariant overlap for Erler-Schnabl’s solution. Actually, using a non-real version of the solution (3.24), the computation of the gauge invariant overlap has been performed in reference [2]. Here we are going to present the
computation for the case of the real solution. Let us write the real solution (3.24) as the following integral representation

\[ \Psi_{\text{Er-Sch}} = \frac{1}{\pi} \left[ (1 - \partial_s) \int_0^\infty dt_1 dt_2 \frac{e^{-t_1 - t_2}}{\sqrt{t_1 t_2}} \Omega t_1 cB \Omega^s c \Omega t_2 \right]_{s=0}. \]  

(4.10)

therefore the gauge invariant overlap for this solution (4.10) will be given by

\[ \langle V | \Psi_{\text{Er-Sch}} \rangle = \frac{1}{\pi} \left[ (1 - \partial_s) \int_0^\infty dt_1 dt_2 \frac{e^{-t_1 - t_2}}{\sqrt{t_1 t_2}} \Omega t_1 cB \Omega^s c \Omega t_2 \right]_{s=0}. \]  

(4.11)

Employing equation (4.4), from equation (4.11), we write

\[ \langle V | \Psi_{\text{Er-Sch}} \rangle = \frac{1}{\pi} \left[ (1 - \partial_s) \int_0^\infty dt_1 dt_2 \frac{e^{-t_1 - t_2}}{\sqrt{t_1 t_2}} (t_1 + t_2) C \right]_{s=0} = \frac{1}{\pi} \int_0^\infty dt_1 dt_2 \frac{e^{-t_1 - t_2}}{\sqrt{t_1 t_2}} (t_1 + t_2) C \]  

(4.12)

As we can see, this result (4.12) is exactly the same as the one obtained for Schnabl’s solution (4.9).

And as the last example of analytical calculation, let us evaluate the gauge invariant overlap for Jokel’s real solution. Since BRST exact terms do not contribute for the evaluation of the gauge invariant overlap, we just need to consider the non-BRST exact terms of the solution. These terms are given on the right hand side of equation (3.27) and they can be written as

\[ \hat{\Psi}_{\text{Jok}} = \frac{1}{4} \left( \frac{1}{1 + K} e + e \frac{1}{1 + K} + c \frac{B}{1 + K} c + \frac{1}{1 + K} c \frac{1}{1 + K} \right) \]  

\[ = \frac{1}{4} \int_0^\infty dt \ e^{-t} (\Omega t c + c \Omega + c \Omega B c) + \frac{1}{4} \int_0^\infty dt_1 dt_2 \ e^{-t_1 - t_2} \Omega t_1 c \Omega t_2. \]  

(4.13)

Therefore the gauge invariant overlap for Jokel’s real solution is given by

\[ \langle V | \hat{\Psi}_{\text{Jok}} \rangle = \frac{1}{4} \int_0^\infty dt \ e^{-t} \langle V | \Omega t c + c \Omega + c \Omega B c \rangle + \frac{1}{4} \int_0^\infty dt_1 dt_2 \ e^{-t_1 - t_2} \langle V | \Omega t_1 c \Omega t_2 \rangle. \]  

(4.14)

Using equations (4.2) and (4.4), from equation (4.14) we obtain

\[ \langle V | \hat{\Psi}_{\text{Jok}} \rangle = \left[ \frac{1}{2} \int_0^\infty dt \ e^{-t} + \frac{1}{4} \int_0^\infty dt_1 dt_2 (t_1 + t_2) e^{-t_1 - t_2} \right] C \]  

\[ = C \left[ \langle V (i\infty) c(0) \rangle \right]_{c_1}. \]  

(4.15)

---

2The reality condition of a string field is defined as \( \Psi^\dagger = \Psi \) where the operation \( \dagger \) means the composition of BPZ and Hermitian conjugation. Since the basic string fields \( K, B \) and \( c \) are real string fields in this sense, the reality condition requires that the string field read the same way from the left as from the right.
Note that this result (4.15) is the same as the ones obtained in the case of Schnabl’s (4.9) and Erler-Schnabl’s solutions (4.12).

It should be nice to obtain the above analytic results by numerical means. For instance, using the traditional Virasoro $L_0$ level truncation scheme, in reference [13], the authors have evaluated the gauge invariant overlap for Schnabl’s solution. However, up to now, using the Virasoro $L_0$ level truncation scheme, the analysis of the gauge invariant overlap for Erler-Schnabl’s and Jokel’s real solution was not performed. Moreover, the analysis of the gauge invariant overlap by means of the curly $L_0$ level truncation scheme has not been carried out neither for Schnabl’s, Erler-Schnabl’s nor for Jokel’s real solution.

In the next two sections, using the curly $L_0$ and the Virasoro $L_0$ level truncation scheme, we are going to present the evaluation of the gauge invariant overlap for solutions constructed out of elements in the $KBC$ algebra.

5 The gauge invariant overlap: $L_0$ level truncation computations

Since from the beginning, we do not know if the result for the gauge invariant overlap obtained by analytical computations will match the result obtained by numerical means (either by using the $L_0$ or the $L_0$ level truncation scheme), it is important for the consistency of the solutions to explicitly check if these different schemes provide the same answer. In this section, using the $L_0$ level expansion of a rather generic solution $\Psi$, we will present the evaluation of the gauge invariant overlap.

As we know, the solution is given in terms of elements in the $KBC$ algebra (which involves the operators $\hat{L}$, $\hat{B}$ and $\hat{c}$), in general, we can write the following $L_0$ level expansion

$$\Psi = \sum_{n,p} f_{n,p} \hat{L}^n \hat{c}_p |0\rangle + \sum_{n,p,q} f_{n,p,q} \hat{L}^n \hat{B} \hat{c}_p \hat{c}_q |0\rangle,$$

where $n = 0, 1, 2, \cdots$, and $p, q = 1, 0, -1, -2, \cdots$. The coefficients of the expansion $f_{n,p}$ and $f_{n,p,q}$ can be regarded as generic ones, obviously these coefficients depend on the solution we choose. For instance, for the case of Schnabl’s solution (3.21), these coefficients are given by

$$f_{n,p} = \frac{1 - (-1)^p}{2} \frac{l^{-p}}{2^{n-2p+1} n!} \frac{1}{n!} (-1)^n B_{n-p+1},$$

and

$$f_{n,p,q} = \frac{1 - (-1)^{p+q}}{2} \frac{l^{-p-q}}{2^{n-2(p+q)+3} n!} \frac{1}{n!} (-1)^{n-q} B_{n-p-q+2},$$

where $B_m$ are the Bernoulli’s numbers.

To compute the gauge invariant overlap for solutions expanded in terms of $L_0$ eigenstates, we start by replacing the string field $\Psi$ with $e^{\mathcal{L}_0} \Psi$, so that states in the $L_0$ level
expansion will acquire different integer powers of \( z \) at different levels. And as usual, at the end, we will simply set \( z = 1 \).

Let us start with the evaluation of the gauge invariant overlap as a formal power series expansion in \( z \). Plugging the expansion (5.1) into the definition of the gauge invariant overlap (4.1), we obtain

\[
\langle \mathcal{V} | z^{\mathcal{L}_0} \Phi \rangle = \sum_{n,p} z^{-p} f_{n,p}(\langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle + \sum_{n,p,q} z^{n+1-p-q} f_{n,p,q}(\langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{B} \hat{c}_p \hat{c}_q | 0 \rangle). \tag{5.4}
\]

As we can see, we need to compute \( \langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle \) and \( \langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{B} \hat{c}_p \hat{c}_q | 0 \rangle \). To evaluate these quantities, we require express \( \hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle \) and \( \hat{\mathcal{L}}^n \hat{B} \hat{c}_p \hat{c}_q | 0 \rangle \) in terms of elements in the \( KBC \) algebra, for this purpose, it will be useful the following relations

\[
\Omega^{s_1} e^{\mathcal{O} s_2} = e^{u \mathcal{L} \hat{c}(x)} | 0 \rangle, \tag{5.5}
\]

\[
B \mathcal{O}^{t_1} e^{\mathcal{O} t_2} e^{\mathcal{O} t_3} - \frac{1}{2} \mathcal{O}^{t_1 + t_2} e^{\mathcal{O} t_3} + \frac{1}{2} \mathcal{O}^{t_1} e^{\mathcal{O} t_2 + t_3} = \frac{1}{l} \hat{B} e^{u \mathcal{L} \hat{c}(x) \hat{c}(y)} | 0 \rangle, \tag{5.6}
\]

where

\[
s_1 = \frac{l}{4} - \frac{lu}{2} - x, \quad s_2 = \frac{l}{4} - \frac{lu}{2} + x, \quad t_1 = \frac{l}{4} - \frac{lu}{2} - x, \quad t_2 = x - y, \quad t_3 = \frac{l}{4} - \frac{lu}{2} + y. \tag{5.7}
\]

Employing the above relations, we can write \( \hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle \) and \( \hat{\mathcal{L}}^n \hat{B} \hat{c}_p \hat{c}_q | 0 \rangle \) in terms of elements in the \( KBC \) algebra

\[
\hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle = n! \int \frac{du \, dx}{2 \pi i 2 \pi i} u^{-n-1} x^{p-2} \mathcal{O}^{s_1} e^{\mathcal{O} s_2}, \tag{5.9}
\]

\[
\hat{\mathcal{L}}^n \hat{B} \hat{c}_p \hat{c}_q | 0 \rangle = n! \int \frac{du \, dx \, dy}{2 \pi i 2 \pi i 2 \pi i} u^{-n-1} x^{p-2} y^{q-2} \left[ B \mathcal{O}^{t_1} e^{\mathcal{O} t_2} e^{\mathcal{O} t_3} - \frac{1}{2} \mathcal{O}^{t_1 + t_2} e^{\mathcal{O} t_3} + \frac{1}{2} \mathcal{O}^{t_1} e^{\mathcal{O} t_2 + t_3} \right]. \tag{5.10}
\]

Now we are in position to evaluate the quantities \( \langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle \) and \( \langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{B} \hat{c}_p \hat{c}_q | 0 \rangle \). For instance, using equations (4.2) and (5.9), let us compute

\[
\langle \mathcal{V} | \hat{\mathcal{L}}^n \hat{c}_p | 0 \rangle = n! \int \frac{du \, dx}{2 \pi i 2 \pi i} u^{-n-1} x^{p-2} \langle \mathcal{V} | \mathcal{O}^{s_1} e^{\mathcal{O} s_2} \rangle \]

\[
= n! \int \frac{du \, dx}{2 \pi i 2 \pi i} u^{-n-1} x^{p-2} \left( \frac{l}{2} - lu \right) C_V
\]

\[
= l \left( \frac{\delta_{p,1} \delta_{n,0}}{2} - \delta_{p,1} \delta_{n,1} \right) C_V. \tag{5.11}
\]

Performing similar calculations as the above, using equations (4.2), (4.3) and (5.10), we obtain

\[
\langle \mathcal{V} | \hat{B} \hat{\mathcal{L}}^n \hat{c}_p \hat{c}_q | 0 \rangle = l \left( \delta_{n,0} \delta_{p,0} \delta_{q,1} - \delta_{n,0} \delta_{q,0} \delta_{p,1} \right) C_V. \tag{5.12}
\]
Finally, plugging the results (5.11) and (5.12) into the definition of the gauge invariant overlap (5.4), and setting \( z = 1 \), we get

\[
\langle \mathcal{V} | \Psi \rangle = l \left( \frac{f_{0,1}}{2} - f_{1,1} - 2f_{0,1,0} \right) C_{\mathcal{V}}. \tag{5.13}
\]

To compute the gauge invariant overlap for a solution expanded in terms of \( L_0 \) eigenstates (5.1), we only need to know the value of the first three coefficients appearing at levels \( z^{-1} \) and \( z^0 \). Remarkably, this result (5.13) is simpler than the one obtained for the case of the energy. Evaluating the energy in the \( L_0 \) level expansion gives a very complicated non-convergent series, though the series can be resummed numerically by means of the so-called Padé approximants to give a good approximation to the brane tension [1, 2, 10].

Let us apply the general result (5.13) for some particular solutions such as the Schnabl’s solution \( \Psi_{\text{Sch}} \). Using the explicit expressions of the coefficients (5.2) and (5.3)

\[
f_{0,1} = 2 \frac{l}{2}, \quad f_{1,1} = \frac{1}{2l}, \quad f_{0,1,0} = -\frac{1}{4l}, \tag{5.14}
\]

which appear in the \( L_0 \) level expansion of Schnabl’s solution, from equation (5.13) we obtain

\[
\langle \mathcal{V} | \Psi_{\text{Sch}} \rangle = C_{\mathcal{V}}. \tag{5.15}
\]

This result does not depend on the parameter \( l \) and is the same result as the one obtained from analytic computations.

In the case of Erler-Schnabl’s solution \( \Psi_{\text{Er-Sch}} \), using its integral representation (4.10), we can compute the first three coefficients appearing in the \( L_0 \) level expansion of the solution

\[
f_{0,1} = 1, \quad f_{1,1} = \frac{1}{2}, \quad f_{0,1,0} = -\frac{1}{4l}. \tag{5.16}
\]

Therefore, plugging these results (5.16) into equation (5.13), we obtain

\[
\langle \mathcal{V} | \Psi_{\text{Er-Sch}} \rangle = C_{\mathcal{V}}. \tag{5.17}
\]

This result also does not depend on the parameter \( l \) and is the same result as the one obtained for the case of Schnabl’s solution.

In the case of Jokel’s real solution, we can also calculate the curly \( L_0 \) level expansion of the non-BRST exact terms of the solution (4.13). The first three coefficients of this \( L_0 \) level expansion are given by

\[
f_{0,1} = 2 \frac{l}{2}, \quad f_{1,1} = -\frac{1}{4l}, \quad f_{0,1,0} = \frac{1}{8l}. \tag{5.18}
\]
Substituting these results (5.18) into equation (5.13), we get
\[ \langle \mathcal{V} | \hat{\Psi}_{\text{Jok}} \rangle = C_\mathcal{V}. \] (5.19)

As we can see, the result (5.19) is the same as the ones obtained in the case of Schnabl’s and Erler-Schnabl’s solutions.

So far, we have computed the gauge invariant overlap by two means: analytically and using the curly \( L_0 \) level expansion of the solutions. In what follows, we are going to evaluate the gauge invariant overlap by a third method, namely, using the traditional Virasoro \( L_0 \) level expansion of the solutions.

6 The gauge invariant overlap: Virasoro \( L_0 \) level truncation computations

In this section, using the \( L_0 \) level truncation scheme, the evaluation of the gauge invariant overlap will be shown. Since the solution \( \Psi \) involves the operators \( \hat{\mathcal{L}}, \hat{\mathcal{B}} \) and \( \hat{c} \), we can write its \( L_0 \) level expansion as follows
\[ \Psi = \sum g_{n_1 n_2 \cdots n_i} L_{n_1} L_{n_2} \cdots L_{n_i} c_p |0\rangle + \sum g_{m_1 m_2 \cdots m_s p q} L_{m_1} L_{m_2} \cdots L_{m_j} b_s c_p c_q |0\rangle, \] (6.1)
where \( n_i, m_j, s \leq -2 \) and \( p, q = 1, 0, -1, -2, \cdots \). The \( L_n \)'s are the ordinary Virasoro generators with zero central charge \( c = 0 \) of the total (i.e. matter and ghost) conformal field theory. For instance, Schnabl’s solution (5.22), with \( l = \pi \), expanded up to level two states is given by
\[ \Psi_{\text{Sch}} = 0.553465 c_1 |0\rangle + 0.043671 c_{-1} |0\rangle + 0.137646 L_{-2} c_1 |0\rangle + 0.131082 b_{-2} c_0 c_1 |0\rangle. \] (6.2)

To compute the gauge invariant overlap by means of the \( L_0 \) level truncation scheme, it is clear that if we insert the expansion (6.1) into the definition of the gauge invariant overlap (4.1), we will need to evaluate the quantities
\[ \langle \mathcal{V} | L_{n_1} L_{n_2} \cdots L_{n_i} c_p |0\rangle, \quad \langle \mathcal{V} | L_{m_1} L_{m_2} \cdots L_{m_j} b_s c_p c_q |0\rangle. \] (6.3)
We are going to calculate these quantities by means of a recursive method based on the evaluation of the following commutation and anti-commutation relations
\[ [L_m, L_n] = (m - n)L_{m+n}, \] (6.4)
\[ [L_m, b_n] = (m - n)b_{m+n}, \] (6.5)
\[ [L_n, c_p] = (-2n - p)c_{n+p}, \] (6.6)
\[ \{b_m, c_n\} = \delta_{m+n,0}. \] (6.7)
As an illustration, suppose we need to calculate \( \langle \mathcal{V} | L_n c_p | 0 \rangle \). Since for \( n \leq -2 \) the operator \( L_n \) does not annihilate the vacuum \( |0\rangle \), and in order to apply the commutator (6.6), we must first express the operator \( L_n \) in terms of annihilation operators. This can be achieved if we use the fact that the on-shell closed string state \( \mathcal{V} = c \bar{c} \mathcal{V}^m \) is invariant by the transformation generated by \( K_n = L_n - (-1)^n L_{-n} \), namely, we have [13]

\[
\langle \mathcal{V} | L_n = \langle \mathcal{V} | (-1)^n L_{-n}. \tag{6.8}
\]

And now, since \( L_{-n} |0\rangle = 0 \) for \( n \leq -2 \), we are able to compute \( \langle \mathcal{V} | L_n c_p | 0 \rangle \) using the commutator (6.6)

\[
\langle \mathcal{V} | L_n c_p | 0 \rangle = (-1)^n \langle \mathcal{V} | [L_{-n}, c_p] | 0 \rangle = (-1)^n (2n - p) \langle \mathcal{V} | c_p | 0 \rangle. \tag{6.9}
\]

Let us comment that for the case of the operator \( b_n \) which corresponds to the modes of the ghost field \( b \), we have a similar result as the one given by equation (6.8) [11, 12, 13, 23]

\[
\langle \mathcal{V} | b_n = \langle \mathcal{V} | (-1)^n b_{-n}. \tag{6.10}
\]

As we have seen, after the use of the commutation and anti-commutation relations (6.4)-(6.7), we can express the quantities (6.3) as linear combinations of terms like

\[
\langle \mathcal{V} | c_p | 0 \rangle. \tag{6.11}
\]

To evaluate (6.11), first let us express the mode \( c_p \) in the \( \tilde{z} \)-coordinate. Using the conformal transformation of the \( c(z) \) ghost, under the map (2.3), we get

\[
c_p = \oint \frac{d\tilde{z}}{2\pi i} \tilde{z}^{p-2} c(\tilde{z}) = \left( \frac{\pi}{l} \right)^2 \oint \frac{d\tilde{z}}{2\pi i} \sec^4 \left( \frac{\pi\tilde{z}}{l} \right) \tan^{p-2} \left( \frac{\pi\tilde{z}}{l} \right) \tilde{c}(\tilde{z}). \tag{6.12}
\]

If we substitute equation (6.12) into equation (6.11), it is clear that we will need to evaluate the quantity \( \langle \mathcal{V} | \tilde{c}(\tilde{z}) | 0 \rangle \). Using equations (4.2) and (5.5), we can compute this quantity

\[
\langle \mathcal{V} | \tilde{c}(\tilde{z}) | 0 \rangle = \langle \mathcal{V} | \Omega^{-\tilde{z}+l/4} c \Omega^{\tilde{z}+l/4} = \frac{l}{2} C_{\mathcal{V}}. \tag{6.13}
\]

Therefore, employing equations (6.12) and (6.13), we obtain

\[
\langle \mathcal{V} | c_p | 0 \rangle = \left( \frac{\pi}{l} \right)^2 \left( \frac{l}{2} \right) C_{\mathcal{V}} \oint \frac{d\tilde{z}}{2\pi i} \sec^4 \left( \frac{\pi\tilde{z}}{l} \right) \tan^{p-2} \left( \frac{\pi\tilde{z}}{l} \right) = \frac{\pi}{2} (\delta_{p,-1} + \delta_{p,1}) C_{\mathcal{V}}. \tag{6.14}
\]

As a first example, let us compute the gauge invariant overlap for Schnabl’s solution expanded up to level two states

\[
\Psi_{\text{Sch}} = t' c_1 |0\rangle + u' c_{-1} |0\rangle + v' L_{-2} c_1 |0\rangle + w' b_{-2} c_0 c_1 |0\rangle, \tag{6.15}
\]
where the values of the coefficients $t'$, $u'$, $v'$ and $w'$ are given in equation (6.2). Using the property that $\langle \mathcal{V}|L_{-2} = \langle \mathcal{V}|L_2$ and $\langle \mathcal{V}|b_{-2} = \langle \mathcal{V}|b_2$, the evaluation of the gauge invariant overlap reads as follows

$$\langle \mathcal{V}|\Psi_{\text{Sch}} \rangle = t' \langle \mathcal{V}|c_1|0\rangle + u' \langle \mathcal{V}|c_{-1}|0\rangle + v' \langle \mathcal{V}|[L_2, c_1]|0\rangle + w' \langle \mathcal{V}|[b_2, c_0 c_1]|0\rangle$$

$$= t' \langle \mathcal{V}|c_1|0\rangle + u' \langle \mathcal{V}|c_{-1}|0\rangle - 5v' \langle \mathcal{V}|c_3|0\rangle = \frac{\pi}{2}(t' + u') C_V.$$

(6.16)

We would like to compare this result (6.16) with the one obtained in reference [13], where Schnabl’s solution has been expanded in a slightly different basis. Instead of considering the Virasoro generators $L_n$ with zero central charge, the authors have used the $\alpha_n$’s oscillators, for instance, up to level two states, they have written the expansion

$$\Psi_{\text{Sch}} = t c_1|0\rangle + u c_{-1}|0\rangle + v (\alpha_{-1} \cdot \alpha_{-1}) c_1|0\rangle + w b_{-2} c_0 c_1|0\rangle,$$

(6.17)

where the coefficients have the following values

$$t = 0.553465, \quad u = 0.456611, \quad v = 0.068823, \quad w = -0.144210.$$

(6.18)

Then by using an explicit oscillator representation for the on-shell closed string state, which can be found in references [13, 24], the gauge invariant overlap for the expanded Schnabl’s solution (6.17) turns out to be

$$\langle \mathcal{V}|\Psi_{\text{Sch}} \rangle = \frac{1}{4} t - \frac{3}{2} v + \frac{1}{4} u = 0.149284.$$

(6.19)

Let us compare this result (6.19) with the one obtained by us (6.16). To get the same answer, we should choose the normalization where $C_V = 1/(2\pi)$, and in fact with this normalization from equation (6.16), we obtain

$$\langle \mathcal{V}|\Psi_{\text{Sch}} \rangle = \frac{1}{4}(t' + u') = 0.149284.$$

(6.20)

Taking into account higher level states, we have performed the computation of the gauge invariant overlap for Schnabl’s solution, and the results we have obtained with the normalization $C_V = 1/(2\pi)$ are in agreement with the ones presented in reference [13]. We can consider this agreement as a test for the method of computing the gauge invariant overlap based on the use of the equations (6.8), (6.10) and the commutation and anti-commutation relations (6.4)-(6.7).

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3We have noted that if we use equation (3.36) of reference [13], the value of the coefficient $v$ turns out to be twice the value presented here (6.13). This means that if the authors want to use the definition of $v$ as given in their equations (3.31) and (3.32), their equation (3.36) should be replaced by a half of it. We have communicated this issue to one of the authors, and he has confirmed this little mistake which nevertheless does not change the main result presented in [13].
The advantage of this method compared to the one presented in [13], is that we do not require to use an explicit oscillator representation for the on-shell closed string state. The implication of this observation will be reflected in the simplification of the evaluation of the gauge invariant overlap. Recall that the $L_0$ level expansion of analytic solutions constructed out of elements in the $KBc$ algebra, as presented in (6.1), is naively given in terms of the total (matter+ghost) Virasoro generators $L_n$, the $b_n$ and $c_p$ modes, and since we do not require to use an explicit oscillator representation for the on-shell closed string state, using the expansion (6.1) we can directly evaluate the gauge invariant overlap without the necessity of reexpressing the expansion in terms of the $\alpha_n$’s oscillators (which will require an additional work).

Before to study the numerical evaluation of the gauge invariant overlap for the case of Erler-Schnabl’s and Jokel’s solutions, we would like to mention some motivations for doing this computation. Firstly, using the $L_0$ level truncation scheme, the numerical analysis of the gauge invariant overlap for Erler-Schnabl’s and Jokel’s solutions has not been carried out. This analysis should be crucial if we want to confirm the analytic result. However, the main motivation for performing such numerical computations is to see whether or not higher level contributions yield to increasingly convergent results which approach to the expected answer. In the case of Schnabl’s solution, it has been shown that every time we increase the level of the truncated solution, the gauge invariant overlap converges to the expected analytical result without the necessity of using any regularization scheme such as Padé approximants [13].

Let us start with the $L_0$ level truncation analysis of the gauge invariant overlap for Erler-Schnabl’s solution. To simplify the computations, it will be useful to write the solution (3.24) in the following way

$$
\Psi_{\text{Er-Sch}} = \frac{1}{\sqrt{1 + K}} c \frac{\sqrt{1 + K}}{\sqrt{1 + K}} + Q\left\{ \frac{1}{\sqrt{1 + K}} Bc \frac{1}{\sqrt{1 + K}} \right\}.
$$

(6.21)

Inserting the solution (6.21) into the definition of the gauge invariant overlap, the BRST exact term does not contribute, and so we only need to consider the first term appearing on the right hand side of equation (6.21), let us denote this term as

$$
\Psi^{(1)} \equiv \frac{1}{\sqrt{1 + K}} c \frac{1}{\sqrt{1 + K}}.
$$

(6.22)

To compare the $L_0$ level expansion of the string field (6.22) with the one presented in reference [2], we choose the value of the parameter $l$, which appears in the definition of the map (2.3), as $l = 2$. The $L_0$ level expansion of the string field (6.22) can be obtained from the following result [2, 25]

$$
\Psi^{(1)} = \frac{1}{2\pi^2} \int_0^\infty ds dt \int \frac{1}{\sqrt{sl}} e^{-s-t-r^2} \cos^2 \left( \frac{\pi x}{r} \right) \tilde{U}_{r} c \left( \frac{2 tan \left( \frac{\pi x}{r} \right)}{r} \right) |0\rangle,
$$

(6.23)
where $r$ and $x$ are given by
\[ r = s + t + 1, \quad x = \frac{s - t}{2}. \]  
(6.24)

The operator $\tilde{U}_r$ is defined as
\[ \tilde{U}_r \equiv \cdots e^{u_{10},r} L_{-10} e^{u_{8},r} L_{-8} e^{u_{6},r} L_{-6} e^{u_{4},r} L_{-4} e^{u_{2},r} L_{-2}. \]  
(6.25)

To find the coefficients $u_{n,r}$ appearing in the exponentials, we use
\[ \frac{r}{2} \tan \left( \frac{r}{2} \arctan z \right) = \lim_{N \to \infty} \left[ f_{2,u_{2,r}} \circ f_{4,u_{4,r}} \circ f_{6,u_{6,r}} \circ f_{8,u_{8,r}} \circ f_{10,u_{10,r}} \circ \cdots \circ f_{N,u_{N,r}} (z) \right] \]
\[ = \lim_{N \to \infty} \left[ f_{2,u_{2,r}} (f_{4,u_{4,r}} (f_{6,u_{6,r}} (f_{8,u_{8,r}} (f_{10,u_{10,r}} (\cdots (f_{N,u_{N,r}} (z) \cdots)))))) \right], \]  
(6.26)

where the function $f_{n,u_{n,r}} (z)$ is given by
\[ f_{n,u_{n,r}} (z) = \frac{z}{(1 - u_{n,r} nz^n)^{1/n}}. \]  
(6.27)

By performing the change of variables
\[ s \to \frac{1}{2} (u - u \eta), \quad t \to \frac{1}{2} (u + u \eta), \quad dsdt \to \frac{u}{2} dud\eta, \]  
(6.28)

where $u \in [0, \infty)$ and $\eta \in (-1, 1)$, we are going to numerically evaluate the double integrals coming from equation (6.23).

Employing the above results, let us write the string field (6.22), expanded up to level four states
\[ \Psi^{(1)} = \Psi^{(1)}_{+} + \Psi^{(1)}_{-} \]
\[ = + 0.509038 c_1 |0\rangle + 0.13231 c_{-1} |0\rangle - 0.001576 L_{-2} c_1 |0\rangle + 0.0893356 c_{-2} |0\rangle \\
- 0.0135795 L_{-4} c_1 |0\rangle - 0.00694968 L_{-2} c_{-1} |0\rangle + 0.0231579 L_{-2} L_{-2} c_1 |0\rangle. \]  
(6.29)

To evaluate the gauge invariant overlap using the $L_0$ level truncation scheme, first we perform the replacement $\Psi^{(1)} \rightarrow z^{L_0} \Psi^{(1)}$ and then using the resulting string field $z^{L_0} \Psi^{(1)}$, we define
\[ \langle \mathcal{V} | \Psi^{(1)} \rangle (z) \equiv \langle \mathcal{V} | z^{L_0} \Psi^{(1)} \rangle. \]  
(6.30)

The value of the gauge invariant overlap is obtained just by setting $z = 1$. As we can see, our problem has been reduced to the computation of quantities like $\langle \mathcal{V} | L_{n_1} L_{n_2} \cdots L_{n_p} c_p |0\rangle$ which can be evaluated using equations (6.4)-(6.7), (6.8) and (6.14).

As an example, plugging the level expansion (6.29) into the definition (6.30), we obtain
\[ \langle \mathcal{V} | \Psi^{(1)} \rangle (z) = \left[ \frac{0.79959514}{z} + 0.20783242z - 0.11276868z^3 \right] c_V. \]  
(6.31)
If we set $z = 1$, from equation (6.31) we get about 89% of the expected result for the gauge invariant overlap (4.12). This result may appear good, however, considering the string field (6.22) expanded up to level twenty four states, we obtain about 116% of the expected result. This behavior is in contrast with the case of Schnabl’s solution, where it has been shown that every time we increase the level of the truncated solution, the gauge invariant overlap converges to the expected analytical result [13]. Therefore, as we suspect, for the case of Erler-Schnabl’s solution, by naively setting $z = 1$, we are obtaining a non-convergent result. Recall that in numerical $L_0$ level truncation computations a regularization procedure based on Padé approximants produces desired results for gauge invariant quantities like the energy [2]. Let us see if after applying Padé approximants, we can obtain the expected answer for the case of the gauge invariant overlap.

To obtain a Padé approximant of order $P_{1+4}^n(z)$ for the gauge invariant overlap, we need to know the series expansion of (6.30) up to the order $z^{2n-1}$. For the numerical evaluation, we have considered the string field $\Psi^{(1)}(1)$ expanded up to level twenty four states, so that we obtain a series expansion for (6.30) truncated up to the order $z^{23}$. The explicit expression for the gauge invariant overlap, truncated up to this order, is given by

$$\langle V | \Psi^{(1)}(1) \rangle(z) = \left[ 0.79959514 \frac{z}{z} + 0.20783242z - 0.11276868z^3 + 0.03183002z^5 
+ 0.1105491863z^7 + 0.003197445654z^9 - 0.14509620056z^{11} 
+ 0.0040708415z^{13} + 0.1939886423z^{15} + 0.002321956902z^{17} 
- 0.2468785966z^{19} + 0.0009635172z^{21} + 0.313942988469z^{23} \right] C_V. \quad (6.32)$$

As an illustration of the numerical method based on Padé approximants, let us compute the value of the gauge invariant overlap using a Padé approximant of order $P_{1+4}^4(z)$. First, we express $\langle V | \Psi^{(1)}(1) \rangle(z)$ as the rational function

$$\langle V | \Psi^{(1)}(1) \rangle(z) = P_{1+4}^4(z) = \frac{1}{z} \left[ a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \right] C_V. \quad (6.33)$$

Expanding the right hand side of (6.33) around $z = 0$ up to seventh order in $z$ and equating the coefficients of $z^{-1}$, $z^0$, $z^1$, $z^2$, $z^3$, $z^4$, $z^5$, $z^6$, $z^7$ with the expansion (6.32), we get a system of algebraic equations for the unknown coefficients $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $b_1$, $b_2$, $b_3$ and $b_4$. Solving those equations we get

$$a_0 = 0.799595, \quad a_1 = 0, \quad a_2 = 3.68919, \quad a_3 = 0, \quad a_4 = 2.55861, \quad (6.34)$$
$$b_1 = 0, \quad b_2 = 4.35389, \quad b_3 = 0, \quad b_4 = 2.20925. \quad (6.35)$$

Replacing the value of these coefficients inside the definition of $P_{1+4}^4(z)$ (6.33), and evaluating this at $z = 1$, we get the following value of the gauge invariant overlap

$$P_{1+4}^4(z = 1) = 0.931807965 C_V. \quad (6.36)$$
The results of our calculations are summarized in Table 6.1. As we can see, the value of the gauge invariant overlap evaluated using Padé approximants confirms the expected analytic result (4.12). Although the convergence to the expected answer gets quite slow, by considering higher level contributions, we will eventually reach to the right value of the gauge invariant overlap $\langle \mathcal{V} | \Psi^{(1)} \rangle \rightarrow 1 C_V$.

Table 6.1: The Padé approximation for the value of the gauge invariant overlap $\langle \mathcal{V} | z^{L_0} \Psi^{(1)} \rangle$ divided by $C_V$ and evaluated at $z = 1$. The third column shows the $P_{1+n}^n$ Padé approximation. In the last column, $P_1^{2n}$ represents a trivial approximation, a naively summed series. At each line, we have considered the string field expanded up to level $2n$ states.

| Level | $P_{1+n}^n$ | $P_1^{2n}$ |
|-------|-------------|-------------|
| $n = 0$ | 0.7995951404 | 0.7995951404 |
| $n = 2$ | 0.9343242915 | 0.8946588687 |
| $n = 4$ | 0.9318079653 | 1.0370380866 |
| $n = 6$ | 0.9644587833 | 0.8951393273 |
| $n = 8$ | 0.9815354429 | 1.0931988113 |
| $n = 10$ | 0.9728969059 | 0.8486421716 |
| $n = 12$ | 0.9757472737 | 1.1635486772 |

Finally, let us show the $L_0$ level truncation analysis of the gauge invariant overlap for Jokel’s solution. In order to expand the string field (4.13) in the state space of Virasoro $L_0$ eigenstates, we need to write this string field as follows [7]

$$
\hat{\Psi}_{\text{jok}} = \int_0^\infty dt \frac{e^{-t} r \sin^2 \left( \frac{\pi t}{2} \right) (2 \pi r - r \sin \left( \frac{\pi t}{2} \right) + \pi)}{16 \pi^2} \tilde{U}_r \left( c \left( - \frac{2 \tan \left( \frac{\pi t}{2} \right)}{r} \right) + c \left( \frac{2 \tan \left( \frac{\pi t}{2} \right)}{r} \right) \right) + \int_0^\infty dt \sum_{k=1}^\infty \frac{e^{-t} (-1)^k 2k-3}{\pi^2 (4k^2 - 1)} \sin^4 \left( \frac{\pi t}{2} \right) \tilde{U}_r b_{-2k} c \left( - \frac{2 \tan \left( \frac{\pi t}{2} \right)}{r} \right) c \left( \frac{2 \tan \left( \frac{\pi t}{2} \right)}{r} \right) + \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{e^{-t_1-t_2} (1 + t_1 + t_2)^2 \cos^2 \left( \frac{\pi (t_1-t_2)}{2(t_1+t_2)} \right)}{8 \pi} \tilde{U}_{t_1+t_2} c \left( \frac{2 \tan \left( \frac{\pi (t_1-t_2)}{2(t_1+t_2)} \right)}{1 + t_1 + t_2} \right),
$$

(6.37)

where $r = 1 + t$.

By writing the $c$ ghost in terms of its modes $c(z) = \sum_m c_m / z^{m-1}$ and employing equations (6.25) and (6.37), the string field $\hat{\Psi}_{\text{jok}}$ can be readily expanded and the individual coefficients can be numerically integrated. For instance, let us write the expansion of $\hat{\Psi}_{\text{jok}}$ up to level four states

$$
\hat{\Psi}_{\text{jok}} = + 0.45457753 c_1 |0\rangle + 0.17214438 c_{-1} |0\rangle - 0.03070678 L_{-2} c_{-1} |0\rangle - 0.01400692 b_{-2} c_0 c_1 |0\rangle - 0.0060589 L_{-4} c_1 |0\rangle + 0.02033379 L_{-2} L_{-2} c_1 |0\rangle + 0.16194599 c_{-3} |0\rangle - 0.00976204 b_{-2} c_{-2} c_1 |0\rangle - 0.01053192 L_{-2} c_{-1} |0\rangle + 0.00976204 b_{-2} c_{-1} c_0 |0\rangle + 0.00465417 b_{-4} c_0 c_1 |0\rangle - 0.00308797 L_{-2} b_{-2} c_0 c_1 |0\rangle.
$$

(6.38)
In order to compute the gauge invariant overlap using the $L_0$ level truncation scheme, we perform the replacement $\hat{\Psi}_{\text{Jok}} \rightarrow z^{L_0} \hat{\Psi}_{\text{Jok}}$ and then using the resulting string field $z^{L_0} \hat{\Psi}_{\text{Jok}}$, we define

$$\langle V|\hat{\Psi}_{\text{Jok}}\rangle(z) \equiv \langle V|z^{L_0}\hat{\Psi}_{\text{Jok}}\rangle.$$  \hspace{1cm} (6.39)

It turns out that if we naively set $z = 1$ in (6.39), we obtain a non-convergent result, therefore in the case of Jokel’s solution, we are also required to use Padé approximants.

We have considered the string field $\hat{\Psi}_{\text{Jok}}$ expanded up to level twenty four states, so that we obtain a series expansion for (6.39) truncated up to the order $z^{23}$. The explicit expression for the gauge invariant overlap, truncated up to this order, is given by

$$\langle V|\hat{\Psi}_{\text{Jok}}\rangle(z) = \left[\frac{0.71404871}{z} + 0.27040377z - 0.11286698z^3 + 0.03411771z^5 \right.\
+ 0.13303393978z^7 + 0.0051412823z^9 - 0.17797842572z^{11}\
+ 0.00302494385z^{13} + 0.24163840461z^{15} + 0.0057271144z^{17}\
- 0.30732930326z^{19} - 0.00086271048z^{21} + 0.3881263427z^{23}\left.\right]C_V. \hspace{1cm} (6.40)$$

Starting from this expression (6.40), we have computed the value of the gauge invariant overlap using Padé approximants of order $P^{n}_{1+n}(z)$. Since these computations are similar to the ones developed in the case of Erler-Schnabl’s solution, at this point we only present the results which are shown in table 6.2. We observed that the value of the gauge invariant overlap evaluated using Padé approximants confirms the expected analytic result.

**Table 6.2:** The Padé approximation for the value of the gauge invariant overlap $\langle V|z^{L_0}\hat{\Psi}_{\text{Jok}}\rangle$ divided by $C_V$ and evaluated at $z = 1$. The third column shows the $P^{n}_{1+n}$ Padé approximation. In the last column, $P^{2n}_1$ represents a trivial approximation, a naively summed series. At each line, we have considered the string field expanded up to level $2n$ states.

| Level | $P^{n}_{1+n}$ | $P^{2n}_1$ |
|-------|--------------|------------|
| $n = 0$ | 0.7140487176 | 0.7140487176 |
| $n = 2$ | 4.0948229924 | 0.8715855076 |
| $n = 4$ | 8.9042818456 | 1.0387366169 |
| $n = 6$ | 12.9506363141 | 0.8658994735 |
| $n = 8$ | 16.9699784236 | 1.1105628220 |
| $n = 10$ | 20.9642533690 | 0.8089606332 |
| $n = 12$ | 24.9715134811 | 1.1962242655 |

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7 Summary and discussion

Through analytical and numerical techniques, we have evaluated the gauge invariant overlap for solutions within the $KBc$ algebra. In order to numerically analyze the gauge invariant overlap, we have used two types of expansions for the truncated solutions, namely, the curly $L_0$ and the Virasoro $L_0$ level expansions.

We have shown that when we expand a solution $\Psi$ in the basis of curly $L_0$ eigenstates, the resulting expression for the gauge invariant overlap $\langle I | V(i) | \Psi \rangle$ is given in terms of a finite series and so the use of Padé approximants was not necessary. This is a quite generic result provided that the solution belongs to the state space constructed out of elements in the $KBc$ algebra. As explicit examples, we have presented the results for the case of Schnabl’s, Erler-Schnabl’s and Jokel’s solutions.

Regarding the Virasoro $L_0$ level truncation analysis of the gauge invariant overlap for Erler-Schnabl’s and Jokel’s solutions, we have shown that the resulting expressions are given in terms of divergent series which nevertheless using Padé approximants can be numerically evaluated. These results are in contrast to the case of Schnabl’s original solution where the expression of the gauge invariant overlap obtained from Virasoro $L_0$ level truncation computations becomes a convergent series, therefore, in that case [13], there was no need for using Padé approximants.

Our original motivation for studying the level truncation analysis of the gauge invariant overlap has been to prepare a numerical background to analyze more cumbersome solutions, such as the multibrane solutions [22], however there are problems that can arise when using the $KBc$ algebra to construct such solutions, for instance, depending on the regularization used to define the solutions, the analytic computation of the energy and the gauge invariant overlap becomes ambiguous [26, 27]. moreover, these solutions are not well defined when expanded in the basis of Virasoro $L_0$ eigenstates [28].

With the hope of constructing well-behaved solutions other than the tachyon vacuum, recently, the $KBc$ algebra has been extended to a larger algebra given as a string field representation of the Virasoro algebra [29]. Since the evaluation of the gauge invariant overlap is simpler than the energy, it should be nice to extend the results presented in our work in order to compute the gauge invariant overlap for solutions constructed within the proposed Mertes-Schnabl’s algebra.

Finally, we would like to comment that the evaluation of the gauge invariant overlap can be generalized for solutions in the context of superstring field theory [30, 31, 32]. For instance, we should analyze the gauge invariant overlap for solutions constructed out of elements in the so-called $GKBc_\gamma$ algebra introduced in references [33, 34, 35, 36, 37]. The analytic computation of this gauge invariant quantity has already been presented for some particular solutions [38, 39, 40], however it remains the numerical analysis.
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