Spin precession of Dirac particles in Kerr geometry

Anusar Farooqui

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal QC H3A 0B9, Canada

E-mail: farooqui@math.mcgill.ca

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Abstract
We isolate and study the transformation of the intrinsic spin of Dirac particles as they propagate along timelike geodesics in Kerr geometry. Reference frames play a crucial role in the definition and measurement of the intrinsic spin of test particles. We show how observers located in the outer geometry of Kerr black holes may exploit the symmetries of the geometry to set up reference frames using purely geometric, locally-available information. Armed with these geometrically-defined reference frames, we obtain a closed-form expression for the geometrically-induced spin precession of Dirac particles in the outer geometry of Kerr black holes. We show that the spin of Dirac particles does not precess on the equatorial place of Kerr geometry; and hence, in Schwarzschild geometry.

Keywords: Kerr geometry, spin precession, Dirac equation, reference frames

(Some figures may appear in colour only in the online journal)

1. Introduction

In [FKP14], we saw how observers located in Kerr geometry can communicate information by exchanging polarized photons. In this paper, we consider the problem of observers in Kerr geometry trying to communicate information by exchanging massive spin-\(\frac{1}{2}\) particles instead.

In the massive spin-\(\frac{1}{2}\) case, one encounters two problems that were not present in the photonic case covered in [FKP14].

Suppose Alice encodes information in the spin of a massive spin-\(\frac{1}{2}\) particle and sends it to Bob. First, she needs to ensure that the particle’s trajectory will intersect Bob’s worldline. Spin-\(\frac{1}{2}\) particles are described by spinor fields that obey the Dirac equation. It is not immediate how to relate a given solution to the Dirac equation, a spinor field, to a timelike geodesic
along which the test particle propagates. Second, whereas a photon’s polarization 4-vector is parallel propagated and remains orthogonal to its 4-velocity, there is no such propagation law for the spin vector of a Dirac particle. Following [Aud81], we address these two problems by using the semiclassical ansatz for the Dirac equation, which allows us to recover a timelike geodesic along which the test particle propagates and derive a propagation law for the spin vector.

Terashima and Ueda’s seminal paper [TU04] outlined a strategy for evaluating the spin precession induced by the motion of spin-$\frac{1}{2}$ particles in a curved spacetime; a strategy that has since been followed by numerous authors including [AF12, AJK09, Lan12, PTW12, RPGC11, SA10]. The strategy outlined in [TU04] is to calculate the Wigner rotation induced by the instantaneous local Lorentz transformation relating the particle’s 4-momentum at nearby events along the particle’s worldline. For a particle moving along a timelike geodesic, the strategy assumes that the precession of the intrinsic spin of the particle is determined solely by the rotation of the observer’s frame with respect to a frame that is parallel propagated along the particle’s geodesic worldline1.

[RPGC11, SA10] follow the strategy of [TU04] to obtain the spin precession of spin-$\frac{1}{2}$ particles on circular orbits confined to the equatorial plane of Kerr–Newman geometry. [Lan12] also follows the strategy of [TU04] to obtain the spin precession of spin-$\frac{1}{2}$ particles on circular and radially-infalling geodesic orbits confined to the equatorial plane of Kerr geometry.

We take a very different approach from [TU04]. We work directly with the Dirac equation and, following [Aud81], use the decomposition of the Dirac current due to Gordon to first define the spin vector of a Dirac particle. Then we introduce the semiclassical ansatz for the Dirac equation and thereby recover the geodesic along which the Dirac particle propagates; a strategy first proposed in [Pau32]. We extend the results of [Aud81] and show that the spin vector is parallel propagated along the aforementioned geodesic to $O(\hbar^2)$ in the semiclassical ansatz (theorem 6.2). We develop a new method of constructing a reference frame on purely geometric criteria that allows us to obtain the proper time-dependent rotation of the spin vector in a coordinate independent manner. Our expression for the geometrically-induced precession of the spin vector is valid for a Dirac particle propagating along an arbitrary timelike geodesic in the outer geometry of a Kerr black hole. We also obtain an expression for the spherical curvature of the curve traced out by the spin vector, which allows us to analyze the dynamical behaviour of the spin vector with a single invariant function. Even though our approach is quite different from [TU04], our qualitative result that the spin vector is parallel propagated along a Dirac particle’s geodesic worldline agrees with the assumption underlying their strategy.

Another approach starts with classical results for spinning particles that obey the Mathisson–Papapetrou equations, the so-called pole-dipole approximation [Mat37, Pap51]. Rüd81 shows how to formally obtain the Mathisson–Papapetrou equations from the Dirac equation using the WKB approximation. The main consequence of using the pole-dipole approximation is that the motion of spinning test particles is no longer geodesic [BdFG04, CM08, dKvdVvH16, Sem99]. This poses an insurmountable problem from the perspective of the present study due to the absence of a meaningful measurement protocol adapted to particles undergoing zitterbewegung. Since our goal is to understand how observers located in the outer geometry of Kerr black holes may exploit the spin of Dirac particles as a resource to exchange quantum

1 If the particle is accelerated, there is an additional term that arises from boosting the 4-momentum along the worldline of the particle.
information, we ignore the effect of spin on motion and instead focus on the effect of geodesic motion on the spin of test particles.

The rest of this paper is organized as follows. In section 2, we recall some of the salient geometric properties of the Kerr metric that will be used in this paper and obtain the equations of motion by quadratures (equivalent to those originally obtained by [Car68]). In section 3, we introduce the Dirac equation, present our choice of representations of the Clifford and Lorentz algebras, provide the decomposition of the Dirac current due to Gordon, and define the spin vector of a Dirac particle. Next, in section 4, we introduce the semiclassical ansatz for the Dirac equation and thereby recover the classical trajectory of a Dirac particle. We construct a parallel propagated frame along arbitrary timelike geodesics in Kerr geometry in section 5, and prove a propagation law for the spin vector in Kerr geometry in section 6. Then, we construct a reference frame on purely geometric criteria in section 7. In section 8, we formally define and provide an explicit expression for the geometrically-induced precession of the spin vector; and obtain a spherical curvature invariant for the curve traced out by the spin vector. We conclude with a discussion of the findings in section 9.

Remark 1.1 (Notation). We shall work throughout in natural units: \( G = c = 1 \). We shall in general follow the notational conventions of [Cha92]. We shall reserve lower case Latin indices, \( a, b, c, \ldots \), for arbitrary orthonormal frames in which the metric takes the form \( \eta^{ab} := \text{diag} \{1, -1, -1, -1\} \); bracketed Latin indices, \( (a), (b), (c), \ldots \), for orthonormal frames that are parallel propagated along a timelike geodesic and in which the metric takes the form \( \eta^{(a)(b)} := \text{diag} \{1, -1, -1, -1\} \); hatted Latin indices, \( \hat{i}, \hat{j}, \hat{k}, \ldots \), for the spacelike components in an orthonormal frame; unhatted Latin indices \( i, j, k, \ldots \), and Greek indices, \( \alpha, \beta, \gamma, \ldots \), for spacetime coordinates in which the line element takes the form \( ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \).

We shall sometimes find it convenient to use semicolons to denote covariant derivatives, e.g. \( \nabla_\alpha \Psi = \nabla_\alpha \Psi \), whereas commas will denote ordinary partial derivatives, e.g. \( f_\alpha := \frac{\partial f}{\partial x^\alpha} \).

We will, on occasion, use index-free notation as follows. Given a metric \( g_{ij} \) and vector field \( X^i := \partial \), the 1-form dual to \( X \) will be denoted by \( X^{\flat} \), whose components are given by \( X^{\flat} = g_{ij} X^j \). Similarly, given a 1-form \( \omega^i := \omega_i dx^i \), the vector field dual to \( \omega \) will be denoted by \( \omega^i \), whose components are given by \( \omega^i := g^{ij} \omega_j \).

2. Kerr geometry

Remark 2.1. This section is adapted in part from [FKP14].

The Kerr metric is a 2-parameter family of solutions to the Einstein’s equations for the vacuum defined on the manifold \( \mathcal{M} = \mathbb{R}^2 \times S^2 \). In Boyer–Lindquist coordinates \( x^i = (t, r, \vartheta, \varphi) \) with \(-\infty < t < +\infty, r_*, < r < +\infty, 0 \leq \vartheta \leq \pi, 0 \leq \varphi < 2\pi \), the Kerr metric takes the form

\[
\text{d} s^2 = \frac{\Delta}{\Sigma} (\text{d} t - a \sin^2 \vartheta \text{d} \varphi)^2 - \frac{\Sigma}{\Delta} \text{d} r^2 - \Sigma \text{d} \vartheta^2 - \frac{\sin^2 \vartheta}{\Sigma} (\text{d} r^2 - (r^2 + a^2) \text{d} \varphi^2),
\]

(2.1)

with

\[
\Sigma(r, \vartheta) := r^2 + a^2 \cos^2 \vartheta, \quad \Delta(r) := r^2 - 2Mr + a^2.
\]

The parameters \( M > 0 \) and \( a \geq 0 \) labeling the solutions within the Kerr family correspond respectively to the mass and angular momentum per unit mass of the black hole, as measured
from infinity. We shall restrict our attention to the non-extreme case \( M > a \geq 0 \), in which case the function \( \Delta(r) \) has two distinct zeros,

\[
r_\pm = M \pm \sqrt{M^2 - a^2}.
\] (2.3)

The metric (2.1) can be analytically continued across the hypersurfaces \( r = r_+ \) and \( r = r_- \) in such a way that these become null hypersurfaces in the maximal extension. The null hypersurface \( r = r_+ \) corresponds to the event horizon, while \( r = r_- \) corresponds to the Cauchy horizon of the black hole. We shall only be interested in the region \( r > r_+ \), which describes the geometry outside the event horizon of the black hole.

The hypersurface determined by the equation,

\[
g_{\vartheta \vartheta} = 1 - \frac{2Mr}{r^2 + a^2 \cos^2 \vartheta} = 0,
\] (2.4)

is called the ergosphere. It is a stationary limit surface in the sense that it is the boundary of the region on which the Killing vector field \( \partial_r \) is timelike. The hypersurface determined by equation (2.4) coincides with the event horizon only at the poles \( \vartheta = 0 \) and \( \vartheta = \pi \).

The Kerr metric enjoys a number of remarkable symmetries. First of all, the Kerr metric admits a two-parameter Abelian isometry group that acts orthogonally transitively on timelike orbits, meaning that the orbits of the group action are timelike 2-surfaces with the property that the distribution of 2-planes orthogonal to the orbits is integrable. The orthogonal transitivity is manifest in the Boyer–Lindquist coordinates since the metric does not admit cross terms mixing the differentials \( dr, d\vartheta \) with the differentials \( d\varphi, dt \). In Boyer–Lindquist coordinates, the action of the continuous part of the isometry group is generated by the flows of the pair of commuting Killing vector fields \( \partial_t \) and \( \partial_\varphi \), and thus given by

\[
(t, r, \vartheta, \varphi) \mapsto (t + c_1, r, \vartheta, \varphi + c_2),
\] (2.5)

where \( c_1, c_2 \) are arbitrary real constants. Furthermore, the isometry group of the Kerr metric admits a discrete subgroup isomorphic to \( \mathbb{Z}_2 \), whose action is not of the form (2.5). More precisely, we say following Carter’s terminology, that the isometry group is invertible, meaning that at every \( x \in M \), there exists a \((1,1)\)-tensor \( L_x \in \text{End}(T_x M) \), which acts as an involutive isometry of \((T_x M, g_x)\) and is such that if \( O_x \) denotes the orbit of the isometry group through \( x \), then

\[
L_x |_{(T_x O_x)^1} = \text{id}|_{(T_x O_x)^1},
\] (2.6)

and for all \( X_x \in T_x O_x \)

\[
L_x (X_x) = -X_x.
\] (2.7)

Carter showed that if an isometry group acts orthogonally transitively on non-null orbits then the action is necessarily invertible [Car69]. In Boyer–Lindquist coordinates, the involution is given by

\[
L_x = f_x \big|_x,
\] (2.8)

where \( f \) is the isometry given by

\[
(t, r, \vartheta, \varphi) \mapsto (-t, r, \vartheta, -\varphi).
\] (2.9)

We will commit an abuse of notation and denote both \( L_x = f_x \big|_x \) and the dual map \( f^* \big|_x \) by \( L \).

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2 This result is not true if the orbits of the isometry group are null.
The Weyl conformal curvature tensor of the Kerr solution is of Petrov type D, meaning that it admits a pair of repeated principal null directions, each of which is defined up to multiplication by a non-zero scalar function. These repeated principal null directions give rise to null congruences which are geodesic and shear-free as a consequence of the Goldberg–Sachs theorem. We choose the scale factors in such a way that the principal null directions are given by

\[ \ell = \ell^i \frac{\partial}{\partial x^i} = \frac{1}{\sqrt{2\Sigma\Delta}} \left( (r^2 + a^2) \frac{\partial}{\partial t} + \sqrt{\Delta} \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right), \] (2.10)

and

\[ n = n^i \frac{\partial}{\partial x^i} = \frac{1}{\sqrt{2\Sigma\Delta}} \left( (r^2 + a^2) \frac{\partial}{\partial t} - \sqrt{\Delta} \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right). \] (2.11)

The choice of scale factors leading to (2.10) and (2.11) will be characterized geometrically through the involutive isometry (2.8) as follows.

**Definition 2.2 (Symmetric frame).** Our null coframe is chosen such that

\[ L\vartheta^1 = -\vartheta^2, \quad L\vartheta^2 = -\vartheta^1, \quad L\vartheta^3 = -\vartheta^4, \quad L\vartheta^4 = -\vartheta^3, \] (2.12)

where \( L \) is the involutive isometry (2.8). We refer to this frame as the *symmetric null coframe*. It is given in Boyer–Lindquist coordinates by

\[ \vartheta^1 = \frac{1}{\sqrt{2\Sigma\Delta}} (\Delta dt + \Sigma dr - a \sin^2 \vartheta \Delta d\varphi), \] (2.13)

\[ \vartheta^2 = \frac{1}{\sqrt{2\Sigma\Delta}} (\Delta dt - \Sigma dr - a \sin^2 \vartheta \Delta d\varphi), \] (2.14)

\[ \vartheta^3 = \frac{1}{\sqrt{2\Sigma}} ((r^2 + a^2) \sin \vartheta d\varphi - i \Sigma d\vartheta - a \sin \vartheta \sin \vartheta d\varphi), \] (2.15)

\[ \vartheta^4 = \frac{1}{\sqrt{2\Sigma}} ((r^2 + a^2) \sin \vartheta d\varphi + i \Sigma d\vartheta - a \sin \vartheta \sin \vartheta d\varphi). \] (2.16)

The definition eliminates the scaling freedom we would have otherwise had in defining a null coframe adapted to the principal null directions of the Weyl tensor. The *orthonormal symmetric coframe* \((\omega^0, \omega^1, \omega^2, \omega^3)\) corresponding to the symmetric null coframe \((\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4)\) is then defined by

\[ \omega^0 = \frac{1}{\sqrt{2}} (\vartheta^1 + \vartheta^2), \quad \omega^3 = \frac{1}{\sqrt{2}} (\vartheta^1 - \vartheta^2), \quad \omega^2 = -\frac{1}{\sqrt{2}} (\vartheta^3 + \vartheta^4), \quad \omega^4 = \frac{i}{\sqrt{2}} (\vartheta^1 - \vartheta^2), \] (2.17)

and given in Boyer–Lindquist coordinates by

\[ \omega^0 = \frac{\Delta}{\Sigma} (dt - a \sin^2 \vartheta d\varphi), \] (2.18)

\[ \omega^1 = \frac{\Sigma}{\Delta} dr, \] (2.19)
\[
\omega^2 = \frac{\sin \vartheta}{\sqrt{\Sigma}} (a \, dt - (r^2 + a^2) \, d\varphi), \quad (2.20)
\]

\[
\omega^3 = \sqrt{\Sigma} \, d\vartheta. \quad (2.21)
\]

We have already seen that the symmetric frame is closely tied to intrinsic geometric properties of the Kerr black hole. We now define observers in terms of the principal null directions of the Weyl tensor and the involution \(L\).

**Definition 2.3 (Carter observers).** The vector field,

\[
U := \frac{1}{\sqrt{2}} (\ell + n) = \frac{1}{\sqrt{\Sigma \Delta}} \left( (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi} \right), \quad (2.22)
\]

where \(\ell\) and \(n\) are given by (2.10) and (2.11), is timelike and future-pointing, and identifies a family of observers, that we call *Carter observers*, whose 4-velocities are symmetric linear combinations of the principal null directions of the Weyl tensor.

Carter observers exist everywhere outside the event horizon including the region between the event horizon and the stationary limit surface defined by equation (2.4) where the stationary Killing field \(\xi = \partial_t\) becomes null. Their angular velocity is \(\frac{a}{r^2 - a^2}\), which is exactly the angular velocity of the event horizon; both as measured at infinity. Therefore, this class of observers is uniquely suited to analyze the behaviour of test particles near the horizon.

We choose the observers’ frames to be duals of the symmetric coframe defined in equations (2.18)–(2.21). These frames can constructed using locally-available geometric data. Specifically, the construction of the symmetric frame only requires knowledge of the principal null directions of the Weyl tensor. [Mar99] observes that, in the Kerr family of solutions, the principal null directions of the Weyl tensor coincide with the principal null directions of the Killing 2-form defined by

\[
\Upsilon_{a,b} := \nabla_a \xi_b + i \star \nabla_a \xi_b, \quad (2.23)
\]

where \(\star\) denotes the Hodge dual and \((\xi^a)\) is the stationary Killing field. The principal null directions of the Weyl tensor \(\ell\) and \(n\), given in Boyer–Lindquist coordinates by (2.10) and (2.11), are thus doubly significant.

In addition to its two-parameter Abelian group of isometries, the Kerr metric possesses further symmetries that are ’hidden’ in the sense that they cannot be represented by Killing vector fields. The existence of these hidden symmetries is closely tied to the fact that all the known massless and massive wave equations are separable in Kerr geometry. The geometric object that generates all these additional symmetries is a rank two *Killing–Yano tensor*, that is, a (0,2) skew-symmetric tensor \((f_{ij})\) satisfying the Killing–Yano equation,

\[
\nabla_{(i} f_{j)k} = 0. \quad (2.24)
\]

In Boyer–Lindquist coordinates and in the symmetric orthonormal coframe, any Killing–Yano 2-form is a constant multiple of

\[
f = -a \cos \vartheta \, \omega^0 \wedge \omega^1 + r \omega^2 \wedge \omega^3. \quad (2.25)
\]

The Hodge dual of a Killing–Yano 2-form is given in the orthonormal symmetric coframe by

\[
h := *f = r \omega^0 \wedge \omega^1 + a \cos \vartheta \omega^2 \wedge \omega^3. \quad (2.26)
\]
The Hodge dual of the Killing–Yano 2-form will prove useful in constructing a parallel propagated frame along null geodesics in Chapter 3.

The role played by this Killing–Yano tensor in the separability properties of the Kerr metric stems from the fact that it appears as a ‘square root’ of the quadratic first integral discovered by Carter in his proof of the separability in Kerr geometry of the Hamilton–Jacobi equation for geodesics and the Klein-Gordon equation for massive scalar fields [Car68]. More precisely, the symmetric (0,2)-tensor \( K_{ij} \) defined by

\[
K_{ij} = f_{jk} f^k_j, \tag{2.27}
\]

satisfies the Killing equation

\[
\nabla_i K_{jkl} = 0, \tag{2.28}
\]

and therefore gives rise to a quadratic first integral for the geodesic flow in Kerr geometry first discovered by Carter,

\[
\kappa = K_{ij} U^i U^j, \tag{2.29}
\]

where \((U^i)\) is the 4-velocity. In Boyer–Lindquist coordinates and in the symmetric orthonormal coframe, the symmetric Killing (0,2)-tensor defined by equation (2.27) is given by

\[
K = a^2 \cos^2 \theta (\omega^0 \otimes \omega^0 - \omega^1 \otimes \omega^1) + r^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3). \tag{2.30}
\]

The quadratic first integral defined by equation (2.29) exists in addition to the two linear first integrals arising from the presence of the two commuting Killing vector fields \( \partial_t \) and \( \partial_\phi \) and therefore reduces the integration of the geodesic flow to quadratures.

We now show how to obtain the solution to the geodesic equations directly from the equations for the integrals of motion. The integration by quadratures of the geodesic equations presented below leads to results equivalent to those originally obtained by [Car68].

The integral curves of a vector field \((U^i)\) are geodesics if and only if

\[
U^i \nabla_i U^j = 0. \tag{2.31}
\]

Equation (2.31) implies that the norm of \((U^i)\),

\[
\mu := g_{ij} U^i U^j, \tag{2.32}
\]

is an integral of motion. We may, without loss of generality, choose the affine parameter such that

\[
\mu = \begin{cases} 
+1 & \text{for timelike geodesics,} \\
-1 & \text{for spacelike geodesics,} \\
0 & \text{for null geodesics.}
\end{cases} \tag{2.33}
\]

The fact that the Kerr metric admits a 2-parameter Abelian group of isometries implies the existence of 2 commuting Killing vector fields which give rise to two more integrals of motion in addition to the norm of the 4-velocity. The integral of motion arising from the stationary Killing field \( \partial_t \) in Boyer–Lindquist coordinates is called energy and denoted by \( E \). The integral of motion associated to the generator of axisymmetry \( \partial_\phi \) in Boyer–Lindquist coordinates is called angular momentum and denoted by \( \Phi \).

Letting dot denote derivatives with respect to an affine parameter and \( U^i := (i, \dot{r}, \dot{\theta}, \dot{\phi}) \) denote the 4-velocity, we immediately obtain the equations for the integrals of motion associated to the Killing vector fields,
\[ E := p_\varphi = \left( 1 - \frac{2Mr}{\Sigma} \right) i + \frac{2Mra \sin^2 \vartheta}{\Sigma} \varphi, \quad (2.34) \]

\[ \Phi := -p_\varphi = -\frac{2Mra \sin^2 \vartheta}{\Sigma} i + \left( r^2 + a^2 + \frac{2Mra^2 \sin^2 \vartheta}{\Sigma} \right) \sin^2 \varphi. \quad (2.35) \]

Equations (2.34) and (2.35) imply that

\[ a \sin \vartheta i - (r^2 + a^2) \sin \vartheta \varphi = aE \sin \vartheta - \frac{\Phi}{\sin \vartheta}, \quad (2.36) \]

\[ \Delta i - a\Delta \sin^2 \vartheta \varphi = E(r^2 + a^2) - a\Phi. \quad (2.37) \]

We thus obtain

\[ \Sigma \Delta i = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \vartheta)E - 2Mra\Phi, \quad (2.38) \]

\[ \Sigma \Delta \varphi = 2MraE + \frac{\Delta - a^2 \sin^2 \vartheta}{\sin^2 \vartheta} \Phi. \quad (2.39) \]

Substituting the expressions for \( i \) and \( \varphi \) given by equations (2.38) and (2.39) into the equation for the conserved norm of the 4-velocity,

\[ \mu = g_{ij} U^i U^j, \quad (2.40) \]

yields

\[ \Sigma \Delta \left( \mu + \frac{\Sigma}{\Delta} r^2 + \Sigma \varphi^2 \right) = [(r^2 + a^2)^2 - a^2 \Delta \sin \vartheta] E^2 - 4MraE \Phi \]

\[ + \frac{a^2 \sin^2 \vartheta - \Delta}{\sin^2 \vartheta} \Phi^2, \quad (2.41) \]

while substituting \( i \) and \( \varphi \) into the equation for Carter’s fourth integral of motion,

\[ \kappa = K_\varphi U^i U^j, \quad (2.42) \]

yields

\[ \Sigma \Delta \left( a^2 \cos^2 \vartheta \frac{\Sigma}{\Delta} r^2 - \kappa - r^2 \Sigma \varphi^2 \right) = [r^2 \Delta \sin^2 \vartheta + \cos^2 \vartheta (r^2 + a^2)] a^2 E^2 \]

\[ - [a^2 \cos^2 \vartheta (r^2 + a^2) + r^2 \Delta] 2aE \Phi \]

\[ + \frac{r^2 \Delta + a^2 \cos^2 \vartheta \sin^2 \varphi}{\sin^2 \varphi} \Phi^2. \quad (2.43) \]

Combining the equations (2.41) and (2.43) we obtain

\[ \Sigma^2 \varphi^2 = (E(r^2 + a^2) - a\Phi)^2 - \Delta(\kappa + r^2 \mu), \quad (2.44) \]

\[ \Sigma^2 \varphi^2 = \kappa - a \cos^2 \vartheta \mu - \left( aE \sin \vartheta - \frac{\Phi}{\sin \vartheta} \right)^2, \quad (2.45) \]
Remark 2.4. The equations (2.44) and (2.45), are not decoupled since $\Sigma(r, \vartheta)$ is a function of both non-ignorable coordinates.

In order to reduce the equations (2.44) and (2.45) to a set of decoupled ordinary differential equations, it is convenient to introduce Mino time, $\lambda$, first discussed in [Min03], which is defined in terms of the affine parameter, $\tau$, by the relation

$$\frac{d\tau}{d\lambda} = r^2 + \cos^2 \vartheta.$$  

(2.46)

Our basic equations for timelike geodesics ($\mu = +1$), then, are

$$\left(\frac{dr}{d\lambda}\right)^2 = (E(r^2 + a^2) - a\Phi)^2 - \Delta(\kappa + r^2),$$  

(2.47)

$$\left(\frac{d\vartheta}{d\lambda}\right)^2 = \kappa - a\cos^2 \vartheta - \left(aE\sin \vartheta - \frac{\Phi}{\sin \vartheta}\right)^2,$$  

(2.48)

$$\frac{dr}{d\lambda} = \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \vartheta E - \frac{2Mra}{\Delta} \Phi,$$  

(2.49)

$$\frac{d\varphi}{d\lambda} = \frac{2MraE}{\Delta} + \Delta - a^2 \sin^2 \vartheta \Phi.$$  

(2.50)

Remark 2.5. One is not free to choose the signs of $\frac{dr}{d\lambda}$ and $\frac{d\vartheta}{d\lambda}$. A choice of integrals of motion fixes a congruence of geodesics determined by equations (2.47)–(2.50). A choice of initial data $(r_0, \vartheta_0)$ fixes the branches of the Riemann surfaces associated to the elliptic equations (2.47) and (2.48) respectively, which in turn determine the signs of $\frac{dr}{d\lambda}$ and $\frac{d\vartheta}{d\lambda}$.

Mino time $\lambda$ is not an affine parameter. It can be eliminated by rewriting equations (2.47) and (2.48) in integral form as

$$\int_{r_0}^{r} \frac{dr}{\sqrt{R(r)}} = \int_{\vartheta_0}^{\vartheta} \frac{d\vartheta}{\sqrt{\Theta(\vartheta)}},$$  

(2.51)

where

$$R(r) := (E(r^2 + a^2) - a\Phi)^2 - \Delta(\kappa + r^2)\mu,$$  

(2.52)

$$\Theta(\vartheta) := \kappa - a\cos^2 \vartheta \mu - \left(aE\sin \vartheta - \frac{\Phi}{\sin \vartheta}\right)^2.$$  

(2.53)

We now turn to the Dirac equation.

3. The Dirac equation

The Dirac equation is given by

$$i\hbar \gamma^a \nabla_a \Psi = m\Psi,$$  

(3.1)

As claimed, for instance, in [Cha92].
where $\Psi$ is a 4-component spinor field, $m > 0$ is the rest mass of the Dirac particle, $\nabla_a$ is the covariant derivative, $h$ is Planck’s constant, and $\gamma^a$ is a representation of the Clifford algebra $C(1, 3)$. 

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \mathbf{1},$$  

(3.2) 

where $\mathbf{1}$ is the unit element of the algebra. In equation (3.1), the frame components of the covariant derivative are defined by

$$\nabla_a := e_a^\mu \nabla_\mu,$$  

(3.3) 

where $e_a$ is an orthonormal frame and the covariant derivative for spinor fields $\Psi$ is defined by

$$\nabla_\mu \Psi \equiv \psi_{\mu} := \psi_{\mu} + \frac{1}{8} \Gamma_{\mu}^{ab} \gamma^a \gamma^b \psi_{\mu},$$  

(3.4) 

where the connection $\Gamma_{\mu}^{ab}$ is given in terms of the Levi-Civita connection $\Gamma^a_{\mu \nu}$ of the metric $g_{\mu \nu}$ by

$$\Gamma_{\mu}^{ab} := e_b^\alpha (e_\mu^\alpha, \partial_\mu e_\beta) + \partial_\mu e_\beta e_\alpha.$$

(3.5) 

Recall that the proper, orthochronous Lorentz group, denoted by $SO^+;(1, 3)$, is the connected component of the Lorentz group containing the identity. The Lie algebra of $SO^+;(1, 3)$ is isomorphic to the Lie algebra of its universal cover,

$$SL(2, \mathbb{C}) \longrightarrow SO^+;(1, 3).$$

(3.6) 

We shall henceforth refer to the Lie algebra of $SL(2, \mathbb{C})$ as the Lorentz algebra and denote it by $\mathfrak{sl}(2, \mathbb{C})$. 

**Remark 3.1.** The Dirac equation (3.1) combines two inequivalent representations of $\mathfrak{sl}(2, \mathbb{C})$, denoted by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, and related to each other by a parity transformation. That is, we are working with a $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation.

As a direct consequence of the Clifford algebra relations (3.2), the commutators of the gamma matrices,

$$\sigma^{ab} := \frac{i}{2} [\gamma^a, \gamma^b],$$

(3.7) 

constitute a representation of the Lorentz algebra $\mathfrak{sl}(2, \mathbb{C})$ defined by the relations

$$[\sigma^{ab}, \sigma^{cd}] = 4i (\eta^{ac} \sigma^{bd} - \eta^{ad} \sigma^{bc} + \eta^{bc} \sigma^{ad} - \eta^{bd} \sigma^{ac}).$$

(3.8) 

The generators of the Lorentz algebra (3.7), satisfying the commutation relations (3.8), will play a key role in our definition of the spin vector for Dirac particles. We shall be using the standard representation of the Clifford algebra given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} (i = 1, 2, 3),$$

(3.9) 

where $I$ is the $2 \times 2$ identity matrix. The $2 \times 2$ Pauli spin matrices $\sigma^I$ are explicitly by

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Given a spinor field \( \Psi \), the adjoint spinor is defined by \( \bar{\Psi} := \Psi^\dagger \gamma^0 \), where the dagger denotes complex conjugation and transposition. If \( \Psi \) satisfies the Dirac equation (3.1), then \( \bar{\Psi} \) satisfies the adjoint Dirac equation,

\[
i \gamma^a \bar{\Psi} \gamma^a = -m \bar{\Psi}.
\]

(3.11)

For a spinor field \( \Psi \) satisfying the Dirac equation (3.1), the quantity

\[
j^a := \bar{\Psi} \gamma^a \Psi,
\]

(3.12)

defines a vector field called the Dirac current. The Dirac current is conserved, that is,

\[
\nabla a^a = 0,
\]

(3.13)
as can be seen by multiplying equation (3.1) on the left by \( \bar{\Psi} \), multiplying equation (3.11) on the right by \( \Psi \), and adding the two terms together to eliminate the mass term.

**Remark 3.2.** Since the vector field given by (3.12) is in general not tangent to a geodesic, it cannot possibly be the classical trajectory of a test particle.

We now turn to Gordon’s result [Gor28], which demonstrates that the Dirac current decomposes into two parts which are separately conserved.

**Theorem 3.3 (Gordon decomposition of the Dirac current).** The Dirac current decomposes into convection and polarization 4-currents which are separately conserved. More precisely,

\[
\Psi \gamma^a \Psi := j^a = j^a_{\text{polar}} + j^a_{\text{con}},
\]

(3.14)

where

\[
j^a_{\text{polar}} := \frac{\hbar}{2m} (\bar{\Psi} \sigma^{ab} \Psi)_{,b},
\]

(3.15)

is the polarization current satisfying \( \nabla a^a_{\text{polar}} = 0 \) and

\[
j^a_{\text{con}} := \frac{\hbar}{2mi} (\bar{\Psi} \gamma^a \Psi - \Psi \gamma^a \bar{\Psi}),
\]

(3.16)

is the convection 4-current satisfying \( \nabla a^a_{\text{con}} = 0 \).

**Proof.** The decomposition follows from noting that

\[
\frac{2}{i} (\bar{\Psi} \sigma^{ab} \Psi)_{,b} = (\bar{\Psi} \gamma^a \gamma^b \Psi)_{,b} - (\bar{\Psi} \gamma^b \gamma^a \Psi)_{,b}
\]

(3.17)

\[
= \frac{2m}{i\hbar} j^a + \bar{\Psi} \gamma^a \gamma^b \Psi - \bar{\Psi} \gamma^b \gamma^a \Psi_{,b}
\]

(3.18)

\[
= \frac{4m}{i\hbar} j^a + 2(\bar{\Psi} \gamma^a \Psi - \Psi \gamma^a \bar{\Psi}),
\]

(3.19)

where we have used the Dirac equation and its conjugate repeatedly, along with the symmetrization and antisymmetrization of the Dirac gamma matrices. The conservation of the convection current follows from recalling Lichnerowicz’ identity [Lic64],

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\[ \Delta = - \nabla^a \nabla_a + \frac{1}{4} R, \quad (3.20) \]

where \( R := \text{trace(Ric)} \) is the scalar curvature and \( \Delta \) is the Laplace operator whose action on the Dirac spinor and its adjoint is given by

\[ \Delta \Psi = \gamma^{abc} \nabla_a \gamma^{bcd} \Psi = \left( \frac{m}{\hbar} \right)^2 \Psi, \quad (3.21) \]

and

\[ \Delta \bar{\Psi} = \nabla_a \gamma^a \gamma^{bcd} \bar{\Psi} = \left( \frac{m}{\hbar} \right)^2 \bar{\Psi}, \quad (3.22) \]

respectively. It follows that

\[ (\bar{\Psi} \gamma^a \Psi, a) \rho a = (\gamma^a \psi \gamma^a \Psi \rho a) - \bar{\Psi} (\nabla^a \nabla_a \Psi) \]

\[ = \frac{1}{4} R \Psi \rho a - (\Delta \Psi) \Psi - \frac{1}{4} R \Psi \rho a + \bar{\Psi} (\Delta \Psi), \quad (3.23) \]

\[ = \left( \frac{m}{\hbar} \right)^2 (- \bar{\Psi} \Psi + \bar{\Psi} \Psi) \]

\[ = 0. \quad (3.25) \]

Since both the Dirac current and the convection current are conserved, the polarization current, defined by equation (3.15), must be conserved as well. \( \square \)

For a given Dirac spinor \( \Psi \), the convection current \( j^a_{\text{con}} \) defines a congruence of timelike curves with unit tangent vector \( K^a \) defined by:

\[ K^a := \frac{j^a_{\text{con}}}{\sqrt{j_{\text{con}} j^a_{\text{con}} j^b_{\text{con}}}}. \quad (3.27) \]

The integral curve of the vector field \( K^a \) is also generically non-geodesic and hence cannot serve as the classical trajectory of a particle described by a spinor field \( \Psi \) that satisfies equation (3.1).

We are now in a position to define the spin vector. The following definition was originally proposed in [Aud81].

**Definition 3.4 (Spin vector).** We define the spin vector associated to the spinor \( \Psi \) by

\[ W^a := \frac{1}{2} \varepsilon^{abcd} K_b \frac{\bar{\Psi} \sigma_{cd} \Psi}{\bar{\Psi} \Psi}, \quad (3.28) \]

where \( \varepsilon^{abcd} \) is the anti-symmetric Levi–Civita symbol given by

\[ \varepsilon^{abcd} := \begin{cases} +1 & \text{if } abcd \text{ is an even permutation of } 0123, \\ -1 & \text{if } abcd \text{ is an odd permutation of } 0123, \\ 0 & \text{for repeated indices.} \end{cases} \quad (3.29) \]
Since the $\sigma$-matrices, defined by equation (3.7), are the generators of the Lorentz algebra (3.8), we may interpret $\bar{\Psi} \sigma^{ab} \Psi$ as spin density. Kirsch $et$ $al$ have shown that the spin operator defined uniquely via the Gordon decomposition corresponds to the Foldy–Wouthuysen mean-spin operator [KRH01]. Meanwhile, Bauke $et$ $al$ have recently shown that, in the Lagrangian formulation, the Foldy–Wouthuysen mean-spin operator is the only known relativistic spin operator that commutes with the free Dirac Hamiltonian, has the eigenvalues $\pm \frac{\hbar}{2}$, and obeys the angular momentum algebra [BAKG14]. Similar results were derived in [CRW13], who further show that the Foldy–Wouthuysen mean-spin operator is the only spin operator proposed so far, that has the right non-relativistic limit and does not convert positive (negative) energy states into negative (positive) energy states (their ‘charge symmetry condition’). These results give us good confidence that the definition 3.4, first proposed in [Aud81], of the spin vector in terms of the Gordon decomposition of the Dirac current is a good one.

We now turn to the semiclassical ansatz which will allow us to recover the classical trajectory of a Dirac particle and derive a propagation law for the spin vector defined by (3.28).

4. Semiclassical ansatz

The semiclassical ansatz for the Dirac equation is a formal power series expansion in Planck’s constant $\hbar$, and given explicitly by

$$\Psi(x) = \exp(iS(x)/\hbar) \sum_{n=0}^{\infty} (-i\hbar)^n a_n(x). \quad (4.1)$$

where $S(x)$ is a scalar field and $a_n (n \geq 0)$ is a countable sequence of 4-component spinor fields. Plugging the ansatz (4.1) into the Dirac equation (3.1) and setting to zero the coefficients of the different powers of $\hbar$, we obtain

$$\gamma^a S_{,a} + mI a_0 = 0, \quad (4.2)$$

$$\gamma^a S_{,a} + mI a_{n+1} = -\gamma^a a_{n+1} (\forall n \in \mathbb{N}), \quad (4.3)$$

The existence of a solution to the homogeneous equation (4.2) requires

$$\det(\gamma^a S_{,a} + mI) = 0. \quad (4.4)$$

Equation (4.4) is equivalent to the Hamilton–Jacobi equation for timelike geodesics,

$$S^a S_{,a} = m^2. \quad (4.5)$$

We define

$$p_a := -S_{,a}, \quad (4.6)$$

and normalize to obtain a unit-norm vector field

$$U^a := \frac{1}{m} p^a = -\frac{1}{m} S^a. \quad (4.7)$$

As a result of the Hamilton–Jacobi equation (4.5), the integral curves of $U$, defined by (4.7), are guaranteed to be timelike geodesics. And since equation (4.2) is the classical limit ($\hbar \to 0$) for the Dirac equation with the semiclassical ansatz, we interpret $p_a$ given by equation (4.6), as the 4-momentum, and $U^a$, given by (4.7), as the 4-velocity of the Dirac particle described by $\Psi$. 
The use of the semiclassical ansatz for the Dirac equation (4.1) and the observation that the homogeneous equation (4.2) implies the Hamilton–Jacobi equation for spinless particles goes back to Pauli [Pau32]. Bargmann et al derived equations describing the classical trajectories of spin-\(\frac{1}{2}\) particles in uniform and constant electric and magnetic fields [BMT59]. Rubinow and Keller showed that the classical equations of Bargmann et al could be obtained from the Dirac equation using the semiclassical ansatz [RK63]. Rafenelli and Schiller obtained essentially the same result soon after, using the so-called ‘squared’ Dirac equation along with the semiclassical ansatz [RS64].

We follow the more recent work by Audretsch who showed that the spin vector, defined by equation (3.28), is parallel propagated along the particle’s trajectory to zeroth order in the asymptotic expansion (4.1) [Aud81]. We will extend the result of [Aud81] and show that, in Kerr geometry, the spin vector is parallel propagated along \(U^{\mu}\) to first-order in \(\hbar\). We begin our analysis with reporting some results from [Aud81].

The matrix acting on \(a_0\) in equation (4.2) is of rank 2, as is manifest by considering a frame that is parallel propagated along the congruence \(U^\mu\) in which the 4-momentum takes the form \(p^{(a)} = (m,0,0,0)\). The matrix acting on \(a_0\) is then given by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2m & 0 & 0 \\
0 & 0 & 0 & 2m
\end{pmatrix},
\]

(4.8)

Since the matrix acting on \(a_0\) in equation (4.2) is of rank 2, the general solution of the homogeneous equation (4.2) takes the form

\[
a_0 = \beta_1 b_{01} + \beta_2 b_{02},
\]

(4.9)

where the basis 4-spinors \(b_{01}\) and \(b_{02}\) are two linearly independent solutions

\[
b_{01} = \frac{1}{\sqrt{E + m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p^1}{E - m} \\ \frac{p^2}{E - m} \end{pmatrix}, \quad b_{02} = \frac{1}{\sqrt{E + m}} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1 - ip^2}{E - m} \\ - \frac{p^1}{E - m} \end{pmatrix},
\]

(4.10)

where \(E = p^0\) is the energy of the Dirac particle and \(p^a\) is defined by equation (4.6). The basis spinors (4.10) are parallel propagated along \(U\),

\[
b_{01,\mu} U^\mu = 0, \quad b_{02,\mu} U^\mu = 0.
\]

(4.11)

Choosing a frame that is parallel propagated along the congruence \(U^\mu\) in which the 4-momentum takes the form \(p^{(a)} = (m, 0, 0, 0)\), the spinors \(b_{01}\) and \(b_{02}\) reduce to

\[
b_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
\]

(4.12)

With the inner product on spinors defined by \(\langle \chi, \psi \rangle := \bar{\chi} \psi = \bar{\psi} \chi\), for arbitrary 4-spinors \(\chi, \psi\), an orthonormal basis for the 2-plane orthogonal to the fundamental solutions (4.10) of the homogeneous system (4.2) is given by the spinors...
The spinor fields \( b_{11}, b_{02}, b_{11}, b_{12} \) constitute a basis for 4-spinors. In a parallel propagated frame along the congruence \( U^{\alpha} \) in which the 4-momentum takes the form \( p^{\alpha} = (m, 0, 0, 0) \), \( b_{11} \) and \( b_{12} \) take the form

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}.
\]

(4.14)

These spinors are parallel propagated along \( U \) as well,

\[
b_{11,\alpha} U^\alpha = 0, \quad b_{12,\alpha} U^\alpha = 0.
\]

(4.15)

The general solution to (4.3) for \( n = 1 \) can be written as

\[
a_1 = v_1 b_{01} + v_2 b_{02} + \lambda b_{11} + \lambda_2 b_{12}.
\]

(4.16)

We are now in a position to report the main result that we will need from [Aud81].

**Lemma 4.1 (Audretsch).** The propagation equation for the scalar coefficients \( \beta = \beta_1, \beta_2, v_1, v_2 \) of the fundamental spinors in the general solution (4.16) to the first-order equation (4.3) with \( n = 1 \) is given by

\[
\partial_{\alpha} U^{\alpha} = -\frac{\theta}{2} \beta.
\]

(4.17)

In order to prove that the spin vector defined by (3.28) is parallel propagated to first-order in the asymptotic expansion (4.1), we will now determine the solutions of the propagation equation for the scalar coefficients in Kerr geometry.

**Theorem 4.2 (Scalar coefficients in Kerr geometry).** Let \( U \) be tangent to a timelike geodesic in Kerr geometry. The general solution to the propagation equation for the scalar coefficients (4.17) is given by

\[
\beta(r, \vartheta) := \frac{c}{\sqrt{R(r) \Theta(\vartheta) \sin \vartheta}},
\]

(4.18)

where \( c \) is a constant of integration.

**Proof.** We may rewrite the propagation equation (4.17) in terms of proper time \( \tau \) as follows.

\[
\frac{d}{d\tau} \ln \beta(\tau) = -\frac{1}{2} \theta(\tau),
\]

(4.19)

\[
= -\frac{1}{2} \left( \frac{R'}{R} + \frac{\Theta'}{\Theta} \cot \vartheta + \cot \vartheta \right).
\]

(4.20)
where the dot denotes $\frac{d}{d\tau}$, prime denotes ordinary partial derivatives, and where the functions $R(r)$ and $\Theta(\vartheta)$ are defined by equations (2.52) and (2.53) respectively. In equation (4.20), we have used the explicit expressions we obtained in section 2 for the 4-velocity ($U^\mu$) of an arbitrary timelike geodesic in Kerr geometry. It follows from equation (4.21) that the general solution to (4.17) is given by the scalar field (4.18).

Let

\[
\beta_1 = \frac{c_1}{\sqrt{R(r)\Theta(\vartheta)\sin \vartheta}},
\]

\[
\beta_2 = \frac{c_2}{\sqrt{R(r)\Theta(\vartheta)\sin \vartheta}},
\]

\[
v_1 = \frac{d_1}{\sqrt{R(r)\Theta(\vartheta)\sin \vartheta}},
\]

\[
v_2 = \frac{d_2}{\sqrt{R(r)\Theta(\vartheta)\sin \vartheta}}.
\]

We can easily verify that

\[
f := \tilde{a}_0 a_0 = \sqrt{\beta_1^2 + \beta_2^2},
\]

satisfies the propagation equation

\[
f_a U^a = -\frac{\theta}{2}.
\]

Define a spinor $b_0$ such that

\[
a_0 = fb_0.
\]

Then, $b_0$ has unit norm,

\[
\tilde{b}_0 b_0 = 1,
\]

and is parallel propagated along $U$,

\[
b_{0,a} U^a = 0.
\]

We now recall our definition of the spin vector given by equation (3.28),

\[
W^a := \frac{1}{2} \varepsilon^{abcd} h_0 \frac{\Psi_{a,c} \Psi_{b,d}}{\Psi^2},
\]

where $\varepsilon^{abcd}$ is the Levi–Civita symbol. In order to derive a propagation equation for the spin vector to first-order in $\hbar$, we note that

\[
\frac{1}{\Psi^2} = \frac{1}{f^2} \left( 1 + \frac{i\hbar}{f^2} (\tilde{a}_0 a_0 - \tilde{a}_0 a_0) + O(\hbar^2) \right).
\]

\footnote{Specifically, equations (2.38)–(2.39) and (2.44)–(2.45) with $\mu = 1$.}
Therefore, the spin vector is given by
\[ W^a = W_0^a + h W_1^a + O(h^2), \] (4.33)
where
\[ W_0^a = \frac{1}{2} \varepsilon^{abcd} U_b \bar{b}_0^c \bar{b}_d^0, \] (4.34)
and
\[ W_1^a = \frac{1}{2} \varepsilon^{abcd} \left[ U_b \left( \bar{a}_d a_1 - \bar{a}_i a_0 - \frac{1}{f^2} (\bar{a}_d a_i a_1 - \bar{a}_i a_d a_0) \right) \right. \\
\left. - \frac{1}{2m} (\bar{b}_0 a_0 \bar{b}_0 - \bar{b}_0 a_0 \bar{b}_0) \bar{b}_0 \right], \] (4.35)

**Lemma 4.3.** Up to first-order in \( h \), the spin vector defined by (3.28) is a unit-norm, space-like vector field. That is,
\[ W^a W_a = -1 + O(h^2). \] (4.36)

**Proof.** The proof can be found in appendix.

In order to prove that the spin vector defined by (3.28) is parallel propagated to first-order in the asymptotic expansion (4.1) in Kerr geometry, we first construct an orthonormal frame that is parallel propagated along the integral curve of \( U \). We will then show that, to first-order in \( h \), the spin vector defined by (3.28) has constant components in the parallel propagated frame.

### 5. Parallel propagated frame along timelike geodesics

In this section we shall work exclusively in the symmetric coframe defined in terms of the principal null directions of the Weyl tensor and given in Boyer–Lindquist coordinates by (2.18)–(2.21). Vector fields will be written as four component row vectors whose entries are understood to be the components of the vector fields in the symmetric frame.

The 4-velocity of an arbitrary timelike geodesic in Kerr geometry can be written as
\[ U := \frac{1}{\sqrt{\Sigma}} \left( \frac{\mathbb{P}}{\sqrt{\Delta}}, \sqrt{\frac{R}{\Delta}}, \sqrt{\Theta} \right), \] (5.1)
where
\[ \mathbb{P}(r) = E(r^2 + a^2) - a\Phi, \] (5.2)
\[ \mathbb{Q}(\vartheta) = aE \sin \vartheta - \Phi \sin \vartheta, \] (5.3)
\[ R(r) = \mathbb{P}^2 - \Delta(\kappa + r^2), \] (5.4)
\[ \Theta(\vartheta) = \kappa - a^2 \cos^2 \vartheta - \mathbb{Q}^2. \] (5.5)

In order to obtain an orthonormal frame \( L_{(a)} \) that is parallel propagated along \( U \), we follow Marck’s elegant construction [Mar83].
Note first that $U$ is parallel propagated along itself so that we can set

$$L^a_{(0)} := U^a.$$  \hspace{1cm} (5.6)

In [FKP14], we constructed a parallel propagated frame along null geodesics using the Killing–Yano 2-form admitted by the Kerr metric and its Hodge dual. One cannot use Killing–Yano theory to obtain a parallel propagated orthonormal frame along timelike geodesics. However, one can still obtain a 4-vector that is parallel propagated along a timelike geodesic.

Recall that the Kerr metric admits a Killing–Yano 2-form, given in the symmetric coframe by

$$f := -a \cos \theta \omega^0 \wedge \omega^1 + r \omega^2 \wedge \omega^3.$$  \hspace{1cm} (5.7)

from which one immediately obtains another 4-vector,

$$L^a := f^a_k U^k,$$  \hspace{1cm} (5.8)

that is parallel propagated along $U$ as a direct consequence of the Killing–Yano equation. We normalize $L$, defined by equation (5.8), to obtain a unit-norm vector given by

$$L^a := \frac{1}{\sqrt{\kappa \Sigma}} \left( \frac{a \cos \theta \sqrt{\kappa}}{\sqrt{\Delta}}, -\frac{\omega_r \sqrt{\rho}}{\sqrt{\Delta}}, -\frac{\omega_\theta \sqrt{\rho}}{\omega_r}, r \sqrt{\Omega}, -r \right).$$  \hspace{1cm} (5.9)

In order to obtain the remaining two vectors of the parallel propagated frame, we start with a basis

$$L_{(1)} = \frac{1}{\sqrt{\kappa \Sigma}} \left( \frac{\omega_r \sqrt{\rho}}{\sqrt{\Delta}}, -\frac{\omega_\theta \sqrt{\rho}}{\omega_r}, -\frac{a \cos \theta \sqrt{\rho}}{\omega_r} \right),$$  \hspace{1cm} (5.10)

$$L_{(2)} = \frac{1}{\sqrt{\Sigma}} \left( \frac{\omega_\rho \sqrt{\rho}}{\sqrt{\Delta}}, -\frac{\omega_\theta \sqrt{\rho}}{\omega_r}, \sqrt{\Omega} \right),$$  \hspace{1cm} (5.11)

where

$$\omega := \frac{\kappa - a^2 \cos^2 \theta}{r^2 + \kappa}.$$  \hspace{1cm} (5.12)

Then we solve the ODE that governs a proper time dependent rotation angle $\Phi(\tau)$ such that the two vectors

$$L_{(1)} = \cos \Phi(\tau) \tilde{\lambda}_{(1)} - \sin \Phi(\tau) \tilde{\lambda}_{(2)},$$  \hspace{1cm} (5.13)

$$L_{(2)} = \sin \Phi(\tau) \tilde{\lambda}_{(1)} + \cos \Phi(\tau) \tilde{\lambda}_{(2)},$$  \hspace{1cm} (5.14)

are parallel propagated along $U$. The solution, originally obtained in [Mar83], is given by

$$\frac{d \Phi}{d \tau} = \frac{1}{\Sigma} \left[ \frac{P}{r^2 + \kappa} - a \sin \theta \frac{\Theta}{\kappa - a^2 \cos^2 \theta} \right],$$  \hspace{1cm} (5.15)

where $\tau$ denotes proper time. The proper time dependent angle $\Phi$ may be obtained by separation of variables,

$$\Phi(r, \theta) := F(r) + G(\theta),$$  \hspace{1cm} (5.16)
where
\[ F(r) := \frac{1}{\kappa} \int_{r'}^r \frac{\mu(r) \, dr}{r^2 + \kappa \sqrt{R}}, \quad G(\vartheta) = a \kappa \frac{1}{2} \int_0^1 \frac{\sin \vartheta \Psi(\vartheta)}{\kappa - a^2 \cos^2 \vartheta} \, d\vartheta. \] (5.17)

**Proposition 5.1** (Parallel propagated orthonormal frames along timelike geodesics in Kerr geometry). Let \( \gamma \) be a timelike geodesic parametrized by proper time \( \tau \), with tangent vector \( U := \frac{d}{d\tau} \gamma \). Then, the orthonormal frame, \( L_{(a)} \), defined by equations (5.6), (5.13), (5.14) and (5.9), is parallel propagated along \( \gamma \).

**Remark 5.2.** If \( K_{(a)} = (K_{(a)}) := U, K_{(1)}, K_{(2)}, K_{(3)} \) is another orthonormal frame that is parallel propagated frame along \( \gamma \), then \( L_{(a)} \) and \( K_{(a)} \) are related by a constant coefficient transformation that does not depend on \( \tau \). More precisely,
\[ L_{(0)} := U =: K_{(0)} \text{ and } L_i = E_{(i)}^j K_{(j)}, \] (5.18)
where \( E_{(i)}^j \) is a \( 3 \times 3 \) rotation matrix with entries that are constant along \( \gamma \).

In the next section we will show that the spin vector defined by (3.28) is parallel propagated to first-order in the formal asymptotic expansion (4.1) in Kerr geometry.

### 6. Propagation law for the spin vector

In order to prove that the spin vector defined by (3.28) is parallel propagated to first-order in the formal asymptotic expansion (4.1) in Kerr geometry, we will need the following lemma.

**Lemma 6.1.** For the general solution to the propagation equation for the scalar coefficients in Kerr geometry \( \beta(r, \vartheta) \), given by (4.18), we have
\[ \frac{\beta_{(0)}}{2} = -\frac{\theta}{2} \beta, \] (6.1)
\[ \frac{\beta_{(1)}}{2\Sigma} = -\frac{\beta}{2\Sigma} \left[ \mp R \left( \cos \Phi \Psi \kappa^{-\frac{1}{2}} - \sin \Phi \right) + \frac{\cot \vartheta + \Theta'/\Theta}{\mp} \left( \cos \Phi a \cos \vartheta \Psi \kappa^{-\frac{1}{2}} - \sin \Phi \Theta \right) \right], \] (6.2)
\[ \frac{\beta_{(2)}}{2\Sigma} = -\frac{\beta}{2\Sigma} \left[ \mp R \left( \sin \Phi \Psi \kappa^{-\frac{1}{2}} + \cos \Phi \right) + \frac{\cot \vartheta + \Theta'/\Theta}{\mp} \left( \sin \Phi a \cos \vartheta \Psi \kappa^{-\frac{1}{2}} + \cos \Phi \Theta \right) \right], \] (6.3)
\[ \frac{\beta_{(3)}}{2\kappa \Sigma} = -\frac{\beta}{2\kappa \Sigma} \left[ -a \cos \vartheta \Psi R' \kappa + r R \left( \cot \vartheta + \frac{\Theta'}{\Theta} \right) \right], \] (6.4)
where the prime denotes ordinary derivatives.

**Proof.** Let \( \langle X, Y \rangle := X^a Y_a \) denote the inner product for arbitrary vector fields \( X \) and \( Y \). Observe that \( \beta_{(a)} = \langle \tilde{d}\beta, L_{(a)} \rangle \). The proof then follows from direct computation using the parallel propagated frame \( L_{(a)} \) constructed in section 5.
\[ \langle \tilde{d}\beta, L_{(0)} \rangle = \langle \tilde{d}\beta, U \rangle = -\frac{\theta}{2} \beta. \] (6.5)
\begin{align}
\langle d\beta, \mathcal{L}_{(1)} \rangle &= -\frac{\beta}{2\kappa^2} \left[ r\varphi^{2} \frac{R'}{R} + a \cos \vartheta \Theta \left( \cot \vartheta \Theta \right) \right], \\
\langle d\beta, \mathcal{L}_{(2)} \rangle &= -\frac{\beta}{2\Sigma} \left[ r\varphi^{2} + \Theta \left( \cot \vartheta \Theta \right) \right], \\
\langle d\beta, \mathcal{L}_{(3)} \rangle &= -\frac{\beta}{2\kappa^2} \left[ -a \cos \vartheta \varphi^{2} \frac{R'}{R} + r \varphi \left( \cot \vartheta \Theta \right) \right].
\end{align}

\textbf{Theorem 6.2.} In Kerr geometry, to first-order in \( \hbar \), the spin vector defined by (3.28) is parallel propagated along \( \mathcal{U} \).

\textbf{Proof.} Theorem 4.2 guarantees that the scalar coefficients of the spinor fields take the form (4.22)–(4.25) for some constants of integration \( c_1, c_1, d_1, d_2 \). Plugging the scalar coefficients (4.22)–(4.25) into the expressions previously established for the spin vector, (4.34) and (4.35), we obtain in the parallel propagated frame

\begin{align}
W_0 &= \frac{1}{c_1^2 c_1 + c_2^2 c_2} \begin{bmatrix}
0 \\
c_1 c_2^* + c_2 c_1^* \\
i(c_1 c_2^* - c_2 c_1^*) \\
c_2 c_2^* - c_1 c_1^*
\end{bmatrix}, \\
W_i &= P + Q + R,
\end{align}

where

\begin{align}
P &= \frac{i}{\bar{\mathcal{F}}} (\ddot{a}_0 \varphi^0 - \ddot{a}_0 \varphi^0) W_0 = i \frac{c_1 d_1 + c_2 d_2 - d_1^* c_1 - d_2^* c_2}{c_1^2 c_1 + c_2^2 c_2} W_0, \\
Q &= \frac{i}{2\bar{\mathcal{F}}} e^{a_0(0)0 \varphi^0} (\ddot{a}_0 \varphi^0 \alpha^0 - \ddot{a}_0 \varphi^0 \alpha^0), \\
R &= \frac{1}{2} e^{a_0(0)0 \varphi^0} (\delta J_0 \hat{b}_0 - \delta J_0 \hat{b}_0).
\end{align}

and where \( \delta J \) is given by

\begin{align}
\delta J_0 = \frac{1}{2m_0} (\hat{b}_0 \omega + \hat{b}_0 \omega).
\end{align}
Remark 6.3. Note that $P$ and $Q$ have constant components in a parallel propagated frame. We will now show that $R$, given by (6.14), vanishes identically.

We begin by evaluating (6.15)\(^5\).

\[
\bar{b}_{0(a)} b_0 - \bar{b}_{b} b_{0(b)} = \frac{(\beta_1 L_{00}(\beta_1') + \beta_2 L_{00}(\beta_2')) - (\beta_1' L_{00}(\beta_1) + \beta_2' L_{00}(\beta_2))}{f^2},
\]

(6.16)

where $L_{00}(\beta) := (d\beta, L_{00})$. Since $L_{00}(\beta) = (d\beta, U) = -\frac{\theta}{2}\beta$, we have

\[
\bar{b}_{0(0)} b_0 - \bar{b}_{b} b_{0(0)} = 0.
\]

(6.17)

And for the spacelike basis vectors of the parallel propagated frame $L_{0i}$,

\[\bar{b}_{0(i)} b_0 - \bar{b}_{b} b_{0(i)} = \frac{\beta_1 \beta_1' + \beta_2 \beta_2' - \beta_1 \beta_1' - \beta_2 \beta_2'}{f^2}
\]

(6.18)

Since

\[
\bar{b}_{0(\alpha(0)} b_0 = 0 \quad \text{and} \quad b_{0(0)} = 0,
\]

(6.19)

at least one of the three terms, $\varepsilon^{(a)b(\rho)} \delta_{0\alpha}$ and $\bar{b}_{0(\beta)c} b_0$, vanishes for any choice of indices $(a),(b),(c)$, one of which has to be (0). Thus, $R^{(0)}$ must vanish for $i = 1, 2, 3$.

The timelike component of $R$ is given by

\[
R^{(0)} = \frac{1}{2mf^4} \left[ (\beta_1' \beta_2 + \beta_2' \beta_1)(\beta_{1,10} \beta_1' - \beta_{1,01} \beta_1) + \beta_{2,10} \beta_2' - \beta_{2,01} \beta_2' \right]
\]

\[
+ \frac{i}{2} (\beta_1' \beta_2 - \beta_2' \beta_1)(\beta_{1,10} \beta_1' - \beta_{1,01} \beta_1) + \beta_{2,10} \beta_2' - \beta_{2,01} \beta_2' \right].
\]

(6.20)

Note that we have a common factor between the derivatives of the scalar fields (6.2)–(6.4). Define

\[
\zeta(i) := \frac{\beta_{1,10}}{\beta_1} = \frac{\beta_{2,01}}{\beta_2} = \frac{\beta_{1,01}}{\beta_1'} = \frac{\beta_{2,01}}{\beta_2'}.
\]

(6.21)

Therefore we may replace each term, $\beta_{k,0i}$, by $\zeta(i)\beta_k$ (for $k = 1, 2, 3$) in the expression for $R^{(0)}$ given by (6.20) as follows.

\[
R^{(0)} = \frac{1}{2mf^4} \left[ (\zeta(1) \beta_1' \beta_2 + \beta_2' \beta_1)(\beta_1 \beta_1' - \beta_1 \beta_1) + \beta_2 \beta_2' - \beta_2 \beta_2' \right]
\]

\[
+ i\zeta(2)(\beta_1' \beta_2 - \beta_2' \beta_1)(\beta_1 \beta_1' - \beta_1 \beta_1) + \beta_2 \beta_2' - \beta_2 \beta_2' \right],
\]

(6.22)

This completes the proof of our claim that $R$ vanishes identically.

\(^5\) Recall that $a_0 = b_0 = \beta \theta b_0 + \beta_2 b_0$ and $f = \sqrt{\beta_1' \beta_1 + \beta_2' \beta_2}$ in accordance with (4.26), (4.28) and (4.9).
Since \((c_1, c_2, d_1, d_2)\) are constant along the integral curve of \(U\), we see that, to first order in \(\hbar\), the spin vector has constant components in the parallel propagated frame, and is thus parallel propagated along \(U\).

For the sake of completeness, we note the expression of the spin vector for a Dirac particle in Kerr geometry to first-order in \(\hbar\).

\[
W^a = W_0^a + \hbar W_1^a + O(\hbar^2),
\]

where

\[
W_0 = \frac{1}{c_1^2 + c_2^2}
\begin{bmatrix}
0 \\
\iota(c_1^2 c_2 - c_2^2 c_1) \\
\iota(c_1 c_2 - c_2 c_1)
\end{bmatrix},
\]

and

\[
W_1 = \frac{\iota}{c_1^2 + c_2^2}
\begin{bmatrix}
0 \\
\iota(c_0 W_0 + \left( (d_1 c_2 + d_2 c_1) - (c_1^2 d_2 + c_2^2 d_1) \right) \\
\iota((c_0 d_1 + d_1 c_2) - (c_1^2 d_2 + c_2^2 c_1)) \\
\iota((d_1 c_2 + d_2 c_1) - (d_1^2 c_2 + d_2^2 c_1))
\end{bmatrix},
\]

where \(c_0 := (c_1^2 d_2 + c_2^2 d_1) - (d_1 c_1 + d_2 c_2)\).

7. Reference frame for Dirac particles

In this section, we will construct a measurement frame on purely geometric criteria and specify the communication protocol. We seek a construction that is coordinate independent and as closely tied to the symmetries of Kerr geometry as possible. In the photonic case, one could use the principal null directions of the Weyl tensor to define a pair of basis vectors for the plane of polarization. Unfortunately, that construction cannot serve us in the case of massive spin-\(\frac{1}{2}\) particles since the spin vector cannot in general be confined to a 2-plane, as was the case with the polarization vector of a photon.

Recall that \(U\) denotes the 4-velocity of the Dirac particle. We first define a volume form for the 3-space \((U)\) by

\[
\Omega := âˆ— U^3,
\]

where \(\ast\) denotes the Hodge duality operator. Since

\[
U^\rho := (\Delta \Sigma)^{-\frac{1}{2}} \left( \mathcal{P} \omega^0 - R \mathcal{T} \omega^1 - \mathcal{D} \omega^2 - \Delta \mathcal{T} \omega^3 \right),
\]

the volume form \(\Omega\), defined by (7.1), is given

\[
(\Delta \Sigma)^{\frac{1}{2}} \Omega = -\Delta \mathcal{T} \omega^0 \land \omega^1 \land \omega^2 + \Delta \mathcal{T} \omega^0 \land \omega^1 \land \omega^3 \\
- R \mathcal{T} \omega^0 \land \omega^2 \land \omega^3 + \mathcal{P} \omega^1 \land \omega^2 \land \omega^3.
\]
Remark 7.1. The volume form $\Omega$ given by (7.3) should be a monomial. This is indeed the case if we consider 1-forms restricted to $(U)^\perp$. Let $\alpha^i := L^i_j$ for $i = 1, 2, 3$, where $L_{(a)}$ is the parallel propagated frame constructed in section 5. Then,

$$\Omega = \alpha^1 \wedge \alpha^2 \wedge \alpha^3.$$  

(7.4)

Consider the symmetric frame dual to the coframe $(2.18)$–$(2.21)$. Since one of the basis 4-vectors, $(e_0)$, is timelike and three $(e_i, i = 1, 2, 3)$ are spacelike, there is still an ambiguity in labeling the spacelike basis 4-vectors corresponding to six permutations. We show how Carter observers can use geometric criteria to eliminate this ambiguity and agree on the labeling of the symmetric frame.

Recall that the principal null directions of the Weyl tensor are given in Boyer–Lindquist coordinates by

$$\ell = \ell^t \frac{\partial}{\partial x^t} = \frac{1}{\sqrt{2\Sigma}} \left( (r^2 + a^2) \frac{\partial}{\partial t} + \sqrt{\Delta} \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right),$$  

(7.5)

and

$$n = n^t \frac{\partial}{\partial x^t} = \frac{1}{\sqrt{2\Sigma}} \left( (r^2 + a^2) \frac{\partial}{\partial t} - \sqrt{\Delta} \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right).$$  

(7.6)

The symmetric linear combination of the principal null directions of the Weyl tensor, $2 \hat{\omega} (\ell + n)$, is timelike, while their difference $2 \hat{\omega} (\ell - n)$ is spacelike. This is a geometrically privileged vector that we can label $e_1$. Since the observers’ 4-velocity vector $e_0 := 2 \hat{\omega} (\ell + n)$ is known, they can identify their spacelike 4-acceleration vector, whose components in the symmetric frame are given by

$$\nabla_{e_0} e_0 = \Sigma^{-1} \left( 0, \Delta^t ((r^2 - a^2 \cos^2 \vartheta)M - ra^2 \sin^2 \vartheta \varphi), 0, -a^2 \cos \vartheta \sin \vartheta \right).$$  

(7.7)

That is, of the two remaining vectors of the symmetric frame to be labeled, the acceleration vector, given by (7.7), is orthogonal to only one of them, which we label $e_2$. The last remaining basis 4-vector is labeled $e_3$. Thus, there is no ambiguity in labeling the indices of the symmetric frame.

We now obtain a basis for $(U)^\perp$ by contracting the spacelike basis 4-vectors with $\Omega$ in an increasing sequence ($i < j$) and performing a Gram–Schmidt orthonormalization. The first step yields

$$\Omega (e_1, e_2)^i = (\Delta \Sigma)^{-1} (-\Delta \Theta \hat{\Theta}^2, 0, 0, -\mathbb{P}),$$  

(7.8)

$$\Omega (e_1, e_3)^i = (\Delta \Sigma)^{-1} (\Delta \hat{\Theta}^2, 0, \mathbb{P}, 0),$$  

(7.9)

$$\Omega (e_2, e_3)^i = (\Delta \Sigma)^{-1} (\hat{\Theta}^2, -\mathbb{P}, 0, 0),$$  

(7.10)

where $\omega^i$ denotes the vector field dual to a 1-form $\omega$ and given by $(\omega^a)^a = \omega_{ab} \rho^{ab}$. Performing Gram–Schmidt orthonormalization yields the three orthonormal vectors

$$X := (P^2 - \Delta \Theta)^{-1} (-\Delta \hat{\Theta}^2, 0, 0, -\mathbb{P}),$$  

(7.11)

$$Y := (e(P^2 - \Delta \Theta)^{-1} (\Delta \hat{\Theta}^2, 0, (P^2 - \Delta \Theta), \Delta \hat{\Theta}^2),$$  

(7.12)
\[ Z := (\Delta \Sigma \vartheta)^{-\frac{1}{2}}(-\mathbb{P} R^4_2, -\vartheta, -\mathbb{P} \Delta^4_2 R^4_2, -\Delta^4_2 \Theta^4_2 R^4_2), \]  
(7.13)

where

\[ \vartheta := \mathbb{P}^2 - \Delta (\kappa - a^2 \cos^2 \vartheta). \]  
(7.14)

**Definition 7.2 (Reference 3-frame).** The set of three spacelike 4-vectors, \{X, Y, Z\}, defined by (7.11)–(7.13), constitutes an orthonormal 3-frame for \((U)^\perp\), that we call the reference 3-frame.

We are now in a position to specify our communication protocol. Let Alice and Bob be two Carter observers located in Kerr geometry. In order to communicate with Bob, Alice polarizes a massive spin-\(\frac{1}{2}\) at event \(x_A\) by choosing a unit-norm 3-vector \(\mathbf{W}_{A}\), given in the reference 3-frame at event \(x_A\) by \(\mathbf{W}_{A}(1)\), and launches it on a timelike geodesic \(\gamma_{\tau}\) that intersects with Bob’s worldline. Bob intercepts the particle and measures its spin vector by projecting it onto the reference 3-frame at event \(x_B\) to obtain components \(\mathbf{W}_{B}(1)\). The timelike geodesic \(\gamma\) must satisfy

\[ \gamma(\tau) = x_A, \quad \text{and} \quad \gamma(\tau) = x_B, \]  
(7.15)

for some \(\tau > \tau_B\). We may without loss of generality set \(\tau_0 = 0\), and suppress the subscript for \(\tau\).

Thus, \(\gamma(0) = x_A\) and \(\gamma(\tau) = x_B\).

In the next section, we define the proper-time dependent, geometrically-induced rotation of a Dirac particle, \(\Lambda(\tau)\); obtain an exact expression for \(\Lambda(\tau)\); and obtain a curvature invariant for the curve traced out by the spin vector on the unit sphere under the action of \(\Lambda(\tau)\).

### 8. Geometrically-induced rotation of the spin vector

Armed with the definitions of the reference 3-frame and the communication protocol, we can define the geometrically-induced precession of a massive spin-\(\frac{1}{2}\) particle as follows.

**Definition 8.1 (Geometrically-induced precession of the spin vector).** The precession of the spin of the Dirac particle is given by the proper time dependent rotation \(\mathbf{SO}(3)\) of the reference 3-frame with respect to the 3-frame that is parallel propagated along \(U\) and also spans \((U)^\perp\). More precisely, we have

\[ \Lambda^i_j(\tau) := L^i_{\hat{k}}(\tau) L^k_j(0), \]  
(8.1)

where \(L^i_{\hat{k}}(0)\) is the change-of-basis matrix from the reference 3-frame \{X, Y, Z\}, given explicitly by (7.11)–(7.13), to the parallel propagated 3-frame \(L_{\hat{k}}(0)\), given explicitly by (5.13), (5.14) and (5.9), at event \(x_A\), whereas \(L^i_{\hat{k}}(\tau)\) is the change-of-basis matrix from the parallel propagated 3-frame \(L_{\hat{k}}(\tau)\) to the reference 3-frame at event \(x_B\).

**Remark 8.2.** In equation (8.1), the hatted indices refer to the reference 3-frame and the bracketed indices refer to the parallel propagated 3-frame. We are thus identifying the tangent spaces along \(\gamma\) using the parallel propagated frame. Under this identification of tangent spaces, the matrix \(\Lambda^i_j(\tau)\), defined by (8.1), rotates Alice’s reference 3-frame to Bob’s reference 3-frame. Thus,
\[ \mathbf{W} = \Lambda(\tau) \mathbf{W}_0. \]  

(8.2)

Plugging in the explicit expressions for the reference 3-frame (7.11)–(7.13) and the parallel propagated 3-frame (5.13), (5.14) and (5.9) into the formula (8.1) yields,

\[
\Lambda(\tau) := \begin{pmatrix}
\frac{r \partial P + a \cos \vartheta \sqrt{R \Theta}}{\sqrt{\kappa \Sigma \xi}} & -\frac{\kappa \Sigma \chi}{\sqrt{\kappa \Sigma \xi}} & \frac{\kappa^2 \cos \chi \Sigma \Theta^2 - c_1 \sin \chi}{\sqrt{\kappa \Sigma \xi}} \\
\frac{r \vartheta \partial P - a \cos \vartheta \partial P R}{\sqrt{\kappa \Sigma \xi}} & \frac{\kappa \Sigma \chi}{\sqrt{\kappa \Sigma \xi}} & \frac{\kappa \Sigma \chi}{\sqrt{\kappa \Sigma \xi}} \\
-a \cos \vartheta \partial P & -\frac{\kappa \Sigma \chi}{\sqrt{\kappa \Sigma \xi}} & \frac{\kappa \Sigma \chi}{\sqrt{\kappa \Sigma \xi}}
\end{pmatrix},
\]

(8.3)

where

\[ c_1 := (\kappa + r^2) \left( r \vartheta \sqrt{R \Theta} - a \cos \vartheta \partial P \right), \quad c_2 := (\kappa + r^2) \left( a \cos \vartheta \Theta^2 + r \vartheta \partial P R \right), \]

(8.4)

\[ \xi := \mathbb{P}^2 - \Delta \Theta, \]

(8.5)

and \( \vartheta \) was defined in (5.12), while \( \vartheta \) was given by (7.14). In the matrix on the RHS of equation (8.3), all entries depends on proper time \( \tau \) through \( r(\tau) \) and \( \vartheta(\tau) \). We verify that \( \Lambda^T \Lambda = I \) and \( \det \Lambda = 1 \). Thus, \( \Lambda \in SO(3) \) indeed.

Let \( \mathbf{W} \) be the initial position of the spin vector prepared by Alice. At proper time \( \tau > 0 \), the position vector is given by

\[ \mathbf{W}(\tau) = \Lambda(\tau) \mathbf{W}_0, \]

(8.6)

where \( \Lambda(\tau) \) is given by (8.3).

The differential equation satisfied by the Darboux frame \((\mathbf{W}, \mathbf{W}', \mathbf{W} \times \mathbf{W}')\) for a curve on the unit sphere can be written as [WW15]

\[
\begin{pmatrix}
\mathbf{W}' \\
\mathbf{W}'' \\
\mathbf{W} \times \mathbf{W}'
\end{pmatrix}' = \begin{pmatrix}
0 & 1 & 0 \\
0 & -k_g & 0 \\
-k_g & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{W}' \\
\mathbf{W}'' \\
\mathbf{W} \times \mathbf{W}'
\end{pmatrix},
\]

(8.7)

where the prime denotes derivatives taken with respect to the arclength parameter \( s \) and \( k_g(s) \) is the spherical curvature\(^6\). Any two curves confined to the unit sphere with the same spherical curvature are related to each other by a constant rotation and conversely\(^7\).

From equation (8.7), we obtain the following straightforward formula for the spherical curvature of curves on the unit sphere parametrized by the arclength \( s \).

\[ k_g(s) = \frac{\mathbf{W}' \cdot (\mathbf{W}'' \times \mathbf{W})}{\mathbf{W}' \cdot \mathbf{W}'}. \]

(8.8)

\(^6\) The spherical curvature \( k_g \) is usually called geodesic curvature. But we shall not that terminology to avoid confusion between geodesics on the sphere and geodesics of Kerr geometry.

\(^7\) More precisely, if \((\alpha(s))_{s \in (a,b)}\) and \((\beta(s))_{s \in (a,b)}\) are two curves on the unit sphere parametrized by arc length \( s \), then there exists \( \Lambda \in SO(3) \) such that \( \alpha(s) = \Lambda \beta(s), \forall s \in (a,b) \) if and only if \( k_g^\alpha(s) = k_g^\beta(s), \forall s \in (a,b) \).
Note that our position vector (8.6) is parametrized by proper time, \( \tau \), not by arclength \( s \) of the curve traced out by the position vector \( W \). We therefore amend equation (8.8) to obtain the following formula for the spherical curvature invariant in terms of proper time \( \tau \).

\[
k_g(\tau) = \frac{\frac{d}{d\tau} W \cdot \left( \frac{d}{d\tau} W \times W \right)}{\left\| \frac{d}{d\tau} W \right\|^3}, \tag{8.9}
\]

where \( \| \cdot \| \) denotes the Euclidean norm. The spherical curvature \( k_g \) defined by the differential equation (8.7) and given by formula (8.9) is independent of the choice of initial position vector \( W_0 \). Another choice of initial position vector merely induces a constant rotation of the curve that leaves the spherical curvature \( k_g \) invariant. We are thus free to choose the initial position vector so as to minimize computational complexity. We fix the initial position vector at \( \tau = 0 \) to be given in the reference frame (7.11)–(7.13) by

\[
W_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{8.10}
\]

At proper time \( \tau > 0 \), the spin vector is then given simply by

\[
\tilde{W}(\tau) = \Lambda(\tau) \tilde{W}_0 = \begin{pmatrix} \frac{r DP + a \cos \vartheta \sqrt{R \Theta}}{\sqrt{\kappa \Sigma \xi}} \\ \frac{r g \sqrt{\Theta} - a \cos \vartheta DP \sqrt{R}}{\sqrt{\kappa \Sigma \xi \vartheta}} \\ a \cos \vartheta \varphi \frac{P}{R} \end{pmatrix}. \tag{8.11}
\]

We are now in a position to prove the following result.

**Proposition 8.3.** There is no spin precession for Dirac particles confined to the equatorial plane, \( \{ -\infty < t < +\infty, r_r = r < +\infty, \vartheta = \pi/2, 0 < \phi < 2\pi \} \), of Kerr geometry.

**Proof.** For timelike geodesics confined to the equatorial plane, we have \( \vartheta = \frac{\pi}{2}, \varrho = P^2 - \kappa \Delta, \quad \varpi = \frac{\varpi}{\tau^\pi \kappa} \) and \( \kappa = (aE - \Phi)^2 \). Thus,

\[
\tilde{W}(\tau) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{8.12}
\]

for arbitrary proper time \( \tau \). \( \square \)

**Corollary 8.4.** There is no spin precession for Dirac particles in Schwarzschild geometry.

**Proof.** Since Schwarzschild geometry is spherically symmetric, geodesics are confined to planes through the origin [Cha92]. Any plane through the origin in Schwarzschild geometry is isometric to the equatorial plane of a degenerate Kerr solution with \( a = 0 \). \( \square \)

In section 9, we will discuss these results as well as present some plots for qualitative analysis.
9. Summary and discussion

In order to isolate the geometrically-induced precession of the spin of Dirac particles we started with the Dirac equation. In general, there is no way to associate a solution of the Dirac equation (3.1) to a classical trajectory; that is, a timelike geodesic along which the nonzero rest mass, spin-$\frac{1}{2}$ test particle propagates. Following the insight of [Pau32], we deploy the semiclassical ansatz for the Dirac spinor. The classical limit ($\hbar \to 0$) for the Dirac equation with the semiclassical ansatz is equivalent to the Hamilton–Jacobi equation for spinless particles. This yields the desired geodesic that we postulate is the classical trajectory of the Dirac particle.

In order to define the spin vector, we appealed to the decomposition of the Dirac current due to Gordon (theorem 3.3). The spin vector defined in terms of the conserved polarization current in the Gordon decomposition (definition 3.4), corresponds to the Foldy–Wouthuysen mean-spin operator [KRH01]. Following the discussion in section 3, we can be confident that this definition of the spin vector, first proposed in [Aud81], is a good one.

Extending the result of [Aud81], we have shown that the spin vector defined in terms of the conserved polarization current in the Gordon decomposition, is parallel propagated along the geodesic obtained from the semiclassical ansatz to $O(\hbar^2)$ in Kerr geometry (theorem 6). This result came as a surprise. We were fully expecting a more involved propagation law for Dirac particles, as opposed to the propagation law for photons that we considered in [FKP14].

We have shown how observers located in the outer geometry of Kerr black holes may set up reference 3-frames in terms of locally-available, purely geometric information (definition 7.2). We were thus able to define the geometrically-induced precession of the spin vector of Dirac particles propagating in the outer geometry of Kerr black holes (definition 8.1). Our geometrically-motivated strategy allows us to obtain a compact expression for the geometrically-induced spin precession for Dirac particles. We have shown how the geometrically-induced precession of the spin of Dirac particles is determined by the rotation of the parallel-propagated frame with respect to the reference frame that the observers use to measure the spin of the test particles. We further showed that the geometrically-induced spin precession of a Dirac particle can be invariantly represented by the spherical curvature of the curve traced out by the spin vector on the unit sphere.

We have shown how there is no spin precession for Dirac particles confined to the equatorial plane of Kerr geometry (proposition 8.3). The significance of this result comes from the fact that many authors restrict attention to the equatorial plane in order to simplify computations (e.g. [RPGC11, SA10]). Note that the non-triviality of the results obtained in [RPGC11, SA10] is due to the non-zero acceleration of their chosen test particles.

Finally, we have shown how, as an immediate consequence of proposition 8.3, spin precession is trivial for Dirac particles in Schwarzschild geometry (corollary 8.4).

In order to qualitatively analyze the expression we have obtained for the spin precession of Dirac particles, we present some plots. In each of 3 sets of figures, the first figure (a) shows the orbital behaviour of the timelike geodesic with $(r(s), \phi(s))$ as polar coordinates, the second figure (b) depicts the same orbits in three dimensions with spherical coordinates $(r(s), \theta(s), \phi(s))$, and the last figure (c) depicts the spherical curvature $k_g$ as a function of propertime $\tau$. Figures 1 and 2 present co-rotating orbits, while figure 3 presents a counter-rotating orbit.
Figure 1. A co-rotating orbit with $E = 2, \Phi = 3, \kappa = 12$ and initial data $r(0) = 20$, $\vartheta(0) = 1.57$, and $\phi(0) = 0$. (a) The orbit in polar coordinates $(x = r \cos \phi, y = r \sin \phi)$. (b) The orbit in 3D spherical coordinates $(x = r \cos \phi \sin \vartheta, y = r \sin \phi \sin \vartheta, z = r \cos \vartheta)$. (c) The spherical curvature $k_\varphi$ as a function of propertime.
Figure 2. A co-rotating orbit with $E = 1.004$, $\Phi = 4$, $\kappa = 16$ and initial data $r(0) = 20$, $\vartheta(0) = 1.57$, and $\phi(0) = 0$. (a) The orbit in polar coordinates $(x = r \cos \phi, y = r \sin \phi)$. (b) The orbit in 3D spherical coordinates $(x = r \cos \phi \sin \vartheta, y = r \sin \phi \sin \vartheta, z = r \cos \vartheta)$. (c) The spherical curvature $k_\vartheta$ as a function of propertime.
Figure 3. A counter-rotating orbit with $E = 1.004, \Phi = -4, \kappa = 60$ and initial data $r(0) = 20, \vartheta(0) = 1.57, \text{ and } \phi(0) = 0$. (a) The orbit in polar coordinates $(x = r \cos \phi, y = r \sin \phi)$. (b) The orbit in 3D spherical coordinates $(x = r \cos \phi \sin \vartheta, y = r \sin \phi \sin \vartheta, z = r \cos \vartheta)$. (c) The spherical curvature $k_s$ as a function of propertime.
Appendix. Proof of lemma 4.3

We want to prove that
\[ W^\alpha W_\alpha = -1 + O(\hbar^2). \]  
(A.1)

We have
\[ b_0^2 \Sigma_{12} b_0 = \frac{\beta^2_2 \beta_1 - \beta^1_1 \beta_2}{f^2} \]  
(A.2)
\[ b_0^2 \Sigma_{13} b_0 = \frac{i \beta^2_2 \beta_1 - \beta^1_1 \beta_2}{f^2} \]  
(A.3)
\[ b_0^2 \Sigma_{23} b_0 = \frac{\beta^2_2 \beta_1 + \beta^1_1 \beta_2}{f^2} \]  
(A.4)
and
\[ b_0^2 \Sigma_{aa} b_0 = 0, \forall a = 1, 2, 3, 4. \]  
(A.5)

Since the Levi–Civita symbol vanishes for repeated indices and \( u_1 = 0, u_4 = 1 \) in the parallel propagated frame, we have
\[ W_0^1 = \epsilon^{4231} b_0 \Sigma_{12} b_0, \]  
(A.6)
\[ W_0^2 = \epsilon^{2413} b_0 \Sigma_{13} b_0, \]  
(A.7)
\[ W_0^3 = \epsilon^{3412} b_0 \Sigma_{12} b_0, \]  
(A.8)
\[ W_0^4 = 0, \]  
(A.9)

which shows that the spin vector is spacelike. Using the fact that
\[ \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta\gamma\delta}, \]  
(A.10)

we obtain
\[ W_{0a} S_{0a} = -\frac{1}{f^4} \left( (\beta^2_2 \beta_1 - \beta^1_1 \beta_2)^2 - (\beta^2_2 \beta_1 - \beta^1_1 \beta_2)^2 + (\beta^2_2 \beta_1 + \beta^1_1 \beta_2)^2 \right), \]  
(A.11)
\[ = \frac{(\beta^1_1 \beta_2 + \beta^2_2 \beta_1)^2}{f^4}, \]  
(A.12)
\[ = -1. \]  
(A.13)

Therefore,
\[ \sqrt{-W_{0a}^2 W_{0a}} = 1, \]  
(A.14)

Now, to first order in \( \hbar \),
\[ W^\alpha S_\alpha = (W_0^\alpha + \hbar W_1^\alpha)(W_0^a + \hbar W_1^a) = -1 + \hbar (W_1^2 W_0^1 + W_0^3 W_1^4) \]  
(A.15)
so we need to compute the second term. In the spin frame one can write the spinors
\[
a_0 = \begin{pmatrix}
\beta_1 \\
\beta_2 \\
0 \\
0
\end{pmatrix}, \quad a_1 = \begin{pmatrix}
v_1 \\
v_2 \\
\lambda_1 \\
\lambda_2
\end{pmatrix}
\] (A.16)

One therefore obtains the following expressions:

\[
\tilde{a}_0 a_1 = \beta_1^v v_1 + \beta_2^v v_2, \quad \tilde{a}_0 a_0 = v_1^v \beta_1 + v_2^v \beta_2,
\] (A.17)

\[
\tilde{a}_0 \Sigma^{12} a_1 = -\beta_1^v v_1 + \beta_2^v v_2
\] (A.19)

\[
\tilde{a}_0 \Sigma^{12} a_0 = -v_1^v \beta_1 + v_2^v \beta_2
\] (A.20)

\[
\tilde{a}_0 \Sigma^{13} a_1 = i(-\beta_1^v v_2 + \beta_2^v v_1)
\] (A.21)

\[
\tilde{a}_0 \Sigma^{13} a_0 = i(-v_1^v \beta_2 + v_2^v \beta_1)
\] (A.22)

\[
\tilde{a}_0 \Sigma^{23} a_1 = \beta_2^v v_1 + \beta_1^v v_2
\] (A.23)

\[
\tilde{a}_0 \Sigma^{23} a_0 = v_2^v \beta_1 + v_1^v \beta_2
\] (A.24)

So that

\[
\tilde{a}_0 a_1 - \tilde{a}_0 a_0 = (\beta_1^v v_1 + \beta_2^v v_2) - (v_1^v \beta_1 + v_2^v \beta_2)
\] (A.25)

\[
\tilde{a}_0 \Sigma^{12} a_1 - \tilde{a}_1 \Sigma^{12} a_0 = (v_1^v \beta_1 + v_2^v \beta_2) - (\beta_1^v v_1 + \beta_2^v v_2)
\] (A.26)

\[
\tilde{a}_0 \Sigma^{13} a_1 - \tilde{a}_1 \Sigma^{13} a_0 = i((\beta_2^v v_1 + v_1^v \beta_2) - (\beta_1^v v_2 + v_2^v \beta_1))
\] (A.27)

\[
\tilde{a}_0 \Sigma^{23} a_1 - \tilde{a}_1 \Sigma^{23} a_0 = (\beta_1^v v_2 + \beta_2^v v_1) - (v_1^v \beta_2 + v_2^v \beta_1)
\] (A.28)

Note that

\[
(\tilde{a}_0 \Sigma^{ij} a_1 - \tilde{a}_1 \Sigma^{ij} a_0) \tilde{b}_0 \Sigma^{ik} b_0 = (\beta_1^i v_1 + \beta_2^i v_2) - (v_1^i \beta_1 + v_2^i \beta_2) = \tilde{a}_0 a_1 - \tilde{a}_0 a_0
\] (A.29)

The first-order velocity correction term

\[
\delta v_i := \frac{\tilde{b}_0 \Sigma^{ij} b_0 - \tilde{b}_0 \Sigma^{ij}_0 b_0}{2m_i}
\] (A.30)

shows up in the first-order term for spin motion \(W_i^\alpha\) as

\[
W_i^\alpha = \frac{1}{2} e^{\alpha \beta \rho \sigma} \left[ \frac{i}{f} u_\beta \left( (\tilde{a}_0 a_1 - \tilde{a}_1 a_0) \tilde{b}_0 \Sigma^{ij}_0 b_0 + (\tilde{a}_1 \Sigma^{ij}_0 a_0 - \tilde{a}_0 \Sigma^{ij}_0 a_1) \right) \right]
\] (A.31)

\[
+ \delta v_\beta \tilde{b}_0 \Sigma^{ij}_0 b_0
\] (A.32)

Consider \(W_0^a W_0^a\). The first term in this expression is
\[
\frac{i}{f^2}(\bar{\alpha}_0 \partial_t \alpha_0 - \bar{\alpha}_0 \alpha_0)W_0^0 W_{00} = - \frac{i}{f^2}(\bar{\alpha}_0 \partial_t \alpha_0 - \bar{\alpha}_0 \alpha_0)
\]
(A.33)

and the second term equals
\[
\frac{1}{f^2}(\bar{\alpha}_0 \Sigma_i^{ij} \alpha_0 - \bar{\alpha}_0 \Sigma_i^{ij} \alpha_0)\bar{b}_0 \Sigma_i^{ij} b_0 = \frac{i}{f^2}(\bar{\alpha}_0 \partial_t \alpha_0 - \bar{\alpha}_0 \alpha_0).
\]
(A.34)

so that only the third, and last, term survives. The last term, corresponding to the velocity correction, is
\[
\frac{1}{2} \epsilon^{\alpha \beta \gamma \delta} \partial_\gamma \bar{b}_0 \Sigma_\alpha \partial_\beta b_0 \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} u^\mu \bar{b}_0 \Sigma^\rho \Sigma^\sigma b_0
\]
(A.35)

But
\[
\delta u^\mu = \frac{\bar{b}_0 b_0 - b_0 \bar{b}_0}{2m} = 0
\]
(A.36)

since \(\bar{b}_0 b_0 = 0\) and \(u^4 = 1, u^i = 0\), so that \(\bar{b}_0 = 0\) in the parallel propagated frame.

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