Multipliers of integrals of Cauchy - Stieltjes type

Peyo Stoilov

Abstract

Let $G$ be a domain with closed rectifiable Jordan curve $\ell$. Let $K(G)$ be the space of all analytic functions in $G$ representable by a Cauchy - Stieltjes integral. Let $\mathcal{M}(K)$ be the class of all multipliers of the space $K(G)$. In this paper we prove that if $f$ is bounded analytic function on $G$ and

$$\text{ess sup}_{\eta \in \ell} \int_{\ell} \frac{|f(\zeta) - f(\eta)|}{|\zeta - \eta|} |d\zeta| < \infty,$$

then $f \in \mathcal{M}(K)$.

If $G = \mathbb{D}$ is the unit disc, this theorem was proved for the first time by V. P. Havin. In particular for a smooth curve $\ell$ we prove that if $f' \in E^p(G)$, $p > 1$, then $f \in \mathcal{M}(K)$, where $E^p(G)$ are the spaces of Smirnov.

1 Introduction

Let $\ell$ be a closed rectifiable Jordan curve bounding the finite and infinite domains $G$ and $G^-$ in the complex plane $\mathbb{C}$. Let $\ell_r$ $(0 < r < 1)$ be the map of circle $|z| = r$ under conformal mapping $\zeta = \varphi(z)$ of the unit disc $\mathbb{D}$ on the domain $G$. Let denote by $A(Q)$ the set of all analytic functions in a domain $Q \subset \mathbb{C}$. For $0 < p \leq \infty$, let $E^p(G)$ be the Smirnov class $[1,2]$ of functions $f \in A(G)$, for which

$$\|f\|_{E^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{\ell_r} |f(\zeta)|^p |d\zeta| < \infty, \quad 0 < p < \infty.$$

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\[ \|f\|_{E^\infty} = \sup_{\zeta \in G} |f(\zeta)| < \infty, \quad p = \infty. \]

If \( G = \mathbb{D}, \mathbb{D} = \{ \zeta : |\zeta| < 1 \} \), then \( E^p(G) = H^p \) is the usual Hardy class.

Let \( N \) denote the Nevanlinna class:

\[ f \in N \iff f \in A(D) : \lim_{r \to 1^-} \int_{|\zeta| = 1} \log^+ |f(r\zeta)||d\zeta| < \infty, \]

and \( N^+ \) - the Smirnov class:

\[ f \in N^+ \iff f \in N : \lim_{r \to 1^-} \int_{|\zeta| = 1} \log^+ |f(r\zeta)||d\zeta| = \int_{|\zeta| = 1} \log^+ |f(\zeta)||d\zeta|. \]

Some basic facts.

1. \( f(\zeta) \in E^p(G) \iff f(\varphi(z)) \sqrt[\varphi'(z)}(z) \in H^p; \)

2. Theorem of Smirnov - Kotchine:

If \( f \in N^+ \) and has \( L^p \) boundary values, then \( f \in H^p. \)

Let \( M(\ell) \) is the space of all finite, complex Borel measures on \( \ell \) with the usual variation norm. We denote by \( K(G) \) the space of all functions \( g \in A(G) \), representable by an integral of Cauchy–Stieltjes type

\[ g(z) = \int_{\ell} \frac{d\mu(\zeta)}{\zeta - z} \overset{df}{=} K_\mu(z), \quad \mu \in M(\ell). \]

If \( g \in K(G) \), then \( M_g \) will denote the set of all measures \( \mu \in M(\ell) \) participating in the representation of \( g \).

Since \( K_\mu(z) = 0, \forall z \in G \iff K_\mu(z) \in E^1(\mathbb{G}^-) \) and

\[ E^1(\mathbb{G}^-) = \left\{ K_\mu : d\mu = f(\zeta)d\zeta, \ f(\zeta) \in L^1(\ell), \ \int_{\ell} \zeta^n f(\zeta)d\zeta = 0, \ n = -1, -2 \ldots \right\}, \]

then the space \( K(G) \) is the Banach space with the natural norm

\[ \|g\|_{K(G)} = \inf \left\{ \|\mu\|_{M(\ell)} : \mu \in M_g \right\}, \]

isometrically isomorphic to \( M(\ell)/E^1(\mathbb{G}^-) \).

Let \( C_A(\mathbb{G}^-) \) denote the space of all functions \( h \), analytic in \( \mathbb{G}^- \) and continuous on \( \overline{\mathbb{G}^-} = \mathbb{G}^- \cup \ell \), with norm
\[ \|h\|_\infty = \sup \{ |h(z)| : z \in \mathbb{G}^- \}. \]

Since \( M(\ell)/E^1(\mathbb{G}^-) \cong C_A(\mathbb{G}^-) \), then \( K(\mathbb{G}) \cong C_A^*(\mathbb{G}^-) \) and

\[ \|g\|_{K(\mathbb{G})} = \sup \left\{ \left| \int h(\eta)d\mu(\eta) \right| : h \in C_A(\mathbb{G}^-), \|h\|_\infty \leq 1 \right\}, \mu \in M_g. \]

Taking into account that the set \( A_0 = A(\overline{\mathbb{G}^-}) \) is dense in \( C_A(\mathbb{G}^-) \), then easily we receive for \( h \in A_0, g = K_\mu : \)

\[ \left| \int_{\ell^*} h(\zeta)g(\zeta)d\zeta \right| = \left| \int_{\ell^*} h(\zeta) \left( \int_{\ell} \frac{d\mu(\eta)}{\eta - \zeta} \right) d\zeta \right| = \left| \int_{\ell} \left( \int_{\ell^*} \frac{h(\zeta)}{\zeta - \eta} d\zeta \right) d\mu(\eta) \right| = \left| \int h(\eta)d\mu(\eta) \right| \]

and consequently

\[ \|g\|_{K(\mathbb{G})} = \sup \left\{ \lim_{r \to 1^-} \left| \int_{\ell^*} h(\zeta)g(\zeta)d\zeta \right| : h \in A_0, \|h\|_\infty \leq 1 \right\}. \]

Let \( \mathfrak{M}(K) \) be the class to all multipliers of the space \( K(\mathbb{G}) : \)

\[ \mathfrak{M}(K) = \{ f \in A(\mathbb{G}) : fg \in K(\mathbb{G}), \forall g \in K(\mathbb{G}) \}. \]

**Lemma.** If \( f \in \mathfrak{M}(K) \), then

\[ \sup \left\{ \|fg\|_{K(\mathbb{G})} : g \in K(\mathbb{G}), \|g\|_{K(\mathbb{G})} \leq 1 \right\} < \infty. \]

**Proof.** Let \( f \in \mathfrak{M}(K) \). We will show that the operator \( g \to fg \) is bounded in \( K(\mathbb{G}) \). By the closed graph theorem it suffices to show that the graph of the operator \( g \to fg \) is closed on \( K(\mathbb{G}) \).

Suppose \( (g_n)_{n \geq 0} \subset K(\mathbb{G}) \) with \( g_n \to g \) and \( fg_n \to \phi \) in the \( K(\mathbb{G}) \) -norm.

If \( F \in K(\mathbb{G}) \), then for all \( z \in \mathbb{G} \),

\[ F(z) = \int_{\ell} \frac{d\mu(\zeta)}{\zeta - z} \Rightarrow |F(z)| \leq (\inf |\zeta - z| : \zeta \in \ell)^{-1} \|F\|_{K(\mathbb{G})}. \]

From this inequality follows that the converse in norm in \( K(\mathbb{G}) \) implies a pointwise converse on \( \mathbb{G} \). Consequently \( \phi = fg \) and the graph of the operator \( g \to fg \) is closed on \( K(\mathbb{G}) \). \( \square \)
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From proved lemma follows that $\mathfrak{M}(K)$ is commutative Banach algebra with norm defined by

$$\|f\|_{\mathfrak{M}(K)} = \sup \left\{ \|fg\|_{K(G)} : g \in K(G), \|g\|_{K(G)} \leq 1 \right\}.$$ 

If $G = \mathbb{D}$, then the study of the space $\mathfrak{M} = \mathfrak{M}(K)$ was started in papers of V. P. Havin [3, 4] and was continued in [5, 6, 7]. For example, in [5, 6] was proved:

If $f$ is bounded analytic function on $\mathbb{D}$ and

$$\text{ess sup}_{|\eta|=1} \int_{|\zeta|=1} \left| f(\zeta) - f(\eta) \right| |d\zeta| < \infty,$$

then $f \in \mathfrak{M}$.

In this paper this result is generalized for the multipliers of the space $K(G)$.

2 Main results

Theorem 1. Let $G$ be a domain with closed rectifiable Jordan curve $\ell$.

If $f \in E^\infty(G)$ and

$$\Lambda(f) \overset{\text{def}}{=} \text{ess sup}_{\eta \in \ell} \int_{\ell} \frac{|f(\zeta) - f(\eta)|}{|\zeta - \eta|} |d\zeta| < \infty,$$

then $f \in \mathfrak{M}(K)$ and

$$\|f\|_{\mathfrak{M}(K)} \leq \|f\|_{E^\infty} + \Lambda(f).$$

Proof. Let $f \in E^\infty(G)$ and $\Lambda(f) < \infty$. Let $E \subseteq \ell$ be a subset with total Lebesgue measure ($m(E) = m(\ell)$) lying on $\ell$ such that

$$\|f\|_{E^\infty} = \sup_{\eta \in E} |f(\eta)|.$$ 

Then

$$\|f\|_{\mathfrak{M}(K)} = \sup \left\{ \|fg\|_{K(G)} : g \in K(G), \|g\|_{K(G)} \leq 1 \right\} =$$

$$= \sup \left\{ \lim_{r \to 1^-} \int_{\ell_r} f(\zeta)g(\zeta)h(\zeta)d\zeta : g \in K(G), \|g\|_{K(G)} \leq 1, \ h \in A_0, \ \|h\|_{\infty} \leq 1 \right\},$$

where $\ell_r$ ($0 < r < 1$) is the map of circle $|z| = r$ under conformal mapping $\zeta = \varphi(z)$ of the unit disc $\mathbb{D}$ on the domain $G$. 

If \( g \in K(\mathbb{G}) \), then
\[
g(z) = \int_{\mathbb{G}} \frac{d\mu(\eta)}{\eta - z}, \quad z \in \mathbb{G}
\]
and
\[
\left| \int_{\ell_r} f(\zeta)g(\zeta)h(\zeta) d\zeta \right| = \left| \int_{\ell_r} f(\zeta)h(\zeta) \left( \int_{\ell_r} \frac{d\mu(\eta)}{\eta - \zeta} \right) d\zeta \right| = \\
\leq \left| \int_{\ell_r} \left( \int_{\ell_r} \frac{f(\zeta)h(\zeta)}{\zeta - \eta} d\zeta \right) d\mu(\eta) \right| \\
\leq \int_{\ell_r} |d\mu(\eta)| \sup \left\{ \left| \int_{\ell_r} \frac{f(\zeta)h(\zeta)}{\zeta - \eta} d\zeta \right| : \eta \in E \right\},
\]

\[ \Rightarrow \]
\[
\left| \int_{\ell_r} f(\zeta)g(\zeta)h(\zeta) d\zeta \right| \leq \|g\|_{K(\mathbb{G})} \sup \left\{ \left| \int_{\ell_r} \frac{f(\zeta)h(\zeta)}{\zeta - \eta} d\zeta \right| : \eta \in E \right\}.
\]

Then
\[
\|f\|_{\gamma_1(\mathbb{G})} \leq \sup \left\{ \lim_{r \to 1^-} \left| \int_{\ell_r} \frac{f(\zeta)h(\zeta)}{\zeta - \eta} d\zeta \right| : \eta \in E, \ h \in A_0, \ \|h\|_{\infty} \leq 1 \right\}.
\]

Further, if \( \eta \in E, \ h \in A_0, \ \|h\|_{\infty} \leq 1 \), we have
\[
\left| \int_{\ell_r} \frac{f(\zeta)h(\zeta)}{\zeta - \eta} d\zeta \right| \leq \left| \int_{\ell_r} \frac{f(\zeta) - f(\eta)}{\zeta - \eta} h(\zeta) d\zeta \right| + |f(\eta)| \left| \int_{\ell_r} \frac{h(\zeta)}{\zeta - \eta} d\zeta \right| \leq \\
\leq \|h\|_{L^\infty(\ell_r)} \int_{\ell_r} \left| \frac{f(\zeta) - f(\eta)}{\zeta - \eta} \right| |d\zeta| + \|f\|_{E^\infty}.
\]

Then
\[
\|f\|_{\gamma_1(\mathbb{G})} \leq \sup \left\{ \lim_{r \to 1^-} \left| \int_{\ell_r} \frac{f(\zeta) - f(\eta)}{\zeta - \eta} |d\zeta| : \eta \in E \right\} + \|f\|_{E^\infty}.
\]

We denote for \( \eta \in E \)
\[
F_\eta(\zeta) = \frac{f(\zeta) - f(\eta)}{\zeta - \eta}, \quad \zeta \in \mathbb{G}.
\]

Then
\[
\|f\|_{\gamma_1(\mathbb{G})} \leq \sup \left\{ \|F_\eta(\zeta)\|_{E^1} : \eta \in E \right\} + \|f\|_{E^\infty}.
\]
To end the proof it is necessary to show that
\[ \Lambda(f) < \infty \Rightarrow \sup_{\eta \in E} \| F_\eta \|_{E^1} < \infty. \]

Let \( \zeta = \varphi(z) \) is conformal mapping of the unit disc \( \mathbb{D} \) on the domain \( \mathcal{G} \), \( \eta \in E \) and \( \eta = \varphi(t) \), \( |t| = 1 \). Then
\[
F_\eta(\zeta) = \frac{f(\zeta) - f(\eta)}{\zeta - \eta} \in E^1(\mathcal{G}) \iff \Omega(z) \overset{\text{def}}{=} \frac{f(\varphi(z)) - f(\varphi(t))}{\varphi(z) - \varphi(t)} \varphi'(z) \in H^1.
\]

In [8] was proved that the functions
\[
1 \in N^+.
\]

Since
\[
f(\varphi(z)) - f(\varphi(t)) \in H^\infty, \varphi'(z) \in H^1, \text{ then } \Omega(z) \in N^+.
\]

Besides
\[
\int_{|z|=1} |\Omega(z)||dz| = \int_\ell |F_\eta(\zeta)||d\zeta| \leq \Lambda(f) < \infty
\]
and according to the Theorem of Smirnov - Kotchine
\[
\Omega(z) \in H^1, \|\Omega(z)\|_{H^1} \leq \Lambda(f) < \infty.
\]

Consequently
\[
\|F_\eta\|_{E^1} = \|\Omega(z)\|_{H^1} \leq \Lambda(f) < \infty
\]
and
\[
\|f\|_{\mathfrak{M}(K)} \leq \Lambda(f) + \|f\|_{E^\infty} < \infty.
\]

\[\Box\]

**Theorem 2.** Let \( \mathcal{G} \) be a domain with a smooth curve \( \ell \).

If \( f \in E^\infty(\mathcal{G}) \) and \( f' \in E^p(\mathcal{G}), \ p > 1 \), then \( f \in \mathfrak{M}(K) \).

**Proof.** Let \( f' \in E^p(\mathcal{G}), \ p > 1 \), \( z = z(s) \) \((0 \leq s \leq s_0)\) is the equation of \( \ell \), where the arc length is as parameter. We shall prove that
\[
\Lambda(f) = \sup_{\sigma \in [0, s_0]} \int_0^{s_0} \frac{|f(s) - f(\sigma)|}{|z(s) - z(\sigma)|} ds < \infty,
\]
where \( f(s) = f \circ z(s) \).

Since \( \ell \) is a smooth curve, then

\[
|z(s) - z(\sigma)| \geq |s - \sigma|,
\]

where \( c_0 \) is a constant. Then

\[
\int_0^{s_0} \frac{|f(s) - f(\sigma)|}{|z(s) - z(\sigma)|} ds \leq \frac{1}{c_0} \int_0^{s_0} \frac{1}{|s - \sigma|} \left( \int_{\sigma}^{s} f'(x) dx \right) ds \leq \frac{1}{c_0} \int_0^{s_0} \left( \int_{\sigma}^{s} |f'(x)|^p dx \right)^{1/p} |s - \sigma|^{-1/p} ds \leq \text{Const.} \|f'\|_{E^p(G)}.
\]

Consequently

\[
\Lambda(f) \leq \text{Const.} \|f'\|_{E^p(G)} < \infty
\]

and by Theorem 1 \( f \in \mathcal{M}(K) \). \( \square \)

**Remark.** It should be noted that if \( G = \mathbb{D} \), Theorem 2 remains valid and for \( p = 1 \) (Theorem of S. A. Vinogradov [5]: \( f'(z) \in H^1 \Rightarrow f \in \mathcal{M} \)).

Whether Theorem 2 for \( p = 1 \) is correct, if \( G \) is domain with a smooth boundary, remains unknown.

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Department of Mathematics  
Technical University  
25, Tsanko Dustabanov,  
Plovdiv, Bulgaria  
e-mail: peyyyo@mail.bg