Comment on ‘An elementary argument for the magnetic field outside a solenoid’

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Abstract
An alternative elementary argument for the magnetic field outside a solenoid is described.

(Some figures may appear in colour only in the online journal)

1. Introduction

Pathak [1] showed that the superposition of magnetic fields produced by the individual coils of a solenoid tend to cancel outside the solenoid. The author stated, ‘Informally, the two ends of the long solenoid behave like two magnetic poles, each of which produce fields that fall off with the inverse square of distance, and thus the fields outside are small when the length of the solenoid is increased, keeping the current fixed’. In this comment, I point out that this argument can be made rigorous and formal.

The magnetic field that is produced by time-independent electric currents is given by the Biot–Savart law

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dV' \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},
\]

where \(\mathbf{J}\) is the charge current density which arises from both free and bound currents. Bound currents arise from the magnetisation \(\mathbf{M}\) of materials and are given by \(\mathbf{J}_b = \nabla \times \mathbf{M}\). In the case of the boundary of a magnetised material, the \(\mathbf{J}_b\) becomes a surface current of \(\mathbf{K}_b = \mathbf{M} \times \mathbf{n}\) where \(\mathbf{n}\) is the unit vector pointing out of and perpendicular to the magnetised surface boundary [2].

The magnetic field \(\mathbf{B}\) produced by a given charge current density does not depend on whether the currents are free or bound. Thus, the magnetic field due to a given free current configuration \(\mathbf{J}_f(\mathbf{r})\) can be obtained as follows. First, one obtains the magnetisation \(\mathbf{M}(\mathbf{r})\) which gives bound currents that reproduce \(\mathbf{J}_f\) (i.e. a \(\mathbf{M}(\mathbf{r})\) such that \(\nabla \times \mathbf{M} = \mathbf{J}_f\)). Then, \(\mathbf{B}\) be found from \(\mathbf{M}\) using techniques borrowed from electrostatics, as described below.
2. Obtaining the magnetic field from the magnetisation

In the absence of free currents and in static situations, the Maxwell’s equations for the auxiliary magnetic field \( \mathbf{H} \), which is defined as

\[
\mathbf{B} \equiv \mu_0 (\mathbf{H} + \mathbf{M}),
\]

are

\[
\nabla \times \mathbf{H} = 0, \tag{2}
\]

\[
\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}. \tag{3}
\]

These equations mirror the Maxwell’s equations for electrostatics in the absence of free charges,

\[
\nabla \times \mathbf{E} = 0, \tag{5}
\]

\[
\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P}, \tag{6}
\]

where \( \mathbf{E} \) is the electric field and \( \mathbf{P} \) is the electric polarisation. Equation (6) comes from

\[
\nabla \cdot \mathbf{D} = 0 \quad \text{(in the absence of free charges)} \quad \text{and} \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}.
\]

The \(-\nabla \cdot \mathbf{P}\) inside the volume of an electrically polarised material represents the bound charge density \( \rho_b \). At the the surface of the polarised object, the polarisation is discontinuous, and \(-\nabla \cdot \mathbf{P}\) reduces to \( \mathbf{P} \cdot \mathbf{n} \) where \( \mathbf{n} \) is the unit vector pointing perpedicular to and out of the surface\(^1\). Therefore, the electric field due to the electric polarisation is given by Coulomb’s law (expressed in terms of the electrostatic potential \( \varphi \)):

\[
\mathbf{E}(\mathbf{r}) = -\nabla \varphi(\mathbf{r}), \tag{7}
\]

\[
\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V dV' \frac{\rho_b(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi\varepsilon_0} \int_S dS' \frac{\sigma_b(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{8}
\]

\[
\rho_b(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}), \tag{9}
\]

\[
\sigma_b(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}. \tag{10}
\]

In equation (8), \( V \) and \( S \) are the volume and the surface of the region containing the polarised material.

Since equations (3) and (4) can be obtained from equations (5) and (6) by replacing \( \mathbf{E} \rightarrow \mathbf{H} \) and \( \varepsilon_0^{-1} \mathbf{P} \rightarrow \mathbf{M} \), it follows that \( \mathbf{H} \) can be obtained from \( \mathbf{M} \) by the same replacement in equations (7)–(10).\(^2\) This together with equation (2) gives

\[
\mathbf{B}(\mathbf{r}) = \mu_0 (-\nabla \phi^m(\mathbf{r}) + \mathbf{M}(\mathbf{r})), \tag{11}
\]

\[
\phi^m(\mathbf{r}) = \frac{1}{4\pi} \int_V dV' \frac{\rho_b^m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi} \int_S dS' \frac{\sigma_b^m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{12}
\]

\[
\rho^m_b(\mathbf{r}) = -\nabla \cdot \mathbf{M}(\mathbf{r}), \tag{13}
\]

\[
\sigma^m_b(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \cdot \mathbf{n}. \tag{14}
\]

These relationships have been derived in alternative ways in [3] and [4].

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\(^1\) See e.g. [2], p 188.

\(^2\) This depends on the uniqueness of the solution of a vector field when its divergence and curl are known (and the assumption that the field goes to zero sufficiently quickly as \(|\mathbf{r}| \rightarrow \infty\)), which is guaranteed by the Helmholtz theorem. See e.g. [2], appendix B.
3. Magnetic field of a finite uniform solenoid

Let us apply the results obtained above to the case examined in [1], that of a finite-length solenoid of uniform surface current density. Assume that one has an idealised solenoid of arbitrarily shaped cross section with the axis in the $z$-direction a surface current $K$ of uniform magnitude. (For a physical solenoid where the current is carried by a wire that is repeatedly wound uniformly around the solenoid, $nI = \hat{z}$, where $I$ is the current through the wire and $n$ is the number of loops of wire per unit length.)

The surface current for the solenoid is reproduced by a magnetised material of constant magnetisation $M = K\hat{z}$ that is the same shape as the solenoid. Since the magnetisation is constant, the bulk magnetic charge $\rho_b = -\nabla \cdot M = 0$. The surface magnetic charge for this configuration is a disc of uniform density $\sigma_b^+ = -|M|$ on the south pole of the magnet and $\sigma^+ = |M|$ on the north pole. This shows rigourously that the magnetic field outside the solenoid is identical in form to that of the electric field of two discs of uniform charge density $\pm \sigma$ at the ends of the solenoid, with $\sigma/\varepsilon$ replaced by $\mu_0 \sigma_b^+ = nI\mu_0$. As indicated by equation (11), inside the solenoid, one needs to add $\mu_0 M$, which in this case is $\mu_0 K\hat{z}$. This procedure for obtaining the magnetic field is shown schematically in figure 1.

For a finite solenoid where the length scale of the cross section is much smaller than other length scales, the magnetic charges at the ends of the solenoids can be approximated by point charges. Using Coulomb’s law, this gives equation (5) in [1] for the magnetic field outside the solenoid. In the case of a solenoid with a circular cross section, a semi-analytic expression for the magnetic field can be obtained as an expansion in spherical harmonics, using the solution
of the electric field of a uniform charged disc\(^3\). This expression was obtained by Muniz et al \cite{5} using a different technique.

\section*{4. Another example}

The method described here assumes that one can come up with \(\mathbf{M}\) which satisfies \(\nabla \times \mathbf{M} = \mathbf{J}_f\), which is not necessarily easy to do. In cases where there are only surface currents, as in solenoids, it is typically easier to guess the appropriate \(\mathbf{M}\) which reproduces the currents. Here is another example, which is related to a paper by \cite{6} on the magnetic field produced by an axial current along a finite segment of a cylindrical tube.

Let us consider two coaxial cylindrical conductors that are capped with conducting materials at both ends, so that the two cylinders are electrically connected, as shown in figure 2. A uniform current flows in the axial direction on the outside cylinder, radially inwards along the cap to the inner cylinder, in the opposite axial direction on the inner cylinder, and then radially outwards back to the outer cylinder at the other cap.

By charge conservation, the magnitude of the surface currents must be inversely proportional to \(s\), the distance from the axis of the cylinders. Let the total current flowing through these cylinders be \(I_r\). Then, the surface current density is \(I_r/(2\pi s)\).

In this case, the currents are reproduced by the magnetisation \(\mathbf{M} = I_r/(2\pi s) \hat{\phi}\) (where \(\hat{\phi}\) is the unit vector in the azimuthal direction) in between the cylinders and zero outside. Since

\[^3\text{See e.g. \cite{2}, p 150.}\]
\[ \nabla \cdot \mathbf{M} = 0 \quad \text{and} \quad \mathbf{M} \cdot \hat{n} = 0, \] there is no magnetic charge. Hence, the magnetic field is \[ \mathbf{B} = \mu_0 \mathbf{M} = \mu_0 \frac{I}{2\pi s} \hat{\phi} \] in between the cylinders and zero everywhere outside.

Reference [6] showed that axial currents in the cylindrical segments alone (i.e. excluding the caps) produce non-zero magnetic fields everywhere in space. Therefore, everywhere outside the cylinders, the radial currents on the caps connecting the two cylinders produce magnetic fields that exactly cancel the magnetic field due to the axial currents along the cylinders. This result might appear at first sight to be surprising—why does this exact cancellation of magnetic fields occur outside the cylinders? This unexpected result can be understood by recognising that this example is in fact a toroid with a rectangular cross section (the axis of symmetry of the toroid is the co-axes of the cylinders). As is well known, the magnetic field of any uniform toroid of arbitrary cross section is confined within the toroid itself.\(^4\)

In fact, the method described in this comment can be used as an alternative way to prove this fact. Consider a toroid with an arbitrary cross section and a uniform surface current. In cylindrical coordinates, the interior of the toroid is given by a certain region in the \(s-z\) plane. The current on the surface of the toroid is reproduced by the magnetization \[ \mathbf{M} = (I/2\pi s) \hat{\phi} \] inside this region and zero outside (here \(I\) is the total current around the toroid, i.e. if there is a current \(I\) in a wire that is wound \(N\) times around the toroid, then \(I = NI\)). Since \(\nabla \cdot \mathbf{M} = 0\) in the interior and \(\mathbf{M} \cdot \hat{n} = 0\) on the surface of the toroid, there is no magnetic charge contribution to the magnetic field, and therefore \[ \mathbf{B} = \mu_0 \mathbf{M} = (\mu_0 I/2\pi s) \hat{\phi} \] inside of the toroid and zero outside.

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\(^4\) See e.g. [2], pp 238–239.