MONOGENIC HULL FOR THE $n$-CAUCHY-FUETER OPERATOR AND TWISTOR THEORY

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Abstract. This is the first part in a series of three articles in which are studied the domains of monogenicity for the $n$-Cauchy-Fueter operator. Using the twistor theory, we will in this article show that for a given open subset $U$ of $\mathbb{H}^n$, there is an open subset $\mathcal{H}(U)$, called the monogenic hull of $U$, of $M_{2n \times 2}^C = \mathbb{H}^n \otimes \mathbb{C}$ such that each monogenic function in $U$ extends to a unique pair of holomorphic functions on $\mathcal{H}(U)$.

In the second part of the series we will exploit the twistor theory furthermore to prove that each pseudoconvex domain in $\mathbb{H}^n$ is a domain of monogenicity. In the third part of the series, we show the other implication and provide a geometric characterization of the domains of monogenicity.

1. Introduction

We will write $\mathbb{H}^n = \{(q_1, \ldots, q_n) : q_\ell = x_\ell^0 + ix_\ell^1 + jx_\ell^2 + kx_\ell^3 \in \mathbb{H}, \ \ell = 1, \ldots, n\}$. Let $U$ be an open subset of $\mathbb{H}^n$ and $\psi : U \to \mathbb{H}$ be smooth. We put

$$\partial_{q_\ell} \psi := \frac{\partial \psi}{\partial x_\ell^0} + i \frac{\partial \psi}{\partial x_\ell^1} + j \frac{\partial \psi}{\partial x_\ell^2} + k \frac{\partial \psi}{\partial x_\ell^3}, \ \ell = 1, \ldots, n$$

and call

$$D \psi := (\partial_{q_1} \psi, \ldots, \partial_{q_n} \psi)$$

the $n$-Cauchy-Fueter operator. If $D \psi = 0$, then $\psi$ is called monogenic (or regular) and we denote by $\mathcal{R}(U)$ the space of monogenic functions in $U$. See [3], [4], [6] and [7] for some background on this operator.

Fixing a linear isomorphism $\mathbb{H} \to \mathbb{C}^2$ as in Section 2.1, the function $\psi$ corresponds to a pair of functions $\psi_A' : U \to \mathbb{C}$, $A = 0, 1$. If $\psi \in \mathcal{R}(U)$, then it is well known that $\psi_0, \psi_1$, are analytic and since $\mathbb{H}^n \otimes \mathbb{C}$ is isomorphic to the space $M_{2n \times 2}^C$ of complex $2n \times 2$ matrices, there is an open set $U_C \subset M_{2n \times 2}^C$ and unique holomorphic functions $\psi_A' : U_C \to \mathbb{C}$ such that $U_C \cap \mathbb{H}^n = U$ and $\psi_A'|_U = \psi_A$, $A = 0, 1$. The main result of this paper is (see Theorem 2.1) that there is an open subset $\mathcal{H}(U)$ of $M_{2n \times 2}^C$, called the monogenic hull of $U$, with $\mathcal{H}(U) \cap \mathbb{H}^n = U$ such that each monogenic function in $U$ extends to a pair of holomorphic functions in $\mathcal{H}(U)$.

2010 Mathematics Subject Classification. Primary: 35N05, 32L25.
Key words and phrases. Twistor theory, Penrose transform, $n$-Cauchy-Fueter operator, monogenic hull.
If \( n = 1 \), then it is easy to see that \( \mathcal{H}(U) \) is maximal among all open subsets of \( M^C_{2 \times 2} \) which satisfy the extension requirement (see Example 2.2). If \( n > 1 \), then the situation is more subtle. This is related to the fact that \( D \) is an overdetermined operator and Hartog’s phenomenon holds for monogenic functions (this is originally due to [13], see also [6, Theorem 3.3.5], [17] and [18]). Hence, the theory of monogenic functions of several quaternionic variables is parallel to the theory of holomorphic functions. Actually, we will use in the third part of the series [16] the main result of this article to show that any domain of monogenicity \(^1\) is pseudoconvex \(^2\).

In order to prove the main result, we will use the twistor theory as in [9], see also [1], [2], [8], [18] and [19]. Recall [5, Section 4.4.9] that there is a fiber bundle \( S^2 \to \tilde{U} \to U \), called the twistor space, associated to the flat almost quaternionic structure over \( U \). The total space \( \tilde{U} \) carries a tautological almost complex structure which is integrable and thus, \( \tilde{U} \) is a complex manifold. In this article we will view \( \tilde{U} \) as an open submanifold of \( \mathbb{C}P^{2n+1} \). The hardest part of the proof of Theorem 2.1 is (see Theorem 5.2) to show that there is an isomorphism

\[
H^1(\tilde{U}, L) \to \mathcal{R}(U)
\]

where \( L \) is a certain holomorphic line bundle over \( \tilde{U} \). This extends results given in ([12]). The isomorphism (1.3) is given by some completely explicit integral formula and is coming from the Penrose transform.

In the second part [15] of the series, we will exploit the twistor theory furthermore. Using \( L^2 \) estimates as in [11], we will show that \( H^2(\tilde{U}, L) = 0 \) when \( U \) is pseudoconvex and from this information we will conclude that \( U \) is a domain of monogenicity.

**Notation**

- \( M^T_{n \times k} \) = matrices of size \( n \times k \) with coefficients in a field \( T \)
- \( T^* = T \setminus \{0\} \)
- \( \text{Sp}(1) = \) the group of unit quaternions

**Acknowledgment**

The author is grateful to Vladimír Souček for stimulating discussions and to Roman Lávička for his help. The author gratefully acknowledges the support of the grant 17-01171S of the Grant Agency of the Czech Republic.

2. \( n \)-Cauchy-Fueter operator

2.1. Some background on quaternions. Let \( \mathbb{RH} \) and \( \mathbb{IH} \) be the subspace of real and imaginary quaternions, respectively. We denote by \( (x, y) := \mathbb{R}(\bar{x}y), x, y \in \mathbb{HH}^n \) the standard real inner product and by \( \|x\| := \sqrt{(x,x)} \) the associated norm. We will always view \( \mathbb{HH}^n \) as a right \( \mathbb{H} \)-vector space and

\(^1\)Loosely speaking, we call the open set \( U \) a domain of monogenicity if there is no open subset \( V \) of \( \mathbb{H}^n \) with \( U \subset V \) such that the restriction map \( \mathcal{R}(V) \to \mathcal{R}(U) \) is surjective. See [15] or [14] for a precise definition.

\(^2\)We call \( U \) pseudoconvex if it admits a smooth exhausting \( \mathbb{H} \)-plurisubharmonic function or equivalently, \( \delta^{-2} \) is \( \mathbb{H} \)-plurisubharmonic where \( \delta \) is the usual distance function to the boundary of \( U \). See [10]
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$\mathbb{C}$ as the subalgebra of $\mathbb{H}$ generated by $i$. Then $\mathbb{H}^n$ is a complex vector space of dimension $2n$ and the map

$$\mathcal{C}^{2n} \to \mathbb{H}^n, \quad (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)^T \mapsto (\alpha_1 + k\beta_1, \ldots, \alpha_n + k\beta_n)^T,$$

is a $\mathbb{C}$-linear isomorphism. Using the notation set in Introduction, we may write

$$\alpha_\ell = x_0^\ell + ix_1^\ell, \quad \beta_\ell = x_2^\ell + ix_3^\ell \quad \text{and} \quad \partial_{\bar{\alpha}_\ell} = 2(\partial_{\bar{x}} + k\partial_{\bar{x}}), \quad \ell = 1, \ldots, n.$$

The map $\mathbb{H}^n \to \mathbb{H}^n$, $w \mapsto wk$ corresponds to

$$\mathbb{K}: \mathcal{C}^{2n} \to \mathcal{C}^{2n}, \quad \mathbb{K}(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)^T = (-\bar{\beta}_1, \bar{\alpha}_1, \ldots, -\bar{\beta}_n, \bar{\alpha}_n)^T.$$

Let $T = \mathbb{C}$ or $\mathbb{H}$. As any $\mathbb{H}$-linear map $\mathbb{H}^k \to \mathbb{H}^n$ is also complex linear, there is an injective homomorphism of algebras $M_{n \times k}^{\mathbb{R}} \hookrightarrow M_{2n \times 2k}^{\mathcal{C}}$. On the other hand, a complex linear map $A : \mathcal{C}^k \to \mathcal{C}^{2n}$ is $\mathbb{H}$-linear if and only if $A \circ \mathbb{K} = \mathbb{K} \circ A$. If $n = k = 1$, then the embedding is

$$\mathcal{C}^2 = \mathbb{H} = M_{1 \times 1}^{\mathbb{H}} \hookrightarrow M_{2 \times 2}^{\mathcal{C}}, \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}$$

and more generally:

$$\mathcal{C}^{2n} = \mathbb{H}^n = M(n, 1, \mathbb{H}) \hookrightarrow M_{2n \times 2n}^{\mathcal{C}}, \quad x \mapsto M(x) := (x|\mathbb{K}(x)).$$

We will use the isomorphisms in (2.5) without further comment. Given $(z_{A\ell}) \in M_{2 \times 2}^\mathcal{C}$, there are unique $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\begin{pmatrix} z_00^\ell & z_01^\ell \\ z_{10}^\ell & z_{11}^\ell \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} + i \begin{pmatrix} \gamma & -\bar{\delta} \\ \delta & \bar{\gamma} \end{pmatrix}.$$

This shows that $M_{2n \times 2k}^{\mathcal{C}} = M_{n \times k}^{\mathbb{R}} \otimes \mathbb{C}$ and thus, we can view a matrix $z \in M_{2n \times 2k}^{\mathcal{C}}$ as a pair $(x, y)$ where $x, y \in \mathbb{H}^n$. We will do that without further comment. We will work with the norm $\|(x, y)\|_C := \sqrt{\|x\|^2 + \|y\|^2}$ on $M_{2n \times 2n}^{\mathcal{C}}$.

2.2. $n$-Cauchy-Fueter operator and the monogenic hull. Given $\psi : U \to \mathbb{H}$, there are unique functions $\psi_{A\ell} : U \to \mathbb{C}$, $A = 0, 1$ so that $\psi = \psi_{\ell} + kv_{\ell}$. Using (2.2) we see that $D\psi = 0$ if and only if

$$\begin{align*}
\partial_{\bar{\alpha}_\ell} \psi_{\ell} &- \partial_{\bar{x}_\ell} \psi_{\ell} = 0, \\
\partial_{x_\ell} \psi_{\ell} + \partial_{\bar{\beta}_\ell} \psi_{\ell} = 0
\end{align*}$$

for every $\ell = 1, \ldots, n$. As $M_{2n \times 2}^{\mathcal{C}}$ is the complexification of the totally real submanifold $\mathbb{H}^n$ and any monogenic function is real analytic, it follows that there is an open subset $U_C$ of $M_{2n \times 2}^{\mathcal{C}}$ with a pair of holomorphic functions $\psi_{\ell}^C : U_C \to \mathbb{C}$ such that:

(i) $U_C \cap \mathbb{H}^n = U$

(ii) $\psi_{A\ell}^C|_U = \psi_{A\ell}$, $A = 0, 1$.

Moreover, it is well known that $(\psi_{A\ell}^C)_{A=0,1}$ are null solutions of

$$D^C(\psi_{\ell}^C, \psi_{\ell}^C) = \left(\partial_{z_{A\ell}} \psi_{\ell}^C - \partial_{z_{A\ell}} \psi_{\ell}^C\right)_{A=0,1, \ldots, 2n-1}.$$
Conversely, if \((\psi^C_{A'})_{A=0,1}\) are holomorphic in \(U_C\) and are null solutions of \((2.8)\), then the restriction of these functions to \(U\) is monogenic. If \(V\) is an open subset of \(M^2_{2n \times 2}\), then we put
\[
\mathcal{R}^C(V) := \{(\psi^C_{A'})_{A=0,1} : D^C(\psi^C_{0}', \psi^C_{1'}) = 0\}.
\]

**Definition 2.1.** Let \(U \subset \mathbb{H}^n\) be open. We call the set
\[
\{(x, y) \in M^2_{2n \times 2} : \forall q \in \text{Sp}(1) \cap \mathbb{H} : x + yq \in U\}
\]
the monogenic hull \(\mathcal{H}(U)\) of \(U\).

It follows from the definition that \(\mathcal{H}(U)\) is open, \(U = \mathcal{H}(U) \cap \mathbb{H}^n\) and that \(\mathcal{H}(U) = \bigcap_{x \in U} \mathcal{H}(\mathbb{H}^n \setminus \{x\})\). Consider also the following example.

**Example 2.1.** Let \(U\) be an open subset of \(\mathbb{H}\). Since \(\det(x, y) := \|x\| - \|y\| + 2i(x, y)\), it is clear that \(\mathcal{H}(\mathbb{H}^n) = \text{GL}(2, \mathbb{C})\). As \(\text{GL}(2, \mathbb{C})\) is the complement of the analytic variety \(\{z_{00'}z_{11'} - z_{01'}z_{10'} = 0\}\) in \(M^2_{2n \times 2}\), it is a domain of holomorphy. By [11] Theorem 2.6.9, also \(\mathcal{H}(U)\) is a domain of holomorphy.

**Remark 2.1.** As we have seen in Example 2.1, \(\mathcal{H}(U)\) is a domain of holomorphy for any open subset \(U\) of \(\mathbb{H}\). We will show in the third part of the series that \(\mathcal{H}(U)\) is a domain of holomorphy if and only if \(U\) is a domain of monogenicity in \(\mathbb{H}^n\).

The main result of this article is the following theorem.

**Theorem 2.1.** Let \(U\) be an open subset of \(\mathbb{H}^n\). Then the restriction map
\[
(2.10) \quad \mathcal{R}^C(\mathcal{H}(U)) \to \mathcal{R}(U), \quad (\psi^C_{A'})_{A=0,1} \mapsto (\psi^C_{A'}|U)_{A=0,1}
\]
is an isomorphism.

The proof of Theorem 2.1 will occupy the rest of this article. Let us consider the following example.

**Example 2.2.** The fundamental solution of the Cauchy-Fueter operator is (up to constant)
\[
E(q) = \frac{\tilde{q}}{|q|^4} = \frac{\tilde{\alpha} - k\beta}{(\alpha\tilde{\alpha} + \beta\tilde{\beta})^2}
\]
which is monogenic in \(\mathbb{H}^n\). Hence, the corresponding complex functions are
\[
\psi_{0'} = \frac{\tilde{\alpha}}{(\alpha\tilde{\alpha} + \beta\tilde{\beta})^2} \quad \text{and} \quad \psi_{1'} = \frac{-\beta}{(\alpha\tilde{\alpha} + \beta\tilde{\beta})^2}.
\]
The holomorphic extension of \(E\) to \(\text{GL}(2, \mathbb{C})\) is
\[
\psi^C_{0'} = \frac{z_{11'}}{(z_{00'}z_{11'} - z_{01'}z_{10'})^2} \quad \text{and} \quad \psi^C_{1'} = \frac{-z_{10'}}{(z_{00'}z_{11'} - z_{01'}z_{10'})^2}.
\]

If \(U \subset \mathbb{H}^n\) is open and \(x \in U\), we put \(\delta(x, U^C) := \inf_{y \in U^C} \|x - y\|\) so that \(\delta(-, U^C)\) is continuous in \(U\). If \(U_C \subset \mathbb{C}^{4n}\) is open, we similarly define \(\delta^C(z, U^C) := \inf_{w \in U^C} \|z - w\|\), \(z \in U_C\).

**Lemma 2.2.** Let \(U\) be an open subset of \(\mathbb{H}^n\). Then, with the notation set above, we have
\[
\delta^C((x, y), \mathcal{H}(U)^C) = \frac{1}{\sqrt{2}} \inf_{q \in \mathbb{H}^n \cap \text{Sp}(1)} \delta(x + yq, U^C), \quad (x, y) \in \mathcal{H}(U).
\]
Proof. Fix \((x, y) \in \mathcal{H}(U)\) and put \(c := \inf_{q \in 3\mathbb{H} \cap \text{Sp}(1)} \delta(x + yq, U^\mathcal{C})\). Then there are \(x_o \in \partial U\) and \(q \in 3\mathbb{H} \cap \text{Sp}(1)\) such that \(\|x + yq - x_o\| = c\). Put \(w := x_o - x - yq\) and consider \(x' = x + \frac{1}{2}w\) and \(y' = y - \frac{1}{2}wq\). Then

\[
\|(x', y') - (x, y)\|_C = \left\| \frac{1}{2}(w, -wq) \right\|_C = \frac{\sqrt{2}}{2} \|w\| = \frac{c}{\sqrt{2}}
\]

and

\[
x' + y'q = x + yq + w = x_o.
\]

It follows that \((x', y') \notin \mathcal{H}(U)\) and thus \(\delta^C((x, y), \mathcal{H}(U) \cap C) \leq \frac{c}{\sqrt{2}}\).

On the other hand, choose \((x'', y'') \in M_{2n \times 2}^\mathbb{C}\) with

\[
\|(x'', y'') - (x, y)\|_C < \frac{c}{\sqrt{2}}.
\]

We put \(w' := x'' - x\) and \(w'' := y'' - y\). If \(q' \in 3\mathbb{H} \cap \text{Sp}(1)\), then

\[
\|(x'' + y''q') - (x + yq')\| = \|w' + w''q'\| \\
\leq \sqrt{\|w'||^2 + \|w''||^2 + 2(w', w'')} \\
\leq \sqrt{\|w'||^2 + \|w''||^2 + 2\|w''\|w''||} \\
\leq \sqrt{2(\|w'||^2 + \|w''||^2)} < c.
\]

It follows that \((x'', y'') \in \mathcal{H}(U)\) and thus \(\delta^C((x, y), \mathcal{H}(U) \cap C) \geq \frac{c}{\sqrt{2}}\).

Hence, we have the following

**Corollary 2.1.** If \(\psi\) is a monogenic function in the ball \(B_r := \{x \in \mathbb{H}^n : \|x\| < r\}\), then the Taylor series of \(\psi\) centered at 0 converges in \(\mathcal{B}_{\frac{c}{\sqrt{2}}}\) to \(\psi\).

Notice that Corollary 2.1 is in accordance with [10].

3. The Penrose transform for the \(n\)-Cauchy-Fueter operator

In Section 3.1 we will review some well known material on sheaf cohomology groups of holomorphic line bundles over the Riemann sphere and provide some elementary proofs which will be used afterwards.

3.1. Complex projective line. We will use the standard homogeneous coordinates on \(\mathbb{CP}^1\) and put \(\mathfrak{X}_i := \{[\pi_0 : \pi_1] : \pi_i \neq 0\}, \ i = 0, 1.\) Then there are biholomorphisms \(\mathfrak{X}_0 \rightarrow \mathbb{C}, \ [\pi_0 : \pi_1] \mapsto z := \pi_1/\pi_0\) and \(\mathfrak{X}_1 \rightarrow \mathbb{C}, \ [\pi_0 : \pi_1] \mapsto w := \pi_0/\pi_1\) with inverses \(\mathbb{C} \ni z \mapsto [1 : z] \in \mathfrak{X}_0\) and \(\mathbb{C} \ni w \mapsto [w : 1] \in \mathfrak{X}_1,\) respectively. We have that \(\mathfrak{X}_0 \cap \mathfrak{X}_1 = \{[\pi_0 : \pi_1] : \pi_0\pi_1 \neq 0\} = \{[1 : z] : \ z \neq 0\} = \{[w : 1] : \ w \neq 0\} \cong \mathbb{C}^*\) and that \(\{[1 : z] = [w : 1]\) if and only if \(z = w^{-1}\). Hence, we can view \(\mathfrak{X}_0\) and \(\mathfrak{X}_1\) as \(\mathbb{C}\) and we will do that without further comment.

We will denote by \(Q_k, \ k \in \mathbb{Z}\) the holomorphic line bundle over \(\mathbb{CP}^1\) with the transition function \(z^{-k}\) in \(\mathfrak{X}_0 \cap \mathfrak{X}_1\). This means that smooth functions \(f_i : \mathfrak{X}_i \rightarrow \mathbb{C}, \ i = 0, 1\) define a smooth section of \(Q_k\) if

\[
f_i(z^{-1}) = z^{-k}f_0(z), \ \forall z \in \mathbb{C}^*.
\]

The section is holomorphic if both functions are holomorphic.

\textsuperscript{3}The Taylor series of \(\psi\) is in the variables \(\alpha, \alpha, \beta, \beta, \ell = 1, \ldots, n.\)
We will denote by $\Lambda^{(0,1)}\mathbb{CP}^1$ the vector bundle of $(0,1)$-forms over $\mathbb{CP}^1$, i.e. the fiber of this bundle over $x \in \mathbb{CP}^1$ is the vector space of all complex anti-linear maps $T_x \mathbb{CP}^1 \to \mathbb{C}$. The bundle $\Lambda^{(0,1)}\mathbb{CP}^1$ is trivialized by $d\bar{z}$ over $\mathcal{X}_0$ and by $dw$ over $\mathcal{X}_1$ with $dw = -\bar{z}^{-2}d\bar{z}$ in $\mathcal{X}_0 \cap \mathcal{X}_1$. It follows that a global smooth section of $\Lambda^{(0,1)}(Q_k) := \Lambda^{(0,1)}\mathbb{CP}^1 \otimes Q^k$ is then given by a pair $(f_0, d\bar{z}, f_1 dw)$ where $f_i : \mathcal{X}_i \to \mathbb{C}$, $i = 0, 1$ are smooth and

$$f_1(z^{-1}) = -z^{-k}z^2f_0(z), \ \forall z \in \mathbb{C}^*.$$ 

We denote by

$$\mathcal{E}(\mathbb{CP}^1, Q_k) := \Gamma(\mathbb{CP}^1, Q_k) \text{ and } \mathcal{E}^{(0,1)}(\mathbb{CP}^1, Q_k) := \Gamma(\mathbb{CP}^1, \Lambda^{(0,1)}(Q_k))$$

the corresponding spaces of global sections. The Dolbeault complex is

$$\bar{\partial} : \mathcal{E}(\mathbb{CP}^1, Q_k) \to \mathcal{E}^{(0,1)}(\mathbb{CP}^1, Q_k), \ \bar{\partial}(f_0, f_1) = (\partial_z f_0, \partial_w f_1 dw).$$

We put $H^0(\mathbb{CP}^1, Q_k) := \ker(\bar{\partial})$ and $H^1(\mathbb{CP}^1, Q_k) := \coker(\bar{\partial})$. By definition, $H^0(\mathbb{CP}^1, Q_k)$ is the space of global holomorphic sections of $Q_k$.

$\mathbb{CP}^1$ can be also viewed as a 1-point compactification of $\mathbb{C} = \mathcal{X}_0$ with the point $\infty = [0 : 1]$ at infinity, i.e. $\mathcal{X}_0$ is an open and dense subset of $\mathbb{CP}^1$ and thus each smooth section of a vector bundle over $\mathbb{CP}^1$ is uniquely determined by its restriction to $\mathcal{X}_0$.

**Lemma 3.1.** Let $k \in \mathbb{Z}$ and $f_0 : U_0 \to \mathbb{C}$ be smooth.

(a) If $f_0$ extends to a global smooth section of $Q_k$, then

$$\lim_{z \to \infty} z^\ell f_0(z) \begin{cases} = 0, & \ell < -k \\ \in \mathbb{C}, & \ell = -k \end{cases}$$

(b) If $f_0$ extends to a global smooth section of $\Lambda^{(0,1)}(Q_k)$, then

$$\lim_{z \to \infty} z^\ell z^n f_0(z) = 0$$

provided that $\ell + n < -k + 2$.

**Proof.** (a) If $(f_0, f_1) \in \mathcal{E}(\mathbb{CP}^1, Q_k)$, it follows that $z^\ell f_0(z) = z^{k+\ell} f_1(z^{-1}) = w^{-k-\ell} f_1(w)$ where $w = z^{-1} \neq 0$ and $\ell \in \mathbb{Z}$. Thus, (3.3) is equal to $\lim_{w \to 0} w^{-k-\ell} f_1(w) = f_1(0) \lim_{w \to 0} w^{-k-\ell}$ and the first claim follows.

(b) If $(f_0 d\bar{z}, f_1 dw) \in \mathcal{E}^{(0,1)}(\mathbb{CP}^1, Q_k)$, then we find that $z^\ell z^n f_0(z) = w^{-k-\ell} w^{-n+2} f_1(w)$ where $w = z^{-1} \neq 0$ and $\ell \in \mathbb{Z}$. It follows that the limit in (3.4) is equal to $f_1(0) \lim_{w \to 0} w^{-k-\ell} w^{-n+2}$. If $-k-\ell-n+2 > 0$, then it is zero. \qed

Assume that $\omega := (h_0(z) \ d\bar{z}, h_1(w) \ dw) \in \mathcal{E}^{(0,1)}(\mathbb{CP}^1, Q_k)$. By Lemma 3.1 it follows that the integral

$$a_\ell := \frac{1}{2\pi i} \int_C z^\ell h_0(z) \ d\bar{z} \wedge dz, \ 0 \leq \ell \leq -k-2$$

converges.

**Lemma 3.2.** If $k \leq -2$, then the map

$$\omega \mapsto (a_0, \ldots, a_{-k-2})$$

defined above descends to linear isomorphism $H^1(\mathbb{CP}^1, Q_k) \to \mathbb{C}^{-k-1}$ while $H^1(\mathbb{CP}^1, Q_k) = \{0\}$ otherwise.
Proof. If $\omega$ is exact, say $\partial_z f_0 = h_0$ where $f_0 : \mathbb{X}_i \to \mathbb{C}$ is smooth, then by Stokes’ theorem (see [11, Section 1.3]):

$$\int_{\mathbb{C}} \tau^k h_0(z) \, dz \wedge d\tau = \lim_{R \to +\infty} \int_{S_R^n} \tau^k f_0 \, dz = \lim_{R \to +\infty} \int_0^{2\pi} f_0(R e^{it}) i R^{l+1} e^{it(\ell+1)} \, dt$$

where $S_R^n = \{ z \in \mathbb{C} : |z| = R \}$. Since $\ell + 1 < -k$, it follows by Lemma 3.1(a) that $\lim_{R \to +\infty} R^{\ell+1} |f_0(R e^{it})| = 0$ and thus the map (3.6) descends to cohomology.

It is easy to see that the map (3.6) is onto and thus, it remains to show that it is injective. Let us assume that $a_0 = \cdots = a_{-k-2} = 0$. By [11, Theorem 1.4.4], there are functions $g_i : \mathbb{X}_i \to \mathbb{C}$, $i = 0, 1$ such that $\partial_z g_0 = h_0$ and $\partial_z g_1 = h_1$. Put $t : \mathbb{X}_0 \cap \mathbb{X}_1 \to \mathbb{C}$, $t(z) := z^k g_1(z^{-1})$. Then $\partial_z t = h_0$ in $\mathbb{X}_0 \cap \mathbb{X}_1$, and thus $g_0 - t$ is analytic in $\mathbb{X}_0 \cap \mathbb{X}_1$, say $g_0 - t = \sum_i b_i z^i$, $b_i \in \mathbb{C}$, $z \neq 0$. By the definition of $t$ and the first part of the proof, it follows that $a_0 = b_{-1-\ell}$, $\ell = 0, \ldots, -k-2$. Put $f_0 := g_0 - t = \sum_i b_i z^i$ and $f_1 := g_1 - \sum_i b_i w^{k-\ell}$. Then $f_1 : \mathbb{X}_i \to \mathbb{C}$, $i = 0, 1$ are smooth with $\partial_z f_0 = h_0$, $\partial_{\bar{w}} f_1 = h_1$ and $f_1(z^{-1}) = z^{-k} f_0(z)$. We have proved that $f = (f_0, f_1) \in \mathcal{E}(\mathbb{C}^2, Q_k)$ and $\omega = \partial f$.

Example 3.1. Notice that the map

$$(3.7) \quad \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \mapsto 2 \left[ \begin{bmatrix} a_0 \bar{z} \bar{d} \bar{z} + a_1 \bar{z} \bar{d} \bar{z} \\ (1 + z \bar{z})^3 \\ -a_0 \bar{w} \bar{d} \bar{w} - a_1 \bar{w} \bar{d} \bar{w} \\ (1 + \bar{w} \bar{w})^3 \end{bmatrix} \right],$$

where [ ] denotes the corresponding cohomology class, is inverse to the isomorphism $H^1(\mathbb{C}^2, Q^{-3}) \to \mathbb{C}^2$ from Lemma 3.2.

3.2. Double fibration diagram and correspondence. Using (2.1), there is a well defined embedding $\iota : \mathbb{H}^n \hookrightarrow V_2(\mathbb{C}^{2n+2})$ where $\mathbb{H}^n$ is the quaternionic projective space in dimension $n$ and $V_2(\mathbb{C}^{2n+2})$ is the Grassmannian of complex 2-dimensional subspaces in $\mathbb{C}^{2n+2}$. We will view $\mathbb{H}^n$ as the standard affine subset $\{ [1 : q_1 : \cdots : q_n] : q_\ell \in \mathbb{H}, \ell = 1, \ldots, n \}$ of $\mathbb{H}^n$. Now consider the map

$$(3.8) \quad (z_{AB'}) \in M_{2n \times 2}^{\mathbb{C}} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ z_{2n-1,0'} & z_{2n-1,1'} \end{bmatrix} \in V_2(\mathbb{C}^{2n+2})$$

where we denote by square brackets the complex linear subspace spanned by the columns of the given $(2n + 2) \times 2$-matrix. The map identifies $M_{2n \times 2}^{\mathbb{C}}$ with an open, dense and affine subset of $V_2(\mathbb{C}^{2n+2})$ which we for brevity also denote by $M_{2n \times 2}^{\mathbb{C}}$ and we will view a $2n \times 2$-matrix as the corresponding 2-plane in $\mathbb{C}^{2n+2}$ without further comment. Altogether, there are inclusions

$$(3.9) \quad \mathbb{H}^n \subset \mathbb{H}^n \cap M_{2n \times 2}^{\mathbb{C}} \subset V_2(\mathbb{C}^{2n+2}).$$

where the embedding $\mathbb{H}^n \hookrightarrow M_{2n \times 2}^{\mathbb{C}}$ is given in (2.5).
Consider the double fibration diagram

\[
\begin{array}{c}
\mathbb{CP}^{2n+1} \\
\downarrow \eta \\
V_2(\mathbb{C}^{2n+2}) \\
\end{array}
\xleftarrow{\tau} F_{1,2}
\]

where \(F_{1,2}\) is the flag manifold of nested subspaces \((\ell, \Sigma)\) where \(\ell \in \mathbb{CP}^n\), \(\Sigma \in V_2(\mathbb{C}^{2n+2})\) and \(\ell \subset \Sigma\). The maps \(\eta\) and \(\tau\) are the obvious projections. The space on the left hand side is called the \textit{twistor space} and the space in the middle upstairs is called the \textit{correspondence space}.

Let \(U_c \subset V_2(\mathbb{C}^{2n+2})\). We put \(\hat{U}_c := \tau^{-1}(U_c)\) and \(\tilde{U}_c := \eta(\hat{U}_c)\) so there is another diagram

\[
\begin{array}{c}
\hat{U}_c \\
\downarrow \\
\tilde{U}_c \\
\end{array}
\xleftarrow{\iota} U_c
\]

We for clarity put \(\tilde{\mathbb{H}}^n := \mathbb{H}^n\), \(\tilde{M}_{2n \times 2}^\Sigma := M_{2n \times 2}^{\Sigma}\) and \(\tilde{\Sigma} := \{\Sigma\}\) where \(\Sigma \in V_2(\mathbb{C}^{2n+2})\). By definition, \(\tilde{\Sigma}\) is the set of all complex projective lines which are contained in \(\Sigma\) and thus, \(\tilde{\Sigma}\) is biholomorphic to \(\mathbb{CP}^1\). If \(\Sigma\) is the 2-plane on the right hand side of \((3.10)\), then \(\tilde{\Sigma}\) is the image of the embedding

\[
\iota_\Sigma : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^{2n+1},
\]

\[
[\pi_0 : \pi_1] \mapsto [\pi_0 : \pi_1 : \pi_0 z_{00'} + \pi_1 z_{01'} : \cdots : \pi_0 z_{2n-1,0'} + \pi_1 z_{2n-1,1'}].
\]

It is easy to see that there is a biholomorphism

\[
(3.13) \quad \mathbb{CP}^1 \times U \rightarrow \hat{U}_c, \, ([\pi_0 : \pi_1], \Sigma) \mapsto (\iota_\Sigma([\pi_0 : \pi_1]), \Sigma)
\]

so that \(\tau|_{\hat{U}_c} : \hat{U}_c \rightarrow U_c\) corresponds to the projection onto the first factor.

It is clear that \(\tilde{M}_{2n \times 2}^\Sigma = W_0 \cup W_1\) where

\[
(3.14) \quad W_0 = \{[1 : z_0 : \cdots : z_{2n}] : z_i \in \mathbb{C}, \, i = 0, 1, \ldots, 2n\}
\]

and

\[
(3.15) \quad W_1 = \{[w_0 : 1 : w_1 : \cdots : w_{2n}] : w_i \in \mathbb{C}, \, i = 0, 1, \ldots, 2n\}
\]

and that \(W_0 \cap W_1\) is the subset \(\{z_0 \neq 0\}\) of \(W_0\) and \(\{w_0 \neq 0\}\) of \(W_1\). The change of coordinates is

\[
w_0 = z_0^{-1}, \, w_i = z_i z_0^{-1}, \, i = 1, \ldots, 2n.
\]

### 3.3. Correspondence over \(\mathbb{H}^n\). Let \(X_0, X_1\) be the open affine subsets of \(\mathbb{CP}^1\) from Section 3.1 and \(U\) be an open subset of \(\mathbb{H}^n\). By \((3.9)\), we shall view \(U\) as an open subset of \(V_2(\mathbb{C}^{n+2})\). By \((3.3)\), there is a diffeomorphism \(\mathbb{CP}^1 \times U \rightarrow \hat{U}\) such that \(\mathbb{CP}^1 \times U \cong \hat{U} \xrightarrow{\pi|_{\hat{U}}} U\) is the canonical projection onto the second factor. We will view \(\hat{U}_i := X_i \times U = \mathbb{C} \times U, \, i = 0, 1\) as open
where \( M(x) \) is a column vector of \( n \times 2 \) entries.

Recall (2.5) that \( \Sigma = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) \).

Using the notation from (3.14) and (3.15), we have

\[
\begin{pmatrix}
\alpha_1 \\
\beta_1 \\
\vdots \\
\alpha_n \\
\beta_n
\end{pmatrix}
\]

respectively. Using the notation from (3.13) and (3.15), we have

\[
\begin{pmatrix}
z \\
\alpha_1 - z\beta_1 \\
\beta_1 + z\alpha_1 \\
\vdots \\
\alpha_n - z\beta_n \\
\beta_n + z\alpha_n
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\alpha_1 \\
\beta_1 \\
\vdots \\
\alpha_n \\
\beta_n
\end{pmatrix}
\]

and conversely

\[
\begin{pmatrix}
z = z_0 \\
\alpha_i = \frac{z_{2i-1} + z_0 z_{2i}}{1 + z_0 z_{2i}} \\
\beta_i = \frac{z_{2i} - z_0 z_{2i-1}}{1 + z_0 z_{2i}}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
w = w_0 \\
\alpha_i = \frac{\bar{w}_{2i} + \bar{w}_0 w_{2i-1}}{1 + w_0 \bar{w}_{0}} \\
\beta_i = \frac{-\bar{w}_{2i-1} + \bar{w}_0 w_{2i}}{1 + w_0 \bar{w}_{0}}
\end{pmatrix}
\]

where \( i = 1, \ldots, n \). It is now clear that the maps \( \tilde{U}_i \rightarrow \tilde{U}_i \), \( i = 0, 1 \) are diffeomorphisms and hence, we get the following important Lemma.

**Lemma 3.3.** Let \( U \) be an open subset of \( \mathbb{H}^n \). Then \( \tilde{U} \) is an open subset of \( \mathbb{C}P^{2n+1} \) and \( \eta : \tilde{U} \rightarrow \tilde{U} \) is a diffeomorphism.

We can now give an equivalent characterization of the monogenic hull associated to \( U \).

**Theorem 3.4.** Let \( U \) be an open subset of \( \mathbb{H}^n \). Then

\[
\mathcal{H}(U) = \{ \Sigma \in M_{2n \times 2}^\mathbb{C} : \tilde{\Sigma} \subset \tilde{U} \}
\]

**Proof.** First of all, it is clear that \( \mathbb{H}^n = M_{2n \times 2}^\mathbb{C} = W_0 \cup W_1 \) and that \( \mathbb{H}^n = \bigcup_{\Sigma \in \mathbb{H}^n} \tilde{\Sigma} \) where the sum is disjoint. Hence, if we denote by \( \Sigma_\ell \in \mathbb{H}P^n \) the unique quaternionic line which contains \( \ell \in \mathbb{C}P^{2n+1} \), then we have

\[
\tilde{\Sigma} \subset \tilde{U} \iff \tilde{\Sigma} \cap \tilde{U}^\mathbb{C} = \emptyset
\]

\[
\iff \tilde{\Sigma} \cap \tilde{U}^\mathbb{C} = \emptyset
\]

\[
\iff \{ \Sigma_\ell : \ell \in \tilde{\Sigma} \} \cap \tilde{U}^\mathbb{C} = \emptyset
\]

\[
\iff \{ \Sigma_\ell : \ell \in \tilde{\Sigma} \} \subset U
\]

where we put \( \tilde{U}^\mathbb{C} := \mathbb{H}^n \setminus \tilde{U} \). Thus if \( \Sigma = (x, y) \) where \( x, y \in \mathbb{C}^{2n} = \mathbb{H}^n \), then it is enough to show that

\[
\{ \Sigma_\ell : \ell \in \tilde{\Sigma} \} = \{ x + yq : q \in \mathbb{H} \cap \text{Sp}(1) \}
\]

Recall (2.5) that \( \Sigma = (x, y) \) by definition means that \( \Sigma = M(x) + iM(y) \)

where \( M(x) = (x|\mathbb{K}(x)) \), i.e. the first column of \( M(x) \in M_{2n \times 2}^\mathbb{C} \) is \( x \) and
the second column is $\mathbb{K}(x)$. Observe that $iM(y) = (iy| - \mathbb{K}(iy))$. It is a straightforward computation to verify that
\[
\{\Sigma_\ell : \ell \in \tilde{\Sigma}\} = \{x + i(\alpha \bar{\alpha} - \beta \bar{\beta})y - 2\beta \bar{\alpha} \mathbb{K}(iy) \in \alpha, \beta \in \mathbb{C}, \alpha \bar{\alpha} + \beta \bar{\beta} = 1\}.
\]
By (4.1), $\mathbb{K}(iy) \in \mathbb{C}^{2n}$ corresponds to $yi \in \mathbb{H}^n$ and thus, we see that $i(\alpha \bar{\alpha} - \beta \bar{\beta})y - 2\beta \bar{\alpha} \mathbb{K}(iy) \in \mathbb{C}^{2n}$ corresponds to $y(\alpha \bar{\alpha} - \beta \bar{\beta})i + 2yj\bar{\alpha} \bar{\beta} \in \mathbb{H}^n$.

Now it is easy to see that (3.19) holds. \hfill \Box

4. Dolbeault complex over the twistor space

4.1. Filtration of the vector bundle of $(0,q)$-forms. By Lemma 3.3 there are diffeomorphisms $\tilde{U} \cong \tilde{U} \cong \mathbb{CP}^1 \times U$ and we for brevity denote the composition $\tilde{U} \cong \tilde{U} \overset{\tau}{\to} U$ also by $\tau$ as there is no risk of confusion. Also recall Section 3.3 that $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$ and that $\tilde{U}_i = \tilde{X}_i \times U \cong \mathbb{C} \times U$, $i = 0, 1$.

The composition $\zeta: \tilde{U} \cong \mathbb{CP}^1 \times U \to \mathbb{CP}^1$, where the second map is the canonical projection, is the restriction of the canonical projection $\mathbb{CP}^{2n+1} \ni [\pi_0 : \pi_1 : \cdots : \pi_{2n+1}] \mapsto [\pi_0 : \pi_1]$. We see that $\zeta$ is holomorphic and thus $L_k := \zeta^* Q_k$, $k \in \mathbb{Z}$ is a holomorphic vector bundle over $\tilde{U}$. By (3.1), it follows that a pair of smooth functions $f_i : \tilde{U}_i = \tilde{X}_i \times U \to \mathbb{C}$, $i = 0, 1$ defines a global smooth section of $L_k$ if and only if
\[
f_i(z^{-1}, x) = z^{-k} f_0(z, x), \quad z \in \mathbb{C}^*, \quad x \in U
\]
holds on $\tilde{U}_0 \cap \tilde{U}_1 = (\tilde{X}_0 \cap \tilde{X}_1) \times U = \mathbb{C}^* \times U$.

As $\tilde{U}$ is an open subset of the complex manifold $\mathbb{CP}^{2n+1}$, then $T\tilde{U}_C := T\tilde{U} \otimes \mathbb{C} = T^{(1,0)} \oplus T^{(0,1)}$ where $T^{(0,1)}$ and $T^{(1,0)}$ is the $(+i)$-eigenspace and $(-i)$-eigenspace with respect to the canonical almost complex structure on $T\tilde{U}_C$, respectively. We denote by $\Lambda^{0,q}$ the vector bundle over $\tilde{U}$ whose fiber over $\ell$ consists of all skew-symmetric complex anti-linear maps $\otimes^q T\ell \tilde{U} \to \mathbb{C}$.

Even though the projection $\tau : \tilde{U} \to U$ is not holomorphic, it is a submersive surjection with fibers diffeomorphic to $\mathbb{CP}^1$. This induces a surjective vector bundle map $T\tau_C : T\tilde{U}_C \to T\tilde{U}_C$. It follows that $\ker(T\tau_C)$ is a subbundle of $T\tilde{U}_C$ of rank 1 and thus, $K := \ker(T\tau_C)^1 \cap \Lambda^{0,1}$ is a subbundle of $\Lambda^{0,1}$ of co-dimension 1 which induces a short exact sequence
\[
0 \to K \to \Lambda^{0,1} \to \Lambda^{0,1}_r \to 0
\]
and more generally,
\[
0 \to \Lambda^{q+1} \to \Lambda^{0,q+1} \to \Lambda^{0,1}_r \wedge \Lambda^q K \to 0, \quad q \geq 1.
\]
Let us now consider the bundles $K$ and $\Lambda^{0,1}_r$ over $\tilde{U}_i$, $i = 0, 1$.

Using (3.17), it is clear that the vector bundle $T^{0,1}$ is over $\tilde{U}_0$ spanned by the vector fields
\[
\partial_\zeta, \quad X^n := z \partial_{\beta^n} - \partial_{\alpha^n}, \quad X^n_i := z \partial_{\alpha^n} - \partial_{\beta^n}, \quad i = 1, \ldots, n.
\]
We denote by $d\tilde{z}, dX^n_i$, $i = 1, \ldots, 2n$ the dual co-framing by $(0,1)$-forms which trivialize $\Lambda^{0,1}$ over $\tilde{U}_0$. From (3.18), it follows that the bundle $T^{0,1}$
is over $\tilde{U}_1$ spanned by the anti-holomorphic vector fields
\[ \partial_{\bar{w}}, \, X_1^{2i-1} := \partial_{\bar{\beta}_i} - w\partial_{\alpha_i}, \, X_1^{2i} := \partial_{\alpha_i} + w\partial_{\beta_i}, \, i = 1, \ldots, n. \]
We denote by $d\bar{w}, dX_i^1, \, i = 1, \ldots, 2n$ the dual co-framing over $\tilde{U}_1$ by $(0,1)$-forms.

We see that $\text{ker}(T\tau_C)$ is over $\tilde{U}_0$ spanned by $\partial_{z}, \partial_{\bar{z}}$ and by $\partial_{\bar{w}}, \partial_{w}$ over $\tilde{U}_1$. Hence, $K$ is over $\tilde{U}_0$ spanned by $dX_0^1, \ldots, dX_0^{2n}$ and by $dX_1^1, \ldots, dX_1^{2n}$ over $\tilde{U}_1$. Notice that over $\tilde{U}_0 \cap \tilde{U}_1$:
\[ (4.5) \quad X_i^1 = z^{-1} X_0^1 \text{ and } dX_i^1 = z dX_0^1, \quad i = 1, \ldots, 2n \]
which implies

**Lemma 4.1.** $K$ is a holomorphic vector bundle isomorphic to $\bigoplus_{1}^{2n} L_1$.

It also follows that the complex line bundle $\Lambda_{\tau}^{0,1}$ is over $\tilde{U}_0$ spanned by $d\bar{z} + K$ and by $d\bar{w} + K$ over $\tilde{U}_1$. As there is no risk of confusion, we will for brevity write $d\bar{z}$ and $d\bar{w}$ instead of $d\bar{z} + K$ and $d\bar{w} + K$, respectively.

4.2. Dolbeault complex. Let us for brevity put $L := L_{-3}$. We will use the following conventions:
\[ \Lambda^{(0,q)}(L) := \Lambda^{0,q}(L) \otimes K \otimes L, \quad \Lambda_{\tau}^{(0,q)}(L) := \Lambda^q K \otimes L, \]
\[ \Lambda_{\tau}^{(0,q)}(L) := \Lambda^{q-1} K \wedge \Lambda^{(0,1)} \otimes L, \quad \mathcal{E}_{\ast}^{(0,q)}(\tilde{U}, L) := \Gamma(\tilde{U}, \Lambda_{\tau}^{(0,q)} \otimes L) \]
where $\ast \in \{ \ , K, \tau \}$. The filtration \((4.3)\) turns the Dolbeault complex into a filtered complex:

\[ (4.6) \]

\[ \begin{array}{ccc}
\mathcal{E}_{K}^{(0,1)}(\tilde{U}, L) & \xrightarrow{\partial} & \mathcal{E}_{K}^{(0,2)}(\tilde{U}, L) \\
\downarrow \nu & & \downarrow \nu \\
\mathcal{E}^{(0,0)}(\tilde{U}, L) & \xrightarrow{\partial} & \mathcal{E}^{(0,1)}(\tilde{U}, L) \\
\downarrow \pi_{\tau} & & \downarrow \pi_{\tau} \\
\mathcal{E}^{(0,1)}(\tilde{U}, L) & \xrightarrow{\partial} & \mathcal{E}^{(0,2)}(\tilde{U}, L) \\
\downarrow 0 & & \downarrow 0 \\
0 & & 0 \\
\end{array} \]

We put:
\[ \partial^0_0 : \mathcal{E}^{(0,0)}(\tilde{U}, L) \xrightarrow{\partial} \mathcal{E}^{(0,1)}(\tilde{U}, L) \xrightarrow{\pi_{\tau}} \mathcal{E}^{(0,1)}(\tilde{U}, L) \]
and
\[ \partial^1_0 : \mathcal{E}_{K}^{(0,1)}(\tilde{U}, L) \xrightarrow{\nu} \mathcal{E}^{(0,1)}(\tilde{U}, L) \xrightarrow{\partial} \mathcal{E}^{(0,2)}(\tilde{U}, L) \xrightarrow{\pi_{\tau}} \mathcal{E}^{(0,1)}(\tilde{U}, L). \]
so that
\[ \partial^0_0(f_0, f_1) = (\partial z f_0, d\bar{z}, \partial_{\bar{w}} f_1, d\bar{w}) \]
Proof. are short exact. one can find functions Lemma 5.1. and so one can use the proof of Lemma 3.2.

Moreover, if $\tau$ is zero, it follows that also $\ker(\bar{\tau}(5.3)) \rightarrow E(5.5) \rightarrow 0$ and (5.6) below, it remains to show that $\ker(\bar{\tau}(5.4)) \rightarrow E \rightarrow 0$.

Let us now assume that $\Sigma \subseteq U$. As (5.1) depends smoothly on $\Sigma$, it induces

$$\tau_* : \mathcal{E}_r^{(0,1)}(\bar{U}, L_k) \rightarrow \mathcal{E}_r^{(0,1)}(\mathbb{C}P^1, Q_k) \rightarrow H^1(\mathbb{C}P^1, Q_k) \rightarrow C^{-k-1},$$

where the first map is the pullback associated to $\iota_{\Sigma}$, the second map is the canonical projection and the last map is the isomorphism from Lemma 5.2. As in Section 3.1, we may assume that $-k - 1 > 0$ so that (5.1) is non-zero.

Let $\omega$ be as in (5.2). Using Cauchy’s integral formula and partition of unity underlying the open cover $\{U_i : i = 0, 1\}$ of $\mathbb{C}P^1$, it is easy to construct functions $g_i : \bar{U}_i \cong \bar{X}_i \times U \rightarrow \mathbb{C}, \ i = 0, 1$ such that $\partial_\bar{z}g_0 = h_0$, $\partial_{\bar{w}}g_1 = h_1$. Moreover, if $\psi_A = 0$, $A = 0, 1$, then arguing as in the proof of Lemma 3.2, one can find functions $t_i : \bar{U}_i \rightarrow \mathbb{C}$ such that $\partial_\bar{z}t_0 = 0$ and $\partial_{\bar{w}}t_1 = 0$ and

$$\tilde{\partial}_i = \tilde{\partial}_i^{(0,1)} = \partial_\bar{w} + \sum_{i=1}^{2n} (f_i^0 \partial_\bar{z} + f_i^1 \partial_{\bar{w}}) \partial_\bar{z} \partial_{\bar{w}}.$$

We will for brevity write $\tilde{\partial}_i := \tilde{\partial}_i^{(0,1)}$, $i = 0, 1, \ldots$

5. Integral Formula for the n-Cauchy-Fueter Operator

Let $U$ be an open subset of $\mathbb{H}^n \subset \mathbb{H}P^n$. Assume that $\Sigma \in \mathcal{H}(U) \subset M_{2n \times 2}$ is by Theorem 3.3 equivalent to $\Sigma \subset \bar{U}$. Recall (3.12) that $\Sigma$ is equal to the image of the embedding $\iota_{\Sigma} : \mathbb{C}P^1 \hookrightarrow \bar{U}$. It is straightforward to verify that $\iota_{\Sigma}^* L_k \cong Q_k$ and $\iota_{\Sigma}^* \Lambda_r^{(0,1)}(L_k) \cong \Lambda^{(0,1)}(Q_k)$. Hence, there is a well defined composition of maps

$$\mathcal{E}_r^{(0,1)}(\bar{U}, L_k) \rightarrow \mathcal{E}_r^{(0,1)}(\mathbb{C}P^1, Q_k) \rightarrow H^1(\mathbb{C}P^1, Q_k) \rightarrow C^{-k-1},$$

where $\tau_\Sigma$ is the pullback associated to $\iota_{\Sigma}$, the second map is the canonical projection and the last map is the isomorphism from Lemma 5.2. As in Section 3.1, we may assume that $-k - 1 > 0$ so that (5.1) is non-zero.

Let $\omega$ be as in (5.2). Using Cauchy’s integral formula and partition of unity underlying the open cover $\{X_i : i = 0, 1\}$ of $\mathbb{C}P^1$, it is easy to construct functions $g_i : \bar{U}_i \cong X_i \times U \rightarrow \mathbb{C}, \ i = 0, 1$ such that $\partial_\bar{z}g_0 = h_0$, $\partial_{\bar{w}}g_1 = h_1$. Moreover, if $\psi_A = 0$, $A = 0, 1$, then arguing as in the proof of Lemma 3.2, one can find functions $t_i : \bar{U}_i \rightarrow \mathbb{C}$ such that $\partial_\bar{z}t_0 = 0$ and $\partial_{\bar{w}}t_1 = 0$ and

$$\tilde{\partial}_i = \tilde{\partial}_i^{(0,1)} = \partial_\bar{w} + \sum_{i=1}^{2n} (f_i^0 \partial_\bar{z} + f_i^1 \partial_{\bar{w}}) \partial_\bar{z} \partial_{\bar{w}}.$$

We will for brevity write $\tilde{\partial}_i := \tilde{\partial}_i^{(0,1)}$, $i = 0, 1, \ldots$
that \( f_i := g_i - t_i, \ i = 0, 1 \) satisfy (4.11). Then \( f = (f_0, f_1) \in \mathcal{E}^{(0,0)}(\tilde{U}, L) \) and \( \tilde{\partial}_f \omega = \omega. \)

Using (3.7), it is clear that

\[
\mathcal{E}^{(0,1)}(U, \mathbb{C}^2) \to \mathcal{E}^{(0,1)}(\tilde{U}, L),
\]

is a splitting of the map \( \tau_* \circ \pi_\tau \). It is then straightforward to verify that there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}^{(0,1)}(U, \mathbb{C}^2) & \xrightarrow{\partial} & \mathcal{E}^{(0,2)}(\tilde{U}, L) \\
\downarrow \tau_* \circ \pi_\tau & & \downarrow \tau_* \circ \pi_\tau \\
\mathcal{E}(U, \mathbb{C}^2) & \xrightarrow{D} & \mathcal{E}(U, \mathbb{C}^{2n^*})
\end{array}
\]

where \( D \) is the \( n \)-Cauchy-Fueter operator.

**Theorem 5.2.** Consider

\[
\tau_* \circ \pi_\tau : \mathcal{E}^{(0,1)}(\tilde{U}, L) \to \mathcal{E}^{(0,1)}(\tilde{U}, L) \to \mathcal{C}^\infty(U, \mathbb{C}^2).
\]

If \( \Omega \in \mathcal{E}^{(0,1)}(\tilde{U}, L) \) is closed, then \( \tau_* \circ \pi_\tau(\Omega) \in \mathcal{R}(U) \) and the composition (5.8) induces isomorphism

\[
\mathcal{P} : H^1(\tilde{U}, L) \to \mathcal{R}(U).
\]

**Proof.** The first claim is an easy consequence of the commutativity of (5.7). By (5.9), it follows that \( \tau_* \circ \pi_\tau(\Omega) = 0 \) provided that \( \Omega \) is exact. We see that (5.8) is injective and it remains to show surjectivity. So assume that \( (\psi_A)_{A=1}^3 \) is monogenic in \( U \) and put \( \Omega := (\psi_A)_{A=1}^3 \) for brevity. Then by the commutativity of (5.7) again, \( \tau_* \circ \pi_\tau \circ \partial(\Omega) = 0 \) and by the exactness of (5.5), it follows that there is \( \theta \in \mathcal{E}^{(0,1)}(\tilde{U}, L) \) such that \( \tilde{\partial}(\theta) = \pi_\tau \circ \partial(\Omega). \)

Hence, \( \tilde{\theta}(\Omega - \theta) \in \mathcal{E}_{\mathcal{K}}^{(0,2)}(\tilde{U}, L) \). But arguing as in the proof Lemma 5.1 it is easy to see that \( \mathcal{E}_{\mathcal{K}}^{(0,2)}(\tilde{U}, L) \hookrightarrow \mathcal{E}^{(0,2)}(\tilde{U}, L) \xrightarrow{\partial} \mathcal{E}^{(0,3)}(\tilde{U}, L) \) is injective. Hence, \( \tilde{\theta}(\Omega - \theta) = 0 \) which completes the proof.

Now we are ready to proof the main result of this article.

**Proof of Theorem 2.1.** As the only assumption on \( \Sigma \) in (5.1) is that \( \Sigma \subset \tilde{U} \), it follows that (5.1) induces a map

\[
\tau_*^C : \mathcal{E}^{(0,1)}(\tilde{U}, L) \to \mathcal{C}^\infty(\mathcal{H}(U), \mathbb{C}^2).
\]

Explicitly, if \( \omega \) is as in (5.2), then the value of \( \tau_*^C(\omega) \) at the point \( \Sigma = (z_A)_{A=0,1,\ldots,2n-1} \) is \( (\psi_A^{\mathcal{C}}(\Sigma), \psi_{1\prime}^{\mathcal{C}}(\Sigma)) \) where

\[
\psi_{A\prime}^{\mathcal{C}}(\Sigma) := \frac{1}{2\pi i} \int_{\Sigma} z_A h_0(z, z_0, z_{2n-1}, \ldots, z_{2n-1,1}) dz \wedge d\bar{z}.
\]

Arguing as in the proof of Theorem 5.2 it follows that \( \tau_*^C \circ \pi_\tau(\Omega) = 0 \) whenever \( \Omega \) is exact. On the other hand, if \( \Omega \) is closed, then it is easy to see that \( \tau_*^C \circ \pi_\tau(\Omega) \) is holomorphic and thus, \( \tau_*^C \) induces a map

\[
\mathcal{P}^C : H^1(\tilde{U}, L) \to \mathcal{O}(\mathcal{H}(U), \mathbb{C}^2).
\]
Differentiating under the integral sign in (5.11), we see that $D^C r^C_\tau \circ \pi r ([\Omega]) = 0$. Hence, the map (5.12) takes values in $R^C (H(U))$ and we obtain a commutative diagram

$$H^1(\tilde{U}, L) \xrightarrow{PC} R^C (H(U)) \xrightarrow{P} R(U)$$

where the vertical arrow is the restriction map. Since the diagonal arrow is an isomorphism, it follows that the restriction map is surjective. As it is obviously injective, it is an isomorphism and thus, Theorem 2.1 follow. □

Notice that we have also shown

**Lemma 5.3.** The map (5.12) induces isomorphism

$$H^1(\tilde{U}, L_k) \to R^C (H(U)).$$

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