Effective resistance of toroidal graphs; some sharper results and applications

Wilbert Samuel Rossi       Paolo Frasca       Fabio Fagnani *

May 11, 2014

Abstract

The average effective resistance of a graph can be computed as a function of its eigenvalues. Using this formula and focusing on toroidal $d$-dimensional graphs, we study the role of the network topology and specifically of the graph dimension in determining the resistance. Considering sequences of graphs of increasing size, we study the asymptotical behavior of the effective resistance, proving that it is (asymptotically) inversely proportional to the dimension. These findings are relevant in many applications, including distributed estimation and control of sensor networks and clock networks.

1 Introduction

Effective resistance in a graph is a classical fundamental concept which naturally comes up when the graph is interpreted as an electrical circuit of resistors. Since several decades it is known to play a key role in the theory of time-reversible Markov chains, because of its connection with escape probabilities and commute times [10, 9, 1, 16]. More generally, effective resistance has broad application in science, for instance in chemistry [15]. In chemistry, the total effective resistance (summed over all pair of nodes) is known as the Kirchhoff index of the graph, where the graph of interest has the atoms as nodes and their bonds as edges. This classical index is linked to the properties of organic

*W. S. Rossi and F. Fagnani are with Dipartimento di Scienze Matematiche, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy; wilbertsamuel.rossi@polito.it; fabio.fagnani@polito.it. P. Frasca is with Department of Applied Mathematics, University of Twente, 7500 AE Enschede, The Netherlands; p.frasca@utwente.nl. The research leading to this paper was performed while P. Frasca was with Politecnico di Torino. The authors wish to thank F. Garin for fruitful conversations on the topic of this paper. Their work has been partly supported by the Italian Ministry MIUR under grant PRIN-20087W5P2K.
macromolecules [6] and to the vibrational energy of the atoms: the latter property has also been interpreted as a measure of vulnerability of a complex network [11].

Recently, effective resistance has appeared as an important performance index in several network-oriented problems of control and estimation, where the nodes (or agents) collectively need to obtain estimates of given quantities with limited communication effort. One instance is the consensus problem, where there is a set of agents, each with a scalar value, and the goal is to drive all agents to reach a common state, which is a weighted average of the initial values. This problem can be solved by a simple linear iterative algorithm, which has become very popular. The performance of this algorithm depends on the graph representing the allowed communication between the agents: more specifically, the average effective resistance of the graph plays two important roles in the analysis. Indeed, the effective resistance determines both the convergence speed during the transient [19, Section 3.4] [13] and the robustness to additive noise affecting the updates [22]: a more detailed discussion is available in [8, 12]. Another issue in network systems is the problem of estimating the values of quantities at the vertices of a network from noisy measurements of differences performed along the available edges. This problem was first introduced in [14] in the context of clock synchronization, and then studied in much detail in [3, 4, 5, 20]: we will come back to it in the next section. Finally, the average effective resistance is closely related to the properties of coherence in large-scale networks of dynamical systems subject to stochastic disturbances [2], a topic that is motivated by problems of vehicle platooning. In all these cases, performance improves by reducing the effective resistance: this observation motivates the minimisation problem in [19].

Average effective resistance can be exactly computed in very few examples (trees, circulant graphs) while estimates can be obtained through the variational Thompson’s principle and the Nash-Williams inequality. By these tools it has been shown that in a grid graph of dimension $d$ and size $N$ (the cardinality of the set of vertices), the average effective resistance $R_{\text{ave}}$ scales in $N \to +\infty$ (and fixed $d$) as follows

$$R_{\text{ave}} = \begin{cases} 
\Theta(N) & d = 1 \\
\Theta(\ln N) & d = 2 \\
\Theta(1) & d \geq 3
\end{cases}$$

To the best of our knowledge, no estimation of constants is available in the literature (except for the case $d = 1$). Particularly significant is the lack of this information when $d \geq 3$ since it is not clear, in particular, what is the behavior of $R_{\text{ave}}$ as a function of $d$ and for $d \to +\infty$.

\textsuperscript{1}Given two sequences $f, g : \mathbb{N} \to \mathbb{R}^+$, let $\ell^+ = \lim \sup_n f(n)/g(n)$ and $\ell^- = \lim \inf_n f(n)/g(n)$. We write that $f = O(g)$ when $\ell^+ < +\infty$; that $f = o(g)$ when $\ell^+ = 0$; that $f \sim g$ when $\ell^+ = \ell^- = 1$, and $f = \Theta(g)$ when $\ell^+, \ell^- \in (0, +\infty)$. Finally, we write $f = \Omega(g)$ when $g = O(f)$. 

2
In this paper we concentrate on $d$-dimensional grids of toroidal type and we sharpen the above statements. We first express, by a classical result, the average effective resistance in terms of the eigenvalues of the Laplacian of the graph and then use the closed formula available for such eigenvalues for toroidal $d$-grids. In order to motivate our approach, we recall that grid graphs have often been chosen as exemplary $d$-dimensional graphs in the study of network systems, thanks to their nice mathematical properties: recent examples include [7, 8, 13, 2]. Remarkably, the scaling properties deduced on “regular” grid graphs can typically be extended, with due care, to “non-regular” geometric graphs: works in this direction include [3, 17, 18]. We envisage that our results, which are described in the next section, can also be suitable for a similar extension.

2 Problem statement and main results

Consider an undirected graph $G = (V, E)$ where $V$ is a finite set of vertices and $E$ is a subset of unordered pairs of distinct elements of $V$ called edges. Assume the graph to be connected. Given two distinct vertices $u, v \in V$ imagine $G$ as an electrical network (with all edges having resistance equal to 1) and with an input current 1 at node $u$ and an output current 1 at node $v$. Using Ohm’s and Kirchoff’s law a potential $W$ is then uniquely defined at every node (up to translation constants). The effective resistance between $u$ and $v$ is formally defined as $R_{\text{eff}}(u, v) := W_u - W_v$. The average effective resistance of $G$ is defined as

$$R_{\text{ave}}(G) := \frac{1}{2N^2} \sum_{u,v \in V} R_{\text{eff}}(u, v)$$

where $N = |V|$ is the size of the graph. Consider the Laplacian of $G$, $L(G) \in \mathbb{R}^{V \times V}$ defined by

$$L(G)_{uu} = |\{v \in V | \{u, v\} \in E\}|, \quad L(G)_{uv} = \begin{cases} -1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise} \end{cases} \quad u \neq v$$

It is well known that its eigenvalues can be ordered to satisfy $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ and the following relation holds true [19, Eq. (15)]

$$R_{\text{ave}}(G) = \frac{1}{N} \sum_{i \geq 2} \frac{1}{\lambda_i}$$

Consider now the cyclic group $\mathbb{Z}_M$ of integers modulo $M$ and the product group $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$. The element $e_j \in \mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$ is the vector with all 0’s except 1 in position $j$. Define $S = \{ \pm e_j | j = 1, \ldots, d \}$. We define the toroidal $d$-grid over $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$, the graph $T_{M_1, \ldots, M_d} = (\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}, E_{M_1, \ldots, M_d})$ where

$$E_{M_1, \ldots, M_d} := \{ \{(x_1, \ldots, x_d), (y_1, \ldots, y_d)\} | (x_1 - y_1, \ldots, x_d - y_d) \in S \}$$
In the special case $M_1 = \cdots = M_d$, we will use the notation $T_{M^d}$ at the place of $T_{M_1,\ldots,M_d}$. Eigenvalues of the Laplacian of $T_{M_1,\ldots,M_d}$ can be exactly computed using Fourier analysis \textsuperscript{[13]}:

$$\lambda_\mathbf{h} = \lambda_{h_1,\ldots,h_d} = 2d - 2 \sum_{i=1}^{d} \cos \frac{2\pi h_i}{M_i}, \quad \mathbf{h} = (h_1,\ldots,h_d) \in \mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$$  \hspace{1cm} (1)

so that

$$R_{\text{ave}} (T_{M_1,\ldots,M_d}) = \frac{1}{M_1 \cdots M_d} \sum_{\mathbf{h} \neq 0} \frac{1}{2d - 2} \sum_{i=1}^{d} \cos \left( \frac{2\pi h_i}{M_i} \right)$$  \hspace{1cm} (2)

### 2.1 Application: the localization problem

To better clarify the role of effective resistance in applications, below we briefly recall the position localization problem, in which such connection is particularly straightforward. Imagine the graph $G$ to denote a network of agents each possessing a certain fixed but unknown position $\bar{x}_v \in \mathbb{R}^q$ (in the sequel for simplicity we will consider $q = 1$). Any pair of agents $v,w \in V$, connected by an edge in $G$, make a cooperative measurement of their relative positions

$$b_{\{v,w\}} = \bar{x}_v - \bar{x}_w + n_{\{v,w\}}$$  \hspace{1cm} (3)

where $n_{\{v,w\}}$ are independent random variables modeling the measurement noise, identically distributed with mean 0 and variance $\sigma^2$. Consider the so-called incidence matrix of $G$ which is a matrix $B \in \mathbb{R}^{E \times V}$ such that $B_{eu} = 0$ if $u \notin e$ while $B_{\{v,w\},v} = 1$ and $B_{\{v,w\},w} = -1$ (choice of signs is arbitrary and we simply follow the convention chosen in \textsuperscript{[3]}). Relations (3) can thus be recasted into a vector form

$$b = B\bar{x} + n$$  \hspace{1cm} (4)

On the basis of the available measurements $b$, the goal is to obtain an estimation $\hat{x}$ of $\bar{x}$. A classical solution is the so called least square estimator, defined as

$$\hat{x} := \arg\min_{x \in \mathbb{R}^V} \|Bx - b\|_2^2$$  \hspace{1cm} (5)

It is immediate to check that minima are the solutions of the equation $L(G)x = B^*b$. Since $L(G)\mathbf{1} = 0$ and $L(G)$ has rank equal to $N - 1$, minima form a 1-dimensional affine space $\hat{x} = \hat{x}_0 + c\mathbf{1}$ where $c \in \mathbb{R}$. The performance of the least square estimator is intimately connected with the average effective resistance of the graph as the following equality shows

$$\frac{1}{N} \mathbb{E}\|\hat{x} - \bar{x}\|^2 = \sigma^2 R_{\text{ave}} (G)$$
(where \( \|\hat{x} - \bar{x}\| \) denotes the Euclidean distance from \( \bar{x} \) to the affine subspace \( \hat{x} = \hat{x}_0 + c1 \)).

The problem formulated above has first been posed and studied in [14] in the context of clock synchronization and then in a series of papers [3, 4, 5]. In all these papers, however, authors assume to have an anchor node \( v_0 \), which knows its position exactly.

This case can be analyzed as above defining the least square estimator \( \hat{x} \) such that \( \hat{x}_{v_0} = \bar{x}_{v_0} \). In this case one can easily show that

\[
\frac{1}{N} \mathbb{E} \|\hat{x} - \bar{x}\|^2 = \sigma^2 \frac{1}{N} \sum_{v \in V} R_{\text{eff}}(v_0, v)
\]

In the case of toroidal grids considered in this paper, notice that straightforward symmetric arguments show that

\[
R_{\text{ave}}(G) = \frac{1}{2N^2} \sum_{u,h \in V} R_{\text{eff}}(u + v_0, u + h) = \frac{1}{2N^2} \sum_{u,h \in V} R_{\text{eff}}(v_0, h) = \frac{1}{2N} \sum_{h \in V} R_{\text{eff}}(v_0, h)
\]

This shows that the average effective resistance plays in this case an analogue role in the analysis of the least square estimator with the presence of an anchor node.

2.2 Main results

As we announced, this paper focuses on the estimation of the average effective resistance of toroidal \( d \)-grids. In the case when \( d = 1 \), the effective resistance can be computed directly in a very simple way. Indeed, using the usual electric properties of series and parallel resistors [16, pages 119-120], we have that

\[
R_{\text{eff}}(v_0, v_0 + l) = \frac{l(M - l)}{M}
\]

Hence, exact calculation gives:

\[
R_{\text{ave}}(T_M) = \frac{1}{2M} \sum_{l=1}^{M-1} \frac{l(M - l)}{M} = \frac{1}{2M^2} \sum_{l=1}^{M-1} [lM - l^2]
\]

\[
= \frac{1}{2M^2} \left[ \frac{(M - 1)M}{2} - \frac{(M - 1)M(2M - 1)}{6} \right] = \frac{1}{2M^2} \frac{M^3 - M}{6}
\]

Then,

\[
R_{\text{ave}}(T_M) \sim \frac{M}{12} \quad \text{for} \quad M \to +\infty
\]

For \( d \geq 2 \), direct computation of the effective resistance is no longer possible. Using formula (2) in this paper we will prove the following asymptotic relations.
Theorem 1.

\[
R_{\text{ave}}(T_{M,M}) \sim \frac{1}{4\pi} \ln M \quad \text{for } M \to +\infty
\]  

(8)

\[
\lim_{M \to +\infty} R_{\text{ave}}(T_{M^d}) = \Theta\left(\frac{1}{d}\right) \quad \text{for } d \to +\infty
\]  

(9)

We conjecture that the second statement can be sharpened to

\[
R_{\text{ave}}(T_{M^d}) = \Theta\left(\frac{1}{d}\right) \quad \text{for } d \to +\infty, \ M \text{ fixed}
\]

but at the moment we are only capable of proving such a result in the special case \( M = 2 \), corresponding to a hypercube.

Our results and conjecture are corroborated by numerical experiments, which are summarized in Figure 1. The left plot regards low-dimensional graphs and confirms the linear/logarithmic/constant scaling with increasing \( N \), depending on the dimension. The right plot instead regards high-dimensional graphs and shows that \( R_{\text{ave}} \) increases with \( M \) and decreases with \( d \): more precisely, for large \( M \) and large \( d \), \( R_{\text{ave}} \) is roughly inversely proportional to \( d \).

2.3 Paper Structure

Section 2 states the problem of interest, reviews the literature, and presents our main results. Section 3 is devoted to a detailed analysis of the bidimensional case. Section 4
analyzes the case when dimension $d \geq 3$, and Section 5 the case of hypercubes. The Appendix contains a part of our analysis, which is based on a mean-field approximation of the graph eigenvalues.

### 3 Effective resistance of bidimensional toroidal grids

In this section we consider the family of toroidal grids in two dimension $T_{M,L} = (\mathbb{Z}_M \times \mathbb{Z}_L, E_{M,L})$ and obtain sharp results on their average effective resistance.

\[
R_{\text{ave}}(T_{M,L}) := \frac{1}{ML} \sum_{(i,j) \neq 0} \frac{1}{\lambda_{i,j}} = \frac{1}{ML} \sum_{(i,j) \neq 0} \frac{1}{4 - 2\cos(2\pi i/M) - 2\cos(2\pi j/L)}.
\]  

(10)

In what follows, we derive upper and lower bounds on $R_{\text{ave}}(T_{M,L})$, depending on $L$ and $M$, with the assumption that $L \geq M \geq 4$. Afterwards, we will fix specific relations between $L$ and $M$ and study the asymptotic behavior when the size of the graph goes to infinity.

**Proposition 2** (Torus $T_{M,L}$ – upper bound). Suppose $L \geq M \geq 4$. Then,

\[
R_{\text{ave}}(T_{M,L}) \leq \frac{1}{2\pi} \log L + \frac{1}{12} \frac{L}{M} + 1
\]

Proof. First, notice that we can write

\[
R_{\text{ave}}(T_{M,L}) = \frac{1}{L} R_{\text{ave}}(T_M) + \frac{1}{M} R_{\text{ave}}(T_L) + \tilde{R}_{\text{ave}}(T_{M,L})
\]

(11)

where

\[
\tilde{R}_{\text{ave}}(T_{M,L}) = \frac{1}{ML} \sum_{i \neq 0} \sum_{j \neq 0} \frac{1}{\lambda_{i,j}}
\]

The first two terms in (11) can easily be bounded using the explicit formula (6) as follows

\[
\frac{1}{L} R_{\text{ave}}(T_M) + \frac{1}{M} R_{\text{ave}}(T_L) \leq \frac{M}{12L} + \frac{L}{12M}
\]

(12)

We now focus on the term $\tilde{R}_{\text{ave}}(T_{M,L})$. First, notice that by symmetry it is always true that:

\[
\tilde{R}_{\text{ave}}(T_{M,L}) = \frac{1}{ML} \sum_{i=1}^{M-1} \sum_{j=1}^{L-1} \frac{1}{\lambda_{i,j}} \leq \frac{4}{ML} \sum_{i=1}^{\lfloor M/2 \rfloor} \sum_{j=1}^{\lfloor L/2 \rfloor} \frac{1}{\lambda_{i,j}}.
\]

(13)
Then, let us consider the function $f(x, y)$ defined as

$$f(x, y) = \frac{1}{4 - 2 \cos(2\pi x) - 2 \cos(2\pi y)}$$  \hspace{1cm} (14)$$

and notice that $\frac{1}{\lambda_{i,j}} = f\left(\frac{i}{M}, \frac{j}{L}\right)$. Since $f$ is decreasing in $x \in (0, 1/2]$ and $y \in (0, 1/2]$, it follows that, for each pair $i, j$ with $1 \leq i \leq [M/2]$ and $1 \leq j \leq [L/2]$,

$$\frac{1}{ML} \sum_{i=1}^{[M/2]} \sum_{j=1}^{[L/2]} \frac{1}{\lambda_{i,j}} \leq \int_{\frac{i}{M}}^{\frac{i+1}{M}} \int_{\frac{j}{L}}^{\frac{j+1}{L}} f(x, y) \, dx \, dy.$$  

If we define $D = [0, 1/2] \times [0, 1/2]$ and $D^* = D \setminus ([0, 1/M] \times [0, 1/L])$ as in Figure 2 (left), we can thus estimate

$$\hat{R}_{ave}(T_{M,L}) = \frac{4}{ML} \sum_{i=1}^{[M/2]} \sum_{j=1}^{[L/2]} \frac{1}{\lambda_{i,j}} \leq \frac{4}{ML} f\left(\frac{1}{M}, \frac{1}{L}\right) + 4 \int_{D^*} f(x, y) \, dx \, dy.$$  \hspace{1cm} (15)$$

Note that the term for $i = 1, j = 1$ has been kept aside, because of the singularity in zero. Next, instead of computing the integral in (15) in closed form, we observe that

$$f(x, y) = \frac{1}{4 - 2 \cos(2\pi x) - 2 \cos(2\pi y)}$$

$$\leq \frac{1}{(2\pi x)^2 + (2\pi y)^2 - \frac{(2\pi x)^4}{12} - \frac{(2\pi y)^4}{12}}$$

$$\leq \frac{1}{(2\pi)^2(x^2 + y^2) - \frac{(2\pi)^4}{12}(x^2 + y^2)^2} = g(\sqrt{x^2 + y^2}),$$

Figure 2: The regions $D$, $D^*$, and $C$, which are useful in the proof of Proposition 2.
where we defined the function \( g : (0, \sqrt{3} \pi) \to \mathbb{R}^+ \) as
\[
g(r) = \frac{1}{4\pi^2 r^2 \left(1 - \frac{\pi^2}{3} r^2\right)}.
\] (16)

Unfortunately, \( g \) does not provide a useful upper bound because \( g \) has a singularity in \( \sqrt{3} \pi \). We instead use the following continuous modification
\[
\tilde{g}(\rho) = \begin{cases} 
\frac{1}{4\pi^2 \rho^2 \left(1 - \frac{\pi^2}{3} \rho^2\right)} & \text{if } 0 < \rho < \frac{1}{2} \\
\frac{1}{\pi^2 \left(1 - \frac{\pi^2}{3} \rho^2\right)} & \text{if } \rho \geq \frac{1}{2}
\end{cases},
\]
which is decreasing in \( (0, \sqrt{\frac{3}{2\pi}}) \) and such that \( f(x, y) \leq \tilde{g}(\sqrt{x^2 + y^2}) \) for all \((x, y) \in D\). We now use this bound to estimate the right hand side of (15). Regarding the first term, using that \( L \geq M \geq 4 \), we obtain
\[
\frac{4}{ML} \tilde{g} \left( \sqrt{\frac{1}{L^2} + \frac{1}{M^2}} \right) \leq \frac{4}{ML} \tilde{g}(1/M) \leq \frac{2}{\pi^2} \frac{M}{L}.
\] (17)

On the other hand, defining \( C = \{(x, y) \in \mathbb{R}^2 : \frac{1}{L^2} \leq x^2 + y^2 \leq \frac{1}{4}\} \) as illustrated in Figure 2 (right), we can estimate the second term passing to polar coordinates:
\[
4 \int_{D^*} f(x, y) \, dx \, dy = 4 \int_{D^*} \tilde{g}(\rho) \rho \, d\rho \, d\theta \leq 4 \int_{C} \tilde{g}(\rho) \rho \, d\rho \, d\theta + 4 \int_{D^* \setminus C} \tilde{g}(\rho) \rho \, d\rho \, d\theta \leq 4 \int_{0}^{\frac{1}{2}} \int_{\frac{1}{L}}^{\frac{1}{M}} \frac{1}{4\pi^2 \rho^2 \left(1 - \frac{\pi^2}{3} \rho^2\right)} \rho \, d\rho \, d\theta + \left(1 - \frac{\pi}{4}\right) \tilde{g} \left( \frac{1}{2} \right) \leq \frac{2}{\pi^2} \frac{M}{L} + \frac{1}{2\pi} \int_{\frac{1}{L}}^{\frac{1}{2}} \frac{1}{\rho - \frac{\pi^2}{3} \rho^3} \, d\rho + \frac{1}{6} \leq \frac{1}{2\pi} \left[ \log \rho - \frac{1}{2} \log \left(1 - \frac{\pi^2}{3} \rho^2\right) \right]_{\frac{1}{L}}^{\frac{1}{2}} + \frac{1}{6} \leq \frac{1}{2\pi} \log L - \frac{1}{4\pi} \log \left(1 - \frac{\pi}{12}\right) + \frac{1}{6} \leq \frac{1}{2\pi} \log L + \frac{1}{5}.
\] (18)

Using bounds (17) and (18) in (15) we obtain
\[
\bar{R}_{ave}(T_{M,L}) \leq \frac{1}{2\pi} \log L + \frac{2}{\pi^2} \frac{M}{L} + \frac{1}{5},
\] (19)

9
Using now (19) and (12) in (11), we finally obtain
\[
R_{\text{ave}}(T_{M,L}) \leq \frac{1}{2\pi} \log L + \frac{L}{12M} + \left(\frac{2}{\pi^2} + \frac{1}{12}\right) \frac{M}{L} + \frac{1}{5}
\]
and the thesis follows since \(\frac{M}{L} \leq 1\).

In order to obtain a tight lower bound, we combine two estimates in the next result.

**Proposition 3** (Torus \(T_{M,L}\) – lower bound). Suppose \(L \geq M \geq 2\). Then,
\[
R_{\text{ave}}(T_{M,L}) \geq \max \left\{ \frac{1}{12} \frac{L}{M} - \frac{1}{24} \log M - \frac{1}{12} \frac{L}{M} - \frac{1}{2} \right\}.
\]

*Proof.* The first lower bound can be proved easily: it is enough to neglect in the expression of \(R_{\text{ave}}(T_{M,L})\) all terms that have \(i > 0\) or \(j > 0\). Then,
\[
R_{\text{ave}}(T_{M,L}) \geq \frac{1}{L} R_{\text{ave}}(T_M) + \frac{1}{M} R_{\text{ave}}(T_L) = \frac{1}{L} \left(\frac{M}{12} - \frac{1}{12M}\right) + \frac{1}{M} \left(\frac{L}{12} - \frac{1}{12L}\right)
\geq \frac{1}{12} \left(\frac{M}{L} + \frac{L}{M}\right) - \frac{1}{6LM} \geq \frac{1}{12} \frac{L}{M} - \frac{1}{24}
\]

In order to prove the second bound, we use an approach based on an integral as in the proof of Proposition 2. To this goal, it is convenient to have a symmetric domain. Define the index set
\[
\Gamma_+ = \mathbb{Z}_M \times \mathbb{Z}_L \setminus \{(0,0)\}
\]
\[
\Gamma^+ = \Gamma_+ \cup \{M\} \times \{1, 2, \ldots, L - 1\} \cup \{1, 2, \ldots, M - 1\} \times \{L\}
\]
It is possible to write
\[
R_{\text{ave}}(T_{M,L}) = \frac{1}{ML} \sum_{\Gamma_+} \frac{1}{\lambda_{i,j}} = \overline{R}_{\text{ave}}(T_{M,L}) - \frac{1}{L} R_{\text{ave}}(T_M) - \frac{1}{M} R_{\text{ave}}(T_L) \quad (20)
\]
where \(\overline{R}_{\text{ave}}(T_{M,L}) = \frac{1}{ML} \sum_{\Gamma^+} \frac{1}{\lambda_{i,j}}\). To estimate \(\overline{R}_{\text{ave}}(T_{M,L})\), we consider the function \(f(x, y)\) as defined in the proof of Proposition 2 and the domain \(E\), defined (Figure 3) as:
\[
E = [0, 1] \times [0, 1] \setminus \left(\left[0, \frac{1}{M}\right] \cup \left[1 - \frac{1}{M}, 1\right]\right) \times \left(\left[0, \frac{1}{L}\right] \cup \left[1 - \frac{1}{L}, 1\right]\right)
\]
and we notice that
\[
\overline{R}_{\text{ave}}(T_{M,L}) \geq \int_E f(x, y) \, dx \, dy = 4 \int_{D^*} f(x, y) \, dx \, dy \quad (21)
\]
where the equality exploits the symmetry of \( f \). Since \( f(x, y) \geq (4\pi^2)^{-1}(x^2 + y^2)^{-1} \), we obtain
\[
R_{\text{ave}}(T_{M,L}) \geq \frac{1}{\pi^2} \int \int_{D^*} \frac{1}{x^2 + y^2} \, dx \, dy \\
\geq \frac{1}{2\pi} \int_0^{\delta} \frac{1}{\rho^2} \, d\rho = \frac{1}{2\pi} \left( \log(\delta^{-1}) - \log 2 \right),
\]
with \( \delta = \sqrt{\frac{1}{M^2} + \frac{1}{L^2}} \). If we observe that \( \frac{1}{M^2} + \frac{1}{L^2} \leq \frac{2}{M^2} \), we get
\[
R_{\text{ave}}(T_{M,L}) \geq \frac{1}{2\pi} \log(M) - \frac{1}{4},
\]
(22)

Using now (22) inside (20) together with the exact calculation (6), we finally obtain
\[
R_{\text{ave}}(T_{M,L}) \geq \frac{1}{2\pi} \log(M) - \frac{L}{12M} - \frac{M}{12L} - \frac{1}{4} \geq \frac{1}{2\pi} \log(M) - \frac{L}{12M} - \frac{1}{2}.
\]

This inequality concludes the proof.

In order to discuss the consequences of Propositions 2 and 3 for sequences of 2-tori of increasing size \( N \), let us consider some examples in which \( N \) grows and there is a
simple relation between $M$ and $L$, respecting $L \geq M \geq 4$. Preliminarily, we remark that in Proposition 3 the bound $m_1$ dominates when $M$ and $L$ grow with different rates, while $m_2$ dominates when $M$ and $L$ have the same rate of growth.

Given $N$ and a constant $c \geq 1$, we consider three possible relations between $M$ and $L$.

1. $M = cn$, $L = N/c$.

$$
\frac{1}{12} \frac{N}{c^2} - \frac{1}{24} \leq R_{\text{ave}} \left( T_{c,N/c} \right) \leq \frac{1}{12} \frac{N}{c^2} + \frac{1}{2\pi} \log N + 1.
$$

In this case, $R_{\text{ave}} \left( T_{c,N/c} \right) \sim \frac{N}{12c^2}$: we may interpret this phenomenon as reminiscent of the behaviour in the one-dimensional case.

2. $c = \frac{L}{M}$, $M = \sqrt{N/c}$, $L = \sqrt{cN}$.

$$
\frac{1}{4\pi} \log N - \frac{1}{4\pi} \log c - \frac{1}{12} - \frac{1}{2} \leq R_{\text{ave}} \left( T_{\sqrt{N/c}, \sqrt{cN}} \right) \leq \frac{1}{4\pi} \log N + \frac{1}{12} c + \frac{1}{4\pi} \log c + 1.
$$

In this case, observe that

$$
R_{\text{ave}} \left( T_{\sqrt{N/c}, \sqrt{cN}} \right) \sim \frac{1}{4\pi} \log N \quad \text{as} \quad N \to +\infty.
$$

3. $M = \sqrt{N}$, $L = \sqrt{N^{c-1}}$ with $c > 2$.

$$
\frac{1}{12} \frac{N^{c-2}}{c} - \frac{1}{24} \leq R_{\text{ave}} \left( T_{\sqrt{N}, \sqrt{N^{c-1}}} \right) \leq \frac{1}{12} \frac{N^{c-2}}{c} + \frac{1}{2\pi} \frac{c}{c} \log N + 1
$$

In this case $R_{\text{ave}} \left( T_{\sqrt{N}, \sqrt{N^{c-1}}} \right) \sim N^{\frac{c-2}{c}}/12$.

The above discussion shows how the choice of taking $M$ proportional to $L$ makes $R_{\text{ave}} \left( T_{M,L} \right)$ to grow logarithmically with $N$: by virtue of the previous bounds this is lowest asymptotic average effective resistance reachable by a bidimensional array. This order of growth must be contrasted against the linear growth which characterizes one-dimensional graphs –as the cycle graphs considered in Section 3, or nearly-one-dimensional graphs as shown above.

4 Effective resistance of $d$-toroidal grids with $d \geq 3$

In this section we consider $d$-tori with $d \geq 3$. We fix the size along each of the $d$ dimensions to be equal to $M$. The total size of the graph is then $N = M^d$. We start with a simple lower bound.
**Proposition 4.** Consider a $d$-torus of dimensions $N = M^d$, with $M \geq 2$ and $d \geq 3$. It holds that:

$$R_{\text{ave}}(T_{M^d}) \geq \frac{1}{4d}$$

*Proof.* Observe that $\forall h \neq 0$, it is true that $\frac{1}{\lambda_h} \geq \frac{1}{4d}$. Moreover, notice that $\frac{1}{\lambda_{(1,0,...,0)}} = \frac{1}{2 - 2 \cos(\frac{2\pi}{M})} \geq \frac{1}{2d}$. Then,

$$R_{\text{ave}}(T_{M^d}) \geq \frac{1}{M^d} \left( (M^d - 2) \frac{1}{4d} + \frac{2}{4d} \right) = \frac{1}{4d}.$$

$\qed$

Instead, finding a tight upper bound is far from being trivial: our result, which relies on a technical estimate provided in the Appendix, is presented below.

**Proposition 5.** Consider a $d$-torus of dimensions $N = M^d$, with $M \geq 4$ and $d \geq 3$. It holds that:

$$R_{\text{ave}}(T_{M^d}) \leq \frac{8}{d + 1} \left( 1 + \frac{1}{M} \right)^{d+1} + \frac{d}{4M^{d-2}} \left( \frac{1}{3} + \frac{(d-1) \log M}{\pi} \right)$$

*Proof.* Let us consider the terms of the sum for which $h > 0$, i.e., those for which all $h_i > 0$. Define

$$\hat{R}_{\text{ave}}(T_{M^d}) = \frac{1}{M^d} \sum_{h > 0} \frac{1}{2d - 2 \sum_{i=1}^{d} \cos \left( \frac{2\pi h_i}{M} \right)}$$

(where $h > 0$ means that $h_i > 0$ for all $i$), and observe that

$$R_{\text{ave}}(T_{M^d}) = \sum_{m=1}^{d} \left( \frac{d}{m} \right) \frac{1}{M^{d-m}} \hat{R}_{\text{ave}}(T_{M^m}).$$

The crucial point is now to observe that, for any $m \geq 1$,

$$\hat{R}_{\text{ave}}(T_{M^m}) \leq \gamma(m)$$

where

$$\gamma(m) := \int_{[0,1]^m} \frac{dx}{2m - 2 \sum_{i=1}^{m} \cos(2\pi x_i)}, \quad (23)$$

since we can see $\hat{R}_{\text{ave}}(T_{M^m})$ as a lower Riemann sum of the integral. This remark allows us to use Lemma in the Appendix, so that if $m \geq 3$, then

$$\hat{R}_{\text{ave}}(T_{M^m}) \leq \frac{4}{m}.$$
The integral (23) does not converge when $m = 1$ or $m = 2$ (2-tori). In those cases we use the results from the previous sections, namely (12), and (19). We thus obtain

$$R_{\text{ave}} (T_M^d) \leq \left(\frac{d}{1}\right) \frac{1}{M^{d-1}} \frac{M}{\pi^2} + \left(\frac{d}{2}\right) \frac{1}{M^{d-2}} \left[ \frac{1}{2\pi} \log M + 1 \right] + \sum_{m=3}^{d} \left(\frac{d}{m}\right) \frac{1}{M^{d-m}} \frac{4}{m}$$

$$\leq \frac{4}{M^d} \sum_{m=1}^{d} \left(\frac{d}{m}\right) M^m \frac{1}{m} + \frac{d}{4M^{d-2}} \left[ \frac{1}{3} + \frac{(d-1) \log M}{\pi} \right].$$

After noting that

$$\frac{4}{M^d} \sum_{m=1}^{d} \left(\frac{d}{m}\right) M^m \frac{1}{m} \leq \frac{4}{M^d} \sum_{m=1}^{d} \left(\frac{d}{m}\right) M^m \frac{2}{m+1}$$

$$\leq \frac{8}{M^{d+1}} \sum_{m=1}^{d} \left(\frac{d+1}{m+1}\right) \frac{M^{m+1}}{d+1}$$

$$\leq \frac{8}{d+1} \frac{1}{M^{d+1}} \sum_{n=0}^{d+1} \left(\frac{d+1}{n}\right) M^n$$

$$= \frac{8}{d+1} \left(1 + \frac{1}{M}\right)^{d+1},$$

the thesis immediately follows.

We now move on to apply the bounds obtained above, in order to infer the asymptotic behavior of $R_{\text{ave}} (T_M^d)$ for $N \to +\infty$ and, in particular prove Theorem 1. Notice, first of all, that if $d \geq 3$ is fixed and $M$ diverges, then Propositions 4 and 5 yield $R_{\text{ave}} (T_M^d) = \Theta(1)$ as $M \to +\infty$. This fact was actually already known in the literature.

**Proof of Theorem 1** Relation (8) follows immediately from Propositions 2 and 3. Instead, relation (9) follows from Propositions 4 and 5.

For finite $M$, instead, we can not claim that $R_{\text{ave}} (T_M^d) = \Theta(1/d)$. Indeed, Proposition 5 leaves the possibility for $R_{\text{ave}} (T_M^d)$ to diverge when $M$ is fixed and $d$ diverges. However, sharper analytical results can be found for the special case of $M = 2$: this study will be performed in the next section.

## 5 Effective resistance of hypercubes

We have noted in the previous sections that the “dimension” of a graph seems to play a significant role in the corresponding error estimate. Hypercube graphs, studied in
this section, are a good way to analyze the role of the graph dimension in more depth. Hypercubes of dimension \( d \), denoted by \( H^d \), are toroidal graphs with respect to the group \( \mathbb{Z}_2^d \); in fact \( H_d = T_{2^d} \). Note that \( N = 2^d \) and the degree of each vertex is \( d \). The eigenvalues of an hypercube are \( \lambda_m = 2m \) for \( m \in \{0,\ldots,d\} \) and the eigenvalue \( \lambda_m \) has multiplicity \( \binom{d}{m} \). The expression of the average effective resistance thus becomes

\[
R_{\text{ave}}(H_d) = \frac{1}{2^d} \sum_{m=1}^{d} \frac{1}{2m} \binom{d}{m},
\]

and by exploiting the properties of the binomial coefficient we obtain the following result.

**Proposition 6.** For a \( d \)-dimensional hypercube, with \( d \geq 2 \), the following estimates hold:

\[
\frac{1}{2} \frac{1}{d+1} \leq R_{\text{ave}}(H_d) \leq \frac{2}{d+1}.
\]

**Proof.** For the lower bound, we have

\[
R_{\text{ave}}(H_d) \geq \frac{1}{2d+1} \sum_{m=1}^{d} \frac{1}{2m+1} \binom{d}{m} = \frac{1}{2d+1} \sum_{m=1}^{d} \frac{1}{2} \binom{d+1}{m+1}
\]

By the change of variables \( m' = m + 1 \) and \( d' = d + 1 \), it is possible to compute

\[
\sum_{m=1}^{d} \binom{d+1}{m+1} = 2^{d+1} - d - 2
\]

and conclude that

\[
R_{\text{ave}}(H_d) \geq \left(1 - \frac{d+2}{2^{d+1}}\right) \frac{1}{d+1}.
\]

(24)

For the upper bound, we instead have

\[
R_{\text{ave}}(H_d) \leq \frac{1}{2d+1} \sum_{m=1}^{d} \frac{2}{2m+1} \binom{d}{m} = \frac{1}{2d+1} \sum_{m=1}^{d} \frac{2}{d+1} \binom{d+1}{m+1} \leq \frac{2}{d+1}.
\]

\[\square\]
The bounds in Proposition 6 imply that \( R_{\text{ave}}(H_d) = \Theta \left( \frac{1}{d} \right) \), as mentioned in the Introduction. A more precise asymptotical analysis is given in the next result.

**Proposition 7 (Asymptotics of \( R_{\text{ave}}(H_d) \)).** It holds true that

\[
R_{\text{ave}}(H_d) \sim \frac{1}{d} \quad \text{as } d \to \infty.
\]

**Proof.** From the definition of \( R_{\text{ave}}(H_d) \) and using Pascal’s rule we can compute

\[
R_{\text{ave}}(H_d) = \frac{1}{2} R_{\text{ave}}(H_{d-1}) + \frac{1}{2d+1} \frac{d}{k} \sum_{k=1}^{d} \left( \frac{d}{k} \right)
\]

We have thus shown that the sequence \( R_{\text{ave}}(H_d) \) can be constructed recursively by the above formula and defining \( R_{\text{ave}}(H_0) = 0 \). This recursion implies that the \( d \)-th term can be written as

\[
R_{\text{ave}}(H_d) = \sum_{i=1}^{d} \frac{1}{2^{d-i}} \frac{1}{2i} \left( 1 - \frac{1}{2i} \right)
\]

Consequently,

\[
R_{\text{ave}}(H_d) \leq \frac{1}{2^{d+1}} \sum_{i=1}^{d} \frac{2^i}{i}
\]

and we claim that

\[
\lim_{d \to +\infty} \frac{1}{2^{d+1}} \sum_{i=1}^{d} \frac{2^i}{i} = 1.
\]

This fact can be shown true as follows. Let \( a_d = \frac{d}{2^{d+1}} \sum_{i=1}^{d} \frac{2^i}{i} \). Then, it is immediate to verify that \( a_d \) satisfies the following recursion

\[
\begin{align*}
 a_0 &= 0 \\
 a_{d+1} &= \frac{1}{2} \left( 1 + \frac{1}{d} \right) a_d + \frac{1}{2} \quad \text{for } d \geq 0
\end{align*}
\]

and –by induction– that if \( d \geq 3 \), then \( a_d > 1 \), and if \( d \geq 5 \), then \( a_{d+1} < a_d \). Then, \( a_d \) must have a finite limit \( \ell \geq 1 \). Also, note that

\[
a_{d+1} = \frac{1}{2} \left( 1 + \frac{1}{d} \right) a_d + \frac{1}{2} \leq \frac{1}{2} a_d + \frac{4}{3} \frac{1}{d} + \frac{1}{2}.
\]

16
By taking the limit on both sides of the inequality, we obtain that $\ell \leq 1$. Finally, the result follows by combining Equations (24) and (25).

References

[1] D. Aldous and J. Fill. *Reversible Markov chains and random walks on graphs*. Available on-line [http://www.stat.berkeley.edu/~aldous/index.html](http://www.stat.berkeley.edu/~aldous/index.html).

[2] B. Bamieh, M. R. Jovanovic, P. Mitra, and S. Patterson. Coherence in large-scale networks: dimension-dependent limitations of local feedback. *IEEE Transactions on Automatic Control*, 57(9):2235–2249, 2012.

[3] P. Barooah and J. P. Hespanha. Estimation from relative measurements: Algorithms and scaling laws. *IEEE Control Systems Magazine*, 27(4):57–74, 2007.

[4] P. Barooah and J. P. Hespanha. Estimation from relative measurements: Electrical analogy and large graphs. *IEEE Transactions on Signal Processing*, 56(6):2181–2193, 2008.

[5] P. Barooah and J. P. Hespanha. Error scaling laws for linear optimal estimation from relative measurements. *IEEE Transactions on Information Theory*, 55(12):5661–5673, 2009.

[6] D. Bonchev, E. J. Markel, and A. H. Dekmezian. Long chain branch polymer chain dimensions: application of topology to the Zimm-Stockmayer model. *Polymer* 43(1):203–222, 2002.

[7] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri. Communication constraints in the average consensus problem. *Automatica*, 44(3):671–684, 2008.

[8] R. Carli, F. Garin, and S. Zampieri. Quadratic indices for the analysis of consensus algorithms. In *Information Theory and Applications Workshop*, San Diego, CA, 2009.

[9] A. K. Chandra, P. Raghavan, W. L. Ruzzo, and R. Smolensky. The electrical resistance of a graph captures its commute and cover times. In *ACM symposium on Theory of computing*, 1989, pages 574-586.

[10] P. G. Doyle and J. L. Snell. *Random Walks and Electric Networks*. Carus Monographs. Mathematical Association of America, 1984.

[11] E. Estrada and N. Hatano. A vibrational approach to node centrality and vulnerability in complex networks. *Physica A* 389:3648–3660, 2010.
[12] F. Garin and L. Schenato. Distributed estimation and control applications using linear consensus algorithms. In Networked control systems, A. Bemporad, M. Heemels, and M. Johansson (eds). Springer, 2011.

[13] F. Garin and S. Zampieri. Mean square performance of consensus-based distributed estimation over regular geometric graphs. SIAM Journal on Control and Optimization, 50(1):306–333, 2012.

[14] A. Giridhar and P. R. Kumar. Distributed clock synchronization over wireless networks: algorithms and analysis. In IEEE Conference on Decision and Control, pages 4915–4920, 2006.

[15] D. J. Klein and M. Randic. Resistance distance. Journal of Mathematical Chemistry, 12(1):81–95, 1993.

[16] D. A. Levin, Y. Peres, and E. L. Wilmer. Markov chains and mixing times. American Mathematical Society, 2008.

[17] E. Lovisari, F. Garin, and S. Zampieri. Resistance-based performance analysis of the consensus algorithm over geometric graphs. SIAM Journal on Control and Optimization, 2013, accepted.

[18] E. Lovisari and S. Zampieri. Performance metrics in the average consensus problem: a tutorial. Annual Reviews in Control, 36(1):26–41, 2012.

[19] A. Ghosh, S. Boyd, and A. Saberi. Minimizing effective resistance of a graph. SIAM Review, 50(1):37–66, 2008.

[20] W. S. Rossi, P. Frasca, and F. Fagnani. Transient and limit performance of distributed relative localization. In IEEE Conference on Decision and Control. Maui, HI, USA, 2012, pages 2744–2748.

[21] A. S. Shabani. Notes on the upper and lower bounds of two inequalities for the gamma function. Hacettepe Journal of Mathematics and Statistics, 39(1):11–15, 2010.

[22] L. Xiao, S. Boyd, and S.-J. Kim. Distributed average consensus with least-mean-square deviation. Journal of Parallel and Distributed Computing, 67(1):33–46, 2007.
A Mean-field approximation

In this appendix we study the “continuous” approximation of $R_{\text{ave}} (T_{M^d})$ defined in (23) as

$$\gamma (d) = \int_{[0,1]^d} \frac{1}{2d - 2 \sum_{i=1}^d \cos(2\pi x_i)} \, dx,$$

which is used to prove Proposition 5.

Lemma 8. If $d \geq 3$, then

$$\frac{1}{4d} \leq \gamma (d) \leq \frac{4}{d}.$$

Proof. The lower bound follows from Proposition 4 while this proof is devoted to prove the upper bound. Observe that by symmetry

$$\gamma (d) = 2^d \int_{[0, \frac{1}{2}]^d} \frac{1}{2d - 2 \sum_{i=1}^d \cos(2\pi x_i)} \, dx,$$  \hfill (26)

and define the following three subsets of $[0, \frac{1}{2}]^d$,

$$A = \{ x \in [0, \frac{1}{2}]^d \text{ s.t. } \|x\|_2 \leq \frac{1}{\pi} \}$$

$$B = \{ x \in [0, \frac{1}{2}]^d \text{ s.t. } \|x\|_2 \geq \frac{1}{\pi} \text{ and } x_i \leq \frac{1}{\pi} \forall i \}$$

$$C = \{ x \in [0, \frac{1}{2}]^d \text{ s.t. } \exists x_i \geq \frac{1}{\pi} \}$$

such that $A \cup B \cup C = [0, \frac{1}{2}]^d$. Correspondingly, we define

$$I_d^A = 2^d \int_A \frac{1}{2d - 2 \sum_{i=1}^d \cos(2\pi x_i)} \, dx$$

$$I_d^B = 2^d \int_B \frac{1}{2d - 2 \sum_{i=1}^d \cos(2\pi x_i)} \, dx$$

$$I_d^C = 2^d \int_C \frac{1}{2d - 2 \sum_{i=1}^d \cos(2\pi x_i)} \, dx,$$

so that $\gamma (d) = I_d^A + I_d^B + I_d^C$.  

19
We begin by studying $\mathcal{I}_d^A$. First, we work on the denominator of the integrand, using the inequality $1 - \cos x \geq \frac{x^2}{2} - \frac{x^4}{24}$ to show

$$2 \sum_{i=1}^{d} (1 - \cos(2\pi x_i)) \geq 4\pi^2 \sum_{i=1}^{d} x_i^2 - \frac{16\pi^4}{12} \sum_{i=1}^{d} x_i^4$$

$$\geq 4\pi^2 \left( \sum_{i=1}^{d} x_i^2 - \frac{\pi^2}{3} \sum_{i=1}^{d} \sum_{j=1}^{d} x_i^2 x_j^2 \right)$$

$$= 4\pi^2 \left( 1 - \frac{\pi^2}{3} \sum_{i=1}^{d} x_i^2 \right) \sum_{i=1}^{d} x_i^2.$$ 

If now we use the last expression and we move to polar coordinates, we obtain

$$\mathcal{I}_d^A \leq 2^d \int_A \frac{1}{4\pi^2 \left( \sum_{i=1}^{d} x_i^2 \right) \left( 1 - \frac{\pi^2}{3} \sum_{i=1}^{d} x_i^2 \right)} \, dx$$

$$= \int_0^{\frac{1}{\pi}} \frac{2\pi^d}{\Gamma \left( \frac{d}{2} \right)} \rho^{d-1} \frac{1}{4\pi^2 \rho^2 \left( 1 - \frac{\pi^2}{3} \rho^2 \right)} \, d\rho$$

$$= \frac{\pi^{\frac{d}{2}}-2}{\Gamma \left( \frac{d}{2} \right)} \int_0^{\frac{1}{\pi}} \frac{1}{1 - \frac{\pi^2}{3} \rho^2} \, d\rho.$$ 

Note that the change of variables involving the Gamma function has eliminated the singularity in zero, and that the new integrand is an increasing function. Then,

$$\mathcal{I}_d^A \leq \frac{\pi^{\frac{d}{2}}-2}{2\Gamma \left( \frac{d}{2} \right)} \int_0^{\frac{1}{\pi}} \left( \frac{1}{\pi} \right)^{d-3} \left( 1 - \frac{\pi^2}{3} \left( \frac{1}{\pi} \right)^2 \right) \, d\rho = \frac{3}{4\pi^2 \Gamma \left( \frac{d}{2} \right)}.$$ 

Since $x^{(1-\gamma)x-1} < \Gamma(x)$ if $x > 1$ (see [21]), where $\gamma \simeq 0.577$ is the Euler-Mascheroni constant, we have

$$\mathcal{I}_d^A \leq \frac{3d}{8\pi^2 \left( \frac{1}{\pi} \right)^{d-1}}.$$ 

(27)

Next, we estimate $\mathcal{I}_d^B$. To this purpose, recall definition (14) and notice that the function

$$f(x) := \frac{1}{2d - 2\sum_{i=1}^{d} \cos(2\pi x_i)}$$
is decreasing in every direction $i$, when $x \in [0, \frac{1}{2}]$. Then, defining $g(\rho)$ as in [16], we have

$$I^B_d \leq 2^d \mu(B) g \left( \frac{1}{\pi} \right) \leq \frac{3}{8} \left( \frac{2}{\pi} \right)^d,$$  \hspace{1cm} (28)

where $\mu(B)$ denotes the measure of $B$, and $B \subset [0, \frac{1}{2}]^d$.

Next, we consider $I^C_d$. Let $\Omega = \{0, 1\}^d$ and for all $\omega \in \Omega$, define the set $C_\omega \subset C$ as $C_\omega = \{x \in C \text{ s.t. } x_i \geq \frac{1}{\pi} \text{ iff } \omega_i = 1\}$. Clearly, $\bigcup_{\omega \neq 0} C_\omega = C$. Then,

$$I^C_d = \sum_{\omega \neq 0} \int_{C_\omega} \frac{1}{2d - 2 \sum_{i=1}^d \cos(2\pi x_i)} \, dx.$$

If for a fixed $\omega \in \Omega$ we denote by $l_\omega$ the number of 1’s in $\omega$ (that is, the so-called Hamming weight of $\omega$), we can remark that

$$\mu(C_\omega) = \left( \frac{1}{\pi} \right)^{d-l_\omega} \left( \frac{1}{2} - \frac{1}{\pi} \right)^{l_\omega}.$$

Moreover, the function $f(x)$ is symmetric under permutations of the components of $x$. Then,

$$f(x) \leq f \left( \frac{1}{\pi} \omega \right) = \frac{1}{2(1 - \cos(2))} \frac{1}{l_\omega} \text{ if } x \in C_\omega.$$

Since clearly there are $\binom{d}{l}$ elements in $\Omega$ with Hamming weight $l$, we can argue that

$$I^C_d \leq 2^d \sum_{l=1}^d \binom{d}{l} \frac{1}{2l(1 - \cos(2))} \left( \frac{1}{\pi} \right)^{d-l} \left( \frac{1}{2} - \frac{1}{\pi} \right)^l$$

\begin{align*}
&= \frac{1}{2(1 - \cos(2))} \sum_{l=1}^d \binom{d}{l} \left( \frac{2}{\pi} \right)^{d-l} \left( 1 - \frac{2}{\pi} \right)^l \frac{1}{l} \\
&\leq \frac{1}{(1 - \cos(2))(1 - \frac{2}{\pi})} \frac{1}{d+1},
\end{align*}

where the last inequality follows from standard manipulations on the binomials. Finally, this bound can be replaced by a simpler

$$I^C_d \leq \frac{3}{d},$$  \hspace{1cm} (29)

and we are able to conclude the proof by combining (27), (28), and (29) to observe that

$$\gamma(d) = I^A_d + I^B_d + I^C_d \leq \frac{4}{d}.$$

\hspace{1cm} \Box