Huge tables and multicommodity flows are fixed-parameter tractable via unimodular integer Carathéodory

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Abstract

The three-way table problem is to decide if there exists an $l \times m \times n$ table satisfying given line sums, and find a table if yes. Recently, it was shown to be fixed-parameter tractable with parameters $l, m$. Here we extend this and show that the huge version of the problem, where the variable side $n$ is encoded in binary, is also fixed-parameter tractable with parameters $l, m$. We also conclude that the huge multicommodity flow problem with a huge number of consumers is fixed-parameter tractable. One of our tools is a theorem about unimodular monoids which is of interest on its own right.

Keywords: integer programming, integer Carathéodory, multiway table, bin packing, cutting stock, fixed-parameter tractable, totally unimodular, multicommodity flow.

1 Introduction

The study of multiway table problems, also known as multi-index transportation problems, goes back to the classical paper of Motzkin [15]. It also has applications in privacy in databases and confidential data disclosure of statistical tables, see the survey [8] by Fienberg and Rinaldo and the references therein. Specifically, the three-way table problem is to decide if there exists a nonnegative integer $l \times m \times n$ table satisfying given line sums, and find a table if there is one. Deciding the existence of such a table is NP-complete already for $l = 3$, see [4]. Moreover, every bounded integer program can be isomorphically represented in polynomial time for some $m$ and $n$ as some $3 \times m \times n$ table problem, see [5]. When both $l$ and $m$ are fixed, the problem can be solved in polynomial time using Graver bases and the theory of $n$-fold integer programming [3]. See the book [16] for further background.

Recently, the problem was shown in [12] to be fixed-parameter tractable when $l$ and $m$ are parameters, solvable in time $O(f(l,m) \cdot n^3 \cdot \text{size(line sums)})$ where

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size(line sums) is the binary-encoding length of all the given line sums and $f(l, m)$ is a suitable computable function. (While we do not need a bound on $f(l, m)$ for our results, it is worthwhile to note that it is known to satisfy $f(l, m) = (lm)^O(lm)$, see [2, 13, 16] for more details on this important so-called Graver complexity function.)

Recall that a parameterized problem with parameter $p$ and input $I$ is called fixed-parameter tractable if it admits an algorithm that runs in time $O(f(p) \cdot \text{size}(I)^k)$ for some computable function $f(p)$ of $p$ which is independent of $I$ and some $k$ which is independent of $p$ and $I$. In particular, if a problem is fixed-parameter tractable, then for each fixed value $p$ of the parameter, it is polynomial-time solvable, but fixed-parameter tractability is much stronger since the degree $k$ of the polynomial running time is independent of the parameter value $p$. See the book [6] by Downey and Fellows for more details on this important branch of complexity theory.

More recently, in [17], the huge version of the problem, where the variable table side $n$ is a huge number encoded in binary, and the $n$ many $l \times m$ layers of the table come in $t$ types, was also shown to be polynomial-time solvable for fixed $l$ and $m$. Here we strengthen this and show that the huge problem is moreover fixed-parameter tractable as well. (All layers of each given type have the same specified row and column sums, see Section 3 for a more detailed description of the problem.)

**Theorem 3.1** The huge $l \times m \times n$ table problem with $t$ types, parameter $l$, and $n$ variable and binary-encoded, is fixed-parameter tractable in the following situations:

1. when $m$ is also a parameter and $t$ is variable and unary-encoded;
2. when $t$ is also a parameter and $m$ is variable and unary-encoded.

One application of this theorem is to multicommodity flows. We show that the problem with a variable number $m$ of suppliers and a huge variable number $n$ encoded in binary of consumers, that come in $t$ types, is fixed-parameter tractable. (All consumers of each given type have the same consumption in each commodity and the same capacity from each supplier, see Section 3 for more details.)

**Corollary 3.2** The huge multicommodity flow problem parameterized by the number $l$ of commodities and the number $t$ of consumer types is fixed-parameter tractable.

One of the tools we use is a theorem about totally unimodular monoids which we discuss next. Let $S \subseteq \mathbb{Z}^d$ be a set of integer points. The monoid generated by $S$ is the set of nonnegative integer combinations of finitely many elements of $S$,

$$\text{mon}(S) := \left\{ \sum_{k=1}^{m} \lambda_k x^k : m, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}_+ , x^1, \ldots, x^m \in S \right\} .$$

The monoid decomposition problem, also called the integer Carathéodory problem, is the following: given a set $S \subseteq \mathbb{Z}^d$ and a vector $a \in \mathbb{Z}^d$, decide if $a \in \text{mon}(S)$, and if yes, find a decomposition $a = \sum \lambda_k x^k$ with all $\lambda_k \in \mathbb{Z}_+$ and all $x^k \in S$. 
Of course, the complexity of the problem depends on the presentation of $S$. When $S$ is given explicitly as a set of vectors, this is simply integer programming, so even the decision problem is already NP-complete. Here we consider the much more difficult situation with $S$ given implicitly as the set $S = \{ x \in \mathbb{Z}^d : Ax \leq b \}$ of integer points satisfying a given system of inequalities, where $A$ is a $c \times d$ integer matrix and $b \in \mathbb{Z}^c$. Under such a presentation, even for fixed dimension $d$, the number of points in $S$ can be infinite or exponential in the encoding length of $A$ and $b$, so it is unclear how to even write down in polynomial time an expression $a = \sum \lambda_k x^k$, let alone find one. In spite of this, Eisenbrand and Shmonin showed in [7] that if $a \in \text{mon}(S)$ then there is an expression $a = \sum_{k=1}^{m} \lambda_k x^k$ with $m \leq 2^d$. Recently, Goemans and Rothvoß showed in [10], using heavy machinery, that for fixed $d$, the problem is polynomial-time solvable, with degree which is exponential in $d$.

Here we show that when $A$ is totally unimodular, which holds in the context of the three-way table problem, the problem can be solved in polynomial time even when $d$ is variable. (Note that $A$ being totally unimodular does not imply that so is the matrix with columns in $S$, just take $d = 1$ and $S = \{ x \in \mathbb{Z} : 2 \leq x \leq 2 \} = \{2\}$.) In fact, we prove in Section 2 a more general result on the monoid problem for sets $S = P \cap \mathbb{Z}^d$ for polyhedra $P$ in an oracle setup, and deduce the following corollary.

**Corollary 2.6** The monoid decomposition problem for any $S = \{ x \in \mathbb{Z}^d : Ax \leq b \}$ and any $a$, with $A$ totally unimodular and $b$ integer, is solvable in time polynomial in the binary-encoding length of $A, b,$ and $a$, even when the dimension $d$ is variable.

We proceed as follows. In Section 2 we prove Corollary 2.6. In Section 3 we discuss multiway tables and multicommodity flows and use Corollary 2.6 to prove Theorem 3.1 and Corollary 3.2. We conclude in Section 4 with some open problems.

## 2 Unimodular integer Carathéodory

As mentioned in the introduction, we solve here the monoid problem for a broad class of sets of the form $S = P \cap \mathbb{Z}^d$ where $P$ are polyhedra presented by suitable oracles. Corollary 2.6 will then follow as a special case. Throughout, all polyhedra $P \subset \mathbb{R}^d$ and all vectors $x \in \mathbb{R}^d$ are rational, and we will not indicate this further for brevity. We will use the algorithmic theory of polyhedra developed in [11]. The description complexity of a polyhedron $P$ is the smallest positive integer $\Delta$ such that $P$ admits a description $P = \{ x \in \mathbb{R}^d : Ax \leq b \}$ with $A$, $b$ integer and $|A_{i,j}|, |b_i| \leq \Delta$ for all $i,j$. (We do not need to know this description explicitly, and the number of inequalities may be exponential.) A separation oracle for a polyhedron $P \subset \mathbb{R}^d$ is one that, queried on $x \in \mathbb{R}^d$, either asserts that $x \in P$ or returns an $h \in \mathbb{R}^d$ such that $hy < hx$ for all $y \in P$. In all algorithmic statements on polyhedra $P \subset \mathbb{R}^d$ involving oracles, an algorithm is said to run in polynomial time if its running time including queries to the oracles involved is polynomial in $d$, the binary-encoding $\log \Delta$ of the description complexity of $P$, and other relevant inputs. See [11] for more details.
We begin with two simple lemmas.

**Lemma 2.1** Given polyhedron $P \subset \mathbb{R}^d$ presented by separation oracle and $a \in \mathbb{R}^d$, we can in polynomial time either find $n \in \mathbb{N}$ with $\frac{1}{n}a \in P$ or asserts none exists.

*Proof.* Using the separation oracle of $P$ it is possible to efficiently realize a separation oracle for the intersection $Q := P \cap \{ \lambda a : 0 \leq \lambda \}$ of $P$ and the ray generated by $a$. Minimizing and maximizing the linear function $ax$ over $Q$ using the algorithmic equivalence of separation and optimization from [11] we conclude with one of the following: $Q = \emptyset$ so there is no $n$; $Q = \{ \lambda a : \alpha \leq \lambda \}$ and then if $\alpha > 1$ then there is no $n$ whereas if $\alpha \leq 1$ then we can take $n = 1$; $Q = \{ \lambda a : \alpha \leq \lambda \leq \beta \}$ and then if there is $n \in \mathbb{N}$ with $\frac{1}{\beta} \leq n \leq \frac{1}{\alpha}$ then we can take it and otherwise there is no $n$. \qed

A polyhedron $P \subset \mathbb{R}^d$ is decomposable if for every $n \in \mathbb{N}$ and every $x \in nP \cap \mathbb{Z}^d$ there are $x^1, \ldots, x^n \in P \cap \mathbb{Z}^d$ with $x = x^1 + \cdots + x^n$, where $nP := \{ ny : y \in P \}$.

Note that for any $S \subseteq \mathbb{Z}^d$, even $S = \emptyset$, we have that $a = 0$ is trivially in $\text{mon}(S)$ with the empty decomposition. The next lemma deals with the case of $a \neq 0$.

**Lemma 2.2** Let $P \subset \mathbb{R}^d$ be a polyhedron and let $S := P \cap \mathbb{Z}^d$. Let $a \in \mathbb{Z}^d$ be a nonzero vector. If $a \in \text{mon}(S)$ then there is an $n \in \mathbb{N}$ such that $a \in nP$. If $P$ is moreover decomposable and there is an $n \in \mathbb{N}$ such that $a \in nP$ then $a \in \text{mon}(S)$.

*Proof.* If $a \in \text{mon}(S)$ then $a = \sum \lambda_k x^k$ with $\lambda_k \in \mathbb{N}$ and $x^k \in S \subseteq P$. Let $n := \sum \lambda_k \geq 1$. Then $x := \sum \frac{\lambda_k}{n} x^k$ is a convex combination of points in $P$ and therefore $x \in P$, so $a = nx \in nP$. If $P$ is decomposable and $a \in nP$ for some $n \in \mathbb{N}$ then there are $x^1, \ldots, x^n \in P \cap \mathbb{Z}^d = S$ with $a = x^1 + \cdots + x^n$, so $a \in \text{mon}(S)$. \qed

A decomposition oracle for a decomposable $P$ is one that, queried on $n \in \mathbb{N}$ given in unary, and on $x \in nP \cap \mathbb{Z}^d$, returns $x^1, \ldots, x^n \in nP \cap \mathbb{Z}^d$ with $x = x^1 + \cdots + x^n$.

We proceed to establish the efficient solution of the monoid problem over polyhedra defined by oracles. For simplicity we provide the statement and proof for pointed polyhedra. (A polyhedron is pointed if it has at least one vertex, which is equivalent to admitting an inequality description with a matrix of full column rank.) The polyhedra appearing in typical applications are indeed pointed. Moreover, in the specializations of the oracle result to concrete polyhedra in Corollaries 2.6 and 2.7 in the sequel, we solve the monoid problem even for non pointed polyhedra.

We will need the following result of [11] (see Corollary 6.5.13 therein).

**Proposition 2.3** Given a pointed polyhedron $P \subset \mathbb{R}^d$ presented by a separation oracle and a point $x \in P$, we can in polynomial time obtain vertices $x^0, \ldots, x^k$ of $P$ for some $0 \leq k \leq d$, a point $y$ (possibly zero) in the recession cone of $P$, and positive rational numbers $\lambda_0, \ldots, \lambda_k$ satisfying $\sum_{i=0}^{k} \lambda_i = 1$ and $x = y + \sum_{i=0}^{k} \lambda_i x^i$.

We can now establish our oracle result.
Theorem 2.4 The monoid decomposition problem over any set which is of the form $S := P \cap \mathbb{Z}^d$ with $P \subset \mathbb{R}^d$ any decomposable pointed polyhedron presented by a separation oracle and endowed with a decomposition oracle is polynomial-time solvable.

Proof. Given any nonzero $a \in \mathbb{Z}^d$, we need to decide if $a \in \text{mon}(S)$ and find a decomposition if yes. We apply Lemma 2.1. If there is no $n \in \mathbb{N}$ with $a \in nP$ then $a \notin \text{mon}(S)$ by Lemma 2.1. So assume we find $n \in \mathbb{N}$ with $a \in nP$. Then $a \in \text{mon}(S)$ by Lemma 2.1 again. We need to find a monoid decomposition of $a$.

But we cannot simply query the decomposition oracle on $n$ in unary and on $a$ to get the decomposition: the crucial difficulty is that the $n$ we got may be very large, and only the binary-encoding length of $n$, not $n$ itself, is guaranteed to be polynomial in the binary-encoding length of $a$ and the description complexity of $P$.

So instead we proceed as follows. We use Proposition 2.3 and obtain vertices $x^0, \ldots, x^k$ of $P$ with $0 \leq k \leq d$, point $y$ (possibly zero) in the recession cone of $P$, and positive rational numbers $\lambda_0, \ldots, \lambda_k$ with $\sum_{i=0}^k \lambda_i = 1$ and $\frac{1}{n}a = y + \sum_{i=0}^k \lambda_i x^i$.

Now, we claim that since $P$ is decomposable, its vertices are integer. Indeed, consider any vertex $v$ of $P$. Since $P$ is rational so is $v$ and so for some $q \in \mathbb{N}$ we have $qv \in qP \cap \mathbb{Z}^d$. Then $qv = z^1 + \cdots + z^q$ for some $z^i \in P \cap \mathbb{Z}^d$ so $v = \frac{1}{q}(z^1 + \cdots + z^q)$ which implies $v = z^1 = \cdots = z^q$ since $v$ is a vertex, and therefore $v \in \mathbb{Z}^d$.

So the vertices $x^0, \ldots, x^k$ that we obtained are in $P \cap \mathbb{Z}^d = S$. Now define

\[
\bar{n} := \sum_{i=0}^k (n\lambda_i - \lfloor n\lambda_i \rfloor) = n - \sum_{i=0}^k \lfloor n\lambda_i \rfloor, \\
\bar{a} := ny + \sum_{i=0}^k (n\lambda_i - \lfloor n\lambda_i \rfloor)x^i = a - \sum_{i=0}^k \lfloor n\lambda_i \rfloor x^i.
\]

Suppose first that $\bar{n} = 0$. Then $n\lambda_i$ is an integer for $i = 0, \ldots, k$ and therefore $ny = a - \sum_{i=0}^k n\lambda_i x^i$ is an integer vector in the recession cone of $P$. Therefore $x^0 + ny \in P \cap \mathbb{Z}^d = S$. Now $n\lambda_0 \geq 1$ and therefore we obtain the decomposition

\[
a = (n\lambda_0 - 1)x^0 + (x^0 + ny) + \sum_{i=1}^k n\lambda_i x^i.
\]

Next suppose $\bar{n} \neq 0$. Then $\bar{n}$ is an integer satisfying $1 \leq \bar{n} \leq d$. Moreover, we have

\[
\frac{1}{\bar{n}}\bar{a} = \frac{n}{\bar{n}}y + \sum_{i=0}^k \frac{n\lambda_i - \lfloor n\lambda_i \rfloor}{\bar{n}}x^i.
\]

So $\frac{1}{\bar{n}}\bar{a}$ is the sum of a convex combination of vertices of $P$ and a vector in the recession cone of $P$, and hence is in $P$. Therefore $\bar{a} \in \bar{n}P \cap \mathbb{Z}^d$. We now query the decomposition oracle of $P$ on $\bar{n}$ and $\bar{a}$ and obtain $\bar{a} = \sum_{i=1}^k z^i$ for suitable $z^i \in P \cap \mathbb{Z}^d = S$. This gives again a decomposition of $a$, and completes the proof,

\[
a = \sum_{i=0}^k \lfloor n\lambda_i \rfloor x^i + \sum_{i=1}^k z^i. \quad \square
\]
We next consider polyhedra defined by totally unimodular matrices. We need an algorithmic version of the decomposition theorem of Baum and Trotter \[1\].

**Lemma 2.5** For any totally unimodular matrix \( A \) and any integer vector \( b \), the polyhedron \( P := \{ x \in \mathbb{R}^d : Ax \leq b \} \) is decomposable. Moreover, there is a polynomial time algorithm that, given such \( A \) and \( b \), realizes a decomposition oracle for \( P \).

**Proof.** We show by induction on \( n \) that given \( n \in \mathbb{N} \) and \( a \in nP \cap \mathbb{Z}^d \) we can find in polynomial time \( x^1, \ldots, x^n \in P \cap \mathbb{Z}^d \) with \( a = x^1 + \cdots + x^n \). For \( n = 1 \) simply take \( x^1 = a \). Next consider \( n > 1 \) and consider the following system in variable vector \( x \),

\[
Ax \leq b, \quad A(a - x) \leq (n - 1)b .
\]

Then \( x := \frac{1}{n} a \) is a real solution of this system, since \( a \in nP \) implies \( A\frac{n}{n}a \leq b \) and \( A(a - \frac{n}{n}a) = (n - 1)A\frac{n}{n}a \leq (n - 1)b \). Now, the defining matrix of the system (2) consists of one block of \( A \) and one block of \(-A\) and hence is totally unimodular since \( A \) is, and the right hand side of this system is integer. So the system also admits an integer solution \( x^n \in \mathbb{Z}^d \) which can be found in polynomial time by linear programming. Then \( Ax^n \leq b \) and hence \( x^n \in P \cap \mathbb{Z}^d \). Moreover, \( A(a - x^n) \leq (n - 1)b \) so \( a - x^n \in (n - 1)P \cap \mathbb{Z}^d \) and hence, by induction, we can find a decomposition \( a - x^n = \sum_{i=1}^{n-1} x^i \) with all \( x^i \in P \cap \mathbb{Z}^d \). This yields the decomposition \( a = \sum_{i=1}^{n} x^i \). \( \square \)

We can now conclude the following corollary mentioned in the introduction.

**Corollary 2.6** The monoid decomposition problem for any \( S = \{ x \in \mathbb{Z}^d : Ax \leq b \} \) and any \( a \), with \( A \) totally unimodular and \( b \) integer, is solvable in time polynomial in the binary-encoding length of \( A, b, \) and \( a \), even when the dimension \( d \) is variable.

**Proof.** Let \( A, b, \) and \( a \neq 0 \) be given input to the problem, so \( P := \{ x \in \mathbb{R}^d : Ax \leq b \} \), \( S := P \cap \mathbb{Z}^d \), and we need to decide if \( a \in \text{mon}(S) \) and find a decomposition if yes. Since \( P \) is not necessarily pointed, we proceed as follows. We let \( T := Q \cap \mathbb{Z}^d \) with

\[
Q := \{ x \in \mathbb{R}^d : Ax \leq b, \ 0 \leq \text{sign}(a_i)x_i \leq |a_i|, \ i = 1, \ldots, d \} ,
\]

where the sign of \( r \in \mathbb{R} \) is \( \text{sign}(r) := 1 \) if \( r \geq 0 \) and \( \text{sign}(r) := -1 \) if \( r < 0 \). Clearly \( Q \) has a separation oracle and description complexity polynomial in the input. Moreover, the system defining \( Q \) is totally unimodular and hence \( Q \) is decomposable with a decomposition oracle by Lemma 2.5. We apply Lemma 2.1 to \( P \). If there is no \( n \in \mathbb{N} \) with \( a \in nP \) then \( a \notin \text{mon}(S) \) by Lemma 2.2. So assume we find \( n \in \mathbb{N} \) with \( a \in nP \). Then also \( a \in nQ \cap \mathbb{Z}^d \) and hence \( a \in \text{mon}(T) \) by Lemma 2.2. Now \( Q \) is a polytope hence pointed. So we can apply Theorem 2.1 to \( T = Q \cap \mathbb{Z}^d \) and obtain a decomposition \( a = \sum \lambda_k x^k \) with \( \lambda_k \in \mathbb{Z} \) and \( x^k \in T \subseteq S \) as desired. \( \square \)

We conclude this section with an extension of Corollary 2.6 to the following broader class of monoids. A *totally unimodular projection* is a polyhedron of the
form $P = \{ x = Ly \in \mathbb{R}^d : y \in Q \}$ which is the linear projection of a polyhedron $Q = \{ y \in \mathbb{R}^c : Ay \leq b \}$, with $b$ an integer vector and $[A^T \, L^T]$ totally unimodular. The special case $L = [I_d \, 0_{d \times (c-d)}]$ with $I_d$ the identity gives variable-eliminating projections and the case $L = I_d$ and $c = d$ gives the polyhedra in Corollary 2.6.

We have the following extension of Corollary 2.6 to such polyhedra.

**Corollary 2.7** The monoid decomposition problem over any totally unimodular projection $P$ can be solved in polynomial time even when the dimension $d$ is variable.

**Proof.** Let $L, A, b,$ and $a \neq 0$ be given input and let $S := P \cap \mathbb{Z}^d$. We need to decide if $a \in \text{mon}(S)$ and find a decomposition if yes. Note that the data gives separation oracles for both $P$ and $Q$ with description complexities polynomial in the input.

We apply Lemma 2.1 to $P$. If there is no $n \in \mathbb{N}$ with $a \in nP$ then $a \notin \text{mon}(S)$ by Lemma 2.2. So assume we find $n \in \mathbb{N}$ with $a \in nP$. Consider the system

$$Ay \leq nb, \quad Ly = a.$$  

Since $\frac{1}{n}a \in P$ there is a $y \in \mathbb{R}^c$ with $Ay \leq b$ and $\frac{1}{n}a = Ly$. Then $ny$ satisfies the system. Since $[A^T \, L^T]$ is totally unimodular, we can find an integer solution $z$ to the system. So $z \in nQ \cap \mathbb{Z}^c$. Let $T := Q \cap \mathbb{Z}^c$. By Lemma 2.2 we have that $z \in \text{mon}(T)$. Since $A$ is totally unimodular we can use Corollary 2.6 and find in polynomial time a decomposition $z = \sum \lambda_k z^k$ for some $\lambda_k \in \mathbb{Z}_+$ and $z^k \in Q \cap \mathbb{Z}^c = T$. Let $x^k := Lz^k$ for all $k$. Then $x^k \in P \cap \mathbb{Z}^d = S$ since $L$ is integer, and we obtain the decomposition

$$a = Lz = L \sum \lambda_k z^k = \sum \lambda_k Lz^k = \sum \lambda_k x^k. \quad \Box$$

### 3 Huge tables are fixed-parameter tractable

As noted in the introduction, the three-way table problem is to decide if the following set of nonnegative integer $l \times m \times n$ tables is nonempty, and find a table if one exists,

$$\left\{ x \in \mathbb{Z}_+^{l \times m \times n} : \sum_i x_{i,j,k} = u_{j,k}, \quad \sum_j x_{i,j,k} = v_{i,k}, \quad \sum_k x_{i,j,k} = w_{i,j} \right\},$$

with the line sums binary-encoded integers $u_{j,k}, v_{i,k}, w_{i,j}$ for $1 \leq i \leq l, 1 \leq j \leq m,$ and $1 \leq k \leq n$, of binary-encoding length $\text{size}(u, v, w)$. This problem was shown in [12] to be fixed-parameter tractable when $l$ and $m$ are parameters, solvable in time $O(f(l, m) \cdot n^3 \cdot \text{size}(u, v, w))$ for suitable computable function $f(l, m) = (lm)^{O(lm)}$.

Regard now each table as a tuple $x = (x^1, \ldots, x^n)$ consisting of $n$ many $l \times m$ layers. Following [17], call the problem huge if the variable number $n$ of layers is encoded in binary. We are then given $t$ types of layers, where each type $k$ has its column sums vector $u^k \in \mathbb{Z}_+^m$ and row sums vector $v^k \in \mathbb{Z}_+^l$. In addition, we are given positive integers $n_1, \ldots, n_t, n$ with $n_1 + \cdots + n_t = n$, all encoded in binary. A feasible table $x = (x^1, \ldots, x^n)$ then must have first $n_1$ layers of type 1, next $n_2$ layers of type 2, and so on, with last $n_t$ layers of type $t$. The special case of $t = 1$ type is the symmetric case, where all layers have the same row and column sums.
Theorem 3.1 The huge $l \times m \times n$ table problem with $t$ types, parameter $l$, and $n$ variable and binary-encoded, is fixed-parameter tractable in the following situations:

1. when $m$ is also a parameter and $t$ is variable and unary-encoded;

2. when $t$ is also a parameter and $m$ is variable and unary-encoded.

Proof. We first formulate the symmetric case as a monoid decomposition problem.

Let

$$S := \left\{ z \in \mathbb{Z}_{+}^{lm} \cong \mathbb{Z}_{+}^{lm} : \sum_{i} z_{i,j} = u_{j}, \sum_{j} z_{i,j} = v_{i} \right\} = \{ z \in \mathbb{Z}_{+}^{d} : Az = b \}$$

with $d = lm$, suitable $b$, and $A$ the $(l+m) \times lm$ vertex-edge incidence matrix of the complete bipartite graph $K_{l,m}$ which is well known to be totally unimodular.

Note that even when $l$ and $m$ are fixed, the number of elements of $S$ is typically exponential in the binary-encoding length $\text{size}(u,v)$ of the row and column sums.

Now, if $x = (x^{1}, \ldots, x^{n})$ is a feasible symmetric table then $x^{k} \in S$ for all $k$ and $\sum_{k=1}^{n} x^{k} = w$ is the vertical line sum vector so $Aw = \sum_{k=1}^{n} Ax^{k} = nb$. So assume $Aw = nb$, which is easy to check, else there is no feasible table and we are done. Assume also $b \neq 0$ else the unique feasible table is zero and we are done again. Then $w \in \text{mon}(S)$ by Lemmas 2.2 and 2.3 so $w = \sum \lambda_{k} z^{k}$ for some $\lambda_{k} \in \mathbb{Z}_{+}$ and $z^{k} \in S$.

We then have

$$\sum \lambda_{k} b = \sum \lambda_{k} Az^{k} = A \sum \lambda_{k} z^{k} = Aw = nb,$$

so $\sum \lambda_{k} = n$. So there is a feasible table $x = (x^{1}, \ldots, x^{n})$ with $\lambda_{k}$ layers equal to $z^{k}$ for all $k$. By Corollary 2.6 we can solve this monoid decomposition problem in time polynomial in $l$ and $m$ and $\text{size}(u,v,w)$ and find the $\lambda_{k} \in \mathbb{Z}_{+}$ and $z^{k} \in S$ which provide a compact representation of a feasible huge symmetric three-way table $x$.

We proceed to the general case of huge tables with $t$ types. The solution has two steps. First, a regular (non huge) compressed problem over $l \times m \times t$ tables is derived from the huge problem data, with the same vertical sums $w$, and column sums $n_{k}u^{k}$ and row sums $n_{k}v^{k}$ for $k = 1, \ldots, t$. We solve the compressed problem by the Graver bases methods of [12], either in time $O(f(l,m) \cdot t^{3} \cdot \text{size}(n_{k}u^{k}, n_{k}v^{k}, w))$, or in time $O(f(l,t) \cdot m^{3} \cdot \text{size}(n_{k}u^{k}, n_{k}v^{k}, w))$, according to which of $m$ and $t$ is chosen to be the parameter and which is chosen to be variable. If the original problem has a feasible table $x = (x^{1}, \ldots, x^{n})$ then, taking $y^{1}$ to be the sum of the first $n_{1}$ layers of $x$, taking $y^{2}$ to be the sum of the next $n_{2}$ layers of $x$, and so on, with lastly taking $y^{t}$ to be the sum of the last $n_{t}$ layers of $x$, we obtain a table $y = (y^{1}, \ldots, y^{t})$ which is feasible in the compressed problem. So assume we found a table $y$ which is feasible in the compressed problem, else the original problem is infeasible and we are done.

Second, for $k = 1, \ldots, t$ we consider the huge symmetric table problem which asks for a huge $l \times m \times n_{k}$ table with vertical sums given by $y^{k}$ and with column
sums $u^k$ and row sums $v^k$. Each of these $t$ huge symmetric table problems is now formulated as a monoid problem as just explained above with the same matrix $A$ and a suitable $b^k$ defined from $u^k$ and $v^k$. Since $y^k$ has column sums $n_k u^k$ and row sums $n_k v^k$, we have $A y^k = n_k b^k$ and so, as explained above, this symmetric problem is feasible and by Corollary 2.6 we can find in polynomial time a compact representation of a feasible $l \times m \times n_k$ table. The concatenation of these compact representations provides a compact representation of an $l \times m \times (n_1 + \cdots + n_t) = l \times m \times n$ table which provides the desired solution of the original huge table problem.

This theorem also has a consequence to the following huge multicommodity flow problem over the complete bipartite graph. There are $l$ commodities, $m$ suppliers, and nonnegative integer numbers $s^k_i$ of units that supplier $i$ is to supply of commodity $k$. There are $t$ consumer types, where, for $r = 1, \ldots, t$, we have $n_r$ consumers of type $r$, nonnegative integer numbers $c^k_r$ of units that each consumer of type $r$ is to consume of commodity $k$, and nonnegative integer capacities $u_{i,r}$ of allowed flow of all commodities from supplier $i$ to each consumer of type $r$. Adding one slack commodity, we may assume that the capacities should be attained with equality. The problem is to find a (compact representation of a) feasible flow $x^k_{i,j}$ from each supplier $i$ to each consumer $j$ of each commodity $k$. The numbers $n^k_r$ of consumers of each type are encoded in binary, so there is a huge number $n_1 + \cdots + n_t$ of consumers. It is then not hard to see that this can be directly encoded as a suitable huge three-way table problem. Theorem 3.1 then implies the following statement.

**Corollary 3.2** The huge multicommodity flow problem with $l$ commodities, $t$ consumer types, unary-encoded number $m$ of suppliers, binary-encoded supplies, consumptions and capacities $s^k_i$, $c^k_r$ and $u_{i,r}$, and binary-encoded numbers $n^k_r$ of consumers of type $r = 1, \ldots, t$, parameterized by $l$ and $t$, is fixed-parameter tractable.

We conclude this section with an extension of Theorem 3.1 to a class of huge $n$-fold integer programming problems defined as follows. The $n$-fold product of a $c \times d$ integer matrix $A$ is the following $(d + cn) \times (dn)$ matrix, with $I_d$ the identity,

$$A^{[n]} := \begin{pmatrix} I_d & I_d & \cdots & I_d \\ A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}.$$  

The $n$-fold integer programming feasibility problem is to decide if the following set,

$$\left\{ x \in \mathbb{Z}^{dn} : A^{[n]} x = b, \ l \leq x \leq u \right\},$$

is nonempty, and find a feasible point if there is one, where $b \in \mathbb{Z}^{d+cn}$ and $l, u \in \mathbb{Z}^{dn}$. See [16] for more details and for the many applications of this class of problems.
In [12] it was shown to be fixed-parameter tractable parameterized by \( d \) and \( a \in \mathbb{N} \) an upper bound on \( |A_{ij}| \) for all \( i, j \), solvable in time \( O(f(a, d) \cdot n^3 \cdot \text{size}(b, l, u)) \), with \( \text{size}(b, l, u) \) the bit size of \( b, l, u \), and \( f(a, d) = (ad)^{O(d^2)} \) a computable function.

The vector ingredients of an \( n \)-fold program are naturally arranged in \emph{bricks}, with \( x = (x^1, \ldots, x^n) \) and likewise \( l \) \( u \) with \( x^i, l^i, u^i \in \mathbb{Z}^d \) for \( i = 1, \ldots, n \), and with \( b = (b^0, b^1, \ldots, b^n) \) with \( b^0 \in \mathbb{Z}^d \) and \( b^i \in \mathbb{Z}^c \) for \( i = 1, \ldots, n \). Following [17], the \( n \)-fold program is called \emph{huge} if \( n \) is encoded in binary. More precisely, we are now given \( t \) \emph{types} of bricks, where each type \( k = 1, \ldots, t \) has its lower and upper bounds \( l^k, u^k \in \mathbb{Z}^d \) and right-hand side \( b^k \in \mathbb{Z}^c \). A brick \( x^i \in \mathbb{Z}^d \) has type \( k \) if \( Ax^i = b^k \) and \( l^k \leq x^i \leq u^k \). Also given are \( b^0 \in \mathbb{Z}^d \) and \( n_1, \ldots, n_t, n \in \mathbb{N} \) with \( n_1 + \cdots + n_t = n \), all encoded in binary. A feasible point \( x = (x^1, \ldots, x^n) \) now must have first \( n_1 \) bricks of type 1, next \( n_2 \) bricks of type 2, and so on, with last \( n_t \) bricks of type \( t \), and also satisfy \( \sum_{i=1}^{n} x^i = b^0 \). When the defining matrix \( A \) is totally unimodular, which in particular implies \( a = 1 \) holds, a proof similar to that of Theorem 3.1, the details of which are omitted, where bricks replace layers, leads to the following theorem.

**Theorem 3.3** The huge \( n \)-fold integer programming problem with \( t \) types over any totally unimodular \( c \times d \) matrix \( A \), parameterized by \( d \), with \( t \) unary-encoded and with \( b^0, n, b^k, l^k, u^k \), and \( n_k \) binary-encoded for \( k = 1, \ldots, t \), is fixed-parameter tractable.

### 4 Open problems

We now raise several remaining open problems. First, the complexity of huge tables of higher dimensions is unsettled. For such tables, the layers are tables of dimension at least three, and therefore the matrix which defines the resulting monoid is no longer totally unimodular. In particular, what is the complexity of deciding the existence of a huge four-way \( k \times l \times m \times n \) table with given sums, with \( k, l, m \) fixed and \( n \) encoded in binary, with \( t \) types? It is known that for fixed \( t \) the problem is in \( \mathbb{P} \), and for variable \( t \) it is in \( \mathbb{NP} \) intersect \( \mathbb{coNP} \) but is not known to be in \( \mathbb{P} \) even for \( 3 \times 3 \times 3 \times n \) tables, see [17]. We also do not know whether the problem, with \( k, l, m \) as parameters, with variable or even fixed \( t \), is fixed-parameter tractable.

Next, we discuss bin packing. We need to pack items of \( d \) types in identical bins. Each item of type \( i \) has a positive integer volume \( v_i \) and there are \( n_i \) items of type \( i \) to be packed. Each bin has a positive integer volume \( v \). The question is, given \( n \), whether \( n \) bins suffice to pack all items. (The minimal possible number of bins needed can then be found by binary search.) We consider the huge version of the problem, usually referred to as the \emph{cutting stock problem}, where all data, including the numbers \( n_i \) of items of each type \( i \), and \( n \), are encoded in binary. The study of this huge version goes back to the classical paper [9] by Gilmore and Gomory. We formulate this problem as a monoid decomposition problem in \( \mathbb{Z}^{d+1} \) as follows. Let

\[
S := \{ z \in \mathbb{Z}_+^{d+1} : z_0 = 1, \sum_{i=1}^{d} v_i z_i \leq v \} = \{ z \in \mathbb{Z}_+^{d+1} : z_0 = 1, \ A z \leq b \}
\]
with \( A = (0, v_1, \ldots, v_d) \), \( b = v \), and variables \( z = (z_0, z_1, \ldots, z_d) \). Then \( z \in S \) if and only if \( z_0 = 1 \) and \( (z_1, \ldots, z_d) \) is an admissible packing pattern which means that it is possible to pack \( z_i \) items of type \( i \) for \( i = 1, \ldots, d \) in a single bin. Now let \( a := (n, n_1, \ldots, n_d) \). Then \( a = \sum \lambda_k z_k \) with \( \lambda_k \in \mathbb{Z}_+ \) and \( z_k \in S \) if and only if there is a packing of all items using \( \sum \lambda_k = n \) bins, where for each \( k \) there are \( \lambda_k \) bins packed in pattern \((z_k^1, \ldots, z_k^d)\). The solution of this monoid decomposition problem allows to decide if there is a packing with \( n \) bins and if there is, to find it. McCormick, Smallwood and Spieksma showed in [14] that the problem is polynomial-time solvable for \( d = 2 \) types, and asked about higher \( d \). This was resolved only very recently by Goemans and Rothvoß in [10] who showed that it can be solved in polynomial time for any fixed \( d \), as a consequence of their solution of the monoid problem for fixed \( d \). Unfortunately, the matrix \( A \) above is not totally unimodular and therefore Corollary 2.6 does not apply. So it remains open whether the cutting stock problem, with the number \( d \) of types as a parameter, is fixed-parameter tractable or not.

Finally, it remains an important open question whether the monoid problem for general matrices, parameterized by the dimension \( d \), is fixed-parameter tractable.

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