Multi-bump positive solutions for a logarithmic Schrödinger equation with deepening potential well

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Abstract This article concerns the existence of multi-bump positive solutions for the following logarithmic Schrödinger equation:

$$
\begin{cases}
-\Delta u + \lambda V(x)u = u \log u^2 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
$$

where $N \geq 1$, $\lambda > 0$ is a parameter and the nonnegative continuous function $V : \mathbb{R}^N \to \mathbb{R}$ has the potential well $\Omega := \text{int } V^{-1}(0)$ which possesses $k$ disjoint bounded components $\Omega = \bigcup_{j=1}^{k} \Omega_j$. Using the variational methods, we prove that if the parameter $\lambda > 0$ is large enough, then the equation has at least $2^k - 1$ multi-bump positive solutions.

Keywords variational methods, logarithmic Schrödinger equation, multi-bump solutions, deepening potential well

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1 Introduction

In this article, we are concerned with the existence of multi-bump positive solutions for the following logarithmic Schrödinger equation:

$$
\begin{cases}
-\Delta u + \lambda V(x)u = u \log u^2 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\quad (P_\lambda)
$$

where $\lambda > 0$ is a parameter, $N \geq 1$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(V1) $V(x) \geq 0$ for all $x \in \mathbb{R}^N$.

(V2) $\Omega := \text{int } V^{-1}(0)$ is a non-empty bounded open subset with smooth boundary and $\overline{\Omega} = V^{-1}(0)$, where $\text{int } V^{-1}(0)$ denotes the set of the interior points of $V^{-1}(0)$.

(V3) $\Omega$ consists of $k$ components:

$$
\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,
$$

and $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ for all $i \neq j$.

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Definition 1.1. A solution of the problem (P_λ) is a function \( u \in H^1(\mathbb{R}^N) \) such that \( u^2 \log u^2 \in L^1(\mathbb{R}^N) \) and
\[
\int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv)dx = \int_{\mathbb{R}^N} uv \log u^2 dx \quad \text{for all } v \in C_0^\infty(\mathbb{R}^N).
\]

In recent years, the logarithmic Schrödinger equation has received considerable attention. This class of equations has some important physical applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation (see [31] and the references therein). On the other hand, the logarithmic Schrödinger equation also raises many difficult mathematical problems, for example, the superfluidity and Bose-Einstein condensation (see [31] and the references therein). On the other hand, the logarithmic Schrödinger equation also raises many difficult mathematical problems, for example, the associated energy functional is not well defined in \( H^1(\mathbb{R}^N) \), because there exists \( u \in H^1(\mathbb{R}^N) \) such that \( \int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty \). Indeed, it is enough to consider a smooth function that satisfies
\[
u(x) = \begin{cases} (|x| N/2 \log(|x|))^{-1}, & |x| \geq 3, \\ 0, & |x| \leq 2. \end{cases}
\]

In order to overcome this technical difficulty, some authors have used different techniques to study the existence, multiplicity and concentration of the solutions under some assumptions on the potential \( V \), which can be seen in [2, 3, 5–7, 14–16], [21, 24–26, 28, 30] and the references therein.

One of the main motivations of this paper goes back to the results for the nonlinear Schrödinger equations with deepening potential well of the type
\[
\begin{align*}
-\Delta u + (\lambda V(x) + Z(x))u &= u^p \quad \text{in } \mathbb{R}^N, \\
u(x) &= 0 \quad \text{in } \mathbb{R}^N,
\end{align*}
\]
by supposing that the first eigenvalue of \(-\Delta + Z(x)\) on \( \Omega_j \) under the Dirichlet boundary condition is positive for each \( j \in \{1, 2, \ldots, k\} \), \( p \in (1, \frac{N+2}{N-2}) \) and \( N \geq 3 \). In [19], Ding and Tanaka showed that the problem (1.1) has at least \( 2^k - 1 \) multi-bump solutions for \( \lambda > 0 \) large enough. These solutions have the following characteristics.

For each non-empty subset \( \Gamma \subset \{1, 2, \ldots, k\} \) and \( \epsilon > 0 \) fixed, there exists \( \lambda^* > 0 \) such that the problem (1.1) possesses a solution \( u_\lambda \) for \( \lambda \geq \lambda^* = \lambda^*(\epsilon) \) satisfying
\[
\left| \int_{\Omega_j} ((\nabla u_\lambda)^2 + (\lambda V(x) + Z(x))|u_\lambda|^2)dx - \left( \frac{1}{2} - \frac{1}{p + 1} \right)^{-1} c_j \right| < \epsilon, \quad \forall j \in \Gamma
\]
and
\[
\int_{\mathbb{R}^N \setminus \Omega_\Gamma} ((\nabla u_\lambda)^2 + (\lambda V(x) + Z(x))|u_\lambda|^2)dx < \epsilon,
\]
where \( \Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j \) and \( c_j \) is the minimax level of the energy functional related to the problem
\[
\begin{align*}
-\Delta u + Z(x)u &= u^p \quad \text{in } \Omega_j, \\
u(x) &= 0 \quad \text{in } \Omega_j, \\
u(x) &= 0 \quad \text{on } \partial \Omega_j.
\end{align*}
\]

Later, for the critical growth case, Alves et al. [4] considered the existence of multi-bump solutions for the following problem:
\[
\begin{align*}
-\Delta u + (\lambda V(x) + Z(x))u &= f(u) \quad \text{in } \mathbb{R}^N, \\
u(x) &= 0 \quad \text{in } \mathbb{R}^N,
\end{align*}
\]
where \( N \geq 3 \). For the case where \( N = 2 \) and \( f \) has an exponential critical growth, Alves and Souto [10] obtained the same results. Moreover, these solutions found in [4, 10] have the same characteristics of those found in [19]. For the further research about the nonlinear Schrödinger equations with deepening potential well, we refer to [1, 8, 9, 12, 13, 20, 22] and their references.
In particular, due to our scope, we would like to mention [26], where Tanaka and Zhang studied the multi-bump solutions for the spatially periodic logarithmic Schrödinger equation

$$\begin{aligned}
-\Delta u + V(x)u &= Q(x)u \log u^2, \quad u > 0 \quad \text{in } \mathbb{R}^N, \\
u &\in H^1(\mathbb{R}^N),
\end{aligned} \tag{LS}$$

where $N \geqslant 1$ and $V(x)$ and $Q(x)$ are spatially 1-periodic functions of class $C^1$. The authors took an approach using spatially $2L$-periodic problems ($L \gg 1$) and showed the existence of infinitely many multi-bump solutions of the equation (LS) which are distinct under $\mathbb{Z}^N$-action. In the present paper, we shall establish the existence of multi-bump solutions by using a different approach from that found in [26]. We also notice that in [3], Alves et al. have studied the problem $(P_{\lambda})$ with $V$ satisfying $(V1), (V2)$ and

$$(V3') \text{ There exists } M_0 > 0 \text{ such that } |\{x \in \mathbb{R}^N; V(x) \leqslant M_0\}| < +\infty, \text{ where } |A| \text{ denotes the Lebesgue measure of a measurable set } A \subset \mathbb{R}^N.$$

On one hand, by using the condition $(V3')$, we see that it is easy to overcome the difficulty of lack of compactness in the whole space $\mathbb{R}^N$. On the other hand, Alves et al. [3] cannot obtain the multi-bump solutions. Recently, Alves and Ji [5] used the variational methods to prove the existence and concentration of positive solutions for a logarithmic Schrödinger equation under a local assumption on the potential $V$. In that paper, in order to prove the Palais-Smale (PS) condition, we modified the nonlinearity in a special way to work on an auxiliary problem. By making some new estimates, we proved that the solutions obtained for the auxiliary problem are solutions of the original problem when $\epsilon > 0$ is sufficiently small. Moreover, since the functional associated with the auxiliary problem lost some other good properties, we developed a new method to prove the boundedness of the (PS) sequence. Inspired by [1,5,19], the main purpose of the present paper is to investigate the existence and multiplicity of multi-bump positive solutions, as in [19], for the problem $(P_{\lambda})$ by adapting the penalization method found in [18].

The main result to be proved is the theorem below.

**Theorem 1.2.** Suppose that $V$ satisfies $(V1)$–$(V3)$. Then for any non-empty subset $\Gamma$ of $\{1, 2, \ldots, k\}$, there exists $\lambda^* > 0$ such that for all $\lambda \geqslant \lambda^*$, the problem $(P_{\lambda})$ has a positive solution $u_{\lambda}$. Moreover, the family $\{u_{\lambda}\}_{\lambda \geqslant \lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence $\lambda_{n_i}$ such that $u_{\lambda_{n_i}}$ converges strongly in $H^1(\mathbb{R}^N)$ to a function $u$ which satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma$ and the restriction $u|_{\Omega_\Gamma}$ is a least energy solution of

$$\begin{aligned}
-\Delta u &= u \log u^2 \quad \text{in } \Omega_\Gamma, \\
u &> 0 \quad \text{in } \Omega_\Gamma, \\
u &= 0 \quad \text{on } \partial \Omega_\Gamma,
\end{aligned}$$

where $\Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j$.

**Corollary 1.3.** Under the assumptions of Theorem 1.2, there exists $\lambda_* > 0$ such that for all $\lambda \geqslant \lambda_*$, the problem $(P_{\lambda})$ has at least $2^k - 1$ positive solutions.

In the above mentioned papers [1,5,19], the method developed in [18] is essential and consists in modifying the nonlinearity to obtain an auxiliary problem, whose associated energy functional satisfies the (PS) condition. After that, by making some estimates, it is possible to prove that the solutions obtained for the auxiliary problem are in fact solutions for the original problem when $\lambda$ is large enough. However, the natural energy functional associated with the problem $(P_{\lambda})$ given by

$$I_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx$$

is not well defined in $H^1(\mathbb{R}^N)$. Because there exists a function $u$ in that space such that $I_{\lambda}(u) = +\infty$, we cannot directly use the critical point theory for $C^1$ functionals, and then we need to use a different
approach from that used to the Schrödinger equation (1.1). Here, we have used strongly the fact that the functional $I_\lambda$ is of class $C^1$ in $H^1(\mathbb{D})$, when $\mathbb{D} \subset \mathbb{R}^N$ is a bounded domain. Based on this observation, for each $R > 0$ and $\lambda > 0$ large enough, we first find a solution $u_{\lambda,R} \in H^1_0(B_R(0))$, and after taking the limit of $R \to +\infty$, we obtain a solution for the original problem. On the other hand, for the nonlinear term $u^p$ in (1.1), it is easy to verify that $\lim_{t \to 0} t^p/|t| = 0$ as $t \to 0$ and the function $t^p/t$ is increasing for $t \in (0, +\infty)$ which are very important to use the method in [18]. But for our problem, the nonlinear term is $u \log u^2 + u$, and it is clear that $t \log t^2 + t \neq o(t)$ as $t \to 0$. Thus, we cannot apply directly del Pino and Felmer’s method in [18] and our problem is more difficult and complicated. The plan of the paper is as follows. In Section 2, we prove the existence of multi-bump solutions for an auxiliary problem in the ball $B_R(0)$ for $R > 0$. In Section 3, we prove Theorem 1.2.

Notation. From now on in this paper, unless otherwise mentioned, we use the following notations:

- $B_r(u)$ is an open ball centered at $u$ with the radius $r > 0$ and $B_r = B_r(0)$.
- If $g$ is a measurable function, the integral $\int_{\mathbb{R}^N} g(x)dx$ will be denoted by $\int g(x)dx$.
- $C$ denotes any positive constant, whose value is not relevant.
- $| \cdot |_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^N)$ for $p \in [1, +\infty]$, and $\| \cdot \|$ denotes the usual norm of the Sobolev space $H^1(\mathbb{R}^N)$.
- $H^1_0(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) : u$ has compact support $\}$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$.

2 An auxiliary problem on the ball $B_R(0)$

We work on the following space of functions:

$$E_\lambda = \left\{ u \in H^1(\mathbb{R}^N) : \int V(x)|u|^2dx < +\infty \right\}$$

endowed with the norm

$$\| u \|_\lambda = \left( \int (|\nabla u|^2 + \lambda V(x) + 1)|u|^2dx \right)^{\frac{1}{2}}.$$

Since $V(x) \geq 0$ for all $x \in \mathbb{R}^N$, the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^N)$ is continuous, so the embedding $E_\lambda \hookrightarrow L^q(\mathbb{R}^N)$ is also continuous for all $q \in [2, 2^*]$.

For each $R > 0$, we define a norm $\| \cdot \|_{\lambda,R}$ on $H^1_0(B_R(0))$ by

$$\| u \|_{\lambda,R} = \left( \int_{B_R(0)} (|\nabla u|^2 + \lambda V(x) + 1)|u|^2dx \right)^{\frac{1}{2}},$$

which is equivalent to the usual norm in that space for all $\lambda, R > 0$. In what follows we denote by $E_{\lambda,R}$ the space $H^1_0(B_R(0))$ endowed with the norm $\| \cdot \|_{\lambda,R}$.

Following the approach explored in [21, 24], for a small $\delta > 0$, let us define the following functions:

$$F_1(s) = \begin{cases} 0, & s = 0, \\ -\frac{1}{2}s^2 \log s^2, & 0 < |s| < \delta, \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{1}{2}\delta^2, & |s| \geq \delta \end{cases}$$

and

$$F_2(s) = \begin{cases} 0, & |s| < \delta, \\ \frac{1}{2}s^2 \log \left( \frac{s^2}{\delta^2} \right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s| \geq \delta. \end{cases}$$
Therefore,
\[ F_2(s) - F_1(s) = \frac{1}{2} s^2 \log s^2, \quad \forall s \in \mathbb{R}. \] (2.1)

It was proved in [21,24] that \( F_1 \) and \( F_2 \) verify the following properties:
\[ F_1, F_2 \in C^1(\mathbb{R}_{+}, \mathbb{R}). \]

If \( \delta > 0 \) is small enough, \( F_1 \) is convex, even, \( F_1(s) \geq 0 \) for all \( s \in \mathbb{R} \) and \( F'_1(s)s \geq 0, \ s \in \mathbb{R} \). For each fixed \( p \in (2, 2^*) \), there exists \( C > 0 \) such that
\[ |F'_2(s)| \leq C|s|^{p-1}, \quad \forall s \in \mathbb{R}. \] (2.2)

By a simple observation, it is easy to see that \( F'_2(s)/s \) is nondecreasing for \( s > 0 \) and \( F'_2(s)/s \) is strictly increasing for \( s > \delta \), \( \lim_{s \to +\infty} F'_2(s)/s = +\infty \), and
\[ F'_2(s) \geq 0 \quad \text{for } s > 0 \quad \text{and} \quad F'_2(s) > 0 \quad \text{for } s > \delta. \]

### 2.1 The auxiliary problem

For each \( j \in \{1, \ldots, k\} \), we fix a bounded open subset \( \Omega'_j \) with smooth boundary such that
(i) \( \overline{\Omega'_j} \subset \Omega'_j \);
(ii) \( \overline{\Omega'_j} \cap \overline{\Omega'_l} = \emptyset \) for all \( j \neq l \).

From now on, we fix a non-empty subset \( \Gamma \subset \{1, \ldots, k\} \) and \( R > 0 \) such that \( \Omega'_\Gamma \subset B_R(0) \) and
\[ \Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega'_j, \quad \Omega'_\Gamma = \bigcup_{j \in \Gamma} \Omega'_j. \]

Let \( l > 0 \) small and \( a_0 > 0 \) such that \( F'_2(a_0)/a_0 = l \). It is clear that \( a_0 > \delta \). We define
\[ \tilde{F}'_2(s) = \begin{cases} F'_2(s), & 0 \leq s \leq a_0, \\ l s, & s \geq a_0 \end{cases} \]
and
\[ G'_2(x,t) = \chi_\Gamma(x) F'_2(t) + (1 - \chi_\Gamma(x)) \tilde{F}'_2(t), \]
where
\[ \chi_\Gamma(x) = \begin{cases} 1, & x \in \Omega'_\Gamma, \\ 0, & x \in B_R(0) \setminus \Omega'_\Gamma. \end{cases} \]

Now we consider the existence of the solution for the problem
\[
\begin{cases}
-\Delta u + (\lambda V(x) + 1)u = G'_2(x,u^+) - F'_1(u) & \text{in } B_R(0), \\
u = 0 & \text{on } \partial B_R(0).
\end{cases} \tag{M_{\lambda,R}}
\]

Notice that if \( u_{\lambda,R} \) is a positive solution of \((M_{\lambda,R})\) with \( 0 < u_{\lambda,R} \leq a_0 \) for all \( x \in B_R(0) \setminus \Omega'_\Gamma \), then \( G'_2(x,u_{\lambda,R}) = F'_2(u_{\lambda,R}) \), so \( u_{\lambda,R} \) is also a positive solution of
\[
\begin{cases}
-\Delta u + \lambda V(x)u = u \log u^2 & \text{in } B_R(0), \\
u > 0 & \text{in } B_R(0), \\
u = 0 & \text{on } \partial B_R(0).
\end{cases} \tag{P_{\lambda,R}}
\]

Moreover, we shall look for the nontrivial critical points for the functional
\[
\Phi_{\lambda,R}(u) = \frac{1}{2} \int_{B_R(0)} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2) dx + \int_{B_R(0)} F_1(u) dx - \int_{B_R(0)} G_2(x,u^+) dx,
\]
where
\[ u^+ = \max\{u(x), 0\} \quad \text{and} \quad G_2(x, t) = \int_0^t G_2'(x, s)ds, \quad \forall (x, t) \in B_R(0) \times \mathbb{R}. \]

It is standard to show that \( \Phi_{\lambda, R} \in C^1(E_{\lambda, R}, \mathbb{R}) \).

Our first lemma establishes that the functional \( \Phi_{\lambda, R} \) satisfies the mountain pass geometry (see [29]).

**Lemma 2.1.** For all \( \lambda > 0 \), the functional \( \Phi_{\lambda, R} \) satisfies the following conditions:
(i) \( \Phi_{\lambda, R}(0) = 0 \);
(ii) there exist \( \alpha, \rho > 0 \) such that \( \Phi_{\lambda, R}(u) \geq \alpha \) for any \( u \in E_{\lambda, R} \) with \( \|u\|_{\lambda, R} = \rho \);
(iii) there exists \( e \in E_{\lambda, R} \) with \( \|e\|_{\lambda, R} > \rho \) such that \( \Phi_{\lambda, R}(e) < 0 \).

**Proof.** (i) It is clear.
(ii) Note that \( \Phi_{\lambda, R}(u) \geq \frac{1}{2}\|u\|^2_{\lambda, R} - \int_{B_R(0)} F_2(u^+)dx \). Hence, from (2.2),
\[
\Phi_{\lambda, R}(u) \geq \frac{1}{2}\|u\|^2_{\lambda, R} - C\|u\|^p_{\lambda, R} \geq C_1 > 0
\]
for some \( C_1 > 0 \) and \( \|u\|_{\lambda, R} > 0 \) small enough. Here, the constant \( C_1 \) does not depend on \( \lambda \) and \( R \).
(iii) Fixing \( 0 < v \in C_0^\infty(\Omega_R) \), by (2.1), we have
\[
\Phi_{\lambda, R}(sv) = s^2\left[ \Phi_{\lambda, R}(v) - \log s \int_{B_R(0)} v^2dx \right] \to -\infty \quad \text{as} \quad s \to +\infty.
\]
Thereby, there exists \( s_0 > 0 \) independent of \( \lambda > 0 \) and \( R > 0 \) large enough such that \( \Phi_{\lambda, R}(sv) < 0 \). \( \Box \)

The mountain pass level associated with \( \Phi_{\lambda, R} \), denoted by \( c_{\lambda, R} \), is given by
\[
c_{\lambda, R} = \inf_{\gamma \in \Gamma_{\lambda, R}} \max_{t \in [0, 1]} \Phi_{\lambda, R}(\gamma(t)),
\]
where \( \Gamma_{\lambda, R} = \{ \gamma \in C([0, 1], E_{\lambda, R}) : \gamma(0) = 0 \text{ and } \Phi_{\lambda, R}(\gamma(1)) < 0 \} \). Notice that by Lemma 2.1,
\[
c_{\lambda, R} \geq \alpha > 0, \quad \forall \lambda > 0, \quad R > 0 \text{ large enough}.
\]

In order to show the boundedness of the (PS) sequence of \( \Phi_{\lambda, R} \), we need a new logarithmic inequality, whose proof can be found in [17, p. 153].

**Lemma 2.2.** There exist constants \( A, B > 0 \) such that
\[
\int |u|^2 \log(|u|^2)dx \leq A + B \log(\|u\|), \quad \forall u \in H^1(\mathbb{R}^N) \setminus \{0\}.
\]

As an immediate consequence we have the corollary.

**Corollary 2.3.** There exist \( C, \Theta > 0 \) such that if \( u \in H^1(\mathbb{R}^N) \) and \( \|u\| \geq \Theta \), then
\[
\int |u|^2 \log(|u|^2)dx \leq C(1 + \|u\|).
\]

By the definition of \( G_2 \), it is easy to see that \( G_2(x, s) \leq F_2(s) \), \( s \geq 0 \). Consequently,
\[
\Phi_{\lambda, R}(u) \geq \frac{1}{2} \int_{B_R(0)} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2)dx - \frac{1}{2} \int_{B_R(0)} (u^+)^2 \log(u^+)^2dx \quad (2.3)
\]
for all \( u \in E_{\lambda, R} \).

**Lemma 2.4.** Let \( (u_n) \subset E_{\lambda, R} \) be a sequence such that \( (\Phi_{\lambda, R}(u_n)) \) is bounded in \( \mathbb{R} \). Then \( (u_n) \) is a bounded sequence in \( E_{\lambda, R} \).

**Proof.** Since \( (\Phi_{\lambda, R}(u_n)) \) is bounded, there exists \( M > 0 \) such that
\[
M \geq \Phi_{\lambda, R}(u_n), \quad \forall n \in \mathbb{N}.
\]
The functional

Without loss of generality, we will assume that

For all

Corollary 2.6. If

By Corollary 2.6, the sequence

From this, assume that there exists

Thus by (2.3),

If 0

As a byproduct of the last lemma, we have the following result.

Our next lemma shows that \( \Phi_{\lambda,R} \) verifies the (PS) condition.

Lemma 2.7. The functional \( \Phi_{\lambda,R} \) satisfies the (PS) condition.

Proof. Let \((u_n) \subset E_{\lambda,R} \) be a (PS) \( q \) sequence for \( \Phi_{\lambda,R} \), i.e.,

\[ \Phi_{\lambda,R}(u_n) \to d \quad \text{and} \quad \Phi'_{\lambda,R}(u_n) \to 0. \]

By Corollary 2.6, the sequence \((u_n)\) is bounded in \( E_{\lambda,R} \). Without loss of generality, we can assume that there exist \( u \in E_{\lambda,R} \) and a subsequence of \((u_n)\), still denoted by itself, such that

\[ u_n \to u \quad \text{in} \quad E_{\lambda,R}, \]

\[ u_n \to u \quad \text{in} \quad L^q(B_R(0)), \quad \forall q \in [1, 2^*) \]

and

\[ u_n(x) \to u(x) \quad \text{a.e. in} \quad B_R(0). \]

For all \( t \in \mathbb{R} \) and by fixing \( p \in (2, 2^*) \), we see that there exists \( C > 0 \) such that

\[ |G_2'(x,t)| \leq t|t| + C|t|^{p-1}, \quad \forall t \in \mathbb{R} \]

and

\[ |F_1'(t)| \leq C(1 + |t|^p), \quad \forall t \in \mathbb{R}. \]

Hence, by the Sobolev embeddings,

\[ \int_{B_R(0)} G_2'(x,u_n^+)u_n^+ dx \to \int_{B_R(0)} G_2'(x,u^+)u^+ dx, \quad \int_{B_R(0)} F_1'(u_n)u_n dx \to \int_{B_R(0)} F_1'(u)u dx \]
and
\[ \int_{B_R(0)} G_2'(x, u^+_n)vdx \to \int_{B_R(0)} G_2'(x, u^+)vdx, \]
\[ \int_{B_R(0)} F_1'(u_n)vdx \to \int_{B_R(0)} F_1'(u)vdx \]
for any \( v \in E_{\lambda,R} \).

Now, using the limits \( \Phi_{\lambda,R}(u_n)u_n = \Phi_{\lambda,R}(u_n)u = o_n(1) \), we obtain
\[ |u_n - u|^2_{\lambda,R} = \int_{B_R(0)} (G_2'(x, u^+_n) - G_2'(x, u^+))(u^+_n - u^+)dx \]
\[ - \int_{B_R(0)} (F_1'(u_n) - F_1'(u))(u_n - u)dx + o_n(1) = o_n(1), \]
showing the desired result. \( \square \)

**Theorem 2.8.** The problem \((M_{\lambda,R})\) has a positive solution \( u_{\lambda,R} \in E_{\lambda,R} \) such that \( \Phi_{\lambda,R}(u_{\lambda,R}) = c_{\lambda,R} \), where \( c_{\lambda,R} \) denotes the mountain pass level associated with \( \Phi_{\lambda,R} \).

**Proof.** The existence of the nontrivial solution \( u_{\lambda,R} \) is an immediate result of Lemma 2.1, Corollary 2.6 and Lemma 2.7. The function \( u_{\lambda,R} \) is nonnegative, because
\[ \Phi_{\lambda,R}(u_{\lambda,R})(u_{\lambda,R}) = 0 \Rightarrow u_{\lambda,R} = 0, \]
where \( u_{\lambda,R}^- = \min \{u_{\lambda,R}, 0\} \). By the maximum principle (see [27, Theorem 1]), we have \( u_{\lambda,R}(x) > 0 \) for a.e. \( x \in B_R(0) \). \( \square \)

### 2.2 The \((PS)_{\infty,R}\) condition

In the sequel, for each \( R > 0 \), we study the behavior of a \((PS)_{\infty,R}\) sequence for \( \Phi_{\lambda,R} \), i.e., a sequence \( \{u_n\} \subset H_0^1(B_R(0)) \) satisfying
\[ u_n \in E_{\lambda,R} \quad \text{and} \quad \lambda_n \to \infty, \]
\[ \Phi_{\lambda_n,R}(u_n) \to c, \]
\[ ||\Phi_{\lambda_n,R}'(u_n)|| \to 0. \]

**Proposition 2.9.** Let \( \{u_n\} \subset H_0^1(B_R(0)) \) be a \((PS)_{\infty,R}\) sequence. Then for some subsequence, still denoted by \( \{u_n\} \), there exists \( u \in H_0^1(\Omega) \) such that
\[ u_n \rightharpoonup u \quad \text{in} \quad H_0^1(B_R(0)). \]
Moreover,
(i) \( u_n \) converges to \( u \) in the strong sense, i.e.,
\[ ||u_n - u||_{\lambda_n,R} \to 0, \]
and hence,
\[ u_n \to u \quad \text{in} \quad H_0^1(B_R(0)); \]
(ii) \( u \equiv 0 \) in \( B_R(0) \setminus \Omega \) and \( u \) is a solution of
\[ \begin{cases} -\Delta u = u \log u^2 & \text{in} \ \Omega, \\ u = 0 & \text{on} \ \partial \Omega; \end{cases} \quad \text{(P_{\infty,\Gamma})} \]
(iii) \( u_n \) also satisfies
\[ \lambda_n \int_{B_R(0)} V(x)|u_n|^2dx \to 0, \]
\[ ||u_n||_{\lambda_n,R(0) \setminus \Omega \Gamma}^2 \to 0, \]
\[ ||u_n||_{\lambda_n,\Omega_j}^2 \to \int_{\Omega_j} (|\nabla u|^2 + |u|^2)dx \quad \text{for all} \ j \in \Gamma. \]
Proof. By Corollary 2.5, there exists $K > 0$ such that
\[ \|u_n\|_{\lambda_n, R}^2 \leq K, \quad \forall n \in \mathbb{N}. \]
Thus, $(u_n)$ is bounded in $H^1_0(B_R(0))$ and we can assume that for some $u \in H^1_0(B_R(0))$,
\[ u_n \rightharpoonup u \quad \text{weakly in } H^1_0(B_R(0)) \]
and
\[ u_n(x) \to u(x) \quad \text{a.e. in } B_R(0). \]
Fixing $C_m = \{ x \in B_R(0) : V(x) \geq \frac{1}{m} \}$, we have
\[ \int_{C_m} |u_n|^2 dx \leq \frac{m}{\lambda_n} \int_{B_R(0)} \lambda_n V(x)|u_n|^2 dx, \]
i.e.,
\[ \int_{C_m} |u_n|^2 dx \leq \frac{m}{\lambda_n} \|u_n\|^2_{\lambda_n, R}. \]
The last inequality together with Fatou’s lemma yields
\[ \int_{C_m} |u|^2 dx = 0, \quad \forall m \in \mathbb{N}. \]
Then $u(x) = 0$ on $\bigcup_{m=1}^{+\infty} C_m = B_R(0) \setminus \Omega$, so $u|_{\Omega_j} \in H^1_0(\Omega_j)$, $j \in \{1, \ldots, k\}$. From this, we are able to prove (i)–(iii).

(i) Since $u = 0$ in $B_R(0) \setminus \Omega$ and $\Phi'_{\lambda_n, R}(u_n)u_n = \Phi'_{\lambda_n, R}(u_n)u = o_n(1)$, we have
\[ \int_{B_R(0)} (|\nabla u_n| - |\nabla u|)^2 + (\lambda_n V(x) + 1)|u_n - u|^2 dx \to 0, \]
which implies that $u_n \rightharpoonup u$ in $H^1_0(B_R(0))$.

(ii) Since $u \in H^1_0(B_R(0))$ and $u = 0$ in $B_R(0) \setminus \Omega$, we deduce that $u \in H^1_0(\Omega)$, or equivalently, $u|_{\Omega_j} \in H^1_0(\Omega_j)$ for $j = 1, \ldots, k$. Moreover, $u_n \rightharpoonup u$ in $H^1_0(B_R(0))$ combined with $\Phi'_{\lambda_n, R}(u_n)\varphi \to 0$ as $n \to +\infty$ for each $\varphi \in C_c^\infty(\Omega_j)$ implies that
\[ \int_{\Omega_j} (\nabla u \nabla \varphi + u \varphi) dx + \int_{\Omega_j} F'_2(u) \varphi dx - \int_{\Omega_j} F'_2(u^+) \varphi dx = 0, \]
from which it follows that $u|_{\Omega_j}$ is a solution for $(P_{\infty, \Gamma})$.

On the other hand, for each $j \in \{1, 2, \ldots, k\} \setminus \Gamma$,
\[ \int_{\Omega_j} (|\nabla u|^2 + u^2) dx + \int_{\Omega_j} F'_2(u) u dx = \int_{\Omega_j} F'_2(u^+) u^+ dx = 0. \]
By the fact that $F'_2(s)s \geq 0$ and $F'_2(s)s \leq ls^2$ for all $s \in \mathbb{R}^+$, we derive that
\[ \int_{\Omega_j} (|\nabla u|^2 + u^2) dx \leq \int_{\Omega_j} F'_2(u^+) u^+ dx \leq l \int_{\Omega_j} u^2 dx. \]
Since $l < 1$, $u = 0$ in $\Omega_j$ for $j \in \{1, 2, \ldots, k\} \setminus \Gamma$ and $u \geq 0$ in $B_R(0)$, which shows (ii).

To prove (iii), note that from (i),
\[ \int_{B_R(0)} \lambda_n V(x)|u_n|^2 dx = \int_{B_R(0)} \lambda_n V(x)|u_n - u|^2 dx \leq C\|u_n - u\|^2_{\lambda_n, R}, \]
so
\[ \int_{B_R(0)} \lambda_n V(x)|u_n|^2 dx \to 0 \quad \text{as } n \to +\infty. \]
Moreover, from (i) and (ii), it is easy to check that
\[ \|u_n\|^2_{L^2(\Omega_{\Gamma})} \to 0, \]
\[ \|u_n\|^2_{L^2(\Omega_j)} \to \int_{\Omega_j} (|\nabla u|^2 + |u|^2) \, dx \quad \text{for all } j \in \Gamma. \]
This completes the proof. \( \square \)

With a few modifications in the arguments in the proof of Proposition 2.9 and using Corollary 2.5, we also have the result below that will be used in Section 3.

**Proposition 2.10.** Let \( u_n \in E_{\lambda_n,R_n} \) be a \((PS)_{\infty,R_n}\) sequence with \( R_n \to +\infty \), i.e.,
\[ u_n \in E_{\lambda_n,R_n} \quad \text{and} \quad \lambda_n \to \infty, \]
\[ \Phi_{\lambda_n,R_n}(u_n) \to c, \]
\[ \|\Phi'_{\lambda_n,R_n}(u_n)\| \to 0. \]
Then for some subsequence, still denoted by \((u_n)\), there exists \( u \in H^1_0(\Omega) \) such that
\[ u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^N). \]
Moreover,
(i) \( \|u_n - u\|_{L^2(\Omega_{\Gamma})} \to 0 \), so \( u_n \to u \quad \text{in } H^1(\mathbb{R}^N); \)
(ii) \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega_{\Gamma} \) and \( u \) is a solution of
\[ \begin{cases} -\Delta u = u \log u^2 & \text{in } \Omega_{\Gamma}, \\ u = 0 & \text{on } \partial\Omega_{\Gamma}; \end{cases} \quad (P_{\infty,\Gamma}) \]
(iii) \( u_n \) also satisfies
\[ \lambda_n \int_{B_{R_n}(0)} V(x)|u_n|^2 \, dx \to 0, \]
\[ \|u_n\|^2_{L^2(B_{R_n}(0) \setminus \Omega_{\Gamma})} \to 0, \]
\[ \|u_n\|^2_{L^2(\Omega_j)} \to \int_{\Omega_j} (|\nabla u|^2 + |u|^2) \, dx \quad \text{for all } j \in \Gamma. \]

**Proof.** First of all, the boundedness of \((\Phi_{\lambda_n,R_n}(u_n))\) implies that there exists \( K > 0 \) such that
\[ \|u_n\|^2_{L^2(\Omega_{\Gamma})} \leq K, \quad \forall n \in \mathbb{N}. \]
Thus, \((u_n)\) is bounded in \( H^1(\mathbb{R}^N) \) and we can assume that for some \( u \in H^1(\mathbb{R}^N), \)
\[ u_n \to u \quad \text{in } H^1(\mathbb{R}^N) \]
and
\[ u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N, \]
and \( u(x) = 0 \) on \( \mathbb{R}^N \setminus \Omega. \)
(i) Let \( 0 < R < R_n \) and \( \phi_R \in C^\infty(\mathbb{R}^N, \mathbb{R}) \) be a cut-off function such that
\[ \phi_R = 0, \quad x \in B_{R/2}(0), \quad \phi_R = 1, \quad x \in B_R(0), \quad 0 \leq \phi_R \leq 1 \quad \text{and} \quad |\nabla \phi_R| \leq C/R, \]
where \( C > 0 \) is a constant independent of \( R \). Since the sequence \((\|\phi_R u_n\|_{L^2(\Omega_{\Gamma})})\) is bounded, we derive that
\[ \Phi'_{\lambda_n,R_n}(u_n)(\phi_R u_n) = o_n(1), \]
i.e.,
\[
\int (|\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2) \phi_R dx = \int_{\Omega_1^c} F'_2(u_n^+) \phi_R u_n dx + \int_{\mathbb{R}^N \setminus \Omega_1^c} \tilde{F}'_2(u_n^+) \phi_R u_n dx \\
- \int u_n \nabla u_n \nabla \phi_R dx - \int F'_1(u_n) \phi_R u_n dx + o_n(1).
\]

By choosing \( R > 0 \) such that \( \Omega_1^c \subset B_{R/2}(0) \), we see that Hölder’s inequality together with the boundedness of the sequence \( (\|u_n\|)_{\lambda_n, R_n} \) in \( \mathbb{R} \) leads to
\[
\int (|\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2) \phi_R dx \leq l \int |u_n|^2 \phi_R dx + \frac{C}{R} \|u_n\|_{\lambda_n, R_n}^2 + o_n(1).
\]

Therefore, by fixing \( \zeta > 0 \) and passing to the limit in the last inequality, we have
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2) dx \leq \frac{C}{R} < \zeta \tag{2.5}
\]
for some \( R \) sufficiently large.

Since \( G'_2 \) has a subcritical growth, the above estimate (2.5) ensures that
\[
\int G'_2(x, u_n^+) dw \to \int G'_2(x, u^+) dw, \quad \forall w \in C_0^\infty(\mathbb{R}^N),
\]
\[
\int G'_2(x, u_n^+) u_n^+ dx \to \int G'_2(x, u^+) u^+ dx
\]
and
\[
\int G_2(x, u_n^+) dx \to \int G_2(x, u^+) dx.
\]

Now, recalling that \( \lim_{n \to \infty} \Phi'_{\lambda_n, R_n}(u_n) w = 0 \) for all \( w \in C_0^\infty(\mathbb{R}^N) \) and \( \|u_n\|_{\lambda_n, R_n}^2 \leq K, \quad \forall n \in \mathbb{N} \), we deduce that
\[
\int (\nabla u \nabla \omega + u \omega) dx + \int F'_1(u) \omega dx = \int G'_2(x, u^+) \omega dx,
\]
so
\[
\int (|\nabla u|^2 + |u|^2) dx + \int F'_1(u) dx = \int G'_2(x, u^+) u^+ dx.
\]

This together with the equality \( \lim_{n \to \infty} \Phi'_{\lambda_n, R_n}(u_n) u_n = 0 \), i.e.,
\[
\int (|\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2) dx + \int F'_1(u_n) u_n dx = \int G'_2(x, u_n^+) u_n^+ dx + o_n(1)
\]
leads to
\[
\lim_{n \to \infty} \left( \int (|\nabla u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2) dx + \int F'_1(u_n) u_n dx \right) \\
= \int (|\nabla u|^2 + |u|^2) dx + \int F'_1(u) u dx,
\]
from which it follows that for some subsequence,
\[
u_n \to u \quad \text{in} \quad H^1(\mathbb{R}^N), \quad \lambda_n \int V(x)|u|^2 dx \to 0
\]
and
\[
F'_1(u_n) u_n \to F'_1(u) u \quad \text{in} \quad L^1(\mathbb{R}^N).
\]

Since \( F_1 \) is convex, even and \( F(0) = 0 \), we know that \( F'_1(t) t \geq F_1(t) \geq 0 \) for all \( t \in \mathbb{R} \). Thus, the last limit together with the Lebesgue dominated convergence theorem yields
\[
F_1(u_n) \to F_1(u) \quad \text{in} \quad L^1(\mathbb{R}^N).
\]
Combining (2.8)–(2.10), we have Lemma 2.11.

Let \( u \) be a family of positive solutions of (\( M_{\lambda,R} \)) such that \( (\Phi_{\lambda,R}(u)) \) is bounded in \( \mathbb{R} \) for any \( \lambda > 0 \) and \( R > 0 \) large enough. Then there exist \( K > 0 \) that does not depend on \( \lambda > 0 \) and \( R > 0 \), and \( R^* > 0 \) such that
\[
|u_{\lambda,R}|_{\infty,R} \leq K, \quad \forall \lambda > 0 \quad \text{and} \quad R \geq R^*.
\]

Proof. For each \( \lambda > 0, L > 0 \) and \( \beta > 1 \), let
\[
\begin{align*}
    u_{L,\lambda} &:= \begin{cases} 
u_{L,\lambda}, & \text{if } u_{L,\lambda} \leq L, \\ L, & \text{if } u_{L,\lambda} \geq L, \end{cases} \\
    z_{L,\lambda} &:= u_{L,\lambda}^{2(\beta-1)} \quad \text{and} \quad \omega_{L,\lambda} = u_{L,\lambda}^{\beta-1}.
\end{align*}
\]
Using the fact that \( u_{L,\lambda} \) is a positive solution to (\( M_{\lambda,R} \)) and taking \( z_{L,\lambda} \) as a test function, we have
\[
\begin{align*}
    &\iint_{B_R(0)} u_{L,\lambda}^{2(\beta-1)}|\nabla u_{L,\lambda}|^2 dx + 2(\beta-1) \iint_{B_R(0)} u_{L,\lambda}^{2\beta-3} u_{L,\lambda} \nabla u_{L,\lambda} \nabla u_{L,\lambda} dx \\
    &+ \iint_{B_R(0)} (\lambda V(x) + 1) u_{L,\lambda}^{2(\beta-1)} |u_{L,\lambda}|^2 dx + \iint_{B_R(0)} F'(u_{L,\lambda}) u_{L,\lambda}^{2(\beta-1)} u_{L,\lambda} dx \\
    &= \iint_{B_R(0)} G'_2(x, u_{L,\lambda}) u_{L,\lambda}^{2(\beta-1)} u_{L,\lambda} dx.
\end{align*}
\] (2.6)

From the definition of \( G_2 \),
\[
G'_2(x, t) \leq F'_2(t) \leq C t^{p-1}, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}^+,
\] (2.7)
where \( p \in (2, 2^*) \). Hence, from (2.6) and (2.7),
\[
\iint_{B_R(0)} (|\nabla \omega_{L,\lambda}|^2 + |\omega_{L,\lambda}|^2) dx \leq C \iint_{B_R(0)} u_{L,\lambda}^p u_{L,\lambda}^{2(\beta-1)} dx = C \iint_{B_R(0)} u_{L,\lambda}^{p-2} \omega_{L,\lambda}^2 dx.
\] (2.8)

Using Hölder’s inequality, we have
\[
\iint_{B_R(0)} u_{L,\lambda}^{p-2} \omega_{L,\lambda}^2 dx \leq C \beta^2 \left( \iint_{B_R(0)} u_{L,\lambda}^p dx \right)^{(p-2)/p} \left( \iint_{B_R(0)} \omega_{L,\lambda}^2 dx \right)^{2/p}.
\] (2.9)

On the other hand, by the Sobolev inequality,
\[
\left( \iint_{B_R(0)} |\omega_{L,\lambda}|^2 dx \right)^{2/2^*} \leq C \iint_{B_R(0)} (|\nabla \omega_{L,\lambda}|^2 + |\omega_{L,\lambda}|^2) dx.
\] (2.10)

Combining (2.8)–(2.10), we have
\[
\left( \iint_{B_R(0)} |\omega_{L,\lambda}|^2 dx \right)^{2/2^*} \leq C \beta^2 \left( \iint_{B_R(0)} u_{L,\lambda}^p dx \right)^{2/p}.
\]
Using Fatou’s lemma in the variable \( L \), one has
\[
\left( \int_{B_R(0)} |u_\lambda|^{2^*} \beta^* \, dx \right)^{2/2^*} \leq C \beta^2 \left( \int_{B_R(0)} u_\lambda^{p^*} \, dx \right)^{2/p},
\]
so
\[
\left( \int_{B_R(0)} |u_\lambda|^{2^*} \beta^* \, dx \right)^{1/2^*} \leq C^{1/\beta} \beta^{1/\beta} \left( \int_{B_R(0)} u_\lambda^{p^*} \, dx \right)^{1/p^*}.
\] (2.11)

Since \((\Phi_{\lambda,R}(u_{\lambda,R}))\) is bounded in \( \mathbb{R} \) for any \( \lambda > 0 \) and \( R > 0 \) large enough and \((u_{\lambda,R})\) is a solution of \((M_{\lambda,R})\), by arguing as in Subsection 2.1, we see that there exists \( C > 0 \) such that
\[
\|u_{\lambda,R}\|_{L^\infty(\mathbb{R}^N)} \leq C
\]
for \( \lambda > 0 \) and \( R > 0 \) large enough.

Fixing any sequences \( \lambda_n \to +\infty \) and \( R_n \to +\infty \), we may see that \((u_{\lambda_n,R_n})\) satisfies the hypotheses from Proposition 2.10, and then \( u_{\lambda_n,R_n} \to u \) strongly in \( H^1(\mathbb{R}^N) \).

Now, since \( 2 < p < 2^* \) and \((u_{\lambda_n,R_n})_{L^{2^*}(\mathbb{R}^N)}\) is bounded in \( \mathbb{R} \), a well-known iteration argument (see [5, Lemma 3.10]) and (2.11) imply that there exists a positive constant \( K_1 > 0 \) such that
\[
|u_{\lambda_n,R_n}|_{L^\infty(\mathbb{R}^N)} \leq K_1, \quad \forall n \in \mathbb{N}.
\]

From the above analysis, it is easy to see that the lemma follows from arguing by contradiction.

\[\square\]

**Lemma 2.12.** Let \((u_{\lambda,R})\) be a family of positive solutions of \((M_{\lambda,R})\) with \(\Phi_{\lambda,R}(u_{\lambda,R})\) bounded in \( \mathbb{R} \) for any \( \lambda > 0 \) and \( R > 0 \) large enough. Then there exist \( \lambda' > 0 \) and \( R' > 0 \) such that
\[
|u_{\lambda,R}|_{\infty,B_R(0)\setminus\Omega_{\lambda}} \leq a_0, \quad \forall \lambda \geq \lambda', \quad R \geq R'.
\]
In particular, \( u_{\lambda,R} \) solves the original problem \((P_{\lambda,R})\) for any \( \lambda \geq \lambda' \) and \( R \geq R' \).

**Proof.** Choose large \( R_0 > 0 \) such that \( \overline{\Omega}_G \subset B_{R_0}(0) \) and fix a neighborhood \( B \) of \( \partial \Omega_{\lambda} \) such that
\[
B \subset B_{R_0}(0) \setminus \Omega_{\lambda}.
\]

The Moser’s iteration technique implies that there exists \( C > 0 \), which is independent of \( \lambda \), such that
\[
|u_{\lambda,R}|_{L^\infty(\partial \Omega_{\lambda})} \leq C |u_{\lambda,R}|_{L^{2^*}(B)}, \quad \forall R \geq R_0.
\]

Fixing two sequences \( \lambda_n \to +\infty \) and \( R_n \to +\infty \) and using Proposition 2.10, we have for some subsequence that \( u_{\lambda_n,R_n} \to 0 \) in \( H^1(B_{R_n}(0)\setminus\Omega_{\lambda}) \), and then \( u_{\lambda_n,R_n} \to 0 \) in \( H^1(B_{R_n}(0)\setminus\Omega_{\lambda}) \), so
\[
|u_{\lambda_n,R_n}|_{L^{2^*}(B)} \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence, there exists an \( n_0 \in \mathbb{N} \) such that
\[
|u_{\lambda_n,R_n}|_{L^\infty(\partial \Omega_{\lambda})} \leq a_0, \quad \forall n \geq n_0.
\]

Now, for \( n \geq n_0 \) we set \( \tilde{u}_{\lambda_n,R_n} : B_{R_n}(0)\setminus\Omega_{\lambda} \to \mathbb{R} \) given by
\[
\tilde{u}_{\lambda_n,R_n}(x) = (u_{\lambda_n,R_n} - a_0)^+(x).
\]

Thereby, \( \tilde{u}_{\lambda_n,R_n}(x) \in H^1_0(B_{R_n}(0)\setminus\Omega_{\lambda}) \). Our goal is to show that \( \tilde{u}_{\lambda_n,R_n}(x) = 0 \) in \( B_{R_n}(0)\setminus\Omega_{\lambda} \), because this will ensure that
\[
|u_{\lambda_n,R_n}|_{\infty,B_{R_n}(0)\setminus\Omega_{\lambda}} \leq a_0.
\]

In fact, extending \( \tilde{u}_{\lambda_n,R_n}(x) = 0 \) in \( \Omega_{\lambda} \) and taking \( \tilde{u}_{\lambda,R} \) as a test function, we obtain
\[
\int_{B_{R_n}(0)\setminus\Omega_{\lambda}} \nabla u_{\lambda,R_n} \nabla \tilde{u}_{\lambda_n,R_n} \, dx + \int_{B_{R_n}(0)\setminus\Omega_{\lambda}} (\lambda_n V(x) + 1) u_{\lambda,R_n} \tilde{u}_{\lambda_n,R_n} \, dx
\]
bounded, and Rabinowitz [11], there exist two positive functions which are the energy functionals associated with the following logarithmic equations:

\[ 2.4 \text{ A special minimax level} \]

\[
\int_{B_{R_n}(0)\setminus \Omega^*_1} \nabla u_{\lambda, R_n} \nabla \tilde{u}_{\lambda, R_n} \, dx = \int_{B_{R_n}(0)\setminus \Omega^*_1} |\nabla \tilde{u}_{\lambda, R_n}|^2 \, dx,
\]

\[
\int_{B_{R_n}(0)\setminus \Omega^*_1} (\lambda_n V(x) + 1) u_{\lambda, R_n} \tilde{u}_{\lambda, R_n} \, dx = \int_{(B_{R_n}(0)\setminus \Omega^*_1)^+} (\lambda_n V(x) + 1)(\tilde{u}_{\lambda, R_n} + a_0) \tilde{u}_{\lambda, R_n} \, dx
\]

and

\[
\int_{B_{R_n}(0)\setminus \Omega^*_1} \tilde{F}_2''(u_{\lambda, R_n}) \tilde{u}_{\lambda, R_n} \, dx = \int_{(B_{R_n}(0)\setminus \Omega^*_1)^+} \frac{\tilde{F}_2''(u_{\lambda, R_n})}{u_{\lambda, R_n}} (\tilde{u}_{\lambda, R_n} + a_0) \tilde{u}_{\lambda, R_n} \, dx,
\]

where

\[
(B_{R_n}(0)\setminus \Omega^*_1)^+ = \{ x \in B_{R_n}(0)\setminus \Omega^*_1 : u_{\lambda, R_n}(x) > a_0 \}.
\]

From the above equalities, we have

\[
\int_{B_{R_n}(0)\setminus \Omega^*_1} |\nabla \tilde{u}_{\lambda, R_n}|^2 \, dx + \int_{(B_{R_n}(0)\setminus \Omega^*_1)^+} \left( (\lambda_n V(x) + 1) - \frac{\tilde{F}_2''(u_{\lambda, R_n})}{u_{\lambda, R_n}} \right)(\tilde{u}_{\lambda, R_n} + a_0) \tilde{u}_{\lambda, R_n} \, dx = 0.
\]

By the definition of \( \tilde{F}_2'' \), we know that

\[
(\lambda_n V(x) + 1) - \frac{\tilde{F}_2''(u_{\lambda, R_n})}{u_{\lambda, R_n}} \geq 1 - l > 0 \text{ in } (B_{R_n}(0)\setminus \Omega^*_1)^+.
\]

Thus, \( \tilde{u}_{\lambda, R_n} = 0 \) in \( (B_{R_n}(0)\setminus \Omega^*_1)^+ \) and \( B_{R_n}(0)\setminus \Omega^*_1 \). From the above arguments, there exist \( \lambda' > 0 \) and \( R' > 0 \) such that

\[
|u_{\lambda, R}|_{2, B_{R_n}(0)\setminus \Omega^*_1} \leq a_0, \quad \forall \lambda \geq \lambda', \quad R \geq R'.
\]

The proof is completed. \( \square \)

2.4 A special minimax level

In this subsection, for any \( \lambda > 0 \) and \( j \in \Gamma \), let us denote by \( I_j : H^1_0(\Omega_j) \to \mathbb{R} \) and \( I_{\lambda, j} : H^1(\Omega'_j) \to \mathbb{R} \) the functionals given by

\[
I_j(u) = \frac{1}{2} \int_{\Omega_j} (|\nabla u|^2 + |u|^2) \, dx - \frac{1}{2} \int_{\Omega_j} u^2 \log u^2 \, dx,
\]

\[
I_{\lambda, j}(u) = \frac{1}{2} \int_{\Omega'_j} (|\nabla u|^2 + (\lambda V(x) + 1)|u|^2) \, dx - \frac{1}{2} \int_{\Omega'_j} u^2 \log u^2 \, dx,
\]

which are the energy functionals associated with the following logarithmic equations:

\[
\begin{cases}
-\Delta u = u \log u^2 & \text{in } \Omega_j, \\
 u = 0 & \text{on } \partial \Omega_j
\end{cases} \tag{D_j}
\]

and

\[
\begin{cases}
-\Delta u + \lambda V(x)u = u \log u^2 & \text{in } \Omega'_j, \\
 \partial_u / \partial \eta = 0 & \text{on } \partial \Omega'_j.
\end{cases} \tag{N_j}
\]

It is immediate to check that \( I_j \) and \( I_{\lambda, j} \) satisfy the mountain pass geometry. Since \( \Omega_j \) and \( \Omega'_j \) are bounded, and \( I_j \) and \( I_{\lambda, j} \) satisfy the (PS) condition, from the mountain pass theorem due to Ambrosetti and Rabinowitz [11], there exist two positive functions \( \omega_j \in H^1_0(\Omega_j) \) and \( \omega_{\lambda, j} \in H^1(\Omega'_j) \) verifying

\[
I_j(\omega_j) = c_j, \quad I_{\lambda, j}(\omega_{\lambda, j}) = c_{\lambda, j} \quad \text{and} \quad I'_j(\omega_j) = I'_{\lambda, j}(\omega_{\lambda, j}) = 0,
\]
Hence, by the definition of $0$ large enough, such that $\rho > 0$ and $\rho > 0$
and
In fact, a simple computation gives

Lemma 2.13. Without loss of generality, we consider $\Gamma = \gamma \in \gamma$ where

In what follows, moreover, by a direct computation, there exists a $\tau > 0$ such that if $u \in N_j$, for any $j \in \Gamma$, then

where $\| \cdot \|_j$ denotes the norm on $H^1_0(\Omega_j)$ given by

In particular, as $\omega_j \in N_j$, we must have $\| \omega_{\lambda,j} \|_j > \tau$, where $\omega_{\lambda,j} = \omega_j |_{\Omega_j}$ for all $j \in \Gamma$.

In what follows, $c_j = \sum_{j=1}^l c_j$ and $T > 0$ is a constant large enough, which does not depend on $\lambda$ and $R > 0$ large enough, such that

Hence, by the definition of $c_j$, one has

Without loss of generality, we consider $\Gamma = \{1, 2, \ldots, l\}$ with $l \leq k$ and fix

We remark that $\gamma_0 \in \Gamma_*$, so $\Gamma_* \neq \emptyset$ and $b_{\lambda,R, \Gamma}$ is well defined.

**Lemma 2.13.** For each $\gamma \in \Gamma_*$, there exists $(t_1, t_2, \ldots, t_l) \in [1/T^2, 1]^l$ such that

\[ I'_{\lambda,j}(\gamma(t_1, \ldots, t_l))\gamma(t_1, \ldots, t_l) = 0 \quad \text{for } j \in \{1, \ldots, l\}. \]
Proof. Given \( \gamma \in \Gamma_* \), consider the map \( \tilde{\gamma} : [1/T^2, 1]^l \rightarrow \mathbb{R}^l \) defined by
\[
\tilde{\gamma}(s_1, \ldots, s_l) = (I_{\lambda_1}(\gamma(s_1, \ldots, s_l)), \ldots, I_{\lambda_l}(\gamma(s_1, \ldots, s_l))) \gamma(s_1, \ldots, s_l).
\]
For \((s_1, \ldots, s_l) \in \partial([1/T^2, 1]^l)\), we know that
\[
\gamma(s_1, \ldots, s_l) = \gamma_0(s_1, \ldots, s_l).
\]
Now, the lemma follows by employing (2.13) and Miranda’s theorem (see [23]).

Lemma 2.14. \(a\) \( \sum_{j=1}^l e_{\lambda,j} \leq b_{\lambda,R,T} \leq c_T \) for any \( \lambda > 0 \) and \( R > 0 \) large enough.
\(b\) For \( \gamma \in \Gamma_* \) and \((s_1, \ldots, s_l) \in \partial([1/T^2, 1]^l)\), we have
\[
\Phi_{\lambda,R}(\gamma(s_1, \ldots, s_l)) < c_T, \quad \forall \lambda > 0.
\]
The proof of the lemma is the same as that of [1, Proposition 4.2], so we omit it.

Corollary 2.15. \(a\) \( b_{\lambda,R,T} \) is a critical value of \( \Phi_{\lambda,R} \) for \( \lambda > 0 \) and \( R > 0 \) large enough.
\(b\) \( b_{\lambda,R,T} \rightarrow c_T \), when \( \lambda \rightarrow +\infty \) uniformly for \( R > 0 \) large enough.
The proof of the corollary is similar to that of [1, Corollary 4.3]. Here, we also omit it.

2.5 A special solution for the auxiliary problem
Hereafter, let us denote by \( \Phi_{\lambda,R}^{cr} \) and \( \Upsilon \) the sets below:
\[
\Phi_{\lambda,R}^{cr} = \{ u \in E_{\lambda,R} : \Phi_{\lambda,R}(u) \leq c_T \}
\]
and
\[
\Upsilon = \left\{ u \in E_{\lambda,R} : \| u \|_{\lambda,\Omega_j} > \frac{\tau}{2T}, \forall j \in \Gamma \right\},
\]
where \( \tau \) and \( T \) were fixed in (2.12) and (2.13), respectively.

Fixing \( \kappa = \frac{\tau}{8T} \) and \( \mu > 0 \), we define
\[
A^{\nu}_{\mu,R} = \{ u \in \Upsilon_2 : \Phi_{\lambda,B(0)\setminus\Omega_T}^j(u) \geq 0, \| u \|_{\lambda,B(0)\setminus\Omega_T}^2 \leq \mu, |I_{\lambda,j}(u) - c_j| \leq \mu, \forall j \in \Gamma \},
\]
where \( \Upsilon_r, r > 0 \) denotes the set
\[
\Upsilon_r = \left\{ u \in E_{\lambda,R} : \inf_{v \in \Upsilon} \| u - v \|_{\lambda,\Omega_j} \leq r, \forall j \in \Gamma \right\}.
\]
Notice that \( w = \sum_{j=1}^l w_j \in A_{\mu,R}^{\nu} \cap \Phi_{\lambda,R}^{cr} \), which shows that \( A_{\mu,R}^{\nu} \cap \Phi_{\lambda,R}^{cr} \neq \emptyset \).

Next, we shall establish a very important uniform estimate of \( \| \Phi_{\lambda,R}'(u) \| \) in the set \((A^{\lambda}_{2\mu,R}) \cap \Phi_{\lambda,R}^{cr}\).

Proposition 2.16. For each \( \mu > 0 \), there exist \( \lambda_\ast > 0, R^* > 0 \) large enough and \( \sigma_0 > 0 \) independent of \( \lambda \) and \( R > 0 \) large enough such that
\[
\| \Phi_{\lambda,R}'(u) \| \geq \sigma_0 \quad \text{for } \lambda \geq \lambda_\ast, \quad R \geq R^* \quad \text{and} \quad u \in (A^{\lambda}_{2\mu,R}) \cap \Phi_{\lambda,R}^{cr}.
\]
Proof. Arguing by contradiction, we assume that there exist \( \lambda_n, R_n \rightarrow \infty \) and \( u_n \in (A^{\lambda}_{2\mu,R}) \cap \Phi_{\lambda,R}^{cr} \) such that
\[
\| \Phi_{\lambda_n,R_n}(u_n) \| \rightarrow 0.
\]
Since \( u_n \in A^{\lambda}_{2\mu,R} \), we know that \((\| u_n \|_{\lambda_n,R_n})\) and \((\Phi_{\lambda_n,R_n}(u_n))\) are both bounded. Then passing to a subsequence if necessary, we can assume that \((\Phi_{\lambda_n,R_n}(u_n))\) is a convergent sequence. Thus, from Proposition 2.10, there exists \( 0 \leq u \in H^1_0(\Omega_T) \) such that \( u \) is a solution for (D) and
\[
\text{As } (u_n) \subset \Upsilon_{2\kappa}, \text{ it occurs that } \quad \| u_n \|_{\lambda_n,\Omega_j}^2 > \frac{\tau}{4T}, \quad \forall j \in \Gamma,
\]
Therefore, \( u_n \rightarrow +\infty \), we obtain the inequality below:
\[
\|u\|_j^2 \geq \frac{\tau}{4T} > 0, \quad \forall j \in \Gamma,
\]
which yields \( u|_{\Omega_j} \neq 0, \ j = 1, \ldots, l \) and \( I'_\tau(u) = 0 \). Consequently, by (2.12),
\[
\|u\|_j^2 > \frac{\tau}{2T} > 0, \quad \forall j \in \Gamma.
\]
In this way, \( I_\tau(u) \geq c_\tau \). However, from the fact that \( \Phi_{\lambda_n, R_n}(u_n) \leq c_\tau \) and \( \Phi_{\lambda_n, R_n}(u_n) \to I_\tau(u) \) as \( n \to +\infty \), we deduce that \( I_\tau(u) = c_\tau \). Thus, for \( n \) large enough,
\[
\|u_n\|_j^2 > \frac{\tau}{2T}, \quad |\Phi_{\lambda_n, R_n}(u_n) - c_\tau| \leq \mu \quad \text{for any } j \in \Gamma.
\]
Therefore, \( u_n \in A_{\mu, R_n}^\lambda \) for large \( n \), which is a contradiction to \( u_n \in (A_{2\mu, R_n}^\lambda \setminus A_{\mu, R_n}^\lambda) \). Thus, we complete the proof. \( \square \)

In the sequel, \( \mu_1 \) and \( \mu^* \) denote the following numbers:
\[
\min_{t \in \partial[1/T^2, 1]} |I_\tau(\gamma_0(t)) - c_\tau| = \mu_1 > 0
\]
and
\[
\mu^* = \min\{\mu_1, \kappa, \tau/2\},
\]
where \( \kappa = \frac{\tau}{2T} \) was given before and \( r > \max\{\|u_j\|_{\Omega_j} : j = 1, \ldots, l\} \). Moreover, for each \( s > 0 \), \( B_s^\lambda \) denotes the set
\[
B_s^\lambda = \{u \in E_\lambda(B_R(0)) : \|u\|_{\lambda, R} \leq s\} \quad \text{for } s > 0.
\]

**Proposition 2.17.** Let \( \mu \in (0, \mu^*) \), \( \Lambda_n > 0 \) and \( R^* > 0 \) large enough as given in Proposition 2.16. Then for \( \lambda \geq \Lambda_n \) and \( R \geq R^* \), there exists a positive solution \( u_{\lambda, R} \) of \((M_{\lambda, R})\) satisfying \( u_{\lambda, R} \in A_{\mu, R}^\lambda \cap \Phi_{\lambda, R}^{CF} \cap B_{r+1}^\lambda \). Seeking for a contradiction, we assume that there exist no critical points for the functional \( \Phi_{\lambda, R}(u) \in A_{\mu, R}^\lambda \cap \Phi_{\lambda, R}^{CF} \cap B_{r+1}^\lambda \) for \( \lambda \geq \Lambda_n \). Since \( \Phi_{\lambda, R}(u) \) verifies the (PS) condition, there exists a constant \( d_\lambda > 0 \) such that
\[
\|\Phi_{\lambda, R}(u)\| \geq d_\lambda \quad \text{for all } u \in A_{\mu, R}^\lambda \cap \Phi_{\lambda, R}^{CF} \cap B_{r+1}^\lambda.
\]
By Proposition 2.16,
\[
\|\Phi_{\lambda, R}(u)\| \geq \sigma_0 \quad \text{for all } u \in (A_{2\mu, R}^\lambda \setminus A_{\mu, R}^\lambda) \cap \Phi_{\lambda, R}^{CF},
\]
where \( \sigma_0 > 0 \) is independent of \( \lambda \). In what follows, \( \Psi : E_{\lambda, R} \to \mathbb{R} \) is a continuous functional verifying
\[
\Psi(u) = 1 \quad \text{for } u \in A_{\mu/2, R}^\lambda \cap \mathcal{Y}_\kappa \cap B_r^\lambda,
\]
\[
\Psi(u) = 0 \quad \text{for } u \notin A_{\mu/2, R}^\lambda \cap \mathcal{Y}_\kappa \cap B_r^\lambda,
\]
\[
0 \leq \Psi(u) \leq 1, \quad \forall u \in E_{\lambda, R},
\]
and \( H : \Phi_{\lambda, R}^{CF} \to E_\lambda(B_R(0)) \) is a function given by
\[
H(u) := \begin{cases} \Psi(u) \frac{Y(u)}{\|Y(u)\|}, & u \in A_{\mu, R}^\lambda \cap B_{r+1}^\lambda, \\ 0, & u \notin A_{\mu, R}^\lambda \cap B_{r+1}^\lambda. \end{cases}
\]
where \( Y \) is a pseudo-gradient vector field for \( \Phi_{\lambda, R} \) on \( \mathcal{K} = \{u \in E_{\lambda, R} : \Phi_{\lambda, R}(u) \neq 0\} \). Observe that \( H \) is well defined, since \( \Phi_{\lambda, R}(u) \neq 0 \) for \( u \in A_{2\mu, R}^\lambda \cap \Phi_{\lambda, R}^{CF} \). The following inequality
\[
\|H(u)\| \leq 1, \quad \forall \lambda \geq \Lambda_n \quad \text{and } u \in \Phi_{\lambda, R}^{CF}.
\]
guarantees that the deformation flow $\eta : [0, \infty) \times \Phi^{c^r}_{\lambda,R} \to \Phi^{c^r}_{\lambda,R}$ defined by

$$\frac{d\eta}{dt} = H(\eta) \quad \text{and} \quad \eta(0,u) = u \in \Phi^{c^r}_{\lambda,R}$$

verifies

$$\frac{d}{dt}\Phi_{\lambda,R}(\eta(t,u)) \leq -\Psi(\eta(t,u))\|\Phi'_{\lambda,R}(\eta(t,u))\| \leq 0, \quad \eta(t,u) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad u \in \Phi^{c^r}_{\lambda,R} \setminus (A_{2\mu,R} \cap B^\lambda_{r+1}). \quad (2.14)$$

We now study two paths, which are relevant for what follows:

1. The path $t \to \eta(t, \gamma_0(t))$, where $t = (t_1, \ldots, t_2) \in [1/T^2, 1]^l$.

   Therefore, if $\mu \in (0, \mu^*)$, we have

   $$\gamma_0(t) \not\in A^\lambda_{2\mu,R}, \quad \forall t \in \partial([1/T^2, 1]^l).$$

   Since $\Phi_{\lambda,R}(\gamma_0(t)) \leq c_\Gamma, \forall t \in \partial([1/T^2, 1]^l)$, from (2.15), it follows that

   $$\eta(t, \gamma_0(t)) = \gamma_0(t), \quad \forall t \in \partial([1/T^2, 1]^l).$$

   Therefore, $\eta(t, \gamma_0(t)) \in \Gamma_*$ for all $t \geq 0$.

2. The path $t \to \eta_0(t)$, where $t = (t_1, \ldots, t_2) \in [1/T^2, 1]^l$.

   Since $\supp(\gamma_0(t)) \subset \Omega_\Gamma$ for all $t \in [1/T^2, 1]^l$, $\Phi_{\lambda,R}(\gamma_0(t))$ does not depend on $\lambda > 0$. On the other hand,

   $$\Phi_{\lambda,R}(\gamma_0(t)) \leq c_\Gamma, \quad \forall t \in [1/T^2, 1]^l$$

   and

   $$\Phi_{\lambda,R}(\gamma_0(t)) = c_\Gamma \quad \text{if and only if} \quad t_j = 1/T, \quad \forall j \in \Gamma.$$ 

   Therefore,

   $$m_0 := \sup\{\Phi_{\lambda,R}(u) : u \in \gamma_0([1/T^2, 1]^l)] \setminus A^\lambda_{\mu} \}$$

   is independent of $\lambda, R > 0$ and $m_0 < c_\Gamma$. Now, observing that there exists a $K_* > 0$ such that

   $$|\Phi_{\lambda,R}(u) - \Phi_{\lambda,R}(v)| \leq K_*\|u - v\|_{\lambda,R}, \quad \forall u, v \in B^\lambda_{r},$$

   we claim that if $T_* > 0$ is large enough, the estimate below holds:

   $$\max_{t \in [1/T^2, 1]^l} \Phi_{\lambda}(\eta(T_*, \gamma_0(t))) < \max\left\{ m_0, c_\Gamma - \frac{1}{2K_*} \sigma_0 \mu \right\}. \quad (2.16)$$

   In fact, write $u = \gamma_0(t), t \in [1/T^2, 1]^l$. If $u \not\in A^\lambda_{\mu,R}$, we must have that by (2.14),

   $$\Phi_{\lambda,R}(\eta(t,u)) \leq \Phi_{\lambda,R}(\eta(0,u)) = \Phi_{\lambda,R}(u) \leq m_0, \quad \forall t \geq 0.$$

   On the other hand, if $u \in A^\lambda_{\mu,R}$, by setting $\tilde{\eta}(t) = \eta(t,u), \tilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}$ and $T_* = \frac{\sigma_0 \mu}{2K_* \tilde{d}_\lambda} > 0$, now we distinguish two cases:

   1. $\tilde{\eta}(t) \in A^\lambda_{\mu/2,R} \cap Y_\kappa \cap B^\lambda_{r}, \forall t \in [0, T_\kappa].$

   2. $\tilde{\eta}(t_0) \not\in A^\lambda_{\mu/2,R} \cap Y_\kappa \cap B^\lambda_{r}$ for some $t_0 \in [0, T_\kappa]$. If (1) holds, we have $\Psi(\tilde{\eta}(t)) = 1$ and $\|\Phi'_{\lambda,R}(\tilde{\eta}(t))\| \geq \tilde{d}_\lambda$ for all $t \in [0, T_\kappa]$. Therefore, by (2.14),

   $$\Phi_{\lambda,R}(\tilde{\eta}(T_*)) = \Phi_{\lambda,R}(u) + \int_0^{T_*} \frac{d}{ds} \Phi_{\lambda,R}(\tilde{\eta}(s))ds$$
From the definition of $K_*$, we have
\[
\begin{align*}
\|\tilde{\eta}(t) - \hat{\eta}(t_1)\|_{\Lambda, R} & \geq \frac{1}{K_*} |\Phi_{\Lambda, R}(\tilde{\eta}(t)) - \Phi_{\Lambda, R}(\hat{\eta}(t_1))| \\
& \geq \frac{1}{K_*} \left( |\Phi_{\Lambda, R}(\tilde{\eta}(t_2)) - c_{j_0}| - |\Phi_{\Lambda, R}(\hat{\eta}(t_1)) - c_{j_0}| \right) \\
& \geq \frac{1}{2K_*} \mu.
\end{align*}
\]

By the mean value theorem and $t_2 - t_1 \geq \frac{1}{2K_*/\mu}$, we find
\[
\Phi_{\Lambda, R}(\tilde{\eta}(T_*)) = \Phi_{\Lambda, R}(u) + \int_{0}^{T_*} \frac{d}{ds} \Phi_{\Lambda, R}(\tilde{\eta}(s)) ds \\
\leq \Phi_{\Lambda, R}(u) - \int_{0}^{T_*} \Psi(\tilde{\eta}(s)) \|\Phi'_{\Lambda, R}(\tilde{\eta}(s))\| ds \\
\leq c_T - \int_{t_1}^{t_2} \sigma_0 ds \\
= c_T - \sigma_0 (t_2 - t_1) \\
\leq c_T - \frac{\sigma_0 \mu}{2K_*},
\]
which proves (2.16).

Fixing $\tilde{\eta}(t) = \eta(T_*, \gamma_0(t))$, we have $\tilde{\eta}(t) \in \mathcal{Y}_{2\kappa}$, so $\tilde{\eta}(t)|_{\Omega'_j} \neq 0$ for all $j \in \Gamma$. Thus, $\tilde{\eta} \in \Gamma_*$ and
\[
b_{\Lambda, R, \Gamma} \leq \max_{s \in [t_1/T_*, t_2/T_*]} \Phi_{\Lambda, R}(\tilde{\eta}(s)) \leq \max \left\{ m_0, c_T - \frac{\sigma_0 \mu}{2K_*} \right\} < c_T.
\]
However, by Corollary 2.15, $b_{\Lambda, R, \Gamma} \rightarrow c_T$ as $\Lambda \rightarrow \infty$ uniformly holds for $R > 0$ large enough, which leads to a contradiction.

Thus, we can conclude that $\Phi_{\Lambda, R}$ has a critical point $u_{\Lambda, R} \in A_\mu^\Lambda$ for $\Lambda > 0$ and $R > 0$ large enough.  \[\square\]
3 Proof of Theorem 1.2

From Proposition 2.17, for $\mu \in (0, \mu^*)$ and $\Lambda > 0$, there exists a positive solution $u_{\lambda,R}$ for the problem $(M_{\lambda,R})$ satisfying $u_{\lambda,R} \in A_{\mu,R}^{\lambda} \cap \Phi_{\lambda,R}^{cr} \cap B_{\lambda+1}$ for all $\lambda \geq \Lambda$, and $R \geq R^*$. Now we shall fix $\lambda \geq \Lambda_*$ and take a sequence $R_n \to +\infty$. Thereby, we have a solution $u_{\lambda,n} = u_{\lambda,R_n}$ for $(M_{\lambda,R_n})$ with $u_{\lambda,n} \in A_{\mu,R_n}^{\lambda} \cap \Phi_{\lambda,R_n}^{cr} \cap B_{\lambda+1}^R, \ \forall \, n \in \mathbb{N}.$ As $(u_{\lambda,n})$ is bounded in $H^1(\mathbb{R}^N)$, we can assume that for some $u_\lambda \in H^1(\mathbb{R}^N)$,

$$
\Phi_{\lambda,R_n}(u_{\lambda,n}) \to d \leq c_T, \\
u_{\lambda,n} \to u_\lambda \text{ in } H^1(\mathbb{R}^N), \\
u_{\lambda,n} \to u_\lambda \text{ in } L^q_{loc}(\mathbb{R}^N) \text{ for any } q \in [1,2^*)
$$

and

$$u_{\lambda,n}(x) \to u_\lambda(x) \text{ a.e. } x \in \mathbb{R}^N.$$

Recalling from Lemma 2.12 that

$$0 \leq u_{\lambda,n}(x) \leq a_0, \ \forall \, x \in \mathbb{R}^N \setminus \Omega,$$

we also have

$$0 \leq u_\lambda(x) \leq a_0, \ \forall \, x \in \mathbb{R}^N \setminus \Omega.$$

The next two lemmas play a fundamental role in the proof of Theorem 1.2. Since their proofs follow from the similar arguments in the proof of Proposition 2.10, we omit them.

Lemma 3.1. For any fixed $\zeta > 0$, there exists an $R > 0$ such that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} (|\nabla u_{\lambda,n}|^2 + (\lambda V(x) + 1)|u_{\lambda,n}|^2)dx \leq \zeta.$$

Lemma 3.2. $u_{\lambda,n} \to u_\lambda$ in $H^1(\mathbb{R}^N)$. Moreover,

$$F_1(u_{\lambda,n}) \to F_1(u_\lambda) \text{ and } F'_1(u_{\lambda,n})u_{\lambda,n} \to F'_1(u_\lambda)u_\lambda \text{ in } L^1(\mathbb{R}^N).$$

As a consequence, by setting the energy functional $\Phi_\lambda : E_\lambda \to (-\infty, +\infty]$ given by

$$\Phi_\lambda(u) = \frac{1}{2} \int ([|\nabla u|^2 + (\lambda V(x) + 1)|u|^2)dx - \frac{1}{2} \int u^2 \log u^2 dx,$$

we see that the function $u_\lambda$ is a critical point of $\Phi_\lambda$ with

$$u_\lambda \in A_\mu^\lambda = \{ u \in (Y_\infty)^{2k} : \Phi_{\lambda,\mathbb{R}^N \setminus \Omega}(u) \geq 0, ||u||_{L^2(\mathbb{R}^N \setminus \Omega)} \leq \mu, |I_{\lambda,j}(u) - c_j| \leq \mu, \forall \, j \in \Gamma \},$$

where

$$Y_\infty = \left\{ u \in E_\lambda : ||u||_{\lambda,\Omega} > \frac{\tau}{2T}, \forall \, j \in \Gamma \right\}$$

and

$$(Y_\infty)_r = \left\{ u \in E_\lambda : \inf_{v \in Y_\infty} ||u - v||_{\lambda,\Omega} \leq r, \forall \, j \in \Gamma \right\}.$$

Here, by a critical point we understand that $u_\lambda$ satisfies the inequality below:

$$\int \nabla u_\lambda \nabla (v - u_\lambda) dx + \int (\lambda V(x) + 1)u_\lambda (v - u_\lambda) dx + \int F_1(v) dx - \int F_1(u_\lambda) dx \geq \int F'_1(u_\lambda)(v - u_\lambda) dx$$

for all $v \in E_\lambda$. Hence, $u_\lambda$ satisfies the equality below:

$$\int_{\mathbb{R}^N} (\nabla u_\lambda \nabla v + \lambda V(x) u_\lambda v) dx = \int_{\mathbb{R}^N} u_\lambda v \log u_\lambda^2 dx \text{ for all } v \in C_0^\infty(\mathbb{R}^N).$$
Now, we are ready to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Now, given $\lambda_n \to +\infty$ and $\mu_n \in (0, \mu^*)$ with $\mu_n \to 0$, there exists a solution $u_n \in A_{\Omega_n}^\lambda$ of the problem $(P_\lambda)$ with $\lambda = \lambda_n$. Therefore, $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$ and satisfies

(a) $\|\Phi'_{\lambda_n}(u_n)\| = 0$, $\forall n \in \mathbb{N}$;
(b) $\|u_n\|_{\lambda_n, \mathbb{R}^N \setminus \Omega} \to 0$;
(c) $\Phi_{\lambda_n}(u_n) \to d \leq c_\Gamma$.

Here,

$$\|\Phi'_{\lambda}(u)\| = \sup \{ \langle \Phi'_{\lambda}(u), z \rangle : z \in H^1(\mathbb{R}^N) \text{ and } \|z\|_{\lambda} \leq 1 \}.$$ 

By arguing as in Proposition 2.10, there exists a $u \in H^1(\mathbb{R}^N)$ such that $u_{\lambda_n} \to u$ strongly in $H^1(\mathbb{R}^N)$, and $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and $u$ is a nontrivial solution of

$$\begin{cases} -\Delta u = u \log u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \quad (P_{\infty, \Gamma})$$

so $I_\Gamma(u) \geq c_\Gamma$. On the other hand, we also know that $\Phi_{\lambda_n}(u_{\lambda_n}) \to I_\Gamma(u)$, so $I_\Gamma(u) = d$ and $d \geq c_\Gamma$. Since $d \leq c_\Gamma$, it yields that $I_\Gamma(u) = c_\Gamma$, showing that $u$ is a least energy solution for $(P_{\infty, \Gamma})$. This completes the proof of the theorem. \qed

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References

1 Alves C O. Existence of multi-bump solutions for a class of quasilinear problems. Adv Nonlinear Stud, 2006, 6: 491–509
2 Alves C O, de Morais Filho D C. Existence and concentration of positive solutions for a Schrödinger logarithmic equation. Z Angew Math Phys, 2018, 69: 144
3 Alves C O, de Morais Filho D C, Figueiredo G M. On concentration of solution to a Schrödinger logarithmic equation with deepening potential well. Math Methods Appl Sci, 2019, 42: 4862–4875
4 Alves C O, de Morais Filho D C, Souto M A S. Multiplicity of positive solutions for a class of problems with critical growth in $\mathbb{R}^N$. Proc Edinb Math Soc (2), 2009, 52: 1–21
5 Alves C O, Ji C. Existence and concentration of positive solutions for a logarithmic Schrödinger equation via penalization method. Calc Var Partial Differential Equations, 2020, 59: 21
6 Alves C O, Ji C. Multiple positive solutions for a Schrödinger logarithmic equation. Discrete Contin Dyn Syst, 2020, 40: 2671–2685
7 Alves C O, Ji C. Existence of a positive solution for a logarithmic Schrödinger equation with saddle-like potential. Manuscripta Math, 2021, 164: 555–575
8 Alves C O, Nóbrega A B. Existence of multi-bump solutions for a class of elliptic problems involving the biharmonic operator. Monats Math, 2017, 183: 35–60
9 Alves C O, Nóbrega A B, Yang M B. Multi-bump solutions for Choquard equation with deepening potential well. Calc Var Partial Differential Equations, 2016, 55: 48
10 Alves C O, Souto M A S. Multiplicity of positive solutions for a class of problems with exponential critical growth in $\mathbb{R}^2$. J Differential Equations, 2008, 244: 1501–1520
11 Ambrosetti A, Rabinowitz P H. Dual variational methods in critical point theory and applications. J Funct Anal, 1973, 14: 349–381
12 Bartsch T, Wang Z-Q. Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$. Comm Partial Differential Equations, 1995, 20: 1725–1741
13 Bartsch T, Wang Z-Q. Multiple positive solutions for a nonlinear Schrödinger equation. Z Angew Math Phys, 2000, 51: 366–384
14 d’Avenia P, Montefusco E, Squassina M. On the logarithmic Schrödinger equation. Commun Contemp Math, 2014, 16: 1350032
15 d’Avenia P, Squassina M, Zenari M. Fractional logarithmic Schrödinger equations. Math Methods Appl Sci, 2015, 38: 5207–5216
16 Degiovanni M, Zani S. Multiple solutions of semilinear elliptic equations with one-sided growth conditions. Math Comput Modelling, 2000, 32: 1377–1393
17 del Pino M, Dolbeault J. The optimal Euclidean $L^p$-Sobolev logarithmic inequality. J Funct Anal, 2003, 197: 151–161
18 del Pino M, Felmer P L. Local mountain passes for semilinear elliptic problems in unbounded domains. Calc Var Partial Differential Equations, 1996, 4: 121–137
19 Ding Y H, Tanaka K. Multiplicity of positive solutions of a nonlinear Schrödinger equation. Manuscripta Math, 2003, 112: 109–135
20 Guo Y X, Tang Z W. Multi-bump bound state solutions for the quasilinear Schrödinger equation with critical frequency. Pacific J Math, 2014, 270: 49–77
21 Ji C, Szulkin A. A logarithmic Schrödinger equation with asymptotic conditions on the potential. J Math Anal Appl, 2016, 437: 241–254
22 Liang S H, Zhang J H. Multi-bump solutions for a class of Kirchhoff type problems with critical growth in $\mathbb{R}^N$. Topol Methods Nonlinear Anal, 2016, 48: 71–101
23 Miranda C. Un’osservazione su un teorema di Brouwer. Boll Unione Mat Ital (9), 1940, 3: 5–7
24 Squassina M, Szulkin A. Multiple solutions to logarithmic Schrödinger equations with periodic potential. Calc Var Partial Differential Equations, 2015, 54: 585–597
25 Squassina M, Szulkin A. Erratum to: Multiple solutions to logarithmic Schrödinger equations with periodic potential. Calc Var Partial Differential Equations, 2015, 56: 56
26 Tanaka K, Zhang C X. Multi-bump solutions for logarithmic Schrödinger equations. Calc Var Partial Differential Equations, 2017, 56: 56
27 Vázquez J L. A strong maximum principle for some quasilinear elliptic equations. Appl Math Optim, 1984, 12: 191–201
28 Wang Z-Q, Zhang C X. Convergence from power-law to logarithm-law in nonlinear scalar field equations. Arch Ration Mech Anal, 2019, 231: 45–61
29 Willem M. Minimax Theorems. Boston: Birkhäuser, 1996
30 Zhang C X, Zhang X. Bound states for logarithmic Schrödinger equations with potentials unbounded below. Calc Var Partial Differential Equations, 2020, 59: 23
31 Zloshchastiev K G. Logarithmic nonlinearity in the theories of quantum gravity: Origin of time and observational consequences. Gravit Cosmol, 2010, 16: 288–297