RIGIDITY OF SPECIAL ANOSOV ENDO MORPHISMS OF TORUS

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Abstract. We investigated rigidity for Anosov endomorphisms. We get smooth conjugacy between non necessarily special Anosov endomorphisms in a particular case. Among other results, we prove that an strongly special $C^\infty$–Anosov endomorphism of $T^2$ and its linearization are smoothly conjugated since they have the same periodic data. Moreover, assuming that for an strongly special $C^\infty$–Anosov endomorphism of $T^2$ every point is regular (in Oseledec’s Theorem sense), then we get again smooth conjugacy with the linearization. In this work, we also present a result of local rigidity of linear Anosov endomorphisms of $d$–torus, where $d \geq 3$.

1. Introduction

In 1970s, the works [18] and [12] generalized the notion of Anosov diffeomorphism for non invertible maps, introducing the notion of Anosov endomorphism. We consider $M$ a $C^\infty$–closed manifold.

Definition 1.1. [18] Let $f : M \to M$ be a $C^1$ local diffeomorphism. We say that $f$ is an Anosov endomorphism if there are constants $C > 0$ and $\lambda > 1$, such that, for every $(x_n)_{n \in \mathbb{Z}}$ an $f$–orbit there is a splitting

$$T_{x_i}M = E^s_{x_i} \oplus E^u_{x_i}, \forall i \in \mathbb{Z},$$

which is preserved by $Df$ and for all $n > 0$ we have

$$\|Df^n(x_i) \cdot v\| \geq C^{-1} \lambda^n \|v\|, \text{ for every } v \in E^u_{x_i} \text{ and for any } i \in \mathbb{Z},$$

$$\|Df^n(x_i) \cdot v\| \leq C\lambda^{-n} \|v\|, \text{ for every } v \in E^s_{x_i} \text{ and for any } i \in \mathbb{Z}.$$  

We denote by $M^f$ the space of all $f$–orbits $\hat{x} = (x_n)_{n \in \mathbb{Z}}$, endowed with me metric

$$\tilde{d}(\hat{x}, \hat{y}) = \sum_{i \in \mathbb{Z}} \frac{d(x_i, y_i)}{2^n},$$

where $d$ denotes the riemannian metric on $M$ and $\hat{x} = (x_n)_{n \in \mathbb{Z}}, \hat{y} = (y_n)_{n \in \mathbb{Z}}$, two $f$–orbits. We denote by $p : M^f \to M$, the natural projection

$$p((x_n)_{n \in \mathbb{Z}}) = x_0.$$  

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The space \((M^f, \tilde{d})\) is compact, moreover \(f\) induces a continuous map \(\tilde{f} : M^f \to M^f\), given by the shift

\[
\tilde{f}((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}.
\]

Anosov endomorphisms can be defined in an equivalent way ([12]).

**Definition 1.2.** ([12]) A \(C^1\) local diffeomorphism \(f : M \to M\) is said an Anosov endomorphism if \(Df\) contracts uniformly a \(Df\)–invariant and continuous sub-bundle \(E^s \subset TM\) into itself and the action of \(Df\) on the quotient \(TM/E^s\) is uniformly expanding.

**Proposition 1.3** ([12]). A local diffeomorphism \(f : M \to M\) is an Anosov endomorphism of \(M\) if and only if the lift \(\tilde{f} : \tilde{M} \to \tilde{M}\) is an Anosov diffeomorphism of \(\tilde{M}\), the universal cover of \(M\).

Sakai, in [21] proved that, in fact, the definitions 1.1 and 1.2 are equivalent.

The definition 1.2 will be particularly important for the proof of Theorem 1.7.

An advantage to work with the definition given in [12] is that in \(M\) we can construct invariant foliations \(F^s_f\) and \(F^u_f\). In Theorem 1.7 we will borrow the transverse structure of \(F^s_f\) and \(F^u_f\).

Let \(f : M \to M\) be a \(C^r\)–Anosov endomorphism with \(r \geq 1\), it is know that \(E^s\) and \(E^u\) are integrable to \(C^r\)–leaves \(W^s_f(\tilde{x})\) and \(W^u_f(\tilde{x})\), which are \(C^r\)–submanifolds, such that

1. \(W^s_f(x) = \{y \in M \mid \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0\}\),
2. \(W^u_f(x) = \{y \in M \mid \exists \tilde{y} \in M^f \text{ such that } y_0 = \tilde{y} \text{ and } \lim_{n \to +\infty} d(x_{-n}, y_{-n}) = 0\}\).

The leaves \(W^s_f(\tilde{x})\) and \(W^u_f(\tilde{x})\) vary \(C^1\–continuously with \(\tilde{x}\), see Theorem 2.5 of [18].

Given an Anosov endomorphism let \(E^u(\tilde{x})\) denotes the bundle \(E^u_{x_0}\). An Anosov endomorphism for which \(E^u(\tilde{x})\) just depends on \(x_0\) (unique unstable direction for each point) is called special Anosov endomorphism. A linear Anosov endomorphism of torus is an example of special Anosov endomorphism. Of course, when \(f\) is an special Anosov endomorphism we have \(W^u_f(\tilde{x}) = W^u_f(\tilde{y})\), for any \(\tilde{x}, \tilde{y}\) such that \(x_0 = y_0\). So makes sense denote in this case \(W^u_f(\tilde{x}) = W^u_f(x_0)\).

A contrast between Anosov diffeomorphisms and Anosov endomorphisms is the non-structural stability of the latter. Indeed, \(C^1\–close to any linear Anosov endomorphism \(A\) of torus, Przytycki [18] constructed Anosov endomorphism which has infinitely many unstable direction for some positive orbit and consequently he showed that \(A\) is not structurally stable. However, it is curious to observe that the topological entropy is locally constant among Anosov endomorphisms. Indeed, take the lift of Anosov endomorphism to the inverse limit space (see preliminaries for the definition). At the level of inverse limit
space, two nearby Anosov endomorphisms are conjugate ([18], [2]) and lifting to inverse limit space does not change the entropy.

**Definition 1.4.** A continuous surjection \( f : \mathbb{T}^n \to \mathbb{T}^n \) is said strongly special Anosov endomorphism map if \( f \) is an special Anosov endomorphism which is not injective and for each point \( x \in \mathbb{T}^n \), the stable leaf \( W^s_f(x) \) is dense in \( \mathbb{T}^n \).

The celebrated theory due Franks, Manning and Newhouse asserts that given \( f : \mathbb{T}^n \to \mathbb{T}^n \) an Anosov diffeomorphism with codimension one, then \( f \) is conjugated with its linearization \( A \). It means that there is a homeomorphism \( h : \mathbb{T}^n \to \mathbb{T}^n \) such that \( h \circ A = f \circ h \).

We understand the linearization \( A \) of \( f \) being the action on \( \mathbb{T}^n \), of the matrix with integer entries \( A = f^* \), the action of \( f \) in \( \Pi_1(\mathbb{T}^n) = \mathbb{Z}^n \).

In 1990 years, R. de la Llave in several works characterized the smooth Anosov diffeomorphisms \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) which are differentiable conjugated with its linearization \( A \). If fact, the condition is known by same periodic data between corresponding points, it means that if \( p \) and \( q \) are periodic points for \( A \) and \( f \) respectively, with period \( n \) and the conjugacy \( h \) is such that \( h(p) = q \), then the Lyapunov exponents of \( p \) and \( q \) coincides, i.e,

\[
\lim_{n \to +\infty} \frac{1}{n} \log(Df^n(q)|E^s_f(q)) = \lim_{n \to +\infty} \frac{1}{n} \log(DA^n(q)|E^s_A(p)), \quad * \in s, u.
\]

Recent advances are made for Anosov diffeomorphisms of \( \mathbb{T}^3 \), see [7]. For \( n \geq 4 \), there are counterexamples, see [11].

Of course a non special Anosov endomorphism can not be conjugated with its linearization. For strongly special Anosov endomorphism we have the Theorem due to [1], that we present shortly.

**Theorem 1.5.** Every strongly special Anosov endomorphism \( f : \mathbb{T}^n \to \mathbb{T}^n \) is conjugated with its linearization.

From this point, a natural question is to find conditions for a \( C^\infty \) strongly special Anosov endomorphism and its linearization to be smooth conjugated. Before we list our main results let us we introduce.

**Definition 1.6 (Hypothesis S).** We say that an Anosov endomorphism \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) satisfies Hypothesis S, if given \( \varepsilon > 0 \), there is \( K > 0 \), such that every stable arc with length bigger than \( K \) is \( \varepsilon \)-dense.

We can prove the following.

**Theorem 1.7.** Let \( f, g : \mathbb{T}^2 \to \mathbb{T}^2 \) be two \( C^\infty \) Anosov endomorphism with degree \( k \geq 1 \), such that \( f_* = g_* = A \), with \( \dim(E^u_A) = \dim(E^s_A) = 1 \), have dense stable leaves. Suppose that the endomorphisms \( f \) and \( g \) preserve measures \( m_f \) and \( m_g \) which are
equivalent to volume $m$ and $\lambda_{m,f}^u = \lambda_{m,g}^u$, where $\lambda_{m,f}^u$ denotes the unstable Lyapunov exponent of $f$. If $f$ and $g$ are absolutely continuous conjugated (the conjugacy $h$ and its inverse $h^{-1}$ are absolutely continuous functions) and $f$ satisfies the Hypothesis S, then they are smooth conjugated.

For the next result we define.

**Definition 1.8.** Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov endomorphisms such that there is a $Df$–invariant splitting

$$T_{x_k} \mathbb{T}^d = \bigoplus_{i=1}^n E_i^s(x_k : x), k \in \mathbb{Z},$$

for any orbit $x = (x_k)_{k \in \mathbb{Z}}$. We say that $f$ is $E_i^s$–special if for any orbits $x = (x_k)_{k \in \mathbb{Z}}$ and $y = (y_k)_{k \in \mathbb{Z}}$, such that $x_0 = y_0$, we have $E_i^s(x_0 : x) = E_i^s(y_0 : y)$.

**Theorem 1.9.** Let $A : \mathbb{T}^d \to \mathbb{T}^d$, $d \geq 3$, be a linear Anosov endomorphisms, such that $\dim E_i^s_A \geq 1, \dim E_i^u_A \geq 1$. Suppose that $A$ is irreducible over $\mathbb{Q}$ and it has simple real spectrum, such that $E_i^s_A = E_1^s \oplus \ldots \oplus E_k^s$ and $E_i^u_A = E_1^u \oplus \ldots \oplus E_n^u$. Consider $f : \mathbb{T}^d \to \mathbb{T}^d$ an smooth Anosov endomorphisms $C^1$–close to $A$, such that $f$ is $E_i^s$ and $E_j^s$ special, for $i = 1, \ldots, k$ and $j = 1, \ldots, n$, the natural continuations of invariant sub bundles of $A$. Suppose that each leaf of the foliations $\mathcal{F}_i^s$ and $\mathcal{F}_i^u$ tangent to $E_i^s$, $i = 1, \ldots, k$ and $E_j^u$, $j = 1, \ldots, n$ is non compact. If for any periodic point of $f$ we have coincidence of Lyapunov exponents $\lambda_i^s = \lambda_i^s$, $i = 1, \ldots, k$ and $\lambda_j^u = \lambda_j^u$, $i = 1, \ldots, n$, then $f$ and $A$ are $C^1$–conjugated.

As consequences of the previous theorems we have.

**Theorem 1.10.** Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a $C^\infty$ strongly special Anosov endomorphism and $A : \mathbb{T}^2 \to \mathbb{T}^2$, its linearization. If the corresponding periodic points of $f$ and $A$ have the same Lyapunov exponents, then $f$ and $A$ are smooth conjugated.

**Theorem 1.11.** Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a $C^\infty$ strongly special Anosov endomorphism. Suppose that for any $x \in \mathbb{T}^2$ are defined all Lyapunov exponents. Then $f$ is smooth conjugated with its linearization $A$.

2. Comments on the proofs

In the proof of Theorem 1.7 since we are not supposing special Anosov endomorphism, we borrow the transverse foliations structure of stable and unstable manifolds of the lift of $f$ on $\mathbb{R}^2$. We use the well established S.R.B theory \cite{20,16} for endomorphisms to after apply an O.D.E argument, similar to one in \cite{11}. Finally, applying Journé’s Lemma \cite{8}, we conclude that $h$ is smooth.
In the proof of Theorem 1.9 we use cohomological equations and Livsic’s Theorem to construct, via conformal metrics on leaves, a function holding the same definition of the conjugacy. We conclude that the conjugacy applies invariant leaves of $A$ to corresponding invariant leaves of $f$. In fact we don’t use the fact that the conjugacy $h$ send invariant leaves of $A$ to corresponding invariant leaves of $f$, this fact is consequence of our construction. Finally, by Livsic’s Theorem, the conjugacy $h$ is $C^{1+\alpha}$ for some $\alpha > 0$. We observe that in [6 7] the authors used a construction of S.R.B measures with absolutely continuous densities on intermediate foliations, in non invertible setting it can be more difficult, mainly when we work with intermediate stable leaves. In the case coincidence periodic data with linear, the technique presented in Theorem 1.9 holds for diffeomorphism context.

In Theorem 1.9 we have $h$ is $C^1$, so by direct application of the method of Theorem 1.9 joint with the result in Theorem 1.7 we get Theorem 1.10.

To prove Theorem 1.11 we use specification to ensure that $f$ has constant periodic data. Using again S.R.B theory, Ruelle’s inequality and Pesin formula we get that $f$ and its linearization $A$ have the same periodic data, so we finalize by applying Theorem 1.10.

3. Preliminaries on S.R.B measures for endomorphisms

At this moment we need work with the concept of S.R.B measures for endomorphisms. In fact S.R.B measures play an important role in ergodic theory of differentiable dynamical systems. For $C^{1+\alpha}$ systems these measures can be characterized such as ones that realize the Pesin Formula or equivalently the measures which the conditional measures are absolutely continuous w.r.t. Lebesgue restricted to local stable/unstable manifolds. We go to focused our attention in the endomorphisms case. Before to proceed with the proof let us to give important and useful definitions and results concerning S.R.B measures for endomorphisms.

First let us recall an important result.

**Theorem 3.1 ([19]).** Let $(M,d)$ be a compact metric space and $f : M \rightarrow M$ a continuous map. If $\mu$ is an $f$–invariant borelian probability measure, the exist a unique $\bar{f}$–invariant borelian probability measure $\bar{\mu}$ on $M^f$, such that $\mu(B) = \bar{\mu}(p^{-1}(B))$.

**Definition 3.2.** A measurable partition $\eta$ of $M^f$ is said to be subordinate to $W^u$–manifolds of a system $(f, \mu)$ if for $\bar{\mu}$-a.e. $\bar{x} \in M^f$, the atom $\eta(\bar{x})$, containing $\bar{x}$, has the following properties:

1. $p|\eta(\bar{x}) \rightarrow p(\eta(\bar{x}))$ is bijective;
2. There exists a $k(\bar{x})$–dimensional $C^1$–embedded submanifold $W(\bar{x})$ of $M$ such that $W(\bar{x}) \subset W^u(\bar{x})$, $p(\eta(\bar{x})) \subset W(\bar{x})$
and $p(\eta(\tilde{x}))$ contains an open neighborhood of $x_0$ in $W(\tilde{x})$. This neighborhood being taken in the topology of $W(\tilde{x})$ as a submanifold of $M$.

**Definition 3.3.** Let $f : M \to M$ be a $C^2$–endomorphism preserving an invariant borelian probability $\nu$. We say that $\nu$ has S.R.B property if for every measurable partition $\eta$ of $M^r$ subordinate to $W^u$–manifolds of $f$ with respect to $\nu$, we have $p(\nu(\tilde{x})) \ll m^u_{p(\tilde{x})}$ for $\tilde{x}$–a.e. $\tilde{x}$, where $\{\nu(\eta(\tilde{x}))\}_{\tilde{x} \in M^r}$ is a canonical system of conditional measures of $\tilde{\nu}$ associated with $\eta$ and $m^u_{p(\eta(\tilde{x}))}$ is the Lebesgue measure on $W(\tilde{x})$ induced by its inherited riemannian metric as a submanifold of $M$.

In the case of above definition, if we denote by $\rho^u_f$ the densities of conditional measures $\nu(\eta(\tilde{x}))$, we have

$$\rho^u_f(\tilde{y}) = \frac{\Delta(\tilde{x}, \tilde{y})}{L(\tilde{x})}, \quad (3.1)$$

for each $\tilde{y} \in \eta(\tilde{x})$, where

$$\Delta(\tilde{x}, \tilde{y}) = \prod_{k=1}^{\infty} \frac{J^u_f(x_{-k})}{J^u_f(y_{-k})}, \tilde{x} = (x_k)_{k \in \mathbb{Z}}, \tilde{y} = (y_k)_{k \in \mathbb{Z}}$$

and

$$L(\tilde{x}) = \int_{\eta(\tilde{x})} \Delta(\tilde{x}, \tilde{y}) L(\tilde{x}) d\tilde{m}^u_{\eta(\tilde{x})}.$$

The measure $\tilde{m}^u_{\eta(\tilde{x})}$ is such that $p(\tilde{m}^u_{\eta(\tilde{x})}(B)) = m^u_{p(\eta(\tilde{x}))}(B)$. Therefore

$$p(\nu(\tilde{x})) \ll m^u_{p(\eta(\tilde{x}))},$$

and

$$\rho^u_f(y) = \frac{\Delta(\tilde{x}, y)}{L(\tilde{x})}, \quad y \in p(\eta(\tilde{x})).$$

**Theorem 3.4.** [15] Let $f : M \to M$ be a $C^2$ endomorphism and $\mu$ an $f$–invariant Borel probability measure on $M$. If $\mu \ll m$, then there holds Pesin’s formula

$$h_\mu(f) = \int_M \sum \lambda^i(x)^+ m_i(x) d\mu. \quad (3.2)$$

**Theorem 3.5** [20]. Let $f$ be a $C^2$ endomorphism on $M$ with an invariant Borel probability measure $\mu$ such that $\log(||f(x)||) \in L^1(M, \mu)$. Then the entropy formula

$$h_\mu(f) = \int_M \sum \lambda^i(x)^+ m_i(x) d\mu \quad (3.3)$$

holds if and only if $\mu$ has S.R.B property.

For Axiom A endomorphisms, including Anosov endomorphims we have.
Theorem 3.6 ([19]). For an Axiom A attractor, in particular Anosov endomorphisms, there is a unique invariant measure which is characterized by each of the following properties:

1. The metric entropy is equal to the sum of positive Lyapunov exponents.
2. The conditional measures of the invariant measure on unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.
3. Lebesgue almost every point in an open neighborhood of the attractor is generic to this measure.

There are analogous formulations concerning subordinate partition with respect to stable manifolds, see [16]. Since the assumption $H'$ of [16] holds for Anosov endomorphisms we have.

Theorem 3.7. Consider $f : M \to M$ an Anosov endomorphism preserving a borelian probability measure $\mu$. The measure $\mu$ have absolutely continuous conditional measures on the stable manifolds for every measurable partition $\eta$ subordinate to the $W^s$-manifolds of $(f, \mu)$ if and only if

$$h_\mu(f) = F_\mu(f) = \int_M \sum \lambda^i(x) m_i(x) d\mu. \quad (3.4)$$

Here $F_\mu(f)$ denotes the conditional entropy $H_\mu(\xi / f^{-1} \xi)$, where $\xi$ is the partition of $M$ into single points.

In the setting of the previous Theorem, if $(f, \mu)$ satisfies the Stable Pesin Formula (3.4) then for a given subordinate partition $\eta$, with respect to stable manifolds, we have

$$\mu_\eta(x) \ll m^s_\eta(x),$$

for $\mu$ a.e $x \in M$. Moreover

$$\rho^s_\eta(x) = \frac{\Delta_s(x, y)}{\int_{\eta(x)} \Delta_s(x, y) dm^s_\eta(x)} , \forall y \in \eta(x). \quad (3.5)$$

Here $\Delta_s(x, y) = \prod_{k=0}^\infty \frac{f^k(f^k(x))}{f^k(f^k(y))} \cdot \frac{f^k(f^k(x))}{f^k(f^k(y))}$.

Lemma 3.8. For a $C^k$, $k \geq 2$, Anosov endomorphism, the conditional measures of stable and unstable S.R.B measures restricted to stable and unstable leaves respectively are $C^{k-1}$. In particular, if $f$ is smooth, then the conditional measures are smooth.

Up to minor adjusts, the proof is essentially the same of Corollary 4.4 of [11].

4. Proof of Theorem 1.7

Lemma 4.1. Consider $A : \mathbb{T}^2 \to \mathbb{T}^2$ the linearization of $f$ and $g$ as in Theorem 1.7 then the matrix $A$ has irrational eigenvalues. Particularly $A$ has a positive orbit dense.
Proof. Let \( \beta^s, \beta^u \) the eigenvalues of \( A \) suppose that \( 0 < |\beta^s| < 1 < |\beta^u| \). Since \( A \) has dense stable leaves, it implies that \( \beta^s \) is irrational. If it was possible \( \beta^u \) be rational, then since \( \beta^u \cdot \beta^s = \det(A) = k \geq 1 \), an integer number, it would imply \( \beta^s \) a rational number, that doesn’t occur.

\[ \square \]

\textbf{Lemma 4.2.} Consider \( f, g \) and \( h \) as in Theorem 1.7 then \( h_*(m_f) = m_g \).

Proof. Since \( h \) is absolutely continuous we have \( \nu = h_*(m_f) \) is an absolutely continuous invariant probability measure for \( g \), so the Pesin formula holds for \( (g, \nu) \). Since \( g \) is Axiom A on \( \mathbb{T}^2 \), then there is a unique measure satisfying the Pesin formula, see [19]. We conclude \( \nu = m_g \). \( \square \)

\textbf{Lemma 4.3.} Given a choice of unstable/stable foliation of an small neighborhood \( Q \) of an arbitrary point \( z \in \mathbb{T}^2 \), the conjugacy \( h \) is smooth restricted to each unstable/stable leaf in \( Q \).

Proof. Pick \( z_0 \in \mathbb{T}^2 \) and consider \( \tilde{z}_0 \in \mathbb{R}^2 \), such that it projects on \( z_0 \). Let \( V \) be an small box containing \( \tilde{z}_0 \) which is \( s \) and \( u \) foliated. Consider \( Q = p(V) \) and \( \tilde{z} = p(O(\tilde{z}_0)) \). Denote by \( W^u(f, \tilde{z}, Q) \) the connected component of the unstable leaf of \( f \) corresponding the orbit \( \tilde{z} \) contained in \( Q \). Since \( \tilde{f} \) and \( \tilde{A} \) are conjugated at inverse limit level we can choose an \( f \)-orbit \( \tilde{x} = (x_n)_{n \in \mathbb{Z}} \) is such that \( (x_n)_{n \geq 0} \) is dense and there is an atom \( \eta(\tilde{x}) \), for which is defined the \( C^\infty \)-conditional measure \( \rho^u_{\eta} \), for a suitable subordinate partition w.r.t unstable leaves. Since \( h_*(m_f) = m_g \), then \( h \) sends conditional measures of \( (f, m_f) \) in conditional measures of \( (g, m_g) \). Since these measures are equivalent to riemannian measures of unstable leaves, so \( h \) sends null sets of \( p(\eta(\tilde{x})) \) in null sets of \( p(\eta(h(\tilde{x}))) \) with respect to riemannian measures of unstable leaves, where \( h \) is the conjugacy at level of limit inverse space between \( \tilde{f} \) and \( \tilde{g} \).

Consider \( B^u_{x_0} \subset \eta(\tilde{x}) \) an small open unstable arc.

\[
\int_{B^u_{x_0}} \rho^u_f(y)dy = \int_{h(B^u_{x_0})} \rho^u_{\tilde{g}}(y)dy = \int_{B^u_{x_0}} \rho^u_{\tilde{g}}(h(y))h'(y)dy,
\]

therefore solving the O.D.E.

\[
x' = \frac{\rho^u_f(t)}{\rho^u_{\tilde{g}}(x)}, x(x_0) = h(x_0),
\]

we find \( h \) is \( C^\infty \) on \( B^u_{x_0} \).

Since \( O(x_0) \) is dense, then there exist a sequence \( n_k, k = 1, 2, \ldots \) such that \( f^{n_k}(\tilde{x}) \rightarrow \tilde{z} \). Since \( f \) is local diffeomorphism, taking forward iterated for \( f \), we conclude that \( h \) is \( C^\infty \) on each arc \( f^n(B^u_{x_0}) \). By continuous dependence of \( W^u \)-leaves, we have

\[
f^{n_k}(B^u_{x_0}) \cap Q \rightarrow W^u(f, \tilde{z}, Q)
\]
in $C^1$-topology, see Theorem 2.5 of [13].

Note that every point $y$ in a connected component of $f^n(B^u_y) \cap Q$ is such that there is a small open ball $B^u_y$ centered in $y$, such that $B^u_y \subset \eta(y)$. Since $\eta$ is a partition, then points in different atoms are contained in disjointed balls. So by connectedness, the connected component of $f^n(B^u_y) \cap Q$ that contains $y$ is contained in an atom of the subordinated partition $\eta$. We denote the connected component of $y \in f^n(B^u_y) \cap Q$ by $W^u_{f, n_k}(y)$.

We can normalize the conditional measures such that
\[
\int_{W^u_{f,n_k}(y)} c_k \cdot \rho^u_{f}(t) dVol_{W^u_{f,n_k}(y)} = 1,
\]
since $h.(\rho^u_{f}(t) dVol_{W^u_{f,n_k}}) = \rho^u_{g}(t) dVol_{W^u_{g,n_k}(h(y))}$, then $h$ send normalized conditional measures into normalized conditional measures. Moreover, once $W^u_{f,n_k}(y)$ have bounded length and $y \mapsto Vol_{W^u_{f,n_k}(y)}$ is continuous, the sequence of positive numbers $c_k$ is bounded away from zero.

For any $y \in W^u_{f,n_k}(y_0)$ holds
\[
\rho^u_{f}(y) = c \cdot \Delta^u(y_0, y),
\]
for some constant $c$. The same constant $c$ is such that
\[
\rho^u_{g}(h(y)) = c \cdot \Delta^u(h(y_0), h(y)).
\]

In this way, by relation (3.1), we get $h$ satisfies the following O.D.E,
\[
x' = \frac{\Delta^u(y_0, t)}{\Delta^u(h(y_0), x)} x(y_0) = h(x_0),
\]
for each pair of connected component $W^u_{f,n_k}(y)$ and $W^u_{g,n_k}(h(y))$.

Denoting by $h_k$ the solution of the above equation, we note that the solution $h_k$ is smooth. The map $h_k$ is the restriction of the conjugacy $h$ on $W^u_{f,n_k}(y)$. By Lemma 4.3 of [11], for each component $W^u_{f,n_k}(y)$ we have a collection $\{h_k : W^u_{f,n_k}(y) \to W^u_{g,n_k}(h(y))\}_{k=1}^{\infty}$, with is uniform bounded as well the collection of their derivatives of order $r = 1, 2, \ldots$ By an Arzela-Ascoli argument type applied to a sequence $h_k$ and the sequence of their derivatives, we conclude that $h$ is $C^\infty$ restricted to $W^u(f, z, Q)$.

For stable leaves we use a similar argument. The stable foliation restricted to $Q$ is an absolutely continuous foliation, then for $m$–a.e. point $t \in Q$, we have that holds (3.5) for $m_s(t)$ a.e. point $y \in W^s_f(t) \cap Q$. The connected components of $f^{-n}(B^s_{x_0})$ grows exponentially. By Hypothesis S, for suitable choice of stable arcs in pre images, we have
\[
f^{-n}(B^s_{x_0}) \cap Q \to W^s(f, z, Q)
\]
in $C^1$–topology.

As in the argument for unstable leaves, via O.D.E, we get $h$ is $C^\infty$ restricted to each component of pre images $f^{-n}(B^s_{i0})$. Since the conditional measures are uniformly Lipschitz, by an Arzela-Ascoli type argument we have $h$ is $C^\infty$ restricted to $W^s(f, z, Q)$.

The same argument can be applied for any point $z \in p(V)$, with unstable and stable manifolds projected of corresponding ones from $V$. 

To finalize the proof of Theorem 1.9 we apply locally the following classic result.

**Theorem 4.4** (Journe’s Theorem). Let $F_s$ and $F_u$ two continuous and transversal foliations with uniformly smooth leaves, of some manifold. If $f$ is uniformly smooth along the leaves of $F_s$ and $F_u$, then $f$ is smooth.

We conclude that $h$ is smooth.

## 5. Proof of Theorem 1.9

In the setting of Theorem 1.9 we can consider the lifts of $f$ and $\tilde{A}$, it is possible by analogous arguments in Pesin [17], we claim that if $f$ is $C^1$–close to $A$, then at universal cover level $\tilde{f}$ admits a similar splitting $E^s_\tilde{f} = E^{s,1}_1 \oplus E^{s,1}_2 \oplus \ldots \oplus E^{s,1}_k$ and $E^u_\tilde{f} = E^{u,1}_1 \oplus E^{u,1}_2 \oplus \ldots \oplus E^{u,1}_n$. As before, define $E^{u,1}_i = E^{u,i}_1 \oplus \ldots \oplus E^{u,i}_i$ and $E^{s,1}_k = E^{s,i}_1 \oplus \ldots \oplus E^{s,i}_k$, analogously, for $j \geq i$, we define $E^{s,j}_i$ and $E^{u,j}_i$.

By [17] of each subbundle is Hölder continuous. We can take the decomposition $E^s \oplus E^{u,1}_1 \oplus E^{u,1}_{(i+1),n}$ such that it is a uniform partially hyperbolic splitting. Moreover, by [17], each $E^{u,i}_i = E^{u,i}_1 \oplus \ldots \oplus E^{u,i}_i$, is uniquely integrable to an invariant foliation $W^{u,i}_i$, with $i = 1, \ldots, n$. An analogous construction holds for stable directions. Note that $W^{u,i}_i(x) \cap W^{u,i}_i(i, n) := W^{u,i}_i(x)$ tangent to $E^{u,i}_i(x)$. The same for stable directions. Define $f$–invariant directions $E^{u,i}_i(x) = D\pi(y) \cdot E^{u,i}_i(y)$, for any $y \in \mathbb{R}^d$ such that $\pi(y) = x$. The same for stable directions. By hypothesis, if $x - y \in \mathbb{Z}^d$ then $W^{u,i}_i(\pi(x)) = \pi(W^{u,i}_i(x)) = \pi(W^{u,i}_i(y))$, the same for stable directions. By assumption of Theorem 1.9 each leaf $W^{u,i}_i(x)$, $W^{u,f}_i(x)$ are non compact leaves.

In the setting of Theorem 1.9 we can suppose that the eigenvalues of $A$ satisfying $0 < |\beta^1_i| < \ldots < |\beta^k_i| < 1 < |\beta^1_i| < \ldots < |\beta^u_i|$. The Lyapunov exponents of $A$, are $\lambda^i_j(A) = \log(|\beta^i_j|), i = 1, \ldots, k$ and $\lambda^u_i(A) = \log(|\beta^u_i|), i = 1, \ldots, n$. For $f$ we denote by $\lambda^i_j(x, f)$ the Lyapunov exponent of $f$ at $x$ in the direction
implies that $f \in E_i$, $i = 1, \ldots, n$ and by $\lambda^i(x, f)$ the Lyapunov exponent of $f$ at $x$ in the direction $E^{u,f}_i$, $i = 1, \ldots, k$, in the cases that Lyapunov exponents are defined.

Let us introduce a notation $E^{s,A}_{(i,)} = E^s_{1} \oplus \ldots \oplus E^s_{i}$, $i = 1, \ldots, k$ and $E^{u,A}_{(i,)} = E^u_{1} \oplus \ldots \oplus E^u_{i}$, $i = 1, \ldots, n$. If $j \geq i$, we denote $E^{s,A}_{(i,j)} = E^s_{i} \oplus \ldots \oplus E^s_{j}$ and $E^{u,A}_{(i,j)} = E^u_{i} \oplus \ldots \oplus E^u_{j}$.

We need an important tool related to Livsic’s Theorem. It is known by [4] a version of shadowing lemma for endomorphisms.

**Proposition 5.1** (Closing Lemma for Endomorphisms, Lemma 3 of [4]). For $0 \leq k \leq \dim M$, $\chi > 0$, $l \geq 1$ and $\rho > 0$ there exists a number $\gamma_l(\rho) = \gamma_l(k, \chi, \rho) > 0$ such that, if $\tilde{x} = (x_n) \in \tilde{\Lambda}^k_{\chi, l}$ satisfies $f^m(\tilde{x}) \in \tilde{\Lambda}^k_{\chi, l}$, $d(f^m(\tilde{x}), \tilde{x}) \leq \gamma_l(\rho)$ for some $m \geq 1$, then there is a hyperbolic periodic point $p = p(\tilde{x}) \in M$ off with $f^m(p) = p$ such that $d(f^j(p), x_j) \leq \rho$ for all $0 \leq j \leq m - 1$.

The point $p$ above is unique. In the context Anosov endomorphisms, there is a suitable choice of constants such that $\tilde{\Lambda}^k_{\chi, l} = M$, in this specific case the above proposition is known by Anosov Closing Lemma. Endowed with the Anosov Closing Lemma we can prove, using the same argument the version for diffeomorphisms, the following version of the Livsic’s Theorem.

**Theorem 5.2** (Livsic’s Theorem). Let $M$ be a riemannian manifold, $f : M \to M$ a transitive smooth Anosov endomorphism and $\varphi : M \to \mathbb{R}$ an $\alpha$– Hölder function.

Suppose that for every $x \in M$ such that $f^n(x) = x$, we have $\sum_{i=0}^{n-1} \varphi(f^i(x)) = 0$. Then there exists a unique $\alpha$– Hölder function $\phi : M \to \mathbb{R}$, such that $\varphi(x) = \phi(f(x)) - \phi(x)$ and $\phi$ is unique up to an additive constant.

For the proof see [13], page 610.

Let us start with the unstable directions $E^{u,f}_i$. Fix $i \in \{1, \ldots, n\}$ and for each $x \in T^d$, consider the tangent leaf $W^{u,f}_i(x)$, projected from $\mathbb{R}^d$. Up to change $f, A$ by $f^2, A^2$, consider on tangent leaves an orientations such that $f$ and $A$ acts increasingly on $W^{u,f}_i(x)$ and $W^{u,A}_i(x)$ respectively. Since $f$ is $E^{s,f}, E^{u,f}$–special, it implies that $f$ is strongly special. In fact, by Proposition 2.5 of [13] we have that $E^{u}_f(x)$, the collection of all unstable directions at $x$ is given by $E^{u}_f(x) = \bigcup_{\pi(y) = x} D\pi(y) \cdot (E^{u}_f(y))$. In the universal cover $\mathbb{R}^d$ we have $E^{u,f}_f(y) = \bigoplus_{j=1}^{n} E^{u,f}_j(y)$, as $f$ is special with respect to each (continuation) bundle $E^{u,f}_j$, so for $y, y'$ such that $\pi(y) = \pi(y') = x$ then $E^{u,f}_j(y) = E^{u,f}_j(y')$. Applying Proposition 2.5 of [13] we get $f$ is special.
Since \(A\) is irreducible over \(\mathbb{Q}\) the leaves \(W_{iA}^u\) are non compact and dense on \(T^d\).

Denote by \(\lambda_i^u\) the common value of the Lyapunov exponents of periodic points of \(f\) and \(A\) in the directions \(E_i^u, f\) and \(E_i^u, A\), respectively. Let us to denote on \(T^d\), the \(f\)–invariant foliations \(\mathcal{F}_i^{u, f}\) tangent to \(E_i^{u, f}, * \in \{s, u\}\) induced by the corresponding ones foliations from \(\mathbb{R}^d\).

We see that \(\log(\|D f(x)\|E_i^{u, f}(x)\|) - \lambda_i^u\) has zero average over every periodic orbit.

Since \(f\) is smooth and \(f\) is Anosov, the map \(x \mapsto \log(\|D f(x)\|E_i^{u, f}(x)\|)\) is uniform Lipschitz on \(T^d\). Hence, by Livsic’s theorem \([10, 3]\), we can find a Lipschitz function \(\phi_i^u : T^d \rightarrow \mathbb{R}\) such that

\[
\log(\|D f(x)\|E_i^{u, f}(x)\|) - \lambda_i^u = \phi_i^u(f(x)) - \phi_i^u(x). \quad (5.1)
\]

Equivalently

\[
e^{-\phi_i^u(x)}\|D f(x)\|E_i^{u, f}(x)\|e^{-\phi_i^u(f(x))} = e^{\lambda_i^u}. \quad (5.2)
\]

We can interpret \((5.2)\) as saying that, if we define a metric, conformal to the standard metric in the torus by a factor \(e^{-\phi_i^u}\), then for a convenient metric \(f\) expands on \(W_{iA}^{u, f}\)–leaves by exactly \(e^{\lambda_i^u}\).

In fact, for any \(a \geq b\) on \(W_{iA}^{u, f}(x)\), consider the metric

\[
d_i^u(a, b) = \int_a^b e^{-\phi_i^u(x)}dx,
\]

where \(dx\) denotes the infinitesimal size on \(W_{iA}^{u, f}(x)\). We have

\[
d_i^u(f(a), f(b)) = \int_{f(a)}^{f(b)} e^{-\phi_i^u(y)}dy = \int_a^b e^{-\phi_i^u(f(x))}\|D f(x)\|E_i^{u, f}(x)\|dx = e^{\lambda_i^u} \int_a^b e^{-\phi_i^u(x)}dx = \lambda_i^u d_i^u(a, b).
\]

Let \(h : T^d \rightarrow T^d\) be the conjugacy between \(f\) and \(A\), such that

\[
h \circ A = f \circ h.
\]

Let us introduce a leaf equivalence on the unstable leaves. We say that two unstable leaves \(L\) and \(L'\), tangent to \(E_i^{u, A}\), are related if there is an orbit \(\tilde{x} = (x_n)_{n \in \mathbb{Z}}\) such that for some integers \(n_1, n_2\) we have \(L = W_i^{u, A}(x_{n_1})\) and \(L' = W_i^{u, A}(x_{n_2})\). For each equivalence \([L]\) class choose a representant \(\tilde{L}\), a point \(x_0 \in L\) and \(\tilde{y}\), an orbit such that \(y_0 = x_0\). We will use \(\tilde{d}_i^u\) to construct a new function \(\tilde{h}\), as following.

First we define \(\tilde{h}(x_0) = h(x_0)\), now if \(a > x_0\) such that \(|x_0 - a| = \alpha > 0\), we define \(\tilde{h}(a)\) the point \(b \in W_{iA}^{u, f}(h(x_0))\), such that \(d_i^u(b, h(x_0)) = \alpha\). Analogously we define \(\tilde{h}(a)\), in the case \(a < x_0\).
For the iterated $A(W^{u,i}(x_0)) = W^{u,i}_i(A(x_0))$, we first define $\tilde{h}(A(x_0)) := h(A(x_0))$, so we have $\tilde{h}(A(x_0)) = h(A(x_0)) = f(h(x_0)) = f(\tilde{h}(x_0))$. Since $f,A$ have the same periodic data, then $A$ expands $W^{u,i}_i(x_0)$ with the factor $e^{\lambda i}f$, the same factor such that $f$ expands $W^{u,f}(h(x_0))$ with the metric $d_i$. So, as before, using distances, we define $\tilde{h}$ on $W^{u,i}_i(A(x_0))$. Note that if $a > x_0$ and $|x_0 - a| = \alpha > 0$, then $|A(a) - A(x_0)| = e^{\lambda i}\alpha$, with $A(a) > A(x_0)$. If $b = \tilde{h}(A(a))$, then $b > h(A(x_0))$ and $d_i(b,h(A(x_0))) = e^{\lambda i}\alpha$, by definition we get $f(\tilde{h}(a)) = \tilde{h}(A(a))$. It is analogous if $a < A(x_0)$. We can proceed inductively for all iterated $A^n(W^{u,i}(x_0)), n \geq 0$.

If $A(y) = x_0$, we define $\tilde{h}$ on $W_A(y)$ using distances as before, with initial condition $\tilde{h}(y) = h(y)$. Following for backward this argument, we have defined a function $\tilde{h} : Q_L := \bigcup_{L' \in [L]} L' \to \mathbb{T}^d$, such that $\tilde{h}(A(x)) = f(\tilde{h}(x))$.

Now running on all $Q_L$, we construct a function $\tilde{h} : \mathbb{T}^d \to \mathbb{T}^d$, such that $\tilde{h}(A(x)) = f(\tilde{h}(x))$, such that it coincides with $h$ at least for one point on each unstable leaf $W^{u,i}_i$. It is not hard to see that each restriction $\tilde{h}|L$ is a homeomorphism, so $\tilde{h} : \mathbb{T}^d \to \mathbb{T}^d$ is a bijection.

For a suitable lift $\tilde{H}$ of $\tilde{h}$, we have $\tilde{H}(x + c) = \tilde{H}(x) + c$, for any $x \in \mathbb{R}^d$ and $c \in \mathbb{Z}^d$, we have

$$\tilde{H} \circ \tilde{A} = \tilde{f} \circ \tilde{H},$$

where $\tilde{A}, \tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ are lifts of $f$ and $A$ respectively.

If $\tilde{h}(x) = y$, define $\tilde{\phi}(y) = x$.

Denote by $\phi$ a suitable lift of $\tilde{\phi}$ such that $\phi(x + c) = \phi(x) + c$ for any $x \in \mathbb{R}^d$ and $c \in \mathbb{Z}^d$. We have

$$\tilde{A} \circ \phi = \phi \circ \tilde{f}. \quad (5.4)$$

For a suitable lift $\tilde{H}$ of $\tilde{h}$, we have $\tilde{H}(x + c) = \tilde{H}(x) + c$, for any $x \in \mathbb{R}^d$ and $c \in \mathbb{Z}^d$.

By compactness of a fundamental domain of $\mathbb{T}^d$, we have

$$\bar{d}(\tilde{H}, Id_{\mathbb{R}^d}) = \sup_{x \in \mathbb{R}^d} ||\tilde{H} - Id_{\mathbb{R}^d}|| \leq \sqrt{d}.$$

In the case $\tilde{H}(x) = y$, changing $x$ by $\phi(y) := \tilde{H}^{-1}(y)$ we have

$$\bar{d}(\phi, Id_{\mathbb{R}^d}) \leq \sqrt{d}.$$
(1) \( L \circ \phi = \phi \circ T \),
(2) \( \bar{d}(\phi, id_{\mathbb{R}^n}) \) is finite.
Furthermore, for \( K > 0 \) there is a constant \( \delta_K > 0 \) such that if \( \bar{d}(L, T) < K \),
then the above map \( \phi \) has the following properties :
(3) \( \bar{d}(\phi, id_{\mathbb{R}^n}) < \delta_K \),
(4) \( \phi \) is a continuous surjection,
(5) \( \phi \) is uniformly continuous under \( \bar{d} \) if so is \( T \).

By the above proposition the maps \( \tilde{H} \) coincides with \( H \), a suitable lift of the
conjugacy \( h \). We conclude \( \tilde{h} = h \).

The construction of \( h \) using distances ensures that each restriction \( h|L \) is an
absolutely continuous function with domain \( L = \mathbb{R} \). We have
\[
x - x_0 = d_{\bar{d}}(\tilde{h}(x), \tilde{h}(x_0)) = \int_{\tilde{h}(x_0)}^{\tilde{h}(x)} e^{-\phi_{\bar{d}}(t)} dt,
\]
where \( dt \) denotes the length form on \( W^{u}_{f}(h(x_0)) \).

By the changing \( t = \tilde{h}(\theta) \), we get
\[
x - x_0 = \int_{x_0}^{x} e^{-\phi_{\bar{d}}(\tilde{h}(\theta))} \tilde{h}'(\theta) d\theta,
\]
a real differential equation. Again, \( \tilde{h}' \) can be obtained by solving \( \tilde{h}'(t) = e^{\phi_{\bar{d}}(\tilde{h}(t))} \) with initial condition \( \tilde{h}(A(x_0)) = h(A(x_0)) = f(h(x_0)). \)

If \( \phi_{\bar{d}} \) is Lipschitz, then \( h \) is uniformly \( C^{1+\alpha} \), for some \( \alpha > 0 \), enough small, on
the unstable leaves \( W^{u}_{i,A} \), since we know that \( W^{u}_{i,f} \) are \( C^{1+\varepsilon} \)-submanifolds.

Note that leaves of type \( W^{u}_{1,A} \) and \( W^{u}_{2,A} \) are transversal and subfoliate \( W^{u}_{1,2} \),
so by Journé’s Lemma \([8]\), we have \( h \) is uniformly \( C^{1+\alpha} \), for some \( \alpha > 0 \), enough small, on
the unstable leaves \( W^{u}_{i,A} \). Inductively \( h \) is uniformly \( C^{1+\alpha} \), for some
\( \alpha > 0 \), enough small, on the unstable leaves \( W^{u}_{1,2,A} = W^{u}_{A} \). Analogously, we have \( h \) is uniformly \( C^{1+\alpha} \), for some \( \alpha > 0 \), enough small, on the stable leaves \( W^{s}_{A} \).
Finally, By Journé’s Lemma \([8]\), we have \( h \) is \( C^{1+\alpha} \), for some \( \alpha > 0 \).

6. Proof of Theorem 1.10

For \( d = 2 \) we can imitate the proof of Theorem 1.9. Indeed we don’t need that \( f, A \) being \( C^{1} \)-close. Since \( f \) and \( A \) are conjugated the conjugacy \( h \) applies
stable/unstable manifolds of \( A \) in stable/unstable manifolds of \( f \) respectively. As the stable/unstable foliations are one dimensional and non compact foliations, we can define \( h \) using convenient distances on stable/unstable leaves. We conclude that \( h \) is \( C^{1} \), it implies that \( f \) preserves an invariant measure absolutely
continuous with respect to $m$, moreover $A$ satisfies Hypothesis S. By applying directly Theorem 1.7 we conclude $h$ is smooth.

7. Proof of Theorem 1.11

For this section we need specification to prove the following lemma.

Lemma 7.1. Consider $f : \mathbb{T}^2 \to \mathbb{T}^2$ an Anosov endomorphism such that every point is regular. So for any point $p, q \in \text{Per}(f)$ we have

$$\lambda_*(p) = \lambda_*(q), * \in \{s, u\}.$$  

We present the proof later.

As in equation (5.2), we have

$$||D f^n(x)||_{E^s(x)} = e^{n \lambda_s f(x)} - \phi_s(x),$$

for some $\phi_s : \mathbb{T}^2 \to \mathbb{R}$ a Lipschitz function. So we get $\lambda_s(x) = \lambda_s$, for any $x \in \mathbb{T}^2$. Since $\phi_s$ is continuous, the convergence $\frac{1}{n} \log(||D f^n(x)||_{E^s(x)}) \to \lambda_s$ is uniform on $\mathbb{T}^2$. Analogously $\lambda_u(x) = \lambda_u$, for any $x \in \mathbb{T}^2$, with uniform convergence. The same idea holds for $J_f = |\det(Df)|$, meaning that there is a Lipschitz function $\phi : \mathbb{T}^2 \to \mathbb{R}$, such that

$$J_f = e^{\phi(f(x)) - \phi(x)}.$$  

(7.1)

By Oseledec’s Theorem $c = \lambda_u + \lambda_s$ in (7.1). Of course, the convergence $\frac{1}{n} \log(J_f(x)) \to \lambda_u + \lambda_s$ is uniform.

Lemma 7.2. Consider $f : \mathbb{T}^2 \to \mathbb{T}^2$ an Anosov endomorphism such that every point is regular. Then $f$ preserves an absolutely continuous measure.

Proof. By [14] there are unique invariant $f$–invariant measures $\mu_+^f$ and $\mu_-^f$, named S.R.B and inverse S.R.B measures for $f$. These measures satisfy:

$$h_{\mu_+^f} = \log(\lambda_u)$$

(7.2)

and denoting $k = \text{deg}(f)$,

$$h_{\mu_-^f} = \log(k) - \log(\lambda_s).$$

(7.3)

Since $J_f = |\det(Df)|$ is cohomologous to constant, by [14] we have $h_{\mu_+^f}(f) = h_{\mu_-^f}(f)$, we conclude that $\lambda_u + \lambda_s = \log(k)$. The formula (7.1) can be rewritten as

$$\log(J_f) - \log(k) = \phi(f(x)) - \phi(x).$$

(7.4)

So we have

$$J_f(x)e^{-\phi(x)} = ke^{-\phi(x)}.$$

Define
Let $B$ be an small open ball and $B_1, B_2, \ldots, B_k$ its mutually disjoint preimages, $f(B_i) = B$.

Define the measure $dv = e^{-\phi(x)}dm$, we have

$$v(B) = v(f(B_i)) = \int_{f(B_i)} e^{-\phi(y)}dm = \int_{B_i} Jf(x)e^{-\phi(f(x))}dm = \int_{B_i} ke^{-\phi(x)}dm = kv(B_i)$$

$$v(B_i) = \frac{1}{k}v(B)$$

$$v(B) = \sum_{i=1}^{k} v(B_i) = v(f^{-1}(B)).$$

Define $\mu(X) = \frac{v(X)}{v(T^2)}$, to obtain an $f$–invariant measure absolutely continuous w.r.t. $m$. □

Let us to end the proof of Theorem 1.11. We know that $\lambda^u(x) = \lambda^u$, for any $x \in T^2$. Using the Ruelle’s inequality we have

$$h_\nu(f) \leq \lambda^u,$$

for any $\nu$ an $f$–invariant, borelian, probability measure. By variational principle

$$h_{top}(f) \leq \lambda^u.$$

By the version of Pesin Theorem for endomorphism, we have

$$h_\nu(f) = \log(\lambda^u).$$

So $\nu = \mu^+_f = \mu^-_f$ the maximal entropy measure of $f$. Since $f$ and $A$ are conjugated, they are same topological entropy, then $\lambda^u = \lambda^u_A$ and $\lambda^s = \lambda^s_A$, using Theorem 1.10 we conclude the proof.

8. Specification Property and Proof of Lemma 7.1

Let us explain about the specification property.

**Definition 8.1** (Specification Property). Let $f : M \to M$ be a diffeomorphism. We say that $f$ has the specification property if given $\varepsilon > 0$ there is a relaxation time $N \in \mathbb{N}$ such that every $N$–spaced collection of orbit segments is $\varepsilon$–shadowed by an actual orbit. More precisely, for points $x_1, x_2, \ldots, x_n$ and lengths $k_1, \ldots, k_n \in \mathbb{N}$ one can find times $a_1, \ldots, a_n$ such that $a_{i+1} \leq a_i + N$ and a point $x$ such that $d(f^{a_i}(x), f^j(x)) < \varepsilon$ whenever $0 \leq j \leq k_i$. Moreover, one can choose $x$ a periodic point with period no more than $a_n + k_n + N$.

**Theorem 8.2** (Bowen, [3]). Every transitive Anosov diffeomorphism has the specification property.
Recently Moriyasu, Sakai and Yamamoto in [22], proved among other things the following result.

**Proposition 8.3** (Corollary 1 of [22]). The set of $C^1$-regular maps of $M$ satisfying the $C^1$-stable specification property is characterized as the set of transitive Anosov maps.

So we can apply specification to sketch prove Lemma 7.1.

**Proof.** Suppose that $n \geq 1$ is an integer number. Suppose that $\lambda(p), \lambda(q)$ denote the Lyapunov exponents corresponding to direction $E^s_j$. Consider $z$ as above for $n \geq 1$ is an integer number. Suppose that $\lambda(p), \lambda(q)$ denote the Lyapunov exponents corresponding to direction $E^s_j$ and $\lambda(p) < \lambda(q)$. Consider $\delta > 0$ such that $(1 + \delta)^2 \lambda(p) < (1 -\delta^2) \lambda(q)$, and $\varepsilon > 0$ such that if $d(x, y) < \varepsilon$, then $1 - \delta < \frac{|D^u f(x)|}{|D^u f(y)|} < 1 + \delta$. Let $N > 0$ be the relaxation time, for the given $\varepsilon > 0$, where $D^u f(x) = D f(x) E^s_j(x)$. For each $j \in \mathbb{N}$ we consider the orbit segments $P_j = \{\theta_j, f^1(\theta_j), \ldots, f^{s_j}(\theta_j)\}$, where $\theta_j = p$, if $j$ is odd and $\theta_j = q$, if $j$ is even. We define inductively $k_j$ as follows. First $k_1 = n, k_{j+1} = (k_1 + \ldots + k_j + j)N^2$, for $j = 1, 2, \ldots$. Consider $O_j$ the concatenation of $P_1, \ldots, P_k$. The length of the sequence $O_j$ is $k_1 + \ldots + k_j$. By specification property of $f$, for any $j$ there is a point $z_j$ and a segment of orbit $\{z_j, f(z_j), \ldots, f^{r_j}(z_j)\}$, with $r_j \leq (k_1 + \ldots + k_j + (j-1)N) + k_j$ which the specification property. Observe that $r_j$ is a natural number of the form $s_j + t_j^2$, with $t_j = (k_1 + \ldots + k_j + (j-1)N) \in \mathbb{N}$ and $0 < s_j \leq t_j$.

Let $x = z_j$, for some $j$. For the integer $s + t_j^2$, with $s = s_j$ and $t = t_j$ as above, we have

$$
\frac{1}{s + t_j^2} \log(|D^u f^{s + t_j^2}(x)|) = \frac{1}{s + t_j^2} \log(\prod_{i=0}^{s-1} |D^u f(f^i(x))|) \cdot \prod_{i=s}^{s + t_j^2 - 1} |D^u f(f^i(x))| \approx \frac{s}{s + t_j^2} \log(K) + (1 \pm \delta) \frac{t_j^2}{s + t_j^2} \log(\lambda(\theta_j)) + \frac{r_j}{s + t_j^2} \log(K),
$$

where $r_j$ is the rest of the division of $s + t_j^2$ by $n$ and $K = \max_{x \in \mathbb{T}} |D f(x)|$.

So, taking $j = 2n - 1 \rightarrow +\infty$, we have $\frac{1}{s + t_j^2} \log(|D^u f^{s + t_j^2}(z_j)|) \approx (1 \pm \delta) \lambda(p)$, analogously taking $j = 2n \rightarrow +\infty$, we have $\frac{1}{s + t_j^2} \log(|D^u f^{s + t_j^2}(z_j)|) \approx (1 \pm \delta) \lambda(q)$.

Consider if $j \geq n$, and $z_n$ obtained by specification as above. There is an integer $0 < s = s_j \leq t_j$, such that for $t = t_j$ we have

$$
\frac{1}{s + t_j^2} \log(|D^u f^{s + t_j^2}(z_n)|) \approx (1 \pm \delta) \lambda(\theta_j),
$$

it is because $z_n$ accompanies $O_j$, according to specification.

By compactness of $T$ we can suppose that $z_n \rightarrow z$. We claim that $z$ is not regular.

By continuity of $D^u f$, if $j$ is odd, taking $z_n$ enough close to $z$, with $n \geq j$. Let $r_j$ be as above for $z_n$, we have $\frac{1}{s + t_j^2} \log(|D^u f^{r_j}(z)|) \approx (1 \pm \delta)(1 \pm \delta) \lambda(p)$, where
\( s = s_j' \) and \( t = t_j \). Analogously if \( j \) is even, then we have 
\[
\frac{1}{d+1} \log(|D^a f^{d+1}(z)|) \approx (1 \pm \delta)(1 \pm \delta)\lambda(q),
\]
since \( \delta \) is small we conclude that \( z \) is not regular.

\( \square \)

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