Backward problem for time fractional reaction-diffusion equation with nonlinear source and discrete data

Nguyen Huy Tuan 1, Tran Ngoc Thach 2, Donal O’Regan3 and Nguyen Huu Can4,*

1 Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam
2 Department of Mathematics and Computer Science, University of Science, VNU-HCM, Ho Chi Minh City, Vietnam
3 School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
4 Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

November 1, 2019

Abstract

In this paper, we study the problem of finding the solution of a multi-dimensional time fractional reaction-diffusion equation with nonlinear source from the final value data. We prove that the present problem is not well-posed. Then regularized problems are constructed using the truncated expansion method (in the case of two-dimensional) and the quasi-boundary value method (in the case of multi-dimensional). Finally, convergence rates of the regularized solutions are given and investigated numerically.

Keywords: Backward problem; Fractional reaction-diffusion equation; Regularization method; Nonlinear source; Discrete data.

Subject Classification: 35K99; 47J06; 47H10; 35K05.

1 Introduction

Fractional partial differential equations (FDEs) arise naturally in physics, biology, chemical, engineering and control theory; see [11, 14, 15, 22, 24, 31]. For definitions we refer the reader to the Riemann-Liouville (R-L) definition and the Caputo definition in [16, 22, 29].

In this paper, we study a problem for the time fractional reaction-diffusion equation with nonlinear source

\[
\begin{align*}
\frac{\partial}{\partial t} u(t,x) - \frac{\partial^{1-\alpha}}{\partial t} \Delta u(t,x) &= f(t,x,u(t,x)), \\
(t,x) &\in (0,T) \times \Omega, \alpha \in (0,1) \\
u(t,x) &= 0, \\
(t,x) &\in (0,T) \times \partial \Omega, \\
u(T,x) &= \varphi(x), x \in \Omega,
\end{align*}
\]

where \( \Omega = (0,\pi)^d \) is a subset of \( \mathbb{R}^d \), \( x = (x_1,x_2,\ldots,x_d) \) is a \( d \)-dimensional variable. The source function \( f \) is given and \( \varphi \) is called the final value status. The notation \( \frac{\partial^{1-\alpha}}{\partial t} \) denotes the R-L fractional derivative

\[
\frac{\partial^{1-\alpha}}{\partial t} v(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} v(s) ds, \quad t > 0,
\]

where \( \Gamma(\cdot) \) is the Gamma function. It should be noted that if \( \alpha = 1 \) then \( \frac{\partial^{1-\alpha}}{\partial t} \Delta u \) becomes \( \Delta u \) and the equation

\[
\frac{\partial}{\partial t} u(t,x) - \frac{\partial^{1-1}}{\partial t} \Delta u(t,x) = f(t,x,u(t,x)),
\]

*Correspondence: nguyenhuucan@tdtu.edu.vn (Nguyen Huu Can). E-mail of other authors: ngocthachtnt@gmail.com (Tran Ngoc Thach), thnguyen2683@gmail.com (Nguyen Huy Tuan), donal.oregan@nuigalway.ie (Donal O’Regan).
reduces to the typical heat equation (note 2 comes from 24). Schneider and Wyss 25 showed that the description of diffusion in special types of porous media is an application of 2 and that the fractional parameter \( \alpha \in (0, 1) \) can represent the “gray” noise instead of the white noise in the case of \( \alpha = 1 \).

Equation (2) with the initial condition \( u(0, x) = \psi(x) \) is known as the direct problem. For more details of this equation we refer the reader to [1, 25, 26] and the references therein. Numerical solutions of the alternative representation of such direct problem have been studied in 35, 36, 33, 32. In contrast, the problem of recovering the function \( u \) at previous time \( t \in [0, T] \) as in [1] is called the backward problem. This kind of equation arises in practical situations in which the initial density of the diffusing substance is not available and we can only measure the density at positive time. We mention the applications of backward in time diffusion equations in the work of A. S. Carraso 7, 8. Two major current applications of the backward problem are hydrologic inversion and image deblurring. Hydrologic inversion seeks to identify sources of groundwater pollution by backtracking contaminant plumes (see 3, 5) and this involves solving the diffusion backward in time, given the contaminant spatial distribution at the current time \( T \). In image analysis, an effective setting for studying 2-D backward diffusion lies in the field of imaging rehabilitation. One can create imaginary fuzzy image data, using a certain sharp image as the initial value in the nonlinear diffusion equation studied and select the corresponding solution in a positive number time \( T \) so successful backward continuation from \( t = T \) to \( t = 0 \), would restore the original sharp image. Until now, there are some interesting papers on inverse problem of fractional diffusion. We can list some well-known results, for example, J. Jia et al 39, J. Liu et al 40, some papers of M. Yamamoto and his group see 43, 44, 47, 46, 45, B. Kaltenbacher et al 37, 38, 50, 51, W. Rundell et al 48, 49, J. Janno see 41, 42, etc. However, to the best of our knowledge, there is no result concerning the backward problem for 2 with random noise.

Motivated by the above, in this paper, we study problem [1] and aim to provide an approximate solution. In reality, it is impossible to get the exact final data \( \varphi \) and we only have the noisy physical measurement \( \tilde{\varphi} \). A difficult point of the backward problem is a small noise between \( \tilde{\varphi} \) and \( \varphi \) can generate a very large error in the solution \( u \). In other words, the solution does not depend continuously on the final value status (which makes (1) not well-posed). Therefore, we must provide some suitable methods to find an approximation for \( u \). When the final value status is measured on the whole space \( \Omega \) many good methods can be applied to establish the approximate regularized solution such as the Tikhonov, the quasi-boundary value (QBV), the quasi-reversibility (QR) and the truncated expansion method (see 10, 12, 19, 23). Here we consider a different situation in which only a finite number of data (instead of data on the whole space) is available. Precisely, we assume that the data \( \varphi \) is measured at \( n_1 \times n_2 \times \cdots \times n_d \) grid points \( x_k = x_{k_1, k_2, \ldots, k_d} \in \Omega, d \geq 2, k = (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d \), as follows

\[
x_k = (X_{k_1}, X_{k_2}, \ldots, X_{k_d}) = \left( \frac{2k_1 - 1}{2n_1}, \frac{2k_2 - 1}{2n_2}, \ldots, \frac{2k_d - 1}{2n_d} \right),
\]

where \( k_i = 1, 2, \ldots, n_i, i = 1, 2, \ldots, d. \) Furthermore, the value of \( \varphi \) at each point \( x_k \) is contaminated by the observation \( \Phi_{obs}^{obs} \)

\[
\varphi(x_k) = \varphi(X_{k_1}, X_{k_2}, \ldots, X_{k_d}) \approx \Phi_{k_1, k_2, \ldots, k_d}^{obs} = \Phi_{obs}^{k_1, k_2, \ldots, k_d}.
\]

The relationship between the two kinds of data is described by the random model

\[
\Phi_{obs}^{k_1, k_2, \ldots, k_d} = \varphi(x_k) + \varepsilon_k W_k,
\]

where \( W_k = W_{k_1, k_2, \ldots, k_d} \) are mutually independent random variables, \( W_k \sim \mathcal{N}(0, 1) \) and \( \varepsilon_k = \varepsilon_{k_1, k_2, \ldots, k_d} \) are positive constants bounded by a positive constant \( \varepsilon_{\text{max}} \). Some inverse problems when \( d = 1 \) were studied in 6, 21, 28.

Our main contributions in this paper is as follows:

- For the two dimensional case, i.e., \( d = 2 \), we apply the Fourier truncation method introduced in 18 to give a regularized problem. The model in 18 is linear. Our problem is nonlinear and we use the Banach fixed point theorem to show the existence of the regularized solution in the space \( \mathcal{X}_T \) (note this space does not appear in 18). Some new estimates of Mittag-Leffler type are used.

- For the multidimensional case with \( d > 2 \), we apply the quasi-boundary value method (QBV). We emphasize that our random model here is a multidimensional case which is a generalization of the results in 6, 21, 28. Our method in this case is new and very different from the methods in 18. First, we approximate \( H \) and \( \varphi \) by the approximating functions \( \tilde{\varphi}^n \) defined in Theorem 5.1. Then, we use the approximation data to establish a regularized solution using the QBV method. Moreover, we also give a new
filter method which contains some results on the truncation and quasi-boundary value method (this filter is a new contribution). In particular in our error estimates we show that the norm of the difference between the regularized solution and the solution of the problem (1) tends to zero when $\sqrt{n_1^2 + \ldots + n_d^2} \to +\infty$.

The structure of this paper is as follows. We first give some preliminaries which are needed for this paper in Section 2. In Section 3, we establish an integral formulation for the solution of problem (1). In Section 4, we prove that the present problem is not well-posed and then we construct an approximate regularized solution for the 2-dimensional problem using the Fourier truncated method. The convergence result is also given there. In Section 5, the multi-dimensional problem is considered and regularized using the quasi-boundary value method. We estimate the error between the approximation and the sought solution in two different spaces. Finally, we provide some numerical results to illustrate the convergence rates.

2 Preliminaries

Before going to the main parts, we present some concepts:

• For $j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d$, we denote $|j| = \sqrt{\sum_{i=1}^d j_i^2}$. It is well-known that the following problem

$$\begin{cases}
-\Delta \xi_j(x) = \lambda_j \xi_j(x), & x \in \Omega, \\
\xi_j(x) = 0, & x \in \partial \Omega.
\end{cases}$$

admits eigenvalues $\{\lambda_j\}$ and eigenvectors $\{\xi_j\}$ as follows

$$\lambda_j = \lambda_{j_1, \ldots, j_d} = \sum_{i=1}^d j_i^2, \quad \xi_j(x) = \xi_{j_1, \ldots, j_d}(x) = \left(\sqrt{\frac{2}{\pi}}\right)^d \prod_{i=1}^d \sin(j_i x_i).$$

• For $\alpha > 0$ and $\beta > 0$, the function defined as follows

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad z \in \mathbb{C}. \quad (4)$$

is called the Mittag-Leffler function. Some properties of this function can be found in [22].

• We introduce the subspace of $L^2(\Omega)$

$$H^\theta(\Omega) = \left\{ g \in L^2(\Omega) : \sum_{j \in \mathbb{N}^d} \lambda_j^\theta \langle g, \xi_j \rangle^2 < \infty \right\}, \quad \theta > 0,$$

with the norm $\|g\|_{H^\theta(\Omega)} = \left(\sum_{j \in \mathbb{N}^d} \lambda_j^\theta \langle g, \xi_j \rangle^2\right)^{1/2}$, in which $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$.

• For an arbitrary Banach space $B$, we set

$$L^\infty(0, T; B) = \left\{ h : (0, T) \to B \text{ measurable s.t. esssup}_{t \in (0, T)} \|h(t, \cdot)\|_B < \infty \right\}.$$

• We denote by $X_T$ (see [3]), the space of all $L^2$-valued predictable processes $w$ such that

$$\|w\|_{X_T} = \sup_{t \in [0, T]} \sqrt{\mathbb{E} \|w(t, \cdot)\|^2_{L^2(\Omega)}} < \infty.$$

For $\sigma > 0$, we denote by $S_{\sigma, T}$, the space of all $H^\sigma$-valued predictable processes $w$ such that

$$\|w\|_{S_{\sigma, T}} = \sup_{t \in [0, T]} \sqrt{\mathbb{E} \|w(t, \cdot)\|^2_{H^\sigma(\Omega)}} < \infty.$$
3 The solution of the backward problem

Let \( u(t, x) = \sum_{j \in \mathbb{N}^d} u_j(t)\xi_j(x) \) be the Fourier series of \( u \) in \( L^2(\Omega) \), where \( u_j(t) := \langle u(t, \cdot), \xi_j \rangle \) are called the Fourier coefficients of \( u \). Similarly, we denote \( h_j := \langle h, \xi_j \rangle \) and \( f_j(u)(t) := \langle f(t, \cdot, u(t, \cdot)), \xi_j \rangle \). Next, we find a representation for the solution \( u \) of problem (1). By taking the inner product on both sides of the first equation in (1) with \( \xi_j \), one has

\[
\frac{\partial}{\partial t} u_j(t) + \lambda_j \frac{1}{\alpha} \partial^\alpha \xi_j u_j(t) = f_j(u)(t).
\]

Using the Laplace transform and \( u_j(T) = \varphi_j \), one obtains

\[
\tilde{u}_j(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda_j} u_j(0) + \frac{s^{\alpha-1}}{s^\alpha + \lambda_j} \tilde{f}(u)(s),
\]

where the notation \( \tilde{g} \) is the Laplace transform of \( g \). In order to solve the equation above, we need some following properties of the Mittag-Leffler function (see [22])

\[
\frac{\partial}{\partial t} (E_{\alpha,1}(-\lambda t^\alpha)) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad \frac{\partial}{\partial t} (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha),
\]

and

\[
\int_0^\infty e^{-st} E_{\alpha,1}(-\lambda t^\alpha) dt = \frac{s^{\alpha-1}}{s^\alpha + \lambda^\alpha}, \quad \text{for } \text{Re}(s) > \lambda^{1/\alpha},
\]

in which \( \lambda \) is a positive constant. In this way, one gets

\[
u_j(t) = E_{\alpha,1}(-\lambda_j t^\alpha) u_j(0) + \int_0^t E_{\alpha,1}(-\lambda_j(t-s)^\alpha) f_j(u)(s) ds.
\]

It follows that

\[
u_j(T) = E_{\alpha,1}(-\lambda_j T^\alpha) u_j(0) + \int_0^T E_{\alpha,1}(-\lambda_j(T-s)^\alpha) f_j(u)(s) ds.
\]

Using the latter equation to determine \( u_j(0) \) based on \( u_j(T) \) and then substituting this quantity into (5), the Fourier coefficients are obtained

\[
u_j(t) = \frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \varphi_j + \int_0^t E_{\alpha,1}(-\lambda_j(t-s)^\alpha) f_j(u)(s) ds \]
\[
- \int_0^T E_{\alpha,1}(-\lambda_j t^\alpha) E_{\alpha,1}(-\lambda_j(T-s)^\alpha) f_j(u)(s) ds.
\]

For convenience, we define the operators

\[
A(t)g := \sum_{j \in \mathbb{N}^d} \left( \frac{E_{\alpha,1}(-\lambda_j t^\alpha)}{E_{\alpha,1}(-\lambda_j T^\alpha)} \langle g, \xi_j \rangle \right) \xi_j,
\]

\[
B(t)g := \sum_{j \in \mathbb{N}^d} \left( E_{\alpha,1}(-\lambda_j t^\alpha) \langle g, \xi_j \rangle \right) \xi_j,
\]

and \( D(t,s)g := A(t)B(s)g \), for \( g \in L^2(\Omega) \), \( t, s \in [0, T] \). Now, we conclude that \( u \) satisfies the equation

\[
u(t, x) = A(t) \varphi(x) + \int_0^t B(t-s) f(s, x, u(s, x)) ds - \int_0^T D(t, T-s) f(s, x, u(s, x)) ds.
\]

The following lemma (see [22]) is useful for estimating the Mittag-Leffler function:
Lemma 3.1. Let $0 < \alpha < 1$ and the function $E_{\alpha, \alpha}$ defined in (4). For a real number $z > 0$, we have
\[
\frac{M_1}{1+z} \leq E_{\alpha, \alpha}(-z) \leq \frac{M_2}{1+z},
\]
in which $M_1 = M_1(\alpha)$ and $M_2 = M_2(\alpha)$ are positive constants depending only on $\alpha$.

Using the latter lemma, we give some properties for the operators $B, D$ as follows:

Lemma 3.2. Let $g$ be a function in $L^2(\Omega)$. For $t \in [0, T]$, $s \in (0, T]$, we have
\[
\|B(t)g\|_{L^2(\Omega)} \leq M_2 \|g\|_{L^2(\Omega)}, \quad \|D(t, s)g\|_{L^2(\Omega)} \leq \frac{M_2^2 T^\alpha}{M_1 s^{\alpha}} \|g\|_{L^2(\Omega)}.
\]

Proof. Using Lemma 3.1, we have
\[
\|B(t)g\|_{L^2(\Omega)}^2 = \sum_{j \in \mathbb{N}^d} \left( \frac{E_{\alpha, 1} \left( -\lambda_j t^\alpha \right) E_{\alpha, 1} \left( -\lambda_j s^\alpha \right)}{E_{\alpha, 1} \left( -\lambda_j T^\alpha \right)} \right)^2 \langle g, \xi_j \rangle^2 \leq \frac{M_2^2 T^\alpha}{M_1 s^{\alpha}} \|g\|_{L^2(\Omega)}^2.
\]

For the second term, we have
\[
\|D(t, s)g\|_{L^2(\Omega)}^2 = \sum_{j \in \mathbb{N}^d} \left( \frac{M_2 \left( 1 + \lambda_j T^\alpha \right)}{M_1 \left( 1 + \lambda_j s^\alpha \right)} \right)^2 \langle g, \xi_j \rangle^2 \leq \frac{M_2^2 T^\alpha}{M_1 s^{\alpha}} \|g\|_{L^2(\Omega)}^2.
\]
This completes the proof.

4 Backward problem in the case of two-dimensional variables

In this section, we study problem (1) when $d = 2$. Here, we recall that $\Omega = (0, \pi) \times (0, \pi)$, $x = (x_1, x_2)$ is a 2-dimensional variable. The value of $\varphi$ at $n_1 \times n_2$ grid points
\[
x_{k_1, k_2} = (x_{k_1}, x_{k_2}) = \left( \frac{2k_1 - 1}{n_1}, \frac{2k_2 - 1}{n_2} \pi \right), \quad k_1 = 1, 2, \ldots, n_1, \quad k_2 = 1, 2, \ldots, n_2,
\]
are contaminated by the observed data $\Phi^{\text{obs}}_{k_1, k_2}$ as in the model
\[
\Phi^{\text{obs}}_{k_1, k_2} = \varphi(x_{k_1, k_2}) + \varepsilon_{k_1, k_2} W_{k_1, k_2}, \quad (10)
\]
where $W_{k_1, k_2}$ are mutually independent random variables, $W_{k_1, k_2} \sim \mathcal{N}(0, 1)$ and $0 < \varepsilon_{k_1, k_2} \leq \varepsilon_{\text{max}}$.

Our main aim is to show that the problem is not well-posed and then to construct an approximate regularized solution using the Fourier truncation method.

4.1 Estimator for the final value data $\varphi$

It should be noted that the final data $\varphi$ on the whole space $\Omega$ is not available. Therefore, we now establish an approximate function for $\varphi$ based on the observations and then estimate the error between them.

Lemma 4.1. Let $N_1 = N_1(n_1)$, $N_2 = N_2(n_2)$ be natural numbers less than $n_1, n_2$ respectively. Assume that there exists a constant $\theta > 2$ such that $\varphi \in H^\theta(\Omega)$. Approximate $\varphi$ by the following function
\[
\tilde{\varphi}^{N_1, N_2} := \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \Phi^{\text{obs}}_{k_1, k_2} \xi_{j_1, j_2}(x_{k_1, k_2}) \right) \xi_{j_1, j_2}.
\]

Then, we have
\[
\mathbb{E} \left\| \tilde{\varphi}^{N_1, N_2} - \varphi \right\|_{L^2(\Omega)}^2 \leq \frac{2\pi^2 \varepsilon_{\text{max}}^2 + 2C_0^2 \|\varphi\|^2_{H^\theta(\Omega)}}{n_1 n_2} \left( N_1 \right) \left( N_2 \right) + 2 \left[ \left( N_1 + 1 \right)^{-2\theta} \left( N_2 + 1 \right)^{-2\theta} \right] \|\varphi\|^2_{H^\theta(\Omega)},
\]
where $C_0$ is a positive constant (see (17)).
Remark 4.1. If we choose \(N_1 = N_1(n_1), N_2 = N_2(n_2)\) satisfying

\[
\lim_{n_1 \to \infty} N_1 = \lim_{n_2 \to \infty} N_2 = \infty, \quad \lim_{n_1, n_2 \to \infty} \frac{N_1 N_2}{n_1 n_2} = 0,
\]

then \(\mathbb{E} \left\| \overline{z}^{N_1, N_2} - \phi \right\|_{L^2(\Omega)}^2\) tends to zero as \(n_1, n_2\) tend to infinity.

The following Lemma is useful for proving Lemma 4.1.

Lemma 4.2. Let \(m_1, m_2\) be natural numbers less than \(n_1, n_2\) respectively. Set

\[
\mathcal{E}_{j_1, j_2, m_1, m_2} := \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \xi_{j_1, j_2}(x_{k_1, k_2}) \xi_{m_1, m_2}(x_{k_1, k_2}).
\]

Then, we have

\[
\mathcal{E}_{j_1, j_2, m_1, m_2} = \begin{cases} 
\frac{(-1)^{j_1+j_2} N_1 N_2}{\pi^2}, & \text{if } (j_1, m_1) = \pm(j_2, m_2) + (2l_1 n_1, 2l_2 n_2), \\
\frac{(-1)^{j_1+j_2}}{\pi^2} N_1 n_2, & \text{if } (j_1, m_1) = \pm(-j_2, m_2) + (2l_1 n_1, 2l_2 n_2), \\
0, & \text{otherwise}.
\end{cases}
\]

Proof of Lemma 4.2. This Lemma can be proved using the method in [13], page 145. \(\square\)

Proof of Lemma 4.1. Recall that \(\varphi_{j_1, j_2} = \langle \phi, \xi_{j_1, j_2} \rangle\), for \(j_1, j_2 \in \mathbb{Z}^+\). Using (11) and the fact that

\[
\varphi(x) = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \varphi_{j_1, j_2} \xi_{j_1, j_2}(x),
\]

we have

\[
\left\| \overline{z}^{N_1, N_2} - \phi \right\|_{L^2(\Omega)}^2 = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \Phi_{k_1, k_2}^{\text{obs}} \xi_{j_1, j_2}(x_{k_1, k_2}) - \varphi_{j_1, j_2} \right)^2 + \sum_{j_1=1}^{N_1} \sum_{j_2=N_2+1}^{\infty} \varphi_{j_1,j_2}^2 + \sum_{j_1=N_1+1}^{\infty} \sum_{j_2=1}^{N_2} \varphi_{j_1,j_2}^2 + \sum_{j_1=N_1+1}^{\infty} \sum_{j_2=N_2+1}^{\infty} \varphi_{j_1,j_2}^2.
\]

(12)

Part A (Estimating \(\mathcal{E}_1\)). Construct estimators for the coefficients \(\varphi_{j_1, j_2}\), for \(j_1 \leq N_1, j_2 \leq N_2\), as follows

\[
\varphi_{j_1, j_2} \approx \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \varphi(x_{k_1, k_2}) \xi_{j_1, j_2}(x_{k_1, k_2}),
\]

and set

\[
\mathcal{E}_{j_1, j_2} := \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \varphi(x_{k_1, k_2}) \xi_{j_1, j_2}(x_{k_1, k_2}) - \varphi_{j_1, j_2}.
\]

(13)

Then, we get

\[
\mathcal{E}_1 = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \varepsilon_{k_1, k_2} W_{k_1, k_2} \xi_{j_1, j_2}(x_{k_1, k_2}) + \mathcal{E}_{j_1, j_2} \right)^2 \leq 2 \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \varepsilon_{k_1, k_2} W_{k_1, k_2} \xi_{j_1, j_2}(x_{k_1, k_2}) \right)^2 + 2 \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \mathcal{E}_{j_1, j_2}^2.
\]

=: \(\mathcal{E}_{1,1} + \mathcal{E}_{1,2}\).
The first quantity can be estimated as follows
\[
\mathbb{E} E_{1,1} \leq \frac{2\pi^4}{n_1^2 n_2^2} \varepsilon_{\text{max}}^2 \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \mathbb{E} \left( \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} W_{k_1,k_2} \xi_{j_1,j_2}(x_{k_1,k_2}) \right)^2.
\]
Using the properties \(\mathbb{E} (W_{k_1,k_2} W_{l_1,l_2}) = \delta_{k_1,l_1} \delta_{k_2,l_2}\) and \(\sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \xi_{j_1,j_2}^2(x_{k_1,k_2}) = E_{j_1,j_2,j_1,j_2} = \pi^{-2} n_1 n_2\) (see Lemma 12), we obtain
\[
\mathbb{E} E_{1,1} \leq \frac{2\pi^4}{n_1^2 n_2^2} \varepsilon_{\text{max}}^2 \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \mathbb{E} W_{k_1,k_2}^2 \xi_{j_1,j_2}^2(x_{k_1,k_2}) \right) = \frac{2\pi^2 N_1 N_2}{n_1 n_2} \varepsilon_{\text{max}}^2.
\]
In order to estimate the term \(\mathbb{E} E_{1,2}\), we need to undergo some steps as follows:

**Step 1** (Finding an explicit form for the error \(\Upsilon_{j_1,j_2}\)). In this step, we will prove that
\[
\Upsilon_{j_1,j_2} = \sum_{l_2=1}^{\infty} (-1)^{l_2} \varphi_{2l_1 n_1 \pm j_1,j_2} + \sum_{l_1=1}^{\infty} (-1)^{l_1} \varphi_{2l_1 n_1 \pm j_1,j_2} + \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} (-1)^{l_1+l_2} \varphi_{2l_1 n_1 \pm j_1,j_2 + j_2} - \varphi_{2l_1 n_1 \pm j_1,j_2},
\]
where we define
\[
\varphi_{2l_1 n_1 \pm j_1,j_2} := \varphi_{2l_1 n_1 + j_1,j_2} - \varphi_{2l_1 n_1 - j_1,j_2}, \quad \varphi_{2l_1 n_1 \pm j_1,j_2} := \varphi_{2l_1 n_1 + j_1,j_2} - \varphi_{2l_1 n_1 - j_1,j_2},
\]
and
\[
\varphi_{2l_1 n_1 \pm j_1,j_2 + j_2} := \varphi_{2l_1 n_1 + j_1,j_2 + j_2} - \varphi_{2l_1 n_1 - j_1,j_2 + j_2} + \varphi_{2l_1 n_1 - j_1,j_2 + j_2} - \varphi_{2l_1 n_1 - j_1,j_2}.
\]
Indeed, since \(\varphi(x_{k_1,k_2}) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \varphi_1 m_1, m_2 \xi_{m_1, m_2}(x_{k_1,k_2})\), we have
\[
\sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \varphi(x_{k_1,k_2}) \xi_{j_1,j_2}(x_{k_1,k_2}) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \varphi_1 m_1, m_2 \left[ \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \xi_{m_1, m_2}(x_{k_1,k_2}) \xi_{j_1,j_2}(x_{k_1,k_2}) \right]
\]
\[
= \sum_{m_1 < n_1, m_2 < n_2} \varphi_1 m_1, m_2 E_{j_1,j_2,m_1,m_2} + \sum_{m_1 < n_1, m_2 \geq n_2} \varphi_1 m_1, m_2 E_{j_1,j_2,m_1,m_2}
\]
\[
+ \sum_{m_1 \geq n_1, m_2 < n_2} \varphi_1 m_1, m_2 E_{j_1,j_2,m_1,m_2} + \sum_{m_1 \geq n_1, m_2 \geq n_2} \varphi_1 m_1, m_2 E_{j_1,j_2,m_1,m_2}.
\]
Applying Lemma 12, we get
\[
\frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \varphi(x_{k_1,k_2}) \xi_{j_1,j_2}(x_{k_1,k_2}) = \varphi_{2l_1 n_1 \pm j_1,j_2} + \sum_{l_2=1}^{\infty} (-1)^{l_2} \varphi_{2l_1 n_1 \pm j_1,j_2} + \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} (-1)^{l_1+l_2} \varphi_{2l_1 n_1 \pm j_1,j_2 + j_2},
\]
which gives us formula (14).

**Step 2** (Estimating \(E_{1,2}\)). From (14), we see
\[
|\Upsilon_{j_1,j_2}| \leq \sum_{l_2=1}^{\infty} |\varphi_{2l_1 n_1 \pm j_1,j_2}| + \sum_{l_1=1}^{\infty} |\varphi_{2l_1 n_1 \pm j_1,j_2}| + \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} |\varphi_{2l_1 n_1 \pm j_1,j_2 + j_2}|, \quad j_1 \leq N_1, j_2 \leq N_2.
\]
Since \(\varphi \in H^\theta(\Omega)\), we have \(\lambda_{j_1,j_2}^\theta |\varphi_{j_1,j_2}| \leq \|\varphi\|_{H^\theta(\Omega)}\) for \(j_1, j_2 \in \mathbb{Z}^+\). It follows that
\[
|\varphi_{j_1,2l_2 n_2 \pm j_2}| \leq \frac{2}{(j_1^2 + (2l_2 n_2 - j_2)^2)^{\theta}} \|\varphi\|_{H^\theta(\Omega)} \leq 2l_2^{-\theta} n_2^{-\theta} \|\varphi\|_{H^\theta(\Omega)}.
\]
Similarly, one gets
\[
|\varphi_{2l_1 n_1 \pm j_1,j_2}| \leq 2l_1^{-\theta} n_1^{-\theta} \|\varphi\|_{H^\theta(\Omega)}, \quad |\varphi_{j_1,2l_2 n_2 \pm j_2}| \leq 4 \left( l_1^2 n_1^2 + l_2^2 n_2^2 \right)^{-\frac{\theta}{2}} \|\varphi\|_{H^\theta(\Omega)}.
\]
Hence
\[
|\mathcal{Y}_{j_1,j_2}| \leq \left( 2n_2^{-\theta} \sum_{l_2=1}^{\infty} l_2^{-\theta} + 2n_1^{-\theta} \sum_{l_1=1}^{\infty} l_1^{-\theta} + 4(n_1^{-\theta} + n_2^{-\theta}) \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} l_1^{-\theta} l_2^{-\theta} \right) \|\varphi\|_{H^s(\Omega)}.
\]

Put
\[
C_0 := 2 \sum_{l_1=1}^{\infty} l_1^{-\theta} + 4 \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} l_1^{-\theta} l_2^{-\theta}.
\]

Then, one has
\[
\mathcal{E}_{1,2} = 2 \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \mathcal{Y}_{j_1,j_2} \leq \frac{2C_0^2 N_1 N_2}{n_1 n_2} \|\varphi\|^2_{H^s(\Omega)},
\]
for \(n_1, n_2\) large enough. Now, we conclude that
\[
\mathbb{E}\mathcal{E}_1 \leq \mathbb{E}\mathcal{E}_{1.1} + \mathcal{E}_{1,2} \leq \frac{\left(2\pi^2 \varepsilon_{\text{max}}^2 + 2C_0^2 \|\varphi\|^2_{H^s(\Omega)}\right) N_1 N_2}{n_1 n_2}.
\]

**Part B** (Estimating \(\mathcal{E}_{2,1}\) to \(\mathcal{E}_{2,3}\)). From the definition of \(\mathcal{E}_{2,1}\), one can see that
\[
\mathcal{E}_{2,1} = \sum_{j_1=1}^{N_1} \sum_{j_2=1+N_2+1}^{\infty} (j_1^2 + j_2^2)^{-\theta} \lambda_{j_1,j_2}^\theta \mathcal{Y}_{j_1,j_2} \leq (N_2 + 1)^{-2\theta} \|\varphi\|^2_{H^s(\Omega)}.
\]
Similarly, one obtains
\[
\mathcal{E}_{2,2} \leq (N_1 + 1)^{-2\theta} \|\varphi\|^2_{H^s(\Omega)}, \quad \mathcal{E}_{2,3} \leq [(N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta}] \|\varphi\|^2_{H^s(\Omega)}.
\]
This leads to
\[
\mathcal{E}_{2,1} + \mathcal{E}_{2,2} + \mathcal{E}_{2,3} \leq 2 [(N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta}] \|\varphi\|^2_{H^s(\Omega)}.
\]
Now, from (12), (13) and (19), we deduce that
\[
\mathbb{E} \|\varphi^{N_1,N_2} - \varphi\|^2_{L^2(\Omega)} \leq \frac{\left(2\pi^2 \varepsilon_{\text{max}}^2 + 2C_0^2 \|\varphi\|^2_{H^s(\Omega)}\right) N_1 N_2}{n_1 n_2} + 2 [(N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta}] \|\varphi\|^2_{H^s(\Omega)}.
\]
This completes the proof. \(\Box\)

### 4.2 The ill-posedness of problem with discrete data

This subsection is aimed to demonstrate that the solution of the present problem is not stable, which follows that our problem is not well-posed. The solution of our problem is called stable if for any sequence \(\varphi^{n_1,n_2}\), we have
\[
\lim_{n_1,n_2 \to \infty} \mathbb{E} \|\varphi^{n_1,n_2} - \varphi\|^2_{L^2(\Omega)} = 0,
\]
implies
\[
\lim_{n_1,n_2 \to \infty} \|u^{n_1,n_2} - u\|_{\mathcal{X}_F} = 0,
\]
in which \(u^{n_1,n_2}\) satisfies the system
\[
\begin{cases}
\frac{\partial}{\partial t} u^{n_1,n_2}(t,x) - \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \Delta u^{n_1,n_2}(t,x) = f(t,x,u^{n_1,n_2}(t,x)), \quad (t,x) \in (0,T) \times \Omega, \\
u^{n_1,n_2}(t,x) = 0, \quad (t,x) \in (0,T) \times \partial \Omega, \\
u^{n_1,n_2}(T,x) = \varphi^{n_1,n_2}(x), \quad x \in \Omega.
\end{cases}
\]
Now, we give an example showing that there exists a sequence \(\varphi^{n_1,n_2}\) such that (20) holds but (21) does not:
Example 4.1. Let $0 < \alpha < \frac{1}{2}$, $\varphi = 0$, $f(t, x, u(t, x)) = \mathcal{R}u(t, x)$ with

$$
\mathcal{R} = \mathcal{R}(\alpha, T) = \frac{M_1}{2M_2 T} \sqrt{\frac{1 - 2\alpha}{(1 - 2\alpha) + M_2^2}},
$$

and the observed discrete data is $\Phi_{k_1, k_2}^{obs} = \frac{1}{\varphi^{n_1, n_2}} W_{k_1, k_2}$. Based on the idea in Subsection 4.1, we construct the sequence $\{\varphi^{n_1, n_2}\}$ as follows

$$
\varphi^{n_1, n_2} = \sum_{j_1=1}^{n_1-1} \sum_{j_2=1}^{n_2-1} \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \Phi_{k_1, k_2}^{obs} \xi_{j_1, j_2}(x_{k_1, k_2}) \right) \xi_{j_1, j_2}. \tag{23}
$$

By a similar calculation as in Subsection 4.1 one can check that

$$
\mathbb{E} \left\| \varphi^{n_1, n_2} - \varphi \right\|_{L^2(\Omega)}^2 = \mathbb{E} \left[ \sum_{j_1=1}^{n_1-1} \sum_{j_2=1}^{n_2-1} \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \Phi_{k_1, k_2}^{obs} \xi_{j_1, j_2}(x_{k_1, k_2}) \right)^2 \right] = \frac{\pi^2(n_1 - 1)(n_2 - 1)}{n_1^{3/2} n_2^{3/2}}, \tag{24}
$$

which implies that $\mathbb{E} \left\| \varphi^{n_1, n_2} - \varphi \right\|_{L^2(\Omega)}^2$ tends to zero as $n_1, n_2 \to \infty$.

Next, we show that $\|u^{n_1, n_2} - u\|_{X_T}$ tends to infinity as $n_1, n_2 \to \infty$. To do this, we first prove that $\mathbb{E} \|u(t, \cdot)\|_{L^2(\Omega)}^2 = 0$, which implies that $u \equiv 0$. Indeed, from (25) and $\varphi = 0$, one has

$$
u(t, x) = \int_0^t \mathcal{B}(t - s) f(s, x, u(s, x)) ds - \int_0^T \mathcal{D}(t, T - s) f(s, x, u(s, x)) ds. \tag{25}
$$

From the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, for $a, b \in \mathbb{R}$, and Hölder’s inequality, one can see that

$$
\mathbb{E} \|u(t, \cdot)\|_{L^2(\Omega)}^2 \\
\leq 2\mathbb{E} \left[ \int_0^t \|\mathcal{B}(t - s) f(s, \cdot, u(s, \cdot))\|_{L^2(\Omega)}^2 ds \right] + 2\mathbb{E} \left[ \int_0^T \|\mathcal{D}(t, T - s) f(s, \cdot, u(s, \cdot))\|_{L^2(\Omega)}^2 ds \right] \\
\leq 2t \int_0^T \mathbb{E} \|\mathcal{B}(t - s) f(s, \cdot, u(s, \cdot))\|_{L^2(\Omega)}^2 ds + 2T \int_0^T \mathbb{E} \|\mathcal{D}(t, T - s) f(s, \cdot, u(s, \cdot))\|_{L^2(\Omega)}^2 ds. \tag{26}
$$

Using Lemma 3.2 and the fact that $f(s, x, u(s, x)) = \mathcal{R}u(s, x)$, one gets

$$
\|\mathcal{B}(t - s) f(s, \cdot, u(s, \cdot))\|_{L^2(\Omega)}^2 = \mathcal{R}^2 \|\mathcal{B}(t - s) u(s, \cdot)\|_{L^2(\Omega)}^2 \\
\leq \mathcal{R}^2 M_2^2 \|u(s, \cdot)\|_{L^2(\Omega)}^2,
$$

and

$$
\|\mathcal{D}(t, T - s) f(s, \cdot, u(s, \cdot))\|_{L^2(\Omega)}^2 = \mathcal{R}^2 \|\mathcal{D}(t, T - s) u(s, \cdot)\|_{L^2(\Omega)}^2 \\
\leq \mathcal{R}^2 \left( \frac{M_2^2 T^\alpha}{M_1 (T - s)^\alpha} \right)^2 \|u(s, \cdot)\|_{L^2(\Omega)}^2.
$$

Hence, we deduce that

$$
\mathbb{E} \|u(t, \cdot)\|_{L^2(\Omega)}^2 \leq 2\mathcal{R}^2 M_2^2 t \int_0^t \mathbb{E} \|u(s, \cdot)\|_{L^2(\Omega)}^2 ds + 2\mathcal{R}^2 \left( \frac{M_2^2}{M_1} \right)^2 \mathcal{T}^2 \int_0^T \frac{T^{2\alpha} \mathbb{E} \|u(s, \cdot)\|_{L^2(\Omega)}^2 ds}{(T - s)^{2\alpha}}.
$$

Since $2\mathcal{R}^2 M_2^2 T^2 \left( 1 + \frac{M_2^2}{(1 - 2\alpha) M_1^2} \right) = \frac{1}{2}$, we conclude that $u \equiv 0$.

Now, we are ready to estimate the error $\|u^{n_1, n_2} - u\|_{X_T}$. Applying the result in Section 3 (see (27)), one has

$$
u^{n_1, n_2}(0, x) = \mathcal{A}(0) \varphi^{n_1, n_2}(x) - \int_0^T \mathcal{D}(0, T - s) f(s, x, u^{n_1, n_2}(s, x)) ds.$$
Since \( \langle \varphi^{n_1,n_2}, \xi_{j_1,j_2} \rangle = 0 \) for \( j_1 \geq n_1 \) or \( j_2 \geq n_2 \) and \( E_{\alpha,1}(0) = 1 \), one can see that
\[
A(0)\varphi^{n_1,n_2} = \sum_{j_1=1}^{n_1-1} \sum_{j_2=1}^{n_2-1} \frac{\left( \varphi^{n_1,n_2}, \xi_{j_1,j_2} \right)}{E_{\alpha,1}(\cdot - \lambda_{j_1,j_2} T^\alpha)} \xi_{j_1,j_2}.
\]
It follows that
\[
2 \left\| u^{n_1,n_2}(0,\cdot) \right\|_{L^2(\Omega)}^2 \geq \left\| A(0)\varphi^{n_1,n_2} \right\|_{L^2(\Omega)}^2 - 2 \left\| \int_0^T D(0, T-s) f(s, \cdot, u^{n_1,n_2}(s, \cdot)) ds \right\|_{L^2(\Omega)}^2
\]
\[
\geq \sum_{j_1=1}^{n_1-1} \sum_{j_2=1}^{n_2-1} \frac{\left( \varphi^{n_1,n_2}, \xi_{j_1,j_2} \right)^2}{E_{\alpha,1}(\cdot - \lambda_{j_1,j_2} T^\alpha)} - 2 \mathcal{R} \left[ \int_0^T \left\| D(0, T-s) u^{n_1,n_2}(s, \cdot) \right\|_{L^2(\Omega)}^2 ds \right]^2
\]
\[
\geq \sum_{j_1=1}^{n_1-1} \sum_{j_2=1}^{n_2-1} \frac{\lambda_{j_1,j_2} T^{2\alpha}}{M_2^2} \left( \varphi^{n_1,n_2}, \xi_{j_1,j_2} \right)^2 - 2 \mathcal{R} \left( \int_0^T \| D(0, T-s) u^{n_1,n_2}(s, \cdot) \|_{L^2(\Omega)}^2 ds \right)^2.
\]
(27)

By a similar calculation as in (24), we have
\[
E \left( \langle \varphi^{n_1,n_2}, \xi_{j_1,j_2} \rangle \right)^2 = E \left( \frac{\pi^2}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \tilde{\Phi}^{obs}_{k_1,k_2} \xi_{j_1,j_2}(x_{k_1,k_2}) \right)^2 = \frac{\pi^2}{n_1^{3/2} n_2^{3/2}}.
\]
(28)

On the other hand
\[
2 \mathcal{R} T \int_0^T E \| D(0, T-s) u^{n_1,n_2}(s, \cdot) \|_{L^2(\Omega)}^2 ds \leq 2 \mathcal{R} \left( \frac{M_2^2}{M_1^2} \right)^2 T \int_0^T \frac{T^{2\alpha}}{(T-s)^{2\alpha}} E \| u(s, \cdot) \|_{L^2(\Omega)}^2 ds
\]
\[
\leq 2 \mathcal{R} T^2 \sup_{t \in [0,T]} E \| u(t, \cdot) \|_{L^2(\Omega)}^2.
\]
(29)

Combining (27)-(29), we deduce that
\[
2 \mathcal{E} \left\| u^{n_1,n_2}(0, \cdot) \right\|_{L^2(\Omega)}^2 \geq \frac{\pi^2 T^{2\alpha}}{M_2^2} \lambda_{n_1,n_2-1} \left( \frac{3}{n_1^{3/2} n_2^{3/2}} \right)^2 - 2 \mathcal{R} T^2 \frac{M_1^4}{(1-2\alpha) M_1^2} \sup_{t \in [0,T]} E \| u(t, \cdot) \|_{L^2(\Omega)}^2.
\]

Since \( 2 \mathcal{R} T^2 \frac{M_1^4}{(1-2\alpha) M_1^2} \leq \frac{1}{2} \) and \( \lambda_{j_1,j_2} = \sqrt{j_1^2 + j_2^2} \), we obtain
\[
\frac{3}{2} \sup_{t \in [0,T]} E \left\| u^{n_1,n_2}(t, \cdot) \right\|_{L^2(\Omega)}^2 \geq \frac{\pi^2 T^{2\alpha} (n_1-1)^2 + (n_2-1)^2}{M_2^2} \left( \frac{3}{n_1^{3/2} n_2^{3/2}} \right)^2.
\]

Using the fact that
\[
\left\| u^{n_1,n_2} - u \right\|_{X_T} = \left\| u^{n_1,n_2} \right\|_{X_T} = \sup_{t \in [0,T]} \sqrt{E \left\| u^{n_1,n_2}(t, \cdot) \right\|_{L^2(\Omega)}^2} \geq \sqrt{\sup_{t \in [0,T]} E \left\| u^{n_1,n_2}(t, \cdot) \right\|_{L^2(\Omega)}^2},
\]
we conclude that \( \left\| u^{n_1,n_2} - u \right\|_{X_T} \) tends to infinity as \( n_1, n_2 \to \infty \).

### 4.3 Fourier truncated method and regularized solution

For \( N_1, N_2 \) as in Lemma 4.11 and the function \( g \in L^2(\Omega) \), we set
\[
A^{N_1,N_2}(t) := \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{E_{\alpha,1}(\cdot - \lambda_{j_1,j_2} T^\alpha)}{E_{\alpha,1}(\cdot - \lambda_{j_1,j_2} T^\alpha)} \right) \langle g, \xi_{j_1,j_2} \rangle \xi_{j_1,j_2},
\]
\[
B^{N_1,N_2}(t) := \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{E_{\alpha,1}(\cdot - \lambda_{j_1,j_2} T^\alpha)}{E_{\alpha,1}(\cdot - \lambda_{j_1,j_2} T^\alpha)} \right) \langle g, \xi_{j_1,j_2} \rangle \xi_{j_1,j_2},
\]

10
and \( \mathbb{D}^{N_1,N_2}(t,s) \) are truncated series of \( A(t)g, B(t)g \) and \( D(t,s)g \) respectively. Based on the approximate function for \( \varphi \) constructed as in Lemma 4.1 (denote by \( \tilde{\varphi}^{N_1,N_2} \)), we give the regularized solution as follows

\[
\tilde{\varphi}^{N_1,N_2}(t,s) = \mathbb{A}^{N_1,N_2}(t)\varphi^{N_1,N_2}(x) + \int_0^t \mathbb{B}^{N_1,N_2}(t-s)f(s,x,\tilde{\varphi}^{N_1,N_2}(s,x))ds
- \int_0^T \mathbb{D}^{N_1,N_2}(t,T-s)f(s,x,\tilde{\varphi}^{N_1,N_2}(s,x))ds.
\]  

(30)

Before presenting the convergence rate between \( \tilde{\varphi}^{N_1,N_2} \) and the solution \( u \) of (1) of the two-dimensional, we show the existence and uniqueness of the solution \( \tilde{\varphi}^{N_1,N_2} \):

**Lemma 4.3.** Assume that \( f \) satisfies the globally Lipschitz property, i.e. there exists a positive constant \( K \) such that

\[
\|f(t,\cdot,u_1(t,\cdot)) - f(t,\cdot,u_2(t,\cdot))\|_{L^2(\Omega)} \leq K \|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\Omega)}, \quad u_1, u_2 \in L^2(\Omega).
\]  

(31)

Assume further that \( K \in \left(0, Q_{\alpha,T}^{-1}\right) \) where

\[
Q_{\alpha,T} = \frac{2M_2T\sqrt{2(M_1(1-\alpha)^2 + M_2^2)}}{M_1(1-\alpha)}.
\]  

(32)

Then, the integral equation (66) has a unique solution \( \tilde{\varphi}^{N_1,N_2} \in \mathbb{X}_T \).

**Proof.** Put

\[
\mathfrak{F}(v(t,x)) := \mathbb{A}^{N_1,N_2}(t)\varphi^{N_1,N_2}(x) + \int_0^t \mathbb{B}^{N_1,N_2}(t-s)f(s,x,v(s,x))ds
- \int_0^T \mathbb{D}^{N_1,N_2}(t,T-s)f(s,x,v(s,x))ds.
\]

We will show that \( \mathfrak{F}(v_1) - \mathfrak{F}(v_2) \leq \frac{KQ_{\alpha,T}}{2} \|v_1 - v_2\|_{\mathbb{X}_T} \), which implies that \( \mathfrak{F} \) is a contraction and thus the equation (66) has a unique solution. We first have

\[
\|\mathfrak{F}(v_1(t,\cdot)) - \mathfrak{F}(v_2(t,\cdot))\|_{L^2(\Omega)} \leq \int_0^t \|\mathbb{B}^{N_1,N_2}(t-s)(f(s,\cdot,v_1(s,\cdot)) - f(s,\cdot,v_2(s,\cdot)))\|_{L^2(\Omega)} ds
+ \int_0^T \|\mathbb{D}^{N_1,N_2}(t,T-s)(f(s,\cdot,v_1(s,\cdot)) - f(s,\cdot,v_2(s,\cdot)))\|_{L^2(\Omega)} ds.
\]

Using Lemma 3.2 and assumption (51), we get

\[
\|\mathfrak{F}(v_1(t,\cdot)) - \mathfrak{F}(v_2(t,\cdot))\|_{L^2(\Omega)} \leq KM_2 \int_0^t \|v_1(s,\cdot) - v_2(s,\cdot)\|_{L^2(\Omega)} ds
+ KM_2 \frac{T^{1-\alpha}}{M_1} \int_0^T (T-s)^{-\alpha} \|v_1(s,\cdot) - v_2(s,\cdot)\|_{L^2(\Omega)} ds.
\]
By Hölder’s inequality and \((a + b)^2 \leq 2(a^2 + b^2)\), for \(a, b \in \mathbb{R}\), we obtain

\[
E\|\mathfrak{F}(v_1(t, \cdot)) - \mathfrak{F}(v_2(t, \cdot))\|_{L^2(\Omega)}^2 \leq 2K^2M_2^2 t \int_0^T E\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^2 \, ds \\
+ 2K^2M_2^2 T^{2\alpha} \int_0^T (T - s)^{-\alpha} ds \int_0^T (T - s)^{-\alpha} E\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^2 \, ds.
\]

It follows that

\[
E\|\mathfrak{F}(v_1(t, \cdot)) - \mathfrak{F}(v_2(t, \cdot))\|_{L^2(\Omega)}^2 \leq 2K^2M_2^2 T^2 \sup_{t \in [0,T]} E\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^2 \\
+ 2K^2M_2^2 \frac{T^2}{(1-\alpha)^2} \sup_{t \in [0,T]} E\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^2.
\]

Thus

\[
\|\mathfrak{F}(v_1) - \mathfrak{F}(v_2)\|_{X_T}^2 \leq 2K^2M_2^2 T^2 \left(1 + \frac{M_2^2}{M_1^2 (1-\alpha)^2}\right) \sup_{t \in [0,T]} E\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^2 \\
= \left(\frac{KQ_{\alpha,T}}{2}\right)^2 \sup_{t \in [0,T]} E\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\Omega)}^2.
\]

We conclude that \(\|\mathfrak{F}(v_1) - \mathfrak{F}(v_2)\|_{X_T} \leq \frac{KQ_{\alpha,T}}{2} \|v_1 - v_2\|_{X_T}\). This completes the proof. \(\square\)

Note in the above result we could assume \(K \in (0, Q_{\alpha,T}^{-1})\). However \(K \in (0, Q_{\alpha,T}^{-1})\) is needed in our next result.

### 4.4 Convergence result

Now, we are ready to state the main result of the present section in the following theorem:

**Theorem 4.1.** Let \(N_1 = N_1(n_1)\), \(N_2 = N_2(n_2)\) be natural numbers less than \(n_1, n_2\) respectively and satisfy

\[
\lim_{n_1 \to \infty} N_1 = \lim_{n_2 \to \infty} N_2 = \infty, \quad \lim_{n_1,n_2 \to \infty} \frac{N_1N_2 \lambda_{N_1,N_2}^2}{n_1n_2} = 0. \quad (33)
\]

Assume the conditions of Lemma 4.1 and Lemma 4.3 hold and \(u \in L^\infty(0,T; H^0(\Omega))\). Then, the error \(\|\tilde{u}^{N_1,N_2} - u\|_{X_T}^2\) is of order

\[
\max \left\{ \frac{N_1N_2}{n_1n_2} (N_1^4 + N_2^4), N_1^{-2\theta}N_2^{-2\theta} \right\}.
\]

**Remark 4.2.** If we choose

\((N_1, N_2) = (\lceil n_1^{1/5} \rceil, \lceil n_2^{1/5} \rceil)\) or \((N_1, N_2) = (\lceil \log n_1 \rceil, \lceil \log n_2 \rceil),\)

where we denote \([p]\) the greatest natural number less than \(p\), then \(N_1, N_2\) satisfy (33).

**Proof.** Let

\[
v^{N_1,N_2}(t, x) = A^{N_1,N_2}(t)\varphi(x) + \int_0^t \mathbb{B}^{N_1,N_2}(t-s)f(s, x, u(s, x)) \, ds \\
- \int_0^T \mathbb{D}^{N_1,N_2}(t, T-s)f(s, x, u(s, x)) \, ds, \quad (34)
\]
which is the truncated series of \( u \). Then, we have

\[
\frac{1}{2} \| \tilde{u}^{N_1, N_2} (t, \cdot) - u (t, \cdot) \|^2_{L^2(\Omega)} \leq \| \tilde{u}^{N_1, N_2} (t, \cdot) - v^{N_1, N_2} (t, \cdot) \|^2_{L^2(\Omega)} + \| v^{N_1, N_2} (t, \cdot) - u (t, \cdot) \|^2_{L^2(\Omega)}.
\]

**Step 1** (Estimating the error between \( \tilde{u}^{N_1, N_2} \) and \( v^{N_1, N_2} \)). From (35) and (36), one can see that

\[
\| \tilde{u}^{N_1, N_2} (t, \cdot) - v^{N_1, N_2} (t, \cdot) \|^2_{L^2(\Omega)} \leq \left\| A^{N_1, N_2} (t) \left( \tilde{\varphi}^{N_1, N_2} - \varphi \right) \right\|^2_{L^2(\Omega)}
\]

\[
+ \int_0^T \left\| B^{N_1, N_2} (t - s) (f (s, \cdot, \tilde{u}^{N_1, N_2} (s, \cdot)) - f (s, \cdot, u (s, \cdot))) \right\|^2_{L^2(\Omega)} ds
\]

\[
+ \int_0^T \left\| D^{N_1, N_2} (t - s) (f (s, \cdot, \tilde{u}^{N_1, N_2} (s, \cdot)) - f (s, \cdot, u (s, \cdot))) \right\|^2_{L^2(\Omega)} ds
\]

\[=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \tag{35}\]

For \( g \in L^2(\Omega) \), we have

\[
\| \tilde{u}^{N_1, N_2} (t) g \|^2_{L^2(\Omega)} \leq \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \frac{M_2 (1 + \lambda_{j_1, j_2} T^\alpha)}{M_1} \right)^2 \left\| g (j_1, j_2) \right\|^2
\]

\[
\leq \left( \frac{M_2 (1 + \lambda_{N_1, N_2} T^\alpha)}{M_1} \right)^2 \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left\| g (j_1, j_2) \right\|^2,
\]

and recall that (see (12) and (18))

\[
\mathbb{E} \mathcal{E}_1 = \mathbb{E} \left[ \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( \left\| \tilde{\varphi}^{N_1, N_2} - \varphi \right\| \right) \right] \leq \frac{\left( 2 \pi^2 \varepsilon^2 + 2 C_0^2 \| \varphi \|^2_{H^0(\Omega)} \right) N_1 N_2}{n_1 n_2},
\]

Hence

\[
\mathbb{E} \mathcal{I}_1^2 \leq \left( \frac{M_2 (1 + \lambda_{N_1, N_2} T^\alpha)}{M_1} \right)^2 \mathbb{E} \mathcal{E}_1 \leq \left( \frac{M_2 (1 + \lambda_{N_1, N_2} T^\alpha)}{M_1} \right)^2 \left( 2 \pi^2 \varepsilon^2 + 2 C_0^2 \| \varphi \|^2_{H^0(\Omega)} \right) N_1 N_2 \tag{36}\]

For the term \( \mathcal{I}_2 + \mathcal{I}_3 \), by a similar technique as in the proof of Lemma 4.3, one arrives at

\[
\mathbb{E} (\mathcal{I}_2 + \mathcal{I}_3)^2 \leq 2 K^2 M_2^2 T^2 \left( 1 + \frac{M_2^2}{M_1^2 (1 - \alpha)^2} \right) \sup_{t \in [0, T]} \mathbb{E} \sum_{t} \| \tilde{u}^{N_1, N_2} (t, \cdot) - u (t, \cdot) \|^2_{L^2(\Omega)} \tag{37}\]

Combining (35)–(37), one obtains

\[
\mathbb{E} \| \tilde{u}^{N_1, N_2} (t, \cdot) - v^{N_1, N_2} (t, \cdot) \|^2 \leq 4 \left( \frac{M_2 (1 + \lambda_{N_1, N_2} T^\alpha)}{M_1} \right)^2 \left( 2 \pi^2 \varepsilon^2 + C_0^2 \| \varphi \|^2_{H^0(\Omega)} \right) N_1 N_2
\]

\[
+ 4 K^2 M_2^2 T^2 \left( 1 + \frac{M_2^2}{M_1^2 (1 - \alpha)^2} \right) \| \tilde{u}^{N_1, N_2} - u \|^2_{L^2(\Omega)}.
\]

**Step 2** (Estimating the error between \( v^{N_1, N_2} \) and \( u \)). From (39) and (41), one can see that

\[
\| v^{N_1, N_2} (t, \cdot) - u (t, \cdot) \|^2_{L^2(\Omega)} = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \| u_{j_1, j_2} (t) \|^2 + \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \| u_{j_1, j_2} (t) \|^2 + \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \| u_{j_1, j_2} (t) \|^2.
\]

The quantity above can be estimated in exactly the same way as in Part B in the proof of Lemma 4.1. In this way, one gets

\[
\| v^{N_1, N_2} - u \|^2_{L^2(\Omega)} \leq 2 \left[ (N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta} \right] \| u (t, \cdot) \|^2_{H^\theta(\Omega)}
\]

\[
\leq 2 \left[ (N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta} \right] \| u \|^2_{L^\infty (0, T; H^\theta(\Omega))}.
\]
Now, using the results of the two steps, we deduce that
\[
\frac{1}{2} \mathbb{E} \left\| \tilde{u}^{N_1,N_2}(t, \cdot) - u(t, \cdot) \right\|_{L^2(\Omega)}^2 \leq 4 \left( \frac{M_2(1 + \lambda N_1 N_2 T^\alpha)}{M_1} \right)^2 \left( \pi^2 \varepsilon_{\max}^2 + C_0^2 \| \varphi \|^2_{H^s(\Omega)} \right) \frac{N_1 N_2}{n_1 n_2} + 4 K^2 M_2^2 T^2 \left( 1 + \frac{M_3^2}{M_1^2 (1-\alpha)^2} \right) \left\| \tilde{u}^{N_1,N_2} - u \right\|_{\mathcal{X}^T}^2 + 2 \left[ (N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta} \right] \| u \|^2_{L^\infty(0,T;H^s(\Omega))},
\]
which gives
\[
\frac{1 - K^2 Q_n^2 T}{2} \left\| \tilde{u}^{N_1,N_2} - u \right\|_{\mathcal{X}^T}^2 \leq 4 \left( \frac{M_2(1 + \lambda N_1 N_2 T^\alpha)}{M_1} \right)^2 \left( \pi^2 \varepsilon_{\max}^2 + C_0^2 \| \varphi \|^2_{H^s(\Omega)} \right) \frac{N_1 N_2}{n_1 n_2} + 2 \left[ (N_1 + 1)^{-2\theta} + (N_2 + 1)^{-2\theta} \right] \| u \|^2_{L^\infty(0,T;H^s(\Omega))},
\]
Hence, we conclude that \( \| \tilde{u}^{N_1,N_2} - u \|^2_{\mathcal{X}^T} \) is of order
\[
\max \left\{ \frac{N_1 N_2}{n_1 n_2} (N_1^4 + N_2^4), N_1^{-2\theta}, N_2^{-2\theta} \right\}.
\]

5 Backward problem in the multi-dimensional case

Based on Subsection 4.2 we claim that the multi-dimensional backward problem (1) with discrete data is not well-posed. Thus, a regularized method is required to construct a stable approximate solution. To do this, in next subsection, we establish an approximation for the final data \( \varphi \).

5.1 Estimator for \( \varphi \) in the multi-dimensional case

For any positive constant \( \gamma_n = \gamma_{n_1,n_2,\ldots,n_d} \) depending on \( n = (n_1, n_2, \ldots, n_d) \), we define
\[
\mathcal{W}_n = \left\{ \mathbf{j} = (j_1,j_2,\ldots,j_d) \in \mathbb{N}^d : \sum_{i=1}^d j_i^2 \leq \gamma_n \right\}.
\]
(38)

For \( \gamma_n \) satisfying \( \lim_{|n| \to \infty} \gamma_n = \infty \), we define an approximation for \( \varphi \) as follows
\[
\tilde{\varphi}^n = \sum_{\mathbf{j} \in \mathcal{W}_n} \frac{\pi^d}{\prod_{i=1}^d n_i} \prod_{i=1}^d \sum_{k_i=1}^{n_i} \prod_{i=1}^d \Phi_k^{d_{\mathbf{j}}} \xi_k x_k \xi_j.
\]
(39)

Theorem 5.1. Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_d) \in \mathbb{R}^d \) in which \( \mu_k > \frac{1}{2} \) for any \( k = 1, 2, \ldots, d \) and \( \mu_0 \in \mathbb{R}^+ \) such that \( \mu_0 \geq d \max(\mu_1, \mu_2, \ldots, \mu_d) \). Then we have

(a) Error estimate in \( L^2(\Omega) \) (see Theorem 2.1 of [27]). If \( \varphi \in H^{\mu_0}(\Omega) \) then
\[
\mathbb{E} \left\| \tilde{\varphi}^n - \varphi \right\|_{L^2(\Omega)}^2 \leq \overline{C}(\mu, \varphi) \gamma_n^{d/2} \prod_{i=1}^d n_i^{-4\mu_i} + 4 \gamma_n^{-\mu_0} \| \varphi \|^2_{H^{\mu_0}(\Omega)},
\]
where
\[
\overline{C}(\mu, \varphi) = 8 \pi^d \varepsilon_{\max}^2 \frac{2 \pi^{d/2}}{dT \left( \frac{d}{2} \right)} + \frac{16 C^2(\mu) \pi^{d/2}}{dT \left( \frac{d}{2} \right)} \| \varphi \|^2_{H^{\mu_0}(\Omega)},
\]
with
\[
C(\mu) = d \max(n_1, n_2, \ldots, n_d) \sum_{i \in \mathbb{N}^d, |i| \neq 0} \prod_{l=1}^d (2l_i - 1)^{-2\mu_i}.
\]
(b) **Error estimate in** $H^\sigma(\Omega)$. If there exists a constant $\sigma > 0$ such that $\varphi \in H^{n+\sigma}(\Omega)$ then

$$
E\|\tilde{\varphi}^n - \varphi\|^2_{H^\sigma(\Omega)} \leq C(\mu, \varphi) \frac{\sigma + \frac{d}{4}}{4} \prod_{i=1}^{d} (n_i)^{-4\mu_i} + \gamma_n^{-\mu_o} \|\varphi\|_{H^{n+\sigma}(\Omega)}^2.
$$

**Proof of Part (b).** First, by Lemma 2.3 of [20], we have

$$
\varphi_j = \frac{\pi^d}{\prod_{i=1}^{d} n_i} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varphi(x_k)\xi_j(x_k) - \sum_{p=2l-n \pm j} \varphi_p.
$$

Thus, we get

$$
\tilde{\varphi}^n(x) - \varphi(x)
= \sum_{j \in \mathcal{W}_n} \left[ \frac{\pi^d}{\prod_{i=1}^{d} n_i} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \Phi_k \varphi_j(x_k) - \varphi_j \right] \xi_j(x) - \sum_{j \notin \mathcal{W}_n} \varphi_j \xi_j(x)
= \sum_{j \in \mathcal{W}_n} \left[ \frac{\pi^d}{\prod_{i=1}^{d} n_i} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varepsilon_k W_k \varphi_j(x_k) + \sum_{p=2l-n \pm j} \varphi_p \right] \xi_j(x) - \sum_{j \notin \mathcal{W}_n} \varphi_j \xi_j(x).
$$

It follows that

$$
\|\tilde{\varphi}^n - \varphi\|^2_{H^\sigma(\Omega)}
= \sum_{j \in \mathcal{W}_n} \lambda^2 j \left[ \frac{\pi^d}{\prod_{i=1}^{d} n_i} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varepsilon_k W_k \varphi_j(x_k) + \sum_{p=2l-n \pm j} \varphi_p \right]^2 + \sum_{j \notin \mathcal{W}_n} \lambda^2 j \varphi^2_j
\leq 2 \sum_{j \in \mathcal{W}_n} \lambda^2 j \left[ \frac{\pi^d}{\prod_{i=1}^{d} n_i} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varepsilon_k W_k \varphi_j(x_k) \right]^2 + 2 \sum_{j \notin \mathcal{W}_n} \lambda^2 j \left[ \sum_{p=2l-n \pm j} \varphi_p \right]^2 + \sum_{j \notin \mathcal{W}_n} \lambda^2 j \varphi^2_j
= : C_I + C_{II} + C_{III}.
$$

(40)

From Lemma 2.2 of [20] and the property of $W_k$, the first term can be estimated as follows

$$
E C_I \leq 2 \sum_{j \in \mathcal{W}_n} \gamma_n \frac{\pi^{2d}}{\prod_{i=1}^{d} n_i} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varepsilon_k W_k \varphi_j(x_k) \leq \frac{2\pi^{2d}}{\prod_{i=1}^{d} n_i} \gamma_n \varepsilon_{\text{max}}(\mathcal{W}_n) \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varphi^2_j(x_k) = \frac{2\pi^{2d}}{\prod_{i=1}^{d} n_i} \gamma_n \varepsilon_{\text{max}}(\mathcal{W}_n).$

In addition, using the inequality (2.50) of [20] that $\varepsilon_{\text{max}}(\mathcal{W}_n) \leq \frac{2\pi^{2d}}{dT(\pi^2 \varepsilon_n)} \gamma_n$, we deduce that

$$
E C_I \leq \frac{2\pi^{2d}}{dT(\pi^2 \varepsilon_n)} \gamma_n \varepsilon_{\text{max}}(\mathcal{W}_n) \leq \frac{4\pi^{2d}}{dT(\pi^2 \varepsilon_n)} \gamma_n \varepsilon_{\text{max}}(\mathcal{W}_n).
$$

(41)

From (2.37) of [20], the second term can be estimated as follows

$$
C_{II} = 2 \sum_{j \in \mathcal{W}_n} \lambda^2 j \left[ \sum_{p=2l-n \pm j} \varphi_p \right]^2
\leq 2\varepsilon_n^2 C^2(\mu) \|\varphi\|_{H^{n+\sigma}(\Omega)}^2 \prod_{i=1}^{d} (n_i)^{-4\mu_i} \varepsilon_{\text{max}}(\mathcal{W}_n) \leq 2\varepsilon_n^2 \frac{4\pi^{2d}}{dT(\pi^2 \varepsilon_n)} \gamma_n \varepsilon_{\text{max}}(\mathcal{W}_n) \prod_{i=1}^{d} (n_i)^{-4\mu_i}.
$$

(42)
Combining (40)-(43), we conclude that
\begin{equation}
C_{III} = \sum_{j \in \mathbb{N}} \lambda_j^{-\mu_{0}} \lambda_j^{\mu_{0}+\sigma} \varphi_j^2 \leq C_{n+\sigma} \|\varphi\|^2_{H^\mu_{0}+\sigma(\Omega)}.
\end{equation}

5.2 Quasi-boundary value method and regularized solution

For the last term, it is clear that
\[ 5.2 \text{ Quasi-boundary value method and regularized solution} \]

For convenience, we define the left-hand side of the last equation. Then, using the result (5), one has
\[ \phi \text{ basic idea is to replace the final value data } \varphi \text{ by its approximation } \hat{\varphi}_n \text{ and add the quantity } \partial_n \hat{\varphi}_n(0, x) \text{ into the left-hand side of the last equation. Then, using the result } (5), \text{ one has} \]
\[ \hat{\varphi}_n(t, x) = \hat{\varphi}_n^0 \hat{\varphi}_n(t, x) + \int_0^t \hat{\varphi}_n(1 - \alpha_j(t)) \hat{\varphi}_n(0, x) ds, \]

where we denote \( \hat{\varphi}_n^0 = \langle \hat{\varphi}_n, \xi_j \rangle \). On the other hand, the last equation of (44) gives us
\[ \hat{\varphi}_n^0(T, x) + \partial_n \hat{\varphi}_n(0, x) = \hat{\varphi}_n^0, \]

where \( \hat{\varphi}_n^0 := \langle \hat{\varphi}_n, \xi_j \rangle \). From the two latter equations, one can see that
\[ \hat{\varphi}_n^0(t) = \frac{E_{\alpha, 1}(-\lambda_j \alpha)}{\partial_n + E_{\alpha, 1}(-\lambda_j \alpha)} \hat{\varphi}_n^0 + \int_0^t E_{\alpha, 1}(-\lambda_j (t-s)^\alpha) f_j(\hat{\varphi}_n^0) ds \]

For convenience, we define
\[ \hat{\varphi}_n(t) := \sum_{j \in \mathbb{N}} \left( \frac{E_{\alpha, 1}(-\lambda_j \alpha)}{\partial_n + E_{\alpha, 1}(-\lambda_j \alpha)} \langle \hat{\varphi}_n, \xi_j \rangle \right) \xi_j, \]

and \( \tilde{B}_n(t, s) := \hat{\varphi}_n(t) B(s) g, \) for \( t, s \in [0, T] \). Then, \( \hat{\varphi}_n^0 \) (called regularized solution) satisfies the equation
\[ \hat{\varphi}_n^0(t, x) = \tilde{B}_n(t, x) \hat{\varphi}_n^0(x) + \int_0^t B(t-s)f(s, x, \hat{\varphi}_n^0(s, x) ds \]

\[ \text{subject to } \tilde{B}_n(t, S) = \hat{\varphi}_n^0(S), \quad S \in [0, T]. \]
5.3 Convergence results

We now estimate the error between the regularized solution \( \hat{u}^{\gamma_n, \vartheta_n} \) and the sought solution \( u \) in two different cases of space under the following assumptions:

\[(H1)\ f \] satisfies the globally Lipschitz property, i.e., there exists a positive constant \( K \) such that
\[
\|f(t, \cdot, u_1(t, \cdot)) - f(t, \cdot, u_2(t, \cdot))\|_{L^2(\Omega)} \leq K \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\Omega)}, \quad u_1, u_2 \in L^2(\Omega).
\]

\[(H2)\ K \in (0, Q_{\alpha,T}^{-1}) \] in which \( Q_{\alpha,T}^{-1} \) is defined in \([32]\).

**Part A (Convergence rate in \( X_T \)).** In this part, we give the error estimate in the space \( X_T \):

**Theorem 5.2.** Let \( \gamma_n, \vartheta_n \), with \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \), satisfying
\[
\lim_{|n| \to \infty} \gamma_n = \infty, \quad \lim_{|n| \to \infty} \vartheta_n = 0, \quad \text{and} \quad \lim_{|n| \to \infty} \frac{\gamma_{n}^{d/2}}{\vartheta_n^2} \prod_{i=1}^{d} n_{i}^{-4\mu_{1}} = \lim_{|n| \to \infty} \frac{\gamma_{n}^{-\mu_{0}}}{\vartheta_n^{2}} = 0.
\]

Assume that \( u(0, \cdot) \in H^1(\Omega), \varphi \in H^{\mu_{0}}(\Omega) \) with \( \mu_{0} \) as in Theorem 5.1 and the assumptions (H1),(H2) are satisfied. Then, \( \|\hat{u}^{N_1,N_2} - u\|^2_{X_T} \) is of order
\[
\max \left\{ \frac{\gamma_{n}^{d/2}}{\vartheta_n^2} \prod_{i=1}^{d} n_{i}^{-4\mu_{1}}, \frac{\gamma_{n}^{-\mu_{0}}}{\vartheta_n^{2}}, \vartheta_n \right\}.
\]

In order to prove the theorem above, we first give some properties for the operators appearing in the equation \([46]\).

**Lemma 5.1.** Let \( g \) be a function in \( L^2(\Omega) \). Then
\[
\left\| \hat{A}_{\vartheta_n}(t)g \right\|_{L^2(\Omega)} \leq \frac{M_2}{\vartheta_n^2} \|g\|_{L^2(\Omega)}, \quad \text{for} \ 0 \leq t \leq T,
\]

and
\[
\left\| \hat{D}_{\vartheta_n}(t,s)g \right\|_{L^2(\Omega)} \leq \frac{M_2^2 T_{\alpha}}{M_1 s^\alpha} \|g\|_{L^2(\Omega)}, \quad \text{for} \ 0 \leq t \leq T, 0 < s \leq T.
\]

**Proof.** Since
\[
\frac{E_{\alpha,1}(-\lambda_2 t^\alpha)}{\vartheta_n + E_{\alpha,1}(-\lambda_2 T^\alpha)} \leq \frac{E_{\alpha,1}(-\lambda_2 t^\alpha)}{\vartheta_n} \leq \frac{M_2}{\vartheta_n},
\]

and
\[
\frac{E_{\alpha,1}(-\lambda_2 t^\alpha)}{\vartheta_n + E_{\alpha,1}(-\lambda_2 T^\alpha)} E_{\alpha,1}(-\lambda_2 s^\alpha) \leq \frac{E_{\alpha,1}(-\lambda_2 t^\alpha)}{E_{\alpha,1}(-\lambda_2 T^\alpha)} E_{\alpha,1}(-\lambda_2 s^\alpha) \leq \frac{M_2^2 T_{\alpha}}{M_1 s^\alpha},
\]

we have
\[
\left\| \hat{A}_{\vartheta_n}(t)g \right\|_{L^2(\Omega)}^2 \leq \frac{M_2^2}{\vartheta_n^2} \sum_{j \in \mathbb{N}^d} \langle g, \xi_j \rangle^2 = \frac{M_2^2}{\vartheta_n^2} \|g\|^2_{L^2(\Omega)}, \quad \text{and} \quad \left\| \hat{D}_{\vartheta_n}(t,s)g \right\|_{L^2(\Omega)} \leq \frac{M_2^2 T_{\alpha}}{M_1 s^\alpha} \|g\|^2_{L^2(\Omega)}.
\]

This completes the proof. \( \square \)

Under assumptions (H1) and (H2), one can check that the integral equation \([46]\) has a unique solution \( \hat{u}^{\gamma_n, \vartheta_n} \in X_T \) using Lemma 3.2, Lemma 5.1 and a similar method as in Lemma 4.3. Now, we are ready to prove Theorem 5.1.
Proof of Theorem 5.1. Let

\[ v^{\alpha_n}(t, x) = \tilde{A}_{\alpha_n}(t) \varphi(x) + \int_0^t \mathcal{B}(t-s)f(s, x, u(s, x))ds - \int_0^T \hat{D}_{\alpha_n}(t, T-s)f(s, x, u(s, x))ds. \]  

Then, we have

\[ \|\tilde{u}^{\alpha_n, \alpha_n}(t, \cdot) - u(t, \cdot)\|_{L^2(\Omega)} \leq \|\tilde{u}^{\alpha_n, \alpha_n}(t, \cdot) - v^{\alpha_n}(t, \cdot)\|_{L^2(\Omega)} + \|v^{\alpha_n}(t, \cdot) - u(t, \cdot)\|_{L^2(\Omega)}. \]  

**Step 1** (Estimate the first term of (53)). It follows from (50) and (47) that

\[ \|\tilde{u}^{\alpha_n, \alpha_n}(t, \cdot) - v^{\alpha_n}(t, \cdot)\|_{L^2(\Omega)} \leq \left\| \tilde{A}_{\alpha_n}(t) (\tilde{\varphi}^{\alpha_n} - \varphi) \right\|_{L^2(\Omega)} + \int_0^t \left\| B(t-s)(f(s, \cdot, \tilde{u}^{\alpha_n, \alpha_n}(s, \cdot)) - f(s, \cdot, u(s, \cdot))) \right\|_{L^2(\Omega)} ds 
\]
\[ + \int_0^T \left\| \hat{D}_{\alpha_n}(t, T-s)(f(s, \cdot, \tilde{u}^{\alpha_n, \alpha_n}(s, \cdot)) - f(s, \cdot, u(s, \cdot))) \right\|_{L^2(\Omega)} ds \]
\[ =: \Theta_1 + \Theta_2 + \Theta_3. \]

From Lemma 5.1, one gets

\[ E\Theta_1^2 = E \left\| \tilde{A}_{\alpha_n}(t) (\tilde{\varphi}^{\alpha_n} - \varphi) \right\|^2_{L^2(\Omega)} \leq \frac{M^2_2}{\eta_n^2} E \|\tilde{\varphi}^{\alpha_n} - \varphi\|^2_{L^2(\Omega)}. \]

For the term \(\Theta_2\) and \(\Theta_3\), by a similar technique as in (57), one arrives at

\[ E (\Theta_2 + \Theta_3)^2 \leq 2K^2M^2_2T^2 \left( 1 + \frac{M^2_2}{M^2_1(1 - \alpha)^2} \right) \sup_{t \in [0, T]} E \|\tilde{u}^{\alpha_n, \alpha_n}(t, \cdot) - u(t, \cdot)\|^2_{L^2(\Omega)}. \]

Hence

\[ E \left\| \tilde{u}^{\alpha_n, \alpha_n}(t, \cdot) - v^{\alpha_n}(t, \cdot) \right\|^2_{L^2(\Omega)} \leq 2E\Theta_2^2 + 2E (\Theta_2 + \Theta_3)^2 \leq 2 \frac{M^2_2}{\eta_n^2} E \|\tilde{\varphi}^{\alpha_n} - \varphi\|^2_{L^2(\Omega)} + \frac{\Theta_3^2}{2} \|\tilde{u}^{\alpha_n, \alpha_n} - u\|^2_{X_T}. \]  

**Step 2** (Estimate the second term of (53)). It follows from (5) and (47) that

\[ \|v^{\alpha_n}(t, \cdot) - u(t, \cdot)\|^2_{L^2(\Omega)} \]
\[ = \left\| \tilde{A}_{\alpha_n}(t) \varphi - A(t) \varphi - \int_0^T \left( \hat{D}_{\alpha_n}(t, T-s)f(s, \cdot, u(s, \cdot)) - D(t, T-s)f(s, \cdot, u(s, \cdot)) \right) ds \right\|^2_{L^2(\Omega)} 
\]
\[ = \left\| \sum_{j \in \mathbb{N}^d} \varphi_j \int_0^T \frac{\varphi_j - \int_0^T E_{\alpha, 1}(-\lambda_j T^\alpha) f_j(u)(s)}{E_{\alpha, 1}(-\lambda_j T^\alpha)} ds \right\|^2_{L^2(\Omega)}.
\]

In addition \(u_0(j) = \frac{\varphi_j - \int_0^T E_{\alpha, 1}(-\lambda_j T^\alpha) f_j(u)(s)}{E_{\alpha, 1}(-\lambda_j T^\alpha)},\)

\[ \frac{\varphi_j^2E_{\alpha, 1}(-\lambda_j T^\alpha)}{(\varphi_j + E_{\alpha, 1}(-\lambda_j T^\alpha))^2} \leq \frac{\varphi_j^2M^2_2}{4E_{\alpha, 1}(-\lambda_j T^\alpha)} \leq \frac{\varphi_j^2M^2_2}{4M^2_1(1 + T^\alpha) \lambda_j}.
\]

Hence

\[ \|v^{\alpha_n}(t, \cdot) - u(t, \cdot)\|^2_{L^2(\Omega)} = \sum_{j \in \mathbb{N}^d} \frac{\varphi_j^2E_{\alpha, 1}(-\lambda_j T^\alpha)}{(\varphi_j + E_{\alpha, 1}(-\lambda_j T^\alpha))^2} u_j^2(0) \leq \frac{\varphi_j^2M^2_2}{4M_1^2(1 + T^\alpha) \|u(0, \cdot)\|^2_{X_T}}. \]  

(50)
Combining (23), (30) and (51), we deduce that
\[
\mathbb{E}\|\tilde{u}^{\gamma_n,\delta_n}(t, \cdot) - u(t, \cdot)\|_{L^2(\Omega)}^2 \\
\leq 4 M_n^2 \mathbb{E} \|\varphi^{\gamma_n} - \varphi\|_{L^2(\Omega)}^2 + Q_n \|\tilde{u}^{\gamma_n,\delta_n} - u\|_{L^2(\Omega)}^2 + \frac{\varphi_n M_n^2}{2M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)},
\]
which follows that
\[
(1 - Q_n) \|\tilde{u}^{\gamma_n,\delta_n} - u\|_{L^2(\Omega)}^2 \leq 4 M_n^2 \mathbb{E} \|\varphi^{\gamma_n} - \varphi\|_{L^2(\Omega)}^2 + \frac{\varphi_n M_n^2}{2M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)}^2.
\]
Now, using Part (a) of Theorem 5.1 we conclude that
\[
(1 - Q_n) \|\tilde{u}^{\gamma_n,\delta_n} - u\|_{L^2(\Omega)}^2 \leq 4 M_n^2 \left[ C(\mu, \varphi) \frac{\gamma_n^{d/2}}{2} \prod_{i=1}^d n_i^{-4\mu_i} + 4 \gamma_n^{\sigma^*} \|\varphi\|_{H^{\sigma^*}(\Omega)}^2 \right] + \frac{\varphi_n M_n^2}{2M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)}^2,
\]
which implies that \(\|\tilde{u}^{\gamma_n,\delta_n} - u\|_{L^2(\Omega)}^2\) is of order
\[
\max \left\{ \frac{\gamma_n^{d/2}}{2} \prod_{i=1}^d n_i^{-4\mu_i}, \frac{\gamma_n^{\sigma^*}}{2}, \varphi_n \right\}.
\]

**Part B (Convergence rate in \(S_{\sigma,T}\)).** The following theorem gives the error estimate in the space \(S_{\sigma,T}\):

**Theorem 5.3.** Let \(\gamma_n, \varphi_n\), with \(n = (n_1, \ldots, n_d) \in \mathbb{N}^d\), satisfying
\[
\lim_{|n| \to \infty} \gamma_n = \infty, \quad \lim_{|n| \to \infty} \varphi_n = 0, \quad \text{and} \quad \lim_{|n| \to \infty} \frac{\gamma_n}{\varphi_n} = \lim_{|n| \to \infty} \frac{\gamma_n^{\sigma^*}}{\varphi_n} = 0.
\]
Assume that there exists a positive constant \(\sigma > 0\) such that \(u(0, \cdot) \in H^{\sigma+1}(\Omega), \varphi \in H^{\sigma^*+\sigma}(\Omega)\) with \(\mu_\sigma\) as in Theorem 5.1 and assumptions (H1), (H2) are satisfied. Then, \(\|\tilde{u}^{N_1,N_2} - u\|_{S_{\sigma,T}}^2\) is of order
\[
\max \left\{ \frac{\gamma_n^{d/2}}{2} \prod_{i=1}^d n_i^{-4\mu_i}, \frac{\gamma_n^{\sigma^*}}{2}, \varphi_n \right\}.
\]

**Proof.** In order to prove the theorem above, we first give a similar estimate as in Lemma 5.1
\[
\|\tilde{A}\varphi(t)\|_{H^s(\Omega)} \leq M_2 \frac{\gamma_n}{\varphi_n} \|g\|_{H^s(\Omega)}, \quad \|B(t)g\|_{H^s(\Omega)} \leq M_2 \|g\|_{H^s(\Omega)}, \quad \text{for } 0 \leq t \leq T, \tag{51}
\]
and
\[
\|\tilde{D}_\varphi(t,s)g\|_{H^s(\Omega)} \leq M_2^2 T^\alpha \frac{\gamma_n}{M_1 \varphi_n} \|g\|_{H^s(\Omega)} \quad \text{for } 0 \leq t \leq T, 0 < s \leq T. \tag{52}
\]
Under assumptions (H1) and (H2), one can check that the integral equation (40) has a unique solution \(\tilde{u}^{\gamma_n,\delta_n} \in S_{\sigma,T}\) using (51) - (52) and a similar method as in Lemma 5.3 Next, we will prove Theorem 5.3 using a similar technique as in Part A. In this way, we arrive at
\[
\mathbb{E} \|\tilde{u}^{\gamma_n,\delta_n}(t, \cdot) - u^{\delta_n}(t, \cdot)\|_{H^s(\Omega)}^2 \leq 2 M_2^2 \mathbb{E} \|\varphi^{\gamma_n} - \varphi\|_{H^s(\Omega)}^2 + Q_n \|\tilde{u}^{\gamma_n,\delta_n} - u\|_{S_{\sigma,T}}^2 + \frac{\varphi_n M_n^2}{2M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)}^2,
\]
and
\[
\|u^{\delta_n}(t, \cdot) - u(t, \cdot)\|_{H^s(\Omega)}^2 \leq \frac{\varphi_n M_n^2}{4M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)}^2,
\]
and
\[
\|\tilde{u}^{\gamma_n,\delta_n}(t, \cdot) - u^{\delta_n}(t, \cdot)\|_{H^s(\Omega)}^2 \leq 2 M_2^2 \mathbb{E} \|\varphi^{\gamma_n} - \varphi\|_{H^s(\Omega)}^2 + Q_n \|\tilde{u}^{\gamma_n,\delta_n} - u\|_{S_{\sigma,T}}^2 + \frac{\varphi_n M_n^2}{2M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)}^2,
\]
and
\[
\|u^{\delta_n}(t, \cdot) - u(t, \cdot)\|_{H^s(\Omega)}^2 \leq \frac{\varphi_n M_n^2}{4M_1} (1 + T^\alpha) \|u(0, \cdot)\|_{H^1(\Omega)}^2,
\]
which gives us
\[
(1 - Q^2_{\alpha,T}) \| \tilde{u}^{\gamma_n,\vartheta_n} - u \|_{S_{n,T}}^2 \leq 4 \frac{M^2_{\varphi}}{q^2_{n}} \| \tilde{\varphi}^{\gamma_n} - \varphi \|_{H^\varphi(\Omega)}^2 + \frac{\vartheta_n M^2_{\varphi}}{2M_1} (1 + T^\vartheta) \| u(0,) \|_{H^{\varphi+\vartheta}(\Omega)}^2.
\]

Finally, using Part (b) of Theorem 5.1, we conclude that
\[
(1 - Q^2_{\alpha,T}) \| \tilde{u}^{\gamma_n,\vartheta_n} - u \|_{S_{n,T}}^2 \leq 4 \frac{M^2_{\varphi}}{q^2_{n}} \left[ C(\mu, \varphi)^{\gamma_n,\vartheta_n} \sum_{i=1}^d n_i^{-4\mu_i} + 4 \gamma_n^{-\mu_n} \| \varphi \|_{H^{\varphi+\vartheta}(\Omega)}^2 \right] + \frac{\vartheta_n M^2_{\varphi}}{2M_1} (1 + T^\vartheta) \| u(0,) \|_{H^{\varphi+\vartheta}(\Omega)}^2,
\]

which implies that \( \| \tilde{u}^{\gamma_n,\vartheta_n} - u \|_{S_{n,T}}^2 \) is of order
\[
\max \left\{ \frac{n^{\gamma_n,\vartheta_n}}{q^2_{n}} \sum_{i=1}^d n_i^{-4\mu_i}, \frac{1}{\vartheta_n^2} \right\}.
\]

\[ \square \]

**Remark 5.1.** The truncation method in this paper is similar to the method in [27]. The quasi-boundary value method in this section is more effective and useful than the one in [28]. The advantage of this method is that it allows us to estimate the norm on the Hilbert scales \( H^\varphi(\Omega) \). As is known, estimates on higher Sobolev spaces such as \( H^\varphi(\Omega) \) is not an easy task.

### 5.4 A general filter method in the multi-dimensional case

Now, we introduce one more regularization method, called a general filter method. The main idea is to replace the quantity \( \frac{E_{\sigma,1}(\lambda \varphi^T)}{E_{\sigma,1}(\lambda \varphi^T)} \) by a new one \( \tilde{L}_j(\vartheta_n) \frac{E_{\sigma,1}(\lambda \varphi^T)}{E_{\sigma,1}(\lambda \varphi^T)} \), with \( \tilde{L}_j(\vartheta_n) \) chosen as in Theorem 5.3. In this way, regularized solutions \( \tilde{w}^{\gamma_n,\vartheta_n} \) are obtained as follows

\[
\tilde{w}^{\gamma_n,\vartheta_n}(t, x) = \tilde{A}_{\vartheta_n}(t) \tilde{\varphi}^{\gamma_n}(x) + \int_0^t \tilde{B}(t - s) f(s, x, \tilde{w}^{\gamma_n,\vartheta_n}(s, x)) \, ds - \int_0^T \tilde{D}_{\vartheta_n}(t, T - s) f(s, x, \tilde{w}^{\gamma_n,\vartheta_n}(s, x)) \, ds,
\]

where \( \gamma_n, \vartheta_n \) satisfy \( \lim_{|n| \to \infty} \gamma_n = \infty, \lim_{|n| \to \infty} \vartheta_n = 0 \) and

\[
\tilde{A}_{\vartheta_n}(t) := \sum_{j \in \mathbb{N}^d} \left( L_j(\vartheta_n) \frac{E_{\sigma,1}(\lambda \varphi^T)}{E_{\sigma,1}(\lambda \varphi^T)} \langle g, \xi_j \rangle \xi_j \right) = \tilde{A}_{\vartheta_n}(t) B(s) g.
\]

**Theorem 5.4** (Error estimate obtained by general filter method). Let \( \tilde{L}_j(\vartheta_n) \) satisfies the following conditions

\[
\tilde{L}_j(\vartheta_n) \frac{E_{\sigma,1}(\lambda \varphi^T)}{E_{\sigma,1}(\lambda \varphi^T)} \leq C_1(\vartheta_n), \quad 0 \leq 1 - \tilde{L}_j(\vartheta_n) \leq C_1(\vartheta_n) \lambda_j^q, \quad \text{for some } q > 0,
\]

where \( C_1(\vartheta_n), C_1(\vartheta_n) \) satisfy

\[
\lim_{|n| \to \infty} C_1(\vartheta_n) = \lim_{|n| \to \infty} C_1(\vartheta_n) \gamma_n^{\sigma+\vartheta} \sum_{i=1}^d n_i^{-4\mu_i} = \lim_{|n| \to \infty} C_1(\vartheta_n) \gamma_n^{-\mu_n} = 0.
\]

Assume that there exists a positive constant \( \sigma > 0 \) such that \( u(0,) \in H^{\sigma+2q}(\Omega), \varphi \in H^{\mu+\vartheta}(\Omega) \) with \( \mu > 0 \) as is in Theorem 5.3 and assumptions (H1),(H2) are satisfied. Then, \( \| \tilde{u}^{\gamma_n,\vartheta_n} - u \|_{S_{n,T}}^2 \) is of order

\[
\max \left\{ C_1^2(\vartheta_n) \gamma_n^{\sigma+\vartheta} \sum_{i=1}^d n_i^{-4\mu_i}, C_1^2(\vartheta_n) \gamma_n^{-\mu_n}, C_1^2(\vartheta_n) \right\}.
\]

20
Proof. Let
\[
\tilde{v}^{\alpha}(t,x) = \tilde{A}_\alpha(t)\varphi(x) + \int_0^t B(t-s)f(s,x,u(s,x))ds - \int_0^T \tilde{D}_\alpha(t,T-s)f(s,x,u(s,x))ds.
\] (56)

From the definition [52], the following estimates hold for \(0 \leq t \leq T, \ 0 < s \leq T\)
\[
\left\|A_\alpha(t)g\right\|_{H^\alpha(\Omega)} \leq C_1(\alpha) \|g\|_{H^\alpha(\Omega)}, \quad \left\|D_\alpha(t,s)g\right\|_{H^\alpha(\Omega)} \leq \frac{M_2^2 T^{\alpha}}{M_1 s^{\alpha}} \|g\|_{H^\alpha(\Omega)}, \quad g \in H^\alpha(\Omega).
\]

By similar techniques as in the proof of Theorem 5.3 and noting that \([1 - L_1(\vartheta_n)] E_{\alpha_1}(-t^{\alpha}) \leq C_1(\vartheta_n)M_2\lambda_j^\alpha\]
one can check that
\[
E \left\|\tilde{u}^{\gamma,\alpha}(t,\cdot) - \tilde{v}^{\alpha}(t,\cdot)\right\|_{H^\alpha(\Omega)}^2 \leq 2C_1^2(\vartheta_n)E \left\|\tilde{\varphi}\gamma - \varphi\right\|_{H^\alpha(\Omega)}^2 + \frac{Q_{\alpha,T}}{2} \left\|\tilde{\varphi}\gamma - \varphi\right\|_{H^\alpha(\Omega)}^2,
\] (57)

and that
\[
\left\|\tilde{v}^{\alpha}(t,\cdot) - u(t,\cdot)\right\|_{L^2(\Omega)}^2 = \sum_{j \in \mathbb{N}^d} \left[1 - L_1(\vartheta_n)\right]^2 E_{\alpha_1}(-t^{\alpha})u_j^2(0) \leq M_2^2 C_1^2(\vartheta_n) \|u(0,\cdot)\|_{H^\alpha(\Omega)}^2,
\] (58)

where \(Q_{\alpha,T}\) is defined as in [32]. From [57]-[58], one arrives at
\[
(1 - Q_{\alpha,T}) \left\|\tilde{\varphi}\gamma - u\right\|_{H^\alpha(\Omega)}^2 \leq 4C_1^2(\vartheta_n) \left[\left(C(\mu, \varphi)\gamma_\pi + d/2\right) \prod_{i=1}^d n_i^{-4\mu_i} + 4\gamma_\varphi \right] \left\|\varphi\right\|_{H^\alpha(\Omega)}^2 + \frac{M_2^2 C_1^2(\vartheta_n)}{\|u(0,\cdot)\|_{H^\alpha(\Omega)}}^2,
\]

which implies that \(\left\|\tilde{\varphi}\gamma - u\right\|_{H^\alpha(\Omega)}\) is of order
\[
\max \left\{C_1^2(\vartheta_n)\gamma_\pi + d/2 \prod_{i=1}^d n_i^{-4\mu_i}, C_1^2(\vartheta_n)\gamma_\varphi \right\}.
\]

This completes the proof. \(\Box\)

**Remark 5.2.** Our problem is restricted to a rectangular geometry for which the eigenvalues and eigenfunctions of the Laplacian are readily available. The analysis here comes from the trigonometric functions (sine, the cosine function) of eigenfunctions. Lemma 3.1 gives the representations of the exact solution which is given by trigonometric functions. However, if we let an arbitrary domain \(\Omega\) with a \(C^2\)-boundary, the analysis in this paper is not applied and such problem is more difficult. This challenge and open problem may be addressed in future works.

6 **Numerical example**

In this section, we describe the Fourier truncated method applied to some examples of finding the function \(u = u(t,x)\) satisfying the following conditions
\[
\frac{\partial}{\partial t}u(t,x) - \frac{\partial^{1-\alpha}}{\partial t}\Delta u(t,x) = f(t,x,u(t,x)), \quad (t,x) \in (0,1) \times \Omega_d, \quad (59)
\]
\[
u(t,x) = 0, \quad (t,x) \in (0,1) \times \partial \Omega_d, \quad (60)
\]
\[
u(T,x) = \varphi(x), \quad x \in \Omega_d, \quad (61)
\]

where \(\alpha \in (0,1), \ t \in (0,1)\) is time variable, \(x \in \Omega_d = (0,\pi)^d\) and \(x = (x_1, x_2, \ldots, x_d)\) is \(d\)-dimensional variable.

The discrete form of the problem \((59),(61)\) is as follows: We divide the domain \((0,T) \times \Omega_d\) into \(N_d\) and \(N_t\) subintervals of equal length \(h_d\) and \(h_t\), where \(h_d = \frac{\pi}{N_d}\) and \(h_t = \frac{1}{N_t}\), respectively, where \(N_d\) is chosen satisfies the random model as follow:
The data $\varphi$ is measured at $n_1 \times n_2 \times \cdots \times n_d$ grid points $x_k = x_{k_1,k_2,\ldots,k_d} \in \Omega$, $k = (k_1,k_2,\ldots,k_d) \in \mathbb{N}^d$, as follows

$$x_k = (X_{k_1},X_{k_2},\ldots,X_{k_d}) = \left(\frac{2k_1-1}{2n_1}\pi,\frac{2k_2-1}{2n_2}\pi,\ldots,\frac{2k_d-1}{2n_d}\pi\right),$$

where $k_i = 1,2,\ldots,n_i$, $i = 1,2,\ldots,d$. Furthermore, the value of $\varphi$ at each point $x_k$ is contaminated by the observation $\Phi_{k_{obs}}^\text{obs}$

$$\varphi(x_k) = \varphi(X_{k_1},X_{k_2},\ldots,X_{k_d}) \approx \Phi_{k_{1,k_2,\ldots,k_d}}^\text{obs}.$$  

The relationship between two kinds of data is described by the random model

$$\Phi_{k_{obs}}^\text{obs} = \varphi(x_k) + \varepsilon_k W_k,$$  

(62)

where $W_k = W_{k_1,k_2,\ldots,k_d}$ are mutually independent random variables, $W_k \sim \mathcal{N}(0,1)$ and $\varepsilon_k = \varepsilon_{k_1,k_2,\ldots,k_d}$ are positive constants bounded by a positive constant $\varepsilon_{\text{max}}$.

The function $\text{randn}$ may be used to generate a random number drawn from the $\mathcal{N}(0,1)$ distribution in Matlab software. In order to simulate a state of randomness, the command $\text{randn('state',n)}$ is used. As an example, one could use $\text{randn}(8)$ to generate a fixed set of random numbers then we get a matrix $8 \times 8$ with the average of the elements is zero (see Table I).

| $\varepsilon$ |
|-------------|
| -2.2207     |
| -0.2391     |
| 0.0687      |
| -0.2020     |
| -0.3641     |
| -0.0813     |
| -1.9797     |
| 0.7882      |
| 0.7366      |
| 0.9553      |
| 1.2995      |
| -0.753     |
| -0.8984     |
| -3.2823     |
| -0.0300     |
| 0.6134      |

Table 1: An example of the function $\text{randn}(8)$

In the following, we discuss two examples to illustrate of our results.

6.1 Case 1: $d = 1, \alpha = 0.3$

In first case, the source function $f$ and the data $\varphi$ are chosen as

$$f := -\sin(x) \left[ \frac{100^{0.3}}{3\Gamma(0.3)} + 1 \right], \quad \varphi = \sin(x), \quad \Phi_{k_{obs}}^\text{obs} = \varphi(x) + 1\% W,$$  

(63)

so that the exact solution of the problem (60)-(61) is given by $u(t,x) = t\sin(x)$.

The eigenvalues $\{\lambda_{j_1}\}$ and the eigenvectors $\{\xi_{j_1}\}$ are given by

$$\lambda_{j_1} = j_1^2, \quad \xi_{j_1} = \sqrt{\frac{2}{\pi}} \sin(j_1 x), \quad \text{for} \ j_1 = 1,2,\ldots.$$  

According to (66), we have the regularized solution as follows

$$\overline{u}^{N_1}(t,x) = A^{N_1}(t)\varphi^{N_1}(x) + \int_0^t \mathcal{B}^{N_1}(t-s)f(s,x)ds - \int_0^T \mathcal{D}^{N_1}(t,T-s)f(s,x)ds,$$  

(64)

where

$$A^{N_1}(t) := \sum_{j_1=1}^{N_1} \left( \frac{E_{\alpha,1}(-\lambda_{j_1}t^\alpha)}{E_{2\alpha,1}(-\lambda_{j_1}T^\alpha)} \langle g, \xi_{j_1} \rangle \right) \xi_{j_1},$$

$$\mathcal{B}^{N_1}(t) := \sum_{j_1=1}^{N_1} \left( \frac{E_{\alpha,1}(-\lambda_{j_1}t^\alpha)}{E_{2\alpha,1}(-\lambda_{j_1}T^\alpha)} \langle g, \xi_{j_1} \rangle \right) \xi_{j_1},$$

$$\mathcal{D}^{N_1}(t) := \sum_{j_1=1}^{N_1} \left( \frac{E_{\alpha,1}(-\lambda_{j_1}T^\alpha)}{E_{2\alpha,1}(-\lambda_{j_1}T^\alpha)} \langle g, \xi_{j_1} \rangle \right) \xi_{j_1}.$$
and $D^N_1(t, s)g := A^N_1(t)B^N_1(s)g$.

Before presenting the results of this subsection, we present an approximate methods to support the calculation as follows.

In numerical analysis, Simpson's rule is a method for numerical integration. Let $\theta \in L^2(0, \pi)$, we have the following approximation

$$\int_0^\pi \theta(z)dz \approx \Delta z \left( \frac{1}{3} \theta(z_1) + \frac{2}{3} \theta(z_2) + \frac{1}{3} \theta(z_{n+1}) \right).$$

Then the errors are estimated by

$$\text{Err}^N_1(t) = \sqrt{\frac{1}{n_1} \sum_{i=1}^{n_1} \left( \tilde{u}^N_1(t, x_i) - u(t, x_i) \right)^2},$$

where we choose $N_1$ equal to greatest natural numbers less than $\log n_1$.

Figure ?? (a) and Figure ?? (b) show the exact and regularized solutions of the problem (59)-(61) with conditions (63) at $t = 0.3$, $n_1 = 50$ and $n_1 = 100$, respectively. In addition, the error between the exact and regularized solutions is shown in Figure ?? (b) and Figure ?? (b). Moreover, we also present the solutions on $(t, x) \in (0, 1) \times (0, \pi)$ in Figure ?? (for $n_1 = 50$) and Figure ?? (for $n_1 = 100$).

6.2 Case 2: $d = 2, \alpha = 0.5$

In this case, the model concerned subjects to the following source function and final data

$$f = -2 \sin(x_1) \sin(x_2) \left[ 1 + \frac{2 t^{1/2}}{\Gamma(0.5)} \right], \quad \varphi(x_1, 2) = \sin(x_1) \sin(x_2), \quad \Phi_{1,2}^{obs} = \varphi(x_1, 2) + 1.5\% W_{1,2}, \quad (65)$$

In order to obtain the solution $u(t, x) = t \sin(x_1) \sin(x_2)$ of our problem in this case, we employ the conditions is give by Eq. (65).

The eigenvalues $\{\lambda_{j_1, j_2}\}$ and the eigenvectors $\{\xi_{j_1, j_2}\}$ are given by

$$\lambda_{j_1, j_2} = j_1^2 + j_2^2, \quad \xi_{j_1, j_2} = \frac{2}{\pi} \sin(j_1 x_1) \sin(j_2 x_2), \quad \text{for} \quad j_1, j_2 = 1, 2, \ldots$$

According to (65), we have the regularized solution as follows

$$\tilde{u}^{N_1, N_2}(t, x) = A^{N_1, N_2}(t)\varphi^{N_1, N_2}(x) + \int_0^t B^{N_1, N_2}(t - s) f(s, x)ds - \int_T^t D^{N_1, N_2}(t, T - s) f(s, x)ds, \quad (66)$$

where

$$A^{N_1, N_2}(t)g := \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( E_{\alpha, 1} \left( -\lambda_{j_1, j_2} t^\alpha \right) \langle g, \xi_{j_1, j_2} \rangle \right) \xi_{j_1, j_2},$$

$$B^{N_1, N_2}(t)g := \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \left( E_{\alpha, 1} \left( -\lambda_{j_1, j_2} t^\alpha \right) \langle g, \xi_{j_1, j_2} \rangle \right) \xi_{j_1, j_2},$$

and $D^{N_1, N_2}(t, s)g := A^{N_1, N_2}(t)B^{N_1, N_2}(s)g$.

Then we have the errors are estimated by

$$\text{Err}_{n_1, n_2}^{N_1, N_2}(t) = \sqrt{\frac{1}{n_1 n_2} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \left( \tilde{u}^{N_1, N_2}(x_{j_1}, x_{j_2}, t) - u(x_{j_1}, x_{j_2}, t) \right)^2},$$

where we choose $N_1$ and $N_2$ equal to greatest natural numbers less than $\log n_1$ and $\log n_2$, respectively.

In this case, we show the results about the regularized solution (see Figure ?? (b)) at $t = 0.3$, $n_1 = n_2 = 50$. We can compare the exact (see Figure ?? (a)) and regularized solutions thanks to the error of these solutions by the contour graph (see Figure ??). In Table 2, we show the comparison of errors between theoretical method and numerical method for the cases 1D and 2D with $t \in \{0.3, 0.5, 0.8\}$. From this table, it shows that errors by the numerical method give better results than errors by the theoretical method. From the aforementioned evidence, we can conclude that the method that we propose is acceptable.
| $\text{Err}(t)$ | $1D \ (n_1 = 50)$ | $2D \ (n_1 = n_2 = 50)$ |
|----------------|-----------------|-----------------|
|                | Numerical method | Theoretical method | Numerical method | Theoretical method |
| $\text{Err}(0.3)$ | 0.011025806961961 | 0.010118040520960 | 0.024935435226306 | 0.054647006945596 |
| $\text{Err}(0.5)$ | 0.010580529848833 | 0.020919970167952 | 0.029741170367171 | 0.031748823894940 |
| $\text{Err}(0.8)$ | 0.009010268605417 | 0.010238297159259 | 0.040756196453124 | 0.031748168670840 |

Table 2: The error between the exact and regularized solutions at $t \in \{0.3, 0.5, 0.8\}$

Acknowledgements

The first author gratefully acknowledge stimulating discussions with Dr Yavar Kian.

References

[1] De Andrade, B., Viana, A. (2017). *On a fractional reaction–diffusion equation*. Zeitschrift Für Angewandte Mathematik Und Physik, 68(3). doi:10.1007/s00033-017-0801-0

[2] Atangana, A., Gómez-Aguilar, J. F. (2017). *Numerical approximation of Riemann-Liouville definition of fractional derivative: From Riemann-Liouville to Atangana-Baleanu*. Numerical Methods for Partial Differential Equations, 34(5), 1502-1523. doi:10.1002/num.22195

[3] Atmadja, J., Bagtzoglou, A. C. (2001). *Pollution source identification in heterogeneous porous media*. Water Resources Research, 37(8), 2113–2125. doi:10.1029/2001wr000223

[4] Baeumer, B., Geisert, M., Kovács, M. (2015). *Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise*. Journal of Differential Equations, 258(2), 535–554. doi:10.1016/j.jde.2014.09.020

[5] Bagtzoglou, A. C., Atmadja, J. (2003). *Marching-jury backward beam equation and quasi-reversibility methods for hydrologic inversion: Application to contaminant plume spatial distribution recovery*. Water Resources Research, 39(2). doi:10.1029/2001wr001021

[6] Bissantz, N., Holzmann, H. (2008). *Statistical inference for inverse problems*. Inverse Problems, 24(3), 034009. doi:10.1088/0266-5611/24/3/034009

[7] Carasso, A. S. (2013). *Hazardous Continuation Backward in Time in Nonlinear Parabolic Equations, and an Experiment in Deblurring Nonlinearly Blurred Imagery*. Journal of Research of the National Institute of Standards and Technology, 118, 199. doi:10.6028/jres.118.010

[8] Carasso, A. S. (2015). Stable explicit time marching in well-posed or ill-posed nonlinear parabolic equations. Inverse Problems in Science and Engineering, 24(8), 1364-1384. doi:10.1080/17415977.2015.1110150

[9] Cavalier, L. (2008). *Nonparametric statistical inverse problems*. Inverse Problems, 24(3), 034004. doi:10.1088/0266-5611/24/3/034004

[10] Cheng, J., Liu, J. J. (2008). A *quasi Tikhonov regularization for a two-dimensional backward heat problem by a fundamental solution*. Inverse Problems, 24(6), 065012. doi:10.1088/0266-5611/24/6/065012

[11] Debnath, L. (2003). *Recent applications of fractional calculus to science and engineering*. International Journal of Mathematics and Mathematical Sciences, 2003(54), 3413–3442. doi:10.1155/s0161171203303148

[12] Denche, M., Bessila, K. (2005). A *modified quasi-boundary value method for ill-posed problems*. Journal of Mathematical Analysis and Applications, 301(2), 419–426. doi:10.1016/j.jmaa.2004.08.001

[13] Eubank, R. L. (1999). *Nonparametric Regression and Spline Smoothing*, 2nd ed., Marcel Dekker, New York.

[14] Gorenflo, R., Mainardi, F. (2009). *Some recent advances in theory and simulation of fractional diffusion processes*. Journal of Computational and Applied Mathematics, 229(2), 400–415. doi:10.1016/j.cam.2008.04.005
[15] Gupta, V., Bora, S. N., Nieto, J. J. (2019). Dhage iterative principle for quadratic perturbation of fractional boundary value problems with finite delay. Mathematical Methods in the Applied Sciences. doi:10.1002/mma.5643

[16] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. (2006). Theory and applications of fractional differential equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam,

[17] Mair, B. A., Ruymgaart, F. H. (1996). Statistical Inverse Estimation in Hilbert Scales. SIAM Journal on Applied Mathematics, 56(5), 1424–1444. doi:10.1137/s0036139994264476

[18] Minh, N. D., To Duc, K., Tuan, N. H., Trong, D. D. (2018). A two-dimensional backward heat problem with statistical discrete data. Journal of Inverse and Ill-Posed Problems, 26(1), 13–31. doi:10.1515/jiip-2016-0038

[19] Nam, P. T. (2010). An approximate solution for nonlinear backward parabolic equations. Journal of Mathematical Analysis and Applications, 367(2), 337–349. doi:10.1016/j.jmaa.2010.01.020

[20] Nane, E., Tuan, N. H. (2018). A random regularized approximate solution of the inverse problem for Burgers’ equation. Statistics & Probability Letters, 132, 46–54. doi:10.1016/j.spl.2017.08.014

[21] Nane, E., Tuan, N. H. (2018). Approximate Solutions of Inverse Problems for Nonlinear Space Fractional Diffusion Equations with Randomly Perturbed Data. SIAM/ASA Journal on Uncertainty Quantification, 6(1), 302–338. doi:10.1137/17m1111139

[22] Podlubny, I. (1990). Fractional Differential Equations, Mathematics in Science and Engineering, vol 198, Academic Press Inc, San Diego, CA.

[23] Qian, A., Mao, J. (2011). Quasi-reversibility regularization method for solving a backward heat conduction problem, Amer. J. Comput. Math. 1, no. 3, 159–162. doi: 10.4236/ajcm.2011.13018.

[24] Samko, S. G., Kilbas, A. A., Marichev, O. I. (1987) Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach Science, Naukai Tekhnika, Minsk.

[25] Schneider, W. R., Wyss, W. (1989). Fractional diffusion and wave equations. Journal of Mathematical Physics, 30(1), 134–144. doi:10.1063/1.528578

[26] Seki, K., Wojcik, M., Tachiya, M. (2003). Fractional reaction-diffusion equation. The Journal of Chemical Physics, 119(4), 2165–2170. doi:10.1063/1.1587126

[27] Tuan, N. H., Nane, E. (2017). Inverse source problem for time-fractional diffusion with discrete random noise. Statistics & Probability Letters, 120, 126–134. doi:10.1016/j.spl.2016.09.026

[28] Yang, F., Zhang, Y., Li, X.-X., Huang, C.-Y. (2018). The quasi-boundary value regularization method for identifying the initial value with discrete random noise. Boundary Value Problems. doi:10.1186/s13661-018-1030-y

[29] Zhou, Y. (2014). Basic Theory of Fractional Differential Equations, World Scientific, Singapore. https://doi.org/10.1142/9069

[30] Zou, G., Wang, B. (2017). Stochastic Burgers’ equation with fractional derivative driven by multiplicative noise. Computers & Mathematics with Applications, 74(12), 3195–3208. doi:10.1016/j.camwa.2017.08.023

[31] B. Jin, B. Li, Z. Zhou; Numerical analysis of nonlinear subdiffusion equations , SIAM J. Numer. Anal., 56 (2018), 1–23.

[32] B. Jin, R. Lazarov, and Z. Zhou, An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data, IMA J. Numer. Anal., 36 (2016), 197—221.

[33] K. Mustapha, An L1 approximation for a fractional reaction-diffusion equation, a second-order error analysis over time-graded meshes, https://arxiv.org/pdf/1909.06739.pdf

[34] W. McLean Regularity of solutions to a time-fractional diffusion equation ANZIAM J. 52 (2010), no. 2, 123–138

[35] C. M. Chen, F. Liu, V. Anh, and I. Turner, Numerical methods for solving a two-dimensional variable-order anomalous subdiffusion equation, Math. Comp., 81, (2012) 345—366.
[36] E. Cuesta and C. Lubich, C. Palencia, Convolution quadrature time discretization of fractional diffusive-wave equations, Math. Comp., 75, (2006) 673–696

[37] B. Kaltenbacher, W. Rundell On an inverse potential problem for a fractional reaction–diffusion equation, Inverse Problems, Volume 35, Number 6, 2019.

[38] B. Kaltenbacher, W. Rundell Regularization of a backward parabolic equation by fractional operators Inverse Probl. Imaging 13 (2019), no. 2, 401–430.

[39] J. Jia, J. Peng, J. Gao, Y. Li, Backward problem for a time-space fractional diffusion equation Inverse Probl. Imaging 12 (2018), no. 3, 773–799.

[40] L. Wang, J. Liu, Total variation regularization for a backward time-fractional diffusion problem Inverse Problems 29 (2013), no. 11, 115013, 22 pp.

[41] J. Janno, N. Kinash; Reconstruction of an order of derivative and a source term in a fractional diffusion equation from final measurements, Inverse Problems, 34 (2018), 19 pp.

[42] J. Janno, K. Kasemets; Uniqueness for an inverse problem for a semilinear time-fractional diffusion equation, Inverse Probl. Imaging, 11 (2017), 125–149.

[43] D. Jiang, Z. Li, Y. Liu, M. Yamamoto; Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations, Inverse Problems, 33 (2017), 21 pp.

[44] Z. Li, O.Y. Imanuvilov, M. Yamamoto; Uniqueness in inverse boundary value problems for fractional diffusion equations, Inverse Problems 32, (2016), 16 pp.

[45] G. Li, D. Zhang, X. Jia, M. Yamamoto; Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation, Inverse Problems, 29 (2013), 36 pp.

[46] Y. Luchko, W. Rundell, M. Yamamoto, L. Zuo; Uniqueness and reconstruction of an unknown semilinear term in a time-fractional reaction-diffusion equation, Inverse Problems, 29 (2013), 16 pp.

[47] L. Miller, M. Yamamoto; Coefficient inverse problem for a fractional diffusion equation, Inverse Problems, 29 (2013), 8 pp.

[48] W. Rundell, Z. Zhang , Recovering an unknown source in a fractional diffusion problem J. Comput. Phys. 368 (2018), 299–314.

[49] W. Rundell, Z. Zhang, Fractional diffusion: recovering the distributed fractional derivative from overposed data Inverse Problems 33 (2017), no. 3, 035008, 27 pp.

[50] B. Kaltenbacher and W. Rundell. On the identification of a nonlinear term in a reaction-diffusion equation. Inverse Problems, 2019.

[51] B. Kaltenbacher and W. Rundell Recovery of multiple coefficients in a reaction-diffusion equation, Vol 481, Issue 1, Journal of Mathematical Analysis and Applications, 2020