VANISHING OF MULTIZETA VALUES OVER $\mathbb{F}_q[t]$ AT NEGATIVE INTEGERS

SHUHUI SHI

Abstract. Let $\mathbb{F}_q$ be the finite field of $q$ elements. In this paper, we study the vanishing behavior of multizeta values over $\mathbb{F}_q[t]$ at negative integers. These values are analogs of the classical multizeta values. At negative integers, they are series of products of power sums $S_d(k)$ which are polynomials in $t$. By studying the $t$-valuation of $S_d(s)$ for $s < 0$, we show that multizeta values at negative integers vanish only at trivial zeros. The proof is inspired by the idea of Sheats in the proof of a statement of “greedy element” by Carlitz.

1. Introduction

Classical multizeta values (i.e. over $\mathbb{Z}$) are defined as the convergent series

$$\zeta(s) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}} \in \mathbb{R}.$$ 

where $s = (s_1, \ldots, s_r) \in \mathbb{Z}^+_r$ with $s_1 > 1$. We call $r$ the depth and $\sum_i s_i$ the weight of $\zeta(s)$. Multizeta values of depth 1 are the usual Riemann zeta values. These values were first considered by Euler in 1776 [Eul75] in the study of $\zeta(3)$. After a long time of oblivion, they were recently rediscovered with renewed interest because of their appearance in many different contexts, including the absolute Galois group, periods of mixed Tate motives, knot invariants and calculations of integrals associated to Feynman diagrams in perturbative quantum field theory (see the introduction of [Tha17] and the reference mentioned there, as well as [BGF]). These various connections with other fields have led to big progresses in the study of classical multizeta values, although some fundamental questions still remain open.

Having learned about the rich interconnections in the classical case, Thakur, in 2002, defined two types of multizeta values over function fields [Tha04, Sec. 5.10], one complex valued (generalizing special values of Artin-Weil zeta functions) and the other with values in Laurent series over finite field (generalizing Carlitz zeta values). The first type was completely evaluated in [Tha04] for $\mathbb{F}_q(t)$ (see [Mas06] for a study in the higher genus case). In this paper, we focus on the second type and stick to the rational function field $\mathbb{F}_q(t)$.

Throughout this paper, $p$ is a prime and $q := p^f$ is a power of $p$. We say an integer is $q$-even if it’s divisible by $q - 1$ and $q$-odd otherwise. Let $K := \mathbb{F}_q(t)$ be the rational function field over the finite field $\mathbb{F}_q$, $\infty$ be the rational place of $K$ with uniformiser $1/t$ and $K_{\infty} := \mathbb{F}_q((1/t))$ be its completion at $\infty$. Let $A := \mathbb{F}_q[t]$ be the polynomial ring in $t$, $A_{\pm} := \{\text{monics in } A\}$ and $A_{d \pm} := \{\text{monic in } A \text{ of degree } d\}$ for $d \geq 0$. For $d \geq 0$ and...
for details). At positive integers, the definition above can be later). See
for definition of “trivial zero”.

\[ \zeta(s) := \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \in K_\infty. \]

The convergence of \( \zeta(s) \) at positive integers, i.e. \( s \in \mathbb{Z}_+ \), is clear from definition of \( S_d \).

At non-positive integers, it follows from the fact that \( S_d(0) = 0 \) for \( d > 0 \) and \( S_d(s) = 0 \) for \( d \gg 0 \) if \( s < 0 \) (see §2 for details). At positive integers, the definition above can be restated as

\[ \zeta(s) = \sum_{a_1, a_2, \ldots, a_r} \frac{1}{a_1^{s_1} a_2^{s_2} \cdots a_r^{s_r}} \in K_\infty, \]

where the sum is over all \( a_i \in A_{d+} \) with \( d_1 > d_2 > \cdots d_r \geq 0 \). Following the classical case, we say \( \zeta(s) \) is of depth \( r \) and weight \( \sum_i s_i \). For general introduction of results on function field multizeta values and comparison with the classical case, we refer the reader to the survey papers [Cha14, Tha17]. In this paper, \( \zeta(s) \) is used to denote multizeta values in both classical and function field case. It should be clear which one we are referring to from the context.

A natural question to ask is when \( \zeta(s) \) vanishes. In classical case, \( \zeta(s) > 0 \) by definition at positive integers with \( s_1 > 1 \). Treating \( s_i \)'s as complex variables, the series defining \( \zeta(s) \) is absolutely convergent in the region \( \{(s_1, \ldots, s_r) \in \mathbb{C}^r : \text{Re}(s_1 + \ldots + s_j) > j \text{ for } 1 \leq j \leq r \} \) and can be meromorphically continued to \( \mathbb{C}^r \) with singular hyperplanes \( \{s_1 = 1, s_1 + s_2 \in \{2, 1, 0, -2, -4, -6, \ldots\}, \sum_{i=1}^k s_i \in \mathbb{Z}_{\leq k} \text{ for } 3 \leq k \leq r \} \). In particular, all the negative integer points, except when \( r = 2 \) or \( s_1 + s_2 \) odd, lie on these hyperplanes. Moreover, they are points of indeterminacy. See [FKMT17] and references mentioned in its “Introduction” for several different approaches to define and determine the multizeta values at these points.

In function field case, Thakur [Tha09] showed that \( \zeta(s) \neq 0 \) at positive integers. At negative integers, the vanishing of multizeta values of depth 1 is completely understood by Goss [Gos79]. Its vanishing behavior is quite similar to that of the Riemann zeta values although lacking a functional equation. In this paper, we study the vanishing of \( \zeta(s) \) at negative integers of higher depth.

Replacing \( a \) by \( t^d + \sum_{i=1}^d \theta_{d-i} t^i \) in (1.1), we can rewrite \( S_d(s) \) as a sum of monomials in \( t \) for negative \( s \), whose sum indices are in \( \mathbb{N}^{d+1} \) satisfying some restrictions. Denote the set of these indices as \( U_d(-s) \). Our main result (restated as Theorem 2.8) gives an explicit description of the \( t \)-valuation of \( S_d(s) \) in terms of elements in \( U_d(-s) \).

**Theorem 1.1.** Assume \( U_d(-s) \neq \emptyset \), then there is a unique monomial in the sum \( S_d(s) \) achieving the lowest degree. Moreover, this term corresponds to the element in \( U_d(-s) \) whose reverse is lexicographically the largest.

This result implies monotonicity of the \( t \)-valuation of \( S_d(s) \) with respect to \( d \), using which we completely solve the vanishing of \( \zeta(s) \) at negative integers (stated as Theorem 2.10 later). See §2.1 for definition of “trivial zero”.

**Theorem 1.2.** At negative integers, \( \zeta(s) \) of depth at least 2 only vanishes at trivial zeros.
Here is the outline of the paper. In §2, we study the behavior of $S_d(s)$ at negative $s$ in detail and discuss how our main result implies Theorem 1.2. §3 gives the proof of Theorem 1.1.

Acknowledgements. The results of this paper are a part of the authors Ph.D. thesis in University of Rochester. The author would like to thank her advisor, Prof. Dinesh Thakur, for suggesting this problem and for all of his guidance and encouragement.

2. Main Result

In this section, we study the vanishing behavior of multizeta values in detail. We continue to use the notations in the previous section. Our main object of study is $S_d(s)$, the building blocks of multizeta values. The reader will see that the vanishing of $\zeta(s)$ is really a reflection of properties of $S_d(s)$.

2.1. Trivial Zeros. Let $s < 0$. We first take a closer look at when $S_d(s)$ vanishes. Writing out the coefficients of $a$ in (1.1), we get

$$S_d(s) = \sum_{\theta \in \mathbb{F}_q} (t^d + \theta_1 t^{d-1} + \cdots + \theta_d)^{-s}$$

$$= \sum_{\theta \in \mathbb{F}_q} \sum_{ m_0 + \cdots + m_d = -s, m_i \geq 0} \left( -s \right)_{m_0, \ldots, m_d} \theta_1^{m_1} \cdots \theta_d^{m_d} t^{dm_0+(d-1)m_1+\cdots+m_{d-1}}$$

$$= (-1)^d \sum_{ m_0 + \cdots + m_d = -s, m_0 \geq 0, m_i > 0 \text{ for } i > 0} \left( -s \right)_{m_0, \ldots, m_d} t^{dm_0+(d-1)m_1+\cdots+m_{d-1}}$$

$$= (-1)^d \sum_{ \oplus_i^d m_i = -s, m_0 \geq 0, m_i > 0 \text{ q-even for } i > 0} \left( -s \right)_{m_0, \ldots, m_d} t^{dm_0+(d-1)m_1+\cdots+m_{d-1}};$$

(2.1)

where $\oplus_i^d m_i$ denotes sum $\sum_{i=0}^d m_i$ with no carry over of digits base $p$. The third equality comes from exchanging the two sum indices and the fact that $\sum_{\theta \in \mathbb{F}_q} \theta^k = -1$ if $k$ is a positive multiple of $q-1$ and $0$ otherwise. The last equality follows from Lucas’ theorem.

For $k > 0$ and $d \geq 0$, let

$$U_d(k) := \{(m_0, m_1, \ldots, m_d) \in \mathbb{N}^{d+1} : k = \oplus_i^d m_i \text{ and } m_i > 0 \text{ is q-even for } 1 \leq i \leq d\}.$$

Let $\mathcal{P}(n)$ be the multiset of $p$-powers adding up to $n$ with no carry over base $p$. Then the condition $k = \oplus_i^d m_i$ is equivalent to

$$\mathcal{P}(k) = \bigsqcup_{i=0}^d \mathcal{P}(m_i).$$

(2.2)

Note that $U_d(-s)$ is the set of sum indices in (2.1). Clearly, $S_d(s)$ vanishes if $U_d(-s) = \emptyset$. In [Car48], Carlitz claimed without proof that the converse also holds. More precisely, he asserted that if $U_d(-s) \neq \emptyset$, the term $t^{dm_0+(d-1)m_1+\cdots+m_{d-1}}$ with $(m_0, \ldots, m_d)$ lexicographically largest among sum indices attains the unique maximal degree. Such $(m_0, \ldots, m_d)$ is called greedy. This claim was not proved until 50 years later. Diaz-Vargas [DV96] gave a proof for the case $q = p$ and a general proof for any $q$ is given by Sheats [She98].
Theorem 2.1 (Calitz, Diaz-Vargas, Sheats). For $s < 0$, $S_d(s) \neq 0$ if and only if $U_d(-s) \neq \emptyset$. Moreover, if $U_d(-s) \neq \emptyset$, the summand in $S_d(s)$ corresponding to the greedy element achieves the unique maximal degree.

Böckle pointed out that with some results in [She98], one gets a more straightforward criterion when $S_d(s)$ vanishes.

Definition 2.2. For $k \in \mathbb{Z}_+$ with base $q$ expansion $k = a_0 + a_1 q + \cdots + a^n q^n$, let $l(k) = \sum a_i$ be the sum of base $q$ digits of $k$. Recall that $q = p^f$. Define
\[ L_k := \min_{i=0, \ldots, f-1} \{l(kp^i)/(q - 1)\}. \]
We note that since $k \equiv l(k) \mod q - 1$, $L_k$ is an integer if and only if $(q - 1)|k$, i.e. $k$ is $q$-even.

Proposition 2.3 ([Böc13, Thm. 1.2(a)]). For $s$ negative, $S_d(s) = 0 \iff d > L_{-s}$.

For reader’s convenience, we provide a proof of the result above. For $d \geq 0$ and $k > 0$, let
\[ V_d(k) := \{(m_0, \ldots, m_d) \in U_d(k) : m_0 > 0\}. \]
The proposition follows from the following lemma of Sheats. We note that the notations and expression of the lemma are slightly different from those in Sheats’ paper, but one can check that they are equivalent.

Lemma 2.4 ([She98, Prop. 4.3(a)]). $V_d(k) = \emptyset \iff d \geq L_k$.

Proof of Proposition 2.3: By Theorem 2.1, it’s enough to show that $U_d(k) = \emptyset$ iff $d > L_k$. We break it up into two cases.

If $k$ is $q$-even, $U_d(k) = V_d(k) \cup \{(0, m_1, \ldots, m_d) \mid (m_1, \ldots, m_d) \in V_{d-1}(k)\}$. $U_d(k) = \emptyset$ iff $V_d(k) = V_{d-1}(k) = \emptyset$, i.e. $d - 1 \geq L_k$ by Lemma 2.4. Since $L_k$ is an integer, $d - 1 \geq L_k \iff d > L_k$.

If $k$ is $q$-odd, $U_d(k) = V_d(k)$, thus $U_d(k) = \emptyset$ iff $d \geq L_k$ by Lemma 2.4. As $L_k$ is not an integer in this case, $d \geq L_k \iff d > L_k$. 

Note that in (1.2), the least $d$ appearing in $S_d(s_i)$ is $r - i$. Thus, if $r - i > L_{-s_i}$, all terms in the sum vanishes and so does the multizeta value. With this observation, we define

Definition 2.5. Let $s_j \in \mathbb{Z}_-$. $\zeta(s_1, \ldots, s_r) = 0$ trivially if there exists some $1 \leq i \leq r - 1$ such that $r - i > L_{-s_i}$. We call such zeros trivial zeros. Other zeros are called nontrivial.

2.2. Existence of Nontrivial Zero. We now investigate nontrivial zeros of $\zeta(s)$ where $s_i < 0$. The depth 1 case is completely understood by Goss [Gos79].

Theorem 2.6 (Goss, see [Tha04, Sec. 5.3]). For $s$ negative, $\zeta(s) = 0$ if and only if $s$ is $q$-even.

Note that multizeta values in this case reduce to Carlitz zeta values. The above theorem shows that the behavior of zeros of Carlitz zeta at negative integers is analogous to that of the trivial zeros of the classical Riemann zeta function. However, unlike a direct implication from the functional equation of the Riemann zeta, the vanishing of $\zeta(s)$ in our case, without any known functional equations, follows from cancellations among monomials.
The proof of the nonvanishing of $\zeta(s)$ at $q$-odd $s$ [Tha04, Thm. 5.3.2] showed that there’s a unique term of least degree, 1, in the polynomial sum of $\zeta(s)$, which could not be canceled. Similarly, the fact that multizeta values at positive integers never vanish [Tha09, Thm. 4] follows from the strict monotonicity in $d$ of the $\infty$-valuation of $S_d(s)$. We use the same strategy to show that there’s no nontrivial zeros in higher depth case.

**Definition 2.7.** $M = (M_0, \ldots, M_d) \in U_d(k)$ is called *modest* if $(M_d, M_{d-1}, \ldots, M_0)$ is lexicographically the largest, i.e. $M_d \geq m_d$ for all $(m_0, \ldots, m_d) \in U_d(k)$, $M_{d-1} \geq m_{d-1}$ for those $(m_0, \ldots, m_d)$ with $m_d = M_d$ and so on. Such element always exists and is unique if $U_d(k) \neq \emptyset$.

Our main result is the following theorem, which characterises the term in $S_d(s)$ with least degree. Its proof is given in §3.

**Theorem 2.8.** Assume $S_d(s) \neq 0$. The term corresponding to the modest element in $U_d(-s)$ attains the unique minimum degree in $t$ among all summands in $S_d(s)$.

Recall that for $d \leq L_s$, elements in $U_d(-s)$ and summands in (2.1) are in one-to-one correspondence. Take $(m_0, m_1, \ldots, m_d) \in U_d(-s)$, its corresponding term in $S_d(s)$ has degree $dm_0 + (d-1)m_1 + \ldots + m_{d-1}$. Define $\nu_d(s) := v_t(S_d(s))$, where $v_t$ is the $t$-valuation. We have the following corollary.

**Corollary 2.9.** Fix $s < 0$, then

$$\nu_{[L_s]}(s) > \nu_{[L_s]-1}(s) > \cdots > \nu_1(s) \geq \nu_0(s).$$

**Proof.** Since $\nu_0(s) = v_t(1) = 0$ for all $s$, the last inequality is obvious. Assume $0 < d \leq L_s$ and let $\mathbf{M} = (M_0, \ldots, M_d)$ be the modest element in $U_d(-s)$, then Theorem 2.8 implies $\nu_d(s) = dM_0 + (d-1)M_1 + \ldots + M_{d-1}$. Consider $\mathbf{N} = (M_0, \ldots, M_{d-2}, M_{d-1} + M_d)$, then $\mathbf{N} \in U_{d-1}(-s)$ and thus $\nu_{d-1}(s) \leq (d-1)M_0 + (d-2)M_1 + \ldots + M_{d-2} \leq \nu_d(s)$, where the second inequality is equality iff $d = 1$ and $M_d = -s$.

With this result, we finish the discussion of vanishing of multizeta values of higher depth at negative integers.

**Theorem 2.10.** For $s = (s_1, \ldots, s_r)$ with $s_i < 0$ and $r > 1$, $\zeta(s) = 0$ if and only if $s$ is a trivial zero.

**Proof.** It’s equivalent to show that $\zeta(s) \neq 0$ if $s$ is not a trivial zero. In this case, the sum $\zeta(s) = \sum_{d_1 \geq \cdots \geq d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r)$ is nonempty. In particular, $S_{r-1}(s_1) \cdots S_0(s_r) \neq 0$ and

$$v_t(S_{r-1}(s_1) \cdots S_0(s_r)) = \sum_{i=1}^r \nu_{r-i}(s_i).$$

For any other term $S_{d_1}(s_1) \cdots S_{d_r}(s_r)$ in the sum, $d_i \geq r - i$ for all $i$ and there exist some $j$ such that $d_j > r - j > 0$, thus by Corollary 2.9,

$$v_t(S_{d_1}(s_1) \cdots S_{d_r}(s_r)) = \sum_{i=1}^r \nu_{d_i}(s_i) > v_t(S_{r-1}(s_1) \cdots S_0(s_r)).$$

By strict triangle inequality, $v_t(\zeta(s)) = v_t(S_{r-1}(s_1) \cdots S_0(s_r)) = \sum_{i=1}^r \nu_{r-i}(s_i)$. In particular, $\zeta(s) \neq 0$. 
\qed
Remark 2.11. We note that the same strategy fails in analysing the vanishing of $\zeta(s)$ at integers of mixed signs. This is because for both places $t$ and $\infty$, $s$ being positive and negative determine opposite monotonicity of the valuation of $S_d(s)$ in $d$.

3. Proof of Theorem 2.8

The proof of Theorem 2.8 is quite complicated and combinatorial. This is because the two conditions on elements of $U_d(-s)$ lie in $p$ and $q$ levels each while $p$ and $q$ are different in general. Major difficulty of the proof arises from how to track these two conditions simultaneously.

3.1. Special case. When $q = p$ is a prime, the problem mentioned above disappears and the theorem can be proved in a way similar to the proof of Theorem 2.1 for $q = p$ case by Diaz-Vargas. Another simple case, without restriction on $q$, is where $s$ is $q$-even, which follows directly from the result on greedy element. We first prove these two special cases.

Proof of Theorem 2.8 for special cases. Let $k = -s$ and $M = (M_0, \ldots, M_d) \in U_d(k)$ be the modest element. For $m = (m_0, \ldots, m_d) \in U_d(k)$, define

$$\text{wt}(m) := dm_0 + (d - 1)m_1 + \cdots + m_{d-1}$$

to be its weight, which equals the degree of its corresponding term in $S_d(s)$. For both cases, we need to show $M$ achieves the unique minimum weight.

1. $q = p$ is a prime: We show that given any non-modest element $m$, one can always adjust it to get another $m'$ of smaller weight. Let $l$ be the largest index such that $M_i > m_i$. Then $M_i = m_i$ for $i > l$ by the choice of $M$. By (2.2), there exist some $p^e \in \mathcal{P}(m_i)$ and $p^{e'} \in \mathcal{P}(m_{i'}) \cap \mathcal{P}(M_l)$ such that $p^e < p^{e'}$ and $l' < l$. Consider

$$m' = (m_0, \ldots, m_{l'} - p^{e'}, \ldots, m_l - p^e + p^{e'}, \ldots, m_d),$$

then it’s easy to check that $m' \in U_d(k)$ and $\text{wt}(m') < \text{wt}(m)$.

2. $s$ is $q$-even: Recall that

$$V_d(k) = \{(m_0, \ldots, m_d) \in U_d(k) : m_0 > 0\}.$$ 

In this case, $M \in U_d(k) \setminus V_d(k)$ since otherwise $(0, M_1, \ldots, M_d + M_0)$ is also contained in $U_d(k)$ whose reverse is lexicographically larger. Similar argument shows that $m \in U_d(k) \setminus V_d(k)$ if $m$ is of minimum weight. Consider the bijective map

$$\varphi : (0, m_1, \ldots, m_d) \mapsto (m_d, \ldots, m_1)$$

between $U_d(k) \setminus V_d(k)$ and $V_{d-1}(k)$. Note that $k = \sum_i m_i$, thus for $m \in U_d(k) \setminus V_d(k)$,

$$\text{wt}(m) = (d - 1)m_1 + \cdots + m_{d-1} = (d - 1)k - \text{wt}(\varphi(m)).$$

$\text{wt}(m)$ being minimum indicates that $\varphi(m) = (m_d, \ldots, m_1)$ achieves the largest weight in $V_{d-1}(k)$. By Theorem 2.1, $\varphi(m)$ has to be the greedy element in $U_{d-1}(k)$. This implies that the reverse of $m$ is lexicographically the largest in $U_d(k) \setminus V_d(k)$, hence $m = M$. \qed
3.2. General case. Our proof for general case is inspired by Sheats’s proof [She98] of
Theorem 2.1 on greedy element. We prove by contradiction. Roughly speaking, assuming
there exists a tuple not modest in $U_d(-s)$ gives a term of lowest degree in $S_d(s)$, we
construct another term with smaller degree.

We fix a prime power $q = p^f$. In this section, $\bar{x}$ denotes a column vector of length $f$,
where $x$ is either an English or Greek letter, with or without subscript. If not mentioning
explicitly, its entries are denoted as $x_i$ with $0 \leq i < f$, e.g. $\bar{u} = [u_0, u_1, \ldots, u_{f-1}]^t$. Note
that the subscripts start from 0. The zero vector is denoted as $\bar{0}$.

3.2.1. Set up and preliminaries. Before the proof, we change to a different notation for
easy expression. A $d$-tuple $(X_1, \ldots, X_d) \in \mathbb{N}^d$ is said to be a composition of $N$ if $N = \sum_{i=1}^d X_i$. For $d > 0$ and $N \in \mathbb{Z}_+$, let

$$W_d(N) = \{(X_1, X_2, \ldots, X_d) \in \mathbb{N}^d : (X_d, X_{d-1}, \ldots, X_1) \in U_{d-1}(N)\}$$

$$= \{(X_1, X_2, \ldots, X_d) \in \mathbb{N}^d : N = \bigoplus_{i=1}^d X_i; \text{ for } i < d, X_i > 0 \text{ and } q\text{-even}\}.$$ 

In this new set up, the modest element in $U_{d-1}(N)$ corresponds to be the lexicographically
largest composition in $W_d(N)$, which we again call it modest.

Definition 3.1. Let $X = (X_1, \ldots, X_d) \in W_d(N)$. Define its weight, denoted as $wt(X)$,
by

$$wt(X) = X_1 + 2X_2 + \cdots + dX_d.$$ 

Any composition $X$ achieving the minimum weight in $W_d(N)$ is called optimal.

One can check that Theorem 2.8 is equivalent to the following.

Theorem 3.2. For $W_d(N) \neq \emptyset$, the modest composition is the only optimal composition.

Remark 3.3. The theorem holds for $d = 1$ trivially since $W_1(N) = \{(N)\}$ has only one
composition. For $d = 2$, $wt(X) = 2N - X_1$ for any $X \in W_2(N)$ and hence the modest
composition is the only optimal element.

The following proposition consists of some observations on how to get new modest or
optimal compositions from old ones.

Proposition 3.4. Suppose $W_d(N) \neq \emptyset$. $X = (X_1, \ldots, X_d)$ is the modest composition in
$W_d(N)$. Then

1. $(X_1, X_2, \ldots, X_{d-1})$ is the modest composition in $W_{d-1}(N - X_d)$;
2. $(X_2, X_3, \ldots, X_d)$ is the modest composition in $W_{d-1}(N - X_1)$;
3. for any $n \geq 0$, $(p^nX_1, \ldots, p^nX_n)$ is the modest composition in $W_d(p^nN)$.

These three statements remain true when replacing “the modest composition” with “an
optimal composition”.

Proof. (1) and (2) are obvious from definition in each case. To show (3), we observe
that all $p$-powers in $\mathcal{P}(p^nN)$ are divisible by $p^n$. Thus, for $(Y_i) \in W_d(p^nN)$, $p^n \mid Y_i$ for
all $i$ since $\mathcal{P}(Y_i) \subset \mathcal{P}(p^nN)$. Moreover, $(Y_i) \mapsto (p^{-n}Y_i)$ gives an 1-to-1 correspondence
between compositions in $W_d(p^nN)$ and $W_d(N)$. (3) follows from this observation easily
in both cases. 

Given base $p$ expansion $n = \sum_{j \geq 0} a_j p^j$, we define $\Gamma(n) \in \mathbb{N}^f$ to be the column vector $[\mu_0, \ldots, \mu_{f-1}]^t$, where
\[ \mu_i = \sum_{j \equiv i \mod f} a_j. \]
Let $\psi_0 := [1, p, \ldots, p^{f-1}]^t$, then
\[ \langle \psi_0, \Gamma(n) \rangle = \mu_0 + \cdots + p^{f-1} \mu_{f-1} \]
is the sum of base $q$ digits of $N$. In particular, $n$ is $q$-even iff $(q - 1) | \langle \psi_0, \Gamma(n) \rangle$. Then $X \in W_d(N)$ if and only if
\begin{enumerate}
  \item $\Gamma(N) = \Gamma(X_1) + \Gamma(X_2) + \cdots + \Gamma(X_d)$,
  \item for $1 \leq i \leq (d - 1)$, $(q - 1) | \langle \psi_0, \Gamma(X_i) \rangle$ \neq 0.
\end{enumerate}
For a composition $X = (X_1, \ldots, X_d)$ of $N$, define $\Gamma(X)$ to be the $f \times d$ matrix with columns $\Gamma(X_1), \ldots, \Gamma(X_d)$.

**Example.** Let $q = 9$ and $N = 11213$. In base $3$, $N = 11213$. Thus $\Gamma(N) = [5, 2]^t$. For any $X \in W_d(N)$, $\Gamma(X)$ is one of the two matrices: $\begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}$. $(128, 3) = (11202_3, 103)$, $(104, 27) = (10212_3, 1000_3) \in W_2(N)$ correspond to the first one, and the rest correspond to the latter.

In the example above, we give a partition of compositions in $W_d(N)$ with respect to matrix representation. Given $B \in \text{Mat}_{f \times d}(\mathbb{N})$, define
\[ W^B_d(N) := \{ X \in W_d(N) : \Gamma(X) = B \}. \]
We call $B$ a valid matrix of $W_d(N)$ if $W^B_d(N) \neq \emptyset$. Let $B_1, \ldots, B_d$ be columns of $B$, then $B$ is valid if and only if
\begin{enumerate}
  \item $\Gamma(N) = B_1 + \cdots + B_d$,
  \item for $1 \leq i \leq (d - 1)$, $(q - 1) | \langle \psi_0, B_i \rangle$ \neq 0.
\end{enumerate}
For $n > 0$, denote $\tau(n)$ the nonincreasing sequence of $p$-powers in $\mathcal{P}(n)$ and $\tau_k(n)$ be its subsequence consisting of those $p^i$ with $i \equiv k \mod f$ for $0 \leq k < f$.

**Example.** Take $q = 9$ and $N = 131 = 11213$. Then
\[ \tau(N) = (3^4, 3^3, 3^2, 3^2, 3^1, 3^0, 3^0), \quad \tau_0(N) = (3^4, 3^2, 3^2, 3^0, 3^0), \quad \tau_1(N) = (3^3, 3^1). \]
Given $X \in W_d(N)$, $\tau_k(X_i)$’s give a partition of $p$-powers in $\tau_k(N)$ for each $k$. We call $X$ is $\tau$-monotonic if the sequence $\tau_k(N)$ is the concatenation of the subsequences $\tau_k(X_1), \tau_k(X_2), \ldots, \tau_k(X_d)$ for all $0 \leq k < f$. Note that there is a unique $\tau$-monotonic composition in $W^B_d(N)$ for each $B$ valid.

**Lemma 3.5.** Suppose $B$ is a valid matrix of $W_d(N)$, then the $\tau$-monotonic composition with respect to $B$ is lexicographically the largest and achieves the unique minimum weight in $W^B_d(N)$. In particular, both modest and optimal compositions are $\tau$-monotonic.

**Proof.** Take $X = (X_1, \ldots, X_d) \in W^B_d(N)$ which is not $\tau$-monotonic. Then there exist some $k, i, j, m, n$ such that $i < j$, $m < n$, with $p^m \in \tau_k(X_i)$, $p^n \in \tau_k(X_j)$. Consider the composition $Y = (X_1, \ldots, X_i - p^m + p^n, \ldots, X_j - p^n + p^m, \ldots, X_d)$. Then $Y \in W^B_d(N)$ since $m \equiv n \equiv k \mod f$. Clearly, $Y$ is lexicographically larger than $X$. Easy computation shows that $wt(Y) = wt(X) - (j - i)(p^n - p^m) < wt(X)$. \qed
Define

\[ \mathcal{J} := \{ \Gamma(n) : n > 0 \text{ is } q\text{-even}\}. \]

Given \( B = [B_1, \ldots, B_d] \) an \( f \times d \) matrix, the conditions for \( B \) being valid for \( W_d(N) \) can be translated as

1. \( \Gamma(N) = B_1 + \cdots + B_d \),
2. \( B_i \in \mathcal{J} \) for \( 1 \leq i \leq (d - 1) \).

We follow Sheats’ discussion in [She98] to give a characterization of vectors in \( \mathcal{J} \). Let \( \bar{e}_0, \ldots, \bar{e}_{f-1} \) be the standard basis of \( \mathbb{R}^f \), i.e. \( [\bar{e}_0, \ldots, \bar{e}_{f-1}] = I \), the identity matrix. Define matrix \( E = [E_0, E_1, \ldots, E_{f-1}] \) where

\[ E_i := p\bar{e}_{i-1} - \bar{e}_i. \]

Here and from now on, subscripts which should range from 0 to \( f - 1 \) are evaluated modulo \( f \), e.g. \( \bar{e}-1 = \bar{e}_{f-1} \) and \( E_0 = p\bar{e}_{f-1} - \bar{e}_0 \). Given vectors \( \bar{u} \) and \( \bar{v} = E\bar{u} \), we have for all \( i \)

\[ v_i = pu_{i+1} - u_i. \]

Let \( R = [\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{f-1}, \bar{e}_0] \) be the permutation matrix such that \( R\bar{e}_i = \bar{e}_{i+1} \). Then \( R^f = I \) and \( \langle R\bar{u}, R\bar{v} \rangle = \langle \bar{u}, \bar{v} \rangle \) for any \( \bar{u} \) and \( \bar{v} \). Recall that \( \bar{\psi}_0 = [1, p, \ldots, p^{f-1}]^t \), define

\[ \bar{\psi}_i := R^i\bar{\psi}_0 = [p^{f-i}, \ldots, p^{f-1}, 1, \ldots, p^{f-1-i}]^t \]

for \( 1 \leq i < f \). Then

\[ \langle \bar{\psi}_i, E_j \rangle = \begin{cases} q - 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \]

which implies

\[ E^{-1} = (q - 1)^{-1}[\bar{\psi}_0, \bar{\psi}_1, \ldots, \bar{\psi}_{f-1}]^t. \]

Given two vectors \( \bar{u} \) and \( \bar{v} \), we denote \( \bar{u} \geq \bar{v} \) if \( u_i \geq v_i \) for all \( i \), \( \bar{u} > \bar{v} \) if \( \bar{u} \geq \bar{v} \) and \( u_i > v_i \) for some \( i \) and \( \bar{u} \gg \bar{v} \) if \( u_i > v_i \) for all \( i \).

**Lemma 3.6.** Let \( \bar{u} = E\bar{a} \) and \( \bar{v} = E\bar{b} \), then

(1) \( \bar{a} > \bar{v} \Rightarrow \bar{a} \gg \bar{b} \). In particular, if \( \bar{a} > \bar{0} \), then \( \bar{a} \gg \bar{0} \).

(2) Let \( \bar{1} = [1, \ldots, 1]^t \). If \( \bar{0} < \bar{u} < (p - 1)\bar{1} \), then \( \bar{0} \ll \bar{a} \ll \bar{1} \).

**Proof.** \( \bar{a} - \bar{b} = E^{-1}(\bar{u} - \bar{v}) \). Since all components of \( E^{-1} \) are positive, \( \bar{u} - \bar{v} > \bar{0} \) implies \( \bar{a} - \bar{b} \gg \bar{0} \). This proves (1). (2) is a direct application of (1) as \( [p - 1, \ldots, p - 1]^t = E[1, \ldots, 1]^t \). \( \square \)

Take a positive integer \( n \). Let \( E\bar{a} = \Gamma(n) \), then we have for each \( i \)

\[ \alpha_i = (q - 1)^{-1}\langle \bar{\psi}_0, \Gamma(p^{f-i}n) \rangle \]

since \( R\Gamma(n) = \Gamma(pn) \) and \( \langle \bar{\psi}_i, \Gamma(n) \rangle = \langle R^i\bar{\psi}_0, \Gamma(n) \rangle = \langle \bar{\psi}_0, R^{f-i}\Gamma(n) \rangle \). In particular,

\[ \bar{a} \in \mathbb{Z}^f \Leftrightarrow n \text{ is } q\text{-even}. \]

The above discussion can be rephrased as following.

**Proposition 3.7 ([She98, Lem. 3.4]).** \( \mathcal{J} = (EZ^f) \cap (N^f \setminus \{\bar{0}\}) \).
3.2.2. Criterion for $W_d(N) \neq \emptyset$. For $d > 0$, define

$$I_d := \{ \Gamma(n): \exists \: \bar{v}_1, \ldots, \bar{v}_{d-1} \in \mathfrak{J} \text{ such that } \Gamma(n) > \bar{v}_1 + \cdots + \bar{v}_{d-1} \},$$

$$J_d := \mathfrak{J} \cap (I_d \setminus I_{d+1}).$$

$J_0 = \emptyset$. By definition, $J_d$ consists of those $\Gamma(n)$ such that $n$ can be written as a sum of $d$ many, but not $d + 1$ many, positive $q$-even numbers without carry over base $p$. Then

$$W_d(N) \neq \emptyset \iff \Gamma(N) \in J_{d-1} \cup I_d.$$ (3.1)

The next proposition by Sheats characterizes elements in $I_m$ and $J_m$.

**Proposition 3.8** ([She98, Prop. 4.3]). For $m \geq 1$,

1. $I_m = \{ E\bar{x} \in \mathbb{N}^f \setminus \{0\}: \bar{x} \in \mathbb{R}^f \text{ and } \min_{0 \leq i < f} (x_i) > m - 1 \}$;
2. $J_m = \{ E\bar{a} \in \mathbb{N}^f \setminus \{0\}: \bar{a} \in \mathbb{R}^f \text{ and } \min_{0 \leq i < f} (a_i) = m \}$.

With (3.1), it implies the following result which is indeed equivalent to Proposition 2.3.

**Corollary 3.9.** Let $\Gamma(N) = E\bar{a}$, then $W_d(N) \neq \emptyset$ iff $\min_{0 \leq i < f} (a_i) \geq d - 1$. \qed

3.2.3. Modest/optimal composition. The following results give estimation on components of the modest and optimal compositions.

**Proposition 3.10.** Let $X = (X_1, \ldots, X_d) \in W_d(N)$ be modest or optimal. Then $\Gamma(X_i) \in J_1$ for $2 \leq i \leq d - 1$, $X_d = 0$ if $N$ is $q$-even or $\Gamma(X_d) \in I_1 \setminus I_2$ if $N$ is $q$-odd.

**Proof.** Suppose not, then either there exists some $i$ with $2 \leq i \leq d - 1$ and vectors $\bar{v}_1, \bar{v}_2 \in \mathfrak{J}$ such that $\Gamma(X_i) = \bar{v}_1 + \bar{v}_2$, or $N$ is $q$-odd and $\Gamma(X_d) = \bar{w}_1 + \bar{w}_2$ with $\bar{w}_1 \in \mathfrak{J}, \bar{w}_2 \in I_1$ (the case where $N$ is $q$-even is discussed in \S 3.1).

In the first case, we can take $a_1, a_2$ such that $\Gamma(a_i) = \bar{v}_i$ and $X_i = a_1 \oplus a_2$. Define

$$Y := (X_1 + a_1, \ldots, X_{i-1} + a_2, X_i, \ldots, X_d).$$

Since $\bar{v}_i \in \mathfrak{J}$, both $a_i$’s are $q$-even. The sum of entries in $Y$ has no carry over base $p$. So $Y \in W_d(N)$. Moreover, $Y$ is lexicographically larger than $X$ and $wt(Y) = wt(X) - (i - 1)a_1 < wt(X)$, which implies $X$ is neither modest or optimal. Contradiction.

The second case is similar. We have $X_d = b_1 \oplus b_2$ with $\Gamma(b_i) = \bar{w}_i$ and $b_1$ $q$-even. Define $Y := (X_1 + b_1, X_2, \ldots, X_{d-1} + b_2)$. Then $Y \in W_d(N)$, $Y$ is lexicographically larger than $X$ and $wt(Y) < wt(X)$. Contradiction again. \qed

Take $N \in \mathbb{Z}_+$, let $\bar{u} = \Gamma(N)$ and $\bar{\beta} = E^{-1} \bar{u}$.

**Lemma 3.11.** Let $\bar{v} = E\bar{a} \in \mathbb{N}^f$ with $0 < \bar{v} < \bar{u}$. Suppose $\min_{0 \leq i < f} ([\beta_i] - [a_i]) = k$ for some $k \in \mathbb{N}$, then there exists some $\bar{w} \in \mathfrak{J}$ with $\bar{v} \leq \bar{w} \leq \bar{u}$ and $\bar{u} - \bar{w} \in J_k \cup I_{k+1}$.

**Proof.** To find such an $\bar{w}$ is equivalent to find a $\bar{\gamma}$ with $\bar{w} = E\bar{\gamma}$. Recall that if $\bar{x} = E\bar{a}$, $x_i = p\gamma_{i+1} - a_i$. By Propositions 3.7 and 3.8, we get the following conditions on $\bar{\gamma}$:

1. $v_i \leq p\gamma_{i+1} - \gamma_i \leq u_i$,
2. $\gamma_i \in \mathbb{Z}$ and $\min_{0 \leq j < f} (\beta_j - \gamma_j) \geq k$.

To construct $\bar{\gamma}$, take an $l$ such that $[\beta_l] - [\alpha_l] = k$. Let $\gamma_l = [\alpha_l]$. For $i = l - 1, l - 2, \ldots, l - f + 1$, define inductively

$$\gamma_i = \min ([\beta_i] - k, p\gamma_{i+1} - v_i).$$
Condition (2) holds automatically by the construction of $\gamma_i$. The construction also implies $v_i \leq p\gamma_{i+1} - \gamma_i$ for $i \neq l$. To prove it for $i = l$, we first show that $\gamma_i \geq \lceil \alpha_i \rceil$ for all $i$. By definition, $\gamma_i = \lceil \alpha_i \rceil$. We prove the rest by backwards induction. Suppose $\gamma_{i+1} \geq \lceil \alpha_{i+1} \rceil$. If $\gamma_i = \lceil \beta_i \rceil - k$, then clearly $\gamma_i \geq \lceil \alpha_i \rceil$ since $\lceil \beta_i \rceil - \lceil \alpha_i \rceil \geq k$; otherwise,

$$\gamma_i = p\gamma_{i+1} - v_i,$$

$$= p\gamma_{i+1} - p\alpha_{i+1} + \alpha_i \geq \alpha_i.$$  

Since $\gamma_i$ is an integer, we have $\gamma_i \geq \lceil \alpha_i \rceil$. Now we have $v_i \leq p\gamma_{i+1} - \gamma_i$ since

$$p\gamma_{i+1} - v_i = p\gamma_{i+1} - p\alpha_{i+1} + \alpha_i \geq \lceil \alpha_i \rceil = \gamma_i.$$  

Next we show $p\gamma_{i+1} - \gamma_i \leq u_i$. For $i = l$,

$$u_i -(p\gamma_{i+1} - \gamma_i) = p\beta_{l+1} - \beta_l - (p\gamma_{l+1} - \gamma_l)$$

$$= p(\beta_{l+1} - \gamma_{l+1}) - (\beta_l - \gamma_l)$$

$$= p(\beta_{l+1} - \gamma_{l+1}) - (\beta_l - \lceil \alpha_l \rceil)$$

$$> p(\beta_{l+1} - \gamma_{l+1}) - k - 1 \geq -1,$$

where the last inequality comes from that $\beta_{l+1} - \gamma_{l+1} \geq k$ and $p \geq 2$. Since the left-hand side is an integer, we have

$$u_i -(p\gamma_{l+1} - \gamma_l) \geq 0.$$  

Now let $i \neq l$. If $\gamma_i = p\gamma_{i+1} - v_i$, then $p\gamma_{i+1} - \gamma_i = v_i \leq u_i$; otherwise, $\gamma_i = \lceil \beta_i \rceil - k$, then a similar computation as in the $i = l$ case shows $p\gamma_{i+1} - \gamma_i \leq u_i$. \hfill $\square$

**Proposition 3.12.** Take $N$ with base $p$-expansion $N = \sum_{i=0}^{n} a_i p^i$, where $a_n \neq 0$. Suppose $W_d(N) \neq \emptyset$. Let $X = (X_1, \ldots, X_d) \in W_d(N)$ be modest or optimal, then

1. $X_1 \geq a_n p^n$. In particular, $X_1 > N/2$.
2. $N \leq wt(X) < 2N$.
3. $W_d(N - X_1) = \emptyset$ if $d \geq 2$. In particular, by (3.1), $\Gamma(N - X_1) \notin I_d$.

**Proof.** We prove each case separately.

$X$ modest: Let $\bar{u} = \Gamma(N)$ and $\bar{\alpha} = E^{-1} \bar{u}$. By Corollary 3.9, $\min_i(\lceil \alpha_i \rceil) = m \geq d - 1$. Let $k = n \mod f$ and $\bar{\beta} = E^{-1}(a_k \bar{e}_k)$. Lemma 3.6 implies $\lceil \beta_i \rceil = 1$ for each $i$. By Lemma 3.11, we can extend $\bar{v} = a_k \bar{e}_k$ to some $\bar{w}_1 \in \mathcal{I}$ with $\bar{u} - \bar{w}_1 \in J_{m-1} \cup I_m$. In particular, we can write $\bar{u} - \bar{w}_1$ as $\bar{u} - \bar{w}_1 = \bar{w}_2 + \cdots + \bar{w}_{d-1} + \bar{w}_d$, where $\bar{w}_i \in \mathcal{I}$ for $2 \leq i \leq d - 1$ and $\bar{w}_d \in \mathcal{N}^d$. Take $B = [\bar{w}_1, \ldots, \bar{w}_d]$, then $B$ is a valid matrix, i.e. $W_d^B(N) \neq \emptyset$. Let $\mathbf{Y}$ be the $\tau$-monotonic element in $W_d^B(N)$, then $Y_1 \geq a_n p^n$ since $\bar{w}_1 \geq a_n \bar{e}_k$. $X$ is modest so $X_1 \geq Y_1 \geq a_n p^n$. This proves (1).

We prove (2) by induction on $d$. For $d = 1$, $X = (N)$ and $wt(X) = N$. Suppose (2) holds for $d - 1$. By Proposition 3.4 (2), $Y = (X_2, \ldots, X_d)$ is modest in $W_{d-1}(N - X_1)$. By induction, $N - X_1 \leq wt(Y) < 2(N - X_1) < N$. Thus $N \leq wt(X) = N + wt(Y) < 2N$.

Suppose (3) fails. Take $(X_1', \ldots, X_d') \in W_d(N - X_1)$, then $(X_1 + X_1', X_2, \ldots, X_d') \in W_d(N)$. But this contradicts that $X$ is modest.

$X$ optimal: (2) holds automatically by the minimum weight property.

To prove (1), we first show that $X_1 \geq p^n$. If $N$ is $q$-even or $d < 3$, by §3.1 and Remark 3.3, optimal is equivalent to modest, thus (1) holds. We assume $N$ is $q$-odd and $d \geq 3$. Then $X_1 + X_2 > p^n$ since otherwise $X_1 + X_2 < p^n < N/2$ and $wt(X) > N + 2(N - X_1 - X_2) > 2N$. For $wt(X)$ being minimal, $X_1 \geq X_2$ which implies $X_1 \geq p^n$. Now suppose $X_1 = \sum_{i=0}^{n} b_i p^i$ with $0 < b_n < a_n$, then $N - X_1 > p^n$. Note that $(X_2, \ldots, X_d)$
is optimal in $W_{d-1}(N-X_1)$, thus $X_2 \geq p^n$ and $N-X_d > (b_n+1)p^n$. But by Proposition 3.4 (1), $(X_1, \ldots, X_{d-1}) \in W_{d-1}(N-X_d)$ is optimal and thus modest since $N-X_d$ is $q$-even. In particular, $X_1 \geq (b_n+1)p^n$. Contradiction.

At last, we show (3) holds. If not, let $(X'_1, \ldots, X'_{d'}) \in W_d(N-X_1)$ be optimal. Then $X'_1 > (N-X_1)/2$ by (1). $(X'_2, \ldots, X'_{d'})$ being optimal in $W_{d-1}(N-X_1 - X'_1)$, wt$(X'_2, \ldots, X'_{d'}) < 2(N-X_1 - X'_1) < N-X_1$. Let $Y = (X_1 + X'_1, X'_2, \ldots, X'_{d'})$, then $Y \in W_d(N)$ and wt$(Y) = N + wt(X'_2, \ldots, X'_{d'}) < 2N-X_1$. However, wt$(X) = N + wt(X_2, \ldots, X_d) \geq 2N-X_1$. Contradiction. □

3.2.4. Constructing composition of smaller weight. Suppose Theorem 3.2 fails. Then $q = p^f$ with $f > 1$. Take the least $d$ and some $N$ such that there exist $M = (M_i), O = (O_i) \in W_d(N)$ with $M$ modest, $O$ optimal and $M \neq O$. Then $N$ is $q$-odd, $d \geq 3$ and $M_1 > O_1$ by §3.1, Remark 3.3 and Proposition 3.4 (2) respectively. Let $p^a$ be the largest $p$-power in $P(M_1) \setminus P(O_1)$. By Proposition 3.4 (3), we may assume $a \equiv f-1 \mod f$. Let

\[
\bar{u} := \Gamma(N), \quad \bar{x} := \Gamma(M_1), \quad \bar{y} := \Gamma(O_1), \\
\bar{\eta} := E^{-1}\bar{u}, \quad \bar{\alpha} := E^{-1}\bar{x}, \quad \bar{\beta} := E^{-1}\bar{y}.
\]

By our construction, $x_{f-1} > y_{f-1}$ since both $M$ and $O$ are $\tau$-monotonic. Define

\[
\bar{v} = [v_i]^t := \min(x_i, y_i), \quad \bar{w} = [w_i]^t := \bar{y} - \bar{v}.
\]

Then $w_{f-1} = 0$. Note that $\bar{w} > 0$, since otherwise $\bar{x} > \bar{y}$ and $\bar{x} - \bar{y} \in \mathfrak{j}$, $\Gamma(N-O_1) = (\bar{x} - \bar{y}) + \Gamma(M_2) + \cdots + \Gamma(M_d) \in I_d$, contradicting Proposition 3.12 (3). Let $0 \leq k \leq f-2$ be the least subscript such that $w_k > 0$. We have the following result.

**Lemma 3.13.**

1. $\langle \bar{v}_i, \bar{w} \rangle < \langle \bar{\psi}_i, \bar{e}_{f-1} \rangle$ for $0 \leq i \leq k$.
2. $|\eta_i| - \beta_i \geq d-2$ for all $i$ and there exists $k < l \leq f-1$ such that $|\eta_i| - \beta_i = d-2$ and $|\eta_i| - \beta_i \geq d-1$ for $l-f < i \leq k$, i.e. $i = l+1, l+2, \ldots, f-1, 0, 1, \ldots, k$.

**Proof.** We show (1) by contradiction. For each $0 \leq i \leq k$, by definition of $\bar{\psi}_i$,

\[
\langle \bar{\psi}_i, \bar{e}_{f-1} \rangle = p^{f-1-i}
\]

and $\langle \bar{\psi}_i, \bar{w} \rangle$ is a sum of $p$-powers less than $p^{f-1-i}$ since $w_j = 0$ for $-1 \leq j < i$. Suppose $\langle \bar{\psi}_i, \bar{w} \rangle \geq \langle \bar{\psi}_i, \bar{e}_{f-1} \rangle$ for some $0 \leq i \leq k$. Then there is a subset of $p$-powers in the sum $\langle \bar{\psi}_i, \bar{w} \rangle$ whose terms add up to $p^{f-1-i}$. In other words, there exists some $\bar{w}' \leq \bar{w}$ such that

\[
\langle \bar{\psi}_i, \bar{w}' \rangle = \langle \bar{\psi}_i, \bar{e}_{f-1} \rangle.
\]

Another observation is that $\bar{w}$ represents those $p$-powers $p^b$ in $P(O_1) \setminus P(M_1)$. In particular, $b < a$ for each $b$ since $O_1 < M_1$. $\bar{w}' \leq \bar{w}$ represents a subset of such $p$-powers. Let $M$ be the sum of $p$-powers represented by $\bar{w}'$, then $M < p^a$ and $\Gamma(M) = \bar{w}'$. Note that $\Gamma(p^a) = \bar{e}_{f-1}$, then (3.2) implies $M \equiv p^a \mod q - 1$.

By the choice of $p^a$, we can find some $j > 1$ such that $p^a \in P(O_j)$. Consider composition

\[
X = (O_1 - M + p^a, \ldots, O_j - p^a + M, \ldots, O_d),
\]

then $X \in W_d(N)$ and wt$(X) < wt(O)$. Contradiction.
To prove (2), we note that $E(\bar{\eta} - \bar{\beta}) = \Gamma(N - O_1) = \Gamma(O_2) + \cdots + \Gamma(O_d) \in I_{d-1}$. Hence, by Proposition 3.8 (1), $\eta_i - \beta_i > d - 2$. Thus

$$[\eta_i] - \beta_i \geq d - 2$$

for all $i$. On the other hand, $\Gamma(N - O_1) \not\in I_d$ implies $\min_i([\eta_i] - \beta_i) = d - 2$. Let $l$ be the largest subscript such that $[\eta_i] - \beta_i = d - 2$. Then for $l < i < f$,

$$[\eta_i] - \beta_i \geq d - 1.$$  

To finish the proof, we show $[\eta_i] - \beta_i \geq d - 1$ for $0 \leq i \leq k$. By construction, we have

$$\bar{x} = \bar{v} + \bar{w}, \quad \bar{y} = \bar{v} + \bar{w},$$

where $\bar{w} = \bar{e}_f - 1$. For $0 \leq i \leq k$,

$$\beta_i = (q - 1)^{-1} \langle \bar{\psi}_i, \bar{v} \rangle = (q - 1)^{-1} \langle \bar{\psi}_i, \bar{v} \rangle + (\bar{\psi}_i, \bar{e}_f - 1)$$

(by (1)) $\leq (q - 1)^{-1} \langle \bar{\psi}_i, \bar{v} \rangle + (\bar{\psi}_i, \bar{e}_f - 1)$

$$\leq (q - 1)^{-1} \langle \bar{\psi}_i, \bar{v} \rangle$$

$$= (q - 1)^{-1} \langle \bar{\psi}_i, \bar{v} \rangle$$

$$\leq \alpha_i.$$  

Note that $E(\bar{\eta} - \bar{\alpha}) = \Gamma(N - M_1) \in I_{d-1}$, by Proposition 3.8 (1), we have $[\eta_i] - \alpha_i > d - 2$ for all $0 \leq i < f$. For $0 \leq i \leq k$, $\beta_i < \alpha_i$ by above calculation, thus $[\eta_i] - \beta_i \geq d - 1$. □

Define for $1 \leq j \leq d$

$$\bar{u}_j = [u_{0,j}, \ldots, u_{f-1,j}]^t := \sum_{s=j}^d \Gamma(O_s), \quad \bar{\theta}_j = [\theta_{0,j}, \ldots, \theta_{f-1,j}]^t := E^{-1} \bar{u}_j.$$  

Note that $\bar{u}_1 = \bar{u}$ and $\bar{\theta}_1 = \bar{\eta}$. By Lemma 3.13 (2), we have

$$[\theta_{i+2}] = d - 2, \quad [\theta_{i+2}] \geq d - 1 \quad \text{for } l - f < i \leq k.$$  

The following construction is to use $\bar{\theta}_j$ to get a new composition $Z \in W_d(N)$ whose weight is less than that of $O$. Let $\phi_1 = [\phi_{i,1}]^t := \bar{\theta}_1$. Define $\phi_2 = [\phi_{i,2}]^t$ inductively as following.

$$\phi_{i,2} = \begin{cases} \theta_{i,2} & \text{for } k < i \leq l \\ \min(\theta_{i,2}, p\phi_{i+1,2}) & \text{for } i = k, k - 1, \ldots, 0, f - 1, f - 2, \ldots, l + 1. \end{cases}$$

For $j = 3, \ldots, d$, define $\phi_j = [\phi_{i,j}]^t$ recursively as

$$\phi_{i,j} = \begin{cases} \theta_{i,j} & \text{for } k < i \leq l \\ \min(\phi_{i,j-1}, p\phi_{i+1,j}) & \text{for } i = k, k - 1, \ldots, 0, f - 1, f - 2, \ldots, l + 1. \end{cases}$$

**Proposition 3.14.** For $1 \leq j \leq d$, let

$$\bar{z}_j = [z_{0,j}, \ldots, z_{f-1,j}]^t := E\phi_j.$$  

Then

1. $\bar{z}_j - \phi_{j+1} \in \mathbb{Z}_+^f$ for $1 \leq j \leq d - 1$.
2. $\bar{z}_j \in \mathbb{Z}_+^f$ for all $j$.
3. $\min_{0 \leq i \leq f-1 ([\phi_{i,j}])} = [\phi_{i,j}] = d - j$ for $2 \leq j \leq d$.
4. $z_{k,2} = u_{k,2} + 1 \leq u_k$. 

(5) $0 \leq z_{i,2} \leq u_{i,2} - p$.

(6) $0 \leq z_{i,2} \leq \max(u_{i,2} - (p - 1), 0)$ for $l - f < i < k$.

(7) $z_{i,j} = u_{i,j}$ for $k < i < l$ and $2 \leq j \leq d$.

(8) $0 \leq z_{i,j} \leq \max(z_{i,j-1} - (p - 1), 0)$ for $l - f \leq i \leq k$ and $3 \leq j \leq d$.

Proof. (1): By construction, $\phi_{i,j} - \phi_{i,j+1} > 0$ for all $i, j$. Hence it's enough to show

(a) $\tilde{\theta}_j - \tilde{\theta}_{j+1} \in \mathbb{Z}^f$ for $1 \leq j \leq d - 1$;

(b) $\{\phi_{i,j}\} = \{\theta_{i,j}\}$ for all $i, j$, where $\{x\}$ is the fractional part of $x$.

We note that for $1 \leq j \leq d - 1$

$$E(\tilde{\theta}_j - \tilde{\theta}_{j+1}) = \Gamma(O_j) \in \mathcal{J}.$$ 

This implies (a) by Proposition 3.7. (a) says $\{\theta_{i,j}\} = \{\theta_{i,j+1}\}$ for all $i$ and $j$. Also $p\{\theta_{i+1,j}\} - \{\theta_{i,j}\} \in \mathbb{Z}$ since $p\theta_{i+1,j} - \theta_{i,j} = u_{i,j} \in \mathbb{N}$. With these two properties in mind, starting with the initial case $\{\phi_{i,1}\} = \{\theta_{i,1}\}$ since $\tilde{\phi}_1 = \theta_1$, following the inductive construction of $\phi_{i,j}$, one can check that (b) holds.

(2): Since $\tilde{\phi}_1 = \bar{n}$, $\bar{z}_1 = E\bar{n} = \bar{u} \in \mathbb{Z}^f$. For $j > 1$,

$$\bar{z}_j = E\tilde{\phi}_1 - \sum_{s=1}^{j-1} E(\tilde{\phi}_s - \tilde{\phi}_{s+1}).$$

By (1), $\tilde{\phi}_s - \tilde{\phi}_{s+1} \in \mathbb{Z}^f$ for each $s$, hence $\bar{z}_j \in \mathbb{Z}^f$.

(3): We first show that $|\phi_{i,j}| \geq d - j$ for $2 \leq j \leq d$ and $k < i \leq l$. This is the same as showing

$$|\theta_{i,j}| \geq d - j$$

for $2 \leq j \leq d$, since $\phi_{i,j} = \theta_{i,j}$ by construction. Note that for each $j$, $E\tilde{\theta}_j = \sum_{s=j}^{d} \Gamma(O_j) \in I_{d-j+1}$. Thus the statement follows from Proposition 3.8.

Next we prove $|\phi_{i,j}| = d - j$ for each $j$. The $j = 2$ case is given by (3.3). For $3 \leq j \leq d$, $\tilde{\theta}_2 - \tilde{\theta}_j = \sum_{s=2}^{j-1} E^{-1}\Gamma(O_s)$. By Propositions 3.10 and 3.8, $\tilde{\theta}_2 - \tilde{\theta}_j \geq j - 2$, which implies $|\theta_{i,j}| \leq |\theta_{i,2}| - (j - 2) = d - j$. Thus the statement follows since $|\theta_{i,j}| \geq d - j$.

Last, we show $|\phi_{i,j}| \geq d - j$ for $l - f < i \leq k$ by induction on $j$. For $j = 2$, we have

$$|\theta_{i,2} - 1| \geq d - 2$$

for $l - f < i \leq k$ by (3.3). Taking $i = k$, since $|p\phi_{k+1,2}| = |p\theta_{k+1,2}| \geq d - 2$, we get

$$|\phi_{k,2}| \geq \min(|\theta_{k,2} - 1|, |p\phi_{k+1,2}|) \geq d - 2.$$ 

Note that

$$|\phi_{i+1,2}| \geq d - 2 \Rightarrow |p\phi_{i+1,2}| \geq d - 2$$

$$\Rightarrow |\phi_{i,2}| = \min(|\theta_{i,2} - 1|, |p\phi_{i+1,2}|) \geq d - 2.$$ 

Hence, a backwards induction on $i$ starting from $k$ implies that for $k \geq i > l - f$,

$$|\phi_{i,2}| \geq d - 2.$$ 

Suppose $|\phi_{i,j-1}| \geq d - j + 1$ for $l - f < i \leq k$. Then $|\phi_{k,j-1} - 1| \geq d - j$ and $|p\phi_{k+1,j}| \geq d - j$ since $|\phi_{k+1,j}| \geq d - j$ by previous statement. This implies $|\phi_{k,j}| = \min(|\phi_{k,j-1} - 1|, |p\phi_{k+1,j}|) \geq d - j$. Similarly, we have

$$|\phi_{i+1,2}| \geq d - j \Rightarrow |p\phi_{i+1,j}| \geq d - j$$

$$\Rightarrow |\phi_{i,j}| = \min(|\phi_{i,j-1} - 1|, |p\phi_{i+1,j}|) \geq d - j.$$
Again, a backwards induction on \( i \) shows that \( |\phi_{i,j}| \geq d - j \) for \( k \geq i > l - f \).

(4): Since
\[
p\phi_{k+1,2} - (\theta_{k,2} - 1) = p\theta_{k+1,2} - \theta_{k,2} + 1 = u_{k,2} + 1 > 0,
\]
\( \phi_{k,2} = \theta_{k,2} - 1 \) and \( z_{k,2} = u_{k,2} + 1 \). By construction,
\[
u_{k,2} = u_k - y_k \leq u_k - w_k \leq u_k - 1,
\]
so \( u_{k,2} + 1 \leq u_k \).

(5): The second inequality is given by
\[
z_{l,2} = p\phi_{l+1,2} - \phi_{l,2} \leq p(\theta_{l+1,2} - 1) - \theta_{l,2} = u_{l,2} - p.
\]
To show \( z_{l,2} \geq 0 \), we have \( \min_{0 \leq i \leq f-1} |\phi_{i,2}| = |\phi_{l,2}| = d - 2 \) by (3). This implies
\[
z_{l,2} = p\phi_{l+1,2} - \phi_{l,2} \geq p(d - 2) - (d - 2) - \{ \phi_{l,2} \} \geq 0.
\]

(6): For \( l - f < i < k \), if \( \phi_{i,2} = p\phi_{i+1,2} \), \( z_{i,2} = p\phi_{i+1,2} - \phi_{i,2} = 0 \); otherwise, \( \phi_{i,2} = \theta_{i,2} - 1 \) and \( z_{i,2} = p\phi_{i+1,2} - (\theta_{i,2} - 1) \leq p(\theta_{i+1,2} - 1) - (\theta_{i,2} - 1) = u_{i,2} - (p - 1) \).

(7): This follows directly from the construction of \( \phi_j \)'s.

(8): We break up the proof into three cases.

For \( l - f < i < k \), we only need to check for the case where \( \phi_{i,j} = \phi_{i,j-1} - 1 \), since otherwise \( z_{i,j} = 0 \). In this case, \( \phi_{i,j} - 1 \leq p\phi_{i+1,j} \) and
\[
0 \leq z_{i,j} = p\phi_{i+1,j} - (\phi_{i,j-1} - 1) \leq p(\phi_{i+1,j-1} - 1) - (\phi_{i,j-1} - 1) = z_{i,j-1} - (p - 1).
\]

For \( i = k \), again, we may assume \( \phi_{k,j} = \phi_{k,j-1} - 1 \), then
\[
0 \leq z_{k,j} = p\theta_{k+1,j} - (\phi_{k,j-1} - 1)
\leq p(\theta_{k+1,j-1} - 1) - (\phi_{k,j-1} - 1) = z_{k,j-1} - (p - 1),
\]
where the second inequality follows from that \( \theta_{j-1} - \bar{\theta}_j = E^{-1}\Gamma(O_{j-1}) \geq 1 \) by Propositions 3.10 and 3.8.

For \( i = l \), by (3), we have
\[
z_{l,j} = p\phi_{l+1,j} - \phi_{l,j} \geq p(d - j) - (d - j) - \{ \phi_{l,j} \} \geq 0.
\]
Finally, \( z_{l,j} = p\phi_{l+1,j} - \theta_{l,j} \leq p(\phi_{l+1,j-1} - 1) - (\theta_{l,j-1} - 1) = z_{l,j-1} - (p - 1). \)

Proposition 3.14 implies that the matrix
\[
B = [\tilde{z}_1 - \tilde{z}_2, \ldots, \tilde{z}_{d-1} - \tilde{z}_d, \tilde{z}_d]
\]
is a valid matrix of \( W_d(N) \). Let \( Z = (Z_1, \ldots, Z_d) \) be the \( \tau \)-monotonic element in \( W_d^B(N) \).
We show that \( wt(Z) < wt(O) \) and hence get a contradiction.

### 3.2.5. Estimation on \( wt(Z) \).

For \( 2 \leq j \leq d \), define
\[
Z'_j := Z_j + Z_{j+1} + \cdots + Z_d,
O'_j := O_j + O_{j+1} + \cdots + O_d.
\]
Then \( \Gamma(Z'_j) = \tilde{z}_j \) and \( \Gamma(O'_j) = \tilde{u}_j \). And weights of \( Z \) and \( O \) can be expressed as
\[
wt(Z) = N + Z'_2 + \cdots + Z'_d,
wt(O) = N + O'_2 + \cdots + O'_d.
\]
To describe these \( Z'_i, O'_i \) explicitly, for each \( 0 \leq i \leq f - 1 \), denote
\[
\tau_i(N) = (\tau_{i,u}, \tau_{i,u-1}, \ldots, \tau_{i,1}).
\]
We recall that $\tau_i(N)$ is defined as the subsequence of the nonincreasing sequence of $p$-powers in $P(n)$, where the exponents of powers in it are congruent to $i$ modulo $f$.

Let $\tau_{i,0} = 0$. Then, by $\tau$-monotonicity, we have

$$Z_j^f = \sum_{i=0}^{f-1} \sum_{s=0}^{z_i,j} \tau_{i,s}, \quad O_j^f = \sum_{i=0}^{f-1} \sum_{s=0}^{u_i,j} \tau_{i,s}.$$ 

By Proposition 3.14 (7),

$$O_j^f - Z_j^f = \sum_{i \in I} \left( \sum_{s=0}^{u_i,j} \tau_{i,s} - \sum_{s=0}^{z_i,j} \tau_{i,s} \right),$$

where

$$I = \{l, \ldots, f - 1, 0, \ldots, k\}.$$ 

For $j = 2$ and $i \in I$, we have the following:

1. By Proposition 3.14 (4, 5, 6), $z_{k,2} = u_{k,2} + 1$ and for $i \in I \setminus \{k\}$, $z_i,2 \leq u_i,2$ where $\tau$ holds iff $z_i,2 = u_i,2 = 0$.
2. $\tau_{f-1,u_{f-1},2} = p^a$ since it is the largest $p$-power not in $P(O_1)$ whose exponent is $f - 1$ mod $n$. In particular, $u_{f-1,2} > 0$ hence $z_{f-1,2} < u_{f-1,2}$ by (1).
3. Let $\tau_{k,z_{k,2}} = p^b$, $z_{k,2} = u_{k,2} + 1$ implies that $p^b$ is the last $p$-power in $\tau_k(O_1)$. By our choice of $k$, $p^b \in P(O_1) \setminus P(M_1)$. In particular, $p^b < p^a$.

With these observations, we have

$$O_2^f - Z_2^f \geq \tau_{f-1,u_{f-1},2} - \tau_{k,z_{k,2}} + \sum_{i \in I \setminus \{k,f-1\}} \tau_{i,z_i,2} = p^a - p^b + \sum_{i \in I \setminus \{k,f-1\}} \tau_{i,z_i,2}.$$ 

Thus

$$O_j^f - Z_j^f \geq p^a - p^b + \sum_{i \in I \setminus \{k,f-1\}} \tau_{i,z_i,2}.$$ 

(3.4) 

$$wt(O) - wt(Z) = \sum_{j=2}^{d} O_j^f - Z_j^f \geq p^a - p^b + \sum_{i \in I \setminus \{k,f-1\}} \tau_{i,z_i,2}.$$

The next lemma gives a lower bound for $\sum_{j=3}^{d} O_j^f - Z_j^f$.

**Lemma 3.15.** Let $I = \{l, \ldots, f - 1, 0, \ldots, k\}$, then

$$\sum_{j=3}^{d} O_j^f - Z_j^f > - \sum_{i \in I} \tau_{i,z_{i,2}}.$$ 

**Proof.** We note that $\tau_{k,z_{k,2}} > 0$ since $z_{k,2} > 0$ by Proposition 3.14 (4). The statement is trivial if $z_{i,j} = 0$ for all $i \in I$ and $3 \leq j \leq d$. Assuming they are not all vanishing, the statement follows from the following calculation.

$$\sum_{j=3}^{d} O_j^f - Z_j^f = \sum_{j=3}^{d} \sum_{i \in I} \left( \sum_{s=0}^{u_i,j} \tau_{i,s} - \sum_{s=0}^{z_i,j} \tau_{i,s} \right)$$

$$> - \sum_{i \in I} \sum_{j=3}^{d} \sum_{s=0}^{z_i,j} \tau_{i,s}$$

$$> - p \sum_{i \in I} \sum_{j=3}^{d} \tau_{i,z_{i,j}}$$
\[ > -p^2 / q \sum_{i \in I} \tau_i, z_{i,2} \]
\[ \geq - \sum_{i \in I} \tau_i, z_{i,2}. \]

The first inequality is trivial. The last one follows from the assumption \( f \geq 2 \). For the second inequality, we note that each \( p \)-power appeared in the sum \( \sum_{i=0}^{z_{i,j}} \tau_{i,s} \) repeats at most \( p - 1 \) times and the largest term is \( \tau_{i, z_{i,j}} \), which indicates \( \sum_{i=0}^{z_{i,j}} \tau_{i,s} < p \tau_{i, z_{i,j}} \).

By a similar argument and Proposition 3.14 (8), for all \( i \in I \) and \( 3 \leq j \leq d \), we have
\[ \tau_{i, z_{i,j}} \leq q^{-1} \tau_{i, z_{i,j-1}}, \]
thus
\[ \sum_{j=3}^{d} \tau_{i, z_{i,j}} \leq q^{-1} \sum_{j=2}^{d-1} \tau_{i, z_{i,j}} < \frac{p}{q} \tau_{i, z_{i,2}}. \]

This gives the third inequality.

Now we are ready to claim the contradiction, which finishes the proof of Theorem 3.2.

**Proposition 3.16.** \( wt(Z) < wt(O) \).

**Proof.** We first show \( \tau_{f-1, z_{f-1,2}} < p^a \). Assume \( z_{f-1,2} > 0 \) since otherwise \( \tau_{f-1, z_{f-1,2}} = 0 \). Then \( \tau_{f-1, z_{f-1,2}} \leq q^{-1} \tau_{f-1, u_{f-1,2}} \) because \( z_{f-1,2} \leq u_{f-1,2} - (p - 1) \) by Proposition 3.14.

Note that \( \tau_{f-1, u_{f-1,2}} = p^a \) as we mentioned earlier, thus \( \tau_{f-1, z_{f-1,2}} < p^a \).

Putting together (3.4) and Lemma 3.15, we have
\[ wt(O) - wt(Z) > p^a - 2p^b - \tau_{f-1, z_{f-1,2}}. \]

Here \( p^b \) and \( \tau_{f-1, z_{f-1,2}} \) are \( p \)-powers less than \( p^a \) and they are distinct since their exponents fall into different residue classes mod \( f \). We break up the proof into 3 cases.

1. \( p \geq 3 \): \( wt(O) - wt(Z) > 0 \) since \( 2p^b + \tau_{f-1, z_{f-1,2}} < p^a \).
2. \( p = 2 \) and \( b + 1 < a \): \( 2p^b + \tau_{f-1, z_{f-1,2}} = p^{b+1} + \tau_{f-1, z_{f-1,2}} \leq p^a \), thus \( wt(O) - wt(Z) > 0 \).
3. \( p = 2 \) and \( b + 1 = a \): In this case, we have \( k + 1 = l = f - 1 \) and \( I = \{0, \ldots, f-1\} \).

By Proposition 3.14 (5), \( u_{f-1,2} - z_{f-1,2} \geq 2 \). Hence
\[ O'_2 - Z'_2 = \left( \sum_{s=0}^{u_{f-1,2}} \tau_{i,s} - \sum_{s=0}^{z_{f-1,2}} \tau_{i,s} \right) + \sum_{i=0}^{k-1} \left( \tau_{i,s} - \sum_{s=0}^{z_{i,s}} \tau_{i,s} \right) \]
\[ \geq p^a + \tau_{f-1,1 + z_{f-1,2}} - p^b + \sum_{i=0}^{k-1} \tau_{i, z_{i,2}} \]
and by Lemma 3.15,
\[ wt(O) - wt(Z) = \sum_{j=2}^{d} O'_j - Z'_j \]
\[ > p^a + \tau_{f-1,1 + z_{f-1,2}} - p^b + \sum_{i=0}^{k-1} \tau_{i, z_{i,2}} - \sum_{i \in I} \tau_{i, z_{i,2}} \]
\[ = p^a + \tau_{f-1,1 + z_{f-1,2}} - p^b - p^b - \tau_{f-1, z_{f-1,2}} \]
\[ = \tau_{f-1,1 + z_{f-1,2}} - \tau_{f-1, z_{f-1,2}} \geq 0. \]
References

[BGF] J. I. Burgos Gil and J. Fresán, Multiple zeta values: from numbers to motives, to appear in Clay Mathematics Proceedings. Available at http://javier.fresan.perso.math.cnrs.fr/mzv.pdf.

[Böc13] G. Böckle, The distribution of the zeros of the Goss zeta-function for $A = \mathbb{F}_2[x, y]/(y^2 + y + x^3 + x + 1)$, Math. Z. 275 (2013), no. 3-4, 835–861.

[Car48] L. Carlitz, Finite sums and interpolation formulas over $GF[q^n, x]$, Duke Math. J. 15 (1948), 1001–1012.

[Cha14] Chieh-Yu Chang, On characteristic $p$ multizeta values, Algebraic number theory and related topics 2012, RIMS Kôkyûroku Bessatsu, B51, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014, pp. 177–202.

[DV96] J. Diaz-Vargas, Riemann hypothesis for $\mathbb{F}_p[T]$, J. Number Theory 59 (1996), no. 2, 313–318.

[Eul75] L. Euler, Meditationes circa singulare serierum genus, Novi Comm. Acad. Sci. Petropol. 20 (1775), 140–186, Reprinted in “Opera Omnia”, ser. 1, vol. 15, B. G. Teubner, Berlin, 1927, pp. 217-267.

[FKMT17] H. Furusho, Y. Komori, K. Matsumoto, and H. Tsumura, Desingularization of complex multiple zeta-functions, Amer. J. Math. 139 (2017), no. 1, 147–173.

[Gos79] D. Goss, $v$-adic zeta functions, $L$-series and measures for function fields, Invent. Math. 55 (1979), 107–119.

[Gos96] Basic structures of function field arithmetic, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 35, Springer-Verlag, Berlin, 1996.

[Mas06] R. Masri, Multiple zeta values over global function fields, Multiple Dirichlet series, automorphic forms, and analytic number theory, Proc. Sympos. Pure Math., vol. 75, Amer. Math. Soc., Providence, RI, 2006, pp. 157–175.

[She98] J. T. Sheats, The Riemann hypothesis for the Goss zeta function for $\mathbb{F}_q[T]$, J. Number Theory 71 (1998), no. 1, 121–157.

[Tha04] D. S. Thakur, Function field arithmetic, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

[Tha09] Power sums with applications to multizeta and zeta zero distribution for $\mathbb{F}_q[t]$, Finite Fields Appl. 15 (2009), no. 4, 534–552.

[Tha17] Multizeta values for function fields: a survey, J. Théor. Nombres Bordeaux 29 (2017), no. 3, 997–1023.

[Zha00] J. Zhao, Analytic continuation of multiple zeta functions, Proc. Amer. Math. Soc. 128 (2000), no. 5, 1275–1283.

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
E-mail address: shuhui@math.tamu.edu