The Modified Kumaraswamy Weibull Distribution: Properties and Applications in Reliability and Engineering Sciences

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Abstract

We introduce the Kumaraswamy alpha power-G (KAP-G) family which extends the alpha power family (Mahdavi and Kundu, 2017) and some other families. We consider the Weibull as baseline for the KAP family and generate Kumaraswamy alpha power Weibull distribution which has right-skewed, left-skewed, symmetrical, and reversed-J shaped densities, and decreasing, increasing, bathtub, upside-down bathtub, increasing-decreasing–increasing, J shaped and reversed-J shaped hazard rates. The proposed distribution can model non-monotone and monotone failure rates which are quite common in engineering and reliability studies. Some basic mathematical properties of the new model are derived. The maximum likelihood estimation method is used to evaluate the parameters and the observed information matrix is determined. The performance and flexibility of the proposed family is illustrated via two real data applications.

Key Words: Alpha power family; Kumaraswamy family; Maximum likelihood estimation; Weibull distribution.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Recently, Mahdavi and Kundu (2017) proposed the alpha power transformation (APT) family. For any baseline cumulative distribution function (CDF) \( G(x) \), the APT family is specified by the CDF (for \( x \in \mathcal{R} \))

\[
H_{APT}(x) = \begin{cases} 
\frac{\alpha^G(x)-1}{\alpha-1} & \text{if } \alpha > 0, \alpha \neq 1 \\
G(x) & \text{if } \alpha = 1 
\end{cases}
\]  

(1)

Its probability density function (PDF) has the form

\[
h_{APT}(x) = \begin{cases} 
\ln \alpha \frac{G(x)\alpha^G(x)}{\alpha-1} & \text{if } \alpha > 0, \alpha \neq 1 \\
G(x) & \text{if } \alpha = 1 
\end{cases}
\]  

(2)

The APT family is considered by several authors to extend some distributions and to propose other extensions to it. For example, Mead et al. (2019) derived the general properties of the APT method and studied the alpha power exponentiated Weibull distribution. Nassar et al. (2019) introduced a new extension of the APT method called Marshall Olkin alpha power-G family.

In this paper, we study a new wider class called the Kumaraswamy alpha power-G (KAP-G) family, with three shape parameters to increase the flexibility to the generated class. The new KAP-G family is constructed based on the Kumaraswamy-G (K-G) family (Cordeiro and de Castro, 2011).
Consider the CDF and PDF of a given random variable namely $G(x)$ and $g(x)$. Then, the CDF and PDF of the K-G family take the forms

$$F(x) = 1 - [1 - G(x)]^b, \ a, b > 0$$

and

$$f(x) = abg(x)G(x)^{a-1}[1 - G(x)]^{b-1}, \ a, b > 0.$$  

The K-G family has been used to construct several extended distributions and families. For example, the Kumaraswamy complementary Weibull geometric distribution (Afify et al., 2017), the Kumaraswamy transmuted-G family (Afify et al., 2016a).

We define a new KAP family by inserting the CDF of the APT as a baseline CDF in Equation (3). The new KAP family is used to construct the new four-parameter KAP-Weibull (KAPW) distribution, which has several desirable properties.

The KAP-G family some useful motivations are: (i) Its special sub-models, for example the KAPW model, contains some lifetime sub-models such as the Weibull, exponentiated Weibull (Mudholkar and Srivastava, 1993) and Kumaraswamy Weibull (Cordeiro et al., 2010), among others; (ii) Its sub-models can be used in modeling all important hazard rate shapes such as bathtub, upside down bathtub, increasing, decreasing, and reversed-J hazard rates; and (iii) The KAPW model compares very well with other eight competing extensions of the Weibull distribution in two real data applications.

The rest of this paper is outlined as follows. We define the KAP-G distribution and its special cases are presented in Section 2. In Section 3, we study the new KAPW distribution. Some mathematical properties of the KAPW distribution are derived in Section 4. The maximum likelihood estimators of the model parameters are obtained, and some simulations are provided to assess the performance of these estimators in Section 5. In Section 6, we analyze two real data sets to illustrate the importance and flexibility of the KAPW model. Finally, some conclusions are given in Section 7.

2. The KAP-G family

The CDF of the KAP-G family follows by replacing $G(x)$ in Equation (1) by the CDF of the APT family (3), $H_{APT}(x)$. Then, the CDF of KAP-G class has the form

$$F(x) = \begin{cases} 1 - \left\{ 1 - \left[ \frac{a^{G(x)}}{a-1} \right]^{\frac{1}{a-1}} \right\}^b & \text{if } a, b > 0, \alpha \neq 1, \\ G(x) & \text{if } a = 1. \end{cases}$$

The KAP-G PDF can be expressed as

$$f(x) = \begin{cases} ab \ln(a) \frac{g(x)}{a-1} \left[ \frac{a^{G(x)}}{a-1} \right]^{a-1} \left\{ 1 - \left[ \frac{a^{G(x)}}{a-1} \right]^{\frac{1}{a-1}} \right\}^{b-1} & \text{if } a, b > 0, \alpha \neq 1, \\ G(x) & \text{if } a = 1. \end{cases}$$

The quantile function (QF) of $X$, $Q(p) = F^{-1}(p)$, can be obtained by inverting (5), as

$$Q_{KAP}(p) = G^{-1} \left( \frac{\log \left[ 1 + (\alpha - 1) \left[ 1 - (1 - p)^{1/b} \right]^{1/a} \right]}{\log(\alpha)} \right), \alpha \neq 1.$$  

A random sample of size $n$ the KAP-G CDF is obtained (for $\alpha \neq 1$), using the above equation, as $X_i = Q_{KAP}(U_i)$, where $U_i \sim \text{Uniform}(0, 1), i = 1, \ldots, n$.

The new KAP-G class contains some special cases which are shown in Table 1.

### Table 1: Sub-families of the KAP-G family

| $\alpha$ | $a$ | $b$ | Reduced family | Authors |
|-------|-----|-----|----------------|---------|
| $\alpha$ | $a$ | $1$ | Exponentiated alpha power-G (EAP-G) | New |
| $1$ | $a$ | $b$ | Kumaraswamy-G (K-G) | Cordeiro and de Castro (2011) |
| $\alpha$ | $1$ | | Alpha power-G (AP-G) | Mahdavi and Kundu (2017) |
| $1$ | $a$ | $1$ | Exponentiated-G (E-G) | Gupta et al. (1998) |

Using the following two power series

$$\alpha^q = \sum_{k=0}^{\infty} \frac{(\ln \alpha)^k}{k!} q^k$$

and
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Inserting (7) in Equation (5), the CDF of the KAPW distribution is

\[ F(x) = \sum_{i=0}^{\infty} \left( \frac{1}{\alpha} \right)^i F_X(x)^i G(x)^i, \]

and the corresponding PDF is

\[ f(x) = a b \ln(\alpha) g(x) \sum_{i=0}^{\infty} \frac{(-1)^i}{\alpha^{i+1}} \left( \begin{array}{c} b - 1 \end{array} \right) a(i + 1) - 1 \alpha^{-j G(x)}. \]

Then, we can write

\[ f(x) = a b \ln(\alpha) g(x) \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}[a(i+1) - j]^k}{k! (\alpha - 1)^{a(i+1)+1}} [\ln(\alpha)]^{-k-1} g(x) G(x)^k \left( \begin{array}{c} b - 1 \end{array} \right) a(i + 1) - 1. \]

Hence

\[ [\alpha^G(x) - 1]^{a(i+1)-1} = \alpha^{a(i+1)-1} \alpha^G(x) \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} a(i + 1) - 1 \end{array} \right) a^{-j G(x)}. \]

Applying the power series

\[ \alpha^q = \sum_{k=0}^{\infty} \left( \frac{\ln(\alpha)}{k!} \right) q^k, \]

the PDF of the KAPW class reduces to

\[ f(x) = a b \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}[a(i+1) - j]^k}{k! (\alpha - 1)^{a(i+1)+1}} g(x) G(x)^k \left( \begin{array}{c} b - 1 \end{array} \right) a(i + 1) - 1. \]

Then, we have

\[ f(x) = \sum_{k=0}^{\infty} \delta_k h_{k+1}(x), \]

where \( h_{k+1}(x) = (k + 1)g(x)G(x)^k \) is the exponentiated-G density with power parameter and \((k + 1) > 0\) and \( \delta_k = a b \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}[\ln(\alpha)]^{a(i+1)+1}}{(k + 1)! (\alpha - 1)^{a(i+1)+1}} \left( \begin{array}{c} b - 1 \end{array} \right) a(i + 1) - 1. \)

3. The KAPW distribution

The Weibull (W) distribution is considered one of the most popular life time distributions in the theory of reliability, and it has many applications in biological, medical, engineering, and meteorology, among others. Recently, many authors have proposed various generalized forms of the W distribution to increase its flexibility in modeling different types of lifetime data. For example, the exponentiated W distribution by Mudholkar and Srivastava (1993), the additive W distribution by Xie and Lai (1995), the modified W distribution by Xie et al. (2002), generalized modified W distribution by Jalmar et al. (2008), Kumaraswamy W distribution by Cordeiro et al. (2010), the exponential W distribution by Cordeiro et al. (2014), the alpha logarithmic transformed W distribution by Nassar et al. (2018) and the Lindley W distribution by Cordeiro et al. (2018).

Consider the two-parameter W distribution with a scale parameter \( \lambda > 0 \) and a shape parameter \( \beta > 0 \), then the CDF of the W random variable \( Y \) has the form

\[ F_W(y; \lambda, \beta) = 1 - e^{-\lambda y^\beta}, y, \lambda, \beta > 0 \] (7)

and the corresponding PDF is

\[ f_W(y; \lambda, \beta) = \lambda \beta y^{\beta-1} e^{-\lambda y^\beta}, y, \lambda, \beta > 0. \] (8)

The \( r \)th ordinary and incomplete moments of \( Y \) are, respectively, expressed as

\[ \mu_r(y) = \lambda \beta \Gamma \left( \frac{r}{\beta} + 1 \right), \quad \mu_r(y; t) = \lambda \beta \Gamma \left( \frac{r}{\beta} + 1, \lambda t^\beta \right), \]

where \( y(a, b) = \int_0^x a^{-1} e^{-x} dx \) refers to lower incomplete gamma function.

Inserting (7) in Equation (5), the CDF of the KAPW distribution is
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The PDF corresponding to (9) is

$$f(x, \varphi) = \begin{cases} \frac{a b \lambda (\ln \alpha)}{\alpha - 1} x^{\beta - 1} e^{-\lambda x^\beta} \alpha^{\gamma - 1} e^{-\gamma x^\beta} \left( \frac{\alpha^{\gamma - 1} e^{-\gamma x^\beta} - 1}{\alpha - 1} \right)^{a - 1} \left[ 1 - \left( \frac{\alpha^{\gamma - 1} e^{-\gamma x^\beta} - 1}{\alpha - 1} \right)^{a - 1} \right] \quad & \text{if } \alpha \neq 1 \\ a b \lambda e^{-\lambda x^\beta} \left( 1 - e^{-\lambda x^\beta} \right)^{a - 1} \left[ 1 - (1 - e^{-\lambda x^\beta})^a \right]^{-1} \quad & \text{if } \alpha = 1 \end{cases}$$

(10)

The hazard rate function (HRF) of $X$ are, respectively, given by

$$h(x, \varphi) = \begin{cases} \frac{a \lambda \beta \log \alpha}{\alpha - 1} x^{\beta - 1} e^{-\lambda x^\beta} \alpha^{\gamma - 1} e^{-\gamma x^\beta} \left( \frac{\alpha^{\gamma - 1} e^{-\gamma x^\beta} - 1}{\alpha - 1} \right)^{a - 1} \left[ 1 - \left( \frac{\alpha^{\gamma - 1} e^{-\gamma x^\beta} - 1}{\alpha - 1} \right)^{a - 1} \right] \quad & \text{if } \alpha \neq 1 \\ a \beta \gamma e^{-\gamma x^\beta} \left( 1 - e^{-\gamma x^\beta} \right)^{a - 1} \left[ 1 - (1 - e^{-\gamma x^\beta})^a \right]^{-1} \quad & \text{if } \alpha = 1 \end{cases}$$

Table 2 lists seventeen important special models of the new distribution.

**Table 2: Sub-models of the KAPW distribution**

| $\alpha$ | $a$ | $b$ | $\lambda$ | $\beta$ | Reduced model | Authors |
|----------|-----|-----|-----------|---------|---------------|---------|
| $\alpha$ | $a$ | $b$ | $\lambda$ | $2$ | KAP-Rayleigh | New |
| $\alpha$ | $a$ | $b$ | $\lambda$ | $1$ | KAP-exponential | New |
| $1$ | $a$ | $b$ | $\lambda$ | $\beta$ | K-Weibull | Cordeiro et al. (2010) |
| $1$ | $a$ | $b$ | $\lambda$ | $2$ | K-Rayleigh | - |
| $1$ | $a$ | $b$ | $\lambda$ | $1$ | K-exponential | - |
| $1$ | $a$ | $1$ | $\lambda$ | $\beta$ | E-Weibull | Mudholkar and Srivastava (1993) |
| $1$ | $a$ | $1$ | $\lambda$ | $2$ | E-Rayleigh | - |
| $1$ | $a$ | $1$ | $\lambda$ | $1$ | E-exponential | - |
| $\alpha$ | $a$ | $1$ | $\lambda$ | $\beta$ | EAP-Rayleigh | - |
| $\alpha$ | $a$ | $1$ | $\lambda$ | $2$ | EAP-Weibull | - |
| $\alpha$ | $a$ | $1$ | $\lambda$ | $1$ | EAP-exponential | - |
| $1$ | $1$ | $1$ | $\lambda$ | $\beta$ | AP-Rayleigh | - |
| $1$ | $1$ | $1$ | $\lambda$ | $2$ | AP-Weibull | - |
| $1$ | $1$ | $1$ | $\lambda$ | $1$ | AP-exponential | - |
| $1$ | $1$ | $1$ | $\lambda$ | $\beta$ | Weibull | - |
| $1$ | $1$ | $1$ | $\lambda$ | $2$ | Rayleigh | - |
| $1$ | $1$ | $1$ | $\lambda$ | $1$ | Exponential | - |
4. Properties of KAPW distribution

4.1 Linear representation

The KAPW density is expressed as a linear mixture of W densities
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Hence, we can write the MGF of

\[
Y
\]

Using the Wright generalized hypergeometric function defined by

By expanding

The MGF of \( Y \) can be expressed as

We now provide the moment generating function (MGF) of the W model as derived by Nadarajah et al. (2013). The

\[
\text{The first incomplete moment of } X \text{ is obtained by replacing } r = 1 \text{ in the above equation.}
\]

4.4 Moment generating function

We now provide the moment generating function (MGF) of the W model as derived by Nadarajah et al. (2013). The MGF of \( Y \) can be expressed as

\[
M_Y(t; \beta, \lambda) = \beta \lambda \int_0^\infty e^{tx} x^{\beta-1} e^{-\lambda x} dx.
\]

By expanding \( e^{tx} \) and calculating the integral, we have

\[
M_Y(t; \beta, \lambda) = \sum_{k=0}^\infty \frac{(t/\lambda)^k}{k!} \Gamma \left( \frac{k}{\beta} + 1 \right).
\]

Using the Wright generalized hypergeometric function defined by

\[
\Psi_p \left( \frac{\lambda_1 A_1}{(\beta_1 B_1)^{\alpha}}, \ldots, \frac{\lambda_p A_p}{(\beta_p B_p)^{\alpha}} \right) x^m = \sum_{m=0}^\infty \frac{\prod_{j=1}^p \Gamma(\lambda_j + A_j m) x^m}{\prod_{j=1}^p \Gamma(\beta_j + B_j m) m!}
\]

Hence, we can write the MGF of \( Y \) as
\[ M_p(t; \beta, \lambda) = \Psi_0 \left[ \frac{1}{1 - \beta}; t/\lambda^{1/\beta} \right]. \]

Combining the above expression and Equation (11), the MGF of \( X \) can be expressed as
\[
M_X(t; \beta, \lambda) = \sum_{m=0}^{\infty} d_m \Psi_0 \left[ \frac{1}{1 - \beta}; \frac{t}{(m + 1)\lambda}^{1/\beta} \right].
\]

### 4.5 Residual and reversed residual lives

For \( n = 1, 2, \ldots \) and \( t > 0 \), the \( n \)th moment of residual life of \( X \) has the form
\[
m_n(t) = \frac{1}{1 - F(t)} \int_t^\infty (x - t)^n dF(x).
\]

Using Equation (11), we can write
\[
m_n(t) = \frac{1}{F(t)} \sum_{m=0}^{\infty} \sum_{i=0}^{n} (n + 1) \frac{t^{n-i}}{(-1)^{n-i}i!} d_m [(m + 1)\lambda]^{-1/\beta} \frac{\Gamma(r)}{\beta + 1, [(m + 1)\lambda]^{1/\beta}},
\]
where \( \rho_i = \Gamma(\rho + 1)/\Gamma(\rho - i + 1) \) refers to falling factorial.

The mean residual life refers to the expected additional life length for a unit which is alive at age \( x \), and it follows for \( X \) from the last equation with \( n = 1 \).

For \( n = 1, 2, \ldots \) and \( t > 0 \), the \( n \)th moment of reversed residual life of the KAPW model takes the form
\[
M_n(t) = \frac{1}{F(t)} \int_0^t (t - x)^n dF(x).
\]

Then, we can write
\[
M_n(t) = \frac{1}{F(t)} \sum_{i=0}^{n} \sum_{m=0}^{\infty} (n + 1) \frac{t^{n-i}}{(-1)^{n-i}i!} d_m [(m + 1)\lambda]^{-1/\beta} \frac{\Gamma(r)}{\beta + 1, [(m + 1)\lambda]^{1/\beta}}.
\]

The mean inactivity time refers to the waiting time elapsed since the failure of an item on condition that this failure had occurred in \((0, x)\), and it follows for \( X \) from the above equation with \( n = 1 \).

### 4.6 Order statistics

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \), and let \( X_{i:n} \) denote the \( i \)th order statistic. The PDF of \( X_{E:n}, f_{i:n}(x) \), is given by
\[
f_{i:n}(x) = \frac{n! f(x)}{(i-1)! (n-i)!} \left[ 1 - F(x) \right]^{n-i} \left[ F(x) \right]^{i-1}. \tag{12}
\]

Substituting (9) and (10) in (12), the \( i \)th order statistic of the KAPW distribution reduces to
\[
f_{i:n}(x) = \frac{a b \lambda_0 \ln \alpha}{\beta(i, n - i + 1)} x^{\beta-1} e^{-\lambda x^\beta} \alpha^1 e^{-\lambda x^\beta} \left( \frac{\alpha^{1-e^{-\lambda x^\beta}} - 1}{\alpha - 1} \right)^{a-1} \times \left[ 1 - \left( \frac{\alpha^{1-e^{-\lambda x^\beta}} - 1}{\alpha - 1} \right)^{a-b} \right]^{b(n-i+1)-1} \left[ 1 - \left( \frac{\alpha^{1-e^{-\lambda x^\beta}} - 1}{\alpha - 1} \right)^{a-b} \right]^{b(n-i+1)-1} \tag{13}
\]

Applying the binomial series, then \( f_{i:n}(x) \) can be in the form
\[
f_{i:n}(x) = \sum_{k=0}^{i-1} \sum_{j, m=0}^{\infty} \frac{(-1)^{k+j+m} ab (ln \alpha)^{s+1} [a(j + 1) - m]^s}{\beta(i, n - i + 1)s!(\alpha - 1)a^{(j+1)}} \left( \frac{b(n + k - i + 1)}{k} \right) \left( \frac{a(j + 1) - 1}{j} \right) \sum_{m=0}^{\infty} \frac{(-1)^{k+j+m}}{(l + 1)} \sum_{l=0}^{\infty} a^l b^l \lambda x^{\beta-1} e^{-(l+1)\lambda x^\beta}.
\]

Or simply in the form
\[
f_{E:n}(x) = \sum_{l=0}^{\infty} d_l g_{l+1}(x; \beta, (l + 1)\lambda),
\]
where \( g_{l+1}(x; \beta, (l + 1)\lambda) \) as before, is \( W \) density with parameters \( \beta \) and \( (l + 1)\lambda \), and
\[
d_l = \sum_{k=0}^{i-1} \sum_{j, m=0}^{\infty} \frac{ab (ln \alpha)^{s+1} [a(j + 1) - m]^s}{\beta(i, n - i + 1)s!(\alpha - 1)a^{(j+1)}} \left( \frac{b(n + k - i + 1)}{k} \right) \left( \frac{a(j + 1) - 1}{j} \right) \frac{(-1)^{k+j+m}}{(l + 1)} \frac{(l + 1)}{(l + 1)}.
\]
The $q$th moments of $X_{i,m}$ has the form
\[ E(X_{i,m}^q) = \sum_{i=0}^{\infty} d_i [(l + 1) \lambda] \frac{r}{\beta} \Gamma\left(\frac{r}{\beta} + 1\right). \]

5. Estimation and simulations
Let $x_1, x_2, ..., x_n$ be a random sample from KAPW distribution then the logarithm of the likelihood function ($\ell$), becomes
\[ \ell = n \ln a + n \ln b + n \ln \lambda + n \ln \beta + n \ln \left(\frac{n \alpha}{\lambda} \right) + (\beta - 1) \sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^\beta + \ln(\alpha) \sum_{i=1}^{n} d_i + (\alpha - 1) \sum_{i=1}^{n} \xi_i \ln(1 - \xi_i) \]
where $d_i = 1 - e^{-\lambda x_i^\beta}$ and $\xi_i = (\alpha d_i - 1)/(\alpha - 1)$.
To obtain the MLEs of $a, b, \alpha, \lambda$ and $\beta$, the first derivatives of $\ell$ are obtained with respect to $a, b, \alpha, \lambda$ and $\beta$. These derivatives are
\[
\begin{align*}
\frac{\partial \ell}{\partial a} &= \frac{n}{a} + \sum_{i=1}^{n} \xi_i^\alpha \left(\frac{(\alpha - 1) d_i e^{-\lambda x_i^\beta} - (\alpha d_i - 1)}{(\alpha - 1) (\alpha d_i - 1)} - 1\right)\left(1 - \xi_i\right) \\
\frac{\partial \ell}{\partial \lambda} &= -n \frac{\alpha}{\lambda} + n \alpha \sum_{i=1}^{n} x_i^\beta e^{-\lambda x_i^\beta} + (\alpha - 1) \sum_{i=1}^{n} \frac{\alpha d_i x_i^\beta e^{-\lambda x_i^\beta} \ln \alpha}{\alpha d_i - 1} \\
\frac{\partial \ell}{\partial \beta} &= -n \frac{\beta}{\beta} + \sum_{i=1}^{n} \xi_i^\alpha \left(\frac{(\alpha d_i \lambda x_i^\beta e^{-\lambda x_i^\beta} \ln \alpha \ln x_i)}{\alpha d_i - 1} \right) \left(1 - \xi_i\right) \frac{1}{\alpha d_i - 1} \\
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha (\alpha - 1) \ln \alpha} - \frac{1}{\alpha - 1} \sum_{i=1}^{n} \xi_i^\alpha \left(\frac{(\alpha - 1) d_i e^{-\lambda x_i^\beta} - (\alpha d_i - 1)}{(\alpha - 1) (\alpha d_i - 1)} \right) - a(b - 1) \sum_{i=1}^{n} \xi_i^\alpha \left(\frac{(\alpha d_i \lambda x_i^\beta e^{-\lambda x_i^\beta} \ln \alpha)}{\alpha d_i - 1} \right) \left(1 - \xi_i\right) \frac{1}{\alpha d_i - 1} \\
\frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^{n} \xi_i^\alpha \left(\frac{(\alpha d_i \lambda x_i^\beta e^{-\lambda x_i^\beta} \ln \alpha \ln x_i)}{\alpha d_i - 1} \right) \left(1 - \xi_i\right) \frac{1}{\alpha d_i - 1} \\
\frac{\partial \ell}{\partial \xi_i} &= \frac{n}{\lambda} \left(\frac{(\alpha d_i \lambda x_i^\beta e^{-\lambda x_i^\beta} \ln \alpha \ln x_i)}{\alpha d_i - 1} \right) \left(1 - \xi_i\right) \frac{1}{\alpha d_i - 1}
\end{align*}
\]

The MLEs of the parameters $a, b, \alpha, \lambda$ and $\beta$ can follow by solving the above system of equations. No explicit form for these estimates, we use a numerical technique like Newton-Raphson method is used to solve the above non-linear equations.

Now we investigate the performance of the of maximum likelihood estimators for the KAPW parameters. We generate 5,000 samples of the KAPW distribution using its QF in Section 4.2, for some sample sizes, $n = (20, 50, 100, 200, 400)$, and for some parameters values, where $\alpha = (0.75, 2.75)$, $\beta = (0.67, 2.00, 2.5)$, $\lambda = (0.5, 0.67, 2.50)$, $\alpha = (0.25, 0.75, 1.5)$ and $b = (0.30, 0.5, 1.50)$.
For each sample and parameters combination, we use the R software to obtain the average values of the estimates (AEs), mean squared errors (MSEs), biases and mean relative estimates (MREs), Tables 3, 4 and 5 show the AEs, MSEs, bias and MREs of the MLEs for the KAPW parameters. It is clear that the values of MSEs, biases and MREs decrease as
increases for all cases, which illustrates the estimators are quite stable and the estimates are very close to the true values of the parameters.

6. Applications
In this section, the importance and flexibility KAPW distribution are studied via two real data applications. The first set of data refers to the actual taxes data that represent monthly actual taxes revenue in Egypt from January 2006 to November 2010 (taxes revenue in 1000 million Egyptian pounds).

The second set of data represents strengths of 1.5 cm glass fibers of 63 observations which originally obtained by workers at the UK National Physical Laboratory. These data were reported in Smith and Naylor (1987) and analyzed by Afify et al. (2016b) and Alizadeh et al. (2020).

For both data sets, we compare the fits of the KAPW distribution with some competitive models called, generalized Burr X W (GBXW) by Aldahlan et al. (2018), exponentiated W (EW) by Mudholkar and Srivastava (1993), odd log-logistic exponentiated W (OLLEW) by Afify et al. (2018), alpha power W (APW) by Nassar et al. (2017), transmuted complementary W geometric (TCWG) by Afify et al. (2014), alpha logarithmic transformed W by Nassar et al. (2018), W-W (WW) by Abouelmagd et al. (2017) and W distributions.

Table 3: The AEs, MSEs, biases and MREs of the KAPW parameters for different values of the parameter and \( n \)

| \( \alpha = 2.75, \beta = 2, \lambda = 0.67, \alpha = 0.25, b = 0.30 \) | AEs | MSEs |
|---|---|---|
| \( n \) | | |
| 20 | 2.23441 2.47810 0.54919 0.21301 0.29707 | 5.71870 0.45532 0.21518 0.01567 0.00294 |
| 50 | 2.74777 2.32618 0.65002 0.23210 0.29467 | 3.64467 0.22538 0.12104 0.00649 0.00149 |
| 100 | 2.75022 2.19460 0.67000 0.24414 0.29299 | 1.48926 0.11826 0.07104 0.00373 0.00083 |
| 200 | 2.75028 2.09111 0.67189 0.25000 0.29291 | 0.06028 0.05596 0.03777 0.00183 0.00049 |
| 400 | 2.75016 2.01812 0.67088 0.25198 0.29317 | 0.00130 0.01556 0.01475 0.00100 0.00032 |

| Bias | MREs |
|---|---|
| 20 | 2.39138 0.67477 0.46388 0.12517 0.05425 | 0.86959 0.33739 0.69236 0.50067 0.18083 |
| 50 | 1.90910 0.47474 0.34790 0.08054 0.03857 | 0.69422 0.23737 0.51926 0.32216 0.12857 |
| 100 | 1.22035 0.34389 0.26653 0.06104 0.02887 | 0.44376 0.17195 0.39781 0.24417 0.09622 |
| 200 | 0.24552 0.23655 0.19435 0.04282 0.02224 | 0.08928 0.11828 0.29008 0.17129 0.07412 |
| 400 | 0.03608 0.12473 0.12145 0.03157 0.01789 | 0.01312 0.06237 0.29008 0.17129 0.07412 |

| \( \alpha = 2.75, \beta = 2, \lambda = 2.5, \alpha = 0.75, b = 1.5 \) | AEs | MSEs |
|---|---|---|
| \( n \) | | |
| 20 | 4.07390 2.71371 0.60728 0.45970 1.52358 | 6.73995 1.60569 0.89592 0.29604 0.05544 |
| 50 | 2.46008 2.25873 2.50734 0.57995 1.51200 | 4.66756 0.61242 0.45178 0.10561 0.01915 |
| 100 | 2.15322 2.19430 2.36130 0.62981 1.50243 | 4.18259 0.39537 0.43094 0.05133 0.01041 |
| 200 | 2.14888 2.17650 2.33636 0.66720 1.50470 | 3.87516 0.27845 0.36050 0.02896 0.00510 |
| 400 | 2.41907 2.13128 2.40820 0.68861 1.50127 | 3.56024 0.20290 0.26650 0.01859 0.00252 |

| Bias | MREs |
|---|---|
| 20 | 2.59614 1.26716 0.94653 0.54409 0.23545 | 0.94405 0.63358 0.37861 0.72546 0.15697 |
| 50 | 2.16045 0.78257 0.67215 0.32498 0.13837 | 0.78562 0.39129 0.26886 0.43331 0.09225 |
| 100 | 2.04514 0.62879 0.65646 0.22657 0.10204 | 0.74369 0.31439 0.26258 0.30209 0.06803 |
| 200 | 1.96854 0.52769 0.60042 0.17018 0.07142 | 0.71583 0.26384 0.24017 0.22690 0.04761 |
| 400 | 1.88686 0.45045 0.51623 0.13635 0.05016 | 0.68613 0.22522 0.20649 0.18180 0.03344 |
Table 4: The AEs, MSEs, biases and MREs of the KAPW parameters for different values of the parameter and $n$

| $n$  | AEs       | MSEs       |
|------|-----------|------------|
| 20   | 0.69667   | 2.90606    | 0.50646   | 0.20696 | 0.50971 | 0.37314 | 0.64446 | 0.14248 | 0.01095 | 0.00643 |
| 50   | 0.70448   | 2.67970    | 0.58261   | 0.22843 | 0.49982 | 0.20621 | 0.22990 | 0.07847 | 0.00413 | 0.00274 |
| 100  | 0.72934   | 2.56191    | 0.63977   | 0.23787 | 0.49862 | 0.13056 | 0.11069 | 0.04260 | 0.00198 | 0.00137 |
| 200  | 0.73950   | 2.50304    | 0.66615   | 0.24514 | 0.49733 | 0.06365 | 0.04431 | 0.01913 | 0.00096 | 0.00072 |
| 400  | 0.74290   | 2.49686    | 0.67027   | 0.24927 | 0.49810 | 0.00990 | 0.00764 | 0.00288 | 0.00036 | 0.00035 |

Table 5: The AEs, MSEs, biases and MREs of the KAPW parameters for different values of the parameter and $n$

| $n$  | AEs       | MSEs       |
|------|-----------|------------|
| 20   | 0.69667   | 2.90606    | 0.50646   | 0.20696 | 0.50971 | 0.37314 | 0.64446 | 0.14248 | 0.01095 | 0.00643 |
| 50   | 0.70448   | 2.67970    | 0.58261   | 0.22843 | 0.49982 | 0.20621 | 0.22990 | 0.07847 | 0.00413 | 0.00274 |
| 100  | 0.72934   | 2.56191    | 0.63977   | 0.23787 | 0.49862 | 0.13056 | 0.11069 | 0.04260 | 0.00198 | 0.00137 |
| 200  | 0.73950   | 2.50304    | 0.66615   | 0.24514 | 0.49733 | 0.06365 | 0.04431 | 0.01913 | 0.00096 | 0.00072 |
| 400  | 0.74290   | 2.49686    | 0.67027   | 0.24927 | 0.49810 | 0.00990 | 0.00764 | 0.00288 | 0.00036 | 0.00035 |

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Tables 6 and 7 provide the MLEs of the model parameters, their standard errors (SEs) and the values of $-\hat{\ell}$, KS and PV for both data sets, respectively.

**Table 6: Estimated values with SEs (in parentheses), and analytical measures of the KAPW distribution and other competitors for actual taxes data**

| Distribution | Estimates (SEs) | $-\hat{\ell}$ | KS | PV |
|--------------|----------------|----------------|----|----|
| KAPW $(\alpha, \beta, \lambda, a, b)$ | 323.173 (17.699) | 0.8279 (0.4678) | 0.9761 (1.0422) | 16.5278 (31.148) | 0.2607 (0.2665) | 187.887 | 0.0617 | 0.9779 |
| GBXW $(\alpha, \beta, a, b)$ | 175.749 (159.08) | 258.722 (433.25) | 5.8501 (8.8959) | 0.0264 (0.0079) | 188.349 | 0.0631 | 0.9727 |
| EW $(\alpha, \beta, \lambda)$ | 2813.00 (11683) | 0.2772 (0.1420) | 4.2640 (3.5913) | 188.241 | 0.0640 | 0.9686 |
| OLLWEW $(\alpha, \beta, \gamma, \theta)$ | 0.0721 (0.1505) | 0.1500 (0.1433) | 5.5155 (6.9616) | 7.0781 (9.9837) | 190.718 | 0.0727 | 0.9134 |
| APW $(\alpha, \beta, \lambda)$ | 3432.25 (4219.6) | 0.8786 (0.0934) | 0.2811 (0.0792) | 192.019 | 0.1055 | 0.5266 |
| TCWG $(\alpha, \beta, \lambda, \delta)$ | 0.9999 (0.6072) | 2.0179 (0.3280) | 0.6436 (0.2337) | 0.0538 (0.0109) | 195.706 | 0.1324 | 0.2518 |
| ALTW $(\alpha, \beta, \lambda)$ | 0.4333 (0.2708) | 1.9431 (0.1099) | 0.0039 (0.0014) | 196.466 | 0.1228 | 0.3353 |
| WW $(\alpha, \beta, a, b)$ | 0.2636 (4.9663) | 77.5484 (23.026) | 0.6699 (0.1279) | 0.0170 (0.0070) | 197.380 | 0.1432 | 0.1774 |
| W $(\beta, \lambda)$ | 1.8403 (0.1711) | 0.0653 (0.0049) | 0.0106 (0.0007) | 197.290 | 0.1431 | 0.1780 |

**Table 7: Estimated values with SEs (in parentheses), and analytical measures of the KAPW distribution and other competitors for glass fibers data**

| Distribution | Estimates (SEs) | $-\hat{\ell}$ | KS | PV |
|--------------|----------------|----------------|----|----|
| KAPW $(\alpha, \beta, \lambda, a, b)$ | 89.2219 (263.60) | 5.4350 (1.5714) | 0.1260 (0.0816) | 0.4975 (0.2354) | 0.8040 (0.8459) | 12.150 | 0.0975 | 0.5864 |
| GBXW $(\alpha, \beta, a, b)$ | 0.4623 (0.7270) | 1.3915 (0.8250) | 0.0690 (0.2522) | 2.9125 (2.7039) | 14.565 | 0.1406 | 0.1653 |
| EW $(\alpha, \beta, \lambda)$ | 0.6712 (0.2209) | 7.2844 (1.4869) | 0.0194 (0.0210) | 14.675 | 0.1462 | 0.1351 |
| OLLWEW $(\alpha, \beta, \gamma, \theta)$ | 1.9919 (0.2971) | 8.7488 (3.9362) | 0.3021 (0.2664) | 1.6872 (0.7428) | 14.024 | 0.1319 | 0.2223 |
| APW $(\alpha, \beta, \lambda)$ | 10.8558 (12.717) | 4.4836 (0.7626) | 0.1947 (0.1082) | 13.474 | 0.1224 | 0.3010 |
| TCWG $(\alpha, \beta, \lambda, \delta)$ | 0.0698 (0.1140) | 3.2035 (0.9403) | -0.1380 (0.9303) | 0.8911 (0.1952) | 12.030 | 0.0995 | 0.5598 |
| ALTW $(\alpha, \beta, \lambda)$ | 22.528 (42.543) | 4.4786 (0.7487) | 0.2549 (0.1913) | 13.575 | 0.1432 | 0.1507 |
| WW $(\alpha, \beta, a, b)$ | 0.0278 (0.0724) | 3.1168 (2.7740) | 0.8617 (0.5162) | 1.0134 (0.6708) | 14.412 | 0.1373 | 0.1852 |
| W $(\beta, \lambda)$ | 5.7807 (5.5760) | 0.6142 (0.0139) | 15.206 | 0.1522 | 0.1078 |

Plots of the fitted KAPW PDF and other fitted densities, for both data sets, are displayed in Figures 4 and 5, respectively. The estimated PDF, CDF, SF and PP plots for both data sets are shown in Figure 6. Based on the values in Tables 3 and 4 and the plots in Figures 4, 5 and 6, we conclude that the KAPW distribution provides a close fit to both data sets as compared to other rival distributions.
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Figure 4: The estimated KAPW PDF and other estimated PDFs for actual taxes data

Figure 5: The estimated KAPW PDF and other estimated PDFs for glass fibers data

Figure 6: Estimated PDF, CDF, SF and PP plots of the KAPW distribution (left) for actual taxes data and (right) for glass fibers data
7. Concluding remarks

In this article, we introduce the Kumaraswamy alpha power-G (KAP-G) family that extends the alpha power-G (Mahdavi and Kundu, 2017) and Kumaraswamy-G (Cordeiro and de Castro, 2011), exponentiated-G (Gupta et al., 1998) families. Based on the KAP-G class, we construct the four-parameter Kumaraswamy alpha power Weibull (KAPW) distribution which has some desirable properties. Some mathematical quantities of the KAPW model are derived. The KAPW parameters are estimated via maximum likelihood method and detailed simulation results are obtained to assess the performance of the estimates. The flexibility of the KAPW model is examined by two real data applications, proved its better fits as compared to several others Weibull extensions.

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