Charged rotating BTZ black holes in noncommutative spaces and torsion gravity

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February 7, 2018

Abstract

We consider charged rotating BTZ black holes in noncommutative space by use of Chern-Simons theory formulation of $2+1$ dimensional gravity. The noncommutativity between the radial and the angular variables is introduced through the Seiberg-Witten map for gauge fields, and the deformed geometry to the first order in the noncommutative parameter is derived. It is found that the deformation also induces nontrivial torsion, and the Einstein-Cartan theory appears to be a suitable framework to investigate the equations of motion. Though the deformation is indeed nontrivial, the deformed and the original Einstein equations are found to be related by a rather simple coordinate transformation.
1 Introduction

It is widely believed that at a very high energy scale, such as Planck or the string scale, the notion of smooth geometry is no longer valid due to the effect of quantum gravity; namely quantum fluctuation of spacetime itself becomes significant and may not be treated as perturbation around a classical geometry. Though we have not yet fully understand such a quantum geometry, among those available proposals, the noncommutative geometry [1] may capture some desired features of it. In the quantum geometry, space-time coordinates are no longer regarded as $c$-numbers but the ones obeying a specific quantum algebra, which naturally introduces a length scale serving as the UV cutoff. Quantum field theory formulated on a noncommutative geometry also exhibits various intriguing behaviors such as the UV/IR mixing [2] and stringy properties [3].

The rich structure emerged in noncommutative quantum field theory has enticed several proposals to consider gravitational theory on noncommutative space [4]. Although it is not easy to investigate concrete solutions due to their complicated structures, gravity in 2 + 1 dimensions may be an exception. For instance, the Poisson brackets of $SL(2, \mathbb{R})$ were studied in Ref.[5] and families of deformation were found leading to a discrete spectrum for time operator. In Ref.[6], an effective metric of a noncommutative geometry was sourced by delocalized mass and charges due to the minimal length. In addition, 3D gravity is known to have a description in terms of Chern-Simons theory [7]. In this case, one may take advantage of the Seiberg-Witten map that relates a theory on commutative space to a corresponding theory on a noncommutative space. To mention a few examples: the authors of [8, 9] used the Seiberg-Witten map to modify algebraic relation. They found no first order correction as expected in the canonical treatment in the noncommutative geometry, as long as classical torsion is excluded. In Ref.[10], the ambiguity in the metric due to gauge transformation was discussed and fixed by introducing nonminimal coupled scalars and a nontrivial potential. In Ref.[11], it was argued that cosmological constant got quantized in the noncommutative Chern-Simons gravity.

On the other hand, the three-dimensional anti-de Sitter (AdS) space admits the well-known black hole solution [12] and its charged counterpart [13, 14]. In Ref.[15], a constant gauge field was introduced in coupled with the Chern-Simon action and it amounted to mix mass and angular momentum in the original BTZ. In Ref.[16], a noncommutative deformation in polar coordinates was introduced via the Seiberg-Witten map and a noncommutative neutral BTZ black hole metric up to the first order in $\theta$ (noncommutative parameter) was obtained. However, this result appeared in conflict with that in Ref.[8, 9] for its first order correction in metric. Before we could solve the puzzle, it is useful to review their construction.

In Ref.[16], a noncommutative deformation of a neutral rotating BTZ black hole solution is investigated based on a commutation relation in the polar coordinates, that is, $[r^2, \varphi] = 2i\theta$. The solution is written in terms of Chern-Simons gauge fields and the noncommutative deformation is introduced by the Seiberg-Witten map. The resultant metric, to the first order in $\theta$, reads\footnote{We will review the noncommutativity deformation in more detail in Sec.2}

$$ds^2 = - f^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 N^\theta dt d\varphi + \left( r^2 - \frac{\theta B}{2}\right) d\varphi^2 + O(\theta^2), \quad (1.1)$$

\footnote{We have corrected the sign mistakes in $f^2$ and $d\varphi^2$ parts in Ref.[16].}
with

\[ N^\phi = - \frac{r_+ r_-}{\ell r^2}, \quad (1.2) \]
\[ f^2 = \frac{r^2 - r_+^2 - r_-^2}{\ell^2} - \frac{\theta B}{2\ell^2}, \quad (1.3) \]
\[ \hat{N}^2 = \frac{1}{\ell^2 r^2} \left[ (r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2} (2r^2 - r_+^2 - r_-^2) \right], \quad (1.4) \]

where \( r_+ \) represents the horizon radius of the undeformed metric (the explicit forms are given in the Appendix A). The noncommutative extension requires two extra \( U(1) \) gauge fields \( B^{(\pm)}_\mu \), which are chosen as \( B^{(\pm)}_\phi = B \) with a constant \( B \). Some properties of this deformed black hole solution are investigated in Ref.[16], for instance, the locations of various types of horizon.

We, however, confirm that this metric satisfies the vacuum Einstein equation to the first order, \( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 + O(\theta^2) \), if the corrections of the metric in Ref.[16] are taken into account. This fact suggests that there should exist another coordinate system in which the metric looks like a pure AdS\(_3\). It turns out that, indeed, by making a coordinate transformation

\[ r \to \tilde{r} + \frac{\theta B}{4\ell}, \quad (1.5) \]

and only keeping terms up to first order in \( \theta \), the metric (1.1) comes back to the undeformed BTZ black hole solution; namely the first order correction can be eliminated.\(^3\) The angular part of the metric becomes \( \tilde{r}^2 d\phi^2 \), where \( \tilde{r} \) is regarded as a standard radial coordinate. Consequently, the deformed BTZ black hole and the undeformed one are equivalent up to the coordinate transformation (1.5). We remark that while the change is only made in \( r \), the geometrical structure near the boundary would not be changed since \( r \) and \( \tilde{r} \) are asymptotically the same. Thus, various mechanical and thermodynamic properties of black holes, such as the Hawking temperature, entropy, and orbital motion of particles, appear to be equivalent. This equivalence may attribute to the fact that the vacuum solution of 2 + 1 dimensional gravity with a negative cosmological constant is essentially unique. This motivates us to investigate a different class of solutions that are not vacuum solutions. In this paper, we shall explore the charged rotating BTZ black hole solution in a noncommutative space.

The organization of the paper is as follows: In the Sec\(^2\) a noncommutative deformation is formulated by use of the Seiberg-Witten map in the Chern-Simons framework of 2 + 1 dimensional gravity. We start with a charged rotating BTZ solution and obtain deformed gauge fields, vielbeins, and spin connections. In the Sec\(^3\) we investigate the properties of noncommutative charged rotating BTZ black hole solutions. There appears nontrivial torsion and the deformed equations of motion are found to be nicely fitted in to the framework of Einstein-Cartan theory of torsion gravity. The relation between the deformed and the original solutions through a coordinate change is expounded. We conclude the paper with discussion and overview in the Sec\(^4\). The appendices are given to summarize our convention and to explain more technical details.

\(^3\)There is a subtle issue about the regions covered by these coordinates, which will be discussed in the Sec\(^3.4\).
2 Three dimensional gravity in noncommutative space

2.1 Noncommutativity in polar coordinates

A noncommutative space is introduced by applying the following commutation relations in the rectangular coordinates,

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu}.
\]  

(2.1)

Since timelike noncommutativity is known to have several difficulties, such as acausality or nonuniqueness [17], we shall restrict our discussion to a purely spatial noncommutativity; for example, \([x, y] = i\theta\) in 2 + 1 dimensions, with \(\theta\) being the parameter of noncommutativity. In this paper, however, the charged rotating BTZ black hole in consideration is conveniently constructed in the polar coordinates \((t, r, \varphi)\) thanks to its azimuthal symmetry, one shall introduce a noncommutativity between \(r\) and \(\varphi\) coordinates instead. As suggested in Ref. [16], the noncommutative relation

\[
[r^2, \varphi] = 2i\theta,
\]  

(2.2)

is a natural choice; this is because the standard spatial noncommutative relation \([x, y] = i\theta\) can be recovered by use of the polar coordinates and (2.2) to the first order in \(\theta\), namely \([x, y] = [r \cos \varphi, r \sin \varphi] = i\theta + O(\theta^2)\). We thus adopt the noncommutative relation (2.2) and will consider a \(\theta\)-deformed charged rotating BTZ black hole solution.

2.2 Charged rotating BTZ black hole solutions and Chern-Simons theory

2.2.1 Chern-Simons formulation of Einstein-Maxwell theory

We start with Einstein-Maxwell theory in (2 + 1) dimensions,

\[
I = I_{\text{gravity}} + I_{\text{gauge}},
\]  

(2.3)

\[
I_{\text{gravity}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g}(R - 2\Lambda), \quad I_{\text{gauge}} = -\frac{1}{4\lambda^2} \int d^3x \sqrt{-g} f_{\mu\nu} f^{\mu\nu},
\]  

(2.4)

where \(\lambda\) is the coupling constant of \(U(1)\) gauge field \(a_\mu\), whose field strength is \(f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\). \(I_{\text{gravity}}\) part of action can be rewritten by use of two \(SU(1, 1) \simeq SO(1, 2)\) connection 1-forms (relevant conventions are summarized in App. A),

\[
A^{(\pm)} = \omega^a \pm \frac{1}{\ell} e^a
\]  

(2.5)

as the Chern-Simons terms

\[
S = I_{\text{CS}}[A^{(+)}] - I_{\text{CS}}[A^{(-)}],
\]  

(2.6)

\[
I_{\text{CS}}[A] = \frac{k}{4\pi} \int \text{tr} \left[ A dA + \frac{2}{3} AAA \right],
\]  

(2.7)

where the Chern-Simons level is given by \(k = -\frac{\pi}{4G}\).

Using the definition of Hodge star, \(*(dx^\mu \wedge dx^\nu) = \sqrt{|g|} e^{\mu\nu} \rho dx^\rho\), the gauge part of action can be written as

\[
I_{\text{gauge}} = -\frac{1}{4\lambda^2} \int d^3x \sqrt{-g} f_{\mu\nu} \hat{f}_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = -\frac{1}{2\lambda^2} \int \hat{f} \wedge *\hat{f},
\]  

(2.8)
The equations of motion with respect to the variation of $A_\mu^a$ are

$$\frac{k}{4\pi} e^{\mu\rho\sigma} \left[ \partial_\rho A_\sigma^{(\pm)a} - \frac{e^a_{\ b^c}}{2} A_\rho^{(\pm)b} A_\sigma^{(\pm)c} \right] = \frac{\ell}{2} \cdot eT^{\mu\rho} e_\rho^a, \quad (2.9)$$

where $T^{\mu\nu}$ is the energy momentum tensor of $U(1)$ gauge field $a_\mu$, given by

$$T^{\mu\nu} = \frac{1}{\lambda^2} \left[ f^{\mu\nu} g^{\rho\sigma} - \frac{1}{4} g^{\mu\nu} f^{\rho\sigma} \right]. \quad (2.10)$$

In terms of the vielbeins and the spin connections, the equations of motion can be also represented as

$$\frac{k}{2\pi} e^{\mu\rho\sigma} \left[ \partial_\rho \omega_\sigma^a - \frac{e^a_{b^c}}{2} \omega_\rho^b \omega_\sigma^c - \frac{e^a_{b^c}}{\ell^2} \omega_\rho^b e_\sigma^c \right] = \ell \cdot eT^{\mu\rho} e_\rho^a, \quad (2.11)$$

$$e^{\mu\rho\sigma} \left( \partial_\rho e_\sigma^a - e^a_{b^c} e_\rho^b \omega_\sigma^c \right) = 0. \quad (2.12)$$

The second equation is nothing but the torsion free condition, $T^a = De^a = de^a + \omega_{ab} e^b = 0$, while it is straightforward to see that the first one is the Einstein equation,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{\ell^2} g^{\mu\nu} = 8\pi G T^{\mu\nu}. \quad (2.13)$$

### 2.2.2 Charged rotating BTZ black hole

The charged rotating BTZ black hole solution is given in Ref. [14] as

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\varphi - \frac{4GJ}{r^2} dt \right)^2 \quad (2.14)$$

$$= -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2 - \frac{2\gamma}{\ell} dt d\varphi, \quad (2.15)$$

$$f(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} - 8\pi G Q^2 \ln r = \frac{1}{\ell^2} \left( -\alpha + r^2 + \frac{\gamma^2}{r^2} - \beta \ln r \right), \quad (2.16)$$

$$f_{lr} = \frac{\lambda Q}{r}, \quad (2.17)$$

where $Q$ is the electric charge of black hole and we have defined

$$\alpha = 8GM\ell^2, \quad \beta = 8\pi G\ell^2 Q^2, \quad \gamma = 4GJ\ell, \quad (2.18)$$

and $h(r) = f(r) - \frac{r^2}{\ell^2}$ for our convenience.

From this metric, we shall choose a set of convenient, but not unique, vielbeins and spin connections as follows,

$$e^0 = \sqrt{h(r)} dt + \frac{\gamma}{\ell \sqrt{h(r)}} d\varphi, \quad e^1 = \frac{1}{\sqrt{f(r)}} dr, \quad e^2 = r \sqrt{\frac{f(r)}{h(r)}} d\varphi, \quad (2.19)$$

$$\omega^0 = -\frac{\gamma h'(r)}{2\ell r \sqrt{h(r)}} dt - \sqrt{h(r)} d\varphi, \quad \omega^1 = \frac{\gamma h'(r)}{2\ell r h(r) \sqrt{f(r)}} dr, \quad \omega^2 = -\frac{h'(r)}{2} \sqrt{\frac{f(r)}{h(r)}} dt. \quad (2.20)$$
where the prime ' denotes the derivative with respect to $r$. With this choice of vielbeins, the Chern-Simons gauge fields are written as (2.5):

$$A^{(\pm)0} = \pm \frac{1}{\ell} \left( \sqrt{h(r)} \mp \frac{\gamma h'(r)}{2r \sqrt{h(r)}} \right) dt - \left( \sqrt{h(r)} \mp \frac{1}{\ell^2} \frac{\gamma}{\sqrt{h(r)}} \right) d\varphi,$$

$$A^{(\pm)1} = \frac{1}{\ell \sqrt{f(r)}} \left( \frac{\gamma h'(r)}{2r h(r)} \mp 1 \right) dr,$$

$$A^{(\pm)2} = \frac{r}{\ell} \sqrt{\frac{f(r)}{h(r)}} \left( - \frac{\ell h'(r)}{2r} dt \mp d\varphi \right).$$

### 2.3 Noncommutative Chern-Simons theory

The Chern-Simons formulation of noncommutative three-dimensional gravity has been considered in Refs. [15, 18]. In the Lorentzian version [18], it has been shown that the theory involves two extra $U(1)$ gauge fields $B^{(\pm)}_\mu$ and the gauge group becomes $U(1,1) \times U(1,1)$ rather than $SO(1,2) \times SO(1,2)$ as in the commutative case, in which extra $U(1)$ fields will be decoupled. The action of noncommutative Chern-Simons theory now reads

$$\hat{I}_{CS}[A^{(\pm)}] = \frac{k}{4\pi} \int \tr \left[ A^{(\pm)} \hat{\wedge} dA^{(\pm)} + \frac{2}{3} A^{(\pm)} \hat{\wedge} A^{(\pm)} \hat{\wedge} A^{(\pm)} \right],$$

where

$$f \hat{\wedge} g = \frac{1}{\pi \ell} \int_{\mu_1 \cdots \mu_k \eta_{\nu_1 \cdots \nu_j}} (dx^{\mu_1} \cdots dx^{\mu_k}) \wedge (dx^{\nu_1} \cdots dx^{\nu_j}),$$

and $\hat{\wedge}$ represents the Moyal product $f(x) \hat{\wedge} g(x) = e^{i \theta^{\mu\nu} \partial_\mu \partial_\nu} f(x)g(y) \bigg|_{y \rightarrow x}$ with an antisymmetric tensor $\theta^{\mu\nu}$. The $SU(1,1)$ gauge fields $\hat{A}^{(\pm)a}$ ($a = 0, 1, 2$) in the commutative Chern-Simons theory, together with two extra $U(1)$ gauge fields $\hat{B}^{(\pm)}_\mu$ form the new $U(1,1)$ gauge fields

$$A^{(\pm)A}_\mu \tau_A = \hat{A}^{(\pm)a}_\mu \tau_a + \hat{B}^{(\pm)}_\mu \tau_3,$$

where $A^{(\pm)3} = \hat{B}^{(\pm)}_\mu$. We summarize convention for the generators in App. [A].

The equations of motion derived from the action $\hat{I}_{CS}$ read

$$\frac{\delta \hat{I}_{CS}}{\delta \hat{A}^{(\pm)a}_\mu} = \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[ \epsilon_{abc} \partial_\rho \hat{A}^{(\pm)b}_\sigma - \frac{\epsilon_{abc}}{2} \hat{A}^{(\pm)b}_\rho \hat{\wedge} \hat{A}^{(\pm)c}_\sigma + \frac{i}{6} \eta_{ab} (\hat{A}^{(\pm)b}_\rho \hat{\wedge} \hat{B}^{(\pm)}_\sigma + \hat{B}^{(\pm)}_\rho \hat{\wedge} \hat{A}^{(\pm)b}_\sigma) \right]$$

$$= 0,$$

$$\frac{\delta \hat{I}_{CS}}{\delta \hat{B}^{(\pm)}_\mu} = \pm \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[ \partial_\rho \hat{B}^{(\pm)}_\sigma - \frac{i}{6} \eta_{ab} \hat{A}^{(\pm)a}_\rho \hat{\wedge} \hat{A}^{(\pm)b}_\sigma + \frac{i}{2} \hat{B}^{(\pm)}_\rho \hat{\wedge} \hat{B}^{(\pm)}_\sigma \right] = 0.$$

In the commutative limit $\theta \rightarrow 0$, these equations boil down to the following decoupled equations of motion,

$$F^{(\pm)a} = 0, \quad dB^{(\pm)} = 0.$$
2.4 Coupling of matter fields to noncommutative gravity

Here we briefly discuss the coupling of the Abelian gauge field $a_\mu$ to the noncommutative gauge field $A_\mu^{(\pm)A}$. The straightforward extension of Maxwell action reads

$$\int d^3x \sqrt{-g} \hat{\Gamma}_{\mu\nu} \hat{\hat{g}}^{\mu\nu} \hat{\hat{g}}^{\rho\sigma} |_{\kappa},$$

(2.30)

where the products between fields are understood to be the star product, and the field strength is defined as

$$\hat{\hat{F}}_{\mu\nu} = \partial_\mu \hat{\hat{a}}_\nu - \partial_\nu \hat{\hat{a}}_\mu + \hat{\hat{a}}_\mu \times \hat{\hat{a}}_\nu,$$

(2.31)

with $\hat{\hat{a}}_\mu$ being a noncommutative extension of $U(1)$ gauge field. The metric $\hat{g}_{\mu\nu} = \hat{\eta}_{ab} \hat{e}_\mu^a \times \hat{e}_\nu^b$ is given by the noncommutative extension of vielbeins, $\hat{e}_\mu^a = \hat{e}_\mu^{(a)} + \hat{A}_\mu^{(a)}$. As a result, the Maxwell action is highly nonlinear in terms of $\hat{A}_\mu^{(\pm)a}$.

On top of this nonlinearity, the standard Maxwell action in a curved background poses a question: general coordinate transformation is given by a field dependent gauge transformation and this action is in general not fully $SU(1,1)$ gauge invariant. If we want to maintain the full gauge invariance, we need to write the matter part coupling in terms of $\hat{A}_\mu^{(\pm)a}$ and this action is in general not fully gauge invariant. Therefore we take a gauge-invariant way, thus we consider the non-minimal coupling. The possible terms derivatives and torsion tensors. It is known that gauge field cannot minimally couple to torsion in gauge invariance, we need to write the matter part coupling in terms of $\hat{A}_\mu^{(\pm)a}$. As a result, the Maxwell action is highly nonlinear in terms of $\hat{A}_\mu^{(\pm)a}$.

In such a way that the obtained action keeps gauge invariance intact and also comes back to $\hat{A}_\mu^{(\pm)a}$ in the commutative limit. This is a fairly nontrivial problem without immediate answer. We thus take the following strategy: the noncommutative gravity is introduced via the Seiberg-Witten map in the formulation of Chern-Simons theory, while the matter part is treated as being coupled to the noncommutative gravity through the gravitational degrees of freedom, namely, $\hat{g}_{\mu\nu}$, $T_{\mu\nu}^{\rho}$ (torsion), and $\hat{B}^{(\pm)}$ instead of $\hat{A}^{(\pm)a}$.

We may try to write down possible coupling terms, which are classified by the numbers of derivatives and torsion tensors. It is known that gauge field cannot minimally couple to torsion in a gauge-invariant way, therefore we shall consider the non-minimal coupling. The possible terms of lower dimensions are (we omit the symbol of the star product)

$$I'_{\text{gauge}} = \hat{I}^{(\pm)}_{0} \int \hat{\hat{a}} \wedge d\hat{\hat{B}}^{(\pm)} + \xi^{(\pm)}_{1} \int d^3x \sqrt{-\hat{\hat{g}}_{\mu\nu}} (d\hat{\hat{B}}^{(\pm)})^{\mu\nu} + \int d^3x \sqrt{-\hat{\hat{g}}_{\mu\nu}} K^{\mu\nu},$$

(2.32)

where the first term is gauge invariant up to a surface term. The tensor $K^{\mu\nu}$ is torsion dependent and reads,

$$K^{\mu\nu} = K_{1}^{\mu\nu} + K_{2}^{\mu\nu},$$

(2.33)

$$K_{1}^{\mu\nu} = \hat{\zeta}_{1} T^{\rho\sigma\mu} T_{\rho\sigma\nu} + \hat{\zeta}_{2} T^{\rho\mu\sigma} T_{\rho\nu}^{\sigma} + \hat{\zeta}_{3} T^{\mu\nu} + \hat{\zeta}_{4} T^{\mu\nu}_{\epsilon} T^{\nu}_{\rho},$$

(2.34)

$$K_{2}^{\mu\nu} = \hat{\zeta}_{1} T^{\rho\sigma\mu} T_{\rho\sigma\nu} + \hat{\zeta}_{2} T^{\rho\sigma\mu} T_{\rho\sigma\nu} + \hat{\zeta}_{3} T^{\mu\nu} + \hat{\zeta}_{4} T^{\mu\nu}_{\epsilon} T^{\nu}_{\rho},$$

(2.35)

We just choose metric and torsion as the fundamental degrees of freedom in noncommutative gravity, though it is also possible to use $\hat{e}_\mu^a$ and $\hat{\omega}_\mu^a b$ instead.
where $T_\mu = \delta^\rho_\sigma T_{\mu \rho}^\sigma$ is the torsion vector (the trace of torsion), $T^\mu_{\nu \rho} = \epsilon^{\mu \rho \sigma} T_{\nu \rho \sigma}$, and $T^\mu_{\nu \rho} = \epsilon^{\mu \rho \sigma} T_{\nu \rho \sigma}$.

Note that for $T^\mu_{\nu \rho}$ and $T^\mu_{\nu \rho}$ the order of indices is important. Here, only the terms up to mass dimension 2 are presented.

In the commutative limit $\theta \to 0$, we expect that $T'_{\text{gauge}} = 0$ or $B^{(\pm)}_\mu$ and torsion are decoupled from the usual Einstein-Maxwell part. Since $K_1^{\mu \nu}$ becomes symmetric in the commutative limit, the coupling constants $\hat{\zeta}_i (i = 1, \cdots, 5)$ can be nonvanishing. On the other hand, the other terms, if not vanished as $\theta \to 0$, would have affected the leading order solution. We may take the coupling constants $\hat{\xi}_0^{(\pm)}$ and $\zeta_i$ to be proportional to $\theta$ as a simple choice.

In the following subsection, we first introduce the noncommutativity to gravity part via Seiberg-Witten map. The desirable deformation of the matter part will be discussed later in the Sec 3.3.

### 2.5 Seiberg-Witten map

The Seiberg-Witten map [19] is introduced as a map between gauge theories on commutative and noncommutative geometries. As shown in Ref. [20], Chern-Simons theory has a peculiar feature under the map; the form of the action remains unchanged (up to surface terms), and we can simply replace the ordinary products with the Moyal products. This property suggests that at least for the part of Chern-Simons action, a solution for the equation of motion can be mapped into its noncommutative counterpart.

We now consider the Seiberg-Witten map based on the radius-angle commutation relation [16]

$$[\hat{R}, \hat{\varphi}] = 2i\theta$$

(2.36)

where $\hat{R} = \hat{r}^2$. Namely, $\theta^{R \varphi} = -\theta^{\varphi R} = 2\theta$ and the other components are all zero. The convention is fixed in App A.1 and the correction term from the Seiberg-Witten map is

$$A'_\mu (A) = -\frac{i}{4} (2\theta) \left[ \frac{1}{2} \eta_{ab} A^a_R (\partial_\varphi A^b_\mu + F^b_{\varphi \mu}) - \frac{1}{4} \eta_{ab} A^a_\varphi (\partial_R A^b_\mu + F^b_{R \mu}) + i (A^a_R \tau_a + B_R \tau_3) (\partial_\varphi B^b_\mu + F^b_{\varphi \mu}) - i (A^a_\varphi \tau_a + B_\varphi \tau_3) (\partial_R B^b_\mu + F^b_{R \mu}) + i B_R (\partial_\varphi A^b_\mu + F^b_{\varphi \mu}) \tau_b - i B_\varphi (\partial_R A^b_\mu + F^b_{R \mu}) \tau_b \right].$$

(2.37)

Since the noncommutative version of Chern-Simons theory has two extra gauge fields $B^{(\pm)}_\mu$, we need to give their forms in the commutative case, where they have vanishing field strength, that is, $dB^{(\pm)} = 0$. We consider the simplest case with $B^{(\pm)}_\mu = B d\varphi$ for a constant $B$. Then the Seiberg-Witten map now reads

$$A^{(\pm) \mu}_\mu = -\frac{\theta B}{2} \left[ \partial_R A^{(\pm) a}_\mu + F^a_{R \mu} \right],$$

(2.38)

$$B^{(\pm) \mu}_\mu = -\frac{\theta}{2} \eta_{ab} \left[ A^{(\pm) a}_R F^b_{\varphi \mu} - A^{(\pm) a}_\varphi F^b_{R \mu} - A^{(\pm) a}_\varphi \partial_R A^{(\pm) b}_\mu \right].$$

(2.39)

By applying this map to the gauge fields (2.21)–(2.23), to the first order in $\theta$, the noncommutative gauge fields are

$$A^{(\pm) 0}_i = \pm \frac{1}{\ell} \left( \sqrt{h} + h' \frac{\gamma}{2r \sqrt{h}} \right) - \theta B \frac{(2r^2 - \beta^2) \gamma \pm 2\ell^2 (2r^4 - \beta r^2 + 4\beta \gamma) h}{16 \ell^2 r^4 h^{3/2} \gamma},$$

(2.40)
\[ A^{(\pm)0} = - \left( \sqrt{h} \pm \frac{1}{\ell^2} \frac{\gamma}{\sqrt{h}} \right) + \theta B \frac{\pm \gamma(2r^2 - \beta) + 2\ell^2(r^2 - \beta)h}{8\ell^4 r^2 h^{3/2}}, \quad (2.41) \]
\[ A^{(\pm)1} = \frac{1}{\ell \sqrt{f}} \left( \frac{\hbar'}{2r(h(r))} \pm 1 \right) + \frac{\theta B}{32\ell^4 r^2 h^2 f^{3/2}} \left[ 4\gamma^3(2r^2 - \beta)^2 + 2\ell^2 h \left[ 3\gamma r^2(2r^2 - \beta)^2 - 4\beta \gamma^3 + 2\ell^2 r^2 h \left( \pm r^2(2r^2 - \beta) + \gamma(2r^2 - 3\beta) \pm 2\ell^2 r^2 h \right) \right] \right], \quad (2.42) \]
\[ A^{(\pm)2} = - \frac{h'}{2} \sqrt{\frac{f}{h}} + \theta B \frac{-\gamma^2(2r^2 - \beta)^2 + 2\ell^2 h [4\beta \gamma^2 + 2\ell^2 r^2 h (r^2 + \beta)]}{16\ell^6 r^5 h^{3/2} f^{1/2}}, \quad (2.43) \]
\[ A^{(\pm)3} = \pm \frac{r}{\ell} \left[ \frac{\hbar'}{h} + \theta B \frac{2\beta \gamma (\pm \ell^2 h + \gamma) + 4r^2 (\ell^4 h^2 - r^2)^2}{16\ell^5 r^3 h^{3/2} f^{1/2}} \right], \quad (2.44) \]
\[ \dot{B}^{(\pm)} = \left( B + \frac{\beta \theta}{4\ell^2 r^2} \right) d\varphi - \theta \beta \frac{r^2 + 2\gamma}{4\ell^3 r^4} dt, \quad (2.45) \]

where the prime ' denotes the \( r \) derivative.

In the following section, we discuss black hole solutions in noncommutative gravity based on these expressions. Note that since gauge fields are all functions of \( r \) only, we can again replace \( \star \) product with a usual product.

## 3 Noncommutative charged rotating BTZ black holes and torsion gravity

### 3.1 \( \theta \)-deformed metric

In Sec\( ^2 \) we have derived the noncommutative Chern-Simons gauge fields \((2.40), (2.45)\). From them we can reconstruct noncommutative vielbeins and spin connections as follows:

\[ e^a = \frac{\ell}{2} (\hat{A}^{(+)}a - \hat{A}^{(-)}a), \quad \omega^a = \frac{1}{2} (\hat{A}^{(+)}a + \hat{A}^{(-)}a), \quad (3.1) \]

and the explicit forms are

\[ e^0 = \left( \sqrt{h} - \theta B \frac{2r^2 - \beta}{8\ell^2 r^2 \sqrt{h}} \right) dt + \frac{\gamma}{\ell \sqrt{h}} \left( 1 + \theta B \frac{2r^2 - \beta}{8h \ell^2 r^2} \right) d\varphi, \quad (3.2) \]
\[ e^1 = \left[ \frac{1}{\sqrt{f}} + \theta B \frac{2\ell^2 h + 2r^2 - \beta}{8\ell^2 r^2 f^{3/2}} \right] dr, \quad (3.3) \]
\[ e^2 = \left[ r \sqrt{\frac{f}{h}} - \theta B \frac{2\ell^4 r^2 h^2 - (2r^2 - \beta)^2}{8\ell^4 r^3 h^{3/2} f^{1/2}} \right] d\varphi, \quad (3.4) \]
\[ \omega^0 = \left[ - \frac{\hbar'}{2\ell r \sqrt{f}} + \theta B \frac{8\ell^2 \beta h - (2r^2 - \beta)^2}{16\ell^6 r^4 h^{3/2}} \right] dt + \left[ - \sqrt{h} + \theta B \frac{r^2 - \beta}{4\ell^2 r^2 \sqrt{h}} \right] d\varphi, \quad (3.5) \]
\[ \omega^1 = \left[ \frac{\hbar'}{2\ell r h \sqrt{f}} + \theta B \frac{2\ell^4 r^2 h^2 (2r^2 - 3\beta) + 2\gamma^2 (2r^2 - \beta)^2 + \ell^2 h (12r^6 - 12\beta r^4 + 3\beta^2 r^2 - 4\beta \gamma^2)}{16\ell^7 r^6 h^2 f^{3/2}} \right] dr, \quad (3.6) \]
\[ \omega^2 = \left[ - \frac{h'}{2} \sqrt{\frac{f}{h}} + \theta B \frac{4\ell^4 r^2 h^2 (r^2 + \beta) + 8\beta \gamma^2 \ell^2 h (2r^2 - \beta)^2 \gamma^2}{16\ell^6 r^5 h^{3/2} f^{1/2}} \right] dt - \theta B \frac{\beta \gamma}{8\ell^3 r^3 \sqrt{f} h} d\varphi. \quad (3.7) \]
From the vielbeins, one can further construct the deformed metric:

\[
\begin{align*}
\text{ds}^2 &= -(e^0)^2 + (e^1)^2 + (e^2)^2 \\
&= -\left[ h(r) - \theta B \frac{2r^2 - \beta}{4\ell^2 r^2} \right] dt^2 + \left[ \frac{1}{f(r)} + \theta B \frac{2h(r)\ell^2 + 2r^2 - \beta}{4\ell^2 f(r)^2} \right] d\varphi^2 + \left[ r^2 - \frac{\theta B}{2} \right] d\varphi^2 - \frac{2\gamma}{\ell} dt d\varphi \\
&\quad + \mathcal{O}(\theta^2).
\end{align*}
\]

(3.8)

In the neutral limit \( Q \to 0 \) (namely \( \beta \to 0 \)), this metric agrees with (1.1). When one applies the same change of coordinates as in (1.5), the metric recovers the undeformed one (2.15) with \( r \) replaced by \( \tilde{r} \). This implies that the Einstein equation

\[
\hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R} = 8\pi G \hat{T}_{\mu\nu},
\]

(3.9)

is satisfied if we apply the same coordinate transformation to the right hand side (the gauge field energy-momentum tensor) simultaneously.

Now we would like to investigate the change of coordinates and the Einstein equation more closely. The Ricci tensor and the scalar curvature are constructed from the deformed metric \( \hat{g}_{\mu\nu} \) and its Levi-Civita connection,

\[
\left\{ \begin{array}{c} \rho \\ \mu
\end{array} \right\} = \frac{1}{2} \hat{g}^{\rho\sigma} \left( \partial_\mu \hat{g}_{\nu\sigma} + \partial_\nu \hat{g}_{\mu\sigma} - \partial_\sigma \hat{g}_{\mu\nu} \right).
\]

(3.10)

We denote the left hand side of (3.9) as \( G^{(\Lambda)}_{\mu\nu}(\hat{g}, \{ \} ) \). On the other hand, the deformed energy-momentum tensor reads

\[
\hat{T}_{\mu\nu} = \frac{1}{\lambda^2} \left[ \hat{f}_{\mu\nu\rho\sigma} \hat{g}^{\rho\sigma} - \frac{1}{4} \hat{g}_{\mu\nu} \hat{f}_{\rho\sigma\xi\zeta} \hat{g}^{\rho\xi} \hat{g}^{\sigma\zeta} \right],
\]

(3.11)

where

\[
\hat{f}_{\mu\nu}(r) = f_{\mu\nu}(r) \bigg|_{r \to r - \frac{\theta B}{2r^2}} = \frac{\lambda Q}{r} \left( 1 + \frac{\theta B}{2r^2} \right) + \mathcal{O}(\theta^2),
\]

(3.12)

is obtained by applying the inverse of the coordinate transformation (1.5) to the undeformed field strength \( f_{\mu\nu}(r) \). As a result, two equations of motions are related as follows,

\[
G^{(\Lambda)}_{\mu\nu}(\hat{g}, \{ \} ) = 8\pi G \hat{T}_{\mu\nu} \quad \xrightarrow{r \to r - \frac{\theta B}{2r^2}} \quad G^{(\Lambda)}_{\mu\nu}(\hat{g}, \{ \} ) = 8\pi G \hat{T}_{\mu\nu}.
\]

(3.13)

It may appear that the deformed metric is again trivial and equivalent to the undeformed one up to a simple coordinate transformation. However, it turns out that there remains a non-vanishing torsion tensor in this charged case and the solution is not related to the undeformed metric just by a coordinate change. This issue will be discussed in the next subsection.

### 3.2 Torsion and Einstein-Cartan gravity

The connection in noncommutative space is calculated by use of the deformed vielbeins and spin connections (3.2)–(3.7) as follows:

\[
\Gamma^\lambda_{\mu\nu} = \hat{e}^\lambda_a \left( \partial_\mu \hat{e}^a_\lambda + \hat{\omega}_\mu^a \hat{e}^b_\lambda \right).
\]

(3.14)
They are asymmetric with respect to $\mu$ and $\nu$ indices, and provide nontrivial torsion:

$$ T_{\mu\nu}^\rho = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. \quad (3.15) $$

To be explicit, the non-vanishing components of torsion are

$$ T^{tr}_0 = -\beta \theta B \frac{r^2 f + \gamma^2}{2\ell^4} + \frac{\gamma}{8\ell^4 r^3 f h^{1/2}}, \quad T^{tr}_0 = -\beta \theta B \frac{\gamma}{8\ell^4} f h^{1/2}, \quad (3.16) $$

$$ T^{tr}_2 = -\beta \theta B \frac{\gamma}{4\ell^3} (f h)^{1/2}, \quad T^{tr}_2 = -\beta \theta B \frac{1}{8\ell^4 r^2 (f h)^{1/2}}. \quad (3.17) $$

Therefore, the connection associated with the deformed solution is not a Levi-Civita connection but a more general Affine connection. Furthermore, the curvature tensors should be calculated by use of $\hat{g}_{\mu\nu}$ and the Affine connection $\Gamma^\rho_{\mu\nu}$. The non-vanishing components of Einstein tensor $G^{(A)}_{\mu\nu}(\hat{g}, \Gamma)$ (including a cosmological constant term) are

$$ G^{(A)}_{tt}(\hat{g}, \Gamma) = \beta \frac{f^2 h^2 + 2\gamma^2}{2\ell^4} + \frac{\gamma}{16\ell^4 r^6}; \quad (3.18) $$

$$ G^{(A)}_{t\varphi}(\hat{g}, \Gamma) = -\beta \frac{\gamma}{2\ell^4} - \beta \gamma B \frac{3}{4\ell^4 r^3}; \quad (3.19) $$

$$ G^{(A)}_{r\varphi}(\hat{g}, \Gamma) = -\beta \frac{1}{2\ell^2} - \beta \gamma B \frac{1}{8\ell^4 r^2 f^2}; \quad (3.20) $$

$$ G^{(A)}_{\varphi t}(\hat{g}, \Gamma) = -\beta \frac{\gamma}{2\ell^2} - \beta \gamma B \frac{3}{8\ell^4 r^4}; \quad (3.21) $$

$$ G^{(A)}_{\varphi\varphi}(\hat{g}, \Gamma) = \beta \frac{1}{2\ell^2} + \beta \theta B \frac{1}{8\ell^4 r^2}. \quad (3.22) $$

Note that the Einstein tensor $G^{(A)}_{\mu\nu}(\hat{g}, \Gamma)$ is also asymmetric due to torsion. Since torsion transforms as a genuine tensor, this solution cannot be related to the undeformed one with vanishing torsion by a mere coordinate change.

The theory of gravity with torsion is known as Einstein-Cartan theory of gravitation. Some features of Einstein-Cartan theory are briefly summarized in the App.A.3. As explained there, the equations of motion has an extra contribution depending on torsion, and they now read

$$ G^{(A)}_{\mu\nu}(\hat{g}, \Gamma) - \frac{1}{2} \nabla_\alpha \left[ -\bar{T}_{\mu\nu}^\alpha + \bar{T}_{\mu\nu}^\alpha + \bar{T}_{\mu\nu}^\alpha \right] = 8\pi G \bar{T}_{\mu\nu}, \quad (3.23) $$

where $\nabla_\alpha \equiv \nabla_\alpha + T_\alpha$. $T_\alpha$ is the trace of the torsion tensor, while $\bar{T}_{\mu\nu}^\rho$ is the deformed one. Note that $(3.17)$ leads to the vanishing trace of the torsion $T_\alpha = 0$ and then $\nabla_\alpha = \nabla_\alpha$.

Now we make an interesting observation that the equations of motion $(3.23)$ are also satisfied if the deformed energy momentum tensor $(3.11)$ is adopted. Namely, we have confirmed the following equivalence under the change of coordinate:

$$ G^{(A)}_{\mu\nu}(\hat{g}, \Gamma) - \frac{1}{2} \nabla_\alpha \left[ -\bar{T}_{\mu\nu}^\alpha + \bar{T}_{\mu\nu}^\alpha + \bar{T}_{\mu\nu}^\alpha \right] = 8\pi G \bar{T}_{\mu\nu}, \quad (3.24) $$

In other words, the effect of torsion at the left hand side of equation $(3.23)$ appears to cancel out.
So far we have observed a part of the set of equations of motion. In Einstein-Cartan theory, there are also equations of motion from the variation with respect to the torsion:

\[ K_\rho^\nu + T^\nu_\rho \delta^\mu_\rho - T^\mu_\rho = - \frac{\delta I_{\text{gauge}}}{\delta T^\mu_\rho}, \quad (3.25) \]

where \( K_\rho^\nu \) is the contortion. The standard action for deformed \( U(1) \) gauge field action does not couple to torsion, therefore the right hand side is zero. Since the left hand side is nonvanishing, the matter part of the action should also be modified such that it couples to torsion. In the next section, we will treat these two equations of motions in a unified way by use of Chern-Simons equations of motion.

### 3.3 The Chern-Simons equations of motion and the matter energy-momentum tensor

After the deformation, we may assume that the matter part action is replaced as

\[ \hat{I}_{\text{gauge}} = - \frac{1}{\lambda^2} \int d^3 x \sqrt{-\hat{g}} \hat{f}_{\mu
u} \hat{g}^{\mu\nu} + I'_{\text{gauge}}, \quad (3.26) \]

where \( I'_{\text{gauge}} \) includes the coupling to \( B^{(\pm)}_\mu \) and torsion (or spin connection), and should vanish in the commutative limit \( \theta \to 0 \) (or to provide decoupled equations of motion). The generic form is argued in the Sec. 2.4. We will calculate the energy-momentum tensor and the spin density tensor from this action, and choose the coupling constant to determine the correction term. In terms of Chern-Simons gauge fields, the correction term should satisfy the following equations of motion,

\[ \frac{\delta I_{CS}}{\delta \hat{A}_\mu^{(\pm)a}} = \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[ \eta_{ab} \partial_\rho \hat{A}^{(\pm)b}_\sigma - \frac{\epsilon^{abc}}{2} \hat{A}^{(\pm)c}_\rho \hat{A}^{(\pm)b}_\sigma + \frac{i}{6} \eta_{ab} \left( \hat{A}^{(\pm)b}_\rho \hat{B}^{(\pm)}_\sigma + \hat{B}^{(\pm)}_\rho \hat{A}^{(\pm)b}_\sigma \right) \right] \]

\[ = - \frac{\delta \hat{I}_{\text{gauge}}}{\delta \hat{A}_\mu^{(\pm)a}}, \quad (3.27) \]

\[ \frac{\delta I_{CS}}{\delta \hat{B}_\mu^{(\pm)}} = \pm \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[ \partial_\rho \hat{B}^{(\pm)}_\sigma - \frac{i}{6} \eta_{ab} \hat{A}^{(\pm)a}_\rho \hat{A}^{(\pm)b}_\sigma + \frac{i}{2} \hat{B}^{(\pm)}_\rho \hat{B}^{(\pm)}_\sigma \right] = - \frac{\delta \hat{I}_{\text{gauge}}}{\delta \hat{B}_\mu^{(\pm)}}, \quad (3.28) \]

\[ \frac{\delta \hat{I}_{\text{gauge}}}{\delta \hat{a}_\mu} = \frac{1}{\lambda^2} \nabla_\rho \hat{g}^{\mu\rho} + \frac{\delta I'_{\text{gauge}}}{\delta \hat{a}_\mu} = 0. \quad (3.29) \]

As for \( \hat{A}_\mu^{(\pm)a} \) and \( \hat{B}_\mu^{(\pm)} \), we consider the explicit solution via the Seiberg-Witten map. We first require that the last equation \( (3.29) \) leads to the solution \( \hat{f}_{\mu
u} \) in \( (3.12) \). We also assume that all the fields are functions of only \( r \) so we can replace \( \star \) product with a usual product and those terms for interaction between \( \hat{A}_\mu^{(\pm)a} \) and \( \hat{B}_\mu^{(\pm)} \) are dropped. At the end, \( (3.27) \) becomes

\[ \frac{\delta I_{CS}}{\delta \hat{A}_\mu^{(\pm)a}} = \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \eta_{ab} \left[ \partial_\rho \hat{A}^{(\pm)b}_\sigma - \frac{\epsilon^{bcd}}{2} \hat{A}^{(\pm)c}_\rho \hat{A}^{(\pm)d}_\sigma \right] \]

\[ = \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \eta_{ab} \left[ \partial_\rho \hat{\omega}^b_\sigma - \frac{\epsilon^{bcd}}{2} \hat{\omega}^c_\rho \hat{\omega}^d_\sigma + \frac{1}{\ell^2} \hat{\epsilon}^b_\rho \hat{\epsilon}^c_\sigma \hat{\omega}^d_\delta \right] \]

\[ = \pm \frac{k}{4\pi} g^{\mu\nu} \hat{\epsilon}_\rho^\delta \delta_\alpha^\delta \sqrt{-g} G^{(\Lambda)}_{\alpha\beta}(\hat{g}, \Gamma) - \frac{k}{8\pi \ell} \epsilon^{\mu\nu\rho} \eta_{ab} \hat{T}^{ab}_\rho \]

\[ (3.30) \]
It is easy to check that only the first term survives in (3.28). Therefore the equations of motion become

\[ \pm \frac{k}{4\pi} \hat{g}^{\mu \nu} \hat{e}_a \hat{\nabla} \sqrt{-\hat{g}} G_{\xi \delta}^{(A)} (\hat{g}, \Gamma) - \frac{k}{8\ell} \hat{g}^{\mu \rho \sigma} \eta_{ab} T_{\rho \sigma}^b = \pm \frac{\ell}{2} \eta_{ac} \cdot \sqrt{-\hat{g}} \hat{T}^{\mu \nu} \hat{e}_{\rho} - \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\mu^{(\pm)a}} , \] (3.31)

\[ \pm \frac{k}{4\pi} \hat{g}^{\mu \rho \sigma} \partial_{\rho} \hat{B}_\sigma^{(\pm)} = - \frac{\delta I'_{\text{gauge}}}{\delta \hat{B}_\mu^{(\pm)}} . \] (3.32)

Or equivalently, one may write it as

\[ G_{\mu \nu}^{(A)} (\hat{g}, \Gamma) = 8\pi G \hat{T}_{\mu \nu} - \frac{2\pi}{k} \hat{g}^{\mu \nu} \hat{e}_a \hat{\nabla} \left( \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(\pm)a}} - \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(-)a}} \right) , \] (3.33)

\[ T_{\mu \nu}^a = 8\pi G \eta_{ab} \hat{e}_{\mu \nu} \hat{\nabla} \left( \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(+)b}} + \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(-)b}} \right) . \] (3.34)

By comparing with (3.28), the first equation implies that

\[ \frac{2\pi}{k} \hat{g}^{\mu \nu} \hat{e}_a \hat{\nabla} \left( \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(+)a}} - \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(-)a}} \right) = - \frac{1}{2} \hat{\nabla} \left[ - \hat{T}_{\mu \nu}^{\alpha} + \hat{T}_{\alpha \mu}^{\nu} + \hat{T}_{\alpha \nu}^{\mu} \right] . \] (3.35)

We therefore find a set of conditions for the correction term in the matter part as follows:

\[ \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\alpha^{(+)a}} = \pm \frac{\ell}{16\pi G} \hat{g}^{\mu \rho \sigma} \eta_{ab} \hat{\nabla} \left[ - \hat{T}_{\mu \rho}^{\alpha} + \hat{T}_{\alpha \rho}^{\mu} + \hat{T}_{\alpha \rho}^{\mu} \right] - \frac{1}{32\pi G} \hat{g}^{\mu \nu} \eta_{ab} T_{\nu \rho}^b , \] (3.36)

\[ \frac{\delta I'_{\text{gauge}}}{\delta \hat{B}_\mu^{(\pm)}} = \pm \frac{\ell}{16\pi G} \hat{g}^{\mu \rho \sigma} \partial_{\rho} \hat{B}_\sigma^{(\pm)} , \] (3.37)

\[ \frac{\delta I'_{\text{gauge}}}{\delta \hat{a}_\mu} = - \frac{1}{\lambda^2} \hat{\nabla} \hat{e}_{\rho} \hat{e}_{\mu} . \] (3.38)

We have not fixed the explicit form of the correction term due to its complexity. Here we simply present the necessary conditions for the \( \theta \) dependent correction term for the matter part of the action.

3.4 Coordinate change

Finally, we briefly comment on the change of coordinates (1.5). There is a subtle point on the regions that the radial coordinate covers. Recall that \( \tilde{r} \) covers \( 0 \leq \tilde{r} < 0 \) and it has one-to-one correspondence to the region \( \sqrt{\theta B}/2 \leq r < \infty \). Except for the vicinity of center in the deformed geometry, \( 0 \leq r \leq \sqrt{\theta B}/2 \), it can be mapped to the undeformed geometry. Now we investigate the angular part of the metric, namely \( \hat{r}^2 d\phi^2 \) or \( (r^2 - \frac{\theta B}{2}) d\phi^2 \). In the deformed metric, the radial coordinate \( r \) makes sense only for the region \( r \geq \sqrt{\theta B}/2 \). Therefore, in the deformed geometry, there appears an effective minimum length scale \( r_{\text{min}} = \sqrt{\theta B}/2 \). This may not be so surprising; in the current formulation, the noncommutative parameter appears only in the combination of \( \theta B \) and we learn that \( \sqrt{\theta B} \) serves a characteristic length scale in the noncommutative geometry. Now

\footnote{By use of the change of the variables (A.11), one can also consider these relations in terms of the variations with respect to the metric and the torsion.}
let $r_+$ be the location of the horizon of noncommutative BTZ measured in $r$ coordinate, that is, the largest root of $\hat{g}_{tt}(r_+) = 0$. As long as $r_+ \geq r_{\text{min}}$, we can see the correspondence to the undeformed BTZ solution. On the other hand, a black hole of the size $r_+ < r_{\text{min}}$ is not well-defined in the noncommutative side.

Finally, we comment on a subtle issue on the coordinate invariance of noncommutative gravity. The action of noncommutative gravity, by using Chern-Simons formulation, is invariant under a deformed coordinate transformation, which is reduced to the usual diffeomorphism in the commutative limit. As studied in Ref. [21], two noncommutative theories obtained by the Seiberg-Witten map with the rectangular and polar coordinates are distinct; this implies that we cannot map one to the other by a simple coordinate change in the noncommutative theory. In the present case, the situation is much simpler; the change is only for the radial coordinate $r$ and it does not change the noncommutative algebra (unlike the change between the rectangular and polar coordinates discussed in Ref. [21]). On top of that, all the relevant function in the metric depends only on $r$. In other words, the noncommutativity is irrelevant when we consider a deformed coordinate transform (star product simply reduces to ordinary product), and the transformed solution satisfies the conventional Einstein equation. Together with the uniqueness of local solution for the vacuum case, this could be the very reason why the deformed solution is related to the undeformed one by the coordinate change (1.5). In the case of Einstein-Maxwell theory, since solutions do not have to be unique and a noncommutative extension involves nontrivial torsion, we cannot see immediately why the simple relation still holds. It may be related to the fact that since the functions in the commutative solution only involve the radial coordinate $r$, it could be sufficient to consider a commutative version of Einstein-Cartan theory while we only concern the equation of motion. We expect a simple coordinate change like (1.5) becomes impossible for a generic geometry whose metric functions depend on both $r$ and $\phi$ coordinates. We, however, leave this complexity for future studies.

4 Conclusion

In this paper, we have explored the charged rotating BTZ black hole geometry by use of Chern-Simons formulation in $2 + 1$ dimensional gravity and the Seiberg-Witten map.

The noncommutativity in question is the one between the radial coordinate and the angular coordinate, namely $[r^2, \phi] = 2i\theta$. The noncommutative deformation for the pure gravity part is introduced by the Seiberg-Witten map for the Chern-Simons gauge fields where two extra $U(1)$ gauge fields are added. The deformation for the matter gauge field part is to be determined to satisfy the deformed equations of motion.

It is found that as with the neutral case, the deformed metric is related to the undeformed one via a simple coordinate transformation. Through this observation, we discover that the deformation of the matter energy-momentum tensor can also be obtained by the same coordinate change. Nevertheless, there appears nonvanishing torsion that is proportional to the noncommutativity parameter and it cannot be eliminated by a coordinate change. We thus analyze the equations of motion in the framework of Einstein-Cartan torsion gravity. It is found that with the same deformed matter energy-momentum tensor, the equations of motion derived from torsion gravity

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6We thank an anonymous referee to raise this point.
are also satisfied. Though we have not yet fixed the action of deformed matter completely, a set of conditions for the correction term are presented.

There are several issues to be clarified. Firstly, we do not fully understand why the noncommutative deformation is represented by a simple change of the radial coordinate. One can verify that the result of the Seiberg-Witten map for the difference of the gauge fields, say $A^{(+)}_\mu - A^{(-)}_\mu$, can be obtained by the same coordinate change, however the sum is not. Therefore, the deformed vielbeins $\hat{e}^a_\mu$ are related to the undeformed ones via the coordinate change, however the spin connections are not. This subtle difference leads to the nontrivial torsion in the deformed background. One may argue that the gauge field representation has some nonphysical degrees of freedom, but the appearance of torsion is physical and cannot be trivially eliminated by the coordinate change.

We observe that the matter gauge field couples to $B^{(\pm)}$ after the deformation. In Ref. [18], the authors argued that in the noncommutative Chern-Simons theory there is a coupling term between $(B^{(+)}) + B^{(-)}$ and torsion. One thus may guess that the coupling to torsion appears via $B^{(\pm)}$. We remark that the noncommutative extension of torsion constructed in Ref. [18] consists of two parts: a standard part (which we call torsion in this paper) and a term like $\eta_{ab}(\omega^a \wedge e^b + e^a \wedge \omega^b)$ (which trivially vanishes in the commutative limit). However, $B^{(\pm)}$ couples only to the latter. Therefore, the coupling between the geometrical part of torsion to the other degrees of freedom remains unclear.

Secondly, the admitted minimal black holes discussed in the Sec. 3.4 may imply that the noncommutative space-time has its own entropy, i.e. $S \propto \theta B$ in a region of Planckian size, presuming the area law still applies. This reminds us of the spin foam model in the loop quantum gravity [22] and we wonder if $e^S$ counts the spin combination.

Thirdly, it is curious which properties of the charged BTZ black hole are changed or unchanged after the deformation. The torsion may affect the property of black holes through the change of metric [23]. However, in our case as long as we look at the metric only, we do not see the difference. It is interesting to see whether this is a peculiar feature of the current solution, or this may happen in a broader setup of $2 + 1$ dimensional gravity with noncommutativity. On top of that, it should also be important to fix the deformation of the matter part action and examine how the matter part action couples to the torsion or the extra $U(1)$ gauge fields $B^{(\pm)}_\mu$.

Finally, we would like to comment on results made in Ref. [8, 9], where noncommutative structures, including Lie algebraic structure ones, are considered in $3 + 1$ dimensional gravity. It was argued that the first order correction vanishes under the condition of vanishing classical torsion. In our construction, however, we include a matter field whose deformation is not completely fixed by the Seiberg-Witten map and the deformed solution has nontrivial torsion. Therefore, our result would not be immediately contradict to their results. Since there appears a simple relation between the deformed geometry with torsion and the undeformed one, it is interesting to investigate the applicability of our argument to generic backgrounds in torsion gravity.

Note added: Upon completing this work, there appeared a paper [24], which considered a noncommutative deformation in four dimensional gravity. They also observed the emergence of torsion.


Acknowledgment

This work is supported in parts by the Taiwan’s Ministry of Science and Technology (grant No. 102-2112-M-033-003-MY4) and the National Center for Theoretical Science.

A Conventions and notations

We summarize our conventions and notations in this paper here.

A.1 Seiberg-Witten map

Seiberg and Witten showed that a field theory on D-branes with a background $B$ field can be formulated as a conventional Yang-Mills theory or a noncommutative Yang-Mills theory depending on the regulator we choose, Pauli-Villars or point-splitting respectively [19]. The gauge transformation is now defined by use of Moyal product as

\[
\hat{\delta}_\xi \hat{A}_\mu = \partial_\mu \hat{\xi} - \hat{\xi} \star \hat{A}_\mu + \hat{A}_\mu \star \hat{\xi} = \partial_\mu \hat{\xi} - \frac{i}{2} \theta^{\mu\rho} (\partial_\nu \hat{\xi} \partial_\rho \hat{A}_\mu - \partial_\nu \hat{A}_\mu \partial_\rho \hat{\xi}) + \mathcal{O}(\theta^2). \tag{A.1}
\]

The Seiberg-Witten map is defined as a compatibility condition of gauge transformation and a mapping between $A$ and $\hat{A}$,

\[
\hat{A}(A) + \hat{\delta}_\xi \hat{A}(A) = \hat{A}(A + \delta_\xi A), \tag{A.2}
\]

for infinitesimal $\xi$ and $\hat{\xi}$. The solution is

\[
\hat{A}_\mu(A) = A_\mu - \frac{i}{4} \theta^{\mu\rho} \{ A_\nu, \partial_\rho A_\mu + F_{\rho\mu} \} + \mathcal{O}(\theta^2), \tag{A.3}
\]

\[
\hat{\xi}(\xi, A) = \xi + \frac{i}{4} \theta^{\mu\nu} \{ \partial_\mu \xi, A_\nu \} + \mathcal{O}(\theta^2), \tag{A.4}
\]

where $\{ f, g \} = fg + gf$ is the anti-commutator with respect to the conventional matrix product.

A.2 Some notations and $U(1, 1)$ generators

The epsilon tensor is $\epsilon_{012} = -\epsilon^{012} = 1$. We define for a spin connection 1-form $\omega^{ab}$,

\[
\omega_a = -\frac{1}{2} \epsilon_{abc} \omega^{bc}. \tag{A.5}
\]

For the neutral BTZ black holes, $r_\pm$ is defined by

\[
r_\pm^2 = 4G\ell^2 \left( M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}} \right), \tag{A.6}
\]

\[
M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad J = \frac{r_+ - r_-}{4G\ell}. \tag{A.7}
\]

Our convention of $U(1, 1)$ generators is

\[
\tau_0 = \frac{i}{2} \sigma_3, \quad \tau_1 = \frac{1}{2} \sigma_1, \quad \tau_2 = \frac{1}{2} \sigma_2, \quad \tau_3 = \frac{i}{2} \sigma_2, \quad \tau_3 = \frac{i}{2} \mathbf{1}_2, \tag{A.8}
\]
with \( a, b = 0, 1, 2 \), \( A, B = 0, 1, 2, 3 \), \( \eta_{AB} = \text{diag}(-1,1,1,-1) \) and they satisfy

\[
g_{AB} = \text{tr}(\tau_A \tau_B) = \frac{1}{2} \eta_{AB}, \quad [\tau_A, \tau_B] = -\varepsilon_{ABC} \tau_C, \quad \varepsilon_{ABC} \begin{cases} \varepsilon_{a b} c \\ \varepsilon_{a b} 3 = 0 \end{cases}
\]

(A.9)

\[
\{\tau_a, \tau_b\} = \frac{1}{2} \eta_{ab} \mathbf{1}_2, \quad \{\tau_A, \tau_3\} = i\tau_A, \quad \text{tr}(\tau_a \tau_b \tau_c) = -\frac{1}{4} \varepsilon_{abc} \tau^c, \quad \varepsilon_{abc} = \frac{1}{3}
\]

(A.10)

By use of the chain rule, we can convert the variation with respect to the gauge fields to those with respect to the metric and the torsion as

\[
\frac{\delta}{\delta A_{\mu}^{(\pm)a}} = \mp \frac{l^2}{2} \left[ 2g^{\mu a} e^\beta a \frac{\delta}{\delta g^{\alpha \beta}} + e^\beta a \left[ \delta^\beta_\sigma T_{\rho \sigma \mu} + \delta^\sigma_\mu \Gamma^{\alpha}_\sigma \beta - \delta^\mu_\alpha \Gamma_{\rho \sigma} \beta \right] \frac{\delta}{\delta T_{\rho \sigma} \alpha} \right] + \frac{1}{2} e^b c a e^a b \left( \delta^\mu_\rho e_\sigma c - \delta^\mu_\sigma e_\rho c \right) \frac{\delta}{\delta T_{\rho \sigma} \alpha}.
\]

(A.11)

### A.3 Einstein-Cartan theory of torsion gravity

The Einstein-Cartan theory of gravitation is a generalization of Einstein’s theory of general relativity to allow torsion in space-time. It can be regarded as a gauge theory of the Poincaré symmetry instead of the Lorentz symmetry\(^{[26]}\). While curvature is related to the energy momentum tensor with Lorentz symmetry, torsion is related to the density of intrinsic angular momentum or spin. For some overview of torsion gravity, see Ref.\(^{[27]}\).

The vielbeins \( e^\mu_a \) relate to the metric by

\[
g_{\mu \nu} = e^\mu_a \eta^{ab} e_\nu b, \quad \text{where} \quad \eta_{ab} = \text{diag}(-1,1,1), \quad \text{and its inverse is} \quad e_\mu^a e_\nu^b = \delta^a_b.
\]

With spin connections \( \omega_\mu^a b \), Affine connections are defined by

\[
\Gamma^\nu_{\mu \lambda} = e^\nu_c \left( \partial_\mu e^a_c + \omega_\mu^a b e^b_c \right),
\]

(A.12)

and the torsion tensor is

\[
T^a_{\mu \nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega_\mu^a b e^b_\nu - \omega_\nu^a b e^b_\mu.
\]

(A.13)

The curvature tensor is

\[
R^\lambda_{\rho \mu \nu} = \partial_\mu \Gamma^\lambda_{\nu \rho} - \partial_\nu \Gamma^\lambda_{\mu \rho} + \Gamma^\lambda_{\mu \beta} \Gamma^\beta_{\nu \rho} - \Gamma^\lambda_{\nu \beta} \Gamma^\beta_{\mu \rho},
\]

(A.14)

and the Ricci tensor and the scalar curvature are defined by

\[
R_{\mu \nu} = R^\rho_{\mu \rho \nu} \quad \text{and} \quad R = g^{\mu \nu} R_{\mu \nu},
\]

respectively.

In Einstein-Cartan theory of torsion gravity, the metric \( g_{\mu \nu} \) and the connection \( \Gamma^\rho_{\mu \nu} \) are treated as independent variables. When we consider the equations of motion, we can take the variation of torsion tensor instead of the connection. The action is given by the usual Einstein-Hilbert form,

\[
I_G = \frac{1}{16\pi G} \int d^3 x \sqrt{-g} R,
\]

(A.15)

and its variations give

\[
\frac{16\pi G}{\sqrt{-g}} \frac{\delta I_G}{\delta g^{\mu \nu}} = G^{(\Lambda)}_{\mu \nu}(g, \Gamma) - \frac{1}{2} \nabla_\alpha \left[ \sqrt{-g} \nabla_\mu \Gamma^\alpha_{\mu \nu} + \sqrt{-g} \nabla_\nu \Gamma^\alpha_{\mu \mu} \right],
\]

(A.16)
\[ 16\pi G \frac{\delta I_G}{\delta T_{\mu\nu}^\rho} = K_{\rho}^{\mu\nu} + T_{\nu}^{\rho} \delta_{\rho}^{\mu} - T_{\mu}^{\rho} \delta_{\rho}^{\nu}, \]  
(A.17)

where $\nabla_\alpha \equiv \nabla_\alpha + T_\alpha$ with $\nabla_\alpha$ being a covariant derivative and $T_\alpha$ the trace of the torsion tensor $T_{\alpha\nu}^a \hat{e}_a^\nu$. The contorsion tensor $K_{\mu\nu\sigma}$ is defined as

\[ K_{\mu\nu\sigma} = \frac{1}{2} (T_{\mu\nu\sigma} - T_{\nu\sigma\mu} + T_{\sigma\mu\nu}), \]  
(A.18)

and $\tilde{T}_{\mu\nu}^\rho$ is known as the deformed torsion tensor:

\[ \tilde{T}_{\mu\nu}^\rho = T_{\mu\nu}^\rho + \delta_{\mu}^\rho T_{\nu} - \delta_{\nu}^\rho T_{\mu}. \]  
(A.19)

Finally, by use of the one forms $e^a = e_\mu^a dx^\mu$ and $\omega^a_{\ b} = \omega^a_{\ \mu b} dx^\mu$, the torsion and the curvature two forms are written as

\[ T^a = De^a = de^a + \omega^a_{\ b} \wedge e^b, \]  
(A.20)

\[ R^a_{\ bc} = d\omega^a_{\ bc} + \omega^a_{\ cd} \wedge \omega^d_{\ bc}. \]  
(A.21)

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