Drawing Bipartite Graphs on Two Parallel Convex Curves

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Abstract

Let $G$ be a bipartite graph, and let $\lambda_e$, $\lambda_i$ be two parallel convex curves; we study the question about whether $G$ admits a planar straight-line drawing such that the vertices of one partite set of $G$ lie on $\lambda_e$ and the vertices of the other partite set lie on $\lambda_i$. A characterization is presented that gives rise to linear time testing algorithm. We also describe a drawing algorithm that runs in linear time if the curves are two concentric circles and the real RAM model of computation is adopted.

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1 Introduction

Common requirements for drawing a bipartite graph are that the bipartition is highlighted in the visualization by representing the vertices on two distinct layers, the edges have as few bends as possible, and the number of edge crossings is minimized. A bipartite graph is a biplanar graph if it has a straight-line crossing-free drawing where the vertices of one partite set are on a horizontal layer and the vertices of the other partite set are on a separate parallel horizontal layer [6]. Biplanar graphs have been independently characterized in [5, 8, 11]. Also, the problem of computing straight-line drawings of bipartite graphs with the vertices on two horizontal layers and minimum number of crossings has been well studied; see, e.g. [2, 3, 7, 9, 10] for some basic references on this and related topics.

This paper studies planar drawings of bipartite graphs where vertices are constrained to be on two parallel convex curves, which generalizes the case of horizontal layers. Let $G$ be a bipartite graph, and let $\lambda_e, \lambda_i$ be two parallel convex curves; we want to answer the question about whether $G$ admits a planar straight line drawing such that the vertices of one partite set of $G$ lie on $\lambda_e$ and the vertices of the other partite set lie on $\lambda_i$.

Our interest in this question is in part motivated by the observation that the class of bipartite graphs that admit a planar straight line drawing on two horizontal lines is quite restricted and that one may hopefully enlarge this class by allowing some curvature on the two layers. Indeed, there is already some evidence in the literature that if the vertices in a drawing are not constrained to be collinear but instead can lie on curves, the family of representable graphs for specific drawing conventions can increase significantly; see, e.g. [4] for drawings of planar graphs with at most one bend per edge and vertices constrained to be on a given curve.

The problem addressed in this paper is also related to the study of radial planarity testing initiated by Bachmaier, Brandenburg and Forster [1]. In [1] the input is a $k$-partite graph $G$ and $k$-concentric circles; the question is whether $G$ has a crossing-free drawing where the vertices of the same partite set are points of the same radial level (circle) and the edges are simple Jordan curves in the outward direction. Here, we study radial planarity testing for bipartite graphs with the additional constraint that the edges are straight-line segments (indeed, two concentric circles are a special case of two parallel convex curves).

Our contribution is as follows. The family of bipartite graphs which admit a planar straight-line drawing with the vertices constrained to be on two parallel convex curves and with no two vertices of the same partite set on different curves is characterized. The characterization gives rise to a linear time testing algorithm. The proof of sufficiency uses a linear time (real RAM) drawing algorithm in the case of two concentric circles.
2 Preliminaries

A graph \( G = (V, E) \) is bipartite if there exists a partition \( V = V_0 \cup V_1 \) of the vertices of \( G \) such that \( E \subseteq V_0 \times V_1 \). The two sets \( V_0 \) and \( V_1 \) are called partite sets of \( G \). A bipartite graph with a given planar embedding is maximal if every internal face of \( G \) consists of four edges.

A simple curve \( \lambda \) in the Euclidean plane is a closed curve if it partitions the plane into two topologically connected regions; \( \lambda \) is an open curve otherwise. Curve \( \lambda \) is convex if any straight line intersects \( \lambda \) in at most two points. Note that a circle is a special case of closed convex curve.

Let \( p, q \) be two distinct points of \( \lambda \). If \( \lambda \) is an open curve we say that \( p \) precedes \( q \) on \( \lambda \) if \( p \) is encountered before \( q \) when traversing \( \lambda \) in the clockwise direction. If \( \lambda \) is a closed curve, let \( p \) and \( q \) be two distinct points of \( \lambda \) such that the portion of \( \lambda \) traversed when going from \( p \) to \( q \) in the clockwise direction is shorter than the portion of \( \lambda \) traversed when going from \( q \) to \( p \); we say that \( p \) precedes \( q \) and that \( q \) follows \( p \) on \( \lambda \).

Two convex curves are parallel if every normal to one curve is also a normal to the other curve and the distance between the points where the normals intersect the two curves is a constant. In the rest of this paper we denote with \( \lambda_e, \lambda_i \) two parallel convex curves such that the curvature of \( \lambda_e \) is less than the curvature of \( \lambda_i \); \( \lambda_e \) is the external curve, \( \lambda_i \) is the internal curve (in the special case of two concentric circles, \( \lambda_e \) is the circle with larger radius). Curves \( \lambda_e, \lambda_i \) are paired if there exist two points \( p \in \lambda_i \) and \( q \in \lambda_e \) such that the straight-line segment \( pq \) intersects \( \lambda_i \) twice. A straight-line segment with the property of \( pq \) is said to cross curve \( \lambda_i \). Observe that two concentric circles are paired. Two curves will be called non-paired if they are parallel, convex, but are not paired. Figure 1 shows an example of two paired and two non-paired curves, respectively. In particular, if the curves are non-paired and \( \ell \) is the straight line through a point \( p \) of \( \lambda_i \), see Figure 1(b), two cases are possible: either \( \ell \) intersects \( \lambda_e \) in a point \( q \) without crossing \( \lambda_i \) or \( \ell \) crosses \( \lambda_i \) in a point \( x \) without intersecting \( \lambda_e \).

Let \( \lambda_e, \lambda_i \) be two parallel convex curves. A bipartite graph \( G \) is curve bipla-
nar on $\lambda_e, \lambda_i$ if it admits a curve biplanar drawing, i.e. a planar straight-line drawing such that all vertices of a partite set of $G$ are represented as points on $\lambda_e$ and the vertices of the other partite set are represented as points on $\lambda_i$. A graph is a caterpillar if deleting all vertices of degree one produces a (possibly empty) path. A leaf is a vertex with degree 1. A 2-claw is the graph consisting of one degree-3 vertex (called the centre) which is adjacent to three degree-2 vertices, each of which is adjacent to the centre and to one leaf. Figure 2 depicts an example of a caterpillar and a 2-claw. Note that a graph $G$ is a forest of caterpillars if and only if $G$ is acyclic and contains no 2-claw, see [5, 8, 11]. As the next theorem shows, if $\lambda_e, \lambda_i$ are non-paired, the family of curve biplanar graphs coincides with the family of biplanar graphs characterized in [5, 8, 11].

Figure 2: (a) A caterpillar. (b) A 2-claw.

**Theorem 1** A bipartite graph admits a curve biplanar drawing on two non-paired curves if and only if it is a forest of caterpillars.

**Proof:** The proof is an easy adaptation of the arguments in [5, 8, 11]. Initially we prove that if a graph $G$ admits a biplanar drawing on two non-paired curves, then it is a forest of caterpillars.

We prove first that $G$ is acyclic. Since the curves are non-paired and the edges are drawn as straight-line segments, every edge is contained in the annulus between the two curves. Let $v_0, u_0, v_1, u_1, \ldots, v_{h-1}, u_{h-1}$ be a path such that $v_j \in V_0$ and $u_j \in V_1$ ($0 \leq j \leq h-1$). In order to avoid crossings in the drawing of this path, each vertex $v_j$ must precede vertex $v_{j+1}$ on one of the two curves, say $\lambda_e$, and each vertex $u_j$ must precede vertex $u_{j+1}$ ($j = 0, 1, \ldots, h-2$) on the other curve, i.e. $\lambda_i$. If one wants to close a cycle by adding edge $(v_0, u_{h-1})$, then this edge would cross all edges of the path, because these edges are completely contained in the annulus between the two curves and the two curves are non-paired. It follows that $G$ is acyclic.

We prove now that $G$ does not contain a 2-claw and hence it is a forest of caterpillars. Assume, by contradiction, that $G$ contains a 2-claw. The centre $v$ of the 2-claw is drawn on one curve, say $\lambda_e$, and it is adjacent to the three
degree-2 vertices that are on $\lambda_i$. Let $u_0, u_1$ and $u_2$ be these degree-2 vertices and assume that $u_1$ follows $u_0$ and precedes $u_2$ on $\lambda_i$. The leaf $v_1$ adjacent to $u_1$ either precedes $v$ on $\lambda_e$, in which case there is a crossing between $(v, u_0)$ and $(u_1, v_1)$, or it follows $v$ on $\lambda_e$, in which case there is a crossing between $(v, u_2)$ and $(u_1, v_1)$. It follows that $G$ cannot contain a 2-claw.

Let $G$ be a forest of caterpillars and let $\lambda_e, \lambda_i$ be two non-paired curves. We prove the sufficiency in the case that $G$ is one caterpillar. If $G$ is not connected, every connected component can be drawn independently. Let $\Pi = v_0, u_0, v_1, u_1, \ldots, v_{h-1}, u_{h-1}$ be the path obtained by removing all leafs from $G$. Draw each vertex $v_j$ on one of the two curves, say $\lambda_e$, and each vertex $u_j$ on the other curve, i.e. $\lambda_i$ ($j = 0, \ldots, h - 1$), so that $v_j$ precedes $v_{j+1}$ on $\lambda_e$ and $u_j$ precedes $u_{j+1}$ on $\lambda_i$ ($j = 0, \ldots, h - 2$). For each vertex $v_j$, draw the leafs adjacent to $v_j$ as points of $\lambda_e$ which follow $u_{j-1}$ and precede $u_j$. For each vertex $u_j$, draw the leafs adjacent to $u_j$ as points of $\lambda_e$ which follow $v_j$ and precede $v_{j+1}$.

Motivated by Theorem 1, we will investigate the family of bipartite graphs that admit a curve biplanar drawing on two paired curves. We first show how to draw a specific family of graphs, namely bipartite fans, and then present a complete characterization of curve biplanar graphs on two paired curves.

3 How to Draw a Bipartite Fan

Let $G$ be a biconnected bipartite graph with a given planar embedding. $G$ is a bipartite fan if it has a vertex $u$, called apex, that is shared by all its faces (including the external one). The edges incident on $u$ are the radial edges of the fan. Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of a fan $G$ in the counterclockwise order they have on the external face.Edges $(u, v_0)$ and $(u, v_{n-2})$ are called first edge and last edge of the fan, respectively. Any three vertices $v_{2j}, v_{2j+1}, v_{2j+2}$ ($0 \leq j \leq \frac{n-4}{2}$) form a fan triplet of $G$. Notice that $v_{2j+1}$ belongs to the same partite set as $u$. See Figure 3(a) for an illustration of a bipartite fan.

We show how to compute a curve biplanar drawing of a bipartite fan on two paired curves such that the drawing is contained in a suitable region of the plane called a wedge and defined as follows. Let $\lambda_e, \lambda_i$ be two paired curves, let $p, q, r$ be three points such that:

(i) $p, r \in \lambda_e$ and $p$ precedes $r$ on $\lambda_e$;

(ii) $q \in \lambda_i$;

(iii) segment $\overline{pq}$ does not cross curve $\lambda_i$;

(iv) segment $\overline{qr}$ crosses $\lambda_i$.

Let $\lambda_{pr}$ be the portion of $\lambda_e$ consisting of all points $x \in \lambda_e$ such that $x$ follows $p$ and precedes $r$. The closed bounded region delimited by $\overline{pq}$, $\overline{qr}$ and $\lambda_{pr}$ is a wedge of $\lambda_e$, $\lambda_i$ and is denoted as $W(p, q, r)$ (see Figure 3(b) for an example).
Lemma 1 Let $G$ be a bipartite fan with $n$ vertices and apex $u$. Let $\lambda_e, \lambda_i$ be two paired curves and let $W(p, q, r)$ be a wedge of $\lambda_e, \lambda_i$. Fan $G$ admits a curve biplanar drawing on $\lambda_e, \lambda_i$ contained inside $W(p, q, r)$ such that:

(i) the first and the last edge of $G$ are represented by segments $qr$ and $pq$, respectively;

(ii) for every fan triplet $v_{2j}, v_{2j+1}, v_{2j+2}$ of $G$ $(0 \leq j \leq \frac{n-4}{2})$, the three points representing the triplet define a wedge of $\lambda_e, \lambda_i$.

Moreover, if $\lambda_e$ and $\lambda_i$ are circles the drawing can be computed in $O(n)$ time.

Proof: We assume that $G$ is maximal, i.e. that every internal face consists of four edges and four vertices; if not, we can split each internal face $f$ having more than four edges by connecting $u$ to all vertices of $f$ that are not adjacent to $u$ and do not belong to the same partite set of $u$ (it is immediate to see that the resulting augmented graph is still a bipartite fan). Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of fan $G$ in the counterclockwise order they have on its external face, (see Figure 3(a)), in what follows we denote as $r_i$ points where vertices $v_i$ $(0 \leq i \leq n - 2)$ are placed and we adopt the following notation for the faces of $G$: $f_0$ is the external face; $f_{j+1}$ with $0 \leq j \leq \frac{n-4}{2}$ is the internal face containing the apex $u$ and the fan triplet $v_{2j}, v_{2j+1}, v_{2j+2}$.

First, we prove that a bipartite fan can be drawn inside a wedge $W(p, q, r)$ in such a way that (see Figure 4 for an example):

- condition (ii) is satisfied;
- the apex $u$ is drawn on the point $q$;
- the first edge of the fan is represented by the segment $qr$;
- points $p, q, r_{n-2}$ define a wedge $W(p, q, r_{n-2})$ entirely contained in $W(p, q, r)$. 
The proof is by induction on the number $n_{fi}$ of the internal faces of $G$.

**Base Case:** $n_{fi} = 1$. $G$ is a 4-cycle and it is sufficient to show how its internal face $f_1$ must be drawn. The apex $u$ is drawn at point $q$ and the vertex $v_0$ is drawn at point $r_0$ coincident with $r$, as depicted in Figure 5(a). Thus, the first edge of $G$ is represented by the segment $qr_0$. Let $x_1$ be the point where segment $qr_0$ crosses $\lambda_i$. Since $\lambda_i$ is convex, there exists a straight line through $r_0$ and tangent to $\lambda_i$ at a point $t_1$ such that $t_1$ follows $q$ and precedes $x_1$ on $\lambda_i$; since $pq$ does not cross $\lambda_i$, then $t_1$ is inside wedge $W(p, q, r)$. Choose a point $q_1$ that follows $q$ and precedes $t_1$ on $\lambda_i$, and let $q_1'$ be the point on $\lambda_i$ that follows $q_1$ and where the tangent to $\lambda_i$ through $r_2$ intersects $\lambda_i$; note that $q_1'$ may not exist. Draw vertex $v_1$ at any point $r_1$ of $\lambda_i$ that follows $q_1$ and precedes $t_1$ and $t_1'$ when it exists.

Segment $r_1r_0$ crosses $\lambda_i$ because $r_1$ precedes $t_1$ on $\lambda_i$ and $\lambda_i$ is convex. Let $t_1'$ be the point of $\lambda_i$ that precedes $q_1$ and where the tangent to $\lambda_i$ through $r_2$ intersects $\lambda_i$; $t_1''$ follows $q$ and precedes $r_1$; since $t_1'$ follows $r_1$ the segment $r_2r_1'$ does not cross $\lambda_i$.

Therefore points $r_0, r_1, r_2$, which represent the vertices of the fan triplet $v_0, v_1, v_2$, define a wedge $W(r_2, r_1, r_0)$. Also, segment $r_0r_2$ crosses $\lambda_i$, and therefore points $p, q, r_2$ define a wedge $W(p, q, r_2)$, as shown in Figure 5(b).

**Inductive Case:** $n_{fi} = k + 1$. Suppose that the internal faces $f_1, f_2, \ldots, f_k$ have already been drawn, and let $v_{2k-2}, v_{2k-1}, v_{2k}$ be the fan triplet of $f_k$. By the inductive hypothesis vertex $v_{2k}$ is drawn at point $r_{2k}$, the segment $r_{2k}r_{2k-1}$ crosses $\lambda_i$ and points $p, q, r_{2k}$ define a wedge $W(p, q, r_{2k})$ (see Figure 6(a) for an example).

In order to draw face $f_{k+1}$, we only need to draw vertices $v_{2k+1}, v_{2k+2}$ be-
Figure 5: Illustration of the first step of the proof of Lemma 1.
Figure 6: Illustration of the inductive step of the proof of Lemma 1.
cause the apex \( u \) and vertex \( v_{2k} \) have already been drawn. It can be done by applying the same technique that we used to draw \( v_1 \) and \( v_2 \) inside wedge \( W(p, q, r) \) in the base case of the inductive proof (see also Figure 6(b)).

To complete the proof, we need to show that the drawing of the fan can be modified in such a way that condition (i) be satisfied. Namely, although the first edge is represented by the segment \( qr \) as required, the last edge of the fan is not represented by the segment \( pq \). However, this can be accomplished by changing the drawing of the last internal face \( f_{n-1} \) as follows (see Figure 7 for an illustration).

![Diagram](a)

![Diagram](b)

Figure 7: Redrawing technique of the last internal face.
Let \( t' \) be the point on \( \lambda_i \) that follows \( q \) and where the tangent to \( \lambda_i \) through \( p \) intersects \( \lambda_i \). Move vertex \( v_{n-2} \) to point \( p \) and move vertex \( v_{n-3} \) to any point \( q' \) that follows \( q \) and precedes both \( t' \) and \( r_{n-3} \). This choice ensures that segment \( \overline{pq} \) does not cross \( \lambda_i \). Instead, since \( q' \) precedes \( r_{n-3} \) it precedes \( t_{n-1} \), thus segment \( \overline{q'r_{n-3}} \) crosses \( \lambda_i \). Hence, condition (i) is satisfied and condition (ii) is preserved.

Note that, moving vertices \( v_{n-2}, v_{n-3} \) as explained above does not introduce crossings with other elements of the drawing, because vertices \( v_{n-2} \) and \( v_{n-3} \) have been moved inside the wedge \( W(p, q, r_{n-4}) \), while all the other vertices (and edges) are drawn in the plane region delimited by the segment \( \overline{q'r_{n-4}} \), the segment \( \overline{fr} \) and the portion of \( \lambda_n \) from point \( r_{n-4} \) to \( r \).

Concerning the time complexity, it is immediate to see that the augmentation that makes \( G \) maximal can be performed by visiting each vertex at most once. Also, computing the intersection and the tangent to the circle requires \( O(1) \) time in the real RAM model of computation (note that this may not be the case for other types of curves).

Finally, the drawing algorithm computes the coordinates for the four vertices of each face of a fan by solving at most four quadratic equations. \( \Box \)

### 4 Curve Biplanar Graphs

We start with a sufficient condition whose proof uses the following definition. Let \( G = (V, E) \) be a connected graph. A subset of vertices \( S \subset V \) is a cut-set if the removal of \( S \) disconnects \( G \). Let \( G_0, \ldots, G_{k-1} \) be the connected components of \( G - S \) (possibly isolated vertices). The \( S \)-components of \( G \) are the subgraphs of \( G \) induced by sets \( V(G_j) \cup S \) (\( 0 \leq j \leq k - 1 \)), where \( V(G_j) \) denotes the set of vertices of \( G \).

**Lemma 2** Let \( G \) be a biconnected bipartite graph with a given planar embedding such that all vertices in one partite set belong to the external face. Then \( G \) is curve biplanar on two paired curves.

**Proof:** We describe now how to compute an embedding preserving curve biplanar drawing of \( G \). To this aim, we decompose it into subgraphs that are bipartite fans and draw each fan by using the technique described in Lemma 1.

Let \( V_0 \) and \( V_1 \) be the two partite sets of \( G \) and assume that all vertices of \( V_0 \) belong to the external face of \( G \) in the given embedding. Since \( G \) is bipartite, there exists a vertex \( u \in V_1 \) such that \( u \) belongs to the external face of \( G \). Let \( F_u \) be the subgraph of \( G \) induced by all vertices that share an internal face with \( u \) (\( F_u \) exists because \( G \) is biconnected). Note that \( F_u \) is a bipartite fan, which we call the fan of \( u \). Let \( \lambda_e \) and \( \lambda_i \) be two paired curves, let \( W(p, q, r) \) be an arbitrarily chosen wedge of \( \lambda_e, \lambda_i \) (such a wedge always exists because \( \lambda_e \) and \( \lambda_i \) are paired). By Lemma 1, \( F_u \) can be drawn inside \( W(p, q, r) \) so that its first and last edge are represented by segments \( \overline{fr} \) and \( \overline{pq} \), respectively.

Let \( u, v_0, v_1, \ldots, v_{n-2} \) be the vertices of \( F_u \) in the counterclockwise order they have on the external face of \( F_u \). Since \( u \in V_1 \), vertices \( v_{2j} \) (\( j = 0, 1, \ldots, \frac{n-2}{2} \))
belong to $V_0$ and are on the external face of $G$. This implies that every fan triplet $\tau = \{v_{2j}, v_{2j+1}, v_{2j+2}\}$ is a cut-set for $G$ unless $v_{2j+1}$ is on the external face. Indeed, since $G$ is biconnected, the boundary of its external face is a cycle $C$, and if $v_{2j+1} \in \tau$ is not on the external face, then the vertices of $\tau$ induce a path that splits $C$ in two paths $C'$ and $C''$; since no edge of $G$ can connect a vertex of $C'$ to a vertex of $C''$ (otherwise $G$ would not be planar), then $\tau$ is a cut-set for $G$ (see Figure 8 for an illustration).

The $\tau$-component of $G$ that does not contain $u$ is a planar bipartite graph that satisfies the condition expressed by the statement. Also, by Lemma 1, points $r_{2j}, r_{2j+1}, r_{2j+2}$ representing the vertices of $\tau$ in the drawing of $F_u$ define a wedge $W(r_{2j}, r_{2j+1}, r_{2j+2})$. Therefore, the $\tau$-component of $G$ that does not contain $u$ can be recursively drawn inside $W(r_{2j}, r_{2j+1}, r_{2j+2})$. \qed

The previous sufficient condition can be extended also to non-biconnected graphs by using the augmentation technique described in the following lemma.

**Lemma 3** Let $G$ be a non-biconnected bipartite graph and let $V_0$ be one of the partite sets of $G$. If $G$ has a planar embedding with all vertices of $V_0$ on the external face, then $G$ can be augmented with dummy vertices and edges so that the resulting graph $G'$ is biconnected, bipartite, has a partite set $V_0'$ such that $V_0 \subseteq V_0'$, and has a planar embedding with all vertices of $V_0'$ on the external face.

**Proof:** Consider the embedding of $G$ such that all vertices of $V_0$ belong to the external face and let $v$ be a cut-vertex of $G$. If $v \in V_0$, then it is on the external face. If $v \in V_1$, all its adjacent vertices are on the external face. It follows that $v$ belongs to the external face also in this case.
Suppose that \( v \) belongs to \( h \) biconnected components. In the cyclic ordering of the edges incident on \( v \) there are \( h \) consecutive pairs that belong to different biconnected components. Consider one such pair \((v, u)\) and \((v, w)\). We augment \( G \) by adding a path \( u, x, w \) for \( h - 1 \) of these pairs, where \( x \) is a dummy vertex. Since \( G \) is bipartite, then \( v \) belongs to one partite set and \( u, w \) belong to the other one. To maintain the bipartiteness \( x \) is assigned to the same partite set containing \( v \). All dummy vertices are added on the external face and each vertex that was on the external face before adding the paths remains on the external face. By repeating this augmentation until there are no more cut-vertices, we obtain a biconnected bipartite graph \( G' \) that satisfies the statement.

**Theorem 2** A bipartite graph \( G \) is curve biplanar on two paired curves if and only if it admits a planar embedding such that all vertices in one partite set belong to the external face. Also, if \( G \) is curve biplanar on two concentric circles, a curve biplanar drawing of \( G \) can be computed in \( O(n) \) time in the real RAM model of computation, where \( n \) is the number of vertices of \( G \).

**Proof:** We first prove the sufficiency. Let \( V_0 \) be a partite set of \( G \) such that all vertices of \( V_0 \) belong to the external face of a planar embedding of \( G \). If \( G \) is biconnected, the sufficiency follows from Lemma 2. Otherwise, as shown by Lemma 3, \( G \) can be augmented by adding dummy vertices and edges such that the augmented graph \( G' \) is biconnected and bipartite, one of its partite sets is \( V'_0 \) with \( V_0 \subseteq V'_0 \), and has a planar embedding with the vertices of \( V'_0 \) on the external face. It follows that \( G' \) has a biplanar drawing on two paired curves by Lemma 2 and hence \( G \) is curve biplanar on two paired curves.

We now prove the necessity. Let \( \Gamma \) be a curve biplanar drawing of a graph \( G \) on two paired curves \( \lambda_e \) and \( \lambda_i \). All vertices drawn as points of \( \lambda_e \) are on the external face of \( \Gamma \) because the curves are convex and the drawing is straight-line. Since all vertices on the same curve are in the same partite set, \( G \) admits a planar embedding such that all vertices in one partite set belong to the external face.

Time complexity. The augmentation technique can be performed in \( O(n) \) time by visiting the vertices on the external face. The fan \( F_u \) of \( u \) can be computed in time proportional to the number \( n_u \) of vertices in \( F_u \) by visiting all faces containing \( u \). As stated in Lemma 3, drawing each fan \( F_u \) requires \( O(n_u) \) time when the curves are two concentric circles and the real RAM model of computation is adopted. Moreover, since \( \sum_{F_u} O(n_u) = O(n) \) it follows that the overall time complexity is \( O(n) \).

**Theorem 3** Let \( G \) be a bipartite planar graph with \( n \) vertices. The curve biplanarity of \( G \) on two paired curves can be tested in \( O(n) \) time.

**Proof:** A curve biplanarity test can be performed by executing at most two planarity tests with the additional constraint that all vertices in one of the partite sets of \( G \) belong to the external face.

We conclude this section by describing how the result of Theorem 2 is related to radial planarity. A \( k \)-partite graph \( G = (V_0, \ldots, V_{k-1}, E) \) is radial \( k \)-level
planar if it admits a planar drawing on \( k \) concentric circles \( C_0, C_1, \ldots, C_{k-1} \), with the vertices of partition \( V_i \) drawn on circle \( C_i \) \((0 \leq i \leq k - 1)\) and the edges drawn as strictly monotone curves from inner to outer circles. Radial planarity has been studied by Bachmaier et al. [1] who present a linear time algorithm for radial planarity testing and embedding. Based on the fact that two concentric circles are two paired curves and that straight-line segments with the end-vertices on two different circles are a special case of strictly monotone curves from inner to outer circles, the following results immediately follows from Theorems 2 and 3.

**Corollary 1** A bipartite graph \( G \) is radial 2-level planar with straight-line edges if and only if it admits an embedding such that all vertices in one partite set belong to the external face. Also, there exists an \( O(n) \)-time algorithm that tests whether a bipartite graph \( G \) is radial 2-level planar with straight-line edges.

## 5 Open Problems

In this paper we have studied the curve biplanar drawability problem of bipartite graphs. It may be considered a generalization of the well known biplanar drawability problem, when parallel layers are allowed to be curves.

We have shown that, if the curves are non-paired, the families of graph that admits a curve biplanar drawing is the forest of caterpillars, e.g. the same family that characterizes the biplanar graphs [5, 8, 11]. Instead, if the curves are paired, we have proved that a bipartite graph is curve biplanar if and only if it admits a planar embedding with all vertices of one partite set on the external face. We have provided also a drawing algorithm that runs in linear time if the curves are two concentric circles and the real RAM model of computation is adopted. Moreover, we have proven that the curve biplanarity can be tested in linear time.

We conclude by listing some open problems that in our opinion should be investigated.

- Extend the study to \( k \)-partite graphs and \( k \) parallel curves with \( k > 2 \). In particular, it would be interesting to study radial planarity testing with straight-line edges and more than two concentric circles.

- Study the complexity of the following problem: Let \( G \) be a planar bipartite graph and let \( c \) be a positive integer. Does \( G \) have a curve biplanar subgraph (not necessarily induced) with at least \( c \) edges?

- Study the complexity of the edge crossing minimization problem for straight-line drawings of bipartite graphs on two parallel convex curves.

- Investigate what happens when non-convex curves are considered: does the class of curve biplanar (bipartite) graphs change? If yes, how does it depend on the profile of the curves?
• The presented algorithm may require very large resolution, expressed as the ratio between the longest edge and the shortest edge. What is a lower bound for the resolution? How does it depend on the distance between the two curves?

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