Spontaneous emission of a moving atom in a waveguide of rectangular cross section

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We study the spontaneous emission (SE) of an excited two-level nonrelativistic system (TLS) interacting with the vacuum in a waveguide of rectangular cross section. All TLS’s transitions and the center-of-mass motion of the TLS are taken into account. The SE rate and the carried frequency of the emitted photon for the TLS initial being at rest is obtained, it is found in the first order of the center of mass (c.m.) that the frequency of the emitted photon could be smaller or larger than the transition frequency of the TLS but the SE rate is smaller than the SE rate \( \Gamma_f \) of the TLS fixed in the same waveguide. The SE rate and the carried frequency of the emitted photon for the TLS initial being moving is also obtained in the first order of the c.m.. The SE rate is larger than \( \Gamma_f \) but it is independent of the initial momentum. The carried frequency of the emitted photon is creased when it travels along the direction of the initial momentum and is decreased when it travels in the opposite direction of the initial momentum.

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I. INTRODUCTION

Photons are desirable for distributing information and transferring entanglement in quantum networks. With the demanding to build a device in a quantum network for controlling single photons, single quantum emitters (Hereafter, we will often use the word “atom” instead of “quantum emitter.”) would be desirable since there is no direct interaction among photons. An two-level system (TLS) act as a quantum switch in a one dimension (1D) waveguide with linear dispersion relation \( [1] \) or a 1D coupled-resonator waveguide (CRW) \( [2] \); A V-type or ladder-type atom functions as the frequency converter for single photons \( [3] \) in a 1D waveguide; A cyclic or a Λ-type atom works as a multichannel quantum router \( [4] \). Atoms are widely proposed to act as quantum nodes in extended communication networks and scalable computational devices \( [5,17] \). Photons propagating along the network, are confined in a 1D waveguide. With the mode volume decreased, the coupling strength of atoms to the waveguide is enhanced. The atomic spontaneous emission (SE) has played an important role on controlling photon transport in quantum network.

SE is a result of the electromagnetic interaction between an atom and a quantum field. It is a process of the following: a system initially in an excited state relaxed to its lower state and emitted a quanta of energy to its surrounding vacuum field, which carries away the difference in energy between the two levels. Different from atoms in a cavity or a free space, atoms used to control single photons in quantum networks interact with its surrounding electromagnetic (EM) field which is confined in a 1D waveguide. SE has been widely studied in free space, a semi-infinite or infinite 1D waveguide, however, most works focus on 1D waveguide without a cross section, and the atom under study is usually assumed to be stationary. A stationary atom possesses an undetermined kinetic energy according to the Heisenberg’s uncertainty relation, and thus, the stationary-atom model is too simple. In this paper, we study the spontaneous radiative decay of an atom moving in an infinite waveguide of rectangular cross section. We analyze the interaction of an initially excited TLS with the waveguide in vacuum. When an initially excited TLS spontaneously emits a photon into the waveguide, the atomic center-of-mass (c.m.) experiences a change in momentum space. The dynamical behavior of the TLS is studied by taking the quantization of TLS’s momentum and position into account.

This paper is organized as follows. In Sec. II we introduce the model and establish the notation. In Sec. III we derive the relevant equations describing the dynamics of the system for the case of the TLS being initially excited and the waveguide mode in the vacuum state. In Sec. IV the spontaneous emission rate has been presented for TLS initially at rest and moving respectively. We make a conclusion in Sec. V.

II. MODEL SETUP

The system we studied is shown in Fig.I. A waveguide made of ideal perfect conducting walls is formed from surfaces at \( x = 0, \ x = a, \ y = 0, \ y = b, \) and is placed along the \( z \) axis. There are two types of modes for the field in the waveguide: TE modes whose electric field has no longitudinal component, and TM modes whose magnetic field has no longitudinal component. Let \( \vec{k} = (k_x, k_y, k) \) be the wave vector. The relations \( k_x = m \pi/a \) and \( k_y = n \pi/a \) with positive integers \( n, m \) can be imposed by the condition that the tangential components of the electric field vanish at all the conducting wall, however, there is no constraint on \( k \). Therefore, the waveguide al-
The interaction between the TLS and field via the dipole TM guiding mode by waveguide, we label the annihilation operator for each specifies the mode function of this air-filled metal pipe where \(\omega_{mnk} = \sqrt{c^2k^2 + \Omega_{mn}^2}\), (1)

where \(c\) is the speed of light in vacuum, the cutoff frequency for a traveling wave \(\Omega_{mn} = \pi c \sqrt{m^2/a^2 + n^2/b^2}\). We note that \(m\) and \(n\) cannot both be zero. If \(a > b\), TE\(_{10}\) is the lowest guiding mode for the waveguide [20], and the lowest TM modes occur for \(m = 1, n = 1\). Obviously, the waveguide modes form a one-dimensional continuum. Each guiding mode provides a quantum channel for photons to travel from one location to the other.

An TLS with mass \(M\) and the transition frequency \(\omega_A\) is inside the waveguide. It has two internal states: the upper level \(|e\rangle\) and the lower level \(|g\rangle\), which introduce the the rising (lowering) atomic operator \(\hat{\sigma}_+ \equiv |e\rangle \langle g|\) \((\hat{\sigma}_- \equiv |g\rangle \langle e|)\). We assume that the atom moves only along the \(z\) axis, which means the atom has unchanged position along the \(xy\) plane. The atomic momentum and position are characterized by the operators \(\hat{p}_z\) and \(\hat{r} = (x_0, y_0, \bar{z})\) respectively, where \(x_0\) and \(y_0\) are constant. The momentum and position of the atomic c.m. obey canonical commutation relations. The free Hamiltonian for the TLS is described by

\[
H_s = \frac{\hat{p}_z^2}{2M} + \omega_A \hat{\sigma}_+ \hat{\sigma}_-. \tag{2}
\]

We consider the atomic electric dipole is oriented along the \(z\) direction, which means that the TLS only interacts with the TM\(_{mn}\) modes. Since the number \((m, n, k)\) specifies the mode function of this air-filled metal pipe waveguide, we label the annihilation operator for each TM guiding mode by \(a_{mnk}\). The free Hamiltonian for the waveguide is described by

\[
H_f = \sum_{mnk} \int_{-\infty}^{\infty} dk \omega_{mnk} \hat{a}_{mnk} \hat{a}_{mnk}^\dagger. \tag{3}
\]

The interaction between the TLS and field via the dipole coupling in the rotating-wave approximation reads

\[
H_I = \sum_{mnk} \int_{-\infty}^{\infty} dk g_{mnk}(\hat{\sigma}_-^\dagger \hat{a}_{mnk} e^{-ikz} - \hat{\sigma}_+ \hat{a}_{mnk} e^{ikz}) \tag{4}
\]

where the coupling strength

\[
g_{mnk} = \frac{id\Omega_{mn}}{\sqrt{\pi A_0\omega_{mn}}} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b} \tag{5}
\]

Here, \(\epsilon_0\) the permittivity of free space, \(d\) the magnitude of the transition dipole moment of the TLS and assumed to be real, \(A = \sqrt{ab}\) the area of the rectangular cross section.

The TLS located at \(x_0 = a/2\) and \(y_0 = b/2\) decouples to the TM\(_{mn}\) guiding mode with even integer \(m\) or \(n\). The total system are described by Hamiltonian \(H = H_s + H_f + H_I\).

### III. Dynamic of the Moving Atom

Let us introduce the operators of the excited number and the atom-field momentum

\[
N_e = \hat{\sigma}_+ \hat{\sigma}_- + \sum_{mn} \int_{-\infty}^{\infty} dk \hat{a}_{mnk}^\dagger \hat{a}_{mnk}, \tag{6a}
\]

\[
N_p = \hat{p}_z + \sum_{mn} \int_{-\infty}^{\infty} dk h \hat{k} \hat{a}_{mnk}^\dagger \hat{a}_{mnk}, \tag{6b}
\]

respectively. It can be found that both \(N_e\) and \(N_p\) commute with the total Hamiltonian of the system, thus, the excited number and atom-field momentum are conservative quantities. It is well-known that an stationary TLS being excited initially will evolves into a superposition of itself and the states in which the TLS is unexcited and has released a photon into the field, however, when the quantization of the atomic momentum and position are taking into account, the conservation of momentum indicates that the c.m. motion of the atom is changed by the recoil caused by the emission of photons.

The initial wave function of the system we consider is in a product state \(|\psi(0)\rangle = |e, p_z, 0\rangle\), where \(p_z\) is an eigenstate of the momentum operator \(\hat{p}_z\) corresponding to the eigenvalue \(p_z\) and the field is in the vacuum state. The time evolution of the state \(|\psi_t\rangle\) is described by the Schrödinger equation. Via the Fourier transformation, the Green operator \(\hat{G} = (q - \hat{H})^{-1}\) relates the state \(|\psi_t\rangle\) to the initial state \(|\psi_0\rangle\), which yields

\[
|\psi_t\rangle = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} dq e^{-iqt} \hat{G}(q) |\psi_0\rangle. \tag{7}
\]

The probability for finding the atom in its excited state equals to the probability for finding the TLS in its initial state \(|\psi_0\rangle\), which is denoted by \(P_A = |\langle \psi_0 | \psi_t |^2\). Since the atom-field coupling energy \(H_f\) is small compared with the free energy \(H_0 = H_s + H_f\), the Green operator \(\hat{G}\) can be expanded into a series of ascending powers of \(H_f\). By defining the free Green operator \(\hat{G}_0 = (q - H_0)^{-1}\), the probability \(P_A\) is rewritten as

\[
P_A(t) = \left| \left\langle \psi_0 | \hat{G}_0 \sum_{l=0}^{+\infty} (H_f \hat{G}_0)^l |\psi_0\rangle \right|^2. \tag{8}
\]
Since the photon’s emission transform the external state of the TLS from \( |p_z\rangle \) to \( |p_z - h\omega\rangle \), we observe that

\[
\langle \psi_0 | H_I \hat{G}_0 | \psi_0 \rangle = 0,
\]

(9a)

\[
\langle \psi_0 | H_I \hat{G}_0 | \psi_0 \rangle = \left( \langle \psi_0 | H_I \hat{G}_0 H_I \hat{G}_0 | \psi_0 \rangle \right)_{\text{I}}
\]

(9b)

Then the amplitude for finding the TLS in its initial state to the coupling to field. If the coupling strength

The principal part is merely included into redefinition of Eq. (13) in terms of the frequency

\[
A(t) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \frac{e^{-iqt}}{q - p_z^2/(2M) - \omega_A - B},
\]

(10)

where the quantity \( B \) is defined as

\[
B = \sum_{mn} \int_{-\infty}^{\infty} dk \frac{|g_{mnk}|^2}{q - (p_z - h\omega)^2/(2M) - \omega_{mnk}}
\]

(11)

Eq. (10) can be solved by the residue theorem, but the solution of the equation

\[
0 = q - p_z^2/(2M) - \omega_A - B
\]

has to be found. Since Eq. (12) is a transcendental equation involving integral, the approximations are used to give an analytic discussion. In this following discussion, for the sake of simplification, we denote the transversely confined propagating modes which couple to the atom as TM\(_j\) with \( j = (m, n) \) according to the ascending order of the cutoff frequencies.

## IV. SPONTANEOUS EMISSION

The roots of the denominator in Eq. (12) can be split into a sum of the singular and principal value parts. The principal part is merely included into redefinition of the energy. The singular part gives the decay rate due to the coupling to field. If the coupling strength \( g_{mnk} \) is small, the solution of Eq. (12) can be expanded into a series of ascending powers of \( g_{jk} \). Under the second-order approximation about the weak coupling, the atomic decay is dominantly exponential with rate

\[
\Gamma = 2\pi \sum_{j} \int dk |g_{jk}|^2 \delta \left( \omega_A + \frac{p_z^2}{2M} - \omega_{jk} - \frac{(p_z - k)^2}{2M} \right)
\]

(13)

The delta function expresses the conservation of energy before and after the emission. We express the integral in Eq. (13) in terms of the frequency

\[
\Gamma = \sum_{j} \int_{\Omega_j} dw \frac{4\pi \omega |g_{jw}|^2}{c\sqrt{\omega^2 - \Omega_j^2}} \delta \left( \omega_A - \omega_{jk} + \frac{p_z k}{M} - \frac{k^2}{2M} \right)
\]

(14)

by using the dispersion relation, where the coupling strength

\[
g_{jw} = \frac{id\Omega_{mn}}{\sqrt{\pi A\epsilon_0 \omega}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

(15)

### A. the TLS being initially at rest

We now assume that the TLS is initially at rest, i.e., \( p_z = 0 \). The spontaneous decay rate becomes

\[
\Gamma_R = \sum_{j} \int_{\Omega_j} dw \frac{4\pi \omega |g_{jw}|^2}{c\sqrt{\omega^2 - \Omega_j^2}} \delta \left( \omega_A - \omega - \frac{\omega^2 - \Omega_j^2}{2Mc^2} \right)
\]

(16)

From the delta function, the frequency that the emitted photon carried is obtained

\[
\omega_R = \sqrt{\left(Mc^2 \right)^2 + 2Mc^2 \omega_A + \Omega_j^2} - Mc^2,
\]

(17)

After some algebra, we obtain the spontaneous rate for a TLS initially at rest

\[
\Gamma_R = \sum_{mn} 2d^2 \Omega_{mn}^2 \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \frac{A\epsilon_0 \omega_A - \Omega_j^2}{\sqrt{\omega_A^2 - \Omega_{mn}^2}} \sqrt{\left(Mc^2 \right)^2 + 2Mc^2 \omega_A + \Omega_{mn}^2} + 2Mc^2
\]

(18)

It can be found that the modal profile affects on the decay rate via location of the TLS. If the atom is located at \( x_0 = a/2 \) and \( y_0 = b/2 \), no photons are radiated into the TM\(_{mn}\) guiding mode with even integer \( m \) or \( n \) since the guiding mode are standing waves in the transverse direction. As the initial atomic energy \( \omega_A \) approaches one of the cutoff frequencies, the TLS loses its energy very quickly. The more transverse modes interact with the TLS, the faster the TLS decays. The mass also contributes to the spontaneous emission rate due to the recoil of the TLS.

To comparing with the TLS fixed inside a waveguide with a rectangular cross section, we consider the mass is larger and keep the first order of \( M^{-1} \). Then the spontaneous rate reduces to

\[
\Gamma_R = \left( 1 - \frac{3\omega_A}{4Mc^2} \right) \sum_{mn} \Gamma_{mn}^S
\]

(19)

which is smaller than spontaneous rate of the TLS fixed in the vacuum of the waveguide with a rectangular cross section. Here,

\[
\Gamma_{mn}^S = \frac{4d^2 \Omega_{mn}^2}{A\epsilon_0} \frac{\sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b}}{\sqrt{\omega_A^2 - \Omega_{mn}^2}}
\]

(20)

is the decay rate of a stationary TLS due to its interacting with the vacuum of the TM\(_{mn}\) transverse mode. The frequency of the emitted photon approximately reads

\[
\omega_R = \omega_A + \frac{\Omega_{mn}^2 - \omega_A^2}{2Mc^2}
\]

(21)

It shows that the cutoff frequencies around the transition frequency \( \omega_A \) determines whether \( \omega_R \) is smaller or larger than the transition frequency of the TLS. If the mass of the TLS is so large that the recoil can be neglect, we recover the decay rate of the stationary TLS and the frequency \( \omega_A \) of the emitted photon.
B. the TLS initially moving

When the atom is not initially at rest, i.e., \( p_z \neq 0 \), the roots of the following equations

\[
0 = \omega_A - \omega \pm \frac{p_z}{Mc} \sqrt{\omega^2 - \Omega_j^2} - \frac{\omega^2 - \Omega_j^2}{2Mc^2} \quad (22)
\]
determine the frequency of the emitted photon as well as the value of spontaneous rate. We note that Hamiltonian \([2]\) is written in the nonrelativistic region where the motion of center of mass of the TLS is very slower than light, so we apply the perturbation approach to Eq. (22) to find the analytic solutions,

\[
\omega = \omega_0 + \lambda \omega_1 + \cdots , \quad (23)
\]

Here, \( \lambda \) is a continuously varying parameter ranging from zero to unity. \( \omega_0 \) is of the zeroth order in \((Mc)^{-1}\), \( \omega_1 \) is of the first order in \((Mc)^{-1}\). We now substitute Eq. (23) into Eq. (22) and retain only terms up to the first order in \((Mc)^{-1}\). We approximately obtained frequencies of the emitted photon

\[
\omega_+ = \omega_A + \frac{p_z}{Mc} (\omega_A^2 - \Omega_j^2) - \frac{\omega_A^2 - \Omega_j^2}{2Mc^2} \quad (24a)
\]
\[
\omega_- = \omega_A - \frac{p_z}{Mc} (\omega_A^2 - \Omega_j^2) - \frac{\omega_A^2 - \Omega_j^2}{2Mc^2} \quad (24b)
\]

The nonvanishing initial C.M. motion of the TLS splits the frequency of the emitted photon in Eq. (21) into two. For the emitted photon traveling along the same direction of the initial TLS momentum, its carried frequency is increased. For the emitted photon traveling in the opposite direction of the initial TLS momentum, its carried frequency is decreased. The emission of the right-going and left-going photon along the \( z \) axis leads to a loss of TLS’s energy with rate

\[
\Gamma_M = 2\pi \sum_j \frac{\omega_j |g_j\omega_j|^2}{c \sqrt{\omega_j^2 - \Omega_j^2}} + \frac{\omega_j |g_j\omega_j|^2}{c \sqrt{\omega_j^2 - \Omega_j^2}} . \quad (25)
\]

For a moving atom, the mass, momentum and the cutoff frequencies all contribute to the decay rate. As \( \omega_w \) approaches one of the cutoff frequencies, the TLS loses its energy very quickly. The more transverse modes interact with the TLS, the faster the TLS decays. For a much larger mass, the decay rate reads

\[
\Gamma_M = \sum_{mn} \Gamma_{mn} (1 + \frac{\omega_A}{2Mc^2}) \quad (26)
\]
in the first order of \( M^{-1} \), where the contribution of the momentum disappeared. However, the spontaneous emission rate definitely increased an amount.

V. CONCLUSION

We consider a TLS of c.m. interacting with a waveguide of rectangular cross section. The spontaneous emission of an atom in vacuum is studied. Both atom and field variables are fully quantized, so that the atomic recoil on the emission of radiation are automatically included in the calculation. Under the second order approximation of the weak coupling, the SE rate and the frequency of the emitted photon for the atom initial being at rest or moving are obtained. It is shown that the modal profile affects on the decay rate via location of the TLS, the more transverse modes interact with the TLS, the faster the TLS decays. For the TLS initial being at rest, the transition frequency \( \omega_A \) of the TLS determines the speed of the SE rate. As \( \omega_A \) approach any of the cutoff frequencies, the TLS decays fast. In the first order of the c.m., the SE rate is smaller than the SE rate of the TLS fixed in the vacuum of the same waveguide, however, the frequency of the emitted photon could be larger or smaller than the initial energy \( \omega_A \) of the total system. For the TLS initial being moving, the carried frequency of the emitted photon is creased when it travels along the direction of the initial momentum of the TLS and is decreased when it travels in the opposite direction of the initial momentum of the TLS. The SE rate is larger than the SE rate of the TLS fixed in the vacuum of the same waveguide but it is independent of the initial momentum in the first order of the c.m..

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