Relaxation oscillations in an idealized ocean circulation model

Andrew Roberts
Cornell University

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Abstract

This work is motivated by a desire to understand transitions between stable equilibria observed in Stommel’s 1961 thermohaline circulation model. In two adaptations of the model, we include forcing parameters as dynamic slow variables. The resulting models are piecewise-smooth three time-scale systems. The models are analyzed using geometric singular perturbation theory to demonstrate the existence of attracting periodic orbits. However, the theory must be adapted to handle a codimension 1 set where the vector field is not differentiable. The stability of the slow manifold changes as it crosses this set, giving rise to an ‘2’-shaped critical manifold. The main results state that, despite a non-smooth critical manifold, the system exhibit classical relaxation oscillations. This shows that a simple modification to Stommel’s model can account for not only multiple meta-stable equilibria but also the rapid transitions between them.

1 Introduction

An important aspect of the climate system is the variability in the climate record. An understanding of this variability, specifically with regard to glacial millennial climate change, has remained elusive (see [5]). In different eras of Earth’s history, the variations themselves change in both period and amplitude. It is even possible to have small oscillations superimposed over larger oscillations. Increasingly, scientists are utilizing improved technology to study the climate system through high-powered computer simulations. Large scale oscillations and critical transitions, however, are often better understood by examining conceptual models that can be studied analytically. Crucifix reviews key dynamical systems concepts and their applications to paleoclimate problems in [6], mentioning relaxation oscillations in particular.

One of the concepts described in [6] is that of the relaxation oscillator, which requires a separation of time scales. The separation of time scales gives rise to a fast/slow system that can be analyzed using geometric singular perturbation theory (GSP). A relaxation oscillator is characterized by (1) equilibration of the fast system to a stable state followed by (2) a slow de-stabilization of the state. Since relaxation oscillators only require a minimum of two variables (one fast, one slow), much of the full climate system might not be relevant to understanding the mechanism(s) behind a particular oscillation. Rather, it should be possible to consider a relevant underlying subsystem that is analytically tractable.

Over the last 100 kyr the climate record shows a series of Dansgaard-Oeschger (D-O) events. In [5], Cronin describes the temperature change corresponding with D-O events as “characterized by an initial sharp increase over only a few decades or less followed by a gradual decline...and finally a sharp drop.” Figure 1 depicts Dansgaard-Oeschger events during the last glacial period. The description provided by [5] along with the plateaus at bottoms of the cycles seen in Figure 1 are reminiscent of relaxation oscillations.

Since the effects of these critical climate transitions are most dramatic in the North Atlantic, scientists have hypothesized that D-O events are accompanied by changes in ocean circulation in the North Atlantic. The bistability of the circulation in the North Atlantic was first demonstrated by Stommel in 1961 [19]. Physical oceanographers have provided a vast array of models capturing various mechanisms that can cause the circulation to oscillate between the two steady states in Stommel’s model. Dijkstra and Ghil surveyed many of these models in [8]. Some models generate oscillations as a result of intrinsic ocean dynamics [4]. Other models generate oscillations due to changes in freshwater forcing, be it periodic [9] or stochastic [1]. In [18], Saltzman, Sutera, and Evenson argue that thermal effects are the driving force behind the oscillations. Additionally, Saha shows that an ocean-ice feedback mechanism can generate oscillations [16] [17].

The aims of this paper are to prove the existence of attracting periodic orbits in two adaptations
of Stommel’s model. In the first adaptation, we incorporate the ‘freshwater forcing’ parameter—actually a ratio of precipitation forcing to thermal forcing—as a dynamic slow variable. The adapted model is a three time-scale model with three variables (1 fast, 1 intermediate, and 1 slow). We use GSP to reduce the model to a 2D fast/slow system and show that, for a certain parameter range, the reduced system has an ‘2’-shaped fast nullcline (similar to an ‘S’-shaped nullcline, where one of the folds is actually a corner). We then find conditions under which the model has either a relaxation oscillation or a canard cycle.

In the second adaptation of the model, we incorporate the precipitation and thermal forcing terms as separate dynamic variables. The model is a three time-scale model with four variables (1 fast, 1 intermediate, and 2 slow). Again, we use GSP to reduce the model by one dimension. The reduced model is a fast/slow system with two slow variables, and the fast nullcline has a bistable, ‘2’-shaped region. The nullcline also has a cusp point where the boundaries of both stable regions meet. The ‘2’-shape of the nullcline allows for a periodic orbit with two relaxation phases, and we find conditions under which the model has an attracting orbit of this type. The cusp theoretically allows for the possibility of a periodic orbit with only one relaxation phase, however we show that this is impossible for physically relevant parameters.

The relevant background material from GSP theory is discussed in section 2. Section 3 describes the physical mechanisms in Stommel’s 1961 model. We also analyze the model using geometric singular perturbation theory, following the analysis of Glendinning [10]. In Section 4, we develop and analyze the first adapted model. Section 5 focuses on the second adapted model. Finally, the paper concludes with further discussion in Section 6.
2 Fast/slow dynamics

A fast/slow system is a system of the form

\[
\begin{align*}
  x' &= f(x, y, \varepsilon) \\
  y' &= \varepsilon g(x, y, \varepsilon),
\end{align*}
\]

where the prime \( ' \) denotes \( d/dt \). Here, \( x \in \mathbb{R}^n \) is the fast variable and \( y \in \mathbb{R}^m \) is the slow variable. Typically, \( f \) and \( g \) are assumed to be smooth functions. If time is rescaled by \( \varepsilon \neq 0 \), i.e. \( \tau = \varepsilon t \), then (1) becomes

\[
\begin{align*}
  \varepsilon \dot{x} &= f(x, y, \varepsilon) \\
  \dot{y} &= g(x, y, \varepsilon),
\end{align*}
\]

where the dot \( \cdot \) denotes \( d/d\tau \). The fast system (1) and the slow system (2) are equivalent as long as \( \varepsilon \neq 0 \). However, insight can be gained by looking at the \( \varepsilon = 0 \) limit. Indeed, the reduction in phase space dimensions from considering the limit may turn an analytically intractable problem into a tractable one.

As \( \varepsilon \to 0 \), (1) approaches

\[
\begin{align*}
  x' &= f(x, y, 0) \\
  y' &= 0,
\end{align*}
\]

which is called the layer problem. The slow system (2) approaches

\[
\begin{align*}
  0 &= f(x, y, 0) \\
  \dot{y} &= g(x, y, 0),
\end{align*}
\]

which is called the reduced problem. In both cases, the set \( M_0 = \{ f(x, y, 0) = 0 \} \)—called the critical manifold—is special. \( M_0 \) is the set of critical points of (3) and the algebraic set on which (4) is defined. One can obtain a caricature of (1) or (2) by allowing the layer problem to equilibrate and then considering the reduced problem. The theory of GSP describes the manner in which the two subsystems are pieced together provided

\[
\det \left( \frac{\partial f}{\partial x} \bigg|_{M_0} \right) \neq 0.
\]

A survey of the basic theory of GSP can be found in [11].

2.1 Relaxation Oscillations and Canards

Since \( M_0 \) is a set of critical points, we can discuss its stability in a natural way. The condition (5) is a non-degeneracy condition that says \( M_0 \) is hyperbolic with respect to the fast dynamics. \( M_0 \) is attracting (resp. repelling) where its points are attracting (resp. repelling) critical points of the layer problem. The interesting and scientifically relevant mathematics often occur due to degenerate points, and that is the case in the models we will examine in this paper.

Perhaps the most common type of degeneracy is a folded critical manifold. A fold is a codimension one set where \( M_0 \) is attracting on one side and repelling on the other. If \( M_0 \) has two folds, we say it is ‘S’-shaped. In planar systems, ‘S’-shaped critical manifolds can produce relaxation oscillations; in higher dimensional systems, even more complicated oscillatory behavior is possible. The relaxation oscillations result from a Hopf bifurcation that occurs as the slow nullcline passes through a fold point. Initially, the periodic orbits born of the bifurcation are small amplitude orbits called canard cycles. Over an exponentially small parameter range, the canard cycles will grow into
large relaxation oscillations. This rapid transition from small orbits to relaxation oscillations is a phenomenon known as a canard explosion [13].

The models we analyze in this paper do not have smooth vector fields due to an absolute value term. The set where the vector field is not differentiable is called the splitting line. As we will see in our models, it is possible for a ‘non-smooth fold point’ of the critical manifold to occur when it intersects the splitting line. In [15], Roberts and Glendinning describe the possible oscillatory behavior in a fast/slow system with a piecewise-defined ‘2’-shaped critical manifold. That is, they consider a system of the form

\[
\begin{align*}
\dot{x} &= -y + F(x) \\
\dot{y} &= \varepsilon(x - \lambda)
\end{align*}
\]  

(6)

where

\[ F(x) = \begin{cases} 
  g(x) & x \leq 0 \\
  h(x) & x \geq 0
\end{cases} \]

with \( g, h \in C^k \), \( k \geq 1 \), \( g(0) = h(0) = 0 \), \( g'(0) < 0 \) and \( h'(0) > 0 \), and assume that \( h \) has a maximum at \( x_M > 0 \). The critical manifold

\[ \mathcal{N}_0 = \{ y = F(x) \} \]

is ‘2’-shaped with a smooth fold at \( x_M \) and a corner along the splitting line \( x = 0 \).

The model analyzed in Section 4 takes a similar form, so we will utilize the theory from [15]. The first theorem provides conditions under which a system of the form (6) exhibits canard behavior at the smooth fold.

**Theorem 1.** Fix \( 0 < \varepsilon \ll 1 \). In system (6), assume \( g(0) = 0 = h(0), h'(0) > 0 \), and \( g'(0) < 0 \). Then there is a Hopf bifurcation when \( \lambda = x_M \). If the Hopf bifurcation is non-degenerate, then it will produce canard cycles.

The second theorem from [15] will be more useful. It describes the bifurcation as the \( y \) nullcline passes through the splitting line.

**Theorem 2.** In system (6), assume \( g(0) = 0 = h(0), h'(0) > 0 \), and \( g'(0) < 0 \). The system undergoes a bifurcation for \( \lambda = 0 \) by which a stable periodic orbit \( \Gamma^n(\lambda) \) exists for \( 0 < \lambda < x_M \). There exists an \( \varepsilon_0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) the nature of the bifurcation is described by the following:

1. If \( 0 < h'(0) < 2\sqrt{\varepsilon} \), then canard cycles \( \Gamma^n(\lambda) \) are born of a Hopf-like bifurcation as \( \lambda \) increases through \( 0 \). The bifurcation is subcritical if \( |g'(0)| < |h'(0)| \) and supercritical if \( |g'(0)| > |h'(0)| \).

2. If \( h'(0) > 2\sqrt{\varepsilon} \), the bifurcation at \( \lambda = 0 \) is a super-explosion. The system has a stable periodic orbit \( \Gamma^n(\lambda) \), and \( \Gamma^n(\lambda) \) is a relaxation oscillation. If \( |g'(0)| \geq 2\sqrt{\varepsilon} \), the bifurcation is supercritical in that no periodic orbits appear for \( \lambda < 0 \). However, if \( |g'(0)| < 2\sqrt{\varepsilon} \) the bifurcation is subcritical, in that a stable periodic and stable critical point coexist simultaneously for some \( \lambda < 0 \).

The remainder of the paper will be devoted to developing and analyzing fast/slow systems modeling large-scale ocean circulation.

### 3 Stommel’s Model

Investigating changes in ocean circulation begins with Stommel’s 2-box model. Stommel modeled the North Atlantic by partitioning it into an equatorial and a polar region. He assumed that water
near the equator would become warmer and saltier due to its interaction with the atmosphere. Water near the pole would lose its heat to the atmosphere and have its salt concentration diluted by incoming freshwater. Oceanic circulation causes the water in the two regions to mix, preventing either box from equilibrating with its surrounding environment. The density difference between the boxes drives the circulation, and it is the salinity and temperature of each box that determines the density. The analysis in this section follows [10] and [12]. It is included here as a reminder of the mechanics of Stommel’s model and a means of setting up the equations for the main results of this paper.

The equations describing the model are

\[
\begin{align*}
\frac{dT_e}{dt} &= R_T(T^a e - T_e) + |\psi| (T_p - T_e) \\
\frac{dT_p}{dt} &= R_T(T^a p - T_p) + |\psi| (T_e - T_p) \\
\frac{dS_e}{dt} &= R_S(S^a e - S_e) + |\psi| (S_p - S_e) \\
\frac{dS_p}{dt} &= R_S(S^a p - S_p) + |\psi| (S_e - S_p).
\end{align*}
\]

(7)

Here, \(T\)'s are temperatures and \(S\)'s are salinities. The subscripts \(e\) and \(p\) denote the box at the equator and pole, respectively, while the superscript \(a\) denotes an atmospheric forcing term. The strength of the circulation is given by \(|\psi|\), where

\[
\psi = \psi_0 \left( \frac{\rho_p - \rho_e}{\rho_0} \right).
\]

The density of box \(i\) is denoted \(\rho_i\), and it is calculated using a linear equation of state with reference values \(T_0, S_0\) and \(\rho_0\). So,

\[
\rho_i = \rho_0 \left[ 1 - \alpha (T_i - T_0) + \beta (S_i - S_0) \right].
\]

Thus \(\psi\) is easily computed to be

\[
\psi = \psi_0 [\alpha(T_e - T_p) - \beta(S_e - S_p)].
\]

(8)

Next, the system is reduced to have only two degrees of freedom by looking at the temperature and salinity differences between the boxes. Defining

\[
\begin{align*}
T &= T_e - T_p, & S &= S_e - S_p, \\
T^a &= T^a e + T^a p, & S^a &= S^a e + S^a p, \\
X &= T_e + T_p, & Y &= S_e + S_p,
\end{align*}
\]

we see that the the equations for \(dX/dt\) and \(dY/dt\) decouple from the others. Therefore, the system reduces (7) to

\[
\begin{align*}
\frac{dT}{dt} &= R_T(T^a - T) - 2|\psi|T, \\
\frac{dS}{dt} &= R_S(S^a - S) - 2|\psi|S,
\end{align*}
\]

(9)

and (8) becomes

\[
\psi = \psi_0 (\alpha T - \beta S).
\]

To non-dimensionalize the system, set

\[
\begin{align*}
x &= \frac{T}{T^a}, & y &= \frac{\beta S}{\alpha T^a}, & \tau &= R_{St}, & \mu &= \frac{\beta S^a}{\alpha T^a}, & A &= \frac{2\psi_0 \alpha T^a}{R_S}.
\end{align*}
\]

Then the system (9) becomes

\[
\begin{align*}
\varepsilon \dot{x} &= 1 - x - \varepsilon A|x - y|x \\
\dot{y} &= \mu - y - A|x - y|y.
\end{align*}
\]

(10)
Figure 2: Graphs of (12) for (a) $A < 1$ and (b) $A > 1$.

where

$$\varepsilon = \frac{R_S}{R_T} \ll 1$$

is a small parameter and the dot ( \( \dot{\cdot} \) ) denotes differentiation with respect to \( \tau \). The system (10) is a fast/slow system set up to be analyzed using GSP. In the limit as \( \varepsilon \to 0 \), \( \{x = 1\} \) is a globally attracting, and therefore normally hyperbolic, critical manifold. The reduced problem has one degree of freedom, so the dynamics are entirely characterized by equilibria. The system is

$$\dot{y} = \mu - y - A|1 - y|y.$$  \hspace{1cm} (11)

Critical points occur at

$$\mu = \begin{cases} 
(1 + A)y - Ay^2 & \text{for } y < 1 \\
(1 - A)y + Ay^2 & \text{for } y > 1
\end{cases}$$  \hspace{1cm} (12)

and the nature of the system depends on \( A \), as seen in Figure 2. Taking a derivative gives

$$\frac{d\mu}{dy} = \begin{cases} 
(1 + A) - 2Ay & \text{for } y < 1 \\
(1 - A) + 2Ay & \text{for } y > 1
\end{cases}.$$

If \( A < 1 \), then the curve of equilibria \( \mu = \mu(y) \) is monotone increasing. The system (11), and consequently (10), has a unique equilibrium solution. The equilibrium is globally attracting, and it is important to remember that the solution corresponds to a unique stable circulation state (i.e., direction and strength).

However, if \( A > 1 \) the system exhibits bistability for a range of \( \mu \) values. While \( \mu(y) \) is still monotone increasing for \( y > 1 \), the curve has a local maximum at \( y = (1 + A)/(2A) < 1 \). Thus for \( 1 < \mu < (1 + A)^2/(4A) \) there are three equilibria. The system is bistable with the outer two equilibria being stable, and the middle equilibrium being unstable. The stable equilibrium for \( \psi < 0 \) is called the haline state, since the circulation is driven by the salinity difference. When \( \psi > 0 \), the circulation is driven by temperature, and the system is in the thermal state. In the bistable regime, there is a stable thermal state as well as an unstable thermal state. As mentioned in the introduction, some oceanographers attempt to explain oscillations using only oceanic processes, however Figure 3 suggests \( \mu \) is the key to generating such oscillations.
4 Dynamic Oscillations with 1 Slow Variable

If \( \mu \) is the key to oscillations, there may be an intrinsic feedback mechanism that causes \( \mu \) to change. Recall that \( \mu \) is the ratio of the effect of atmospheric salinity forcing on density to that of atmospheric temperature forcing on density. The idea is to consider how the state of the ocean affects its interaction with the atmosphere. Typically, in a coupled ocean-atmosphere model, the atmosphere is the fast component and the ocean is the slow component (see [21], for example). However, in Stommel’s model, \( \mu \) is considered constant. Therefore, if a model is going to incorporate \( \mu \) as a dynamic variable, it should vary on a slower time scale than the other variables in the model. The physical intuition is to consider the variation of long term average behavior in the atmosphere. A general system of this form is

\[
\begin{align*}
\psi' &= 1 - x - \varepsilon A|\psi - y|x \\
y' &= \varepsilon(\mu - y - A|\psi - y|y) \\
\mu' &= \varepsilon \delta f(x, y, \mu, \delta, \varepsilon),
\end{align*}
\]

(13)

where \( \delta \ll 1 \) is another small parameter. This system is a three time-scale model where \( x \) is fast, \( y \) is intermediate, and \( \mu \) is slow. If \( f(x, y, \mu, 0, 0) \) behaves in a desirable manner, one can turn the hysteresis loop in the bifurcation diagram into a true periodic orbit. The important question to answer is “how do variations in \( x \) and \( y \) affect \( \mu \)?” One way the ocean can affect the atmosphere is through clouds, and here we list (reasonable) assumptions about long-term ocean-atmosphere feedback. An increase in ocean temperature should lead to increased evaporation, and thus increased cloud formation. An increase in salinity should cause a decrease in evaporation and decreased cloud formation. Clouds are important because they reflect sunlight back to space before it reaches the Earth’s surface. Therefore an increase in clouds will decrease the effect of heat forcing near the equator where the sun’s effect is strongest. Near the poles, clouds serve as a blanket, preventing heat from escaping. Thus, more clouds mean a decrease in \( T^\alpha \) (from [9]). Having more clouds is a result of more evaporation at the equator and leads to more precipitation at high latitudes. So more clouds also mean an increase in \( S^\alpha \) (also from [9]). Mathematically, the
The effects are
\[
\frac{\partial T^a}{\partial T} < 0 \quad \frac{\partial T^a}{\partial S} > 0
\]
\[
\frac{\partial S^a}{\partial T} > 0 \quad \frac{\partial S^a}{\partial S} < 0
\]
since \(T\) larger means a greater temperature difference between boxes (for the same average temperature) and thus a warmer equator. Similarly, larger \(S\) means a saltier equator. Recalling that
\[
\mu = \frac{\beta S_a}{\alpha T^a},
\]
the dependence of \(f\) on \(x\) and \(y\) must be
\[
\frac{\partial f}{\partial x} > 0 \quad \text{and} \quad \frac{\partial f}{\partial y} < 0. \tag{14}
\]

If this condition is implemented in the simplest possible way, then there is a parameter regime in which the system has a unique periodic orbit. Taking \(f\) to be a linear function of \(x\) and \(y\), (13) becomes
\[
x' = 1 - x - \varepsilon A|x - y|x
\]
\[
y' = \varepsilon(\mu - y - A|x - y|y)
\]
\[
\mu' = \epsilon \delta(1 + ax - by). \tag{15}
\]

As in the previous section, \(\{x = 1\}\) is still an attracting critical manifold. However, the reduced problem,
\[
\dot{y} = \mu - y - A|1 - y|y
\]
\[
\dot{\mu} = \epsilon \delta(1 + a - by), \tag{16}
\]
is now itself a fast/slow system which is analyzed using GSP. The shape of the critical manifold of (16) depends on the parameter \(A\), similar to the previous section. In fact, the critical manifolds will be precisely the curves in Figure 2. Above the curve, the fast dynamics (fixed \(\mu\) trajectories) will move to the right. Below the curve, the fast dynamics moves to the left. As expected, the sections of the curves which corresponded to stable equilibria in (10) (and consequently (11)) are now attracting branches of a critical manifold in (16). The decreasing portion of the graph in Figure 2b, which contains the unstable equilibrium for the bistable regime in (10) is now a repelling branch of a critical manifold. Let
\[
M_0 = \{ \mu = y + A|1 - y|y \} \tag{17}
\]
denote the critical manifold.

The dynamics on \(M_0\) depend only on parameters and \(y\), which is fast in the reduced problem. Therefore, the key to resolving the slow flow is the location of the \(\mu\) nullcline. The cases \(A < 1\) and \(A > 1\) are treated separately, due to the different shape of the critical manifold as depicted in Figure 2.

### 4.1 Globally Attracting Critical Manifold

Here we consider the case where \(A < 1\). Since the only two branches of \(M_0\) are both attracting for \(A < 1\), one expects that it should behave as a 1-dimensional system. The intersection of the \(\mu\) nullcline and the critical manifold should be a globally attracting critical point, however GSP cannot be applied “out of the box” due to the non-differentiability of the vector field. Instead, we consider two distinct smooth dynamical systems: (1) where \(|1 - y| = 1 - y\) and (2) where
Figure 4: Possible phase spaces of (16) for $A < 1$ and $1 + a \neq b$. The red line is the $\mu$ nullcline. The black arrows indicate fast dynamics, and the blue arrows indicate slow dynamics.

$|1 - y| = y - 1$. The system where $|1 - y| = 1 - y$ agrees with (16) when $y < 1$. Similarly the system where $|1 - y| = y - 1$ agrees with (16) when $y > 1$.

The dynamics of (16) is obtained by taking trajectories from the two smooth systems and cutting them along the line $y = 1$—called the splitting line—where both smooth systems agree with the system of interest. We then paste the relevant pieces together along the splitting line. Since the right-hand side in (16) is Lipschitz, trajectories pass through the splitting line in a well-defined manner due to uniqueness of solutions.

Now, GSP can be applied to both of the smooth systems, which will produce two critical manifolds that intersect (in the singular limit) when $y = 1$. $M_0$ defined in (17) is obtained by taking the relevant critical manifold on either side of the splitting line. The $\mu$ nullcline is the vertical line $y = \frac{1 + a}{b}$.

We see that $\mu$ is increasing to the left of this line, and decreasing to the right of this line as in Figure 4. Therefore the system has a globally attracting equilibrium if $A < 1$.

### 4.2 Bistable Critical Manifold

If $A > 1$, the system is much more interesting due to the ‘2’-shaped critical manifold. To simplify the analysis, we rewrite (16) as

\[
\begin{align*}
\dot{y} &= \mu - y - A|1 - y|y \\
\dot{\mu} &= \delta_0(\lambda - y),
\end{align*}
\]

where $\delta_0 = \delta b$ and $\lambda = (1 + a)/b$.

**Theorem 3.** Consider the system (18) with $A > 1$, $0 < \delta_0 \ll 1$, and $\lambda > 0$ fixed. Then the following statements hold:

(A) For $\lambda \geq 1$, there is a globally attracting equilibrium in the haline state.

(B) For $(1 + A)/(2A) < \lambda < 1$ the equilibrium is unstable and surrounded by a unique stable periodic orbit created through a non-smooth bifurcation at $\lambda = 1$.

(i) When $A < 1 + 2\sqrt{\delta_0}$, the bifurcation creates non-smooth canard cycles.
(ii) When $A > 1 + 2\sqrt{\delta_0}$, the bifurcation is a super-explosion and the periodic orbit is a relaxation oscillation for
\[ \frac{1 + A + 2\sqrt{\delta_0}}{2A} < \lambda < 1. \]

(C) For $\lambda \leq (1 + A)/(2A)$ there is an attracting equilibrium in the thermal state.

Proof. Define $F_\pm(y) = y \pm A(1 - y)y$, and
\[ F(y) = \begin{cases} F_+(y) & y < 1 \\ F_-(y) & y > 1 \end{cases} \]
Then (18) always has a unique equilibrium at $(y_0, \mu_0) = (\lambda, F(\lambda))$. Direct computation shows that the Jacobian of (18) is
\[ J(\lambda, F(\lambda)) = \begin{pmatrix} -F'(\lambda) & 1 \\ \delta_0 & 0 \end{pmatrix}, \quad (19) \]
We see that $\text{det} J > 0$ everywhere, and $\text{Tr}(J) < 0$ when $F'(\lambda) > 0$. Therefore, for $\lambda > 1$, we have an attracting equilibrium. When $\lambda = 1$ the equilibrium is attracting from the right (haline state), but repelling from the left (thermal state). All trajectories in the thermal state will eventually be returned to the haline state above the $y$ nullcline. Since $F'_+(1) > 1$, we can assume $|F'_+(1)|^2 > 4\delta_0$, so the equilibrium will be a node. The proof of Theorem 2 (which is Theorem 3.5 in [15]) shows that all trajectories entering the haline state above the nullcline will be attracted to the equilibrium along the splitting line. This proves assertion (A). Also, when $\lambda < 1$ we see that $F'(\lambda) > 0$ if and only if $\lambda < (1 + A)/(2A)$. This proves assertion (C).

When $(1 + A)/(2A) < \lambda < 1$, $F'(\lambda) < 0$ and the equilibrium is unstable. If $A < 1 + 2\sqrt{\delta_0}$, then $F'_+(1) < 2\sqrt{\delta_0}$ and the equilibrium will be an unstable focus near the fold. By Theorem 2, the bifurcation will create canard cycles as shown in Figure 5c. However, if $A < 1 + 2\sqrt{\delta_0}$, then $F'_+(1) > 2\sqrt{\delta_0}$. The bifurcation turns a stable node into an unstable node. Theorem 2 from the previous section indicates that this will be a super-explosion whereby a stable relaxation orbit (bounded away from the equilibrium) appears instantaneously upon bifurcation. The relaxation oscillation resulting from the super-explosion is depicted in Figure 5d.

Remark 1. The bifurcation at $\lambda = (1 + A)/(2A)$ is degenerate, and therefore Theorem 1 does not apply. When the slow nullcline intersects the critical manifold on the unstable branch, we will still have an attracting periodic orbit guaranteed by the Poincaré-Bendixson Theorem. However, whether the orbit is a canard cycle or a relaxation oscillation is undetermined.

5 Separating the Forcing Terms

In the previous section we explored how incorporating a parameter as a slow dynamic variable can produce relaxation oscillations. In actuality, the parameter $\mu$ was a ratio of two forcing parameters from system (9). In this section we explore what happens if the two forcing parameters are allowed to vary independently, with the goal of finding conditions under which the system exhibits relaxation oscillations. Using a similar nondimensionalization as the one that produces (10), we will consider
a system of the form

\[
\begin{align*}
\frac{dx}{dt} & = z - x - \varepsilon A |x - y| x \\
\frac{dy}{dt} & = \varepsilon (u - y - A |x - y| y) \\
\frac{dz}{dt} & = \varepsilon \delta (ay - bx + c) \\
\frac{du}{dt} & = \varepsilon \delta (px - qy + r),
\end{align*}
\]

where \( 0 < \varepsilon, \delta \ll 1 \). The new model (20) is still a three time-scale system, with \( x \) fast, \( y \) intermediate, and \( z, u \) slow. As in the case with only one slow variable (15), we will be able to perform two reductions using GSP. The first reduction occurs as \( \varepsilon \to 0 \). We see that the critical manifold is now the set

\[ \{x = z\} \]

so the fast variable is slave to a slow variable. Similar to (15), this critical manifold is globally attracting, and the dynamics on the manifold are described by

\[
\begin{align*}
y' & = u - y - A |z - y| y \\
z' & = \delta (ay - bz + c) \\
u' & = \delta (px - qy + r).
\end{align*}
\]

Figure 5: Oscillatory behavior in (18).
Note that in this reduced system, the circulation variable $\psi = z - y$, and the set $\{y = z\}$ is the splitting surface.

5.1 The Critical Manifold

Define the function

$$F(y, z, u) = u - y - A|z - y|y.$$ 

Then the critical manifold

$$S = \{F(y, z, u) = 0\} = \{u = y + A|z - y|\}.$$ 

To determine where $S$ is attracting, we consider

$$F_y = \begin{cases} -1 - A(z - 2y) & z > y \\ -1 - A(2y - z) & z < y \end{cases}.$$ 

Wherever $F_x < 0$ (resp. $F_x > 0$), the critical manifold $S$ will be attracting (resp. repelling). When $z < y$, which corresponds to the haline circulation state ($\psi < 0$), we have

$$-1 - A(2y - z) < -1 - A(2z - z) < 0.$$ 

So the haline state is always attracting. When $z > y$, corresponding to the thermal state ($\psi > 0$), we have

$$F_y < 0 \iff z > 2y - \frac{1}{A}.$$ 

The quantity $A^{-1}$ will become important later, and we now define $\rho = A^{-1}$. The lines $z = y$ and $z = 2y - \rho$ intersect at the point $(\rho, \rho)$, producing a cusp on $S$. For all $z < \rho$, $S$ is stable. However for all $z > \rho$, $S$ is a ‘2’-shaped manifold, with two attracting branches and an unstable branch. We denote

$$S_A^- = \{z > 2y - \rho\} \cap \{z > \rho\}$$
$$S_R = \{y < z < 2y - \rho\}$$
$$S_A^+ = \{y > z > \rho\},$$

where the subscript $A$ (resp. $R$) indicates an attracting (resp. repelling) branch of $S$. The ‘fold lines’ $L^\pm$ separate $S_A^\pm$ and $S_R$, where $L^- = \{z = 2y - \rho : z > \rho\} \cap S$ is a smooth fold and $L^+ = \{z = y : z > \rho\} \cap S$ is a corner. Therefore, we can write

$$S|_{z>\rho} = S_A^- \cup L^- \cup S_R \cup L^+ \cup S_A^+.$$ 

5.2 The Reduced Problem and Singularities

Since $F_y$ is either zero or undefined on $L^\pm$, we cannot describe the critical manifold as $\{y = h(z, u)\}$ where $h$ is globally defined. Thus we cannot formulate the reduced problem in terms of only the variables $z$ and $u$. However, we can utilize the fact that $F_u \equiv 1$ to formulate the reduced problem in terms of $y$ and $z$. That is, the reduced problem can be written as

$$-F_y\dot{y} = F_z\dot{z} + F_u\dot{u}$$
$$\dot{z} = ay - bz + c.$$
Still, $F_y$ is defined piecewise, so we obtain different equations of the reduced problem for the thermal and haline states. In the thermal state ($z > y$) we have the subsystem

$$[1 + A(z - 2y)] \dot{y} = -Ay(ay - b + c) + (pz - qy + r)$$

and in the haline state ($z \leq y$) we have the subsystem

$$[1 + A(2y - z)] \dot{y} = Ay(ay - b + c) + (pz - qy + r)$$

Note that these systems are defined on all of $S$, not just where $z \leq \rho$. These systems may have five different types of singularities:

- ordinary singularities—these are equilibria of (21)
- regular fold points—also called jump points,
- folded equilibria—these behave like equilibria in (22) but are not equilibria of (21)
- corner points—these behave like jump points for $z > \rho$.
- the cusp at $(\rho, \rho)$.

Note that fold points (either regular or folded equilibria) happen when the coefficient of $\dot{y}$ is zero; this only happens in (22). We can rescale the time variable by $[1 + A(z - 2y)]p^{-1}$ in the thermal subsystem and $p[1 + A(2y - z)]p^{-1}$ in the haline subsystem to obtain the desingularized problem,

$$\dot{y} = -\frac{A}{p} y(ay - b + c) + \left(z - \frac{q}{p} y + \frac{r}{p}\right)$$

and

$$\dot{z} = \frac{1}{p}[1 + A(z - 2y)](ay - bz + c),$$

and

$$\dot{y} = \frac{A}{p} y(ay - b + c) + \left(z - \frac{q}{p} y + \frac{r}{p}\right)$$

$$\dot{z} = \frac{1}{p}[1 + A(2y - z)](ay - bz + c).$$

Defining

$$\gamma = \frac{Ab}{p}, \quad \alpha = \frac{a}{b}, \quad \beta = \frac{q}{p},$$

$$k = \frac{r}{p}, \quad m = \frac{c}{b},$$

we can reformulate the subsystems (24) and (25) as

$$\dot{y} = -\gamma y(\alpha y - z + m) + (z - \beta y + k)$$

and

$$\dot{z} = \gamma (\rho - 2y + z)(\alpha y - z + m),$$

and

$$\dot{y} = \gamma y(\alpha y - z + m) + (z - \beta y + k)$$

$$\dot{z} = \gamma (\rho + 2y - z)(\alpha y - z + m),$$
respectively. In these systems it is easy to classify the different types of singularities. Ordinary
singularities occur when \((\alpha y - z + m) = 0 = (z - \beta y + k)\). Folded equilibria occur when \(\dot{z} = 0\) due
to the rescaling. Thus they are only possible in the thermal subsystem (27) (or equivalently (24)),
occuring when \((\rho - 2y + z) = 0\) and \(\dot{y} = 0\). If \((\rho - 2y + z) = 0\) but \(\dot{y} \neq 0\), then the fold point is
a regular fold point. Note that regular fold points are no longer singularities in (27), which is why
we call it the desingularized problem. The haline subsystem has no fold points, so (28) is not truly
a desingularized problem. We perform the rescaling in the haline subsystem so that the analysis is
similar to that of the thermal subsystem.

Corner points occur when \(z = y\). Even in the desingularized systems, the corner points are
still singularities for \(z \leq \rho\) since the vector fields have different limits on either side of the split.
The theory has not been developed to analyze exactly what happens in the case of a ‘corner
equilibrium.’ That is, if the desingularized flow on either side of the corner is tangent to the corner,
the dynamics are unclear. However, in the case that the desingularized flow crosses the corner in
the same direction, then we know what trajectories will do along the corner. If \(\dot{z} - \dot{y} > 0\) in both
systems, the desingularized reduced flow sends trajectories from the haline to the thermal state
along the fold. If \(\dot{z} - \dot{y} < 0\) in both systems, then the desingularized reduced flow sends trajectories
from the thermal to haline state along the fold. The case we would like to avoid is the one where
\(\dot{z} = \dot{y}\). In the case of both systems (27) and (28), we have
\[
(\dot{z} - \dot{y})|_{z=y} = \gamma \rho[(\alpha - 1)z + m] + (\beta - 1)z - k. \tag{29}
\]

Having trajectories cross the fold in the desingularized flow has different implications for dif-
ferent regions of \(S\). When \(z < \rho\), trajectories will cross in the way we expect. However for \(z > \rho\),
the corner is adjacent to \(S_R\) where trajectories have been reversed. Along the corner in this region,
crossing in the desingularized system means trajectories are either directed towards the fold on
both sides or away from the fold on both sides in the actual system (21). Therefore, the corner will
behave as a standard jump point.

5.3 Strategy

In order to show that the model (21) exhibits relaxation oscillations, we need to construct a singular
periodic orbit \(\Gamma\), consisting of heteroclinic orbits of the layer problem and a segment on each of
the stable branches \(S^+_A\). The heteroclinic orbits consist of trajectories that connect a fold \(L^\pm\) to its
projection \(P(L^\pm)\) on the opposite stable branch. An example of a singular periodic orbit \(\Gamma\) is shown
in Figure 6. The following theorem due to Szmolyan and Wechselberger [20] provides conditions
under which \(\Gamma\) perturbs to a relaxation oscillation in a smooth system of 3 variables.

**Theorem 4.** Assume a fast/slow system with small parameter \(0 < \varepsilon \ll 1\) satisfies the following
conditions:

(A1) The critical manifold is ‘S’-shaped,

(A2) the fold curves \(L^\pm\) are given as graphs \((y^\pm(z), z, u^\pm(z))\) for \(y \in I^\pm\) for certain intervals \(I^\pm\)
where the points on the fold curves \(L^\pm\) are jump points,

(A3) the reduced flow near the fold curves is directed towards the fold curves,

(A4) the reduced flow is transversal to the curve \(P(L^\pm)|_{I^\pm} \subset S^+_A\), and

(A5) there exists a hyperbolic singular periodic orbit \(\Gamma\).
Then there exists a locally unique hyperbolic relaxation orbit close to the singular orbit $\Gamma$ for $\varepsilon$ sufficiently small.

Currently, there is no analog for this theorem that applies to piecewise-smooth systems. We will find conditions such that (21) satisfies ($A_1$) by showing that the dynamics appear as in Figure 7. While this will demonstrate the existence of a singular periodic orbit $\Gamma$, it will not constitute a rigorous proof that (21) exhibits relaxation oscillations because the vector field of (21) is not smooth.

We have already shown that the critical manifold $S$ is ‘2’-shaped for $z > \rho$, so ($A_1$) is satisfied. Clearly, we can write the fold curves as graphs for $z > \rho$. Next, we find conditions so that there are no folded equilibria on the smooth fold $L^-$.  

**Lemma 5.** Assume that  

(a) $0 < \alpha < 1$ and  

(b) $k > \gamma \rho^2 (\alpha - 1) + (\gamma m + \beta - 1) \rho$.

Then (27) has no folded equilibria for $z > \rho$.

**Proof.** Folded equilibria occur when the line $z = 2y - \rho$ intersects the parabola $-\gamma y(\alpha y - z + m) + (z - \beta y + k) = 0$ for some $z > \rho$. Therefore, at folded equilibrium we have 

$$
\gamma(2 - \alpha)y^2 + [2 - \beta - \gamma(\rho + m)]y + k - \rho = 0.
$$

So, the $y$ coordinates of the intersections are 

$$
y_{\pm} = \frac{\gamma(\rho + m) + \beta - 2 \pm \sqrt{[\gamma(\rho + m) + \beta - 2]^2 - 4\gamma(2 - \alpha)(k - \rho)}}{2\gamma(2 - \alpha)}.
$$

Algebraic manipulation shows that $y_+ < \rho$ precisely when 

$$
k > \gamma \rho^2 (\alpha - 1) + (\gamma m + \beta - 1) \rho.
$$

\[\square\]
Figure 7: Existence of a stable singular periodic orbit. The blue line is the $z$ nullcline. The regions between $P(L^\pm)$ and $L^\mp$ are locally positively invariant, so there must be a stable singular periodic orbit.

Remark 2. In fact, we only need $\alpha < 2$ to prove Lemma 5. The stricter condition $\alpha < 1$ will be required for the next lemma.

Lemma 5 provides conditions so that (A2) is satisfied. Next, we find conditions so that (A3) – (A4) are satisfied. In order to do so, we need to find equations for $P(L^\pm)$. $P(L^-)$ is the projection of the line $y = (z + \mu)/2$ onto $S_A^+ = \{u = y + \rho^{-1}(y - z)y\}$. Therefore we want

$$u = \frac{z + \rho}{2} + \frac{z + \rho}{2\rho} \left( \frac{z - \rho}{2} \right) = \frac{(z + \rho)^2}{4\rho}.$$  

Plugging this back into the equation for $S_A^+$ we see that $P(L^-)$ is the the curve

$$y = \frac{z - \rho + \sqrt{2(z^2 + \rho^2)}}{2}$$  

on $S_A^+$. Similarly, $P(L^+)$ is the projection of the line $y = z$ onto $S_A^- = \{u = y + \rho^{-1}(z - y)y\}$. Therefore, on $P(L^+)$ we have $u = z$. From this we see $P(L^+)$ is the line $y = \rho$ on $S_A^-$. We define $z_*$ to be the $z$ coordinate of the intersection of the $z$ nullcline $z = \alpha y + m$ with the curve $P(L^-)$, which will play an important role in constructing locally invariant sets.

Lemma 6. Assume conditions (a)-(b) from Lemma 5. Define $R^-$ to be the region bounded by $P(L^+)$ on the left, $L^-$ on the right, the line $z = \alpha \rho + m$ below, and the line $z = z_*$ above. Also assume

(c) $m > (1 - \alpha)\rho$.

Then, the vector field of (27) is traverse to $P(L^+)$, and $R^-$ is locally positively invariant.
Proof. Since $z > \rho$ on $P(L^+)$

$$\dot{y}|_{y=\rho} = -\gamma \rho (\alpha \rho - z + m) + z - \beta \rho + k$$
$$> -\gamma \rho [\alpha - 1) \rho + m] + (1 - \beta) \rho + k.$$ 

Therefore, (b) implies $\dot{y}|_{y=\rho} > 0$ and the vector field is transverse to $P(L^+)$. Conditions (a) and (c) guarantee that the $z$ nullcline will intersect $L^-$. Below the nullcline the vector field points upward, and above it points downward. Therefore, the vector field points into $R^-$ along all boundaries except (possibly) the fold $L^-$. The only way to leave $R^-$ is by hitting the fold and jumping to $S_A^+$, so $R^-$ is locally positively invariant.

Next, we show a similar result for the analogous region $R^+$ on $S_A^+$.

Lemma 7. Assume conditions (a)-(c) from Lemmas 5-6. Define $R^+$ to be the region bounded by $P(L^-)$ on the right, $L^+$ on the left, the line $z = \alpha \rho + m$ below, and the line $z = z_*$ above. Also assume

(d) $k < (\beta - 1) (\alpha \rho + m)$, and

(e) $1 < \beta < 2$.

Then the vector field of (28) is transverse to $P(L^-)$, and $R^+$ is locally positively invariant.

Proof. Conditions (a)-(c) imply that the line $z = \alpha \rho + m$ lies below the $z$ nullcline on $S_A^+$, so the vector field points into $R^+$ here. The line

$$z = \frac{m}{1 - \alpha}$$

is the $z$ coordinate of the intersection of the $z$ nullcline with $P(L^-)$. Therefore the line lies above the nullcline and the vector field points down into $R^+$.

Showing that the vector field points into $R^+$ on $P(L^-)$ is slightly more difficult, and to do so we need to use the slopes of the tangent lines along $P(L^-)$. Differentiating (30) we get

$$2 \frac{dy}{dz} = \left(1 + \frac{2z}{\sqrt{2(z^2 + \rho^2)}} \right) dz$$

$$= \frac{2y + \rho + z}{2y + \rho - z} dz.$$ 

This tells us that the vector field points upward into $R^+$ if

$$(2y + \rho + z) \dot{z} - 2(2y + \rho - z) \dot{y} > 0.$$ 

Plugging in $\dot{z}$ and $\dot{y}$ from (28), this condition simplifies to

$$\gamma (\alpha y - z + m)(\rho + z) - 2(z - \beta y + k) > 0.$$ 

The first term on the LHS is positive since the $z$ nullcline lies above $P(L^-)$ on the interval under consideration. Therefore it is sufficient to show that the line $z = \beta y - k$ lies above the $P(L^-)$ as well. Since $\beta > 1$ this can be done by showing that the line $z = \beta - k$ crosses the corner $L^+$ for some $y < \alpha \rho + m$ (or equivalently $z < \alpha \rho$). This happens precisely when $k < (\beta - 1)(\alpha \rho + m)$. \qed
Remark 3. Again, the conditions of Lemma 7 are more strict than required to prove the lemma. In particular, we only need $\beta > 1$—not $\beta < 2$. The stricter conditions will be used to show the lack of a stable equilibrium on $S_A^\pm$ in the following lemma.

We have constructed two locally positively invariant regions as shown in Figure 7. Once a trajectory reaches $R^\pm$ the only way it can leave is by hitting the fold at a jump point, where the fast dynamics will take it from $R^\pm$ to $R^\mp$. In fact, unless there is a stable critical point in $R^\pm$, the only possibility is for trajectories to reach the folds.

Lemma 8. Under the conditions (a)-(e) of Lemmas 5-7, the only critical points of the systems (27)-(28) will lie on $S_R$.

Proof. If the line $z = \beta y - k$ crosses the corner $z = y$ to the right of the cusp (i.e. for $y, z > \rho$), then it will be trapped in the unstable region since $\beta < 2$. Furthermore, it will enter the unstable region below the $z$ nullcline, so its only intersection with the $z$ nullcline will be on the unstable branch. This happens when

$$k > (\beta - 1)\rho,$$

but that is already guaranteed by conditions (b) and (c).

5.4 Evidence of Stable Relaxation Orbit

Conjecture 9. Assume (21) satisfies (a)-(e) of Lemmas 5-7. Then (21) there exists a singular periodic orbit $\Gamma$ and an attracting relaxation orbit close to $\Gamma$ for $\delta$ sufficiently small.

Lemmas 5, 6, and 8 imply that $R^-$ is a locally positively invariant set containing no critical points. Since there are no equilibria along the fold (where $\dot{z} = 0$), we know $\dot{y} \neq 0$ and that the fold consists only of jump points in $R^-$. When combined with the fact that $\dot{y} > 0$ at the intersection of the fold with the line $y = \rho$, we know that the flow must be directed at the fold in $R^-$. Also, from Lemmas 5, 7, and 8 we know that $R^+$ is a positively locally invariant set with no critical points. Using equation (29), we see that condition (e) implies the flow will be directed at the corner in $R^+$ and that the corner will behave like regular fold points (i.e. jump points). We define the intervals $I^\pm = L^\pm \cap R^\pm$

We define the following maps:

$$\pi_- : P(L^+) |_{R^-} \to L^-$$
$$\pi^- : \text{Im}(\pi_-) \to P(L^-)$$
$$\pi_+ : \text{Im}(\pi^-) \to L^+$$
$$\pi^+ : \text{Im}(\pi_+) \to P(L^+)$$

where $\pi_\pm$ are the maps induced by the reduced flow, and $\pi^\pm$ are the maps induced by the layer problem. Furthermore we define $\Pi : P(L^+) \to P(L^+)$ by

$$\Pi = \pi^+ \circ \pi_+ \circ \pi^- \circ \pi_-.$$

Through the use of Wazewski maps, we see that $\Pi(L^+) \subset L^+$. By uniqueness of solutions of the flow gives, we see that for any two points $P, Q \in P(L^+) |_{R^-}$, there is a $K < 1$ such that

$$|\Pi(P) - \Pi(Q)| < K|P - Q|.$$
Therefore $\Pi$ is a contraction, so it has a unique fixed point. The fixed point corresponds to a singular periodic orbit $\Gamma$ and $\Pi$ is the Poincaré map. Furthermore, since $\Pi$ is a contraction,

$$\lim_{P \to Q} \Pi'(Q) \leq K < 1,$$

so $\Gamma$ is hyperbolic.

This shows that under the conditions (a)-(e), (21) satisfies the assumptions (A1) – (A5) of Theorem 4. In order to prove our conjecture, we would need to show that Theorem 4 generalizes to piecewise-smooth systems. We leave this to future work and provide further justification by simulating the model. Figure 9 depicts a stable periodic orbit near the singular orbit in Figure 6. Figure 8 shows portions of phase space that satisfy conditions (a)-(e).
5.5 Lack of a Periodic Orbit with only One Relaxation Phase

Since the critical manifold $S$ has a cusp, it is theoretically possible to have a periodic orbit $\Gamma'$ with exactly one relaxation phase, where the slow dynamics flow the orbit around the cusp. In order for such an orbit to relate to D-O events, the relaxation phase should be a rapid transition from the haline to the thermal state. However this is not possible in our model if parameters have physically meaningful values (i.e., $\alpha, \beta, k,$ and $m$ positive), and we will demonstrate that here.

**Theorem 10.** There is no periodic orbit in system (21) that contains exactly one relaxation phase characterized by a rapid transition from the haline state to the thermal state.

**Proof.** Such a periodic orbit requires that the vector field point left somewhere on $P(L^+) = \{y = \rho\}$. Since the restriction of $\dot{y}$ to $P(L^+)$ is an increasing function of $z$, a necessary condition for $\Gamma'$ is that the vector field (27) points left at the cusp $(\rho, \rho)$. This happens when

$$0 < k < \gamma \rho [ (\alpha - 1) \rho + m ] + (\beta - 1) \rho.$$

Next, we consider the position of the $z$ nullcline. The requirement that $k, m > 0$ implies that the line $z = \alpha y + m$ lies above the line $z = \beta y - k$ near $y = 0$. They will intersect if and only if $\beta > \alpha$. In the event they do intersect, the point of intersection will be a critical point of the reduced flow (and the full system), and it will occur for some $y = y_0 > 0$. We can calculate the stability of this critical point by looking at the Jacobian of the subsystem where the intersection occurs. The Jacobian in the thermal subsystem (27) is

$$J_T(y_0, z_0) = \begin{pmatrix} -\gamma \alpha y_0 - \beta & \gamma y_0 + 1 \\ \gamma \alpha (z_0 - 2 y_0 + \rho) & -\gamma (z_0 - 2 y_0 + \rho) \end{pmatrix}.$$ 

If the intersection happens on the attracting branch of $S$ in the thermal state, then $z_0 - 2 y_0 + \rho > 0$. Therefore it is easy to see that the trace $\text{Tr}(J_T) < 0$. Also the determinant $\text{det}(J_T) = \gamma (\beta - \alpha) (z_0 - 2 y_0 + \rho) > 0$, so the equilibrium is attracting if it lies on the stable branch. A similar calculation shows that any critical points on in the haline state will be attracting as well. Therefore, in order to obtain the orbit $\Gamma'$, we either need all critical points (if any exist) to lie on the unstable branch. This has important consequences for the dynamics on the stable branches.

There are two possibilities for the $z$ nullcline: either it intersects the line $y = \rho$ above the cusp or it intersects below the cusp. We will show that both possibilities preclude the existence of an
First, suppose the $z$ nullcline intersects $y = \rho$ above the cusp. We know from the lack of critical points on $S^+_A$ that the line $z = \beta - k$ lies below the $z$ nullcline for all $y < \rho$. Therefore the vector field points to the right along the $z$ nullcline. This implies that somewhere between the cusp and the $z$ nullcline, the vector field switches from pointing left to point right along $P(L+)$. All trajectories that land on $P(L+)$ in the region where the vector field points left lie below the $z$ nullcline, and therefore the flow takes them up and the left. However, they are unable to turn downwards since they are bounded below the $z$ nullcline, so they forced to cross $P(L+)$ above the cusp. Upon doing so they enter the invariant region $R^-$ and are prevented from ever crossing from the thermal state to the haline state below the cusp.

Second, we consider the case where the the $z$ nullcline crosses the line $y = \rho$ below the cusp. This implies that $\alpha < 1$, and the $z$ nullcline never enters the unstable region. Since any critical point will be attracting, we require $\beta < \alpha < 1$ to ensure that no critical points will exist. However, this contradicts condition (a'), since the RHS is negative.

6 Discussion

Incorporating the environmental forcing parameters as dynamic variables in Stommel’s 1961 thermohaline circulation model produces a relaxation oscillator. Stommel would not have seen this since his model is based on a tangible experiment—a rarity in climate science—in which the environment that forced the system was not actually a gaseous atmosphere but rather baths of water with prescribed temperatures and salinities. As demonstrated by Theorems 3 and 9, the relaxation oscillation is present regardless of whether the forcing parameters are incorporated together as a single slow variable or allowed to vary independently. Indeed, the time series for $\psi$ with only one slow variable, shown in Figure 5b, is remarkably similar the time series found in Figure 9b. Both figures appear to be qualitatively similar to Figure 1, including the asymmetric nature of the behavior within the stable states.

In both cases, the key to generating relaxation oscillations in the model is the ‘2’-shaped $y$ nullcline. In the model with a 1D critical manifold, the ‘2’-shape relies on the parameter $A > 1$. In the model (21) where the critical manifold is a surface, it will always have a bistable, ‘2’-shaped region. However, we were unable to find any parameter regime with $\rho > 1$ (corresponding to $A < 1$) in which the model exhibited relaxation oscillations. Although it is not entirely clear from conditions (a)-(e) of Theorem 9 that this should be the case, the picture for $\rho > 1$ analogous to those of Figure 8 is just an empty box.

In some sense, this paper is a case study for the usefulness of conceptual models. We have demonstrated that two conceptual models produce a qualitative pattern similar to the paleoclimate data. The qualitative difference in output between our two models appears negligible, suggesting that the simpler model is ‘good enough.’ Additionally, we are able to find the requirement that $A > 1$ analytically. Furthermore, we see why it is required; there is only one stable state when $A < 1$.

Since the limit cycle is seen in the reduced system (16), the oscillation can be described by a system with two degrees of freedom, which is the minimum requirement for an oscillator. Aside from the non-differentiability due to the absolute value term, the equations are relatively simple. In reality, the non-smooth nature of the vector field is fortuitous since the necessary ‘S’ shape of the critical manifold comes from the lack of differentiability of the absolute value function. If the critical manifold were a smooth cubic, the system would be indistinguishable from the van der Pol system. In fact, that is the way scientists have assumed (13) would behave, despite the GSP theory breaking down at one of the jump points. In essence, this provides rigorous justification for that
assumption.

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