The Onset of Chaotic Motion of a Spinning Particle around the 

Schwarzchild Black Hole 

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Abstract 

In the Schwarzchild black hole spacetime, we show that chaotic motion can be triggered by the spin of a particle. Taking the spin of the particle as a perturbation and using the Melnikov method, we find that the perturbed stable and unstable orbits are entangled with each other and that illustrates the onset of chaotic behavior in the motion of the particle.

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In astrophysics, the angular momentum or spin of a particle is one of the most important elements for consideration because nearly all astrophysical objects possess it. It may also be the crucial ingredient in the study of chaotic behavior as shown, for example, by Suzuki and Maeda [1] that the motions of spinning particles in the Schwarzschild black hole spacetime could be chaotic by analyzing the Lyapunov exponents [2]. Moreover, this is an important result due to its relevancy to the detection of gravitation waves from black hole coalescences [3, 4] as it may jeopardize the use of matching templates in interpreting the upcoming data from the LIGO experiment.

In this Letter we would like to use a different method, called the Melnikov technique [5], to investigate the chaotic motions of spinning particles in the Schwarzschild black hole spacetime. In [6], we have successfully applied this technique to detect the onset of chaotic motions in the simple dynamical system of the one-dimensional Duffing hamiltonian with a spinning particle. In this method one consider homoclinic orbits emanating and terminating at the same unstable fixed point in an integrable system. When time-dependent perturbations are added to the system, the stable and the unstable orbits in the perturbed system would split in general. The Melnikov function, which is an integral evaluated along the unperturbed homoclinic orbit, calculates the transversal distance between the perturbed stable and unstable orbits on the Poincaré section. The isolated zeros in the Melnikov function indicated complicated entanglements between the two perturbed orbits and therefore the presence of chaotic behaviors.

Here we first illustrate that there is a homoclinic orbit in the two-dimensional phase space of the radial motion of a spinless particle in the Schwarzschild spacetime. Suppose that the mass of the Schwarzschild black hole is $M$, and the line element is described in the usual Schwarzschild coordinates by

\[
d s^2 = -f \, dt^2 + f^{-1} \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

\[f(r) = 1 - \frac{2M}{r}.\quad (1)\]

For a relativistic particle of mass $m$ moving in a curved background described by the spacetime metric $g_{\mu\nu}$, the action can be taken to be

\[
S[x] = \frac{m}{2} \int g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \, d\tau.
\]

$x^\mu(\tau)$ denotes the spacetime coordinates of the particle where $\tau$ is the proper time, and we
must in addition impose the mass shell constraint

\[ g^{\mu\nu}p_\mu p_\nu = -m^2, \]

\[ p_\mu = m g_{\mu\nu} \dot{x}^\nu, \] (3)

where \( p_\mu \) is the momentum canonically conjugate to \( x^\mu \). Because \( g_{\mu\nu} \) is stationary and axisymmetric, the particle has two additional conserved quantities, the momenta conjugate to \( \tau \) and \( \phi \),

\[ E = -p_t = mf \frac{dt}{d\tau}, \]

\[ L = p_\phi = mr^2 \sin^2 \theta \frac{d\phi}{d\tau}. \] (4)

Let us restrict the particle motion, with a fixed angular momentum \( L \), to lie in the \( \theta = \pi/2 \) plane. Then the only effective degree of freedom is \( r \), and we will find that there is exactly one homoclinic orbit in the reduced phase space \( (r, p_r) \) with appropriately chosen \( E \) and \( L \).

Substitute the expressions for \( E \) and \( L \) into Eq. (3), a first-order equation of motion for \( r(\tau) \) can be obtained

\[ m^2 \left( \frac{dr}{d\tau} \right)^2 + f(r) \left( m^2 + \frac{L^2}{r^2} \right) = E^2 \]

\[ \Rightarrow m^2 \left( \frac{dr}{d\tau} \right)^2 - \frac{2Mm^2}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3} = E^2 - m^2. \] (5)

This equation can be interpreted as describing the one-dimensional problem of a non-relativistic particle with energy \( E^2 - m^2 \) moving in the potential

\[ U_{e,f} = \frac{2Mm^2}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3}, \] (6)

which has an attractive \( 1/r \) term and a repulsive \( 1/r^2 \) term as in the corresponding Newtonian system, plus an additional attractive \( 1/r^3 \) term which is a purely general relativistic effect. It is the latter which is responsible for the existence of the homoclinic orbit.

It is convenient to change the dynamical variable to \( u \equiv 2M/r \). The equation of motion in Eq. (5) then becomes

\[ \left( \frac{du}{d\phi} \right)^2 - \frac{4M^2m^2}{L^2} u + u^2 - u^3 = \frac{4M^2(E^2 - m^2)}{L^2}. \] (7)
It is easy to see that $U_{\text{eff}}$ has extrema at real values of $u$, provided that $12M^2m^2/L^2 < 1$, and they are located at

$$u_{st} = \frac{1}{3}(1 - \beta),$$

$$u_{un} = \frac{1}{3}(1 + \beta).$$

(8)

where $\beta = \sqrt{1 - 12M^2m^2/L^2}$ is the parameter in the problem to characterize the existence and the properties of the homoclinic orbit. $u_{st}$ is the stable fixed point, while $u_{un}$ is the unstable one. Figure 1 shows the potential with $\beta = 0.4$.

From Eq. (8), we see that at the asymptotic value $r = \infty$, the potential $U_{\text{eff}}(\infty) = 0$. To obtain a homoclinic orbit, we must therefore require the condition $U_{\text{eff}}(u_{un}) < 0$,

$$U_{\text{eff}}(u_{un}) = -\frac{1}{27}(1 + \beta)^2(1 - 2\beta) < 0$$

$$\Rightarrow \quad 0 < \beta < \frac{1}{2}.$$  

(9)

There could be a homoclinic orbit in this range of $\beta$, and we will use the $\beta = 0.4$ to illustrate the chaotic behavior be in the homoclinic orbit. Using the equations (3) and (5) we have the express for the momentum

$$p_r = \frac{m}{f} \frac{dr}{d\tau} = \pm \frac{1}{f} \left[ E^2 - f \left( m^2 + \frac{L^2}{r^2} \right) \right]^{1/2}. \quad (10)$$

In Figure 2, we obtain a homoclinic orbit in the $(r, p_r)$ reduced phase space, the parameters are $m = 10^{-6}, M = 1$ and $L = 3.78 \times 10^{-6}$ for $\beta = 0.4$. 

![Image of Figure 1: $U_{\text{eff}}(r)$, $\beta = 0.4$.](image-url)
Next, we add spin to the particle on this homoclinic orbit to show that chaos may occur in the motion of a spinning particle around a black hole. The equations of motion of a spinning test particle in a general spacetime were first derived by Papapetrou [9]. Those are a set of equations:

$$\frac{dx^\mu}{d\tau} = v^\mu,$$
$$\frac{Dp^\mu}{D\tau} = -\frac{1}{2} R^\mu_{\nu\rho\sigma} v^\nu S^{\rho\sigma},$$
$$\frac{DS^{\mu\nu}}{D\tau} = p^\mu v^\nu - p^\nu v^\mu,$$ \hspace{1cm} (11)

$D/D\tau$ denotes a covariant derivative with respect to $\tau$. Here $v^\mu$, $p^\mu$, and $S^{\mu\nu}$ are the 4-velocity of the particle, the momentum, and the spin tensor, respectively. The multipole moments of the particle higher than the mass monopole and the spin dipole are ignored. It is called the pole-dipole approximation.

There is a supplementary condition which gives a relation between $v^\mu$ and $p^\mu$ because $p^\mu$ is no longer parallel to $v^\mu$ in this case. We adopt the following condition [11],

$$p_\mu S^{\mu\nu} = 0.$$ \hspace{1cm} (12)

This condition is related to how to choose the center of mass in an extended body, and we
can write down the relation between $v^\mu$ and $p^\mu$ explicitly,

$$v^\mu = \frac{N}{m} \left[ g^{\mu\nu} p_\nu + \frac{1}{2m^2 \Delta} S^{\mu\nu} p_\lambda g^{\lambda\xi} R_{\nu\xi\rho\sigma} S^{\rho\sigma} \right],$$

$$N = \left[ 1 - \frac{1}{4 \Delta^2 m^4} S_{\mu\nu} p_\lambda S_{\rho\sigma} R^{\mu\lambda\rho\sigma} S^{\alpha\beta} p_\gamma S_{\gamma\delta} R_{\alpha\beta\gamma\delta} \right]^{1/2},$$

$$\Delta = 1 + \frac{1}{4m^2} R_{\alpha\beta\gamma\delta} S^{\alpha\beta} S_{\gamma\delta}. \quad (13)$$

The equation shows that $p^\mu$ is not parallel to $v^\nu$ in general due to the presence of the spin tensor. From Eq. (11) we get the equation of motion of the momentum $p_\mu$,

$$\frac{dp_\mu}{d\tau} = \Gamma_{\mu\nu} p_\nu - \frac{1}{2} R_{\mu\alpha\beta\gamma} v^\alpha p_\beta S^{\gamma\delta}, \quad (14)$$

which will be used in the analysis of the behavior of chaotic motions.

Because the spacetime is static and spherically symmetric, there are two Killing vector fields, $\zeta^\mu_\nu$ and $\zeta^\mu_\phi$. The constants of motion are as given in \[1\]

$$C_\equiv \zeta^\mu p_\mu - \frac{1}{2} \zeta_{\mu\nu} S^{\mu\nu},$$

$$E = -C_\nu(t) = -p_t - \frac{M}{r^2} S^{tr},$$

$$J_z = C_\phi = p_\phi + r^2 \sin \theta \cos \theta S^{\theta\phi} - r \sin^2 \theta S^{\phi r}. \quad (15)$$

$E$ and $J_z$ are interpreted as the energy of particle and the $z$ component of the total angular momentum, respectively. The $x$ and $y$ components of the total angular momentum are also conserved,

$$J_x = -p_\theta \sin \phi - p_\phi \cot \theta \cos \phi$$

$$-r \sin \phi S^{r\theta} + r^2 \sin^2 \theta \cos \phi S^{\theta\phi} + r \sin \theta \cos \theta \cos \phi S^{\phi r},$$

$$J_y = p_\theta \cos \phi - p_\phi \cot \theta \sin \phi$$

$$+r \cos \phi S^{r\theta} + r^2 \sin^2 \theta \sin \phi S^{\theta\phi} + r \sin \theta \cos \theta \sin \phi S^{\phi r}. \quad (16)$$

Because the background is spherically symmetric, we can choose the $z$ axis to be in the direction of total angular momentum, then we have the constraint equations of the spin tensor,

$$S^{\theta\phi} = \frac{J}{r^2} \cot \theta,$$

$$S^{r\theta} = -\frac{p_\theta}{r},$$

$$S^{\phi r} = \frac{1}{r} \left( -J + \frac{p_\phi}{\sin^2 \theta} \right). \quad (17)$$
From the constraint equation (12), we have the time-space components of the spin tensor,

\[ S^{tr} = -\frac{1}{rp_t} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} - J p_\phi \right), \]
\[ S^{t\theta} = \frac{1}{rp_t} \left( p_r p_\theta + \frac{J}{r} p_\phi \cot \theta \right), \]
\[ S^{t\phi} = -\frac{1}{rp_t} \left( J p_r - \frac{p_r p_\phi}{\sin^2 \theta} + \frac{J}{r} p_\theta \cot \theta \right). \]  

(18)

Here we treat the spin tensor as a perturbation. Suppose that without these perturbative terms, the particle is originally in a homoclinic orbit on the equatorial plane \( \theta = \pi/2 \). When spin terms are turned on, we write

\[ \theta = \frac{\pi}{2} - \delta \theta. \]  

(19)

Using \( \delta \theta \) to qualify the degree of perturbation, we suppose that

\[ p_\theta = \delta p_\theta + O(\delta \theta^2) \]
\[ p_\phi = J + \delta p_\phi + O(\delta \theta^2) \]  

(20)

where \( \delta p_\theta \sim O(\delta \theta) \) and \( \delta p_\phi \sim O(\delta \theta^2) \). With these we can expand the components of the spin tensor in Eqs. (17) and (18).

\[ S^{\theta\phi} = \frac{J}{r^2} \delta \theta + O(\delta \theta^2), \]
\[ S^{r\theta} = -\frac{\delta p_\theta}{r} + O(\delta \theta^2), \]
\[ S^{\phi r} = \frac{1}{r} \left( J \delta \theta^2 + \delta p_\phi \right) + O(\delta \theta^3), \]
\[ S^{tr} = \frac{1}{r E} \left( \delta p_\theta^2 + J^2 \delta \theta^2 + \delta p_\phi \right) + O(\delta \theta^3), \]
\[ S^{t\theta} = -\frac{1}{r E} \left( p_r \delta p_\theta + \frac{J^2}{r} \delta \theta \right) + O(\delta \theta^2), \]
\[ S^{t\phi} = \frac{1}{r E} \left( -J p_r \delta \theta^2 + \frac{J}{r} \delta p_\theta \delta \theta - p_r \delta p_\phi \right) + O(\delta \theta^3). \]  

(21)

We have shown explicitly only the lowest order term in each spin tensor component in these equations.

Similarly, we also expand the 4-velocity \( v^\mu \) and the momentum of the particle \( p_\mu \) in Eqs. (13) and (14). In particular, we have

\[ \frac{d \phi}{d \tau} = \frac{J}{mr^2} + \frac{3M J^2}{m^2 r^3} \delta \theta^2 + \frac{1}{m r^2} \delta p_\phi + O(\delta \theta^3). \]  

(22)
Using this equation we change the parameter of the equations of motion from $\tau$ to $\phi$. The main equations of motion of the reduced phase space of the radial motion can then be written as

\[
\begin{align*}
\frac{dr}{d\phi} &= \frac{1}{J} \left( 1 - \frac{2M}{r} \right) r^2 p_r - \frac{1}{J} \left( 1 - \frac{3M J^2}{r^3} \right) \left( 1 - \frac{2M}{r} \right) r^2 p_r \delta \theta^2 \\
&\quad - \frac{3MJ}{m^2 r^2} \delta p_\theta \delta \theta - \frac{1}{J^2} \left( 1 - \frac{2M}{r} \right) r^2 p_r \delta p_\phi + O(\delta \theta^3), \\
\frac{dp_r}{d\phi} &= -\frac{ME^2}{J} \left( 1 - \frac{2M}{r} \right)^{-2} - \frac{M}{J} p_r^2 + \frac{J}{r} \\
&\quad + \left[ \frac{M}{J} \left( 1 - \frac{3MJ^2}{r^3} \right) p_r^2 - \frac{3MJ}{r^2} \left( 1 - \frac{MJ^2}{r^3} \right) \left( 1 - \frac{2M}{r} \right)^{-1} \\
&\quad + \frac{ME^2}{J} \left( 1 - \frac{3MJ^2}{r^3} \right) \left( 1 - \frac{2M}{r} \right)^{-2} - \frac{2M^2 J}{r^3} \left( 1 - \frac{2M}{r} \right)^{-2} \right] \delta \theta^2 \\
&\quad + \left[ \frac{6M^2 J}{m^2 r^4} \left( 1 - \frac{2M}{J} \right)^{-1} p_r \right] \delta p_\theta \delta \theta \\
&\quad + \left[ \frac{1}{Jr} - \frac{3MJ}{Jr^2} \left( 1 - \frac{2M}{r} \right)^{-1} - \frac{2M^2 J}{J r^3} \left( 1 - \frac{2M}{r} \right)^{-2} \right] \delta p_\theta^2 \\
&\quad + \left[ \frac{1}{r} + \frac{M}{J^2} p_r^2 - \frac{3M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1} \\
&\quad + \frac{ME^2}{J^2} \left( 1 - \frac{2M}{r} \right)^{-2} - \frac{2M^2 J}{r^3} \left( 1 - \frac{2M}{r} \right)^{-2} \right] \delta p_\phi \\
&\quad + O(\delta \theta^3). \\
\end{align*}
\]

(23)

There are still the other three equations of motion from Eqs. (13) and (14),

\[
\begin{align*}
\frac{d\delta \theta}{d\phi} &= -\frac{1}{J} \delta p_\theta + O(\delta \theta^3), \\
\frac{d\delta p_\theta}{d\phi} &= J \left( 1 - \frac{3M}{r} \right) \delta \theta + O(\delta \theta^2), \\
\frac{d\delta p_\phi}{d\phi} &= \left[ -2M + \frac{3MJ^2}{m^2 r^2} \left( 1 - \frac{2M}{r} \right) \right] p_r \delta \theta^2 + \frac{3M}{r} \left( 1 - \frac{J^2}{m^2 r^2} \right) \delta p_\theta \delta \theta \\
&\quad + \frac{2M}{J} p_r \delta p_\phi + O(\delta \theta^3). \\
\end{align*}
\]

(25)

These equations are used to calculate the three perturbations $\delta \theta$, $\delta p_\theta$, and $\delta p_\phi$. We need these quantities as inputs in the main equations of motion, Eqs. (23) and (24). Since we want only the lowest order terms of these perturbations, we can take $r$ and $p_r$ in these equations to be in their zeroth order, that is, corresponding to the unperturbed homoclinic orbit.
Moreover, if we define $\epsilon \sim O(\delta \theta)$, or $\delta \theta = \epsilon \delta \tilde{\theta}$, we can write $\delta \tilde{p}_\theta = \epsilon \delta \tilde{p}_\theta$ and $\delta \tilde{p}_\phi = \epsilon^2 \delta \tilde{p}_\phi$, with $\delta \tilde{\theta}$, $\delta \tilde{p}_\theta$, and $\delta \tilde{p}_\phi$ all of $O(1)$. Then the solutions with the initial conditions,

$$
\delta \tilde{\theta}(0) = 1, \quad \delta \tilde{p}_\theta(0) = \delta \tilde{p}_\phi(0) = 0,
$$

are plotted in Figs. 3-5, and the corresponding spin tensor components are shown in Figs. 6-8. We find them all to be oscillatory. Note that the choice of the initial conditions is quite arbitrary. We have checked the behaviors of those quantities for other sets of initial conditions like $\delta \tilde{\theta}(0) = 0; \delta \tilde{p}_\theta(0) = 1; \delta \tilde{p}_\phi(0) = 0$, and $\delta \tilde{\theta}(0) = 0; \delta \tilde{p}_\theta(0) = 0; \delta \tilde{p}_\phi(0) = 1$. The results are also oscillatory as time evolves.

We have illustrated that external oscillatory perturbative spin terms may induce chaotic
behaviors in simple systems [6] by calculating the Melnikov function [5, 12]. Now in the case
FIG. 7: $\tilde{S}_{\phi \theta}(\phi)$.

FIG. 8: $\tilde{S}_{\phi \phi}(\phi)$.
of a spinning particle in the Schwarzschild black hole spacetime, the Melnikov function,

\[ M(\phi_0) = \int_{-\infty}^{\infty} d\phi \left\{ -\left[ \frac{ME^2}{J} \left( 1 - \frac{2M}{r} \right)^{-2} - \frac{M}{J} p_r^2 + \frac{J}{r} \right] g_1 + \left[ \frac{1}{J} \left( 1 - \frac{2M}{r} \right) r^2 p_r \right] g_2 \right\}. \]  

(27)

\[ g_1 = -\frac{1}{J} \left( 1 - \frac{3MJ^2}{m^2 r^3} \right) \left( 1 - \frac{2M}{r} \right) r^2 p_r \delta \theta^2 - \frac{3MJ}{m^2 r^2} \delta \rho \delta \theta \]

\[ g_2 = \left[ \frac{M}{J} \left( 1 - \frac{3MJ^2}{m^2 r^3} \right) p_r^2 - \frac{3MJ}{r^2} \left( 1 - \frac{MJ^2}{m^2 r^3} \right) \left( 1 - \frac{2M}{r} \right)^{-1} \right. \]

\[ + \frac{ME^2}{J} \left( 1 - \frac{3MJ^2}{m^2 r^3} \right) \left( 1 - \frac{2M}{r} \right)^{-2} - \frac{2M^2J}{r^3} \left( 1 - \frac{2M}{r} \right)^{-2} \left[ \delta \theta^2 \right. \]

\[ + \left[ \frac{6M^2J}{m^2 r^4} \left( 1 - \frac{2M}{J} \right)^{-1} p_r \right] \delta \rho \delta \theta \]

\[ + \left[ \frac{1}{J^2} - \frac{3M}{J^2 r^2} \left( 1 - \frac{2M}{r} \right)^{-1} - \frac{2M^2}{J^3 r^3} \left( 1 - \frac{2M}{r} \right)^{-2} \right] \delta \rho^2 \]

\[ + \left[ \frac{1}{r} + \frac{M}{J^2} p_r^2 - \frac{3M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1} \right. \]

\[ + \frac{ME^2}{J^2} \left( 1 - \frac{2M}{r} \right)^{-2} - \frac{2M^2}{r^3} \left( 1 - \frac{2M}{r} \right)^{-2} \left\] \delta \phi \]

Here \( r \) and \( p_r \) are again in their zeroth order corresponding to the unperturbed homoclinic orbit, with \( \phi \to \phi + \phi_0 \), where \( \phi_0 \) is used to parametrize the location along the unperturbed homoclinic orbit. Moreover, \( g_1 \) and \( g_2 \) involve only the oscillatory perturbative terms. The Melnikov function measures the transversal distance on the Poincaré section between the perturbed stable and unstable orbits emanating from the unstable fixed point. The infinite number of discrete zeros of the Melnikov function as shown in Fig. 9 indicates that the two orbits entangle with each other, and this indicates the occurrence of chaotic behavior for the perturbed system.

In addition, we have also calculated the Melnikov functions for the parameter \( \beta = 0.1, 0.2, \) and \( 0.3 \). Similar results as shown in Fig. 9 with \( \beta = 0.4 \) are obtained. These values of \( \beta \) satisfy the condition, \( 0 < \beta < 0.5 \), for which unperturbed homoclinic orbits exist. We have thus proved the onset of chaotic motion for spinning particles near the homoclinic orbit in the Schwarzschild black hole spacetime, even with small spin terms. This is complementary to the results obtained by Suzuki and Maeda \[1\], in which chaotic behavior is detected
numerically by calculating the Lyapunov exponent in the case of large spin terms. It is interesting to see if our method can be extended to consider other parts of the phase space and cases with finite spin terms. We plan to explore that in the future.

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