F-theory and the Witten Index

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Abstract

We connect the fermionic fields, localized on the intersection curve Σ of two D7-branes with zero background flux, to a $N=2$ supersymmetric quantum mechanics algebra, within the theoretical framework of F-theory.

Introduction

F-theory [1-79], is a strongly coupled formulation of type IIB superstring theory with 7-branes. The most interesting achievement is that it provides a consistent non-perturbative theoretical apparatus for consistent GUT model building [4-72], with gravity being decoupled from the theory. The fields of the Standard Model arise from local F-theory GUT geometries which constrain the phenomenological outcomes of the models.

Perturbative local D-branes models offer the possibility of generating the Standard Model fields existing on branes, in the absence of gravity (see e.g., references in [9]). For example in type IIB, GUT groups like $SO(10)$ or $SU(5)$ can be obtained on a stack of D7-branes or their orientifolds. However couplings such as $5_H \times 10_M \times 10_M$ for $SU(5)$, or representations like the spinor 16 of $SO(10)$, cannot be realized within the perturbative framework of superstring theories. F-theory provides a consistent theoretical framework (actually a UV completion of type IIB) on which GUT theories can be built.

Realistic GUT models must necessarily include mechanisms that break the GUT group, to the Standard Model one. In addition, proton stability must be ensured, and the various flavor mass hierarchies must be explained. Furthermore a coupling to a supersymmetry breaking sector must be included. F-theory compactifications deal with all the aforementioned problems and offers phenomenologically interesting, geometry based answers. Moreover the geometry of the compactifications provides rich alternatives in model building and different approaches to the same problem [1,4,6,9].

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The most interesting F-theory compactifications are those which preserve $N = 1$ supersymmetry in four dimensions [4,5]. This condition is met for compactifications on a complex dimension four Calabi-Yau manifold. Nevertheless the six physical internal dimensions are a threefold, say $B_3$, that needs not to be Calabi-Yau.

D7-branes naturally arise when Calabi-Yau fourfolds are considered. Actually the locations of the seven branes are given by the roots of the discriminant of a complex dimensional equation in $B_3$ [4,7,10]. The singularities in turn, are classified according to the Kodaira classification of singular fibers (ADE singularities). It is this classification that realizes gauge groups.

Since the originating papers on F-theory [1–3], a vast amount of work has been carried out [4,79]. For comprehensive reviews see for example [6–9,19,22]. For recent work on realistic GUT models see [4–72].

In this work, we consider F-theory elliptic $K3$ fibrations over a complex dimension two surface $S$. Particularly we focus our study on the localized fields that are generated on the intersection curve $\Sigma$ of two D7-branes in the absence of external gauge fluxes. As we shall see, the fields that have localized solutions along a matter curve $\Sigma$, are connected to an $N = 2$ supersymmetric quantum mechanics algebra [80,81]. We relate the number of the zero modes with the Witten index of the susy algebra.

This paper is organized as follows: In section 1 we describe the F-theory setup, that is, D7-branes intersections, matter curves and the eight dimensional Super-Yang-Mills (SYM) theory. In section 2 we give in brief, a self-contained review of the supersymmetric quantum mechanics algebra. In section 3 we connect the localized solutions of the BPS equations of motion, with an $N = 2$ supersymmetric quantum mechanics algebra, and relate the total number of solutions to the Witten index. In section 4 we generalize the latter to the case of three matter curves. Finally in section 5 we present the conclusions.

1 F-theory, D7-Branes Intersections and 8-dimensional SYM

Singularities on complex manifolds play an important role in string theory phenomenology [1,4,6,11]. Actually exceptional gauge groups are realized by the geometry of singularities. For instance, real codimension four theories with $N = 1$ supersymmetry descend from M-theory defined on a seven dimensional manifold with $G_2$ holonomy [1,9,11]. The compactified wrapped space is $C^2/\Gamma$, with the last being a singularity. Within the F-theory setup, $N = 1$, $D = 4$ supersymmetric gauge theories arise when F-theory is compactified on Calabi-Yau fourfolds [3,4,9,11]. The geometric techniques that can be defined on complex manifolds enable us to build theories with rich phenomenology [4,66,69,72]. We shall take into account F-theory compactifications on Calabi-Yau fourfold that is actually an elliptic $K3$ fibration over a complex dimension two surface $S$. Locally the theory is described by the worldvolume of a D7 brane of ADE type which wraps $R^{1,3} \times S$ over the Calabi-Yau fourfold. The Kähler surface $S$ is wrapped by the seven brane and it’s geometry can shrink to zero size inside a threefold base so we can assume that gravity decouples from the theory. The resulting theory in four dimensions is an $N = 1$ supersymmetric theory [4,11]. Our analysis is based on references [4,11].
The gauge group \(G_S\) in F-theory is generated by codimension one singularities which are the world volume of D7-branes. The D7-branes wrap a complex dimension two surface \(S\). Let \(H_S\) be the gauge group of a supersymmetric gauge field on \(S\), and also suppose \(H_S \subset G_S\). This gauge field configuration breaks the gauge group \(G_S\) down to \(H_S\), thus making possible to reduce the symmetries down, close to the (usual) GUT ones. Specifically when the gauge group is of the form \(H_S' \times U(1)\), the resulting spectrum contains chiral matter that come from the zero modes of bulk fields of the surface \(S\). Of course the way to generate the \(H_S\) supersymmetric gauge configuration is to give a vacuum expectation value to an appropriate adjoint scalar of \(G_S\).

Gauge groups are realized as ADE singularities on the complex surface \(S\). These singularities can be enhanced over complex curves \(\Sigma\) on the complex surfaces \(S\). The curves \(\Sigma\) are actually Riemann surfaces and arise as the intersection locus of the surface \(S\) and another surface \(S'\). As a result bifundamental matter fields are defined on the intersection \(\Sigma\). Moreover a non-trivial background gauge field on \(\Sigma\) can induce chiral four dimensional matter.

We now describe in brief the field configurations that reside in the complex surface \([4,11]\).

The physics of the D7-branes wrapping \(S\) is given in terms of \(D=8\) twisted Super Yang-Mills on \(R^3,1 \times S\). The supersymmetric multiplets include the gauge field plus a complex scalar \(\varphi\) and the adjoint fermions \(\eta, \psi, \chi\).

Let \((z_1, z_2)\) be the local coordinates that parameterize complex surface \(S\). Then the field content of the supermultiplets is:

\[
A = A_\mu dx^\mu + A_m dz^m + A_\bar{m} d\bar{z}^\bar{m}, \quad \varphi = \varphi_{12} dz^1 \wedge dz^2
\]

and also,

\[
\psi_a = \psi_{a1} dz^1 + \psi_{a2} dz^2, \quad \chi_a = \chi_{a12} dz^1 \wedge dz^2
\]

In the above, \(\psi\) is a \((0,1)\)-form while \(\varphi\) and \(\chi\) are \((2,0)\)-forms (we dropped the subscript \(a\) for simplicity). The fermion \(\eta\) is a \((0,0)\)-form. Also \(a = 1,2\) and \(m = 1,2\).

The \(N=1, D=4\) supersymmetric theory consists of the gauge multiplet \((A_\mu, \eta)\) together with the chiral multiplets \((A_m, \psi_m)\) and \((\varphi_{12}, \chi_{12})\), plus their complex conjugates.

The \(D=8\) effective action can be integrated over the compact surface \(S\) thus obtaining the \(D=4\) multiplet dynamics. In reference to the Yukawa couplings, the most interesting term is the superpotential,

\[
W = M_*^4 \int_S \text{Tr}(F_S^{(0,2)} \wedge \Phi) = W = M_*^4 \int_S \text{Tr}(\partial A \wedge \Phi) + M_*^4 \int_S \text{Tr}(A \wedge A \wedge \Phi)
\]

with \(M_*\) the mass scale of the supergravity low energy limit of F-theory. The chiral superfields \(A\) and \(\Phi\) have components,

\[
A_{\bar{m}} = A_{\bar{m}} + \sqrt{2} \theta \psi_{\bar{m}} + \cdots
\]

and also,

\[
\Phi_{\bar{m}} = \phi_{12} + \sqrt{2} \theta \chi_{12} + \cdots
\]

where \(\cdots\) involves auxiliary fields. Only the \((0,2)\) component of the superstrength appears in \([11]\) and \([13]\). In order to address the zero mode problem, we must find the equations.
of motion that stem from the $D = 8$ effective action. Omitting any kinetic terms, the bilinear in the fermions, part of the action is equal to,

$$I_F = \int_{R^{1,3} \times S} dx^4 \text{Tr}\left( \chi \wedge \partial_A \psi + 2 i \sqrt{2} \omega \wedge \partial_A \eta \wedge \psi + \frac{i}{2} \psi \wedge [\varphi, \psi] + \sqrt{2} \eta [\bar{\varphi}, \chi] + \text{h.c.} \right)$$  \hspace{1cm} (6)

where $\omega$ is the fundamental Kähler form of the complex surface $S$. The variations of $\eta$, $\psi$ and $\chi$, yield the equations of motion \[4, 11\]:

$$\omega \wedge \partial_A \psi + \frac{i}{2} [\bar{\varphi}, \chi] = 0$$ \hspace{1cm} (7)

$$\bar{\partial}_A \chi - 2 i \sqrt{2} \omega \wedge \partial \eta - [\varphi, \psi] = 0$$

$$\bar{\partial}_A \psi - \sqrt{2} [\bar{\varphi}, \eta] = 0$$

In regard to the bosonic fields, the field strength $F_s$ must have vanishing $(2,0)$ and $(0,2)$ components and hence it must satisfy the BPS condition,

$$\omega \wedge F_s + \frac{i}{2} [\varphi, \bar{\varphi}] = 0$$ \hspace{1cm} (8)

Additionally, the complex scalar field must satisfy the holomorphicity condition $\bar{\partial}_A \varphi = 0$.

The charged massless multiplets in $D = 4$ (localized zero modes) are specified by the vacuum expectation value of the adjoint scalar $\varphi$ and also from the background gauge field. When $\langle \varphi \rangle = 0$, the equations of motion imply that the number of zero modes of $\psi$ and $\chi$ are determined by topological invariants that depend both on $S$ and the background gauge bundle.

In the next section we briefly present the supersymmetric quantum mechanics algebra which we shall frequently use in the subsequent sections.

2 Supersymmetric Quantum Mechanics

We review some issues related to supersymmetric quantum mechanics \[80, 81\] relevant to our analysis. Consider a quantum system, described by a Hamiltonian $H$ and characterized by the set \{ $H, Q_1, ..., Q_N$ \}, with $Q_i$ self-adjoint operators. The quantum system is called supersymmetric, if,

$$\{ Q_i, Q_j \} = H \delta_{ij}$$ \hspace{1cm} (9)

with $i = 1, 2, ..., N$. The $Q_i$ are called supercharges and the Hamiltonian “$H$” is called SUSY Hamiltonian. The algebra (9) constitutes the so called N-extended supersymmetry. Due to the anti-commutativity one has,

$$H = 2 Q_1^2 = Q_2^2 = \ldots = 2 Q_N^2 = \frac{2}{N} \sum_{i=1}^{N} Q_i^2.$$ \hspace{1cm} (10)

A supersymmetric quantum system \{ $H, Q_1, ..., Q_N$ \} is said to have “good susy” (unbroken supersymmetry), if its ground state vanishes, that is $E_0 = 0$. In the case $E_0 > 0$, that is for a positive ground state energy, susy is said to be broken.
For good supersymmetry, the Hilbert space eigenstates must be annihilated by the supercharges,

\[ Q_i |\psi_j^{0}\rangle = 0 \]  

(11)

for all \( i, j \).

In this paper we shall use the \( N = 2 \) supersymmetric quantum mechanics algebra. For convenience we shall refer to it as “\( N = 2 \) SUSY QM”, or simply “SUSY QM”. We present in brief the basic features of it. The \( N = 2 \) algebra consists of two supercharges \( Q_1 \) and \( Q_2 \) and a Hamiltonian \( H \), which obey the following relations,

\[ \{ Q_1, Q_2 \} = 0, \quad H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2 \]  

(12)

It is convenient for the our purposes to introduce the operator,

\[ Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \]  

(13)

and the adjoint,

\[ Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \]  

(14)

The above two satisfy the following equations,

\[ Q^2 = |Q|^2 = 0 \]  

(15)

and also are related to the Hamiltonian as,

\[ \{ Q, Q^\dagger \} = H \]  

(16)

The Witten parity, \( W \) for a \( N = 2 \) algebra is defined as,

\[ [W, H] = 0 \]  

(17)

and

\[ \{ W, Q \} = \{ W, Q^\dagger \} = 0 \]  

(18)

Also \( W \) satisfies,

\[ W^2 = 1 \]  

(19)

By using \( W \), we can span the Hilbert space \( \mathcal{H} \) of the quantum system to positive and negative Witten parity spaces. The last are defined as, \( \mathcal{H}^\pm = P^\pm \mathcal{H} = \{ |\psi\rangle : W|\psi\rangle = \pm |\psi\rangle \} \). Therefore the quantum system Hilbert space \( \mathcal{H} \) is decomposed into the eigenspaces of \( W \), hence \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \). Each operator acting on the vectors of \( \mathcal{H} \) can be represented by \( 2N \times 2N \) matrices. We use the representation:

\[ W = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]  

(20)
with \( I \) the \( N \times N \) identity matrix. Recall that \( Q^2 = 0 \) and \( \{Q, W\} = 0 \), hence the supercharges are of the form,

\[
Q = \begin{pmatrix}
0 & A \\
0 & 0
\end{pmatrix}
\]  
(21)

and

\[
Q^\dagger = \begin{pmatrix}
0 & 0 \\
A^\dagger & 0
\end{pmatrix}
\]  
(22)

which imply,

\[
Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & A \\
A^\dagger & 0
\end{pmatrix}
\]  
(23)

and also,

\[
Q_2 = \frac{i}{\sqrt{2}} \begin{pmatrix}
0 & -A \\
A^\dagger & 0
\end{pmatrix}
\]  
(24)

The \( N \times N \) matrices \( A \) and \( A^\dagger \), are generalized annihilation and creation operators. The action of \( A \) is defined as \( A : \mathcal{H}^- \rightarrow \mathcal{H}^+ \) and that of \( A^\dagger \) as, \( A^\dagger : \mathcal{H}^+ \rightarrow \mathcal{H}^- \). In the representation (20), (21), (22) the quantum mechanical Hamiltonian \( H \), can be cast in a diagonal form,

\[
H = \begin{pmatrix}
AA^\dagger & 0 \\
0 & A^\dagger A
\end{pmatrix}
\]  
(25)

Therefore for a \( N = 2 \) supersymmetric quantum system, the total supersymmetric Hamiltonian \( H \), consists of two superpartner Hamiltonians,

\[
H_+ = AA^\dagger, \quad H_- = A^\dagger A
\]  
(26)

We can define an operator \( P^\pm \). The eigenstates of \( P^\pm \), denoted as \( |\psi^\pm\rangle \) are called positive and negative parity eigenstates. These satisfy,

\[
P^\pm |\psi^\pm\rangle = \pm |\psi^\pm\rangle
\]  
(27)

Using the representation (20), the parity eigenstates are represented in the form,

\[
|\psi^+\rangle = \begin{pmatrix}
|\phi^+\rangle \\
0
\end{pmatrix}
\]  
(28)

and also,

\[
|\psi^-\rangle = \begin{pmatrix}
0 \\
|\phi^-\rangle
\end{pmatrix}
\]  
(29)

with \( |\phi^\pm\rangle \in H^\pm \).

“Good supersymmetry” has some implications on the ground states. For “good supersymmetry”, there must be at least one state in the Hilbert space with vanishing energy eigenvalue, that is \( H|\psi_0\rangle = 0 \). The Hamiltonian commutes with the supercharges, \( Q \) and \( Q^\dagger \), hence it is obvious that, \( Q|\psi_0\rangle = 0 \) and \( Q^\dagger|\psi_0\rangle = 0 \). For a ground state with negative parity,

\[
|\psi_0^-\rangle = \begin{pmatrix}
|\phi_0^-\rangle \\
0
\end{pmatrix}
\]  
(30)
this would imply that $A|\phi_0^+\rangle = 0$, while for a positive parity ground state,

$$|\psi_0^+\rangle = \begin{pmatrix} 0 \\ |\phi_0^+\rangle \end{pmatrix}$$

it would imply that $A^\dagger|\phi_0^+\rangle = 0$. Generally a ground state can have either positive or negative Witten parity. In the case the ground state is degenerate both cases can occur. When it happens $E \neq 0$, the number of positive parity eigenstates is equal to the negative parity eigenstates. This does not occur for the zero modes. A rule to decide if there are zero modes is given by the Witten index. Let $n_\pm$ the number of zero modes of $H_\pm$ in the subspace $\mathcal{H}^\pm$. For finite $n_+$ and $n_-$ the quantity,

$$\Delta = n_- - n_+$$

is called the Witten index. When the Witten index is non-zero integer, supersymmetry is good-unbroken. If the Witten index is zero, it is not clear whether supersymmetry is broken (which would mean $n_+ = n_- = 0$) or not ($n_+ = n_- \neq 0$).

The Fredholm index of the operator $A$ defined as,

$$\text{ind}A = \dim \ker A - \dim \ker A^\dagger = \dim \ker A^\dagger A - \dim \ker AA^\dagger$$

is obviously related to the Witten index as follows,

$$\Delta = \text{ind}A = \dim \ker H_- - \dim \ker H_+$$

The Fredholm index is a topological invariant and we shall only consider Fredholm operators.

3 Fields at D7-branes Intersections

As we mentioned in the previous sections, the localized fields are on a matter curve $\Sigma$ which is the intersection of the complex surfaces $S$ and $S'$. The preservation of $N = 1$ supersymmetry in $D = 4$, requires the theory (that actually describes matter) defined on $\mathbb{R}^{1,3} \times \Sigma$ to be $D = 6$ twisted super Yang-Mills [4,11]. The six dimensional twisted supermultiplets include two complex scalars and a Weyl spinor. In four dimensions this decomposes into the dimension four chiral multiplets $(\sigma, \lambda)$ and $(\sigma^c, \lambda^c)$ plus the CPT-conjugate. The number of zero modes is given by topological invariants inherent to the curve $\Sigma$ and the gauge bundle in $G_\Sigma$ (we shall say more on this, at the conclusion section). In this section we shall relate the number of zero modes to the Witten index of an $N = 2$ SUSY QM algebra. Let the $D = 8$ theory on $S$ have a gauge group $G_\Sigma$. We give a coordinate dependent vacuum expectation value to the adjoint scalar, of the form,

$$\langle \varphi \rangle = m^2 z_1 Q_1$$

with $z_1$ a complex coordinate on $S$, $Q_1$ a $U(1)$ generator of the Cartan subalgebra of $G_\Sigma$ and $\varphi = \varphi_{12}$. The parameter “$m$” is a mass parameter we used, in order $\varphi$ has the correct
dimensions. When the vacuum expectation value of $\varphi$ is non zero, the $D = 8$ fields have zero modes localized at $z_1 = 0$ that are connected to the fields appearing in $\Sigma$ \cite{4}. When $z_1 = 0$, the gauge group is unbroken, but when $z_1 \neq 0$, the group breaks to,

$$G_S \times U(1) \subset G_\Sigma$$  \hspace{1cm} (36)

and the generators of $G_S$ commute with $Q_1$. The locus $z_1 = 0$ defines the intersection curve $\Sigma$ and hence at $z_1 = 0$ the symmetry enhances to $G_\Sigma$.

In reference to D7-branes, an adjoint scalar corresponds to degrees of freedom in the transverse direction, and thus a non zero vacuum expectation value means that some branes are separated and the gauge group is broken. For example if a brane is moved from a stack of $K + 1$ D7-branes the original $SU(K+1)$ symmetry is broken to $SU(K) \times U(1)$. This way, the resulting theory at the intersection contains localized massless bifundamentals of the form $(K, -1) \oplus (\bar{K}, +1)$. The massless bifundamentals coming from the adjoint of $G_\Sigma$ are classified in terms of the irreducible representations of $G_S \times U(1)$. For interesting examples of this see \cite{4}.

In order to find how localized fermion matter on $\Sigma$ results from zero modes of the $D = 8$ bulk theory, we must solve the $D = 8$ equations of motion for the twisted fermions. The assumptions are that $\varphi$, has a non-trivial $z_1$-dependent vacuum expectation value, like in (35) and that no background gauge fields (usually $U(1)$’s) are present. The localized fermions correspond to the $\lambda$ and $\lambda^c$ fermions of the twisted Super Yang-Mills on $R^{1,3} \times \Sigma$.

Let the Kähler form of $\Sigma$ be of the form,

$$\omega = \frac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2)$$  \hspace{1cm} (37)

The coordinates of $S$, $z_1$ and $z_2$, parameterize the intersection $\Sigma$ in transverse and tangent directions respectively. With $\omega$ being of the form (37) and neglecting the $z_2$ derivatives, the equations of motion can be cast as \cite{4}:

$$\sqrt{2}\partial_1 \eta - m^2 z_1 q_1 \psi_2 = 0 \hspace{1cm} \partial_1 \psi_1 - m^2 z_1 q_1 \chi = 0$$  \hspace{1cm} (38)

$$\partial_1 \psi_2 - \sqrt{2} m^2 z_1 q_1 \eta = 0 \hspace{1cm} \bar{\partial}_1 \chi - m^2 z_1 q_1 \bar{\psi}_1 = 0$$

with $\chi = \chi_{12}$. In the above, $q_1$ stands for the $U(1)$ fermion charge belonging to a representation $(R, q_1)$ of $G_S \times U(1)$. It is obvious that there are no localized solutions for the fermions $\eta$ and $\psi_2$ \cite{11}. On the contrary the solutions for $\chi$ and $\psi_1$ are:

$$\chi = f(z_2)e^{-q_1 m^2 |z_1|^2}, \hspace{0.5cm} \psi_1 = -f(z_2)e^{-q_1 m^2 |z_2|^2}.$$  \hspace{1cm} (39)

The solutions are peaked around $z_1 = 0$. Also the constant $q_1 m^2$ is of the order of the F-theory mass scale $M_5^2$. There are similar equations of motion stemming from the hermitian conjugate terms in relation (37) which we omitted for simplicity. These act on the conjugate fermions $\bar{\psi}$ and $\bar{\chi}$.

Let us see how the number of the zero modes corresponding to the fields $\psi_1, \chi$, is related to the Witten index. We define,

$$D = \begin{pmatrix} \partial_1 & -m^2 z_1 q_1 \\ -m^2 z_1 q_1 & \bar{\partial}_1 \end{pmatrix}$$  \hspace{1cm} (40)
and additionally,

\[
D^\dagger = \begin{pmatrix}
\bar{\partial}_1 & -m^2 \bar{z}_1 q_1 \\
-m^2 z_1 q_1 & \partial_1
\end{pmatrix}
\]  

(41)

acting on,

\[
\begin{pmatrix}
\psi_1 \\
\chi
\end{pmatrix}
\]  

(42)

The solutions of the equations of motion (38) for \(\psi\) and \(\chi\) are the zero modes of \(D\). The Fredholm index \(I_D\) of the operator \(D\), is equal to,

\[
\text{ind}_D = I = \dim \ker(D^\dagger) - \dim \ker(D)
\]  

(43)

which is the number of zero modes of \(D\) minus the number of zero modes of \(D^\dagger\).

Using \(D\) we can define an \(N = 2\) supersymmetric quantum mechanical system. Indeed we can write,

\[
Q = \begin{pmatrix}
0 & D \\
0 & 0
\end{pmatrix}
\]  

(44)

and additionally,

\[
Q^\dagger = \begin{pmatrix}
0 & 0 \\
D^\dagger & 0
\end{pmatrix}
\]  

(45)

Also the Hamiltonian of the system can be written,

\[
H = \begin{pmatrix}
DD^\dagger & 0 \\
0 & D^\dagger D
\end{pmatrix}
\]  

(46)

It is obvious that the above matrices obey, \(\{Q, Q^\dagger\} = H\), \(Q^2 = 0\), \(Q^\dagger^2 = 0\), \(\{Q, W\} = 0\), \(W^2 = I\) and \([W, H] = 0\). Thus we can relate the Witten index of the \(N = 2\) supersymmetric quantum mechanics system, to the index \(I_D\) of the operator \(D\). Indeed we have \(I_D = -\Delta\), because,

\[
I_D = \dim \ker D^\dagger - \dim \ker D = \dim \ker DD^\dagger - \dim \ker D^\dagger D = -\text{ind} D = -\Delta = n_- - n_+
\]  

(47)

with \(n_-\) and \(n_+\) defined above equation (32). Owing to the supersymmetric quantum mechanical structure of the system, the zero modes of the operators \(D\) and \(D^\dagger\) are related to the zero modes of the operators \(DD^\dagger\) and \(D^\dagger D\). The above imply that the zero modes of the operator \(DD^\dagger\) and also of \(D^\dagger D\) can be classified according to the Witten parity, to parity positive and parity negative solutions.

If the theory defined on \(\Sigma\) can be viewed as a cosmic string defect on \(S^4\), the classification of states in parity even and parity odd states, could be useful. This is due to the fact that, if someone could define a movement of the localized modes, the classification in parity even and odd states could be an analogue of L-movers and R-movers of the superconducting string case (of course this assumption could be true if some restrictions are imposed on the function \(f(z_2)\)). Actually GUT inspired fermionic zero modes in superconducting strings backgrounds, can be classified to parity even and parity odd states, owing to an underlying SUSY QM algebra [82].
4 Intersecting Matter Curves, Fermionic Zero Modes and SUSY QM

Consider now three matter curves. These matter curves can intersect at a point. We denote the three matter curves as $\Sigma_i$, with $i = 1, 2, 3$. Each matter curve has a group $G_i$, that on the intersection point further enhances to a higher group $G_p$. In order to find the localized fermionic solutions of the eight dimensional theory on $S$ one must have a non-trivial background for the adjoint scalar, as in the previous section. Next, the solutions must be found for each matter curve [4,11].

In the case we are studying, a more involved vacuum expectation value is needed, which looks like [4,11]:

$$\langle \varphi \rangle = m^2 z_1 Q_1 + m^2 z_2 Q_2$$

(48)

In the above, $Q_1$ and $Q_2$ are the $U(1)$ generators that are included in the enhancement group $G_p$, at the intersection point, and “$m_1$” and “$m_2$” are mass scales that are related to the F-theory mass scale $M_*$. Taking $m_1 = m_2 = m$ will simplify things but will not change the results.

The above form of the vacuum expectation value of the adjoint scalar field resolves the $G_p$ singularity at the intersection point. We suppose the intersection point is $(z_1, z_2) = (0, 0)$. The three different curves $\Sigma_1, \Sigma_2, \Sigma_3$ are defined by the loci $z_1 = 0$, $z_2 = 0$ and $z_1 + z_2 = 0$ respectively. Each curve represents a fermion and we can say that under the $U(1)$ charges, the curves are classified according to the following table,

| matter curve | $(q_1, q_2)$ | locus     |
|--------------|-------------|-----------|
| $\Sigma_1$  | $(q_1, 0)$  | $z_1 = 0$ |
| $\Sigma_2$  | $(0, q_2)$  | $z_2 = 0$ |
| $\Sigma_3$  | $(-q_1, -q_2)$ | $z_1 + z_2 = 0$ |

Classification of the matter curves

where the constants $(q_1, q_2)$ are the $U(1)$ charges of the fermions belonging to an irreducible representation $(R, q_1, q_2)$ of $G_S \times U(1)_1 \times U(1)_2$ (note that $Q_1$ generates $U(1)_1$ and $Q_2$ generates $U(1)_2$). With the adjoint vacuum expectation value being that of relation (48) the equations of motion are:

$$\partial_2 \psi_2 + \partial_1 \psi_1 - m^2 (z_1 q_1 + z_2 q_2) \chi = 0$$

(49)

$$\partial_1 \chi - m^2 (z_1 q_1 + z_2 q_2) \psi_1 = 0$$

$$\partial_2 \chi - m^2 (z_1 q_1 + z_2 q_2) \psi_2 = 0$$

4.1 Localized fermion around $z_1 = 0$

In the case of the curve $\Sigma_1$, we have $q_2 = 0$. The localized fermions are at $z_1 = 0$. Localized solutions to the equations of motion (49) are [11]:

$$\psi_2 = 0, \quad \chi = f(z_2) e^{-q_1 m^2 |z_1|^2}, \quad \psi_1 = -\chi.$$  

(50)
with $f(z_2)$ a holomorphic function of $z_2$. We can associate a $N = 2$ SUSY QM algebra to this matter curve. Using the notation of the previous section, we can define the matrix $D_1$ and also $D_1^\dagger$ as follows,

$$D_1 = \left(\begin{array}{cc}
\frac{\partial}{\partial_1} & -m^2 (\bar{z}_1 q_1 + \bar{z}_2 q_2) \\
-m^2 (z_1 q_1 + z_2 q_2) & \frac{\partial}{\partial_1}
\end{array}\right)$$

and,

$$D_1^\dagger = \left(\begin{array}{cc}
\bar{\partial}_{\bar{1}} & -m^2 (\bar{z}_1 q_1 + \bar{z}_2 q_2) \\
-m^2 (z_1 q_1 + z_2 q_2) & \bar{\partial}_{\bar{1}}
\end{array}\right)$$

acting on,

$$\left(\begin{array}{c}
\psi_\bar{1} \\
\chi
\end{array}\right)$$

Then all the results of the previous section, apply in this case. We must note that the SUSY QM structure exists if $\psi_2 = 0$ on this matter curve. In the converse case, we cannot define a matrix like $D_1$.

### 4.2 Localized fermion around $z_2 = 0$

In the case of the curve $\Sigma_2$, we have $q_1 = 0$. The localized fermions are at $z_2 = 0$. Localized solutions to the equations of motion (49) are:

$$\psi_2 = -\chi, \quad \chi = g(z_2) e^{-q_2 m^2 |z_1|^2}, \quad \psi_{\bar{1}} = 0.$$  

with $g(z_1)$ a holomorphic function of $z_1$. Thus the $N = 2$ SUSY QM algebra can be defined in terms of the $D_2$ matrix, which is equal to:

$$D_2 = \left(\begin{array}{cc}
\bar{\partial}_2 & -m^2 (\bar{z}_1 q_1 + \bar{z}_2 q_2) \\
-m^2 (z_1 q_1 + z_2 q_2) & \bar{\partial}_2
\end{array}\right)$$

### 4.3 Localized fermion around $z_1 + z_2 = 0$

The matter curve $\Sigma_3$, has generic charges $q_1$ and $q_2$. To make things easier we make the following transformations:

$$w = z_1 + z_2, \quad \psi_w = \frac{1}{2} (\psi_{\bar{1}} + \psi_2)$$

$$u = z_1 - z_2, \quad \psi_u = \frac{1}{2} (\psi_{\bar{1}} - \psi_2)$$

Then the equations of motion (49) can be cast as:

$$2 \partial_w \psi_w + 2 \partial_u \psi_u - m^2 \left(\bar{w}(q_1 + q_2) + u(q_1 - q_2)\right) \chi = 0$$

$$2 \bar{\partial}_w \chi - m^2 \left(w(q_1 + q_2) + u(q_1 - q_2)\right) \psi_w = 0$$

$$2 \bar{\partial}_u \chi - m^2 \left(w(q_1 + q_2) + u(q_1 - q_2)\right) \psi_u = 0$$
In the same way as in the two previous cases a $N = 2$ SUSY QM algebra underlies the fermion system when $\bar{\psi}_u = 0$. Indeed we can define the matrices $D_3$ and $D_3^\dagger$ as:

$$D_3 = \begin{pmatrix} 2 \partial_w & -m^2 \left( \bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2) \right) \\ -m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) & 2 \bar{\partial}_w \end{pmatrix}$$

(58)

and,

$$D_3^\dagger = \begin{pmatrix} 2 \bar{\partial}_w & -m^2 \left( \bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2) \right) \\ -m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) & 2 \partial_w \end{pmatrix}$$

(59)

acting on,

$$\begin{pmatrix} \psi_\bar{w} \\ \chi \end{pmatrix}$$

(60)

The localized solutions to the equations of motion (57) around $z_1 + z_2 = 0$ are:

$$\psi_\bar{w} = \frac{1}{\sqrt{2}} \chi, \quad \chi = g(u)e^{-\frac{m^2 |w|^2}{\sqrt{2}}}$$

(61)

In conclusion each matter curve corresponds to an underlying $N = 2$ SUSY QM algebra that can be built using the three matrices $D_1$, $D_2$ and $D_3$. In addition the zero modes of $D_1$, $D_2$ and $D_3$ correspond to the solutions of (57).

Before closing we must note that an $N = 2$ supersymmetric structure was expected in the system since, the dimensional reduction of the intersection of two D7-branes gives rise to a four-dimensional $N = 2$ sector of the full effective field theory.

5 Conclusions

In this article we have connected the system of fermion matter fields that exist at the intersection locus of D7-branes, with a supersymmetric quantum mechanics algebra, within the theoretical framework of F-theory. We found that only the localized fields along the complex dimension one intersection (Riemann surface) are related to the SUSY QM algebra, and we examined three (phenomenologically important) matter curves and their localized solutions. The zero modes solutions of the equations of motion are related to the Witten index of the underlying SUSY QM algebra.

Mention that in order to obtain non-trivial fermion mass hierarchies, non-constant fluxes are needed. We have not checked whether there is an underlying $N = 2$ SUSY QM algebra in the presence of non-constant and also constant fluxes. An interesting case, although we shall not pursuit here.

Before closing, we must mention that there are cohomological techniques that determine the massless spectrum localized on the intersection $\Sigma$ [4]. Indeed (following [4]) if $\mathcal{H}$ denotes the group that remains at the intersection $\Sigma$, $\mathcal{V}_i$ the gauge bundle that transforms
as a representation of $\mathcal{H}$ and $K_\Sigma$ the canonical bundle on $\Sigma$, the net chirality of the zero modes is [4]:

$$n_{\nu_i} - n_{\nu_i}^* = (1 - g)\text{rk}(K_\Sigma^{1/2} \otimes V_i) + \int_\Sigma c_1(K_\Sigma^{1/2} \otimes V_i)$$

(62)

In the above equation, $n_{\nu_i}$ and $n_{\nu_i}^*$ stand for the number of generations and anti-generations respectively, while $g$ is the genus of the curve $\Sigma$.

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