Exact algebraic solution for a quantum harmonic oscillator with time-dependent frequency

D. M. Tibaduiza, L. B. Pires, C. A. D. Zarro, and C. Farina
Instituto de Física, Universidade Federal do Rio de Janeiro, 21941-972 Rio de Janeiro, RJ, Brazil

D. Szilard
Centro Brasileiro de Pesquisas Físicas, 22290-180 Rio de Janeiro, RJ, Brazil

A. L. C. Rego
Instituto de Aplicação Fernando Rodrigues da Silveira, Universidade do Estado do Rio de Janeiro, 20261-232 Rio de Janeiro, Rio de Janeiro, Brazil

Using algebraic methods and appropriate BCH-like relations of Lie algebras we present an alternative way of solving exactly the problem of a harmonic oscillator (HO) with an arbitrary time-dependent frequency. Assuming the HO is initially in its fundamental state, we explicitly show that a time-dependent frequency gives rise to a squeezed state. In our method we develop an iterative approach which allows one to obtain the HO state at any instant and for any time-dependent frequency. Our method is very well suited for numerical calculations used in the analysis of the squeezing parameter and the variances of quadrature operators at any instant. To get confidence in the method, we recover some important results found in the literature. Then, we discuss two frequency modulations that produce strong squeezing, namely, the parametric resonance modulation and the Janszky-Adam scheme. A direct comparison of these two squeezing procedures shows that the latter is the most efficient one.

I. INTRODUCTION

The importance of the harmonic oscillator with time-dependent frequency (HOTDF) cannot be underestimated since it is a natural scenario for the study of many important topics as for instance time-dependent hamiltonians, foundations of quantum mechanics, mathematical theorems involving Lie algebras and the increasingly important squeezed states. Such states appear in quantum optics [18], in cosmology [9-14] and even in some approaches to the dynamical Casimir effect, particularly, those based on analogue models [15-22]. The main property of these states consists in the fact that they provide variances of certain quadratures smaller than the minimum value associated to the coherent states [23-24], making possible the reduction of the quantum noise/signal ratio. This remarkable property has had a great impact in many areas of physics, from metrology [25, 26] and telecommunications [27] until the improvements of interferometric gravitational-wave detectors recently used in the first direct observations of gravitational waves [28, 29].

One could naively think that solving the HOTDF would be as simple as solving the HO under an external arbitrary time-dependent force, both described by time-dependent hamiltonians. But this is far from the truth by the following reason: while the latter problem can be easily solved with the aid of the simple composition law satisfied by the Glauber operator, the problem of a HOTDF involves the more complex squeezing operator and, as shown by Rhodes [30], their composition law is much more intricate. Rhodes solved formally the driven HOTDF expressing the time evolution operator (TEO) at any instant as a product of a squeezing operator, a Glauber operator and a rotation operator, apart from an overall phase factor.

However, though rigorous and very elegant, Rhodes’ solution is not very practical for numerical implementations, since the final expressions are written in terms of functions that satisfy non-linear differential equations involving the time-dependent frequency \( \omega(t) \) under consideration. There is a vast list of authors that have tackled this problem with distinct purposes and approaches, from situations where the mass can also depend on time [37-41] to those cases where driven forces were included and different initial states were considered [42, 43]. Particular cases for sudden or linear frequency variations have been considered [40, 44-48] and some pioneering papers on this subject should also be mentioned [49, 50].

In this work our main purpose is to establish an iterative procedure for solving a HOTDF in such a way that no matter which time-dependent frequency is considered, one will be able to write the quantum state of the system at any instant and with the desired precision. Using algebraic methods and appropriate Baker-Campbell-Hausdorff (BCH)-like relations as in Refs. [36, 37] we establish an iterative approach to obtain the TEO in terms of a recurrence relation that is suitable for numerical calculations. Conveniently, this recurrence relation can be expressed as generalized continuous fractions. To get confidence in our method, we reobtain the variances of quadratures for specific time-dependent fre-
quencies already considered in the literature. Then, we use our method to discuss and compare two frequency modulations that produce strong squeezing, namely, the parametric resonance modulation and the Janszky-Adam scheme. By a direct comparison of these two squeezing procedures we show that the latter is the most efficient one.

This paper is organized as follows. In Section II, we present a detailed discussion of our method and establish the main theoretical result of our paper. In Section III, we illustrate and apply our method in a variety of situations, including an original discussion regarding different but efficient squeezing procedures. Section IV is left for some final remarks and conclusions.

II. HARMONIC OSCILLATOR WITH TIME-DEPENDENT FREQUENCY

Let us consider a one-dimensional harmonic oscillator of unit mass with an arbitrary time-dependent frequency, $\omega(t)$, whose hamiltonian is given by

$$\hat{H}(t) = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \omega^2(t) \hat{q}^2.$$  

(1)

Suppose that, initially ($t = 0$), the HO is in its fundamental state $|0\rangle$. The main purpose of this work is to determine the state of the harmonic oscillator at an arbitrary subsequent time. However, this is not an easy task, since the hamiltonian is a time-dependent one and hence the computation of the TEO is quite intricate. In fact, from its definition

$$|\psi(t)\rangle = \hat{U}(t, 0) |\psi(0)\rangle,$$  

(2)

and the Schrödinger equation (we are using $\hbar = 1$),

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle,$$  

(3)

it is immediate to see that $\hat{U}(t, 0)$ satisfies the following differential equation,

$$i \frac{\partial}{\partial t} \hat{U}(t, 0) = \hat{H}(t) \hat{U}(t, 0),$$  

(4)

with the initial condition $\hat{U}(0, 0) = \mathbb{I}$, whose solution is

$$\hat{U}(t, 0) = \mathbb{I} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \hat{H}(t_1) \hat{H}(t_2) \cdots \hat{H}(t_n).$$  

(5)

The previous equation can be cast into the formal expression

$$\hat{U}(t, 0) = T \left\{ \exp \left[ -i \int_0^t \hat{H}(t') dt' \right] \right\},$$  

(6)

where $T$ means time ordering operator. This expression is known as Dyson series and its application to the problem at hand is extremely difficult. Instead of using Dyson series to solve the HOTDF it is more convenient, as we shall see, to appeal to the composition property of the TEO which follows directly from definition [2], namely,

$$\hat{U}(t, 0) = \hat{U}(t, t_{N-1}) \hat{U}(t_{N-1}, t_{N-2}) \cdots \hat{U}(t_1, 0).$$  

(7)

Although for finite time intervals the expressions of $\hat{U}(t_j, t_{j-1})$, with $j = 1, 2, \ldots, N - 1$, are quite involved since the problem under consideration has a time-dependent hamiltonian (they are given, essentially by Dyson series), if we take $\tau \rightarrow 0$ and $N \rightarrow \infty$ with $N\tau = t$, we can write the TEO as an infinite product of simple infinitesimal time evolution operators, namely,

$$\hat{U}(t, 0) = \lim_{N \rightarrow \infty} e^{-i\hat{H}(N\tau)\tau} e^{-i\hat{H}((N-1)\tau)\tau} \cdots e^{-i\hat{H}(\tau)\tau}.$$  

(8)

Our iterative method is based on the above equation. However, before applying the previous equation to calculate the TEO, we shall write for convenience the hamiltonian given by Eq. (1) in terms of the “annihilation” and “creation” operators, defined by

$$\hat{a} = \sqrt{\frac{\omega_0}{2}} \left( \hat{q} + i \frac{\hat{p}}{\omega_0} \right); \quad \hat{a}^\dagger = \sqrt{\frac{\omega_0}{2}} \left( \hat{q} - i \frac{\hat{p}}{\omega_0} \right).$$  

(9)

where $\omega_0 = \omega(t = 0)$, which satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Inverting these equations and substituting the expressions of operators $\hat{p}$ and $\hat{q}$ in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$ into Eq. (1), we obtain after a straightforward calculation

$$\hat{H}(t) = 2\omega(t) \cosh[2\rho(t)] \hat{K}_c + \omega(t) \sinh[2\rho(t)] \left( \hat{K}_+ + \hat{K}_- \right),$$  

(10)

where we defined

$$\rho(t) := \frac{1}{2} \ln \left( \frac{\omega(t)}{\omega_0} \right),$$  

(11)

as well as the operators

$$\hat{K}_+ := \frac{\hat{a}^\dagger^2}{2}, \quad \hat{K}_- := \frac{\hat{a}^2}{2} \quad \text{and} \quad \hat{K}_c := \frac{\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger}{4}.\quad \text{(12)}$$

It is straightforward to show that the above operators satisfy the following commutation relations

$$[\hat{K}_+, \hat{K}_-] = -2\hat{K}_c \quad \text{and} \quad [\hat{K}_c, \hat{K}_\pm] = \pm \hat{K}_\pm,$$  

(13)

so that they can be identified as the three generators of the $su(1,1)$ Lie algebra. This fact will allow us to
use appropriate BCH-like formulas for this Lie algebra to obtain iteratively the state of the system for any instant of time. As we shall show in the next subsection, the HOTDF will evolve to a squeezed state. Hence, for future convenience, we briefly introduce these states (a detailed discussion on squeezed states can be found in Ref. [53]).

A squeezed state $|z\rangle$ can be obtained by application of the squeezing operator $\hat{S}(z)$ on the fundamental state, $|z\rangle = \hat{S}(z)|0\rangle$, with $\hat{S}(z)$ defined by

$$
\hat{S}(z) \equiv \exp \left\{ \frac{z}{2} \hat{a}^\dagger - \frac{z^*}{2} \hat{a} \right\},
$$

where $z = re^{i\varphi}$ is a complex number. With the aid of ordering theorems the squeezed state can be written as a superposition of the even energy eigenstates [53]

$$
|z\rangle = \sqrt{\text{sech}(r)} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left[ -\frac{1}{2} e^{i\varphi} \tanh(r) \right]^n |2n\rangle.
$$

Note that $z$, and hence $r$ and $\varphi$, determines uniquely the squeezed state. In order to interpret $r$ and $\varphi$, it is convenient to introduce the quadrature operator $\hat{Q}_\lambda$, defined by [53]

$$
\hat{Q}_\lambda = \frac{1}{\sqrt{2}} \left[ e^{i\lambda} \hat{a}^\dagger + e^{-i\lambda} \hat{a} \right].
$$

The quadrature operators satisfy the commutation relation $[\hat{Q}_\lambda, \hat{Q}_{\lambda+\pi/2}] = i$. It is evident from the previous definition that $\hat{Q}_{\lambda=0} = (\hat{a}^\dagger + \hat{a})/\sqrt{2} \propto \hat{q}$ and $\hat{Q}_{\lambda=\pi/2} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2} \propto \hat{p}$. The harmonic oscillator is squeezed if the variance of one of the quadratures is smaller than $\frac{1}{2}$. It can be shown that the variance of the operator $\hat{Q}_\lambda$ in the generic squeezed state $|z\rangle$ written in Eq. (15) is given by [53]

$$
\langle \Delta Q^2_\lambda \rangle = \langle z | \hat{Q}^2_\lambda | z \rangle - (\langle z | \hat{Q}_\lambda | z \rangle)^2
= \frac{1}{2} \left[ e^{2r} \sin^2 (\lambda - \varphi/2) + e^{-2r} \cos^2 (\lambda - \varphi/2) \right].
$$

Note the explicit dependence of $\langle \Delta Q^2_\lambda \rangle$ with $r$ and $\varphi$. Further, from the previous equation we see that

$$
e^{-2r} \leq \langle \Delta Q^2_\lambda \rangle \leq e^{2r},
$$

which justifies the interpretation of $r$ as the squeezing parameter (SP). Parameter $\varphi$ is referred to as the squeezing phase (SP).

### A. Time Evolution

In order to construct an iterative method to compute the TEO of our problem, let us consider a time discretization in small intervals of equally size $\tau$ and let the time-dependent frequency $\omega(t)$ be considered constant in each of these intervals as follows

$$
\omega(t) = \begin{cases}
\omega_0 & \text{for } t \leq 0 \\
\omega_1 & \text{for } 0 < t \leq \tau \\
\omega_j & \text{for } (j-1)\tau < t \leq j\tau \\
\omega_N & \text{for } (N-1)\tau < t \leq N\tau
\end{cases}
$$

where $\omega_j$ can be taken as any value assumed by $\omega(t)$ with $t_{j-1} < t \leq t_j$. For convenience, we choose $\omega_j := \omega(j\tau)$. Recall that $N\tau = t$ and an exact result is obtained only in the limit $N \to \infty$ ($\tau \to 0$). Note also that we chose time-dependent frequencies that are constant and equal to $\omega_0$ from $-\infty$ to $t = 0$, so that the HO remained in its fundamental state from the remote past until $t = 0$. Once $t_{j-1} \leq t \leq \tau$, for any $j$, Eq. (7) takes the form

$$
\hat{U}(t,0) = \hat{U}(N\tau, (N-1)\tau) \cdots \hat{U}(2\tau, \tau) \hat{U}(\tau,0).
$$

Assuming $N$ is as large as we want, we may approximate the hamiltonian in each time interval $t_{j-1} < t \leq t_j$, denoted by $\hat{H}_j$, as a constant one. Hence, from Eq. (10) we may write

$$
\hat{H}_j = 2\omega_j \cosh(2\rho_j) \hat{K}_c + \omega_j \sinh(2\rho_j) \left( \hat{K}_+ + \hat{K}_- \right),
$$

where from Eq. (11) it is clear that

$$
\rho_j = \frac{1}{2} \ln \left( \frac{\omega_j}{\omega_0} \right).
$$

Since all $\hat{H}_j$ are now considered as time-independent hamiltonians, the TEO for each time interval, $\hat{U}(t_j, t_{j-1})$, with $j = 1, 2, \ldots, N$, can be written as

$$
\hat{U}_j := \hat{U}(t_j, t_{j-1}) = e^{-i\hat{H}_j \tau}.
$$

Inserting Eq. (21) into the previous equation, we obtain

$$
\hat{U}_j = e^{\lambda_j^+ \hat{K}_c + \lambda_j^- \hat{K}_-},
$$

where we defined

$$
\lambda_j^+ = \lambda_j^- = -i \omega_j \tau \sinh(2\rho_j),
$$

$$
\lambda_{jc} = -2i \omega_j \tau \cosh(2\rho_j).
$$

Using well known BCH relations of the $su(1,1)$ Lie algebras as given in Ref. [53], it is possible to write Eq. (24) as a product of exponentials of the Lie algebra generators in a suitable order, namely,

$$
\hat{U}_j = e^{\Lambda_{jc} \hat{K}_c + \ln(\lambda_{jc}) \hat{K}_c} e^{\lambda_j^- \hat{K}_-},
$$

where

$$
\Lambda_{jc} = \left( \cosh(\nu_j) - \frac{\lambda_{jc}}{2\nu_j} \sinh(\nu_j) \right)^{-2},
$$

$$
\Lambda_{j\pm} = \frac{2\lambda_j \pm \sinh(\nu_j)}{2\nu_j \cosh(\nu_j) - \lambda_{jc} \sinh(\nu_j)}.
$$
with \( \nu_j \) given by
\[
\nu_j^2 = \frac{1}{4} \lambda_j^2 - \lambda_j + \lambda_j^0.
\] (30)

Inserting Eqs. (25) and (26) into Eq. (30) and substituting the obtained result into Eqs. (28) and (29), it is straightforward to show that
\[
\nu_j = \pm i \omega_j \tau,
\] (31)
\[
\lambda_{j c} = (\cos(\omega_j \tau) + i \cosh(2 \rho_j) \sin(\omega_j \tau))^{-2},
\] (32)
\[
\lambda_{j \pm} = \frac{-i \sinh(2 \rho_j) \sin(\omega_j \tau)}{\cos(\omega_j \tau) + i \cosh(2 \rho_j) \sin(\omega_j \tau)}.
\] (33)

Therefore, using Eqs. (2), (20) and (27) the state of the HOTDF at an arbitrary instant \( t > 0 \) can be written as the following product of operators
\[
|\psi(t)\rangle = e^{\Lambda_n \hat{K}_+} e^{i n \Lambda_c \hat{K}_c} e^{\Lambda_{-n} \hat{K}_-} e^{\Lambda_{n-1} \hat{K}_+} e^{i n \Lambda_{c-1} \hat{K}_c} e^{\Lambda_{-n-1} \hat{K}_-} \ldots \ldots e^{\Lambda_2 \hat{K}_+} e^{i 2 \Lambda_c \hat{K}_c} e^{\Lambda_{-2} \hat{K}_-} e^{\Lambda_1 \hat{K}_+} e^{i \Lambda_{c-1} \hat{K}_c} e^{\Lambda_{-1} \hat{K}_-} |0\rangle.
\] (34)

This formula is not yet suitable for numerical applications but, as we shall see in the next subsection, it is the starting point for the deduction of a very convenient recurrence relation which will be the core of our iterative method.

**B. Recurrence Formula**

In this section we present a recurrence formula which allows us to calculate iteratively the state of the HOT at any instant \( t > 0 \). As we shall show, this state will be a squeezed state. In other words, we shall show that Eq. (34) can be written in the form of Eq. (15). Once we have determined the final squeezed state of the HOT we can immediately write an expression for the squeezing parameter for an arbitrary instant of time, \( r(t) \).

In order to obtain a recurrence formula, we shall proceed inductively. We assume \( \omega(t) \) is constant by parts, as defined in Eq. (19) and start by analyzing the first two frequency abrupt changes, namely, from \( \omega_0 \) to \( \omega_1 \) at \( t = 0 \), and from \( \omega_1 \) to \( \omega_2 \) at \( t = \tau \). For \( t = \tau \) we have
\[
|\psi(\tau)\rangle = U_1(\tau, 0) |0\rangle = e^{\Lambda_1 \hat{K}_+} e^{i \Lambda_{c-1} \hat{K}_c} e^{\Lambda_{-1} \hat{K}_-} |0\rangle.
\] (35)

From Eq. (12), we can calculate
\[
\hat{K}_- |n\rangle = \frac{1}{2} \sqrt{n(n-1)} |n-2\rangle,
\] (36)
\[
\hat{K}_+ |n\rangle = \frac{1}{2} \sqrt{(n+1)(n+2)} |n+2\rangle,
\] (37)
\[
\hat{K}_0 |n\rangle = \frac{1}{2} (n+\frac{1}{2}) |n\rangle.
\] (38)

Using the above equations and the well-known formula
\[
e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^n,
\] (39)
valid for a general operator \( \hat{A} \), it can be shown that
\[
e^{\Lambda_{-1} \hat{K}_-} |0\rangle = |0\rangle
\] (40)
\[
e^{i \Lambda_{c-1} \hat{K}_c} |0\rangle = (\Lambda_{c-1})^{1/4} |0\rangle.
\] (41)

Therefore, Eq. (35) takes the form
\[
|\psi(\tau)\rangle = (\Lambda_{c-1})^{1/4} e^{\Lambda_{c-1} \hat{K}_c} e^{\Lambda_{-1} \hat{K}_-} |0\rangle.
\] (42)

For later convenience, let us rewrite last equation as
\[
|\psi(\tau)\rangle = (\alpha_1)^{1/4} e^{\gamma_1 \hat{K}_c} e^{\lambda_1 \hat{K}_-} |0\rangle,
\] (43)
where we defined
\[
\alpha_1 = \Lambda_{c-1} \quad \text{and} \quad \gamma_1 = \Lambda_{-1}.
\] (44)

Using Eqs. (39) and (48), Eq. (43) reads
\[
|\psi(\tau)\rangle = \sqrt{|\alpha_1|^{1/2}} \sum_{n=0}^{\infty} \sqrt{\frac{2n!}{n!}} \left[ \frac{1}{2} \left| \gamma_1 e^{i \delta_1} \right|^n \right] |2n\rangle,
\] (45)
where the overall phase has been removed by the following definitions:
\[
\alpha_1 = |\alpha_1| e^{i \delta_1}, \quad \text{and} \quad \gamma_1 = |\gamma_1| e^{i \delta_1}.
\] (46)

Comparing Eqs. (45) and (15) we see that if the following identifications are possible,
\[
|\alpha_1|^{1/2} = \text{sech } r; \quad |\gamma_1| = \text{tanh } r \quad \text{and} \quad \delta_1 = \varphi + \pi,
\] (47)
then, we can conclude that \( |\psi(\tau)\rangle \) is a squeezed state with squeezing parameter \( r \). In fact, this is indeed the case and it is straightforward to show that
\[
|\alpha_1| + |\gamma_1|^2 = \text{sech}^2 r + \tanh^2 r = 1.
\] (48)

In order to find the squeezing parameter at instant \( \tau \), \( r(\tau) \), we use the first equation in (47) as well as the first equation in (44) to write
\[
r(\tau) = \text{arccosh} \left( |\Lambda_{c-1}|^{-1/2} \right).
\] (49)

Then, using Eq. (32), with \( j = 1 \), we obtain the squeezing parameter in terms of \( \rho_j \) and \( \omega_1 \) as
\[
r(\tau) = \text{arccosh} \left[ \frac{1}{2} \sinh^2 (2 \rho_j) \sin^2 (\omega_1 \tau) \right]^{1/2}.
\] (50)

The next step is to calculate \( |\psi(2\tau)\rangle \). Applying the TEO for the second infinitesimal interval in Eq. (43), we obtain
\[
|\psi(2\tau)\rangle = (\alpha_1)^{1/4} e^{\Lambda_{c-2} \hat{K}_c} e^{\Lambda_{-2} \hat{K}_-} e^{\gamma_1 \hat{K}_c} e^{\lambda_1 \hat{K}_-} |0\rangle.
\] (51)
In order to proceed, we shall rewrite the exponentials involving $K_+,$ $K_-$ and $\hat{K}_c$ in a convenient order. As it is shown in the Appendix A, Eq. (51) can be written as

$$|\psi(2\tau)| = (\alpha_1)^{1/4} e^{(\Lambda_2 + \Sigma_{2c} \Lambda_2^c)K_c} \cdot e^{i\ln(\Lambda_2 + \Sigma_{2c})K_c} e^{(\Sigma_{2c} + \Lambda_2^c)K_c} |0\rangle,$$

(52)

where

$$\Sigma_{2c} = (1 - \gamma_1 \Lambda_{2c})^{-2},$$

(53)

$$\Sigma_{2+} = \frac{\gamma_1}{1 - \gamma_1 \Lambda_{2c}},$$

(54)

$$\Sigma_{2-} = \frac{\gamma_1 (\Lambda_{2c})^2}{1 - \gamma_1 \Lambda_{2c}}.$$  

(55)

Using Eqs. (40) and (41) into Eq. (52), the state $|\psi(2\tau)|$ can be cast into the form

$$|\psi(2\tau)| = (\alpha_2)^{1/4} e^{(\gamma_2)K_c} |0\rangle,$$

(56)

where we defined

$$\alpha_2 = \frac{\alpha_1 \Lambda_{2c}}{1 - \gamma_1 \Lambda_{2c}},$$

(57)

$$\gamma_2 = \Lambda_{2c} + \frac{\gamma_1 \Lambda_{2c}}{1 - \gamma_1 \Lambda_{2c}}.$$  

(58)

Since Eq. (56) is similar to Eq. (42), the state $|\psi(2\tau)|$ is a squeezed state with the same structure as the state $|\psi(\tau)|$, given by Eq. (45), except by the fact that the subscript has changed from 1 to 2.

In order to apply finite induction, we now assume that the state at instant $(j-1)\tau$ is given by

$$|\psi((j-1)\tau)| = (\alpha_{j-1})^{1/4} e^{(\gamma_{j-1})K_c} |0\rangle.$$  

(59)

If this is so, the state after the next abrupt frequency variation is obtained by applying on Eq. (59) the appropriate TEO, $\hat{U}_j$:

$$|\psi(j\tau)| = (\alpha_{j-1})^{1/4} e^{(\Lambda_{j-1} K_c + \ln(\Lambda_{j-1}) K_c)} \cdot e^{\Lambda_{j-1} K_c} e^{i\ln(\Lambda_{j-1}) K_c} |0\rangle.$$  

(60)

Following the same steps as before, we can write the above equation as

$$|\psi(j\tau)| = (\alpha_{j-1})^{1/4} e^{(\Lambda_2 + \Sigma_{2c} \Lambda_2^c)K_+} \cdot e^{i\ln(\Lambda_2 + \Sigma_{2c})K_+} e^{(\Sigma_{2c} + \Lambda_2^c)K_+} |0\rangle,$$

(61)

where we defined

$$\Sigma_{jc} = (1 - \gamma_{j-1} \Lambda_{j-1})^{-2},$$

(62)

$$\Sigma_{j+} = \frac{\gamma_{j-1}}{1 - \gamma_{j-1} \Lambda_{j-1}},$$

(63)

$$\Sigma_{j-} = \frac{\gamma_{j-1} (\Lambda_{j-1})^2}{1 - \gamma_{j-1} \Lambda_{j-1}}.$$  

(64)

Again, using Eqs. (40) and (41) into Eq. (61), last equation reads

$$|\psi(j\tau)| = (\alpha_j)^{1/4} e^{(\gamma_j)K_c} |0\rangle,$$

where we defined

$$\alpha_j = \frac{\alpha_{(j-1)} \Lambda_{j-1}}{1 - \gamma_{(j-1)} \Lambda_{j-1}},$$

(66)

$$\gamma_j = \Lambda_{j-1} + \frac{\gamma_{(j-1)} \Lambda_{j-1}}{1 - \gamma_{(j-1)} \Lambda_{j-1}}.$$  

(67)

Therefore it has been demonstrated that the final state after $j$ abrupt frequency changes can be written in the closed form

$$|\psi(j\tau)| = \sqrt{|(|\alpha_j|^{1/2} \sum_{n=0}^{\infty} \sqrt{(2n)!} \left[ \frac{1}{2} \frac{\gamma_j}{e^{i\varphi_j}} \right]^n |2n\rangle},$$

(68)

where, again, the overall phase was removed by the redefinitions

$$\alpha_j = |\alpha_j| e^{i\chi_j}, \quad \text{and} \quad \gamma_j = |\gamma_j| e^{i\theta_j}.$$  

(69)

Summing up: we have shown that $|\psi(j\tau)|$ is a squeezed state. The corresponding squeezing parameter and squeezing phase can be computed by comparing Eqs. (68) and (15), a procedure which leads to

$$r(j\tau) = \tanh^{-1} |\gamma_j|, \quad \text{and} \quad \varphi(j\tau) = \varphi_j \pm \pi.$$  

(70)

Observe that the state is totally defined by the complex parameter $\gamma_j$, since its modulus gives the squeezing parameter $r(j\tau)$ and its phase $\varphi_j$ gives $\varphi(j\tau)$. Note also that, for any $j$, the relation

$$|\alpha_j| + |\gamma_j|^2 = 1$$  

(71)

must be satisfied, as it can be straightforwardly checked.

Finally, we can restate our results into a recurrence formula, suitable for numerical implementations. For convenience, we define

$$a_j = \Lambda_{j+} \quad \text{and} \quad b_j = \Lambda_{j-}.$$  

(72)

Hence, Eqs. (60) and (67) are written generically for the $m$-th jump as

$$\alpha_m = \frac{\alpha_{(m-1)} b_m}{(1 - \gamma_{(m-1)} a_m)^2},$$

(73)

$$\gamma_m = a_m + b_m \gamma_{m-1} \frac{1 - \gamma_{m-1} a_m}{1 - \gamma_{m-1} a_m}.$$  

(74)

Notice that Eq. (74) enables an easy numerical implementation, therefore allowing us to calculate the dynamics of the main properties of the final state for any frequency function at any time. Interestingly, Eq. (74) is a generalized continued fraction (GCF):

$$\gamma_m = a_m - \frac{b_m}{a_m - \frac{b_m}{a_m - \frac{b_m}{a_m - \ldots}}}$$  

(75)
The above expression is a new kind of GCF for which some topics such as convergence can be investigated. Generalized continued fractions lie in the context of complex analysis and are specially useful to study analyticity of functions as well as number theory among other fields. For an interested reader we suggest Ref.[54].

III. RESULTS AND DISCUSSIONS

In this section, we apply our iterative method described previously in a variety of situations. Initially, in order to check the consistency of our method, we recover in Subsection III A some interesting results obtained by Adams and Janszky [41]. In this reference the authors considered time-dependent frequencies that return asymptotically to their original values as $t \to \infty$. In the other two subsections, we consider time-dependent frequencies corresponding to two very efficient ways of generating squeezing states, namely, the parametric resonance and the so-called Janszky-Adam (J-A) scheme [45]. Choosing appropriately the parameters for both methods, we compare them and show that squeezing in the Adams-Janszky scheme is stronger than in the parametric resonance model. Despite our results are valid for any initial frequency, for convenience we choose in all our numerical calculations $\omega_0 = 1$.

A. Checking the method

In order to get confidence in our method, in this subsection we use our method to recover a well known result of the literature involving a harmonic oscillator with a time-dependent frequency. Our main purpose here is to obtain the squeezing parameter as well as the variance of a quadrature operator for the system discussed in Ref. [41]. Following Janszky’s paper [41] we consider a non-oscillatory frequency function of the type

$$
\omega(t) = \begin{cases} 
\omega_0 & \text{for } t \leq 0 \\
\omega_0 \left[1 + \frac{\omega_0 t}{2} \exp \left(-\frac{\omega_0 t}{B}\right)\right] & \text{for } t > 0,
\end{cases}
$$

(76)

where $B$ is a positive parameter. In Fig.1.a we plot the previous frequency as a function of time for different values of $B$. Note that these frequencies are non-monotonic functions that start increasing but after passing by their maximum values return monotonically and asymptotically to their original values. Also, note that the larger the parameter $B$, the longer it takes to the frequency to return to its initial value $\omega_0$.

In Fig.1.b, applying the method developed previously, we plot the squeezing parameter $r(t)$ as a function of time for the three frequencies plotted in Fig.1.a. We see that $r(t)$ has an oscillatory behavior which crudely follows the shape of the corresponding time-dependent frequency. Notice that the oscillations in $r(t)$ tend to cease as the frequency asymptotically returns to its origi-
inal value and the squeezing parameter evolves to a constant value which depends on \(B\). This is a direct consequence of the fact that the final value of the frequency is the same as the initial one.

In Fig.(1(c) applying again our method we plot the time evolution of the quadrature variance. This variance has an oscillatory behavior even after the oscillations in the squeezing parameter tends to cease. This oscillatory behavior even after \(r(t)\) has achieved a constant value is due solely to the squeezing phase term present in Eq. (17). As it can be checked by a direct comparison, our results are in total agreement with those appearing in Janszky’s paper [11]. Since the time-dependent frequencies considered in this subsection are quite non-trivial, we can be very confident with our method. It is worth mentioning that in Janszky’s paper [11] only the time-dependence of the quadrature is plotted, but not the time-dependence of the squeezing parameter.

**B. Parametric Resonance**

The parametric resonance condition in a harmonic oscillator can be studied, for instance, by using the following frequency function

\[
\omega(t) = \begin{cases} 
\omega_0 & \text{for } t \leq 0 \\
\frac{\omega_0}{1.02 - 0.02 \cos (\epsilon \omega_0 t)} & \text{for } t > 0
\end{cases}
\]  

(77)

where some numerical values have been fixed for latter convenience. In Fig.(2(a) we plot the above time-dependent frequency as a function of time for different values of the parameter \(\epsilon\). This parameter allows one to tune the parametric resonance phenomenon which occurs when the time-dependent term in the previous equation varies sinusoidally with time with a frequency which is precisely twice the value of the constant term, denoted by reference frequency \(\omega_R = 1.02 \omega_0\). Frequency \(\omega_R\) is the time average value of the harmonic oscillator frequency \(\omega(t)\). The parametric resonance condition is achieved with \(\epsilon = 2.04\). The other values of \(\epsilon\) were chosen so that they are close but smaller than the resonant value.

In Fig.(2(b), we plot the SP as a function of time for different values of \(\epsilon\). The main characteristic shown in this figure is that at resonance condition the average value of the SP grows linearly with time, indefinitely, in contrast to what happens in the non-resonance cases, where the average value of the SP starts growing, achieves a maximum value and then diminishes until it vanishes and then starts the process again, presenting a periodic behavior. Note that as \(\epsilon\) approaches the resonant value, \(\epsilon = 2.04\), the period of oscillation of \(r(t)\) becomes larger, tending to infinite as \(\epsilon \to 2.04\).

\[
\begin{align*}
\text{FIG. 2. (a) The frequency functions and, in the same time interval, the time evolution of (b) the SP and (c) the quadrature variance. In (b), it is also shown the behavior of SP for the initial dynamics. The curves associated to the three different values of the parameter are } & \epsilon = 1.96 \text{ (dashed line), } \\
& \epsilon = 2.0 \text{ (dotted line) and } \epsilon = 2.04 \text{ (solid line and resonance case)}
\end{align*}
\]

\[
\begin{align*}
\text{We could have chosen values for } & \epsilon \text{ close but greater than 2.04, but the results would have been the same.}
\end{align*}
\]
modulation of the oscillations is exponentially increasing.

In Fig. 3 we used Eq. (70) to plot in the complex plane the dynamics of $z = r e^{i\varphi} = \tanh^{-1} |\gamma| e^{i(\varphi \pm \pi)}$, where $|z| = |r e^{i\varphi}|$ is the squeezed state of the system, as it was similarly done in Ref. 35. This is a geometric representation containing all relevant information of the dynamics of the system. The non-resonant cases are plotted in Figs. 3(a) and 3(b). For these cases, the curves are limited and bounded by maximum radii in the complex plane of $z$, since out of resonance the squeezing parameter ($r = |z|$) has a maximum value. The closer to the resonant condition, the bigger the radius in the complex plane of the curves that describe the dynamics of the system. This can be seen by inspection of Figs. 3(a) and 3(b), since the bigger radius corresponds to the bigger value of $\epsilon$. At resonance condition, as shown in Fig. 3(c), there is no enclosing circle and the curve is given by a growing spiral since $|z|$ increases exponentially indefinitely.

C. Janszky-Adam scheme × parametric resonance

The Janszky-Adam (J-A) scheme is known as a very strong squeezing model by frequency modulation in the harmonic oscillator [55]. This is so because it uses sudden jumps between two fixed frequencies appropriately synchronized [45, 46] and these abrupt frequency changes produce a high degree of squeezing [40]. In Fig. (4.a), the time dependent frequency of the HO in the Janszky-Adam model is plotted. It consists, as mentioned above, of periodic sudden jumps between two constant frequencies, namely $\omega_0$ and $\omega_1$ (chosen to be 1.0 and 1.5 in arbitrary units in Fig. (4.a)). The respective time intervals in each frequency are suitable chosen to optimize the increasing in the SP.

In Fig. (4.b), we apply our method to plot the SP as a function of time corresponding to such a frequency modulation. First, note that there is an increasing of $r(t)$ only when the frequency jumps from $\omega_0$ to $\omega_1$, but not when the frequency jumps back from $\omega_1$ to the initial frequency $\omega_0$. In fact, after the frequency abruptly changes from $\omega_1$ to its original value $\omega_0$ the SP remains constant in time until the next jump from $\omega_0$ to $\omega_1$. This can be understood in the following way: after the jump from $\omega_0$ to $\omega_1$ we showed that the state of the HO is a squeezed state of the original hamiltonian $\hat{H}_0$ (a HO with constant frequency $\omega_0$), and it is known that the time evolution described by $e^{-i\hat{H}_0 t/\hbar}$ of a squeezed state with respect to hamiltonian $\hat{H}_0$ does not change the value of the SP (though the variance of a quadrature operator oscillates with time due to its dependence on the squeezing phase $\varphi$). That is why in Fig. (4.b) we have plateaus whenever $\omega(t) = \omega_0$. Our description should be contrasted with that appearing in Refs. [45, 46], where it is suggested that the SP suffer abrupt (discontinuous) changes as the frequency jumps from $\omega_0$ and $\omega_1$ and from $\omega_1$ back to $\omega_0$. However, this is only an apparent disagreement since here the SP is always considered with respect to the original hamiltonian $\hat{H}_0$, while in the above mentioned papers, though not explicitly stated, squeezing is considered with respect to the instantaneous hamiltonian.

It is worth mentioning that since finite changes in the HO frequency cause only finite changes in the corresponding hamiltonian, the physical state of the HO evolves continuously in time since

$$\lim_{\delta \to 0} e^{i\hat{H}_0 \delta/\hbar} |\psi(t)\rangle = |\psi(t)\rangle.$$
FIG. 4. (a) Plot of the frequency modulation function of the J-A scheme. (b) The time-evolution of the SP. (c) The dynamics of the complex number $z$ characterizing the final state.

Hence, the same thing occurs with the SP, it can not suffer discontinous changes, unless it is defined with respect to the instantaneous hamiltonian (which is not our case). In Ref. [48], we analyze a simplified version of the Janszky-Adam model which consists of one sudden frequency change from $\omega_0$ to $\omega_1 > \omega_0$ (at $t = 0$) followed by another sudden change from $\omega_1$ back to the initial frequency $\omega_0$ after a time interval $T$. We obtain an exact analytical solution with the aid of algebraic methods based on Lie algebras and use this problem to unveil some qualitative aspects of squeezing processes by abrupt frequency changes. Particularly, we show why there is no change in the SP when the frequency jumps back to its original.

In Fig. (4.c) the dynamics of $|\psi(t)\rangle = |z = re^{i\phi}\rangle$ is depicted through the time evolutions of the real and imaginary parts of $z$. As expected, in the complex $z$-plane, the curve exhibits a spiral like behavior, since the modulus of $z$ increases without bound.

Finally, in order to compare which process between the parametric resonance model and the Janszky-Adam scheme is more effective in squeezing the HO, we plot in Fig. (5.a) the frequency modulations corresponding to these two models. Of course, in order to our comparison make sense, we must choose appropriately the parameters in both models. Since the parametric resonance model to be used is that described by Eq. (77), it is natural to choose both modulations between the same minimum and maximum frequency values given by 1.00 and 1.04 (in arbitrary units). In Fig. (5.b) we plot the SP as a function of time for both models with the above choices for the parameters involved. Although both curves have the same general form and show squeezing parameters that in-
crease without bound, it is evident from In Fig. (b) that
the Janszky-Adam scheme is more efficient to squeeze than
the parametric resonance.

IV. CONCLUSIONS

In this paper, using algebraic methods and appropri-
ate BCH-like relations of Lie algebras we developed an
iterative method for solving the problem of a harmonic
oscillator with an arbitrary time-dependent frequency.
Although the problem of a harmonic oscillator with a
time dependent frequency had already been solved ex-
actly (see, for instance, Ref.[35]), our method has the
advantage of being very well adapted for numerical calcu-
lations no matter the time dependence of the frequency.
In other methods, only a few particular cases of time-
dependent frequencies can be handled easily. As it was
already known in the literature, we have shown that a
time-dependent frequency gives rise to a squeezed state.
Our results enable us to follow the state of the system at
time and with the desired precision. As a consistence
test, and to get more confidence in our method, we first
recovered some important results found in the literature
[11]. Then, we considered other important cases, namely,
(i) the parametric resonance model and (ii) the Janszky-
Adam scheme. By computing the squeezing parameter
and the variances of quadrature operators for these mod-
els, we showed that the latter is the most efficient method
for squeezing. We think our method may be useful for
a deeper understanding of squeezing procedures. More-
ever, since the HO with a time-dependent frequency ap-
pears in different areas in physics, from quantum optics
to quantum field theory in flat space-time (for instance in
the dynamical Casimir effect) as well as in curved space-
times (for instance in cosmological particle creation), we
hope our method may inspire alternative ways of attacking
the problem of particle creation in time-dependent back-
grounds in general.

ACKNOWLEDGMENTS

The authors acknowledge R. Acosta Diaz, D. R. Her-
rera, L. Garcia, C. M. D. Solano, Reinaldo F. de Melo e
Souza, M. V. Congo-Pinto and P.A. Maia Neto for en-
tlightening discussions. The authors thank the brazilian
agencies for scientific and technological research CAPES,
CNPq and FAPERJ for partial financial support.

Appendix A: Mathematical details of Subsection 11B

In this appendix, we show how to obtain Eq. (56) from
Eq. (51). First, we must reorder the exponentials in Eq.

\[ e^{A_2 - K_- e^{\gamma_1 K_+}} = \left\{ e^{A_2 - K_- e^{\gamma_1 K_+}} e^{-A_2 - K_-} \right\} e^{A_2 - K_-} \]  

(A1)

From the BCH relations [53], the above brace triple op-
erator product is written as

\[ e^{A_2 - K_- e^{\gamma_1 K_+}} = e^{(\sigma_2 + \hat{K}_+ + \sigma_2 - \hat{K}_-)} e^{A_2 - K_-} \]  

(A2)

where

\[ \sigma_2^+ = \gamma_1, \]
\[ \sigma_2^- = 2 \gamma_1 A_{2-}, \]
\[ \sigma_2 = \gamma_1 (A_{2-})^2. \]  

(A3)

Proceeding similarly as done to obtain Eq. (27), we get

\[ e^{A_2 - K_- e^{\gamma_1 K_+}} = e^{\Sigma_2^+ \hat{K}_+ e^{\ln(\Sigma_2^c)} K_+ e^{\Sigma_2^- \hat{K}_- e^{A_2 - \hat{K}_-}}} \]  

(A4)

where

\[ \Sigma_2^c = \left( \cosh(\beta_2) - \frac{\sigma_{2c}}{2 \beta_2} \sinh(\beta_2) \right)^2, \]  

(A5)

\[ \Sigma_2^+ = \frac{2 \sigma_{2+} \sinh(\beta_2)}{2 \beta_2} \cosh(\beta_2) - \sigma_{2c} \sinh(\beta_2), \]  

(A6)

\[ \beta_2^2 = \frac{1}{4} \sigma_{2c}^2 - \sigma_{2+} \sigma_{2-}, \]  

(A7)

Substituting Eq. (A3) into Eqs. (A5), (A6) and (A7),
we find:

\[ \beta_2 = 0, \]  

(A8)

\[ \Sigma_2^c = (1 - \gamma_1 A_{2-})^2, \]  

(A9)

\[ \Sigma_2^+ = \frac{\gamma_1}{1 - \gamma_1 A_{2-}} = \gamma_1 (\Sigma_2^c)^2 \]  

and \( \Sigma_2^- = \frac{\gamma_1 (A_{2-})^2}{1 - \gamma_1 A_{2-}} = \gamma_1 (A_{2-})^2 (\Sigma_2^c)^2. \)  

(A10)

(A11)

Hence Eq. (51) becomes

\[ |\psi(2\tau)\rangle = (\alpha_1)^{1/4} e^{A_{2+} \hat{K}_+ e^{\ln(\Sigma_2^c)} K_+ e^{\Sigma_2^+ \hat{K}_+}} e^{\Sigma_2^- \hat{K}_- e^{A_2 - \hat{K}_-}} |0\rangle. \]  

(A12)

\[ e^{\ln(\Sigma_2^c)} K_+ e^{\ln(\Sigma_2^- \hat{A}_- \hat{K}_-)} |0\rangle. \]  

(A13)

Using Eqs. (40) and (41) into Eq. (13), we have

\[ |\psi(2\tau)\rangle = (\alpha_1 \Sigma_2^c)^{1/4} e^{A_{2+} \hat{K}_+ e^{\ln(\Sigma_2^c)} K_+ e^{\Sigma_2^+ \hat{K}_+}} |0\rangle. \]  

(A14)

Now, following the same steps performed in Eqs. (A1) and
(A2), we obtain

\[ e^{\ln(\Sigma_2^c)} K_+ e^{\Sigma_2^+ \hat{K}_+} = \left\{ e^{\ln(\Sigma_2^c)} K_+ e^{\Sigma_2^+ \hat{K}_+} e^{-\ln(\Sigma_2^c)} K_+ \right\} e^{\ln(\Sigma_2^c)} K_+ \]  

(A15)

So finally we get

\[ |\psi(2\tau)\rangle = (\alpha_1 A_{2c} \Sigma_2^c)^{1/4} e^{(\Sigma_2^+ + \Sigma_2^c \Sigma_2^c) K_+} |0\rangle, \]  

which is exactly Eq. (50).
