Superunitary operator and BRST transformations of a non-Abelian two-form

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Abstract – Using the superunitary operator approach, we derive (anti-)BRST symmetry transformations for the non-Abelian 2-form gauge theories by studying the gauge symmetries of a new Lagrangian which involves coupling of matter fields not only to 1-form field but also to a 2-form field.

Introduction. – Just as 1-form (4-vector potential) couples to charged point particle, \(p\)-forms couple to higher-dimensional objects, such as strings, membranes, etc. A generalisation of the 1-form gauge theories to \(p\)-form gauge theories has been a long discussed problem in theoretical physics [1–8]. Since a 2-form couples to surface, it is natural to think of it as a gauge field for (open or closed) strings. Although this is a consistent picture for Abelian 2-form, it is problematic for its non-Abelian counterpart owing to the difficulties in defining surface-ordered exponentials, which appear when a 2-form is coupled to the world surface of a string [1]. Nevertheless, non-Abelian 2-forms have appeared in the context of the nonlinear sigma model [3,4], in the loop space formulations of Yang-Mills theory [5,6] and gravity as a gauge theory [7,8].

Within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism [9–12], one approach to \(p\)-form gauge theories is the superfield approach [13–17]. A superfield is a function of superspace, which is a 4-dimensional Minkowski spacetime augmented with two additional Grassmann coordinates \(\theta\) and \(\bar{\theta}\). In the superfield approach, BRST transformations for the (non-)Abelian 1-form gauge and corresponding ghost and anti-ghost fields can be derived exploiting the so-called horizontality condition. This condition is basically equating the supercurvature 2-form defined on the 6-dimensional superspace to the ordinary curvature 2-form defined on the 4-dimensional Minkowski spacetime. The local superfield quantisation of gauge theories can be found in [18–20]. To include interacting systems where the gauge field couples to matter fields, the superfield approach has been consistently generalised to obtain the BRST transformations for the matter fields as well, which is called the augmented superfield approach [21–24], where, in addition to the horizontality condition, some gauge-invariant restrictions are also exploited. The mapping of ordinary fields on the Minkowski spacetime to the superspace on the superspace can also be carried out via a superspace unitary operator [13–15,25,26], or superunitary operator for short. The superunitary operator upgrades the fields and gauge connections to their superspace counterparts in the same fashion as the unitary gauge operator maps the fields and gauge connections to their gauge-transformed counterparts. This superunitary operator is determined from the horizontality condition and gauge-invariant restrictions.

Our goal in this paper is to deduce the BRST transformations for the Kalb-Ramond 2-form \(B\) field, following the superunitary operator approach. In order to consider the interacting theory where the matter fields interact with the 1-form as well as the 2-form gauge field, we introduce a 2-form covariant object \(\Omega\). The two gauge fields, too, interact with each other. A particular choice of \(\Omega\) is needed to reproduce the BRST transformations of the Kalb-Ramond 2-form field given in [27–29].

Although the transformations of the \(B\) field have earlier appeared in the literature, we present here a new way of deducing these transformations based on the superunitary operator approach.

We start with a brief review, in the next section, of 1-form gauge theories, BRST transformations and the idea...
of superfields and superunitary operator. In the third section, we extend the superunitary operator approach to 2-form non-Abelian gauge theories. We consider the interaction of the matter fields with both the 1-form and the 2-form gauge fields, which interact with each other as well. We also include the kinetic term for the 2-form field and obtain the BRST transformations for the various fields. We summarise our results in the last section.

BRST transformations and superunitary operator approach for 1-form gauge theories. – In this section we present a brief review of the BRST transformations, the augmented superfield approach [1,2,21] and the superunitary operator approach [13–15,25,26], which also helps to set up notations. We outline the general procedure and state important results without going into detailed derivations.

We start with the BRST-invariant Lagrangian density for the interacting Abelian 1-form gauge theory:

\[ L_1 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \]
\[ + \frac{1}{2} B^2 + B(\partial_\mu A^\mu) - i\partial_\mu \bar{C}\partial^\mu C, \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \) \( D_\mu \psi = \partial_\mu \psi + i A_\mu \psi, \) \( B(x) \) is the Nakamichi-Lautrup auxiliary scalar field and the Fadeev-Popov (anti-)ghost fields \((C)C\) are virtual scalar fields satisfying the Grassmann-odd properties \( C^2 = 0, C^2 = 0, CC = C \). This theory is invariant under the following (anti-)BRST symmetry transformations:

\[ s_b A_\mu = \partial_\mu C, \quad s_b C = 0, \quad s_b \bar{C} = iB, \quad s_b B = 0, \]
\[ s_b \bar{\psi} = -iC\psi, \quad s_b \psi = -i\bar{\psi}C, \]
\[ s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} \bar{C} = 0, \]
\[ s_{ab} B = 0, \quad s_{ab} \psi = -i\bar{C}\psi, \quad s_{ab} \bar{\psi} = -i\bar{\psi}C. \]

These transformations are anti-commuting, \( s_b s_{ab} + s_{ab} s_b = 0, \) and off-shell nilpotent, \( s_b^2 = s_{ab}^2 = 0. \)

One way to obtain the symmetry transformations (2) is the augmented superfield approach [1,2,13–15,21,25,26]. This approach relies on the extension of the ordinary spacetime to a superspace which has a space of Grassmann-odd coordinates, \( \theta \) and \( \bar{\theta} \) (satisfying \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \)), attached to each and every point of ordinary spacetime. Any field or operator in ordinary spacetime gets upgraded to its superspace-counterpart. A general expansion of the upgraded spinor fields is written as

\[ \psi(x) \rightarrow \Psi(x, \theta, \bar{\theta}) = \psi(x) + \theta \bar{p}(x), \]
\[ \bar{\psi}(x) \rightarrow \bar{\Psi}(x, \theta, \bar{\theta}) = \bar{\psi}(x) + \bar{\theta} p(x), \]
\[ \bar{\psi}(x) \rightarrow \bar{\Psi}(x, \theta, \bar{\theta}) = \bar{\psi}(x) + \bar{\theta} \bar{p}(x), \]
\[ \bar{\psi}(x) \rightarrow \bar{\Psi}(x, \theta, \bar{\theta}) = \bar{\psi}(x) + \bar{\theta} \bar{p}(x), \]

where the auxiliary fields \( p, \bar{p}, r \) and \( \bar{r} \) have Grassmann-even character while the auxiliary fields \( q \) and \( t \) are of the Grassmann-odd character. The 1-form gauge field \( A = A_\mu(x) dx^\mu \) is upgraded as

\[ A(x) \rightarrow \bar{A}(x, \theta, \bar{\theta}) = d\bar{\psi}(x) + d\bar{\theta} \bar{F}(x, \theta, \bar{\theta}) \]
\[ = E(x, \theta, \bar{\theta}) + d\bar{\theta} \bar{F}(x, \theta, \bar{\theta}) \]
\[ + d\bar{\theta} \bar{F}(x, \theta, \bar{\theta}), \]

where the superfields \( E, F \) and \( \bar{F} \) can be further decomposed as

\[ E(x, \theta, \bar{\theta}) = A(x) + \bar{\theta} \bar{R}(x) + \theta \bar{R}(x) + \theta \bar{H}(x), \]
\[ F(x, \theta, \bar{\theta}) = C(x) + \theta \bar{K}(x) + \bar{\theta} K(x) + \theta \bar{S}(x), \]
\[ \bar{F}(x, \theta, \bar{\theta}) = \bar{C}(x) + \theta L(x) + \bar{\theta} L(x) + \theta \bar{C}(x), \]

Obviously, the auxiliary fields \( R, \bar{R}, S \) and \( T \) have Grassmann-odd character while the fields \( H, K, \bar{K}, L \) and \( \bar{L} \) have Grassmann-even character. Various auxiliary fields \( R, \bar{R}, S, T \) have Grassmann-odd character while the fields \( H, K, \bar{K}, L \) and \( \bar{L} \) have Grassmann-even character.

Now comparing eqs. (7) with the generic expansion of a superfield \( G(x, \theta, \bar{\theta}) \) in terms of its ordinary counterpart \( G(x) \),

\[ G(x, \theta, \bar{\theta}) = G(x) + \theta s_{ab} G(x) + \bar{\theta} s_b G(x) + \theta \bar{\theta} (s_b s_{ab} G(x)), \]

and keeping in mind the correspondence \( \Psi(x, \theta, \bar{\theta}) \rightarrow E(x, \theta, \bar{\theta}), \) \( C(x) \rightarrow F(x, \theta, \bar{\theta}), \) \( \bar{C}(x) \rightarrow \bar{F}(x, \theta, \bar{\theta}), \) \( \psi(x) \rightarrow \Psi(x, \theta, \bar{\theta}), \) \( \bar{\psi}(x) \rightarrow \bar{\Psi}(x, \theta, \bar{\theta}) \), the (anti-)BRST transformations (2) follow. It is worthwhile to mention here that the (anti-)BRST transformations \( s_b \) and \( s_{ab} \) are connected with the translation generators \( \partial_\theta \) and \( \partial_{\bar{\theta}} \), respectively, along the Grassmannian directions of the superspace.

It is often convenient and more intuitive to introduce the so-called superunitary operator [13–15,25,26] which upgrades the fields and gauge connections on ordinary spacetime to their counterparts on the superspace in the same fashion as the ordinary fields and gauge connection are mapped to their gauge-transformed counterparts by a unitary operator. Thus, the fields \( \psi, \bar{\psi} \) and \( A \) upgrade as

\[ \psi(x) \rightarrow \Psi(x, \theta, \bar{\theta}) = \bar{U}(x, \theta, \bar{\theta}) \psi(x), \]
\[ \bar{\psi}(x) \rightarrow \bar{\Psi}(x, \theta, \bar{\theta}) = \bar{\psi}(x) \bar{U}(x, \theta, \bar{\theta}), \]
\[ A(x) \rightarrow \bar{A}(x, \theta, \bar{\theta}) = \bar{U}(x, \theta, \bar{\theta}) A(x) \bar{U}(x, \theta, \bar{\theta}), \]

where \( \tilde{\phi}(x, \theta, \bar{\theta}) = i \tilde{A}(x, \theta, \bar{\theta}) \tilde{U}(x, \theta, \bar{\theta}) \) and \( \tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}} \) is the extension of the ordinary exterior derivative \( d \). From eqs. (5), (7) and (9) one can identify
the superunitary operator $\hat{U}(x, \theta, \bar{\theta})$ as

$$\hat{U}(x, \theta, \bar{\theta}) = 1 - i\theta \partial C + i\theta C + \theta \partial (B + C)C = \exp[-i(\theta \partial C + \partial C + i\theta \partial B)],$$

$$\hat{U}^\dagger(x, \theta, \bar{\theta}) = 1 + i\theta \partial C + i\partial C - \theta \partial (B - C)C = \exp[i(\theta \partial C + \partial C + \theta \partial B)],$$

which expectedly satisfies the unitarity condition, $\hat{U}\hat{U}^\dagger = 1 = \hat{U}\hat{U}^\dagger$.

For the non-Abelian case the BRST-invariant Lagrangian density can be written as

$$\mathcal{L}_2 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) + B \cdot (\partial_\mu A^\mu) - i\partial_\mu \bar{C} \cdot D^\mu C,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu \times A_\nu$. The coupled but equivalent Lagrangian density can be obtained from the above Lagrangian density by using the celebrated Curuci-Ferrari condition, $B + \bar{B} + i(C \times C) = 0$, as

$$\mathcal{L}_3 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \bar{B} \cdot (\partial_\mu A^\mu) - iD_\mu \bar{C} \cdot \partial^\mu C.$$

These theories are invariant under the following (anti-)BRST transformations:

$$s_B A_\mu = D_\mu C, \quad s_B C = -iC, \quad s_B = iB, \quad s_\bar{B} = 0, \quad s_\bar{\psi} = i\bar{\psi} C, \quad s_\bar{\psi} = -i\bar{\psi} C,$$

where $\mathcal{L}_\alpha = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \bar{B} \cdot (\partial_\mu A^\mu) - iD_\mu \bar{C} \cdot \partial^\mu C.$

Following the same procedure as in above Abelian case, the form of the superunitary operator in this case turns out to be

$$\hat{U}(x, \theta, \bar{\theta}) = 1 - i\theta \partial C + i\partial C + \theta \partial (C + B)C = \exp[-i(\theta \partial C + \partial C + i\theta \partial B)],$$

$$\hat{U}^\dagger(x, \theta, \bar{\theta}) = 1 + i\theta \partial C + i\partial C - \theta \partial (C + B)C = \exp[i(\theta \partial C + \partial C + \theta \partial B)].$$

BRST transformations for non-Abelian 2-form gauge field. – A string is a 1-dimensional object which when moves in the background spacetime, traces out a 2-dimensional surface in spacetime. (For bosonic string, mathematical consistency of the theory requires the number of spacetime dimensions to be $D = 26$.) The Nambu-Goto action for a free string is given by

$$S_f = -\int d\tau d\sigma \left[ \left( \frac{\partial X^\mu}{\partial \tau} \right)^2 - \left( \frac{\partial X^\nu}{\partial \tau} \right) \left( \frac{\partial X^\nu}{\partial \sigma} \right) \right]^{1/2},$$

where $\tau$ and $\sigma$ are the usual world-sheet parameters and $X^\mu(\tau, \sigma)$ are the string coordinates in the target space. The action for an interacting string, in the background of a Kalb-Ramond 2-form gauge field $B = \frac{1}{2}(dx^\mu \wedge dx^\nu)B_{\mu\nu}$, can be written as

$$S = S_f - \frac{1}{2}\int d\tau d\sigma B_{\mu\nu}(X(\tau, \sigma)) \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} - \frac{1}{2(3!)^2}\int d\tau d\sigma d\lambda H_{\mu\nu\lambda} H^\mu_{\nu\lambda},$$

where $H = dB = \frac{1}{2}(dx^\mu \wedge dx^\nu \wedge dx^\lambda)H_{\mu\nu\lambda}$, with $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$, is the curvature 3-form corresponding to the $B$-field, while $M^{[\nu\mu]}$ is a shorthand notation for $M^{\mu\nu} - M^{\nu\mu}$. In light of this, the Kalb-Ramond Lagrangian density, for the non-Abelian case, can be written as

$$\mathcal{L}_4 = \frac{1}{12}H_{\mu\nu\lambda} H^\mu_{\nu\lambda}.$$

This is one way of introducing a 2-form in a theory. Another way, which does not necessarily require higher-dimensional spacetime, is to introduce a 2-form field interacting with Dirac spinors. This 2-form must involve the derivative of the 1-form gauge field, i.e., $dA$. Let us call this new object $\hat{\mathcal{U}}$. Then we can construct a Lagrangian density of the form

$$\mathcal{L}_5 = -\frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + i\bar{\psi} \Sigma^{\mu\nu} \bar{\Omega}^\mu_\nu \psi$$

$$+ \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) + B \cdot \partial_\mu A^\mu - i\bar{D}_\mu \bar{C} \cdot \partial^\mu C,$$

where $\Sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Exploiting the Curuci-Ferrari condition, $B + \bar{B} + i(C \times C) = 0$, a coupled but equivalent Lagrangian density can be written as

$$\mathcal{L}_6 = -\frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + i\bar{\psi} \Sigma^{\mu\nu} \bar{\Omega}^\mu_\nu \psi$$

$$+ \frac{1}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - \bar{B} \cdot \partial_\mu A^\mu - iD_\mu \bar{C} \cdot \partial^\mu C.$$
The 2-form $\mathcal{U}$ here is interacting with the Dirac spinors and at this stage there is no reason to ignore its interaction with the 1-form gauge connection $A$. As is obvious, $\bar{\psi} U \psi$ must transform as $(\bar{\psi} U \psi)' = U(\bar{\psi} U \psi)$ for the interaction term (the term involving $\mathcal{U}$) to be invariant under the usual SU(2) gauge symmetry transformations:

$$\psi(x) \rightarrow \psi'(x) = U(x) \psi(x),$$
$$\bar{\psi}'(x) = \bar{\psi}'(x) U^T(x),$$
$$A \rightarrow A' = U A U^\dagger + \phi, \quad \mathcal{F}' = U \mathcal{F} U^\dagger, \quad \phi = i d U U^\dagger. \tag{21}$$

The transformation of the $B$-field given in [27–29] can be obtained in our approach by choosing a suitable combination of 2-form and 1-form connections. As we demonstrate now, the particular combination $\mathcal{U} = B - i d A$, up to a multiplicative parameter, meets this requirement. We see that

$$(\bar{\psi} U \psi)' = \left( B' - i d A' \right) \bar{\psi}' = \left[ B' U - i d (U A U^\dagger + \phi) U \right] \psi, \tag{22}$$

where we have made use of (21) in the second step. Equating this with

$$U(\bar{\psi} U \psi) = U(B - i d A) \psi = [U B - i U d A] \psi, \tag{23}$$

we get $B' U - i d (U A U^\dagger + \phi) U = U B - i U d A$ which yields the gauge transformation of the $B$-field:

$$B' = U B U^\dagger + \phi \wedge U A U^\dagger + U A U^\dagger \wedge \phi + \phi \wedge \phi, \tag{24}$$

where we have used $\phi = -i d U d^\dagger$ and $d \phi + i \phi \wedge \phi = 0$, which follows from the unitarity condition $U U^\dagger = U^\dagger U = 1$.

Now we include the kinetic term for the 2-form fields and also the interaction between the 1-form and the 2-form in the Lagrangian density:

$$\mathcal{L}_7 = \frac{1}{12} W_{\mu \nu \lambda} \cdot \mathcal{F}^{\mu \nu \lambda} - \frac{1}{4} \mathcal{F}^{\mu \nu} \cdot \mathcal{F}^{\mu \nu}$$
$$+ \bar{\psi} (i \gamma^\mu D_{\mu} + i \Sigma^{\mu \nu} B_{\mu \nu} - m) \psi$$
$$+ \frac{1}{2} (B \cdot B + B \cdot \dot{B} + B \cdot \partial_\mu \mathcal{A}^\mu)$$
$$- i \partial_\mu C \cdot D^\mu C, \tag{25}$$

where

$$W = dB + i (dF + A \wedge B - B \wedge A) \tag{26}$$

is the usual field strength 3-form, which in component form reads

$$W_{\mu \nu \lambda} = \partial_\mu B_{\nu \lambda} + i A_{\mu} \times \partial_\nu A_{\lambda} - A_{\mu} \times B_{\nu \lambda} + \text{cyclic terms}$$
$$= \partial_\mu B_{\nu \lambda} + i (B_{\mu \nu} - i \partial_\nu A_{\mu} A_{\lambda}) \times A_{\lambda} + \text{cyclic terms}$$
$$= \partial_\mu B_{\nu \lambda} + i \partial_\mu \nu \lambda \times A_{\lambda} + \text{cyclic terms}. \tag{27}$$

It follows that

$$(D \wedge \mathcal{U} - \mathcal{U} \wedge D) \psi = W \psi. \tag{28}$$

Thus the field strength 3-form $W$, given in (26), can be written as $W = D \wedge \mathcal{U} - \mathcal{U} \wedge D$, which is analogous to the corresponding expression for the field strength 2-form $\mathcal{F}$ in the 1-form non-Abelian gauge theory: $D \wedge D = i \mathcal{F}$.

The transformation of the 3-form $\mathcal{W}$ follows from the definition (26) and eqs. (21) and (24). We provide some intermediate steps here.

$$d B' = d (U B U^\dagger + \phi \wedge U A U^\dagger + U A U^\dagger \wedge \phi + \phi \wedge \phi)$$
$$= d U \wedge B U^\dagger + U d B U^\dagger + U B \wedge d U^\dagger$$
$$- i d U \wedge d U^\dagger \wedge U A U^\dagger$$
$$+ i U d U^\dagger \wedge U A U^\dagger - i d U \wedge d A U^\dagger$$
$$- i U d A \wedge d U^\dagger$$
$$- i U A \wedge d U^\dagger \wedge d U U^\dagger$$
$$+ i U A U^\dagger \wedge d U \wedge d U^\dagger, \tag{29}$$

$$i d \mathcal{F}' = i d (U F U^\dagger) = i d (U (d A + i A \wedge A) U^\dagger)$$
$$= i d U \wedge d A U^\dagger + i U d A \wedge d U^\dagger$$
$$- d U \wedge A \wedge A U^\dagger$$
$$- U d A \wedge A U^\dagger + U A \wedge d A U^\dagger$$
$$- U A \wedge A \wedge d U^\dagger, \tag{30}$$

$$A' \wedge B' = (U A U^\dagger + \phi) \wedge (U B U^\dagger + \phi \wedge U A U^\dagger)$$
$$+ U A U^\dagger \wedge \phi \wedge \phi$$
$$= U A \wedge B U^\dagger + U A \wedge \phi \wedge U A U^\dagger$$
$$+ U A \wedge U A U^\dagger \wedge \phi$$
$$+ U A U^\dagger \wedge \phi \wedge \phi$$
$$+ U U A U^\dagger \wedge \phi$$
$$+ U U A U^\dagger \wedge \phi \wedge \phi \wedge \phi, \tag{31}$$

$$B' \wedge A' = (U B U^\dagger + \phi \wedge U A U^\dagger)$$
$$+ U A U^\dagger \wedge \phi \wedge \phi \wedge \phi \wedge (U A U^\dagger + \phi)$$
$$= U B \wedge A U^\dagger + U B \wedge \phi \wedge \phi \wedge U A U^\dagger$$
$$+ U A U^\dagger \wedge \phi \wedge \phi$$
$$+ U U A U^\dagger \wedge \phi \wedge \phi$$
$$+ U U A U^\dagger \wedge \phi \wedge \phi \wedge \phi \wedge \phi. \tag{32}$$

From (31) and (32) it follows that

$$i (A' \wedge B' - B' \wedge A') = i U (A \wedge A - B \wedge A) U^\dagger$$
$$+ i d U \wedge A \wedge A U^\dagger - U B \wedge U^\dagger, \tag{33}$$

which along with (29) and (30) finally gives the transformation of the 2-form $\mathcal{W}$:

$$\mathcal{W}' = d B' + i (d F' + A' \wedge B' - B' \wedge A')$$
$$= U [d B + i (d F + A \wedge B - B \wedge A)] U^\dagger$$
$$= U \mathcal{W} U^\dagger. \tag{34}$$

In order to obtain the (anti-)BRST symmetry transformations for the $B$-field, we again apply the superunitary operator approach:

$$\tilde{B} = \tilde{U} B U^\dagger + \tilde{\phi} \wedge \tilde{U} A U^\dagger + \tilde{U} A U^\dagger \wedge \tilde{\phi} + \tilde{\phi} \wedge \tilde{\phi},$$
$$\tilde{\phi} = i \tilde{U} d \tilde{U}^\dagger = -i \tilde{U} d \tilde{U}^\dagger. \tag{35}$$
Being a 2-form, $\tilde{B}$ can be written as
\begin{equation}
\begin{aligned}
\tilde{B}(x, \theta, \bar{\theta}) &= \left( dx^\mu \wedge dx^\nu \right) M_{\mu \nu}(x, \theta, \bar{\theta}) + \cdots \\
&= \mathcal{M}(x, \theta, \bar{\theta}) + \cdots ,
\end{aligned}
\end{equation}
where, on the right-hand side, we have written only the term which is relevant for our purpose\(^3\). It is easy to see that if we use (36) on the left-hand side of (35), then we get
\begin{equation}
\begin{aligned}
\mathcal{M} &= \tilde{U} \tilde{B} \tilde{U}^\dagger + \phi' \wedge \tilde{U} \tilde{A} \tilde{U}^\dagger + \tilde{U} \tilde{A}^\dagger \wedge \phi' + \phi' \wedge \phi',
\end{aligned}
\end{equation}
where $\phi' = i d \tilde{U} \tilde{U}^\dagger = -i \tilde{U} d \tilde{U}^\dagger$. With the help of (15), we now compute the right-hand side of (37). We find
\begin{equation}
\begin{aligned}
\tilde{U} \tilde{B} \tilde{U}^\dagger &= B + i \theta [B, \bar{C}] + i \bar{\theta} [B, C] \\
&+ \theta \bar{\theta} \left( [B, B] + \{ [C, B], \bar{C} \} \right),
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\phi' &= \theta \bar{\theta} C + \theta d \bar{\theta} B + \bar{\theta} d \theta C + d d C \bar{C},
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\phi' \wedge \tilde{U} \tilde{A} \tilde{U}^\dagger &= \theta (d \bar{C} \wedge A) + \bar{\theta} (d C \wedge A) \\
&+ i \theta \bar{\theta} (d B \wedge A + \{ d C \wedge A, \bar{C} \}) \\
&+ d \bar{C} \wedge [C, A]),
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\tilde{U} \tilde{A}^\dagger \wedge \phi' &= \theta (A \wedge d \bar{C} + \bar{\theta} (A \wedge d C) \\
&+ i \theta \bar{\theta} (A \wedge d B + \{ A \wedge d C, \bar{C} \}) \\
&+ i \{ A, C \wedge d \bar{C} \},
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\phi' \wedge \phi' &= \theta \bar{\theta} (d C \wedge d C - d \bar{C} \wedge d \bar{C}).
\end{aligned}
\end{equation}
We thus find the final relation between $\mathcal{M}$ and $B$:
\begin{equation}
\begin{aligned}
\mathcal{M} &= B + \theta (d \bar{C} \wedge A + A \wedge d \bar{C} + i [B, \bar{C}]) \\
&+ \bar{\theta} (d C \wedge A + A \wedge d C + i [B, C]) \\
&+ \theta \bar{\theta} ([B, B] + \{ [C, B], i d C \wedge A \\
&+ i A \wedge d C \}) \\
&+ i A \wedge d B + i d B \wedge A \\
&+ DC \wedge d \bar{C} - d C \wedge DC).
\end{aligned}
\end{equation}
The comparison of this equation with the standard expansion
\begin{equation}
\begin{aligned}
\mathcal{M} &= B + \theta (s_b B) + \bar{\theta} (s_b B) + \theta \bar{\theta} (s_{ab} s_b B)
\end{aligned}
\end{equation}
yields the following (anti-)BRST transformations of the B-field:
\begin{equation}
\begin{aligned}
s_b B &= d C \wedge A + A \wedge d C + i [B, C],
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
s_{ab} B &= d \bar{C} \wedge A + A \wedge d \bar{C} + i [B, \bar{C}],
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
s_{ab} s_b B &= [B, B] + \{ [C, B], i d C \wedge A + i A \wedge d C \},
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
&+ i A \wedge d B + i d B \wedge A \\
&+ DC \wedge d \bar{C} - d C \wedge DC.
\end{aligned}
\end{equation}
As a consistency check, we now use (45), (46) and (14) to compute $s_{ab} s_b B$. Equation (47) is then reproduced.

\(^3\)Other terms hidden inside “...” are those which involve $dx^\mu \wedge d \theta$, $dx^\mu \wedge d \bar{\theta}$, $d \theta \wedge d \bar{\theta}$, $d \bar{\theta} \wedge d \theta$ or $d \theta \wedge d\theta$.

In analogy with the 1-form operator $\tilde{U}_B$ and BRST transformations of the non-Abelian two-form $\tilde{B}$, for any vector in Lie space, $P = P^a T^a$, this is given as
\begin{equation}
\begin{aligned}
\tilde{U}_B P = i [B, P] + d P \wedge A + A \wedge d P = \\
i [B, P] - d [A, P].
\end{aligned}
\end{equation}
Then (45), (46) and (47) can be rewritten as
\begin{equation}
\begin{aligned}
s_b B &= \tilde{U}_B C, \\
s_{ab} B &= \tilde{U}_B C, \\
s_{ab} s_b B &= i \tilde{U}_B B + i \{ C, \tilde{U}_B C \} \\
&+ DC \wedge d \bar{C} - d C \wedge DC.
\end{aligned}
\end{equation}
It follows that
\begin{equation}
\begin{aligned}
s_{ab} s_b B &= i \tilde{U}_B B + i \{ C, \tilde{U}_B C \} \\
&+ DC \wedge d \bar{C} - d C \wedge DC,
\end{aligned}
\end{equation}
which, along with eq. (51), implies
\begin{equation}
\begin{aligned}
\{ s_b, s_{ab} \} B &= i \tilde{U}_B (B + \bar{B}) + i \{ C, \tilde{U}_B C \} + i \{ C, \tilde{U}_B \bar{C} \} \\
&+ DC \wedge d \bar{C} - d C \wedge DC \\
&+ DC \wedge d C - d C \wedge DC.
\end{aligned}
\end{equation}
We also obtain
\begin{equation}
\begin{aligned}
\tilde{U}_B (CC) &= (\tilde{U}_B \bar{C}) C + \bar{C} (\tilde{U}_B C) \\
&+ id \bar{C} \wedge DC - i d C \wedge d C,
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\tilde{U}_B (C \bar{C}) &= (\tilde{U}_B C) \bar{C} + \bar{C} (\tilde{U}_B \bar{C}) \\
&+ id C \wedge DC - i d C \wedge d C,
\end{aligned}
\end{equation}
which yields
\begin{equation}
\begin{aligned}
i \tilde{U}_B \{ C, \bar{C} \} &= i \{ \tilde{U}_B \bar{C}, C \} + i \{ \bar{C}, \tilde{U}_B C \} \\
&+ DC \wedge d \bar{C} - d C \wedge DC \\
&+ DC \wedge d C - d C \wedge DC.
\end{aligned}
\end{equation}
Now from eqs. (53) and (56), we see that
\begin{equation}
\begin{aligned}
\{ s_b, s_{ab} \} B &= i \tilde{U}_B (B + \bar{B} + i \bar{C} \times C).
\end{aligned}
\end{equation}
Similarly it follows that
\begin{equation}
\begin{aligned}
s_b^2 B &= 0, \\
s_{ab}^2 B &= 0,
\end{aligned}
\end{equation}
Thus, the absolute anti-commutativity and the nilpotency of order 2 of (anti-)BRST operators follow once we use the Curci-Ferrari condition. It is worth mentioning here that the operator $\tilde{U}_B$ does not follow the standard Leibniz rule, as is obvious from eqs. (54) and (55).

**Conclusion.**— We have exploited the superunitary operator approach to derive the BRST transformations of the Kalb-Ramond B-field. Although these transformations of the B-field have earlier appeared in the literature, the use of superunitary operator approach to derive these transformations in a comprehensive and systematic manner is completely new.
We considered a new Lagrangian, where we introduced a 2-form field $\Omega$, in addition to the 1-form field $A$, interacting with the matter fields. To reproduce the appropriate BRST transformations of the Kalb-Ramond $B$ field given in [27–29], a suitable combination of the 1-form gauge field $A$ and the $B$-field was chosen: $\Omega = B - i dA$. The transformation of $B$ followed naturally from this choice of $\Omega$. The two gauge fields further coupled mutually. In order to include the kinetic term for the 2-form field, we introduced the field strength 3-form $W$ which helped to complete this work.

It would be interesting to consider the superfield form for the extended to arbitrary (anti-)BRST symmetry in Yang-Mills theories was extended to arbitrary $N$-parametric BRST symmetry [31]. It would be interesting to consider the superfield form for these $N$-parametric BRST symmetries.

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