Robust Point-to-Set Control of Hybrid Systems with Uncertainties Using Constraint Tightening

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Abstract: This paper proposes a solution to the problem of computing optimized point-to-set control strategies for hybrid systems with mixed inputs and uncertainties in the continuous-valued dynamics as well as the reset functions. The solution is based on the idea to account for the uncertainties by tightening the guard and invariant sets, and thus construct a substitute hybrid automaton with deterministic transition dynamics. It can be ensured that a solution for this new model also solves the problem for the original system, i.e. the latter is robustly transferred between initial point and target set.

Keywords: Hybrid systems, uncertain systems, constrained control, robust control, optimization.

1. INTRODUCTION

The optimal control of hybrid systems with transitions conditioned upon both guard sets and external triggers has attracted interest recently (Liu and Stursberg (2018, 2019); Adamek et al. (2008)). In contrast to the relative popular class of piecewise affine (PWA) systems, where the change of the discrete state is only an immediate consequence of the continuous state crossing hyperplanes which partition the state space, the class of hybrid systems considered here is more general: As in the definition of hybrid automata in Henzinger (1996); Lynch et al. (2003), for this type of system the change of the discrete state is enabled by the fact that the continuous state lies inside of a guard set. Whether and when the transition indeed takes place is decided by a discrete input. In addition, the transition may cause an instantaneous change of the continuous state through reset functions. The optimal control problem of such hybrid systems requires not only to determine the optimal continuous controls, but also to determine the optimal times to trigger enabled transitions by discrete inputs. The work in Liu and Stursberg (2019) has proposed a method to cast the transition logic into a set of linear constraints among binary and real variables, and thus to solve the optimal control problem quite efficiently (by solvers like CPLEX (2009)).

The important extension considered in this paper is that the continuous as well as discrete dynamics (more precisely the reset functions) are subject to uncertainties. So far robust control of hybrid systems with focus on optimization-based approaches has only considered simpler system classes: In Kerrigan and Mayne (2002), PWA systems with additive disturbances were under study, and robust controllers were obtained by backward computation of robust controllable sets. The work in Lin and Antsaklis (2003) extended the idea of backward computation for the case of parametric model uncertainties. For the same class of PWA systems, the authors in Necoa et al. (2004) and Silva et al. (2003) aimed at setting up min-max problems, or to employ multi-parametric linear programming (see Borrelli et al. (2003)), or to use a branch-and-bound strategy tailored to worst-case uncertainty and thus ensure robustness. Specific to the considered class of hybrid systems with transitions conditioned by guard sets, the work in Moor and Davoren (2001) proposed a robust switching law for the case when no continuous input is applied. The work in Adamek et al. (2008) focused on another class of uncertainty, where the discrete transition structure is probabilistic.

In this work, however, both parametric uncertainties and additive disturbances affecting the the continuous dynamics are considered, as well as uncertain reset functions assigned to transitions. For the task of computing robust control trajectories, we employ and extend ideas that were used for simpler systems in Langson et al. (2004); Mayne et al. (2005); Ghasemi and Afzalian (2017), namely to construct reachable tubes around nominal trajectories. Through tightening of the invariants and guard sets of the hybrid systems by use of such tubes, and by optimizing the nominal trajectories for a modified hybrid automaton, we show that optimal point-to-set control tasks can be solved reliably despite the presence of the uncertainties.

This paper is organized as follows: in Sec. 2, the class of uncertain hybrid system and the robust controller synthesis problem are defined. Sec. 3 explains the procedure of set tightening to obtain robustness by consideration of a single transitions. Then, based on the findings of Sec. 3, a deterministic hybrid system is constructed in Sec. 4, and we prove that the robust controller synthesis problem of the uncertain hybrid system can be reduced to an optimal control problem of a deterministic substitute model. A
numerical example is presented in Sec. 5, followed by conclusions.

2. PROBLEM FORMULATION

This section describes the type of system and the control task under consideration. First, let \( \mathcal{P} \) denote the set of all polytopes \( P \in \mathbb{R}^n \) with \( P = \{ x \mid C \in \mathbb{R}^{n \times n}, d \in \mathbb{R}^{n \times 1} : C \cdot x \leq d \} \).

Now consider a class of uncertain hybrid systems formulated in discrete time and with mixed inputs according to \( H A = (T, U, X, Z, I, T, G, V, r, f) \) containing:

- the discrete time domain \( T = \{ t_k \mid k \in \mathbb{N} \cup \{0\}, \Delta \in \mathbb{R}^+ : t_k := k \cdot \Delta \} \);
- the continuous inputs \( u(t_k) \in U \subseteq \mathbb{R}^n \);
- the continuous states \( x(t_k) \in X \subseteq \mathbb{R}^n \);
- the finite set of discrete states \( Z = \{ z_{(i)} : i = 1, \ldots, l \} \);
- the finite set of transitions \( \mathcal{T} \subseteq Z \times Z \), where the transition from \( z_{(i)} \) to \( z_{(j)} \) is denoted by \( \tau_{(i,j)} \in \mathcal{T} \);
- the set \( G \) of guard sets, of which any element \( G_{(i,j)} \in \mathcal{P} \) is assigned to \( \tau_{(i,j)} \in \mathcal{T} \), and we require that \( G_{(i,j)} \subseteq (i) \) and that for no pairs of transitions from \( z_{(i)} \) the guard sets intersect;
- the finite set \( V \) of discrete inputs, where any element \( v_{(i,j)} \in \{ 0, 1 \} \) in \( V \) refers to one transition \( \tau_{(i,j)} \in \mathcal{T} \); we use a vector \( v_t := v(t_k) \) to denote the choice of discrete inputs at time \( t_k \);
- an uncertain reset function \( r : \mathcal{T} \times X \rightarrow X \) to update the continuous states \( \tilde{x} \) upon a transition \( \tau_{(i,j)} \in \mathcal{T} \) according to the following scheme:

\[
x = F_{(i,j)} \cdot \tilde{x} + e_{(i,j)}.
\]

with a matrix \( F_{(i,j)} \) and a vector \( e_{(i,j)} \), both being uncertain; \( F_{(i,j)} \) is randomly selected from a given polytope \( \mathcal{F}_{(i,j)} \in \mathcal{P} \) defined by:

\[
F_{(i,j)} = \left\{ F_{(i,j)} = \sum_{l=1}^{\rho_{(i,j)}} \gamma_l \cdot F_{(i,j)}^{(l)} \big| \sum_{l=1}^{\rho_{(i,j)}} \gamma_l > 0 = 1 \right\}.
\]

with \( F_{(i,j)}^{(l)} \) denoting the vertices of \( F_{(i,j)} \), and \( \rho_{(i,j)} \) is the number of these vertices, and \( \gamma_l \in [0, 1] \). The vector \( e_{(i,j)} \) is randomly taken from a given polytope \( \mathcal{E}_{(i,j)} \in \mathcal{P} \);
- an uncertain flow function \( f : X \times U \times Z \rightarrow X \) defining the discrete-time continuous-valued dynamics to:

\[
x_{k+1} = A_{(i)} \cdot x_k + B_{(i)} \cdot u_k + W_{(i,k)}.
\]

with \( x_{k+1} = x(t_{k+1}) \) and \( z_{(i)} \in Z \). The matrices \( A_{(i)} \) and \( B_{(i)} \) are uncertain, but are randomly selected from given polytopic sets \( \mathcal{A}_{(i)} \) and \( \mathcal{B}_{(i)} \):

\[
\mathcal{A}_{(i)} = \left\{ A_{(i)} = \sum_{l=1}^{\rho_{(i,A)}} \gamma_l \cdot A_{(i)}^{(l)} \big| \sum_{l=1}^{\rho_{(i,A)}} \gamma_l > 0 = 1 \right\},
\]

\[
\mathcal{B}_{(i)} = \left\{ B_{(i)} = \sum_{l=1}^{\rho_{(i,B)}} \gamma_l \cdot B_{(i)}^{(l)} \big| \sum_{l=1}^{\rho_{(i,B)}} \gamma_l > 0 = 1 \right\}.
\]

with coefficients \( \gamma_l \in [0, 1] \). The additive disturbances \( W_{(i,k)} \) can assume arbitrary values from a set \( \mathcal{W}_{(i)} \in \mathcal{P} \) containing the origin.

The set of admissible executions of the model \( HA \) considering the uncertain components of the continuous dynamics and reset functions is defined as follows:

**Definition 1. (Admissible execution of the uncertain model \( HA \))** Let a finite time domain \( T_N = \{ 0, 1, \ldots, N \} \subseteq T \) and an initial hybrid state \( (x_0, z_0) \) with \( z_0 := z_{(s_0)} \in Z \), \( x_0 \in I_{(s)} \), and \( x_0 \notin G_{(s)} \) for all \( \tau_{(s)} \in \mathcal{T} \) be given. Then, for given input sequences \( \phi_u = \{ u_0, u_1, \ldots, u_{N-1} \} \) and \( \phi_v = \{ v_0, v_1, \ldots, v_{N-1} \} \), an admissible execution is a pair of state sequences \( \phi_x = \{ x_0, x_1, \ldots, x_N \} \) and \( \phi_z = \{ z_0, z_1, \ldots, z_N \} \) complying to the following rules: for \( k \in \{ 0, 1, \ldots, N-1 \} \), \( x_k \in \phi_x \), and \( z_k = z_{(i)} \), the successor states \( x_{k+1} \in \phi_x \) and \( z_{k+1} \in \phi_z \) must satisfy:

- if \( v_k = 0 \), then there must exist \( A_{(i)} \in \mathcal{A}_{(i)}, B_{(i)} \in \mathcal{B}_{(i)} \), and \( w_{(i,k)} \in \mathcal{W}_{(i)} \) to obtain \( x_{k+1} \in I_{(i)} \) according to (3) and \( z_{k+1} = z_{(i)} \);
- if \( v_k = v_{(i,j)} = 1 \) and if \( A_{(i)} \in \mathcal{A}_{(i)}, B_{(i)} \in \mathcal{B}_{(i)} \), and \( w_{(i,k)} \in \mathcal{W}_{(i)} \) exist to obtain an intermediate state \( x_{k+1} \in G_{(i,j)} \) according to (3), then there must exist \( F_{(i,j)} \in \mathcal{F}_{(i,j)} \) and \( e_{(i,j)} \in \mathcal{E}_{(i,j)} \) to have \( x_{k+1} = F_{(i,j)} \); \( z_{k+1} = z_{(i,j)} + e_{(i,j)} \) according to (1), and \( z_{k+1} = z_{(i)} \).

Obviously, for given inputs at time \( k \), any value of the uncertain components \( A_{(i)}, B_{(i)}, w_{(i,k)} \), \( F_{(i,j)}, \) and \( e_{(i,j)} \) may contribute to and determine the hybrid successor state, as long as the necessary containment in the invariants and guard sets are observed – thus, any controller designed for a model of type \( HA \) must consider the complete set of possible executions according to Def. 1.

Turning to the control task to be addressed within this paper, now consider the optimal transfer of \( HA \) from an initial hybrid state \( (x_0, z_0) \) into a set of goal states while taking the uncertainties into account. For the hybrid goal states, assume the pair \( (X_g, z_g) \) with \( z_g \in Z \) and \( X_g \in P \) as well as \( X_g \subseteq I_g \). Furthermore, let a continuous state \( x_g \in X \) denote the volumetric center of \( X_g \).

Furthermore, let a cost functional \( J(x_0, x_g, N) \) be specified to quantify the performance of transferring the system into the goal within a finite time domain of \( T_N \).

**Definition 2. (Point-to-set control task for \( HA \))** For an initial hybrid state \( (x_0, z_0) \) of \( HA \), a time domain \( T_N \), as well as a set of goal states \( (X_g, z_g) \), find the pair of sequences of continuous inputs \( \phi_u = \{ u_0, u_1, \ldots, u_{N-1} \} \) and discrete inputs \( \phi_v = \{ v_0, v_1, \ldots, v_{N-1} \} \) such that:

- the resulting pair of state sequences \( \phi_x = \{ x_0, x_1, \ldots, x_N \} \) and \( \phi_z = \{ z_0, z_1, \ldots, z_N \} \) satisfy Def. 1,
- the terminal states satisfy \( x_N \subseteq X_g \) and \( z_N = z_g \),
- and the cost functional \( J(x_0, x_g, N) \) is minimized for a nominal evolution.

Note that an equivalent problem for the case without uncertainties was already addressed in Liu and Stursberg (2018, 2019), proposing a particular structure to take care of the theoretically exponential increase of the number of possible \( \phi_z \) and \( \phi_v \) over \( N \). For the variant of hybrid systems with uncertainties considered in this paper, the additional challenge is to guarantee that the selected pair of \( \phi_u \) and \( \phi_v \) realizes the path into \( (X_g, z_g) \) for all possible realizations of the uncertainties. To succeed in...
this task, we employ principles inspired the so-called tube-based predictive control for continuous-valued dynamics, see Langson et al. (2004); Mayne et al. (2005). Along this line, we show that, based on tubes of reachable sets, the control problem in Def. 2 can be cast into a problem for a modified model $\mathcal{HA}$ with deterministic sequence of discrete states, to which the method in Liu and Stursberg (2019) can be applied.

3. REACHABILITY TUBES TO HANDLE UNCERTAIN TRANSITIONS

In order to explain the principle of using reachability tubes to robustly control $\mathcal{HA}$, this section focuses first on a single transition as part of the sequence $\phi_2$ which solves the problem of Def. 2. More precisely, we consider a transition $\tau_{(i,j)} \in \mathcal{T}$, and show how to ensure the semantics in Def. 1 in terms of: 1) the continuous state evolving inside of $z_{(i)}$ (called the pre-transition phase), 2) transitioning from $z_{(i)}$ to $z_{(j)}$, and 3) further evolving inside of $z_{(j)}$ (the post-transition phase). If this procedure is later applied to all transitions in $\mathcal{T}$, this will lead to a substitute hybrid system $\mathcal{HA}$ with deterministic sequence $\phi_2$.

3.1 Pre-Transition Phase

First, let the polytopic sets leading to the uncertainty in both, reset function and flow function, be decomposed into two parts, namely a nominal part (indicated by $\mathcal{A}$), and a disturbance set containing the origin:

$$F_{(i,j)} = \tilde{F}_{(i,j)} \oplus \mathcal{F}_{(i,j)}, \quad E_{(i,j)} = \tilde{E}_{(i,j)} \oplus \mathcal{E}_{(i,j)};$$

$$\mathcal{A}_{(i)} = \mathcal{A}_{(i)} \oplus \mathcal{A}_{(i)} \oplus \mathcal{B}_{(i)} = \tilde{B}_{(i)} \oplus \mathcal{B}_{(i)}. \quad (5)$$

Here, the symbol $\oplus$ denotes the Minkowski addition. Note that the set $\mathcal{W}_{(j)}$ contains the origin by definition and needs not to be decomposed. Next, a nominal flow function as well as a nominal reset function is defined (excluding all uncertainties in $f$ and $r$) to obtain the nominal continuous state and input:

$$\tilde{x}_{k+1} = \tilde{A}_{(i)} \cdot \tilde{x}_{k} + \tilde{B}_{(i)} \cdot \tilde{u}_{k}; \quad (6)$$

$$\tilde{x} = F_{(i,j)} \cdot \tilde{x} + E_{(i,j)}. \quad (7)$$

Now assume for step $k$ that the state $x_k$ from (3) and the nominal state $\tilde{x}_k$ from (6) are located inside of invariant $I_{(i)}$. Then, by applying a continuous input $u_k$ and a nominal $\tilde{u}_k$ in (3) and (6) respectively, the difference between $x_{k+1}$ and $\tilde{x}_{k+1}$ can be determined according to:

$$x_{k+1} - \tilde{x}_{k+1} = A_{(i)}(x_k - \tilde{x}_k) + B_{(i)}(u_k - \tilde{u}_k) + w_{(i),k} + \Delta_{(A_{(i)},k)} \cdot x_k + \Delta_{(B_{(i)},k)} \cdot u_k \quad (8)$$

with $\Delta_{(A_{(i)},k)} \in \mathcal{A}_{(i)}$ and $\Delta_{(B_{(i)},k)} \in \mathcal{B}_{(i)}$.

Suppose further that a closed-loop controller $K_i \in \mathbb{R}^{n_u \times n_x}$ is defined such that $\tilde{A}_{K_{(i)}} := \tilde{A}_{(i)} + B_{(i)} \cdot K_i$ is stable. If $x_k$ is measurable and $u_k$ chosen to:

$$u_k = \tilde{u}_k + K_i \cdot (x_k - \tilde{x}_k), \quad (9)$$

then the difference between $x_{k+1}$ and $\tilde{x}_{k+1}$ in (8) can be written as:

$$x_{k+1} - \tilde{x}_{k+1} = A_{K_{(i)}}(x_k - \tilde{x}_k) + w_{(i),k} + \Delta_{(A_{(i)},k)} \cdot x_k + \Delta_{(B_{(i)},k)} \cdot u_k. \quad (10)$$

Furthermore, as $x_k \in I_{(i)}$ and $\Delta_{(A_{(i)},k)} \in \mathcal{A}_{(i)}$, and both $I_{(i)}$ and $\mathcal{A}_{(i)}$ are polytopic, the following applies according to Bünger (2014):

$$\Delta_{(A_{(i)},k)} \cdot x_k \in \text{Conv} \left( \left\{ A_{(i)}^{(l)} \cdot x_k \mid l \in \{1, \ldots, \rho_{(A_{(i)})}\} \right\} \right), \quad (11)$$

where $l$ and $q$ are the indices running over the $\rho_{(A_{(i)})}$ vertices, or respectively $\rho_{(B_{(i)})}$ vertices of the respective polytopes, and $\text{Conv}$ is the function to determine the convex hull over the combinations of vertices. Similarly, the value of $\Delta_{(B_{(i)},k)} \cdot u_k$ in (10) can be bounded by:

$$\Delta_{(B_{(i)},k)} \cdot u_k \in \text{Conv} \left( \left\{ B_{(i)}^{(l)} \cdot U(q) \mid l \in \{1, \ldots, \rho_{(B_{(i)})}\} \right\} \right). \quad (12)$$

For abbreviation, we write $\text{Conv}(A_{(i)} I_{(i)})$ and $\text{Conv}(B_{(i)} U)$ to denote the convex set on the right hand side of (11), and (12) respectively. With it, a disturbance invariant set $\mathcal{D}_{i}$ can be determined according to (10) by satisfying:

$$\Delta_{K_{(i)}} D_i \oplus \left( W_{(i)} \oplus \text{Conv}(A_{(i)} I_{(i)}) \oplus \text{Conv}(B_{(i)} U) \right) \subseteq \mathcal{D}_{i}. \quad (13)$$

The set $\mathcal{D}_{i}$ exists, since $\Delta_{K_{(i)}}$ is stable and contains the origin, see Mayne et al. (2005). The relation (13) means that, if $x_k - \tilde{x}_k \in \mathcal{D}_{i}$ applies, then $x_{k+1} - \tilde{x}_{k+1} \in \mathcal{D}_{i}$ also follows from using the control law (9) despite all the uncertainties in the flow function, and the following holds:

**Lemma 1.** If for two given states $x_k$ and $\tilde{x}_k$ in $I_{(i)}$ applies that $x_k \in \mathcal{E}_{(i)} \oplus \mathcal{D}_{i}$, then the control law (9) implies that the relation $x_{k+1} \in \tilde{x}_{k+1} \oplus \mathcal{D}_{i}$ holds for all $\Delta_{(A_{(i)},k)} \in \mathcal{A}_{(i)}$, $\Delta_{(B_{(i)},k)} \in \mathcal{B}_{(i)}$ and $w_{(i),k} \in \mathcal{W}_{(i)}$.

By use of Lemma 1 and when denoting the Pontryagin difference by $\ominus$, the following fact can also be established:

**Proposition 1.** If the nominal state satisfies $x_k \in I_{(i)} \ominus \mathcal{D}_{i}$ and if the nominal input satisfies $\tilde{u}_k + K_i(x_k - \tilde{x}_k) \in U$, then there always exists $u_k := \tilde{u}_k + K_i(x_k - \tilde{x}_k) \in U$ such that $x_{k+1} \in I_{(i)}$ for all $\Delta_{(A_{(i)},k)} \in \mathcal{A}_{(i)}$, $\Delta_{(B_{(i)},k)} \in \mathcal{B}_{(i)}$ and $w_{(i),k} \in \mathcal{W}_{(i)}$.

The proofs of Lemma 1 and Proposition 1 follow the lines in Mayne et al. (2005) and the fact that $I_{(i)} \ominus \mathcal{D}_{i} \subseteq I_{(i)}$. The proposition means that, as long as the evolution of the nominal $x_k$ lies inside of $I_{(i)} \ominus \mathcal{D}_{i}$ (i.e. inside of a tightened invariant), and if the nominal continuous input $\tilde{u}_k$ is selected from set $U \ominus \mathcal{D}_{i}$ (i.e. from a tightened input set), then we can always find $u_k \in U$ to obtain the next continuous state $x_{k+1}$ inside of $I_{(i)}$ despite all the uncertainties. This is also illustrated in Fig. 1.

In addition, as all the sets in (13) are in $\mathcal{P}$, the computation of $\mathcal{D}_{i}$ can be assumed to be tractable according to Blanchini (1999). Furthermore, it is desired that $\mathcal{D}_{i}$ is as small as possible in order to reduce conservatism.

With respect to the semantics of $\mathcal{HA}$ in Def. 1, the state $x$ can be kept inside of $I_{(i)}$ during the evolution in $z_{(i)}$ by forcing the nominal state $\tilde{x}$ to be inside of $I_{(i)} \ominus \mathcal{D}_{i}$.

3.2 The Transition

Next, if the transition $\tau_{(i,j)}$ is triggered in step $k$, it must hold that $v_{(i,j),k} = 1$ and the following two conditions must be satisfied:
For the first condition, it is known from Proposition 1 that the difference between the states $\bar{x}$ and $\bar{x}$ is inside of $G_{(i,j)}$; the guard $G_{(i,j)}$ is fully contained in $I_{(i)}$, the tube around the nominal state $\bar{x}$ during evolution in $I_{(j)}$. As the guard $G_{(i,j)}$ is guaranteed to be triggered correctly. The blue polytope represents the disturbance invariant set $D_j$.

The proof implies that $\bar{x} \in G_{(i,j)}$ exists despite the uncertainties, if a nominal intermediate state $\bar{x}$, as obtained from (7), is inside of $G_{(i,j)} \ominus D_i$. Thus, the first condition is guaranteed to be satisfied by restricting the position of the nominal intermediate state $\bar{x}$.

Next, due to the uncertainties on the reset function (1), a difference between the states $\bar{x}_{k+1}$ and $x_{k+1}$ may be obtained according to:

$x_{k+1} - \bar{x}_{k+1} = \bar{F}_{(i,j)}(\bar{x} - \tilde{x}) + \Delta \bar{F}_{(i,j)} \cdot \bar{x} + \Delta e_{(i,j)}$, \hspace{0.5cm} (14)

where $\Delta \bar{F}_{(i,j)} \in \Phi_{(i,j)}$ and $\Delta e_{(i,j)} \in E_{(i,j)}$. By Lemma 2, the difference between $\bar{x}$ and $\tilde{x}$ is bounded to $D_i$. As $\bar{x}$ must be inside of $G_{(i,j)}$, the value of $\Delta \bar{F}_{(i,j)} \cdot \bar{x}$ must be bounded by $\text{Conv}(\Phi_{(i,j)} G_{(i,j)})$. Hence, the difference of $x_{k+1}$ and $\bar{x}_{k+1}$ in (14) is bounded by:

$x_{k+1} - \bar{x}_{k+1} \in \bar{F}_{(i,j)} D_i \oplus \text{Conv}((\Phi_{(i,j)} G_{(i,j)}) \oplus E_{(i,j)})$. \hspace{0.5cm} (15)

With the help of this bound, Lemma 2 can be extended to:

**Lemma 3.** If the nominal state after the transition $\tau_{(i,j)}$ satisfies $\bar{x}_{k+1} \in I_{(j)} \ominus (\bar{F}_{(i,j)} D_i \oplus \text{Conv}(\Phi_{(i,j)} G_{(i,j)} \oplus E_{(i,j)}))$, then the relation $x_{k+1} \in I_{(j)}$ holds for the state obtained from $\tau_{(i,j)}$ despite all the uncertainties in (1).

3.3 The Post-Transition Phase

After reaching the discrete state $z_{(j)}$, the further evolution inside of $I_{(j)}$ has to be ensured. Obviously, this can be again achieved by applying a similarly control law as in (9), considering a disturbance invariant set $D_j$ for $z_{(j)}$. However, such a recursive robustness guaranty relies on the condition that the difference of the states $x_{k+1}$ and $\bar{x}_{k+1}$ is in $D_j$ according to Lemma 1. But according to (15), the difference is known to be inside of $\bar{F}_{(i,j)} D_i \oplus \text{Conv}(\Phi_{(i,j)} G_{(i,j)} \oplus E_{(i,j)})$, which is not depending on $D_j$, see Fig. 3.

To avoid that robustness is lost at this point, the following criterion is formulated.

**Definition 3.** (Robust Transition) Any transition $\tau_{(i,j)} \in T$ of $HA$ may only be executed, if the condition: \hspace{0.5cm} $\bar{F}_{(i,j)} D_i \oplus \text{Conv}(\Phi_{(i,j)} G_{(i,j)}) \oplus E_{(i,j)} \subseteq D_j$ \hspace{0.5cm} (16)

is satisfied, otherwise it is prohibited.

Fig. 1. By applying (9), the difference between $x$ and $\bar{x}$ is recursively lying inside of $D_i$ (denoted by the blue polytope) despite all the uncertainties.

Fig. 2. For the case that $\bar{x} \ominus D_i$ is not fully contained in $G_{(i,j)}$ (upper part), the transition $\tau_{(i,j)}$ may not be correctly triggered due to the deviation between $\bar{x}$ and $\tilde{x}$; If $\bar{x} \ominus D_i \subseteq G_{(i,j)}$ (lower part), then the transition $\tau_{(i,j)}$ is guaranteed to be triggered correctly. The blue polytope represents the disturbance invariant set $D_i$.

Fig. 3. For the case $x_{k+1} \notin \bar{x}_{k+1} \ominus D_j$ due to (15), the further evolution of the real continuous state in $z_{(j)}$ may not be bounded by the $D_j$ around the nominal state (upper part); for $x_{k+1} \in \bar{x}_{k+1} \ominus D_j$ (lower part), such a bound applies. The blue polytope represents the disturbance invariant set $D_j$. 

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Lemma 4. Let the transition $\tau(i,j)$ satisfy (16) and let the control law (9) be applied in both $z(i)$ and $z(j)$. If $x - \bar{x}$ is then bounded to $D_i$ before the transition $\tau(i,j)$, then $x - \bar{x}$ is also bounded to the set $D_j$ after the transition $\tau(i,j)$. □

Proof. If $x_k - \bar{x}_k \in D_i$ applies before the transition $\tau(i,j)$, then $x_{k+1} - \bar{x}_{k+1} \in \bar{F}(i,j)D_i \oplus \text{Conv}(E_{(i,j)}G(i,j)) \oplus E_{(i,j)} \subseteq D_j$ holds true after the transition according to (15) and (16). Then, using Lemma 1, it follows that the difference will be recursively bounded to $D_j$ by applying the control law (9) in the further steps. □

Eventually at this stage, the sequence of system evolution in $z(i)$, of the transition $\tau(i,j)$, and of the further evolution in $z(j)$ was successfully cast into a set of constraints on the nominal state and inputs as well as the condition (16).

In the next section, a substitute hybrid automaton $\mathcal{HA}$ with deterministic behavior for the nominal trajectory is constructed based on these constraints for all transitions in $T$. It is then shown that the control task for $HA$ in Def. 2 can be transformed into a corresponding problem for $\mathcal{HA}$.

4. HYBRID AUTOMATON WITH DETERMINISTIC BEHAVIOR

We construct a modified hybrid automaton $\mathcal{HA} = (T, U, X, Z, I, T, G, V, \bar{r}, f)$ based on the original one $HA$. The sets $T, U, X$ and $Z$ are identical to those in $HA$, while the other components of $\mathcal{HA}$ are determined according to the following rules:

- a set $\tilde{U} = \{\tilde{U}(1), \ldots, \tilde{U}(n_z)\}$ of continuous input sets, where for any $z(i) \in Z$, the continuous input set is $\tilde{U}(i) := U \cap K_i D_i$;
- a set $I = \{I(1), \ldots, I(n_z)\}$ of invariants, where the invariant of any discrete state $z(i) \in Z$ is obtained to $I(i) := I(i) \cap D_i$;
- the finite set of transitions $T$, obtained from deleting those of $T$, which do not satisfy (16);
- the set $G$ of guard sets contains one polytopic set $G(i,j) := G(i,j) \cap D_i$ assigned to any transition $\tau(i,j) \in T$;
- the finite set $V$ of discrete input variables, where any element $v(i,j) \in \{0,1\}$ in $V$ refers to one transition $\tau(i,j) \in T$;
- a deterministic reset function $\bar{r}$, which updates the continuous state $\bar{x}$ according to (7);
- a deterministic flow function $f$ defines the continuous dynamics according to (6).

Note that all the uncertain components of $HA$ are excluded here, and an admissible execution of the deterministic automaton $\mathcal{HA}$ is defined as:

Definition 4. (Admissible Execution of $\mathcal{HA}$) For $\mathcal{HA}$, let a finite time set $T_N = \{0,1,\ldots,N\} \subseteq T$ and an initial hybrid state $(\bar{x}_0, z_0) \in Y$ satisfying $z_0 := z(s) \in Z$, $x_0 \in \bar{I}(s)$, and $x_0 \notin \bar{G}(s)$ for all $\tau(s,j) \in T$ be given. For selected input sequences $\phi_0 = \{\tilde{u}_0, u_1, \ldots, u_{N-1}\}$ and $\phi_e = \{v_0, v_1, \ldots, v_{N-1}\}$, the pair of state sequences $\phi_x = \{x_0, x_1, \ldots, x_N\}$ and $\phi_z = \{z_0, z_1, \ldots, z_N\}$ is admissible, if and only if for any $k \in \{0, \ldots, N\}$ the pair $(x_{k+1}, z_{k+1})$

follows from $(\bar{x}_k, z_k)$, $x_k \in \bar{I}(i), z_k := z(i)$ according to the following semantics:

1.) $\bar{x} := A(i) \cdot \bar{x} + \tilde{B}(i) \cdot \tilde{u}_k \in \bar{I}(i)$,
2.) $G(i,j) \subseteq G(i,j)$ such that $\bar{x} \in \bar{G}(i,j)$ and if $\bar{v}(i,j) = 1$, then $\bar{x}_{k+1} = \tilde{F}(i,j) \cdot \bar{x} + \bar{e}(i,j)$.

Eventually, $\bar{x}$ and $z(k)$ are all satisfying the condition (16). Thus, according to Proposition 1 as well as Lemma 2, 3, 4, the satisfaction

Then, by defining the new hybrid goal state $(X_g, z_g)$, where $X_g := X_g \cap D_g$, as well as assigning $\bar{x}_0 := x_0$, the following cost functional $J(x_0, x_g, N)$ is selected:

$$
J(x_0, x_g, N) = \sum_{k=1}^{N} \{ (\bar{x}_k - x_g)^T Q(\bar{x}_k - x_g) + (\bar{u}_k - u_0)^T R(\bar{u}_k - u_0) \} + q_g \cdot N_g,
$$

where $Q$ and $R$ are semi-positive-definite weighting matrices, and $q_g \in \mathbb{R}^R$. The variable $N_g := \min\{k \in \{0,1,\ldots,N\} | x_k \in \bar{X}_g, z_k = z(g)\}$ encodes the first point of time at which the continuous state has reached the goal set. Note if $Q = R = 0$ and $q_g \neq 0$, such cost functional matches the concept of time optimal control.

The control problem of the deterministic $\mathcal{HA}$ can then be defined as:

Definition 5. For $\mathcal{HA}$ initialized to $(\bar{x}_0, z_0) := (z(s), x_0)$, let a time set $T_N$ and a goal $(X_g, z_g)$ be given. Then, determine input sequences $\phi_x$ and $\phi_e$ as the solution of:

$$
\phi_x, \phi_e, \phi_z
$$

s.t.: For all $k \in \{0, \ldots, N-1\}$:

- $\phi_x$ with $u_k \in \bar{U}(i)$, when $x_k \in \bar{I}(i), \forall z(i) \in Z$,
- $\phi_e$ with $v(l,j), k \in \{0,1\}$;
- $\phi_x, \phi_z$ admissible for $\mathcal{HA}$;
- $\bar{x}_N \in \bar{X}_g, z_N = z_g$.

If now the parameters of the initial and goal sets as well as that of the cost functional in Def. 5 are chosen identical to that of Def. 2, a solution of the previous one can be referred to the latter problem, see below. Note first that the problem in Def. 5 is of the same type as the one considered in Liu and Stursberg (2018, 2019), i.e., it can also be solved by using the methods proposed there (also briefly reviewed at the end of this section). The following result can be obtained:

Theorem 1. If, for the same parameterization of Def. 2 and Def. 5, a feasible solution is obtained for the problem in Def. 5, then this solution also solves the control task in Def. 2. □

Proof. For the initial continuous state, the difference $x_0 - \bar{x}_0 = 0 \in D_i$ applies. The optimized inputs $\bar{u}_k$ in $\phi_x$ obtained from the solution of the problem in Def. 5 are all selected from the tightened input sets $\bar{U}$, the optimal continuous states $\bar{x}_N$ in $\phi_x$ are all located in the tightened invariant sets $I$, as well as the transitions to be executed in $\phi_e$ are all satisfying the condition (16). Thus, according to Proposition 1 as well as Lemma 2, 3, 4, the satisfaction
of the semantics in Def. 1 is always ensured during the evolution in each $z^*_k$, as well as for each transition in $\phi^*_i$ when using the through control law (9). In addition, for the last state $\bar{x}_N^*$ in $\phi^*_i$, as $x_N - \bar{x}_N^* \in D_g$ applies according to Lemma 1, and $\bar{x}_N^* \in X_g \oplus D_g$ applies according the last constraint in the problem in Def. 2, it is ensured that $x_N \in X_g$. □

Accordingly, by controlling the original $HA$ over the time domain $T_N$, we only have to: 1) solve the problem in Def. 5; 2) assign $v_{(i,j),k} := \bar{v}_{(i,j),k}$ and calculate $u_k$ according to (9) at each step $k$. Then the goal states $(X_g, z_0)$ are ensured to be reached at the end of the horizon, while satisfying the semantics in Def. 1. For the solution of the problem in Def. 5 according to the method in Liu and Stursberg (2018, 2019), we first decompose the semantics in Def. 4 into two parts, namely:

(1) for discrete state evolution:

If $z_k = z_{(i)}, \bar{x} \notin G_{(i,j)}$, then $z_{k+1} := z_{(i)}$;

If $z_k = z_{(i)}, \bar{x} \in G_{(i,j)}, \bar{v}_{(i,j),k} = 0$, then $z_{k+1} := z_{(i)}$;

If $z_k = z_{(i)}, \bar{x} \in G_{(i,j)}, \bar{v}_{(i,j),k} = 1$, then $z_{k+1} := z_{(i)}$.

(18)

(2) for continuous state evolution:

If $z_k = z_{(i)}, \bar{x} \notin G_{(i,j)}$, then dynamic (6) applies;

If $z_k = z_{(i)}, \bar{x} \in G_{(i,j)}, \bar{v}_{(i,j),k} = 0$, then (6) applies;

If $z_k = z_{(i)}, \bar{x} \in G_{(i,j)}, \bar{v}_{(i,j),k} = 1$, then (7) applies.

(19)

Next, by introducing one binary variable per time step for each invariant set, guard set, and the terminal set in $HA$, both of the (18) and (19) can be equivalently cast into a set of linear constraints formulated for the binary and real variables, employing the modeling principles introduced in Williams (2013). The last term in the cost functional $J(x_0, x_N, N)$ can also be reformulated into a sum over a set of binary variables without any approximation. See the specifications in Liu and Stursberg (2018, 2019) for details on this modeling procedure, and on the solution as mixed-integer programming problem with linear constraints.

5. NUMERICAL EXAMPLE

The uncertain $HA$ considered here for illustration of the procedure consists of 4 discrete states with a set of possible transitions, as shown in the lower part of Fig. 4. In the lower part of this figure, the invariants of the model are marked in red, the guards are in green, and the selected terminal set $X_g \subseteq I_{(3)}$ is marked in yellow. The flow function $f$ and reset function $r$ of the states and transitions are parametrized with different bounds of uncertainties. The initial state is $x_0 = [10, 4.5]^T \in I_{(1)}$ and a horizon of $N = 20$ is available to realize the transition from the initial state into the goal set.

The controller $K_i$ for each discrete state is selected to be the minimum time controller according to Langson et al. (2004), such that $A_{R_{(i)}} = 0$ applies. The minimal disturbance invariant set $D_g$ is computed with the help of the Matlab Invariant Set Toolbox proposed in Kerrigan (2001). When constructing the deterministic model $\overline{HA}$, each invariant set, guard set, the terminal set as well as the continuous input set are tightened according to the disturbance invariant set, see Fig. 5. By evaluating the condition (16) for each transition in $HA$, only the transition $\tau_{(1,2)}$ fails to satisfy it, and is thus prohibited. Thereafter, by solving the problem in Def. 5 for the resulting deterministic hybrid automaton $\overline{HA}$, the optimal trajectory is found by first transferring from $z_{(1)}$ to $z_{(4)}$, and then to $z_{(2)}$, see Fig. 5. The time required for the computation is 0.66sec on a 3.4GHz processor using Matlab-2015a.

Based on the solution of the problem in Def. 5, the original automaton $HA$ is controlled by the control law (9) in each step, and for 50 different realizations of the uncertainties the trajectories is shown in Fig. 6 are obtained. It can be
seen in this figure that all continuous states along these trajectories evolve inside of the invariant and that the transitions are correctly triggered in the guard set, despite the uncertainties. In addition, all 50 trajectories reach the terminal set at the end of the horizon.

6. CONCLUSION

This work has extended principles of tube-based control and constraint tightening, which were established for purely continuous dynamics, to hybrid systems with guarded transitions and mixed inputs. As the main result, point-to-set transitions for this class of hybrid systems can be computed by the proposed techniques in an optimized fashion, while reaching the goal set is guaranteed despite uncertainties of the flow and reset functions. The key idea is to optimize the nominal trajectory, compute control laws to let the continuous dynamics converge towards the nominal trajectory (thus counteracting the uncertainties), and to reduce the size of guards and invariant sets to ensure that the sequence of discrete states (corresponding to the nominal trajectory) is realized for any possible disturbance. The computational procedure involves to compute a modified hybrid automaton $\mathcal{HA}$ with tightened sets, for which an optimized trajectory is obtained, that in turn is feasible and goal-attaining for the original system.

In current work, we aim at extending the control technique to uncertain nonlinear flow functions, and to hybrid automata with invariants and guard sets which can vary over time.

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