A NEW SMOOTHING APPROACH TO EXACT PENALTY FUNCTIONS FOR INEQUALITY CONSTRAINED OPTIMIZATION PROBLEMS

Ahmet Sahiner*, Gulden Kapusuz and Nurullah Yilmaz

Suleyman Demirel University, Department of Mathematics
Isparta, 32100, TURKEY

Abstract. In this study, we introduce a new smoothing approximation to the non-differentiable exact penalty functions for inequality constrained optimization problems. Error estimations are investigated between non-smooth penalty function and smoothed penalty function. In order to demonstrate the effectiveness of proposed smoothing approach the numerical examples are given.

1. Introduction. We consider the following constrained optimization problem

\[
(P) \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m.
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g_i(x) : \mathbb{R}^n \to \mathbb{R} \), \( i \in I = \{1, 2, \ldots, m\} \) are continuously differentiable functions.

Let us define the set of feasible solution by \( G_0 \) (\( G_0 := \{ x \in \mathbb{R}^n | g_i(x) \leq 0, \quad i = 1, 2, \ldots, m \} \)) and we assume that \( G_0 \) is not empty.

The penalty function methods have been proposed in order to transform a constrained optimization problem to an unconstrained optimization problem. The following function is one of the well-known penalty function

\[
F_2(x, \rho) = f(x) + \rho \sum_{i=1}^{m} (g_i^+(x))^2,
\]

where \( \rho > 0 \) is a penalty parameter and \( g_i^+(x) = \max\{0, g_i(x)\} \), \( i \in I \). Clearly, \( F_2(x, \rho) \) is continuously differentiable exact penalty function. According to Zangwill [19], an exact penalty function has been defined by

\[
F_1(x, \rho) = f(x) + \rho \sum_{i=1}^{m} g_i^+(x).
\]

The obvious difficulty in minimization of \( F_1 \) is the non-differentiability of \( F_1 \) which originates from the presence of “max” operator (when the power of max is equal 1).

2010 Mathematics Subject Classification. Primary: 90C30, 57R12; Secondary: 53C35.

Key words and phrases. Constrained optimization, penalty function, smoothing approach.

* Corresponding author: ahmetsahiner@sdu.edu.tr, ahmetnur32@gmail.com.

The reviewing process of the paper was handled by Gerhard-Wilhelm Weber and Herman Mawengkang as Guest Editors.
In the last decade, the lower order penalty functions
\[ F_p(x, \rho) = f(x) + \rho \sum_{i=1}^{m} (g_i^+(x))^p, \quad p \in (0, 1). \]
have been investigated [11]. One of the important tools for solving these types of non-smooth (non-Lipschitz) problems is the smoothing approach. The smoothing approach is based on making some modification on the objective function or approximate the objective function by smooth functions. The first study on smoothing approach is the Bertsekas’s famous paper [4]. In order to improve the smoothing approaches, different types of valuable techniques and algorithms are developed [18, 16, 3, 5]. In recent years, the smoothing approach protects its popularity among the scientist who faced non-smooth optimization problems. Smoothing approach for the exact penalty function start with [10], for other important studies we refer to the following references [7, 8, 15, 9, 12, 2].

In this paper we aim to develop more efficient methods for smoothing each of the exact penalty functions \( F_1(x, \rho) \) and \( F_p(x, \rho) \) and solve the corresponding unconstrained optimization problems
\[
\begin{align*}
(P_1) & \quad \min_{x \in \mathbb{R}^n} F_1(x, \rho), \\
(P_p) & \quad \min_{x \in \mathbb{R}^n} F_p(x, \rho), \quad p \in (0, 1),
\end{align*}
\]
by using the differentiation based methods.

The next section is devoted to some preliminary knowledge about smoothing approaches. In Section 3, we propose a new smoothing approach and we prove some results for error estimates among the optimal objective function values of the smoothed penalty problem, non-smooth penalty problem and original optimization problem. In Section 4, we present minimization algorithms for problems \((P_1)\) and \((P_p)\) in order to find an approximate solution for the problem \((P)\). In Section 5, we apply the algorithms on the important test problems and compare the results obtained from \((P_1)\) and \((P_p)\) and, show the convergence of the algorithms. In sections 6, we present some concluding remarks.

2. Preliminaries. The differentiability of the optimization problems is one of the useful property in terms of the optimality conditions. As is mentioned above, in order to exploiting the good properties of the differentiability the smoothing approach has been arisen. The smoothing studies starts with the smoothing of the following non-smooth problem:
\[ \theta(x) = \max\{x, 0\} \]
for \( x \in \mathbb{R}^n \). One of the effective smoothing approach is hyperbolic smoothing method proposed in [13, 17, 14] and it is defined by
\[ \phi_\tau(x) = \frac{x + \sqrt{x^2 + \tau^2}}{2} \]
where \( \tau > 0 \) is parameter. That smoothing approach is used for solving min-max problems in [1] and it is used for solving non-smooth and non-Lipschitz regularization problems in [6].

In this study, we construct a new smoothing approach for non-smooth \( \max\{x, 0\} \) and non-lipschitzian \( (\max\{x, 0\})^p, \) \( 0 < p < 1 \) and we use this smoothing approach
3. A New Smoothing Approach for \( l_1 \) Exact Penalty Functions. Let \( q(t) = \max\{t, 0\} \). We define the following smoothing function \( \bar{q}_r(t, \tau) \) by

\[
\bar{q}_r(t, \tau) = t + (|t|^r + \tau^r)^{\frac{1}{r}}
\]

where \( r > 1 \) and \( \tau > 0 \). The function \( \bar{q}_r(t, \tau) \) is continuously differentiable on \( \mathbb{R} \) and

\[
\lim_{\tau \to 0} \bar{q}_r(t, \tau) = q(t).
\]

**Lemma 3.1.** Let \( r > 1 \) and \( \tau > 0 \) then

\[
0 < \bar{q}_r(t, \tau) - q(t) \leq \frac{\tau}{2}
\]

for any \( t \in \mathbb{R} \).

**Proof.** For \( r > 1 \) and \( \tau > 0 \)

\[
0 \leq \bar{q}_r(t, \tau) - q(t) = \begin{cases} 
\frac{t+(|t|^r+\tau^r)^{\frac{1}{r}}}{2}, & t \leq 0 \\
\frac{-t+(|t|^r+\tau^r)^{\frac{1}{r}}}{2}, & t > 0.
\end{cases}
\]

Since the difference equation \( \bar{q}_r(t, \tau) - q(t) \) is increasing for \( t \leq 0 \) and decreasing for \( t > 0 \), it takes the maximum value at \( t = 0 \).

**Lemma 3.2.** For a fixed \( \tau \), let us assume \( r_1 > r_2 > 1 \) then,

\[
q(t) \leq \bar{q}_{r_1}(t, \tau) \leq \bar{q}_{r_2}(t, \tau).
\]

**Corollary 1.** If \( \tau \to 0 \) and \( r \to \infty \) then the smoothing function \( \bar{q}_r(t, \tau) \to q(t) \).

Depending on above smoothing approach we consider the differentiable objective function

\[
\tilde{F}_1(x, \rho, r, \tau) := f(x) + \rho \sum_{i=1}^{m} \bar{q}_r(g_i(x), \tau),
\]

and the corresponding minimization problem

\[
(\tilde{P}_1) \quad \min_{x \in \mathbb{R}^n} \tilde{F}_1(x, \rho, r, \tau).
\]

**Theorem 3.3.** Let \( x \in \mathbb{R}^n \), \( \tau > 0 \) and \( r > 1 \)

\[
0 \leq \tilde{F}_1(x, \rho, r, \tau) - F_1(x, \rho) \leq \frac{m}{2} \rho \tau.
\]

**Proof.** From Lemma 3.1 we obtain

\[
\tilde{F}_1(x, \rho, r, \tau) - F_1(x, \rho) = \rho \sum_{i=1}^{m} \bar{q}_r(g_i(x), \tau) - \rho \sum_{i=1}^{m} q(g_i(x))
\]

\[
= \rho \sum_{i=1}^{m} (q_i(g_i(x), \tau) - q(g_i(x)))
\]

\[
\leq \frac{m}{2} \rho \tau.
\]

\( \square \)
Theorem 3.4. For a fixed \( r \), let \( \{ \tau_j \} \to 0 \) and \( x^j \) be a solution of \((\tilde{P}_1)\) for \( \rho > 0 \). Assume that \( \bar{x} \) is an accumulation point of \( \{ x^j \} \). Then \( \bar{x} \) is an optimal solution for \((P_1)\).

Proof. By considering the Corollary 1 and Theorem 3.3, the proof is obtained. \( \square \)

Theorem 3.5. For a fixed \( \tau \), let \( \{ r_j \} \to \infty \) and \( x^j \) be a solution of \((\tilde{P}_1)\) for \( \rho > 0 \). Assume that \( \bar{x} \) is an accumulation point of \( \{ x^j \} \). Then \( \bar{x} \) is an optimal solution for \((P_1)\).

Proof. By considering the Corollary 1 and Theorem 3.3, the proof is obtained. \( \square \)

Theorem 3.6. Let \( x^* \) be an optimal solution for the problem \((P_1)\) and \( \bar{x} \) be an optimal solution for the problem \((\tilde{P}_1)\). Then we have the following:

\[
0 \leq \bar{F}_1(\bar{x}, \rho, r, \tau) - F_1(x^*, \rho) \leq \frac{m \rho \tau}{2}.
\]

Proof. From the Theorem 3.3 we have the following:

\[
0 \leq \bar{F}_1(\bar{x}, \rho, r, \tau) - F_1(\bar{x}, \rho, r, \tau) - F_1(x^*, \rho) \\
\leq \bar{F}_1(x^*, \rho, r, \tau) - F_1(x^*, \rho) \leq \frac{m \rho \tau}{2}.
\]

\( \square \)

Figure 1. (a) The green and solid one is the graph \( q(x) = \max\{x, 0\} \), the red and dotted one is the graph of \( \tilde{q}_4(x, 0.3) \), the blue and dashed one is the graph of \( \tilde{q}_4(x, 0.8) \). (b) The green and solid one is the graph \( q(x) \), the red and dotted one is the graph of \( \tilde{q}_4(x, 0.5) \), the blue and dashed one is the graph of \( \tilde{q}_{1.5}(x, 0.5) \).

Definition 3.7. Let \( \tau > 0 \), a point \( x_\tau \) is called \( \tau \)-feasible solution of problem \((P)\), if

\[
g_i(x) \leq \tau, \quad i = 1, 2, \ldots, m.
\]

Theorem 3.8. Let \( x^* \) be an optimal solution for \((P_1)\), \( \bar{x} \) be an optimal solution for \((\tilde{P}_1)\) and let \( x^* \) be a feasible solution for \((P)\) and \( \bar{x} \) be an \( \tau \)-feasible solution for \((P)\), then we have

\[
-\frac{m \rho \tau}{2} \leq f(\bar{x}) - f(x^*) \leq 0.
\]
Proof. Since $\sum_{i=1}^{m} q(g_i(x^*)) = 0$, we have

$$0 \leq \tilde{F}_1(\tilde{x}, \rho, r, \tau) - F_1(x^*, \rho) = f(\tilde{x}) + \rho \sum_{i=1}^{m} \tilde{q}_r(g_i(\tilde{x})) - \left(f(x^*) + \rho \sum_{i=1}^{m} q(g_i(x^*))\right) \leq \frac{m \rho \tau}{2}$$

and we have

$$-\rho \sum_{i=1}^{m} \tilde{q}_r(g_i(\tilde{x})) \leq f(\tilde{x}) - f(x^*) \leq -\rho \sum_{i=1}^{m} q(g_i(\tilde{x})) + \frac{m \rho \tau}{2}.$$ 

From Lemma 3.1, we have

$$-\frac{m \rho \tau}{2} \leq f(\tilde{x}) - f(x^*) \leq 0.$$

4. A New Smoothing for $l_p$ ($0 < p \leq 1$) Exact Penalty Functions. Let us consider the non-lipschitz function $q^p(t) = \max\{t, 0\}^p$ for $0 < p \leq 1$ we define the following smoothing function:

$$\tilde{q}^p_r(t, \tau) = \left((q^p(t))^r + \tau^r\right)^{\frac{1}{r}}$$

where $r > 1$ is a parameter such that $pr > 1$ and $\tau > 0$.

**Lemma 4.1.** For $0 < p \leq 1$, let $r > 1$ such that $pr > 1$, and $\tau > 0$ then

$$0 < \tilde{q}^p_r(t, \tau) - q^p(t) \leq \tau$$

for any $t \in \mathbb{R}$.

**Proof.** For $r > 1$ and $\tau > 0$

$$0 \leq \tilde{q}^p_r(t, \tau) - q^p(t) = \left\{ \begin{array}{ll} \tau & , t \leq 0 \\ \left((tp^r + \tau^r)^{\frac{1}{r}} - tp^r\right)^r & , t > 0. \end{array} \right.$$ 

Since the difference equation $\tilde{q}^p_r(t, \tau) - q^p(t)$ is constant for $t \leq 0$ and decreasing for $t > 0$. Thus, the difference between $\tilde{q}^p_r(t, \tau)$ and $q^p(t)$ can be at most as the parameter $\tau > 0$. 

**Lemma 4.2.** For a fixed $\tau$, let us assume $r_1 > r_2 > 1$ such that $pr_1 > 1$ and $pr_2 > 1$ then,

$$\tilde{q}^p_{r_1}(t, \tau) \leq \tilde{q}^p_{r_2}(t, \tau).$$

**Corollary 2.** If $\tau \to 0$ and $r \to \infty$ then the smoothing function $\tilde{q}^p_r(t, \tau) \to q^p(t)$.

Depending on above smoothing approach we consider the differentiable objective function

$$\tilde{F}_p := f(x) + \rho \sum_{i=1}^{m} \tilde{q}_r^p(g_i(x), \tau),$$

and the corresponding minimization problem

$$(\tilde{P}_p) \min_{x \in \mathbb{R}^n} \tilde{F}_p(x, \rho, r, \tau).$$

**Theorem 4.3.** Let $x \in \mathbb{R}^n$, $\tau > 0$, $p \in (0, 1)$ and $r > 1$

$$0 \leq \tilde{F}_p(x, \rho, r, \tau) - F_p(x, \rho) \leq m \rho \tau.$$
Proof. From Lemma 4.1 we obtain

\[ \tilde{F}_p(x, \rho, r, \tau) - F_p(x, \rho) = \rho \sum_{i=1}^{m} q_i(g_i(x), \tau) - \rho \sum_{i=1}^{m} q(g_i(x)) \]

\[ \leq \rho \sum_{i=1}^{m} (q_i(g_i(x), \tau) - q(g_i(x))) \leq m\rho \tau. \]

\[ \square \]

Theorem 4.4. For a fixed \( r \), let \( \{\tau_j\} \to 0 \) and \( x^j \) be a solution of \((\tilde{P}_p)\) for \( \rho > 0 \). Assume that \( \bar{x} \) is an accumulation point of \( \{x^j\} \). Then \( \bar{x} \) is an optimal solution for \((P_\rho)\).

Proof. By considering the Corollary 2 and Theorem 4.3, the proof is obtained. \( \square \)

Theorem 4.5. Let \( x^* \) be an optimal solution of \((P_\rho)\) and \( \bar{x} \) be an optimal solution of \((\tilde{P}_p)\). Then we have

\[ 0 \leq \tilde{F}_p(\bar{x}, \rho, r, \tau) - F_p(x^*, \rho) \leq m\rho \tau. \]

Proof. From the Theorem 4.3 we have the following:

\[ 0 \leq \tilde{F}_p(\bar{x}, \rho, r, \tau) - F_p(\bar{x}, \rho) \leq \tilde{F}_p(\bar{x}, \rho, r, \tau) - F_p(x^*, \rho) \leq \tilde{F}_p(x^*, \rho, r, \tau) - F_p(x^*, \rho) \leq m\rho \tau. \]

\[ \square \]

Figure 2. (a) The green and solid one is the graph \( f(x) = q_4^{1/2}(x) \), the red and dotted one is the graph of \( \tilde{q}_4^{1/2}(x, 0.1) \), the blue and dashed one is the graph of \( \tilde{q}_4^{1/2}(x, 0.4) \). (b) The green and solid one is the graph \( q^{1/2}(x) \), the red and dotted one is the graph of \( \tilde{q}_4^{1/2}(x, 0.5) \), the blue and dashed one is the graph of \( \tilde{q}_4^{1/2}(x, 0.5) \).

Theorem 4.6. Let \( x^* \) be an optimal solution of \((P_\rho)\), \( \bar{x} \) be an optimal solution of \((\tilde{P}_p)\) and let \( x^* \) be a feasible solution of \((P)\) and \( \bar{x} \) be an \( \tau - \) feasible solution of \((P)\), then we have

\[ -m\rho \tau \leq f(\bar{x}) - f(x^*) \leq 0. \]
Proof. Since \( \sum_{i=1}^{m} q(g_i(x^*)) = 0 \), we have
\[
0 \leq \tilde{F}_p(x, \rho, r, \tau) - F_p(x^*, \rho) \\
= f(x) + \rho \sum_{i=1}^{m} \tilde{q}^p_i(g_i(x)) - \left( f(x^*) + \rho \sum_{i=1}^{m} q^p_i(g_i(x^*)) \right) \\
\leq m \rho \tau
\]
and we have
\[
-\rho \sum_{i=1}^{m} \tilde{q}^p_i(g_i(x)) \leq f(x) - f(x^*) \leq -\rho \sum_{i=1}^{m} q^p_i(g_i(x)) + m \rho \tau.
\]
From Lemma 4.1, we have
\[
-m \rho \tau \leq f(x) - f(x^*) \leq 0.
\]

5. Algorithms for Minimization Procedure. In this section, we propose algorithms to find the global optimal point by considering above smoothing approach. The first algorithm is proposed for the problem \((P_1)\) and the second algorithm is proposed for the problem \((P_p)\).

Algorithm I:

Step 1 Determine the initial point \(x^0\). Determine \(\tau_0 > 0, \rho_0 > 0, r_0 > 1, 0 < \eta < 1,\) and \(N > 1\), let \(j = 0\) and go to Step 2.

Step 2 Use \(x^j\) as the starting point to solve \((\tilde{P}_1)\). Let \(x^{j+1}\) be the solution.

Step 3 If \(x^{j+1}\) is \(\tau\)–feasible for \((P)\), then stop and \(x^{j+1}\) is the optimal solution. If not, determine \(\rho_{j+1} = N \rho_j, \tau_{j+1} = \eta \tau_j, r_{j+1} = N r_j\) and \(j = j + 1\), then go to Step 2.

In order to guaranteed that the algorithm is worked straightly, we have to prove the following theorem.

Theorem 5.1. Assume that for \(\rho \in [\rho_0, \infty), \tau \in (0, \tau_0]\) and \(r \in (1, r_0]\) the set
\[
\arg\min_{x \in \mathbb{R}^n} \tilde{F}_1(x, \rho, r, \tau) \neq \emptyset.
\]
Let \(x^j\) is generated by Algorithm I when \(\eta N < 1\). If \(\{x^j\}\) has a limit point, then the limit point of \(x^j\) is the solution for \((P)\).

Proof. Assume \(\vec{x}\) is a limit point of \(\{x^j\}\). Then there exists set \(J \subset \mathbb{N}\), such that \(x^j \to \vec{x}\) for \(j \in J\). We have to show that \(\vec{x}\) is the optimal solution for \((P)\). Thus, it is sufficient to show (i) \(\vec{x} \in G_0\) and (ii) \(f(\vec{x}) \leq \inf_{x \in G_0} f(x)\).

i. Let us consider the contrary that \(\vec{x} \not\in G_0\), i.e. for sufficiently large \(j \in J\), there exist \(\delta_0 > 0\) and \(i_0 \in \{1, 2, \ldots, m\}\) such that
\[
g_{i_0}(x^j) \geq \delta_0 > 0.
\]
Since \( x^j \) is the global minimum according j-th values of the parameters \( \rho_j, \tau_j, r_j \), for any \( x \in G_0 \) we have

\[
F_1(x^j, \rho_j, r_j, \tau_j) = f(x^j) + \rho_j(\frac{\tau_j}{2}) + \frac{(m - 1)}{2} \rho_j \tau_j \\
= f(x^j) + \rho_j \delta_0 + \frac{m}{2} \rho_j \tau_j \\
\leq f(x) + \frac{m}{2} \rho_j \tau_j.
\]

If \( j \to \infty \) then, \( \rho \to \infty, \rho_j \tau_j \to 0 \) and \( \rho_j \delta_0 \to \infty \). Thus, \( f(x) \) takes infinite values on \( G_0 \) and it contradicts with the boundedness of \( f \) on \( G_0 \).

ii. By considering the Step 2 in Algorithm I and for any \( x \in G_0 \),

\[
\tilde{F}_1(x^j, \rho_j, r_j, \tau_j) \leq \tilde{F}_1(x, \rho_j, r_j, \tau_j) = f(x) + \frac{1}{2} m \rho_j \tau_j
\]

When \( j \to \infty \), we have \( f(\bar{x}) \leq f(x) \).

\[\square\]

**Algorithm II**

Step 1 Determine the initial point \( x^0 \). Determine \( \tau_0 > 0, \rho_0 > 0, r_0 > 1, 0 < \eta < 1 \), and \( N > 1 \), let \( j = 0 \) and go to Step 2.

Step 2 Use \( x^j \) as the starting point to solve \((\tilde{P}_p)\). Let \( x^{j+1} \) be the solution.

Step 3 If \( x^{j+1} \) is \( \tau \)-feasible for \((P)\), then stop and \( x^{j+1} \) is the optimal solution. If not, determine \( \rho_{j+1} = N \rho_j \), \( \tau_{j+1} = 1 + \eta \tau_j \), \( r_{j+1} = r_j + 2 \) and \( j = j + 1 \), then go to Step 2.

**Theorem 5.2.** Assume that for \( \rho \in [\rho_0, \infty), \tau \in (0, \tau_0] \) and \( r \in (1, r_0] \) the set

\[
\arg\min_{x \in \mathbb{R}^n} \tilde{F}_p(x, \rho, r, \tau) \neq \emptyset.
\]

Let \( x^j \) is generated by Algorithm II when \( \eta N < 1 \). If \( \{x^j\} \) has a limit point, then the limit point of \( x^j \) is the solution for \((P)\).

**Proof.** The proof is very similar to the proof of the Theorem 5.1. \(\square\)

5.1. **Numerical Examples.** In this section, we apply our algorithm to test problems. The proposed algorithm is programmed in Matlab R2011A. Numerical results shows the efficiency of this method. The detailed results are presented in the tables for all problems. For these tables we use some symbols in order to abbreviate the expressions. The meaning of these symbols are as the following:

- \( j \) : The number of iterations.
- \( x^j \) : The local minimum point of the jth iteration.
- \( \rho_j \) : Penalty functions parameter of the jth iteration.
- \( \tau_j \) and \( r_j \) : Smoothing parameter of the jth iteration.
- \( g_i(x^j) \) : The value of local minimum point \( x^j \) under the constraint functions.
- \( \tilde{F}_p(x^j, \rho_j, r_j, \tau_j) \) : The value of local minimum point \( x^j \) under \( \tilde{F}_p \).
- \( f(x^j) \) : The value of local minimum point \( x^j \) under \( f \).
Problem 1. Let us consider the Example in [7]
\[ \min f(x) = x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3 \]
\[ \text{s.t. } g_1(x) = (x_1 - 2)^2 + x_2^2 - 1.6^2 \leq 0, \]
\[ g_2(x) = x_1^2 + (x_2 - 3)^2 - 2.7^2 \leq 0, \]
\[ 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2. \]

We choose \( x^0 = (1, 1) \) as a starting point \( \rho_0 = 10, \tau_0 = 0.01, r = 2, \eta_0 = 0.1 \) and \( N = 3 \). The results are shown in the Table 1 and 2. By considering both \( \bar{P}_1 \) and \( \bar{P}_p \) the global minimum is obtained at a point \( x^* = (0.7254, 0.3993) \) with the corresponding value 1.8376. In the papers [7, 15], the obtained global minimum point is \( x^* = (0.72540669, 0.3992805) \) with the corresponding value 1.837623. Both of our algorithms find the correct point as \([7, 15]\).

| Table 1: Table of minimization process of the Problem 1 by considering Algorithm I |
|---|
| \( j \) & \( x^{j+1} \) & \( \rho_j \) & \( \tau_j \) & \( r_j \) & \( g_1(x^j) \) & \( g_2(x^j) \) & \( F_\rho(x^j, \rho_j, r_j, \tau_j) \) & \( f(x^j) \) |
| 0 & (0.7253, 0.3994) & 10 & 0.01 & 2 & -0.7757 & -0.0008 & 1.8414 & 1.8390 |
| 1 & (0.7254, 0.3993) & 30 & 0.001 & 6 & -0.7759 & -0.0001 & 1.8378 & 1.8378 |
| 2 & (0.7254, 0.3993) & 90 & 0.0001 & 18 & -0.7759 & -0.0000 & 1.8376 & 1.8376 |

| Table 2: Table of minimization process of the Problem 1 by considering Algorithm II for \( p = 2/3 \) |
|---|
| \( j \) & \( x^{j+1} \) & \( \rho_j \) & \( \tau_j \) & \( r_j \) & \( g_1(x^j) \) & \( g_2(x^j) \) & \( F_\rho(x^j, \rho_j, r_j, \tau_j) \) & \( f(x^j) \) |
| 0 & (0.7254, 0.3993) & 10 & 0.01 & 2 & -0.7759 & 0.0000 & 1.8915 & 1.8375 |
| 1 & (0.7254, 0.3993) & 30 & 0.001 & 6 & -0.7759 & -0.0000 & 1.8538 & 1.8376 |
| 2 & (0.7254, 0.3993) & 90 & 0.0001 & 18 & -0.7765 & -0.0000 & 1.8376 & 1.8376 |

Problem 2. Let us consider the example in [15],
\[ \min f(x) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3 \]
\[ \text{s.t. } g_1(x) = x_1^2 + x_2^2 + x_3^2 - 25 = 0, \]
\[ g_2(x) = (x_1 - 5)^2 + x_2^2 + x_3^2 - 25 = 0 \]
\[ g_3(x) = (x_1 - 5)^2 + (x_2 - 5)^2 + (x_3 - 5)^2 - 25 \leq 0 \]

We choose \( x^0 = (2, 2, 2) \) as a starting point \( \rho_0 = 10, \tau_0 = 0.1, r = 2, \eta_0 = 0.1 \) and \( N = 3 \). The results are shown in the Table 3 and 4. By considering Algorithm I and Algorithm II the global minimum is obtained at a points \( x^* = (2.5000, 4.2196, 0.9721) \) and \( x^* = (2.5000, 4.2210, 0.9661) \) respectively. The corresponding value of both points is 944.2157. In the papers [15], the obtained global minimum point is \( x^* = (2.500000, 4.221305, 0.964666) \) with the corresponding value 944.2157.

Both of our algorithms find the correct solutions with the lower iteration numbers than [15].
Problem 3. The Rosen-Suzski problem in [7]

\[
\begin{align*}
\min_{x} f(x) &= x_1^2 + x_2^2 + 2x_3^3 + x_4^2 - 5x_1 - 21x_3 + 7x_4 \\
\text{s.t.} & \quad g_1(x) = 2x_1^2 + x_2^2 + x_3^2 + 2x_1 + x_2 + x_3 - 5 \leq 0, \\
& \quad g_2(x) = x_1^2 + x_2^2 + x_3^2 + x_1 - x_2 + x_3 - x_4 - 8 \leq 0, \\
& \quad g_3(x) = x_1^2 + 2x_2^2 + x_3^2 + 2x_1^2 - x_1 - x_4 - 10 \leq 0.
\end{align*}
\]

First, we choose \(x^0 = (0, 0, 0, 0)\), \(\rho_0 = 10\), \(\tau_0 = 0.01\), \(r = 2\), \(\eta_0 = 0.1\) and \(N = 3\). The results are shown in the Tables 5 and 6. By considering (\(P_1\)) the global minimum is obtained at a point \(x^* = (0.1697, 0.8358, 2.0084, -0.9651)\) with the corresponding value \(-44.2383\). By considering (\(P_2\)) the global minimum is obtained at a point \(x^* = (0.1788, 0.8189, 2.0119, -0.9588)\) with the corresponding value \(-44.2322\). In the paper [7], the obtained global minimum point is \(x^* = (0.1684, 0.8539, 0.0016, -0.9755)\) with the corresponding value \(-44.2304\). In [15], the obtained global minimum point is \(x^* = (0.1701, 0.8356, 0.0028, -0.9524)\) with the corresponding value \(-44.2338\).

It can be seen that our algorithms present numerically better results than [7] and they find the approximate solutions with the lower iteration numbers in comparison with [15].

#### Table 3. Table of minimization process of the Problem 2 by considering Algorithm I

| \(j\) | \(x^{j+1}\) | \(\rho_j\) | \(r_j\) | \(g_1(x^j)\) | \(g_2(x^j)\) | \(g_3(x^j)\) | \(F_j(x^j, \rho_j, r_j, \tau_j)\) | \(f(x^j)\) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|----------------|-------------|
| 0     | (2.4998, 4.1199, 1.3332) | 10 | 0.01 | 2 | 0.0062 | 0.0022 | -4.5292 | 944.4283 | 944.4943 |
| 1     | (2.5000, 4.2165, 0.9856) | 30 | 0.001 | 6 | 0.0001 | 0.0000 | -2.0204 | 944.2221 | 944.2161 |
| 2     | (2.5000, 4.2193, 0.9732) | 90 | 0.0001 | 18 | 0.0000 | -0.0000 | -1.9255 | 944.2175 | 944.2157 |
| 3     | (2.5000, 4.2196, 0.9721) | 270 | 0.00001 | 54 | 0.0000 | 0.0000 | -1.9171 | 944.2157 | 944.2157 |

#### Table 4. Table of minimization process of the Problem 2 by considering Algorithm II for \(p = 2/3\)

| \(j\) | \(x^{j+1}\) | \(\rho_j\) | \(r_j\) | \(g_1(x^j)\) | \(g_2(x^j)\) | \(g_3(x^j)\) | \(F_j(x^j, \rho_j, r_j, \tau_j)\) | \(f(x^j)\) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|----------------|-------------|
| 0     | (2.5000, 4.2157, 0.9890) | 10 | 0.01 | 2 | 0.0000 | -0.0000 | -2.0468 | 944.2176 | 944.2164 |
| 1     | (2.5000, 4.2210, 0.9658) | 30 | 0.001 | 6 | 0.0000 | 0.0000 | -1.8687 | 944.2159 | 944.2157 |
| 2     | (2.5000, 4.2210, 0.9661) | 90 | 0.0001 | 18 | 0.0000 | 0.0000 | -1.8708 | 944.2157 | 944.2157 |

#### Table 5. Table of minimization process of the Problem 3 by considering Algorithm I

| \(j\) | \(x^{j+1}\) | \(\rho_j\) | \(r_j\) | \(g_1(x^j)\) | \(g_2(x^j)\) | \(g_3(x^j)\) | \(F_j(x^j, \rho_j, r_j, \tau_j)\) | \(f(x^j)\) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|----------------|-------------|
| 0     | (0.1694, 0.8354, 2.0085, -0.9650) | 10 | 0.01 | 2 | -0.0016 | -0.0008 | -1.8836 | 44.2272 | -44.2311 |
| 1     | (0.1698, 0.8352, 2.0087, -0.9648) | 30 | 0.001 | 6 | -0.0002 | 0.0001 | -1.8849 | 44.2334 | -44.2334 |
| 2     | (0.1697, 0.8358, 2.0084, -0.9651) | 90 | 0.0001 | 18 | -0.0000 | 0.0000 | -1.8821 | 44.2338 | -44.2338 |
Problem 4. Let us consider the Example in [15, 10]

\[
\begin{align*}
\min f(x) &= 10x_2 + 2x_3 + x_4 + 3x_5 + 4x_6 \\
\text{s.t.} & \quad g_1(x) = x_1 + x_2 - 10 = 0, \\
& \quad g_2(x) = -x_1 + x_3 + x_4 + x_5 = 0, \\
& \quad g_3(x) = -x_2 - x_3 + x_5 + x_6 = 0, \\
& \quad g_4(x) = 10x_1 - 2x_3 + 3x_4 - 2x_5 - 16 \leq 0, \\
& \quad g_5(x) = x_1 + 4x_3 + x_5 - 10 \leq 0, \\
& \quad 0 \leq x_1 \leq 12, \quad 0 \leq x_2 \leq 18, \\
& \quad 0 \leq x_3 \leq 5, \quad 0 \leq x_4 \leq 12, \\
& \quad 0 \leq x_5 \leq 1, \quad 0 \leq x_6 \leq 16,
\end{align*}
\]

We choose \( x^0 = (0, 0, \ldots, 0) \) as a starting point \( \rho_0 = 300, \) \( \tau_0 = 0.01, \) \( \eta_0 = 0.1, \) \( r_0 = 8 \) and \( N = 3 \) for both Algorithm I and Algorithm II. The results are shown in the Table 7 and 8. By considering \( (\hat{P}_g) \) the global minimum is obtained at a point \( x^* = (1.6227, 8.3773, 0.0216, 0.6037, 0.9974, 7.4015) \) with the corresponding value 117.0182 and by considering \( (\hat{P}_p) \) the global minimum is obtained at a point \( x^* = (1.7593, 8.2407, 0.3756, 0.3837, 1.0000, 7.6164) \) with the corresponding value 117.0071. In the paper [15], the obtained global minimum point is \( x^* = (1.650682, 8.349318, 0.091775, 0.558907, 1.00000, 7.441092) \) with the corresponding value 117.000004.

In [15] in which three algorithms are offered for a new smoothing technique, approximate solution is found with 4, 3 and 13 iterations in the Algorithms I, II and III, respectively. We note that the solution is not found in Algorithm II of [15]. Whereas, approximate solution is found with 3 iterations in both of our Algorithms I and II for our smoothing technique. Of course the lower iteration numbers in no way mean the lesser computational times.

### Table 7. Table of minimization process of the Problem 4 by considering Algorithm I

| \( j \) | \( x^{(j)} \) | \( \rho_j \) | \( \tau_j \) | \( \eta_j \) | \( f(x^{(j)}) \) | \( f(x^{(j)}) \) | \( f(x^{(j)}) \) | \( f(x^{(j)}) \) | \( f(x^{(j)}) \) |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0   | (1.6227, 8.3773, 0.0216, 0.6037, 0.9974, 7.4015) | 300  | 0.01  | 2.0000 | -0.0000 | -0.0000 | -0.0000 | -0.0000 | -0.0000 |
| 1   | (1.6227, 8.3773, 0.0216, 0.6037, 0.9974, 7.4015) | 900  | 0.01  | 6.0000 | 0.0000  | 0.0000  | 0.0000  | 0.0000  | 0.0000  |
| 2   | (1.6227, 8.3773, 0.0216, 0.6037, 0.9974, 7.4015) | 2700 | 0.005 | 18.000 | 0.0000  | 0.0000  | 0.0000  | 0.0000  | 0.0000  |

6. Conclusion. In this study, we propose two new smoothing approaches for \( l_1 \) and \( l_p \) exact penalty functions. Both of our smoothing approaches present lower errors among non-smooth penalty problems, smoothed penalty problems and original...
optimization problem. By considering these smoothing approaches, we construct the minimization algorithms. We apply these algorithms on test problems and obtain satisfactorily results.

Our smoothing techniques provide good approximations to the non-smooth function. In fact, both of smoothing techniques can be used for non-smooth and non-Lipschitz functions by controlling the parameter \( r \). Moreover, they have easy formulations and they are easy applicable.

The algorithms are effective for both medium scale and large scale optimization problems. The Algorithm I reaches the optimum value rapidly and Algorithm II presents high accuracy in finding of the optimum point.

For future works, we plan to introduce new smoothing techniques for min-max, regularization problems and penalty function approach with smaller errors and use these smoothing techniques inside the new global optimization algorithms.

REMARKS

REFERENCES

1. A. M. Bagirov, A. I. Nazhim and N. Sultanova, Hyperbolic smoothing functions for non-smooth minimization, *Optimization*, 62 (2013), 759–782.

2. F. S. Bai, Z. Y. Wu and D. L. Zhu, Lower order calmness and exact penalty function, *Optimization Methods ans Software*, 21 (2006), 515–525.

3. A. Ben-Tal and M. Teboule, Smoothing technique for nondifferentiable optimization problems, Lecture notes in mathematics, 1405, Springer-Verlag, Heidelberg, (1989), 1–11.

4. D. Bertsekas, Nondifferentiable optimization via approximation, *Mathematical Programming Study*, 3 (1975), 1–25.

5. C. Chen and O. L. Mangasarian, A Class of smoothing functions for nonlinear and mixed complementarity problem, *Computational Optimization and Application*, 5 (1996), 97–138.

6. X. Chen, Smoothing Methods for nonsmooth, nonconvex minimization, *Mathematical Programming Serie B*, 134 (2012), 71–99.

7. S. J. Lian, Smoothing approximation to \( l_1 \) exact penalty for inequality constrained optimization, *Applied Mathematics and Computation*, 219 (2012), 3113–3121.

8. B. Liu, On smoothing exact penalty function for nonlinear constrained optimization problem, *Journal of Applied Mathematics and Computing*, 30 (2009), 259–270.

9. Z. Meng, C. Dang, M. Jiang and R. Shen, A smoothing objective penalty function algorithm for inequality constrained optimization problems, *Numerical Functional Analysis and Optimization*, 32 (2011), 806–820.

10. M. C. Pinar and S. Zenios, On smoothing exact penalty functions for convex constrained optimization, *SIAM Journal on Optimization*, 4 (1994), 468–511.

11. Z. Y. Wu, H. W. J. Lee, F. S. Bai and L. S. Zhang, Quadratic smoothing approximation to \( l_1 \) exact penalty function in global optimization, *Journal of Industrial and Management Optimization*, 53 (2005), 533–547.

12. Z. Y. Wu, F. S. Bai, X. Q. Yang and L. S. Zhang, An exact lower order penalty function and its smoothing in nonlinear programming, *Optimization*, 53 (2004), 51–68.

13. A. E. Xavier, The hyperbolic smoothing clustering method, *Pattern Recognition*, 43 (2010), 731–737.

14. A. E. Xavier and A. A. F. D. Oliveira, Optimal covering of plane domains by circles via hyperbolic smoothing, *Journal of Global Optimization*, 31 (2005), 493–504.

15. X. Xu, Z. Meng, J. Sun and R. Shen A penalty function method based on smoothing lower order penalty function, *Journal of Computational and Applied Mathematics*, 235 (2011), 4047–4058.
[16] N. Yilmaz and A. Sahiner, A New Global Optimization Technique Based on the Smoothing Approach for Non-smooth, Non-convex Optimization, Submitted.

[17] N. Yilmaz and A. Sahiner, Smoothing Approach for Non-lipschitz Optimization, Submitted.

[18] I. Zang, A smoothing out technique for min-max optimization, Mathematical Programming, 19 (1980), 61–77.

[19] W. I. Zangwill, Nonlinear programing via penalty functions, Management Science, 13 (1967), 344–358.

Received December 2015; 1st revision March 2016; final revision May 2016.

E-mail address: ahmetnur32@gmail.com
E-mail address: nurullahyilmaz@sdu.edu.tr
E-mail address: guldenkapusuzz@gmail.com