**INCIDENCE ESTIMATES FOR WELL SPACED TUBES**

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**Abstract.** We prove analogues of the Szemerédi–Trotter theorem and other incidence theorems using δ-tubes in place of straight lines, assuming that the δ-tubes are well spaced in a strong sense.

**1 Introduction**

In a series of papers in the late 90s, Tom Wolff explored the connection between incidence geometry and Kakeya-type problems in harmonic analysis. By adapting techniques from the incidence geometry literature, he was able to prove some striking results in harmonic analysis (cf. [Wol96, Wol97, Wol00]). Incidence geometry is about the intersection patterns of lines, and the Kakeya problem is about the intersection patterns of thin tubes, and so it sounds reasonable that they should be related. It turns out, however, that it is quite a subtle problem to adapt theorems from the setting of lines to the setting of thin tubes, and a lot remains unknown. In order to get non-trivial estimates in the setting of tubes, it is necessary to add some assumption about how the tubes are spaced. There are many possible assumptions, and so there are many different problems to consider. In this paper, we consider very strong spacing assumptions on the tubes, and under those assumptions we prove nearly sharp incidence estimates.

Our first main result is an analogue of the Szemerédi–Trotter theorem from incidence geometry. We first recall the Szemerédi–Trotter theorem. Suppose that $\mathcal{L}$ is a set of lines in the plane. For $r \geq 2$, let $P_r(\mathcal{L})$ denote the $r$-rich points of $\mathcal{L}$—the set of points that lie in at least $r$ lines of $\mathcal{L}$. The Szemerédi–Trotter theorem gives sharp bounds for $|P_r(\mathcal{L})|$: $|P_r(\mathcal{L})| \lesssim r^{-3}|\mathcal{L}|^2 + r^{-1}|\mathcal{L}|$ (ST)

where $|\cdot|$ denotes the cardinality of a set and for $A, B > 0$, $A \lesssim B$ means $A \leq CB$ for an absolute positive constant $C$.

Now suppose that $\mathbb{T}$ is a set of $\delta \times 1$ tubes (i.e. rectangles) in $[0, 1]^2$, which we call δ-tubes for short. The set of all δ-balls that intersect at least $r$ tubes of $\mathbb{T}$ is infinite, and so we define $P_r(\mathbb{T})$ to be the set of $r$-rich δ-balls, the δ-balls that have centers in the lattice $\delta\mathbb{Z}^2$ and intersect at least $r$ tubes of $\mathbb{T}$. The bound (ST) does...
not hold for tubes in this generality. We begin with a few simple examples to show that some spacing conditions are necessary. First of all, if all the tubes of \( T \) are tiny perturbations of a fixed tube, with the size of the perturbation less than \( \delta \), then we can get \( \delta^{-1} \) -rich \( \delta \)-balls for \( r \sim |T| \). We say that two \( \delta \)-tubes, \( T_1 \) and \( T_2 \), are essentially distinct if

\[
|T_1 \cap T_2| \geq (1/2)|T_1|.
\]

From now on we assume that the tubes of \( T \) are essentially distinct. But the bound (ST) does not hold for essentially distinct tubes either. Let \( R \) be an \( r \delta \times 1 \) rectangle. There are \( \sim r^2 \) essentially distinct \( \delta \)-tubes in \( R \), and we let \( T_R \) denote such a set of \( \delta \)-tubes. Then \( P_r(T_R) \sim r \delta^{-1} \), which is often much bigger than the right-hand side of (ST). In the context of the Kakeya problem, one sometimes considers tubes that point in distinct directions. For instance, suppose that \( T \) consists of \( \delta^{-1} \) -tubes, all going through the origin, and pointing in \( \delta \)-separated directions. In this case, it’s not hard to check that \( |P_r(T)| \sim r^{-2}|T|^2 \), which is still bigger than the right-hand side of (ST) for all \( 1 \ll r \ll \delta^{-1} \).

To get an analogue of (ST) for tubes, we need to make a stronger hypothesis about how the tubes are spaced. We will consider the following hypothesis, which is the strongest spacing condition that we can make. Fix some \( W \geq 1 \). There are \( \sim W^2 \) essentially distinct \( W^{-1} \times 1 \) rectangles in \([0,1]^2\). Then fix some \( \delta < 1/W \) and let \( T \) be a set of \( W^2 \) \( \delta \)-tubes, one contained in each of these \( W^{-1} \times 1 \) rectangles. Even under this spacing condition (ST) does not always hold. The reason is that an average \( \delta \)-ball in \([0,1]^2\) is \( r \)-rich for \( r \sim \delta |T| \). If \( r \leq \delta |T| \), then for a typical choice of \( T \), we have \( |P_r(T)| \sim \delta^{-2} \), which often violates (ST). Our first theorem says that if \( T \) is well-spaced in this sense, and if \( r \) is bigger than the threshold \( \delta |T| \), then the (ST) bound holds up to small errors.

**Theorem 1.1.** Suppose that \( 1 \leq W \leq \delta^{-1} \). Suppose that \( T \) is a set of \( \sim W^2 \) \( \delta \)-tubes in \([0,1]^2\) with at most one \( \delta \)-tube of \( T \) in each \( W^{-1} \times 1 \) rectangle.

If \( r > \max(\delta^{-1-\epsilon}|T|, 1) \),

then \( |P_r(T)| \leq C(\epsilon)\delta^{-\epsilon}r^{-3}|T|^2 \)

for all \( \epsilon > 0 \). Here \( C(\epsilon) \) is a constant only depending on \( \epsilon \), in particular, independent of \( W \).

Another variation of our argument estimates the incidences for a set of tubes with many well-spaced tubes in every direction.

**Theorem 1.2.** Let \( 1 \leq W \leq \delta^{-1} \) and \( 1 \leq N_1 \leq (W \delta)^{-1} \). Divide the circle into arcs \( \theta \) of length \( \delta \). For each \( \theta \), and each \( 1 \leq j \leq W \), let \( T_{\theta,j} \subset [0,1]^2 \) be a \( \delta \)-tube. Suppose that for each \( \theta \), and each \( W^{-1} \times 1 \) rectangle in direction \( \theta \), there are uniformly \( \sim N_1 \) tubes \( T_{\theta,j} \) in the rectangle. Let \( T \) be the set of all the tubes \( T_{\theta,j} \). Then for any \( \epsilon > 0 \)

if \( r \geq C_1(\epsilon)\delta^{1-\epsilon}|T| \),
then \( |P_r(T)| \leq C_2(\epsilon)\delta^{-4}W^{-1}r^{-2}|T|^2, \)

where \( C_1(\epsilon) \) and \( C_2(\epsilon) \) are two constants only depending on \( \epsilon \), in particular, independent of \( W \) and \( N_1 \).

Theorem 1.2 can be interpreted as stating that, under the separation hypothesis involving the parameter \( W \), one obtains an improvement over Cordoba’s theorem [Cor77] by a gain of \( W^{-1} \) in the constant.

This estimate is also sharp, as we will see below. This problem came up in conversations with Ciprian Demeter about decoupling theory. We will discuss the connection with decoupling in a paper [DGW19] with him.

In Theorem 1.2, the \( \delta \)-tubes in direction \( \theta \) statistics is roughly the same for all \( \theta \), but this is not necessarily the case for Theorem 1.1. In fact, the assumptions on Theorems 1.1 and 1.2 are different even when \( N_1 = 1 \). In Theorem 1.1, there are about \( W^2 \delta \)-tubes, while in Theorem 1.2, there are about \( \delta^{-1}N_1W \delta \)-tubes.

At this point, let us discuss a little further the circle of problems raised in [Wol96] and how our result fits into the literature. In [Wol96], Wolff explained how incidence estimates for rectangles are related to the dimension of sets of Furstenberg type. In [KT01], Katz and Tao showed that the dimension of Furstenberg type sets is connected with discretized versions of Falconer’s distance set conjecture and to discretized sum-product problems. For all three problems, the known methods as of 2000 gave some partial results, and [KT01] showed that further progress on any one problem would imply progress on the others. A little later, Bourgain [Bou10] proved a discretized sum-product theorem, giving some small further progress on the Furstenberg type problem and Falconer problem as well. The further progress is quantitatively small, but meaningful. All three problems involve spacing conditions. The exact statement of the spacing conditions is a little different in each problem, but morally the spacing conditions in all three problems are much weaker than the spacing conditions we consider here. There is related work in progress by Bateman–Lie which gives an incidence estimate for rectangles under spacing conditions that are much more general than in our theorems, but the bounds they prove are weaker. In summary, all these previous results work with rather general spacing conditions, and prove estimates that are small but meaningful improvements over standard methods. In contrast, this paper considers very strong spacing conditions, but it proves sharp estimates.

We were also able to push our method to three dimensions. In [GK15], the first author and Nets Katz proved an incidence estimate for lines in \( \mathbb{R}^3 \), which says that if \( \mathcal{L} \) is a set of lines in \( \mathbb{R}^3 \) with at most \( |\mathcal{L}|^{1/2} \) lines in any plane or degree 2 algebraic surface, then

\[
|P_r(\mathcal{L})| \lesssim r^{-2}|\mathcal{L}|^{3/2} + r^{-1}|\mathcal{L}|. \tag{1}
\]

We prove an analogue of this estimate for well-separated tubes in three dimensions.
**Theorem 1.3.** Suppose that \( 1 \leq W \leq \delta^{-1} \). Suppose that \( \mathbb{T} \) is a set of \( \sim W^4 \) \( \delta \)-tubes in \([0, 1]^3\) with at most one \( \delta \)-tube of \( \mathbb{T} \) in any tube of radius \( W^{-1} \) and length 1.

If \( r > \max(\delta^{2-\epsilon}|\mathbb{T}|, 1) \),
then \( |P_r(\mathbb{T})| \leq C(\epsilon)\delta^{-\epsilon}r^{-2}|\mathbb{T}|^{3/2} \).

This theorem gives a very special case of the Kakeya conjecture in \( \mathbb{R}^3 \). The Kakeya maximal function conjecture in \( \mathbb{R}^3 \) says that if \( \mathbb{T} \) is a set of \( \delta^{-2} \) \( \delta \)-tubes pointing in \( \delta \)-separated directions, then \( |P_r(\mathbb{T})| \leq C(\epsilon)\delta^{-\epsilon}r^{-3/2}|\mathbb{T}|^{3/2} \). Our bound is stronger than this one, but it only applies if the tubes of \( \mathbb{T} \) obey our very strong spacing condition.

The incidence estimate for lines in \( \mathbb{R}^3 \) in [GK15] was motivated by the Erdős distinct distance problem in the plane. The problem asks for the minimal number of distinct distances determined by \( N \) points in the plane. In [ES10], Elekes and Sharir proposed an interesting approach to the distinct distance problem which connects it to incidences between points and lines in three dimensions. Combining their approach with the bound (1), the paper [GK15] proved that \( N \) points in the plane determine \( \gtrsim N/\log N \) distinct distances, which is sharp up to logarithmic factors. Using the Elekes–Sharir framework, Theorem 1.3 implies a similar distance estimate for well-spaced \( \delta \)-balls or points.

**Theorem 1.4.** If \( E \) is a set of \( \delta^{-1} \) \( \delta \)-balls in \([0, 1]^2\) with \( \lesssim 1 \) \( \delta \)-balls in each \( \delta^{1/2} \)-ball, then the number of \( \delta \)-intervals needed to cover \( \Delta(E) \) is \( \gtrsim C(\epsilon)\delta^{-1+\epsilon} \).

This theorem is relevant to the Falconer problem which is a kind of continuous analogue of the Erdős distinct distance problem. Falconer asked for the smallest Hausdorff dimension of a compact set \( E \subset [0, 1]^2 \) which guarantees that \( \Delta(E) \) has positive measure. In [Fal86], FALCONER proved that \( \dim_H(E) > 3/2 \) suffices, and he conjectured that \( \dim_H(E) > 1 \) suffices. In [Mat87], MATTLA proposed a Fourier analytic approach to the problem which connects it to restriction theory. Using that connection, WOLFF [Wol99] proved that \( \dim_H(E) > 4/3 \) suffices. Recently, using decoupling, the paper [GIOW18] proved that \( \dim_H(E) > 5/4 \) suffices. Falconer’s conjecture is closely related to the following conjecture about finite sets of balls.

**Conjecture 1.5.** Suppose that \( \alpha > 1 \). Suppose that \( E \) is a set of \( \delta^{-\alpha} \) \( \delta \)-balls in \([0, 1]^2\), and that any ball of radius \( S\delta \) contains \( \lesssim \delta^{-\epsilon}S^{\alpha} \) balls of \( E \). Then the number of \( \delta \)-intervals needed to cover \( \Delta(E) \) is \( \gtrsim C(\epsilon)\delta^{-1} \).

Theorem 1.4 proves this conjecture up a factor of \( \delta^\epsilon \) for sets \( E \) that are as widely spaced as possible. In the other direction, there has been some remarkable work by ORPONEN [Orp17a] and KELETI–SHMERKIN [KS18] on the case when \( E \) is tightly spaced. We say that \( E \) is an Ahlfors–David regular set of \( \delta \)-balls if, for each ball of \( E \), the concentric \( S\delta \) ball contains about \( S^{\alpha} \) balls of \( E \). ORPONEN’s paper [Orp17a] implies that this conjecture holds up to a factor of \( \delta^\epsilon \) for Ahlfors–David regular sets.
Let us now describe the sharp examples for Theorem 1.2, because these examples indicate an important structure that plays a role in the proofs. We pick $W$ balls of radius $A\delta$, for a parameter $A$ to be determined later, with centers evenly spaced along the line segment from $(0,0)$ to $(1,0)$. For each $\theta$, we choose one tube $T_{\theta,j}$ passes through the center of each $A\delta$-ball and let $T = \{T_{\theta,j}\}$ So $\sim \delta^{-1}$ tubes in $T$ pass through each $A\delta$-ball. We call these $A\delta$-balls heavy balls.

Now we choose $A$ such that $r = A^{-1}\delta^{-1}$, any $\delta$-ball inside those heavy balls has $\geq r$ tubes in $T$ passing through it.

We compute

$$|P_r(T)| \gtrsim W A^2 = W r^{-2} \delta^{-2} = W^{-1} r^{-2} |T|^2.$$  

These heavy balls play an important role in the proof. One key tool in our proof is a Fourier analysis argument which shows that if there are too many $r$-rich $\delta$-balls, then they have to be organized into larger heavy balls like in this example. This Fourier analysis argument is based on arguments in the literature on projection theory, especially the recent paper by ORPONEN [Orp17b]. We combine this heavy ball lemma with the idea of partitioning, which comes from the incidence geometry literature. In [CEGSW90], CLARKSON, EDELSBRUNNER, GUIBAS, SHARIR and WELZL used the idea of partitioning to give a new proof of the Szemerédi–Trotter theorem and prove new theorems in incidence geometry, and Wolff in turn built on this partitioning idea in the papers mentioned above.

**Notation.** In the following, $|X|$ denotes the cardinality of $X$ if $X$ is a finite set and the Lebesgue measure of $X$ if $X$ is an infinite measurable set. $C(x)$ means an absolute constant only depending on $x$. If $B$ is a ball or rectangle, then we use $\lambda B$ to denote a $\lambda$-rescaling of $B$ with the same center.

We use the notation $A \lesssim B$ in several sections. Morally, it always means that $A$ is bounded by something slightly larger than $B$. The precise meaning is defined in each section the first time that $\lesssim$ appears.

## 2 Finding Heavy Balls

**Proposition 2.1.** Suppose that $P$ is a set of unit balls in $[0,D]^n$ and $T$ is a set of essentially distinct tubes of length $D$ and radius 1 in $[0,D]^n$. Suppose that each ball of $P$ lies in about $E$ tubes of $T$. Let $S = D^{\epsilon/10n}$ for a tiny $\epsilon > 0$. Then either

**Thin case.** $|P| \lesssim_n S^n E^{-2} |T|D^{n-1}$, or

**Thick case.** There is a set of finitely overlapping $2S$-balls $Q_j$ (heavy balls) such that

1. $\cup_j Q_j$ contains a fraction $\gtrsim_n 1$ of the balls of $P$,
2. Each $Q_j$ intersects $\gtrsim_n S^{n-1} E$ tubes of $T$.

Here $\lesssim_n$ means $\leq C(\epsilon, n) D^{10n\epsilon^3}$.  

We choose the name “thick case” because in this case, we usually thicken the balls and tubes and go to a next scale. We use the name “thin case” to represent when it is “non-thick”.

Before giving the proof, let us discuss the numerology. $D > 1$ is a large number. Since $S$ is small compared to $D$, the $S$ factor in the thin case is minor, and it will be negligible in our later analysis. To give a sense of the thin case, let us focus on dimension $n = 2$. First consider the case that $|T| = D$, which is the case if we have one tube in each direction in the sense of the Kakeya conjecture. If $|T| = D$, then we get the bound $|P| \lesssim E^{-2}|T|^2$ (we write $\lesssim$ instead of $\lesssim_\alpha$ when the $n$ is a bounded number clear from the context), which is the same as the bound in Cordoba’s theorem for the Kakeya maximal function [Cor77]. If $|T|$ is much bigger than $D$, then the bound in the thin case represents a savings. Now we turn to the thick case. Without loss of generality, we can think of $P$ as $P_E(T)$. In the thick case, a typical unit ball in one of the $2S$-balls $Q_j$ lies in $\sim E$ tubes of $T$, and so morally all the unit balls of each $Q_j$ lie in $P_E(T)$. This structure matches the heavy balls we saw in the sharp example for Theorem 1.2 in the introduction. However, since we use small $S$, applying this Proposition does not immediately find the whole heavy ball in the example—only a smaller heavy sub-ball. In the full proof of Theorem 1.2, we will use induction, implicitly using this lemma many times at different scales.

**Proof.** The proof of Proposition 2.1 is based on Fourier analysis. We first do some basic reduction. Define

$$W_S(q) = \#\{T \in T : T \cap N_S(q) \neq \emptyset\}.$$ 

We choose a subset $P' \subset P$ such that $|P'| \gtrsim |P|$ and $W_S(q_1) \sim W_S(q_2)$ for any $q_1, q_2 \in P'$. We work with $P'$ for the rest of the proof. To ease the notation, we write $P'$ as $P$.

For each unit ball $q$ of $P$, we let $\psi_q$ be a smooth bump function approximating $\chi_q$ in the sense that $\text{supp} \psi_q \subset 2q$ and $\psi_q = 1$ on $q$. Let $f = \sum_{q \in P} \psi_q$. For each tube $T$ in $T$, let $\psi_T$ be a smooth bump approximating $\chi_T$. Let $g = \sum_{T \in T} \psi_T$. Let $I(P, T)$ denote the cardinality of the set $\{(q, T) : |q \cap T| \geq |q|/2, q \in P, T \in T\}$. If $q$ intersects $T$, then $\int \psi_q \psi_T \gtrsim 1$, and so

$$I(P, T) \lesssim \int fg.$$

We apply Plancherel: $\int fg = \int \hat{f} \hat{g}$. Next we decompose Fourier space into high-frequency and low-frequency pieces. Let $\rho$ be a real number that is slightly larger than $S^{-1}$. For example, we can take $\rho = D^{\epsilon^2} S^{-1}$. Let $\eta_0$ be a smooth bump function which is equal to 1 on the unit ball, and which is supported in the ball of radius 2. Then we take $\eta(\omega) = \eta_0(\rho^{-1} \omega)$.

$$I(P, T) \lesssim \int \eta \hat{f} \hat{g} + \int (1 - \eta) \hat{f} \hat{g}.$$
If the high frequency piece dominates, we will show that the conclusion of the thin case holds, and if the low frequency piece dominates, then we will show that the conclusion of the thick case holds.

**The high frequency case.** If the high-frequency term dominates, then we have

$$I(P, \mathbb{T}) \lesssim \int (1 - \eta) \hat{f} \hat{g} \leq \left( \int (1 - \eta) |\hat{f}|^2 \right)^{1/2} \left( \int (1 - \eta) |\hat{g}|^2 \right)^{1/2}.$$  

We bound the factor involving \( f \) by \( \|\hat{f}\|_{L^2} = \|f\|_{L^2} \sim |P|^{1/2} \).

To bound the factor involving \( g \), we take advantage of the support of the Fourier transform of \( \psi_T \). Cover the unit sphere \( S^{n-1} \) by \( 1/D \)-caps \( \theta \). For each \( \theta \), we call the outer normal direction of the center of \( \theta \) on \( S^{n-1} \) as the direction of \( \theta \).

Let \( \mathbb{T}_\theta \) be the set of \( T \in \mathbb{T} \) in direction \( \theta \), and let \( g_\theta = \sum_{T \in \mathbb{T}_\theta} \psi_T \). If \( T \) is a \( 1 \times D \) tube in direction \( \theta \), then \( \hat{\psi}_T \) is rapidly decaying outside of \( \theta^* \), where \( \theta^* \) is a \( D^{-1} \times 1 \times \cdots \times 1 \) slab through the origin perpendicular to the direction of \( \theta \). Now we consider the integral

$$\int (1 - \eta) |\hat{g}|^2 = \int (1 - \eta(\omega)) \left| \sum_{\theta} \hat{g}_\theta(\omega) \right|^2 \, d\omega. \quad (2)$$

If \( 1 - \eta(\omega) \neq 0 \), then \( |\omega| \geq \rho \). In that case, \( \omega \) belongs to \( D^c \theta^* \) for \( \lesssim \rho^{-n} D^{n-2+ne^3} \) different \( \theta \). (For comparison, note that the total number of \( \theta \) is \( D^{n-1} \).) Note that the \( \rho^{-n} \) might not be optimal, but since we take \( \rho^{-1} \) to be very small compared to \( D \), it does not matter as long as it is a polynomial power. The function \( \hat{g}_\theta \) is essentially supported in \( \theta^* \) with a rapidly decaying tail. In other words, outside of \( D^c \theta^* \), we have \( |\hat{g}_\theta(\omega)| \leq C_N D^{-N} \).

Applying Cauchy–Schwarz, we see that for any \( N \),

$$\left| \sum_{\theta} \hat{g}_\theta(\omega) \right|^2 \lesssim \rho^{-n} D^{n-2+ne^3} \sum_{\theta} |\hat{g}_\theta(\omega)|^2 + C_N D^{-N}.$$  

The term \( C_N D^{-N} \) accounts for the rapidly decaying tails of the functions \( \hat{g}_\theta \). This term is negligible, and we ignore it in the sequel. Plugging our bound into (2), we see that

$$\int (1 - \eta) |\hat{g}|^2 \lesssim \rho^{-n} D^{n-2+ne^3} \sum_{\theta} \int |\hat{g}_\theta|^2 = \rho^{-n} D^{n-2+ne^3} \sum_{\theta} \int |g_\theta|^2.$$  

Now for each \( \theta \), the tubes \( T \in \mathbb{T}_\theta \) are essentially distinct. Since they share the same direction, the tubes in \( T \in \mathbb{T}_\theta \) are finitely overlapping, and so

$$\rho^{-n} D^{n-2} \sum_{\theta} \int |g_\theta|^2 \sim \rho^{-n} D^{n-2} \sum_{\psi_T} \int |\psi_T|^2 \sim \rho^{-n} D^{n-1} \mathbb{T}.$$
Combining what we have done so far, we see that in the high-frequency case

$$I(P, T) \lesssim \rho^{-n/2} D^{n-1} |P|^{1/2} |T|^{1/2}.$$ 

On the other hand, we know that

$$I(P, T) \approx E|P|.$$ 

Rearranging, we get

$$|P| \lesssim \rho^{-n} D^{3E} E^{-2} D^{n-1} |T| \lesssim S^n E^{-2} D^{n-1} |T|$$

because $\rho^{-n} D^{3E} \leq S^n$.

**The low frequency case.** If the low frequency case dominates, then we have

$$I(P, T) \lesssim \int \eta \hat{f} \hat{g} = \int f(g \ast \eta^\vee) = \sum_{q \in P} \sum_{T \in \mathbb{T}} \int \psi_q(\psi_T \ast \eta^\vee).$$

Now $\psi_T \ast \eta^\vee$ is rapidly decaying outside of the $\rho^{-1} \times D$ tube around $T$, and $|\psi_T \ast \eta^\vee| \lesssim \rho^{n-1}$. We write $N_S(q)$ for the $S$-neighborhood of $q$, which is essentially a ball of radius $S$. Since $S = D^{3E} \rho^{-1}$, $\psi_T \ast \eta^\vee$ is negligible outside of the $S \times D$ tube around $T$. Therefore,

$$\sum_{T \in \mathbb{T}} \int \psi_q(\psi_T \ast \eta^\vee) \lesssim \rho^{(n-1)} W_S(q) \leq S^{-(n-1)} W_S(q).$$

Recall that $W_S(q) = \#\{T \in \mathbb{T} | T \cap N_S(q) \neq \emptyset\}$. Now $I(P, T) \approx E|P|$, so

$$E|P| \lesssim \sum_{q \in P} S^{-(n-1)} W_S(q).$$

Since the $W_S(q)$ are approximately the same for all $q \in P$,

$$W_S(q) \gtrsim S^{n-1} E.$$ (3)

This is the desired estimate in the thick case.

We explain how the $Q_j$ are defined here. We cover $[0, D]^2$ with $2S$-balls $Q$ centered at $(SZ)^2$. Then each $q$ satisfies $N_S(q) \subset Q$ for some $Q$. Since we have selected a fraction $\geq 1$ of $q \in P$ with the property (3), we choose $\{Q_j\} = \{Q : N_S(q) \subset Q$ for some $q$ with property (3)\}. 

\[\square\]
3 Proof of Theorem 1.2

We start by proving Theorem 1.2 because the argument is slightly less complicated and because the role of the heavy balls is clearest. We recall the statement.

**Theorem.** Let $1 \leq W \leq \delta^{-1}$ and $1 \leq N_1 \leq (W\delta)^{-1}$. Divide the circle into arcs $\theta$ of length $\delta$. For each $\theta$, and each $1 \leq j \leq W$, let $T_{\theta,j} \subset [0,1]$ be a $\delta$-tube. Suppose that for each $\theta$, and each $W^{-1} \times 1$ tube in direction $\theta$, there are $\sim N_1$ tubes $T_{\theta,j}$ in the $W^{-1} \times 1$ tube. Let $\mathbb{T}$ be the set of all the tubes $T_{\theta,j}$. Then for any $\epsilon > 0$:

$$\text{if } r \geq C_1(\epsilon)\delta^{1-\epsilon}|\mathbb{T}|,$$

$$\text{then } |P_r(\mathbb{T})| \leq C_2(\epsilon)\delta^{-\epsilon}W^{-1}r^{-2}|\mathbb{T}|^2,$$

where $C_1(\epsilon)$ and $C_2(\epsilon)$ are two constants only depending on $\epsilon$, in particular, independent of $W$ and $N_1$.

**Proof of Theorem 1.2.** The proof is by induction, and there are two base cases. The first base case is when $r \gtrsim \delta^{-1}$, the number of distinct $\delta$-tubes through a $\delta$-ball is $\delta^{-1}$, so $P_r(\mathbb{T})$ is empty.

The second base case is when $W$ is almost as big as $\delta^{-1}$. If $W = \delta^{-1}$, then $\mathbb{T}$ must consist of essentially all of the $\delta^{-2}$ distinct $\delta$-tubes in $[0,1]^2$. Formally, the second base case is when $W \geq \delta^{-1}+\epsilon/2$. In this case, we see that $r \geq \delta^{1-\epsilon}W \gtrsim \delta^{-1}+\epsilon/2$. So $P_r(\mathbb{T})$ is empty by the first base case.

Now we begin the inductive argument. The induction hypothesis is the following:

assume that Theorem 1.2 holds with $\bar{r} > r$ or $\bar{\delta} > \delta$.

(4)

Let $P \subset P_r(\mathbb{T})$ be the set of $\delta$-balls lying in $\sim r$ tubes of $\mathbb{T}$. If $|P_r(\mathbb{T})| \geq 10|P|$, then by the induction hypothesis (4),

$$|P_r(\mathbb{T})| \leq \frac{10}{9}|P_{2r}(\mathbb{T})| < C_2(\epsilon)\delta^{-\epsilon}W^{-1}r^{-2}|\mathbb{T}|^2.$$

So we assume that $|P_r(\mathbb{T})| \leq 10|P|$.

We apply Proposition 2.1 with $D = \delta^{-1}$ and $E = r$.

3.1 Thin case. If we are in the thin case, then

$$|P| \lesssim \frac{|\mathbb{T}|}{r^2}\delta^{-1} \leq \frac{|\mathbb{T}|^2}{r^2}W^{-1}\delta^{-1}W N_1 \geq \delta^{-1}W.$$

(5)

because $|\mathbb{T}| = \delta^{-1}W N_1 \geq \delta^{-1}W$.

Because we applied Proposition 2.1 with $D = \delta^{-1}$ and dimension $n = 2$, the symbol $\lesssim$ here means $\leq C(\epsilon,2)\delta^{-20\epsilon^2}$. We use $\lesssim$ this way throughout this section of the paper. Taking this into account, the last equation gives

$$|P| \leq C(\epsilon,2)\delta^{-20\epsilon^2}\frac{|\mathbb{T}|^2}{r^2}W^{-1},$$

which gives the desired bound.
3.2 Thick case. Otherwise we are in the thick case, meaning that there is a set of finitely overlapping $2S\delta$-balls $Q_j$, such that

(1) $\cup_j Q_j$ contains a fraction $\gtrsim 1$ of the balls of $P$,
(2) Each $Q_j$ intersects $\gtrsim S\epsilon$ tubes of $T$.

We thicken our $\delta$-tubes to $2S\delta$-tubes. Let $\tilde{P}$ denote the set of $2S\delta$-balls $Q_j$. For a given $N$, define $\tilde{T}_N$ to be the set of essentially distinct $2S\delta \times 1$ tubes containing $\approx N$ tubes of $T$. By pigeonholing, we can choose $N$ such that the tubes of $\tilde{T}_N$ contain a fraction $\gtrsim 1$ of the incidences between $T$ and $\tilde{P}$. We fix $N$, and we define $\tilde{T} = \tilde{T}_N$. We have $|\tilde{T}| \lesssim N^{-1}|T|$. A typical $2S\delta$-ball of $\tilde{P}$ is $\tilde{r} \gtrsim N^{-1}S\epsilon$-rich for $\tilde{T}$. Since each such $2S\delta$-ball contains only $\sim S^2\delta$-balls,

$$|P_r(\tilde{T})| \lesssim S^2|P_r(\tilde{T})|, \text{ where } \tilde{r} \gtrsim N^{-1}S\epsilon. \quad (6)$$

Now we will apply induction on $\delta$ to bound $P_r(\tilde{T})$. We check if the set $\tilde{T}$ obeys the hypothesis of Theorem 1.2. Since $2S\delta < W^{-1}$, $1 \leq W \leq \delta^{-1}$, which checks the first hypothesis. Divide the circle into arcs $\theta$ of length $2S\delta$. For each $\delta\theta$, and each $W^{-1} \times 1$ tube in direction $\tilde{\theta}$, there are at most $\lesssim N^{-1}N_1$ tubes in $\tilde{T}$ in the $W^{-1} \times 1$ tube. If in direction $\theta$, there are fewer than $N^{-1}N_1$ tubes in $\tilde{T}$ in the $W^{-1} \times 1$ tube, we add additional tubes to $\tilde{T}$ to compensate that. Now the set $\tilde{T}$ obeys the hypotheses of Theorem 1.2 with $\tilde{\delta} = 2S\delta$ and $W = W$. Furthermore, $|\tilde{T}| \sim N^{-1}|T|$.

Finally we have to check that $\tilde{r}$ is big enough:

$$\tilde{r} \gtrsim N^{-1}S\epsilon \geq N^{-1}SC_1(\epsilon)\delta^{1-\epsilon}|\tilde{T}| \sim S^\epsilon C_1(\epsilon)(S\delta)^{1-\epsilon}|\tilde{T}|.$$

Recall that $\lesssim$ means $\gtrsim C(\epsilon, 2)\delta^{20k^3}$. Since $S = \delta^{-\epsilon/20}$, when $\delta$ is small enough, $S^\epsilon \delta^{20k^3} > 1$. Now we can inductively apply Theorem 1.2 at scale $S\delta$ to get

$$|P_r(\tilde{T})| \lesssim C_2(\epsilon)(S\delta)^{-\epsilon}W^{-1}(N^{-1}S\epsilon)^{-2}|\tilde{T}|^2 \lesssim C_2(\epsilon)(S\delta)^{-\epsilon}S^{-2}W^{-1}\epsilon^{-2}|\tilde{T}|^2.$$

Plugging this into equation (6), we get

$$|P_r(\tilde{T})| \lesssim C_2(\epsilon)(S\delta)^{-\epsilon}W^{-1}|\tilde{T}|^2.$$

We claim that this gives the desired bound for $|P_r(\tilde{T})|$ and closes the induction. To check this, we have to see that $S$ is big enough such that $S^{-\epsilon}$ dominates the implicit factor in the $\lesssim$. This indeed happens, because $S = \delta^{-\epsilon/20} \geq \delta^{-20k^2}$. \hfill \Box

4 Proof of Theorems 1.1 and 1.3

In this section, we prove Theorems 1.1 and 1.3. It uses the ideas from the last proof, but there are also some new ideas needed especially for the 3-dimensional result, Theorem 1.3. For one thing, we bring into play the idea of partitioning from [CEGSW90].

The following theorem combines Theorems 1.1 and 1.3.
Theorem 4.1. Let $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a collection of distinct $\delta$-tubes in $B^n(0, 2)$, for $n = 2, 3$. If $\mathbb{T}$ is a set of $\sim W^{2(n-1)}$ tubes with at most one tube of $\mathbb{T}$ in each $1/W$-tube, then for $r > \max(\delta^{-n-1}/4|\mathbb{T}|, 1)$ the number of $r$-rich $\delta$-balls is bounded by

$$|P_r(\mathbb{T})| \lesssim \delta^{-\epsilon} |\mathbb{T}| \frac{n-1}{r^{n-1}}. \quad (7)$$

Proof. We will prove the theorem by induction. There are two base cases for our induction.

The first base case is when $W = O(1)$ and $r > 1$. Since the tubes through one point are $1/W \sim 1$-separated, we have $|P_r(\mathbb{T})| \lesssim |\mathbb{T}|^2 \lesssim 1$.

The second base case is when $W \sim \delta^{-1}$. In this case, $\mathbb{T}$ is essentially the set of all distinct $\delta$-tubes in $B^n$. More precisely, the second base case is when $W > \delta^{-1+\epsilon/10n}$. In this case $r \gtrsim \delta^{-\epsilon/4}W^{n-1} > \delta^{-(n-1)-\epsilon/10}$. But a $\delta$-ball can essentially lie in at most $\lesssim \delta^{-(n-1)}$ distinct $\delta$-tubes, and so $|P_r(\mathbb{T})| = 0$.

In the inductive argument, we distinguish between the case when $r$ is small and the case when $r$ is large. The main new ingredient in this proof is a way to handle the small $r$ case.

Our induction hypothesis is

$$\text{if } \tilde{\delta} > \delta \text{ or } \tilde{r} > r, \text{ then Theorem 4.1 holds.} \quad (8)$$

4.1 The narrow case when $r < \delta^{-\epsilon^3}$. After dyadic pigeonholing, we can assume that the maximum angle between tubes passing through a typical ball of $P_r(\mathbb{T})$ is $\alpha \gtrsim 1/W$. If two tubes $T_1, T_2 \in \mathbb{T}$ intersect, then the angle between $T_1$ and $T_2$ is larger than $\gtrsim 1/W$. Otherwise, there is a $1/W$-tube containing both $T_1$ and $T_2$, which violates our assumption.

If $\alpha$ is much smaller than 1, then we cover the sphere $S^{n-1}$ by caps $\tau$ of radius $\alpha$. For each $\tau$, we divide the unit ball into cells $\square_{\tau}$, where each $\square_{\tau}$ is a thick tube of length 1 and radius $\alpha$, pointing in the direction defined by $\tau$. The number of $\square_{\tau}$ is $\alpha^{2(n-1)}$. Let $\mathbb{T}_{\square}$ denote the collection of tubes contained $\square_{\tau}$. Then $|\mathbb{T}_{\square}| \sim (\alpha W)^{2(n-1)}$ because $\square_{\tau}$ contains about $(\alpha W)^{2(n-1)}$ essentially distinct $1/W$-tubes, and each $1/W$-tube contains one $T \in \mathbb{T}$.

We rescale $\square_{\tau}$ to the unit ball. The set of $\delta$-tubes $\mathbb{T}_{\square}$ becomes a set of $\tilde{\delta}$-tubes $\tilde{\mathbb{T}}$. These tubes $\tilde{\mathbb{T}}$ obey the hypotheses of Theorem 4.1 with $\tilde{W} = \alpha W$ and $\tilde{\delta} = \delta/\alpha$. The maximal angle between tubes in a typical $2$-rich $\tilde{\delta}$-ball of $\tilde{\mathbb{T}}$ is $\gtrsim 1$.

By our induction hypothesis (8),

$$P_2(\tilde{\mathbb{T}}) \lesssim \tilde{\delta}^{-\epsilon} |\tilde{\mathbb{T}}| \frac{n}{\tilde{\delta}^{n-1}}. \quad (9)$$

We rescale back $\tilde{\mathbb{T}}$ to be $\mathbb{T}_{\square}$, the 2-rich $\tilde{\delta}$-balls of $\tilde{\mathbb{T}}$ become a tube segment of radius $\delta$ and length $\alpha^{-1}\delta$, which corresponds to $\alpha^{-1}$ $\delta$-balls in $P_2(\mathbb{T}_{\square})$. Therefore,

$$|P_2(\mathbb{T})| \lesssim \alpha^{-2(n-1)} \alpha^{-1} |P_2(\tilde{\mathbb{T}})| \lesssim \tilde{\delta}^{-\epsilon} \alpha^{-2n+1} |\tilde{\mathbb{T}}| \frac{n}{\tilde{\delta}^{n-1}} \lesssim \delta^{-\epsilon} \alpha |\mathbb{T}| \frac{n}{\delta^{n-1}}.$$

Since $r < \delta^{-\epsilon^3}$, the induction closes if $\alpha < \delta^{10\epsilon^3}$. 
4.2 The broad case when $r < \delta^{-e^3}$. Next suppose that $\alpha \geq \delta^{-10e^3}$. Now we have two subcases. If $W > \delta^{-1/2+\epsilon/10n}$, then the trivial bound $|P_r(\mathbb{T})| \leq \delta^{-n}$ suffices. Indeed the desired upper bound for $|P_r(\mathbb{T})|$ is

$$\delta^{-\epsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}} \geq \delta^{-\epsilon} \delta^{-10e^3} W^{2n} \geq \delta^{-n-\epsilon/2}. $$

So now suppose that $W \leq \delta^{-1/2+\epsilon/10n}$, which implies that $\delta < W^{-2}$, and indeed that

$$\delta \leq \delta \frac{\pi}{W} W^{-2}. $$

Let $\rho = W^{-2}$, and let $\mathbb{T}$ be the set of essentially distinct tubes formed by thickening each $\delta$-tube of $\mathbb{T}$ to a $\rho$-tube. (Since each $1/W$-tube contains one tube of $\mathbb{T}$, each $\rho$-tube of $\mathbb{T}$ also contains $\sim 1$ tubes of $\mathbb{T}$ and $|\mathbb{T}| \sim |\mathbb{T}|$.) Cover $B^n(0, 2)$ with $\rho$-balls, and let $Q_{X,M}$ be the set of $\rho$-balls containing $\sim X$ balls of $P_r(\mathbb{T})$ and intersecting $\sim M$ tubes of $\mathbb{T}$. We can choose $X, M$ such that $\cup_{Q \in Q_{X,M}} Q$ contains a $\geq \log |\mathbb{T}|^{-2}$ fraction of $P_r(\mathbb{T})$. Because each $\delta$-ball in $P_r(\mathbb{T})$ is intersected by two tubes with angle about $\alpha \geq \delta^{-10e^3}$, the number of $\delta$-balls in $Q$ is at most $\alpha^{-1} M^2$. Therefore,

$$X \lesssim \delta^{-10e^3} M^2. $$

The number of $M$-rich $\rho$-balls is, by induction on scale $\rho > \delta$,

$$P_M(\mathbb{T}) \lesssim \rho^{-\epsilon} \frac{|\mathbb{T}|^{\frac{n}{n-1}}}{M^{\frac{n+1}{n-1}}}. $$

Therefore,

$$|P_r(\mathbb{T})| \lesssim |\log \delta|^{2} \cdot \delta^{-O(e^3)} M^2 |P_M(\mathbb{T})| \lesssim \rho^{-\epsilon} M^2 \frac{\rho^{-\epsilon} \delta^{n+1}}{\rho^{-\epsilon} \delta^{n-1}} |\mathbb{T}|^{\frac{n}{n-1}}. $$

If $n = 2$ or $3$, then $\frac{n+1}{n-1} \geq 2$, and the power of $M$ is $\leq 0$. To close the induction, we need to check that $\delta^{-O(e^3)} \rho^{-\epsilon} \geq \delta^{-\epsilon} r^3$. Since $r = \delta^{-O(e^3)}$ it suffices to check that $\rho^{-\epsilon} \leq \delta^{-\epsilon} O(e^3)$. But $\rho / \delta \geq \delta^{-\epsilon/5n}$, and so this is true. This finishes the induction in the small $r$ case.

4.3 The thin case when $r \geq \delta^{-e^3}$. Now we turn to the induction in the large $r$ case.

Let $1 \leq D \leq W$ be a parameter. In this proof, we will eventually choose $D$ to be a small power of $\delta$. In the rest of the proof, $A \lesssim B$ means that $A \lesssim \delta^{-\epsilon} B$. We cover the unit square with finitely overlapping $D\delta$-balls $Q$. We let $P$ be the set of $r$ rich balls in $P_r(\mathbb{T})$, and we can assume by induction on $r$ that $|P| \sim |P_r(\mathbb{T})|$. A tube $T$ intersects $Q$ in a tube segment $T_D$ of radius $\delta$, length $D\delta$. One $T_D$ could lie in many tubes $T \in \mathbb{T}$. Let $\mathbb{T}_{Q,M}$ be the set of distinct $T_D$ in $Q$ which essentially lie in $\sim M$ tubes of $\mathbb{T}$. Here $1 \leq M \lesssim W$, otherwise there is a $1/W$-tube containing two
tubes in $\mathbb{T}$ intersecting $T_D$. We choose $M$ to preserve most of the incidences. More precisely, we choose $M$ so that

$$\sum_Q MI(P \cap Q, \mathbb{T}_{Q,M}) \gtrsim I(P, \mathbb{T}). \quad (10)$$

We can do so because $|P| \lesssim \sum_Q |P \cap Q|$ and

$$I(P, \mathbb{T}) \sim r|P| \lesssim \sum_Q \sum_{M \text{ dyadic}} MI(P \cap Q, \mathbb{T}_{Q,M}).$$

Once we fix $M$, we abbreviate $\mathbb{T}_{Q,M}$ to $\mathbb{T}_Q$. Let $P_{Q,E}$ be the set of $\delta$-balls of $P \cap Q$ that lie in $\sim E$ tube segments of $\mathbb{T}_Q$. Since $\sum_{E \text{ dyadic}} I(P_{Q,E}, \mathbb{T}_Q) \geq I(P \cap Q, T_Q)$, we choose $E$ such that

$$\sum_Q MI(P_{Q,E}, \mathbb{T}_Q) \gtrsim I(P, \mathbb{T}). \quad (11)$$

Once we fix $E$, we abbreviate $P_Q = P_{Q,E}$. Because each $q \in P$ lies in $\sim r$ tubes of $\mathbb{T}$, (11) implies that

$$|P| \lesssim \sum_Q |P_Q|.$$

Also, the left-hand side of (11) is about $ME \sum_Q |P_Q| \lesssim ME|P|$, and the right-hand side if $\sim r|P|$, so

$$ME \gtrsim r.$$

Since the directions of $T_D$ intersecting at a common point are $1/D$-separated, we have $E \leq D^{n-1}$.

Next we apply Proposition 2.1 to bound each $|P_Q|$. We set the scale $S$ to be $D^{\epsilon/10n}$ and rescale $\mathbb{T}_Q$ to be tube segments of length $D$, radius 1. We apply Proposition 2.1 to the rescaled set $\mathbb{T}_Q$. For each $Q$, we will be in either the thin case or the thick case depending on the result of Proposition 2.1.

In the thin case, we have

$$|P_r(\mathbb{T})| \lesssim \sum_{Q \text{ thin}} |P_Q| \lesssim \sum_Q S^n E^{-2} D^{n-1} |T_Q| \sim E^{-2} D^{n-1+\epsilon/10} \sum_Q |T_Q|.$$
To estimate $\sum_{Q} |T_Q|$, we cover the sphere $\mathbb{S}^{n-1}$ by caps $\tau$ of radius $1/D$. For each $\tau$, we divide the unit ball into cells $\Box_{\tau}$, where each $\Box_{\tau}$ is a thick tube of length 1 and radius $1/D$, pointing in the direction defined by $\tau$. The number of cells $\Box_{\tau}$ is $D^{2(n-1)}$. A tube $S$ in $T_Q$ lies in $M$ different tubes of $T$, and they must all lie in the same cell $\Box_{\tau}$.

Let $T_{\Box_{\tau}}$ be the set of $T \in T$ contained in $\Box_{\tau}$. Rescale $\Box_{\tau}$ to the unit ball, and let $\tilde{T}_{\Box_{\tau}}$ be the resulting set of tubes. For each $\Box_{\tau}$, $\tilde{T}_{\Box_{\tau}}$ obeys the hypotheses of Theorem 4.1 with

1. $\tilde{\delta} = D\delta$,
2. $\tilde{\tau} = M \approx E^{-1}r$,
3. $\tilde{W} = W/D$.

More precisely, $\tilde{T}_{\Box_{\tau}}$ is a collection of distinct $\tilde{\delta}$-tubes, and each $1/\tilde{W}$-tube contains one $\tilde{\delta}$-tube in $\tilde{T}_{\Box_{\tau}}$. (since they are the $1/W$-tube and $\delta$-tube in $T$ before rescaling).

We have

$$\sum_{Q} |T_Q| \lesssim D^{2(n-1)} \max_{\Box_{\tau}} |P_M(\tilde{T}_{\Box_{\tau}})|.$$

We fix a $\Box_{\tau}$ and abbreviate $\tilde{T} = \tilde{T}_{\Box_{\tau}}$. We will apply induction to bound $|P_M(\tilde{T})|$. Before we can apply the induction hypothesis, we have to verify that $\tilde{\tau} = M$ is sufficiently large: $M > \tilde{\delta}^{n-1-\epsilon/4} |\tilde{T}|$ and $M \geq 2$. We check the first bound on $M$ by calculation:

$$M \gtrsim E^{-1}r > E^{-1}\delta^{n-1-\epsilon/4} |T| \sim E^{-1}D^{n-1+\epsilon/4}\tilde{\delta}^{n-1-\epsilon/4} |\tilde{T}| > \tilde{\delta}^{n-1-\epsilon/4} |\tilde{T}|.$$

(The last inequality is because $E \leq D^{n-1}$.) To check $M \geq 2$, we recall that $r$ is big, and we choose $D$ small. Recall that we are in the case $r > \delta^{-\epsilon^2}$. We set $D \sim \delta^{-\epsilon^2}$, and then $M \approx E^{-1}r \geq D^{-(n-1)}r$ and so $M \geq 2$. We have to deal with the small $r$ case separately because of this step of the argument.

We have now confirmed that $M$ is sufficiently large, and we can apply induction, giving:

$$\sum_{Q} |T_Q| \lesssim (D\delta)^{-\epsilon} D^{2(n-1)} M^{-\frac{n+1}{n-1}} (D^{-2(n-1)} |T|)^{\frac{n}{n-1}} \sim (D\delta)^{-\epsilon} D^{-2} M^{-\frac{n+1}{n-1}} |T|^{\frac{n}{n-1}}.$$

Hence,

$$|P_r(T)| \lesssim \delta^{-\epsilon} D^{n-1-2-\epsilon/2} E^{-2} M^{-\frac{n+1}{n-1}} |T|^{\frac{n}{n-1}}.$$

We now check that this closes the induction. If $n = 3$, the right-hand side is

$$D^{-\epsilon/2} \delta^{-\epsilon} (ME)^{-2} |T|^{3/2} \lesssim D^{-\epsilon/2} \delta^{-\epsilon} r^{-2} |T|^{3/2}.$$
If \( n = 2 \), the right-hand side is

\[
D^{-\epsilon/2} \delta^{-\epsilon} D^{-1} E^{-2} M^{-3} |\mathbb{T}|^2 \leq D^{-\epsilon/2} \delta^{-\epsilon} E^{-3} M^{-3} |\mathbb{T}|^2 \approx D^{-\epsilon/2} \delta^{-\epsilon} r^{-3} |\mathbb{T}|^2.
\]

4.4 The thick case when \( r \geq \delta^{-\epsilon^3} \). In the thick case, we have a set of \( \delta \) balls \( \tilde{P} \) such that \( \tilde{P} \) covers a fraction \( \gtrapprox 1 \) of \( P \), and each ball of \( \tilde{P} \) intersects \( \gtrapprox S^{n-1} r \) tubes of \( \mathbb{T} \). Let \( \tilde{\mathbb{T}} \) be the set of tubes formed by thickening each \( \delta \) tube of \( \mathbb{T} \) to a \( \rho \)-tube. We see that \( \tilde{\mathbb{T}} \) ⊆ \( S^{n-1} \tilde{r} \). The tubes \( \tilde{\mathbb{T}} \) obey the hypotheses of our theorem with \( \tilde{\delta} = S \delta \), \( \tilde{W} = W \), and \( |\tilde{\mathbb{T}}| = |\mathbb{T}| \). (We just have to check that \( W \leq S \delta^{-1} \). But by the second base case, we know that \( W \leq \delta^{-1+\epsilon/10n} \) and we chose \( S \leq D = \delta^{-\epsilon^4} \), so this holds.)

Therefore,

\[
|P_r(\mathbb{T})| \lesssim S^n |\tilde{P}| \leq S^n |P_r(\tilde{\mathbb{T}})| \lesssim (S\delta)^{-\epsilon} S^n (S^{n-1} r)^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}} \leq S^{-1} \delta^{-\epsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}}.
\]

This closes the induction in the thick case and finishes the proof. \( \square \)

Remark The statement of Theorem 4.1 makes sense for all dimensions \( n \). We do not know any counterexamples, and it seems plausible to us that it is true for all \( n \). In our proof, we used \( n \leq 3 \) in the calculation in several places.

There is an example in the appendix of [GK15] showing that Theorem 4.1 is sharp and leads to the conjectured statement for all dimensions \( n \). We modify the example slightly to accommodate the \( \delta \)-tube version. Let \( G \) be the grid \( (\mathbb{Z}/W)^n \cap [0,1]^{n-1} \). If \( a, b \in G \), we take the lines in \( \mathcal{L} \) to be the lines from \((a,0)\) to \((b,1)\). And we take the tubes in \( \mathbb{T} \) to be the \( \delta \)-neighborhood of line segments from \((a,0)\) to \((b,1)\). For any pair of tubes \( T_1, T_2 \in \mathbb{T} \), either they have distance \( 1/W \) or their angle is \( 1/W \)-separated. So we have verified the assumption of Theorem 4.1 for the tubes in \( \mathbb{T} \).

The calculations in the appendix of [GK15] shows that

\[
P_r(\mathcal{L}) \gtrsim \frac{|\mathcal{L}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}.
\]

An \( r \)-rich point \( x \) is of the form

\[
x = \left( \frac{q-p}{q} a + \frac{p}{q} b, \frac{p}{q} \right),
\]

where \( q/10 \leq p < q \) are co-prime positive integers with \( q \sim W r^{-\frac{1}{n-1}} \) and the first \( n-1 \) coordinates have value in \([1/4, 3/4]\).

Given \( x = (\frac{q-p}{q} a + \frac{p}{q} b, \frac{p}{q}) \) and \( x' = (\frac{q-p'}{q} a' + \frac{p'}{q} b', \frac{p'}{q}) \) in the form of equation (12), if \( x \neq x' \) then \( |x-x'| \gtrsim \min(W^{-1}, W^{-2} r^{\frac{1}{n-1}}) \). The reason is the following: if \( x_n \neq x'_n \), then \( |x_n - x'_n| \geq |x_n - x'_n| \geq W^{-1} r^{\frac{1}{n-1}} \); otherwise \( x_n = x'_n \), we have \( |x-x'| = |(1-\frac{p}{q})(a-a') + \frac{p}{q} (b-b')| \geq W^{-2} r^{\frac{1}{n-1}} \) because \( x-x' \in (W^{-1} q^{-1} \mathbb{Z})^n \). Since \( r > W^{-2(n-1)} \delta^{1-\epsilon} \) and \( 1/W > \delta \), the distance between two distinct points \( |x-x'| \gtrsim \min(W^{-1}, W^{-2} r^{\frac{1}{n-1}}) > \delta \).
Now we have showed that the points in $P_r(L)$ are $\delta$-separated. So we can thicken the points in $P_r(L)$ and they become disjoint $r$-rich $\delta$-balls in $P_r(T)$.

5 An Application to the Falconer Problem

In this section, we consider a distinct distances type problem for $\delta$-balls in $\mathbb{R}^2$, which is related to the Falconer distance problem in $\mathbb{R}^2$. As we mentioned in the introduction, ORPONEN [Orp17a] and KELETTI–SHMERKIN [Shm17, Shm19, KS18] essentially solved the Falconer distance problem for sets that are close to Ahlfors–David regular. Here we consider the opposite type of set: Ahlfors–David regular sets of a given dimension are packed as tightly as possible, and we consider here sets that are as spread out as possible.

If $E$ is a set in the plane, recall that $\Delta(E)$ is the distance set

$$\Delta(E) = \{|x - y| : x, y \in E\},$$

where $|x - y|$ denote the Euclidean distance between two points $x$ and $y$.

**Theorem 5.1.** Fix $1 < s < 2$. Suppose that $E$ is a set of $\delta^{-s}$ $\delta$-balls in $[0, 1]^2$, with $\leq 1$ $\delta$-balls in each ball of radius $\delta^{s/2}$. Then the number of disjoint $\delta$-intervals contained in $\Delta(E)$ is $\gtrsim \delta^{-1+\epsilon}$ for all $\epsilon > 0$.

We can choose two balls $B_1$ and $B_2$ of radius $1/1000$ and with centers about $1/3$ part such that each $E_j = E \cap B_j$ contains about $1/10^6$ of $E$. It suffices to show that $\Delta(E_1, E_2) = \{|x - y|, x \in E_1, y \in E_2\}$ contains $\gtrsim \delta^{-1+\epsilon}$ many $\delta$-intervals.

We recall the ELEKES–SHARIR framework [ES10], which was used in the Erdős distinct distance problem [GK15].

If $|x_1 - y_1| = |x_2 - y_2|$ for points $x_1, x_2 \in E_1$ and $y_1, y_2 \in E_2$, then there exists a unique (orientation-preserving) rigid motion $g$ on the plane sending $x_1$ to $y_2$ and $y_1$ to $x_2$. A rigid motion $g = (c, \theta)$ is uniquely determined by the center $c \in \mathbb{R}^2$ and the rotation angle $\theta$. We could represent $g$ by a point $\rho(g) = (c, \cot \frac{\theta}{2})$ in $\mathbb{R}^3$. Let $g_{xy}$ denote the collection of rigid motions sending a point $x$ to $y$. Then $\rho(g_{xy})$ a line in $\mathbb{R}^3$:

$$l_{xy} = \rho(g_{xy}) = \left\{ \left( \frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, 0 \right) + t \left( -\frac{y_2 - x_2}{2}, \frac{y_1 - x_1}{2}, 1 \right), t \in \mathbb{R} \right\}. \quad (13)$$

In particular, the centers of those $g$ lie on the perpendicular bisector of $x$ and $y$. We can also read the coordinates of $x$ and $y$ from the parameterized equation of $l_{xy}$. If a line $l$ is parametrized by

$$l = \{(c_1, c_2, 0) + t(k_1, k_2, 1), t \in \mathbb{R}\},$$

then there exists $x = (x_1, x_2)$ and $y = (y_1, y_2)$ such that $l_{xy} = l$. To find $x, y$, it suffices to solve the linear equations system:

$$x_1 + y_1 = 2c_1, \quad x_2 + y_2 = 2c_2, \quad y_2 - x_2 = -2k_1, \quad y_1 - x_1 = 2k_2.$$
Lemma 5.2. Assume that and \( \rho \) in \( B_\delta \) the set of such rigid motions to a \( \delta \)-thickening variation, we need some few notation. Let \( p \) be a \( \delta \)-ball in \( E_1 \) and \( q \) be a \( \delta \)-ball in \( E_2 \). We say that a rigid motion \( g \) sends \( p \) to \( q \) if \( g(p) \cap q \neq \emptyset \).

Proof. Let \( x \) be the center of \( p \) and \( y \) be the center of \( q \), then \( \rho(p_{pq,B}) \) lies inside and contains

\[
\left\{ \left( \frac{x_1 + y_1}{2} + O(\delta), \frac{x_2 + y_2}{2} + O(\delta), 0 \right) + t \left( -\frac{y_2 - x_2}{2} + O(\delta), \frac{y_1 - x_1}{2} + O(\delta), 1 \right), t \in \mathbb{R} \right\},
\]

where \( O(\delta) \) means a constant \( \sim \delta \).

By equation (13), the angle between \( l_{xy} \) and the \( \{ z = 0 \} \)-plane is \( \arctan \left( \frac{|x - y|}{2} \right) \).

Since \( \text{dist}(p, q) \sim 1 \), for any \( x' \in p \) and \( y' \in q \), \( \frac{|x' - y'|}{2} \sim 1 \) and

\[
\left| \arctan \left( \frac{|x - y|}{2} \right) - \arctan \left( \frac{|x' - y'|}{2} \right) \right| \lesssim \delta.
\]

So \( \rho(p_{pq,B}) \) is approximately a \( \delta \)-tube \( T_{pq,B} \) of length about 1. \( \square \)

If \( p_1, p_2 \) are two \( \delta \)-balls in \( E_1 \) and \( q_1, q_2 \) are two \( \delta \)-balls in \( E_2 \) such that

\[
|\text{dist}(p_1, q_1) - \text{dist}(p_2, q_2)| < \delta,
\]

then there exists a rigid motion \( g \) sending \( p_1 \) to \( q_2 \) and \( q_1 \) to \( p_2 \). Moreover, \( \rho \) maps the set of such rigid motions to a \( \delta \)-ball in \( \mathbb{R}^3 \).

Lemma 5.3. Suppose that \( p_j, q_j \) are disjoint \( \delta \)-balls satisfying: \( \text{dist}(p_1, p_2) \leq 1/1000 \), \( \text{dist}(q_1, q_2) \leq 1/1000 \), \( \text{dist}(p_1, q_1) \geq 1/3 \) and

\[
|\text{dist}(p_1, q_1) - \text{dist}(p_2, q_2)| < \delta.
\]

Then \( \rho(p_{q_1q_2}) \) is roughly a ball of radius \( \delta \). More precisely, we can find a ball \( B_{\delta/10} \) and a ball \( B_{10\delta} \) such that \( B_{\delta/10} \subset \rho(p_{q_1q_2} \cap q_{1q_2}) \subset B_{10\delta} \).

Proof. Let \( B_1 \) be a ball of radius 1/1000 containing \( p_1 \) and \( p_2 \), and \( B_2 \) be another ball of radius 1/1000 containing \( q_1 \) and \( q_2 \). Then \( \text{dist}(B_1, B_2) \geq 1/4 \).

If a rigid motion \( g \) sends \( p_1 \) to \( q_2 \) and \( q_1 \) to \( p_2 \), then \( g \) is roughly a reflection between \( B_1 \) and \( B_2 \): \( g(B_1) \cap B_2 \neq \emptyset \) and \( g(B_2) \cap B_1 \neq \emptyset \). So the center of \( g \) must lie in a ball \( B_3 \) of radius 1/5 with center in the midpoint of \( p_1 \) and \( q_1 \).

Let \( B = 20B_3 \). By Lemma 5.2, \( \rho \) maps the collection of rigid motions sending \( p_1 \) to \( q_2 \) with centers in \( B \) to approximately a tube \( T_{p_1, q_2, B} \) of radius \( \delta \).
Now we would like to understand how \( T_{p_1q_2,B} \) and \( T_{q_1p_2,B} \) intersect. If \( x, y \) are centers of \( p_1 \) and \( q_2 \), and \( x', y' \) are centers of \( p_2 \) and \( q_1 \), then \( l_{xy} \) and \( l_{y'x'} \) form an angle \( \geq 1 \) because

\[
\left| \left( \frac{-y_2 - x_2}{2}, \frac{y_1 - x_1}{2}, 1 \right) \times \left( \frac{-x'_2 - y_2}{2}, \frac{x'_1 - y'_1}{2}, 1 \right) \right| \\
\geq \left| \left( \frac{-y_2 - x_2}{2}, \frac{y_1 - x_1}{2}, 1 \right) \times \left( \frac{-x_2 - y_2}{2}, \frac{x_1 - y_1}{2}, 1 \right) \right| \\
- \left| \left( \frac{-y_2 - x_2}{2}, \frac{y_1 - x_1}{2}, 1 \right) \times \left( \frac{-x'_2 - x_2 + y_2 - y'_2}{2}, \frac{x'_1 - x_1 + y_1 - y'_1}{2}, 0 \right) \right| \\
\geq \frac{1}{36} - \frac{3}{1000} \geq 1. \quad \square
\]

To prove Theorem 5.1, it suffices to bound the number of distance quadruples.

**Proposition 5.4.** Let \( E \) be as in Theorem 5.1. Let \( E_1 \) and \( E_2 \) be two subsets of \( E \) satisfying: \( E_1 \) and \( E_2 \) are in two balls of radius \( 1/1000 \), \( |E_1|, |E_2| \geq |E| \), and \( \text{dist}(E_1, E_2) \geq 1/4 \). Set \( W = \delta^{-s/2} \), such that \( E \) contains \( \leq 1 \) \( \delta \)-ball in each \( 1/W \)-ball in \( [0,1]^2 \). Let \( Q \) denote the collection of distance quadruples \((p_1, p_2, q_1, q_2)\) such that

\[
|\text{dist}(p_1, q_1) - \text{dist}(p_2, q_2)| < \delta,
\]

and \( p_1, p_2 \in E_1, q_1, q_2 \in E_2 \). Then

\[
|Q| \leq W^8 \delta^{1-s}.
\]

Proposition 5.4 implies Theorem 5.1 because by Cauchy–Schwartz. Here are the details. Let \( \#\Delta(E_1, E_2) \) denote the number of disjoint \( \delta \)-intervals in \( \Delta(E_1, E_2) \) and \( |E_j| \) denote the cardinality of \( E_j \). Since \( \Delta(E_1, E_2) \subset \Delta(E) \), the number of disjoint \( \delta \)-intervals contained in \( \Delta(E) \) is

\[
\geq \#\Delta(E_1, E_2)^{1/2} \geq \frac{|E_1||E_2|}{\#Q^{1/2}}.
\]

Now we turn to the proof of Proposition 5.4.

**Proof.** Let \( B \) be the ball of radius 1 containing \( E_1 \) and \( E_2 \). A distance quadruple \((p_1, p_2, q_1, q_2)\) \( \in Q \) corresponds to the event that \( T_{p_1q_2,B} \) and \( T_{q_1p_2,B} \) intersects on a \( \delta \)-ball with an angle \( \geq 1 \). Let \( T \) denote the collection of \( T_{pq,B} \) and \( T_{qp,B} \) for all \( p \in E_1 \) and \( q \in E_2 \).

Since each \( \delta \)-ball in \( P_r(T) \) corresponds to at most \( r^2 \) distance quadruples

\[
\#Q \leq \sum_{r \text{ dyadic}} r^2 |P_r(T)|
\]
and $W^2 = \delta^{-2s}$, it suffices to show that

$$|P_r(T)| \leq \delta^{1-s-\epsilon} \frac{|T|^{3/2}}{r^2}. \quad (14)$$

When $r \lesssim \delta^2 - \epsilon/4 |T|$, the estimate is true because $|P_r(T)| \lesssim \delta^{-3}$, which is the maximum number of $\delta$-balls in $[0, 1]^3$. When $r \gtrsim \delta^2 - \epsilon/4 |T|$, we want to apply Theorem 4.1. Once we check that $T$ obeys the spacing hypotheses in Theorem 4.1, the theorem will give the bound (14).

To finish the proof, we check that tubes in $\mathbb{T}$ have the good spacing property. We decompose the sphere $S^2$ into union of caps $\tau$ of radius $1/W$. For each $\tau$, we can cover $[0, 1]^3$ by finitely overlapping tubes of radius $1/W$ pointing on the direction in $\tau$. Each $\delta$-tube in $\mathbb{T}$ corresponds to a unique pair of $\delta$-balls $(p, q)$. This correspondence holds for essentially the same reason as the (almost) one-to-one correspondence between the lines $l_{xy}$ and the pairs $(x, y)$.

Each $1/W$-tube in $[0, 1]^3$ corresponds to a unique pair of $1/W$-balls $(Q_1, Q_2)$, $Q_i \subset [0, 1]^2$. Now the $\delta$-tube of $\mathbb{T}$ corresponding to $(p, q)$ lies inside the $1/W$ tube corresponding to $(Q_1, Q_2)$ if and only if $p \in Q_1$ and $q \in Q_2$. Since $E$ contains $\lesssim 1$ $\delta$-balls in any $1/W$-ball, each $1/W$-tube contains $\lesssim 1$ tubes of $\mathbb{T}$. Moreover, $|T| \sim |E|^2 \sim \delta^{-2s} \sim W^4$. So $\mathbb{T}$ verifies the spacing hypotheses of Theorem 4.1. \qed

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Incidence Estimates for Well Spaced Tubes

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