MULTI-LINEAR MULTIPLIERS ASSOCIATED TO SIMPLEXES OF ARBITRARY LENGTH

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ABSTRACT. In this article we prove that the \( n \)-linear operator whose symbol is the characteristic function of the simplex \( \Delta_n = \xi_1 < \ldots < \xi_n \) is bounded from \( L^2 \times \ldots \times L^2 \) into \( L^{2/n} \), generalizing in this way our previous work on the “bi-est” operator [12], [13] (which corresponds to the case \( n = 3 \)) as well as the Lacey - Thiele theorem on the bi-linear Hilbert transform [7], [8] (which corresponds to the case \( n = 2 \)).

1. Introduction

The present paper is a natural continuation of our previous work in [12] and [13]. In those articles we studied the \( L^p \) boundedness properties of a tri-linear operator \( T_3 \) defined by the formula

\[
T_3(f_1, f_2, f_3)(x) = \int_{\xi_1 < \xi_2 < \xi_3} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x (\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3 \tag{1}
\]

for \( f_1, f_2, f_3 \) Schwartz functions on the real line. A particular case of our main theorem there is

\begin{itemize}
  \item Theorem 1.1. \( T_3 \) extends to a bounded tri-linear operator from \( L^2 \times L^2 \times L^2 \) into \( L^{2/3} \).
\end{itemize}

Related to \( T_3 \) is the well known bi-linear Hilbert transform \( T_2 \) essentially defined by

\[
T_2(f_1, f_2)(x) = \int_{\xi_1 < \xi_2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x (\xi_1 + \xi_2)} d\xi_1 d\xi_2 \tag{2}
\]

From the work of Lacey and Thiele [7], [8] we know in particular the following result

\begin{itemize}
  \item Theorem 1.2. \( T_2 \) extends to a bounded bi-linear operator from \( L^2 \times L^2 \) into \( L^1 \).
\end{itemize}

The main task of the current paper is to generalize these theorems and prove similar estimates for multi-linear multipliers whose symbols are given by characteristic functions of simplexes of arbitrary length. More precisely, for any \( n \geq 2 \) denote by \( T_n \) the \( n \)-linear operator defined by

\[
T_n(f_1, \ldots, f_n)(x) = \int_{\xi_1 < \ldots < \xi_n} \hat{f}_1(\xi_1) \ldots \hat{f}_n(\xi_n) e^{2\pi i x (\xi_1 + \ldots + \xi_n)} d\xi_1 \ldots d\xi_n \tag{3}
\]

The reader familiar with our earlier work in [11] should notice that our main result in that paper could not handle the case of \( T_3 \) since the singularity of the symbol \( \chi_{\xi_1 < \xi_2 < \xi_3} \) being two dimensional, is too big; roughly speaking, Theorem 1.1. in [11] allows one to prove estimates for \( n \)-linear multipliers as long as the dimension \( k \) of the singularity of the symbol satisfies the inequality \( k < \frac{n+1}{2} \).
where as before $f_1, \ldots, f_n$ are Schwartz functions on $\mathbb{R}$. Our main result is the following \footnote{Clearly, the correct estimates one is looking for are those of H"older type, since when one ignores the symbol $\chi_{\xi_1 \ldots \xi_n}$, the corresponding formula in (3) becomes the product of the $n$ functions.}

**Theorem 1.3.** $T_n$ extends to a bounded $n$-linear operator from $L^2 \times \ldots \times L^2$ into $L^{2/n}$.

The initial motivation to consider and study such multi-linear operators came from the work of Christ and Kiselev on eigenfunctions of Schrödinger operators [3], [4]. Since their theory is quite relevant to our discussion here, we should pause and recall several aspects of it.

For every positive real number $\lambda$ consider the following eigenfunction differential equation

$$- \Delta u(x) + V(x) u(x) = \lambda u(x) \quad (4)$$

for $x$ on the real line, where $V$ is a real valued potential function. Clearly, when $V$ is identically equal to zero every solution of the equation (4) can be written as a linear combination of the fundamental solutions $u_j^\pm(x) = e^{\pm i \sqrt{\lambda} x}$ and as a consequence, it is a bounded function. The main question addressed in the papers [3], [4] was how much can one perturb the free Laplacian $-\Delta$ by a potential function $V$ and still get bounded corresponding solutions $u_1$ of (4) for almost every $\lambda > 0$?

It was known (and not difficult to prove) that the case $V \in L^1(\mathbb{R})$ is true and it was also known that the case $V \in L^p(\mathbb{R})$ for $p > 2$ is in general false, [20]. Christ and Kiselev showed in their papers that the answer to the above question is still affirmative if one considers potential functions in the class $L^p(\mathbb{R})$ for any $1 \leq p < 2$. Roughly speaking (and oversimplifying a lot) the starting point of their proof was to realize that every solution $u_1$ of the Schrödinger equation (4) can be essentially written as a series of expressions of the type

$$\int_{x_1 < \ldots < x_n < x} V(x_1) \ldots V(x_n) e^{i \sqrt{\lambda} (x_1 - x_2 + x_3 - \ldots + (-1)^p x_n)} \, dx_1 \ldots dx_n. \quad (5)$$

Clearly, in order to prove that every such a formula is bounded (as a function of $x$) for almost every $\lambda > 0$, it is enough to prove $L^p$ bounds for the corresponding maximal operator $\overline{M}_n$ defined by

$$\overline{M}_n(f_1, \ldots, f_n)(x) = \sup_N \left| \int_{x_1 < \ldots < x_n < N} f_1(x_1) \ldots f_n(x_n) e^{i \sqrt{\lambda} (x_1 - x_2 + x_3 - \ldots + (-1)^p x_n)} \, dx_1 \ldots dx_n \right|. \quad (6)$$

One of the main results in [3], [4] says that $\overline{M}_n$ is indeed a bounded operator from $L^p \times \ldots \times L^p$ into $L^{p'/n}$ where $1/p + 1/p' = 1$ and $1 \leq p < 2$, which means that the following inequalities hold

$$||\overline{M}_n(f_1, \ldots, f_n)||_{p'/n} \leq C_{n,p} ||f_1||_p \ldots ||f_n||_p. \quad (7)$$

One should also observe that in the simplest case of $L^1$ potentials, one has the trivial pointwise bound

$$||\overline{M}_n(f_1, \ldots, f_n)||_{\infty} \leq \frac{1}{n!} ||V||_1. \quad (8)$$
As it turned out [3] such a small constant appears also in (7) in the place of \( C_{n,p} \) in the particular case when \( f_1 = \ldots = f_n = V \) and this essentially allowed the authors of [3] to carefully sum up the contributions of all these expressions in (5) and prove the boundedness of the eigenfunctions.

The case \( p = 2 \) remained open and it still is today. \(^3\) If one would like to follow the same strategy, one is naturally led to considering (after using Plancherel) the following sequence of maximal operators (still denoted by \( \widetilde{M}_n \)) defined by

\[
\widetilde{M}_n(f_1, \ldots, f_n)(x) = \sup \left| \int_{\xi_1 < \cdots < \xi_n < N} \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i x (\xi_1 + \cdots + (-1)^n \xi_n)} d\xi_1 \cdots d\xi_n \right| \tag{9}
\]

and their simplified multi-linear variants \( \widetilde{T}_n \) given by

\[
\widetilde{T}_n(f_1, \ldots, f_n)(x) = \int_{\xi_1 < \cdots < \xi_n} \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i x (\xi_1 + \cdots + \xi_n)} d\xi_1 \cdots d\xi_n \tag{10}
\]

and proving at least \( L^2 \times \ldots \times L^2 \to L^{2/n,\infty} \) bounds for each of them.

This was precisely our initial attempt of understanding the \( L^2 \) question, but before doing anything else we first “fixed” their phases and replaced the \( (\widetilde{T}_n)_n \) and \( (\widetilde{M}_n)_n \) with \( (T_n)_n \) and \( (M_n)_n \), respectively, defined by

\[
T_n(f_1, \ldots, f_n)(x) = \int_{\xi_1 < \cdots < \xi_n} \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i x (\xi_1 + \cdots + \xi_n)} d\xi_1 \cdots d\xi_n \tag{11}
\]

and

\[
M_n(f_1, \ldots, f_n)(x) = \sup \left| \int_{\xi_1 < \cdots < \xi_n < N} \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i x (\xi_1 + \cdots + \xi_n)} d\xi_1 \cdots d\xi_n \right| \tag{12}
\]

since these new operators looked “more symmetric” to us and at the time, we believed that their \( L^2 \) boundedness properties should be similar to the \( L^2 \) boundedness properties of the original operators.

In the series of papers [12], [13], [15], [16] we understood completely the cases of \( T_3 \) and \( M_2 \). However, later on when we returned to the study of \( \widetilde{T}_3 \) and \( \widetilde{M}_2 \) we surprisingly realized that not only they do not satisfy the necessary weak-\( L^2 \) estimates, but they don’t satisfy any \( L^p \) estimates whatsoever [14]; and the same is true for all the operators \( (\widetilde{T}_n)_n \) and \( (\widetilde{M}_n)_n \) with the only exceptions of \( \widetilde{M}_1 \) (which is the Carleson operator [2]) and \( \widetilde{T}_2 \) (which essentially coincides with \( H(f_1, f_2) \) where \( H \) is the Hilbert transform [21]). \(^4\)

Since at least heuristically, the corresponding counterexample for \( \widetilde{T}_3 \) is not difficult to explain, we will briefly describe it in what follows.

First, let us recall that if one replaces the symbol \( \chi_{\xi_1 < \xi_2} \) by \( \text{sgn}(\xi_1 - \xi_2) \) one obtains the kernel representation of the bi-linear Hilbert transform [7] given by

\[^{3}\]This also explains our predilection for proving \( L^2 \) estimates only for our operators, even though one can in principle prove many other \( L^p \) estimates, as we did in [12] and [13].

\[^{4}\]We also showed in [14] that in spite of all of these, the corresponding eigenfunctions are still bounded functions! After observing this, our strategy towards proving the \( L^2 \) Schrödinger conjecture changed and we eventually proved a discrete Cantor group model of it, by completely different means.
\[ T_2(f_1, f_2)(x) = \int_{\mathbb{R}} f_1(x-t) f_2(x+t) \frac{dt}{t}. \]  

Similarly, if one replaces the symbol \( \chi_{\xi_1 < \xi_2 < \xi_3} \) by \( sgn(\xi_1 - \xi_2) \cdot sgn(\xi_2 - \xi_3) \) in (10) and (11) when \( n = 3 \), one can rewrite the modified \( T_3 \) and \( \tilde{T}_3 \) as

\[ T_3(f_1, f_2, f_3)(x) = \int_{\mathbb{R}^2} f_1(x-t_1) f_2(x+t_1 + t_2) f_3(x-t_2) \frac{dt_1 dt_2}{t_1 t_2}, \]

and

\[ \tilde{T}_3(f_1, f_2, f_3)(x) = \int_{\mathbb{R}^2} f_1(x-t_1) f_2(x-t_1 - t_2) f_3(x-t_2) \frac{dt_1 dt_2}{t_1 t_2}. \]

This time, these are all harmless modifications, since the new resulted operators behave similarly. Now, if one takes \( f_1(x) = f_3(x) = e^{ix^2} \) and \( f_2(x) = e^{-ix^2} \) one observes that formally,

\[ \tilde{T}_3(f_1, f_2, f_3)(x) = e^{ix^2} \int_{\mathbb{R}^2} e^{i|x|^2} \frac{dt_1 dt_2}{t_1 t_2} = C e^{ix^2} \int_{\mathbb{R}} \frac{dt}{|t|}. \]

In other words, we have \( \tilde{T}_3(f_1, f_2, f_3) = C f_1 \cdot f_2 \cdot f_3 \cdot \int_{\mathbb{R}} \frac{dt}{|t|} \). One can then quantify this equality by restricting the functions \( f_1, f_2, f_3 \) to an interval of the form \([-N, N] \). Roughly speaking, one obtains in this way that

\[ \tilde{T}_3(f_1 \chi_{[-N,N]}, f_2 \chi_{[-N,N]}, f_3 \chi_{[-N,N]}) \sim f_1 f_2 f_3 \chi_{[-N,N]} \log N \]  

and it is precisely this logarithmic factor which determines the failure of any attempt of proving \( L^p \) estimates for \( \tilde{T}_3 \) (see [14] for details).

It is also interesting and worth mentioning the fact that if one replaces the bi-parameter kernel \( \frac{1}{t_1 t_2} \) with a classical Calderón-Zygmund kernel \( K(t_1, t_2) \) of two variables [21], the corresponding trilinear operators (14) and (15) behave quite similarly and they both satisfy many \( L^p \) estimates, including the \( L^2 \times L^2 \times L^2 \to L^{2/3} \) one (see [11]).

Because of these counterexamples, we stopped for a while our study of \( (T_n)_n \) and \( (M_n)_n \) thinking that maybe their invention was a bit artificial \( ((\tilde{T}_n)_n \) and \( (\tilde{M}_n)_n \) were, after all, the operators which appeared “naturally”). However, more recently, our interest in them has been rekindled by the discovery that they really do appear in connection to a very similar but more general problem related to the behaviour of solutions of the so-called AKNS systems which play an important role in nuclear physics [1]. We will explain all these connections in detail, later on. The desired \( L^2 \) boundedness properties for \( T_n \) will be described in this paper, while the corresponding theorem for \( M_n \) will be postponed and presented in a future, forthcoming work.

\[ ^5 \text{We thus decided to continue our initial program and study not only the sequence } (M_n)_n \text{ but also the sequence } (T_n)_n \text{ since they are all very interesting objects from a purely Fourier analytic point of view (after all, the simplest operator in the } (M_n)_n \text{ sequence is the Carleson operator [2] while the simplest operator in the } (T_n)_n \text{ sequence is the bi-linear Hilbert transform.)} \]
We still don’t have any news regarding the “$L^2$- Schrödinger conjecture” but we have some interesting (we think) results related to the analogous “$L^2$-AKNS conjecture” which we now can prove in the case of upper (and lower) triangular matrices.

The article is organized as follows. In the next section we introduce the AKNS systems and describe their connection with our operators $(T_n)_n$ and $(M_n)_n$. Then, the rest of the paper is devoted to the proof of the main Theorem 1.3. We should warn the reader already familiar with our previous $T_3$ papers [12] and [13] that Theorem 1.3 is not a routine generalization of our previous work, since the complexity of $T_n$ for $n \geq 4$ adds some fundamentally new features which did not appear in the $T_3$ case. We will unravel them as we move along.

In Section 3, we present a way of decomposing the symbol $\chi_{\xi_1 < \ldots < \xi_n}$ naturally, into finitely many slightly smoother pieces. These pieces are intimately connected with several subregions of the simplex $\Delta_n = \{\xi \mid \xi_1 < \ldots < \xi_n\}$ which will be described with the help of certain combinatorial rooted trees. As a consequence, our operator $T_n$ will be decomposed as

$$T_n = \sum_G T^n_G$$

where the sum in (17) runs over a certain subclass of rooted trees having precisely $n$ leaves. Then, in Section 4 we show how to discretize all these operators $T^n_G$ and also show that in order to prove our main theorem it is enough to prove it in the case of these discretized model operators.

Section 5 contains the main part of the actual proof of the theorem and the paper ends with Section 6, in which we prove the “delicate Bessel” Lemma which plays an important role in the argument.

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2. AKNS systems and Fourier analysis

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and consider the system of differential equations

$$u' = i\lambda Du + Nu$$

where $u = [u_1, \ldots, u_n]'$ is a vector valued function defined on the real line, $D$ is a diagonal $n \times n$ constant matrix with real and distinct entries $d_1, \ldots, d_n$ and $N = (a_{ij})_{i,j=1}^n$ is a matrix valued function defined also on the real line and having the property that $a_{ii} \equiv 0$ for every $i = 1, \ldots, n$. These systems play a fundamental role in nuclear physics and they are called AKNS systems [1]. The particular case $n = 2$ is also known to be deeply connected to the classical theory of Schrödinger operators [3], [4].

If $N \equiv 0$ it is easy to see that our system (18) becomes a union of independent single equations

$$u_k' = i\lambda d_k u_k$$

for $k = 1, \ldots, n$ whose solutions are

$$u_k^\lambda(x) = C_{k,\lambda} e^{i\lambda d_k x}$$
and they are all $L^\infty(\mathbb{R})$-functions.

As in the case of the Schrödinger equation mentioned in the introduction, it is natural to ask how much can one perturb the $N \equiv 0$ case and still obtain bounded solutions $(u_k^j)^n_k$ for almost every real $\lambda$. As before, the answer is affirmative and easy for $L^1$ entries, very likely to hold true for $L^p$ entries when $1 \leq p < 2$ (we have not checked this carefully but we believe that the arguments of [3], [4] should be able to be adapted in this setting also) and is false for $L^p$ entries if $p > 2$ [20].

Thus, one is left with the following

**Question 2.1.** *Is it true that as long as the entries of the potential matrix $N$ are $L^2(\mathbb{R})$ functions, the corresponding solutions $(u_k^j)^n_k$ of the AKNS system (18) are all bounded functions for almost every real number $\lambda$?*

When $N \neq 0$ one can use a simple variation of constants argument and write $u_k(x)$ as

$$u_k(x) := e^{id_kx}v_k(x)$$

for $k = 1, ..., n$. As a consequence, the column vector $v = [v_1, ..., v_n]'$ becomes the solution of the following system

$$v' = Wv$$

(19)

where the entries of $W$ are given by $w_{lm}(x) := a_{lm}(x)e^{i(d_l - d_m)x}$. It is therefore enough to prove that the solutions of (19) are bounded as long as the entries $a_{lm}$ are square integrable.

To get a feeling of the difficulties of the problem, let us first consider the easiest possible case, that of $2 \times 2$ upper triangular matrices. This means that $n = 2$ and $a_{11} = a_{22} = a_{21} \equiv 0$ while $a_{12}(x) := f(x)$ is an arbitrary $L^2(\mathbb{R})$ function. The system (19) then becomes

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & f(x)e^{i(d_1 - d_2)x} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(20)

which implies that

$$v_1' = v_2(x)f(x)e^{i(d_1 - d_2)x}$$

$$v_2' = 0.$$  

Clearly, $v_2$ is bounded since it is constant (which we call $C_1$), while $v_1(= v_1^d)$ can be written as

$$v_1^d(x) = C_1 \int_{-\infty}^x f(y)e^{i(d_1 - d_2)y}dy + \bar{C}_1$$

for some other constant $\bar{C}_1$. In particular, we have

$$\|v_1^d\|_\infty \leq |C_1| \sup_x \left| \int_{-\infty}^x f(y)e^{i(d_1 - d_2)y}dy \right| + |\bar{C}_1|.$$  

(21)

We now recall the Carleson operator $C$ defined by

$$Cf(x) := \sup_N \left| \int_{\xi \in N} \hat{f}(\xi)e^{2\pi i x \xi}d\xi \right|.$$
A celebrated theorem of Carleson [2] says that $C$ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ boundedly and in particular this means that $Cf(x) < \infty$ for almost every $x \in \mathbb{R}$, as long as $f$ is an $L^2(\mathbb{R})$-function. Using this fact and Plancherel we see from (21) that indeed $\|v^3\|_\infty$ is finite for almost every $\lambda$ which means that the conjecture is true in this particular case.

Let us similarly consider now the case of $3 \times 3$ upper triangular systems. So this time $n = 3$ and $a_{12}(x) := f_1(x)$, $a_{13}(x) := f_2(x)$, $a_{23}(x) := f_3(x)$ and all the other entries are identically equal to zero. Our system (19) becomes

$$
\begin{bmatrix}
v'_1 \\
v'_2 \\
v'_3
\end{bmatrix} =
\begin{bmatrix}
0 & f_1(x)e^{i\lambda(d_1-d_2)x} & f_2(x)e^{i\lambda(d_1-d_3)x} \\
0 & 0 & f_3(x)e^{i\lambda(d_2-d_3)x} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
$$

(22)

which implies that

$$
\begin{align*}
v'_1 &= v_2(x)f_1(x)e^{i\lambda(d_1-d_2)x} + v_3(x)f_2(x)e^{i\lambda(d_1-d_3)x} \\
v'_2 &= v_3(x)f_3(x)e^{i\lambda(d_2-d_3)x} \\
v'_3 &= 0.
\end{align*}
$$

Clearly, $v_3$ is bounded since it is constant (say $C_\lambda$) and exactly as before $v_2(= v'_2)$ is also bounded for almost every $\lambda$, as a consequence of the same theorem of Carleson. Since

$$v_2(x) = C_\lambda \int_{-\infty}^{x} f_3(y)e^{i\lambda(d_2-d_3)y} \, dy + \widetilde{C}_\lambda,$$

it follows that

$$v'_1(x) = C_\lambda \left( \int_{-\infty}^{x} f_3(y)e^{i\lambda(d_2-d_3)y} \, dy \right) f_1(x)e^{i\lambda(d_1-d_2)x} + C_\lambda f_2(x)e^{i\lambda(d_1-d_3)x}.$$

By taking one more antiderivative, $v_1(= v'_1)$ becomes

$$v'_1(x) = C_\lambda \int_{-\infty}^{x} f_1(y)e^{i\lambda(d_1-d_2)y} \left( \int_{-\infty}^{y} f_3(z)e^{i\lambda(d_2-d_3)z} \, dz \right) \, dy + \widetilde{C}_\lambda \int_{-\infty}^{x} f_1(y)e^{i\lambda(d_1-d_2)y} \, dy + C_\lambda \int_{-\infty}^{x} f_2(y)e^{i\lambda(d_1-d_3)y} \, dy + \widetilde{C}_\lambda := I + II + III + \widetilde{C}_\lambda.$$

The terms $II$ and $III$ are bounded for almost every $\lambda$ as before, while the first one can be rewritten as

$$C_\lambda \int_{y \leq x} f_3(z)f_1(y)e^{i\lambda[(d_2-d_3)z+(d_1-d_2)y]} \, dz \, dy.$$

(23)

Let us now recall the bi-Carleson operator operator $M^{\alpha}_2$ introduced in [16] and defined by

$$M^{\alpha}_2(f, g)(x) = \sup_N \left| \int_{\xi_1 \leq \xi_2 \leq N} \tilde{f}(\xi_1)\tilde{g}(\xi_2)e^{2\pi i(\alpha_1\xi_1+\alpha_2\xi_2)} \, d\xi_1d\xi_2 \right|.$$
A recent theorem in [16] says that if \( \alpha_1 + \alpha_2 \neq 0 \) then \( M_2^\alpha \) maps \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) into \( L^1(\mathbb{R}) \) and as a consequence this means that \( M_2^\alpha(f, g)(x) < \infty \) for almost every \( x \in \mathbb{R} \), as long as \( f \) and \( g \) are \( L^2(\mathbb{R}) \) functions. Using this fact and Plancherel again, we see from (23) (since \( d_2 - d_3 + d_1 - d_2 = d_1 - d_3 \neq 0 \)) that \( v_1 \) is also bounded for almost every \( \lambda \), which means that the conjecture is also true for upper triangular \( 3 \times 3 \) potential matrices \( N \).

The case of general upper triangular \( n \times n \) matrices for \( n \geq 2 \) is similar and can be reduced to proving \( L^2(\mathbb{R}) \) estimates for maximal operators of the form

\[
M_k^\alpha(f_1, ... f_k)(x) := \sup_{N} \left| \int_{\xi_1 < ... < \xi_N \in N} \hat{f}_1(\xi_1) ... \hat{f}_k(\xi_k) e^{2\pi i \lambda (\alpha_1 \xi_1 + ... + \alpha_k \xi_k)} d\xi_1 ... d\xi_k \right|,
\]

where \( \alpha_1, ..., \alpha_k \) satisfy the nondegeneracy condition

\[
\sum_{j \neq j_1} \alpha_j \neq 0
\]

for every \( 1 \leq j_1 < j_2 \leq k \). It will be clear from the method of proof that our Theorem 1.3 holds not only for the operators \( T_n \) but also for the operators \( T_k^\alpha \) defined by

\[
T_k^\alpha(f_1, ... f_k)(x) := \int_{\xi_1 < ... < \xi_k} \hat{f}_1(\xi_1) ... \hat{f}_k(\xi_k) e^{2\pi i \lambda (\alpha_1 \xi_1 + ... + \alpha_k \xi_k)} d\xi_1 ... d\xi_k,
\]

as long as \( \alpha_1, ..., \alpha_k \) satisfy the nondegeneracy condition (24). \(^8\)

3. Rooted trees and the decomposition of the symbol \( \chi_{\xi_1 < ... < \xi_n} \)

The goal of this section is to carefully decompose the symbol \( \chi_{\xi_1 < ... < \xi_n} \) as a finite sum of several well localized multipliers associated with various subregions of the simplex \( \Delta_n = \xi_1 < ... < \xi_n \), which will be best described by using the combinatorial language of rooted trees.

To motivate this decomposition procedure we shall briefly revisit the already understood cases of \( \Delta_2 \) and \( \Delta_3 \).

The \( \Delta_2 \) case.

This case corresponds to the bi-linear Hilbert transform [7]. Let us first recall that by a shifted dyadic interval we simply mean any interval of the form \( 2^k (k + (0, 1) + (-1)^j) \alpha \) for any \( k, j \in \mathbb{Z} \) and \( \alpha \in [0, \frac{1}{3}, \frac{2}{3}] \). Then, for any integer \( d \) greater or equal than 1 a shifted dyadic quasi-cube of dimension \( d \) is defined to be any \( d \) - dimensional set of the form \( Q = Q_1 \times ... \times Q_d \) having the property that \( |Q_1| \sim ... \sim |Q_d| \) where each \( Q_i \) is a shifted dyadic interval. Observe as in [13] that for any arbitrary cube \( \bar{Q} \subseteq \mathbb{R}^d \) there always exists a shifted dyadic quasi-cube \( Q \) so that \( \bar{Q} \subseteq \frac{7}{10} Q \) and satisfying \( l(\bar{Q}) \sim l(Q) \). \(^9\) Let us then denote by \( \Gamma \) the singularity set

\(^*\)It is also known that if \( \alpha_1 + \alpha_2 = 0 \) the \( M_2^\alpha \) does not satisfy any \( L^p \) estimates [14].

\(^8\)It is also interesting to remark that in the general \( n = 2 \) case, a standard iterative procedure of Picard type will produce multi-linear expansions where the phases are of the form \( (d_1 - d_2)\xi_1 + (d_2 - d_1)\xi_2 + ... \) as in the Schrödinger case. Since one has \( d_1 - d_2 + d_2 - d_1 = 0 \), all the corresponding maximal operators are unbounded.

\(^9\)\( \frac{7}{10} Q \) is the parallelepiped having the same center as \( Q \) but \( \frac{7}{10} \) times smaller than it. The center of the quasi-cube is defined to be the \( d \) - dimensional point whose \( j \)th coordinate is the midpoint of the interval \( Q_j \). By \( l(Q) \) we simply denote the length of the first interval \( |Q_1| \) since they are all of comparable size. Finally, we will write \( A \leq B \) whenever \( A \leq CB \) for some fixed constant \( C > 0 \) and also \( A \sim B \) whenever \( A \leq B \) and \( B \leq A \).
\[
\Gamma = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 = \xi_2\}
\]
and consider the collection \(Q\) of all shifted dyadic quasi-cubes \(Q\) of dimension 2 having the property that \(Q \subseteq \Delta_2\) and also satisfying\(^{10}\)

\[
\operatorname{dist}(Q, \Gamma) \sim \operatorname{diam}(Q).
\]

Since the set of parallelepipeds \(\{\frac{9}{10}Q : Q \in Q\}\) forms a finitely overlapping cover of \(\Delta_2\), by a standard partition of unity we can write the symbol \(\chi_{\xi_1 < \xi_2}\) as

\[
\chi_{\xi_1 < \xi_2} = \sum_Q \Phi_Q(\xi_1, \xi_2)
\]

where \(\Phi_Q\) is a bump function adapted to \(\frac{9}{10}Q\). By splitting further each \(\Phi_Q\) as a double Fourier series in \(\xi_1, \xi_2\) we can rewrite the above expression as

\[
\sum_{n \in \mathbb{Z}^2} C_n \sum_Q \Phi_{Q,n,1}(\xi_1) \Phi_{Q,n,2}(\xi_2)
\]

where \((C_n)\) is a rapidly decreasing sequence and \(\Phi_{Q,n,j}\) is a bump function adapted to \(\frac{9}{10}Q_j\) uniformly in \(n\) for \(j = 1, 2\), (see also \([13]\)).

Since \(\xi_1 \in \frac{9}{10}Q_1\) and \(\xi_2 \in \frac{9}{10}Q_2\) it follows that \(\xi_1 + \xi_2 \in \frac{9}{10}Q_1 + \frac{9}{10}Q_2\) and as a consequence, one can find a shifted dyadic interval \(Q_3\) with the property \(\frac{9}{10}Q_1 + \frac{9}{10}Q_2 \subseteq \frac{2}{5}Q_3\) and also satisfying \(|Q_1| \sim |Q_2| \sim |Q_3|\). In particular, there exists a bump function \(\Phi_{Q,n,3}\) adapted to \(\frac{9}{10}Q_3\) uniformly in \(n\) such that \(\Phi_{Q,n,3} \equiv 1\) on \(\frac{9}{10}Q_1 + \frac{9}{10}Q_2\). This means that the expression (28) can also be written as

\[
\sum_{n \in \mathbb{Z}^2} C_n \sum_Q \Phi_{Q,n,1}(\xi_1) \Phi_{Q,n,2}(\xi_2) \Phi_{Q,n,3}(\xi_1 + \xi_2),
\]

where this time \(Q\) runs over the corresponding set of shifted dyadic quasi-cubes of dimension 3. Generic multipliers of the type \(m(\xi_1, \xi_2) = \Phi_1(\xi_1) \Phi_2(\xi_2) \Phi_3(\xi_1 + \xi_2)\) are well localized and they allow one to decompose the corresponding bi-linear operator \(T_m\) nicely\(^{11}\), as one can see from the following sequence of equalities.

\[
\int_{\mathbb{R}} T_m(f_1, f_2)(x)f_3(x)dx = \\
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \Phi_1(\xi_1) \Phi_2(\xi_2) \Phi_3(\xi_1 + \xi_2) e^{2\piix(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right) f_3(x)dx = \\
\int_{\mathbb{R}^2} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \Phi_1(\xi_1) \Phi_2(\xi_2) \Phi_3(\xi_1 + \xi_2) f_3(-\xi_1 - \xi_2)d\xi_1 d\xi_2 :=
\]

\(^{10}\)Here the understanding is that there exists a fixed large constant \(C > 0\) so that \(C \operatorname{diam}(Q) \leq \operatorname{dist}(Q, \Gamma) \leq 100C \operatorname{diam}(Q)\).

\(^{11}\)In general, if \(m(\xi_1, ..., \xi_k)\) is a multiplier, by \(T_m\) we denote the \(k\)-linear operator defined by \(T_m(f_1, ..., f_k)(x) := \int_{\mathbb{R}^k} m(\xi_1, ..., \xi_k) f_1(\xi_1) ... f_k(\xi_k) e^{2\piix(\xi_1 + ... + \xi_k)} d\xi_1 ... d\xi_k\).
\[ \int_{\mathbb{R}^2} \mathcal{F}_1(\xi_1) \Phi_1(\xi_1) \mathcal{F}_2(\xi_2) \Phi_2(\xi_2) \mathcal{F}_3(-\xi_1 - \xi_2) \Phi_3(-\xi_1 - \xi_2) d\xi_1 d\xi_2 = \]

\[ \int_{\lambda_1 + \lambda_2 + \lambda_3 = 0} (f_1 * \hat{\Phi}_1)(\lambda_1) (f_2 * \hat{\Phi}_2)(\lambda_2) (f_3 * \hat{\Phi}_3)(\lambda_3) d\lambda = \]

\[ \int_{\mathbb{R}} (f_1 * \hat{\Phi}_1)(x) (f_2 * \hat{\Phi}_2)(x) (f_3 * \hat{\Phi}_3)(x). \]  

(30)

This also implies that

\[ T_m(f_1, f_2)(x) = [ (f_1 * \hat{\Phi}_1)(f_2 * \hat{\Phi}_2) ] * \tilde{\Phi}_3. \]

If one discretizes further the expression (30) in the \( x \)-variable for each of the similar multipliers appearing in (29), one obtains the usual model for the bi-linear Hilbert transform [7]. For reasons that will become clearer later on, we would like to associate to this simplex \( \Delta_2 \) the simplest rooted tree having precisely two leaves, as in Figure 1.

**Figure 1.** The rooted tree of the bi-linear Hilbert transform.

The \( \Delta_3 \) case.

This case corresponds to the “bi-est” operator [12], [13]. If \( \xi_1 < \xi_2 < \xi_3 \) then clearly there are three possibilities: either \( |\xi_1 - \xi_2| << |\xi_2 - \xi_3| \) or \( |\xi_2 - \xi_3| << |\xi_1 - \xi_2| \) or \( |\xi_1 - \xi_2| \sim |\xi_2 - \xi_3| \) each defining different types of regions inside the simplex \( \Delta_3 \).

The idea of [13] was to split the symbol \( \chi_{\xi_1 < \xi_2 < \xi_3} \) as a sum of three distinct symbols

\[ \chi_{\xi_1 < \xi_2 < \xi_3} = m_I + m_{II} + m_{III} \]

well adapted to these regions described above.

To define \( m_I \) properly, let us first observe that as before, the symbol \( \chi_{\xi < \eta} \) can be also decomposed as

\[ \chi_{\xi < \eta} = \sum_{n' \in \mathbb{Z}^2} C'_{n'} \sum_{Q'} \Phi_{Q',n',1}(\xi) \Phi_{Q',n',2}(\eta) \]

(31)

where this time \( Q' \) are shifted dyadic quasi-cubes of dimension 2 having the property that \( Q' \subseteq \{ (\xi, \eta) \in \mathbb{R}^2 : \frac{\xi}{2} < \eta \} \) and also that

\[ \text{dist}(Q', \Gamma') \sim \text{diam}(Q') \]

where \( \Gamma' \) denotes the singularity line.

\[ A << B \] denotes the statement that there exists a large fixed constant \( C > 0 \) so that \( CA \leq B. \)
\[ \Gamma' = \{(\xi, \eta) \in \mathbb{R}^2 : \frac{\xi}{2} = \eta\}. \]

In particular, if \((\xi_1, \xi_2, \xi_3)\) is a fixed point in the simplex \(\Delta_3\) which belongs to the first region \(|\xi_1 - \xi_2| \ll |\xi_2 - \xi_3|\), one clearly has not only \(\xi_2 < \xi_3\) but also \(\frac{\xi_1 + \xi_3}{2} < \xi_3\) and this implies that (using (28) and (31))

\[ 1 = \chi_{\xi_1 \ll \xi_2} \cdot \chi_{\xi_1 + \xi_3 < \xi_3} (\xi_1, \xi_2, \xi_3) = \sum_{n, n', Q} C_n C_{n'} \Phi_Q, n, 1(\xi_1) \Phi_{Q', n, 2}(\xi_2) \Phi_{Q', n', 1}(\xi_1 + \xi_2) \Phi_{Q', n', 2}(\xi_3) \quad (32) \]

It is also not difficult to see that the last expression (32) is also equal to

\[ \sum_{n, n', Q} C_n C_{n'} \Phi_Q, n, 1(\xi_1) \Phi_{Q, n, 2}(\xi_2) \Phi_{Q', n', 1}(\xi_1 + \xi_2) \Phi_{Q', n', 2}(\xi_3) \]

with the implicit constants independent on the fixed point \((\xi_1, \xi_2, \xi_3)\) and dependent only on the corresponding constants defining the first region.

Then, one defines the symbol \(m_I\) simply by

\[ m_I(\xi_1, \xi_2, \xi_3) := \sum_{n, n', Q} C_n C_{n'} \Phi_Q, n, 1(\xi_1) \Phi_{Q, n, 2}(\xi_2) \Phi_{Q', n', 1}(\xi_1 + \xi_2) \Phi_{Q', n', 2}(\xi_3) \quad (33) \]

for any \((\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3\). Clearly, by construction, \(m_I\) is identically equal to 1 on the first region and is also supported on another larger region of the same type (defined by different constants) which is contained inside the simplex \(\Delta_3\). Similarly, one defines the symbol \(m_{II}\) adapted to the second region \(|\xi_2 - \xi_3| < |\xi_2 - \xi_2|\) and in the end one sets

\[ m_{II} := \chi_{\Delta_3} - m_I - m_{II} \]

which clearly is supported inside a region of the third type \(|\xi_1 - \xi_2| \sim |\xi_2 - \xi_3|\).

As before (when we passed to (29) from (28)), one can “complete” the expressions in (33) obtaining products of type

\[ m(\xi_1, \xi_2, \xi_3) = \Phi_1(\xi_1) \Phi_2(\xi_2) \Phi_3(\xi_1 + \xi_2) \Phi'_1(\xi_1 + \xi_2) \Phi'_2(\xi_2) \Phi'_3(\xi_1 + \xi_2 + \xi_3). \quad (34) \]

The reason for which we preferred to describe the region \(|\xi_1 - \xi_2| \ll |\xi_2 - \xi_3|\) as rather being \(|\xi_1 - \xi_2| \ll |\frac{\xi_1 + \xi_2}{2} - \xi_3|\) will be clearer when we calculate the tri-linear form associated to the tri-linear operator \(T_m\) given by symbols of type (34). This time we can write

\[ \int_{\mathbb{R}} T_m(f_1, f_2, f_3)(x) f_4(x) dx = \]

\[ \int_{\mathbb{R}^3} \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{f}_3(\xi_3) \Phi_1(\xi_1) \Phi_2(\xi_2) \Phi_3(\xi_1 + \xi_2) \Phi'_1(\xi_1 + \xi_2) \Phi'_2(\xi_2) \Phi'_3(\xi_1 + \xi_2 + \xi_3) \tilde{f}_4(-\xi_1 - \xi_2 - \xi_3) d\xi := \]

\[ \int_{\mathbb{R}^3} \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{f}_3(\xi_3) \Phi_1(\xi_1) \Phi_2(\xi_2) \Phi_3(\xi_1 + \xi_2) \Phi'_1(\xi_1 + \xi_2) \Phi'_2(\xi_2) \Phi'_3(-\xi_1 - \xi_2 - \xi_3) \tilde{f}_4(-\xi_1 - \xi_2 - \xi_3) d\xi = \]
\[
\int_{\lambda_1, \lambda_2, \lambda_3 = 0} \left( \int_{\xi_1, \xi_2 = \lambda_1} \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \Phi_1(\xi_1) \Phi_2(\xi_2) d\xi_1 d\xi_2 \right) \Phi_3(\lambda_1) \Phi'_1(\lambda_1) = \\
\int_{\lambda_1, \lambda_2, \lambda_3 = 0} \left( \left[ (f_1 * \tilde{\Phi}_1)(f_2 * \tilde{\Phi}_2) \right] * \tilde{\Phi}_3 \right) \Phi_1(\lambda_1) (f_3 * \tilde{\Phi}_2)(\lambda_2) (f_4 * \tilde{\Phi}_3)(\lambda_3) d\lambda = \\
\int_{\mathbb{R}} \left[ (f_1 * \tilde{\Phi}_1)(f_2 * \tilde{\Phi}_2) \right] * \tilde{\Phi}_3 \Phi_1(x) (f_3 * \tilde{\Phi}_2)(x) (f_4 * \tilde{\Phi}_3)(x) dx.
\]

As before, if one discretizes further this expression in the \(x\) - variable for each of the similar multipliers appearing in (33), one obtains the discretized model for the “bi-est” operator in [13]. To each of the three regions described above, we associate a rooted tree as in the Figure 2.

Figure 2. The rooted trees of the “bi-est”.

The \(\Delta_u\) case.

We denote generically by \(G\) an arbitrary rooted tree. Let us recall that each vertex of \(G\) has a level. There is precisely one vertex, the root, which has level 0. All adjacent vertices differ by exactly one level and each vertex at level \(i + 1\) is adjacent to exactly one vertex at level \(i\). All the vertices adjacent to a fixed vertex \(u\) and at levels below \(u\) are called the sons of \(u\). Also, all the vertices that are joined to \(u\) by a chain of vertices at levels below \(u\), are called descendants of \(u\). The vertices which do not have sons are called leaves.

We will consider only those rooted trees having precisely \(n\) leaves which we label with numbers from 1 to \(n\) from the most left one to the most right one. Also, our trees have the property that every vertex of the tree which is not a leaf has at least two sons. We denote the class of all such rooted trees by \(G_n\).

The maximal possible level in a tree is called the height of the tree. We will also denote by \(V_G\) the set of all vertices of \(G\) which are not leaves.

If \(u \in V_G\), we denote by \(I_u\) the collection of all the integers \(1 \leq i \leq n\) having the property that the leaf labeled “\(i\)” is a descendant of \(u\). It is not difficult to see that there exist two integers \(1 \leq l_u < r_u \leq n\) so that

\[
I_u = \{l_u, l_u + 1, ..., r_u\}.
\]

In case \(u\) is a leaf labeled \(i_0\), we simply set \(I_u := \{i_0\}\). Now, if \(\xi_1 < ... < \xi_n\), we denote by \(I_1, I_2, ... I_{n-1}\) the intervals \([\xi_1, \xi_2], [\xi_2, \xi_3], ..., [\xi_{n-1}, \xi_n]\) respectively.
Fix now $G \in \mathcal{G}_n$ and $u \in V_G$. Denote also by $u_1, u_2, \ldots, u_\#$ all the sons of $u$. To this vertex $u$ we associate a region $R_u \subseteq \Delta_n$ defined to be the set all all vectors $(\xi_1, \ldots, \xi_n) \in \Delta_n$ having the property that

$$|I_{l_{u_1}}| \sim |I_{l_{u_2}}| \sim \ldots \sim |I_{l_{u_{\#}-1}}| \gg |I_l|$$

(35)

for every $l_u \leq l \leq r_u-1$ and $l \neq r_{u_1}, r_{u_2}, \ldots, r_{u_{\#}-1}$.

Then, we define the region $R_G \subseteq \Delta_n$ by

$$R_G := \bigcap_{u \in V_G} R_u.$$  

(36)

By a region $R(= R_G)$ of type $G$ we simply mean from now on a region defined in this way for various implicit constants.

For instance, if $G$ is the rooted tree on the left in Figure 3 then by a region of type $G$ we mean one defined by the inequalities $(|I_3| \sim |I_4| \gg |I_1|, |I_2|, |I_5|) \cap (|I_1| \gg |I_2|)$ and if $G$ is the tree on the right in Figure 3, then by a region of type $G$ we mean one given by the inequalities $(|I_4| \gg |I_1|, |I_2|, |I_3|, |I_5|) \cap (|I_2| \gg |I_1|, |I_3|)$.

It is now easy to see that if the implicit constants are chosen carefully, one can decompose the simplex $\Delta_n$ as

$$\Delta_n = \bigcup_{G \in \mathcal{G}_n} R_G.$$

We would now like to associate to each $G \in \mathcal{G}_n$ a multiplier $m_G$ adapted to such a region $R_G$ constructed before. We first need the following definition.

13In other words, the region $R_u$ is the subregion of $\Delta_n$ defined by the constraints in (35). Note that sometimes it may happen that some (or all!) of the constraints “do not make sense”, in which case they should be simply disregarded. For instance, if all the sons of $u$ are leaves (as in Figure 4), then there is no $l$ with the property $l_u \leq l \leq r_u-1$ and $l \neq r_{u_1}, r_{u_2}, \ldots, r_{u_{\#}-1}$. In this case, the constraints defining the region are only the first ones, namely $|I_{r_{u_1}}| \sim |I_{r_{u_2}}| \sim \ldots \sim |I_{r_{u_{\#}-1}}|$. If in addition $u$ has only two sons (and they are both leaves as before), then one can see that the sequence of inequalities $|I_{r_{u_1}}| \sim |I_{r_{u_2}}| \sim \ldots \sim |I_{r_{u_{\#}-1}}|$ becomes redundant since $\#-1 = 1$. In this case, there are simply no constraints and as a consequence the corresponding region $R_u$ coincides with the whole simplex $\Delta_n$.
Definition 3.1. Let \( d \geq 2 \) and \( \vec{a} = (a_1, \ldots, a_d) \) be a vector with real and strictly positive entries. Denote by \( R_{\vec{a}} \) the region of all vectors \((x_1, \ldots, x_d) \in \mathbb{R}^d\) satisfying
\[
a_1x_1 < a_2x_2 < \cdots < a_dx_d
\]
and also
\[
|a_1x_1 - a_2x_2| \sim |a_2x_2 - a_3x_3| \sim \cdots \sim |a_{d-1}x_{d-1} - a_dx_d|.
\]
A shifted dyadic quasi-cube \( Q = (Q_1, Q_2, \ldots, Q_d) \) of dimension \( d \) is said to be adapted to the region \( R_{\vec{a}} \) if and only if for every \( j = 1, \ldots, d-1 \) the quasi-cubes \( Q^j := (Q_j, Q_{j+1}) \) have the property that
\[
Q^j \subseteq \{(x_j, x_{j+1}) \in \mathbb{R}^2 : a_jx_j < a_{j+1}x_{j+1}\}
\]
and also that
\[
\text{dist}(Q^j, \Gamma^j) \sim \text{diam}(Q^j)
\]
where
\[
\Gamma^j := \{(x_j, x_{j+1}) \in \mathbb{R}^2 : a_jx_j = a_{j+1}x_{j+1}\}.
\]

We are now ready to define a standard symbol associated to \( G \) as follows. First, for \( u \in V_G \) denote by \( u_1, \ldots, u_\# \) all its sons and consider the region \( R_{\vec{a}} \) associated to the vector
\[
\vec{a} := \left(\frac{1}{|I_{u_1}|}, \ldots, \frac{1}{|I_{u_\#}|}\right)
\]
as in Definition 3.1.

We denote by \( m_u \) any expression of the form
\[
m_u((\xi_l)_{l \in I_u}) = \sum_{Q_u} \Phi_{Q_u,1}(\sum_{l \in I_{x_1}} \xi_l) \cdots \Phi_{Q_u,\#}(\sum_{l \in I_{x_\#}} \xi_l)
\]
where the sum in taken over shifted dyadic quasi-cubes adapted to the region \( R_{\vec{a}} \) above in the sense of Definition 3.1, and \( \Phi_{Q_u,j} \) are bumps adapted to \( \frac{9}{10}Q_u^j \) for any \( j = 1, \ldots, \# \).

The expression (37) can also be written as
\[
m_u := \sum_{k \in \mathbb{Z}} m_u^k
\]
where \( m_u^k \) is defined by the same formula (37) but with the aditional constraint that \( l(Q_u) \sim 2^k \).

In the end, a multiplier \( m_G \) corresponding to the tree \( G \) is defined to be any expression of the form
\[
m_G := \sum_n C_n \sum_{(k_1, k_2, \ldots, k_{|V_G|}) \in S_G} \left( \prod_{l=1}^{|V_G|} m_{v_l,n}^{k_l} \right)
\]
where we assumed that \( V_G = \{v_1, \ldots, v_{|V_G|}\} \), the sequence \((C_n)_n\) is a rapidly decreasing sequence indexed over a countable set, and by \( S_G \) we denoted the collection of all tuples of integers.
(k_1, ..., k_{|\Gamma|}) having the property that \( k_r >> k_{r'} \) as long as the vertices \( v_r \) and \( v_{r'} \) are adjacent and \( v_{r'} \) is the son of \( v_r \). \(^{14}\)

The following Lemma will be very helpful.

**Lemma 3.2.** Let \( d, \bar{a}, R_d \) be as in Definition 3.1. Then, there exists a symbol \( m_{\bar{d}} \) of the form

\[
m_{\bar{d}}(x_1, ..., x_d) = \sum_n C_n \sum_{Q} \Phi_{Q_1,n,1}(x_1) \cdots \Phi_{Q_d,n,d}(x_d) \tag{39}
\]

so that \( m_{\bar{d}}|R_d \equiv 1 \), where \((C_n)_n\) is a rapidly decreasing sequence, \( \Phi_{Q,n,j} \) is a bump adapted to \( \frac{9}{10}Q_j \) uniformly in \( n \) and the sum in (39) is taken over \( n \) belonging to a certain countable set and \( Q \) belonging to a certain collection of shifted dyadic quasi-cubes adapted to \( R_d \).

**Proof** Using our standard procedure, for each \( i = 1, ..., d - 1 \) one can decompose as before the symbol \( \chi_{a_i < a_{i+1} < x_i} \) as

\[
\chi_{a_i < a_{i+1} < x_i} = \sum_{n \in \mathbb{Z}^2} C_n^{i} \sum_{Q} \Phi_{Q_1,n,1}(x_i) \Phi_{Q_2,n,2}(x_{i+1}) \tag{40}
\]

where as usual \((C_n)_n\) is a rapidly decreasing sequence, \( \Phi_{Q_1,n,1}, \Phi_{Q_2,n,2} \) are bumps adapted to \( \frac{9}{10}Q_1 \) and \( \frac{9}{10}Q_2 \) respectively and the sum is taken over shifted dyadic quasi-cubes \( Q^i \) of dimension 2 inside the region \( \{(x_i, x_{i+1}) \in \mathbb{R}^2 : a_i x_i < a_{i+1} x_{i+1}\} \) and satisfying

\[\text{dist}(Q^i, \Gamma^i) \sim \text{diam}(Q^i)\]

where \( \Gamma^i \) has been defined in Definition 3.1.

Now, if one takes the product of all the \( d - 1 \) expressions in (40) and if one restricts the summation to those \( Q^1, ..., Q^{d-1} \) for which \( l(Q^1) \sim ... \sim l(Q^{d-1}) \), one gets an expression (named “\( m_{\bar{d}} \)”) which can clearly be written in the form required by the Lemma and which also satisfies \( m_{\bar{d}}|R_d \equiv 1 \) if the implicit constants are chosen appropriately.

The following Lemma will also play an important role.

**Lemma 3.3.** Let \( G \in \mathcal{G}_n \) and \( R_G \) be a fixed region of type \( G \). Then, there exists a standard symbol \( m_G \) of type \( G \) having the property that

\[m_G|R_G \equiv 1\]

and also that \( \text{supp}(m_G) \) is included inside a larger region of the same type \( G \).

**Proof** We will prove this by induction with respect to the height of the rooted tree. First, let us consider the case of a rooted tree \( G \) having height 1 and an arbitrary number of leaves \( L \) for \( 2 \leq L \leq n \) (see Figure 4).

The region \( R_G \) is then described as being the set of vectors \((\xi_1, ..., \xi_L)\) so that \( \xi_1 < ... < \xi_L \) and satisfying

\(^{14}\) Also, by \( m_{\bar{d},n} \) we clearly mean an expression of the same type with the one in (37) but with the corresponding bumps \( \Phi_{Q_{n,i}} \) (for which \( l(Q_n) \sim 2^n \)) being replaced by similar ones \( \Phi_{Q_{n,i}} \) which have the same properties, uniformly in \( n \).
Figure 4. Rooted tree of height 1.

\[ |\xi_1 - \xi_2| \sim |\xi_2 - \xi_3| \sim ... \sim |\xi_{L-1} - \xi_L|. \]

The fact that a multiplier \( m_G \) of type \( G \) having the property that \( m_G|R_G \equiv 1 \) exists, is in this case a simple consequence of the previous Lemma 3.2.

Let us now assume that our statement holds for any rooted tree \( G \) having height smaller or equal than \( h \) and any number of leaves \( L \) and we would like to prove that it holds for trees of height \( h + 1 \) and any number of leaves \( L \).

Fix \( G \) a rooted tree with height \( h + 1 \) and \( L \) leaves, for some \( 2 \leq L \leq n \). Denote by \( u \) the root of it and as usual by \( u_1, ..., u_# \) the sons of \( u \). If \( u_j \) is not a leave, denote also by \( G_j \) the corresponding sub-tree of \( G \) whose root is \( u_j \). Clearly, all of these sub-trees have heights smaller or equal than \( h \).

Fix also \( R_G \) a region of type \( G \). This region defines the “projected regions” \( R_{G_j} \) given by

\[ R_{G_j} := \{ (\xi_l)_{l \in I_{u_j}} : (\xi_l)_{l=1}^L \in R_G \}. \]

By the induction hypothesis, there exist multipliers \( m_{G_j} \) of type \( G_j \) having the property that \( m_{G_j}|R_{G_j} \equiv 1 \) and so that their supports are also included into a slightly larger region of the same corresponding type \( G_j \).

Fix now \( (\xi_l)_{l=1}^L \in R_G \). Clearly, since \( (\xi_l)_{l \in I_{u_j}} \in R_{G_j} \) it follows that \( m_{G_j}((\xi_l)_{l \in I_{u_j}}) = 1 \).

On the other hand, from the definitions of the regions \( R_G \) of type \( G \) we also know that

\[ |I_{u_1}| \sim |I_{u_2}| \sim ... \sim |I_{u_{L-1}}| >> |I_l| \]  \hspace{1cm} (41)

for every \( l_u \leq l \leq r_u - 1 \) and \( l \neq r_{u_1}, r_{u_2}, ..., r_{u_{L-1}} \).

In particular, this implies that the following inequalities hold

\[ \frac{1}{|I_{u_1}|} \sum_{l \in I_{u_1}} \xi_l < \frac{1}{|I_{u_2}|} \sum_{l \in I_{u_2}} \xi_l < ... < \frac{1}{|I_{u_{L-1}}|} \sum_{l \in I_{u_{L-1}}} \xi_l \]  \hspace{1cm} (42)

and also that

\[ \left| \frac{1}{|I_{u_1}|} \sum_{l \in I_{u_1}} \xi_l - \frac{1}{|I_{u_2}|} \sum_{l \in I_{u_2}} \xi_l \right| \sim ... \sim \left| \frac{1}{|I_{u_{L-1}}|} \sum_{l \in I_{u_{L-1}}} \xi_l - \frac{1}{|I_{u_{L-1}}|} \sum_{l \in I_{u_{L-1}}} \xi_l \right|. \]  \hspace{1cm} (43)

As a consequence of Lemma 3.2, if we denote as before by \( \bar{a} := (\frac{1}{|I_{u_1}|}, ..., \frac{1}{|I_{u_{L-1}}|}) \) we know that we can find a multiplier \( m_{\bar{a}} \) having the property that \( m_{\bar{a}}|R_{\bar{a}} \equiv 1 \). In particular, from (42) and (43) we see that
\[
m_d \left( \sum_{l \in I_{u_1}} \xi_l, \ldots, \sum_{l \in I_{u_g}} \xi_l \right) = 1
\]

which implies that

\[
1 = \left( \prod_j m_{G_j}(\xi_l)_{l \in I_{u_j}} \right) \cdot m_d \left( \sum_{l \in I_{u_1}} \xi_l, \ldots, \sum_{l \in I_{u_g}} \xi_l \right).
\] (44)

Denote now by \( j_1, \ldots, j_s \) those indices “\( l \)” between 1 and \# for which the corresponding son \( u_l \) of \( u \) is not a leave. It is not difficult to see that this last expression (44) is also equal to

\[
\sum_{k \gg k_{j_1}, \ldots, k_{j_s}} m_{G_{j_1}}^k(\xi_l)_{l \in I_{u_{j_1}}} \ldots m_{G_{j_s}}^k(\xi_l)_{l \in I_{u_{j_s}}} m_d \left( \sum_{l \in I_{u_1}} \xi_l, \ldots, \sum_{l \in I_{u_g}} \xi_l \right),
\] (45)

where in general, by \( m_{\tilde{G}}^k \) we denote the multiplier defined by the same formula (38) but with the additional constraint that the summation index corresponding to the root of \( \tilde{G} \) is kept constant and equal to \( k \). Similarly, by \( m_{\bar{G}}^k \) we denote the multiplier defined by the same expression in Lemma 3.2 but again with the additional constraint that the summation over \( Q \) is restricted to those for which \( l(Q) \sim 2^k \).

It is also important to note that the implicit constants in (45) are independent on the previously fixed vector \( (\xi_l)_{l=1}^L \in R_G \) (and dependent only on the constants defining the region). Then, one simply defines the desired multiplier \( m_G \) by the same similar expression in (45), and then one can observe that \( m_G \) has all the desired properties.

Having all these constructions at our disposal, we can actually start describing the decomposition of the symbol \( \chi_{\Delta_n} \). Assume for simplicity that \( \mathcal{G}_n = \{ G_1, \ldots, G_N \} \). Clearly, there exist regions \( R_{G_1}, \ldots, R_{G_N} \) corresponding to these rooted trees so that

\[ \chi_{\Delta_n} = R_{G_1} \cup \ldots \cup R_{G_N}. \]

Let \( m_{G_1} \) be a symbol of type \( G_1 \) satisfying \( m_{G_1}|R_1 \equiv 1 \) as in Lemma 3.3 and define

\[
m^1 := \chi_{\Delta_n} - m_{G_1}.
\] (46)

Observe that \( m^1|R_1 \equiv 0 \) and as a consequence, we have that

\[ \text{supp}(m^1) \subseteq R_2 \cup \ldots \cup R_N. \]

Then, consider \( m_{G_2} \) a symbol of type \( G_2 \) satisfying \( m_{G_2}|R_2 \equiv 1 \) again as in Lemma 3.3 and define

\[
m^2 := m^1 - m^1 \cdot m_{G_2}.
\] (47)

Observe as before that

\[ \text{supp}(m^2) \subseteq \text{supp}(m^1) \subseteq R_2 \cup \ldots \cup R_N \]

and since \( m^2|R_2 \equiv 0 \) it follows that actually

\[ \text{supp}(m^2) \subseteq R_3 \cup \ldots \cup R_N. \]
One can continue in this way and define \( m^1, \ldots, m^{N-1} \) recursively and in the end one observes that \( m^N \) defined by

\[
m^N := m^{N-1} - m^{N-1} \cdot m_{G_N}
\]

has the property that \( m_N \equiv 0 \) since \( \text{supp}(m_N) \subseteq R_N \) and \( m_{G_N}R_N \equiv 1 \).

Adding all these equalities (46), (47), (48) together we obtain that \( \chi_{\Delta_n} \) can be written as

\[
\chi_{\Delta_n} = m_{G_1} + m^1 \cdot m_{G_2} + m^2 \cdot m_{G_3} + \ldots + m^{N-1} \cdot m_{G_N}.
\]

Moreover, since \( m^1 \) has an explicit formula, all the symbols \( m^i \) are explicit since they have been defined recursively. It is easy to remark that our symbol \( \chi_{\Delta_n} \) can as a result be written as a finite sum of products of multipliers of type

\[
m_{G_1} \cdot m_{G_2} \cdot \ldots \cdot m_{G_N}.
\]

However, the next Proposition shows that these product symbols are essentially of the same type as the standard ones considered above.

Before stating it, we first need to recall the following definition from [13].

**Definition 3.4.** A subset \( Q \) of shifted dyadic quasi-cubes is said to be sparse if and only if for any two quasi-cubes \( Q, Q' \in Q \) with \( Q \neq Q' \) we have \( |Q| < |Q'| \) implies \( |CQ| < |Q'| \) and \( |Q| = |Q'| \) implies \( CQ \cap CQ' = \emptyset \), where \( C > 0 \) is a fixed large constant.

We would like to assume from now on, without losing the generality, that all our families of quasi-cubes that implicitly enter the formulas (38) are sparse.

The following simple Lemma will also play an important role later on.

**Lemma 3.5.** Let \( d \geq 2 \), \( Q = (Q_1, \ldots, Q_d) \) be a shifted dyadic quasi-cube, \( I \) be a shifted dyadic interval so that \( |I| \sim \text{diam}(Q) \) and \( 0 < \alpha < \beta < 1 \). Consider also bump functions \( \Phi_{Q_1}, \ldots, \Phi_{Q_d}, \Phi_I \) adapted to \( \alpha Q_1, \ldots, \alpha Q_d \) and \( \alpha I \) respectively. Then, there exists a sequence \( (C_n)_n \) of rapidly decreasing complex numbers (independent on \( Q, I \) !) and bump functions \( \widetilde{\Phi}_{Q_1,n}, \ldots, \widetilde{\Phi}_{Q_d,n} \) uniformly adapted to \( \beta Q_1, \ldots, \beta Q_d \) respectively, so that

\[
\Phi_{Q_1}(x_1) \ldots \Phi_{Q_d}(x_d) \cdot \Phi_I(x_1 + \ldots + x_d) = \sum_n C_n \Phi_{Q_1,n}(x_1) \ldots \Phi_{Q_d,n}(x_d).
\]

**Proof** First, consider bump functions \( \widetilde{\Phi}_{Q_1}, \ldots, \widetilde{\Phi}_{Q_d} \) adapted to \( \beta Q_1, \ldots, \beta Q_d \) and having the property that

\[
\widetilde{\Phi}_{Q_j} \equiv 1
\]

on the support of \( \Phi_{Q_j} \) for every \( j = 1, \ldots, d \). In particular, the left hand side of (50) can be rewritten as

\[
\Phi_{Q_1}(x_1) \ldots \Phi_{Q_d}(x_d) \cdot [\Phi_{Q_1}(x_1) \ldots \Phi_{Q_d}(x_d) \cdot \Phi_I(x_1 + \ldots + x_d)] := \Phi_{Q_1}(x_1) \ldots \Phi_{Q_d}(x_d) \cdot m(x_1, \ldots, x_d).
\]

Then, one just has to write \( m \) as a multiple Fourier series in the variables \( x_1, \ldots, x_d \) on \( Q \) and to take advantage of the smoothness of \( m \).

\[ \blacksquare \]
**Proposition 3.6.** Let $G_1, G_2 \in \mathcal{G}_n$ and $m_{G_1}, m_{G_2}$ symbols associated to $G_1$ and $G_2$ respectively. Then, there exists a rooted tree $G \in \mathcal{G}_n$ so that
\[ m_{G_1} \cdot m_{G_2} = m_G \]
for a certain symbol $m_G$ of type $G$.

**Proof** Fix $G_1, G_2 \in \mathcal{G}_n$. If $m_{G_1}$ and $m_{G_2}$ are symbols of type $G_1$ and $G_2$ respectively, then one can write
\[ m_{G_1} \cdot m_{G_2} = \left( \sum_k m_{G_1}^k \right) \left( \sum_{\tilde{k}} m_{G_2}^{\tilde{k}} \right) = \sum_{k, \tilde{k}} m_{G_1}^k m_{G_2}^{\tilde{k}} = \sum_{k < \tilde{k}} m_{G_1}^k m_{G_2}^{\tilde{k}} \]  
(51)
since $m_{G_1}^k m_{G_2}^{\tilde{k}} \equiv 0$ unless $k \sim \tilde{k}$. Clearly, since both $k$ and $\tilde{k}$ run inside sparse sets, for any fixed $k$ there is a unique $\tilde{k}$ for which $k \sim \tilde{k}$. By abuse of notation we will denote from now on the corresponding $m_{G_1}^k$, simply by $m_{G_2}^k$.

We will prove by induction over $n$ that there exists $G \in \mathcal{G}_n$ so that
\[ m_{G_1}^k \cdot m_{G_2}^k = m_G^k \]  
(52)
for every $k$, for a certain symbol $m_G$ of type $G$. If we accept for a moment (52), then (51) becomes
\[ \sum_k m_{G_1}^k \cdot m_{G_2}^k = \sum_k m_G^k = m_G \]
which would complete our proof.

Denote by $u$ the root of $G_1$, by $v$ the root of $G_2$ and by $w$ the root of $G$. Denote also by $u_1, \ldots, u_t$ the sons of $u$, by $v_1, \ldots, v_s$ the sons of $v$ and by $w_1, \ldots, w_q$ the sons of $G$. We will in fact prove that the rooted tree $G$ we are looking for has also the following refinement property (with respect to $G_1$ and $G_2$) which says that the sets of indices $I_u$, $I_v$, and $I_w$ can each be written as a disjoint union of various sets of indices of type $I_i$.

It remains to prove the inductive claim.

Clearly, the case $n = 2$ is completely obvious, since there is only one type of rooted trees in $\mathcal{G}_2$. Assume now that our statement holds for indices up to $n - 1$ and we will prove it for $n$.

**Case 1**

Let us first assume that we are in the easier case when there exist $r_{u_0}$ for $1 \leq i_0 \leq \# - 1$ and $r_{v_0}$ for $1 \leq j_0 \leq \# - 1$ so that
\[ r_{u_0} = r_{v_0} := l_0. \]
In this case, define $G'_1$ and $G'_2$ to be the minimal subtrees of $G_1$ and $G_2$ respectively whose leaves are only those indexed from 1 to $l_0$. Similarly, define $G''_1$ and $G''_2$ to be the minimal subtrees whose leaves are those indexed from $l_0 + 1$ to $n$. It is not difficult to remark that the roots of $G'_1$ and $G''_1$ are either equal to the root $u$ of $G_1$ or they are sons of $u$. Similarly, the roots of $G'_2$ and $G''_2$ are either equal to the root $v$ of $G_2$, or they are sons of $v$. As a consequence of this fact, we are facing several subcases.

\[ ^{15} \text{And we add this "refinement property" to the induction hypothesis.} \]
Case 1. Assume that $G'_1$ and $G''_1$ have the same root with $G_1$ and that $G'_2$ and $G''_2$ have the same root with $G_2$.

Then, for a fixed $k$, one can write (using (38))

$$m^k_{G_1, n_1}((\xi_i)_{i=1}^n) = \sum_{n_1} C_{n_1} m^k_{G_1, n_1}((\xi_i)_{i=1}^n)$$

and similarly

$$m^k_{G_2, n_2}((\xi_i)_{i=1}^n) = \sum_{n_2} C_{n_2} m^k_{G_2, n_2}((\xi_i)_{i=1}^n).$$

Using (37) (in the case when the vertex is either the root of $G_1$ or $G_2$) and (38) one can further split $m^k_{G_1, n_1}$ and $m^k_{G_2, n_2}$ naturally as

$$m^k_{G_1, n_1} = \sum_{Q_u} m^k_{G_1, n_1}$$

and

$$m^k_{G_2, n_2} = \sum_{Q_v} m^k_{G_2, n_2}.$$

Fix now $n_1, n_2, Q_u, Q_v$ and consider the corresponding product term

$$m^k_{G_1, n_1} \cdot m^k_{G_2, n_2}. \quad (53)$$

Observe that for every fixed $Q_u$ there are at most $O(1)$ quasi-cubes $Q_v$ for which the above product is not identically equal to zero. One can then rewrite $m^k_{G_1, n_1}$ and $m^k_{G_2, n_2}$ more explicitly as

$$m^k_{G_1, n_1} = \left[ \Phi^{n_1}_{Q_u, 1} \left( \sum_{l \in I_{l_1}} \xi_l \right) \ldots \Phi^{n_1}_{Q_u, 0} \left( \sum_{l \in I_{l_0}} \xi_l \right) \right] \cdot \left[ \Phi^{n_1}_{Q_u, 1} \left( \sum_{l \in I_{l_0+1}} \xi_l \right) \ldots \Phi^{n_1}_{Q_u, #} \left( \sum_{l \in I_{l_q}} \xi_l \right) \right] \cdot \left[ \Phi^{n_2}_{Q_u, 1} \left( \sum_{l \in I_{l_1}} \xi_l \right) \ldots \Phi^{n_2}_{Q_u, 0} \left( \sum_{l \in I_{l_0}} \xi_l \right) \right] \cdot \left[ \Phi^{n_2}_{Q_u, 1} \left( \sum_{l \in I_{l_0+1}} \xi_l \right) \ldots \Phi^{n_2}_{Q_u, #} \left( \sum_{l \in I_{l_q}} \xi_l \right) \right]. \quad (54)$$

and

$$m^k_{G_2, n_2} = \left[ \Phi^{n_2}_{Q_v, 1} \left( \sum_{l \in I_{l_1}} \xi_l \right) \ldots \Phi^{n_2}_{Q_v, 0} \left( \sum_{l \in I_{l_0}} \xi_l \right) \right] \cdot \left[ \Phi^{n_2}_{Q_v, 1} \left( \sum_{l \in I_{l_0+1}} \xi_l \right) \ldots \Phi^{n_2}_{Q_v, #} \left( \sum_{l \in I_{l_q}} \xi_l \right) \right] \cdot \left[ \Phi^{n_2}_{Q_v, 1} \left( \sum_{l \in I_{l_1}} \xi_l \right) \ldots \Phi^{n_2}_{Q_v, 0} \left( \sum_{l \in I_{l_0+1}} \xi_l \right) \right] \cdot \left[ \Phi^{n_2}_{Q_v, #} \left( \sum_{l \in I_{l_q+1}} \xi_l \right) \right]. \quad (55)$$

In particular, the product (53) can be written as

$$m^k_{G_1, n_1} \cdot m^k_{G_2, n_2}.$$
Observe now as before that for our fixed and 

Using these, the expression (56) becomes

for which the previous expression (57) does not vanish.

By using the induction hypothesis, there exist two trees $G'$ and $G''$ and symbols associated to them with the property that

and

for every $k \in \mathbb{Z}$. Denote now by $G$ the rooted tree obtained by concatenating $G'$ and $G''$ together and let $w$ denote the root of $G$.

Using these, the expression (56) becomes

Observe now as before that for our fixed $Q_u$ and $Q_v$ there exist at most $O(1)$ quasi-cubes $Q'_u$ and $Q'_v$ for which the previous expression (57) does not vanish.

Also, by using the fact that $G'$ and $G''$ have the refinement property (with respect to $(G'_1, G'_2)$ and $(G''_1, G''_2)$ respectively) one can successively apply the previous “fixing” Lemma 3.5 and rewrite (57) in the form

$$
\sum_n \tilde{C}_n \tilde{m}_{G',n \rightarrow n} \cdot \tilde{m}_{G'',n \rightarrow n} := \sum_n \tilde{C}_n \tilde{m}_{G',n \rightarrow n} \cdot \tilde{m}_{G'',n \rightarrow n}.
$$
Summing now over all the previously fixed parameters \( n_1, n_2, Q_u, Q_v \) means summing over \( n_1, n_2 \) and \( Q_w \times Q_w' \) and as a consequence, our original product \( m_{G_1}^k \cdot m_{G_2}^k \) can be clearly written as \( m_G^k \) for a certain multiplier \( m_G \) of type \( G \).

Case 1. Assume that the roots of \( G'_1, G'_2 \) and the roots of \( G''_1, G''_2 \) are all sons of \( u \) and \( v \) respectively.

It is then not difficult to see that this can only happen if both \( u \) and \( v \) have precisely two sons \( u_1, u_2 \) and \( v_1, v_2 \). And this means that the root of \( G'_1 \) is \( u_1 \), the root of \( G''_1 \) is \( u_2 \), the root of \( G'_2 \) is \( v_1 \) and the root of \( G''_2 \) is \( v_2 \). Using the same notations as before, this time one can write \(^{17}\)

\[
m_{G_1, n_1}^k = \left[ \Phi_{Q_1, 1}^{n_1} \left( \sum_{l=1}^{l_0} \xi_l \right) \right] \cdot \left[ \Phi_{Q_2, 2}^{n_1} \left( \sum_{l=l_0+1}^{n} \xi_l \right) \right] \cdot \left[ \sum_{k_1' < k} m_{G'_1, n_1}^k (\xi_{l_1}^{(1)}) \right] \cdot \left[ \sum_{k_1' < k} m_{G''_1, n_1}^k (\xi_{l_1}^{(n)}) \right]
\]

and

\[
m_{G_2, n_2}^k = \left[ \Phi_{Q_1, 1}^{n_2} \left( \sum_{l=1}^{l_0} \xi_l \right) \right] \cdot \left[ \Phi_{Q_2, 2}^{n_2} \left( \sum_{l=l_0+1}^{n} \xi_l \right) \right] \cdot \left[ \sum_{k_2' < k} m_{G'_2, n_2}^k (\xi_{l_2}^{(1)}) \right] \cdot \left[ \sum_{k_2' < k} m_{G''_2, n_2}^k (\xi_{l_2}^{(n)}) \right].
\]

In this case, it is very easy to remark that when one considers the product \( m_{G_1, n_1}^k \cdot m_{G_2, n_2}^k \) the terms of the previous two expressions match each other perfectly (there is no need of the “fixing” lemma this time) and the induction hypothesis can be applied twice, solving the problem.

Case 1. Assume that we are in a “mixed case” when the roots of \( G'_1, G'_2 \) are sons of \( u \) and \( v \) respectively and the roots of \( G''_1, G''_2 \) coincide with \( u \) and \( v \) respectively. It is not difficult to see that this case can be solved by combining the arguments used for the previous two cases.

Finally, we are left with

Case 1. Assume that we are in a “skewed situation” now, when for instance the root of \( G'_1 \) is a son of \( u \), the root of \( G'_2 \) is a son of \( v \), the root of \( G''_1 \) is \( u \) and the root of \( G''_2 \) is \( v \). It is easy to see that in fact the root of \( G'_1 \) is \( u_1 \) while the root of \( G'_2 \) is \( v_1 \). As a consequence, the two multipliers \( m_{G_1, n_1}^k \) and \( m_{G_2, n_2}^k \) become

\[
m_{G_1, n_1}^k = \left[ \Phi_{Q_1, 1}^{n_1} \left( \sum_{l=1}^{l_0} \xi_l \right) \right] \cdot \left[ \Phi_{Q_2, 2}^{n_1} \left( \sum_{l=l_0}^{n} \xi_l \right) \right] \cdot \left[ \sum_{k_1' < k} m_{G'_1, n_1}^k (\xi_{l_1}^{(1)}) \right] \cdot \left[ \sum_{k_1' < k} m_{G''_1, n_1}^k (\xi_{l_1}^{(n)}) \right]
\]

and

\[
m_{G_2, n_2}^k = \left[ \Phi_{Q_1, 1}^{n_2} \left( \sum_{l=1}^{l_0} \xi_l \right) \right] \cdot \left[ \Phi_{Q_2, 2}^{n_2} \left( \sum_{l=l_0}^{n} \xi_l \right) \right] \cdot \left[ \sum_{k_2' < k} m_{G'_2, n_2}^k (\xi_{l_2}^{(1)}) \right] \cdot \left[ \sum_{k_2' < k} m_{G''_2, n_2}^k (\xi_{l_2}^{(n)}) \right].
\]

Then, one observes that in order for the product between \( m_{G_1, n_1}^k \) and \( m_{G_2, n_2}^k \) to be nonzero, one must have \( k_1' \sim k \) and also \( k_2' \sim k \). But then, the whole case becomes similar to the previously studied Case 1.\(^{18}\)

\(^{17}\)This time by \( m_{G, n}^k \) we denote the symbol obtained from the formula (38) corresponding to \( m_{G, n}^k \), by taking only the sums and products associated to the vertices of \( G' \); and all the other symbols are defined similarly.

\(^{18}\)Of course, all the other possible “skewed” cases can be treated similarly.
Case 2

Assume now that \( r_{u_i} \neq r_{v_j} \) for any \( 1 \leq i \leq \# - 1 \) and \( 1 \leq j \leq \# - 1 \). In this case we claim that one can construct two other rooted trees \( \text{Ret}(G_1) \) and \( \text{Ret}(G_2) \) (a “retract” of \( G_1 \) and a “retract” of \( G_2 \)) having the property that the pair \( (\text{Ret}(G_1), \text{Ret}(G_2)) \) satisfies the condition of Case 1 and also such that

\[
m_{G_1} \cdot m_{G_2} = m_{\text{Ret}(G_1)} \cdot m_{\text{Ret}(G_2)}
\]

for certain symbols \( m_{\text{Ret}(G_1)} \) and \( m_{\text{Ret}(G_2)} \) of type \( \text{Ret}(G_1) \) and \( \text{Ret}(G_2) \) respectively. Clearly, (58) allows us to reduce the general Case 2 to the Case 1 discussed before.

The idea of proving the claim is very natural. In order for \( m_{G_1} \cdot m_{G_2} \) to be non-zero at a given point \( (\xi_l)_{l=1}^n \) one must have

\[
|I_{r_{u_1}}| \sim ... \sim |I_{r_{u_{\#-1}}}| \gg |I|
\]

for any \( l \neq r_{u_1}, ..., r_{u_{\#-1}} \) and also

\[
|I_{r_{v_1}}| \sim ... \sim |I_{r_{v_{\#-1}}}| \gg |I|
\]

for any other \( l \neq r_{v_1}, ..., r_{v_{\#-1}} \). In particular, one has to have

\[
|I_{r_{u_1}}| \sim ... \sim |I_{r_{u_{\#-1}}}| \sim |I_{r_{v_1}}| \sim ... \sim |I_{r_{v_{\#-1}}}| \gg |I|
\]

for any other \( l \) different than all these indices. Intuitively, it is clear that all these conditions should induce many equivalences between the summation indices which appear in the definitions of \( m_{G_1} \) and \( m_{G_2} \) in (38) and this should allow one to simplify the trees.

Denote by \( S_1 \) the set

\[
S_1 := \{r_{u_1}, ..., r_{u_{\#-1}}\}
\]

and by \( S_2 \) the set

\[
S_2 := \{r_{v_1}, ..., r_{v_{\#-1}}\}.
\]

We will describe the construction of \( \text{Ret}(G_1) \), the one for \( \text{Ret}(G_2) \) being similar.

The root of \( \text{Ret}(G_1) \) will be the same as the one of \( G_1 \) itself, but the sons will be different. They are selected from the former vertices of \( G_1 \) as follows.

First, we look at all the sons of \( u \), namely \( u_1, ..., u_\# \) and select those \( u_j \) having the property that the sets \( \overline{I}_{u_j} := \{I_{u_j}, r_{u_j}\} \cap \mathbf{N} \) do not contain any element from \( S_2 \), i.e. \( \overline{I}_{u_j} \cap S_2 = \emptyset \).

At the second step we are left with those not selected sons of \( u \). Consider the sons of all of them. If \( v \) is such a new son (therefore a grandson of \( u \)) having the property that \( \overline{I}_{v} \cap S_2 = \emptyset \) then we select it, if not, we do not. Clearly, this selection procedure ends after a finite number of steps producing the selected vertices \( V_{\text{select}}^{G_1} \).

As we already mentioned, the vertices in this set will be (by definition) the sons of the top of \( \text{Ret}(G_1) \) and then the rest of the tree is constructed by simply “copying and pasting” the subtrees of \( G_1 \) whose roots are these selected vertices in \( V_{\text{select}}^{G_1} \), see Figure 5 for a particular case.

It is not difficult to observe that by construction, one has that the sets \( (I_v)_{v \in V_{G_1}} \) form a partition of \( \{1, ..., n\} \) and also that
$S_1 \cup S_2 \subseteq \{ r_v : v \in V_{G_1}^{select}, r_v \neq n \}$
and a similar inclusion holds for $G_2$. This shows that $Ret(G_1)$ and $Ret(G_2)$ are indeed as in Case 1.

If in the definition of $m_{G_1}$ one takes into account the new constraints induced by the relations (59), one obtains a new formula which is clearly a multiplier of type $Ret(G_1)$ which we call $m_{Ret(G_1)}$. Similarly, one defines $m_{Ret(G_2)}$. There is only one technical issue left to be solved. If one looks carefully at the products corresponding to the top of $Ret(G_1)$ (in the definition (38)), one sees not only the standard expressions of the form

$$\prod_{v \in V_{G_1}^{select}} \Phi^k_v \left( \sum_{l \in I_v} \xi_l \right)$$

but also extra terms of type

$$\Phi^k_w \left( \sum_{l \in I_w} \xi_l \right)$$

coming from various other vertices $w$. However, it is not difficult to see that by construction, one always has that the sets $I_w$ can be written as unions of $I_v$ for various $v \in V_{G_1}^{select}$ and as
a consequence, the issue is easily solved by repeatedly applying the previous “fixing” Lemma 3.5.

All of these show that indeed our initial symbol $\chi_{\xi_1<\ldots<\xi_n}$ can be written as

$$
\chi_{\xi_1<\ldots<\xi_n} = \sum_{G} m_{G}
$$

for various multipliers $m_{G}$ of type $G$. The reason for which we like this decomposition will be clearer in the next section.

![Figure 6. The rooted trees of $T_4$.](image)

4. Models and the main reduction

In this section we introduce some discrete model operators deeply related to our original operators $T_n$ in (3) and show that in order to prove our main Theorem 1.3 it is enough to prove it for these model operators.

We first need to recall certain definitions from earlier papers [11] - [17].

**Definition 4.1.** Let $d \geq 1$. A tile is a rectangle $P = I_P \times \omega_P$ of area one where $I_P$ is a dyadic interval and $\omega_P$ is a shifted dyadic interval. A vector tile of dimension $d$ is a $d$-tuple $P = (P_1, \ldots, P_d)$ where each $P_i$ for $1 \leq i \leq d$ is a tile and with $I_{P_1} = \ldots = I_{P_d} (= I_P)$. We will
sometimes refer to the tiles in the \( i \) th position as being \( i \)-tiles. The intervals \( I_P \) are called the
time intervals of the tiles \( P \) and the intervals \( \omega_P \) are called the frequency intervals of the tiles \( P \).
Similarly, the quasi-cube \( \omega_P := \omega_{P_1} \times \ldots \times \omega_{P_d} \) is called the frequency cube of the vector tile \( P \).

**Definition 4.2.** A set \( \mathcal{P} \) of vector tiles of dimension \( d \) is said to be sparse if and only if the
collection \( \{ \omega_P : P \in \mathcal{P} \} \) of quasi-cubes is sparse.

**Definition 4.3.** Let \( P \) and \( P' \) be tiles. We write \( P' < P \) if \( I_{P'} \subseteq I_P \) and \( 3 \omega_{P'} \subseteq 3 \omega_P \), and \( P' \leq P \) if \( P' < P \) or \( P' = P \). We also write \( P' \leq P \) if \( I_{P'} \subseteq I_P \) and \( C \omega_{P'} \subseteq C \omega_P \) where \( C > 0 \) is a large
fixed constant. Finally, we write \( P' \lesssim P \) if \( P' \leq P \) but \( P' \not\leq P \).

**Definition 4.4.** A collection \( \mathcal{P} \) of vector tiles of dimension \( d \) is said to have rank 1 if one has the
following properties for all \( P, P' \in \mathcal{P} \):
\begin{enumerate}
  \item If \( P \neq P' \) then \( P_j \neq P'_j \) for all \( 1 \leq j \leq d \).
  \item If \( P'_j \leq P_j \) for some \( 1 \leq j \leq d \), then \( P'_i \leq P_i \) for all \( 1 \leq i \leq d \).
  \item If in addition to \( P'_j \leq P_j \) one also assumes that \( C|I_{P'}| < |I_P| \) (for a fixed large constant
    \( C > 0 \)) then we have \( P'_i \lesssim P_i \) for every \( i \neq j \).
\end{enumerate}

Finally, we also recall

**Definition 4.5.** Let \( P \) be a tile. A wave packet adapted to \( P \) is a function \( \Phi_P \) which has Fourier
support inside the interval \( \frac{9}{10} \omega_P \) and satisfies the estimate
\[
|\Phi_P(x)| \leq |I_P|^{-1/2} \left( 1 + \frac{|x - c_P|}{|I_P|} \right)^{-m} \tag{62}
\]
(where \( c_P \) is the center of the interval \( I_P \)) for all \( m > 0 \), with the implicit constants depending on \( m \).

Thus, heuristically, \( \Phi_P \) is \( L^2 \)-normalized and supported in \( P \).

Sometimes we will also use the notation \( \tilde{\chi}_I \) for the bump function defined by
\[
\tilde{\chi}_I(x) := (1 + \frac{\text{dist}(x, I)}{|I|})^{-10}. \tag{63}
\]

Having all these definitions at our disposal, we can start the description of our model operators.
They will be associated to arbitrary rooted trees \( G \in \mathcal{G}_n \). We will define them inductively,
with respect to their height \( h \).

Let \( G \) be a rooted tree of height 1. Then, a discrete model operator of type \( G \), is an \( n \)-linear
operator of the form
\[
T^G(f_1, \ldots, f_n) := \int_0^1 \sum_{P \in \mathcal{P}_G} \frac{1}{|I_P|^{a+1}} \langle f_1, \Phi_{P_1}^{1,\alpha} \rangle \cdots \langle f_n, \Phi_{P_n}^{n,\alpha} \rangle \Phi_{P_{n+1}}^{n+1,\alpha} \, d\alpha \tag{64}
\]
where \( \mathcal{P}_G \) is an arbitrary finite collection of rank 1 of vector tiles of dimension \( n + 1 \) and \( \Phi_{P_i}^{i,\alpha} \)
are wave packets adapted to the tiles \( P_i \) for \( i = 1, \ldots, n+1 \), uniformly for \( \alpha \in [0, 1] \).

Similarly, one defines model operators associated to rooted trees of height 1 having an\n
arbitrary number of leaves. Then, if \( I \) is a dyadic interval, we define \( T^G_{\mathcal{P}_I} \) to be the multi-linear
operator defined by

26
prove this inductively, let us first assume that

define the model operators

various wave packets considered.

that the following inequalities hold

all of these trees have height at most

denoted the set of indices corresponding to the leaves which are descendents of the root of

We also adopt the convention that if a certain
model operator of type

precisely # sons. Denote by

Suppose now that we know how to define model operators

are wave packets adapted to

similarly by

In particular,

To see why this claim is true, fix

We now claim that our main Theorem 1.3 can be in fact reduced to the following

\begin{align}
T^G_i(f_1, \ldots, f_n) := \int_0^1 \sum_{p \in \mathcal{P}_G; |p| \leq |I|} \frac{1}{|p|^2} \langle f_1, \Phi^{1,\alpha}_{p_1} \rangle \cdots \langle f_n, \Phi^{n,\alpha}_{p_n} \rangle \Phi^{n+1,\alpha}_{p_{n+1}} \, d\alpha.
\end{align}

(65)

Suppose now that we know how to define model operators $T^G$ and $T^G_{|I|}$ associated to rooted trees of height $h - 1$ and an arbitrary number of leaves and we will describe the definition of a model operator associated to a rooted tree of height $h$.

Fix $G$ of height $h$ having an arbitrary number of leaves $L$. Assume that the root of it has precisely # sons. Denote by $G_1, \ldots, G_\#$ the subtrees of $G$ whose roots are all these sons. Clearly, all of these trees have height at most $h - 1$ and by the induction hypothesis we know how to define the model operators $T^{G_i}((f_i)_{i \in I_i})$ and also $T^{G_i}_{|I_i|}((f_i)_{i \in I_i})$ for $1 \leq i \leq \#$. By $I_j$ we simply denoted the set of indices corresponding to the leaves which are descendents of the root of $G_j$.

We also adopt the convention that if a certain $G_j$ is a leave then $T^G = T^G_{|I|} = \text{id}$. Then, by a model operator of type $G$ we mean an expression of the form

\begin{align}
T^G(f_1, \ldots, f_L) := \int_0^1 \sum_{p \in \mathcal{P}_G} \frac{1}{|p|^2} (T^G_i((f_i)_{i \in I_1}), \Phi^{1,\alpha}_{p_1}) \cdots (T^G_n((f_i)_{i \in I_n}), \Phi^{n,\alpha}_{p_n}) \Phi^{n+1,\alpha}_{p_{n+1}} \, d\alpha
\end{align}

(66)

where as before $\mathcal{P}_G$ is a finite collection of rank 1 of vector tiles of dimension $\# + 1$ and $\Phi_{p_i}^{i,\alpha}$ are wave packets adapted to $P_i$ for every $1 \leq i \leq \#$, uniformly in $\alpha$. Finally, one defines $T^G_{|I|}$ similarly by

\begin{align}
T^G_{|I|}(f_1, \ldots, f_L) := \int_0^1 \sum_{p \in \mathcal{P}_G; |p| \leq |I|} \frac{1}{|p|^2} (T^G_i((f_i)_{i \in I_1}), \Phi^{1,\alpha}_{p_1}) \cdots (T^G_n((f_i)_{i \in I_n}), \Phi^{n,\alpha}_{p_n}) \Phi^{n+1,\alpha}_{p_{n+1}} \, d\alpha
\end{align}

(67)

We now claim that our main Theorem 1.3 can be in fact reduced to the following

**Theorem 4.6.** Let $G \in \mathcal{G}_n$, let $\epsilon > 0$ be a small number and let $\alpha_1, \ldots, \alpha_n \in (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$. Then, for every measurable sets $F_1, \ldots, F_n$ of finite measure, every functions $f_i$ having the property that $f_i \leq \chi_{F_i}$ for $1 \leq i \leq n$, and every $F$ with $|F| \sim 1$, there exists a subset $F'$ of $F$ with $|F'| \sim 1$ such that the following inequalities hold

\begin{align}
| \int_{\mathbb{R}} T^G(f_1, \ldots, f_n)(x) \chi_{F'}(x) \, dx | \lesssim |F_1|^{\alpha_1} \cdots |F_n|^{\alpha_n}
\end{align}

(68)

where the implicit constants can be chosen to be independent on all the cardinalities of the finite sets of vector-tiles which implicitly appear in the definition of $T^G$ and also independent on the various wave packets considered.

To see why this claim is true, fix $G \in \mathcal{G}_n$ and pick $m_G$ a multiplier of type $G$ as defined in the previous section. We will see in what follows that the corresponding multi-linear operator $T_{m_G}$ can in fact be written as an weighted average of model operators of type $T^G$. Since we will prove this inductively, let us first assume that $G$ is a rooted tree of height 1.

In particular, $m_G$ is of the form

\begin{align}
m_G(\xi_1, \ldots, \xi_n) = \sum_n C_n \sum_Q \Phi_{Q_1,n_1}(\xi_1) \cdots \Phi_{Q_n,n_n}(\xi_n)
\end{align}

(67)
where as usual \((C_n)_n\) is a rapidly decreasing sequence and the inner summation runs over shifted dyadic quasi-cubes adapted to the region defined by the inequalities \(\xi_1 < \ldots < \xi_n\) and \(|\xi_1 - \xi_2| \sim \ldots \sim |\xi_{n-1} - \xi_n|\) in the sense of Definition 3.1.

Fix now \(n\) and consider only the inner sum. As before, it can be “completed” and rewritten as

\[
\sum_n \Phi_{Q,n,1}(\xi_1) \ldots \Phi_{Q,n,n}(\xi_n) \Phi_{Q_{n+1},n+1}(\xi_1 + \ldots + \xi_n)
\]

for an appropriate choice of a wave packet \(\Phi_{Q_{n+1},n+1}\). As a consequence, the expression

\[
\int_\mathbb{R} T_{mc}(f_1, \ldots, f_n)(x) f_{n+1}(x) dx
\]

becomes

\[
\sum_n C_n \sum_Q \int_\mathbb{R} \tilde{f}_1(\xi_1) \ldots \tilde{f}_n(\xi_n) e^{2\pi i (\xi_1 + \ldots + \xi_n)} d\xi_1 \ldots d\xi_n f_{n+1}(x) dx =
\]

\[
\sum_n C_n \sum_Q \int_\mathbb{R} \tilde{f}_1(\xi_1) \Phi_{Q,1,1}(\xi_1) \ldots \tilde{f}_n(\xi_n) \Phi_{Q,n,n}(\xi_n) \cdot 
\]

\[
\Phi_{Q_{n+1},n+1}(\xi_1 + \ldots + \xi_n) \tilde{f}_{n+1}(\xi_1 + \ldots + \xi_n) d\xi_1 \ldots d\xi_n :=
\]

\[
\sum_n C_n \sum_Q \int_\mathbb{R} \tilde{f}_1(\xi_1) \Phi_{Q,1,1}(\xi_1) \ldots \tilde{f}_n(\xi_n) \Phi_{Q,n,n}(\xi_n) \cdot 
\]

\[
\tilde{\Phi}_{Q_{n+1},n+1}(\xi_1 + \ldots + \xi_n) \tilde{f}_{n+1}(\xi_1 + \ldots + \xi_n) d\xi_1 \ldots d\xi_n =
\]

\[
\sum_n C_n \sum_Q \int_{\lambda_1 + \ldots + \lambda_{n+1} = 0} (f_1 * \Phi_{Q,1,1}(\lambda_1)) \ldots (f_n * \Phi_{Q,n,n}(\lambda_n))(f_{n+1} * \Phi_{Q_{n+1},n+1}(\lambda_{n+1})) d\lambda =
\]

\[
\sum_n C_n \sum_Q \int_\mathbb{R} (f_1 * \Phi_{Q,1,1}(x)) \ldots (f_n * \Phi_{Q,n,n}(x))(f_{n+1} * \Phi_{Q_{n+1},n+1}(x)) dx.
\]

Fix now \(Q\) with \(l(Q) \sim 2^k\) and look at the corresponding inner term in the previous expression. By making the change of variables \(x = 2^{-k}y\) that term becomes

\[
2^{-k} \int_\mathbb{R} (f_1 * \Phi_{Q,1,1}^v)(2^{-k}y) \ldots (f_n * \Phi_{Q,n,n}^v)(2^{-k}y)(f_{n+1} * \Phi_{Q_{n+1},n+1}^v)(2^{-k}y) dy =
\]

\[
2^{-k} \int_0^1 \sum_{l \in \mathbb{Z}} (f_1 * \Phi_{Q,1,1}^v)(2^{-k}(l + \alpha)) \ldots (f_n * \Phi_{Q,n,n}^v)(2^{-k}(l + \alpha))(f_{n+1} * \Phi_{Q_{n+1},n+1}^v)(2^{-k}(l + \alpha)) d\alpha.
\]

Now, every generic term of the form \((f_j * \Phi_{j,n,j}^v)(2^{-k}(l + \alpha))\) can be written as

\[
(f_j * \Phi_{j,n,j}^v)(2^{-k}(l + \alpha)) = \int_\mathbb{R} f_j(y) \Phi_{j,n,j}^v(2^{-k}l + 2^{-k}\alpha - y) dy := \int_\mathbb{R} f_j(y) \Phi_{j,n,j}^\sim(y - 2^{-k}l - 2^{-k}\alpha) dy =
\]
\[
\int_{\mathbb{R}} f_j(y) \Phi_{j,n}^\gamma(y - 2^{-k}l - 2^{-k}a) dy := 2^{k/2} \langle f_j, \Phi_{p_j}^{jn,a} \rangle
\]

where \( P_j \) is the tile \( P_j := 2^{-k}[l, l + 1] \times Q \) for \( j = 1, ..., n \) and \( P_j := 2^{-k}[l, l + 1] \times (-Q) \) for \( j = n + 1 \). It is now clear that putting together all these calculations, \( T_m \) can be written indeed as an weighted average of discrete model operators of the type described before.

Assume now that this is true for every multiplier \( m_G \) associated with trees of height \( h - 1 \) and we want to demonstrate it for symbols \( m_G \) associated to trees of height \( h \). Using (38) we can write any such a multiplier as

\[
m_G = \sum_{n} C_n \sum_{Q} \sum_{k_1, \ldots, k_\# < Q} m_{G_1, n}^k ((\xi_1)_{i \in I_1}) \ldots m_{G_\#, n}^k ((\xi_1)_{i \in I_\#}) \Phi_{Q_1, 1}^1 (\sum_{l \in I_1} \xi_l) \ldots \Phi_{Q_\#, 1} (\sum_{l \in I_\#} \xi_l)
\]

where we implicitly assumed that \( G \) has \# sons.\(^{19}\)

Fix \( n \) and consider only the inner summation where we suppress for simplicity the dependence on \( n \). Fix also \( Q \) so that \( l(Q) \sim 2^k \) then fix \( k_1, \ldots, k_\# < k \) and consider only the corresponding term determined by these fixed indices. As before, the \( n + 1 \)-linear form associated with the \( n \)-linear operator given by such a symbol is equal to (after the usual “completion”)

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} m_{G_1}^k ((\xi_1)_{i \in I_1}) \ldots m_{G_\#}^k ((\xi_1)_{i \in I_\#}) \Phi_{Q_1, 1}^1 (\sum_{l \in I_1} \xi_l) \ldots \Phi_{Q_\#, 1}^1 (\sum_{l \in I_\#} \xi_l).
\]

\[
\cdot \tilde{f_1}(\xi_1) \ldots \tilde{f_n}(\xi_n) e^{2\pi i (\xi_1 + \ldots + \xi_n)} d\xi_1 \ldots d\xi_n f_{n+1}(x) dx =
\]

\[
\int_{\mathbb{R}^n} m_{G_1}^k ((\xi_1)_{i \in I_1}) \ldots m_{G_\#}^k ((\xi_1)_{i \in I_\#}) \Phi_{Q_1, 1}^1 (\sum_{l \in I_1} \xi_l) \ldots \Phi_{Q_\#, 1}^1 (\sum_{l \in I_\#} \xi_l).
\]

\[
\cdot \tilde{f_1}(\xi_1) \ldots \tilde{f_n}(\xi_n) \tilde{f}_{n+1}(\xi_1 + \ldots + \xi_n) d\xi_1 \ldots d\xi_n :=
\]

\[
\int_{\mathbb{R}^n} m_{G_1}^k ((\xi_1)_{i \in I_1}) \ldots m_{G_\#}^k ((\xi_1)_{i \in I_\#}) \Phi_{Q_1, 1}^1 (\sum_{l \in I_1} \xi_l) \ldots \Phi_{Q_\#, 1}^1 (\sum_{l \in I_\#}) \Phi_{Q_{n+1}, 1}^1 (\xi_1 + \ldots + \xi_n),
\]

\[
\cdot \tilde{f_1}(\xi_1) \ldots \tilde{f_n}(\xi_n) \tilde{f}_{n+1}(\xi_1 + \ldots + \xi_n) d\xi_1 \ldots d\xi_n =
\]

\[
\int_{A_1 + \ldots + A_{n+1} = 0} T_{m_{G_1}}^k ((f_1)i \in I_1) (A_1) \ldots T_{m_{G_\#}}^k ((f_1)i \in I_\#) (A_\#) \Phi_{Q_1, 1}(A_1) \ldots \Phi_{Q_\#, 1}(A_\#) \tilde{f}_{n+1}(A_{n+1}) \tilde{\Phi}_{Q_{n+1}, 1}(A_{n+1}) d\lambda =
\]

\[
\int_{A_1 + \ldots + A_{n+1} = 0} (T_{m_{G_1}}^k ((f_1)i \in I_1) * \Phi_{Q_1, 1}^\gamma(A_1) \ldots T_{m_{G_\#}}^k ((f_1)i \in I_\#) * \Phi_{Q_\#, 1}^\gamma(A_\#)) \cdot (f_{n+1} * \tilde{\Phi}_{Q_{n+1}, 1}^\gamma(A_{n+1})) d\lambda =
\]

\[
\int_{\mathbb{R}} (T_{m_{G_1}}^k ((f_1)i \in I_1) * \Phi_{Q_1, 1}^\gamma(x) \ldots T_{m_{G_\#}}^k ((f_1)i \in I_\#) * \Phi_{Q_\#, 1}^\gamma(x)) \cdot (f_{n+1} * \tilde{\Phi}_{Q_{n+1}, 1}^\gamma(x)) d\lambda.
\]

\(^{19}\)As usual, if \( u \) is the root of \( G \) and \( u_1, \ldots, u_\# \) are the sons of \( u \), we denote by \( G_1, \ldots, G_\# \) the subtrees of \( G \) whose roots are all these sons. Then, for simplicity, we also denoted by \( I_j \) the previously defined sets of indices \( I_{u_j} \) for \( j = 1, \ldots, \# \).
At this point, one can discretize as usual once again in the $x$ variable, to obtain an average (over $\alpha$) of expressions of type

$$
\sum_{\vec{P}} \frac{1}{|I_{\vec{P}}|} \langle T_{m_{G1}} (\{f_i\}_{i \in I_1}), \Phi_{P_1}^{1,\alpha} \rangle \cdots \langle T_{m_{G#}} (\{f_i\}_{i \in I_#}), \Phi_{P_#}^{#,\alpha} \rangle \cdot \langle f_{n+1}, \Phi_{P_{#+1}}^{#+1,\alpha} \rangle
$$

(71)

where the sum runs over vector tiles $\vec{P}$ so that $|\omega_{\vec{P}}| \sim 2^k$ for every $j = 1, \ldots, # + 1$. Using now the induction hypothesis and the fact that $k_1, \ldots, k_# << k$, it follows that indeed our operator $T_{mg}$ can be written as an weighted average of discrete operators of the form (66) as desired.

More specifically, we have seen that every $T_{mg}$ can be written as $T_{mg} = \sum_n D_n T_{nG}$ where $(D_n)_n$ is a rapidly decreasing sequence indexed over a countable set, while $T_{nG}$ is a discrete operator of the type (66). The only difference is that in its case, the corresponding sum in (66) may be infinite. Using now Theorem 4.6, scaling invariance, the interpolation theory from [11] and a standard limiting argument, it follows that each $T_{nG}$ is bounded from $L^2 \times \ldots \times L^2$ into $L^2/n$ with bounds which are independent on $n$. This shows that $T_{mg}$ itself satisfies the same estimates, which proves our main Theorem 1.3. It is therefore enough to prove Theorem 4.6 only.

5. Proof of Theorem 4.6

First, we need to recall several definitions from some of our earlier work [11] - [16]. We will also assume from now on that all our collections of vector-tiles are sparse.

**Definition 5.1.** Let $d \geq 3$ and $\vec{P}$ be a collection of rank 1 vector tiles of dimension $d$. Let also $1 \leq j \leq d$. A subcollection $T \subseteq \vec{P}$ is said to be a $j$-tree if and only if there exists a vector tile $P_T \in \vec{P}$ such that $P_j \leq P_{T,j}$ for all $P \in T$, where $P_{T,j}$ is the $j$th component of $P_T$. The vector tile $P_T$ is called the top of the tree. We write $I_T$ for $I_{P_T}$ and $\omega_{T,j}$ for $\omega_{P_T,j}$ respectively.

Note also that a tree $T$ does not necessarily have to contain its top $P_T$.

**Definition 5.2.** Using the same notations in the previous definition, two trees $T$ and $T'$ are said to be strongly $i$-disjoint ($1 \leq i \leq d$) if and only if

1. $P_i \neq P'_i$ for all $P \in T$ and $P' \in T'$.

2. Whenever $P \in T$ and $P' \in T'$ are such that $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$ then one has $I_{P_i} \cap I_{P'_i} = \emptyset$ and similarly with $T$ and $T'$ reversed.

Note also that if $T$ and $T'$ are strongly $i$-disjoint, then $(I_P \times 2\omega_{P_i}) \cap (I_{P'} \times 2\omega_{P'_i}) = \emptyset$ for all $P \in T$ and $P' \in T'$.

It is also important to point out that if $T$ is an $i$-tree, then for all $P, P' \in T$ and $j \neq i$, either

$$
\omega_{P_j} = \omega_{P'_j}
$$

or
\[ 2\omega_{P_j} \cap 2\omega'_{P_j} = \emptyset. \]

It is now clear from the previous sections that in order to prove our main theorem we need to be able to estimate generic expressions of the form

\[ \sum_{P \in \vec{P}} \frac{1}{|I_p|^{d/4}} |a_{P_1}^{j} \ldots a_{P_d}^{j}| \]

(72)

where \(\vec{P}\) is a finite collection of rank 1 vector tiles of dimension \(d\) and \((a_{P_j}^{j})_{P \in \vec{P}}\) are complex numbers of the form

\[ a_{P_j}^{j} = \langle T_{|I_p|}^G((f_i)_{i \in I_j}), \Phi_{P_j}^{j} \rangle. \]

The usual way to do this, is by using certain sizes and energies which are very helpful to describe the local behaviour of expressions of type (72). We recall first the following definition from [11].

**Definition 5.3.** Let \(\vec{P}\) be a rank 1 collection of vector tiles of dimension \(d\), \(1 \leq j \leq d\) and let also \((a_{P_j}^{j})_{P \in \vec{P}}\) be a sequence of complex numbers. We define the size of this sequence by

\[ \text{size}_j((a_{P_j}^{j})_{P \in \vec{P}}) := \sup_{T \subseteq \vec{P}} \left( \frac{1}{|I_T|} \sum_{P \in T} |a_{P_j}^{j}|^2 \right)^{1/2} \]

where \(T\) ranges over all trees in \(\vec{P}\) which are \(i\)-trees for some \(i \neq j\).

The following John-Nirenberg type lemma is also very useful (see for instance [11] for a complete proof).

**Lemma 5.4.** Under the same hypothesis of the previous definition, one has

\[ \text{size}_j((a_{P_j}^{j})_{P \in \vec{P}}) \sim \sup_{T \subseteq \vec{P}} \left( \frac{1}{|I_T|} \sum_{P \in T} |a_{P_j}^{j}|^2 \right)^{1/2} \]

where again \(T\) ranges over all trees in \(\vec{P}\) which are \(i\)-trees for some \(i \neq j\).

The following lemma is also known (see for instance [11] for a proof).

**Lemma 5.5.** Let \(f\) be a measurable function. Then, one has

\[ \text{size}_j((f, \Phi_{P_j}^{j})_{P \in \vec{P}}) \leq \sup_{P \in \vec{P}} \frac{\int_{I_p} |f| |\vec{\chi}_{I_p}^M|}{|I_p|} \]

for any positive real number \(M\), where the implicit constant depends on \(M\).

Let us also recall the following definition from [13].

**Definition 5.6.** Using the same notations as before, one defines the energy of the sequence \((a_{P_j}^{j})_{P \in \vec{P}}\) by

\[ \text{energy}_j((a_{P_j}^{j})_{P \in \vec{P}}) := \sup_{n \in \mathbb{Z}} \sup_{T \subseteq \vec{P}} \left( \sum_{T \subseteq \vec{P}} |I_T| \right)^{1/2} \]

for any positive real number \(M\), where the implicit constant depends on \(M\).
where \( \Pi \) ranges over all collections of strongly \( j \)-disjoint trees in \( \tilde{\mathbf{P}} \) (which are \( i \)-trees for some \( i \neq j \)) such that

\[
\left( \frac{1}{|T|} \sum_{P \in T} |a_{P}^{j}|^{2} \right)^{1/2} \geq 2^{n}
\]

for all \( T \in \Pi \) and also satisfying

\[
\left( \frac{1}{|T'|} \sum_{P \in T'} |a_{P}^{j}|^{2} \right)^{1/2} \leq 2^{n+1}
\]

for all sub-trees \( T' \subseteq T \in \Pi \).

It is also not difficult to observe the following lemma [13].

**Lemma 5.7.** For any sequence \( (a_{P}^{j})_{P \in \tilde{\mathbf{P}}} \) there exists a collection \( \Pi \) of strongly \( j \)-disjoint trees (which are \( i \)-trees for some \( i \neq j \)) and complex numbers \( c_{P}^{j} \) for all \( P \in \bigcup_{T \in \Pi} T \) such that

\[
\text{energy}_{j}( (a_{P}^{j})_{P \in \tilde{\mathbf{P}}} ) \sim \sum_{T \in \Pi} \sum_{P \in T} a_{P}^{j} \overline{c_{P}^{j}}
\]

and such that

\[
\sum_{P \in T} |c_{P}^{j}|^{2} \leq \frac{|T|}{\sum_{T \in \Pi} |T|}
\]

for all \( T \in \Pi \) and all subtrees \( T' \subseteq T \).

The following lemma is also well known (see for instance [13]).

**Lemma 5.8.** For any \( f \in L^{2}(\mathbb{R}) \) one has

\[
\text{energy}_{j}( (f, \Phi_{P}^{j})_{P \in \tilde{\mathbf{P}}} ) \leq \|f\|_{2}.
\]

(73)

The following lemma will also play an important role when estimating the energies of various general sequences, later on. For a proof of it see [13].

**Lemma 5.9.** Let \( d_{1}, d_{2} \geq 3 \) and \( \tilde{\mathbf{P}}, \tilde{\mathbf{Q}} \) be rank 1 collections of vector tiles of dimensions \( d_{1} \) and \( d_{2} \) respectively. Let also \( 1 \leq i \leq d_{1} \) and \( 1 \leq j \leq d_{2} \). Consider also two sequences of complex numbers \( (c_{P}^{i})_{P} \) and \( (c_{Q}^{j})_{Q} \) where \( P \) runs inside a collection of strongly \( i \)-disjoint trees which are \( l \)-trees for some \( l \neq i \) and \( Q \) runs inside a collection of strongly \( j \)-disjoint trees which are \( l \)-trees for some \( l \neq j \). Assume also that both of these sequences satisfy the conclusion of the previous Lemma 5.7. Then, one has

\[
\left| \sum_{P,Q:|P| \leq |Q|} c_{P}^{i} c_{Q}^{j} \langle \Phi_{P}^{i}, \Phi_{P}^{i} \rangle \right| \lesssim 1.
\]

(74)

In addition to the above lemma, we need also the following result, which will play a crucial role later on.
Lemma 5.10. Assume that the sequences of complex numbers \((c^i_P)_P\) and \((c^j_Q)_Q\) are precisely as in the previous Lemma 5.9. Assume in addition that there are two subsets \(S_P\) and \(S_Q\) of the real line, so that \(S_P \subseteq S_Q\) and so that every \(P\) satisfies

\[
\frac{\text{dist}(I_P, S_P)}{|I_P|} \sim 2^{k_1}
\]

and every \(Q\) satisfies

\[
\frac{\text{dist}(I_Q, S_Q)}{|I_Q|} \sim 2^{k_2}
\]

for two fixed numbers \(k_1, k_2\) so that \(k_2 >> k_1\).

Then, the corresponding estimate for the left hand side of (74) can be improved to

\[
\left| \sum_{P, Q \mid |I_P| \leq |I_Q|} c^i_P c^j_Q \langle \Phi^j_P, \Phi^j_Q \rangle \right| \lesssim 2^{-Mk_2} \tag{75}
\]

for any positive constant \(M\), where the implicit constant depends on \(M\).

The proof of this lemma is quite delicate and will be presented in the last section of the paper. The main proposition used to estimate expressions of the form (72) is the following.

Proposition 5.11. Let \(\vec{P}\) be a rank 1 collection of vector tiles of dimension \(d \geq 3\). Let also consider arbitrary sequences of complex numbers \((a^j_P)_P\) for \(1 \leq j \leq d\). Then, one has the inequality

\[
\left| \sum_{P \in \vec{P}} \frac{1}{|I_P|} a^1_P \cdots a^d_P \right| \lesssim \prod_{j=1}^{d} \text{size}_j((a^j_P)_{P \in \vec{P}})^{\theta_j} \cdot \text{energy}_j((a^j_P)_{P \in \vec{P}})^{1-\theta_j}
\]

for any \(0 \leq \theta_1, \ldots, \theta_d < 1\) with \(\theta_1 + \ldots + \theta_d = d - 2\) with the implicit constants depending on \((\theta_j)_j\).

Proof

The proof of it is based on the following lemma and its corollary, which have been proven in [13].

Lemma 5.12. Let \(1 \leq j \leq d\), \(\vec{P}'\) be a subset of \(\vec{P}\), \(n \in \mathbb{Z}\) and assume that

\[
\text{size}_j((a^j_P)_{P \in \vec{P}'}) \leq 2^{-n} \text{energy}_j((a^j_P)_{P \in \vec{P}}).
\]

Then, one can decompose \(\vec{P}'\) as \(\vec{P}' = \vec{P}'' \cup \vec{P}'''\) such that

\[
\text{size}_j((a^j_P)_{P \in \vec{P}''}) \leq 2^{-n} \text{energy}_j((a^j_P)_{P \in \vec{P}})
\]

and also such that \(\vec{P}'''\) can be written as a disjoint union of trees in \(\vec{P}\) with the property that

\[
\sum_{T \in \vec{P}} |I_T| \leq 2^n.
\]

By iterating the above lemma we immediately obtain the following Corollary.
Corollary 5.13. Fix $1 \leq j \leq d$. Then, there exists a partition

$$\vec{P} = \bigcup_{n \in \mathbb{Z}} \vec{P}_n^j$$

where for each $n \in \mathbb{Z}$ one has

$$\text{size}_j((a_{P_j}^j)_{P_j \in \vec{P}_n^j}) \leq \min(2^{-n} \text{energy}_j((a_{P_j}^j)_{P_j \in \vec{P}}), \text{size}_j((a_{P_j}^j)_{P_j \in \vec{P}})).$$

Also, we can cover $\vec{P}_n^j$ by a disjoint union $\Pi_{n}^j$ of trees such that

$$\sum_{T \in \Pi_{n}^j} |I_T| \leq 2^n.$$

We can now start the actual proof of our Proposition 5.11.

First, let us observe that for every $l$-tree $T$ one can estimate the corresponding term in the inequality by

$$|\sum_{P \in T} \frac{1}{|P|^d} a_{P_1}^1 \ldots a_{P_d}^d| \leq \sum_{P \in T} \frac{1}{|P|^d} |a_{P_1}^1| |a_{P_d}^d| \leq$$

$$\prod_{k \neq l_1, l_2} (\sup_{P \in T} |a_{P_k}^k|) \cdot (\sum_{P \in T} |a_{P_1}^1|^2)^{1/2} \cdot (\sum_{P \in T} |a_{P_2}^2|^2)^{1/2}$$

for any $l_1 \neq l$ and $l_2 \neq l$. But this is clearly smaller than

$$\prod_{j=1}^d \text{size}_j((a_{P_j}^j)_{P \in T}) \cdot |I_T|.$$ 

Using this simple “tree estimate” and applying $d$ times Corollary 5.13, we can estimate our general left hand side of our inequality by

$$E_1 \ldots E_d \sum_{n_1, \ldots, n_d} 2^{-n_1} \ldots 2^{-n_d} \sum_{T \in \Pi_{n_1, \ldots, n_d}} |I_T|$$

(76)

where $\Pi_{n_1, \ldots, n_d}$ is just the intersection of the collections of trees $\Pi_{n_j}^j$ for $j = 1, \ldots, d$ provided by the Corollary 5.13 and we denoted for simplicity by $E_j := \text{energy}_j((a_{P_j}^j)_{P_j \in \vec{P}})$ and we will also use the notation $S_j$ for $\text{size}_j((a_{P_j}^j)_{P_j \in \vec{P}})$.

One should also observe that as a consequence of the same corollary, the above summations run inside the set of integers $n_1, \ldots, n_d$ for which

$$2^{-n_j} \leq \frac{S_j}{E_j}$$

for $j = 1, \ldots, d$. On the other hand, we also know that

$$\sum_{T \in \Pi_{n_1, \ldots, n_d}} |I_T| \leq \sum_{T \in \Pi_{n_j}^j} |I_T| \leq 2^{n_j}$$

for any $j = 1, \ldots, d$ and as a consequence we can write
\[
\sum_{T \in \Pi_{n_1,...,n_d}} |I_T| \lesssim \min(2^{2n_1}, ..., 2^{2n_d}). \tag{77}
\]

To prove that the proposition holds for any \(0 < \theta_1, ..., \theta_d < 1\), one can use instead of (77) the weaker inequality

\[
\sum_{T \in \Pi_{n_1,...,n_d}} |I_T| \lesssim 2^{2\alpha_1} ... 2^{2\alpha_d} \tag{78}
\]

for any \(0 < \alpha_1, ..., \alpha_d < 1\) so that \(\alpha_1 + ... + \alpha_d = 1\).

In particular, this allows us to estimate further our expression by

\[
E_1...E_d \sum_{n_1,...,n_d} 2^{-n_1} ... 2^{-n_d} 2^{2\alpha_1} ... 2^{2\alpha_d} =
\]

\[
E_1...E_d \sum_{n_1,...,n_d} 2^{-n_1(1-2\alpha_1)} ... 2^{-n_d(1-2\alpha_d)}.
\]

Now, assuming that in addition we also have \(1 - 2\alpha_j > 0\), for every \(j = 1, ..., d\) we obtain the upper bound

\[
E_1...E_d \frac{S_1^{(1-2\alpha_1)}}{E_1} ... \frac{S_d^{(1-2\alpha_d)}}{E_d} =
\]

\[
S_1^{(1-2\alpha_1)} ... S_d^{(1-2\alpha_d)} E_1^{2\alpha_1} ... E_d^{2\alpha_d}.
\]

Since we observe that \((1 - 2\alpha_1) + ... + (1 - 2\alpha_d) = d - 2(\alpha_1 + ... + \alpha_d) = d - 2\), this proves our assertion.

In the case that one of our \(\theta_j\)'s is equal to zero (note that at most one can be zero !), say \(\theta_d = 0\), we can estimate our expression in (76) by

\[
E_1...E_d \sum_{n_1,...,n_d} 2^{-n_1} ... 2^{-n_d} \min(2^{2n_1},...,2^{2n_d}) =
\]

\[
E_1...E_d \sum_{n_1,...,n_d} 2^{-n_1} ... 2^{-n_d-1} \min(2^{n_1},2^{-n_d} \min(2^{2n_1}, ..., 2^{2n_{d-1}})).
\]

Now, if we fix \(n_1, ..., n_{d-1}\) and first sum over \(n_d\) using the elementary inequality

\[
\sum_{n \in \mathbb{Z}} \min(2^n, 2^{-n}) \lesssim a^{1/2}, \tag{79}
\]

we obtain the bound

\[
E_1...E_d \sum_{n_1,...,n_{d-1}} 2^{-n_1} ... 2^{-n_{d-1}} \min(2^{n_1},...,2^{n_{d-1}}) \lesssim
\]

\[
E_1...E_d \sum_{n_1,...,n_{d-1}} 2^{-n_1} ... 2^{-n_{d-1}} 2^{\alpha_1 n_1} ... 2^{\alpha_{d-1} n_{d-1}}
\]

for every \(0 < \alpha_1, ..., \alpha_{d-1} < 1\) with the property that \(\alpha_1 + ... + \alpha_{d-1} = 1\).

After summing the above expression, we obtain the upper bound.
\[ E_1 \ldots E_d \left( \frac{S_1}{E_1} \right)^{1-\alpha_1} \ldots \left( \frac{S_{d-1}}{E_{d-1}} \right)^{1-\alpha_{d-1}} = \]

\[ S_1^{1-\alpha_1} \ldots S_{d-1}^{1-\alpha_{d-1}} E_1^{\alpha_1} \ldots E_{d-1}^{\alpha_{d-1}} E_d \]

which coincides with the desired estimate. 

Having Proposition 5.11 at our disposal, we can now start the proof of Theorem 4.6. Fix \( F_1, \ldots, F_n \) and \( F \) arbitrary measurable sets of finite measure and \( f_1 \leq \chi_{F_1}, \ldots, f_n \leq \chi_{F_n} \). Assume also that \( |F| \sim 1 \). Our goal is to construct a subset \( F' \subseteq F \) with \( |F'| \sim |F| \), so that the corresponding inequality in Theorem 4.6 holds.

First, for every rooted tree \( G \) we will construct inductively an exceptional set \( \Omega_G \) as follows. Assume that \( G \) has height 1 and an arbitrary number of leaves \( L \). Then, the exceptional set \( \Omega_G \) is defined by \(^{20}\)

\[ \Omega_G := \{ x : M(\chi_{F_1})(x) > C|F_1| \} \cup \ldots \cup \{ x : M(\chi_{F_L})(x) > C|F_L| \} \]  \quad \text{(80)}

where \( C > 0 \) is big enough to guarantee that \( |\Omega_G| << 1 \).

Suppose now that we know how to construct such exceptional sets for arbitrary rooted trees of height smaller or equal than \( h \) and we describe the construction of \( \Omega_G \) in the case of an arbitrary rooted tree of height \( h+1 \). Fix such a tree \( G \). Assume that the root of it is \( u \) and that the sons of \( u \) are \( u_1, \ldots, u_\# \). Denote also, as usual, by \( G_i \) the subtree of \( G \) whose root is \( u_i \) for \( 1 \leq i \leq \# \).

Clearly, either \( G_i \) is a leaf or it is a tree of a strictly smaller height. Fix \( 1 \leq i \leq \# \) so that \( G_i \) is not a leaf. Fix also \( (k_v)_{v \in V_{G_i}} \) a vector indexed over the vertices of \( G_i \) whose entries are all positive integers. Denote by \( T_G^{k_v} \) the model operator defined by the same formula as \( T_G^{k_v} \), but where the implicit sums run over the sets \( \bar{P}^{k_v} \) instead of \( \bar{P}^{k_v} \), where by \( \bar{P}^{k_v} \) we denote the collection of all vector-tiles \( P \in \bar{P}^{k_v} \) having the property that \(^{21}\)

\[ 1 + \frac{\text{dist}(I_p, \Omega_{G_i}^c)}{|I_p|} \sim 2^{k_v} \]

Denote also by \( \bar{\Omega}_{G_i} \) the set

\[ \bar{\Omega}_{G_i} := \bigcup_{(k_v)_{v \in V_{G_i}}} \left\{ x : M(T_G^{k_v}((f_i)_{v \in I})) (x) > C \left( \prod_{v \in V_{G_i}} 2^{k_v} \right) \|T_G^{k_v}((f_i)_{v \in I})\|_2 \right\} \]  \quad \text{(81)}

Clearly, \( |\bar{\Omega}_{G_i}| << 1 \) if \( C > 0 \) is a big enough constant. Similarly, one defines \( \bar{\Omega}_{G_j} \) for any other index \( 1 \leq j \leq \# \) for which \( G_j \) is not a leaf.

In case \( G_j \) is a leaf, then instead, we define \( \bar{\Omega}_{G_j} \) by

\[ \bar{\Omega}_{G_j} := \{ x : M(\chi_{F_j})(x) > C|F_j| \} \]

In the end, we define the exceptional set associated to \( G \) by

\(^{20}\)M is the classical Hardy-Littlewood maximal operator

\(^{21}\)As usual, \( G_v \) is the rooted sub-tree whose root is the vertex \( v \) and \( \Omega_G \) is the exceptional set associated to \( G_v \) which exists by the induction hypothesis.
Denote also by \( \Omega_G := \left( \bigcup_{j=1}^{\#} \Omega_{G_j} \right) \bigcup \left( \bigcup_{j:G_j \neq \text{leave}} \Omega_{G_j} \right) \). (82)

Then, we simply define \( F' \) by

\[
F' := F \setminus \Omega_G
\]

and we observe that indeed, \( |F'| \sim 1 \) if all the constants \( C \) involved are large enough.

We are therefore left with estimating the following expression

\[
\int_0^1 \sum_{P \in \mathcal{P}_G} \frac{1}{|P|^{1+\frac{1}{2}}} \langle T^{G_1}_{|P|}((f_i)_{i \in I_1}), \Phi^{1,\alpha}_{P_1} \rangle \cdots \langle T^{G_s}_{|P|}((f_i)_{i \in I_s}), \Phi^{s,\alpha}_{P_s} \rangle \cdot \langle \chi_{F'}, \Phi^{s+1,\alpha}_{P_{s+1}} \rangle d\alpha.
\] (83)

Fix now \((k_v)_{v \in G} \) positive integers. We will assume from now on that all the implicit inner sums in (83) are taken over collections of the form \( \mathcal{P}^{k_v}_{G_v} \) for every \( v \in G \). We will estimate the corresponding term under these restrictions and in the end we will sum over all the vectors \((k_v)_{v \in G} \).

By applying Proposition 5.11 we can estimate (83) by

\[
\sup_{0 \leq \alpha \leq 1} \left[ \text{size}_1 \left( \langle (T^{G_1}_{|P|}((f_i)_{i \in I_1}), \Phi^{1,\alpha}_{P_1} \rangle_p \right) \sup_{0 \leq \alpha \leq 1} \text{energy}_1 \left( \langle (T^{G_1}_{|P|}((f_i)_{i \in I_1}), \Phi^{1,\alpha}_{P_1} \rangle_p \right)^{1-\theta_1} \right] \cdots \left[ \text{size}_s \left( \langle (T^{G_s}_{|P|}((f_i)_{i \in I_s}), \Phi^{s,\alpha}_{P_s} \rangle_p \right) \sup_{0 \leq \alpha \leq 1} \text{energy}_s \left( \langle (T^{G_s}_{|P|}((f_i)_{i \in I_s}), \Phi^{s,\alpha}_{P_s} \rangle_p \right)^{1-\theta_s} \right].
\] (84)

\[
\sup_{0 \leq \alpha \leq 1} \left[ \text{size}_{s+1} \left( \langle \chi_{F'}, \Phi^{s+1,\alpha}_{P_{s+1}} \rangle_p \right) \sup_{0 \leq \alpha \leq 1} \text{energy}_{s+1} \left( \langle \chi_{F'}, \Phi^{s+1,\alpha}_{P_{s+1}} \rangle_p \right)^{1-\theta_{s+1}} \right],
\]

for every positive numbers \( 0 \leq \theta_1, ..., \theta_{s+1} < 1 \) so that \( \theta_1 + \cdots + \theta_{s+1} = \# - 1 \).

We write for simplicity the previous expression as

\[
[S^{\theta_1}_1 E^{1-\theta_1}_1] \cdots [S^{\theta_s}_s E^{1-\theta_s}_s] \cdots [S^{\theta_{s+1}}_{s+1} E^{1-\theta_{s+1}}_{s+1}].
\] (85)

Estimates for \([S^{\theta_1}_1 E^{1-\theta_1}_1] \cdots [S^{\theta_s}_s E^{1-\theta_s}_s] \cdots [S^{\theta_{s+1}}_{s+1} E^{1-\theta_{s+1}}_{s+1}] \).

We concentrate now on estimating the term \([S^{\theta_1}_1 E^{1-\theta_1}_1] \). It will be later on clear that in exactly the same way one can estimate every other term of type \([S^{\theta_j}_j E^{1-\theta_j}_j] \) for \( 1 \leq j \leq \# \).

To fix the notations, we also assume that the sons of \( u_1 \) (which is the root of \( G_1 \)) are \( u_{1,1}, ..., u_{1,1} \). Denote also by \( G'_1 \) the subtree of \( G_1 \) whose root is \( u'_1 \) for \( 1 \leq i \leq \#_1 \).

Estimates for \( E_1 \).

Fix \( \alpha \) for which the supremum in the definition of \( E_1 \) is attained and consider the corresponding expression. We will also suppress for simplicity the dependence on \( \alpha \) in the next formulas since its presence is irrelevant.

By duality, we know that there exists a sequence of complex numbers \((C_{P_i}^1)_{P_i}\) as in Lemma 5.7 so that
\[
E_1 \sim \sum_{p} \langle T_{|p|}^{G_1}((f_i)_{i \in I_1}), \Phi_P \rangle C_P = \\
\sum_{p} \left( \int_0^1 \sum_{Q: |Q| \leq |p|} |I_Q|^{\gamma} \langle T_{|Q|}^{G_1}((f_i)_{i \in I_1}), \Phi_{Q_1} \rangle \cdots \langle T_{|Q|}^{G_1}((f_i)_{i \in I_1}), \Phi_{Q_{k_1}} \rangle \cdot \langle \sum_{p: |p| \leq |Q|} C_P \Phi_P, \Phi_{Q_{k_1+1}} \rangle \rangle \right) C_P = \\
\int_0^1 \sum_{Q: |Q| \leq |p|} \sum_{|Q_1| \leq |Q|} \frac{1}{|Q_1|^{\gamma}} \langle T_{|Q_1|}^{G_1}((f_i)_{i \in I_1}), \Phi_{Q_1} \rangle \cdots \langle T_{|Q_1|}^{G_1}((f_i)_{i \in I_1}), \Phi_{Q_{k_1}} \rangle \cdot \langle \sum_{p: |p| \leq |Q_1|} C_P \Phi_P, \Phi_{Q_{k_1+1}} \rangle \rangle C_P. \
\]  

By applying the same Proposition 5.11, we can estimate this further by

\[
[(S_1^1)^{\beta_1} (E_1^1)^{1-\beta_1}] \cdots [(S_1^{\#_1})^{\beta_{\#_1}} (E_1^{\#_1})^{1-\beta_{\#_1}}] \cdot 1 \cdot E_1^{\#_1+1},
\]

for any \(0 < \beta_1, \ldots, \beta_{\#_1} < 1\) so that \(\beta_1 + \cdots + \beta_{\#_1} = 1 - 1\). By \(S_1^{j}\) and \(E_1^{j}\) we denoted the expressions

\[
S_1^{j} := \sup_{0 \leq \alpha \leq 1} \text{size}_1((T_{|Q|}^{G_1}((f_i)_{i \in I_1}), \Phi_{Q_1}^{j\alpha}))_Q)
\]

and

\[
E_1^{j} := \sup_{0 \leq \alpha \leq 1} \text{energy}_1((T_{|Q|}^{G_1}((f_i)_{i \in I_1}), \Phi_{Q_1}^{j\alpha}))_Q),
\]

for any \(1 \leq j \leq \#_1\). Clearly, \(E_1^{\#_1+1}\) is defined similarly and corresponds to the last term in (86).

To estimate \(E_1^{\#_1+1}\), we fix as before the index \(\alpha\) for which the supremum is attained and consider the corresponding expression. We suppress again the dependence on \(\alpha\) for simplicity, since it is irrelevant to the argument.

By using Lemma 5.7 there exists a sequence of complex numbers \((C_{Q_{k+1}})_{Q}\) having the property that

\[
E_1^{\#_1+1} \sim \sum_{p: |p| \leq |Q|} C_P \Phi_P, \Phi_{Q_{k+1}} \rangle C_P.
\]

Let us also recall that the summation above runs over \(P \in \mathcal{P}_{G_1}^{k_1}\) and \(Q \in \mathcal{P}_{G_1}^{k_1}\). On the other hand, by construction, we also know that \(\Omega_{G_1} \subseteq \Omega_G\) and so \(\Omega_G \subseteq \Omega_{G_1}\).

In particular, this means that by using the previous Lemmas 5.9 and 5.10 we have that \(E_1^{\#_1+1}\) is \(O(1)\) in general, but in the case when \(k_{\#_1} >> k_1\), we have that \(E_1^{\#_1+1} \leq 2^{-Mk_{\#_1}}\) for arbitrary constants \(M > 0\) (with the implicit constant depending on \(M\)).

Estimates for \(S_1^{\#_1}\).

Fix \(\alpha\) for which the supremum in the definition of \(S_1\) is attained and consider the corresponding expression with \(\alpha\) suppressed. We have

\[
S_1 \leq \text{size}_1((T_{|p|}^{G_1}((f_i)_{i \in I_1}), \Phi_P)_p) + \text{size}_1((T_{|p|}^{G_1}((f_i)_{i \in I_1}), \Phi_P)_p) := I + II,
\]

where \(T_{|p|}^{G_1} = \sum_{p: |p| \leq |Q|} (T_{|Q|}^{G_1}((f_i)_{i \in I_1}), \Phi_P)_Q\).

To estimate \(I\), from the definition of the exceptional sets, it is easy to see that one has
\[ I \lesssim (\prod_{v \in V_{G_1}} 2^L_v \cdot 2^{L_v} \cdot ||T^{G_1}(f_{1})_{\ell \in I_1}||_2) . \]

To estimate \( ||T^{G_1}(f_{1})_{\ell \in I_1}||_2 \) we pick \( g \in L^2, \ |g|_2 = 1 \) so that this term becomes equivalent with

\[
\int_0^1 \sum_{Q} {1 \over |Q|^{1/2}} \langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{Q_1} \rangle \cdots \langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{Q_{n}} \rangle \cdot \langle g, \Phi_{Q_{n+1}} \rangle d\alpha
\]

(88)

and as before by using the same Proposition 5.11 together with Lemma 5.8 we get an estimate of the form

\[
[(S_1^{1})^{\beta_1}(E_1^{1})^{1-\beta_1}] \cdots [(S_1^{1})^{\beta_n}(E_1^{1})^{1-\beta_n}] \cdot 1 \cdot 1,
\]

for every \( \beta_1, \ldots, \beta_n \), exactly as before.

To estimate \( II \), pick \( T \) a tree where the corresponding supremum is attained. Then, observe that \( II \) becomes equivalent with

\[
{1 \over |T|} \| (\sum_{P \in T} |\langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{P} \rangle|^2 \over |P|)^{1/2} ||_{1, \infty}
\]

where \( T^{G_1}_{I_0} \) is defined to be the corresponding sum over vector-tiles \( Q \) having the property that there exists a \( P \in T \) so that \( |\omega_{P_{1}}| < |\omega_{Q_{n+1}}| \) and \( \omega_{P_{1}} \cap \omega_{Q_{n+1}} \neq \emptyset \). We denote this set of vector-tiles by \( Q_{T} \). By using Lemma 5.5 we see that the above term is smaller than

\[
{1 \over |T|} \| T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}) \|_{L^1(Q_{T}^{\infty})} =
\]

\[
{1 \over |T|} \int_0^1 \sum_{Q \in Q_{T}} \frac{1}{|Q|^{1/2}} \langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{Q_{1}} \rangle \cdots \langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{Q_{n}} \rangle \cdot \langle h \chi_{T_{1}}, \Phi_{Q_{n+1}} \rangle d\alpha
\]

for some well chosen function \( h \in L^\infty \) with \( ||h||_{\infty} = 1 \).

This can be also written as

\[
{1 \over |T|} \sum_{m=1}^{\infty} \int_0^1 \sum_{Q \in Q_{T}^{m}} \frac{1}{|Q|^{1/2}} \langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{Q_{1}} \rangle \cdots \langle T^{G_1}_{I_0}((f_{1})_{\ell \in I_1}), \Phi_{Q_{n}} \rangle \cdot \langle h \chi_{T_{1}}, \Phi_{Q_{n+1}} \rangle d\alpha
\]

where \( Q_{T}^{m} \) denotes the set of all vector-tiles \( Q \in Q_{T} \) having the property that

\[
m - 1 \leq \text{dist}(I_T, I_Q) \leq m.
\]

Since it is not difficult to see that all these \( Q_{T}^{m} \) sets are trees, we deduce that the above expression can be estimated by

\[
{1 \over |T|} \sum_{m=1}^{\infty} S_1^{1} \cdots S_1^{\beta_m} \sup_{0 \leq \alpha \leq 1} \sup_{Q \in Q_{T}^{m}} \frac{|\langle h \chi_{T_{1}}, \Phi_{Q_{n+1}} \rangle|}{|Q|^{1/2}} \cdot |I_T| \lesssim
\]

(89)
\[
\frac{1}{|I_T|} \sum_{m=1}^{\infty} \frac{1}{m^{10}} S_{m}^{1} \cdots S_{m}^{\#} |I_T| \leq S_{1}^{1} \cdots S_{1}^{\#}.
\]

Putting all these estimates together, we can finish our original estimate for \(S_{1}^{\theta} \cdot E_{1}^{1-\theta}\) as follows:

\[
S_{1}^{\theta} \cdot E_{1}^{1-\theta} \leq (I + II)^{\theta_{1}} \cdot E_{1}^{1-\theta_{1}} \leq I^{\theta_{1}} \cdot E_{1}^{1-\theta_{1}} + II^{\theta_{1}} \cdot E_{1}^{1-\theta_{1}} \leq
\]

\[
C \cdot [(S_{1}^{\theta_{1}} \cdot (E_{1}^{1})^{1-\beta_{1}]} \cdots [(S_{1}^{\theta_{1}} \cdot (E_{1}^{1})^{1-\theta_{1} - v_{1}}) \cdot (E_{1}^{\theta_{1}})^{1-\beta_{1}}]^{1-\theta_{1}} +
\]

\[
C \cdot [(S_{1}^{\theta_{1}} \cdot (E_{1}^{1})^{1-\beta_{1}]} \cdots [(S_{1}^{\theta_{1}} \cdot (E_{1}^{1})^{1-\theta_{1} - v_{1}}) \cdot (E_{1}^{\theta_{1}})^{1-\beta_{1}}]^{1-\theta_{1}} +
\]

\[
C \cdot [(S_{1}^{\gamma_{1}} \cdot (E_{1}^{1})^{1-\gamma_{1}]} \cdots [(S_{1}^{\gamma_{1}} \cdot (E_{1}^{1})^{1-\gamma_{1} - v_{1}}) \cdot (E_{1}^{\gamma_{1}})^{1-\gamma_{1}}],
\]

for some numbers \(0 < \gamma_{1}, ..., \gamma_{\theta_{1}} < 1\).

In other words we showed that the expression \([S_{1}^{\theta_{1}} \cdot E_{1}^{1-\theta_{1}}]\), which corresponds to the subtree \(G_{1}\) (whose root is \(u_{1}\)), can be estimated by a finite sum of products of similar expressions involving terms which correspond to the sons of \(u_{1}\), namely \(u_{1}^{1}, ..., u_{1}^{\#}\). It is not difficult to observe that one can estimate in exactly the same way all the other expressions of the form \([S_{j}^{\theta_{1}} \cdot E_{j}^{1-\theta_{1}}]\) in (85) for \(1 \leq j \leq \#\). Also, it is important to observe that the constants \(C\) above depend as we have seen on the integers \((k_{j})_{v \in V_{j}}\), and also on \(k_{u}\).

Clearly, one can then iterate this procedure further, eventually arriving at estimating the expressions corresponding to the leaves of the rooted tree \(G\). It is then easy to see that in the case when the sequence \((a_{P})_{P}\) is indeed of the form \((\langle f_{j}, \Phi_{P} \rangle)_{P}\), one clearly has

\[
S := \text{size}((\langle f_{j}, \Phi_{P} \rangle)_{P}) \leq 2^{v} \min(|F_{j}|, 1) \leq 2^{\beta} |F_{j}|^{\beta}
\]

for every \(0 < \beta < 1\), where \(v\) is the vertex whose son is the leave indexed “\(j\)”, while

\[
E := \text{energy}((\langle f_{j}, \Phi_{P} \rangle)_{P}) \leq |F_{j}|^{1/2}.
\]

In particular, this implies that any product of the form \(S^{\theta} \cdot E^{1-\theta}\) for some \(0 < \theta < 1\), becomes smaller than

\[
|F_{j}|^{\theta \beta} \cdot |F_{j}|^{(1-\theta)/2} \leq |F_{j}|^{1/2 + \theta(\beta - 1/2)}
\]

and clearly, the exponent \(1/2 + \theta(\beta - 1/2)\) can be made arbitrarily close to \(1/2\) by taking \(\beta\) close to \(1/2\) (no matter which \(\theta\) we face).

Putting all these estimates together and also using the fact that the size of the last sequence \((\langle \chi_{F^{*}}, \Phi_{P}^{\#} \rangle)_{P}\) in (83) is smaller than \(2^{-MK_{u}}\) (for any \(M\) arbitrarily large, with the implicit constants depending on it) while the energy of it is \(O(1)\), we obtain an upper bound of the form
for \( \alpha_1, \ldots, \alpha_n \) arbitrarily close to \( \frac{1}{2} \) (as required in Theorem 4.6), where the constant \( C \) depends on all the integers \((k_v)_{v \in V_G}\) fixed before.

However, it is not difficult to see that \( C \) is actually of the form

\[
C_n \prod_{v \in V_G} 2^{C_{s,k_v}} \cdot 2^{C_{e,k_v}}
\]

where \( C_s \) is the constant coming from estimating the various sizes which can be chosen to be dependent only on \( n \), while \( C_e \) is the constant coming from estimating the various energies which is either zero, or it is of the form \(-M\) for an arbitrarily big \( M \), as we have seen.

In order to see that this big geometric series is convergent when we sum over all the positive integers \((k_v)_{v \in V_G}\), it is enough to observe that every time \( v \) and \( w \) are adjacent and say \( w \) is the son of \( v \), we have always three possibilities: either \( k_w << k_v \) or \( k_w \sim k_v \) or \( k_w >> k_v \). In the first two cases one should first sum over \( k_w \) and get a bound of the form \( 2^{C_{s,k_v}} \), while in the third one should also sum over \( k_w \) first since this time the corresponding energy estimate comes with a factor of type \( 2^{-M k_w} \) and in this case we also obtain an upper bound of the same type \( 2^{C_{s,k_v}} \).

Hence, if one starts summing from the vertices having the highest levels (those vertices whose sons are the leaves of the tree) and continues until one reaches the root of the tree, one sees that the geometric sum is indeed convergent. This ends the proof of Theorem 4.6.

We are therefore left with proving Lemma 5.10.

6. Proof of Lemma 5.10

Fix \( \vec{P}, \vec{Q} \) and \( S_{\vec{F}} \subseteq S_{\vec{G}} \), fix also \( k_2 >> k_1 \) and to simplify the notation suppress from now on the dependence on the indices \( i, j \) which appear in Lemma 5.10. We would therefore like to prove that

\[
\left| \sum_{P \cap Q \subseteq \vec{I}_{\vec{Q}}} C_P C_Q \langle \Phi_P, \Phi_Q \rangle \right| \leq 2^{-M k_2}
\]

(92)

for any \( M > 0 \) with the implicit constants depending on \( M \). We also know from hypothesis that the collections \( \vec{P} \) and \( \vec{Q} \) can be written as unions of strongly disjoint trees which we call \( \Pi \) and \( \Pi' \) respectively. We also denote by \( S \) and \( S' \) the expressions

\[
S := \sum_{T \in \Pi} |I_T|
\]

and

\[
S' := \sum_{T \in \Pi'} |I_T|.
\]

We should note from the very beginning the crucial fact that for any \( Q_1, Q_2 \) with \( |I_{Q_1}| \neq |I_{Q_2}| \) one has \( I_{Q_1} \cap I_{Q_2} = \emptyset \) and also that \( 2^{k_2-5} I_{Q} \cap I_P = \emptyset \) for any \( P \) and \( Q \). It is also important to note that since \( k_2 >> k_1 \) one has that all the trees in \( \Pi' \) are “one-tile trees” and as a consequence we have \( S' = \sum_Q |I_Q| \).
Using these, the left hand side of the above inequality (92) can be estimated by

\[
\sum_{P, Q} |C_P||C_Q|\langle \Phi_P, \Phi_Q \rangle \leq \sum_{Q} |C_Q| \left( \sum_{P: |I_P| \leq |I_Q|, \omega_P \cap \omega_Q \neq \emptyset} |C_P||\langle \Phi_P, \Phi_Q \rangle| \right) \leq
\]

\[
\frac{1}{(S')^{1/2}} \sum_{Q} |I_Q|^{1/2} \left( \sum_{P: |I_P| \leq |I_Q|, \omega_P \cap \omega_Q \neq \emptyset} |C_P||\langle \Phi_P, \Phi_Q \rangle| \right) \leq
\]

\[
\frac{1}{(S')^{1/2}} \sum_{Q} |I_Q|^{1/2} \left( \sum_{P: |I_P| \leq |I_Q|, \omega_P \cap \omega_Q \neq \emptyset} |I_P|^{1/2} S^{1/2} |\langle \Phi_P, \Phi_Q \rangle| \right) \leq
\]

\[
\frac{1}{(S')^{1/2}} \frac{1}{S^{1/2}} \sum_{Q} \left( \sum_{P: |I_P| \leq |I_Q|, \omega_P \cap \omega_Q \neq \emptyset} |G_{\chi_P}^{\otimes N}, \chi_Q^{\otimes N}| \right),
\]

for any \( N > 0 \) with the implicit constants depending on \( N \).

Using now the fact that all the \( P \) tiles are disjoint together with the previous observation that \( 2^{k_2 - 5} I_Q \cap I_P = \emptyset \) one can estimate the previous expression further by

\[
2^{-Nk_2} \frac{1}{(S')^{1/2}} \frac{1}{S^{1/2}} \sum_{Q} \sum_{P: |I_P| \leq |I_Q|, \omega_P \cap \omega_Q \neq \emptyset} \left( 1 + \frac{\text{dist}(I_P, I_Q)}{|I_Q|} \right)^{-N} |I_P| \leq
\]

\[
2^{-Nk_2} \frac{1}{(S')^{1/2}} \frac{1}{S^{1/2}} \sum_{Q} |I_Q| = 2^{-Nk_2} \frac{1}{(S')^{1/2}} \frac{1}{S^{1/2}} S' = 2^{-Nk_2} \frac{(S')^{1/2}}{S^{1/2}}.
\]

Let us assume now that \( N = 10M \) where \( M \) is the generic number in the hypothesis. If we knew that

\[
\frac{(S')^{1/2}}{S^{1/2}} \approx 2^{9Mk_2}
\]

then we would be done.

We are therefore left with understanding the opposite case when one has

\[
\frac{S^{1/2}}{(S')^{1/2}} \approx 2^{-9Mk_2} \quad (93)
\]

In this case, we write the left hand side of (92) as

\[
\sum_{P, Q: |I_P| \leq |I_Q|} C_P C_Q \langle \Phi_P, \Phi_Q \rangle = - \sum_{P, Q: |I_P| > |I_Q|} C_P C_Q \langle \Phi_P, \Phi_Q \rangle + \sum_{P, Q} C_P C_Q \langle \Phi_P, \Phi_Q \rangle =
\]

\[
- \sum_{P, Q: |I_P| > |I_Q|} C_P C_Q \langle \Phi_P, \Phi_Q \rangle + \sum_{P} C_P \Phi_P, \sum_{Q} C_Q \Phi_Q \rangle. \quad (94)
\]

To estimate the first term in (94), we can write

\[
\left| \sum_{P, Q: |I_P| > |I_Q|} C_P C_Q \langle \Phi_P, \Phi_Q \rangle \right| = \left| \sum_{P, Q: |I_P| > |I_Q|, \omega_P \cap \omega_Q \neq \emptyset} C_P C_Q \langle \Phi_P, \Phi_Q \rangle \right| =
\]
by using (93).

Then, the second term in (95) can be estimated by

\[
\sum_T \sum_{P \in T} C_P \left( \sum_{Q: |I_Q| > |I_T|, \omega_Q \cap \omega_Q \neq \emptyset, I_T \cap I_Q \neq \emptyset} C_Q \langle \Phi_P, \Phi_Q \rangle \right)^2 \\
\leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T|^{1/2} \left( \sum_{Q \in Q_T} |I_Q| \right)^{1/2} \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T| \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = \frac{S^{1/2}}{(S')^{1/2}} \leq 2^{-9Mk_2}
\]

by using (93).

Then, the second term in (95) can be estimated by

\[
\sum_T \left( \sum_{P \in T} |C_P|^2 \right)^{1/2} \left( \sum_{Q: |I_Q| > |I_T|, \omega_Q \cap \omega_Q \neq \emptyset, I_T \cap I_Q \neq \emptyset} |C_Q| \langle \Phi_P, \Phi_Q \rangle \right)^2 \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T|^{1/2} \left( \sum_{Q \in Q_T} |I_Q| \right)^{1/2} \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T| \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = \frac{S^{1/2}}{(S')^{1/2}} \leq 2^{-9Mk_2}
\]

by using (93).

Then, the second term in (95) can be estimated by

\[
\sum_T \left( \sum_{P \in T} |C_P|^2 \right)^{1/2} \left( \sum_{Q: |I_Q| > |I_T|, \omega_Q \cap \omega_Q \neq \emptyset, I_T \cap I_Q \neq \emptyset} |C_Q| \langle \Phi_P, \Phi_Q \rangle \right)^2 \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T|^{1/2} \left( \sum_{Q \in Q_T} |I_Q| \right)^{1/2} \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T| \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = \frac{S^{1/2}}{(S')^{1/2}} \leq 2^{-9Mk_2}
\]

by using (93).

Then, the second term in (95) can be estimated by

\[
\sum_T \left( \sum_{P \in T} |C_P|^2 \right)^{1/2} \left( \sum_{Q: |I_Q| > |I_T|, \omega_Q \cap \omega_Q \neq \emptyset, I_T \cap I_Q \neq \emptyset} |C_Q| \langle \Phi_P, \Phi_Q \rangle \right)^2 \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T|^{1/2} \left( \sum_{Q \in Q_T} |I_Q| \right)^{1/2} \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T| \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = \frac{S^{1/2}}{(S')^{1/2}} \leq 2^{-9Mk_2}
\]
\[ |I_p|^{-1/2} \sum_{Q: |I_p| > |I_Q|, \omega_p \cap \omega_Q \neq \emptyset, I_p \cap I_Q = 0} \left( 1 + \frac{\text{dist}(I_Q, I_p)}{|I_p|} \right)^{-N} |I_Q| \leq |I_p|^{-1/2} \left( 1 + \frac{\text{dist}(I_p, I_T)}{|I_p|} \right)^{-N} |I_p| = \]

\[ |I_p|^{1/2} \left( 1 + \frac{\text{dist}(I_p, I_T)}{|I_p|} \right)^{-N} \]

since the time intervals \( I_Q \) which contribute to the above sum are all disjoint.

In particular, (97) becomes smaller than

\[ \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T|^{1/2} \left( \sum_{P \in T} |I_P| \left( 1 + \frac{\text{dist}(I_P, I_T)}{|I_P|} \right)^{-2N} \right)^{1/2} = \]

\[ \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T|^{1/2} \left( \sum_{k=0}^{\infty} \sum_{P \in T: |I_P| = 2^{-k} |I_T|} 2^{-k} |I_T| \left( 1 + \frac{\text{dist}(I_P, I_T)}{|I_P|} \right)^{-2N} \right)^{1/2} \leq \]

\[ \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T| = \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = S^{1/2} \frac{1}{(S')^{1/2}} \approx 2^{-9M_k}, \]

as before.

In conclusion, we are left with estimating the second term in (94) namely the expression

\[ |\langle \sum_{P} C_P \Phi_P, \sum_{Q} C_Q \Phi_Q \rangle|. \]

One should first observe that by simply applying Cauchy-Schwartz, we get a bound of the form

\[ \| \sum_{P} C_P \Phi_P \|_2 \cdot \| \sum_{Q} C_Q \Phi_Q \|_2 \]

and this is \( O(1) \) by a result from [13]. Unfortunately, this is not enough since we need this time to get an extra factor of type \( 2^{-M_k} \). We need to introduce a few notations and definitions to proceed further.

If \( I \) is an arbitrary dyadic interval, we say that a smooth function \( \Phi_I \) is a \textit{relaxed bump} adapted to \( I \) if and only if one has

\[ \left| \frac{d^l}{dx^l} (\Phi_I(x)) \right| \leq |I|^{-l} \left( 1 + \frac{\text{dist}(x, I)}{|I|} \right)^{-10} \]

for any \( 1 \leq l \leq 10 \).

Then, if \( Q \) is an arbitrary tile \( Q = I_Q \times \omega_Q \) we say that \( \Phi_Q \) is a \textit{relaxed wave packet} adapted to \( Q \) if and only if \( \Phi_Q(x) = \Phi_{I_Q}(x) \cdot 2^{2n_k\xi_Q} \) where \( \xi_Q \) is the center of the frequency interval \( \omega_Q \) and \( \Phi_{I_Q} \) is any \textit{relaxed bump} adapted to the interval \( I_Q \). Note that this time we do not assume arbitrary decay and also we do not assume that the Fourier transform of \( \Phi_Q \) has compact support.

The following Lemma will be useful.
Lemma 6.1. Let $Q_1$ and $Q_2$ be two tiles so that $|I_{Q_1}| \geq |I_{Q_2}|$. Then, if $\Phi_{Q_1}$ and $\Phi_{Q_2}$ are relaxed wave packets adapted to $Q_1$ and $Q_2$ respectively, one has the estimate

$$\left|\langle \Phi_{Q_1}, \Phi_{Q_2} \rangle \right| \lesssim \left(1 + \frac{\text{dist}(\omega_{Q_1}, \omega_{Q_2})}{|\omega_{Q_2}|}\right)^{-10} \cdot \int_{\mathbb{R}} \tilde{I}_{Q_1}(x) \overline{\tilde{I}_{Q_2}(x)} dx \lesssim$$

$$\left(1 + \frac{\text{dist}(\omega_{Q_1}, \omega_{Q_2})}{|\omega_{Q_2}|}\right)^{-10} \cdot \left(1 + \frac{\text{dist}(I_{Q_1}, I_{Q_2})}{|I_{Q_1}|}\right)^{-10} \cdot |I_{Q_2}|.$$

Proof Clearly, if both $\Phi_{Q_1}$ and $\Phi_{Q_2}$ would be "real wave packets" (therefore compactly supported in frequency) then the first factor $\left(1 + \frac{\text{dist}(\omega_{Q_1}, \omega_{Q_2})}{|\omega_{Q_2}|}\right)$ has to be equal to 1, otherwise the scalar product would be zero. In that case, the estimate simply becomes the usual estimate of the scalar product of two bump functions. Since this is not the case, one has instead to take advantage of the oscillation of $\Phi_{Q_2}$, in the situation when $\left(1 + \frac{\text{dist}(\omega_{Q_1}, \omega_{Q_2})}{|\omega_{Q_2}|}\right)$ is a big number by the usual integration by parts argument which should be performed ten times.

The straightforward details are left to the reader.

The following lemma will also be important.

Lemma 6.2. Consider for each $Q \in \tilde{Q}$ a relaxed $L^2$-normalized wave packet $\Phi_Q$ adapted to the tile $Q$. Then, if $(C_Q)_Q$ is a sequence of complex numbers as before, then one has

$$\| \sum_Q C_Q \Phi_Q \|_2 \lesssim 1.$$

Proof First of all, let us recall that since any $Q$ has the property that

$$\frac{\text{dist}(I_Q, S_Q)}{|I_Q|} \sim 2^{k_2}$$

and $k_2$ is a large positive integer, one has that every time $Q$ and $Q'$ are so that $|I_Q| \neq |I_{Q'}|$ then one must have $I_Q \cap I_{Q'} = \emptyset$. This also implies that our collection of tiles can contain only one-tile trees and so $S' = \sum_{T \in \prod^* |I_T| = \sum_Q |I_Q|}$.

Using this, one can write

$$\| \sum_Q C_Q \Phi_Q \|_2^2 = \left| \sum_Q C_Q \Phi_Q, \sum_Q C_Q \Phi_Q' \right| =$$

$$\left| \sum_{Q, Q'} C_Q C_{Q'} \langle \Phi_Q, \Phi_{Q'} \rangle \right| \leq \sum_{Q, Q'} |I_Q|^{1/2} |I_{Q'}|^{1/2} \left| \langle \Phi_Q, \Phi_{Q'} \rangle \right| :=$$

$$\frac{1}{(S')^{1/2}} \frac{1}{(S')^{1/2}} \sum_{Q, Q'} |\langle \tilde{\phi}_Q, \tilde{\phi}_{Q'} \rangle| \leq \frac{1}{S'}, \sum_Q \left( \sum_{Q', |I_{Q'}| \leq |I_Q|} |\langle \tilde{\phi}_Q, \tilde{\phi}_{Q'} \rangle| \right).$$

Using now our previous Lemma 6.1 together with the observations made at the beginning of the proof, it is not difficult to see that the last expression is smaller than
\[ \frac{1}{S'} \sum_Q |I_Q| = 1 \]

which ends the proof. 

Coming back now to our expression \( \langle \sum_P C_P \Phi_P, \sum_Q C_Q \Phi_Q \rangle \) we will do the following. For each \( Q \), split the corresponding \( \Phi_Q \) as

\[ \Phi_Q = \sum_{l \in \mathbb{Z}} \Phi'_Q \]

where each \( \Phi'_Q \) is defined to be the old function \( \Phi_Q \) multiplied by a cut-off bump function supported on an interval of comparable length with \( I_Q \) but \( l \) units of length \( |I_Q| \) away from \( I_Q \). Since \( \Phi_Q \) is a Schwartz function, we can further write \( \Phi_Q \) as

\[ \Phi_Q = \sum_{l \in \mathbb{Z}} \frac{1}{(1 + |l|)^N} \tilde{\Phi}'_Q. \]

As a consequence, our expression splits as

\[ \sum_{l \in \mathbb{Z}} \frac{1}{(1 + |l|)^N} \langle \sum_Q C_Q \tilde{\Phi}'_Q, \sum_P C_P \Phi_P \rangle := I + II \]

where

\[ I := \sum_{l \in \mathbb{Z}, |l| \leq 2^{k_2-5}} \frac{1}{(1 + |l|)^N} \langle \sum_Q C_Q \tilde{\Phi}'_Q, \sum_P C_P \Phi_P \rangle \]

and

\[ II := \sum_{l \in \mathbb{Z}, |l| > 2^{k_2-5}} \frac{1}{(1 + |l|)^N} \langle \sum_Q C_Q \tilde{\Phi}'_Q, \sum_P C_P \Phi_P \rangle. \]

Now, it is not difficult to remark that for each fixed \( l \) with \( |l| > 2^{k_2-5} \), the function \( \tilde{\Phi}'_Q \) is also a relaxed wave packet adapted to \( Q \). In particular, this implies that one can simply apply Cauchy-Schwartz for term II, together with Lemma 6.2 to bound it by \( 2^{-Mk_2} \) as desired.

It is therefore enough to estimate term \( I \). Fix \( l \in \mathbb{Z} \) so that \( |l| \leq 2^{k_2-5} \) and consider the expression

\[ \langle \sum_Q C_Q \tilde{\Phi}'_Q, \sum_P C_P \Phi_P \rangle. \]

This time, we will take advantage of the fact that the functions \( \tilde{\Phi}'_Q \) are all compactly supported.

First of all, let us denote by \( I \) the collection of all dyadic intervals \( I \) for which there exists a \( Q \) with \( I_Q = I \). As we already remarked, since every \( Q \) has the property that

\[ 2^{k_2} \leq \frac{\text{dist}(I_Q, S_Q)}{|I_Q|} \leq 2^{k_2+1} \]
all these intervals \( I \) are disjoint. Denote also by \( I^l \) the collection of all dyadic intervals \( I' \) defined by \( I' := I + h |l| \) for some \( I \in I \). It is also not difficult and important to observe that the intervals in \( I^l \) have bounded overlap. Also, for any \( Q \) we denote by \( I_Q^l \) the interval \( I_Q^l := I_Q + h |I_Q| \).

Clearly, each function \( \Phi_Q^l \) is supported on a certain fixed enlargement (with a factor of 3 say) of the interval \( I_Q^l \). We will use these observations and notations later on.

Using these, we can estimate (99) by

\[
\langle \sum_Q C_Q \Phi_Q^l, \sum_p C_p \Phi_p \rangle = \sum_{p,Q} C_p C_Q \langle \Phi_Q^l, \Phi_p \rangle = \\
\sum_T \sum_{p \in T} C_p \langle \sum_Q C_Q \Phi_Q^l, \Phi_p \rangle = \\
\sum_T \sum_{p \in T} C_p \langle \sum_{Q : I_Q^l \subseteq \tau} C_Q \Phi_Q^l, \Phi_p \rangle + \sum_T \sum_{p \in T} C_p \langle \sum_{Q : I_Q^l \not\subseteq \tau} C_Q \Phi_Q^l, \Phi_p \rangle := \alpha + \beta.
\]

Estimates for \( \beta \).

To understand \( \beta \) we write

\[
\beta \lesssim \sum_T \left( \sum_{p \in T} |C_p|^2 \right)^{1/2} \left( \sum_T \left( \sum_{Q : I_Q^l \subseteq \tau} C_Q \Phi_Q^l \right)^2 \right)^{1/2} \lesssim \\
\frac{1}{S^{1/2}} \sum_T |I_T|^2 \left( \sum_T \left( \sum_{Q : I_Q^l \subseteq \tau} C_Q \Phi_Q^l \right)^2 \right)^{1/2}.
\]

We will show next that

\[
\left( \sum_T \left( \sum_{Q : I_Q^l \subseteq \tau} C_Q \Phi_Q^l \right)^2 \right)^{1/2} \lesssim \frac{1}{(S')^{1/2}} |I_T|^{1/2}.
\]

If (101) were true, then the estimate on \( \beta \) could be completed as follows

\[
\beta \lesssim \frac{1}{S^{1/2}} \sum_T |I_T|^{1/2} \cdot \frac{1}{(S')^{1/2}} |I_T|^{1/2} = \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_T |I_T| = \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = \frac{S^{1/2}}{(S')^{1/2}} \lesssim 2^{-9M^2}
\]
as desired. It is therefore enough to show (101).

Fix \( P \in T \). Then, the corresponding inner term in (101) can be estimated by

\[
|\langle \sum_{Q : I_Q^l \subseteq \tau} C_Q \Phi_Q^l, \Phi_P \rangle| \lesssim \sum_{Q : I_Q^l \subseteq \tau} |C_Q||\langle \Phi_P, \Phi_Q^l \rangle| \lesssim \\
\frac{1}{(S')^{1/2}} \sum_{Q : I_Q^l \subseteq \tau} |I_Q|^{1/2} |\langle \Phi_P, \Phi_Q^l \rangle| = \frac{1}{(S')^{1/2}} \sum_{Q : I_Q^l \subseteq \tau} |I_P|^{-1/2} |\langle \Phi_Q^l, \Phi_Q^\infty \rangle| =
\]

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\[
\frac{1}{(S')^{1/2}} |I_P|^{-1/2} \sum_{Q: I_Q' \subseteq I_T'} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle|,
\]

where \( \Phi_P^\infty := |I_P|^{1/2} \Phi_P \) and \( \Phi_Q^{l,\infty} := |I_Q|^{1/2} \Phi_Q^l \) and they are both \( L^\infty \) normalized functions.

We claim now that

\[
\sum_{Q: I_Q' \subseteq I_T'} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle| \lesssim \left(1 + \frac{\text{dist}(I_P, I_T')}{|I_P|}\right)^{-m} |I_P|
\]

(102)

for any positive integer \( m \), with the implicit constants depending on it.

If we assume the claim, the corresponding last term can be estimated further by

\[
\frac{1}{(S')^{1/2}} |I_P|^{1/2} \left(1 + \frac{\text{dist}(I_P, I_T')}{|I_P|}\right)^{-m}
\]

and as a consequence, the left hand side of (101) becomes smaller than

\[
\frac{1}{(S')^{1/2}} \left(\sum_{P \in T} \left(1 + \frac{\text{dist}(I_P, I_T')}{|I_P|}\right)^{-2m} |I_P|\right)^{1/2}
\]

and this as we have already seen before is smaller than \( \frac{1}{(S')^{1/2}} |I_T|^{1/2} \) as desired.

It is therefore enough to prove the previous claim (102).

Split the left hand side of it as

\[
\sum_{Q: I_Q' \subseteq I_T'} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle| + \sum_{Q: I_Q' \subseteq I_T', |I_Q'| > |I_P|} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle| := C_1 + C_2.
\]

To estimate \( C_1 \), let us assume that our collection \( T' \) defined before can be listed as

\[
T' = \{I_1, ..., I_K\}.
\]

Using this and also Lemma 6.1 one can write

\[
C_1 = \sum_{j=1}^{K} \sum_{Q: I_Q' \subseteq I_T', |I_Q'| \leq |I_P|} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle| \lesssim \sum_{j=1}^{K} \left(1 + \frac{\text{dist}(I_P, I_j)}{|I_P|}\right)^{-m} |I_j|
\]

and the last sum is clearly smaller than

\[
\left(1 + \frac{\text{dist}(I_P, I_T')}{|I_P|}\right)^{-m} |I_P|
\]

as required by (102).

To estimate \( C_2 \), this time we can write

\[
\sum_{Q: I_Q' \subseteq I_T', |I_Q'| > |I_P|} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle| = \sum_{l=1}^{\infty} \sum_{Q: I_Q' \subseteq I_T', |I_Q'| > |I_P|, |I_Q'| = 2^l |I_P|} |\langle \Phi_P^\infty, \Phi_Q^{l,\infty} \rangle|.
\]
Fix $l$ and look at the corresponding inner sum. It is not difficult to remark that for every $Q$ as there, one has $2^{k_2+1-5}I_P \cap I_Q = \emptyset$. Tacking also into account the fact that all the functions $\Phi_Q^{l,m}$ have compact support and applying carefully several times Lemma 6.1 one obtains for $C_2$ the upper bound

$$2^l \frac{1}{2^{(k_2+1)m}} \left(1 + \frac{\text{dist}(I_P, I_T)}{|I_P|}\right)^{-m} |I_P|$$

which is fine, since it is an expression summable over $l$. This ends the discussion on $\beta$, we start now estimating term $\alpha$.

Estimates for $\alpha$.

We now write

$$\alpha = \sum_{T} \sum_{P \in T} C_P \left( \sum_{Q, I_Q \subseteq I_T} C_Q \Phi_Q^{l} \cdot \Phi_P \right) = \sum_{T} \sum_{P \in T} C_P C_Q (\Phi_Q^{l}, \Phi_P).$$

Fix $T$. We will show that

$$\left| \sum_{P \in T, Q, I_Q \subseteq I_T} C_P C_Q (\Phi_Q^{l}, \Phi_P) \right| \lesssim \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} |I_T|. \quad (103)$$

If we accept for a moment (103) then $\alpha$ becomes smaller than

$$\frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} \sum_{T} |I_T| = \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} S = \frac{S^{1/2}}{(S')^{1/2}} \lesssim 2^{-9Mk_2}$$

which would be the desired upper bound. We are therefore left with understanding (103).

We split the left hand side of it as

$$\left| \sum_{P \in T, Q, I_Q \subseteq I_T, |I_Q| \leq |I_P|} C_P C_Q (\Phi_Q^{l}, \Phi_P) \right| + \left| \sum_{P \in T, Q, I_Q \subseteq I_T, |I_Q| > |I_P|} C_P C_Q (\Phi_Q^{l}, \Phi_P) \right| := A + B.$$ Estimate for $A$.

Pick an index $1 \leq j \leq K$ with the property that $I_j \subseteq I_T$ and consider the corresponding sum

$$\left| \sum_{P \in T, Q, I_Q \subseteq I_T, |I_Q| = |I_P|} C_P C_Q (\Phi_Q^{l}, \Phi_P) \right|. $$

It can be further majorized by

$$\sum_{1} \left| \sum_{P \in T, Q, I_Q \subseteq I_T, |I_Q| > |I_P|, |I_Q| = 2^l |I_P|} C_P C_Q (\Phi_Q^{l}, \Phi_P) \right|. $$

Arguing as before (by using Lemma 6.1) we deduce that the previous expression can be estimated by
where \( h \) is supported in frequency. Since for every \( Q \) the function \( \tilde{\Phi}_Q^j \) is a bump adapted to the interval \( \omega_Q \), we can split it accordingly as

\[
\tilde{\Phi}_Q^j = \sum_l \frac{1}{(1 + \|l\|)^{10}} \Phi_Q^{l1}
\]

where clearly, \( \Phi_Q^{l1} \) is a wave packet adapted to the tile \( Q^{l1} := I_Q^l \times \omega_Q \) where \( \omega_Q \) is the interval defined by \( \omega_Q := \omega_Q + |\omega_Q| \).

As a consequence, our term \( A \) can be majorized by

\[
\sum_l \frac{1}{(1 + \|l\|)^{10}} \left| \sum_{P \in T: I_P \subset I_T \cap |I_P| \leq |I_T|} C_P C_Q(\Phi_Q^{l1}, \Phi_P) \right|.
\]

It is important to observe now that for each fixed \( I \) the corresponding set of tiles \( Q^{l1} \) is still strongly disjoint. Also, since both \( \Phi_Q^{l1} \) and \( \Phi_P \) have compact Fourier support, it is clear that

\[
\langle \Phi_Q^{l1}, \Phi_P \rangle = 0 \text{ unless } \omega_Q \cap \omega_P \neq \emptyset.
\]

Fix \( I \) now. The inner sum above can also be written as

\[
\left| \sum_{P \in T: I_P \subset I_T, \omega_Q \cap \omega_P \neq \emptyset, |\omega_Q^{l1}| > |\omega_P|} C_P C_Q(\Phi_Q^{l1}, \Phi_P) \right|.
\]

Denote also as before by \( Q_T \) the set of all tiles \( Q \) with \( I_Q^l \subset I_T \) for which there exists a tile \( P \in T \) so that \( \omega_Q \cap \omega_P \neq \emptyset \) and \( |\omega_Q^{l1}| > |\omega_P| \). Note also that those \( Q \) inside \( Q_T \) must have disjoint \( l \) intervals. It is also not difficult to see that the above expression can also be written as

\[
\left| \sum_{P \in T} C_P(h_T, \Phi_P) \right|,
\]

where \( h_T := \sum_{Q \in Q_T} C_Q \Phi_Q^{l1} \). Then, (104) is smaller than

\[
\left( \sum_{P \in T} |C_P|^2 \right)^{1/2} \left( \sum_{P \in T} |\langle h_T, \Phi_P \rangle|^2 \right)^{1/2} \leq \frac{|I_T|^{1/2}}{S^{1/2}} \cdot \|h_T\|_2.
\]

Now, one also has

\[
\frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} 2^{l_j} \sum_{P \in T: |I_P| = 2^{|I_P|}} \left( 1 + \frac{\text{dist}(I_j, I_P)}{|I_P|} \right)^{-m} |I_j| \leq \frac{1}{S^{1/2}} \frac{1}{(S')^{1/2}} 2^{l_j} \frac{1}{2^{(k_2+1)m}} |I_j|.
\]
\[
\|h_T\|_2 \leq (\sum_{Q \in Q_r} |C_Q|^2)^{1/2} \leq \frac{1}{(S')^{1/2}} (\sum_{Q \in Q_r} |I_Q|)^{1/2} = \\
\frac{1}{(S')^{1/2}} (\sum_{Q \in Q_r} |I_Q|^2)^{1/2} \leq \frac{1}{(S')^{1/2}} |I_T|^{1/2}
\]

Using this in (105) finishes the proof.

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