Interacting staggered domain wall fermions
Pavlos M. Vranas and George T. Fleming

a IBM T.J. Watson Research Center, Route 134, Yorktown Heights, NY 10598, USA.
b Physics Department, The Ohio State University, Columbus, OH 43210-1168, USA.

The behavior of staggered domain wall fermions in the presence of gauge fields is presented. In particular, their
response to gauge fields with nontrivial topology is discussed.

1. Introduction

I remember it very clearly: It was a sunny spring Northern California afternoon in 1988. I
had just passed my Ph.D. exams and was about to
start research. My Ph.D. advisor took his other
student (also my collaborator) and me out to
lunch to the place across from the physics depart-
ment. This was unusual, so it naturally weighed
a lot. I do not remember his exact words, but he
basically told us about the many difficulties that
would lay ahead should we decide to follow re-
search in lattice gauge theory. Now, as if that was
not enough, the next year, at the lattice confer-
cence at Capri, the father of this all, K. Wilson,
resigned with a farewell-good-luck-I-am-out-of-here
talk. I am not sure if I heard this directly from K.
Wilson or from anectodal rumors, but whatever
might be the case, it was something like this: “To
be able to do QCD we will need lattice volumes
$V \approx 128^3$, and by the time computers will reach
this capability I will be too old... And there are
so many more interesting things in science to just
wait for this...” Now I see what the great man
meant. Having been involved in building super-
computers for the last 8 years it is my estimate
that that volume will be reached around 2012.

Now, these were dire warnings. Needless to
say, I did not listen and the price has been high.
Nevertheless, nearly 14 years later, I am writing
this from a Boston cafe while Lattice 2002 is in
progress. But it is all my advisor’s fault... He in-
troduced me to the lattice fermion doubling prob-
lem; and it was love at first sight...

Where else in theoretical physics can you find
a problem that appears so painfully simple and
yet runs so deep? Well, there are a few more but
this is definitely one of them.

And, yes, an extra dimension came in nat-
urally to cater to this problem. Domain wall
fermions (DWF), a revolutionary technique, were
introduced in [1–3] (for reviews and references see
[4]). And, the extra dimension did not come from
string theory, nor from any other theory-beyond-
the-standard-model, but from this silly little tech-
nical lattice problem. And I still feel that we
have not yet grasped its full meaning. Because,
at the end of the day, it is the problem of non-
perturbative regularization of chiral gauge theo-
ries, which in turn are at the boundary of the
standard model.

And, to add insult to injury, this is one of the
main reasons for the very slow progress in numer-
ical simulations of QCD. The fastest supercom-
puters ever built have been traditionally used by
QCD only to feel in their guts of gates and wires
the difficulties of the doubling problem.

What I am trying to say is that the lure is still
strong, the problem is still theoretically very in-
teresting and numerical simulations can still ben-
efit a great deal from improved lattice fermion
methods. So, for better or for worse, here are
staggered domain wall fermions, SDWF.[4,5]

2. It is not just about doubling

As is well known, even naive lattice fermions
are not equivalent to 16 diagonal flavors. Other-
wise there would be 255 naive pions. But there
are only 15. Even in naive lattice fermions there is inherent flavor mixing. Traditional Wilson fermions “hide” this mixing by raising the doublet masses but staggered fermions “retain” the mixing. Depending on your point of view this is interesting or plain annoying or perhaps both.

3. SDWF

SDWF are a cross between DWF and staggered fermions. They have an exact \( \text{U}(1) \times \text{U}(1) \) chiral symmetry for any \( L_s \) where \( L_s \) is the size of the fifth dimension). The full \( \text{SU}(4) \times \text{SU}(4) \) is recovered at the \( L_s \to \infty \) limit. SDWF should offer an advantage for simulations of the finite temperature QCD phase transition.

The SDWF Dirac operator in the Saclay basis is given in [4]. The free theory exhibits local-structure QCD phase transition. The symmetry content for any \( \text{U}(1) \times \text{U}(1) \) axial symmetry is inherent flavor mixing. Traditional WilsonDirac operator is \( \lambda \). The propagator is given in [4]. The effective mass \( m_{\text{eff}} \) in \( 2n \) dimensions is similar to DWF. For \( L_s \) odd:

\[
m_{\text{eff}} = (1 - \frac{2n}{4} m_0^2) (m_f + |1 - m_0| L_s) \tag{2}
\]

4. The transfer matrix in the Saclay basis

The SDWF transfer matrix is:

\[
T = - \begin{pmatrix}
B^{-1} \lambda \, B^{-1} C & B^{-1} C
\end{pmatrix}
\begin{pmatrix}
\lambda_i B & a_5 [C B^{-1} C - B]
\end{pmatrix}. \tag{3}
\]

One can easily check that:

\[
[C, \xi_5] = 0, \quad \{B, \xi_5\} = 0, \quad \{T, \xi_5\} = 0 \tag{4}
\]

\[
B^\dagger = -B \Rightarrow T^\dagger = -T \tag{5}
\]

This is anti-hermitian. This is different from DWF. Standard transfer matrix manipulations should be done with the hermitian transfer matrix \( T^2 \).

5. Surprise?

The \( a_5 \to 0 \) Hamiltonian is proportional to the identity in flavor. No flavor mixing at all...

\[
- T^2 = e^{-2a_5 (H \otimes I)} \tag{6}
\]

\[
H = - \begin{pmatrix}
\frac{1}{2} \sum_\mu \Delta_\mu + m_0 & C
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} \sum_\mu \Delta_\mu - m_0
\end{pmatrix} \tag{7}
\]

This \( H \) is very similar to DWF. However, a zero eigenvalue for \( H \) does not imply an eigenvalue of magnitude 1 for \( T \) at any \( a_5 \). Only at \( a_5 \to 0 \). This is different from DWF. For \( a_5 = 0 \), all crossings are in \( 0 < m_0 < 2 \).

6. Pseudo - Hamiltonian

To investigate the \( m_0 \) dependence of \( |\lambda(T)| = 1 \), one can eliminate \( B^{-1} \) as in DWF. This leads to a pseudo-Hamiltonian \( H_p \). For \( a_5 = 1 \) all crossings are in \( 0 < m_0 < 4 \).

\[
\lambda(H_p) = 0 \Rightarrow \lambda(T) = i\lambda, \quad \lambda = \pm 1 \tag{8}
\]

\[
H_p = \begin{pmatrix}
1 + a_5 \lambda_i B & a_5 C
\end{pmatrix}
\begin{pmatrix}
a_5 C^\dagger & -1 - a_5 \lambda_i B
\end{pmatrix} \tag{9}
\]
7. The spectrum of $T$

The spectrum of $T$ is doubly degenerate because $\{T, \xi_5\} = 0$. For a degenerate four-flavor theory, $\log(-T^2)$ must have four zero crossings at the same $m_0$ with the same chirality. For any SU(2) field the degeneracy is always four-fold.

For a smooth non-trivial SU(3) instanton configuration (plaquette = 0.05) the eigenvalues are almost exactly four-fold degenerate. This can be seen in Fig. 2. At every crossing, four eigenvalues of the same chirality cross. For a very rough SU(3) gauge field configuration (plaquette = 0.85) the four-fold degeneracy of $\log(-T^2)$ splits to two-fold, but only by a small amount. This is shown in Table 1 for one of the worst cases on a $2^4$ lattice.

\[
\begin{array}{ccc}
m_0 & \log\lambda(-T^2) & m_0 \\
0.3 & -0.262638 & 0.3 \\
0.3 & -0.262638 & 0.3 \\
0.3 & -0.261551 & 0.3 \\
0.3 & -0.261551 & 0.3 \\
\end{array}
\]

Table 1

The near zero spectrum of $\log(-T^2)$ for $m_0 = 0.3$.

8. Going back to where we came from ...

Numerical simulations are done in the single component basis. For various approaches see [4]. For example, the standard transcription from the Saclay basis to the single component basis can be used. When gauge fields are present, the transcription carries a Jacobian that is a function of the gauge fields. Since the gauge field dependent part of the fermionic and PV actions is identical, the Jacobians must cancel (for details see [4]).

9. Still...

1) For QCD the nearly four-fold crossing degeneracy must be investigated more thoroughly.
2) SDWF in the single component basis must be tested.
3) A full Hamiltonian analysis is needed.
4) The Kogut-Sinclair 4-fermion interaction with SDWF can be used to span the finite temperature QCD phase transition at zero quark mass.
5) Is there something new that SDWF have revealed about the inherent lattice fermion flavor mixing?

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