Topological Field Theories and the Period Integrals

TOHRU EGUCHI

Department of Physics, Faculty of Science,
University of Tokyo, Tokyo 113, Japan

YASUHIKO YAMADA

National Laboratory for High Energy Physics
(KEK), Tsukuba, Ibaraki 305, Japan

and

SUNG-KIL YANG

Institute of Physics, University of Tsukuba, Ibaraki 305, Japan

ABSTRACT

We discuss topological Landau-Ginzburg theories coupled to the 2-dimensional topological gravity. We point out that the basic recursion relations for correlation functions of the 2-dimensional gravity have exactly the same form as the Gauss-Manin differential equations for the period integrals of superpotentials. Thus the one-point functions on the sphere of the Landau-Ginzburg theories are given exactly by the period integrals. We discuss various examples, A-D-E minimal models and the $c = 3$ topological theories.
In this article we study the Landau-Ginzburg description of the topological field theories coupled to the 2-dimensional topological gravity. We point out that the basic recursion relations for correlation functions in the 2-dimensional gravity have the identical structure as the Gauss-Manin differential equations of the period integrals of superpotentials. Then it is possible to show that the one-point functions of the Landau-Ginzburg models are given exactly by the periods of superpotentials. We will check our observation in various examples, the A-D-E minimal series with $c < 3$ as well as the $c = 3$ topological field theories.

Let us start our discussion with the $A_{k+1}$-type minimal theories which are described by a superpotential \[ W(x, t_0, \ldots, t_k) = \frac{1}{k+2} x^{k+2} + \sum_{i=0}^{k} g_i(t) x^i. \] (1)
The perturbation parameters $t_0, \ldots, t_k$ are chosen to be the flat coordinates \([2,3]\) of the space of deformations of the superpotential. Primary fields conjugate to these variables are defined by
\[ \phi_i(x, t) = \frac{\partial}{\partial t_i} W(x, t), \quad i = 0, 1, \ldots, k. \] (2)
Flat coordinates are characterized by the condition
\[ \frac{\partial}{\partial t_i} \phi_j(x, t) = \frac{\partial}{\partial x} Q_{ij}(x, t), \] (3)
where $Q_{ij}$ is defined by
\[ \phi_i(x, t) \phi_j(x, t) = \sum_{\ell=0}^{k} \eta^\ell_{im} c_{ij\ell}(t) \phi_m(x, t) + \frac{\partial}{\partial x} W(x, t) Q_{ij}(x, t), \] (4)
$c_{ij\ell}(t)$ is the deformed fusion ring coefficient and is given by
\[ c_{ij\ell} = \langle \phi_i \phi_j \phi_\ell \rangle \]
\[ = \int d^2x \frac{\phi_i(x, t) \phi_j(x, t) \phi_\ell(x, t)}{\partial_x W(x, t)}. \] (5)
$\eta^\ell_{im} = \delta_{\ell+m,k}$ is the metric of the space of deformations written in the flat co-
ordinate. In (5) the residue integral in $x$ is taken at $x = \infty$. Note that in (1) $g_i(t)$ ($i = 1, \ldots, k$) do not depend on the variable $t_0$ while $g_0(t)$ has a form $g_0(t) = t_0 + \tilde{g}_0(t_1, t_2, \ldots, t_k)$. Hence $\partial W/\partial t_0 = 1$. It is convenient to introduce an analogue of a pseudo-differential operator

$$W = \frac{1}{k+2} L^{k+2}. \quad (6)$$

$L(x, t)$ has an expansion, $L = x + \sum_{i=1}^{\infty} a_i(t)x^{-i}$. Then the primary field (2) is also expressed as

$$\phi_i = [L^i \partial_x L]_+ , \quad i = 0, 1, \ldots, k, \quad (7)$$

where $[L^i \partial_x L]_+$ means taking the non-negative powers of the Laurent series $L^i \partial_x L$.

The basic formulas for the one-point function on the sphere are obtained by integrating the three-point function (5) [1],

$$\langle \phi_i \rangle = \frac{1}{(i+1)(i+k+3)} \oint dx L^{k+i+3}$$

$$= \frac{(k+2)^{i+1} \cdot \cdots \cdot (i+1)(i+k+3)}{(i+1)(i+k+3)} \oint dx W^{i+1} \cdot \cdots \cdot \frac{1}{i+1}(i+k+2) \cdot \cdots \cdot (i+1+(n-1)(k+2))^{-1} , \quad c_0,i = 1. \quad (8)$$

Here again the residue integral is taken at $\infty$.

When the topological matter is coupled to the topological gravity new physical observables, i.e. gravitational descendants appear in the system. Let us denote the $n$–th descendant of $\phi_i$ as $\sigma_n(\phi_i)$ ($n = 0, 1, 2, \ldots$). In [4, 5] it was shown that the representative of the BRST cohomology class of $\sigma_n(\phi_i)$ can be expressed using only the matter fields. The basic formulas for the gravitational descendants is given by [5]

$$\sigma_n(\phi_i) \equiv c_{n,i} [L^{(k+2)n+i} \partial_x L]_+ , \quad i = 0, 1, \ldots, k, \quad n = 0, 1, \ldots$$

$$c_{n,i} = \left( (i+1)(i+1+k+2) \cdots (i+1+(n-1)(k+2)) \right)^{-1} , \quad c_{0,i} = 1. \quad (9)$$
We note a recurrence relation

$$\frac{\partial}{\partial t_0} \sigma_n(\phi_i) = \sigma_{n-1}(\phi_i).$$  \hfill (10)

It is also possible to derive an identity

$$\sigma_n(\phi_i(x)) = \partial_x W(x) \int x dy \sigma_{n-1}(\phi_i(y)) + \sum_{\ell=0}^k \frac{\partial}{\partial t_\ell} R_{n,i}(t) \phi_{k-\ell}(x).$$  \hfill (11)

Here $R_{n,i}$ are the (dispersionless limits of the) Gelfand-Dikii potentials of the KdV hierarchy,

$$R_{n,i} = c_{n+1,i} \oint dx L^{n+2+i+1}.$$  \hfill (12)

One-point functions of the gravitational descendants are then given by [5,6]

$$\langle \sigma_n(\phi_i) \rangle = R_{n+1,i} = c_{n+2,i} \oint dx L^{(n+1)(k+2)+i+1}$$

$$= (k+2)^{n+1+i+1} c_{n+2,i} \oint dx W^{n+1+i+1+1/k+2}.$$  \hfill (13)

(13) is the direct generalization of (8).

After coupling to gravity the metric $\eta_{ij}$ is identified as the 3-point function $\eta_{i,j} = \langle P \phi_i \phi_j \rangle$ where $P$ is the puncture operator coupled to the parameter $t_0$. By integrating $\eta_{i,j} = \delta_{i+j,k}$ over $t_j$ we have $t_{k-i} = \langle P \phi_i \rangle$. Then combining with the $t_0$ derivative of (8) we obtain

$$t_{k-i} = \frac{(k+2)^{i+1}}{i+1} \oint dx W^{i+1}.$$  \hfill (14)

Using (9),(10),(11) it is easy to recover the basic recursion relations for the correlation functions in 2-dimensional gravity. We note that (10) immediately
leads to the puncture equation [7]

\[ \langle \frac{\partial}{\partial t_0} \sigma_n(\phi_i) \rangle = \langle \sigma_{n-1}(\phi_i) \rangle = \langle P \sigma_n(\phi_i) \rangle. \] (15)

\( P \) is the puncture operator coupled to the parameter \( t_0 \). Using (11) we also obtain

\[ \langle \sigma_n(\phi_i) \phi_j \phi_m \rangle = k \sum_{\ell=0}^{k} \frac{\partial}{\partial t_\ell} R_{n,i} \langle \phi_{k-\ell} \phi_j \phi_m \rangle \]

\[ = k \sum_{\ell=0}^{k} \frac{\partial}{\partial t_\ell} \langle \sigma_{n-1}(\phi_i) \rangle \langle \phi_{k-\ell} \phi_j \phi_m \rangle \] (16)

This is the topological recursion relation of Witten [8].

Now we note that (16) may be rewritten as a differential equation for the one-point function

\[ \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_m} \langle \sigma_n(\phi_i) \rangle = k \sum_{\ell=0}^{k} c_{j,m} \frac{\partial}{\partial t_\ell} \frac{\partial}{\partial t_0} \langle \sigma_n(\phi_i) \rangle. \] (17)

Our crucial observation is that (17) has exactly the same form as the well-known Gauss-Manin differential equations for period integrals expressed in flat coordinates [9,10]. Let us consider the general case of \( N \) variables \( x_1, \ldots x_N \) and an integral

\[ u^{(\lambda)}(t) = \oint dx_1 \cdots dx_N W(x,t)^{-\lambda} \] (18)

over a suitably chosen cycle \( \gamma \). \( \lambda \) is an arbitrary parameter. By taking derivatives
of (18) we obtain
\[
\frac{\partial}{\partial t_j} \frac{\partial}{\partial t_m} u^{(\lambda)}(t)
= -\lambda \frac{\partial}{\partial t_j} \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N \phi_m(x, t) W(x, t)^{-\lambda - 1}
= \lambda(\lambda + 1) \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N \phi_j(x, t) \phi_m(x, t) W(x, t)^{-\lambda - 2}
- \lambda \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N \frac{\partial}{\partial t_j} \phi_m(x, t) W(x, t)^{-\lambda - 1}.
\]

Here the primary fields are again defined by
\[
\phi_i(x, t) = \frac{\partial}{\partial t_i} W(x, t).
\]

The condition for the flat coordinates now reads
\[
\frac{\partial}{\partial t_i} \phi_j(x, t) = \sum_{a=1}^{N} \frac{\partial}{\partial x_a} Q_{ij}^a(x, t)
\]
where \(Q_{ij}^a\) is defined by
\[
\phi_i(x, t) \phi_j(x, t) = \sum c_{ij}^\ell(t) \phi_\ell(x, t) + \sum_{a=1}^{N} \frac{\partial}{\partial x_a} W(x, t) Q_{ij}^a(x, t).
\]

Then (19) is rewritten as
\[
\frac{\partial}{\partial t_j} \frac{\partial}{\partial t_m} u^{(\lambda)}(t)
= \lambda(\lambda + 1) \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N \left( \sum_\ell c_{jm}^\ell(t) \phi_\ell(x, t) + \sum_{a=1}^{N} \frac{\partial}{\partial x_a} W(x, t) Q_{jm}^a(x, t) \right) W(x, t)^{-\lambda - 2}
- \lambda \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N \frac{\partial}{\partial t_j} \phi_m(x, t) W(x, t)^{-\lambda - 1}
= -\lambda \sum_\ell c_{jm}^\ell \frac{\partial}{\partial t_l} \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N W(x, t)^{-\lambda - 1} - \lambda \int_{\gamma} dx_1 \cdot \cdot \cdot dx_N \frac{\partial}{\partial x_a} \left( Q_{ij}^a(x, t) W(x, t)^{-\lambda - 1} \right).
\]
After discarding the second term which is a total derivative we finally obtain

$$\frac{\partial}{\partial t_j} \frac{\partial}{\partial t_m} u^{(\lambda)}(t) = \sum_{\ell} c_{jm} \frac{\partial}{\partial t_\ell} \frac{\partial}{\partial t_0} u^{(\lambda)}(t). \quad (24)$$

The Gauss-Manin system (24) is a set of differential equations of the regular singular type and has been studied extensively in the mathematical literature. We see that it has exactly the same form as (17). Therefore the period integrals (18) provide solutions to the recursion relations of the 2-dimensional gravity! The exponent $\lambda$ is adjusted to the $U(1)$-charge of the operator $\sigma_n(\phi_i)$. In the case of the $A_{k+1}$ minimal series (13) the one-point functions are given by the integral of the superpotential with the exponents

$$\lambda = - \left( n + 1 + \frac{i + 1}{k + 2} \right). \quad (25)$$

The integral part of $\lambda$ refers to the gravitational excitations while the fractional part has the form $(\text{Coxeter exponent } +1)/(\text{dual Coxeter number})$. In the A-type theory the period integral becomes simply a residue integral at infinity.

The free-energy $F$ of a topological theory of a central charge $c$ on the sphere has a $U(1)$-charge (weight) $[F] = 3 - \frac{c}{3}$. If an operator $\sigma_n(\phi_i)$ carries a weight $[\sigma_n(\phi_i)] = q_{n,i}$, its conjugate parameter $t_{n,i}$ has a weight $[t_{n,i}] = 1 - q_{n,i}$ and then the one-point function $\langle \sigma_n(\phi_i) \rangle$ carries a weight $3 - \frac{c}{3} - (1 - q_{n,i}) = 2 - \frac{c}{3} + q_{n,i}$.

When the variables $\{x_j\}$ have $U(1)$-charges $[x_j] = q_j, j = 1, \ldots, N$, the exponent $\lambda_{n,i}$ is given by

$$\lambda_{n,i} = - \left( 2 - \frac{c}{3} + q_{n,i} - \sum_{j=1}^{N} q_j \right) = -2 + \frac{c}{3} + \frac{N}{2} - q_{n,i}, \quad (26)$$

where we have used $c = 3 \sum_j (1 - 2q_j)$. We note that when a variable, say, $x_j$, occurs quadratically in the superpotential being decoupled from the other variables, $x_j$ may be trivially eliminated from the superpotential with a shift of $\lambda$ by 1/2.
Therefore if the $A_{k+1}$-type singularity is considered as a 3-variable singularity, 
$W(x, y, z) = \frac{1}{k+2} x^{k+2} + \sum g_i x^i + y^2 + z^2$, $\lambda$ of (25) is replaced by $\lambda = -n - \frac{i+1}{k+2}$.

It is easy to check that if the $A, D, E$ minimal theories are all regarded as three-variable singularities, (26) is uniformly written as
\[
\lambda_{n,i} = -\frac{1}{g^*} - q_{n,i} = -\frac{ng^* + i + 1}{g^*},
\tag{27}
\]
where $g^*$ is the dual Coxeter number of the Lie algebra and $i$ is its Coxeter exponent.

We conjecture that the integral
\[
\oint_{\gamma} dx_1 \cdots dx_N W(x, t)^{-\lambda_{n,i}}
\tag{28}
\]
over a suitable cycle $\gamma$ gives a correct one-point function on the sphere of an operator with a weight $q_{n,i}$ of a general Landau-Ginzburg theory described by the superpotential $W(x, t)$. When the weights of the observables $\{q_{n,i}\}$ are all distinct there should be a unique cycle $\gamma$ of integration for each $\lambda_{n,i}$. In the case of a degeneracy among the weights $\{q_{n,i}\}$, there must be a corresponding multiplicity in the choice of integration contours $\gamma$.

Let us now check our conjecture and illustrate some computations using examples of the $D_\ell, E_6$ minimal models and the $\tilde{E}_6, c = 3$ theory.

$\ell$ theory

The superpotential for the $D_\ell$-series is given by
\[
W(x, y) = \frac{x^{\ell-1}}{2(\ell - 1)} + \frac{1}{2} xy^2 + \sum_{i=0}^{\ell-1} g_{2i}(t) x^i + t_* y.
\tag{29}
\]
This case may be reduced to the case of a single-variable theory by eliminating $y$ using the equation of motion $\partial_y W = 0$ [1]. The elimination of the variable $y$
introduces a Jacobian factor $\sqrt{x}$ which is cancelled by means of a change of variable $x = z^2$. Then the superpotential is given by

$$W(z) = \frac{z^{2(\ell-1)}}{2(\ell-1)} + \sum_{i=0}^{\ell-2} g_{2i}(t) z^{2i} - \frac{1}{2} t^2 z^{-2}. \quad (30)$$

We introduce a pair of Lax-type operators

$$W(z) = \frac{L(z)^{2\ell-2}}{(2\ell-2)}, \quad L(z) = z + \sum_{i=1}^{\infty} a_i(t, t^*) z^{-2i+1},$$

$$W(z) = -\frac{1}{2} M(z)^2, \quad M(z) = t_* z^{-1} + \sum_{j=1}^{\infty} b_j(t, t^*) z^{2j-1}. \quad (31)$$

Primary fields are given by

$$\phi_{2i} = [L^{2i} \partial_z L]_+ = \frac{\partial}{\partial t_{2i}} W, \quad i = 0, 1, \ldots, \ell - 2$$

$$\phi_* = [\partial_z M]_- = \frac{\partial}{\partial t_*} W \quad (32)$$

and their metric is equal to $\eta_{2i,2j} = \delta_{i+j, \ell-2}$, $\eta_{*,2i} = 0$, $\eta_{*,*} = -1$. Their gravitational descendants are

$$\sigma_n(\phi_{2i}) = c_{n,2i}[L^{2i+(2\ell-2)n} \partial_z L]_+, \quad c_{n,2i} = \left( (2i+1)(2i+1+2\ell-2) \cdots (2i+1+(n-1)(2\ell-2)) \right)^{-1},$$

$$\sigma_n(\phi_*) = c_{n,*}[M^{2n} \partial_z M]_-, \quad c_{n,*} = \frac{(-1)^n}{(2n-1)!!}. \quad (33)$$

The structure of the operators $\{\sigma_n(\phi_i)\}$ are exactly the same as in the $A_{2\ell-3}$-theory. One-point functions are then given by

$$\langle \sigma_n(\phi_{2i}) \rangle = (2\ell - 2)^{n+1+\frac{2i+1}{2\ell-1}} c_{n+2,2i} \int_{\infty} dW^{n+1+\frac{2i+1}{2\ell-1}},$$

$$\langle \sigma_n(\phi_*) \rangle = (-2)^{n+\frac{3}{2}} c_{n+2,*} \int_0 dW^{n+1+\frac{3}{2}}. \quad (34)$$

Note that the residue integrals for the exponents $(0, 2, 4, \ldots, 2\ell - 4)$ are taken at $\infty$ while the integrals for the exponent $\ell - 2$ is taken at the origin. When $\ell =$even, the
exponent $\ell - 2$ has a multiplicity 2 and we distinguish these two fields by taking different cycles of integration.

**$E_6$-theory**

The perturbed superpotential of the $E_6$ theory is given by

$$W(x_1, x_2) = \frac{1}{3} x_1^3 + \frac{1}{4} x_2^4 + s_{10} x_1 x_2^2 + s_7 x_1 x_2 + s_6 x_2^2 + s_4 x_1 + s_3 x_2 + s_0.$$  \hspace{1cm} (35)

The metric has a simple form in the flat coordinates $(t_0, t_3, t_4, t_6, t_7, t_{10})$

$$\eta_{ij} = P \langle \phi_i \phi_j \rangle = \delta_{i,j,10}, \quad \frac{\partial W}{\partial t_i} = \phi_i.$$  \hspace{1cm} (36)

The flat coordinates $\{t_i\}$ are related to the parameters $\{s_i\}$ as $t_i = s_i +$ (higher orders), $(i = 0, 3, 4, 6, 7, 10)$. Integrating the metric in $t_j$ we have $\langle \phi_i P \rangle = t_{10-i}$. Then our conjecture

$$\langle \phi_i \rangle = c_i \oint dx_1 dx_2 W(x_1, x_2, s)^{-\lambda_i}, \quad \lambda_i = -\frac{1}{2} - \frac{i + 1}{12}$$  \hspace{1cm} (37)

implies

$$t_{10-i} = c'_i \oint dx_1 dx_2 W(x_1, x_2, s)^{-\lambda_i-1}, \quad c'_i = -c_i \lambda_i.$$  \hspace{1cm} (38)

$c_i, c'_i$ are numerical constants independent of $\{t_j\}$. (38) expresses flat coordinates as the period integrals of the superpotential and is the counterpart of (14) of the $A_{k+1}$-theory. Let us examine (38) as a check of our conjecture. Integrals of the $E_6$-theory has been evaluated by Noumi in [10]. We can make use of his results and express $\{t_i\}$'s as functions of $\{s_i\}$'s. We adjust $\{c'_i\}$ such that the coefficients of $s_i$ in $t_i = s_i +$ (higher orders) is unity. Integrals are first evaluated in the case $s_6 = s_7 = s_{10} = 0$ and are expressed as solutions of generalized hypergeometric differential equations. One then applies the shift operator to recover their dependence on
\(s_6, s_7, s_{10}\)

\[
u^{(\lambda)}(s_0, s_3, s_4, s_6, s_7, s_{10}) = \exp \left( s_6 \frac{\partial}{\partial s_3} \left( \frac{\partial}{\partial s_0} \right)^{-1} + s_7 \frac{\partial^2}{\partial s_3 \partial s_4} \left( \frac{\partial}{\partial s_0} \right)^{-1} + s_{10} \frac{\partial^3}{\partial s_4 \partial s_3^2} \left( \frac{\partial}{\partial s_0} \right)^{-2} \right) u^{(\lambda)}(s_0, s_3, s_4, s_6 = 0, s_7 = 0, s_{10} = 0) .
\] (39)

When \(\lambda's\) assume special values as in (37), these solutions become simple polynomials. Furthermore for each value of \(\lambda\) there exists a unique solution which is regular (has no branch cut) at \(t_0 = 0\) and is physically acceptable. The result is

\[
\begin{align*}
t_{10} &= s_{10}, \\
t_7 &= s_7, \\
t_6 &= s_6 + \frac{1}{2} s_{10}^3, \\
t_4 &= s_4 - s_6 s_{10} - \frac{5}{12} s_{10}^4, \\
t_3 &= s_3 + s_7 s_{10}^2, \\
t_0 &= s_0 - \frac{1}{2} s_6^2 - \frac{5}{6} s_6 s_{10}^3 + \frac{1}{2} s_7^2 s_{10} - \frac{1}{4} s_{10}^6 + \frac{1}{2} s_4 s_{10}^2 .
\end{align*}
\] (40)

(40) agree with the formulas of the flat coordinates [1,11] which are obtained by demanding the constancy of the metric given by the integral

\[
\eta_{ij} = \oint dx_1 dx_2 \frac{\phi_i(x_1, x_2, t) \phi_j(x_1, x_2, t)}{\partial_{x_1} W \partial_{x_2} W} .
\] (41)

\(\hat{E}_6\)-theory

Let us consider a \(c = 3\) theory described by the potential

\[
\begin{align*}
W(x_1, x_2, x_3) &= \frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \alpha_1(t)x_1 x_2 x_3 - (\alpha_2 t_4 x_1 x_2 + \alpha_2 t_5 x_1 x_3 + \alpha_2 t_6 x_2 x_3) \\
&- (\alpha_3(t)t_1 x_1 + \alpha_5(t)t_6^2 x_1 + \alpha_4(t)t_4 t_5 x_1 + \alpha_3(t)t_2 x_2 + \alpha_5(t)t_6^2 x_2 + \alpha_4(t)t_4 t_6 x_2 \\
&+ \alpha_3(t)t_3 x_3 + \alpha_5(t)t_6^2 x_3 + \alpha_4(t)t_5 t_6 x_3) \\
&- (\alpha_6(t)(t_1 t_6 + t_2 t_5 + t_3 t_4) + \alpha_7(t)(t_4^3 + t_5^3 + t_6^3) + \alpha_8(t)t_4 t_5 t_6) + t_0 .
\end{align*}
\] (42)

This system possesses a marginal perturbation described by the parameter \(t\) in addition to the relevant perturbations due to parameters \(t_i, i = 1, \cdots, 6\). Dependence
of the superpotential on the variables \( \{ t_i \} \) in (42) is dictated by their \( U(1) \) charges and the discrete \( \mathbb{Z}_3 \) symmetry. Functions \( \alpha_i, (i = 1, \ldots, 8) \) are infinite series in the marginal parameter \( t \) since it carries a vanishing \( U(1) \) charge.

A novel feature of the \( c = 3 \) theories (elliptic singularities) is the appearance of the primitive factor in period integrals [12,13,11,14]. It is known that the flatness condition (21) no longer has a solution in the presence of marginal operators and one introduces a certain integration factor to the period integral. Then the flatness equation is modified to possess a solution and the Gauss-Manin equations (24) continue to hold. For the sake of illustration we examine the 2-point function

\[
t_0 = \langle P \phi_7 \rangle = \sqrt{h(t)} \int_{\gamma} dx_1 dx_2 dx_3 \log W(x, t) ,
\]

where \( \phi_7 \) is the marginal operator coupled to \( t \) and \( \sqrt{h(t)} \) is the primitive factor. We denote the scale-invariant part of the superpotential as \( W_0 \)

\[
W_0(x_1, x_2, x_3) = \frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - sx_1x_2x_3, \quad s \equiv \alpha_1(t)
\]

and expand \( W \) around \( W_0 \) in the RHS of (43). A finite number of terms remain in the expansion due to the scaling properties of the integral and we obtain

\[
\text{RHS of (43)} = t_4 t_5 t_6 (-2\alpha_2^3 I_2 - 3\alpha_2^2 \alpha_4 I_1) \\
+ (t_4^3 + t_5^3 + t_6^3) (-\frac{1}{3} \alpha_2^3 (\frac{1}{2} I_0 + \frac{3}{2} s I_1) / (1 - s^3) - \alpha_2^2 \alpha_5 I_1) \\
+ (t_3 t_4 + t_2 t_5 + t_1 t_6) (-\alpha_2^3 I_1) \\
- (\alpha_3^2 t_4 t_5 t_6 + \alpha_6(t_3 t_4 + t_2 t_5 + t_1 t_6) + \alpha_7(t_4^3 + t_5^3 + t_6^3)) I_0 + t_0 I_0 .
\]

Here

\[
I_i = \sqrt{h} \int_{\gamma} dx_1 dx_2 dx_3 (x_1 x_2 x_3)^i W_0(x_1, x_2, x_3)^{-i-1}, \quad i = 0, 1, 2
\]

and we have used

\[
\int (x_1 x_2)^2 / W_0^3 = \int (x_1 x_3)^2 / W_0^3 = \int (x_2 x_3)^2 / W_0^3 = (1/2 I_0 +
\]
Making use of the identity

\[ I_2 = \frac{s}{1 - s^3} (\frac{1}{2} I_0 + \frac{3}{2} s I_1) \]  

(47)

we find that the terms proportional to \( t_4 t_5 t_6, (t_3^2 + t_5^2 + t_6^2), (t_3 t_4 + t_2 t_5 + t_1 t_6) \) in (45) all cancel and we recover (43) if the following relations are obeyed

\[
\begin{align*}
\alpha_8 + \alpha_2^3 \frac{s}{1 - s^3} &= 0, \\
\alpha_2 \alpha_4 + \alpha_3^2 \frac{s^2}{1 - s^3} &= 0, \\
\alpha_2 \alpha_5 + \frac{1}{2} \alpha_2^3 \frac{s}{1 - s^3} &= 0, \\
\alpha_7 + \frac{1}{6} \alpha_2^3 \frac{1}{1 - s^3} &= 0, \\
\alpha_6 I_0 + \alpha_2 \alpha_3 I_1 &= 0
\end{align*}
\]

(48)

and

\[ I_0 = 1. \]  

(49)

Making use of the results of [13,11]

\[
\begin{align*}
\alpha_2 &= \sqrt{s}(1 - s^3)^{1/6}, \\
\alpha_3 &= \sqrt{s}(1 - s^3)^{-1/6}, \\
\alpha_4 &= -s' s^2 (1 - s^3)^{-2/3}, \\
\alpha_5 &= -\frac{1}{2} s' s (1 - s^3)^{-2/3}, \\
\alpha_6 &= -\frac{1}{2} \left( \frac{s''}{s} + \frac{3 s^2 s'}{(1 - s^3)} \right), \\
\alpha_7 &= -\frac{1}{6} s'^3 / (1 - s^3)^{-1/2}, \\
\alpha_8 &= -s'^3 / s (1 - s^3)^{-1/2},
\end{align*}
\]

\[ h(t) = s'^{-1} (1 - s^3) = \left( F\left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; s^3\right] \right)^{-2}, \quad t = s F\left[\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; s^3\right] / F\left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; s^3\right] \]

(50)

it is easy to check that the above relations are in fact satisfied. Here \( F\left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; s^3\right] \) and \( s F\left[\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; s^3\right] \) are the hypergeometric functions and solutions of the differential equation \((z(1 - z) \partial^2 / \partial z^2 + (2/3 - 5z/3) \partial / \partial z - 1/9) I = 0\) satisfied by the integral \( I = \oint dx_1 dx_2 dx_3 W_0^{-1} \). Due to (49) we have

\[ \oint_{\gamma} dx_1 dx_2 dx_3 W_0^{-1} = F\left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; s^3\right] \]

(51)

and hence the cycle \( \gamma \) diagonalizes the monodromy around the origin \( s = 0 \). If we denote the cycle corresponding to the solution \( s F\left[\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; s^3\right] \) as \( \gamma' \), the integral of
\( W^{-1} \) along \( \gamma' \) is identified as the 2-point function \( \langle PP \rangle \)

\[
\langle PP \rangle = t = \sqrt{h} \oint_{\gamma'} dx_1 dx_2 dx_3 W^{-1} = \sqrt{h} \oint_{\gamma'} dx_1 dx_2 dx_3 W_0^{-1} = sF\left[ \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{4}{3} ; s^3 \right] / F\left[ \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} ; s^3 \right].
\]

(52)

It is easy to see that \( t \) undergoes a fractional linear transformation when \( s^3 \) rotates around the discriminant \( s^3 = 1 \) of the elliptic curve \( W_0 = 0 \). It will be interesting to study implications of such a global transformation law of physical correlation functions.

So far the detailed study of 2-dimensional topological field theories has been largely limited to the case of topological minimal models of the \( A \)-type where the efficient method of pseudo-differential operators and the KP hierarchy is available. We hope that our method of period integrals may become their substitute and play an equally efficient role in the study of a more general class of topological field theories.

We would like to thank Profs. K. Saito, M. Noumi and Dr. I. Satake for discussions on singularity theory.

The researches of T.E. and S.-K.Y. are partly supported by the Grant-in-Aid for Scientific Research on Priority Area "Infinite Analysis".
REFERENCES

1. R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B352** (1991) 59.

2. K. Saito, "On the Periods of Primitive Integrals", Harvard Lecture Notes, 1980.

3. B. Blok and A. Varchenko, *Int. J. Mod. Phys.* **A7** (1992) 1467.

4. A. Losev, "Descendants Constructed from Matter Field in Topological Landau-Ginzburg Theories Coupled to Topological Gravity", ITEP preprint Nov. 1992.

5. T. Eguchi, H. Kanno, Y. Yamada and S.-K. Yang, "Topological Strings, Flat Coordinates and Gravitational Descendants", Phys. Lett. **B**, in press.

6. B. Dubrovin, "Integrable Systems and Classification of 2-Dimensional Topological Field Theories", SISSA preprint, Sept. 1992.

7. R. Dijkgraaf and E. Witten, *Nucl. Phys.* **B342** (1990) 486.

8. E. Witten, *Nucl. Phys.* **B340** (1990) 281.

9. K. Saito, Publ. RIMS, Kyoto University **19** (1983) 1231.

10. M. Noumi, Tokyo J. Math. **7** (1984) 1.

11. A. Klemm, S. Theisen and M. Schmidt, *Int. J. Mod. Phys.* **A7** (1992) 6215.

12. K. Saito, Publ. RIMS, Kyoto University **21** (1985) 75.

13. E. Verlinde and N.P. Warner, *Phys. Lett.* **B269** (1991) 96.

14. W. Lerche, D.-J. Smit and N.P. Warner, *Nucl. Phys.* **B372** (1992) 87.