Four-Loop Tadpoles: Applications in QCD

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Recent applications of single-scale four-loop tadpoles are briefly reviewed. An algorithm for the evaluation of current correlators based on differential equations is described and applied to obtain high moments of the vacuum polarization function at $\mathcal{O}(\alpha_s^2)$ as a preparation of $\mathcal{O}(\alpha_s^3)$ calculations.

1. INTRODUCTION

There are several reasons for the appearance of vacuum integrals in perturbative calculations. This type of integrals may, in fact, be introduced for any graph with any number of external legs if large mass or large momentum expansions are applied. This case is equivalent to the calculation of Wilson (matching) coefficients as part of the construction of an Effective Field Theory or to the determination of vacuum expectation values of composite operators (condensates) in the Operator Product Expansion. Iterative use of these techniques may produce integrals where all lines are either massless or carry just one mass. Another possibility is the calculation of renormalization group anomalous dimensions in the MS scheme. Here, the polynomial dependence of counterterms on dimensionful parameters is exploited and the relevant Green functions are first made dimensionless by deriving with respect to masses and momenta and then evaluated at vanishing external momenta and with all masses set to unity in order to avoid spurious infrared divergences. The problem is thus again reduced to the evaluation of tadpoles, albeit only to divergent parts.

Recently, problems involving tadpoles at the four-loop level have become tractable. This impressive progress has been made possible to a large extent by the Laporta algorithm for the reduction of integrals to masters described in \cite{1}, and by the difference equation method for the numerical evaluation of the masters proposed in the same publication. The first sets of integrals have been evaluated precisely using these principles \cite{2,3} (see also these proceedings \cite{4}) and confirmed by independent techniques in \cite{5}. At present, other methods are also available, see \cite{6} and \cite{7}. We should note, that lower precision values, which can be used for tests, can be obtained by the sector decomposition method \cite{8} and by Mellin-Barnes techniques \cite{9,10} implemented in the \texttt{MB} package \cite{11}, see also \cite{12}.

The first application of four-loop vacuum integrals concerned the determination of renormalization group parameters, in particular the four-loop QCD $\beta$-function \cite{13,14} and the mass anomalous dimension in the MS scheme \cite{15}. As far as the running of couplings and masses in theories with different mass scales is concerned, decoupling relations have also been determined at the same level of precision \cite{16,17,18}.

Another set of applications is related to the vacuum polarization function, because of its connection to the hadronic production cross section. In fact, the Taylor expansion coefficients, which are equivalent through dispersion relations to mo-
ments of the cross section allow one to obtain charm and bottom quark masses with good precision [18]. Such an analysis has been performed at four-loops using the first physical moment in [19,20] resulting in a large reduction of the scale dependence.

Four-loop vacuum integrals appear also when QCD corrections to electroweak observables, in particular the $W$ boson mass [21] and the effective weak mixing angle [22] are computed, through the corrections to the $\rho$ parameter of Veltman [23]. After many three-loop contributions have been evaluated [24,25,26], the pure $O(\alpha_s^3)$ part could finally be done in [27,28].

Finally, we should mention that the set of solved problems is not confined to the usual perturbation theory, since observables in high temperature QCD could also have been evaluated using similar techniques [29].

2. CURRENT CORRELATORS

One of the computations mentioned in the Introduction concerned the vacuum polarization function and in particular its Taylor expansion coefficients. It turned out that the evaluation of the first physical moment (expansion up to the second order in the external momentum squared) is already at the edge of the present techniques. Phenomenology requires, however, further moments to be computed.

Let us first explain the origin of the difficulties. Since the expansion operator is $\partial^2/\partial q_\mu \partial q^\mu$ and the evaluation point is at the origin, every term of the expansion introduces two dots (additional powers of the denominators) and one irreducible numerator. This leads to a steep growth of the number of integrals to reduce. When using the Laporta algorithm this means solving systems of equations involving millions of variables already for the first physical moment, and thus the next moment is in practice unreachable in any reasonable amount of time. It might seem that a better algorithm for tadpole reduction could solve this problem, but the experience made at the three-loop level speaks for the contrary. In fact, highly optimized software [30] used in [31] could only generate seven terms in a calculation that took several months.

Obviously, an algorithm of different complexity is needed here. Fortunately, the expansion coefficients are not given by unrelated integrals, but can be generated with linear complexity from differential equations. In fact, the vacuum polarization function is expressed through propagator integrals $P_i(z,\epsilon)$, with $z = q^2/4m^2$ and $d = 4 - 2\epsilon$ the dimension of space-time (a factor of $m^{-8\epsilon}$ has been taken out). Since a reduction to masters exists, we have a system of linear equations

$$\frac{d}{dz} P_i(z,\epsilon) = A_{ij}(z,\epsilon) P_j(z,\epsilon),$$

where the matrix $A_{ij}(z,\epsilon)$ has a block-triangular form with elements being rational functions of $z$ and $\epsilon$. The structure of the matrix allows one to reduce the problem to a solution of several small systems of differential equations, which are easily integrated after expansion in both variables. Vacuum integrals are now only the boundary conditions, and will thus have little dots and/or irreducible numerators. On the other hand, to a good approximation, every next term of the Taylor expansion is obtained by repeating the same procedure within the same amount of time (linear complexity).

There is of course, a price to pay for the improvement of computational complexity. First of all, what is needed is a reduction of massive four-loop propagator integrals. Even though, they will be much less numerous (of the order of $10^4$ for the four-loop vacuum polarization), they involve two variables. Second, one needs an automated approach to the treatment of spurious poles in both $z$ and $\epsilon$. We have constructed such an implementation the sketch of which can be found in Fig. 1.

We should note at this point that our method is an evolution of the one proposed for the two-loop sunrise graph in [32]. As a first application, we have applied our solution to the three-loop vacuum polarization function. With a prefactor of $(\alpha_s/\pi)^3 3Q^2/16\pi^2$, we have obtained the following 30 terms in a matter of hours (zeroth order term is zero)
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Figure 1. Schematic view of an implementation of automatic expansions of two-point functions.

\[ + C_F^2 \left( 0.15413 z + 0.30248 z^2 + 0.27107 z^3 + 0.27581 z^4 + 0.34237 z^5 + 0.45346 z^6 + 0.60069 z^7 + 0.77496 z^8 + 0.96941 z^9 + 1.1789 z^{10} + 1.3996 z^{11} + 1.6287 z^{12} + 1.8639 z^{13} + 2.1036 z^{14} + 2.3465 z^{15} + 2.5917 z^{16} + 2.8382 z^{17} + 3.0855 z^{18} + 3.3331 z^{19} + 3.5805 z^{20} + 3.8276 z^{21} + 4.074 z^{22} + 4.3195 z^{23} + 4.5639 z^{24} + 4.8072 z^{25} + 5.0492 z^{26} + 5.2899 z^{27} + 4.5292 z^{28} + 5.767 z^{29} + 6.0034 z^{30} \right) \]

\[ - C_A C_F \left( 0.0069711 z - 0.22146 z^2 + 0.015317 z^3 + 1.22184 z^4 + 0.37748 z^5 + 0.49364 z^6 + 0.58131 z^7 + 0.64841 z^8 + 0.70043 z^9 + 0.74121 z^{10} + 0.77344 z^{11} + 0.79907 z^{12} + 0.81954 z^{13} + 0.83592 z^{14} \right) + 0.84902 z^{15} + 0.85946 z^{16} + 0.86774 z^{17} + 0.87423 z^{18} + 0.87925 z^{19} + 0.88304 z^{20} + 0.88579 z^{21} + 0.88768 z^{22} + 0.88884 z^{23} + 0.88938 z^{24} + 0.88939 z^{25} + 0.88894 z^{26} + 0.88812 z^{27} + 0.88696 z^{28} + 0.88553 z^{29} + 0.88385 z^{30} \right) \]

\[ + C_F T_F \left( -2.39562 z - 0.016957 z^2 - 0.10262 z^3 - 0.13008 z^4 - 0.13959 z^5 - 0.14211 z^6 - 0.14155 z^7 - 0.13957 z^8 - 0.13694 z^9 - 0.13434 z^{10} - 0.13105 z^{11} - 0.12809 z^{12} - 0.12522 z^{13} - 0.12246 z^{14} - 0.11982 z^{15} - 0.11731 z^{16} - 0.11492 z^{17} - 0.11264 z^{18} - 0.11048 z^{19} - 0.10843 z^{20} - 0.10647 z^{21} - 0.10461 z^{22} - 0.10284 z^{23} - 0.10114 z^{24} - 0.099528 z^{25} - 0.097983 z^{26} - 0.096504 z^{27} - 0.095088 z^{28} - 0.09373 z^{29} - 0.092427 z^{30} \right) \]
\[ + C_F T_F n_1 (-1.99342 \, z + 0.68237 \, z^2 \]
\[ + 0.64329 \, z^3 + 0.63725 \, z^4 + 0.63623 \, z^5 \]
\[ + 0.63507 \, z^6 - 1.63287 \, z^7 + 0.62966 \, z^8 \]
\[ + 0.62566 \, z^9 + 0.6211 \, z^{10} + 0.61616 \, z^{11} \]
\[ + 0.61097 \, z^{12} + 0.60564 \, z^{13} + 0.60024 \, z^{14} \]
\[ + 0.59482 \, z^{15} + 0.58943 \, z^{16} + 0.58409 \, z^{17} \]
\[ + 0.57883 \, z^{18} + 0.57365 \, z^{19} + 0.56858 \, z^{20} \]
\[ + 0.5636 \, z^{21} + 0.55874 \, z^{22} + 0.55398 \, z^{23} \]
\[ + 0.54933 \, z^{24} + 0.54479 \, z^{25} + 0.54035 \, z^{26} \]
\[ + 0.53603 \, z^{27} + 0.5318 \, z^{28} + 0.52768 \, z^{29} \]
\[ + 0.52365 \, z^{30} \]

Let us note finally that the most costly part of the calculation is not the preparation of differential equation or their solution, but the reduction of the integrals which occur in the actual diagrams. The reduction at the four-loop level is under way.

3. CONCLUSIONS

We have reviewed the current status of calculations involving four-loop tadpoles. Further progress will depend crucially on the development of more efficient algorithms of reduction of integrals to masters. We have made a first step in this direction and presented a method to evaluate high order expansions of current-current correlators. As an illustration, we have applied this approach to the vacuum polarization function at the three-loop level.

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