Multisymplectic structures and invariant tensors for Lie systems

X Gràcia, J de Lucas, M C Muñoz-Lecanda and S Vilarino

1 Department of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Spain
2 Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, Warszawa, Poland
3 Centro Universitario de la Defensa & IUMA, Zaragoza, Spain

E-mail: javier.de.lucas@fuw.edu.pl

Received 19 December 2018, revised 13 March 2019
Accepted for publication 4 April 2019
Published 26 April 2019

Abstract

A Lie system is a non-autonomous system of differential equations describing the integral curves of a non-autonomous vector field taking values in a finite-dimensional Lie algebra of vector fields, a so-called Vessiot–Guldberg Lie algebra. This work pioneers the analysis of Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a multisymplectic structure: the multisymplectic Lie systems. Geometric methods are developed to consider a Lie system as a multisymplectic one. By attaching a multisymplectic Lie system via its multisymplectic structure with a tensor coalgebra, we find multisymplectic and coalgebra methods to derive superposition rules, constants of motion, and invariant tensor fields relative to its evolution. Our results are illustrated with examples occurring in physics and mathematics such as Schwarz equations, control systems, or diffusion-type equations. This also shows the interest of multisymplectic techniques in ordinary differential equations, which represents a not much investigated field of application of multisymplectic geometry.

Keywords: Casimir element, multisymplectic structure, Lie system, nonlinear superposition rule, tensor coalgebra, Grassmann algebra, invariant tensor

1. Introduction

A Lie system is a non-autonomous first-order system of ordinary differential equations in normal form whose general solution can be expressed as an autonomous function, a so-called superposition rule [21, 23, 50, 66], depending on a generic finite family of particular
solutions and a set of constants. Standard examples of Lie systems are most types of Riccati equations \[2, 35, 66\] and non-autonomous first-order affine systems of ordinary differential equations \[23\].

The Lie–Scheffers theorem \[22, 50, 55\] states that a Lie system amounts to a \(t\)-dependent vector field taking values in a Vessiot–Guldberg Lie algebra \[23, 43\]. The later property gave rise to a number of methods for the determination of superposition rules based upon the integration of systems of ordinary and/or partial differential equations that are simpler to solve than the Lie systems they describe \[19, 22, 23, 66\]. At the same time, the Lie–Scheffers theorem also showed that being a Lie system is rather the exception than the rule \[23\]. Despite this, Lie systems admit relevant physical and mathematical applications, as witnessed by the many works on the topic \[9, 17, 21, 33, 39, 43, 60, 63, 66\]. This is due, in part, to the fact that Lie systems can be applied to the study of higher-order systems of ordinary differential equations and partial differential equations (PDEs) such as Wess–Zumino–Novikov–Witten equations, \(\mathbb{CP}^N\)-sigma models, Schwarz equations, Toda lattices, Bäcklund transformations for KdV equations, and other relevant physical systems (see \[19, 22, 32, 38\]).

Recently, a lot of attention has been paid to Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields and/or Lie symmetries relative to geometric structures: Poisson and symplectic \[4, 7, 9, 17, 21, 25, 33\], Dirac \[17, 20\], \(k\)-symplectic \[52\], Jacobi \[41\], Riemann \[42, 49\], and others \[17, 49\]. We say that these Lie systems admit compatible geometric structures. Although such Lie systems represent a relatively small subclass of Lie systems \[4, 37, 39, 49\], they seem to admit more applications than Lie systems without compatible geometric structures \[9, 49\]. For instance, Lie systems with compatible geometric structures appear in the study of the chaotic or integrable character of classical systems \[3\] and in the analysis of a real analogue of the Schwarz derivative \[20\], which occurs in the theory of Teichmüller spaces and modular forms \[48\].

Geometric structures compatible with Lie systems allow for the algebraic construction of superposition rules, constants of motion, and other evolution invariants of Lie systems without solving generally complicated systems of partial and/or ordinary differential equations \[7, 20, 52\] as in standard methods \[22, 23, 66\]. Such geometric structures also unveil the geometric properties of superposition rules and constants of motion \[7, 9\], lead to the investigation of Hamiltonian systems through Poisson–Hopf algebras \[6\], and allow for the analysis of physical and mathematical problems \[9, 17, 39, 49\].

The first aim of this paper is to introduce Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a multisymplectic structure \[16\]. Such systems are called multisymplectic Lie systems and the associated multisymplectic form is referred to as a compatible multisymplectic form. To motivate the relevance of multisymplectic Lie systems, we provide examples appearing in physical and mathematical problems, e.g. in the study of the Schwarz derivative \[8, 46, 58\], control systems \[54, 60\], and diffusion equations \[62\].

The theory of multisymplectic Lie systems plays a more relevant role than Lie systems related to other geometric structures \[7, 20, 25, 49, 52\]. For instance, since symplectic structures are multisymplectic ones, the Lie–Hamilton systems admitting a compatible symplectic structure \[25\] can be considered as multisymplectic Lie systems. Moreover, we prove that the hereafter called locally automorphic Lie systems, which are Lie systems that are locally diffeomorphic to the very relevant automorphic Lie systems \[11, 21, 45\], can always be studied through multisymplectic Lie systems (see theorem 4.9). This illustrates that multisymplectic Lie systems admit much more applications than other types of Lie systems with compatible geometric structures that cannot be used to study all above-mentioned types of Lie systems, e.g. Lie–Hamilton ones (see \[20, 52\]). As a byproduct of our techniques, we find that multisymplectic Lie systems can frequently be endowed with other compatible geometric
structures, e.g. Dirac and $k$-symplectic structures \cite{20, 52}, which can be used to apply previously known techniques, for instance, to obtaining their constants of motion or superposition rules.

More specifically, theorems 4.9, 4.11, and corollary 4.12, give methods to endow locally automorphic Lie systems with a compatible multisymplectic structure and other invariants, e.g. compatible presymplectic structures which can also be understood as Dirac structures \cite{20}. As the local diffeomorphisms mapping locally automorphic Lie systems into automorphic ones are difficult to be obtained explicitly and they are generally locally defined, it is unlikely that automorphic Lie systems can be used to study directly locally automorphic Lie systems. Despite that, the existence of the local diffeomorphism (see theorem 4.2) is at the core of all methods in section 4 to infer the existence of invariant geometric structures, e.g. multisymplectic volume forms, for locally automorphic Lie systems. Such compatible multisymplectic forms can be obtained algebraically in an easy manner under mild conditions, e.g. when the locally automorphic Lie system admits a unimodular Vessiot–Guldberg Lie algebra (see corollary 4.12). Additionally, other accessory results concerning the properties of locally automorphic Lie systems are detailed in corollaries 4.3 and 4.4.

Next, compatible multisymplectic forms are employed to study superposition rules and constants of motion for multisymplectic Lie systems in geometric and algebraic terms. Our methods do not require the integration of systems of partial or ordinary differential equations as most standard methods in the literature \cite{22, 66}. Our procedures also avoid the transformation of a Lie system onto a normal form, as employed in several works \cite{37, 43, 61}. As a byproduct, our approach also retrieves algebraically and geometrically invariants and geometric structures related to Lie systems appearing in previous works \cite{7, 9, 20, 52}. These structures were obtained in the previous literature in an ad hoc manner or by solving systems of PDEs. Then, our work simplifies their derivation.

It is worth noting that multisymplectic geometry is mostly concerned with the description of classical field theories and partial differential equations \cite{15, 16}. Meanwhile, this paper is focused upon the development of a rather non-standard field of application of multisymplectic structures: systems of first-order ordinary differential equations.

Remarkably, the coalgebra method to derive superposition rules and constants of motion for Lie–Hamilton systems developed in \cite{7, 9} can be retrieved as a consequence of our techniques when it concerns to Lie–Hamilton systems related to symplectic forms. Moreover, our methods give rise to tensor field invariants for multisymplectic Lie systems from invariants of tensor algebras, which are more general than the invariant structures appearing in the standard coalgebra method, e.g. Casimir elements and invariant functions \cite{7}.

More specifically, a multisymplectic Lie system $(N, \Theta, X)$, where $X$ is a Lie system on a manifold $N$ with a compatible multisymplectic form $\Theta$, is endowed with a finite-dimensional Lie algebra $\mathfrak{M}$ of Hamiltonian forms for one of its Vessiot–Guldberg Lie algebras: a so-called Lie–Hamilton algebra of $X$. If $\mathfrak{g}$ is an abstract Lie algebra isomorphic to $\mathfrak{M}$, then the adjoint representation of $\mathfrak{g}$ can be extended to a Lie algebra representation on the tensor algebra, $T(\mathfrak{g})$, which makes the latter into a $\mathfrak{g}$-module \cite{64}. Similarly, the symmetric and Grassmann algebras, $S(\mathfrak{g})$ and $\Lambda(\mathfrak{g})$, can be considered as $\mathfrak{g}$-submodules of $T(\mathfrak{g})$. Moreover, we endow $T(\mathfrak{g})$, $S(\mathfrak{g})$, $\Lambda(\mathfrak{g})$ with coalgebra structures (see \cite{7, 27} for details), which are extended to the tensor products $T^{(m)}(\mathfrak{g}) = T(\mathfrak{g}) \otimes \cdots \otimes T(\mathfrak{g})$, $S^{(m)}(\mathfrak{g}) = S(\mathfrak{g}) \otimes \cdots \otimes S(\mathfrak{g})$, and $\Lambda^{(m)}(\mathfrak{g}) = \Lambda(\mathfrak{g}) \otimes \cdots \otimes \Lambda(\mathfrak{g})$. Previous structures are then represented as covariant tensor fields on $N$ and $N^m$ in such a way that the $\mathfrak{g}$-invariants in $T(\mathfrak{g})$ (or its $\mathfrak{g}$-submodules $\Lambda(\mathfrak{g})$ and $S(\mathfrak{g})$) give rise to tensor invariants for $X$ and its so-called diagonal prolongations \cite{22}; see diagrams (5.3) and (5.4) for details. This is employed to obtain constants of motion and superposition rules for $X$ \cite{23}.
Our approach shows that invariants and superposition rules for multisymplectic Lie systems can be obtained through Casimir elements of universal enveloping algebras, which can be understood as symmetric tensors in $T(g)$, or co-cycles and other types of elements of the Chevalley–Eilenberg cohomology of $g$ [64], which are understood as antisymmetric tensors of $T(g)$. Moreover, this method gives rise to obtaining $k$-symplectic or presymplectic structures compatible with Lie systems, which allows for the application of the techniques in [20, 52] to study multisymplectic Lie systems.

As an application, our methods are employed to study superposition rules for multisymplectic Lie systems related to locally automorphic Lie systems. In particular, the cases of Schwarz equations and Riccati-type diffusion systems are studied in detail, while control and Darboux–Brioschi–Halphen systems are used to illustrate some results and/or techniques [40, 54, 60].

The structure of the paper goes as follows. Section 2 surveys several fundamental concepts on Lie systems and multisymplectic structures to be used hereupon. Section 3 is devoted to motivating the definition of multisymplectic Lie systems and to illustrating some of its applications in the physics and mathematics literature. Methods for the calculation of compatible multisymplectic forms for locally automorphic Lie systems are described in section 4. The use of multisymplectic structures and tensor coalgebras for the determination of invariants, constants of motion, and superposition rules for multisymplectic Lie systems is developed in section 5. Section 6 summarises the results of the work and provides some hints on future research. Additionally, the appendix contains the proof of some technical results.

2. Some basic concepts and notations

Unless otherwise stated, we assume all mathematical objects to be real, smooth, and globally defined. This permits us to omit minor technical problems so as to highlight the main aspects of our theory. $N$ will hereafter represent an $n$-dimensional connected manifold. All remaining manifolds are considered to be finite-dimensional and connected if not stated otherwise.

2.1. Generalised distributions and $t$-dependent vector fields

Let $V$ be a Lie algebra. Given two subsets $A, B \subset V$, we write $[A, B]$ for the linear space spanned by the Lie brackets between elements of $A$ and $B$. Meanwhile, $\text{Lie}(B)$ stands for the smallest Lie subalgebra of $V$ containing $B$.

Given a vector bundle $\rho: P \to N$, we denote by $\Gamma(\rho)$ its $C^\infty(N)$-module of sections. In particular, if $\tau_N: TN \to N$ is the tangent bundle projection, then $X(N) = \Gamma(\tau_N)$ designates the $C^\infty(N)$-module of vector fields on $N$.

Remember that a generalised distribution $\mathcal{D}$ on a manifold $N$ is a function mapping each $x \in N$ to a linear subspace $\mathcal{D}_x \subset T_xN$. A vector field $Y$ on $N$ is said to take values in $\mathcal{D}$, in short $Y \in \mathcal{D}$, when $Y_x \in \mathcal{D}_x$ for all $x \in N$.

An arbitrary set $V$ of vector fields on $N$ generates a generalised distribution $\mathcal{D}^V$ on $N$ by considering, at each point $x \in N$, the linear span of all of its vector fields: $\mathcal{D}^V_x = \text{span}\{X_x \mid X \in V\}$. As these vector fields are smooth by assumption, the generalised distribution $\mathcal{D}^V$ is smooth [29]. Along the paper all distributions are assumed to be smooth.

The dimension of $\mathcal{D}_x$ is called the rank of $\mathcal{D}$ at $x$. A generalised distribution $\mathcal{D}$ is regular at $x' \in N$ when, in a neighbourhood of $x'$, the distribution has constant rank. The generalised distribution $\mathcal{D}$ is called regular when its rank is constant on the whole $N$.  


A \( t \)-dependent vector field on \( N \) is a map \( X : (t, x) \in \mathbb{R} \times N \mapsto X(t, x) \in TN \) such that \( \tau_N \circ X = \pi_2 \), where \( \pi_2 : (t, x) \in \mathbb{R} \times N \mapsto x \in N \). An integral curve of \( X \) is a curve \( \gamma : \mathbb{R} \to N \) such that
\[
\frac{d\gamma}{dt}(t) = X(t, \gamma(t)), \quad \forall t \in \mathbb{R}.
\] (2.1)

Then, \( \tilde{\gamma}(t) = (t, \gamma(t)) \) is an integral curve of the suspension \( \tilde{X} \) of \( X \), namely the vector field \( \tilde{X} = \partial/\partial t + X \) on the product manifold \( \mathbb{R} \times N \) [1]. Conversely, if \( \tilde{\gamma} : \mathbb{R} \to \mathbb{R} \times N \) is an integral curve of the suspension \( \tilde{X} \) satisfying \((\pi_1 \circ \tilde{\gamma})(t) = t \) for every \( t \), where \( \pi_1 : (t, x) \in \mathbb{R} \times N \mapsto t \in \mathbb{R} \), then \( \gamma = \pi_2 \circ \tilde{\gamma} \) is an integral curve of \( X \).

Every \( t \)-dependent vector field \( X \) gives rise to a unique system (2.1) describing its integral curves. Also, every system (2.1) describes the integral curves \( \tilde{\gamma} : t \in \mathbb{R} \to (t, \gamma(t)) \in \mathbb{R} \times N \) of the suspension of a unique \( t \)-dependent vector field \( X \). This motivates to use \( X \) to designate both a \( t \)-dependent vector field and its associated system (2.1), indistinctly.

Notice that giving a \( t \)-dependent vector field \( X \) amounts to providing a family of vector fields \( \{X_t\}_{t \in \mathbb{R}} \) on \( N \), with \( X_t : x \in N \mapsto X(t, x) \in TN \) [23]. This enables us to relate \( t \)-dependent vector fields to several geometric structures given in the following definition.

**Definition 2.1.** Let \( X \) be a \( t \)-dependent vector field on \( N \). The **smallest Lie algebra** of \( X \) is the smallest real Lie algebra, \( V^X \), containing the vector fields \( \{X_t\}_{t \in \mathbb{R}} \), namely \( V^X = \text{Lie}(\{X_t\}_{t \in \mathbb{R}}) \).

The **associated distribution** of \( X \) is the generalised distribution on \( N \) spanned by the vector fields of the smallest Lie algebra \( V^X \), that is, \( D^{V^X} \).

It can be proved that the rank of \( D^{V^X} \) must only be constant on the connected components of an open and dense subset of \( N \), where the distribution becomes regular, involutive, and integrable (see [25]). The most relevant instance for us is when \( D^{V^X} \) is determined by a finite-dimensional \( V^X \) and hence the distribution becomes integrable on the whole \( N \) in the sense of Stefan–Sussmann [56, p 63]. Among other reasons, the associated distribution is important to study superposition rules for Lie systems [23].

2.2. Lie systems

Let us now turn to some fundamental notions appearing in the theory of Lie systems (see [23] for details).

**Definition 2.2.** A **superposition rule** depending on \( m \) particular solutions for a system \( X \) in \( N \) is a function \( \Phi : N^m \times N \to N \), \( x = \Phi(x_1, \ldots, x_m; \lambda) \), such that the general solution, \( x(t) \), of \( X \) can be brought into the form \( x(t) = \Phi(x_1(t), \ldots, x_m(t); \lambda) \), where \( x_1(t), \ldots, x_m(t) \) is any generic family of particular solutions and \( \lambda \) is a point of \( N \) to be related to initial conditions. A **Lie system** is a system of first-order ordinary differential equations which admits a superposition rule.

The conditions ensuring that a \( t \)-dependent system possesses a superposition rule are stated by the **Lie–Scheffers theorem** [22, 50].

**Theorem 2.3.** A \( t \)-dependent vector field \( X \) admits a superposition rule if and only if \( X \) can be written as \( X = \sum_{\alpha=1}^{r} b_{\alpha}(t)X_{\alpha} \) for a certain family \( b_1(t), \ldots, b_r(t) \) of functions and a collection \( X_1, \ldots, X_r \) of vector fields spanning an \( r \)-dimensional real Lie algebra.

In other words, \( X \) admits a superposition rule if and only if its smallest Lie algebra \( V^X \) is finite-dimensional. Normally, a Lie system is given in the form \( X = \sum_{\alpha=1}^{r} b_{\alpha}(t)X_{\alpha} \), where
the vector fields \( X_\alpha \), with \( \alpha = 1, \ldots, r \), span a Lie algebra \( V \) that may strictly contain \( V^X \). Then, \( V \) is called a Vessiot–Guldberg Lie algebra of \( X \).

In view of the preceding theorem and comments, from now on we will denote a Lie system as a triple \( (N, X, V) \), where \( N \) is a manifold, \( X \) is a \( t \)-dependent vector field on \( N \) as given by Lie–Scheffers theorem, and \( V \) is a Vessiot–Guldberg Lie algebra of \( X \).

Now we will see how the integration of a Lie system on a manifold can be reduced to the integration of a Lie system on a Lie group [21, 65]. To this end, we need to recall some facts about Lie group actions.

Consider a (left) Lie group action \( \phi \colon G \times N \to N \), or more generally a local Lie group action \( \phi \colon D \subset G \times N \to N \), and denote by \( \phi_x = \phi(., x) \) the partial map defined, in general, on an open neighbourhood of the neutral element of \( G \) (see [59] for details). Then one defines, for every \( \xi \in T_x G \), the fundamental vector field \( \xi_N \in \mathfrak{X}(N) \), given by \( \xi_N(x) = T_x \phi_x(\xi) \). The tangent space \( T_x G \) is in bijection with the left-invariant, \( \mathfrak{X}_L(G) \), and right-invariant, \( \mathfrak{X}_R(G) \), vector fields on \( G \); let us denote by \( \xi^L \) and \( \xi^R \) the respective invariant vector fields associated with \( \xi \in T_x G \). By convention, \( g = T_x G \) inherits its Lie algebra structure from \( \mathfrak{X}_L(G) \). Then (see [47, chapter 20])

1. The map \( \hat{\phi} : g \to \mathfrak{X}(N) \), \( \xi \mapsto \xi_N \), is a Lie algebra antihomomorphism (the infinitesimal generator of \( \phi \)).
2. For every \( x \in N \), the right-invariant vector field \( \xi^R \) and the fundamental vector field \( \xi_N \) are \( \phi \)-related.

In general, an antihomomorphism \( g \to \mathfrak{X}(N) \) is called a (left) Lie algebra action, and, when \( g \) is finite-dimensional, it can be integrated to a Lie group action. More precisely:

**Theorem 2.4.** Let \( g \) be a finite-dimensional Lie algebra, and let \( G \) be a Lie group with Lie algebra \( g \). Given a Lie algebra action \( \hat{\phi} \) of \( g \) on a manifold \( N \), there exists a local Lie group action \( \phi \) of \( G \) on \( N \) such that \( \hat{\phi} \) is the infinitesimal generator of \( \phi \). If \( G \) is simply connected and the vector fields of the image of \( \hat{\phi} \) are complete, then \( \phi \) can be supposed to be a global Lie group action.

The proof of these results, and related facts, can be found, for instance, in [59, p.58] [47, p.529] [12, p.207]. This Lie group action is the device that relates the Lie system on \( N \) to a Lie system on \( G \).

**Theorem 2.5.** Let \( (N, X, V) \) be a Lie system of the form \( X = \sum_{\alpha=1}^r b_\alpha(t)X_\alpha \), where \( X_1, \ldots, X_r \) is a basis of the Vessiot–Guldberg Lie algebra \( V \). Let \( G \) be a Lie group whose Lie algebra is isomorphic to \( V \), and let \( \varphi \) be a local Lie group action of \( G \) on \( N \) as given by the preceding theorem. Let \( X^G_\alpha \) be the right-invariant vector fields on \( G \) related to the vector fields \( X_\alpha \) through the action \( \varphi \). Then:

1. The triple \( (G, X^G, V^G) \), where
   \[
   X^G(t, g) = \sum_{\alpha=1}^r b_\alpha(t)X^G_\alpha(g)
   \]
   and \( V^G = \mathfrak{X}_G(G) \), is a Lie system on \( G \).
2. For every \( x_0 \in N \) and \( t \in \mathbb{R} \), the vector field \( X^G \) is \( \varphi_{x_0} \)-related with \( X_\alpha \); namely, the \( t \)-dependent vector fields \( X^G \) and \( X \) are \( \varphi_{x_0} \)-related.
3. If \( g(t) \) is the integral curve of \( X^G \) with \( g(0) = e \), and \( x_0 \in N \), then \( x(t) = \varphi(g(t), x_0) \) is the integral curve of \( X \) with \( x(0) = x_0 \).
The proof of this result is almost immediate: the vector fields \( X^G_p \) span the Lie algebra \( \mathfrak{X}_G(G) \), the \( t \)-dependent vector field \( X^G \) is \( \varphi_{\omega_0} \)-related with \( X \), and this implies that \( \varphi_{\omega_0} \) maps integral curves of \( X^G \) to integral curves of \( X \) (see \([21, 23]\) for details).

In this manner, if \( \varphi \) is explicitly known, then finding all the integral curves of \( X \) reduces to finding one particular integral curve of (2.2). Conversely, the general solution of \( X \) enables us to construct the integral curve of (2.2) with \( g(0) = e \) by solving an algebraic system of equations obtained through \( \varphi \); this can be used to solve other Lie systems \([23]\).

Lie systems of the form (2.2) are sometimes called *automorphic Lie systems* in the literature \([10, 65]\). Due to their specific structure, these systems admit invariant forms relative to their evolution; this will be studied later in section 4. We will say that the automorphic Lie system

\[
(g; X^G, \Theta^k) = \Phi^G_{VR}(G) \tag{2.2}
\]

is *globally Hamiltonian* (or simply *Hamiltonian*) if

\[
\omega^\flat_{\varphi_{\omega_0}} : \mathfrak{X}(N) \to \Omega^{k-1}(N),
\]

is injective. In this case, the corresponding morphism of \( \mathcal{C}^\infty(N) \)-modules \( \hat{\omega} : \mathfrak{X}(N) \to \Omega^{k-1}(N) \), \( X \mapsto \iota_X \omega \), is also injective.

**Definition 2.6.** A multisymplectic \( k \)-form on \( N \) is a closed and 1-nondegenerate differential \( k \)-form \( \Theta \in \Omega^k(N) \). A multisymplectic manifold of degree \( k \) is a pair \((N, \Theta)\), where \( \Theta \) is a multisymplectic \( k \)-form on \( N \).

Thus, multisymplectic two-forms are just symplectic forms. Multisymplectic \( n \)-forms on \( N \) coincide with volume forms. In what follows, we assume that \( \dim N \geq 2 \). In this case, every multisymplectic \( k \)-form has degree \( k \geq 2 \).

**Definition 2.7.** Let \((N, \Theta)\) be a multisymplectic manifold of degree \( k \). A vector field \( X \) on \( N \) is *locally Hamiltonian* if \( \iota_X \Theta \) is closed; this amounts to saying that \( \Theta \) is invariant by \( X \), that is, the Lie derivative of \( \Theta \) relative to \( X \) vanishes,

\[
\mathcal{L}_X \Theta = 0.
\]

A vector field \( X \) is *globally Hamiltonian* (or simply *Hamiltonian*) if \( \iota_X \Theta \) is exact; that is, there exists a differential \((k - 2)\)-form \( \Upsilon_X \) on \( N \) such that

\[
\iota_X \Theta = d\Upsilon_X.
\]
In previous cases, \( d \Omega X \) and \( \Omega X \) are called \textit{Hamiltonian} \((k - 1)\)- and \((k - 2)\)-forms associated with \( X \), respectively. For simplicity, we will generally call \( d \Omega X \) and \( \Omega X \) Hamiltonian forms associated with \( X \). The degree is obvious from context.

For locally Hamiltonian vector fields \( X, Y \), we have \( \iota_{[Y,X]} \Theta = d_{\Omega Y X} \Theta \), that is, their Lie bracket is a Hamiltonian vector field. Therefore, the space of (locally) Hamiltonian vector fields of \((\mathcal{N}, \Theta)\) is a Lie algebra.

\textbf{Definition 2.8.} Let \((\mathcal{N}, \Theta)\) be a multisymplectic manifold of degree \( k \).

- Let \( \xi, \zeta \in \text{Im} \hat{\Theta} \subset \Omega^{k-1}(\mathcal{N}) \), and let \( X, Y \in \mathfrak{X}(\mathcal{N}) \) be the unique vector fields such that \( \iota_X \Theta = \xi \) and \( \iota_Y \Theta = \zeta \). The bracket between \( \xi \) and \( \zeta \) is defined by

\[ \{ \xi, \zeta \} = \iota_{[Y,X]} \Theta \in \text{Im} \hat{\Theta}. \]

(2.3)

It is immediate from its definition that this bracket satisfies the Jacobi identity and becomes a Lie bracket.

- The bracket between Hamiltonian \((k - 2)\)-forms is defined in the following way: let \( \Omega X, \Omega Y \in \Omega^{k-2}(\mathcal{N}) \) be Hamiltonian forms, corresponding to the Hamiltonian vector fields \( X, Y \in \mathfrak{X}(\mathcal{N}) \). Then, we define

\[ \{ \Omega X, \Omega Y \} = \iota_{Y[X]} \Theta. \]

Finally we will recall the notion of multivector field, which will be used to find constants of motion of multisymplectic Lie systems (see [30, 34] for more details). An \( \ell \)-multivector field on \( \mathcal{N} \) is a section of \( \Lambda^\ell(T\mathcal{N}) \). An \( \ell \)-multivector field \( Y \) is said to be \textit{decomposable} if there is a family of vector fields \( Y_1, \ldots, Y_\ell \in \mathfrak{X}(\mathcal{N}) \) such that \( Y = Y_1 \wedge \ldots \wedge Y_\ell \).

Let \((\mathcal{N}, \Theta)\) be a multisymplectic manifold of degree \( k \). Generalising the notion of Hamiltonian vector field, we say that an \( \ell \)-multivector field \( Y \) is \textit{Hamiltonian} (with respect to \( \Theta \)) if there exists a \((k - \ell - 1)\)-form \( \Theta \) such that \( \iota_Y \Theta = d\Theta \). Additionally, \( Y \) is \textit{locally Hamiltonian} or \textit{multisymplectic} if \( \mathcal{L}_Y \Theta = 0 \) (see [15, 16]).

\textbf{2.4. Unimodular Lie algebras}

This section surveys the notions of unimodular Lie algebra and unimodular Lie group. These two concepts are necessary in the following parts of the paper, when the existence of multisymplectic structures compatible with Lie systems is addressed.

Consider a Lie group \( G \) with a Lie algebra \( \mathfrak{g} = T_0 G \). Recall that a (left) \textit{Haar measure} on \( G \) is given by a left-invariant volume form on \( G \) [44]. Every Lie group admits a Haar measure
given by a left-invariant volume form, and it is unique up to a non-zero multiplicative constant (see [14]).

Let $X^1_L, \ldots, X^r_L$ be a basis of the Lie algebra $\mathfrak{X}_L(G)$ of left-invariant vector fields on $G$ and let $\eta^1_L, \ldots, \eta^r_L$ be the dual basis of left-invariant differential one-forms. Then any left-invariant volume form on $G$ is a nonzero scalar multiple of

$$\Theta = \eta^1_L \wedge \ldots \wedge \eta^r_L.$$  

If $X^k$ is any left-invariant vector field on $G$, then

$$\mathcal{L}_{X^k}\Theta = -\text{Tr}(\text{ad}_{X^k})\Theta;$$

(2.5)

here $\text{Tr}$ denotes the trace of an endomorphism, and $\text{ad}: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, $v \mapsto \text{ad}_v$, denotes the adjoint representation of a Lie algebra $\mathfrak{g}$, given by $\text{ad}_v w = [v, w]$.

Remember that a Lie group is called unimodular if its Haar measure is also right-invariant [53]. All Abelian Lie groups, as well as all compact and semi-simple Lie groups, are unimodular [67]. In this work we are mainly concerned with the Lie algebras of unimodular Lie groups, whose main properties are detailed in the following definition and proposition.

**Definition 2.9.** A finite-dimensional Lie algebra $\mathfrak{g}$ is called unimodular when the maps $\text{ad}_v \in \text{End}(\mathfrak{g})$ are traceless—we say that the adjoint representation is traceless.

**Proposition 2.10.** A (connected) Lie group $G$ is unimodular if and only if its Lie algebra is unimodular.

**Proof.** Let $\Theta$ be a left-invariant volume form on $G$. The Lie algebra is unimodular if and only if the adjoint representation is traceless, namely the right-hand side of equation (2.5) is zero for any $X^k \in \mathfrak{X}_L(G)$. But this amounts to saying that $\Theta$ is also right-invariant: a tensor field $T$ on a connected Lie group $G$ is right-invariant if and only if it is invariant with respect to all left-invariant vector fields. □

**Remark 2.11.** A comment about the proof of proposition 2.10 is pertinent. It is well-known that each left-invariant vector field on $G$ admits a flow of the form $\phi : t \in \mathbb{R} \mapsto R_{\exp(tv)} \in \text{Diff}(G)$ for a certain $v \in \mathfrak{g}$. From this it follows that a vector field $Y$ on a connected Lie group $G$ is right-invariant if and only if it commutes with every left-invariant vector field $X^k$, namely $\mathcal{L}_{X^k}Y = 0$. This also applies to tensor fields on $G$.

### 3. Multisymplectic Lie systems

This section shows that there exist physical models whose dynamic can be studied through Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a multisymplectic structure. This suggests us to introduce the hereafter called *multisymplectic Lie systems*. Next, the most fundamental properties of these systems are detailed.

#### 3.1. Example: Schwarz equation

Consider a Schwartz equation [8, 46, 58] of the form

$$\frac{d^3 x}{dt^3} = \frac{3}{2} \left( \frac{dx}{dt} \right)^{-1} \left( \frac{d^2 x}{dt^2} \right)^2 + 2b(t) \frac{dx}{dt}. \quad (3.1)$$
The relevance of this differential equation is due to its appearance in the study of Milne–Pinney equations and the Schwarz derivative (see [9, 18, 20] and references therein).

The differential equation (3.1) is known to be a higher-order Lie system [18]. This means that the associated system of first-order differential equations obtained by adding the variables \( v = dx/dt \) and \( a = d^2x/dt^2 \), i.e.

\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a, \quad \frac{da}{dt} = \frac{3}{2} \frac{a^2}{v} + 2b_1(t)v,
\]

is a Lie system. Indeed, it is associated with the \( t \)-dependent vector field on \( \mathcal{O} = \{ (x,v,a) \in \mathbb{R}^3 \mid v \neq 0 \} \) of the form

\[
X^S = X_3 + b_1(t)X_1,
\]

where the vector fields given by

\[
X_1 = 2v \frac{\partial}{\partial v}, \quad X_2 = v \frac{\partial}{\partial v} + 2a \frac{\partial}{\partial a}, \quad X_3 = v \frac{\partial}{\partial v} + a \frac{\partial}{\partial a} + \frac{1}{2} \frac{a^2}{v} \frac{\partial}{\partial a},
\]

satisfy the commutation relations

\[
[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.
\]

As a consequence, \( X_1, X_2, \) and \( X_3 \) span a three-dimensional Lie algebra of vector fields \( V^S \) isomorphic to \( \mathfrak{sl}_2 \), and \( V^3 \) becomes a \( t \)-dependent vector field taking values in \( V^3 \), i.e. \( (\mathcal{O}, X^S, V^S) \) is a Lie system.

Let us determine a multisymplectic structure on \( \mathcal{O} \) so that the vector fields of \( V^S \) become locally Hamiltonian relative to it. Since \( X_1, X_2, X_3 \) span \( \text{to} \), then \( X_1 \wedge X_2 \wedge X_3 \neq 0 \) and \( X_1, X_2, X_3 \) admit a family of dual forms \( \eta_1, \eta_2, \) and \( \eta_3 \). This proves that \( \eta_\alpha(X_\beta) = \delta_\alpha^\beta \) for \( \delta_\alpha^\beta \) being the Kronecker delta function and \( \alpha, \beta = 1, 2, 3 \). In local coordinates,

\[
\eta_1 = \frac{a^2}{4v^3} dx - \frac{a}{v^2} dv + \frac{1}{2v} da, \quad \eta_2 = -\frac{a}{v^2} dx + \frac{1}{v} dv, \quad \eta_3 = \frac{1}{v} dx.
\]

By means of them, one can construct the volume form

\[
\Theta_S = \eta_1 \wedge \eta_2 \wedge \eta_3 = \frac{1}{2v^3} da \wedge dv \wedge dx.
\]

It is well known that, if a frame \( X_\alpha \) of \( TN \) satisfies \( [X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma \), then the Lie derivatives of the dual frame \( \eta_\alpha \) are given by \( \mathcal{L}_{X_\alpha} \eta_\beta = -c^\gamma_{\alpha\beta} \eta_\gamma \). In our example we have

\[
\mathcal{L}_{X_1} \eta_1 = -\eta_2, \quad \mathcal{L}_{X_1} \eta_2 = -2 \eta_3, \quad \mathcal{L}_{X_1} \eta_3 = 0, \\
\mathcal{L}_{X_2} \eta_1 = \eta_1, \quad \mathcal{L}_{X_2} \eta_2 = 0, \quad \mathcal{L}_{X_2} \eta_3 = -\eta_3, \\
\mathcal{L}_{X_3} \eta_1 = 0, \quad \mathcal{L}_{X_3} \eta_2 = 2 \eta_1, \quad \mathcal{L}_{X_3} \eta_3 = \eta_2,
\]

from which it is easily proved that

\[
\mathcal{L}_{X_\alpha} \Theta_S = 0, \quad \alpha = 1, 2, 3.
\]

This shows that the \( X_\alpha \) are locally Hamiltonian with respect to \( \Theta_S \). But indeed

\[
\iota_{X_1} \Theta_S = \frac{1}{v} (a \frac{\partial}{\partial v} - \frac{1}{2v} \frac{\partial}{\partial a}) \wedge dx = \frac{1}{2} \frac{\partial}{\partial a}, \\
\iota_{X_2} \Theta_S = \frac{1}{v^2} (\frac{a^2}{4v^3} dx + \frac{1}{v^2} \frac{\partial}{\partial a} \wedge dx = \frac{1}{2} \frac{\partial}{\partial a}, \\
\iota_{X_3} \Theta_S = -\frac{3a^2}{4v^3} dx \wedge dv - \frac{a}{v^2} \frac{\partial}{\partial a} \wedge dx + \frac{1}{2v^2} \frac{\partial}{\partial a} \wedge dv = -\frac{1}{v} \frac{\partial}{\partial a};
\]

\[\text{J. Phys. A: Math. Theor. 52 (2019) 215201}\]

X Gràcia et al
therefore, \(X_1, X_2,\) and \(X_3\) are Hamiltonian vector fields with respect to the multisymplectic structure \((\mathcal{O}, \Theta_3)\), with Hamiltonian one-forms \(\theta_1 = -\eta_1, \theta_2 = \frac{1}{2}\eta_1, \theta_3 = -\eta_1\).

As a consequence of the above, independently of the \(t\)-dependent coefficients in (3.2), the evolution of \(X^O\) preserves the volume form \(\Theta_3\). Since \(D^{V^O} = T\mathcal{O}\) and in view of (3.7), the value of \(\Theta_3\) at a point \(o \in \mathcal{O}\) determines the value of \(\Theta_3\) on the connected component of \(o\) in \(\mathcal{O}\). Moreover, \(\Theta_3\) is, up to a multiplicative constant on each connected component of \(\mathcal{O}\), the only volume form satisfying the equation (3.7). Since every one-form and two-form on a three-dimensional manifold are one-degenerate, the system under study has a unique, up to a non-zero proportional constant, multisymplectic form which is invariant under the action of \(V^S\).

### 3.2. Definition and main properties of multisymplectic Lie systems

The example given in section 3.1, along with the other multisymplectic Lie systems detailed throughout the rest of this work, motivate the following definition.

**Definition 3.1.** A (locally) multisymplectic Lie system is a triple \((N, \Theta, X)\), where \(X\) is a Lie system whose smallest Lie algebra \(V^X\) is a finite-dimensional real Lie algebra of (locally) Hamiltonian vector fields relative to a multisymplectic structure \(\Theta\) on \(N\). If \(\Theta\) has degree \(k\), we say that \((N, \Theta, X)\) is a multisymplectic Lie system of degree \(k\).

In view of the above, the Schwarz equation (written as a first-order system) defines a multisymplectic Lie system \((\mathcal{O}, \Theta_3, X^O)\) of degree 3.

A relevant family of multisymplectic Lie systems is provided by automorphic Lie systems, as stated by the following proposition.

**Proposition 3.2.** Every automorphic Lie system \((G, X, V^R)\), where \(V^R\) is the Lie algebra of right-invariant vector fields on a connected Lie group \(G\), is a locally multisymplectic Lie system relative to any left-invariant volume form.

**Proof.** The Lie system \(X\) has as a Vessiot–Guldberg Lie algebra the set of right-invariant vector fields. Therefore, any left-invariant differential form on \(G\) is invariant with respect to them. In particular, this is true for a left-invariant volume form \(\Theta\), which is also a multisymplectic form. From \(L_X \Theta = 0\), the \(X^R \in V^R\) are locally Hamiltonian vector fields with respect to \(\Theta\). \(\square\)

Next it will be shown that every multisymplectic Lie system is related to a finite-dimensional Lie algebra of Hamiltonian forms induced by a Vessiot–Guldberg Lie algebra. This Lie algebra will be a key structure for the determination of superposition rules for multisymplectic Lie systems.

Consider a multisymplectic Lie system \((N, \Theta, X)\), and let \(X_1, \ldots, X_r\) be a basis of its minimal Lie algebra \(V^X\). By assumption, \(X_1, \ldots, X_r\) are Hamiltonian vector fields relative to \(\Theta\), so \(\iota_{X_i} \Theta = d\theta_i\). This implies that the \(t\)-dependent vector field \(X = \sum_{i=1}^r b_i(t)X_i\) satisfies \(\iota_X \Theta = d\theta_i\) for \(\theta_i = \sum_{j=1}^r b_i(t)\theta_j\), i.e. \(X\) is a \(t\)-dependent Hamiltonian vector field with respect to the \(t\)-dependent differential \((k-2)\)-form given by \(\theta = \sum_{i=1}^r b_i(t)\theta_i\). So we are lead to the following definition.

**Definition 3.3.** Let \((N, \Theta, X)\) be a multisymplectic Lie system of degree \(k\). A Lie–Hamilton differential form for it is a \(t\)-dependent Hamiltonian differential \((k-2)\)-form \(\theta\) for \(X\), i.e. \(\iota_X \Theta = d\theta_i\) for every \(i\). A Lie–Hamilton algebra for the system is a finite-dimensional Lie algebra of Hamiltonian \((k-1)\)-forms (relative to the Lie bracket (2.3)) containing all the differentials \(d\theta_i\).
The same considerations apply to the locally multisymplectic case, since it is multisymplectic on small star-shaped open sets. In most practical applications one is concerned with local properties of dynamical systems. Hence, one can assume that the dynamical system under study is a multisymplectic Lie system.

Recall that every locally Hamiltonian vector field relative to a multisymplectic form of degree $k$ gives rise to a closed differential $(k−1)$-form and this correspondence is an injective Lie algebra anti-homomorphism relative to the Lie bracket of vector fields and the Lie bracket of differential $(k−1)$-forms (2.3). One has the following trivial consequence:

**Proposition 3.4.** Every multisymplectic Lie system $(N, Θ, X)$ possesses a smallest Lie–Hamilton algebra, namely

$$\mathfrak{M} = \{\iota_Z Θ \mid Z \in V^X\}.$$  

3.2.1 Example. As seen in section 3.1, the multisymplectic Lie system $(O, Θ_S, X_S)$ associated with the Schwarz equation is such that the vector fields $X_1, X_2, X_3$ spanning its Vessiot–Guldberg Lie algebra, $V^S$, are Hamiltonian with Hamiltonian forms

$$\theta_1 = -\frac{1}{v}dx, \quad \theta_2 = -\frac{a}{2v^2}dx + \frac{1}{2v}dv, \quad \theta_3 = -\frac{a^2}{4v^3}dx + \frac{a}{v^2}dv - \frac{1}{2v}da.$$  

(3.9)

Therefore, their differentials

$$d\theta_1 = \frac{dv \wedge dx}{v^2}, \quad d\theta_2 = \frac{2adv \wedge dx}{v^3} - \frac{da \wedge dx}{2v^2}, \quad d\theta_3 = -\frac{3a^2dx \wedge dv}{4v^4} - \frac{4a^2dv \wedge dx}{2v^3} + \frac{da \wedge dv}{2v^2}.$$  

(3.10)

satisfy the commutation relations (for the bracket of differentials of Hamiltonian forms)

$$\{d\theta_1, d\theta_2\} = -d\theta_1, \quad \{d\theta_1, d\theta_3\} = -2d\theta_2, \quad \{d\theta_2, d\theta_3\} = -d\theta_3,$$  

(3.11)

and span a Lie–Hamilton algebra $\mathfrak{M}$ of the system. Moreover, the Hamiltonian one-form for Schwarz equation reads

$$\theta_t = \theta_3 + b_1(t)\theta_1 = \left( -\frac{b_1(t)}{v} - \frac{a^2}{4v^3} \right)dx + \frac{a}{v^2}dv - \frac{1}{2v}da.$$

△

4. Locally automorphic Lie systems and invariant forms

In this section we analyse conditions under which one can ensure that there is a multisymplectic form $Θ$ invariant with respect to the elements of a Vessiot–Guldberg Lie algebra $V$.

It may be difficult to find multisymplectic forms compatible with a Lie system $X$ admitting a Vessiot–Guldberg Lie algebra $V$ as this requires to search for appropriate solutions, namely $Θ$, of a system of partial differential equations $L_Y Θ = 0$ for every $Y \in V$. Nevertheless, we can devise several simpler methods to find compatible invariant forms for a particular class of Lie systems with relevant physical applications: the hereafter locally automorphic Lie systems.

4.1. Locally automorphic Lie systems

**Definition 4.1.** A locally automorphic Lie system on $N$ is a triple $(N, X, V)$, where $X$ is a Lie system on $N$ with a Vessiot–Guldberg Lie algebra $V$ such that $\dim V = \dim N$ and $D^V = TN$. 
Locally automorphic Lie systems are called in this way because they are locally diffeomorphic to automorphic Lie systems. The following theorem proves this fact.

**Theorem 4.2.** Let \((N, X, V)\) be a locally automorphic Lie system, let \(G\) be a Lie group whose Lie algebra is isomorphic to \(V\), let \(\varphi\) be a local action of \(G\) on \(N\) obtained from the integration of \(V\), and let \((G, X^G, V^G)\) be the corresponding automorphic Lie system on \(G\) given by theorem 2.5. For every \(x \in N\) the map \(\varphi_x = \varphi(\cdot, x)\) is a local diffeomorphism mapping \(X^G\) to \(X\).

**Proof.** Recall that \(N\) is assumed to be an \(n\)-dimensional manifold. As stated in theorem 2.5, given \(x \in N\), the map \(\varphi_x\) relates a basis of right-invariant vector fields \(X^G_\alpha\) of \(G\) with a basis \(X_\alpha\) of \(V\), i.e. \(T_{g} \varphi_x : T_g G \rightarrow T_{\varphi_x(g),x} N\) maps \(X^G_\alpha(g)\) onto \(X_\alpha(\varphi(g, x))\). But due to the definition of locally automorphic system, the \(X^G_\alpha(g)\) are \(n\) linearly independent vectors constituting a basis of \(T_g G\), and the \(X_\alpha(\varphi(g, x))\) are also \(n\) linearly independent vectors constituting a basis of \(T_{\varphi_x(g),x} N\). Thus, \(T_{g} \varphi_x\) is a linear isomorphism, and \(\varphi_x\) becomes a local diffeomorphism. Therefore, when \(\varphi_x\) is restricted to open sets yielding a diffeomorphism, it sends \(X^G\) onto \(X\).

Remember that the action \(\varphi\) can be ensured to be globally defined only if \(G\) is simply connected and \(V\) consists of complete vector fields [59]. For simplicity, we will hereafter assume that \(\varphi\) is globally defined.

The mapping \(\varphi\) not only allows us to establish local diffeomorphisms \(\varphi_x : G \rightarrow N\), with \(x \in N\), but also maps certain geometric structures related to the locally automorphic Lie system \((N, X, V)\) with the associated automorphic one \((G, X^G, V^G)\).

In view of theorem 4.2, one obtains the following corollaries.

**Corollary 4.3.** Let \((N, X, V)\) be a locally automorphic Lie system. Then, \(X\) admits a superposition rule depending only on one particular solution of \(X\).

**Proof.** This is a consequence of theorem 4.2 and the fact that automorphic Lie systems admit a superposition rule depending on one particular solution, as mentioned at the end of section 2.2.

**Corollary 4.4.** If \((N, X, V^X)\) is a locally automorphic Lie system on a (connected) manifold, then all \(t\)-independent constants of motion of \(X\) are constants.

**Proof.** Recall that \(V^X\) is the smallest Lie algebra containing the vector fields \(X_t\). In particular, \(V^X\) is spanned by the vector fields \(X_t\) and all the linear combinations of their successive Lie brackets [23]. Hence, if \(f\) is a constant of motion of \(X\), then for every \(Z \in V^X\) we have \(L_Z f = 0\). As by hypothesis the vector fields of \(V^X\) span \(T_x N\) at every \(x \in N\), one has that \(df = 0\), and \(f\) is locally constant. If the manifold is connected, then \(f\) is constant.

The existence of the local diffeomorphism \(\varphi_x\) is useful to obtain theoretical properties of locally automorphic Lie systems, as for instance corollary 4.3. Nevertheless, its practical use to map them into automorphic Lie systems is quite limited because the locality of \(\varphi_x\) and the difficulties to obtain an explicit expression, which must be obtained by solving a system of nonlinear ordinary differential equations determined by \(V\). This is illustrated in the following two examples of locally automorphic Lie systems.

**Example 4.5 (The generalized Darboux–Brioschi–Halphen (DBH) system).** Consider the system of differential equations [28]:
\[
\begin{align*}
\frac{dw_1}{dr} &= w_3w_2 - w_1w_3 - w_1w_2 + \tau^2, \\
\frac{dw_2}{dr} &= w_1w_3 - w_2w_1 - w_2w_3 + \tau^2, \\
\frac{dw_3}{dr} &= w_2w_1 - w_3w_2 - w_3w_1 + \tau^2,
\end{align*}
\]

\[
\tau^2 = \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3),
\]

\[\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.\]

(4.1)

The DBH system with \(\tau = 0\) appears in the description of triply orthogonal surfaces and the vacuum Einstein equations for hyper-Kähler Bianchi-IX metrics [26, 28, 40]. Meanwhile, the generalized DBH system for \(\tau \neq 0\) is a reduction of the self-dual Yang–Mills equations corresponding to an infinite-dimensional gauge group of diffeomorphisms of a three-dimensional sphere [26].

Although the DBH system is autonomous, it is more appropriate, e.g. to obtain its Lie symmetries [31], to consider it as a Lie system related to a Vessiot–Guldberg Lie algebra \(V^{DBH}\) spanned by

\[
X_1^{DBH} = \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} + \frac{\partial}{\partial w_3}, \quad X_2^{DBH} = w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2} + w_3\frac{\partial}{\partial w_3} - (w_1w_3 - w_2(w_1 + w_3) + \tau^2)\frac{\partial}{\partial w_2} - (w_2w_1 - w_3(w_2 + w_1) + \tau^2)\frac{\partial}{\partial w_3}.
\]

In fact,

\[
[X_1^{DBH}, X_2^{DBH}] = X_3^{DBH}, \quad [X_1^{DBH}, X_3^{DBH}] = 2X_2^{DBH}, \quad [X_2^{DBH}, X_3^{DBH}] = X_1^{DBH}.
\]

Hence, \(\dim V^{DBH} = \dim \mathcal{O}\) and \(X_1^{DBH} \wedge X_2^{DBH} \wedge X_3^{DBH} \neq 0\) on an open submanifold \(\mathcal{O}\) of \(\mathbb{R}^3\). Thus, \((\mathcal{O}, X_1^{DBH}, V^{DBH})\) is a locally automorphic Lie system. To obtain a local diffeomorphism mapping this system into an automorphic one, we need to integrate the vector fields of \(V^{DBH}\). Their analytic form makes it clear that it is very hard to provide such a local diffeomorphism. \(\triangle\)

**Example 4.6 (A control system [54, 60]).** Consider the system of differential equations on \(\mathbb{R}^5\) given by

\[
\frac{dx_1}{dr} = b_1(t), \quad \frac{dx_2}{dr} = b_2(t), \quad \frac{dx_3}{dr} = b_2(t)x_1, \quad \frac{dx_4}{dr} = b_2(t)x_2^2, \quad \frac{dx_5}{dr} = 2b_2(t)x_1x_2, \tag{4.2}
\]

where \(b_1(t)\) and \(b_2(t)\) are arbitrary \(t\)-dependent functions.

This system is defined by the \(t\)-dependent vector field \(X^{CS} = b_1(t)X_1 + b_2(t)X_2\) on \(\mathbb{R}^5\), where the vector fields

\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_1^2\frac{\partial}{\partial x_4} + 2x_1x_2\frac{\partial}{\partial x_5}, \quad X_3 = \frac{\partial}{\partial x_3} + 2x_1\frac{\partial}{\partial x_4} + 2x_2\frac{\partial}{\partial x_5}, \quad X_4 = \frac{\partial}{\partial x_4}, \quad X_5 = \frac{\partial}{\partial x_5}, \tag{4.3}
\]

are such that their only non-vanishing commutation relations read

\[
[X_1, X_2] = X_3, \quad [X_1, X_3] = 2X_4, \quad [X_2, X_3] = 2X_5. \tag{4.4}
\]
Hence, $X_1, \ldots, X_5$ span a five-dimensional nilpotent algebra $V^{CS}$. Since $X^{CS}$ takes values in $V^{CS}$, then $(\mathbb{R}^5, X^{CS}, V^{CS})$ is a Lie system as already noticed in [60]. The vector fields of $V^{CS}$ span a distribution $\mathcal{D}^{CS} = T\mathbb{R}^5$ and $\text{dim } V^{CS} = \text{dim } \mathbb{R}^5$. Hence, $(\mathbb{R}^5, X^{CS}, V^{CS})$ is a locally automorphic Lie system.

4.2. Invariants for locally automorphic Lie systems

Recall that a Lie symmetry of a Lie system $(N, X, V)$ is a vector field $Y$ on $N$ such that $\mathcal{L}_Y Z = 0$ for every vector field $Z \in V$. If $V = \langle X_1, \ldots, X_r \rangle$, this is equivalent to saying that $Y$ has to satisfy the system of partial differential equations

$$\mathcal{L}_Y Y = 0, \quad i = 1, \ldots, r. \quad (4.5)$$

The set $\text{Sym}(V)$ of Lie symmetries of $(N, X, V)$ is a Lie algebra. Let us study this set for the case of locally automorphic Lie systems.

If $(N, X, V)$ is a locally automorphic Lie system, then each mapping $\varphi_i$ maps it onto an automorphic Lie system $(G, X^R, V^R)$. It is immediate that $\text{Sym}(V^R) = V^L$. Since $\varphi_i$ is a local diffeomorphism mapping $V$ onto $V^R$, then it also maps $\text{Sym}(V)$ onto $V^L$. Hence, one obtains the following lemma, whose implications will be illustrated in example 4.8.

**Lemma 4.7.** Let $(N, X, V)$ be a locally automorphic Lie system. The Lie algebra $\text{Sym}(V)$ of symmetries of $V$ is isomorphic to $V$.

**Example 4.8.** We reconsider example 4.6 studying the control system given by (4.2). By solving the linear system of partial differential equation (4.5) in the unknown coefficients of $Y$ in the basis $\partial/\partial x_1, \ldots, \partial/\partial x_5$, which demands a very long and tedious calculation, one gets that every Lie symmetry $Y$ of an arbitrary control system (4.2) must be a linear combination with constant coefficients of the vector fields

$$Y_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + 2x_3 \frac{\partial}{\partial x_4} + x_2^2 \frac{\partial}{\partial x_5}, \quad Y_2 = \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_5},$$
$$Y_3 = \frac{\partial}{\partial x_3}, \quad Y_4 = \frac{\partial}{\partial x_4}, \quad Y_5 = \frac{\partial}{\partial x_5}. \quad (4.6)$$

A straightforward calculation shows that the vector fields $\{Y_i, i = 1, \ldots, 5\}$, generate a Lie algebra with the same structure constants as $X_1, \ldots, X_5$.

Since every locally automorphic Lie system $(N, X, V)$ is locally diffeomorphic to an automorphic Lie system $(G, X^R, V^R)$, one has that every differential form on $N$ invariant with respect to the Lie derivative of elements of $V$ must be locally diffeomorphic to a left-invariant differential form on $G$. Since $\text{Sym}(V)$ is also diffeomorphic to $V^L$, in view of remark 2.11 one obtains the following theorem:

**Theorem 4.9.** Let $(N, X, V)$ be a locally automorphic Lie system and let $Y_1, \ldots, Y_r$ be a basis of $\text{Sym}(V)$, with dual frame $\nu^1, \ldots, \nu^r$. Then, a differential form on $N$ is invariant with respect to the Lie algebra $V$ if and only if it is a linear combination with real coefficients of exterior products of $\nu^1, \ldots, \nu^r$.

**Example 4.10.** We consider again example 4.6 studying the control system given by (4.2). Its Lie symmetries are given by the vector fields $Y_1, \ldots, Y_5$ described in (4.6), whose dual frame reads
\[ \eta_1 = dx_1, \quad \eta_2 = dx_2, \quad \eta_3 = -x_2 dx_1 + dx_3, \]
\[ \eta_4 = -2x_3 dx_1 + dx_4, \quad \eta_5 = -x_3^2 dx_1 - 2x_3 dx_2 + dx_5. \]  
(4.7)

Therefore, according to the preceding theorem, all the invariant differential forms of this control system are linear combinations, with real coefficients, of the exterior products of the \( \eta_i \); this space is \( \Lambda(\text{Sym}(V^{CS}))^* \).

It is easy to obtain multisymplectic forms invariant with respect to \( V^{CS} \) within \( \Lambda(\text{Sym}(V^{CS}))^* \). It follows from linear algebra considerations that all non-degenerate differential forms on \( \mathbb{R}^5 \) must have rank five or three. For instance, if we consider the invariant five-form

\[ \Theta_{vol} = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4 \wedge \eta_5, \]

then we obtain a left-invariant volume form. Hence, this is a multisymplectic form satisfying that

\[ \mathcal{L}_Y \Theta_{vol} = 0, \quad \forall Z \in V^{CS}. \]

Therefore, \((\mathbb{R}^5, \Theta_{vol}, X^{CS})\) is a multisymplectic Lie system.

We can obtain other examples of multisymplectic forms compatible with this Lie system. Since

\[ d\eta_1 = 0, \quad d\eta_2 = 0, \quad d\eta_3 = \eta_1 \wedge \eta_2, \quad d\eta_4 = 2\eta_1 \wedge \eta_3, \quad d\eta_5 = 2\eta_2 \wedge \eta_3, \]

we consider now the closed three-form

\[ \Theta = d(\eta_3 \wedge \eta_4) + d(\eta_4 \wedge \eta_5) = \eta_1 \wedge \eta_2 \wedge \eta_4 + 2\eta_1 \wedge \eta_3 \wedge \eta_5 - 2\eta_4 \wedge \eta_2 \wedge \eta_3 = (1 - 2x_2)dx_{124} + 8x_3 dx_{123} + 2dx_{135} - 2dx_{234}. \]  
(4.8)

where we use the notation \( dx_{ijk} = dx_i \wedge dx_j \wedge dx_k \). It is easy to prove that this three-form is non-degenerate, then it is a multisymplectic form of degree 3. Therefore \((\mathbb{R}^5, \Theta, X^{CS})\) is a new multisymplectic Lie system. It is worth noting that \( \Theta \) is not a volume form. \( \triangle \)

Although lemma 4.7 guarantees the existence of the Lie algebra \( \text{Sym}(V) \), its computation may become computationally complicated. For this reason, we will provide theorem 4.11, which gives a family of invariant differential forms under the action of the elements of a Lie algebra \( V \) of a locally automorphic Lie system without using the Lie algebra of Lie symmetries \( \text{Sym}(V) \).

We need to introduce previously some additional structures. First, every linear representation \( \rho : g \to \text{End}(E) \) of a Lie algebra \( g \) on a vector space \( E \) can be extended to a linear representation on the exterior algebra of \( E \); its elements are indeed derivations, so this yields a homomorphism \( g \to \text{Der}(\Lambda E) \) \([36, \text{p} 110]\). Let us consider the adjoint representation \( \text{ad}_e \) of \( g \), and denote by \( \text{coad}_e \) its contragradient representation, which is the linear representation on the dual space \( g^* \) given by \( \text{coad}_e := -(\text{ad}_e)^\top \). We apply the preceding remark to this representation, thus obtaining a map \( g \to \text{Der}(\Lambda g^*) \) that we denote by \( v \mapsto D_v \).

**Theorem 4.11.** Let \((N, X, V)\) be a locally automorphic Lie system, and let \( \phi : g \to V \) be a Lie algebra isomorphism. Then we have:

1. The isomorphism \( \phi : g \to V \) maps the adjoint endomorphism \( \text{ad}_e \) of \( g \) to the Lie derivative \( \mathcal{L}_\phi(v) \) of the vector fields in \( V \).
2. The dual space \( V^* \) can be identified with the set \( \{ \theta \in \Omega^1(N) \mid \forall X \in V, \ i_X \theta \) is constant \}. With this identification, the contragradient isomorphism \( \phi^* : g^* \rightarrow V^* \) also maps the co-adjoint endomorphism \( \text{coad}_g \) to the Lie derivative \( L_{\phi^*(\varepsilon)} \) of one-forms.

3. The exterior power \( \Lambda^p V^* \) can be identified with the set of differential \( p \)-forms \( \theta \) on \( N \) whose contractions with \( p \) vector fields of \( V \) are constant.

4. If an element \( \omega \in \Lambda g^* \) satisfies that \( D_v \omega = 0 \) for each \( v \in g \), then its image, \( \Lambda \phi^*(\omega) \), is a differential form in \( N \) which is invariant with respect to the elements of \( V \); namely, the Lie derivative of \( \Lambda \phi^*(\omega) \) relative to elements of \( V \) vanishes.

**Proof.** The first assertion is immediate: \( \phi([\text{ad}_g u]) = \theta([\phi(v)], \phi(u)] = L_{\phi^*(v)} \phi(u) \).

For the second, if \( X_i \) is a basis of \( V \), it is also a frame of the tangent bundle \( TN \); then, if \( \theta_i \) is its dual frame, the \( \theta_i \) can be considered as a basis of \( V^* \). On the other hand, every one-form on \( N \) can be written as \( \theta = \sum_i g_i \theta_i \), and the functions \( g_i \) are constant if and only if, for every \( X \in V \), one has that \( i_X \theta \) is constant. The Lie derivative of one-forms satisfies \( L_{Y i_X \theta} = i_X L_Y \theta + i_L \varepsilon X \theta \). When \( X, Y \in V \) and \( \varepsilon \in V^* \), this implies that \( i_X L_Y \theta = -i_L \varepsilon X \theta \), which means that \( L_Y \theta \) is the minus the transpose of \( L_Y \varepsilon \) on \( V^* \); this completes the proof of the second statement.

Third statement proceeds in a similar way. The correspondence between \( D_v \) and \( L_{\phi^*(v)} \) is a consequence of the fact that in their respective algebras both operators are derivations, and, by the preceding statement, they agree when applied to the subspaces \( g^* \) and \( V^* \), both of which generate the corresponding exterior algebras.

From this we obtain \( \Lambda \phi^*(D_v \omega) = L_{\phi^*(v)} \Lambda \phi^*(\omega) \). Therefore, the invariance of \( \omega \) with respect to every \( v \in g \) implies the invariance of \( \Lambda \phi^*(\omega) \) with respect to every \( Y \in V \).

Let us show now that certain conditions on \( X \) allow us to easily construct a multisymplectic form turning the Vessiot–Guldberg Lie algebra for \( X \) into locally Hamiltonian vector fields.

The idea is to find \( g \)-invariant elements in \( \Lambda g^* \) whose image under \( \Lambda \phi^* \) is a multisymplectic form. In particular, every unimodular Lie algebra gives rise to an invariant element of \( \Lambda g^* \) of maximal degree whose representation is the volume differential form.

Recall that the local diffeomorphism \( \varphi \), that maps \( (N, X, V) \) onto an automorphic system \( (G, X^R, V^R) \) maps also \( V^* \) onto the \( (V^R)^* \), which consists of the right-invariant differential one-forms on \( G \). In view of this, one immediately obtains the following corollary.

**Corollary 4.12.** Let \( (N, X, V) \) be a locally automorphic Lie system. If \( V \) is unimodular, then \( V \) admits an invariant volume form given by

\[
\Theta = \eta^1 \wedge \ldots \wedge \eta^r,
\]

where \( \eta^1, \ldots, \eta^r \) is any basis of elements of \( V^* \). Then, \( (N, \Theta, X) \), where \( X \) takes values in \( V \), is a multisymplectic Lie system. Moreover, \( \Theta \) is invariant with respect to the Lie derivatives with elements of the Lie algebra \( \text{Sym}(V) \).

**Remark 4.13.** Theorem 4.11 and corollary 4.12 give us a procedure to easily construct a compatible multisymplectic form certain class of Lie systems. In fact, example 3.1 was carried out by using this procedure to obtain a compatible multisymplectic structure.
4.2.1. Example. The Schwarz equation, whose first-order system $X^S$ is given by (3.2), admits a Lie algebra of symmetries given by (see [46, 57])

$$
Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = x \frac{\partial}{\partial x} + v \frac{\partial}{\partial v} + a \frac{\partial}{\partial a}, \quad Y_3 = x^2 \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v} + 2(ax + v^2) \frac{\partial}{\partial a}.
$$

(4.9)

These are therefore the vector fields commuting with the Lie algebra $V^S = \langle X_1, X_2, X_3 \rangle$, where $X_1, X_2, X_3$ are given by (3.2). The dual forms to (4.9) read

$$
\eta_1 = dx - \frac{2v^2 x + ax^2}{2v^3} dv + \frac{x^2}{2v^3} da, \quad \eta_2 = \frac{v^2 + ax}{v^3} dv - \frac{x}{v^3} da, \quad \eta_3 = -\frac{a}{2v^3} dv + \frac{1}{2v^3} da.
$$

According to theorem 4.9, every differential form invariant with respect to the vector fields $X_1, X_2, X_3$, which span the Vessiot–Guldberg Lie algebra $V^S$ of $X^S$, is a linear combination of the exterior products of $\eta_1, \eta_2, \eta_3$. For instance,

$$
-2\eta_1 \wedge \eta_3 = \frac{a dx \wedge dv + v da \wedge dx + x dv \wedge da}{v^3}, \quad 2\eta_2 \wedge \eta_3 = \frac{dv \wedge da}{v^3},
$$

are invariant with respect to the elements of $V^S$. In fact, these are indeed the invariant presymplectic forms obtained in [20, 52] in an ad hoc manner. △

It may be difficult to find multisymplectic forms compatible with Lie systems, but it turns out to be quite easy to find compatible closed invariant forms. The knowledge of the latter can be used to easily find compatible multisymplectic forms, as will be showed in following sections.

**Proposition 4.14.** Every locally automorphic Lie system $(N, X, V)$ has non-zero closed invariant forms.

**Proof.** The conditions of the Vessiot–Guldberg Lie algebra allow us to assume that $X$ along with $N$ are locally diffeomorphic to a $t$-dependent right-invariant vector field $X^G$ on a connected and simply connected Lie group $G$. Hence, every left-invariant differential form is invariant under right-invariant vector fields. It is obvious that there exist closed left-invariant forms on $G$, e.g. volume forms or the differentials of left-invariant forms. The diffeomorphism between $G$ and $N$ maps these left-invariant forms onto invariant closed differential forms on $N$ compatible with $X$. □

In view of the above, there are plenty of multisymplectic compatible forms. The main point is to determine closed multilinear forms on the Chevalley–Eilenberg cohomology of the Lie algebra, which is a purely algebraic problem. We will not give any precise procedure to construct non-degenerate closed elements of the Chevalley–Eilenberg cohomology. Nevertheless, every non-zero decomposable $k$-form $\eta \in \Lambda g^*$ is such that the rank of the mapping $\eta : \mathfrak{g} \mapsto \ker \eta$ is $k$ and its kernel has dimension $n-k$. This fact suggests that appropriately chosen closed $k$-forms will have eventually a zero-dimensional kernel and they will become 1-non-degenerate. This will be enough to accomplish our aims in this work. Indeed this was used in the control system example in order to obtain the invariant multisymplectic form of degree 3 given by equation (4.8).

5. **Superposition rules for multisymplectic lie systems**

Let us employ the multisymplectic form of a multisymplectic Lie system $(N, \Theta, X)$ so as to construct superposition rules for $X$. In short our idea consists in constructing an abstract tensor algebra through a Lie–Hamilton algebra of Hamiltonian differential forms of $(N, \Theta, X)$. Then,
we use the algebraic properties of this tensor algebra and their representation as geometric objects to obtain invariant tensor fields of the diagonal prolongation \((N^{[m]}, \Theta^{[m]}, X^{[m]})\). From these invariant tensor fields and, eventually, the Lie symmetries of the Lie system \(X\), we will obtain constants of motion that will finally lead to the superposition rule for \(X\). As a byproduct, many other invariants of \(X\) appear, e.g. symplectic forms invariant under the action of the elements of \(\mathcal{V}_X\).

5.1. Coalgebras, \(g\)-modules, and tensor fields

The following methods rely on considering the structures related to multisymplectic Lie systems as realizations of tensor algebras and \(g\)-modules [64]. The properties of such algebraic structures will be then employed to obtain superposition rules for multisymplectic Lie systems. We refer to [13, 27, 64] for further details on the algebraic structures appearing in this section.

A coalgebra is a linear space \(A\) along with two mappings \(\Delta : A \to A \otimes A\), the coproduct, and \(\epsilon : A \to \mathbb{R}\), the counit, such that

\[
(\Delta \circ \text{Id}_A) \circ \Delta = (\text{Id}_A \otimes \Delta) \circ \Delta, \quad (\text{Id}_A \otimes \epsilon) \circ \Delta = (\epsilon \otimes \text{Id}_A) \circ \Delta = \text{Id}_A.
\]

(5.1)

Here \(\otimes\) refers to the tensor product used to define a coalgebra and \(\text{Id}_A\) is the identity map on \(A\).

If \(A\) is an associative algebra with unit element, then \(A^{(m)} = A \otimes \cdots \otimes A\) admits a canonical associative algebra structure with unit. In particular, the tensor algebra \(T(g)\) related to the Lie algebra \(g\) has a unital associative algebra structure that induces a canonical new one in \(T^{(m)}(g)\) [13]. In view of this, a simple calculation leads to prove the following proposition (see [13, chapter 3, section 11]).

**Proposition 5.1.** The tensor algebra \(T(g)\) admits a coalgebra structure relative to the coproduct, \(\Delta\), and the counit, \(\epsilon\), given by the unique morphisms of associative algebras satisfying

\[
\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \epsilon(v) = 0, \quad \forall v \in g.
\]

Analogously, we also define a higher-order coproduct \(\Delta^{(m)} : T(g) \to T^{(m+1)}(g)\) recurrently

\[
\Delta^{(m+1)} = (\text{Id} \otimes \cdots \otimes \text{Id} \otimes \Delta) \circ \Delta^{(m)}, \quad \forall m \in \mathbb{N}, \quad \Delta^{(1)} = \Delta,
\]

which is a morphism of associative algebras.

Recall that a \(g\)-module is a linear space \(E\) along with a Lie algebra morphism \(\rho : g \to \text{End}(E)\). The adjoint representation \(\text{ad} : v \in g \mapsto \text{ad}_v \in \text{Der}(g)\) is such that each \(\text{ad}_v\) can be uniquely extended to a derivation relative to the tensor product in \(T(g)\). In this manner, \(T(g)\) becomes a \(g\)-module relative to \(\text{ad} : v \in g \mapsto \text{ad}_v \in \text{Der}(T(g))\). From now on, we will denote the adjoint representation of \(g\) and several of its generalizations and/or extensions in the same way as this will not lead to confusion.

The Lie algebra morphism \(\text{ad} : g \to \text{Der}(T(g))\) induces a second one \(\text{ad} : g \to \text{Der}(T^{(m)}(g))\) by requiring

\[
\text{ad}_v(t_1 \otimes \cdots \otimes t_m) = \text{ad}_v(t_1) \otimes \cdots \otimes \text{ad}_v(t_m), \quad \forall t_1, \ldots, t_m \in T(g).
\]

This turns the spaces \(T^{(m)}(g)\), with \(m \in \mathbb{N}\), into \(g\)-modules.

Previous structures have a special relevance for the theory of multisymplectic Lie systems. In particular, the space to be defined next plays a significative role in the determination of their superposition rules and tensorial invariants.
Definition 5.2. Let $E$ be the $\mathfrak{g}$-module relative to a Lie algebra representation $\rho: \mathfrak{g} \to \text{End}(E)$. We write $E^g$ for the space of $\mathfrak{g}$-invariant elements of $E$, namely

$$E^g = \{ e \in E : \rho_v(e) = 0, \ \forall v \in \mathfrak{g} \}.$$ 

Proposition 5.3. The mappings $\Delta^{(m)} : T(\mathfrak{g}) \to T^{(m+1)}(\mathfrak{g})$, with $m \in \mathbb{N}$, are $\mathfrak{g}$-module morphisms between the natural $\mathfrak{g}$-module structures of $T(\mathfrak{g})$ and $T^{(m+1)}(\mathfrak{g})$, namely

$$\Delta^{(m)} \circ \text{ad}_v = \text{ad}_v \circ \Delta^{(m)}, \quad \forall v \in \mathfrak{g}. \quad (5.2)$$

Moreover, $\Delta^{(m)}(T(\mathfrak{g})^g) \subset [T^{(m+1)}(\mathfrak{g})]^g$.

Proof. Let us prove (5.2) by induction. In fact, $\text{ad}_v \circ \Delta = \Delta \circ \text{ad}_v$ for every $v \in \mathfrak{g}$ and (5.2) holds for $m = 1$. If (5.2) is obeyed for a fixed $m$, then

$$\Delta^{(m+1)} \circ \text{ad}_v = (\text{Id} \otimes \ldots \otimes \text{Id} \otimes \Delta) \circ \Delta^{(m)} \circ \text{ad}_v = (\text{Id} \otimes \ldots \otimes \text{Id} \otimes \Delta) \circ \text{ad}_v \circ \Delta^{(m)} = \text{ad}_v \circ \Delta^{(m+1)}$$

for every $v \in \mathfrak{g}$. By induction (5.2) is valid for any $m \in \mathbb{N}$ and the relation $\Delta^{(m)}(T(\mathfrak{g})^g) \subset [T^{(m+1)}(\mathfrak{g})]^g$ follows trivially. 

Let $\mathfrak{T}(N)$ be the unital associative algebra of covariant tensor fields on $N$. We define $N^m = N \times \ldots \times N$ and let $(\xi_1, \ldots, \xi_m)$ be a point of $N^m$. We write $\mathfrak{T}^{(m)}(N) = \mathfrak{T}(N) \otimes_{N^m} \ldots \otimes_{N^m} \mathfrak{T}(N)$ for the space of covariant tensor fields on $N^m$ being linear combinations of tensor fields of the form

$$T_1(\xi_1) \otimes_{N^m} \ldots \otimes_{N^m} T_m(\xi_m),$$

where $T_1, \ldots, T_m \in \mathfrak{T}(N)$.

Theorem 5.4. Consider the $\mathfrak{g}$-module structure on $\mathfrak{T}(N)$ given by $\rho : \mathfrak{g} \to \text{Der}(\mathfrak{T}(N))$. If $\iota : \mathfrak{g} \to \mathfrak{T}(N)$ is a $\mathfrak{g}$-module morphism, then $\iota$ can be extended uniquely to a $\mathfrak{g}$-module and associative algebra morphism $\Upsilon : T(\mathfrak{g}) \to \mathfrak{T}(N)$ by requiring that

$$\Upsilon(v_1 \otimes \ldots \otimes_v v_r) = \iota(v_1) \otimes_N \ldots \otimes_N \iota(v_r), \quad \forall v_1, \ldots, v_r \in \mathfrak{g}, \quad \forall r \in \mathbb{N},$$

where $\otimes_N$ is the tensor product in $T(\mathfrak{g})$ whereas $\otimes$ is the tensor field product on $N$.

Moreover, for every $m \in \mathbb{N}$, $\rho$ gives rise to a Lie algebra representation $\rho^{(m)} : \mathfrak{g} \to \text{Der}(\mathfrak{T}^{(m)}(N))$ such that

$$\rho^{(m)}(T_1 \otimes_{N^m} \ldots \otimes_{N^m} T_m) = \rho_v(T_1) \otimes_{N^m} \ldots \otimes_{N^m} T_m + \ldots + T_1 \otimes_{N^m} \ldots \otimes_{N^m} \rho_v(T_m),$$

for all $T_1, \ldots, T_m \in \mathfrak{T}(N)$ and every $v \in \mathfrak{g}$. Additionally, there exists a $\mathfrak{g}$-module morphism $\Upsilon^{(m)} : T^{(m)}(\mathfrak{g}) \to \mathfrak{T}^{(m)}(N) \subset \mathfrak{T}(N^m)$ such that

$$\Upsilon^{(m)}(t_1 \boxtimes \ldots \boxtimes t_m) = \Upsilon(t_1) \otimes_{N^m} \ldots \otimes_{N^m} \Upsilon(t_m), \quad \forall t_1, \ldots, t_m \in T(\mathfrak{g}).$$

Proof. The map $\Upsilon$ is a well-defined algebra morphism. Let us prove that it is a $\mathfrak{g}$-module morphism also. Since $\iota$ is a $\mathfrak{g}$-module morphism and $\rho_v$ is a derivation for every $v \in \mathfrak{g}$, one obtains that, on decomposable elements of $T(\mathfrak{g})$, 

---

J. Phys. A: Math. Theor. 52 (2019) 215201
X Gràcia et al.
As $\Upsilon \circ \text{ad}_v = \rho_v \circ \Upsilon$ on decomposable elements of $T(\mathfrak{g})$, this equality is obeyed on the whole $T(\mathfrak{g})$ and $\Upsilon \circ \text{ad}_v = \rho_v \circ \Upsilon$ for every $v \in \mathfrak{g}$ and $\Upsilon$ becomes a morphism of $\mathfrak{g}$-modules.

Let us verify that $\rho(v)$ is a Lie algebra morphism. As previously, we will start by proving this fact on decomposable elements of $\mathfrak{g}$:

$$\rho(v)(T_1 \otimes \cdots \otimes T_m) = \sum_{i=1}^{m} T_i \otimes \cdots \otimes T_j \rho(v)(T_i) \otimes \cdots \otimes T_j \rho(v)(T_j) \otimes \cdots \otimes T_m,$$

for all $v, w \in \mathfrak{g}$ and $T_1, \ldots, T_m \in \mathfrak{g}$. Subtracting from this the value of $\rho(v)\rho(w)(T_1 \otimes \cdots \otimes T_m)$, all elements with $v$ and $w$ acting on elements $T_i, T_j$ with $i \neq j$ disappear and we are left with

$$[\rho(v), \rho(w)](T_1 \otimes \cdots \otimes T_m) = \sum_{i=1}^{m} T_1 \otimes \cdots \otimes T_i \rho(v)(T_i) \otimes \cdots \otimes T_i \rho(w)(T_i) \otimes \cdots \otimes T_m.$$

Since $\rho$ is a Lie algebra representation, it follows that

$$[\rho(v), \rho(w)](T_1 \otimes \cdots \otimes T_m) = \sum_{i=1}^{m} T_1 \otimes \cdots \otimes T_i \rho([v, w])(T_i) \otimes \cdots \otimes T_i \rho([v, w])(T_i) \otimes \cdots \otimes T_m,$$

on decomposable elements of $\mathfrak{g}$. The equality for the whole $\mathfrak{g}$ is therefore also satisfied.

The fact that $\Upsilon(m)$ is a morphism of $\mathfrak{g}$-modules results immediately from using the previous ideas.

It is useful now to consider two particular cases described in the following lemmas.
Lemma 5.5. The space \( S(\mathfrak{g}) \) of totally symmetric tensors over \( \mathfrak{g} \) and the space \( \Lambda(\mathfrak{g}) \) of totally antisymmetric tensors over \( \mathfrak{g} \) are \( \mathfrak{g} \)-submodules of \( T(\mathfrak{g}) \). Similarly, \( S^{(m)}(\mathfrak{g}) \) and \( \Lambda^{(m)}(\mathfrak{g}) \) are \( \mathfrak{g} \)-submodules of \( T^{(m)}(\mathfrak{g}) \).

Its proof is an immediate consequence of the fact that if \( \mathfrak{t}_1 \) is an element totally antisymmetric (symmetric) of \( T^{(m)}(\mathfrak{g}) \), then \( \text{ad}_v(\mathfrak{t}_1) \) is totally antisymmetric (symmetric) for every \( v \in \mathfrak{g} \).

The following lemma shows that the mappings \( \Delta^{(m)} \) can be restricted to symmetric and antisymmetric tensors.

Lemma 5.6. For every Lie algebra \( \mathfrak{g} \) and \( m \in \mathbb{N} \), one has that \( \Delta^{(m)}S(\mathfrak{g}) \subset S^{(m+1)}(\mathfrak{g}) \) and \( \Delta^{(m)}\Lambda(\mathfrak{g}) \subset \Lambda^{(m+1)}(\mathfrak{g}) \).

The proof of this result is rather technical; it will be done in the appendix.

Lemma 5.6 allows us to extend proposition 5.3 and theorem 5.4 to \( S(\mathfrak{g}) \) and \( \Lambda(\mathfrak{g}) \). All these results can be summarised through the commutative diagrams displayed in figure 2.

5.2. Application to multisymplectic Lie systems: calculus of tensor invariants

Let us use the algebraic structures developed in the previous section to derive superposition rules for multisymplectic Lie systems without solving systems of PDEs or ordinary differential equations as in most of the literature [22–24, 66].

A relevant tool for obtaining a superposition rule is given by diagonal prolongations of vector fields [22]. If \( X \) is a vector field on \( N \), its diagonal prolongation to \( N^m \) is the vector field \( X^{[m]}(x_1, \ldots, x_m) = X(x_{11}) + \cdots + X(x_{mm}) \). Let us recall the distributional method to obtain superposition rules. For further details we refer to [23, sections 1.5 and 1.6]. Given a Lie system on \( N \) with a Vessiot–Guldberg Lie algebra \( V \) spanned by a basis of vector fields \( X_1, \ldots, X_r \), a superposition rule for \( X \) can be obtained by determining the smallest \( m \) so that \( X_1^{[m]}, \ldots, X_r^{[m]} \) are linearly independent at a generic point. Then, we must obtain \( n \) common first-integrals \( I_1, \ldots, I_n \) for \( X_1^{[m+1]}, \ldots, X_r^{[m+1]} \) satisfying that

\[
\frac{\partial(I_1, \ldots, I_n)}{\partial(x_{11}, \ldots, x_{1m})} \neq 0. \tag{5.5}
\]

Let \( \lambda_1, \ldots, \lambda_n \) be real numbers. By assuming \( I_1 = \lambda_1, \ldots, I_n = \lambda_n \), condition (5.5) allows us to express \( x_1^{(1)}, \ldots, x_{1m}^{(1)} \) as functions of \( \lambda_1, \ldots, \lambda_n \) and the variables \( x_1^{(i)}, \ldots, x_{1m}^{(i)} \) for \( i = 2, \ldots, m + 1 \), which gives a superposition rule depending on \( m \) particular solutions.

Our algebraic/geometric methods to obtain superposition rules rely on obtaining \( I_1, \ldots, I_n \) through \( \mathfrak{g} \)-invariant elements of the spaces \( T^{(q)}(\mathfrak{g}) \), with \( q \in \mathbb{N} \). We propose two methods, one relying on the Casimir elements of the universal enveloping algebra \( U(\mathfrak{g}) \), and another one basing on the invariant elements of the Grassmann algebra \( \Lambda(\mathfrak{g}) \) of the linear space \( \mathfrak{g} \).

Proposition 5.7. If \( (N, \Theta, X) \) is a multisymplectic Lie system, then \( (X^{[m]}, \Theta^{[m]}, X^{[m]}) \) is also a multisymplectic Lie system.

Proof. By assumption, the Vessiot–Guldberg Lie algebra \( V^X \) related to \( X \) consists of Hamiltonian vector fields relative to the multisymplectic form \( \Theta \), i.e. \( \iota_X \Theta = d\theta \), for certain differential forms \( \theta \), with \( t \in \mathbb{R} \). Since \( \Theta^{[m]}(x_1, \ldots, x_{mn}) = \sum_{a=1}^{mn} \Theta(x_{a1}) \), then \( \Theta^{[m]} \) is closed and 1-nondegenerate. Additionally,
are Hamiltonian relative to $\forall \in L$ allows us to define a linear morphism $\rho: T \rightarrow L$, then $g g [\rightarrow T L \rightarrow L \forall: \in]$

ture on the space of Hamiltonian degree Hamiltonian Proposition 5.8.

J. Phys. A: Math. Theor. 52 (2019) 215201

X Gràcia et al

Figure 2. Commutative diagrams summarising the results of theorem 5.4 and lemma 5.6. Both diagrams are the restrictions of figure 1 to the submodules of symmetric and antisymmetric tensors.

\[ \iota_{\chi^{[m]}}(\Theta^{[m]}) = \sum_{a=1}^{m} d\theta(x(a)), \]

and the vector fields $\chi^{[m]}$ are Hamiltonian relative to $\Theta^{[m]}$ for every $t \in \mathbb{R}$. Hence, $(N^{[m]}, \Theta^{[m]}, X^{[m]})$ is a multisymplectic Lie system. 

It has been shown in proposition 3.4 that every multisymplectic Lie system $(N, \Theta, X)$ of degree $k$ Assume that $\Theta$ is a multisymplectic form of degree $k$, induces on the set $\mathfrak{W}$ of the Hamiltonian $(k-1)$-forms of $V$ a Lie algebra structure inherited from the Lie algebra structure on the space of Hamiltonian $(k-1)$-forms defined in definition 2.8. Let us apply the results of the preceding section to $\mathfrak{W}$.

**Proposition 5.8.** Let $(N, \Theta, X)$ be a multisymplectic Lie system with an induced Lie–Hamilton algebra $\mathfrak{W}$. Consider a Lie algebra isomorphism $\phi: \mathfrak{g} \simeq \mathfrak{M}$. Then, $\Sigma^{(m)}(N)$ becomes a $\mathfrak{g}$-module with respect to the Lie algebra representation $\rho^{(m)}: v \in \mathfrak{g} \mapsto L_{-\chi^{[m]}} \in \text{Der}(\Sigma^{(m)}(N)),$

where $X_v$ is the unique Hamiltonian vector field (relative to $\Theta$) associated with $\phi(v) \in \mathfrak{M}$. Moreover, $\phi$ can be extended to a $\mathfrak{g}$-module morphism $\Upsilon^{(m)}: \Upsilon^{(m)}(\mathfrak{g}) \rightarrow \Upsilon^{(m)}(N)$, i.e.

\[ \Upsilon^{(m)}(\text{ad}_v(t)) = L_{-\chi^{[m]}} \Upsilon^{(m)}(t), \quad \forall v \in \mathfrak{g}, \forall t \in \Upsilon^{(m)}(\mathfrak{g}). \] (5.6)

**Proof.** The Lie algebra isomorphism $\phi: \mathfrak{g} \simeq \mathfrak{M}$ allows us to define a linear morphism $\rho: v \in \mathfrak{g} \mapsto L_{-X_v} \in \text{Der}(\Upsilon(N)),$

where $X_v$ is the unique Hamiltonian vector field such that $\iota_{X_v} \Theta = \phi(v)$. Let us show that $\rho$ is a Lie algebra morphism. It is immediate that $\rho$ is linear. From the properties of the Lie bracket (2.3), $\phi([v, \check{v}]) = \{\phi(v), \phi(\check{v})\}$ is the Hamiltonian form of $- [X_v, X_{\check{v}}]$. Hence,

\[ \rho([v, \check{v}]) = L_{X_v, \check{v}} = L_{-X_v, -\check{v}} = [\rho_v, \rho_{\check{v}}], \quad \forall v, \check{v} \in \mathfrak{g}. \]

and $\rho$ is a Lie algebra morphism. Moreover, $\phi$ can be considered as an injection $\phi: \mathfrak{g} \simeq \mathfrak{M} \subset \Sigma(N)$ of $\mathfrak{g}$ in $\Sigma(N)$ and it is also a $\mathfrak{g}$-module morphism. Hence, all assumptions of theorem 5.4 hold and we can apply it to our particular case. More specifically, $\phi$ can be extended to a mapping $\Upsilon^{(m)}: \Upsilon^{(m)}(\mathfrak{g}) \rightarrow \Upsilon^{(m)}(N)$.

Theorem 5.4 states that $\rho$ can be extended to a Lie algebra morphism $\rho^{(m)}: \mathfrak{g} \rightarrow \text{Der}(\Sigma^{(m)}(N))$. It is interesting to observe that on decomposable elements of $\Sigma^{(m)}(N)$, namely elements of the
form $T_1 \otimes N^m \ldots \otimes N^m T_m$ with $T_1, \ldots, T_m \in \Sigma(N)$, one has that

$$\rho_v^{(m)}(T_1 \otimes N^m \ldots \otimes N^m T_m) = \mathcal{L}_{-X_v^{(m)}}(T_1 \otimes N^m \ldots \otimes N^m T_m),$$

where $X_v^{(m)}$ is the prolongation to $N^m$ of the Hamiltonian vector field $X_v$, associated with $v \in \mathfrak{g}$. As a consequence, one can assume that $\rho_v^{(m)} = L_{-X_v^{(m)}}$. Since theorem 5.4 ensures that $\Upsilon^{(m)}$ is a morphism of $\mathfrak{g}$-modules, one obtains the relation (5.6).

The Poincaré–Birkhoff–Witt theorem may be employed to prove that every element of the enveloping universal algebra $U(\mathfrak{g})$ can be understood as a unique symmetric element of the tensor algebra $T(\mathfrak{g})$ and vice versa (see [64]). In other words, there exists a linear isomorphism $\lambda : U(\mathfrak{g}) \rightarrow S(T(\mathfrak{g}))$ identifying both linear spaces. Moreover, $\lambda$ is also a $\mathfrak{g}$-module morphism [64] and $S(T(\mathfrak{g}))$ is isomorphic (as a $\mathfrak{g}$-module) to the symmetric algebra of $\mathfrak{g}$, namely the algebra of commutative polynomials in the elements of $\mathfrak{g}$ (see [64] for details). The elements of $U(\mathfrak{g})$ that commute with any other element of $\mathfrak{g}$ relative to its $\mathfrak{g}$-module structure, the so-called Casimir elements, give rise via $\lambda$ to elements of $[S(T(\mathfrak{g}))]^\mathfrak{g}$. From these comments, lemma 5.6 and proposition 5.8, we will obtain the following result:

**Corollary 5.9.** Let $(N, \Theta, X)$ be a multisymplectic Lie system with a Lie–Hamilton algebra $\mathfrak{M}$. Let $\phi : \mathfrak{g} \simeq \mathfrak{M}$ be a Lie algebra isomorphism and let $\Upsilon^{(m)} : T^{(m)}(\mathfrak{g}) \rightarrow T^{(m)}(N)$ be its induced morphism of $\mathfrak{g}$-algebras given in proposition 5.8. If $C$ is a Casimir element of $U(\mathfrak{g})$ or an element of $\Lambda(\mathfrak{g})^\mathfrak{g}$, then $\Upsilon^{(m)}(\Delta^{(m-1)} C)$ is an invariant relative to the evolution of $X^{(m)}$.

**Proof.** Let $V$ be the Lie algebra of Hamiltonian vector fields of $\mathfrak{M}$. If $Y \in V$, then $Y^{(m)} \in V^{(m)}$ and $V^{(m)}$ is spanned by the vector fields $Y^{(m)}$, where $Y$ is an arbitrary element of $V$. It is immediate that $V^{(m)}$ is a Lie algebra isomorphic to $V$.

If $\theta_Y = t_Y \Theta$, then $\theta_Y^{(m)} = t_Y \Theta^{(m)} = \theta_Y^{[m]}$. As a consequence $\mathfrak{M}^{[m]}$ is isomorphic to $\mathfrak{M}$ relative to the Lie brackets (2.3) induced by $\Theta^{[m]}$ and $\Theta$, respectively. Moreover, $\mathfrak{M}^{[m]}$ becomes a Lie–Hamilton algebra for $(X^{[m]}, \Theta^{[m]}, X^{[m]})$.

Let $v \in \mathfrak{g}$ be such that $\theta_V = \phi(v)$. Thus, $\Upsilon^{(m)}(\Delta^{(m-1)}(v)) = \theta_V^{[m]} = \theta_Y^{[m]}$. In other words, $\Upsilon^{(m)}(\Delta^{(m-1)}(v))$ is the Hamiltonian form corresponding to $Y^{(m)}$. If $C$ is a Casimir of $U(\mathfrak{g})$ or an element of $\Lambda(\mathfrak{g})^\mathfrak{g}$, then its symmetric or antisymmetric representative in $T(\mathfrak{g})$ is a $\mathfrak{g}$-invariant element of $T(\mathfrak{g})$. By using diagram (5.4), we obtain that

$$\mathcal{L}_{Y^{[m]}}[\Upsilon^{(m)}(\Delta^{(m-1)} C)] = \Upsilon^{(m)} \circ \rho_v^{(m)}(\Delta^{(m-1)}(C)) = \Upsilon^{(m)} \circ \Delta^{(m-1)}(\mathrm{ad}_v(C)) = 0 \quad (5.8)$$

for every $Y \in V$. Hence, $\Upsilon^{(m)}(\Delta^{(m-1)}(C))$ is an invariant relative to the Vessiot–Guldberg Lie algebra $V^{(m)}$ of $X^{(m)}$ and it becomes invariant under the evolution of $X^{(m)}$.

It is worth noting that $\Upsilon^{(m)}$ is not an algebra morphism relative to the product in $T^{(m)}(\mathfrak{g})$ and the tensor product on $N^m$. For instance,

$$\Upsilon^{(2)}([1 \otimes v_1](v_2 \otimes 1)) = \Upsilon^{(2)}(v_2 \otimes v_1) = \Upsilon(v_2)(x_1) \otimes_{N^2} \Upsilon(v_1)(x_2),$$

whereas

$$\Upsilon^{(2)}(1 \otimes v_1) \otimes_{N^2} \Upsilon^{(2)}(v_2 \otimes 1) = \Upsilon(v_1)(x_2) \otimes_{N^2} \Upsilon(v_2)(x_1).$$
Therefore, both expressions coincide only when
\[ \Upsilon(v_2)(x_1) \otimes_{\mathcal{N}^2} \Upsilon(v_1)(x_2) = \Upsilon(v_1)(x_2) \otimes_{\mathcal{N}^2} \Upsilon(v_2)(x_1). \]

It may happen that a Vessiot–Guldberg Lie algebra \( V \) of a multisymplectic Lie system \((N, \Omega, X)\) with a multisymplectic \( k \)-form \( \Omega \) also gives rise to a Lie algebra \( \mathcal{M}_{k-2} \) of Hamiltonian \((k-2)\)-forms relative to the bracket of Hamiltonian \((k-2)\)-forms. This occurs, for instance, when \( \Omega \) is a symplectic form and the Hamiltonian \((k-2)\)-forms of the elements of \( V \) become mere functions (see [7] for details). In such cases, proposition 5.8 and corollary 5.9 can easily be extended to create new invariants by replacing \( \mathcal{M} \) with \( \mathcal{M}_{k-2} \). This also retrieves the method to obtain superposition rules for Lie–Hamilton systems devised in [7] for the particular case of Lie–Hamilton systems with compatible symplectic forms. Since every Lie–Hamilton system can be reduced to the study of Lie–Hamilton systems with compatible symplectic forms [25], our new method can also be applied to any Lie–Hamilton system.

### 5.3. Casimir elements and superposition rules

Let us illustrate in this section how to apply the formalism devised in the previous one to obtain, via the Casimir of \( \mathfrak{sl}_2 \), a superposition rule for the multisymplectic Lie system related to Schwarz equations.

Consider the Schwarz equation given by (3.1). Its first-order system of differential equation (3.2) is related to a Vessiot–Guldberg Lie algebra \( V^S = \langle X_1, X_2, X_3 \rangle \), where \( X_1, X_2, X_3 \) are given in (3.4). As shown in section 3.2, this Lie system is related to a multisymplectic Lie system \((\mathcal{O}, \Theta_S, X^S)\) with a Lie–Hamilton algebra of differential two-forms \( \mathcal{M}_S = \langle dh_1, dh_2, dh_3 \rangle \), with \( dh_1, dh_2, dh_3 \) given by (3.10), isomorphic to \( \mathfrak{sl}_2 \) relative to the Lie bracket (2.3) induced by \( \Theta_S \).

Let \( \{e_1, e_2, e_3\} \) be a basis of \( \mathfrak{sl}_2 \) satisfying the commutation relations
\[ [e_1, e_2] = -e_1, \quad [e_1, e_3] = -2e_2, \quad [e_2, e_3] = -e_3. \] (5.9)

The associated universal enveloping algebra \( U(\mathfrak{sl}_2) \) has essentially a unique Casimir element \([7, 27]\) whose symmetric tensorial form is given by
\[ C = e_1 \otimes e_3 + e_3 \otimes e_1 - 2e_2 \otimes e_2. \]

The corresponding morphism of algebras \( \Upsilon : U(\mathfrak{sl}_2) \to T(\mathcal{O}) \) gives rise to an invariant (relative to the Lie derivatives with elements of \( V^S \)) \( 4 \)-covariant tensor field on \( \mathcal{O} \) given by
\[ \Upsilon(C) = dh_1 \otimes dh_1 + dh_3 \otimes dh_1 - 2dh_2 \otimes dh_2. \]

It is simple to verify by direct computation that this tensor field is invariant relative to the Lie derivatives with respect to elements of the Vessiot–Guldberg Lie algebra \( V^S \).

The triple \((\mathcal{O}, X^S, V^S)\) is an automorphic Lie system. Therefore, it satisfies the conditions of lemma 4.7, which ensures that \( X^S \) admits a Lie algebra \( \text{Sym}(V^S) \) isomorphic to \( V^S \) of Lie symmetries of \( V^S \) and \( X^S \). In view of lemma 4.7, the multisymplectic form \( \Theta_S \) is also invariant relative to the elements of \( \text{Sym}(V^S) \). It is known that \( V^S \) admits a Lie algebra of Lie symmetries spanned by \([46, 57]\) the vector fields \( Y_1, Y_2, Y_3 \) given by (4.9).

The contractions of the tensor field \( \Upsilon(C) \) with four elements of \( \text{Sym}(V^S) \) are constants of motion for \( X^S \). In particular, a long but simple calculation shows that
\[ \iota_{Y_1,Y_2,Y_3,Y_4} \Upsilon(C) = -2, \quad \iota_{Y_2,Y_3,Y_4,Y_5} \Upsilon(C) = -1. \]
To obtain a superposition rule for the system, we have to obtain three functionally independent constants of motion for the diagonal prolongation of $X^3$ to $\mathcal{O}^3$ (see [23]). Let us write $d\theta_i^{(j)} = d\theta_i(x_{i(j)})$. Then, the extended invariant $\mathcal{I}^{(3)} = \mathcal{Y}^{(3)}(\Delta(C))$ reads

$$\mathcal{Y}(C)^{(3)} + 2(d\theta_i(\xi_1) \otimes d\theta_i(\xi_2) + d\theta_i(\xi_1) \otimes d\theta_j(\xi_2) - 2d\theta_2(\xi_1) \otimes d\theta_2(\xi_2)).$$

It is worth noting that $\mathcal{I}^{(3)} \neq \mathcal{Y}(C)^{(3)}$.

In virtue of corollary 5.9, the contractions of the tensor field $\mathcal{Y}(\Delta(C))$ with $Y_1^{(3)}, Y_2^{(3)}, Y_3^{(3)}$ are also invariants of $(X^3)^{(3)}$. This fact also allows us to obtain constants of motion for $(X^3)^{(3)}$ and, since the vector fields $X_1, X_2, X_3$ are linearly independent at a generic point, to obtain superposition rules for $X^3$ in view of the distributional method (see [22, 23]).

The contractions of $\mathcal{Y}(C)^{(3)}$ with four arbitrary vector fields of $(Y_1^{(3)}, Y_2^{(3)}, Y_3^{(3)})$ satisfy that

$$\tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} \mathcal{Y}(C)^{(3)} = \tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} \mathcal{Y}(C)(x_{i(1)}) + \tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} \mathcal{Y}(C)(x_{i(2)}),$$

for all $Y_a, Y_b, Y_c, Y_d \in \text{Sym}(V^3)$. As a consequence, such contractions are constants and therefore useless for our purposes as they will lead to trivial constants that cannot be used to obtain a superposition rule. Meanwhile, if $C = \mathcal{I}^{(3)} - \mathcal{Y}(C)^{(3)}$, then the contractions

$$I_1 = \left(\frac{a_2 v_1 - a_1 v_2}{v_1^2 v_2^2}\right)^2 = 2\tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} \tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} C,$$

$$I_2 = \left(\frac{2a_1 v_2(v_1 - v_2)}{a_2 v_1 - v_2 a_1}\right) + x_1 + x_2 = 2\tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} \tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} C I_1,$$

$$I_3 = \left(x_2 - \frac{2a_1 v_2^2}{a_2 v_1 - a_1 v_2}\right) \left(I_2 + x_2 + \frac{2a_1 v_2^2}{a_2 v_1 - a_1 v_2}\right) = \tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} \tau_{y_1^{(3)}} \tau_{y_2^{(3)}} \tau_{y_3^{(3)}} C I_1,$$

give rise to three constants of motion for $(X^3)^{(3)}$. Remarkably, the contractions of $C$ with elements of $\text{Sym}(V^3)$ offer many other constants of motion for the diagonal prolongation $(X^3)^{(3)}$, which allows us to select those having a simpler or more appropriate form to obtain a superposition rule. In particular, $I_1, I_2, I_3$ are chosen so as that

$$\partial(I_1, I_2, I_3)/\partial(x_1, x_2, a_1) \neq 0,$$

which implies that $I_1, I_2, I_3$ are functionally independent and enable us to obtain a superposition rule (see [22, 23]). Since $X_1^{(3)}, X_2^{(3)}, X_3^{(3)}$ span a distribution or rank three almost everywhere on the six-dimensional manifold $\mathcal{O}^3$, all remaining constants of motion for a generic $(X^3)^{(3)}$ are of the form $F = F(I_1, I_2, I_3)$ for a certain function $F: \mathbb{R}^3 \to \mathbb{R}$.

The above procedure is more powerful than the methods devised in [20, 51] to obtain the constants of motion for $(X^3)^{(3)}$. Indeed, the constants of motion obtained in [51] were derived via the characteristics method for $X_1^{(3)}, X_2^{(3)}, X_3^{(3)}$, which is very tedious. Meanwhile, the Dirac structure method employed in [20] is simpler to be applied than the procedure in [51] but it still demands to obtain a Dirac structure, which may be long, along with some Lie symmetries of $\text{Sym}(V^3)$ and some other symmetries chosen in an ad hoc way. Instead, our present methods give rise to the multisymplectic form in an immediate manner and, knowing several Lie symmetries, one can get the superposition rule. Observe that other contractions of $\mathcal{Y}(C)$ or $\mathcal{Y}^{(3)}(\Delta(C))$ with the Lie symmetries (4.9) and their prolongations to $\mathcal{O}^3$ can be used to obtain other tensorial invariants to study the Schwarz equation, e.g. presymplectic forms given by $\tau_{y_i^{(3)}} \mathcal{Y}(C)$ with $i, j = 1, 2, 3$.  

26
To obtain the searched superposition rule for the Schwarz equation, we use the functions \( I_1, I_2, I_3 \), which satisfy condition (5.10). Therefore, the following functions

\[
\begin{align*}
\Upsilon_1 &= I_1, \\
\Upsilon_2 &= I_2 \pm \frac{\sqrt{I_2^2 - 4I_3}}{2} = x_2 - \frac{2v_1v_2^2}{a_2v_1 - a_1v_2}, \\
\Upsilon_3 &= I_2 - I_1 = x_1 + \frac{2v_1^2v_2}{a_2v_1 - a_1v_2},
\end{align*}
\]

also satisfy (5.10) and the equations \( \Upsilon_1 = k_1, \ Upsilon_2 = k_2, \ Upsilon_3 = k_3 \) can be employed to obtain \( x_1, v_1, a_1 \) from \( x_2, v_2, a_2 \) and \( k_1, k_2, k_3 \). Indeed, these are the equations employed in [20] to obtain the superposition rule for Schwarz equations. Here, they appear without the necessity of using symmetries apart from the Lie symmetries of \( \text{Sym}(V) \).

More specifically, the superposition rule can be obtained by using the equation \( \Upsilon_1 = k_1 \), which enables us to obtain the value of \( a_2v_1 - v_2a_1 \) in terms of \( k_1, v_1, v_2 \). Substituting this into equations \( \Upsilon_2 = k_2 \) and \( \Upsilon_3 = k_3 \), we obtain two algebraic equations concerning the variables \( v_1, v_2, x_1, x_2 \) and the constants \( k_1, k_2, k_3 \). This allows us to obtain \( x_1, v_1 \) in terms of \( x_2, v_2, k_1, k_2, k_3 \). In particular, one finds that

\[
x_1 = \frac{\alpha x_2 + \beta}{\gamma x_2 + \delta},
\]

for certain constants \( \alpha, \beta, \gamma, \delta \) satisfying that \( \alpha\delta - \beta\gamma = 1 \) and whose form can be expressed as a function of \( k_1, k_2, k_3 \). The expression of \( v_1 \) and \( a_1 \) in terms of \( x_2, v_2, a_2, \alpha, \beta, \gamma, \delta \) can be then easily obtained from (5.11). The resulting expressions become a superposition rule for the Schwarz equation (see [22, 23]).

### 5.4. Invariant forms and superposition rules

Let us study an application of the methods of the previous section to derive, via invariant elements of the Grassmann algebra \( \Lambda(\mathfrak{s}l_2) \), a superposition rule for the multisymplectic Lie system related to a control system.

Consider the Riccati-type diffusion system

\[
\begin{align*}
\frac{dv}{dt} &= a(t)v^2, \\
\frac{dv}{dt} &= -b(t) + 2c(t)u + 4a(t)u^2, \\
\frac{dv}{dt} &= (c(t) + 4a(t)u) v, 
\end{align*}
\]

\( (5.12) \)

where \( a(t), b(t), \) and \( c(t) \) are arbitrary \( t \)-dependent functions. This system appears as a reduction of a system of differential equations that is used to solve diffusion-type equations, Burger’s equations, and other PDEs [62]. Moreover, its relation to Dirac structures has been studied in [20]. Let us apply our methods to obtain its properties.

For simplicity, we restrict ourselves to analysing the system (5.12) on \( N = \{(u, v, w) \in \mathbb{R}^3 \mid v \neq 0\} \). This highlights the main points of our presentation by avoiding secondary technical details, e.g., all hereafter given structures are well-defined on \( N \).

The system (5.12) describes the integral curves of the \( t \)-dependent vector field

\[
X^R_{\tau} = a(t)X_1 - b(t)X_2 + c(t)X_3,
\]

on \( N \), where

\[
\begin{align*}
X_1 &= 4u^2 \frac{\partial}{\partial u} + 4uv \frac{\partial}{\partial v} + v^2 \frac{\partial}{\partial w}, \\
X_2 &= 2u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\
X_3 &= \frac{\partial}{\partial u},
\end{align*}
\]

(5.13)

span a Lie algebra \( V^R \) isomorphic to \( \mathfrak{s}l_2 \). In fact,
\[ [X_1, X_2] = -2X_1, \quad [X_2, X_3] = -2X_3, \quad [X_1, X_3] = -4X_2. \] (5.14)

Hence, \( V^R \) is a Vessiot–Guldberg Lie algebra for the system (4.1) which becomes a Lie system.

Since \((du \wedge dv \wedge dw)(X_1, X_2, X_3) = v^4\), one has that \( D^R_p = T_pN \) for any \( p \in N \) and \( \dim V^R = \dim N \). Therefore, \((N, X, V^R)\) is a locally automorphic Lie system. Since the vector fields \( X_1, X_2, X_3 \) are linearly independent at a generic point, \( X \) admits a superposition rule depending on a unique particular solution. Recall that lemma 4.7 ensures that all Lie symmetries for the elements of \( V^R \) span a Lie algebra \( \text{Sym}(V^R) \) isomorphic to \( V^R \). A long but simple calculation allows us to obtain that \( \text{Sym}(V^R) \) is spanned by:

\[
Y_1 = v^2 \frac{\partial}{\partial u} + 4v dw \frac{\partial}{\partial v} + 4w^2 \frac{\partial}{\partial w}, \quad Y_2 = v \frac{\partial}{\partial v} + 2w \frac{\partial}{\partial w}, \quad Y_3 = \frac{\partial}{\partial w}.
\] (5.15)

The corresponding dual one-forms to the vector fields in (5.13) read

\[
\eta_1 = dw \frac{\partial}{\partial v^2}, \quad \eta_2 = dv \frac{\partial}{\partial v} - 4udw \frac{\partial}{\partial v^2}, \quad \eta_3 = dw - \frac{2udv}{v} + \frac{4u^2 dw}{v^2}.
\] (5.16)

Since \( V^R \) is semisimple, it is therefore unimodular and to obtain a multisymplectic form invariant under the Lie derivatives with elements of \( V^R \), one has just to define the multisymplectic form

\[
\Theta^{RS} = \eta_1 \wedge \eta_2 \wedge \eta_3 = \frac{1}{5} dw \wedge dv \wedge du
\]

turning \((N, \Theta^{RS}, X^{RS})\) into a multisymplectic Lie system.

The differential forms \( \omega_\alpha \Theta^{RS} \), with \( \alpha = 1, 2, 3 \), span a \( \text{Lie–Hamilton algebra} \mathfrak{m} \) isomorphic to \( V^R \). In view of the bracket (2.3) and (5.14), their commutation relations are

\[
[d\theta_1, d\theta_2] = 2d\theta_1, \quad [d\theta_1, d\theta_3] = 4d\theta_2, \quad [d\theta_2, d\theta_3] = 2d\theta_3.
\] (5.17)

To obtain invariants and superposition rules for \( X^{RS} \), we will derive invariants of the Grassmann algebra \( \Lambda(\mathfrak{sl}_2) \). More precisely, we will use an element of \( \Lambda(\mathfrak{sl}_2)^{Sym} \), e.g. \( C = e_1 \wedge e_2 \wedge e_3 \), where \( \{e_1, e_2, e_3\} \) is a basis of the real Lie algebra \( \mathfrak{sl}_2 \) satisfying the commutation relations (5.9). Then \( \Upsilon(2) \Delta(C) \) will be invariant under the evolution of \((X^{RS})[2]\), in view of corollary 5.9.

More specifically,

\[
\Delta(e_1 \wedge e_2 \wedge e_3) = (e_1 \wedge e_2 \wedge e_3) \boxtimes 1 + 1 \boxtimes (e_1 \wedge e_2 \wedge e_3) + (e_1 \wedge e_2) \boxtimes e_3 + (e_2 \wedge e_3) \boxtimes e_1 +
+(e_1 \wedge e_1) \boxtimes e_2 + e_2 \boxtimes (e_1 \wedge e_2) + e_3 \boxtimes (e_3 \wedge e_1) + e_1 \boxtimes (e_2 \wedge e_3).
\]

It is worth noting that, as proved in lemma 5.6, we have that \( \Delta(e_1 \wedge e_2 \wedge e_3) \subset \Lambda(\mathfrak{sl}_2) \boxtimes \Lambda(\mathfrak{sl}_2) \).

A simple calculation gives that

\[
\Upsilon_{1,1,1,1,1,1} \Upsilon(2) \Delta(e_1 \wedge e_2 \wedge e_3) = -2 \frac{[v_1^2 + v_2^2 - 4(u_1 - u_2)(w_1 - w_2)]^2}{v_1 v_2}.
\]

This can be simplified to

\[
f_1 := \frac{v_1^2 + v_2^2 - 4(u_1 - u_2)(w_1 - w_2)}{v_1 v_2}.
\]

The application of the Lie symmetries \( Y_1, Y_2, Y_3 \) to \( f_1 \) gives rise to the following two functionally independent invariants:
\[ f_2 = \frac{u_2 - u_1}{v_1 v_2}, \quad f_3 = \frac{v_1^2 - v_2^2 - 4(u_1 - u_2)(w_1 + w_2)}{v_1 v_2}. \]

Since \( \partial(f_1, f_2, f_3)/\partial(v_1, u_1, w_1) \neq 0 \), by equating \( f_1, f_2, f_3 \) to constants \( k_1, k_2, k_3 \), it is possible to obtain \( u_1, v_1, w_1 \) as functions of \( u_2, v_2, w_2 \) and the constants \( k_1, k_2, k_3 \) and to give rise to a superposition rule for the system under study.

As commented before, the above system admits a superposition rule depending on just one particular solution, natural generalization of Lie systems to the realm of partial differential equations. A PDE Lie system is a system of PDEs of the form

6. Conclusions and outlook

This work has illustrated the existence of multisymplectic Lie systems in the literature and has provided tools to endow Lie systems with a compatible multisymplectic structure. This has lead us to relate multisymplectic Lie systems with certain algebraic structures that enable the obtention of their superposition rules and invariants, retrieving as particular cases invariants found in the previous literature on the topic. Our new techniques do not involve the solution of complicated systems of PDEs, which circumvents some difficulties of previous approaches. Our methods also seem to have applications to extend the coalgebra method in the literature [5], which would have applications in the theory of integrable systems. Our work also illustrates the relevance of multisymplectic structures in the study of ordinary differential equations, which has been so far a topic of scarce analysis. Results have been illustrated by examples of physical and mathematical interest.

In the future, we aim to extend the approach given in multisymplectic Lie systems to arbitrary Lie systems by attaching to them an associative tensor algebra obtained by tensor products of the elements of a Vessiot–Guldberg Lie algebra and, eventually, their dual forms. The Vessiot–Guldberg Lie algebra acts then on this associative algebra and its invariants should give rise to invariant structures for Lie systems. We believe that this approach will recover all results of the whole literature on geometric structures for Lie systems as particular cases [4, 7, 20, 37, 49, 52]. Finally, it seems that the ideas of this work can be applied to generalise not only the coalgebra formalism for obtaining superposition rules for Lie–Hamilton systems but also the coalgebra method itself (see [5]).

Acknowledgments

The authors acknowledge fruitful discussions on the topic of the paper with our colleague N Román-Roy. We acknowledge partial financial support from the Polish National Science Centre project 2016/22/M/ST1/00542 (HARMONIA); from the Spanish Ministerio de Economía y Competitividad projects MTM2014–54855–P and MTM2015–64166–C2-1-P; from the Catalan Government project 2017–SGR–932; and from the Aragon Government grant E38_17R. The authors appreciate the careful review of the referees and their comments.

Appendix. Proof of lemma 5.6

Lemm A.1. Let \( \mathfrak{g} \) be a Lie algebra and \( m \in \mathbb{N} \). Then, \( \Delta^{(m)} S(\mathfrak{g}) \subset S^{(m+1)}(\mathfrak{g}) \) and \( \Delta^{(m)} \Lambda(\mathfrak{g}) \subset \Lambda^{(m+1)}(\mathfrak{g}) \).
Proof. Let us prove the inclusion for $S(g)$. It will be enough to prove it for homogeneous elements of $S(g)$. We first consider the case $m = 1$.

Consider elements $v_1, \ldots, v_r$ of $g$. Let us write $v_{i_1 \cdots \leq i_k} = v_{i_1} \otimes \cdots \otimes v_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq r$, the $r$ is any natural number, and $v_{i_1 \cdots \leq i_k}$ is the exterior product of the elements in $\{v_1, \ldots, v_r\}$, where $e$ stands for the complementary set in $\{v_1, \ldots, v_r\}$, ordered with respect to their indexes from the lower to the higher one. Using the fact that $\Delta$ is a morphism of associative algebras, we get that

$$\Delta(v_1 \otimes \cdots \otimes v_r) = \Delta(v_1) \cdots \Delta(v_r) = (v_1 \otimes 1 + 1 \otimes v_1) \cdots (v_r \otimes 1 + 1 \otimes v_r).$$

It follows by induction that the product on the right-hand side can be written as

$$\sum_{k=0}^{r} \sum_{1 \leq i_1 < \cdots < i_k \leq r} v_{i_1 \cdots < i_k} \otimes v_{i_1' \cdots < i_k'}.$$

If we write $\text{Alt}(v_1 \otimes \cdots \otimes v_r) = \sum_{\sigma \in S_r} \sigma(v_1 \otimes \cdots \otimes v_r)$, where $S_r$ is the permutation group of $r$ elements, then

$$\Delta(\text{Alt}(v_1 \otimes \cdots \otimes v_r)) = \sum_{\sigma \in S_r} \left[ \sum_{k=0}^{r} \sum_{1 \leq i_1 < \cdots < i_k \leq r} \left( v_{i_1 \cdots < i_k} \otimes v_{i_1' \cdots < i_k'} \right) \right]. \quad (A.1)$$

To prove that the right-hand side belongs to $S(g) \otimes S(g)$, we aim to decompose the elements of $\sigma \in S_r$ in a specific way so as to control how they act on the elements $v_{i_1 \cdots < i_k} \otimes v_{i_1' \cdots < i_k'}$.

Let $G_{i_1 \cdots < i_k}$ be the subgroup of $S_r$ whose elements leave the subset $\{i_1, \ldots, i_k\}$ invariant and let $G_{i_1 \cdots < i_k}$ be the subgroup of $G_{i_1 \cdots < i_k}$ whose elements fix all the elements $v_{i_1}, \ldots, v_{i_k}$. Let $\bar{\sigma}_1, \ldots, \bar{\sigma}_r$ and $\bar{\sigma}_1', \ldots, \bar{\sigma}_r'$ be representatives of the equivalence classes of the cosets $S_r/G_{i_1 \cdots < i_k}$ and $G_{i_1 \cdots < i_k}/G_{i_1 \cdots < i_k}$, respectively. Then, $S_r = \bigcup_{r=1}^{r} \bigcup_{\beta \in G_{i_1 \cdots < i_k}} \bar{\sigma}_r \circ \beta \circ \sigma'$. Every $\sigma' \in G_{i_1 \cdots < i_k}$ fixes all the elements of $\{i_1, \ldots, i_k\}$ and leaves invariant the subset $\{i_1, \ldots, i_k\}$. Hence, $\sigma'(v_{i_1 \cdots < i_k}) = v_{i_1 \cdots < i_k}$ whereas $\sum_{\sigma' \in G_{i_1 \cdots < i_k}} \sigma'(v_{i_1' \cdots < i_k'}) = \text{Alt}(v_{i_1' \cdots < i_k'})$. Using the previous decomposition of $S_r$ and (A.1), we obtain

$$\Delta(\text{Alt}(v_1 \otimes \cdots \otimes v_r)) = \sum_{k=0}^{r} \sum_{1 \leq i_1 < \cdots < i_k \leq r} \sum_{\alpha, \beta} \bar{\sigma}_r \circ \beta \alpha [v_{i_1 \cdots < i_k}] \otimes \text{Alt}(v_{i_1' \cdots < i_k'})]. \quad (A.2)$$

The element $\bar{\sigma}_r \circ \beta$ leaves stable $\{v_{i_1}, \ldots, v_{i_k}\}$ and its complementary set. Since $\text{Alt}(v_{i_1' \cdots < i_k'})$ is totally symmetric, one has that $\bar{\sigma}_r \circ \beta \alpha [\text{Alt}(v_{i_1' \cdots < i_k'})] = \text{Alt}(v_{i_1' \cdots < i_k'})$. Since $G_{i_1 \cdots < i_k}/G_{i_1 \cdots < i_k} \cong S_k$, one has that

$$\sum_{\beta} \bar{\sigma}_r \circ \beta [v_{i_1 \cdots < i_k}] = \text{Alt}(v_{i_1 \cdots < i_k}).$$

Using previous results in (A.2), one gets

$$\Delta(\text{Alt}(v_1 \otimes \cdots \otimes v_r)) = \sum_{\alpha} \bar{\sigma}_r \alpha \text{Alt}(v_{i_1 \cdots < i_k}) \otimes \text{Alt}(v_{i_1' \cdots < i_k'}) \in S^2(g)$$

and $\Delta S(g) \subset S^2(g)$. 

30
Once the case for \( m = 1 \) has been proved, the case for general \( m \) can be proved by induction. Assume that the result is true for \( m \) and therefore \( \Delta^{(m)}(S(g)) \subset S^{(m+1)}(g) \). Let us prove that our lemma holds true for \( m + 1 \). By the recurrence relation for \( \Delta^{(m+1)} \) and the induction hypothesis, one has that

\[
\Delta^{(m+1)}(S(g)) = (\text{Id} \otimes \cdots \otimes \text{Id} \otimes \Delta) \circ \Delta^{(m)}(S(g)) = (\text{Id} \otimes \cdots \otimes \text{Id} \otimes \Delta)(S^{(m)}(g)) \subset S^{(m+1)}(g).
\]

The proof for \( \Lambda(g) \) follows analogously by defining \( \overline{\Lambda}(v_{i_1, \ldots, i_k}) = \sum_{\sigma \in \Sigma} (-1)^{\text{sign}(\sigma)} \sigma(v_{i_1, \ldots, i_k}) \) and using the following equalities:

\[
\sum_{\sigma' \in \Omega} (-1)^{\text{sign}(\sigma')} \sigma'(v_{i_1, \ldots, i_k}) = \overline{\Lambda}(v_{i_1, \ldots, i_k}),
\]

\[
\sum_{\beta = 1}^{v_{i_1, \ldots, i_k}} (-1)^{\text{sign}(\sigma)} \sigma(\beta)v_{i_1, \ldots, i_k}) \otimes \overline{\Lambda}(v_{i_1, \ldots, i_k}) \big) = \sum_{\beta = 1}^{v_{i_1, \ldots, i_k}} \overline{\Lambda}(v_{i_1, \ldots, i_k}) \otimes \overline{\Lambda}(v_{i_1, \ldots, i_k}).
\]

\( \square \)

**References**

[1] Abraham R and Marsden J E 1987 *Foundations of Mechanics* (Reading, MA: Addison-Wesley)
[2] Anderson R L, Harnad J and Winternitz P 1981 Group theoretical approach to superposition rules for systems of Riccati equations *Lett. Math. Phys.* **5** 143–8
[3] Angelo R M, Duzzioni E I and Ribeiro A D 2012 Integrability in \( t \)-dependent systems with one degree of freedom *J. Phys. A: Math. Theor.* **45** 055101
[4] Ballesteros A, Blasco A, Herranz F, de Lucas J and Sardón C 2015 Lie–Hamilton systems on the plane: properties, classification and applications *J. Differ. Equ.* **258** 873–907
[5] Ballesteros A, Blasco A, Herranz F J, Musso F and Ragnisco O 2009 (Super)integrability from coalgebra symmetry: formalism and applications *J. Phys.: Conf. Ser.* **175** 012004
[6] Ballesteros A, Campoamor-Stursberg R, Fernandez-Saiz E, Herranz F J and de Lucas J 2018 Poisson–Hopf algebra deformations of Lie–Hamilton systems *J. Phys. A: Math. Theor.* **51** 065202
[7] Ballesteros A, Cariñena J F, Herranz F, de Lucas J and Sardón C S 2013 From constants of motion to superposition rules for Lie–Hamilton systems *J. Phys. A: Math. Theor.* **46** 285203
[8] Berkovich L M 2007 Method of factorization of ordinary differential operators and some of its applications *Appl. Anal. Discrete Math.* **1** 122–49
[9] Blasco A, Herranz F J, de Lucas J and Sardón C 2015 Lie–Hamilton systems on the plane: applications and superposition rules *J. Phys. A: Math. Theor.* **48** 345202
[10] Blázquez Sanz D 2008 *Differential Galois Theory and Lie–Vessiot Systems* (Berlin: VDM Verlag)
[11] Blázquez-Sanz D and Morales-Ruiz J J 2012 Lie’s reduction method and differential Galois theory in the complex analytic context *Discrete Continuous Dyn. Syst.* **32** 353–79
[12] Bourbaki N 1968 *Groupes et Algèbres de Lie* (Paris: Hermann)
[13] Bourbaki N 1974 *Elements of Mathematics. Algebra, Part I: Chapters 1–3* (Paris: Hermann)
[14] Bump D 2013 *Lie Groups* (Graduate Texts in Mathematics vol 225) (New York: Springer)
[15] Cantrijn F, Ibort L A and de León M 1996 Hamiltonian structures on multisymplectic manifolds (Geometrical structures for physical theories, I (Vietri, 1996)) Rend. Sem. Mat. Univ. Politech. Torino 54 225–36
[16] Cantrijn F, Ibort A and de León M 1999 On the geometry of multisymplectic manifolds and J. Aust. Math. Soc. A 66 303–30
[17] Cariñena J F, Clemente-Gallardo J, Jover-Galtier J A and de Lucas J 2016 Lie systems and Schrödinger equations (60 Years Alberto Ibort Fest Classical and Quantum Physics) (Berlin: Springer)
[18] Cariñena J F, Grabowski J and de Lucas J 2012 Superposition rules for higher-order systems and their applications J. Phys. A: Math. Theor. 45 185202
[19] Cariñena J F, Grabowski J and de Lucas J 2019 Quasi-Lie schemes for PDEs Int. J. Geom. Methods Mod. Phys. accepted (arXiv:1712.02238v2)
[20] Cariñena J F, Grabowski J, de Lucas J and Sardón C 2014 Dirac–Lie systems and Schwarzian equations J. Differ. Equ. 257 2303–40
[21] Cariñena J F, Grabowski J and Marmo G 2000 Lie–Scheffers Systems: a Geometric Approach (Naples: Bibliopolis)
[22] Cariñena J F, Grabowski J and Marmo G 2007 Superposition rules, Lie theorem and partial differential equations Rep. Math. Phys. 60 237–58
[23] Cariñena J F and de Lucas J 2011 Lie systems: theory, generalisations, and applications Dissertationes Math. 479 1–162
[24] Cariñena J F and de Lucas Araujo J 2011 Superposition rules and second-order Riccati equations J. Geom. Mech. 3 1–22
[25] Cariñena J F, de Lucas J and Sardón C 2013 Lie–Hamilton systems: theory and applications Int. J. Geom. Methods Mod. Phys. 10 1350047
[26] Chakravarty S and Halburd R 2003 First integrals of a generalized Darboux–Halphen system J. Math. Phys. 44 1751–62
[27] Chari V and Pressley A 1994 A Guide to Quantum Groups (Cambridge: Cambridge University Press)
[28] Darboux G 1878 Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux Ann. Sci. École Norm. Super. 7 101–50
[29] Drager L D, Lee J M, Park E and Richardson K 2012 Smooth distributions are finitely generated Ann. Global Anal. Geom. 41 357–69
[30] Echeverría-Enríquez A, Muñoz-Lecanda M C and Román-Roy N 1998 Multivector fields and connections: setting Lagrangian equations in field theories J. Math. Phys. 39 4578–603
[31] Estévez P G, Herranz F J, de Lucas J and Sardón C 2016 Lie symmetries for Lie systems: applications to systems of ODEs and PDEs Appl. Math. Comput. 273 435–52
[32] Ferreira L A, Gomes J F, Razumov A V, Saveliev M V and Zimerman A H 1999 Riccati-type equations, generalised WZW equations, and multidimensional Toda systems Commun. Math. Phys. 203 649–66
[33] Flores-Espinoza R 2011 Periodic first integrals for Hamiltonian systems of Lie type Int. J. Geom. Methods Mod. Phys. 8 1169–77
[34] Forger M and Römer H 2001 A Poisson bracket on multisymplectic phase space Rep. Math. Phys. 48 211–8
[35] Grabowski J and de Lucas J 2013 Mixed superposition rules and the Riccati hierarchy J. Differ. Equ. 254 179–98
[36] Greub W 1978 Multilinear Algebra (New York: Springer)
[37] Grundland A M and de Lucas J 2017 A Lie systems approach to the Riccati hierarchy and partial differential equations J. Differ. Equ. 263 299–337
[38] de Lucas J and Grundland A M 2018 A cohomological approach to immersed submanifolds via integrable systems Sel. Math. 24 4749–80
[39] Grundland A M, Martina L and Rideau G 1997 Partial differential equations with differential constraints Advances in Mathematical Sciences: 25 years (Proc. and Lecture Notes vol 11) (Providence, RI: American Mathematical Society) pp 135–54
[40] Halphen G 1881 Sur un système d’équations différentielles C. R. Acad. Sci., Paris 92 1101–3
[41] Herranz F J, de Lucas J and Sardón C 2015 Jacobi–Lie systems: fundamentals and low-dimensional classification (Dynamical systems, differential equations and applications) (10th AIMS Conf.) Discrete Contin. Dyn. Syst. Suppl. 605–14
[42] Herranz F J, de Lucas J and Tobolski M 2017 Lie–Hamilton systems on curved spaces: a geometrical approach J. Phys. A: Math. Theor. 50 405201
[43] Ibragimov N H 2009 Integration of systems of first-order equations admitting nonlinear superposition J. Nonlinear Math. Phys. 16 137–47
[44] James I M 1996 Reflections of the history of topology Rend. Sem. Mat. Fis. Milano 66 87–96
[45] Komrakov B, Churyumov A and Doubrov B 1993 Two-dimensional homogeneous spaces Pure Math. 17 1–142
[46] Leach P G L and Govinder K S 1999 On the uniqueness of the Schwarzian and linearisation by nonlocal contact transformation J. Math. Anal. Appl. 235 84–107
[47] Lee J M 2013 Introduction to Smooth Manifolds 2nd edn (New York: Springer)
[48] Lehto O 1987 Univalent Functions and Teichmüller Spaces (Graduate Texts in Mathematics vol 109) (New York: Springer)
[49] Lewandowski M M and de Lucas J 2017 Geometric features of Vessiot–Guldberg Lie algebras of conformal and Killing vector fields on $\mathbb{R}^2$ Banach Center Publ. 113 243–62
[50] Lie S and Scheffers G 1893 Vorlesungen über Continuierliche Gruppen Mit Geometrischen und Anderen Anwendungen (Leipzig: Teubner)
[51] de Lucas J and Sardón C 2013 On Lie systems and Kummer–Schwarz equations J. Math. Phys. 54 033505
[52] de Lucas J and Vilariño S 2015 $k$-symplectic Lie systems: theory and applications J. Differ. Equ. 258 2221–55
[53] Milnor J 1976 Curvatures of left invariant metrics on Lie groups Adv. Math. 21 293–329
[54] Nikitin S 2000 Control synthesis for Caplygin polynomial systems Acta Appl. Math. 60 199–212
[55] Odzijewicz A and Grundland A M 2000 The superposition principle for the Lie type first-order PDEs Rep. Math. Phys. 45 293–306
[56] Ortega J P and Ratiu T S 2004 Momentum Maps and Hamiltonian Reduction (Progress in Mathematics vol 222) (Boston, MA: Birkhäuser)
[57] Ovsienko V and Tabachnikov S 2005 Projective Differential Geometry Old and New: from the Schwarzian Derivative to Cohomology of Diffeomorphism Groups (Cambridge: Cambridge University Press)
[58] Ovsienko V and Tabachnikov S 2009 What is ... the Schwarzian derivative? Not. AMS 56 34–6
[59] Palais R S 1957 A global formulation of the Lie theory of transformation groups Mem. Am. Math. Soc. 22 1–123
[60] Ramos A 2006 New links and reductions between the Brockett nonholonomic integrator and related systems Rend. Semin. Mat. Univ. Politech. Torino 64 39–54
[61] Shnider S and Winternitz P 2006 Classification of systems of nonlinear ordinary differential equations with superposition principles J. Math. Phys. 25 3155–65
[62] Suazo E, Suslov S K and Vega-Guzmán J M 2014 The Riccati system and a diffusion-type equation Mathematics 2014 96–118
[63] Temple G 1960 A superposition principle for ordinary nonlinear differential equations Lectures on Topics in Nonlinear Differential Equations (Carderock: David Taylor Model Basin) pp 1–15 (report 1415)
[64] Varadarajan V S 1984 Lie Groups, Lie Algebras, and Their Representations (Graduate Texts in Mathematics vol 102) (New York: Springer)
[65] Vessiot E 1904 Sur la théorie de Galois et ses diverses généralisations Ann. Sci. l’École Norm. Supér. 21 9–85
[66] Winternitz P 1983 Lie groups and solutions of nonlinear differential equations Nonlinear Phenomena (Lecture Notes in Physics vol 189) (Berlin: Springer) pp 263–331
[67] Yoo W 2015 The automorphisms of a Lie algebra Appl. Math. Sci. 9 121–7