COMMENTS ON SAMPSON’S APPROACH TOWARD HODGE CONJECTURE ON ABELIAN VARIETIES

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Abstract. Let $A$ be an Abelian variety of dimension $n$. For $0 < p < 2n$ an odd integer, Sampson constructed a surjective homomorphism $\pi : J^p(A) \to A$, where $J^p(A)$ is the higher Weil Jacobian variety of $A$. Let $\omega$ be a fixed form in $H^{1,1}(J^p(A), \mathbb{Q})$, and $N = \dim(J^p(A))$. He observes that if the map $\pi_*(\omega^{N-p-1} \wedge \cdot) : H^{1,1}(J^p(A), \mathbb{Q}) \to H^{n-p,n-p}(A, \mathbb{Q})$ is injective (and hence surjective, by dimension considerations), then the Hodge conjecture is true for $A$ in bidegree $(p,p)$.

In this paper, we show that the map above is not surjective if $\dim_{\mathbb{Q}} H^{p,p}(A, \mathbb{Q}) > \dim_{\mathbb{Q}} H^{1,1}(A, \mathbb{Q})$. The proof uses that because $\pi^* H^1(A, \mathbb{R})$ is exactly

$$\{ \alpha \in H^1(J^p(A), \mathbb{R}) : \alpha(u) = 0 \forall u \in \ker(\pi) \},$$

we can write conveniently $\hat{\omega} = \omega_1 + \pi^*(\alpha)$, here $\alpha \in H^{1,1}(A, \mathbb{Q})$ and $\omega_1^{N-n+1} = 0$. Here $N = \dim(J^p(A))$.

This result is valid for any surjective homomorphism $\pi : \hat{A} \to A$ between Abelian varieties.

1. Introduction and Results

A compact complex manifold $X$ is projective if it is a submanifold of a complex projective space $\mathbb{P}^N$. Hodge conjecture is the following statement

**Hodge conjecture.** Let $X$ be a projective manifold. If $u \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ then $u$ is a linear combination with rational coefficients of the classes of algebraic cycles on $X$.

There have been a lot of works on the conjecture, however, it is still very largely open (see [2, 4]). The case of Abelian varieties, on which the cohomology groups are explicitly described, have been extensively studied, see Appendix 2 in [2]. In this case, also, the Hodge conjecture is still open, even though many partial results have been obtained.

Sampson [3] (see also Appendix 2 in [2]) proposed one approach toward proving the Hodge conjecture for Abelian varieties using Weil Jacobians. He suggested that the Hodge conjecture would follow if a certain map is injective (and hence surjective). In this paper we show that in general this is not the case.

We will first recall the construction of the map $\pi$, then will show that it is not surjective (and hence not injective), in general.

1.1. **Abelian varieties.** Let $A = V/L$ be an Abelian variety of dimension $n$. Here $V = \mathbb{R}^{2n}$ is equipped with a complex structure $J : V \to V$ with $J^2 = -1$, and $L$ is a lattice of rank $2n$. There is one alternating bilinear form $E : V \times V \to \mathbb{R}$ such that
$E(Jx, Jy) = E(x, y)$, $E(x, Jy)$ is a symmetric and positive definite bilinear form on $V$, and $E(L, L) \subset \mathbb{Z}$. There associated an integral Kähler form on $A$, given by the following formula

$$\omega = \sum_{i,j} E(e_i, e_j) dx^i \wedge dx^j.$$ 

Here $e_1, \ldots, e_{2n}$ are a basis for $V$, and $x^i$ is the real coordinate corresponding to $e_i$. The Kähler form $\omega$ does not depend on the choice of the basis.

There also associated a Hermitian metric

$$H(x, y) = E(x, Jy) - i E(x, y).$$

For more on Abelian varieties, see [1].

### 1.2. Weil Jacobians

Let $e_1, \ldots, e_{2n}$ be a basis for the lattice $L$. Let $0 < p < 2n$ be an odd integer. Define $\hat{V} = \bigwedge^p V$.

We define $\hat{L} \subset \hat{V}$ to be the lattice generated by the elements $e_I = \wedge_{i \in I} e_i$, where $I$ is a multi-index of length $p$.

$J$ defines a complex structure $\hat{J}$ on $\hat{V}$ by the formula $\hat{J}(e_I) = \wedge_{i \in I} J e_i$.

$E$ defines a bilinear form $\hat{E}$ on $\hat{V}$ by the formula: $\hat{E}(e_I, e_J) = \det(E(e_i, e_j))_{i \in I, j \in J}$.

It can then be checked that $\hat{E}$ is alternating, $\hat{E}(Jx, Jy) = \hat{E}(x, y)$, $\hat{E}(\hat{L}, \hat{L}) \subset \mathbb{Z}$, and $\hat{E}(e_I, \hat{J} e_J)$ is symmetric and positive definite. Thus $J^p(A) = \hat{V}/\hat{L}$ is an Abelian variety.

There is an isomorphism $f : H^{1,1}(J^p(A), \mathbb{Z}) \rightarrow H^{p,p}(A, \mathbb{Q})$, see Proposition 7 in [3].

### 1.3. Sampson’s construction

Starting from the Kähler form $\omega$ associated with the bilinear form, Sampson defines a surjective homomorphism $\pi : \hat{V} \rightarrow V$, which is $\mathbb{C}$-linear and preserves the lattice $\hat{L}$. Thus it descends to a homomorphism $\pi : J^p(A) \rightarrow A$.

The construction of Sampson is to assign directly

$$\pi(e_I) = \sum_{j=1}^{2n} b_I^j e_j,$$

where $b_I^j$ comes from the coefficients of the form $\omega^{(p+1)/2}$, and the inverse of the matrix $(E(e_i, e_j))$. Then he uses explicit computations to show that the map $\pi$ is surjective and $\mathbb{C}$-linear.

If we consider what happens with the pullback map $\pi^* : H^1(A, \mathbb{R}) \rightarrow H^1(J^p(A), \mathbb{R})$, then the above construction will look more transparent. In fact, let $x^i$ be the coordinate corresponding to $e_i$, and $x^I$ the coordinate corresponding to $e_I$. Then we have

$$\pi(\sum_I x^I e_I) = \sum_{j=1}^{2n} (\sum_I b_I^j x^I) e_j.$$
Hence $x^j = \sum_I b^j_I x^I$. From this, we obtain
\[ \pi^*(dx^j) = \sum_I b^j_I dx^I. \]

Here we recall that given a basis $(v_I)$ for a vector space, with corresponding coordinates $z^I$, then the form $dz^j$ is given by $dz^j(v_I) = \delta^I_j$.

Now we make the following identification $\psi : H^1(Jp(A), \mathbb{R}) \to H^p(\hat{A}, \mathbb{R})$. We assign $\psi(dx^I) = \wedge_{i \in I} dx^i$. Then, by using a quasi-symplectic basis $e_1, \ldots, e_{2n}$ for $L$, we obtain a very simple formula
\[ \psi \circ \pi^*(dx^j) = c_j dx^j \wedge \omega^{(p-1)/2}. \]

Here $c_j$ is a non-zero constant. Thus we see that $\psi \circ \pi^*$ is, up to a multiplicative constant, the Lefschetz map.

By Lefschetz isomorphism theorem (see Lecture 11 in [2]), $\psi \circ \pi^*(dx^j)$ is injective, and hence $\pi$ is surjective. The property that $\pi$ is $\mathbb{C}$-linear can also be checked by choosing the basis $Je_1, \ldots, Je_{2n}$ in the definition of the map $\psi \circ \pi^*$.

**1.4. Non-surjectivity of the pushforward $\pi_*$.** Let notations be as in the previous subsections. Let $\tilde{\omega}$ be the integral Kähler form on $\tilde{V}$ corresponding to the bilinear form $\hat{E}$. Sampson observed that if the map $\pi_* (\tilde{\omega}^{N-p-1} \wedge .) : H^{1,1} (J^p(A), \mathbb{Q}) \to H^{n-p,n-p}(A, \mathbb{Q})$ is injective (and hence surjective, by dimension considerations), then the Hodge conjecture is true for $A$ in bidegree $(p, p)$. We will show that in general this is not the case. The result is valid in a more general setting. In the remark after the proof of the result we will discuss how the result still holds under the optimal condition $\dim \mathbb{Q} H^{p,p}(A, \mathbb{Q}) > \dim \mathbb{Q} H^{1,1}(A, \mathbb{Q})$, if a strong Poincaré duality holds for $A$.

**Proposition 1.1.** Let $\pi : \hat{A} = \tilde{V}/\tilde{\Lambda} \to A = V/L$ be a surjective homomorphism of Abelian varieties. Let $\tilde{\omega}$ be a fixed form in $H^{1,1}(\hat{A}, \mathbb{Q})$. If $\dim \mathbb{Q} H^{p,p}(A, \mathbb{Q}) > \dim \mathbb{Q} H^{1,1}(\mathbb{Q})$, then the map
\[ \pi_* (\tilde{\omega}^{N-p-1} \wedge .) : H^{1,1}(\hat{A}, \mathbb{Q}) \to H^{n-p,n-p}(A, \mathbb{Q}) \]
is not surjective.

**Proof.** Let $N = \dim(\hat{A})$ and $n = \dim(A)$. Let $\hat{J}$ be the complex structure on $\hat{A}$ and $J$ the complex structure on $A$. Let $\hat{E}$ be the corresponding bilinear form of $\tilde{\omega}$. First we consider the case $\hat{E}(x, \hat{J}x) \neq 0$ for all $0 \neq x \in \hat{V}$. The general case will be dealt with at the end of the proof.

1) We define $W \subset \hat{V}$ to be the kernel of the map $\pi : \hat{V} \to V$. Because the map $\pi$ is $\mathbb{C}$-linear, it follows that $JW = W$. Moreover, since $\pi$ is surjective, $\dim(W) = 2N - 2n$.

2) We observe that if $\pi^*(du) \in \pi^* H^1(A, \mathbb{R})$ and $v \in W$, then $\pi^*(du)(v) = du(\pi(v)) = du(0) = 0$.

3) We let $\hat{W}^\perp$ to be the orthogonal complement of $W$, with respect to $\hat{E}$. Because $\hat{E}(x, \hat{J}x) > 0$ for all $0 \neq x \in \hat{V}$, we have $W \cap W^\perp = 0$. Therefore, we have the
decomposition

\[ \hat{V} = W \oplus W^\perp. \]

We note that \( \dim(W^\perp) = 2n. \)

4) We choose a basis \( e_1, \ldots, e_{2N-2n} \) for \( W \), and \( f_1, \ldots, f_{2n} \) a basis for \( W^\perp \). We let \( x^1, \ldots, x^{2N-2n} \) and \( y^1, \ldots, y^{2n} \) be the corresponding coordinates. Then we have the corresponding 1-forms \( dx^1, \ldots, dx^{2N-2n} \) and \( dy^1, \ldots, dy^{2n} \) on \( \hat{V} \).

5) By definition, we have

\[ dy^j(e_i) = 0 \]

for all \( i, j \). Comparing with point 2) and taking dimensions into consideration, we conclude that \( \pi^*H^1(A, \mathbb{R}) \) is generated by \( dy^1, \ldots, dy^{2n} \).

6) By point 3), the form

\[
\hat{\omega} = \sum \hat{E}(e_i, e_j)dx^i \wedge dx^j + \sum \hat{E}(f_i, f_j)dy^i \wedge dy^j + \sum \hat{E}(e_i, f_j)dx^i \wedge dy^j
\]

has no cross term. By point 5) we see that we can write \( \hat{\omega} = \omega_1 + \omega_2 \), where \( \omega_1 \) involves only \( dx^i \), and \( \omega_2 = \pi^*(\alpha) \in \pi^*H^2(A, \mathbb{R}) \).

Moreover, we see that \( \omega_1 \) is not other than the restriction of \( \hat{\omega} \) to \( W \), and \( \pi^*(\alpha) \) is not other than the restriction of \( \hat{\omega} \) to \( W^\perp \). Since \( W \) and \( W^\perp \) are both invariant under the complex structure \( \hat{J} \), both forms \( \omega_1 \) and \( \pi^*(\alpha) \) are of type \((1, 1)\). Then \( \alpha \) is of bidegree \((1, 1)\) also.

We also have that both \( \omega_1 \) and \( \alpha \) are rational. This again follows easily from that \( \omega_1 \) and \( \pi^*(\alpha) \) are the restrictions of \( \hat{\omega} \) to \( W \) and \( W^\perp \), and both \( \hat{L} \cap W \) and \( \hat{L} \cap W^\perp \) have maximal ranks.

7) We now show that the map

\[ \pi_*(\hat{\omega}^{N-p-1} \wedge \cdot) : H^{1,1}(\hat{A}, \mathbb{Q}) \to H^{n-p,n-p}(A, \mathbb{Q}) \]

is not surjective. Let \( u_0 \in H^{p,p}(X, \mathbb{Q}) \). Then, for any divisor \( D \in H^{1,1}(A, \mathbb{Q}) \)

\[ \pi_*(\hat{\omega}^{N-p-1} \wedge D) \wedge u_0 = \pi_*(\hat{\omega}^{N-p-1} \wedge \pi^*(u_0) \wedge D). \]

Using \( \omega = \omega_1 + \pi^*(\alpha) \), we have

\[ \hat{\omega}^{N-p-1} \wedge \pi^*(u_0) \wedge D = (\omega_1 + \pi^*(\alpha))^{N-p-1} \wedge \pi^*(u_0) \wedge D \]

\[ = \sum_j c_j \omega_1^{N-p-1-j} \wedge \pi^*(\alpha)^j \wedge \pi^*(u_0) \wedge D. \]

Here \( c_j \in \mathbb{N} \) are constants. From points 5) and 6), we have that the \( j \)-th summand in the above sum is zero, unless \( j + p \leq n \) and \( N - p - 1 - j \leq N - n \). Hence there are only two terms left

\[ \hat{\omega}^{N-p-1} \wedge \pi^*(u_0) \wedge D = c_1 \omega_1^{N-n-1} \wedge D \wedge \pi^*(\alpha^{n-p} \wedge u_0) + c_2 \omega_1^{N-n} \wedge D \wedge \pi^*(\alpha^{n-p-1} \wedge u_0). \]

This shows that

\[ \pi_*(\hat{\omega}^{N-p-1} \wedge D) = \alpha^{n-p-1} \wedge [\pi_*(c_1 \omega_1^{N-n-1} \wedge D) \wedge \alpha + c_2 \omega_1^{N-n} \wedge D]. \]
We note that 
\[ \pi_*(c_1 \omega_1^{N-n-1} \wedge D) \wedge \alpha + c_2 \omega_1^{N-n} \wedge D \in H^{1,1}(A, \mathbb{Q}). \]

From this it follows that \( \pi_*(\hat{\omega}^{N-p-1} \wedge H^{1,1}(\hat{A}, \mathbb{Q})) \) is contained in the image of the linear map 
\[ \alpha^{n-p-1} \wedge : H^{1,1}(A, \mathbb{Q}) \to H^{n-p,n-p}(A, \mathbb{Q}). \]

Therefore 
\[ \dim_Q \pi_*(\hat{\omega}^{N-p-1} \wedge H^{1,1}(\hat{A}, \mathbb{Q})) \leq \dim_Q H^{1,1}(A, \mathbb{Q}). \]

Hence, by the assumption of the proposition, \( \pi_*(\hat{\omega}^{N-p-1} \wedge H^{1,1}(\hat{A}, \mathbb{Q})) \) can not be the whole \( H^{n-p,n-p}(A, \mathbb{Q}) \).

8) Now we consider a general form \( \hat{\omega} \in H^{1,1}(\hat{A}, \mathbb{Q}) \). We can write 
\[ \hat{\omega} = \lim_{t \to 0} \hat{\omega}(t). \]

Here for \( t \neq 0 \), then the bilinear form \( \hat{E}(t)(\hat{\omega}(t)) \) satisfies the condition \( \hat{E}(t)(x, \hat{J}x) \neq 0 \) for \( 0 \neq x \in \hat{V} \). We can write for each \( t \neq 0 \):
\[ \hat{\omega}(t) = \omega_1(t) + \pi^*(\alpha(t)). \]

Since \( \omega_1(t) = \omega(t)|_W \), by point 6), the following limit exists
\[ \lim_{t \to 0} \omega_1(t) = \omega_1. \]

Moreover, \( \omega_1 \) is also of bidegree \((1, 1)\). Since \( \omega_1(t)^{N-n+1} = 0 \) for all \( t \neq 0 \), it follows that \( \omega_1^{N-n+1} = 0 \). We deduce that \( \lim_{t \to 0} \alpha(t) = \alpha \) also exists, and of bidegree \((1, 1)\).

We need to check that both \( \omega_1 \) and \( \alpha \) are rational. To see this, we can first check that \( \omega_1 \) is rational, which is clear using \( \alpha|_W = 0 \), and \( W \cap \hat{L} \) has maximal rank. Then it also follows that \( \alpha \) is also rational. Then we can proceed as before.

**Remark 1.2.** The following modification of the approach, requiring that 
\[ \pi_*(\wedge^{N-p} H^{1,1}(J^p(A), \mathbb{Q})) = H^{n-p,n-p}(A, \mathbb{Q}) \]

may be working.

**References**

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