Wandering continua for rational maps
Guizhen Cui, Yan Gao

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Abstract

We prove that a Lattès map admits an always full wandering continuum if and only if it is flexible. The full wandering continuum is a line segment in a bi-infinite or one-side-infinite geodesic under the flat metric.

1 Introduction

Let $f$ be a rational map of the Riemann sphere $\hat{\mathbb{C}}$ with $\deg f \geq 2$. Denote by $J_f$ and $F_f$ the Julia set and the Fatou set of $f$ respectively. One may refer to [12] for their definitions and basic properties. By a continuum we mean a connected compact set consisting of more than one point. A continuum $K \subset \hat{\mathbb{C}}$ is called a wandering continuum for $f$ if $K \subset J_f$ and $f^n(K) \cap f^m(K) = \emptyset$ for any $n > m \geq 0$.

The existence of wandering continua for polynomials has been studied by many authors. It was proved that all wandering components of the Julia set of a polynomial with disconnected Julia set are points [1, 8, 15]. For polynomials with connected Julia sets, it was proved that a polynomial without irrational indifferent periodic cycles has no wandering continuum if and only if the Julia set is locally connected [2, 5, 6, 9, 16].

The situation for non-polynomial rational maps is different. There are hyperbolic rational maps which have non-degenerate wandering components of their Julia sets. The first example was given by McMullen, where the wandering Julia components are Jordan curves [11]. In fact, it was proved that for a geometrically finite rational map, a wandering component of its Julia set is either a Jordan curve or a single point [14].

In this work we study wandering continua for rational maps with connected Julia sets. A continuum $K \subset \hat{\mathbb{C}}$ is called full if $\hat{\mathbb{C}} \setminus K$ is connected. A wandering continuum $K$ for a rational map $f$ is always full if $f^n(K)$ is full for all $n \geq 0$. Refer to [3] for the following theorem and the definition of Cantor multicurves.

**Theorem A.** Let $f$ be a post-critically finite rational map and $K \subset J_f$ be a wandering continuum. Then either $K$ is always full or there exists an integer $N \geq 0$ such that $f^n(K)$ is a Jordan curve for $n \geq N$. The latter case happens if and only if $f$ has a Cantor multicurve.
Problem: Under what condition does a post-critically finite rational map $f$ admit an always full wandering continuum?

In this paper, we solve this problem for Lattès maps (refer to §2 for its definition). Here is the main theorem:

**Theorem 1.1.** A Lattès map $f$ admits an always full wandering continuum if and only if it is flexible. In this case the wandering continuum is a line segment in an infinite geodesic under the flat metric.

## 2 Lattès maps

This section is a review about Lattès maps. Refer to [10, 12, 13] for details. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with $\deg f \geq 2$. Denote by $\deg_z f$ the local degree of $f$ at a point $z \in \hat{\mathbb{C}}$, 

$$\Omega_f = \{ z : \deg_z f > 1 \},$$

the critical set and 

$$P_f = \bigcup_{n>0} f^n(\Omega_f)$$

the post-critical set of $f$. The rational map $f$ is called **post-critically finite** if $\#P_f < \infty$.

Let $f$ be a post-critically finite rational map. Define $\nu_f(z)$ for each point $z \in \hat{\mathbb{C}}$ to be the least common multiple of the local degrees $\deg_y f^n$ for all $n > 0$ and $y \in \hat{\mathbb{C}}$ with $f^n(y) = z$. By convention $\nu_f(z) = \infty$ if the point $z$ is contained in a super-attracting cycle. The **orbifold** of $f$ is defined by $O_f = (\hat{\mathbb{C}}, \nu_f)$. Note that $\nu_f(z) > 1$ if and only if $z \in P_f$. The **signature** of the orbifold $O_f$ is the list of the values of $\nu_f$ restricted to $P_f$. The Euler Characteristic of $O_f$ is given by

$$\chi(O_f) = 2 - \sum_{z \in \hat{\mathbb{C}}} \left( 1 - \frac{1}{\nu_f(z)} \right).$$

It turns out in [10] that $\chi(O_f) \leq 0$. The orbifold $O_f$ is **hyperbolic** if $\chi(O_f) < 0$, and **parabolic** if $\chi(O_f) = 0$. It is easy to check that the signature of a parabolic orbifold $O_f$ can only be $(\infty, \infty)$, $(2, 2, \infty)$, $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$.

Suppose that the signature of $O_f$ is $(\infty, \infty)$. Then $f$ is Möbius conjugate to a power map $z \mapsto z^d$ with $|d| \geq 2$. Suppose that the signature of $O_f$ is $(2, 2, \infty)$. Then $f$ is Möbius conjugate to $\pm \Psi_d$, where $\Psi_d$ is the **Chebyshev polynomial** of degree $d$ defined by the equation

$$\Psi_d(z + \frac{1}{z}) = z^d + \frac{1}{z^d}.$$ 

Note that the Julia set of the map $\pm \Psi_d$ is the interval $[-2, 2]$. Thus in both cases, there exist no wandering continua for $f$.

A post-critically finite rational map $f$ with parabolic orbifold is called a **Lattès map** if $\nu_f(z) \neq \infty$ for any point $z \in \hat{\mathbb{C}}$. Let $\nu(O_f) = \max\{ \nu_f(z) : z \in \hat{\mathbb{C}} \}$. Refer to [13, Theorem 3.1] for the following theorem.
Theorem 2.1. Let \( f \) be a Lattès map. Then there exist a lattice \( \Lambda = \{ n + m \omega, n, m \in \mathbb{Z} \} \) (\( \text{Im} \, \omega > 0 \)), a finite holomorphic cover \( \Theta : \mathbb{C}/\Lambda \to \mathcal{O}_f \), a finite cyclic group \( G \) of order \( \nu(\mathcal{O}_f) \) generated by a conformal self-map \( \rho \) of \( \mathbb{C}/\Lambda \) with fixed points, and an affine map \( A(z) = az + b \mod \Lambda : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \), such that

\[
\Theta(z_1) = \Theta(z_2) \iff z_1 = \rho^n(z_2) \quad \text{for} \quad n \in \mathbb{Z},
\]

and the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\
\downarrow \Theta & & \downarrow \Theta \\
\mathcal{O}_f & \xrightarrow{f} & \mathcal{O}_f.
\end{array}
\]

A Lattès map \( f \) is called **flexible** if \( \mathcal{O}_f \) has the signature \( (2, 2, 2, 2) \) and the affine map \( A(z) = az + b \mod \Lambda : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \) defined in Theorem 2.1 has an integer derivative \( A' = a \in \mathbb{Z} \). A Lattès map admits a non-trivial quasiconformal deformation if and only if it is flexible by the following discussion.

Let \( f \) be a Lattès map. If \( \#P_f = 3 \) and \( f \) is topologically conjugate to another rational map \( g \), then \( f \) and \( g \) are Möbius conjugate.

Now we assume that \( \#P_f = 4 \). Then the signature of \( \mathcal{O}_f \) is \( (2, 2, 2, 2) \) and \( \nu(\mathcal{O}_f) = 2 \). Let \( \tilde{\rho} : \mathbb{C} \to \mathbb{C} \) be a lift of the generator \( \rho \) of \( G \) under the natural projection \( \pi : \mathbb{C} \to \mathbb{C}/\Lambda \). Let \( z_0 \in \mathbb{C} \) be the unique fixed point of \( \tilde{\rho} \). Then \( \tilde{\rho}(z) = 2z_0 - z \). Denote by \( Q \subset \mathbb{C}/\Lambda \) the set of fixed points of \( \rho \). Then \( \#Q = 4 \) and \( \Theta(Q) = P_f \). Therefore

\[
\pi^{-1}(Q) = \{ n/2 + m \omega/2 + z_0, n, m \in \mathbb{Z} \}.
\]

(1)

Let \( A(z) = az + b \mod \Lambda : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \) be the affine map defined in Theorem 2.1. Write \( \alpha(z) = az + b \). Since \( f(P_f) \subset P_f \), we have \( \alpha(\pi^{-1}(Q)) \subset \pi^{-1}(Q) \). Equivalently, there exist integers \( (p, q, r, s) \) such that

\[
a = p + q \omega, \quad a \omega = r + s \omega.
\]

(2)

It follows that

\[
q \omega^2 + (p - s) \omega - r = 0.
\]

(3)

If \( a \) is a real number, then \( q = r = 0 \) and \( a = p = s \). Thus the real number \( a \) must be an integer and equations (2) hold for any complex number \( \omega \). This shows that one can make a quasiconformal deformation for the map \( f \) to get another rational map such that they are not Möbius conjugate.

If \( a \) is not real, then \( q \neq 0 \) and thus the complex number \( \omega \) with \( \text{Im} \, \omega > 0 \) is uniquely determined by the integers \( (p, q, r, s) \) from equation (3). This shows that if the map \( f \) is topologically conjugate to another rational map \( g \), then \( f \) and \( g \) are Möbius conjugate.

**Remark.** A Lattès map is flexible if and only if it has a Cantor multicurve. Therefore a Lattès map admits a wandering Jordan curve if and only if it is flexible by Theorem A.

### 3. Wandering continua for torus coverings

Let \( \Lambda = \{ n + m \omega : n, m \in \mathbb{Z} \} \) (\( \text{Im} \, \omega > 0 \)) be a lattice. Then \( \mathbb{C}/\Lambda \) is a torus. A continuum \( E \subset \mathbb{C}/\Lambda \) is **full** if there exists a simply connected domain \( U \subset \mathbb{C}/\Lambda \) such that \( E \subset U \).
and $U \setminus E$ is connected. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the natural projection. If $E \subset \mathbb{C}/\Lambda$ is a full continuum, then so is each component of $\pi^{-1}(E)$. In this section, we will prove the following theorem.

**Theorem 3.1.** Let $A(z) = az + b \pmod{\Lambda} : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be a covering of the torus with $\deg A \geq 2$. Then the map $A$ admits an always full wandering continuum $E$ if and only if its derivative $a$ is an integer. In this case, the wandering continuum $E$ is a line segment in an infinite geodesic under the flat metric of $\mathbb{C}/\Lambda$.

The proof of Theorem 3.1 is based on the following lemmas.

**Lemma 3.2.** Let $E \subset \mathbb{C}/\Lambda$ be a full continuum. For any line $L \subset \mathbb{C}$ and any connected component $B$ of $\pi^{-1}(E)$, if $I$ is a bounded component of $L \setminus B$, then $\pi$ is injective on $I$.

**Proof.** Let $I$ be a bounded component of $L \setminus B$. Then there are exactly two components $U, V$ of $\mathbb{C} \setminus (L \cup B)$ such that their boundaries contain the interval $I$. We claim that at least one of them, denoted it by $U$, is bounded. Otherwise one may find a Jordan curve $\gamma$ in $U \cup V \cup I$ such that $\gamma$ separates the two endpoints $x_1$ and $y_1$ of $I$. Since $\gamma$ is disjoint from $B$, and both $x_1$ and $y_1$ are contained in $B$, this contradicts the fact that $B$ is connected.

Assume by contradiction that $\pi$ is not injective on $I$, i.e. there exist two distinct points $x, y \in I$ such that $\pi(x) = \pi(y)$. For each connected component $G$ of $B \cap \partial U$, the set $G \cap L$ is non-empty. Denote by $H(G)$ the closed convex hull of $G \cap L$, i.e. $H(G)$ is the minimal closed interval in $L$ with $H(G) \supset G \cap L$. Then for any two components $G_1, G_2$ of $B \cap \partial U$, $H(G_1)$ and $H(G_2)$ are either disjoint or one contains another. In particular, there exists a component $G_0$ of $B \cap \partial U$ such that $H(G_0) \supset H(G)$ for any component $G$ of $B \cap \partial U$. Moreover, $H(G_0) \supset I$.

![Figure 1. Lifting of a full continuum.](image)
\( C \setminus L \) that contains \( U \), then \( T^n(G_0) \) is a continuum in \( H \cup L \) joining \( T^n(x_2) \) with \( T^n(y_2) \), whereas \( G_0 \) is a continuum in \( H \cup L \) joining \( x_2 \) with \( y_2 \). Thus \( G_0 \) must intersect \( T^n(G_0) \).

On the other hand, since \( \pi(x) = \pi(y) \), we have \( (x - y) \in \Lambda \). Thus \( T^n(z) = z \mod \Lambda \) and \( T^n(B) \) is another component of \( \pi^{-1}(E) \) and hence is disjoint from \( B \). This contradicts the facts that \( G_0 \subset B \) and \( G_0 \) intersects \( T^n(G_0) \).

\( \square \)

**Lemma 3.3.** Let \( A(z) = az + b \mod \Lambda : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \) be a covering with \( \deg A \geq 2 \) and \( E \subset \mathbb{C}/\Lambda \) be an always full wandering continuum. Then \( E \) must be a line segment.

**Proof.** Let \( B \) be a component of \( \pi^{-1}(E) \). Assume by contradiction that \( B \) is not a line segment. We claim that there exists a line \( L \subset \mathbb{C} \) such that \( L \setminus B \) has a bounded component \( I \). Otherwise, each line segment joining two points in \( B \) must be contained in \( B \). Thus \( B \) is convex and hence has positive measure since it is not a line segment. This is impossible since \( A \) is expanding and \( E \) is wandering.

As in the proof of Lemma 3.2, there exists a bounded component \( U \) of \( \mathbb{C} \setminus (L \cup B) \) such that \( I \subset \partial U \). Since \( \deg A \geq 2 \), we have \( a \neq 1 \). Thus the map \( \alpha(z) = az + b : \mathbb{C} \to \mathbb{C} \) has a unique fixed point \( z_0 \in \mathbb{C} \). Denote by \( \Gamma_0 = \{ n + m\omega + z_0 : n, m \in \mathbb{Z} \} \) and \( \Gamma_n = \alpha^{-n}(\Gamma_0) \). Then there exists two distinct points \( x_n, y_n \in U \cap \Gamma_n \) for some integer \( n \geq 0 \) such that for the line \( L_n \) that passes through the points \( x_n, y_n \), the set \( L_n \cap U \) has a component \( I_n \) which contains both \( x_n \) and \( y_n \), and the two endpoints of \( I_n \) are contained in \( B \).

![Figure 2. A wandering continuum is a line segment.](image)

Now consider the full continuum \( \alpha^n(B) \) and the line \( \alpha^n(L_n) \). The set \( \alpha^n(L_n) \setminus \alpha^n(B) \) has a component \( \alpha^n(I_n) \), which contains \( x = \alpha^n(x_n) \) and \( y = \alpha^n(y_n) \). Since \( x_n, y_n \in \Gamma_n \), we have \( x, y \in \Gamma_0 \) and hence \( \pi(x) = \pi(y) \). This contradicts Lemma 3.2. \( \square \)

**Lemma 3.4.** Let \( A(z) = az + b \mod \Lambda : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda \) be a torus covering with \( \deg A \geq 2 \). If \( a \) is not real, then any line segment in \( \mathbb{C}/\Lambda \) is not wandering.
Proof. Let \( E \subset \mathbb{C}/\Lambda \) be a line segment. We want to show that there exists an integer \( n > 0 \) such that \( A^n(E) \) intersects \( A^{n+1}(E) \).

Let \( R \) be the full parallelogram with vertices 0, 1, \( \omega \) and 1+\( \omega \). Then \( R \) is a fundamental domain of the group \( \Lambda \). Thus for any \( n \geq 0 \), the set \( \pi^{-1}(A^n(E)) \) has a component \( B_n \) such that the midpoint \( m(B_n) \) of the line segment \( B_n \) is contained in the closure of \( R \). Since the diameter of \( R \) is less than 1 + \( |\omega| \), for any \( n \geq 0 \), the Euclidean distance
\[
|m(B_n) - m(B_{n+1})| \leq 1 + |\omega|.
\]

Denote by \( a = |a| \exp(i\theta) \). Then 0 < \( |\theta| < \pi \) since \( a \) is not real. Let \( L_n \) be the line containing \( B_n \) for \( n \geq 0 \). Then \( L_n \) and \( L_{n+1} \) must intersect at a point \( O_n \) and the angle formed by these two lines is \( |\theta| \). If \( B_n \) is disjoint from \( B_{n+1} \), then \( O_n \notin B_n \) or \( O_n \notin B_{n+1} \).

In the former case, we have
\[
|O_n - m(B_n)| \geq \frac{|B_n|}{2},
\]
where \( |B_n| \) is the length of \( B_n \). Therefore the Euclidean distance from \( m(B_n) \) to \( L_{n+1} \) satisfies
\[
d(m(B_n), L_{n+1}) \geq \frac{|B_n|}{2} \sin |\theta|.
\]
It follows that
\[
1 + |\omega| \geq |m(B_n) - m(B_{n+1})| \geq d(m(B_n), L_{n+1}) \geq \frac{|B_n|}{2} \sin |\theta|.
\]
So
\[
|B_n| \leq \frac{2(1 + |\omega|)}{|\sin \theta|}. \tag{5}
\]

In the latter case, we have:
\[
|B_{n+1}| \leq \frac{2(1 + |\omega|)}{|\sin \theta|}. \tag{6}
\]
Noticing that \( \deg A = |a|^2 \geq 2 \), we have \( |B_n| = |a|^n |B_0| \to \infty \) as \( n \to \infty \). Thus both cases are impossible. \( \square \)
Now suppose that $A(z) = az + b \ (\text{mod } \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is a covering with $\deg A \geq 2$ and $a$ is an integer. Let $L \subset \mathbb{C}$ be a line. Then either $\pi(L)$ is a Jordan curve on $\mathbb{C}/\Lambda$ or $\pi$ is injective on $L$. Write $\alpha(z) = az + b$. Then for any $n, m \geq 0$, $\alpha^n(L)$ and $\alpha^m(L)$ either coincide or are parallel. Thus if $\pi$ is injective on $L$, then $\pi(L)$ is either eventually periodic or a wandering line, i.e. $A^n(\pi(L)) \cap A^m(\pi(L)) = \emptyset$ for any $n > m \geq 0$.

**Lemma 3.5.** Let $L \subset \mathbb{C}$ be a line and $B \subset L$ be a line segment.

(a) If $\pi(L)$ is a Jordan curve, then $A^n(\pi(B))$ is not full when $n$ is large enough.

(b) If $\pi(L)$ is a wandering line, then $\pi(B)$ is a wandering continuum.

(c) If $\pi(L)$ is an eventually periodic line, then there exists a line segment $B_0 \subset B$ such that $\pi(B_0)$ is a wandering continuum.

**Proof.** (a) Since $\pi(L)$ is a Jordan curve, there exist two distinct points $x, y \in L$ such that $\pi(x) = \pi(y)$. Since $\deg A = |a|^2 \geq 2$, there exists an integer $n_0 > 0$ such that the Euclidean length $|\alpha^n(B)| \geq |x - y|$ when $n \geq n_0$. Thus $A^n(\pi(B)) = \pi(\alpha^n(B)) = \pi(L)$, which is a Jordan curve, since $a$ is real.

(b) This is obviously.

(c) Assume that $\pi(L)$ is periodic with period $p \geq 1$ for simplicity. Since $\deg A \geq 2$, there exists a unique point $x_0 \in L$ such that $A^p(\pi(x_0)) = \pi(x_0)$. Pick a point $y_0$ in the interior of $B$ with $y_0 \neq x_0$. Then for any $n \geq 1$, there exists a unique point $y_n \in L$ such that $A^{np}(\pi(y_0)) = \pi(y_n)$. Moreover, $y_n \to \infty$ as $n \to \infty$.

Suppose that the integer $a$ is positive. Then all points $y_n$ are contained in the same component of $L \setminus \{x_0\}$. Since $y_0$ is contained in the interior of $B$, there exists a closed line segment $B_0 \subset B$ such that $B_0 \subset (y_0, y_1)$. Then $\pi(B_0)$ is a wandering continuum.

Now suppose that the integer $a$ is negative. Then the points $y_{2k}$ are contained in the same component of $L \setminus \{x_0\}$ for $k \geq 0$. Since $y_0$ is contained in the interior of $B$, there exists a closed line segment $B_0 \subset B$ such that $B_0 \subset (y_0, y_2)$. Then $\pi(B_0)$ is a wandering continuum for $A$. \(\square\)

**Proof of Theorem 3.1.** The result follows directly from Lemmas 3.2, 3.3, 3.4 and 3.5. \(\square\)

### 4 Proof of Theorem 1.1

**Proof of Theorem 1.1.** Let $f$ be a Lattès map. By Theorem 2.1, there exist a lattice $\Lambda = \{n + m\omega, n, m \in \mathbb{Z}\}$ with $\text{Im } \omega > 0$, a finite holomorphic cover $\Theta : \mathbb{C}/\Lambda \to O_f$, a finite cyclic group $G$ of order $\nu(O_f)$ generated by a conformal self-map $\rho$ of $\mathbb{C}/\Lambda$ with fixed points, and an affine map $A(z) = az + b \ (\text{mod } \Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, such that $\Theta(z_1) = \Theta(z_2) \iff z_1 = \rho^n(z_2)$ for $n \in \mathbb{Z}$, and the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\
\Theta \downarrow & & \downarrow \Theta \\
O_f & \xrightarrow{f} & O_f.
\end{array}
$$

Suppose that $K$ is an always full wandering continuum of the map $f$. Then for each $n \geq 0$, every component of $\Theta^{-1}(f^n(K))$ is a full continuum in $\mathbb{C}/\Lambda$ since $f^n(K)$ is disjoint.
from $P_f$. Let $E$ be a component of $\Theta^{-1}(K)$. It is an always full wandering continuum for the map $A$. Therefore the derivative $a$ is an integer and $K$ is a line segment in an infinite geodesic under the flat metric on $\mathbb{C}/\Lambda$ by Theorem 3.1.

Let $E_n$ be a component of $\Theta^{-1}(f^n(K))$. Then $\rho(E_n)$ is also a component of $\Theta^{-1}(f^n(K))$, where $\rho$ is the generator of the group $G$. Let $R$ be the full parallelogram with vertices $0, 1, \omega$ and $1 + \omega$. Then $R$ is a fundamental domain of the group $\Lambda$. Thus there are components $I_n$ and $J_n$ of $\pi^{-1}(E_n)$ and $\pi^{-1}(\rho(E_n))$ respectively, such that the midpoints of $I_n$ and $J_n$ are contained in $R$.

Assume that $\#P_f = 3$. Let $\bar{\rho}$ be a lift of the map $\rho$ under the projection $\pi$. Then $\bar{\rho}$ is a rotation around its fixed point with angle $(2\pi)/\nu$, where $\nu = 3, 4$ or $6$. Thus the angle formed by the two lines containing $I_n$ and $J_n$ respectively, is $(2\pi)/\nu$. As in the proof of Lemma 3.4, we have:

$$|I_n| \leq \frac{2(1 + |\omega|)}{\sin(\pi/3)},$$

where $|I_n|$ is the length of $I_n$. This leads to a contradiction since $|I_n| = |a|^n|I_0| \to \infty$ as $n \to \infty$. Therefore $\#P_f = 4$ and $f$ is a flexible Lattès map.

Conversely, suppose that the map $f$ is flexible. Denote by $Q$ the set of fixed points of $\rho$. Then $A(Q) \subset Q$ since $f(P_f) \subset P_f$.

Let $L \subset \mathbb{C}$ be a line. If $L$ passes through at least two points $z_1, z_2 \in \pi^{-1}(Q)$, then it passes through the point $(z_2 + (z_2 - z_1))$. Note that $z_2 - z_1 \equiv 0 \mod (\Lambda/2)$ by (1). So we have $2(z_2 - z_1) \equiv 0 \mod \Lambda$, i.e., $\pi(z_2 + (z_2 - z_1)) = \pi(z_1)$. So $\pi(L)$ is a Jordan curve on $\mathbb{C}/\Lambda$. If $L$ passes through exactly one point $z_0 \in \pi^{-1}(Q)$, then $\pi$ is injective on $L$, $\Theta(\pi(L))$ is a ray from a point in $P_f$, and $\Theta : \pi(L) \to \Theta(\pi(L))$ is a folding with exactly one fold point at $\pi(z_0) \in Q$. If $L$ is disjoint from $\pi^{-1}(Q)$, then $\Theta \circ \pi$ is injective on $L$.

Suppose that $\pi(L) \subset \mathbb{C}/\Lambda$ is a wandering line. Then $A^n(\pi(L))$ is disjoint from $Q$ for all $n \geq 0$. Thus $\Theta$ is injective on each line $A^n(\pi(L))$. On the other hand, if $\Theta(A^n(\pi(L)))$ intersects $\Theta(A^{n+p}(\pi(L)))$ for some integer $p > 0$, then they coincide since the map $\rho$ in $G$ preserves the slopes of the lines. Thus $A^{n+p}(\pi(L)) = \rho(A^n(\pi(L)))$. Therefore $A^{n+2p}(\pi(L)) = \rho(A^{n+p}(\pi(L)))$ since $A \circ \rho = \rho \circ A$. But $\rho^2$ is the identity. So $A^{n+2p}(\pi(L)) = A^n(\pi(L))$. This is a contradiction. Therefore $\Theta(A^n(\pi(L)))$ is pairwise disjoint. Thus for any line segment $E \subset \pi(L)$, the set $\Theta(E)$ is an always full wandering continuum for $f$.

Now suppose that $\pi(L) \subset \mathbb{C}/\Lambda$ is an eventually periodic line with period $p \geq 1$. Then $\Theta(A^n(\pi(L)))$ are either bi-infinite or one-side-infinite geodesics depending on whether $A^n(\pi(L))$ passes through a point in $Q$. Since $\rho^2$ is the identity, either they are disjoint and have the same period, or two of them coincide and the period is $p/2$. Let $E \subset \pi(L)$ be a wandering line segment. In the former case, $\Theta(E)$ is an always full wandering continuum of $f$. In the latter case, there exists a line segment $E_0 \subset E$ such that $\Theta(E_0)$ is an always full wandering continuum for $f$.

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Guizhen Cui  
Academy of Mathematics and Systems Science  
Chinese Academy of Sciences, Beijing 100190  
P. R. China.  
Email: gzcui@math.ac.cn

Yan Gao  
Mathematical School  
Sichuan University  
P. R. China  
Email: gyan@mail.ustc.edu.cn