BLOWUPS FOR A CLASS OF SECOND ORDER NONLINEAR HYPERBOLIC EQUATIONS: A REDUCED MODEL OF NONLINEAR JEANS INSTABILITY

CHAO LIU

Abstract. Due to the importance of understanding the formation of the nonlinear structures in the universe and stellar systems, the nonlinear Jeans instability which plays the key role in these formation problems is a long-standing open problem presented by astrophysics more than one hundred years. In this article, we focus on a reduced model of the nonlinear Jeans instability for the expanding Newtonian universe which is characterized by a class of second order nonlinear hyperbolic equations

\[ \Box \rho(x^\mu) + \frac{a}{t} \partial_\mu \rho(x^\mu) - \frac{b}{t^2} \rho(x^\mu)(1 + \rho(x^\mu)) - \frac{c - k}{1 + \rho(x^\mu)} (\partial_\mu \rho(x^\mu))^2 = k F(t). \]

We establish a family of nonlinear self-increase blowup solution (the solution itself becomes infinite, stable ODE-type blowup) to this equation and give estimates on the growth rate of \( \rho \) which may explain the much faster growth rate of the nonlinear structures of universe in astrophysical observations than the rate given by the classical Jeans instability.

Keywords: blowup, ODE blowup, Jeans instability, self-increase blowup, second order nonlinear hyperbolic equations

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1. Introduction

The gravitational instability is extremely important in astrophysics since it characterizes the mass accretions of self-gravitating systems and further contributes to understanding the formations of stellar systems and the nonlinear structures in the universe. However, up to now, the gravitational instability was only studied in the linear regime since the first linearized gravitational instability was studied by Jeans \[9\] for Newtonian gravity in 1902 (thus it is also known as Jeans instability), and then generalized to general relativity by Lifshitz \[11\] in 1946. Due to the expanding universe, the Jeans instability was extended to the expanding background universe in Bonnor \[4\] (also see \[19, 26\]). With the accretions of the mass, the derivations of the linear Jeans instability will be completely spoiled since the increasing density leads to the derivations significantly deviate the linear regime. Moreover, the growth rate of the density predicted by the classical linearized Jeans instability can not yield the observed large inhomogeneities of the universe nowadays and formations of galaxies, because this growth rate is too slow and thus is much less efficient. In fact, \[4, 19, 26\] and our previous paper \[15\] indicate the growth rate of density for the linear or certain partially nonlinear Jeans instability in expanding Newtonian models is of order \( \sim t^{\frac{2}{3}} \). Because of the incompleteness of these astrophysical theories and lots of mysterious phenomenons potentially relying on nonlinear Jeans instability (e.g., the space charges in conductors and hair formations of some black holes), we have to study the nonlinear analysis of Jeans instability. However, as pointed out by Rendall \[22\] in 2002, there are no results on Jeans instability available for the fully nonlinear case, and it becomes a long-standing open problem presented by astrophysics.
In this article, we focus on a reduced model of the nonlinear Jeans instability for expanding Newtonian universes, which can be characterized by the nonlinear self-increase blowups (the solution itself becomes infinite at some time, known as “ODE-type blowup” according to Alinhac [1]) of the solution itself to the following type of second order nonlinear hyperbolic equations,

\[
\Box \varrho(x^\mu) + \frac{a}{t} \partial_t \varrho(x^\mu) - \frac{b}{t^2} \varrho(x^\mu)(1 + \varrho(x^\mu)) - \frac{c - \kappa}{1 + \varrho(x^\mu)} \left( \partial_t \varrho(x^\mu) \right)^2 = \kappa F(t),
\]

(1.1)

\[
\varrho|_{t=t_0} = \hat{\varrho}(x^i) > 0 \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \hat{\varrho}_0(x^i) > 0,
\]

(1.2)

where\(^1\), for simplicity, we take \(\Box := \partial^2_t - \Delta_g = \partial^2_{tt} - g^{ij} \partial_i \partial_j\), \(g^{ij}(t)\) is a given Riemannian metric, \(x^\mu := (t, x^1, \cdots, x^n) \in [t_0, \infty) \times \mathbb{T}^n\), \(t_0 > 0\), \(\hat{\varrho}(x^i)\) and \(\hat{\varrho}_0(x^i)\) are given positive valued functions, \(a, b, c, \kappa\) are constants satisfying

\[
a > 1, \quad b > 0, \quad 1 < c < 3/2 \quad \text{and} \quad 3c - \sqrt{2} \sqrt{8c - 5} < \kappa < 3c + \sqrt{2} \sqrt{8c - 5}. \]

(1.3)

In addition, in order to capture some crucial features of velocity fields in the fully nonlinear Jeans instability, in this article, we focus on the following two specific functions \(g^{ij}(t)\) and \(F(t)\),

\[
g^{ij}(t) := \frac{m^2 \partial_t f(t)^2}{(1 + f(t))^2} \delta^{ij} \quad \text{and} \quad F(t) := \left( \frac{\partial_t f(t)}{1 + f(t)} \right)^2.
\]

(1.4)

where \(m \in \mathbb{R}\) is a given constant and \(f(t)\) solves an ordinary differential equation (ODE),

\[
\partial^2_t f(t) + \frac{a}{t} \partial_t f(t) - \frac{b}{t^2} f(t)(1 + f(t)) - \frac{c - \kappa}{1 + f(t)} \left( \partial_t f(t) \right)^2 = 0, \quad (1.5)
\]

\[
f(t_0) = \hat{f} > 0 \quad \text{and} \quad \partial_t f(t_0) = \hat{f}_0 > 0. \quad (1.6)
\]

where \(\hat{f}, \hat{f}_0 > 0\) are positive constants.

If letting

\[
a = 4/3, \quad b = 2/3, \quad c = 4/3 \quad \text{and} \quad 3c - \sqrt{2} \sqrt{8c - 5} < \kappa < 3c + \sqrt{2} \sqrt{8c - 5},
\]

(1.7)

this system (1.1)–(1.2) can be regarded as a poor man’s model of the nonlinear Jeans instability of the expanding Newtonian universe for the Euler–Poisson system\(^2\) by ignoring the velocity field equations, and the effects of the velocity fields can be captured by the source term \(F(t)\) and \(g^{ij}(t)\). It intimately related to the problem of the classical linearized Jeans instability of expanding Newtonian universe (see the references [19, 26] and our previous paper [15]). In astrophysics literature, the variable \(\varrho\) represents the relative density of the fluids, thus the increasing of the relative density \(\varrho\) itself means the mass accretions and formations of nonlinear structures.

As the conclusion, we will prove: (1) the solution of the main equation (1.1)–(1.2), if the initial data \(\hat{\varrho}(x^i)\) and \(\hat{\varrho}_0(x^i)\) deviate from positive constants \(\hat{f}\) and \(\hat{f}_0\) in very small amplitudes of perturbations, undergoes a nonlinear self-increase and itself (in fact \(\varrho\) and \(\varrho_0\) becomes infinite (blowups). (2) Furthermore, if the initial data additionally satisfies

\[
\hat{f}_0 > \frac{(1 - a)(1 + \hat{f})}{(1 - c)t_0}, \quad (1.8)
\]

then this self-increase solution blowups at a finite time \(t_m < \infty\) (i.e., the finite time blowup).

(3) Along with this, we also estimate the growth rate of \(\varrho\) is way faster than that for the classical linearized Jeans instability (recalling the growth rate for the linear Jeans instability

\(^1\)In this article, we use the index convention given in §1.4, i.e., \(\mu = 0, \cdots, n\) and \(i = 1, \cdots, n\), \(x_0 = t\).

\(^2\)The fully nonlinear Jeans instability of the expanding Newtonian universe for the Euler–Poisson equations is in preparations.
of expanding Newtonian universe is of order $\sim t^2$, see [4, 15, 19, 26]). For example, in the special case (1.7), the growth rate of $\varrho$ is, at least, of order $\sim \exp(t^2)$, and if data satisfies (1.8), the solution blowup at finite time with way faster rate than this exponential rate and we also have (see Theorems 1.1 and 1.2 for detailed statements) a lower bound estimate on the growth rate even for this case. All in all, from this reduced model, it hints that the growth rate of the nonlinear Jeans instability may way faster than the rate of the classical (linear) one. This consists with the astrophysical observations and provides theoretical reasons for these observations. That is, this article implies the deviations of the growth rates of $\varrho$ from results in classical Jeans instability come from the nonlinear effects.

The basic idea of proving above stated ODE-type blowups (i.e., the solution itself becomes infinite at some time $t_m$ by a process of self-increase) is to treat the target equations (1.1) as nonlinear perturbations of the ODE (1.5), and try to conclude the solution to the main equation (1.1) is dominated by the solution to the ODE (1.5) (see Theorem 1.2 for precise expressions). In other words, we try to claim the solution to (1.1) behaves like a small perturbation of the solution $f(t)$ to the ODE (1.5) for given data functions $\hat{\varrho}$ and $\hat{\varrho}_0$ perturbed around $f$ and $f_0$ in very small amplitudes of perturbations. This idea indicates the basic blowup mechanism of (1.1). To achieve this aim, we use the approach of the Cauchy problem for Fuchsian system (see (C.1)–(C.2) in Appendix C for details) firstly introduced by Oliynyk [20].

On the other hand, according to this idea, in order to acquire the behavior of $\varrho$ and determine its growth rate, as a preparation of this idea, we have to firstly have a complete picture on the reference solution $f$ which solves the ODE (1.5)–(1.6). However, the ODE (1.5)–(1.6) can not be easily and directly solved. Thus we have to estimate the behavior of its solution and bound this solution $f$ and $\partial_t f$ below and above by suitable increasing functions. Eventually, these bounds also imply the behaviors of $\varrho$ (see Theorem 1.1 for precise statements).

In the followings, let us first state the main theorems of this article, then briefly overview the proofs of the main theorems and outline the structures of this article. After that, a very brief review on the relevant works are given and we fix the notations used in this article.

1.1. Main theorems. We, in this section, present the main theorems of this article which are given in the following Theorem 1.1 and 1.2. As we mentioned above, in order to obtain the behaviors on $\varrho$, we have to obtain a clear picture on the reference solution $f$. Therefore, Theorem 1.1 gives a description on the reference solution $f$ to equations (1.5)–(1.6). Then, we can state Theorem 1.2 on the solution $\varrho$ to the main equation (1.1)–(1.2) as perturbations of $f$.

From now on, in order to simplify the notations, we denote (noting (1.3) for the regions of parameters $a$, $\xi$ and $c$)

$$\triangle := \sqrt{(1 - a)^2 + 4\xi} > -\bar{a}, \quad \bar{a} = 1 - a < 0, \quad \bar{c} = 1 - c < 0$$

and we introduce constants $A$, $B$, $C$, $D$ and $E$ depending on the initial data $\hat{f}$ and $\hat{f}_0$ to (1.5)–(1.6) and parameters $a$, $\xi$ and $c$,

$$A := t_0^{-\frac{\bar{a} + \triangle}{\triangle}} \left( \frac{t_0 f_0}{(1 + f)^2} - \frac{\bar{a} + \triangle}{2} \frac{\hat{f}}{1 + \hat{f}} \right),$$

$$B := t_0^{-\frac{\bar{a} - \triangle}{\triangle}} \left( \frac{\bar{a} - \triangle}{2} \frac{\hat{f}}{1 + \hat{f}} - \frac{t_0 f_0}{(1 + f)^2} \right) < 0,$$
\[ C := \frac{2}{2 + \bar{a} + \triangle} \left( \ln(1 + \hat{f}) + \frac{\bar{a} + \triangle}{26} \frac{t_0 \hat{f}_0}{1 + \hat{f}} \right) t_0^{\frac{\bar{a} + \triangle}{2 + \bar{a} + \triangle}} > 0, \]
\[ D := \frac{\bar{a} + \triangle}{2 + \bar{a} + \triangle} \left( \ln(1 + \hat{f}) - \frac{1}{6} \frac{t_0 \hat{f}_0}{1 + \hat{f}} \right) t_0, \]
\[ E := \frac{\hat{c} \hat{f}_0^{1 - \bar{a}}}{\bar{a}(1 + \hat{f})} > 0. \]

We define the following two critical times \( t_* \) and \( t^* \).

**Definition 1.1.** Suppose \( A, B, E, \bar{a} \) and \( \triangle \) are defined above, then

1. Let \( \mathcal{R} := \{ t_r > t_0 \mid At_r^{\frac{\bar{a} + \triangle}{\bar{a} + \triangle}} + Bt_r^{\frac{\bar{a} + \triangle}{\bar{a} + \triangle}} + 1 = 0 \} \) and define \( t_* := \min \mathcal{R} \).
2. If \( t_0^\alpha > E^{-1} \), we define \( t^* := (t_0^\alpha - E^{-1})^{1/\bar{a}} \in (0, \infty) \), i.e., \( t = t^* \) solves \( 1 - E\hat{f}_0^\alpha + E\hat{f}^\alpha = 0 \).

We are now in a position to state the main theorems of this article.

**Theorem 1.1.** Suppose constants \( \alpha, \beta, c \) and \( \epsilon \) are defined by (1.3), \( t_* \) and \( t^* \) are defined by Definition 1.1 and the initial data \( \hat{f}, \hat{f}_0 > 0 \), then

1. \( t_* \in [0, \infty) \) exists and \( t_* > t_0 \);
2. there is a constant \( t_m \in [t_*, \infty) \), such that there is a unique solution \( f \in C^2([t_0, t_m]) \) to the equation (1.5)–(1.6), and
\[
\lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty. \tag{1.10}
\]
3. \( f \) satisfies upper and lower bound estimates,
\[
1 + f(t) > \exp \left( C t^{\frac{\bar{a} + \triangle}{2}} + D t^{-1} \right) \quad \text{for} \quad t \in (t_0, t_0^\alpha);
\]
\[
1 + f(t) < \left( C t^{\frac{\bar{a} + \triangle}{2}} + D t^{\frac{\bar{a} + \triangle}{2}} + 1 \right)^{-1} \quad \text{for} \quad t \in (t_0, t_0^\alpha). \]

Furthermore, if the initial data satisfies (1.8), i.e., \( \hat{f}_0 > \bar{a}(1 + \hat{f})/(\epsilon t_0) \), then

4. \( t_* \) and \( t^* \) exist and finite, and \( t_0 < t_* < t^* < \infty \);
5. there is a finite time \( t_m \in [t_*, t^*) \), such that there is a solution \( f \in C^2([t_0, t_m]) \) to the equation (1.5) with the initial data (1.6), and
\[
\lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty. \]

6. the solution \( f \) has improved lower bound estimates, for \( t \in (t_0, t_m) \),
\[
(1 + \hat{f})(1 - E\hat{f}_0^\alpha + E\hat{f}^\alpha)^{1/\epsilon} < 1 + f(t). \]

**Theorem 1.2.** Suppose \( s \in \mathbb{Z}_{\geq \frac{\alpha + \beta}{2} + 3} \), \( \alpha, \beta, c, k \) are constants satisfying (1.3), \( f \in C^2([t_0, t_m]) \) given by Theorem 1.1 solves equation (1.5)–(1.6) where \( \hat{f} > 0 \) and \( \hat{f}_0 > 0 \) are given, and assume \( t_m > t_0 \) such that \([t_0, t_m]\) is the maximal interval of existence of \( f \) given by Theorem 1.1. Then there are small constants \( \sigma_* > 0 \), such that if the initial data satisfies
\[
\left\| \frac{\partial}{\hat{f}} - 1 \right\|_{H^s(T^n)} + \left\| \frac{\partial}{\hat{f}_0} - 1 \right\|_{H^s(T^n)} + \left\| \frac{m\hat{f}_0}{1 + \hat{f}} \right\|_{H^s(T^n)} \leq \frac{1}{2} \sigma_* \sigma, \tag{1.11}
\]
then there is a solution \( \varrho \in C^2([t_0, t_m] \times T^n) \) to the equation (1.1)–(1.2) and \( \varrho \) satisfies the estimate
\[
\left\| \frac{\varrho(t)}{f(t)} - 1 \right\|_{H^s(T^n)} + \left\| \frac{\partial \varrho(t)}{f_0(t)} - 1 \right\|_{H^s(T^n)} + \left\| \frac{m\delta \varrho(t)}{1 + f(t)} \right\|_{H^s(T^n)} \leq C \sigma < 1 \tag{1.12}
\]
for \( t \in [t_0, t_m] \) and some constant \( C > 0 \). Moreover, \( \varrho \) blowups at \( t = t_m \), i.e.,
\[
\lim_{t \to t_m} \varrho(t, x^i) = +\infty \quad \text{and} \quad \lim_{t \to t_m} \varrho_0(t, x^i) = +\infty,
\]
with the rate estimates \((1 - C\sigma)f \leq \varrho \leq (1 + C\sigma)f\) and \((1 - C\sigma)f_0 \leq \varrho_0 \leq (1 + C\sigma)f_0\) for \( t \in [t_0, t_m] \).

1.2. Overviews and outlines. According to the basic idea we mentioned before §1.1, it is clear that this article will be stated in the following two sections:

In §2, we have two tasks: The first one is to study the reference solutions \( f(t) \) of the ODE (1.5)–(1.6) in details and prove Theorem 1.1. We bound \( f(t) \) and \( \partial_\tau f(t) \) above and below by some increasing functions. On the other hand, the second task is to introduce three special functions \( g(t) \), \( \chi(t) \) and \( \xi(t) \) (defined by (2.1), (2.48) and (2.51), respectively) and study their certain properties such as bounds and limits as \( t \) approaching to \( t_m \) (\( t_m \) is the maximal time of existence of \( f \) as Theorem 1.1). We study these quantities since they are important, in §3, when rewriting the main equation (1.1) into the Fuchsian formulation (see (C.1)–(C.2) in Appendix C). For example, \( g(t) \) will provide a time transform \( \tau = -g(t) \) which compactifies the time to \([-1, 0)\) from \([t_0, t_m]\) and mapping the maximal time \( t = t_m \) to \( \tau = 0 \). The Fuchsian system will be formulated in the compactified time \( \tau \)-coordinate. \( \chi(t) \) and \( \xi(t) \) are crucial for distinguishing the singular terms from regular ones in the Fuchsian formulations (i.e., the singular term \( \frac{1}{4}B(t, x, u)Pu \) and the regular term \( H(t, x, u) \) in Fuchsian system (C.1)–(C.2)).

In §3, we start to analyze the main equation (1.1) or, equivalently, the nonlinear perturbation equations (subtracting (1.5) from (1.1)) to prove Theorem 1.2. We proceed by seven steps. As we mentioned above, the fundamental idea is to use the approach of the Cauchy problem for Fuchsian system given in Appendix C. Hence, the first difficulty is to rewrite this perturbation equation to a Fuchsian formulation (see Step 1–Step 3 in §3). As we remarked in our previous works [12–16], one key and most difficult step to achieve this rewriting is to introduce suitable new variables (i.e., suitable Fuchsian fields) and proper time compactifications. In this article, it turns out if we take
\[
\tau := -g(t) = -\exp\left(-A \int_{t_0}^{t} \frac{f(s)(f(s) + 1)}{s^2 f_0(s)} ds\right) = -\left(1 + 6B \int_{t_0}^{t} s^{d-2} f(s)(1 + f(s))^{1-d} ds\right)^{-\frac{1}{4}} \in [-1, 0),
\]
given in (3.10), where \( A \in (0, 2\delta/(3 - 2\epsilon)) \) and \( B := (1 + \hat{\delta})/(\hat{t}_0^\epsilon \hat{f}_0) \), to define the compactified time and introducing the Fuchsian fields by weighting suitable increasing functions on the perturbation variables (see (3.11)–(3.13)), then the main equation (1.1) will eventually, near \( \tau = 0 \) (i.e. near time infinity \( t = \infty \)), become a Fuchsian system which is verified in Step 4. In Step 5, we will conclude by global existence theorem for the Fuchsian system given by Theorem C.2, if \( \tau \) is enough close to \( 0 \) (i.e., there is a small \( \tau_\delta < 0 \) close to \( 0 \), or equivalently, \( t \) is large enough), the solution exists if the data at \( \tau = \tau_\delta \) is small. Then in Step 6, by letting the data at \( \tau = -1 \) much smaller than the small data at \( \tau = \tau_\delta \), standard energy estimates leads to that the local solution can be extended to time \( \tau_\ast > \tau_\delta \) (i.e., the local solution for \( \tau \in [-1, \tau_\ast) \) where \( \tau_\ast > \tau_\delta \)). We then, with the help of the solution near \( \tau = 0 \) above (i.e., the solution for \( \tau \in [\tau_\delta, 0) \)), obtain a global solution at \( \tau \in [-1, 0) \) (i.e., \( t \in [t_0, \infty) \)). At the end, let us transform the Fuchsian fields back to the original variables \((\varrho, \partial_\varrho, \partial_t \varrho)\), we conclude the self-increase blowups on \( \varrho \) and \( \partial_t \varrho \), and then complete the proof.

\(^3\)This \( \tau = -g(t) \) is the most key and difficult construction.
1.3. Related works. The blowup problem has been widely studied for various hyperbolic equations, especially for the wave type. For example, Speck [23, 24] recently studied stable ODE-type blowup result for quasilinear wave equations featuring a Riccati-type derivative-quadratic semilinear term and a form of stable Tricomi-type degeneracy formation for a certain wave equation, respectively. We refer readers to [1, 10] for an introduction on blowups for various hyperbolic system. We also recommend readers to consult Speck [23, 24] for a detailed reviews on various blowup results and their basic proof techniques.

The key idea for the global solutions taken in this article is the Cauchy problem for Fuchsian systems. The idea using Fuchsian formulations for the global solution of the Cauchy problem was firstly introduced by Oliynyk [20], and then developed and applied for various problems by a series of works, for example, [2, 3, 12–16].

In addition, we point out recently there are works on the gravitational collapse [5, 6], and formations of the implosion singularity [18] which both lead to mass accretions and may have some relations to the Jeans instability.

1.4. Notations. Unless stated otherwise, we will apply the following conventions of notations throughout this article without recalling their meanings in the following sections.

1.4.1. Indices and coordinates. Unless stated otherwise, our indexing convention will be as follows: we use lower case Latin letters, e.g. $i, j, k$, for spatial indices that run from 1 to $n$, and lower case Greek letters, e.g. $\alpha, \beta, \gamma$, for spacetime indices that run from 0 to $n$. We will follow the Einstein summation convention, that is, repeated lower and upper indices are implicitly summed over. We use $x^i$ ($i = 1, \cdots, n$) to denote the standard periodic coordinates on the $n$-torus $\mathbb{T}^n$ and $t = x^0$ a time coordinate on the interval $[t_0, \infty)$.

This article involves two different time coordinates: the open time $t \in [t_0, \infty)$ and the compactified time $\tau = -g(t) \in [-1, 0)$ given by (1.13) (see details in §3.2). For scalar functions $f(t, x^i)$, we always use
\[
\tilde{f}(\tau, x^i) := f(g^{-1}(-\tau), x^i)
\] (1.14)
to denote the representation of $f$ in compactified time coordinate $\tau$ throughout this article.

1.4.2. Derivatives. Partial derivatives with respect to the coordinates $(x^\mu) = (t, x^i)$ will be denoted by $\partial_\mu = \partial/\partial x^\mu$ and we use $Du = (\partial_j u)$ and $\partial u = (\partial_\mu u)$ to denote the spatial and spacetime gradients, respectively, with respect to the coordinates. We also use Greek letters to denote multi-indices, e.g. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, and employ the standard notation $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$ for spatial partial derivatives. It will be clear from context whether a Greek letter stands for a spacetime coordinate index or a multi-index.

1.4.3. Function spaces, inner-products and matrix inequalities. Given a finite dimensional vector space $V$, we let $H^s(\mathbb{T}^n, V)$, $s \in \mathbb{Z}_{\geq 0}$, denote the space of maps from $\mathbb{T}^n$ to $V$ with $s$ derivatives in $L^2(\mathbb{T}^n)$. When the vector space $V$ is clear from context, for example, $V = \mathbb{R}^N$, we write $H^s(\mathbb{T}^n)$ instead of $H^s(\mathbb{T}, V)$. Letting
\[
\langle u, v \rangle = \int_{\mathbb{T}^n} (u(x), v(x)) \, dx,
\]
where $(\cdot, \cdot)$ is the Euclidean inner product on $\mathbb{R}^N$ (i.e., $(\xi, \zeta) = \xi^T \zeta$ for any $\xi, \zeta \in \mathbb{R}^N$), denote the standard $L^2$ inner product, the $H^s$ norm is defined by
\[
\|u\|_{H^s}^2 = \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, D^\alpha u \rangle.
\]
For matrices $A, B \in \mathbb{M}_{N \times N}$, we define

$$A \leq B \iff (\zeta, A\zeta) \leq (\zeta, B\zeta), \quad \forall \zeta \in \mathbb{R}^N.$$ 

2. The analysis of the reference solutions

This section contributes to two tasks: the first one is to prove Theorem 1.1 which prepares the reference solutions $f$ of the ODE (1.5)–(1.6); and the second one is to study three important quantities $g(t), \chi(t)$ and $\xi(t)$, the preparations for the Fuchsian formulations.

2.1. The time transform function $g(t)$ and rough estimates of $f$ and $\partial_t f(t)$. Let us define a useful function $g(t)$ which will repeatedly appear in later calculations, and in fact, it will play an important role in the Fuchsian formulation (see §3) since it serves as a compactified time transformation. Let

$$g(t) := \exp\left(-A \int_{t_0}^t \frac{f(s)(f(s) + 1)}{s^2 f_0(s)} ds\right) > 0 \quad (2.1)$$

where $A \in (0, 2\delta/(3-2c))$ is a constant.

The following lemma gives another representation of $g(t)$ only involving $f$ without $f_0$, fundamental properties of the function $g(t)$ and expresses $f_0$ in terms of $f$ and $g$. These useful representations help us remove $f_0$ in later calculations.

**Lemma 2.1.** Suppose $f \in C^2([t_0, t_1])$ ($t_1 > t_0$) solves the equation (1.5)–(1.6), $g(t)$ is defined by (2.1), and denote $f_0(t) := \partial_t f(t)$, then

1. $f_0$ can be expressed by

$$f_0(t) = B^{-1}t^{-a}g^{-\frac{a}{2}}(t)(1 + f(t)) \quad (2.2)$$

for $t \in [t_0, t_1]$ where $B := (1 + \dot{f})/(t_0^a \dot{f_0}) > 0$ is a constant depending on the data;

2. If the data $\dot{f} > 0$, then $f(t) > 0$ for $t \in [t_0, t_1]$;

3. $g(t)$ can be represented by

$$g(t) = \left(1 + 6B \int_{t_0}^t s^{a-2} f(s)(1 + f(s))^{1-c} ds\right)^{-\frac{2}{3}} \in (0, 1), \quad (2.3)$$

for $t \in [t_0, t_1]$, and $g(t_0) = 1$;

4. $g(t)$ is strictly decreasing and invertible in $[t_0, t_1]$.

**Proof.** (1)&(2): Multiplying (1.5) by $1/(1 + f)$ on both sides yields for $t \in [t_0, t_1)$ yields

$$\partial_t \left(\frac{f_0}{1 + f}\right) = -\frac{a}{t} \left(\frac{f_0}{1 + f}\right) + 6 \frac{f}{t^2} f - \left(1 - c\right) \frac{f_0^2}{(1 + f)^2}. \quad (2.4)$$

Then multiplying above equation (2.4) by $(1 + f)/f_0$, we arrive at

$$\partial_t \ln\left(\frac{f_0}{1 + f}\right) = -\frac{a}{t} + \frac{6 f (1 + f)}{t^2 f_0} + (c - 1) \frac{f_0}{1 + f}.$$ 

Due to the identity

$$\frac{f_0}{1 + f} = \partial_t f = \partial_t \ln(1 + f),$$

we obtain

$$\partial_t \ln\left(\frac{f_0}{1 + f}\right) = -\frac{a}{t} + \frac{6 f (1 + f)}{t^2 f_0} + \partial_t \ln(1 + f)^{c-1}.$$
Let us prove \( (2.6) \) and using the data \( (1.6) \), with the help of the definition \( (2.1) \), we obtain

\[
\frac{f_0(t)}{(1 + f(t))^c} = \frac{\dot{f}_0}{(1 + f(t))^c} \left( \frac{t}{t_0} \right)^{-a} \exp \left( b \int_{t_0}^t \frac{f(s)(1 + f(s))}{s^2 f_0(s)} \, ds \right) = B^{-1} t^{-\frac{a}{1-c}} g^{-\frac{a}{1-c}}(t) \geq 0, \tag{2.6}
\]

for \( t \in [t_0, t_1] \). Therefore, this means \( f_0(t) = B^{1-a} t^{-\frac{a}{1-c}} g^{-\frac{a}{1-c}}(t)(1 + f(t))^c \).

Noting \( f_0 = \partial_t f \) and using \( (2.6) \) yield a rough estimate for \( t \in [t_0, t_1] \),

\[
\partial_t \left[ \frac{1}{1-c} (1 + f)^{1-c} \right] \geq 0
\]

which, by integrating this inequality and noting \( c > 1 \), leads to \( (1 + f)^c \leq (1 + \dot{f})^c \) for \( t \in [t_0, t_1] \).

Since \( f \in C^2([t_0, t_1]) \) is the solution to \( (1.5)-(1.6) \), let us prove \( f(t) > 0 \) for any \( t \in [t_0, t_1] \) by contradiction. If there is a time \( t_s \in [t_0, t_1] \) such that \( f(t_s) \in [-1, 0] \), then \( 0 \leq (1 + f(t_s))^c \leq (1 + \dot{f})^c \) (note \( c < 0 \)) which implies \( f(t_s) > \dot{f} > 0 \). This leads to a contradiction, thus \( f(t) \notin [-1, 0] \) for any \( t \in [t_0, t_1] \). On the other hand, if there is a time \( t_r \in [t_0, t_1] \) such that \( f(t_r) < -1 \). By the continuity of \( f \) (since \( f \in C^2 \)) and \( \dot{f} > 0 \), then there is a time \( t_s \in (t_0, t_r) \), such that \( f(t_s) \in [-1, 0] \). This contradicts with the fact \( f(t) \notin [-1, 0] \) for any \( t \in [t_0, t_1] \). Hence, \( f(t) > 0 \) for all \( t \in [t_0, t_1] \). Therefore, using \( (2.2) \), we further conclude \( f_0(t) > 0 \) and \( f(t) > \dot{f} > 0 \) for \( t > t_0 \).

(3)&(4): Let us prove \( (2.3) \). Differentiating \( (2.1) \) with respect to \( t \) yields

\[
\partial_t g(t) = -A g(t) \left( \frac{f(t)(f(t)+1)}{t^2 f_0(t)} \right).
\]

Then using \( (2.2) \) to replace \( f_0 \), we arrive at

\[
\partial_t g(t) = -A B g^{\frac{a}{1-c}}(t) t^a f(t)(1 + f(t))^{1-c}.
\]

This leads to

\[
\partial_t g^{\frac{1}{1-c}} = -\frac{b}{A} g^{\frac{a}{1-c}-1} \partial_t g = B t^a f(t)(1 + f(t))^{1-c}.
\]

Integrating this equation \( (2.8) \), noting \( g(t_0) = 1 \) (by \( (2.1) \)) and \( f(t) > 0 \) for \( t > t_0 \), we obtain

\[
g^{\frac{1}{1-c}}(t) = 1 + B t^a \int_{t_0}^t s^{a-2} f(s)(1 + f(s))^{1-c} \, ds \geq 1
\]

for \( t \in [t_0, t_1] \), which implies \( (2.3) \). Since \( s^{a-2} f(s)(1 + f(s))^{1-c} \in (0, \infty) \) and is continuous for \( s \in [t_0, t_1] \), by \( (2.9) \), \( g^{\frac{1}{1-c}}(t) \) is a strictly increasing function for \( t \in [t_0, t_1] \) and so \( g(t) \) strictly decreasing and invertible, and \( g(t) \in (0, 1] \) for \( t \in [t_0, t_1] \). The proof is complete.

2.2. A priori estimates of \( f(t) \) and the first order system. For the ODE \( (1.5)-(1.6) \), we claim the following Proposition \( (2.2) \), a priori estimates for the solution \( f(t) \). We postpone the lengthy proof to §2.4 in order to maintain the coherence of this article, but first rewrite the ODE \( (1.5) \) to a first order system for later use.

Let us restate the equation \( (1.5) \) by a variable transform

\[
y(t) := 1 + f(t). \tag{2.10}
\]
Then (1.5) becomes
\[ \partial_t^2 y(t) + \frac{a}{t} \partial_t y(t) - \frac{6}{t^2} (y(t) - 1)y(t) - \frac{c}{y(t)} (\partial_t y(t))^2 = 0. \] (2.11)

By Lemma 2.1, the solution \( f \in C^2 \) to (1.5)–(1.6) must satisfy \( f(t) > 0 \) and \( f_0(t) > 0 \) for \( t > t_0 \) (i.e., \( y(t) > 1 \) and \( y_0(t) := \partial_t y(t) = f_0(t) > 0 \) for \( t > t_0 \)). Hence, we are able to introduce the following variables to rewrite (1.5). We denote
\[ q(t) := \ln y(t), \quad \text{and} \quad q_0(t) := \partial_t q(t). \] (2.12)

Then (2.11) (i.e., (1.5)) becomes
\[ \partial_t^2 q(t) + \frac{a}{t} \partial_t q(t) - \frac{6}{t^2} (e^{q(t)} - 1) + (1 - c)(\partial_t q)^2 = 0. \]

Further, let us rewrite it, in terms of \((q, q_0)\), as a first order system,
\[ \partial_t q(t) = q_0(t), \] (2.13)
\[ \partial_t q_0(t) = -\frac{a}{t} q_0(t) + \frac{6}{t^2} (e^{q(t)} - 1) - (1 - c)q_0^2. \] (2.14)

By (2.12)–(2.14), we can also reexpress (2.13)–(2.14) in terms of \((y, q_0)\),
\[ \partial_t y = e^q \partial_t q = y q_0, \] (2.15)
\[ \partial_t q_0 = -\frac{a}{t} q_0 + \frac{6}{t^2} (y - 1) - (1 - c)q_0^2. \] (2.16)

In the following, for the simplicity of the statements, we introduce two functions
\[ \mathcal{F}(t) := A t^{\frac{\alpha}{2}} + B t^{\frac{\alpha + \Delta}{2}} + 1 \quad \text{and} \quad \mathcal{L}(t) := (1 + \hat{f}) \hat{c} (1 - E t_0^\hat{a} + E t^\hat{a}). \] (2.17)

Now let us state the a priori estimates for the solution \( f(t) \).

**Proposition 2.2.** Suppose \( f \in C^2([t_0, t_1]) \) \((t_1 > t_0)\) solves ODE (1.5)–(1.6) and \( A, B, C, D, E, \hat{a}, \hat{c} \) and \( \Delta \) are defined in §1.1, then \( f \) satisfies, for \( t \in (t_0, t_1) \), estimates
\[ \exp (\mathcal{O} t^{\frac{\alpha + \Delta}{2}} + D t^{-1}) < 1 + f(t), \] (2.18)
\[ (1 + f(t))^\hat{c} < (1 + \hat{f})^\hat{c} (1 - E t_0^\hat{a} + E t^\hat{a}) = \mathcal{L}(t) \] (2.19)

and
\[ \mathcal{F}(t) = A t^{\frac{\alpha}{2}} + B t^{\frac{\alpha + \Delta}{2}} + 1 < (1 + f(t))^{-1}. \] (2.20)

and for \( t \in (t_0, t_1) \), the derivative \( f_0(t) \) can be bounded by
\[ 0 < f_0(t) < -B \Delta t^{\frac{\Delta + \hat{a}}{2} - 1}(1 + f)^2. \] (2.21)

Moreover, if \( t_* \) is defined by Definition 1.1, then the maximal existence interval \([t_0, t_m]\) of \( f \) satisfies \( t_m \geq t_* \), and a direct corollary of above results is the function \( \mathcal{L} \) is positive at \( t = t_* \), i.e., \( \mathcal{L}(t_*) = (1 + \hat{f})^\hat{c} (1 - E t_0^\hat{a} + E t_*^\hat{a}) > 0 \).
2.3. Reference solutions and proof of Theorem 1.1. Before proving the main Theorem 1.1, we give a Lemma 2.3 on the critical times \( t^* \) and \( t_* \) defined by Definition 1.1.

**Lemma 2.3.** Suppose \( t_* \) and \( t^* \) are defined by Definition 1.1, then

1. \( t_* \in [0, \infty) \) exists and \( t_* > t_0 \);
2. if, furthermore, the initial data satisfies (1.8), i.e., \( f_0 > \tilde{a}(1 + \hat{f})/(\chi_0) \), then \( t^* \) exists and \( 0 < t_* < t^* \).

**Proof.** (1) We note that there is an infimum \( \bar{t} \) of \( f \). Let us next prove the local existence of solution \( f \) in the interval \( (0, \bar{t}) \). By contradiction. Assume \( f \) is a solution of (1.6) and \( f \) exists and \( \lim_{\bar{t} \to \infty} f(t) = -\infty \) due to \((\tilde{a} - \Delta)/2 < 0 \) and \((\tilde{a} + \Delta)/2 > 0 \). Then there is, at least, a time \( t_\infty \in (t_0, \infty) \) such that \( L(t_\infty) = \frac{2x}{2t} + Bt_\infty > 1 = 0 \), thus the set \( R = \emptyset \) (recall \( R \) is defined in Definition 1.1). Since \( t_\infty > t_0 \) for all \( t_\infty \in R \), then there is an infimum \( t_\infty \) of \( R \).

We first claim \( t_\infty > t_0 \), otherwise, if \( t_\infty = t_0 \), then \( L(t_\infty) = L(t_0) > 0 \). Due to the definition of \( t_\infty \), for any \( n \in \mathbb{N}_+ \) there is \( t_r, n \in R \), such that \( t_\infty < t_r, n < t_\infty + 1/n \). We then have a sequence \( \{t_r, n\} \) and \( \lim_{n \to \infty} t_r, n = t_\infty \). According to the continuity of \( L \), we obtain

\[
0 = \lim_{n \to \infty} L(t_r, n) = L(t_\infty) > 0.
\]

This contradiction confirms \( t_\infty > t_0 \).

We then claim \( t_\infty \in R \), otherwise, i.e., \( t_\infty \notin R \), then \( L(t_\infty) = \frac{2x}{2t} + Bt_\infty > 1 > 0 \) (if \( L(t_\infty) = \frac{2x}{2t} + Bt_\infty < 0 \), there is at least another zero of \( L \) in the interval \( (t_0, t_\infty) \), which contradicts with the meaning of \( t_\infty \)). By the continuity of \( L \), we similarly arrive at (2.22), a contradiction that leads to \( t_\infty \in R \) that means \( t_\infty = \min R = t_* \). Hence, \( L(t_*) = 0 \) and \( t_* \) exists and \( t_* > t_0 \).

(2) Firstly, since the condition (1.8) (i.e., \( \hat{f}_0 > \tilde{a}(1 + \hat{f})/(\chi_0) \)) is equivalent, by the definition of \( E \) in §1.1, to \( E^{-1} < t_0^{\tilde{a}} \), then by Definition 1.1, we can define \( t^* := (t_0^{\tilde{a}} - E^{-1})^0 > 0 \). By further noting \( \tilde{a} = 1 - a < 0 \) and \( E > 0 \), we obtain \( t^* > t_0 \). We note \( L(t) := (1 + \hat{f})^2(1 - E^{t_0} + Et^\tilde{a}) \) is a decreasing function for \( t > t_0 \) (due to \( \tilde{a} < 0 \)), \( t^* \) is a zero of \( L(t) \), i.e., \( L(t^*) = 0 \) and note \( L(t_0) = (1 + \hat{f})^2 > 0 \).

Let us next prove \( t^* > t_* \) by contradiction. That is, assume \( t_* \geq t^* \), then since \( L \) is decreasing function, we obtain \( L(t_* \leq L(t^*) = 0 \). However, by Proposition 2.2, we have a corollary that \( L(t_*) > 0 \). This leads to a contradiction. Therefore, \( t^* > t_* \). It completes the proof. \( \square \)

We are now in the position to prove Theorem 1.1 for the solution \( f(t) \) to (1.5)–(1.6).

**Proof of Theorem 1.1.** By Lemma 2.3, we conclude (1) and (4).

(2) In order to conclude (2), let us first prove the local existence of solution \( f \). We consider the first order system (2.15)–(2.16) for \( y, q_0 \) and use Theorem A.1 (Existence and uniqueness of ODEs), there is a small time interval \( [t_0, t_1] \), such that, for \( t \in [t_0, t_1] \) \((t_0 < t_1 < t_* \)) there is a unique \( C^1 \) solution \( (y, q_0) \) and so we have a local solution \( f \) in \([t_0, t_1] \). By Proposition 2.2, the maximal existence interval \( [t_0, t_m] \) of \( f \) can be extended to a time \( t_m \geq t_* \). Thus \( f \in C^2([t_0, t_m]) \) is a solution of (1.5)–(1.6).

If \( t_m = \infty \), by taking the limits on the both sides of (2.18), we readily obtain \( \lim_{t \to t_m} f(t) = \infty \). Let us next prove \( \lim_{t \to t_m} f(t) = \infty \) for finite \( t_m < \infty \) by contradiction. Assume \( \lim_{t \to t_m} f(t) < \infty \) then by (2.21) in Proposition 2.2, \( f_0(t) \to \infty \), and further

\[
\lim_{t \to t_m} (y, q_0) = \lim_{t \to t_m} \left(1 + f, \frac{f_0}{1 + f}\right) < +\infty.
\]
By the continuation principle (see Theorem A.2), we can continue the solution beyond \( t_m \) which contradicts with the meaning of \( t_m \). Therefore, \( \lim_{t \to t_m} f(t) = +\infty \). On the other hand, since, by Lemma 2.1.(4), \( g \) is a decreasing function which leads to \( g^{-\frac{1}{\alpha}}(t) \geq 1 \), with the help of (2.2) and (2.18), we then obtain
\[
 f_0(t) = B^{-1}t^{-a}g^{-\frac{1}{\alpha}}(t)(1 + f(t))^c \geq B^{-1}t^{-a}(1 + f(t))^c \\
 \geq B^{-1}t^{-a} \exp((c - 1)(\alpha t^{\frac{2+\alpha}{\alpha}} + D t^{-1}))(1 + f(t)), \tag{2.23}
\]
then by taking the limit on the both sides, we claim
\[
 \lim_{t \to t_m} f_0(t) = +\infty. \tag{2.24}
\]

We point out if \( t_m < \infty \), we can use \( f_0(t) \geq B^{-1}t^{-a}(1 + f(t))^c \) directly obtain (2.24) and if \( t_m = \infty \), we use (2.23) and note the fact \( \lim_{x \to \infty} (x^a/e^x) = 0 \) and \( \lim_{t \to \infty} \exp((c - 1)Dt^{-1}) = 1 \) to conclude (2.24). We then proved (2).

3. Because \( f \in C^2([t_0, t_m]) \) is a solution of (1.5)–(1.6), by (2.18) in Proposition 2.2, \( 1 + f(t) > \exp(\alpha t^{\frac{2+\alpha}{\alpha}} + Dt^{-1}) \) for \( t \in (t_0, t_m) \). By (2.19) and (2.20), we recall them here, with the help of Lemma 2.1.(2) (\( f(t) > 0 \) in \( [t_0, t_m] \)) and the definition of \( t_\text{*} \), \( f \) in Definition 1.1 (\( \mathcal{F}(t) > 0 \) for \( t \in [t_0, t_\text{*}] \)), otherwise, \( t_\text{*} \) is not the minimum,
\[
 0 < (1 + f(t))^{\bar{c}} < (1 + \bar{f})^{\bar{c}}(1 - Et_0^{\bar{a}} + Et^{\bar{a}}) = \mathcal{L}(t) \quad \text{for} \quad t \in (t_0, t_m), \\
 0 < \mathcal{F}(t) = A t^{\frac{2+\alpha}{2}} + B t^{\frac{2+\alpha}{2}} + 1 < (1 + f(t))^{-1} \quad \text{for} \quad t \in (t_0, t_\text{*}).
\]
Then noting \( \bar{c} < 0 \),
\[
 1 + f(t) > (1 + \bar{f})^{\frac{c}{\bar{c}}}(1 - Et_0^{\bar{a}} + Et^{\bar{a}})^{1/\bar{c}} \quad \text{for} \quad t \in (t_0, t_m), \\
 (A t^{\frac{2+\alpha}{2}} + B t^{\frac{2+\alpha}{2}} + 1)^{-1} > 1 + f(t) \quad \text{for} \quad t \in (t_0, t_\text{*}).
\]

It then completes the proof of (3).

4. Now let us turn to (5). By (2.19) in Proposition 2.2, \( (1 + f(t))^{\bar{c}} < (1 + \bar{f})^{\bar{c}}(1 - Et_0^{\bar{a}} + Et^{\bar{a}}) =: \mathcal{L}(t) \) for \( t \in (t_0, t_m) \). If the initial data satisfies (1.8) which is equivalent to \( 1 - Et_0^{\bar{a}} < 0 \). In this case, we can define \( t_\text{*} \) according to Definition 1.1. Let us prove \( t_m < t_\text{*} \). Otherwise, assume \( t_m \geq t_\text{*} \). Since, by Lemma 2.1.(2) (\( f(t) > 0 \) in \( [t_0, t_m] \)), \( f(t_\text{*}) > 0 \) for \( t_\text{*} \in (t_0, t_m) \), we obtain
\[
 0 < (1 + f(t_\text{*}))^{\bar{c}} < (1 + \bar{f})^{\bar{c}}(1 - Et_0^{\bar{a}} + Et^{\bar{a}}) =: \mathcal{L}(t_\text{*}) = 0,
\]
which leads to a contradiction. Hence, \( t_m < t_\text{*} \).

5. In the end, we prove (6). For \( t \in (t_0, t_m) \) where \( t_m < t_\text{*} \), we have \( \mathcal{L}(t) = (1 + \bar{f})^{\bar{c}}(1 - Et_0^{\bar{a}} + Et^{\bar{a}}) > 0 \) since \( \mathcal{L}(t) \) is decreasing and \( \mathcal{L}(t_\text{*}) = 0 \). Then
\[
 1 + f(t) = y(t) > (1 + \bar{f})^{\frac{c}{\bar{c}}}(1 - Et_0^{\bar{a}} + Et^{\bar{a}})^{1/\bar{c}}.
\]

We complete the proof.

\[\square\]

2.4. Proof of Proposition 2.2. In this section, let us prove Proposition 2.2.

\textbf{Proof.} Firstly by Lemma 2.1.(1) and (2), we note \( f(t) > 0 \) and \( f_0(t) > 0 \) for any \( t \in (t_0, t_1) \), then with the help of (2.12) and (2.15), we obtain
\[
 y(t) = 1 + f(t) > 1, \quad y_0(t) > 0, \quad q(t) > 0 \quad \text{and} \quad q_0(t) = y_0(t)/y(t) > 0
\]
for \( t \in (t_0, t_1) \). Next, let us prove these inequalities, respectively.

1. Proof of (2.18): Since the \( f(t) \) solves equation (1.5), by (2.10)–(2.14) and introducing a variable
\[
 \alpha(t) := q(t)/t > 0, \tag{2.25}
\]
in terms of \((\alpha, q_0)\), (2.13)–(2.14) become
\[
\partial_t \alpha(t) = \frac{1}{t}(q_0(t) - \alpha(t)),
\tag{2.26}
\]
\[
\partial_t q_0(t) = -\frac{a}{t}q_0(t) + \frac{6}{t^2}(e^{\alpha(t)}t - 1) - (1 - c)q_0^2.
\tag{2.27}
\]
By (2.26) and (2.27), we obtain
\[
\partial_t(\alpha + kq_0) = \frac{1}{t}(q_0(t) - \alpha(t)) - \frac{ka}{t}q_0(t) + \frac{k\delta}{t^2}(e^{\alpha(t)}t - 1) + k(c - 1)q_0^2
\tag{2.28}
\]
where we take
\[
k = \frac{\sqrt{(a - 1)^2 + 4\delta} - a + 1}{2\delta} = \frac{\bar{a} + \Delta}{2\delta} > 0.
\]
Since \(e^{\alpha(t)}t - 1 > \alpha(t)t\) and \(c > 1\), with the help of a direct fact \(1 - ka = k(k\delta - 1)\), (2.28) and (2.27) have lower bounds,
\[
\partial_t(\alpha + kq_0) > \frac{1}{t}(q_0(t) - \alpha(t)) - \frac{ka}{t}q_0(t) + \frac{k\delta}{t^2}(\alpha(t) + kq_0(t)) = \frac{q_0}{t}k(k\delta - 1) + \frac{\alpha}{t}(k\delta - 1)
\tag{2.29}
\]
\[
\partial_0 q_0(t) > -\frac{a}{t}q_0(t) + \frac{6}{t^2}\alpha(t).
\tag{2.30}
\]
Then (2.29) and (2.30) imply
\[
\partial_t[t^{1-k\delta}(\alpha + kq_0)] > 0 \quad \text{and} \quad \partial_0(t^\alpha q_0) > \delta\alpha(t)t^{\alpha - 1}.
\tag{2.31}
\]
Integrating the first differential inequality in (2.31) yields, for \(t \in (t_0, t_1)\),
\[
\alpha + q_0 > (\alpha(t_0) + kq_0(t_0))t_0^{1-k\delta}t^{k\delta - 1}.
\tag{2.32}
\]
According to the definition (2.12) and (2.25) of \(q_0\) and \(\alpha\), we note above (2.32) implies
\[
\alpha + \partial_t q = \alpha + \partial_t(\alpha t) > (\alpha(t_0) + kq_0(t_0))t_0^{1-k\delta}t^{k\delta - 1},
\]
which in turn implies
\[
\partial_t(t^2\alpha) > (\alpha(t_0) + kq_0(t_0))t_0^{1-k\delta}t^{k\delta}.
\tag{2.33}
\]
Then integrating (2.33) and noting \(\alpha(t_0) = \ln(1 + \hat{f})/t_0\) and \(q_0(t_0) = \hat{f}_0/(1 + \hat{f})\) yield, for \(t \in (t_0, t_1)\),
\[
\alpha(t) > \frac{1}{k\delta + 1}(\alpha(t_0) + kq_0(t_0))t_0^{1-k\delta}t^{k\delta - 1} + \frac{k\delta}{k\delta + 1}\left[\alpha(t_0) - \frac{1}{\delta}q_0(t_0)\right]t_0^{2}t^{-2} = Ct^\frac{\alpha + \Delta}{k\delta} + Dt^{-1}.
\]
Then by (2.12) and (2.25), we have \(1 + f(t) = y(t) = \exp(\alpha(t)t)\), and this, for \(t \in (t_0, t_1)\), concludes
\[
1 + f(t) = y(t) > \exp(Ct^\frac{\alpha + \Delta}{k\delta} + Dt^{-1}).
\]

(2) Proof of (2.20): Multiplying \(1/y\) on the both sides of (2.14) yields
\[
\partial_t\left(\frac{q_0}{y}\right) = -\frac{a}{t}q_0\frac{1}{y} + \frac{6}{t^2}\left[1 - \frac{1}{y}\right] + (c - 2)\frac{q_0^2}{y}.
\]
Then recalling \(c \in (1, 3/2)\) and since \(y > 1\) and \(q_0 > 0\), we obtain
\[
\partial_t\left(\frac{q_0}{y}\right) + \frac{a}{t}q_0\frac{1}{y} < \frac{6}{t^2}\left[1 - \frac{1}{y}\right].
\]
which in turn implies
\[ \partial_t \left( \frac{t^a q_0}{y} \right) < \beta t^{a-2} \left( 1 - \frac{1}{y} \right). \]  
(2.34)

On the other hand, by (2.12), we have \( \partial_t y = y q_0 \), then direct calculations leads to
\[ \partial_t \left( \frac{1}{y} \right) = -y^{-2} \partial_t y = -\frac{q_0}{y}. \]  
(2.35)

Let
\[ Q := \frac{q_0}{y} \quad \text{and} \quad Y := \frac{1}{y}. \]  
(2.36)

Then (2.34) and (2.35) become
\[ \partial_t (t^a Q) < \beta t^{a-2} (1 - Y) \quad \text{and} \quad \partial_t Y = -Q. \]  
(2.37)

By (2.37), with direct calculations, we obtain
\[ \partial_t (t^a Q + \ell t^{a-1} Y) < \beta t^{a-2} - \ell t^{a-1} Q + [\ell (a - 1) - 6] t^{a-2} Y \]  
(2.38)

where we take
\[ \ell := \frac{1 - a - \sqrt{(1 - a)^2 + 4\beta}}{2} = \frac{\bar{a} - \Delta}{2} < 0. \]

In this case, we have
\[ \ell (a - 1) - 6 = -\ell^2. \]  
(2.39)

Then (2.38), using (2.39), becomes
\[ \partial_t (t^a Q + \ell t^{a-1} Y) + \ell t^{a-1} (t^a Q + \ell t^{a-1} Y) < \beta t^{a-2} \]

which in turn leads to
\[ \partial_t (t^{\ell+a} Q + \ell t^{\ell+a-1} Y) < \beta t^{\ell+a-2}. \]

Then integrating it yields, for \( t \in (t_0, t_1) \),
\[ Q + \ell t^{-1} Y < \left( C_2 - \frac{\beta t_0^{\ell+a-1}}{\ell + a - 1} \right) t^{-\ell-a} + \frac{6}{\ell + a - 1} t^{-1} \quad \text{where} \quad C_2 = \frac{\ell t_0^{\ell+a} f_0}{(1 + f)^2} + \frac{\ell t_0^{\ell+a-1}}{1 + f}. \]

By (2.36), with the help of (2.35), above inequality becomes
\[ -\partial_t \left( \frac{1}{y} \right) + \ell t^{-1} \frac{1}{y} = \frac{q_0}{y} + \ell t^{-1} \frac{1}{y} < \left( C_2 - \frac{\beta t_0^{\ell+a-1}}{\ell + a - 1} \right) t^{-\ell-a} + \frac{6}{\ell + a - 1} t^{-1}. \]  
(2.40)

This inequality implies
\[ \partial_t \left( t^{-\ell} \frac{1}{y} \right) > \left( \frac{\beta t_0^{\ell+a-1}}{\ell + a - 1} - C_2 \right) t^{-2\ell-a} - \frac{6}{\ell + a - 1} t^{-1-\ell}. \]

Then its integration for \( t \in (t_0, t_1) \), with the help of (2.39) (i.e., \( \ell (\ell + a - 1) = 6 \)), is
\[ \frac{1}{y} > \frac{1}{1 + f} t_0^{-\ell} + \frac{1}{1 - 2\ell - a} \left( \frac{\beta t_0^{\ell+a-1}}{\ell + a - 1} - C_2 \right) (t^{-\ell-a} - t_0^{-2\ell-a} + (1 - t_0^{-\ell}) t^{\ell}) \]
\[ = \left( -\frac{t_0^{-\ell} f_0}{1 + f} - \frac{1}{1 - 2\ell - a} \left( \ell t_0^{-\ell} - \frac{t_0^{1-\ell} f_0}{(1 + f)^2} - \frac{t_0^{-\ell}}{1 + f} \right) \right) t^{\ell} \]
\[ + \frac{1}{1 - 2\ell - a} \left( \ell t_0^{\ell+a-1} - \frac{t_0^{\ell+a} f_0}{(1 + f)^2} - \frac{\ell t_0^{\ell+a-1}}{1 + f} \right) t^{1-\ell-a} + 1 \]
\[ = \frac{t^{\ell}}{1 - 2\ell - a} \left( \frac{t_0^{1-\ell} f_0}{(1 + f)^2} - (1 - \ell - a) f t_0^{-\ell} \right) + \frac{t^{1-\ell-a}}{1 - 2\ell - a} \left( \ell f t_0^{\ell+a-1} - \frac{t_0^{\ell+a} f_0}{(1 + f)^2} \right) + 1 \]
= At^{\frac{\gamma}{2}} + Bt^{\frac{\gamma}{2}} + 1.

In terms of $f$, we obtain for $t \in (t_0, t_1)$,

$$\left(1 + f\right)^{-1} = y^{-1} > At^{\frac{\gamma}{2}} + Bt^{\frac{\gamma}{2}} + 1. \quad (2.41)$$

(3) Proof of (2.21): Using (2.40), with the help of $f = y - 1$ and (1.9) (recalling $\Delta > -\bar{a}$, then $\bar{a} - \Delta < 2\bar{a} < 0$), yields,

$$q_0 < \left(\frac{t_0^{t_0 + \bar{a}}}{(1 + f)^2} - \frac{t_0^{t_0 + \bar{a} - 1}}{1 + f}\right) t^{t - a} y + \ell t^{-1} (y - 1) = -B\Delta t^{\frac{\Delta}{2}} - 1 y + \frac{\bar{a} - \Delta}{2} t^{-1} f < -B\Delta t^{\frac{\Delta}{2}} - 1 y. \quad (2.42)$$

Then by (2.15), we conclude

$$f_0 = \partial_t y = y q_0 < -B\Delta t^{\frac{\Delta}{2}} - 1 (1 + f)^2 + \frac{\bar{a} - \Delta}{2} t^{-1} f (1 + f) < -B\Delta t^{\frac{\Delta}{2}} - 1 (1 + f)^2.$$

(4) Proof of (2.19): Multiplying $1/y^{c-1}$ on the both sides of (2.16), we arrive at

$$\partial_t \left(\frac{q_0}{y^{c-1}}\right) - (1 - c) \frac{q_0^2}{y^{c-1}} = -\frac{\bar{a}}{t} \frac{q_0}{y^{c-1}} + \frac{\bar{a}}{t^2} \left(y^{2 - c} - y^{1 - c}\right) - (1 - c) \frac{q_0^2}{y^{c-1}},$$

that is,

$$\partial_t \left(\frac{q_0}{y^{c-1}}\right) + \frac{\bar{a}}{t} \frac{q_0}{y^{c-1}} = \frac{\bar{a}}{t^2} \left(y^{2 - c} - y^{1 - c}\right).$$

This in turn implies

$$\partial_t \left(t^a \frac{q_0}{y^{c-1}}\right) = \frac{\bar{a}}{t^2} \left(y^{2 - c} - y^{1 - c}\right). \quad (2.43)$$

On the other hand, by using (2.15), we have

$$\partial_t y^{1 - c} = (1 - c) y^{-c} \partial_t y = (1 - c) y^{1 - c} q_0. \quad (2.44)$$

Denote

$$X := q_0 y^{1 - c} \quad \text{and} \quad Y := y^{1 - c},$$

then (2.43) and (2.44) become

$$\partial_t (t^a X) = \frac{\bar{a}}{t^2} (y - 1) Y \quad \text{and} \quad \partial_t Y = (1 - c) X. \quad (2.45)$$

By (2.46), with direct calculations and using $f = y - 1 > 0$, we obtain

$$\partial_t (t^a X + \lambda t^{a-1} Y) = \left[\frac{\bar{a}}{t} (y - 1) + \lambda (a - 1)\right] t^{a-2} Y + \lambda (1 - c) t^{a-1} X \quad \Rightarrow \lambda (a - 1) t^{a-2} Y + \lambda (1 - c) t^{a-1} X = \lambda (a - 1) t^{-1} (t^a X + \lambda t^{a-1} Y),$$

where $\lambda$ is defined by $\lambda := (a - 1)/(1 - c) = -\bar{a}/c$. Then

$$\partial_t (t^a X + \lambda t^{a-1} Y) - \lambda (1 - c) t^{-1} (t^a X + \lambda t^{a-1} Y) > 0,$$

that is

$$\partial_t (t^a X - \lambda (1 - c) Y + \lambda t^{a-1} X) > 0.$$ Integrating it yields for $t \in (t_0, t_1)$,

$$t^{a-\lambda (1 - c) Y + \lambda t^{a-1} (1 - c) Y} > C_0 := \frac{t_0^{\bar{a} f_0}}{(1 + f)^{c}} + \frac{a - 1}{1 - c} (1 + f)^{1 - c},$$

which in turn implies

$$X + \lambda t^{-1} Y > C_0 t^{a + \lambda (1 - c)}.$$
By (2.15), (2.45), (2.12) and noting $c > 1$, the above inequality becomes
\[ \partial_t y^{1-c} + (1 - c)\lambda t^{-1} y^{1-c} < C_0(1 - c)t^{-\alpha + \lambda(1-c)}, \]
which yields
\[ \partial_t (t^{1-c}\lambda y^{1-c}) < C_0(1 - c)t^{-\alpha + 2\lambda(1-c)}. \]
Integrating this inequality, noting $c > 1$ and using $\lambda = -\alpha / \bar{c}$, after a lengthy calculation, we arrive at for $t \in (t_0, t_1)$,
\[ (1 + f(t))^\bar{c} = y^\bar{c}(t) < (1 + \bar{f})^\bar{c}\left(1 - \frac{c\bar{t}_0 \bar{f}_0}{\bar{a}(1 + f)} + \frac{c\bar{f}_0 t_0^{1-\bar{a}}}{\bar{a}(1 + f)}t^{\bar{a}}\right) = (1 + \bar{f})^\bar{c}\left(1 - E \bar{t}_0^\bar{a} + E t^{\bar{a}}\right). \]
Note $(1 + \bar{f})^\bar{c}(1 - E \bar{t}_0^\bar{a} + E t^{\bar{a}}) = (1 + \bar{f})^\bar{c}$. 

(5) extension of solutions and $\mathcal{L}(t_*) > 0$: Suppose $t_*$ is defined by Definition 1.1, the smallest zero which is greater than $t_0$ of $\mathcal{F}$. Since $f \in C^2(\mathcal{F})$ solves (1.5) for any $t \in [t_0, t_1)$, then by (2.10)–(2.16), $(y, q_0)$ solves the system (2.15)–(2.16) for any $t \in [t_0, t_1)$ Let’s us confirm that if $t_m$ is the maximal time of the existence of $(y, q_0)$, then $t_m \geq t_*$. Otherwise, assume $t_m < t_*$, since solution $f \in C^2([t_0, t_m))$ exists, by (2.41), $(y(t))^{-1} = (1 + f(t))^{-1} > \mathcal{F}(t) > 0$ for $t \in (t_0, t_m)$ (recall $\mathcal{F}(t)$ is defined by (2.17) and $\mathcal{F}(t) > 0$ for all $t \in (t_0, t_1)$). Then $1 < y(t) < \mathcal{F}(t) < (1+f(t))^{-1} < \infty$ for $t \in (t_0, t_m)$. Along with the bound of $y$, we also, by (2.15) and (2.42), have $0 < y_0(t)/y(t) = q_0(t) < -B\Delta t^{\frac{\lambda}{2-\alpha}} y(t) < \infty$ (recall $B < 0$ in §1.1) for $t \in (t_0, t_m)$. By the continuation principle A.1, the solution $(y, q_0)$ can be continued to the right passing through the point $t = t_m$ which contradicts with the fact that $t_m$ is the maximal time of the existence of $(y, q_0)$. Therefore, $t_m \geq t_*$. In the end since $f \in C^2([t_0, t_m))$ for $t_m > t_*$ solves the ODE (1.5)–(1.6), by (2.47), we obtain $(1 + f(t))^\bar{c} < (1 + \bar{f})^\bar{c}(1 - E \bar{t}_0^\bar{a} + E t^{\bar{a}}) = \mathcal{L}(t)$ for $t \in (t_0, t_m)$. Then $\mathcal{L}(t_*) > (1 + f(t_*))^\bar{c} > 0$. We complete the proof. 

2.5. Estimates of two crucial quantities $\chi(t)$ and $\xi(t)$. In this section, we estimate two important quantities $\chi(t)$ and $\xi(t)$ which will be frequently used in rewriting the main equation (1.1)–(1.2) into the Fuchsian formulation (C.1)–(C.2) from Appendix C. The first quantity is defined by
\[ \chi(t) := \frac{t^{2-\alpha} f_0(t)}{(1 + f(t))^{2-\alpha} f(t) y(t)^{\frac{\alpha}{\gamma}(t)} = \frac{g^{-\frac{\alpha}{\gamma}(t)} t^{2(1-\alpha)}}{B f(t)(1 + f(t))^{2(1-\alpha)}}. \]
We point out the second equality holds by directly using identity (2.2) in Lemma 2.1 to recall $B$ is defined there.

Before proceeding, we first give the following useful lemma.

Lemma 2.4. Suppose $g(t)$ is defined by (2.1) and $f \in C^2([t_0, t_m))$ (where $[t_0, t_m)$ is the maximal interval of existence of $f$) solves ODE (1.5)–(1.6), then $g^{\frac{\alpha}{\gamma}(t)}t^{a-1}(1 + f(t))^{1-c} > 0$ is bounded in $[t_0, t_m)$, and
\[ \lim_{t \to t_m} \left(g^{\frac{\alpha}{\gamma}(t)}t^{a-1}(1 + f(t))^{1-c}\right) = 0. \]

Proof. (1) If $t_m < \infty$, since, by Lemma 2.1.(3), $g(t) \in (0, 1]$ for $t \in [t_0, t_m)$, we obtain
\[ 0 \leq g^{\frac{\alpha}{\gamma}} t^{a-1}(1 + f)^{1-c} \leq t_m ^{a-1}(1 + f)^{1-c}. \]
According to Theorem 1.1.(2) and $c > 1$, we obtain $\lim_{t \to t_m} (1 + f)^{1-c} = 0$, which leads to this lemma.
(2) If \( t_m = \infty \), then by using Proposition 2.2 and since \( g(t) \in (0, 1] \) for \( t \in [t_0, t_m) \), we have the estimate
\[
0 \leq g^{\frac{a}{t}} t^{a-1} (1 + f)^{1-c} \leq t^{a-1} \exp\left((1-c) (\text{Li}^{\frac{a+\Delta}{2}} + \Delta t^{-1})\right).
\]
Since, using the fact \( \lim_{t \to \infty} \left( x^c/x^c \right) = 0 \) and noting \( -\bar{a} > 0 \),
\[
\lim_{t \to \infty} \left[ t^{a-1} \exp\left((1-c) (\text{Li}^{\frac{a+\Delta}{2}} + \Delta t^{-1})\right) \right] = \lim_{t \to \infty} \left[ t^{-\bar{a}} \exp\left((1-c) \text{Li}^{\frac{a+\Delta}{2}}\right) \right] = 0.
\]

Since \( g^{\frac{a}{t}} t^{a-1} (1 + f)^{1-c} \) is a continuous function in \([t_0, t_m)\), by Proposition B.2 (see Appendix B), we conclude the boundness of it and further this lemma. □

**Lemma 2.5.** Suppose \( f \in C^2([0, t_0]) \) (where \([0, t_0]\) is the maximal interval of existence of \( f \) given by Theorem 1.1) solves ODE (1.5)–(1.6) and assume \( t_m < \infty \) is finite. Then
\[
\lim_{t \to t_m} [(t_m - t) f^{2-c}(t)] = +\infty.
\]

**Proof.** Noting \( c \in (1, 3/2) \) and using the representation of \( f_0 \) (2.2), with the help of L'Hospital rules since \( \lim_{t \to t_m} f^{2-c}(t) = \infty \) (by Theorem 1.1.(5)) and \( \lim_{t \to t_m} (t_m - t) = 0 \), straightforward calculations yield
\[
\lim_{t \to t_m} [(t_m - t) f^{2-c}(t)] = \frac{1}{2-c} \lim_{t \to t_m} \frac{1}{f_0^{t-3}} (2.2) \quad \lim_{t \to t_m} \frac{B(t_m)}{2-c} \frac{f^{3-2c}}{g^{\frac{a}{t}}(t)}. \tag{2.49}
\]

Let us discuss this limit in two cases by assuming \( \lim_{t \to t_m} g(t) = 0 \) or \( = 0 \). Firstly if \( g(t_m) := \lim_{t \to t_m} g(t) > 0 \) (since \( g(t) \in (0, 1) \) for \( t \in [t_0, t_m) \) by Lemma 2.1.(3)), by (2.49), we have \( \lim_{t \to t_m} [(t_m - t) f^{2-c}(t)] = +\infty \) by Theorem 1.1.(5). On the other hand, if \( g(t_m) := \lim_{t \to t_m} g(t) = 0 \),
\[
\lim_{t \to t_m} [(t_m - t) f^{2-c}(t)] = (2.8) B(t_m) (3 - 2c) \lim_{t \to t_m} g^{-\frac{a}{t}}(t) = +\infty.
\]

We complete the proof. □

**Lemma 2.6.** Suppose \( g(t) \) is defined by (2.1) and \( c \in (1, 3/2) \), and \( f \in C^2([0, t_0]) \) (where \([0, t_0]\) is the maximal interval of existence of \( f \) given by Theorem 1.1) solves ODE (1.5)–(1.6). Then \( \lim_{t \to t_m} g(t) = 0 \).

**Remark 2.5.1.** Due to this lemma, it is convenience to continuously extend \( g(t) \) from \([t_0, t_m]\) to \([t_0, t_m]\) by letting \( g(t_m) := \lim_{t \to t_m} g(t) = 0 \), then \( g^{-1}(0) = t_m \).

**Proof.** By Lemma 2.1.(3), for \( t \in [t_0, t_m] \), we have
\[
g^{-\frac{a}{t}}(t) = 1 + 6g \int_{t_0}^{t} s^{a-2} f(s)(1 + f(s))^{1-c} ds.
\]
Hence, in order to conclude \( \lim_{t \to t_m} g(t) = 0 \), we only need to prove \( \lim_{t \to t_m} g^{-\frac{a}{t}}(t) = 0 \). Furthermore, we need to verify \( \int_{t_0}^{t} s^{a-2} f(s)(1 + f(s))^{1-c} ds = \infty \).

First if \( t_m < +\infty \), let us prove \( \int_{t_0}^{t} s^{a-2} f(s)(1 + f(s))^{1-c} ds = \infty \), i.e., we will prove it is a divergence improper integrals. Note the limit, with the help of Lemma 2.5,
\[
\lim_{t \to t_m} \frac{t^{a-2} f(t)(1 + f(t))^{1-c}}{1/(t_m - t)} = \lim_{t \to t_m} \frac{t_m^{a-2}}{(t_m-t)^2} \lim_{t \to t_m} [(t_m - t) f(t)(1 + f(t))^{1-c}] = \lim_{t \to t_m} [(t_m - t) f^{2-c}(t)] = +\infty,
\]
and since the improper integral
\[ \int_{t_0}^{t_m} \frac{1}{t_m - s} ds = +\infty, \]
i.e., it is a divergent improper integral. Then by the comparison test for improper integrals, we obtain
\[ \int_{t_0}^{t_m} s^{a-2} f(s)(1 + f(s))^{1-c} ds = +\infty. \]
Thus, by (2.9), we arrive at \( \lim_{t \to t_m} g^{-\frac{c}{a}}(t) = \infty \).

On the other hand, if \( t_m = +\infty \), Let us prove \( \int_{t_0}^{\infty} s^{a-2} f(s)(1 + f(s))^{1-c} ds = \infty \). If \( a \geq 2 \), then, \( \lim_{t \to +\infty} [t^{a-2} f(t)(1 + f(t))^{1-c}] = \infty \) by Theorem 1.1.(2). If \( a \in (1,2) \), then by (2.18) in Proposition 2.2 and the fact \( \lim_{x \to \infty} (x^a/e^x) = 0 \), we obtain
\[ \lim_{t \to +\infty} [t^{a-2} f(t)(1 + f(t))^{1-c}] = \lim_{t \to +\infty} \left[ t^{a-2} (1 + f(t))^{1-c} f^{2^{1-c}}(t) \right] > \lim_{t \to +\infty} \left[ t^{a-2} \exp \left( \frac{1}{2} (Ct^{a-2} + D t^{-1}) \right) f^{2^{1-c}}(t) \right] = \lim_{t \to +\infty} \left[ t^{a-2} \exp \left( \frac{1}{2} Ct^{a-2} \right) f^{2^{1-c}}(t) \right] = \infty. \]
That is, \( \lim_{t \to +\infty} [t^{a-2} f(t)(1 + f(t))^{1-c}] = \infty \) for any \( a > 1 \). This leads to, since \( t^{a-2} f(t)(1 + f(t))^{1-c} \in C^2([t_0, t_m]) \) and it is strictly positive for all \( t \in [t_0, t_m] \), by the property of continuous function, that there is a constant \( M > 0 \), such that \( t^{a-2} f(t)(1 + f(t))^{1-c} \geq M > 0 \) for all \( t \in [t_0, +\infty) \). Therefore,
\[ \int_{t_0}^{t_m} s^{a-2} f(s)(1 + f(s))^{1-c} ds \geq M \int_{t_0}^{\infty} ds = \infty, \]
which, by (2.9), in turn implies \( \lim_{t \to +\infty} g(t) = 0 \). We complete the proof.

\[ \square \]

**Proposition 2.7.** Suppose \( c \in (1,3/2), \ b > 0, \ a > 1, \ \chi \) is defined by (2.48) and \( f \in C^2([t_0, t_m]) \) (where \( [t_0, t_m] \) is the maximal interval of existence of \( f \) given by Theorem 1.1) solves ODE (1.5)–(1.6). Then there is a function \( \mathfrak{G} \in C^1([t_0, t_m]), \) such that for \( t \in [t_0, t_m], \)
\[ \chi(t) = \frac{2\mathfrak{G}}{3 - 2c} + \mathfrak{G}(t). \]
where \( \lim_{t \to t_m} \mathfrak{G}(t) = 0. \) Moreover, there is a constant \( C_\chi > 0 \) such that \( 0 < \chi(t) \leq C_\chi \) in \( [t_0, t_m], \) and there are continuous extensions of \( \chi \) and \( \mathfrak{G} \) such that \( \chi \in C^0([t_0, t_m]) \) and \( \mathfrak{G} \in C^0([t_0, t_m]) \) by letting \( \chi(t_m) := 2\mathfrak{G}/(3-2c) \) and \( \mathfrak{G}(t_m) := 0. \)

**Proof.** Using (2.2), we first calculate
\[ \partial_t(Bt^{2(a-1)} f(1 + f)^{2(1-c)}) = (3-2c) t^{a-2} (1 + f)^{2-c} g^{-\frac{a}{4}} - 2(1-c) t^{a-2} (1 + f)^{1-c} g^{-\frac{a}{4}} \]
\[ + 2B(a-1) t^{2a-3} f(1 + f)^{2(1-c)}. \]
(2.50)

By Theorem 1.1 and due to \( c \in (1,3/2) \) leading to \( 2(1-c) \in (-1,0) \) and \( 3 - 2c \in (0,1) \), we have limit,
\[ \lim_{t \to t_m} [f(1 + f)^{2(1-c)}] = \lim_{t \to t_m} [(1 + f)^{3-2c} - (1 + f)^{2(1-c)}] = \infty. \]

With the help of this limit and \( \lim_{t \to t_m} g^{-\frac{2c}{a}}(t) = \infty \) due to Lemma 2.6, we can apply the L’Hospital rule, with the help of (2.8), (2.48), (2.2) and Proposition 2.4,
\[ \lim_{t \to t_m} \chi(t) = \lim_{t \to t_m} \frac{g^{-\frac{2c}{a}}}{Bt^{2(a-1)} f(1 + f)^{2(1-c)}} \overset{\text{L’Hospital}}{=} \lim_{t \to t_m} \frac{2g^{-\frac{2c}{a}} \partial_t g^{-\frac{2c}{a}}}{\partial_t (Bt^{2(a-1)} f(1 + f)^{2(1-c)})} = \frac{2B}{3 - 2c}. \]

Lemma 2.4
which means \( \lim_{t \to t_m} \Theta(t) = 0 \). Then since \( \chi(t) > 0 \) (positivity by its definition (2.48)) is a continuous function in \([t_0, t_m]\), by Proposition B.2 (see Appendix B), we conclude the boundness and then complete the proof.

The second crucial quantity in the Fuchsian formulation is
\[
\xi(t) := 1/(g(t)(1 + f(t))),
\]
the next proposition prove \( \xi \) is bounded and its limit vanishes as \( t \) tends to \( t_m \).

**Proposition 2.8.** Suppose \( f \in C^2([t_0, t_m]) \) (where \([t_0, t_m]\) is the maximal interval of existence of \( f \) given by Theorem 1.1) solves ODE (1.5)–(1.6), \( g(t) \) is defined by (2.1) and \( \xi(t) \) is given by (2.51), then \( \xi \in C^1([t_0, t_m]) \) and
\[
\lim_{t \to t_m} \xi(t) = 0.
\]
Moreover, there is a constant \( C_* > 0 \), such that \( 0 < \xi(t) \leq C_* \) for every \( t \in [t_0, t_m] \), and there is a continuous extension of \( \xi \) such that \( \xi \in C^0([t_0, t_m]) \) by letting \( \xi(t_m) := 0 \).

**Proof.** By the definition (2.48) of \( \chi(t) \) and Proposition 2.7 (i.e., \( 0 < \chi \leq C_\chi \)),
\[
0 < g^{-\frac{2b}{\chi}}(t) \leq C_\chi Bf(t)(1 + f(t))^{2(1-\epsilon)t^{2(a-1)}}.
\]
Since \( 1 + f > f \), using (2.53), we arrive at
\[
\xi^{\frac{2b}{\chi}} = g^{-\frac{2b}{\chi}}(1 + f) < \frac{(1 + f)^{1-\frac{2b}{\chi}}}{g^{\frac{2b}{\chi}} f} \leq BC_\chi t^{2(a-1)}(1 + f)^{3-2c-\frac{2b}{\chi}}.
\]
That is,
\[
0 < \xi < B\frac{A^4}{\chi} C_\chi^{\frac{A}{\chi}} t^{(a-1)\epsilon}(1 + f)^{\frac{A}{\chi}(3-2c-\frac{2b}{\chi})}.
\]
Since \( A \in (0, 2\delta/(3-2c)) \) (recalling it in (2.1)) and \( c \in (1, 3/2) \), we have \( 3-2c-\frac{2b}{\chi} < 0 \). Then (1) if \( t_m < \infty \), directly using (2.54) and Theorem 1.1 (i.e., \( \lim_{t \to t_m} f(t) = +\infty \)), we conclude (2.52). (2) If \( t_m = \infty \), the right hand of (2.54) can be estimated, due to \( 3-2c-\frac{2b}{\chi} < 0 \) and (2.18) in Proposition 2.2, by
\[
 t^{A(a-1)\epsilon}(1 + f)^{\frac{A}{\chi}(3-2c-\frac{2b}{\chi})} < t^{A(a-1)\epsilon} \exp \left[ \frac{A}{2\delta} \left( 3 - 2c - \frac{2b}{\delta} \right) (t^{2\epsilon} + Dt^{-1}) \right].
\]
Then, using the fact \( \lim_{x \to \infty} (x^a / e^x) = 0 \),
\[
\lim_{t \to \infty} \left( t^{A(a-1)\epsilon} \exp \left[ \frac{A}{2\delta} \left( 3 - 2c - \frac{2b}{\delta} \right) (t^{2\epsilon} + Dt^{-1}) \right] \right) = \lim_{t \to \infty} \left( \frac{t^{A(a-1)\epsilon}}{\exp \left[ \frac{-A}{2\delta} \left( 3 - 2c - \frac{2b}{\delta} \right) (t^{2\epsilon} + Dt^{-1}) \right] } \right) = 0.
\]
This means the right hand side of (2.54) tends to 0 as \( t \to t_m \), which implies (2.52). Then since \( \xi(t) \) is a continuous function in \([t_0, t_m]\), by Proposition B.2 (see Appendix B), we conclude the boundness and then complete the proof.

3. The analysis of the perturbed solutions

This section contributes to the analysis of the main second order nonlinear hyperbolic equations (1.1)–(1.2) with parameters satisfying (1.3). Recalling it, this equation is
\[
\square \varrho(x^\mu) + \frac{a}{t} \partial_t \varrho(x^\mu) - \frac{\delta}{t^2} \varrho(x^\mu)(1 + \varrho(x^\mu)) - \frac{c}{1 + \varrho(x^\mu)} (\partial_t \varrho(x^\mu))^2 = \kappa F(t),
\]
\[
\varrho|_{t=t_0} = \hat{\varrho}(x^i) \quad \text{and} \quad \partial_t \varrho|_{t=t_0} = \hat{\partial}_0(x^i).
\]
The aim of this section is to prove the Theorem 1.2. Let us recall it here.
Theorem 3.1. Suppose \( s \in \mathbb{Z}_{\geq 2} + 3, a, b, c, k \) are constants satisfying (1.3), \( f \in C^2([t_0, t_m]) \) solves equation (1.5)–(1.6) where \( f > 0 \) and \( f_0 > 0 \) are given, and assume \( t_m > t_0 \) such that \([t_0, t_m]\) is the maximal interval of existence of \( f \) given by Theorem 1.1. Then there are small constants \( \sigma_*, \sigma > 0 \), such that if the initial data satisfies
\[
\left\| \frac{\partial \rho}{\rho} - 1 \right\|_{H^s(T^n)} + \left\| \frac{\partial \rho_0}{f_0} - 1 \right\|_{H^s(T^n)} + \left\| \frac{m \partial \rho}{1 + f} \right\|_{H^s(T^n)} \leq \frac{1}{2} \sigma_*, \sigma,
\]
then the solution to the equation (1.1) with data (1.2) exists in \( t \in [t_0, t_m] \) and \( \rho \) satisfies an estimate
\[
\left\| \frac{\partial \rho}{\rho} - 1 \right\|_{H^s(T^n)} + \left\| \frac{\partial \rho_0}{f_0} - 1 \right\|_{H^s(T^n)} + \left\| \frac{m \partial \rho}{1 + f} \right\|_{H^s(T^n)} \leq C \sigma < 1 \tag{3.3}
\]
for \( t \in [t_0, t_m] \) and some constant \( C > 0 \). Moreover, \( \rho \) blowups at \( t = t_m \), i.e.,
\[
\lim_{t \to t_m} \rho(t, x^i) = +\infty \quad \text{and} \quad \lim_{t \to t_m} \rho_0(t, x^i) = +\infty,
\]
with the rate estimates \((1 - C \sigma)f \leq \rho \leq (1 + C \sigma)f \) and \((1 - C \sigma)f_0 \leq \rho_0 \leq (1 + C \sigma)f_0 \).

Remark 3.0.1. The regularity \( s \in \mathbb{Z}_{\geq 2} + 3 \) on the initial data in this Theorem 3.1 may not be optimal since the proof does not take into account the semilinearity of this equations (3.1). As pointed out by [3, Remark 4.3], regularity improvements can be obtained by establishing global existence theorems of Fuchsian systems in semilinear version (analog to Appendix C) that is adapted to semilinear equations.

The basic idea of proving this theorem is to transform (3.1)–(3.2) to a Fuchsian formulation (see the model (C.1)–(C.2) in Appendix C), then, roughly speaking, using the results of this formulation (i.e., Theorem C.2 in Appendix C) to obtain the estimate (3.3). We will state the transform in next three steps (Step (1)–Step (3)) in the subsequent sections.

3.1. Step 1: Equations of the Perturbations. Let \( f(t) \) solves equation (1.5)–(1.6), then we denote
\[
w(t, x^i) := \rho(t, x^i) - f(t), \tag{3.4}
w_0(t, x^i) := \partial_i w(t, x^i) = \partial_i \rho(t, x^i) - f_0(t), \tag{3.5}
w_i(t, x^i) := \partial_i w(t, x^i) = \partial_i \rho(t, x^i). \tag{3.6}
\]
Substituting these new variables (3.4)–(3.6) into (3.1) and using the equation (1.5) of \( f \), we arrive at the following equation of perturbations
\[
\begin{align*}
\partial_i w_0 - g^{ij} \partial_j w_i + \frac{a}{t} w_0 - \frac{b}{l^2} (w + w^2 + 2f w) - \frac{c - k}{1 + w + f} w_0^2 \\
- \frac{2(c - k) f_0 w_0}{1 + w + f} + \frac{(c - k) f_0^2 w}{(1 + w + f)(1 + f)} = 0. \tag{3.7}
\end{align*}
\]
With direct calculations from (3.5) and (3.6), we obtain
\[
\begin{align*}
\partial_i w_i &= \partial_i \partial_i w = \partial_i w_0, \tag{3.8}
\partial_i w &= w_0. \tag{3.9}
\end{align*}
\]
Then, equations (3.7)–(3.9) constitute a first order system.
3.2. Step 2: Time transforms and Fuchsian formulations. In this section, we intend to rewrite the above (3.7)–(3.9) into the Fuchsian formulations (see (C.1) in Appendix C). In order to achieve this, since the Fuchsian formulations rely\(^4\) on a certain compactified time coordinate, we first have to introduce a proper time transform (the function \(g(t)\) given in §2.1 provides a decent candidate of this time transform). In addition, the Fuchsian formulations strongly depend on the variable selections (Fuchsian fields). A necessary condition to obtain the Fuchsian formulations is to select decent decaying Fuchsian fields.

To serve these aims, it turns out we have to introduce (i) a time transform,

\[
\tau := -g(t) = -\exp\left(-A\int_{t_0}^{t} \frac{f(s)(f(s)+1)}{s^2f_0(s)} ds\right)
\]

\[
= - \left(1 + B\int_{t_0}^{t} s^{a-2}f(s)(1+f(s))^{1-\epsilon} ds\right)^{-\frac{\phi}{2}} \in [-1, 0), \quad (3.10)
\]

where, recall that in §2.1, \(g(t)\) is a function defined by (2.1) (equivalently, by (2.3)) and \(A \in (0, 2B/(3-2\epsilon))\) is an arbitrary constant. This time transform \(\tau\) maps the initial time \(t = t_0\) to \(\tau = -1\) and the maximal time of existence \(t = t_m\) to \(\tau = 0\). In addition, we introduce (ii) rescaled Fuchsian fields as follows (recall \(m\) is a constant given in (1.4)),

\[
u(t, x^i) = \frac{1}{f(t)} w(t, x^i), \quad (3.11)
\]

\[
u_0(t, x^i) = \frac{1}{f_0(t)} w_0(t, x^i), \quad (3.12)
\]

\[
u_i(t, x^i) = \frac{m}{1+f(t)} w_i(t, x^i). \quad (3.13)
\]

Recall the notation conventions (1.14) in §1.4, \(u\) represents \(u\) in the compactified time coordinate \(\tau\), and in the following we will frequently use this notations, for example, we denote

\[
\underline{u}(\tau, x^i) = u(g^{-1}(-\tau), x^i), \quad \underline{u}_0(\tau, x^i) = u_0(g^{-1}(-\tau), x^i) \quad \text{and} \quad \underline{u}_i(\tau, x^i) = u_i(g^{-1}(-\tau), x^i). \quad (3.14)
\]

Under this time transform \((3.10)\), using (2.7), for later use, we arrive at\(^5\)

\[
\partial_\tau \underline{u}_\mu = -[g'(t)]^{-1} - \partial_t u_\mu = \frac{t^{2-a}(1+f)^{\epsilon^{-1}}}{ABg^{0.5}f^{0.5+1}} \partial_t u_\mu. \quad (3.15)
\]

(1) Rewrite (3.7): By (3.11)–(3.13), we obtain

\[
w = fu, \quad w_0 = f_0 u_0, \quad w_i = m^{-1}(1+f)u_i. \quad (3.16)
\]

Substituting \((3.16)\) into (3.7), we derive

\[
u_0 \partial_t f_0 + f_0 \partial_t u_0 - m^{-1}(1+f)g^{ij} \partial_j u_i + \frac{a}{t} f_0 u_0 - \frac{\epsilon}{t^2}(fu + (fu)^2 + 2f^2u)
\]

\[
- \frac{(c-K)(f_0u_0)^2}{1+fu+f} - \frac{2(c-K)f_0^2 u_0}{1+fu+f} + \frac{(c-K)uf_0^2}{(1+fu+f)(1+f)} = 0.
\]

\(^4\)We have remarked in our previous works [12–16], the key and most difficult steps to use this approach are to construct proper time compactifications and Fuchsian fields.

\(^5\)Note the underlines here have used the notation convention (1.14) in §1.4.
Then using the ODE (1.5) of \( f \) to substitute \( \partial_t f_0 \) in the above first term and noting the cancellation of terms \( at^{-1}f_0u_0 \), we obtain
\[
\frac{6}{t^2}f(1+f)u_0 + c\frac{f_0^2u_0}{1+f} + f_0\partial_t u_0 - m^{-1}(1+f)g^{ij}\partial_j u_i - \frac{6}{t^2}(fu + f^2u^2 + 2f^2u)
\]
\[
- \frac{(c-k)(f_0u_0)^2}{1+fu+f} - \frac{2(c-k)f_0^2u_0}{1+fu+f} + \frac{(c-k)uf f_0^2}{(1+fu+f)(1+f)} = 0. \tag{3.17}
\]

Multiplying \(1/(f_0g^{\frac{2}{a}+1})\) on the both sides of (3.17), it leads to
\[
-g^{\frac{2}{a}-1}\partial_t u_0 + \frac{(1+f)g^{ij}}{m f_0 g^{\frac{2}{a}+1}}\partial_j u_i = \frac{6f(1+f)u_0 + c\frac{f_0^2u_0}{1+f}}{g^{\frac{2}{a}+1}u_0} - \frac{6f}{g^{\frac{2}{a}+1}u_0}(u + fu^2 + 2fu)
\]
\[
- \frac{(c-k)f_0 g^{\frac{2}{a}-1}u_0^2}{1+fu+f} - \frac{2(c-k)f_0 g^{\frac{2}{a}-1}u_0^2}{1+fu+f} + \frac{(c-k)uf f_0 g^{\frac{2}{a}-1}}{(1+fu+f)(1+f)}. \tag{3.18}
\]

According to the definition (2.48) of \( \chi \) and the expression (2.2) of \( f_0 \), respectively, we obtain
\[
f_0 g^{\frac{2}{a}} = (1+f)^{2c-1}f^{a-2} \chi \quad \text{and} \quad 1/(f_0g^{\frac{2}{a}}) = B t^a (1+f)^{-c}. \tag{3.19}
\]

Multiplying \( t^{2-a}(1+f)^{c-1}/(ABf) \) on the both sides of (3.18) and using (3.19) to replace the corresponding terms in the equation (3.18), it reduces to
\[
- \frac{t^{2-a}(1+f)^{c-1}}{AB g^{\frac{2}{a}+1}f} \partial_t u_0 + \frac{t^{2}g^{ij}}{Ag f g}\partial_j u_i = \frac{6}{Ag} u_0 + c \frac{\chi u_0}{ABg} - \frac{6}{Ag(1+f)}u \frac{6f}{Ag(1+f)}u^2
\]
\[
- \frac{2\delta f}{Ag(1+f)u} - \frac{(c-k)(1+f)\chi u_0^2}{AB(1+f+fu)g} - \frac{2(c-k)(1+f)\chi u_0}{AB(1+f+fu)g} + \frac{(c-k)uf \chi}{AB(1+f+fu)g}. \tag{3.20}
\]

Then note the following simple identities from straightforward calculations,
\[
\frac{f}{1+f} = 1 - \frac{1}{1+f}, \quad \frac{1+f}{(1+f+fu)g} = 1 - \frac{\mathcal{R}(t,u)}{g}, \quad \frac{1+f}{(1+f)g} = 1 + \frac{\mathcal{R}(t,u)}{g},
\]
where
\[
C^0([t_0, t_m], C^\infty(B_R(0))) \ni \mathcal{R}(t,u) := \frac{u}{1+1/f+u}
\]
for some constant \( R > 0 \), and \( \mathcal{R}(t,0) = 0 \). Using these identities, with the help of the definition (2.51) of \( \xi \), the equation (3.20) becomes
\[
- \frac{t^{2-a}(1+f)^{c-1}}{AB g^{\frac{2}{a}+1}f} \partial_t u_0 + \frac{t^{2}g^{ij}}{Ag f g}\partial_j u_i = \frac{1}{Ag} \left( \delta + \frac{(2k-c)\chi}{B} \right) u_0 - \frac{(c-k)\chi u_0^2}{ABg}
\]
\[
+ \frac{(c-k)\chi u_0^2\mathcal{R}(t,u)}{ABg} + \frac{2(c-k)\chi u_0\mathcal{R}(t,u)}{ABg} + \frac{1}{Ag} \left( \frac{(c-k)\chi}{B} - 2\delta \right) u - \delta u^2
\]
\[
- \frac{(c-k)u \chi \mathcal{R}(t,u)}{ABg} + \frac{6\xi}{A} u^2 + \frac{6\xi}{A} u - \frac{(c-k)u \chi \xi}{AB} + \frac{(c-k)u \chi \xi \mathcal{R}(t,u)}{AB}. \tag{3.21}
\]

Using the definition (1.4) of \( g^{ij} \), the expression (2.2) of \( f_0 \) and the definition (2.48) of \( \chi \), we reach
\[
g^{ij} = m^2 f_0^2 \frac{1}{(1+f)^2} \delta^{ij} = m^2 B^{-2} t^{-2a} g^{-\frac{2}{a}}(1+f)^{2c-2} \delta^{ij} = m^2 B^{-1} f t^{-2} \chi \delta^{ij}. \tag{3.22}
\]
Then, with the help of Proposition 2.7, replacing $\chi$ by $\frac{2kB}{3-2} + \mathcal{G}(t)$ and using the expression (3.22) of $g^{ij}$, the equation (3.21) becomes

$$\begin{align*}
- \frac{t^{2-a}(1+f)^{-1}}{ABg^{\frac{k}{k+1}}f} \partial_t u_0 + \frac{m\chi}{ABg} \delta^{ij} \partial_j u_i \\
= \frac{1}{Ag} \left( \frac{26(2k-c)}{3-2c} + B^{-1}(2k-c)\mathcal{G}(t) + \mathcal{S}(t, u_0, u) \right) u_0 \\
+ \frac{1}{Ag} \left( \frac{26(c-k)}{3-2c} - 2\delta + B^{-1}(c-k)\mathcal{G}(t) + \mathcal{F}(t, u) \right) u + \mathcal{L}(t, u),
\end{align*}$$

(3.23)

where we denote

$$\begin{align*}
\mathcal{S}(t, u_0, u) := & - \frac{(c-k)\chi}{B} \left( u_0 + u_0\mathcal{H}(t, u) + 2\mathcal{H}(t, u) \right), \\
\mathcal{F}(t, u) := & - 6u + (c-k)B^{-1}\mathcal{H}(t, u), \\
\mathcal{L}(t, u) := & \frac{6\xi}{A} u^2 + \frac{6\xi}{A} - \frac{(c-k)u\chi\xi}{AB} + \frac{(c-k)u\chi\xi\mathcal{H}(t, u)}{AB}.
\end{align*}$$

(3.24) (3.25) (3.26)

We can verify, by using Propositions 2.7 and 2.8 on $\chi(t)$ and $\xi(t)$, that $\mathcal{S} \in C^0([t_0, t_m], C^\infty(B_R(0) \times B_R(0)))$ and $\mathcal{F}, \mathcal{L} \in C^0([t_0, t_m], C^\infty(B_R(0)))$, and $\mathcal{S}(t, 0, 0) = 0, \mathcal{F}(t, 0) = 0, \mathcal{L}(t, 0) = 0$.

Now let us transform the above equation (3.23) which is in terms of the independent variable $t$ to the one with the independent variable $\tau$ via $t = g^{-1}(-\tau)$. Using notations given in (3.14) and $\partial_{\tau} u_0$ given by (3.15), we arrive at

$$\begin{align*}
\partial_{\tau} u_0 + \frac{m\chi}{AB} \delta^{ij} \partial_j u_i = & \frac{1}{A\tau} \left( \frac{26(2k-c)}{3-2c} + B^{-1}(2k-c)\mathcal{G}(\tau) + \mathcal{S}(\tau, u_0, u) \right) u_0 \\
+ & \frac{1}{A\tau} \left( \frac{26(c-k)}{3-2c} - 2\delta + B^{-1}(c-k)\mathcal{G}(\tau) + \mathcal{F}(\tau, u) \right) u + \mathcal{L}(\tau, u),
\end{align*}$$

(3.27)

where, by (3.24)–(3.26) and notation conventions in §1.4.1, $\mathcal{G}(\tau) := \mathcal{G}(g^{-1}(-\tau)), \mathcal{H}(\tau, u) = \mathcal{H}(g^{-1}(-\tau), u(g^{-1}(-\tau), x^i))$ and

$$\begin{align*}
\mathcal{S}(\tau, u_0, u) := & - \frac{(c-k)\chi}{B} \left( u_0 + u_0\mathcal{H}(\tau, u) + 2\mathcal{H}(\tau, u) \right), \\
\mathcal{F}(\tau, u) := & - 6u + (c-k)B^{-1}\mathcal{H}(\tau, u), \\
\mathcal{L}(\tau, u) := & \frac{6\xi}{A} u^2 + \frac{6\xi}{A} - \frac{(c-k)u\chi\xi}{AB} + \frac{(c-k)u\chi\xi\mathcal{H}(\tau, u)}{AB}.
\end{align*}$$

(3.28) (3.29) (3.30)

We can verify, again by using Propositions 2.7 and 2.8 that $\xi(t)$ and $\chi(t)$ are continuous and bounded, that $\mathcal{S} \in C^0([-1, 0], C^\infty(B_R(0) \times B_R(0)))$, and $\mathcal{F}, \mathcal{L} \in C^0([-1, 0], C^\infty(B_R(0)))$, and $\mathcal{S}(\tau, 0, 0) = 0, \mathcal{F}(\tau, 0) = 0, \mathcal{L}(\tau, 0) = 0$.

(2) Rewrite (3.8): Inserting (3.16) into (3.8), we derive

$$\partial_{\tau} u_i - \frac{mf_0}{1+f} \partial_{\tau} u_0 = - \frac{f_0}{1+f} u_i.$$ 

Multiplying it by $-1/(ABt^{a-2}g^\frac{k-1}{k+1}(1+f)^{1-c})$ and using (2.2) to substitute $f_0$ yield

$$\begin{align*}
- \frac{t^{2-a}(1+f)^{1-c}}{ABg^{\frac{k}{k+1}}f} \partial_{\tau} u_i + \frac{mt^{2-a}(1+f)^{2c-2}}{AB^2g^{\frac{k}{k+1}}f} \partial_{\tau} u_0 = \frac{t^{2-a}(1+f)^{2c-2}}{AB^2g^{\frac{k}{k+1}}f} u_i.
\end{align*}$$
This, by applying the definition (2.48) of \( \chi \), implies
\[
- \frac{t^{2-a}(1+f)^{c-1}}{ABg^\frac{a}{a+1}f} \partial_t u_i + \frac{mX}{ABg} \partial_i u_0 = \frac{\chi}{ABg} u_i.
\]
Then, with the help of (3.15), transforming into the independent variable \( \tau \) via \( t = g^{-1}(-\tau) \), representing above equation in terms of variables (3.14), using the definition (2.48) of \( \chi \) and multiplying it by \( \delta^{ij} \), we arrive at
\[
\delta^{ij} \partial_{\tau} u_i + \frac{mX}{AB\tau} \delta^{ij} \partial_t u_0 = \frac{1}{\tau AB} \delta^{ij} u_i.
\] (3.31)

(3) Rewrite (3.9): Inserting (3.16) into (3.9), straightforward calculations yield
\[
\partial_t u = \frac{f_0}{f} u_0 - \frac{f_0}{f} u.
\]
Multiplying it by \(-1/(AB^{a-2}g^\frac{a}{a+1}f(1+f)^{-c})\) and using the definition (2.48) of \( \chi \) to substitute corresponding terms yield
\[
- \frac{t^{2-a}(1+f)^{c-1}}{ABg^\frac{a}{a+1}f} \partial_t u = \frac{1}{g} \left( - \frac{1}{AB} \frac{f + 1}{f} \lambda u_0 + \frac{1}{AB} \frac{f + 1}{f} \chi u \right).
\] (3.32)

Then according to the definition (2.51) of \( \xi \), we derive
\[
\frac{1}{fg} = \xi \left( 1 + \frac{1}{f} \right).
\] (3.33)

With the help of (3.33) and Proposition 2.7, the equation (3.32) becomes
\[
- \frac{t^{2-a}(1+f)^{c-1}}{ABg^\frac{a}{a+1}f} \partial_t u = \frac{1}{Ag} \left( \left( - \frac{2\xi}{3-2c} - \frac{1}{B} \mathcal{G}(t) \right) u_0 + \left( \frac{2\xi}{3-2c} + \frac{1}{B} \mathcal{G}(t) \right) u \right) + \mathcal{R}(t, u_0, u)
\]
where
\[
\mathcal{R}(t, u_0, u) := - \frac{1}{AB} \chi \xi \left( 1 + \frac{1}{f} \right) u_0 + \frac{1}{AB} \chi \xi \left( 1 + \frac{1}{f} \right) u.
\]
Then, with the help of (3.15), transforming into the independent variable \( \tau \) via \( t = g^{-1}(-\tau) \), representing above equation in terms of variables (3.14), we arrive at
\[
\partial_{\tau} u = \frac{1}{A\tau} \left( \left( - \frac{2\xi}{3-2c} - \frac{1}{B} \mathcal{G}(\tau) \right) u_0 + \left( \frac{2\xi}{3-2c} + \frac{1}{B} \mathcal{G}(\tau) \right) u \right) - \mathcal{R}(\tau, u_0, u)
\] (3.34)
where
\[
\mathcal{R}(\tau, u_0, u) := - \frac{1}{AB} \chi \xi \left( 1 + \frac{1}{f} \right) u_0 + \frac{1}{AB} \chi \xi \left( 1 + \frac{1}{f} \right) u.
\] (3.35)

We can also verify, again by using Propositions 2.7 and 2.8 on \( \xi(t) \) and \( \chi(t) \), that \( \mathcal{R} \in C^0([-1, 0], C^\infty(B_R(0) \times B_R(0))) \).

3.3. Step 3: Fuchsian formulations. Gather (3.27), (3.31) and (3.34) together, we arrive at the following system,
\[
\mathcal{B}^0 \partial_{\tau} u + \mathcal{B}^i \partial_j u = \frac{1}{\tau} \mathcal{B} \mathcal{P} u + \mathcal{H}
\] (3.36)
where \( u := (u_0, u_1, u)^T \), \( \mathcal{H} := \mathcal{H}(\tau, u_0, u) = (-\mathcal{G}(\tau, u), 0, -\mathcal{R}(\tau, u_0, u))^T \), \( \mathcal{P} := 1 \),
\[
\mathcal{B}^0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^{ki} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{B}^i := \frac{mX}{AB\tau} \begin{pmatrix} 0 & \delta^{ij} & 0 \\ \delta^{kj} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
\[ B := \frac{1}{A} \left( \begin{array}{ccc}
\delta + (2k - c)(\frac{26}{3-2k} + \frac{\delta}{B}) + \delta & 0 & -2\delta + (c - k)(\frac{26}{3-2k} + \frac{\delta}{B}) + \delta \\
0 & (\frac{26}{3-2k} + \frac{\delta}{B})\delta^{ki} & 0 \\
-(\frac{26}{3-2k} + \frac{\delta}{B}) & 0 & (\frac{26}{3-2k} + \frac{\delta}{B})
\end{array} \right), \]

\[ \mathcal{B} := \mathcal{B}(\tau), \quad \mathfrak{G} := \mathfrak{G}(\tau, u), \quad \mathfrak{B} := \mathfrak{B}(\tau, u_0, u), \quad \mathfrak{C} := \mathfrak{C}(\tau, u) \text{ and } \mathfrak{R} := \mathfrak{R}(\tau, u_0, u) \text{ are defined by (3.28)–(3.30) and (3.35).} \]

3.4. Step 4: Verifications of Fuchsian system. In this section (Step 4), let us first verify (3.36) satisfies all the conditions (i.e., Conditions (I)–(VI) in Appendix C) of a Fuchsian system (C.1) for \( \tau > \tau_0 \) close to 0. Once we verified these conditions, we can then apply Theorem C.2 to conclude the solution exists near \( \tau = 0 \), i.e., \( \tau \in [\tau_0, 0) \) (Step 5). Then by letting the initial data small enough, we, by the continuation principle, extend the local solution beyond \( \tau_0 \), then combing with above near \( \tau = 0 \) solution, we obtain a global solution (Step 6). At the end, by transforming the Fuchsian fields to the origin variables \( \varrho \) and \( \partial_t \varrho \) (Step 7), we conclude Theorem 1.2.

In (3.36), corresponding to the system (C.1)–(C.2), we have \( P = \mathbb{P} = 1 \), \( B^0 = \mathfrak{B}^0 = 1 \) and \( B^i_2(\tau) = \tau B^i(\tau) \) is independent of \( (x', u) \) and \( B^0_0 = 0 \), and there is

\[ \mathfrak{B} = \mathfrak{B}^i = \frac{1}{A} \left( \begin{array}{ccc}
\delta + (2k - c)(\frac{26}{3-2k} + \frac{\delta}{B}) & 0 & -2\delta + (c - k)(\frac{26}{3-2k} + \frac{\delta}{B}) \\
0 & (\frac{26}{3-2k} + \frac{\delta}{B})\delta^{ki} & 0 \\
-(\frac{26}{3-2k} + \frac{\delta}{B}) & 0 & (\frac{26}{3-2k} + \frac{\delta}{B})
\end{array} \right), \]

such that

\[ \mathfrak{B}(\tau, u) - \mathfrak{B}(\tau) \equiv \mathfrak{B}(\tau, u) - \mathfrak{B}(\tau) = O(1) \]

for all \( (t, u) \in [T_0, 0] \times B_R(\mathbb{R}^{n+2}) \). In this case, we find the parameter \( b \) defined in Theorem C.2 satisfying \( b = 0 \) (since \( \mathfrak{B} \) and \( \mathfrak{B} \) are only \( \tau \) dependent). Using this settings, there is a constant \( R > 0 \) (shrinking it if necessary), such that it is straightforward that the conditions (I)–(III) and (V) are verified by noting the fact \( \mathbb{P} = \mathfrak{B}^i = \mathfrak{B}_0 = 0 \) (since \( \mathfrak{B}_0 \) is independent of \( u, \Theta(s) \equiv B^0 \equiv 1 \), further \( \Theta'(0) \equiv 0 \), we then verify (VI). Moreover, from this, we conclude \( \beta_{2k+1} = 0 \) for all \( \ell = 0, 1, 2, 3 \). Further, we obtain \( \gamma_1 \max\{\sum_{\ell=0}^{3} \beta_{2k+1}, \beta_1 + 2k(k + 1)b\} = 0 \), then we only need \( \kappa > 0 \) to make sure \( \kappa > (1/2)\gamma_1 \max\{\sum_{\ell=0}^{3} \beta_{2k+1}, \beta_1 + 2k(k + 1)b\} \).

In the following, let us verify Condition (IV). In order to do so, we have to verify that there exists constants \( \kappa, \gamma_2 \) and \( \gamma_1 = 1 \) such that

\[ 1 = \mathfrak{B}^0 \leq \frac{1}{\kappa} \mathfrak{B} \leq \gamma_2 1 \quad \text{i.e.,} \quad \zeta^T 1 \zeta \leq \frac{1}{\kappa} \zeta^T \mathfrak{B} \zeta \leq \gamma_2 \zeta^T 1 \zeta \]

for all \( (\tau, u) \in [-1, 0] \times B_R(\mathbb{R}^{n+2}) \) and \( \zeta \in \mathbb{R}^{n+2} \). However, in the following we will see this can not be achieved but we can still prove it for a much smaller region of \( (\tau, u) \).

Let us first note there is a symmetric matrix \( B \) defined by

\[ B := \frac{1}{A} \left( \begin{array}{ccc}
\delta + (2k - c)(\frac{26}{3-2k} + \frac{\delta}{B}) & 0 & -2\delta + (c - k)(\frac{26}{3-2k} + \frac{\delta}{B}) + \delta \\
0 & (\frac{26}{3-2k} + \frac{\delta}{B})\delta^{ki} & 0 \\
-(\frac{26}{3-2k} + \frac{\delta}{B}) & 0 & (\frac{26}{3-2k} + \frac{\delta}{B})
\end{array} \right), \]

such that

\[ \zeta^T \mathfrak{B} \zeta = \zeta^T B \zeta. \]
Since $B$ is a real symmetric matrix, there is an orthogonal matrix $\Omega$, such that

$$B = \Omega^T \Lambda \Omega$$

where $\Lambda := \frac{1}{A} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 \delta^{ij} & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$,

and the eigenvalues are

$$\lambda_1 := \tilde{\lambda}_1 + 3_1(G, \tilde{\Phi}, \tilde{\Phi}) = \frac{6(4c - 4k - 5 - \tilde{\Delta})}{2(2c - 3)} + 3_1(G, \tilde{\Phi}, \tilde{\Phi}),$$

$$\lambda_2 := \tilde{\lambda}_2 + 3_2(G, \tilde{\Phi}, \tilde{\Phi}) = \frac{b(4c - 4k - 5 + \tilde{\Delta})}{2(2c - 3)} + 3_2(G, \tilde{\Phi}, \tilde{\Phi}),$$

$$\lambda_3 := \tilde{\lambda}_3 + \frac{26}{3 - 2c} + \frac{G}{B},$$

where $3_\ell := 3_\ell(G, \tilde{\Phi}, \tilde{\Phi})$ ($\ell = 1, 2$) are lengthy expressions satisfying $3_\ell$ are analytic in all their variables and $3_\ell(0, 0, 0) = 0$. Since $G \in C^0([-1, 0]),$ $\tilde{\Phi} \in C^0([-1, 0], C^\infty(B_R(0) \times B_R(0)))$, $\tilde{\Phi} \in C^0([-1, 0], C^\infty(B_R(0)))$, and $3_\ell(\tau, 0, 0) = 0$, $\tilde{\Phi}(\tau, 0) = 0$, and by Lemma 2.6 and Proposition 2.7 (recall $\lim_{t \to t_m} G(t) = 0$ and by continuity of $g$ and $g^{-1}$, $\lim_{t \to 0} g^{-1}(-\tau) = g^{-1}(0) = t_m$),

$$\lim_{\tau \to 0^-} G(-\tau) = 0,$$

we conclude $\tilde{3}(\tau, u_0, u) := 3_\ell(G(\tau), \tilde{\Phi}(\tau, u_0, u), \tilde{\Phi}(\tau, u))$ is continuous in $(\tau, u_0, u) \in [-1, 0] \times B_R(0) \times B_R(0)$ and

$$\lim_{(\tau, u_0, u) \to (0, 0, 0)} \tilde{3}(\tau, u_0, u) = 0.$$

Since $\delta > 0$, $1 < c < 3/2$ and $3c - \sqrt{2\sqrt{8c - 5} - k} < k < 3c + \sqrt{2\sqrt{8c - 5}}$, we have

$$\tilde{\lambda}_{1,2} = \frac{6(4c - 4k - 5 + \tilde{\Delta})}{2(2c - 3)} > 0 \quad \text{and} \quad \tilde{\lambda}_3 = \frac{26}{3 - 2c} > 0.$$

Therefore, by (3.37) and (3.38), if we take $\varepsilon$ satisfying

$$0 < \varepsilon < \min\{\tilde{\lambda}_m | m = 1, 2, 3\},$$

then there are small enough constants $\tau_0 \in (-1, 0)$ and $\tilde{R} \in (0, R)$, such that if $\tau \in (\tau_0, 0)$ and $|(u_0, u)| < \tilde{R}$, then $|3_\ell| < \varepsilon$ and $|G/B| < \varepsilon$. Furthermore, let a set

$$\Gamma := \{(\tau, u_0, u) | \tau \in (\tau_0, 0), \; |(u_0, u)| < \tilde{R}\},$$

we obtain if $(\tau, u_0, u) \in \Gamma$, then $0 < \tilde{\lambda}_m - \varepsilon < \lambda_m < \tilde{\lambda}_m + \varepsilon$ $(m = 1, 2, 3)$. In this case, if taking

$$0 < \kappa \leq \frac{1}{A} \min \{\tilde{\lambda}_m - \varepsilon\} \quad \text{and} \quad \gamma_2 \geq \frac{1}{\kappa A} \max \{\tilde{\lambda}_m + \varepsilon\},$$

we have

$$(\Omega \zeta)^T \mathbb{1} \Omega \zeta \leq \frac{1}{\kappa} (\Omega \zeta)^T \Lambda \Omega \zeta \leq \gamma_2 (\Omega \zeta)^T \Omega \zeta,$$

i.e.,

$$\zeta^T \mathbb{1} \zeta \leq \frac{1}{\kappa} \zeta^T \mathbb{B} \zeta \leq \gamma_2 \zeta^T \mathbb{1} \zeta$$

for any $\tau \in (\tau_0, 0), \; |(u_0, u)| < \tilde{R}$ and $\zeta \in \mathbb{R}^{n+2}$. Therefore, we verified the system (3.36) is a Fuchsian system given by (C.1) in Appendix C for $(\tau, x') \in (\tau_0, 0) \times \mathbb{T}^n$ and $u \in B_{\tilde{R}}(\mathbb{R}^N)$. 
3.5. Step 5: The existence of the solution near $\tau = 0$. After verifying that (3.36) is a Fuchsian system for $(\tau, x^i) \in (\tau_0, 0) \times \mathbb{T}^n$ and $U \in B \mathcal{R}(\mathbb{R}^N)$, we are able to directly use Theorem C.2, that is, there exist small constants $\sigma, \sigma_0 \in (0, R/\kappa_s)$ and $\sigma < \sigma_0$, such that if

$$
\| U(\tau_0) \|_{H^s} \leq \sigma \quad \text{(i.e.} \quad \| U(\tau_0) \|_{L^\infty} \leq \kappa_s \| U(\tau_0) \|_{H^s} \leq \kappa_s \sigma < \bar{R},
$$

then there exists a unique solution

$$
U \in C^0([\tau_0, 0), H^s(\mathbb{T}^n)) \cap C^1([\tau_0, 0), H^{s-1}(\mathbb{T}^n)) \cap L^\infty([\tau_0, 0), H^k(\mathbb{T}^n))
$$

(3.39)
of the initial value problem (C.1)–(C.2). Moreover, for $\tau_0 \leq \tau < 0$, the solution $U$ satisfies the energy estimate

$$
\| U(\tau) \|_{H^s} \leq C(\sigma_0, \sigma_0^{-1}) \| U(\tau_0) \|_{H^s} < C_1 \sigma
$$

(3.40)

where we take $C_1 > \max\{C(\sigma_0, \sigma_0^{-1}), 1\}$. By shrinking $\sigma > 0$, we are able to make sure $C_1 \sigma < 1$ and $\| U(\tau) \|_{L^\infty} \leq \kappa_s C_1 \sigma < \bar{R}$.

3.6. Step 6: The existence of the solution for $\tau \in [-1, 0)$. Noting when $\tau \in [-1, \tau_0)$, the system (3.36) may not be a Fuchsian system since, from Step 4, Condition (IV) may be violated and thus we can not use Theorem C.2 to prove $U$ exists for $\tau \in [-1, 0)$. In order to solve this problem, in this step, let us prove the following lemma to conclude the global solution in $\tau \in [-1, 0)$.

**Lemma 3.1.** Let the constant $\sigma > 0$ and $C_1$ given by above Step 5 and $\kappa_s$ is the Sobolev constant from Theorem B.1, then there is a small constant $\sigma_\ast \in (0, 1)$, such that if the data $\| U(-1) \|_{H^s} \leq \sigma_\ast$, there exists a classical solution $U \in C^1([-1, 0) \times \mathbb{T}^n)$ of the system (3.36), and

$$
U \in C^0([-1, 0), H^s) \cap C^1([-1, 0), H^{s-1}).
$$

Moreover, there is a constant $C > 1$, such that, for $\tau \in [-1, 0)$,

$$
\| U(\tau) \|_{W^{1, \infty}} \leq 2\kappa_s \| U(\tau) \|_{H^s} \leq C \sigma.
$$

(3.41)

**Proof.** Firstly, according to the assumption of this lemma, note the initial data $\| U(-1) \|_{H^s} < \sigma/2$. Then by using Theorem B.1 (by shifting the time coordinate properly), for the constant $R_1 := \max\{\sigma, 2\kappa_s\sigma\} > \kappa_s \sigma/2 > \kappa_s \| U(-1) \|_{H^s}$, there is a $\tau^\ast \in (-1, 0]$ which we assume it is the maximal time, such that there exists a classical solution $U \in C^1([-1, \tau^\ast) \times \mathbb{T}^n)$ of the system (3.36) with the bound

$$
\sup_{(\tau, x^i) \in [\tau, \tau^\ast) \times \mathbb{T}^n} | U(\tau, x^i) | \leq R_1,
$$

and

$$
U \in C^0([-1, \tau^\ast), H^s) \cap C^1([-1, \tau^\ast), H^{s-1}).
$$

We can claim (1) either $\| U(\tau) \|_{H^s} < \sigma$ for all $\tau \in [-1, \tau^\ast)$ or (2) there exists a first time $\tau_\ast \in [-1, \tau^\ast)$ such that $\| U(\tau_\ast) \|_{H^s} = \sigma$. Setting $\tau_\ast = \tau^\ast$ if the first case holds, then, by Sobolev’s inequality ($\| U(\tau) \|_{W^{1, \infty}} \leq 2\kappa_s \| U(\tau) \|_{H^s}$), we have, for $\tau \in [-1, \tau_\ast)$, that

$$
\max\{\| U(\tau) \|_{W^{1, \infty}}, \| U(\tau) \|_{H^s}\} < R_1.
$$

(3.42)

For the first case (1), we claim $\tau^\ast = 0$, otherwise, we assume $\tau^\ast < 0$, and since $\| U(\tau) \|_{W^{1, \infty}} \leq 2\kappa_s \| U(\tau) \|_{H^s} < R_1 < \infty$ for all $\tau \in [-1, \tau^\ast)$, then by the continuation principle (see [17, Theorem 2.2]) $U$ can be extended beyond $\tau^\ast$ to a larger interval $[-1, \tau')$ where $\tau' > \tau^\ast$. This contradicts with the maximality of $\tau^\ast$. Thus $\tau^\ast = 0$. In this case, we have proved this lemma.
Let us focus on the second case (2) in the rest of the proof. Since (3.36) is a special symmetric hyperbolic system satisfying conditions given by Theorem C.1 in Appendix C, using this theorem, we conclude the solution \( U \in C^1([-1, \tau_*] \times \mathbb{T}^n) \) to (3.36) satisfies an energy estimate

\[
\|U(\tau)\|_{H^s} \leq \|U(-1)\|_{H^s} + c_1 e^{c_2(\tau+1)}
\]

for \( \tau \in [-1, \tau_*] \) where \( c_1 := c_1(\|U\|_{L^\infty([-1, \tau_*], W^{1, \infty})}) \) and \( c_2 := c_2(\|U\|_{L^\infty([-1, \tau_*], L^\infty)}) \) are two finite positive constants due to the bound (3.42). Take \( \sigma_* \) satisfying

\[
0 < \sigma_* \leq e^{-c_2((1+\tau_\delta)(-\tau_\delta))c_1(R_1)}
\]

where \( \tau_\delta \) is defined in Step 4 and 5. Then if we further let the initial data \( \|U(-1)\|_{H^s} \leq (1/2)\sigma_*\sigma \), then let us claim:

\[
\tau_* > \tau_\delta.
\]

We prove it by contradiction and assume \( \tau_* \leq \tau_\delta \), then for \( -1 < \tau < \tau_* \leq \tau_\delta < 0 \), by the Sobolev embedding theorem, the estimate (3.43) for \( \tau \in [-1, \tau_*] \) and noting \( c_\ell = c_\ell(\|U\|_{L^\infty([-1, \tau_*], W^{1, \infty})}) \leq c_\ell(R_1)(\ell = 1, 2) \), i.e., \( (-\tau_\delta)c_1(R_1) \leq (-\tau_\delta)c_1(R_1) \leq (-\tau)^c_1 \) and \( e^{-c_2((1+\tau_\delta))c_1(R_1)} \leq e^{-c_2((1+\tau_\delta))c_1(R_1)} \leq e^{-c_2((1+\tau_\delta))c_1(R_1)} \leq \frac{1}{2} \sigma \)

for \( \tau \in [-1, \tau_*] \). Then \( \|U(\tau_*)\|_{H^s} \leq \sigma/2 \neq \sigma \). This result contradicts with the definition of \( \tau_* \). Hence, \( \tau_* > \tau_\delta \). From this we know \( \|U(\tau_\delta)\|_{H^s} < \sigma \) by the definition of \( \tau_* \). Combining the existence interval in last Step 5 that there is a solution \( U \) for \( \tau \in [\tau_\delta, 0] \)

Moreover, there is a constant \( C > 1 \), such that, for \( \tau \in [-1, 0] \),

\[
\|U(\tau)\|_{W^{1, \infty}} \leq 2\kappa_s \|U(\tau)\|_{H^s} \leq C\sigma.
\]

We complete the proof.

3.7. Step 7: Proofs of Theorem 1.2.

Proof of Theorem 1.2. Using (3.4)–(3.6) and (3.11)–(3.13), we arrive at the transforms

\[
\varrho(t, x^i) = f(t) + f(t)u(t, x^i), \quad \partial_t \varrho(t, x^i) = f_0(t) + f_0(t)u_0(t, x^i)
\]

and

\[
\partial_t \varrho(t, x^i) = \frac{1 + f(t)}{m} u_i(t, x^i).
\]

Taking \( \sigma, \sigma_* > 0 \) defined as above, if the data satisfies (1.11), then by using this transform (3.44)–(3.45), with the help of (3.14) and \( U = (u_0, u_i, u)^T \), (1.11) implies

\[
\|(u_0, u_i, u)\|_{t=t_0, H^s} = \left\| \frac{\partial \varrho}{f} - 1 \right\|_{H^s(T^n)} + \left\| \frac{\partial \varrho}{f_0} - 1 \right\|_{H^s(T^n)} + \left\| \frac{m \varrho_i}{1 + f} \right\|_{H^s(T^n)} \leq \frac{1}{2} \sigma_* \sigma,
\]

that is, (1.11) is equivalent to \( \|U(-1)\|_{H^s} \leq (1/2)\sigma_* \sigma \). By Lemma 3.1 in Step 6 and using (3.41), (3.14) and \( U = (u_0, u_i, u)^T \), we obtain for \( t \in [t_0, t_m] \),

\[
\|(u_0, u_i, u)\|_{W^{1, \infty}} \leq 2\kappa_s \|(u_0, u_i, u)\|_{H^s} \leq C\sigma.
\]

\[\text{Note} \ \tau_\delta \in (-1, 0) \text{ is close to 0. Hence } \sigma_* \text{ is also small.}\]
Then by using transforms (3.44)–(3.45), Under this initial data, the estimate (3.46) becomes
\[
\begin{align*}
\|\frac{\partial}{f} - 1\|_{W^{1,\infty}(T^n)} + \|\frac{\partial^2}{f_0} - 1\|_{W^{1,\infty}(T^n)} + \|\frac{m\partial_t}{1 + f}\|_{W^{1,\infty}(T^n)} \\
\leq 2\kappa \left(\|\frac{\partial}{f} - 1\|_{H^s(T^n)} + \|\frac{\partial^2}{f_0} - 1\|_{H^s(T^n)} + \|\frac{m\partial_t}{1 + f}\|_{H^s(T^n)}\right) \leq C\sigma
\end{align*}
\]
for \( t \in [t_0, t_m] \). This further implies
\[
\|\frac{\partial}{f} - f\|_{L^\infty} \leq C\sigma f, \quad \|\partial_t f - f_0\|_{L^\infty} \leq C\sigma f_0 \quad \text{and} \quad \|\partial_t\|_{L^\infty} \leq C\sigma f.
\]
As we mentioned before, by shrinking \( \sigma \), we make sure \( C\sigma < 1 \), then
\[
0 < (1 - C\sigma)f \leq \frac{\partial}{f} \leq (1 + C\sigma)f \quad \text{and} \quad (1 - C\sigma)f_0 \leq \partial_t f \leq (1 + C\sigma)f_0.
\]
By taking the limit as \( t \to t_m \) and by (1.10) in Theorem 1.1, we reach
\[
\lim_{t \to t_m} \phi(t, x^i) = +\infty \quad \text{and} \quad \lim_{t \to t_m} \partial_t \phi(t, x^i) = +\infty.
\]
We then complete the proof of this theorem. \( \square \)

APPENDIX A. PRELIMINARIES ON ORDINARY DIFFERENTIAL EQUATIONS

In this section, we review some fundamental theorems on the existences, uniqueness and continuation theorems of the ordinary differential equations (ODEs) without proofs which can be found in various references of ODEs, for example, see [7,8]. Let \( f \) be a continuous function in a neighborhood of \((t_0, y_0) \subset \mathbb{R} \times \mathbb{R}^n \), with values in \( \mathbb{R}^n \). We focus on the following initial value problem
\[
\frac{dy(t)}{dt} = f(t, y(t)); \quad y(t_0) = y_0 \tag{A.1}
\]
for \( t \) in a neighborhood of \( t_0 \). For given \( a > 0, b > 0 \) and \( M > 0 \), let
\[
D := \{(t, y) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \leq a, |y - y_0| \leq b\}.
\]
We assume \( f \) is defined in \( D \) and there is a constant \( M > 0 \), such that
\[
|f(t, y)| \leq M, \quad \text{when} \quad (t, y) \in D. \tag{A.2}
\]

**Theorem A.1** (Existences and uniqueness of ODEs). Assume (A.2) and the Lipschitz condition
\[
|f(t, y) - f(t, z)| \leq C|y - z|, \quad \text{if} \quad |t - t_0| \leq a, |y - y_0| \leq b, |z - y_0| \leq b.
\]
Then there is a unique \( C^1 \) solution of the initial value problem (A.1) for \( |t - t_0| \leq T \) if \( T \leq \min\{a, b/M\} \).

**Theorem A.2** (Continuation of solutions). Let \( f \in C(D) \) and satisfy (A.2). Suppose \( \phi \) is a solution of (A.1) on the interval \( J = (a, b) \). Then
\begin{enumerate}
  \item \( \lim_{t \to a^+} \phi(t) = \phi(a) \) and \( \lim_{t \to b^-} \phi(t) = \phi(b) \) both exist and finite.
  \item if \( (b, \phi(b)) \in D \), then the solution \( \phi \) can be continued to the right passing through the point \( t = b \).
\end{enumerate}

**Corollary A.1** (Continuation principle). Let \( f \in C(D) \). Suppose \( \phi \) is a solution of (A.1) on the interval \( J = (a, b) \), and if there is a finite constant \( M > 0 \), such that for every \( t \in (a, b) \),
\[
|f(t, \phi(t))| \leq M < +\infty,
\]
then the solution \( \phi \) can be continued to the right passing through the point \( t = b \).
APPENDIX B. TOOLS OF ANALYSIS

B.1. Calculus. The following lemma and proposition generalize the well-known result “continuous functions on a compact set are bounded”.

Lemma B.1. If \( y = f(x) \) is continuous in \( x \in [a, \infty) \) and \( \lim_{x \to \infty} f(x) = A \) where \( A < \infty \) is a finite constant, then \( y = f(x) \) is bounded in \( [a, \infty) \).

Proof. The proof of the following Lemma B.1 is straightforward, for example, by the definition of continuity or introducing a new variable \( x = 1/\sqrt{y-(1-a)} \) for \( y \in (0, 1) \) and letting \( f(x(y))|_{y=0} := A \) to transform this statement to continuous functions on the compact set. We omit the detailed proof.

Combining above Lemma B.1 and the result that “continuous functions on a compact set are bounded” together, we obtain the following proposition.

Proposition B.2. If \( y = f(x) \) is continuous in \( x \in [a, b) \) where the constant \( b > a \) is finite or infinite, and \( \lim_{x \to b-} f(x) = A \) where \( A < \infty \) is a finite constant, then \( y = f(x) \) is bounded in \( [a, b) \).

B.2. The local existence for quasilinear symmetric hyperbolic systems. This appendix contributes to the local existence theorem for quasilinear symmetric hyperbolic systems without proofs. Detailed proofs can be found in, for example, [17, 21, 25]. The following theorem is given by [21, §5].

Let us consider the initial value problem,

\[
A^0(t,x,u)\partial_t u + A^j(t,x,u)\partial_j u + B(t,x,u)u = 0, \tag{B.1}
\]

\[
u|_{t=0} = u_0 \tag{B.2}
\]

where \( u = (u_1, \ldots, u_N) \in \mathbb{C}^N \), \( u = u(t,x) \), \( t \in \mathbb{R} \), \( x \in \mathbb{M} \) (\( \mathbb{M} = \mathbb{R}^n \) or \( \mathbb{T}^n \)), \( A^\mu, B \) are complex \( N \times N \)-matrices and \( C^\infty \)-functions of their arguments \( v \in \mathbb{C}^N \). \( A^\mu \) is Hermitian and \( A^0 \) is positive definite, uniformly in each compact set with respect to \( u \).

Theorem B.1 (Local existence theorem). Suppose \( u_0 \in H^s \), \( s \in \mathbb{Z}_{>n/2+1} \). Let \( q_1 := \kappa_s \|u_0\|_{H^s} \) where \( \kappa_s \) denote the Sobolev constant\(^7 \) and \( q_2 > q_1 \) arbitrary but fixed. Then there is a \( T > 0 \) such that there exists a classical solution \( u \in C^1([0,T] \times \mathbb{M}) \) of the initial value problem (B.1)–(B.2) with the bound

\[
\sup_{(t,x) \in [0,T] \times \mathbb{M}} |u(t,x)| \leq q_2,
\]

and

\[ u \in C^0([0,T],H^s) \cap C^1([0,T],H^{s-1}). \]

where \( T \) is a function of \( \|u_0\|_{H^s} \) and \( q_2 \).

APPENDIX C. CAUCHY PROBLEMS FOR FUCHSIAN SYSTEMS

In this Appendix, we introduce the main tool for the analysis of this article which is a variation of the theorem originally established in [20, Appendix B] and significantly developed in [3]. The proof has been omit, but readers can find the detailed proofs in [3] for a very general case beyond the original theorem. Its other generalizations and applications can be find in, for example, [2, 12–16].

\(^7\)\( \kappa_s \) denote the Sobolev constant characterizing the continuous imbedding of \( H^s \) into the space of uniformly bounded, continuous function if \( s > n/2 \), i.e., \( |u(x)| \leq \kappa_s \|w\|_{H^s} \) for \( w \in H^s \). see details in [21, §5].
Consider the following symmetric hyperbolic system:

\[ B^\mu(t, x, u) \partial_\mu u = \frac{1}{t} B(t, x, u) P u + H(t, x, u) \quad \text{in } [T_0, T_1) \times \mathbb{T}^n, \quad (C.1) \]

\[ u = u_0 \quad \text{in } \{T_0\} \times \mathbb{T}^n, \quad (C.2) \]

where \( T_0 < T_1 \leq 0 \) and we require the following Conditions\(^8\):

(I) \( P \) is a constant, symmetric projection operator, i.e., \( P^2 = P, \quad P^T = P \) and \( \partial_\mu P = 0 \).

(II) \( u = u(t, x) \) and \( H(t, x, u) \) are \( \mathbb{R}^N \)-valued maps, \( H \in C^0([T_0, 0], C^\infty(\mathbb{T}^n \times B_R(\mathbb{R}^N), \mathbb{R}^N)) \) and satisfies \( H(t, x, 0) = 0 \).

(III) \( B^\mu = B^\mu(t, x, u) \) and \( B = B(t, x, u) \) are \( \mathbb{M}_{N,N} \)-valued maps, and there is a constant \( R > 0 \), such that \( B^i \in C^0([T_0, 0], C^\infty(\mathbb{T}^n \times B_R(\mathbb{R}^N), \mathbb{M}_{N,N}), B \in C^0([T_0, 0], C^\infty(\mathbb{T}^n \times B_R(\mathbb{R}^N), \mathbb{M}_{N,N})), B^0 \in C^1([T_0, 0], C^\infty(\mathbb{T}^n \times B_R(\mathbb{R}^N), \mathbb{M}_{N,N})) \) and they satisfy

\[ (B^\mu)^T = B^\mu, \quad [P, B] = PB - BP = 0. \]

and \( B^i \) can be expanded as

\[ B^i(t, x, u) = B^i_0(t, x, u) + \frac{1}{t} B^i_2(t, x, u) \]

where \( B^i_0, B^i_2 \in C^0([T_0, 0], C^\infty(\mathbb{T}^n \times B_R(\mathbb{R}^N), \mathbb{M}_{N,N})). \)

Suppose there is \( \tilde{B}_0, \tilde{B} \in C^0([T_0, 0], C^\infty(\mathbb{T}^n, \mathbb{M}_{N,N})), \) such that

\[ [P, \tilde{B}] = 0, \quad B^0(t, x, u) - \tilde{B}^0(t, x) = O(u), \quad B(t, x, u) - \tilde{B}(t, x) = O(u) \]

for all \( (t, x, u) \in [T_0, 0] \times \mathbb{T}^n \times B_R(\mathbb{R}^N). \)

Moreover, there is \( \tilde{B}_2 \in C^0([T_0, 0], C^\infty(\mathbb{T}^n, \mathbb{M}_{N,N})), \) such that

\[ PB^i_2(t, x, u) P^\perp = O(P u), \quad P^\perp B^i_2(t, x, u) P = O(P u), \]

\[ P^\perp B^i_2(t, x, u) P^\perp = O(P u \otimes P u), \quad P(B^i_2(t, x, u) - \tilde{B}^i_2(t, x)) P = O(u), \]

for all \( (t, x, u) \in [T_0, 0] \times \mathbb{T}^n \times B_R(\mathbb{R}^N) \) where \( P^\perp = 1 - P \) is the complementary projection operator.

(IV) There exists constants \( \kappa, \gamma_1, \gamma_2 \) such that

\[ \frac{1}{\gamma_1} 1 \leq B^0 \leq \frac{1}{\kappa} B \leq \gamma_2 1 \]

for all \( (t, x, u) \in [T_0, 0] \times \mathbb{T}^n \times B_R(\mathbb{R}^N). \)

(V) For all \( (t, x, u) \in [T_0, 0] \times \mathbb{T}^n \times B_R(\mathbb{R}^N), \) assume

\[ P^\perp B^0(t, P^\perp u) P = PB^0(t, P^\perp u) P^\perp = 0. \]

(VI) For each \( (t, x, u) \in [T_0, 0] \times \mathbb{T}^n \times B_R(\mathbb{R}^N), \) there exists a \( s_0 > 0 \) such that

\[ \Theta(s) = B^0 \left( t, x, u + s [B^0(t, x, u)]^{-1} \left( \frac{1}{t} B(t, x, u) P v + H(t, x, u) \right) \right), \quad |s| < s_0, \]

defines smooth curve in\(^9\) \( L(\mathbb{R}^N) \). There exist constants \( \theta \) and \( \beta_{2\ell+1} \geq 0, \ell = 0, \cdots, 3, \) such that the derivative \( \Theta'(0) \) satisfies\(^10\)

\[ (u, P \Theta'(0) P u) = \Theta(\theta u \otimes u + |t|^{-1} \beta_1 P u \otimes P u), \]

---

\(^8\)The notations in this Appendix, for example, \( O(\cdot), \Theta(\cdot) \) and \( B_R(\mathbb{R}^N) \), are defined in [3, §2.4].

\(^9\)\( L(\mathbb{R}^N) \) denotes all the linear maps from \( \mathbb{R}^N \) to itself.

\(^10\)This condition is a reformulation of [3, §3.1.v]. It is straightforward to check that it implies the condition [3, §3.1.v] for the Fuchsian equation (C.1)–(C.2) that we are considering here.
\[ (u, P' \Theta'(0) P^\perp u) = \Theta \left( \theta u \otimes u + \frac{|t|^{-\beta_3}}{R} P u \otimes Pu \right), \]
\[ (u, P^\perp \Theta'(0) Pu) = \Theta \left( \theta u \otimes u + \frac{|t|^{-\beta_5}}{R} Pu \otimes Pu \right) \]
and
\[ (u, P^\perp \Theta'(0) P^\perp u) = \Theta \left( \theta u \otimes u + \frac{|t|^{-\beta_7}}{R^2} Pu \otimes Pu \right) \]
for all \((t, x, u) \in [T_0, 0) \times \mathbb{T}^n \times B_R(\mathbb{R}^N)\).

Let us first give rough estimates on the solutions to equations of form (C.1) which, however, only satisfies parts of above conditions (II)–(III).

**Theorem C.1.** Suppose that \(k \in \mathbb{Z}_{>\frac{n}{2}+1}, u_0 \in H^k(\mathbb{T}^n)\) and conditions (II)–(III) are fulfilled. In addition, \(B^0 = 1, B^i = 0, P = 1, B(t)\) and \(B^i(t)\) are only \(t\) dependent. If there exists a classical solution \(u \in C^1_b([T_0, T_*] \times \mathbb{T}^n)\) \((T_* < 0)\) to (C.1)–(C.2), then it satisfies the energy estimate
\[ \|u\|_{H^s} \leq e^{-c_2 T_0} (T_0)^{c_1} \|u_0\|_{H^s} e^{-c_1 t} e^{c_2 t} \]
for all \(t \in [T_0, T_*]\), where \(c_1 := c_1(\|u\|_{L^\infty([T_0, T_*], W^{1, \infty})})\) and \(c_2 := c_2(\|u(t)\|_{L^\infty([T_0, T_*], L^\infty)})\) are constants\(^{11}\).

**Proof.** According the assumptions of this theorem, since \(B^0 = 1, P = 1\) and \(B^0 = 0\), (C.1) reduces to
\[ \partial_0 u + \frac{1}{t} B^i_2(t) \partial_i u = \frac{1}{t} B(t, u) u + H(t, u). \] (C.3)
Differentiating (C.3) by \(D^\alpha\) yields
\[ \partial_0 D^\alpha u + \frac{1}{t} B^i_2(t) \partial_i D^\alpha u = \frac{1}{t} B(t, u) D^\alpha u + \frac{1}{t} [D^\alpha, B(t, u)] u + D^\alpha H(t, u). \] (C.4)
Then using \(\sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, \cdot \rangle \sim \sum_{0 \leq |\alpha| \leq s} \int_{\mathbb{T}^n} (D^\alpha u, \cdot) d^n x\) to act on the both sides of (C.4),
\[ \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, \partial_0 D^\alpha u \rangle + \frac{1}{t} \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, B^i_2(t) \partial_i D^\alpha u \rangle = \frac{1}{t} \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, B(t, u) D^\alpha u \rangle \]
\[ + \frac{1}{t} \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, [D^\alpha, B(t, u)] u \rangle + \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, D^\alpha H(t, u) \rangle. \]
Note, by recalling notations in §1.4.3,
\[ \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, \partial_0 D^\alpha u \rangle = \frac{1}{2} \partial_0 \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, D^\alpha u \rangle = \frac{1}{2} \partial_0 \|u\|_{H^s}^2, \]
and
\[ \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, B^i_2(t) \partial_i D^\alpha u \rangle = 0. \]
Using Hölder inequality and standard Moser’s estimates of compositions (see, for example, [13, Lemma A.3] and [25]), we arrive at
\[ \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, D^\alpha H(t, u) \rangle \leq \|u\|_{H^s} \|H(t, u)\|_{H^s} \leq C(\|H\|_{C^1_b(\mathbb{R}^N)}, \|u\|_{L^\infty}) \|u\|_{H^s}^2, \]
\(^{11}\)This type of constants are non-negative, non-decreasing, continuous functions in all their arguments.
\[
\sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, B(t, u) D^\alpha u \rangle \leq C \|B(t, u)\|_{L^\infty} \|u\|_{H^s}^2,
\]
and using the wellknown inequality \(\|D^\alpha, f\|_{L^2} \leq C(\|Df\|_{L^\infty} \|g\|_{H^{s-1}} + \|Df\|_{H^{s-1}} \|g\|_{L^\infty})\),
\[
\sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, [D^\alpha, B(t, u)]u \rangle \leq \sum_{0 \leq |\alpha| \leq s} \|D^\alpha u\|_{L^2} \|D^\alpha, B(t, u)\|_{L^2}
\leq C \|u\|_{H^s}(\|DB(t, u)\|_{L^\infty} \|u\|_{H^{s-1}} + \|DB(t, u)\|_{H^{s-1}} \|u\|_{L^\infty})
\leq C \|u\|_{H^s}(\|DB(t, u)\|_{L^\infty} \|u\|_{H^{s-1}} + C(\|D_u B\|_{C^{s-1} \overline{(B_R(\mathbb{R}^N))}}, \|u\|_{L^\infty}) \|u\|_{H^s} \|u\|_{L^\infty})
\leq C(\|D_u B\|_{C^{s-1} \overline{(B_R(\mathbb{R}^N))}}, \|u\|_{H^s}) \|u\|_{L^\infty}^2.
\]
Then, gathering above estimates together and noting \(t < T, t < 0\), we obtain for any \(t \in [T_0, T_*]\),
\[
\partial_t \|u\|_{H^s}^2 \leq -\frac{2c_1}{t} \|u\|_{H^s}^2 + 2c_2 \|u\|_{H^s}^2.
\]
where \(c_1 := c_1(\|u\|_{L^\infty([T_0, T_*], W^{1, \infty})})\) and \(c_2 := c_2(\|u\|_{L^\infty([T_0, T_*], L^\infty)})\) are constants depending on \(\|u\|_{L^\infty([T_0, T_*], W^{1, \infty})}\). This inequality is equivalent to
\[
\partial_t ((-t)^{2c_1} e^{-2c_2 t} \|u\|_{H^s}^2) \leq 0.
\]
Integrating this inequality, we conclude for any \(t \in [T_0, T_*]\),
\[
\|u\|_{H^s} \leq e^{-c_2 T_0} (T_0)^{c_1} \|u_0\|_{H^s} (-t)^{-c_1} e^{c_2 t}.
\]
We complete the proof. \(\square\)

Now let us present the global existence theorem for the Fuchsian system (see [3] for detailed proofs).

**Theorem C.2.** Suppose that \(k \in \mathbb{Z}_{\geq \frac{n}{2}+3}\), \(u_0 \in H^k(\mathbb{T}^n)\) and conditions (I)-(VI) are fulfilled, and the constants \(\kappa, \gamma_1, \beta_1, \beta_3, \beta_5, \beta_7\) from the conditions (I)-(VI) satisfy
\[
\kappa > \frac{1}{2} \gamma_1 \max \left\{ \sum_{\ell=0}^3 \beta_{2\ell+1}, \beta_1 + 2k(k+1)b \right\}
\]
where
\[
b := \sup_{T_0 \leq \ell < 0} \left( \|P \tilde{B} D(\tilde{B}^{-1} \tilde{B}_0)(\tilde{B}_0)^{-1} P \tilde{B}_2^2 P \|_{L^\infty} + \|P \tilde{B} D(\tilde{B}^{-1} \tilde{B}_2^2) P \|_{L^\infty} \right).
\]
Then there exist constants \(\delta_0, \delta > 0\) satisfying \(\delta < \delta_0\), such that if
\[
\|u_0\|_{H^k} \leq \delta,
\]
then there exists a unique solution
\[
u \in C^0([T_0, 0), H^k(\mathbb{T}^n)) \cap C^1([T_0, 0), H^{k-1}(\mathbb{T}^n)) \cap L^\infty([T_0, 0), H^k(\mathbb{T}^n))
\]
of the initial value problem (C.1)-(C.2) such that \(P^{-1} u(0) := \lim_{t \to 0} P^{-1} u(t)\) exists in \(H^{s-1}(\mathbb{T}^n)\).
Moreover, for \(T_0 \leq t < 0\), the solution \(u\) satisfies the energy estimate
\[
\|u(t)\|_{H^k}^2 - \int_{T_0}^t \frac{1}{\tau} \|Pu(\tau)\|_{H^k}^2 d\tau \leq C(\delta_0, \delta_0^{-1}) \|u_0\|_{H^k}^2.
\]

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REFERENCES

1. Serge Alinhac, *Blowup for nonlinear hyperbolic equations*, Birkhäuser Boston, 1995.
2. Florian Beyer and Todd A. Oliynyk, *Localized big bang stability for the Einstein-scalar field equations*, arXiv:2112.07730 (2021).
3. Florian Beyer, Todd A. Oliynyk, and J. Arturo Olvera-Santamaría, *The Fuchsian approach to global existence for hyperbolic equations*, Communications in Partial Differential Equations 46 (2020), no. 5, 1–82.
4. W. B. Bonnor, *Jeans’ formula for gravitational instability*, Monthly Notices of the Royal Astronomical Society 117 (1957), no. 1, 104–117.
5. Yan Guo, Mahir Hadžić, and Juhi Jang, *Continued gravitational collapse for Newtonian stars*, Archive for Rational Mechanics and Analysis (2020).
6. Yan Guo, Mahir Hadžić, Juhi Jang, and Matthew Schrecker, *Gravitational collapse for polytropic gaseous stars: Self-similar solutions*, arxiv.org/abs/2107.12056 (2021).
7. Lars Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques et Applications, Springer Berlin Heidelberg, 1997.
8. Sze-Bi Hsu, *Ordinary differential equations with applications*, World scientific, jan 2013.
9. J. H. Jeans, *The stability of a spherical nebula*, Philos. Trans. R. Soc. Lond. A 199 (1902), 1–53.
10. Satyanad Kichenassamy, *Nonlinear wave equations*, CRC Press, may 2021.
11. E. M. Lifshitz, *On the gravitational stability of the expanding universe*, Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki 16 (1946), 587–602.
12. Chao Liu and Todd A. Oliynyk, *Cosmological newtonian limits on large spacetime scales*, Communications in Mathematical Physics 364 (2018), no. 3, 1195–1304.
13. , *Newtonian limits of isolated cosmological systems on long time scales*, Annales Henri Poincaré 19 (2018), no. 7, 2157–2243.
14. Chao Liu, Todd A. Oliynyk, and Jinhua Wang, *Global existence and stability of de Sitter-like solutions to the Einstein-Yang-Mills equations in spacetime dimensions n ≥ 4*, (2022).
15. Chao Liu and Yiqing Shi, *Rigorous proof of the slightly nonlinear Jeans instability in the expanding Newtonian universe*, Physical Review D 105 (2022), no. 4, 043519.
16. Chao Liu and Changhua Wei, *Future stability of the FLRW spacetime for a large class of perfect fluids*, Annales Henri Poincaré 22 (2021), 715–770.
17. Andrew Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, Springer New York, 2012.
18. Frank Merle, Pierre Raphaël, Igor Rodnianski, and Jeremie Szeftel, *On the implosion of a compressible fluid I: Smooth self-similar inviscid profiles*, Annals of Mathematics 196 (2022), no. 2.
19. Viatcheslav Mukhanov, *Physical foundations of cosmology*, Cambridge University Press, November 2013.
20. Todd A. Oliynyk, *Future stability of the FLRW fluid solutions in the presence of a positive cosmological constant*, Communications in Mathematical Physics 346 (2016), 293–312.
21. Reinhard Racke, *Lectures on nonlinear evolution equations*, Springer International Publishing, 2015.
22. Alan D. Rendall, *Theorems on existence and global dynamics for the Einstein equations*, Living Reviews in Relativity 5 (2002), no. 1.
23. Jared Speck, *Finite-time degeneration of hyperbolicity without blowup for quasilinear wave equations*, Analysis & PDE 10 (2017), no. 8, 2001–2030.
24. , *Stable ODE-type blowup for some quasilinear wave equations with derivative-quadratic nonlinearities*, Analysis & PDE 13 (2020), no. 1, 93–146.
25. Michael E. Taylor, *Partial differential equations III: Nonlinear equations*, second ed., Applied Mathematical Sciences, Springer New York, 2010.
26. I. D. Novikov Ya. B. Zel’dovich, *Relativistic astrophysics 2: The structure and evolution of the universe*, University of Chicago Press, Chicago, 1971.

(Chao Liu) Center for Mathematical Sciences and School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, Hubei Province, China.
Email address: chao.liu.math@foxmail.com