Monoidal Rigidity for Free Wreath Products

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Abstract

In this note we observe that any compact quantum group monoidally equivalent, in a nice way, to a free wreath product of a compact quantum group $G$ by the quantum automorphism group of a finite dimensional C*-algebra with a $\delta$-form is actually isomorphic to a free wreath product of $G$ by the quantum automorphism group of another finite dimensional C*-algebra with a $\delta$-form.

Introduction

The theory of compact quantum groups, which includes the fundamental constructions of Drinfeld and Jimbo as a deformation of the universal enveloping algebras of classical Lie algebras, was introduced by Woronowicz in the eighties. Later, Wang introduced the quantum automorphism group $G(B,\psi)$ of a finite dimensional C*-algebra $B$ with a state $\psi$ and Banica studied its representation category when $\psi$ is a $\delta$-form.

Bichon introduced the construction of free wreath product $G \wr S^n_+$ of a compact quantum group $G$ by the permutation quantum group $S^n_+$ by using an action of $S^n_+$ on $n$ copies of $G$. In analogy with the classical case, the free wreath product allows to describe the quantum symmetry group of $n$ copies of a finite graph in terms of the symmetry group of the graph and of $S^n_+$. The representation category of a free wreath product $G \wr S^n_+$, when $G$ is Kac, was computed in [LT14] by Lemeux and Tarrago. The free wreath product $H^+(B,\psi)(G)$ of a compact quantum and a discrete group has been introduced by the second author in the case of $G = \hat{\Gamma}$, the dual of a discrete group $\Gamma$, and the representation category has been computed in [Pit16]. For a general $G$, the construction and the computation of the representation category has been studied in [FP15].

This note is a continuation of the general study of free wreath product initiated in [FP15]. Since we know from [FP15] the rigid C*-tensor category given by the representation theory of a free wreath product $H^+(B,\psi)(G)$, we will focus now on compact quantum groups which are monoidally equivalent to such a free wreath product. The purpose of this note is to prove the following rigidity result.

**Theorem A.** Let $G, G_0$ be compact quantum groups and $(B, \psi)$ be a finite dimensional C*-algebra with a $\delta$-form $\psi$. If $G$ is monoidally equivalent to $H^+(B,\psi)(G_0)$ via a nice enough unitary tensor functor then there exists a finite dimensional C*-algebra $D$ with a $\delta$-form $\psi_D$ on $D$ such that $G$ is isomorphic to $H^+(D,\psi_D)(G_0)$.

The precise meaning of nice enough will be given in Section 2 just before the statement of Theorem 2.3. When $G_0 = \{e\}$ is the trivial (quantum) group, any unitary tensor functor is nice enough. However, when $G_0 = \{e\}$, a stronger statement than the one of Theorem A is known to hold: it is proved in [Mro15] that any compact quantum group having the same fusion rules as the one of $SO(3)$ is actually isomorphic to the quantum automorphism group of some finite dimensional C*-algebra.

The paper has two sections: Section 1 is a preliminary section in which we introduce the notations of this paper and recall some useful results and Section 2 contains the proof of Theorem A (see Theorem 2.3).

1 Preliminaries

In this paper, we always assume the scalar products on Hilbert spaces to be linear in the first variable. For Hilbert spaces $H, K$, we denote by $L(H, K)$ the Banach space of bounded linear maps from $H$ to $K$ and we write $L(H) := L(H,H)$. The same symbol $\otimes$ is used to denote the tensor product of Hilbert spaces or the minimal tensor product of C*-algebras.
Given a unital C*-algebra $A$ and unitaries $u \in \mathcal{L}(H) \otimes A$ and $v \in \mathcal{L}(K) \otimes A$, for $H, K$ Hilbert spaces, we define $\text{Mor}(u, v) := \{T \in \mathcal{L}(H, K) : (T \otimes 1)u = v(T \otimes 1)\}$. We also use the notation
$$u \otimes v := u_{13}v_{23} \in \mathcal{L}(H \otimes K) \otimes A,$$
where $v_{23} := \text{id}_H \otimes v$ and $u_{13} := (\sigma \otimes \text{id}_A)(\text{id}_K \otimes u)$ and $\sigma : \mathcal{L}(K) \otimes \mathcal{L}(H) \to \mathcal{L}(H) \otimes \mathcal{L}(K)$ is the flip automorphism.

For $G$ a compact quantum group in the sense of [Wor98] we denote by $C(G)$ the maximal C*-algebra of $G$ i.e. the enveloping C*-algebra of the $*$-algebra given by the linear span of the coefficients of the irreducible representations of $G$. We will also denote by $\varepsilon$ the trivial representation of $G$, $\text{Irr}(G)$ the set of equivalence classes of irreducible representations of $G$ and by $\mathcal{R}(G)$ the rigid C*-tensor category of finite dimensional unitary representations of $G$. We refer to the chapter 2 of the book [NT13] for all the details concerning rigid C*-tensor categories and monoidal equivalence.

Let $B$ be a finite dimensional C*-algebra and $\psi$ a faithful state on $B$. Then $B$ is a Hilbert space with the scalar product defined by $\langle a, b \rangle := \psi(b^*a)$. Let $m_B : B \otimes B \to B$ be the multiplication on $B$ and $\eta_B : C \to B$ be the unit. We call $\psi$ a $\delta$-form when $m_Bm_B^* = \text{id}_B$, where $m_B^* : B \to B \otimes B$ is the adjoint of $m_B$ with respect to the scalar product on $B$ defined above.

Let $G_0$ be a compact quantum group and for $\alpha \in \text{Irr}(G_0)$, we choose a representative $u^\alpha$ acting on the Hilbert space $H_\alpha$. The free wreath product of $G_0$ by the quantum automorphism group $G_{(B, \psi)}$ of $(B, \psi)$, denoted by $H_{(B, \psi)}^+(G_0)$, is the compact quantum group defined in [FP15] as follows. The full C*-algebra $C(H_{(B, \psi)}^+(G_0))$ of this quantum group is the universal unital C*-algebra generated by the coefficients of matrices $a(\alpha) \in \mathcal{L}(B \otimes H_\alpha) \otimes C(H_{(B, \psi)}^+(G_0))$, for $\alpha \in \text{Irr}(G_0)$, such that:

- $\alpha(\alpha)$ is unitary for all $\alpha \in \text{Irr}(G_0)$,
- $(m_B \otimes S)\Sigma_{23} \in \text{Mor}(a(\alpha) \otimes a(\beta), a(\gamma))$ for all $\alpha, \beta, \gamma \in \text{Irr}(G_0)$ and all $S \in \text{Mor}(\alpha \otimes \beta, \gamma)$,
- $\eta_B \in \text{Mor}(\varepsilon, a(\varepsilon))$.

Moreover, there exists a unique unital $*$-homomorphism $\Delta : C(H_{(B, \psi)}^+(G_0)) \to C(H_{(B, \psi)}^+(G_0)) \otimes C(H_{(B, \psi)}^+(G_0))$ for which the elements $a(\alpha)$ are representations for all $\alpha \in \text{Irr}(G_0)$. The pair $(C(H_{(B, \psi)}^+(G_0)), \Delta)$ is a compact quantum group, called the free wreath product.

When $\psi$ is a $\delta$-form, the representation theory of $H_{(B, \psi)}^+(G_0)$ is totally understood [FP15]. In particular, the dimension of the spaces $\text{Mor}(\varepsilon, a(\alpha_1) \otimes \ldots \otimes a(\alpha_n))$ only depends on $G_0$ and $\alpha_1, \ldots, \alpha_n$ and not on $(B, \psi)$.

## 2 Proof of Theorem A

Suppose that $G$ and $G_0$ are compact quantum groups and $G$ is unitary monoidally equivalent to $H_{(B, \psi)}^+(G_0)$, where $\psi$ is a $\delta$-form. This means (see [NT13]) that there exist unitary tensor functors $F : \mathcal{R}(H_{(B, \psi)}^+(G_0)) \to \mathcal{R}(G)$, $H : \mathcal{R}(G) \to \mathcal{R}(H_{(B, \psi)}^+(G_0))$ such that $FH$ and $HF$ are naturally monoidally unitarily isomorphic to the identity functors, meaning that the natural isomorphisms $FH \simeq \iota$ and $HF \simeq \iota$ are both implemented by unitaries.

For $\alpha \in \text{Irr}(G_0)$, we choose a representative $u^\alpha$ acting on the Hilbert space $H_\alpha$. Recall that the representation $a(\alpha)$ is acting on $B \otimes H_\alpha$. We define the unitary representation $v_\alpha := F(a(\alpha))$ of $G$ acting on the finite dimensional Hilbert space $K_\alpha$. For all $u, v$ finite dimensional unitary representations of $H_{(B, \psi)}^+(G_0)$, we denote by the same symbol $F_2 \in \text{Mor}(F(u) \otimes F(v), F(u \otimes v))$ the unitary given by the definition of the unitary tensor functor $F$. Since the category is strict we have:
$$F_2 \circ (F_2 \otimes \text{id}) = F_2 \circ (\text{id} \otimes F_2) \in \text{Mor}(F(u) \otimes F(v), F(u \otimes v \otimes w)) \text{ for all } u, v, w. \quad (2.1)$$
Moreover, the naturally of $F_2$ means that, for all $u, v, u', v'$ finite dimensional unitary representations we have:
$$F(f \otimes g) \circ F_2 = F_2 \circ (F(f) \otimes F(g)) \in \text{Mor}(F(u) \otimes F(v), F(u' \otimes v')) \text{ for all } f \in \text{Mor}(u, u'), g \in \text{Mor}(v, v'). \quad (2.2)$$


Finally, we note that, since $F$ is unitary, we may and will assume that

$$F(\varepsilon) = \varepsilon \quad \text{and} \quad F(\text{id}) = \text{id} \text{ for all } \text{id} \in \text{Mor}(u, u). \quad (2.3)$$

Define the finite dimensional Hilbert space $D = K_x$. Since $m_B \in \text{Mor}(a(\varepsilon) \otimes a(\varepsilon), a(\varepsilon))$ and $\eta_B \in \text{Mor}(\varepsilon, a(\varepsilon))$, we may define the linear maps

$$m_D : D \otimes D \to D, \quad m_D := F(m_B)F_2 \quad \text{and} \quad \eta_D : C \to D, \quad \eta_D := F(\eta_B).$$

**Proposition 2.1.** $(D, m_D, \eta_D)$ is a unital algebra and $m_D m_D^* = \delta \text{id}_D$.

**Proof.** Using Equations (2.1), (2.2), (2.3) and the associativity of $m_B$ we get:

$$m_D \circ (m_D \circ \text{id}) = F(m_B)F_2 \circ (F(m_B)F_2 \circ \text{id}) = F(m_B) \circ F_2 \circ F(m_B) \circ \text{id} \circ (F_2 \otimes 1)$$

$$= F(m_B) \circ F(m_B \circ \text{id} \otimes F_2 \circ (F_2 \otimes 1) = F(m_B) \circ (m_B \circ \text{id}) \circ F_2 \circ (\text{id} \otimes F_2)$$

$$= F(m_B \circ \text{id} \otimes m_B) \circ F_2 \circ (\text{id} \otimes F_2) = F(m_B) \circ F(\text{id} \otimes m_B) \circ F_2 \circ (\text{id} \otimes F_2)$$

$$= F(m_B) \circ F_2 \circ (\text{id} \otimes F(m_B)) \circ (\text{id} \otimes F_2) = F(m_B)F_2 \circ (\text{id} \otimes F(m_B))F_2 = m_D \circ (m_D \circ \text{id}).$$

Hence, $m_D$ is associative. Moreover, using Equations (2.2), (2.3) and the fact that $\eta_B$ is the unit of $(B, m_B)$, which means that $m_B(\eta_B \otimes \text{id}_B) = m_B(\text{id}_B \otimes \eta_B)$, we find:

$$m_D(\eta_D \otimes \text{id}_D) = F(m_B)F_2(F(\eta_B) \otimes \text{id}_D) = F(m_B)F_2F_2(\eta_B \otimes \text{id}_B) = F(m_B(\eta_B \otimes \text{id}_B)) = F(\text{id}_D) = \text{id}_D \text{ and,}$$

$$m_D(\text{id}_D \otimes \eta_D) = F(m_B)F_2(\text{id}_D \otimes F(\eta_B)) = F(m_B)F_2F_2(\text{id}_D \otimes \eta_B) = F(m_B(\text{id}_B \otimes \eta_B)) = F(\text{id}_D) = \text{id}_D.$$ 

Finally, $m_D m_D^* = F(m_B)F_2F_2 = m_D \circ (m_D \circ \text{id}) = m_D^*$. 

We are now ready to turn $D$ into a $^*$-algebra. Define $t = m_D^* \circ \eta_D = F_2^* F(m_B^* \eta_B) \in \mathcal{L}(D, D \otimes D)$. We may and will view $t \in D \otimes D$. Define the anti-linear map $S_D : D \to D$ by $S_D(x) = (x^* \otimes \text{id})(t)$. Denote by $L_x \in \mathcal{L}(D)$ the bounded operator given by left multiplication by $x \in D$ and write $1_D := \eta_D(1) \in D$.

**Proposition 2.2.** $S_D$ is an involution on the unital algebra $D$. Moreover, equipped with this involution and the norm defined by $\|x\|_D := \|L_x\|$ for $x \in D$, $D$ is a unital $C^*$-algebra and the following holds.

1. $(L_x)^* = L_{S_D(x)}$ for all $x \in D$.
2. $\psi_D : D \to \mathbb{C}, \ x \mapsto (x, 1_D)$ is a faithful state on $D$ satisfying $\psi_D(S_D(y)x) = \langle x, y \rangle$ for all $x, y \in D$.
3. $\psi_D$ is a $\delta$-form on $D$.

**Proof.** (1) Using diagrams computations one can easily check that:

$$(\text{id} \otimes m_B) \circ (m_B^* \circ \eta_B \otimes \text{id}) = m_B.$$

Applying the functor $F$ we find $(\text{id} \otimes m_D) \circ (m_D^* \circ \eta_D \otimes \text{id}) = m_D$. Hence, for all $x, y, z \in D$, one has

$$L_{S_D(x)}(y, z) = (m_D(S_D(y) \otimes \eta_B), z) = (m_D((x^* \otimes \text{id} \circ m_B^* \circ \eta_B \otimes y), z)$$

$$= ((x^* \otimes \text{id} \circ m_D \circ \eta_B \otimes \text{id})(y), z) = (L_x^*(y), z) = (m_D^*(y), x) = (y, m_D(x \otimes z)) = (y, L_x(z)) = ((L_x)^*(y), z).$$

This concludes the proof of 1 and shows that $S_D$ is anti-multiplicative since it implies that, for all $x, y, z \in D$, $S_D(xy) = ((L_x)^* \circ L_y)^* = (L_y)^*(L_x)^* = L_{S_D(y)}(S_D(x)) = S_D(y)S_D(x)$. 

Assertion (1) also shows that $S_D$ is involutive since it implies that $L_{S_D(x)} = L_{S_D(x)}^* = (L_x)^*$, hence, $S_D^2(x) = x$. It follows that $S_D$ turns $(D, m_D, \eta_D)$ into an involutive unital algebra. It is moreover
clear that \(|x|_D := \|L_x\|_{\mathcal{L}(D)}\), for \(x \in D\), defines a norm on \(D\) which satisfies the C*-condition since we have

\[\|S_D(x)\|_D = \|L_{S_D(x)}\| = \|L_{S_D(x)}L_x\| = \|L_x\|_D = \|x\|_D\]

for all \(x \in D\).

(2) One has \(\psi_D(S_D(y)x) = \langle S_D(y)x, 1_D \rangle = \langle L_{S_D(y)}(x), 1_D \rangle = \langle (L_y)^*x, 1_D \rangle = \langle x, L_y(1_D) \rangle = \langle x, y \rangle\). It follows directly from this relation and the non degeneracy of the scalar product on \(D\) that \(\psi_D\) is faithful. It also follows that the scalar product induced by \(\psi_D\) on \(D\) is the same as the initial scalar product on \(D\). Hence, taking the adjoint with the \(\psi_D\)-scalar product we have, from Proposition 2.1, \(m_{Dm_{D}}^D = \delta_{id_D}\). \(\square\)

We assume from now that the functor \(F\) is nice enough meaning that for any \(\alpha \in \text{Irr}(G_0)\), there exists a unitary \(V_\alpha \in \mathcal{L}(K_\alpha, K_\alpha \otimes H_\alpha)\), with \(V_\varepsilon = \text{id}_D\), and such that the following diagram is commutative for all \(\alpha, \beta, \gamma \in \text{Irr}(G_0)\) and all \(S \in \text{Mor}(\alpha \otimes \beta, \gamma)\):

\[
\begin{array}{ccc}
(K_\varepsilon \otimes H_\alpha) \otimes (K_\varepsilon \otimes H_\beta) & \xrightarrow{\text{id} \otimes \pi} & (K_\varepsilon \otimes H_\gamma) \\
\uparrow V_\alpha \otimes V_\beta & & \uparrow V_\gamma \\
K_\alpha \otimes K_\beta & \xrightarrow{F((m_\beta \otimes S)\Sigma_{23})F_2} & K_\gamma
\end{array}
\]

Observe that any unitary tensor functor is nice enough if \(G_0\) is the trivial quantum group. We are now ready to state our main result. Let us denote by \(\tilde{\alpha}(\alpha) \in \mathcal{L}(D \otimes H_\alpha) \otimes C(H_{(D,\psi_D)}(G_0))\) the canonical generators.

**Theorem 2.3.** There exists a unique *-isomorphism \(\pi : C(H^+_{(D,\psi_D)}(G_0)) \rightarrow C(G)\) such that

\[(\text{id} \otimes \pi)(\tilde{\alpha}(\alpha)) = (V_\alpha \otimes 1)\nu_\alpha(V_\alpha^* \otimes 1) \in \mathcal{L}(D \otimes H_\alpha) \otimes C(G)\]

for all \(\alpha \in \text{Irr}(G_0)\).

Moreover, \(\pi\) intertwines the comultiplications.

**Proof. Step 1.** There exists a surjective unital *-homomorphism \(\phi : C(H^+_{(D,\psi_D)}(G_0)) \rightarrow C(G)\) such that

\[(\text{id} \otimes \pi)(\tilde{\alpha}(\alpha)) = (V_\alpha \otimes 1)\nu_\alpha(V_\alpha^* \otimes 1) \in \mathcal{L}(D \otimes H_\alpha) \otimes C(G)\]

for all \(\alpha \in \text{Irr}(G_0)\).

From the universal property of the C*-algebra \(C(H^+_{(D,\psi_D)}(G_0))\) it suffices to check that the unitary representations \(u_\alpha := (V_\alpha \otimes 1)\nu_\alpha(V_\alpha^* \otimes 1)\) of \(G\) on the Hilbert space \(D \otimes H_\alpha\) satisfy the following relations:

1. \((m_\alpha \otimes S)\Sigma_{23} \in \text{Mor}(u_\alpha \otimes u_\beta, u_\gamma)\) for all \(\alpha, \beta, \gamma \in \text{Irr}(G_0)\) and all \(S \in \text{Mor}(\alpha \otimes \beta, \gamma)\),

2. \(\eta_D \in \text{Mor}(\epsilon, u_\varepsilon)\).

(1) Since \(F\) is nice enough we have, for all \(\alpha, \beta, \gamma \in \text{Irr}(G_0)\) and all \(S \in \text{Mor}(\alpha \otimes \beta, \gamma)\),

\[V_\gamma^*(m_\alpha \otimes S)\Sigma_{23}(V_\alpha \otimes V_\beta) = F((m_\beta \otimes S)\Sigma_{23})F_2 \in \text{Mor}(\nu_\alpha \otimes \nu_\beta, \nu_\gamma)\]

It follows that \(((m_\alpha \otimes S)\Sigma_{23}) \circ (u_\alpha \circ u_\beta)\) is equal to:

\[
(((m_\alpha \otimes S)\Sigma_{23}) \circ (u_\alpha \circ u_\beta) = ((m_\alpha \otimes S)\Sigma_{23})(V_\alpha \otimes V_\beta) \circ 1(u_\alpha \otimes \nu_\beta)(V_\gamma^* \otimes 1)
\]

\[= (V_\gamma \otimes 1)v_\gamma((m_\alpha \otimes S)\Sigma_{23})(V_\alpha \otimes V_\beta)(V_\alpha^* \otimes V_\beta^* \otimes 1) = u_\gamma((m_\alpha \otimes S)\Sigma_{23}).\]

(2) It is obvious since \(\eta_D \in \text{Mor}(\epsilon, u_\varepsilon)\) implies that \(\eta_D = F(\eta_B) \in \text{Mor}(\epsilon, \nu_\varepsilon)\) and \(u_\varepsilon = \nu_\varepsilon\).

Note that \(\pi\) automatically intertwines the comultiplications and \(\pi\) is surjective since \(F\) is essentially surjective.

**Step 2.** \(\pi\) is an isomorphism.

It suffices to check that \(\pi\) intertwines the Haar measures \(h\) on \(C(H^+_{(D,\psi_D)}(G_0))\) and \(h_G\) on \(C(G)\). Since the linear span of the coefficients of representations of the form \(\tilde{\alpha}(\alpha_1) \otimes \ldots \otimes \tilde{\alpha}(\alpha_n)\), for \(\alpha_1, \ldots, \alpha_n \in \text{Irr}(G_0)\) and \(n \geq 1\), is dense in \(C(H^+_{(D,\psi_D)}(G_0))\), it suffices to check that, for all \(n \geq 1, \alpha_1, \ldots, \alpha_n \in \text{Irr}(G_0)\) one has

\[(\text{id} \otimes h_G)(u_{\alpha_1} \otimes \ldots \otimes u_{\alpha_n}) = (\text{id} \otimes h)(\tilde{\alpha}(\alpha_1) \otimes \ldots \otimes \tilde{\alpha}(\alpha_n)),\]
which is equivalent to \( \dim(\text{Mor}(\varepsilon, u_{\alpha_1} \otimes \ldots \otimes u_{\alpha_n})) = \dim(\text{Mor}(\varepsilon, \tilde{a}(\alpha_1) \otimes \ldots \otimes \tilde{a}(\alpha_n))) \). Since \( u_\alpha \simeq v_\alpha = F(a(\alpha)) \) for all \( \alpha \in \text{Irr}(G_0) \) and \( F \) is a monoidal equivalence, the left hand side is equal to
\[
\dim(\text{Mor}(\varepsilon, v_{\alpha_1} \otimes \ldots \otimes v_{\alpha_n})) = \dim(\text{Mor}(\varepsilon, a(\alpha_1) \otimes \ldots \otimes a(\alpha_n))).
\]
Moreover since, \( \psi_D \) is a \( \delta \)-form we also know from [FP15, Theorem 3.5] an explicit formula for the number \( \dim(\text{Mor}(\varepsilon, a(\alpha_1) \otimes \ldots \otimes a(\alpha_n))) \) which only depends on \( G_0 \) and not on \((B, \psi)\) or \((D, \psi_D)\) and we have
\[
\dim(\text{Mor}(\varepsilon, a(\alpha_1) \otimes \ldots \otimes a(\alpha_n))) = \dim(\text{Mor}(\varepsilon, \tilde{a}(\alpha_1) \otimes \ldots \otimes \tilde{a}(\alpha_n))).
\]

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