A NOTE ON THE SUPPORT AND COSUPPORT CONDITIONS FOR A PERVERSE SHEAF

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ABSTRACT. We give a characterization of the support and cosupport conditions for a perverse sheaf in terms of the Whitney filtration.

1. INTRODUCTION

Let $R$ be a regular, Noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$), and let $A^\bullet$ be a bounded, constructible complex of sheaves of $R$-modules on a complex analytic space $X$, i.e., let $A^\bullet \in D^b_c(X)$.

The complex of sheaves $A^\bullet$ is, by definition, a perverse sheaf (using the singular form “sheaf” is standard) if and only if it satisfies two conditions: the support and cosupport conditions. There are two equivalent well-known characterizations of these conditions, a characterization that does not refer to a Whitney stratification and one that does. See [1], [3], [4], [2], and [6].

To give these two characterizations, let us first select a Whitney stratification $S$ of $X$ with respect to which $A^\bullet$ is constructible and, for each $S \in S$, let $r_S : S \hookrightarrow X$ denote the inclusion. Note that there is no requirement that the strata of $S$ be connected, so in fact, by replacing the strata of $S$ with a stratification where the strata are the unions of the strata of $S$ of each given dimension, we may assume if we wish that there is (at most) one stratum of each dimension.

We also need to define the $k$-th support and cosupport of $A^\bullet$. For all $x \in X$, let $j_x : \{x\} \hookrightarrow X$ denote the inclusion. The $k$-th support of $A^\bullet$ is
\[
\text{supp}^k(A^\bullet) := \{x \in X \mid H^k(j^*_x A^\bullet) \neq 0\}
\]
and the $k$-th cosupport is
\[
\text{cosupp}^k(A^\bullet) := \{x \in X \mid H^k(j^!_x A^\bullet) \neq 0\}.
\]

Now we give the standard descriptions of support and cosupport conditions on $A^\bullet$. The dimensions here are the complex dimensions and, as usual, a negative dimension indicates that a set is empty.

Definition 1.1. The support and cosupport conditions are defined in either/both of the following two ways:

(1) \hspace{1cm}• \hspace{0.5cm} S1: (support) For all $k$, $\dim(\text{supp}^{-k}(A^\bullet)) \leq k$.
\hspace{1cm}• \hspace{0.5cm} C1: (cosupport) For all $k$, $\dim(\text{cosupp}^k(A^\bullet)) \leq k$.

(2) \hspace{1cm}• \hspace{0.5cm} S2: (support) For all $S \in S$, for all $k > -\dim S$, $H^k(r_S^* A^\bullet) = 0$.
\hspace{1cm}• \hspace{0.5cm} C2: (cosupport) For all $S \in S$, for all $k < -\dim S$, $H^k(r_S! A^\bullet) = 0$. 

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Remark 1.2. It is easy to see that the two different characterizations of the support condition are equivalent, but it is significantly more difficult to see that the two different cosupport conditions are equivalent. The two variants are connected by the following general result. If \( g : Y \hookrightarrow X \) is the inclusion of an orientable submanifold into another orientable manifold, and \( r \) is the real codimension of \( Y \) in \( X \), and \( F^* \in D^*_b(X) \) has locally constant cohomology on \( X \), then \( g^* F^* \) has locally constant cohomology on \( Y \) and \( g^* F^*[-r] \cong g^* F^* \).

Referring now to Definition 1.1, if we let \( F^* = r^*_S A^* \), let \( x \in S \), let \( g : \{ x \} \hookrightarrow S \) be the inclusion, and \( r = 2 \dim S \) (where \( \dim \) means the complex dimension), we obtain

\[
\text{H}^k(r^*_S A^*) \cong \text{H}^k(g^* r^*_S A^*) \cong \text{H}^k(g^* r^*_S A^*[2 \dim S]) \cong \text{H}^{k+2 \dim S}(j^*_x A^*).
\]

Therefore, \( \text{H}^k(r^*_S A^*) = 0 \) if and only if, for all \( x \in S \), \( \text{H}^{k+2 \dim S}(j^*_x A^*) = 0 \) and so:

for all \( k < -\dim S \), \( \text{H}^k(r^*_S A^*) = 0 \) if and only if, for all \( k < -\dim S \), \( \dim \), for all \( k < -\dim S \), \( \text{H}^k(r^*_S A^*) = 0 \).

That is, for all \( S \in \mathcal{S} \), for all \( k < -\dim S \), \( \text{H}^k(r^*_S A^*) = 0 \) if and only if, for all \( k \), \( \text{cosupp}^k(A^*) \subseteq \bigcup_{\dim S \leq k} S \).

There is only one result in this short paper, one that may be of use as a lemma for other results; we give the support and cosupport conditions in terms of the open and closed filtrations of \( X \) which come from the stratification \( \mathcal{S} \).

2. SUPPORT AND COSUPPORT IN TERMS OF FILTRATIONS

We continue with all of our notation from the introduction. In particular, we have a Whitney stratification \( \mathcal{S} \) of \( X \) with respect to which \( A^* \) is constructible and, for each \( S \in \mathcal{S} \), \( r_S : S \hookrightarrow X \) is the inclusion.

We define the upper filtration of \( X \) by: for all \( m \geq 0 \), let \( U^m := \bigcup_{S \in \mathcal{S}, \dim S \geq m} S \). We define the lower filtration of \( X \) by: for all \( m \geq 0 \), let \( L^m := \bigcup_{S \in \mathcal{S}, \dim S \leq m} S \).

Then, our new characterizations of the support and cosupport condition are:

Lemma 2.1. For all \( m \geq 0 \), let \( u_m : U^m \hookrightarrow X \) and \( \ell_m : L^m \hookrightarrow X \) denote the inclusions.

- The support condition is equivalent to: (new support) for all \( m \geq 0 \), for all \( k > -m \), \( \text{H}^k(u^*_m A^*) = 0 \).
- The cosupport condition is equivalent to: (new cosupport) for all \( m \geq 0 \), for all \( k < -m \), \( \text{H}^k(\ell^*_m A^*) = 0 \).

Proof.

Support:

This is easy. Both (S2) and the new support condition are trivially equivalent to: for all \( k \), for all \( S \in \mathcal{S} \) such that \( \dim S > -k \), for all \( x \in S \), \( \text{H}^k(j^*_x A^*) = 0 \).

Cosupport:

For notational convenience, we assume that our Whitney stratification has, at most, one stratum for each dimension. We can do this by replacing the original individual strata of dimension \( m \) by one stratum which is the union of all of the original \( m \)-dimensional strata. This has no effect on Version 2 in Definition 1.1 as each original \( m \)-dimensional stratum is an open subset of the union of all of the \( m \)-dimensional strata (since the union is locally connected). We denote the unique \( m \)-dimensional stratum by \( S^m \).
Now, let \( q_m : S^m \hookrightarrow L^m \) denote the inclusion; as \( S^m \) is open in \( L^m \), we have a natural isomorphism \( q_m^* \cong q_m^! \). We write \( r_m \) in place of \( r_{S^m} \) for the inclusion of \( S^m \) into \( X \). Thus, \( r_m = \ell_m q_m \).

Note that with the above assumption and notation, the cosupport condition \((C2)\) becomes: \((C2)\) for all \( m \), for all \( k < -m \), \( H^k(\ell_m^! A^\bullet) = 0 \).

New cosupport \( \implies \) \((C2)\):

Suppose the new cosupport condition holds, so that, for all \( m \geq 0 \), for all \( k < -m \), \( H^k(\ell_m^! A^\bullet) = 0 \). Then, it is immediate that, for all \( m \geq 0 \), for all \( k < -m \),

\[
0 = H^k(q_m^* \ell_m^! A^\bullet) \cong H^k(q_m^! \ell_m^! A^\bullet) \cong H^k(\ell_m^! A^\bullet),
\]

which is \((C2)\).

\((C2)\) \( \implies \) new cosupport:

Suppose that \((C2)\) holds. So assume that, for all \( m \geq 0 \), for all \( k < -m \), \( H^k(\ell_m^! A^\bullet) = 0 \). We shall prove by induction on \( m \) that, for all \( m \geq 0 \), for all \( k < -m \), \( H^k(\ell_m^! A^\bullet) = 0 \).

\( m = 0 \):

Since \( S^0 = L^0 \), \((C2)\) immediately implies that, for all \( k < 0 \), \( H^k(\ell_0^! A^\bullet) = 0 \).

Inductive step:

Now assume that \( m_0 \geq 1 \) and that the new cosupport conditions holds for all \( m \) such that \( 0 \leq m \leq m_0 - 1 \). We wish to show that, for all \( k < -m_0 \), \( H^k(\ell_m^! A^\bullet) = 0 \).

Let \( p_{m_0} : L^{m_0-1} = L^{m_0} \setminus S^{m_0} \hookrightarrow L^{m_0} \) denote the closed inclusion. Then we have the fundamental distinguished triangle

\[
(p_{m_0})_! p_{m_0}^! (\ell_m^! A^\bullet) \to \ell_m^! A^\bullet \to (q_{m_0})_* q_{m_0}^! (\ell_m^! A^\bullet) \xrightarrow{[1]} \]

and the associated long exact sequence on stalk cohomology.

Suppose that \( k < -m_0 \) and \( x \in L^{m_0} \). We will show that the stalk cohomology \( H^k(\ell_m^! A^\bullet)_x = 0 \) by showing that \( H^k((p_{m_0})_! p_{m_0}^! (\ell_m^! A^\bullet))_x = 0 \) and \( H^k((q_{m_0})_* q_{m_0}^! (\ell_m^! A^\bullet))_x = 0 \).

As \( \ell_m p_{m_0} = \ell_{m_0} \) and \( k < -m_0 < -m_0 - 1 \), we have

\[
H^k((p_{m_0})_! p_{m_0}^! (\ell_m^! A^\bullet))_x = H^k((p_{m_0})_! \ell_{m_0}^! A^\bullet)_x,
\]

which equals 0 by our inductive step if \( x \in L^{m_0-1} \) and equals 0 otherwise since \((p_{m_0})_!\) is the extension by zero.

As \( q_{m_0}^* \cong q_{m_0}^! \) and \( r_{m_0} = \ell_{m_0} q_{m_0} \), we have \( H^k((q_{m_0})_* q_{m_0}^* (\ell_m^! A^\bullet))_x \cong H^k((q_{m_0})_* r_m^! A^\bullet)_x \), and note that, by \((C2)\), for all \( k < -m_0 \), \( H^k(r_m^! A^\bullet) = 0 \). By constructibility, there is an open neighborhood \( U \) of \( x \) in \( L^{m_0} \) such that

\[
H^k((q_{m_0})_* r_m^! A^\bullet)_x \cong \mathbb{H}^k(U; (r_m^! A^\bullet)|_U),
\]

but now either the canonical injective resolution of \( (r_m^! A^\bullet)|_U \) or the \( E_2 \) spectral sequence for hypercohomology tells us that this is 0. \( \square \)

Why did we want to prove that there is a third way of describing the support and cosupport conditions, as is given in Lemma 2.1?

It is our hope that these new support and cosupport conditions will become as equally well-known as the two versions given in Definition 1.1, mainly for the new version of the cosupport condition. We, ourselves, and others have instead had to essentially prove that \((C2)\) implies the
new cosupport condition in the midst of other results; this is the case, for instance, in the proof of Theorem 3.1 in [5] where $A^\bullet$ is the perverse sheaf of shifted vanishing cycles along a function $f$.

Using the notation from Lemma 2.1, we can now easily prove the following proposition.

**Proposition 2.2.** Let $S$ be a Whitney stratification of $X$ with respect to which $A^\bullet$ is constructible. For all $m \geq 0$, let $U^m := \bigcup_{S \in S, \dim S \geq m} S$. Suppose that $A^\bullet$ satisfies the cosupport condition.

Then, for all $m \geq 0$,

1. for all $k \leq -m - 2$, the canonical morphism yields an isomorphism $\mathbb{H}^k(X; A^\bullet) \cong \mathbb{H}^k(U^{m+1}; A^\bullet|_{U^{m+1}})$, and
2. the canonical morphism yields an injection $\mathbb{H}^{-m-1}(X; A^\bullet) \hookrightarrow \mathbb{H}^{-m-1}(U^{m+1}; A^\bullet|_{U^{m+1}})$.

**Proof.** As before, for all $m \geq 0$, let $L^m := \bigcup_{S \in S, \dim S \leq m} S$, and let $u_m : U^m \hookrightarrow X$ and $\ell_m : L^m \hookrightarrow X$ denote the inclusions.

Consider the distinguished triangle

$$(\ell_m)_! \ell_m^! A^\bullet \to A^\bullet \to (u_{m+1})_! u_{m+1}^* A^\bullet \to [1].$$

Then, combining Lemma 2.1 with the long exact sequence on hypercohomology yields the stated conclusions (once again using, as in the proof of the lemma, that zero sheaf cohomology below a given dimension implies zero hypercohomology below that dimension). $\square$

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