Sufficient Lie Algebraic Conditions for Sampled-Data Feedback Stabilization of Affine in the Control Nonlinear Systems

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Abstract

For general nonlinear autonomous systems, a Lyapunov characterization for the possibility of semi-global asymptotic stabilization by means of a time-varying sampled-data feedback is established. We exploit this result in order to derive a Lie algebraic sufficient condition for sampled-data feedback semi-global stabilization of affine in the control nonlinear systems.

Keywords: Stabilization, Sampled-data, Time-Varying Feedback, Lie Algebra, Nonlinear Systems

1. Introduction

Several important results towards stabilization of nonlinear autonomous systems by means of sampled-data feedback control have appeared in the literature (see for instance [1]-[2], [4]-[14], [17],[19]-[20] and relative references therein). In the recent works [19]-[20], the concept of Weak Global Asymptotic Stabilization by Sampled-Data Feedback (SDF-WGAS) is presented for systems:

\[ \dot{x} = f(x, u), \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \]
\[ f(0, 0) = 0 \]  \hspace{1cm} (1.1)

and various Lyapunov-like sufficient characterizations of this property have been examined. Particularly, in Proposition 2 in [20], a Lie algebraic sufficient condition for SDF-WGAS is established for the case of affine in the control systems

\[ \dot{x} = f(x) + ug(x), \quad x \in \mathbb{R}^n \]
\[ f(0) = 0 \]  \hspace{1cm} (1.2)

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This condition constitutes an extension of the well-known “Artstein-Sontag” sufficient condition for asymptotic stabilization of systems (1.2) by means of an almost smooth feedback; (see [3], [16] and [18]). In order to provide the precise statement of [20, Proposition 2], we first need to recall the following standard notations. For any pair of $C^1$ mappings $X : \mathbb{R}^n \to \mathbb{R}^k$, $Y : \mathbb{R}^k \to \mathbb{R}^\ell$ we adopt the notation $XY := DYX$, $DY$ being the derivative of $Y$. By $\lbrack \cdot, \cdot \rbrack$ we denote the Lie bracket operator, namely, $\lbrack X, Y \rbrack = DYX - DXY$ for any pair of $C^1$ mappings $X, Y : \mathbb{R}^n \to \mathbb{R}^n$, and we recall the well known property $\lbrack X, Y \rbrack \Phi = XY\Phi - YX\Phi$ for any $C^2$ function $\Phi : \mathbb{R}^n \to \mathbb{R}$.

The precise statement of [20, Proposition 2] is the following. Assume that $f, g \in C^2$ and there exists a $C^2$, positive definite and proper function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that the following implication holds:

$$
(gV)(x) = 0, x \neq 0 \quad \Rightarrow \quad \begin{cases} 
\text{either } (fV)(x) < 0, \text{("Artstein - Sontag" implication)} \\
\text{or } (fV)(x) = 0; \ (\lbrack f, g \rbrack V)(x) \neq 0 
\end{cases} 
$$

(1.3)

Then system (1.2) is SDF-WGAS.

Proposition 2 of present work establishes that for systems (1.1) the same Lyapunov characterization of SDF-WGAS, originally proposed in [19], implies Semi-Global Asymptotic Stabilization by means of a time-varying Sampled-Data Feedback (SDF-SGAS), which is a stronger type of SDF-WGAS. Proposition 3 is the main result of our present work. It constitutes a generalization of [20, Proposition 2] mentioned above and provides a Lie algebraic sufficient condition for SDF-SGAS(WGAS) for the case of affine in the control systems (1.2). This condition is weaker than (1.3) and involves a particular Lie sub-algebra of the dynamics $f, g$ of the system (1.2).

The paper is organized as follows. Section 2 contains the precise definitions of the concepts of SDF-WGAS and SDF-SGAS and the statements of our results (Propositions 2 and 3). Section 3 contains the proofs of these results and in Section 4 illustrative examples are provided.

2. Definitions and Main Results

Consider system (1.1) and assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous. We denote by $x(\cdot) = x(\cdot, s, x_0, u)$ the trajectory of (1.1) with initial $x(s, s, x_0, u) = x_0 \in \mathbb{R}^n$ corresponding to certain (measurable and essentially bounded) control $u : [s, T_{\max}) \to \mathbb{R}^m$, where $T_{\max} = T_{\max}(s, x_0, u)$ is the corresponding maximal existing time of the trajectory.

**Definition 1.** We say that system (1.1) is Weakly Globally Asymptotically Stabilizable by Sampled-Data Feedback (SDF-WGAS), if for any constant $\sigma > 0$ there exist mappings $T : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^+ \backslash \{0\}$ satisfying

$$
T(x) \leq \sigma, \ \forall x \in \mathbb{R}^n \backslash \{0\} 
$$

(2.1)
and 

\[ k(t, x; x_0) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m \]

such that for any fixed \( (x, x_0) \in \mathbb{R}^2 \) the map \( k(\cdot, x; x_0) : \mathbb{R}^+ \rightarrow \mathbb{R}^m \) is measurable and essentially bounded and such that for every \( x_0 \neq 0 \) there exists a sequence of times

\[ t_1 := 0 < t_2 < t_3 < \ldots < t_\nu < \ldots \text{, with } t_\nu \rightarrow \infty \]

(2.2)

in such a way that the trajectory \( x(\cdot) \) of the sampled-data closed loop system:

\[ \dot{x} = f(x, k(t, x(t_i); x_0)), \quad t \in [t_i, t_{i+1}], \quad i = 1, 2, \ldots \]

\[ x(0) = x_0 \in \mathbb{R}^n \]

(2.3)

satisfies:

\[ t_{i+1} - t_i = T(x(t_i)), \quad i = 1, 2, \ldots \]

(2.4)

and the following properties:

- **Stability:**

\[ \forall \varepsilon > 0 \Rightarrow \exists \delta = \delta(\varepsilon) > 0 : |x(0)| \leq \delta \]

\[ \Rightarrow |x(t)| \leq \varepsilon, \forall t \geq 0 \]

(2.5)

- **Attractivity:**

\[ \lim_{t \rightarrow \infty} x(t) = 0, \forall x(0) \in \mathbb{R}^n \]

(2.6)

where \( |x| \) denotes the Euclidean norm of the vector \( x \).

Next we give the Lyapunov characterization of SDF-WGAS proposed in [19]-[20] that constitutes a generalization of the concept of the control Lyapunov function (see Definition 5.7.1 in [15]).

**Assumption 1:** There exist a positive definite \( C^0 \) function \( V : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) and a function \( a \in K \) (namely, \( a(\cdot) \) is continuous, increasing with \( a(0) = 0 \)) such that for every \( \xi > 0 \), a constant \( \varepsilon_0 \in (0, \xi] \) can be found such that for every \( x_0 \neq 0 \) and \( \varepsilon \in (0, \varepsilon_0] \), a control \( u_\varepsilon, x_0 : [0, \varepsilon] \rightarrow \mathbb{R}^m \) can be determined satisfying

\[ V(x(\varepsilon, 0, x_0, u_\varepsilon, x_0)) < V(x_0); \]

(2.7a)

\[ V(x(s, 0, x_0, u_\varepsilon, x_0)) \leq a(V(x_0)), \forall s \in [0, \varepsilon] \]

(2.7b)

The following result was established in [19].

**Proposition 1.** Under Assumption 1, system (1.1) is SDF-WGAS.

We now present the concept of SDF-SGAS, which is a stronger version of SDF-WGAS:

**Definition 2.** We say that system (1.1) is Semi-Globally Asymptotically Stabilizable by Sampled-Data Feedback (SDF-SGAS), if for every \( R > 0 \) and for any given partition of times

\[ T_1 := 0 < T_2 < T_3 < \ldots < T_\nu < \ldots \text{ with } T_\nu \rightarrow \infty \]

(2.8)

there exist a neighborhood \( \Pi \) of zero with \( B[0, R] := \{ x \in \mathbb{R}^n : |x| \leq R \} \subset \Pi \) and a map \( k : \mathbb{R}^+ \times \Pi \rightarrow \mathbb{R}^m \) such that for any \( x \in \Pi \) the map \( k(\cdot, x) : \mathbb{R}^+ \rightarrow \mathbb{R}^m \) is
measurable and essentially bounded and the trajectory $x(\cdot)$ of the sampled-data closed loop system

$$\dot{x} = f(x, k(t, x(T_i))), \ t \in [T_i, T_{i+1}), \ i = 1, 2, \ldots$$

satisfies:

- **Stability:**
  \[\forall \varepsilon > 0 \Rightarrow \exists \delta = \delta(\varepsilon) > 0 : x(0) \in \Pi, \ |x(0)| \leq \delta \Rightarrow |x(t)| \leq \varepsilon, \ \forall t \geq 0 \] (2.10)

- **Attractivity:**
  \[\lim_{t \to \infty} x(t) = 0, \ \forall x(0) \in \Pi \] (2.11)

The following proposition is one of our main results and its proof is given in Section 3.

**Proposition 2.** Under Assumption 1, system \( \mathbf{1} \) is SDF-SGAS.

We next present the precise statement of the central result of present work, which provides a Lie algebraic sufficient condition for SDF-SGAS(WGAS) for the affine in the control single-input system \( \mathbf{12} \). Assume that its dynamics $f$, $g$ are smooth ($C^\infty$) and let $\Delta \in \text{Lie}\{f, g\}$. We define

$$\text{order}_{(f,g)} \Delta := \begin{cases} 1, & \text{if } \Delta \in \text{span}\{f, g\} \\ k > 1, & \text{if } \Delta = [[[\omega_1, \omega_2], \omega_3], \ldots, \omega_k] \\ & \text{for some } 0 \neq \omega_i \in \text{span}\{f, g\} \end{cases}$$ (2.12)

**Proposition 3.** Suppose that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$, being positive definite and proper, such that for every $x_0 \neq 0$, either

\[ (gV)(x_0) = 0 \Rightarrow (fV)(x_0) < 0 \] (2.13)

or there exists an integer $N = N(x_0) \geq 1$ such that

\[ (gV)(x_0) = 0, \ (f^iV)(x_0) = 0, \ i = 1, 2, \ldots, N \] (2.14a)

\[ (\Delta_{i_1} \Delta_{i_2} \ldots \Delta_{i_k} V)(x_0) = 0 \]

\[ \forall \Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_k} \in \text{Lie}\{f, g\} \setminus \{g\} \]

with

\[ \sum_{p=1}^{k} \text{order}_{(f,g)} \Delta_{i_p} \leq N \] (2.14b)

and in such a way that one of the following properties hold:
(P1) \[(f^{N+1}V)(x_0) < 0 \tag{2.15}\]

(P2) \(N\) is odd and
\[(\underbrace{[[[f,g],g],\ldots,g],g]}_{N\ text{ times}}V)(x_0) \neq 0 \tag{2.16}\]

(P3) \(N\) is even and either
\[(\underbrace{[[[f,g],g],\ldots,g],g]}_{N\ text{ times}}V)(x_0) < 0 \tag{2.17}\]
or
\[(f^{N+1}V)(x_0) = 0, \tag{2.18a}\]
\[(\underbrace{[[[g,f],f],\ldots,f],f]}_{N\ text{ times}}V)(x_0) \neq 0 \tag{2.18b}\]

Then system (1.2) is SDF-SGAS.

Remark. (i) For the particular case of \(N = 1\) examined in [20], condition (2.14a) is equivalent to \((gV)(x_0) = 0\) and \((fV)(x_0) = 0\), the previous equality is equivalent to (2.14b) and obviously (2.16) is equivalent to \((|f,g|V)(x_0) \neq 0\).

(ii) The result of Proposition 2 can be extended to multi-input affine in the control systems; for reasons of simplicity, only the single-input is considered here.

3. Proof of Main Results

Proof of Proposition 2. Let \(R, \rho\) be a pair of constants with \(R > \rho \geq 0\) and define \(S[\rho,R] := \{x \in \mathbb{R}^n : \rho \leq V(x) < R\}\). By exploiting (2.14a) and (2.7b) and applying similar arguments with those in proof of Proposition 1 in [20], it follows that for any \(\xi > 0\) there exist \(\varepsilon_0 \in (0,\xi]\) such that for every \(\varepsilon \in (0,\varepsilon_0]\), a constant \(L = L(\rho, R) > 0\) can be found in such a way that for every \(\varepsilon \in (0,\varepsilon_0]\) and \(x_0 \in S[\rho, R]\) there exists a control \(u^t_{\varepsilon,x_0}(s) := u_{\varepsilon,x_0}(s-t) : [t,t+\varepsilon] \to \mathbb{R}^m\), (where the control \(u_{\varepsilon,x_0}(\cdot)\) is determined in (2.7)), such that the trajectory \(x_{\varepsilon,t,x_0}(\cdot), x(t,t,x_0,u^t_{\varepsilon,x_0}) = x_0\) satisfies:
\[V(x(t+\varepsilon,t,x_0,u^t_{\varepsilon,x_0})) \leq V(x_0) - L; \tag{3.1a}\]
\[V(x(t,t,x_0,u^t_{\varepsilon,x_0})) \leq 2a(V(x_0)), \quad \forall s \in [t,t+\varepsilon] \tag{3.1b}\]

Let \(R > 0\) arbitrary and let \(\bar{R} > 0\) be a constant such that \(B[0, \bar{R}] \subset S[0, \bar{R}]\). Consider a partition of constants \(\{R_n, n = 1, 2, \ldots\}\) with
\[R_1 = \bar{R}, \quad R_{n+1} < R_n, \quad \forall n = 1, 2, \ldots \quad \text{with} \quad \lim_{n \to \infty} R_n = 0 \tag{3.2}\]
Also, let \( \{ T_\nu, \nu = 1, 2, \ldots \} \) be a given partition of times satisfying (3.3). For each \( i = 1, 2, \ldots \) and constants \( \varepsilon_i > 0, i = 1, 2, \ldots \) consider the following partition of times:

\[
P_i := \{ t_{i,1} := 0, t_{i,2}, t_{i,3}, \ldots \} \quad \lim_{p \to \infty} t_{i,p} = \infty, \ i = 1, 2, \ldots \quad (3.3)
\]

satisfying the following properties:

\[
t_{i,p} < t_{i,p+1}; \quad (3.4a)
\]

\[
\{ T_\nu, \nu = 1, 2, \ldots \} \subset P_i \subset P_{i+1}; \quad (3.4b)
\]

\[
\varepsilon_i \geq t_{i,p+1} - t_{i,p}, \ \forall i, p \in \mathbb{N} \quad (3.4c)
\]

By using (3.1a) and (3.1b) with \( \rho = R_{i+1} \) and \( R = R_i, i = 1, 2, \ldots \), we may find a constant \( L_i > 0 \), a partition of times and sufficiently small constant \( \varepsilon_i > 0 \) such that (3.4) holds and simultaneously for \( x_0 \in S[R_{i+1}, R_i] \) and any pair of integers \( (i, p) \in \mathbb{N} \times \mathbb{N} \), a control \( u_{(i,p),x_0} : [t_{i,p}, t_{i,p} + \varepsilon_i] \to \mathbb{R}^m \) can be found satisfying:

\[
V(x(t_{i,p+1}, t_{i,p}, x_0, u_{(i,p),x_0})) \leq V(x_0) - L_i; \quad (3.5a)
\]

\[
V(x(s, t_{i,p}, x_0, u_{(i,p),x_0})) \leq 2a(V(x_0)), \forall s \in [t_{i,p}, t_{i,p+1}] \quad (3.5b)
\]

We conclude that, for given \( \{ T_\nu, \nu = 1, 2, \ldots \} \), a partition of times (3.3) can be determined in such a way that (3.5a), (3.5b) hold and simultaneously (3.5) is fulfilled, provided that \( x_0 \in S[R_{i+1}, R_i] \). For each initial \( x(0) \in \Pi \equiv S[0, R_1] \) consider the map \( x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n \) defined as follows:

\[
x(t) = x(t_{i,p+1}, x_{(i,p), x(t_{i,p})}) \quad \forall t \in [t_{i,p}, t_{i,p+1}], \ x(t_{i,p}) \in S[R_{i+1}, R_i], \ i, p \in \mathbb{N} \quad (3.6a)
\]

where the map \( \pi(t) := \pi(t, s, z, u) \) satisfies:

\[
\dot{\pi} = f(\pi, u), \ 0 \leq s \leq t \quad (3.6b)
\]

An immediate consequence of (3.5a), (3.5b), (3.6a) and (3.6b) is the following fact:

**Fact 1:** The map \( x(\cdot) \) as defined by (3.6a) satisfies:

\[
V(x(t_{i,p+1})) \leq V(x(t_{i,p})) - L_i; \quad (3.7a)
\]

\[
V(x(s)) \leq 2a(V(x(t_{i,p}))), \forall s \in [t_{i,p}, t_{i,p+1}], \ i, p \in \mathbb{N} \quad (3.7b)
\]

provided that \( x(t_{i,p}) \in S[R_{i+1}, R_i] \)

and as a consequence of (3.7a) we get:

**Fact 2:**

\[
V(x(t_k)) \leq V(x(t_1)) - (k - 1) \min\{L_j, j = \nu, \nu + 1, \ldots, m\}, \ \forall k, m, \nu \in \mathbb{N}; m > \nu, \ t_1 \in P_m, i = 1, 2, \ldots, k; \quad t_1 < t_2 < \ldots < t_k \quad (3.8)
\]

provided that \( x(t_1), x(t_2), \ldots, x(t_k) \in S[R_{m+1}, R_\nu] \).
and

\[ V(x(t_2)) \leq V(x(t_1)), \forall t_2 < t_1; \quad t_2, t_1 \in \bigcup_{i=1}^{\infty} P_i, x(t_1) \in \Pi \quad (3.9) \]

Moreover, by taking into account (3.4b), (3.7b) and (3.9), it follows:

**Fact 3**: For any \( \tau \in \bigcup_{i=1}^{\infty} P_i \) with \( x(\tau) \in \Pi \), there exists a sequence \( \{ t_k, k = 1, 2, \ldots \} \) with \( t_k \in \bigcup_{i=1}^{\infty} P_i \) and \( t_{k+1} < t_k < \tau \), \( k = 2, 3, \ldots \), \( t_1 := \tau \) such that \( \lim_{k \to \infty} t_k = \infty \) and

\[ V(x(s)) \leq 2a(V(x(t_k))), \forall s \in [t_k, t_{k+1}) \quad (3.10) \]

which by virtue of (3.9) implies:

\[ V(x(s)) \leq 2a(V(x(t_1))), \forall s \geq t_1 \quad (3.11) \]

We next show that the map \( x(\cdot) \) satisfies both (2.10) and (2.11). Since \( V \) is positive definite and proper, in order to establish (2.11), it suffices to show that for initial nonzero \( x(0) \in \Pi(= S[0, R_1]) \) and sufficiently small constant \( \sigma > 0 \) there exists a time \( \tau \in \bigcup_{i=1}^{\infty} P_i \) such that

\[ V(x(t)) \leq \sigma, \forall t \geq \tau \quad (3.12) \]

Let \( \xi, \sigma > 0 \) with \( 2a(\xi) < \sigma \) and \( \xi \leq R_1 \) and let \( m \) be an integer with

\[ R_{m+1} \leq \xi < R_m \quad (3.13) \]

We claim that there exists an integer \( \bar{p} \in \mathbb{N} \) such that \( t_{m, \bar{p}} \in P_m \) and

\[ V(x(t_{m, \bar{p}})) \leq \xi \quad (3.14) \]

Indeed, otherwise we would have

\[ \{ x(t_{m, p}) : p = 1, 2, \ldots \} \cap S[0, R_{m+1}) = \emptyset \]

and since \( t_{m, p} \in P_m \), we obtain from (3.8) that

\[ R_{m+1} < V(x(t_{m, p})) \leq V(x(0)) - (p - 1) \min_{\nu = 1, \ldots, m} \{ L_\nu, \nu = 1, \ldots, m \} \]

\[ \forall p = 1, 2, \ldots \]

a contradiction, hence (3.14) is fulfilled. The latter, in conjunction with (3.11) and the definition of \( \xi \) and \( \sigma \), implies \( 2a(V(x(t_{m, p}))) \leq 2a(\xi) < \sigma \), which by virtue of (3.11), asserts that for given \( x(0) \in \Pi \) and sufficiently small constant \( \sigma > 0 \) there exists a time \( \tau \in \bigcup_{i=1}^{\infty} P_i \) such that the map \( x(\cdot) \) satisfies \( V(x(t)) < 2a(V(x(\tau))) < \sigma \) for all \( t \geq \tau \), which establishes (2.11). Likewise, by using (3.11) with \( t_1 = 0 \) we can establish that (2.10) also holds for the map \( x(\cdot) \). We
are now in a position to establish that there exists a map \( k : \mathbb{R}^+ \times \Pi \to \mathbb{R}^m \) such that the trajectory of the sampled-data closed loop system \((2.9)\) satisfies both \((2.10)\) and \((2.11)\). Indeed, due to the first inclusion of \((3.4b)\), for each given \( T_i \) and vector \( z \in \Pi \) there exist times \( t_{ik,p_k} \in \bigcup_{i=1}^{\infty} P_i, k = 1, 2, ..., \nu \) and inputs \( \omega_k : [t_{ik,p_k}, t_{ik+1,p_k+1}] \to \mathbb{R}^m \), \( k = 1, 2, ..., \nu - 1 \) such that

\[
\begin{align*}
& t_{ik,p_k} < t_{ik+1,p_k+1}; i_k \leq i_{k+1};
& i_k = i_{k+1} \Rightarrow p_k+1 = p_k + 1;
& t_{i1,p_1} := T_i, \ t_{i\nu,p_{\nu}} := T_{i+1} \quad (3.15a)
\end{align*}
\]

\[
\begin{align*}
& x_1 := z; \ \omega_1(t) := u_{(i_1,p_1)}(t), t \in [t_{i_1,p_1}, t_{i_2,p_2}]
& x_2 := x(t_{i_2,p_2}, t_{i_1,p_1}, x_1, \omega_1), \ \omega_2(t) := u_{(i_2,p_2)}(t),
& t \in [t_{i_2,p_2}, t_{i_3,p_3}]
& x_3 := x(t_{i_3,p_3}, t_{i_2,p_2}, x_2, \omega_2), \ \omega_3(t) := u_{(i_3,p_3)}(t),
& t \in [t_{i_3,p_3}, t_{i_4,p_4}]
& \vdots
& x_{\nu-1} := x(t_{i_{\nu-1},p_{\nu-1}, t_{i_{\nu-2},p_{\nu-2}}}, x_{\nu-2}, \omega_{\nu-2});
& \omega_{\nu-1}(t) := u_{(i_{\nu-1},p_{\nu-1})}(\nu-1)(t), t \in [t_{i_{\nu-1},p_{\nu-1}}, t_{i_{\nu-1},p_{\nu}}]
\end{align*}
\]

Then, obviously, if we define:

\[
\phi_i(t, z) := \omega_k(t), t \in [t_{ik,pk}, t_{ik+1,pk+1}], z \in \Pi, \ k = 1, 2, ..., \nu - 1, \ t_{i_1,p_1} = T_i, \ t_{i\nu,p\nu} = T_{i+1} \quad (3.16a)
\]

\[
k(t, z) := \phi_i(t, z), t \in [T_i, T_{i+1}], i = 1, 2, ..., \ z \in \Pi \quad (3.16b)
\]

the map \( x(\cdot) \) as defined in \((3.6)\) coincides with the solution of the closed-loop \((2.9)\) with \( k : \mathbb{R}^+ \times \Pi \to \mathbb{R}^m \) as defined by \((3.15)\) and \((3.16)\), provided that their initial values at \( t = 0 \) are the same. It turns out, according to stability analysis made for \( x(\cdot) \), that \((2.10)\) and \((2.11)\) also hold for the trajectory of the system \((2.9)\) with \( k : \mathbb{R}^+ \times \Pi \to \mathbb{R}^m \) as defined above.

**Proof of Proposition 3.** Let \( 0 \neq x_0 \in \mathbb{R}^n \) and suppose first that either \((gV)(x_0) \neq 0, \) or the “Artstein-Sontag” implication in \((1.3)\) is fulfilled, namely, assume that \((gV)(x_0) = 0 \) and \((fV)(x_0) < 0. \) Then there exists a constant input \( u \) such that both \((2.7a)\) and \((2.7b)\) hold; particularly, for every sufficiently small \( \varepsilon > 0 \) we have:

\[
V(x(s, 0, x_0, u)) < V(x_0), \ \forall s \in (0, \varepsilon) \quad (3.17)
\]

Assume next there exists an integer \( N = N(x_0) \geq 1 \) satisfying \((2.14)\), as well as one of the properties \((P1), (P2), (P3)) \). In order to derive the desired conclusion we proceed as follows. We define:

\[
X := f + u_1g, \ Y := f + u_2g \quad (3.18)
\]

and let as denote by \( X_t(z) \) and \( Y_t(z) \) the trajectories of the systems \( \dot{x} = X(x) \) and \( \dot{y} = Y(y) \), respectively, initiated at time \( t = 0 \) from some \( z \in \mathbb{R}^n \). Also, for any constant \( a > 0 \) consider the time-varying map:

\[
R(t) := (X_{at} \circ Y_t)(x_0), t \geq 0, R(0) = x_0 \quad (3.19)
\]
Also, define:

\[ m(t) := V(R(t)), t \geq 0 \quad (3.20) \]

and denote in the sequel by \( (\nu) m(\cdot), \nu = 1, 2, \ldots \) its \( \nu \)-time derivative. We prove that, under previous assumptions concerning the integer \( N = N(x_0) \), there exist a constant \( a = a(x_0) > 0 \) and a pair of constant inputs \( u_1 \) and \( u_2 \) such that \( m(0) = 0, n = 1, 2, \ldots, N \) and \( m(N+1)(0) < 0 \). This would imply that \( m(t) < m(0) = V(x_0) \) for every \( t > 0 \) near zero and the latter in conjunction with (3.19) and (3.20) will lead to the validity of both inequalities (2.7a) and (2.7b) guaranteeing, according to Proposition 2, that (1.2) is SDF-SGAS. In order to get the desired result, we express the time derivatives \( (\nu) m(0), \nu = 1, 2, \ldots \) of the map \( m(\cdot) \) in terms of the elements of the Lie algebra of \( \{f, g\} \) and the function \( V \) evaluated at \( x_0 \). We apply the Campbell-Baker-Hausdorff formula for the right hand side map of (3.19). Then for every \( k \in \mathbb{N} \) we find:

\[
\dot{R}(t) = aX(R(t)) + (DX_{at}Y) \circ X_{-at}(R(t)) \\
= aX(R(t)) + Y(R(t)) + at[Y, X] \circ R(t) \\
+ \frac{a^2 t^2}{2!} [[Y, X], X] \circ R(t) + \ldots \\
+ \frac{a^k t^k}{k!} [\ldots[[Y, X], X], \ldots, X] \circ R(t) + O(t^k) 
\]

(3.21)

where \( \lim_{t \to 0^+} (O(t)/t) = 0 \). Let

\[
A_0 := aX + Y, \\
A_{\nu} := [\ldots[[Y, X], X], \ldots, X], \nu = 1, 2, \ldots 
\]

(3.22)

Notice that, since \( A_{\nu} \in \text{Lie}\{X, Y\} \), we may define, according to (2.12) the order of each \( A_{\nu} \) with respect to the Lie algebra of \( \{X, Y\} \); particularly, in our case, we have:

\[
\text{order}_{(X,Y)} A_{\nu} = \nu + 1, \quad \forall \nu = 0, 1, 2, \ldots
\]

(3.23)

Now, (3.21) is rewritten:

\[
\dot{R}(t) = (A_0 + atA_1 + \frac{1}{2!} a^2 t^2 A_2 + \ldots + \frac{1}{k!} a^k t^k A_k) \circ R(t) + O(t^k)
\]

(3.24)

thus by invoking (3.20) it follows that for any \( k \in \mathbb{N} \) we have:

\[
(1)^{(1)} \dot{m}(t) = (A_0 V + atA_1 V + \frac{1}{2!} a^2 t^2 A_2 V + \ldots \\
+ \frac{1}{k!} a^k t^k A_k V)(R(t)) + O(t^k)
\]

(3.25)

Since we have assumed that \( (fV)(x_0) = (gV)(x_0) = 0 \), it follows from (3.18), (3.22) and (3.26) that:

\[
(1)^{(1)} \dot{m}(0) = 0
\]

(3.26)
From (3.24) and (3.25) we find:

\[
\begin{align*}
(2) \dot{m}(t) &= (DA_0 V + atDA_1 V + \frac{a^2 t^2}{2!} DA_2 V + \ldots \\
&+ \frac{a^{k+1} t^k}{k!} DA_k V)(R(t)) \times \dot{R}(t) \\
&+ (aA_1 V + a^2 tA_2 V + \frac{a^3 t^2}{2!} A_3 V + \ldots \\
&+ \frac{a^{k+1} t^k}{(k+1)!} A_{k+1} V)(R(t)) + O(t^{k-1}) \\
= & (A_0^2 V)(R(t)) + ta \times \text{span} \{ A_1 A_0 V, A_0 A_1 V \} (R(t)) \\
&+ t^2 a^2 \times \text{span} \{ A_2 A_0 V, A_0^2 V, A_0 A_2 V \} (R(t)) \\
&+ t^3 a^3 \times \text{span} \{ A_3 A_0 V, A_2 A_1 V, A_1 A_2 V, A_3 A_0 V \} (R(t)) \\
&+ t^4 a^4 \times \text{span} \{ A_0 A_4 V, A_1 A_3 V, A_2^2 V, A_3 A_1 V, A_4 A_0 V \} (R(t)) \\
&+ \ldots + t^k a^k \times \{ A_k A_0 V, A_{k-1} A_1 V, \ldots, A_0 A_k V \} (R(t)) \\
&+ a(A_1 V)(R(t)) \\
&+ \text{span} \{ a^2 tA_2 V, a^3 t^2 A_3 V, \ldots, a^{k+1} t^k A_{k+1} V \}(R(t)) + O(t^{k-1}) \\
&\quad = (3.27)
\end{align*}
\]

We show by induction that for every pair of integers \(n, k\) with \(2 \leq n \leq k\), the \(n\)-time derivative \(m^{(n)}(\cdot)\) of \(m(\cdot)\) satisfies:

\[
\begin{align*}
&\quad m^{(n)}(t) \in S_n(t, x_0) := (A_0^2 V)(R(t)) \\
&\quad \quad + \sum_{j=0}^{n-k} \frac{t^j \text{span} \left\{ \sum_{\nu=1}^{\nu} \text{order}(X,Y) A_{ij} = n + j; \right.} \\
&\quad \quad \quad \quad \quad \left. \sum_{s=1}^{\nu} r^j = \sum_{s=1}^{\nu} i^j \in \{1, 2, \ldots, n + j - 2\} \right\} \\
&\quad \quad \quad + a^{n-1}(A_{n-1} V)(R(t)) \\
&\quad \quad \quad + \text{span} \{ a^n t(A_n V)(R(t)), a^{n+1} t^2(A_{n+1} V)(R(t)), \ldots, \\
&\quad \quad \quad \quad a^{n+k-1} t^k(A_{k+n-1} V)(R(t)) \} + O(t^{k-n+1}) \\
&\quad = (3.28)
\end{align*}
\]

with \(i^j, i^j, \ldots, i^j \in \mathbb{N}_0, j = 0, 1, 2, \ldots, k\). By taking into account (3.27), it can be easily verified that inclusion (3.28) is indeed fulfilled for \(n = 2\). Suppose that (3.28) holds for some integer \(n, 2 \leq n < k\). We show that it is also fulfilled for \(n = n + 1 \leq k\). Indeed, from (3.28) the \((n + 1)\)-time derivative of \(m(\cdot)\) is
\[
\frac{(n+1)m}{m} (t) = \frac{d}{dt} \left( \frac{n}{m}(t) \right) \in D(A_0^n V)(R(t)) \dot{R}(t)
\]

\[
+ \sum_{j=0}^{k} \sum_{\nu=1}^{m} \sum_{s=1}^{\nu} \text{span} \left\{ \begin{array}{l}
D(a_t A_{i_1} \ldots A_{i_\nu} V)(R(t)) : \nu \geq 2;
\sum_{s=1}^{\nu} \text{order}_{(X,Y)} A_{i_\nu} = n + j;

r_n = \sum_{s=1}^{\nu} i^s \in \{1, 2, \ldots, n + j - 2\}
\end{array} \right\} \times \dot{R}(t)
\]

\[
+ \sum_{j=1}^{k} j^{j-1} \times \text{span} \left\{ \begin{array}{l}
a_t^r (A_{i_1} \ldots A_{i_\nu} V)(R(t)) : \nu \geq 2;
\sum_{s=1}^{\nu} \text{order}_{(X,Y)} A_{i_\nu} = n + j;

r_n = \sum_{s=1}^{\nu} i^s \in \{1, 2, \ldots, n + j - 2\}
\end{array} \right\}
\]

\[
+ a^{n-1} D(A_{n-1} V)(R(t)) \times \dot{R}(t)
\]

\[
+ \text{span} \{ a^n t D(A_n V)(R(t)), a^{n+1} t^2 D(A_{n+1} V)(R(t)), \ldots, a^{n+k-1} t^k D(A_{k+n-1} V)(R(t)) \} \times \dot{R}(t)
\]

\[
+ \text{span} \{ a^n(A_n V)(R(t)), a^{n+1} t(A_{n+1} V)(R(t)), \ldots, a^{n+j} t^j (A_{n+j} V)(R(t)), j = 0, 1, 2, \ldots, k \} + O(t^{k-n})
\]

Hence, by invoking (3.24) we have:

\[
\frac{(n+1)m}{m} (t) \in (A_0^{n+1} V)(R(t))
\]

\[
+ \text{span} \{ a^q t^q (A_q A_0^n V)(R(t)), q = 1, n, n + 1, \ldots, k \}
\]

\[
+ \sum_{q=0, \ldots, k} \sum_{j=0, \ldots, k} j^{j+q} \times \text{span} \left\{ \begin{array}{l}
a^{r+q} (A_{i_1} \ldots A_{i_\nu} V)(R(t)) : \nu \geq 2;
\sum_{s=1}^{\nu} \text{order}_{(X,Y)} A_{i_\nu} = n + j;

r_n = \sum_{s=1}^{\nu} i^s \in \{1, 2, \ldots, n + j - 2\}
\end{array} \right\}
\]

\[
+ a^n(A_n V)(R(t))
\]

\[
a^{n-1} \text{span} \{ a^q t^q (A_q A_{n-1} V)(R(t)) ; q = 0, 1, 2, \ldots, n, n + 1, \ldots, k \}
\]

\[
+ \text{span} \{ a^{n+1} t^{j+q} (A_q A_{j+n-1} V)(R(t)) ; j = 1, 2, \ldots, n, n + 1, \ldots, k, q = 0, 1, \ldots, k ; j + q \leq k \}
\]

\[
+ \text{span} \{ a^{n+1} t^j (A_{n+1} V)(R(t)), \ldots, a^{n+j} t^j (A_{n+j} V)(R(t)), j = 1, 2, \ldots, k \} + O(t^{k-n})
\]

(3.30)
Notice that each new term \( t^K a^I A'_1 \ldots A'_{r_M} V \) that appears above satisfies
\[
\sum_{s=1}^{s=M} \text{order}_{\{X,Y\}} A'_s = (n+1) + K; \tag{3.31}
\]
\[
L = \sum_{s=1}^{s=M} \tau_s \in \{1,2,\ldots,(n+1) + K - 2\} \tag{3.32}
\]
For completeness we note that for the terms \( q a^I t^Q (A_q A'_0 V) \), \( q = 1,\ldots,k \) it follows, by taking into account \( \{A_q A'_0 V\} \) and \( \{A_q A'_0 V\} \), that \( \text{order}_{\{X,Y\}} A'_q \) + \( \sum_{s=1}^{s=M} \text{order}_{\{X,Y\}} A'_s = (n+1) + q \) and obviously \( \{A_q A'_0 V\} \) holds as well. For the terms \( t^j a'^{r^j+q} (A_q A'_j \ldots A'_{r_M} V) \)
we have:
\[
\text{order}_{\{X,Y\}} A'_q + \sum_{j=1}^{\nu} \text{order}_{\{X,Y\}} A'_{r^j} = (n+1) + q + j
\]
and, since \( r^j_0 \in \{1,\ldots,n + j - 2\} \) as imposed in \( \{A_q A'_0 V\} \), we have:
\[
r^j_0 + q \in \{1,2,\ldots,n + q + j - 2\} \subset \{1,2,\ldots,(n+1) + (q + j) - 2\}.
\]
Also, for the terms \( t^j a'^{r^j_0} (A_j A'_j \ldots A'_{r_M} V) \) in \( \{A_q A'_0 V\} \) we have:
\[
\sum_{j=1}^{\nu} \text{order}_{\{X,Y\}} A'_{r^j} = (n+1) + j - 1
\]
and obviously \( r^j_0 \in \{1,\ldots,n + j - 2\} \subset \{1,2,\ldots,(n+1) + j - 2\} \). Likewise, we handle the rest terms in the right hand side of \( \{A_q A'_0 V\} \) and show that both \( \{A_q A'_0 V\} \) and \( \{A_q A'_0 V\} \) hold. These conditions imply that the right hand set in \( \{A_q A'_0 V\} \) is included in \( S_{n+1}(t,x_0) \) as the latter is defined in \( \{A_q A'_0 V\} \), which guarantees that inclusion \( \{A_q A'_0 V\} \) holds for \( n := n + 1 \) and therefore is fulfilled for every pair of integers \( 2 \leq n \leq k \). It follows from \( \{A_q A'_0 V\} \) and \( \{A_q A'_0 V\} \) that
\[
\bar{m}(0) = (A_0^q V)(x_0) + (a A_1 V)(x_0) \tag{3.33}
\]
for the case \( n = 2 \) and generally for \( n \geq 2 \):
\[
\bar{m}(0) = (A_0^q V)(x_0) + \text{span} \left\{ a^\nu (A_{q_1} A_{q_2} \ldots A_{q_{\nu}} V)(x_0) : \nu \geq 2; \begin{array}{c} a^\nu (A_{q_1} A_{q_2} \ldots A_{q_{\nu}} V)(x_0) = n; \\ q_1, q_2, \ldots, q_{\nu} \in \mathbb{N}_0; \sum_{j=1}^{\nu} \text{order}_{\{X,Y\}} A'_{r^j} = n; \\ r^j_0 = \sum_{j=1}^{\nu} q^j_0 \in \{1,2,\ldots,n - 2\} \end{array} \right\} + a^{n-1} (A_{n-1}^1 V)(x_0) \tag{3.34}
\]
By taking into account definition \( \{A_q A'_0 V\} \) of the vector fields \( X \) and \( Y \) and by setting
\[
u_2 = -a u_1 \tag{3.35}
\]
we get

\[ A_0 = (a + 1)f, \quad A_1 = (a + 1)u_1[f, g] \]
\[ A_2 = (a + 1)(u_1^2[[[f, g], g], f]) \]
\[ \vdots \]
\[ A_n = (a + 1)u_1^n[[[f, g], g], \ldots, [g, f], f] + \ldots \]
\[ + (a + 1)u_1^n([[[[f, g], f], \ldots, [f, f], g] + \ldots\]
\[ + [[[f, g], f], \ldots, [f, f], g] + \ldots + [[[f, g], [f], \ldots, f], f]) \]
\[ - (a + 1)u_1[[[g, f], f], \ldots, f]] \]

\[ n = 3, 4, \ldots \] \hspace{1cm} (3.36)

Obviously, (3.36) implies:

\[ A_k \in \text{span}\{\Delta \in \text{Lie} \{f, g\} \setminus \{g\} : \text{order}\Delta_{(f, g)} = k + 1\} \]
\[ k = 0, 1, 2, \ldots \] \hspace{1cm} (3.37)

Also, we recall from (3.23) and (3.34) that

\[ r_0^n = \sum_{s=1}^{n} t_s \in \{1, 2, \ldots, n - 2\} \text{ and } \sum_{j=1}^{n} \text{order}_{(x, y)}A_{ij} = r_0^n + \nu = n \text{ with } \nu \geq 2 \text{ and therefore } \nu \leq n - 1. \]

By (3.34) - (3.37) and the previous facts we get:

\[ (n) \]
\[ m(0) \in (a + 1)^n(f^nV)(x_0) + u_1 \pi_1(a, a + 1; x_0) \]
\[ + \text{span}\{u_1^k \pi_k(a, a + 1; x_0), k = 2, \ldots, n - 2\} \]
\[ + a^{n-1}(a + 1)u_1^{n-1}([[[[f, g], [f], \ldots, f], f)V(x_0) \]
\[ - a^{n-1}(a + 1)u_1([[[[g, f], f], \ldots, f], f)V(x_0) \]
\[ n = 2, 3, \ldots \] \hspace{1cm} (3.38)

for \( n = 2, 3, \ldots \) and for certain smooth functions

\[ \pi_k : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}, \quad k = 1, 2, \ldots, n - 2 \]

satisfying the following properties:

(S1) For each \( x_0 \in \mathbb{R}^n \) each map \( \pi_k(\alpha, \beta; x_0) : \mathbb{R}^2 \to \mathbb{R} \) is a polynomial with respect to the first two variables in such a way that

\[ \text{span}\{\pi_k(\alpha, \beta; x_0), k = 1, 2, \ldots, n - 2\} \subset \]
\[ \text{span}\{\Delta_{i_1, i_2, \ldots, i_k}V(x_0) : i_1, i_2, \ldots, i_k \in \mathbb{N}_0, \]
\[ \Delta_{i_1, i_2, \ldots, i_k} \in \text{Lie}\{f, g\} \setminus \{g\}; \]
\[ \sum_{j=1}^{k} \text{order}_{(f, g)}\Delta_{ij} = n \} \]

(3.39)
For each \( x_0 \in \mathbb{R}^n \) there exist integers \( \lambda_i, \mu_i, i = 1, 2, ..., L \in \mathbb{N} \) with \( 1 \leq \lambda_i \leq n - 2, 2 \leq \mu_i \leq n - 1 \) such that the map \( \pi_1(\alpha, \beta; x_0) : \mathbb{R}^2 \to \mathbb{R} \) satisfies:

\[
\pi_1(\alpha, \beta; x_0) \in \text{span} \{ a^{\lambda_1} \beta^{\mu_1}, a^{\lambda_2} \beta^{\mu_2}, ..., a^{\lambda_L} \beta^{\mu_L} \}
\]  
(3.40)

The latter implies that for each fixed \( x_0 \in \mathbb{R}^n \) the polynomials \( \pi_1(a, a + 1; x_0) \) and

\[-a^{n-1}(a + 1)(\underbrace{[\ldots [g, f], f], \ldots, fV}_{n-1 \text{ times}})(x_0)\]

are linearly independent, provided that

\[
(\underbrace{[\ldots [g, f], f], \ldots, fV}_{n-1 \text{ times}})(x_0) \neq 0
\]  
(3.41)

If we define:

\[
\xi_n(a; x) := \pi_1(\alpha, a + 1; x_0) - a^{n-1}(a + 1)(\underbrace{[\ldots [g, f], f], \ldots, fV}_{n-1 \text{ times}})(x_0)
\]  
(3.42)

the inclusion (3.38) is rewritten:

\[
\begin{align*}
&\pi_1^{(n)}(0) \in (a + 1)^n(f^nV)(x_0) + u_1 \xi_n(a; x_0) \\
&+ \text{span} \{ u_1^k \pi_k(a, a + 1; x_0), k = 2, ..., n - 2 \} \\
&+ a^{n-1}(a + 1) u_1^{n-1}(\underbrace{[\ldots [f, g], g], \ldots, gV}_{n-1 \text{ times}})(x_0)
\end{align*}
\]  
(3.43)

and a constant \( a = a(x_0) > 0 \) can be found with

\[
\xi_n(a; x_0) \neq 0
\]  
(3.44)

provided that (3.41) holds. Suppose now that there exists an integer \( N = N(x_0) \geq 1 \) satisfying (2.14), as well as one of the properties (P1), (P2), (P3). By (3.26) and by taking into account (2.14), (3.38) and (3.39) it follows:

\[
(\pi_1^{(n)}(0) = 0, n = 1, 2, ..., N
\]  
(3.45)

and we distinguish four cases:

**Case 1:** (2.15) holds. Then by using (3.43) with \( n := N + 1 \) and by setting \( u_1 = 0 \) we find:

\[
m^{(N+1)}(0) < 0
\]  
(3.46)

**Case 2:** \( N \) is even and (2.17) holds. Then by using (3.43) with \( n := N + 1 \) it follows that there exists a sufficiently large constant \( u_1 = u_1(x_0) > 0 \) such that (3.46) holds.
Case 3: \( N \) is even and both (2.18a) and (2.18b) are satisfied. Then, due to assumption (2.18b), it follows that (3.41) is fulfilled with \( n := N + 1 \) hence it follows that there exists a constant \( a = a(x_0) > 0 \) satisfying (3.44) with \( n := N \). By invoking again (3.43) with \( n := N + 1 \) and by taking into account assumption (2.18a), it follows that there exists a sufficiently small constant \( u_1 = u_1(x_0) \neq 0 \) such that (3.46) holds.

Case 4: \( N \) is odd and (2.16) holds. We again invoke (3.43) with \( n := N + 1 \) and our assumption that \( N \) is odd. It follows that there exists a sufficiently large constant \( u_1 = u_1(x_0) \) such that again (3.46) is fulfilled. We conclude, by taking into account (3.19), (3.20), (3.35), (3.44) and (3.45), that in all previous cases, there exists a constant \( u_1 \) such that, if we define:

\[
\begin{align*}
  u_{t,x_0}(s) := & \begin{cases} 
    u_2 = -a u_1, & s \in [0, t] \\
    u_1, & s \in (t, t + at]
  \end{cases} 
\end{align*}
\]  

(3.47)

with \( a = a(x_0) > 0 \) as considered in the third case above, then for every sufficiently small \( \varepsilon_0 > 0 \) we have:

\[
m(t) < m(0), \forall t \in (0, \varepsilon_0]
\]  

(3.48)

where \( m(t) := V((X_{s\cdot} \circ Y_t)(x_0)) = V(x(t + at, 0, x_0, u_t,x_0)) \) and \( x(\cdot, 0, x_0, u_t,x_0) \) is the trajectory of (1.2) corresponding to the input \( u_{t,x_0} \). Equivalently:

\[
V(x(t, 0, x_0, u_{t,x_0})) < V(x_0), \forall t \in (0, \varepsilon_0]
\]  

(3.49)

Since the constant \( a = a(x_0) \) is independent of \( t \), we may pick \( \varepsilon \in (0, \varepsilon_0] \) sufficiently small in such a way that inequality in (3.49) holds for \( t := \varepsilon \), namely:

\[
V(x(\varepsilon, 0, x_0, u_{\varepsilon,x_0})) < V(x_0)
\]  

(3.50)

and simultaneously:

\[
V(x(s, 0, x_0, u_{\varepsilon,x_0})) \leq 2V(x_0), \forall s \in (0, \varepsilon]
\]  

(3.51)

We conclude, by taking into account (3.17) and (3.50) together with (3.51), that for every \( x_0 \neq 0 \) and every sufficiently small \( \varepsilon_0 > 0 \) there exist \( \varepsilon \in (0, \varepsilon_0] \) and a measurable and essentially bounded control \( u_{\varepsilon,x_0} : [0, \varepsilon] \to \mathbb{R} \) such that (2.46) and (2.7b) hold with \( a(s) := 2s \). Therefore, according to Proposition 2, the system (1.2) is SDF-SGAS.

4. Examples

The following examples illustrate the nature of Proposition 2. The first example below generalizes Example 12 in [19].

Example 1. Consider the planar case:

\[
\dot{x}_1 = F(x_1, x_2), \dot{x}_2 = u, \ (x_1, x_2) \in \mathbb{R}^2
\]  

(4.1)
where \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) is \( C^\infty \) and assume that for every \( x_1 \neq 0 \) one of the following properties hold: (H1)

\[
x_1 F(x_1, 0) < 0
\]

(H2) there exists an odd integer \( N = N(x_1) \geq 1 \) with

\[
\frac{\partial^i F}{\partial x_2^i}(x_1, 0) = 0, \quad i = 0, 1, \ldots, N - 1; \quad \frac{\partial^N F}{\partial x_2^N}(x_1, 0) \neq 0
\]

(H3) there exists an even integer \( N = N(x_1) \geq 1 \) with

\[
\frac{\partial^i F}{\partial x_2^i}(x_1, 0) = 0, \quad i = 0, 1, \ldots, N - 1; \quad x_1 \frac{\partial^N F}{\partial x_2^N}(x_1, 0) < 0
\]

Then by setting \( x := (x_1, x_2)^T, \ V(x) := \frac{1}{2}(x_1^2 + x_2^2), \ f(x) := (F(x_1, x_2), 0)^T \) and \( g(x) := (1, 0)^T \) it follows from (4.2)-(4.3) that, either (2.13) holds, or (2.14) together with one of the properties (P2), (P3) of Proposition 2 are fulfilled. We conclude that system (4.1) is SDF-SGAS.

**Example 2.** Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2 a(x_3) \\
\dot{x}_2 &= -x_1 b(x_3) \\
\dot{x}_3 &= u
\end{align*}
\]

where \( a(\cdot), b(\cdot) \in C^2(\mathbb{R}, \mathbb{R}) \), which satisfy:

\[
a(0) = b(0) \neq 0 \tag{4.6}
\]

\[
(1) a(0) \neq (1) b(0) \tag{4.7}
\]

where \((1) a( \cdot )\) and \((1) b( \cdot )\) denote the first derivatives of the functions \( a(\cdot) \) and \( b(\cdot) \), respectively. Define:

\[
\begin{align*}
x &:= (x_1, x_2, x_3)^T, \ f(x) := (x_2 a(x_3), -x_1 b(x_3), 0)^T, \\
g(x) := (0, 0, 1)^T, \ V(x) := \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).
\end{align*}
\]

Then we find:

\[
(gV)(x) = 0 \Leftrightarrow x_3 = 0 \tag{4.8}
\]

\[
([f, g]V)(x) = x_1 x_2 \left( (1) b(0) - (1) a(0) \right) \tag{4.9}
\]

It follows by (4.6) that

\[
(gV)(x) = 0, \ x \neq 0 \Rightarrow (fV)(x) = (f^2V)(x) = (f^3V)(x) = 0 \tag{4.10}
\]

We distinguish two cases:
Case 1: Suppose that
\[(gV)(x) = 0 \text{ for certain } x = (x_1, x_2, x_3)^T \neq 0 \quad (4.11)\]
and both \(x_1\) and \(x_2\) are nonzero. Then due to our assumption (4.7), it follows by (4.9):
\[[f, g](V)(x) \neq 0 \quad (4.12)\]
which in conjunction with (4.10) and (4.11) assert that system (4.5) satisfies the assumptions (2.14) and (2.16) of Proposition 2 with \(N = 1\).

Case 2: Suppose next that (4.11) holds and further
\[[g, f](V)(x) = 0 \quad (4.13)\]
for certain \(x = (x_1, x_2, x_3)^T \neq 0\), which by virtue of (4.7), (4.9) and (4.11) is equivalent to
\[x_1x_2 = 0, \ (x_1, x_2) \neq 0 \quad (4.14)\]
We finally evaluate:
\[([f, g], g)(V)(x_1, x_2, 0) = x_1x_2 ((2)\ a(0) - (2)\ b(0)) \quad (4.15a)\]
\[([g, f], f)(V)(x_1, x_2, 0) = x_1^2 ((1)\ a(0)b(0) - a(0) (1)\ b(0)) + x_2^2 (a(0) (1)\ b(0) - (1)\ a(0)b(0)) \quad (4.15b)\]
where \((2)\ a(\cdot)\) and \((2)\ b(\cdot)\) denote the second derivatives of the functions \(a(\cdot)\) and \(b(\cdot)\), respectively, hence, by (4.6), (4.7) and (4.13) we find \([f, g](V)(x) = 0\) and \([g, f](V)(x) \neq 0\). The latter in conjunction with (4.10), (4.11) and (4.13)-(4.15) assert that assumptions (2.14) and (2.18) are satisfied with \(N = 2\) for the above nonzero vector \(x = (x_1, x_2, x_3)^T\). We conclude that in both cases, all hypotheses of Proposition 2 are satisfied, therefore system (4.5) is SDF-SGAS.

References

[1] F. Ancona, A. Bressan, Patchy vector fields and asymptotic stabilization, ESAIM-COCV 4 (1999) 445-471.

[2] A. Anta, P. Tabuada, To sample or not to sample: self-triggered control for nonlinear systems, IEEE Transactions on Autom. Control 55 (2010) 2030-2042.

[3] Z. Artstein, Stabilization with relaxed controls, Nonlinear Analysis TMA 7 (1983) 1163-1173.

[4] A. Bacciotti, L. Mazzi, From Artstein-Sontag Theorem to the min-projection strategy, Trans. of the Institute of Measurement and Control 32 (6) (2010) 571-581.
[5] A. Bacciotti, L. Mazzi, Stabilizability of nonlinear systems by means of time-depended switching rules, Int. J. Control 83 (4) (2010) 810-815.

[6] F.H. Clarke, Y.S. Ledyaev, E.D. Sontag and A.I. Subbotin, Asymptotic controllability implies feedback stabilization, IEEE Trans. Autom. Control 42 (10) (1997) 1394-1407.

[7] R. Goebel, A.R. Teel, Direct design of robustly asymptotically stabilizing hybrid feedback, ESAIM-COCV 15 (1) (2009) 205-213.

[8] L. Grüne, D. Nešić, Optimization based stabilization of sampled-data nonlinear systems via their approximate discrete-time models, SIAM J. Control Optim. 42 (2003) 98-122.

[9] N. Marchand, M. Alamir, Asymptotic controllability implies continuous discrete-time feedback stabilization, in: Nonlinear Control in the Year 2000, vol. 2, Springer, Berlin, Heidelberg, New York, 2000.

[10] D. Nešić, A.R. Teel, P.V. Kokotovic, Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations, Systems and Control Lett. 38 (4-5) (1999) 259-270.

[11] D. Nešić, A.R. Teel, A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models, IEEE Trans. Autom. Control 49 (2004) 1103-1122.

[12] C. Prieur, Asymptotic controllability and robust asymptotic stabilizability, SIAM J. Control Optim. 43 (2005) 1888-1912.

[13] C. Prieur, R. Goebel, A.R. Teel, Hybrid feedback control and robust stabilization of nonlinear systems, SIAM J. Control Optim. 43 (2005) 1888-1912

[14] H. Shim, A.R. Teel, Asymptotic controllability and observability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback, Automatica 39 (2003) 441-454.

[15] E.D. Sontag, Mathematical control theory, 2nd edn., Springer, Berlin, Heidelberg, New York, 1998.

[16] E.D. Sontag, A "universal" construction of Artstein's theorem on nonlinear stabilization, Systems and Control Lett. 13 (1989) 117-123.

[17] P. Tabuada, Event-triged real-time scheduling of stabilizing control tasks, IEEE Trans. Autom. Control 52 (2007) 1680-1685.

[18] J. Tsinias, Sufficient Lyapunov-like conditions for stabilization, Math. Contr. Sign. Syst. 2 (1989) 343-357.

[19] J. Tsinias, Remarks on asymptotic controllability and sampled-data feedback stabilization for autonomous systems, IEEE Trans. Autom. Control 55 (2010) 721-726.
[20] J. Tsinias, New results on sampled-data feedback stabilization for autonomous nonlinear systems, Systems and Control Lett. 61 (2012) 1032-1040.