Training Well-Generalizing Classifiers for Fairness Metrics and Other Data-Dependent Constraints

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October 2, 2018

Abstract

Classifiers can be trained with data-dependent constraints to satisfy fairness goals, reduce churn, achieve a targeted false positive rate, or other policy goals. We study the generalization performance for such constrained optimization problems, in terms of how well the constraints are satisfied at evaluation time, given that they are satisfied at training time. To improve generalization performance, we frame the problem as a two-player game where one player optimizes the model parameters on a training dataset, and the other player enforces the constraints on an independent validation dataset. We build on recent work in two-player constrained optimization to show that if one uses this two-dataset approach, then constraint generalization can be significantly improved. As we illustrate experimentally, this approach works not only in theory, but also in practice.

1 Introduction

It is useful to train classifiers with data-dependent constraints in order to achieve certain guarantees on the training set, such as statistical parity or other fairness guarantees, specified recall, or a desired positive classification rate [e.g. \cite{scott2005fairness, zafar2015fairness, goh2016fairness, woodworth2017fairness, narasimhan2018fairness}]. However, a key question is whether the achieved constraints will \textit{generalize}. For example: will a classifier trained to produce 80\% statistical parity on training examples still achieve 80\% statistical parity at evaluation time?

Unfortunately, the answer is “not quite.” Because such constraints are data-dependent, overfitting can occur, and constraints that were satisfied on the training set should be expected to be slightly violated on an \textit{i.i.d.} test set. This is particularly problematic in the context of fairness constraints, which will typically be chosen based on real-world requirements (e.g. the 80\% rule of some US laws \cite{biddle2005fairness, vuolo2013fairness, zafar2015fairness,hardt2016equality}). In this paper, we investigate how well constraints generalize, and propose algorithms to improve the generalization of constraints to new examples.

Specifically, we consider problems that minimize a loss function subject to data-dependent constraints, expressed in terms of \textit{expectations} over a data distribution $\mathcal{D}$:

\begin{equation}
\min_{\theta \in \Theta} \mathbb{E}_{x \sim \mathcal{D}} [\ell_0 (x; \theta)] \\
\text{s.t.} \quad \mathbb{E}_{x \sim \mathcal{D}} [\ell_i (x; \theta)] \leq 0
\end{equation}
where \( x \in \mathcal{X} \) is a feature vector, \( D \) is the data distribution over \( \mathcal{X} \), \( \Theta \) is a space of model parameters for the function class of interest, and \( \ell_0, \ell_1, \ldots, \ell_m : \mathcal{X} \times \Theta \to \mathbb{R} \) are loss functions associated with the objective and the \( m \) constraints. We do not require these loss functions to be convex. Appendix A contains two examples of how Equation 1 can be used to express certain data-dependent constraints (see Goh et al. [2016] and Narasimhan [2018] for more).

One typically trains a classifier on a finite training set drawn from \( D \), but the true goal is to satisfy constraints in expectation over \( D \), as in Equation [1]. To this end, we build on a long line of prior work that treats constrained optimization as a two-player game [e.g. Christiano et al., 2011; Arora et al., 2012; Rakhlin and Sridharan, 2013; Kearns et al., 2017; Narasimhan, 2018; Agarwal et al., 2018]. In this setting, the first player optimizes the model parameters \( \theta \), and the second player enforces the constraints, e.g. using the Lagrangian formulation:

\[
\mathcal{L}(\theta, \lambda) := \mathbb{E}_{x \sim D} \left[ \ell_0(x; \theta) + \sum_{i=1}^{m} \lambda_i \ell_i(x; \theta) \right]
\]

In practice, one would approximate the Lagrangian with a finite i.i.d. training sample from \( D \), and the first player would minimize over the model parameters \( \theta \in \Theta \) while the second player maximizes over the Lagrange multipliers \( \lambda \in \mathbb{R}_+^m \).

Our key idea is to treat constrained optimization similarly to hyperparameter optimization: just as one typically chooses hyperparameters based on a validation set, instead of the training set, to improve classifier generalization, we would like to choose the Lagrange multipliers on a validation set to improve constraint generalization. In “inner” optimizations we would, given a fixed \( \lambda \), minimize the empirical Lagrangian on the training set. Then, in an “outer” optimization, we would choose a \( \lambda \) that results in the constraints being satisfied on the validation set. Such an approach, could it be made to work, would not eliminate the constraint generalization problem completely—hyperparameter overfitting [e.g. Ng, 1997] is a real problem—but would mitigate it, since constraint generalization would no longer depend on size of the training sample and the complexity of \( \Theta \) (which could be extremely large, e.g. for a deep neural network), but rather on the size of the validation sample and the effective complexity of \( \mathbb{R}_+^m \ni \lambda \), which, being \( m \)-dimensional, is presumably much simpler than \( \Theta \).

While the above approach is intuitive, challenges arise when attempting to analyze it. The most serious is that since \( \theta \) is chosen based on the training set, and \( \lambda \) on the validation set, the \( \theta \)-player is minimizing a different function than the \( \lambda \)-player is maximizing, so the corresponding two-player game is non-zero-sum (the players have different cost functions). To handle this, we must depart from the typical Lagrangian formulation, but the key idea remains: improving generalization by using a separate validation set to enforce the constraints.

Fortunately, the recent work of Cotter et al. [2018] gives a strategy for dealing with a non-zero-sum game in the context of constrained supervised learning. We adapt their approach to this new setting to give bounds on constraint generalization that are agnostic to model complexity. After some preliminary definitions in Section 3, in Section 4 we present two algorithms for which we can provide theoretical bounds.

In Section 5 we perform a set of experiments demonstrating that our two-dataset approach successfully improves constraint generalization even when our theoretical results do not hold. In other words, providing independent datasets to the \( \theta \)- and \( \lambda \)-players seems to work well as a heuristic for improving constraint generalization.

2 Related Work

While several recent papers have proved generalization bounds for constrained problems [e.g. Goh et al. [2016]; Agarwal et al. [2018]; Donini et al. [2018]], the problem of improving constraint generalization is a fairly new one, having, so far as we know, only been previously considered in the work of Hardt et al. [2016], who handled generalization subject to “equalized odds” constraints in the setting of Hardt et al. [2016]. Specifically, their approach is to first learn a predictor on \( S^{(train)} \), and then to learn a “correction” on \( S^{(valid)} \) to more tightly satisfy the fairness constraints. The second stage requires estimating only a constant number of parameters, and the final predictor consequently enjoys a
generalization guarantee for the fairness constraints which is independent of the predictor’s complexity, with only a modest penalty to the loss. However, their approach relies heavily upon the structure of equalized odds constraints: it requires that any classifier can be modified to satisfy the fairness constraints and have low loss on a validation set by tuning only a small number of parameters.

Woodworth et al. [2017]’s overall approach can be summarized as “train a complicated model on a training set, and then a simple correction on a validation set”. If, as they show to be the case for equalized odds constraints, the “simple correction” is capable of satisfying the constraints without significantly compromising on quality, then this technique results in a well-performing model for which the validation constraint generalization depends not on the complexity of the “complicated model”, but rather of that of the “simple correction”. In this paper, we extend Woodworth et al. [2017]’s two-dataset idea to work on data-dependent constraints in general.

Our primary baseline is Agarwal et al. [2018]’s recently-proposed algorithm for fair classification using the Lagrangian formulation. Their proposal, like our Algorithm 1, uses an oracle to optimize w.r.t. θ (they use the terminology “best response”), and, like all of our algorithms, results in a stochastic classifier. However, our setting differs slightly from theirs—they focus on fair classification, while we work in the slightly more general inequality-constrained setting (Equation 1). For this reason, in Appendix D we provide an analysis of the Lagrangian formulation for inequality constrained optimization.

3 Background & Definitions

Our algorithms are based on the non-zero-sum two-player game proposed by Cotter et al. [2018], which they call the “proxy-Lagrangian” formulation. The key novelty of their approach is the use of “proxy” constraint losses, which are (non-surrogate) constraints. Our work differs in that we use a non-zero-sum game to provide different datasets (Equation 1). For this reason, in Appendix D we provide an analysis of the Lagrangian formulation for inequality constrained optimization.

Despite this difference, the use of proxy-constraints is perfectly compatible with our proposal, so we permit the approximation of each of our constraint losses ℓi with a (presumably differentiable) upper-bound ˜ℓi. These are used only by the θ-player; the λ-player uses the original constraint losses. The use of proxy constraint losses is entirely optional: one is free to choose ˜ℓi := ℓi for all i.

Definition 1. Let S^{(train)} and S^{(val)} be two random datasets each drawn i.i.d. from a data distribution D. Given proxy constraint losses ˜ℓi(x; θ) ≥ ℓi(x; θ) for all x ∈ X, θ ∈ Θ and i ∈ [m], the empirical proxy-Lagrangians ˆLθ, ˆLλ : Θ × Λ → R of Equation 7 are:

\[ ˆL_θ(θ, λ) := \frac{1}{|S^{(train)}|} \sum_{x \in S^{(train)}} \left( \lambda_1 \ell_0(x; θ) + \sum_{i=1}^{m} \lambda_{i+1} ˜\ell_i(x; θ) \right) \]

\[ ˆL_λ(θ, λ) := \frac{1}{|S^{(val)}|} \sum_{x \in S^{(val)}} \sum_{i=1}^{m} \lambda_{i+1} ˜\ell_i(x; θ) \]

where Λ := Δ^m+1 is the (m + 1)-dimensional simplex.

The difference between the above, and Definition 2 of Cotter et al. [2018], is that ˆLθ is an empirical average over the training set, while ˆLλ is over the validation set. The θ-player seeks to minimize ˆLθ over θ, while the λ-player seeks to maximize ˆLλ over λ. In words, the λ-player will attempt to satisfy the original constraints on the validation set by choosing how much to penalize the proxy constraints on the training set.
3.1 Generalization

Our ultimate interest is in generalization, and our bounds will be expressed in terms of both the training and validation generalization errors, defined as follows:

**Definition 2.** Define the training generalization error $\tilde{G}^{(\text{train})}(\Theta)$ such that:

$$\left| \mathbb{E}_{x \sim D} [\ell(x, \theta)] - \frac{1}{|S^{(\text{train})}|} \sum_{x \in S^{(\text{train})}} \ell(x, \theta) \right| \leq \tilde{G}^{(\text{train})}(\Theta)$$

for all $\theta \in \Theta$ and all $\ell \in \{\ell_0, \tilde{\ell}_1, \ldots, \tilde{\ell}_m\}$ (the objective and proxy constraint losses, but not the original constraint losses).

Likewise, define the validation generalization error $G^{(\text{val})}(\hat{\Theta})$ to satisfy the analogous inequality in terms of $S^{(\text{val})}$:

$$\left| \mathbb{E}_{x \sim D} [\ell(x, \theta)] - \frac{1}{|S^{(\text{val})}|} \sum_{x \in S^{(\text{val})}} \ell(x, \theta) \right| \leq G^{(\text{val})}(\hat{\Theta})$$

for all $\theta \in \hat{\Theta} \subseteq \Theta$ and all $\ell \in \{\ell_1, \ldots, \ell_m\}$ (the original constraint losses, but not the objective or proxy constraint losses).

Throughout this paper, $\hat{\Theta} := \{\theta^{(1)}, \ldots, \theta^{(T)}\} \subseteq \Theta$ is the set of $T$ iterates found by one of our proposed algorithms. Each of our guarantees will be stated for a particular stochastic model $\bar{\theta}$ supported on $\hat{\Theta}$ (i.e. $\bar{\theta}$ is a distribution over $\hat{\Theta}$), instead of for a single deterministic $\theta \in \hat{\Theta}$. Notice that the above definitions of $\tilde{G}^{(\text{train})}(\Theta)$ and $G^{(\text{val})}(\hat{\Theta})$ also apply to such stochastic models: by the triangle inequality, if every $\theta \in \hat{\Theta}$ generalizes well, then any $\bar{\theta}$ supported on $\Theta$ generalizes equally well, in expectation.

4 Algorithms

We seek a solution that (i) is nearly-optimal, (ii) nearly-feasible, and (iii) generalizes well on the constraints. The optimality and feasibility goals were already tackled by Cotter et al. [2018] in the context of the proxy-Lagrangian formulation of Definition 1. They proposed having the $\theta$-player minimize ordinary external regret, and the $\lambda$-player minimize swap regret using an algorithm based on Gordon et al. [2008]. Rather than finding a single solution (a pure equilibrium of Definition 1), they found a distribution over solutions (a mixed equilibrium). Our proposed approach follows this same pattern, but we build on top of it to address challenge (iii): generalization.

To this end, we draw inspiration from Woodworth et al. [2017] (see Section 2), and isolate the constraints from the complexity of $\Theta$ by using two independent datasets: $S^{(\text{train})}$ and $S^{(\text{val})}$. The “training” dataset will be used to choose a good set of model parameters $\theta$, and the “validation” dataset to choose $\lambda$, and thereby impose the constraints. Like Woodworth et al. [2017], the resulting constraint generalization bounds will be independent of the complexity of the function class.

We’ll begin, in Section 4.1, by proposing and analyzing an oracle-based algorithm that improves generalization by discretizing the candidate set, but makes few assumptions (not even convexity). Next, in Section 4.2, we give an algorithm that is more “realistic”—there is no oracle, and no discretization—but requires stronger assumptions, including strong convexity of the objective and proxy-constraint losses $\ell_0, \tilde{\ell}_1, \ldots, \tilde{\ell}_m$ (but not of the original constraint losses $\ell_1, \ldots, \ell_m$).

In Section 5 we will present and perform experiments on simplified “practical” algorithms with no guarantees, but that incorporate our key idea: having the $\lambda$-player use an independent validation set.
We will take the discretized set of possible to bound optimality and feasibility requirements. The simplest way to attack the generalization problem, and the first that we propose, is to for a finite function class, which will be tightest when we take solution. Additionally, since the set of discretized classifiers is finite, we can apply the standard generalization bound then for any achievable objective function value and corresponding constraint violations, there will be a \( \theta \), while the \( \lambda \)-player uses a swap-regret minimizing algorithm in the style of [Gordon et al. 2008], using the left-stochastic state matrices \( M^{(t)} \in \mathbb{R}^{(m+1) \times (m+1)} \).

**Algorithm 1.** Finds an approximate equilibrium of the empirical proxy-Lagrangian game (Definition[1]), with Theorem[1] being its convergence and generalization guarantee. This is essentially a discretized version of Algorithm 4 of [Cotter et al. 2018]—like that algorithm, because of its dependence on an oracle, this algorithm does not require convexity. Here, \( \mathcal{O}_\rho \) is a deterministic Bayesian optimization oracle (Definition[3]), and \( C_r \) is a radius-\( r \) (external) covering of \( \Lambda := \Delta^{m+1} \) w.r.t. the 1-norm. The \( \theta \)-player uses oracle calls to approximately minimize \( \hat{L}_\theta (\cdot, \lambda) \), while the \( \lambda \)-player uses a swap-regret minimizing algorithm in the style of [Gordon et al. 2008], using the left-stochastic state matrices \( M^{(t)} \in \mathbb{R}^{(m+1) \times (m+1)} \).

DiscreteTwoDataset \((\hat{L}_\theta, \hat{L}_\lambda : \Theta \times \Delta^{m+1} \rightarrow \mathbb{R}, \mathcal{O}_\rho : (\Theta \rightarrow \mathbb{R}) \rightarrow \Theta, C_r \subseteq \mathbb{R}^{m+1}, T \in \mathbb{N}, \eta_\lambda \in \mathbb{R}_+ \)

1. Initialize \( M^{(1)} \in \mathbb{R}^{(m+1) \times (m+1)} \) with \( M_{i,j} = 1/ (m+1) \)
2. For \( t \in [T] \):
3. Let \( \lambda^{(t)} = \text{fix} \, M^{(t)} \) // Fixed point of \( M^{(t)} \), i.e. a stationary distribution
4. Let \( \check{\lambda}^{(t)} = \arg\min_{\lambda \in C_r} \| \lambda^{(t)} - \lambda \|_1 \) // Discretization to closest point in \( C_r \)
5. Let \( \theta^{(t)} = \mathcal{O}_\rho (\check{L}_\lambda (\cdot, \check{\lambda}^{(t)})) \)
6. Let \( \tilde{\lambda}^{(t)} \) be a supergradient of \( \hat{L}_\lambda (\theta^{(t)}, \check{\lambda}^{(t)}) \) w.r.t. \( \lambda \)
7. Update \( \tilde{M}^{(t+1)} = M^{(t)} \odot \exp \left( \eta_\lambda \tilde{\lambda}^{(t)} (\theta^{(t)})^T \right) \) // \( \odot \) and \( \exp \) are element-wise
8. Project \( M_{i,j}^{(t+1)} = \tilde{M}_{i,j}^{(t+1)}/ \| \tilde{M}_{i,j}^{(t+1)} \|_1 \) for \( i \in [m+1] \) // Column-wise projection w.r.t. KL divergence
9. Return \( \theta^{(1)}, \ldots, \theta^{(T)} \) and \( \lambda^{(1)}, \ldots, \lambda^{(T)} \)

4.1 Covering-based Algorithm

The simplest way to attack the generalization problem, and the first that we propose, is to discretize the space of allowed \( \lambda \)-s, and associate each \( \lambda \) with a unique \( \theta \in \Theta \), where this association is based only on the training set. If the set of discretized \( \lambda \)-s is sufficiently small, then the set of discretized \( \theta \)-s will likewise be small, and since it was chosen independently of the validation set, its validation performance will generalize well.

Specifically, we take \( C_r \) to be a radius-\( r \) (external) covering of \( \Lambda := \Delta^{m+1} \) w.r.t. the 1-norm. The set of allowed \( \lambda \)-s is exactly the covering centers, while, following [Chen et al. 2017], [Agarwal et al. 2018] and [Cotter et al. 2018], the associated \( \theta \)-s are found using an approximate Bayesian optimization oracle:

**Definition 3.** A \( \rho \)-approximate Bayesian optimization oracle is a function \( \mathcal{O}_\rho : (\Theta \rightarrow \mathbb{R}) \rightarrow \Theta \) for which:

\[
\inf_{\theta^* \in \Theta} f(\mathcal{O}_\rho(f)) \leq f(\mathcal{O}_\rho(f)) + \rho
\]

for any \( f : \Theta \rightarrow \mathbb{R} \) that can be written as \( f(\theta) = \hat{L}_\theta (\theta), \lambda) \) for some \( \lambda \). Furthermore, every time it is given the same \( f \), \( \mathcal{O}_\rho \) will return the same \( \theta \) (i.e. it is deterministic).

We will take the discretized set of \( \theta \)-s to be the oracle solutions corresponding to the covering centers, i.e. \( \Theta_{C_r} := \{ \mathcal{O}_\rho(\hat{L}_\theta (\cdot, \check{\lambda})) : \check{\lambda} \in C_r \} \). The proof of the upcoming theorem shows that if the radius parameter \( r \) is sufficiently small, then for any achievable objective function value and corresponding constraint violations, there will be a \( \theta \in \Theta_{C_r} \) that is almost as good. Hence, despite the use of discretization, we will still be able to find a nearly-optimal and nearly-feasible solution. Additionally, since the set of discretized classifiers is finite, we can apply the standard generalization bound for a finite function class, which will be tightest when we take \( r \) to be as large as possible while still satisfying our optimality and feasibility requirements.

Algorithm 1 combines our proposed discretization with the oracle-based proxy-Lagrangian optimization procedure proposed by [Cotter et al. 2018]. As desired, it finds a sequence of solutions \( \hat{\Theta} := \{ \theta^{(1)}, \ldots, \theta^{(T)} \} \) for which it is possible to bound \( \mathcal{G}(\hat{\Theta}) \) independently of the complexity of the function class parameterized by \( \Theta \), and finds a random parameter vector \( \theta \) supported on \( \Theta \) that is nearly-optimal and nearly-feasible.
Table 1: Simplified comparison of our suboptimality and infeasibility bounds (Theorems 1 and 2) to those for the Lagrangian formulation trained only on $S^{(\text{train})}$. The “one-dataset” row is the result of an analysis of this one-dataset Lagrangian approach (essentially the same algorithm as [Agarwal et al. 2018]—see Appendix D for details). The “two-dataset” row contains the results for our algorithms, which use the proxy-Lagrangian formulation on two independent datasets. In both cases, $\varepsilon$ measures how far the sequence of iterates is from being the appropriate type of equilibrium ($\varepsilon = \rho + 2\varepsilon$ for Theorem 1 and $\varepsilon = 2\varepsilon$ for Theorem 2). The big-Os absorb only constants which are properties of the constrained problem (Equation 1) and choice of proxy-constraint losses: $\gamma$ and $B_{t_0}$. The key difference is in the “Infeasibility” column: our proposal depends on $G^{(\text{val})}(\hat{\Theta})$—which we bound independently of the model complexity (see Table 2) rather than $\tilde{G}^{(\text{train})}(\Theta)$.

| # Datasets | Suboptimality | Infeasibility | Assuming |
|------------|---------------|---------------|----------|
| One        | $O(\varepsilon + \tilde{G}^{(\text{train})}(\Theta))$ | $O(\varepsilon + \tilde{G}^{(\text{train})}(\Theta))$ | $\tilde{G}^{(\text{train})}(\Theta) \leq \gamma/2$ |
| Two        | $O(\varepsilon + \tilde{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\Theta))$ | $O(\varepsilon + G^{(\text{val})}(\Theta))$ | $\varepsilon + 2\tilde{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\Theta) \leq \gamma/2$ |

**Theorem 1.** Given any $\varepsilon > 0$, there exists a covering $C_\varepsilon$, such that, if we take $T \geq 4B_{\Delta}^2 (m + 1) \ln (m + 1) / \varepsilon^2$ and $\eta = \sqrt{(m + 1) \ln (m + 1) / TB_{\Delta}^2}$, where $B_{\Delta} = \max_{t \in [T]} \| \Delta^{(t)} \|$, $\| \Delta^{(t)} \|$ is a bound on the gradients, then the following hold, where $\hat{\Theta} := \{ \theta^{(1)}, \ldots, \theta^{(T)} \}$ is the set of results of Algorithm 1:

**Optimality and Feasibility:** Let $\theta$ be a random variable taking values from $\hat{\Theta}$, defined such that $\theta = \theta^{(t)}$ with probability $\lambda_1^{(t)}/\sum_s \lambda_1^{(s)}$, and let $\lambda := \left( \sum_{t=1}^T \lambda_1^{(t)} \right) / T$. Then $\theta$ is nearly-optimal in expectation:

$$
\mathbb{E}_{x \sim D} \left[ \ell_0 \left( x; \theta \right) \right] \leq \inf_{\theta \in \Theta : \forall i \in [m] \mathbb{E}_{x \sim D} \left[ \ell_i \left( x; \theta \right) \right] \leq 0} \mathbb{E}_{x \sim D} \left[ \ell_0 \left( x; \theta^* \right) \right] + 1 / \lambda_1 \left( \rho + 2\varepsilon + 2\tilde{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\Theta) \right)
$$

(3)

and nearly-feasible:

$$
\max_{i \in [m]} \mathbb{E}_{x \sim D} \left[ \ell_i \left( x; \theta \right) \right] \leq \varepsilon + G^{(\text{val})}(\hat{\Theta})
$$

(4)

Additionally, if there exists a $\theta' \in \Theta$ that satisfies all of the constraints with margin $\gamma$ (i.e. $\mathbb{E}_{x \sim D} \left[ \ell_i \left( x; \theta' \right) \right] \leq -\gamma$ for all $i \in [m]$), then:

$$
\lambda_1 \geq \frac{1}{\gamma + B_{t_0}} \left( \gamma - \rho - 2\varepsilon - 2\tilde{G}^{(\text{train})}(\Theta) - G^{(\text{val})}(\hat{\Theta}) \right)
$$

(5)

where $B_{t_0} \geq \sup_{\theta \in \Theta} \mathbb{E}_{x \sim D} \left[ \ell_0 \left( x; \theta \right) \right] - \inf_{\theta \in \Theta} \mathbb{E}_{x \sim D} \left[ \ell_0 \left( x; \theta \right) \right]$ is a bound on the range of the objective loss.

**Generalization:** With probability $1 - \delta$ over the sampling of $S^{(\text{val})}$:

$$
G^{(\text{val})}(\hat{\Theta}) < B_{\ell} \sqrt{\frac{m \ln \left( 10 B_{\ell} / \varepsilon \right) + \ln (2m / \delta)}{2 |S^{(\text{val})}|}}
$$

(6)

where $B_{\ell} \geq |\ell(x, \theta)|$ for all $\ell \in \{ \ell_0, \ell_1, \ldots, \ell_m \}$, and $B_{\ell} \geq \max_{i \in [m]} (b_i - a_i)$ assuming that the range of each $\ell_i$ is the interval $[a_i, b_i]$.

**Proof.** In Appendix C.1

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This condition could be removed by defining the feasibility margin $\gamma$ in terms of $S^{(\text{train})} \setminus D$ instead of $D$, causing $\gamma$ to depend on the particular training sample, instead of being solely a property of the constrained problem and choice of proxy-constraint losses.
Table 2: Comparison of the standard Rademacher complexity-based generalization bound (of the function class \( \mathcal{F} \) parameterized by \( \Theta \)) to our bounds on \( G^{(\text{val})}(\hat{\Theta}) \). All bounds hold with probability \( 1 - \delta \), and we assume that \( |S^{(\text{train})}| \propto n \) and \( |S^{(\text{val})}| \propto n \) (e.g. if the data is split 50/50). The big-\( O \)s absorb only constants which are properties of the constrained problem (Equation 1) and choice of proxy-constraint losses: \( B_i, B_j, L \) and \( \mu \). For our algorithms, the validation generalization performance of the constraints is independent of the model complexity.

| \( G^{(\text{train})}(\Theta) \) | \( G^{(\text{val})}(\Theta) \) (Theorem 1) | \( G^{(\text{val})}(\hat{\Theta}) \) (Theorem 2) |
|----------------------|---------------------------------|---------------------------------|
| \( O \left( R_n(\mathcal{F}) + \sqrt{\frac{\ln(1/\delta)}{n}} \right) \) | \( O \left( \sqrt{\frac{m \ln(1/\epsilon) + \ln(m/\delta)}{n}} \right) \) | \( O \left( \sqrt{\frac{m \ln n + \ln(m/\delta)}{n}} + \epsilon \right) \)

When reading the above result, it’s natural to wonder about the role played by \( \bar{\lambda}_1 \). Recall that, unlike the Lagrangian formulation, the proxy-Lagrangian formulation (Definition 1) has a weight \( \lambda_1 \) associated with the objective, in addition to the \( m \) weights \( \lambda_2, \ldots, \lambda_{m+1} \) associated with the constraints. When the \( i \)th constraint is violated, the corresponding \( \lambda_{i+1} \) will grow, pushing \( \bar{\lambda}_1 \) towards zero. Conversely, when the constraints are satisfied, \( \bar{\lambda}_1 \) will be pushed towards one. In other words, the magnitude of \( \lambda_1 \) encodes the \( \lambda \)-player’s “belief” about the feasibility of the solution. Just as, when using the Lagrangian formulation, Lagrange multipliers will tend to be small on a feasible problem, the proxy-Lagrangian objective weight \( \lambda_1 \) will tend to be large on a feasible problem, as shown by Equation 5, which guarantees that \( \bar{\lambda}_1 \) will be bounded away from zero provided that there exists a margin-feasible solution with a sufficiently large margin \( \gamma \). In practice, of course, one need not rely on this lower bound: one can instead simply inspect the behavior of the sequence of \( \lambda_1^{(t)} \)'s during optimization.

Equation 5 causes our results to be gated by the feasibility margin. Specifically, it requires the training and validation datasets to generalize well enough for \( \rho + 2\epsilon + 2G^{(\text{train})}(\Theta) + G^{(\text{val})}(\hat{\Theta}) \) to stay within the feasibility margin \( \gamma \). Past this critical threshold, \( \lambda_1 \) can be lower-bounded by a constant, and can therefore be essentially ignored. To get an intuitive grasp of this condition, notice that it is similar to requiring \( \gamma \)-margin-feasible solutions on the training dataset to generalize well enough to also be margin-feasible (with a smaller margin) on the validation dataset, and vice-versa.

Table 1 contains a comparison of our bounds, obtained with the proxy-Lagrangian formulation and two datasets, versus bounds for the standard Lagrangian on one dataset. The “Assuming” column contains a condition resulting from the above discussion. There are two key ways in which our results improve on those for the one-dataset Lagrangian: (i) in the “Infeasibility” column, our approach depends on \( G^{(\text{val})}(\hat{\Theta}) \) instead of \( G^{(\text{train})}(\Theta) \), and (ii): as shown in Table 2, for our algorithms the generalization performance \( G^{(\text{val})}(\hat{\Theta}) \) of the constraints is bounded independently of the complexity \( \Theta \).

It’s worth emphasizing that this generalization bound (Table 2) is distinct from the feasibility bound (the “Infeasibility” column of Table 1). When using our algorithms, testing constraint violations will always be close to the validation violations, regardless of the value of \( \bar{\lambda}_1 \). The “Assuming” column is only needed when asking whether the validation violations are close to zero.

### 4.2 Gradient-based Algorithm

Aside from the unrealistic requirement for a Bayesian optimization oracle, the main disadvantage of Algorithm 1 is that it relies on discretization. Our next algorithm instead makes much stronger assumptions—strong convexity of the objective and proxy constraint losses, and Lipschitz continuity of the original constraint losses—enabling us to dispense with discretization entirely in both the algorithm and the corresponding theorem statement.

The proof of the upcoming theorem, however, still uses a covering. The central idea is the same as before, with one extra step: thanks to strong convexity, every (approximate) minimizer of \( \hat{\mathcal{L}}_{\mu}(\cdot, \lambda) \) is close to one of the discretized parameter vectors \( \theta \in \Theta_{C_{\cdot}} \). Hence, the set of such minimizers generalizes as well as \( \Theta_{C_{\cdot}} \), plus an additional term measuring the cost that we pay for approximating the minimizers with elements of \( \Theta_{C_{\cdot}} \).
Additionally, if there exists a \( \theta \) that satisfies all of the constraints with margin \( \gamma \) (i.e. \( \mathbb{E}_{x \sim \mathcal{D}}[\ell_i(x; \theta')] \leq -\gamma \) for all \( i \in [m] \)), then:

\[
\lambda_1 \geq \frac{1}{\gamma + B_{\ell_0}} \left( \gamma - 2\epsilon - 2\hat{G}^{(\text{train})}(\Theta) - G^{(\text{val})}(\hat{\Theta}) \right)
\]

where \( B_{\ell_0} \) is as in Theorem 7.

Theorem 2. Suppose that \( \Theta \) is compact and convex, and that \( \ell(x; \theta) \) is \( \mu \)-strongly convex in \( \theta \) for all \( \ell \in \{ \ell_0, \ell_1, \ldots, \ell_m \} \). Given any \( \epsilon > 0 \), if we take \( T_\theta \geq (B_\Delta^2 / \mu_\epsilon) \ln (B_\Delta^2 / \mu_\epsilon), T_\lambda \geq 4B_\Delta^2 (m + 1) \ln (m + 1) / \epsilon^2 \) and \( \eta_\lambda \geq \sqrt{(m + 1) \ln (m + 1) / T_\lambda B_\Delta^2} \), where \( B_\Delta \) is as in Theorem 1 and \( B_\Delta \geq \max_{s,t \in [T_\theta] \times [T_\lambda]} \| \Delta_{\theta}^{(t,s)} \|_2 \) is a bound on the subgradients, then the following hold, where \( \hat{\Theta} := \{ \theta^{(1)}, \ldots, \theta^{(T_\lambda)} \} \) is the set of results of Algorithm 2.

Optimality and Feasibility: Let \( \hat{\theta} \) be a random variable taking values from \( \hat{\Theta} \), defined such that \( \hat{\theta} = \theta^{(t)} \) with probability \( \lambda_1^{(t)} / \sum_{s=1}^{T_\lambda} \lambda_1^{(s)} \), and let \( \lambda := \left( \sum_{t=1}^{T_\lambda} \lambda_1^{(t)} \right) / T_\lambda \). Then \( \hat{\theta} \) is nearly-optimal in expectation:

\[
\mathbb{E}_{\hat{\theta} \sim \mathcal{D}}[\ell_0(x; \hat{\theta})] \leq \inf_{\theta^* \in \Theta} \mathbb{E}_{x \sim \mathcal{D}}[\ell_0(x; \theta^*)] + \frac{1}{\lambda_1} \left( 2\epsilon + 2\hat{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\hat{\Theta}) \right)
\]

and nearly-feasible:

\[
\max_{i \in [m]} \mathbb{E}_{\hat{\theta} \sim \mathcal{D}}[\ell_i(x; \hat{\theta})] \leq \frac{\epsilon}{\lambda_1} + G^{(\text{val})}(\hat{\Theta})
\]

Additionally, if there exists a \( \theta^* \in \Theta \) that satisfies all of the constraints with margin \( \gamma \) (i.e. \( \mathbb{E}_{x \sim \mathcal{D}}[\ell_i(x; \theta^*)] \leq -\gamma \) for all \( i \in [m] \)), then:

\[
\lambda_1 \geq \frac{1}{\gamma + B_{\ell_0}} \left( \gamma - 2\epsilon - 2\hat{G}^{(\text{train})}(\Theta) - G^{(\text{val})}(\hat{\Theta}) \right)
\]
We present two sets of experiments, the first on simulated data, and the second on real data. In both cases, each dataset is split into thee parts: training, validation and testing. We compare our proposed two-dataset approach, in which

Algorithm 3 “Practical” algorithm for optimizing the empirical proxy-Lagrangian game (Definition 1). This is essentially Algorithm 2 of [Cotter et al. 2018], differing only in that it is applied to the two-dataset formulation of Definition 1.

```
PracticalTwoDataset \( \hat{\mathcal{L}}_\theta, \hat{\mathcal{L}}_\lambda : \Theta \times \Delta^{m+1} \rightarrow \mathbb{R}, T \in \mathbb{N}, \eta_\theta, \eta_\lambda \in \mathbb{R}_+ \):
1. Initialize \( \theta^{(1)} = 0 \) \hspace{1cm} // Assumes 0 \in \Theta
2. Initialize \( M^{(1)} \in \mathbb{R}^{(m+1) \times (m+1)} \) with \( M_{i,j} = 1 / (m + 1) \)
3. For \( t \in [T] \):
   4. Let \( \lambda^{(t)} = \text{fix} \ M^{(t)} \) \hspace{1cm} // fixed point of \( M^{(t)} \), i.e. a stationary distribution
   5. Let \( \hat{\Delta}_\theta^{(t)} \) be a stochastic subgradient of \( \hat{\mathcal{L}}_\theta (\theta^{(t)}, \lambda^{(t)}) \) w.r.t. \( \theta \)
   6. Let \( \Delta_\lambda^{(t)} \) be a stochastic gradient of \( \hat{\mathcal{L}}_\lambda (\theta^{(t)}, \lambda^{(t)}) \) w.r.t. \( \lambda \)
   7. Update \( \theta^{(t+1)} = \Pi_\Theta \( \theta^{(t)} - \eta_\theta \hat{\Delta}_\theta^{(t)} \) \)
   8. Update \( M^{(t+1)} = M^{(t)} \odot \exp \left( \eta_\lambda \Delta_\lambda^{(t)} \left( \lambda^{(t)} \right)^T \right) \) \hspace{1cm} // \( \odot \) and \( \exp \) are element-wise
   9. Project \( M^{(t+1)}_{i,i} = \tilde{M}_{i,i}^{(t+1)} / \left\| \tilde{M}_{i,i}^{(t+1)} \right\|_1 \) for \( i \in [m + 1] \) \hspace{1cm} // Column-wise projection w.r.t. KL divergence
10. Return \( \theta^{(1)}, \ldots, \theta^{(T)} \) and \( \lambda^{(1)}, \ldots, \lambda^{(T)} \)
```

**Generalization:** If, in addition to the above requirements, \( \ell(x; \theta) \) is \( L \)-Lipschitz continuous in \( \theta \) for all \( \ell \in \{\ell_1, \ldots, \ell_m\} \), then with probability \( 1 - \delta \) over the sampling of \( S^{(\text{val})} \):

\[
\mathcal{G}^{(\text{val})}(\Theta) < B_\ell \sqrt{\frac{2m}{|S^{(\text{val})}|}} \max \left\{ 1, \ln \left( \frac{160L^2B_\ell |S^{(\text{val})}|}{m\mu B_\ell^2} \right) \right\} + B_\ell \sqrt{\frac{\ln (2m/\delta)}{2 |S^{(\text{val})}|}} + 2L_\ell \sqrt{\frac{\eta}{\mu}} \tag{7}
\]

where \( B_\ell \) and \( B_\ell \) are as in Theorem 7.

**Proof.** In Appendix C.2 \( \square \)

The above optimality and feasibility guarantees are very similar to those of Theorem 1 as is shown in Table 1 (in which the only difference is the definition of \( \varepsilon \)). Algorithm 2’s generalization bound (Equation 7) is more complicated than that of Algorithm 1 (Equation 6), but Table 2 shows that the two are roughly comparable. Hence, the overall theoretical performance of Algorithm 2 is very similar to that of Algorithm 1 and, while it does rely on stronger assumptions, it neither uses discretization, nor does it require an oracle.

## 5 Experiments

While Section 4 has demonstrated the theoretical performance of Algorithms 1 and 2, we believe that our proposed two-dataset approach is useful as a heuristic for improving constraint generalization performance, even when one is not using a theoretically-justified algorithm. For this reason, we experiment with two “practical” algorithms. The first, Algorithm 3, is a bare-bones version of Algorithm 2 in which \( \theta \) and \( \lambda \) are updated simultaneously using stochastic updates, instead of in an inner and outer loop. This algorithm implements our central idea—imposing constraints using an independent validation dataset—without compromising on simplicity or speed. The purpose of the second, Algorithm 4, is to explore how well our two-dataset approach can be applied to the usual Lagrangian formulation. For this algorithm, proxy-constraints and the use of two independent datasets are essentially “tacked on” to the Lagrangian. Neither of these algorithms enjoys the theoretical guarantees of Section 4 but, as we will see, both are still successful at improving constraint generalization.

We present two sets of experiments, the first on simulated data, and the second on real data. In both cases, each dataset is split into thee parts: training, validation and testing. We compare our proposed two-dataset approach, in which
\textbf{Algorithm 4} “Practical” algorithm for optimizing a variant of the standard Lagrangian game, modified to support proxy constraints and two datasets. Here, instead of using the proxy-Lagrangian formulation of Definition 4, we take \( \hat{L}_\theta := \mathbb{E}_{x \sim S^{(\text{train})}} [\ell_0(x; \theta) + \sum_{i=1}^{m} \lambda_i \hat{\ell}_i(x; \theta)] \) and \( \hat{L}_\lambda := \mathbb{E}_{x \sim S^{(\text{val})}} [\ell_0(x; \theta) + \sum_{i=1}^{m} \lambda_i \ell_i(x; \theta)] \), with \( \lambda \in \Lambda := \mathbb{R}_+^m \). Compared to Algorithm 3, this algorithm is further from those for which we can prove theoretical results (Algorithms 1 and 2), but is much closer to the Lagrangian-based approach of Agarwal et al. [2018]. We include it to demonstrate that our two-dataset proposal works as a heuristic.

\begin{algorithm}
\caption{LagrangianTwoDataset \((\hat{L}_\theta, \hat{L}_\lambda : \Theta \times \Delta^m \rightarrow \mathbb{R}, T \in \mathbb{N}, \eta_0, \eta_\lambda \in \mathbb{R}_+)\):}
\begin{algorithmic}[1]
\State Initialize \( \theta^{(1)} = 0, \lambda^{(1)} = 0 \) \hspace{1cm} // Assumes \( 0 \in \Theta \)
\State For \( t \in [T] \):
\State \hspace{0.5cm} Let \( \Delta_\theta^{(t)} \) be a stochastic subgradient of \( L \left( \theta^{(t)}, \lambda^{(t)} \right) \) w.r.t. \( \theta \)
\State \hspace{0.5cm} Let \( \Delta_\lambda^{(t)} \) be a stochastic gradient of \( L \left( \theta^{(t)}, \lambda^{(t)} \right) \) w.r.t. \( \lambda \)
\State \hspace{0.5cm} Update \( \theta^{(t+1)} = \Pi_\Theta \left( \theta^{(t)} - \eta_\theta \Delta_\theta^{(t)} \right) \) \hspace{1cm} // Projected SGD updates . . .
\State \hspace{0.5cm} Update \( \lambda^{(t+1)} = \Pi_\Lambda \left( \lambda^{(t)} + \eta_\lambda \Delta_\lambda^{(t)} \right) \) \hspace{1cm} // ... 
\State Return \( \theta^{(1)}, \ldots, \theta^{(T)} \) and \( \lambda^{(1)}, \ldots, \lambda^{(T)} \)
\end{algorithmic}
\end{algorithm}

\( S^{(\text{train})} \) is the training dataset and \( S^{(\text{val})} \) is the validation dataset, to the the natural baseline one-dataset approach of using the union of the training and validation sets to define both \( S^{(\text{train})} \) and \( S^{(\text{val})} \). Hence, both approaches “see” the same total amount of data during training.

This difference between the data provided to the two algorithms leads to a slight complication when reporting “training” error rates and constraint violations. For the two-dataset approach, the former are reported on \( S^{(\text{train})} \) (used to learn \( \theta \)), and the latter on \( S^{(\text{val})} \) (used to learn \( \lambda \)). For the baseline one-dataset algorithm, both quantities are reported on the full dataset (i.e. the union of the training and validation sets). “Testing” numbers are always reported on the testing dataset.

Our implementation uses TensorFlow, and is based on Cotter et al. [2018]’s open-source constrained optimization library. To avoid a hyperparameter search, we replace the stochastic gradient updates of Algorithms 3 and 4 with ADAM [Kingma and Ba, 2014], using the default parameters. For both our two-dataset algorithm and the one-dataset baseline, the result of training is a sequence of iterates \( \theta^{(1)}, \ldots, \theta^{(T)} \), but instead of keeping track of the full sequence, we only store a total of 100 evenly-spaced iterates for each run. Rather than using the weighted predictor of Theorems 1 and 2, we use the “shrinking” procedure of Cotter et al. [2018] (see Appendix B) to find the best stochastic classifier supported on the sequence of 100 iterates.

In all of our experiments, the objective and proxy constraint functions \( \ell_0, \hat{\ell}_1, \ldots, \hat{\ell}_m \) are hinge upper bounds on the quantities of interest, while the \textit{original} constraint functions \( \ell_1, \ldots, \ell_m \) are precisely what we claim to constrain (in these experiments, proportions, represented as linear combinations of indicator functions).

\section{Simulated-data Experiments}

Our first experiment uses a simulated binary classification problem designed to be especially prone to overfitting. To generate the dataset, we first draw \( n = 1000 \) points \( z_1, \ldots, z_n \) from two overlapping Gaussians in \( \mathbb{R}^2 \), and another \( n \) points \( w_1, \ldots, w_n \) from the same distribution. For each \( i \), we let the classification label \( y_i \) indicate which of the two Gaussians \( z_i \) was drawn from, and generate a feature vector \( x_i \in \mathcal{X} := \mathbb{R}^n \) such that the \( j \)th feature satisfies \( x_{i,j} := \exp(-\|z_i - w_j\|^2 / 2\sigma^2) \). Our results are averaged over ten runs, with different random splits of the data into equally-sized training, validation and testing datasets.

The classification task is learn a classifier on \( \mathcal{X} \) that determines which of the two Gaussian distributions generated the example, with the model’s recall constrained to be at least 97\%. The \( \sigma \) parameter partly controls the amount of overfitting: as \( \sigma \rightarrow 0 \), a linear classifier on \( \mathcal{X} \) approaches a 1-nearest neighbor classifier over \( w_1, \ldots, w_n \), which one would expect to overfit badly.
Figure 1: Results of the experiment of Section 5.1 using Algorithm 5 on a linear model. The left-hand plot demonstrates both that overfitting increases as $\sigma$ decreases, as expected, and that our model performs worse than the baseline, in terms of accuracy. In the right-hand plot, however, we can see that our approach generalizes better on the constraints (the two red curves are closer together than the two blue ones), and also does a better job of satisfying the constraints on the testing set (the solid red curve is below the solid blue one).

Table 3: Properties of datasets used in Section 5.2. For the one-dataset experiments, the entire training set was used for training. For the two-dataset experiments, the training dataset was split in half between $S^{(\text{train})}$ and $S^{(\text{val})}$.

| Dataset               | Model         | Training examples | Testing examples | Features |
|-----------------------|---------------|-------------------|------------------|----------|
| Communities and Crime | Linear        | 1 459             | 499              | 140      |
| Business Entity Resolution | Lattice      | 11 560            | 3 856            | 37       |
| Adult                | Neural Network | 32 561           | 16 281            | 122      |
| COMPAS               | Neural Network | 4 110             | 2 026            | 32       |

We trained four sets of models using Algorithm 5: linear, and one-hidden-layer neural networks with 5, 10 and 100 hidden ReLU units. We also varied $\sigma$ between 0 and 1. Figures 1 and 2 show that our approach consistently comes closer to satisfying the constraints on the testing set, but that, as one would expect, this comes at a slight cost in testing accuracy. Unsurprisingly, our approach is most advantageous for the most complex models (100-hidden unit), and less so for the simplest (linear).

5.2 Real-data Experiments

Our next set of experiments were performed on four real datasets, which we summarize in Table 3 and describe below. On each dataset, we trained one of three different types of models: linear, calibrated lattice [Gupta et al., 2016], or a neural network with one hidden layer containing 50 ReLU neurons. The neural network models are more complex than necessary for these datasets, and are used here because they overfit, and therefore illustrate the improved generalization performance of the two-dataset approach.

Communities and Crime: This UCI dataset [Dheeru and Karra Taniskidou, 2017] includes features aggregated from census and law enforcement data, on which we train a linear model. The binary classification task is to predict whether a community has a high (above the 70th percentile) or low crime rate, as in [Kearns et al., 2017]. To form protected groups, we use four racial percentage features as real-valued protected attributes. Each is thresholded at the 50th percentile to form eight protected groups: low-Asian, high-Asian, low-Black, high-Black, low-Hispanic, high-Hispanic,
Table 4: Error rates and maximum constraint violations for all compared algorithms, on the datasets of Table 3, as described in Section 5.2. The “Unconstrained” columns contain the results for entirely-unconstrained models. All quantities are averaged over 100 runs. The training constraint violations are occasionally exactly zero thanks to our use of Cotter et al.
[2018]’s “shrinking” procedure (Appendix B).

| Dataset            | Unconstrained | Algorithm 3 | Algorithm 4 |
|--------------------|---------------|-------------|-------------|
|                    | Error Viol.   | One-dataset | Two-dataset | One-dataset | Two-dataset |
| Communities and Crime |    | Train       | .121 .231  | .153 0       | .161 0       | .163 -.001  | .165 0       |
|                    |               | Test        | .142 .300  | .173 .022    | .199 -.008  | .181 .001   | .195 -.012   |
| Entity Resolution  |    | Train       | .148 .309  | .216 .026    | .215 .040   | .225 0       | .261 .003    |
|                    |               | Test        | .156 .278  | .222 .073    | .221 .072   | .232 .042   | .267 .041    |
| Adult              |    | Train       | .102 .077  | .132 0       | .110 0       | .131 0       | .113 0       |
|                    |               | Test        | .156 .075  | .156 .011    | .169 .005   | .156 .013   | .165 .008    |
| COMPAS             |    | Train       | .216 .004  | .216 -.005   | .154 -.003  | .216 -.005   | .151 -.003   |
|                    |               | Test        | .353 .046  | .353 .038    | .378 .004   | .349 .029    | .378 .006    |

low-White, and high-White. There is one fairness constraint for each of the eight protected groups, which constrains the group’s false positive rate to be no larger than the overall false positive rate.

**Business Entity Resolution:** This is a proprietary Google dataset for which the task is to predict whether a pair of business descriptions describe the same real business. Features include measures of similarity between the two business titles, phone numbers, and so on. We impose several constraints: (i) for each of the 16 most common countries, the recall must be at least 95%; (ii) for the set of all chain businesses, and likewise for the set of all non-chain businesses, the recall must be at least 95%; (iii) the accuracy on non-chain businesses must be no more than 10% higher than that on chain businesses. The purpose of this final constraint is to attempt to treat small businesses and large businesses comparably.

**Adult:** This is a version of the UCI Adult dataset, preprocessed to include only binary features (using one-hot encodings for categorical features, and bucketing for continuous features). The classification task is to predict whether a person’s yearly income is greater than $50,000, subject to the 80% rule for demographic parity: for each of four overlapping protected classes (Black, White, Female and Male), the positive prediction rate must be at least 80% of the overall positive prediction rate.

**COMPAS:** This is the ProPublica COMPAS dataset analyzed by Angwin et al.
[2016], preprocessed similarly to the Adult dataset. The classification task is to predict recidivism, subject to equal opportunity [Hardt et al., 2016] fairness constraints: for each of four overlapping protected classes (Black, White, Female and Male), the positive prediction rate on the positively-labeled examples must be at most 5% higher than the overall positive prediction rate on positively-labeled examples.

All of these datasets have designated training/testing splits. For the one-dataset experiments, both $S^{(train)}$ and $S^{(val)}$ were taken to be the entire training set. For the two-dataset experiments, the training dataset was split in half, into $S^{(train)}$ and $S^{(val)}$. All reported numbers are averaged over 100 such random splits, with random permutations applied to the data.

Table 4 summarizes the results of these experiments. On three of the four datasets (all but Business Entity Resolution), and for both Algorithms 3 and 4, the two-dataset experiments have a clear and significant advantage in terms of constraint generalization performance, although it comes at a cost: the error rates are, as expected, somewhat higher. While one must be mindful of this trade-off, it seems that providing independent datasets to the $\theta$- and $\lambda$-players does improve constraint generalization in practice, even when our proofs do not apply.
Figure 2: Same as Figure 1 but for one-hidden-layer neural networks with 5, 10 and 100 hidden units. The results follow the same general trend as those of Figure 1 but, just as one would expect, the benefits of our proposed two-dataset approach become more pronounced as the model complexity increases.
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Table 5: Key notation.

| Symbol | Description |
|--------|-------------|
| \( m \) | Number of constraints in Equation 1 |
| \( \ell_0 \) | Objective loss function in Equation 1 |
| \( \ell_i \) | \( i \)th constraint loss function in Equation 1 |
| \( \hat{\ell}_i \) | \( i \)th proxy-constraint loss function (corresponding to \( \ell_i \)) in Definition 1 |
| \( x \) | Feature vector for an example |
| \( \mathcal{X} \) | Space of feature vectors \( x \in \mathcal{X} \) |
| \( \mathcal{D} \) | Data distribution over feature vectors |
| \( S^{(\text{train})} \) | An i.i.d. training sample from \( \mathcal{D} \) |
| \( S^{(\text{val})} \) | An i.i.d. “validation” sample from \( \mathcal{D} \) |
| \( \Theta \) | Parameter defining a model |
| \( \tilde{\Theta} \) | Space of parameter vectors (\( \theta \in \Theta \)) |
| \( \lambda \) | Vector of Lagrange-multiplier-like “hyperparameters” (Definition 1) |
| \( \Lambda \) | Space of Lagrange-multiplier-like “hyperparameter” vectors (\( \lambda \in \Lambda := \Delta^{m+1} \)) |
| \( \bar{\lambda} \) | Average “hyperparameters” \( \bar{\lambda} := (\sum_{i=1}^{m+1} \lambda_i) / T \) (Theorems 1 and 2) |
| \( \lambda_1 \) | First coordinate of \( \lambda \) measuring our “belief” that \( \theta \) is feasible |
| \( \mathcal{L}_\theta \) | In-expectation proxy-Lagrangian function minimized by \( \theta \)-player (Definition 4) |
| \( \hat{\mathcal{L}}_\lambda \) | In-expectation proxy-Lagrangian function maximized by \( \lambda \)-player (Definition 4) |
| \( \hat{\mathcal{L}}_\theta \) | Empirical proxy-Lagrangian function minimized by \( \theta \)-player (Definition 1) |
| \( \hat{\mathcal{L}}_\lambda \) | Empirical proxy-Lagrangian function maximized by \( \lambda \)-player (Definition 1) |
| \( \hat{G}^{(\text{train})}(\Theta) \) | Generalization bound for \( \ell_0, \hat{\ell}_1, \ldots, \hat{\ell}_m \) on \( S^{(\text{train})} \) for \( \theta \in \Theta \) (Definition 5) |
| \( G^{(\text{val})}(\Theta) \) | Generalization bound for \( \ell_1, \ldots, \ell_m \) on \( S^{(\text{val})} \) for \( \theta \in \Theta \) (Definition 2) |
| \( \mathcal{O}_\rho \) | Bayesian oracle of Definition 3 |
| \( \rho \) | Additive approximation of the Bayesian oracle of Definition 3 |
| \( C_r \) | A radius-\( r \) external covering of \( \Lambda := \Delta^{m+1} \) w.r.t. the 1-norm |
| \( r \) | Radius of the covering \( C_r \) |
| \( \Theta_{C_r} \) | Oracle evaluations at the covering centers (\( \Theta_{C_r} := \{ \mathcal{O}_\rho(\hat{\mathcal{L}}_\theta(\cdot, \bar{\lambda})) : \bar{\lambda} \in C_r \} \)) |
| \( T \) | Number of iterations performed in Algorithms 1, 2, 3 and 4 |
| \( T_\lambda \) | Number of iterations performed in the outer loop of Algorithm 2 |
| \( T_\theta \) | Number of iterations performed in the inner loop of Algorithm 2 |
| \( \eta_\lambda \) | Step size associated with \( \lambda \)-player in Algorithms 1, 2, 3 and 4 |
| \( \eta_\theta \) | Step size associated with \( \theta \)-player in Algorithms 3 and 4 |
| \( M \) | A left-stochastic \((m + 1) \times (m + 1)\) matrix |
| \( \mathcal{M} \) | Space of all left-stochastic \((m + 1) \times (m + 1)\) matrices (\( M \in \mathcal{M} \)) |
| \( \gamma \) | Maximum margin by which the proxy-constraints can be satisfied in Theorems 1 and 2 |
| \( \mu \) | Strong convexity parameter of \( \ell_0, \hat{\ell}_1, \ldots, \hat{\ell}_m \) in Theorem 2 |
| \( L \) | Lipschitz constant of \( \ell_1, \ldots, \ell_m \) in Theorem 2 |
| \( B_{\ell_0} \) | Upper bound on \( b_0 - a_0 \), where \( \text{range}(\ell_0) = [a_0, b_0] \) |
| \( B_\ell \) | Upper bound on \( b_i - a_i \) for all \( i \in [m] \), where \( \text{range}(\ell_i) = [a_i, b_i] \) |
| \( B_{\hat{\ell}} \) | Upper bound on \( |\ell(x; \theta)| \) for all \( \ell \in \{ \ell_0, \hat{\ell}_1, \ldots, \hat{\ell}_m \} \) |
| \( B_\Delta \) | Upper bound on the 2-norms of subgradients of \( \hat{\mathcal{L}}_\theta \) w.r.t. \( \theta \) |
| \( B_{2\Delta} \) | Upper bound on the \( \infty \)-norms of gradients of \( \hat{\mathcal{L}}_\lambda \) w.r.t. \( \lambda \) |
A  Examples of Constraints

In this appendix, we’ll provide some examples of constrained optimization problems in the form of Equation 1.

A.1  Neyman-Pearson

The first example we’ll consider is Neyman-Pearson classification [Davenport et al., 2010, Gasso et al., 2011]. Imagine that we wish to learn a classification function \( f : \mathcal{X} \times \Theta \rightarrow \mathbb{R} \) parameterized by \( \theta \), with the goal being to minimize the false positive rate, subject to the constraint that the false negative rate be at most 10%:

\[
\begin{align*}
\min_{\theta \in \Theta} & \quad \mathbb{E}_{x,y|y=-1} \left[ 1 \{ f(x; \theta) \geq 0 \} \right] \\
\text{s.t.} & \quad \mathbb{E}_{x,y|y=1} \left[ 1 \{ f(x; \theta) \leq 0 \} \right] \leq 0.1
\end{align*}
\]

One way to convert this problem into the form of Equation 1 is to define \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) as the marginal distributions over \( x \)s for which \( y = +1 \) and \( y = -1 \) (respectively), and take \( \mathcal{D} := \mathcal{D}_+ \times \mathcal{D}_- \) so that \( \mathcal{D} \) is a distribution over pairs of feature vectors, the first having a positive label, and the second a negative label. Defining \( \ell_0(x_+, x_-; \theta) := 1 \{ f(x) \geq 0 \} \) and \( \ell_1(x_+, x_-; \theta) := 1 \{ f(x) \leq 0 \} - 0.1 \) puts the original Neyman-Pearson problem in the form of Equation 1.

In practice, the fact that \( \ell_0 \) and \( \ell_1 \) are defined in terms of indicator functions, and are therefore discontinuous, will problematic. To fix this, using the formulation of Definition 1, one could instead define \( \ell_0(x_+, x_-; \theta) := \max \{0, 1 + f(x_-)\} \) as a hinge upper bound on the false positive rate, and leave \( \ell_1 \) as-is while defining the corresponding proxy-constraint to be \( \tilde{\ell}_1(x_+, x_-; \theta) := \max \{0, 1 - f(x_+)\} \).

A.2  Equal Opportunity

The second example we’ll consider is a fairness-constrained problem. As before, we’ll take \( f : \mathcal{X} \times \Theta \rightarrow \mathbb{R} \) to be a classification function, but we’ll imagine that each \( x \in \mathcal{X} \) contains a feature \( x_k \in \{1, 2, 3\} \) indicating to which of three protected classes the corresponding example belongs. We will seek to minimize the overall error rate, subject to the constraint that, for each of the three protected classes, the false negative rate is at most 110% of the false negative rate across all three classes (this is essentially an equal opportunity constraint [Hardt et al., 2016]):

\[
\begin{align*}
\min_{\theta \in \Theta} & \quad \mathbb{E}_{x,y} \left[ 1 \{ y f(x; \theta) \leq 0 \} \right] \\
\text{s.t.} & \quad \mathbb{E}_{x,y|x_k=i} \left[ 1 \{ f(x; \theta) \leq 0 \} \right] \leq 1.1 \cdot \mathbb{E}_{x,y|y=1} \left[ 1 \{ f(x; \theta) \leq 0 \} \right]
\end{align*}
\]

While we could use the same approach as in the Neyman-Pearson example, i.e. taking marginals and crossing them to define a data distribution over tuples of examples, we’ll instead take \( \mathcal{D} \) to be the data distribution over \( \mathcal{X} \times \{\pm1\} \) pairs, and use the indicator feature \( x_k \) to define:

\[
\begin{align*}
\ell_0(x, y; \theta) := 1 \{ y f(x; \theta) \leq 0 \} \\
\ell_i(x, y; \theta) := 1 \{ y = 1 \land x_k = i \} 1 \{ f(x; \theta) \leq 0 \} - 1.1 \cdot \frac{1 \{ y = 1 \} 1 \{ f(x; \theta) \leq 0 \}}{\Pr(y = 1 \mid x,y \sim \mathcal{D})}
\end{align*}
\]

for all \( i \in \{1, 2, 3\} \), where we assume that the probabilities in the denominators of the ratios defining \( \ell_i \) are constants known a priori.

As in the Neyman-Pearson example, in practice the indicators in the objective function could be replaced with differentiable upper bounds, and a differentiable proxy-constraint \( \tilde{\ell}_i \) could be introduced for each \( \ell_i \).
B Shrinking

Cotter et al. [2018] introduced a procedure for “shrinking” the support size of a \( \tilde{\theta} \). When adapted to our setting, the first step is to evaluate the objective and constraints for every iterate (in practice, this is overkill; one should subsample the iterates):

\[
\ell^{(t)}_0 = \frac{1}{|S(\text{train})|} \sum_{x \in S(\text{train})} \ell_0(x; \theta^{(t)})
\]

\[
\ell^{(t)}_i = \frac{1}{|S(\text{val})|} \sum_{x \in S(\text{val})} \ell_i(x; \theta^{(t)})
\]

Next, we optimize a linear program (LP) that seeks a distribution \( p \) over \( \hat{\Theta} \) that minimizes the objective while violating no constraint by more than \( \epsilon \):

\[
\min_{p \in \Delta^T} \left< p, \ell^0 \right> \quad \text{s.t.} \quad \forall i \in [m], \left< p, \ell_i \right> \leq \epsilon
\]

Notice that the \( p \in \Delta^T \) condition adds several implicit simplex constraints to this LP. The key to this procedure is that, as Cotter et al. [2018] show, every vertex \( p \) of this linear program has at most \( m + 1 \) nonzero elements, where \( m \) is the number of constraints.

In particular, if \( \epsilon \) is chosen to be the maximum validation constraint violation of the “original” stochastic classifier \( \tilde{\theta} \), then an optimal vertex \( p^* \) will have a training objective function value and maximum validation constraint violation that are no larger than those of \( \tilde{\theta} \), and \( p^* \) will be supported on only \( m + 1 \) \( \theta^{(t)} \)'s. Furthermore, since the resulting stochastic classifier is still supported on a subset of \( \Theta \), the generalization definitions of Section 3.1 apply to it just as well as they did to \( \tilde{\theta} \).

While it would be possible to provide optimality and feasibility guarantees for the result of this “shrinking” procedure, we will only use it in our experiments (Section 5). There, instead of taking the LP’s \( \epsilon \) parameter to be the maximum validation constraint violation of \( \tilde{\theta} \), we follow Cotter et al. [2018]’s suggestion to use an outer bisection search to find the smallest \( \epsilon \geq 0 \) for which the LP is feasible.

C Proofs

We’ll begin by reproducing a definition from Cotter et al. [2018]:

**Definition 4.** (Definition 2 of Cotter et al. [2018]) Given proxy loss functions \( \tilde{\ell}_i(x; \theta) \geq \ell_i(x; \theta) \) for all \( x \in \mathcal{X} \) and \( i \in [m] \), the proxy-Lagrangians \( \mathcal{L}_\theta, \mathcal{L}_\lambda : \Theta \times \Lambda \rightarrow \mathbb{R} \) of Equation 1 are:

\[
\mathcal{L}_\theta (\theta, \lambda) := \mathbb{E}_{x \sim \mathcal{D}} \left[ \lambda_1 \ell_0(x; \theta) + \sum_{i=1}^{m} \lambda_{i+1} \tilde{\ell}_i(x; \theta) \right]
\]

\[
\mathcal{L}_\lambda (\theta, \lambda) := \mathbb{E}_{x \sim \mathcal{D}} \left[ \sum_{i=1}^{m} \lambda_{i+1} \ell_i(x; \theta) \right]
\]

where \( \Lambda := \Delta^{m+1} \) is the \((m + 1)\)-dimensional simplex.

This definition differs from Definition 1 in that the former is the in-expectation version of the latter, which is written in terms of separate i.i.d. training and validation sets.

Theorem 2 of Cotter et al. [2018] characterizes the optimality and feasibility properties of a particular type of \( \Phi \)-correlated equilibrium of Definition 4. We adapt it to our setting, giving the analogous result for such an equilibrium of Definition 1.
Theorem 3. Define \( \mathcal{M} \) as the set of all left-stochastic \((m + 1) \times (m + 1)\) matrices, \( \Lambda := \Delta^{m+1} \) as the \((m + 1)\)-dimensional simplex, and assume that each \( \ell_i \) upper bounds the corresponding \( \ell_i \). Let \( \theta(1), \ldots, \theta(T) \in \Theta \) and \( \lambda(1), \ldots, \lambda(T) \in \Lambda \) be sequences satisfying:

\[
\frac{1}{T} \sum_{i=1}^{T} \hat{L}_{\theta} \left( \theta(t), \lambda(t) \right) - \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{i=1}^{T} \hat{L}_{\theta} \left( \theta^*, \lambda(t) \right) \leq \epsilon_{\theta} \tag{8}
\]

\[
\max_{M^* \in \mathcal{M}} \frac{1}{T} \sum_{i=1}^{T} \hat{L}_{\lambda} \left( \theta(t), M^* \lambda(t) \right) - \frac{1}{T} \sum_{i=1}^{T} \hat{L}_{\lambda} \left( \theta(t), \lambda(t) \right) \leq \epsilon_{\lambda}
\]

Define \( \hat{\Theta} := \{ \theta(1), \ldots, \theta(T) \} \). Let \( \bar{\theta} \) be a random variable taking values from \( \hat{\Theta} \), defined such that \( \bar{\theta} = \theta(t) \) with probability \( \lambda_1(t) / \sum_{i=1}^{T} \lambda_i(t) \), and let \( \bar{\lambda} := \left( \sum_{i=1}^{T} \Lambda(t) \right) / T \). Then \( \bar{\theta} \) is nearly-optimal in expectation:

\[
E_{\bar{\theta}, x \sim D} [\ell_0 (x; \bar{\theta})] \leq \inf_{\theta^* \in \Theta: \forall i \in [m], E_{x \sim D} [\ell_i (x; \theta^*)] \leq 0} \left[ E_{x \sim D} [\ell_0 (x; \theta^*)] + \frac{1}{\bar{\lambda}_1} \left( \epsilon_{\theta} + \epsilon_{\lambda} + 2 \hat{G}^{(\text{train})(\Theta)} + G^{(\text{val})(\hat{\Theta})} \right) \right]
\]

and nearly-feasible:

\[
\max_{i \in [m]} E_{\bar{\theta}, x \sim D} [\ell_i(x; \bar{\theta})] \leq \frac{\epsilon_{\lambda}}{\bar{\lambda}_1} + G^{(\text{val})(\hat{\Theta})} \tag{10}
\]

Additionally, if there exists a \( \theta' \in \Theta \) that satisfies all of the constraints with margin \( \gamma \) (i.e. \( E_{x \sim D} [\ell_i(x; \theta')] \leq -\gamma \) for all \( i \in [m] \)), then:

\[
\bar{\lambda}_1 \geq \frac{\gamma - \epsilon_{\theta} - \epsilon_{\lambda} - 2 \hat{G}^{(\text{train})(\Theta)} - G^{(\text{val})(\hat{\Theta})}}{\gamma + B_{\ell_0}} \tag{11}
\]

where \( B_{\ell_0} \geq \sup_{\theta \in \Theta} E_{x \sim D} [\ell_0(x; \theta)] - \inf_{\theta \in \Theta} E_{x \sim D} [\ell_0(x; \theta)] \) is a bound on the range of the objective loss.

Proof. This proof closely follows those of Theorem 4 and Lemma 7 in [Cotter et al. 2018]—the only addition is that, in this proof, we use the empirical proxy-Lagrangian formulation with separate training and validation datasets (Definition 1), instead of the in-expectation proxy-Lagrangian (Definition 4), and therefore need to account for generalization.

Optimality: If we choose \( M^* \) to be the matrix with its first row being all-one, and all other rows being all-zero, then \( \hat{L}_{\lambda}(\theta, M^* \lambda) = 0 \), which shows that the first term in the LHS of the second line of Equation (8) is nonnegative. Hence:

\[
-E_{t \sim [T]} \left[ \hat{L}_{\lambda} \left( \theta(t), \lambda(t) \right) \right] \leq \epsilon_{\lambda}
\]

so by the definitions of \( \hat{L}_{\lambda} \) (Definition 1) and \( G^{(\text{val})(\hat{\Theta})} \) (Definition 2), and the facts that \( \hat{\ell}_i \geq \ell_i \) and \( \lambda(t) \in \Delta^{m+1} \):

\[
E_{t \sim [T], x \sim D} \left[ \sum_{i=1}^{m} \lambda_{i+1} \hat{\ell}_i(x; \theta(t)) \right] \geq -\epsilon_{\lambda} - G^{(\text{val})(\hat{\Theta})}
\]

Notice that \( \hat{L}_{\theta} \) is linear in \( \lambda \), so the first line of Equation (8) combined with the definition of \( \hat{G}^{(\text{train})(\Theta)} \), becomes:

\[
E_{t \sim [T], x \sim D} \left[ \lambda_1(t) \ell_0(x; \theta(t)) + \sum_{i=1}^{m} \lambda_{i+1} \hat{\ell}_i(x; \theta(t)) \right] - \inf_{\theta^* \in \Theta} E_{x \sim D} \left[ \bar{\lambda}_1 \ell_0(x; \theta^*) + \sum_{i=1}^{m} \bar{\lambda}_{i+1} \hat{\ell}_i(x; \theta^*) \right] \leq \epsilon_{\theta} + 2 \hat{G}^{(\text{train})(\Theta)}
\]
Combining the above two results:

\[
\mathbb{E}_{t \sim [T], x \sim D} \left[ \lambda_1^{(t)} \ell_0 \left( x; \theta^{(t)} \right) \right] - \inf_{\theta^* \in \Theta} \mathbb{E}_{x \sim D} \left[ \bar{\lambda}_1 \ell_0 \left( x; \theta^* \right) + \sum_{i=1}^{m} \bar{\lambda}_{i+1} \hat{\ell}_i \left( x; \theta^* \right) \right] \\
\leq \epsilon_{\theta} + \epsilon_{\lambda} + 2\tilde{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\hat{\Theta})
\]

(12)

Choose \( \theta^* \) to be the optimal solution that satisfies the proxy constraints, so that \( \mathbb{E}_{x \sim D} \left[ \hat{\ell}_i \left( x; \theta^* \right) \right] \leq 0 \) for all \( i \in [m] \).

Then:

\[
\mathbb{E}_{t \sim [T], x \sim D} \left[ \lambda_1^{(t)} \ell_0 \left( x; \theta^{(t)} \right) \right] - \mathbb{E}_{x \sim D} \left[ \bar{\lambda}_1 \ell_0 \left( x; \theta^* \right) \right] \leq \epsilon_{\theta} + \epsilon_{\lambda} + 2\tilde{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\hat{\Theta})
\]

which is the optimality claim.

**Feasibility:** We’ll begin by simplifying our notation: define \( g_1 (\theta) := 0 \) and \( g_{i+1} (\theta) := \mathbb{E}_{x \sim S^{(\text{val})}} [\ell_i (x; \theta)] \) for \( i \in [m] \), so that \( \hat{\mathcal{L}}_\lambda (\theta, \lambda) = \langle \lambda, g : (\theta) \rangle \). Consider the first term in the LHS of the second line of Equation 8:

\[
\max_{M^* \in \mathcal{M}} \mathbb{E}_{t \sim [T]} \left[ \hat{\mathcal{L}}_\lambda \left( \theta^{(t)}, M^* \lambda^{(t)} \right) \right] = \max_{M^* \in \mathcal{M}} \mathbb{E}_{t \sim [T]} \left[ \langle M^* \lambda^{(t)}, g : \left( \theta^{(t)} \right) \rangle \right]
\]

\[
= \max_{M^* \in \mathcal{M}} \mathbb{E}_{t \sim [T]} \left[ \sum_{i=1}^{m} \sum_{j=1}^{m+1} M^*_{j,i} \lambda^{(t)}_i g_j \left( \theta^{(t)} \right) \right]
\]

\[
= \sum_{i=1}^{m+1} \max_{M^* \in \mathcal{M}} \sum_{j=1}^{m+1} \mathbb{E}_{t \sim [T]} \left[ M^*_{j,i} \lambda^{(t)}_i g_j \left( \theta^{(t)} \right) \right]
\]

\[
= \sum_{i=1}^{m+1} \max_{j \in [m+1]} \mathbb{E}_{t \sim [T]} \left[ \lambda^{(t)}_i g_j \left( \theta^{(t)} \right) \right]
\]

where we used the fact that, since \( M^* \) is left-stochastic, each of its columns is a \((m+1)\)-dimensional multinomial distribution. For the second term in the LHS of the second line of Equation 8, we can use the fact that \( g_1 (\theta) = 0 \):

\[
\mathbb{E}_{t \sim [T]} \left[ \sum_{i=2}^{m+1} \lambda^{(t)}_i g_i \left( \theta^{(t)} \right) \right] \leq \sum_{i=2}^{m+1} \max_{j \in [m+1]} \mathbb{E}_{t \sim [T]} \left[ \lambda^{(t)}_i g_j \left( \theta^{(t)} \right) \right]
\]

Plugging these two results into the second line of Equation 8, the two sums collapse, leaving:

\[
\max_{i \in [m+1]} \mathbb{E}_{t \sim [T]} \left[ \lambda^{(t)}_i g_i \left( \theta^{(t)} \right) \right] \leq \epsilon_{\lambda}
\]

Substituting the definitions of \( g_i \) and \( G^{(\text{val})}(\hat{\Theta}) \) (since the \( g_i \)s are defined on \( S^{(\text{val})} \), but we want our result to hold on \( \mathcal{D} \)) then yields the feasibility claim.

**Bound on \( \bar{\lambda}_1 \):** Choosing \( \theta^* = \theta' \) in Equation 12 (recall that \( \theta' \) satisfies all of the proxy constraints with margin \( \gamma \)) and substituting the definition of \( B_{\bar{\lambda}} \):

\[
\epsilon_{\theta} + \epsilon_{\lambda} + 2\tilde{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\hat{\Theta}) \geq \mathbb{E}_{t \sim [T], x \sim \mathcal{D}} \left[ \lambda_1^{(t)} \ell_0 \left( x; \theta^{(t)} \right) \right] - \lambda_1^{(t)} \ell_0 \left( x; \theta' \right) + (1 - \bar{\lambda}_1 \gamma)
\]

\[
\geq - \bar{\lambda}_1 B_{\bar{\lambda}} + (1 - \bar{\lambda}_1) \gamma
\]

Solving for \( \bar{\lambda}_1 \) yields the claim. \( \square \)

Before moving on to the convergence and generalization properties of our actual algorithms, we need to state some (fairly standard) elementary results:
**Definition 5.** We say that $C_r \subseteq \mathbb{R}^{m+1}$ is a radius-$r$ external covering of $\Lambda := \Delta^{m+1}$ w.r.t. the 1-norm if for every $\lambda \in \Lambda$ there exists a $\hat{\lambda} \in C_r$ for which $\|\lambda - \hat{\lambda}\|_1 \leq r$. Notice that we do not require $C_r$ to be a subset of $\Lambda$—this is why it’s an external covering.

**Lemma 1.** Assuming that $r \leq 1$, there exists a radius-$r$ external covering of $\Lambda := \Delta^{m+1}$ w.r.t. the 1-norm of size no larger than $(5/r)^m$.

**Proof.** Consider the $m$-dimensional unit ball $\hat{B} := \left\{ \hat{\lambda} \in \mathbb{R}^m : \|\hat{\lambda}\|_1 \leq 1 \right\}$ w.r.t. the 1-norm (note that we could instead consider only the positive orthant, which would improve the constant in the overall result). There exists a radius-$r/2$ covering $\hat{C}_r \subseteq \mathbb{R}^m$ of $\hat{B}$ with $|\hat{C}_r| \leq (1 + 4/r)^m \leq (5/r)^m$ [Bartlett, 2013].

Define $C_r \subseteq \mathbb{R}^{m+1}$ as:

$$C_r = \left\{ \left[ \frac{\hat{\lambda}}{1 - \|\hat{\lambda}\|_1} \right] : \hat{\lambda} \in \hat{C}_r \right\}$$

Notice that we do not necessarily have that $C_r \subseteq \Delta^{m+1}$, i.e. this will be an external covering.

From any $\lambda \in \Delta^{m+1}$, we can define $\lambda' \in \mathbb{R}_+^m$ by dropping the last coordinate of $\lambda$, and we’ll have that $\|\lambda\|_1 \leq \|\lambda\|_1 = 1$, so there will exist a $\hat{\lambda} \in \hat{C}_r$ such that $\|\hat{\lambda} - \lambda'\|_1 \leq r/2$, which implies that the corresponding element of $C_r$ is $r$-far from $\lambda$, showing that $C_r$ is a radius-$r$ covering of $\Delta^{m+1}$ w.r.t. the 1-norm.

**Lemma 2.** Let $S$ be an i.i.d. sample from a distribution $\mathcal{D}$ supported on $\mathcal{X}$, and $\hat{\Theta} \subseteq \Theta$ the finite set of permitted model parameters, which defines a a finite function class ($\Theta$ may be a random variable, but must be independent of $S$). Suppose that $\ell : \mathcal{X} \times \Theta \rightarrow [a, b]$ with $B_t := b - a$. Then:

$$\left| \frac{1}{|S|} \sum_{x \in S} \ell(x; \theta) - \mathbb{E}_{x \sim \mathcal{D}} [\ell(x; \theta)] \right| < B_t \sqrt{\frac{\ln \left( \frac{2|\hat{\Theta}|}{\delta} \right)}{2|S|}}$$

for all $\theta \in \hat{\Theta}$, with probability at least $1 - \delta$ over the sampling of $S$.

**Proof.** Allowing $\hat{\Theta}$ to be a random variable independent of $S$, instead of a constant set, doesn’t significantly change the standard proof [e.g. Srebro, 2016]. By Hoeffding’s inequality:

$$\Pr \left\{ \left| \frac{1}{|S|} \sum_{x \in S} \ell(x; \theta) - \mathbb{E}_{x \sim \mathcal{D}} [\ell(x; \theta)] \right| \geq \epsilon \right\} \leq 2 \exp \left( -\frac{2|S| \epsilon^2}{B_t^2} \right)$$

the above holding for any $\theta \in \Theta$. Since $\hat{\Theta}$ is independent of $S$, we can apply the union bound over all $\theta \in \hat{\Theta}$:

$$\Pr \left\{ \exists \theta \in \hat{\Theta}. \left| \frac{1}{|S|} \sum_{x \in S} \ell(x; \theta) - \mathbb{E}_{x \sim \mathcal{D}} [\ell(x; \theta)] \right| \geq \epsilon \right\} \leq 2 |\theta| \exp \left( -\frac{2|S| \epsilon^2}{B_t^2} \right)$$

Rearranging terms yields the claimed result.
C.1 Algorithm 1

Lemma 3. If we take $\eta_\lambda := \sqrt{(m+1) \ln (m+1) / TB^2_\Delta}$, then the result of Algorithm 1 satisfies the conditions of Theorem 3 with:

$$\epsilon_\theta = \rho + 2rB_\ell$$

$$\epsilon_\lambda = 2B_\Delta \sqrt{(m+1) \ln (m+1)}$$

where $B_\ell \geq |\ell(x, \theta)|$ for all $\ell \in \{\ell_0, \ell_1, \ldots, \ell_m\}$, and $B_\Delta \geq \max_{t \in [T]} \|\Delta_\lambda^{(t)}\|_\infty$ is a bound on the gradients.

Proof. Since $C_r$ is a radius-$r$ external covering of $\Lambda := \Delta^{m+1}$ w.r.t. the 1-norm, we must have that $\|\hat{\lambda}^{(t)} - \lambda^{(t)}\|_1 \leq r$, which implies by Definition 3 that:

$$|\hat{\mathcal{L}}_\theta (\theta, \hat{\lambda}^{(t)}) - \hat{\mathcal{L}}_\lambda (\theta, \lambda^{(t)})| \leq rB_\ell$$

the above holding for all $\theta \in \Theta$, and all $t$. In particular, this implies that:

$$\hat{\mathcal{L}}_\theta (\theta^{(t)}, \lambda^{(t)}) \leq \hat{\mathcal{L}}_\theta (\theta^{(t)}, \hat{\lambda}^{(t)}) + rB_\ell$$

$$\leq \inf_{\theta^* \in \Theta} \hat{\mathcal{L}}_\theta (\theta^*, \hat{\lambda}^{(t)}) + \rho + rB_\ell$$

$$\leq \inf_{\theta^* \in \Theta} \hat{\mathcal{L}}_\theta (\theta^*, \lambda^{(t)}) + \rho + 2rB_\ell$$

Therefore:

$$\frac{1}{T} \sum_{t=1}^{T} \hat{\mathcal{L}}_\theta (\theta^{(t)}, \lambda^{(t)}) \leq \frac{1}{T} \sum_{t=1}^{T} \inf_{\theta^* \in \Theta} \hat{\mathcal{L}}_\theta (\theta^*, \lambda^{(t)}) + \rho + 2rB_\ell$$

$$\leq \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \hat{\mathcal{L}}_\theta (\theta^*, \lambda^{(t)}) + \rho + 2rB_\ell$$

so the first condition of Theorem 3 is satisfied with the claimed $\epsilon_\theta$. The second condition, on $\epsilon_\lambda$, follows immediately from Lemma 8 of Appendix C.1 of Cotter et al. [2018], taking $\bar{m} = m + 1$.

Lemma 4. If we take $\hat{\Theta} := \{\theta^{(1)}, \ldots, \theta^{(T)}\}$ as in Theorem 3, where $\theta^{(1)}, \ldots, \theta^{(T)}$ are the result of Algorithm 1, then with probability $1 - \delta$ over the sampling of $S^{(val)}$:

$$G^{(val)}(\hat{\Theta}) < B_\ell \sqrt{\ln (2m |C_r| / \delta)}$$

where $B_\ell \geq \max_{i \in [m]} (b_i - a_i)$ assuming that the range of each $\ell_i$ is the interval $[a_i, b_i]$.

Proof. Since each $\theta^{(t)}$ is uniquely associated with a $\hat{\lambda}^{(t)} \in C_r$ (Definition 3), we will have that $\hat{\Theta} \subseteq \Theta_{C_r}$, where:

$$\Theta_{C_r} := \left\{ \mathcal{O}_\rho \left( \hat{\mathcal{L}}_\theta \left( \cdot, \hat{\lambda} \right) \right) : \hat{\lambda} \in C_r \right\}$$

Because the oracle call defining $\Theta_{C_r}$ depends only on $\hat{\mathcal{L}}_\theta$, which itself depends only on $S^{(train)}$, we can apply Lemma 2 to $\Theta_{C_r}$, yielding that, for each $i \in [m]$, the following holds with probability $\delta / m$ for all $\theta \in \Theta_{C_r}$:

$$\left| \frac{1}{|S^{(val)}|} \sum_{x \in S^{(val)}} \ell_i (x; \theta) - \mathbb{E}_{x \sim D} [\ell_i (x; \theta)] \right| < B_\ell \sqrt{\ln (2m |\Theta_{C_r}| / \delta)}$$

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The claimed result on $G^{(\text{val})} (\check{\Theta})$ then follows from the union bound and the facts that $\check{\Theta} \subseteq \Theta_{C_r}$ and $|\Theta_{C_r}| = |C_r|$.

**Theorem 1.** Given any $\epsilon > 0$, there exists a covering $C_r$ such that, if we take $T \geq 4B^2_\Delta (m + 1) / r \epsilon^2$ and $\eta_\lambda = \sqrt{(m + 1) \ln (m + 1) / T B^2_\Delta}$, where $B_\Delta = \max_{t \in [T]} \left\| \Delta_t^{(i)} \right\|$ is a bound on the gradients, then the following hold, where $\check{\Theta} := \{ \Theta^{(1)}, \ldots, \Theta^{(T)} \}$ is the set of results of Algorithm 1.

**Optimality and Feasibility:** Let $\check{\Theta}$ be a random variable taking values from $\check{\Theta}$, defined such that $\check{\Theta} = \Theta^{(i)}$ with probability $\lambda_i^{(i)}/ \sum_{s=1}^T \lambda_s^{(i)}$, and let $\check{\lambda} := \left( \sum_{s=1}^T \lambda_s^{(i)} \right) / T$. Then $\check{\Theta}$ is nearly-optimal in expectation:

\[
\mathbb{E}_{\check{\Theta}, x \sim D} [\ell_0 (x; \check{\Theta})] \leq \inf_{\theta^* \in \Theta : \forall i \in [m] \ell_i (x; \theta^*)} \mathbb{E}_{x \sim D} [\ell_0 (x; \theta^*)] + \frac{1}{\lambda_1} (\rho + 2\epsilon + 2\check{G}^{(\text{train})}(\Theta) + G^{(\text{val})}(\check{\Theta}))
\]

and nearly-feasible:

\[
\max_{i \in [m]} \mathbb{E}_{\check{\Theta}, x \sim D} [\ell_i (x; \check{\Theta})] \leq \epsilon \gamma + G^{(\text{val})}(\check{\Theta})
\]

Additionally, if there exists a $\theta' \in \Theta$ that satisfies all of the constraints with margin $\gamma$ (i.e. $\mathbb{E}_{x \sim D} [\ell_i (x; \theta')] \leq -\gamma$ for all $i \in [m]$), then:

\[
\check{\lambda} \geq \frac{1}{\gamma + B_\theta} \left( \gamma - \rho - 2\epsilon - 2\check{G}^{(\text{train})}(\Theta) - G^{(\text{val})}(\check{\Theta}) \right)
\]

where $B_\theta \geq \sup_{\Theta_0 \in \Theta} \mathbb{E}_{x \sim D} [\ell_0 (x; \Theta_0)] - \inf_{\Theta \in \Theta} \mathbb{E}_{x \sim D} [\ell_0 (x; \Theta)]$ is a bound on the range of the objective loss.

**Generalization:** With probability $1 - \delta$ over the sampling of $S^{(\text{val})}$:

\[
G^{(\text{val})} (\check{\Theta}) < B_\ell \sqrt{\frac{m \ln (10B_\ell / \epsilon) + \ln (2m / \delta)}{2 |S^{(\text{val})}|}}
\]

where $B_\ell \geq |\ell (x, \theta)|$ for all $\ell \in \{ \ell_0, \ell_1, \ldots, \ell_m \}$, and $B_\ell \geq \max_{i \in [m]} (b_i - a_i)$ assuming that the range of each $\ell_i$ is the interval $[a_i, b_i]$.

**Proof.** The particular values we choose for $T$ and $\eta_\lambda$ come from Lemma 3, taking $r = \epsilon / 2B_\ell$, $\epsilon_\theta = \rho + 2rB_\ell = \rho + \epsilon$, and $\epsilon_\lambda = \epsilon$. The optimality and feasibility results then follow from Theorem 1.

For the bound on $G^{(\text{val})} (\check{\Theta})$, notice that by Lemma 4 there exists a radius-$r$ covering $C_r$ w.r.t. the 1-norm with $|C_r| \leq (5/\epsilon)^m = (10B_\ell / \epsilon)^m$. Substituting this, and the definition of $r$, into the bound of Lemma 4 yields the claimed bound.

**C.2 Algorithm 2**

**Lemma 5.** Suppose that $\Theta$ is compact and convex, and that $\ell (x; \theta)$ is $\mu$-strongly convex in $\theta$ for all $\ell \in \{ \ell_0, \ell_1, \ldots, \ell_m \}$. If we take $\eta_\lambda := \sqrt{(m + 1) \ln (m + 1) / T_\lambda B^2_\Delta}$, then the result of Algorithm 2 satisfies the conditions of Theorem 1 with:

\[
\epsilon_\theta = \frac{B_\Delta^2 (1 + \ln T_0)}{2\mu T_\theta}, \quad \epsilon_\lambda = 2B_\Delta \sqrt{\frac{(m + 1) \ln (m + 1)}{T_\lambda}}
\]
where $B_\Delta \geq \max_{s,t \in \{T_\Delta \}} \left\| \Delta^{(t,s)}_\theta \right\|_2$ is a bound on the subgradients, and $B_\Delta \geq \max_{t \in \{T_\Delta \}} \left\| \Delta^{(t)}_\lambda \right\|_\infty$ is a bound on the gradients.

**Proof.** By Lemma 1 of Shalev-Shwartz et al. [2011], the fact that $\mathcal{L}_\theta (\theta, \lambda)$ is $\mu$-strongly convex in $\theta$ (because $\ell(x; \theta)$ is $\mu$-strongly convex in $\theta$ for $\ell \in \{\ell_0, \ell_1, \ldots, \ell_m\}$, and $\lambda \in \Lambda : = \Delta^{m+1}$), and Jensen’s inequality:

$$\mathcal{L}_\theta (\theta^{(t)}, \lambda^{(t)}) \leq \frac{1}{T_\theta} \sum_{s=1}^{T_\theta} \mathcal{L}_\theta (\tilde{\theta}^{(t,s)}, \lambda^{(t)}) \leq \min_{\tilde{\theta}^* \in \Theta} \mathcal{L}_\theta (\tilde{\theta}^*, \lambda^{(t)}) + \frac{B_\Delta^2 (1 + \ln T_\theta)}{2\mu T_\theta}$$

the above holding for all $t$. Therefore:

$$\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_\theta (\theta^{(t)}, \lambda^{(t)}) \leq \min_{\tilde{\theta}^* \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_\theta (\tilde{\theta}^*, \lambda^{(t)}) + \frac{B_\Delta^2 (1 + \ln T_\theta)}{2\mu T_\theta}$$

so the first condition of Theorem 3 is satisfied with the claimed $\epsilon_\theta$.

As in the proof of Lemma 3, the second condition, on $\epsilon_\lambda$, follows immediately from Lemma 8 of Appendix C.1 of Cotter et al. [2018], taking $m = m + 1$. \hfill \square

**Lemma 6.** In addition to the conditions of Lemma 3, suppose that $\ell(x; \theta)$ is $L$-Lipschitz continuous in $\theta$ for all $\ell \in \{\ell_1, \ldots, \ell_m\}$. If we take $\hat{\Theta} := \{\theta^{(1)}, \ldots, \theta^{(T)}\}$ as in Theorem 3 where $\theta^{(1)}, \ldots, \theta^{(T)}$ are the result of Algorithm 2, then with probability $1 - \delta$ over the sampling of $S^{(val)}$:

$$G^{(val)}(\hat{\Theta}) \leq 2L \sqrt{\frac{4r B_\ell^2}{\mu}} + 2LB_\Delta \sqrt{\frac{1 + \ln T_\theta}{T_\theta}} + B_\ell \sqrt{\frac{\ln \left(2m |C_r| / \delta\right)}{2 |S^{(val)}|}}$$

where $B_\ell \geq |\ell(x, \theta)|$ for all $\ell \in \{\ell_0, \ell_1, \ldots, \ell_m\}$, $B_\ell \geq \max_{s \in [m]} (b_i - a_i)$ assuming that the range of each $\ell_i$ is the interval $[a_i, b_i]$, and $C_r$ is a radius-$r$ covering of $\Lambda : = \Delta^{m+1}$ w.r.t. the 1-norm.

**Proof.** Define $\hat{\lambda}^{(t)} := \arg\min_{\hat{\lambda} \in C_r} \left\| \lambda^{(t)} - \hat{\lambda} \right\|_1$ for all $t$. Since $C_r$ is a radius-$r$ covering of $\Lambda : = \Delta^{m+1}$ w.r.t. the 1-norm, we must have that $\left\| \hat{\lambda}^{(t)} - \lambda^{(t)} \right\|_1 \leq r$, which implies by Definition 1 that:

$$\left| \mathcal{L}_\theta \left( \theta, \hat{\lambda}^{(t)} \right) - \mathcal{L}_\lambda \left( \theta, \lambda^{(t)} \right) \right| \leq r B_\ell$$

the above holding for all $\theta \in \Theta$, and all $t$. Take $\theta^{(t*)} := \arg\min_{\tilde{\theta}^* \in \Theta} \mathcal{L}_\theta (\tilde{\theta}^*, \lambda^{(t)})$ and $\hat{\theta}^{(t*)} := \arg\min_{\hat{\theta}^* \in \Theta} \mathcal{L}_\theta (\hat{\theta}^*, \hat{\lambda}^{(t)})$. Then, by the above result and the triangle inequality:

$$\left| \mathcal{L}_\theta \left( \theta^{(t*)}, \lambda^{(t)} \right) - \mathcal{L}_\theta \left( \hat{\theta}^{(t*)}, \hat{\lambda}^{(t)} \right) \right| \leq r B_\ell$$
$$\left| \mathcal{L}_\theta \left( \theta^{(t*)}, \hat{\lambda}^{(t)} \right) - \mathcal{L}_\theta \left( \hat{\theta}^{(t*)}, \hat{\lambda}^{(t)} \right) \right| \leq 2r B_\ell$$

so by the fact that $\mathcal{L}_\theta (\theta, \lambda)$ is $\mu$-strongly convex in $\theta$ for all $\lambda$:

$$\left\| \theta^{(t*)} - \hat{\theta}^{(t*)} \right\|_2 \leq \sqrt{\frac{4r B_\ell^2}{\mu}}$$

Again by strong convexity, but applied to Equation 13 in the proof of Lemma 5:

$$\left\| \theta^{(t)} - \theta^{(t*)} \right\|_2 \leq \sqrt{\frac{B_\Delta^2 (1 + \ln T_\theta)}{\mu^2 T_\theta}}$$
Additionally, if there exists a

Optimality and Feasibility:

Suppose that

Theorem 2.

\( t \)

the above holding for all \( i \in [m] \).

Define \( \tilde{\Theta} := \{ \tilde{\theta}^{(1)}, \ldots, \tilde{\theta}^{(T)} \} \). Observe that since \( \tilde{\theta}^{(t)} \) is uniquely associated with a \( \tilde{\lambda}(t) \in C_r \), we will have that \( \tilde{\Theta} \subseteq \Theta_{C_r} \), where:

\[
\Theta_{C_r} := \left\{ \hat{\Theta} \in \Theta \mid \inf_{\tilde{\theta}^* \in \Theta} \hat{\lambda}(\tilde{\theta}^*) : \hat{\lambda} \in C_r \right\}
\]

Because the argmins defining \( \Theta_{C_r} \) depend only on \( \hat{\lambda}(\tilde{\theta}) \), which itself depends only on \( S^{(train)} \), we can apply Lemma 2 to \( \Theta_{C_r} \), yielding that, for each \( i \in [m] \), the following holds with probability \( \delta/m \) for any \( \theta \in \Theta_{C_r} \):

\[
\left| \frac{1}{|S^{(val)}|} \sum_{x \in S^{(val)}} \ell_i(x; \theta) - \mathbb{E}_{x \sim D} [\ell_i(x; \theta)] \right| < B_{\ell_i} \sqrt{\ln (2m |\Theta_{C_r}| / \delta) / 2 |S^{(val)}|}
\]

By the union bound, we could instead take the above to hold uniformly for all \( i \in [m] \) with probability \( 1 - \delta \). Substituting Equation 14 and using the facts that \( \tilde{\Theta} \subseteq \Theta_{C_r} \) and \( |\Theta_{C_r}| = |C_r| \) yields the claimed result. \( \square \)

**Theorem 2** Suppose that \( \Theta \) is compact and convex, and that \( \ell(x; \theta) \) is \( \mu \)-strongly convex in \( \theta \) for all \( \ell \in \{ \ell_0, \ell_1, \ldots, \ell_m \} \). Given any \( \epsilon > 0 \), if we take \( T_\theta \geq (B_\Delta^2 / \mu) \ln \left( B_\Delta^2 / \mu \right) \), \( T_{\lambda} \geq 4B_\Delta^2 (m + 1) \ln (m + 1) / \epsilon^2 \) and \( \eta_\lambda = \sqrt{(m + 1) \ln (m + 1) / T_{\lambda} B_\Delta^2} \), where \( B_\Delta \) is as in Theorem 1 and \( B_\Delta \geq \max_{s \in \Theta} \left\| \Delta_s^{(t)} \right\|_2 \) is a bound on the subgradients, then the following hold, where \( \hat{\Theta} := \{ \theta^{(1)}, \ldots, \theta^{(T)} \} \) is the set of results of Algorithm 1:

**Optimality and Feasibility:** Let \( \tilde{\theta} \) be a random variable taking values from \( \tilde{\Theta} \), defined such that \( \tilde{\theta} = \theta^{(t)} \) with probability \( \lambda_1^{(t)} / \sum_{s=1}^{T} \lambda_1^{(s)} \), and let \( \tilde{\lambda} := \left( \sum_{t=1}^{T} \lambda_1^{(t)} \right) / T_{\lambda} \). Then \( \tilde{\theta} \) is nearly-optimal in expectation:

\[
\mathbb{E}_{\tilde{\theta}, x \sim D} [\ell_0(x; \tilde{\theta})] \leq \inf_{\theta^* \in \Theta : \mathbb{E}_{x \sim D} [\ell_i(x; \theta^*)] \leq 0} \mathbb{E}_{x \sim D} [\ell_0(x; \theta^*)] + \frac{1}{\lambda_1} \left( 2\epsilon + 2\hat{G}_{\text{train}}(\Theta) + G_{\text{val}}(\hat{\Theta}) \right)
\]

and nearly-feasible:

\[
\max_{i \in [m]} \mathbb{E}_{\tilde{\theta}, x \sim D} [\ell_i(x; \tilde{\theta})] \leq \frac{\epsilon}{\lambda_1} + G_{\text{val}}(\hat{\Theta})
\]

Additionally, if there exists a \( \theta' \in \Theta \) that satisfies all of the constraints with margin \( \gamma \) (i.e. \( \mathbb{E}_{x \sim D} [\ell_i(x; \theta')] \leq -\gamma \) for all \( i \in [m] \)), then:

\[
\tilde{\lambda}_1 \geq \frac{1}{\gamma + B_{\ell_0}} \left( \gamma - 2\epsilon - 2\hat{\Delta}_{\text{train}}(\Theta) - G_{\text{val}}(\hat{\Theta}) \right)
\]

where \( B_{\ell_0} \) is as in Theorem 1.

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Generalization: If, in addition to the above requirements, $\ell(x; \theta)$ is $L$-Lipschitz continuous in $\theta$ for all $\ell \in \{\ell_1, \ldots, \ell_m\}$, then with probability $1 - \delta$ over the sampling of $S^{(\text{val})}$:

$$
G^{(\text{val})}(\hat{\Theta}) < B_{\ell} \sqrt{\frac{2m}{|S^{(\text{val})}|}} \max \left\{ 1, \ln \left( \frac{160L^2B_{\ell} |S^{(\text{val})}|}{m\mu B_{\ell}^2} \right), \frac{\ln (2m/\delta)}{2|S^{(\text{val})}|} \right\} + \sqrt{\frac{L}{\mu}} \ln \left( \frac{2m}{\delta} \right)
$$

where $B_{\ell}$ and $B_{\ell}$ are as in Theorem 7.

Proof. The particular values we choose for $T_{\theta}, T_{\lambda}$ and $\eta_{\lambda}$ come from Lemma 5, taking $\epsilon_{\theta} = \epsilon_{\lambda} = \epsilon$. The optimality and feasibility results then follow from Theorem 3.

For the bound on $G^{(\text{val})}(\hat{\Theta})$, notice that by Lemma 1, there exists a radius-$r$ external covering $C_r$ w.r.t. the 1-norm with $|C_r| \leq \max \{1, (5/r)^m\}$. Substituting into the bound of Lemma 6:

$$
G^{(\text{val})}(\hat{\Theta}) < 2L \sqrt{\frac{4rB_{\ell}}{\mu}} + \frac{2LB_{\Delta}}{\mu} \sqrt{\frac{1 + \ln T_{\theta}}{T_{\theta}}} + B_{\ell} \sqrt{\frac{m \max \{0, \ln (5/r)\}}{2|S^{(\text{val})}|}} + B_{\ell} \sqrt{\frac{1 + \ln T_{\theta}}{T_{\theta}}} + B_{\ell} \sqrt{\frac{1 + \ln T_{\theta}}{T_{\theta}}}
$$

Taking $r := (m\mu B_{\ell}^2) / (32L^2B_{\ell} |S^{(\text{val})}|)$:

$$
G^{(\text{val})}(\hat{\Theta}) < 2B_{\ell} \sqrt{\frac{m \max \{0, \ln (5/r)\}}{2|S^{(\text{val})}|}} + \frac{2LB_{\Delta}}{\mu} \sqrt{\frac{1 + \ln T_{\theta}}{T_{\theta}}} + B_{\ell} \sqrt{\frac{1 + \ln T_{\theta}}{T_{\theta}}} + B_{\ell} \sqrt{\frac{1 + \ln T_{\theta}}{T_{\theta}}}
$$

Substituting the definition of $T_{\theta}$ then yields the claimed result.

D One-dataset Lagrangian Baseline Approach

In this appendix, we’ll analyze the most natural theoretical baseline for our proposed approach, namely using a single training dataset, and optimizing the empirical Lagrangian:

$$
\hat{L}(\theta, \lambda) := \frac{1}{|S^{(\text{train})}|} \sum_{x \in S^{(\text{train})}} \left( \ell_0(x; \theta) + \sum_{i=1}^{m} \lambda_i \ell_i(x; \theta) \right)
$$

This is essentially an extension of the approach proposed by Agarwal et al. [2018]—who proposed using the Lagrangian formulation in the particular case of fair classification—to the slightly more general setting of inequality constrained optimization.

Theorem 1 of Cotter et al. [2018] characterizes the optimality and feasibility properties of Nash equilibria of the in-expectation Lagrangian (Equation 2). The analogous result for the empirical Lagrangian of Equation 15 is given in the following theorem:
Theorem 4. Define $\Lambda = \{ \lambda \in \mathbb{R}^m : \| \lambda \|_1 \leq R \}$, let $\hat{\ell}_i := \ell_i$ for all $i \in [m]$, and consider the empirical Lagrangian of Equation 15. Let $\theta^{(1)}, \ldots, \theta^{(T)} \in \Theta$ and $\lambda^{(1)}, \ldots, \lambda^{(T)} \in \Lambda$ be sequences satisfying:

$$\max_{\lambda^* \in \Lambda} \frac{1}{T} \sum_{t=1}^T \ell(\theta^{(t)}, \lambda^*) - \inf_{\theta^* \in \Theta} \frac{1}{T} \sum_{t=1}^T \ell(\theta^*, \lambda^{(t)}) \leq \epsilon$$

(16)

Define $\bar{\lambda}$ as a random variable for which $\bar{\lambda} = \theta^{(t)}$ with probability $1/T$, and let $\bar{\lambda} := \left( \sum_{t=1}^T \lambda^{(t)} \right) / T$. Then $\bar{\lambda}$ is nearly-optimal in expectation:

$$\mathbb{E}_{\bar{\lambda}, x \sim \mathcal{D}} \left[ \ell_0(x; \bar{\lambda}) \right] \leq \inf_{\theta^* \in \Theta : \forall i \in [m] \mathbb{E}_{x \sim \mathcal{D}} [\ell_i(x; \theta^*)] \leq 0} \mathbb{E}_{x \sim \mathcal{D}} \left[ \ell_0(x; \theta^*) \right] + \epsilon + 2\bar{G}^{\text{(train)}}(\Theta)$$

and nearly-feasible:

$$\max_{i \in [m]} \mathbb{E}_{\bar{\lambda}, x \sim \mathcal{D}} \left[ \ell_i(x; \bar{\lambda}) \right] \leq \frac{\epsilon}{R - \| \bar{\lambda} \|_1} + \bar{G}^{\text{(train)}}(\Theta)$$

Additionally, if there exists a $\theta^* \in \Theta$ that satisfies all of the constraints with margin $\gamma$ (i.e. $\mathbb{E}_{x \sim \mathcal{D}} [\ell_i(x; \theta^*)] \leq -\gamma$ for all $i \in [m]$), then:

$$\| \bar{\lambda} \|_1 \leq \frac{\epsilon + B_{\ell_0}}{\gamma - \bar{G}^{\text{(train)}}(\Theta)}$$

assumimg that $\gamma > \bar{G}^{\text{(train)}}(\Theta)$, where $B_{\ell_0} \geq \sup_{\theta \in \Theta} \mathbb{E}_{x \sim \mathcal{D}} [\ell_0(x; \theta)] - \inf_{\theta \in \Theta} \mathbb{E}_{x \sim \mathcal{D}} [\ell_0(x; \theta)]$ is a bound on the range of the objective loss.

Proof. The empirical Lagrangian is nothing but the in-expectation Lagrangian over the finite training sample $S^{(\text{train})}$, so by Theorem 1 of Cotter et al. [2018], $\bar{\lambda}$ is nearly-optimal in expectation:

$$\mathbb{E}_{\bar{\lambda}, x \sim S^{(\text{train})}} \left[ \ell_0(x; \bar{\lambda}) \right] \leq \inf_{\theta^* \in \Theta : \forall i \in [m] \mathbb{E}_{x \sim S^{(\text{train})}} [\ell_i(x; \theta^*)] \leq 0} \mathbb{E}_{x \sim S^{(\text{train})}} \left[ \ell_0(x; \theta^*) \right] + \epsilon$$

and nearly-feasible:

$$\max_{i \in [m]} \mathbb{E}_{\bar{\lambda}, x \sim S^{(\text{train})}} \left[ \ell_i(x; \bar{\lambda}) \right] \leq \frac{\epsilon}{R - \| \bar{\lambda} \|_1}$$

Since $\theta^*$ satisfies the constraints with margin $\gamma$ in expectation, it will satisfy them with margin $\gamma - \bar{G}^{\text{(train)}}(\Theta)$ on the training dataset, so the same theorem gives the claimed upper-bound $\| \bar{\lambda} \|_1 \leq (\epsilon + B_{\ell_0}) / (\gamma - \bar{G}^{\text{(train)}}(\Theta))$ when $\gamma > \bar{G}^{\text{(train)}}(\Theta)$.

Notice that the above expectations are taken over the finite training sample $S^{(\text{train})}$, rather than the data distribution $\mathcal{D}$. To fix this, we need only define $\hat{\ell}_i = \ell_i$, and appeal to the definition of $\bar{G}^{\text{(train)}}(\Theta)$ (Definition 3), yielding the claimed results.

Here, $R$ is the maximum allowed 1-norm of the vector of Lagrange multipliers (such a bound is necessary for Cotter et al. [2018]’s proof to work out). Notice that we have assumed that $\hat{\ell}_i := \ell_i$ for all $i$. This is purely for notational reasons (the Lagrangian does not involve proxy constraints at all)—it allows us to re-use the definition of $\bar{G}^{\text{(train)}}(\Theta)$ in the above Theorem, and causes $\gamma$ to have the same definition here, as it did in Appendix C. □