Research Article

Sunghoon Kim* and Ki-Ahm Lee

Properties of generalized degenerate parabolic systems

https://doi.org/10.1515/anona-2022-0236
received October 1, 2021; accepted January 13, 2022

Abstract: In this article, we consider the parabolic system

\[(u_i)_t = \nabla \cdot (m U^{m-1} A(u^i, x, t) + B(u^i, x, t)), \quad (1 \leq i \leq k)\]

in the range of exponents \(m > \frac{n-2}{n}\) where the diffusion coefficient \(U\) depends on the components of the solution \(u = (u^1, \ldots, u^k)\). Under suitable structure conditions on the vector fields \(A\) and \(B\), we first showed the uniform \(L^\infty\) boundedness of the function \(U\) for \(t \geq \tau > 0\). We also proved the law of \(L^1\) mass conservation and the local continuity of solution \(u\). In the last result, all components of the solution \(u\) have the same modulus of continuity if the ratio between \(U\) and \(u_i\), \((1 \leq i \leq k)\), is uniformly bounded above and below.

Keywords: local continuity of degenerate parabolic systems, uniform boundedness, law of mass conservation

MSC 2020: 35K40, 35K45, 35K65, 35K67

1 Introduction and main results

We consider a closed system in which various species exist. Let \(k \in \mathbb{N}\) be the number of different species and let \(u_i, (1 \leq i \leq k)\) represent the population density of \(i\)th species in that system. Since the system is closed, it can be expected that the diffusion of population of each species will be governed by some quantity depending on the total population of the system.

Next two parabolic systems would be the considerable mathematical modelings which describe the evolution of population density of each species in a closed system:

\[(u_i)_t = \nabla \cdot (m U^{m-1} A(u^i, x, t) + B(u^i, x, t)) = \nabla \cdot \left( \left( \sum_{i=1}^{k} u_i \right)^{m-1} \nabla u^i \right), \quad i = 1, \ldots, k, \tag{1.1}\]

where the diffusion coefficient \(U\) depends on the total population \(\sum_{i=1}^{k} u_i\), and

\[(u_i)_t = \nabla \cdot (m U^{m-1} \nabla u^i) = \nabla \cdot \left( \left( \sum_{i=1}^{k} |u^i|^2 \right)^{m-1} \nabla u^i \right) = \nabla \cdot (|u|^m \nabla u^i), \quad i = 1, \ldots, k, \tag{1.2}\]

where \(U\) depends on the size of the vector \(u = (u^1, \ldots, u^k)\).

Many studies on the parabolic systems (1.1) and (1.2) can be found. We refer the reader to the papers [1,2] for the local Hölder continuity and asymptotic large time behaviour in the range of exponents \(m > 1\).

* Corresponding author: Sunghoon Kim, Department of Mathematics, The Catholic University of Korea, 43 Jibong-ro, Bucheon-si, Gyeonggi-do, 14662, Republic of Korea, e-mail: math.s.kim@catholic.ac.kr

Ki-Ahm Lee: Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Gwanak-ro 1, Gwanak-Gu, Seoul, 08826, Republic of Korea, e-mail: kiahm@snu.ac.kr

Open Access. © 2022 Sunghoon Kim and Ki-Ahm Lee, published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.
Compared to the divergence-type parabolic system, there are also some studies on the non-divergence-type parabolic system. We refer the reader to the papers [3–5] for the mathematical theories on the degenerate parabolic system
\[
\mathbf{u}_i = \Delta |\mathbf{u}|^{m-1} \mathbf{u}, \quad m > 1, \quad \mathbf{u} = (u^1, \ldots, u^k).
\]

In this article, we are going to consider the parabolic system which is a generalized version of parabolic systems (1.1) and (1.2). More precisely, let \( \mathbf{u} = (u^1, \ldots, u^k) \) be a solution of the parabolic system
\[
(u^i)_t = \nabla \cdot (mU^{m-1} \mathcal{A}(\nabla u^i, u^i, x, t) + \mathcal{B}(u^i, x, t)) \quad (1 \leq i \leq k),
\] (GPS)
with the conditions
\[
u^i \geq 0 \quad \text{and} \quad 0 \leq \lambda_i(u^i)^\beta \leq U = U(u^1, \ldots, u^k) \quad \forall 1 \leq i \leq k
\]
in \( \mathbb{R}^n \times (0, \infty) \) where the constants \( m, \lambda_i, \beta_i \) are such that
\[
m > \frac{n-2}{n} \quad \text{and} \quad \lambda_i > 0, \quad \beta_i \geq 0.
\]
In (1.3), the first condition comes from the non-negativity of population of each species and the second one represents the relations between diffusion coefficient and the population of each species.

In recent years, there has been a lot of studies on non-Newtonian fluids because of their wide applications. It appears in a variety of situations such as geometric one [6], quasiregular mapping [7], and fluid dynamics [8]. The mathematical modelling of non-Newtonian fluids was obtained by equations having a stress tensor determined by the symmetric part of the gradient of the velocity. One of the well-known examples of such models is the modified Navier-Stokes equations, which is considered by Ladyzhenskaya in the mid-1960s [9,10]:
\[
\nu_i = \nabla \cdot \mathcal{A}(\nabla \nu) + \nabla p = \nabla (\nu \otimes \nu) - f, \quad \nabla \nu = 0,
\]
where \( \nu = (\nu^1, \nu^2, \nu^3) \) denotes the velocity, \( D(\nu) \) is the symmetric part of \( \nabla \nu \), \( p \) is the pressure, and \( \mathcal{A} \) is a monotone vector field.

Since the non-Newtonian fluids appear in many physical and mechanical situations, it is very important to provide mathematical theories of such fluids. Due to the similarity between equations in system (GPS) and in the mathematical modelling of non-Newtonian fluids, the study of system (GPS) is expected to play a key role in the research of non-Newtonian fluids.

The aim of this article is to provide the regularity theory of the diffusion coefficient \( U^{m-1} \) and the solution \( \mathbf{u} = (u^1, \ldots, u^k) \) of the system (GPS) when the vectors \( \mathcal{A} \) and \( \mathcal{B} \) are assumed to be measurable in \( (x, t) \in \mathbb{R}^n \times [0, \infty) \) and continuous with respect to \( u \) and \( \nabla u \) for almost all \( (x, t) \).

To control effects from diffusion in the energy-type inequality for (GPS), suitable structure assumptions are needed to be imposed on the function \( U \) and vector fields \( \mathcal{A} \) and \( \mathcal{B} \): The function \( U = U(u^1, \ldots, u^k) \) satisfies
\[
U^m U_i(t) \in H^1_0(\mathbb{R}^n) \quad \forall 1 \leq i \leq k, \quad t > 0,
\]
where \( U_i \) is the derivative of \( U \) with respect to \( u^i \), and
\[
U(0, \ldots, 0) = 0, \quad \sum_{i=1}^k U_i'(\xi^i, \ldots, \xi^k) \xi^i \leq C_i U_i, \quad U_i'(\xi^i, \ldots, \xi^k) \geq 0 \quad \forall \xi = (\xi^1, \ldots, \xi^k) \in \mathbb{R}^k.
\]
\[
(1.2)
\]

The vectors \( \mathcal{A}(p, z, x, t) \) and \( \mathcal{B}(z, x, t) \), \( (z, p) \in \mathbb{R}^+ \times \mathbb{R}^n \), are assumed to have the following structures:
\[
\sum_{i=1}^k (mU^{m-1} \mathcal{A}(\nabla u^i, u^i, x, t) + \mathcal{B}(u^i, x, t)) \nabla U_i \geq 0,
\]
\[
(1.3)
\]
\[
\nabla U \cdot \left[ \sum_{i=1}^k U_i' \mathcal{A}(\nabla u^i, u^i, x, t) \right] \geq c \|\nabla U\|^2 - C_2 U^2,
\]
\[
(1.4)
\]
and, for any positive function \( u \geq 0 \)
\[
\mathcal{A}(\nabla u, u, x, t) \nabla u \geq c |\nabla u|^2 - C_2 u^2, \\
|\mathcal{A}(\nabla u, u, x, t)| \leq C_3 |\nabla u| + C_4 u, \\
|\mathcal{B}(u, x, t)| \leq C_5 u^q,
\]
where
\[
0 < c \leq 1 \leq C_1, C_2, C_3, C_4, C_5, < \infty
\]
and
\[
1 < q < \left( m \left( 1 + \frac{1 + m}{mn} \right) - 1 \right) \min_{1 \leq i \leq k} \beta_i + 1 = \left( m \left( 1 + \frac{1 + m}{mn} \right) - 1 \right) \beta_1 + 1 = m(\beta_1). \tag{1.4}
\]

**Remark 1.1.**
- Let \( k = 1, \mathcal{A} = \nabla u' \), and \( \mathcal{B} = 0 \). Then the system (GPS) is called the porous medium equation (or slow diffusion equation) for \( m > 1 \), heat equation for \( m = 1 \), and fast diffusion equation for \( m < 1 \).
- \( \frac{n-2}{n} \) is the critical number of porous medium equation and \( m > \frac{n-2}{n} \) gives the conservation law of \( L^1 \) mass in the standard porous medium equation.
- Condition (A2) is a natural growth condition and the monotonicity of \( U \) with respect to \( u' \), \( 1 \leq i \leq k \).
- Condition (A5) is coercivity condition to get parabolicity. Conditions (A6) and (A7) are growth conditions.
- We refer the reader to the papers [11–13] for the structure conditions of parabolic partial differential equations (PDEs).
- If \( m = 1 \) and \( \beta_1 = 1 \), then the constant \( m(\beta_1) \) in the condition (1.4) will be \( \frac{n+2}{n} \), which becomes critical number in the energy estimates of the standard heat equation.

Let \( \Omega \) be an open set in \( \mathbb{R}^n \), and for \( T > 0 \) let \( \Omega_T \) denote the parabolic domain \( \Omega \times (0, T] \). We say that \( u = (u', ..., u^k) \) is a weak (energy) solution of (GPS) in \( \Omega_T \) if the component \( u' \), \( 1 \leq i \leq k \), is a locally integrable function satisfying

1. \( u' \) belongs to function space:
\[
U^{m-1} |\mathcal{A}(\nabla u', u', x, t)| \in L^2(0, T : L^2(\Omega)). \tag{1.5}
\]
2. \( u' \) satisfies the identity:
\[
\int_0^T \int_{\Omega} \left( mU^{m-1} \mathcal{A}(\nabla u', u', x, t) \nabla \varphi + \mathcal{B}(u', x, t) \nabla \varphi + u' \varphi \right) dxdt = 0, \tag{1.6}
\]
which holds for any test function \( \varphi \in H^1(0, T : L^2(\Omega)) \cap L^2(0, T : H^1_0(\Omega)) \).

To take care of the difficulties from \( (u')_h \), we introduce the Lebesgue-Steklov average \( (u')_h \) of the function \( u' \), for \( h > 0 \):
\[
(u')_h(\cdot, t) = \frac{1}{h} \int_t^{t+h} u'(\cdot, \tau) d\tau.
\]

\( (u')_h \) is well-defined and it converges to \( u' \) as \( h \to 0 \) in \( L^p \) for all \( p \geq 1 \). In addition, it is differentiable in time for all \( h > 0 \) and its derivative is
\[
\frac{u'(t + h) - u'(t)}{h}.
\]
Fix $t \in (0, T)$ and let $h$ be a small positive number such that $0 < t < t + h < T$. Then for every compact subset $\mathcal{K} \subset \mathbb{R}^n$ the following formulation is equivalent to (1.6):

$$
\int_{\mathcal{K} \times [t]} \left[ ((u^i)_h) \phi + m(U^{m-1})_h(\nabla u^i, u^i, x, t)_h \phi + (B(u^i, x, t))_h \phi \right] \, dx = 0, \quad \forall 0 < t < T - h
$$

(1.7)

for any $\phi \in H^1_0(\mathcal{K})$. Thus, from now on, we consider the weak formulation (1.6) of (GPS) as the limit of formulation (1.7) with respect to $h$.

Generally in Cauchy problem, the solutions lose their initial information in a large time, and diffuse only under the laws governed by the systems. Hence, the evolution of the solutions is determined by the diffusion coefficients and external forces after the large time. By (1.3), the diffusion coefficient $U^{m-1}$ is bounded from below by $A^{m-1}(u^i)^{\beta(m-1)}$. Thus, we need to control the function $U^{m-1}$ from above to improve the regularity theories of (GPS). The first part of the article is about a priori estimates of the function $U$. The statement is as follows.

**Theorem 1.2.** (Uniform $L^{\infty}$ boundedness of $U$) Let $n \geq 2$, $m > \frac{n-2}{n+2}$ and let $q$ be given by (1.4). Let $u = (u^1, \ldots, u^k)$ be a solution of (GPS) satisfying conditions (A3)–(A7). Suppose that $U = U(u^1, \ldots, u^k)$ satisfies (A1). Then, for a small $t_0 > 0$ there exists a constant $K(t_0) > 0$ such that

$$
\sup_{x \in \mathbb{R}^n, t \in [t_0]} |U(x, t)| \leq K(t_0).
$$

(1.8)

Moreover, if the constant $C_q = 0$ in the structure (A7), then the range of exponents $m$ is independent of $q$, i.e., (1.8) holds for all $m > \frac{n-2}{n+2}$.

**Remark 1.3.**
- The bound $K(t_0)$ blows up as $t_0 \to 0$.
- When $\mathcal{A} = \nabla u^i, B = 0$, and $k = 1$, it is well known that $K(t_0) = \frac{c(m,n)\|u_0\|_{L^2}}{t_0^{m-1}}$, which is optimal for the Barenblatt solution of the porous medium equations.

As the second result of this article, we will deal with the mass conservation of the parabolic system (GPS) under the assumption that

$$
B(u^i, x, t) = 0.
$$

The concept of $L^1$ mass conservation appears widely in many fields, such as mechanics and fluid dynamics. In the theory of PDEs, it also plays an important role in the study of the asymptotic behaviour of solutions. Although the law of mass conservation is considered as a part of assumptions in classical mechanics, but in general, mass is not always preserved in systems. Thus, it is very important to check the initial mass is maintained for all time in the parabolic system. In the second part of this article, we are going to show that the $L^1$ mass of each component $u^i$, $(i = 1, \ldots, k)$, is preserved under structure assumptions (A1)–(A6).

In the study of parabolic equations, the law of mass conservation is well known in the case where the constants $C_2, C_3, C_5$ in the structures (A4)–(A7) are all zeros or $m = 1$, but it is non-trivial for general constants $C_2, C_3, C_5$ and $m$. We refer the reader to the paper [13] for the mass conservation of singular parabolic equations.

By (1.3),

$$
0 \leq \lambda_i^{m-1}(u^i)^{\beta(m-1)} \leq U^{m-1} \quad (m > 1).
$$

Thus, the degeneracy of the parabolic system (GPS) is determined by $U$, not the solution $u^i$. If we take

$$
\mathcal{A}(\nabla u^i, u^i, x, t) = \nabla u^i \quad \text{and} \quad u^i = \left( \frac{U}{\lambda_i} \right)^{\frac{1}{\beta}},
$$

...
then the parabolic system (GPS) can be expressed in the form of

$$\left( u^i \right)_t = \nabla \cdot \left( U^{m-1} \mathcal{A} (\nabla u^i, u^i, x, t) \right) = \nabla \cdot \left( U^{m-1} \nabla u^i \right) = \frac{1}{\beta_i (m - 1 + \frac{1}{\beta_i})} \Delta U^{m-1+\frac{1}{\beta_i}}.$$

Thus, the system becomes degenerate when \( U = 0 \) and \( m - 1 + \frac{1}{\beta_i} > 1 \). Therefore, we need to consider both the ranges

$$m - 1 + \frac{1}{\beta_i} > 1 \quad \text{and} \quad m > 1$$

for the study of degenerate parabolic system (GPS). Under this observation, we are going to state the first result of the second part: \( L^1 \) mass conservation of components to the degenerate parabolic system.

**Theorem 1.4.** (Mass conservation in degenerate range) Let \( n \geq 2, \ 1 \leq i \leq k, \ m > \max \left( 1, 2 - \frac{1}{\beta_i} \right) \) and

$$U_0 \in L^1_t(\mathbb{R}^n) \cap L^{m-1+\frac{1}{\beta_i}}(\mathbb{R}^n). \quad (1.9)$$

Let \( u^0_i \in L^1(\mathbb{R}^n) \) be a positive function and let \( u = (u^1, \ldots, u^k) \) be a weak solution to the degenerate parabolic system (GPS) with initial data \( u_0 = (u^0_1, \ldots, u^0_k) \in \mathbb{R}^n \times [0, \infty) \). Suppose that conditions (1.3), (A1)–(A6) hold, \( \mathcal{B} \equiv 0 \) and

$$\int_0^t \mathcal{K}^{-1}(r) dr \to 0 \quad \text{as} \quad t \to 0, \quad (1.10)$$

where \( \mathcal{K}(t) \) is given by Theorem 1.2. Then for any \( t > 0 \)

$$\int_{\mathbb{R}^n} u^i(x, t) dx = \int_{\mathbb{R}^n} u^i_0(x) dx.$$

**Remark 1.5.** As mentioned in Remark 1.3, \( \mathcal{K}(t_0) = \frac{c(m, n)|u^i_0|^\frac{1}{\beta_i}}{t^{\frac{m-1}{\beta_i}}} \) when \( \mathcal{A} = \nabla u^i, \ \mathcal{B} = 0 \) and \( k = 1 \). Thus, the assumption (1.10) is easily obtained for the standard porous medium equation.

In the singular case \( (m < 1) \), the diffusion coefficient of the parabolic system (GPS) is bounded from above by \( \lambda_i^{m-1}(u^i)^{\beta_i(m-1)} \), i.e.

$$U^{m-1} \leq \lambda_i^{m-1}(u^i)^{\beta_i(m-1)}.$$

Hence, the main controller of the parabolic system (GPS) will be the component \( u^i \) itself. If

$$\mathcal{A} (\nabla u^i, u^i, x, t) = \nabla u^i \quad \text{and} \quad U = \lambda_i(u^i)^{\beta_i},$$

then we can have

$$(u^i)_t = \nabla \cdot \left( U^{m-1} \mathcal{A} (\nabla u^i, u^i, x, t) \right) = \nabla \cdot \left( U^{m-1} \nabla u^i \right) = \frac{\lambda_i^{m-1}}{\beta_i (m - 1 + 1)} \Delta (u^i)^{\beta_i}$$

where \( m_i = \beta_i(m - 1) + 1 \). This becomes singular when \( u^i = 0 \) in the range of exponents \( \beta_i(m - 1) + 1 < m_i \). Hence, we have to consider the range \( m_i < 1 \) as well as the range \( m < 1 \) for the study of singular parabolic system (GPS). The last result of the second part is as follows.

**Theorem 1.6.** (\( L^1 \) mass conservation in supercritical range) Let \( n \geq 2, \ 1 \leq i \leq k \). Let \( \beta_i \) be given by (1.3) and let \( 0 < m < 1 \) satisfy

$$\frac{n^2 + n + 4 + \sqrt{2n(7n + 11)}}{n^2 + 5n + 8} < m_i = \beta_i(m - 1) + 1 < 1, \quad (1.11)$$
Let $u_0^i$ be a positive, integrable function with compact support in $\mathbb{R}^n$ and let $u = (u^1, \ldots, u^k)$ be a weak solution to the singular parabolic system (GPS) with initial data $u_0 = (u_0^1, \ldots, u_0^k)$ in $\mathbb{R}^n \times [0, \infty)$. Suppose that conditions (1.3), (A1)–(A6) hold, $\mathcal{B} \equiv 0$ and there exist constants $R^*$ and $\Lambda$ such that

$$U(x, t) \leq \Lambda \quad \text{a.e. on } \{|x| \geq R^*, \ t \geq 0\}.$$  

Then for any $t > 0$

$$\int_{\mathbb{R}^n} u^i(x, t) dx = \int_{\mathbb{R}^n} u_0^i(x) dx. \quad (1.12)$$

Moreover, if the constants $C_2$ and $C_4$ in the structures (A4)–(A6) are all zeros, then $L^1$-mass conservation (1.12) holds for

$$\frac{n^2 - n + 3 + \sqrt{n^2 + 2n - 7}}{n^2 + 2n + 4} < m_i = \beta_i(m - 1) + 1 < 1.$$  

Remark 1.7.

- By the diffusion coefficient $U(u^1, \ldots, u^k)$, each component of the system (GPS) could not evolve in a way determined by the component itself, i.e., they interact each other. Thus, in order for a component of the system to have the law of $L^1$-mass conservation, the influence of other components must be governed, i.e., to control the effects from constants $c, C_1, \ldots, C_5$, and the ratio between $U$ and $u_i (1 \leq i \leq k)$, the lower bound of the exponents $m$ in the diffusion coefficients of the parabolic system must be more restrictive. The interval (1.11) gives the minimal range that the exponents $m$ must have for the existence of $L^1$-mass conservation of each component in the parabolic system (GPS).

- If there exist some constants $0 < c \leq C < \infty$ such that

$$c\lambda_i(u^i)^b \leq U_i \leq C\lambda_i(u^i)^b \quad \text{in } Q(\mathbb{R}, \theta = R^2),$$

and constants $C_2$ and $C_4$ are all zeros, then $L^1$-mass conservation (1.12) holds for

$$\frac{n-2}{n} < m_i = \beta_i(m - 1) + 1 < 1.$$  

In this case, the lower bound of $m_i$ is the critical number of standard porous medium equation.

By structure condition (A2) and the assumption of $u_0^i, (1 \leq i \leq k)$, in Theorem 1.6,

$$U(x, 0) = 0, \quad \forall |x| > 1,$$

e.g., $U$ is bounded above at $t = 0$ on the region far away from $0$, and this condition, “boundedness of $U$ far away from $0$,” persists for a short time $t_0 > 0$. Moreover, by Theorem 1.2, it is also obtained that $U(t)$ is bounded for $t \geq t_0$ if the exponents $m$ satisfy

$$m > \frac{n - 2}{n + 2}.$$  

From these two observations, we can get rid of the condition “Boundedness of $U$” from Theorem 1.6 if the constant $m$ is sufficiently close to $1$. Therefore, we can guarantee the law of $L^1$ mass conservation without the condition “Boundedness” for some special $m$ in the super critical range.

Corollary 1.8. Let $n \geq 2, 1 \leq i \leq k$. Let $\beta_i$ be given by (1.3) and let

$$m \in \left(\frac{n - 2}{n + 2}, 1\right) \cap \left(1 - \frac{4(n + 1)}{\beta_i(n^2 + 5n + 8)} + \frac{2n(7n + 11)}{\beta_i(n^2 + 5n + 8)}, 1\right).$$

Let $u_0^i$ be a positive, integrable function with compact support in $\mathbb{R}^n$ and let $u = (u^1, \ldots, u^k)$ be a weak solution to the singular parabolic system (GPS) with initial data $u_0 = (u_0^1, \ldots, u_0^k)$ in $\mathbb{R}^n \times [0, \infty)$. Suppose that conditions (1.3), (A1)–(A6) hold and $\mathcal{B} \equiv 0$. Then for any $t > 0$

$$\int_{\mathbb{R}^n} u^i(x, t) dx = \int_{\mathbb{R}^n} u_0^i(x) dx. \quad (1.13)$$
If the constants $C_2$ and $C_4$ in the structures (A4)–(A6) are all zeros, then $L^1$-mass conservation (1.13) holds for

$$m \in \left( \frac{n-2}{n+2}, 1 \right) \cap \left( 1 - \frac{3n+1}{\beta(n^2+2n+4)} + \frac{\sqrt{7n^2+2n-7}}{\beta(n^2+2n+4)}, 1 \right).$$

The proofs for mass conservation in the singular case are based on a system version of integral Harnack estimate which controls the speed of the propagation of the solution $u = (u^1, \ldots, u^k)$. We remark that Harnack-type estimates are known for the general parabolic equations but, as far as we know, not for general parabolic systems.

At the end of this article, we are going to give an explanation about the local continuity of the weak solution $u = (u^1, \ldots, u^k)$ to the parabolic system (GPS) by showing that the differences between supremum and infimum of components $u^i, (1 \leq i \leq k)$, on a chosen set decrease as the radius of the set shrinks. If the diffusion coefficient $U^{m-1}$ is uniformly parabolic, then De Giorgi and Moser’s technique [14,15] on regularity theory for uniformly elliptic and parabolic PDE’s are enough to show the local continuity of the solution $u$. Otherwise, we need to take care of the difficulties coming from the diffusion coefficient. To overcome it, we use the well-known technique called intrinsic scaling whose parameters are determined by the size of oscillation of $u^i, (1 \leq i \leq k)$. The statement of the last result is as follows.

**Theorem 1.9.** Let $n \geq 2$, $m > 1$ and let $\beta_i$ be given by (1.3). Suppose that

$$B = 0 \quad \text{or} \quad \frac{1}{2}(m-1) \min_{1 \leq i \leq k} \beta_i + 1 < q < \left( m \left( 1 + \frac{1 + m}{mn} \right) - 1 \right) \min_{1 \leq i \leq k} \beta_i + 1 \quad (1.14)$$

in the structure condition (1.4). Then, any weak solution of degenerate parabolic system (GPS) is locally continuous in $\mathbb{R}^n \times (0, \infty)$.

A brief outline of the article is as follows. In Section 2, we will prove uniform $L^\infty$ boundedness of the function $U$. Section 3 is devoted to the proofs of $L^1$ mass conservation in degenerate case (Theorem 1.4) and in singular case (Theorem 1.6). Finally in Section 4, we investigate the local continuity of components $u^i, (1 \leq i \leq k)$ of weak solution of (GPS) with the intrinsic scaling technique developed by [16] and [17].

**Notations.** Before we deal with the main idea of the article, let us summarize the notations and definitions that will be used.

- We denote by $B_R(x)$ the ball centred at $x \in \mathbb{R}^n$ of radius $R > 0$. Let $B_R = B_R(0)$.
- $Q(R, r) = B_R \times (-r, 0)$.
- Let $E$ be an open set in $\mathbb{R}^n$. We denote by $E_T$ the parabolic domain $E \times (0, T)$ for $T > 0$.
- Numbers: $\lambda_i, \beta_i$ are given by (1.3), $m_i = \beta_i(m-1) + 1$, and $C_1, \ldots, C_4$ are given by structure conditions (A1)–(A7).

## 2 Uniform boundedness of the function $U$

This section is aimed at providing the proof of Theorem 1.2. The proof is based on a recurrence relation between a series of truncations of $U$.

**Proof of Theorem 1.2.** We will use a modification of the proof of Theorem 1 of [18]. Let

$$L_j = K \left( 1 - \frac{1}{2^j} \right) \quad \text{and} \quad U_j = (U - L_j),$$

for a constant $K > 2$, which will be determined later. Then

$$U \geq \frac{K}{2} > 1 \quad \text{on} \quad \{ U_j > 0 \} \quad \forall j \in \mathbb{N}.$$
By structures (A1) and (A3), we have the following energy-type inequality for the truncation $U_j$:

\[ \frac{1}{1 + m} \frac{\partial}{\partial t} \int_{\mathbb{R}^n} U_j^{t+m} dx + m \int_{\mathbb{R}^n} \left( \sum_{i=1}^{k} (mU_{m}^i U_j A(\nabla u^i, u^i, x, t) + U_j B(u^i, x, t) \cdot \nabla U) \right) dx \]

\[ = \frac{1}{1 + m} \frac{\partial}{\partial t} \int_{\mathbb{R}^n} U_j^{t+m} dx + m^2 \int_{\mathbb{R}^n} \left( \sum_{i=1}^{k} U_j A(\nabla u^i, u^i, x, t) \right) \cdot \nabla U dx \]

\[ + m \int_{\mathbb{R}^n} \left( \sum_{i=1}^{k} U_j B(u^i, x, t) \right) \cdot \nabla U dx \leq 0. \]  

(2.2)

Let $\beta = \min_{1 \leq i \leq k} \beta_i$ and $A^2 = \min_{1 \leq i \leq k} A_i$. By (1.3), (A2), (A4), (A7), (2.2), and Hölder inequality, we can obtain

\[ \frac{1}{1 + m} \frac{\partial}{\partial t} \int_{\mathbb{R}^n} U_j^{t+m} dx + c \int_{\mathbb{R}^n} |\nabla U_j|^2 dx \leq \int_{\mathbb{R}^n} \left( m^2 C_2 U_j^{2m} + C_5 \left( \sum_{i=1}^{k} (u^i)^{q-1} \right) |\nabla U_j|^2 \right) \chi_{(U_j > 0)} dx \]

\[ \leq \int_{\mathbb{R}^n} \left( m^2 C_2 U_j^{2m} + \frac{C_5}{A^2} U_j^{2m-1} |\nabla U_j|^2 \right) \chi_{(U_j > 0)} dx \]

\[ \leq \int_{\mathbb{R}^n} \left( m^2 C_2 U_j^{2m} + \frac{C_5}{4cA^2} U_j^{2m-1} + \frac{c}{2} |\nabla U_j|^2 \right) \chi_{(U_j > 0)} dx. \]  

(2.3)

By (2.1) and (2.3), there exists constants $C_1, C_2$ depending on $C_1, C_2, C_5, c, m, q, \text{ and } A^2$ such that

\[ \Rightarrow \frac{\partial}{\partial t} \left( \int_{\mathbb{R}^n} U_j^{t+m} dx \right) + \int_{\mathbb{R}^n} |\nabla U_j|^2 dx \]

\[ \leq C_2 \left( \int_{\mathbb{R}^n} U_j^{2m} + L_j^{2m} \chi_{(U_j > 0)} dx \right) + \int_{\mathbb{R}^n} \left( \frac{2m-1}{2} U_j^{\frac{2m-1}{2}} \chi_{(U_j > 0)} dx \right) \]

\[ \leq C_2 \left( 2^{2m} J_{m-1} dx + \int_{\mathbb{R}^n} \left( \frac{2m-1}{2} U_j^{\frac{2m-1}{2}} \chi_{(U_j > 0)} dx \right) \right) \]

\[ \text{since} \]

\[ U = U_j + L_j \quad \text{and} \quad U_{j+1} \geq \frac{L_j}{2} \quad \text{on} \quad \{U_j > 0\}. \]

For fixed $t_0 > 0$, let $T_j = t_0 \left( 1 - \frac{1}{2^{j+1} \lambda_j \gamma} \right)$ and

\[ A_j = \sup_{t \leq T_j} \left( \int_{\mathbb{R}^n} U_j^{t+m} dx \right) + \int_{T_j}^{t} \int_{\mathbb{R}^n} |\nabla U_j|^2 dx dt. \]

We first choose the constant $K$ so large that

\[ A_j \leq 1. \]
Then

\[ A_j \leq 1 \quad \forall j \in \mathbb{N}. \]  

(2.5)

Integrating (2.4) over \((s, t)\) and \((s, \infty), (T_{j-1} < s < T_j, \ t > T_j)\), we have

\[ A_j \leq \int_{\mathbb{R}^n} U_j^{j+m}(x, s)dx + C_2 \left( \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} 2^{2m}U_j^{j+m}dxdt + \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} 2^{2m}U_j^{2m}dxdt \right). \]

Taking the mean value in \(s \) on \([T_{j-1}, T_j]\), we have

\[ A_j \leq \left[ \frac{2^{1+m}j}{t_0} \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} U_j^{j+m}dxdt + C_2 \left( \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} 2^{2m}U_j^{j+m}dxdt + \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} 2^{2m}U_j^{2m}dxdt \right) \right]. \]

(2.6)

By Proposition 3.1 of Chap. I of [16], there exists constant \(C_3 > 0\) such that

\[ \left( \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} (U_j^{j+m})^{\frac{1}{j}} \frac{1}{\mathbb{R}^n} \right) \leq \frac{1}{C_3} \left( \sup_{T_{j-1}} \left( \int_{\mathbb{R}^n} (U_j^{m})^{\frac{1}{m}} \frac{1}{\mathbb{R}^n} \right) \right)^{\frac{m}{j}} \left( \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} |\nabla U_j|^2 \frac{1}{\mathbb{R}^n} \right)^{\frac{1}{2}}. \]

Combining this with (2.5), we can obtain

\[ \left( \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} 2^{2m}U_j^{j+m}dxdt \right)^{\frac{1}{j+m}} \leq C_4 A_{j-1}^{K}, \quad \mathcal{K} = \min \left\{ 1 + \frac{m+1}{nm}, \frac{2m}{n} + \frac{2m+1}{1+m} \right\} > 1 \]  

(2.7)

for some constant \(C_4 > 0\). By (2.6) and (2.7), there exists a constant \(C_5(t_0, K) > 0\) such that

\[ A_j \leq 2^{2m(1+ \frac{m+1}{nm})} \left( \frac{1}{t_0 K^{n-1}} + \frac{2}{K^{2m+1}} + \frac{C_2}{K^{2m+1}} \right)^{\frac{1}{j}} \int_{T_{j-1}}^{\infty} \int_{\mathbb{R}^n} U_j^{j+m}dxdt \leq \frac{2^{2m(1+ \frac{m+1}{nm})}}{C_5(t_0, K)} A_{j-1}^{K}. \]

(2.8)

Since

\[ U_{j-1} \geq \frac{K}{2} \quad \text{on } \{U_j > 0\}. \]

By (1.4), the constant \(C_6(t_0, K)\) satisfies

\[ C_6(t_0, K) \rightarrow \infty \quad \text{as } K \rightarrow \infty. \]

Choose the constant \(K > 0\) so large that

\[ A_1 \leq (C_6(t_0, K))^{\frac{1}{m} - \frac{1}{m+1}(\frac{1}{m} - \frac{1}{m+1})^2}. \]

Then, by Lemma 4.1 of Chap. I of [16] we have

\[ A_j \rightarrow 0 \quad \text{as } j \rightarrow \infty. \]

Therefore,

\[ \sup_{x \in \mathbb{R}^n, t \geq t_0} |U(x, t)| \leq K = K(t_0) \]

and the theorem follows. \qed
3 Law of mass conservation in $L^1$

In this section, we will deal with the $L^1$ mass conservation of the parabolic system (GPS) under the assumption that

$$\mathcal{B}(u', x, t) = 0 \quad \forall (x, t) \in \mathbb{R}^n \times [0, \infty),$$

i.e., we consider the law of mass conservation for the parabolic system

$$(u^i)_t = \nabla \cdot (mU^{m-1}A(\nabla u^i, u^i, x, t)) \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \quad (1 \leq i \leq k)$$

(3.1)

with the structural conditions (1.3) and (A1)–(A6). We divide this section into two subsections with respect to the range of the exponents $m$. The first one is about the $L^1$ mass conservation on the degenerate parabolic systems.

3.1 Degenerate case: $m > \max\left(1, 2 - \frac{1}{\beta_i}\right)$

The proof of Theorem 1.4 is based on suitable energy-type estimates. Global boundedness estimates (Theorem 1.2) and condition (1.10) play important roles in controlling the remainder term of the weak energy inequality, such as $\|U\|_{L^p}$ for some $p > 1$.

Let $0 < \theta < 1$ and $\eta_n \in C^\infty(\mathbb{R}^n)$ be a cutoff function such that

$$\eta_n(x) = 1 \quad \text{for} \quad |x| \leq n, \quad \eta_n(x) = 0 \quad \text{for} \quad |x| > n + 1, \quad 0 < \eta_n(x) < 1 \quad \text{for} \quad n < |x| < n + 1.$$

and

$$\|\nabla \eta_n\|_{L^\infty} \leq 2 \quad \forall n \in \mathbb{N}.$$  

Take $\varphi = (u^i)^\theta \eta_n^2$ as a test function in the weak formulation (1.6) for (3.1) and integrate it over $\mathbb{R}^n \times (0, t]$. Then, by structure assumptions (A5) and (A6), we have

$$\sup_{0 < t \leq T} \int_{\mathbb{R}^n} (u^i)^{1+\theta}(x, \tau)dx + \int_0^t \int_{\mathbb{R}^n} U^{m-1}(u^i)^{1+\theta} \nabla u^2 dx \, dt \leq C_1 \left( \int_{\mathbb{R}^n} (u^i)^{1+\theta}(x, 0)dx + \int_0^t \int_{\mathbb{R}^n} U^{m-1}(u^i)^{1+\theta}dx \, dt \right)$$  

(3.2)

for some constant $C_1 > 0$ depending on $m$, $\beta_i$, $c$, and $C_2$. We also take $\varphi = (U)^{m-2-\frac{1}{\beta_i}} \eta_n^2$ as a test function in the weak formulation (1.6) for (3.1), and sum it over $1 \leq i \leq k$, and integrate it over $\mathbb{R}^n \times (0, t]$. Then, by structure assumptions (A3) and (A4), we obtain

$$\sup_{0 < t \leq T} \int_{\mathbb{R}^n} U^{m-1-\frac{1}{\beta_i}}(x, \tau)dx + \int_0^t \int_{\mathbb{R}^n} \left| \nabla U^{m-1-\frac{1}{\beta_i}} \right|^2 dx \, dt \leq C_2 \left( \int_{\mathbb{R}^n} U^{m-1-\frac{1}{\beta_i}}(x, 0)dx + \int_0^t \int_{\mathbb{R}^n} U^{2m-2-\frac{1}{\beta_i}}dx \, dt \right)$$

(3.3)

for some constant $C_2 > 0$ depending on $m$, $\beta_i$, $c$, and $C_2$.

To control last terms in (3.2) and (3.3), we suppose that the bound $K(t)$ in the Theorem 1.2 satisfies the following condition:

$$\int_0^t K^{m-1}(\tau) \, d\tau \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.$$

Then there exists a constant $t_0 > 0$ such that

$$\int_0^{t_0} K^{m-1}(\tau) \, d\tau < \frac{1}{2(C_1 + C_2)}.$$
Then,
\[ \int \int_{0}^{t_0} U^{m-\frac{1}{n}}(x, r)dxdr \leq \left( \int_{0}^{t_0} K^{m-\frac{1}{n}}(\tau)d\tau \right) \left( \sup_{0 < \tau < t_0} \int_{\mathbb{R}^n} U^{m-1}(x, r)dx \right) \leq \frac{1}{2(C_1 + C_2)} \left( \sup_{0 < \tau < t_0} \int_{\mathbb{R}^n} U^{m-1}(x, r)dx \right) \] (3.4)

and
\[ \int \int_{0}^{t_0} U^{m-1}r(x, r)dxdr \leq \left( \int_{0}^{t_0} K^{m-1}(\tau)d\tau \right) \left( \sup_{0 < \tau < t_0} \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx \right) \leq \frac{1}{2(C_1 + C_2)} \left( \sup_{0 < \tau < t_0} \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx \right). \] (3.5)

By (3.2), (3.3), (3.4), and (3.5), we have
\[ \sup_{0 < \tau < t_0} \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx + \int_{0}^{t_0} \int_{\mathbb{R}^n} U^{m-1}(u^{\theta-1}d\theta dxdt \leq 2C_1 \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx(0) \] and
\[ \sup_{0 < \tau < t_0} \int_{\mathbb{R}^n} U^{m-1}(x, r)dx + \int_{0}^{t_0} \int_{\mathbb{R}^n} |\nabla U|^{m-1} \frac{1}{\frac{1}{\theta}} dxdr \leq 2C_2 \int_{\mathbb{R}^n} U^{m-1}(x, r)dx. \]

Since the constants $C_1$ and $C_2$ are all independent of $t$ and $K(\tau_1) \geq K(\tau_2) \ \forall \tau_1 < \tau_2$,
applying above arguments on $(t_0, 2t_0)$, we also have
\[ \sup_{0 < \tau < 2t_0} \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx + \int_{0}^{2t_0} \int_{\mathbb{R}^n} U^{m-1}(u^{\theta-1}d\theta dxdt \leq 2C_1(1 + 2C_1) \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx(0) \] and
\[ \sup_{0 < \tau < 2t_0} \int_{\mathbb{R}^n} U^{m-1}(x, r)dx + \int_{0}^{2t_0} \int_{\mathbb{R}^n} |\nabla U|^{m-1} \frac{1}{\frac{1}{\theta}} dxdr \leq 2C_2(1 + 2C_2) \int_{\mathbb{R}^n} U^{m-1}(x, r)dx. \]

Continuing in this manner, we finally obtain
\[ \sup_{0 < \tau < t} \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx + \int_{0}^{t} \int_{\mathbb{R}^n} U^{m-1}(u^{\theta-1}d\theta dxdt \leq 2C_1(1 + 2C_1)^n \int_{\mathbb{R}^n} (u^{1+\theta})^{\frac{1}{\theta}} dx \] (3.6)

and
\[ \sup_{0 < \tau < t} \int_{\mathbb{R}^n} U^{m-1}(x, r)dx + \int_{0}^{t} \int_{\mathbb{R}^n} |\nabla U|^{m-1} \frac{1}{\frac{1}{\theta}} dxdr \leq 2C_2(1 + 2C_2)^n \int_{\mathbb{R}^n} U^{m-1}(x, r)dx \] (q > 0) (3.7)

for any $t > 0$ where $n'$ is the natural number satisfying
\[ (n' - 1)t_0 < t \leq n't_0. \]

We are now ready for the proof of Theorem 1.4.
Proof of Theorem 1.4. For $R > 1$, let $\zeta_0 \in C^\infty(\mathbb{R}^n)$ be a cut-off function such that
\[ \zeta_0(x) = 1 \quad \text{for} \ |x| \leq 1, \quad \zeta_0(x) = 0 \quad \text{for} \ |x| \geq 2, \quad 0 < \zeta_0(x) < 1 \quad \text{for} \ 1 < |x| < 2 \]
and let $\zeta_R(x) = \zeta_0\left(\frac{x}{R}\right)$ for any $R > 1$. By weak formulation (1.7) for (3.1) we have
\[
\int_{\mathbb{R}^n} u'(x, t)\zeta_R(x) \, dx - \int_{\mathbb{R}^n} u_0'(x, t)\zeta_R(x) \, dx = -m \int_0^t \int_{\mathbb{R}^n} U^{m-1}\mathcal{A}(\nabla u', u', x, t) \nabla \zeta_R \, dx \, dt.
\]
(3.8)

Let $0 < \theta < 1$ be a constant satisfying
\[
m > 1 + \frac{\theta}{\beta_i} \quad \text{and} \quad m - 1 + \frac{1}{\beta_i} > \frac{n\theta}{\beta_i}.
\]
(3.9)

Then, by (A6), (3.8), and Hölder inequality, we have
\[
\left| \int_{\mathbb{R}^n} u'(x, t)\zeta_R(x) \, dx - \int_{\mathbb{R}^n} u_0'(x, t)\zeta_R(x) \, dx \right| \leq \frac{mC_4 \|\nabla \zeta_0\|_{L^\infty(\mathbb{R}^n)}}{(\lambda_i)^{\frac{n}{m}R}} \int_0^t \int_{B_{2R}\setminus B_R} U^{m-1}\frac{1}{\lambda} \, dx \, dt
\]
\[
+ \frac{mC_1 \|\nabla \zeta_0\|_{L^\infty(\mathbb{R}^n)}}{(\lambda_i)^{\frac{n}{m}R}} \left( \int_0^t \int_{B_{2R}\setminus B_R} U^{m-1}(u')^{\theta-1} |\nabla u'|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^t \int_{B_{2R}\setminus B_R} U^{m-1}(u')^{\theta-1} |\nabla u'|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\]
(3.10)

By (3.7) and (3.6), there exists a constant $C_3 > 0$ such that
\[
\left\| U^{m-1}\frac{1}{\lambda}(\cdot, t) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_3 \left\| U_0^{m-1}\frac{1}{\lambda} \right\|_{L^\infty(\mathbb{R}^n)}
\]
and
\[
\left\| U^{m-1}(u')^{\theta-1} |\nabla u'|^2 \right\|_{L^1(\mathbb{R}^n \times (0, t))} \leq C_3 \left\| U_0^{\frac{1+\theta}{\beta_i}} \right\|_{L^1(\mathbb{R}^n)} \quad \forall t > 0.
\]
(3.11)

Thus, by (1.9) and (3.11), the right hand side of (3.10) converges to zero as $R \to \infty$. Therefore,
\[
\left| \int_{\mathbb{R}^n} u'(x, t)\eta_j(x) \, dx - \int_{\mathbb{R}^n} u_0'(x, t)\eta_j(x) \, dx \right| \to 0 \quad \text{as} \ j \to \infty
\]
and the theorem follows.
3.2 Singular case: \( \max \left(0, 1 - \frac{1}{\beta} \right) < m < 1 \)

Integrate (3.1) in a ball of radius \( R \) at time \( t \). Then we have

\[
\left| \frac{d}{dt} \left( \int_{|x| \leq R} u'(x, t) dx \right) \right| = m \left| \int_{|x| = R} U^{m-1}A(x, t) \cdot v dS \right| \leq m \int_{|x| = R} U^{m-1}(C_i v u') d\sigma = \Psi(R, t),
\]

where \( v \) is the unit outward normal vector and \( d\sigma \) is the area element on \( S^{n-1} \). Thus, the claim of mass conservation in the singular case is completed if we can show that

\[
\Psi(R, t) \to 0 \quad \text{as} \quad R \to \infty. \quad (3.12)
\]

Next four lemmas will be used to get an upper bound for the component \( u_i \), which plays a key role in the proof for (3.12). We first are going to get the following energy estimate for the system (3.1).

**Lemma 3.1.** Let \( \alpha \in (-1, 0) \) be a constant such that

\[
0 < \beta_i(m-1) + 1 + \alpha, \quad 1 + \alpha, \quad \beta_i(m-1) + 1 - \alpha < 1. \quad (3.13)
\]

For \( T > 0 \), let \( u = (u^1, \ldots, u^d) \) be a weak solution to the singular parabolic system (GPS) in \( \Omega_T \) with structures (1.3), (A1)–(A7). Suppose that

\[
0 < m < 1 \quad \text{and} \quad 1 - \frac{1}{\beta_i} < m < 1.
\]

Then there exists a positive constant \( \gamma \) depending on the data \( \alpha, m, N, \lambda_i, c, \) and \( C_j \) such that for all cylinder \( B_0(y) \times (s, t) \subset \Omega_T, \) all \( \sigma \in (0, 1) \),

\[
\int_s^t \int_{B_0(y)} U^{m-1}(u^{a-1} v u')^2 dx \, dr \leq \gamma \left( C_i + C_4 + \frac{1 + C_6}{\sigma^2 \rho^2} \left( S_0^i \right)^{\beta_i(m-1)+\alpha+1} (t-s)^{\rho^{m(\alpha-\beta_i(m-1))}} + \frac{\left( S_0^j \right)^{\alpha+1}}{\rho^{m(\alpha+1)}} \right), \quad (3.14)
\]

where

\[
S_0 = \sup_{s < r < t} \int_{B_1(0, \sigma y)} u'(x, r) dx.
\]

**Proof.** We will use a modification of the proof of Lemma B.1.1 of [11] to prove the lemma. Without loss of generality we let \((y, s) = (0, 0)\). Consider a non-negative, piecewise smooth cut off function \( \xi(x) \) such that

\[
\begin{align*}
0 \leq \xi & \leq 1, \quad \text{in} \quad B_{1+\sigma y} \\
\xi & = 1 \quad \text{in} \quad B_{\rho} \\
\xi & = 0 \quad \text{on} \quad \partial B_{1+\sigma y} \\
|\nabla \xi| & \leq \frac{1}{\rho} \quad \text{in} \quad B_{1+\sigma y}.
\end{align*}
\]

Take \( \varphi = (u^i + \varepsilon)^2 \xi^2 \) as a test function in the weak formulation (1.6) and integrate it over \( B_{1+\sigma y} \times (0, t) \). Then, letting \( \varepsilon \to 0 \), we have

\[
\frac{1}{1 + \alpha} \int_{B_{1+\sigma y}} (u^{1+\alpha} \xi^2) dx(t) = \frac{1}{1 + \alpha} \int_{B_{1+\sigma y}} (u^{1+\alpha} \xi^2) dx(0) - \int_0^t \int_{B_{1+\sigma y}} mU^{m-1}A(\nabla v u^i, u^i, x, t) \cdot \nabla (u^i)^{\alpha+1} dV. \quad (3.15)
\]

Sunghoon Kim and Ki-Ahm Lee
By Hölder inequality,
\[
\left| \frac{1}{1 + \alpha} \int_{B_{1+\epsilon \rho}} (u^\alpha)^{1+\alpha} \xi^2 dx(t) - \frac{1}{1 + \alpha} \int_{B_{1+\epsilon \rho}} (u^\alpha)^{1+\alpha} \xi^2 dx(0) \right| \leq \frac{2^{2n+1}}{1 + \alpha} (S_1^{\alpha^n})^{p - m}. \tag{3.16}
\]

By structure conditions (A5), (A6), and Young’s inequality, we have
\[
\int_0^t \int_{B_{1+\epsilon \rho}} m U^{m-1} \mathcal{A}(\nabla u^\alpha, u^\alpha, x, t) \cdot \nabla ((u^\alpha)^{\alpha^n}) dx dr
\]
\[
\leq -cm|a| \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{a-1} |\nabla u^\alpha|^2 \xi^2 dx dr + m|a| C_2 \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{a+1} \xi^2 dx dr
\]
\[
+ 2C_1 \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{\alpha^n} \xi^2 dx dr + 2mC_4 \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{a+1} \xi^2 dx dr
\]
\[
\leq -\frac{cm|a|}{2} \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{a-1} |\nabla u^\alpha|^2 \xi^2 dx dr + \left( m|a| C_2 + \frac{2mC_4}{\alpha_\rho} + \frac{2mC_1}{c|a| \alpha_\rho^2} \right) \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{a+1} \xi^2 dx dr
\]
\[
\leq -\frac{cm|a|}{2} \int_0^t \int_{B_{1+\epsilon \rho}} U^{m-1}(u^\alpha)^{a-1} |\nabla u^\alpha|^2 \xi^2 dx dr + \left( m|a| C_2 + \frac{C_4}{\alpha_\rho} + \frac{1}{\alpha_\rho^2} \right) \left( S_1 \beta_\rho \right) \left( S_2 \beta_\rho \right) (m-1)^{1+1} \rho^{-(a-\beta)(m-1)}, \tag{3.17}
\]

where
\[
\gamma_0 = \frac{1}{(\lambda_1)^{1-m}} \max \left\{ 2m, \frac{2mC_1}{c|a|} \right\}.
\]

By (3.15), (3.16), and (3.17), (3.14) holds and the lemma follows. \(\square\)

**Lemma 3.2.** Let \(\alpha \in (-1, 0)\) be a constant given by (3.13) and let \(0 < m < 1\) satisfy
\[
1 - \frac{1}{\beta_i} < m < 1.
\]

For \(T > 0\), let \(u = (u^1, \ldots, u^d)\) be a weak solution to the singular parabolic system (GPS) in \(\Omega_T\) with structure conditions (1.3), (A1)–(A7). Then, there exists a positive constant \(\gamma\) depending on the data \(\alpha, m, N, \lambda_1, c,\) and \(c_3\) such that for all cylinder \(B_{2\rho}(y) \times (s, t) \in \Omega_T\)
\[
\sup_{s < t < \tau} \int_{B_{\tau \rho}(y)} u^\alpha(x, \tau) dx \leq \gamma \inf_{s < t < \tau} \int_{B_{\tau \rho}(y)} u^\alpha(x, \tau) dx + \left( 1 + C_6 + \sqrt{C_8 + C_6} \right) \rho^{-1/m} \left( \frac{t - s}{\rho^\theta} \right)^{1/2}, \tag{3.18}
\]

where \(\theta_i = \beta_i n(m - 1) + 2\).
Proof. We will use a modification of the proof of Proposition B.1.1 of [11] to prove the lemma. Without loss of
genemerity let \((y, s) = (0, 0)\). For each \(j \in \mathbb{N}\), set
\[
\rho_j = \sum_{i=1}^{j} \frac{1}{2} \rho, \quad B_j = B_{\rho_j}, \quad \tilde{\rho}_j = \frac{\rho_j + \rho_{j+1}}{2}, \quad \tilde{B}_j = B_{\tilde{\rho}_j}.
\]
We also consider a non-negative, piecewise smooth cut off function \(\xi_j(x)\) such that
\[
\begin{aligned}
0 \leq \xi_j \leq 1 & \quad \text{in } \tilde{B}_j \\
\xi_j = 1 & \quad \text{in } B_j \\
\xi_j = 0 & \quad \text{on } \partial \tilde{B}_j \\
|\nabla \xi_j| \leq \frac{2^{j+2}}{\rho} & \quad \text{in } \tilde{B}_j.
\end{aligned}
\]
Taking \(\varphi = \xi_j\) as a test function in the weak formulation (1.6) and integrating it over \(\tilde{B}_j \times (\eta, t)\), we have
\[
\int_{\tilde{B}_j} u^i(x, t)dx \leq \int_{\tilde{B}_j} u^i(x, \eta)dx + \int_{B_j} m^\mu |\nabla u^i| dx
\]
\[
\leq \int_{\tilde{B}_j} u^i(x, \eta)dx + \int_{\tilde{B}_j} \left( \frac{m^\mu |\nabla u^i|}{\rho} \right) dx.
\]
By (1.3), (3.19), and Hölder inequality,
\[
\int_{\tilde{B}_j} u^i(x, t)dx \leq \int_{\tilde{B}_j} u^i(x, \eta)dx + \int_{\tilde{B}_j} \left( \frac{m^\mu |\nabla u^i|}{\rho} \right) dx
\]
\[
+ \frac{m^\mu}{\rho} \int_{\tilde{B}_j} \left( \int_{\eta}^{t} U^m |\nabla u^i| dx \right) \left( \int_{\eta}^{t} \left( \frac{m^\mu |\nabla u^i|}{\rho} \right) dx \right)^{\frac{1}{2}}.
\]
where the constant \(\alpha\) is given by (3.13). By Hölder inequality, Lemma 3.1, and (3.20), there exists a constant
\(\gamma_0 > 0\) depending on \(a, m, N, \lambda_i, c,\) and \(C_3\) such that
\[
\int_{\tilde{B}_j} u^i(x, t)dx \leq \gamma_0 \left( \frac{\sqrt{C_2 + C_4}}{\rho} + \frac{2 \sqrt{1 + C_4}}{\rho^2} \right)
\]
\[
\times \left( \frac{\rho}{\rho^0} \right)^{2 j (m+1) + 1} \left( \frac{\rho}{\rho^0} \right)^{1/2} + \left( \frac{S_{l+1}^{j-m} \rho^0}{\rho} \right)^{1/2} \left( \frac{S_{l+1}^{j-m} \rho^0}{\rho} \right)^{1/2}
\]
\[
= \gamma_0 \left( \frac{\sqrt{C_2 + C_4}}{\rho} + \frac{2 \sqrt{1 + C_4}}{\rho^2} \right)^2 \left( \frac{t}{\rho^0} \right)^{1/2} + \gamma_0 \left( \frac{\sqrt{C_2 + C_4}}{\rho} + \frac{2 \sqrt{1 + C_4}}{\rho^2} \right) \rho^2 \left( \frac{S_{l+1}^{j-m} \rho^0}{\rho} \right)^{1/2} \left( \frac{t}{\rho^0} \right)^{1/2}
\]
\[
\leq \gamma_0 \left( \frac{\sqrt{C_2 + C_4}}{\rho} + \frac{2 \sqrt{1 + C_4}}{\rho^2} \right) \left( \frac{t}{\rho^0} \right)^{1/2} + \gamma_0 \left( \frac{\sqrt{C_2 + C_4}}{\rho} + \frac{2 \sqrt{1 + C_4}}{\rho^2} \right) \rho^2 \left( \frac{S_{l+1}^{j-m} \rho^0}{\rho} \right)^{1/2} \left( \frac{t}{\rho^0} \right)^{1/2}
\]

where

\[ S_j^i = \sup_{0 \leq r \leq t} \int_{B_{r_0}} u^i(x, r) \, dx. \]

We now choose \( \tau_i \) such that

\[ \int_{B_{r_0}} u^i(x, \tau_i) \, dx = \inf_{0 \leq r \leq t} \int_{B_{r_0}} u^i(x, r) \, dx = I^i \]

and let \( m_i = \beta_i(m - 1) + 1 \in (0, 1) \). Then we have

\[ S_j^i \leq I^i + \gamma_0(\mathbf{C}_4 + \sqrt{\mathbf{C}_2 + \mathbf{C}_4})\rho S_{j+1}^m \left( \frac{t}{\rho_0^i} \right) + \gamma_0 \left( 4^i \sqrt{1 + \mathbf{C}_4} S_{j+1}^m \left( \frac{t}{\rho_0^i} \right) + 2^i S_{j+1}^m \left( \frac{t}{\rho_0^i} \right)^{\frac{1}{m}} \right). \] (3.21)

By Young's inequality,

\[ 2^i \rho S_{j+1}^m \left( \frac{t}{\rho_0^i} \right) \leq m_i \varepsilon^{\frac{i}{m}} S_{j+1}^m + (1 - m_i) \left( \frac{1}{2} \right) \left( \frac{t}{\rho_0^i} \right)^{\frac{1}{m}}, \] (3.22)

\[ 4^i \rho S_{j+1}^m \left( \frac{t}{\rho_0^i} \right) \leq m_i \varepsilon^{\frac{i}{m}} S_{j+1}^m + (1 - m_i) \left( \frac{1}{2} \right) \left( \frac{t}{\rho_0^i} \right)^{\frac{1}{m}}, \] (3.23)

and

\[ 2^i S_{j+1}^m \left( \frac{t}{\rho_0^i} \right)^{\frac{1}{m}} \leq m_i + 1 \varepsilon^{\frac{i}{m}} S_{j+1}^m + (1 - m_i) \left( \frac{1}{2} \right) \left( \frac{t}{\rho_0^i} \right)^{\frac{1}{m}}, \] (3.24)

for any \( \varepsilon \in (0, 1) \). By (3.21), (3.22), (3.23), and (3.24), we have

\[ S_j^i \leq \gamma_0 \left( \mathbf{C}_4 + \sqrt{\mathbf{C}_2 + \mathbf{C}_4} \right) \varepsilon^{\frac{i}{m}} + \varepsilon^{\frac{i}{m}} \right) S_{j+1}^m + \gamma_0 \left( \frac{1}{\varepsilon^{\frac{i}{m}}} + \frac{1}{\varepsilon^{\frac{i}{m}}} \right) \left( 4^i \sqrt{1 - m_i} \right)^{\frac{1}{m}} \]

\[ \times \left( I^i + \left( \sqrt{1 + \mathbf{C}_4} + (\mathbf{C}_4 + \sqrt{\mathbf{C}_2 + \mathbf{C}_4})\rho_0^i \right) \left( \frac{t}{\rho_0^i} \right) \right)^{\frac{1}{m}} \]

\[ = \varepsilon_0 S_{j+1}^m + \gamma_0 \left( \frac{1}{\varepsilon^{\frac{i}{m}}} + \frac{1}{\varepsilon^{\frac{i}{m}}} \right) \left( I^i + \left( \sqrt{1 + \mathbf{C}_4} + (\mathbf{C}_4 + \sqrt{\mathbf{C}_2 + \mathbf{C}_4})\rho_0^i \right) \left( \frac{t}{\rho_0^i} \right) \right)^{\frac{1}{m}} \left( 4^i \sqrt{1 - m_i} \right)^{\frac{1}{m}}. \]

By iteration,

\[ S_j^i \leq \varepsilon_0 S_{j+1}^m + \gamma_0 \left( \frac{1}{\varepsilon_0} \right) \left( I^i + \left( \sqrt{1 + \mathbf{C}_4} + (\mathbf{C}_4 + \sqrt{\mathbf{C}_2 + \mathbf{C}_4})\rho_0^i \right) \left( \frac{t}{\rho_0^i} \right) \right)^{\frac{1}{m}} \sum_{i=0}^{j-1} \varepsilon_0 \left( 4^i \sqrt{1 - m_i} \right)^{\frac{1}{m}}. \] (3.25)

Now we choose the constant \( \varepsilon_0 \) so small that the last term is bounded by a convergence series. Then, letting \( j \to \infty \) in (3.25) we have the inequality (3.18) and the lemma follows. \( \square \)

Through the De Giorgi iteration, we will get the following \( L^\infty \) estimate of the system (3.1).

**Lemma 3.3.** Let \( \alpha \in (-1, 0) \) be a constant given by (3.13) and let \( 0 < m < 1 \) satisfy

\[ 1 - \frac{1}{\beta_i} < m < 1, \quad \theta_i = n(m_i - 1) + 2 = n\beta_i(m - 1) + 2 > 0. \]
For $T > 0$, let $\mathbf{u} = (u^1, \ldots, u^k)$ be a weak solution to the singular parabolic system (GPS) in $\Omega_T$ with structures (1.3), (A1)–(A7). Suppose that there exists a constant $\Lambda > 0$ such that

$$U(x, t) \leq \Lambda \quad \text{a.e. on } \Omega_T.$$

Then, there exists a positive constant $\gamma$ depending on the data $\alpha, m, N, c,$ and $C_3$ such that for all cylinder $B_{2\rho}(y) \times (2s - t, t) \subset \Omega_T$

$$\sup_{B_{2\rho}(y) \times (s, t)} u^j \leq \gamma (2 + (C_2 + C_4)\rho^2) \left( \frac{\rho^2}{t - s} \right)^{\frac{n}{2}} \left( 1 + \frac{\rho^2}{t - s} \right)^{\frac{n+2}{2n}} \left( \int_{2s - t}^t \int_{B_{2\rho}(y)} u^j(\xi_1, t) \, d\xi_1 \right)^{\frac{1}{n}} + \left( \frac{t - s}{\rho^2} \right)^{\frac{1}{2(1 - \sigma)}}.$$  

(3.26)

**Proof.** We will use a modification of the proof of Proposition B.4.1 of [11] to prove the lemma. Without loss of generality let $(y, s) = (0, 0)$. For fixed $\sigma \in (0, 1)$ and each $j \in \mathbb{N}$, set

$$\rho_j = \sigma \rho + \frac{1 - \sigma}{2}\rho, \quad t_n = -\sigma t - \frac{1 - \sigma}{2} t, \quad B_j = B_{\rho_j}, \quad Q_i = B_j \times (t_j, t).$$

Then we first observe that

$$Q_0 = B_{\rho} \times (-t, t) \quad \text{and} \quad Q_{\infty} = B_{\rho_{\infty}} \times (-\sigma t, t).$$

Set

$$M^i = \text{esssup}_{Q_0} \max \{u^i, 0\}, \quad \frac{i}{\sigma} = \text{esssup}_{Q_{\infty}} \max \{u^i, 0\}.$$ 

Let $\xi_j(x, t) = \xi_j^1(x)\xi_j^2(t)$ be a non-negative, piecewise smooth cut-off function in $Q_j$ such that

$$\begin{cases}
\xi_j^1 = 1 & \text{in } B_{r_1} \\
\xi_j^1 = 0 & \text{on } \mathbb{R}^n \setminus B_{r_1} \\
|\nabla \xi_j^1| \leq \frac{2i+1}{(1 - \sigma)\rho} & \text{in } \mathbb{R}^n.
\end{cases}$$

$$\frac{i}{\sigma} = \frac{2i+1}{(1 - \sigma)\rho}$$

Then $\xi_j^1$ equals one on $Q_{j+1}$. Consider the increasing sequence

$$l_j = \left(1 - \frac{1}{2}\right)l_i,$$

where $k > 2$ will be determined later. Let

$$m_i = \beta_i(m - 1) + 1.$$ 

Taking $\varphi = ((u^m - l_{i-1}^m), \xi_j^2)$ as a test function in the weak formulation (1.6) and integrating it over $Q_j$, we have

$$0 = \int_{Q_j} (u^m) (u^m - l_{i-1}^m), \xi_j^2 \, dx \, dt + \int_{Q_j} mU^{m-1}(A(\nabla u^m, u^m, x, t) \cdot \nabla ((u^m - l_{i-1}^m), \xi_j^2) \, dx \, dt$$

$$+ 2 \int_{Q_j} mU^{m-1}((u^m - l_{i-1}^m), (A(\nabla u^m, u^m, x, t) \cdot \nabla \xi_j) \, dx \, dt$$

$$= I_1 + I_2 + I_3.$$  

(3.27)
By direct computation,

\[
I_1 = \frac{\partial}{\partial t} \int_0^\infty \left( (u^m)_{t}^m, \frac{1}{m} \left[ (s + l_{j+1}^m) \delta_{m-1}^m \right] \frac{d^2}{dx^2} \right) \xi_j^2 dx dt - \frac{2}{m_1} \int_0^\infty \left( (u^m)^m, \frac{1}{m_1} \left[ (s + l_{j+1}^m) \delta_{m-1}^m \right] \right) \xi_j^2 dx dt
\]

\[
\geq \frac{1}{m} \int_0^\infty \left( (u^m)^m, \frac{1}{m} \left[ (s + l_{j+1}^m) \delta_{m-1}^m \right] \right) \xi_j^2 dx dt - \frac{2}{m_1} \int_0^\infty \left( (u^m)^m, \frac{1}{m_1} \left[ (s + l_{j+1}^m) \delta_{m-1}^m \right] \right) \xi_j^2 dx dt
\]

\[
\geq \frac{1}{m} \int_0^\infty \left( (u^m)^m - l_{j+1}^m \frac{m+1}{m} \xi_j^2 dx dt - \frac{2}{m_1} \int_0^\infty \left( (u^m)^m, \frac{1}{m_1} \left[ (s + l_{j+1}^m) \delta_{m-1}^m \right] \right) \xi_j^2 dx dt.
\]

Observe that

\[u' \geq \frac{1}{2} \text{ on } \{u' - l_{j+1}, > 0\}.
\]

Thus, we can obtain

\[
I_2 \geq \frac{mcN^{1-m}}{4m_1} \int_0^\infty \left| \nabla ((u^m)^m - l_{j+1}^m), \xi_j^2 \right| dx dt - \frac{mcN^{1-m}}{m_1} \int_0^\infty \left| ((u^m)^m - l_{j+1}^m)^2 \right| nx_j^2 dx dt
\]

\[
- \frac{m m_2 C_2}{(\lambda_3)^{1-m}} \int_0^\infty \left( u'^2 \eta^2 \chi_{[u' > \Lambda]} \right) dx dt
\]

and

\[
|I_2| \leq \frac{mcN^{1-m}}{8m_1} \int_0^\infty \left| \nabla ((u^m)^m - l_{j+1}^m), \xi_j^2 \right| dx dt + \frac{mcN^{1-m}}{m_1} \int_0^\infty \left| ((u^m)^m - l_{j+1}^m)^2 \right| nx_j^2 dx dt
\]

\[
+ \frac{m}{(\lambda_3)^{1-m}} \left( 2C_2 + \frac{8C_3}{m_2 cN^{1-m} N^{1-m}} \right) \int_0^\infty \left| ((u^m)^m - l_{j+1}^m)^2 \right| nx_j^2 dx dt.
\]

By (3.27), (3.28), (3.29), and (3.30), there exists a constant \(C_4 > 0\) depending on \(m, \lambda, \beta, c, C, C_s, C, \) and \(\Lambda\) such that

\[
\sup_{t \leq \tau_1} \int_0^\infty \left| ((u^m)^m - l_{j+1}^m), \xi_j^2 \right| dx dt + \int_0^\infty \left| \nabla ((u^m)^m - l_{j+1}^m), \xi_j^2 \right| dx dt
\]

\[
\leq C_4 \left( 1 + \frac{1}{\lambda_3} \right) \left[ \int_0^\infty \left| ((u^m)^m - l_{j+1}^m), \xi_j^2 \right| dx dt + \int_0^\infty \left| ((u^m)^m - l_{j+1}^m)^2 \right| nx_j^2 dx dt
\]

\[
+ \int_0^\infty \left( u'^2 \eta^2 \chi_{[u' > \Lambda]} \right) dx dt \right].
\]

Since

\[
((u^m)^m - l_{j+1}^m) \geq \frac{1}{C_2} \left( \frac{l_j}{2^{1+1}} \right)^m \geq \frac{1}{C_2} \left( \frac{l_j}{2^{1+1}} \right)^m \text{ on } \{u' > l_{j+1}\}
\]
for some constant $C_2 > 0$, we can obtain

\[
\int_{Q_j} \left( (u')^{m+1} \chi_{[u > h_j + \xi]} \xi |(\xi)| x dx \right) dr \leq C_2 \left( \frac{m+1}{m} \right) \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) x dx dr,
\]

\[
\int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) |\nabla \xi|^2 dx dr \leq C_3 \left( \frac{m+1}{m} \right) \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) x dx dr,
\]

\[
\int_{Q_j} \left( (u')^{2m+2} \chi_{[u > h_j + \xi]} \xi |(\xi)| x dx \right) dr \leq C_4 \left( \frac{m+1}{m} \right) \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) x dx dr
\]

for some constant $C_3 > 0$. By (3.31) and (3.32), there exists a constant $C_4 > 0$ such that

\[
\sup_{t \leq t \leq T} \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) |\nabla \xi| \frac{m+1}{m} dx dr + \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) \xi |(\xi)| x dx dr
\]

since $\frac{n-2}{n} < m_i < 1$. If we choose the constant $l$ to be

\[
l \geq \left( \frac{t}{\rho^2} \right)^{\frac{1}{m}},
\]

then

\[
\sup_{t \leq t \leq T} \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) |\nabla \xi| \frac{m+1}{m} dx dr + \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) \xi |(\xi)| x dx dr
\]

\[
\leq C_4 \left( \frac{m+1}{m} \right) \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) x dx dr
\]

On the other hand, by Hölder inequality and Proposition 3.1 of Chap. I of [16], there exists a constant $C_5 > 0$ such that

\[
\frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} \left( (u')^{m} - l_j^{m+1} \right) x dx dr
\]

\[
\leq C_5 \left( \sup_{t \leq t \leq T} \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) \xi |(\xi)| x dx dr \right) \left( \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) x dx dr \right) \left( \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) |\nabla \xi| \frac{m+1}{m} dx dr \right)
\]

\[
\times \frac{1}{|Q_j|^\frac{1}{2\alpha}} \left( \frac{m+1}{m} \right) \frac{1}{|Q_j|} \int_{Q_j} \left( (u')^{m} - l_j^{m+1} \right) x dx dr
\]

where

\[
p = \frac{2(nm_i + m_i + 1)}{nm_i}.
\]
By (3.33) and (3.34), there exists a constant $C_6 > 0$ such that

$$A_{j+1} \leq \frac{C_6}{|Q_l|^{m+l}} \bigg( \frac{2^{m+l}}{(1-\sigma)^2} \left( 1 + \frac{\rho^2}{t} \right) \left( 2 + (C_2 + C_4)^2 |Q|A_j \right) \bigg) \frac{m+l}{m}$$

$$\times \left( \frac{2^{m+l}/(1-\sigma)^2}{l^{m+1}} \left( 1 + \frac{\rho^2}{t} \right) \left( 2 + (C_2 + C_4)^2 \right) |Q|A_j \right) \frac{m+l}{m}$$

$$\leq \frac{C_6 b_1}{(1-\sigma)^2 |Q_l|^{m+1}} \left( 2 + (C_2 + C_4)^2 \right) \frac{m+l}{m} \left( 1 + \frac{\rho^2}{t} \right) \left( \frac{m+l}{m} \right) \frac{1}{n} \frac{m+l}{m} \frac{1}{n} \frac{m+l}{m} \frac{1}{n} \frac{m+l}{m}$$

where

$$b_1 = 2 \left( \frac{m+l}{m} \right) \left( 1 + \frac{\rho^2}{t} \right)$$

and

$$A_j = \frac{1}{|Q_l|} \int_{Q_l} ((u^d)^m - l^m)^{m+l-1} dx.$$

Let

$$C_7 = C_6 \frac{1}{(1-\sigma)^2} \frac{2}{m}, \quad C_8 = C_7 \frac{2}{m(m+l-1)+2m+2}, \quad b_2 = b_1 \left( \frac{1}{(1-\sigma)^2} \right)^{1/2}, \quad b_3 = b_2 \frac{2}{m(m+l-1)+2m+2}.$$

If we take the constant $l$ such that

$$A_0 = \frac{1}{|Q_l|} \int_{Q_l} ((u^d)^m + dx) \leq C_7 b_3 (1-\sigma)^{-1/2} \left( 2 + (C_2 + C_4)^2 \right) \frac{\rho^2}{t} \left( 1 + \frac{\rho^2}{t} \right)$$

$$\Rightarrow l \geq \frac{C_7 b_3}{(1-\sigma)^{m+l} + 2m+2} \left( 2 + (C_2 + C_4)^2 \right) \frac{\rho^2}{t} \left( 1 + \frac{\rho^2}{t} \right)$$

$$\times \left( \frac{1}{|Q_l|} \int_{Q_l} ((u^d)^m + dx) \right)^{2/3}$$

then, by Lemma 5.1 in Chap. 2 of [11], we have

$$A_j \to 0 \quad \text{as} \quad j \to \infty.$$  \hspace{1cm} (3.35)

If

$$\frac{C_7 b_3}{(1-\sigma)^{m+l} + 2m+2} \left( 2 + (C_2 + C_4)^2 \right) \frac{\rho^2}{t} \left( 1 + \frac{\rho^2}{t} \right) \left( \frac{1}{|Q_l|} \int_{Q_l} ((u^d)^m + dx) \right)^{2/3}$$

$$\leq \left( \frac{t}{\rho^2} \right)^{1/m},$$

then we take

$$l = \left( \frac{t}{\rho^2} \right)^{1/m}.$$

Then by (3.35) we have

$$u^d \leq l \left( \frac{t}{\rho^2} \right)^{1/m} \quad \text{on} \quad Q_{co},$$

and (3.26) holds with $\sigma = \frac{1}{2}$.
Otherwise, we take
\[
  l = \frac{C_{10}^2 (2 + (C_2 + C_0) \beta_1^2)^{\frac{n+2}{2(n-1)+2m+2}}}{(1 - \sigma)^{\frac{n+2}{2n-1} + \frac{2m+2}{2}} \int (u')^m + 1 \, dx \, dt}.
\]

Then
\[
  M_i^j \leq \frac{C_{10}^2 (2 + (C_2 + C_0) \beta_1^2)^{\frac{n+2}{2(n-1)+2m+2}}}{(1 - \sigma)^{\frac{n+2}{2n-1} + \frac{2m+2}{2}} \int (u')^m + 1 \, dx \, dt} \times \left( \frac{1}{|Q_0|} \int_{Q_0} (u')^m + 1 \, dx \, dt \right) \quad \text{(3.36)}
\]

Set
\[
  p_j = \sigma + (1 - \sigma) \sum_{j'=1}^j 2^{-j'}, \quad \tilde{t}_j = -at - (1 - \sigma) t \sum_{j'=1}^j \frac{1}{2^{j'}},
\]

and
\[
  \bar{Q}_j = B_{\rho} \times (\tilde{t}_j, t], \quad \bar{Q}_{\infty} = B_{\rho} \times (-t, t], \quad \bar{Q}_0 = B_{\rho} \times (-at, t]
\]

and
\[
  \bar{M}_j^i = \operatorname{esssup}_{\bar{Q}_j} \max \{u', 0\}.
\]

By (3.36) with \(\sigma\) being replaced by \(\sigma + (1 - \sigma) \sum_{j'=1}^j 2^{-j'}\),
\[
  \bar{M}_j^i \leq C_{10} \frac{2^{2(n-2)+2m+2}}{(1 - \sigma)^{\frac{n+2}{2n-1} + \frac{2m+2}{2}} \int (u')^m + 1 \, dx \, dt} \times \left( \frac{1}{|Q_\infty|} \int_{Q_\infty} (u')^m + 1 \, dx \, dt \right) \quad \text{(3.36)}
\]

for some constant \(C_{10} > 0\). Then by an interpolation lemma (Lemma 5.2 in Chap. 2 of [11]),
\[
  \bar{M}_0^i = \operatorname{sup}_{B_{\rho} \times (-at, t]} u' \leq C_{10} \frac{2^{2(n-2)+2m+2}}{(1 - \sigma)^{\frac{n+2}{2n-1} + \frac{2m+2}{2}} \int (u')^m + 1 \, dx \, dt} \times \left( \frac{1}{|Q_\infty|} \int_{Q_\infty} (u')^m + 1 \, dx \, dt \right) \quad \text{(3.36)}
\]

for some constant \(C_{10} > 0\). This immediately implies the inequality (3.26) and the lemma follows.

As a consequence of Lemmas 3.2 and 3.3, we can obtain the following full version of Harnack-type estimate for system (3.1).

**Lemma 3.4.** Let \(\alpha \in (-1, 0)\) be a constant given by (3.13) and let \(0 < m < 1\) satisfy
\[
  1 - \frac{1}{\beta_1} < m < 1, \quad \theta_1 = m(n - 1) + 2 = n\beta_1(m - 1) + 2 > 0.
\]

For \(T > 0\), let \(u = (u', \ldots, u^k), (u' \geq 0\) is bounded for all \(1 \leq i \leq k\), be a weak solution to the singular parabolic system (GPS) in \(\Omega_T\) with structures (1.3), (A1)–(A7). Then, there exists a positive constant \(\gamma\) depending on the data \(\alpha, m, N,\) and \(c\) such that for all cylinder \(B_{\rho}(y) \times (2s - t, t) \in \Omega_T\)
We are now ready for the proof of Theorem 1.6.

**Proof of Theorem 1.6.** We will use a modification of the proof of Theorem 1.1 of [13] to prove the theorem. By (3.1) and divergence theorem,

\[
\frac{d}{dt} \int_{B_R} u(x, t) dx = m \int_{\partial B_R} U^{m-1} \mathcal{A}(\nabla u^i, u^i, x, t) \cdot \nu dx
\]  

for any \( R > 0 \) where \( \nu \) is the unit outward normal vector on \( \partial B_R \). Let \( m_i = \beta_i (m - 1) + 1 \). Then, by (A6) and (3.38) we have

\[
\sup_{0 \leq t \leq T} \int_{B_R} u(x, t) dx - \int_{B_R} u_0(x) dx \\
\leq m \left[ C_3 \int_0^T \int_{\partial B_R} U^{m-1} |\nabla u^i| d\sigma dt + C_4 \int_0^T \int_{\partial B_R} U^{m-1} u^i d\sigma dt \right] \\
\leq m \left[ C_3 \int_0^T \int_{\partial B_R} U^{m-1} |\nabla u^i| d\sigma dt + \frac{C_4}{(\lambda_0)^{m-1}} \int_0^T \int_{\partial B_R} u^i d\sigma dt \right] \\
\leq R^{2m/T^2} \left[ C_3 \left( \int_0^T \int_{\partial B_R} (U^{m-1} |\nabla u^i|)^2 d\sigma dt \right)^{\frac{1}{2}} + C_4 \left( \int_0^T \int_{\partial B_R} (u^i)^2 d\sigma dt \right)^{\frac{1}{2}} \right].
\]  

Suppose that

\[
\text{supp } u_0^i \subset B_R
\]

and let \( R_0 > \max\{8, 8R\} \). Then

\[
\text{supp } u_0^i \subset B_{R_0}.
\]  

Let \( \xi \in C^\infty(\mathbb{R}^n) \) be a cutoff function such that

\[
\begin{cases}
0 \leq \xi \leq 1 & \text{in } \mathbb{R}^n \\
\xi = 1 & \text{in } B_{2R_0} \setminus B_{\frac{3R_0}{2}} \\
\xi = 0 & \text{in } B_{R_0} \cup \{\mathbb{R}^n \setminus B_{4R_0}\} \\
|\nabla \xi| \leq \frac{C_1}{R_0} & \text{in } \mathbb{R}^n
\end{cases}
\]

\[
\sup_{B_{R_0}(y) \times (s, t)} u^i \leq \left[ \frac{2 + (C_2 + C_4) p^{\frac{n+2}{n}} (t - s + \rho^2)^{\frac{n-1}{n}}}{(t - s)^{\frac{n+1}{n}}} \left( \inf_{B_{R_0}(y)} \int_{B_{R_0}(y)} u^i(x, \tau) d\tau \right)^{\frac{1}{2}} \right] + \left[ 1 + \frac{2 + (C_2 + C_4) p^{\frac{n+2}{n}} (t - s + \rho^2)^{\frac{n-1}{n}}}{(t - s)^{\frac{n+1}{n}}} \left( 1 + C_4 + (C_2 + C_4) \rho \beta_i (1-m) \right)^{\frac{1}{2}} \right] \left( t - s \right)^{\frac{1}{1+m}}
\]  

(3.37)
for some constant \( c_1 > 0 \). Let \( T > 0 \). Taking \((u^m)\xi^2\) as a test function in the weak formulation (1.6) and integrating it over \( \mathbb{R}^n \times [0, T] \), we have

\[
\frac{1}{1 + m} \int_{\mathbb{R}^n} (u^m)\xi^2(x, T)\xi^2 dx - \frac{1}{1 + m} \int_{\mathbb{R}^n} (u^0)\xi^2 dx
= \frac{-mm}{T} \int_0^T \int_{\mathbb{R}^n} U^{-\frac{1}{2}}(u^m)\xi^2 \xi^2 dx dt - 2m \int_0^T \int_{\mathbb{R}^n} U^{-\frac{1}{2}}(u^m) (\mathcal{A} \cdot \nabla \xi) \xi dx dt.
\]

By (A5), (A6), (3.40), (3.41), and Young’s inequality,

\[
\frac{\langle \lambda \rangle^{1-m} m c}{2} \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt
\leq \frac{m}{\langle \lambda \rangle^{1-m}} \left( m C_2 + C_6 \right) \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt + \frac{m}{\langle \lambda \rangle^{1-m}} \left( C_3 + C_4 \right) \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt
\]

\[
+ \frac{m}{\langle \lambda \rangle^{1-m}} \left( C_4 + \frac{C_3}{cm} \right) \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt.
\]

Since \( |\nabla \xi| \leq \frac{\rho}{R_0} \),

\[
\int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt
\leq \frac{2}{c \langle \lambda \rangle^{3(1-m)} R_0^2} \left( C_2 + \frac{C_4}{m} \right) \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt
\]

\[
+ \frac{2C_4}{m c \langle \lambda \rangle^{3(1-m)} R_0^2} \left( C_4 + \frac{C_3}{cm} \right) \left[ \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt + \int_0^T \int_{\mathbb{R}^n} (u^m)\xi^2 dx dt \right].
\]

We now put \( t = T, s = \frac{T}{2}, \) and \( \rho = \frac{|x|}{8} \) in (3.37). Then, there exists a constant \( c_2 > 0 \) such that

\[
\begin{aligned}
\sup_{y \in B_{\rho}(x)} u^m(y, T)
\leq & \left[ \left( 2 + (C_2 + C_6) \left( \frac{|x|}{8} \right) \right) \left( \frac{T}{2} + \left( \frac{|x|}{8} \right)^2 \right) \frac{\langle \lambda \rangle^{1-m}}{2} \right] \left( \inf_{0 < \tau < T} \int_{B_{\rho}(x)} u^m(x, \tau) dx \right) \frac{\langle \lambda \rangle^{1-m}}{2} \\
+ & \left[ \left( 2 + (C_2 + C_6) \left( \frac{|x|}{8} \right) \right) \left( \frac{T}{2} + \left( \frac{|x|}{8} \right)^2 \right) \frac{\langle \lambda \rangle^{1-m}}{2} \right] \left( \inf_{0 < \tau < T} \int_{B_{\rho}(x)} u^m(x, \tau) dx \right) \frac{\langle \lambda \rangle^{1-m}}{2}
\end{aligned}
\]

\[
\times \left( \sqrt{1 + C_4 + (C_4 + \sqrt{C_2 + C_6}) \left( \frac{|x|}{8} \right)^{\frac{1}{1-m}}} \right) \left( \frac{T}{2} \right) \frac{\langle \lambda \rangle^{1-m}}{2}.
\]

\[
\frac{1}{1 + m} \int_{\mathbb{R}^n} (u^m)\xi^2(x, T)\xi^2 dx - \frac{1}{1 + m} \int_{\mathbb{R}^n} (u^0)\xi^2 dx
= \frac{-mm}{T} \int_0^T \int_{\mathbb{R}^n} U^{-\frac{1}{2}}(u^m)\xi^2 \xi^2 dx dt - 2m \int_0^T \int_{\mathbb{R}^n} U^{-\frac{1}{2}}(u^m) (\mathcal{A} \cdot \nabla \xi) \xi dx dt.
\]
By (3.40) and (3.43), we have
\[
\begin{align*}
u(x, T) \leq c_0 & \left( 1 + (1 + |x|^2)^{\frac{m_2}{2m}}(1 + (C_2 + c_4)|x|^2)^{\frac{m_2}{2m}} \left( \sqrt{1 + C_4} + (C_4 + \sqrt{C_2 + C_4})|x|^\frac{1}{R_t^{1-m}} \right)^\frac{1}{2m} \right) \\
& \times \left( \frac{1}{|x|^2} \right)^\frac{1}{m-1}, \quad \forall |x| \geq \frac{R_0}{4},
\end{align*}
\]  

(3.44)

for some constant \(c_4 = c_4(T) > 0\). By (3.42) and (3.44), there exists a constant \(c_4 > 0\) depending on \(c_1, c_2, c_3, m_i, m, \lambda_i,\) and \(T\) such that

\[
\begin{align*}
\iint_T^T \frac{(U_m - \nabla u)^2}{v^2} & dx \, dt \\
& \leq c_0 \left( (C_2 + c_4)R_0^{n-2(1+m)}R_0^{\frac{1}{1-m}} + (C_4 + c_4)R_0^{n-2(1+m)} \right) \\
& \times \left[ 1 + R_0^{\frac{4m(n+2)}{m_1}} \left( 1 + (C_2 + c_4)R_0^{\frac{2m(n+2)}{m_1}} \right)^{\frac{m}{m_1}} \right].
\end{align*}
\]  

(3.45)

By (3.45) and mean value theorem for integrals, there exists \(\frac{R_0}{2} \leq R_0 \leq 2R_0\) such that

\[
\begin{align*}
\iint_T^T \frac{(U_m - \nabla u)^2}{v^2} & dx \, dt \\
& \leq c_0 \left( (C_2 + c_4)R_0^{n-2(1+m)}R_0^{\frac{1}{1-m}} + (C_4 + c_4)R_0^{n-2(1+m)} \right) \\
& \times \left[ 1 + R_0^{\frac{4m(n+2)}{m_1}} \left( 1 + (C_2 + c_4)R_0^{\frac{2m(n+2)}{m_1}} \right)^{\frac{m}{m_1}} \right].
\end{align*}
\]  

Combining this with (3.39), we have

\[
\begin{align*}
\sup_{0 \leq \tau \leq T} \left| \int_{B_R} u(x, \tau) \, dx - \int_{B_R} u_0(x) \, dx \right| \\
& \leq c_0R_0^{n-1-\frac{1+m}{1-m}}(1 + (C_2 + c_4)R_0)^2 \left[ 1 + R_0^{\frac{2m(n+2)}{m_1}} \left( 1 + (C_2 + c_4)R_0^{\frac{2m(n+2)}{m_1}} \right)^{\frac{m}{m_1}} \right] \\
& \leq c_0R_0^{n-1-\frac{1+m}{1-m}}(1 + (C_2 + c_4)R_0)^2 \left[ 1 + R_0^{\frac{2m(n+2)}{m_1}} \left( 1 + (C_2 + c_4)R_0^{\frac{2m(n+2)}{m_1}} \right)^{\frac{m}{m_1}} \right] \\
& \leq c_0R_0^{n-1-\frac{1+m}{1-m}}(1 + (C_2 + c_4)R_0)^2 \left[ 1 + R_0^{\frac{2m(n+2)}{m_1}} \left( 1 + (C_2 + c_4)R_0^{\frac{2m(n+2)}{m_1}} \right)^{\frac{m}{m_1}} \right]
\end{align*}
\]  

(3.46)

for some constant \(c_5 > 0\) and \(c_6 > 0\). Thus, if

\[
\frac{n^2 + n + 4 + \sqrt{2m(n+11)}}{n^2 + 5n + 8} < m_i < 1,
\]

then the right hand side of (3.46) converges to zero as \(R_t \to \infty\). Therefore, letting \(R_t \to \infty\) in (3.46) we have

\[
\begin{align*}
\int_{R^n} u(x, T) \, dx = \int_{R^n} u_0(x) \, dx
\end{align*}
\]  

for any \(T > 0\) and the theorem follows. \(\Box\)
Remark 3.5.

(1) Suppose that the constants $C_2$ and $C_4$ in the structures (A4)–(A6) are all zeros. Then, by (3.46) we have

$$\sup_{0 \leq t \leq T} \left| \int_{B_{r_0}} u'(x, t) \, dx - \int_{B_{r_0}} u_0'(x) \, dx \right| \leq c_d R_0 \left( \frac{\lambda u_{m_1}}{1 + \frac{1}{m_1}} \right)^{2m_1/2} \cdot$$

Therefore, $L^1$-mass conservation (1.12) holds for $m_i = \beta(m - 1) + 1 < 1$.

(2) Suppose that $U$ is equivalent to $\lambda(u')^{\beta}$, i.e., there exists some constants $0 < C \leq C < \infty$ such that $c_{\lambda}(u')^{\beta} \leq U \leq C_{\lambda}(u')^{\beta}$ in $Q(R, \theta_0^\infty \Theta^2)$.

Then the term $(1 + |x|^2)^{n/2}$ in (3.44) can be replaced by an uniform constant. Therefore, if $U$ is equivalent to $\lambda(u')^{\beta}$, and constants $C_2$ and $C_4$ are all zeros, then $L^1$-mass conservation (1.12) holds for $\frac{n - 2}{n} < m_i = \beta(m - 1) + 1 < 1$.

4 Local continuity (Proof of Theorem 1.9)

In Section 2, we discussed the $L^\infty$ boundedness of the function $U$ which makes the diffusion coefficient under control. It is a very useful tool for investigating the regularity theories of solutions to the parabolic systems. With this observation, we are going to prove the local continuity of the parabolic system (GPS) under the structural assumptions (1.3), (A1)–(A7), (1.4). We start by stating a well-known result, Sobolev-type inequality, which plays an important role for the local continuity.

Lemma 4.1. (cf. Lemma 3.1 of [19]) Let $u(x, t)$ be a cut-off function compactly supported in $B$, and let $u$ be a function defined in $R^n \times (t_1, t_2)$ for any $t_1 > t_2 > 0$. Then $u$ satisfies the following Sobolev inequalities:

$$\|u\|_{L^\infty(R^n)} \leq C \|\nabla(u)\|_{L^2(R^n)}$$

and

$$\|u\|_{L^2(R^n)} \leq C \left( \sup_{k \leq \beta \leq b} \|u\|_{L^2(R^n)} + \|\nabla(u)\|_{L^2(R^n)} \right) \|u > 0\|^{\frac{n}{2}}$$

(4.2)

for some $C > 0$.

For the local continuity of the parabolic system (GPS), we will use a modification of the technique introduced in [1,16,17,19]. Choose a point $(x_0, t_0) \in R^n \times (0, \infty)$ and a constant $R_0 > 0$ such that

$$(x_0, t_0) + Q(R_0, R_0^2) = (x_0, t_0) + B_R \times (-R_0^2, 0) \subset R^n \times (0, \infty),$$

where $0 < \varepsilon < 1$ is a small number which is determined by (4.24). After translation, we may assume without loss of generality that

$$(x_0, t_0) = (0, 0).$$

By Theorem 1.2, there exists a constant $1 < \Lambda < \infty$ such that

$$U(x, t) \leq \Lambda \quad \forall (x, t) \in Q(R_0, R_0^2).$$

(4.3)

Thus we can set, for each $1 \leq i \leq k$,

$$(\mu^i)^+ = \text{ess sup}_{Q(R_0, R_0^2)} u^i, \quad (\mu^i)^- = \text{ess inf}_{Q(R_0, R_0^2)} u^i, \quad \omega^i = \text{osc}_{Q(R_0, R_0^2)} u^i = (\mu^i)^+ - (\mu^i)^-.$$
By (1.3), the equation (GPS) is non-degenerate on the region where $u^i > 0$. Thus, if $(\mu^i)^- > 0$ for some $1 \leq i \leq k$, then the equation is uniformly parabolic in $Q(R_0, R_0^{2-\epsilon})$. By standard regularity theory for the parabolic equation [20], the local Hölder continuity follows. Hence from now on, we assume that

$$(\mu^i)^- = 0 \quad \forall 1 \leq i \leq k.$$ 

If $(\mu^i)^+ = 0$ for some $1 \leq i \leq k$, then

$$u^i = 0 \quad \text{on} \quad Q(R_0, R_0^{2-\epsilon}).$$

This immediately implies the local continuity of solution $u^i$. Hence, we also assume that

$$(\mu^i)^+ > 0 \quad \forall 1 \leq i \leq k.$$ 

For the intrinsic scaling technique, let

$$\omega_M = \max_{1 \leq i \leq k} \omega^i \quad \text{and} \quad \theta = \frac{\omega_M}{4}.$$ 

and construct the cylinder

$$Q(R, \theta^{-\alpha}R^2) = B_R \times (-\theta^{-\alpha}R^2, 0) \quad (\alpha_0 = \beta_i(m - 1)), \quad (4.4)$$

where $\beta_i$ is given by (1.3). We will assume that the radius $0 < R < R_0$ is sufficiently small that

$$\theta^{\alpha_0} > R^\epsilon. \quad (4.5)$$

By (4.4) and (4.5),

$$Q(R, \theta^{-\alpha}R^2) \subset Q(R, R^{2-\epsilon}) \subset Q(R_0, R_0^{2-\epsilon})$$

and

$$\text{osc}_{Q(R, \theta^{-\alpha}R^2)} u^i \leq \omega_M = 4\theta \quad \forall 1 \leq i \leq k.$$ 

For the proof of local continuity of the component $u^i$, two separated cases are considered. The first one is to find a parabolic cylinder of the form of (4.4) where $u^i$ is mostly large, and the other one is when such a cylinder cannot be found. In both cases, we are going to show that the (essential) oscillation of $u^i$ in a smaller cylinder decreases in a way that can be measured quantitatively.

### 4.1 The first alternative

Suppose that there exists a cylinder such that $u^i$ is mostly large. Then, through the first alternative, we will prove that the component $u^i$ is above a small level in a smaller cylinder. The statement of the first alternative is as follows.

**Lemma 4.2.** There exists a positive number $\rho_0$ depending on $m$, $q$, $\lambda$, $\Lambda$, and $\frac{\Lambda}{\partial \mu}$, such that if

$$\left| \left\{ (x, t) \in Q(R, \theta^{-\alpha}R^2) : u^i(x, t) < \frac{\omega_M}{2} \right\} \right| \leq \rho_0|Q(R, \theta^{-\alpha}R^2)|, \quad (4.6)$$

then

$$u^i(x, t) > \frac{\omega_M}{4} \quad \forall (x, t) \in Q \left( \frac{R}{2}, \theta^{-\alpha} \left( \frac{R}{2} \right)^2 \right). \quad (4.7)$$

**Proof.** For $j \in \mathbb{N}$, we set

$$R_j = \frac{R}{2^j} + \frac{R}{2^j} \quad \text{and} \quad l_j = (\mu^i)^- + \left( \frac{\omega_M}{4} + \frac{\omega_M}{2^{j+1}} \right) = \frac{\omega_M}{4} + \frac{\omega_M}{2^{j+1}}.$$
Consider a cut-off function \( \eta(x, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}) \) such that

\[
\begin{align*}
0 \leq \eta & \leq 1 \quad \text{in } Q(R_i, \theta^{-\alpha_0}R_i^2) \\
\eta & = 1 \quad \text{in } Q(R_{i+1}, \theta^{-\alpha_0}R_{i+1}^2) \\
\eta & = 0 \quad \text{on } \{B_{R_i} \times \{t = -\theta^{-\alpha_0}R_i^2\} \cup \{\partial B_{R_i} \times [-\theta^{-\alpha_0}R_i^2, 0]\}\}
\end{align*}
\]

\( |\nabla \eta| \leq \frac{2^{j+1}R_j}{R}, \quad |(\eta,j)| \leq \frac{2^{2(j+1)}\theta^{\alpha_0}}{R^2} \) in \( Q(R_i, \theta^{-\alpha_0}R_i^2) \).

Let \( u = u^j, \beta_j = \beta \) and \( \lambda_j = \lambda \) for the convenience. We first take \( \varphi = (u - l_j) \cdot \eta_j^2 \) as a test function in the weak formulation (1.6) and integrating it over \( (-\theta^{-\alpha_0}R_i^2, t) \) for \( t \in (-\theta^{-\alpha_0}R_i^2, 0) \). Then, we have

\[
0 = \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} u_i[(u - l_j) \cdot \eta_j^2] \, dx \, dr + m \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} U^{-1} \mathcal{A}(\nabla u, u, x, t) \nabla [(u - l_j) \cdot \eta_j^2] \, dx \, dr \\
+ \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} B(u, x, t) \nabla [(u - l_j) \cdot \eta_j^2] \, dx \, dr \\
= I + II + III.
\]

Observe that

\[
u \leq \frac{\omega_M}{2} = \theta \quad \text{on } \{u \leq l_j\}, \quad (u - l_j) \cdot \frac{\omega_M}{2} = \theta.
\]

By (4.9), we have

\[
-I = \frac{1}{2} \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} [(u - l_j)^2] \eta_j^2 \, dx \, dr \\
= \frac{1}{2} \int_{B_{R_i} \times [t]} (u - l_j)^2 \eta_j^2 \, dx - \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} (u - l_j)^2 \eta_j^2 \, dx \, dr \\
\geq \frac{1}{2} \int_{B_{R_i} \times [t]} (u - l_j)^2 \eta_j^2 \, dx - \frac{2^{2(j+1)}\theta^{\alpha_0}(2\theta)^2}{R^2} \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} \chi_{(u \leq l_j)} \, dx \, dr.
\]

By (A5), (A6), (4.9), (4.14), and Young’s inequality

\[
-II \geq m \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} \eta_j^2 U^{-1} \mathcal{A}((\nabla u, u, x, t), \nabla u) \chi_{(u \leq l_j)} \, dx \, dr \\
- 2m \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} \eta_j U^{-1}(u - l_j) \cdot |\mathcal{A}(\nabla u, u, x, t)| \nabla \eta_j \, dx \, dr \\
\geq mc \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} U^{-1} |\nabla (u - l_j)| \eta_j^2 \, dx \, dr - \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} U^{-1} \eta_j^2 \, dx \, dr \\
- 2mC_4 \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} \eta_j U^{-1}(u - l_j) \cdot |\nabla u| \nabla \eta_j \, dx \, dr - 2mC_4 \int_{-\theta^{-\alpha_0}R_i^2}^{t} \int_{B_{R_i}} \eta_j U^{-1}(u - l_j) \cdot u \nabla \eta_j \, dx \, dr
\]

Observe that

\[
u \leq \frac{\omega_M}{2} = \theta \quad \text{on } \{u \leq l_j\}, \quad (u - l_j) \cdot \frac{\omega_M}{2} = \theta.
\]
\[ t \geq \frac{mc}{2} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} U_{m-1} |\nabla(u - l_j)|^2 \eta_j^2 \, dx \, dt - mC_2 \Lambda^{m-1}(2\theta)^2 \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt \]

\[ - mC_2 \Lambda^{m-1}(2\theta)^2 \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt. \]

By (A7), (1.14), and (4.9), we can obtain

\[ III \leq \frac{mc}{4} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} U_{m-1} |\nabla(u - l_j)|^2 \eta_j^2 \, dx \, dt + \frac{1}{mc} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt \]

\[ \leq C_0 \theta^2 \left[ \frac{2^{2j+1} \Lambda^{m-1}}{R^2} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt + \frac{\Lambda^{m-1} + \Lambda^{2x-1}}{\theta^{\alpha_0}} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt \right] \]

(4.12)

where \( \theta = \min_{t \leq t < 0} R \). By (4.8), (4.10), (4.11), and (4.12), there exists a constant \( C_0 \) depending on \( m, \lambda, \beta, C_2, C_3, C_4, \) and \( C_5 \) such that

\[ \sup_{-\theta - \omega R^2 < t < 0} \int_{B_{B_R} \times [t]} (u - l_j)^2 \eta_j^2 \, dx \, dt \]

\[ \leq C_0 \theta^2 \left[ \frac{2^{2j+1} \Lambda^{m-1}}{R^2} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt + \frac{\Lambda^{m-1} + \Lambda^{2x-1}}{\theta^{\alpha_0}} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_l|} \, dx \, dt \right] \]

(4.13)

since

\[ 4\theta = \omega M \leq \left( \frac{\lambda}{\Lambda} \right)^{1/2} \Rightarrow \theta^{\alpha_0} \leq \frac{1}{4^{\alpha_0} \Lambda^{m-1}}. \]

To control the diffusion coefficient \( U_{m-1} \), we consider the function \( u_\omega = \max\{u, \frac{\omega M}{\theta} \} \) which is introduced in [17]. Then,

\[ \chi_{|u_l|} = \chi_{|u_\omega|} \] and \( \theta = \frac{\omega M}{4} \leq u \leq \left( \frac{\omega M}{4} \right)^{1/2} \text{ on } \{|\nabla(u_\omega - l_j)| = 0\}. \]

(4.14)

Thus, by (4.13) and (4.14) we can have

\[ \sup_{-\theta - \omega R^2 < t < 0} \int_{B_{B_R} \times [t]} (u_\omega - l_j)^2 \eta_j^2 \, dx \, dt + \frac{1}{\theta^{\alpha_0}} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_\omega|} \, dx \, dt \]

\[ \leq C_1 \theta^2 \left[ \frac{2^{2j+1} \Lambda^{m-1}}{R^2} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_\omega|} \, dx \, dt + \frac{\Lambda^{m-1} + \Lambda^{2x-1}}{\theta^{\alpha_0}} \int_{-\theta - \omega R^2}^t \int_{B_{B_R}} \chi_{|u_\omega|} \, dx \, dt \right] \]

(4.15)

for some constant \( C_1 > 0 \). To control the last term in (4.15), let \( q_1, q_2 \geq 1, \) and \( 0 < \kappa_1 < 1 \) be constants satisfying

\[ \frac{n}{2q_1} + \frac{1}{q_2} = 1 - \kappa_1. \]
Let 
\[ \tilde{q} = \frac{2q(1 + \kappa)}{q_1 - 1}, \quad \tilde{r} = \frac{2q(1 + \kappa)}{q_2 - 1}, \quad \text{and} \quad \kappa = \frac{2}{n} \kappa_1. \] (4.17)

Then, by Hölder inequality and conditions on \( R \) and \( \theta \), there exists a constant \( C_2 > 0 \) such that
\[
\int_{-\theta - a_0 R_0^2}^{\theta - a_0 R_0^2} \int_{|x| \leq l} |A_{i,j}(r)|^\frac{p}{q} dr \leq \frac{R^2}{\theta a_0^2} \left( \int_{-\theta - a_0 R_0^2}^{\theta - a_0 R_0^2} \int_{|x| \leq l} |A_{i,j}(r)|^\frac{p}{q} dr \right)^{\frac{p}{q}}
\]
\[
\leq C_2 \left( \int_{-\theta - a_0 R_0^2}^{\theta - a_0 R_0^2} \int_{|x| \leq l} |A_{i,j}(r)|^\frac{p}{q} dr \right),
\]
where \( A_{i,j}(t) = \{ x \in B_t : (u - l)_+ \geq 0 \} \). By (4.15) and (4.18), we have
\[
\sup_{-\theta - a_0 R_0^2 < t < 0} \int_{B_{R_0} \times \{ t \}} (u_\omega - l_j)^2 \eta^2 \, dx + \theta a_0 \int_{-\theta - a_0 R_0^2}^{\theta - a_0 R_0^2} \int_{B_{R_0}} |\nabla (u_\omega - l)_- \eta|^2 \, dx \, dt
\leq C_3 \theta^2 \left[ \frac{2^{2j+1} \Lambda^{m-1}}{R^2} \int_{-\theta - a_0 R_0^2}^{\theta - a_0 R_0^2} \int_{B_{R_0}} \chi_{|x| \leq l_j} \, dx \, dt + \left( \frac{\Lambda^{2(m-1)} + \frac{2^{2j+1}}{\theta a_0^2}}{\theta a_0} \right) \left( \int_{-\theta - a_0 R_0^2}^{\theta - a_0 R_0^2} \int_{|x| \leq l} |A_{i,j}(r)|^\frac{p}{q} \, dr \right) \right]^{\frac{2}{1+k}}
\]
(4.19)
for some constant \( C_3 \) depending on \( C_1 \) and \( C_2 \). We now take the change of variables
\[ z = \theta a_0 t \]
(4.20)
and set the new functions 
\[ \eta_\omega(z, \hat{\eta}_\omega(z)) = u_\omega(\cdot, \theta^{-a_0 z}) \quad \text{and} \quad \eta_\eta(z, \hat{\eta}_\eta(z)) = \eta_\omega(\cdot, \theta^{-a_0 z}). \]

Then, by (4.19) and (4.20) we have
\[
\sup_{-R_0^2 < z < 0} \int_{B_{R_0} \times \{ z \}} (\eta_\omega - l_j)^2 \hat{\eta}_\omega^2 \, dx + \int_{-R_0^2}^{0} \int_{B_{R_0}} |\nabla (\eta_\omega - l)_- \hat{\eta}|^2 \, dx \, dz
\leq C_3 \theta^2 \left[ \frac{2^{2j+1} \Lambda^{m-1}}{R^2} \left( \frac{\Lambda^{2(m-1)} + \frac{2^{2j+1}}{\theta a_0^2}}{\theta a_0} \right) \theta^{-a_0 (2^{2j+1})} \left( \int_{-R_0^2}^{0} |A_j(z)|^\frac{p}{q} \, dz \right) \right]^{\frac{2}{1+k}}
\]
(4.21)
where
\[ A_j = \int_{-R_0^2}^{0} \int_{|x| \leq l_j} \chi_{|x| \leq l} \, dx \, dz \quad \text{and} \quad A_j(z) = \{ x \in B_{R_0} : \eta_\omega(x, z) < l_j \}. \]

By Lemma 4.1 and (4.21),
\[ \| (\pi_{\omega} - l_j)^2 \mathbf{u}_{j}^2 \|_{L^2(Q(R_j, R_j'))} \leq C_4 \theta^2 \left[ \frac{\Lambda^2}{R_j^2} \left( \frac{\Lambda}{\theta^p} \right)^{m-1} A_j + \left( \Lambda^{2(m-1)} + \Lambda^2 \right) \theta^{-a_0(2-\frac{m}{p})} \left( \int_{-R_j}^{0} \left( \mathcal{A} \right) \frac{\mathcal{F}^2}{|A_j(z)|} \, dz \right) \right] \frac{2^{(1+\kappa)}}{A_j^{1+\kappa}} \]  

for some constant \( C_4 > 0 \). This immediately implies

\[ A_{j+1} \leq C_4 2^{q+1} \left[ \frac{\Lambda}{\theta^p} \right]^{m-1} \left( \frac{\Lambda}{\theta^p} \right) A_j + \left( \Lambda^{2(m-1)} + \Lambda^2 \right) \theta^{-a_0(2-\frac{m}{p})} \left( \int_{-R_j}^{0} \left( \mathcal{A} \right) \frac{\mathcal{F}^2}{|A_j(z)|} \, dz \right) \right] \frac{2^{(1+\kappa)}}{A_j^{1+\kappa}} \]  

since

\[ \| (\pi_{\omega} - l_j)^2 \mathbf{u}_{j}^2 \|_{L^2(Q(R_j, R_j'))} \geq (l_{j+1} - l_j)^2 \int_{-R_j}^{0} \| (x, t) \in B_{R_j} : \pi_{\omega} \leq l_{j+1} \| \, dt = \left( \frac{\theta}{2} \right)^2 A_{j+1}. \]

Choose the number \( \varepsilon > 0 \) sufficiently small that

\[ \varepsilon < \frac{nq}{2q_2 - 1}. \]  

Then, by (4.23) there exists a constant \( C_j > 0 \) depending on \( \frac{\Lambda}{\theta^p} \) such that

\[ X_{j+1} \leq C_3 16 \left( X_j^{1+\kappa} + \theta^{-a_0(2-\frac{m}{p})} \theta^{1+\kappa} \right) Y_j^{1+\kappa} \leq C_4 16 \left( X_j^{1+\kappa} + Y_j^{1+\kappa} \right) \]  

where

\[ X_j = \frac{A_j}{|Q(R_j, R_j')|} \quad \text{and} \quad Y_j = \frac{1}{|B_{R_j}|} \left( \int_{-R_j}^{0} \left( \mathcal{A} \right) \frac{\mathcal{F}^2}{|A_j(z)|} \, dz \right)^{1/2}. \]

By an argument similar to the one presented in the proof of Lemma 3.5 of [19], we also have

\[ Y_{j+1} \leq C_6 \left( X_j + Y_j^{1+\kappa} \right) \quad \forall j \in \mathbb{N} \]  

for some constant \( C_6 > 0 \). By (4.25) and (4.26), there exists a constant \( C_7 > 0 \) such that

\[ L_{j+1} \leq C_7 16^{(1+\kappa)} L_j^{1+\kappa} \quad \forall j \in \mathbb{N}, \]

where \( L_j = X_j + Y_j^{1+\kappa} \) and \( \kappa = \min \left\{ \kappa, \frac{2}{p+2} \right\} \). If we take the constant \( \rho_0 > 0 \) in (4.6) sufficiently small that

\[ L_0 \leq C_7 \left( \frac{1+\kappa}{16^{\frac{1+\kappa}{p}}} \right) \]

holds, then

\[ L_j \leq C_7 \left( \frac{(1+\kappa)}{16^{\frac{(1+\kappa)}{2}}} \right)^i \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty \]

and the lemma follows. \( \Box \)

**Remark 4.3.** Let the constants \( \Lambda, \theta \) be given by (1.3). If \( U \) is equivalent to \( \lambda(u)^{\beta} \), i.e., there exists some constants \( 0 < c \leq C < \infty \) such that

\[ c \lambda(u)^{\beta} \leq U \leq C \lambda(u)^{\beta} \quad \text{in} \quad Q(R, \theta^{-a_0} R^2), \]

then the constant \( \rho_0 \) in (4.6) is independent of \( U \) and \( \omega^1, \ldots, \omega^k \).
4.2 The second alternative

At this moment, we have shown that if the measure of the set
\[ \{(x, t) \in Q(R, \theta^{-a_0}R^2) : u'(x, t) < \frac{\omega_M}{2} \} \]
is very small, then the component \( u' \) is strictly bounded below away from zero (essential infimum) in a smaller cylinder of \( Q(R, \theta^{-a_0}R^2) \). That is, their essential oscillation will decrease. We now need to get rid of assumption “very small.”

Suppose that the assumption of Lemma 4.2 does not hold, i.e., for every sub-cylinder \( Q(R, \theta^{-a_0}R^2) \), \( \theta R^2 \in (0, \theta^2) \),
\[ \left| \left\{(x, t) \in Q(R, \theta^{-a_0}R^2) : u'(x, t) < \frac{\omega_M}{2} \right\} \right| > \rho_0|Q(R, \theta^{-a_0}R^2)|. \]

Then
\[ \left| \left\{(x, t) \in Q(R, \theta^{-a_0}R^2) : u'(x, t) > \frac{\omega_M}{2} \right\} \right| \leq (1 - \rho_0)|Q(R, \theta^{-a_0}R^2)| \quad (4.27) \]
is valid for all cylinders $Q(R, \theta^{-a_0}R^2) \subset Q(R, R^{2-\epsilon})$.

In the second alternative, we are going to show that the essential oscillation of $u'$ decreases in a smaller cylinder by showing that the essential supremum of $u'$ decreases. We start this alternative by stating the following lemma.

Lemma 4.4. If (4.6) is violated, then there exists a time level
\[ t^* \in \left[-\theta^{-a_0}R^2, -\frac{\rho_0}{2}\theta^{-a_0}R^2 \right] \]
such that
\[ |\mathcal{A}_0| = \left| \left\{ x \in B_R : u'(x, t^*) > \frac{\omega_M}{2} \right\} \right| \leq \left( \frac{1 - \rho_0}{1 - \frac{\rho_0}{2}} \right)|B_R|. \]

Proof. Suppose not. Then
\[
\left| \left\{ (x, t) \in Q(R, \theta^{-a_0}R^2) : u'(x, t) > \frac{\omega_M}{2} \right\} \right| \geq \int_{-\theta^{-a_0}R^2}^{rac{\rho_0}{2}\theta^{-a_0}R^2} \left| \left\{ x \in B_R : u'(x, t) > \frac{\omega_M}{2} \right\} \right| dt \\
> \left( 1 - \frac{\rho_0}{1 - \frac{\rho_0}{2}} \right)|B_R| \left( 1 - \frac{\rho_0}{2} \right) \theta^{-a_0}R^2 \\
= (1 - \rho_0)|Q(R, \theta^{-a_0}R^2)|,
\]
which contradicts (4.27). \qed

By Lemma 4.4, there exists a time $t^* < 0$ such that the region $\mathcal{A}_0$ takes a portion of the ball $B_R$. The next lemma shows that this occurs for all $t \geq t^*$.

Lemma 4.5. There exists a positive integer $s_1 > 1$ depending on $\frac{A}{\rho_0}$ such that
\[ \left| \left\{ x \in B_R : u'(x, t) > \left( 1 - \frac{1}{2^n} \right) \omega_M \right\} \right| < \left( 1 - \left( \frac{\rho_0}{2} \right)^2 \right)|B_R|, \quad \forall t \in [t^*, 0]. \quad (4.28) \]
Proof. We will use a modification of the proof of Lemma 3.7 of [19] to prove the lemma. Let \( u = u', \beta_i = \beta, \gamma_i = \gamma \) for the convenience and let

\[
H = \sup_{B_{Rt}(t^*, 0)} \left( u - \frac{\omega_M}{2} \right) \leq \frac{\omega_M}{2}
\]

and assume that there exists a constant \( 1 < s_2 \in \mathbb{N} \) such that

\[
0 < \frac{\omega_M}{2^{s_2+1}} < H.
\]

If there is no such integer \( s_2 \), (4.28) holds for any \( s_1 > 1 \) and the lemma follows.

We now introduce the logarithmic function which appears in Section 2 of [16] by

\[
\Psi(H, (u - k)_+, c) = \max\left\{ 0, \log\left( \frac{H}{H - (u - k)_+ + c} \right) \right\}
\]

for \( k = \frac{\omega_M}{2} \) and \( c = \frac{\omega_M}{2^{s_2+1}} \). Note that

\[
\Psi(H, (u - k)_+, c) = 0 \quad \text{if} \quad u \leq k = \frac{\omega_M}{2}.
\]

(4.29)

For simplicity, let \( \psi(u) = \Psi(H, (u - k)_+, c) \). Then \( \psi \) satisfies

\[
\psi \leq s_2 \log 2, \quad 0 \leq (\psi)' \leq \frac{2s_2 + 1}{\omega_M}, \quad \text{and} \quad \psi'' = (\psi')^2 \geq 0.
\]

(4.30)

Set

\[
\varphi = (\psi^2(u))^{\xi^2}
\]

and take it as a test function in (1.6) where \( \xi(x) \geq 0 \) is a smooth cut-off function such that

\[
\xi = 1 \quad \text{in} \quad B_{(1-v)R}, \quad \xi = 0 \quad \text{on} \quad \partial B_R, \quad \text{and} \quad |\nabla \xi| \leq \frac{C}{\sqrt{R}}
\]

(4.31)

for some constants \( 0 < v < 1 \) and \( C > 0 \). Then integrating (1.6) over \((t^*, t)\) for all \( t \in (t^*, 0) \), we have

\[
0 = \int_{t^*}^t \int_{B_R} (\psi^2(u)\xi^2) \delta_1 dxd\tau + m \int_{t^*}^t \int_{B_R} U^{m-1}(\Delta u, u, x, t) \cdot \nabla((\psi^2(u))^2) dxd\tau
\]

\[
+ \int_{t^*}^t \int_{B_R} \delta_2(u, x, t) \cdot \nabla((\psi^2(u))^2) dxd\tau
\]

\[
= I + II + III
\]

(4.32)

Then we have

\[
I = \int_{B_{R*}(t)} \psi^2(u)\xi^2 dx - \int_{B_{R*}(t^*)} \psi^2(u)\xi^2 dx
\]

(4.33)

and

\[
II \geq 2cm \int_{t^*}^t \int_{B_R} U^{m-1}(1 + \psi)(\psi')^2 \xi^2 |\nabla u|^2 dxd\tau - 2cm \int_{t^*}^t \int_{B_R} U^{m-1}(1 + \psi)(\psi')^2 \xi^2 u^2 dxd\tau
\]

\[
- \left( \frac{C^2}{c} + C_m \right) \int_{t^*}^t \int_{B_R} U^{m-1}\psi |\nabla \xi|^2 dxd\tau - C_m \int_{t^*}^t \int_{B_R} U^{m-1}\psi |\psi'|^2 |\xi^2 u|^2 dxd\tau
\]

(4.34)

and
\[ -III = 2C_t \int_{t'}^t \int_{B_R} u^2(1 + \psi)(\psi')^2 \xi^2 |\nabla u| \, dx \, dt' + 4C_t \int_{t'}^t \int_{B_R} u^2 \psi \psi' |\nabla \xi| \, dx \, dt' \leq cm\Lambda^{m-1} \int_{t'}^t \int_{B_R} (1 + \psi)(\psi')^2 |\nabla u|^2 \, dx \, dt' + \frac{2C_2}{cm\Lambda^{m-1}} \int_{t'}^t \int_{B_R} u^2(1 + \psi)(\psi')^2 \xi^2 \, dx \, dt' \] (4.35)

\[ + 4cm\Lambda^{m-1} \int_{t'}^t \int_{B_R} \psi |\nabla \xi|^2 \, dx \, dt'. \]

By (4.30), (4.31), (4.32), (4.33), (4.34), (4.35), and Lemma 4.4,

\[ \int_{B_{u^2(t)}} \psi^2 (u) \xi^2 \, dx \leq \left[ \frac{s_2^2}{(\log 2)^2} \left( \frac{1 - \rho_0}{1 - \rho_0} \right) + C \left( \frac{s_1 \log 2}{n^2} \left( \frac{\Lambda}{\theta^2} \right)^{m-1} + \left( \Lambda^{m-1} + \Lambda^{\frac{m-1}{4}} \right) 4^{s_1+1} R^{2-s_2} \log 2 \right) \right] |B_t| \] (4.36)

holds for all \( t \in (t', 0) \) with some constant \( C > 0 \) depending on \( m, q, \lambda, \beta, c, C_3, C_4, \) and \( C_5 \). Let

\[ S = \left\{ x \in B_{(1-v)R} : u(x, t) > \left( 1 - \frac{1}{2^{s_1+1}} \right) \omega_M \right\}. \]

Then, the left hand side of (4.36) is bounded below by

\[ \int_{B_{u^2(t)}} \psi^2 (u) \xi^2 \, dx \geq \int_S \psi^2 (u) \xi^2 \, dx \geq (s_2 - 1)^2 (\log 2)^2 |S| \quad \forall t \in (t', 0). \] (4.37)

On the other hand,

\[ \left| \left\{ x \in B_R : u(x, t) > \left( 1 - \frac{1}{2^{s_1+1}} \right) \omega_M \right\} \right| \leq |S| + Nv |B_t|. \] (4.38)

By (4.36), (4.37), and (4.38),

\[ \left| \left\{ x \in B_R : u(x, t) > \left( 1 - \frac{1}{2^{s_1+1}} \right) \omega_M \right\} \right| \leq \left[ \left( \frac{s_2 - 1}{s_2} \right)^2 \left( \frac{1 - \rho_0}{1 - \rho_0} \right) + n v + C \left( \frac{s_2}{\nu^2 (s_2 - 1)^2 \log 2} \left( \frac{\Lambda}{\theta^2} \right)^{m-1} + \left( \Lambda^{m-1} + \Lambda^{\frac{m-1}{4}} \right) 4^{s_1+1} R^{2-s_2} \log 2 \right) \right] |B_t|. \]

To complete the proof, we choose \( v \) so small that \( n v \leq \frac{3}{8} \rho_0^2 \) and then \( s_2 \) so large that

\[ \left( \frac{s_2}{s_2 - 1} \right)^2 \leq \left( 1 - \frac{1}{2} \rho_0 \right) (1 + \rho_0) \quad \text{and} \quad C \frac{s_2}{\nu^2 (s_2 - 1)^2 \log 2} \left( \frac{\Lambda}{\theta^2} \right)^{m-1} \leq \frac{1}{4} \rho_0^2. \]

With such \( v \) and \( s_2 \), we choose the radius \( R \) sufficiently small that

\[ \frac{C \left( \Lambda^{m-1} + \Lambda^{\frac{m-1}{4}} \right) 4^{s_1+1} R^{2-s_2} \log 2}{(s_2 - 1)^2 \log 2} \leq \frac{3}{8} \rho_0^2. \] (4.39)

Then (4.28) holds for \( s_1 = s_2 + 1 \) and the lemma follows. \( \square \)

Since \( t' \in \left[ -\theta^{-q_0} R^2, -\frac{\rho_0}{2} \theta^{-q_0} R^2 \right] \), the previous lemma implies the following result.
Corollary 4.6. There exists a positive integer \( s_1 > s_0 \) such that for all \( t \in \left( -\frac{\rho_0}{2} \theta^{-\alpha_0} R^2, 0 \right) \)

\[
\left\{ x \in B_R : u'(x, t) > \left( 1 - \frac{1}{2^{k_1}} \right) \omega_M \right\} < \left( 1 - \left( \frac{\rho_0}{2} \right)^2 \right) B_R. \tag{4.40}
\]

To control the measure of the region where \( u' \) is close to the value \( \omega_M \), we are going to use the following De Giorgi’s isoperimetric inequality.

Lemma 4.7. (De Giorgi [14]) If \( f \in W^{1, 1}(B) \) \((B \subset \mathbb{R}^n)\) and \( l_1, l_2 \in \mathbb{R}, l_1 < l_2, \) then

\[
(l_2 - l_1) \int_{|x| \in [l_1, l_2]} |f(x)| \, dx \leq C_{n+1} \int_{B} \int_{|x| < l_1, l_2} |\nabla f(x)| \, dx,
\]

where \( C \) depends only on \( n \).

By Corollary 4.6 and Lemma 4.7, we have the following lemma which controls the measure of upper level sets.

Lemma 4.8. If \((4.6)\) is violated, for every \( \nu \in (0, 1) \) there exists a natural number \( s^* > s_1 > 1 \) depending on \( \frac{\Lambda}{\theta} \) such that

\[
\left\{ (x, t) \in Q(R, \frac{\rho_0}{2} \theta^{-\alpha_0} R^2) : u'(x, t) > \left( 1 - \frac{1}{2^{k_1}} \right) \omega_M \right\} \leq \nu \int_{|x| > R} Q(R, \frac{\rho_0}{2} \theta^{-\alpha_0} R^2). \tag{4.41}
\]

Proof. We will use a modification of the proof of Lemma 8.1 of Section III of [16] to prove the lemma. Let \( l_1 = \left( 1 - \frac{1}{2} \right) \omega_M \) and \( l_2 = \left( 1 - \frac{1}{2^{k_1}} \right) \omega_M \) for \( s \geq s_1 \) and let \( \eta(x, t) \in C_{\infty}(Q(2R, \rho_0 \theta^{-\alpha_0} R^2)) \) be a cut-off function such that

\[
\begin{align*}
0 & \leq \eta \leq 1 & \text{in } Q(2R, \rho_0 \theta^{-\alpha_0} R^2) \\
\eta & = 1 & \text{in } Q(R, \rho_0 \theta^{-\alpha_0} R^2) \\
\eta & = 0 & \text{on the parabolic boundary of } Q(2R, \rho_0 \theta^{-\alpha_0} R^2) \\
|\nabla \eta| & \leq \frac{1}{R^2} & |\eta_t| \leq \frac{2 \theta^{\alpha_0}}{\rho_0 R^2}.
\end{align*}
\]

Let \( u = u', \lambda = \lambda_t, \beta = \beta_t \) for the convenience and put \( \varphi = (u_t - k), \xi^2 \) in the weak formulation (1.7). Integrate it over \((-\rho_0 \theta^{-\alpha_0} R^2, t)\) for \( t \in (-\rho_0 \theta^{-\alpha_0} R^2, 0) \) and take the limit as \( h \to 0 \). Then, by an argument similar to the proof of energy-type inequality (4.19) there exists a constant \( C > 0 \) depending on \( m, \lambda, \) and \( A \) such that

\[
\int_{-\rho_0 \theta^{-\alpha_0} R^2}^{t} \int_{B_{\rho}} |\nabla (u') - l_1|^2 \, dx \, d\tau \leq C \left( \frac{\omega}{2^k} \right)^2 \frac{1}{\rho_0 R^2} \left( 1 + \left( \frac{\Lambda}{\theta} \right)^{m-1} + 2^{-1} R^{m-1} e^{(2 - \frac{1}{k_1})} \right) \int_{Q(R, \rho_0 \theta^{-\alpha_0} R^2)} \left( \frac{\rho_0}{2} \theta^{-\alpha_0} R^2 \right). \tag{4.42}
\]

where constants \( \kappa, q_2 \) are given by (4.16) and (4.17). Let

\[
A_{s}(t) = \left\{ x \in B_R : u(x, t) > \left( 1 - \frac{1}{2^{k_1}} \right) \omega \right\}, \quad \forall t \in \left( -\frac{\rho_0}{2} \theta^{-\alpha_0} R^2, 0 \right)
\]

and

\[
A_{s} = \int_{-\frac{\rho_0}{2} \theta^{-\alpha_0} R^2}^{0} |A_{s}(t)| \, dt.
\]
Then, by Corollary 4.6, Lemma 4.7, and (4.42) we have

\[
\left( \frac{\omega_M}{2^{s+1}} \right) |A_{s_1}(t)| \\
\leq \frac{CR}{\rho_0^2} \int_0^1 |\nabla u| dx \\
\quad \forall s = s_1, \ldots, s^* - 1
\]

\[
\Rightarrow \left( \frac{\omega_M}{2^{s+1}} \right) A_{s_1} \leq \frac{CR}{\rho_0^2} \left( \int_0^1 |\nabla (u - l_i)|^2 dx dt \right)^{1/2} |A_s \setminus A_{s_1}|^{1/2}
\]

\[
\Rightarrow A_{s_1}^2 \leq \frac{C}{\rho_0^2} \left( 1 + \left( \frac{\Lambda}{\psi^2} \right)^{m-1} + 2\epsilon R^{m-1} \right) Q(R, \frac{\rho_0}{2} \psi^{-1} R^2) |A_s \setminus A_{s_1}|
\]

\[
\Rightarrow (s^* - s_1)A_{s_1}^2 \leq \sum_{s=s_1}^{s^*-1} A_{s_1}^2 \leq \frac{C}{\rho_0^2} \left( 1 + \left( \frac{\Lambda}{\psi^2} \right)^{m-1} + 2\epsilon R^{m-1} \right) Q(R, \frac{\rho_0}{2} \psi^{-1} R^2) \sum_{s=s_1}^{s^*-1} |A_s \setminus A_{s_1}|
\]

\[
\Rightarrow A_{s_1}^2 \leq \frac{C}{\rho_0^2} (s^* - s_1) \left( 1 + \left( \frac{\Lambda}{\psi^2} \right)^{m-1} + 2\epsilon R^{m-1} \right) Q(R, \frac{\rho_0}{2} \psi^{-1} R^2)
\]

Thus, if we choose \( s^* \in \mathbb{N} \) sufficiently large that

\[
\frac{C}{\rho_0^2} (s^* - s_1) \left( 2 + \left( \frac{\Lambda}{\psi^2} \right)^{m-1} \right) \leq \nu_i^2
\]

and then \( R \) sufficiently small that

\[
2\epsilon R^{m-1} \leq 1,
\]

then (4.41) holds and the lemma follows.

**Remark 4.9.** Let the constants \( \lambda_i, \beta_i \) be given by (1.3). If \( U \) is equivalent to \( \lambda_i(u)^{\beta_i} \), i.e., there exists some constants \( 0 < c \leq C < \infty \) such that

\[
c\lambda_i(u)^{\beta_i} \leq U \leq C\lambda_i(u)^{\beta_i} \quad \text{in } Q(R, \theta^{-\alpha_0} R^2),
\]

then the constant \( s^* \) is independent of \( U \) and \( \omega^i, \ldots, \omega^k \).

By Lemma 4.8, we have a similar assumption to the one in Lemma 4.2 for sufficiently small number \( \nu_i > 0 \). Therefore, by an argument similar to the proof of Lemma 4.2, we can have the following result.

**Lemma 4.10.** The number \( \nu_i \in (0, 1) \) can be chosen (and hence \( s^* \)) such that

\[
u_i \leq \left( 1 - \frac{1}{2^{s+1}} \right) \omega_M \quad \text{a.e. on } Q \left( R, \frac{\rho_0}{2} \psi^{-1} R^2 \right).
\]

### 4.3 Local continuity

By Lemmas 4.2 and 4.10, we have the following Oscillation lemma.

**Lemma 4.11.** (Oscillation lemma) Let \( 1 \leq i \leq k \). There exist numbers \( \rho_0, \sigma_0 \in (0, 1) \) depending on \( \frac{\Lambda}{\psi^2} \) such that if

\[
\operatorname{osc}_{Q(R, \theta^{-\alpha_0} R^2)} u^i \leq \omega_M,
\]
then we have

\[
\left( \frac{R_{n+1}}{2} \right)^{\alpha_n} \leq \sigma \omega_{n+1}.
\]

**Proof of Theorem 1.9.** By Lemma 4.11, a family of nest and shrinking cylinders \( \{Q_{n,i}\}_{i=1}^{\infty} \), whose radius is \( R_n \), and a decreasing sequence \( \{\omega_{n+1}\}_{i=1}^{\infty} \) can be constructed recursively such that

\[
\frac{R_{n+1}}{R_n} < c, \quad \forall n \in \mathbb{N},
\]

for some constant \( 0 < c < 1 \) and

\[
\text{osc} u^i_{\partial_n} \leq \omega_n, \quad \forall n \in \mathbb{N},
\]

and

\[
\omega_{n+1} \leq \sigma(\omega_n)\omega_n
\]

for some decreasing function \( \sigma : (0, 1) \to (0, 1) \). Thus, by an argument similar to the one presented in the proof of Theorem 2 in Section 7 of [21], we have

\[
\omega_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Therefore, the local continuity of \( u^i \) holds and the theorem follows. \( \square \)

**Remark 4.12.** Since the constant \( \sigma_0 \) in (4.44) depends on the ratio \( \frac{\Lambda}{\rho_i} \) at each step of iteration, we cannot find the modulus of continuity at this stage. We refer the reader to the paper [21] for the details on the local continuity of the PDEs.

**Remark 4.13.** Let the constants \( \lambda_i, \beta_i \) be given by (1.3). If \( U \) is equivalent to \( \lambda_i(u^i)^{\beta_i} \) in \( \mathbb{R}^n \times (0, \infty) \), i.e., there exist some constants \( 0 < c \leq C < \infty \) such that

\[
c\lambda_i(u^i)^{\beta_i} \leq U \leq C\lambda_i(u^i)^{\beta_i} \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),
\]

then each components of the solution \( u \) is locally Hölder continuous in \( \mathbb{R}^n \times (0, \infty) \). Moreover, all components have the same modulus of continuity. We refer the reader to the paper [1,2] for the details.

**Funding information:** Ki-Ahm Lee was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1701-03. Ki-Ahm Lee also holds a joint appointment with Research Institute of Mathematics of Seoul National University. Sunghoon Kim was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1F1A1A01048334). Sunghoon Kim was also supported by the Research Fund, 2021 of The Catholic University of Korea.

**Conflict of interest:** The authors state no conflict of interest.

**References**

[1] S. Kim and K-A. Lee, *Local continuity and asymptotic behaviour of degenerate parabolic systems*, Nonlinear Anal. 192 (2020), 111702.

[2] S. Kim and K-A. Lee, *System of porous medium equations*, J. Differ. Equ. 272 (2021), 433–472.

[3] T. Kuusi, L. Monsaingeon, and J. Videman, *Systems of partial differential equations in porous medium*, Nonlinear Anal. 133 (2016), 79–101.
4. T. A. Sanikidze and A. F. Tedeev, *On the temporal decay estimates for the degenerate parabolic system*, Commun. Pure Appl. Anal. 12 (2013), no. 4, 1755–1768.

5. H. M. Yin, *On a degenerate parabolic system*, J. Differ. Equ. 245 (2008), 722–736.

6. K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. 138 (1977), no. 3–4, 219–240.

7. V. Iwaniec, *Projections onto gradient fields and L^p-estimates for degenerated elliptic operators*, Studia Math. 75 (1983), no. 3, 293–312.

8. A. S. Kalashnikov, *The heat equation in a medium with nonuniformly distributed nonlinear heatsources or absorbers* (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1983, no. 3, 20–24.

9. O. Ladyzhenskaya, *New equations for the description of the viscous incompressible fluids and global solvability in the range of the boundary value problems to these equations*, Trudy Steklovas Math. Inst. 102 (1967), 85–104.

10. O. Ladyzhenskaya, *Modifications of the Navier-Stokes equations for large gradients of velocity*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 7 (1968), 126–154.

11. E. DiBenedetto, U. Gianazza, and V. Vespri, *Harnack’s inequality for degenerate and singular parabolic equations*, Springer Monographs in Mathematics, Springer, New York, p. xiv+278.

12. E. DiBenedetto, U. Gianazza, and V. Vespri, *Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 9 (2010), no. 2, 385–422.

13. A. Z. Fino, F. G. Düzgün, and V. Vespri, *Conservation of the mass for solutions to a class of singular parabolic equations*, Kodai Math. J. 37 (2014), no. 3, 519–531.

14. E. De Giorgi, *Sulla differenziabilitá e la analiticitá delle estremali degli integrali multipli regolari*, Mem. Acc. Sci. Torino, Cl. Sci. Fis. Mat. Natur. 3 (1957), 25–34.

15. J. Moser, *A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. 13 (1960), 457–468.

16. E. DiBenedetto, *Degenerate Parabolic Equations*, Univertext, Springer-Verlag, New York, 1993, p. xvi, ISBN: 0-387-94020-0.

17. E. Henriques and J. M. Urbano, *Intrinsic scaling for PDE’s with an exponential nonlinearity*, Indiana Univ. Math. J. 55 (2006), no. 5, 1701–1721.

18. L. A. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. Math. 171 (2010), no. 3, 1903–1930.

19. S. Kim and K.-A. Lee, *Hölder regularity and uniqueness theorem on weak solutions to the degenerate Keller-Segel system*, Nonlinear Anal. 138 (2016), 229–252.

20. O. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, U.S.A, 1968.

21. J. M. Urbano, *Continuous solutions for a degenerate free boundary problem*, Ann. Mat. Pura Appl. 178 (2000), 195–224. MR 1849386 (2002h:35353).