COEFFICIENT AND RADIUS ESTIMATES OF STARLIKE
FUNCTIONS WITH POSITIVE REAL PART

ADIBA NAZ, SUSHIL KUMAR, AND V. RAVICHANDRAN

Abstract. Let $\mathcal{S}_e^*$ and $\mathcal{S}_R^*$ denote the classes of analytic functions $f$ in the open
unit disk normalized by conditions $f(0) = 0$ and $f'(0) = 1$ satisfying the subordination $zf'(z)/f(z) \prec e^z$ and $zf'(z)/f(z) \prec 1 + z(k + z)/(k(k - z)) =: \varphi_R(z)$ where
$k = \sqrt{2} + 1$ respectively. In this paper, we obtain the sharp bound for the fifth coeffi-
cient for the functions in the class $\mathcal{S}_e^*$. The upper bound for certain types of Hankel
determinant for the classes $\mathcal{S}_e^*$ and $\mathcal{S}_R^*$ is also investigated. In addition, some radius
estimates associated with the subclasses $\mathcal{S}_e^*$ and $\mathcal{S}_R^*$ are also computed.

1. Introduction

The class of all analytic functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in the open disk
$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ is denoted
by $\mathcal{A}$. Denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of univalent functions. Let the class
$\mathcal{P}$ consists of all analytic functions $p$ in $\mathbb{D}$ with positive real part that are normalized
by $p(0) = 1$. For any two analytic functions $f$ and $g$, we say that $f$ is subordinate to
g, written as $f \prec g$, if there exists a Schwarz function $w$ with $w(0) = 1$ and $|w(z)| < 1$
that satisfies $f(z) = g(w(z))$ for $z \in \mathbb{D}$. In particular, if $g$ is univalent in $\mathbb{D}$, then
$f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. In terms of subordination, Ma and Minda [18] gave a
unified representation of various geometric subclasses of $\mathcal{A}$ which is as follows:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad (1.1)$$

where $\varphi$ is any analytic univalent function with positive real part mapping $\mathbb{D}$ onto
domains which are symmetric with respect to the real axis and starlike with respect
to $\varphi(0) = 1$ such that $\varphi'(0) > 0$. For $-1 \leq B < A \leq 1$, $\mathcal{S}^*(A, B) := \mathcal{S}^*((1 + A\z)/\z)$ is a well-known class consisting of Janowski [11] starlike functions.

The special case when $A = 1 - 2\alpha$ and $B = -1$ reduces to $\mathcal{S}^*(\alpha)$ ($0 \leq \alpha \leq 1$)
consisting of starlike functions of order $\alpha$ [30]. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ is the
class of starlike functions. In the similar fashion, several authors defined many new
interesting subclasses of starlike functions by altering the superordinate function $\varphi$.
However this paper aims to consider the cases $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ consisting of functions
$f \in \mathcal{S}$ such that $zf'(z)/f(z)$ lies in the domain bounded by $|\log(zf'(z)/f(z))| < 1$,
introduced by Mendiratta et al. [19] and the class $\mathcal{S}_R^* := \mathcal{S}^*(\varphi_R)$ where

$$\varphi_R(z) := 1 + \frac{z}{k} \left( \frac{k + z}{k - z} \right) = 1 + \frac{1}{k^2}z^2 + \frac{2}{k^3}z^3 + \cdots \quad (1.2)$$
such that $k = \sqrt{2} + 1$ and $z \in \mathbb{D}$ discussed in [13].

2010 Mathematics Subject Classification. 30C45, 30C50, 30C80.

Key words and phrases. Univalent functions, Starlike functions, Rational function, Exponential
function, Fekete-Szegö inequality, Hankel determinants, Radius problems.
In 1914, Gronwall proved an area theorem related to coefficient estimates. In 1916, Bieberbach [3] established bound for the second coefficient of an analytic univalent function. Further, Bieberbach gave a conjecture that \(|a_n| \leq n\) for all \(n \in \mathbb{N} \setminus \{1\}\) for the function \(f \in \mathcal{S}\) and the sharpness follows by Koebe function and its rotation. This conjecture was later proved by Louis de Branges in 1985. In an attempt to resolve Bieberbach conjecture for various subclasses of univalent functions, researchers followed many research areas. For \(f \in \mathcal{S}^*(\varphi)\), Ma and Minda [18] determined the sharp bound for the second and the third coefficients. Later, Ali et al. [2] determined the sharp bound for the fourth coefficient of the functions in the class \(\mathcal{S}\). In 1985, Pommerenke [23,24] first studied the Hankel determinant for the class \(S\) and by Noonam and Thomas [20] for mean univalent functions. Noor [21,22] studied the Hankel determinant for close-to-convex and Bazilevic functions. Similarly, the majority of the sharp results were obtained by several authors for the second Hankel determinant given by \(H_2(2) = a_2a_4 - a_2^2\) (cf. [4,11,16]). Fekete and Szegö [9] considered the second Hankel determinant \(H_2(1) = a_3 - a_2^2\) for the class \(\mathcal{S}\). They estimated the upper bound for a more general and well-known Fekete-Szegö functional \(|a_3 - \mu a_2^2|\) where \(\mu\) is any real number. However, very few papers discuss the third Hankel determinant \(H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)\). Babalola [3] investigated the upper bound on \(H_3(1)\) for the well-known classes of bounded turning, starlike and convex functions while Prajapat et al. [25] investigated same for a class of close-to-convex functions. Recently, Raza and Malik [29] obtained the third Hankel determinant for the class \(\mathcal{S}_L^*\) and Zhang et al. [35] for the class \(\mathcal{S}_e^*\). For more details, see [4,14,16].

Set \(D_r := \{z \in \mathbb{C}: |z| < r\}\). Let \(M\) be a set of functions and \(P\) be a property. Then a real number \(\mathcal{R}_P = \sup\{r > 0: f\) has the property \(P\) in the disk \(D_r\) for all \(f \in M\}\) is called as the radius of property for the set \(M\). If there exists \(F_0 \in M\) such that \(F_0\) has the property \(P\) in \(D_{\mathcal{R}_P}\), then sharpness follows for the function \(F_0\). For instance, the radius of convexity for the class \(\mathcal{S}\) is \(2 - \sqrt{3}\) and the Koebe function \(k(z) = z/(1 - z)^2\) shows the sharpness of this result [8].

Motivated by the above said work, in the following section, we estimate the sharp bound for the absolute value of the fifth coefficient and sharp estimates of some second Hankel determinant for the functions in the class \(\mathcal{S}_e^*\). We also determine the upper bound for \(|H_3(1)|\) for the class of functions in the class \(\mathcal{S}_R^*\). In the last section, we estimate the sharp \(\mathcal{S}_R^*\)-radius, \(\mathcal{R}_{\mathcal{M}}(\beta)\), \(\mathcal{R}_{\mathcal{L}}^*\) and \(\mathcal{S}_e^*\)-radius for various well-known classes of functions.

The Hankel determinant for a given function \(f \in \mathcal{S}\) is defined as follows:

\[
H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}
\]

where \(a_1 = 1\) and \(n, q\) are fixed positive integers. Problem of finding the exact bounds of \(|H_q(n)|\) for various subclasses of analytic functions is investigated by many authors. Pommerenke [23,24] first studied the Hankel determinant for the class \(\mathcal{S}\) of univalent functions. Later, \(H_2(n)\) was studied by Hayman [10] for mean univalent functions and by Noonam and Thomas [20] for mean \(p\)-valent functions. Noor [21,22] studied the Hankel determinant for close-to-convex and Bazilevic functions. Similarly, the majority of the sharp results were obtained by several authors for the second Hankel determinant given by \(H_2(2) = a_2a_4 - a_2^2\) (cf. [4,11,16]). Fekete and Szegö [9] considered the second Hankel determinant \(H_2(1) = a_3 - a_2^2\) for the class \(\mathcal{S}\). They estimated the upper bound for a more general and well-known Fekete-Szegö functional \(|a_3 - \mu a_2^2|\) where \(\mu\) is any real number. However, very few papers discuss the third Hankel determinant \(H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)\). Babalola [3] investigated the upper bound on \(H_3(1)\) for the well-known classes of bounded turning, starlike and convex functions while Prajapat et al. [25] investigated same for a class of close-to-convex functions. Recently, Raza and Malik [29] obtained the third Hankel determinant for the class \(\mathcal{S}_L^*\) and Zhang et al. [35] for the class \(\mathcal{S}_e^*\). For more details, see [4,14,16].
2. COEFFICIENT ESTIMATES

For any function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\varphi) \), let

\[
p(z) := \frac{zf''(z)}{f'(z)} = 1 + B_1 z + B_2 z^2 + \cdots.
\]

In particular, we have

\[
a_2 = B_1, \quad a_3 = \frac{1}{2}(B_1^2 + B_2), \quad a_4 = \frac{1}{6}(B_1^3 + 3B_1 B_2 + 2B_3)
\]

and

\[
a_5 = \frac{1}{24}(B_1^4 + 6B_1^2 B_2 + 3B_2^2 + 8B_1 B_3 + 6B_4).
\]

We now express the coefficients \( a_n \) \((n = 2, 3, 4, 5)\) of \( f \in \mathcal{S}^*(\varphi) \) in terms of the coefficient of the function \( \varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots \) and of a function with the positive real part in \( \mathbb{D} \). Since \( \varphi \) is univalent and \( p < \varphi \), the existence of a Schwartz function \( w \) will imply that \( q \in \mathcal{P} \), where

\[
q(z) := \frac{1 + w(z)}{1 - w(z)} = \frac{1 + \varphi^{-1}(p(z))}{1 - \varphi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots.
\]

Equivalently

\[
p(z) = \varphi\left(\frac{q(z) - 1}{q(z) + 1}\right)
\]

and therefore we can express the coefficients \( B_i \) in terms of \( c_i \) and \( b_i \). From the expansion of \( q \), it follows that for \( f \in \mathcal{S}^*(\varphi) \)

\[
a_2 = \frac{1}{2} b_1 c_1 \quad \text{(2.1)}
\]

\[
a_3 = \frac{1}{8} \left( (b_1^2 - b_1 + b_2) c_1^2 + 2b_1 c_2 \right) \quad \text{(2.2)}
\]

\[
a_4 = \frac{1}{48} \left( (b_1^3 - 3b_1 b_2 + 3b_1 b_2 + 2b_1 - 4b_2 + 2b_3) c_1^3 + 2(3b_1^2 - 4b_1 + 4b_2) c_1 c_2 
+ 8b_1 c_3 \right) \quad \text{(2.3)}
\]

\[
a_5 = \frac{1}{384} \left( (b_1^4 - 6b_1^3 + 6b_1^2 b_2 + 11b_1^2 - 22b_1 b_2 + 3b_2^2 + 8b_1 b_3 - 6b_1 + 18b_2 
- 18b_3 + 6b_4) c_1^5 + 4(3b_1^3 - 11b_1^2 b_2 + 11b_1 b_2 + 9b_1 - 18b_2 + 9b_3) c_1^2 c_2 
+ 12(b_1^2 - 2b_1 + 2b_2) c_2^2 + 16(2b_1^2 - 3b_1 + 3b_2) c_1 c_3 + 48b_1 c_4 \right) \quad \text{(2.4)}
\]

We first estimate the well-known Fekete-Szegö functional for the class \( \mathcal{S}_e^* \).

**Theorem 2.1.** Let \( f \in \mathcal{S}_e^* \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then for any complex number \( \mu \), we have

\[
|a_3 - \mu a_2^2| \leq \frac{1}{2} \max \left\{ 1, \frac{1}{2} |4 \mu - 3| \right\}.
\]

The result obtained is sharp.

The following lemma is needed in proving the result:

**Lemma 2.2.** [18] Let \( p \in \mathcal{P} \) and \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \), then for any complex number \( \nu \), we have

\[
|p_2 - \nu p_1^2| \leq 2 \max\{1, |2 \nu - 1|\}.
\]
The result is sharp.

Proof of Theorem 2.1. Let \( f \in \mathcal{S}_e^* \). Then \( zf'(z)/f(z) < e^z \). Use of the taylor series expansion of \( e^z \), (2.1) and (2.2) show that the coefficients \( a_2 \) and \( a_3 \) are given by

\[
a_2 = \frac{1}{2}c_1 \quad \text{and} \quad a_3 = \frac{1}{16}(c_1^2 + 4c_2)
\]

respectively. Therefore using Lemma 2.2, we have

\[
|a_3 - \mu a_2^2| = \frac{1}{16} \left| (1 - 4\mu)c_1^2 + 4c_2 \right|
\]

and hence the required result follows. The sharpness of the result of the functional in the result follows from the functions

\[
\frac{zf'(z)}{f(z)} = e^z \quad \text{or} \quad \frac{zf'(z)}{f(z)} = e^{2z}.
\]

\[\square\]

Remark 2.3. Taking \( \mu = 1 \) in Theorem 2.1, we obtain \( |a_3 - a_2^2| \leq 1/2 \) which is same as obtained in [35, Theorem 1] and [19, p. 372].

Now we estimate the sharp bound on the absolute value of \( H_2(2) \) for the functions in the class \( \mathcal{S}_e^* \).

Theorem 2.4. Let \( f \in \mathcal{S}_e^* \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{4}
\]

The bound obtained is sharp.

To prove our result, we need the following two lemmas.

Lemma 2.5 (Carathéodory’s Lemma). [8] Let \( p \in \mathcal{P} \) and \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \), then \( |p_n| \leq 2 \) for \( n = 1, 2, \ldots \). This inequality is sharp for each \( n \).

Lemma 2.6. [17] Let \( p \in \mathcal{P} \) and \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \), then

\[
2p_2 = p_1^2 + x(4 - p_1^2)
\]

\[
4p_3 = p_1^3 + 2p_1 x(4 - p_1^2) - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2) z
\]

for some \( |x| \leq 1 \) and \( |z| \leq 1 \).

Proof of Theorem 2.4. Using (2.3) for the expansion of \( \varphi(z) = e^z \), we have

\[
a_4 = \frac{1}{288}(-c_1^2 + 12c_1 c_2 + 48c_3).
\]

(2.6)

From (2.5) and (2.6), it follows that

\[
a_2a_4 - a_3^2 = \frac{-1}{2304}(13c_1^4 + 24c_1^2 c_2 + 144c_2^2 - 192c_1 c_3).
\]
By Lemma 2.5, we have \(|c_1| \leq 2\). Therefore substituting the values of \(c_2\) and \(c_3\) from Lemma 2.6 and assuming \(c_1 = c \in [0, 2]\) without loss of generality, we get
\[
a_2a_4 - a_3^2 = \frac{-1}{2304} (13c^4 + 36(4 - c^2)x^2 - 12(4 - c^2)x + 48(4 - c^2)x^2 - 96(4 - c^2)(1 - |x|^2)c).
\]
Applying triangle inequality and replacing \(|x|\) by \(\mu\), we have
\[
|a_2a_4 - a_3^2| \leq \frac{1}{2304} (13c^4 + 36(4 - c^2)^2\mu^2 + 12(4 - c^2)\mu^2 + 48(4 - c^2)c^2\mu^2 + 96(4 - c^2)(1 - \mu^2)c) =: F(c, \mu).
\]
Since \(\partial F/\partial \mu > 0\) for \((c, \mu) \in [0, 2] \times [0, 1]\), \(F(c, \mu)\) is an increasing function of \(\mu\) in the closed interval \([0, 1]\) which implies \(F(c, \mu)\) attains its maximum value at \(\mu = 1\), that is,
\[
\max F(c, \mu) = F(c, 1) =: G(c)
\]
where
\[
G(c) = \frac{1}{2304} (13c^4 + 36(4 - c^2)^2 + 60c^2(4 - c^2)).
\]
The second derivative test shows that maximum value of \(G\) occurs at \(c = 0\), therefore
\[
|a_2a_4 - a_3^2| \leq G(0) = \frac{1}{4}.
\]
The bound is sharp for the function \(f\) such that
\[
\frac{zf'(z)}{f(z)} = e^{z^2}.
\]
\[\square\]

Note that the upper bound on the second Hankel determinant is an improvement of obtained bound in [35, Theorem 3]. Mendiratta et al. [19, Theorem 2.3 p. 372] estimated the sharp upper bounds \(|a_2| \leq 1, |a_3| \leq 3/4\) and \(|a_4| \leq 17/36\) for the functions in \(\mathcal{S}_e^*\). However authors were not able to maximizes \(|a_n|\) for \(n \geq 5\). Here we obtain the sharp bound for the absolute value of fifth coefficient for the functions in the class \(\mathcal{S}_e^*\).

**Theorem 2.7.** If \(f \in \mathcal{S}_e^*\) and \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\), then \(|a_5| \leq 1/4\). The estimate is sharp.

We will make use of the following lemma to prove our desired estimation.

**Lemma 2.8.** [28] Let \(\alpha, \beta, \gamma\) and \(\delta\) satisfy the inequalities \(0 < \alpha < 1\), \(0 < \delta < 1\) and
\[
8\delta(1-\delta)((\alpha\beta - 2\gamma)^2 + (\alpha(\alpha + \delta) - \beta)^2) + \alpha(1 - \alpha)(\beta - 2\alpha\delta)^2 \leq 4\alpha^2(1 - \alpha)^2\delta(1 - \delta),
\]
then
\[
|\gamma a_4^4 + \delta a_2^2 + 2\alpha a_1 a_3 - (3/2)\beta a_2^2 a_2 - a_4| \leq 2.
\]

**Proof of Theorem 2.7.** Let \(f \in \mathcal{S}_e^*\). By making use of the taylor series expansion of \(e^z\) and (2.4), the coefficient \(a_5\) is given by
\[
a_5 = -\frac{1}{8} \left( \frac{1}{144} c_1^4 + \frac{1}{12} c_1^2 c_2 - \frac{1}{6} c_1 c_3 - c_4 \right).
\]
Since $\alpha = -1/12$, $\beta = -1/18$, $\gamma = -1/144$ and $\delta = 0$ satisfy (2.7), applying Lemma 2.8 we get $|a_5| \leq 1/4$. Let the function $f_e : \mathbb{D} \to \mathbb{C}$ be defined by

$$f_e(z) = z \exp \left( \int_0^z \frac{e^{t^2} - 1}{t} \, dt \right) = z + \frac{1}{4}z^5 + \frac{1}{32}z^9 + \cdots.$$  

Then $f_e(0) = 0$, $f'_e(0) = 1$, $zf'_e(z)/f_e(z) = e^{z^2}$ and therefore the function $f_e \in \mathcal{S}_R^*$ which completes the sharpness part of the result.

Now if $f \in \mathcal{S}_R^*$ and $f(z) = z + \sum_{n=2}^\infty a_n z^n$, then $zf'(z)/f(z) \prec \varphi_R(z)$ where $\varphi_R$ is given by (1.2). Using the taylor series expansion of $\varphi_R$ given by (1.2) and equations (2.1), (2.2), (2.3), precisely we get

$$a_2 = \frac{1}{2k} c_1 \quad (2.8)$$

$$a_3 = \frac{1}{8k^2} (2kc_2 + (3-k)c_1^2) \quad (2.9)$$

$$a_4 = \frac{1}{48k^3} \left( (11 - 11k + 2k^2)c_1^3 + 2(11 - 4k)kc_1c_2 + 8k^2c_3 \right). \quad (2.10)$$

Here we estimate the Fekete-Szegö functional for the class $\mathcal{S}_R^*$.

**Theorem 2.9.** Let $f \in \mathcal{S}_R^*$ and $f(z) = z + \sum_{n=2}^\infty a_n z^n$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{2k} \max \left\{ 1, \frac{1}{k} |2\mu - 3| \right\}$$

where $k = \sqrt{2} + 1$. The result obtained is sharp.

**Proof.** Since

$$|a_3 - \mu a_2^2| = \frac{1}{4k} \left| c_2 - \frac{1}{2k} (2\mu - 3 + k)c_1^2 \right|$$

using Lemma 2.2 we get the required result. The sharpness of the result follows from the functions

$$\frac{zf'(z)}{f(z)} = 1 + \frac{z}{k} \left( \frac{k + z}{k - z} \right) \quad \text{or} \quad \frac{zf'(z)}{f(z)} = 1 + \frac{z^2}{k} \left( \frac{k + z^2}{k - z^2} \right).$$

**Remark 2.10.** If the function $f(z) = z + \sum_{n=2}^\infty a_n z^n$ belongs to the class $\mathcal{S}_R^*$, then using Theorem 2.9 we obtain

$$|a_2| \leq \frac{1}{k} \quad \text{and} \quad |a_3| \leq \frac{3}{2k^2}.$$  

The function $h$ defined by

$$h(z) := \frac{k^2 z^2}{(k - z)^2} e^{-z/k}$$

$$= z + \frac{1}{k}z^2 + \frac{3}{2k^2}z^3 + \frac{11}{6k^3}z^4 + \cdots + \frac{1}{k^{n+1}} \left( \sum_{p=0}^{n-1} (-1)^p \frac{n - p}{p!} \right) z^n + \cdots$$

(2.11)

plays the role of extremal function for the class $\mathcal{S}_R^*$ and hence we conclude following Conjecture.
Lemma 2.6 and assuming $c \neq 1$.

Using the triangle inequality and substituting Conjecture 2.11.

The second derivative test shows that maximum value of $G$ occurs at $\mu = 1$ and therefore we have

$$|a_n| \leq \frac{1}{k^{n-1}} \left( \sum_{p=0}^{n-1} (-1)^p \frac{n! - p!}{p!} \right).$$

Substituting $\mu = 1$ in Theorem 2.9 gives the following bound on the coefficients $a_2$ and $a_3$.

Corollary 2.12. Let $f \in \mathcal{S}_R^*$ and $f(z) = z + \sum_{n=2}^\infty a_n z^n$, then

$$|a_3 - a_2^2| \leq \frac{1}{2k} \approx 0.207107.$$  

The result is sharp.

Now, we estimate the sharp bound on the second Hankel determinant $H_2(2)$ for the class $\mathcal{S}_R^*$.

Theorem 2.13. Let $f \in \mathcal{S}_R^*$ and $f(z) = z + \sum_{n=2}^\infty a_n z^n$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{4k^2} \approx 0.0428932.$$  

The bound obtained is sharp.

Proof. Using (2.8), (2.9) and (2.10), we have

$$a_2a_4 - a_3^2 = \frac{1}{192k^4} \left( (-5 - 4k + k^2)c_1^4 + 4k(2 - k)c_1^2c_2 - 12k^2c_2^2 + 16k^2c_1c_3 \right).$$

In view of Lemma 2.5 $|c_1| \leq 2$, therefore substituting the values of $c_2$ and $c_3$ from Lemma 2.6 and assuming $c_1 = c \in [0, 2]$, we have

$$a_2a_4 - a_3^2 = \frac{1}{192k^4} \left( -5c^4 + 4k(4 - c^2)xc^2 - 3k^2(4 - c^2)^2x^2 - 4k^2(4 - c^2)x^2c^2 + 8k^2(4 - c^2)(1 - |x|^2)c \right).$$

Using the triangle inequality and substituting $|x|$ by $\mu$, we get

$$|a_2a_4 - a_3^2| \leq \frac{1}{192k^4} \left( 5c^4 + 4k(4 - c^2)\mu c^2 + 3k^2(4 - c^2)^2\mu^2 + 4k^2(4 - c^2)\mu^2c^2 + 8k^2(4 - c^2)(1 - \mu^2)c \right) =: F(c, \mu).$$

Since

$$\frac{\partial F}{\partial \mu} = \frac{1}{192k^4} \left( 4k(4 - c^2)c^2 + 2k^2(4 - c^2)(c^2 - 8c + 12)\mu \right) > 0$$

for $(c, \mu) \in [0, 2] \times [0, 1]$, $F(c, \mu)$ is an increasing function of $\mu$ in $[0, 1]$, that is, $F(c, \mu)$ attains its maximum value at $\mu = 1$ and therefore we have

$$\max F(c, \mu) = F(c, 1) =: G(c)$$

where

$$G(c) = \frac{1}{192k^4} \left( 5c^4 + 4k(4 - c^2)c^2 + 3k^2(4 - c^2)^2 + 4k^2(4 - c^2)c^2 \right).$$

The second derivative test shows that maximum value of $G$ occurs at $c = 0$ and hence

$$|a_2a_4 - a_3^2| \leq G(0) = \frac{1}{4k^2}.$$
The bound is sharp for the function $f$ satisfying
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{z^2}{k} \left( \frac{k + z^2}{k - z^2} \right).
\]

**Theorem 2.14.** Let $f \in S_R^*$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then
\[
|a_2 a_3 - a_4| \leq \frac{5220 + 3683\sqrt{2} + 359\sqrt{359} + 246\sqrt{2} + 246\sqrt{718} + 492\sqrt{2}}{1458(1 + \sqrt{2})^5} 
\approx 0.244395.
\]

**Proof.** Again by making use of (2.8), (2.9) and (2.10), we have
\[
a_2 a_3 - a_4 = -\frac{1}{24k^3} \left( (1 - 4k + k^2)c_1^3 + 4k(2 - k)c_1c_2 + 4k^2c_3 \right).
\]
Using Lemma 2.6 assuming $c > 0$ and letting $c_1 = c \in [0, 2]$, we have
\[
a_2 a_3 - a_4 = -\frac{1}{24k^3} (c^3 + 4k(4 - c^2)xc - k^2(4 - c^2)x^2c + 2k^2(4 - c^2)(1 - |x|^2)z).
\]
With the help of same technique as used in previous theorem, an application of triangle inequality and the fact that $1 - |x|^2 \leq 1$ give
\[
|a_2 a_3 - a_4| \leq \frac{1}{24k^3} (c^3 + 4k(4 - c^2)\mu c + k^2(4 - c^2)\mu^2c + 2k^2(4 - c^2)) =: F(c, \mu).
\]
Since $\partial F/\partial \mu > 0$ for any fixed $c \in [0, 2]$ and for all $\mu \in [0, 1]$, we can say that $F(c, \mu)$ is an increasing function of $\mu$ and hence
\[
\max F(c, \mu) = F(c, 1) =: G(c)
\]
where
\[
G(c) = \frac{1}{24k^3} (c^3 + 4k(4 - c^2)c + k^2(4 - c^2)c + 2k^2(4 - c^2)).
\]
Since
\[
G'' \left( \frac{2(-k^2 + M)}{3N} \right) = -0.31492 < 0
\]
where $M = (-12k + 45k^2 + 24k^3 + 4k^4)^{1/2}$ and $N = -1 + 4k + k^2$, the maximum value of $G$ occurs at $c = 2(-k^2 + M)/(3N)$. Therefore
\[
|a_2 a_3 - a_4| \leq G \left( \frac{2(-k^2 + M)}{3N} \right)
\]
\[
= \frac{144k^4 + 16k^5 - 24M + 12k^2(-15 + 4M) + k^3(243 + 8M)}{1458(1 + \sqrt{2})^5} + 9k(3 + 10M)
\]
\[
= \frac{81k^2N^2}{5220 + 3683\sqrt{2} + 359\sqrt{359} + 246\sqrt{2} + 246\sqrt{718} + 492\sqrt{2}}.
\]

By making use of Conjecture 2.11, Corollary 2.12, Theorem 2.13 and Theorem 2.14 we obtain the following bound on the third Hankel determinant for the functions in the class $S_R^*$. 
Theorem 2.15. Let the function \( f \in \mathscr{S}_{R}^{*} \), then
\[
|H_{3}(1)| \leq \frac{4293 + 1458k + \frac{88(144k^{4} + 16k^{5} - 24M + 12k^{2}(-15 + 4M)+k^{2}(243 + 8M) + 9k(3 + 10M))}{N^{2}}}{3888k^{5}}
\]
\[
\approx 0.0563448
\]
where \( M = (-12k + 45k^{2} + 24k^{3} + 4k^{4})^{1/2} \) and \( N = -1 + 4k + k^{2} \).

3. RADIUS ESTIMATES

By using the Ma-Minda relation (1.1), Sokół and Stankiewicz [33] introduced the subclass \( \mathcal{S}_{L}^{*} := \mathcal{S}^{*}(\sqrt{1 + z}) \) associated with lemniscate of Bernoulli, Raina and Sokól [26] defined the subclass \( \mathcal{S}_{q}^{*} := \mathcal{S}^{*}(z + \sqrt{1 + z^{2}}) \) associated with lune and Sharma et al. [32] investigated the class \( \mathcal{S}_{c}^{*} := \mathcal{S}^{*}(1 + 4z/3 + 2z^{2}/3) \) associated with cardioid. For \( \alpha \in (0, 1) \), Kargar et al. [12] (see also [6]) introduced the class \( \mathcal{B}S^{*}(\alpha) := \mathcal{S}^{*}(G_{\alpha}(z) = 1 + z/(1 - \alpha z^{2})) \) associated with Booth lemniscate. The interesting class \( \mathcal{M}(\beta) \) where \( \beta > 1 \), defined by
\[
\mathcal{M}(\beta) = \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \beta, \ z \in \mathbb{D} \right\}
\]
was investigated by Uraleagaddi et al. [34].

Let \( \mathcal{C}S^{*}(\alpha) \) be the class of close-to-star functions of type \( \alpha \) which is defined by
\[
\mathcal{C}S^{*}(\alpha) = \left\{ f \in \mathcal{A} : \frac{f}{g} \in \mathcal{P}, g \in S^{*}(\alpha) \right\}.
\]

Sokół and Stankiewicz [33] estimated the radius of convexity for functions in the class \( \mathcal{S}_{L}^{*} \). Recently, Kumar and Ravichandran [13] and Mendiratta et al. [19] estimated the sharp \( \mathcal{S}_{R}^{*} \)-radii and \( \mathcal{S}_{c}^{*} \)-radii, respectively for various well-known classes of functions. For example, they estimated the radius of convexity, \( \mathcal{S}_{R}^{*} \)-radius and \( \mathcal{S}_{c}^{*} \)-radius for the class \( \mathcal{S}^{*}[A, B] \), \( W = \{ f \in \mathcal{A} : \text{Re}(f(z)/z) > 0, z \in \mathbb{D} \} \), \( \mathcal{S}_{L} = \{ f \in \mathcal{A} : f/g \in \mathcal{P} \) for some \( g \in W \}, \mathcal{F}_{2} := \{ f \in \mathcal{A} : |f(z)/g(z) - 1| < 1 \) for some \( g \in W \} \) and so forth. In this section, we compute the sharp \( \mathcal{S}_{R}^{*} \)-radius, \( \mathcal{M}(\beta) \)-radius, \( \mathcal{S}_{L}^{*} \) and \( \mathcal{S}_{c}^{*} \)-radius for various other well-known classes of functions.

Theorem 3.1. The \( \mathcal{S}_{R}^{*} \)-radii for the classes \( \mathcal{C}S^{*}(\alpha) \), \( \mathcal{S}_{q}^{*} \) and \( \mathcal{B}S^{*}(\alpha) \), \( \mathcal{M}(\beta) \)-radius for the class \( \mathcal{S}_{R}^{*} \) and \( \mathcal{S}_{L}^{*} \)-radius for the class \( \mathcal{S}_{c}^{*} \) are given by:

(a) \( \mathcal{R}_{\mathcal{S}}^{*}(\mathcal{C}S^{*}(\alpha)) = p_{0} := (2 - \alpha + \sqrt{7 - 6\alpha + \alpha^{2}})/(-3 + 2\alpha) \)

(b) \( \mathcal{R}_{\mathcal{S}}^{*}(\mathcal{S}_{q}^{*}) = (-2 + \sqrt{2 + \sqrt{4 - 4\sqrt{2}}})/2 \approx 0.350701 \) which is the smallest positive root of the equation \( 4r^{4} - 4r^{2} + (57 - 40\sqrt{2}) = 0 \)

(c) \( \mathcal{R}_{\mathcal{S}}^{*}(\mathcal{B}S^{*}(\alpha)) = (-3 + 2\sqrt{2}) + \sqrt{4\alpha + 17 + 12\sqrt{2}})/2\alpha \)

(d) \( \mathcal{R}_{\mathcal{M}(\beta)}(\mathcal{S}_{R}^{*}) = \left\{ \begin{array}{ll}
1 & \text{if } \beta \geq 2 \\
N(\beta + \sqrt{\beta^{2} + 4\beta - 4})/2 & \text{if } \beta \leq 2
\end{array} \right. \)

(e) \( \mathcal{R}_{\mathcal{S}}^{*}(\mathcal{S}_{c}^{*}) = (-1 + \sqrt{2})(-4 - 3\sqrt{2} + \sqrt{62 + 44\sqrt{2}})/2 \approx 0.601232 \) respectively. The radii obtained are sharp.

The subclass of \( \mathcal{P} \) which satisfies \( \text{Re} p(z) > \alpha \) where \( 0 \leq \alpha < 1 \) is denoted by \( \mathcal{P}(\alpha) \). In general for \( |B| \leq 1 \) and \( A \neq B \), the class \( \mathcal{P}[A, B] \) consists of all those functions \( p \)
Lemma 3.2. [31] If \( p \in \mathcal{P}(\alpha) \), then
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r(1 - \alpha)}{(1 - r)(1 + (1 - 2\alpha)r)}, \quad |z| = r < 1.
\]

Lemma 3.3. [27] If \( p \in \mathcal{P}[A, B] \), then
\[
\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{|A - B|r}{1 - B^2r^2}, \quad |z| = r < 1.
\]

Lemma 3.4. [13] For \( 2(\sqrt{2} - 1) < a < 2 \), let \( r_a \) be defined by
\[
\begin{cases}
a - 2(\sqrt{2} - 1), & \text{if } 2(\sqrt{2} - 1) < a \leq \sqrt{2}; \\
2 - a, & \text{if } \sqrt{2} \leq a < 2.
\end{cases}
\]
Then \( \{w \in \mathbb{C} : |w - a| < r_a \} \subset \varphi_R(\mathbb{D}) \) where
\[
\varphi_R(\mathbb{D}) := \{w \in \mathbb{C} : |w + (w^2 + 4w - 4)^{1/2}| < 2/k\}.
\]

Proof of Theorem 3.1 (a) Let \( f \in \mathcal{C} \mathcal{S}^*(\alpha) \) and \( g \in \mathcal{S}^*(\alpha) \) such that \( p(z) = f(z)/g(z) \in \mathcal{P} \). Then \( zg'(z)/g(z) \in \mathcal{P}(\alpha) \) and Lemma 3.3 gives
\[
\left| \frac{zg'(z)}{g(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2}.
\]
Since \( p \in \mathcal{P} \), applying Lemma 3.2 yields
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1 - r^2}.
\]
Using the above estimates in the identity
\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}
\]
we can see that
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(2 - \alpha)r}{1 - r^2}.
\]
(3.1)
Let \( 0 \leq r \leq \rho_0 \). Then it can be easily seen that if \( a := (1 + (1 - 2\alpha)r^2)/(1 - r^2) \), then \( a \leq 2 \). Therefore from Lemma 3.4, we can see that the disk (3.1) lies inside the domain \( \varphi_R(\mathbb{D}) \) if and only if
\[
\frac{2(2 - \alpha)r}{1 - r^2} \leq 2 - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2}.
\]
The last inequality reduces to \( -1 + 2(2 - \alpha)r + (3 - 2\alpha)r^2 \leq 0 \). Since \( r \leq \rho_0 \), the result follows. Consider the functions \( f \) and \( g \) defined by
\[
f(z) = \frac{z(1 + z)}{(1 - z)^{3 - 2\alpha}} \quad \text{and} \quad g(z) = \frac{z}{(1 - z)^{2 - 2\alpha}}.
\]
Since
\[
\text{Re} \frac{f(z)}{g(z)} = \text{Re} \frac{1 + z}{1 - z} > 0 \quad \text{and} \quad \text{Re} \frac{zg'(z)}{g(z)} = \text{Re} \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} > \alpha
\]
then $g \in S^*(\alpha)$ and hence $f \in \mathcal{C} S^*(\alpha)$. Also at the point $z = \rho_0$, we see that

$$\frac{zf'(z)}{f(z)} = \frac{1 + 2(2 - \alpha)z + (1 - 2\alpha)z^2}{1 - z^2} = 1 + \frac{(1 - 2\alpha)\rho_0^2}{1 - \rho_0^2} - \frac{2(2 - \alpha)\rho_0}{1 - \rho_0^2} = 2.$$  

This proves the sharpness of the result.

(b) Since $f$ is in the class $\mathcal{S}_q^*$, we have $zf'(z)/f(z) \prec z + \sqrt{1 + z^2} := q(z)$. For $|z| = r$, note that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq |z + \sqrt{1 + z^2} - 1| 
\leq 1 - r - \sqrt{1 - r^2}. \quad (3.2)$$

In view of Lemma 3.4, the disk $(3.2)$ lies in the domain $\varphi_R(\mathbb{D})$ if $1 - r - \sqrt{1 - r^2} \leq 3 - 2\sqrt{2}$ or $4r^4 - 4r^2 + (57 - 40\sqrt{2}) \leq 0$ which gives the desired radius estimate and this estimate is best possible for the function

$$f_q(z) := z \exp(q(z) - \log(1 - z + q(z)) + \log 2 - 1). \quad (3.3)$$

(c) Let $f \in B S^*(\alpha)$. Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{1 - \alpha z^2}.$$  

A simple calculation yields

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{z}{1 - \alpha z^2} \right| 
\leq \frac{r}{1 - \alpha r^2}. \quad (3.4)$$

Using Lemma 3.4, note that the disk $(3.4)$ is contained in the domain $\varphi_R(\mathbb{D})$ if

$$\frac{r}{1 - \alpha r^2} \leq 3 - 2\sqrt{2}$$

or $\alpha r^2 + (3 + 2\sqrt{2})r - 1 \leq 0$. This gives the required radius estimate

$$r \leq -\frac{(3 + 2\sqrt{2}) + \sqrt{4\alpha + 17 + 12\sqrt{2}}}{2\alpha}.$$  

The function defined by

$$f_B(z) := z \exp\left(\frac{\tanh^{-1}(\sqrt{\alpha}z)}{\sqrt{\alpha}}\right) \quad (3.5)$$

proves that the estimation is sharp.

(d) Let $f \in \mathcal{S}_R^*$. Then $zf'(z)/f(z) \prec \varphi_R(z)$.

Case 1. Let $\beta \geq 2$. For $|z| = r < 1$, using the definition of subordination, it is easy to see that

$$\Re \frac{zf'(z)}{f(z)} \leq \max_{|z|=r} \varphi_R(z) \leq 1 + \frac{r}{k} \left(\frac{k + r}{k - r}\right) < \frac{k^2 + 1}{k(k - 1)} \leq \beta.$$
Case 2. Let \( \beta \leq 2 \). For \( |z| = r < k(-\beta + \sqrt{\beta^2 + 4\beta - 4})/2 \), using the same technique as in Case 1, it follows that
\[
\text{Re} \frac{zf'(z)}{f(z)} \leq 1 + \frac{r}{k} \left( \frac{k + r}{k - r} \right) < \beta.
\]
This proves the desired result. Sharpness follows by considering the function
\[
f_r(z) = \frac{kz}{(k - z)^2} e^{-z/k}, \tag{3.6}
\]
(e) Since \( f \in \mathcal{S}_R^* \), we have \(zf'(z)/f(z) < \varphi_R(z) \). Then
\[
|\varphi_R(z) - 1|^2 = \frac{r^2}{k^2} \left( \frac{k^2 + r^2 + 2kr \cos t}{k^2 + r^2 - 2kr \cos t} \right) < (\sqrt{2} - 1)^2
\]
if
\[
\frac{|zf'(z)|}{f(z)} - 1 < \sqrt{2} - 1, \quad |z| = r < \frac{1}{2}(-1 + \sqrt{2})(-4 - 3\sqrt{2} + \sqrt{62 + 44\sqrt{2}}).
\]
Therefore the result follows from [11] Lemma 2.2, p. 6559. The radius estimate is sharp for the function \( f_r(z) \) defined by (3.6).

Next result yields the sharp radius estimates related to the class \( \mathcal{S}_e^* \).

**Theorem 3.5.** The \( \mathcal{S}_e^* \)-radii for the classes \( \mathcal{S}_L^*, \mathcal{S}_q^*, \mathcal{S}_R^*, \mathcal{S}_C^* \) and \( B_1^*(\alpha) \) are given as:

(a) \( R_{\mathcal{S}_L^*}(\mathcal{S}_L^*) = (e^2 - 1)/e^2 \approx 0.864665 \)
(b) \( R_{\mathcal{S}_q^*}(\mathcal{S}_q^*) = (-2e + \sqrt{-4e^2 + 8e^3})/(4e^2) \approx 0.498824 \) which is the smallest positive root of the equation \( 4r^4 - 4r^2 + ((e^2 - 1)/e^2)^2 = 0 \)
(c) \( R_{\mathcal{S}_R^*}(\mathcal{S}_R^*) = (k - 2ek + k\sqrt{1 - 8e + 8e^2})/(2e) \approx 0.780444 \)
(d) \( R_{\mathcal{S}_C^*}(\mathcal{S}_C^*) = (-2e + \sqrt{10e^2 - 4e})/2e \approx 0.395772 \)
(e) \( R_{\mathcal{S}_e^*}(B_1^*(\alpha)) = (-e + \sqrt{e^2 + 4(e - 1)^2\alpha})/(2\alpha(e - 1)) \)
respectively. The results are all sharp.

To prove our estimations, we will make use of the following result.

**Lemma 3.6.** [11] For \( 1/e < a < e \), let \( r_a \) be defined by
\[
r_a = \begin{cases} 
a - e^{-1}, & \text{if } e^{-1} < a \leq (e + e^{-1})/2; \\
e - a, & \text{if } (e + e^{-1})/2 \leq a < e.
\end{cases}
\]
Then \( \{w \in \mathbb{C} : |w - a| < r_a\} \subset \{w \in \mathbb{C} : |\log w| < 1\} \).

**Proof of Theorem 3.5.** Let \( |z| = r \).
(a) Since \( f \in \mathcal{S}_L^* \), we have \(zf'(z)/f(z) < \sqrt{1 + z} \). Therefore we get
\[
\frac{|zf'(z)|}{f(z)} - 1 = |\sqrt{1 + z} - 1| \leq 1 - \sqrt{1 - r}.
\]
By applying Lemma 3.6 we note that \( f \in \mathcal{S}_e^* \) if \( 1 - \sqrt{1 - r} \leq 1 - 1/e \) which leads to the inequality \( r \leq e^2 - 1/e^2 \). The obtained radius estimate is sharp for the function
\[
\frac{zf_L'(z)}{f_L'(z)} = \sqrt{1 + z}.
\]
(b) In view of Lemma 3.6, we see that the disk (3.2) is contained in the domain $e^z(D) := \{ w \in \mathbb{C} : |\log w| < 1 \}$ provided $1 - r - \sqrt{1 - r^2} \leq 1 - 1/e$ or $r + \sqrt{1 - r^2} \geq 1/e$ or equivalently

$$4r^4 - 4r^2 + \left( \frac{e^2 - 1}{e^2} \right)^2 \leq 0.$$  

The last inequality yields the desired estimate for the radius. The sharpness follows for the function $f_q$ defined by (3.3).

(c) Let $f \in S^*_R$. Then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \frac{|z|}{k} \left( \frac{k + z}{k - z} \right) \leq \frac{r}{k} \left( \frac{k + r}{k - r} \right).$$

Using Lemma 3.6 we see that the disk (3.7) lies in the domain $\{ w \in \mathbb{C} : |\log w| < 1 \}$ if

$$\frac{r}{k} \left( \frac{k + r}{k - r} \right) \leq 1 - \frac{1}{e}.$$  

The above inequality simplifies to

$$er^2 + k(2e - 1)r - k^2(e - 1) \leq 0$$

which gives the required radius estimate. Sharpness follows for the function defined by (3.6).

(d) Let $f \in S^*_C$. Then a simple calculation gives

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{4z}{3} + \frac{2z^2}{3} \right| \leq \frac{1}{3} (4r + 2r^2).$$

Using Lemma 3.6 we note that the disk (3.4) is contained in the domain $\{ w \in \mathbb{C} : |\log w| < 1 \}$ if $(4r + 2r^2)/3 \leq 1 - 1/e$ or $2er^2 + 4er - 3(e - 1) \leq 0$. The last inequality gives

$$r \leq \frac{-2e + \sqrt{10e^2 - 6e}}{2e}.$$  

The result is sharp for the function

$$f_C(z) := z \exp \left( \frac{4z}{3} + \frac{2z^2}{3} \right).$$

(e) Using Lemma 3.6 note that the disk (3.4) lies in the domain $e^z(D)$ provided

$$\frac{r}{1 - \alpha r^2} \leq 1 - \frac{1}{e}.$$  

By a simple computation, the last inequality becomes $\alpha(e - 1)r^2 + er - (e - 1) \leq 0$ which gives

$$r \leq \frac{-e + \sqrt{e^2 + 4(e - 1)^2 \alpha}}{2\alpha(e - 1)}.$$  

The function $f_B(z)$ defined by (3.5) shows the sharpness of this radius estimate. □
References

[1] R. M. Ali, N. K. Jain, and V. Ravichandran, *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, 218 (2012), pp. 6557–6565.

[2] R. M. Ali, V. Ravichandran and N. Seenivasagan, *Coefficient bounds for p-valent functions*, Appl. Math. Comput., 187 (2007), no. 1, 35–46.

[3] K. O. Babalola, *On H3(1) Hankel determinant for some classes of univalent functions*, Inequality Theory and Applications, 6 (2010), pp. 1–7.

[4] D. Bansal, *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett., 26 (2013), pp. 103–107.

[5] L. Bieberbach, *Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. 38 (1916), 940955.

[6] N. E. Cho, S. Kumar, V. Kumar and V. Ravichandran, *Differential subordination and radius estimates for starlike functions associated with the Booth lemniscate*, Turkish J. Math. 42 (2018), pp. 1380–1399.

[7] N. E. Cho, O. S. Kwon, A. Lecko, and Y. J. Sim, *Sharp estimates of generalized Zalcman functional of early coefficients for Ma-Minda type functions*, Filomat 32 (2018), no. 18, 6267–6280.

[8] P. L. Duren, *Univalent functions*, vol. 259 of Grundlehren der Mathematischen Wissenschaften, Fundamental Principles of Mathematical Sciences, Springer-Verlag, New York, 1983.

[9] M. Fekete and G. Szegő, *Eine Bemerkung Uber Ungerechte Schlichte Funktionen*, J. London Math. Soc., 8 (1933), pp. 85–89.

[10] W. K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc. (3), 18 (1968), pp. 77–94.

[11] W. Janowski, *Some extremal problems for certain families of analytic functions. I*, Ann. Polon. Math., 28 (1973), pp. 297–326.

[12] R. Kargar, A. Ebadian, and J. Sokół, *On booth lemniscate and starlike functions*, Analysis and Mathematical Physics, (2017), pp. 1–12.

[13] S. Kumar and V. Ravichandran, *A subclass of starlike functions associated with a rational function*, Southeast Asian Bull. Math., 40 (2016), pp. 199–212.

[14] V. Kumar, S. Kumar and V. Ravichandran, *Third Hankel Determinant For Certain Classes of Analytic Functions*, preprint.

[15] S. Kumar, V. Ravichandran and S. Verma, *Initial coefficients of starlike functions with real coefficients*, Bull. Iranian Math. Soc. 43 (2017), no. 6, 1837–1854.

[16] S. K. Lee, V. Ravichandran, and S. Supramaniam, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl., (2013), pp. 2013:281, 17.

[17] R. J. Libera and E. J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivative in p*, Proc. Amer. Math. Soc., 87 (1983), pp. 251–257.

[18] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994, pp. 157–169.

[19] R. Mendiratta, S. Nagpal, and V. Ravichandran, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc., 38 (2015), pp. 365–386.

[20] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., 223 (1976), pp. 337–346.

[21] K. I. Noor, *On certain analytic functions related with strongly close-to-convex functions*, Appl. Math. Comput., 197 (2008), pp. 149–157.

[22] K. I. Noor and S. A. Al-Bany, *On Bazilevic functions*, Internat. J. Math. Math. Sci., 10 (1987), pp. 79–88.

[23] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc., 41 (1966), pp. 111–122.

[24] ———, *On the Hankel determinants of univalent functions*, Mathematika, 14 (1967), pp. 108–112.

[25] J. K. Prajapat, D. Bansal, A. Singh, and A. K. Mishra, *Bounds on third Hankel determinant for close-to-convex functions*, Acta Univ. Sapientiae Math., 7 (2015), pp. 210–219.

[26] R. K. Raina and J. Sokół, *Some properties related to a certain class of starlike functions*, C. R. Math. Acad. Sci. Paris, 353 (2015), pp. 973–978.
[27] V. Ravichandran, F. Ronning, and T. N. Shanmugam, Radius of convexity and radius of starlikeness for some classes of analytic functions, Complex Variables Theory Appl., 33 (1997), pp. 265–280.

[28] V. Ravichandran and S. Verma, Bound for the fifth coefficient of certain starlike functions, C. R. Math. Acad. Sci. Paris, 353 (2015), pp. 505–510.

[29] M. Raza and S. N. Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, J. Inequal. Appl., (2013), pp. 2013:412, 8.

[30] M. S. Robertson, Certain classes of starlike functions, Michigan Math. J., 32 (1985), pp. 135–140.

[31] G. M. Shah, On the univalence of some analytic functions, Pacific J. Math., 43 (1972), pp. 239–250.

[32] K. Sharma, N. K. Jain, and V. Ravichandran, Starlike functions associated with a cardioid, Afr. Mat., 27 (2016), pp. 923–939.

[33] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, Zeszyty Nauk. Politech. Rzeszowskiej Mat., (1996), pp. 101–105.

[34] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi, Univalent functions with positive coefficients, Tamkang J. Math., 25 (1994), pp. 225–230.

[35] H. Y. Zhang, H. Tang, and X. M. Niu, Third-order hankel determinant for certain class of analytic functions related with exponential function, Symmetry, 10 (2018), p. 501.

Department of Mathematics, University of Delhi, Delhi–110 007, India
E-mail address: adibanaz81@gmail.com

Bharati Vidyapeeth’s College of Engineering, Delhi–110 063 India
E-mail address: sushilkumar16n@gmail.com

Department of Mathematics, National Institute of Technology, Tiruchirappalli–620 015, India
E-mail address: vravi68@gmail.com