Let \( n \) be a positive integer. The aim of this paper is to study two local moves \( V(n) \) and \( V^n \) on welded links, which are generalizations of the virtualization move. We show that a \( V(n) \)-move is an unknotting operation on welded knots for any \( n \), and give a classification of welded links up to \( V(n) \)-move. On the other hand, we give a necessary condition for which two welded links are equivalent up to \( V^n \)-move. This leads to show that a \( V^n \)-move is not an unknotting operation for welded knots except \( n = 1 \). We also discuss relations among \( V^n \)-moves, associated core groups and the multiplexing of crossings.

1. Introduction

A \( \mu \)-component virtual link diagram is an immersion of \( \mu \) circles in the plane, whose singularities are transverse double points divided into classical crossings and virtual crossings shown in Figure 1.1. Note that we do not use here the usual drawing convention for virtual crossings, which is a small circle around the corresponding double point.

A virtual link is an equivalence class of virtual link diagrams under generalized Reidemeister moves, which consist of Reidemeister moves R1–3 and virtual moves VR1–4 shown in Figure 1.2. In virtual context, there are two forbidden local moves OC and UC (meaning over-crossings and under-crossings commute, respectively) shown in Figure 1.3. An extension of the generalized Reidemeister moves which also allows the OC move is called welded Reidemeister moves, and a sequence of welded Reidemeister moves is called a welded isotopy. A welded link is an equivalence class of virtual link diagrams under welded isotopy.

M. Goussarov, M. Polyak and O. Viro [7] essentially proved that welded isotopic classical link diagrams are equivalent, that is, they can be related by a sequence of Reidemeister moves R1–3. Therefore, we can consider welded links as a natural extension of classical links. We remark that Goussarov, Polyak and Viro [7], T. Kanenobu [10] and S. Nelson [20] independently proved that any virtual knot diagram can be unknotted by a sequence of UC moves and welded Reidemeister moves. This result is one reason why the UC move is still forbidden in welded context.

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In classical knot theory, local moves have played important roles and hence have been studied widely, see for example [17, 16, 8, 14, 4, 5]. Recently, some ‘classical’ local moves have been studied for welded knots and links [2, 22, 18]. We will study ‘non-classical’ local moves for welded links. A typical non-classical local move is the virtualization move. The virtualization move is a local move on a virtual link diagram replacing a classical crossing by a virtual one or vice versa, see the left-hand side of Figure 1.4. We remark that any virtual link diagram can be deformed into a diagram of a trivial link by a sequence of virtualization moves.

The virtualization move is equivalent to the local move shown in the right-hand side of Figure 1.4, that is, each move is realized by a sequence of the other moves (and welded Reidemeister moves). In this paper, we introduce two local moves $V(n)$ and $V^n$ shown in the upper and lower sides of Figure 1.5 respectively, for each positive integer $n$. They are considered as generalizations of the virtualization move. Note that both $V(1)$- and $V^1$-moves are equivalent to the virtualization move, and that if $n$ is even then a $V(n)$-move may change the number of components. Two welded links are $V(n)$-equivalent (resp. $V^n$-equivalent) if their diagrams are related by a sequence of $V(n)$-moves (resp. $V^n$-moves) and welded Reidemeister moves.
We have that a $V(n)$-move is an unknotting operation on welded knots for any $n$ because a UC move is realized by a sequence of $V(n)$-moves and welded Reidemeister moves (Proposition 1.4). Moreover, we give a complete classification of welded links up to $V(n)$-equivalence in the sense of Theorems 1.1 and 1.2.

**Theorem 1.1.** Let $n$ be an even number. Any welded link is $V(n)$-equivalent to the unknot.

Let $D$ be an ordered oriented virtual link diagram. For any $i, j$ ($i \neq j$) let $\ell_{ij}(D)$ denote the sum of the signs of all classical crossings of $D$ whose overpasses and underpasses belong to the $i$th and $j$th component, respectively. $\ell_{ij}(D)$ is a welded link invariant, and is also preserved by UC moves. For an ordered oriented welded link $L$, the ordered linking number $\ell_{ij}(L)$ between the $i$th and $j$th components is defined to be $\ell_{ij}(D)$ for a diagram $D$ of $L$.

It is not hard to see that if $n$ is odd then the modulo-$n$ reduction of $\ell_{ij}(L) + \ell_{ji}(L)$ is preserved by $V(n)$-moves. Using these invariants, we have the following theorem.

**Theorem 1.2.** Let $n$ be an odd number. Two ordered oriented $\mu$-component welded links $L$ and $L'$ are $V(n)$-equivalent if and only if $\ell_{ij}(L) + \ell_{ji}(L) \equiv \ell_{ij}(L') + \ell_{ji}(L') \pmod{n}$ for any $i, j$ ($1 \leq i < j \leq \mu$).

On the other hand, $V^n$-moves preserve the modulo-$n$ reduction of $\ell_{ij}(L)$ for any positive integer $n$. However, these invariants are not strong enough to classify welded links up to $V^n$-equivalence since a $V^n$-move is not an unknotting operation for welded knots except $n = 1$ (Proposition 5.2). By combining with a UC move being an unknotting operation for welded knots, we have the following theorem.

**Theorem 1.3.** Let $n$ be a positive integer. Two ordered oriented $\mu$-component welded links $L$ and $L'$ are $(V^n + UC)$-equivalent if and only if $\ell_{ij}(L) \equiv \ell_{ij}(L') \pmod{n}$ for any $i, j$ ($1 \leq i \neq j \leq \mu$).

Here, two welded links are $(V^n + UC)$-equivalent if their diagrams are related by a sequence of $V^n$-moves, UC moves and welded Reidemeister moves.

We also discuss relations among $V^n$-moves, associated core groups and the multiplexing of crossings. The associated core group is known as a classical link invariant [6, 9, 12, 23]. This group is naturally extended to a welded link invariant. Furthermore, the associated core group of a welded link is preserved by $V^n$-moves for any even number $n$ (Proposition 6.2). The authors [15] introduced the notion of multiplexing of crossings, which yields a new welded link $L(m_1, \ldots, m_\mu)$ for a $\mu$-tuple $(m_1, \ldots, m_\mu)$ of integers and an ordered $\mu$-component welded link $L$. For any $\mu$-tuple $(m_1, \ldots, m_\mu)$ of even numbers, $L(m_1, \ldots, m_\mu)$ is $V^2$-equivalent to the $\mu$-component trivial link (Proposition 6.3). As a consequence, we have that there exist infinitely many nontrivial welded knots whose associated core groups are isomorphic to that of the unknot (Theorem 7.1).

The main tool for the proofs is Arrow calculus, introduced by J.-B. Meilhan and the third author in [15], which is similar to Gauss diagrams but more 'interacts' with welded links.

2. Arrow calculus

In this section, we will briefly recall the basic notions of Arrow calculus from [15]. We only need the notion of $w$-arrow, and refer the reader to [15] for more details of Arrow calculus.

**Definition 2.1.** Let $D$ be an oriented virtual link diagram. A $w$-arrow $\gamma$ for $D$ is an oriented arc immersed in the plane of the diagram such that:

1. the endpoints of $\gamma$ are contained in $D \setminus \{\text{crossings of } D\}$,
2. for all $i, j$ for which the endpoints of $\gamma$ belong to the $i$th and $j$th components of $D$, we have $\ell_{ij}(D) + 2 \cdot \text{sign}(\gamma) = 0 \pmod{n}$.
all singularities of $\gamma$ are virtual crossings,
(3) all singularities between $D$ and $\gamma$ are virtual crossings, and
(4) $\gamma$ has a number (possibly zero) of decorations $\bullet$ on the interior of $\gamma$, called twists, which are disjoint from all crossings.

The initial and terminal points of $\gamma$ are called the tail and the head, respectively.

For a union of w-arrows for $D$, all crossings among w-arrows are assumed to be virtual.

We note that diagrams are drawn with bold lines while w-arrows are drawn with thin lines from now on.

Let $\mathcal{A}$ be a union of w-arrows for $D$. We next define surgery along $\mathcal{A}$ on $D$ which yields a new diagram, denoted by $D_{\mathcal{A}}$, as follows. Suppose that there exists a disk in the plane which intersects $D \cup \mathcal{A}$ as shown in Figure 2.1. Then the figure indicates the result of surgery along a w-arrow of $\mathcal{A}$ on $D$. We emphasize that the surgery move depends on the orientation of the strand of $D$ containing the tail of the w-arrow.

\begin{center}
\begin{align*}
D_{\cup A} & = D_A \\
D \cup A & = D_A
\end{align*}
\end{center}

\textbf{Figure 2.1.} Surgery along a w-arrow of $\mathcal{A}$ on $D$

If a w-arrow of $\mathcal{A}$ intersects a (possibly the same) w-arrow (resp. $D$), then the result of surgery is essentially same as above but each intersection introduces virtual crossings shown in the left-hand side (resp. center) of Figure 2.2. Moreover, if a w-arrow of $\mathcal{A}$ has some twists, then each twist is converted to a half-twist whose crossing is virtual, see the right-hand side of Figure 2.2.

\begin{center}
\begin{align*}
\mathcal{A} & = D_A \\
D & = D_A \\
D_A & = D_A
\end{align*}
\end{center}

\textbf{Figure 2.2.}

An Arrow presentation for an oriented virtual link diagram $D$ is a pair $(T, \mathcal{A})$ of a virtual link diagram $T$ without classical crossings and a union $\mathcal{A}$ of w-arrows for $T$ such that $T_{\mathcal{A}}$ is welded isotopic to $D$. We remark that any virtual link diagram has an Arrow presentation because any classical crossing can be replaced by a virtual one with a w-arrow, see Figure 2.3.

\begin{center}
\begin{align*}
VR2 & =
\end{align*}
\end{center}

\textbf{Figure 2.3.} Any classical crossing can be replaced by a virtual one with a w-arrow.

Two Arrow presentations are equivalent if the surgeries yield welded isotopic virtual link diagrams. Arrow moves consist of virtual moves VR1–3 involving w-arrows and/or strands of $D$ and the local moves AR1–10 on Arrow presentations.
shown in Figure 2.4. Here, each vertical strand in the AR1–3 moves is either a strand of D or a w-arrow, and the symbol on a w-arrow in the AR8 and AR10 moves denotes that the w-arrow may or may not contain a twist. It is shown that two Arrow presentations are equivalent if and only if they are related by a sequence of Arrow moves [13, Theorem 4.5].

In the rest of this section, we will introduce several local moves on Arrow presentations. We first define local moves AR11 and AR12 as shown in Figures 2.5 and 2.6, respectively. They are realized by a sequence of Arrow moves as shown in Figure 2.7, where in the figure denotes a sequence of Arrow moves. While Figure 2.7 shows that the left-hand side moves in Figures 2.5 and 2.6 are realized by Arrow moves, the other cases are similarly shown.

A heads exchange move is a local move on an Arrow presentation exchanging positions of consecutive two heads of w-arrows, see Figure 2.8. While there exist
several kinds of heads exchange moves depending on the orientation of the strand containing the tail and existence or nonexistence of a twist for a w-arrow, we have the following.

\[\begin{array}{c}
\text{Figure 2.7. AR11 and AR12 moves are realized by a sequence of Arrow moves.}
\end{array}\]

\[\begin{array}{c}
\text{Figure 2.8. Heads exchange move}
\end{array}\]

**Sublemma 2.2.** A heads exchange move is realized by a sequence of the $H$ move shown in Figure 2.9 and Arrow moves.

\[\begin{array}{c}
\text{Figure 2.9. H move}
\end{array}\]

**Proof.** We demonstrate that two moves of them are realized by sequences of $H$ moves and Arrow moves. The upper side of Figure 2.10 indicates the proof when the orientation of the strand containing the tail of a single w-arrow is opposite to that of the $H$ move. The lower side of Figure 2.10 indicates the proof for the case where a w-arrow has a twist. It is not hard to show the other cases. \hfill \square

\[\begin{array}{c}
\text{Figure 2.10. A heads exchange move is realized by a sequence of an $H$ move and Arrow moves.}
\end{array}\]

A *head-tail exchange move* is a local move on an Arrow presentation exchanging positions of consecutive a head and a tail of w-arrows, see Figure 2.11.
Sublemma 2.3. A head-tail exchange move is realized by a sequence of a heads exchange move and Arrow moves.

Proof. See Figure 2.12. □

Three kinds of moves, AR7, heads exchange and head-tail exchange moves, are called ends exchange moves. From Sublemmas 2.2 and 2.3, we obtain the following lemma.

Lemma 2.4. An ends exchange move is realized by a sequence of an H move and Arrow moves.

3. Local moves on diagrams and Arrow presentations

In this section, we will study some local moves on virtual link diagrams and those on Arrow presentations. We start with the following lemma concerning a UC move.

Lemma 3.1. An Arrow presentation for a UC move is realized by a sequence of a heads exchange move and Arrow moves. Conversely, surgery along a heads exchange move is realized by a sequence of a UC move and welded Reidemeister moves.

Proof. Figure 3.1 shows that an Arrow presentation for a UC move is realized by a heads exchange move and Arrow moves. In the figure, we choose certain orientations of two strands at the virtual crossing. The other cases are similarly shown.

Conversely, Figure 3.2 shows that surgery along an H move is realized by a sequence of a UC move and welded Reidemeister moves, where \( \sim \) in the figure denotes a welded isotopy. This and Sublemma 2.2 complete the proof. □
Figure 3.2. Surgery along an H move is realized by a sequence of a UC move and welded Reidemeister moves.

We define an $A(n)$-move as a local move on an Arrow presentation depending on the parity of $n$. An $A(n)$-move is shown in Figure 3.3 (resp. Figure 3.4) when $n$ is odd (resp. even). It is easy to see that surgery along an $A(n)$-move is realized by a $V(n)$-move.

Remark 3.2. Two local moves shown in Figure 3.3 are equivalent. Figure 3.5 shows that the right-hand side move in Figure 3.3 is realized by a sequence of the left-hand side move in Figure 3.3 and Arrow moves. The proof for the other case can be done by deformations similar to those in Figure 3.5. We can also show that two local moves in Figure 3.4 are equivalent.

We have the following lemma.

Lemma 3.3. An ends exchange move is realized by a sequence of $A(n)$-moves and Arrow moves.

Proof. By Lemma 2.4, it suffices to show that an H move is realized by a sequence of $A(n)$-moves and Arrow moves for any $n$. The upper (resp. lower) side of Figure 3.6 indicates the proof for the case where $n$ is odd (resp. even), while the figure describes only when $n = 3$ (resp. $n = 2$).
Figure 3.5.

As a consequence of Lemmas 3.1 and 3.3, we have the following proposition.

**Proposition 3.4.** Let $n$ be a positive integer. A UC move is realized by a sequence of $V(n)$-moves and welded Reidemeister moves. Hence, a $V(n)$-move is an unknotting operation for welded knots.

Next we define an $A_n$-move as a local move on an Arrow presentation shown in Figure 3.7. Note that surgery along an $A_n$-move is realized by a $V_n$-move. We have the following lemma.

**Lemma 3.5.** Let $n$ be an odd number. An $A_n$-move is realized by a sequence of $A(n)$-moves and Arrow moves.

*Proof.* We consider the local move $M$ shown in Figure 3.8 which is realized by a sequence of $A(n)$-moves and AR9 moves. (Figure 3.9 shows that one of the $M$ moves
is realized by a sequence of $A(n)$-moves and AR9 moves. The other case is similarly shown.) Combining M moves with an $A(n)$-move, we obtain an $A^n$-move. □

**Figure 3.8.** M move

![M move diagram]

**Figure 3.9.** An M move is realized by a sequence of $A(n)$-moves and AR9 moves.

**Remark 3.6.** When $n$ is odd we may consider oriented $V(n)$-moves. Then we have four kinds of oriented $V(n)$-moves depending on the orientations of two parallel strands. We note that there exists one-to-one correspondence between the four oriented $V(n)$-moves and the four local moves in Figures 3.3 and 3.11. We can show that two local moves in Figure 3.11 are equivalent by deformations similar to those in Figure 3.3. These facts and Remark 3.2 imply that there are essentially two kinds of oriented $V(n)$-moves as in the upper and lower sides of Figure 3.11 which correspond to two moves in the left-hand sides of Figures 3.3 and 3.10, respectively. Figure 3.12 shows that the move in the left-hand side of Figure 3.10 is realized by a sequence of the move in that of Figure 3.3 and Arrow moves. While the proof is described only when $n = 3$ in the figure, it is essentially the same for any positive integer $n$. It follows that oriented $V(n)$-moves can be realized by the moves in the upper side of Figure 3.11.

**Figure 3.10.**
In this section, we will give proofs of Theorems 1.1, 1.2 and 1.3.

**Proof of Theorem 1.1.** If \( n \) is even then any welded link can be deformed into a welded knot by a sequence of \( V(n) \)-moves since \( V(n) \)-moves can change the number of components of the welded link. Therefore, Theorem 1.1 follows from Proposition 3.4. \( \square \)

Let \( \mathbf{1} \) be the ordered oriented \( \mu \)-component trivial string link diagram without crossings such that all strands are oriented upwards. For an integer \( a \), \((\mathbf{1}, H_{ij}(a))\) is an Arrow presentation as in Figure 4.1, that is, \( H_{ij}(a) \) consists of \(|a|\) horizontal w-arrows whose tails (resp. heads) are attached to the \( i \)th (resp. \( j \)th) strand of \( \mathbf{1} \) (\( 1 \leq i < j \leq \mu \)) such that each w-arrow has exactly one twist if \( a \geq 0 \) or no twist if \( a < 0 \). Note that, for Arrow presentations \((\mathbf{1}, H_{ij}(a))\) and \((\mathbf{1}, H_{kl}(a'))\), the stacking products \((\mathbf{1}, H_{ij}(a)) * (\mathbf{1}, H_{kl}(a'))\) and \((\mathbf{1}, H_{kl}(a')) * (\mathbf{1}, H_{ij}(a))\) are related by a sequence of ends exchange moves and Arrow moves, hence, by a sequence of \( \Lambda(n) \)-moves and Arrow moves. Let \( \prod_{1 \leq i < j \leq n} (\mathbf{1}, H_{ij}(a_{ij})) \) denote the stacking products of \((\mathbf{1}, H_{ij}(a_{ij}))\) for integers \( a_{ij} \) (\( 1 \leq i < j \leq \mu \)). We remark that the ordered linking numbers \( \ell_{ij} \) and \( \ell_{ji} \) of the closure of the stacking products \( \prod_{1 \leq i < j \leq \mu} 1_{H_{ij}(a_{ij})} \) are equal to \( a_{ij} \) and 0, respectively (\( 1 \leq i < j \leq \mu \)).

To prove Theorem 1.2, we need the following lemma.
Lemma 4.1. Let $n$ be an odd number. For any ordered oriented $\mu$-component virtual string link diagram $D$, there are integers $a_{ij}$ with $0 \leq a_{ij} < n$ ($1 \leq i < j \leq \mu$) such that an Arrow presentation for $D$ and $\prod_{1 \leq i < j \leq n}(1,H_{ij}(a_{ij}))$ are related by a sequence of $A(n)$-moves and Arrow moves.

Proof. Let $(1,\bigcup_{1 \leq i,j \leq \mu}W_{ij})$ be an Arrow presentation for an ordered oriented $\mu$-component virtual string link diagram where $W_{ij}$ is a set of w-arrows for 1 whose tails (resp. heads) are attached to the $i$th (resp. $j$th) strand ($1 \leq i,j \leq \mu$, possibly $i = j$). We show that $(1,\bigcup_{1 \leq i,j \leq \mu}W_{ij})$ can be deformed into the desired form by a sequence of $A(n)$-moves and Arrow moves (including ends exchange moves, M moves and $A^n$-moves). First, the ends of each w-arrow in $W_{ii}$ ($1 \leq i \leq \mu$) can be moved into position to be removed by an AR8 move. Hence, all w-arrows in $W_{ii}$ are removed for any $i$. Next, $(1,\bigcup_{1 \leq i,j \leq \mu}W_{ij})$ can be deformed into $\prod_{1 \leq i < j \leq \mu}(1,H_{ij}(a_{ij}))$ for some integers $a_{ij}$ by combining M moves, ends exchange moves and AR9 moves. Finally, we have the desired form by performing $A^n$-moves and AR9 moves. \hfill \Box

Proof of Theorem 1.2. It suffices to show the ‘if’ part. Let $D$ and $D'$ be virtual string link diagrams of $L$ and $L'$, respectively. For any virtual string link diagram, there exists a virtual string link diagram whose closure is welded isotopic to the virtual link diagram. Hence, by Lemma 4.1, Arrow presentations $(T,A)$ for $D$ and $(T',A')$ for $D'$ can be related to the closures of $\prod_{1 \leq i < j \leq n}(1,H_{ij}(a_{ij}))$ and $\prod_{1 \leq i < j \leq n}(1,H_{ij}(a'_{ij}))$ respectively, by a sequence of $A(n)$-moves and Arrow moves for some non-negative integers $a_{ij}, a'_{ij}$ ($\leq n$). Then we have that for any $i,j$ ($1 \leq i < j \leq \mu$),

$$a_{ij} \equiv \ell_{ij}(D) + \ell_{ji}(D) \equiv \ell_{ij}(D') + \ell_{ji}(D') \equiv a'_{ij} \pmod{n}.$$ 

Since $0 \leq a_{ij}, a'_{ij} < n$, we have that $a_{ij} = a'_{ij}$. Therefore $(T,A)$ and $(T',A')$ are related by a sequence of $A(n)$-moves and Arrow moves. Consequently, $D(= T_A)$ and $D'(= T'_A)$ are related by a sequence of $V(n)$-moves and welded Reidemeister moves. \hfill \Box

For an integer $b$, $(1,\overline{P}_{ij}(b))$ is an Arrow presentation as in Figure 4.2 that is, $\overline{P}_{ij}(b)$ consists of $|b|$ horizontal w-arrows whose heads (resp. tails) are attached to the $i$th (resp. $j$th) strand of 1 ($1 \leq i < j \leq \mu$) such that each w-arrow has no twist if $b \geq 0$ or exactly one twist if $b < 0$. We remark that, for integers $a_{ij}$ and $b_{ij}$, the ordered linking numbers $\ell_{ij}$ and $\ell_{ji}$ of the closure of the stacking products $\prod_{1 \leq i < j \leq \mu}(1,H_{ij}(a_{ij}) \ast 1,\overline{P}_{ij}(b_{ij}))$ are equal to $a_{ij}$ and $b_{ij}$, respectively ($1 \leq i < j \leq \mu$).

Then we have the following lemma.

**Figure 4.2. Arrow presentation $(1,\overline{P}_{ij}(b))$**

Lemma 4.2. Let $n$ be a positive integer. For any ordered oriented $\mu$-component virtual string link diagram $D$, there are integers $a_{ij}, b_{ij}$ with $0 \leq a_{ij}, b_{ij} < n$ ($1 \leq i \leq \mu$).
$i < j \leq \mu$ such that an Arrow presentation for $D$ and $\prod_{1 \leq i < j \leq n}(1, H_{ij}(a_{ij})) * (1, T_{ij}(b_{ij}))$ are related by a sequence of $A^\mu$-moves, ends exchange moves and Arrow moves.

The proof of Lemma 4.2 can be done by a similar way to the proof of Lemma 4.1. Note that we are not permitted to use $M$ moves. This is the reason why we need not only $H_{ij}(a)$ but also $H_{ij}(b)$.

Proof of Theorem 5.3. It suffices to show the ‘if’ part. Let $D$ and $D'$ be virtual link diagrams of $L$ and $L'$, respectively. By Lemma 4.2, Arrow presentations $(T, A)$ for $D$ and $(T', A')$ for $D'$ can be related to the closures of $\prod_{1 \leq i < j \leq n}(1, H_{ij}(a_{ij})) * (1, T_{ij}(b_{ij}))$ and $\prod_{1 \leq i < j \leq n}(1, H_{ij}(a'_{ij})) * (1, T_{ij}(b'_{ij}))$, respectively, by a sequence of $A^\mu$-moves, ends exchange moves and Arrow moves for some non-negative integers $a_{ij}, b_{ij}, a'_{ij}, b'_{ij} (< n)$. Then we have that for any $i, j (1 \leq i < j \leq \mu)$,

$$a_{ij} \equiv \ell_{ij}(D) \equiv \ell_{ij}(D') \equiv a'_{ij} \pmod{n}$$

and

$$b_{ij} \equiv \ell_{ji}(D) \equiv \ell_{ji}(D') \equiv b'_{ij} \pmod{n}.$$  

Since $0 \leq a_{ij}, b_{ij}, a'_{ij}, b'_{ij} < n$, we have that $a_{ij} = a'_{ij}$ and $b_{ij} = b'_{ij}$. Therefore $(T, A)$ and $(T', A')$ are related a sequence of by $A^\mu$-moves, ends exchange moves and Arrow moves. It follows from Lemmas 2.4 and 5.1 that $D (= T_A)$ and $D' (= T_{A'})$ are related by a sequence of $V^n$-moves, UC moves and welded Reidemeister moves. \hfill \Box

5. $V^n$-moves and UC moves

As mentioned in Section 1, a $V^n$-move is not an unknotted operation except $n = 1$. To prove this, we use the **elementary ideals** (in the sense of [9]) obtained from the group of a welded link by using the Fox free derivative. Here, the **group** of an oriented virtual link diagram is known to be a welded link invariant [11, Section 4], and hence the elementary ideals are naturally extended to welded link invariants. By a similar way to the proof of Theorem 1 in [19], we can show the following proposition.

**Proposition 5.1.** Let $n$ be a positive integer. If two oriented welded links $L$ and $L'$ are $V^n$-equivalent, then their $k$th elementary ideals are congruent modulo $I(1 - t^n)$ for a non-negative integer $k$, where $I(1 - t^n)$ is the ideal generated by $1 - t^n$ in $\mathbb{Z}[t^{\pm 1}]$.

**Proof.** Let $D$ and $D'$ be virtual link diagrams of $L$ and $L'$, respectively. It suffices to show that if $D$ and $D'$ are related by a single $V^n$-move then for their properly chosen Alexander matrices $A_D(t)$ and $A_{D'}(t)$,

$$A_D(t) \equiv A_{D'}(t) \pmod{I(1 - t^n)}.$$  

Suppose that $D'$ is obtained from $D$ by an R1 move and a $V^n$-move, and put labels $x_1, x_2$ and $x_3$ on arcs of $D$ and $D'$ as shown in Figure 5.1 and labels $x_4, \ldots, x_t$ on the other arcs outside the figure.

![Figure 5.1](https://example.com/figure5.1.png)
Then, we have group presentations of the groups $G(D)$ and $G(D')$ of $D$ and $D'$, respectively, as follows:

$$G(D) = \langle x_1, x_2, x_3, x_4, \ldots, x_l \mid x_1x_2^{-1}\{r_i\}\rangle,$$

$$G(D') = \langle x_1, x_2, x_3, x_4, \ldots, x_l \mid x_1x_2^{-1}x_3^{-1}x_3^{-1}\{r_i\}\rangle,$$

where $\{r_i\}$ is the set of relations corresponding to the other crossings. By using the Fox free derivative [3], we have the Alexander matrices $A_D(t)$ and $A_{D'}(t)$ of $D$ and $D'$, respectively, as follows:

$$A_D(t) = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ a_{\gamma} & (r_i \ x_j) \end{pmatrix}, \quad A_{D'}(t) = \begin{pmatrix} 1 & -t^n & t^n-1 & 0 & \cdots & 0 \\ a_{\gamma} & (r_i \ x_j) \end{pmatrix}.$$ 

Therefore, $A_D(t) - A_{D'}(t)$ is a zero matrix modulo $I(1-t^n)$.

Now we have the following proposition.

**Proposition 5.2.** A $V^n$-move is not an unknotting operation on welded knots for $n \geq 2$.

**Proof.** We show that the trefoil knot is not $V^n$-equivalent to the unknot for $n \geq 2$. The first elementary ideal (the Alexander polynomial) of the trefoil knot is $1-t+t^2$, and that of the unknot is 1. We regard $\mathbb{Z}[t^{\pm 1}]$ and $I(1-t^n)$ as abelian groups. Since the first elementary ideal of a knot is determined within $\pm t^k$ ($k \in \mathbb{Z}$), it suffices to show that $1-t+t^2 + \varepsilon t^k \notin I(1-t^n)$ for any $n \geq 2$ ($\varepsilon \in \{\pm 1\}$) by Proposition 5.1.

Suppose that $n \geq 2$. We define a group homomorphism $f_n : \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ by $f_n(\sum t^i a_it^i) = \sum_{i=0}^{n} a_i$, where $a_i \in \mathbb{Z}$. Then we have that $f_n(\delta t^i(1-t^n)) = 0$ for any $\delta t^i(1-t^n) \in I(1-t^n)$ ($l \in \mathbb{Z}, \delta \in \{\pm 1\}$). On the other hand, we have that $f_n(1-t+t^2 + \varepsilon t^k)$ is not equal to 0 for any $n \geq 2$. This completes the proof. 

The proposition above immediately implies the following corollary.

**Corollary 5.3.** A UC move is realized by a sequence of $V^n$-moves and welded Reidemeister moves if and only if $n = 1$.

6. $V^n$-MOVES AND ASSOCIATED CORE GROUPS

In this section, we will discuss relations among $V^n$-moves, associated core groups and the multiplexing of crossings.

For an unoriented classical link diagram $D$, the associated core group $\Pi_D^{(2)}$ is defined as follows. Each arc of $D$ yields a generator, and each classical crossing gives a relation $yx^{-1}yz^{-1}$, where $x$ and $z$ correspond to the underpasses and $y$ corresponds to the overpass at the crossing. This group $\Pi_D^{(2)}$ is known as a classical link invariant [9, 12, 16, 23].

**Remark 6.1.** Let $L$ be a classical link in the 3-sphere and $D$ a diagram of $L$. M. Wada [23] proved that $\Pi_D^{(2)}$ is isomorphic to the free product of the fundamental group of the double branched cover $M_L^{(2)}$ of the 3-sphere branched along $L$ and the infinite cyclic group $\mathbb{Z}$. Namely, $\Pi_D^{(2)} \cong \pi_1(M_L^{(2)}) * \mathbb{Z}$.

We similarly define the associated core group $\Pi_L^{(2)}$ of an unoriented virtual link diagram $D$ by generators and relations as described above. (Note that virtual crossings do not produce any generator or relation.) It is not hard to see that $\Pi_D^{(2)}$ is a welded link invariant, and we define the associated core group $\Pi_L^{(2)}$ of a welded link $L$ to be $\Pi_D^{(2)}$ of a diagram $D$ of $L$. Moreover, we have the following proposition.
Proposition 6.2. If \( n \) is even, then \( \Pi_L^{(2)} \) is preserved by \( V^n \)-moves.

Proof. \( \Pi_L^{(2)} \) is preserved by \( V^2 \)-moves as shown in Figure 6.1 and furthermore, a \( V^n \)-move is realized by a sequence of \( V^2 \)-moves for any even number \( n \). \( \square \)

There exist welded knots whose associated core groups are nontrivial, for example, all knots having nontrivial Fox colorings, see [21, Proposition 4.5]. Therefore, the proposition above gives an alternative proof for that a \( V^n \)-move is not an unknotting operation on welded knots for any even number \( n \).

In [15], the authors introduced the multiplexing of crossings for a virtual link diagram, which yields a new virtual link diagram. Let \( (m_1, \ldots, m_\mu) \) be a \( \mu \)-tuple of integers, and let \( D = \bigcup_{i=1}^\mu D_i \) be an ordered \( \mu \)-component virtual link diagram. For a classical crossing of \( D \) whose overpass belongs to \( D_j \), we define the multiplexing of the crossing associated with \( m_j \) as a local change shown in Figure 6.2. When \( m_j = 0 \), the multiplexing of the crossing is the virtualization move of it. The number of the classical crossings that appear in the multiplexing of the crossings is the absolute value of \( m_j \). Let \( D(m_1, \ldots, m_\mu) \) denote the virtual link diagram obtained from \( D \) by the multiplexing of all classical crossings of \( D \) associated with \( (m_1, \ldots, m_\mu) \). For welded isotopic virtual link diagrams \( D \) and \( D', D(m_1, \ldots, m_\mu) \) and \( D'(m_1, \ldots, m_\mu) \) are also welded isotopic for any \( (m_1, \ldots, m_\mu) \in \mathbb{Z}^\mu \) [15, Theorem 2.1]. For an ordered \( \mu \)-component welded link \( L \), we define \( L(m_1, \ldots, m_\mu) \) to be \( D(m_1, \ldots, m_\mu) \) of a diagram \( D \) of \( L \).

It is not hard to see that \( L(m_1, \ldots, m_\mu) \) is \( V^2 \)-equivalent to \( L(0, \ldots, 0) \) for any \( \mu \)-tuple \( (m_1, \ldots, m_\mu) \) of even numbers. Since \( L(0, \ldots, 0) \) is trivial, we have the following proposition.

Proposition 6.3. Let \( (m_1, \ldots, m_\mu) \) be a \( \mu \)-tuple of even numbers. For any ordered \( \mu \)-component welded link \( L \), \( L(m_1, \ldots, m_\mu) \) is \( V^2 \)-equivalent to the \( \mu \)-component trivial link.
In [15, Theorem 3.2], the authors proved that classical knots $K$ and $K'$ are equivalent up to mirror image if and only if $K(m)$ and $K'(m)$ are welded isotopic up to mirror image for any fixed non-zero integer $m$. Hence, we have that if a classical knot $K$ is nontrivial then $K(m)$ is also nontrivial. By Propositions 6.2 and 6.3, if $m$ is even then $\Pi^2(K(m))$ is isomorphic to that of the unknot, that is, $\Pi^2(K(m)) \cong \mathbb{Z}$. Therefore, we have the following theorem although by Remark 6.1 the associated core groups seem to be very strong invariants.

**Theorem 6.4.** Let $m \neq 0$ be an even number. For any nontrivial welded knot $K$, $K(m)$ is nontrivial and $\Pi^2(K(m)) \cong \mathbb{Z}$.

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