Determining implicit equation of conic section from quadratic rational Bézier curve using Gröbner basis

Y R Anwar, H Tasman, and N Hariadi

Department of Mathematics, Faculty of Mathematics and Natural Science, Universitas Indonesia, Depok 16424, Indonesia
E-mail: htasman@sci.ui.ac.id

Abstract. The Gröbner Basis is a subset of finite generating polynomials in the ideal of the polynomial ring \( k[x_1, \ldots, x_n] \). The Gröbner basis has a wide range of applications in various areas of mathematics, including determining implicit polynomial equations. The quadratic rational Bézier curve is a rational parametric curve that is generated by three control points \( P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2) \) in \( \mathbb{R}^2 \) and weights \( \omega_0, \omega_1, \omega_2 \), where the weights \( \omega_i \) are corresponding to control points \( P_i(x_i, y_i) \), for \( i = 0, 1, 2 \). According to Cox et al (2007), the quadratic rational Bézier curve can represent conic sections, such as parabola, hyperbola, ellipse, and circle, by defining the weights \( \omega_0 = \omega_2 = 1 \) and \( \omega_1 = \omega \) for any control points \( P_0(x_0, y_0), P_1(x_1, y_1), \) and \( P_2(x_2, y_2) \). This research is aimed to obtain an implicit polynomial equation of the quadratic rational Bézier curve using the Gröbner basis. The polynomial coefficients of the conic section can be expressed in the term of control points \( P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2) \) and weight \( \omega \), using Wolfram Mathematica. This research also analyzes the effect of changes in weight \( \omega \) on the shape of the conic section. It shows that parabola, hyperbola, and ellipse can be formed by defining \( \omega = 1, \omega > 1 \), and \( 0 < \omega < 1 \), respectively.

1. Introduction
The Gröbner basis is a subset of an ideal in the polynomial ring \( k[x_1, \ldots, x_n] \), which contains finite generating polynomials [1]. Bruno Buchberger first introduced it in 1965 [2]. Currently, it has a wide range of applications in various areas of mathematics, including determining implicit polynomial equations [3–7].

In the early 1960’s, Dr. Pierre Bezier developed the Bezier curves to construct a curve formulation for use in shape design. The Bézier curves can be either polynomial or rational function. The quadratic rational Bézier curve generated by three control points \( P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2) \) in \( \mathbb{R}^2 \) and weights \( \omega_0, \omega_1, \omega_2 \), which are corresponding to control points \( P_i(x_i, y_i) \), for \( i = 0, 1, 2 \). The advantage of the quadratic rational Bézier curve is it can represent the conic sections, such as parabola, hyperbola, ellipse, and circle [1,8,9].

In this paper, the Gröbner basis is implemented to determine the implicit polynomial equation of the conic section from the quadratic rational Bézier curve using Wolfram Mathematica. The polynomial coefficients of the conic section can be expressed in the term of control points \( P_0, P_1, P_2 \), and weight \( \omega \). Furthermore, we also analyze the effect of changes in weight \( \omega \) on the shape of the conic section.
2. Preliminaries
First, we recall the definition of polynomial, the ideal of the polynomial ring, Gröbner basis, and rational Bézier curve.

2.1. Polynomial
A polynomial \( f \) is a linear combination of monomials \( x^\alpha \) with coefficients in field \( k \), denoted by

\[
f = \sum_{\alpha} a\alpha x^\alpha, \quad a\alpha \in k.
\]

Furthermore, \( a\alpha x^\alpha \) is called a term of \( f \), where \( a\alpha \neq 0 \). The set of all polynomials with variables \( x_1, \ldots, x_n \) and coefficients in field \( k \) form a polynomial ring, denoted by \( k[x_1, \ldots, x_n] \) [1].

The terms of a polynomial can be ordered in an unambiguous way using lexicographic order. A term \( x^\alpha \) is greater than \( x^\beta \) in lexicographic order if the leftmost nonzero entry of \( \alpha - \beta \) is positive. The greatest term of the polynomial \( f \) in lexicographic order is called a leading term, denoted by \( \text{LT}(f) \) [1].

2.2. Ideal of Polynomial Ring
An ideal of the polynomial ring \( k[x_1, \ldots, x_n] \) is a nonempty subset \( I \) that satisfy [1]:

(i) \( 0 \in I \);
(ii) For every \( f, g \in I \), then \( f + g \in I \);
(iii) For every \( f \in I \) and \( h \in k[x_1, \ldots, x_n] \), then \( hf \in I \).

An ideal \( I \subset k[x_1, \ldots, x_n] \) is called finitely generated by polynomials \( f_1, \ldots, f_s \in k[x_1, \ldots, x_n] \) if every polynomial in \( I \) can be expressed as a combination of polynomials \( f_1, \ldots, f_s \), denoted by \( I = \langle f_1, \ldots, f_s \rangle \), where

\[
\langle f_1, \ldots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, \ldots, h_s \in k[x_1, \ldots, x_n] \right\}.
\]

Furthermore, \( f_1, \ldots, f_s \) are called a basis of an ideal \( I \subset k[x_1, \ldots, x_n] \) [1].

2.3. Gröbner Basis
A Gröbner basis is a subset \( G \) of an ideal \( I \subset k[x_1, \ldots, x_n] \), which contains finite generating polynomials \( g_1, \ldots, g_t \), that satisfies

\[
\langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle,
\]

where \( \text{LT}(I) \) is a set of leading terms of all polynomials in \( I \) [1].

The Gröbner basis can be constructed using Buchberger’s algorithm from any generating polynomial set \( F \). A brief explanation of Buchberger’s algorithm can be found in [1]. There is also much research that has been conducted to solve the complexity problem of Buchberger’s algorithm, see [12–25].

If given an ideal \( I \subset k[x_1, \ldots, x_n] \) that generated by polynomials \( f_1, \ldots, f_s \) with lexicographic order \( x_1 > \cdots > x_n \), then we can find a polynomial \( f = h_1 f_1 + \cdots + h_s f_s \) in the Gröbner basis \( G \subset I \), where \( h_1, \ldots, h_s \in k[x_1, \ldots, x_n] \), that eliminate the first \( \ell \)-th variables \( x_1, \ldots, x_\ell \), so the polynomial \( f \in G \) only contains variables \( x_{\ell+1}, \ldots, x_n \). Based on this statement, we can determine the implicit polynomial equation using Gröbner basis [1].
2.4. Rational Bézier Curve
Given the control points \( P_i \) and the weights \( \omega_i \) corresponding to the control points \( P_i \), for \( i = 0, 1, \ldots, n \). A rational Bézier curve with degree \( n \) given by [10]:

\[
Q(t) = \frac{\sum_{i=0}^{n} \omega_i B^n_i(t) P_i}{\sum_{i=0}^{n} \omega_i B^n_i(t)},
\]

for \( t \in \mathbb{R} \), where \( B^n_i(t) \) are the Bernstein polynomials, defined as

\[
B^n_i(t) = \binom{n}{i} t^i (1 - t)^{n-i}.
\]

A brief explanation of the rational Bézier curve can be found in [10].

3. Determining the Implicit Equation of Conic Section
A conic section is a collection of points (locus) that forms a curve in two-dimensional space, represented by a quadratic equation [11]:

\[
ax^2 + bxy + cx^2 + dx + ey + f = 0.
\] (1)

According to [1], a conic section can be constructed from any three points \( P_0(x_0, y_0) \), \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) in \( \mathbb{R}^2 \) using quadratic rational Bézier curve. By defining the weights \( \omega_0 = \omega_2 = 1 \) and \( \omega_1 = \omega \), the parametric representation of conic section given by:

\[
x = \frac{(1 - t)^2 x_0 + 2t(1 - t)\omega x_1 + t^2 x_2}{(1 - t)^2 + 2t(1 - t)\omega + t^2},
\]

\[
y = \frac{(1 - t)^2 y_0 + 2t(1 - t)\omega y_1 + t^2 y_2}{(1 - t)^2 + 2t(1 - t)\omega + t^2},
\] (2)

where \( \omega \in \mathbb{R} \), \( \omega > 0 \), for \( t \in \mathbb{R} \).

The Gröbner basis can be used to obtain the implicit polynomial equation of the conic section generated by equation (2), so the coefficient in equation (1) can be represented in the form of \( \omega, \omega_i \), for \( i = 0, 1, 2 \).

First, we define an ideal \( J \subset \mathbb{R}[r, t, x, y, x_i, y_i, \omega] \) in the lexicographic order \( r > t > x > y > x_i > y_i > \omega \) as follows:

\[
J = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[r, t, x, y, x_i, y_i, \omega],
\]

where

\[
f_1 = ((1 - t)^2 + 2t(1 - t)\omega + t^2)x - ((1 - t)^2 x_0 + 2t(1 - t)\omega x_1 + t^2 x_2),
\]

\[
f_2 = ((1 - t)^2 + 2t(1 - t)\omega + t^2)y - ((1 - t)^2 y_0 + 2t(1 - t)\omega y_1 + t^2 y_2),
\]

\[
f_3 = ((1 - t)^2 + 2t(1 - t)\omega + t^2)^2 r - 1.
\]

Note that, \( f_1 \) and \( f_2 \) are polynomials generated from (2), while \( f_3 \) is a polynomial that is generated from rational function

\[
r = \frac{1}{((1 - t)^2 + 2t(1 - t)\omega + t^2)^2}.
\]

Based on Section 2.3, we can find a polynomial \( g \), which is a combination of polynomials \( f_1, f_2, f_3 \), in the Gröbner basis \( G \subset J \), that eliminates the first few variables. In this case, we want to eliminate the variables \( r \) and \( t \), so the polynomial \( g \) only contain the variables
variables $x, y, x_i, y_i, \omega$. In particular, the polynomial $g$ is an implicit polynomial equation derived from the quadratic rational Bézier curve.

Using Buchberger’s algorithm in Wolfram Mathematica, there exist 61 polynomials with 11 variables $r, t, x, y, x_i, y_i, \omega$ in the Gröbner basis $G$ that construct an ideal $J$.

However, there is only one polynomial that eliminates unnecessary variables $r$ and $t$. The polynomial with 9 variables $x, y, x_i, y_i, \omega$ in the Gröbner basis $G$ given by

$$
g = y^2 x_0^2 - 4 y^2 \omega^2 x_0 x_1 + 4 y^2 \omega^2 x_1^2 - 2 y^2 x_0 x_2 + 4 y^2 \omega^2 x_0 x_2 - 4 y^2 \omega^2 x_1 x_2
+ y^2 x_2^2 - 2 xy y_0 y_0 + 4 xy \omega^2 x_1 y_0 - 4 y^2 \omega^2 x_1 y_0 - 4 xy x_2 y_0 - 4 xy \omega^2 x_2 y_0
+ 2 y y_0 x_2 y_0 + 4 y \omega^2 x_1 x_2 y_0 - 2 y^2 y_0^2 + 2 x^2 y_0^2 + 2 x^2 y_0^2
+ 4 xy y_0 y_0 y_1 - 8 xy y_0^2 x_1 y_1 + 4 y \omega^2 x_0 x_1 y_1 + 4 xy y_0^2 x_0 x_2 + 8 y^2 y_0 x_2 y_1
+ 4 y \omega^2 x_0 x_2 y_1 - 4 y \omega^2 x_1 y_1 y_1 + 4 xy^2 x_1 y_1 y_1 + 4 \omega^2 x_1 y_1 y_1 - 4 \omega^2 x_1 x_2 y_1 y_1
+ 4 x^2 \omega^2 y_1^2 - 4 x^2 x_0 y_1^2 - 4 x^2 x_2 y_1^2 + 4 \omega^2 x_0 x_2 y_1 + 4 xy x_0 y_2
- 4 y \omega^2 x_0 y_2 - 2 y \omega^2 y_1 y_2 + 4 y \omega^2 x_1 y_1 y_2 + 4 \omega^2 x_0 x_1 y_2 - 4 \omega^2 x_1^2 y_2
- 2 xy x_2 y_2 + 8 y x_0 x_2 y_2 - 2 x^2 y_0 y_2 + 4 \omega^2 x_0 y_2 + 8 x y x_0 y_2 - 8 x \omega^2 x_1 y_0 y_2 + 4 \omega^2 x_1 y_0 y_2 - 2 x x_2 y_0 y_2 - 4 \omega^2 x_1 y_0 y_2
+ 4 \omega^2 x_0 y_1 y_2 + 4 \omega^2 x_1 y_1 y_2 - 4 \omega^2 x_0 x_1 y_1 y_2 + x^2 y_2^2 - 2 x x y_0^2 + 2 x_0 y_2^2.
$$

(3)

According to [1], if we set $g = 0$, then it forms an implicit polynomial equation of conic section. Furthermore, if we rearrange the 54 terms of the polynomial $g$ in (3) into (1), then we get the relation between the coefficients in (1) and $x_i, y_i, \omega$ in (2) which is described in the following proposition.

**Proposition 3.1.** The relation between the coefficients $a, b, c, d, e, f$ in equation (1) and $x_i, y_i, \omega$ in equation (2), for $i=0,1,2$, is described in Table 1.

Table 1: The relation between the coefficients in equation (1) and the implicit polynomial equation generated by (2).

| Coefficient in (1) | Implicit polynomial equation generated by (2) |
|--------------------|---------------------------------------------|
| $a$                | $y^2 - 4 \omega^2 y_0 y_1 + 4 \omega^2 y_1^2 - 2 y_0 y_2 + 4 \omega^2 y_0 y_2 - 4 \omega^2 y_1 y_2 + y_2^2$ |
| $b$                | $-2 x_0 y_0 + 4 \omega^2 x_0 y_0 + 2 x_0 y_0 + 4 \omega^2 x_2 y_0 + 4 \omega^2 x_0 y_1 - 8 \omega^2 x_1 y_1 + 4 \omega^2 x_1 y_2 + 2 x_0 y_2 - 2 x_2 y_2$ |
| $c$                | $x_0^2 - 4 x_0 x_1 + 4 x_1^2 - 2 x_0 x_2 + 4 \omega^2 x_0 x_2 - 4 \omega^2 x_1 x_2 + x_2^2$ |
| $d$                | $-2 x_2 y_0^2 + 4 \omega^2 x_1 y_0 y_1 + 4 \omega^2 x_2 y_0 y_1 - 4 \omega^2 x_0 y_2^2 - 4 \omega^2 x_0 y_2^2 + 2 x_0 y_0 y_2 - 8 \omega^2 x_1 y_0 y_2 + 2 x_2 y_0 y_2 + 4 \omega^2 x_0 y_0 y_2 + 4 \omega^2 x_1 y_1 y_2 - 2 x_0 y_2^2$ |
| $e$                | $-4 \omega^2 x_1 y_0^2 + 8 x_0 x_2 y_0 + 4 \omega^2 x_1 x_2 y_0 - 2 x_2 y_0 + 4 \omega^2 x_0 x_1 y_1 - 8 \omega^2 x_0 x_2 y_1 + 4 \omega^2 x_0 x_1 y_2 - 2 x_0 y_1 y_2 + 4 \omega^2 x_0 x_1 y_2 - 4 \omega^2 x_0 x_1 x_2 y_2 + 4 \omega^2 x_1 y_0 y_2 - 2 x_0 x_2 y_0 y_2 - 4 \omega^2 x_0 x_1 y_1 y_2 + x_0 y_2^2$ |
| $f$                | $x_0^2 y_0^2 - 4 \omega^2 x_1 x_2 y_0 y_1 + 4 \omega^2 x_0 x_2 y_1 + 4 \omega^2 x_1 y_0 y_2 + 2 x_0 x_2 y_0 y_2 - 4 \omega^2 x_0 x_1 y_1 y_2$ |
4. The Effect of Weight $\omega$

First, we recall the following two lemmas cited from [11].

**Proposition 4.1.** [11] A conic section in equation (1) can be uniquely characterized as:

- Parabola if and only if $\text{det}(A) \neq 0$ and $\text{det}(B) = 0$;
- Hyperbola if and only if $\text{det}(A) \neq 0$ and $\text{det}(B) < 0$;
- Ellipse if and only if $\text{det}(A) \neq 0$ and $\text{det}(B) > 0$,

where

\[
A = \begin{pmatrix}
  a & \frac{1}{2}b & \frac{1}{2}d \\
  \frac{1}{2}b & c & \frac{1}{2}e \\
  \frac{1}{2}d & \frac{1}{2}e & f
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
  a & \frac{1}{2}b \\
  \frac{1}{2}b & c
\end{pmatrix}.
\]

**Proposition 4.2.** [11] Three points $(x_0, y_0), (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ lie on a straight line if and only if satisfies

\[
\begin{vmatrix}
  x_0 & y_0 & 1 \\
  x_1 & y_2 & 1 \\
  x_2 & y_2 & 1
\end{vmatrix} = 0.
\]

Based on Proposition 3.1, the implicit polynomial equation of the conic section $ax^2 + bxy + cy^2 + ex + dy + f = 0$ can be generated from equation (2) with the coefficients $a, b, c, d, e, f$ given in Table 1.

By applying Proposition 4.1 in Wolfram Mathematica, the conditions for a conic section to form a parabola are

$$(\omega = -1 \text{ or } \omega = 1) \text{ and } x_2(y_0 - y_1) + x_0(y_1 - y_2) + x_1(-y_0 + y_2) \neq 0.$$  \hspace{1cm} (4)

Note that, the second statement, $x_2(y_0 - y_1) + x_0(y_1 - y_2) + x_1(-y_0 + y_2) \neq 0$, means that the points $P_0, P_1, P_2 \in \mathbb{R}^2$ do not lie in a straight line, by Proposition 4.2.

Assuming that $\omega > 0$ in equation (2), then the conditions for the conic section to form a parabola are $\omega = 1$ and the points $P_0, P_1, P_2 \in \mathbb{R}^2$ do not lie in the straight line.

Similarly, we can get the conditions for the conic section to form a hyperbola, and an ellipse is $\omega > 1$ and $0 < \omega < 1$ respectively, provided that the control points $P_0, P_1, P_2 \in \mathbb{R}^2$ do not lie in the straight line.

Based on these statements, we get the following proposition.

**Proposition 4.3.** Given three arbitrary points $P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2) \in \mathbb{R}^2$ which do not lie in a straight line. The implicit polynomial equation of the conic section with the coefficients given in Table 1 is uniquely characterized as:

- Parabola if and only if $\omega = 1$;
- Hyperbola if and only if $\omega > 1$;
- Ellipse if and only if $0 < \omega < 1$.

**Example 4.4.** Given the control points $P_0(-1, 1), P_1(0, 0), P_2(1, 1) \in \mathbb{R}^2$. By applying Proposition 4.1, the implicit polynomial equation of a conic section that defined by the control points $P_0, P_1, P_2 \in \mathbb{R}^2$, can be expressed as

$$\omega^2 x^2 + 0xy + (1 - \omega^2)y^2 + 0x - 2y + 1 = 0.$$  \hspace{1cm} (4)

Note that, the change of weight $\omega$ in equation (4) results in a different shape of a conic section.
• If \( \omega = 2 \), then the conic section form a vertical hyperbola with the following equation:

\[
4x^2 - 3y^2 - 2y + 1 = 0 \quad \text{or} \quad \left( \frac{y + \frac{1}{3}}{\frac{4}{9}} \right)^2 - \left( \frac{x}{\frac{1}{9}} \right)^2 = 1.
\]

• If \( \omega = 1 \), then the conic section form a parabola with a minimum point at \((0, \frac{1}{2})\) given by the following equation:

\[
x^2 - 2y + 1 = 0 \quad \text{or} \quad y = \frac{1}{2}x^2 + \frac{1}{2}.
\]

• If \( \omega = 0.8 \), then the conic section form a vertical ellipse with the following equation:

\[
16x^2 + 9y^2 - 50y + 25 = 0 \quad \text{or} \quad \frac{x^2}{\left( \frac{25}{9} \right)} + \frac{(y - \frac{25}{9})^2}{\left( \frac{400}{81} \right)} = 1.
\]

• In particular, if \( \omega = \frac{1}{\sqrt{2}} \approx 0.707107 \), then the conic section form a circle with center point at \((0, 2)\) and radius \(\sqrt{2}\) given by the following equation:

\[
x^2 + y^2 - 4y + 2 = 0 \quad \text{or} \quad x^2 + (y - 2)^2 = 2.
\]

Illustration of a conic section with control points \( P_0(-1, 1), P_1(0, 0), P_2(1, 1) \in \mathbb{R}^2 \) and weights \( \omega = 2, \omega = 1, \omega = 0.8, \omega = \frac{1}{\sqrt{2}} \approx 0.7071 \) is given in Figure 1.

![Illustration of conic sections with control points](image_url)

**Figure 1:** The conic section with control points \( P_0(-1, 1), P_1(0, 0), P_2(1, 1) \in \mathbb{R}^2 \) and various weights \( \omega \).
5. Conclusion

The Gröbner basis can be applied to determine the implicit polynomial equation of a conic section from a quadratic rational Bézier curve, by defining the weights $\omega_0 = \omega_2 = 1$ and $\omega_1 = \omega$, for any control points $P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2) \in \mathbb{R}^2$. The polynomial coefficients of the conic section can be expressed in the form of control points $P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2)$ and weight $\omega$ as given in Table 1. The shape of the conic section changes with respect to the weight $\omega$. A parabola, a hyperbola, and an ellipse can be formed by defining $\omega = 1$, $\omega > 1$, and $0 < \omega < 1$, respectively.

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