Breaking a “poor man” RG approach in the Luttinger liquid with one impurity.

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Abstract

The method of summation of infrared logarithms for reflection coefficient in one-dimensional channel with impurity is suggested. The method is based on the original formulation of Gell-Mann-Low renormalization group. On the level of 2 loops results are different from the so-called “poor man” renorm-group used previously. The reasons for this discrepancy are analyzed.
I. INTRODUCTION

Study of one-dimensional interacting fermions has a very long history [1]. Pure 1D systems with arbitrary electron-electron interactions which are described by the Luttinger model [2] are completely understood. Electron transport in these systems is ballistic with the Fermi speed renormalized by the interaction. A single impurity doped into 1D channel drastically changes the conductivity. It remains ballistic at $\omega \to 0$ ($\omega$ is the frequency of an external electrical field) for attractive electron-electron interaction, but goes to zero for repulsive interaction. This behavior is due to infrared divergencies arising at small $\omega$. As a result summation of an infinite number of perturbative contributions becomes necessary [3].

The renormalization group (RG) approach is a natural way to sum the infrared logarithms in calculation of conductivity. First time this was done in [4] where the effective reflection coefficient of impurity $|R_\omega|^2$ was calculated in the leading log approximation:

$$\nu \log \frac{M}{\omega} \leq 1, \quad \nu \equiv v_c^{-1} - 1 \ll 1$$

$v_c$ being the renormalized Fermi speed (in units of bare one) and $M$ is an ultraviolet cut-off.

The conductance obtained in this approach coincides with the results of Kane and Fischer [3] who have used a different approximation. They considered the cases of small bare reflection coefficient $|R|^2 \ll 1$ (for attraction) or transition coefficient $|K|^2 \ll 1$ (for repulsion). It is well known [5] that the approach suggested in [4] does not work beyond the leading log approximation. Another RG approach was developed in [6] where contributions of two and three loops were calculated.

Unfortunately, all papers dealing with RG in the Luttinger liquid (including [4] and [6]) are using the so called “poor man” RG approach [7] which is a simplified version of the original Gell-Mann-Low RG formulation [8] (for a systematic modern review see, e.g., [9]). In the case of the Luttinger liquid with an impurity this approach is based on two assumptions. First, it should be assumed that the theory has only one “charge” which should be renormalized and, second, this charge coincides with an effective reflection (or transition) coefficient. In other words $|R_\omega|^2$ should be the only quantity which determines all low energy properties of the Luttinger model with impurity. These assumptions simplify greatly summation of logs in the model. It is sufficient to calculate all diagrams of the given order in $\nu$ that contain $\log M$ linearly, substitute the effective reflection coefficient $|R_\omega|^2$ instead of the bare reflection coefficient $|R|^2$ and plug the result into the Gell-Mann-Low
equation. It is this procedure which was used in [4] in the first order in $\nu$ and in [6] in a few subsequent orders in order to sum the infrared logs.

We will show below that both assumptions do not hold in the Luttinger model with an impurity. We will use the effective 0 + 1-dimensional theory which we developed previously [10, 11]. This theory reduces the original model to the dynamics of the electron phase at the position of the impurity. It simplifies greatly the calculation of conductivity being still exactly equivalent to the Luttinger model with an impurity for an arbitrary electron-electron interaction potential.

The quantum field $\alpha$ (electron phase) in the effective theory is dimensionless and the renormalized Lagrangian contains an infinite number of terms. This means that we deal with a theory with an infinite number of coupling constants. Nevertheless, the renormalized Lagrangian does not depend on small distances and the Lagrangian reproduces itself after renormalization, just as it happens in a renormalizable theory.

Only one coupling constant is involved in renormalization in the leading approximation and the effective reflection (transition) coefficient $|R_\omega|^2$ is proportional to this coupling constant. Requirements of the “poor man” approach are satisfied in this approximation, and our approach gives results coinciding with [4]. However, already in the next-to-leading order (contributions of the type $\nu^{(n+1)} \log^n M/\omega$ are collected) more coupling constants enter the Gell-Mann-Low equation. In fact, in 2- and 3-loop approximations it is still possible to introduce new charges (functions of the previous ones) and diagonalize the Gell-Mann-Low equation. But the “poor man” approach does not work any more, since the effective reflection coefficient in these approximations does not coincide with the renormalized charge. For this reason, while the 2-loop and 3-loop Gell-Mann-Low equation below is the same as in [6] our results for $|R_\omega|^2$ and conductance are completely different. Starting with the 4 loops the “poor man” scaling fails completely. We will derive the correct expression for $|R_\omega|^2$ below.

Our findings are necessary for comparison of the perturbative results with the exact solution known at some specific values of $\nu$ [12]. An explicit example of breakdown of the "poor man" scaling seems to be even more important since this approach is used without derivation in many areas of condensed matter physics.
II. RG APPROACH TO EFFECTIVE THEORY

As it was shown in [10], the Luttinger liquid in equilibrium with a point-like impurity is exactly equivalent to the 0+1 dimensional theory:

$$\langle\langle \ldots \rangle\rangle = \frac{1}{Z} \int D\alpha \ldots \text{Det}_{imp} \exp \left[ - \int \frac{d\omega}{2\pi} \frac{\alpha(-\omega)\alpha(\omega)}{2W(\omega)} \right], \quad (2)$$

where $\alpha(t)$ is the electron phase (more precisely, difference of the phases of the $R$- and $L$-components) resulting from the scattering on the impurity. The quadratic form ("kinetic" energy) of the $\alpha$-field for the point-like electron-electron interaction $V(x-y) = V_0 \delta(x-y)$ has the form

$$W(\omega) = \frac{2\pi}{|\omega|} \left[ \frac{1}{v_c} - 1 \right] \equiv \frac{2\pi\nu}{|\omega|}. \quad (3)$$

Here $v_c = \sqrt{1 + \frac{V_0}{\pi}}$ is a renormalized Fermi speed. Theories with $V_0 < 0$ ($e-e$ attraction) and $V_0 > 0$ ($e-e$ repulsion) are dual to each other [3, 10]. For this reason we consider only the theory of mutually attracting one-component (i.e. without spin) electrons and calculate the effective reflection coefficient $|R_\omega|^2$.

The factor $\text{Det}_{imp}$ is a sum of multiloop fermion diagrams and describes interaction in the effective theory. It has the form [11]

$$S_{int}[\alpha] = - \log \text{Det}_{imp}[\alpha] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \frac{|R|}{|K|} \right)^{2n} B_{2n-1}[\alpha],$$

$$B_n = \int \frac{d\tau_0 \ldots d\tau_n}{(2\pi)^{n+1}} \frac{1 - \cos[\alpha(\tau_0) - \alpha(\tau_1) + \ldots - \alpha(\tau_n)]}{(\tau_0 - \tau_1 - i\delta) \ldots (\tau_n - \tau_0 - i\delta)} \quad (4)$$

where $R$ ($K$) is bare reflection (transition) coefficient. The total effective action is the sum $S_{eff} = S_{kin}[\alpha] + S_{int}[\alpha]$.

The effective reflection coefficient in our theory in terms of the Green function of the $\alpha$-field has the form

$$|R_\omega|^2 = \nu + 1 - \frac{1}{\nu^2} \frac{|\omega|}{2\pi} |G_0(\omega) - G(\omega)|. \quad (5)$$

where $G(\tau) = \langle\langle \alpha(\tau)\alpha(0)\rangle\rangle$ and $G_0(\omega)$ is the free Green function.

The reflection coefficient is a physical observable, conductance $C(\omega)$ is simply related to $|R_\omega|^2$:

$$C(\omega) = \frac{e^2 |K_\omega|^2}{2\pi v_c}, \quad |K_\omega|^2 = 1 - |R_\omega|^2. \quad (6)$$

We will use the RG method in order to sum up logarithmically infrared divergent corrections to the reflection coefficient. In the original formulation of the renormalization group
one has to find counter-terms compensating all ultraviolet divergencies and include them into the renormalized action

\[ S_r[\alpha] = \sum_{n=1} g_{2n}(\mu) \int \frac{d\omega_1...d\omega_{2n}}{(2\pi)^{2n}(2n)!} \Gamma_{2n}(\omega_1...\omega_{2n}) \cdot \alpha(\omega_1)...\alpha(\omega_{2n})\delta(\omega_1+...+\omega_{2n}), \] (7)

Here \( g_{2n}(\mu) \) are the renormalized coupling constants

\[ g_{2n}(\mu) = g_{2n}^{(0)} + \delta g_{2n}(\mu) \] (8)

where \( g_{2n}^{(0)} \) is a bare coupling constant and \( \delta g_{2n}(\mu) \) is a sum of counter-terms normalized at some point \( \mu \). In what follows, we will use the Pauli-Villars regularization. The physical quantities (in particular, the reflection coefficient) do not depend on the regularization scheme. The renormalized coupling constants should be normalized by the condition \( g_{2n}(\mu=M) = 1 \) which means that at this point the renormalized Lagrangian coincides with the original one, eq. (4).

Duality between theories with electron-electron attraction and repulsion puts very strict restrictions on the vertices \( \Gamma_n \). In fact, up to a constant, this requirement fixes vertices completely

\[ \Gamma_{2n}(\omega_1,\ldots) = \left[ S_n(\prod_i \omega_i) + A_n(\prod_i \omega_i) \right] \gamma_{2n}(\omega_1,\ldots), \] (9)

where \( \gamma_{2n} \) is a function of external frequencies,

\[ \gamma(\omega_1,\ldots\omega_{2n}) = \sum_i |\omega_i| - \sum_{i<j} |\omega_i + \omega_j| + \sum_{i<j<k} |\omega_i + \omega_j + \omega_k| - \ldots, \] (10)

and \( S_n, A_n \) are functions of bare reflection (\(|R|^2\)) and transition (\(|K|^2\)) coefficients, \( S_n \) being symmetric and \( A_n \) antisymmetric under \(|R|^2 \leftrightarrow |K|^2\) exchange. Functions \( S_n \) and \( A_n \) obey the following recurrent relations

\[ S_n(x) = -\frac{\partial A_{n-1}(x)}{\partial \log x}, \quad A_n(x) = -\frac{\partial S_{n-1}(x)}{\partial \log x} \] (11)

1 We are using a bit non-standard version of the Gell-Mann-Low theory. Besides other coupling constants we introduce also the coupling constant \( g_2 \) for vertices with only two legs. Respectively, there is no need in \( Z \)-factors renormalizing field \( \alpha \) itself.
FIG. 1. Diagrams contributing to renormalization of coupling constant $g_{2n}$: A)—order $\nu$; B),C)—order $\nu^2$; D)—order $\nu^3$

and

$$S_1 = \frac{x}{1+x}, \quad S_2 = -\frac{\partial S_1}{\partial \log x} = -S_3, \quad A_1 = A_2 = 0;$$

where $x = |R|^2/|K|^2$. Respectively, in the given order $n > 2$ there are two constants $g^{(s)}_{2n}$ and $g^{(a)}_{2n}$ in front of symmetric and antisymmetric structures which should be renormalized independently.

Now let us proceed with the renormalization program. In the first order in $\nu \ll 1$ there is only one diagram renormalizing the vertex $\Gamma_{2n}$ which is presented on Fig.1A. This diagram is equal

$$\{1A\} = \frac{g_{2n+2}}{4\pi} \int \frac{d\Omega}{2\pi} G_{P.V.}(\Omega) \Gamma_{2n+2}(\Omega, -\Omega, \omega_1, \ldots, \omega_{2n}) \times$$

$$\times \alpha(\omega_1) \ldots \alpha(\omega_{2n})$$

(12)

where

$$G_{P.V.}(\omega) = \frac{2\pi \nu M_{P.V.}}{|\omega|(|\omega| + M_{P.V.})}$$

is the bare Green function regularized according to Pauli-Villars. The integral in eq. (12) is logarithmically divergent for $|\Omega| \gg |\omega|$. To compensate this divergence one has to add to the action a counter-term

$$\delta g^{(s)}_{2n}(\mu) = -2\nu g^{(a)}_{2n+2}(\mu) \log \left( \frac{M_{P.V.}}{\mu} \right) \frac{1}{S_n} \frac{\partial S_n}{\partial \log x},$$

(13)

(analogously for $\delta g^{(a)}_{2n}(\mu)$ with substitution $S \rightarrow A$).

In the second order in $\nu$ three diagrams contribute to renormalization. They are 1B, 1C, and the diagram 1A with a vertex substituted by counter-term eq. (13). The sum of 1A and 1B has divergencies proportional to $\log^2 M_{P.V.}$ and $\log M_{P.V.}$. It is important that the
coefficients before these terms do not depend on external frequencies $\omega$. For this reason, these divergencies can be compensated by local counter-terms:

$$\delta g_{2n}^{(s)}[IB+IA] = -2\nu^2 g_{2n+4}^{(s)} \log^2 \left( \frac{M_{PV}}{\mu} \right) \frac{1}{S_n} \frac{\partial^2 S_n}{\partial \log x^2}$$

as it should be in renormalizable theory. Moreover, the same counter-terms arise when one iterates the Gell-Mann-Low equation with a $\beta$-function constructed from the first order counter-terms, eq. (13). Only the diagram in Fig. 1C, which is essentially new, contributes to the Gell-Mann-Low equation. This diagram is proportional to the first power of logarithm and the corresponding counter-term differs from the one in eq. (13) only by the factor $-\nu g_2 S_1$ or $-\nu g_2 |R|^2$.

Differentiating counter-terms in $\log \mu$ we obtain the Gell-Mann-Low equation

$$\frac{\partial g_{2n}^{(s)}(\mu)}{\partial \log \mu} = \frac{2\nu g_{2n+2}(\mu)}{1 + \nu g_2(\mu)|R|^2} \frac{\partial \log S_n}{\partial \log x}$$

and analogously for $g_{2n}^{(a)}(\mu)$. This equation takes into account the sum of diagrams with an arbitrary number of $\Gamma_2$-vertices in the loop. In the order $\nu^2$ it is the diagram in Fig.1C, in the order $\nu^3$ it is the diagram in Fig.1D. We have seen that in order $\nu^2$ it is the only contribution, the same is true for $\nu^3$. Thus eq. (15) should be modified only in the order $\nu^4$.

Eq. (15) can be viewed as a system of recurrence relations which express subsequent coupling constants $g_{2n+2}(\mu)$ in terms of derivatives of the previous ones. The first constant $g_2(\mu)$ is not determined by this equation. Nevertheless, boundary conditions for all coupling constants $g_{2n}(\mu=M) = 1$ are sufficient to determine it completely at all $\mu$.

It is possible to derive a closed equation for $g_2$ with the accuracy used in eq. (15). Dividing each equations in the system in (15) with $n > 1$ by the first equation we obtain the following system

$$h_4 \left( \frac{\partial S_1}{\partial z} \right)^{-1} \frac{\partial h_{2n}}{\partial z} = h_{2n+2}, \quad h_2 = \frac{z}{1+z}$$

We introduced here a notation

$$h_{2n} = \left\{ S_1 g_2, \frac{\partial S_1}{\partial \log x} g_4, \frac{\partial^2 S_1}{(\partial \log x)^2} g_6^{(a)}, \frac{\partial^3 S_1}{(\partial \log x)^3} g_8^{(s)}, \ldots \right\}$$

$$f_{2n} = \left\{ \frac{\partial S_1}{\partial \log x} g_6^{(s)}, \frac{\partial^2 S_1}{(\partial \log x)^2} g_8^{(a)}, \frac{\partial^3 S_1}{(\partial \log x)^3} g_{10}^{(s)}, \ldots \right\}$$

Equations for $f_{2n}$ look similar.
It is easy to find solution of eq. (16):

\[ h_{2n} = \frac{\partial^{n-1} S_1(z)}{(\partial \log z)^{n-1}} = f_{2n+2}, \quad n > 1 \]  

(18)

Indeed, as can be seen from the eq. (17), boundary conditions for \( g_2 \) \((g_2(M) = 1)\) mean that at \( \mu = M \) one has \( z = x \). Using the definition in eq. (17) we see from eq. (18) that the boundary conditions for other coupling constants \((g_{2n}(M) = 1)\) are also satisfied at \( z = x \). At last, the expression in eq. (18) explicitly obeys eq. (16).

The solution in eq. (18) allows to find dependence of the coupling constant on \( h_2 \) and hence write the first equation of the system (15) as a closed nonlinear equation for \( h_2 \).

\[
\frac{\partial h_2}{\partial \log \mu} = \frac{2\nu}{1 + \nu h_2} \frac{\partial S_1(z)}{\partial \log z} = \frac{2\nu}{1 + \nu h_2} h_2 (1 - h_2)
\] 

(19)

which determines dependence of \( h_2 \) on the normalization point.

Let us notice now that this solution of the Gell-Mann-Low equation literally coincides with the one in the “poor man” RG approach. Indeed, the coupling constant \( h_2(\mu) \) at \( \mu = M \) coincides with the reflection coefficient squared: \( h_2(M) = S_1(x) = |R|^2 \). At smaller \( \mu \) the Gell-Mann-Low function (r.h.s. of eq. (19)) is determined by the perturbative one-logarithmic diagrams, in which one has to substitute \(|R|^2 \to h_2(\mu)\) (i.e. \( x \to z \)). This procedure is literally the “poor man” one, if one identifies \(|R_\omega|^2 = h_2(\omega)\). For this reason eq. (19) exactly coincides with the Gell-Mann-Low equations derived in [4] (in the leading order) and \([6]\) (in order \( \nu^3 \)) which used this method. It is convenient to write solutions of eq. (19) in terms of \( z(\mu) \), it reduces to an algebraic equation:

\[
z(\mu) = x \left( \frac{\mu}{M} \right)^{2\nu} \left[ |K|^2 (1 + z(\mu)) \right]^{-\nu}
\] 

(20)

Iterating this equation one can find renormalized coupling constants in a few lowest orders:

\[
g_2(\mu) = \frac{1}{|R|^2 + |K|^2(M/\mu)^{2\nu}} - \\
-\nu |K|^2(M/\mu)^{2\nu} \log[|K|^2 + |R|^2(M/\mu)^{2\nu}] + \ldots
\]

\[
g_4(\mu) = \left( \frac{M/\mu}{|R|^2 + |K|^2(M/\mu)^{2\nu}} \right)^2 + \ldots
\] 

(21)

and so on.

We see that applicability of the “poor man” scaling to our problem is related mainly to the fact that higher orders come as multiplicative renormalization of the leading order, the
FIG. 2. “Water melon” diagram $O(\nu^4)$, the first contribution to the charges not described by “poor man” approach.

same for all coupling constants and depending only on $h_2$. This property breaks already in
the order $\nu^4$. In this order there is a “water melon” diagram (see Fig. 2) which diverges as
the first power of $\log M_{PV}$ and modifies eq. (16) to

$$\frac{\partial h_{2n}}{\partial \log \mu} = \frac{2\nu h_{2n+2}}{1 + \nu h_2} - \frac{\pi^2}{3} \nu^3 h_4 h_{2n+4}$$ (22)

(it is the only diagram in order $\nu^4$ that modifies this equation). Denoting $\delta h_{2n}$ the correction
due to this diagram, we can linearize eq. (22)

$$\frac{\partial \delta h_{2n}}{\partial \log \mu} - 2\nu \delta h_{2n+2} = -\frac{\pi^2}{3} \nu^3 h_4 h_{2n+4}$$ (23)

where the r.h.s. should be calculated in the leading order. The solution of this system for
$\delta h_2$ is:

$$\delta h_2 = -\frac{\pi^2}{3} \nu^3 \frac{\partial^2 h_2}{(\partial \log (\mu^2)^2}[h_2(\mu) - h_2(M)]$$ (24)

Meanwhile, the “poor man” recipe implies that we calculate the r.h.s. of eq. (22) in the
leading order and substitute $h_2$ instead of each $|R|^2$. This would lead to the equation

$$\frac{\partial \delta h_2}{\partial \log \mu} - 2\nu \delta h_2 = -\frac{\pi^2}{3} \nu^3 h_2(1 - h_2)(h_2 - 3h_2^2 + 2h_3^2)$$

which is different from eq. (22) and has a solution different from the one in eq. (24). Hence
the “poor man” RG for charges fails at the order $\nu^4$ and one is left with a complete system
of the Gell-Mann-Low equations accounting for an infinite number of coupling constants.

To calculate the effective reflection coefficient one has to use eq. (5). The lowest diagrams
for the Green function are presented in Fig. 3. We use the renormalized perturbation theory
but have to calculate diagrams exactly, including the finite contributions (not only the
divergent parts as for the Gell-Mann-Low function). Green functions defined in this way depend on an arbitrary normalization point \( \mu \) where the renormalized coupling constants are defined. On the other hand, the effective reflection coefficient as a physical observable cannot depend on \( \mu \). We choose \( \mu = \omega \) where \( \omega \) is the external frequency. Then the renormalized Green functions at \( \omega \sim \mu \) do not contain infrared logs at all. All infrared logs are summed up by the system of the Gell-Mann-Low equations.

In order \( \nu^2 \) the expression for \( |R_\omega|^2 \) is equal to

\[
|R_\omega|^2 = (1 + \nu)|R|^2 g_2(\omega) - \nu|R|^2 g_2^2(\omega) - 2\nu g_4(\omega)|K|^2
\]

(25)

We have calculated the coupling constants in order \( \nu^4 \), and obtained an expression for \( |R_\omega|^2 \) with accuracy \( \nu^4 \) but it is too cumbersome and will be published elsewhere.

We see that the identification of \( |R_\omega|^2 \) with the charge \( h_2(\omega) = |R|^2 g_2(\omega) \) is valid only in the leading order. In the next order this second main assumption of the “poor man” RG breaks down and the effective reflection coefficient depends in some way on all charges in the theory.

Expressions for the r.h.s. of Gell-Mann-Low equations (15) and (22) depends on the regularization scheme starting from the second loop (terms \( O(\nu^2) \)). The same is true for renormalized perturbation theory (25) — coefficient in front of \( g_4(\omega) \) is scheme-dependent. However, the complete expression for the effective reflection coefficient \( R^2_\omega \) does not depend on the scheme as it should be for any physical observable.

To summarize: we proved that in the Luttinger liquid with one impurity the “poor man” RG fails in order \( \nu^4 \) for the Gell-Mann-Low equations. It fails already in order \( \nu^2 \) for the effective reflection coefficient \( |R_\omega|^2 \) and conductance \( C(\omega) \). The expressions above for physical observables are different from those obtained earlier by other authors, starting from this order.
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[1] S.Tomonaga, Prog. Theor. Phys. 5 (1950), 544.
[2] J. M. Luttinger, J. Math. Phys. 4 (1963), 1154.
[3] C.L. Kane, M.P.A. Fisher, Phys.Rev. B46 (1992), 15233; A. Furusaki and N. Nagaosa, Phys.Rev. B47 (1993), 4531.
[4] L.I. Glazman, K.A. Matveev, D. Yue, Phys.Rev.Lett. 71(1994); Phys.Rev. B49 (1994), 1966.
[5] D. G. Polyakov, I. V. Gornyi, Transport of interacting electrons through a double barrier in quantum wires, arXiv:cond-mat/0212355 [cond-mat.str-el]; Phys. Rev. B, 68 (2003) 035421.
[6] D.N. Aristov, P. Woelfle, EPL 82 (2008), 27001; Phys.Rev. B 80 (2009), 045109.
[7] P.W. Anderson J.Phys.C:Solid St.Phys. 3(1970), 2436.
[8] Murray Gell-Mann, F.E. Low, Phys.Rev. 95 (1954), 1300.
[9] J.Collins, “Renormalization”, Cambridge, Cambridge press, 1998.
[10] V. V. Afonin and V. Y. Petrov Pis’ma v ZhETF 97 (2013), 587.
[11] V.V. Afonin and V.Y. Petrov Pis’ma v ZhETF 101 (2015), 697.
[12] P.Fendley, A.W.W.Ludwig and H.Saluer, Phys.Rev. Lett. 74(1995), 3005; Phys.Rev. B52(1995), 8934.