REDUCTIONS OF HECKE CORRESPONDENCES 
ON ANDERSON MODULAR OBJECTS

A. GRISHKOV, D. LOGACHEV

Abstract. We formulate some properties of a conjectural object \(X_{\text{fun}}(r,n)\) parametrizing Anderson t-motives of dimension \(n\) and rank \(r\). Namely, we give formulas for \(p\)-Hecke correspondences of \(X_{\text{fun}}(r,n)\) and its reductions at \(p\) (where \(p\) is a prime of \(\mathbb{F}_q[\theta]\)). Also, we describe their geometric interpretation. These results are analogs of the corresponding results of reductions of Shimura varieties. Finally, we give conjectural formulas for Hodge numbers (over the fields generated by Hecke correspondences) of middle cohomology submotives of \(X_{\text{fun}}(r,n)\).

0. Introduction. Let \(X\) be a Shimura variety of PEL-type. Its points parametrize abelian varieties with some fixed endomorphism rings, polarization and level. There is a problem to describe relations between the rings \(\mathbb{H}_p(X)\) of \(p\)-Hecke correspondences of \(X\) (\(p\) is a prime) and of \(\tilde{X}_p\) - the reduction of \(X\) at \(p\), particularly, to find the characteristic polynomial of the Frobenius correspondence of \(\tilde{X}_p\) over \(\mathbb{H}_p(X)\). Also, we can ask what are geometric interpretations of reductions of \(p\)-Hecke correspondences.

There is also a problem of extreme complexity — to prove Langlands theorems for Shimura varieties (relations between \(L\)-functions of submotives of \(X\) and of automorphic representations of the corresponding reductive group \(G\)). It is solved only in a few cases of \(X\) of low dimension. Even exact statements of theorems giving these relations is a complicated problem.

Analogs of abelian varieties for the function field case are Anderson t-motives (or Anderson modules — their categories are anti-equivalent). It is natural to consider an object (function field analog of a Shimura variety) whose points parametrize
these t-motives (for example, Anderson t-motives of a fixed dimension \( n \), rank \( r \), type of a nilpotent operator \( N \), see Section 2.4, and of an analog of PEL-type). We denote this object by \( X_{\text{fun}} = X_{\text{fun}}(r, n) \).

Unfortunately, at the moment for \( n > 1 \) these objects are conjectural (only for the case \( n = 1 \) we have a good theory of moduli spaces of Drinfeld modules). For example, if we restrict ourselves by pure uniformizable Anderson t-motives and assume that there is \( 1 \rightarrow 1 \) (or near \( 1 \rightarrow 1 \)) correspondence between these t-motives and their lattices (which is rather likely, see [GL17]), then the moduli variety of lattices would be the quotient set of Siegel matrices by an (almost) action of \( GL_r(\mathbb{F}_q[\theta]) \). But this (almost) action does not have desired properties, see [GL17], Proposition 1.7.1.

So, the whole contents of the present paper concerns conjectural objects \( X_{\text{fun}}(r, n) \) that we shall call Anderson modular objects. Nevertheless, we can get some information about them. Let us give more definitions. Naively, \( X_{\text{fun}}(r, n) \) parametrize pure abelian t-motives of rank \( r \) and dimension \( n \) whose nilpotent operator \( N \) is 0. The corresponding reductive group \( G \) is \( GL_r \) and the dominant coweight \( \mu \) is \((1, \ldots, 1, 0, \ldots, 0) \) (\( n \) ones and \( r - n \) zeroes). For more information for any \( G \) on \( X_{\text{fun}} \) and \( \mu \) see [V].

Let \( p \) be a prime ideal of \( \mathbb{F}_q[\theta] \) (\( q \) is a power of \( p \) and \( \theta \) an abstract variable, see Section 2). The contents of the present paper is the following.

1. We formulate (for some cases) in Section 2 the theorems concerning reductions at \( p \) of Hecke correspondences \( T_{p,i} \) \((i = 0, \ldots, r)\) on \( X_{\text{fun}} \), and their geometric interpretation. These results are functional analogs of [FCh], Chapter 7, [BR], Chapter 6, and [W]. The case \( n = 1 \) (Drinfeld varieties) is treated with more details and explicit formulas.

2. Sketches of the proofs of these theorems are given in Section 3.

We see that the function field case — Anderson varieties of rank \( r \) and dimension \( n \), \( G_{\text{fun}} = GL_r(\mathbb{F}_q(\theta)) \) is analogous to the number field case where \( G_{\text{num}} = GU(r-n,n) \) corresponds to Shimura varieties (called unitary for brevity) of PEL-type parametrizing abelian \( r \)-folds with multiplication by an imaginary quadratic field \( K \), of signature \((r-n,n)\). We indicate in Section 4 that really, properties of unitary Shimura varieties are similar to the properties of \( X_{\text{fun}} (r,n) \). By the way, this analogy is a source of more results, see for example [GL21].

3. Finally, in Section 5 we state conjectural formulas for Hodge numbers (over fields generated by Hecke correspondences) of submotives of middle cohomology of \( X_{\text{fun}} \). They are analogs of the corresponding formulas for Shimura varieties ([BR], Section 4.3, p. 548). In Section 6 we consider the action of Hecke correspondences on some non-ordinary Drinfeld modules. These results will be useful for a future proof of analog of Kolyvagin’s theorem (finiteness of Tate-Shafarevich group) for the case of Drinfeld varieties.

In order to show the analogy between the number field and the function fields cases, we give in Section 1 some well-known results on Hecke correspondences and their reductions for the case of Siegel varieties.
1. Definitions and results for the number field case (Siegel varieties).

1.1. Reductions of correspondences.

For a comparison with the number field case, here we formulate well-known results for reductions of Siegel modular varieties. References for the results of this section are: [FCh], Section 7, and [W].

Let $X$ be a Siegel variety of genus $g$ (of any fixed level), i.e. a quotient of the Siegel upper half plane by a congruence subgroup of $GSp_{2g}(\mathbb{Z})$, or, equivalently, a set of principally polarized abelian varieties of dimension $g$ together with some level structure. Let the congruence subgroup be such that $X$ is defined over $\mathbb{Q}$. We have $G = G_X = GSp_{2g}(\mathbb{Q})$ is the corresponding reductive group. Let $p$ be a fixed prime which does not divide the level, and $\bar{X}$ the reduction of $X$ at $p$.

Let $\text{Corr}(X) = \text{Corr}_p(X)$ (resp. $\text{Corr}(\bar{X}) = \text{Corr}_p(\bar{X})$) be the algebra of $p$-Hecke correspondences on $X$ (resp. $\bar{X}$). We have the Frobenius map on $\bar{X}$; considering it as a correspondence we get an element $fr_X \in \text{Corr}(\bar{X})$.

There is a map $\gamma : \text{Corr}(X) \to \text{Corr}(\bar{X})$ — the reduction of a correspondence at $p$. There is a problem of description of $\gamma$ and of finding of the characteristic polynomial of $fr$ over $\text{im}(\gamma)$. The answer is the following. Let $M$ be the following block diagonal subgroup of $G$:

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & \lambda \cdot (A^t)^{-1} \end{pmatrix} \right\} \subset G,$$

(1.1.1)

(blocks have size $g$), and let $T$ be the subgroup of $M$ consisting of diagonal matrices.

The abstract $p$-Hecke algebras $\mathbb{H}(G) = \mathbb{H}(G)(\mathbb{Q}_p)$ (resp. $\mathbb{H}(M) = \mathbb{H}(M)(\mathbb{Q}_p)$, $\mathbb{H}(T) = \mathbb{H}(T)(\mathbb{Q}_p)$) consist of double cosets $K\alpha K$, where $K = K_G = G(\mathbb{Z}_p)$ (resp. $K = K_M = M(\mathbb{Z}_p)$, $K = K_T = T(\mathbb{Z}_p)$) and $\alpha \in G(\mathbb{Q}_p)$, resp. $\alpha \in M(\mathbb{Q}_p)$, $\alpha \in T(\mathbb{Q}_p)$. There are the Satake inclusions $S^G_M : \mathbb{H}(G) \to \mathbb{H}(M)$, $S^M_T : \mathbb{H}(M) \to \mathbb{H}(T)$.

The Hecke algebra $\mathbb{H}(T)$ is the subalgebra of $\mathbb{Z}[U_1^{\pm 1}, \ldots, U_g^{\pm 1}, V_1^{\pm 1}, \ldots, V_g^{\pm 1}]$ (here $U_i$, $V_i$ are abstract variables) generated by $(U_1 V_1^{-1})^{\pm 1}, \ldots, (U_g V_g^{-1})^{\pm 1}$ and $(U_1 \cdots U_g)^{\pm 1}$. The Weyl group $W_G$ of $G$ is the semidirect product of the permutation group $S_g$ and of $(\mathbb{Z}/2\mathbb{Z})^g = (\pm 1)^g$, where $S_g$ permutes coordinates in $(\pm 1)^g$. There is a section $S_g \hookrightarrow W_G$, we denote its image by $W_{G,M}$. Further, $W_G$ acts on $\mathbb{H}(T)$ in the obvious manner: $S_g$ permutes indices and $(\pm 1)^g$ interchanges $U_*, V_*$. We have:

$$S^G_T(\mathbb{H}(M)) = \mathbb{H}(T)^{W_{G,M}}, \quad S^G_T(\mathbb{H}(G)) = \mathbb{H}(T)^{W_G}.$$ (1.1.2)

It is known that there are surjections

$$\beta_1 : \mathbb{H}(G) \to \text{Corr}(X), \quad \beta_2 : \mathbb{H}(M) \to \text{Corr}(\bar{X})$$

(1.1.3) (1.1.4)

whose kernel is generated by the relation $KpK = id$, where $p = pl_{2g}$ is the scalar matrix in both $G$ and $M$, and $id$ is the trivial correspondence on both $X$, $\bar{X}$. 


Theorem 1.1.5. There exists a commutative diagram:

\[
\begin{array}{ccc}
S^G_M : & \mathbb{H}(G) & \rightarrow & \mathbb{H}(M) \\
\beta_1 \downarrow & & & \beta_2 \downarrow \\
\gamma : & \text{Corr}(X) & \rightarrow & \text{Corr}(\bar{X})
\end{array}
\]  

(1.1.6)

We denote by $\tau_p$ the matrix \[
\begin{pmatrix}
1 & 0 \\
0 & p
\end{pmatrix}
\]
where entries are scalar $g \times g$-blocks, and we denote the corresponding elements $K_G \tau_p K_M$ (resp. $K_M \tau_p K_M$) of $\mathbb{H}(G)$ (resp. $\mathbb{H}(M)$) by $T_p$ (resp. $fr_M$).

Theorem 1.1.7. $\beta_2(fr_M) = fr_X$.

Remark. Formulas 1.1.2 and theorems 1.1.5, 1.1.7 permit us to find the Hecke polynomial of $X$ (the characteristic polynomial of $fr_X$ over $\text{Corr}(X)$). Really, 1.1.2 implies that $H(M)$ is a free module over $S^G_M(\mathbb{H}(G))$ of dimension $#(W_G)/#(W_G, M) = 2^g$. An explicit description of $fr_M \in \mathbb{H}(M)$ (see below) permits us to find easily its characteristic polynomial over $\mathbb{H}(G)$. Theorem 1.1.5 implies that it is also the characteristic polynomial of the Frobenius correspondence on $\bar{X}$ over the algebra $\text{Corr}X$.

1.2. Geometric interpretation.

For $i = 0, \ldots, g$ we consider diagonal matrices in a block form

$$
\varphi_i = \begin{pmatrix}
I_i & 0 & 0 & 0 \\
0 & pI_{g-i} & 0 & 0 \\
0 & 0 & pI_i & 0 \\
0 & 0 & 0 & I_{g-i}
\end{pmatrix} \in M
$$

(sizes of diagonal blocks are $i$, $g - i$, $i$, $g - i$), and we denote the corresponding elements $K_M \varphi_i K_M \in \mathbb{H}(M)$ by $\Phi_i$. Particularly, $\Phi_g = fr_M$.

Let $I$ be a subset of $\{1, \ldots, g\}$. We denote $U_I := \prod_{i \in I} U_i \prod_{i \notin I} V_i \in \mathbb{H}(T)$. We have

\[
S_T^M(\Phi_i) = \sum_{\#(I) = i} U_I
\]

and

\[
S^G_M(T_p) = \Phi_0 + \Phi_1 + \cdots + \Phi_g \in \mathbb{H}(M).
\]  

(1.2.1)

We denote $\beta_1(T_p) \in \text{Corr}(X)$, $\beta_2(\Phi_i) \in \text{Corr}(\bar{X})$ again by $T_p$, $\Phi_i$ respectively, so (1.2.1) and Theorem 1.1.5 give us the following equality on $\text{Corr}(\bar{X})$:

$$
\gamma(T_p) = \Phi_0 + \Phi_1 + \cdots + \Phi_g.
$$  

(1.2.2)

For any algebraic variety $Z$ there exists an involution on $\text{Corr}(Z)$ (symmetry with respect the coordinates). Also, there exist involutions on $\mathbb{H}(M)$, $\mathbb{H}(G)$ commuting with involutions on $\text{Corr}(X)$, $\text{Corr}(\bar{X})$ with respect to (1.1.6). We denote these involutions by hat; we have $\hat{T}_p = T_p$, $\hat{\Phi}_i = \Phi_{g-i}$.
The geometric interpretation of (1.2.2) is the following. Let $t \in X(\overline{\mathbb{Q}})$ be a generic point, $A_t$ the corresponding principally polarized abelian $g$-fold with a fixed polarization form and $(A_t)_p$ the $\mathbb{F}_p$-space of its $p$-torsion points. The polarization on $A_t$ defines a skew form on $(A_t)_p$. $T_p(t)$ is a finite set of points; we have: $t' \in T_p(t)$ iff there exists an isogeny $\alpha_{t,t'} : A_t \rightarrow A_{t'}$ of type $(1,\ldots,1,p,\ldots,p)$. The kernel of $\alpha_{t,t'}$ is an isotropic $g$-dimensional subspace of $(A_t)_p$. So, we have a

**Theorem 1.2.3.** The set $T_p(t)$ is in 1–1 correspondence with the set of isotropic $g$-dimensional subspaces of $(A_t)_p$.

Now let $t \in X(\mathbb{Q})$ be a generic point such that $A_t$ has a good ordinary reduction at $p$. Let $(\hat{A}_t)_{p,points}$ be the set of closed points of $\hat{A}_t$ of order $p$, and $\text{red} : (A_t)_p \rightarrow (\hat{A}_t)_{p,points}$ the reduction map. We denote by $D = D_{\text{Siegel}}$ the kernel of $\text{red}$, it is an isotropic $g$-dimensional subspace of $(A_t)_p$.

Let $t' \in T_p(t)$ and $\tilde{t} \in \hat{X}$ its reduction. (1.2.2) shows that $\tilde{t}'$ belongs to one of $\Phi_i(\tilde{t})$.

**Theorem 1.2.4.** Number $i$ is defined as follows:

$$i = \dim \left( \text{Ker} (\alpha_{t,t'}) \cap D_{\text{Siegel}} \right).$$

(1.2.5)

Particularly, $i = g \iff \tilde{t}' = fr(\tilde{t}) \iff \text{Ker} (\alpha_{t,t'}) = D_{\text{Siegel}}$.

Further, we have the following

**Theorem 1.2.6.** Let $t'$, $t''$ be 2 points of $T_p(t)$. Then

$$\tilde{t}' = \tilde{t}'' \iff \text{Ker} \left( \alpha_{t,t'} \right) \cap D_{\text{Siegel}} = \text{Ker} \left( \alpha_{t,t''} \right) \cap D_{\text{Siegel}}.$$

Recall that any correspondence $C$ on $X$ has the bidergee $d_1(C), d_2(C)$ — the degrees of 2 projections $\pi_1, \pi_2$ of its graph $\Gamma(C)$ to $X$. By definition, $\pi_i(C) = \pi_i(C)$ (here $1 = 2, 2' = 1$). Further, $C$ has the separable (resp. non-separable) bidergee $d^s_1(C), d^s_2(C)$ (resp. $d^{ns}_1(C), d^{ns}_2(C)$) — the separable (resp. non-separable) degrees of $\pi_1, \pi_2$. We have $d_i(C) = d^s_i(C)d^{ns}_i(C)$ and

$$d^s_i(\hat{C}) = d^s_i(C), \quad i = 1, 2, \quad * = s, ns.$$

(1.2.6a)

We denote by $g(k,l)$ the cardinality of the Grassmann variety $Gr(k,l)(\mathbb{F}_p)$:

$$g(k,l) = \prod_{i=1}^{k} \frac{p^l - p^{i-1}}{p^k - p^{i-1}}.$$  

(1.2.7)

**Theorem 1.2.8.**

$$d^s_1(\Phi_i) = g(i,g), \quad d^{ns}_1(\Phi_i) = p^{(g+1-i)(g-i)/2}, \quad d^s_2(\Phi_i) = g(i,g), \quad d^{ns}_2(\Phi_i) = p^{(i+1)i/2}.$$  

2. Definitions and statement of conjectures for the case of Anderson modular objects.

We use standard notations for Anderson $t$-motives. Let $q$ be a power of a prime $p$, $\mathbb{F}_q$ the finite field of order $q$. The function field analog of $\mathbb{Z}$ is the ring of polynomials
\[ \mathbb{F}_q[\theta] \] where \( \theta \) is an abstract variable. The analog of the archimedean valuation on \( \mathbb{Q} \) is the valuation at infinity of the fraction field \( \mathbb{F}_q(\theta) \) of \( \mathbb{F}_q[\theta] \); it is denoted by \( \text{ord} \), it is uniquely determined by the property \( \text{ord}(\theta) = -1 \). The completion of an algebraic closure of the completion of \( \mathbb{F}_q(\theta) \) with respect the valuation ”ord” is the function field analog of \( \mathbb{C} \). It is denoted by \( \mathbb{C}_\infty \).

The definition of a t-motive \( \mathcal{M} \) is given in [G], Definitions 5.4.2, 5.4.12 (Goss uses another terminology: ”abelian t-motive” of [G] = ”t-motive” of the present paper). Particularly, \( \mathcal{M} \) is a free \( \mathbb{C}_\infty[T] \)-module of dimension \( r \) (this number \( r \) is called the rank of \( \mathcal{M} \)) endowed by a \( \mathbb{C}_\infty \)-skew-linear operator \( \tau \) satisfying some properties. Its dimension \( n \) is defined in [G], Remark 5.4.13.2 (Goss denotes the dimension by \( \rho \)). A nilpotent operator \( N = N(\mathcal{M}) \) associated to a t-motive is defined in [G], Remark 5.4.3.2. Condition \( N = 0 \) implies \( n \leq r \). Except Section 2.4, we shall consider only the case \( N = 0 \).

As it was written in the introduction, the main object of the present paper is conjectural. It is called an Anderson modular object, it is denoted by \( \mathcal{X}_{fun} = X_{fun}(r,n) \), it is the function field analog of \( X \). Naively, it parametrizes Anderson t-motives of rank \( r \) and dimension \( n \).

An analog of \( p \) of Section 1 is a valuation (distinct of ord) of \( \mathbb{F}_q(\theta) \) = a prime ideal of \( \mathbb{F}_q[\theta] \). We denote by \( \mathfrak{p} \) both its generator and the prime ideal itself, and we denote \( q = \#(\mathbb{F}_q[\theta]/\mathfrak{p}) \). The corresponding algebraic group \( G_{fun} \) — the function field analog of \( \text{GSp}_{2g}(\mathbb{Q}) \) — is \( \text{GL}_r(\mathbb{F}_q(\theta)) \). Hence, the analogs of \( \mathbb{Q}_p, \mathbb{Z}_p \) for the functional case are \( \mathbb{F}_q(\theta)_p, \mathbb{F}_q[\theta]_p \) respectively, and the analog of \( K_G \) of Section 1 is \( \text{GL}_r(\mathbb{F}_q[\theta]_p) \) (it will be denoted by \( K_G \) as well).

In order to simplify the present version of the text, for the case \( n > 1 \) we state conjectures of Section 2.3 only for uniformizable Anderson t-motives. Analytically, an uniformizable Anderson t-motives of rank \( r \) and dimension \( n \) over \( \mathbb{C}_\infty \) is the quotient \( \mathbb{C}_\infty/L \), where \( L \) is a free \( r \)-dimensional \( \mathbb{F}_q[\theta] \)-module. Since not all Anderson t-motives are uniformizable, the exact statements of these conjectures must be slightly changed, see Remark 2.3.4a.

**Theorem 2.1.** The analog of \( M \) for this case is the group \( M_{r-n,n} \) of block diagonal matrices \( \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset \text{GL}_r(\mathbb{F}_q(\theta)) \), sizes of blocks are \( r - n, n \). We denote \( K_M := M(\mathbb{F}_q[\theta]_p) \). The analogs of Hecke algebras and of algebras of correspondences are defined like in Section 1. The analogs of Theorems 1.1.5, 1.1.7 hold for this case. Particularly, \( \text{Corr}(\mathcal{X}_{fun}) \) is the quotient of \( \mathbb{H}(M) \) by the trivial relation \( K_M p K_M = \text{id} \).

**2.2. Description of \( \mathbb{H}(G_{fun}), \mathbb{H}(M) \) and of the Satake inclusions.**

Let like in Section 1, \( T \) be the subgroup of \( G_{fun} \) of diagonal matrices. We have \( \mathbb{H}(T) = \mathbb{Z}[U_1^{\pm 1}, \ldots, U_r^{\pm 1}] \), and the Weyl group \( W_{G_{fun}} = S_r \), it acts on \( \mathbb{H}(T) \) permuting indices. An analog of \( W_{G, M} \) is the subgroup \( W_{G_{fun}, M} = S_{r-n} \times S_n \hookrightarrow S_r = W_{G_{fun}} \) with the obvious inclusion. Formulas (1.1.2) hold for our case, explicit formulas are the following.

**2.2.1.** \( \mathbb{H}(G_{fun}) \): For \( i = 0, \ldots, r \) we denote by \( \tau_{p,i} \) the diagonal matrix \( \begin{pmatrix} I_{r-i} & 0 \\ 0 & p I_i \end{pmatrix} \in G_{fun} \), where sizes of blocks are \( r - i, i \), and we denote the corresponding elements \( K_G \tau_{p,i} K_G \in \mathbb{H}(G_{fun}) \) by \( T_{p,i} \). We have \( T_{p,0} = 1 \), \( T_{p,r} \) is the
trivial correspondence, and other $T_{p,i}$ are free generators of $\mathbb{H}(G_{fun})$. We have

$$S_T^{G_{fun}}(T_{p,i}) = q^{-i(i-1)/2} \sigma_i(U_1, ..., U_r),$$  \hspace{1cm} (2.2.2)

where $\sigma_i$ is the $i$-th symmetric polynomial, and

$$\hat{T}_{p,i} = T_{p,r-i}.$$ \hspace{1cm} (2.2.2a)

2.2.3. $\mathbb{H}(M)$: (a) For $i = 0, \ldots, r - n$ we denote by $\varphi_i$ the diagonal matrix

$$
\begin{pmatrix}
I_{r-n-i} & 0 & 0 \\
0 & pI_i & 0 \\
0 & 0 & I_n
\end{pmatrix}
\in M
$$

where sizes of blocks are $r - n - i, i, n$, and we denote the corresponding elements $K_M \varphi_i K_M \in \mathbb{H}(M)$ by $\Phi_i$.

(b) For $i = 0, \ldots, n$ we denote by $\psi_i$ the diagonal matrix

$$
\begin{pmatrix}
I_{r-n} & 0 & 0 \\
0 & pI_i & 0 \\
0 & 0 & I_{n-i}
\end{pmatrix}
\in M
$$

where sizes of blocks are $r - n, i, n - i$, and we denote the corresponding elements $K_M \psi_i K_M \in \mathbb{H}(M)$ by $\Psi_i$. We have $\Phi_0 = \Psi_0 = 1$, $\Phi_i = 0$ (resp. $\Psi_i = 0$) if $i$ is out of the range $0, \ldots, r - n$ (resp. $0, \ldots, n$) and other $\Phi_i, \Psi_i$ are free generators of $\mathbb{H}(M)$. Obviously

$$S_M^T(\Phi_i) = q^{-i(i-1)/2} \sigma_i(U_1, ..., U_{r-n}), \quad S_M^T(\Psi_i) = q^{-i(i-1)/2} \sigma_i(U_{r-n+1}, ..., U_r).$$ \hspace{1cm} (2.2.4)

Formulas (2.2.2), (2.2.4) imply immediately that

$$S_M^{G_{fun}}(T_{p,j}) = \sum_{i=0}^{j} q^{-i(i-j)} \Psi_i \Phi_{j-i}.$$ \hspace{1cm} (2.2.5)

Further, we have:

$$\Psi_n \Phi_{r-n} = q^{n(r-n)},$$ \hspace{1cm} (2.2.5a)

$$\hat{\Phi}_i = q^{-n(r-n-i)} \Psi_n \Phi_{r-n-i},$$ \hspace{1cm} (2.2.5b)

$$\hat{\Psi}_i = q^{-(n-i)(r-n)} \Psi_{n-i} \Phi_{r-n}.$$ \hspace{1cm} (2.2.5c)

particularly, $\hat{\Phi}_{r-n} = \Psi_n$. Coefficients of (2.2.5b,c) can be found from the property that equations (2.2.2a) and (2.2.5) are concordant with respect to the duality.

Remark. (2.2.5), (2.1), (1.1.5) imply that for the case $n = 1$ (Drinfeld modules) the explicit formulas are the following ($fr = \Psi_1$):

$$\tilde{T}_{p,1} = fr + \Phi_1;$$

$$\tilde{T}_{p,2} = \frac{1}{q} fr \Phi_1 + \Phi_2;$$

$$\ldots$$ \hspace{1cm} (2.2.6)

$$\tilde{T}_{p,r-1} = \frac{1}{q^{r-2}} fr \Phi_{r-2} + \Phi_{r-1};$$
\[ \tilde{T}_{p,r} = \frac{1}{q^{r-1}} f_r \Phi_{r-1}. \]

\( \mathbb{H}(M) \) is a free module over \( S_{M}^{G_{\text{fun}}} (\mathbb{H}(G_{\text{fun}})) \) respectively the Satake inclusion. Its dimension is \( \#(W_{G_{\text{fun}}})/\#(W_{G_{\text{fun}},M}) = \binom{r}{n}. \)

We denote \( \Psi_n \) by \( f_r M \). Its characteristic polynomial over \( S_{M}^{G_{\text{fun}}} (\mathbb{H}(G_{\text{fun}})) \) (the Hecke polynomial) can be easily found by elimination of \( \Phi_1, ..., \Phi_{r-n}, \Psi_1, ..., \Psi_{n-1} \) in the system (2.2.5). We denote it by \( P_{r,n} \), it belongs to \( \mathbb{H}_{p}(G_{\text{fun}})[f_r M] \). For \( n = 1 \) we have

\[ P_{r,1} = \sum_{i=0}^{r} (-1)^i q^{(i-1)/2} T_{p,i} f_r^{r-i} \]

\[ = f_r^r - T_{p,1} f_r^{r-1} + q T_{p,2} f_r^{r-2} \pm \cdots + (-1)^r q^{r(r-1)/2} T_{p,r}. \] (2.2.7)

2.3. Statements of results.

Theorems 1.2.4, 1.2.6 can be rewritten almost word-to-word as conjectures for the functional case. Let us do it. It is more convenient to use Anderson modules ([G], 5.4.5; Goss calls them Anderson T-modules) instead of Anderson t-motives. The categories of Anderson t-motives and modules are anti-isomorphic, so there is no essential difference which object to use.

Let \( t \in X_{\text{fun}} \) be such that the corresponding Anderson module \( E_t \) is uniformizable: \( E_t = \mathbb{C}_{\infty}^n / L \) (as earlier, \( n \) is the dimension and \( r \) is the rank). This condition is “closed under Hecke correspondences”: if \( t' \in T_{p,j}(t) \) then \( E_{t'} \) is also uniformizable.

\[ \text{Theorem 2.3.1. } (E_t)_p \] — the set of \( p \)-torsion points of \( E_t \) — is \( p^{-1}L/L \) and hence is an \( r \)-dimensional \( \mathbb{F}_p[\theta]/p \)-vector space.

\[ \text{Theorem 2.3.2. } t' \in T_{p,j}(t) \text{ iff there exists an isogeny } \alpha_{t,t'} : E_t \to E_{t'} \text{ of type } (1, ..., 1, p, ..., p) \text{ (} r - j \text{ 1’s and } j \text{ p’s).} \]

\[ \text{Theorem 2.3.3. } \text{The set } T_{p,j}(t) \text{ is in 1–1 correspondence with the set of } j \text{-dimensional subspaces of } (E_t)_p. \]

Now we can formulate the conjecture on reductions at \( p \). Let \( (\tilde{E}_t)_{p, \text{points}} \) and \( \text{red} : (E_t)_p \to (\tilde{E}_t)_{p, \text{points}} \) be analogs of the corresponding objects in Section 1.2. We denote by \( D_{\text{fun}} \) the kernel of red. Firstly, we have a

\[ \text{Theorem 2.3.4. } \text{For a generic } t \in X_{\text{fun}} \text{ } D_{\text{fun}} \text{ is an } n \text{-dimensional subspace of } (E_t)_p. \]

\[ \text{Remark 2.3.4a. } \text{For an arbitrary } t \in X_{\text{fun}} \text{ (such that } E_t \text{ is not uniformizable) statements of the above and below theorems and conjectures require minor changes. For example, Theorem 2.3.1 becomes} \]

\[ \text{Theorem 2.3.4b. } (E_t)_p \text{ is an } r \text{-dimensional } \mathbb{A}/p \text{-vector space.} \]

(we cannot claim that it is \( p^{-1}L/L \) because \( L \) does not exist).

Geometric interpretation.

Here we formulate analogs of (1.2.4) – (1.2.8) for the function field case. Let \( t' \in T_{p,j}(t) \) and \( \tilde{t} \in \tilde{X}_{\text{fun}} \) its reduction. Conjecture 2.1 and (2.2.5) show that \( \tilde{t} \) belongs to one of \( q^{-i(j-i)} \Phi_{j-i}(\tilde{t}). \)
Theorem 2.3.5. Number \( i \) is defined as follows:

\[ i = \dim (\text{Ker} (\alpha_{t,t'}) \cap D_{\text{fun}}). \tag{2.3.6} \]

Particularly, \( j = i = n \iff \tilde{t}' = f_{rX_{\text{fun}}}(\tilde{t}) \iff \text{Ker} (\alpha_{t,t'}) = D_{\text{fun}}. \)

Now \( g(k,l) \) will mean the cardinality of Grassmann variety \( Gr(k,l) \) over \( \mathbb{F}_q[\theta]/\mathfrak{p} \), i.e. \( p \) in (1.2.7) must be replaced by \( q \). Obviously we have

\[ d_1(T_{p,i}) = d_2(T_{p,i}) = g(i, r). \]

For the reader’s convenience, we formulate the following conjecture separately for the case \( n = 1 \).

Theorem 2.3.7. For \( n = 1 \) we have:

\[ d_1^s(\Psi_1) = d_1^{ns}(\Psi_1) = 1, \quad d_2^s(\Psi_1) = 1, \quad d_2^{ns}(\Psi_1) = q^{r-1}; \]

\[ d_1^s(\Phi_i) = g(i, r - 1), \quad d_1^{ns}(\Phi_i) = q^i, \quad d_2^s(\Phi_i) = g(i, r - 1), \quad d_2^{ns}(\Phi_i) = 1. \]

Corollary 2.3.7.1. For correspondences \( \frac{1}{q}I \cdot \Phi_i \) — summands in the right hand side of (2.2.6) — we have

\[ d_1^s\left(\frac{1}{q} \cdot \Phi_i\right) = g(i, r - 1), \quad d_1^{ns}\left(\frac{1}{q} \cdot \Phi_i\right) = 1, \]

\[ d_2^s\left(\frac{1}{q} \cdot \Phi_i\right) = g(i, r - 1), \quad d_2^{ns}\left(\frac{1}{q} \cdot \Phi_i\right) = q^{r-1-i}. \]

An analog of the Theorem 1.2.6 is the following. Let \( t', t'' \) be 2 points of \( T_{p,i}(t) \). Firstly we consider the case when

\[ \text{Ker} (\alpha_{t,t'}) \cap D_{\text{fun}} = \text{Ker} (\alpha_{t,t''}) \cap D_{\text{fun}} = 0, \]

i.e. both \( \tilde{t}', \tilde{t}'' \in \Phi_i(\tilde{t}) \) (the second summand in (2.2.6)).

Theorem 2.3.8. Under this condition we have (for any \( n \)):

\( \tilde{t}' = \tilde{t}'' \) as closed points iff the linear spans coincide:

\[ < \text{Ker} (\alpha_{t,t'}), D_{\text{fun}} > = < \text{Ker} (\alpha_{t,t''}), D_{\text{fun}} >. \]

If (for \( n = 1 \))

\[ \text{Ker} (\alpha_{t,t'}) \supset D_{\text{fun}}, \quad \text{Ker} (\alpha_{t,t''}) \supset D_{\text{fun}}, \]

i.e. both \( \tilde{t}', \tilde{t}'' \in \frac{1}{q^{i-1}} \Psi_i \Phi_{i-1}(\tilde{1}) = \frac{1}{q^{i-1}} I \cdot \Phi_{i-1}(\tilde{1}) \) (the first summand in (2.2.6)) then (2.3.7.1) implies that \( \tilde{t}', \tilde{t}'' \) are always different.

Now let us consider the case of arbitrary \( n \).

Theorem 2.3.10.

\[ d_1^s(\Psi_i) = g(i, n), \quad d_1^{ns}(\Psi_i) = 1; \]

\[ d_1^s(\Phi_i) = g(i, r - n), \quad d_1^{ns}(\Phi_i) = q^{in}. \]

\[ 9 \]
Remark 2.3.10a. Numbers $d_2^s(\Phi_i)$, $d_2^n(\Psi_i)$ ($* = \emptyset, s, ns$) can be found from the above formulas using (1.2.6a), (2.2.5b,c). Particularly, for the Frobenius $\Psi_n$ we have

\[ d_2^s(\Psi_n) = 1, \quad d_2^n(\Psi_n) = q^{n(r-n)}. \]

Corollary 2.3.11. For correspondences $\frac{1}{q^{(j-i)}} \Psi_i \Phi_{j-i}$ — summands in the right hand side of (2.2.5) — we have

\[ d_1^s\left(\frac{1}{q^{(j-i)}} \Psi_i \Phi_{j-i}\right) = g(i,n)g(j-i,r-n), \quad d_1^n\left(\frac{1}{q^{(j-i)}} \Psi_i \Phi_{j-i}\right) = q^{(j-i)(n-i)}. \]

Let $t'$, $t'' \in T_{p,j}(t)$ be as above such that

\[ \dim (\ker (\alpha_{t,t'}) \cap D_{fun}) = \dim (\ker (\alpha_{t,t''}) \cap D_{fun}) = i. \]

According Theorem 2.3.5, both $\tilde{t}', \tilde{t}'' \in \frac{1}{q^{(j-i)}} \Psi_i \Phi_{j-i}$.

Theorem 2.3.12. $\tilde{t}' = \tilde{t}''$ as closed points iff both intersections and linear spans coincide:

\[ \ker (\alpha_{t,t'}) \cap D_{fun} = \ker (\alpha_{t,t''}) \cap D_{fun}; \]
\[ <\ker (\alpha_{t,t'}), D_{fun}>, = <\ker (\alpha_{t,t''}), D_{fun}>. \]

Remark 2.3.13. (a) If we fix an $i$-dimensional subspace $V_i \subset D_{fun}$ and a $(n+j-i)$-dimensional overspace $V_{n+j-i} \supset D_{fun}$ then the quantity of $j$-dimensional spaces $\alpha$ such that

\[ \alpha \cap D_{fun} = V_i, \quad <\alpha, D_{fun}>, = V_{n+j-i}, \]

is equal to $q^{(j-i)(n-i)} = d_1^{ns}\left(\frac{1}{q^{(j-i)}} \Psi_i \Phi_{j-i}\right)$, as it is natural to expect.

(b) Thanks to existence of skew pairing in the number case, we have

\[ <\ker (\alpha_{t,t'}), D_{fun}>, = (\ker (\alpha_{t,t'}) \cap D_{fun})^{\text{dual}}, \]

(dual is with respect to the skew pairing), so in Theorem 1.2.6 it is sufficient to claim only coincidence of intersections.

(c) For $n = 1$ (Theorem 2.3.8) intersections always coincide, so it is sufficient to claim only coincidence of linear spans.

2.4. Case of $N \neq 0$. This is a subject of further research. Here we do not even give statements of results, we indicate only the discrete invariants of t-motives having $N \neq 0$ and explain the methods how these statements can be obtained.

Let $M$ be a uniformizable Anderson t-motive such that its $N$ is not (necessarily) 0, of dimension $n$ and of rank $r$. $M$ is a $C_\infty[T]$-module with a skew map $\tau : M \to M$ such that $M/\tau M$ is annihilated by a power of $T - \theta$. We have Lie($M$) = $C_\infty^n$, and $N$ is a nilpotent operator acting on Lie($M$). Also, $T$ acts on Lie($M$); we have $T = \theta I_n + N$ on End (Lie($M$)). Particularly, we can consider Lie($M$) as a $F_q[T]$-module.
The lattice $L(\mathfrak{M})$ of $\mathfrak{M}$ is a free $\mathbb{F}_q[T]$-submodule of $\text{Lie}(\mathfrak{M})$ considered as a $\mathbb{F}_q[T]$-module. We have a natural inclusion $\mathbb{F}_q[T] \hookrightarrow \mathbb{C}_\infty[[N]]$ ($T \mapsto N + \theta$). Hence, the tautological inclusion $L(\mathfrak{M}) \hookrightarrow \text{Lie}(\mathfrak{M})$ defines a surjection

$$L(\mathfrak{M}) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \twoheadrightarrow \text{Lie}(\mathfrak{M}).$$

Its kernel is denoted by $q_{\mathfrak{M}}$; the exact sequence

$$0 \rightarrow q_{\mathfrak{M}} \rightarrow L(\mathfrak{M}) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \rightarrow \text{Lie}(\mathfrak{M}) \rightarrow 0$$

(see [P]; [Gl18], (1.8.3); [HJ], Example 2.5, (2.2) for the case of Drinfeld modules) is a particular case of a Hodge-Pink structure.

Discrete invariants of $\mathfrak{M}$ are the discrete invariants of the pair of lattices $(q_{\mathfrak{M}}; L(\mathfrak{M}) \otimes \mathbb{C}_\infty[[N]])$ over a discrete valuation ring $\mathbb{C}_\infty[[N]]$. Let us give a description of these invariants from [GL18], (3.3). Let $\nu$ be the minimal number such that $N^\nu = 0$. These invariants are numbers $k_1 \geq 0, \ldots, k_{\nu+1} \geq 0$ defined as follows. There exists a basis $l_1, \ldots, l_r$ of $L(\mathfrak{M})$ over $\mathbb{F}_q[T]$ and its partition on $\nu + 1$ sets of lengths $k_1, \ldots, k_{\nu+1}$ (if some $k_i = 0$ then the $i$-th set is empty); the $i$-th set is denoted by $l_{i,1}, \ldots, l_{i,k_i}$, having the following properties:

- $N^{\nu-1}(l_{\nu+1,i}), i = 1, \ldots, k_{\nu+1}$, form a $\mathbb{C}_\infty$-basis of $N^{\nu-1}\text{Lie}(\mathfrak{M})$ ([GL18], (3.6));
- $N^{\nu-2}(l_{\nu,i}), i = 1, \ldots, k_{\nu}, N^{\nu-2}(l_{\nu+1,i}), i = 1, \ldots, k_{\nu+1}$;
- $N^{\nu-1}(l_{\nu+1,i}), i = 1, \ldots, k_{\nu+1}$, form a $\mathbb{C}_\infty$-basis of $N^{\nu-2}\text{Lie}(\mathfrak{M})$ ([GL18], (3.8));
- $N^{\nu-3}(l_{\nu-1,i}), i = 1, \ldots, k_{\nu-1}, N^{\nu-3}(l_{\nu,i}), N^{\nu-2}(l_{\nu,i}), i = 1, \ldots, k_{\nu}$;
- and $N^{\nu-3}(l_{\nu+1,i}), N^{\nu-2}(l_{\nu+1,i}), N^{\nu-1}(l_{\nu+1,i}), i = 1, \ldots, k_{\nu+1}$,

form a $\mathbb{C}_\infty$-basis of $N^{\nu-3}\text{Lie}(\mathfrak{M})$ ([GL18], (3.8)); etc., until

- $l_{2,i}, i = 1, \ldots, k_2, \ldots, N^{\nu-1}(l_{\nu+1,i}), i = 1, \ldots, k_{\nu+1}$,

form a $\mathbb{C}_\infty$-basis of $\text{Lie}(\mathfrak{M})$.

See [GL18], (3.3) - (3.10) for more details.

Particularly, we have:

$$r = k_1 + \ldots + k_{\nu+1}; \quad n = k_2 + 2k_3 + 3k_4 + \ldots + \nu k_{\nu+1}.$$

([GL18], (3.10) and (3.5)). If $\nu = 1$, i.e. $N = 0$ — this is the case considered above, then $k_1 = r - n$, $k_2 = n$. Therefore, numbers $k_1, \ldots, k_{\nu+1}$ are $N \neq 0$-generalizations of numbers $r - n$, $n$.

**Conjecture 2.4.1.** The analog of the subgroup $M$ of $G_{fun} = GL_r$ for the set of Anderson t-motives having $N \neq 0$ and invariants $k_1, \ldots, k_{\nu+1}$ is the subgroup of $GL_r$ of block diagonal matrices with block sizes $k_1, \ldots, k_{\nu+1}$. 
Remark 2.4.2. Some of $k_i$ can be 0. In this case we cannot distinguish between $M$ for different sets of $k_1, \ldots, k_{\nu+1}$. Hence, maybe it is necessary to modify the statement of Conjecture 2.4.1.

As it was written, finding of analogs of statements of Sections 2.2, 2.3 for the sets of Anderson t-motives having invariants $k_1, \ldots, k_{\nu+1}$ is a subject of further research.

Remark 2.4.3. Hartl and Juschka use some other invariants of $\mathfrak{M}$, see [HJ], Section 2. First, they consider slightly more general objects, namely, their $q = q_{\mathfrak{M}}$ is a subset not of $L(\mathfrak{M}) \otimes \mathbb{C}_\infty [[N]]$ but of $L(\mathfrak{M}) \otimes \mathbb{C}_\infty ((N))$ (also, they consider a weight filtration on $L(\mathfrak{M})$). Further, their Hodge-Pink weights $\omega_1, \ldots, \omega_r$ are related with $k_1, \ldots, k_{\nu+1}$ as follows: for all $i = 1, \ldots, \nu+1$ the number $-i+1$ occurs $k_i$ times among $\omega_1, \ldots, \omega_r$ (i.e. among $\omega_1, \ldots, \omega_r$ there are $k_1$ zeroes, $k_2$ minus ones etc.).

3. Proofs. We follow [FCh], Ch. 7, Section 4 using the same notations if possible, and indicating results that are not completely analogous to the number field case.

Recall that $p$ is a prime ideal of $\mathbb{F}_q[\theta]$. We denote by $\mathbb{F}_q[\theta]_p$, $\mathbb{F}_q(\theta)_p$ the completions at $p$ of $\mathbb{F}_q[\theta]$, $\mathbb{F}_q(\theta)$ respectively, and by $\mathbb{F}_q[\theta]^nr_p$ the ring of integers of the maximal unramified extension of $\mathbb{F}_q(\theta)_p$. As usual, bar means an algebraic closure.

There are maps $\mathbb{F}_q[\theta]^nr_p \hookrightarrow \mathbb{F}_q(\theta)_p$, $\mathbb{F}_q[\theta]^nr_p \surject \mathbb{F}_q(\theta)/\mathfrak{p}$. The corresponding maps of schemes Spec $\overline{\mathbb{F}_q(\theta)}_p \rightarrow$ Spec $\mathbb{F}_q[\theta]^nr_p$, Spec $\overline{\mathbb{F}_q(\theta)}_p/p \rightarrow$ Spec $\mathbb{F}_q[\theta]^nr_p$ are denoted by $\xi_k$, $\xi_p$ respectively. The inverse image $\xi_p^*$ of an object (i.e. the reduction of this object) is denoted by tilde.

We fix $i$, and let $\Gamma$ be the graph of $T_{p,i}$ over Spec $\overline{\mathbb{F}_q[\theta]^nr}_p$.

It is known that it exists. For $t \in \Gamma$ (resp. $t \in \tilde{\Gamma}$) let $\phi_t : E_t \rightarrow E'_t$ be the corresponding map of Anderson modules over Spec $\overline{\mathbb{F}_q[\theta]^nr}_p$ (resp. Spec $\overline{\mathbb{F}_q(\theta)}_p$).

We consider the ordinary locus $\Gamma^0$ of $\Gamma$:

$$t \in \Gamma^0 \iff \xi_k(E_t), \xi_k(E'_t)$$

are ordinary.

Lemma 3.1. $\tilde{\Gamma}^0$ is dense in $\tilde{\Gamma}$. □

3.1a. Now let $\tau_p \in G_{fun}$ be any diagonal matrix, $T_{G,p}$ the element of Hecke algebra $\mathbb{H}(G_{fun})$ corresponding to the double coset $K_G \tau_p K_G$, $\Gamma$ the graph of $T_{G,p}$ over Spec $\overline{\mathbb{F}_q[\theta]^nr}_p$, and $\tilde{\Gamma}^0$ for this $\Gamma$ is defined as earlier.

Let $c$ be the highest power of $p$ that appears in the diagonal entries of $\tau_p$ (for example, if $\tau_p = \tau_{p,i}$ then $c = 1$). Let $s \in \tilde{\Gamma}^0$ and $E_s$, $E'_s$ the corresponding Anderson t-motives over Spec $\overline{\mathbb{F}_q[\theta]}/\mathfrak{p}$. This means that we have a direct sum decomposition of the finite $\mathbb{F}_q[\theta]$-module scheme $(E_s)[p^c]$ over Spec $\overline{\mathbb{F}_q[\theta]}/\mathfrak{p}$ on its multiplicative and etale part:

$$(E_s)[p^c] = (E_s)[p^c]_{mult} \oplus (E_s)[p^c]_{et}, \quad (3.1.1)$$
where
\[(E_s)[p^c]_{\text{mult}} = (\mu_p)^n, \quad (E_s)[p^c]_{\text{et}} = (\text{Spec } \mathbb{F}_q[\theta]/p^c)^{r-n}.\]  

(3.1.2)  
(3.1.3)  

We can restrict \(\phi_s\) to \((E_s)[p^c]\) getting a map
\[(\phi_s)[p^c] : (E_s)[p^c] \rightarrow (E_s')[p^c].\]  

(3.1.4)  

In its turn, this map is restricted to both etale and multiplicative parts:
\[(\phi_s)[p^c]_{\text{mult}} : (E_s)[p^c]_{\text{mult}} \rightarrow (E_s')[p^c]_{\text{mult}} \quad (3.1.5)\]
and
\[(\phi_s)[p^c]_{\text{et}} : (E_s)[p^c]_{\text{et}} \rightarrow (E_s')[p^c]_{\text{et}}.\]  

(3.1.6)  

Taking into consideration (3.1.2), (resp. (3.1.3)), we see that \(\phi_s\) defines elements in \(H(GL_n)\) (resp. \(H(GL_{r-n})\)). In concordance of notations of [FCh], we denote them by \(a\) (resp. \(d\)). This pair \((a, d)\) defines us an element of \(H(M)\). It is called the type of \(s\).

**Remark.** Unlike in the number case, here the elements \(a, d\) are independent.

In order to formulate the below proposition 3.4, we need the following notations:

3.2. Let \(\delta : \mathcal{E}_1 \rightarrow \mathcal{E}_2\) be a map of Anderson modules over \(\text{Spec } \mathbb{F}_q[\theta]/p\) of type \(\Psi_i \Phi_j\).

This means that \(c\) of (3.1.1) is 1, and kernels of the map (3.1.5) (resp. (3.1.6)) is isomorphic to \((\mu_p)^i\) (resp. \((\text{Spec } \mathbb{F}_q[\theta]/p)^j\)). We denote them by \(\tilde{K}_m, \tilde{K}_e\) respectively.

Further, let \(E_1\) be a Anderson module over \(\text{Spec } \mathbb{F}_q[\theta]/p\) such that \(\tilde{E}_1 = \mathcal{E}_1\).

**Lemma 3.3.** We can identify \(\tilde{K}_m\) (resp. \(\tilde{K}_e\)) with some \(i\) (resp. \(j\))-dimensional subspaces in \(D_{\text{fun}}(E_1)\) (resp. \((E_1)_p/D_{\text{fun}}(E_1)\)).

We denote these subspaces by \(K_m, K_e\) respectively.

Now let us consider the set of pairs \((\phi, E_2)\) where \(\phi : E_1 \rightarrow E_2\) is a map of Anderson modules over \(\text{Spec } \mathbb{F}_q[\theta]/p\), such that \(\tilde{\phi} = \delta\) (and hence \(\tilde{E}_2 = \mathcal{E}_2\)).

**Proposition 3.4.** The set of the above \((\phi, E_2)\) is isomorphic to the set of subspaces \(W \subset (E_1)_p\) such that
\[W \cap D_{\text{fun}}(E_1) = K_m, \quad W + D_{\text{fun}}(E_1)/D_{\text{fun}}(E_1) = K_e.\]  

(3.4.1)  

**Proof.** We need the function field analog of [K], Th. 2.1. Let \(R\) be an Artinian local ring with residue field \(\mathbb{F}_q[\theta]/p\) and the maximal ideal \(m\). We consider only the case \(R = R_\eta = \mathbb{F}_q[\theta]^{nr}/p^n\) for some \(\eta\). Let \(\mathcal{E}\) be an ordinary Anderson module over \(\mathbb{F}_q[\theta]/p\). Let us consider (3.1.1) for \(\mathcal{E}\), and let \(T_p(\mathcal{E})\) be the Tate module of the etale part:
\[T_p(\mathcal{E}) = \lim_{c \to \infty} \mathcal{E}[p^c]_{\text{et}}.\]
The dual Anderson module $\mathcal{E}^t$ is defined in [L], [F], it is of rank $r$ and dimension $n$.

Let $\mathcal{E}_R$ be an Anderson module over $R$ such that its reduction to $\overline{\mathbb{F}_q[\theta]/p}$ is $\mathcal{E}$ (a lift of $\mathcal{E}$ on $R$).

The function field analog of [K], Th. 2.1, (1) is the following:

**Theorem 3.4.1a.** The set of $\mathcal{E}_R$ is in 1–1 correspondence with the set of maps

$$\text{Hom} \left( T_p(\mathcal{E}) \otimes T_p(\mathcal{E}^t), m \right),$$

(3.4.2)

where Hom is of $\mathbb{F}_q[T]/p$-modules.

**Notation.** For a fixed $\mathcal{E}_R$ we denote this map by $q_{\mathcal{E}_R}$.

**Idea of the proof.** First, we define the analog of the map $\varphi_{A/R}$, [K], p. 151 for the present situation. Here it is $\varphi_{\mathcal{E}_R} : T_p(\mathcal{E}) \to m^\oplus n$.

Recall that $\eta$ satisfies $m^n = 0$. We choose $k$ such that $q^k \geq \eta$, and we consider formulas of multiplication by $p^k$ for $\mathcal{E}$:

$$p^k(X) = \sum_{i=k}^{\eta} C_i X^{q^i}$$

(3.4.2a)

where $X \in \mathbb{C}^{\oplus n}_\infty$ is a column vector and $C_i \in M_{n \times n}(\mathbb{C}_\infty)$. Condition $X \in \mathcal{E}[p^k]$ means that $\sum_{i=k}^{\eta} C_i X^{q^i} = 0$.

Let $\tilde{X} \in R^{\oplus n}$ be a lift of $X \in \mathcal{E}[p^k]_{\text{et}}$. Since for the first term $C_k X^{q^k}$ of 3.4.2a we have $q^k \geq N$, we get that $\varphi_{\mathcal{E}_R}(X) := \sum_{i=k}^{\eta} C_i \tilde{X}^{q^i} \in m^\oplus n$ does not depend on the choice of $\tilde{X}$. □

Now we need the function field analog of [K], Th. 2.1, (4). Let $\mathcal{E}_1$, $\mathcal{E}_2$ be ordinary Anderson modules over $\overline{\mathbb{F}_q[\theta]/p}$, $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$ a map and $\mathcal{E}_{1,R}$, $\mathcal{E}_{2,R}$ lifts of $\mathcal{E}_1$, $\mathcal{E}_2$ on $R$. We denote by

$$T_p(\alpha) : T_p(\mathcal{E}_1) \to T_p(\mathcal{E}_2)$$

(3.4.3)

$$T_p(\alpha^t) : T_p(\mathcal{E}_2^t) \to T_p(\mathcal{E}_1^t)$$

(3.4.4)

the maps obtained by functoriality.

**Lemma 3.4.5.** A map $\alpha_R : \mathcal{E}_{1,R} \to \mathcal{E}_{2,R}$ such that its reduction is $\alpha$ exists iff for any $x \in T_p(\mathcal{E}_1)$, $y \in T_p(\mathcal{E}_2^t)$ we have

$$q_{\mathcal{E}_{2,R}}(T_p(\alpha)(x), y) = q_{\mathcal{E}_{1,R}}(x, T_p(\alpha^t)(y)).$$

(3.4.6)

and moreover if this condition is satisfied then $\alpha_R$ is unique. □

**Lemma 3.4.6a.** (Conjectural statement). To define $E_2$ over $\mathbb{F}_q[\theta]/p$ is the same as to define a concordant system of $(E_2)_\eta$ over $R_\eta$ (the concordance condition is clear).

**Remark.** Obviously $E_2$ defines a concordant system of $(\mathcal{E}_2)_\eta$. But is the inverse really true? Maybe non-trivial automorphisms of $(\mathcal{E}_2)_\eta$ give obstacles?
Now we return to the proof of Proposition 3.4. We fix \( \eta \), we take \( \mathcal{E}_1 = \mathcal{E}_2 \). According (3.2), there exist bases

\[
e_1^t, \ldots, e_n^t, e_{n+1}, \ldots, e_r, f_1^t, \ldots, f_n^t, f_{n+1}, \ldots, f_r\]

of \( T_p(\mathcal{E}_1^t), T_p(\mathcal{E}_2^t), T_p(\mathcal{E}_2^t) \) respectively such that the maps \( T_p(\delta), T_p(\delta^t) \) in these bases are the following:

\[
T_p(\delta)(e_{n+1}) = pf_{n+1}
\]
\[
T_p(\delta)(e_{n+j}) = pf_{n+j},
\]
\[
T_p(\delta)(e_{n+j+1}) = f_{n+j+1}
\]
\[
T_p(\delta)(e_r) = f_r,
\]
\[
T_p(\delta^t)(f_1^t) = pe_1^t
\]
\[
T_p(\delta^t)(f_i^t) = pe_i^t,
\]
\[
T_p(\delta^t)(f_{i+1}^t) = e_{i+1}^t
\]
\[
T_p(\delta^t)(f_n^t) = e_n^t.
\]

Now we apply formula (3.4.6) to these formulas. We consider 4 types of \( x, y \):

**Type 13.** \( x \) of type 1, \( y \) of type 3 \( (\lambda \in [n + 1, \ldots, n + j], \mu \in [1, \ldots, i]) \):

We get:

\[
p \cdot qe_2, R(f_\lambda, f_\mu^t) = p \cdot qe_1, R(e_\lambda, e_\mu^t).
\]

(3.4.7)

If \( m \) had no \( p \)-torsion then we can divide the above equality by \( p \) and to get

\[
qe_2, R(f_\lambda, f_\mu^t) = qe_1, R(e_\lambda, e_\mu^t),
\]

(3.4.8)

this means that \( qe_2, R \) on these \( f_\lambda, f_\mu^t \) is defined uniquely.

We think that in order to prove that we can really divide by \( p \), we must consider not one fixed \( \eta \), but all the values of them. The similar problem exists for the next type:

**Type 14.** \( x \) of type 1, \( y \) of type 4 \( (\lambda \in [n + 1, \ldots, n + j], \mu \in [i + 1, \ldots, n]) \):

We get:

\[
p \cdot qe_2, R(f_\lambda, f_\mu^t) = qe_1, R(e_\lambda, e_\mu^t).
\]

(3.4.9)

If \( m \) were \( p \)-divisible and had the \( p \)-torsion isomorphic to \( \mathbb{F}_q[\theta]/\mathfrak{p} \) then OK: we have \( q^{(n-i)j} \) possibilities for \( (\mathcal{E}_2)_R \) as it should be.

For other types of \( x, y \) there is no such problem. Really:

**Type 23.** \( x \) of type 2, \( y \) of type 3 \( (\lambda \in [n + j + 1, \ldots, r], \mu \in [1, \ldots, i]) \):

We get:

\[
qe_2, R(f_\lambda, f_\mu^t) = p \cdot qe_1, R(e_\lambda, e_\mu^t).
\]

(3.4.10)
This means that $q_{e_{2,R}}$ on these $f_\lambda, f_\mu^i$ is defined uniquely;

**Type 24.** $x$ of type $2$, $y$ of type $4$ ($\lambda \in [n+j+1, \cdots, r], \mu \in [i+1, \cdots, n]$):

We get:

$$q_{e_{2,R}}(f_\lambda, f_\mu^i) = q_{e_{1,R}}(e_\lambda, e_\mu^i).$$

(3.4.11)

This means that $q_{e_{2,R}}$ on these $f_\lambda, f_\mu^i$ is defined uniquely;

**3.4.12.** We get that we have $q^{(n-i)j}$ modules $E_2$, this number is equal to the quantity of $W$ satisfying (3.4.1).

**3.4.13.** Now we need to prove that these $W$ really satisfy (3.4.1). □

Now we can define the map $\beta_2 : \mathbb{H}(M) \to \text{Corr}(\tilde{X})$ from (1.1.4). Idea of the definition: let $\tau_p$ have the form $\begin{pmatrix} p^A & 0 \\ 0 & p^B \end{pmatrix}$ where $A = (a_1, \ldots, a_{r-n}), B = (b_1, \ldots, b_n), p^A = \text{diag} (p^{a_1}, \ldots, p^{a_{r-n}})$, $p^b = \text{diag} (p^{b_1}, \ldots, p^{b_n})$. We denote by $T_{M,p} = T_{M,p}(A, B)$ the element of Hecke algebra $\mathbb{H}(M)$ corresponding to the double coset $K_M \tau_p K_M$. Explicit formula for $S_M^{G_{\text{fun}}}$ shows that

$$S_M^{G_{\text{fun}}}(T_{G,p}) = q^{-m_{A,B}}T_{M,p} + \text{other terms},$$

(3.5)

where these other terms are linear combinations of $T_{M,p}(A', B')$ for pairs $(A', B')$ distinct from $(A, B)$. Coefficient $m_{A,B} \geq 0$ can be easily found explicitly; for $T_{M,p}(A, B) = \Psi_i \Phi_j$ we have $m_{A,B} = ij$.

Now we consider the reduction of the correspondence $\beta_1(T_{G,p})$. Let $\Gamma_{\text{irr}}$ be an irreducible component of its graph, $\phi : E_1 \to E_2$ a map of Anderson modular objects over $A/p$ corresponding to a point of $\Gamma_{\text{irr}}$, and $t \in \mathbb{H}(M)$ its type. $t$ depends only on $\Gamma_{\text{irr}}$ but not on $\phi : E_1 \to E_2$ because it is a discrete invariant, so we can call it the type of $\Gamma_{\text{irr}}$.

First, we denote by $\mathcal{C}(A, B)$ the correspondence on $\tilde{X}$ whose graph is the sum of all the irreducible components of the graph of reduction of the correspondence $\beta_1(T_{G,p})$ whose type is $T_{M,p}(A, B)$ (really, for each $(A, B)$ there exists only one such component). By abuse of notations we denote by $\mathcal{C}(\Psi_i \Phi_j)$ the $\mathcal{C}(A, B)$ where $A, B$ are from 2.2.3 a.b. Finally, we define

$$\beta_2(T_{M,p}(A, B)) = q^{m_{A,B}}\mathcal{C}(A, B),$$

(3.6)

hence

$$\beta_2(\Psi_i \Phi_j) = q^{ij}\mathcal{C}(\Psi_i \Phi_j).$$

(3.7)

(3.5) and (3.6) show immediately that the function field analog of the diagram (1.1.6) is commutative.

**Corollary 3.8.** $d_1^{n_s}(\mathcal{C}(\Psi_i \Phi_j)) = q^{(n-i)j}$.

**Proof.** Follows immediately from 3.4.12. □

3.7 and 3.8 imply that

$$d_1^{n_s}(\beta_2(\Psi_i \Phi_j)) = q^{nj}.$$

(3.9)

**Proposition 3.10.** $\beta_2$ is a ring homomorphism.
Idea of the proof. Let \((A, B), (A', B')\) be 2 pairs of multiindices as above and let \(T_{M,p}(A, B) \cdot T_{M,p}(A', B') = \sum_i \kappa_i T_{M,p}(A_i, B_i)\) for some pairs \((A_i, B_i)\) and coefficients \(\kappa_i\).

Lemma 3.10.1. For all \(i\) we have
\[
q^{m_A,B}d_1^{ns}(\mathcal{C}(A, B)) \cdot q^{m_{A'},B'}d_1^{ns}(\mathcal{C}(A', B')) = \kappa_i q^{m_{A_i,B_i}}d_1^{ns}(\mathcal{C}(A_i, B_i)).
\]

Proof. Explicit calculation. For a particular case corresponding to \((A, B) = \Psi_i, (A', B') = \Phi_j, (A_i, B_i) = \Psi_i \Phi_j\) this follows from the above results.

We have:
\[
T_{G,p}(A, B) \cdot T_{G,p}(A', B') = \sum_i \kappa_i T_{G,p}(A_i, B_i) + \text{other terms}. \tag{3.10.2}
\]

Since the reduction is a ring homomorphism, we see that:

(a) (3.5) applied to the pairs \((A, B), (A', B'), (A_i, B_i)\);
(b) (3.10.2) and Lemma 3.10.1;
(c) Commutativity of the the function field analog of the diagram (1.1.6)

imply that
\[
\beta_2(T_{M,p}(A, B)) \cdot \beta_2(T_{M,p}(A', B')) + \text{other terms} =
\]
\[
= \sum_i \kappa_i \beta_2(T_{M,p}(A_i, B_i)) + \text{other terms}.
\]

3.10.3. Now naive considerations show us that “other terms” in both sides of the above equality are equal. Really, let us denote by \(ST(A, B)\) the support of the graph \(\Gamma(\beta_2(T_{M,p}(A, B))) \subset \tilde{X} \times \tilde{X}\), and analogically for the pairs \((A', B'), (A_i, B_i)\).

We have:
\[
(t_1, t_2) \in ST(A, B) \iff \text{there is a map } E_{t_1} \to E_{t_2} \text{ of type } T_{M,p}(A, B).
\]

By definition of the product of correspondences,
\[
(t_1, t_3) \in \cup_i ST(A_i, B_i) \iff \text{there exists } t_2 \text{ such that}
\]
\[
(t_1, t_2) \in ST(A, B), \quad (t_2, t_3) \in ST(A', B').
\]

Since the type of the composition of maps of Anderson varieties is concordant with the multiplication in \(\mathbb{H}(M)\), we get 3.10.3. □

3a. Conjectural form of Langlands correspondence.

According Langlands, L-function \(L(M, s)\) of an irreducible submotive \(M\) of a Shimura variety is related with \(L(\pi, \varpi, s)\), where \(\pi\) is an automorphic representation of \(G(\mathbb{A}_\mathbb{Q})\) and \(\varpi : L^1 G \to GL(2\mathbb{W})\) a finite-dimensional representation of \(L^1 G\):
\[
L(M, s) \sim L(\pi, \varpi, s). \tag{3a.0}
\]

Conjectural construction of \(\varpi\) is given for example in [BR], Section 5.1, p. 550.
Let us formulate an analog of this result for Anderson modular objects. For this case an analog of \( G(\mathbb{A}_\mathbb{Q}) \) is \( G_{fun}(\mathbb{A}_{\mathbb{F}_q(\theta)}) \).

**Theorem 3a.1.** If an analog of (3a.0) is true for Anderson modular objects \( X_{fun}(r,n) \) of any level then the restriction of \( \mathfrak{r} \) to \( \hat{G}_{fun} \) is the \( n \)-th skew power representation of \( GL_r \).

This theorem follows from the below Theorem 3a.3.

Let \( \pi = \otimes \pi_l \) be a representation of \( G_{fun}(\mathbb{A}_{\mathbb{F}_q(\theta)}) \) corresponding (according Langlands) to an irreducible submotive of an Anderson modular object, and \( \theta_p \in L^G \) a Langlands element of \( \pi_p \) (we consider the case of \( p \) such that \( \pi_p \) is non-ramified). Let \( \alpha_i, i = 1, \ldots, r \), be eigenvalues of \( \theta_p \) and \( a_i \) the eigenvalues of \( T_{p,i} \) (analogs of Fourier coefficients of an automorphic form for the classical case). Standard formalism of Langlands elements for \( GL_r \) in the non-ramified case together with (2.2.2) shows that

\[
a_i = q^{-i(i-1)/2} \sigma_i(\alpha_s). \tag{3a.2}
\]

We denote by \( P'_{r,n} \) the characteristic polynomial of \( \mathfrak{r}(\theta_p) \), it belongs to \( \mathbb{Z}[a_1, \ldots, a_r][T] \) where \( T \) is an abstract variable. The following theorem follows immediately from (2.2.2), (3a.2) (like in the number case):

**Theorem 3a.3.** \( P_{r,n} = P'_{r,n} \) (after identification of \( T \) and \( fr, a_i \) and \( T_{p,i} \)). \( \square \)

4. Unitary Shimura varieties.

We consider abelian varieties with multiplication by an imaginary quadratic field (abbreviation: MIQF). Let \( K \) be such field, \( X_{num} \) the corresponding Shimura variety parametrizing abelian \( r \)-folds with multiplication by \( K \), of signature \( (r-n,n) \). We shall call them unitary Shimura varieties. The corresponding reductive group over \( \mathbb{Q} \) is \( G = G_{num} = GU(r-n,n) \). We have \( \dim X_{num} = (r-n)n \). Let \( p \) be a prime inert in \( K \); we shall consider \( p \)-Hecke correspondences and the reduction at \( p \).

**Theorem 4.1.** \( M \) for this case is the same as in Theorem 2.1.

**Corollary 4.2.** Satake maps for this case coincide with the ones for the functional case (formulas (2.2.4), (2.2.5)).

Let \( A_t \) be as in Subsection 1.2. \( (A_t)_p \) is an \( r \)-dimensional vector space over \( \mathbb{F}_{p^2} \). Let \( D = D_{\text{unitary}} \) be as in Subsection 1.2.

**Theorem 4.3.** \( D_{\text{unitary}} \) is a vector space over \( \mathbb{F}_{p^2} \) of dimension \( \max (r-n,n) \).

**Remark 4.4.** There exists a symmetry between \( n \) and \( r-n \). Nevertheless, here the analogy between functional and unitary case apparently is not complete.

**Theorem 4.5.** Analog of the Theorem 2.3.5 (i.e. formula 2.3.6) holds for the unitary case (dimension is taken over \( \mathbb{F}_{p^2} \)).

**Theorem 4.6.** ([BR], Section 5.1, p. 550, example (b)). Restriction of \( \mathfrak{r} \) on \( \hat{G} \subset L^G \) is the same as in Theorem 3a.1.

We think that analogs of Theorems 2.3.7, 2.3.8, 2.3.10 also hold for this case.
5. Conjectural values of Hodge numbers.

There are conjectural formulas for values of Hodge numbers $h^{ij}$ (over the fields of multiplications coming from Hecke correspondences) of irreducible submotives of Shimura varieties (see, for example, [BR], Section 4.3, p. 548). For example, for the case of Siegel modular varieties of genus $g$ (their dimension is $d_g = g(g + 1)/2$) and for a generic pure submotive of weight $d_g$ they are the following:

**Theorem 5.1.** $h^{i,d_g-i} = \{\text{the quantity of subsets } (j_1, \ldots, j_\alpha) \text{ of the set } 1, 2, \ldots, g \text{ such that } j_1 + \ldots + j_\alpha = i\}$, where $\alpha$ is arbitrary.

For other types of submotives the formulas for $h^{ij}$ are similar but more long.

For example, for the case of unitary Shimura variety of Section 4 and for the same type of submotives the formula is the following:

**Theorem 5.2.** $h^{i,(r-n)n-i} = \{\text{the quantity of subsets } (j_1, \ldots, j_n) \text{ of the set } (1, 2, \ldots, r) \text{ such that } j_1 + \ldots + j_n - (1 + \ldots + n) = i\}$.

By analogy between functional and unitary case we can conjecture that the same formula holds for the functional case.

6. Non-ordinary Drinfeld modules.

For further applications we shall state two problems and give their conjectural answers. Let us restrict ourselves by the case $n = 1$ and the correspondence $T_{p,1}$. These problems are related with the description of intersection of two irreducible components of the graph of $T_{p,1}$ in characteristic $p$. Let $E$ be a Drinfeld module such that its reduction is a generic non-ordinary, i.e. $\dim_{\mathbb{F}_p}(D_{\text{fun}}(E)) = 2$ is the least possible. Let $t$ be the point on $X_{\text{fun}}$ corresponding to $E$ and $t', t'' \in T_{p,1}(t)$.

**Question 6.1.** Formulate analogs of conjectures 2.3.5, 2.3.8 for this $t$.

**Conjectural answer.**

(a) $\tilde{t}'$ is (the only) closed point of $\Psi_1(\bar{t})$ iff $\ker(\alpha_{t,t'}) \subset D_{\text{fun}}$.

All such $\tilde{t}'$ also belong to $\Phi_1(\bar{t})$.

(b) $\tilde{t}' = \tilde{t}''$ as closed points iff the $\mathbb{F}_p$-linear spans coincide:

$$< \ker(\alpha_{t,t'}), D_{\text{fun}} > = < \ker(\alpha_{t,t''}), D_{\text{fun}} > .$$

Now we consider a more special situation. Let $r$ be even, $L$ a quadratic extension of $\mathbb{F}_q(\theta)$ such that $p$ inert in $L/\mathbb{F}_q(\theta)$, and $E$ a generic Drinfeld module with multiplication by $L$. In this case $D_{\text{fun}}(E)$ is a 1-dimensional $\mathbb{F}_{p^2}$-vector space. Let $t, t', t''$ be as above.

**Question 6.2.** Formulate analogs of conjectures 2.3.5, 2.3.8 for this $t$.

**Conjectural answer.** (a) is the same as above, and in (b) we consider $\mathbb{F}_{p^2}$-linear spans:

$$\tilde{t}' = \tilde{t}'' \iff < \ker(\alpha_{t,t'}), D_{\text{fun}} >_{\mathbb{F}_{p^2}} = < \ker(\alpha_{t,t''}), D_{\text{fun}} >_{\mathbb{F}_{p^2}}.$$
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First author: Departamento de Matemática e estatistica Universidade de São Paulo. Rua de Matão 1010, CEP 05508-090, São Paulo, Brasil, and Omsk State University n.a. F.M.Dostoevski. Pr. Mira 55-A, Omsk 644077, Russia.

Second author: Departamento de Matemática, Universidade Federal do Amazonas, Manaus, Brasil.