POLARIZABILITY OF MICROPARTICLES IN RELATIVISTIC FIELD THEORY

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The relativistic second-rank tensor containing electric and magnetic polarization vectors of a medium is obtained based on the Maxwell equations and the determination of the charge density and polarization current of a structural microparticle. Using this tensor and the electromagnetic field tensor, the relativistic Lagrangian of the electromagnetic field interaction with the structural microparticle is obtained taking into account the polarization of its structural elements. Using this tensor, the relativistic Lagrangian of electromagnetic field interaction with the structural microparticle has been constructed considering the electric and magnetic polarizabilities.

Keywords: Lagrangian, tensor, electric polarizability, magnetic polarizability.

INTRODUCTION

One of the effective theoretical methods for studying electrodynamic processes is the use of Lagrangians obtained within the framework of relativistic field approaches and consistent with low-energy theorems [1]. The construction of the effective relativistic invariant Lagrangians allows not only electromagnetic characteristics of microparticles to be physically interpreted, but also information on the mechanisms of the electromagnetic field interaction with these particles to be obtained. The most important electromagnetic characteristics of the microparticles are their polarizabilities the magnitude of which is determined by both structural and quantum properties of these microparticles [2]. Many electrodynamic processes have recently been used based on which data on the polarizability of such microparticles, as hadrons, nuclei, etc. are obtained. In this regard, the problem arises: How can we successively take into account the contribution of polarizabilities to the amplitudes and cross sections of the electrodynamic processes on the microparticles by the covariant method? This problem can be solved by constructing a field relativistic formalism of electromagnetic field interaction with the microparticles taking into account their polarizabilities. In work [3], the effective relativistic Lagrangian of electromagnetic field interaction with the microparticles possessing electric and magnetic dipole moments was constructed. In the present work, using methods of works [3–5], the relativistic Lagrangian has been obtained taking into account the electric and magnetic polarizabilities based on the induced dipole moments and some consequences from this Lagrangian have been established.

MAIN PART

Consider the polarizability of the microparticles in the context of relativistic field theory. By definition, the electromagnetic field tensor $F^{\mu\nu}$ is expressed through the 4D electromagnetic field potential
as follows:

\[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (1) \]

where

\[ \partial^\mu \{ \partial_0 = \partial_t, -\nabla \}, \quad \partial_\mu \{ \partial_0 = \partial_t, \nabla \}. \]

The explicit form of the tensor \( F^{\mu\nu} \) given by Eq. (1) is represented in the form of the matrix

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E^1 & -E^2 & -E^3 \\
E^1 & 0 & -B^3 & B^2 \\
E^2 & B^3 & 0 & -B^1 \\
E^3 & -B^2 & B^1 & 0
\end{pmatrix}, \quad (2)
\]

where \( E^i \) are the components of the electric field strength vector, and \( B^i \) are the components of the magnetic induction vector. Using the definition of the metric tensor

\[
g^{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad (3)
\]

we obtain the following definition for the electromagnetic field tensor:

\[
F_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} F^{\rho\sigma},
\]

which in the matrix form is written as

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E^1 & E^2 & E^3 \\
-E^1 & 0 & -B^3 & B^2 \\
-E^2 & B^3 & 0 & -B^1 \\
-E^3 & -B^2 & B^1 & 0
\end{pmatrix}. \quad (4)
\]
Using the Levi–Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ (in this work, we take advantage of the condition $\epsilon^{0123} = +1$ and $\epsilon_{0123} = -1$), we define the dual electromagnetic tensors:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix},$$  \hspace{1cm} (5)

and

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & E^3 & -E^2 \\ -B^2 & -E^3 & 0 & E^1 \\ -B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \hspace{1cm} (6)$$

From definitions (2)–(6), the expression for the electromagnetic field Lagrangian follows

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{2} (B^2 - E^2),$$

and the relationship

$$\tilde{F}_{\rho\kappa} \tilde{F}^{\rho\sigma} = F_{\rho\kappa} F^{\rho\sigma} \frac{1}{2} \delta^{\sigma}_{\kappa} F^2,$$$$

where $F^2 = F_{\mu\nu} F^{\mu\nu}$.

On the basis of the matrix representation of electromagnetic tensors (2)–(6) and the 4D vector of the particle (the system of units $\hbar = c = 1$ is used), we obtain

$$u^\mu \{u^0, u\}, \quad u^2 = (u^0)^2 - u^2 = 1,$$

where $u^0 = \gamma = \frac{1}{\sqrt{1 - \beta^2}}$, $u = \gamma \nu$, $\nu$ is the particle velocity, and $\beta^2 = \nu^2$, and define the following 4D vectors [6]:

$$e^\mu = F^{\mu\nu} u_\nu, \quad e_\mu = F_{\mu\nu} u^\nu, \quad b^\mu = \tilde{F}^{\mu\nu} u_\nu, \quad b_\mu = \tilde{F}_{\mu\nu} u^\nu. \hspace{1cm} (7)$$

These vectors are expressed through the vectors $\mathbf{B}$ and $\mathbf{E}$ and the particle velocity vector as follows:

$$e^\mu \{uE, (u^0 E + [uB])\}, \hspace{1cm} (8)$$
In turn, electromagnetic field tensors (2)–(6) can be defined through vectors (7)–(10) and the vector \( u^\mu \). For example,

\[
F^{\mu\nu} = e^{\mu} \nu - e^{\nu} \mu + e^{\nu\rho\sigma} u_\rho b_\sigma.
\]  

Let us now proceed to a relativistic field description of the electromagnetic field interaction with particles possessing nonzero electric and magnetic polarizabilities. In nonrelativistic electrodynamics, the electric, \( \alpha_E \), and magnetic, \( \beta_M \), polarizabilities determine the proportionality between the applied electric and magnetic fields and the induced dipole moments [7]:

\[
d = 4\pi\alpha_E E,
\]

\[
m = 4\pi\beta_M H.
\]

Hence, the polarizability possesses the fundamental property and is determined by the microparticle structure and the mechanism of electromagnetic field interaction with these microsystems.

The energy of the applied electromagnetic field interaction with the microparticle has the form

\[
U = -\frac{4\pi}{2} \alpha_E E^2 - \frac{4\pi}{2} \beta_M H^2.
\]

If the microsystem consists of \( N \) particles in unit volume, the dielectric permittivities are related to the polarizabilities by the formulas [7]

\[
\varepsilon = 1 + N4\pi\alpha_E , \quad \mu = 1 + N4\pi\beta_M.
\]

From these formulas, it can be seen that the polarizabilities \( \alpha_E \) and \( \beta_M \) are measured in cm\(^2\). In turn, the electric and magnetic polarizations of the macrosystem are expressed through the polarizability as follows:

\[
P = N4\pi\alpha_E E , \quad M = N4\pi\beta_M H.
\]

To proceed to relativistic representation of relationship (17), we take advantage of the Maxwell equations with allowance for the polarization of the medium [8]:

\[
\text{rot} \, B = \frac{\partial E}{\partial t} + j + j_P,
\]

\[
\text{div} \, E = \rho + \rho_P ,
\]

\[
\text{div} \, B = 0 .
\]
In Eqs. (18) and (19), \( \rho \) and \( j \) are the free charge density and current, and \( \rho_p \) and \( j_p \) are the charge density and current induced by polarization. Here \( \rho_p \) and \( j_p \) are expressed through \( P \) and \( M \) as follows:

\[
\rho_p = - (\nabla P),
\]

\[
\mathbf{j}_p = - \frac{\partial \mathbf{P}}{\partial t} + \text{rot} \mathbf{M}.
\]

Formulas (22) and (23) can be represented in the covariant form

\[
\mathbf{j}_p^\mu = \delta_\mu \mathbf{M}^{\mu\nu},
\]

where \( \mu \) is equal to 0, 1, 2, and 3.

In the definition of the 4D current induced by polarization, the tensor \( \mathbf{M}^{\mu\nu} \) has the form

\[
\mathbf{M}^{\mu\nu} = \begin{pmatrix}
0 & P^1 & P^2 & P^3 \\
-P^1 & 0 & -M^3 & M^2 \\
-P^2 & M^3 & 0 & -M^1 \\
-P^3 & -M^2 & M^1 & 0
\end{pmatrix}.
\]

In 3D representation, formula (24) with allowance for formula (25) takes the form

\[
j_p^{(0)} = \rho_p = - (\nabla P), \quad \mathbf{j}_p = - \partial_\mu \mathbf{P} + [\nabla \mathbf{M}].
\]

Using the metric tensor \( g_{\mu\nu} \) and the Levi–Civita tensor, the matrix representation of the tensors \( \mathbf{M}^{\mu\nu} \), \( \tilde{\mathbf{M}}^{\mu\nu} \), and \( \tilde{\tilde{M}}_{\mu\nu} \) can be obtained. The results of calculations have the form

\[
\mathbf{M}^{\mu\nu} = \begin{pmatrix}
0 & -P^1 & -P^2 & -P^3 \\
P^1 & 0 & -M^3 & M^2 \\
P^2 & M^3 & 0 & -M^1 \\
P^3 & -M^2 & M^1 & 0
\end{pmatrix},
\]

\[
\tilde{\mathbf{M}}^{\mu\nu} = \begin{pmatrix}
0 & -M^1 & -M^2 & -M^3 \\
M^1 & 0 & -P^3 & P^2 \\
M^2 & P^3 & 0 & -P^1 \\
M^3 & -P^2 & P^1 & 0
\end{pmatrix},
\]

(26)
Based on the tensors $M_{\mu\nu}$, $M^{\mu\nu}$, $\tilde{M}^{\mu\nu}$, and $\tilde{M}_{\mu\nu}$, we define the 4D vectors:

$$d^\mu = M^{\mu\nu} u_\nu, \quad d_\mu = M_{\mu\nu} u^\nu, \quad m^\mu = \tilde{M}^{\mu\nu} u_\nu, \quad m_\mu = \tilde{M}_{\mu\nu} u^\nu.$$  \hspace{1cm} (27)

By analogy with the electromagnetic field tensor $F^{\mu\nu}$, using vectors $e^\mu$, $e_\mu$, $b^\mu$, and $b_\mu$, tensor $M_{\mu\nu}$ (12) can be represented as

$$M^{\mu\nu} = d^\mu u^\nu - d^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} u_\rho m_\sigma.$$  \hspace{1cm} (28)

From formulas (25) and (26), it follows that

$$d^\mu = M^{\mu\nu} u_\nu = \{(-P u), (-u_\nu P - [Mu])\},$$  \hspace{1cm} (29)

$$m_\sigma = \tilde{M}_{\sigma\rho} u^\rho = \{(Mu), (-u_\nu M + [Pu])\}.$$  \hspace{1cm} (30)

If we take advantage of formulas (29)–(30) in representation (29), we obtain the matrix elements of matrix (25).

Let us define the Lagrangian of electromagnetic field interaction with the medium possessing the polarizations as follows:

$$L_I = \frac{1}{4} M^{\mu\nu} F_{\mu\nu}.$$  \hspace{1cm} (31)

and substitute $M_{\mu\nu}$ in matrix representation (25) and $F_{\mu\nu}$ in matrix representation (4) into definition (31). As a result, we obtain

$$L_I = \frac{1}{4} M^{\mu\nu} F_{\mu\nu} = \frac{1}{2} [(PE) + (MB)].$$  \hspace{1cm} (32)

From formula (32) it follows that $L_I$ is invariant under the Lorentz transformations. The Lagrangian consisting of the sum of the electromagnetic field Lagrangians and the Lagrangian of interaction of this field with the polarizing medium has the form

$$L_I = \frac{1}{2} \left( B^2 - E^2 \right) + \frac{1}{2} (PE + MB) = \frac{1}{2} (DE + BH),$$  \hspace{1cm} (33)

where $D = E + P$, $H = B + M$, $D$ is the electric induction vector, $H$ is the field strength, and $B$ is the magnetic induction vector.

We now write down the interaction Hamiltonian $H_I$ through the polarization current and the electromagnetic field potential:
Using the relationship

\[ \partial_{\rho} \left( M^{\rho \mu} A_{\mu} \right) = \left( \partial_{\rho} M^{\rho \mu} \right) A_{\mu} + M^{\rho \mu} \left( \partial_{\rho} A_{\mu} \right) \]

and the asymptotic condition, we obtain

\[ H_I = -\frac{1}{4} M^{\rho \mu} F_{\rho \mu}^{\nu} = -\frac{1}{2} \left[ (PE) + (MB) \right]. \] (35)

With allowance for formulas (12) and (28), Lagrangian (32) takes the form

\[ L = \frac{1}{2} \left[ d^{\mu} e_{\mu} - m^{\mu} b_{\mu} \right]. \] (36)

Let us now establish the relationship of \( d^{\mu} \) and \( m^{\mu} \) with the vectors \( e_{\mu} \) and \( b_{\mu} \) in terms of the polarizabilities \( \alpha_E \) and \( \beta_M \). For this purpose, we take advantage of the relativistic generalization of the material equations \( D = \varepsilon E \) and \( B = \mu H \) [6, 8]. If we consider definitions (22) and (23) for \( \rho_{\mu} \) and \( j_{\mu} \), the Maxwell equations (18) and (19) in the relativistic representation take the form

\[ \partial_{\mu} H^{\mu \nu} = j^{\nu}, \] (37)

where \( H^{\mu \nu} = F^{\mu \nu} - M^{\mu \nu} \). 4D generalization of the formula \( D = \varepsilon E \) is described by the equation [8]

\[ H^{\mu \nu} u_\nu = \varepsilon F^{\mu \nu} u_\nu. \] (38)

Equation (38) can be written in terms of the vectors \( d^{\mu} \) and \( e^{\mu} \):

\[ \left( e^{\mu} - d^{\mu} \right) = \varepsilon e^{\mu}. \] (39)

Considering the definition \( \varepsilon = 1 + 4\pi\alpha_E \), from Eq. (39) we obtain

\[ d^{\mu} = -4\pi\alpha_E e^{\mu}. \] (40)

From Eqs. (7), (29), and (40) for the particle velocity equal to zero, it follows that

\[ d = -4\pi\alpha_E E. \]

The relativistic generalization of the formula \( B = \mu H \) is represented by the equation [6, 8]

\[ F^{\mu \nu} u_\nu = \mu H^{\mu \nu} u_\nu, \] (41)

where \( \tilde{H}^{\mu \nu} \) is the dual tensor of the tensor \( H^{\mu \nu} \) expressed in terms of the dual tensors \( \tilde{F}^{\mu \nu} \) and \( \tilde{M}^{\mu \nu} \):
\[ \tilde{H}^{\mu\nu} = \tilde{F}^{\mu\nu} - \tilde{M}^{\mu\nu}. \]  

Substituting formula (42) into (41) and using the relationship

\[ \mu = 1 + 4\pi\beta_M, \]  

we obtain

\[ m^\mu = 4\pi\beta_M b^\mu. \]  

According to Eqs. (10) and (30), for the microparticle velocity equal to zero, from Eq. (44) it follows that the magnetic dipole moment is

\[ \mathbf{m} = 4\pi\beta_M \mathbf{B}. \]

Let us now represent interaction Lagrangian (31) using definitions (40) and (44) of the vectors \( d^\mu \) and \( m^\mu \). Then taking into account formulas (40) and (44), interaction Lagrangian (36) assumes the form

\[ L_I = -\frac{4\pi}{2} \left[ \alpha_E \left( e^\mu e_\mu \right) + \beta_M \left( b^\mu b_\mu \right) \right] = -2\pi \left[ \alpha_E e^2 + \beta_M b^2 \right]. \]  

Using the definition of vectors \( e^\mu \) and \( b^\mu \) in terms of the tensors of electromagnetic field (7), it is easy to be convinced that

\[ L_I = -2\pi \left[ \alpha_E \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} + \beta_M \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} \right] u_{\mu} u^\nu. \]  

In turn, since from formula (7) it follows that

\[ e^2 = e^\mu e_\mu = \left[ \left( uE \right)^2 - \left( u^0 \right)^2 E^2 + 2u^0 \left( u[EB] \right) - u^2 B^2 + \left( uB \right)^2 \right], \]  

\[ b^2 = b^\mu b_\mu = \left[ \left( uB \right)^2 - \left( u^0 \right)^2 B^2 + 2u^0 \left( u[EB] \right) - u^2 E^2 + \left( uE \right)^2 \right], \]

then Lagrangian (45) is expressed through the polarizability as follows:

\[ L_I = -2\pi \left\{ \left( \alpha_E + \beta_M \right) \left[ \left( uE \right)^2 + \left( uB \right)^2 + 2u^0 \left( u[EB] \right) \right] - \left( \alpha_E \left( u^0 \right)^2 + \beta_M u^2 \right) E^2 \right\} \]

\[ - \left( \alpha_E u^2 + \left( u^0 \right)^2 \beta_M \right) B^2 \}. \]  

From Eq. (49) it follows that in the zero order with respect to the velocity,

\[ L_I^0 = 2\pi \left[ \alpha_E E^2 + \beta_M B^2 \right], \]

and in decomposition (49) to the first order in the velocity, we obtain
$$L_J = 2\pi \left[ \alpha_E E^2 + \beta_M B^2 - (\alpha_E + \beta_M) 2(\mathbf{a}[\mathbf{EB}]) \right].$$

(50)

This formula is in agreement with $L_J$ obtained in [3] with allowance for the polarizability of the microparticle.

CONCLUSIONS

Thus, based on the Maxwell equations and the definition of the charge density and the polarization current of the structural microparticle, the second-rank relativistic tensor has been obtained the components of which are the electric, $\mathbf{P}$, and magnetic, $\mathbf{M}$, polarization vectors of the medium. With the help of this tensor and the electromagnetic field tensor, the relativistic Lagrangian of electromagnetic field interaction with the structural microparticle was constructed taking into account the polarization of its structural elements. Using the relativistic material equations, the Lagrangian of electromagnetic field interaction with the structural microparticle was obtained with allowance for its electric and magnetic polarizabilities, and the consequences of this Lagrangian were given.

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