ON THE EXISTENCE OF PARADOXICAL MOTIONS OF GENERICALLY RIGID GRAPHS ON THE SPHERE

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Abstract. We interpret realizations of a graph on the sphere up to rotations as elements of a moduli space of curves of genus zero. We focus on those graphs that admit an assignment of edge lengths on the sphere resulting in a flexible object. Our interpretation of realizations allows us to provide a combinatorial characterization of these graphs in terms of the existence of particular colorings of the edges. Moreover, we determine necessary relations for flexibility between the spherical lengths of the edges. We conclude by classifying all possible motions on the sphere of the complete bipartite graph with 3 + 3 vertices where no two vertices coincide or are antipodal.

1. Introduction

The study of mobile spherical mechanisms is an active area of research, as confirmed by several recent publications on the topic, see for example [3, 6, 21, 22, 23, 34]. One of the reasons for this interest is that serial manipulators for which all the axes of the joints are concurrent can be considered as spherical mechanisms. In this paper, we focus on the combinatorial structure of these mechanisms, namely on the graphs whose vertices correspond to joints and whose edges correspond to bars. We will always suppose that the graphs are connected, without multiedges or self-loops. We describe combinatorial properties of these graphs ensuring that there exist assignments of edge lengths that make them flexible on the sphere. By the latter we mean that there are infinitely many essentially distinct ways to realize the vertices of the graph on the sphere so that the distance of adjacent vertices is the given edge length. Here by “essentially distinct” we mean that any two such realizations do not differ by an isometry of the sphere. Notice that only non-adjacent vertices are allowed to coincide or to be antipodal. Moreover, we deal with complex realizations, namely we allow the coordinates of the points defining a realization of a graph to be complex numbers. This opens the door to tools from algebraic geometry that can be employed in the study of this problem.

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A lot of the current research on rigidity and flexibility of graphs, polyhedra, and mechanisms analyzes the situation in two- or three-dimensional Euclidean space. However, there have been already substantial contributions concerning rigidity or flexibility in other ambient spaces, for example the torus \[ \text{[26, 28, 29]} \], the cylinder \[ \text{[24]} \], revolute surfaces \[ \text{[25]} \], and other geometries (spherical, hyperbolic, projective) \[ \text{[1, 7, 8, 16, 30, 31, 32]} \].

Although the combinatorial characterization we obtain applies to all graphs, it is of particular interest in the case of generically rigid graphs. These are graphs such that for a general realization on the sphere there are only finitely many essentially distinct realizations having the same edge lengths. It is known that generically rigid graphs on the sphere coincide with generically rigid graphs in the plane; for a modern account on this topic, see for example \[ \text{[5]} \]. Pollaczek-Geiringer \[ \text{[27]} \] and Laman \[ \text{[19]} \] proved that generically rigid graphs are precisely those spanned by graphs \[ G = (V, E) \] such that \[ |E| = 2|V| - 3 \] and \[ |E'| \leq 2|V'| - 3 \] for every nontrivial induced subgraph \[ G' = (V', E') \] of \[ G \]. Graphs of the latter kind are called Laman graphs.

The two papers \[ \text{[12, 13]} \] study flexible assignments of edge lengths of graphs in the plane. The authors provide there a combinatorial description of those graphs admitting flexible assignments in terms of special colorings of their edges. In the spherical case, we use a subclass of the edge colorings for the planar case. A fundamental construction in \[ \text{[12]} \] is based on parallelograms, which are not available on the sphere, hence we need to rely on new ideas.

The main technique we use takes inspiration from bond theory, which has been developed to study paradoxical motions of serial manipulators, for example \(5R\) closed chains with two degrees of freedom, or mobile \(6R\) closed chains (see \[ \text{[14, 15, 20]} \]). The core idea is to consider a compactification of the space of configurations of a manipulator which admits some nice algebraic properties. This introduces elements “on the boundary”, namely limits of configurations that are, in turn, not configurations. The analysis of these elements, called bonds, provides information about the geometry of the manipulator.
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In our case, we reduce the study of realizations of a graph on the unit sphere, up to rotations, to the study of elements of a so-called moduli space of stable curves with marked points. The latter is a well-known object in algebraic geometry, and we use it to define the notion of bond for graphs that allow a flexible assignment of edge lengths, and to prove the main results of this paper. The reduction has been already explained and used in [9], and it is based on the observation that the spherical distance between two points on the unit sphere can be expressed in terms of the cross-ratio of four points on a conic. Moreover, the action of rotations on the points of the sphere is translated into an action of the general linear group on the points on the conic. Once this reduction is established, we can apply the general philosophy of bond theory: if a graph admits a flexible assignment, then there is at least a one-dimensional set of realizations in the moduli space; this one-dimensional set intersects the boundary of the moduli space, originating bonds. It turns out that bonds have a rich combinatorial structure, and they can be used to define a special class of bichromatic edge colorings on the graph we started with. We call these colorings NAP ("No Alternating Path") since no 3-path in the graph has an alternating coloring. Our main result is then (see theorem 3.14):

**Theorem.** Let $G$ be a connected graph without multiedges, or self-loops. Then $G$ admits a flexible assignment of edge lengths on the sphere if and only if it admits a NAP-coloring.

Notice that, as we have already remarked, although the previous theorem holds in full generality, it is of particular interest when we consider generically rigid graphs. In fact, not all Laman graphs admit NAP-colorings; when they do, there is a way to realize them on the sphere so that they are flexible. This is an example of what we mean by paradoxical motion in the title of this paper.

We then delve into a particular case, the one of the bipartite graph $K_{3,3}$, which is the smallest “interesting” graph that is at the same time generically rigid and admits NAP-colorings. Here “interesting” means that we have realizations such that no two vertices collapse or are antipodal. We obtain a classification of all the possible motions of $K_{3,3}$ on the sphere. Here we stress that, unlike the rest of the paper, we focus on real motions. The classification is obtained by carefully analyzing how the subgraphs of $K_{3,3}$ isomorphic to $K_{2,2}$ move during a motion of $K_{3,3}$. This analysis employs both constraints imposed by the possible bonds that may arise for $K_{3,3}$, and elementary considerations in sphere geometry: these two approaches combined narrow the cases to be considered so that they can be examined one by one. In contrast to the planar situation — where there are only two such motions, described by Dixon more than a hundred years ago [4] — on the sphere there are three possibilities (see theorem 4.32):

**Theorem.** All the possible motions of $K_{3,3}$ on the sphere, for which no two vertices coincide or are antipodal, are the spherical counterparts of the planar Dixon motions, and a third new kind of motion, in which the angle between the two diagonals of a quadrilateral in $K_{3,3}$ stays constant during the motion.
Structure of the paper. Section 2 recalls from [9] how (complex) realizations on the sphere up to rotations can be interpreted as elements of a moduli space of stable curves with marked points. Section 3 introduces the notion of NAP-coloring, and proves the characterization of graphs admitting a flexible (complex) assignment on the sphere via NAP-colorings. The section ends by explaining how to obtain algebraic relations between the spherical lengths of a flexible assignment of a graph that has “enough” edges (e.g., a graph spanned by a Laman graph). Section 4 analyzes all possible (real) motions of the graph $K_{3,3}$ on the sphere for which no two vertices collapse or are antipodal.

2. Realizations on the sphere as elements of a moduli space

The aim of this section is to recall how we can interpret a realization of a graph on the sphere, up to sphere isometries, as an element of the moduli space of stable curves of genus zero with marked points. This interpretation is described in [9, Sections 1–3] and here we recall its main aspects to make the paper more self-contained; no new results are present in this section.

First of all, let us define what we mean by realization of a graph, which we suppose connected, and without multiedges or self-loops. As we pointed out in the introduction, we work over the complex numbers, and we denote by $S^2 \subset \mathbb{C}^3$ the complexification of the unit sphere $S^2$, namely

$$S^2 \subset \mathbb{C}^3 := \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = 1\}.$$  

On the real sphere $S^2 \subset \mathbb{R}^3$ we can compute the distance between two points as the cosine of the angle they form with respect to the center of the sphere. Since this function is algebraic, we can use it also in the complex setting.

**Definition 2.1.** Given two points $T, U \in S^2 \subset \mathbb{C}^3$, we define

$$\delta_{S^2}(T, U) := \langle T, U \rangle = \sum_{i=1}^{3} T_i U_i.$$  

Notice that, if $\bar{U} := -U$ denotes the antipodal point of $U$, then $\delta_{S^2}(T, U) = -\delta_{S^2}(T, \bar{U})$. For technical reasons, we define the spherical distance as

$$d_{S^2}(T, U) := \frac{1 - \langle T, U \rangle}{2}.$$  

If $T$ and $U$ are real points, then $\delta_{S^2}$ ranges from $-1$ to $1$, and attains the value $0$ when the vectors corresponding to $T$ and $U$ are orthogonal, while $d_{S^2}$ ranges from $0$ (when $T$ and $U$ coincide) to $1$ (when $T$ and $U$ are antipodal).

Realizations of graphs on the sphere are then assignments of points on the sphere to the vertices of the graph.

**Definition 2.2.** Let $G = (V, E)$ be a graph. A realization of $G$ is a function $\rho: V \rightarrow S^2 \subset \mathbb{C}^3$. Given an assignment $\lambda: E \rightarrow \mathbb{C} \setminus \{0, 1\}$ of spherical distances for the edges, we say that a realization $\rho$ is compatible with $\lambda$ if

$$d_{S^2}(\rho(a), \rho(b)) = \lambda\{a, b\} \quad \text{for all} \quad \{a, b\} \in E.$$
Recall that we allow non-adjacent vertices to coincide/be antipodal in such a realization, whereas the adjacent ones cannot coincide/be antipodal since the spherical distance is not allowed to be 0 or 1.

Once we know what a realization is, we can define when an assignment of spherical distances is flexible.

**Definition 2.3.** Let $G = (V,E)$ be a graph, and let $\lambda: E \rightarrow C \setminus \{0,1\}$ be an assignment of edge lengths. We say that $\lambda$ is a flexible assignment if there exist infinitely many realizations $\rho: V \rightarrow S^2$ compatible with $\lambda$ that are essentially distinct. We say that two realizations $\rho_1, \rho_2$ of $G$ are essentially distinct if there exists no element $\sigma \in SO_3(C)$ such that $\rho_1 = \sigma \circ \rho_2$, where

$$SO_3(C) := \{\sigma \in C^{3\times3} : \sigma^t \sigma = \sigma = id, \det(\sigma) = 1\}$$

is the complexification of the group of rotations on the unit sphere.

The first step towards the moduli space is to associate to each point in $S^2_C$ two points on a rational curve. We do this by considering the projective closure of $S^2_C$:

$$S^2_C := \{(x : y : z : w) \in P^3_C : x^2 + y^2 + z^2 - w^2 = 0\}$$

and defining the conic $A$ as the intersection

$$A := S^2_C \cap \{w = 0\}.$$  

Since $S^2_C$ is a smooth quadric, there are two families of lines in $S^2_C$, and each point on $S^2_C$ lies on exactly one line in one family and on exactly one line in the other family.

**Definition 2.4.** Let $T \in S^2_C$ and let $L_1$ and $L_2$ be the two lines in $S^2_C$ passing through $T$. We define the left and the right lifts of $T$, denoted by $T_l$ and $T_r$, as the intersections of $L_1$ and $L_2$ with the conic $A$.

Since the conic $A$ is a smooth rational curve, it is isomorphic to $P^1_C$, hence the cross-ratio of four points in $A$ is defined. If the four points are the left and right lifts of two points $T, U \in S^2_C$, one finds that the spherical distance between $T$ and $U$ can be expressed in terms of the cross-ratio of the lifts (see [9, Lemma 3.3]):

$$d_{S^2}(T,U) = \frac{cr(T_l,T_r,U_l,U_r)}{cr(T_l,T_r,U_l,U_r) - 1} = cr(T_l,U_r,U_l,T_r).$$

The key step for moving from realizations to elements of a moduli space is that two $n$-tuples of points in $S^2_C$ differ by an element of $SO_3(C)$ if and only if the corresponding two $2n$-tuples of lifts differ by an element of $PGL_2(C)$ when all left and right lifts are distinct (see [9, Proposition 3.4]).

The previous results clarify that dealing with realizations of a graph on the sphere, up to $SO_3(C)$, is equivalent to dealing with tuples of points on a smooth rational curve, up to automorphisms, on which constraints for the cross-ratios are imposed, provided that the realizations give rise to distinct left and right lifts. Tuples of
distinct \( m \) points on a smooth rational curve up to automorphisms are parametrized by the moduli space \( \mathcal{M}_{0,m} \), which is isomorphic to

\[
(\mathbb{P}^1_\mathbb{C} \setminus \{(0 : 1), (1 : 0), (1 : 1)\})^{m-3} \setminus \left\{ \text{tuples } (r_1, \ldots, r_{m-3}) \text{ where } r_i = r_j \text{ for some } i \neq j \right\}.
\]

To see why \( \mathcal{M}_{0,m} \) is isomorphic to the latter space, recall that all smooth rational curves are isomorphic to \( \mathbb{P}^1_\mathbb{C} \), and that its automorphism group \( \mathbb{P}\text{GL}_2(\mathbb{C}) \) is 3-transitive on points, namely if we have an \( m \)-tuple of distinct points in \( \mathbb{P}^1_\mathbb{C} \), up to isomorphism we can suppose that the first three points are equal to \( (0 : 1) \), \( (1 : 0) \), and \( (1 : 1) \). For our purposes, however, we need a compact moduli space, and this is not the case for \( \mathcal{M}_{0,m} \). We could take \( (\mathbb{P}^1_\mathbb{C})^{m-3} \) as a candidate for such a compactification, but in this way we would not know how to interpret the elements of \( (\mathbb{P}^1_\mathbb{C})^{m-3} \setminus \mathcal{M}_{0,m} \) as tuples of points on a rational curve. As in [9], we use a smooth compactification of \( \mathcal{M}_{0,m} \), proposed by Knudsen [18] and denoted by \( \overline{\mathcal{M}}_{0,m} \). This space is composed of the so-called stable curves of genus zero with \( m \) marked points (see [17, Introduction]). These are reduced, at worst nodal curves \( X \) with \( m \) distinct marked points \( O_1, \ldots, O_m \) such that the irreducible components of \( X \) form a tree of rational curves, the points \( \{O_i\}_{i=1}^m \) lie on the smooth locus of \( X \), and for each irreducible component of \( X \), there are at least 3 points which are either singular, or marked. The moduli space \( \mathcal{M}_{0,m} \) sits inside \( \overline{\mathcal{M}}_{0,m} \) as an open set, since every rational curve with distinct marked points is naturally a stable curve. The complement of \( \mathcal{M}_{0,m} \) in \( \overline{\mathcal{M}}_{0,m} \) can be interpreted as a “boundary” for \( \mathcal{M}_{0,m} \). The boundary of \( \overline{\mathcal{M}}_{0,m} \) will play a prominent role in the next sections. For example, \( \mathcal{M}_{0,4} \) is isomorphic to \( \mathbb{P}^1_\mathbb{C} \setminus \{(0 : 1), (1 : 0), (1 : 1)\} \) and \( \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1_\mathbb{C} \), where the boundary is constituted of the following three stable curves:

The boundary makes \( \overline{\mathcal{M}}_{0,m} \) a compact space, so in particular there exist limits of sequences of stable curves where two marked points get closer and closer: the limit of this sequence is a stable curve where a new component is added, and the two marked points are located on this new component (see [17, pages 546–547]). Notice that the elements of \( \overline{\mathcal{M}}_{0,m} \) are (stable) curves, but of course, being \( \overline{\mathcal{M}}_{0,m} \) an algebraic variety, we can speak of curves inside \( \overline{\mathcal{M}}_{0,m} \), meaning one-dimensional subvarieties of \( \overline{\mathcal{M}}_{0,m} \). Actually, we will see that flexible assignments of graphs determine curves in \( \overline{\mathcal{M}}_{0,m} \), in the sense of one-dimensional subvarieties. To prevent confusion and distinguish these two concepts, whenever we think of an element of \( \overline{\mathcal{M}}_{0,m} \) as a curve, we always attach the adjective stable, while when we speak about subvarieties, we just say curves.

We will encode realizations of a graph on the sphere as fibers of a map between moduli spaces. The discussion after Definition 2.4 explains that, when realizations determine distinct left and right lifts, we can associate them to elements of \( \mathcal{M}_{0,2n} \subset \overline{\mathcal{M}}_{0,2n} \).
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When this is not the case (for example, when we have a realization that is not injective) this association is not straightforward. However, we are going to see (lemma 3.6) that the formalism of $\mathcal{M}_{0,2n}$ allows us to deal with all cases in a uniform way.

**Notation.** We denote by $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ the marked points of stable curves of the moduli space $\mathcal{M}_{0,2n}$. The labeling here reflects the fact that we think about the $2n$ marked points as the $2n$ lifts of $n$ points on the sphere, listing first the left lifts, and then the right lifts.

**Definition 2.5** ([9, Definition 5.1]). Let $G = (V, E)$ be a graph, and suppose that $V = \{1, \ldots, n\}$. We define the following morphism

$$\Phi_G: \mathcal{M}_{0,2n} \rightarrow \prod_{\{a,b\} \in E} \mathcal{M}_{0,4}^{a,b} \cong \left(\mathbb{P}^1_\mathbb{C}\right)^{|E|}$$

whose components are the maps $\pi_{a,b}: \mathcal{M}_{0,2n} \rightarrow \mathcal{M}_{0,4}^{a,b}$ that forget all the marked points except for the ones labeled by $P_a$, $P_b$, $Q_a$, and $Q_b$. A complex assignment $\lambda: E \rightarrow \mathbb{C} \setminus \{0, 1\}$ of $G$ corresponds to an element $\Lambda$ in the codomain of $\Phi_G$; due to the correspondence between elements of $\mathcal{M}_{0,2n}$ and points on the sphere, the preimage $\Phi_G^{-1}(\Lambda) \cap \mathcal{M}_{0,2n}$ encodes realizations of $G$ on the sphere compatible with the assignment $\lambda$, up to rotations.

We conclude this section by describing some divisors in the moduli space $\mathcal{M}_{0,2n}$ which are crucial for our investigations.

**Definition 2.6** ([17, Introduction]). Let $(I, J)$ be a partition of the marked points $\{P_1, \ldots, P_n, Q_1, \ldots, Q_n\}$ where $|I| \geq 2$ and $|J| \geq 2$. We define the divisor $D_{I,J}$ in $\mathcal{M}_{0,2n}$ to be the divisor whose general element is a stable curve with two irreducible components such that the marked points in $I$ lie on one component, while the marked points in $J$ lie on the other component. Hence $D_{I,J}$ contains all those stable curves, together with stable curves with more components, where the marked points are distributed according to a refinement of the partition $(I, J)$.

**Proposition 2.7** (Knudsen, [17, Introduction and Fact 2]). Any divisor $D_{I,J}$ as in definition 2.6 is isomorphic to the product $\mathcal{M}_{0,|I|+1} \times \mathcal{M}_{0,|J|+1}$. Under this isomorphism, the point of intersection of the two components of a general stable curve in $D_{I,J}$ counts as an extra marked point in each of the two factors of the product.

**Remark 2.8.** Recall from [17] that if $a, b \in \{1, \ldots, n\}$, then there is a way to identify $\mathbb{P}^1_\mathbb{C}$ with $\mathcal{M}_{0,4}$ so that the preimages of the points $(0 : 1)$, $(1 : 0)$, and $(1 : 1)$ under the forgetful map $\pi_{a,b}: \mathcal{M}_{0,2n} \rightarrow \mathcal{M}_{0,4}$ are given by

$$\bigcup_{P_a, Q_b \in I} D_{I,J}, \quad \bigcup_{P_a, Q_b \in J} D_{I,J}, \quad \text{and} \quad \bigcup_{P_a, Q_b \in J} D_{I,J}.$$
3. Bonds and NAP-colorings

In this section we characterize combinatorially the property for a graph of admitting a flexible assignment of spherical distances for the edges. We do so by proving that flexibility is equivalent to the existence of particular colorings. This equivalence is shown by considering how configuration curves of mobile graphs intersect the boundary of $\mathcal{M}_{0,m}$.

**Definition 3.1.** Consider a stable curve $X$ of genus zero with $2n$ marked points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ that has at least two irreducible components, namely $X$ belongs to the boundary $\mathcal{M}_{0,2n} \setminus \mathcal{M}_{0,2n}$. Pick a singular point $z$ of $X$. Then $X \setminus \{z\}$ has exactly two connected components $X_1$ and $X_2$. We define the cut of $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ determined by $(X, z)$ to be the partition $(I, J)$ of the marked points $\{P_1, \ldots, P_n, Q_1, \ldots, Q_n\}$ given by

$I := \{P_1, \ldots, P_n, Q_1, \ldots, Q_n\} \cap X_1, \quad J := \{P_1, \ldots, P_n, Q_1, \ldots, Q_n\} \cap X_2$.

**Remark 3.2.** Notice that if $(X, z)$ induces a cut $(I, J)$, then $X \in D_{I,J}$. Moreover, if in a cut $(I, J)$ we swap $P_k$ with $Q_k$ for all $k$, obtaining another cut $(\bar{I}, \bar{J})$, then the divisor $D_{\bar{I}, \bar{J}}$ is the complex conjugate of $D_{I,J}$.

As we explained in the introduction, one of the key ideas of this paper is to look at limit elements of the configuration set of a graph with a flexible assignment of spherical edge lengths.

**Definition 3.3.** Let $G = (V, E)$ be a graph. A bond of $G$ is any stable curve with at least two irreducible components in a fiber $\Phi_G^{-1}(\Lambda)$ (see definition 2.5) where no component of $\Lambda \in (\mathbb{P}^1)^{|E|}$ is $(0 : 1), (1 : 0)$, or $(1 : 1)$, where the identification $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ is made according to remark 2.8.

Bonds determine special cuts:

**Lemma 3.4.** Let $G = (V, E)$ be a graph and let $X$ be a bond of $G$. Pick a singular point $z$ of $X$. Then from definition 3.1 we get a cut $(I, J)$, which is a partition of the $2n$-tuple of the marked points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ of the stable curves in $\overline{\mathcal{M}}_{0,2n}$. Such a cut satisfies the following condition: for every edge $\{a, b\} \in E$, the cardinality of the intersection $I \cap \{P_a, P_b, Q_a, Q_b\}$ is never 2, and the same holds for $J \cap \{P_a, P_b, Q_a, Q_b\}$.

**Proof.** This lemma follows from remark 2.8: in fact, if the statement did not hold for an edge $\{a, b\}$, we would have that $\pi_{a,b}$ takes one of the three values $(0 : 1), (1 : 0)$, or $(1 : 1)$ on $X$. On the other hand, since $X$ is a bond and $\{a, b\}$ is an edge, $\pi_{a,b}(X)$ cannot be one of these three values. \hfill \Box

Our first goal now is to show that a graph admitting a flexible assignment has a bond with good properties (proposition 3.7). To do so, we introduce the Cayley-Menger variety. For $n \geq 2$, we define $CM_n$ to be the Zariski closure in $((\mathbb{P}^1)^2)^n$ of

$$\left\{(d_{a,b}(W_1, \ldots, W_n))_{1 \leq a < b \leq n} \text{ for } (W_1, \ldots, W_n) \in (S^2)^n\right\},$$
where \( d_{a,b} : (S^2)^n \longrightarrow \mathbb{P}_C^L \) maps a tuple \((W_1, \ldots, W_n)\) to the point \((d_{a2}(W_a, W_b) : 1)\) in \(\mathbb{P}_C^L\). Hence, by definition, the Cayley-Menger variety \(CM_n\) comes with \binom{n}{2}\) projection maps \(\tau_{a,b} : CM_n \longrightarrow \mathbb{P}_C^L\). Given a graph \(G = (V, E)\) where \(V = \{1, \ldots, n\}\), we denote the product \(\prod_{(a,b) \in E} \tau_{a,b}\) by \(\tau_G\).

**Lemma 3.5.** Let \(\lambda \in \mathbb{C}^E \setminus \{0, 1\}\) be a flexible assignment of a graph \(G = (V, E)\). Then there exists a non-edge \(\{c, d\}\) such that the restriction of \(\tau_{c,d}^{-1}(\Lambda)\) is surjective, where \(\Lambda\) is the point in \((\mathbb{P}_C^1)^{|E|}\) corresponding to \(\lambda\).

**Proof.** This follows from the definition of flexible assignment: in fact, we know that the image of \(\tau_{c,d}^{-1}(\Lambda)\) is composed of infinitely many elements for some non-edge \(\{c, d\}\), because the distance of at least two vertices must vary among the infinitely many essentially distinct realizations. Hence the map must be surjective being a morphism between projective varieties. \(\square\)

**Lemma 3.6.** If a graph \(G = (V, E)\) has a flexible assignment on the sphere, then there exists \(\Lambda \in \prod_{(a,b) \in E} \mathcal{M}_{0.4} \) and a non-edge \(\{c, d\}\) such that the restriction of \(\pi_{c,d}\) to \(\Phi_G^{-1}(\Lambda)\) is surjective.

**Proof.** By lemma 3.5 there exists \(\Lambda \in (\mathbb{P}_C^1)^{|E|}\) and a non-edge \(\{c, d\}\) such that \(\tau_{c,d}^{-1}(\Lambda)\) is surjective. We construct a map \(\eta : \mathcal{M}_{0.2n} \longrightarrow CM_n\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{M}_{0.2n} & \xrightarrow{\tau_{c,d}} & \mathcal{M}_n \\
\eta \downarrow & & \downarrow \tau_G \\
\mathcal{M}_{0.2n} & \cong & \prod_{(a,b) \in E} \mathcal{M}_{0.4}
\end{array}
\]

The morphism \(\eta\) is defined as follows: take the product of all maps \(\pi_{a,b}\) where \(a, b \in \{1, \ldots, n\}\) with \(a \neq b\). This gives a map \(\mathcal{M}_{0.2n} \longrightarrow (\mathbb{P}_C^1)^{|E|}\), once we identify each \(\mathcal{M}_{0.4}\) with \(\mathbb{P}_C^1\). The image of \(\mathcal{M}_{0.2n}\) is contained in \(CM_n\) and contains an open subset of \(CM_n\), because from an element in \(\mathcal{M}_{0.2n}\) we can construct a set of \(n\) points in \(S^2\), and conversely a general choice of \(n\) distinct points in \(S^2\) determines a general element of \(CM_n\). Since both \(\mathcal{M}_{0.2n}\) and \(CM_n\) are irreducible and of the same dimension, the map \(\eta\) has the desired codomain and is surjective. By construction, the previous diagram is commutative. Now the statement follows by inspecting the diagram. \(\square\)

**Proposition 3.7.** If \(G\) has a flexible assignment \(\lambda\) on the sphere, then it admits a bond. Moreover, we can choose the bond so that it induces a cut \((I, J)\) such that there exists a non-edge \(\{c, d\}\) for which \(P_c, Q_c \in I\) and \(P_d, Q_d \in J\).

**Proof.** By lemma 3.6, there exists a non-edge \(\{c, d\}\) such that \(\pi_{c,d}\) is surjective on \(\Phi_G^{-1}(\Lambda)\). Hence, there exists an element in \(\Phi_G^{-1}(\Lambda) \cap \pi_{c,d}^{-1}(1)\). This is a bond fulfilling the conditions in the statement (see also remark 2.8). \(\square\)
The special cuts arising from bonds allow us to define colorings of the edges of a graph.

**Definition 3.8.** Let \( G = (V, E) \) be a graph and let \((I, J)\) be a cut induced by a bond of \( G \). We define a coloring \( \varepsilon_{I,J} : E \to \{ \text{red}, \text{blue} \} \) as follows: the edge \( \{a, b\} \) is colored red if at least three among \( P_a, P_b, Q_a, Q_b \) are in \( I \); it is colored blue if at least three among \( P_a, P_b, Q_a, Q_b \) are in \( J \). Lemma 3.4 guarantees that every edge has a color.

**Remark 3.9.** Notice that if in definition 3.8 we swap \( I \) and \( J \), then we swap red and blue in the coloring.

We now describe a crucial property of these colorings \( \varepsilon_{I,J} \). To do so, we start with a little detour to the planar situation.

No almost cycle (NAC) colorings were introduced in [12] as a combinatorial tool to characterize graphs having flexible assignments in the plane. We recall their definition.

**Definition 3.10.** Let \( G = (V, E) \) be a graph, and let \( \varepsilon : E \to \{ \text{red}, \text{blue} \} \) be a coloring of its edges. We say that a cycle is an almost red cycle if exactly one of its edges is blue, and analogously for almost blue cycles. We say that \( \varepsilon \) is a NAC-coloring if it is surjective and there are no almost red cycles or almost blue cycles in \( G \). In other words, every cycle is either monochromatic or has at least two edges for each color.

The main property of NAC-colorings can be stated as follows (see [12, Theorem 3.1]): a graph admits a flexible assignment in the plane if and only if it admits a NAC-coloring.

Here we introduce another kind of coloring, called NAP-coloring.

**Definition 3.11.** Let \( G = (V, E) \) be a graph, and let \( \varepsilon : E \to \{ \text{red}, \text{blue} \} \) be a coloring of its edges. We say that a path \((v, w, z, t)\) in \( G \) is an alternating path if \( \{v, w\} \) and \( \{z, t\} \) have the same color and \( \{w, z\} \) has the opposite color. We say that \( \varepsilon \) is a NAP-coloring (“no alternating path”) if it is surjective, all the 3-cycles in \( G \) are monochromatic, and there are no alternating paths in \( G \). In other words, every edge has an incident vertex such that all its incident edges have the same color.

![Figure 2. On the left, an alternating path. On the right, a path which is not alternating.](image)

**Remark 3.12.** Every NAP-coloring is a NAC-coloring.
Proposition 3.13. Let \((I,J)\) be a cut induced by a bond of a graph \(G\). The coloring \(\varepsilon_{I,J}\) from definition 3.8 is a NAP-coloring if and only if there exists a non-edge \(\{c,d\}\) for which \(P_c, Q_c \in I\) and \(P_d, Q_d \in J\).

Proof. Assume that there exists such a non-edge \(\{c,d\}\). Let us suppose that there exist edges \(\{v,w\}, \{w,z\}, \) and \(\{z,t\}\) in \(G\) such that \(\{v,w\}\) is red, \(\{w,z\}\) is blue, and \(\{z,t\}\) is red. Notice that here it may happen that \(v = t\). Then at least three out of \(P_v, P_w, Q_v, Q_w\) are in \(I\), and the same holds for \(P_z, P_t, Q_z, Q_t\). This contradicts the fact that at least three of \(P_w, P_z, Q_w, Q_z\) are in \(J\). The same argument works if we swap red and blue. To conclude the proof, we just need to prove that the coloring is surjective. Consider now the non-edge \(\{c,d\}\). Since the graph is connected, there exist edges \(\{c,e\}\) and \(\{d,f\}\). By construction, \(\varepsilon_{I,J}(\{c,e\}) \neq \varepsilon_{I,J}(\{d,f\})\), so the coloring is surjective.

Now assume that \(\varepsilon_{I,J}\) is a NAP-coloring. Then there exist two incident edges \(\{c,e\}\) and \(\{d,e\}\) of different colors, say \(\varepsilon_{I,J}(\{c,e\}) = \text{red}\) and \(\varepsilon_{I,J}(\{d,e\}) = \text{blue}\). Because of the NAP hypothesis, \(\{c,d\}\) cannot be an edge of the graph. Then we must have that either \(P_c, Q_c, P_e \in I\) and \(P_d, Q_d, Q_e \in J\), or \(P_c, Q_c, Q_e \in I\) and \(P_d, Q_d, P_e \in J\). In both cases the statement holds.

Theorem 3.14. Let \(G\) be a connected graph without multiedges, or self-loops. Then \(G\) has a flexible assignment on the sphere if and only if it admits a NAP-coloring.

Proof. Suppose that \(G\) has a flexible assignment on the sphere. Then proposition 3.7 combined with proposition 3.13 shows that \(G\) has a NAP-coloring.

Suppose now that \(G\) admits a NAP-coloring. Let \(\mathcal{T}\) be the set of vertices which are incident to a red as well as a blue edge. No two vertices in \(\mathcal{T}\) are adjacent, because this would violate the NAP condition. Let \(\rho : V \to S^2\) be any realization mapping each vertex from \(\mathcal{T}\) either to the North pole \((0,0,1)\) or to the South pole \((0,0,-1)\), i.e., \(\rho(\mathcal{T}) = \{(0,0,1), (0,0,-1)\}\). Then the blue part of the graph can rotate around the polar axis, while keeping the red part fixed (see fig. 4). This shows that \(G\) has a flexible assignment on the sphere.

Remark 3.15. Theorem 3.14 holds also over the reals in the following sense: if we have a flexible assignment with infinitely many, up to rotations, real compatible realizations, then in particular we have a complex one, so we obtain a NAP-coloring; vice versa, the construction of the flexible assignment starting from a NAP-coloring can be performed also on the real unit sphere \(S^2 \subset \mathbb{R}^3\).
Figure 4. The NAP-colorings on the right induce flexible assignments: the two (respectively, three) vertices incident to both blue and red edges are mapped to two antipodal points on the sphere. For the second graph, this leads to identification of two points: the vertex on top is sent to the North Pole, and the vertices in the middle and at the bottom are sent to the South Pole.

From remark 3.12, we get:

**Corollary 3.16.** If a graph has a flexible assignment on the sphere, then it has a flexible assignment in the plane.

In the next section, we describe the real motions on the sphere of the bipartite graph $K_{3,3}$ for which no two vertices coincide or are antipodal. The graph $K_{3,3}$ is the smallest “interesting” Laman graph admitting a NAP-coloring. In fact, the graph in fig. 5 is the smallest minimally rigid graph with a NAP-coloring but, as we can easily see, its flexible assignments force some of its vertices to be either coincident or antipodal.

We conclude this section by pointing out how our setup allows us to determine necessary conditions for the spherical lengths of the edges of a graph having a flexible assignment.

Let us assume that a graph $G = (V, E)$ has a flexible assignment on the sphere, and suppose that $|E| > 2|V| - 4$. By assumption, $G$ admits a bond. Let $(I, J)$ be a cut determined by this bond, and consider the coloring $\varepsilon_{I,J}$. Consider the divisor $D_{I,J}$
from definition 2.6. By proposition 2.7, the divisor $D_{I,J}$ is isomorphic to $\mathcal{M}_{0,|I|+1} \times \mathcal{M}_{0,|J|+1}$. For each red edge $\{a,b\}$ in $G$, the cross-ratio $\text{cr}(P_a, P_b, Q_a, Q_b)$ is defined on $\mathcal{M}_{0,|I|+1}$. Therefore, we have a map $\mathcal{M}_{0,|I|+1} \rightarrow \prod_{\epsilon_{I,J}(\{a,b\})=\text{red}} \mathcal{M}_{0,4}^{a,b}$.

If we assume that the number of red edges is bigger than $|I| - 2$, the previous map cannot be dominant. Hence, its image satisfies at least one algebraic equation. Analogously, if the number of blue edges is bigger than $|J| - 2$, we obtain an algebraic equation between the spherical lengths of blue edges.

By assumption, the number of edges is bigger than $|I| + |J| - 4$, so at least one of the previous conditions is met, hence we get an equation for the spherical lengths of edges. Notice that a class for which the condition $|E| > 2|V| - 4$ holds is the one of Laman graphs (i.e. minimally rigid graphs in the plane or, equivalently, on the sphere).

4. Flexibility of $K_{3,3}$

The goal of this section is to analyze assignments for the spherical lengths of the edges of $K_{3,3}$, the complete bipartite graph with $3 + 3$ vertices, that are flexible on the sphere. In contrast to the first part of the paper, here we only consider real realizations, and this hypothesis will be crucial in several steps of our analysis. We want to classify all motions on the sphere of $K_{3,3}$ for which no two vertices coincide or are antipodal. After some preliminary discussions, we analyze the possible motions of the subgraphs of $K_{3,3}$ isomorphic to $K_{2,2}$. After that, we prove that there is only a limited number of cases that we need to consider in order to classify all motions of $K_{3,3}$. Eventually, we analyze the possible cases one by one.

We fix once and for all the following labeling for the vertices of $K_{3,3}$:
This choice has as a consequence, that any subgraph of $K_{3,3}$ that is isomorphic to $K_{2,2}$ has exactly two vertices with even label and two vertices with odd label. These are called even and odd vertices, respectively.

Recall that Dixon described two motions of $K_{3,3}$ in the plane more than a hundred years ago [4], and Walter and Husty proved that these are the only possible ones [35]. For the first motion, the odd vertices 1, 3, and 5 are placed on a line and the even vertices 2, 4, and 6 are placed on another line, perpendicular to the first one. Then, the induced distances between adjacent vertices are taken as edge lengths, and this constitutes a flexible assignment. The second construction works as follows: we consider two rectangles with the same intersection of the diagonals such that the edges are parallel/orthogonal to each other. Consider the realization of $K_{4,4}$ where the partition of the vertex set is given by the vertices of the two rectangles. This provides a flexible assignment for $K_{4,4}$, from which one can extract an assignment for $K_{3,3}$ by forgetting a pair of vertices. This construction can be described in terms of symmetry: take two orthogonal lines, and consider two points in a quadrant determined by the two lines; then construct the eight points given by reflecting the two original ones around the two lines. Dixon’s motions can be also described on the sphere [36] (compare also [2, 33]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The two motions of the complete graph $K_{3,3}$ discovered by Dixon. In the motion on the left, three points are moving on the $x$-axis, and three points are moving on the $y$-axis. The right is a symmetric motion of $K_{4,4}$ consisting of the vertices of two rectangles sharing their symmetry axes, with two points omitted.}
\end{figure}
The main result we obtain in this section is the following (see theorem 4.32):

**Classification.** The only (real) motions of $K_{3,3}$ on the sphere such that no two vertices coincide, or are antipodal, are the two analogous motions of the planar ones, and a third new motion.

The classification follows from a careful analysis of the relations between the curves of realizations in the moduli spaces $\overline{\mathcal{M}}_{0,2n}$ of $K_{3,3}$ and of its subgraphs.

**Remark 4.1.** One can check that all the NAP-colorings (up to swapping the two colors) of $K_{3,3}$ are the ones listed in fig. 7. Theorem 3.14 ensures the existence of flexible assignments, but the construction provided there yields non-injective realizations.

![Figure 7. The complete bipartite graph $K_{3,3}$ has exactly 6 NAP-colorings.](image)

We start by defining what we mean by “motion”. Here we make clear that we consider real assignments of edge lengths, and real points on the sphere. Furthermore, we focus on motions for which no two vertices coincide or are antipodal.

**Definition 4.2.** Given a flexible assignment $\lambda: E \to (0,1)$ for the spherical lengths of $K_{3,3}$, let $D$ be the corresponding fiber of the map $\Phi_{K_{3,3}}$ from definition 2.5, which we call the configuration curve of $K_{3,3}$ determined by that assignment. Any positive-dimensional real irreducible component $C$ of $D$ with infinitely many real elements is called a motion of $K_{3,3}$. For every subgraph $G' \subset K_{3,3}$ with vertex set $V' \subseteq \{1, \ldots, 6\}$, let $\{i_1, \ldots, i_{6-|V'|}\} := \{1, \ldots, 6\} \setminus V'$. We denote by $\mathcal{C}_{i_1\cdots i_{6-|V'|}}$ the projection of $C$ via the forgetful map $\overline{\mathcal{M}}_{0,12} \to \overline{\mathcal{M}}_{0,2|V'|}^1$.

We are going to use extensively some particular maps, hence we introduce a special notation for them.

**Definition 4.3.** Let $C$ be a motion of $K_{3,3}$. For all distinct $i, j, k, \ell \in \{1, \ldots, 6\}$, we define the forgetful maps:

\[
\begin{align*}
p_i & : C \to C_i, \\
p_{ij} & : C \to C_{ij}, \\
q^k_i & : C_k \to C_{ki}, \\
q^k_{ij} & : C_k \to C_{kij}, \\
r^\ell_i & : C_{k\ell} \to C_{k\ell i}, \\
q^{ij}_k & : C_k \to C_{ij}.
\end{align*}
\]

**Definition 4.4.** We say that a motion $C$ of $K_{3,3}$ is **proper** if the intersection of $C$ with $\overline{\mathcal{M}}_{0,12}$ has infinitely many real elements corresponding to real realizations of $K_{3,3}$ on the sphere. Notice that here we ask that the points come from $\overline{\mathcal{M}}_{0,12}$, the open subset of $\overline{\mathcal{M}}_{0,12}$ whose elements are stable curves with a single component.
Remark 4.5. Being proper for a motion is equivalent to the fact that there are infinitely many essentially distinct realizations of $K_{3,3}$ where no two vertices coincide or are antipodal. In fact, the properness of the motion is equivalent to the fact that we have infinitely many realizations of $K_{3,3}$ on the (real) sphere where 6 vertices determine 12 distinct left and right lifts. Now, two points on the sphere have the same left (or right) lift if and only if they belong to the same line on $S^2$. Moreover, let $T$ and $U$ be real points on the sphere such that the left lift of $T$ coincides with the right lift of $U$. Let $T'$ be the antipodal of $T$. Since taking antipodals swaps left and right lifts, the points $U$ and $T'$ have the same right lift, so by what we have just proved they must coincide. Therefore $U$ is the antipodal of $T$. Hence, asking for a realization of 6 real points on the sphere to have 12 distinct left and right lifts is equivalent to asking that no two points coincide or are antipodal.

Remark 4.6. For proper motions, the maps $p_i$, $q_i^k$, and $r_i^{k\ell}$ from definition 4.3 are either birational or 2:1. In fact, the possible positions of vertex $i$ in their fibers — whose number is the degree of the map — are obtained by intersecting circles on the sphere, and this may give either one or two intersections.

We define the spherical analogues of the two motions described by Dixon. For this, we consider $K_{4,4}$ to be the graph with vertices $\{1, \ldots, 8\}$ and edges $\{\{i,j\}: i \in \{1, 3, 5, 7\}, j \in \{2, 4, 6, 8\}\}$.

Definition 4.7. A Dixon 1 motion of $K_{3,3}$ on the sphere is a proper motion in which the vertices of the graph move along two orthogonal great circles. A Dixon 2 motion of $K_{4,4}$ on the sphere is a proper motion in which, up to relabeling the vertices and swapping any vertex with its antipode, there exist three involutions (i.e. isometries whose square is the identity) $o_1$, $o_2$ and $o_3$ such that $o_1 o_2 o_3 = id_{S^2}$ and

\[
\begin{align*}
o_1: & \quad R_1 \leftrightarrow R_3, \quad R_2 \leftrightarrow R_4, \\
& \quad R_5 \leftrightarrow R_7, \quad R_6 \leftrightarrow R_8,
\end{align*}
\[
\begin{align*}
o_2: & \quad R_1 \leftrightarrow R_7, \quad R_2 \leftrightarrow R_8, \\
& \quad R_4 \leftrightarrow R_5, \quad R_4 \leftrightarrow R_6,
\end{align*}
\[
\begin{align*}
o_3: & \quad R_1 \leftrightarrow R_5, \quad R_2 \leftrightarrow R_6, \\
& \quad R_3 \leftrightarrow R_7, \quad R_4 \leftrightarrow R_8,
\end{align*}
\]

where $(R_1, \ldots, R_8)$ is any realization of $K_{4,4}$ in the motion, see fig. 8. A Dixon 2 motion of $K_{3,3}$ is any motion obtained by removing a pair of vertices in a Dixon 2 motion of $K_{4,4}$.

Lemma 4.8. Let $C$ be a proper motion of $K_{3,3}$. If $\deg p_{56} \geq 3$, then $C$ is a Dixon 1 motion.

Proof. Since $p_{56}$ is the composition of $p_5$ and $q_6^0$, and since both of these maps can be either birational or 2:1, it follows that the only possibility if $\deg p_{56} \geq 3$ is that
deg \( p_{56} = 4 \). Pick a realization \((R_1, \ldots, R_6) \in (S^2)^6\) of the vertices 1, 2, 3, 4, 5, 6 in the motion \( \mathcal{C} \). Let \( R'_5 \) and \( R'_6 \) be the two other points appearing in the four realizations given by \( p^{-1}_{56}(p_{56}(c)) \). Then we have

\[
\begin{align*}
    d_{S^2}(R_1, R_6) &= d_{S^2}(R_1, R'_6), \\
    d_{S^2}(R_3, R_6) &= d_{S^2}(R_3, R'_6), \\
    d_{S^2}(R_5, R_6) &= d_{S^2}(R_5, R'_6),
\end{align*}
\]

and this shows that \( R_1, R_3, \) and \( R_5 \) are cocircular\(^1\), because this is the only case where three distinct and not antipodal points can be at the same distances from two other points. By a symmetric argument, \( R_2, R_4, \) and \( R_6 \) are cocircular as well. Moreover, since

\[
d_{S^2}(R'_5, R_6) = d_{S^2}(R'_5, R'_6) = d_{S^2}(R_5, R'_6),
\]

then also \( R'_6 \) is cocircular with \( R_2, R_4, \) and \( R_6 \). We now need to prove that these two great circles are orthogonal. Notice that the two positions \( R_6 \) and \( R'_6 \) must be equidistant from the great circle \( \overline{R_1R_3R_5} \); hence the great circle \( \overline{R_6R'_6} \) is orthogonal to \( \overline{R_1R_3R_5} \). This shows then that \( \overline{R_2R_4R_6} \) is orthogonal to \( \overline{R_1R_3R_5} \), and so we have a Dixon 1 motion. \( \square \)

So from now on we can (and we will) suppose \( \deg p_{ij} \leq 2 \) for all \( i, j \). Before proceeding further in the analysis of the mobility of the whole \( K_{3,3} \), we focus on how its subgraphs isomorphic to \( K_{2,2} \) may move.

### 4.1. Mobility of quadrilaterals

A thorough discussion of the mobility of quadrilaterals on the sphere, and in particular of their configuration spaces and their irreducible components, is provided in [10, 11]. We introduce the following nomenclature for mobile subgraphs of \( K_{3,3} \) isomorphic to \( K_{2,2} \). Recall from definition 2.1 that for \( T, U \in S^2 \) we denote \( \delta_{S^2}(T, U) := \langle T, U \rangle \).

\(^1\)Here by saying that the three points are *cocircular* we mean that they lie on a common geodesic on the sphere. The terminology is motivated by the fact that the geodesics are exactly the great circles.
Definition 4.9. Given a proper motion $C \subseteq \mathcal{M}_{0,12}$ of $K_{3,3}$ with edge length assignment $\lambda$, we define five families of proper motions of subgraphs of $K_{3,3}$ isomorphic to $K_{2,2}$. We call these graphs quadrilaterals. To give the definitions, we focus for simplicity on the subgraph $H_{56}$ induced by the vertices $\{1, 2, 3, 4\}$. Let $D_{56} \subseteq \mathcal{M}_{0,8}$ be the configuration curve of $H_{56}$ defined by the edge lengths induced by $\lambda$. Note that $D_{56}$ may have several irreducible components. We call the motion $C_{56} := p_{56}(C)$ of $K_{2,2}$ (which is irreducible by construction):

\textbf{g:} general if and only if $D_{56} = C_{56}$. Here, all the maps $r_{1}^{56}, r_{2}^{56}, r_{3}^{56},$ and $r_{4}^{56}$ are $2:1$.

\textbf{o:} odd deltoid if and only if $D_{56}$ has two components, $D_{56} = C_{56} \cup Z$, and $Z$ is a (degenerate) motion. The component $Z$ entirely lies on the boundary; its elements correspond to realizations where the odd vertices coincide or are antipodal (see fig. 9). Here, we have $\deg r_{i}^{56} = 1$ for $i$ odd, and $\deg r_{i}^{56} = 2$ for $i$ even. Moreover, for all realizations $(R_{1}, \ldots, R_{4}) \in (S^{2})^{4}$ corresponding to elements of $C_{56} \cap \mathcal{M}_{0,8}$, we have that:

$$\delta_{S^{2}}(1, 2) = \alpha \delta_{S^{2}}(2, 3), \quad \delta_{S^{2}}(3, 4) = \alpha \delta_{S^{2}}(1, 4),$$

where $\alpha \in \{1, -1\}$, and $\delta_{S^{2}}(i, j)$ is a shorthand for $\delta_{S^{2}}(R_{i}, R_{j})$. The two possible cases are depicted in the following figure, where edges are labeled by the values of $\delta_{S^{2}}$ on the corresponding realizations. Notice that we can pass from one case to the other by swapping the realization of vertex 3 or 1 with its antipode. For the degenerate component we can see these two cases in fig. 9.

\begin{center}
\includegraphics[width=0.8\textwidth]{fig9.png}
\end{center}

Figure 9. An odd deltoid (left) and two degenerate deltoids where vertices 1 and 3 coincide (middle) or are antipodal (right). Vertex 2 may move along the dashed circle.
For an odd deltoid there exists an involution of the sphere fixing the even vertices and swapping the odd ones.

\textbf{r:} \textit{even deltoid} if and only if the condition for the odd deltoid holds after interchanging even and odd vertices.

For a rhomboid there exists an involution of the sphere swapping both the even and the odd vertices. This involution, however, has different nature for the two components $\mathcal{Y}_1$ and $\mathcal{Y}_2$: for one it is given by a rotation, while for the other by a reflection.

\textsuperscript{2}Here we appeal to the terminology used by Euclid in Book I, Definition 22 of the Elements, where \textit{rhomboid} denotes a quadrilateral whose opposite sides and angles are equal.
Figure 10. Three rhomboids: The rhomboid on the left has a symmetry plane, which intersects the sphere in the green circle. The rhomboid in the middle has a symmetry line, which meets the sphere in the intersection of the two diagonals. The rhomboid on the right is obtained from the one in the middle by replacing one of its vertices by its antipodal point.

\[ \text{lozenge} \text{ if and only if the configuration space } D_{56} = C_{56} \cup Z_1 \cup Z_2, \text{ and both } Z_1 \text{ and } Z_2 \text{ are degenerate, namely either the even or the odd vertices coincide. Here, all the maps } r_{56}^1, r_{56}^2, r_{56}^3, r_{56}^4 \text{ are birational.} \]

Moreover, for the realizations corresponding to the elements of \( C_{56} \cap M_{0,8} \), we have that:

\[ \delta_{S^2}(1, 2) = \alpha \cdot \delta_{S^2}(2, 3) = \beta \cdot \delta_{S^2}(3, 4) = \gamma \cdot \delta_{S^2}(1, 4) \neq 0, \]

where \( (\alpha, \beta, \gamma) \in \{(1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1)\} \). Notice that the condition on the lengths \( \delta_{S^2} \) being nonzero follows from the hypothesis that no two vertices coincide or are antipodal. The four possible cases are depicted in the following figure: we can pass from one case to the others by swapping the realizations of some vertices with their antipodes.

Figure 11. Two lozenges with four equal side lengths. The diagonals are orthogonal and bisect each other.
Remark 4.10. A lozenge is not a special case of a rhomboid: in fact, in a rhomboid the configuration curve has only two components, and this excludes the situation where all edge lengths are equal (up to sign).

The fact that the classification provided by definition 4.9 is complete relies on the hypothesis of considering only real proper motions.

Remark 4.11. A case by case analysis in the classification of definition 4.9 reveals that \( \deg r_i^{k\ell} = \deg r_j^{k\ell} \) if \( i \) and \( j \) are vertices of the same parity of a 4-cycle.

Cuts for \( K_{2,2} \). Due to proposition 3.13, there exist four possible cuts \((I, J)\) for \( K_{2,2} \) (up to swapping \( I \) with \( J \), and the \( P \)'s with \( Q \)'s) that induce a NAP-coloring. They are the following:

\[
\begin{align*}
T_{ou} & := (P_1Q_1, P_2P_3, P_3Q_3, Q_2Q_4), \\
T_{eu} & := (P_2Q_2, P_1P_3, P_4Q_4, Q_1Q_3), \\
T_{om} & := (P_1Q_1, P_2Q_4, P_3Q_3, Q_2P_4), \\
T_{em} & := (P_2Q_2, P_1Q_3, P_2Q_4, Q_1P_3).
\end{align*}
\]

Here, “o” stands for odd vertices being separated, namely \( P_1, Q_1 \) belong to \( I \), while \( P_3, Q_3 \) belong to \( J \). Analogously, “e” stands for even vertices being separated. The letters “m” and “u” stand for mixed and unmixed, respectively, and describe the relation between the \( P \)'s and \( Q \)'s in the remaining indices. We use the notation \( D_{ou} \) for the divisor corresponding to \( T_{ou} \), and similarly for the other three cuts.

Our first goal is to understand how a proper motion of a subgraph of \( K_{3,3} \) isomorphic to \( K_{2,2} \) intersects the divisors \( D_{ou}, \ldots, D_{em} \) in \( \mathcal{M}_{0,8} \). To this end, given a proper motion \( C' \subseteq \mathcal{M}_{0,8} \) of \( K_{2,2} \), we define the numbers:

\[
\begin{align*}
\mu(T_{ou}) := \deg(D_{ou}|C'), & \quad \mu(T_{om}) := \deg(D_{om}|C'), \\
\mu(T_{eu}) := \deg(D_{eu}|C'), & \quad \mu(T_{em}) := \deg(D_{em}|C').
\end{align*}
\]

Here, the symbol \( D_{ou}|C' \) denotes the restriction of the divisor \( D_{ou} \) to \( C' \), and analogously for the other similar symbols. We are going to collect all the possible values for the numbers \( \mu \) in table 1.

Let us suppose that \( C' \) is general, namely it is the whole configuration space of \( K_{2,2} \). Then the curve \( C' \) is defined by the four equations

\[
\begin{align*}
\cr(P_1, Q_1, P_2, Q_2) & = \lambda_{12}, & \cr(P_1, Q_1, P_4, Q_4) & = \lambda_{14}, \\
\cr(P_2, Q_2, P_3, Q_3) & = \lambda_{23}, & \cr(P_3, Q_3, P_4, Q_4) & = \lambda_{34},
\end{align*}
\]

where the numbers \( \lambda_{12}, \ldots, \lambda_{34} \) are the lengths of the corresponding edges of \( K_{2,2} \). We restrict these equations to the divisor \( D_{ou} \). This divisor is isomorphic to \( \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \), more precisely \( \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \) (see proposition 2.7). The four equations for the restriction of \( C' \) to \( \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \times \mathcal{M}_{0,5} \) become two for the left factor, and two for the right one:

left: \[
\begin{align*}
\cr(P_1, Q_1, P_2, *) & = \lambda_{12}, & \cr(P_1, Q_1, P_4, *) & = \lambda_{14},
\end{align*}
\]

right: \[
\begin{align*}
\cr(*, Q_2, P_3, Q_3) & = \lambda_{23}, & \cr(P_3, Q_3, *, Q_4) & = \lambda_{34}.
\end{align*}
\]
This system of equations admits a single solution for general values of \(\lambda_{12}, \ldots, \lambda_{34}\).
In fact, let us focus on the left factor, and let us first look for solutions in \(M_{0,8}\) (namely, not on the boundary). Then by the action of \(\mathbb{P}GL(2, \mathbb{C})\) we can suppose that \(P_1, Q_1\) and \(*\) are fixed. The two equations determine uniquely \(P_2\) and \(P_4\), and similarly for the right factor. One can then check that there are no solutions on the boundary.

We can also use the algorithm developed in [9] to compute the degree of the intersection of these equations for general values of \(\lambda_{12}, \ldots, \lambda_{34}\), which gives \(\mu(T_{ou}) = 1\).

Similarly, we conclude that \(\mu(T_{ou}) = \mu(T_{om}) = \mu(T_{eu}) = \mu(T_{em}) = 1\).

Let us now consider the odd deltoid case. Here, we have a degenerate component \(Z\) in the configuration space where the vertices 1 and 3 coincide or are antipodal. This component \(Z\) is then contained in the boundary of \(\mathcal{M}_{0,8}\), and in particular, when 1 and 3 coincide, in the subvariety of the boundary whose general element is:

\[
\begin{array}{ccccccc}
P_2 & Q_2 & P_3 & Q_3 & P_4 & Q_4 & \\
| & | & | & | & | & |
\end{array}
\]

Recall in fact from page 6 that when two marked points come together (as, in this case, the left lifts of the realizations of vertices 1 and 3, and also the right lifts) then we get a stable curve with a new component containing the two marked points. Here, this happens for the marked points \(P_1\) and \(P_3\), and \(Q_1\) and \(Q_3\).

Notice that the subvariety from above is the intersection of the two divisors

\[
D_{P_1P_3|P_2Q_2P_4Q_4P_1Q_1Q_3} \cap D_{P_1P_3P_2Q_2P_4Q_4Q_1Q_3}.
\]

We want to understand how \(Z\) intersects \(D_{ou}\). We have that \(Z \cdot D_{ou} = 0\) as a consequence of Keel’s quadratic relations in the description of the Chow ring of \(\mathcal{M}_{0,8}\) (see [17, Section 4, Theorem 1]). In fact, we have that \(D_{ou} \cdot D_{P_1P_3P_2Q_2P_4Q_4Q_1Q_3} = 0\) since, for example, \(P_1\) and \(P_3\) are split by the partition of \(D_{ou}\), but they are not by the partition of \(D_{P_1P_3P_2Q_2P_4Q_4Q_1Q_3}\). Similarly, we get \(D_{ou} \cdot D_{P_1P_3|P_2Q_2P_4Q_4Q_1Q_3} = 0\). For the same reason, we have \(Z \cdot D_{em} = Z \cdot D_{ou} = 0\). The only number left to compute is \(Z \cdot D_{eu}\). Notice that \(Z \cap D_{eu}\) is a subvariety of

\[
D_{eu} \cap D_{P_1P_3|P_2Q_2P_4Q_4Q_1Q_3} \cap D_{P_1P_3P_2Q_2P_4Q_4Q_1Q_3}
\]

defined by two equations. The last intersection is a surface whose general element has the form:
We can then again use the algorithm from [9] to compute $Z \cdot D_{eu}$, which turns out to be 1, or do it via a direct computation, as we did before. Now that we know how $Z$ intersects the divisors, we can determine how $C'$ intersects them by relying on the following fact:

**Lemma 4.12.** The configuration space of any $K_{2,2}$ in $K_{3,3}$ is a complete intersection; in particular, the class of its intersection with the divisors $D_{I,3}$ does not depend on the type of quadrilateral.

**Proof.** The configuration space of any subgraph of $K_{3,3}$ isomorphic to $K_{2,2}$ given by edge lengths induced by the ones of a flexible assignment of $K_{3,3}$ is always a curve. At the same time, the configuration space is determined by four equations in $\mathcal{M}_{0,8}$, which is five-dimensional. Hence, it is a complete intersection. $\square$

Denote by $\mathcal{D}'$ the configuration curve of $K_{2,2}$. Using the result about general quadrilaterals, we know that the product of $\mathcal{D}'$ with all the four special divisors $D_{ou}, \ldots, D_{em}$ in $\mathcal{M}_{0,8}$ is 1. Hence, when 1 and 3 coincide in the degenerate component of an odd deltoid, then for the non-degenerate component we have:

- $C' \cdot D_{ou} = D' \cdot D_{ou} - Z \cdot D_{ou} = 1 - 0 = 1$
- $C' \cdot D_{om} = D' \cdot D_{om} - Z \cdot D_{om} = 1 - 0 = 1$
- $C' \cdot D_{em} = D' \cdot D_{em} - Z \cdot D_{em} = 1 - 0 = 1$
- $C' \cdot D_{eu} = D' \cdot D_{eu} - Z \cdot D_{eu} = 1 - 1 = 0$

If, instead, the points 1 and 3 are antipodal, then the degenerate component $Z$ is contained in the subvariety whose general element is
Arguing as before, we get
\[ C' \cdot D_{ou} = 1, \quad C' \cdot D_{om} = 1, \]
\[ C' \cdot D_{eu} = 0, \quad C' \cdot D_{em} = 1. \]

The case of the even deltoid is dealt similarly.

The rhomboid case is different, since there we have two non-degenerate components:
\[ \mathcal{D}' = \mathcal{Y}_1 \cup \mathcal{Y}_2. \]
We know that
\[ \mathcal{Y}_1 \cdot D_{ou} + \mathcal{Y}_2 \cdot D_{ou} = \mathcal{D}' \cdot D_{ou} = 1 \]
and similarly for the other three divisors \( D_{om}, D_{eu}, \) and \( D_{em}. \) In fact, the degrees of the intersections of the whole configuration curve with the divisors are the same as in the general case due to lemma 4.12. Consider the situation where the edge lengths are
\[ \delta S_2(1, 2) = a, \quad \delta S_2(2, 3) = b, \quad \delta S_2(3, 4) = a, \quad \delta S_2(1, 4) = b. \]

The other situation for the edge lengths, where \( \delta S_2(1, 2) = -\delta S_2(3, 4) \) and \( \delta S_2(1, 4) = -\delta S_2(2, 3), \) can be analyzed similarly. Consider the automorphism \( \sigma \) of \( M_{0, 8} \) that acts as follows on the marked points:
\[ P_1 \leftrightarrow Q_3, \quad P_2 \leftrightarrow Q_4, \quad Q_1 \leftrightarrow P_3, \quad Q_2 \leftrightarrow P_4. \]

Look at the equations in \( \overline{M}_{0, 8} \) that define the configuration curve \( \mathcal{D}' \) for the situation of edge lengths under consideration. They are of the form:
\[ \text{cr}(P_1, Q_1, P_2, Q_2) = \alpha, \quad \text{cr}(P_2, Q_2, P_3, Q_3) = \beta, \]
\[ \text{cr}(P_3, Q_3, P_4, Q_4) = \alpha, \quad \text{cr}(P_1, Q_1, P_4, Q_4) = \beta, \]
where \( \alpha \) and \( \beta \) depend on \( a \) and \( b. \) Then one sees that these equations are invariant under the action of \( \sigma. \) Hence \( \sigma \) leaves \( \mathcal{D}' \) invariant. Now consider the component of \( \mathcal{D}' \), say \( \mathcal{Y}_1, \) whose corresponding realizations \( (R_1, \ldots, R_4) \) on the sphere are symmetric with respect to a rotation of the sphere, which swaps \( R_1 \) with \( R_3 \), and \( R_2 \) with \( R_4. \) Since, by construction, both a realization and its symmetric with respect to the rotation determine the same element on the moduli space \( \overline{M}_{0, 8} \), we get that \( \sigma \) leaves the component \( \mathcal{Y}_1 \) pointwise invariant. By inspecting the action of \( \sigma \) on the four cuts \( T_{ou}, T_{om}, T_{eu}, \) and \( T_{em}, \) and taking into account remark 3.2, one finds that
\[ \sigma(D_{ou}) = \overline{D_{ou}}, \quad \sigma(D_{om}) = D_{om}, \]
\[ \sigma(D_{eu}) = \overline{D_{eu}}, \quad \sigma(D_{em}) = D_{em}, \]
where \( (\cdot) \) denotes complex conjugation. Consider now the automorphism \( \rho \) of \( \overline{M}_{0, 8} \) that acts as follows on the marked points:
\[ P_1 \leftrightarrow Q_3, \quad P_2 \leftrightarrow Q_4, \quad Q_1 \leftrightarrow P_4, \quad Q_2 \leftrightarrow P_2. \]

With similar arguments as before, we see that \( \rho \) leaves \( \mathcal{D}' \) invariant. Moreover, we know that the component \( \mathcal{Y}' \) has a symmetry given by a reflection which swaps \( R_1 \) with \( R_3, \) and \( R_2 \) with \( R_4. \) Composing this symmetry with the antipodal map yields a direct isometry of the sphere. Again, a realization and its image under this direct
isometry give the same element in the moduli space \( \overline{\mathcal{M}}_{0,8} \), hence we get that \( \rho \) fixes \( \mathcal{Y}_2 \) pointwise. As we did for \( \sigma \), we find that

\[
\rho(\text{Dou}) = \text{Dou}, \quad \rho(\text{Dom}) = \text{Dom}, \quad \rho(\text{Deu}) = \text{Deu}, \quad \rho(\text{Dem}) = \text{Dem}.
\]

Now, let us suppose that the curve \( \mathcal{C}' \) giving the rhomboid motion is \( \mathcal{Y}_1 \). We claim that \( \mathcal{C}' \cdot \text{Dou} = 0 \). In fact, if we had \( \mathcal{C}' \cdot \text{Dou} = 1 \), then by applying \( \sigma \) we would get \( \mathcal{C}' \cdot \overline{\text{Dou}} = 1 \). However, due to the relations in the Chow ring of \( \mathcal{M}_{0,8} \) (see [17, Section 4, Theorem 1]), we know that \( \text{Dou} \cdot \overline{\text{Dou}} = 0 \), and so this situation cannot happen. Hence we get \( \mathcal{C}' \cdot \text{Dou} = 0 \). Using the map \( \rho \) on the other component \( \mathcal{Y}_2 \) one can find that \( \mathcal{C}' \cdot \text{Dom} = 1 \), and similarly for the intersection with the other two divisors. In this way we recover the row of Type 1 for the rhomboid in table 1. If the curve \( \mathcal{C}' \) giving the rhomboid motion is \( \mathcal{Y}_2 \), we recover the row of Type 4 for the rhomboid. In the situation where the edge lengths are

\[
\delta_{S^2}(1,2) = a, \quad \delta_{S^2}(2,3) = b, \quad \delta_{S^2}(3,4) = -a, \quad \delta_{S^2}(1,4) = -b,
\]

we perform analogous arguments, this time using the two automorphisms of \( \overline{\mathcal{M}}_{0,8} \):

\[
\sigma: P_1 \leftrightarrow Q_3, \quad P_2 \leftrightarrow P_4, \quad Q_1 \leftrightarrow P_3, \quad Q_2 \leftrightarrow Q_4,
\]

and

\[
\rho: P_1 \leftrightarrow P_3, \quad P_2 \leftrightarrow Q_4, \quad Q_1 \leftrightarrow Q_3, \quad Q_2 \leftrightarrow P_4.
\]

In this way we compute all the values in table 1 for the rhomboid.

In the case of lozenges, we have that the configuration curve splits into three connected components, two of which are degenerate because either the odd or the even vertices coincide. We can then adopt the same technique we used in the case of the odd deltoid to compute the contributions provided by degenerate components, and so obtaining the \( \mu \)-numbers for the non-degenerate one. We have four cases, each for a possible configuration of edge lengths as shown in definition 4.9.

We can sum up what we have obtained so far in the table 1.

4.2. Classification of forgetful maps. The goal of this subsection is to determine all the possible cases for the degrees of the maps \( \{ p_i \}, \{ q^k_i \}, \) and \( \{ r^{k\ell}_i \} \) once a proper motion \( \mathcal{C} \) of \( K_{3,3} \) is fixed. The first result we want to show (proposition 4.17) is that, for a proper motion of \( K_{3,3} \) which is not a Dixon 1 motion, all the maps \( p_1, \ldots, p_6 \) are birational.

Proposition 4.17 will follow once we prove a couple of auxiliary results. We start by stating an elementary fact.

**Lemma 4.13.** Let \( T \) and \( U \) be points on the sphere \( S^2 \subset \mathbb{R}^3 \). If an involution swaps \( T \) and \( U \), then \( T, U \) and the axis of the involution are coplanar.

To keep notation simple, in the next lemmas we focus on the degree of specific maps from definition 4.3, but the results still hold once indices are modified according to automorphisms of \( K_{3,3} \).

**Lemma 4.14.** \( \deg q^6_{113} = \deg q^6_{115} = \deg q^6_{325} \).
Table 1. Intersections of the projection of a configuration curve on a quadrilateral in $K_{3,3}$ with the boundary divisors of $\mathcal{M}_{0,8}$. In the deltoid cases, the subcases are distinguished by properties of the respective residual degenerate components.

| case | subcase     | $\mu(T_{em})$ | $\mu(T_{ou})$ | $\mu(T_{em})$ | $\mu(T_{ou})$ |
|------|-------------|----------------|----------------|----------------|----------------|
| $g$  | 1, 3 coincide | 1              | 1              | 1              | 0              |
|      | 1, 3 antipodal| 1              | 1              | 0              | 1              |
| $\epsilon$ | 2, 4 coincide | 1              | 0              | 1              | 1              |
|      | 2, 4 antipodal| 0              | 1              | 1              | 1              |
| $\tau$ | Type 1     | 1              | 0              | 1              | 0              |
|      | Type 2     | 0              | 1              | 1              | 0              |
|      | Type 3     | 1              | 0              | 0              | 1              |
|      | Type 4     | 0              | 1              | 0              | 1              |
| $l$  | Type 1     | 1              | 0              | 1              | 0              |
|      | Type 2     | 0              | 1              | 1              | 0              |
|      | Type 3     | 1              | 0              | 0              | 1              |
|      | Type 4     | 0              | 1              | 0              | 1              |

Proof. Notice that we have the following factorizations:

$$
\begin{align*}
\mathcal{C}_0 & \rightarrow \mathcal{C}_{156} \\
\mathcal{C}_6 & \rightarrow \mathcal{C}_{16} \\
\mathcal{C}_15 & \rightarrow \mathcal{C}_{r_3^{16}} \\
\mathcal{C}_16 & \rightarrow \mathcal{C}_{r_3^{16}} \\
\end{align*}
$$

Since by remark 4.11 we have $\deg r_3^{16} = \deg r_3^{16}$, it follows that $\deg q_6^{15} = \deg q_6^{15}$. The other equality is proved similarly.

Lemma 4.15. If $r_3^{56}$, $r_3^{16}$ and $r_5^{36}$ are 2:1 maps, then $q_1^6$, $q_3^6$ and $q_5^6$ are not all birational.

Proof. Since $r_3^{56}$ is 2:1, then the subgraph 1234 is either $g$ or $\epsilon$, and similarly for the subgraphs 1245 and 2345. Suppose that all $q_1^6$, $q_3^6$, and $q_5^6$ are birational. We show that this configuration is not possible, and to do so we use cuts. We consider the cut $T_{em}$ for 1234, namely:

$$
T_{em} = (P_2Q_2, P_1Q_3|P_4Q_4, Q_4P_3).
$$

From table 1, we get $\mu^{56}(T_{em}) = 1$, where $\mu^{56}(\cdot)$ are the $\mu$-values for the quadrilateral 1234 as defined in Equation (1). Similar relations hold for the other two
Lemma 4.16. Suppose that 1, 3, and 5 are cocircular. Then not all three maps $q_1^6$, $q_3^6$, and $q_5^6$ can be birational.

Proof. Suppose, for a contradiction, that all the maps $q_1^6$, $q_3^6$, and $q_5^6$ are birational. Lemma 4.14 tells us that the degrees of the maps $q_{13}^6$, $q_{15}^6$, $q_{35}^6$ are the same. We distinguish two cases, depending on $\deg q_{13}^6$.

$\deg q_{13}^6 = 1$: Looking at the commutative diagrams as in the proof of lemma 4.14, we see that $\deg r_{1}^{16} = \deg r_{3}^{16} = \deg r_{5}^{36} = 1$. Then, by the classification of moving $K_{2,2}$ we get that $C_{56}$, $C_{16}$, and $C_{36}$ can be $\sigma$, $\tau$, or $I$. Let us suppose that $C_{56}$ is $I$. This forces $C_{16}$ to be $I$ as well. However, this contradicts the general assumption that no two vertices coincide or are antipodal (in fact, in this case either 2 and 4, or 1 and 3 must coincide or be antipodal). Hence, none of $C_{56}$, $C_{16}$, and $C_{36}$ can be $I$. A direct inspection shows that not all three of them can be $\tau$, and if one of them is $\sigma$, then the other two must be $\tau$, so the latter is
the only possibility left. Recall from definition 4.9 that in this case there exist
isometries \( o, p_1, \) and \( p_2 \) of the sphere such that:
1. \( o \) swaps 1 and 3, and fixes 2 and 4;
2. \( p_1 \) swaps 3 and 5, and 2 and 4;
3. \( p_2 \) swaps 1 and 5, and 2 and 4.
Using lemma 4.13 we see that all the points 1, 2, 3, 4, and 5 are cocircular,
and this violates the hypothesis of non-collapsing and non-antipodal vertices.
Hence we reached a contradiction when \( \deg q_{13} = 1 \).
\[
\deg q_{13} = 2: \quad \text{From the diagram}
\]
\[
\begin{array}{c}
C_6 \xrightarrow{o} C_{56} \\
\downarrow \quad r_i^{56} \\
C_{156}
\end{array}
\]
we have that \( r_i^{56} \) is a 2:1 map, and similarly for \( r_3^{16} \) and \( r_5^{36} \). Hence by
lemma 4.15 we get that not all maps \( q_1^6, q_3^6 \) and \( q_5^6 \) are birational.
\( \square \)

With this result at hand, we can prove proposition 4.17.

**Proposition 4.17.** Suppose that \( \mathcal{C} \) is a proper motion of \( K_{3,3} \) which is not a
Dixon 1 motion. Then all the maps \( p_1, \ldots, p_6 \) are birational.

**Proof.** Without loss of generality, we prove that \( p_6 \) is birational. By lemma 4.8, the
maps \( p_{16}, p_{36} \) and \( p_{56} \) can only be 2:1 or birational. If any of \( q_1^6, q_3^6, q_5^6 \) is 2:1, then
\( p_6 \) is birational, so the proof is concluded. If all maps \( \{q_i^6\} \) are birational, then the
vertices 1, 3, and 5 are not cocircular by lemma 4.16. Hence, in this case \( p_6 \) must
be birational, because it is not possible on the sphere that two distinct points are
equidistant from three non-co-circular points, which would be the case for the two
elements in a fiber of \( p_6 \) if this map were 2:1.
\( \square \)

We proceed now by classifying the possible degrees of the maps \( \{q_i^k\} \) and \( \{r_i^{k\ell}\} \).

**Lemma 4.18.** Suppose that \( \mathcal{C} \) is a proper motion of \( K_{3,3} \). The numbers \( \{\deg r_i^{k\ell}\} \)
are completely determined by the numbers \( \{\deg q_i^k\} \).

**Proof.** Consider the \( K_{3,2} \) subgraph of \( K_{3,3} \) spanned by the vertices 1, 2, 3, 4, 5: there
are exactly three maps from the motion \( C_6 \) to a motion of a subgraph of this \( K_{3,2} \)
isomorphic to \( K_{2,2} \), namely:

\[
\begin{align*}
q_1^6 &: C_6 \longrightarrow C_{16}, \\
q_3^6 &: C_6 \longrightarrow C_{36}, \\
q_5^6 &: C_6 \longrightarrow C_{56}.
\end{align*}
\]
For each \( q \)-map, say \( q_1^6 \), we consider two \( r \)-maps, namely \( r_1^{16} \) and \( r_3^{16} \). We can
arrange all these maps in a hexagonal diagram:
We can now start with the proof. Let us first suppose that not all $q$-maps have the same degree. Then without loss of generality we can suppose that $\deg q_{36} = 1$, $\deg q_{356} = 2$. This implies that $\deg r_{36} = 2$ and $\deg r_{56} = 1$. We know that $\deg r_{36} = \deg r_{356}$, and similarly for the other pairs of $r$-maps (see remark 4.11). If we denote by $x$ the number $\deg q_{36}^6$, and by $y$ the number $\deg r_{16} = \deg r_{156}$, we find the equation $xy = 2$ from the commutativity of the diagram. Hence the missing degrees of the $r$-maps are determined by the degrees of the $q$-maps.

Let us now suppose that all the $q$-maps have the same degree. Then we know from remark 4.11 that all the $r$-maps have the same degree, but we do not know whether it equals the degree of the $q$-maps or not. We claim that in this case the degrees of the $q$- and $r$-maps are the same. Assume that the $r$-maps have degree two. Then, the $q$-maps cannot be all birational by lemma 4.15, namely they all must have degree two. On the other hand, assume that all $r$-maps are birational. Looking at the preimages of the elements of $C_{356}$, the positions of vertices 3 and 5 are uniquely determined by 1, 2 and 4. Therefore, $q_{36}^6$ and $q_{56}^6$ are birational. Similarly for the map $q_{16}^6$.

**Remark 4.19.** As a consequence of the proof of lemma 4.18 we have that, for a fixed $k \in \{1, \ldots, 6\}$:

$$\deg q_{36}^k = \deg q_{56}^k = \deg q_{156}^k = \vartheta(\deg q_{16}^k, \deg q_{36}^k, \deg q_{56}^k),$$

where

$$\vartheta(d_1, d_2, d_3) := \begin{cases} 
1 & \text{if } d_1 = d_2 = d_3 = 1, \\
4 & \text{if } d_1 = d_2 = d_3 = 2, \\
2 & \text{otherwise,}
\end{cases}$$

and $i, j, \ell$ have parity different from $k$.

### 4.3. Degree tables and type tables.

Lemma 4.18 implies that, if we suppose that we are not in the Dixon 1 case (so that all maps $p_i$ are birational by proposition 4.17), then the degrees of the maps $\{p_{ij}\}$ determine the degrees of all the other maps. Hence, we have $2^9$ possible situations for the degrees of all maps determined by a configuration curve of $K_{3,3,3}$. We can fit the numbers $\{\deg p_{ij}\}$ into a table, which we call a degree table:

![Diagram of Configuration Curves]

**Degree table**
Notice that the set of \(2^9\) possibilities admits several symmetries: we can permute the indices 1, 3, 5 and 2, 4, 6 obtaining an isomorphic configuration, and we can also swap the roles of the odd and even indices. Hence we obtain an action by the group \((S_3 \times S_3) \rtimes Z_2\) on the set of possible degree tables.

**Lemma 4.20.** The group action has 26 orbits.

*Proof.* To each degree table we associate a subgraph of \(K_{3,3}\) as follows: two vertices \(i\) and \(j\) are connected by an edge if and only if there is a 2 in the corresponding entry of the table. In this way we see that the group of symmetries of the degree tables gets translated into the group of automorphisms of \(K_{3,3}\). Therefore, the number of orbits is the number of subgraphs of \(K_{3,3}\) up to automorphisms, which is 26, as one can see, for example, from a direct computation using computer algebra tools. □

By recalling that we are supposing that the maps \(\{p_i\}\) are birational and using remark 4.19, we get the following result.

**Lemma 4.21.** Consider the column \(w_i\) of a degree table corresponding to a label \(i \in \{2, 4, 6\}\). Then \(\vartheta(w_i)\) is the degree of the maps \(p_{i13}, p_{i15}, p_{i35}\) (see definition 4.3). A similar statement holds for the rows of a degree table.

Taking into account lemma 4.21, we can enhance the degree tables by the degrees of the maps \(\{p_{ijk}\}\). For example, we have a table of the form:

\[
\begin{array}{ccc|c}
2 & 4 & 6 & 4 \\
1 & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot \\
5 & \cdot & \cdot & \cdot \\
\end{array}
\]

The crucial fact is that, once a degree table is given, then it prescribes the types of all quadrilaterals in \(K_{3,3}\), with only one ambiguity, as the following proposition clarifies.

**Proposition 4.22.** Given an enhanced degree table, the types of the 9 subgraphs of \(K_{3,3}\) isomorphic to \(K_{2,2}\) are determined as follows. For every entry of the table, associate a type according to the following rules:
The table that we obtain listing the types of the 9 quadrilaterals is called a type table.

Proof. The proof follows from a direct inspection of the classification of quadrilaterals in definition 4.9.

However, only few configurations of quadrilaterals are allowed.

**Proposition 4.23.** Up to permutations, the only allowed rows of a type table are the following:

\[
gg^*, tte, ooo, ool, oog,  
\]

where \( * \) denotes any symbol in \{g, o, e, r, l\}. Up to permutations, the only allowed columns of a type table are the following:

\[
gg^*, tte, eee, eel, eeg,  
\]

where \( * \) denotes any symbol in \{g, o, e, r, l\}.

Proof. We only prove the statement about the columns of a type table; the statement about the rows follows analogously. Suppose that both 1234 (\( _\) ) and 2345 (\( _\) ) are \( e \). Then the situation is as follows, where the edges are labeled by the value of the function \( \delta_{S^2} \):
with $\alpha, \beta \in \{1, -1\}$. We now try to determine the type of $1245$ ($\_\_\_\_\_\_$). We distinguish two cases. Suppose that $\alpha = \beta$. If $c = a$, then $1245$ is $l$; if $c \neq a$, then $1245$ is $e$. Suppose now that $\alpha \neq \beta$. This forces $b = 0$, and so we must have $a \neq 0$ and $c \neq 0$, therefore $1245$ is $g$. These three cases are depicted here:

We now show that if $1234$ is $e$ and $2345$ is $l$, then $1245$ is $e$. The situation is:

with $\alpha, \beta, \gamma, \delta \in \{1, -1\}$. Since $b$ cannot be 0, we have $\alpha = \beta$. If $\gamma = \delta$, then the lozenge condition forces $\alpha = 1$, so $1245$ is $e$. If $\gamma = -\delta$, then the lozenge condition forces $\alpha = -1$, so again $1245$ is $e$. We now exclude that $1234$ is $e$ and $2345$ is $o$. In this case:
Here we must have $b \neq 0$. This is not possible since 2345 would be actually a lozenge for any $\alpha, \beta \in \{1, -1\}$. In a similar way we exclude the case where 1234 is $\tau$ and 2345 is $\tau$, and 1234 is $l$ and 2345 is $r$.

We now exclude that 1234 is $o$ and 2345 is $o$. In this case:

By swapping 3 and 5 with their antipodes, we can suppose $\alpha = \beta = 1$. However, in this case we get three points that are at the same distance from two other ones, and this forces vertices 2 and 4 to be antipodal, which is not allowed. Similarly, we can exclude the cases where 1234 is $o$ and 2345 is $l$, 1234 and 2345 are both $l$.

We now show that if both 1234 and 2345 are $r$, then 1245 must be $o$. We have:

By swapping 3 and 5 with their antipodes we can suppose that $\alpha = \beta = 1$, so 1245 is $o$. With the same technique we can prove that if 1234 is $o$ and 2345 is $r$, then 1245 must be $r$. 
To conclude the proof, one notices that if 1245 is of any of the five types \( g, o, e, r, \) or \( l \), then by picking any two general edge lengths for the edges 23 and 34, one can construct an instance where both 1234 and 2345 are \( g \). □

By inspecting the 26 different cases for the degree table in the light of proposition 4.23, one finds that the only allowed type tables are the following four:

\[
\begin{array}{c|c|c|c}
2 & 2 & 2 & 4 \\
2 & 2 & 2 & 4 \\
2 & 2 & 2 & 4 \\
4 & 4 & 4 & 4 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
r & r & e \\
r & r & e \\
o & o & l \\
o & o & l \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
2 & 1 & 1 & 2 \\
1 & 2 & 1 & 2 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
r & g & g \\
r & g & g \\
g & g & r \\
g & g & r \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
\end{array}
\Rightarrow
\begin{array}{c|c|c|c}
\emptyset & \emptyset & e \\
\emptyset & \emptyset & e \\
o & o & g \\
o & o & g \\
\end{array}
\]

We analyze the four possible cases one by one.

We introduce some notation to deal with nontrivial cuts of \( K_{3,3} \), namely cuts that induce surjective colorings. In order to make the notation lighter, we will always suppose that the cuts are in “normal form”, as described by the following lemma.

**Lemma 4.24.** Let \((I, J)\) be a nontrivial cut of \( K_{3,3} \), namely the coloring \( \varepsilon_{I,J} \) is surjective. Then, up to swapping \( I \) and \( J \), and the \( P' \)s and \( Q' \)s, we can always suppose that \( I \) is of the following form:

\[
\begin{array}{c|c|c|c|c|c|c}
P_i, Q_i, & P_{j_1}, Q_{j_1}, & P_{j_2}, Q_{j_2}, & P_{j_3}, Q_{j_3}, \\
\end{array}
\]

where \( i \in \{1, \ldots, 6\} \) and \( j_1 < j_2 < j_3 \) are the three numbers in \( \{1, \ldots, 6\} \) of parity different from the one of \( i \).

**Proof.** By assumption, the coloring \( \varepsilon_{I,J} \) is NAP. By symmetry, we can suppose that the three edges incident with vertex 1 are red and all other edges are blue. This implies that \( P_1, Q_1 \in I \), since otherwise we would have, in particular, \( P_2, Q_2 \in I \), and this is not compatible with the fact that \( \{2, 3\} \) is blue. The same argument shows that for \( j \in \{2, 4, 6\} \), either \( P_j \in I \) or \( Q_j \in I \). This concludes the proof. □

**Notation.** According to lemma 4.24, we denote by \((i, T_1 T_2 T_3)\), where \( T_k \in \{P, Q\} \), the cut for which \( I = \{P_i, Q_i, (T_{j_1})_{j_1}, (T_{j_2})_{j_2}, (T_{j_3})_{j_3}\} \). For example, the cut \((3, PQP)\)
is the one in which $I = \{P_3, Q_3, P_2, Q_4, P_6\}$, while $(2, PPQ)$ is the one in which $I = \{P_2, Q_2, P_1, P_3, Q_5\}$.

**Definition 4.25.** As we did for motions of $K_{2,2}$, given a proper motion $\mathcal{C}$ of $K_{3,3}$ and a cut $(i, T_1 T_2 T_3)$, we define the number $\mu(i, T_1 T_2 T_3)$ as the degree of the restriction of $\mathcal{C}$ to the divisor corresponding to $(i, T_1 T_2 T_3)$. Notice that, if $(i, V_1 V_2 V_3)$ is the cut obtained from $(i, T_1 T_2 T_3)$ by swapping $P$’s and $Q$’s, then $\overline{D_{i,T_1 T_2 T_3}}$ is complex conjugated to $D_{i,V_1 V_2 V_3}$; since $\mathcal{C}$ is real, we have that $\mu(i, T_1 T_2 T_3) = \mu(i, V_1 V_2 V_3)$.

**Proposition 4.26.** Case (1) cannot happen.

**Proof.** We compute the pullbacks under the maps $p_{ij}$ of the divisors $D_{om}$ on each of the subgraphs isomorphic to $K_{2,2}$. For example, the pullback of $D_{om}$ under $p_{56}$ consists of the four divisors $D_{1,PPP}, D_{1,PPP}, D_{3,PPP}$ and $D_{3,PPP}$. By restricting this pullback to the proper motion $\mathcal{C}$ and taking the degree, we get the equation

$$\mu(1, PPP) + \mu(1, QPP) + \mu(3, PQP) + \mu(3, QPP) = \mu^{T_{om}}(T_{om}) \cdot \deg p_{56} = 2.$$ 

Summing all these equations gives

$$\sum_{i \in \{1,3,5\}} 4\mu(i, PPQ) + 4\mu(i, PQP) + 4\mu(i, QPP) = 18,$$

which does not have integer solutions, so this case cannot happen. 

**Lemma 4.27.** A rhomboid cannot have orthogonal diagonals.

**Proof.** For a contradiction, let us suppose that 1234 is a rhomboid with orthogonal diagonals. Then we can take coordinates for the vertices of the rhomboid as follows:

- $\left( x_1, 0, z_1 \right)$ vertex 1
- $\left( x_2, y_2, 0 \right)$ vertex 2
- $\left( x_3, 0, z_3 \right)$ vertex 3
- $\left( x_4, y_4, 0 \right)$ vertex 4

By assumption, we have

- $x_2 x_3 = a$,
- $x_1 x_2 = b$,
- $x_3 x_4 = \alpha a$,
- $x_1 x_4 = \alpha b$,

with $\alpha \in \{1, -1\}$. From this it follows that $a^2 = b^2$, hence the contradiction.

**Proposition 4.28.** Case (2) cannot happen.

**Proof.** The type table implies that 1234 is a lozenge, 1245 is an even deltoid, and 1346 is an odd deltoid. This forces the vertices 1, 3, and 5 to be cocircular, and similarly for 2, 4, and 6; moreover the two great circles are orthogonal. However, by assumption 2356 is a rhomboid, which would have orthogonal diagonals. This contradicts lemma 4.27.

**Proposition 4.29.** Case (3) is an example of a Dixon 2 motion.

**Proof.** In this case we have three rhomboids, namely 1234, 1256 and 3456, and the motions of the other $K_{2,2}$-subgraphs are general ones. At first sight, we would need to analyze $4^3 = 64$ cases, since each of the rhomboids may have type from 1 to 4. We employ some automorphisms of $\mathcal{A}_{0,12}$ to decrease the total number of cases.
to analyze. For a given \( i \in \{1, \ldots, 6\} \), consider the automorphism \( \sigma_i \) of \( \mathcal{M}_{0,12} \) that swaps \( P_i \) and \( Q_i \) and fixes the other marked points. This automorphism does not, in general, preserve the proper motion \( \mathcal{C} \) of \( K_{3,3} \) we fixed once and for all. However, it preserves the fact that 1234, 1256, and 3456 are rhomboids, as well as the types of the other quadrilaterals. Hence we can employ it to simplify our setting. For example, consider the automorphism \( \sigma_1 \) swapping \( P_1 \leftrightarrow Q_1 \) and its action on \( C_{06} \subseteq \mathcal{M}_{0,8} \). A direct computation shows that the divisors \( D_{ou} \) and \( D_{om} \) are preserved, while \( D_{em} \) is mapped to \( D_{eu} \). Thus the type of the rhomboid 1234 is changed by \( \sigma_1 \) as follows:

\[
\text{Type 1} \leftrightarrow \text{Type 3}, \quad \text{Type 2} \leftrightarrow \text{Type 4}.
\]

Instead, if we use the automorphism \( \sigma_2 \) swapping \( P_2 \leftrightarrow Q_2 \), we get that the type of 1234 changes as follows:

\[
\text{Type 1} \leftrightarrow \text{Type 2}, \quad \text{Type 3} \leftrightarrow \text{Type 4}.
\]

Therefore, here we have an action of the group \( G = \langle \sigma_1, \ldots, \sigma_6 \rangle \cong (\mathbb{Z}_2)^6 \) on the set of triples of types of the rhomboids 1234, 1256, and 3456. We then use this action to simplify the type of the three rhomboids as much as possible. We see that the action is not transitive, since every triple admits as stabilizer the subgroup \( \langle \text{id}, \sigma_1 \circ \sigma_3 \circ \sigma_5, \sigma_2 \circ \sigma_4 \circ \sigma_6 \rangle \), which has cardinality 4. There are then four triples of types of rhomboids that are not equivalent under \( G \), and these are:

\[
\begin{align*}
(\text{Type 1}, \text{Type 1}, \text{Type 1}), & \quad (\text{Type 2}, \text{Type 2}, \text{Type 2}), \\
(\text{Type 3}, \text{Type 3}, \text{Type 3}), & \quad (\text{Type 4}, \text{Type 4}, \text{Type 4}).
\end{align*}
\]

We want to show that only the situation where all the three rhomboids are of Type 1 can happen. To do so, we first consider the equations for the numbers \( \mu_{T,T_1T_2T_3} \) coming from taking the pullbacks under the maps \( p_{ij} \) of the divisors \( D_{ou} \) and \( D_{om} \) on the subgraphs of the three rhomboids. For example, the pullback \( p_{56}^* (D_{ou}) \) equals the sum of the four divisors \( D_1, P_{PQ}, D_1, P_{PP}, D_3, Q_{QP}, \) and \( D_3, Q_{QQ} \), and we get the equation:

\[
\mu(1, PPQ) + \mu(1, PPP) + \mu(3, QQP) + \mu(3, QQQ) = \mu^{56}(T_{ou}) \cdot \deg p_{56}.
\]

When all the three rhomboids are of Type 1 we have \( \mu(T_{ou}) = 0 \) and \( \mu(T_{om}) = 1 \), and so we get the system of equations:

\[
\begin{align*}
\mu(1, PPQ) &= \mu(1, PPP) + \mu(3, QQQ) = \mu(3, QQP) = \mu(3, QQQ) = 0, \\
\mu(1, PQQ) &= \mu(1, PPP) + \mu(5, QQQ) = \mu(5, QPP) = 0, \\
\mu(3, PQQ) &= \mu(3, PPP) + \mu(5, QQQ) = \mu(5, PQQ) = 0,
\end{align*}
\]

and

\[
\begin{align*}
\mu(1, PQQ) + \mu(1, PQP) + \mu(3, QPQ) + \mu(3, QPP) &= 2, \\
\mu(1, PPQ) + \mu(1, PQP) + \mu(5, QPQ) + \mu(5, QPP) &= 2, \\
\mu(3, QPQ) + \mu(3, PPQ) + \mu(5, QQP) + \mu(5, PQP) &= 2.
\end{align*}
\]
Recalling from definition 4.25 that we have equalities \( \mu(1,PPQ) = \mu(1,QQP) \) and similar ones, we see that these equations admit the unique solution
\[
\mu(1,PQQ) = \mu(3,QPQ) = \mu(5,QQP) = 1
\]
(and all other non-conjugated quantities are zero). One can then check that this solution satisfies also all similar equations coming from considering other subgraphs in \( K_{3,3} \) that are isomorphic to \( K_{2,2} \) and that are not rhomboids. Instead, if all the three rhomboids are of Type 2, the situation is different. In that case we have \( \mu(T_{on}) = 1 \) and \( \mu(T_{om}) = 0 \), and the system of equations coming from the three rhomboids has solution
\[
\mu(1,PPP) = \mu(3,PPP) = \mu(5,PPP) = 1.
\]
This solution, however, does not satisfy all the equations coming from the other subgraphs. For example, if we consider the subgraph \( 1236 \), we have that the pull-back \( p_{45}^*(D_{on}) \) equals the sum of the four divisors \( D_{1,PPQ}, D_{1,PQQ}, D_{3,QPQ}, \) and \( D_{3,QQP} \). Hence the subgraph determines the equation
\[
\mu(1,PPQ) + \mu(1,PQQ) + \mu(3,PPP) + \mu(3,QQP) = \mu^{45}(T_{om}) \cdot \deg p_{45} = 1,
\]
which is not satisfied by the previous solution. Hence the situation where all the three rhomboids are of Type 2 cannot occur. The same argument shows that neither Type 4 can occur. To exclude Type 3, instead of taking the divisors \( D_{on} \) and \( D_{om} \), we consider \( D_{eu} \) and \( D_{em} \). Therefore we can suppose that all the three rhomboids are of Type 1. Then we know that there exist three rotations \( o_{1234}, o_{1256}, \) and \( o_{3456} \) of the sphere \( S^2 \), each of which is a symmetry of the corresponding rhomboid. Moreover, as we clarified in Section 4.1, these three rotations are related to automorphisms \( \tau_{56}, \tau_{34}, \) and \( \tau_{12} \) of \( \mathcal{M}_{0,12} \) that preserve the proper motions of the rhomboids. Since \( \tau_{56} \circ \tau_{34} = \tau_{12} \), the same relation holds for the rotations, and so these three rotations commute and are involutions. Thus we are in the situation of a Dixon 2 motion.

To describe the motion that we find in Case (4), let us recall a duality in spherical geometry between pairs of antipodal points and great circles. Given a pair of antipodal points, we can consider the plane orthogonal to the line connecting these two points: we say that the great circle determined by this plane on \( S^2 \) is the dual to the original pair of points. Conversely, the axis of a great circle determines a pair of antipodal points.

**Proposition 4.30.** In Case (4), there exists a proper motion of \( K_{3,3} \) described as follows. We consider a quadrilateral 1234 with three sides of equal spherical length \( (1 - a)/2 \), and the fourth of length \( (1 + a)/2 \). This quadrilateral has the property that, while it moves, the angle between its diagonals stays the same. We pick as realizations of 5 and 6 the duals of the diagonals of 1234. By duality, the spherical distance \( (1 - e)/2 \) between the realizations of 5 and 6 is the cosine of the angle between the diagonals of 1234. We call this motion a constant diagonal angle motion.
Proof. The last row and the last column of the type table imply that certain values of the lengths $\delta_{S^2}$ must be zero. The two even deltoids in the last column, namely 2345 and 1245, imply that we have the following situation, where the edges are labeled by the value of the function $\delta_{S^2}$, and $\alpha, \beta \in \{1, -1\}$:

By swapping the realization of vertex 4 with its antipode, we can suppose without loss of generality that $\delta_{S^2}(1, 4) = a$, so $\alpha = 1$ in the diagram. This forces $\delta_{S^2}(3, 4) = -b$, so $\beta = -1$ in the diagram, because otherwise 1234 would be $e$, while from the table we know that it is $g$. Hence $c = 0$.

A similar argument, using the last row of the type table, implies that $\delta_{S^2}(1, 6) = \delta_{S^2}(3, 6) = 0$. Moreover, since 1346 is $a$, we have $\delta_{S^2}(1, 4) = \pm \delta_{S^2}(3, 4)$, so $a = \pm b$.

By swapping the realization of vertex 3 we can assume that $a = b$. The situation for the whole $K_{3,3}$ is then as follows, where $\delta_{S^2}(5, 6) = e$:

Due to the general type of several quadrilaterals, we know that $e \neq a$ and $e \neq -a$.

The fact that this kind of realizations of $K_{3,3}$ is mobile and that, during the motion, the angle between the diagonals of 1234 stays constant (namely, the distance $e$ stays constant), will be proven via symbolic computation techniques. In particular, the fact that $e$ is constant follows from the fact that $a$ and $e$ satisfy an algebraic relation.

We challenge the reader to find a “geometric” proof (we could not find one).

We set up a symbolic computation by Gröbner bases that will give us the result.

We assume that $R_1, \ldots, R_6 \in S^2$ are the realizations on the sphere of the vertices of $K_{3,3}$. Without loss of generality, we can assume that $R_1 = (1, 0, 0)$ and $R_6 = (0, 1, 0)$, while we keep symbolic coordinates $(x_i, y_i, z_i)$ for all other points.
R_2,\ldots,R_5$. These 12 variables satisfy the sphere equations
\[ x_i^2 + y_i^2 + z_i^2 = 1 \quad \text{for } i \in \{2, 3, 4, 5\}. \]

More equations come from the pieces of information we have about the lengths \( \delta_{S^2} \):
from \( \delta_{S^2}(3, 6) = 0 \) we derive \( y_3 = 0 \), and from \( \delta_{S^2}(5, 6) = e \) we get \( y_5 = e \). Similarly, we obtain \( x_2 = x_4 = a \). We have four other equations:
\[
\begin{align*}
    x_4x_5 + y_4y_5 + z_4z_5 &= 0, \\
    x_2x_5 + y_2y_5 + z_2z_5 &= 0, \\
    x_3x_4 + y_3y_4 + z_3z_4 &= -a, \\
    x_2x_3 + y_2y_3 + z_2z_3 &= a.
\end{align*}
\]

Altogether, we have 12 equations in 14 variables. Via Gröbner bases we see that their solution space \( B \subseteq \mathbb{C}^{14} \) has dimension 2. If we eliminate the variables \( \{x_i, y_i, z_i\}_{i=2}^5 \) from these equations, we obtain the relation \( a^3e^2 + a^3 - ae^2 \) between the parameters \( a \) and \( e \), which defines a curve \( A \subseteq \mathbb{C}^2 \). Hence we have a map \( B \to A \) which, by dimension reasons, must have infinite fibers. This implies that there are mobile instances of \( K_{3,3} \) in this case. \(\square\)

**Example 4.31.** If, in the construction provided by proposition 4.30, we set \( a = 3/5 \) and \( e = 3/4 \), we can take the following (radical) parametrization of the constant diagonal angle motion:
\[
\begin{align*}
    R_1 &= (1, 0, 0), \\
    R_i &= (x_i(t), y_i(t), z_i(t)) \quad \text{for } i \in \{2, 3, 4, 5\}, \\
    R_6 &= (0, 1, 0),
\end{align*}
\]

where
\[
\begin{align*}
    y_3(t) &= 0, & y_5(t) &= 3/4, \\
    x_2(t) &= 3/5, & x_4(t) &= 3/5, \\
    x_3(t) &= \frac{2t}{t^2 + 1}, & z_3(t) &= \frac{t^2 - 1}{t^2 + 1}, \\
    z_2(t) &= \frac{3}{5} \cdot \frac{t - 1}{t + 1}, & z_4(t) &= -\frac{3}{5} \cdot \frac{t + 1}{t - 1}.
\end{align*}
\]
\[ y_2(t) = \pm \frac{\sqrt{(t+7)(7t+1)}}{5t+5}, \]
\[ z_5(t) = -5y_2t^2 + 5y_2 \pm \sqrt{25t^4y_2^2 - 50t^2y_2^2 - 72t^3 + 25y_2^2 - 72}, \]
\[ x_5(t) = \frac{t(16z_5^2 + 9)}{8z_5(t^2 - 1)}, \]
\[ y_4(t) = y_2 + \frac{8(t^2 + 1)z_5}{5(t^2 - 1)}. \]

Figure 12 shows some realizations of \( K_{3,3} \) during this motion.

Now that the analysis is concluded, we sum up the results we have obtained in the following theorem:

**Theorem 4.32.** All the possible (real) motions of \( K_{3,3} \) on the sphere, for which no two vertices coincide or are antipodal, are the following:

- spherical Dixon 1;
- spherical Dixon 2;
- constant diagonal angle motion.

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