LMI REPRESENTATIONS OF CONVEX SEMIALGEBRAIC SETS
AND DETERMINANTAL REPRESENTATIONS OF ALGEBRAIC
HYPERSURFACES: PAST, PRESENT, AND FUTURE

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To Bill Helton, on the occasion of his 65th birthday

ABSTRACT. 10 years ago or so Bill Helton introduced me to some mathematical problems arising from semidefinite programming. This paper is a partial account of what was and what is happening with one of these problems, including many open questions and some new results.

1. Introduction

Semidefinite programming (SDP) is probably the most important new development in optimization in the last two decades. The (primal) semidefinite program is to minimize an affine linear functional $\ell$ on $\mathbb{R}^d$ subject to a linear matrix inequality (LMI) constraint

$$A_0 + x_1 A_1 + \cdots + x_d A_d \geq 0;$$

here $A_0, A_1, \ldots, A_d \in \mathbb{S}^n_R$ (real symmetric $n \times n$ matrices) for some $n$ and $Y \geq 0$ means that $Y \in \mathbb{S}^n_R$ is positive semidefinite (has nonnegative eigenvalues or equivalently satisfies $y^\top Y y \geq 0$ for all $y \in \mathbb{R}^n$). This can be solved efficiently, both theoretically (finding an approximate solution with a given accuracy $\epsilon$ in a time that is polynomial in $\log(1/\epsilon)$ and in the input size of the problem) and in many concrete situations, using interior point methods. Notice that semidefinite programming is a far reaching extension of linear programming (LP) which corresponds to the case when the real symmetric matrices $A_0, A_1, \ldots, A_d$ commute (i.e., are simultaneously diagonalizable). The literature on the subject is quite vast, and we only mention the pioneering book [40], the surveys [52] and [39], and the book [51] for applications to systems and control.

One very basic mathematical question is which convex sets arise as feasibility sets for SDP? In other words, given a convex set $C$, do there exist $A_0, A_1, \ldots, A_d \in \mathbb{S}^n_R$ for some $n$ such that

$$C = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : A_0 + x_1 A_1 + \cdots + x_d A_d \geq 0 \}?$$

We refer to (1.1) as a LMI representation of $C$. Sets having a LMI representation are also called spectrahedra. This notion was introduced and studied in [49], and

\footnote{We can also consider a (complex) self-adjoint LMI representation of $C$, meaning that $A_0, A_1, \ldots, A_d \in \mathbb{H}^n_C$ (complex hermitian $n \times n$ matrices) for some $n$. If $A = B + iC \in \mathbb{H}^n_C$, with $B, C \in \mathbb{R}^{n \times n}$, and we set $\tilde{A} = \begin{bmatrix} B & -C \\ C & B \end{bmatrix} \in \mathbb{S}^{2n \times 2n}$, then $A \geq 0$ if and only $\tilde{A} \geq 0$ and $\det \tilde{A} = (\det A)^2$. So a self-adjoint LMI representation gives a real symmetric LMI representation as defined in the main text with the size of matrices doubled and the determinant of the linear matrix polynomial squared, see [49 Section 1.4] and [49 Lemma 2.14].}
the above question — which convex sets admit a LMI representation, i.e., are spectrahedra — was formally posed in [45]. A complete answer for $d = 2$ was obtained in [28], though there are still outstanding computational questions, see [29, 40, 47]; for $d > 2$, no answer is known, though the recent results of [6, 43, 42] shed some additional light on the problem. It is the purpose of this paper to survey some aspects of the current state of the affairs.

Since a real symmetric matrix is positive semidefinite if and only if all of its principal minors are nonnegative, the set on the right-hand side of (1.1) coincides with the set where all the principal minors of $A_0 + x_1 A_1 + \cdots + x_d A_d$ are nonnegative. Therefore if a convex set $C$ admits a LMI representation then $C$ is a basic closed semialgebraic set (i.e., a set defined by finitely many nonstrict polynomial inequalities). However, as shown in [28], $C$ is in fact much more special: it is a rigidly convex algebraic interior, i.e., an algebraic interior whose minimal defining polynomial satisfies the real zero (RZ) condition with respect to any point in the interior of $C$. Furthermore, LMI representations are (essentially) positive real symmetric determinantal representations of certain multiples of the minimal defining polynomial of $C$. This reduces the question of the existence (and a construction) of LMI representations to an old problem of algebraic geometry — we only mention here the classical paper [12] and refer to [5], [13, Chapter 4], and [32] for a detailed bibliography — but with two additional twists: first, we require positivity; second, there is a freedom provided by allowing multiples of the given polynomial.

This paper is organized as follows. In Section 2 we define rigidly convex sets and RZ polynomials, and explain why LMI representations are determinantal representations. In Section 3 we discuss some of what is currently known and unknown about determinantal representations, with a special emphasis on positive real symmetric determinantal representations. In Section 4 we review some of the ways to (re)construct a determinantal representation starting from its kernel sheaf, especially the construction of the adjoint matrix of a determinantal representation that goes back to [12] and was further developed in [53, 3, 32]. In Section 5 we show how this construction yields positive self-adjoint determinantal representations in the case $d = 2$ by using a RZ polynomial that interlaces the given RZ polynomial. This provides an alternative proof of the main result of [28] (in a slightly weaker form since we obtain a representation that is self-adjoint rather than real symmetric) which is constructive algebraic in that it avoids the use of theta functions.

We have concentrated in this paper on the non-homogenous setting (convex sets) rather than on the homogeneous setting (convex cones). In the homogeneous setting, RZ polynomials correspond to hyperbolic polynomials and rigidly convex algebraic interiors correspond to their hyperbolicity cones, see, e.g., [17, 18, 36, 44, 21, 7, 50]. Theorem 5.1 then provides a solution the Lax conjecture concerning homogeneous hyperbolic polynomials in three variables, see [53], whereas Conjecture 3.3, which may be called the generalized Lax conjecture, states that any hyperbolicity cone is a semidefinite slice, i.e., equals the intersection of the cone of positive semidefinite matrices with a linear subspace.

Finally, the LMI representation problem considered here is but one of the several important problems of this kind arising from SDP. Other major problems have to do with lifted LMI representations (see [35, 25, 26]) and with the free noncommutative setting (see [24, 23]).
Acknowledgments. Apart from my joint work with Bill Helton, a lot of what is
described here is based on earlier joint work with Joe Ball, as well as on more recent
collaboration with Dmitry Kerner. It is a pleasure to thank Didier Henrion, Tim
Netzer, Daniel Plaumann, and Markus Schweighofer for many useful discussions.

2. From LMI Representations of Convex Sets to Determinantal
Representations of Polynomials

2.1. A closed set \( C \) in \( \mathbb{R}^d \) is called an algebraic interior [28, Section 2.2] if there is
a polynomial \( p \in \mathbb{R}[x_1, \ldots, x_d] \) such that \( C \) equals the closure of a connected
component of

\[
\{ x \in \mathbb{R}^d : p(x) > 0 \}.
\]

In other words, there is a \( p \in \mathbb{R}[x_1, \ldots, x_d] \) which vanishes on the boundary \( \partial C \) of
\( C \) and such that \( \{ x \in C : p(x) > 0 \} \) is connected with closure equal to \( C \). (Notice
that in general \( p \) may vanish also at some points in the interior of \( C \); for example,
look at \( p(x_1, x_2) = x_2^2 - x_1^2(x_1 - 1) \).) We call \( p \) a defining polynomial of \( C \). It is
not hard to show that if \( C \) is an algebraic interior then a minimal degree defining
polynomial \( p \) of \( C \) is unique (up to a multiplication by a positive constant); we
call it a minimal defining polynomial of \( C \), and it is simply a reduced (i.e., without
multiple irreducible factors) polynomial such that the real affine hypersurface

\[
V_p(\mathbb{R}) = \{ x \in \mathbb{R}^d : p(x) = 0 \}
\]

equals the Zariski closure \( \overline{\partial C}^{\text{zar}} \) of the boundary \( \partial C \) in \( \mathbb{R}^d \) (normalized to be positive
at an interior point of \( C \)). Any other defining polynomial \( q \) of \( C \) is given by \( q = ph \)
where \( h \) is an arbitrary polynomial which is strictly positive on a dense connected
subset of \( C \). An algebraic interior is a semialgebraic set (i.e., a set defined by a
finite boolean combination of polynomial inequalities) since it is the closure of a
connected component of a semialgebraic set.

Let now \( C \) be a convex set in \( \mathbb{R}^d \) that admits a LMI representation [11,1]. We
will assume that \( \text{Int} C \neq \emptyset \); it turns out that by restricting the LMI representation
(i.e., the matrices \( A_0, A_1, \ldots, A_d \)) to a subspace of \( \mathbb{R}^n \), one can assume without
loss of generality that \( A_0 + x_1 A_1 + \cdots + x_d A_d > 0 \) for one and then every point
of \( \text{Int} C \) (\( Y > 0 \) means that \( Y \in \mathbb{S}^{n\times n} \) is positive definite, i.e., \( Y \) has strictly
positive eigenvalues or equivalently satisfies \( y^\top Y y > 0 \) for all \( y \in \mathbb{R}^n, y \neq 0 \)). It is
then easy to see that \( C \) is an algebraic interior with defining polynomial \( \det(A_0 + 
A_1 + \cdots + x_d A_d) \). Conversely, if \( C \) is an algebraic interior with defining polynomial
\( \det(A_0 + x_1 A_1 + \cdots + x_d A_d) \), and \( A_0 + x_1 A_1 + \cdots + x_d A_d > 0 \) for one point of \( \text{Int} C \),
then it follows easily that \( [11,1] \) is a LMI representation of \( C \). (See [28, Section 2.3]
for details.)

Let \( q(x) = \det(A_0 + x_1 A_1 + \cdots + x_d A_d) \), let \( x^0 = (x^0_1, \ldots, x^0_d) \in \text{Int} C \), and let
us normalize the LMI representation by \( A_0 + x^0_1 A_1 + \cdots + x^0_d A_d = I \). We restrict
the polynomial \( q \) to a straight line through \( x^0 \), i.e., for any \( x \in \mathbb{R}^d \) we consider the
univariate polynomial \( q_x(t) = q(x^0 + tx) \). Because of our normalization, we can write

\[
q_x(t) = \det(I + t(x_1 A_1 + \cdots + x_d A_d)),
\]

and since all the eigenvalues of the real symmetric matrix \( x_1 A_1 + \cdots + x_d A_d \) are
real, we conclude that \( q_x \in \mathbb{R}[t] \) has only real zeroes.

A polynomial \( p \in \mathbb{R}[x_1, \ldots, x_d] \) is said to satisfy the real zero (RZ) condition
with respect to \( x^0 \in \mathbb{R}^d \), or to be a RZ_{x^0} polynomial, if for all \( x \in \mathbb{R}^d \) the univariate
polynomial \( p_x(t) = p(x^0 + tx) \) has only real zeroes. It is clear that a divisor of a \( RZ_{x^0} \) polynomial is again a \( RZ_{x^0} \) polynomial. We have thus arrived at the following result of [28].

**Theorem 2.1.** If a convex set \( C \) with \( x^0 \in \text{Int} \, C \) admits a LMI representation, then \( C \) is an algebraic interior whose minimal defining polynomial \( p \) is a \( RZ_{x^0} \) polynomial. (1.1) is a LMI representation of \( C \) (that is positive definite on \( \text{Int} \, C \)) if and only if \( A_0 + x_0^0 A_1 + \cdots + x_d^0 A_d > 0 \) and
\[
\det(A_0 + x_1 A_1 + \cdots + x_d A_d) = p(x)h(x),
\]
where \( h \in \mathbb{R}[x_1, \ldots, x_d] \) satisfies \( h > 0 \) on \( \text{Int} \, C \).

2.2. The definition of a \( RZ_{x^0} \) polynomial has a simple geometric meaning ([28 Section 3]). Assume for simplicity that \( p \) is reduced (i.e., without multiple irreducible factors) of degree \( m \). Then \( p \) is a \( RZ_{x^0} \) polynomial if and only if a general straight line through \( x^0 \) in \( \mathbb{R}^d \) intersects the corresponding real affine hypersurface \( \mathcal{V}_p(\mathbb{R}) \) (see (2.1)) in \( m \) distinct points. Alternatively, every straight line through \( x^0 \) in the real projective space \( \mathbb{P}^d(\mathbb{R}) \) intersects the projective closure \( \mathcal{V}_p(\mathbb{R}) \) of \( \mathcal{V}_p(\mathbb{R}) \),
\[
\mathcal{V}_p(\mathbb{R}) = \{ [X] \in \mathbb{P}^d : P(X) = 0 \},
\]
in exactly \( m \) points counting multiplicities. Here we identify as usual the \( d \) dimensional real projective space \( \mathbb{P}^d(\mathbb{R}) \) with the union of \( \mathbb{R}^d \) and of the hyperplane at infinity \( X_0 = 0 \), so that the affine coordinates \( x = (x_1, \ldots, x_d) \) and the projective coordinates \( X = (X_0, X_1, \ldots, X_d) \) are related by \( x_1 = X_1/X_0, \ldots, x_d = X_d/X_0 \); we denote by \( [X] \in \mathbb{P}^d(\mathbb{R}) \) the point with the projective coordinates \( X \); and we let \( P \in \mathbb{R}[X_0, X_1, \ldots, X_d] \) be the homogenization of \( p \),
\[
P(X_0, X_1, \ldots, X_d) = X_0^m p(X_1/X_0, \ldots, X_d/X_0).
\]
Notice that if \( X = (1, x) \) and \( X^0 = (1, x^0) \),
\[
P(X + sx^0) = (s + 1)^m p(x_0 + (s + 1)^{-1}(x - x^0)).
\]

It turns out that if \( p \) is a \( RZ_{x^0} \) polynomial with \( p(x^0) > 0 \), and if \( x' \) belongs to the interior of the closure of the connected component of \( x^0 \) in \( \{ x \in \mathbb{R}^d : p(x) > 0 \} \), then \( p(x') > 0 \) and \( p \) is also a \( RZ_{x'} \) polynomial ([28 Section 5.3]). We call an algebraic interior \( C \) whose minimal defining polynomial satisfies the \( RZ \) condition with respect to one and then every point of \( \text{Int} \, C \) a rigidly convex algebraic interior.

As simple examples, we see that the circle \( \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} \) is a rigidly convex algebraic interior, while the “flat TV screen” \( \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} \) is not. Theorem 2.1 tells us that a necessary condition for \( C \) to admit a LMI representation is that \( C \) is a rigidly convex algebraic interior, and the size \( n \) of the matrices in a LMI representation is greater than or equal to the degree \( m \) of a minimal defining polynomial \( p \) of \( C \).

Rigidly convex algebraic interiors are always convex sets ([28 Section 5.3]). They are also basic closed semialgebraic sets, as follows ([11 Remark 2.6] following [30]). Let \( p \) be a minimal defining polynomial of a rigidly convex algebraic interior \( C \), of degree \( m \), and let \( x^0 \in \text{Int} \, C \). We set
\[
P_{x^0}(k)(X) = \frac{d^k}{ds^k} P(X + sx^0)|_{s = 0}, \quad p_{x^0}(k)(x) = P_{x^0}(k)(1, x_1, \ldots, x_d),
\]
where \( P \) is the homogenization of \( p \) (see (2.3)) and \( X^0 = (1, x^0) \); \( p_{x^0}(k) \) is called the \( k \)th Renegar derivative of \( p \) with respect to \( x^0 \). Then \( p_{x^0}(k) \) is a \( RZ_{x^0} \) polynomial.
with \( p^{(k)}_{x^0}(x^0) > 0 \) for all \( k = 1, \ldots, m - 1 \). The rigidly convex algebraic interiors \( C^{(k)} \) containing \( x^0 \) with minimal defining polynomials \( p^{(k)}_{x^0} \) (i.e., the closures of the connected components of \( x^0 \) in \( \{ x \in \mathbb{R}^d : p^{(k)}_{x^0}(x) > 0 \} \)) are increasing: \( C = C^{(0)} \subseteq C^{(1)} \subseteq \cdots \subseteq C^{(m-1)} \), and

\[
C = \{ x \in \mathbb{R}^d : p(x) \geq 0, p^{(1)}_{x^0}(x) \geq 0, \ldots, p^{(m-1)}_{x^0}(x) \geq 0 \}.
\]

RZ polynomials can be also characterized by a very simple global topology of the corresponding real projective hypersurface \( \mathcal{V}_p(\mathbb{R}) \) (see (2.2)); readers who prefer can assume that the corresponding real affine hypersurface \( \mathcal{V}_p^a(\mathbb{R}) \) is compact in \( \mathbb{R}^d \) — this implies that the degree \( m \) of \( p \) is even — and replace in the following the real projective space \( \mathbb{P}^d(\mathbb{R}) \) by the affine space \( \mathbb{R}^d \). We call \( W \subseteq \mathbb{P}^d(\mathbb{R}) \) an ovaloid if \( W \) is isotopic in \( \mathbb{P}^d(\mathbb{R}) \) to a sphere \( S \subset \mathbb{R}^m \subset \mathbb{P}^m(\mathbb{R}) \), i.e., there is a homeomorphism \( F \) of \( \mathbb{P}^d(\mathbb{R}) \) with \( F(S) = W \), and furthermore \( F \) is homotopic to the identity, i.e., there is a homeomorphism \( H \subset [0,1] \times \mathbb{P}^d(\mathbb{R}) \) such that \( H_t = H|_{\{t\} \times \mathbb{P}^d(\mathbb{R})} \) is a homeomorphism of \( \mathbb{P}^d(\mathbb{R}) \) for every \( t \), \( H_0 = \text{Id}_{\mathbb{P}^d(\mathbb{R})} \), and \( H_1 = F \). Notice that \( \mathbb{P}^d(\mathbb{R}) \setminus S \) consists of two connected components only one of which is contractible, hence the same is true of \( \mathbb{P}^d(\mathbb{R}) \setminus W \); we call the contractible component the **interior** of the ovaloid \( W \), and the non-contractible component the **exterior**. We call \( W \subseteq \mathbb{P}^d(\mathbb{R}) \) a pseudo-hyperplane if \( W \) is isotopic in \( \mathbb{P}^d(\mathbb{R}) \) to a (projective) hyperplane \( H \subseteq \mathbb{P}^d(\mathbb{R}) \). In the case \( d = 2 \) we say oval and pseudo-line instead of ovaloid and pseudo-hyperplane. We then have the following result; we refer to [28] Sections 5 and 7] for proof, discussion, and implications.

**Proposition 2.2.** Let \( p \in \mathbb{R}[x_1, \ldots, x_d] \) be reduced of degree \( m \) and assume that the corresponding real projective hypersurface \( \mathcal{V}_p(\mathbb{R}) \) is smooth. Then \( p \) satisfies RZ\(_{x^0} \) with \( p(x^0) \neq 0 \) if and only if

a. if \( m = 2k \) is even, \( \mathcal{V}_p(\mathbb{R}) \) is a disjoint union of \( k \) ovaloids \( W_1, \ldots, W_k \), with \( W_i \) contained in the interior of \( W_{i+1} \), \( i = 1, \ldots, k-1 \), and \( x^0 \) lying in the interior of \( W_1 \);

b. if \( m = 2k + 1 \) is odd, \( \mathcal{V}_p(\mathbb{R}) \) is a disjoint union of \( k \) ovaloids \( W_1, \ldots, W_k \), with \( W_i \) contained in the interior of \( W_{i+1} \), \( i = 1, \ldots, k-1 \), and \( x^0 \) lying in the interior of \( W_1 \), and a pseudo-hyperplane \( W_{k+1} \) contained in the exterior of \( W_k \).

Let us denote by \( \mathcal{I} \) the interior of \( W_1 \), let us normalize \( p \) by \( p(x^0) > 0 \), and let \( H_\infty = \{ X_0 = 0 \} \) be the hyperplane at infinity in \( \mathbb{P}^d(\mathbb{R}) \). If \( \mathcal{I} \cap H_\infty = \emptyset \), then the closure of \( \mathcal{I} \) in \( \mathbb{R}^d \) is a rigidly convex algebraic interior with a minimal defining polynomial \( p \). If \( \mathcal{I} \cap H_\infty \neq \emptyset \), then \( \mathcal{I} \setminus \mathcal{I} \cap H_\infty \) consists of two connected components, the closure of each one of them in \( \mathbb{R}^d \) being a rigidly convex algebraic interior with a minimal defining polynomial \( p \) (if \( m \) is even) or \( p \) for one component and \( -p \) for the other component (if \( m \) is odd).

### 3. Determinantal representations of polynomials: some of the known and of the unknown

3.1. The following is proved in [28] Section 5 (based on the results of [54] and [4], see also [14]).
Theorem 3.1. Let \( p \in \mathbb{R}[x_1, x_2] \) be a \( RZ_{x^0} \) polynomial of degree \( m \) with \( p(x^0) = 1 \). Then there exist \( A_0, A_1, A_2 \in \mathbb{S} \mathbb{R}^{m \times m} \) with \( A_0 + x_1^0 A_1 + x_2^0 A_2 = I \) such that

\[
\text{(3.1)} \quad \det(A_0 + x_1 A_1 + x_2 A_2) = p(x).
\]

We will review the proof of Theorem 3.1 given in [28] in Section 4 below, and then present in Section 5 an alternate proof for positive self-adjoint (rather than real symmetric) determinantal representations that avoids the transcendental machinery of Jacobian varieties and theta functions (though it still involves, to a certain extent, meromorphic differentials on a compact Riemann surface).

Theorem 3.1 tells us that a necessary and sufficient condition for \( C \subseteq \mathbb{R}^2 \) to admit a LMI representation is that \( C \) is a rigidly convex algebraic interior, and the size of the matrices in a LMI representation can be taken equal to be the degree \( m \) of a minimal defining polynomial \( p \) of \( C \).

There can be no exact analogue of Theorem 3.1 for \( d > 2 \). Indeed, we have

Proposition 3.2. A general polynomial \( p \in \mathbb{C}[x_1, \ldots, x_d] \) of degree \( m \) does not admit a determinantal representation

\[
\text{(3.2)} \quad \det(A_0 + x_1 A_1 + \cdots + x_d A_d) = p(x),
\]

with \( A_0, A_1, \ldots, A_d \in \mathbb{C}^{m \times m} \), for \( d > 3 \) and for \( d = 3, m \geq 4 \).

Since for any fixed \( x^0 \in \mathbb{R}^d \) the set of \( RZ_{x^0} \) polynomials of degree \( m \) with \( p(x^0) > 0 \) such that the corresponding real projective hypersurface \( \mathcal{V}_p(\mathbb{R}) \) is smooth is an open subset of the vector space of polynomials over \( \mathbb{R} \) of degree \( m \) (see [28, Sections 5 and 7] following [44]), it follows that a general \( RZ_{x^0} \) polynomial \( p \in \mathbb{R}[x_1, \ldots, x_d] \) of degree \( m \) with \( p(x^0) > 0 \) does not admit a determinantal representation (3.2) with \( m \times m \) matrices — even without requiring real symmetry or positivity — for \( d > 3 \) and for \( d = 3, m \geq 4 \). (For the remaining cases when \( d = 3 \), the case \( m = 2 \) is straightforward and the case \( m = 3 \) is treated in details in [8] when the corresponding complex projective cubic surface \( \mathcal{V}_p \) in \( \mathbb{P}^3(\mathbb{C}) \) is smooth; in both cases there are no positive real symmetric determinantal representations of size \( m \) as in Theorem 3.1 but there are positive self-adjoint determinantal representations of size \( m \), i.e., representations (3.2) with \( m \times m \) self-adjoint matrices such that \( A_0 + x_1^0 A_1 + x_2^0 A_2 + x_3^0 A_3 = I \).

Proposition 3.2 follows by a simple count of parameters, see [11]. It also follows from Theorem 3.1 below using the Noether–Lefschetz theory [37, 22, 20], since for a general homogeneous polynomial \( P \in \mathbb{C}[X_0, X_1, \ldots, X_d] \) of degree \( m \) with \( d > 3 \) or with \( d = 3, m \geq 4 \), the only line bundles on \( \mathcal{V}_P \) are of the form \( \mathcal{O}_{\mathcal{V}_P}(j) \) and these obviously fail the conditions of the theorem.

The following is therefore the “best possible” generalization of Theorem 3.1 to the case \( d > 2 \).

Conjecture 3.3. Let \( p \in \mathbb{R}[x_1, \ldots, x_d] \) be a \( RZ_{x^0} \) polynomial of degree \( m \) with \( p(x^0) = 1 \). Then there exists a \( RZ_{x^0} \) polynomial \( h \in \mathbb{R}[x_1, \ldots, x_d] \) of degree \( \ell \) with \( h(x^0) = 1 \) and with the closure of the connected component of \( x^0 \) in \( \{ x \in \mathbb{R}^d : h(x) > 0 \} \) containing the closure of the connected component of \( x^0 \) in \( \{ x \in \mathbb{R}^d : p(x) > 0 \} \), and \( A_0, A_1, \ldots, A_d \in \mathbb{S} \mathbb{R}^{n \times n} \), \( n \geq m + \ell \), with \( A_0 + x_1 A_1 + \cdots + x_d A_d = I \), such that

\[
\text{(3.3)} \quad \det(A_0 + x_1 A_1 + \cdots + x_d A_d) = p(x)h(x).
\]
Notice that is enough to require that $h$ is a polynomial that is strictly positive on the connected component of $x^0$ in $\{x \in \mathbb{R}^d; h(x) > 0\}$, since it then follows from (3.3) that $h$ is a $RZ_{\mathbb{C}}$ polynomial with $h(x^0) = 1$ and with the closure of the connected component of $x^0$ in $\{x \in \mathbb{R}^d; h(x) > 0\}$ containing the closure of the connected component of $x^0$ in $\{x \in \mathbb{R}^d; p(x) > 0\}$.

Conjecture (3.3) tells us that a necessary and sufficient condition for $C \subseteq \mathbb{R}^d$ to admit a LMI representation is that $C$ is a rigidly convex algebraic interior.

We can also homogenize (3.3),

$$\text{(3.4)}\quad \det(X_0A_0 + X_1A_1 + \cdots + X_dA_d) = P(X)\hat{H}(X),$$

where $\hat{H}(X) = H(X)X_0^{n-m-\ell}$ and $P$ and $H$ are the homogenizations of $P$ and $H$ respectively (see (2.3)).

3.2. The easiest way to establish Conjecture (3.3) would be to try taking $h = 1$ in (3.3) bringing us back to (3.2); in the homogeneous version, $\hat{H} = X_0^{n-m}$ in (3.4). This was the the form of the conjecture stated in [28]. It was given further credence by the existence of real symmetric determinantal representations without the requirement of positivity.

**Theorem 3.4.** Let $p \in \mathbb{R}[x_1, \ldots, x_d]$. Then there exist $A_0, A_1, \ldots, A_d \in \mathbb{SR}^{n \times n}$ for some $n \geq m$ such that $p$ admits the determinantal representation (3.2).

Theorem (3.4) was first established in [27] using free noncommutative techniques. More precisely, the method was to take a lifting of $p$ to the free algebra and to apply results of noncommutative realization theory to first produce a determinantal representation with $A_0, A_1, \ldots, A_d \in \mathbb{R}^{n \times n}$ and then to show that it is symmetrizable; see [27] Section 14 for details and references. An alternate proof of Theorem (3.4) that uses more elementary arguments was given in [48]. As it turns out, determinantal representations also appear naturally in algebraic complexity theory, and a proof of Theorem (3.4) from this perspective was given in [19].

Unfortunately, the analogue of Theorem (3.4) for positive real symmetric (or positive self-adjoint) determinantal representations fails. Counterexamples were first established in [6], and subsequently in [43]. Indeed we have

**Proposition 3.5.** A general $RZ_{\mathbb{C}}$ polynomial $p \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $m$ with $p(x^0) = 1$ does not admit a determinantal representation (3.2), where $A_0, A_1, \ldots, A_d \in \mathbb{HC}^{n \times n}$ for some $n \geq m$ with $A_0 + x_0A_1 + \cdots + x_dA_d = I$, for any fixed $m \geq 4$ and $d$ large enough or for any fixed $d \geq 3$ and $m$ large enough.

Here “large enough” means that $d^3m^2 < \left(\frac{m+d}{m}\right) - 1$. We refer to [48] Section 3 for details and numerous examples of $RZ$ polynomials that do not admit a positive self-adjoint determinantal representation as in Proposition (3.5). One simple example is

$$\text{(3.5)}\quad p = (x_1 + 1)^2 - x_2^2 - \cdots - x_d^2$$

for $d \geq 5$ (for $d = 4$ this polynomial admits a positive self-adjoint determinantal representation but does not admit a positive real symmetric determinantal representation). The proofs are based on the fact that a positive self-adjoint (or real symmetric) determinantal representation of size $n$ always contains, after a unitary (or orthogonal) transformation of the matrices $A_0, A_1, \ldots, A_d$, a direct summand $I_{n-n'} + x_1R_{n-n'} + \cdots + x_dR_{n-n'}$ — yielding a determinantal representation of size $n'$ — for relatively small $n'$: one can always take $n' \leq md$ and in many instances one
can actually take \( n' = m \), see \[43\] Theorems 2.4 and 2.7. It would be interesting to compare these results with the various general conditions for decomposability of determinantal representations obtained in \[33, 32\].

3.3. The next easiest way to establish Conjecture 3.3 is to try taking \( h \) in (3.3) to be a power of \( p, h = p^{r-1}, \) so that we are looking for a positive real symmetric determinantal representation of \( p^r \),

\[
\text{det}(A_0 + x_1 A_1 + \cdots + x_d A_d) = p(x)^r;
\]

in the homogeneous version, \( H = P^{r-1} \cdot X_0^{n - mr} \) in \( \mathbb{R}^d \). If we do not require positivity or real symmetry, then at least for \( p \) irreducible, \( p^r \) admits a determinantal representation \eqref{eq:3.6} with \( A_0, A_1, \ldots, A_d \in \mathbb{C}^{n \times n}, n = mr, \) for some \( r \in \mathbb{N}; \) this follows by the theory of matrix factorizations \[16\], since \( pI \) can be written as a product of matrices with linear entries, see \[2, 30\] (and also the references in \[13\]).

As established in \[6\], the answer for positive real symmetric determinantal representations is again no. Namely, let \( p \) be a polynomial of degree 4 in 8 variables labeled \( x_a, x_b, x_c, x_d, x_a', x_b', x_c', x_d' \), defined by

\[
p = \sum_{S \in \mathcal{B}(V_8)} \prod_{j \in S} (x_j + 1),
\]

where \( \mathcal{B}(V_8) \) is the set consisting of all 4-element subsets of \( \{a, b, c, d, a', b', c', d'\} \) except for

\[
\{a, a', b, b', \} \cup \{b, b', c, c', \} \cup \{c, c', d, d', \} \cup \{d, d', a, a'\} \cup \{a, a', c, c'\}.
\]

\( \mathcal{B}(V_8) \) is the set of bases of a certain matroid \( V_8 \) on the set \( \{a, b, c, d, a', b', c', d'\} \) called the Vamos cube. Then

**Theorem 3.6.** \( p \) is RZ with respect to 0, and for all \( r \in \mathbb{N}, \) the polynomial \( p^r \) does not admit a determinantal representation \eqref{eq:3.6} where \( A_0, A_1, \ldots, A_d \in \mathbb{R}^{n \times n} \) for some \( n \geq mr \) with \( A_0 = I \).

This follows since on the one hand, \( V_8 \) is a half-plane property matroid, and on the other hand, it is not representable over any field, more precisely its rank function does not satisfy Ingleton inequalities. See \[6\] Section 3] for details. Notice that it turns out that one can take without loss of generality \( n = mr \) in Theorem 3.6, see the paragraph following Proposition 3.5 above. Notice also that because of the footnote on page \[4\] it does not matter here whether we are considering real symmetric or self-adjoint determinantal representations.

The polynomial \( (3.7) \) remains so far the only example of a RZ polynomial no power of which admits a positive symmetric determinantal representation\(^2\). For instance, we have

**Theorem 3.7.** Let \( p \in \mathbb{R}[x_1, \ldots, x_d] \) be a RZ polynomial of degree 2 with \( p(x^0) = 1. \) Then there exists \( r \in \mathbb{N} \) and \( A_0, A_1, \ldots, A_d \in \mathbb{S}^{n \times n}, n = mr, \) with \( A_0 + x_0 A_1 + \cdots + x^0 A_d = I, \) such that \( p^r \) admits the determinantal representation \eqref{eq:3.6}.

**Remark 3.7** has been established in \[43\] using Clifford algebra techniques. More precisely, one associates to a polynomial \( p \in \mathbb{R}[x_1, \ldots, x_d] \) of degree \( m \) a unital *-algebra as follows. Let \( \mathbb{C}(z_1, \ldots, z_d) \) be the free *-algebra on \( d \) generators, i.e.,

\(^2\) Peter Brändén noticed (see [http://www.e.uni-magdeburg.de/ragc/talks/branden.pdf](http://www.e.uni-magdeburg.de/ragc/talks/branden.pdf)) that one can use the symmetry of the polynomial \( (14) \) to produce from it a RZ polynomial in 4 variables no power of which admits a positive real symmetric determinantal representation.
z_1, \ldots, z_d are noncommuting self-adjoint indeterminates. For the homogenization $P$ of $p$, we can write

$$P(-x_1z_1 - \cdots - x_dz_d, x_1, \ldots, x_d) = \sum_{k \in \mathbb{Z}_d^+, |k|=m} q_k(z)x^k,$$

for some $q_k \in \mathbb{C}(z_1, \ldots, z_d)$, where $k = (k_1, \ldots, k_d)$, $|k| = k_1 + \cdots + k_d$, and $x^k = x_1^{k_1} \cdots x_d^{k_d}$. We define the generalized Clifford algebra associated with $p$ to be the quotient of $\mathbb{C}(z_1, \ldots, z_d)$ by the two-sided ideal generated by $\{q_k\}_{|k|=m}$. It can then be shown that at least if $p$ is irreducible, $p^r$ admits a self-adjoint determinantal representation \((3.6)\) of size $mr$ with $A_0 = I$ for some $r \in \mathbb{N}$ if and only if the generalized Clifford algebra associated with $p$ admits a finite-dimensional unital $^*$-representation. In case $m = 2$ and $p$ is an irreducible $RZ_0$ polynomial, the generalized Clifford algebra associated with $p$ turns out to be “almost” the usual Clifford algebra, yielding the proof of Theorem \(3.7\). For details and references, see [43, Sections 4 and 5]. It would be interesting to investigate the generalized Clifford algebra associated with the polynomial \((3.7)\).4

A new obstruction to powers of $p$ admitting a positive real symmetric determinantal representation has been recently discovered in [32], it is closely related to the question of how to test a polynomial for the $RZ$ condition, see [29]. For any monic polynomial $f \in \mathbb{R}[t]$ of degree $m$ with zeroes $\lambda_1, \ldots, \lambda_m$, let us define the matrix $H(f) = [h_{ij}]_{i,j = 1, \ldots, m}$ by $h_{ij} = \sum_1^d \lambda_i^{j+2};$ notice that $h_{ij}$ are actually polynomials in the coefficients of $f$. $H(f)$ is called the Hermite matrix of $f$, and it is positive semidefinite if and only if all the zeroes of $f$ are real. Given $p \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $m$ with $p(x^0) = 1$, we now consider $H(\hat{p}_x)$ where $\hat{p}_x(t) = t^mp(x^0 + t^{-1}x)$; it is a polynomial matrix that we call the Hermite matrix of $p$ with respect to $x^0$ and denote $H(p; x^0)$. $p$ is a $RZ_{x^0}$ polynomial if and only if $H(p; x^0)(x) \geq 0$ for all $x \in \mathbb{R}^d$. Now, it turns out that if there exists $r \in \mathbb{N}$ such that $p^r$ admits a determinantal representation \((3.6)\) with $A_0, A_1, \ldots, A_d \in \mathbb{R}^{n \times n}, n = mr$, and $A_0 + x_1^0A_1 + \cdots + x_d^0A_d = I$, then $H(p; x^0)$ can be factored: $H(p; x^0) = Q^tQ$ for some polynomial matrix $Q$, i.e., $H(p; x^0)$ is a sum of squares. We notice that $H(p; x^0)$ can be reduced by homogeneity to a polynomial matrix in $d - 1$ variables, implying that the sum of squares decomposition (factorization) is not an obstruction in the case $d = 2$, but it is in the case $d > 2$. In particular, there is numerical evidence that for the polynomial $p$ of \((3.7)\) the Hermite matrix (with respect to 0) is not a sum of squares. We refer to [42] for details. It would be very interesting to use these ideas in the case $d = 2$ to obtain a new proof of a weakened version of Theorem \(3.1\) that gives a positive real symmetric determinantal representation of $p^r$ (of size $mr$) for some $r \in \mathbb{N}$.

3.4. There have been so far no attempts to pursue Conjecture \(3.3\) with other choices of $h$ than 1 or a power of $p$. Two natural candidates are products of (not necessarily distinct) linear forms (that are nonnegative on the closure of the connected component of $x^0$ in $\{x \in \mathbb{R}^d: p(x) > 0\}$), and products of powers of Renegar derivatives of $p$ with respect to $x^0$ (see \(2.9\)).

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4 Tim Netzer recently reported (see http://www-e.uni-magdeburg.de/rags/talks/netzer.pdf) that for an irreducible $RZ_0$ polynomial $p$ with $p(0) = 1$, Conjecture 5 holds (with $x^0 = 0$) if and only if $-1$ is not a sum of hermitian squares in the generalized Clifford algebra associated with $p$. 
Conjecture 3.3 is a reasonable generalization of Theorem 3.1 for the purposes of LMI representations of convex sets (provided the solution gives a good hold of the extra factor $h$ and of the size $n$). It is less satisfactory as a means of describing or generating $RZ$ polynomials. The following alternative conjecture, that was proposed informally by L. Gurvits, might be more useful for that purpose. It is based on the fact that we have two systematic ways of generating $RZ$ polynomials: positive real symmetric (or self-adjoint) determinantal representations and Renegar derivatives.

**Conjecture 3.8.** Let $p \in \mathbb{R}[x_1, \ldots, x_d]$ be a $RZ_{x_0}$ polynomial of degree $m$ with $p(x_0) = 1$. Then there exist $k \in \mathbb{Z}_+$, a $RZ_{x_0}$ polynomial $q \in \mathbb{R}[x_1, \ldots, x_d]$ of degree $m + k$ such that $p = q^{(k)}$, and $A_0, A_1, \ldots, A_d \in \mathbb{S}\mathbb{R}^{(m+k) \times (m+k)}$ with $A_0 + x_0^0 A_1 + \cdots + x_d^0 A_d = I$, such that

$$\det(A_0 + x_1 A_1 + \cdots + x_d A_d) = q(x).$$

4. Determinantal representations of homogeneous polynomials and sheaves on projective hypersurfaces

The kernel of a determinantal representation of a homogeneous polynomial is a sheaf on the corresponding projective hypersurface from which the representation itself can be reconstructed. We consider here the ways to do so that use the duality between the kernel and the left kernel; this gives the only known approaches to the proof of Theorem 3.1. For a different way using the resolution of the kernel sheaf see [5]; we refer also to the bibliography in [5, 32] and to [13, Chapter 4] and the references therein for more about this old topic in algebraic geometry.

4.1. Let $P \in \mathbb{C}[X_0, X_1, \ldots, X_d]$ ($d > 1$) be a reduced (i.e., without multiple irreducible factors) homogeneous polynomial of degree $m$, and let

$$V_P = \{[X] \in \mathbb{C}^d : P(X) = 0\}$$

be the corresponding complex projective hypersurface. Notice that when $P$ is a polynomial over $\mathbb{R}$, $V_P$ is naturally endowed with an antiholomorphic involution $\tau$ (the complex conjugation or the Galois action of $\text{Gal}(\mathbb{C}/\mathbb{R})$) and the set of fixed points of $\tau$ is exactly the real projective hypersurface $V_P(\mathbb{R})$ as in [22]. Let

$$\begin{align*}
\det(X_0 A_0 + X_1 A_1 + \cdots + X_d A_d) &= P(X), \\
A_\alpha &= [A_{\alpha,ij}]_{i,j=1,\ldots,m} \in \mathbb{C}^{m \times m}, \alpha = 0, 1, \ldots, d,
\end{align*}$$

be a determinantal representation of $P$, and let

$$U = X_0 A_0 + X_1 A_1 + \cdots + X_d A_d, \quad V = [V_{ij}]_{i,j=1,\ldots,m} = \text{adj} \ U,$$

where $\text{adj} Y$ denotes the adjoint matrix of a $m \times m$ matrix $Y$, i.e., the matrix whose $(i, j)$ entry is $(-1)^{i+j}$ times the determinant of the matrix obtained from $Y$ by removing the $j$th row and the $i$th column, so that $Y \cdot \text{adj} Y = \det Y \cdot I$. Notice that

$$\begin{align*}
\det V &= P^{m-1}, \\
\text{adj} V &= P^{m-2} \cdot U.
\end{align*}$$
Notice also that using the formula for the differentiation of a determinant and row expansion,

\[
\frac{\partial P}{\partial x_\alpha} = \sum_{l,k=1}^n A_{\alpha,lk} V_{kl}.
\]

In particular, \(V(X)\) is not zero for a smooth point \([X]\) of the hypersurface \(V_p\), so that \(V(X)\) has rank 1 there and \(U(X)\) has rank \(m-1\).

4.2. We restrict our attention now to the case \(d = 2\), i.e., \(V_p\) is a projective plane curve. Let us assume for a starter that \(P\) is irreducible and that \(V_p\) is smooth — we will explain how to relax this assumption in Section 4.7 below. Then we conclude that \(\mathcal{L}(X) = \ker U(X)\) is a one-dimensional subspace of \(\mathbb{C}^m\) for all points \([X]\) on \(V_p\), and these subspaces glue together to form a line bundle \(\mathcal{L}\) on \(V_p\); more precisely, \(\mathcal{L}\) is a subbundle of the trivial rank \(m\) vector bundle \(V_p \times \mathbb{C}^m\) whose fiber at the point \([X]\) equals \(\mathcal{L}([X])\). It is convenient to twist and define \(\mathcal{E} = \mathcal{L}(m-1)\).

More algebraically, \(\mathcal{E}\) is determined by the exact sequence of sheaves on \(V_p\)

\[
0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}^\oplus(m-1) \xrightarrow{U} \mathcal{O}_{\mathbb{P}^2}^\oplus(m) \to \ker(U) \to 0,
\]

where \(U\) denotes the operator of right multiplication by the matrix acting on columns. The following are some of the properties of the kernel line bundle.

1. The determinantal representation is determined up to a natural equivalence (multiplication on the left and on the right by constant invertible matrices) by the isomorphism class of the line bundle \(\mathcal{E}\).
2. The columns \(F_j = [V_{ij}]_{i=1,\ldots,m}\) of the adjoint matrix \(V\) form a basis for the space \(H^0(\mathcal{E}, V_p)\) of global sections of \(\mathcal{E}\).
3. \(\mathcal{E}\) satisfies \(h^0(\mathcal{E}(-1)) = h^1(\mathcal{E}(-1)) = 0\).

See [9, 53, 5] for details. By the Riemann–Roch theorem, \(\mathcal{E}(-1)\) is a line bundle of degree \(g - 1\) on \(V_p\) (where \(g\) denotes the genus), and it is general in that it has no global sections, i.e., it lies on the complement of the theta divisor in the Jacobian of \(V_p\).

There is a similarly defined line bundle \(\mathcal{L}_f\) on \(V_p\) with fibres \(\mathcal{L}_f([X]) = \ker_f U(X)\), where \(\ker_f\) denotes the left kernel of a matrix (a subspace of \(\mathbb{C}^{1 \times m}\)); we set, analogously, \(\mathcal{E}_f = \mathcal{L}_f(m-1)\). \(\mathcal{E}_f\) is defined by an exact sequence similar to (4.7) except that \(U\) is now acting as the operator of left multiplication by the matrix on rows. The rows \(G_i = [V_{ij}]_{j=1,\ldots,m}\) of the adjoint matrix \(V\) form a basis for the space \(H^0(\mathcal{E}_f, V_p)\) of global sections of \(\mathcal{E}_f\). There is furthermore a nondegenerate pairing \(\mathcal{E} \times \mathcal{E}_f \to \mathcal{K}_{V_p}(2)\) (here \(\mathcal{K}_{V_p} \cong \mathcal{O}_{V_p}(m - 3)\) is the canonical line bundle on \(V_p\), i.e., \(\mathcal{E}_f(-1)\) is isomorphic to the Serre dual \((\mathcal{E}(-1))^* \otimes \mathcal{K}_{V_p}\) of \(\mathcal{E}(-1)\), which is key to the reconstruction of the determinantal representation from the corresponding line bundle.

Notice that if \(P\) is a polynomial over \(\mathbb{R}\) and the determinantal representation is self-adjoint then \(\mathcal{E}_f \cong \mathcal{E}^\ast\), whereas if the determinantal representation is real symmetric, then \(\mathcal{E}_f \cong \mathcal{E}^\ast \cong \mathcal{E}\). (In fact, in the real symmetric case the line bundle \(\mathcal{E}\) is defined over \(\mathbb{R}\) which is a somewhat stronger condition than \(\mathcal{E}^\ast \cong \mathcal{E}\) but the two actually coincide if \(\mathcal{K}_{V_p}(\mathbb{R}) \neq 0\), see [54] and the references there.)
4.3. There are two ways to define the pairing $\mathcal{E} \times \mathcal{E}_\ell \to \mathcal{K}_{\mathcal{V}_p}(2)$. One way, originating in multivariable operator theory and multidimensional system theory, simply pairs the right and left kernels of the matrix $U(X)$ against appropriate linear combinations of the coefficient matrices $A_0$, $A_1$, $A_2$; see [3]. This gives the line bundle $\mathcal{E}(-1)$ on $\mathcal{V}_p$ with $h^0(\mathcal{E}(-1)) = h^1(\mathcal{E}(-1)) = 0$, see [4]. It is obvious from these formulae that choosing $\mathcal{E}(-1)$ with $(\mathcal{E}(-1))^* \otimes \mathcal{K}_{\mathcal{V}_p} \cong \mathcal{E}(-1)^\tau \cong \mathcal{E}(-1)$ (i.e., $\mathcal{E}(-1)$ is a real theta characteristic on $\mathcal{V}_p$) yields a real symmetric determinantal representation (at least in the case $\mathcal{V}_p(\mathbb{R}) \neq \emptyset$). [28, Section 4] verifies (using the tools developed in [54]) that in case the dehomogenization $p(x_1, \ldots, x_d) = P(1, x_1, \ldots, x_d)$ of the original polynomial $P$ is $RZ$, appropriate choices of $\mathcal{E}(-1)$ will yield a positive determinantal representation (to be more precise, the positivity is “built in” [28, (4.1)–(4.3)]). “Appropriate choices” means that the line bundle $\mathcal{E}(-1)$ of degree $g − 1$ (more precisely, its image under the Abel–Jacobi map) has to belong to a certain distinguished real $g$-dimensional torus $T_0$ in the Jacobian of $\mathcal{V}_p$, see [54] Sections 3 and 4; accidentally, this already forces $\mathcal{E}(-1)$ to be in the complement of the theta divisor, i.e., the condition $h^0(\mathcal{E}(-1)) = h^1(\mathcal{E}(-1)) = 0$ becomes automatic. It is interesting to notice that recent computational advances in theta functions on Riemann surfaces make this approach possibly suitable for computational purposes, see [47].

4.4. Another way to define the pairing $\mathcal{E} \times \mathcal{E}_\ell \to \mathcal{K}_{\mathcal{V}_p}(2)$ is more algebraic and goes back to the classical paper [12]; it uses the adjoint matrix $V$ of the determinantal representation. This leads to the following construction of the determinantal representation given a line bundle $\mathcal{E}(-1)$ on $\mathcal{V}_p$ with $h^0(\mathcal{E}(-1)) = h^1(\mathcal{E}(-1)) = 0$, see [12] [53] [3]. Take bases $\{F_1, \ldots, F_m\}$ and $\{G_1, \ldots, G_m\}$ for the spaces of global sections of $\mathcal{E}$ and of $\mathcal{E}_\ell$, respectively, where $\mathcal{E}_\ell(-1) := (\mathcal{E}(-1))^* \otimes \mathcal{K}_{\mathcal{V}_p}$ is the Serre dual. Then $V_{ij} := \langle F_j, G_i \rangle$ is a global section of $\mathcal{K}_{\mathcal{V}_p}(2) \cong \mathcal{O}_{\mathcal{V}_p}(m − 1)$, hence a homogeneous polynomial in $X_0, X_1, X_2$ of degree $m − 1$. It can be shown that the matrix $V = [V_{ij}]_{i,j=1,\ldots,m}$ has rank 1 on $\mathcal{V}_p$, implying that (4.4) holds, up to a constant factor $c$, and that every entry of $\text{adj} \, V$ is divisible by $P^{m−2}$. We can now define a matrix $U$ of linear homogeneous forms by (4.5), and it will be a determinantal representation of $P$, up to the constant factor $c^{m-1}$. It remains only to show that the constant factor is not zero, i.e., that $\det V$ is not identically zero. This follows by choosing the bases for the spaces of global sections adapted to a straight line, so that $V$ becomes diagonal along that line, and uses essentially the condition $h^0(\mathcal{E}(-1)) = h^1(\mathcal{E}(-1)) = 0$.

It is quite straightforward that if $\mathcal{E}$ satisfies $(\mathcal{E}(-1))^* \otimes \mathcal{K}_{\mathcal{V}_p} \cong (\mathcal{E}(-1))^\tau \cong \mathcal{E}(-1)$ we obtain a real symmetric determinantal representation (at least in the case $\mathcal{V}_p(\mathbb{R}) \neq \emptyset$, since we really need $\mathcal{E}$ to be defined over $\mathbb{R}$), whereas if $(\mathcal{E}(-1))^* \otimes \mathcal{K}_{\mathcal{V}_p} \cong \mathcal{E}(-1)^\tau$ we obtain a self-adjoint determinantal representation.

4.5. The above procedure can be written down more explicitly in terms of divisors and linear systems. We recall that for a homogeneous polynomial $F \in \mathbb{C}[X_0, X_1, X_2]$, the divisor $(F)$ of $F$ on $\mathcal{V}_p$ is the formal sum of the zeroes of $F$ on $\mathcal{V}_p$ with the orders of the zeroes as coefficients (the order of the zero equals also the intersection multiplicity of the curves $\mathcal{V}_Q$ and $\mathcal{V}_p$ — here $Q$ can have multiple irreducible factors so that the curve $\mathcal{V}_Q$ can have multiple components, i.e., it may be a non-reduced subscheme of $\mathbb{P}^2$ over $\mathbb{C}$).
Let $Q \in \mathbb{C}[X_0, X_1, X_2]$ be an auxiliary homogeneous polynomial of degree $m - 1$, together with a decomposition $(Q) = D + D_\ell$, $\deg D = \deg D_\ell = m(m - 1)/2$. We assume that $D$ and $D_\ell$ satisfy the condition that $D - (L)$ or equivalently $D_\ell - (L)$ is not linearly equivalent to an effective divisor on $\mathcal{V}_P$, where $L$ is a linear form.

Take a basis $\{V_{11}, \ldots, V_{m1}\}$ of the vector space of homogeneous polynomials of degree $m - 1$ that vanish on $D$, with $V_{11} = Q$, and a basis $\{V_{11}, \ldots, V_{1m}\}$ of the vector space of homogeneous polynomials of degree $m - 1$ that vanish on $D_\ell$. Write $(V_{i1}) = D + D_{\ell,i}$ and $(V_{ij}) = D_j + D_\ell$, where $D_1 = D$ and $D_{\ell,1} = D_\ell$. Define homogeneous polynomials $V_{ij}$ of degree $m - 1$ for $i > 1$ and $j > 1$ by $(V_{ij}) = D_j + D_{\ell,i}$. We then set $V = [V_{ij}]_{i,j=1,\ldots,m}$, and obtain a determinantal representation $U$ of $P$ by \(13\).

To be able to obtain a real symmetric determinantal representation of a polynomial $P$ over $\mathbb{R}$, we need $\mathcal{V}_Q$ to be a real contact curve of $\mathcal{V}_P$, i.e., to be defined by a polynomial $Q$ over $\mathbb{R}$ and to have even intersection multiplicity at all points of intersection (in this case $D = D_\ell$ is uniquely determined). To be able to obtain a self-adjoint determinantal representation we need $\mathcal{V}_Q$ to be a real curve that is contact to $\mathcal{V}_P$ at all real points of intersection (in this case the real points of $D$ and of $D_\ell = D^\tau$ are uniquely determined whereas the non-real points can be shuffled between the two).

4.6. Unlike the approach of Section 4.3, the approach of Sections 4.4–4.5 does not produce directly the coefficient matrices of the determinantal representation, so it is not clear a priori how to obtain a real symmetric or self-adjoint representation that is positive. A delicate calculation with differentials carried out in \(44\) shows that this will happen exactly in case the original polynomial $P$ is $RZ$ and $E(-1)$ (more precisely, its image under the Abel–Jacobi map) belongs to the distinguished real $g$-dimensional torus $T_0$ in the Jacobian of $\mathcal{V}_P$. We will obtain a corresponding result in terms of the auxiliary curve $\mathcal{V}_Q$ in Section 5 below by elementary methods.

4.7. We consider now how to relax the assumption that $\mathcal{V}_P$ is irreducible and smooth. A full analysis of determinantal representations for a general reduced polynomial $P$ involves torsion free sheaves of rank 1 on a possibly reducible and singular curve; see \(32\) and the references therein. However one can get far enough to obtain a full proof of Theorem 3.1 by considering a restricted class of determinantal representations.

Let $\nu: \tilde{\mathcal{V}}_P \to \mathcal{V}_P$ be the normalization or equivalently the desingularization. $\tilde{\mathcal{V}}_P$ is a disjoint union of smooth complex projective curves (or compact Riemann surfaces) corresponding to the irreducible factors of $P$ (the irreducible components of $\mathcal{V}_P$) and

$$\nu \big|_{\tilde{\mathcal{V}}_P \setminus \nu^{-1}(\mathcal{V}_P)_{\text{sing}}} : \tilde{\mathcal{V}}_P \setminus \nu^{-1}(\mathcal{V}_P)_{\text{sing}} \to \mathcal{V}_P \setminus (\mathcal{V}_P)_{\text{sing}}$$

is a (biregular or complex analytic) isomorphism, where $(\mathcal{V}_P)_{\text{sing}}$ denotes the set of singular points of $\mathcal{V}_P$. Let $\lambda \in (\mathcal{V}_P)_{\text{sing}}$; we assume that $\lambda$ lies in the affine plane $\mathbb{C}^2 \subseteq \mathbb{P}^2(\mathbb{C})$ (otherwise we just choose different affine coordinates near $\lambda$). For every $\mu \in \nu^{-1}(\lambda)$ (i.e., for every branch of $\mathcal{V}_P$ at $\lambda$), the differential

$$\nu^* \left( \frac{\partial x_1}{\partial p/\partial x_1} \right) = -\nu^* \left( \frac{\partial x_2}{\partial p/\partial x_2} \right)$$
on $\tilde{V}_P$ has a pole at $\mu$; we denote the order of the pole by $m_\mu$. We define

$$\Delta_\lambda = \sum_{\mu \in \nu^{-1}(\lambda)} m_\mu \mu$$

(the adjoint divisor of $\lambda$), and

(4.8) $$\Delta = \sum_{\lambda \in (\nu P)_{\text{sing}}} \Delta_\lambda$$

(the adjoint divisor, or the divisor of singularities, of $V_P$); see, e.g., [1, Appendix A2].

A determinantal representation $U$ of $P$ is called fully saturated (or $\tilde{V}_P/V_P$ saturated) if all the entries of the adjoint matrix $V$ vanish on the adjoint divisor: $$(\nu^*V_{ij}) \geq \Delta$$ for all $i, j = 1, \ldots, m$. This is a somewhat stronger condition than being a maximal (or maximally generated) determinantal representation, which means that for every $\lambda \in (V_P)_{\text{sing}}$, dim $U(\lambda)$ has the maximal possible dimension equal to the multiplicity of $\lambda$ on $V_P$. We refer to [3, 33] for details. If $P$ is reducible than a fully saturated determinantal representation always decomposes, up to equivalence, as a direct sum of determinantal representations of the irreducible factors of $P$; hence we can assume that $P$ is irreducible.

For a fully saturated determinantal representation $U$ of $P$, we can define a line bundle $\mathcal{L}$ on $\tilde{V}_P \setminus \nu^{-1}((\nu P)_{\text{sing}})$ with fibres $\mathcal{L}(X) = \ker U(X)$ and then extend it uniquely to all of $\tilde{V}_P$; we then define $\mathcal{E} = \mathcal{L}(m-1)(-\Delta)$, see [4] — here $\mathcal{L}(m-1) = \mathcal{L} \otimes \nu^*\mathcal{O}_{\nu P}(m-1)$. Alternatively, we can define $\mathcal{E} = \nu^*\mathcal{E}$, where the sheaf $\mathcal{E}$ on $\nu P$ is still defined by (4.7), see [32]. We introduce similarly the left kernel line bundle $\mathcal{E}_l$. Most of Sections 4.2–4.4 and 4.6 now carry over for a fully saturated determinantal representation $U$ of $P$ and line bundles $\mathcal{E}$ and $\mathcal{E}_l$ on $\tilde{V}_P$; notice that the canonical line bundle on $\tilde{V}_P$ is given by $K_{\tilde{V}_P} \cong \nu^*\mathcal{O}_{\nu P}(m-3)(-\Delta)$.

In Section 4.5 we have to take the auxiliary polynomial $Q$ to vanish on the adjoint divisor: $$(\nu^*Q) \geq \Delta,$$ with a decomposition $$(\nu^*Q) = D + D_l + \Delta.$$ We then take a basis $\{V_{i1}, \ldots, V_{im}\}$ of the vector space of homogeneous polynomials of degree $m - 1$ that vanish on $D$ and on the adjoint divisor, with $V_{i1} = Q$, and a basis $\{V_{i1}, \ldots, V_{im}\}$ of the vector space of homogeneous polynomials of degree $m - 1$ that vanish on $D_l$ and on the adjoint divisor; we write $(V_{i1}) = D + D_l + \Delta$ and $(V_{ij}) = D_j + D_l + \Delta$, where $D_1 = D$ and $D_{l,i} = D_l$; and we define homogeneous polynomials $V_{ij}$ of degree $m - 1$ for $i > 1$ and $j > 1$ by $(V_{ij}) = D_j + D_{l,i} + \Delta$.

4.8. The recent work [32] extends the construction of the adjoint matrix of a determinantal representation outlined in Section 4.1 to the most general higher dimensional situation. Let $P = P^1 \cdots P^k \in \mathbb{C}[X_0, X_1, \ldots, X_d]$, where $P_1, \ldots, P_k$ are (distinct) irreducible polynomials, and let

(4.9) $$V_P = \text{Proj } \mathbb{C}[X_0, X_1, \ldots, X_d]/(P)$$

be the corresponding closed subscheme of $\mathbb{P}^n$ over $\mathbb{C}$; of course $V_P$ is in general highly non-reduced. Let $U$ be a determinantal representation of $P$ as in (4.2)–(4.3); we define the kernel sheaf $\mathcal{E}$ on $V_P$ by the exact sequence (4.7), as before. $\mathcal{E}$ is a torsion-free sheaf on $V_P$ of multirank $(r_1, \ldots, r_k)$ (these notions have to be somewhat carefully defined), and we have

(4.10) $$h^0(\mathcal{E}(-1)) = h^{d-1}(\mathcal{E}(1-d)) = 0,$$ $$h^i(\mathcal{E}(j)) = 0, \ i = 1, \ldots, d-2, \ j \in \mathbb{Z}.$$
Conversely, 

**Theorem 4.1.** Let $\mathcal{E}$ be a torsion-free sheaf on $\mathcal{V}_P$ of multirank $(r_1, \ldots, r_k)$ satisfying the vanishing conditions \((4.10)\); then $\mathcal{E}$ is the kernel sheaf of a determinantal representation of $P$.

As in Section 4.4 Theorem 4.1 is proved by taking bases of $H^0(\mathcal{E}, \mathcal{V}_P)$ and of $H^0(\mathcal{E}_i, \mathcal{V}_P)$, $\mathcal{E}_i = \mathcal{E}^* \otimes \omega_{\mathcal{V}_P}(d)$ (here $\omega_{\mathcal{V}_P} = \mathcal{O}_{\mathcal{V}_P}(m - d - 1)$ is the dualizing sheaf), pairing these bases to construct a matrix $V$ of homogeneous polynomials of degree $m - 1$, and then defining the determinantal representation $U$ by \((1.5)\); there are quite a few technicalities, especially because the scheme is non-reduced. For $P$ a polynomial over $\mathbb{R}$, the determinantal representation can be taken to be self-adjoint if (and only if) $\mathcal{E}^* \cong \mathcal{E}$ representations. (Complex symmetric determinantal representations correspond to conjugation. It should be also possible to characterize real symmetric determinantal representations. (Complex symmetric determinantal representations correspond to $\mathcal{E} \cong \mathcal{E}^* \otimes \omega_{\mathcal{V}_P}(d)$.)

Theorem 4.1 provides a new venue for pursuing Conjecture 3.3. To make it effective requires progress in two directions:

1. Given a reduced homogeneous polynomial $P$, characterize large classes of homogeneous polynomials $\tilde{H}$ such that the scheme $\mathcal{V}_{\tilde{H}}$ admits torsion free sheaves of correct multirank satisfying the vanishing conditions \((4.10)\).
2. If $P$ is $\mathcal{RZ}$, characterize positive real symmetric or self-adjoint determinantal representations of $P$ in terms of the kernel sheaf $\mathcal{E}$. This is interesting not only for the general conjecture but also for special cases, compare the recent paper [10] dealing with singular nodal quartic surfaces in $\mathbb{P}^3$. It could be that the results of Section 5 below admit some kind of a generalization.

5. **Interlacing $\mathcal{RZ}$ Polynomials and Positive Self-Adjoint Determinantal Representations**

5.1. Let $p \in \mathbb{R}[x_1, \ldots, x_d]$ be a reduced (i.e., without multiple factors) $\mathcal{RZ}_{x_0}$ polynomial of degree $m$ with $p(x_0) \neq 0$, and let $P$ be the homogenization of $p$ (see \((2.3)\)). Let $Q \in \mathbb{R}[X_0, X_1, \ldots, X_d]$ be a homogeneous polynomial of degree $m - 1$ that is relatively prime with $P$. We say that $Q$ *interlaces* $P$ if for a general $X \in \mathbb{R}^{d+1}$, there is a zero of the univariate polynomial $Q(X + sX^0)$ in the open interval between any two zeroes of the univariate polynomial $P(X + sX^0)$, where $X^0 = (1, x^0)$. Alternatively, for any $X \in \mathbb{R}^{d+1}$,

\[
(5.1) \quad s_1 \leq s'_1 \leq s_2 \leq \cdots \leq s_{m-1} \leq s'_{m-1} \leq s_m,
\]

where $s_1, \ldots, s_m$ are the zeroes of $P(X + sX^0)$ and $s'_1, \ldots, s'_{m-1}$ are zeroes of $Q(X + sX^0)$, counting multiplicities. Notice (see \((2.3)\)) that we can consider instead the zeroes of the univariate polynomials $\tilde{q}_x(t) = t^m p(x_0 + t^{-1} x)$ and $\tilde{q}_x(t) = t^m \tilde{q}(x_0 + t^{-1} x)$ for a general or for any $x \in \mathbb{R}^d$, where $\tilde{q}(x) = Q(1, x_1, \ldots, x_d)$. It follows that $\tilde{q}$ is a $\mathcal{RZ}_{x_0}$ polynomial with $\tilde{q}(x^0) \neq 0$, and (upon normalizing $p(x^0) > 0$, $\tilde{q}(x^0) > 0$) the closure of the connected component of $x^0$ in $\{x \in \mathbb{R}^d; \tilde{q}(x) > 0\}$ contains the closure of the connected component of $x^0$ in $\{x \in \mathbb{R}^d; p(x) > 0\}$. The degree of $\tilde{q}$ is either $m - 1$ (in which case $Q$ is the homogenization of $\tilde{q}$) or $m - 2$ (in which case $Q$ is the homogenization of $\tilde{q}$ times $X_0$).

Geometrically, let $\mathcal{L}$ be a general straight line through $[X^0]$ in $\mathbb{P}^d(\mathbb{R})$. Then $Q$ interlaces $P$ if and only if any there is an intersection of $\mathcal{L}$ with the real projective
hypothesis $V_Q(\mathbb{R})$ in any open interval on $\mathcal{L} \setminus \{[X^0]\}$ between two intersections of $\mathcal{L}$ with the real projective hypersurface $V_P(\mathbb{R})$. If $Q$ does not contain $X_0$ as a factor, we can consider instead of $\mathcal{L} \setminus \{[X^0]\}$ the two open rays $\mathcal{L}_\pm$ starting at $x^0$ of a general straight line through $x^0$ in $\mathbb{R}^d$ and their intersections with the real affine hypersurfaces $V_q(\mathbb{R})$ and $V_p(\mathbb{R})$.

An example of a polynomial $Q$ interlacing $P$ is the first directional derivative $P_x^{(1)}$, see 2.5. (in this case $q = P_x^{(1)}$ is the first Renegar derivative).

It is not hard to see that (upon normalizing $p(x^0) > 0$) the definition of interlacing is independent of the choice of a point $x^0$ in a rigidly convex algebraic interior with a minimal defining polynomial $p$. In case the real projective hypersurfaces $V_P(\mathbb{R})$ and $V_Q(\mathbb{R})$ are both smooth, the interlacing of polynomials simply means the interlacing of ovaloids, see Proposition 2.2. More precisely, in this case $Q$ interlaces $P$ if and only if

a. If $m = 2k$ is even and $V_P(\mathbb{R}) = W_1 \sqcup \cdots \sqcup W_k$ and $V_Q(\mathbb{R}) = W'_1 \sqcup \cdots \sqcup W'_k$ are the decompositions into connected components, then the ovaloid $W_j'$ is contained in the “shell” obtained by removing the interior of the ovaloid $W_j$ from the closure of the interior of the ovaloid $W_{j+1}$, $i = 1, \ldots, k - 1$, and the pseudo-hyperplane $W'_k$ is contained in the closure of the exterior of the ovaloid $W_k$;

b. If $m = 2k + 1$ is odd and $V_P(\mathbb{R}) = W_1 \sqcup \cdots \sqcup W_k \sqcup W_{k+1}$ and $V_Q(\mathbb{R}) = W'_1 \sqcup \cdots \sqcup W'_k$ are the decompositions into connected components, then the ovaloid $W_j'$ is contained in the “shell” obtained by removing the interior of the ovaloid $W_j$ from the closure of the interior of the ovaloid $W_{j+1}$, $i = 1, \ldots, k - 1$, and the ovaloid $W'_k$ is contained in the closure of the exterior of the ovaloid $W_k$ and the pseudo-hyperplane $W_{k+1}$ is contained in the closure of the exterior of $W_k'$.

Interlacing can be tested via the Bezoutiant, similarly to testing the $RZ$ condition via the Hermite matrix. For polynomials $f, g \in \mathbb{R}[t]$ with $f$ of degree $m$ and $g$ of degree at most $m$, we define the Bezoutiant of $f$ and $g$, $B(f, g) = [b_{ij}]_{i,j=1,\ldots,m}$, by the identity

$$\frac{f(t)g(s) - f(s)g(t)}{t-s} = \sum_{i,j=0}^{m-1} b_{ij} t^i s^j;$$

notice that the entries of $B(f, g)$ are polynomials in the coefficients of $f$ and of $g$. The nullity of $B(f, g)$ equals the number of common zeroes of $f$ and of $g$ (counting multiplicities), and (assuming that the degree of $g$ is at most $m - 1$), $B(f, g) > 0$ if and only if $f$ has only real and distinct zeroes and there is a zero of $g$ in the open interval between any two zeroes of $f$; see, e.g., 3.4. Given $p \in \mathbb{R}[x_1, \ldots, x_d]$ a reduced polynomial of degree $m$ with $p(x^0) \neq 0$, with homogenization $P$, and $Q \in \mathbb{R}[X_0, X_1, \ldots, X_d]$ an homogeneous polynomial of degree $m - 1$ that is relatively prime with $P$, we now consider $B(\tilde{p}_x, \tilde{q}_x)$, where $\tilde{p}_x, \tilde{q}_x$ are as before; it is a polynomial matrix that we call the Bezoutiant of $P$ and $Q$ with respect to $x^0$ and denote $B(P, Q; x^0)$. We see that $p$ is a $RZ_{x^0}$ polynomial and $Q$ interlaces $P$ if and only if $B(P, Q; x^0)(x) \geq 0$ for all $x \in \mathbb{R}^d$.

5.2. Before stating and proving the main result of this section, we make some preliminary observations.
Let \( P \in \mathbb{C}[X_0, X_1, \ldots, X_d] \) be a reduced homogeneous polynomial of degree \( m \) with the corresponding complex projective hypersurface \( \mathcal{V}_P \) (see (4.1), and let \( U \) be a determinantal representation of \( P \) with the adjoint matrix \( V \) as in (4.3). Since \( \dim \ker \mathcal{U}(X) = 1 \) for a general point \([X]\) of any irreducible component of \( \mathcal{V}_P \), the rows of \( V \) are proportional along \( \mathcal{V}_P \) and so are the columns. An immediate consequence is that no element of \( V \) can vanish along \( \mathcal{V}_P \): otherwise, because of the proportionality of the rows, a whole row or a whole column of \( V \) would vanish along \( \mathcal{V}_P \), hence be divisible by \( P \), hence be identically 0 (since all the elements have degree \( m - 1 \) which is less than the degree of \( P \)), implying that \( \det V \) is identically 0, a contradiction. Another consequence is that every minor of order 2 in \( V \) vanishes along \( \mathcal{V}_P \), i.e., the eigenvalues of the generalized eigenvalue problem

\[
\lambda = \frac{\partial \mathcal{U}(X)}{\partial X_i} \mathcal{V}_P = \frac{\partial \mathcal{U}(X)}{\partial X_i} \mathcal{V}_P
\]

is positive definite for \( i = 1, \ldots, m \).

Lemma 5.1. Let \( F_j = [V_{ij}]_{i=1,\ldots,m} \), \( j = 1, \ldots, m \), and \( G_i = [V_{ij}]_{j=1,\ldots,m} \), \( i = 1, \ldots, m \), be the columns and the rows of the adjoint matrix \( V \), respectively, let \( X^0 = (X_0^0, X_1^0, \ldots, X_d^0) \in \mathbb{C}^{d+1} \setminus \{0\} \), and let

\[
P'_{X^0}(X) = \frac{d}{ds} \mathcal{U}(X + sX^0)|_{s=0} = \sum_{\alpha=0}^{d} X_\alpha^0 \frac{\partial \mathcal{U}}{\partial X_\alpha}(X)
\]

be the directional derivative. Then

\[
G_i \mathcal{U}(X^0) F_j = V_{ij} P'_{X^0}
\]

along \( \mathcal{V}_P \).

The result follows immediately by substituting (4.6) into (5.2) to calculate the directional derivative in terms of the entries of the adjoint matrix and of the coefficient matrices of the determinantal representation, and using the vanishing of the minors of order 2 in \( V \) along \( \mathcal{V}_P \). A version of (5.3) was established in [54, Corollary 5.8] in case \( d = 2 \) and \( \mathcal{V}_P \) is smooth (the proof given there works verbatim for fully saturated determinantal representations, see Section 4.7 when \( \mathcal{V}_P \) is possibly singular and / or reducible) using essentially the pairing between the kernel and the left kernel alluded to in Section 4.3.

Assume now that the dehomogenization \( p(x_1, \ldots, x_d) = P(1, x_1, \ldots, x_d) \) is a \( \mathbb{R}^*_0 \) polynomial with \( p(x^0) \neq 0 \), let \( X^0 = (1, x^0) \), and let \( U \) be a self-adjoint determinantal representation. Let \( \mathcal{L} \) be a straight line through \([X^0]\) in \( \mathbb{P}^d(\mathbb{R}) \) intersecting \( \mathcal{V}_P(\mathbb{R}) \) in \( m \) distinct points \([X^1], \ldots, [X^m]\). Then we have

Lemma 5.2. \( U(X^0) > 0 \) if and only if the compression of \( U(X^0) \) to \( \ker \mathcal{U}(X^i) \) is positive definite for \( i = 1, \ldots, m \).

This is just a special case of [54, Proposition 5.5]: the statement there is for \( d = 2 \) but the proof for general \( d \) is exactly the same (it amounts to restricting the determinantal representation \( U \) to the straight line \( \mathcal{L} \), and looking at the canonical form of the resulting hermitian matrix pencil). We give a direct argument in our situation.

Proof of Lemma 5.2. Choose \( X \in \mathbb{R}^{d+1} \) so that \( \mathcal{L} \setminus \{[X^0]\} = \{[X - sX^0]\}_{s \in \mathbb{R}} \). Then \( X^i = X - s_i X^0 \), where \( s_i, i = 1, \ldots, m \), are the zeroes of the univariate polynomial \( P(X - sX^0) \), i.e., the eigenvalues of the generalized eigenvalue problem

\[
(U(X) - sU(X^0)) v = 0.
\]
Lemma 5.2 then shows that $U_1 = \cdots = \ker U(X^t)$. The corresponding eigenspaces are precisely $\ker U(X^t)$. The lemma now follows since the different eigenspaces are orthogonal with respect to $U(X^t)$: if $v_i \in \ker U(X^t)$, $v_j \in \ker U(X^t)$, $i \neq j$, then
\[ s_i v_j^* U(X^t) v_i = v_j^* U(X) v_i = s_j v_j^* U(X^t) v_i \]
(since $s_j \in \mathbb{R}$), implying that $v_j^* U(X^t) v_i = 0$ (since $s_i \neq s_j$). \hfill $\square$

We notice that Lemma 5.2 remains true for non-reduced polynomials provided the determinantal representation $U$ is generically maximal (or generically maximally generated) \cite{22}; if $P = P_1^r \cdots P_k^r$, where $P_1, \ldots, P_k$ are distinct irreducible polynomials, this means that that dim $\ker U(X) = r_i$ at a general point $[X]$ of $V_{P_i}$, $i = 1, \ldots, k$. Since positive self-adjoint determinantal representations are always generically maximal, this may open the possibility of generalizing Theorem 5.3 below to the non-reduced setting.

**Theorem 5.3.** Let $p \in \mathbb{R}[x_1, \ldots, x_d]$ be an irreducible $\mathbb{R}Z_{d,0}$ polynomial of degree $m$ with $p(x^d) \neq 0$, let $P$ be the homogenization of $p$, and let $X^0 = (1, x^0)$. Let $U$ be a self-adjoint determinantal representation of $P$ with adjoint matrix $V$, as in \cite{13}. Then $U(X^0)$ is either positive or negative definite if and only if the polynomial $V_{jj}(X)$ interlaces $P$; here $j$ is any integer between 1 and $m$.

**Proof.** The fact that $U(X^0) > 0$ implies the interlacing follows immediately from Cauchy’s interlace theorem for eigenvalues of Hermitian matrices, see, e.g., \cite{31}. We provide a unified proof for both directions.

Let $L$ be a straight line through $[X^0]$ in $\mathbb{R}^d(\mathbb{R})$ intersecting $V_P(\mathbb{R})$ in $m$ distinct points $[X^1], \ldots, [X^m]$ none of which is a zero of $V_{jj}$. Lemma 5.1 implies that for any $[X] \in V_P(\mathbb{R})$,
\[ F_j(X)^* U(X^0) F_j(X) = P_{X^0}(X) V_{jj}(X). \]
Lemma 5.2 then shows that $U(X^0)$ is positive or negative definite if and only if $P_{X^0} V_{jj}$ has the same sign (positive or negative, respectively) at $X^i$ for $i = 1, \ldots, m$.

Similarly to the proof of Lemma 5.2, let us choose $x \in \mathbb{R}^{d+1}$ so that $L \setminus \{[X^0]\} = \{(X + s X^0) \mid s \in \mathbb{R}\}$, so that $X^i = X + s_i X^0$, where $s_1 < \cdots < s_m$ are the zeroes of the univariate polynomial $P(X + s X^0)$. It follows from Rolle’s Theorem that $\frac{d}{ds} P(X + s X^0) = P_{X^0}(X + s X^0)$ has exactly one zero in each open interval $(s_i, s_{i+1})$, $i = 1, \ldots, m - 1$, hence has opposite signs at $s_i$ and at $s_{i+1}$. Therefore $U(X^0)$ is positive or negative definite if and only if $V_{jj}(X + s X^0)$ has opposite signs at $s_i$ and at $s_{i+1}$, i.e., if and only if $V_{jj}$ interlaces $P$. \hfill $\square$

It would be interesting to find an analogue of Theorem 5.3 for other signatures of a self-adjoint determinantal representation, similarly to \cite{54} Section 5.

Combining Theorem 5.3 with the construction of determinantal representations that was sketched in Section 4.5 (see also Section 4.7 for the extension of the construction to the singular case) then yields the following result.

**Theorem 5.4.** Let $p \in \mathbb{R}[x_1, x_2]$ be an irreducible $\mathbb{R}Z_{d,0}$ polynomial of degree $m$ with $p(x^0) = 1$, let $P$ be the homogenization of $p$, let $\nu : \tilde{V}_P \to V_P$ be the desingularization of the corresponding complex projective curve, and let $\Delta$ be the adjoint divisor on
two basic open questions here are: V

convex algebraic interior containing x can take the directional derivative there certainly exist interlacing polynomials vanishing on the adjoint divisor: we version of Theorem 3.1 for positive self-adjoint determinantal representations since that so is the meromorphic differential ω either everywhere nonnegative or everywhere nonpositive) on V

where L i.e., D was already implied when we wrote, e.g., V

polynomial, implies that V

Proof of Theorem 5.4. It is not hard to see that Q interlacing P implies that VQ is contact to VP at real points of intersection, and that we can write (Q) = D + D′ + Δ. It only remains to show that D − (L) is not linearly equivalent to an effective divisor, where L is a linear form.

Notice that τ lifts to an antiholomorphic involution on the desingularization (this was already implied when we wrote, e.g., D′). Furthermore, the fact that p is a RZ polynomial, implies that VP is a compact real Riemann surface of dividing type, i.e., VP \ VP(R) consists of two connected components interchanged by τ, where VP(R) is the fixed point set of τ, see [54] and the references therein and [28]. We orient VP(R) as the boundary of one of these two connected components.

It is now convenient to change projective coordinates so that [X0] = [1, x0] becomes [0, 0, 1]. It is not hard to see that in the new coordinates, both the meromorphic differential ν* dx1 and the function ν* Q(1, x1, x2) have constant sign (are either everywhere nonnegative or everywhere nonpositive) on VP(R). It follows that so is the meromorphic differential ω = ν* Q(1, x1, x2) dx1. We have (see, e.g., Appendix A2]) (ω) = (Q) − Δ − 2(X0) = D + D′ − 2(X0). If there existed a rational function f and an effective divisor E on VP so that (f) + D − (X0) = E, we would have obtained that (fωf′) = E + E′, i.e., fωf′ is a nonzero holomorphic differential that is everywhere nonnegative or everywhere nonpositive on VP(R), a contradiction since its integral over VP(R) has to vanish by Cauchy’s Theorem.

Notice that this proof is essentially an adaptation of [54] Proposition 4.2 which is itself an adaptation of [15]; it would be interesting to find a more elementary argument.

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