Massive Dual Spinless Fields Revisited

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Abstract

Massive dual spin zero fields are reconsidered in four spacetime dimensions. A closed-form Lagrangian is presented that describes a field coupled to the curl of its own energy-momentum tensor.

In tribute to Peter George Oliver Freund (1936-2018)

Introduction

As indicated in the Abstract, the point of this paper is to find an explicit Lagrangian for the dual form of a massive scalar field self-coupled in a particular way to its own energy-momentum tensor. This boils down to a well-defined mathematical problem whose solution is given here, thereby completing some research initiated and published long ago in this journal [1].

After first presenting a concise mathematical statement of the problem, and then giving a closed-form solution in terms of elementary functions, the field theory that led to the problem is re-examined from a fresh perspective. The net result is a very direct approach that leads to both the problem and its solution.

Some History

Here I reconsider research first pursued in collaboration with Peter Freund, in an effort to tie up some loose ends. In the spring of 1980, when I was a post-doctoral fellow in Yoichiro Nambu’s theory group at The Enrico Fermi Institute, Peter and I were confronted by a pair of partial differential equations (see [1] p 417).

\[ m^2 \frac{\partial^2 \mathcal{L}}{\partial u^2} + 2 \left( \frac{\partial \mathcal{L}}{\partial u} \right) = 4g \left( \frac{\partial \mathcal{L}}{\partial u} \right) \left( u \frac{\partial^2 \mathcal{L}}{\partial u \partial v} + v \frac{\partial^2 \mathcal{L}}{\partial v^2} - \frac{\partial \mathcal{L}}{\partial v} \right) , \]

\[ m^2 \frac{\partial^2 \mathcal{L}}{\partial u \partial v} = 4g \left( \frac{\partial \mathcal{L}}{\partial u} \right) \left( u \frac{\partial^2 \mathcal{L}}{\partial u^2} + v \frac{\partial^2 \mathcal{L}}{\partial u \partial v} - \frac{\partial \mathcal{L}}{\partial u} \right) , \]

where \( m \) and \( g \) are constants. We noticed in passing that these PDEs imply the secondary condition

\[ \left( \frac{\partial^2 \mathcal{L}}{\partial u \partial v} \right)^2 = \left( \frac{\partial^2 \mathcal{L}}{\partial u^2} \right) \left( \frac{\partial^2 \mathcal{L}}{\partial v^2} \right) , \]

and we then looked for a solution to (1-3) as a series in \( g \) beginning with

\[ \mathcal{L}(u,v) = \frac{1}{2} v^2 - \frac{1}{2} m^2 u + \frac{g}{m} \left( \frac{1}{3} v^3 - m^2 uv \right) + O(g^2) . \]
To simplify the equations to follow, I will rescale \( g = m \kappa \) so that the constant \( m \) always appears in (1-3) only in the combination \( v/m \). Thus I may as well set \( m = 1 \), and hence \( \kappa = g \). I can then restore the parameter \( m \) in any subsequent solution for \( \mathcal{L} \) by the substitution \( \mathcal{L}(u, v) \to m^2 \mathcal{L}(u, v/m) \).

Clearly, there is a two-parameter family of exact solutions to these PDEs which depends only on \( v \), namely,

\[
\mathcal{L}_0(v) = a + b v ,
\]

where \( a \) and \( b \) are constants. However, for the model field theory that gave rise to the partial differential equations (1,2), this linear function of \( v \) amounts to a topological term in the action and therefore gives no contribution to the bulk equations of motion. Moreover, \( \mathcal{L}_0(v) \) contributes only a (cosmological) constant term to the canonical energy-momentum tensor. So, in the context of our 1980 paper \([1]\) where solutions of (1,2) were sought which gave more interesting contributions, this \( \mathcal{L}_0(v) \) was not worth noting. Nevertheless, it reappeared in another context, somewhat later \([2]\).

**Completing Some Unfinished Business**

It so happened in 1980 that Peter and I did not find an exact \( \mathcal{L}(u, v) \) to solve the PDEs (1,2). In fact, we reported then only the terms given in \([4]\). Here I wish to present an exact, closed-form solution to all orders in \( g \).

The crucial feature leading to this particular solution is that the dependence on \( v \) is only through the linear combination \( v - gu \). The result is

\[
\mathcal{L}(u, v) = -\frac{1}{2} u + \frac{1}{2} (v - gu)^2 + \frac{1}{3} g (v - gu)^3 \binom{n}{g} F_2 \left(1, \frac{1}{2}; \frac{3}{2}, \frac{5}{2}; -4g^2 (v - gu)^2\right) ,
\]

where as a series

\[
F(w) = \frac{1}{3} w^3 \binom{n}{g} F_2 \left(1, \frac{1}{2}; \frac{3}{2}, \frac{5}{2}; -4w^2\right) = \sum_{n=1}^{\infty} \frac{(2n - 2)!}{(n-1)! n!} \frac{(-1)^{n+1}}{2n+1} w^{2n+1}
\]

\[
= \frac{1}{3} w^3 - \frac{1}{5} w^5 + \frac{2}{7} w^7 - \frac{5}{9} w^9 + \frac{14}{11} w^{11} - \frac{42}{13} w^{13} + \frac{44}{5} w^{15} + O(w^{17}) .
\]

Fortunately, the \( \binom{n}{g} F_2 \) hypergeometric function in \([7]\) reduces to elementary functions. For real \( w \),

\[
F(w) = -\frac{1}{2} w + \frac{1}{4} w \sqrt{1 + 4w^2} + \frac{1}{8} \ln \left(2w + \sqrt{1 + 4w^2}\right) .
\]

Nevertheless, the solution \([6]\) was first obtained in its series form \([7]\) and only afterwards was it expressed as a special case of the hypergeometric \( \binom{n}{g} F_2 \), with its subsequent simplification to elementary functions.

More generally, it is not so difficult to establish that solutions to (1-3) necessarily have the form

\[
\mathcal{L}(u, v) = -\frac{1}{2g} v + G \left(v + 2g \int u H(s) ds\right) ,
\]

where the function \( G \) is differentiable, and \( H \) is integrable, but otherwise not yet determined, as befits the general solution of a more easily solvable 1st-order PDE, albeit nonlinear:

\[
\frac{\partial}{\partial v} \ln \left(\frac{\partial \mathcal{L}}{\partial u}\right) = \frac{\partial}{\partial v} \ln \left(1 + 2g \frac{\partial \mathcal{L}}{\partial v}\right) .
\]

Note in \([9]\) the return of an explicit term linear in \( v \). This term arises as the particular solution of the inhomogeneous 1st-order PDE that results from integrating \([10]\) and exponentiating, namely,

\[
\frac{1}{H(u)} \frac{\partial \mathcal{L}}{\partial u} - 2g \frac{\partial \mathcal{L}}{\partial v} = 1 .
\]

The functions \( G \) and \( H \) are now constrained by additional conditions that lie hidden within (1) and (2).

I will leave it to the reader to flesh out those additional conditions. I will not go through that analysis here. Instead, I will reconsider the model field theory that led to the partial differential equations (1,2) in light of the exact solution \([6]\). That solution provides a good vantage point to view and analyze the model.
The Model Revisited

Consider a Lagrangian density \( \mathcal{L}(u, v) \) depending on a vector field \( V^\mu \) through the two scalar variables,

\[
u = V_\mu V^\mu, \quad v = \partial_\mu V^\mu.
\] (12)

This vector field is to be understood in terms of an antisymmetric, rank 3, tensor gauge field, \( V_{\alpha\beta\gamma} \), i.e. the four-dimensional spacetime dual of a massive scalar [1], with its corresponding gauge invariant field strength,

\[
F_{\mu\alpha\beta\gamma} = \partial_\mu V_{\alpha\beta\gamma} - \partial_\alpha V_{\beta\gamma\mu} + \partial_\beta V_{\gamma\mu\alpha} - \partial_\gamma V_{\mu\alpha\beta}.
\]

Thus

\[
V^\mu = \frac{1}{6} \varepsilon^{\mu\alpha\beta\gamma} V_{\alpha\beta\gamma}, \quad \partial_\mu V^\mu = \frac{1}{24} \varepsilon^{\mu\alpha\beta\gamma} F_{\mu\alpha\beta\gamma}.
\] (13)

The bulk field equations that follow from the action of \( \mathcal{L}(u, v) \) by varying \( V_\mu \) are simply

\[
\partial_\mu \mathcal{L}_v = 2 V_\mu \mathcal{L}_u,
\] (14)

where the partial derivatives of \( \mathcal{L} \) are designated by \( \mathcal{L}_u \equiv \partial \mathcal{L}(u, v)/\partial u \) and \( \mathcal{L}_v \equiv \partial \mathcal{L}(u, v)/\partial v \). An obvious inference from these field equations is that the on-shell vector \( V_\mu \) is a gradient of a scalar \( \Phi \),

\[
V_\mu = \partial_\mu \Phi,
\] (15)

if and only if \( \mathcal{L}_u \) is a function of \( \mathcal{L}_v \). For example, if \( \mathcal{L}_u \) has a linear relation to \( \mathcal{L}_v \) with \( \mathcal{L}_u = a + b \mathcal{L}_v \) for constants \( a \) and \( b \), the field equations give

\[
\Phi = \frac{1}{2b} \ln (a + b \mathcal{L}_v),
\] (16)

More generally, if \( \mathcal{L}_u = \Psi(\mathcal{L}_v) \), then

\[
\Phi = \frac{1}{2} \int \mathcal{L}_v \frac{dz}{\Psi(z)}.
\] (17)

But in any case, on-shell the combination \( U_\mu = V_\mu \mathcal{L}_u \) is a spacetime gradient.

An additional gradient of the field equations then gives

\[
\partial_\lambda \partial_\mu \mathcal{L}_v = 2 (\partial_\lambda V_\mu) \mathcal{L}_u + 2 V_\mu \partial_\lambda \mathcal{L}_u.
\] (18)

From \( \partial_\lambda \partial_\mu \mathcal{L}_v = \partial_\mu \partial_\lambda \mathcal{L}_v \) it follows that

\[
(\partial_\mu V_\lambda - \partial_\lambda V_\mu) \mathcal{L}_u = V_\mu \partial_\lambda \mathcal{L}_u - V_\lambda \partial_\mu \mathcal{L}_u.
\] (19)

Thus the vector \( V_\mu \) is a gradient of a scalar, as in (15), such that

\[
\partial_\mu V_\lambda = \partial_\lambda V_\mu,
\] (20)

if and only if for some scalar function \( \Omega \),

\[
\partial_\lambda \mathcal{L}_u = V_\lambda \Omega.
\] (21)

Simplification

Now for simplicity, demand that \( \mathcal{L}_u = a + b \mathcal{L}_v \) for constants \( a \) and \( b \), in accordance with \( V_\mu \) being a gradient, as in (15) and (20). This linear condition is immediately integrated to obtain

\[
\mathcal{L}(u, v) = au + L(v + bu),
\] (22)

where \( L(v + bu) \) is a differentiable function of the linear combination \( v + bu \). The field equations (14) are now

\[
\partial_\lambda \mathcal{L}_v = \partial_\lambda L' = 2(a + b L') V_\lambda = 2 V_\lambda \mathcal{L}_u.
\] (23)

That is to say, the scalar in (21) is \( \Omega = 2ab + 2b^2 L' \).
Energy-momentum tensors

In [1] Peter and I say that, given (1-3), the field equations for $V_\mu$ amount to (20) along with the “simple, indeed elegant” statement

$$(\Box + m^2) V_\mu = \frac{g}{m} \partial_\mu \theta ,$$

where $g$ has units of length, and $\theta$ is the trace of the conformally improved energy-momentum tensor.

Be that as it may, there is a less oracular method to reach this form for the field equations in light of the simplification (22). As is well-known, there may be two distinct expressions for energy-momentum tensors that result from any Lagrangian. From (22) the canonical results for $\Theta_{\mu \nu}$, and its trace $\Theta = \Theta_{\mu}^{\mu}$, are immediately seen to be

$$\Theta^{[\text{canon}]}_{\mu \nu} = (\partial_\mu V_\nu) L' - g_{\mu \nu} (au + L) , \quad \Theta^{[\text{canon}]} = v L' - 4 (au + L) .$$

(25)

Although not manifestly symmetric, it is nonetheless true that $\Theta^{[\text{canon}]}_{\mu \nu} = \Theta^{[\text{canon}]}_{\nu \mu}$ on-shell in light of the condition (20).

Surprisingly different results follow from covariantizing (22) with respect to an arbitrary background metric $g_{\mu \nu}$, varying the action for $\sqrt{-\det g_{\alpha \beta}} L$ with respect to that metric, and then taking the flat-space limit. This procedure gives the “gravitational” energy-momentum tensor and its trace:

$$\Theta^{[\text{grav}]}_{\mu \nu} = -2 (a + bL') V_\mu V_\nu - g_{\mu \nu} (L - au - (v + 2bu) L') , \quad \Theta^{[\text{grav}]} = 4v + 6bu) L' + 2au - 4L .$$

(26)

The unusual structure exhibited in this tensor follows because in curved spacetime $V^\mu$ as defined by (13) is a relative contravariant vector of weight +1 with no dependence on the metric, so $\partial_\mu V^\mu$ is a relative scalar of weight +1 also with no dependence on $g_{\mu \nu}$, and $V_\mu V^\mu = g_{\mu \nu} V^\mu V^\nu$ is a relative scalar of weight +2 where all dependence on the metric is shown explicitly. Hence the absolute scalar version of $\mathcal{L}(u, v)$ is given by

$$\mathcal{L} = ag_{\mu \nu} V^\mu V^\nu / (-\det g_{\alpha \beta}) + L \left( (\partial_\mu V^\mu) / \sqrt{-\det g_{\alpha \beta}} + bg_{\mu \nu} V^\mu V^\nu / (-\det g_{\alpha \beta}) \right) ,$$

(27)

where again all the metric dependence is shown explicitly.

It is straightforward to check on-shell conservation of either (25) or (26), separately. However, it turns out the flat-space equations of motion can now be written in the form (24) provided a linear combination of $\Theta^{[\text{canon}]}_{\mu \nu}$ and $\Theta^{[\text{grav}]}_{\mu \nu}$ is used for the system’s energy-momentum tensor. Let

$$\Theta_{\mu \nu} = \frac{2}{3} \Theta^{[\text{canon}]}_{\mu \nu} + \frac{1}{3} \Theta^{[\text{grav}]}_{\mu \nu} .$$

(28)

The trace is then

$$\Theta = \Theta_{\mu}^{\mu} = 2 (v + bu) L' - 4L - 2au .$$

(29)

Field equation redux

Since various scales have been previously chosen to set $m = 1$, the field equations (20) and (24) give for the left-hand side of (24)

$$(\Box + 1) V_\mu = \left( 1 + \frac{1}{2} \frac{L''}{a + bL'} \right) \partial_\mu (v + bu) - b \partial_\mu u ,$$

(30)

where (24) implies $\Box V_\mu = \partial^\lambda \partial_\lambda V_\mu = \partial^\lambda \partial_\lambda \partial_\mu u = \partial_\mu v$. On the other hand, from (24) for any constant $c$,

$$c \partial_\mu \Theta = 2c ((v + bu) L'' - L') \partial_\mu (v + bu) - 2ac \partial_\mu u .$$

(31)

The choice $2ac = b$ reconciles the spurious $\partial_\mu u$ term to give the desired form

$$(\Box + 1) V_\mu = c \partial_\mu \Theta$$

(32)

provided the function $L$ satisfies the second-order nonlinear equation

$$1 + \frac{1}{2} \frac{L''(z)}{a + bL'(z)} = 2c (zL''(z) - L'(z)) .$$

(33)

But note, the constant $c$ can be set to a convenient nonzero value by further rescalings.
For example, if \((a, L) \to \left(\frac{a}{2c}, \frac{L}{2b}\right)\), along with the previous choice \(2ac = b \to a = 1\), the equation for \(L\) becomes
\[
1 + \frac{1}{2b} \frac{L''}{b} = \frac{1}{b} \left(zL'' - L'\right).
\] (34)
Finally, rescaling \(z \to w/b\) gives
\[
1 + \frac{1}{2} \frac{L''}{1 + L'} = (wL'' - L')
\] (35)
The solution of this equation for \(L'\) with initial condition \(L'(0) = 0\) is
\[
L'(w) = -1 - 2w + \sqrt{1 + 4w^2}.
\] (36)
Imposing the additional initial condition \(L(0) = 0\), this integrates immediately to
\[
L(w) = -w - w^2 + \frac{1}{2} w\sqrt{1 + 4w^2} + \frac{1}{4} \ln \left(2w + \sqrt{1 + 4w^2}\right).
\] (37)
Comparison with \(\Box\) shows that
\[
L(w) = -w^2 + 2F(w).
\] (38)
Given the previous rescalings, namely, \(\mathcal{L}(u, v) = au + L(v + bu) \to \left[\frac{a}{2c}u + \frac{1}{2b}L(w = bz)\right]_{a=1}\), the Lagrangian density for the model becomes
\[
\mathcal{L}(u, v) = \frac{b}{2c}u + \frac{1}{2bc} \left(-bz - (bz)^2 + \frac{1}{2} (bz) \sqrt{1 + 4(bz)^2} + \frac{1}{4} \ln \left(2(bz) + \sqrt{1 + 4(bz)^2}\right)\right)
\] (39)
\[
= \frac{b}{2c}u - \frac{b}{2c}z^2 + \frac{b^2}{3c}z^3 + O(z^4).
\] (40)
As before, \(v = \partial_\mu V^\mu, u = V^\mu V_\mu\), and \(z = v + bu\). Note that the term linear in \(z\) in (39) cancels out upon power series expansion, so the result agrees with (4) up to and including all terms of \(O(V^3)\).
To comport to the conventions in \([1]\), choose \(b = -g\) and \(c = g\), so that \(z = v - gu\), to find
\[
\mathcal{L}(u, v) = -\frac{1}{2} - \frac{1}{2g^2} \left(\frac{g(v - gu) - g^2(v - gu)^2 - \frac{1}{2} g(v - gu)\sqrt{1 + 4g^2(v - gu)^2}}{\sqrt{1 + 4g^2(v - gu)^2}}\right)
\] (41)
\[
= -\frac{1}{2} u + \frac{1}{2} (v - gu)^2 + \frac{1}{3} g(v - gu)^3 + O((v - gu)^4),
\] (42)
Now restore \(m\) via the coordinate rescaling \(x_\mu \to mx_\mu\), hence \(v \to v/m\) and \(\mathcal{L}(u, v) \to m^2 \mathcal{L}(u, v/m)\), thereby converting (42) into the form (21), with \(\theta = m^2 \Theta\).

**Discussion**

The conventional integral equation form of (21), including a free-field term with \(\Box + m^2 V^{(0)}\mu\) = 0, is given by
\[
V_\mu(x) = V^{(0)}_\mu(x) + \frac{g}{m} \int G(x - y) \frac{\partial \Theta(y)}{\partial y^\mu} \, d^4 y,
\] (43)
where \(\Theta(y)\) depends implicitly on the field \(V_\mu(y)\) and \(G\) is the usual isotropic, homogeneous, Dirichlet boundary condition Green function that solves \((\Box + m^2) G(x - y) = \delta^4(x - y)\). The free-field term must be a gradient, \(V^{(0)}_\mu(x) = \partial_\mu \Phi^{(0)}(x)\) with \((\Box + m^2) \Phi^{(0)} = 0\), to ensure that \(V_\mu(x) = \partial_\mu \Phi(x)\) is also a gradient. Integration by parts followed by an overall integration then gives
\[
\Phi(x) = \Phi^{(0)}(x) + \frac{g}{m} \int G(x - y) \Theta(y) \, d^4 y,
\] (44)
where now \(\Theta(y)\) depends implicitly on \(\Phi(y)\). That is to say, \((\Box + m^2) \Phi = \frac{a}{m} \Theta [\Phi(x)]\).
On the one hand, this is not surprising, since there is a long-known construction of an explicit local Lagrangian that leads directly to this form for the scalar field equations [6]. (It amounts to the Goldstone model after scalar field redefinition.) Taking a gradient to reverse the steps above then leads back to [43]. On the other hand, it is far from obvious that Θ [Φ (x)] can be re-expressed as a local function of V μ = ∂μΦ, and that Θ [V μ (x)] follows in turn from a local, closed-form Lagrangian for V μ. The main point of this paper was to show that, indeed, there is an L such that all this is true.

Were Θ due to anything other than V μ, field equations of the form (24) would easily follow from $L_{\text{easy}} = \frac{1}{2} (\partial_\mu V_\nu \partial^\mu V^\nu - m^2 V_\mu V^\mu) + \frac{g}{m} V^\mu \partial_\mu \Theta^\text{other}$,  
\[ (45) \]
i.e. a simple direct coupling of the vector to the gradient of any other traced energy-momentum tensor. With a pinch of plausibility, this calls to mind the axion coupling, albeit without the group theoretical and topological underpinnings, not to mention the phenomenology.

In any case, Peter and I certainly did not have axions in mind in 1980 when we wrote [1]. As best I can recall, we had only some embryonic thoughts about massive gravity. In that context we speculated (see [1] p 418) that \( g/m \sim L_{\text{Hubble}}L_{\text{Planck}} = (4.7 \times 10^{-5} \text{ m})^3 = 1/(4.2 \times 10^{-3} \text{ eV})^2 \). In retrospect, we were both struck by the fact that this guess is approximately the same as phenomenological lower limits for \( 1/m^2_{\text{axion}} \).

There is one more noteworthy piece of unfinished business in [1], namely, a closed-form Lagrangian for a massive spin 2 field coupled to the four-dimensional curl of its own energy-momentum tensor, where the spin 2 field is not the usual symmetric tensor, but rather the rank three tensor \( T_\lambda^{\mu \nu} \). For progress on this additional unfinished business, please see [8]. With enough effort, perhaps a complete formulation of this spin 2 model will also be available soon, along with a few other variations on the theme of fields coupled to Θ_μν.

In closing, so far as I can tell, Peter had little if any interest in totally antisymmetric tensor gauge fields prior to our paper [1]. But he quickly pursued the subject in stellar fashion with his subsequent work on dimensional compactification [2]. While all this work is still conjectural, at the very least it provided and continues to provide fundamental research problems in theoretical physics, especially for doctoral students.

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[2] P.G.O. Freund and M. Rubin, “Dynamics of Dimensional Reduction” Phys.Lett. 97B (1980) 233-235
[3] In other words, \( W[\mathcal{L}_u, \mathcal{L}_v] = 0 \), where \( W[f, g] = \det \left( \begin{array}{cc} \partial f/\partial u & \partial g/\partial u \\ \partial f/\partial v & \partial g/\partial v \end{array} \right) \). Note that \( W[f, g] = 0 \) is a necessary (but not sufficient) condition for a linear dependence between \( f \) and \( g \) [4]. In fact, \( W[f, g] = 0 \) for any (differentiable) functional dependence between \( f \) and \( g \). For the case at hand, \( W \) is actually a determinant of a 2 × 2 Hessian matrix, an expression familiar from galileon models (e.g. see [5]).
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