Primitive divisors of elliptic divisibility sequences with 
\[ j = 1728 \]
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Abstract

Take a rational elliptic curve defined by the equation \( y^2 = x^3 + ax \) in minimal form and consider the sequence \( B_n \) of the denominators of the abscissas of the iterate of a non-torsion point; we show that \( B_{5m} \) has a primitive divisor for every \( m \). Then, we show how to generalize this method to the terms in the form \( B_{mp} \) with \( p \) a prime congruent to 1 modulo 4.

1 Introduction

Definition 1.1. A sequence of integers \((x_n)_{n \in \mathbb{N}}\) is a divisibility sequence if
\[ m \mid n \implies x_m \mid x_n. \]
Given a sequence of integers \((x_n)_{n \in \mathbb{N}}\), we say that the \( n \)-th term has a primitive divisor if there exists a prime \( p \) such that
\[ p \mid x_n \text{ and } p \nmid x_1 \cdots x_{n-1}. \]

Definition 1.2. Take an elliptic curve \( E \), defined over \( \mathbb{Q} \). Consider \( P \) a rational non-torsion point on \( E \) and take
\[ x(nP) = \frac{A_n}{B_n} \text{ with } (A_n, B_n) = 1 \text{ and } B_n > 0. \]
We will say that the sequence of positive integers \( \{B_n\}_{n \in \mathbb{N}} \) is an elliptic divisibility sequence. The sequence of the \( B_n \) depends on \( E \) and \( P \) and we will sometimes denote it with \( B_n(E, P) \).

Thanks to [5, Proposition 10], we know that for every elliptic curve in minimal form and for every non-torsion point \( P \in E(\mathbb{Q}) \), \( B_n(E, P) \) has a primitive divisor for \( n \) large enough. Computational evidence suggests that it is only when \( n \) is very small that \( B_n \) does not have a primitive divisor. As far as I know, the example of the \( B_n(E, P) \) without a primitive divisor for the largest \( n \) for \( E \) in minimal form is at \( n = 39 \) and it is given at the beginning of page 476 of [2]. Given the curve \( E \) defined by the equation
\[ y^2 + xy + y = x^3 + x^2 - 125615x + 61201397 \]

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and \( P = (7107, -602054) \), \( B_{10}(E, P) \) does not have a primitive divisor. For some classes of curves, there are some effectivity results. For example, in [1] it is proved that, if \( E \) has a non-trivial rational 2-torsion point, then \( B_n \) has a primitive divisor for \( n \) even and greater than an effective computable constant. Also in [1, Theorem 2.2], an unconditional result for primitive divisors of elliptic divisibility sequences associated with elliptic curves of the form \( y^2 = x^3 - T^2x \) is obtained. The work in [9] both improves and generalizes this result, proving that, if \( E \) is defined by \( y^2 = x^3 + ax \) with a fourth-power-free integer, then the sequence of the \( B_n \) has a primitive divisor for every \( n \geq 3 \) even.

The first aim of this paper is to correct an error in the proof of this fact. In order to prove the main result of [9], it is necessary to show that \( B_{5m} \) has a primitive divisor for every \( m \). Yabuta and Voutier prove this in their Lemma 5.1. In the proof of this lemma there is a mistake at the end of page 181 that we want to fix. Putting \( C_k = B_{mk} \), the authors assume that if \( B_{5m} \) does not have a primitive divisor, then \( C_5 \) does not have a primitive divisor too. However, this is not necessarily true. It is possible that \( C_5 \) has a primitive divisor which divides \( B_n \) for some \( n \mid 5m \) with \( m \nmid n \). This means that their use of their Lemma 3.4 to obtain an upper bound for \( \log B_{5m} \) is not correct. This same mistake also seems to affect the proof of Lemma 7 in [2]. We will fix this issue, proving the following.

**Theorem 1.3.** Let \( E_a \) be the elliptic curve defined by the equation

\[
y^2 = x^3 + ax
\]

with \( a \) an integer fourth-power-free and \( P \) be a non-torsion point in \( E(Q) \). Then, \( B_n(E_a, P) \) has a primitive divisor, if \( n \) is a multiple of 5.

Observe that, up to isomorphism over \( \mathbb{Q} \), every elliptic curve in minimal form and with \( j \)-invariant equal to 1728 is defined by the equation \( y^2 = x^3 + ax \) with \( a \) an integer fourth-power-free.

Finally, we show how to generalize the proof of Theorem 1.3 to the case when we consider the terms in the form \( B_{mp} \), for \( p \equiv 1 \mod 4 \), proving the following theorem.

**Theorem 1.4.** Let \( E_a \) be as before and \( P \) be a non-torsion point in \( E_a(Q) \). Take a prime \( p \) congruent to 1 modulo 4. Then \( B_n(E_a, P) \) has a primitive divisor if \( n \) is square-free, \( n > p \) and \( p \) is the smallest divisor of \( n \).

In [9, Remark 1.5], it is conjectured that for every sequence \( B_n(E_a, P) \), every term has a primitive divisor for \( n \geq 4 \). The work of this paper made one step forward in order to prove this conjecture.

## 2 Preliminaries

We start by recalling the hypothesis of [9] and the facts that we will use. Let \( a \) be a fourth-power-free integer and \( E_a \) be the elliptic curve defined by the equation \( y^2 = x^3 + ax \). We will denote by \( \Delta \) the discriminant of the curve, that is \( \Delta := -64a^3 \). We define the height of a rational number as

\[
H\left(\frac{u}{v}\right) = \max\{|u|, |v|\}
\]

if \( u \) and \( v \) are coprime and the logarithmic height as

\[
h\left(\frac{u}{v}\right) = \log H\left(\frac{u}{v}\right).
\]
Given $P \in E(\mathbb{Q})$, we define $H(P) = H(x(P))$ and $h(P) = h(x(P))$. We consider the canonical height of a point as defined in [6, Proposition VIII.9.1]. Observe that if $P$ is in $E_a(\mathbb{Q})$ and $x(P) = u/v$ with $u$ and $v$ two coprime integers, then

$$y^2 = x^3 + ax = \frac{u^3 + auv^2}{v^3}.$$ 

Since $(u^3 + auv^2, v^3) = 1$, then $v^3$ is a square and therefore also $v$ it is. In particular, every term of the sequence of the $B_n(E, P) > 0$ is a square. Moreover, if $A_n = 0$, then $x(nP) = 0$ and so $y(nP)^2 = x(nP)^3 + ax(P) = 0$. If $y(nP) = 0$, then $2nP = O$ thanks to [6, III.2.3] and this is absurd since we are assuming that $P$ has infinite order. So, we have $A_n \neq 0$ for every $n \geq 1$ and in particular $|A_n| \geq 1$.

**Lemma 2.1.** Let

$$C = 0.26 + \frac{\log |a|}{4}.$$ 

Then, 

$$|h(P) - 2\hat{h}(P)| \leq 2C$$

for every $P$ in $E_a(\mathbb{Q})$.

**Proof.** This is proved in [10, Theorem 1.4].

**Lemma 2.2.** Let $P \in E_a(\mathbb{Q})$ be a point of infinite order. Consider the elliptic divisibility sequence $(B_n) = (B_n(E_a, P))$. Then, 

$$\log B_{5m} \geq 18m^2\hat{h}(P) - 29 \log |a| - 32.863.$$ 

**Proof.** This was established in the proof of [9, Lemma 5.1]. See the inequality near the bottom of page 181 in [9].

**Lemma 2.3.** Let $a$ be a fourth-power-free integer. For every non-torsion point $P$ in $E_a(\mathbb{Q})$,

$$\hat{h}(P) \geq \frac{\log |a| - \log 4}{16}.$$ 

**Proof.** This is proved in [10, Theorem 1.2].

**Lemma 2.4.** For every positive integer $n$ define

$$\rho(n) = \sum_{p|n} \frac{1}{p^2}.$$ 

Then, 

$$\rho(n) < \sum_{p \text{ prime}} \frac{1}{p^2} < 0.45225.$$ 

**Proof.** This was proved at the top of page 178 of [9].
Lemma 2.5. Suppose that $B_n$ does not have a primitive divisor. Then,
\[
\log B_n \leq 2\log n + 2n^2 \rho(n)\tilde{h}(P) + 2C\omega(n),
\]
where $\omega(n)$ is the number of prime divisors of $n$.

Proof. Define
\[
\eta(n) = \sum_{p|n} 2\log p.
\]
Then, as was proved in [9, Lemma 3.4],
\[
\log B_n \leq \eta(n) + 2n^2 \rho(n)\tilde{h}(P) + 2C\omega(n).
\]
We conclude the proof by observing that
\[
\eta(n) \leq 2\log n.
\]

Lemma 2.6. Let $\psi_n$ and $\phi_n$ be the polynomials in $\mathbb{Z}[x,y,a]$ as defined in [6, Exercise 3.7]. We recall the properties of these polynomials that we will use in this paper. The $\psi_n$ are the so-called division polynomials.

(a) For every $n > 0$ and every $P \in E(\mathbb{Q})$,
\[
x(nP) = \frac{\phi_n(x(P))}{\psi_n^2(x(P))}.
\]

(b) The polynomial $\phi_n$ is in $\mathbb{Z}[x,a]$.

(c) If $n$ is odd, then the polynomial $\psi_n$ is in $\mathbb{Z}[x,a]$. Instead, if $n$ is even, then $\psi_n$ is a polynomial in $\mathbb{Z}[x,a]$, multiplied by $y$. Therefore, using $y^2 = x^3 + ax$, we can assume that $\psi_n^2 \in \mathbb{Z}[x,a]$ for every $n$.

(d) The polynomial $\phi_n(x)$ is monic and has degree $n^2$. Instead, the polynomial $\psi_n^2(x)$ has degree $n^2 - 1$ and its leading coefficient is $n^2$. The zeros of this polynomial are the $x$-coordinates of the non-trivial $n$-torsion points of $E(\mathbb{Q})$.

Proof. See [6, Exercise 3.7].

Let $x(P) = u/v$ with $(u,v) = 1$. We define, with a little abuse of notation, $\phi_n(u,v)$ and $\psi_n^2(u,v)$ the homogenization of the polynomials evaluated in $u$ and $v$. Then,
\[
x(nP) = \frac{\phi_n(u)}{\psi_n^2(u)} = \frac{v^n \phi_n(u)}{v^n \psi_n^2(u)} = \frac{\phi_n(u,v)}{v\psi_n^2(u,v)}.
\]

Observe that $\phi_n(u,v)$ and $v\psi_n^2(u,v)$ are both integers and take
\[
g_n := \gcd(\phi_n(u,v), v\psi_n^2(u,v)).
\]
Therefore,
\[
B_n = \frac{v\psi_n^2(u,v)}{g_n}.
\]
Lemma 2.7. For every $n$ and $m$,
\[
|\phi_n \psi_m^2 - \psi_n^2 \phi_m| = \psi_{n+m}^2 \psi_{|n-m|}.
\]

Proof. Observe that both sides have degree $2(n^2 + m^2 - 1)$. The leading term of both sides is $(n^2 - m^2)^2$. Then, we just need to check that the zeros of the two polynomials are the same. Using the definition,
\[
\phi_m(x)\psi_n^2(x) - \phi_n(x)\psi_m^2(x) = (x(mP) - x(nP))\psi_n^2(x)\psi_m^2(x).
\]
Thanks to the group law, for every point $R \in E(Q)$, $x(R) = x(-R)$, as shown for example in [6, III.2.3]. If $Q$ is a point of $(n + m)$-torsion, then $x(nQ) = x(-mQ) = x(mQ)$ and so the left side is annihilated in the $x$-coordinates of the $(n + m)$-torsion points. If $Q$ is a point of $|n - m|$-torsion, then $x(nQ) = x(mQ)$ and so the left side is also annihilated in the $x$-coordinates of the $|n - m|$-torsion points. The non-trivial $(n + m)$-torsion points are $(n + m)^2 - 1$ and the non-trivial $|n - m|$-torsion points are $(n - m)^2 - 1$. Therefore, the union of these two sets has $2(n^2 + m^2 - 1)$ elements, that is the degree of the polynomial. So, the roots of both polynomials are the abscissas of the non-trivial $(n + m)$-torsion points and the non-trivial $|n - m|$-torsion points.

Lemma 2.8. Let $u$ and $v$ be two coprime integers. Then,
\[
\gcd(\phi_k(u, v), v\psi_k^2(u, v)) | \Delta k^2(k^2-1)/6.
\]

Proof. Let $R_k := \text{Res}(\phi_k(x), \psi_k^2(x))$, where with Res we denote the resultant of the two polynomials. Then, there exist two polynomials $P_k$ and $Q_k$ with integer coefficients such that
\[
P_k(x)\phi_k(x) + Q_k(x)\psi_k^2(x) = R_k.
\]
Evaluating the equation in $x = u/v$ and multiplying by an appropriate power of $v$, we have
\[
P'_k(u, v)\phi_k(u, v) + vQ'_k(u, v)\psi_k^2(u, v) = R_kv^s
\]
where $P'_k$ and $Q'_k$ are two bivariate polynomials. Thus,
\[
\gcd(\phi_k(u, v), v\psi_k^2(u, v)) | R_kv^s.
\]
If $p$ is a prime divisor of $v$, then
\[
\phi_k(u, v) \equiv u^{k^2} \not\equiv 0 \mod (p)
\]
since $\phi_k$ is monic and then the gcd does not divide any prime divisor of $v$. So,
\[
\gcd(\phi_k(u, v), v\psi_k^2(u, v)) | R_k.
\]
Using [4, Theorem 1.1] we know that
\[
R_k = \Delta \frac{k^2(k^2-1)}{6}
\]
and so we conclude.

Remark 2.9. We can assume $|a| \geq 2$. Indeed, if $|a| = 1$, then $E_a$ has rank 0 and so there are no non-torsion points.
Now, we briefly show how we perform some of the computations. We will use PARI/GP 2.11.1 [7] and SAGE 8.2 [8].

- A Thue equation can be solved using the command "thue" on the software PARI/GP.

- At some point we will need to compute a lower bound for the canonical height of every non-torsion point of a given curve $E$. We can use the command "E.height function ().min(.0001, 20)" of SAGE. This gives a lower bound with an error less than 0.01 for the curves that we will consider. This means that the canonical height of every non-torsion point can be bounded from below by this value minus 0.01.

- With the command "ellheight" of PARI/GP we can compute the canonical height of a point on an elliptic curve.

Remark 2.10. The definition of the canonical height used by PARI/GP and SAGE is slightly different from our definition. Indeed, our canonical height is half the height of the two software. So, every value computed with PARI/GP and SAGE has to be divided by 2.

Lemma 2.11. If $|a| \leq 100$, then

$$\hat{h}(P) \geq \frac{\log |a| + \log 16}{42} > 0.023 \log |a| + 0.066$$

for every non-torsion point $P \in E_a(\mathbb{Q})$.

Proof. If $a \not\equiv 4 \mod 16$, then, using [10, Theorem 1.2],

$$\hat{h}(P) \geq \frac{\log |a| + 4 \log 2}{16} > \frac{\log |a| + \log 16}{42}.$$ 

If $a \equiv 4 \mod 16$ and $|a| \leq 100$, then we have to study only 13 curves. If $a \neq 68$, then, using the ecdb database [3], we find that these curves have rank 0 or 1. If the curve has rank 0, then the lemma is trivial since there are no non-torsion points. If the curve has rank 1, then the minimum for $\hat{h}(P)$ is at the generator of the curve and then, using the ecdb database we can check that the inequality holds. Observe that the database [3] uses the definition of $\hat{h}$ as PARI/GP, so every value of $\hat{h}$ taken from the ecdb database has to be divided by 2. It remains to deal with the curve with $a = 68$. Using Sage 8.2, it is possible to find a lower bound for the canonical height over an elliptic curve, using the command "E.height function ().min(.0001, 20)". For the curve $E_{68}$ this bound is $\hat{h} \geq 0.32$ and so the inequality still holds. The lowest value for

$$\frac{\hat{h}(P)}{\log |a| + \log 16}$$

is at $a = -12$ and $x(P) = -2$, where it is $0.2383... > 1/42$. 

Lemma 2.12. For every a fourth-power-free and for every non-torsion point $P \in E_a(\mathbb{Q})$, it holds

$$\hat{h}(P) \geq \frac{1}{10}.$$
Proof. If \(|a| \geq 100\), then using Lemma 2.3
\[
\hat{h}(P) \geq \frac{\log 100 - \log 4}{16} \geq \frac{1}{5}.
\]
If \(5 \leq |a| \leq 100\), then from Lemma 2.11,
\[
\hat{h}(P) \geq \frac{\log 5 + \log 16}{42} \geq \frac{1}{10}.
\]
For \(1 \leq |a| \leq 4\), the ranks of the curves are zero except for \(a = -2\) and \(a = 3\), where the ranks are both 1. These two cases can be checked using the ecdb database.

\section{Proof of Theorem 1.3}

Recall that we are considering the elliptic divisibility sequence \(B_n(E_a, P)\), where \(E_a\) is defined by the equation \(y^2 = x^3 + ax\) with a fourth-power-free integer. With the next lemma, we show that we need to prove the theorem only for \(n\) square-free. Observe that the next lemma holds for every elliptic curve, not only for \(j(E) = 1728\).

\begin{lemma}
Let \(E\) be a rational elliptic curve and let \(P\) be a non-torsion point. Given a natural number \(n > 1\), let \(r := \prod_{p\mid n} p\) denote the radical of \(n\). If \(B_n(E, P)\) does not have a primitive divisor, then neither does \(B_{r'}(E, (n/r)P)\).
\end{lemma}

\begin{proof}
Suppose that \(B_{r'}(E, (n/r)P)\) has a primitive prime divisor \(q\). We want to show that \(q\) is also a primitive divisor for \(B_n(E, P)\). Observe that \(B_n(E, P) = B_{r'}(E, (n/r)P)\). Suppose that \(q\) divides \(B_{r'}(E, (n/r)P)\) with \(n'\) a proper divisor of \(n\). So, there exists a prime \(p\) such that \(n' \mid n/p\). Using that we are considering divisibility sequences, we have that \(q\) divides \(B_{n'/p}(E, P)\). Take \(r' = r/p\). Hence, \(B_{r'}(E, (n/r)P)\) is divisible by \(q\) since it is equal to \(B_{n'/p}(E, P)\). This is absurd considering that we assumed that \(q\) was a primitive divisor of \(B_{r'}(E, (n/r)P)\).
\end{proof}

So by the contrapositive, it follows that if \(B_{r'}(E_a, (n/r)P)\) always has a primitive divisor, then so does \(B_n(E_a, P)\). Since \(r\) is square-free, then in order to prove Theorem 1.3, we just need to prove that \(B_n(E_a, P)\) has always a primitive divisor for \(n\) square-free.

\begin{proposition}
Let \(n = 5m\) be a square-free integer with \(m \geq 11\). Then \(B_n\) has a primitive divisor.
\end{proposition}

\begin{proof}
Suppose that \(B_n\) does not have a primitive divisor. Then,
\[
\log B_n \leq 2\log n + 2n^2\rho(n)\hat{h}(P) + 2C\omega(n),
\]
thanks to Lemma 2.5. Therefore, using Lemma 2.2,
\[
2n^2\hat{h}(P)\left(\frac{9}{25} - \rho(n)\right) \leq 29\log |a| + 32.863 + 2C\omega(n) + 2\log n.
\]
If we show that the inequality does not hold, then \(B_n\) has a primitive divisor. Suppose that 2 does not divide \(n\). So,
\[
\left(\frac{9}{25} - \rho(n)\right) > \left(\frac{9}{25} - \left(\sum_{p \text{ prime}} \frac{1}{p^2} - \frac{1}{4}\right)\right) > 0.36 - 0.45225 + 0.25 > \frac{1}{6.34}
\]

and
\[ \omega(n) \leq \frac{\log n}{\log 3} < 0.92 \log n \] (3.2)
since \(3^{\omega(n)} \leq n\). Thus, using Lemma 2.3, we obtain
\[ n^2 \leq \frac{6.34(29 \log |a| + 32.863 + 2C\omega(n) + 2 \log n)}{2h(P)} \leq 50.8 \left( \frac{2C(0.92 \log n) + 29 \log |a| + 32.863 + 2 \log n}{\log |a| - \log 4} \right) \]
and so
\[ n^2 \leq \log n \left( \frac{126 + 23.4 \log |a|}{\log |a| - \log 4} \right) + \left( \frac{1670 + 1473.2 \log |a|}{\log |a| - \log 4} \right). \] (3.3)
Suppose \(|a| \geq 100\). Then,
\[ \left( \frac{126 + 23.4 \log |a|}{\log |a| - \log 4} \right) \leq 73 \]
and
\[ \left( \frac{1670 + 1473.2 \log |a|}{\log |a| - \log 4} \right) \leq 2627. \]
Hence, (3.3) becomes
\[ n^2 \leq \log n \left( \frac{126 + 23.4 \log |a|}{\log |a| - \log 4} \right) + \left( \frac{1670 + 1473.2 \log |a|}{\log |a| - \log 4} \right) \leq 73 \log n + 2627. \]
If \(n \geq 55\), it is easy to check that
\[ n^2 \geq 73 \log n + 2627 \]
and so the inequality does not hold and we have a primitive divisor for \(|a| > 100\) and \(n \geq 55\).
Suppose now \(|a| \leq 100\). Then, if \(B_n\) does not have a primitive divisor, we know from (3.1) that
\[ 2n^2 \hat{h}(P) \left( \frac{9}{25} - \rho(n) \right) \leq 29 \log |a| + 32.863 + 2C\omega(n) + 2 \log n. \]
Therefore, from Lemma 2.11 and (3.2), we obtain
\[ n^2 \leq \frac{6.34(29 \log |a| + 32.863 + 2C\omega(n) + 2 \log n)}{2h(P)} \leq 133.2 \left( \frac{2C(0.92 \log n) + 29 \log |a| + 32.863 + 2 \log n}{\log |a| + \log 16} \right). \]
Proceeding as in the case \(|a| \geq 100\), we have
\[ n^2 \leq \left( \frac{330.2 + 61.4 \log |a|}{\log |a| + \log 16} \right) \log n + \left( \frac{3863 \log |a| + 4377.4}{\log |a| + \log 16} \right). \]
For \(2 \leq |a| \leq 100\),
\[ \frac{330.2 + 61.4 \log |a|}{\log |a| + \log 16} \leq 108 \]
and
\[ \frac{3863 \log |a| + 4377.4}{\log |a| + \log 16} \leq 3005 \]
using Lemma 2.11. It is easy to check that for \( n \geq 65 \),
\[ n^2 \geq 108 \log n + 3005 \]
and so the inequality does not holds and we have a primitive divisor for \(|a| \leq 100\) and \( n \geq 65 \).

Now, we want to deal with the case \( n \) even. We will use the ideas used in the proof of [1, Theorem 2.4]. Let \( n = 2k \). Then, using (2.2),
\[ x(nP) = x(2(kP)) = \frac{\phi_2(x(kP))}{\psi_2(x(kP))} = \frac{\phi_2(A_k, B_k)}{B_k \psi_2^2(A_k, B_k)} \]
and therefore
\[ B_n = \frac{B_k \psi_2^2(A_k, B_k)}{\gcd(\phi_2(A_k, B_k), B_k \psi_2^2(A_k, B_k))} \geq \frac{4 \| A_k \| B_k (A_k^2 + aB_k^2)}{\Delta^2} \] (3.4)
since
\[ \gcd(\phi_2(A_k, B_k), B_k \psi_2^2(A_k, B_k)) \mid \Delta \frac{x^2(z^2 - 1)}{2} = \Delta^2, \]
thanks to Lemma 2.8. If \(|A_k| \geq 2 \| a \| B_k\), then put \( z = \| A_k \| / B_k\) and so
\[ |A_k^2 + aB_k^2| = B_k^2 |z^2 + a| \geq B_k^2 (z^2 - |a|) \geq B_k^2 z = |A_k B_k| \]
since \( z \geq 2 \| a \| \geq 4 \). If \(|A_k| \leq B_k\), then
\[ |A_k^2 + aB_k^2| \geq |B_k^2| \geq |A_k B_k| \]
and so, in both cases,
\[ B_n \geq \frac{4 \| A_k \|^2 B_k^2}{\Delta^2} \geq \frac{4H(kP)^2}{\Delta^2} \]
considering that \(|A_k|\) and \(B_k\) are greater than 1. Otherwise, \(B_k \leq |A_k| \leq 2 \| a \| B_k\) and then \(H(kP) = A_k\). Thus,
\[ B_k \geq \frac{H(kP)}{2 \| a \|} \]
and so, using (3.4),
\[ B_n \geq \frac{4 \| A_k \| B_k (A_k^2 + aB_k^2)}{\Delta^2} \geq \frac{4H(kP)^2}{2 \| a \| \Delta^2}. \]
Here we are using that \(A_k^2 + aB_k^2 \neq 0\) and so \(|A_k^2 + aB_k^2| \geq 1\) since it is a non-zero integer. Indeed, if it is 0, then \(y^2(kP) = x(kP)^3 + ax(kP) = 0\) and, thanks to the group law, \(kP\) would be a 2-torsion point. This is absurd considering that \(P\) is not a torsion point. Hence,
\[ \log B_n \geq \log 4 + 2h(kP) - \log(\| 2a \| \Delta^2) \geq \log 4 + 4k^2 \hat{h}(P) - 4C - \log(\| 2a \| \Delta^2) \]
where the second inequality follows from Lemma 2.1. Therefore, if \(B_n\) does not have a primitive divisor, then by Lemma 2.5,
\[ \log 4 + 2n^2 \hat{h}(P) \left( \frac{1}{2} - \rho(n) \right) \leq 2C(\omega(n) + 2) + \log(\| 2a \| (64a^3)^2) + 2 \log n. \] (3.5)
Using Lemma 2.4, we know $\rho(n) < 0.46$. Moreover, $\omega(n) \leq \log n / \log 2$ since $n \geq 2^\omega(n)$ and so

$$n^2 \hat{h}(P) \leq 12.5 \left( 2C(\omega(n) + 2) + 2 \log n + 11 \log 2 + 7 \log |a| \right)$$

$$\leq 12.5((2.76 + 0.73 \log |a|) \log n + 8.67 + 8 \log |a|)$$

$$= 34.5 \log n + 9.125 \log |a| \log n + 108.375 + 100 \log |a|.$$  

Now, we proceed as in the case odd. If $|a| \geq 100$, then

$$\hat{h}(P) \geq \frac{\log |a| - \log 4}{16}$$

and therefore

$$n^2 \leq \frac{552 \log n + 146 \log n \log |a| + 1734 + 1600 \log |a|}{\log |a| - \log 4}.$$  

This equation does not hold for $n \geq 70$ and $|a| \geq 100$. If $2 \leq |a| \leq 100$, then

$$\hat{h}(P) \geq \frac{\log |a| + \log 16}{42}$$

and therefore

$$n^2 \leq \frac{1449 \log n + 383.25 \log(n) \log |a| + 4551.75 + 4200 \log |a|}{\log |a| + \log 16}.$$  

This equation does not hold for $n \geq 80$ and $2 \leq |a| \leq 100$. So, we have proved the proposition for $n \geq 65$ odd and $n \geq 80$ even. Since we are considering only $n \geq 55$ square-free, it remains only the cases $n = 55$ and $n = 70$. Substituting $n = 55$ in (3.1) we obtain

$$\hat{h}(P) \leq \frac{41.92 + 30 \log |a|}{1886}.$$  

If $|a| \geq 100$, then from Lemma 2.3,

$$\frac{\log |a| - \log 4}{16} \leq \frac{41.92 + 30 \log |a|}{1886}$$

and this inequality never holds. If $|a| \leq 100$, then

$$\frac{\log |a| + \log 16}{42} \leq \frac{41.92 + 30 \log |a|}{1886}$$

and this inequality never holds. So, for $n = 55$ there is always a primitive divisor. The case $n = 70$ is analogous. We substitute $n = 70$ in (3.5), obtaining

$$\hat{h}(P) \leq \frac{18.73 + 9.5 \log |a|}{1858}.$$  

This inequality never holds.  

Thanks to Lemma 3.1, we know that we have to prove the theorem for $n$ square-free. We know that $B_n(E_a, P)$ has always a primitive divisor for $n \geq 55$. So, it remains to deal with the cases $n = 5, 10, 15, 30$ and 35. We begin with the case $n = 35$.  

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Proposition 3.3. The term $B_n$ has always a primitive divisor for $n = 35$.

Proof. Suppose that $B_{35}(E_a, P)$ does not have a primitive divisor. Then (3.1) must hold for $n = 35$. Substituting $n = 35$ in (3.1), we obtain

$$\hat{h}(P) \leq \frac{30 \log |a| + 41.1}{734}.$$  \hspace{1cm} (3.6)

Using Lemma 2.3,

$$\frac{\log |a| - \log 4}{16} \leq \hat{h}(P)$$

and the inequality

$$\frac{\log |a| - \log 4}{16} \leq \frac{30 \log |a| + 41.1}{734}$$

does not hold for $|a| \geq 732$. So, $B_{35}(E_a, P)$ has a primitive divisor if $|a| \geq 732$. Using [10, Theorem 1.2], for all $a$ except those satisfying $a \equiv 4 \mod (16),$

$$\hat{h}(P) \geq \frac{\log |a| + \log 16}{16}$$

and for this class of curves the inequality

$$\frac{\log |a| + \log 16}{16} \leq \hat{h}(P) \leq \frac{30 \log |a| + 41.1}{734}$$

does not hold. In conclusion, $B_{35}(E_a, P)$ has always a primitive divisor except for $a \equiv 4 \mod (16)$ and such that

$$732 > a > -732.$$ 

Take a point $P$ on $E_a$ such that (3.6) holds. From Lemma 2.1,

$$h(P) \leq 2\hat{h}(P) + 2C \leq 0.59 \log |a| + 0.64.$$ 

Consider all the couples $(x, a)$ with $x \in \mathbb{Q}$ such that $a \equiv 4 \mod (16), 732 > a > -732, h(x) \leq 0.59 \log |a| + 0.64$ and $x^3 + ax$ is a rational square. This is a finite set that can be easily compute. If a point $P \in E_a(\mathbb{Q})$ satisfies (3.6), then $(x(P), a)$ must belong to this finite set. Using PARI/GP, we can check for such points if the inequality (3.6) holds. It turns out that the only non-torsion points (up to inverse) where the inequality holds is for $(6, 36), (30, 180)$ in $E_{180}$ and $(6, 12), (-2, 4)$ in $E_{-12}$. So, we need to check that $B_{35}(E_a, P)$ has a primitive divisor for each of the previous 4 cases. For such cases, we explicitly compute $B_{35}$ and we check that there is a primitive divisor.

In order to do so we use PARI/GP. If $x(P) = u/v$ and $p$ divides $\psi_{35}(u, v)/(\psi_{35}(u, v)\psi_{7}(u, v))$ but does not divide $\Delta$, then thanks to (2.2) this is a primitive divisor of $B_{35}(E_a, P)$. To compute $\psi_{35}(u, v)/(\psi_{7}(u, v)\psi_{7}(u, v))$ we use the command "elldivpol" of PARI/GP. For example, 139 is a primitive divisor of $B_{35}(E_{-12}, (6, 12))$. The other cases are analogous.

Now, it remains to study the case $n = 30$. The cases with $n \leq 25$ are proved at the beginning of the proof of [9, Lemma 5.1]. We want to use the same ideas for the case $n = 30$ too, but we need some preliminary lemmas. Recall that $g_n = \gcd(\phi_n(u, v), v\psi_n^2(u, v))$ with $x(P) = u/v$.

The strategy for proving the case $n = 30$ is the following. Firstly, we study the sequence of the $g_n$. Secondly, we define a sequence of polynomials $\Psi_n(X, Y)$ and we show that if $B_n$ does not have a primitive divisor, then the equation $\Psi_n(X, Y) = d$ has a solution, for $d$ that ranges in a finite set that depends only on $n$. Finally, we show that the equations $\Psi_{30}(X, Y) = d$ does not have any solution and then $B_{30}$ has a primitive divisor.
Lemma 3.4. For every \( n \) odd,
\[
|A_n B_2 - A_2 B_n|^2 = 4^\delta B_{n+2} B_{n-2}|
\]
and for every \( n \),
\[
|A_n B_4 - A_4 B_n|^2 = B_{n+4} B_{n-4}|
\]
where \( \delta \in \{0,1\} \) is a constant that depends only on \( E \) and \( P \).

Proof. See [9, Lemma 3.5].

Lemma 3.5. For every \( n \),
\[
g_{4+n} g_{4-n} = g_n^2
\]
and for every \( n \) odd
\[
g_{2+n} g_{2-n} = 4^\delta g_2 g_n^2.
\]

Proof. Thanks to (2.2),
\[
g_{n+4} g_{n-4} B_{n+4} B_{n-4} = v^2 \psi_{n+4}^2(u,v) \psi_{n-4}^2(u,v)
\]
and using Lemma 2.7
\[
v^2 \psi_{n+4}^2(u,v) \psi_{n-4}^2(u,v) = |v\phi_n(u,v)\psi_3^2(u,v) - v\psi_n^2(u,v)\phi_4(u,v)|^2.
\]
Using again (2.2),
\[
|v\phi_n(u,v)\psi_3^2(u,v) - v\psi_n^2(u,v)\phi_4(u,v)|^2 = g_n^2 g_4^2 |A_n B_4 - A_4 B_n|^2
\]
and we conclude using Lemma 3.4 since
\[
g_n^2 g_4^2 |A_n B_4 - A_4 B_n|^2 = g_n^2 g_4^2 B_{n+4} B_{n-4}|
\]
The other case is analogous.

Lemma 3.6. For \( n \) odd,
\[
g_n = (2^\delta g_2)^{\frac{n^2-1}{4}}
\]
and, for \( n \equiv 2 \mod 4 \),
\[
g_n = g_2 g_4^2 \frac{n^2-1}{4}
\]
where \( \delta \) is as in Lemma 3.4.

Proof. We will prove the first equation by induction. Thanks to the definition, \( g_1 = \gcd(A_1,B_1) = 1 \) and so the lemma holds for \( n = 1 \). If it holds until \( n \), then
\[
g_{n+2} = \frac{4\delta g_2 g_n^2}{g_{n-2}} = (2^\delta g_2)^2 (2^\delta g_2)^{\frac{n^2-1}{4}} = (2^\delta g_2)^{\frac{(n+2)^2-1}{4}}.
\]
The other case is analogous.
In conclusion, it remains the case when \( \psi_g \) when \( u \).

Using [6, Exercise 3.7], we can explicitly compute \( \psi_4(u,v) \).

**Proof.** Using [6, Exercise 3.7], we can explicitly compute \( \phi_2, \psi_2, \phi_4 \) and \( \psi_4 \). We have

\[
\psi_2(u,v) = u^3 + auv^2,
\]

\[
\phi_2(u,v) = u^4 - 2au^2v^2 + a^2v^4,
\]

\[
\psi_4(u,v) = 4(u^3 + auv^2)(u^6 + 5au^4v^2 - 5a^2u^2v^4 - a^3v^6)^2,
\]

and

\[
\phi_4(u,v) = (u^8 - 20au^6v^2 - 26a^2u^4v^4 - 20a^3u^2v^6 + a^4v^8)^2.
\]

Observe that, if \( u \) and \( a \) are not both even, then

\[
\ord_2(g_4) \leq 6
\]

since the equation

\[
\psi_4^2(u,v) \equiv \phi_4(u,v) \equiv 0 \mod 64
\]

does not have non-trivial solutions by direct computation of all the possible cases modulo 64. Define

\[
k := \min\{\ord_2(a)/2, \ord_2(u)\}.
\]

Therefore, by definition \( \ord_2(\phi_2(u,v)) \geq 3k \) and \( \ord_2(\psi_4(u,v)) \geq 3k \). So,

\[
\ord_2(g_2) \geq 3k.
\]

If \( k = \ord_2(u) \neq \ord_2(a)/2 \), then \( \ord_2(\psi_4^2) = 2 + 15k \). If \( k = \ord_2(a)/2 \neq \ord_2(u) \), then \( \ord_2(\psi_4^2) = 2 + 14k \). In both cases

\[
\ord_2(g_4) \leq \ord_2(\psi_4^2) \leq 15k + 2.
\]

It remains the case when \( k = \ord_2(a)/2 = \ord_2(u) \). Put \( a' = a/2^{2k} \) and \( u' = u/2^k \) and hence \( g_4 = 2^{15k} \gcd(\phi_4(u',v,a'),\psi_4^2(u',v,a')) \) where with \( \phi_4(u',v,a') \) and \( \psi_4^2(u',v,a') \) we denote \( \phi_4 \) and \( \psi_4^2 \) where we substitute \( a \) with \( a' \). In this case we can use the previous result on the gcd in the case when \( u \) and \( a \) are not even, concluding that

\[
\ord_2(\gcd(\phi_4(u',v,a'),\psi_4^2(u',v,a'))) \leq 6.
\]

In conclusion,

\[
\ord_2(g_4) \leq 15k + 6 \leq 5 \ord_2(g_2) + 6.
\]

The aim of next lemmas is to replicate the work of Ingram in [2, Section 2]. We want to improve [2, Lemma 5]. We will use the ideas of Ingram and our work on the sequence of the \( g_n \).

**Lemma 3.8.** Fix \( n > 2 \). Consider the polynomial

\[
\Psi_n(x) := \frac{\psi_n(x,y)}{\text{lcm}_{\mid n}(x,y)}.
\]

This polynomial depends only on \( x \) and if a prime \( p \) divides \( \Psi_n(1)_{|a=-1} \), then \( p \) divides \( 2n \). Moreover, if a prime \( p \) divides \( \Psi_n(0)_{|a=1} \), then \( p \) divides \( 2n \).
Proof. We start by showing that the polynomial depends only on $x$. If $n$ is odd, then we conclude easily observing that $\psi_k$ depends only on $x$ if $k$ is odd. If $n = 2^k$, then $k \geq 2$ for the hypothesis $n > 2$ and so $\psi_n = \psi_n/\psi_{n/2}$. We conclude by using that for $n$ even the polynomial $\psi_n$ is in the form $yp_n(x)$, where $p_n$ depends only on $x$, thanks to part (c) of Lemma 2.6. If $n = 2^kd$ with $d$ odd and greater than 1, then $y$ divides $\text{lcm}_{l|n} \psi_n/l(x, y)$ since it divides $\psi_{n/l}(x, y)$ for every prime divisor $l$ of $d$ and then we argue as in the previous case. Define $p_n(x) = \psi_n(x)$ for odd $n$ and $p_n(x) = \psi_n(x, y)/y$ for even $n$. Hence,

$$\Psi_n(x) := \frac{p_n(x)}{\text{lcm}_{l|n} p_{n/l}(x)}.$$ 

Now we prove that if $p$ divides $p_n(1)|_{a=1}$, then $p$ divides $2n$. Define $h_k = p_k(1)|_{a=-1}$. Using the recurrence law on the $\psi_n$, we obtain, for $k \geq 1$,

- $h_{4k+1} = -h_{2k-1}h_{2k+1}^3$;
- $h_{4k+3} = h_{2k+3}h_{2k+1}^3$;
- $h_{4k} = \frac{h_k}{2}(h_{2k+2}h_{2k} - h_{2k-2}h_{2k+1})$ if $k \neq 1$;
- $h_{4k+2} = \frac{h_{2k+1}}{2}(h_{2k+3}h_{2k} - h_{2k-1}h_{2k+2})$.

We briefly show how to obtain the first equality, all the others are analogous. Using [6, Exercise 3.7] and the definition of $p_n$, we know that

$$p_{4k+1}(x) = \psi_{4k+1}(x) = \psi_{2k+2}(x, y)\psi_{2k}^3(x, y) - \psi_{2k-1}(x)\psi_{2k+1}^3(x) = y^3p_{2k+2}(x)p_{2k}^3(x) - p_{2k-1}(x)p_{2k+1}^3(x) = (x^3 + ax)^3p_{2k+2}(x)p_{2k}^3(x) - p_{2k-1}(x)p_{2k+1}^3(x).$$

Evaluating the equation in $x = 1$ and $a = -1$ we obtain

$$h_{4k+1} = -h_{2k-1}h_{2k+1}^3.$$ 

Now, explicitly writing the first terms on the sequence of the division polynomials, we have $h_1 = 1$, $h_2 = 2$, $h_3 = -4$ and $h_4 = -32$. By induction, it is easy to check that $h_{2k} = (-1)^k k2^k$ and $h_{2k+1} = (-1)^{k+1} 2^{k+1}$. For example,

$$h_{4k+1} = -h_{2k-1}h_{2k+1}^3 = (-1)^k 2^k(2k-1 + 3k+1) = (-1)^k 2^{k+1}2^{k+1}$$

where the second equality follows by induction. The other cases are analogous. So, if $p$ divides $p_n(1)|_{a=-1} = h_n$, then it divides $2n$. Therefore, if $p$ divides $\Psi_n(1)|_{a=-1}$, then it divides $2n$.

Now we want to study $\Psi_n(0)|_{a=1}$. Define $j_n = p_n(0)|_{a=1}$, that satisfies the same recurrence relations as $h_n$. By induction, it is easy to prove that $j_{2k} = (-1)^{k-1}2^k$ and $j_{2k+1} = (-1)^k$. For example,

$$j_{4k+1} = -j_{2k-1}j_{2k+1}^3 = (-1)^{1+k-1+3(k)} = 1 = (-1)^{2k}.$$ 

Hence, we conclude as before. \qed
Define the polynomial $F_n(u, v, a)$ as the homogenization of $\Psi_n(x)$, i.e.

$$F_n(u, v, a) = v^{\deg(\Psi_n(x))}\Psi_n\left(\frac{u}{v}\right).$$

So, $F_n \in \mathbb{Z}[u, v, a]$. We put $a$ in the variables to emphasize that $F_n$ depends also on $a$.

**Lemma 3.9.** Let $n > 2$. The polynomial $F_n(u, v, a)$ can be written as a homogeneous polynomial in the variables $u^2$ and $av^2$. This means that there exists a homogeneous polynomial $\Psi_n(X, Y) \in \mathbb{Z}[X, Y]$ such that

$$F_n(u, v, a) = \Psi_n(u^2, av^2).$$

**Remark 3.10.** The definition of $\Psi_n(X, Y)$ is an abuse of notation since we defined before the polynomial $\Psi_n(x)$. Anyway, we did it because the two polynomials are strictly related. Indeed, the polynomial $\Psi_n(X, Y)$ is the homogeneization of $\Psi_n(x)$ composed with a change of variables. Observe that $\Psi_n(x) = \Psi_n(x^2, a)$. For example, $\Psi_3(x) = 3x^4 + 6ax^2 - a^2$ and $\Psi_3(X, Y) = 3X^2 + 6XY - Y^2$. We are following the notation used by Ingram in [2].

**Proof.** As in the previous lemma, we define $p_n(x) = \psi_n(x)$ for $n$ odd and $p_n(x) = \psi_n(x, y)/y$ for $n$ even. We start by showing that, for all $n \geq 1$, $p_n$ can be written as a homogeneous polynomial with integral coefficients in $x^2$ and $a$. These homogeneous polynomials have degree $(n^2 - 1)/4$ if $n$ is odd and $(n^2 - 4)/4$ for $n$ even. For $1 \leq n \leq 4$ this follows from the definition. For example,

$$p_3(x) = \psi_3(x) = 3x^4 + 6ax^2 - a^2$$

and this can be written as a polynomial of degree 2 in the variable $x^2$ and $a$. Now, we proceed by induction. By definition,

$$p_{4k+1}(x) = (x^3 + ax)^2p_{2k+2}(x)p_{2k}(x) - p_{2k-1}(x)p_{2k+1}^3(x)$$

and, by induction, both addends have degree $((4k+1)^2 - 1)/4$ in the variable $x^2$ and $a$. For example, the degree of the first addend is

$$3 + \frac{(2k+2)^2 - 4}{4} + \frac{(2k)^2 - 4}{4} = \frac{12 + 4k^2 + 8k + 4 - 4 + 12k^2 - 12}{4} = \frac{(4k+1)^2 - 1}{4}.$$

Moreover, every term involved can be written as a homogeneous polynomial with integral coefficients in $x^2$ and $a$ observing that

$$(x^3 + ax)^2 = x^6 + 2ax^4 + a^2x^2$$

and using the induction. So, $p_{4k+1}$ can be written as a homogeneous polynomial in the variable $x^2$ and $a$ with degree $((4k+1)^2 - 1)/4$. The cases $n = 4k$, $n = 4k + 2$ and $n = 4k + 3$ are analogous. Therefore, $p_n$ can be written as a homogeneous polynomial with integral coefficients in $x^2$ and $a$. We know, thanks to the work in the previous lemma, that

$$\Psi_n(x) = \frac{p_n(x)}{\text{lcm}_{i}p_{n/i}(x)}$$

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for \( n \geq 3 \) and so \( \Psi_n(x) \) can be written as a homogeneous polynomial with integral coefficients in \( x^2 \) and \( a \). Let \( \Psi_n(X,Y) \in \mathbb{Z}[X,Y] \) be this polynomial and then
\[
\Psi_n(x^2, a) = \Psi_n(x).
\]

So, taking the homogeneization,
\[
\Psi_n(u^2, av^2) = v^{\deg(\Psi_n(x))} \Psi_n((u/v)^2, a)
= v^{\deg(\Psi_n(x))} \Psi_n(u/v)
= F_n(u, v, a).
\]

Let \( P \) be a rational point on the elliptic curve \( E_a \) and put \( x(P) = u/v \) with \( u \) and \( v \) coprime. Let \( B_n = B_n(E_a, P) \). Define, as in [2, Lemma 5], \( X = u^2/(u^2, av^2) \) and \( Y = av^2/(u^2, av^2) \). So, for \( n \geq 3 \),
\[
(u^2, av^2)^{\deg \Psi_n(x)/2} \Psi_n(X,Y) = \Psi_n(u^2, av^2) = F_n(u, v, a).
\]

Raising to the square, we have
\[
(u^2, av^2)^{\deg \Psi_n(x)} \Psi_n^2(X,Y) = F_n^2(u, v, a) = \frac{\psi_n^2(u, v)}{\operatorname{lcm}_{l|n} \psi_{n/l}^2(u, v)}.
\]

The last equality follows from the fact that the homogenization commutes with the lcm.

**Lemma 3.11.** Let \( n \in \mathbb{N}_{\geq 3} \). If \( B_n \) does not have a primitive divisor, then \( \Psi_n(X, Y) \) divides \( ng_n^{1/2} \), where \( g_n \) is defined in the equation (2.1).

**Proof.** Recall that
\[
B_n = \frac{v\psi_n^2(u, v)}{g_n}
\]
and hence \( \psi_n^2(u, v) \) divides \( B_n g_n \). Moreover, thanks to (3.7), \( \Psi_n^2(X, Y) \) divides \( \psi_n^2(u, v) \). Consider a prime \( q \). If \( q \) does not divide \( B_n \), then
\[
\ord_q(\Psi_n^2(X, Y)) \leq \ord_q(\psi_n^2(u, v)) \leq \ord_q(B_n g_n) = \ord_q(g_n).
\]

If \( q \) divides \( B_n \), then it divides \( B_n/p \) for some prime divisor \( p \) of \( n \) considering that \( B_n \) does not have a primitive divisor. So, using [9, Lemma 3.1], we have
\[
\ord_q(B_n) = \ord_q(B_{n/p}) + 2 \ord_q(p).
\]

Observe that \( \Psi_n^2(X, Y) \) divides \( \psi_n^2(u, v)/\psi_{n/p}^2(u, v) \) thanks to (3.7) since \( \psi_{n/p} \) divides \( \operatorname{lcm}_{l|n} \psi_{n/l} \). Hence, we have
\[
\ord_q(\Psi_n^2(X, Y)) \leq \ord_q(\psi_n^2(u, v)) - \ord_q(\psi_{n/p}^2(u, v))
= \ord_q \left( \frac{g_n}{g_{n/p}} \right) + \ord_q \left( \frac{B_n}{B_{n/p}} \right)
\leq \ord_q(g_n) + 2 \ord_q(p)
\leq \ord_q(g_n) + 2 \ord_q(n).
\]

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So, for every prime \( q \), we have
\[
\text{ord}_q(\Psi_n^2(X, Y)) \leq \text{ord}_q(g_n) + 2 \text{ord}_q(n)
\]
and then \( \Psi_n^2(X, Y) \) divides \( n^2g_n \). Observe that \( g_n \) is a square since, by definition,
\[
g_n = \frac{uv_n^2(u, v)}{B_n}
\]
and every term involved here is a square. \( \square \)

**Lemma 3.12.** Let \( n \geq 3 \). If \( B_n \) does not have a primitive divisor and a prime \( q \) divides \( \Psi_n(X, Y) \), then \( q \) divides \( 2n \).

**Proof.** Take \( q \) a prime divisor of \( \Psi_n(X, Y) \) and then \( q \) divides \( ng_n^{1/2} \) for the previous lemma. Suppose that \( q \) does not divide \( 2n \). So, it divides \( a \) since the prime divisors of \( g_n \) are the prime divisors of \( \Delta = -64a^3 \), thanks to Lemma 2.8. We assume that \( q \) divides \( a \) but does not divide \( 2n \) and we find a contradiction. Recall that \( X = a^2/(u^2, av^2) \) and \( Y = av^2/(u^2, av^2) \). If \( q \) does not divide \( u \), then does not divide \( (u^2, av^2) \). Therefore, \( q \) divides \( Y \) and does not divide \( X \). Hence,
\[
\Psi_n(X, Y) \equiv n^*X \mod q
\]
where \( n^* \) is the coefficient of \( X^{\deg(\Psi_n(X, Y))} \). It is easy to show that \( n^* \) divides \( n \). This follows from the fact that the leading coefficient of \( \psi_n \) is \( n \). Thus, since \( (q, n) = 1 \), we have that \( n^*X \) is coprime with \( q \), that is absurd considering that \( q \) divides \( \Psi_n(X, Y) \). So, \( q \) divides \( u \) and \( a \). Observe that, since \( q \) divides \( u \), then does not divide \( v \). If \( \text{ord}_q(av^2) > \text{ord}_q(u^2) \), then we conclude as before since \( q \) divides \( Y \) and does not divide \( X \). If \( \text{ord}_q(av^2) < \text{ord}_q(u^2) \), then \( q \) divides \( X \) and does not divide \( Y \). Therefore,
\[
\Psi_n(X, Y) \equiv \Psi_n(0, 1)Y^{\deg(\Psi_n(X, Y))} \mod q.
\]
Using the definition,
\[
\Psi_n(0, 1) = \Psi_n(0)|_{\alpha=1}
\]
and hence, from Lemma 3.8, we have that \( q \) does not divide \( \Psi_n(0, 1) \). So, \( q \) does not divide \( \Psi_n(X, Y) \), that is absurd. It remains the case \( \text{ord}_q(u^2) = \text{ord}_q(a) > 0 \). Considering that \( a \) is fourth-power-free, then \( \text{ord}_q(a) \leq 3 \) and it is even since it is equal to \( 2\text{ord}_q(u) \). Hence, \( \text{ord}_q(u) = 1 \) and \( \text{ord}_q(a) = 2 \). Since \( P \) belongs to the curve, then
\[
u^3 + av^2 = v^3(x(P)^3 + ax(P)) = v^3y(P)^2
\]
and so \( u^3 + av^2 \) is a square. Therefore, \( \text{ord}_q(u^3 + av^2u) \) is even. We know
\[
u^3 + av^2 = u(u^2 + av^2) = u(u^2, av^2)(X + Y)
\]
and hence
\[
\text{ord}_q(u^3 + av^2) = 1 + 2 + \text{ord}_q(X + Y).
\]
Since the LHS is even, we have that \( \text{ord}_q(X + Y) \) is odd. So,
\[
X + Y \equiv 0 \mod q.
\]
Therefore, \( X \equiv -Y \mod q \) and then
\[
\Psi_n(X, Y) = X^{\deg(\Psi_n(X, Y))}\Psi_n(1, -1) \mod q.
\]
Using again Lemma 3.8, we conclude. \( \square \)
Thanks to our computations of the $g_n$, we are now able to improve the exponents of [2, Lemma 5]. For the convenience of the reader, we write here the result of Ingram.

**Lemma 3.13.** [2, Lemma 5] Let $n \geq 5$ be square-free. Consider the elliptic divisibility sequence $B_n = B_n(E_a, P)$ and suppose that $B_n$ does not have a primitive divisor. Then,

$$
\Psi_n(X, Y) = 2^\alpha \prod_{l|n} l^{\beta_l}
$$

with $\alpha \leq 2d$ and $\beta_l \leq 3d + 1$ with $d = n^2(n^2 - 1)/4$.

**Proposition 3.14.** Let $n \geq 3$. Let us consider the elliptic divisibility sequence $B_n = B_n(E_a, P)$ and suppose that $B_n$ does not have a primitive divisor for $n$ square-free. Then,

$$
\Psi_n(X, Y) = 2^{\alpha_2} \prod_{l|n} l^{\beta_l}
$$

with $\alpha_2 \leq \frac{75}{4} n^2 - 59$ and $\beta_l \leq \frac{45}{4} n^2 - 35$.

**Proof.** If $n$ is odd, then

$$
\Psi_n(X, Y) \mid ng_n^{1/2} = n(2^6 g_2^{n^2-1})^{\frac{n^2-1}{8}} \mid n(2^2 \Delta) \frac{n^2-1}{8}
$$

using Lemma 2.8, 3.6 and 3.11. If $n$ is even, in the same way,

$$
\Psi_n(X, Y) \mid ng_2^{1/2} g_4^{\frac{n^2-4}{8}} \mid n\Delta(\Delta) \frac{n^2-4}{8} \Delta(\Delta) \frac{n^2-16}{8}
$$

Here we are using that $n$ is square-free and so $n \equiv 2 \mod 4$ if $n$ is even. Since $\Delta = -64a^3$, then

$$
\Psi_n(X, Y) \mid n(4a) \frac{n^2-48}{8}.
$$

Indeed, for $n$ even this is simply the definition and for $n$ odd we have

$$
\Psi_n(X, Y) \mid n(2\Delta) \frac{n^2-4}{8} = n(2^{13} a^6) \frac{n^2-1}{8} \mid n(4a) \frac{15n^2-48}{8}.
$$

Take $p \neq 2$ a prime. If $p$ does not divide $n$, then $\text{ord}_p(\Psi_n(X, Y)) = 0$ thanks to the previous lemma. If $p$ divides $n$, then

$$
\text{ord}_p(\Psi_n(X, Y)) \leq \text{ord}_p(n) + \frac{15n^2 - 48}{4} \text{ord}_p(a)
$$

$$
\leq 1 + 3 \frac{15n^2 - 48}{4}
$$

$$
= \frac{45}{4} n^2 - 35
$$

since $a$ is fourth-power-free and $n$ is square-free. Moreover, in the same way,

$$
\text{ord}_2(\Psi_n(X, Y)) \leq 1 + 5 \frac{15n^2 - 48}{4} = \frac{75}{4} n^2 - 59.
$$

\[\square\]
Remark 3.15. The previous Proposition is an improvement of [2, Lemma 5], since the exponents grow as \( n^2 \) and so, for \( n \) large enough, they are smaller than the exponents of Lemma 5, that grow as \( n^4 \).

Now we are ready to conclude the proof of Theorem 1.3. We need to show that \( B_{30}(E_a, P) \) always has a primitive divisor. We will use our bound on the sequence of the \( g_n \) and the work on the divisors of \( \Psi_n(X, Y) \).

Lemma 3.16. Let \( (B_n(E_a, P))_{n \in \mathbb{N}} \) be an elliptic divisibility sequence and suppose that \( B_{30} \) does not have a primitive divisor. Then,

\[
z := \frac{B_{30}B_5B_2}{B_1B_{15}B_{10}B_6}
\]

is an integer that divides \( 30^2 \).

Proof. Thanks to [9, Lemma 3.1], if \( p \) divides \( B_k \), then

\[
\text{ord}_p(B_{mk}) = \text{ord}_p(B_k) + 2\text{ord}_p(m). \tag{3.8}
\]

Take \( p \) a prime that does not divide \( B_{30} \). So, \( \text{ord}_p(z) = 0 \) since \( p \) does not divide any of the terms involved. If \( p \) divides \( B_{30} \), then it must divide one of the other factors. Suppose \( p \) divides \( B_1 \). Hence, thanks to (3.8),

\[
\text{ord}_p(z) = \text{ord}_p(B_{30}) + \text{ord}_p(B_3) + \text{ord}_p(B_2) + \text{ord}_p(B_5) - \text{ord}_p(B_{15}) - \text{ord}_p(B_6) - \text{ord}_p(B_{10}) - \text{ord}_p(B_1) = 4\text{ord}_p(B_1) + 2\text{ord}_p(30) + 2\text{ord}_p(3) + 2\text{ord}_p(2) + 2\text{ord}_p(5) - 4\text{ord}_p(B_1) - 2\text{ord}_p(15) - 2\text{ord}_p(6) - 2\text{ord}_p(10) - 2\text{ord}_p(1) = 0.
\]

If \( p \) divides \( B_2 \) and does not divide \( B_1 \), then

\[
\text{ord}_p(z) = \text{ord}_p(B_{30}) + \text{ord}_p(B_3) + \text{ord}_p(B_2) + \text{ord}_p(B_5) - \text{ord}_p(B_{15}) - \text{ord}_p(B_6) - \text{ord}_p(B_{10}) - \text{ord}_p(B_1) = 2\text{ord}_p(B_2) + 2\text{ord}_p(30) - 2\text{ord}_p(B_2) - 2\text{ord}_p(5) - 2\text{ord}_p(3) = 0.
\]

The cases when \( p \) divides \( B_3 \) and \( B_5 \) are analogous. If \( p \) divides \( B_6 \) but does not divide \( B_3 \) and \( B_2 \), then

\[
\text{ord}_p(z) = \text{ord}_p(B_{30}) + \text{ord}_p(B_3) + \text{ord}_p(B_2) + \text{ord}_p(B_5) - \text{ord}_p(B_{15}) - \text{ord}_p(B_6) - \text{ord}_p(B_{10}) - \text{ord}_p(B_1) = \text{ord}_p(B_6) + 2\text{ord}_p(5) - \text{ord}_p(B_6) = 2\text{ord}_p(5).
\]

The cases with \( B_{10} \) and \( B_{15} \) are analogous. Finally, for every prime \( p \),

\[
0 \leq \text{ord}_p(z) \leq 2\text{ord}_p(2) + 2\text{ord}_p(3) + 2\text{ord}_p(5) = 2\text{ord}_p(30).
\]

\[\square\]
Proposition 3.17. The term $B_{30}$ has always a primitive divisor.

Proof. Take

$$z = \frac{B_{30}B_3B_5B_2}{B_1B_6B_{10}B_{15}}.$$

Then, using (2.2),

$$z = \frac{B_{30}B_3B_5B_2}{B_1B_6B_{10}B_{15}} = \frac{v_{15}^2(u, v) \cdot v_{10}^2(u, v) \cdot v_{12}^2(u, v) \cdot v_{12}^2(u, v)}{v_{15}^2(u, v) \cdot v_{12}^2(u, v) \cdot v_{12}^2(u, v) \cdot v_{12}^2(u, v) \cdot \frac{g_{10}^2}{g_{30}^2} \cdot \frac{g_{10}^2}{g_{30}^2}}.$$

Observe that

$$\Psi_{30}^2(u^2, av^2) = \frac{v_{15}^2(u, v) \cdot v_{10}^2(u, v) \cdot v_{12}^2(u, v) \cdot v_{12}^2(u, v)}{v_{15}^2(u, v) \cdot v_{12}^2(u, v) \cdot v_{12}^2(u, v) \cdot v_{12}^2(u, v) \cdot \frac{g_{10}^2}{g_{30}^2} \cdot \frac{g_{10}^2}{g_{30}^2}}.$$

thanks to (3.7) and then

$$|\Psi_{30}(u^2, av^2)| \cdot \sqrt{\left(\frac{g_{30}g_{10}g_{6}g_{1}}{g_{90}g_{90}g_{92}}\right)} = \sqrt{2}.$$

If $B_{30}$ does not have a primitive divisor, then $z | (30)^2$ by Lemma 3.16. Thus, thanks to Lemma 3.6,

$$\Psi_{30}(u^2, av^2) | 30 \sqrt{\left(\frac{g_{30}g_{10}g_{6}g_{1}}{g_{90}g_{90}g_{92}}\right)}$$

$$= 30 \cdot \frac{\frac{g_{10}^2}{g_{30}^2} \cdot \frac{g_{10}^2}{g_{30}^2} \cdot \frac{g_{10}^2}{g_{30}^2} \cdot \frac{g_{10}^2}{g_{30}^2} \cdot \frac{g_{10}^2}{g_{30}^2}}{(2^8 g_2)^{\frac{2^{2^2-1}}{2^2-1}} g_2 \cdot (g_4)^{\frac{2^{15}-1}{2^{8}-1}} g_2}.$$

So,

$$\Psi_{30}(u^2, av^2) \mid 30 g_{4}^{24} (2^8 g_2)^{-24}. \quad (3.9)$$

By direct computation, $\Psi_{30}$ is a homogeneous polynomial of degree 144. Recall that $X = u^2/(u^2, av^2)$ and $Y = av^2/(u^2, av^2)$. Hence, $X$ and $Y$ are coprime. Using Lemma 3.12, we obtain

$$\Psi_{30}(X, Y)^{2^{a_1} 3^{a_2} 5^{a_3}}.$$

We know, by (3.9), that

$$a_1 \leq 2 + 24(\text{ord}_2(g_4) - \text{ord}_2(g_2)) \leq 146 + 96 \text{ord}_2(g_2)$$

where the last inequality follows from Lemma 3.7. If $2 | v$, then $\phi_2(u, v) \equiv u^4 \not\equiv 0 \mod 2$ and hence $\text{ord}_2(g_2) \leq \text{ord}_2(\phi_2(u, v)) = 0$. If $2 \nmid v$, then

$$\text{ord}_2(g_2) \leq \text{ord}_2(\gcd(\phi_2(x), \psi_2^2(x))) \leq 3$$

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where the last inequality follows from the fact that the equation
\[ \phi_2(x) \equiv \psi_2^3(x) \equiv 0 \mod 16 \]
has solution only for \( a \equiv 0 \mod 16 \), that is absurd since \( a \) is fourth-power-free. This can be checked evaluating the equation for all the pairs \((x, a)\) modulo 16. Therefore,
\[ a_1 \leq 146 + 96 \text{ord}_2(g_2) \leq 146 + 96 \cdot 3 = 434. \]
The equation \( \Psi_{30}(X,Y) \equiv 0 \mod 3 \) has no solution for \( X \) and \( Y \) coprime and so \( a_2 = 0 \). This follows from the fact that \( \Psi_{30}(x, y) \equiv 0 \mod 3 \) has only the solution \((x, y) \equiv (0, 0) \mod 3 \) and this can be checked by direct computation. In the same way \( a_3 = 0 \). In conclusion, if \( B_{30}(E_a, P) \) does not have a primitive divisor and \( x(P) = u/v \), then \( \Psi_{30}(X, Y)|2^{434} \) with \( X = u^2/(u^2, av^2) \) and \( Y = av^2/(u^2, av^2) \). Hence, we have to solve finitely many Thue equations of the form
\[ \Psi_{30}(X, Y) = d \]
with \( d|2^{434} \). Using PARI/GP, we want to show that this equation has solutions only for \( X = \pm Y, X = 0 \) or \( Y = 0 \) (we will call them the trivial solutions). The polynomial can be factored in four irreducible terms, that we will call \( p_1, p_2, p_3, p_4 \). The polynomial \( p_1 \) has degree 16, the polynomial \( p_2 \) has degree 32 and the coefficient of \( X^{31}Y \) is \(-4256\), the polynomial \( p_3 \) has degree 32 and the coefficient of \( X^{31}Y \) is \(-416\) and the polynomial \( p_4 \) has degree 64. This factorization is obtained using PARI/GP. If \( \Psi_{30}(X, Y)|2^{434} \), then \( p_1(X, Y) = \pm 2^k \) for \( 0 \leq k \leq 217 \) or \( p_2(X, Y) = \pm 2^k \) for \( 0 \leq k \leq 217 \). Using PARI/GP one can check that these two equations have only trivial solutions. This calculation took 3 minutes using PARI/GP 2.11.1 on a Windows 10 desktop with an Intel i7-7500 processor and 8gb of RAM. If \( X = 0 \), then \( u = 0 \) and hence \( P \) is a 2-torsion point, that is absurd since we assumed that \( P \) is non-torsion. If \( Y = 0 \), then \( a = 0 \) or \( v = 0 \) and neither of which we consider here. If \( X = -Y \), then \( u^2 = -av^2 \). Therefore \( x(P)^2 = -a \) and so again \( P \) is a 2-torsion point. If \( X = Y \), then \( x(P)^2 = a \) and hence \( y(P)^2 = x(P)^3 + ax(P) = 2a^{3/2} \). Thus, \( 2a^{1/2} = (y(P)/x(P))^2 \) is a square. Suppose \( p \neq 2 \) divides \( a \). Therefore \( \text{ord}_p(a) = 2 \text{ord}_p(2a^{1/2}) \geq 2 \cdot 2 \) and this is absurd since \( a \) is fourth-power-free. Hence, it remains only the case when \( a \) is a power of 2. Since \( 2a^{1/2} \) is a square and \( a \) is fourth-power-free, then \( a = 4 \). This curve has rank 0. So, there is always a primitive divisor. \( \square \)

4 Application to a more general case

We want to generalize the techniques of the previous section to a more general case. The proof of Theorem 1.4 will follow from the ideas of [9, Lemma 5.1]. As the authors of [9] pointed out in Remark 5.2, \( \psi_k \) is reducible for \( k = 13 \) and 17. We will show that indeed it is reducible for every prime congruent to 1 modulo 4 and therefore we will apply their ideas to prove Theorem 1.4.

Recall that \( E_a \) is the elliptic curve defined by the equation \( y^2 = x^3 + ax \) where \( a \) is a fourth-power-free integer.

Lemma 4.1. Fix \( k \geq 2 \). Let \( T_1 \) and \( T_2 \) be two non-trivial k-torsion points of \( E_a(\mathbb{Q}) \) such that \( T_1 \neq \pm T_2 \). There exists \( K_k \), depending only on \( k \), such that
\[ |x(T_1) - x(T_2)| \geq K_k |a|^{-1/2} \]
and 

\[ |x(T_1)^{-1} - x(T_2)^{-1}| \geq K_k |a|^{-1/2}. \]

In the second inequality we are assuming that \( x(T_1)x(T_2) \neq 0 \).

**Proof.** Fix \( k \) and consider the curve \( E_1 \) defined by the equation \( y^2 = x^3 + x \). Define

\[ K_k := \min_{T_1, T_2 \in E_1(k) \setminus \{O\}} \{ |x(T_1) - x(T_2)|, |x(T_1)^{-1} - x(T_2)^{-1}| \} \]  \hspace{1cm} (4.1)

where \( E_1(k) \) is the set of the \( k \)-torsion points of \( E_1(k) \). There is a complex isomorphism \( \varphi \) between \( E_1 \) and \( E_a \) given by the map

\[ \varphi : (x, y) \rightarrow (a^{\frac{k}{2}}x, a^{\frac{k}{4}}y). \]

If \( R_1 \) is a \( k \)-torsion point of \( E_1 \), then \( \varphi(R_1) \) is a \( k \)-torsion point of \( E_a \). So, if \( R_1 \) and \( R_2 \) are two \( k \)-torsion points of \( E_a(k) \), then

\[ x(R_1) - x(R_2) = a^{1/2}x(T_1) - a^{1/2}x(T_2) \]

for \( T_1 \) and \( T_2 \) two \( k \)-torsion points of \( E_1(k) \). Therefore,

\[ \min_{T_1, T_2 \in E_1(k) \setminus \{O\}} \{ |x(T_1) - x(T_2)| \} = |a|^{1/2} \min_{T_1, T_2 \in E_1(k) \setminus \{O\}} \{ |x(T_1) - x(T_2)| \} \]

and

\[ \min_{T_1, T_2 \in E_1(k) \setminus \{O\}} \left\{ \left| \frac{1}{x(T_1)} - \frac{1}{x(T_2)} \right| \right\} = \min_{T_1, T_2 \in E_1(k) \setminus \{O\}} \left\{ \left| \frac{1}{x(T_1)} - \frac{1}{x(T_2)} \right| \right\} \cdot |a|^{-1/2}. \]

Again we are assuming that \( x(T_1)x(T_2) \neq 0 \) in the last equation. \( \square \)

**Lemma 4.2.** Suppose that \( \psi_k^2(x) = f(x)g(x) \) with \( f \) and \( g \) two coprime polynomials with integer coefficients. Let \( u \) and \( v \) be two coprime integers with \( v \geq 1 \) and \( \psi_k^2(u, v) \) be the homogenization of \( \psi_k^2(x) \) evaluated in \( u \) and \( v \). Let \( d = \min\{\deg f, \deg g\} \) and suppose \( \max\{|u|, v\} \geq (K_k |a|^{-1/2} / 2)^{-1} \). Therefore,

\[ \psi_k^2(u, v) \geq \max\{|u|, v\}^d \left( \frac{K_k |a|^{-1/2}}{2} \right)^d, \]

if \( \psi_k^2(u, v) \neq 0 \).
Proof. Define \( x = u/v \). Suppose that \(|u| \leq v \) and that the root \( \zeta \) of \( \psi_k^2 \) closest to \( u/v \) is a root of \( g \). Then, \( |x - x_0| \geq K_k |a|^{-1/2} / 2 \) for every root \( x_0 \) of \( f \). Indeed, otherwise,
\[
|\zeta - x_0| \leq |\zeta - x| + |x - x_0| \leq 2 |x - x_0| < K_k |a|^{-1/2}
\]
and this is absurd thanks to Lemma 4.1 since \( \zeta \) and \( x_0 \) are different abscissas of non-trivial \( k \)-torsion points. Observe that \( x_0 \neq \zeta \) considering that \( x_0 \) is a root of \( f \), \( \zeta \) is a root of \( g \) and \( (f, g) = 1 \). Denote with \( f(u, v) \) the homogenization of \( f \), evaluated in \( u \) and \( v \). Hence,
\[
|f(u, v)| = v^{\deg f} |f(x)| \geq \left( v \frac{K_k |a|^{-1/2}}{2} \right)^{\deg f} \geq \left( \max\{|u|, v|} \frac{K_k |a|^{-1/2}}{2} \right)^d.
\]
Here, we are using that \( \max\{|u|, |v|\} \geq (K_k |a|^{-1/2} / 2)^{-1} \). Observe that \( g(u, v) \) is a non-zero integer since \( 0 \neq \psi_k^2(u, v) = f(u, v)g(u, v) \) and then \( |g(u, v)| \geq 1 \). In conclusion,
\[
\psi_k^2(u, v) = |f(u, v)g(u, v)| \geq \max\{|u|, v\}^d \left( \frac{K_k |a|^{-1/2}}{2} \right)^d.
\]
If \(|u| \leq v \) and the root \( \psi_k^2 \) closest to \( u/v \) is a root of \( f \), then the proof is identical.

Suppose now \(|u| \geq v \) and that the root \( \zeta \neq 0 \) of \( \psi_k^2 \) that minimize \( |x^{-1} - \zeta^{-1}| \) is a root of \( g \). Then, using again the triangle inequality,
\[
|x^{-1} - x_0^{-1}| \geq K_k |a|^{-1/2} / 2
\]
for every root \( x_0 \neq 0 \) of \( f \). Therefore,
\[
|f(u, v)| \geq |u|^{\deg f} \left( \frac{K_k |a|^{-1/2}}{2} \right)^{\deg f} \geq \left( \max\{|u|, v|} \frac{K_k |a|^{-1/2}}{2} \right)^d.
\]
As above, we have \(|g(u, v)| \geq 1 \). In conclusion,
\[
\psi_k^2(u, v) \geq \max\{|u|, v\}^d \left( \frac{K_k |a|^{-1/2}}{2} \right)^d.
\]
The case when \(|u| \geq v \) and the root \( \zeta \neq 0 \) of \( \psi_k^2 \) that minimize \( |x^{-1} - \zeta^{-1}| \) is a root of \( f \) is identical. \( \square \)

Let \( P \) be a non-torsion point of an elliptic curve \( E_a(\mathbb{Q}) \). Consider the sequence \( B_n = B_n(E_a, P) \). We will focus on the study of the terms \( B_{mk} \) for a fixed \( k \).

**Proposition 4.3.** Let us fix \( k \) such that \( \psi_k^2 = f(x) \cdot g(x) \) with \( f \) and \( g \) two coprime polynomials in \( \mathbb{Z}[x] \). Let \( d = \min\{\deg f, \deg g\} \). Suppose that \( B_{mk} \) does not have a primitive divisor, that \( d > k^2 \rho(mk) \) and that
\[
2m^2 \geq - \log(K_k |a|^{-1/2} / 2) + 2C.
\]
Then,
\[
2(mk)^2 \leq \frac{\log |g_k| - d \log \left( \frac{K_k |a|^{-1/2}}{2} \right) + 2dC + 2 \log mk + 2C \omega(mk)}{\left( \frac{h}{\rho(mk)} \right) h(P)},
\]
with
\[
g_k := \gcd(\phi_k(A_m, B_m), \psi_k^2(A_m, B_m)).
\]
Remark 4.4. If we fix $k$ and we let $m$ grow, then the inequality of the proposition does not hold, since the LHS is quadratic in $m$ and the RHS is logarithmic. So, if we take $k$ so that $\psi_k^2$ is reducible and $d > k^2 \rho(mk)$, then $B_{mk}$ has a primitive divisor for every $m$ large enough. This is the key point for the proof of Theorem 1.4.

Proof. Thanks to the hypotheses, $H(mP) \geq (K_k |a|^{-1/2} / 2)^{-1}$ since

$$h(mP) \geq 2m^2 \hat{h}(P) - 2C \geq -\log(K_k |a|^{-1/2} / 2).$$

Moreover, $\psi_k^2(A_m, B_m) \neq 0$ since $P$ is a non-torsion point. So, we can apply Lemma 4.2 to $\psi_k^2$.

Using (2.2), Lemma 4.2 and that $B_m \geq 1$, we have

$$B_{mk} \geq \frac{B_m \psi_k^2(A_m, B_m)}{\gcd(\phi_k(A_m, B_m), \psi_k^2(A_m, B_m))} \geq \frac{H(mP)d(K_k |a|^{-1/2})^d}{|g_k|}.$$

Considering the logarithms,

$$2dm^2 \hat{h}(P) \leq dh(mP) + 2dC$$

$$\leq \log B_{mk} + \log |g_k| - d \log \left(\frac{K_k |a|^{-1/2}}{2}\right) + 2dC.$$

Thanks to Lemma 2.5, if $B_{mk}$ does not have a primitive divisor, then

$$\log B_{mk} \leq 2 \log mk + 2(mk)^2 \rho(mk) \hat{h}(P) + 2C\omega(mk)$$

and so

$$2m^2k^2 \left(\frac{d}{k^2} - \rho(mk)\right) \hat{h}(P) \leq \log |g_k| - d \log \left(\frac{K_k |a|^{-1/2}}{2}\right)$$

$$+ 2dC + 2 \log mk + 2C\omega(mk).$$

Remark 4.5. If $k = m_1m_2$ with $(m_1, m_2) = 1$, then $f := \psi_{m_1}^2 \psi_{m_2}^2$ divides $\psi_k^2$. Anyway, we can never apply the previous proposition using only this observation. Indeed, in this case, the hypothesis $d > k^2 \rho(mk)$ fails. We have $d \leq \deg(f) = m_1^2 - 1 + m_2^2 - 1$ and

$$\rho(mk) \geq \rho(k) = \rho(m_1) + \rho(m_2) \geq \frac{1}{m_1} + \frac{1}{m_2}.$$

So,

$$d \leq m_1^2 - 1 + m_2^2 - 1 < m_1^2 + m_2^2 = k^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \leq k^2 \rho(mk).$$

Now, we want to find some cases where $\psi_p$ is reducible for $p$ a prime. We will show that this happens for a lot of primes. We briefly recall some classical facts on the elliptic curves, for the details see [6, Section III.9]. We will denote by $\text{End}(E)$ the ring of the endomorphisms (defined over $\overline{\mathbb{Q}}$) of $E$. Since the map given by the multiplication by a rational integer is an endomorphism, it follows that $\mathbb{Z} \subseteq \text{End}(E)$. If $\text{End}(E) \setminus \mathbb{Z}$ is not empty, we say that $E$ has complex multiplication.
Consider the embedding \( \text{End}(E) \hookrightarrow \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} \). If \( E \) has complex multiplication, then every element \( \varphi \) of \( \text{End}(E) \) can be written as \( a + \gamma b \) with \( a \) and \( b \) in \( \mathbb{Q} \subseteq \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} \) and \( \gamma \) such that \( \gamma^2 < 0 \) and \( \gamma^2 \in \mathbb{Q} \). We say that an endomorphism \( \varphi \) splits if there exist \( \alpha \) and \( \beta \) that are not isomorphisms so that \( \varphi = \alpha \beta \). We define the norm as in \([6, \text{Section III.9}]\). If \( \text{Norm}(\varphi) = 1 \), then \( \varphi \) is an isomorphism and \( \text{Norm}(n) = n^2 \) for \( n \in \mathbb{Z} \). Moreover, \( \text{Norm}(\alpha \beta) = \text{Norm}(\alpha) \text{Norm}(\beta) \) for every \( \alpha \) and \( \beta \) in \( \text{End}(E) \). The group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) has a natural action over \( \text{End}(E) \).

**Lemma 4.6.** Suppose that \( E \) is a rational elliptic curves that has complex multiplication (not necessarily with \( j(E) = 1728 \)). Take \( p \neq 2 \) a prime that splits in \( \text{End}(E) \). Hence, \( \psi_p^2(x) = f(x)g(x) \) with \( f \) and \( g \) two coprime polynomials with integer coefficients and

\[
d = \min\{\deg f, \deg g\} = 2p - 2.
\]

**Proof.** Let \( p \) be a prime that splits in \( \text{End}(E) \), i.e. there exist \( \alpha \) and \( \beta \) in \( \text{End}(E) \setminus \text{Aut}(E) \) such that

\[
\alpha \beta = p.
\]

Then \( \alpha = a + \gamma b \) with \( \gamma \) defined as before. Let \( \overline{\alpha} = a - b\gamma \) and take \( g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Hence,

\[
(\gamma g)^2 = (\gamma^2)g^2 = \gamma^2
\]

since \( \gamma^2 \) is rational and we conclude \( \gamma g = \pm \gamma \). So, \( \alpha g = a^g + \gamma^g b^g = a \pm \gamma b \) that is \( \alpha \) or \( \overline{\alpha} \). We denote with \( E[p] \) the set of the \( p \)-torsion points of \( E(\mathbb{Q}) \) that has \( p^2 \) elements and it is invariant under the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Consider the set \( \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \); we will show that it is invariant under the action of the Galois group and has \( 2p - 1 \) elements. If \( P = \alpha(Q) \), then \( P^g = \overline{\alpha}(Q^g) \) or \( P^g = \alpha(Q^g) \) and then \( P^g \) is in the set \( \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \). Therefore, the set \( \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \) is invariant under the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Observe that \( \text{Norm}(\alpha) = p \) since it must be a divisor of \( p^2 \) and cannot be \( 1 \) and \( p^2 \) since \( \alpha \) and \( \beta \) are not isomorphisms. So,

\[
\text{Norm}(\alpha \overline{\alpha}) = \text{Norm}(\alpha) \text{Norm}(\overline{\alpha}) = p^2
\]

and then \( \alpha \overline{\alpha} \) is a rational with norm \( p^2 \), that can be only \( p \) or \( -p \). Moreover, for the properties of the Norm, \( \# \text{Ker}(\alpha) = \text{Norm}(\alpha) = p \) and \( \overline{\alpha}(E[p]) \subseteq \text{Ker}(\alpha) \) since

\[
O = \pm p(E[p]) = \alpha(\overline{\alpha}(E[p])).
\]

Suppose that \( \text{Ker}(\alpha) \cap \text{Ker}(\overline{\alpha}) \neq \{O\} \). Take \( P \neq O \) in the intersection. Hence \( 2\alpha(P) = \alpha(P) + \overline{\alpha}(P) = O \) and in the same way \( 2b(P) = O \), that implies \( p|\text{Norm}(2a) \), \( \text{Norm}(2b) \). Since \( 2a \) and \( 2b \) are both integers, then \( \text{Norm}(2a) \) and \( \text{Norm}(2b) \) are both squares. So, \( p^2 \) must divide their norm and then

\[
p^2|\text{Norm}(2\alpha) = 4p,
\]

that is absurd. Hence, \( \text{Ker}(\alpha) \cap \text{Ker}(\overline{\alpha}) = \{O\} \) and therefore \( \overline{\alpha}(E[p]) \) and \( \alpha(E[p]) \) have trivial intersection. The map \( \alpha : E[p] \rightarrow E[p] \) has kernel with \( p \) elements and then the image has \( p \) elements (recall that \( E[p] \) has \( p^2 \) elements). We conclude that \( \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \) has \( 2p - 1 \) elements. So, if \( f \) is the polynomial with roots the abscissas of the points of the set \( \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \setminus \{O\} \), then \( f \) has degree \( 2p - 2 \) and \( f \in \mathbb{Q}[x] \) since the set of the roots is invariant under the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). We conclude by observing that the roots of \( f \) are roots of \( \psi_p^2 \) and then \( f \) divides \( \psi_p^2 \).
Let \( g = \psi^2_p/f \). The polynomial \( g \) has integral coefficients and has degree \( p^2 - 1 - (2p - 2) = p^2 - 2p + 1 \). So, \( d = \min\{\deg f, \deg g\} = 2p - 2 \) considering that
\[
p^2 - 2p + 1 \geq 2p - 2
\]
for \( p \geq 3 \). It remains to prove that the two polynomials are coprime. Otherwise, there exists \( x_0 \in \mathbb{Q} \) that is a root of \( f \) and \( g \). Hence, there are \( P, P' \in E[p] \setminus \{O\} \) such that \( x_0 = x(P) \) with \( P \in \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \) and \( x_0 = x(P') \) with \( P' \notin \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \). Since \( x(P) = x_0 = x(P') \), then \( P = -P' \) and this is absurd since \( \{\alpha(E[p]) \cup \overline{\alpha}(E[p])\} \) is invariant under the multiplication by \(-1\).

**Corollary 4.7.** Take \( p \neq 2 \) a prime that splits in \( \text{End}(E) \). Then, \( \psi_p(x) \) is reducible.

**Proof.** Let \( \psi^2_p(x) = \prod_i p_i(x)^{a_i} \) be the factorization in prime factors of \( \psi^2_p \). Thanks to the previous lemma we know that \( \psi^2_p \) has at least two prime divisors. Since \( \psi^2_p \) is the square of \( \psi_p \), then \( a_i \) is even for every \( i \) and so \( a_i = 2b_i \). So,
\[
\psi_p(x) = \pm \prod_i p_i(x)^{b_i}
\]
and then \( \psi_p \) is reducible.

We use the previous lemma to prove that \( \psi_p(x) \) is reducible when \( p \equiv 1 \mod 4 \), if we consider the curve \( E_a \).

**Lemma 4.8.** Let \( p \) be a prime congruent to 1 modulo 4. So, \( p \) splits in the ring \( \text{End}(E_a) \).

**Proof.** Let \( i \) be the endomorphism of \( E_a \) so that
\[
i(x, y) = (-x, iy).
\]
This is an endomorphism since
\[
(iy)^2 = -y^2 = -x^3 - ax = (-x)^3 + a(-x)
\]
and then the points in the image of the map are still in \( E_a \). Observe that \( i^2 = [-1] \), where \([-1]\) is the inverse endomorphism. This shows that \( E_a \) has complex multiplication. Thanks to the Fermat’s theorem on sums of two squares, we know that there exist \( a \) and \( b \) integers so that
\[
a^2 + b^2 = p.
\]
Here we are using the hypothesis \( p \equiv 1 \mod 4 \). Let \( \varphi_1 \) be the endomorphism \( a + ib \) and \( \varphi_2 \) be the endomorphism \( a - ib \). Hence,
\[
\varphi_1 \varphi_2 = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 + b^2 = p.
\]
Since \( \varphi_1 \) and \( \varphi_2 \) are conjugate, then they have the same norm. So,
\[
p^2 = \text{Norm } p = \text{Norm } \varphi_1 \text{ Norm } \varphi_2 = \text{Norm } \varphi_1^2 = \text{Norm } \varphi_2^2
\]
and then \( \varphi_1 \) and \( \varphi_2 \) are not isomorphisms. Therefore, \( p \) splits.
We sum up the present situation: Thanks to Lemma 4.6 and 4.8, we know that $\psi_p$ is reducible for $p \equiv 1 \mod 4$. We want to show that $B_{mp}$ has a primitive divisor, under some hypothesis on $m$. Using Proposition 4.3, we know that, in order to prove that $B_{mp}$ has a primitive divisor, we have to show that an inequality does not hold. In the inequalities of Proposition 4.3 every term that appears has been studied, except for $K_k$. So, in the next pages, we will compute an explicit bound for $K_k$ as defined in Lemma 4.1.

Consider the lattice $\Lambda \subseteq \mathbb{C}$ generated by 1 and $i$. As is shown in [6, Chapter VI] there is an isomorphism $\varphi$ between $\mathbb{C}/\Lambda$ and $E(\mathbb{C})$ given by the map
\[
\varphi(z) = (\varphi(z), \frac{\varphi'(z)}{2}, 1)
\]
where
\[
\varphi(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}.
\]
The curve $E$ is defined by the equation $y^2 = x^3 - 15G_4x - 35G_6$ with
\[
G_4 := \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^4}
\]
and
\[
G_6 := \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^6}.
\]
Thanks to [6, Proposition VI.3.6] and [6, Exercise 6.6], we have $j(E) = 1728$. So, $G_6 = 0$ and hence $E$ is defined by the equation
\[
y^2 = x^3 - 15G_4x.
\]
Define
\[
\sigma(z) := z \prod_{\omega \in \Lambda \setminus 0} \left(1 - \frac{z}{\omega}\right)^{\frac{1}{2}} + \frac{1}{z^2}
\]
and
\[
\Lambda^* := \{\lambda \in \Lambda | |\lambda| > 2\}.
\]
For the details on $\sigma$, see [6, Lemma VI.3.3]. Thanks to [6, Exercise 6.3], given $z_1$ and $z_2$ two complex numbers, we have
\[
x(\varphi(z_1)) - x(\varphi(z_2)) = -\frac{\sigma(z_1 + z_2)\sigma(z_1 - z_2)}{\sigma(z_1)^2\sigma(z_2)^2}.
\]
For this reason, in order to compute $K_k$, we need to study the function $\sigma$.

We start by computing an upper and a lower bound for the absolute value of $\sigma$ evaluated at $z$, for $z \in \mathbb{C}$ such that $nz \in \Lambda$ and such that 0 is the element of $\Lambda$ closest to $z$. So, $\varphi(z)$ is a $n$-torsion point of $E(\mathbb{C})$. In order to do so, we need a preliminary lemma.

**Lemma 4.9.** Let $k \geq 3$. Then,
\[
\sum_{\omega \in \Lambda^*} \left| \frac{1}{\omega^k} \right| \leq \frac{21}{k}.
\]
Proof. Every element of \( \Lambda^* \) can be written as \( a + ib \) with \( a \) and \( b \) two integers such that \( a^2 + b^2 \geq 5 \). Fix \( a \) and \( b \) two strictly positive integers such that \( a^2 + b^2 \geq 5 \). Consider the square \( Q_{a,b} \) on the complex plane with vertices \((a - 1, b - 1), (a - 1, b), (a, b - 1)\) and \((a, b)\). Observe that, for \( x \in Q_{a,b} \), 
\[
|a + ib|^k \leq \frac{1}{|x|^k}.
\]
Moreover, the intersection of two different squares has measure 0. For every \( a \) and \( b \), if \( x \in Q_{a,b} \), then 
\[
|a + ib|^k \leq \int_{Q_{a,b}} \frac{1}{|x|^k} \, dx.
\]
and
\[
\sum_{a,b > 0 \atop a^2 + b^2 \geq 5} \frac{1}{a + ib} \leq \sum_{a,b > 0 \atop a^2 + b^2 \geq 5} \int_{Q_{a,b}} \frac{1}{|x|^k} \, dx \leq \int_{\mathbb{R}x \geq 0, |x| \geq 1} \frac{1}{|x|^k} \, dx = \frac{\pi}{2(k-2)} \tag{4.3}
\]
where \( \mathbb{R}x \) represents the real part of the complex number \( x \) and \( \mathbb{I}x \) represents the imaginary part. Moreover,
\[
\sum_{a^2 \geq 5} \frac{1}{a} = 2 \sum_{a=3}^{\infty} \frac{1}{a^k} \leq 2 \int_{2}^{\infty} \frac{1}{a^k} \, da = \frac{2}{2^{k-1}(k-1)} \tag{4.4}
\]
Finally, using (4.3) and (4.4),
\[
\sum_{\omega \in \Lambda^*} \left| \frac{1}{\omega^k} \right| = \sum_{a^2 + b^2 \geq 5} \frac{1}{a + ib}^k = \sum_{a^2 \geq 5} \frac{1}{|a|^k} + \sum_{b^2 \geq 5} \frac{1}{|b|^k} + \sum_{a,b \neq 0 \atop a^2 + b^2 \geq 5} \frac{1}{|a + ib|^k} \leq \frac{2}{2^{k-1}(k-1)} + \frac{2}{2^{k-1}(k-1)} + 4 \int_{\mathbb{R}x \geq 0, |x| \geq 1} \frac{1}{|x|^k} \, dx \leq \frac{1}{k - 1} + \frac{2\pi}{k - 2} \leq \frac{21}{k}
\]
where the last inequality follows from the fact that \( k \geq 3 \).

Given a complex number \( x \) with \( |x| < 1 \), we define
\[
\log(1 - x) = - \sum_{i=1}^{\infty} \frac{x^i}{i}.
\]
Lemma 4.10. Let \( z \neq 0 \in \mathbb{C} \) be such that the element of \( \Lambda \) closest to \( z \) is 0. Suppose that \( nz \in \Lambda \).

Hence,

\[
\frac{0.14}{n} \leq |\sigma(z)| \leq 4.04.
\]

Proof. Observe that \( z = (a + ib)/n \) with \( a \) and \( b \) two integers such that \( |a| \leq n/2 \) and \( |b| \leq n/2 \).

We start by finding a bound for the product of \( \sigma \) considering only the terms in \( \Lambda^* \). Put

\[
z_1 := \prod_{\omega \in \Lambda^*} \left( 1 - \frac{z}{\omega} \right) e^{\frac{1}{2} \left( \frac{z}{\omega} \right)^2}
\]

and observe that

\[
\log \left[ \left( 1 - \frac{z}{\omega} \right) e^{\frac{1}{2} \left( \frac{z}{\omega} \right)^2} \right] = \log \left( 1 - \frac{z}{\omega} \right) + \log \left( e^{\frac{1}{2} \left( \frac{z}{\omega} \right)^2} \right)
\]

\[
= - \left( \sum_{i=1}^{\infty} \frac{z^i}{i\omega^i} \right) + \frac{z}{\omega} + \frac{1}{2} \left( \frac{z}{\omega} \right)^2
\]

\[
= - \sum_{i=1}^{\infty} \frac{z^i}{i\omega^i}.
\]

Taking the logarithm,

\[
|\log z_1| = \left| \log \prod_{\omega \in \Lambda^*} \left( 1 - \frac{z}{\omega} \right) e^{\frac{1}{2} \left( \frac{z}{\omega} \right)^2} \right| = \left| \sum_{i=3}^{\infty} \sum_{\omega \in \Lambda^*} \frac{z^i}{i\omega^i} \right| \leq \sum_{i=3}^{\infty} \sum_{\omega \in \Lambda^*} \left| \frac{z^i}{i\omega^i} \right|.
\]

Since the element of \( \Lambda \) closest to \( z \) is 0, then \(|z| \leq (2)^{-1/2} \) and therefore, using Lemma 4.9,

\[
\sum_{i=3}^{\infty} \sum_{\omega \in \Lambda^*} \left| \frac{z^i}{i\omega^i} \right| \leq \sum_{i=3}^{\infty} \sum_{\omega \in \Lambda^*} \left( \frac{\sqrt{2}}{i} \right)^{-i} \frac{1}{|\omega|^i}
\]

\[
= \sum_{i=3}^{\infty} \sum_{\sigma^2 + b^2 \geq 5} \frac{1}{i |a^2 + b^2|^{i/2}}
\]

\[
= \sum_{i=3}^{\infty} \frac{1}{i (\sqrt{2})^{i}} \sum_{a^2 + b^2 \geq 5} \frac{1}{|a^2 + b^2|^{i/2}}
\]

\[
\leq \sum_{i=3}^{\infty} \frac{21}{2^{i/2} 3^2}
\]

\[
\leq \frac{21}{2^{4/2} 3^2} + \frac{21}{2^{4/2} 4^2} + \frac{21}{2^{5/2} 5^2} + \frac{21}{36\sqrt{2}} \sum_{i=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^i
\]

\[
= \frac{21}{2^{4/2} 3^2} + \frac{21}{2^{4/2} 4^2} + \frac{21}{2^{5/2} 5^2} + \frac{21}{36\sqrt{2}} \sqrt{2} - 1
\]

\[
\leq 1.56
\]

and then

\[
|\Re(\log z_1)| \leq |\log z_1| \leq 1.56.
\]
Therefore, 
\[ e^{-1.56} \leq e^{-|\Re(\log z_1)|} \leq |e^{\log z_1}| = |z_1| \leq e^{\Re(\log z_1)} \leq e^{1.56} \]
since, for \( x \in \mathbb{C} \),
\[ e^{\Re(x)} = |e^x|. \]

Let \( \Lambda_1 = \Lambda \setminus \Lambda^* \setminus \{0\} \). This is a set of 12 complex numbers. Then, by direct computation,

\[ \sum_{\omega \in \Lambda_1} \frac{1}{\omega} = 0, \]
\[ \sum_{\omega \in \Lambda_1} \frac{1}{\omega^2} = 0 \]
and
\[ \prod_{\omega \in \Lambda_1} \frac{1}{\omega} = \frac{1}{64}. \]

Now we need to deal with \( \prod_{\omega \in \Lambda_1} (z - \omega) \). Recalling that 0 is the element of \( \Lambda \) closest to \( z \), then \(-0.5 \leq \Re z \leq 0.5 \) and \(-0.5 \leq \Im z \leq 0.5 \). We have

\[ \min_{-0.5 \leq \Re z \leq 0.5} \left| \prod_{\omega \in \Lambda_1} \frac{(z - \omega)}{\omega} \right| \geq 0.94. \]

This follows from the calculation of the minimum of the absolute value of the polynomial, that can be seen as a real polynomial in two variables, writing \( z \) as \( x + iy \). In the same way

\[ \max_{-0.5 \leq \Re z \leq 0.5} \left| \prod_{\omega \in \Lambda_1} \frac{(z - \omega)}{\omega} \right| \leq 1.2. \]

Observe that

\[ \frac{1}{n} \leq |z| \leq \frac{1}{\sqrt{2}} \]

since every \( n \)-torsion point of \( \mathbb{C}/\Lambda \) is in the form \( (a + ib)/n \) and \( z \neq 0 \). Moreover,

\[ \sigma(z) = z \cdot z_1 \cdot \prod_{\omega \in \Lambda_1} \left( \frac{\omega - z}{\omega} \right) \cdot e^{\sum_{\omega \in \Lambda_1} \omega} \cdot e^{\sum_{\omega \in \Lambda_1} \frac{z^2}{\omega^2}} \]
\[ = z \cdot z_1 \cdot \prod_{\omega \in \Lambda_1} \left( \frac{\omega - z}{\omega} \right). \]

Using all the inequalities before, we conclude

\[ \frac{0.19}{n} \leq \frac{1}{n} \cdot e^{-1.56} \cdot 0.94 \leq \left| z \cdot z_1 \cdot \prod_{\omega \in \Lambda_1} \left( \frac{\omega - z}{\omega} \right) \right| = |\sigma(z)| \]
and

\[ |\sigma(z)| \leq \left| z \cdot z_1 \cdot \prod_{\omega \in \Lambda_1} \left( \frac{\omega - z}{\omega} \right) \right| \leq \frac{1}{\sqrt{2}} \cdot e^{1.56} \cdot 1.2 \leq 4.04. \]

\( \square \)
Now, we need to bound $|\sigma(z)|$ for $z$ so that $nz \in \Lambda$ but with $z$ in a larger region of the complex plane compared to the previous lemma.

**Lemma 4.11.** Let $z = (a + ib)/n \notin \Lambda$ with $a$ and $b$ two integers such that $|a| \leq n$ and $|b| \leq n$. So,

$$|\sigma(z)| \geq \frac{1}{2.1 \cdot 10^{16} \cdot n}.$$  

**Proof.** Let $\omega \in \Lambda$ be such that the element of $\Lambda$ closest to $z + \omega$ is 0. The element $\omega$ can be written in the form $x + iy$ for $x$ and $y$ two integers so that $-1 \leq x, y \leq 1$. Thanks to [6, Exercise 6.4.e],

$$\sigma(z) = \pm \sigma(z + \omega)e^{-\eta(\omega)(z + \omega/2)} = \pm \sigma(z + \omega)e^{-\frac{\eta(\omega)}{2}},$$

where $\eta(\omega)$ defined in [6, Exercise 6.4.b]. Since we are interested in the absolute value, the sign is not important. The function $\eta$ is linear (see [6, Exercise 6.4.c]) and then

$$|\eta(\omega)| \leq |\eta(1)| + |\eta(I)|.$$  

Using the command "elleta" of PARI/GP, it is possible to compute the value of eta, and we have $|\eta(1)| \leq 3.142$ and $|\eta(I)| \leq 9.426$. Recalling that $|z| \leq \sqrt{2}$, we have

$$|-z\eta(\omega)| \leq \sqrt{2}(3.142 + 9.426) \leq 17.8$$

and, in the same way,

$$|-\omega\eta(\omega)/2| \leq 17.8.$$  

So, using Lemma 4.10,

$$|\sigma(z)| = |\sigma(z + \omega)|e^{-\frac{z\eta(\omega)}{2}} \geq \frac{0.14}{n} e^{-|z\eta(\omega)| - |z\eta(\omega)|} \geq \frac{1}{2.1 \cdot 10^{16} \cdot n}$$

since

$$0.14 \cdot e^{-17.8} \cdot e^{-17.8} \geq \frac{1}{2.1 \cdot 10^{16}}.$$  

**Lemma 4.12.** Let $P$ be a non-trivial $n$-torsion point in $E(\mathbb{Q})$, where $E$ is defined by $y^2 = x^3 - 15G_4x$. Then,

$$|x(P)| \leq n^2 + 53.$$  

**Proof.** We want to bound

$$\varphi(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \backslash 0} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}$$

for $z = (a + ib)/n \neq 0$ with $a$ and $b$ two integers such that $|a| \leq n/2$ and $|b| \leq n/2$. Using the previous notation, we observe that $|\omega| > 2|z|$ for $\omega \in \Lambda^*$. So,

$$|z - 2\omega| \leq 2|\omega| + |z| \leq 2|\omega| + \frac{|\omega|}{2} = \frac{5}{2}|\omega|$$

and

$$|z - \omega| \geq |\omega| - |z| \geq |\omega| - \frac{|\omega|}{2} = \frac{|\omega|}{2}.$$
Observe that $|z| \leq \sqrt{2}/2$ and using Lemma 4.9 we have

$$\left| \sum_{\omega \in \Lambda^*} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \sum_{\omega \in \Lambda} \frac{z(z-2\omega)}{\omega^2(z-\omega)^2} \right|$$

$$\leq |z| \sum_{\omega \in \Lambda^*} \frac{5}{4} \frac{|\omega|}{|\omega|^4}$$

$$\leq 5\sqrt{2} \sum_{\omega \in \Lambda^*} \frac{1}{|\omega|^4}$$

$$\leq 5\sqrt{2} \frac{21}{3}$$

$$\leq 50.$$

Furthermore,

$$\sum_{\omega \in \Lambda_1^*} \frac{1}{\omega^2} = 0$$

and

$$\left| \max_{-0.5 \leq x \leq 0.5} \sum_{-0.5 \leq z \leq 0.5, \omega \in \Lambda_1} \frac{1}{(z-\omega)^2} \right| \leq 3$$

by direct computation. Then, using $|z| \geq 1/n$ since $z \neq 0$, we obtain

$$|\varphi(z)| \leq \frac{1}{|z|^2} + \sum_{\omega \in \Lambda^*} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} + \left| \sum_{\omega \in \Lambda_1} \frac{1}{(z-\omega)^2} \right| + \sum_{\omega \in \Lambda_1} \frac{1}{\omega^2}$$

$$\leq n^2 + 50 + 3.$$

We conclude by observing that every $n$-torsion point in $\mathbb{C}/\Lambda$ can be written as $z = a + ib$ with $|a| \leq n/2$ and $|b| \leq n/2$. So, every non-trivial $n$-torsion point $P$ of $E(\mathbb{C})$ is equal to $\varphi(z)$ for $z = (a + ib)/n \neq 0$ with $|a| \leq n/2$ and $|b| \leq n/2$. In conclusion,

$$|x(P)| = |\varphi(z)| \leq n^2 + 53.$$

\[\square\]

**Lemma 4.13.** Let $G_4$ be defined as before. Hence,

$$1 \leq |15G_4| \leq 128.$$

**Proof.** By direct computation,

$$\sum_{\omega \in \Lambda_1} \omega^{-4} = \frac{13}{4}. \quad (4.5)$$
Thus, using Lemma 4.9,

\[ |15G_4| \leq 15 \sum_{\omega \in \Lambda_1} \omega^{-4} + 15 \sum_{a^2 + b^2 \geq 5} \frac{1}{(a^2 + b^2)^2} \]
\[ \leq 15 \cdot \frac{13}{4} + 15 \cdot \frac{21}{4} \]
\[ \leq 128. \]

Now, we focus on the lower bound. Observe that, using (4.5),

\[ \sum_{\omega \in \Lambda_1} \omega^{-4} + \sum_{b^2 \geq 5} \frac{1}{(ib)^4} + \sum_{a^2 \geq 5} \frac{1}{(a)^4} = \sum_{\omega \in \Lambda_1} \omega^{-4} + 4 \sum_{b \geq 3} \frac{1}{b^2} \geq \sum_{\omega \in \Lambda_1} \omega^{-4} = \frac{13}{4} \]

and

\[ \sum_{\omega \in \Lambda} \omega^{-4} = \left( \sum_{\omega \in \Lambda_1} \omega^{-4} + \sum_{b^2 \geq 5} \frac{1}{(ib)^4} + \sum_{a^2 \geq 5} \frac{1}{(a)^4} \right) + 4 \sum_{a,b > 0} \frac{1}{(a^2 + b^2)^2} \]

As we showed in the proof of Lemma 4.9,

\[ 4 \left| \sum_{a,b \geq 0} \frac{1}{(a^2 + b^2)^2} \right| \leq \frac{2\pi}{4-2} = \pi. \]

So,

\[ \left| \sum_{\omega \in \Lambda} \omega^{-4} \right| \geq \left| \left( \sum_{\omega \in \Lambda_1} \omega^{-4} + \sum_{b^2 \geq 5} \frac{1}{(ib)^4} + \sum_{a^2 \geq 5} \frac{1}{(a)^4} \right) - 4 \sum_{a,b > 0} \frac{1}{(a^2 + b^2)^2} \right| \]
\[ \geq \frac{13}{4} - \pi \]
\[ \geq 0.1. \]

Finally,

\[ |15G_4| = 15 \sum_{\omega \in \Lambda} \omega^{-4} \geq 1. \]

\[ \square \]

**Lemma 4.14.** Let \( n \geq 2 \) and \( K_n \) be as in (4.1). So, we have

\[ K_n \geq \frac{1}{2.5 \cdot 10^{36} \cdot n^6}. \]

**Remark 4.15.** This bound is not sharp. Computational evidence shows that we could do much better. Indeed, using the command "polroots" of PARI/GP we can effectively compute \( K_n \) for \( n \) small. It seems that we could take \( K_n \geq 6/n^2 \). Anyway, for our goal, the bound is good enough.
Proof. Using [6, Exercise 6.3], given \( z_1 \) and \( z_2 \) two points in \( \mathbb{C}/\Lambda \), we have

\[
\psi(z_1) - \psi(z_2) = \frac{\sigma(z_1 + z_2)\sigma(z_2 - z_1)}{\sigma(z_1)^2\sigma(z_2)^2}.
\]

Let now \( z_1 \) and \( z_2 \) be two non-zero complex numbers such that the element of \( \Lambda \) closest to \( z_1 \) and \( z_2 \) is 0 and such that \( nz_1 \) and \( nz_2 \) belong to \( \Lambda \). Hence, \( z_1 + z_2 \) and \( z_1 - z_2 \) satisfy the hypothesis of Lemma 4.11, if \( z_1 \neq \pm z_2 \). So, using Lemma 4.10,

\[
|\sigma(z_i)| \leq 4.04
\]

for \( i = 1, 2 \) and using Lemma 4.11

\[
|\sigma(z_1 \pm z_2)| \geq \frac{1}{2.1 \cdot 10^{16} \cdot n}.
\]

Therefore,

\[
|\psi(z_1) - \psi(z_2)| = \left| \frac{\sigma(z_1 + z_2)\sigma(z_2 - z_1)}{\sigma(z_1)^2\sigma(z_2)^2} \right|
\]

\[
\geq \frac{1}{(2.1)^2 \cdot (10)^{32} \cdot (4.04)^4 n^2}
\]

\[
\geq \frac{1}{1.18 \cdot 10^{35} \cdot n^2}
\]

if \( z_1 \neq \pm z_2 \).

Recall that \( E \) is the elliptic curve defined by the equation \( y^2 = x^3 - 15G_4x \). Given \( T_1 \) and \( T_2 \) two non-trivial \( n \)-torsion points on the curve \( E \), there are \( z_1 \) and \( z_2 \) as before such that \( \psi(z_1) = T_1 \) and \( \psi(z_2) = T_2 \). If \( T_1 \neq \pm T_2 \), then we obtain

\[
|x(T_1) - x(T_2)| = |\psi(z_1) - \psi(z_2)| \geq \frac{1}{1.18 \cdot 10^{35} \cdot n^2}.
\]

Let \( E_1 \) be the elliptic curve defined by \( y^2 = x^3 + x \) and so, if \( T_1 \) and \( T_2 \) are two non-trivial \( n \)-torsion points for \( E_1 \), then, thanks to the work in Lemma 4.1 and 4.13,

\[
|x(T_1) - x(T_2)| \geq \frac{1}{\sqrt{15G_4} \cdot 1.18 \cdot 10^{35} \cdot n^2}
\]

\[
\geq \frac{1}{\sqrt{128} \cdot 1.18 \cdot 10^{35} \cdot n^2}
\]

\[
\geq \frac{1}{1.4 \cdot 10^{36} \cdot n^2}.
\]

Let \( T \) be a non-trivial \( n \)-torsion point on \( E_1 \). Using again the work in the proof of Lemma 4.1, we know that

\[
|x(T)| \leq \frac{\max_{R \in E(\mathbb{Q})\setminus \{O\}} |x(R)|}{\sqrt{15G_4}}.
\]

Thanks to Lemma 4.12 and 4.13 we have

\[
|x(T)| \leq \frac{n^2 + 53}{1}.
\]
If $T_1$ and $T_2$ are two non-trivial $n$-torsion points in $E_1(\mathbb{Q})$ with $T_1 \neq \pm T_2$, then

$$|x(T_1)^{-1} - x(T_2)^{-1}| = \frac{|x(T_1) - x(T_2)|}{|x(T_1)x(T_2)|} \geq \frac{1}{1.4 \cdot 10^{36} \cdot n^2(n^2 + 53)^2} \geq \frac{1}{2.5 \cdot 10^{36} \cdot n^6}.$$ 

The last inequality holds only if $n \geq 13$. Here we assumed $x(T_1)x(T_2) \neq 0$. For the cases $2 \leq n \leq 13$ we prove the lemma computing effectively the constant $K_n$ using the command "polroots" of PARI/GP. For example, the roots of $\psi_3(x)$ when $a = 1$ are $\pm 0.3933\ldots$ and $\pm 1.46789\ldots$. So, $K_3 \sim 0.7866$ and then $K_3 \geq 0.75 \geq \frac{1}{2.5 \cdot 10^{36} \cdot 3^6}$.

Thanks to Lemma 4.6 and 4.8 we have that $\psi_p$ is reducible. Therefore, we would like to apply Proposition 4.3. In order to do so, we need to verify the hypothesis of the proposition. For this reason, we prove the following two lemmas.

**Lemma 4.16.** Fix $p \geq 13$ a prime congruent to 1 modulo 4 and let $P$ be a non-torsion point in $E_a(\mathbb{Q})$. Suppose $m \geq p + 2$. Then,

$$2m^2 \geq \frac{2C + \log 2 |a|^{1/2} - \log K_p}{\hat{h}(P)}.$$

**Proof.** Using Lemma 2.1 and the bound of $K_p$ in Lemma 4.14,

$$\frac{2C + \log 2 |a|^{1/2} - \log K_p}{\hat{h}(P)} \leq \frac{6 \log p}{\hat{h}(P)} + \frac{\log |a| + 85.1}{\hat{h}(P)}.$$ 

Thanks to Lemma 2.3, 2.11 and 2.12 we have

$$\frac{6 \log p}{\hat{h}(P)} + \frac{\log |a| + 85.1}{\hat{h}(P)} \leq \frac{3 \log p}{5} + 1039.7.$$ 

Observe that, for $p \geq 23$,

$$2(p + 2)^2 \geq \frac{3 \log p}{5} + 1039.7.$$ 

Therefore,

$$2m^2 \geq 2(p + 2)^2 \geq \frac{3 \log p}{5} + 1039.7 \geq \frac{2C + \log 2 |a|^{1/2} - \log K_p}{\hat{h}(P)}.$$ 

If $p = 13$, then computing $K_{13}$ with PARI/GP we have $K_p \geq 0.04$. So,

$$\frac{2C + \log 2 |a|^{1/2} - \log K_p}{\hat{h}(P)} \leq \frac{\log |a| + 4.5}{\hat{h}(P)}.$$ 

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and using the usual inequalities
\[
\frac{\log |a| + 4.5}{h(P)} \leq 63.
\]
Therefore,
\[
2m^2 \geq 2(p + 2)^2 = 450 \geq 63 \geq \frac{2C + \log 2 |a|^{1/2} - \log K_p}{h(P)}.
\]
The case \(p = 17\) is analogous. \(\square\)

**Lemma 4.17.** Let \(n\) be an integer such that the smallest divisor of \(n\) is \(p \geq 13\). Hence,
\[
\frac{0.8}{p} \leq \frac{2p - 2}{p^2} - \rho(n).
\]

**Proof.** Since the smallest prime divisor of \(n\) is \(p\),
\[
\frac{2p - 2}{p^2} - \rho(n) \geq \frac{2p - 2}{p^2} - \frac{1}{p^2} - \sum_{n \geq p + 2} \frac{1}{n^2} \geq \frac{2p - 3}{p^2} - \int_{p+1}^{\infty} \frac{dx}{x^2} = \frac{2p - 3}{p^2} - \frac{1}{p + 1} \geq \frac{0.8}{p},
\]
where the last inequality holds since \(p \geq 13\). \(\square\)

Now, we are ready to prove Theorem 1.4. Fix a prime \(p \equiv 1 \mod 4\). We want to use Proposition 4.3. We know that \(\psi_p\) is reducible and the hypothesis of the proposition is satisfied thanks to Lemmas 4.16 and 4.17. So, if \(B_{mp}\) does not have a primitive divisor, then inequality (4.2) must hold. Hence, in order to prove the theorem, we will show that this inequality does not hold, using the bound on \(K_n\).

**Proof of Theorem 1.4.** Suppose now that \(n = mp\) is a square-free positive integer with \(p \equiv 1 \mod 4\) and \(m\) such that the smallest divisor of \(n\) is larger than \(p\). We want to show that \(B_n\) always has a primitive divisor. So we may assume that \(n\) is odd, and since \(n\) is square-free, it follows that \(m \geq p + 2\). Furthermore, the case of \(p = 5\) is handled by Theorem 1.3 here, so we can assume that \(p \geq 13\). We will assume that \(B_n\) does not have a primitive divisor and obtain a contradiction.

Since \(p\) is odd, from Lemmas 2.8 and 3.6, we have
\[
g_p | (2\Delta^2)^{\frac{p^2 - 1}{4}} = (2^{13}a^6)^{\frac{p^2 - 1}{4}}.
\]
Thanks to Lemma 4.6 and 4.8, we know that \(\psi_p\) is reducible. Under our conditions here, Lemma 4.16 holds and hence we can apply Proposition 4.3, along with the above upper bound for \(g_p\), to obtain
\[
2(mp)^2 \left( \frac{d}{p^2} - \rho(mp) \right) h(P) \leq \frac{p^2 - 1}{4} \log (2^{13}a^6) - d \log \left( \frac{K_p|a|^{-1/2}}{2} \right) + 2dC + 2 \log(mp) + 2C\omega(mp).
\]
Here we can take \( d = 2p - 2 \), thanks to Lemma 4.6. Also, from Lemma 4.14, we find that the left-hand side of (4.6) is at most

\[
\frac{p^2 - 1}{4} \log \left( \left(2^{13} |a|^6\right) \right) + 4(p-1)C + 2(\log(n) + C \omega(n)) + 2(p-1) \log \left( 5 \cdot 10^{36} p^6 \sqrt{a} \right).
\]

It will help us in what follows to express the right-hand side of this inequality solely in terms of \( a, m \) and \( p \). To do so, we will collect together like terms. Since \( \omega(n) \leq \log(mp)/\log(p) = \log(m)/\log(p) + 1 \leq \log(m)/\log(13) + 1 \) and by the definition of \( C \) in Lemma 2.1, we obtain

\[
2C \omega(n) + 2 \log(n) \leq \left( 0.52 + \frac{\log |a|}{2} \right) \left( \frac{\log(m)}{\log(13)} + 1 \right) + 2 \log(m) + 2 \log(p)
= \log(m) \left( 0.52 + \frac{\log |a|}{2 \log(13)} \right) + 2 \log(p)
+ 0.52 + \frac{\log |a|}{2} + 2 \log(p),
\]

and

\[
4(p-1)C + \frac{p^2 - 1}{4} \log \left( \left(2^{13} |a|^6\right) \right) + 2(p-1) \log \left( 5 \cdot 10^{36} p^6 \sqrt{a} \right)
\leq 4(p-1)(0.26 + \log |a|/4) + \frac{p^2 - 1}{4} \log \left( 2^{13} \right) + (3/2) \left( p^2 - 1 \right) \log |a|
+ 12(p-1) \log(p) + (p-1) \log |a| + 2(p-1) \log \left( 5 \cdot 10^{36} \right)
= \log |a| \left( (3/2) \left( p^2 - 1 \right) + 2(p-1) \right) + \frac{p^2 - 1}{4} \log \left( 2^{13} \right)
+ 12(p-1) \log(p) + 2(p-1) \log \left( 5 \cdot 10^{36} \right) + 1.04(p-1).
\]

Combining these two inequalities, we find that

\[
2m^2 \rho^2 \hat{h}(P) \left( \frac{2p^2 - 2}{p^2} - \rho(n) \right)
\leq \log(m) \left( 2.21 + 0.2 \log |a| \right) + \log |a| \left( (3/2) \left( p^2 - 1 \right) + 2p - 3/2 \right)
+ 2.26 (p^2 - 1) + (12p - 10) \log(p) + 170.1(p-1) + 0.52.
\]

Since \( p \geq 13 \), with some basic analysis we obtain

\[
(3/2) \left( p^2 - 1 \right) + 2p - 3/2 < 1.64p^2 \quad \text{and} \quad 2.26 (p^2 - 1) + (12p - 10) \log(p) + 170.1(p-1) + 0.52 < 16.55p^2.
\]

Therefore,

\[
2m^2 \left( \frac{2p^2 - 2}{p^2} - \rho(n) \right) \leq \frac{\log(m) \left( 2.21 + 0.2 \log |a| \right) + p^2 (1.64 \log |a| + 16.55)}{\hat{h}(P)p^2}.
\]
Using the inequalities for $\hat{h}(P)$ in Lemmas 2.3 and 2.11,

\[ \frac{0.2 \log |a| + 2.21}{\hat{h}(P)} \leq 28.7 \]

and

\[ \frac{1.64 \log |a| + 16.55}{\hat{h}(P)} \leq 214.4. \]

Therefore, in order for the inequality in (4.6) to hold, we must also have

\[ 2m^2 \left( \frac{2p - 2}{p^2} - \rho(n) \right) \leq 28.7 \frac{\log(m)}{p^2} + 214.4. \quad (4.7) \]

Using Lemma 4.17, we obtain the inequality

\[ \frac{1.6m^2}{p} \leq 2m^2 \left( \frac{2p - 2}{p^2} - \rho(n) \right) \leq 28.7 \frac{\log(m)}{p^2} + 214.4. \quad (4.8) \]

If we show that this inequality does not hold for some $m$ and $p$, then $B_{mp}$ always has a primitive divisor. If we fix $p \geq 13$, then the function

\[ \frac{1.6m^2}{p} - 28.7 \frac{\log(m)}{p^2} - 214.4 \]

is increasing for $m \geq p + 2$. Take $p > 129$, then

\[ \frac{1.6m^2}{p} - 28.7 \frac{\log(m)}{p^2} - 214.4 \geq \frac{1.6(p + 2)^2}{p} - 28.7 \frac{\log(p + 2)}{p^2} - 214.4 \geq 0 \]

where the last inequality follows from the assumption $p > 129$. So, there is always a primitive divisor for $p > 129$ and $m \geq p + 2$ since (4.8) does not hold. If $m \geq p + 34$ and $p \geq 13$, then we have

\[ \frac{1.6m^2}{p} - 28.7 \frac{\log(m)}{p^2} - 214.4 \geq \frac{1.6(p + 34)^2}{p} - 28.7 \frac{\log(p + 34)}{p^2} - 214.4 \geq 0. \]

Therefore, it remains to check only finitely many cases in the form $p \cdot m$ with $p \leq 129$ and $p < m < p + 34$. Observe that, since $p$ is the smallest divisor of $n$, then in these cases $m$ is prime. Hence, it remains to check the 85 cases where $n = pq$ with $p$ and $q$ primes satisfying $13 \leq p \leq 129$ with $p \equiv 1 \mod 4$ and $p < q < p + 34$. For these cases we substitute the values in (4.7), taking $m = q$ and we obtain

\[ 2q^2 \left( \frac{2p - 3}{p^2} - \frac{1}{q^2} \right) \leq 28.7 \frac{\log q}{p^2} + 214.4. \]

If $n \neq 13 \cdot 17, 13 \cdot 19, 13 \cdot 23, 17 \cdot 19, 17 \cdot 23, 17 \cdot 29, 17 \cdot 31, 29 \cdot 31, 29 \cdot 37, 37 \cdot 41, 37 \cdot 43, 41 \cdot 43$ or $41 \cdot 47$ then this inequality does not hold and so $B_n$ has always a primitive divisor.

Now, we show how to deal with the case $n = 13 \cdot 17$. Using the command "polroots" of PARI/GP, we can compute the constant $K_{13}$. We have $K_{13} \geq 0.04$. We substitute the value $n = 13 \cdot 17$ in (4.6) using this bound and we obtain

\[ \hat{h}(P) \leq \frac{497 + 277 \ln |a|}{12956}. \]

Using again the usual inequalities on $\hat{h}$, we have that this inequality does not hold and then $B_{13 \cdot 17}$ has always a primitive divisor. The other cases are analogous.
Remark 4.18. An analogue of Theorem 1.4 can also be obtained for any elliptic curve, $E$, with complex multiplication and for all the primes that split in $\text{End}(E)$. In order to find uniform bounds, it is necessary to have inequalities similar to the inequalities in Lemma 2.3 and 2.11.

Remark 4.19. We cannot apply the technique used in the proof of the last theorem in order to prove an analogue for the term in the form $B_p$ with $p \equiv 1 \mod 4$. Indeed, substituting $m = 1$ in the inequality (4.6) we would like to show that the inequality does not hold. But the LHS grows as $p^2$ and the RHS grows as $p$, so the inequality holds for $p$ large enough. Hence, for $p$ large this method does not work.

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