Research Article

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General \((p,q)\)-mixed projection bodies

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Abstract: In this article, the general \((p,q)\)-mixed projection bodies are introduced. Then, some basic properties of the general \((p,q)\)-mixed projection bodies are discussed, and the extreme values of volumes of the general \((p,q)\)-mixed projection bodies and their polar bodies are established.

Keywords: projection bodies, general \((p,q)\)-mixed projection bodies, extreme value

MSC 2020: 52A20, 52A40

1 Introduction

Projection bodies, introduced by Minkowski at the end of nineteenth century, are a central notion in the classical Brunn-Minkowski theory. In recent years, projection bodies and their polars have received considerable attention, see, e.g., [1–9].

The classical Brunn-Minkowski theory has a natural extension called the \(L_p\) Brunn-Minkowski theory, which was initiated in the early 1960s when Firey introduced his concept of \(L_p\) composition of convex bodies (see [10]) and was born in the works of Lutwak [11,12]. Since Lutwak’s two seminal works, this theory has attracted increasing interest in recent years, see, e.g., [13–20]. One of the central concepts in the \(L_p\) Brunn-Minkowski theory is the \(L_p\) projection bodies introduced by Lutwak et al. [17]. Afterward, Ludwig [21] (see also Haberl and Schuster [22]) extended Lutwak, Yang and Zhang’s \(L_p\) projection bodies to an entire class which may be called general \(L_p\) projection bodies. See, e.g., [23–30] for the \(L_p\) and general \(L_p\) projection bodies and their extensions.

The dual Brunn-Minkowski theory, developed by Lutwak [31] in 1975, is a dual concept of the classical Brunn-Minkowski theory. It is a theory of dual mixed volumes of star bodies and has already attracted considerable interest, see, e.g., [32–40]. In particular, a recent groundbreaking work of Huang et al. [41] showed that there exists a new family of geometry measures, in the dual Brunn-Minkowski theory, called dual curvature measures which are the long-sought duals of Federer’s curvature measures. Now, the dual curvature measures become the core of the dual Brunn-Minkowski theory.

Very recently, a unification of the \(L_p\) Brunn-Minkowski theory and the dual Brunn-Minkowski theory was discovered by Lutwak et al. [42], where they introduced the \(L_p\) dual curvature measures which include \(L_p\) surface area measures and \(L_p\) integral curvatures in the \(L_p\) Brunn-Minkowski theory as well as the dual curvature measures in the dual Brunn-Minkowski theory. In the same way that the \(L_p\) surface area and dual curvature measures, respectively, play a critical role in the \(L_p\) and dual Brunn-Minkowski theories, the \(L_p\) dual curvature measures can be seen to be a central concept within the unifying theory. See, e.g., [43–46] for some recent developments related to the \(L_p\) dual curvature measures.

In this article, motivated by the works of Lutwak et al. [42], Ludwig [21] and Haberl and Schuster [22], we introduce a more general definition of projection bodies according to the \(L_p\) dual curvature measures,
which is called the general \((p,q)\)-mixed projection bodies. Special cases include the classical projection bodies, \(L_p\) projection bodies and general \(L_p\) projection bodies. Then, we establish the extreme values of volumes for the general \((p,q)\)-mixed projection bodies and their polar bodies. The detailed descriptions for the definition and main results are provided below.

Throughout \(\mathbb{R}^n\) denotes the \(n\)-dimensional Euclidean space. A convex body is a compact convex subset of \(\mathbb{R}^n\) with nonempty interior. We denote by \(\mathcal{K}^n\) the set of convex bodies and by \(\mathcal{K}_o^n\) the set of convex bodies containing the origin in their interiors. For a convex body \(K \in \mathcal{K}^n\), let \(\partial K\) and \(V(K)\) be its boundary and \(n\)-dimensional volume, respectively. The unit sphere in \(\mathbb{R}^n\) will be denoted by \(S^{n-1}\). For \(x \in \mathbb{R}^n\), \(|x| = \sqrt{x^T x}\) denotes the Euclidean norm of \(x\), and for \(x \in \mathbb{R}^n \setminus \{0\}\), the unit vector \(x/|x| \in S^{n-1}\) will be abbreviated by \(\langle x \rangle\).

For all \(x \in \mathbb{R}^n\), the support function of \(K \in \mathcal{K}^n\) is defined by
\[
h(K, x) = h_K(x) = \max\{|x \cdot y : y \in K\},
\]
where \(x \cdot y\) denotes the standard inner product of \(x\) and \(y\).

The radial function, \(\rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}\), of a compact and star-shaped set, with respect to the origin, \(K \subset \mathbb{R}^n\), is defined by
\[
\rho(K, x) = \max\{\lambda : \lambda x \in K\}.
\]
If \(\rho_K\) is positive and continuous, then \(K\) is called a star body with respect to the origin. The set of all star bodies about the origin in \(\mathbb{R}^n\) is denoted by \(\mathcal{K}_o^n\). Two star bodies \(K\) and \(L\) are dilates (of one another) if \(\rho_K(u)/\rho_L(u)\) is independent of \(u \in S^{n-1}\). For a star body \(Q \in \mathcal{K}_o^n\), \(\|\cdot\|_Q : \mathbb{R}^n \to [0, \infty)\) is a continuous and positively homogeneous function of degree 1, which was defined, in [42], by
\[
\|x\|_Q = \begin{cases} 1/\rho_Q(x) & x \neq 0; \\ 0 & x = 0. \end{cases}
\]

For a convex body \(K \in \mathcal{K}_o^n\), its polar body \(K^\circ\) is defined by
\[
K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.
\]

Suppose \(p, q \in \mathbb{R}\). If \(K \in \mathcal{K}_o^n\) while \(Q \in \mathcal{S}^n_o\), then the \(L_p\) dual curvature measure, \(\check{c}_{p,q}(K, Q, \cdot)\), on \(S^{n-1}\) was defined, in [42], by
\[
\int_{S^{n-1}} g(v) \, dq_{p,q}(K, Q, v) = \frac{1}{n} \int_{S^{n-1}} g(\alpha_K(u)) h_K(\alpha_K(u))^{-p} \rho_K(u)^{-p} \rho_Q(u)^{-q} \, du
\]
for each continuous \(g : S^{n-1} \to \mathbb{R}\), where \(\alpha_K\) is the radial Gauss map that associates with almost each \(u \in S^{n-1}\) the unique outer unit normal at the point \(\rho_K(u) u \in \partial K\).

In [42], the \(L_p\) surface area measures were shown to be special cases of the \(L_p\) dual curvature measures:
\[
\check{c}_{p,q}(K, K, \cdot) = \frac{1}{n} S_p(K, \cdot); \quad \check{c}_{p,q}(K, Q, \cdot) = \frac{1}{n} S_p(K, \cdot).
\]

For \(\tau \in [-1, 1]\), the function \(\varphi_\tau : \mathbb{R} \to [0, \infty)\) was defined, in [22], by
\[
\varphi_\tau(t) = |t| + \tau t. \quad (1.4)
\]

Now, we begin to define the general \((p,q)\)-mixed projection bodies.

**Definition 1.1.** Suppose \(p \geq 1\), \(q \in \mathbb{R}\) and \(\tau \in [-1, 1]\). If \(K \in \mathcal{K}_o^n\) while \(Q \in \mathcal{S}^n_o\), then for \(u \in S^{n-1}\), define the general \((p,q)\)-mixed projection body, \(\Pi_{p,q}^\tau(K, Q)\), to be the convex body whose support function is given by
\[
\left[\left\{h_{\Pi_{p,q}^\tau(K, Q)}(u)\right\}^p\right] = n c_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \, dq_{p,q}(K, Q, v).
\]
Here,
\[ c_{n,p}(\tau) = \frac{c_{n,p}}{(1 + \tau)^p + (1 - \tau)^p}, \]
where \( c_{n,p} = \frac{\Gamma\left(\frac{n+p}{2}\right)}{\pi^{(n-1)/2}\Gamma\left(\frac{1+p}{2}\right)}. \)

If we take \( Q = K \) or \( q = n \) in (1.5), then from (1.2) or (1.3) we have
\[
\Pi_{p,q}(K, Q) = \Pi_{p,q}(K, K),
\]
where \( \Pi_{p,q}(K, Q) \) is the general \( L_p \) projection body [22] of \( K \), whose support function is given by
\[
h^\tau(\Pi_{p,q}(K, K), u) = c_{n,p}(\tau) \int_{S^{n-1}} q_1(u \cdot v)^p dS_p(K, v).
\]

(1.6)

For \( \tau = 0 \) in (1.6), \( \Pi_{p,q}(K, Q) = \Pi_{p,q}(K, K) \) which is the \( L_p \) projection body [17]. For \( \tau = 0 \) and \( p = 1 \) in (1.6), the body \( \Pi_{p,q}(K, K) \) is the classical projection body, \( \Pi_K \), of \( K \).

Our main results are as follows for the general \((p,q)\)-mixed projection bodies. The first result is to establish the extreme values of \( V(\Pi_{p,q}^+, \Phi) \). Note that \( \Pi_{p,q}^+ \) is used to denote the polar body of \( \Pi_{p,q} \).

**Theorem 1.1.** Suppose \( p \geq 1 \) and \( q \in \mathbb{R} \). If \( K \in \mathcal{K}_n^0 \) while \( L \in S_n^0 \), then for every \( \tau \in [-1, 1] \),
\[
V\left(\Pi_{p,q}^+, \Phi(K, Q) \right) \leq V\left(\Pi_{p,q}^+(K, Q) \right) \leq V\left(\Pi_{p,q}^+(K, Q) \right).
\]

(1.7)

Suppose that both \( K \) and \( Q \) are not origin-symmetric. If \( \tau \neq 0 \), then equality holds in the left inequality if and only if \( \Pi_{p,q}(K, Q) \) is origin-symmetric; if \( \tau \neq \pm 1 \), then equality holds in the right inequality if and only if both \( \Pi_{p,q}(K, Q) \) are origin-symmetric.

Here, \( \Pi_{p,q}(K, Q) \) is the nonsymmetric \((p,q)\)-mixed projection body, which is the convex body defined by
\[
h^\tau(\Pi_{p,q}(K, Q), u) = c_{n,p}(\tau) \int_{S^{n-1}} (u \cdot v)^p d\tau_{p,q}(K, Q, v).
\]

(1.8)

for \( u \in S^{n-1} \), where \( (u \cdot v)_+ = \max\{u \cdot v, 0\} \). Moreover, \( \Pi_{p,q}(K, Q) \) and \( \Pi_{p,q}^+(K, Q) \) are, respectively, defined by
\[
\Pi_{p,q}(K, Q) = \Pi_{p,q}(K, Q)
\]

(1.9)

and
\[
\Pi_{p,q}^+(K, Q) = \Pi_{p,q}^+(K, -Q).
\]

(1.10)

The special case \( Q = K \) or \( q = n \) of Theorem 1.1 can be found in [22], which gives rise to the strongest \( L_p \) Petty projection inequality. \( L_p \) Petty projection inequality is one of the crucial tools used for establishing the sharp affine \( L_p \) Sobolev inequalities and the affine Pólya-Szegő principle which are stronger than the usual sharp Sobolev inequalities and the usual Pólya-Szegő principle in the Euclidean space, see, e.g., [24–27].

The following is to provide the extreme values of \( V(\Pi_{p,q}^-) \).

**Theorem 1.2.** Suppose \( p \geq 1 \) and \( q \in \mathbb{R} \). If \( K \in \mathcal{K}_n^0 \) while \( L \in S_n^0 \), then for every \( \tau \in [-1, 1] \),
\[
V\left(\Pi_{p,q}^-\right) \geq V\left(\Pi_{p,q}^-(K, Q) \right) \leq V\left(\Pi_{p,q}^-(K, Q) \right).
\]

(1.11)
Suppose that both $K$ and $Q$ are not origin-symmetric. If $\tau \neq 0$, then equality holds in the left inequality if and only if $\Pi^\tau_{p,q}(K, Q)$ is origin-symmetric; if $\tau \neq \pm 1$, then equality holds in the right inequality if and only if both $\Pi^\tau_{p,q}(K, Q)$ are origin-symmetric.

When $Q = K$ or $q = n$, Theorem 1.2 was given by Haberl and Schuster [22].

For quick later reference, we list in Section 2 some basic and well-known facts of convex and star bodies, radial and reverse radial Gauss images and $L_p$ dual curvature measures. The basic properties of the general and nonsymmetric $(p,q)$-mixed projection bodies are developed in Section 3. Section 4 is devoted to prove Theorems 1.1 and 1.2.

## 2 Preliminaries

### 2.1 Basics regarding convex and star bodies

Good general references about the theory of convex bodies are the books of Gardner [47] and Schneider [10].

Let $SL(n)$ denote the group of special linear transformation. If $\phi \in SL(n)$, then we write $\phi^t$ for the transpose of $\phi$, $\phi^{-1}$ for the inverse of $\phi$ and $\phi^{-t}$ for the inverse of the transpose of $\phi$. From the definitions of the support function, radial function and polar body, it is easy to see that for $\phi \in SL(n)$, $K \in \mathcal{K}_0^n$ and $L \in S^n_0$,

\[ h(\phi K, x) = h(K, \phi^t x), \quad x \in \mathbb{R}^n; \]

\[ \rho(\phi K, x) = \rho(K, \phi^{-1} x), \quad x \in \mathbb{R}^n \setminus \{0\}; \]

\[ (\phi K)^* = \phi^{-t} K^*. \]  

Moreover, it is easy to verify that for $K \in \mathcal{K}_0^n$ and $c > 0$,

\[ (cK)^* = \frac{1}{c} K^*. \]

Suppose $p \in \mathbb{R}$. If $\mu$ is a Borel measure on $S^{n-1}$ and $\phi \in SL(n)$, then the $L_p$ image of $\mu$ under $\phi$, $\phi_p \mu$, is a Borel measure defined, in [42], by

\[ \int_{S^{n-1}} f(u) \, d\phi_p \mu(u) = \int_{S^{n-1}} |\phi^{-1}u|^p f(\langle \phi^{-1}u \rangle) \, d\mu(u), \]

for each Borel $f : S^{n-1} \to \mathbb{R}$.

The support and radial functions of a convex body $K \in \mathcal{K}_0^n$ and its polar body are related by

\[ \rho_K = 1/h_K^* \quad \text{and} \quad h_K = 1/\rho_K^*. \]

If $K_1 \in \mathcal{K}^n$, we say that $K_1 \to K_0 \in \mathcal{K}^n$ provided

\[ h_{K_1} - h_{K_0} \xrightarrow{\text{loc}} \max_{u \in S^{n-1}} |h_{K_1}(u) - h_{K_0}(u)| \to 0. \]

For $K, L \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$ (both not zero), the Minkowski combination $aK + \beta L$ is defined by

\[ aK + \beta L = \{ax + \beta y : x \in K, y \in L\} \]

whose support function is

\[ h_{aK + \beta L} = ah_K + \beta h_L. \]
In the early 1960s, Firey [48] introduced the $L_p$ Minkowski combination, which is also known as the Minkowski-Firey combination. For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\alpha, \beta \geq 0$ (both not zero), the $L_p$ Minkowski combination, $\alpha \cdot K +_p \beta \cdot L$, was defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = ah(K, \cdot)^p + \beta h(L, \cdot)^p,$$

(2.8)

where $\alpha \cdot K = \alpha \tilde{K}$.

In [48], Firey also established the $L_p$ Brunn-Minkowski inequality: if $p \geq 1$ and $K, L \in \mathcal{K}_o^n$, then

$$V(K +_p L)^\frac{p}{n} \geq V(K)^\frac{p}{n} + V(L)^\frac{p}{n},$$

(2.9)

with equality if and only if $K$ and $L$ are dilates.

The $L_p$ harmonic radial combination of two star bodies was introduced by Lutwak [12]. For $p \geq 1$, $K, L \in S^n_o$ and $\alpha, \beta \geq 0$ (both not zero), the $L_p$ harmonic radial combination, $\alpha \ast K +_p \beta \ast L$, is a star body whose radial function is given by

$$p(\alpha \ast K +_p \beta \ast L, \cdot)^p = ap(K, \cdot)^p + \beta p(L, \cdot)^p.$$

(2.10)

Note that $\alpha \ast K = \alpha \tilde{K}$.

Lutwak’s $L_p$ dual Brunn-Minkowski inequality, see [12], is as follows: if $p \geq 1$ and $K, L \in S^n_o$, then

$$V(K +_p L)^\frac{p}{n} \geq V(K)^\frac{p}{n} + V(L)^\frac{p}{n},$$

(2.11)

with equality if and only if $K$ and $L$ are dilates.

### 2.2 Radial and reverse radial Gauss images

In the following, we state some necessary facts with regard to the radial and reverse radial Gauss images, cf. [41,42].

For $K \in \mathcal{K}_o^n$ and $\nu \in S^{n-1}$, the set

$$H_\nu(v) = \{x \in \mathbb{R}^n : x \cdot \nu = h_\nu(v)\}$$

(2.12)

is called the supporting hyperplane of $K$ with the outer unit normal $\nu$.

The spherical image of $\sigma \in \partial K$, $K \in \mathcal{K}_o^n$, is defined by

$$\nu_\sigma(\sigma) = \{v \in S^{n-1} : x \in H_\nu(v) \text{ for some } x \in \sigma \} \subset S^{n-1}.$$

Let $\sigma_K \subset \partial K$ be the set consisting of all $x \in \partial K$ for which the set $\nu_\sigma(\sigma)$, often abbreviated as $\nu_\sigma(x)$, contains more than a single element. Then, $\mathcal{H}^{n-1}(\sigma_K) = 0$ based on Schneider [10, p. 84], where $\mathcal{H}^{n-1}$ is the $(n - 1)$-dimensional Hausdorff measure on $\partial K$. For $x \in \partial K \setminus \sigma_K$, $\nu_\sigma(x)$ has the unique element denoted by $\nu_\sigma(x)$ called the spherical image map of $K$. For convenience, we abbreviate $\partial K \setminus \sigma_K$ as $\partial' K$. If the integration is with respect to $\mathcal{H}^{n-1}$, then it follows from $\mathcal{H}^{n-1}(\sigma_K) = 0$ that it will be immaterial over subsets of $\partial' K$ or $\partial K$.

For $K \in \mathcal{K}_o^n$ and $\omega \subset S^{n-1}$, the radial Gauss image of $\omega$, $\alpha_\omega(\omega)$, is defined by

$$\alpha_K(\omega) = \{v \in S^{n-1} : \rho_K(u) u \in H_\nu(v) \text{ for some } u \in \omega\}.$$

(2.13)

Thus for $u \in S^{n-1}$,

$$\alpha_K(u) = \{v \in S^{n-1} : \rho_K(u) u \in H_\nu(v)\}.$$

(2.13)

Let $\omega_K = \{u \in S^{n-1} : \rho_K(u) u \in \omega_K \} \subset S^{n-1}$. Clearly, if $u \in S^{n-1} \setminus \omega_K$, then $\alpha_K(u)$ contains only a single element denoted by $\alpha_K(u)$, i.e.,

$$\alpha_K : S^{n-1} \setminus \omega_K \rightarrow S^{n-1},$$

which is named as the radial Gauss map of $K$. Combining (2.12) and (2.13), we obtain two essential facts needed: for $\lambda > 0$,

$$\alpha_{\lambda K} = \alpha_K.$$

(2.14)
\[ a_K(u) = -a_K(-u), \quad \text{for } u \in S^{n-1}. \] (2.15)

Let \( K \in \mathcal{K}_o^n \) and \( \eta \subset S^{n-1} \). The reverse radial Gauss image of \( \eta \), \( a_K^*\eta \), is defined by
\[ a_K^*\eta = \{u \in S^{n-1} : \rho_K(u)u \in H_\nu(v) \quad \text{for some } v \in \eta\}. \] (2.16)
Together (2.12) with (2.16), it is easy to see that for \( \lambda > 0 \),
\[ a_K^* = a_K^\lambda. \] (2.17)

### 2.3 \( L_p \) dual curvature measures

The following are some key properties, shown in [42] and needed in next section, of the \( L_p \) dual curvature measures.

In [42], Lutwak et al. showed that for \( p, q \in \mathbb{R} \), \( K \in \mathcal{K}_o^n \) and \( Q \in S_o^n \), definition (1.1) can also be written as:
\[
\int_{S^{n-1}} g(v) \, d\tilde{C}_{p,q}(K, Q, v) = \frac{1}{n} \int_{\partial K} h_K(a_K(u))^{-p} \rho_K^p(u) \rho_Q^{n-q}(u) \, du
\] (2.18)

The \( L_p \) dual curvature measures have the following integral representation (see [42]): if \( p, q \in \mathbb{R} \), \( K \in \mathcal{K}_o^n \) and \( Q \in S_o^n \), then
\[
\tilde{C}_{p,q}(K, Q, \eta) = \frac{1}{n} \int_{a_K(\eta)} h_K(a_K(u))^{-p} \rho_K^p(u) \rho_Q^{n-q}(u) \, du
\] (2.19)
for each Borel set \( \eta \subset S^{n-1} \). Glancing (2.14) and (2.17), it immediately follows from (2.19) that for \( \lambda > 0 \),
\[ \tilde{C}_{p,q}(\lambda K, Q, \eta) = \lambda^{n-p} \tilde{C}_{p,q}(K, Q, \eta). \] (2.20)

It was shown, in [42], that if \( p \neq 0 \) and \( q \neq 0 \), then for \( K \in \mathcal{K}_o^n \), \( Q \in S_o^n \) and all \( \phi \in SL(n) \),
\[ \tilde{C}_{p,q}(\phi K, \phi Q, \cdot) = \phi^\top \tilde{C}_{p,q}(K, Q, \cdot). \] (2.21)

The \( L_p \) dual curvature measures are weakly convergent (see [42]): if \( p, q \in \mathbb{R} \), \( Q \in S_o^n \) and \( K_i \in \mathcal{K}_o^n \) with \( K_i \to K_o \in \mathcal{K}_o^n \), then for each continuous function \( g : S^{n-1} \to \mathbb{R} \),
\[
\lim_{i \to \infty} \int_{S^{n-1}} g(v) \, d\tilde{C}_{p,q}(K_i, Q, v) = \int_{S^{n-1}} g(v) \, d\tilde{C}_{p,q}(K_o, Q, v).
\] (2.22)
Moreover, they are a valuation (see [42]), i.e., if \( K, L \in \mathcal{K}_o^n \) are such that \( K \cup L \in \mathcal{K}_o^n \), then
\[
\tilde{C}_{p,q}(K, Q, \cdot) + \tilde{C}_{p,q}(L, Q, \cdot) = \tilde{C}_{p,q}(K \cap L, Q, \cdot) + \tilde{C}_{p,q}(K \cup L, Q, \cdot).
\] (2.23)

### 3 General and nonsymmetric \((p,q)\)-mixed projection bodies

In this section, we first show the relation between general and nonsymmetric \((p,q)\)-mixed projection bodies. Then, some of their basic properties are established.

Let \( g(v) = \varphi_x(u \cdot v)^p \), with \( p \geq 1 \), in (2.18). Then, it will be easy to see that definition (1.5) can be rewritten as:
\[
h\left[ \tilde{\Pi}_{p,q}(K, Q, u) \right] = c_{n,p}(r) \int_{\partial K} \varphi_x(u \cdot v_K(x))^p(x \cdot v_K(x))^{1-p}\|x\|^n d\mathcal{H}^{n-1}(x).
\] (3.1)
From the definition of $\Pi_{p,q}(K, Q)$, (1.8), (1.1) and (2.15), it follows that for $p \geq 1$, $K \in \mathcal{K}^n_0$ and $Q \in S^n_0$,
\[
h\left(\tilde{\Pi}_{p,q}(K, Q), u\right)^p = n c_{n,p} \int_{S^{n-1}} (-u \cdot v)^p d\mathcal{C}_{p,q}(K, Q, v), \quad u \in S^{n-1}.
\] (3.2)

Since for any $u, v \in S^{n-1}$,
\[q(u \cdot v)^p = (1 + \tau)^p(u \cdot v)^p + (1 - \tau)^p(-u \cdot v)^p,
\] by (1.5), (1.8), (3.2) and (2.8) we have
\[
\tilde{\Pi}_{p,q}(K, Q) = f_1(\tau) \cdot \Pi_{p,q}(K, Q) + f_2(\tau) \cdot \tilde{\Pi}_{p,q}(K, Q),
\] where
\[
f_1(\tau) = \frac{(1 + \tau)^p}{(1 + \tau)^p + (1 - \tau)^p} \quad \text{and} \quad f_2(\tau) = \frac{(1 - \tau)^p}{(1 + \tau)^p + (1 - \tau)^p}.
\]

Note that
\[f_1(\tau) + f_2(\tau) = 1, \quad f_1(-\tau) = f_2(\tau) \quad \text{and} \quad f_2(-\tau) = f_1(\tau).
\]

From (3.3), we see that if $\tau = \pm 1$, then
\[
\tilde{\Pi}_{p,q}(K, Q) = \tilde{\Pi}_{p,q}(K, Q) \quad \text{and} \quad \tilde{\Pi}_{p,q}(K, Q) = \tilde{\Pi}_{p,q}(K, Q);
\] (3.4)

if $\tau = 0$, then
\[
\Pi_{p,q}(K, Q) = \frac{1}{2} \tilde{\Pi}_{p,q}(K, Q) + p \frac{1}{2} \tilde{\Pi}_{p,q}(K, Q).
\] (3.5)

We first show the property of $\Pi_{p,q}^+$. 

**Proposition 3.1.** Suppose $p \geq 1$, $q \in \mathbb{R}$. If $K \in \mathcal{K}^n_0$ while $Q \in S^n_0$, then
\[
-\tilde{\Pi}_{p,q}(K, Q) = \tilde{\Pi}_{p,q}(K, Q) = \tilde{\Pi}_{p,q}(-K, -Q).
\] (3.6)

**Proof.** We just need to prove the left equality. From (3.2) and (1.8), it follows that for any $u \in S^{n-1}$,
\[
h\left(\tilde{\Pi}_{p,q}(K, Q), u\right)^p = n c_{n,p} \int_{S^{n-1}} (-u \cdot v)^p d\mathcal{C}_{p,q}(K, Q, v) = h\left(\tilde{\Pi}_{p,q}(K, Q), -u\right)^p = h\left(-\tilde{\Pi}_{p,q}(K, Q), u\right)^p,
\]
i.e.,
\[
-\tilde{\Pi}_{p,q}(K, Q) = -\tilde{\Pi}_{p,q}(K, Q).
\]

This finishes the proof. \qed

The following is to show the properties of $\Pi_{p,q}^-$. 

**Proposition 3.2.** Suppose $p \geq 1$, $q \in \mathbb{R}$, $K \in \mathcal{K}^n_0$ while $Q \in S^n_0$.
\[\begin{enumerate}
\item[(i)] If $\tau \in [-1, 1]$, then
\[
\tilde{\Pi}_{p,q}(-K, -Q) = \tilde{\Pi}_{p,q}(K, Q) = \tilde{\Pi}_{p,q}(K, Q);
\] (3.7)
\end{enumerate}
(ii) If \( \tau \neq 0 \), then
\[
\Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^\tau(K, Q) \iff \Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^\tau(K, Q).
\] (3.8)

**Proof.**

(i) By (3.3) and (3.6), we have
\[
\Pi_{p,q}^\tau(K, Q) = f_2(\tau) \cdot \Pi_{p,q}^\tau(K, Q) + f_1(\tau) \cdot \Pi_{p,q}^\tau(K, Q)
\]
\[
= f_2(\tau) \cdot \Pi_{p,q}^\tau(-K, -Q) + f_1(\tau) \cdot \Pi_{p,q}^\tau(-K, -Q)
\] (3.9)

Moreover, from (3.3), (2.8), (3.6) and (2.8) again, we get that for any \( u \in S^{n-1} \),
\[
h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p = f_2(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p + f_1(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p
\]
\[
= f_2(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), -u \right]^p + f_1(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), -u \right]^p
\]
\[
= h\left[ f_1(\tau) \cdot \Pi_{p,q}^\tau(K, Q) + f_2(\tau) \cdot \Pi_{p,q}^\tau(K, Q), -u \right]^p
\]
\[
= h\left[ -\Pi_{p,q}^\tau(K, Q), u \right]^p,
\]
i.e.,
\[
\Pi_{p,q}^\tau(K, Q) = - \Pi_{p,q}^\tau(K, Q).
\] (3.10)

Together (3.9) with (3.10) gives
\[
\Pi_{p,q}^\tau(-K, -Q) = - \Pi_{p,q}^\tau(K, Q).
\]

(ii) Let \( \tau \neq 0 \) and \( \Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^\tau(K, Q) \). We know that for any \( u \in S^{n-1} \),
\[
h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p = f_2(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p + f_1(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p
\] (3.11)

and
\[
h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p = f_2(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p + f_1(\tau) h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p.
\] (3.12)

Together (3.11) with (3.12), it follows that
\[
\Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^\tau(K, Q).
\]

Moreover, let \( \Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^\tau(K, Q) \). Combining (3.11) with (3.12), we have
\[
[f_1(\tau) - f_2(\tau)] h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p = [f_1(\tau) - f_2(\tau)] h\left[ \Pi_{p,q}^\tau(K, Q), u \right]^p.
\] (3.13)

Since \( f_1(\tau) - f_2(\tau) \neq 0 \) when \( \tau \neq 0 \), it follows from (3.13) that
\[
\Pi_{p,q}^\tau(K, Q) = \Pi_{p,q}^\tau(K, Q).
\]
Next, we will show that the operator $\Pi_{p,q}(\cdot, \cdot)$ is $SL(n)$ contravariant and homogeneous of $q/p - 1$ with respect to the first part.

**Proposition 3.3.** Suppose $p \geq 1$, $q \in \mathbb{R}$, $\tau \in [-1, 1]$, $K \in \mathcal{K}_n$ while $Q \in S^n$.

(i) For all $\phi \in SL(n)$,

$$\Pi_{p,q}(\phi K, \phi Q) = \phi^{-t} \Pi_{p,q}(K, Q);$$  \hspace{1cm} (3.14)

(ii) For $\lambda > 0$,

$$\Pi_{p,q}(\lambda K, Q) = \lambda^{q/p-1} \Pi_{p,q}(K, Q).$$  \hspace{1cm} (3.15)

**Proof.**

(i) From (1.5), (2.21), (2.5), (1.5) again and (2.1), we have that for any $u \in S^{n-1}$ and all $\phi \in SL(n)$,

$$h \left( \Pi_{p,q}(\phi K, \phi Q), u \right)^p = nc_{n,p}(r) \int_{S^{n-1}} \varphi_s(u \cdot v)^p d\tilde{C}_{p,q}(\phi K, \phi Q, v)$$

$$= nc_{n,p}(r) \int_{S^{n-1}} \varphi_s(u \cdot v)^p d\varphi_s^p \tilde{C}_{p,q}(K, Q, v)$$

$$= nc_{n,p}(r) \int_{S^{n-1}} |\varphi^{-4v}|^p \varphi_s(u \cdot (\varphi^{-4v}))^p d\tilde{C}_{p,q}(K, Q, v)$$

$$= nc_{n,p}(r) \int_{S^{n-1}} \varphi_s(u \cdot \varphi^{-4v})^p d\tilde{C}_{p,q}(K, Q, v)$$

$$= nc_{n,p}(r) \int_{S^{n-1}} \varphi_s(\varphi^{-4v} u \cdot v)^p d\tilde{C}_{p,q}(K, Q, v)$$

$$= h \left( \Pi_{p,q}(K, Q), \varphi^{-4v} u \right)^p$$

$$= h \left( \Pi_{p,q}(K, Q), u \right)^p,$$

i.e.,

$$\Pi_{p,q}(\phi K, \phi Q) = \phi^{-t} \Pi_{p,q}(K, Q).$$

(ii) By (1.5) and (2.20), it follows that for any $u \in S^{n-1}$ and $\lambda > 0$,

$$h \left( \Pi_{p,q}(\lambda K, Q), u \right)^p = nc_{n,p}(r) \int_{S^{n-1}} \varphi_s(u \cdot v)^p d\tilde{C}_{p,q}(\lambda K, Q, v) = \lambda^{q/p} h \left( \Pi_{p,q}(K, Q), u \right)^p = h \left( \lambda^{q/p-1} \Pi_{p,q}(K, Q), u \right)^p.$$

That is,

$$\Pi_{p,q}(\lambda K, Q) = \lambda^{q/p-1} \Pi_{p,q}(K, Q).$$  \hspace{1cm} (3.16)

For a fixed $Q \in S^n$, we will prove that $\Pi_{p,q}(\cdot, \cdot) : \mathcal{K}_n \to \mathcal{K}_n$ is continuous.

**Proposition 3.4.** Suppose $p \geq 1$, $q \in \mathbb{R}$ and $Q \in S^n$. If $K_i \in \mathcal{K}_n$ with $K_i \to K_0 \in \mathcal{K}_n$, then for every $\tau \in [-1, 1]$,

$$\Pi_{p,q}(K_0, Q) \to \Pi_{p,q}(K_0, Q).$$  \hspace{1cm} (3.16)
Proof. Suppose \( u_0 \in S_{n-1} \). From (1.5) and (2.22), we get for a fixed \( Q \in S^n \),
\[
h \left( \tau \frac{r}{|p_q|} (K_0, Q), u_0 \right) \right) \to h \left( \tau \frac{r}{|p_q|} (K_0, Q), u_0 \right), \quad \text{as } i \to \infty.
\]
The support functions \( h_{\Pi_{r,q}(K_0, Q)} \to h_{\Pi_{r,q}(K_0, Q)} \) pointwise on \( S^{n-1} \) imply that they converge uniformly (see [10, Theorem 1.8.15]). This finishes the proof. \( \Box \)

An operator \( Z : \mathcal{K}_0^\pi \to \mathcal{K}_0^\pi \) is called an \( L_p \) Minkowski valuation if
\[
Z K_1 +_p Z K_2 = Z(K_1 \cup K_2) +_p Z(K_1 \cap K_2),
\]
whenever \( K_1, K_2, K_1 \cup K_2 \in \mathcal{K}_0^\pi \).

The theory of real valued valuations lies at the very core of geometry, see, e.g., [49,50]. In the 1970s, Schneider [51] first obtained the results on a special class of Minkowski valuations. Recently, since the seminal work of Ludwig [3,21], the investigations of these Minkowski valuations have become the focus of increased attention, see, e.g., [52–59].

An immediate result of (2.23) is that for a fixed \( Q \in S^n_0, \Pi_{r,q}(\cdot, Q) : \mathcal{K}_0^\pi \to \mathcal{K}_0^\pi \) is an \( L_p \) Minkowski valuation.

**Proposition 3.5.** Suppose, \( q \in \mathbb{R} \) and \( Q \in S^n_0 \). Then, for every \( r \in [-1, 1],
\[
\frac{r}{|p_q|} (\cdot, Q) : \mathcal{K}_0^\pi \to \mathcal{K}_0^\pi
\]
is an \( L_p \) Minkowski valuation, i.e., if \( K, L \in \mathcal{K}_0^\pi \) are such that \( K \cup L \in \mathcal{K}_0^\pi \), then
\[
\frac{r}{|p_q|} (K, Q) +_p \frac{r}{|p_q|} (L, Q) = \frac{r}{|p_q|} (K \cup L, Q) +_p \frac{r}{|p_q|} (K \cap L, Q).
\]

### 4 Proofs of Theorems 1.1–1.2

This section is dedicated to give the proofs of Theorems 1.1–1.2.

**Proof of Theorem 1.1.** Suppose that both \( K \) and \( Q \) are not origin-symmetric (otherwise the result is trivial). We first show the right inequality. Let \( -1 < \tau < 1 \). From (3.3), (2.8), (2.6) and (2.10), it follows that
\[
\tau \frac{r}{|p_q|} (K, Q) = f_1(\tau) \circ \frac{r}{|p_q|} (K, Q) +_p f_2(\tau) \circ \frac{r}{|p_q|} (K, Q).
\]
Using the \( L_p \) dual Brunn-Minkowski inequality (2.11), (3.6) and (2.4), and note that \( \alpha \circ K = \alpha \circ \hat{K} \), we have
\[
V \left( \frac{r}{|p_q|} (K, Q) \right) \leq V \left( \frac{r}{|p_q|} (K, Q) \right),
\]
with equality if and only if \( \Pi_{r,q}(K, Q) = \Pi_{r,q}(K, Q) \) and \( \Pi_{r,q}(K, Q) = \Pi_{r,q}(K, Q) \) are dilates, which is equivalent to \( \Pi_{r,q}(K, Q) = \Pi_{r,q}(K, Q) \). Since \( \Pi_{r,q}(K, Q) = -\Pi_{r,q}(K, Q) \) and \( \Pi_{r,q}(K, Q) = -\Pi_{r,q}(K, Q) \), \( \Pi_{r,q}(K, Q) = \Pi_{r,q}(K, Q) \) implies that both \( \Pi_{r,q}(K, Q) \) and \( \Pi_{r,q}(K, Q) \) are origin-symmetric and vice versa.

Next, we prove the left inequality. Let \( \tau > 0 \). From (4.1) and (2.10), it follows that for any \( u \in S^n_0, \)
\[
\rho \left( \frac{r}{|p_q|} (K, Q), u \right)^p = f_1(\tau) \rho \left( \frac{r}{|p_q|} (K, Q), u \right)^p +_p f_2(\tau) \rho \left( \frac{r}{|p_q|} (K, Q), u \right)^p
\]
\[ \rho \left( \frac{\tau \ast}{pq}(K, Q), u \right)^p = f_2(\rho) \frac{\tau \ast}{pq}(K, Q), u \right)^p + f_1(\tau) \rho \left( \frac{\tau \ast}{pq}(K, Q), u \right)^p. \] (4.3)

Combining (4.2) and (4.3), we get
\[ \frac{1}{2} \rho \left( \frac{\tau \ast}{pq}(K, Q), u \right)^p + \frac{1}{2} \rho \left( \frac{\tau \ast}{pq}(K, Q), u \right)^p = \frac{1}{2} \rho \left( \frac{\tau \ast}{pq}(K, Q), u \right)^p + \frac{1}{2} \rho \left( \frac{\tau \ast}{pq}(K, Q), u \right)^p. \]

This together (2.10), (2.8) with (3.5) yields
\[ \ast \Pi (K, Q) = \frac{1}{2} \ast \Pi (K, Q) + \frac{1}{2} \ast \Pi (K, Q). \] (4.4)

By inequalities (2.11), (3.7) and (2.4), this has
\[ V \left( \ast \Pi (K, Q) \right) \leq V \left( \ast \Pi (K, Q) \right), \]
with equality if and only if \( \Pi_{\ast}^{\ast} (K, Q) \) and \( \Pi_{\ast}^{\ast} (K, Q) \) are dilates, which is equivalent to \( \Pi_{\ast}^{\ast} (K, Q) = \Pi_{\ast}^{\ast} (K, Q) \). From (3.7), \( \Pi_{\ast}^{\ast} (K, Q) = \Pi_{\ast}^{\ast} (K, Q) \) yields that \( \Pi_{\ast}^{\ast} (K, Q) \) is origin-symmetric and vice versa. \( \square \)

We will use the following consequence (see [22]).

**Lemma 5.1.** If \( K \in \mathcal{K}^n_o \) and \( p \) is not an odd integer, then
\[ \Pi_{\ast}^{\ast} (K, Q) = \Pi_{\ast}^{\ast} (K, Q) \triangleq K \text{ is origin-symmetric.} \]

Here, \( \Pi_{\ast}^{\ast} K \) denote the nonsymmetric \( L_p \) projection bodies (see [22]).

Let \( Q = K \) or \( q = n \) in Theorem 1.1. Then, we immediately have the following result given by Haberl and Schuster [22], where the equality conditions follow from Lemma 5.1 and the case \( Q = K \) or \( q = n \) of (3.8).

**Corollary 5.1.** [22] For \( p \geq 1, \tau \in [-1, 1] \) and \( K \in \mathcal{K}^n_o \),
\[ V \left( \Pi_{\ast}^{\ast} (K, Q) \right) \leq V \left( \Pi_{\ast}^{\ast} (K, Q) \right) \leq V \left( \Pi_{\ast}^{\ast} (K, Q) \right). \]

If \( K \) is not origin-symmetric and \( p \) is not an odd integer, equality holds in the left inequality if and only if \( \tau = 0 \) and equality holds in the right inequality if and only if \( \tau = \pm 1 \).

**Proof of Theorem 1.2.** Assume that both \( K \) and \( Q \) are not origin-symmetric. First, we deduce the right inequality. Let \( -1 < \tau < 1 \). Then it follows from (3.3), the \( L_p \) Brunn-Minkowski inequality (2.9) and the fact that \( a \cdot K = a^{\tau} K \) that
\[ \left( \Pi_{\ast}^{\ast} (K, Q) \right) \geq \left( \Pi_{\ast}^{\ast} (K, Q) \right), \]
with equality if and only if \( \Pi_{\ast}^{\ast} (K, Q) \) and \( \Pi_{\ast}^{\ast} (K, Q) \) are dilates, which is equivalent to \( \Pi_{\ast}^{\ast} (K, Q) = \Pi_{\ast}^{\ast} (K, Q) \), i.e., \( \Pi_{\ast}^{\ast} (K, Q) \) are origin-symmetric.

In order to see the left inequality, we suppose \( \tau \neq 0 \). From (4.4), (2.6) and (2.8), we have
\[ \Pi_{\ast}^{\ast} (K, Q) = \frac{1}{2} \Pi_{\ast}^{\ast} (K, Q) + \frac{1}{2} \Pi_{\ast}^{\ast} (K, Q). \]

Using inequality (2.9) and equality (3.7), this gets
\[ V \left( \Pi_{\ast}^{\ast} (K, Q) \right) \geq V \left( \Pi_{\ast}^{\ast} (K, Q) \right), \]
with equality if and only if \( \Pi_{p,q}(K, Q) \) and \( \Pi_{p,q}^{-}(K, Q) \) are dilates, which is equivalent to \( \Pi_{p,q}^{+}(K, Q) = \Pi_{p,q}^{-}(K, Q) \). That is, \( \Pi_{p,q}(K, Q) \) is origin-symmetric.

As a special case of Theorem 1.2, the following is a direct result. Note that the proof of the equality conditions is similar to that of Corollary 5.1.

**Corollary 5.2.** [22] For \( p \geq 1, \tau \in [-1, 1] \) and \( K \in \mathcal{K}_{n}^{+} \),

\[
V \left( \Pi_{p}^{\tau} K \right) \geq V \left( \Pi_{p}^{\tau} K \right) \geq V \left( \Pi_{p}^{\tau} K \right).
\]

If \( K \) is not origin-symmetric and \( p \) is not an odd integer, equality holds in the left inequality if and only if \( \tau = 0 \) and equality holds in the right inequality if and only if \( \tau = \pm 1 \).

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