Casimir-Polder energy for axially symmetric systems

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Abstract

We develop a formalism suitable for studying Maxwell’s equations in the presence of a medium that is axially symmetric, in particular with respect to Casimir-Polder interaction energies. As an application, we derive the Casimir-Polder interaction energy between an electric $\delta$-function plate and an anisotropically polarizable molecule for arbitrary orientations of the principal axes of polarizabilities of the molecule. We show that in the perfect conductor limit for the plate the interaction is insensitive to the orientation of the polarizabilities of the molecule. We obtain the Casimir-Polder energy between an electric $\delta$-function sphere and an anisotropically polarizable molecule, again for arbitrary orientations of the principal axes of polarizabilities of the molecule. We derive results when the polarizable molecule is either outside the sphere, or inside the sphere. We present the perfectly conducting limit for the $\delta$-function sphere, and also the interaction energy for the special case when the molecule is at the center of the sphere. Our general proposition is that the Casimir-Polder energy between a dielectric body with axial symmetry and an unidirectionally polarizable molecule placed on the axis with its polarizability parallel to the axis gets non-zero contribution only from the $m = 0$ azimuth mode. This feature in conjunction with the property that the $m = 0$ mode separates into transverse electric and transverse magnetic modes allows the evaluation of Casimir-Polder energies for axially symmetric systems in a relatively easy manner.

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1. Introduction

It is well known that in three dimensions the Helmholtz equation

$$\left[ \nabla^2 + \omega^2 \right] \phi(x) = 0 \quad (1.1)$$

is exactly solvable in only eleven coordinate systems [1]. Specifically, if the surface upon which the boundary conditions are to be satisfied is not one of these coordinate surfaces, or if the boundary conditions are not the simple Dirichlet or Neumann type, the method of separation of variables fails [1]. The static version of the Helmholtz equation is the Laplace equation, obtained by setting $\omega = 0$ in Eq. (1.1), and leads to the plethora of exactly solvable problems in electrostatics. The vector Helmholtz equation is more stringent and allows exact solutions only in five coordinate systems [1]. A direct consequence of this is that problems in electrodynamics are often harder to solve even for seemingly simple geometries.

The Maxwell equations that govern classical electrodynamics in the presence of medium with permittivity and permeability are tractable whenever the electric and magnetic fields separate into transverse electric (TE) and transverse magnetic (TM) modes. A class of problems in electrodynamics in the presence of a medium, for example, the Casimir effect, often requires the knowledge of the Green dyadic, which is a lot more unyielding when it comes to separation into TE and TM modes. A medium with planar symmetry and a medium with spherical symmetry are among the two classic examples in which the Green dyadic, in the presence of a medium, separate into TE and TM modes.

The electric and magnetic fields, governed by the Maxwell equations in the presence of a medium, are separable into TE and TM modes for axially symmetric systems. This statement appears as an exercise in Chapter VII of Stratton’s textbook [2] in which it is credited to Max Abraham. However, this does not necessarily imply that the corresponding Green’s dyadic also separates into TE and TM modes in axially symmetric systems. We develop a formalism, closely based on the methods developed in Chapter 46 of Ref. [3], for studying Maxwell’s equations in the presence of a medium, which has the potential of being extended to other coordinate systems. We study a medium with planar symmetry using rectangular coordinates, and then again using cylindrical coordinates. We construct the corresponding rectangular and cylindrical vector eigenfunctions conducive for investigations of separation of TE and TM modes. Later, we study a medium with spherical symmetry using spherical coordinates and construct the spherical vector eigenfunctions. Our study of these relatively simpler geometries will serve as a precursor to a future study of non-trivial geometries with axial symmetry.

Our premise is that the Casimir-Polder interaction energy for axially symmetric systems gets non-zero contributions only from the $m = 0$ azimuth mode. Further, the Green dyadic for the $m = 0$ azimuth mode separates into TE and TM modes even though $m \neq 0$ modes do not necessarily separate into TE and TM modes. These observations suggest that the Casimir-Polder energies for axially symmetric systems can be calculated fairly easily.
As an application of the formalism presented here, we calculate the Casimir-Polder interaction energy between a polarizable molecule, for arbitrary orientations of the principal axes of polarizabilities of the molecule, and an electric $\delta$-function plate. We show that in the perfect conductor limit this interaction is insensitive to the orientation of the principal axes of polarizabilities of the molecule. We repeat the calculation for a polarizable molecule, again for arbitrary orientations of the principal axes of polarizabilities of the molecule, and an electric $\delta$-function sphere, for the case when the molecule is either inside or outside the sphere.

2. Maxwell’s equations

In Heaviside-Lorentz units the monochromatic components of Maxwell’s equations, proportional to $\exp(-i\omega t)$, in the absence of net charges and currents, and in the presence of electric and magnetic materials with boundaries, are

$$\nabla \times E = i\omega B, \quad (2.1a)$$

$$-\nabla \times H = i\omega (D + P). \quad (2.1b)$$

These equations imply $\nabla \cdot B = 0$ and $\nabla \cdot (D + P) = 0$. Here $P$ is an external source of polarization, in addition to the polarization of the material in response to the fields $E$ and $H$. The external source $P$ serves as a convenient mathematical tool, and is set to zero in the end. In the following we neglect non-linear responses and assume that the fields $D$ and $B$ respond linearly to the electric and magnetic fields $E$ and $H$:

$$D(r, \omega) = \varepsilon(r; \omega) \cdot E(r, \omega), \quad (2.2a)$$

$$B(r, \omega) = \mu(r; \omega) \cdot H(r, \omega). \quad (2.2b)$$

Using Eq. (2.1a) in Eq. (2.1b) we construct the following differential equation for the electric field,

$$\left[\frac{1}{\omega^2} \nabla \times \mu^{-1} \cdot \nabla \times +1 - \chi\right] \cdot E(r, \omega) = P(r, \omega), \quad (2.3)$$

where

$$\chi(r; \omega) = \varepsilon(r; \omega) - 1. \quad (2.4)$$

The differential equation for the Green’s dyadic $\Gamma(r, r')$, that will play a central role in the following discussions, is guided by Eq. (2.3),

$$\left[\frac{1}{\omega^2} \nabla \times \mu^{-1} \cdot \nabla \times +1 - \chi\right] \cdot \Gamma(r, r'; \omega) = \delta^{(3)}(r - r'), \quad (2.5)$$

and defines the relation between the electric field and the polarization source

$$E(r; \omega) = \int d^3r' \Gamma(r, r'; \omega) \cdot P(r'; \omega). \quad (2.6)$$
The corresponding dyadic for vacuum, obtained by setting $\chi = 0$, is called the free Green’s dyadic and satisfies the equation

$$\left[ \frac{1}{\omega^2} \nabla \times \mu^{-1} \cdot \nabla \times + 1 \right] \cdot \Gamma_0(r, r'; \omega) = 1 \delta^{(3)}(r - r'). \quad (2.7)$$

3. Casimir-Polder interaction energy: Hypothesis

A polarizable molecule, or an atom, or a nanoparticle, positioned at $r_0$, in our model, is suitably described by the dielectric function

$$\varepsilon_{\text{atom}}(r; \omega) - 1 = 4\pi \alpha(\omega) \delta^{(3)}(r - r_0). \quad (3.1)$$

In general, for an anisotropically polarizable molecule, we have the polarizability tensor

$$\alpha(\omega) = \alpha_1(\omega) \hat{e}_1 \cdot \hat{e}_1 + \alpha_2(\omega) \hat{e}_2 \cdot \hat{e}_2 + \alpha_3(\omega) \hat{e}_3 \cdot \hat{e}_3, \quad (3.2)$$

where $\alpha_i(\omega)$, $i = 1, 2, 3$, are the polarizabilities along the principal axes of polarizabilities $\hat{e}_i$, which are the eigenvalues and eigenvectors of the polarizability tensor. The dispersion in the polarizabilities $\alpha_i(\omega)$ can be modeled if necessary, though mostly we will not restrict our analysis to any specific dispersion model. The Casimir-Polder limit of the interaction energy gets contribution from the static limit of polarizabilities $\alpha_i(0)$, which is usually independent of any specific dispersion model. The Casimir-Polder interaction energy between a polarizable molecule and a second dielectric body described by the susceptibility tensor $\chi(r; \omega)$, (restricting our analysis to $\mu(r; \omega) = 1$), is given by the expression

$$E = - \int^{\infty}_{-\infty} d\zeta \int d^3r \text{tr} \Gamma(r_0, r; \omega) \cdot \chi(r; \omega) \cdot \Gamma_0(r, r_0; \omega) \cdot \alpha(\omega), \quad (3.3)$$

where $\Gamma_0$ is the free Green’s dyadic satisfying Eq. (2.7) and $\Gamma$ is the Green’s dyadic in the presence of the susceptibility tensor $\chi$ satisfying Eq. (2.5). Here $\zeta = -i\omega$, is obtained after a Euclidean rotation, where $\omega$ is a frequency.

It is sometimes convenient to define

$$\tilde{\Gamma}(r_1, r_2) = \int d^3r \Gamma(r_1, r; \omega) \cdot \chi(r; \omega) \cdot \Gamma_0(r, r_2; \omega). \quad (3.4)$$

Using the differential equation for the Green’s dyadic in Eq. (2.5) we can deduce the formal relation $\tilde{\Gamma} = \Gamma - \Gamma_0$. One can, then, in principle, drop $\Gamma_0$ to write $\tilde{\Gamma} = \Gamma$, arguing that the dropped term without the information of the second body can not contribute to the interaction energy. This subtraction gives the relatively simpler form for the energy, $E = - \int d\zeta \text{tr} \Gamma(r_0, r_0) \cdot \alpha(\omega)$. But, this introduces spurious divergences in the energy, associated to dropping of $\Gamma_0$. We shall not replace $\tilde{\Gamma} \rightarrow \Gamma$, to appreciate the fact that the expression for the interaction energy in Eq. (3.3), by construction, for disjoint objects, has no divergences.
Let us choose the polarizable molecule to be positioned on the $z$ axis, $r_0 = h \hat{z}$, and let it be unidirectionally polarizable only along the $z$ axis, corresponding to the choice
\[ \alpha(\omega) = \hat{z} \alpha(\omega) \] (3.5)
in Eq. (3.2). Let the second dielectric body be described by the susceptibility tensor
\[ \chi(r;\omega) = 1_\perp \chi(\xi;\omega) \] (3.6)
extpressed in terms of coordinates $(\xi, \eta, \phi)$ that satisfy the completeness relations,
\[ 1 = \hat{\phi} \hat{\phi} + \hat{\eta} \hat{\eta} + \hat{\xi} \hat{\xi}, \quad 1_\perp = \hat{\phi} \hat{\phi} + \hat{\eta} \hat{\eta}, \] (3.7)
where $\phi$ is the azimuth coordinate angle. Let us choose the vector $s$ to represent the projection of vector $r$ perpendicular to the azimuth direction $\hat{\phi}$, such that we can write $r = (s, \phi)$.

The dielectric body described by Eq. (3.6) and the unidirectionally polarizable molecule of Eq. (3.5) placed on the axis of the dielectric body constitutes an axially symmetric configuration. The interaction energy of Eq. (3.3) for this system involves the volume element $d^3r = d^2s \bar{s}d\phi$, where $\bar{s}d\phi$ is the arc length in the direction of $\phi$. The axial symmetry of the configuration requires the dependence of the Green dyadics in the azimuth angles to have the form $(\phi - \phi_0)$, see Appendix A, which implies the Fourier transformations for the Green dyadics in Eq. (3.3) to be,
\[ \Gamma(r_0, r; \omega) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \Gamma_m(s_0, s; \omega) e^{-im(\phi - \phi_0)}, \] (3.8a)
\[ \Gamma_0(r, r_0; \omega) = \frac{1}{2\pi} \sum_{m'=-\infty}^{\infty} \Gamma_{0m}(s, s_0; \omega) e^{im'(\phi - \phi_0)}. \] (3.8b)

Using these Fourier transformations in the expression for the interaction energy in Eq. (3.3) we have
\[ E = -\int_{-\infty}^{\infty} d\zeta \alpha(\omega) \int \frac{s\,d^2\bar{s}}{2\pi} \chi(\xi;\omega) \sum_{m=-\infty}^{\infty} \hat{z} \cdot \Gamma_m(s_0, s; \omega) \cdot 1_\perp \cdot \Gamma_{0m}(s, s_0; \omega) \cdot \hat{z}. \] (3.9)

The suggestion seems to be that all the azimuth modes $m$ contribute to the interaction energy through the sum over $m$ in Eq. (3.9). But, we shall unravel that due to the additional restriction that position of the molecule $s_0$ is on the $z$ axis, only the $m = 0$ term gives non-zero contribution to the energy in Eq. (3.9).

To elucidate on the proposition that for axially symmetric systems only the $m = 0$ mode contributes to the Casimir-Polder energy, we shall proceed here to calculate the Casimir-Polder energy for a polarizable molecule interacting with a dielectric body having planar symmetry. We shall explicate the formalism using rectangular coordinates, and then again using cylindrical coordinates. We shall then expound the formalism for the case of a polarizable molecule interacting
with a dielectric body having spherical symmetry. The transverse electric (TE) and transverse magnetic (TM) modes separate for both these geometries, and arguably leads to significant simplifications that conceals complex interplay of the modes in other geometries. But, the simplifications in these geometries lets us consider more general scenarios than the confines of axially symmetric cases, and contrasting these scenarios with the axially symmetric cases allows us to discover that our hypothesis seems to be true. We shall present our proposition in Sec. 7.

4. Planar systems in rectangular coordinates

Our primary concern in this article is to elucidate on a formalism that is useful in the studies of electromagnetic systems with axial symmetry. A central feature in our formalism involves the construction of vector eigenfunctions suitable for the problem. We illustrate this construction for a system with planar symmetry in rectangular coordinates. To gain insight we shall discuss this same system again using cylindrical polar coordinates in Sec. 5.

We first consider systems with planar symmetry. As a specific example we consider a $\delta$-function plate sandwiched between two dielectric mediums of infinite extent, see Fig. 1. We choose the direction of $\hat{z}$ to be in the direction perpendicular to the plane of $\delta$-function plate. The gradient operator $\nabla$ in rectangular coordinates is

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}.$$  \hspace{1cm} (4.1)

We choose the perpendicular component of the gradient operator to be

$$\nabla_\perp = \nabla - \hat{z} \frac{\partial}{\partial z} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}.$$  \hspace{1cm} (4.2)

Further, let us construct the operator

$$\hat{z} \times \nabla_\perp = \hat{y} \frac{\partial}{\partial x} - \hat{x} \frac{\partial}{\partial y}.$$  \hspace{1cm} (4.3)
Given an arbitrary function \( f(\mathbf{r}) \) the vector field constructions

\[
\hat{z} \frac{\partial f}{\partial z}, \quad \nabla_\perp f, \quad (\hat{z} \times \nabla_\perp)f,
\]

are orthogonal to each other at every point in space. But, this construction is too constrained and not suitable for expressing electric and magnetic fields, which have three independent components each. With the goal of expressing the electric and magnetic vector fields, we express an arbitrary vector field \( \mathbf{E}(\mathbf{r}) \) as

\[
\mathbf{E}(\mathbf{r}) = \nabla_\perp E^{(u)}(\mathbf{r}) + (\hat{z} \times \nabla_\perp)E^{(v)}(\mathbf{r}) + \hat{z}E^{(w)}(\mathbf{r}),
\]

where \( E^{(u)}(\mathbf{r}), \ E^{(v)}(\mathbf{r}), \) and \( E^{(w)}(\mathbf{r}) \), are arbitrary functions describing the vector field \( \mathbf{E}(\mathbf{r}) \). Observe that

\[
[\hat{z}E^{(w)}(\mathbf{r})] \cdot [\nabla_\perp E^{(u)}(\mathbf{r})] = 0, \quad (4.6a)
\]
\[
[
\nabla_\perp E^{(u)}(\mathbf{r})\] \cdot [(\hat{z} \times \nabla_\perp)E^{(v)}(\mathbf{r})] = 0, \quad (4.6b)
\]
\[
[\nabla_\perp E^{(u)}(\mathbf{r})] \cdot [(\hat{z} \times \nabla_\perp)E^{(v)}(\mathbf{r})] \neq 0, \quad (4.6c)
\]

which implies that the decomposition of the vector field in Eq. \((4.5)\) is not an orthogonal construction at every point in space. However, we shall soon be able to construct suitable orthonormal vector eigenfunctions starting from Eq. \((4.5)\).

To this end, we construct the Laplacian on the surface of a plane perpendicular to \( \hat{z} \),

\[
\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (4.7)
\]

This Laplacian satisfies the eigenvalue equation

\[
\nabla_\perp^2 [e^{i\mathbf{K}_\perp \cdot \mathbf{r}_\perp}] = -k_\perp^2 [e^{i\mathbf{K}_\perp \cdot \mathbf{r}_\perp}], \quad (4.8)
\]

where

\[
\mathbf{K}_\perp \cdot \mathbf{r}_\perp = k_x x + k_y y, \quad k_\perp^2 = k_x^2 + k_y^2. \quad (4.9)
\]

The eigenfunctions in Eq. \((4.8)\) are the Fourier eigenfunctions and satisfy the completeness relation

\[
\delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_\perp') = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{K}_\perp \cdot \mathbf{r}_\perp} e^{-i\mathbf{K}_\perp' \cdot \mathbf{r}_\perp'}, \quad (4.10)
\]

and the orthogonality relation

\[
\int d^2r_\perp e^{-i\mathbf{K}_\perp \cdot \mathbf{r}_\perp} e^{i\mathbf{K}_\perp' \cdot \mathbf{r}_\perp'} = (2\pi)^2 \delta^{(2)}(\mathbf{K}_\perp - \mathbf{K}_\perp'). \quad (4.11)
\]

The completeness of the eigenfunctions of the Laplacian on the plane lets us expand the functions \( E^{(i)}(\mathbf{r}), \ i = u, v, w, \) in terms of these eigenfunctions. Thus, we can write

\[
E^{(i)}(\mathbf{r}) = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{K}_\perp \cdot \mathbf{r}_\perp} E^{(i)}(z, k_\perp), \quad (4.12)
\]
functions in Eq. (4.12) the decomposition of the vector field in the basis of the Fourier vector eigenfunctions after the redefinitions, $\hat{v}_k$ of Eq. (4.15) takes the form

$$E(z, k) = \left[ i \hat{v}_k E^{(u)}(z, k) + (\hat{z} \times \hat{v}_k) E^{(v)}(z, k) + \hat{z} E^{(w)}(z, k) \right].$$

(4.14)

4.1. Vector eigenfunctions

We define the Fourier vector eigenfunctions as

$$U(k, r) = \frac{1}{ik} \nabla_i e^{ik \cdot r} = \hat{k} e^{ik \cdot r},$$

(4.15a)

$$V(k, r) = \frac{1}{ik} (\hat{z} \times \nabla_i) e^{ik \cdot r} = (\hat{z} \times \hat{k}) e^{ik \cdot r},$$

(4.15b)

$$W(k, r) = \hat{z} e^{ik \cdot r}.$$  

(4.15c)

The $ik$’s in the denominator of the Fourier vector eigenfunctions in Eqs. (4.15a) and (4.15b) is interpreted as the eigenvalue of the operator $\sqrt{\nabla_i^2}$, in the context of Eq. (4.18). It is introduced for keep the dimensions of $U$, $V$, and $W$, the same. Additionally, it serves to normalize the vector eigenfunctions. We point out that the $k = 0$ mode does not exist in the construction of $U$ and $V$, because the derivative operations return zero for this mode. This does not cause complications in our considerations that involve integrations over $k$, because a single point inside an integral is irrelevant. In terms of the Fourier vector eigenfunctions of Eqs. (4.15) the decomposition of a vector field in Eq. (4.14) takes the form

$$E(r) = \int \frac{d^2 k}{(2\pi)^2} \left[ U(k, r) E^{(u)}(z, k) + V(k, r) E^{(v)}(z, k) + W(k, r) E^{(w)}(z, k) \right],$$

(4.16)

after the redefinitions, $ik E^{(u)}(z, k) \rightarrow E^{(u)}(z, k)$ and $ik E^{(v)}(z, k) \rightarrow E^{(v)}(z, k)$. Using the notations

$$X^{(u)}(k, r) = U(k, r),$$

(4.17a)

$$X^{(v)}(k, r) = V(k, r),$$

(4.17b)

$$X^{(w)}(k, r) = W(k, r),$$

(4.17c)

the decomposition of the vector field in the basis of the Fourier vector eigenfunctions in Eq. (4.16) can be written in the compact form

$$E(r) = \int \frac{d^2 k}{(2\pi)^2} X(i(k, r) E^{(i)}(z, k), \quad i = u, v, w,$$

(4.18)
where we used summation convention in the repeated index \( i \), that is, repeated indices are summed. The completeness relation of the Fourier vector eigenfunctions of Eqs. (4.15) is constructed as

\[
\delta^{(2)}(r_\perp - r'_\perp) = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i k_\perp \cdot r_\perp} \left[ \mathbf{k}_\perp \cdot \mathbf{k}_\perp + (\mathbf{\hat{z}} \times \mathbf{\hat{k}}_\perp)(\mathbf{\hat{z}} \times \mathbf{\hat{k}}_\perp) + \mathbf{\hat{z}} \mathbf{\hat{z}} \right] e^{-i k_\perp \cdot r_\perp}
\]

The orthogonality of the Fourier vector eigenfunctions is stated as

\[
\int d^2 r_\perp X^\ast(i)(k_\perp, r_\perp) \cdot X(j)(k'_\perp, r'_\perp) = \delta^{ij}(2\pi)^2 \delta^{(2)}(k_\perp - k'_\perp).
\]

The divergence of a vector field in the basis of the Fourier vector eigenfunctions of Eqs. (4.15) is evaluated as

\[
\nabla \cdot \mathbf{E} = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i k_\perp \cdot r_\perp} \left[ i k_\perp E^{(u)}(z, k_\perp) + \frac{\partial}{\partial z} E^{(v)}(z, k_\perp) \right],
\]

which uses the identities,

\[
\begin{align*}
\mathbf{\hat{z}} \cdot \mathbf{U} &= 0, & \nabla_\perp \cdot \mathbf{U} &= i k_\perp e^{i k_\perp \cdot r_\perp}, & \nabla \cdot \mathbf{U} &= i k_\perp e^{i k_\perp \cdot r_\perp}, \\
\mathbf{\hat{z}} \cdot \mathbf{V} &= 0, & \nabla_\perp \cdot \mathbf{V} &= 0, & \nabla \cdot \mathbf{V} &= 0, \\
\mathbf{\hat{z}} \cdot \mathbf{W} &= e^{i k_\perp \cdot r_\perp}, & \nabla_\perp \cdot \mathbf{W} &= 0, & \nabla \cdot \mathbf{W} &= e^{i k_\perp \cdot r_\perp} \frac{\partial}{\partial z}.
\end{align*}
\]

Similarly, the curl of a vector field in the basis of the Fourier vector eigenfunctions of Eqs. (4.15) is evaluated as

\[
\nabla \times \mathbf{E} = \int \frac{d^2 k_\perp}{(2\pi)^2} \left[ - \mathbf{U}(k_\perp, r_\perp) \frac{\partial}{\partial z} E^{(v)}(z, k_\perp) + \mathbf{V}(k_\perp, r_\perp) \left\{ \frac{\partial}{\partial z} E^{(v)}(z, k_\perp) - i k_\perp E^{(v)}(z, k_\perp) \right\} \right. \\
+ \left. \mathbf{W}(k_\perp, r_\perp) i k_\perp E^{(v)}(z, k_\perp) \right]
\]

using

\[
\begin{align*}
\mathbf{\hat{z}} \times \mathbf{U} &= \mathbf{V}, & \nabla_\perp \times \mathbf{U} &= 0, \\
\mathbf{\hat{z}} \times \mathbf{V} &= -\mathbf{U}, & \nabla_\perp \times \mathbf{V} &= i k_\perp \mathbf{W}, \\
\mathbf{\hat{z}} \times \mathbf{W} &= 0, & \nabla_\perp \times \mathbf{W} &= -i k_\perp \mathbf{V},
\end{align*}
\]

and

\[
\begin{align*}
\nabla \times \mathbf{U} &= \mathbf{V} \frac{\partial}{\partial z}, \\
\nabla \times \mathbf{V} &= i k_\perp \mathbf{W} - \mathbf{U} \frac{\partial}{\partial z}, \\
\nabla \times \mathbf{W} &= -i k_\perp \mathbf{V}.
\end{align*}
\]
The Maxwell equations in the presence of a medium involve the electric and magnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ together with the respective macroscopic fields $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$. We assume the macroscopic fields to be linearly related to the electric and magnetic fields, given by Eq. (2.2). We also require the electric and magnetic properties of the medium to have planar symmetry. To this end, we shall restrict our analysis to mediums described by

$$\varepsilon(\mathbf{r}; \omega) = \varepsilon^\perp(z; \omega)1_\perp + \varepsilon^\parallel(z; \omega)\hat{z}\hat{z},$$

(4.26a)

$$\mu(\mathbf{r}; \omega) = \mu^\perp(z; \omega)1_\perp + \mu^\parallel(z; \omega)\hat{z}\hat{z}.$$  

(4.26b)

Using the form for the material properties in Eqs. (4.26) and the decomposition of a vector field in terms of Fourier vector eigenfunctions in Eq. (4.16) together in Eq. (2.2) we obtain the following decomposition for the macroscopic fields in terms of Fourier vector eigenfunctions,

$$\mathbf{D}(\mathbf{r}) = \int \frac{d^2k_\perp}{(2\pi)^2} \left[ U(k_\perp, r_\perp)\varepsilon^\perp(z; \omega)E^{(u)}(z, k_\perp) 
+ V(k_\perp, r_\perp)\varepsilon^\parallel(z; \omega)E^{(v)}(z, k_\perp) + W(k_\perp, r_\perp)\varepsilon^\parallel(z; \omega)E^{(w)}(z, k_\perp) \right],$$

(4.27a)

$$\mathbf{B}(\mathbf{r}) = \int \frac{d^2k_\perp}{(2\pi)^2} \left[ U(k_\perp, r_\perp)\mu^\perp(z; \omega)H^{(u)}(z, k_\perp) 
+ V(k_\perp, r_\perp)\mu^\parallel(z; \omega)H^{(v)}(z, k_\perp) + W(k_\perp, r_\perp)\mu^\parallel(z; \omega)H^{(w)}(z, k_\perp) \right].$$

(4.27b)

A dielectric medium with planar symmetry was discussed in Ref. [4]. A highlight of this discussion was that we derived the boundary conditions on the electric and magnetic fields at the interface of two mediums directly from the Maxwell equations in first order form, and deduced the boundary conditions on the related Green’s functions unambiguously. For convenience we chose $k_y = 0$ in the discussion, using the rotational symmetry of a plane. In our present discussion using the construction of vector eigenfunctions we can avoid making this choice. Nevertheless, the first order equations, the boundary conditions, and the related Green’s functions are similar in construction. We shall present these discussions using vector eigenfunctions in the following subsections.

4.2. **Separation of modes**

We consider a dielectric medium with planar symmetry pictured in Fig. 1. We assume that a $\delta$-function plate given by

$$\varepsilon(\mathbf{r}; \omega) - 1 = \lambda_\epsilon(\omega)\delta(z - a),$$

(4.28a)

$$\mu(\mathbf{r}; \omega) - 1 = \lambda_\mu(\omega)\delta(z - a),$$

(4.28b)
resides on the interface of the two dielectric mediums. Physical interpretation of this model was presented in Refs. [4] and [5]. Nevertheless, the results obtained here for the perfect conducting cases will be independent of this model.

Using the expression for curl operator in Eq. (4.23), in conjunction with Eqs. (4.27), the projections of Maxwell’s equations in Eqs. (2.1) along the vector eigenfunctions, \( \mathbf{U} \), \( \mathbf{V} \), and \( \mathbf{W} \), are

\[
\nabla \times \mathbf{E} = i\omega \mathbf{B} \\
\mathbf{U} : \quad -\frac{\partial}{\partial z} E^{(v)} = i\omega \mu^\parallel H^{(u)}, \\
\mathbf{V} : \quad \frac{\partial}{\partial z} E^{(u)} - ik_\perp E^{(w)} = i\omega \mu^\parallel H^{(v)}, \\
\mathbf{W} : \quad ik_\perp E^{(v)} = i\omega \mu^\parallel H^{(w)},
\]

and

\[
\nabla \times \mathbf{H} = -i\omega (\mathbf{D} + \mathbf{P}) \\
\mathbf{U} : \quad -\frac{\partial}{\partial z} H^{(v)} = -i\omega (\epsilon^\parallel E^{(u)} + P^{(u)}), \\
\mathbf{V} : \quad \frac{\partial}{\partial z} H^{(u)} - ik_\perp H^{(w)} = -i\omega (\epsilon^\parallel E^{(v)} + P^{(v)}), \\
\mathbf{W} : \quad ik_\perp H^{(v)} = -i\omega (\epsilon^\parallel E^{(w)} + P^{(w)}).
\]

We have suppressed the variable dependence of the components of the fields \( E^{(i)}(z, k_\perp) \), \( i = u, v, w \), in Eqs. (4.29) and (4.30). The boundary conditions satisfied by the electric field components are derived by integrating Eqs. (4.29) across the planar interface, at \( z = a \), to yield

\[
E^{(u)}(z, k_\perp) \bigg|_{z=a^-}^{z=a^+} = i\omega \lambda_\parallel^g H^{(v)}(a, k_\perp), \\
E^{(v)}(z, k_\perp) \bigg|_{z=a^-}^{z=a^+} = -i\omega \lambda_\parallel^v H^{(u)}(a, k_\perp), \\
D^{(w)}(z, k_\perp) \bigg|_{z=a^-}^{z=a^+} = -ik_\perp \lambda_\parallel^w E^{(u)}(a, k_\perp).
\]

Similarly, the boundary conditions satisfied by the magnetic field components are derived by integrating Eqs. (4.30) across the planar interface, at \( z = a \), to yield

\[
H^{(u)}(z, k_\perp) \bigg|_{z=a^-}^{z=a^+} = -i\omega \lambda_\parallel^e E^{(v)}(a, k_\perp), \\
H^{(v)}(z, k_\perp) \bigg|_{z=a^-}^{z=a^+} = i\omega \lambda_\parallel^v E^{(u)}(a, k_\perp), \\
B^{(w)}(z, k_\perp) \bigg|_{z=a^-}^{z=a^+} = -ik_\perp \lambda_\parallel^w H^{(u)}(a, k_\perp).
\]
We also obtain the constraints
\[ \lambda|E^{(w)}(a, k_\perp)| = 0, \quad (4.33a) \]
\[ \lambda|H^{(w)}(a, k_\perp)| = 0. \quad (4.33b) \]
Here, Eqs. (4.31a) and (4.32a) are obtained by integrating Eqs. (4.29b) and (4.30b), and Eqs. (4.31b) and (4.32b) are obtained by integrating Eqs. (4.29a) and (4.30a). Eqs. (4.31c) and (4.32c) are obtained by substituting Eqs. (4.29c) and (4.30c) in Eqs. (4.29a) and (4.30a) respectively and then integrating across the interface. The constraints in Eqs. (4.33a) and (4.33b) are obtained by integrating Eqs. (4.29c) and (4.30c).

We construct electric and magnetic Green’s functions of Refs. [6, 4] that satisfy the differential equations, \( g^E \) refers to TE mode and \( g^H \) to TM mode as pointed out in footnote 1 of Ref. [4].

\[
\begin{align*}
\left(-\frac{\partial}{\partial z} \frac{1}{\mu}(z) \frac{\partial}{\partial z} + \frac{k_\perp^2}{\mu(z)} - \omega^2 \varepsilon(z)\right) g^E(z, z'; k_\perp, \omega) &= \delta(z - z'), \quad (4.34a) \\
\left(-\frac{\partial}{\partial z} \frac{1}{\varepsilon}(z) \frac{\partial}{\partial z} + \frac{k_\perp^2}{\varepsilon(z)} - \omega^2 \mu(z)\right) g^H(z, z'; k_\perp, \omega) &= \delta(z - z'), \quad (4.34b)
\end{align*}
\]

and deduce the boundary conditions on these Green’s functions, without having any choice to impose them [4]. In Ref. [4] we have shown that all the components of electric and magnetic fields can be expressed in terms of the Green’s functions of Eqs. (4.34). They can be organized together such that we can write them in terms of a Green’s dyadic of the form in Eq. (2.6). Similarly, the magnetic field can be expressed as
\[ \mathbf{H}(r) = \int d^3r' \Phi(r, r'; \omega) \cdot \mathbf{P}(r'), \quad (4.35) \]
where \( \Gamma \) and \( \Phi \) are Green’s dyadics.

A vector field is decomposed in the basis of the vector eigenfunctions as per Eq. (4.16). A Green’s dyadic is a tensor. Further, the Green’s dyadic under consideration has the symmetry of a plane, and the translational symmetry in the directions perpendicular to \( \hat{z} \) imposes a functional dependence of the form \( r_\perp - r'_\perp \). Due to this feature their decomposition in the basis of vector eigenfunctions is given in terms of a single Fourier variable \( k_\perp \),

\[
\begin{align*}
\Gamma(r, r'; \omega) &= \int \frac{d^2k_\perp}{(2\pi)^2} \mathbf{X}^{(i)}(k_\perp, r_\perp) \gamma_{ij}^{ij}(z, z' ; k_\perp, \omega) \mathbf{X}^{(j)*}(k_\perp, r'_\perp), \quad (4.36a) \\
\Phi(r, r'; \omega) &= \int \frac{d^2k_\perp}{(2\pi)^2} \mathbf{X}^{(i)}(k_\perp, r_\perp) \phi_{ij}^{ij}(z, z' ; k_\perp, \omega) \mathbf{X}^{(j)*}(k_\perp, r'_\perp), \quad (4.36b)
\end{align*}
\]
with the matrix components of the respective dyadics given by

\[
\gamma^{ij}(z, z'; k_\perp, \omega) = \begin{bmatrix}
\frac{1}{\varepsilon^+(z)} \frac{\partial}{\partial z} \frac{1}{\varepsilon^+(z')} \frac{\partial}{\partial z'} g^H & 0 & \frac{1}{\varepsilon^+(z)} \frac{\partial}{\partial z} \frac{i k_\perp}{\varepsilon^+(z)} g^H \\
0 & \omega^2 g^E & 0 \\
-\frac{i k_\perp}{\varepsilon^+(z)} \frac{1}{\varepsilon^+(z')} \frac{\partial}{\partial z'} g^H & 0 & -\frac{i k_\perp}{\varepsilon^+(z)} \frac{i k_\perp}{\varepsilon^+(z')} g^H
\end{bmatrix},
\]

(4.37a)

and

\[
\phi^{ij}(z, z'; k_\perp, \omega) = i \omega \begin{bmatrix}
0 & \frac{1}{\mu^+(z)} \frac{\partial}{\partial z} g^E & 0 \\
\frac{1}{\varepsilon^+(z)} \frac{\partial}{\partial z} g^H & 0 & \frac{i k_\perp}{\varepsilon^+(z)} g^H \\
0 & -\frac{i k_\perp}{\mu^+(z)} g^E & 0
\end{bmatrix}.
\]

(4.37b)

In Eq. (4.37a) we omitted the term

\[
- \delta(z - z') \begin{bmatrix}
\frac{1}{\varepsilon^+(z)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{\mu^+(z)}
\end{bmatrix},
\]

(4.38)

that contains a \(\delta\)-function and thus never contributes to interaction energies between two bodies unless they are overlapping. We will confine our discussions to disjoint bodies and drop this term, because it does not contribute. The Green’s dyadics are completely determined in terms of the magnetic and electric Green’s functions that satisfy Eqs. (4.34), explicit solutions for which were presented in Ref. [4].

4.3. Casimir-Polder interaction energy

We shall now evaluate the interaction energy between a polarizable molecule, described by Eqs. (3.1) and (3.2), and a \(\delta\)-function dielectric plate with a dielectric tensor given by Eq. (4.28a) with

\[
\lambda_e(\omega) = 1_\perp \lambda(\omega), \quad 1_\perp = 1 - \hat{z}\hat{z}
\]

(4.39)

and \(\lambda_d(\omega) = 0\). See Fig. 2. Since the dielectric plate thus constructed has translational symmetry in the directions transverse to \(\hat{z}\), the dyadic structure in Eq. (3.4) constructed out of \(\Gamma, \chi, \) and \(\Gamma_0\), also has this symmetry. Thus, in the basis of the vector eigenfunctions of Eqs. (4.15) we can write

\[
\bar{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) = \int \frac{d^2k_\perp}{(2\pi)^2} X^{(i)}(\mathbf{k}_\perp, \mathbf{r}_\perp) \tilde{\gamma}^{ij}(z, z'; k_\perp, \omega) X^{(j)*}(\mathbf{k}_\perp, \mathbf{r}_\perp'),
\]

(4.40)

where

\[
\tilde{\gamma}^{ij}(z, z'; k_\perp, \omega) = \gamma^{ij}(z, a; k_\perp, \omega) \lambda^{i'j'}(\omega) \gamma_0^{j'j}(a, z'; k_\perp, \omega).
\]

(4.41)

Observe that, due to rotational symmetry, the dependence in \(k_\perp\) is of the form \(k_\perp = |k_\perp|\). We allow the principal axes for the polarizabilities of the molecule

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to be arbitrarily oriented while the molecule is at \( z = h \) above the plate. The expression for Casimir-Polder interaction energy in Eq. (3.3) for this scenario, after using the cyclic property of trace, can be expressed in the form

\[
E = -\int_{-\infty}^{\infty} d\zeta \int \frac{d^2k}{(2\pi)^2} \tilde{\gamma}^{ij}(h, a; k, \omega) \lambda^{ij}(\omega) \gamma_0^{ij}(a, h; k, \omega) \alpha^{ij}(\omega),
\]  

(4.42)

where, using the shorthand notation \( \tilde{q} \equiv \hat{z} \times \hat{k} \),

\[
\alpha^{ij}(\omega) = X^{(j)*}(k, r, \alpha(\omega)) \cdot X^{(i)}(k, r)
\]

(4.43)

\[
\gamma^{ij}(z, z'; k, \omega) = \begin{bmatrix}
    \partial \partial' g^H & 0 & i k \partial g^H \\
    0 & \omega^2 g^E & 0 \\
    -i k \partial g^H & 0 & k^2 g^H
\end{bmatrix},
\]

(4.45)

and the matrix components of the free Green dyadic are given by

\[
\gamma_0^{ij}(z, z'; k, \omega) = \begin{bmatrix}
    \partial \partial' g_0 & 0 & i k \partial g_0 \\
    0 & \omega^2 g_0 & 0 \\
    -i k \partial g_0 & 0 & k^2 g_0
\end{bmatrix},
\]

(4.46)

where \( \partial \) represents derivative with respect to the first variable, and \( \partial' \) represents derivative with respect to the second variable. The free Green function \( g_0(z, z') \) in Eqs. (4.45) and (4.46) is given by

\[
g_0(z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|},
\]

(4.47)

The electric and magnetic Green functions for a \( \delta \)-function plate are given by

\[
g^E(z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|} + r_E \frac{1}{2\kappa} e^{-\kappa|z-a| e^{-\kappa|z'-a|}},
\]

(4.48)
and
\[ g^H(z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|} + \eta(z-a)\eta(z'-a)r^H_e \frac{1}{2\kappa} e^{-\kappa|z-a|} e^{-\kappa|z'-a|}, \quad (4.49) \]
respectively, which are expressed in terms of the reflection coefficients
\[ r^E_e = -\frac{\lambda}{\lambda + \frac{2\kappa}{\zeta}}, \quad r^H_e = \frac{\lambda}{\lambda + \frac{2\kappa}{\zeta}}, \quad (4.50) \]
and
\[ \kappa^2 = k_1^2 + \zeta^2, \quad (4.51) \]
where
\[ \eta(z) = \begin{cases} 1, & z > 0, \\ -1, & z < 0. \end{cases} \quad (4.52) \]

Using the matrix components for the dyadics in Eqs. (4.43) to (4.46), in the expression for Casimir-Polder interaction energy in Eq. (4.42) we obtain five terms, characterized by \( \hat{\mathbf{q}}_{\perp} \cdot \alpha \cdot \hat{\mathbf{q}}_{\perp}, \hat{\mathbf{k}}_{\perp} \cdot \alpha \cdot \hat{\mathbf{k}}_{\perp}, \hat{\mathbf{z}} \cdot \alpha \cdot \hat{\mathbf{z}}, \hat{\mathbf{z}} \cdot \alpha \cdot \hat{\mathbf{k}}_{\perp}, \) and \( \hat{\mathbf{k}}_{\perp} \cdot \alpha \cdot \hat{\mathbf{z}}. \)

The coefficients of these terms involve derivatives of the Green functions, given in Eqs. (4.47), (4.48), and (4.49), and we obtain, using Appendix B of Ref. [4], for three of the terms,
\[ \partial \partial' g^H(h, a)\partial \partial' g_0(a, h) = \frac{1}{2\kappa} \frac{\kappa^2}{\lambda + \frac{2\kappa}{\zeta}} e^{-2\kappa|h-a|}, \quad (4.53a) \]
\[ \omega^4 g^H(h, a)g_0(a, h) = \frac{1}{2\kappa} \frac{\zeta^2}{\lambda + \frac{2\kappa}{\zeta}} e^{-2\kappa|h-a|}, \quad (4.53b) \]
\[ k_1^2 \partial \partial' g^H(h, a)\partial g_0(a, h) = \frac{1}{2\kappa} \frac{k_1^2}{\lambda + \frac{2\kappa}{\zeta}} e^{-2\kappa|h-a|}. \quad (4.53c) \]

The remaining two (cross) terms characterized by \( \hat{\mathbf{z}} \cdot \alpha(\omega) \cdot \hat{\mathbf{k}}_{\perp} \) and \( \hat{\mathbf{k}}_{\perp} \cdot \alpha(\omega) \cdot \hat{\mathbf{z}} \) cancel each other, because their coefficients are equal and opposite in sign, using
\[ \partial \partial' g^H(h, a)\partial g_0(a, h) = \partial' g^H(h, a)\partial \partial' g_0(a, h) = -\frac{1}{2\kappa} \frac{1}{\lambda + \frac{2\kappa}{\zeta}} e^{-2\kappa|h-a|}. \quad (4.54) \]

Thus, we obtain the Casimir-Polder interaction energy between a polarizable molecule, with polarizability tensor \( \alpha \), distance \( h \) above a \( \delta \)-function plate, choosing \( a = 0 \), with electric and magnetic reflection coefficients \( r^E_e \) and \( r^H_e \), to be
\[ E = -\int \frac{d\zeta d^2 k_{\perp}}{(2\pi)^2} \frac{e^{-2\kappa h}}{2\kappa} \left[ \kappa^2 r^H_e \hat{\mathbf{k}}_{\perp} \cdot \alpha \cdot \hat{\mathbf{k}}_{\perp} - \zeta^2 r^E_e \hat{\mathbf{q}}_{\perp} \cdot \alpha \cdot \hat{\mathbf{q}}_{\perp} + k_1^2 r^H_e \hat{\mathbf{z}} \cdot \alpha \cdot \hat{\mathbf{z}} \right], \quad (4.55) \]
where the integration limits on all three integrals are from \( -\infty \) to \( \infty \). A Casimir-Polder interaction energy of a polarizable molecule and a plate involves orientations of three independent basis vectors: \( (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \) for the principal axes.
Figure 2: An electric δ-function plate described by polarizability $\lambda(\omega)$ interacting with a polarizable molecule of polarizability $\alpha(\omega)$, placed height $h$ above the plate. The three vectors on the molecule illustrate the directions of the principal axes of polarizabilities, $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. 

of polarizabilities of the molecule; $(\hat{g}_1, \hat{g}_2, \hat{g}_3)$ for the principal axes of polarizabilities of the plate; and $(\hat{x}, \hat{y}, \hat{z})$ for the geometrical orientation of the plate, which is Fourier transformed in the $\hat{x}$ and $\hat{y}$ directions to lead to $(\hat{k}_\perp, \hat{q}_\perp, \hat{z})$. The requirement of planar symmetry for the plane allows the orientations $(\hat{g}_1, \hat{g}_2, \hat{g}_3)$ to be chosen identical to $(\hat{k}_\perp, \hat{q}_\perp, \hat{z})$, which has been used in deriving the expression in Eq. (4.55). In the presence of planar symmetry for the plane the integrals in Eq. (4.55) are insensitive to the difference between $\hat{k}_\perp \cdot \alpha \cdot \hat{k}_\perp = \hat{q}_\perp \cdot \alpha \cdot \hat{q}_\perp$. Thus, inside the integral, we can use

$$\hat{k}_\perp \cdot \alpha \cdot \hat{k}_\perp = \hat{q}_\perp \cdot \alpha \cdot \hat{q}_\perp = \frac{1}{2} \text{tr}\{\alpha(\omega) \cdot 1\}, \quad 1_\perp = 1 - \hat{z}\hat{z}. \quad (4.56)$$

Using this feature of planar symmetry in Eq. (4.55) we have

$$E = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \kappa_\perp dk_\perp \frac{e^{-2\kappa h}}{2\kappa} \times \left[ (\kappa^2 r_H^H - \zeta^2 r_E^E) \text{tr}\alpha(\omega) + (r_H^H \kappa^2 - 2r_H^H \zeta^2 + r_E^E \zeta^2) \hat{z} \cdot \alpha(\omega) \cdot \hat{z} \right]. \quad (4.57)$$

The energy of Eq. (4.57) depends on the orientations of the principal axes of polarizabilities of the molecule completely through the term containing $\hat{z} \cdot \alpha(\omega) \cdot \hat{z}$, because $\text{tr}\alpha(\omega)$ is independent of these orientations. We shall not explore the expression in Eq. (4.57) in its full generality. A counterpart of the expression in Eq. (4.57), for the case of a dielectric slab of finite thickness, was reported and analyzed in Ref. [7].

Let us now consider the δ-function plate to be a perfect conductor. The energy for this case is obtained by taking the limit $\lambda \to \infty$, entailing $r_H^H \to 1$ and $r_E^E \to -1$, in the expression for energy in Eq. (4.57). Further, in the Casimir-Polder limit only the low frequency fluctuations are dominant, which is obtained in the static limit $\alpha(\omega) \to \alpha(0)$. In these ideal limits all the integrals in Eq. (4.57) can be completed to yield

$$E = -\frac{\text{tr}\alpha(0)}{8\pi h^4}. \quad (4.58)$$
It is remarkable, and not a priori intuitive, that in the static and perfect conductor limit the interaction energy should be insensitive to the orientation of the polarizabilities. We emphasize that translational symmetry and static polarizability alone is not sufficient to render the energy independent of the orientation. It is in fact the fine cancellations in the contributions from the electric and magnetic modes in the perfect conductor limit that makes the energy orientation independent. The analogous result of Eq. (4.58), for isotropic polarizabilities, is the well known result by Casimir and Polder, first derived in 1947 [8].

At the loss of considerable generality, but to gain insight from the substantial simplicity that follows, we narrow down our attention to the case of a unidirectionally polarizable atom polarizable in the direction of the axis, described by

\[ \alpha(\omega) = \alpha(\omega) \hat{z} \hat{z}. \]  (4.59)

For this case Eq. (4.57) takes the form

\[ E = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} k_{\perp} dk_{\perp} e^{-2\kappa h} \frac{k_{\perp}^{2} e^{H}}{2\kappa} \alpha(\omega). \]  (4.60)

This is the scenario we are focussing on in this article, because it is an axially symmetric system. If we further assume the static and perfect conductor limit, we can complete the integrals to obtain

\[ E = -\frac{\alpha(0)}{8\pi h^{4}}, \]  (4.61)

which is a corollary of Eq. (4.58).

5. Planar systems in cylindrical coordinates

In Sec. 4 we discussed planar systems using rectangular coordinates in considerable detail. Here we discuss planar systems using cylindrical polar coordinates,

\[ (z, \rho, \phi), \quad -\infty < z < \infty, \quad 0 \leq \rho < \infty, \quad 0 \leq \phi < 2\pi. \]  (5.1)

Here we go through most of the construction from the last section, using cylindrical coordinates, to gain insight into how the azimuth modes contribute in axially symmetric systems. The construction of vector eigenfunctions using cylindrical coordinates is significantly different and is now described by the azimuth mode \( m \). Developments in the last section related to separation of modes, derivation of boundary conditions and the related Green’s dyadic, broadly follows the same steps and we shall suffice our discussion by giving a transcription to go from rectangular coordinates to cylindrical coordinates.

The gradient operator in cylindrical coordinates is given by

\[ \nabla = \hat{z} \frac{\partial}{\partial z} + \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi}. \]  (5.2)
After choosing $\hat{z}$ to be the direction of normal to the plane, the following operators yield two orthogonal vectors perpendicular to $\hat{z}$,

\[
\nabla_\perp = \rho \frac{\partial}{\partial \rho} + \phi \frac{\partial}{\partial \phi}, \tag{5.3a}
\]

\[
\hat{z} \times \nabla_\perp = -\rho \frac{\partial}{\partial \phi} + \phi \frac{\partial}{\partial \rho}. \tag{5.3b}
\]

An arbitrary vector field $\mathbf{E}(\mathbf{r})$ can be expressed in the form of Eq. (4.5), now with the respective gradient operators in cylindrical coordinates, where $E^{(i)}(\mathbf{r})$, $i = u, v, w$, are arbitrary functions. For systems with rotational symmetry about the normal direction $\hat{z}$ it is convenient to express the functions $E^{(i)}(\mathbf{r})$ in terms of the eigenfunctions of the Laplacian on the surface of the planar interface, now in cylindrical coordinates. This Laplacian satisfies

\[
\nabla_\perp^2 \left[ e^{im\phi} J_m(k_\perp \rho) \right] = \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) \left[ e^{im\phi} J_m(k_\perp \rho) \right] = e^{im\phi} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} \right) J_m(k_\perp \rho) = -k_\perp^2 \left[ e^{im\phi} J_m(k_\perp \rho) \right], \tag{5.4}
\]

where we used the differential equation satisfied by the Bessel functions,

\[
\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} + k_\perp^2 \right) J_m(k_\perp \rho) = 0. \tag{5.5}
\]

Thus, we learn that trigonometric functions and Bessel functions are suitable eigenfunctions of the Laplacian on the surface of a plate. These eigenfunctions, together, satisfy the completeness relation

\[
\frac{1}{\rho} \delta(\rho - \rho') \delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m = -\infty}^{\infty} \int_{0}^{\infty} k_\perp dk_\perp e^{-im(\phi - \phi')} J_m(k_\perp \rho) J_m(k_\perp \rho') \tag{5.6}
\]

and the orthogonality relation

\[
\int_{0}^{2\pi} d\phi \int_{0}^{\infty} \rho d\rho e^{-i\phi(m-m')} J_m(k_\perp \rho) J_m(k_\perp \rho') = 2\pi \delta_{mm'} \frac{\delta(k_\perp - k'_\perp)}{k_\perp}. \tag{5.7}
\]

This implies that the functions $E^{(i)}(\mathbf{r})$ can be expressed in the form

\[
E^{(i)}(\mathbf{r}) = \frac{1}{2\pi} \sum_{m = -\infty}^{\infty} e^{im\phi} \int_{0}^{\infty} k_\perp dk_\perp J_m(k_\perp \rho) E^{(i)}_m(z, k_\perp), \tag{5.8}
\]

where the components are given by

\[
E^{(i)}_m(z, k_\perp) = \int_{0}^{2\pi} d\phi e^{-im\phi} \int_{0}^{\infty} \rho d\rho J_m(k_\perp \rho) E^{(i)}(\mathbf{r}). \tag{5.9}
\]
5.1. Vector eigenfunctions

Using Eqs. (5.8) in Eq. (4.5) we can express the vector field \( \mathbf{E}(\mathbf{r}) \) as

\[
\mathbf{E}(\mathbf{r}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} \left[ U_{m}(k_{\perp} \rho, \phi) E_{m}^{u}(z, k_{\perp}) + V_{m}(k_{\perp} \rho, \phi) E_{m}^{v}(z, k_{\perp}) + W_{m}(k_{\perp} \rho, \phi) E_{m}^{w}(z, k_{\perp}) \right], \tag{5.10}
\]

where the cylindrical vector eigenfunctions are defined as

\[
U_{m}(k_{\perp} \rho, \phi) = \frac{1}{ik_{\perp}} \nabla_{\|} e^{im\phi} J_{m}(k_{\perp} \rho), \tag{5.11a}
\]
\[
V_{m}(k_{\perp} \rho, \phi) = \frac{1}{ik_{\perp}} (\hat{z} \times \nabla_{\|}) e^{im\phi} J_{m}(k_{\perp} \rho), \tag{5.11b}
\]
\[
W_{m}(k_{\perp} \rho, \phi) = \hat{z} e^{im\phi} J_{m}(k_{\perp} \rho). \tag{5.11c}
\]

The functions \( E_{m}^{u}(z, k_{\perp}) \) and \( E_{m}^{v}(z, k_{\perp}) \) in Eq. (5.10) are redefinitions of the ones in Eqs. (5.8) to accommodate factors of \( ik_{\perp} \). We note that the mode corresponding to \( k_{\perp} = 0 \) and \( m = 0 \) does not exist in \( U_{m}(k_{\perp} \rho, \phi) \) and \( V_{m}(k_{\perp} \rho, \phi) \), for the same reason mentioned after Eqs. (4.15). Thus, we define the vector eigenfunctions

\[
X_{m}^{u}(k_{\perp} \rho, \phi) = U_{m}(k_{\perp} \rho, \phi), \tag{5.12a}
\]
\[
X_{m}^{v}(k_{\perp} \rho, \phi) = V_{m}(k_{\perp} \rho, \phi), \tag{5.12b}
\]
\[
X_{m}^{w}(k_{\perp} \rho, \phi) = W_{m}(k_{\perp} \rho, \phi). \tag{5.12c}
\]

The vector field of Eq. (5.10) can be written in the form

\[
\mathbf{E}(\mathbf{r}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} \mathbf{X}_{m}^{(i)}(k_{\perp} \rho, \phi) \mathbf{E}_{m}^{(i)}(z, k_{\perp}). \tag{5.13}
\]

The vector eigenfunctions satisfy the completeness relation

\[
\frac{1}{\rho} \delta(\phi - \phi') \delta(\rho - \rho') 1 = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} \mathbf{X}_{m}^{(i)}(k_{\perp} \rho, \phi) \mathbf{X}_{m}^{(i)\ast}(k_{\perp}' \rho, \phi') \tag{5.14}
\]

and the orthogonality relation

\[
\int_{0}^{2\pi} d\phi \int_{0}^{\infty} \rho d\rho \mathbf{X}_{m}^{(i)\ast}(k_{\perp} \rho, \phi) \cdot \mathbf{X}_{m'}^{(j)}(k_{\perp}' \rho, \phi) = 2\pi \delta^{ij} \delta_{mm'} \frac{1}{k_{\perp}} \delta(k_{\perp} - k_{\perp}'). \tag{5.15}
\]

The unit dyadic appearing in Eq. (5.14) is

\[
1 = \nabla \mathbf{r} = \hat{z} \hat{z} + \hat{\rho} \hat{\rho} + \hat{\phi} \hat{\phi}. \tag{5.16}
\]
The divergence of a vector field in the basis of the vector eigenfunctions of Eq. (5.11) is evaluated as
\[ \nabla \cdot E = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} e^{im\phi} J_m(k_{\perp}) \left[ ik_{\perp} E_{m}^{(u)}(z, k_{\perp}) + \frac{\partial}{\partial z} E_{m}^{(w)}(z, k_{\perp}) \right], \]
(5.17)
which uses the identities,
\[ \hat{z} \cdot U_m = 0, \quad \nabla_\perp \cdot U_m = ik_{\perp} e^{im\phi} J_m(k_{\perp} \rho), \]
(5.18a)
\[ \hat{z} \cdot V_m = 0, \quad \nabla_\perp \cdot V_m = 0, \]
(5.18b)
\[ \hat{z} \cdot W_m = e^{im\phi} J_m(k_{\perp} \rho), \quad \nabla_\perp \cdot W_m = 0, \]
(5.18c)
and
\[ \nabla \cdot U_m = ik_{\perp} e^{im\phi} J_m(k_{\perp} \rho), \]
(5.19a)
\[ \nabla \cdot V_m = 0, \]
(5.19b)
\[ \nabla \cdot W_m = e^{im\phi} J_m(k_{\perp} \rho) \frac{\partial}{\partial z}. \]
(5.19c)
Similarly, the curl of a vector field in the basis of the vector eigenfunctions of Eq. (5.11) is evaluated as
\[ \nabla \times E = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} \left[ -U_m(k_{\perp} \rho, \phi) \frac{\partial}{\partial z} E_{m}^{(u)}(z, k_{\perp}) + V_m(k_{\perp} \rho, \phi) \left\{ \frac{\partial}{\partial z} E_{m}^{(w)}(z, k_{\perp}) - ik_{\perp} E_{m}^{(w)}(z, k_{\perp}) \right\} + W_m(k_{\perp} \rho, \phi) ik_{\perp} E_{m}^{(v)}(z, k_{\perp}) \right], \]
(5.20)
using
\[ \hat{z} \times U_m = V_m, \quad \nabla_\perp \times U_m = 0, \]
(5.21a)
\[ \hat{z} \times V_m = -U_m, \quad \nabla_\perp \times V_m = ik_{\perp} W_m, \]
(5.21b)
\[ \hat{z} \times W_m = 0, \quad \nabla_\perp \times W_m = -ik_{\perp} V_m, \]
(5.21c)
and
\[ \nabla \times U_m = V_m \frac{\partial}{\partial z}, \]
(5.22a)
\[ \nabla \times V_m = ik_{\perp} W_m - U_m \frac{\partial}{\partial z}, \]
(5.22b)
\[ \nabla \times W_m = -ik_{\perp} V_m. \]
(5.22c)
Using the form for the material properties in Eqs. (4.26) and the decomposition of a vector field in terms of the Fourier vector eigenfunctions in Eq. (4.14),
together, in Eq. (2.2) we obtain the following decomposition of the macroscopic fields in terms of the Fourier vector eigenfunctions,

\[
D(r) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} \left[ U_{m}^{\perp}(z;\omega)E_{m}(z,k_{\perp}) + V_{m}^{\perp}(z;\omega)E_{m}(z,k_{\perp}) + W_{m}^{\perp}(z;\omega)E_{m}(z,k_{\perp}) \right],
\]

(5.23a)

\[
B(r) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp} \left[ U_{m}^{\parallel}(z;\omega)H_{m}(z,k_{\perp}) + V_{m}^{\parallel}(z;\omega)H_{m}(z,k_{\perp}) + W_{m}^{\parallel}(z;\omega)H_{m}(z,k_{\perp}) \right],
\]

(5.23b)

where we have suppressed the arguments in the vector eigenfunctions \( U_{m}(k_{\perp}\rho,\phi) \), \( V_{m}(k_{\perp}\rho,\phi) \), and \( W_{m}(k_{\perp}\rho,\phi) \), to save typographic space.

5.2. Separation of modes

We derived the curl of a vector field in Eq. (4.23) in the basis of rectangular vector eigenfunctions of Eq. (4.17), and again in Eq. (5.20) in the basis of cylindrical vector eigenfunctions of Eq. (5.12). Comparing the two expressions for the curl in Eqs. (4.23) and (5.20) it is remarkable that both have the same structure, and the differences are completely contained inside the vector eigenfunctions. The transcription to switch from rectangular to cylindrical coordinates is

\[
X^{(i)}(k_{\perp},r_{\perp}) \rightarrow X^{(i)}_{m}(k_{\perp}\rho,\phi),
\]

(5.24a)

\[
E^{(i)}(z,k_{\perp}) \rightarrow E^{(i)}_{m}(z,k_{\perp}),
\]

(5.24b)

\[
\int \frac{d^{2}k_{\perp}}{(2\pi)^{2}} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k_{\perp} dk_{\perp}.
\]

(5.24c)

This feature is not merely a consequence of the system of planar geometry being expressed in two different coordinate systems. In Sec. 6 we shall show that for spherical vector eigenfunctions the curl operator again has a similar structure. Since the Maxwell equations in the absence of free charges and currents are completely given by the curl of the fields, we conclude that a decomposition of the curl of a vector field in the particular form of Eq. (4.23), in a particular basis of vector eigenfunctions, guarantees the separation of electromagnetic modes into transverse electric and magnetic modes.

Thus, all the developments leading to the separation of modes in Eqs. (4.29) to (4.33), and the construction of the Green’s dyadic in Eqs. (4.34) to (4.37b), goes through identically, with the transcription in Eq. (5.24). We observe that because the Green functions \( g^{E,H}(z,z';k_{\perp},\omega) \) and the associated Green dyadics \( \gamma^{ij}(z,z';k_{\perp},\omega) \) and \( \phi^{ij}(z,z';k_{\perp},\omega) \) have dependence of the form \( k_{\perp} = |k_{\perp}| \), the corresponding functions and dyadics in cylindrical coordinates have no \( m \) dependence. That is, only the \( m = 0 \) mode contributes.
5.3. Casimir-Polder interaction energy

The interaction energy between a polarizable molecule of Eq. (3.1) and a δ-function dielectric plate of Eq. (4.39), illustrated in Fig. 2, is given by, using the transcription in Eq. (5.24) in Eq. (4.42),

\[ E = - \int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} k_\perp dk_\perp \gamma^{ji}(h, a; k_\perp, \omega) \chi^{ji'}(\omega) \gamma^{j'i}(a, h; k_\perp, \omega) \alpha^{ji}(\omega), \]

(5.25)

where the dependence in \( m \) is completely inside \( \alpha^{ji} \), given in terms of the cylindrical vector eigenfunctions,

\[ \alpha^{ji}(\omega) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} X_m^{(j)*}(k_\perp \rho_0, \phi_0) \cdot \alpha(\omega) \cdot X_m^{(i)}(k_\perp \rho_0, \phi_0) \]

\[ = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left[ U_m^{*} \cdot \alpha(\omega) \cdot U_m \quad V_m^{*} \cdot \alpha(\omega) \cdot V_m \quad W_m^{*} \cdot \alpha(\omega) \cdot W_m \right], \]

(5.26)

where the arguments of \( U_m(k_\perp \rho_0, \phi_0), V_m(k_\perp \rho_0, \phi_0), \) and \( W_m(k_\perp \rho_0, \phi_0) \) has been suppressed.

We can use the addition theorem for Bessel functions,

\[ \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_m(k_\perp \rho) J_m(k_\perp \rho') = J_0(k_\perp P), \]

(5.27)

where \( P = |\mathbf{r}_\perp - \mathbf{r}'_\perp| \), to derive the following addition theorems for the vector eigenfunctions,

\[ \frac{1}{2\pi} = \sum_{m=-\infty}^{\infty} U_m^{*}(k_\perp \rho, \phi) \cdot U_m(k_\perp \rho, \phi) \]

\[ = \sum_{m=-\infty}^{\infty} V_m^{*}(k_\perp \rho, \phi) \cdot V_m(k_\perp \rho, \phi) \]

\[ = \sum_{m=-\infty}^{\infty} W_m^{*}(k_\perp \rho, \phi) \cdot W_m(k_\perp \rho, \phi). \]

(5.28)

For an isotropically polarizable molecule we have \( \alpha(\omega) = \mathbf{1} \alpha(\omega) \). For this scenario, using the addition theorems in Eq. (5.28), and the orthogonality feature of the vector eigenfunctions when they are evaluated at the same position, based on Eq. (4.6), we have

\[ \alpha^{ji}(\omega) = \delta^{ji} \frac{\alpha(\omega)}{2\pi}, \]

(5.29)

which leads to the Casimir-Polder result [8] for isotropically polarizable molecule,

\[ E = -\frac{3\alpha(0)}{8\pi h^4}, \]

(5.30)
a corollary of Eq. (4.58).
Without any loss of generality we can choose the position of the polarizable molecule to be on the \( z \) axis, such that we have \( \rho = 0 \) and \( \phi = 0 \). For this case, in terms of a unit vector \( \hat{t} \) evaluated on the axis,

\[
\hat{t} = \frac{1}{\sqrt{2}} (\hat{\rho} + i \hat{\phi}), \tag{5.31}
\]

we have

\[
U_m(0, 0) = -\frac{i}{\sqrt{4\pi}} \left[ \delta_{m,1} \hat{t} - \delta_{m,-1} \hat{t}^* \right], \tag{5.32a}
\]

\[
V_m(0, 0) = -\frac{1}{\sqrt{4\pi}} \left[ \delta_{m,1} \hat{t} + \delta_{m,-1} \hat{t}^* \right], \tag{5.32b}
\]

and

\[
W_m(0, 0) = \frac{\delta_{m0}}{\sqrt{2\pi}} \hat{z}, \tag{5.33}
\]

using

\[
J'_m(0) = \frac{1}{2} \left[ \delta_{m,1} - \delta_{m,-1} \right], \tag{5.34a}
\]

\[
\lim_{x \to 0} \frac{mJ_m(x)}{x} = \frac{1}{2} \left[ \delta_{m,1} + \delta_{m,-1} \right]. \tag{5.34b}
\]

Using Eqs. (5.32) we can evaluate

\[
\alpha^{ji}(\omega) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} X^{(j)}(0, 0) \cdot \alpha(\omega) \cdot X^{(i)}(0, 0) \]
\[
= \frac{1}{2\pi} \left[ \frac{1}{2} \text{tr} \{ \alpha(\omega) \cdot \mathbf{1}_0 \} \begin{array}{cc} 0 & 0 \\ 0 & \frac{1}{2} \text{tr} \{ \alpha(\omega) \cdot \mathbf{1}_0 \} \end{array} \right]. \tag{5.35}\]

Using Eq. (5.35) in Eq. (5.25) we reproduce the result for the energy in Eq. (4.57) between a polarizable molecule and \( \delta \)-function plate. All the related results associated with Eq. (4.57) follow.

Thus, we have derived the Casimir-Polder energy of Eq. (4.57) using cylindrical coordinates and the respective cylindrical vector eigenfunctions. The planar case, of course, is simple, and it is not surprising that all the \( m \) modes separate in this case. But, it has given us insight into how the \( m = 0 \) mode alone contributes to the Casimir-Polder energy.

6. Spherically symmetric systems

We next study systems with spherical symmetry. For example, here, we will consider the system described in Fig. 3. The following study closely follows our
study of systems with planar symmetry in Sec. 5. We shall notice that it is possible to transcribe from the equations for planar symmetry to the corresponding equations for spherical symmetry. But, for pedagogical purpose, at the cost of plausible repetition of the concepts, we illustrate the procedure for the the case of spherically symmetric systems.

6.1. Spherical vector eigenfunctions

We shall use the spherical polar coordinates,

\[(r, \theta, \phi), \quad 0 \leq r < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi.\] (6.1)

The gradient operator in spherical coordinates is

\[\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}.\] (6.2)

We can construct the operators

\[\nabla_\perp = \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi},\] (6.3a)

\[\hat{r} \times \nabla_\perp = -\hat{\theta} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \theta},\] (6.3b)

which together with the radial direction \(\hat{r}\) forms an orthogonal set of vectors.

An arbitrary vector field can thus be expressed in the form

\[\mathbf{E}(\mathbf{r}) = \nabla_\perp E^{(n)}(\mathbf{r}) + \hat{r} \times \nabla_\perp E^{(v)}(\mathbf{r}) + \hat{r} E^{(w)}(\mathbf{r}).\] (6.4)
The Laplacian on the surface of a sphere, transverse to \( \hat{\mathbf{r}} \), is given by
\[
\nabla^2_{\perp} = \frac{1}{r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right).
\]  
(6.5)

It satisfies the eigenvalue equation
\[
\nabla^2_{\perp} Y_{lm}(\theta, \phi) = -\frac{l(l+1)}{r^2} Y_{lm}(\theta, \phi),
\]  
(6.6)

where the spherical harmonics \( Y_{lm}(\theta, \phi) \) are defined as
\[
Y_{lm}(\theta, \phi) = e^{im\phi} \frac{1}{\sqrt{2\pi}} \Theta_{lm}(\theta),
\]  
(6.7)

where
\[
\Theta_{lm}(\theta) = \sqrt{\frac{2l+1}{2}} \sqrt{\frac{(l+m)!}{(l-m)!}} \frac{1}{(\sin \theta)^m} \left[ \frac{d}{d \cos \theta} \right]^{l-m} \frac{(\cos^2 \theta - 1)^l}{2^l l!}.
\]  
(6.8)

The spherical harmonics satisfy the completeness relation
\[
\frac{1}{\sin \theta} \delta(\theta - \theta') \delta(\phi - \phi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi')
\]  
(6.9)

and the orthonormality relation
\[
\int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta Y^*_{lm}(\theta, \phi) Y_{lm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'},
\]  
(6.10)

which together implies that any function on the surface of a sphere can be expanded in terms of the spherical harmonics. Thus, we can express the functions \( E^{(i)}(\mathbf{r}) \) appearing in Eq. (6.4) in terms of the spherical harmonics as
\[
E^{(i)}(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) E_{lm}^{(i)}(\mathbf{r}),
\]  
(6.11a)

\[
E_{lm}^{(i)}(\mathbf{r}) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta Y^*_{lm}(\theta, \phi) E^{(i)}(\mathbf{r}),
\]  
(6.11b)

where \( i = u, v, w \). Using Eqs. (6.11) in Eq. (6.4) we can express the vector field \( \mathbf{E}(\mathbf{r}) \) as
\[
\mathbf{E}(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ U_{lm}(\theta, \phi) E_{lm}^{(u)}(\mathbf{r}) + V_{lm}(\theta, \phi) E_{lm}^{(v)}(\mathbf{r}) + W_{lm}(\theta, \phi) E_{lm}^{(w)}(\mathbf{r}) \right],
\]  
(6.12)
where the spherical vector eigenfunctions are defined as

\[ U_{lm}(\theta, \phi) = \frac{1}{ik_{\perp}} \nabla_{\perp} Y_{lm}(\theta, \phi), \quad (6.13a) \]

\[ V_{lm}(\theta, \phi) = \frac{1}{ik_{\perp}} \hat{r} \times \nabla_{\perp} Y_{lm}(\theta, \phi), \quad (6.13b) \]

\[ W_{lm}(\theta, \phi) = \hat{r} Y_{lm}(\theta, \phi). \quad (6.13c) \]

The \( ik_{\perp} \)'s in the denominators of the vector eigenfunctions in Eqs. (6.13a) and (6.13b) are defined as

\[ k_{\perp}^2 = \frac{l(l+1)}{r^2}, \quad (6.14) \]

which are introduced to normalize the vector eigenfunctions. This normalization seems to involve division by zero for \( l = 0 \). But, the mode corresponding to \( l = 0 \) and \( m = 0 \) does not exist in the constructions of \( U_{lm} \) and \( V_{lm} \), because the gradient operator gives zero in their operations for this mode. Unlike the similar situations encountered in the rectangular and cylindrical vector eigenfunctions, refer discussions following Eqs. (4.15), we have to be careful about this fact for spherical vector eigenfunctions. That is, the sum over \( l \) runs from 0 to \( \infty \) for the terms involving \( W_{lm} \), but \( l \) runs from 1 to \( \infty \) for terms involving \( U_{lm} \) and \( V_{lm} \). We emphasize on this by explicitly writing \( l = 0/1 \) for the initial value of index \( l \) in the summation in Eq. (6.12) and in what follows. The functions \( E_{lm}^{(u)}(z, k_{\perp}) \) and \( E_{lm}^{(v)}(z, k_{\perp}) \) in Eq. (6.12), relative to the ones in Eqs. (6.11), are suitably redefined to accommodate this factor of \( ik_{\perp} \). Then, using the notations

\[ X_{lm}^{(u)}(\theta, \phi) = U_{lm}(\theta, \phi), \quad (6.15a) \]

\[ X_{lm}^{(v)}(\theta, \phi) = V_{lm}(\theta, \phi), \quad (6.15b) \]

\[ X_{lm}^{(w)}(\theta, \phi) = W_{lm}(\theta, \phi), \quad (6.15c) \]

the vector field \( \mathbf{E}(r) \) of Eq. (6.12) can be written in the form

\[ \mathbf{E}(r) = \sum_{l=0/1}^{\infty} \sum_{m=-l}^{l} X_{lm}^{(i)}(\theta, \phi) E_{lm}^{(i)}(r), \quad (6.16) \]

where summation convention in the index \( i \) is assumed. The spherical vector eigenfunctions of Eq. (6.13) satisfy the completeness relation

\[ \frac{1}{\sin \theta} \delta(\theta - \theta') \delta(\phi - \phi') 1 = \sum_{l=0/1}^{\infty} \sum_{m=-l}^{l} X_{lm}^{(i)}(\theta, \phi) X_{lm}^{(i)}(\theta', \phi') \quad (6.17) \]

and the orthonormality relation

\[ \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta X_{lm}^{(i)}(\theta, \phi) \cdot X_{lm}^{(j)}(\theta, \phi) = \delta^{ij} \delta_{l'l'} \delta_{m'm'} \quad (6.18) \]
where the unit dyadic appearing in Eq. (6.17) is
\[ 1 = \nabla r = \hat{r} \hat{r} + \hat{\theta} \hat{\theta} + \hat{\phi} \hat{\phi}. \] (6.19)

The divergence of a vector field in the basis of spherical vector eigenfunctions of Eq. (6.13) is evaluated as
\[ \nabla \cdot E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) \left[ i k_{\bot} E_{lm}^{(u)}(r) + \left( \frac{2}{r} + \frac{\partial}{\partial r} \right) E_{lm}^{(w)}(r) \right], \] (6.20)
which uses the identities,
\[ \hat{r} \cdot U_{lm} = 0, \quad \nabla_{\bot} \cdot U_{lm} = i k_{\bot} Y_{lm}(\theta, \phi), \] (6.21a)
\[ \hat{r} \cdot V_{lm} = 0, \quad \nabla_{\bot} \cdot V_{lm} = 0, \] (6.21b)
\[ \hat{r} \cdot W_{lm} = Y_{lm}(\theta, \phi), \quad \nabla_{\bot} \cdot W_{lm} = 2 \frac{1}{r} Y_{lm}(\theta, \phi) \] (6.21c)
and
\[ \nabla \cdot U_{lm} = i k_{\bot} Y_{lm}(\theta, \phi), \] (6.22a)
\[ \nabla \cdot V_{lm} = 0, \] (6.22b)
\[ \nabla \cdot W_{lm} = Y_{lm}(\theta, \phi) \left( \frac{2}{r} + \frac{\partial}{\partial r} \right). \] (6.22c)

Similarly, the curl of a vector field in the basis of spherical vector eigenfunctions of Eq. (6.13) is evaluated as
\[ \nabla \times E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) \left[ - U_{lm}(\theta, \phi) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) E_{lm}^{(u)}(r) \right.
\[ + V_{lm}(\theta, \phi) \left( \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) E_{lm}^{(u)}(r) - i k_{\bot} E_{lm}^{(w)}(r) \right) \]
\[ \left. + W_{lm}(\theta, \phi) i k_{\bot} E_{lm}^{(w)}(r) \right] \] (6.23)
using
\[ \hat{r} \times U_{lm} = V_{lm}, \quad \nabla_{\bot} \times U_{lm} = V_{lm} \frac{1}{r}, \] (6.24a)
\[ \hat{r} \times V_{lm} = - U_{lm}, \quad \nabla_{\bot} \times V_{lm} = - U_{lm} \frac{1}{r} + i k_{\bot} W_{lm}, \] (6.24b)
\[ \hat{r} \times W_{lm} = 0, \quad \nabla_{\bot} \times W_{lm} = - i k_{\bot} V_{lm}, \] (6.24c)
and
\[ \nabla \times U_{lm} = V_{lm} \left( \frac{1}{r} + \frac{\partial}{\partial r} \right), \] (6.25a)
\[ \nabla \times V_{lm} = i k_{\bot} W_{lm} - U_{lm} \left( \frac{1}{r} + \frac{\partial}{\partial r} \right), \] (6.25b)
\[ \nabla \times W_{lm} = - i k_{\bot} V_{lm}. \] (6.25c)
We observe that the curl operator in the basis of spherical vector eigenfunctions, given in Eq. (6.23), can be obtained from the curl operator in the basis of planar vector eigenfunctions, given in Eq. (5.20), by the replacement
\[
\frac{\partial}{\partial z} \rightarrow \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} r
\] (6.26)
which can be used to transcribe the equations between the two geometries because the Maxwell equations of Eqs. (2.1) only involves the curl operation. This correspondence should be noted in light of the observations and discussions around Eq. (5.24). Our notion is that the the specific form and structure of the curl operation in Eqs. (4.23), (5.20), and (6.23), is a generic feature that guarantees the separation of electric and magnetic modes in the Maxwell equations.

6.2. Separation of modes

Restricting our study to spherically symmetric materials, described by the components of dyadics for dielectric permittivity and magnetic permeability being diagonal matrices,

\[
\varepsilon_{ij}(r; \omega) = \begin{bmatrix}
\varepsilon^\perp(r; \omega) & 0 & 0 \\
0 & \varepsilon^\perp(r; \omega) & 0 \\
0 & 0 & \varepsilon^{||}(r; \omega)
\end{bmatrix}
\] (6.27a)

and

\[
\mu_{ij}(r; \omega) = \begin{bmatrix}
\mu^\perp(r; \omega) & 0 & 0 \\
0 & \mu^\perp(r; \omega) & 0 \\
0 & 0 & \mu^{||}(r; \omega)
\end{bmatrix}
\] , (6.27b)
leads to the construction of the fields, using Eq. (2.2),

\[
\mathbf{D}(r) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ U_{lm}(\theta, \phi) \varepsilon^\perp(r; \omega) E_{lm}^{(u)}(r) + V_{lm}(\theta, \phi) \varepsilon^{||}(r; \omega) E_{lm}^{(w)}(r) \right]
\]

\[
+ W_{lm}(\theta, \phi) \mu^\perp(r; \omega) H_{lm}^{(u)}(r) + V_{lm}(\theta, \phi) \mu^{||}(r; \omega) H_{lm}^{(w)}(r) \right]. 
\] (6.28a)

\[
\mathbf{B}(r) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ U_{lm}(\theta, \phi) \mu^\perp(r; \omega) H_{lm}^{(u)}(r)
\]

\[
+ V_{lm}(\theta, \phi) \mu^\perp(r; \omega) H_{lm}^{(w)}(r) + W_{lm}(\theta, \phi) \mu^{||}(r; \omega) H_{lm}^{(u)}(r) \right]. 
\] (6.28b)

Using the expression for curl operator in Eq. (6.23), in conjunction with Eqs. (6.28), the projections of Maxwell’s equations in Eqs. (2.1) along the spher-
ical vector eigenfunctions, $U_{lm}$, $V_{lm}$, and $W_{lm}$, are

$$\nabla \times E = i \omega B$$

$$U_{lm} : \quad \frac{1}{r} \frac{\partial}{\partial r} r E_{lm}^{(v)} = i \omega \mu \perp H_{lm}^{(u)},$$

$$V_{lm} : \quad \frac{1}{r} \frac{\partial}{\partial r} r E_{lm}^{(w)} - i k \lambda \perp E_{lm}^{(v)} = i \omega \mu \perp H_{lm}^{(v)},$$

$$W_{lm} : \quad i k \lambda \perp E_{lm}^{(v)} = i \omega \mu \parallel H_{lm}^{(w)},$$

and

$$\nabla \times H = -i \omega (D + P)$$

$$U_{lm} : \quad \frac{1}{r} \frac{\partial}{\partial r} r H_{lm}^{(v)} = -i \omega (\varepsilon \perp E_{lm}^{(u)} + P_{lm}^{(u)}),$$

$$V_{lm} : \quad \frac{1}{r} \frac{\partial}{\partial r} r H_{lm}^{(w)} - i k \lambda \perp E_{lm}^{(v)} = -i \omega (\varepsilon \perp E_{lm}^{(v)} + P_{lm}^{(v)}),$$

$$W_{lm} : \quad i k \lambda \perp E_{lm}^{(v)} = -i \omega (\varepsilon \parallel E_{lm}^{(w)} + P_{lm}^{(w)}).$$

6.3. Boundary conditions on a $\delta$-function sphere

The boundary conditions satisfied by the fields are derived by integrating the Maxwell equations across the spherical interface, at $r = a$, to yield [9]

$$E_{lm}^{(v)}(r) \bigg|_{r=a^+}^{r=a^-} = i \omega \lambda \mu \perp H_{lm}^{(u)}(a),$$

$$E_{lm}^{(w)}(r) \bigg|_{r=a^+}^{r=a^-} = -i \omega \lambda \mu \perp H_{lm}^{(v)}(a),$$

$$D_{lm}^{(w)}(r) \bigg|_{r=a^+}^{r=a^-} = -i k \lambda \mu \perp E_{lm}^{(u)}(a),$$

and

$$H_{lm}^{(u)}(r) \bigg|_{r=a^+}^{r=a^-} = -i \omega \lambda \mu \perp E_{lm}^{(v)}(a),$$

$$H_{lm}^{(w)}(r) \bigg|_{r=a^+}^{r=a^-} = i \omega \lambda \mu \perp E_{lm}^{(u)}(a),$$

$$B_{lm}^{(w)}(r) \bigg|_{r=a^+}^{r=a^-} = -i k \lambda \mu \perp H_{lm}^{(u)}(a).$$

We also obtain the constraints

$$\lambda \parallel E_{lm}^{(w)}(a) = 0,$$

$$\lambda \parallel H_{lm}^{(w)}(a) = 0.$$
6.4. Spherical Green’s dyadic

In terms of the magnetic and electric spherical Green’s functions defined using

\[
\begin{bmatrix}
\frac{1}{r} \frac{\partial}{\partial r} \mu(r) r - \omega^2 \varepsilon(r) + \frac{l(l+1)}{r^2} \mu(r) r - \omega^2 \varepsilon(r) \\
0
\end{bmatrix}
\begin{bmatrix}
g^E(l,r,r';\omega) \\
g^H(l,r,r';\omega)
\end{bmatrix} = \frac{1}{r^2} \delta(r-r'),
\]

(6.34a)

\[
\begin{bmatrix}
\frac{1}{r} \frac{\partial}{\partial r} \varepsilon(r) r + \frac{l(l+1)}{r^2} \mu(r) r - \omega^2 \mu(r) r + \frac{l(l+1)}{r^2} \varepsilon(r) \\
0
\end{bmatrix}
\begin{bmatrix}
g^H(l,r,r';\omega) \\
g^E(l,r,r';\omega)
\end{bmatrix} = \frac{1}{r^2} \delta(r-r'),
\]

(6.34b)

the electric and magnetic fields are expressed in terms of the Green dyadic using Eqs. (2.6) and (4.35), in the basis of the spherical vector eigenfunctions, as

\[
\Gamma(r,r';\omega) = \sum_{l=0/1}^{\infty} \sum_{m=-l}^{l} X^{(i)}_{lm}(\theta,\phi) \gamma^{(j)}_{lm}(r,r';\omega) X^{(j)*}_{lm}(\theta',\phi'),
\]

(6.35a)

\[
\Phi(r,r';\omega) = \sum_{l=0/1}^{\infty} \sum_{m=-l}^{l} X^{(i)}_{lm}(\theta,\phi) \phi^{(j)}_{lm}(r,r';\omega) X^{(j)*}_{lm}(\theta',\phi'),
\]

(6.35b)

where the matrices \(\gamma^{(j)}_{lm}(r,r';\omega)\) and \(\phi^{(j)}_{lm}(r,r';\omega)\) represent the components of the dyadics in the basis of spherical vector eigenfunctions given by

\[
\gamma^{(j)}_{lm}(r,r';\omega) = \begin{bmatrix}
\varepsilon^{(r)}(r) \frac{1}{r} \frac{\partial}{\partial r} \varepsilon^{(r)}(r) \frac{1}{r} \frac{\partial}{\partial r} r' g^H l \omega^2 g^E l & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
g^E(l,r,r';\omega) \\
g^H(l,r,r';\omega)
\end{bmatrix},
\]

(6.36a)

\[
\phi^{(j)}_{lm}(r,r';\omega) = i\omega \begin{bmatrix}
0 & \frac{\mu^{(r)}(r)}{\varepsilon^{(r)}(r)} \frac{1}{r} \frac{\partial}{\partial r} r' g^H l \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
g^H(l,r,r';\omega) \\
g^E(l,r,r';\omega)
\end{bmatrix},
\]

(6.36b)

with \(k_\perp\) defined using Eq. (6.14) and

\[
k_\perp^2 = \frac{l(l+1)}{r^2}.
\]

(6.37)

We have omitted a term containing a \(\delta\)-function in Eq. (6.36a), which does not contribute to interaction energies between disjoint bodies. The explicit form of this term is

\[
-\frac{1}{r^2} \delta(r-r') \begin{bmatrix}
\varepsilon^{(r)}(r) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{\varepsilon^{(r)}(r)}
\end{bmatrix},
\]

(6.38)

which is presented here for completeness.
6.5. Magnetic spherical Green's function

After a Euclidean rotation the magnetic spherical Green’s function \( g^H_1(r, r'; i\zeta) \) of Eq. (6.34b) satisfies the differential equation

\[
\left[ -\frac{1}{r} \frac{\partial}{\partial r} r \frac{1}{\epsilon_s(r)} \frac{\partial}{\partial r} r \frac{l(l+1)}{r^2 \epsilon_s(r)} + \epsilon^2 \mu^\perp(r) \right] g^H_1(r, r'; i\zeta) = \frac{\delta(r - r')}{r^2}. \tag{6.39}
\]

The continuity conditions on the spherical radial magnetic Green’s functions are obtained by translating the boundary conditions on the fields given by Eqs. (6.31) and (6.32) into the components of the Green’s dyadics in Eqs. (2.6) and (4.35), which are given in terms of the magnetic and electric spherical Green’s functions using Eq. (6.35). For two dielectric mediums separated by a spherical \( \delta \)-function sphere, described by

\[
\begin{align*}
\epsilon^\perp_\perp(r) &= 1 + (\epsilon^\perp_\perp - 1) \theta(a - r) + (\epsilon^\perp_\perp - 1) \theta(r - a) + \lambda_e^\perp \delta(r - a), \tag{6.40a} \\
\mu^\perp_\perp(r) &= 1 + (\mu^\perp_\perp - 1) \theta(a - r) + (\mu^\perp_\perp - 1) \theta(r - a) + \lambda_g^\perp \delta(r - a), \tag{6.40b}
\end{align*}
\]

illustrated in Fig. 3, the continuity conditions for the spherical radial magnetic Green’s functions are derived to be

\[
\frac{1}{\epsilon_s(r)} \frac{\partial}{\partial r} r g^H_1(r, r'; i\zeta) \bigg|_{r=a}^{r=a+} = \zeta^2 \lambda_g^\perp g^H_1(r, r'; i\zeta) \bigg|_{r=a-}^{r=a+}, \tag{6.41a}
\]

\[
\frac{1}{\epsilon_s(r)} \frac{\partial}{\partial r} r g^H_1(r, r'; i\zeta) \bigg|_{r=a}^{r=a-} = \lambda_e^\perp \frac{1}{\epsilon^\perp_\perp(r)} \frac{\partial}{\partial r} r g^H_1(r, r'; i\zeta) \bigg|_{r=a}^{r=a-}. \tag{6.41b}
\]

The solution to the spherical radial magnetic Green’s function in Eq. (6.39) satisfying the boundary conditions in Eqs. (6.41) is

\[
g^H_1(r, r'; i\zeta) =
\begin{cases}
2 \zeta < \epsilon_\perp \left[ i_{l^<_<}(\bar{\zeta} < r) k_{l^<_<}(\bar{\zeta} < r') + \sigma_{l^<_<}^{\text{scatt}, H} i_{l^<_<}(\bar{\zeta} < r) i_{l^<_<}(\bar{\zeta} < r') \right], & r, r' < a, \\
2 \zeta < \epsilon_\perp \left[ i_{l^<_>}(\bar{\zeta} > r) k_{l^<_>}(\bar{\zeta} > r') + \sigma_{l^<_>}^{\text{scatt}, H} k_{l^<_>}(\bar{\zeta} > r) k_{l^<_>}(\bar{\zeta} > r') \right], & a < r, r', \\
2 \zeta < \epsilon_\perp \sigma_{l^<_<}^{\text{abs}, H} i_{l^<_<}(\bar{\zeta} < r) k_{l^<_<}(\bar{\zeta} < r'), & r < a < r', \\
2 \zeta < \epsilon_\perp \sigma_{l^<_<}^{\text{abs}, H} k_{l^<_>}(\bar{\zeta} > r) i_{l^<_<}(\bar{\zeta} < r'), & r' < a < r.
\end{cases} \tag{6.42}
\]
where the scattering and absorption coefficients are

\[
\sigma_{1,<>}^{\text{scatt},H} = -\frac{\lambda_\xi (\lambda_\xi k_{<} + k_{<})(k_{>} - \lambda_\xi^2 k_{<}) + \lambda_\xi (k_{<} + \lambda_\xi^2 k_{<})(\lambda_\xi k_{<} - k_{>})}{\lambda_\xi (\lambda_\xi i_{<} + i_{<})(k_{>} - \lambda_\xi^2 k_{<}) + \lambda_\xi (i_{<} + \lambda_\xi^2 i_{<})(\lambda_\xi k_{<} - k_{>})},
\]

\[(6.43a)\]

\[
\sigma_{1,<>}^{\text{scatt},H} = -\frac{\lambda_\xi (i_{<} + \lambda_\xi^2 i_{<})(\lambda_\xi k_{<} - k_{<}) - \lambda_\xi (\lambda_\xi i_{<} + i_{<})(\lambda_\xi k_{<} - k_{<})}{\lambda_\xi (\lambda_\xi i_{<} + i_{<})(k_{<} - \lambda_\xi^2 k_{<}) + \lambda_\xi (i_{<} + \lambda_\xi^2 i_{<})(\lambda_\xi k_{<} - k_{<})},
\]

\[(6.43b)\]

\[
\sigma_{1,<>}^{\text{abs},H} = -\frac{\lambda_\xi (\lambda_\xi i_{<} + i_{<})(\lambda_\xi k_{<} - k_{<}) - \lambda_\xi (\lambda_\xi i_{<} + i_{<})(\lambda_\xi k_{<} + k_{<})}{\lambda_\xi (\lambda_\xi i_{<} + i_{<})(k_{<} - \lambda_\xi^2 k_{<}) + \lambda_\xi (i_{<} + \lambda_\xi^2 i_{<})(\lambda_\xi k_{<} - k_{<})},
\]

\[(6.43c)\]

\[
\sigma_{1,<>}^{\text{abs},H} = -\frac{\lambda_\xi (\lambda_\xi i_{<} + i_{<})(\lambda_\xi k_{<} + k_{<}) - \lambda_\xi (\lambda_\xi i_{<} + i_{<})(\lambda_\xi k_{<} - k_{<})}{\lambda_\xi (\lambda_\xi i_{<} + i_{<})(k_{<} - \lambda_\xi^2 k_{<}) + \lambda_\xi (i_{<} + \lambda_\xi^2 i_{<})(\lambda_\xi k_{<} - k_{<})},
\]

\[(6.43d)\]

We used the notations

\[
I_{\xi}(l_{\xi} + 1) = l(l + 1) \frac{\epsilon_{\xi}^+}{\epsilon_{\xi}^-}, \quad \tilde{c}_{\xi}^2 = \frac{\xi^2 \mu_{\xi}^+ \epsilon_{\xi}^-}{\epsilon_{\xi}^+},
\]

\[(6.44)\]

and

\[
\tilde{\lambda}_{\xi}^+ = \frac{\zeta \lambda_{\xi}^+}{2} \sqrt{\frac{\mu_{\xi}^-}{\epsilon_{\xi}^+}}, \quad \tilde{\lambda}_{\xi}^- = \frac{\zeta \lambda_{\xi}^-}{2} \sqrt{\frac{\epsilon_{\xi}^-}{\mu_{\xi}^+}},
\]

\[(6.45)\]

and the shorthand notations

\[
i_{\xi} = i_{\xi}(\zeta_{\xi} a),
\]

\[(6.46a)\]

\[
k_{\xi} = k_{\xi}(\zeta_{\xi} a),
\]

\[(6.46b)\]

where \(i_{\xi}(t)\) and \(k_{\xi}(t)\) are modified spherical Bessel functions that are related to the modified Bessel functions by the relations

\[
i_{\xi}(t) = \sqrt{\frac{\pi}{2t}} I_{l+\frac{1}{2}}(t),
\]

\[(6.47a)\]

\[
k_{\xi}(t) = \sqrt{\frac{\pi}{2t}} K_{l+\frac{1}{2}}(t).
\]

\[(6.47b)\]

In particular \(i_{\xi}(t) = i_{\xi}^{(1)}(t)\), the modified spherical Bessel function of the first kind, together with \(k_{\xi}(t)\) are suitable pair of solutions in the right half of the complex plane [10, 11]. We used bars to define the following operations on the modified spherical Bessel functions

\[
\tilde{i}_{\xi}(t) = \left(\frac{1}{t} + \frac{\partial}{\partial t}\right) i_{\xi}(t),
\]

\[(6.48a)\]

\[
\tilde{k}_{\xi}(t) = \left(\frac{1}{t} + \frac{\partial}{\partial t}\right) k_{\xi}(t).
\]

\[(6.48b)\]
In Eqs. (6.43), using the scattering interpretation, the first symbol after comma in the subscript of $\sigma$ denotes the medium of origin of the (incident) beam and the second symbol denotes the medium of the transmitted beam. Thus, for example, $\sigma_{l,>\text{scatt},H}^{\text{cast},H}$ is the TM scattering amplitude for the case when the incident beam strikes the sphere from outside of the sphere and $\sigma_{l,>\text{abs},H}^{\text{abs},H}$ is the corresponding TM absorption coefficient. The analogs for a planar interface are the reflection and transmission coefficients, respectively. Using the Wronskian for the modified spherical Bessel functions,

$$k_i^l - ik_i' = \frac{\pi}{2t^2},$$  \hspace{1cm} (6.49)

where primes denote differentiation, we have used the related Wronskian

$$k_i^l - ik_i' = \frac{\pi}{2t^2}.$$  \hspace{1cm} (6.50)

The electric Green’s function $g_l^E(r, r'; i\zeta)$ is obtained from the magnetic Green’s function $g_l^H(r, r'; i\zeta)$ by swapping $\varepsilon \leftrightarrow \mu$ and $e \leftrightarrow g$ everywhere.

6.5.1. Free Green’s dyadic in spherical polar coordinates

After observing that

$$\left( \frac{1}{r} + \frac{\partial}{\partial r} \right) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r},$$  \hspace{1cm} (6.51)

the free radial Green’s function in vacuum is obtained by setting $\varepsilon^{\perp,||}(r) = 1$ and $\mu^{\perp,||}(r) = 1$ in Eq. (6.39) to yield

$$\left[ -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l(l + 1)}{r^2} + \zeta^2 \right] g_l^{(0)}(r, r'; i\zeta) = \frac{\delta(r - r')}{r^2},$$  \hspace{1cm} (6.52)

whose solution in terms of modified spherical Bessel functions is

$$g_l^{(0)}(r, r'; i\zeta) = \frac{2\zeta}{\pi} i_1(\zeta r_<)k_l(\zeta r>).$$  \hspace{1cm} (6.53)

The free Green’s dyadics in spherical coordinates are

$$\gamma_{lm}^{ij}(0)(r, r'; \omega) =$$

$$\begin{bmatrix}
\zeta^2 i_1(\zeta r_<)k_l(\zeta r>) & 0 & i k_i^l \zeta \left\{ \frac{i_1(\zeta r)k_l(\zeta r')}{i_1(\zeta r')k_l(\zeta r)} \right\} \\
0 & -\zeta^2 i_1(\zeta r_<)k_l(\zeta r>) & 0 \\
-i k_i \zeta \left\{ \frac{i_1(\zeta r)k_l(\zeta r')}{i_1(\zeta r')k_l(\zeta r)} \right\} & 0 & k_i k_i^l \zeta \left\{ \frac{i_1(\zeta r)k_l(\zeta r')}{i_1(\zeta r')k_l(\zeta r)} \right\}
\end{bmatrix},$$  \hspace{1cm} (6.54)
and

\[
\phi_{im}^{(0)}(r, r'; \omega) = -\frac{2\zeta^2}{\pi} \begin{bmatrix}
0 & \zeta \left\{ \bar{i}_l(\zeta r)k_l(\zeta r') \right\} & 0 \\
\zeta \left\{ i_l(\zeta r')\bar{k}_l(\zeta r) \right\} & 0 & i k'_{\perp} i_l(\zeta r_<)k_l(\zeta r_>)
\end{bmatrix}, \quad (6.55)
\]

for \( \{ r < r' \} \).

### 6.5.2. Magneto-electric ball in another magneto-electric medium

\((\lambda_+^\perp = 0 \text{ and } \lambda_-^\perp = 0)\)

Coefficients in Eqs. (6.43) for this case take the form

\[
\sigma_{\text{scatt,}H}^{\text{scatt,}H} = -\frac{\pi}{2} \frac{1}{(\zeta_0 a)^2} \sqrt{\frac{\mu_+^\perp}{\varepsilon_-^\perp}} \varepsilon_\perp \rightarrow \infty \frac{\mu_\perp}{\varepsilon_-^\perp} \rightarrow \frac{\bar{k}_<}{i_<} \rightarrow -\frac{\bar{k}_<}{i_<}, \quad (6.56a)
\]

\[
\sigma_{\text{scatt,}H}^{\text{scatt,}H} = -\frac{\pi}{2} \frac{1}{(\zeta_0 a)^2} \sqrt{\frac{\mu_+^\perp}{\varepsilon_-^\perp}} \varepsilon_\perp \rightarrow \infty \frac{\mu_\perp}{\varepsilon_-^\perp} \rightarrow \frac{i_<}{k_>} \rightarrow -\frac{i_<}{k_>}, \quad (6.56b)
\]

\[
\sigma_{\text{abs,}H}^{\text{abs,}H} = -\frac{\pi}{2} \frac{1}{(\zeta_0 a)^2} \sqrt{\frac{\mu_+^\perp}{\varepsilon_-^\perp}} \varepsilon_\perp \rightarrow \infty \frac{\mu_\perp}{\varepsilon_-^\perp} \rightarrow \frac{1}{2} \frac{\bar{k}_<}{i_<} k_> = 0, \quad (6.56c)
\]

\[
\sigma_{\text{abs,}H}^{\text{abs,}H} = -\frac{\pi}{2} \frac{1}{(\zeta_0 a)^2} \sqrt{\frac{\mu_+^\perp}{\varepsilon_-^\perp}} \varepsilon_\perp \rightarrow \infty \frac{\mu_\perp}{\varepsilon_-^\perp} \rightarrow \frac{1}{2} \frac{i_<}{k_>} \bar{k}_< = 0. \quad (6.56d)
\]

where \( i = i_l(\zeta a) \) and \( k = k_l(\zeta a) \).

### 6.5.3. Semitransparent electric \( \delta \)-function sphere

\((\varepsilon_\perp^\perp = 1, \mu_\perp^\perp = 1, \text{ and } \lambda_\perp^\perp = 0)\)
Coefficients in Eqs. (6.43) for this case take the form

\[ \sigma_{l,<>}^{\text{scatt}} = \frac{\zeta \lambda_+^+ k \bar{k}}{2 \zeta \pm a^2 - \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} \frac{k}{i}, \]  
(6.57a)

\[ \sigma_{l,<>}^{\text{scatt}} = \frac{\zeta \lambda_+^+ i i}{2 \zeta \pm a^2 - \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} -\frac{i}{k}, \]  
(6.57b)

\[ \sigma_{l,<>}^{\text{abs}} = \sigma_{l,<>}^{\text{abs}} = \frac{\pi}{2 \zeta \pm a^2 + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} 0. \]  
(6.57c)

6.5.4. Semitransparent magnetic \( \delta \)-function sphere

\( (\varepsilon_{\|} = 1, \mu_{\perp} = 1, \text{and} \lambda_+^+ = 0) \)

Coefficients in Eqs. (6.43) for this case take the form

\[ \sigma_{l,<>}^{\text{scatt}} = \frac{-\zeta \lambda_+^+ k k}{2 \zeta \pm a^2 + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} -\frac{k}{i}, \]  
(6.58a)

\[ \sigma_{l,<>}^{\text{scatt}} = \frac{-\zeta \lambda_+^+ i i}{2 \zeta \pm a^2 + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} -\frac{i}{k}, \]  
(6.58b)

\[ \sigma_{l,<>}^{\text{abs}} = \sigma_{l,<>}^{\text{abs}} = \frac{\pi}{2 \zeta \pm a^2 + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} 0. \]  
(6.58c)

6.5.5. Semitransparent magneto-electric \( \delta \)-function sphere

\( (\varepsilon_{\|} = 1 \text{ and} \mu_{\perp} = 1) \)

Coefficients in Eqs. (6.43) for this case take the form

\[ \sigma_{l,<>}^{\text{scatt}} = \frac{\left[ \zeta \lambda_+^+ k k - \zeta \lambda_+^+ k k \right]}{(1 + \frac{1}{4} \zeta^2 \lambda_+^+ \lambda_+^+) \frac{\pi}{2 \zeta \pm a^2} - \zeta \lambda_+^+ \pm i k + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} 0, \]  
(6.59a)

\[ \sigma_{l,<>}^{\text{scatt}} = \frac{\left[ \zeta \lambda_+^+ i i - \zeta \lambda_+^+ i i \right]}{(1 + \frac{1}{4} \zeta^2 \lambda_+^+ \lambda_+^+) \frac{\pi}{2 \zeta \pm a^2} - \zeta \lambda_+^+ \pm i k + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} 0, \]  
(6.59b)

\[ \sigma_{l,<>}^{\text{abs}} = \frac{(1 - \frac{1}{4} \zeta^2 \lambda_+^+ \lambda_+^+) \frac{\pi}{2 \zeta \pm a^2} - \zeta \lambda_+^+ \pm i k + \zeta \lambda_+^+ \pm i k} \xrightarrow{\lambda_+^+ \to \infty} -1, \]  
(6.59c)

\[ \sigma_{l,<>}^{\text{abs}} = \sigma_{l,<>}^{\text{abs}}. \]  
(6.59d)
6.6. Casimir-Polder interaction energy

Our framework has been kept very general up till now. It is possible, in our set-up, to consider the system illustrated in Fig. 3 of a δ-function sphere sandwiched between two mediums. But, to limit the extent of our discussion here, we limit ourselves to an electric δ-function sphere in vacuum, described by

\[
\varepsilon(r; \omega) - 1 = \mathbf{1}_\perp \lambda(\omega) \delta(r - a), \quad \mathbf{1}_\perp = \mathbf{1} - \mathbf{r} \mathbf{r}
\]  

(6.60)

and \( \lambda(\omega) = 0 \). In particular, we shall proceed to evaluate the interaction energy between a polarizable molecule described by Eqs. (3.1) and (3.2) and an electric δ-function sphere of Eq. (6.60). The three dyadics in Eq. (3.3), \( \Gamma, \chi, \) and \( \Gamma_0 \), presented as a single dyadic in Eq. (3.4), now has spherical symmetry and thus is isotropic. But, the polarizable molecule is still completely anisotropic. A spherically symmetric dyadic can be expressed in terms of the spherical vector eigenfunctions as per the decomposition of the Green dyadic in Eq. (6.36a). Using this feature the Casimir-Polder energy between an anisotropically polarizable molecule positioned a distance \( h \) from the center of a spherically symmetric electric δ-function sphere of radius \( a \) is given by the expression

\[
E = -a^2 \int_{-\infty}^{\infty} d\zeta \sum_{l=0/1} \gamma^{ij'}_l(h, a; \omega) \lambda^{j'j}(\omega) \gamma^{j'j}_l(0)(a, h; \omega) \alpha^{ji}_l(\omega),
\]  

(6.61)

where the Green’s dyadic matrices are given in terms of the free Green’s function in Eq. (6.53) as

\[
\gamma^{j'j}_l(0)(a, h; \omega) = \frac{2\zeta^3}{\pi} \left[ \begin{array}{ccc}
\tilde{i}_l(\zeta a) \tilde{k}_l(\zeta h) & 0 & \frac{ik'_l}{\zeta} \tilde{i}_l(\zeta a) k_l(\zeta h) \\
0 & -\tilde{i}_l(\zeta a) k_l(\zeta h) & 0 \\
-\frac{ik'_l}{\zeta} \tilde{i}_l(\zeta a) \tilde{k}_l(\zeta h) & 0 & \frac{k'_l k'_l}{\zeta^2} \tilde{i}_l(\zeta a) k_l(\zeta h)
\end{array} \right],
\]  

(6.62)

where we used the notations, overriding the notation in Eq. (6.14),

\[
k^2_l = \frac{l(l+1)}{a^2}, \quad k^2'_l = \frac{l(l+1)}{h^2}.
\]  

(6.63)

The implementation of the initial value for the summation on index \( l \) in Eq. (6.61) will be explicitly clarified after Eq. (6.72). The Green functions for a δ-function sphere, \( g^{H,E}_{l,h}(h, a) \), requires the evaluation of the Green function on the surface of the δ-function sphere, which is deduced as the average of the limiting values of the Green functions while approaching the surface from the inside and outside, and is determined to be

\[
g^{H,E}_{l,h}(h, a) = \frac{\zeta}{\pi} \begin{cases} 
1 + \sigma_{l,h < a}^{\text{abs},H,E} & i_l(\zeta h) k_l(\zeta a) + \sigma_{l,h < a}^{\text{scatt},H,E} i_l(\zeta a) k_l(\zeta h), \quad h < a, \\
1 + \sigma_{l,h > a}^{\text{abs},H,E} & i_l(\zeta h) k_l(\zeta a) + \sigma_{l,h > a}^{\text{scatt},H,E} k_l(\zeta a) k_l(\zeta h), \quad h > a,
\end{cases}
\]  

(6.64)
where we have rewritten the Green’s functions here after releasing most of the compact notation used earlier in Sec. 6.5 to incorporate the generality in the solutions there. The scattering and absorption coefficients that appeared in Sec. 6.5 will be presented in this notation subsequently for \( h > a \) and \( h < a \). We further introduce the notation

\[
\gamma_{ii'}(h, a; \omega) = \zeta^2 \begin{bmatrix}
g_H^{H}(\bar{h}, \bar{a}) & 0 & \frac{i k_i'}{\zeta} g_E^{H}(h, a) \\
0 & -g_E^{E}(h, a) & 0 \\
-\frac{i k_i}{\zeta} g_H^{H}(h, a) & 0 & \frac{k_{i'} k'}{\zeta^2} g_H^{H}(h, a)
\end{bmatrix},
\tag{6.65}
\]

and the corresponding notation for the primed coordinate, which are obtained from the solutions in Eq. (6.64) by replacing the modified Bessel functions containing the variable with the bar by the respective barred modified Bessel functions of Eqs. (6.48). In terms of the Green functions and the corresponding derivative operations in Eq. (6.65) the Green dyadic matrix for the Green dyadic in the presence of \( \delta \)-function sphere is given by

\[
\alpha_{ij}^{(0, 0)}(\omega) = \sum_{m=-l}^{l} X_{lm}^{(j)}(0, 0) \cdot \alpha(\omega) \cdot X_{lm}^{(i)}(0, 0),
\tag{6.67}
\]

where the matrix structure is similar to the cylindrical counterpart in Eq. (5.26) with the vector eigenfunctions now being \( U_{lm}(0, 0), V_{lm}(0, 0), \) and \( W_{lm}(0, 0) \). In terms of the unit vector

\[
\mathbf{t} = \frac{1}{\sqrt{2}} (\theta + i \phi)
\tag{6.68}
\]
evaluated on the axis we have

\[
U_{lm}(0, 0) = i \sqrt{\frac{2l + 1}{8\pi}} \left[ \delta_{m,1} \mathbf{t} - \delta_{m,-1} \mathbf{t}^* \right],
\tag{6.69a}
\]

\[
V_{lm}(0, 0) = \sqrt{\frac{2l + 1}{8\pi}} \left[ \delta_{m,1} \mathbf{t} + \delta_{m,-1} \mathbf{t}^* \right],
\tag{6.69b}
\]

38
and
\[ W_{lm}(0,0) = \hat{z} \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}, \]  
(6.70)

using
\[ Y_{lm}(0,0) = \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}. \]  
(6.71)

Using Eqs. (6.69) we can evaluate,
\[ \alpha_{ji}^l(\omega) = \sum_{m=-l}^{l} X_{lm}^{(i)}(0,0) \cdot \alpha(\omega) \cdot X_{lm}^{(j)}(0,0) \]
\[ = \frac{2l+1}{4\pi} \left[ \begin{array}{ccc} \frac{1}{2} \text{tr} \alpha_\perp(\omega) & 0 & 0 \\ 0 & \frac{1}{2} \text{tr} \alpha_\perp(\omega) & 0 \\ 0 & 0 & \hat{z} \cdot \alpha(\omega) \cdot \hat{z} \end{array} \right], \]  
(6.72)

where \( \text{tr} \alpha_\perp(\omega) = \text{tr} \alpha(\omega) \cdot \mathbf{1}_\perp \), with \( \mathbf{1}_\perp = 1 - \hat{r} \hat{r} \) is evaluated on the axis to yield \( \mathbf{1}_\perp = 1 - \hat{z}\hat{z} \). Thus, in Eqs. (6.66), (4.44), (6.62), and (6.72), we have the explicit forms for the matrices \( \gamma_{ii}'l, \lambda_{ij}'j' \), \( \gamma_{jj}(0)l \), and \( \alpha_{ji} \), respectively, required for the evaluation of the Casimir-Polder energy in Eq. (6.61). The summation over index \( l \) in Eq. (6.61) start from \( l = 1 \) for terms containing \( \text{tr} \alpha_\perp(\omega) \) because its evaluation involves \( U_{lm} \) and \( V_{lm} \), while it starts from \( l = 0 \) for the terms involving \( \hat{z} \cdot \alpha(\omega) \cdot \hat{z} \).

We shall now proceed to evaluate the energy for the cases when the polarizable molecule is either outside \( (h > a) \) or inside \( (h < a) \) the electric \( \delta \)-function sphere.

6.7. Polarizable molecule outside a \( \delta \)-function sphere: \( (h > a) \)

The scattering coefficients for the case when the polarizable molecule is outside the \( \delta \)-function sphere, \( h > a \), as illustrated in Fig. 4, are

\[ \sigma_s^H \equiv \sigma_{scatt,H} = -\frac{\zeta \lambda \lambda_i(\zeta a) \bar{T}_l(\zeta a) \lambda \rightarrow \infty}{\pi \frac{\pi}{2k^2a^2} - \zeta \lambda \lambda_i(\zeta a) \bar{k}_l(\zeta a)} \rightarrow \frac{-\bar{i}_i(\zeta a)}{k_l(\zeta a)}, \]  
(6.73a)

\[ \sigma_s^E \equiv \sigma_{scatt,E} = -\frac{\zeta \lambda \lambda_i(\zeta a) \bar{T}_l(\zeta a) \lambda \rightarrow \infty}{\pi \frac{\pi}{2k^2a^2} + \zeta \lambda \lambda_i(\zeta a) k_l(\zeta a)} \rightarrow \frac{-\bar{i}_i(\zeta a)}{k_l(\zeta a)}, \]  
(6.73b)

and the absorption coefficients, which are independent of whether the molecule is inside or outside, are

\[ \sigma_{l}^{abs,H} = -\frac{\pi \frac{\pi}{2k^2a^2}}{\pi \frac{\pi}{2k^2a^2} - \zeta \lambda \lambda_i(\zeta a) \bar{k}_l(\zeta a)} \lambda \rightarrow \infty \rightarrow 0, \]  
(6.74a)

\[ \sigma_{l}^{abs,E} = -\frac{\pi \frac{\pi}{2k^2a^2} + \zeta \lambda \lambda_i(\zeta a) k_l(\zeta a)}{\pi \frac{\pi}{2k^2a^2} + \zeta \lambda \lambda_i(\zeta a) k_l(\zeta a)} \lambda \rightarrow \infty \rightarrow 0. \]  
(6.74b)
The evaluation of the Casimir-Polder energy in Eq. (6.61) involves the products

\begin{align*}
\lambda \ g_l^H(h, a) \bar{u}(\zeta a) \bar{k}_l(\zeta h) &= \frac{\sigma_s^H}{\zeta^2 a^2} \bar{k}_l^2(\zeta h), \quad (6.75a) \\
\lambda \ g_l^E(h, a) \bar{u}(\zeta a) k_l(\zeta h) &= -\frac{\sigma_s^E}{\zeta^2 a^2} \bar{k}_l^2(\zeta h), \quad (6.75b) \\
\lambda \ g_l^H(h, \bar{a}) \bar{u}(\zeta a) k_l(\zeta h) &= \frac{\sigma_s^H}{\zeta^2 a^2} k_l^2(\zeta h), \quad (6.75c)
\end{align*}

in terms of which the Casimir-Polder interaction energy of Eq. (6.61) between an anisotropically polarizable molecule outside an electric \( \delta \)-function sphere is given by the expression

\begin{align*}
E &= -\int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \zeta^{-1} \sum_{l=0/1} \frac{(2l+1)}{4\pi} \left[ \left( \sigma_s^H k_l^2(\zeta h) - \sigma_s^E k_l^2(\zeta h) \right) \text{tr} \alpha(\omega) \\
+ \left( 2 \frac{l(l+1)}{\zeta^2 ah} \sigma_s^H k_l^2(\zeta h) - \sigma_s^H k_l^2(\zeta h) + \sigma_s^E k_l^2(\zeta h) \right) \bar{z} \cdot \alpha(\omega) \cdot \bar{z} \right], \quad (6.76)
\end{align*}

where we have used

\begin{align*}
\text{tr} \alpha(\omega) &= \text{tr} \alpha_{\perp}(\omega) + \bar{z} \cdot \alpha(\omega) \cdot \bar{z} \quad (6.77)
\end{align*}

to bring out the correspondence between energy in Eq. (6.76) for a molecule interacting with a sphere and the energy in Eq. (4.57) for a molecule interacting with a plane.
In the perfect conductor limit the energy in Eq. (6.76) takes the form, expressed here in terms of \( \text{tr} \, \alpha_{\perp}(\omega) \),
\[
E = - \int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \zeta^3 \sum_{l=0/1}^{\infty} \frac{(2l+1)}{4\pi} \left[ A_l(\zeta a, \zeta h) \text{tr} \, \alpha_{\perp}(\omega) + B_l(\zeta a, \zeta h) \hat{z} \cdot \alpha(\omega) \cdot \hat{z} \right],
\]  
(6.78)
where
\[
A_l(x, y) = -\frac{i}{k_l(x)} k_l^2(y) + i \frac{y}{k_l(x)} k_l^2(y), \quad (6.79a)
\]
\[
B_l(x, y) = -2l(l+1) \frac{1}{xy} k_l^2(y). \quad (6.79b)
\]

Further, in the Casimir-Polder limit, where only the static polarizabilities are relevant, in addition to the perfect conductor limit, the energy can be expressed in the form
\[
E = - \frac{\text{tr} \, \alpha_{\perp}(0)}{8\pi(h-a)^4} f \left( \frac{h}{a} \right) - \frac{\hat{z} \cdot \alpha(0) \cdot \hat{z}}{8\pi(h-a)^4} g \left( \frac{h}{a} \right),
\]  
(6.80)

where
\[
f \left( \frac{h}{a} \right) = \frac{4}{\pi} \int_0^{\infty} y^3 dy \sum_{l=1}^{\infty} (2l+1) A_l \left( \frac{ay}{h-a}, \frac{hy}{h-a} \right) \frac{h \to a}{h \to a} \to 1, \quad (6.81a)
\]
\[
g \left( \frac{h}{a} \right) = \frac{4}{\pi} \int_0^{\infty} y^3 dy \sum_{l=0}^{\infty} (2l+1) B_l \left( \frac{ay}{h-a}, \frac{hy}{h-a} \right) \frac{h \to a}{h \to a} \to 0. \quad (6.81b)
\]

We remind that the index \( l \) begins from \( l = 1 \) for \( f(h/a) \), because it is built out of vector spherical eigenfunctions \( U_{lm} \) and \( V_{lm} \), while it starts from \( l = 0 \) for \( g(h/a) \) because it is built out of \( W_{lm} \). The functions \( f(h/a) \) and \( g(h/a) \) are chosen such that the interaction energy of a sphere reduces to that of a plate in the limit \( h \to a \). Unlike in the case of the plate, for a sphere the interaction is sensitive to the orientation of the polarizabilities of the molecule even in the perfect conductor limit. We shall investigate the orientation dependence in the energy in Eq. (6.76) elsewhere.

Within the confines of the Casimir-Polder limit and the perfect conductor limit let us further narrow down our attention to the special case when the polarizable molecule is unidirectionally polarizable. In addition, to consider an axially symmetric configuration, let us place the molecule on the \( z \) axis with its direction of polarizability aligned with the \( z \) axis, such that we have
\[
\alpha(\omega) = \alpha(\omega) \hat{z} \hat{z}. \quad (6.82)
\]
For this, axially symmetric, scenario the interaction energy in Eq. (6.76) takes the form
\[
E = - \frac{\alpha(0)}{8\pi(h-a)^4} g \left( \frac{h}{a} \right). \quad (6.83)
\]
6.8. Polarizable molecule inside a δ-function sphere: \((h < a)\)

The scattering coefficients for the case when the polarizable molecule is inside the δ-function sphere, \(h > a\), illustrated Fig. 5, are

\[
\sigma_{\text{scatt}, H}^{\text{scatt}, H, h < a} = \frac{\zeta \lambda \tilde{k}_l(\zeta a)\tilde{k}_l(\zeta a)}{2 \pi a^2 - \zeta \lambda i_l(\zeta a)\tilde{i}_l(\zeta a)} \quad \lambda \rightarrow \infty \quad - \frac{i_l(\zeta a)}{\tilde{i}_l(\zeta a)}.
\]

\[
\sigma_{\text{scatt}, E}^{\text{scatt}, E, h < a} = \frac{\zeta \lambda k_l(\zeta a)k_l(\zeta a)}{2 \pi a^2 - \zeta \lambda i_l(\zeta a)k_l(\zeta a)} \quad \lambda \rightarrow \infty \quad - \frac{k_l(\zeta a)}{\tilde{k}_l(\zeta a)}.
\]

and the absorption coefficients, independent of whether the molecule is inside or outside the sphere, is given by Eqs. (6.74).

The Casimir-Polder interaction energy of Eq. (6.61) between an anisotropically polarizable molecule and an electric δ-function sphere, when the molecule is inside the sphere, is obtained from the energy for the molecule outside the sphere by swapping the modified spherical Bessel functions \(i_l \leftrightarrow k_l\) everywhere, including the barred Bessel functions, inside the expression for energy in Eq. (6.76). For the special case when the polarizable molecule is at the center of the perfectly conducting sphere, obtained for \(h = 0\), the interaction energy in the Casimir-Polder limit has the form

\[
E = -\frac{\text{tr} \alpha_\perp(0)}{8\pi a^4} f(0),
\]

where \(f(0)\) is transcribed from Eqs. (6.81) by swapping the modified spherical Bessel functions,

\[
f(0) = \frac{4}{\pi} \int_0^\infty x^3 dx \sum_{l=1}^\infty (2l + 1) \left[ \frac{k_l(x)}{i_l(x)} i_l^2(0) - \frac{k_l(x)}{i_l(x)} \tilde{i}_l^2(0) \right].
\]
Using the asymptotic form for zero argument of modified spherical Bessel functions we have
\[ f(0) = \frac{16}{3\pi} \int_0^\infty x^3 dx \frac{\bar{k}_1(x)}{\bar{i}_1(x)} \sim 8.48049. \] (6.87)

The term in the energy containing \( \hat{z} \cdot \alpha(0) \cdot \hat{z} \) does not appear in the expression of energy when \( h = 0 \) because
\[ g(0) = -\frac{8}{\pi} \int_0^\infty x^3 dx \sum_{l=0}^\infty (2l + 1) \frac{\bar{k}_l(x)}{\bar{i}_l(x)} \lim_{z \to 0} \frac{i_{2l}^2(z)}{z} = 0, \] (6.88)

using \( \lim_{z \to 0} i_{2l}^2(z)/z = 0 \), if \( l \neq 0 \), and since \( l = 0 \) term is zero explicitly.

When the molecule is at the center of a perfectly conducting sphere the energy is expected to be insensitive to the orientation of the polarizabilities of the molecule. The apparent dependence on only two of the principal polarizabilities, of the three, is misleading, because we chose the polarizable molecule to be positioned on the \( z \) axis.

7. Casimir-Polder interaction energy: Proposition

To follow up on the proposition in Sec. 3, let us consider an unidirectionally polarizable molecule described by Eqs. (3.1) and (6.82) interacting with a \( \delta \)-function spheroid described by
\[ \chi(r; \omega) = 1_\perp \frac{\lambda(\omega)}{c} \delta(\xi - \xi_2), \quad 1_\perp = \hat{\eta} \hat{\eta} + \hat{\phi} \hat{\phi}, \] (7.1)
in terms of spheroidal coordinates \([10, 11]\) \((\xi, \eta, \phi)\) and positive constant \(c\), the interaction energy for which is given by
\[ E = -\alpha(0) \lambda(0) \int_{-\infty}^\infty d\zeta \int d^2r_\perp \hat{z} \cdot \Gamma(r_0, r_\perp) \cdot 1_\perp \cdot \Gamma_0(r_\perp, r_0) \cdot \hat{z}. \] (7.2)

The Green dyadics of Eq. (7.2) have decompositions of the form similar to the ones in Eqs. (6.35), but now in terms of spheroidal vector eigenfunctions \( \chi_{lm}^{(i)}(\eta, \phi) \). Unlike for the case of a plate and a sphere, now the reduced Green’s functions are dependent on the \( m \) mode and the interaction energy is given by
\[ E = -\int_{-\infty}^\infty d\zeta \sum_{m=0}^\infty \sum_{m'=m}^\infty \alpha_{lm}^{m'}(\xi_1, \xi_2; \omega) \lambda^{m'j} (\omega) \chi_{lm}^{(0)}(\xi_2; \xi_1; \omega) \alpha_{lm}^{m'}(\omega), \] (7.3)

where \( \xi_1 \) and \( \xi_2 \) specify the position of the molecule and the surface of the spheroid, respectively. The matrix elements of the polarizability of the molecule is given in terms of the angular spheroidal vector eigenfunctions \( W_{lm}(\eta, \phi) \) as
\[ \alpha_{lm}^{m'}(\omega) = W_{lm}^* (1, 0) \cdot \hat{z} \alpha(\omega) \cdot W_{lm} (1, 0) = \delta_{m0} \alpha(\omega), \] (7.4)
where $d$ is a constant given in terms of the product of angular (oblate) spheroidal functions, the spheroidal functions of the first kind \([10, 11]\), $P_{\iota}^{m}(\eta, \zeta^{2}a^{2})$, evaluated at $\eta = 1$. This renders the energy to get non-zero contribution only from the $m = 0$ mode. And, that is not all, there is a bonus. In general, the transverse electric (TE) and transverse magnetic (TM) modes do not separate in spheroidal coordinates, but, the $m = 0$ azimuth mode does indeed separate into TE and TM modes. Our proposition is that this is a generic feature for axially symmetric systems, that is, only the $m = 0$ azimuth mode contributes to the Casimir-Polder interaction energy. The result for the interaction energy between an unidirectionally polarizable molecule and a $\delta$-function spheroid, when the molecule is placed on the axis of symmetry of an oblate spheroid such that the direction of polarizability is aligned with the axis, is given by an expression similar to Eq. (6.83) obtained by replacing modified spherical Bessel functions with radial spheroidal functions \([10, 11]\), $m = 0$,

\[
ir(\zeta r) \rightarrow S_{l}^{m}(1)(i\xi, \zeta c), \quad (7.5a)
\]

\[
k_{l}(\zeta r) \rightarrow S_{l}^{m}(2)(i\xi, \zeta c). \quad (7.5b)
\]

We shall present the details of the calculation for the spheroid in a sequel.

8. Conclusions

We have developed a formalism suitable for studying electrodynamics in the presence of a medium having the symmetry of a coordinate surface. It involves the decomposition of vector fields in the basis of vector eigenfunctions specific to the coordinate surface. The decomposition facilitates a systematic inspection of the separation of electric and magnetic fields into TE and TM modes. Even though we have not presented the details for any non-trivial coordinate surfaces here, we have put forward the proposition that the $m = 0$ azimuth mode for axially symmetric surfaces always separates into TE and TM modes. We have further argued that the Casimir-Polder interaction energy between an axially symmetric dielectric body and an unidirectionally polarizable molecule, when the molecule is positioned on the symmetry axis of the dielectric body with its polarizability in the direction of the axis, gets non-zero contribution only from the $m = 0$ azimuth mode. Together, the suggestion is that Casimir-Polder energies for axially symmetric configurations can be easily evaluated.

As an application of the methods developed here, we have derived the Casimir-Polder energy between an anisotropically polarizable molecule and an electric $\delta$-function surface, the surfaces considered being a plate and a sphere. We have also outlined a procedure that can be used to derive the energy when the coordinate surface is a spheroid. We believe, these methods can be extended for a half hyperboloid, which for spheroidal coordinate $\eta = 0$ corresponds to a plate with an aperture. We intend to develop and present these results in a follow-up paper.
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A. Axial symmetry

A rotation is unambiguously described by a vector $\omega$, that is, the angle representing the amount of rotation is contained in the magnitude of the vector, $\omega$, and the axis of rotation is represented by the direction $\hat{\omega}$ of the vector. The transformation of a position vector $r$ under an infinitesimal rotation $\delta \omega$ is given by

$$\bar{r} = r - \delta r, \quad \delta r = \delta \omega \times r,$$

(A.1)

in the passive description. In particular, a rotation about the direction $\hat{z}$ by an infinitesimal (azimuth) angle $\delta \phi$ is described by

$$\delta \omega = \hat{z} \delta \phi.$$

(A.2)

The corresponding infinitesimal transformation in $r$ is given by

$$\delta r = \delta \omega \hat{z} \times r = \hat{\phi} \rho \delta \phi,$$

(A.3)

where $\rho$ and $\phi$ are the cylindrical coordinates defined as

$$\hat{z} \times \hat{r} = \hat{\phi} \quad \text{and} \quad |\hat{z} \times \hat{r}| = \rho.$$

(A.4)

Observe that, in rectangular coordinates $\rho \hat{\phi} = x \hat{y} - y \hat{x}$.

A (scalar, vector, or tensor,) field $M(r)$ has axial (azimuthal) symmetry about the $\hat{z}$ axis if it does not change under azimuthal variations of $\delta \phi$ in Eqs. (A.2) and (A.3). That is, the response of the field, given by

$$M(r) \rightarrow M(r) + \delta M(r),$$

(A.5)

satisfies the variational statement

$$\frac{\delta M}{\delta \phi} = 0.$$

(A.6)

The transformation of a scalar field $\Phi(r)$ under infinitesimal rotation is given by [12]

$$\delta \Phi(r) = \delta \omega \cdot (r \times \nabla) \Phi,$$

(A.7)

where $\nabla$ is the gradient operator. A scalar field with axial symmetry about the $\hat{z}$ axis satisfies, using Eq. (A.2) in Eq. (A.7),

$$\frac{\delta \Phi}{\delta \phi} = \hat{z} \cdot (r \times \nabla) \Phi = \frac{\partial \Phi}{\partial \phi} = 0,$$

(A.8)
where we used Eq. (A.3) and the cylindrical coordinate representation of gradient to write the second equality. The constraint imposed by Eq. (A.8) is the statement of axial symmetry for a scalar field as per Eq. (A.6).

The transformation of a vector field \( \mathbf{E}(\mathbf{r}) \) under infinitesimal rotation is given by \(^{[12]}\)

\[
\delta \mathbf{E}(\mathbf{r}) = \delta \omega \cdot (\mathbf{r} \times \nabla) \mathbf{E} - \delta \omega \times \mathbf{E}.
\]

(A.9)

A vector field with axial symmetry about the \( \hat{z} \) axis satisfies, using Eq. (A.2) in Eq. (A.9),

\[
\frac{\delta \mathbf{E}}{\delta \phi} = \hat{z} \cdot (\mathbf{r} \times \nabla) \mathbf{E} - \hat{z} \times \mathbf{E} = 0.
\]

(A.10)

Using

\[
\hat{z} \cdot (\mathbf{r} \times \nabla) = \frac{\partial}{\partial \phi}
\]

and the completeness relation for the basis vectors in cylindrical coordinates,

\[
\hat{z} \hat{z} + \hat{\rho} \hat{\rho} + \hat{\phi} \hat{\phi} = 1,
\]

(A.11)

we can rewrite the statement of axial symmetry for vector field in Eq. (A.10) as

\[
\frac{\delta \mathbf{E}}{\delta \phi} = \frac{\partial \mathbf{E}}{\partial \phi} + (\hat{\rho} \hat{\phi} - \hat{\phi} \hat{\rho}) \cdot \mathbf{E} = 0.
\]

(A.14)

In the representation

\[
\mathbf{E}(\mathbf{r}) = \hat{\rho} E_\rho(\mathbf{r}) + \hat{\phi} E_\phi(\mathbf{r}) + \hat{z} E_z(\mathbf{r})
\]

(A.15)

the statement of Eq. (A.14) concerning axial symmetry of a vector field reads

\[
\frac{\delta \mathbf{E}}{\delta \phi} = \frac{\partial E_\rho}{\partial \phi} + \hat{\phi} \frac{\partial E_\phi}{\partial \phi} + \hat{z} \frac{\partial E_z}{\partial \phi} = 0,
\]

(A.16)

which implies that the projection of the vector field along all the basis vectors is independent of the azimuth angle \( \phi \),

\[
\frac{\partial E_\rho}{\partial \phi} = 0, \quad \frac{\partial E_\phi}{\partial \phi} = 0, \quad \frac{\partial E_z}{\partial \phi} = 0.
\]

(A.17)

The independence in \( \phi \) of the components of an axially symmetric vector field given in Eq. (A.17) can be also expressed as

\[
\frac{\partial}{\partial \phi} \left[ \hat{e}_i \cdot \mathbf{E}(\mathbf{r}) \right] = 0, \quad \hat{e}_1 = \hat{z}, \quad \hat{e}_2 = \hat{\rho}, \quad \hat{e}_3 = \hat{\phi}.
\]

(A.18)
As an example of an axially symmetric field we have \( \mathbf{E}(r) = \hat{\rho} \), corresponding to \( E_\rho = 1, E_\phi = 0, \) and \( E_z = 0 \). Another example is \( \mathbf{E}(r) = \hat{\phi} \), corresponding to \( E_\rho = 0, E_\phi = 1, \) and \( E_z = 0 \). Thus, using Eq. (A.15),

\[
\frac{\delta \hat{\rho}}{\delta \phi} = \hat{\rho}, \quad \frac{\delta \hat{\phi}}{\delta \phi} = -\hat{\rho}, \quad \frac{\delta \hat{\phi}}{\delta \phi} = 0.
\] (A.19a)

Further, \( \mathbf{E}(r) = \hat{z} \), corresponding to \( E_\rho = 0, E_\phi = 0, \) and \( E_z = 1 \), is also an axially symmetric field. The distinction between vectors and vector fields in these examples is exemplified by contrasting the above variations with the vector identities

\[
\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}, \quad \frac{\partial \hat{z}}{\partial \phi} = 0.
\] (A.20)

Similarly, the transformation of a dyadic field \( \mathbf{M}(r) \) under infinitesimal rotation is given by [12]

\[
\delta \mathbf{M}(r) = \delta \omega \cdot (r \times \nabla) \mathbf{M} - \delta \omega \times \mathbf{M} + \mathbf{M} \times \delta \omega,
\] (A.21)

and an axially symmetric dyadic field corresponds to all its components being independent of the azimuth angle \( \phi \),

\[
\frac{\partial}{\partial \phi} \left[ \hat{e}_i \cdot \mathbf{M}(r) \cdot \hat{e}_j \right] = 0,
\] (A.22)

for \( \hat{e}_i, j \)'s defined in Eq. (A.18).

A.1. Green’s dyadic

A scalar Green’s function \( G(r, r') \) depends on two positions, \( r \) and \( r' \), often referred to as position of source and observer. A scalar Green’s function, in our study here, will be related to the vacuum expectation value of the product of two scalar fields by the correspondence

\[
\frac{1}{i} G(r, r') = \langle \Phi(r) \Phi(r') \rangle.
\] (A.23)

Under an infinitesimal rotation \( \delta \omega \) the transformation of a scalar Green’s function is dictated by the product structure of fields Using Eq. (A.7) for the individual fields in the product structure we deduce that

\[
\delta G(r, r') = \delta \omega \cdot (r \times \nabla) G + \delta \omega \cdot (r' \times \nabla') G.
\] (A.24)

An axially symmetric scalar Green’s function thus satisfies, using Eqs. (A.8) and (A.24),

\[
\frac{\delta G}{\delta \phi} = \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi'} \right) G(r, r') = 0.
\] (A.25)
If the dependence in the two position \( r \) and \( r' \) in the scalar Green’s is of the form \(|r - r'|\), then, using
\[
|r - r'| = \sqrt{r^2 + r'^2 - 2rr'\{\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')\}} \tag{A.26}
\]
and
\[
\left(\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi'}\right) \cos(\phi - \phi') = 0, \tag{A.27}
\]
we deduce the axial symmetry of such scalar Green’s functions. Another way a scalar Green’s function could be axially symmetric is when the source point \( r' \) is on the symmetry axis \( \hat{z} \) of rotation, that is,
\[
r' = \hat{z} h, \tag{A.28}
\]
because then we have the response
\[
\delta \omega \times r' = \delta \phi \hat{z} \times \hat{z} h = 0. \tag{A.29}
\]
Thus, in this scenario, for axial symmetry, it is sufficient if the dependence of the scalar Green’s function is independent of \( \phi \), that is,
\[
\frac{\delta G}{\delta \phi} = \frac{\partial}{\partial \phi} G(r, r') = 0. \tag{A.30}
\]

Our study in this article will be centered around the Green’s dyadic \( \Gamma(r, r') \) which is related to the vacuum expectation value of the product of two vector fields by the correspondence
\[
\frac{1}{i} \Gamma(r, r') = \langle E(r)E(r') \rangle. \tag{A.31}
\]
The transformation of a Green’s dyadic under an infinitesimal rotation \( \delta \omega \) is given by
\[
\delta \Gamma(r, r') = \delta \omega \cdot (r \times \nabla) \Gamma + \delta \omega \cdot (r' \times \nabla') \Gamma - \delta \omega \times \Gamma + \Gamma \times \delta \omega. \tag{A.32}
\]
The Green’s dyadic for the vacuum, which is termed the free Green’s dyadic, satisfies the dyadic differential equation
\[
- \left[\frac{1}{\zeta^2} \nabla \times \nabla \times + 1\right] \cdot \Gamma_0(r, r') = 1 \delta^{(3)}(r - r'), \tag{A.33}
\]
which has a formal solution of the form
\[
\Gamma_0(r, r') = \left[\nabla \nabla - \zeta^2 1\right] G_0(|r_0 - r'|). \tag{A.34}
\]
For an infinitesimal rotation about the \( \hat{z} \) axis we have
\[
\frac{\delta \Gamma_0}{\delta \phi} = \left(\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi'}\right) \Gamma_0(r, r') - \hat{z} \times \Gamma + \Gamma \times \hat{z}
= \left[\nabla \nabla - \zeta^2 1\right] \left(\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi'}\right) G_0(|r_0 - r'|) = 0, \tag{A.35}
\]
48
using
\[
\frac{\partial}{\partial \phi} \nabla G = \hat{z} \times \nabla G + \nabla \frac{\partial}{\partial \phi} G. \tag{A.36}
\]

Thus, the free Green’s dyadic is axially symmetric.

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