I. INTRODUCTION

Scalar n-point integrals in dimensions $D > 4$ are interesting objects for a number of reasons. They appear in the $O(\epsilon)$ part of $(D = 4 - 2\epsilon)$-dimensional one-loop amplitudes [1], which are required for computations at higher-loop orders.

Quite generally, higher-dimensional scalar integrals are related to tensor integrals in $D = 4$ dimensions [2]. In particular, the $D = 6$ dimensional hexagons are related to finite tensor integrals [3] that appear in $\mathcal{N} = 4$ super Yang-Mills (SYM). More precisely, they appear as derivatives of four-dimensional two-loop tensor integrals. Moreover, applying a further differential operator, the integrals reduce to four-dimensional one-loop tensor integrals [4]. See Ref. [5] for related work on differential equations relevant for integrals in $\mathcal{N} = 4$ SYM.

Finite dual conformal invariant functions [6,7] are also prototypes of functions that can appear in the remainder function of maximally helicity-violating (MHV) amplitudes and the ratio function of non-MHV amplitudes in $\mathcal{N} = 4$ SYM [8–10]. Recently, the massless and one-mass hexagon integrals in $D = 6$ dimensions were computed in Refs. [4,11,12]. It was noted that the massless hexagon integral in $D = 6$ resembles very closely the analytical result of the two-loop remainder function for $n = 6$ points [13–15]. In this note, we extend the computations of hexagon integrals in $D = 6$ dimensions to the case of three nonadjacent external masses.

Our strategy is the following. We derive simple differential equations that relate the three-mass hexagon to known pentagon integrals. These differential equations, together with a boundary condition, completely determine the answer in principle. We find it convenient to first compute the symbol [16] of the answer, and then reconstruct the function from that symbol.

II. INTEGRAL REPRESENTATION AND DIFFERENTIAL EQUATIONS

We consider the hexagon integral with three massive corners,

$$H_0 = \int \frac{d^6x}{i\pi^3} \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{61}^2},$$

where we used dual (or region) coordinates $p_j^\mu = x_j^\mu - x_{j+1}^\mu$ (with indices being defined modulo 9), and $x_i^\mu$, $x_j^\mu$.

The on-shell conditions read $x_{12}^2 = 0$, $x_{23}^2 = 0$, and $x_{34}^2 = 0$. As a scalar integral, $H_0$ is a function of the (nonzero) external Lorentz invariants $x_{jk}^2$. We work in signature $(-++++)$, so that the Euclidean region has all (nonzero) $x_{jk}^2$ positive.

Dual conformal covariance [6,7] of $H_0$, particularly under the inversion of all dual coordinates, $x^\mu \rightarrow x^\mu / x^2$, allows us to write
where the cross ratios
\begin{align}
  u_1 &= \frac{x_2^3 x_7^4}{x_1^3 x_7^2}, & u_2 &= \frac{x_3^2 x_4^2}{x_1^3 x_4^2}, & u_3 &= \frac{x_5^2 x_6^2}{x_1^3 x_6^2},
  
  u_4 &= \frac{x_2^3 x_5^4}{x_1^3 x_5^2}, & u_5 &= \frac{x_3^2 x_7^2}{x_1^3 x_7^2}, & u_6 &= \frac{x_4^2 x_6^2}{x_1^3 x_6^2}.
\end{align}

are invariant under dual conformal transformations. Furthermore, the one-loop hexagon integral with three nonadjacent masses is invariant under the action of the dihedral symmetry group $D_3 \simeq S_3$, generated by the cyclic rotation $c$ and the reflection $r$ acting on the dual coordinates via
\begin{equation}
  x_j^\mu \to x_{j+3}^\mu \quad \text{and} \quad x_j^\mu \to x_{j-3}^\mu.
\end{equation}

where as usual all indices are understood modulo 9. It is easy to see that under the symmetry the six conformal cross ratios group into two orbits of three elements,
\begin{align}
  u_1 &\equiv u_6 c, & u_2 &\equiv u_3 c, & u_4 &\equiv u_5 c, & u_6 &\equiv u_1 c, \\
  u_1 &\equiv u_3, & u_2 &\equiv u_5, & u_6 &\equiv u_4.
\end{align}

One can easily derive a differential equation for $H_9$ by noting that
\begin{equation}
  (x_{21} \cdot \partial_{x_2} + 1) \frac{1}{x_{12}^2} = \frac{1}{(x_{21}^2)^2}.
\end{equation}

Applying this differential operator to Eq. (1), we find
\begin{equation}
  (x_{21} \cdot \partial_{x_2} + 1) H_9 = \int \frac{d^6 x_i}{i \pi^3} \frac{1}{(x_{12}^2)^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2} \equiv P_8.
\end{equation}

The one-loop pentagon integral $P_8$ appearing as an inhomogeneous term in this equation is equivalent to a known four-dimensional pentagon integral [4].
\begin{equation}
  P_8 = \frac{1}{x_{23}^2 x_{27}^2 x_{48}} \Psi_8(u_3, u_4 u_2, u_5).
\end{equation}

The latter is given by
\begin{equation}
  \Psi_8(u, v, w) = \frac{1}{1 - u - v + u v w} \log u \log v + \log L_2(1 - u) + \log L_2(1 - v) + \log L_2(1 - w) - \log L_2(1 - u v w).
\end{equation}

We can rewrite Eq. (7) as a differential equation for the rescaled hexagon integral $\Phi_9(u_1, \ldots, u_6)$ that depends on cross ratios only,
\begin{equation}
  D_1 \Phi_9(u_1, \ldots, u_6) = \Psi_8(u_3, u_4 u_2, u_5),
\end{equation}

where
\begin{equation}
  D_1 = u_1 + u_1 u_6 (u_6 - 1) \partial_6 + (u_4 - 1) \partial_4 + u_1 (u_1 - 1) \partial_1
  + u_1 (1 - u_6) u_3 \partial_3,
\end{equation}

with $\partial_i = \partial/\partial u_i$. By cyclic and reflection symmetry, we have a total of six differential equations. It turns out that only five of them are independent. The remaining freedom can be fixed, e.g., by the boundary condition $H_9(u_1, u_2, u_3, 0, 0, 0) = H_0(u_1, u_2, u_3)$, with $H_0$ given explicitly in Refs. [4,11]. (Alternatively, one could derive further differential equations, as in Ref. [4]). Therefore, the set of equations and the boundary condition completely determine $H_0$.

In the next section, we will use this set of differential equations to determine the symbol $S(\Phi_9)$, where $\Phi_9$ is obtained from $\Phi_9$ by a simple rescaling, see Eq. (16).

Then, we will reconstruct the function $\Phi_9$ (and equivalently $H_0$) from its symbol.

We note that there is a simple line integral representation of $H_9$ [4], see Fig. 1(a),
\begin{equation}
  H_9 = \int d \xi_1 d \xi_4 d \xi_7 \frac{1}{(y_1 - y_4)^2 (y_4 - y_7)^2 (y_7 - y_1)^2}.
\end{equation}

where $y_1^\mu = x_1^\mu + \xi_1 x_2^\mu$, $y_4^\mu = x_4^\mu + \xi_4 x_5^\mu$ and $y_7^\mu = x_7^\mu + \xi_7 x_8^\mu$. The pentagon integral $P_8$ can be expressed in a similar way, which allows us to write
\begin{equation}
  H_9 = \int d \xi_1 P_8(y_1(\xi_1), x_4, x_5, x_7, x_8).
\end{equation}

In this form, the differential equation (7) has the interpretation of localizing one of the line integrals, in this case $y_1(\xi_1) \to x_2$, see Fig. 1(b). It is interesting that similar integrals where certain propagators are localized at cusp
points have also appeared in computations of two-loop Wilson loops [17].

From this discussion it is also clear that the integral reduces further in degree under the action of other differential operators, until one eventually obtains a rational function. More explicitly, the operator \( \langle x_{34} \cdot \partial_{x_3} + 1 \rangle \) acting on \( P_8 \) similarly gives a first-order differential equation relating \( \Psi_8 \) to a single-log function, namely, a 3-mass box integral with two doubled propagators,

\[
X_7 = \int \frac{d^d x_i}{i \pi^2} \frac{1}{(x_{23}^2 + x_{45}^2 + x_{78}^2)} = \frac{1}{x_{23}^2 x_{45}^2 x_{78}^2} \chi_7(u_3 u_5)
\]

where \( \chi_7(y) = \log(y)/(y - 1) \). Acting further on \( X_7 \) with \( \langle x_{87} \cdot \partial_{x_8} + 1 \rangle \) gives the 3-mass triangle with three doubled propagators, which is a constant up to the usual prefactors, \( 1/(x_{23}^2 x_{45}^2 x_{78}^2) \).

The representation (12) may also be useful for numerical checks. For future reference, it can be rewritten as

\[
\Phi_9(u_1, \ldots, u_6) = \int_0^1 \frac{d \xi_1 d \xi_4 d \xi_7}{(u_2 \xi_1 \xi_4 + u_4 \xi_1 \xi_4 + \xi_4)(u_3 \xi_4 \xi_7 + u_5 \xi_3 \xi_4 \xi_7 + \xi_7)(u_1 \xi_7 \xi_1 + u_6 u_1 \xi_7 \xi_1 + \xi_1)},
\]

where \( \tilde{\xi}_i = 1 - \xi_i \).

## III. SYMBOLS FROM DIFFERENTIAL EQUATIONS

We find that the following definition

\[
\Phi_9(u_1, \ldots, u_6) \equiv \frac{1}{\sqrt{\Delta_9}} \Phi_9(u_1, \ldots, u_6).
\]

leads to a pure function \( \tilde{\Phi}_9(u_i) \), i.e., a function that can be written as a linear combination of transcendental functions, with numerical coefficients only. Here

\[
\tilde{D}_1 = \frac{1}{\sqrt{\Delta_9}} (1 - u_3 - u_2 u_4 + u_2 u_4 u_5) \times [u_1 u_6 (u_6 - 1) \partial_6 + (u_4 - 1) \partial_4 + u_1 (u_1 - 1) \partial_1 + u_1 (1 - u_6) u_5 \partial_5]
\]

and

\[
\tilde{\Psi}_8(u, v, w) \equiv (1 - u - v + uvw) \Psi_8(u, v, w).
\]

We find it convenient to convert (18) into a differential equation for the symbol of \( \tilde{\Phi}_9 \), which reads

\[
\tilde{D}_1 \tilde{S}(\tilde{\Phi}_9)(u_1, \ldots, u_6) = \tilde{S}(\tilde{\Phi}_9)(u_3, u_4 u_2, u_5).
\]

Here the differentiation of a symbol is defined by

\[
\partial_x(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n) = \partial_x \log(a_n) \times a_1 \otimes \ldots \otimes a_{n-1}.
\]

The following set of variables is useful to describe the solution,

\[
W_i \equiv \frac{g_i - \sqrt{\Delta_9}}{g_i + \sqrt{\Delta_9}}, \quad i = 1 \ldots 6,
\]

where \( g_i \equiv 1 - u_1 - u_2 + u_3 + u_1 u_2 u_4 - u_2 u_3 u_5 - 2 u_3 u_6 + u_1 u_3 u_6 + 2 u_2 u_3 u_5 u_6 - u_1 u_2 u_3 u_4 u_5 u_6 \).
Moreover, here the quantities

\[ T_i = \frac{1}{2} \log \left( \frac{u_i u_{i+1}}{u_{i+1} u_i} \right) \]

where \( T_i \) is defined as the image of \( D_i \) under the reflection \( u_4 \leftrightarrow u_6 \) and \( u_5 \leftrightarrow u_7 \). Given these variables, we can write the solution to Eq. (21) as

\[ S(\tilde{\Phi}_9)(u_1, \ldots, u_6) = -S(\tilde{\Psi}_8)(u_3, u_4 u_2, u_5) \otimes W_6 + T, \]

(25)

where \( T \) satisfies \( D_i T = 0 \). Taking into account the differential equations related to (21) by symmetry further restricts the form of \( T \). The particular solution we obtained is in general not an integrable symbol. We therefore proceed and add a particular \( T_h \) satisfying \( D_i T_h = 0 \) (for \( i = 1 \ldots 5 \)) to obtain an integrable symbol. Finally, additional terms satisfying the homogeneous equations \( D_i T = 0 \) are fixed by demanding that the symbol for \( \tilde{\Phi}_9 \) for the massless hexagon \([4,11]\) is reproduced when \( u_4 = u_5 = u_6 = 0 \).

Following this procedure, we find that the symbol \( S(\tilde{\Phi}_9) \) can then be written as

\[ S(\tilde{\Phi}_9) = \sum_{i=1}^{6} S(f_i) \otimes W_i, \]

(26)

where \( f_i \) are the following degree two functions,

\[ f_1 = \tilde{\Psi}_8(u_2, u_1 u_6, u_4) + \tilde{\Psi}_8(u_1, u_2 u_5, u_4) \]
\[ + \tilde{\Psi}_8(u_2, u_3 u_5, u_6) - F(u_1, u_2, u_3, u_4, u_5, u_6), \]
\[ f_4 = -\tilde{\Psi}_8(u_1, u_3 u_5, u_6). \]

(27)

Here the quantities \( f_2, f_3 (f_5, f_6) \) are obtained from \( f_1 (f_4) \) by cyclic mappings \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1; 4 \rightarrow 5 \rightarrow 6 \rightarrow 4 \).

Moreover, \n
\[ F = 2\tilde{\Psi}_8(u_1, u_2, u_4) + \log u_1 \log u_5 + \log u_2 \log u_6 \]
\[ - \log u_3 \log u_4. \]

(28)

Note that one can rearrange terms in Eq. (27) because of the identity,

\[ 0 = \tilde{\Psi}_8(u_3, u_2 u_4, u_5) + \tilde{\Psi}_8(u_1, u_2 u_5, u_6) \]
\[ + \tilde{\Psi}_8(u_2, u_1 u_6, u_4) - \tilde{\Psi}_8(u_3, u_1 u_4, u_6) \]
\[ - \tilde{\Psi}_8(u_1, u_2 u_6, u_4) - \tilde{\Psi}_8(u_2, u_3 u_6, u_5). \]

(29)

IV. TWISTOR GEOMETRY ASSOCIATED TO A THREE-MASS HEXAGON

The differential equation technique allowed us to obtain the symbol of the one-loop three-mass hexagon integral. If we want to find the analytic expression for the integral, we need to integrate the symbol to a function. We follow here the approach of Ref. [18], which, after making a suitable choice for the functions that should appear in the answer, allows us to reduce the problem of integrating the symbol to a problem of linear algebra. The algorithm of Ref. [18], however, requires the arguments of the symbol to be rational functions (of some parameters). From Eq. (26) it is clear that in our case this requirement is not immediately fulfilled, because the variables \( W_i \) are algebraic functions of the cross ratios \( u_i \). In order to bypass this problem, we have to parametrize the six cross ratios such that \( \Delta_0 \) becomes a perfect square.

A convenient way to find a parametrization that turns \( \Delta_0 \) into a perfect square is to write the six cross ratios as ratios of twistor brackets. Indeed, even though we work in \( D = 6 \) dimensions where the link to twistor space is not immediately obvious, we can nevertheless consider the cross ratios as being parametrized by cross ratios in twistor space \( \mathbb{CP}^3 \), because the functional dependence of \( \Phi_9 \) is only through the six conformally invariant quantities \( u_i \), which do not make reference to the six-dimensional space. In other words, we can consider the external momenta to lie in a four-dimensional subspace, even as we integrate over six components of loop momentum. Furthermore, in Ref. [15] it was noted that in terms of momentum twistor variables, the equivalent of \( \Delta_0 \) in the massless case becomes a perfect square. Hence, momentum twistors seem to provide a natural framework to search for a suitable parametrization. We therefore briefly review the geometry of a three-mass hexagon configuration in momentum twistor space.

In order to describe this geometry, we assume that the dual coordinates \( x_i \) are elements of four-dimensional Minkowski space \( \mathbb{M}^4 \). As the dependence of \( \Phi_9 \) is solely through cross ratios, we can assume that this condition is satisfied, as long as the “projection” to the four-dimensional space leaves the cross ratios invariant. The twistor correspondence then associates to each point \( x_i \) in \( \mathbb{M}^4 \) a projective line \( X_i \) in momentum twistor space, and two points \( x_i \) and \( x_j \) in \( \mathbb{M}^4 \) are lightlike separated if and only if the corresponding lines \( X_i \) and \( X_j \) intersect. In our case this implies that the six lines must intersect pairwise (see Fig. 2). Denoting the intersection points by \( Z_1, Z_4 \) and \( Z_7 \), we can define six more twistors by

\[ X_i = Z_i \wedge Z_{i-1}, \quad i \in \{1, 2, 4, 5, 7, 8\}. \]

(30)

Note that the only points in twistor space that have an intrinsic geometric meaning are \( Z_1, Z_4 \) and \( Z_7 \), whereas the other six points are defined through Eq. (30), which is left invariant by the redefinitions.
Having obtained a parametrization that makes \( \xi \) function of the twistors induce an action on the lines 
\[
\begin{align*}
Z_2 &\rightarrow Z_2 + \alpha_2 Z_1, \\
Z_5 &\rightarrow Z_5 + \alpha_3 Z_4, \\
Z_8 &\rightarrow Z_8 + \alpha_8 Z_7, \\
Z_9 &\rightarrow Z_9 + \alpha_9 Z_1, \\
Z_3 &\rightarrow Z_3 + \alpha_3 Z_4, \\
Z_6 &\rightarrow Z_6 + \alpha_6 Z_7,
\end{align*}
\] (31)
where \( \alpha_i \) are nonzero complex numbers. These shifts simply express the fact that we can move the points along the line without altering the geometric configuration. Furthermore, the intersection of two lines \( X_i \) and \( X_j \) can be expressed through the condition,
\[
\langle X_i X_j \rangle = \langle (i-1)(j-1) \rangle = Z_{i-1} Z_{j-1} Z_{ij} = 0.
\] (32)
Using the twistor brackets, the cross ratios \( u_i \) can be parametrized as
\[
\begin{align*}
u_1 &= \langle X_2 X_3 \rangle \langle X_1 X_7 \rangle \\
u_3 &= \langle X_3 X_2 \rangle \langle X_1 X_4 \rangle \\
u_5 &= \langle X_4 X_3 \rangle \langle X_1 X_8 \rangle \\
u_4 &= \langle X_5 X_8 \rangle \langle X_3 X_4 \rangle \\
u_6 &= \langle X_4 X_5 \rangle \langle X_2 X_8 \rangle, \\
u_2 &= \langle X_3 X_8 \rangle \langle X_4 X_1 \rangle.
\end{align*}
\] (33)
It is clear that the dihedral symmetry of the integral is reflected at the level of the twistors by
\[
Z_i \overset{c}{\longrightarrow} Z_{i+3} \quad \text{and} \quad Z_i \overset{r}{\longrightarrow} Z_{i-9},
\] (34)
where again all indices are understood modulo 9. This action on the twistors induces an action on the lines \( X_i \) and the planes \( Z_i = Z_{i-1} \wedge Z_i \wedge Z_{i+1} \) via
\[
\Delta_9 = \frac{((x_6 - x_8)(y_9 - y_2)(z_3 - z_5) + (x_5 - x_3)(y_8 - y_9)(z_2 - z_9))}{(x_3 - x_5)^2(y_6 - y_2)^2(z_9 - z_5)^2},
\] (40)
and Eq. (40) is manifestly invariant under the transformations (39). Having obtained a parametrization that makes \( \Delta_9 \) into a perfect square, we can write the symbol in a form in which all the entries are rational functions of the variables we just defined, and hence the symbol now takes a form which allows it to be integrated using the algorithm of Ref. [18]. Furthermore, using this parametrization it is trivial to check that the symbol of \( \Phi_9 \) obtained in the
previous section has the correct dihedral symmetry. In particular, we find that
\[ \sigma[S(\hat{\Phi}_9)] = S(\hat{\Phi}_9) \quad \text{and} \quad r[S(\hat{\Phi}_9)] = -S(\hat{\Phi}_9). \] (41)
The parametrization (38) also makes it very easy to check the various soft limits of \( H_9 \). Indeed, we have
\begin{align*}
  u_4 &\rightarrow 0 \iff z_3 \rightarrow z_2, \\
  u_5 &\rightarrow 0 \iff x_6 \rightarrow x_5, \\
  u_6 &\rightarrow 0 \iff y_9 \rightarrow y_8.
\end{align*}
(42)
We checked that in taking these limits \( S(\hat{\Phi}_9) \) reduces to the symbols for the massless and one-mass hexagon integrals [4,11,12].

V. INTEGRATING THE SYMBOL: THE ONE-LOOP THREE-MASS HEXAGON INTEGRAL

As the parametrization of the cross ratios in terms of momentum twistors introduced in the previous section turns \( \Delta_9 \) into a perfect square, we can now integrate the symbol using the algorithm of Ref. [18]. However, even though the parametrization (38) makes all the symmetries manifest, it uses a redundant set of parameters. We therefore choose a minimal set of parameters by breaking the \( S_3 \) symmetry down to its alternating subgroup \( A_3 \approx \mathbb{Z}_3 \) by fixing six of the 12 parameters,
\[ x_6 = y_9 = z_3 = 0 \quad \text{and} \quad x_3 = y_6 = z_9 = 1. \] (43)
The cross ratios then take the form
\begin{align*}
  u_1 &= \frac{z_2 - z_5}{(1 - y_2)(1 - z_5)}, \\
  u_2 &= \frac{x_5 - x_8}{(1 - x_5)(1 - z_5)}, \\
  u_3 &= \frac{y_8 - y_2}{(1 - x_8)(1 - y_2)}, \\
  u_4 &= \frac{z_2(1 - z_5)}{z_2 - z_5}, \\
  u_5 &= \frac{x_5(1 - x_8)}{x_5 - x_8}, \\
  u_6 &= \frac{y_8(1 - y_2)}{y_8 - y_2},
\end{align*}
(44)
and \( \Delta_9 \) can now be written as
\[ \Delta_9 = \frac{(y_8 z_2 + (1 - x_3)(1 - y_8)(1 - z_2))^2}{(1 - x_8)^2(1 - y_2)^2(1 - z_5)^2}. \] (45)
We note in passing that the Jacobian of the parametrization (44) is nonzero for generic values of the parameters.

In a nutshell, the algorithm of Ref. [18] proceeds in two steps:
1. Given the symbol of \( \Phi_9 \) computed in Section III, it constructs a set of rational functions \( \{ R_i(x_5, x_8, y_2, y_8, z_2, z_5) \} \) such that, e.g., symbols of the form \( S(\text{Li}_{1,g}(R_i)) \) span the vector space of which \( S(\hat{\Phi}_9) \) is an element.
2. Once a suitable set of rational functions has been obtained, it makes an ansatz.

We have implemented the algorithm of Ref. [18] into a Mathematica code, which we have applied to the function \( \Phi_9(x_5, x_8, y_2, y_8, z_2, z_5) \). The result we obtain takes a strikingly simple form,
\[ \Phi_9(u_1, \ldots, u_6) = \frac{1}{\sqrt{\Delta_9}} \sum_{i=1}^{4} \sum_{g \in S_3} \sigma(g) L_3(x_{i,g}^+, x_{i,g}^-), \] (48)
where \( \sigma(g) \) denotes the signature of the permutation (+1 for \( [1, c, c^2] \), -1 for \( [r, rc, rc^2] \)), and where we defined
\[ L_3(x^+, x^-) = \frac{1}{18} (\ell_1(x^+) - \ell_1(x^-))^3 + L_3(x^+, x^-), \] (49)
and
\[ L_3(x^+, x^-) = \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!!} \log^k(x^+ x^-)(\ell_{3-k}(x^+) - \ell_{3-k}(x^-)), \] (50)
with
\[ \ell_n(x) = \frac{1}{2} (\text{Li}_n(x) - \text{Li}_n(1/x)). \] (51)
The arguments appearing in the polylogarithms can be written in the form \( x_{i,g}^\pm \equiv g(x_i^\pm) \), for \( g \in S_3 \), with
\begin{align*}
  x_1^+ &= \chi(1, 4, 7), \\
  x_2^+ &= \chi(2, 5, 7), \\
  x_3^+ &= \chi(2, 4, 8), \\
  x_4^+ &= \chi(1, 5, 8), \\
  x_1^- &= \tilde{\chi}(1, 4, 7), \\
  x_2^- &= \tilde{\chi}(2, 5, 7), \\
  x_3^- &= \tilde{\chi}(2, 4, 8), \\
  x_4^- &= \tilde{\chi}(1, 5, 8),
\end{align*}
(52)
where we defined
\[ \chi(i, j, k) = -\langle \hat{\Phi}_9 \rangle(X_i X_k)(X_j 17), \] (53)
with \( (ij) = (i - 1)(j + 1) \). The function \( \chi \) is related to \( \tilde{\chi} \) by Poincaré duality,
\[ \tilde{\chi}(i, j, k) = -\langle \hat{\Phi}_9 \rangle(X_i X_k)(X_j 4 \cap \hat{7}), \] (54)
The function $\Phi_0$ manifestly has the cyclic symmetry. The reflection symmetry however needs some explanation, because $\Phi_0$ is odd under reflection. In twistor variables, $\Delta_0$ becomes a perfect square, and so we can remove the square root and rewrite $\sqrt{\Delta_0}$ as a rational function of twistor brackets. This procedure however introduces an ambiguity for the sign of the square root. In particular, the rational function we obtained is now odd under the reflection (34), so that $\Phi_0$ is again even.

We stress that Eq. (48) is only valid in the region where $\Delta_0 < 0$. In this region, since $\chi$ and $\bar{\chi}$ are related by Poincaré duality, the function Eq. (48) is manifestly real, and we checked numerically that Eq. (48) agrees with the parametric integral representation for $\Phi_0$ given in Eq. (15). Note that, as multiple zeta values are in the kernel of the symbol map, we could a priori add to Eq. (48) terms proportional to $\zeta_2$ without altering its symbol. The numerical agreement with the integral representation (15) however shows that such terms are absent in the present case.

VI. CONCLUSION

Using a differential equation method to determine the symbol of a function, and an algorithm to reconstruct the function from its symbol, we have computed analytically the one-loop nonadjacent three-mass hexagon integral in $D = 6$ dimensions. Just as for the massless and one-mass hexagon integrals, the result is given in terms of classical polylogarithms of uniform transcendental weight three, which are functions of six dual conformally invariant cross ratios. Because of the high degree of symmetry of the integral, the result is extremely compact: it can be expressed as a sum of 24 terms involving only one basic function, which is a simple linear combination of logarithms, dilogarithms, and trilogarithms. Given the relation between one-loop hexagon integrals in $D = 6$ dimensions and higher-loop amplitudes in $D = 4$ dimensions, we expect that our result will help to understand the structure of $\mathcal{N} = 4$ SYM amplitudes and Wilson loops, particularly at two loops.

Note added:—After this calculation was completed, we were informed of an independent computation of the symbols of hexagon integrals, using a different method [19].

APPENDIX A: SPECIAL CASES

For $u_4 = u_5 = u_6 = 1$, the differential equations simplify considerably. We have

$$[u_1 + u_1(u_1 - 1)\partial_1]\Phi_0(u_1, u_2, u_3, 1, 1, 1) = \Psi_8(u_2, u_3, 1),$$

(A1)

where $\Psi_8(u, v, 1) = \log u \log v/(u - 1)/(v - 1)$, and the two cyclically related equations. The solution is simply

$$\Phi_0(u_1, u_2, u_3, 1, 1, 1) = \prod_{i=1}^3 \frac{\log u_i}{u_i - 1}.$$ (A2)

The case $u_5 = u_6 = 1$ is also very simple,

$$\Phi_0(u_1, u_2, u_3, u_4, 1, 1) = \frac{\log u_3}{u_3 - 1} \Psi_8(u_1, u_2, u_4).$$ (A3)

APPENDIX B: ARGUMENTS IN TERMS OF SPACE-TIME CROSS RATIOS

In this appendix we present the expressions of the functions $x^+_i$ defined in Eq. (52) in terms of the space-time cross ratios $u_i$.\(^2\)

\[^2\]Note that a constant term proportional to $\zeta_3$ is excluded because of the reality condition on the function.
The variables \( x_i^- \) are obtained from \( x_i^+ \) by replacing \( \sqrt{\Delta_0} \) by \(-\sqrt{\Delta_0}\). Also, in Eq. (48) we define the action of the odd permutations \( g \) to include the replacement \( \sqrt{\Delta_0} \rightarrow -\sqrt{\Delta_0} \).

The twistor variables \( x_i, y_i \) and \( z_i \) rationalize the \( x_i^- \), so that they take the form,

\[
\begin{align*}
  x_1^+ &= \frac{x_8}{1-y_8}, \quad x_2^+ = -\frac{x_8(y_2-y_8)}{(1-x_8)(1-y_8)}, \\
  x_3^+ &= \frac{x_8(1-y_2)}{x_3(1-y_8)}, \quad x_4^+ = \frac{x_8 y_3}{(1-y_8)(x_3 - x_8)}, \\
  x_1^- &= \frac{(1-x_3)(1-x_8(1-x_2)-y_8-z_2(1-x_8-y_8))}{y_2[(1-x_3)(1-y_8) - z_5(1-x_8-y_8)]}, \\
  x_2^- &= -\frac{(1-x_3)(y_2-y_8)[1-x_8(1-y_2) - y_8 - z_2(1-x_8-y_8)]}{y_2(1-x_8)[(z_2(1-x_3) - z_5)(1-y_8) + z_5 x_8(1-y_2)]}, \\
  x_3^- &= \frac{(1-y_2)[(1-x_3)[(x_3(1-y_8) - x_8)(1-z_2) + x_8 y_2(1-z_5)]]}{y_2 x_3[(z_2(1-x_3) - z_5)(1-y_8) + z_5 x_8(1-y_2)]}, \\
  x_4^- &= \frac{y_8(1-x_3)[(x_3(1-y_8) - x_8)(1-z_2) + x_8 y_2(1-z_5)]}{y_2(x_5 - x_8)[(1-x_3)(1-y_8) - z_5(1-x_8-y_8)]}.
\end{align*}
\]

(B2)

Note that these expressions correspond to a particular choice for the sign of the square root.

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