Fréchet Borel Ideals with Borel orthogonal

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December 17, 2013

Abstract

We study Borel ideals \( I \) on \( \mathbb{N} \) with the Fréchet property such its orthogonal \( I^\perp \) is also Borel (where \( A \in I^\perp \) iff \( A \cap B \) is finite for all \( B \in I \) and \( I \) is Fréchet if \( I = I^{\perp \perp} \)). Let \( \mathcal{B} \) be the smallest collection of ideals on \( \mathbb{N} \) containing the ideal of finite sets and closed under countable direct sums and orthogonal. All ideals in \( \mathcal{B} \) are Fréchet, Borel and have Borel orthogonal. We show that \( \mathcal{B} \) has exactly \( \aleph_1 \) non isomorphic members. The family \( \mathcal{B} \) can be characterized as the collection of all Borel ideals which are isomorphic to an ideal of the form \( I_{wf} \upharpoonright A \), where \( I_{wf} \) is the ideal on \( \mathbb{N}^{<\omega} \) generated by the wellfounded trees. Also, we show that \( A \subseteq \mathbb{Q} \) is scattered iff \( \text{WO}(\mathbb{Q}) \upharpoonright A \) is isomorphic to an ideal in \( \mathcal{B} \), where \( \text{WO}(\mathbb{Q}) \) is the ideal of well founded subset of \( \mathbb{Q} \).

Keywords: Borel ideals on countable sets, Fréchet property, scattered subsets of \( \mathbb{Q} \), analytic sequential spaces of uncountable sequential rank.

Subjclass[2000] Primary and Secondary: 03E15, 03E05.

1 Introduction

Given a collection \( \mathcal{A} \) of subsets of \( \mathbb{N} \), the orthogonal of \( \mathcal{A} \) is the following family of sets

\[
\mathcal{A}^\perp = \{ B \subseteq \mathbb{N} : (\forall A \in \mathcal{A})(A \cap B \text{ is finite}) \}.
\]

In this paper we study some structural properties of pairs \((I, I^\perp)\) where \( I \) is an ideal of subsets of \( \mathbb{N} \), that is to say, \( I \) is a non empty collection of subsets of \( \mathbb{N} \) closed under finite unions and taking subsets of its elements.

There have been quite of interest in the study of pairs \((\mathcal{A}, \mathcal{B})\) of orthogonal families (i.e, \( \mathcal{A} \subseteq \mathcal{B}^\perp \)) because its natural connection with gaps in the quotient algebra \( \mathcal{P}(\mathbb{N})/\text{Fin} \) and its applications to problems in analysis and topology (see for instance [1, 4, 15, 16] and the references therein). The motivation of our work comes from some results about definable pairs of orthogonal families. The word definable refers to the descriptive complexity of the family as a subset of the Cantor cube \( 2^{\mathbb{N}} \). It is in that sense that we will talk about Borel, analytic or coanalytic ideals. For instance, if \( I \) is Borel (or analytic), then \( I^\perp \) is (at most) coanalytic.
Todorčević [15, Theorem 7] showed that an analytic $p$-ideal $I$ is countably generated iff $I^\perp$ is analytic (and hence Borel). Dodos and Kanellopoulos [4, Remark 5], extending the work of Krawczyk about Rosenthal compacta [9] and Todorčević [15], proved that if $I$ is a selective analytic ideal not countably generated, then $I^\perp$ is a complete coanalytic set.

An ideal $I$ is said to be Fréchet if $I = I^{\perp\perp}$. We will recall later the connection of this definition with the more familiar notion of a Fréchet topological space. Mathias [10] showed that every selective analytic ideal is Fréchet (see also [17, Theorem 7.53]). But the converse is not true. So our initial motivation was to study Fréchet ideals such that $I$ and $I^\perp$ are both analytic (and hence Borel). For example, a countably generated ideal satisfies these conditions. The next observation is that a countable direct sum $\bigoplus_n I_n$ of Fréchet Borel ideals is also Borel and Fréchet, moreover, its orthogonal $(\bigoplus_n I_n)^\perp$ is also Borel. Our first result is the following

**Theorem I.** The smallest collection $\mathcal{B}$ of ideals on $\mathbb{N}$ containing the ideal of finite sets and closed under countable direct sums and the operation of taking orthogonal has exactly $\aleph_1$ non-isomorphic Fréchet ideals. Moreover, all ideals in $\mathcal{B}$ have complexity $F_{\sigma\delta}$.

We will define a sequence $P_\alpha$ for $\alpha < \omega_1$ of ideals such that an ideal belongs to $\mathcal{B}$ iff it is isomorphic to one of the following: $P_\alpha$, $P_\alpha^\perp$ or $P_\alpha \oplus P_\alpha^\perp$. To give an example, let denote by $\mathrm{FIN}$ the ideal of finite sets and by $I_\omega$ the countable direct sum of copies of an ideal $I$. Then $P_0 = \mathcal{P}(\mathbb{N})$, $P_1 = \mathrm{FIN}^\omega$ and $P_2 = (\mathrm{FIN})^{\omega \perp \omega}$. $P_2$ is the simplest example of a Fréchet ideal $J$ such that $J$ and $J^\perp$ are non-isomorphic Borel ideals and neither one is countably generated. We should also mention that the ideals in $\mathcal{B}$ are the only examples we know of Fréchet Borel ideals with Borel orthogonal.

The problem of constructing Fréchet ideals with special properties or uncountable families of pairwise non isomorphic Fréchet ideals on $\mathbb{N}$ has been addressed in the literature [6, 7, 12, 13, 16]. Those constructions usually make use of almost disjoint families of size the continuum and, in general, the filters produced are not definable or at least non Borel. For instance, a typical Fréchet ideal is given by $\mathcal{A}^\perp$ where $\mathcal{A}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$. When $\mathcal{A}$ is analytic, Mathias [10] showed that the ideal generated by $\mathcal{A}$ is selective and by a result of Todorcevic [17, Theorem 7.53] it is also biseparable and therefore, by a result of Krawczyk [9], $\mathcal{A}^\perp$ is Borel only if $\mathcal{A}$ is countable (see also [4]). In contrast with this, the collection $\mathcal{B}$ consists of Fréchet Borel ideals.

It is known that a Fréchet analytic ideal fails to be selective iff some restriction of it is isomorphic to $P_1 = \mathrm{FIN}^\omega$ [16, Corollary 4.5]. Moreover, if $I$ is a selective ideal and $I \upharpoonright A$ is isomorphic to an ideal in $\mathcal{B}$, then $I \upharpoonright A$ is isomorphic to either $\mathcal{P}(\mathbb{N})$, $\mathrm{FIN}$ or $\mathrm{FIN}^\omega$ (these three ideals and their direct finite sums are the only selective ideals in $\mathcal{B}$). Thus it seems natural to investigate, for a given Fréchet ideal $I$, which ideals in $\mathcal{B}$ appear as a restriction of $I$. To illustrate further this idea we need to recall the definition of two well known ideals.

Let $I_{w^f}$ denote the ideal on $\mathbb{N}^{<\omega}$ (the collection of finite sequences of integers) generated by the well founded trees on $\mathbb{N}$. In [4] it was made clear the role played by $I_{w^f}$ in the study

\[\text{An ideal is selective [10] if for all decreasing sequence of sets } A_n \notin I \text{ for } n \in \mathbb{N}, \text{ there is } B \notin I \text{ such that } B \setminus \{0, \cdots, n-1\} \subseteq A_n \text{ for all } n \in B.\]
of the descriptive complexity of orthogonal families. Recall that \( I_{wf} \) is a complete coanalytic subset of \( 2^\mathbb{N} \).

**Theorem II.** Every member of \( \mathcal{B} \) is isomorphic to some restriction of \( I_{wf} \).

In addition to the last result we also proved:

**Theorem III.** For \( A \subseteq \mathbb{N}^{<\omega} \), the following are equivalent:

(i) \( I_{wf} \upharpoonright A \) is isomorphic to an ideal in \( \mathcal{B} \).

(ii) \( I_{wf} \upharpoonright A \) is Borel.

(iii) \( I_{wf} \not\hookrightarrow I_{wf} \upharpoonright A \) (where \( \hookrightarrow \) means an isomorphic embedding).

The second ideal is the collection \( WO(\mathbb{Q}) \) of well founded subsets of \( \mathbb{Q} \). We shall see that a result analogous to the previous theorem holds for \( WO(\mathbb{Q}) \):

**Theorem IV.** For every \( A \subseteq \mathbb{Q} \), the following are equivalent:

(i) \( WO \upharpoonright A \) is isomorphic to an ideal in \( \mathcal{B} \).

(ii) \( WO \upharpoonright A \) is Borel.

(iii) \( WO \not\hookrightarrow WO \upharpoonright A \).

(iv) \( A \) is scattered.

Thus there is an analogy between the collection \( \mathcal{B} \) and the hierarchy of countable scattered linear orders given by the classical Hausdorff’s theorem. However, the ideals \( WO(\mathbb{Q}) \) and \( I_{wf} \) are structurally very different, since \( WO(\mathbb{Q}) \) is isomorphic to its orthogonal and this is no true for \( I_{wf} \).

Finally, we would like to recall the reason for calling Fréchet an ideal \( I \) such that \( I = I^\perp \). To each ideal \( I \) on \( \mathbb{N} \) we associate a topology on \( X = \mathbb{N} \cup \{\infty\} \) where each \( n \in \mathbb{N} \) is isolated and the nbhds of \( \infty \) are all sets of the form \( V \cup \{\infty\} \) with \( V \) in the dual filter of \( I \). Recall that a topological space \( Z \) is said to be Fréchet, if whenever \( A \subseteq Z \) and \( z \in \overline{A} \), there is a sequence \( (z_n)_n \) in \( A \) converging to \( z \). It is easy to verify that \( X \), with the topology defined above, is Fréchet iff \( I \) is Fréchet: Just notice that a sequence \( S \subseteq \mathbb{N} \) converges to \( \infty \) iff \( S \in I^\perp \).

When both the topology of a space \( X \) (in the example above, given by the ideal \( I \)) and the convergence relation (given by \( I^\perp \)) are Borel, we could say that the space \( X \) is definable in a strong sense. The result of Krawczyk about Rosenthal compacta \cite{Krawczyk} says that this is not the case when the compacta is not first countable. Nevertheless, Debs \cite{Debs1, Debs2} has shown that there is a Borel set that codes the convergence relation in a Rosenthal compacta. In the last section we shall see how the ideals in \( \mathcal{B} \) can be used to construct a special kind of sequential spaces, where both the topology and the convergence relation are Borel and yet the space has sequential order \( \omega_1 \). This will answer a question possed in \cite{Gruenhage}. 

3
2 Preliminaries and notation

The set $\mathbb{N}^{<\omega}$ will denote the set of finite sequences of integers and $|s|$ denotes the length of the sequence $s \in \mathbb{N}^{<\omega}$. The set $\mathbb{N}^\omega$ will denote the set of infinite sequences of integers. We will say that a sequence $s \in \mathbb{N}^{<\omega}$ extends a sequence $t \in \mathbb{N}^{<\omega}$, denote $t \preceq s$, if for all $i < |t|$ we have that $s(i) = t(i)$. A tree is a collection of sequences downward closed under $\preceq$. Given a finite sequence $t$ and an integer $n$ we denote the sequence $(t(0), \ldots, t(|t| - 1), n)$ by $t^\perp n$. If $A$ is a subset of $\mathbb{N}^{<\omega}$, $(A)$ denotes the tree generated $A$, whereas $A_n$ denotes the set $\{s \in A : a \preceq s\}$, where $a \in A$. Let $\mathcal{N}_t = \{s \in \mathbb{N}^{<\omega} : t \preceq s\}$.

An ideal over a set $X$ is a family $I$ of subsets of $X$ that contains the empty set, it is closed under taking subsets and finite unions. For convenience, we will allow the trivial ideal $\mathcal{P}(X)$ (i.e. when $X \in I$). $\text{FIN}$ denotes the ideal of finite subsets of $\mathbb{N}$ and $\mathbb{N}^{[\omega]}$ the family of infinite subsets of $\mathbb{N}$.

An ideal $I$ on $X$ is isomorphic to an ideal $J$ on $Y$ if there is a bijection $f : X \to Y$ such that $A \in I$ iff $f[A] \in J$; this will be denoted by $I \cong J$. If $f$ is just an injection, we will write $I \hookrightarrow J$ and say that $J$ has a copy of $I$ and we will write $I \hookrightarrow J$ to denote that $f$ is an isomorphic embedding that witness $I \hookrightarrow J$. If $K$ is a subset of $X$, the restriction of $I$ to $K$, denoted by $I \restriction K$, is the ideal on $K$ consisting of all the subsets of $K$ belonging to $I$.

A subset of a topological space is called analytic if it is a continuous image of a Borel subset of a Polish space. It is called co-analytic when its complement is analytic. By the usual identification of subsets with characteristic functions, we can identify an ideal on $\mathbb{N}$ with a subset of the Cantor cube $2^\mathbb{N}$ and thus it makes sense to say that an ideal is Borel, analytic, co-analytic, etc.

Let $\mathcal{A}$ be a collection of subsets of $X$, the orthogonal $\mathcal{A}^\perp$ of $\mathcal{A}$ was defined in the introduction. This terminology is taken from [15]. Two families of sets $\mathcal{A}$ and $\mathcal{B}$ are orthogonal if $\mathcal{A} \subseteq \mathcal{B}^\perp$. It is easy to verify that $I^\perp = I^\perp \perp$ and $I \subseteq I^\perp$. An ideal $I$ has the Fréchet property or just is a Fréchet ideal, if $I = I^\perp$.

Let $\{K_n : n \in F\}$ be a partition of $X$, where $F \subseteq \mathbb{N}$. For $n \in F$, let $I_n$ be an ideal on $K_n$. The direct sum, denoted by $\bigoplus_{n \in F} I_n$, is defined by

$$A \in \bigoplus_{n \in F} I_n \iff (\forall n \in F)(A \cap K_n \in I_n).$$

In general, given a sequence of ideals $I_n$ over a countable set $X_n$, we define $\bigoplus_n I_n$ by taking a partition $\{K_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ and an isomorphic copy $I'_n$ of $I_n$ on $K_n$ and let $\bigoplus_n I_n$ be $\bigoplus_n I'_n$. It should be clear that $\bigoplus_n I_n$ is, up to isomorphism, independent of the partition and the copy used. If all $I_n$ are equal to $I$ we will write $I^\omega$ instead of $\bigoplus_n I_n$.

For example, if we sum infinite many times the ideal $\text{FIN}$ we get $\text{FIN}^\omega$ a well known ideal, sometimes denoted by $\emptyset \times \text{FIN}$. Its orthogonal $\text{FIN}^{\omega \perp}$, sometimes is denoted by $\text{FIN} \times \emptyset$. Those ideals play a crucial role for the general study of analytic ideals ([5, 14]). Moreover, the topological space associate to $\text{FIN}^\omega$ (as explained in the introduction) is the sequential fan, which is the prototypical example of a non first countable Fréchet space.

The proof of the following result is straightforward.
Lemma 2.1 Let $I$, $J$ and $K$ be ideals.

(i) $I \oplus J = J \oplus I$.

(ii) $(I \oplus J) \oplus K \cong I \oplus (J \oplus K)$.

(iii) Parts (i) and (ii) also hold for infinite sums.

(iv) $(I \oplus J)\perp \cong I \perp \oplus J\perp$.

Lemma 2.2 Let $(I_n)_{n}$ be a sequence of ideals on a countable set $X$.

(i) If $(I_n)_{n}$ is a sequence of Borel ideals, then $\oplus_n I_n$ is also Borel.

(ii) $A \in (\oplus_n I_n)\perp \iff (\exists k \in \mathbb{N}) (A \subseteq \bigcup_{i \leq k} B_i$ and $\forall i \leq k) A \cap B_i \in I_i\perp$.

Conversely, if an ideal $J$ is defined by $A \in J \iff (\exists k \in \mathbb{N}) (A \subseteq \bigcup_{i \leq k} B_i$ and $\forall i \leq k) A \cap B_i \in I_i\perp$, then $J = (\oplus_n I_n)\perp$.

(iii) If $(I_n\perp)_{n}$ is a sequence of Borel ideals, then $(\oplus_n I_n)\perp$ is also Borel.

Proof: It is straightforward and is left to the reader.

Lemma 2.3 If $I_n$ is Fréchet for all $n$, then $\oplus_n I_n$ is Fréchet.

Proof: Let $(K_n)_{n}$ be the partition of $\mathbb{N}$ that defines $\oplus_n I_n$. Take an infinite set $A \subseteq \mathbb{N}$ that is not in $\oplus_n I_n$. Then, there is $n_0 \in \mathbb{N}$ such that $A \cap K_{n_0} \notin I_{n_0}$. Since $I_{n_0}$ is Fréchet, there is an infinite set $B \subseteq A \cap K_{n_0}$ belonging to $I_{n_0}\perp$. It is clear that $B$ is also in $(\oplus_n I_n)\perp$.

Now we define two ideals that play an important role in our results. Consider the ideal $I_{wf}$ generated by the well founded trees on $\mathbb{N}$. We will call a set $A \subseteq \mathbb{N}^{<\omega}$ well founded if it belongs to $I_{wf}$, that is to say, if there is a wellfounded tree $T$ such that $A \subseteq T$. Obviously, this is equivalent to say that the tree generated by $A$ is well founded. The orthogonal of $I_{wf}$ is the ideal $I_d$ generated by the finitely branched trees on $\mathbb{N}$ [4], or equivalently, $I_d$ consists of sets which are dominated by a branch:

$$A \in I_d \iff \exists \alpha \in \mathbb{N}^{\omega} \forall s \in A \forall i < |s| (s(i) \leq \alpha(i))$$

The ideal $I_{wf}$ is a complete coanalytic set [4] while the ideal $I_d$ is easily seen to be $F_{\sigma\delta}$.

3 The family $B$

One of the main purposes of this work is to study the smallest collection $B$ of ideals containing FIN and closed under the operation of taking countable sums and orthogonal. In this section we give a precise characterization of the ideals belonging to $B$ and in particular we show that $B$ has exactly $\aleph_1$ non isomorphic elements. All ideals in $B$ are Borel and Fréchet; moreover, we will see later that the members of $B$ have Borel complexity at most $F_{\sigma\delta}$.

An inductive definition of $B$ is given next.
Definition 3.1 Let $B_0 = \{\mathcal{P}(\mathbb{N}), \text{FIN}, \mathcal{P}(\mathbb{N}) \oplus \text{FIN}\}$. Suppose we have defined $B_\xi$, for every ordinal $\xi < \alpha < \omega_1$. We define the family $B_\alpha$ as follow:

\[
B_\alpha = \{P(N) \oplus \text{FIN}, P(N) \oplus \text{FIN}\}.
\]

Suppose we have defined $B_\xi$, for every ordinal $\xi < \alpha < \omega_1$. We define the family $B_\alpha$ as follow:

\[
B_\alpha = \{I_0 \oplus \cdots \oplus I_n : n \in \mathbb{N} \text{ and } (\forall i \leq n)(I_i \in B_\alpha)\}.
\]

$B = \bigcup_{\alpha < \omega_1} B_\alpha$.

The members of $B$ are, by definition, ideals on $\mathbb{N}$, but we will regard $B$ as if it were closed under isomorphism, in other words, when we say that an ideal $I$ belongs to $B$, we actually mean that $I$ is isomorphic to an ideal in $B$. Notice that, by lemma 2.3, every member of $B$ is a Fréchet Borel ideal. To state our results we need to introduce a collection of ideals.

Put $P_0 = \mathcal{P}(\mathbb{N})$ and $Q_0 = P_0^\perp = \text{FIN}$. For every limit ordinal $\alpha < \omega_1$ we fix an increasing sequence $(\nu_\alpha^n)_n$ of ordinals such that $\sup_n (\nu_\alpha^n) = \alpha$. We put

\[
P_\alpha = \bigoplus_n P_{\nu_\alpha^n},
\]

for $\alpha$ limit and,

\[
P_{\alpha+1} = \bigoplus_n I_n
\]

where $I_n = P_\alpha^\perp$ for all $n$, i.e., $P_{\alpha+1} = P_\alpha^\perp$. Finally, $Q_\alpha$ is defined as $P_\alpha^\perp$. The definition of $P_\alpha, Q_\alpha$ for all $\alpha$ is up to isomorphism independent of the partitions used.

For instance, a standard copy of $P_1 = \text{FIN}^\omega$ is defined on $\mathbb{N}^2$ as follows: $A \in P_1$ iff $\{m \in \mathbb{N} : (n,m) \in A\}$ is finite for all $n \in \mathbb{N}$. Therefore $A \in P_1$ iff there is $n$ such that $A \subseteq \bigcup_{k=0}^{n-1}\{k\} \times \mathbb{N}$.

Our first result about $B$ is the following.

Theorem 3.2 Let $\alpha < \omega_1$.

1. Every ideal in the class $B_\alpha \setminus \bigcup_{\xi < \alpha} B_\xi$ is isomorphic to either $P_\alpha, Q_\alpha$ or $P_\alpha \oplus Q_\alpha$.

2. If $I \in B_\alpha \setminus \bigcup_{\xi < \alpha} B_\xi$ and $J \in B_\beta$ with $\beta < \alpha$, then $I \oplus J \cong I$.

For the proof we need the following lemma.

Lemma 3.3 (i) $P_\alpha \oplus P_\beta \cong P_\alpha$, if $\beta \leq \alpha$.

(ii) $Q_\alpha \oplus Q_\beta \cong Q_\alpha$, if $\beta \leq \alpha$.

(iii) $P_\alpha \oplus Q_\beta \cong P_\alpha$, if $\beta < \alpha$.

(iv) $Q_\alpha \oplus P_\beta \cong Q_\alpha$, if $\beta < \alpha$. 6
Proof: By passing to the orthogonal and using lemma 2.4.1, we get that (i) and (ii) are equivalent. The same occurs with (iii) and (iv). The rest of the proof is by induction on $\alpha$. Suppose now that $f: \mathbb{N} \to \mathbb{N}^2$ given by $f(n) = (n, 0), f(n, m) = (n, m + 1)$. It is left to the reader to check that $f$ is an isomorphism between $P_0 \oplus P_1$.

Suppose the result holds for all ordinals smaller than $\alpha$.

(i). Let $\beta < \alpha$. We show that $P_\alpha \oplus P_\beta \simeq P_\alpha$. If $\alpha$ is a limit ordinal, then $P_\alpha = \bigoplus_n Q_{\alpha n}$. Therefore, there must be $n_0$ such that $\beta < \nu_{n_0}^\alpha < \alpha$. By the induction hypothesis $Q_{\nu_{n_0}^\alpha} \oplus P_\beta \simeq Q_{\nu_{n_0}^\alpha}$. Hence

$$(\bigoplus_n Q_{\nu_{n_0}^\alpha}) \oplus P_\beta = \bigoplus_{n \neq n_0} Q_{\nu_{n_0}^\alpha} \oplus (Q_{\nu_{n_0}^\alpha} \oplus P_\beta) \simeq (\bigoplus_n Q_{\nu_{n_0}^\alpha})$$

(1)

Suppose now that $\alpha = \mu + 1$, then $P_\alpha = Q_\mu^\omega$ and, we have two possibilities for $\beta$. If $\beta < \mu$, by the induction hypothesis, we have that $Q_\mu \oplus P_\beta \simeq Q_\mu$ and as in (1) we get that $Q_\mu^\omega \oplus P_\beta \simeq Q_\mu^\omega$.

Now, if $\beta = \mu$, we have that $P_\beta = \bigoplus_n Q_{\xi_n}$, where $\xi_n < \mu$ (no matter if $\mu$ is limit or not). By the inductive hypothesis, $Q_\mu \oplus Q_{\xi_n} \simeq Q_\mu$ and therefore

$$Q_\mu^\omega \oplus P_\beta = Q_\mu^\omega \oplus \bigoplus_n Q_{\xi_n} \simeq \bigoplus_n (Q_\mu \oplus Q_{\xi_n}) \simeq Q_\mu^\omega$$

(2)

Thus, we have shown that $P_\alpha \oplus P_\beta \simeq P_\alpha$ for $\beta < \alpha$. Now we show $P_\alpha \oplus P_\alpha \simeq P_\alpha$. The argument is similar. If $\alpha$ is limit, we use equivalences as in (2). And for $\alpha = \mu + 1$, we use that $J^\perp \oplus J^\perp \simeq (J \oplus J)^\perp$.

(iii). The proof is entirely similar and is left to the reader.

Proof of theorem 3.2: We will say that the rank of an ideal $I$ is $\beta$, if $I \in B_\beta$ and $I \notin B_\rho$ for all $\rho < \beta$. We will prove (1) and (2) simultaneously by induction. It is easy to see that they hold for $\alpha = 1$. Suppose the result holds for all $\rho < \alpha$.

Claim 3.4 Suppose $\alpha$ is limit. Let $\xi_n, \eta_n < \alpha$ not decreasing with $\sup_n \xi_n = \sup_n \eta_n = \alpha$, $I_n \in B_{\xi_n}$ with rank $\xi_n$ and $J_n \in B_{\eta_n}$ with rank $\eta_n$.

(i) Let $(n_k)_k$ be an increasing sequence of integers such that $\xi_m < \xi_{n_k}$ for $m < n_k$. Then $\bigoplus_n I_n \cong \bigoplus_k I_{n_k}$.

(ii) $\bigoplus_n I_n \cong \bigoplus_n J_n$.

Proof: (i) Using the inductive hypothesis (2) and the condition on $\xi_{n_k}$ we have that for all $k$

$$\bigoplus \{I_m: n_{k-1} < m \leq n_k\} \cong I_{n_k} \quad \text{(where } n_{-1} = -1)$$

Therefore

$$\bigoplus_n I_n \cong \bigoplus_k \bigoplus \{I_m: n_{k-1} < m \leq n_k\} \cong \bigoplus_k I_{n_k}.$$
loss of generality that the original sequences \((\xi_n)_n\) and \((\eta_n)_n\) have those properties. Using
the inductive hypothesis (2) we have that \(I_{n+1} \oplus J_n \cong I_{n+1}\) and \(I_2 \oplus I_3 \oplus J_4 \cong J_4\). Therefore
\(\Omega_n I_n \cong \Omega_n I_{n+1} \cong \Omega_n (I_{n+1} \oplus J_n) \cong J_1 \oplus J_2 \oplus J_3 \oplus (I_2 \oplus I_3 \oplus J_4) \oplus \Omega_n \geq 5 (I_{n-1} \oplus J_n) \cong \Omega_n J_n\)

Now we start the proof of (3.2)
We show that (1) holds for \(\alpha\). Let \(I \in \mathcal{B}_\alpha \setminus \bigcup_{\xi < \alpha} \mathcal{B}_\xi\). By definition of \(\mathcal{B}_\alpha\), there are three
cases to be considered. First we treat the case \(\alpha\) limit.

(a) Suppose that \(I = \Omega_n I_n\), with \(I_n \in \mathcal{B}_{\xi_n}\), where \(\xi_n\) has rank \(\xi_n < \alpha\). Then \(\sup(\xi_n) = \alpha\),
otherwise \(I \in \mathcal{B}_\beta\) for some \(\beta < \alpha\). By lemma 2.1 we can assume that \((\xi_n)_n\) is not decreasing. From the claim 3.4 we conclude that \(I \cong \Omega_n Q_{\alpha_n} = P_\alpha\).

(b) If \(I = (\Omega_n I_n)_{\perp}\), with \(I_n \in \mathcal{B}_{\xi_n}\), \(\xi_n < \alpha\) and \(\sup(\xi_n) = \alpha\). By the previous case, we
can assume, without loss of generality, that \(\Omega_n I_n \cong P_\alpha\) and hence \(I = Q_\alpha\).

(c) If \(I = I_0 \oplus \cdots \oplus I_\alpha\) where \(I_i \in \mathcal{B}_{\alpha_n}^{\perp}\). Then by the previous cases we know that \(I_i\) is
either \(P_\alpha\) or \(Q_\alpha\). From lemma 3.3 we conclude that \(I\) is either \(P_\alpha, Q_\alpha\) or \(P_\alpha \oplus Q_\alpha\).

Now suppose \(\alpha = \mu + 1\). As before, there are three cases to be considered.

(a) Suppose that \(I = \Omega_n I_n\), with \(I_n \in \mathcal{B}_\mu\), for all \(n\). We will show that \(I \cong P_\alpha\).

We can assume that at least one \(I_n\) has rank \(\mu\), otherwise \(I\) would have rank less than \(\alpha\). Let \(A\) be the set of all \(n\) such that \(I_n\) has rank smaller than \(\mu\). Then \(J = \Omega_n \in A I_n\)
has rank at most \(\mu\). Let \(K = \Omega_n \notin A I_n\). Then \(I = J \oplus K\). From the inductive hypothesis (2), we conclude that \(I \cong K\). In summary, we can assume, without loss of generality, that \(I_n\) has rank \(\mu\) for all \(n\). From the inductive hypothesis (1), we can also assume that \(I_n\) is \(P_\mu\) or \(Q_\mu\). Let \(B\) the set of all \(n\) such that \(I_n \cong P_\mu\). Notice that if \(B\) is not empty, then \(\Omega_n \in B I_n \cong P_\mu\). Therefore the set \(C\) of all \(n\) such that \(I_n \cong Q_\mu\) is infinite
and moreover \(\Omega_n C I_n \cong P_{\mu+1} = P_\alpha\). Then using lemma 3.3 we obtain
\[I = (\Omega_n \in B I_n) \oplus (\Omega_n \in C I_n) \cong P_\mu \oplus P_\alpha \cong P_\alpha\]

If \(B\) is empty, the result is the same.

(b) If \(I = (\Omega_n I_n)_{\perp}\), with \(I_n \in \mathcal{B}_\mu\), for all \(n\). From the previous case, we conclude that
\(I \cong P_\alpha_{\perp} = Q_\alpha\).

(c) If \(I = I_0 \oplus \cdots \oplus I_\alpha\) where \(I_i \in \mathcal{B}_{\alpha_n}^{\perp}\). Then by the previous cases we know that \(I_i\) is
either \(P_\alpha\) or \(Q_\alpha\). From (3) we conclude that \(I\) is either \(P_\alpha, Q_\alpha\) or \(P_\alpha \oplus Q_\alpha\).

Finally, (2) follows directly from part (1) and lemma 3.3.

From theorem 3.2 we know there are, up to isomorphism, at most \(\aleph_1\) ideals in \(\mathcal{B}\). Now
we will show that the ideals \(P_\alpha, Q_\alpha\) and \(P_\alpha \oplus Q_\alpha\) are all non isomorphic. This is a long
inductive proof which we present it split on several lemmas.
**Lemma 3.5** \( \mathcal{B} \) is closed under restriction, that is to say, for all \( I \in \mathcal{B}_\alpha \) and all infinite \( K \subseteq \mathbb{N}, I \upharpoonright K \) belongs to \( \mathcal{B}_\alpha \).

**Proof:** By induction on \( \alpha \). It suffices to show the result for the ideals \( P_\alpha \) and \( Q_\alpha \). The result is obvious for \( \alpha = 0 \). The rest of the proof follows from the following two straightforward facts. Let \( \{K_n : n \in \mathbb{N}\} \) be a partition of \( \mathbb{N} \), \( I_n \) an ideal over \( K_n \) and \( K \subseteq \mathbb{N} \) infinite. Then

\[
(\oplus_n I_n) \upharpoonright K \cong \oplus_n (I_n \upharpoonright K \cap K_n).
\]

Let \( I \) be an ideal over \( \mathbb{N} \), then \( I^\perp \upharpoonright K \cong (I \upharpoonright K)^\perp \). □

**Lemma 3.6**

(i) Let \( \rho < \omega_1 \) and \( K \subseteq \mathbb{N} \) infinite. Assume that \( Q_\rho \not\cong Q_\xi \upharpoonright E \), for every \( \xi < \rho \) and every infinite set \( E \subseteq \mathbb{N} \). Then, \( Q_\rho \not\cong P_\rho \upharpoonright K \).

(ii) Let \( \rho < \omega_1 \) and \( K \subseteq \mathbb{N} \) infinite. Assume that \( Q_\rho \not\cong Q_\xi \upharpoonright E \), for every \( \xi < \rho \) and every infinite set \( E \subseteq \mathbb{N} \). Then, \( Q_{\rho+1} \not\cong P_\rho \upharpoonright K \).

(iii) Let \( \rho < \omega_1 \). Assume that \( Q_{\rho+1} \not\cong Q_\rho \upharpoonright E \), for all infinite set \( E \subseteq \mathbb{N} \). Then, \( Q_{\rho+1} \not\cong P_\rho \oplus Q_\rho \).

**Proof:** (i) Suppose, towards a contradiction, that \( f : \mathbb{N} \to K \) is an isomorphism witnessing \( Q_\rho \cong P_\rho \upharpoonright K \). Let \( \{K_n : n \in \mathbb{N}\} \) be a partition of \( \mathbb{N} \) used to define \( Q_\rho \). For every \( n \in \mathbb{N} \), let’s denote by \( L_n \) the set \( K_n \cap K \). Note that \( \{L_n : n \in \mathbb{N}\} \) is a partition defining \( P_\rho \upharpoonright K \). Let \( (\xi_n)_n \) be a sequence of ordinals such that \( P_\rho = \bigoplus_n Q_{\xi_n} \) (note that every \( \xi_n \) is less than \( \rho \) regardless \( \rho \) is limit). We are going to define sequences of integers \( (p_k)_k, (n_k)_k, \) and \( (l_k)_k \) with the following properties:

1. \( p_k \in K_{n_k} \), for all \( k \in \mathbb{N} \),
2. \( (n_k)_k \) and \( (l_k)_k \) are increasing, and
3. \( p_k \in K_{n_k} \) and \( f(p_k) \in L_{l_k} \), for all \( k \in \mathbb{N} \).

Assume we have constructed the sequences with the properties listed above and we deduce the required contradiction. Put \( A = \{p_k : k \in \mathbb{N}\} \). Since \( (n_k)_k \) is increasing, by lemma [2.2] (ii), \( A \not\subseteq Q_\rho \). On the other hand, \( (l_k)_k \) is increasing, then \( f[A] \cap L_{l_k} = \{f(p_k)\} \) for all \( k \in \mathbb{N} \), and hence \( f[A] \subseteq P_\rho \upharpoonright K \). This contradicts that \( f \) is an isomorphism.

We are going to define the sequences mentioned above by induction. First we fix \( p_0 \in K_0 \). \( l_0 \) is the integer satisfying that \( p_0 \in L_{l_0} \). Put \( n_0 = 0 \), and \( m_0 = \max\{n_0, l_0\} \). Suppose have chosen \( n_k, l_k > m_{k-1} \) and \( p_k \in K_{n_k} \) such that

\[
f(p_k) \notin \bigcup_{i=0}^{m_{k-1}} C_i \text{ and } f(p_k) \in L_{l_k}.
\]

Let \( m_k = \max\{n_k, l_k\} \). We claim that

\[
(\exists n > m_k)(\exists p \in K_n)(f(p) \notin \bigcup_{i=0}^{m_k} L_i).
\]
Otherwise, we have that
\[ f[\bigcup_{n>m_k} K_n] \subseteq \bigcup_{i=0}^{m_k} L_i. \]

Let \( D = \bigcup_{n>m_k} K_n \). Notice that \( Q_\rho \upharpoonright D \cong Q_\rho \). Then
\[ Q_\rho \cong (Q_{\xi_0} \oplus \cdots \oplus Q_{\xi_{m_k}}) \upharpoonright f[D] \cong Q_\gamma \upharpoonright f[D], \]
where \( \gamma = \max\{\xi_0, \ldots, \xi_{m_k}\} < \rho \). Taking orthogonal we get
\[ P_\rho \upharpoonright D \cong \bigoplus_{n>m_k} Q_{\xi_n} \cong (Q_\gamma \upharpoonright f[D])^\perp = P_\gamma \upharpoonright f[D]. \]

Taking orthogonal we get
\[ Q_\rho = Q_\gamma \upharpoonright f[D], \]
which is a contradiction, as \( \xi < \rho \).

Let \( n_{k+1} \) and \( p_{k+1} \) as in the claim and take \( l_{k+1} \), which is greater than \( m_k \), such that \( f(p_{k+1}) \in L_{l_{k+1}} \). Put \( m_{k+1} = \max\{n_{k+1}, l_{k+1}\} \).

(ii) It is treated as case (i).

(iii) Let \( \{K_n : n \in \mathbb{N}\} \) be a partition of \( \mathbb{N} \) use in the definition of \( Q_{\rho+1} \) and let \( \{L_n : n \in \mathbb{N}\} \) be a partition of \( \mathbb{N} \) so that \( Q_\rho \) is defined on \( L_0 \) and \( \{L_n : n \geq 1\} \) is the partition used in the definition of \( P_\rho \). Notice that \( Q_\rho \oplus P_\rho \) is defined on \( \{L_n : n \in \mathbb{N}\} \). Assuming that \( Q_\rho \cong P_\rho \oplus Q_\rho \) through a function \( f \), we can find a set \( A \) that is negative for \( Q_{\rho+1} \) whose image is in \( P_\rho \). For this, we start by choosing \( p_0 \) in some \( K_{n_0} \) so that \( f(p_0) \in L_1 \). Then we can complete the proof by following the proof done in part (i) step by step.

Lemma 3.7

(i) \( P_\alpha \not\cong P_\beta \) for all \( \beta < \alpha \).

(ii) \( P_\alpha \not\cong Q_\beta \upharpoonright K \) for all \( \beta \leq \alpha \) and all \( K \subseteq \mathbb{N} \) infinite.

(iii) \( P_\alpha \not\cong P_\beta \oplus Q_\beta \) for all \( \beta \leq \alpha \).

Proof: The proof is by induction on \( \alpha \). It is easy to check that the result holds for \( \alpha = 1 \). Suppose that (i)-(iii) hold for all \( \gamma < \alpha \) and we show it for \( \alpha \).

First we treat the case \( \alpha \) limit.

(i) Let \( \beta < \alpha \) and \( K \subseteq \mathbb{N} \) infinite. Suppose, towards a contradiction, that \( f \) is an isomorphism witnessing \( P_\alpha \cong P_\beta \upharpoonright K \). Let \( \{K_n : n \in \mathbb{N}\} \) be the partition used in the definition of \( P_\alpha = \bigoplus Q_{v_n} \). Let \( m \) be such that \( \beta < v_m \). Then \( Q_{v_m} = P_\alpha \upharpoonright K_m \cong P_\beta \upharpoonright f[K_m] \).

By lemma 3.3, we know that \( P_\beta \upharpoonright f[K_m] \) belongs to \( B_\beta \), but this contradicts the inductive hypothesis.

For parts (ii) and (iii), we first fix \( \beta < \alpha \) and \( K \subseteq \mathbb{N} \) infinite. Arguing as in part (i) we conclude that \( P_\alpha \not\cong Q_\beta \upharpoonright K \) for all \( \beta < \alpha \) and \( P_\alpha \not\cong P_\beta \oplus Q_\beta \).

Now we are going to show (ii) and (iii) for \( \alpha = \beta \).
(ii) Suppose $\alpha = \beta$. In part (i) we just proved that $P_\alpha \not\cong P_\beta \upharpoonright E$, for all $\beta < \alpha$ and all $E \subseteq \mathbb{N}$ infinite. After taking orthogonal, we get the hypothesis of lemma 3.6 (i). Thus, $Q_\alpha \not\cong P_\alpha \upharpoonright K$. Taking orthogonal again we get $P_\alpha \not\cong Q_\alpha \upharpoonright K$.

(iii) Suppose $\alpha = \beta$ and, towards a contradiction that $P_\alpha \cong P_\alpha \oplus Q_\alpha$. Then $Q_\alpha \cong P_\alpha \upharpoonright C$, for some infinite set $C \subseteq \mathbb{N}$. Taking orthogonal we get $P_\alpha \cong Q_\alpha \upharpoonright C$. This contradicts what we just proved in part (ii).

Now we treat the case $\alpha = \mu + 1$.

(i) Let $\beta < \alpha$ and $K \subseteq \mathbb{N}$ infinite. Suppose, towards a contradiction, that $f$ is an isomorphism witnessing $P_\alpha \cong P_\beta \upharpoonright K$. Let $\{K_n : n \in \mathbb{N}\}$ be the partition used in the definition of $P_\alpha = (Q_\mu)^\omega$. Then $P_\alpha \upharpoonright K_0 \cong Q_\mu$. Hence, $Q_\mu \cong Q_\beta \upharpoonright f[K_0]$. Therefore, by taking orthogonal, $P_\mu \cong P_\beta \upharpoonright f[K_0]$. This contradicts part (i) of our inductive hypothesis. Second, suppose now that $\beta = \mu$. Using part (ii) of lemma 3.6 we get that $P_{\mu+1} \not\cong Q_\mu \upharpoonright K$. Finally, suppose $\beta = \mu + 1$. In part (i) we just proved that $P_\alpha \not\cong P_\beta \upharpoonright E$, for all $\beta < \alpha$ and all $E \subseteq \mathbb{N}$ infinite, which is (after taking orthogonal) the hypothesis of lemma 3.6(i). Thus, $Q_\alpha \not\cong P_\alpha \upharpoonright K$. Taking orthogonal we get $P_\alpha \not\cong Q_\alpha \upharpoonright K$.

(ii) Let $\beta \leq \alpha$ and $K \subseteq \mathbb{N}$ infinite. First, we suppose $\beta < \mu < \alpha$. Assume, towards a contradiction, that $f$ is an isomorphism witnessing that $P_{\mu+1} \cong Q_\beta \upharpoonright K$. Let $\{K_n : n \in \mathbb{N}\}$ be the partition used in the definition of $P_\alpha = (Q_\mu)^\omega$. Then $P_\alpha \upharpoonright K_0 \cong Q_\mu$. Hence, $Q_\mu \cong Q_\beta \upharpoonright f[K_0]$. Therefore, by taking orthogonal, $P_\mu \cong P_\beta \upharpoonright f[K_0]$. This contradicts part (i) of the inductive hypothesis. Second, suppose now that $\beta = \mu$. Using part (ii) of lemma 3.6 we get that $P_{\mu+1} \not\cong Q_\mu \upharpoonright K$. Finally, suppose $\beta = \mu + 1$. In part (i) we just proved that $P_\alpha \not\cong P_\beta \upharpoonright E$, for all $\beta < \alpha$ and all $E \subseteq \mathbb{N}$ infinite, which is (after taking orthogonal) the hypothesis of lemma 3.6(i). Thus, $Q_\alpha \not\cong P_\alpha \upharpoonright K$. Taking orthogonal we get $P_\alpha \not\cong Q_\alpha \upharpoonright K$.

(iii) Let $\beta \leq \alpha$ and $K \subseteq \mathbb{N}$ infinite. First, we suppose $\beta < \mu$. Assume, towards a contradiction, that $P_\alpha \cong P_\beta \oplus Q_\beta$. Let $f$ be a map witnessing this fact. Let $\{K_n : n \in \mathbb{N}\}$ be the partition used in the definition of $P_\alpha = (Q_\mu)^\omega$. Then, $Q_\mu \cong (P_\beta \oplus Q_\beta) \upharpoonright f[K_0]$. By lemma 3.5 we have that $(P_\beta \oplus Q_\beta) \upharpoonright f[K_0] \in B_\beta$. Hence $Q_\mu$ is isomorphic to either $P_\gamma$, $Q_\gamma$ or $P_\gamma \oplus Q_\gamma$, for some $\gamma \leq \beta < \mu$. This contradicts the inductive hypothesis.

Second, suppose $\beta = \mu$. In part (i) we just proved that $P_{\mu+1} \not\cong P_\mu \upharpoonright E$, for all $E \subseteq \mathbb{N}$ infinite. After taking orthogonal, we get the hypothesis of the lemma 3.6 (iii). Hence, $Q_{\mu+1} \not\cong P_\mu \oplus Q_\mu$. Again, we take orthogonal to get $P_{\mu+1} \not\cong P_\mu \oplus Q_\mu$.

Finally, suppose $\beta = \mu + 1$. Assume, towards a contradiction that $Q_\alpha \oplus P_\alpha \cong P_\alpha$ and denote by $g$ a function witnessing this fact. Let $C$ be an infinite set $C \subseteq \mathbb{N}$ such that $(P_\alpha \oplus Q_\alpha) \upharpoonright C \cong Q_\alpha$. Then, $Q_\alpha \cong P_\alpha \upharpoonright g[C]$. But in part (ii) we just proved that this is impossible.

From the previous results we immediately get the following

**Theorem 3.8** The family $B$ has $\aleph_1$ pairwise non isomorphic ideals.

## 4 Complexity of ideals in $B$

In this section we study the Borel complexity of the elements of $B$. Our proof is based in a representation of each ideal in $B$ as a restriction of the $F_{\sigma\delta}$ ideal $I_\delta$. Next theorem shows a link between the family $B$ and this ideal.
**Theorem 4.1** Every member of $\mathcal{B}$ is isomorphic to some restriction of $I_{wf}$ and also to some restriction of $I_d$. In particular, every member of $\mathcal{B}$ is $F_{\alpha\delta}$.

The proof is based in the following facts.

**Lemma 4.2** (i) Let $s \in \mathbb{N}^{<\omega}$ and $B_n$ be an infinite subset of $\mathcal{N}_{s^n}$ for every $n \in \mathbb{N}$. Then

$$I_{wf} \upharpoonright (\bigcup_n B_n) \cong \bigoplus_n I_{wf} \upharpoonright B_n.$$  

(ii) Let $\theta \in \mathbb{N}^\omega$, put $s_n = \theta \upharpoonright (n-1)((\theta(n)+1)$ and fix an infinite subset $B_n$ of $\mathcal{N}_{s_n}$ for each $n \in \mathbb{N}$. Then

$$I_{wf} \upharpoonright (\bigcup_n B_n) \cong (\bigoplus_n I_{wf} \upharpoonright B_n)^\perp = (\bigoplus_n I_d \upharpoonright B_n)^\perp.$$  

**Proof:** (i) Notice that if $A \in I_{wf} \upharpoonright (\bigcup_n B_n)$, then $A \cap B_n$ is also a well founded set, for all $n \in \mathbb{N}$. Therefore $A \in \bigoplus_n I_{wf} \upharpoonright B_n$. Conversely, if $A \cap B_n \in I_{wf}$ for all $n \in \mathbb{N}$, since \{$s^n : n \in \mathbb{N}$\} is an antichain, we have that $A$ is a well founded set. So $A \in \bigoplus_n I_{wf} \upharpoonright B_n$. Thus

$$A \in I_{wf} \upharpoonright (\bigcup_n B_n) \Leftrightarrow A \in \bigoplus_n I_{wf} \upharpoonright B_n.$$  

Therefore, $I_{wf} \upharpoonright (\bigcup_n B_n) \cong \bigoplus_n I_{wf} \upharpoonright B_n$.

(ii) Take $A \subseteq \bigcup_n B_n$. If $A$ is well founded, it can only have non empty intersection with finitely many $B_n$’s (otherwise, any tree containing $A$ will have $\theta$ as a branch). So there is $n_0 \in \mathbb{N}$ such that $A \subseteq \bigcup_{i \leq n_0} B_i$. From this and lemma 2.2 we have that $A \in (\bigoplus_n I_{wf} \upharpoonright B_n)^\perp$.

Conversely, if $A \in (\bigoplus_n I_d \upharpoonright B_n)^\perp$, by lemma 2.2 there is $n_0 \in \mathbb{N}$ such that $A \subseteq \bigcup_{i \leq n_0} B_i$ and $A \cap B_i \in I_d \upharpoonright B_i = I_{wf} \upharpoonright B_i$ for every $i \leq n_0$. Therefore, $A \in I_{wf} \upharpoonright (\bigcup_n B_n)$. Thus

$$A \in I_{wf} \upharpoonright (\bigcup_n B_n) \Leftrightarrow A \in (\bigoplus_n I_d \upharpoonright B_n)^\perp.$$  

Hence $I_{wf} \upharpoonright (\bigcup_n B_n) \cong (\bigoplus_n I_d \upharpoonright B_n)^\perp$.

**Proof of theorem 4.1** From theorem 3.2 it suffices to show that the ideals $P_\alpha$, $Q_\alpha$ and $P_\alpha \oplus Q_\alpha$ are isomorphic to a restriction of $I_{wf}$. Since $I_d$ is the orthogonal of $I_{wf}$ and $\mathcal{B}$ is closed under taking orthogonal, then the result also holds for $I_d$. The proof will be by transfinite induction on $\alpha$.

For $\alpha = 0$, take infinite sets $A \in I_{wf}$ and $B \in I_d$ we have that $I_{wf} \upharpoonright A \cong \mathcal{P}(\mathbb{N})$, $I_d \upharpoonright A \cong \mathcal{FIN}$, $I_{wf} \upharpoonright B \cong \mathcal{FIN}$ and, $I_d \upharpoonright B \cong \mathcal{P}(\mathbb{N})$.

Suppose that the result holds for all $\xi < \alpha$. By definition, $P_\alpha = \bigoplus_n Q_{\nu_n}$, where $\nu_n < \alpha$ for all $n \in \mathbb{N}$. Notice that $I_{wf} \upharpoonright \mathcal{N}_{(n)} \cong I_{wf}$ for each $n \in \mathbb{N}$. So, by the inductive hypothesis, there is $B_n \subseteq \mathcal{N}_{(n)}$ such that $I_{wf} \upharpoonright B_n \cong Q_{\nu_n}$. From this and lemma 4.2(i) we conclude

$$I_{wf} \upharpoonright (\bigcup_n B_n) \cong \bigoplus_n I_{wf} \upharpoonright B_n \cong \bigoplus_n Q_{\nu_n} \cong P_\alpha.$$  

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Now we will show it for $Q_\alpha$. Notice that $I_d \cong I_d \upharpoonright \mathcal{N}_{0^{\omega+1}}$ for each $n \in \mathbb{N}$. By the inductive hypothesis, there is $B_n \subseteq \mathcal{N}_{0^{\omega+1}}$ such that $I_d \upharpoonright B_n \cong P_{\nu_n}$. Hence, by lemma \ref{lem:inclusion}(ii), where $\theta$ is the constantly equal to cero sequence, we have that

$$I_{wf} \upharpoonright (\bigcup_B B_n) \cong (\bigoplus_B B_n) \cong (\bigoplus_B P_{\nu_n}) \cong Q_\alpha.$$

We have already showed that $P_\alpha$ and $Q_\alpha$ are isomorphic to a restriction of $I_{wf}$. Now, since $I_{wf} \cong I_{wf} \upharpoonright \mathcal{N}_{(0)}$ and $I_{wf} \cong I_{wf} \upharpoonright \mathcal{N}_{(1)}$, there are infinite sets $C \subseteq \mathcal{N}_{(0)}$ and $D \subseteq \mathcal{N}_{(1)}$ such that $I_{wf} \upharpoonright C \cong P_\alpha$ and $I_{wf} \upharpoonright D \cong Q_\alpha$. Thus

$$I_{wf} \upharpoonright (C \cup D) \cong I_{wf} \upharpoonright C \oplus I_{wf} \upharpoonright D \cong P_\alpha \oplus Q_\alpha.$$

The last statement follows from the fact that $I_d$ is $\Pi^0_3$. \hfill \blacksquare

Most of ideals in $B$ are complete $F_\sigma \delta$. Clearly all ideals in $B_0$ are $F_\sigma$. It is well known that $P_1 = \text{FIN}^{\omega}$ is $F_{\sigma \delta}$-complete \cite[pag. 179]{8} and $Q_1 = \text{FIN}^{\omega \perp}$ is $F_\sigma$ using lemma \ref{lem:inclusion}. Now we take $J \in B \setminus B_1$, from theorem \ref{thm:inclusion} we have that $\text{FIN}^{\omega \perp} \oplus J \cong J$. Hence, $\text{FIN}^{\omega} \hookrightarrow J$ and therefore $\text{FIN}^{\omega} \leq_W J$ (where $\leq_W$ is the Wadge reducibility relation \cite{8}). Thus $J$ is also $F_{\sigma \delta}$-complete.

### 5 Borel restrictions of $I_{wf}$

In view of theorem \ref{thm:inclusion} an immediate question arises: which restrictions of $I_{wf}$ belong to $B$? Next theorem answers this question.

**Theorem 5.1** For every $A \subseteq \mathbb{N}^{<\omega}$, the following are equivalent:

(i) $I_{wf} \upharpoonright A$ belongs to $B$.

(ii) $I_{wf} \upharpoonright A$ is Borel.

(iii) $I_{wf} \not\hookrightarrow I_{wf} \upharpoonright A$.

Since $I_{wf}$ is a complete coanalytic set and each ideal in $B$ is Borel, then it is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). We will need several auxiliary results for proving the other implication.

The first step is to show that we can reduce the problem to the case when $A$ is a tree. Recall that a set $D \subseteq \mathbb{N}^{<\omega}$ is said to be dense, if for all $t \in \mathbb{N}^{<\omega}$, there is $d \in D$ such that $t \leq d$. The following result is probably known, we include its proof for the sake of completeness.

**Lemma 5.2** If $D \subseteq \mathbb{N}^{<\omega}$ is dense, then $I_{wf} \hookrightarrow I_{wf} \upharpoonright D$.

**Proof:** Fix a bijection $\varphi : \mathbb{N}^{<\omega} \to \mathbb{N}$ such that $\varphi(\emptyset) = 0$ and $u \leq t \Rightarrow \varphi(u) \leq \varphi(t)$. Let $\psi$ be the inverse of $\varphi$. Inductively, we are going to define a function $h : \mathbb{N}^{<\omega} \to D$ such that $u \leq t \iff h(u) \leq h(t)$. If such function exists, it is easy to see that $C \notin I_{wf} \iff h[C] \notin I_{wf}$. Therefore, $h$ is an isomorphism between $I_{wf}$ and $I_{wf} \upharpoonright D$. 

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We will define $h(\psi(n))$ by induction on $n$. First, we fix any $d_0 \in D$ and we put $h(\psi(0)) = d_0$. Now, suppose $h$ has been defined on $\psi(j)$ for $j \leq k$. Let $u \in \mathbb{N}^{<\omega}$ and $i \in \mathbb{N}$ such that $\psi(k+1) = \psi(i) \cdot u$. Put $D_k = \{h(\psi(i)) : i \leq k\} \cup \mathcal{N}_{h(t)} : u < t$ and $\varphi(t) \leq k\}$. Since $D_k$ is an $\prec$-downward closed and $D$ is dense, then we can choose $d \in D \cap \mathcal{N}_{h(u)}$ that is not in $D_k$. Put $d = h(\psi(k+1))$. It is routine to verify that $t \prec \psi(k+1) \Leftrightarrow h(t) \preceq h(\psi(k+1))$. □

**Lemma 5.3** Let $A \subseteq N^{<\omega}$ and $T$ be the tree generated by $A$. If $I_{wf} \hookrightarrow I_{wf} \upharpoonright T$, then $I_{wf} \hookrightarrow I_{wf} \upharpoonright A$.

**Proof:** We fix a bijection $\psi : \mathbb{N} \to \mathbb{N}^{<\omega}$ such that $\psi(0) = \emptyset$ and $f : \mathbb{N}^{<\omega} \to T$ witnessing that $I_{wf} \hookrightarrow I_{wf} \upharpoonright T$. We will define functions $h : \mathbb{N}^{<\omega} \to \mathbb{N}^{<\omega}$ and $g : D \to A$ where $D$ is the range of $h$. Since $\psi$ is a bijection, to simplify the notation, let $d_n = h(\psi(n))$, for $n \in \mathbb{N}$. Thus $D = \{d_n : n \in \mathbb{N}\}$. The functions $h$ and $g$ will satisfy the following properties:

(a) $\psi(n) \preceq h(\psi(n))$, for all $n \in \mathbb{N}$,

(b) $f(d_n) \preceq g(d_n)$, for all $n \in \mathbb{N}$,

(c) $g(d_{n+1}) \not\preceq g(d_i)$, for all $i \leq n$.

From property (a) we get that $D$ is dense in $\mathbb{N}^{<\omega}$ and hence, by lemma [5.2], we get that $I_{wf} \hookrightarrow I_{wf} \upharpoonright D$.

We claim that properties (b) and (c) implies that $g I_{wf} \upharpoonright D \hookrightarrow g I_{wf} \upharpoonright A$. By (c), it is clear that $g$ is $\mathcal{I} - 1$. Now we show that $D \supseteq C \notin I_{wf}$ iff $g[C] \notin I_{wf}$:

($\Rightarrow$) if $C \subseteq D$ is not in $I_{wf}$, then $f[C] \notin I_{wf}$ and so there are a sequence $\lambda \in \mathbb{N}^{\omega}$ and an infinite set $\{c_l : l \in \mathbb{N}\} \subseteq C$ such that for all $l \in \mathbb{N}$, $\lambda \upharpoonright l \preceq f(c_l)$. But $f(c_l) \preceq g(c_l)$ for all $l \in \mathbb{N}$. Thus, $g[C] \notin I_{wf}$.

($\Leftarrow$) if $g[C] \notin I_{wf}$ there are a sequence $\eta \in \mathbb{N}^{<\omega}$ and an infinite set $\{c_l : l \in \mathbb{N}\} \subseteq C$ such that for all $l \in \mathbb{N}$, $\eta \upharpoonright l \preceq g(c_l)$. Notice that $\{f(c_l) : l \in \mathbb{N}\}$ is infinite (as $f$ is $\mathcal{I} - 1$). Since $f(c_l) \preceq g(c_l)$ for all $l \in \mathbb{N}$, then the length of the $f(c_l)$’s must increase with $l$. Hence, $\eta \in \{f(c_l) : l \in \mathbb{N}\}$ and so $f[C] \notin I_{wf}$. Therefore $C \notin I_{wf}$, as $f$ is an isomorphism.

In summary, assuming that such functions $h$ and $g$ exist, then $I_{wf} \hookrightarrow I_{wf} \upharpoonright D$ and $I_{wf} \upharpoonright D \hookrightarrow I_{wf} \upharpoonright A$. Thus, $I_{wf} \hookrightarrow I_{wf} \upharpoonright A$.

So it remains to show the construction of $h$ and $g$. We will define $h(\psi(n))$ and $g(d_n)$ by induction on $n$.

First of all, for $n = 0$, it is easy to see that there is $a \in A$ such that $f(\emptyset) \preceq a$. Pick an $a \in A$ with that property. Put $h(\psi(0)) = d_0 = h(\emptyset) = \emptyset$ and $g(d_0) = a$. Note that $\emptyset = \psi(0) \preceq h(\psi(0))$ and $f(d_0) \preceq g(d_0)$. Suppose we have defined $h(\psi(n))$ and $g(d_n)$. We claim that

$$\exists s \in \mathcal{N}_{\psi(n+1)} \exists a \in A \{f(s) \preceq a \text{ and } (\forall i \leq n)(a \not\preceq g(d_i))\}. \quad (3)$$
In fact, since the set $\mathcal{N}_{\psi(n+1)}$ is infinite and $f$ is $1-1$, then $f[\mathcal{N}_{\psi(n+1)}]$ is infinite. In addition, $f[\mathcal{N}_{\psi(n+1)}] \subseteq \{a \in A : (\exists s \geq \psi(n+1)) (f(s) \leq a)\} = P$. Hence, $P$ is infinite. Pick $a \in P$ and $s$ as in (3). Finally, we put $h(\psi(n+1)) = d_{n+1} = s$ and $g(d_{n+1}) = a$.

To deal with trees we will define a derivative on subsets of $\mathbb{N}^{<\omega}$. Let $A \subseteq \mathbb{N}^{<\omega}$, we define

$$A^{(\ell)} = \{a \in A : A_a \notin I_d\}.$$  

For a successor ordinal, we put $A^{\beta+1} = (A^\beta)'$ and for a limit ordinal $\alpha$ we put $A^\alpha = \bigcap_{\xi < \alpha} A^\xi$.

The rank of $A$, denoted $rk(A)$, is the first ordinal $\alpha$ such that $A^\alpha = A^{\alpha+1}$.

**Lemma 5.4** Suppose $\alpha = \mu + 1$ and let $T$ be a tree with rank $\alpha$ and such that $T^{(\alpha)} = \emptyset$.

(i) If $t \in T$, then $rk(T_t) = rk(T_t \cup \{s : s \leq t\})$.

(ii) The tree $H = \{t \in T : rk(T_t) = \alpha\}$ is in $I_d$.

**Proof:** (i) Put $\gamma = rk(T_t)$ and notice that the condition $T^{(\alpha)} = \emptyset$ implies that $T^{(\gamma)} = \emptyset$. Now, since $t \in T^{(\xi)}$ for all $\xi < \gamma$, we have that $\gamma$ must be successor. Put $\gamma = \eta + 1$ and put $S = T_t \cup \{s : s \leq t\}$. Then $S^{(\eta)} = T^{(\eta)} \cup \{s : s \leq t\}$ and the result follows.

(ii) Suppose $H \notin I_d$. Then there is $t \in H$ such that $K = \{n : t \cap n \in H\}$ is infinite. For every $n \in K$, the set $T_{t-n}$ has rank $\alpha$; so by part (i), $T_{t-n}^* = \{s : s \leq t \cap n\} \cup T_{t-n}$ has rank $\alpha$. Consider the tree $L = \bigcup_{n \in K} T_{t-n}^* \subseteq H$. We claim that $t \in L^{(\alpha)}$. In fact, as $T_{t-n}^{(\mu)} \subseteq L_{t-n}^{(\mu)}$, then $t \cap n \in L_{t-n}^{(\mu)}$ for all $n \in K$ and thus $L_t^{(\mu)} \notin I_d$. Hence $\alpha < rk(L) \leq rk(H) = \alpha$ and this is a contradiction.

**Lemma 5.5** Let $H$ be an infinite tree in $I_d$. For every $s \in H$, let $P_s \subseteq \mathbb{N}^{<\omega}$ be a set consisting of extensions of $s$ such that $P_s \cap H = \emptyset$ and $(P_s)_{s \in H}$ is pairwise disjoint. Let $P = \bigcup_{s \in H} P_s$ and $R = H \cup P$. If $I_{w_f} \upharpoonright P_s \in \mathcal{B}$ for all $s \in H$, then $I_{w_f} \upharpoonright R \in \mathcal{B}$.

**Proof:** We will first proof that $I_{w_f} \upharpoonright P \in \mathcal{B}$. We claim that

$$A \in I_{w_f} \upharpoonright P \iff (\exists t_0, \ldots, t_p \in H)(A \subseteq \bigcup_{i \leq p} P_{t_i} \text{ and } (\forall i \leq p) A \cap P_{t_i} \in I_{w_f}). \quad (4)$$

In fact, let $A \subseteq \bigcup_{s \in H} P_s$ with $A \in I_{w_f}$. By the definition of $I_{w_f}$, the tree generated by $A$, denoted $\langle A \rangle$, belongs to $I_{w_f}$ and notice also that $H = \langle \bigcup_{s \in H} P_s \rangle$. If $A$ meets infinitely many $P_s$’s, then $\langle A \rangle$ has infinite many elements of $H$ (because $P_s \cap P_r = \emptyset$ for $s \neq r$) and then $\langle A \rangle \notin I_{w_f}$ (because $H \in I_d = I_{w_f}^\perp$). Thus, $\{s \in H : A \cap P_s \neq \emptyset\}$ is finite. Put $\{t_0, \ldots, t_p\} = \{s \in H : A \cap P_s \neq \emptyset\}$. Then, $A \subseteq \bigcup_{i \leq p} P_{t_i}$. The reverse implication is trivial as $I_{w_f}$ is an ideal.

So we have established (4). By lemma 2.2 and the fact that $J_s = I_{w_f} \upharpoonright P_s \in \mathcal{B}$ for $s \in H$, we have that

$$I_{w_f} \upharpoonright P \cong (\bigoplus_{s \in H} J_s^\perp)^\perp \in \mathcal{B}.$$
Finally, since $H \in I_d$ and $H \cap P = \emptyset$ we have that
\[ I_{wf} \upharpoonright R \cong I_{wf} \upharpoonright H \oplus I_{wf} \upharpoonright P \cong \text{FIN} \oplus I_{wf} \upharpoonright P \in \mathcal{B}. \]

**Lemma 5.6** Let $T$ be tree on $\mathbb{N}$ and $\alpha = \text{rk}(T)$.

(i) If $T^\alpha \neq \emptyset$, then $I_{wf} \hookrightarrow I_{wf} \upharpoonright T$.

(ii) If $T^\alpha = \emptyset$, then $I_{wf} \upharpoonright T \in \mathcal{B}$.

**Proof:** (i) Suppose $T^\alpha \neq \emptyset$. Since $T^\alpha = T^{\alpha+1}$, then $T^\alpha$ is a tree without terminal nodes. Moreover, given $t \in T^\alpha$, as $(T^\alpha)_t \notin I_d$, there is $s \supseteq t$ such that $A(t,s) = \{n : s\upharpoonright n \in (T^\alpha)_t\}$ is infinite.

We define a function $f : \mathbb{N}^{<\omega} \to \mathbb{N}^{<\omega}$ by induction on the length of the sequence. (a) $f(\emptyset) = \emptyset$. (b) Let $s$ be such that $A(\emptyset, s)$ is infinite. Let $f$ maps bijectively all sequences of length one onto the set of all $s\upharpoonright n$ for $n \in A(\emptyset, s)$. For all $n$, $f$ maps bijectively $\{(n,m) : m \in \mathbb{N}\}$ onto $\{s\upharpoonright k : k \in A(f(\langle n \rangle), s)\}$ where $s$ is chosen such that $A(f(\langle n \rangle), s)$ is infinite. And so on.

It is easy to check that $f$ is 1-1 and $A \subseteq \mathbb{N}^{<\omega}$ is well founded iff $f[A] \in I_{wf}$. Therefore $I_{wf} \hookrightarrow I_{wf} \upharpoonright T$.

(ii) We will see that $I_{wf} \upharpoonright T \in \mathcal{B}$, whenever $T^\alpha = \emptyset$, by induction on $\alpha$. If $T^\alpha = \emptyset$, then $T \in I_d$ and therefore $I_{wf} \upharpoonright T \cong \text{FIN} \in \mathcal{B}$. Now suppose that for every $\xi < \alpha$ and for every tree $S$ with $\text{rk}(S) = \xi$, if $S^{(\xi)} = \emptyset$ then $I_{wf} \upharpoonright S \in \mathcal{B}$. Take a tree $T$ with rank $\alpha$ and such that $T^{(\alpha)} = \emptyset$. Notice that $\alpha$ cannot be a limit ordinal, so let $\beta$ be such that $\alpha = \beta + 1$. Consider the set

\[ H = \{t \in T : \text{rk}(T_t) = \alpha\}. \]

By lemma 5.4, $H$ is a tree in $I_d$. For every $s \in H$, let

\[ M_s = \{n \in \mathbb{N} : \text{rk}(T_{s\upharpoonright n}) < \alpha\}. \]

By lemma 5.4, the tree $T_{s\upharpoonright n} \cup \{u : u \preceq s\upharpoonright n\}$ has rank smaller than $\alpha$ for every $s \in H$ and $n \in M_s$. Therefore, by the inductive hypothesis, $I_{wf} \upharpoonright T_{s\upharpoonright n} \in \mathcal{B}$ for every $s \in H$ and $n \in M_s$.

Put $P_s = \bigcup_{n \in M_s} T_{s\upharpoonright n}$. From lemma 4.2(i) we have that $I_{wf} \upharpoonright P_s \in \mathcal{B}$. We claim that

\[ T = H \cup \bigcup_{s \in H} P_s. \]

In fact, only one inclusion needs a proof. Let $t \in T$ and $\text{rk}(T_t) = \alpha$ then $t \in H$. If $\text{rk}(T_t) < \alpha$, let $t'$ be the minimal restriction of $t$ such that $\text{rk}(T_{t'}) < \alpha$. Put $l_0 = |t'| > 0$. By definition of $t'$ it is clear that $s = t' \upharpoonright (l_0 - 1) \in H$. Then $t \in P_s$ and $s \in H$.

Thus, $T = H \cup \bigcup_{s \in H} P_s$, where $P_s \cap H = \emptyset$ and $I_{wf} \upharpoonright P_s \in \mathcal{B}$ for every $s \in H$. If $H$ is a finite set, then

\[ I_{wf} \upharpoonright T \cong I_{wf} \upharpoonright \bigcup_{s \in H} P_s \cong \bigoplus_{s \in H} I_{wf} \upharpoonright P_s \in \mathcal{B}. \]
If $H$ is infinite, applying lemma 5.5 we get that $I_{wf} \upharpoonright T \in \mathcal{B}$. ■

**Proof of theorem 5.1** It remains to show that (iii) implies (i). Suppose $I_{wf} \not\hookrightarrow I_{wf} \upharpoonright A$. Let $T$ be the tree generated by $A$. By lemma 5.3 we have $I_{wf} \not\hookrightarrow I_{wf} \upharpoonright T$. Thus, from lemma 5.6 we conclude that $I_{wf} \upharpoonright T \in \mathcal{B}$. Hence, by lemma 3.5, $I_{wf} \upharpoonright A \sim I_{wf} \upharpoonright (T \cap A) \in \mathcal{B}$. ■

By taking orthogonal, we get the following immediate consequence of theorem 5.1.

**Corollary 5.7** Let $A$ be a subset of $\mathbb{N}^{<\omega}$, then $I_d \not\hookrightarrow I_d \upharpoonright A$ iff $I_d \upharpoonright A \in \mathcal{B}$.

From theorem 4.1 we get the following.

**Corollary 5.8** If $I_{wf} \upharpoonright K$ is Borel, then $I_{wf} \upharpoonright K$ is $F_{\sigma\delta}$.

### 6 Borel restrictions of $WO(\mathbb{Q})$

In this section we will show a result analogous to theorem 5.1 for the ideal $WO(\mathbb{Q})$ of the well founded subsets of $WO(\mathbb{Q})$. For simplicity, we will write $WO$ instead of $WO(\mathbb{Q})$. We first observe that $WO^\perp$ is the ideal of well founded subsets of $(\mathbb{Q}, <^*)$ where $<^*$ is the reversed order of $\mathbb{Q}$. In fact, the map $x \mapsto -x$ from $\mathbb{Q}$ onto $\mathbb{Q}$ is an isomorphism between $WO$ and $WO^\perp$. In particular, $WO$ is a Fréchet ideal.

We recall that linear order $(L, <)$ is said to be *scattered*, if it does not contain a order-isomorphic copy of $\mathbb{Q}$. The main result is the following.

**Theorem 6.1** For every $A \subseteq \mathbb{Q}$, the following are equivalent:

(i) $A$ is scattered.

(ii) $WO \upharpoonright A$ belongs to $\mathcal{B}$.

(iii) $WO \upharpoonright A$ is Borel.

(iv) $WO \not\hookrightarrow WO \upharpoonright A$.

Since $WO$ is a complete coanalytic set (see [8, 33.2]) and each ideal in $\mathcal{B}$ is Borel, then it is clear that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). To see (iv) $\Rightarrow$ (i), suppose $A \subseteq \mathbb{Q}$ is not scattered. Any embedding of $(\mathbb{Q}, <)$ inside $(A, <)$ is also an embedding of $WO$ inside $WO \upharpoonright A$. So it only remains to show that (i) implies (ii). For that end we need to recall a well known result of Hausdorff about countable scattered orders.

Given a sequence of linear orders $(L_n, <_n)$ over a disjoint collection of sets $(L_n)_{n \in \mathbb{N}}$, the sum $\sum_{n \in \mathbb{N}} L_n$ is defined as the lexicographical order on $L = \bigcup_n L_n$. That is to say, for $x, y \in L$, $x <_L y$ iff either $x, y \in L_n$ for some $n$ and $x <_n y$ or $x \in L_n$ and $y \in L_m$ with $n < m$. The sum of two (or finitely many) linear orders is defined in a similar manner. If $L$ is a linear order, then $L^*$ denotes the reversed order.
We denote by $SC$ the closure of $\{ (\mathbb{N}, <) \}$ under the operations of taking countable or finite sums and reversal of an order. The collection $SC$ is naturally presented as an increasing union of of families $SC_{\alpha}$ with $\alpha < \omega_1$. Where $SC_0$ consists of $\mathbb{N}$, $\mathbb{N}^*$ and the sums of them $\mathbb{N} + \mathbb{N}^*$ and $\mathbb{N}^* + \mathbb{N}$. Then $SC_{\alpha}$ consists of sums of orders of rank smaller than $\alpha$ and its reversed orders. We say that $L$ has rank $\alpha$, if $L \in SC_{\alpha}$ and $L \not\in SC_{\beta}$ for all $\beta < \alpha$.

**Theorem 6.2** *(Hausdorff [11])*: A countable linear order is scattered iff it is isomorphic to an order in $SC$.

Notice that if $L \subseteq \mathbb{Q}$, then $(L, <_Q)^*$ is isomorphic to $(-L, <_Q)$ (where $-L = \{-x : x \in L\}$). The following simple observation is the key fact to prove our result.

**Lemma 6.3** Let $L \subseteq \mathbb{Q}$.

(i) If $L$ is order isomorphic to a sum $\sum_{n \in \mathbb{N}} L_n$, for some disjoint sequence of sets $L_n \subseteq \mathbb{Q}$, then $WO|L \cong \bigoplus_n WO|L_n$.

(ii) If $L$ is isomorphic to a sum $L_1 + L_2$ where $L_1$ and $L_2$ are disjoint subsets of $\mathbb{Q}$, then $WO|L \cong WO|L_1 \oplus WO|L_2$.

(iii) $(WO|L)^\perp \cong WO|L^*$.

**Proof of Theorem 6.1**: It only remains to show that (i) implies (ii). This will be done by induction on the scattered order. The base of the induction is trivial since it is clear that if $L \subseteq \mathbb{Q}$ is order isomorphic to $\mathbb{N}$, then $WO|L \cong \mathcal{P}(\mathbb{N})$. The rest follows from lemma 6.3. ■

7 Examples of sequential analytic spaces

As it was explained in the introduction, any ideal can be identify with a topological space on $X = \mathbb{N} \cup \{\infty\}$ such that the space is Fréchet iff the ideal is Fréchet. This idea can be extended to construct other more complex topological spaces. In [16] was presented a construction of a topology $\tau_\mathcal{F}$ on $\mathbb{N}^{<\omega}$ where $\mathcal{F}$ is a filter over $\mathbb{N}$, such that $(\mathbb{N}^{<\omega}, \tau_\mathcal{F})$ is a sequential space (see the definition below) iff $\mathcal{F}$ is a Fréchet filter (i.e. its dual ideal es Fréchet). In fact, they constructed a family of size bigger than the continuum of Fréchet filters such that the corresponding sequential spaces $(\mathbb{N}^{<\omega}, \tau_\mathcal{F})$ are pairwise non homeomorphic. They ask if there is an uncountable family of analytic Fréchet filters with the same property. The purpose of this section is to give a positive answer to that question.

Let us recall that a topological space $X$ is *sequential* if whenever $A \subseteq X$ is non closed, then there is a sequence $(x_n)_n$ in $A$ converging to a point not in $A$. Clearly, any Fréchet space is sequential, but the reciprocal is not true.

Let $\mathcal{F}$ be a filter on $\mathbb{N}$ containing the cofinite sets. Define a topology $\tau_\mathcal{F}$ over $\mathbb{N}^{<\omega}$ by letting a subset $U$ of $\mathbb{N}^{<\omega}$ be open if, and only if, $\{ n \in \mathbb{N} : s \sim n \in U \} \in \mathcal{F}$, for all $s \in U$. The prototypical sequential space of sequential rank $\omega_1$ is the well known Arkhangle'skii-Franklin
space $S_\omega$ which turns out to be homeomorphic to $(N^{<\omega}, \tau_{\text{FIN}})$. The main result of this section is that the topological spaces corresponding to the dual filters of the ideals in $B$ are pairwise nonhomeomorphic. We need some preliminaries results.

**Lemma 7.1** ([16]) Let $F$ be a filter on $N$ containing the cofinite sets. Then

(i) $(N^{<\omega}, \tau_F)$ is $T_2$, zero dimensional and has no isolated points.

(ii) $(N^{<\omega}, \tau_F)$ is sequential if, and only if, $F$ is a Fréchet filter.

(iii) If $(N^{<\omega}, \tau_F)$ is sequential, then $S_\omega$ embeds into it as a closed subspace and therefore $(N^{<\omega}, \tau_F)$ has sequential order $\omega_1$.

(iv) The space $(N^{<\omega}, \tau_F)$ is homogeneous.

(v) If $F$ is Borel, then $\tau_F$ is Borel (as a subset of $2^{N^{<\omega}}$).

We also need the following fact.

**Lemma 7.2** Every ideal $I$ in $B$ is isomorphic to any restriction of itself to a set in its dual filter.

**Proof:** Let $I$ be an ideal in $B$ and $K$ such that $N \setminus K \in I$. We have that

$$I \cong I \upharpoonright (K \cup N \setminus K) \cong I \upharpoonright K \oplus I \upharpoonright (N \setminus K) \cong I \upharpoonright K \oplus P(N) \cong I \upharpoonright K.$$

The last equivalence follows from Theorem 3.2.

We denote by $F_\alpha$ the dual filter of $P_\alpha$, by $\tau_\alpha$ the topology $\tau_{F_\alpha}$, and by $N^{[1]} \subseteq N^{<\omega}$ the set of sequences of length 1.

**Proposition 7.3** If $\alpha \neq \beta$, then $(N^{<\omega}, \tau_\alpha) \not\cong (N^{<\omega}, \tau_\beta)$.

**Proof:** Suppose that $(N^{<\omega}, \tau_\alpha) \cong (N^{<\omega}, \tau_\beta)$ and let $h : N^{<\omega} \to N^{<\omega}$ be an homeomorphism witnessing this fact. By part (iv) of Lemma 7.1 we can assume that $h(\emptyset) = \emptyset$. Consider the sets $A = \{h(n) : n \in N\} \cap N^{[1]}$, $B = \{s(0) : s \in A\}$, and $C = \{h^{-1}(s)(0) : s \in B\}$. Using Lemma 7.1(ii) it is easy to see that $B \in F_\beta$, $C \in F_\alpha$, and $F_\alpha \upharpoonright B \cong F_\beta \upharpoonright C$. Thus, by Lemma 7.2 $F_\alpha \cong F_\beta$ and by Lemma 3.7 $\alpha = \beta$.

The next result gives a positive answer to question 6.9 of [16].

**Corollary 7.4** There is an uncountable family of pairwise nonhomeomorphic analytic sequential spaces of sequential order $\omega_1$. 

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