On the monoidality of Saito reflection functors

Syu Kato *

September 5, 2018

Abstract

We extend the definition of the Saito reflection functor of the Khovanov-Lauda-Rouquier algebras to symmetric Kac-Moody algebra case and prove that it defines a monoidal functor.

Introduction

In [6], the Saito reflection functors for the Khovanov-Lauda-Rouquier algebras of type $ADE$ are introduced. It categorifies Lusztig’s braid group action [11, §39] on (a subalgebra of) of the positive half of the quantum groups in the sense of Khovanov-Lauda-Rouquier [8, 16]. They are main ingredients to construct PBW bases in the spirit of Lusztig [11], and provided a certain role in the representation theory of the Khovanov-Lauda-Rouquier algebras.

The goal of this paper is to develop it little bit further, and provide some basic properties in more general setting than that of [6]. Let $A := \mathbb{Z}[t^{\pm 1}]$. Let $g$ be a symmetric Kac-Moody Lie algebra, and let $U^+$ be the positive half of the $A$-integral version of the quantum group of $g$ (see e.g. Lusztig [11] §1). Let $Q^+ := \mathbb{Z}_{>0}I$, where $I$ is the set of positive simple roots. We have a weight space decomposition $U^+ = \bigoplus_{\beta \in Q^+} U^+_{\beta}$. We have the Weyl group $W$ of $g$ with its set of simple reflections $\{s_i\}_{i \in I}$. For each $\beta \in Q^+$, we have a finite set $B(\infty)_\beta$ which parameterizes a pair of distinguished bases $\{G^{up}(b)\}_{b \in B(\infty)_\beta}$ and $\{G^{low}(b)\}_{b \in B(\infty)_\beta}$ of $\mathbb{Q}(t) \otimes_A U^+_{\beta}$. The Khovanov-Lauda-Rouquier algebra $R_\beta$ is a certain graded algebra whose grading is bounded from below with the following properties:

- The set of isomorphism classes of simple graded $R_\beta$-modules (up to grading shifts) is also parameterized by $B(\infty)_\beta$;

- For each $b \in B(\infty)_\beta$, we have a simple graded $R_\beta$-module $L_b$ and its projective cover $P_b$. Let $L'_\beta \langle k \rangle$ be the grade $k$ shift of $L'_\beta$, and let $[P_b : L'_\beta \langle k \rangle]$ be the multiplicity of $L'_\beta \langle k \rangle$ in $P_b$ (that is finite). Then, we have

$$G^{low}(b) = \sum_{b' \in B(\infty)_\beta, k \in \mathbb{Z}} t^k [P_b : L'_\beta \langle k \rangle]_0 G^{up}(b');$$
For each $\beta, \beta' \in Q^+$, there exists an induction functor
\[ \star : R_{\beta^\cdot \text{gmod}} \times R_{\beta'}^\cdot \text{gmod} \ni (M, N) \mapsto M \star N \in R_{\beta^+ \beta'}^\cdot \text{gmod}; \]

\[ K := \bigoplus_{\beta \in Q^+} \mathbb{Q}(t) \otimes_A K(R_{\beta^\cdot \text{gmod}}) \text{ is an associative algebra isomorphic to } \mathbb{Q}(t) \otimes_A U^+ \text{ with its product inherited from } \star \text{ (and the t-action is a grading shift).} \]

For each $i \in I$ and $\beta \in Q^+$, we have certain quotients $iR_{\beta}^\cdot \text{gmod}$ and $iR_{s_i \beta}^\cdot \text{gmod}$ of $R_{\beta}^\cdot \text{gmod}$. In case $s_i \beta \in Q^+$, an interpretation of Lusztig’s geometric construction yields that $iR_{\beta}^\cdot \text{gmod}$ and $iR_{s_i \beta}^\cdot \text{gmod}$ must be Morita equivalent. This naturally enables us to define a right exact functor
\[ T_i : R_{\beta^\cdot \text{gmod}} \rightarrow iR_{\beta}^\cdot \text{gmod} \rightarrow iR_{s_i \beta}^\cdot \text{gmod} \rightarrow R_{s_i \beta}^\cdot \text{gmod} \]

that we call the Saito reflection functor. Under this setting, our main results read:

**Theorem A** (Theorems 3.8 + 3.9 + 4.1). The functors $\{T_i\}_{i \in I}$ satisfies the following:

1. There exist a right adjoint functor $T^*_i$ of $T_i$;
2. For each $M \in iR_{\beta}^\cdot \text{gmod}$ and $N \in iR_{s_i \beta}^\cdot \text{gmod}$, we have
   \[ \text{ext}^*_R s_i (T_i M, N) \cong \text{ext}^*_R (M, T^*_i N); \]
3. They satisfy the braid relations;
4. For each $\beta_1, \beta_2 \in Q^+ \cap s_i Q^+$ and $M_1 \in iR_{\beta_1}^\cdot \text{gmod}$, $M_2 \in iR_{\beta_2}^\cdot \text{gmod}$, we have a natural isomorphism
   \[ T_i (M_1 \star M_2) \cong (T_i M_1) \star (T_i M_2). \]

Here we understand $M_1, M_2$ as modules of $R_{\beta_1}$ and $R_{\beta_2}$ through the pullbacks.

We remark that Theorem A confirms a conjecture in [5] and provides one way to correct an error in [6] (see Remark 4.2 or the arXiv version of [6]). Also, the above result should extend to the positive characteristic case at least when $g$ is of type ADE by using [14].

We note that Peter McNamara sent me a version of [15] during the preparation of this paper that partly overlaps with the content of the paper.

**1 Conventions and recollections**

An algebra $R$ is a (not necessarily commutative) unital $\mathbb{C}$-algebra. A variety $X$ is a separated reduced scheme $X_0$ of finite type over some localization $\mathbb{Z}_S$ of $\mathbb{Z}$ specialized to $\mathbb{C}$. It is called a $G$-variety if we have an action of a connected affine algebraic group scheme $G$ flat over $\mathbb{Z}_S$ on $X_0$ (specialized to $\mathbb{C}$). As in [11] §6 and [2] (see also [7]), we transplant the notion of weights to the derived category of ($G$-equivariant) constructible sheaves with finite monodromy on $X$. Let us
denote by $D^b(\mathfrak{X})$ (resp. $D^+(\mathfrak{X})$) the bounded (resp. bounded from the below) derived category of the category of constructible sheaves on $\mathfrak{X}$, and denote by $D^b_G(\mathfrak{X})$ the $G$-equivariant derived category of $\mathfrak{X}$. We have a natural forgetful functor $D^b_G(\mathfrak{X}) \to D^+(\mathfrak{X})$, whose preimage of $D^b(\mathfrak{X})$ is denoted by $D^b_G(\mathfrak{X})$. For an object of $D^b_G(\mathfrak{X})$, we may denote its image in $D^b(\mathfrak{X})$ by the same letter.

2 Quivers and the KLR algebras

Let $\Gamma = (I, \Omega)$ be an oriented graph with the set of its vertex $I$ and the set of its oriented edges $\Omega$. Here $I$ is fixed, and $\Omega$ might change so that the underlying graph $\Gamma_0$ of $\Gamma$ is fixed. We have a symmetric Kac-Moody algebra $g$ with its Dynkin diagram $\Gamma_0$. We refer $\Omega$ as the orientation of $\Gamma$. We form a path algebra $\mathbb{C}[\Gamma]$ of $\Gamma$.

For $h \in \Omega$, we define $h' \in I$ to be the source of $h$ and $h'' \in I$ to be the sink of $h$. We denote $i \leftrightarrow j$ for $i, j \in I$ if and only if there exists $h \in \Omega$ such that $(h', h'') = \{i, j\}$. A vertex $i \in I$ is called a sink of $\Gamma$ (or $\Omega$) if $h' \neq i$ for every $h \in \Omega$. A vertex $i \in I$ is called a source of $\Gamma$ (or $\Omega$) if $h'' \neq i$ for every $h \in \Omega$.

Let $Q^+$ be the free abelian semi-group generated by $\{\alpha_i\}_{i \in I}$, and let $Q^+ \subset Q$ be the free abelian group generated by $\{\alpha_i\}_{i \in I}$. We sometimes identify $Q$ with the root lattice of $g$ with a set of its simple roots $\{\alpha_i\}_{i \in I}$. Let $W = W(\Gamma_0)$ denote the Weyl group of type $\Gamma_0$ with a set of its simple reflections $\{s_i\}_{i \in I}$. The group $W$ acts on $Q$ via the above identification. Let $R^+ := W\{\alpha_i\}_{i \in I} \cap Q^+$ be the set of positive roots of $g$.

An $I$-graded vector space $V$ is a vector space over $\mathbb{C}$ equipped with a direct sum decomposition $V = \bigoplus_{i \in I} V_i$.

Let $V$ be an $I$-graded vector space. For $\beta \in Q^+$, we declare $\dim V = \beta$ if and only if $\beta = \sum_{i \in I} (\dim V_i) \alpha_i$. We call $\dim V$ the dimension vector of $V$. Form a vector space $E^I_{\beta} := \bigoplus_{h \in \Omega} \text{Hom}_{\mathbb{C}}(V_h, V_{h'})$.

We set $G_V := \prod_{i \in I} GL(V_i)$. The group $G_V$ acts on $E^I_{\beta}$ through its natural action on $V$. The space $E^I_{\beta}$ can be identified with the based space of $\mathbb{C}[\Gamma]$-modules with its dimension vector $\beta$.

For each $k \geq 0$, we consider a sequence $m = (m_1, m_2, \ldots, m_k) \in I^k$. We abbreviate this as $\text{ht}(m) = k$. We set $\text{wt}(m) := \sum_{j=1}^{k} \alpha_{m_j} \in Q^+$. For $\beta = \text{wt}(m) \in Q^+$, we set $\text{ht} \beta = k$. For a sequence $m' := (m'_1, \ldots, m'_k) \in I^k$, we set $m + m' := (m_1, \ldots, m_k, m'_1, \ldots, m'_k) \in I^{k+k'}$.

For $i \in I$ and $k \geq 0$, we understand that $ki = (i, \ldots, i) \in I^k$.

For each $\beta \in Q^+$, we set $Y^\beta$ to be the set of all sequences $m$ such that $\text{wt}(m) = \beta$. For each $\beta \in Q^+$ with $\text{ht} \beta = n$ and $1 \leq i < n$, we define an action of $\{\sigma_i\}_{i=1}^{n-1}$ on $Y^\beta$ as follows: For each $1 \leq i < n$ and $m = (m_1, \ldots, m_n) \in Y^\beta$, we set $\sigma_i m := (m_1, \ldots, m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots, m_n)$.

It is clear that $\{\sigma_i\}_{i=1}^{n-1}$ generates an $\Sigma_n$-action on $Y^\beta$. In addition, $\Sigma_n$ naturally acts on a set of integers $\{1, 2, \ldots, n\}$.

For $m \in Y^\beta$ and $1 \leq i < \text{ht} \beta$, we set $h_{m,i} := \#\{h \in \Omega \mid h' = m_i, h'' = m_{i+1}\}$ and $a_{m,i} := h_{m,i} + h_{\sigma_i m, i}$. 

3
Definition 2.1 (Khovanov-Lauda [8], Rouquier [16]). Let $\beta \in Q^+$ so that $n = \text{ht} \beta$. We define the KLR algebra $R_\beta$ as a unital algebra generated by the elements $z_1, \ldots, z_n$, $\tau_1, \ldots, \tau_{n-1}$, and $e(m) \ (m \in Y^\beta)$ subject to the following relations:

1. $\deg z_i e(m) = 2$ for every $i$, and

\[
\deg \tau_i e(m) = \begin{cases} 
-2 & (m_i = m_{i+1}) \\
0 & (m_i \leftrightarrow m_{i+1}) \\
0 & (\text{otherwise}) 
\end{cases}
\]

2. $[z_i, z_j] = 0$, $e(m) e(m') = \delta_{m,m'} e(m)$, and $\sum_{m \in Y^\beta} e(m) = 1$;

3. $\tau_i e(m) = e(\sigma_i m) \tau_i e(m)$, and $\tau_i \tau_j e(m) = \tau_j \tau_i e(m)$ for $|i - j| > 1$;

4. $\tau_i^2 e(m) = Q_{m,i}(z_i, z_{i+1}) e(m)$;

5. For each $1 \leq i < n$, we have

\[
\tau_{i+1} \tau_i \tau_{i+1} e(m) - \tau_i \tau_{i+1} \tau_i e(m) = \begin{cases} 
Q_{m,i}(z_{i+1}, z_i - z_{i+1}) e(m) & (m_{i+2} = m_i) \\
0 & (\text{otherwise}) 
\end{cases}
\]

6. $\tau_i z_i e(m) - z_i \tau_i e(m) = \begin{cases} 
-e(m) & (i = k, m_i = m_{i+1}) \\
e(m) & (i = k - 1, m_i = m_{i+1}) \\
0 & (\text{otherwise}) 
\end{cases}$.

Here we set

\[
Q_{m,i}(u, v) = \begin{cases} 
1 & (m_i \neq m_{i+1}, m_i \neq m_{i+1}) \\
(-1)^{\delta m_i} (u - v)^{\alpha m_i} & (m_i \leftrightarrow m_{i+1}) \\
0 & (\text{otherwise}) 
\end{cases},
\]

where $u, v$ are indeterminants.

\[\square\]

Remark 2.2. Note that the algebra $R_\beta$ a priori depends on the orientation $\Omega$ through $Q_{m,i}(u, v)$. Since the graded algebras $R_\beta$ are known to be mutually isomorphic for any two choices of $\Omega$ (cf. [16] §3.2.4 and Theorem 2.3), we suppress this dependence in the below.

For an $I$-graded vector space $V$ with $\dim V = \beta$, we define

\[
F^\Omega_\beta := \left\{ \{ F_j \}_{j=0}^{\text{ht} \beta} \bigg| \begin{array}{l} 
\text{x} \in E^\Omega_\beta. \text{ For each } 0 < j \leq \text{ht} \beta, \\
F_j \subset V \text{ is an } I \text{-graded vector subspace,} \\
F_{j+1} \subset \subseteq F_j, \text{ and satisfies } x F_j \subset \subseteq F_{j+1}. 
\end{array} \right\}
\]

and

\[
B^\Omega_\beta := \left\{ \{ F_j \}_{j=0}^{\text{ht} \beta} \bigg| \begin{array}{l} 
F_j \subset V \text{ is an } I \text{-graded vector subspace s.t. } F_{j+1} \subset \subseteq F_j. 
\end{array} \right\}.
\]

We have a projection

\[
\varphi^\Omega_\beta : F^\Omega_\beta \ni \{ F_j \}_{j=0}^{\text{ht} \beta} \mapsto \{ F_j \}_{j=0}^{\text{ht} \beta} \in B^\Omega_\beta,
\]

4
which is $G_V$-equivariant. For each $m \in Y^\beta$, we have a connected component

$$F^\Omega_m := \{(F_j, j \geq 0, x) \in F^\Omega_{\beta, 2} | \dim F_j/F_{j+1} = \alpha_{m+j}, \forall j \} \subset F^\Omega_{\beta, 2},$$

that is smooth of dimension $d^\Omega_m$. We set $B^\Omega_m := \pi^\Omega_m(F^\Omega_m)$, that is an irreducible component of $B^\Omega_{\beta, 2}$. Let

$$\pi^\Omega_m : F^\Omega_m \ni (\{F_j, j \geq 0, x\}) \mapsto x \in E^\Omega_V$$

be the second projection that is also $G_V$-equivariant. The map $\pi^\Omega_m$ is projective, and hence

$$L^\Omega_m := (\pi^\Omega_m)_! \mathbb{L}[d^\Omega_m]$$

decomposes into a direct sum of (shifted) irreducible perverse sheaves with their coefficients in $D^b(pt)$ (Gabber’s decomposition theorem, [11] 6.2.5). Let us denote by $Q^\Omega_{\beta, m}$ be the set of isomorphism classes of simple irreducible perverse sheaves that appear as a direct summand of $L^\Omega_m$ (with some shifts). We set $L^\Omega_{\beta, 2} := \bigoplus_{m \in Y^\beta} L^\Omega_m$ and $Q^\Omega_{\beta, 2} := \bigcup_{m \in Y^\beta} Q^\Omega_{\beta, m}$. Let $e(m)$ be the idempotent in $\text{End}(L^\Omega_{\beta, 2})$ so that $e(m)L^\Omega_{\beta, 2} = L^\Omega_m$. Since $\pi^\Omega_m$ is projective, we conclude that $\mathbb{D}L^\Omega_m \cong L^\Omega_m$ for each $m \in Y^\beta$, and hence

$$\mathbb{D}L^\Omega_{\beta, 2} \cong L^\Omega_{\beta, 2}. \quad (2.1)$$

**Theorem 2.3** (Varagnolo-Vasserot [17]). Under the above settings, we have an isomorphism of graded algebras:

$$R_{\beta} \cong \bigoplus_{i \in Z} \text{Ext}^i_{G_V}(L^\Omega_{\beta, 2}, L^\Omega_{\beta, 2}).$$

In particular, the RHS does not depend on the choice of an orientation $\Omega$ of $\Gamma_0$.

For each $m, m' \in Y^\beta$, we set

$$R_{m, m'} := e(m)R_{\beta}e(m') = \bigoplus_{i \in Z} \text{Ext}^i_{G_V}(L^\Omega_{m', m}).$$

We set $S_\beta \subset R_\beta$ to be a subalgebra which is generated by $e(m)$ ($m \in Y^\beta$) and $z_1, \ldots, z_n$.

For each $\beta_1, \beta_2 \in Q^+$ with $\text{ht} \beta_1 = n_1$ and $\text{ht} \beta_2 = n_2$, we have a natural inclusion:

$$R_{\beta_1} \otimes R_{\beta_2} \ni e(m) \otimes e(m') \mapsto e(m + m') \in R_{\beta_1 + \beta_2}.$$

$$R_{\beta_1} \otimes 1 \ni z_i \otimes 1, \tau_i \otimes 1 \mapsto z_i, \tau_i \in R_{\beta_1 + \beta_2}$$

$$1 \otimes R_{\beta_2} \ni 1 \otimes z_i, 1 \otimes \tau_i \mapsto z_{i+n_1}, \tau_{i+n_1} \in R_{\beta_1 + \beta_2}$$

This defines an exact functor

$$\star : R_{\beta_1} \otimes R_{\beta_2} - \text{gmod} \ni M_1 \otimes M_2 \mapsto R_{\beta_1 + \beta_2} \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (M_1 \otimes M_2) \in R_{\beta_1 + \beta_2} - \text{gmod}.$$

The functor $\star$ restricts to an exact functor in the category of graded projective modules (see e.g. [8] 2.16):

$$\star : R_{\beta_1} \otimes R_{\beta_2} - \text{proj} \ni M_1 \otimes M_2 \mapsto R_{\beta_1 + \beta_2} \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (M_1 \otimes M_2) \in R_{\beta_1 + \beta_2} - \text{proj}.$$
If $i \in I$ is a source of $\Gamma$ and $f = (f_h)_{h \in \Omega} \in E^0_V$, then we define
\[
\epsilon^*_i(f) := \dim \ker \bigoplus_{h \in \Omega, h' = i, f_h \leq \dim V_i} f_h.
\]

If $i \in I$ is a sink of $\Gamma$ and $f = (f_h)_{h \in \Omega} \in E^0_V$, then we define
\[
\epsilon_i(f) := \dim \coker \bigoplus_{h \in \Omega, h'' = i, f_h \leq \dim V_i} f_h.
\]

Each of $\epsilon^*_i(f)$ or $\epsilon_i(f)$ do not depend on the choice of a point in a $G_V$-orbit, and is a constructible function on $E^0_V$. Hence, $\epsilon_i$ or $\epsilon^*_i$ induces a function on $E^0_V$ that is constant on each $G_V$-orbit, and a function on $Q^0_\beta$ through its value on an open dense subset of the support of its element whenever $i$ is a source or a sink.

**Proposition 2.4** (Lusztig [13]). For each $i \in I$, the functions $\epsilon_i$ and $\epsilon^*_i$ descend to functions on $Q^0_\beta$ for each $\beta \in Q^+$. In particular, it gives rise to functions on the set of isomorphism classes of simple graded $R_\beta$-modules (up to degree shifts).

**Proof.** Note that [13, Proposition 6.6] considers only $\epsilon_i$, but $\epsilon^*_i$ is obtained by swapping the order of the convolution operation.

**Proposition 2.6** ([6]). The sheaf $L^0_\beta$ can be equipped with the structure of pure weight 0. In particular, the graded algebra $R_\beta$ itself is pure of weight 0.

**Proof.** The statement of [6, Proposition 2.7] is only when $\Gamma_0$ is a Dynkin quiver, but the argument works in general.
Thanks to Theorem \ref{main.thm} and Theorem \ref{main.thm} \ref{main.thm.5}, we have an identification $B(\infty,\beta) \cong Q_{\beta}^{\Omega}$. Via this identification, each $b \in B(\infty,\beta)$ defines a $G_{\Omega}$-equivariant simple perverse sheaf $IC^\Omega(b)$ on $E^\Omega_{\beta}$, where $\dim V = \beta$. Each $b \in B(\infty,\beta)$ defines an indecomposable graded projective module $P_b$ of $R_{\beta}$ with simple head $L_b$ that is isomorphic to its graded dual $L_b^\vee$.

Let $\beta \in Q^+$ so that $ht \beta = n$. For each $i \in I$ and $k \geq 0$, we set

$$Y^\beta_{k,i} := \{ m = (m_j) \in Y^\beta \mid m_1 = \cdots = m_k = i \} \text{ and } Y^\beta_{k,i} := \{ m = (m_j) \in Y^\beta \mid m_n = \cdots = m_{n-k+1} = i \}.$$  

In addition, we define two idempotents of $R_{\beta}$ as:

$$e_i(k) := \sum_{m \in Y^\beta_{k,i}} e(m), \quad \text{and} \quad e^*_i(k) := \sum_{m \in Y^\beta_{k,i}^*} e(m).$$

**Theorem 2.7** (Lusztig \cite{Lusztig90} \S 6, Lauda-Vazirani \cite{LaudaVazirani13} \S 2.5.1). Let $\beta \in Q^+$. For each $b \in B(\infty,\beta)$ and $i \in I$, we have

$$e_i(b) = \max \{ k \mid e_i(k)L_b \neq \{0\} \} \quad \text{and} \quad e^*_i(b) = \max \{ k \mid e^*_i(k)L_b \neq \{0\} \}.$$  

Moreover, $e_i(e_i(b))L_b$ and $e^*_i(e^*_i(b))L_b$ are irreducible $R_{e_i(b)} \oplus R_{\beta - e_i(b)}$-module and $R_{\beta - e^*_i(b)} \oplus R_{e^*_i(b)}$-module, respectively. In addition, if we have distinct $b, b' \in B(\infty,\beta)$ so that $e_i(b) = k = e_i(b')$ with $k \geq 0$, then $e_i(k)\tilde{L}_b$ and $e_i(k)\tilde{L}_{b'}$ are not isomorphic as an $R_{\kappa_\alpha} \oplus R_{\beta - \kappa_\alpha}$-module. \hfill \blacksquare

## 3 Saito reflection functors

Let $\Omega_i$ be the set of edges $h \in \Omega$ with $h'' = i$ or $h' = i$. Let $s_i\Omega_i$ be a collection of edges obtained from $h \in \Omega_i$ by setting $(s_i h')' = h''$ and $(s_i h)' = h'$. We define $s_i\Omega := (\Omega \setminus s_i \Omega_i) \cup s_i \Omega_i$, and set $s_i \Gamma := (I, s_i \Omega)$. Note that $\Gamma_0 = (s_i \Gamma)_0$.

Let $V$ be an $I$-graded vector space with $\dim V = \beta$. For a sink $i$ of $\Gamma$, we define

$$iE^\Omega_V := \{ (f_h)_{h \in \Omega} \in E^\Omega_V \mid \ker \left( \bigoplus_{h \in \Omega, h'' = i} f_h : \bigoplus_{h'} V_{h'} \rightarrow V_i \right) = \{0\} \}.$$  

For a source $i$ of $\Gamma$, we define

$$i' E^\Omega_V := \{ (f_h)_{h \in \Omega} \in E^\Omega_V \mid \ker \left( \bigoplus_{h \in \Omega, h'' = i} f_h : \bigoplus_{h'} V_{h'} \right) = \{0\} \}.$$  

Let $\Omega$ be an orientation of $\Gamma$ so that $i \in I$ is a sink. Let $\beta \in Q^+ \cap s_i Q^+$. Let $V$ and $V'$ be $I$-graded vector spaces with $\dim V = \beta$ and $\dim V' = \beta$ respectively. We fix an isomorphism $\phi : \bigoplus_{j \neq i} V_j \rightarrow \bigoplus_{j \neq i} V'_j$ as $I$-graded vector spaces. We define:

$$Z^\Omega_{V, V'} := \left\{ \left( (f_h)_{h \in \Omega}, (f'_h)_{h \in s_i \Omega}, \psi \right) \left\| \begin{array}{l} (f_h) \in iE^\Omega_V, (f'_h) \in i'E^\Omega_V, \\ \phi f_h = f'_h \phi \text{ for } h \in \Omega, \\ \psi : V'_i \xrightarrow{\cong} \ker \left( \bigoplus_{h \in \Omega} f_h : \bigoplus_{h} V_{h'} \rightarrow V_i \right) \end{array} \right. \right\}.$$  

7
We have a diagram:

\[
\begin{array}{ccc}
E^\Omega_{V \downarrow} & \overset{j^\vee}{\longrightarrow} & E^\Omega_{V \downarrow} \\
& \overset{q^\dagger}{\longrightarrow} & Z^\Omega_{V \downarrow} \\
& \overset{i_!}{\longrightarrow} & E^\Omega V, j^\vee & \overset{j^\vee}{\longrightarrow} & E^\Omega V, \\
\end{array}
\]  \quad (3.1)

If we set

\[
G_{V, V'} := GL(V_\downarrow) \times GL(V'_\downarrow) \times \prod_{j \neq i} GL(V_j) \cong GL(V_\downarrow) \times GL(V'_\downarrow) \times \prod_{j \neq i} GL(V'_j),
\]

then the maps \( p^\dagger_{V, V'} \) and \( q^\dagger_{V, V'} \) are \( G_{V, V'} \)-equivariant.

**Proposition 3.1** (Lusztig [12]). The morphisms \( p^\dagger_{V, V'} \) and \( q^\dagger_{V, V'} \) in \( (3.1) \) are \( Aut(V) \)-torsor and \( Aut(V') \)-torsor, respectively.

When \( \beta = \dim V \), we set

\[
i^R_{k, \beta} := \text{Ext}^r_{G_V}(j'_V \mathcal{L}^\Omega \vert_{V'_\downarrow}, j'_V \mathcal{L}^\Omega \vert_{V'_\downarrow}) \quad \text{and} \quad i^R_{s, \beta} := \text{Ext}^s_{G_V}(j'_V \mathcal{L}^\Omega \vert_{V'_\downarrow}, j'_V \mathcal{L}^\Omega \vert_{V'_\downarrow}).
\]

For each \( k > 0 \), we fix an \( I \)-graded vector subspace \( U_k \subset V \) so that \( \dim U_k = \beta - k \alpha_i \) and an \( I \)-graded vector subspace \( U'_k \subset V' \) so that \( \dim U'_k = s_i \beta - k \alpha_i \). We have natural embeddings \( \kappa_k : E^\Omega_{U_k} \subset E^\Omega_V \) and \( \eta_k : E^\Omega_{U'_k} \subset E^\Omega_{V'_k} \) by adding a trivial \( C[I] \)-module of its dimension vector \( k \alpha_i \).

**Theorem 3.2** (Lusztig [13]). Let \( k > 0 \). The restriction \( \kappa_k^* \mathcal{L}^\Omega_{\beta} \) is a direct sum of shifted perverse sheaves in \( Q_{\beta-k \alpha_i} \). Similarly, the restriction \( \eta_k^* \mathcal{L}^\Omega_{\beta} \) is a direct sum of shifted perverse sheaves in \( Q_{s_i, \beta-k \alpha_i} \).

**Proof.** The assertion is exactly [13] Proposition 4.2 since the projection map \( p \) (in the notation of [13]) is an isomorphism if we appropriately arrange \( W \) and \( T \) in [13] \S4.1].

We set \( i^E_{V, k} := G_V E^\Omega_{V, k} \), \( i^E_{V, (k)} := i^E_{V, k} \setminus i^E_{V, k+1} \), \( i_k : i^E_{V, k} \hookrightarrow E^\Omega_V \) and \( j_k : i^E_{V, (k)} \hookrightarrow E^\Omega_{V, k} \) for each \( k > 0 \). We have \( i^E_{V, k} = \bigcup_{k' \geq k} i^E_{V, (k')} \), and we have \( e_i(x) = k \) for \( x \in i^E_{V, (k)} \). The map \( i_k \) is closed immersion, and the map \( j_k \) is an open embedding. We set \( i^E_{V, k} := G_V E^\Omega_{V, k} \), and we define similar maps \( i_k, j_k \) for them that we use only as “an analogous” situation.

**Proposition 3.3.** Let \( k > 0 \). The sheaf \( i_k^* \mathcal{L}^\Omega_{\beta} \) is the direct sum of shifted perverse sheaves in \( Q_{\beta} \) supported on \( i^E_{V, k} \) if we restrict them to \( i^E_{V, (k)} \). Similarly, the restriction \( i_k^* \mathcal{L}^\Omega_{\beta} \) is a direct sum of shifted perverse sheaves in \( Q_{s_i, \beta} \) supported on \( i^E_{V, (k)} \) along the loci with \( e_i^* = k \).

**Proof.** As the proofs of the both cases are completely parallel, we concentrate to the case of \( i_k^* \mathcal{L}^\Omega_{\beta} \).

The map \( \kappa_k \) factors through \( i_k \) as

\[
E^\Omega_{U_k} \overset{\kappa_k}{\longrightarrow} i^E_{V, k} \overset{j_k}{\longrightarrow} E^\Omega_V
\]

for each \( k \). Thus, Theorem 3.2 asserts that \( (\kappa_k^*)^* i_k^* \mathcal{L}^\Omega_{\beta} \) is a direct sum of shifted perverse sheaves in \( Q_{\beta-k \alpha_i} \). We set \( n := \dim V_i \). Let \( P_k \subset GL(n) \cong GL(V_i) \) be
the parabolic subgroup so that its Levi part is $GL(n - k) \times GL(k)$ and stabilizes $E^\Omega_U \subset E^\Omega_V$. Then, we have a map

$$\pi_k : GL(n) \times_{P_k} E^\Omega_U \to E^\Omega_V,$$

that is projective over the image. Note that $\pi_k$ is locally trivial fibration over $iE_{V,(k)}^\Omega$ with its fiber isomorphic to $\text{Gr}(k, n)$.

The sheaf $(\pi_k)_* \pi_k^* E^\Omega_V$ can be regarded as the induction of the sheaf $\kappa^\Omega_{\beta} \mathcal{L}_\beta^\Omega$, and hence it is a direct sum of shifted perverse sheaves in $Q^\Omega_{\beta}$. The above argument tells us that $i_k^* \mathcal{L}_\beta^\Omega$ is a direct summand of $(\pi_k)_* \pi_k^* E^\Omega_V$ when restricted to $iE_{V,(k)}^\Omega$. Therefore, we conclude that $i_k^* \mathcal{L}_\beta^\Omega$ is a direct sum of shifted perverse sheaves in $Q^\Omega_{\beta}$ supported on $iE_{V,k}^\Omega$ restricted to $iE_{V,(k)}^\Omega$ as required. \qed

For each $k > 0$, we define

$$iR^\Omega_{\beta,k} := \text{Ext}^{•}_G (j_{V,k}^!, j_{V,k}^! \mathcal{L}^\Omega_V),$$

where $j_{V,k} : E^\Omega_V \setminus E^\Omega_{V,k} \to E^\Omega_V$. By definition, we have $iR^\Omega_{\beta,1} = iR^\Omega_{\beta}$. By convention, we have $j_{V,k} = \text{id}$ for $k > \dim V$, and we have $iR^\Omega_{\beta,k} = iR^\Omega_{\beta}$ in this case. We also define $iR^\Omega_{s_{\beta},k}$ in a similar fashion, that we use only as “an analogous” situation.

**Theorem 3.4.** For each $k > 0$, we have an algebra isomorphism

$$iR^\Omega_{\beta,k} \cong R_{\beta}/(R_{\beta} c_i(k) R_{\beta}).$$

Moreover, $iR^\Omega_{\beta,k+1} e_i(k)iR^\Omega_{\beta,k+1}$ is projective as a $iR^\Omega_{\beta,k+1}$-module. Similarly, the algebra $iR^\Omega_{s_{\beta},k+1} e_i(k)iR^\Omega_{s_{\beta},k+1}$ is projective as a $iR^\Omega_{s_{\beta},k+1}$-module. In particular, the algebras $iR^\Omega_{\beta,k}$ and $iR^\Omega_{s_{\beta},k}$ do not depend on the choice of $\Omega$.

**Proof.** Since the case of $iR^\Omega_{s_{\beta},k}$ is completely parallel, we concentrate to the case of $iR^\Omega_{\beta,k}$. The case $k > 0$ is clear, and hence we prove the assertion by the downward induction on $k$. In particular, we assume that

$$iR^\Omega_{\beta,k+1} \cong R_{\beta}/(R_{\beta} e_i(k + 1) R_{\beta})$$

to prove our assertion. We denote $iR^\Omega_{\beta,k+1}$ by $R_{\beta,k+1}$ for simplicity.

We have

$$\text{Ext}^{•}_G (j_{V,k}^!, j_{V,k}^! \mathcal{L}^\Omega_V) \cong \text{Ext}^{•}_G (j_{V,k}^!, j_{V,k}^! \mathcal{L}^\Omega_V)$$

$$\cong \text{Ext}^{•}_G ((j_{V,k}) j_{V,k}^! \mathcal{L}^\Omega_V, \mathcal{L}^\Omega_V).$$

We set $E_k := (E^\Omega_V \setminus E^\Omega_{V,k+1})$. By assumption, we can restrict ourselves to $E_k$ to compute the Ext-groups. Hence, we freely assume that our maps are restricted to $E_k$ unless otherwise stated.

We have a distinguished triangle

$$(j_{V,k}) j_{V,k}^! \mathcal{L}^\Omega_V \to \mathcal{L}^\Omega_V \to (i_{V,k}) i_{V,k}^! \mathcal{L}^\Omega_V \to +1,$$  \hspace{1cm} (3.2)
where \( i_{V,k} : iE^0_{V,(k)} \rightarrow E_k \) is the complement inclusion. This yields an exact sequence

\[
\Ext^*_{G_V}(i_{V,k}^! L^0_k, \mathcal{L}_V^0) \rightarrow \Ext^*_{G_V}(L^0_k, \mathcal{L}_V^0) \xrightarrow{\psi} \Ext^*_{G_V}(j_{V,k}^! j_{V,k}^! L^0_k, \mathcal{L}_V^0) \\
\rightarrow \Ext^*_{G_V}(i_{V,k}^! i_{V,k}^! L^0_k, \mathcal{L}_V^0)
\]

as \( R_{\delta,k+1} \)-modules. Note that \( L_0 \) is the coefficient of \( IC^0(b) \) in \( L_0 \), and hence its support is contained in \( iE^0_{V,k} \) when \( \epsilon_i(k)L_0 \neq \{0\} \). In particular, the simple \( R^0_\delta \)-module \( L_0 \) contributes to \( \Ext^*_{G_V}(j_{V,k}^! j_{V,k}^! L^0_k, \mathcal{L}_V^0) \) by (graded) Jordan-Hölder multiplicity zero when \( \epsilon_i(k)L_0 \neq \{0\} \). It follows that \( R_{\delta,k+1} \subset \ker \psi \).

The action of \( H^*_G(pt) \) on \( \Ext^*_{G_V}(L^0_k, \mathcal{L}_V^0) \) is through the center of \( R_\delta \) (see e.g. [L7]), and it is torsion-free. Hence, the action of \( H^*_G(V) \mathbb{Q}(pt) \) on \( R_{\delta,k+1} \cong \Ext^*_{G_V}(j_{V,k+1}^! \mathcal{L}_V^0, j_{V,k+1}^! \mathcal{L}_V^0) \) factors through the center of \( R_{\delta,k+1} \).

Since \( (i_{V,k})_* = (i_{V,k}) ! \), we have

\[
\Ext^*_{G_V}(i_{V,k}^! L^0_k, \mathcal{L}_V^0) \cong \Ext^*_{G_V}(i_{V,k}^! L^0_k, \mathcal{L}_V^0).
\]

By our convention, \( i_{V,k}^! L^0_k \) and \( i_{V,k}^! \mathcal{L}_V^0 \) are supported on \( iE^0_{V,(k)} \). In addition, we have \( iE^0_{V,(k)} \cong GL(V) \times GL(V) \cap iE^0_{V,(k)} \) for a parabolic subgroup \( P_k \subset GL(V) \) borrowed from the proof of Proposition 3.3. Here, the subgroup \( GL(V) \subset P_k \) acts on \( iE^0_{V,(k)} \) trivially. From this and the induction equivalence (2.6.3), we obtain a free action of \( H^*_G(V) \mathbb{Q}(pt) \) on \( \Ext^*_{G_V(V)}((i_{V,k})_* i_{V,k}^! L^0_k, \mathcal{L}_V^0) \). The image of the pullback map \( H^*_G(V) \mathbb{Q}(pt) \rightarrow H^*_G(V) \mathbb{Q}(pt) \) contain \( k \)-algebraically independent elements (over the base field \( \mathbb{C} \)). From these, we conclude that the \( H^*_G(V) \mathbb{Q}(pt) \)-action on \( \Ext^*_{G_V(V)}((i_{V,k})_* i_{V,k}^! L^0_k, \mathcal{L}_V^0) \) contains at least \( k \) algebraically independent elements that act torsion-freely.

On the other hand, the action of \( H^*_G(V) \mathbb{Q}(pt) \) on \( \Ext^*_{G_V(V)}(j_{V,k}^! j_{V,k}^! L^0_k, \mathcal{L}_V^0) \) arises from the \( GL(V) \)-action on some algebraic stratification of \( E_{k-1} \) (see e.g. Chriss-Ginzburg [G 3.2.23 and 8.4.8]) so that the stalks of elements of \( Q_\delta^0 \) are constant (by the construction of \( Q_\delta^0 \); note that our stratification is finite). In other words, we have a finite \( G_V \)-stable stratification

\[
\bigcup_{\lambda \in \Lambda} S_{\lambda} \subseteq \bigcup_{\lambda \in \Lambda} E^0_{V,k} \subseteq \bigcup_{\lambda \in \Lambda} S_{\lambda}
\]

and a complex of locally constant sheaves \( \mathcal{E}_\lambda \) (obtained by a successive application of recollements) over \( S_{\lambda} \) so that \( \Ext^*_{G_V(V)}(j_{V,k}^! L^0_k, \mathcal{L}_V^0) \) is written as a finite successive distinguished triangles using \( H^*_G(V) \mathbb{Q}(S_{\lambda}, \mathcal{E}_\lambda) \). Moreover, \( \Ext^*_{G_V(V)}(j_{V,k}^! \mathcal{L}_V^0, j_{V,k}^! L^0_k) \) must be a finitely generated \( H^*_G(V) \mathbb{Q}(pt) \)-module as a result of the the fact that \( \mathcal{L}_V^0 \) is a finite direct sum of constructible complexes over \( E^0_{\delta} \).

The rank of the stabilizer of the \( GL(V) \)-action on a point of \( E_{k-1} \) is always \( < k \). As a consequence, the action of \( H^*_G(V) \mathbb{Q}(pt) \) on \( H^*_G(V) \mathbb{Q}(S_{\lambda}, \mathcal{E}_\lambda) \) (for every \( \lambda \in \Lambda \)) cannot carry \( k \)-algebraically independent elements that act torsion-free. Therefore, the same holds for \( \Ext^*_{G_V(V)}(j_{V,k}^! L^0_k, j_{V,k}^! L^0_k) \). Thus, the map

\[
\Ext^*_{G_V(V)}(j_{V,k}^! L^0_k, \mathcal{L}_V^0) \rightarrow \Ext^*_{G_V(V)}(i_{V,k}^! L^0_k, \mathcal{L}_V^0)
\]
must be nullity as we do not have enough number of algebraically independent
elements of $H^\bullet_{GL(V_i)}(pt)$ that acts on the LHS without torsion. By imposing the
$G_V$-equivariance, we obtain a map

$$H^\bullet_{GL(V_i)}(pt) \otimes \text{Ext}^\bullet_{GL(V_i)}(j_{V,k}^i \cdot \mathcal{L}_0^\beta, \mathcal{L}_V^\beta) \rightarrow H^\bullet_{GL(V_i)}(pt) \otimes \text{Ext}^{\bullet+1}_{GL(V_i)}((i_{V,k})_* \mathcal{L}_0^\beta, \mathcal{L}_V^\beta)$$

that induces a map

$$\text{Ext}^\bullet_{GL(V_i)}(j_{V,k}^i \cdot \mathcal{L}_0^\beta, \mathcal{L}_V^\beta) \rightarrow \text{Ext}^{\bullet+1}_{GL(V_i)}((i_{V,k})_* \mathcal{L}_0^\beta, \mathcal{L}_V^\beta)$$

through the spectral sequence (pulling back to the classifying space of $G_V$). This map must be also nullity as it is induced from the nullity.

Hence, we conclude a short exact sequence

$$0 \rightarrow \text{Ext}^\bullet_{GL(V_i)}((i_{V,k})_* \mathcal{L}_0^\beta, \mathcal{L}_V^\beta) \rightarrow R_{\beta,k+1} \xrightarrow{\psi} R_{\beta,k} \rightarrow 0$$

as left $R_{\beta,k+1}$-modules.

By Proposition 3.3 the sheaf $(i_{V,k})_* \mathcal{L}_0^\beta$ is a direct sum of shifted perverse sheaves on $E_k$, that is supported on $j_{V,k}^i \cdot \mathcal{L}_0^\beta$ (or $i_{V,k}^i \cdot \mathcal{L}_0^\beta$). It follows that the graded

$R_{\beta,k+1}$-module $\text{Ext}^\bullet_{GL(V_i)}((i_{V,k})_* \mathcal{L}_0^\beta, \mathcal{L}_V^\beta)$ is the direct sum of projective covers of $L_0$ with $\epsilon_i(b) = k$. Since $R_{\beta,k+1} \epsilon_i(b) R_{\beta,k+1}$ is the maximal left $R_{\beta}$-submodule of $R_{\beta,k+1}$ generated by irreducible constituents $\{L_0\}_{b \epsilon_i(k)}$, we deduce

$$R_{\beta,k+1} \epsilon_i(k) R_{\beta,k+1} \cong \text{Ext}^\bullet_{GL(V_i)}((i_{V,k})_* \mathcal{L}_0^\beta, \mathcal{L}_V^\beta) \cong \text{Ext}^\bullet_{GL(V_i)}(\mathcal{L}_V^\beta, (i_{V,k})_* \mathcal{L}_0^\beta)$$

where the latter modules are actually calculated on $E_k$. Therefore, we conclude the assertions for $R_{\beta,k}$ as required.

This proceeds the induction step, and we conclude the assertion.

**Corollary 3.5.** The set of isomorphism classes of graded simple modules of

$i_{R^\beta_1} \otimes i_{R^\beta_2}$ and $i_{R^\beta_1} \otimes i_{R^\beta_2}$ are $\{L_0 \langle j \rangle\}_{\epsilon_i(b) = 0, j \in \mathbb{Z}}$ and $\{L_0 \langle j \rangle\}_{\epsilon_i(b) = 0, j \in \mathbb{Z}}$, respectively.

**Theorem 3.6 (12).** The maps $q_{V,i}^j$ and $p_{V,i}^j$, give rise to a bijective correspondence between perverse sheaves corresponding to $\{b \in B(\infty)_{s \beta} | \epsilon_i(b) = 0\}$ and $\{b \in B(\infty)_{s \beta} | \epsilon_i(b) = 0\}$.

**Proof.** In view of Theorem 3.3, the combination of [12, Theorem 8.6] and Proposition 2.3 implies the result (see also [11, Proposition 38.1.6]).

**Proposition 3.7 (6).** In the setting of Proposition 2.1, two graded algebras

$i_{R^\beta_1}$ and $i_{R^\beta_2}$ are Morita equivalent to each other. In addition, this Morita equivalence is independent of the choice of $\Omega$ (as long as $i$ is a sink).

**Proof.** Although the original setting in [6, Proposition 3.5] is only for types ADE, the arguments carry over to this case in view of Theorem 3.6.

For each $b \in B(\infty)_{s \beta}$, we denote by $T_i(b) \in B(\infty)_{s \beta} \cup \{\emptyset\}$ the element so that

$$(p_{V,i}^j)^* \text{IC}^i_{s \beta}(b) \cdot ((\text{dim } V_i)^2)^2 \cong (q_{V,i}^j)^* \text{IC}^i_{s \beta}(T_i(b)) \cdot ((\text{dim } V_i)^2)^2,$$

(we understand that $T_i(b) = \emptyset$ if $\text{supp} \text{IC}^i_{s \beta}(b) \not\subseteq \text{Im } p_{V,i}^j$. Note that $T_i(b) = \emptyset$ if and only if $\epsilon_i(b) > 0$. In addition, we have $\epsilon_i(T_i(b)) = 0$ if $T_i(b) \neq \emptyset$. We set $T_i^{-1}(b') : b$ if $b' = T_i(b) \neq \emptyset$.}
Thanks to Theorem 3.4, we can drop Ω or $s_i\Omega$ from $iR^\Omega_\beta$ and $iR^s_\beta\Omega$. We define a left exact functor

$$T^*_i : R_\beta\text{-}gmod \to iR_\beta\text{-}gmod \to iR_\beta\text{-}gmod \leftrightarrow R_{s_i}\text{-}gmod,$$

where the first functor is Hom$_{R_\beta}(iR_\beta, \bullet)$, the second functor is Proposition 3.7, and the third functor is the pullback. Similarly, we define a right exact functor

$$T_i : R_\beta\text{-}gmod \to iR_\beta\text{-}gmod \to iR_\beta\text{-}gmod \leftrightarrow R_{s_i}\text{-}gmod \leftrightarrow R_{s_i}\text{-}gmod,$$

where the first functor is $iR_\beta \otimes_{R_\beta} \bullet$. We call these functors the Saito reflection functors ([6, §3]). By the latter part of Proposition 3.7, we see that these functors are independent of the choices involved.

**Theorem 3.8** ([6] Theorem 3.6). Let $i \in I$. We have:

1. For each $b \in B(\infty)_\beta$, we have

$$T_iL_b = \begin{cases} \{L_{T_i}(b) \mid \epsilon^*_i(b) = 0 \} & \text{and} \end{cases} \begin{cases} \{0 \} & \text{and} \end{cases} \begin{cases} \{L_{T_i^{-1}}(b) \mid \epsilon_i(b) = 0 \} & \text{and} \end{cases} \begin{cases} \{0 \} & \text{and} \end{cases} \begin{cases} \{\epsilon_i(b) > 0 \} & \text{and} \end{cases} \begin{cases} \{0 \} & \text{and} \end{cases} \begin{cases} \{\epsilon_i(b) > 0 \} & \text{and} \end{cases} \begin{cases} \{0 \} & \text{and} \end{cases} \begin{cases} \{\epsilon_i(b) > 0 \} \end{cases}.$$

2. The functors $(T^*_i, T_i)$ form an adjoint pair;

3. For each $M \in iR_\beta\text{-}gmod$ and $N \in iR_{s_i}\text{-}gmod$, we have

$$\text{ext}^{\ast}_{R_{s_i}}(T^*_iM, N) \cong \text{ext}^{\ast}_{R_\beta}(M, T_iN).$$

**Proof.** The proof of the first assertion is the same as [6, Theorem 3.6] if we replace standard modules with projective modules, that involves only simple perverse sheaves. The proof of the second assertion is exactly the same as [6, Theorem 3.6]. The third assertion requires the second part of Theorem 3.4 instead of [6, Corollary 1.6] (and also we need to repeat projective resolutions inductively on $\epsilon_i$ and $\epsilon^*_i$ in a downward fashion).

**Theorem 3.9.** Let $i, j \in I$. We have:

- If $i \not\leftrightarrow j$, then we have $T^*_iT_j \cong T_jT_i$;
- If $\# \{h \in \Omega \mid \{h', h'' \} = \{i, j\} \} = 1$, then we have $T^*_iT_jT^*_i \cong T_jT_iT_j$.

The same is true for $T^*_i$ and $T_j$.

**Proof.** By [12, §9.4], the functor $T_i$ induces an isomorphism described in [11, Lemma 38.1.3] (see also [18]). Hence, [11, Theorem 39.4.3] (cf. Theorem 3.8 1)) implies that the both sides give the same correspondence between simple modules. As each of $T_i$ transplants the simple modules and annihilates all the submodule that contains some specific simple modules (that induces an equivalence between some Serre subcategories), the same is true for their composition. Therefore, we conclude the result.
4 Monoidality of the Saito reflection functor

We work in the setting of [2]. The goal of this section is to prove the following:

**Theorem 4.1.** Let $i \in I$, and let $\beta_1, \beta_2 \in Q^+$ so that $s_i \beta_1, s_i \beta_2 \in Q^+$. There exists a natural transformation

$$T_i^*(\bullet \bullet) \longrightarrow T_i^*(\bullet) \times T_i^*(\bullet)$$

as functors from the category of $R_{\beta_1} \boxtimes R_{\beta_2}$-modules that gives rise to an isomorphism of functors. The same holds for $T_i$ if we consider functors from the category of $^t R_{\beta_1} \boxtimes ^t R_{\beta_2}$-modules.

**Remark 4.2.** Theorem 4.1 or rather its T-version, corrects a mistake in the proof of [6] Lemma 4.2.2). Note that another correction was made for the arXiv version of [6].

The rest of this section is devoted to the proof of Theorem 4.1 and the main body of the proof is at the end of this section.

Let $\beta_1, \beta_2 \in Q^+$ and set $\beta := \beta_1 + \beta_2$. The induction functor $\bullet$ is represented by a bimodule $R_{\beta} \otimes \beta$, where

$$e_{\beta_1, \beta_2} = \sum_{m_i \in \mathbb{H} \otimes m_2 \in \mathbb{H}^2} e(m_1) \otimes e(m_2).$$

We fix an orientation $\Omega$, and we might drop the superscript $\Omega$ freely if the meaning is clear from the context. We fix $I$-graded vector spaces $V(1)$ and $V(2)$ so that $\dim V(i) = \beta_i = \sum_{j \in I} d_j(i) \alpha_i$ for $i = 1, 2$, and $V := V(1) \oplus V(2)$.

We consider two varieties with natural $G_V$-actions:

$$\text{Gr}^\Omega_{V(1), V(2)}(V) := \left\{ (F, x, \psi_1, \psi_2) \biggm| \begin{array}{l} F \subset V : \text{I-graded vector subspace} \\
F \subset E_V, \text{ s.t. }xF \subset F \\
\psi_1 : V/F \cong V(1), \psi_2 : F \cong V(2) \end{array} \right\},$$

$$\text{Gr}^\Omega_{V_1, V_2}(V) := \left\{ (F, x) \biggm| F \subset V : \text{I-graded vector subspace} \right\}.$$ 

We have a $G_{V(1)} \times G_{V(2)}$-torsor structure $\theta^\Omega : \text{Gr}_{V(1), V(2)}(V) \longrightarrow \text{Gr}_{V_1, V_2}(V)$ given by forgetting $\psi_1$ and $\psi_2$. We have two maps

$$p^\Omega : \text{Gr}_{V_1, V_2}(V) \ni (F, x) \mapsto x \in E_V$$

and

$$q^\Omega : \text{Gr}_{V(1), V(2)}(V) \ni (F, x, \psi_1, \psi_2) \mapsto (\psi_1(x \mod F), \psi_2(x | F)) \in E_{V(1)} \oplus E_{V(2)}.$$

Notice that $\theta$ and $q$ are smooth of relative dimensions $\dim G_{V(1)} + \dim G_{V(2)}$ and $\frac{1}{2}(\dim G_V + \dim G_{V(1)} + \dim G_{V(2)}) + \sum_{n \in \mathbb{N}} d_1(h')d_2(h'')$, respectively. The map $p$ is projective. We set $N_{\beta, \beta}^\beta := \frac{1}{4}(\dim G_V - \dim G_{V(1)} - \dim G_{V(2)}) + \sum_{n \in \mathbb{N}} d_1(h')d_2(h'')$. For $G_{V(1)}$-equivariant constructible sheaves $\mathcal{F}_i$ on $E_{V(i)}$ for $i = 1, 2$, we define their convolution product as

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_! p^* \mathcal{F}_1 \boxtimes q_! q^* \mathcal{F}_2 \text{ in } D_{G_V}^b(\text{Gr}_{V(1), V(2)}(V)).$$

By construction, the convolution of $\mathcal{L}_{\beta_1}^\beta$ and $\mathcal{L}_{\beta_2}^\beta$ yields the direct summand of $\mathcal{L}_{\beta}^\beta$ corresponding to the idempotent $e_{\beta_1, \beta_2}$. Hence, we have

$$R_{\beta_1, \beta_2} \cong \text{Ext}_{G_V}^1(\mathcal{L}_{\beta_1}^\beta \boxtimes \mathcal{L}_{\beta_2}^\beta, \mathcal{L}_{\beta}^\beta)$$

(4.1)
as $(R_\beta, R_\beta \boxtimes R_\beta)$-bimodule.

Let $\mathcal{L}_{\beta_1, \beta_2}^\Omega$ be a complex so that $\vartheta^* \mathcal{L}_{\beta_1, \beta_2}^\Omega \cong q^* (\mathcal{L}_{\beta_1}^\Omega \boxtimes \mathcal{L}_{\beta_2}^\Omega)$. Then, we have

$$\text{Ext}_{G_V}^\bullet (\mathcal{L}_{\beta_1}^\Omega \boxtimes \mathcal{L}_{\beta_2}^\Omega) \cong \text{Ext}_{G_V}^\bullet (\mathcal{L}_{\beta_1, \beta_2}^\Omega, p^! \mathcal{L}_{\beta}^\Omega).$$

Since we have

$$\text{Ext}_{G_V}^\bullet (\mathcal{L}_{\beta_1, \beta_2}^\Omega, \mathcal{L}_{\beta_2}^\Omega) \cong \text{Ext}_{G_V(1) \times G_V(2)}^\bullet (\mathcal{L}_{\beta_1}^\Omega \boxtimes \mathcal{L}_{\beta_2}^\Omega, \mathcal{L}_{\beta_1}^\Omega \boxtimes \mathcal{L}_{\beta_2}^\Omega),$$

we have a (right) $R_{\beta_1} \boxtimes R_{\beta_2}$-module structure of $R_{\beta_1, \beta_2}$.

From now on, we assume that $i \in I$ is a sink of $\Omega$ and employ the setting of $\mathcal{L}_{\beta_1, \beta_2}^\Omega$. We find $\mathcal{L}_{\beta_1, \beta_2}^\Omega$ so that

$$\vartheta^* \mathcal{L}_{\beta_1, \beta_2}^\omega \cong q^* ((j_1 V(1)) j_1^! \mathcal{L}_{\beta_1}^\Omega \boxtimes (j_2 V(2)) j_2^! \mathcal{L}_{\beta_2}^\Omega),$$

and $\mathcal{O} := \vartheta(q^{-1}(E_{V(1)}(1) \times i E_{V(2)}^\omega(2)))$. The graded vector space

$$\text{Ext}_{G_V}^\bullet (p_! \mathcal{L}_{\beta_1, \beta_2}^\omega, \mathcal{L}_{\beta_2}^\Omega) \cong \text{Ext}_{G_V}^\bullet (\mathcal{L}_{\beta_1, \beta_2}^\Omega, p^! \mathcal{L}_{\beta}^\Omega)$$

admits an $(R_{\beta}, i R_{\beta} \boxtimes i R_{\beta})$-bimodule structure.

By restricting each components to the open set $i E_{V}^\omega$ by $j_V^* = j_V^!$, we deduce that $\text{Ext}_{G_V}^\bullet (j_V^* p_! \mathcal{L}_{\beta_1, \beta_2}^\omega, j_V^! \mathcal{L}_{\beta_2}^\Omega)$ is a left $i R_{\beta}$-module. Applying adjunctions, this module is isomorphic to

$$\text{Ext}_{G_V}^\bullet (p_! \mathcal{L}_{\beta_1, \beta_2}^\omega, (j_V^* \mathcal{L}_{\beta_2}^\Omega)) \cong \text{Ext}_{G_V}^\bullet (\mathcal{L}_{\beta_1, \beta_2}^\Omega, p^! (j_V^* \mathcal{L}_{\beta_2}^\Omega)), \quad (4.2)$$

which admits a right $i R_{\beta} \boxtimes i R_{\beta}$-structure. Hence, $(\mathcal{L}_{\beta_1, \beta_2}^\Omega)$ is an $(i R_{\beta}, i R_{\beta} \boxtimes i R_{\beta})$-bimodule.

We fix $I$-graded vector spaces $V(i)$ so that $\dim V(i) = s_i \beta_i$ for $i = 1, 2$. A similar construction as above implies that we have a sheaf $\mathcal{L}_{s_i \beta_i, s_i \beta_2}$ so that

$$\vartheta^* \mathcal{L}_{s_i \beta_1, s_i \beta_2}^\omega \cong q^* ((j_V^\omega(1)) j_V^*(1) \mathcal{L}_{s_i \beta_1}^\omega \boxtimes (j_V^\omega(2)) j_V^*(2) \mathcal{L}_{s_i \beta_2}^\omega).$$

It yields an $(i R_{\beta_1}, i R_{\beta_1} \boxtimes i R_{\beta_2})$-bimodule

$$\text{Ext}_{G_V}^\bullet (p_! \mathcal{L}_{s_i \beta_1, s_i \beta_2}^\omega, (j_V^\omega)^* \mathcal{L}_{s_i \beta_2}^\Omega) \cong \text{Ext}_{G_V}^\bullet (\mathcal{L}_{s_i \beta_1, s_i \beta_2}^\omega, p^! (j_V^\omega)^* \mathcal{L}_{s_i \beta_2}^\Omega), \quad (4.3)$$

**Theorem 4.3.** Under the above setting, the image of the natural restriction map

$$\text{Ext}_{G_V}^\bullet (p_! \mathcal{L}_{\beta_1, \beta_2}^\Omega, \mathcal{L}_{\beta_2}^\Omega) \longrightarrow \text{Ext}_{G_V}^\bullet (p_! \mathcal{L}_{\beta_1, \beta_2}^\Omega, \mathcal{L}_{\beta_2}^\Omega)$$

is a submodule of the RHS, and is equal to

$$i R_{\beta} \boxtimes R_{\beta_1} \boxtimes R_{\beta_2} (i R_{\beta_1} \boxtimes i R_{\beta_2}).$$

In addition, it is the pure part of weight zero in $\text{Ext}_{G_V}^\bullet (p_! \mathcal{L}_{\beta_1, \beta_2}^\Omega, \mathcal{L}_{\beta_2}^\Omega)$. The same is true if we replace $\Omega$ with $s_i \Omega, \beta_i$ by $s_i \beta_j$, and $i R_{\beta}$ with $i R_{\beta_j}$ ($j = \emptyset, 1, 2$).
Proof. Since the proofs of the both assertions are similar, we prove only the case of \( \Omega \). We have \( \mathcal{O} \subset p^{-1}(iE^0_{1}) \), and hence the restriction map factors through the restriction to \( iE^0_{1} \). By unwinding the definition, we have a factorization

\[
\Ext_{G^c}(pL^{0}_{\beta_1, \beta_2}, L^{0}_{\beta}) \rightarrow \Ext_{G^c}(j_V pL^{0}_{\beta_1, \beta_2}, \beta V L^{0}_{\beta})
\]

\[
\cong \Ext_{G^c}((j_V pL^{0}_{\beta_1, \beta_2}, \beta V L^{0}_{\beta}) \rightarrow \Ext_{G^c}(pL^{0}_{\beta_1, \beta_2}, L^{0}_{\beta})
\]

of \( (R_\beta, R_\beta, \boxtimes R_\beta) \)-bimodule map, where the first map (that is surjection by Theorem 3.4) is the restriction to the open set, the second isomorphism is the adjunction, and the third morphism is obtained by the base change using \( j_V \) and the composition.

We have a distinguished triangle

\[
(j_V(2))^j_L L^{0}_{\beta_2} \rightarrow L^{0}_{\beta_2} \rightarrow \Ker \rightarrow .
\]

Since \( L^{0}_{\beta_2} \) is pure of weight 0, it follows that \( (j_V(2))^j_L L^{0}_{\beta_2} \) must have weight \( \leq 0 \) (15.1.14). Taking account into the fact that \( (j_V(2))^j_L L^{0}_{\beta_2} \) and \( L^{0}_{\beta} \) share the same stalk along \( iE^0_{1} \) and the stalk of \( (j_V(2))^j_L L^{0}_{\beta_2} \) vanishes outside of \( iE^0_{1} \), we conclude that \( \Ker \) has weight \( \leq 0 \). We set \( K := pK' \), where \( pK' \geq q^*(L^{0}_{\beta_2} \boxtimes \Ker) \).

From now on, we make all computations over \( iE^0_{1} \) by using \( j_V^j = j_V \). The above construction gives us a distinguished triangle

\[
p_{\beta}L^{0}_{\beta_1, \beta_2} \rightarrow L^{0}_{\beta_1, \beta_2} \rightarrow \Ker \rightarrow .
\]

Moreover, \( K \) has weight \( \leq 0 \) by \( p_{\ast} = p_{\ast} \).

Hence, we deduce an exact sequence of \( iR_\beta \)-modules

\[
\Ext_{G^c}(K, L^{0}_{\beta}) \rightarrow \Ext_{G^c}(p_{\ast}L^{0}_{\beta_1, \beta_2}, L^{0}_{\beta}) \rightarrow \Ext_{G^c}(p_{\ast}L^{0}_{\beta_1, \beta_2}, L^{0}_{\beta})
\]

Note that the middle term has weight 0 by Theorem 3.4 as both \( L^{0}_{\beta_1, \beta_2} \) and \( L^{0}_{\beta} \) are pure of weight 0. Since \( \Ext_{G^c}(K, L^{0}_{\beta}) \) has weight \( \geq 0 \) (15.1.14), we conclude that \( \text{Im } \rho \) is precisely the weight 0-part of \( \Ext_{G^c}(p_{\ast}L^{0}_{\beta_1, \beta_2}, L^{0}_{\beta}) \) (see also the arguments in 17).

Since the \( (iR_\beta, iR_\beta, \boxtimes iR_\beta) \)-action preserves the weight, it follows that \( \text{Im } \rho \) is an \( (iR_\beta, iR_\beta, \boxtimes iR_\beta) \)-submodule of \( \Ext_{G^c}(p_{\ast}L^{0}_{\beta_1, \beta_2}, L^{0}_{\beta}) \). Since we have \( \Ext_{G^c}(p_{\ast}L^{0}_{\beta_1, \beta_2}, L^{0}_{\beta}) \cong iR_\beta e_{\beta_1, \beta_2} \), we have a surjection

\[
\pi : iR_\beta e_{\beta_1, \beta_2} \rightarrow \text{Im } \rho.
\]

By Proposition 3.5, the sheaf \( \text{Ker } \) is obtained by successive constructions of cones of shifted perverse sheaves on \( Q^0_{\beta_2} \) that are supported outside of \( iE^0_{1} \).

Therefore, we deduce that \( \text{ker } \rho \) admits a surjection from the direct sum of \( iR_\beta \)-modules of the form

\[
R_{\beta_1} \ast R_{\beta_2} \quad b_1 \in B(\infty)_{\beta_1}, b_2 \in B(\infty)_{\beta_2}, e_i(b_2) > 0,
\]

that corresponds to \( IC^0(b_1) \oplus IC^0(b_2) \) with \( e_i(b_2) > 0 \). Let us write \( E \) the sum of the image of all such \( iR_\beta \)-modules in \( iR_\beta e_{\beta_1, \beta_2} \) arises as the above induction.

In view of the construction of \( \text{Ker } \), we have \( \text{ker } \pi \subset E \).
On the other hand, $E$ is precisely the kernel of the natural quotient map
\[ iR_\beta e_{\beta_1, \beta_2} \longrightarrow iR_\beta \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (R_{\beta_1} \boxtimes iR_{\beta_2}). \]

As a consequence, we have a quotient map
\[ \text{Im } \rho \longrightarrow iR_\beta \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (R_{\beta_1} \boxtimes iR_{\beta_2}). \]

The module $\text{Im } \rho$ is a $(iR_{\beta_1}, iR_{\beta_1} \boxtimes iR_{\beta_2})$-bimodule whose bimodule structure is induced from the $(iR_{\beta_1}, iR_{\beta_1} \boxtimes iR_{\beta_2})$-bimodule structure on $iR_{\beta_1} e_{\beta_1, \beta_2}$ by construction (through Theorem 3.4). Thus, $\text{Im } \rho$ admits a surjection from $iR_{\beta} \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (iR_{\beta_1} \boxtimes iR_{\beta_2})$, that is the maximal $(iR_{\beta_1}, iR_{\beta_1} \boxtimes iR_{\beta_2})$-bimodule quotient of $iR_{\beta_1} e_{\beta_1, \beta_2}$ (regarded as a $(iR_{\beta_1}, iR_{\beta_1} \boxtimes iR_{\beta_2})$-bimodule). Therefore, we conclude
\[ \text{Im } \rho \cong iR_{\beta} \otimes_{R_{\beta_1} \otimes R_{\beta_2}} (iR_{\beta_1} \boxtimes iR_{\beta_2}) \]
as required.

**Proof of Theorem 4.1.** Note that the open subset $\mathcal{O} \subset \text{Gr}_0^{\beta_1, \beta_2}(V)$ is precisely the set of points $(F, x)$ so that $x \mid_F \in \iota E_0^{\beta_1, \beta_2}(V)$ and $x \mod F' \in \iota E_0^{\beta_1, \beta_2}(V)$. We set $\mathcal{O}' := \varphi^{-1}(iE_0^{\beta_1, \beta_2}(V))$. The open subset $\mathcal{O}' \subset \text{Gr}_0^{\beta_1, \beta_2}(V)$ is precisely the set of points $(F', x')$ so that $x' \mid_{F'} \in \iota E_0^{\beta_1, \beta_2}(V)$ and $x' \mod F' \in \iota E_0^{\beta_1, \beta_2}(V)$. Therefore, (5.1) yields a variety $\mathcal{O}$ with the $G_{V', V}$-action defined as:
\[
\left\{ (W_i, W_i') \mid (f_h)_{h \in \Omega}, (f_h')_{h \in \Omega}, \psi \in Z_0^{\beta_1, \beta_2}, \phi : W_j = W_j' \text{ for } j \neq i \right\},
\]
\[
\left\{ (W_i, W_i') \mid (f_h)_{h \in \Omega}, (f_h')_{h \in \Omega} \in \text{Gr}_0^{\beta_1, \beta_2}(V), (W_i, W_i') \in \text{Gr}_0^{\beta_1, \beta_2}(V') \right\},
\]
\[
\psi : W_i' \cong \ker(\bigoplus_{h \in \Omega} f_h : \bigoplus_{h} W_h' \to W_i)
\]

Note that the condition $(f_h \mid W_i)_{h \in \Omega} \in iE_0^{\beta_1, \beta_2}$ guarantees that
\[ \dim W_i' = \dim \ker(\bigoplus_{h \in \Omega} f_h : \bigoplus_{h} W_h' \to W_i) \]
and similarly the condition $(f_h' \mid W_i)_{h \in \Omega} \in iE_0^{\beta_1, \beta_2}$ guarantees that
\[ \dim W_i = \dim \text{coker}(\bigoplus_{h \in \Omega} f_h : W_i' \to \bigoplus_{h} W_h'). \]

that actually asserts the same thing. Since we have an isomorphism
\[ \psi : V_i' \cong \ker(\bigoplus_{h \in \Omega} f_h : \bigoplus_{h} V_h' \to V_i) \]
from the definition of $Z_0^{\beta_1, \beta_2}$, taking quotients yield
\[
(f_h \mod \{W_i\})_{h \in \Omega} \in iE_0^{\beta_1, \beta_2} \text{ and } (f_h' \mod \{W_i'\})_{h \in \Omega} \in iE_0^{\beta_1, \beta_2}.
\]
Hence, the quotients of $\mathcal{O}$ by $G_{V'}$ and $G_V$ gives $\overline{q}_V$ and $\overline{p}_{V'}$ in the commutative diagram in the below:

\[
\begin{array}{cccc}
\text{Gr}_{\beta_1, \beta_2}(V) & \xrightarrow{\gamma} & \mathcal{O} & \xrightarrow{\varphi} & \text{Gr}^*_{\beta_1, \beta_2}(V') \\
p^1 & & i & & p^1 \\
E_V^\Omega & \xrightarrow{j_V} & E_{V'}^\Omega & \xrightarrow{q_V} & E_{V'}^\Omega
\end{array}
\]

Therefore, we have an equivalence of the category of $G_{V'}$-equivariant sheaves on $\mathcal{O}$, and the category of $G_V$-equivariant sheaves on $\mathcal{O}'$ (cf. \cite{2} §2.6.3). With an aid of Proposition \ref{3.7} we conclude that

\[\text{Ext}^*_G(V, \mathcal{L}^\Omega_{\beta_1, \beta_2}, \mathcal{L}^\Omega_{\beta}) \cong \text{Ext}^*_G(V, \mathcal{L}^\Omega_{\beta_1, \beta_2}, (p^1)^* \mathcal{L}^\Omega_{\beta})\]

up to amplifications of direct summands (i.e. we allow to duplicate direct summand of both terms). By Theorem \ref{4.3} the comparison of their weight zero parts identifies

\[i^* R_{\beta} \boxtimes_R i^* R_{\beta_2} \text{ and } (i R_{\beta_1} \boxtimes i R_{\beta_2}) \text{ and } i^* R_{\beta_1} \boxtimes_R i^* R_{\beta_2} \text{ and } (i R_{\beta_1} \boxtimes i R_{\beta_2})\]

through the Morita equivalences in Proposition \ref{3.7}. This is actually an identification of bimodules by construction.

In other words, we have an isomorphism

\[\text{T}^*_i(\text{Ext}^*_G(V, (j_V^* \mathcal{L}^\Omega_{\beta_1, \beta_2}, j_V^* \mathcal{L}^\Omega_{\beta}))) \cong \text{Ext}^*_G(V, (L^\Omega_{\beta_1, \beta_2}, L^\Omega_{\beta}), \text{Ext}^*_G(V, (L^\Omega_{\beta_1, \beta_2}, L^\Omega_{\beta})), \text{Ext}^*_G(V, (L^\Omega_{\beta_1, \beta_2}, L^\Omega_{\beta})))\]

where the amplification of direct summands is subsumed in the constructions of $\text{T}^*_i$. This isomorphism commutes with the Morita equivalence of $i R_{\beta_j}$ and $i^* R_{\beta_j}$ for $j = 1, 2$ by the above. Hence, taking their weight 0 part yields the desired natural transformation

\[\text{T}^*_i(\bullet \star \bullet) \longrightarrow \text{T}^*_i(\bullet) \star \text{T}^*_i(\bullet)\]

of functors, and it must be an equivalence. The case of $\text{T}^*_i$ is obtained similarly.

\begin{acknowledgement}

The author would like to thank Masaki Kashiwara and Myungho Kim for helpful correspondences. He also thanks Peter McNamara who kindly sent me a version of \cite{15}. This research is supported in part by JSPS Grant-in-Aid for Scientific Research (B) JP26287004.

\end{acknowledgement}

References

\begin{enumerate}
\item A. A. Beilinson, J. Bernstein, and P. Deligne. \textit{Faisceaux pervers}, volume 100 of \textit{Astérisque}. Soc. Math. France, Paris, 1982.
\item Joseph Bernstein and Valery Lunts. \textit{Equivariant sheaves and functors}, volume 1578 of \textit{Lecture Notes in Mathematics}. Springer-Verlag, Berlin, 1994.
\item Neil Chriss and Victor Ginzburg. \textit{Representation theory and complex geometry}. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition.
\item Masaki Kashiwara. On crystal bases of the $\mathfrak{s}\mathfrak{g}\mathfrak{s}$-analogue of universal enveloping algebras. \textit{Duke Mathematical Journal}, 63(2):465–516, 1991.
\end{enumerate}
[5] Masaki Kashiwara, Myungho Kim, Se Jin Oh, and Euiyong Park. Monoidal categories associated with strata of flag manifolds. Adv. in Math., 328:959–1009, 2018. arXiv:1708.04428.

[6] Syu Kato. Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras. Duke Math. J., 163(3):619–663, 2014. Correction abridged version: arXiv:1203.5254v5.

[7] Syu Kato. An algebraic study of extension algebras. Amer. J. Math., 139(3):567–615, 2017.

[8] M. Khovanov and A. Lauda. A diagrammatic approach to categorification of quantum groups I. Represent. Theory, 13:309–347, 2009.

[9] Aaron D. Lauda and Monica Vazirani. Crystals from categorified quantum groups. Adv. Math., 228(2):803–861, 2011.

[10] G. Lusztig. Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc., 3(2):447–498, 1990.

[11] G. Lusztig. Introduction to quantum groups, volume 110 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1993.

[12] G. Lusztig. Canonical bases and Hall algebras. In Representation theories and algebraic geometry (Montreal, PQ, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, pages 365–399. Kluwer Acad. Publ., 1998.

[13] George Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. Journal of the American Mathematical Society, 4(2):365–421, 1991.

[14] Peter J. McNamara. Representation theory of geometric extension algebras. arXiv:1701.07949.

[15] Peter J. McNamara. Monoidality of Kato’s reflection functors. arXiv:1712.00173, 2017.

[16] R Rouquier. 2-Kac-Moody algebras. arXiv:0812.5023.

[17] Michela Varagnolo and Eric Vasserot. Canonical bases and KLR-algebras. Journal für die reine und angewandte Mathematik (Crelles Journal), 659:67–100, 2011.

[18] Jie Xiao and Minghui Zhao. Geometric realizations of Lusztig’s symmetries. J. Algebra, 475:392–422, 2017.