Bivariate fractal interpolation functions on triangular domain for numerical integration and approximation

Aparna M.P\textsuperscript{a,1}, P. Paramanathan\textsuperscript{b,*,2}

\textsuperscript{a}Department of Mathematics, Amrita School of Physical Sciences, Coimbatore, Amrita Vishwa Vidyapeetham, India.
\textsuperscript{b}Department of Mathematics, Amrita School of Physical Sciences, Coimbatore, Amrita Vishwa Vidyapeetham, India.

ARTICLE INFO

Keywords:
Bivariate fractal interpolation function (BFIF)
Chromatic number
Double integration

ABSTRACT

The primary objectives of this paper are to present the construction of bivariate fractal interpolation functions over triangular interpolating domain using the concept of vertex coloring and to propose a double integration formula for the constructed interpolation functions. Unlike the conventional constructions, each vertex in the partition of the triangular region has been assigned a color such that the chromatic number of the partition is 3. A new method for the partitioning of the triangle is proposed with a result concerning the chromatic number of its graph. Following the construction, a formula determining the vertical scaling factor is provided. With the newly defined vertical scaling factor, it is clearly observed that the value of the double integral coincides with the integral value calculated using fractal theory. Further, a relation connecting the fractal interpolation function with the equation of the plane passing through the vertices of the triangle is established. Convergence of the proposed method to the actual integral value is proven with sufficient lemmas and theorems. Sufficient examples are also provided to illustrate the method of construction and to verify the formula of double integration.

1. Introduction

Fractal interpolation functions (FIF) are constructed using iterated function system (IFS)\cite{1}. The IFS consists of a complete metric space together with a finite set of contraction mappings \cite{11}. Normally, each contraction map in the IFS is composed of two types of functions, the former one responsible for the contraction of the entire interpolating domain and the latter one for defining the Read-Bejractarevic operator whose fixed point is the required fractal interpolation function. The fundamental problem in the theory of fractal interpolation is to establish the well definiteness of this operator. In \cite{2}, L. Dalla imposed a restrictive condition on the interpolation points for tackling this problem, when the interpolating domain is a rectangle. For the same purpose, a piecewise function, defined in terms of the usual IFS, is proposed in \cite{3}. For the triangular interpolating domain, the problem of well definiteness was dealt by Geronimo and Hardin and they proposed the method of vertex coloring to solve the problem \cite{4}. The present paper further connects this approach to define bivariate fractal interpolation functions and to derive double integration for such functions.

Barnsley in \cite{5} introduced the idea of fractal interpolation functions using iterated function systems for the first time where single variable interpolation functions were generated as the attractors of the IFS. The construction of bivariate fractal interpolation functions considered by L. Dalla required the interpolation points on the boundary of the rectangle to be collinear \cite{2}. Malysz used the fold-out technique for constructing bivariate fractal interpolation function over rectangles by taking the same vertical scaling factor \cite{6}. In \cite{7}, considering the vertical scaling factor as a function, Metzler and Yun generalised this construction. In \cite{8}, Massopust constructed bivariate fractal interpolation functions on a triangular region with a restrictive condition on the interpolation points. The construction proposed in his work was further modified in \cite{4}. \cite{4} introduces the idea of the coloring of the vertices to solve the problem of well definiteness. The construction of the bivariate fractal interpolation function is again considered in \cite{9}. The paper, however, fails to establish the well definiteness of the fractal interpolation operator. The numerical integration of fractal interpolation functions was first carried out by Navascues in \cite{10}. The derived formula for integration is then compared with the compound trapezoidal rule there.
The present paper aims to define double integration for two variable fractal interpolation functions constructed over a triangular domain. By proposing a method for the partition of the triangle and introducing a new vertical scaling factor, this paper provides a detailed explanation for the construction of these functions. Following the derivation of double integration, the constructed fractal interpolation function is approximated to the equation of the plane passing through the vertices of the triangle. After proving the theorems in error analysis using this approximation, the paper shows the attractors of the IFS’s and the double integral values obtained for some functions.

The organization of the paper is as follows: The second section presents the formulation of the IFS and proves the corresponding theorems with the results concerning the partitioning of the triangle and its chromatic number. The derivation of the double integration formula is proposed in the third section. In the fourth section, it is established that the bivariate fractal interpolation functions defined over triangular regions can be approximated to the equation of the plane passing through the vertices of the triangle. The fifth section provides the formula for the vertical scaling factor and its upper bound. Using, the newly obtained approximating function, the propositions and theorems are proved for the error analysis. The paper concludes by displaying the tables and graphs describing the results.

2. Construction of bivariate fractal interpolation function over triangular regions

2.1. Method of Partition

Consider the triangle $D$ with vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$. The algorithm for the partition is as follows.

1. Divide the height of the triangle $D$ into $d$ number of equal parts, thereby getting $d+1$ new points $(x_1, y_1), (x_2, y_2), ..., (x_d, y_d)$ along the height of $D$, where $y_1 = y$ coordinate of the point $A$ or $B$, $y_d+1 = y$ coordinate of the point $C$.

2. Draw lines $y = y_j, j = 1, 2, ..., d$ parallel to $X$-axis from $AC$ to $BC$.

3. Divide each of the horizontal lines $y = y_j, j = 1, 2, ..., d$ into $d$ number of equal parts, generating $d+1$ new points denoted by $(x_1, y_j), (x_2, y_j), ..., (x_{d+1}, y_j)$ along each horizontal line $y = y_j$ for $j = 1, 2, ..., d$ where $(x_1, y_j), (x_{d+1}, y_j)$ lies on the sides $AC$ and $BC$ respectively.

4. Then, join the new points as shown in Figure 1.

![Figure 1: Partition of $D$ when $d = 4$](image)

In Figure 1, red corresponds to color 1, blue stands for color 2, and green for color 3.
Lemma 2.1. For the partition defined above, if the height of the triangle $D$ is divided into $d$ number of equal parts, then, there are

i) $2d^2 - 2d + 3$ subtriangles and

ii) $d^2 + d + 1$ vertices,

in the partition.

Proof. i) According to the partition defined, the height of the triangle is divided into $d$ number of equal parts, resulting $d + 1$ 'y' values, where $y_{d+1}$ is the $y$ coordinate of the top vertex of $D$. Along each horizontal line $y = y_j, j = 1, 2, ..., d$ the line is divided into $d$ number of equal parts. The coordinates of the newly obtained points along the line $y = y_j$ are denoted by $(x_1, y_j), (x_2, y_j), ..., (x_{d+1}, y_j)$. It is observed from the partition that between two consecutive points on the line $y = y_j$, two subtriangles are obtained (a normal and an inverted triangle). Hence, along each line $y = y_j$, there are $2d$ subtriangles for $j = 2, ..., d$. Considering the top most three subtriangles, which are fixed irrespective of $d$, there are $(2d)(d - 1) + 3$ subtriangles in the partition. i.e, the total number of subtriangles in the partition is $2d^2 - 2d + 3$. Hence the proof.

ii) Following the partition, since each horizontal line $y = y_j$ is divided into $d$ equal parts, there are $d + 1$ new vertices along each line $y = y_j, j = 1, 2, ..., d$. Now, including the top most vertex $C$, of the triangle $D$, there are $[(d + 1)d] + 1 = d^2 + d + 1$ vertices in the partition. Hence the proof.

Lemma 2.2. If $d = 3n + 1, n \in N$, then the graph of the partition defined above has chromatic number 3.

Proof. According to the partition, each horizontal line $y = y_j$ is divided into $d$ equal parts, generating $d + 1$ points along that line. Let the points be denoted by $(x_1, y_j), (x_2, y_j), ..., (x_{d+1}, y_j)$. Among these points, $(x_1, y_j)$ and $(x_{d+1}, y_j)$ are on the two sides of the triangle $D$. The remaining points are intermediate points and there $d - 1$ such points along that line, where $d - 1$ is a multiple of 3. Now, considering the coloring of these points with the least number of colors, since the points $(x_1, y_j)$ and $(x_{d+1}, y_j)$ are adjacent with respect to the triangle $D$, they have to be colored differently. Without loss of generality, let $(x_1, y_j)$ and $(x_{d+1}, y_j)$ be colored with colors '1' and '2' respectively. Now, since the point $(x_2, y_j)$ is adjacent to $(x_1, y_j)$, it should be colored '2'. Similarly, $(x_3, y_j)$ is colored '1'. Proceeding in this manner, the point $(x_{d-1}, y_j)$ will be colored '1'. Then, the point $(x_d, y_j)$ has to be colored with a different color other than '1' and '2', since it is adjacent to both $(x_{d-1}, y_j)$ and $(x_{d+1}, y_j)$. Hence, a minimum of 3 colors are needed to color the points along this line. The same reasoning can be applied along each of the horizontal lines $y = y_j$ and along the slanting sides of $D$. Thus, it can be established that at least 3 different colors are needed to properly color the graph. Therefore, the chromatic number of the graph is 3. \hfill \Box

2.2. Construction

Consider a triangular domain $D$ with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, colored '1', '2' and '3' respectively. Let the triangle be partitioned into $N$ number of subtriangles $D_1, D_2, ..., D_N$ such that $D = \cup_{n=1}^{N} D_n$ and $\cap_{n=1}^{N} D_n = \emptyset$. The partitioning is done such that the chromatic number of their corresponding graph is 3. Each subtriangle be numbered from 1 to $N$ as shown in Figure 1. Set $P = \{(x_{nj}, y_{nj}) : j = 1, 2, 3, n = 1, 2, ..., N\}$ to be the set of all vertices of the subtriangles $D_n, n = 1, 2, ..., N$. Let $z_{nj} = f(x_{nj}, y_{nj})$ be the corresponding function values. Let $R = \{(x_{nj}, y_{nj}, z_{nj}) : j = 1, 2, 3, n = 1, 2, ..., N\}$ be the data set. Without loss of generality, let $(x_{n1}, y_{n1})$ denotes the vertex colored '1', $(x_{n2}, y_{n2})$ be the vertex with color '2', and $(x_{n3}, y_{n3})$ be the vertex colored '3' in $D_n$. Consider an invertible, affine map $L_n : D \to D_n$ such that

$$L_n(x, y) = (x_{nj}, y_{nj}), \quad \text{for } j = 1, 2, 3. \tag{1}$$

i.e, $L_n$ maps $(x, y)$ to the vertex in $D_n$, which is colored $j$, for $j = 1, 2, 3$. The map $L_n$ used here is given by,

$$L_n(x, y) = \begin{bmatrix} a_{n1} & a_{n2} \\ a_{n3} & a_{n4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \beta_{n1} \\ \beta_{n2} \end{bmatrix} \tag{2}$$
Choose a vertical scaling factor \( \alpha_n \) between -1 and 1 for \( n = 1, 2, ..., N \) with the scale vector \( \overline{\alpha_n} \).

Set \( F = D \times R \) and consider \( N \) maps \( F_n \), contractive in the third variable such that

\[
F_n(x_j, y_j, z_j) = z_{nj}, \quad \text{for } j = 1, 2, 3, \quad n = 1, 2, ..., N.
\]

The map \( F_n \) is given by,

\[
F_n(x, y, z) = Q_n(x, y) + \alpha_n z, \quad n = 1, 2, ..., N,
\]

where \( Q_n(x, y) = \alpha_n x + \alpha_n y + \beta_n, \quad n = 1, 2, ..., N. \)

Then, the IFS becomes

\[
w_n(x, y, z) = (L_n(x, y), F_n(x, y, z)), n = 1, 2, ..., N.
\]

In matrix notation,

\[
W_n(x, y, z) = \begin{bmatrix} \alpha_{n1} & \alpha_{n2} & 0 \\ \alpha_{n3} & \alpha_{n4} & 0 \\ \alpha_{n5} & \alpha_{n6} & \alpha_n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \beta_{n1} \\ \beta_{n2} \\ \beta_{n3} \end{bmatrix}
\]

(5)

The constants in the matrix are obtained by solving the endpoint conditions (1) and (3). Then,

\[
\begin{align*}
\alpha_{n1} &= \frac{(x_n - x_n x_2)(y_1 - y_3) - (x_n - x_3)(y_1 - y_2)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\alpha_{n2} &= \frac{(x_n - x_3 x_2)(x_1 - x_2) - (x_n - x_2)(x_1 - x_3)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\alpha_{n3} &= \frac{(y_n - y_3 x_2)(y_1 - y_3) - (y_n - y_3)(y_1 - y_2)}{(y_1 - y_3)(y_1 - y_3) - (y_1 - y_2)(y_1 - y_2)} \\
\alpha_{n4} &= \frac{(y_n - y_3 x_2)(x_1 - x_2) - (y_n - y_3)(x_1 - x_3)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\alpha_{n5} &= \frac{(z_n - a_n z_2)(y_1 - y_3) - (z_n - a_n x_3)(y_1 - y_2)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\alpha_{n6} &= \frac{-(z_n - a_n z_2)(x_1 - x_2) - (z_n - a_n x_3)(x_1 - x_3)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\beta_{n1} &= \frac{(x_n x_2 - x_1)(y_3) + (x_n x_3 - x_3 x_1)(y_2)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\beta_{n2} &= \frac{(x_1 y_2 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)} \\
\beta_{n3} &= \frac{(x_1 y_2 - x_2)(z_1) + (z_n - a_n z_2)(x_1 y_1 - x_1 y_3)}{(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)}
\end{align*}
\]

**Proposition 2.1.** There is a metric \( \sigma \) equivalent to the Euclidean metric on \( R^3 \) such that the above defined IFS is hyperbolic with respect to the new metric. Moreover, there is a unique, non-empty, compact set \( G \) such that

\[
G = \bigcup_{n=1}^{N} w_n(G),
\]

called the attractor of the IFS.

**Proof.** Let \((x, y, z), (x', y', z')\) be two points in \( D \). Consider the metric \( \sigma \) given by

\[
\sigma(x, y, z) = |x - x'| + |y - y'| + \theta|z - z'|,
\]

where \( \theta \) is to be specified later. Clearly, \( \sigma \) is equivalent to Euclidean metric on \( R^3 \).

Now,
\[
\sigma(w_n(x, y, z), w_n(x', y', z'))
\]
\[
= \sigma \left( (L_n(x, y), F_n(x, y, z)), (L_n(x', y'), F_n(x', y', z')) \right)
\]
\[
= \sigma \left( (a_{n1}x + \alpha_{n2}y + \beta_{n1}, \alpha_{n3}x + \alpha_{n4}y + \beta_{n2}), (\alpha_{n5}x + \alpha_{n6}y + \alpha_{n7}z + \beta_{n3}) \right),
\]
\[
(\alpha_{n1}x' + \alpha_{n2}y' + \beta_{n1}, \alpha_{n3}x' + \alpha_{n4}y' + \beta_{n2}), (\alpha_{n5}x' + \alpha_{n6}y' + \alpha_{n7}z' + \beta_{n3}) \right)
\]
\[
\leq \left( |a_{n1}| + |a_{n3}| \right) |x - x'| + \left( |a_{n2}| + |a_{n4}| + \theta |a_{n5}| \right) |y - y'| + \theta |a_{n7}| |z - z'|
\]
\[
= \left( |a_{n1}| + |a_{n3}| + \theta |a_{n5}| \right) |x - x'| + \left( |a_{n2}| + |a_{n4}| \right) |y - y'| + \theta |a_{n7}| |z - z'|
\]
Choose \( \theta = \min \left\{ \frac{\min\{0.5 - |a_{n1}| + |a_{n5}|\}}{\max\{|a_{n5}|\}}, \frac{\min\{0.5 - |a_{n2}| + |a_{n4}|\}}{\max\{|a_{n6}|\}} \right\} \). Then, the above expression becomes,
\[
\sigma(w_n(x, y, z), w_n(x', y', z'))
\]
\[
\leq \left( |a_{n1}| + |a_{n3}| + \theta |a_{n5}| \right) |x - x'| + \left( |a_{n2}| + |a_{n4}| \right) |y - y'| + \theta |a_{n7}| |z - z'|
\]
\[
\leq 0.5|x - x'| + 0.5|y - y'| + \theta |z - z'| |a_{n7}|
\]
\[
\leq 0.5|x - x'| + 0.5|y - y'| + \theta |z - z'| \max_n\{|a_{n7}|\}
\]
\[
\leq \max\{0.5, \max_n\{|a_{n7}|\}\} \sigma((x, y, z), (x', y', z'))
\]
\[
< 1.
\]
since \( |a_{n7}| < 1, \max\{0.5, \max_n\{|a_{n7}|\}\} < 1 \). Hence the IFS is hyperbolic. Then, trivially, there exists a unique, nonempty, compact set \( G \) in \( R^3 \) such that
\[
G = \bigcup_{n=1}^{N} w_n(G).
\]

\[
\text{Theorem 2.1. Let the given data set be } R = \{(x_{nj}, y_{nj}, z_{nj}) : j = 1, 2, 3, n = 1, 2, ..., N\}. \text{ Consider the IFS defined above, associated with the data set, } w_n(x, y, z) = (L_n(x, y), F_n(x, y, z)) \text{ for } n = 1, 2, ..., N. \text{ Choose } -1 < a_{n7} < 1 \text{ so as to make the IFS hyperbolic. Let } G \text{ denote the attractor of the IFS. Then, } G \text{ is the graph of a continuous function } f : D \to R \text{ such that } f \text{ interpolates the given data set.}
\]

\[
\text{Proof. Let } F \text{ denote the set of all continuous functions } f : D \to R \text{ such that } f(x_{nj}, y_{nj}) = z_{nj}, \text{ for } n = 1, 2, ..., N \text{ and } j = 1, 2, 3. \text{ Then, it is easily verified that } F \text{ is a complete metric space with the supnorm metric. Consider the operator } T : F \to F \text{ defined by}
\]
\[
(Tf)(x, y) = F_n(L_n^{-1}(x, y), f o L_n^{-1}(x, y)), \quad (x, y) \in D_n, \quad n = 1, 2, ..., N.
\]

\[
\text{(6)}
\]

Trivially, \( T \) is well defined at the interiors of each subtriangle. Now, it remains to establish the well definiteness of \( T \) at the common edges. Let \( (x_{nj}, y_{nj}, y_{nj'}) \) be an arbitrary edge of the subtriangle \( D_n \). Let it be shared by the subtriangle \( D_m \). Considering the vertex \( (x_{nj}, y_{nj}) \) in \( D_n \). Then, \( L_n^{-1}(x_{nj}, y_{nj}) = (x_j, y_j) \). Similarly, considering \( (x_{nj}, y_{nj}) \) as in \( D_n \), it will be denoted by \( (x_{mj}, y_{mj}) \). Also, \( L_m^{-1}(x_{nj}, y_{nj}) = (x_j, y_j) \). The same can be done for the vertex \( (x_{nj'}, y_{nj'}) \). Now, since \( L_n^{-1}, L_m^{-1} \) are affine, they give the same output along the edge \( (x_{nj}, y_{nj}), (x_{nj'}, y_{nj'}) \). To verify the same for \( T \), consider the vertex \( (x_{nj}, y_{nj}) \) in the edge \( ((x_{nj}, y_{nj}), (x_{nj'}, y_{nj'})) \). Let this edge be shared by \( D_n \) and \( D_m \). Now, considering \( (x_{nj}, y_{nj}) \) as a point in \( D_n \).

\[
\text{(6)}
\]

First Author et al.: Preprint submitted to Elsevier
Similarly, considering \((x_{nj}, y_{nj})\) as a point in \(D_m\), then, it will be denoted by \((x_{mj}, y_{mj})\). Then,

\[
(Tf)(x_{nj}, y_{nj}) = (Tf)(x_{mj}, y_{mj}) \\
= F_n\left( L_m^{-1}(x_{mj}, y_{mj}), f \circ L_m^{-1}(x_{mj}, y_{mj}) \right) \\
= F_n\left( (x_j, y_j), f(x_j, y_j) \right) \\
= F_n(x_j, y_j, z_j) \\
= z_{mj} \\
= f(x_{mj}, y_{mj}) \\
= f(x_{nj}, y_{nj}) \\
= z_{nj}
\]

Similarly, \((Tf)(x_{nj}', y_{nj}') = z_{nj}'\), while taking the point \((x_{nj}', y_{nj}')\) as in \(D_n\) and \(D_m\). Eventually, since \(T\) is defined in terms of \(L_n\) and \(L_m\) is affine, invertible map, it follows that \(T\) is well defined along each edge of the subtriangles.

Thus, \(T\) is a well defined map. Hence the operator \(T\) is continuous.

In order to show that the operator \(T\) satisfies the endpoint conditions, consider an arbitrary point \((x_{nj}, y_{nj})\). Then,

\[
(Tf)(x_{nj}, y_{nj}) = F_n\left( L_n^{-1}(x_{nj}, y_{nj}), f \circ L_n^{-1}(x_{nj}, y_{nj}) \right) \\
= F_n\left( (x_j, y_j), f(x_j, y_j) \right) \\
= F_n(x_j, y_j, z_j) \\
= z_{nj}
\]

Since \(T\) is well defined, it also gives the same value while considering \((x_{nj}', y_{nj}')\) as a point in \(D_m\).

In order to prove contractivity of \(T\), let \(f, g\) be two points in \(F\). Then,

\[
d(Tf(x, y), Tg(x, y)) = \sup \{|Tf(x, y) - Tg(x, y)| : (x, y) \in D\} \\
= \max_n \sup \{|Tf(x, y) - Tg(x, y)| : (x, y) \in D_n\} \\
= \max_n \sup \{|F_n(L_n^{-1}(x, y), f \circ L_n^{-1}(x, y)) - F_n(L_n^{-1}(x, y), g \circ L_n^{-1}(x, y))| : (x, y) \in D_n\} \\
= \max_n \sup \{|\alpha_n (f \circ L_n^{-1}(x, y) - g \circ L_n^{-1}(x, y))| : (x, y) \in D_n\} \\
\leq \max_n \{||\alpha_n|| \sup \{|(f(x, y) - g(x, y))| : (x, y) \in D\} \\
\leq ad \left( f(x, y), g(x, y) \right),
\]

where \(a = \max_n \{||\alpha_n||\} < 1\).

By contraction mapping theorem, \(T\) has a unique, fixed point \(f\) such that \(Tf(x, y) = f(x, y)\), i.e,

\[
f(x, y) = F_n\left( L_n^{-1}(x, y), f \circ L_n^{-1}(x, y) \right) \\
= \alpha_n f \circ L_n^{-1}(x, y) + Q_n \circ L_n^{-1}(x, y)
\tag{7}
\]

Clearly, \(f\) passes through the interpolation points. Now, let \(G\) be the unique, attractor of the IFS and \(G'\) be the graph of \(f\). Then,

\[
W(G') = \bigcup_{n=1}^{N} \{w_n(G')\} \\
= \bigcup_{n=1}^{N} \{w_n\left( \{ (x, y, f(x, y)) : (x, y) \in D \} \right) \} \\
= \bigcup_{n=1}^{N} \{ \{ L_n(x, y), F_n(x, y, f(x, y)) \} : (x, y) \in D \} \\
= \bigcup_{n=1}^{N} \{ \{ L_n(x, y), f \circ L_n(x, y) \} : (x, y) \in D \} \\
= G'.
\]

Thus, graph of \(f\) is the unique, attractor of the IFS.
3. Double integration

Let $M$ denote the double integral of a bivariate fractal interpolation function $f$ over the triangular region $D$. Then,

\begin{align*}
M &= \int_D f(x, y) \, dx \, dy \\
&= \sum_{n=1}^{N} \int_{D_n} f(x, y) \, dx \, dy \\
&= \sum_{n=1}^{N} \int_{D_n} T f(x, y) \, dx \, dy \\
&= \sum_{n=1}^{N} \int_{D_n} F_n(L_n^{-1}(x, y), f \circ L_n^{-1}(x, y)) \, dx \, dy \\
&= \sum_{n=1}^{N} \int_{D_n} F_n(L_n^{-1}(x, y), f \circ L_n^{-1}(x, y)) \, dx \, dy \\
&= \sum_{n=1}^{N} \int_{D_n} F_n(u, v, f(u, v)) \, du \, dv \\
&= \sum_{n=1}^{N} \int_{D_n} F_n(u, v) \, du \, dv \\
&= \sum_{n=1}^{N} \int_{D_n} (\alpha_n u + \beta_n v) \, du \, dv \\
&= B + \sum_{n=1}^{N} \alpha_n \beta_n \int_{D_n} f(u, v) \, du \, dv \\
&= B + AM
\end{align*}

Using coordinate transformation, $(u, v) = L_n^{-1}(x, y)$, the above expression becomes,

\begin{align*}
M &= \sum_{n=1}^{N} \int_{D_n} F_n(L_n^{-1}(x, y), f \circ L_n^{-1}(x, y)) \, dx \, dy \\
&= \sum_{n=1}^{N} \int_{D_n} F_n(u, v) \, du \, dv \\
&= \sum_{n=1}^{N} \int_{D_n} (\alpha_n u + \beta_n v) \, du \, dv \\
&= B + AM
\end{align*}

where $B = \sum_{n=1}^{N} \int_{D_n} (\alpha_n u + \beta_n v) \, du \, dv$, $A = \sum_{n=1}^{N} \alpha_n \beta_n$, and $\delta_n = \text{det}(L_n)$, which implies that

\begin{align*}
M &= B + AM \\
M(1 - A) &= B \\
M &= \frac{B}{1 - A}
\end{align*}

(8)

4. Relation of the BFIF with the equation of the plane through the vertices of $D_n$

**Theorem 4.1.** Let $f$ be the bivariate fractal interpolation function to the given data set. Then, $f$ satisfies the relation

\begin{equation}
f(x, y) = h(x, y) + \alpha_n(f - b)\circ L_n^{-1}(x, y)
\end{equation}

where $h$ is the equation of the plane passing through the vertices of $D_n$ and $b$ is the equation of the plane passing through the vertices of $D$.

**Proof.** Consider $h, b$ as the equations of the planes passing through the vertices of $D_n$ and $D$ respectively. i.e.,

\begin{align*}
h(x, y) &= \left(\frac{\sum_{i=1}^{3} x_i z_i}{\sum_{i=1}^{3} y_i z_i} - \frac{x_2 z_2}{y_2 z_2}
+ \frac{x_3 z_3}{y_3 z_3} - \frac{x_1 z_1}{y_1 z_1} - \frac{x_3 z_2}{y_3 z_2}
+ \frac{x_2 z_1}{y_2 z_1} - \frac{x_1 z_3}{y_1 z_3} - \frac{x_2 z_3}{y_2 z_3}
+ \frac{x_1 z_2}{y_1 z_2} - \frac{x_3 z_3}{y_3 z_3}ight) y \\
&= \left(\frac{x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1}{x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1}\right) y
\end{align*}

(10)
Let \( R \subseteq \mathbb{R} \times \mathbb{R} \) be a set, \( X = \{ x_1, x_2, \ldots, x_n \} \) be a subset of \( R \), and \( T(x) \) be a function defined on \( R \) with \( T(x) = (y_1, y_2, \ldots, y_n) \). Suppose that \( T(x) \) satisfies the following conditions:

1. \( T(x) \) is continuous on \( X \).
2. \( T(x) \) is differentiable on \( X \).
3. \( T(x) \) is bounded on \( X \).

Then, the function \( T(x) \) is differentiable on \( X \), and its derivative is given by

\[
T'(x) = \left( \frac{y'_1}{x'_1}, \frac{y'_2}{x'_2}, \ldots, \frac{y'_n}{x'_n} \right),
\]

where \( x'_i \) and \( y'_i \) are the partial derivatives of \( x_i \) and \( y_i \) with respect to \( x \), respectively.

Now, approximating \( T(x) \) by \( h(x) \), where \( h(x) \) is a linear function, we have

\[
f(x) = h(x) + \alpha_n g(x, y)\]

with \( g(x, y) = \sum_{i=1}^{n} \left( y_i - y_{i+1} \right) (x - x_i) / (x_{i+1} - x_i) \), where \( x_i \) and \( x_{i+1} \) are the endpoints of the subinterval \( (x_i, x_{i+1}) \) in \( X \).

Hence, the proof.

\[
Q_n(x, y) = h o L_n(x, y) - \alpha_n b(x, y),
\]

which implies

\[
Q_n o L_n^{-1}(x, y) = h(x, y) - \alpha_n b o L_n^{-1}(x, y).
\]

Substituting (12) in (7), it is obtained that

\[
f(x, y) = h(x, y) + \alpha_n (f - b) o L_n^{-1}(x, y).
\]

Hence the proof.

5. Selection of scaling factors

Let \( R = \{ (x_{nj}, y_{nj}, z_{nj}) : n = 1, 2, \ldots, N, j = 1, 2, 3 \} \) be the given set of data. Consider the points \( (x_{nl}, y_{nl}) \) in the subtriangle \( D_n \) where \( l = 1, 2, \ldots, S, S = (d' + 1)^2, d' \) is the number of subdivisions along the height of the triangle \( D_n \) for \( n = 1, 2, \ldots, N \).

If \( g \) is the fractal interpolation function to this data set, then

\[
z_{nl} = g(x_{nl}, y_{nl})
\]

\[
= \alpha_n g o L_n^{-1}(x_{nl}, y_{nl}) + Q_n o L_n^{-1}(x_{nl}, y_{nl})
\]

\[
= \alpha_n g o L_n^{-1}(x_{nl}, y_{nl}) + h(x_{nl}, y_{nl}) - \alpha_n b o L_n^{-1}(x_{nl}, y_{nl})
\]

Now, approximating \( g \) by \( h \) and considering the optimization problem

\[
\min_{\alpha_n} E(\alpha_n) = \sum_{i=1}^{S} \left[ z_{nl} - h(x_{nl}, y_{nl}) - \alpha_n (h - b) o L_n^{-1}(x_{nl}, y_{nl}) \right]^2
\]

with \(-0.9 \leq \alpha_n \leq 0.9\), then, the minimum value of \( \alpha_n \) is:

\[
\alpha_n = \frac{\sum_{i=1}^{S} \left[ z_{nl} - h(x_{nl}, y_{nl}) \right] u_{nl}}{\sum_{i=1}^{S} (u_{nl})^2}.
\]
\text{where } u_{nl} = (h - b)\alpha L_n^{-1}(x_{nl}, y_{nl}).

The upper bound for } \alpha_{n7} \text{ can be calculated by applying Cauchy-Schwartz inequality to the above value of } \alpha_{n7}.

\[
|\alpha_{n7}| \leq \left[ \sum_{l=1}^{S} \left| z_{nl} - h(x_{nl}, y_{nl}) \right|^2 \right]^{1/2} / \left[ \sum_{l=1}^{S} |u_{nl}|^2 \right]^{1/2}
\]

Put } k_h = \left[ \sum_{l=1}^{S} |u_{nl}|^2 \right]^{1/2}. \text{ Then,}

\[
|\alpha_{n7}| \leq \left[ \sum_{l=1}^{S} \left| z_{nl} - h(x_{nl}, y_{nl}) \right|^2 \right]^{1/2} / k_h
\]

However, the formula given below has been used for the computation purpose.

\[
\alpha_{n7} = \frac{Z_G - \frac{1}{3}(z_{n1} + z_{n2} + z_{n3})}{Z_H - \frac{1}{3}(z_1 + z_2 + z_3)},
\]

where } Z_G = \text{ value of } f \text{ at the centroid of the subtriangle } D_n, Z_H = \text{ value of } f \text{ at the centroid of } D.

\section{Error analysis}

Let } f \text{ be a continuous function on } D. \text{ Then,}

\[
||f||_\infty = \max\{|f(x, y)| : (x, y) \in D\}
\]

Modulus of continuity of } f \text{ is defined as

\[
w_f(\delta) = \sup\{||f(x, y) - f(x', y')|| : (x, y), (x', y') \in D, d((x, y), (x', y')) \leq \delta\}
\]

\textbf{Lemma 6.1.} \text{ If } f \text{ is a continuous function providing the data } R = \{(x_{nj}, y_{nj}, z_{nj}) : n = 1, 2, ..., N, j = 1, 2, 3\} \text{ and } g \text{ be the corresponding fractal interpolation function with scale vector } \bar{\alpha}_{n7}. \text{ Then,}

\[
||f - g||_\infty \leq w_f(\delta) k' + ||h - b||_\infty |\alpha_{n7}|_\infty / (1 - |\alpha_{n7}|_\infty).
\]

\textbf{Proof.} \text{ Let } h \text{ be the equation of a plane passing through } (x_{n1}, y_{n1}, z_{n1}), (x_{n2}, y_{n2}, z_{n2}), (x_{n3}, y_{n3}, z_{n3}). \text{ Then,}

\[
||f - g||_\infty \leq ||f - h||_\infty + ||h - g||_\infty
\]

Consider the first part.

\text{Let } w_f(\delta) \text{ be the modulus of continuity of } f, \text{ i.e.,}

\[
w_f(\delta) = \sup\{||f(x, y) - f(x', y')|| : (x, y), (x', y') \in D, d((x, y), (x', y')) \leq \delta\}
\]

Now, rearranging } h(x, y),

\[
h(x, y) = \frac{z_{n1}(x_{n3}y - xy_{n3} + x_{n2}y_{n3} - x_{n3}y_{n2} + xy_{n2} - x_{n2}y)}{(x_{n2}y_{n3} - x_{n3}y_{n2} + x_{n3}y_{n1} - x_{n1}y_{n3} + x_{n1}y_{n2} - x_{n2}y_{n1})}
\]

\[
+ \frac{z_{n2}(xy_{n3} - x_{n3}y + x_{n3}y_{n1} - x_{n1}y_{n3} + x_{n1}y - xy_{n1})}{(x_{n2}y_{n3} - x_{n3}y_{n2} + x_{n3}y_{n1} - x_{n1}y_{n3} + x_{n1}y_{n2} - x_{n2}y_{n1})}
\]
Now, \( |f(x, y) - h(x, y)| \)
\[
\leq 2\sup\{ |f(x, y) - f(x_{n_1}, y_{n_1})| \} \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
+ 2\sup\{ |f(x, y) - f(x_{n_2}, y_{n_2})| \} \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
+ 2\sup\{ |f(x, y) - f(x_{n_3}, y_{n_3})| \} \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
+ 2\sup\{ |f(x_{n_1}, y_{n_1}) - f(x_{n_2}, y_{n_2})| \} \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
+ 2\sup\{ |f(x_{n_1}, y_{n_1}) - f(x_{n_3}, y_{n_3})| \} \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
+ 2\sup\{ |f(x_{n_2}, y_{n_2}) - f(x_{n_3}, y_{n_3})| \} \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
\]
(16)

Here, \( x, y, x_{nj}, y_{nj} \) for \( n = 1, 2, ..., N, j = 1, 2, 3 \) lie in \( D_n \).

Take \( \delta \) to be the maximum of the distance between any two points in \( D_n \). Then, by definition,
\[
w_f(\delta) = \sup\{ |f(x, y) - f(x_{nj}, y_{nj})| : (x, y), (x_{nj}, y_{nj}) \in D_n, d((x, y), (x_{nj}, y_{nj})) \leq \delta \}
\]

Therefore, (16) becomes
\[
|f(x, y) - h(x, y)| \leq 12w_f(\delta) \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \}
\]

Put \( k' = 12 \max_{n,j} \{ |x_{nj}| \} \max_{n,j} \{ |y_{nj}| \} \). Then, \( |f(x, y) - h(x, y)| \leq w_f(\delta) k' \). Hence,
\[
\|f - h\|_{\infty} \leq w_f(\delta) k'.
\]

Now, considering the second part, \( \|h - g\|_{\infty} \).

Using the equation,
\[
g(x, y) = h(x, y) + a_{nj}(g - b)\alpha L_n^{-1}(x, y)
\]

Now,
\[
\|g - h\|_{\infty} \leq |a_{nj}|_{\infty} \|g - b\|_{\infty}
\]
\[
\leq \left[ \|g - h\|_{\infty} + \|h - b\|_{\infty} \right] |a_{nj}|_{\infty}
\]
\[
\leq \frac{\|h - b\|_{\infty} \|a_{nj}\|_{\infty}}{1 - |a_{nj}|_{\infty}}
\]

Therefore,
\[
\|f - g\|_{\infty} \leq w_f(\delta) k' + \frac{\|h - b\|_{\infty} |a_{nj}|_{\infty}}{1 - |a_{nj}|_{\infty}}.
\]

Hence the proof. \( \square \)

**Theorem 6.1.** If \( f \) is a continuous function providing the data \( R = \{(x_{nj}, y_{nj}, z_{nj}) : n = 1, 2, ..., N, j = 1, 2, 3\} \) and \( g \) be the corresponding fractal interpolation function with scale vector \( \alpha_{nj} \). Then, the double integral calculated by the proposed method converges to the actual integral value as \( \delta \to 0 \).

**Proof.**
\[
|E| = \left| \int_D \int_D f - \int_D \int_D g \right| \leq \Delta \|f - g\|_{\infty}
\]
where \( \Delta \) is the area of the triangle \( D \) and
\[
\|f - g\|_{\infty} \leq w_f(\delta) k' + \frac{\|h - b\|_{\infty} |a_{nj}|_{\infty}}{1 - |a_{nj}|_{\infty}}.
\]
Since $f$ is an interpolation function to the data set,

$$\left| z_{nl} - h(x_{nl}, y_{nl}) \right| = \left| f(x_{nl}, y_{nl}) - h(x_{nl}, y_{nl}) \right|$$

$$\leq \sup_{f} \left\{ |f(x_{nl}, y_{nl}) - h(x_{nl}, y_{nl})| \right\}$$

$$= \|f - h\|_{\infty}$$

$$\leq w_f(\delta)k'$$

Now,

$$|\alpha_{n7}|_{\infty} \leq \left[ \frac{\sum_{l=1}^{S} |z_{nl} - h(x_{nl}, y_{nl})|^2}{k_h} \right]^{1/2}$$

$$\leq \left[ \frac{\sum_{l=1}^{S}(w_f(\delta)k')^2}{k_h} \right]^{1/2}$$

$$= \frac{w_f(\delta)k'S}{k_h}$$

Writing $k' = \frac{S k'}{k_h}$, implies $|\alpha_{n7}|_{\infty} \leq w_f(\delta) k'_h$.

Since $\alpha_{n7}$ is bounded by $w_f(\delta) k'_h$,

$$||f - g||_{\infty} \leq w_f(\delta)k' + \frac{||h - b||_{\infty}w_f(\delta)k'_{h}}{1 - w_f(\delta)k'_{h}}.$$  

For a continuous function $f$ on $D$, as $\delta \to 0$, $w_f(\delta) \to 0$.

\textit{i.e,} $||f - h|| \leq w_f(\delta)k' \to 0$, implies $h$ uniformly converges to $f$ and

$$|E| \leq \Delta ||f - g||_{\infty}$$

$$= \Delta w_f(\delta) \left[ k' + \frac{k_h||h - b||_{\infty}}{1 - w_f(\delta)k'_{h}} \right] \to 0$$

as $w_f(\delta) \to 0$. Hence the proof. 

\hfill \blacksquare

### 7. Examples

The double integral values and the attractors of the IFS are provided for two functions. The computation of the attractor and the integral value is done using amrita-hpc matlab 2019.

**Example 1 : Matyas Function**

Consider Matyas function

$$f(x, y) = 0.26(x^2 + y^2) - 0.48xy \text{ where } -10 \leq x \leq 10, -10 \leq y \leq 10.$$  

The actual integral value of the double integral ($I$) is compared with the numerical integration method proposed. The comparison is given in Table 1. The attractor of the IFS is shown in Figure 2.
Figure 2: Attractor of the IFS for Matyas function

| d (No of subdivisions) | N (No of subtriangles) | M     | I     | Error (M-I) |
|-----------------------|------------------------|-------|-------|-------------|
| 4                     | 27                     | 2.4299e+03 | 2.6000e+03 | -170.0594   |
| 7                     | 87                     | 2.5401e+03 | 2.6000e+03 | -59.8738    |
| 10                    | 183                    | 2.5696e+03 | 2.6000e+03 | -30.3787    |
| 13                    | 315                    | 2.5818e+03 | 2.6000e+03 | -18.2392    |
| \vdots                | \vdots                 | \vdots | \vdots | \vdots      |
| 73                    | 10515                  | 2.5993e+03 | 2.6000e+03 | -0.6064     |
| 76                    | 11403                  | 2.5994e+03 | 2.6000e+03 | -0.5598     |
| 79                    | 12327                  | 2.5994e+03 | 2.6000e+03 | -0.5182     |
| \vdots                | \vdots                 | \vdots | \vdots | \vdots      |
| 148                   | 43515                  | 2.5999e+03 | 2.6000e+03 | -0.1292     |
| 151                   | 45303                  | 2.5999e+03 | 2.6000e+03 | -0.0786     |
| 154                   | 47127                  | 2.5999e+03 | 2.6000e+03 | -0.0523     |

Table 1
Matyas function
Example 2: Three-hump Camel Function

Consider Three-hump Camel function

\[ f(x, y) = 2x^2 - 1.05x^4 + \frac{x^6}{6} + xy + y^2 \text{ where } -5 \leq x \leq 5, -5 \leq y \leq 5. \]

A similar comparison of the integral value as in example one is given in Table 2. The attractor of the IFS is given in Figure 3.

![Figure 3: Attractor of the IFS for Three-hump camel function](image-url)
8. Conclusion

This paper describes the construction of bivariate fractal interpolation functions using the coloring technique and derives the formula for double integration. In between, a novel method is also proposed for the partition of the triangle. Since the IFS itself induces a well-defined fractal operator, the paper establishes that the graph of the function coincides with the attractor of the IFS. Instead of choosing the vertical scaling factor randomly, this paper provides a useful formula for it, based on the shape of the interpolating domain. Further, this paper shows that the approximating function for the bivariate fractal interpolation functions over triangular regions is nothing but the equation of the plane passing through the vertices of a triangle. This function is then used to prove the theorems in error analysis. Finally, the results of the double integration are tabulated and the method of construction is explained with appropriate graphs.

References

[1] M.F. Barnsley, Fractals Everywhere, second ed., Academic Press Professional, Newyork, 1988.
[2] L. Dalla, Bivariate fractal interpolation functions on grids, Fractals 10 (2002) 53-58.
[3] Vasileios Drakopoulos and Polychronis Manousopoulos, On Non-tensor product bivariate fractal interpolation surfaces on rectangular grids, Mathematics 8 (2020) 525.
[4] J.S. Geronimo, D. Hardin, Fractal interpolation surfaces and a related 2-D multiresolution analysis, J. Math. Anal. Appl 176 (1993) 561-586.
[5] M. F. Barnsley, Fractal functions and interpolation, Constr. Approx 2 (1986) 303-329.
[6] R. Małysz, The Minkowski dimension of the bivariate fractal interpolation surfaces, Chaos Solitons Fractals 27 (2006) 1147–1156.
[7] W. Metzler, C. Yun, Construction of fractal interpolation surfaces on rectangular grids, Internat. J. Bifur. Chaos 20 (2010) 4079–4086.
[8] P.R. Massopust, Fractal surfaces, J. Math. Anal. Appl 151 (1990) 275-290.
[9] Zekeriya Sari, Gizem Kalender, Serkan Gunel, Fractal interpolation and integration over two-dimensional triangular meshes, J.Physics, Conference series 1391 (2019) 012143.
[10] M.A. Navascués, M.V. Sebastián, Numerical integration of affine fractal functions, Journal of computational and applied mathematics 252 (2013) 169-176.
[11] Ri Songil, A new idea to construct fractal interpolation function, Indagationes mathematicae 29 (2018) 962-971.