Conditioning of Quantum Open Systems

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Abstract

The underlying probabilistic theory for quantum mechanics is non-Kolmogorovian. The order in which physical observables will be important if they are incompatible (non-commuting). In particular, the notion of conditioning needs to be handled with care and may not even exist in some cases. Here we layout the quantum probabilistic formulation in terms of von Neumann algebras, and outline conditions (non-demolition properties) under which filtering may occur.

1 Introduction

The mathematical theory of quantum probability (QP) is an extension of the usual Kolmogorov probability to the setting inspired by quantum theory [1][2][3][4]. In this article, we will emphasize the quantum analogues to events (projections), random variables (operators), sigma-algebras (von Neumann algebras), probabilities (states), etc. The departure from Kolmogorov’s theory is already implicit in the fact that the quantum random variables do not commute. It is advantageous to set up a non-commutative theory of probability and one of the requirements is that Kolmogorov’s theory is contained as the special case when we restrict to commuting observables. The natural analogue of measure theory for operators is the setting of von Neumann algebras [5].

2 Keywords

Quantum Filter, von Neumann algebra, quantum probability, Bell’s Theorem

3 Classical Probability

The standard approach to probability in the classical world is to associate with each model a Kolmogorov triple \((\Omega, \mathcal{F}, \mathbb{P})\) comprising a measurable space \(\Omega\), a sigma-algebra, \(\mathcal{F}\), of subsets of \(\Omega\), and probability measure \(\mathbb{P}\) on the sigma-algebra. \(\Omega\) covers the entirety of all possible things that may enfold in our
model, the elements of $\mathcal{F}$ are distinguished subsets of $\Omega$ referred to as events, and $P[A]$ is the probability of event $A$ occurring. From a practical point of view we need to frame the model in terms of what we may hope to observe, and these constitute the “events”, $\mathcal{F}$, and from a mathematical point of view, restricting to a sigma-algebra clarifies all the ambiguities, while also resolving all the technical issues, pathologies, etc., that would otherwise plague the subject.

A random variable is then defined as a function on $\Omega$ to a value space, another measurable space $(\Gamma, \mathcal{G})$: so, for each $G \in \mathcal{G}$, the set $X^{-1}[G] = \{\omega \in \Omega : X(\omega) \in G\}$ is an event, i.e., in $\mathcal{F}$. The probability that $X$ takes a value in $G$ is then $K_X[G] = P[X^{-1}[G]]$ which produces the distribution of $X$ as $K_X = P \circ X^{-1}$. Let $X_1, \ldots, X_n$ be random variables then their joint probabilities are also well-defined:

$$K_{X_1,\ldots,X_n}[G_1,\ldots,G_n] = P\left[\bigcap_{i=1}^n X_i^{-1}[G_i]\right].$$  (1)

Let $\mathcal{A}$ be a sub-sigma algebra of $\mathcal{F}$. The collection of bounded $\mathcal{A}$-measurable (complex-valued) functions will be denoted as $\mathcal{A} = L^\infty(\Omega,\mathcal{A})$. This is an example of a *-algebra of functions. We now show that there is a natural identification between $\sigma$-algebras of subsets of $\Omega$ and *-algebras of functions on $\Omega$. First we need some definitions.

**Definition 1** A sequence $(f_n)_n$ of functions on $\Omega$ is said to be a **positive bounded monotone sequence** if there exists a finite constant $c > 0$ such that $0 \leq f_n \leq c$ and $f_n \leq f_{n+1}$ for each $n$. A *-algebra of functions $\mathcal{A}$ on $\Omega$ is said to be **monotone class** if every positive bounded monotone sequence in $\mathcal{A}$ has its limit in $\mathcal{A}$.

The next result can be found in Protter [5].

**Theorem 2 (Monotone Class Theorem)** Given a monotone class *-algebra of functions, $\mathcal{A}$, on $\Omega$. Then $\mathcal{A} = L^\infty(\Omega,\mathcal{A})$ where $\mathcal{A}$ is precisely the $\sigma$-algebra generated by $\mathcal{A}$ itself.

Conditional probabilities are natural defined: the probability that event $A$ occurs given that event $B$ occurs is

$$P[A|B] \triangleq \frac{P[A \cap B]}{P[B]},$$  (2)

This is a simple “renormalization” of the probability: one restricts the outcomes in $A$ which also lie in $B$, weighted as a proportion out of all $B$, rather than $\Omega$.

### 3.1 Quantum Probability Models

The standard presentation of quantum mechanics takes physical quantities (observables) to be self-adjoint operators on a fixed Hilbert space $\mathfrak{h}$ of wavefunctions. The normalized elements of $\mathfrak{h}$ are the **pure states**, and the expectation of an observable $X$ for a pure state $\psi \in \mathfrak{h}$ is given $E[X] = \langle \psi | X \psi \rangle$. As such,
observables play the role of random variables. More generally, we encounter quantum expectations of the form
\[ E[X] = \text{tr} \{ \rho X \} \]
where \( \rho \geq 0 \) is a trace-class operator normalized so that \( \text{tr} \{ \rho \} = 1 \). The operator \( \rho \) is called a density matrix and in the pure case corresponds to \( \rho = |\psi\rangle \langle \psi| \).

**Definition 3** A quantum probability space \((\mathcal{A}, E)\) consists of a von Neumann algebra \(\mathcal{A}\) and a state \(E\) (assumed to be continuous in the normal topology).

When \(\mathcal{A}\) is commutative, then the quantum probability space is isomorphic to a Kolmogorov model. Let us motivate now why von Neumann algebras are the appropriate object. Positive operators are well defined, and we may say \(X \leq Y\) if \(Y - X \geq 0\). In particular, the concept of a positive bounded monotone sequence of operators makes sense, as does a monotone class algebra of operators.

**Theorem 4** (van Handel [7]) A collection of bounded operators over a fixed Hilbert space, is a von Neumann algebra if and only if it is a monotone class *-algebra.

Specifically, we see that this recovers the usual monotone class Theorem when we further impose commutativity of the algebra. A state on a von Neumann algebra, \(\mathcal{A}\), is then a normalized positive linear map from \(\mathcal{A}\) to the complex numbers, that is, \(E[1] = 1\), \(E[A + \beta B] = \alpha E[A] + \beta E[B]\), \(E[X] \geq 0\), whenever \(X \geq 0\). Note that if \((X_n)\) is a positive bounded monotone sequence with limit \(X\), then \(E[X_n]\) converges to \(E[X]\) - this in fact equivalent to the condition of continuity in the normal topology, and implies that \(E\) to take the form (3), for some density matrix \(\rho\), [6].

**Definition 5** An observable is referred to as a quantum event if it is an orthogonal projection. If a quantum event \(A\) corresponds to a projection \(P_A\) then its probability of occurring is \(\Pr\{A\} = \text{tr} \{ \rho P_A \} \).

The requirement that \(P_A\) is an orthogonal projection means that \(P_A^2 = P_A = P_A^\dagger\). The complement to the event \(A\), that is not \(A\), will be denoted as \(\bar{A}\), and is the orthogonal projection given by the orthocomplement \(P_{\bar{A}} = 1 - P_A\), where \(1\) is the identity operator on \(\mathcal{H}\). A von Neumann algebra is a subalgebra of the bounded operators on \(\mathcal{H}\) with good closure properties: crucially it will be generated by its projections. So the von Neumann algebra generated by a collection of quantum events is the natural analogue of a sigma-algebra of classical events.

However, the new theory of quantum probability will have features not present classically. For instance, the notion of a pair of events, \(A\) and \(B\), occurring jointly is not generally meaningful. In fact, \(P_A P_B\) is in general not
self-adjoint, and so does not correspond to a quantum event. We therefore cannot interpret \( \text{tr}\{\rho P_A P_B\} \) as the joint probability for quantum events \( A \) and \( B \) to occur.

**Proposition 6** The product \( P_A P_B \) is an event (that is, an orthogonal projection) if and only if \( P_A \) and \( P_B \) commute.

**Proof.** Self-adjointness of \( P_A P_B \) is enough to give the commutativity since then \( P_A P_B = (P_A P_B)^* = P_B P_A \) as both \( P_A \) and \( P_B \) are self-adjoint. We then see that \((P_A P_B)^2 = P_A P_B P_A P_B = P_A^2 P_B^2 = P_A P_B \).

As such, given a pair of quantum events \( A \) and \( B \), it is usually meaningless to speak of their joint probability \( \text{Pr}\{A, B\} \) for a fixed state \( \rho \). An exception is made when the corresponding projectors commute in which case we say the events are compatible and take the probability to be \( \text{Pr}\{A, B\} \equiv \text{tr}\{\rho P_A P_B\} \).

By the spectral theorem, every observable may be written as

\[
X = \int_{-\infty}^{\infty} x P_X(dx)
\]

where \( P_X(dx) \) is a projection valued measure, normalized so that \( P_X(\mathbb{R}) = 1 \), the identity operator. The measure is supported on the spectrum of \( X \) which is, of course real by self-adjointness. In particular, if \( G_1 \) and \( G_2 \) are non-overlapping Borel subsets of \( \mathbb{R} \) then \( P_X[G_1] \) and \( P_X[G_2] \) project onto mutually orthogonal projections.

The orthogonal projection \( P_X[G] \) then corresponds to the quantum event that \( X \) is measured to have a value in the subset \( G \). The smallest von Neumann algebra containing all the projections \( P_X[G] \) will be denoted as \( \mathfrak{F}_X \) and plays an analogous role to the sigma-algebra generated by a random variable.

Once we fix the density matrix \( \rho \), the spectral decomposition leads to the probability distribution of observable \( X \): \( \mathbb{K}_X \{dx\} = \text{tr}\{\rho P_X\{dx\}\} \). We say that observables \( X_1, \ldots, X_n \) are compatible if the quantum events they generate are compatible. In this case

\[
\text{tr}\left\{\rho e^{i \sum_{k=1}^{n} u_k X_k}\right\} = \int e^{i \sum_{k=1}^{n} u_k x_k} \mathbb{K}_{X_1, \ldots, X_n}\{dx_1, \ldots, dx_n\},
\]

where \( \mathbb{K}_{X_1, \ldots, X_n}\{dx_1, \ldots, dx_n\} \) defines a probability measure on the Borel sets of \( \mathbb{R}^n \). This may be no longer true if we drop the compatibility assumption!

We remark that, given a collection of observables \( X_1, \ldots, X_n \), we can construct the smallest von Neumann algebra containing all their individual quantum events; this will typically a non-commutative algebra and effectively plays the role of a sigma algebra generated by random variables.

Positivity preserving measurable mappings are the natural morphisms between Kolmogorov spaces. The situation in quantum probability is rather more prescriptive.

First note that if \( \mathfrak{A} \) and \( \mathfrak{B} \) are von Neumann algebras, then so too is their formal tensor product. A map \( \Phi : \mathfrak{F} \to \mathfrak{F} \) between von Neumann algebras is
positive is $\Phi (A) \geq 0$ whenever $A \geq 0$. However, we need a stronger condition. The mapping has the extension $\Phi_\delta : \mathfrak{A} \otimes \mathfrak{F} \rightarrow \mathfrak{B} \otimes \mathfrak{F}$ for a given von Neumann algebra by $\Phi_\delta (A \otimes F) = \Phi (A) \otimes F$. We say that $\Phi$ is completely positive (CP) if $\Phi_\delta$ is positive for any $\mathfrak{F}$.

A morphism between a pair of quantum probability spaces $[\mathfrak{I}, \Phi : (\mathfrak{A}_1, \mathfrak{E}_1) \rightarrow (\mathfrak{A}_2, \mathfrak{E}_2)$ is a completely positive map with the properties $\Phi (1_{\mathfrak{A}_1}) = 1_{\mathfrak{A}_2}$ and $\mathfrak{E}_2 \circ \Phi = \mathfrak{E}_1$. Despite its rather trivial looking appearance, the CP property is actually quite restrictive.

4 Quantum Conditioning

The conditional probability of event $A$ occurring given that $B$ occurs is defined by $\Pr \{A|B\} = ^{\Pr\{A,B\}}_{\Pr\{B\}}$. In quantum probability, $\Pr \{A, B\}$ may make sense as a joint probability only if $A$ and $B$ are compatible, otherwise there is the restriction that $A$ is measured before $B$.

Let $X$ be an observable with a discrete spectrum (eigenvalues). If we start in a pure state $|\psi_{in}\rangle$ and measure $X$ to record a value $x$ then this quantum event has corresponding projector $P_X (x)$. Von Neumann’s projection postulate states that the state after measurement is proportional to

$$|\psi_{out}\rangle = P_X (x) |\psi_{in}\rangle$$

and that the probability of this event is $\Pr \{X = x\} = \langle \psi_{in}|P_X (x) |\psi_{in}\rangle = \langle \psi_{out}|\psi_{out}\rangle$. Note that $|\psi_{out}\rangle$ is not normalized! A subsequent measurement of another discrete observable $Y$, leading to eigenvalue $y$, will result in $|\psi_{out}\rangle = P_Y (y) P_X (x) |\psi_{in}\rangle$ and so (ignoring dynamics form the time being)

$$\Pr \{X = x; Y = y\} = \langle \psi_{out}|\psi_{out}\rangle = \langle \psi_{in}|P_Y (y) P_X (x) P_Y (y) |\psi_{in}\rangle.$$  

This needs to be interpreted as a sequential probability - event $X = x$ occurs first, and $Y = y$ second - rather than a joint probability. If $X$ and $Y$ do not commute then the order in which they are measured matters as $\Pr \{X = y; Y = x\}$ may differ from $\Pr \{Y = y; X = x\}$.

**Lemma 7** Let $P$ and $Q$ be orthogonal projections then the properties $QPQ = PQP$ and $PQP = QP$ are equivalent to $[Q, P] = 0$.

**Proof.** We begin by noting that $[P, Q]^* |P, Q\rangle = QPQ - PQP + PQP - QPQ$. This may be rewritten as $(Q - P) (QPQ - PQP)$ so if $QPQ = PQP$ then $[P, Q] = 0$. If we have $PQP = QP$, then $[P, Q]^* |P, Q\rangle \equiv QPQ - QPQ + PQP - QP = PQP - QP = 0$ and so $[P, Q] = 0$. Conversely, if $P$ and $Q$ commute then $QPQ = PQP$ and $PQP = QP$ hold true. $\blacksquare$

**Corollary 8** If we have $\Pr \{X = x; Y = y\}$ equal to $\Pr \{Y = y; X = x\}$ for all states $|\psi_{in}\rangle$ and all eigenvalues $x$ and $y$ are equal, then $X$ and $Y$ are compatible.

The symmetry implies that $P_Y (y) P_X (x) P_Y (y)$ equals $P_X (x) P_Y (y) P_X (x)$ and so the spectral projections of $X$ and $Y$ commute.

5
Corollary 9 If for all states $|\psi_{in}\rangle$ whenever we measure $X$, then $Y$ and then $X$ again, we always record the same value for $X$, then $X$ and $Y$ are compatible.

Setting $|\psi_{out}\rangle = P_Y (y) P_X (x) |\psi_{in}\rangle$ we must have that $P_X (x) |\psi_{out}\rangle = |\psi_{out}\rangle$, if this is true for all $|\psi_{in}\rangle$ then $P_X (x) P_Y (y) = P_X (x) P_Y (y) P_X (x)$ which likewise implies that $[P_X (x), P_Y (y)] = 0$.

4.1 Conditioning Over Time

The dynamics under a (possibly time-dependent) Hamiltonian $H (t)$ is described by the two-parameter family of unitary operators

$$U (t, s) = 1 - i \int_s^t H (\tau) U (\tau, s) d\tau, \quad t \geq s. \tag{8}$$

This is the solution to the flow identity

$$U (t_3, t_2) U (t_2, t_1) = U (t_3, t_1) \tag{9}$$

whenever $t_3 \geq t_2 \geq t_1$. In the special case where $H$ is constant, we have $U (t, s) = e^{-i (t-s) H}$. It is convenient to introduce the maps $(t_2 > t_1)$

$$J_{(t_1, t_2)} (X) = U (t_2, t_1)^* X U (t_2, t_1) \tag{10}$$

and, from the flow identity (9), $J_{(t_1, t_2)} \circ J_{(t_2, t_3)} = J_{(t_1, t_3)}$ whenever $t_3 > t_2 > t_1$.

We now consider an experiment where we measure observables $Z_1, \ldots, Z_n$ at times $t_1 < t_2 < \cdots < t_n$ during a time interval $0$ to $T$. At the end of the experiment, if we measure $Z_1 \in G_1, \ldots, Z_n \in G_n$, then the output state should be

$$|\psi_{out}\rangle = U (T, t_n) P_{Z_n} [G_n] U (t_{n-1}, t_n) \cdots U (t_2, t_1) P_{Z_1} [G_1] U (t_1, 0) |\psi_{in}\rangle. \tag{11}$$

It is convenient to introduce the observables

$$Y_k = J_{(0, t_k)} (Z_k). \tag{12}$$

The quantum event $Y_k \in G_k$ at time $t_k$ is then $P_{Y_k} [G_k] = J_{(0, t_k)} (P_{Z_k} [G_k])$. The flow identity implies $U (t_k, t_{k-1}) = U (t_k, 0) U (t_{k-1}, 0)^*$, and we find

$$|\psi_{out}\rangle = U (T, 0) J_{(0, t_n)} (P_{Z_n} [G_n]) \cdots J_{(0, t_2)} (P_{Z_1} [G_1]) |\psi_{in}\rangle = U (T, 0) P_{Y_n} [G_n] \cdots P_{Y_1} [G_1] |\psi_{in}\rangle. \tag{13}$$

We therefore have the probability

$$\Pr \{Y_1 \in G_1; \cdots; Y_n \in G_n \} = \langle \psi_{out} | \psi_{out}\rangle = \text{tr} \{P_{Y_n} [G_n] \cdots P_{Y_1} [G_1] \rho P_{Y_1} [G_1] \cdots P_{Y_n} [G_n] \}, \tag{14}$$

where $\rho$ is the initial state. Note that this takes the pyramidal form.
Here the $Z_k$ are understood as observables specified in the Schrödinger picture at time 0 - what we measure are the $Y_k$ which are the $Z_k$ at respective times $t_k$. The answer depends on the chronological order $t_1 < t_2 < \cdots < t_n$.

It is tempting to think of $\{Y_k : k = 1, \cdots, n\}$ as a discrete time stochastic process, but some caution is necessary. We cannot generally permute the events $Y_1 \in G_1; \cdots; Y_n \in G_n$ so we do not have the symmetry usually associated with Kolmogorov’s Reconstruction Theorem.

**Proposition 10** The finite-dimensional distributions satisfy the marginal consistency for the most recent variable. Specifically, this means that

$$\sum_G \Pr \{Y_1 \in G_1; \cdots; Y_n \in G\} = \Pr \{Y_1 \in G_1; \cdots; Y_{n-1} \in G_{n-1}\}$$

where the sum is over a collectively exhaustive mutually exclusive set $\{G\}$.

**Proof.** We have

$$\sum_G \Pr \{Y_1 \in G_1; \cdots; Y_n \in G\}$$

$$= \sum_G \text{tr} \{P_{Y_{n-1}} [G_{n-1}] \cdots P_{Y_1} [G_1] \rho P_{Y_1} [G_1] \cdots P_{Y_{n-1}} [G_{n-1}] P_{Y_n} [G]\}$$

but $\sum_G P_{Y_n} [G] \equiv 1$ so we obtain the desired reduction.

As an example, suppose that $\{A_k\}$ is a collection of quantum events occurring at a fixed time $t_1$ which are mutually exclusive (that is, their projections project onto orthogonal subspaces). Their union $\cup_k A_k$ makes sense and corresponds to the projection onto the direct sum of these subspaces. Now if $B$ is an event at a later time $t_2$, then typically $\sum_k \Pr \{A_k; B\}$ is not the same as $\Pr \{\cup_k A_k; B\}$. The well known two-slit experiment fits into this description, with $A_k$ the event that an electron goes through slit $k$ ($k = 1, 2$), and $B$ the subsequent event that it hits a detector.

Marginal consistency is a property we take for granted in classical stochastic processes and is an essential requirement for Kolmogorov’s Reconstruction Theorem. However, in the quantum setting it is only guaranteed to work for the last measured observable. For instance, it may not apply to $Y_{n-1}$ unless we can commute its projections with $P_{Y_n} [G_n]$. The most recent measured observable has the potential to demolish all the measurements beforehand.

### 4.2 Bell’s Inequalities

A Bell inequality is any constraint that applies to classical probabilities but which may fail in quantum probability. We look at one example due to Eugene Wigner.

**Proposition 11 (A Bell Inequality)** Given three (classical) events $A, B, C$ then we always have

$$\Pr \{A, C\} \leq \Pr \{A, \overline{B}\} + \Pr \{B, C\}.$$
Proof. From the marginal property we have the following three classical identities

\[Pr\{A, B\} = Pr\{A, B, C\} + Pr\{A, B, C\}, Pr\{B, C\} = Pr\{A, B, C\} + Pr\{A, B, C\}, \text{ and } Pr\{A, C\} = Pr\{A, B, C\} + Pr\{A, B, C\}.\]

We therefore have that

\[Pr\{A, B\} + Pr\{B, C\} - Pr\{A, C\} = Pr\{A, B, C\} + Pr\{A, B, C\} \geq 0. \quad (19)\]

The proof relies on the fact marginal consistency is always valid for classical events. Suppose that the events are quantum events, say \(Y_k\) taking a value in \(G_k\) at time \(t_k\) \((k = 1, 2, 3)\), and chronologically ordered \((t_1 < t_2 < t_3)\). Then only the first of the three classical identities is guaranteed in quantum theory (marginal consistency for the latest event at time \(t_3\) only). The remaining two may fail, and it is easy to construct a quantum system where inequality (18) is violated.

5 Quantum Filtering

Given the issues raised above, one may ask whether it is actually possible to track a quantum system over time?

We shall say that the process \(\{Y_k : k = 1, \ldots, n\}\) is essentially classical whenever all the observables are compatible. In this case its finite dimensional distributions satisfy all the requirements of Kolmogorov’s Theorem and so we can model it as a classical stochastic process. The von Neumann algebra \(\mathcal{N}_n\) they generate will be commutative, and we have \(\mathcal{N}_n \subset \mathcal{N}_{n+1}\) (a filtration of von Neumann algebras!).

For the tracking over time to be meaningful, we ask for the observed process to be essentially classical - this is the self-non-demolition property of the observations.

Let \(X_n = J_{(0, t_n)}(X)\) be an observable \(X\) at time \(t_n\). Suppose that it is compatible with \(\mathcal{N}_n\) then we are lead to a well-defined classical joint distribution \(K_{X_n, Y_1, \ldots, Y_n}\) from which we may compute the conditional probability \(K_{X_n|Y_1, \ldots, Y_n}\). This means that \(\pi_{t_n}(X) = \mathcal{E}[X_n|\mathcal{N}_n]\) is well-defined.

We only try to condition those observables that are compatible with the measured observables! This is known as the non-demolition property.

5.1 Conditioning in Quantum Theory

The natural analogue of conditional expectation in quantum theory would be a projective morphism (CP map) from a von Neumann algebra into a sub-von Neumann algebra. However, this does not always exist in the noncommutative case. Given our discussions above, this is not surprising.

In general, be \(\mathcal{N}\) be a sub-von Neumann algebra of a von Neumann algebra \(\mathcal{B}\), then its commutant is the set of all operators in \(\mathcal{B}\) which commute with each element of \(\mathcal{N}\), that is

\[\mathcal{N}' = \{X \in \mathcal{B} : [X, Y] = 0, \forall Y \in \mathcal{N}\}.\]
Why this is possible is easy to explain. The von Neumann algebra generated by $\mathcal{Y}$ and fixed element $X \in \mathcal{Y}'$ will again be a commutative, so the conditional expectation of $X$ onto $\mathcal{Y}$ is well defined. Physically, it means that $X \in \mathcal{Y}'$ is compatible with $\mathcal{Y}$ so standard classical probabilistic constructs are valid.

As an illustration, suppose that the von Neumann algebra of a system of interest is $\mathfrak{A}$ and its environment’s is $\mathfrak{E}$. Let $X$ be a simple observable of the system, say with spectral decomposition $X = \sum_x xR_x$ where $\{R_x\}$ is a set of mutually orthogonal projections. Similarly, let $Z = \sum_y yQ_y$ be an environment observable. We entangle the system and environment with a unitary $U$ (acting on $\mathfrak{A} \otimes \mathfrak{E}$) and measure the observable $Y = U^* (1_{\mathfrak{A}} \otimes Z)U \equiv \sum_y P_y$. Here, the event corresponding to measuring eigenvalue $y$ of $Y$ is $P_y = U^* (1_{\mathfrak{A}} \otimes Q_y)U$. The observables $X_1 = U^* (X \otimes 1_{\mathfrak{E}})U$ and $Y$ commute and so they have a well-defined joint probability to have eigenvalues $x$ and $y$ respectively for a fixed state $E$.

$$p(x, y) = E[U^* (R_x \otimes Q_y)U]$$. We may think of $X_1$ as $X$ evolved by one time step. Its conditional expectation given the measurement of $Y$ is then $E[X_1|\mathcal{Y}] = \sum_{x, y} x p(x|y) P_y$ where $p(x|y) = p(x, y)/p(y)$, with $p(y) = \sum_{x} p(x, y)$.

6 Summary and Future Directions

Quantum theory can be described in a systematic manner, despite the frequently sensationalist presentations on the subject. Getting the correct mathematical setting allows one to develop an operational approach to quantum measurements and probabilities. We described the quantum probabilistic framework which uses von Neumann algebras in place of measurable functions and shown how some of the usual concepts from Kolmogorov’s theory carry over. However, we were also able to highlight which features of the classical world may fail to be valid in quantum theory.

What is of interest to control theorists is that the extraction of information from quantum measurements can be addressed and, under appropriate conditions, filtering can be formulated. As we have seen, measuring quantum systems over time is problematic. However, the self-non-demolition property of the observations, and the non-demolition principle are conditions which guarantee that the filtering problem is well-posed. These conditions are met in continuous time quantum Markovian models (by causality, prediction turns out to be well-defined, though not necessarily smoothing!). Explicit forms for the filter were given by Belavkin [8], see also [9].

7 Cross References

Article by H.I. Nurdin
8 Recommended Reading

The mathematical program of “quantizing” probability emerged in the 1970’s and has produced a number of technical results used by physicists. However, it the prospect technological applications that has seen QP adopted as the natural framework for quantum analogues of engineering such as filtering and control. The philosophy of QP is given in [1], for instance, with the main tool - quantum Ito calculus - in [3, 4]. The theory of quantum filtering was pioneered by V.P. Belavkin [8] with modern accounts in [7, 9].

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