Dynamical random multiplicative cascade model in 1+1 dimensions

Jürgen Schmiegel\textsuperscript{a,b}, Hans C. Eggers\textsuperscript{a}, and Martin Greiner\textsuperscript{b,c,d}

\textsuperscript{a}Department of Physics, University of Stellenbosch, 7600 Stellenbosch, South Africa
\textsuperscript{b}Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, D–01187 Dresden, Germany
\textsuperscript{c}Department of Physics, Duke University, Durham, NC 27708, USA
\textsuperscript{d}Fysisk Institutt, Universitetet i Bergen, Allegaten 55, N-5007 Bergen, Norway

Abstract

Geometrical random multiplicative cascade processes are often used to model positive-valued multifractal fields such as for example the energy dissipation field of fully developed turbulence. A dynamical generalisation of these models is proposed, which describes the continuous and homogeneous stochastic evolution of the field in one space and one time dimension. Two-point correlation functions are calculated.
Whenever strong anomalous, intermittent fluctuations, long-range correlations, multi-scale structuring and selfsimilarity go hand in hand, the label ‘multifractality’ is attached to the underlying process. Although we have fully developed turbulence of fluid mechanics [1] in mind, such processes occur in various diverse fields such as formation of cloud and rain fields of geophysics [2], internet traffic of signal and communication engineering [3], and stock returns of econometrics [4], just to name a few. The maybe most widely used textbook [5] visualisation for multifractality is represented by a multiplicative cascade process. It reproduces the above mentioned properties by introducing a scale-independent cascade generator, which produces a nested hierarchy of scales and redistributes the local mass multiplicatively.

In fully developed turbulence, random multiplicative cascade models (RMCM) are often employed to model the energy cascade, describing the energy flux through inertial range scales. Due to their multiplicative nature, it is straightforward for them to reproduce multifractal scaling exponents associated with the energy dissipation field [6], the latter representing the intermittency corrections to the K41-theory [1]. Although the justification of RMCMs in terms of the Navier-Stokes equation is far from being clear, these phenomenological models appear to contain more truth than originally anticipated, as recent investigations on multiplier distributions [7] and scale correlations [8] have revealed. – On the other side RMCMs are purely geometrical in nature and are not capable of describing causal dynamical effects of the turbulent energy cascade like backscattering of energy flux from small to larger scales. A generalisation in this direction is clearly called for.

In this Letter we present such a generalisation and construct a dynamical RMCM in 1+1 space-time dimensions, respecting causality. Multifractal scaling is recovered for \( n \)-point correlation functions.

In a discrete RMCM in 1+0 dimensions, the amplitude

\[
\varepsilon(\eta) = \prod_{j=1}^{J} q(l_j) = \exp \left( \sum_{j=1}^{J} \ln q(l_j) \right)
\]

(1)

of the positive-valued energy-dissipation field, resolved at the dissipation scale \( \eta \), is equal to the product of independently and identically distributed random weights \( q(l_j) \), which come with \( \langle q \rangle = 1 \) and a nested hierarchy of scales \( \eta = l_J \leq l_j = L/\lambda^j \leq l_0 = L \) with \( 0 \leq j \leq J \). The integral length \( L \) and the dissipation length \( \eta \) represent the largest and smallest length scale involved, whereas \( \lambda > 1 \) is the discrete scale step.

It is the second step of (1), which we now exploit to propose a generalisation
of the cascade field, now defined in 1+1 space-time dimensions:

\[
\varepsilon(x,t) = \exp \left\{ \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' f(x-x',t-t')\gamma(x',t') \right\}. \tag{2}
\]

Due to the point-resolution the function \(\gamma(x,t) \sim S_\alpha((dxdt)^{\alpha-1}/(\cos(\pi\alpha/2))+\mu n)\), which fixes the parameter \(\mu = \sigma^n/\cos(\pi\alpha/2)\) in order to fulfill the expectation \(\langle \exp\{\gamma\} \rangle = 1\).

Imposing causality, i.e. the field amplitude \(\varepsilon(x,t)\) is only influenced by the past and not by the future, requires \(f(x,t) = 0\) for \(t < 0\). This is satisfied with the following symmetric index function

\[
f(x,t) = \begin{cases} 
1 & (0 \leq t \leq T, -g(t) \leq x \leq g(t)) \\
0 & \text{(otherwise)} \end{cases} , \tag{3}
\]

where the window function \(g(t)\) introduces a correlation time \(T\) and a correlation length \(L\) with \(g(0) \approx 0\) and \(g(T) = L/2\); see Fig. 1 for an illustration. The exponent of the construction (2) can be thought of as a moving average over the stable white-noise field.

According to (3), the time integration in (2) goes from \(-T \leq t' \leq 0\), where \(x = t = 0\) has been set for simplicity. Upon assigning \(2g(-t_j) = l_j\) the hierarchy of length scales \(l_j\) is translated into a hierarchy of time scales, so that (2) then factorises into contributions

\[
q(l_j) = \exp \left\{ \int_{t_{j-1}}^{t_j} dt' \int_{-g(-t')}^{g(-t')} dx' \gamma(x',t') \right\} ; \tag{4}
\]

see again Fig. 1. In order to interpret this as a random multiplicative weight, the probability distribution density of \(q(l_j)\) needs to be independent of scale. Since the \(\gamma(x',t')\) are i.i.d., the integration domain of (4) then has to be independent of the scale index \(j\). This fixes the window function within \(-T \leq t' \leq -t_\eta:\)

\[
g(-t') = \frac{(L/2)}{1 + \frac{(L-\eta)(T+t')}{\eta(T-t_\eta)}}, \tag{5}
\]

which also satisfies the boundary conditions \(g(T)=L/2\) and \(g(t_\eta)=\eta/2\). So far, no specification of \(g(-t')\) for \(-t_\eta \leq t' \leq 0\) has been given; since \(t_\eta \ll T\) should
hold on physical grounds, the simplest choice would be \( t_\eta = 0 \).

The construction proposed in Eqs. (2)-(5) guarantees that the one-point statistics of the dynamical RMCN is identical to its geometrical counterpart. In order to qualify for a complete dynamical generalisation, it is not only the one-point statistics, but the \( n \)-point statistics in general, which should match. Hence, we now consider the equal-time two-point correlator

\[
C_{n_1, n_2}(l = x_2 - x_1 \geq \eta) = \frac{\langle \varepsilon^{n_1}(x_1, t) \varepsilon^{n_2}(x_2, t) \rangle}{\langle \varepsilon^{n_1}(x_1, t) \rangle \langle \varepsilon^{n_2}(x_2, t) \rangle} = \frac{\langle D(l)^{n_1+n_2} \rangle}{\langle D(l)^{n_1} \rangle \langle D(l)^{n_2} \rangle}. \tag{6}
\]

The correlation between the two points with distance \( l < L \) stems from the overlap region of the two index functions \( f(x_1, t) \) and \( f(x_2 = x_1 + l, t) \); see Fig. 2a. This explains the second step of (6), where

\[
D(l \geq \eta) = \exp \left( -g^{-1}(l/2) g(-t') \int_{-T}^{T} dt' \int_{l-g(-t')}^{l} dx' \gamma(x', t') \right) \tag{7}
\]

represents the contribution from the overlap region. The contributions from the non-overlapping regions are statistically independent, hence factorise and cancel. Introducing the spatio-temporal overlap volume

\[
V(l \geq \eta) = \int_{-T}^{T} dt' \int_{l-g(-t')}^{l} dx' \gamma(x', t') = \eta L(T - t_\eta) \ln \left( \frac{L}{l} \right) - \eta (T-t_\eta)(L-l) \tag{8}
\]

and employing basic properties of stable distributions [9], the expectation of the \( n \)-th power of expression (7) can be transformed into

\[
\langle D(l)^n \rangle = \exp \left( \frac{\sigma^\alpha}{\cos \frac{\pi \alpha}{2}} V(l)(n-n^\alpha) \right). \tag{9}
\]

Defining the multifractal scaling exponents \( \tau(n) = \tau(2)(n-n^\alpha)/(2-2^\alpha) \) with \( \tau(2) = (\sigma^\alpha / \cos \frac{\pi \alpha}{2}) (2-2^\alpha) \eta L(T-t_\eta)/(L-\eta) \) as well as \( \tau[n_1, n_2] = \tau(n_1+n_2) - \tau(n_1) - \tau(n_2) \), insertion of (9) into (6) leads to the final expression for the two-point correlator:

\[
C_{n_1, n_2}(l) = \left( \frac{L}{l} \right)^{\tau[n_1, n_2]} \exp \left( -\tau[n_1, n_2] \left( 1 - \frac{l}{L} \right) \right). \tag{10}
\]
It reveals multiscaling behaviour for $\eta < l \ll L$, which shows that the equal-time two-point statistics of the dynamical RMCM in 1+1 dimensions is completely analogous to the findings of the geometrical RMCM in 1+0 dimensions.

In view of this analogy, the large-scale deviation appearing in the last expression is reminiscent of similar recent findings on scaling functions within the geometrical RMCM[10]. With the introduction of a suitably tuned large-scale fluctuation this large-scale deviation can be cancelled. In our present context this implies a small extension of the window function (5) beyond the time interval $-T \leq t' \leq 0$: $g(T \leq -t' \leq T + \Delta T) = L/2$; consult again Fig. 1. It is then straightforward to show that for $\Delta T = (T - t')\eta/(L - \eta)$, which is much smaller than the correlation time $T$, the two-point correlator exactly becomes $C_{n_1,n_2}(l) = (L/l)^{\tau[n_1,n_2]}$ for all $\eta \leq l \leq L$.

Besides equal-time two-point correlations temporal two-point correlations are also of interest. These correlations arise due to a modified spatio-temporal overlap volume as illustrated in Fig. 2b. A straightforward calculation, analogous to (7)-(9), yields

$$C_{n_1,n_2}(t = t_2 - t_1) = \frac{\langle \varepsilon^{n_1}(x, t_1)\varepsilon^{n_2}(x, t_2) \rangle}{\langle \varepsilon^{n_1}(x, t_1) \rangle \langle \varepsilon^{n_2}(x, t_2) \rangle} = \left( \frac{t}{T} + \frac{\eta}{L} \left( 1 - \frac{t}{T} \right) \right)^{-\tau[n_1,n_2]} \approx \left( \frac{T}{t} \right)^{\tau[n_1,n_2]};$$ (11)

for simplicity, the parameter of the window function (5) has been set $t_\eta = 0$. The last step, only holding for $0 \ll t < T$ and $\eta \ll L$, reveals that the temporal two-point correlator comes with scaling exponents identical to those of the equal-time two-point correlators.

Within this dynamical RMCM investigations on several other observables are nearly as straightforward [11]. Here we mention only spatio-temporal ($n \geq 2$)-point correlations in connection with fusion rules, and moments and multiplier distributions based on coarse-grained field amplitudes. All of these observables are found to be in good agreement with experimental results derived from the surrogate energy dissipation field of fully developed turbulence.

The dynamical RMCM, presented here, is a generalisation of the geometrical RMCM. By construction it is continuous and homogeneous, does not make use of a discrete hierarchy of scales and stochastically evolves a positive-valued field in one space and one time dimension. Several generalisations of this new model immediately come to mind: stochastic evolution in $n+1$ dimensions with the optional inclusion of spatial anisotropy, discretisation of space-time into smallest cells to model dissipation and deviation from log-stability, and
dynamical RMCM for vector fields to model the turbulent velocity field. Work in these directions is in progress.

The authors acknowledge fruitful discussions with Jahanshah Davoudi. This work has been supported in parts by DAAD and by BCPL in the framework of the European Community-Access to Research Infrastructure action of the Improving Human Potential Program.

References

[1] U. Frisch, *Turbulence*, (Cambridge University Press, Cambridge, 1995).

[2] D. Schertzer and S. Lovejoy, J. Geophys. Res. 92 (1987) 9693.

[3] R. Riedi, M. Crouse, V. Ribeiro and R. Baraniuk, IEEE Trans. Inf. Theory 45 (1999) 992.

[4] J. Muzy, J. Delour and E. Bacry, Eur. Phys. J. B 17 (2000) 537.

[5] J. Feder, *Fractals*, (Plenum Press, New York, 1988).

[6] C. Meneveau and K. Sreenivasan, J. Fluid Mech. 224 (1991) 429.

[7] B. Jouault, P. Lipa and M. Greiner, Phys. Rev. E 59 (1999) 2451; B. Jouault, M. Greiner and P. Lipa, Physica D 136 (2000) 125.

[8] J. Cleve and M. Greiner, Phys. Lett. A 273 (2000) 104.

[9] G. Samorodnitsky and M. Taqqu, *Stable non-Gaussian random processes*, (Chapman & Hall, New York, 1994).

[10] J. Schmiegel, T. Dziekan, J. Cleve, B. Jouault and M. Greiner, preprint *Scaling functions in a random multiplicative energy-cascade model of turbulence*, submitted to Phys. Rev. E.

[11] J. Schmiegel and M. Greiner, in preparation.
Figure 1: Illustration of the causal index function (3) used to construct the positive-valued multifractal field $\varepsilon(x, t)$. 

\begin{align} 
\varepsilon(x, t) & \quad \text{positive-valued multifractal field} \\
T & \quad \text{time axis} \\
L & \quad \text{space axis} \\
\Delta T & \quad \text{time interval} \\
x & \quad \text{space coordinate} \\
x - g(t - t') & \quad \text{left boundary} \\
x + g(t - t') & \quad \text{right boundary} \\
t_\eta & \quad \text{time delay} \\
l_j & \quad \text{spatial scales} \\
l_j + 1 & \quad \text{next spatial scale}
Figure 2: Spatio-temporal overlap volumes (shaded) producing the correlation for the (a) equal-time and (b) temporal two-point correlator.