Finite sections of the Fibonacci Hamiltonian

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Abstract. We study finite but growing principal square submatrices $A_n$ of the one- or two-sided infinite Fibonacci Hamiltonian $A$. Our results show that such a sequence $(A_n)$, no matter how the points of truncation are chosen, is always stable – implying that $A_n$ is invertible for sufficiently large $n$ and $A_n^{-1} \to A^{-1}$ pointwise.

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1 Introduction

The 1D Schrödinger operator $-\Delta + b$ with a bounded potential $b \in L^\infty(\mathbb{R})$ can be discretized, via finite differences on a uniform grid on $\mathbb{R}$, by the second order difference operator

$$ (Ax)_n = x_{n-1} + v_n x_n + x_{n+1}, \quad n \in \mathbb{Z}, $$

acting on a sequence space like $L^p(\mathbb{Z})$. The discrete potential $v = (v_n) \in L^\infty(\mathbb{Z})$ corresponds to evaluations of the potential $b$ on the grid (subtracted by a two that comes from the discretization of the Laplace operator). $A$ is commonly referred to as a discrete 1D Schrödinger operator.

A particularly beautiful example, the so-called Fibonacci Hamiltonian, arises when the discrete potential $v$ is given by the formula

$$ v_n = \chi_{[1-\alpha,1)}(n\alpha \mod 1), \quad n \in \mathbb{Z}, $$

where $\alpha = \frac{\sqrt{5} - 1}{2}$ is the golden ratio and $\chi_I$ is the characteristic function of an interval $I$.

The sequence $v$ from (2) is not periodic (as $\alpha$ is irrational); it displays a so-called quasiperiodic pattern. Here are its values $v_1, \ldots, v_{55}$ and three attempts to identify basic building blocks of the sequence (one with normal/bold face, one with separation by minus signs and one with under/overlines):

$$ \text{10110-10110-110-10110-110-10110-110} \rightarrow \text{10110-110-10110-110-10110-110} \rightarrow \text{10110-110-10110-110-10110-110}. $$

The global pattern of these building blocks (on each scale) is the same as the pattern formed by 1 and 0 on the finest scale. The Fibonacci potential shows self-similarity on many levels.

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The Fibonacci Hamiltonian is the standard model in 1D for physical properties of so-called quasicrystals and is therefore heavily studied in mathematical physics. Most of the research deals with the spectrum of $A$, which is a Cantor set of measure zero without any eigenvalues (purely singular continuous spectrum).

Our focus is different. The operator (1) acts via multiplication with a two-sided infinite tridiagonal matrix $(a_{ij})_{i,j \in \mathbb{Z}}$. The main diagonal carries the sequence $v$, the first sub- and super-diagonal are constant to one, and the rest is zero. We study the applicability of the so-called finite section method to that infinite matrix.

The finite section method (FSM) looks at finite submatrices

$$A_n = (a_{ij})_{i,j=l_n}^{r_n}, \quad n \in \mathbb{N}$$

of an infinite matrix $A = (a_{ij})_{i,j \in \mathbb{Z}}$ with integer cut-off points

$$l_n \rightarrow -\infty \quad \text{and} \quad r_n \rightarrow +\infty$$

and asks whether

a) the matrices $A_n$ are invertible for all sufficiently large $n$ and

b) their inverses (after embedding them into an infinite matrix again) converge pointwise in $\ell^p(\mathbb{Z})$ to the inverse of $A$.

Assuming invertibility of $A$ on $\ell^p(\mathbb{Z})$, property b) is equivalent to the uniform boundedness of the inverses $A_n^{-1}$.

As a consequence, one can solve an infinite system $Ax = b$ approximately by solving large but finite systems $A_n x_n = b_n$. For one-sided infinite matrices $A = (a_{ij})_{i,j \in \mathbb{N}}$, all of the above remains true but $l_n$ should be fixed at 1.

**Our result.** For the Fibonacci Hamiltonian $A$ from (1) with potential (2) as well as for its one-sided infinite submatrix $A_N := (a_{ij})_{i,j \in \mathbb{N}}$, we first show that both operators are invertible on every space $\ell^p(\mathbb{Z})$, resp. $\ell^p(\mathbb{N})$, with $p \in [1, \infty]$ before proving that the FSM with arbitrary cut-off points is applicable for $A$ as well as for $A_N$.

**Historic remarks.** Quasicrystals are materials that show features of periodicity (so-called Bragg peaks in diffraction experiments) and aperiodicity (symmetries that rule out periodicity) at the same time – so-called quasiperiodicity. They have first been observed by D. Shechtman in 1982 in his laboratory [29] but are meanwhile also found to be occurring in nature. Physicists and mathematicians quickly developed an interest in this topic. In particular the spectrum of the corresponding Hamiltonian is of huge interest for the understanding of electrical properties of the material [1, 13, 30]. The most famous quasiperiodic ensemble in 2D is the Penrose tiling [4]. The understanding of the corresponding spectrum seems however currently out of reach, so that one resorts to 1D ensembles, the most common of which is the Fibonacci sequence (2). A detailed history and the current state of the art on the extensively studied spectral analysis of the Fibonacci Hamiltonian can be found in [9].

The idea of the FSM is so natural that it is difficult to give a historical starting point. First rigorous treatments are from Baxter [2] and Gohberg & Feldman [12] on Wiener-Hopf and convolution operators in dimension $N = 1$ in the early 1960’s. For convolution equations in higher dimensions $N \geq 2$, the FSM goes back to Kozak & Simonenko [14, 15], and for general band-dominated operators with scalar [23] and operator-valued [24, 25] coefficients, most results are due to Rabinovich, Roch & Silbermann. For the state of the art see e.g. [3, 27, 20, 28].
2 The finite section method

As usual, for an index set $\mathbb{I} \subset \mathbb{Z}$, let $\ell^p(\mathbb{I})$ denote the set of all complex sequences $(x_k)_{k \in \mathbb{I}}$ with $\sum_{k \in \mathbb{I}} |x_k|^p < \infty$ for $p \in [1, \infty)$, and $\ell^\infty(\mathbb{I})$ be the set of all bounded complex sequences over $\mathbb{I}$.

Let $A = (a_{ij})_{i,j \in \mathbb{Z}}$ be a band matrix (i.e., a matrix with only finitely many nonzero diagonals) with uniformly bounded complex entries. Then $A$ acts, via matrix-vector multiplication, as a bounded linear operator on all spaces $\ell^p(\mathbb{Z})$ with $p \in [1, \infty]$. Denote that operator again by $A$.

For integer cut-off points $l_1, l_2, \ldots$ and $r_1, r_2, \ldots$ with

$$l_n \to -\infty \quad \text{and} \quad r_n \to +\infty,$$

look at the finite submatrices

$$A_n = (a_{ij})_{i,j = l_n}^{r_n}, \quad n \in \mathbb{N}$$

of $A$ and call the sequence $(A_n)_{n \in \mathbb{N}}$ stable if there exists an $n_0 \in \mathbb{N}$ such that $A_n$ is invertible for all $n \geq n_0$ and $\sup_{n \geq n_0} \|A_n^{-1}\| < \infty$.

Invertibility of $A$ and stability of $(A_n)$ together are sufficient and necessary for the applicability of $(A_n)$, that is the pointwise convergence (i.e., column-wise convergence of the matrices) $A_n^{-1} \to A^{-1}$, when $A_n^{-1}$ is extended to an infinite matrix again. This approximation of $A^{-1}$ can be used for solving equations $Ax = b$ approximately via the solutions of growing finite systems.

We see that it is crucial to know about the stability of $(A_n)$. This stability is closely connected to a family of one-sided infinite matrices that are associated to $A$ and to the cut-off sequences $(l_n)$ and $(r_n)$. Those associated one-sided infinite matrices are partial limits of the upper left and the lower right corner of the finite matrix $A_n$ as $n \to \infty$. Precisely, the associated matrices are the entrywise limits

$$(a_{i+r_n',j+r_n'}, i,j = 0)_{i,j=0}^\infty \to B_+ \quad \text{and} \quad (a_{i+r_n',j+r_n'}, i,j = -\infty)_{i,j=-\infty}^\infty \to C_- \quad \text{as} \quad n \to \infty$$

of one-sided infinite submatrices of $A$, where $(l_n')_{n=1}^\infty$ and $(r_n')_{n=1}^\infty$ are subsequences of $(l_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$, respectively, such that the limits (4) exist. The boundedness of the diagonals of $A$ ensures (by Bolzano-Weierstrass and a Cantor diagonal argument) the existence of such subsequences and the corresponding limits (4). Here is the result.

Lemma 2.1. [Lemma 1.2 of [7]] For a band matrix $A = (a_{ij})_{i,j \in \mathbb{Z}}$ and two cut-off sequences $(l_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ in $\mathbb{Z}$ with $l_n \to -\infty$ and $r_n \to +\infty$, the following are equivalent:

(i) The FSM $(A_n)_{n=1}^\infty$ with $A_n$ from (3) is applicable to $A$,

(ii) The FSM $(A_n)_{n=1}^\infty$ with $A_n$ from (3) is stable,

(iii) $A$ and the limits $B_+$ and $C_-$ from (4) are invertible for all subsequences $(l_n')$ of $(l_n)$ and $(r_n')$ of $(r_n)$.

So by the choice of the cut-off sequences $(l_n)$ and $(r_n)$, one can control the selection of associated matrices $B_+$ and $C_-$ and hence control the stability and applicability of the FSM.

The construction of the $B_+$ and $C_-$ brings us to the notion of a limit operator [23, 26, 19].

Definition 2.2. Let $\mathbb{I}$ be either $\mathbb{Z}$ or $\mathbb{N}$. For a bounded one- or two-sided infinite band matrix $A = (a_{ij})_{i,j \in \mathbb{I}}$ and a sequence $h_1, h_2, \ldots$ in $\mathbb{I}$ with $|h_n| \to \infty$ we say that $B = (b_{ij})_{i,j \in \mathbb{I}}$ is a limit operator of $A$ if, for all $i,j \in \mathbb{I}$,

$$a_{i+h_n,j+h_n} \to b_{ij} \quad \text{as} \quad n \to \infty.$$

We write $A_h$ instead of $B$. 

3
Note that limit operators are always given by a two-sided infinite matrix, no matter if the matrix $A$ to start with is one- or two-sided infinite.

So in this language, our associated matrices $B_+$ and $C_-$ from (4) are one-sided truncations of limit operators of $A$: each $B_+ = (b_{ij})_{i,j=0}^\infty$ is a submatrix of a limit operator $B = (b_{ij})_{i,j\in\mathbb{Z}}$ of $A$ w.r.t. a subsequence $h$ of $(l_n)$, and each $C_- = (c_{ij})_{i,j=-\infty}^0$ is a submatrix of a limit operator $C = (c_{ij})_{i,j\in\mathbb{Z}}$ of $A$ w.r.t. a subsequence $h$ of $(r_n)$. To be able to rephrase Lemma 2.1 in that language, we introduce the following notations.

**Definition 2.3.** a) For a bounded one- or two-sided infinite band matrix $A = (a_{ij})_{i,j\in\mathbb{I}}$ with $\mathbb{I} \in \{\mathbb{Z}, \mathbb{N}\}$ and a sequence $g_1, g_2, \ldots$ in $\mathbb{I}$ with $|g_n| \to \infty$ we write $\text{Lim}_g(A)$ for the set of all limit operators $A_h$ with respect to a subsequence $h$ of $g$, and we write $\text{Lim}(A)$ for the set of all limit operators of $A$. Moreover, put $\text{Lim}_+(A) := \text{Lim}_{(1,2,3,\ldots)}(A)$ and $\text{Lim}_-(A) := \text{Lim}_{(-1,-2,-3,\ldots)}(A)$.

b) For a two-sided infinite matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$, write $A_{\pm} := (a_{ij})_{i,j\in\mathbb{Z}_\pm}$, respectively, where $\mathbb{Z}_- := \{\ldots,-2,-1,0\}$ and $\mathbb{Z}_+ := \{0,1,2,\ldots\}$.

Note that $A_+$ and $A_-$ overlap in $a_{00}$. Here is the announced reformulation of Lemma 2.1.

**Corollary 2.4.** For a bounded band matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$ and two cut-off sequences $l = (l_n)_{n=1}^\infty$ and $r = (r_n)_{n=1}^\infty$ in $\mathbb{Z}$ with $l_n \to -\infty$ and $r_n \to +\infty$, the following are equivalent:

(i) the FSM $(A_n)_{n=1}^\infty$ with $A_n$ from (3) is applicable to $A$,

(ii) the FSM $(A_n)_{n=1}^\infty$ with $A_n$ from (3) is stable,

(iii) $A$ and all operators $B_+$ and $C_-$ with $B \in \text{Lim}_+(A)$ and $C \in \text{Lim}_-(A)$ are invertible.

If stability holds for $l = (-1,-2,-3,\ldots)$ and $r = (1,2,3,\ldots)$ then it holds for arbitrary cut-off sequences $(l_n)$ and $(r_n)$.

**Corollary 2.5.** For a bounded band matrix $A = (a_{ij})_{i,j\in\mathbb{Z}}$, the following are equivalent:

(i) the FSM $(A_n)_{n=1}^\infty$ with $A_n$ from (3) is applicable for arbitrary cut-offs $(l_n)$ and $(r_n)$,

(ii) the FSM $(A_n)_{n=1}^\infty$ with $A_n$ from (3) is stable for arbitrary cut-offs $(l_n)$ and $(r_n)$,

(iii) $A$ and all operators $B_+$ and $C_-$ with $B \in \text{Lim}_-(A)$ and $C \in \text{Lim}_+(A)$ are invertible.

The one-sided infinite case, $A = (a_{ij})_{i,j\in\mathbb{N}}$, only requires minimal changes to what was written above: The sequence $(l_n)$ is then constant at 1 and therefore the operators $B_+$ do not appear in (iii) of Lemma 2.1 and Corollary 2.4.

Limit operators are not only good for detecting stability\(^1\) of the FSM. Their primary purpose is to characterize the coset $A + \mathcal{K}(X)$ of $A$ modulo the ideal of all compact operators $\mathcal{K}(X)$, where we abbreviate $\mathcal{L}(\mathbb{I}) =: X$.

Recall that a bounded linear operator $A$ on $X$, we write $A \in \mathcal{L}(X)$, is a Fredholm operator if its coset $A + \mathcal{K}(X)$ is invertible in the so-called Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, which holds iff the nullspace of $A$ has finite dimension and the range of $A$ has finite codimension in $X$.

**Lemma 2.6.** For a bounded band matrix $A = (a_{ij})_{i,j\in\mathbb{I}}$ with $\mathbb{I} \in \{\mathbb{Z}, \mathbb{N}\}$ and $X = \mathcal{L}(\mathbb{I})$ with any $p \in [1, \infty]$, it holds that the following are equivalent:

(i) $A$ is a Fredholm operator on $X$,

(ii) all limit operators of $A$ are invertible on $\mathcal{L}(\mathbb{Z})$ [23, 21],

(iii) all limit operators of $A$ are injective on $\mathcal{L}(\mathbb{Z})$ [5, 6].

\(^1\)They only come into play here because the stability of the sequence $(A_n)$ is equivalent to the operator $D := \text{Diag}(A_1, A_2, \ldots)$ being a Fredholm operator. Then Lemma 2.6 below is applied to $D$. 

3 The Fibonacci word

Recall the infinite sequence \( v = (v_n)_{n \in \mathbb{Z}} \) of zeros and ones from (2). In this section we interpret \( v \) as an infinite word over the alphabet \( \Sigma = \{0, 1\} \). Let us recall some basic notions on words. For a detailed discussion, including on the Fibonacci word, see e.g. [22].

3.1 Some words on words

An alphabet is a nonempty set \( \Sigma \). A finite vector \( w = (w_1, \ldots, w_n) \in \Sigma^n \) is called a word of length \( n \) over \( \Sigma \). We write \( |w| = n \). Sequences \( (w_1, w_2, \ldots) \) and \( (\ldots, w_{-2}, w_{-1}) \) are one-sided infinite words over \( \Sigma \) and \( (\ldots, w_{-2}, w_{-1}, w_0, w_1, w_2, \ldots) \) is a two-sided infinite word over \( \Sigma \) when \( w_i \in \Sigma \) for all \( i \). The word of length zero is denoted by \( \varepsilon \) and is called the empty word.

Let \( \Sigma^* := \bigcup_{n=0}^{\infty} \Sigma^n \) denote the set of all finite words over \( \Sigma \). Moreover, for an infinite index set \( I \in \{\mathbb{Z}, \mathbb{N}, -\mathbb{N}, \mathbb{Z}^+, \mathbb{Z}^-\} \), let \( \Sigma^I \) denote the set of all infinite words \( (w_n)_{n \in I} \) over \( \Sigma \).

The word \( (w_1, w_2, \ldots, w_n) \) is often simply written as \( w_1 w_2 \ldots w_n \). Similarly for infinite words. For two words \( u = u_1 \ldots u_m \) and \( v = v_1 \ldots v_n \), the word \( u_1 \ldots u_m v_1 \ldots v_n \) is denoted by \( u v \) or just \( uv \). This operation, called concatenation, is associative on \( \Sigma^* \), with \( \varepsilon \) as the neutral element of \( \Sigma^* \). Concatenation is also defined between two oppositely directed one-sided infinite words (at their finite endpoints) and between finite and one-sided infinite words in the natural way.

A word \( w \) is called a subword (or factor) of a word \( u \) if \( u \) can be written as \( xwy \) with (possibly empty) words \( x \) and \( y \). We write \( w \preceq u \) if \( w \) is a subword of \( u \). \( \varepsilon \preceq u \) holds for all words \( u \).

The reversed word of \( u = u_1 \ldots u_m \) and \( w = w_1 w_2 \ldots \) is \( u^R := u_m \ldots u_1 \) and \( w^R := \ldots w_2 w_1 \), respectively.

3.2 Finite Fibonacci words: substitution, recursion and limit

Let \( \Sigma = \{0, 1\} \) and \( \varphi : \Sigma^* \rightarrow \Sigma^* \) be the homomorphism (w.r.t. concatenation \( \circ \)) with \( \varphi : 0 \mapsto 1 \) and \( \varphi : 1 \mapsto 10 \). Then put \( f_1 := 1, f_2 := \varphi(f_1), f_3 := \varphi(f_2), \) etc. In particular, we get

\[
\begin{align*}
  f_1 &= \varphi(0) = 0, \\
f_2 &= \varphi(1) = 1, \\
f_3 &= \varphi(10) = \varphi(1)\varphi(0) = 10, \\
f_4 &= \varphi(101) = \varphi(1)\varphi(0)\varphi(1) = 110, \\
f_5 &= \varphi(1011) = \varphi(1)\varphi(0)\varphi(1)\varphi(1)\varphi(0) = 11101, \\
f_6 &= \varphi(101101) = \varphi(1)\varphi(0)\varphi(1)\varphi(0)\varphi(1)\varphi(0)\varphi(1)\varphi(0)\varphi(1) = 11010110110, \\
&\vdots
\end{align*}
\]

This leads to the list of finite Fibonacci words \( f_1, f_2, \ldots \). It is easy to see (by induction) that

\[
f_{n+1} = f_n f_{n-1}
\]

holds for \( n \geq 2 \), so that the length of \( f_n \) is the \( n \)-th Fibonacci number; let us denote it by \( F_n \).

The pointwise limit of this sequence \( (f_n) \) is the one-sided infinite Fibonacci word \( v_+ = (v_n)_{n \in \mathbb{N}} \) with each \( v_n \) from (2). More precisely, equip \( \Sigma \) with the discrete topology, \( \Sigma^\mathbb{N} \) with the product topology and extend each \( f_n \) (by anything) to the right to a word in \( \Sigma^\mathbb{N} \); then \( (f_n) \) converges, by (6), and the limit is \( v_+ = 1011010110110101101101101011011010110 \cdots \).
3.3 The rotation formula and symmetry

The above mechanisms define the positive half $v_+ = v_1 v_2 \cdots$ of the two-sided infinite Fibonacci word $v = (v_n)_{n \in \mathbb{Z}}$. The missing entries $\cdots, v_{-2}, v_{-1}, v_0$ can, of course, be computed from the “rotation formula” (2) but they can also be expressed in terms of $v_+$:

For all $n \in \mathbb{Z}$, put

$$t_n := n\alpha \mod 1 \in [0, 1),$$

where $\alpha = \frac{\sqrt{5} - 1}{2}$ is the golden ratio, so that $v_n = \chi_{[1-\alpha, 1)}(t_n)$ by (2). For arithmetics modulo 1 it is of course useful to think of the interval $[0, 1)$ as a circle with $0 \sim 1$.

Because $t_{-1} = 1 - \alpha$ and $t_0 = 0 \equiv 1$ exactly mark the two endpoints of the interval $[1-\alpha, 1)$ and the sequences $(t_n)_{n \leq -1}$ and $(t_n)_{n \geq 0}$ evolve from there, equispaced in opposite directions along our circle, one observes the symmetry $v_{-2} = v_1, v_{-3} = v_2, \ldots$ in short:

$$v = v_+^{R_{10}} v_+,$$

(7)

where the 10 in the middle refers to $v_{-1} v_0$.

Note that, by the irrationality of $\alpha$, all $t_n$ are pairwise distinct. So the asymmetry that is caused by the different brackets of the interval $[1-\alpha, 1)$ only shows for $n = -1$ and $n = 0$, where $t_n$ exactly hits the two interval endpoints. For $n \in \mathbb{Z} \setminus \{-1, 0\}$, one has $v_n = v_{-1-n}$.

3.4 Subwords of length $n$

Another intriguing feature of the Fibonacci word is its small number of subwords.

Let $\Sigma = \{0, 1\}$. A random word $u \in \Sigma^\mathbb{Z}$ would, almost surely, contain every one of the $2^n$ words $w \in \Sigma^n$ as a subword, for every $n \in \mathbb{N}$. For the Fibonacci word $v \in \Sigma^\mathbb{Z}$, the situation is very different:

| length | subwords of $v$ of that length | count |
|--------|--------------------------------|-------|
| 1      | 0, 1                          | 2     |
| 2      | 01, 10, 11                    | 3     |
| 3      | 010, 011, 101, 110            | 4     |
| 4      | 0101, 0110, 1010, 1011, 1101  | 5     |
| $\vdots$ | $\vdots$                    | $\vdots$ |
| $n$   | $\ldots$                     | $n+1$ |

(8)

The number, say $\text{sub}_v(n)$, of subwords of $v$ of any length $n \in \mathbb{N}$ is exactly $n + 1$.

For general words $u \in \Sigma^\mathbb{Z}$, it is easy to see that $\text{sub}_u$ grows monotonically, and if $\text{sub}_u(n) = \text{sub}_u(n + 1)$ for some $n$ then $\text{sub}_u(m)$ will remain at that value, say $p$, for all $m \geq n$. The latter says that $v$ is $p$-periodic (up to a finite perturbation).

So for an aperiodic word $u$, the function $\text{sub}_u$ grows strictly monotonically (by at least 1 for each $n$), starting from $\text{sub}_u(1) = |\Sigma| = 2$. So the subword count function with minimal growth (among the unbounded functions) is given by $\text{sub}_u(n) = n + 1$. This is exactly what is observed for the Fibonacci word $u = v$. 

6
4 Finite sections of the Fibonacci Hamiltonian

Let \( v = (v_n)_{n \in \mathbb{Z}} \) be the Fibonacci sequence (2) and let
\[
A := S_{-1} + M_v + S_1 : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})
\]
be the Fibonacci Hamiltonian (1), where
\[
S_k : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}), \quad (S_k x)_n = x_{n+k}, \quad n \in \mathbb{Z}
\]
denotes the shift by \( k \in \mathbb{Z} \) components and
\[
M_b : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}), \quad (M_b x)_n = b_n x_n, \quad n \in \mathbb{N}
\]
denotes the operator of pointwise multiplication by \( b = (b_n)_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}) \).

We identify \( A \) with its two-sided infinite matrix \((a_{ij})_{i,j \in \mathbb{Z}}\) with \( a_{nn} = v_n \) and \( a_{n,n\pm 1} = 1 \) for all \( n \in \mathbb{Z} \) and zeros everywhere else. Corollary 2.5 connects the FSM of \( A \) with the limit operators of \( A \). So we need to get a hand on these limit operators.

4.1 Limit operators of the Fibonacci Hamiltonian

Let \( h = (h_1, h_2, \ldots) \) be a sequence in \( \mathbb{Z} \) with \( h_k \to \pm \infty \), so that the limit operator \( A_h \) of the Fibonacci Hamiltonian \( A \) from (9) exists. Then
\[
A_h = (S_{-1})_h + (M_v)_h + (S_1)_h =: S_{-1} + M_{v_h} + S_1
\]
with a new potential
\[
v_h := \lim_{k \to \infty} S_{-h_k} v,
\]
where the limit is taken w.r.t. pointwise convergence on \( \Sigma^\mathbb{Z} \) for \( \Sigma = \{0, 1\} \).

The set \( \mathcal{F} \) of all such potentials \( v_h \) is translation invariant (translations of limit operators of \( A \) are limit operators of translations of \( A \)) and closed under pointwise convergence. \( \mathcal{F} \) is the so-called Fibonacci subshift. By our definition of \( \mathcal{F} \),
\[
\text{Lim}(A) = \{ S_{-1} + M_{v_h} + S_1 : v_h \in \mathcal{F} \}.
\]
The set \( \mathcal{F} \) is explicitly known (see e.g. Theorem 2.14 in [8] and the appendix of [11]):
\[
\mathcal{F} = \{ v^\theta, w^\theta : \theta \in [0, 1) \},
\]
(10)
where
\[
v_n^\theta := \chi_{[1-\alpha, 1)}(\theta + n\alpha \mod 1), \quad w_n^\theta := \chi_{(1-\alpha, 1]}(\theta + n\alpha \mod 1), \quad n \in \mathbb{Z}.
\]
In particular,
\[
A \in \text{Lim}(A),
\]
(11)
since \( v = v^0 \in \mathcal{F} \). In fact, we do not need this explicit description (10) of \( \mathcal{F} \). The following lemma is sufficient (and much more handy) for us. It expresses the well-known minimality of the Fibonacci subshift.
Lemma 4.1. Every \( v_h \in \mathcal{F} \) has the same list (8) of subwords as \( v \). So for every \( w \in \Sigma^* \) and every \( v_h \in \mathcal{F} \) it holds that

\[ w \prec v \iff w \prec v_h. \]

Proof. Take arbitrary \( w \in \Sigma^* \) and \( v_h \in \mathcal{F} \). So there is a sequence \( h = (h_1, h_2, \ldots) \) in \( \mathbb{Z} \) with \( h_k \to \pm \infty \) and \( v_h = \lim_{k \to \infty} S_{-h_k} v \), pointwise.

\[ \iff \text{If } w \prec v \text{ then } w \prec S_{-h_k} v \text{ for large } k \text{ (strict topology on } \Sigma) \text{, so that } w \prec v. \]

\[ \implies \text{Now let } w \prec v. \text{ W.l.o.g. assume } w \prec v_+. \text{ Choose } n \in \mathbb{N} \text{ so that } w \text{ appears in the first } F_n \text{ letters of } v_+, \text{ i.e. } w \prec f_n \prec f_{n+1} \text{ (recall the notations from } \S 3.2). \]

By (6), we have \( f_{n+2} = f_{n+1} f_n \) and \( f_{n+3} = f_{n+2} f_{n+1} = f_{n+1} f_n f_{n+1} \). By induction, every \( f_m \) with \( m \geq n \), and hence \( v_+ \), is composed of \( f_n \) and \( f_{n+1} \). Since \( w \) appears as a subword in \( f_n \) and \( f_{n+1} \), it appears infinitely often in \( v_+ \), where two appearances of \( w \) are at most \( |f_{n+1}| = F_{n+1} \) letters away from each other. So every translate \( S_{-h_k} v \) of \( v \) contains \( w \) in an \( F_{n+1} \)-neighbourhood of zero. Hence, every limit \( v_h \in \mathcal{F} \) contains \( w \) (in an \( F_{n+1} \)-neighbourhood of zero).

4.2 Main results

Now we are ready to state and prove our two main results.

Theorem 4.2. The FSM of the two-sided infinite Fibonacci Hamiltonian (9) is stable for any choice of cut-off points and in every space \( \ell^p(\mathbb{Z}) \) with \( p \in [1, \infty] \).

The compression \( A_N \) of \( A \) from (9) to \( \ell^p(\mathbb{N}) \) is called one-sided infinite Fibonacci Hamiltonian. Its matrix \( (a_{ij})_{i,j \in \mathbb{N}} \) is the submatrix of \( A \) consisting of all rows and columns with \( i, j \in \mathbb{N} \).

Theorem 4.3. The FSM of the one-sided infinite Fibonacci Hamiltonian \( A_N \) is stable for any choice of cut-off points and in every space \( \ell^p(\mathbb{N}) \) with \( p \in [1, \infty] \).

The rest of this paper is devoted to the proof of these two theorems. The main ingredient, besides Corollary 2.5 and Lemma 2.6, is the following lemma.

Lemma 4.4. For the Fibonacci Hamiltonian \( A \) from (9), the following statements hold:

a) All \( B \in \text{Lim}(A) \) are injective on \( \ell^\infty(\mathbb{Z}) \).

b) For all \( B \in \text{Lim}_+(A) \), the compression \( B_- \) is injective on \( \ell^\infty(\mathbb{Z}_-) \).

c) For all \( B \in \text{Lim}_-(A) \), the compression \( B_+ \) is injective on \( \ell^\infty(\mathbb{Z}_+) \).

Here we use the notations \( B_\pm \) and \( \mathbb{Z}_\pm \) from Definition 2.3 b). We now show how this lemma implies Theorems 4.2 and 4.3 before we come to its proof (in Section 4.3).

Proof of Theorem 4.2. Let \( p \in [1, \infty] \). By Corollary 2.5, we have to show that

1) \( A \) is invertible on \( \ell^p(\mathbb{Z}) \),

2) for all \( B \in \text{Lim}_+(A) \), the compression \( B_- \) is invertible on \( \ell^p(\mathbb{Z}_-) \), and

3) for all \( B \in \text{Lim}_-(A) \), the compression \( B_+ \) is invertible on \( \ell^p(\mathbb{Z}_+) \).

It is sufficient to study the case \( p = 2 \) as \( A \) and all \( B_+ \) and \( B_- \) are band matrices, and so their invertibility is independent of \( p \in [1, \infty] \) (see e.g. [16, §5.2.7]).

Property a) of Lemma 4.4 implies the invertibility of all \( B \in \text{Lim}(A) \), by Lemma 2.6. Since \( A \in \text{Lim}(A) \), by (11), also \( B = A \) is invertible. So 1) is shown.
To show (2), take an arbitrary \( B \in \text{Lim}_+(A) \) and look at \( B_- \) as an operator on \( \ell^2(\mathbb{Z}_-) \). Since \( B_- \) is injective on \( \ell^\infty(\mathbb{Z}_-) \), by Lemma 4.4 b), it is also injective on the subset \( \ell^2(\mathbb{Z}_-) \) of \( \ell^\infty(\mathbb{Z}_-) \). Its adjoint is also injective on \( \ell^2(\mathbb{Z}_-) \) since \( B_- \) is self-adjoint (by \( A = A^* \)). So it remains to show that the range of \( B_- \) is closed: From (1) it follows that \( A \) is Fredholm. By Lemma 2.6, \( B \) is invertible, hence Fredholm. By Lemma 2.6 again, all operators in \( \text{Lim}(B) \supset \text{Lim}(B_-) \) are invertible, whence also \( B_- \) is Fredholm (by Lemma 2.6 again) and hence has a closed range.

3) follows from Lemma 4.4 c) and Lemma 2.6 in the very same way. ■

**Proof of Theorem 4.3.** This time we have to show that

4) \( A_N \) is invertible on \( \ell^p(\mathbb{Z}) \).

5) for all \( B \in \text{Lim}_+(A_N) \), the compression \( B_- \) is invertible on \( \ell^p(\mathbb{Z}_-) \).

Statement 4) follows from 3) because \( A_N = B_+ \) for \( B = S_1 AS_1 \in \text{Lim}_-(A) \).

Statement 5) follows from 2) because \( \text{Lim}_+(A_N) = \text{Lim}_+(A) \), by the construction of \( A_N \). ■

Let us point out that the presence of (iii) in Lemma 2.6 is vital here. With only (i) and (ii) at hand, we would be stuck in a vicious circle. The study of the invertibility of \( A \) can be reduced to the following, presumably easier, problems: injectivity of \( A \), injectivity of \( A^* \), Fredholmness of \( A \). The latter again splits into many, presumably easier, problems: invertibility of all limit operators \( B \) of \( A \), by Lemma 2.6 (ii). But now \( A \) is one of those operators \( B \), by (11), which brings us back to the original problem! So it is good to have – and use – Lemma 2.6 (iii) instead of (ii) here.

Now all that remains to be done is the proof of Lemma 4.4.

### 4.3 Proof of Lemma 4.4

First notice that one can restrict consideration to real sequences in both the one- and two-sided infinite case. Since \( B \) (and the compressions \( B_+ \) and \( B_- \)) correspond to real matrices, it holds

\[
Bx = 0 \iff 0 = \text{Re}(Bx) = B(\text{Re}(x)) \quad \text{and} \quad 0 = \text{Im}(Bx) = B(\text{Im}(x))
\]

with \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) denoting the real and imaginary part of a sequence. So the injectivity of \( B \) on the space of real bounded sequences implies the injectivity on the space \( \ell^\infty(\mathbb{Z}) \) of complex bounded sequences. One is left with proving \( Bx = 0 \Rightarrow x = 0 \) for all bounded real sequences.

The idea is most transparent in the one-sided infinite case. So let us start with the proof of c).

To show that an operator \( B_+ \) is injective on \( \ell^\infty(\mathbb{Z}_+) \), derive the entries \( x_1, x_2, \ldots \) of a solution \( x = (x_n)_{n \in \mathbb{Z}_+} \) of the homogeneous equation \( B_+ x = 0 \), starting from a nonzero initial entry \( x_0 \), and prove that some entry \( x_n \) will eventually exceed (in modulus) any previously given bound \( r > 0 \). Because, for every \( r > 0 \), this computation will only take finitely many steps \( x_1, \ldots, x_n \), it is enough to know about finite subwords of the potential of \( B_+ \). (Our proof does not use the explicit formula (10).)

Identify \( B_+ \) with its matrix \((b_{ij})_{i,j \in \mathbb{Z}_+}\). Because of the tridiagonal structure, the value of \( x_0 \) is sufficient to calculate the whole solution vector \( x \). More precisely, \( x_1 = -b_{00} x_0 \) and \( x_{n+1} = -b_{nn} x_n - b_{n-1} \) for \( n \in \mathbb{N} \). As usual, rewrite this recurrence with transfer matrices:

\[
T_{bn} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}, \quad \text{where} \quad T_{bn} = \begin{pmatrix} 0 & 1 \\ -1 & -b_{nn} \end{pmatrix} \quad \text{with} \quad b_{nn} \in \Sigma = \{0,1\}.
\]
W.l.o.g. we can assume \( x_0 = 1 \). Here is an example computation for a certain diagonal \((b_{nn})\):

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | \ldots |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( b_{nn} \) | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | \ldots |
| \( x_n \) | 1 | \(-1\) | \(-1\) | 2 | \(-1\) | \(-2\) | 3 | 2 | \(-5\) | 3 | 5 | \(-8\) | 3 | 8 | \(-11\) | \(-8\) | 19 | \ldots |

In this example, with a bit of optimism, we seem to observe that
- the diagonal \((b_{nn})\) of \( B \) is composed of blocks “101” and “01”, and
- the entries \( x_n \) at the beginning of each block grow unboundedly in modulus.

We will prove that this is always the case. The following lemma is a special case of a partition of general Sturmian words as introduced in [17] (also see [10, 18]):

**Lemma 4.5.** The diagonal \( b := (b_{nn})_{n \in \mathbb{Z}_+} \) of \( B_+ \) with \( B \in \text{Lim}_-(A) \) is of the form

\[
b = p w_1 w_2 w_3 \ldots \quad \text{with} \quad p \in \{\varepsilon, 1\} \quad \text{and} \quad w_i \in \{01, 01\} \quad \text{for all} \quad i \in \mathbb{N}.
\]

**Proof.** By Lemma 4.1 and (8), \( b \) contains neither 00 nor 111 as a subword. So 0 is always followed by 1, and 1 is always followed by 101 or 01.

So we are particularly interested in the patterns “101” and “01” and their corresponding transfer matrices

\[
T_{101} := T_1 T_0 T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_{01} := T_1 T_0 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.
\]

Let us say that a vector \((y_1, y_2)\) is \( \in \mathbb{R}^2 \) has property \( C \) if \( y_1 \cdot y_2 < 0 \) and \( |y_1| < |y_2| \).

**Lemma 4.6.** Both \( T_{101} \) and \( T_{01} \) preserve property \( C \). More precisely, if \((y_1, y_2)\) \( \in \mathbb{R}^2 \) has property \( C \) then \((z_1, z_2) := T_w (y_1, y_2)\) with \( w \in \{101, 01\} \) has properties

\[
\begin{align*}
\text{A)} \quad & |z_2| > |y_2| \quad \text{with} \quad |z_2| - |y_2| \geq \min\{|y_1 + y_2|, |y_1|\} > 0, \\
\text{B)} \quad & |z_1 + z_2| \geq |y_1 + y_2| \quad \text{and} \quad |z_1| \geq |y_1|, \quad \text{and} \\
\text{C)} \quad & z_1 \cdot z_2 < 0 \quad \text{and} \quad |z_1| < |z_2|.
\end{align*}
\]

**Proof.** This is a straightforward computation using

\[
T_{101} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_2 + (y_1 + y_2) \end{pmatrix} \quad \text{and} \quad T_{01} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ y_1 - y_2 \end{pmatrix}.
\]

Property \( \text{A} \) shows a growth (in modulus) of the second vector component after applying \( T_{101} \) or \( T_{01} \). By property \( \text{B} \), the amount of growth is non-decreasing when applying \( T_{101} \) or \( T_{01} \) again. The fact that property \( C \) is preserved keeps the argument working for the next application of \( T_{101} \) or \( T_{01} \), leading to unbounded growth of the second vector component, by induction.

So all that we need is one first occurrence of property \( C \) for a vector \((x_n)\) in our computation of a sequence \( x = (x_0, x_1, x_2, \ldots) \) that solves \( B_+ x = 0 \).

We start with the case \( p = \varepsilon \). Besides \( x_0 = 1 \) (see above), we put \( x_{-1} := 0 \) to start our recurrence and account for the non-existence of column number \(-1\) in the matrix \( B_+ \). Depending on which of \( T_{101} \) and \( T_{01} \) we apply to \((x_{-1}) = (0, 0)\), we get

\[
T_{01} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{or} \quad T_{101} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\]
So repeated application of $T_{01}$ leads to $\pm \binom{0}{1}$ but after the first application of $T_{101}$, which will eventually happen since $b$ is not periodic, one gets $(\frac{x_n}{x_{n+1}}) = \pm \binom{-1}{2}$ for some $n \in \mathbb{N}$. This vector has property $C$. From our arguments above it follows that the sequence $x$ is unbounded.

If the prefix $p$ of $b$ is $1$, it follows from $x_0 = 1$ that $x_1 = -1$, so that our recurrence starts with $(\frac{x_0}{x_1}) = \binom{1}{-1}$. The application of the transfer matrices yields

$$T_{01} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = \left( \begin{array}{cc} -1 & 0 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = \left( \begin{array}{c} -1 \\ 2 \end{array} \right) \quad \text{or} \quad T_{101} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 2 \end{array} \right) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = \left( \begin{array}{c} 1 \\ -1 \end{array} \right).$$

So repeated application of $T_{101}$ leads to $\binom{1}{-1}$ but after the first application of $T_{01}$, which will eventually happen since $b$ is not periodic, one gets $(\frac{x_n}{x_{n+1}}) = \binom{-1}{2}$ for some $n \in \mathbb{N}$. This vector has property $C$. From our arguments above it follows that the sequence $x$ is unbounded.

For both possibilities of the prefix $b \in \{\varepsilon, 1\}$ and all possibilities of the following blocks $w_i \in \{101, 01\}$ (recall Lemma 4.5), we have shown that all nontrivial solutions $x$ of the homogeneous system $B_+x = 0$ are unbounded, so that every $B_+$ with $B \in \operatorname{Lim}_- (A)$ is injective on $\ell^\infty (Z_+)$.

To see b), consider the three flip operators

$$J_- : \ell^\infty (Z_+) \to \ell^\infty (Z_-), \quad J_+ : \ell^\infty (Z_-) \to \ell^\infty (Z_+), \quad \text{and} \quad J_\leftrightarrow : \ell^\infty (Z) \to \ell^\infty (Z),$$

all three acting by the rule $x \mapsto y$ with $y_n = x_{-n}$ for $n$ in $Z_-, Z_+$ and $Z$, respectively.

The formula $v = v^R_+ 10 v_+$ from (7) implies that $\operatorname{Lim}_+ (A)$ exactly consists of the reflections $C = J_+ B J_+ \leftrightarrow$ of operators $B \in \operatorname{Lim}_- (A)$, so that

$$\{ C_- : C \in \operatorname{Lim}_+ (A) \} = \{(J_+ B J_+ \leftrightarrow) : B \in \operatorname{Lim}_- (A) \}.$$

So, clearly, b) follows from c).

Finally, for the proof of a), we use a two-sided version of the proof of c).

Let $B \in \operatorname{Lim} (A)$ and again let $b \in \{0, 1\}^Z$ be the diagonal of $B$. Let $x$ be a nontrivial solution of the homogeneous equation $Bx = 0$. We will prove that the sequence $x$ grows unboundedly in at least one direction, left or right. The growth to the right is studied as in the proof of c) above – growth to the left by symmetric arguments. Here is the analogue of Lemma 4.5.

**Lemma 4.7.** The diagonal $b := (b_{nm})_{n \in Z}$ of $B \in \operatorname{Lim} (A)$ is of the form

$$b = \cdots w_{-4} w_{-3} \overset{101}{\underbrace{\frac{101}{w_{-2}} \frac{101}{w_{-1}} w_0}} \overset{101}{\underbrace{\frac{101}{w_1} \frac{101}{w_2}}} w_3 w_4 \cdots$$

with $w_{-i} \in \{101, 10\}$ and $w_i \in \{101, 01\}$ for all $i \in \mathbb{N}$.

**Proof.** The word $101101101101101$ is contained in the Fibonacci word $v$ (as $v_{-6} \cdots v_7$) and therefore, by Lemma 4.1, also in $b$. By Lemma 4.1 and (8), $b$ contains neither 00 nor 111 as a subword. So, as argued in Lemma 4.5, 0 is always followed by 1, and 1 is always followed by 101 or 01. Moreover, 0 is always preceded by 1, and 1 is always preceded by 101 or 10. ■

So besides $T_{01}$ and $T_{101}$, we now also look at the transfer matrix $T_{10} := T_0 T_1$. Note that, when we study the asymptotics of $x$ towards $-\infty$ (going backward in “time”), we will have to look at inverses of the transfer matrices.

Therefore, let us say that a vector $\left( \frac{y_1}{y_2} \right) \in \mathbb{R}^2$ has property $F$ if $y_1 \cdot y_2 < 0$ and $|y_1| > |y_2|$. Here is the “leftward” analogue of Lemma 4.6.
Lemma 4.8. Both $T_{101}^{-1}$ and $T_{10}^{-1}$ preserve property $\mathbf{F}$. More precisely, if $(w_{y_2}) \in \mathbb{R}^2$ has property $\mathbf{F}$ then $(z_{y_2}) := T^{-1}_w(y_{y_2})$ with $w \in \{101, 10\}$ has properties

\begin{align*}
\text{D)} & \quad |z_1| > |y_1| \text{ with } |z_1| - |y_1| \geq \min\{|y_1 + y_2|, |y_2|\} > 0, \\
\text{E)} & \quad |z_1 + z_2| \geq |y_1 + y_2| \text{ and } |z_2| \geq |y_2|, \quad \text{and} \\
\text{F)} & \quad z_1 \cdot z_2 < 0 \text{ and } |z_1| > |z_2|.
\end{align*}

Proof. This is again a straightforward computation using

$$
T_{101}^{-1}(y_1) = \begin{pmatrix} y_1 + y_2 + y_1 \\ -y_1 \end{pmatrix} \quad \text{and} \quad T_{10}^{-1}(y_1) = \begin{pmatrix} -y_1 + y_2 \\ -y_2 \end{pmatrix}.
$$

As before, property $\text{D}$ states a growth (in modulus) of the first component. Property $\text{E}$ ensures that the amount of this growth is non-decreasing in further applications of $T_w^{-1}$ with $w \in \{101, 10\}$, and the fact that property $\mathbf{F}$ is preserved makes sure that the same argument keeps working for further applications of $T_w^{-1}$, leading to unbounded growth.

So again, we just need a first occurrence of property $\mathbf{F}$ for a vector $(x_n)$ with $n < 0$ or a first occurrence of property $\mathbf{C}$ for a vector $(x_n)$ with $n \geq 0$ in our computation of a sequence $x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ that solves $Bx = 0$. Then $x$ will be unbounded.

This time, one entry, say $x_0$, does not determine the whole sequence $x$, but two entries do. Let the two entries of $x$ that are associated to the entries 0 and 1 of $w_0$ in (12) be equal to $\alpha$ and $\beta$, respectively, with arbitrary $\alpha, \beta \in \mathbb{R}$. W.l.o.g label them as $x_0$ and $x_1$.

Using the adjacent entries of $w_0 = 01$ in $b$, see (12), the corresponding entries in $x$ turn out to be

$$
(\begin{array}{c}
x_{-4} \\
\cdots \\
x_5
\end{array}) = (\begin{array}{ccccccc}
\alpha - 2\beta & \beta & -\alpha + \beta & -\beta & \alpha & \beta & -\alpha - \beta & \alpha + \beta & -2\alpha - \beta
\end{array}).
$$

With respect to $\alpha$ and $\beta$, we have to distinguish the following cases:

1. If $\alpha = \beta = 0$ then $x = 0$ follows.
2. If $\alpha = 0$ and $\beta \neq 0$ then $(\begin{array}{c}
x_{-4} \\
\cdots \\
x_{-3}
\end{array}) = (\begin{array}{c}
-2\beta
\end{array})$ has property $\mathbf{F}$.
3. If $\alpha \neq 0$ and $\beta = 0$ then $(\begin{array}{c}
x_4 \\
\cdots \\
x_5
\end{array}) = (\begin{array}{c}
\alpha \\
\cdots \\
-2\alpha
\end{array})$ has property $\mathbf{C}$.
4. If $\alpha \neq 0$ and $\beta \neq 0$ we have to look at two more cases:
   (a) If $\alpha \cdot \beta > 0$ then $(\begin{array}{c}
x_4 \\
\cdots \\
x_5
\end{array}) = (\begin{array}{c}
\beta \\
\cdots \\
-\alpha - \beta
\end{array})$ has property $\mathbf{C}$.
   (b) If $\alpha \cdot \beta < 0$ then $(\begin{array}{c}
x_{-4} \\
\cdots \\
x_{-3}
\end{array}) = (\begin{array}{c}
\alpha - 2\beta \\
\cdots \\
\beta
\end{array})$ has property $\mathbf{F}$.

This completes the study of all cases. Each nontrivial solution of the homogenous equation $Bx = 0$ is unbounded, thus $B$ is injective on $\ell^\infty(\mathbb{Z})$. This completes the proof of Lemma 4.4.

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