Genuine covariant description of Hamiltonian dynamics

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Abstract. After reviewing the covariant description of Hamiltonian dynamics, some
applications are done to the non-relativistic isotropic three-dimensional harmonic
oscillator, Rovelli’s model, and $SO(3,1)$ BF theories.

1. Introduction
In Ref. [1] the classical phase space of a given physical theory is defined as the space of solutions
of its classical equations of motion. Using this definition, the authors of Ref. [1] analyze the
phase spaces of the Klein-Gordon scalar field, Yang-Mills, and general relativity theories and
find that they can be endowed with certain symplectic structures that come from the handling
of the equations of motion of each of these theories. This fact might leave the impression
that the classical equations of motion of the theory under consideration, in a certain sense,
uniquely determine the symplectic geometry of the phase space. However, such a conclusion is
not correct. In fact, the classical equations of motion do not uniquely fix a symplectic structure
on the classical phase space even though its points are in one-to-one correspondence with the
solutions of the classical equations of motion. Thus, the phase space is not endowed with a
natural or preferred symplectic structure but rather there exists a freedom in the choice of the
symplectic structure on it, which is relevant both classical and quantum mechanically [2]. In
this sense, a genuine covariant description of Hamiltonian dynamics consists in choosing different
couples $(\omega, H)$, where $\omega$ is a symplectic structure and $H$ is a Hamiltonian defined on the same
phase space $\Gamma$, for the same dynamical system. In this paper a brief review of the covariant
description of Hamiltonian systems is performed. The paper contains some new applications,
including gauge systems.

2. Covariant description of nonparametrized Hamiltonian systems
2.1. Canonical viewpoint
In the standard treatment of Hamiltonian dynamics, the equations of motion are written in the
form [3]

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2, \ldots, n,$$

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where $H$ is the Hamiltonian of the system, the variables $(q^i, p_i)$ are canonically conjugate to each other in the sense that

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0,$$

(2)

where $\{,\}$ is the Poisson bracket defined by

$$\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}.$$

(3)

2.2. Covariant viewpoint

The symplectic geometry involved in the Hamiltonian description of mechanics can clearly be appreciated if Eqs. (1) are written in the form

$$\dot{x}^\mu \frac{\partial}{\partial x^\mu} = \left( \omega^{\mu \nu} \frac{\partial H}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu},$$

(4)

where the independent coordinates $(x^\mu) = (q^1, \ldots, q^n; p_1, \ldots, p_n)$ label the points $x$ of the phase space $\Gamma$, $\{ \frac{\partial}{\partial x^\nu} \}$ is a basis of the tangent space of $\Gamma$ at the point $x$, and

$$(\omega^{\mu \nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

(5)

where $0$ and $I$ are $n \times n$ matrices. Moreover, Eq. (3) acquires the form

$$\{f, g\} = \frac{\partial f}{\partial x^\mu} \omega^{\mu \nu} \frac{\partial g}{\partial x^\nu},$$

(6)

from which Eq. (2) can be rewritten as

$$\{x^\mu, x^\nu\} = \omega^{\mu \nu}.$$  

(7)

Equations (4) are covariant in the sense that they maintain their form if the canonical coordinates are replaced by a completely arbitrary set of coordinates in terms of which $(\omega^{\mu \nu})$ need not be given by Eq. (5). Similarly, it is possible to retain the original coordinates $(q^i, p_i)$ and still write the original equations of motion (1) in the Hamiltonian form (4), but now employing alternative symplectic structures $\omega^{\mu \nu}(x)$, distinct to that given in Eq. (5), and taking as Hamiltonian any real function on $\Gamma$ which is a constant of motion for the system [2]. This means that the writing of the equations of motion of a dynamical system in Hamiltonian form is not unique [2, 4, 5, 6, 7, 8, 9] (see also Refs. [10, 11]). To better understand the covariant description of Hamiltonian dynamics, it is convenient to write Eqs. (4) in the form

$$\dot{x}^\mu \frac{\partial}{\partial x^\mu} = v.$$

(8)

The richness of the covariant description of Hamiltonian systems relies on the freedom to express the vector field $v$ on the RHS of Eq. (8) as $(\omega^{\mu \nu} \frac{\partial H}{\partial x^\nu}) \frac{\partial}{\partial x^\mu}$ using different couples $(\omega, H)$ and keeping the same manifold $\Gamma$. Therefore, it is clear that the integral curves (and the tangent vector field $v$ to them) of any set of equations of motion are not modified.

The writing of the equations of motion in Hamiltonian form is just half of the story. The other half concerns with the 1) algebra of observables and 2) observables in involution. An observable $O$ is any real-valued function defined on the phase space $\Gamma$, $O : \Gamma \rightarrow \mathbb{R}$. If the observable $O$ is
a constant of motion, then \( \frac{dO}{dt} = 0 \), independently of the choice of the symplectic structure \( \omega \) and Hamiltonian \( H \), which also means that \( O \) is constant along the vector field \( v \)

\[
0 = \frac{dO}{dt} = \frac{\partial O}{\partial x^\mu} \dot{x}^\mu = \left( \dot{x}^\mu \frac{\partial}{\partial x^\mu} \right) O = v O .
\]

Nevertheless, even though the analytic form of observables \( O \) is the same in any Hamiltonian formulation of a given dynamical system, the algebra of observables does directly depend on the symplectic structure chosen. As far as the author knows, this point has been explicitly worked out only in Ref. [9]. In fact, suppose that the triple \((\Gamma, \omega, H)\) is a Hamiltonian formulation of a given dynamical system. Suppose that \( \{O_1, \ldots, O_n\} \) is a set of constants of motion which form a Lie algebra \( \mathcal{A} \) with respect to the symplectic structure \( \omega \) of the triple \((\Gamma, \omega, H)\), \( \{O_a, O_b\}_\omega = c_{ab}^c O_c \). Now, suppose that the triple \((\Gamma, \Omega, h)\) were another, different from \((\Gamma, \omega, H)\), alternative Hamiltonian formulation of the same dynamical system. In the generic case, the constants of motion of \( \{O_1, \ldots, O_n\} \) will not satisfy \( \{O_a, O_b\}_\Omega = c_{ab}^c O_c \) with respect to the symplectic structure \( \Omega \) of the triple \((\Gamma, \Omega, h)\). However, it might be possible to find a new set of constants of motion \( \{O_1, \ldots, O_n\} \) which might form a new Lie algebra \( \mathcal{A}_{new} \) with respect to the symplectic structure \( \Omega \), \( \{O_a, O_b\}_\Omega = f_{ab}^c O_c \). Therefore, the ‘algebra of observables’ has no an absolute connotation for a dynamical system. Similarly, if \( \{I_1, \ldots, I_n\} \) is a set of constants of motion which are in involution with respect to the symplectic structure \( \omega \), i.e., \( \{I_i, I_j\}_\omega = 0 \), then the observables of \( \{I_1, \ldots, I_n\} \) are not, in the generic case, in involution with respect to the symplectic structure \( \Omega \). However, it might be possible to find a new set \( \{I_1, \ldots, I_n\} \) which are in involution with respect to the symplectic structure \( \Omega \), \( \{I_i, I_j\}_\Omega = 0 \). On the other hand, in the same sense that the integral action

\[
S[q^i, p_i] = \int_{t_1}^{t_2} dt \left[ q^i p_i - H(q, p, t) \right] ,
\]

provides the usual equations of motion (1), the covariant form of Hamilton’s equations given in Eq. (4) can be obtained from the integral action

\[
S[x^\mu] = \int_{t_1}^{t_2} dt \left( \theta_\mu(x) \frac{dx^\mu}{dt} - H(x, t) \right) ,
\]

provided \( \delta S = 0 \) and \( \delta x^\mu(t_1) = 0 = \delta x^\mu(t_2) \) under the arbitrary configurational (or form) variation of the variables \( x^\mu \) at \( t \) fixed, \( \delta x^\mu \). In fact,

\[
\delta S = \int_{t_1}^{t_2} dt \left( \omega_{\nu\mu}(x) \delta x^\nu - \frac{\partial H}{\partial x^\nu} \delta x^\nu + \left( \theta_\mu(x) \delta x^\mu \right) \right|_{t_1}^{t_2} ,
\]

where \( \omega \) is the symplectic two-form, \( \omega = \frac{1}{2} \omega_{\nu\mu}(x) dx^\nu \wedge dx^\nu \) with \( \omega_{\nu\mu} = \partial_\nu \theta_\mu - \partial_\mu \theta_\nu \). Equivalently, \( \omega = d\theta \) where \( \theta = \theta_\mu(x) dx^\mu \) is the symplectic potential. It is convenient to make clear some aspects involved with the boundary conditions \( \delta x^\mu(t_1) = 0 = \delta x^\mu(t_2) \) employed in Hamilton’s principle. Due to the fact that there are \( 2n \) coordinates \( x^\mu \), one must fix only \( 2n \) conditions at the time boundary, otherwise the system might be over-determined, in which case the system might not evolve from \( t_1 \) to \( t_2 \). For instance, in the case when \( \theta = p_i dq^i \), it is clear that one can arbitrarily choose \( \delta q^i(t_1) = 0 \) and \( \delta q^i(t_2) = 0 \). However, in the generic case, namely, when \( \theta = \theta_\mu(x) dx^\mu \) one can still arbitrarily choose \( \delta x^\mu(t_1) = 0 \) at \( t = t_1 \). Nevertheless, even though \( \delta x^\mu(t_2) = 0 \) still holds, \( x^\mu(t_2) \) cannot be arbitrarily chosen, but it is fixed by the conditions on \( x^\mu \) at \( t_1 \) in order for the system to evolve from \( t_1 \) to \( t_2 \).
Finally, it is important to emphasize that to write a dynamical system in Hamiltonian form it is not necessary that the symplectic two-form has a particular symmetry. This constitutes the richness of the Hamiltonian description of classical dynamics. The relationship between the symmetries of the action (11) and constants of motion of the dynamical system associated to this action is, as usual, through Noether’s theorem. In fact, if under the transformation $t' = t + \delta t$ and $x^\mu(t') = x^\mu(t) + \delta x^\mu$ with $\delta = \delta + \delta \frac{dt}{dt}$ the action (11) transforms as $\delta S = \int_{t_1}^{t_2} \frac{dE}{dt} dt$ then $\theta^\mu \delta x^\mu - H \delta t - F$ is a constant of motion. Further analysis on the implications in the quantum theory of classical symmetries in the context of the covariant description of Hamiltonian dynamics can be found in Ref. [12].

2.3. Example: three-dimensional isotropic harmonic oscillator

The phase space of the system is $\Gamma = \mathbb{R}^6$ whose points can be labelled with the Cartesian coordinates $(x^\mu) = (x, y, z, p_x, p_y, p_z)$. The equations of motion for the system are

$$\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{z} = \frac{p_z}{m}, \quad \dot{p}_x = -m\omega^2 x, \quad \dot{p}_y = -m\omega^2 y, \quad \dot{p}_z = -m\omega^2 z, \quad (13)$$

or, equivalently,

$$\dot{x}^\mu \frac{\partial}{\partial x^\mu} = v, \quad v = \frac{p_x}{m} \frac{\partial}{\partial x} + \frac{p_y}{m} \frac{\partial}{\partial y} + \frac{p_z}{m} \frac{\partial}{\partial z} - m\omega^2 \left( x \frac{\partial}{\partial p_x} + y \frac{\partial}{\partial p_y} + z \frac{\partial}{\partial p_z} \right). \quad (14)$$

The canonical viewpoint consists in rewriting the vector field $v$ (14) tangent to the integral curves as $v = (\omega^{\mu\nu} \frac{\partial H}{\partial x^\nu}) \frac{\partial}{\partial x^\mu}$ where $\omega^{\mu\nu}$ is that of Eq. (5) and $H$ is the energy for the system,

$$H = \frac{1}{2m} \left( (p_x)^2 + (p_y)^2 + (p_z)^2 \right) + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2). \quad (15)$$

However, according to the covariant viewpoint of Hamiltonian dynamics, it is possible to give alternative descriptions of the dynamics of the system, one of which, for instance, consists in rewriting the vector field $v$ of Eq. (14) as

$$v = (\omega^{\mu\nu} \frac{\partial H}{\partial x^\nu}) \frac{\partial}{\partial x^\mu} \quad \text{with}$$

$$\omega^{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad H = \frac{p_x p_z}{m} + m\omega^2 x z + \frac{(p_y)^2}{2m} + \frac{1}{2} m\omega^2 y^2, \quad (15)$$

among many other possibilities.

3. Covariant description of gauge systems

Even though the covariant description of nonparametrized Hamiltonian systems has been analyzed, the covariant description of gauge systems has not been systematically studied. As far as the author knows, the first paper dealing with the covariant Hamiltonian description, in the sense of the present paper, of constrained systems is Ref. [13], where parametrized nonrelativistic Hamiltonian systems were analyzed in detail and where also the first steps towards the covariant Hamiltonian description of arbitrary gauge systems (in the context of constraints) were done. So, we continue the analysis of the generic case of gauge systems. Moreover, we make clear some aspects of gauge systems that were not clearly stated in such a paper. Also, some minor corrections to that part of the paper are done here. After all, one has in mind realistic theories, so it is relevant to understand how the covariant formulation of gauge systems works. This is the issue of this section.
3.1. Canonical viewpoint

Suppose one has a dynamical system defined by the dynamical equations of motion

\[ \dot{q}^i = \lambda^a \frac{\partial \gamma_a}{\partial p_i}, \quad \dot{p}_i = -\lambda^a \frac{\partial \gamma_a}{\partial q^i}, \]  

(16)

where \( \dot{\cdot} = \frac{d}{d\tau} \) and subject to the constraints

\[ \gamma_a \approx 0, \]  

(17)

which satisfy

\[ \{\gamma_a, \gamma_b\} = C_{ab}^{\cdot \gamma_c}, \]  

(18)

with respect to the canonical symplectic structure of Eq. (3), and so the \( \gamma \)'s are first class [14]. If the coordinates \( q \)'s and \( p \)'s which label the points of the extended phase space \( \Gamma_{ext} \) are grouped in the new variable \( x^\mu, \mu = 1, \ldots, 2n \) such that \( (x^1, x^2, \ldots, x^n, x^{n+1}, x^{n+2}, \ldots, x^{2n}) = (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n) \) then the dynamical equations of motion acquire the form

\[ \dot{x}^\mu \frac{\partial}{\partial x^\mu} = \lambda^a X_a, \]  

(19)

where the vector fields \( X_a \) are given by

\[ X_a = X_a^\mu \frac{\partial}{\partial x^\mu} = \left( \omega^{\mu \nu} \frac{\partial \gamma_a}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}, \]  

(20)

with \( (\omega^{\mu \nu}) \) given by Eq. (5).

3.2. Covariant viewpoint

There exists also the possibility of making a different choice of both the symplectic structure \( \omega \) and the analytical form of the constraints \( \gamma_a \) in Eqs. (19) and (20). Let \( \Omega \) and \( G_a \) be a new symplectic structure and a different analytical form of the constraints; respectively. It is important to emphasize that the extended phase space \( \Gamma_{ext} \) as a manifold is the same as the one before. In particular, the points \( p \in \Gamma_{ext} \) can still be labelled with the same coordinates \( x^\mu \) as before. Of course, one can choose another set of coordinates, but this is not relevant for the present discussion and the introduction of a different set of coordinates is avoided just to emphasize that the alternative symplectic structures and the form of the constraints have a different analytical form with respect to the ones of the standard viewpoint. In summary, \( \Omega \neq \omega \) and \( G_a \neq \gamma_a \) on \( \Gamma_{ext} \), which is a statement which holds independently of the particular set of coordinates chosen. Now, the idea is to express the vector fields \( X_a \) of Eq. (20) in terms of the new symplectic structure \( \Omega \) and the new constraints \( G_a \) in such a way that the vector fields \( X_a \) remain the same, i.e.,

\[ X_a = X_a^\mu \frac{\partial}{\partial x^\mu} = \left( \Omega^{\mu \nu} \frac{\partial G_a}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}. \]  

(21)

What happens to the dynamical equations of motion given in Eq. (16)? These equations of motion are the same ones than those given in Eq. (19). Therefore, due to the fact that the vector fields \( X_a \) are the same, the dynamical equations are not modified, i.e., they are also the same in the present case. What about the constraint surface \( \Sigma \)? Well, in the present case the constraint surface \( \Sigma \) is defined by

\[ G_a \approx 0. \]  

(22)
Thus, it is obvious that the analytical form of the equations that define the constraint surface of this case are completely different to those equations of the starting point, given in Eq. (17).

It is also important to emphasize that new constraints $G_a$ are not, in the generic case, a linear combination of the original set of constraints $\gamma_a$. So, the freedom in the choice of the constraint surface has nothing to do with the usual abelianization of the constraints $\gamma_a$ simply because the abelianization procedure is not accompanied with a change in the symplectic structure as it happens in the present context. This fact is clearly appreciated even in the case of parametrized nonrelativistic Hamiltonian systems [13]. So, the new constraints $G_a$ are first class with respect to the new symplectic structure $\Omega$. An action principle, which gives the covariant form of the equations of motion is

$$S[x^\mu, \lambda^a] = \int_{\tau_1}^{\tau_2} d\tau \left[ \theta_\mu(x) \dot{x}^\mu - \lambda^a \gamma_a(x) \right],$$  \hspace{1cm} (23)$$

with $\omega_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu$ and the constraints $\gamma_a$ are first class with respect to $\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu$.

3.3. Examples

3.3.1. Rovelli’s model. The extended phase space of this model is $\Gamma_{ext} = \mathbb{R}^4$. The points $p \in \Gamma_{ext}$ are labelled with the cartesian coordinates $(x^\mu) = (q^1, q^2, p_1, p_2)$. In these coordinates, the symplectic structure $\omega$ on $\Gamma_{ext}$ is defined by

$$\omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2.$$  \hspace{1cm} (24)$$

The dynamical equations are

$$\dot{q}^1 = \lambda p_1, \quad \dot{q}^2 = \lambda p_2, \quad \dot{p}_1 = -\lambda q^1, \quad \dot{p}_2 = -\lambda q^2.$$  \hspace{1cm} (25)$$

where $\lambda$ is a Lagrange multiplier [15]. On the other hand, the constraint surface $\Sigma$ is

$$\gamma := \frac{1}{2} ((p_1)^2 + (p_2)^2 + (q_1)^2 + (q_2)^2) - M \approx 0,$$  \hspace{1cm} (26)$$

where $M$ is a positive constant. From the dynamical equations, the vector field $X$ is read off

$$X = p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} - q_1 \frac{\partial}{\partial p_1} - q_2 \frac{\partial}{\partial p_2} = \left( \omega_{\mu\nu} \frac{\partial \gamma}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}.$$  \hspace{1cm} (27)$$

Using the discussion of the preceding subsection as well as Ref. [13], Rovelli’s model can, alternatively, be described, for instance, as follows:

1) The extended phase space remains the same, $\Gamma_{ext} = \mathbb{R}^4$. If the symplectic structure on $\Gamma_{ext}$ is chosen to be

$$\omega_1 = dp_2 \wedge dq^1 + dp_1 \wedge dq^2,$$  \hspace{1cm} (28)$$

then the new constraint surface is defined by

$$\gamma_1 := (p_1 p_2 + q_1 q_2) - N \approx 0,$$  \hspace{1cm} (29)$$

where $N$ is a constant. The vector field $X$ is rewritten as

$$X = p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} - q_1 \frac{\partial}{\partial p_1} - q_2 \frac{\partial}{\partial p_2} = \left( \omega_{1\mu} \frac{\partial \gamma_1}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}.$$  \hspace{1cm} (30)$$
2) The extended phase space remains the same, \( \Gamma_{\text{ext}} = \mathbb{R}^4 \). If the symplectic structure on \( \Gamma_{\text{ext}} \) is chosen to be
\[
\omega_2 = -dp_1 \wedge dq^1 + dp_2 \wedge dq^2,
\]
then the new constraint surface is defined by
\[
\gamma_2 := \frac{1}{2} ((p_2)^2 - (p_1)^2 + (q^2)^2 - (q^1)^2) - L \approx 0,
\]
where \( L \) is a constant. The vector field \( X \) is rewritten as
\[
X = p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} - q_1 \frac{\partial}{\partial p_1} - q_2 \frac{\partial}{\partial p_2} = \left( \omega_2^{\mu \nu} \frac{\partial \gamma_2}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}.
\]

3) The extended phase space remains the same, \( \Gamma_{\text{ext}} = \mathbb{R}^4 \). If the symplectic structure on \( \Gamma_{\text{ext}} \) is chosen to be
\[
\omega_3 = dq^1 \wedge dq^2 + dp_1 \wedge dp_2,
\]
then the new constraint surface is defined by
\[
\gamma_3 := (q^1 p_2 - q^2 p_1) - R \approx 0,
\]
where \( R \) is a constant. The vector field \( X \) is rewritten as
\[
X = p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} - q_1 \frac{\partial}{\partial p_1} - q_2 \frac{\partial}{\partial p_2} = \left( \omega_3^{\mu \nu} \frac{\partial \gamma_3}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}.
\]

4. Field theory

In this section, an application to field theory is done. The system is defined by the equations of motion
\[
F^I J [A] = 0, \quad DB^{IJ} = 0,
\]
where \( F^I J [A] = dA^I J + A^I K \wedge A^K J \) is the curvature of the Lorentz connection \( A^I J = A^I J d^4x^\mu \) and \( DB^{IJ} = dB^{IJ} + A^I K \wedge B^{KJ} + A^K J \wedge B^{IJ} \) where \( B^{IJ} \) is a set of six two-forms on account of \( B^{IJ} = -B^{JI} \). The equations of motion (37) are usually obtained from the BF action
\[
S[A, B] = a_1 \int_\mathcal{M} B^{IJ} \wedge F_{IJ} [A],
\]
where the Lorentz indices \( I, J, \ldots = 0, 1, 2, 3 \) are raised and lowered with the Minkowski metric \( \eta_{IJ} \), \( (\eta_{IJ}) = \text{diag}(-1, +1, +1, +1) \). Defining \( B^i = -\frac{1}{2} \varepsilon^{ijk} B^{jk} \) and \( \Gamma^i = -\frac{1}{2} \varepsilon^{ijk} A^{jk} \) where the latin indices \( i, j, \ldots \) are raised and lowered with the Euclidean metric \( \delta_{ij} \), \( (\delta_{ij}) = \text{diag}(+1, +1, +1) \), the Hamiltonian analysis of the action (38) leads to
\[
S = \int dx^0 \int_\Sigma d^3 x \left[ \Gamma_a \overset{i}{\gamma}^a + \hat{A}_a \overset{0i}{\gamma}^a - \lambda \breve{\Gamma}_i - \Lambda^i \breve{H}_i - \lambda \breve{C}^{ai} - \rho \breve{D}^{ai} \right],
\]
where it has been assumed that \( \mathcal{M} \) has the topology \( \Sigma \times R \), with \( \Sigma \) closed (compact and without boundary) to avoid boundary terms. The four-dimensional coordinates \( (x^\mu) = (x^0, x^a) \), \( a, b, \ldots = 1, 2, 3 \) are such that \( x^a \) label points on \( \Sigma \). The totally anti-symmetric Levi-Civita density of weight +1, \( \breve{\gamma}^{0123} = +1 \). Also \( \breve{\gamma}^{0abc} = \breve{\gamma}^{abc} \). The other definitions
involved are \( \tilde{\pi}^a_i := a_i \tilde{\eta}^{abc} B_{bc i} \), \( \tilde{\rho}^a_i := a_i \tilde{\eta}^{abc} B_{bc i} \), \( \lambda^i := -\Gamma_0^i \), \( \Lambda^i := -A_0^i \), \( \lambda_{ai} := -2a_1 B_{0ai} \), and

\[
\tilde{G}_i := \partial_i \tilde{\pi}^a_i + \varepsilon_{ij}^k \Gamma_a^j \tilde{\pi}^a_k + \varepsilon_{ij}^k A_a^0 \tilde{\rho}^a_k ,
\]

\[
\tilde{H}_i = \partial_i \tilde{\rho}^a_i + \varepsilon_{ij}^k \Gamma_a^j \tilde{\rho}^a_k - \varepsilon_{ij}^k A_a^0 \tilde{\pi}^a_k ,
\]

\[
\tilde{C}^{ai} := \frac{1}{2} \tilde{\eta}^{abc} F_{bc i} ,
\]

\[
\tilde{D}^{ai} := \frac{1}{2} \tilde{\eta}^{abc} F_{bc 0i} ,
\]

(40)

where \( F_{bc}^i := -\frac{1}{2} \varepsilon^{ijk} F_{bc jk} \). Finally, the symplectic structure is given by (see also Ref. [16])

\[
\{ \Gamma_a^i (x^0, x), \tilde{\pi}^b_j (x^0, y) \} = \delta^b_a \delta^i_j \delta^3 (x, y) , \quad \{ A_a^0 (x^0, x), \tilde{\rho}^b_j (x^0, y) \} = \delta^b_a \delta^3_j (x, y) .
\]

(41)

Alternatively, it is also possible to take the action principle

\[
S_2 [A, B] = a_2 \int_M *B^{IJ} \wedge F_{IJ} [A] ,
\]

(42)

with \( *B^{IJ} = \frac{1}{2} \varepsilon^{IJ}_K L B^{KL} \) and \( \varepsilon_{0123} = +1 \). The variation of the action (42) with respect to the connection \( A^{IJ}_J \) and the \( B^{IJ} \) fields yields \( D *B^{IJ} = 0 \) and \( *F^{IJ} = 0 \), which after the application of the dual operation \( *^* \) reduce to those given in Eq. (37). So, both actions (38) and (42) give rise to the same equations of motion. In spite of this, the symplectic structures defined by both actions are different from each other. In fact, the Hamiltonian analysis of the action (42) leads to

\[
S_2 = \int dx_0 \int_\Sigma \left[ \frac{a_2}{a_1} \Gamma^i_a \tilde{\rho}^a_i - \frac{a_2}{a_1} A_0^i \tilde{\pi}^a_i + \frac{a_2}{a_1} \Lambda^i \tilde{G}_i - \frac{a_2}{a_1} \rho_{ai} \tilde{C}^{ai} + \frac{a_2}{a_1} \lambda_{ai} \tilde{D}^{ai} \right] .
\]

(43)

The symplectic structure in this case is given by

\[
\{ \Gamma_a^i (x^0, x), \frac{a_2}{a_1} \tilde{\rho}^b_j (x^0, y) \} = \delta^b_a \delta^i_j \delta^3 (x, y) , \quad \{ A_a^0 (x^0, x), -\frac{a_2}{a_1} \tilde{\rho}^b_j (x^0, y) \} = \delta^b_a \delta^3_j (x, y) .
\]

(44)

[cf. Eq. (41)].

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