VIBRATION OF SANDWICH BEAMS REINFORCED BY CARBON NANOTUBES UNDER A MOVING LOAD

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Abstract. This work studies the vibration of sandwich beams reinforced with carbon nanotubes (CNTs) under moving point loads. The cores of the beams are homogeneous while their two sides are made of carbon nanotube reinforced composites. The effective properties of two face sheets are estimated through a micromechanical model. A uniform distribution (UD) and four different types of functionally graded (FG) distributions, namely FG-X, FG-Λ, FG-V, FG-O, are considered. Based on a third-order shear deformation theory, a finite element formulation is derived and used to investigate the vibration characteristics of the beams. The effects of carbon nanotube volume fraction, carbon nanotube distribution pattern and moving load velocity on beam vibration behavior are investigated and highlighted. The influence of layer thickness and span-to-height ratio on beam vibration is also examined and discussed.

Keywords: Sandwich beam; carbon nanotube reinforcement, third-order shear deformation theory; moving load; vibration analysis.

Classification numbers: 2.9.4, 5.4.2, 5.4.5.

1. INTRODUCTION

Carbon nanotubes (CNTs) with high strength, high stiffness, high aspect ratio and low density are excellent reinforcement for composite materials. The analysis of functionally graded carbon nanotube-reinforced composite (FG-CNTRC) beams has drawn considerable attention from researchers in recent years. Ke et al. [1, 2] investigated the nonlinear free vibration and dynamic stability of FG nano beams reinforced by single-walled carbon nanotubes (SWCNTs) using Timoshenko theory. Their results show that an increase in CNT volume fraction leads to higher natural frequencies for both uniformly distributed CNT (UD-CNT) and FG-CNTRC beams. Yas and Heshmati [3] studied free and forced vibrations of an FG nanocomposite beam reinforced by randomly straight SWCNTs under a moving load. Free vibration and buckling analysis of FG-CNTRC Timoshenko beams resting on an elastic foundation are also described by Yas and Samadi [4]. Shen and Xiang [5] presented the nonlinear bending and the thermal
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postbuckling analyses of CNTRC beams. The obtained results show that the CNT volume fraction has a significant influence on the load-deflection curves of the beam. Based on Timoshenko beam theory, Ansari et al. [6] studied forced vibration of nanocomposite beams reinforced by SWCNTs. The third-order shear deformation theory was adopted by Lin and Xiang [7] in determining vibration frequencies of UD- and FG-CNTRC beams with various boundary conditions. Nejati and Eslampanah [8] employed the two dimensional (2D) elasticity theory to obtain buckling loads and natural frequencies of cantilever FG-CNTRC beams under axial load. Based on the first-order shear deformation beam theory and von Kármán nonlinearity, Wu et al. [9] investigated nonlinear vibration of imperfect shear deformable FG-CNTRC beams. Nonlinear free vibration and post-buckling of FG-CNTRC beams resting on nonlinear foundation were studied by Shafiei and Setoodeh [10]. Recently, Mohseni and Shakouri [11] investigated the free vibration and buckling of FG-CNTRC beams with variable thickness resting on elastic foundations.

Using FG-CNTRCs as facing composition in sandwich constructions to increase strength and stiffness, Wu and Kitipornchai [12] investigated free vibration and elastic buckling of sandwich beams with FG-CNTRC face sheets, giving a detail on the effects of CNT volume fraction, core-to-face sheet thickness ratio, slenderness ratio, and end supports on the free vibration and buckling behavior of sandwich beams. Ebrahimi and Farzamand Nia [13] proposed a higher-order shear deformation beam theory for free vibration analysis of FG-CNTRC sandwich beams in thermal environment. The effects of carbon nanotube volume fractions, slenderness ratio and core-to-face sheet thickness ratio on the vibration of the sandwich beams have been examined.

The influence of material gradation on the vibration of beams carrying a moving load has been investigated in recent years [14, 15]. It has been shown that the variation of material properties in spatial directions has a significant influence on both free and forced vibrations of the beams. This topic is further explored in the present work by studying vibration of FG sandwich beams reinforced by CNTs. The core of the sandwich beams is homogeneous while its two face sheets are a FG-CNT reinforced material. The effective properties of the two face sheets are determined by an extended rule of mixture. Five types of CNT distribution, namely UD, FG-X, FG-Λ, FG-V, FG-O, are considered. A third-order shear deformation finite element formulation is derived and employed to compute natural frequencies and investigate dynamic response of the beams. A parametric study is carried out to highlight the effects of carbon nanotube volume fraction, the type of carbon nanotube distribution, the beam geometry and moving load velocity on dynamic behavior of the sandwich beams.

2. FG-CNTRC SANDWICH BEAM

Figure 1 shows a sandwich beam with FG-CNTRC face sheets subjected to a concentrated load $F_0$, moving from the left end to the right end of the beam at a constant speed $v$. In the figure, the Cartesian coordinate system $(x,z)$ is chosen such that the $x$-axis lies on the beam mid-plane. The beam consists of three layers, a homogeneous core and two face sheets of CNTRC material. Denoting $h_0 = -h/2$, $h_1 = h/2$ are the vertical coordinates of the bottom surface, two interfaces between the layers, and the top surface. Five types of distribution of CNTs in the beam cross-section, as shown in Figure 2 and given in Table 1, namely the UD, FG-X, FG-Λ, FG-V, FG-O, are considered.
Figure 1. Geometry and coordinate system of FG-CNTRC sandwich beams under moving load

Figure 2. Cross-section of sandwich beam with five types of CNT distribution

Table 1. Volume fraction $V_{\text{CNT}}$ of CNTs in face sheets of sandwich beam

| Distribution type | Bottom face sheet ($h_i \leq z \leq h_1$) | Top face sheet ($h_1 \leq z \leq h_3$) |
|-------------------|------------------------------------------|---------------------------------------|
| UD                | $2 \left[ \frac{h_i + h_1 - 2z}{h_i - h_1} \right] V_{\text{CNT}}$ | $2 \left[ \frac{h_i + h_1 - 2z}{h_i - h_1} \right] V_{\text{CNT}}$ |
| FG-X              | $2 \frac{z - h_i}{h_i - h_1} V_{\text{CNT}}$ | $2 \frac{z - h_i}{h_i - h_1} V_{\text{CNT}}$ |
| FG-Λ              | $2 \frac{h_i - z}{h_i - h_1} V_{\text{CNT}}$ | $2 \frac{z - h_i}{h_i - h_1} V_{\text{CNT}}$ |
| FG-V              | $2 \left( 1 - \frac{2z - h_i - h_1}{h_i - h_0} \right) V_{\text{CNT}}$ | $2 \left( 1 - \frac{2z - h_i - h_1}{h_i - h_0} \right) V_{\text{CNT}}$ |
| FG-O              | $2 \left( 1 - \frac{2z - h_i - h_1}{h_i - h_0} \right) V_{\text{CNT}}$ | $2 \left( 1 - \frac{2z - h_i - h_1}{h_i - h_0} \right) V_{\text{CNT}}$ |

In Table 1, $V_{\text{CNT}}$ is the total CNT volume fraction in face sheets and it is the same for the five types of the CNT distribution; $V_{\text{CNT}}$ defined by $V_{\text{CNT}} = \frac{1}{w_{\text{CNT}}} \left( \rho \left( \frac{E}{E} \right) - \left( \frac{E}{E} \right) \rho \right)$.
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where $w_{CN}$ is the mass fraction of nanotube, $\rho^{CNT}$ and $\rho^m$ are the densities of carbon nanotube and matrix, respectively.

The material properties of CNTRC layers are determined according to an extended rule of mixture as [12]

$$E_{11} = \eta_1 V_{CNT} E^{CNT}_{11} + V_m E^m; \quad \frac{E_2}{E_{22}} = \frac{V_{CNT} E^{CNT}_{22}}{E^m}; \quad \eta_3 = \frac{V_{CNT} G^{CNT}_{12}}{G^m}$$

(1)

In Eq. (1), $E^{CNT}_{11}$, $E^{CNT}_{22}$ and $G^{CNT}_{12}$ are Young’s moduli and shear modulus, respectively, of the CNT; $E^m$, $G^m$ and $V_m = 1 - V_{CNT}$ are Young’s modulus, shear modulus and volume fraction of matrix phase, respectively; $\eta_1, \eta_2, \eta_3$ are the CNT efficiency parameters.

The Poisson’s ratio of the FG-CNTRC face sheets are determined as

$$\nu_{12} = V_{CNT} \nu_{12}^{CNT} + V_m \nu^m; \quad \nu_{21} = \frac{V_{12}}{E_{11}} E_{22};$$

(2)

where $\nu_{12}^{CNT}$, $\nu^m$ are Poisson’s ratio of the CNT and matrix, respectively.

The effective elastic and shear moduli of the $k$th layer are calculated as follows [12]

$$E^{(k)}(z) = \frac{E_{11}}{1 - \nu_{12}^{k} \nu_{21}}; \quad G^{(k)}(z) = G_{12} \quad (k = 1, 3)$$

$$E^{(2)} = E^c; \quad G^{(2)} = G^c$$

(3)

in which $E^c$, $G^c$ are the elastic and shear moduli of the core material. The effective mass density of the $k$th layer is defined as

$$\rho^{(k)}(z) = V_{CNT} \rho^{CNT} + V_m \rho^m \quad (k = 1, 3); \quad \rho^{(2)} = \rho^c$$

(4)

where $\rho^c$ is mass density of the core material.

3. MATHEMATICAL FORMULATION

The Shi’s third-order shear deformation theory [16] is adopted herewith to formulate the governing equations for the beam. This theory is derived from an elasticity formulation, rather than displacement hypothesis, which gives better results than the first-order and other simple higher order shear deformation theories. The displacements of a point in the beam in the $x$ and $z$ directions, $u(x, z, t)$ and $w(x, z, t)$, respectively, are given by

$$u(x, z, t) = u_0(x, t) + \frac{z}{4} (5 \theta + w_{0,z}) - \frac{5z^3}{3\eta^2}(\theta + w_{0,x}); \quad w(x, z, t) = w_0(x, t)$$

(5)

where $u_0(x, t)$ and $w_0(x, t)$ are, respectively, the displacements in the $x$ and $z$ directions of a point on the $x$-axis; $\theta$ is the cross-sectional rotation, and $t$ is the time variable. In Eq. (5) and hereafter, the subscript comma is used to indicate the derivative with respect to the variable that follows.

By using the transverse shear rotation $\gamma_0$, defined as $\gamma_0 = \theta + w_{0,z}$ [17], the axial and transverse displacements in Eq. (5) can be rewritten as follows
\[ u(x, z, t) = u_0(x, t) + z \left( \frac{5}{4} \gamma_0 - w_{0,xx} \right) - \frac{5z^3}{3h^2} \gamma_0; \quad w(x, z, t) = w_0(x, t) \]  

Eq. (6) gives the axial strain \( \varepsilon_{ax} \) and shear strains \( \gamma_{ax} \) in the forms

\[ \varepsilon_{ax} = u_{0,xx} + z \left( \frac{5}{4} \gamma_{0,xx} - w_{0,xxx} \right) - \frac{5z^3}{3h^2} \gamma_{0,xx}; \quad \gamma_{ax} = 5 \left( \frac{1}{4} - \frac{1}{h^2} z^2 \right) \gamma_0 \]  

The normal and shear stresses are given by linear elastic constitutive law as

\[ \sigma^{(k)}_{xx} = E^{(k)}(z) \varepsilon_{ax}^k; \quad \tau^{(k)}_{ax} = G^{(k)}(z) \gamma_{ax}^k \quad (k = 1, 2, 3) \]  

Using Eqs. (7) and (8) one can write the strain energy in the form

\[ U = \frac{1}{2} \int_A \left( \sigma_{xx} e_{xx} + \tau_{ax} \gamma_{ax} \right) dAdx \]

\[ = \frac{1}{2} \int_A \left[ A_{11} u_{0,xx}^2 + 2A_{12} u_{0,xx} + A_{22} \left( \frac{5}{4} \gamma_{0,xx} - w_{0,xxx} \right)^2 + \frac{10}{3h^2} A_{44} u_{0,xx} \gamma_{0,xx} \right] \]

\[ - \frac{10}{3h^2} A_{44} \gamma_{0,xx} \left( \frac{5}{4} \gamma_{0,xx} - w_{0,xxx} \right) + \frac{25}{9h^4} A_{66} \gamma_{0,xx}^2 + 25 \left( \frac{1}{16} B_{11} - \frac{1}{2h^2} B_{22} + \frac{1}{h^4} B_{44} \right) \gamma_0^2 \]  

where \( A = bh \) is the cross-section area; \( A_{11}, A_{12}, \ldots, A_{66} \) and \( B_{11}, B_{22}, B_{44} \) are the beam rigidities, defined as

\[ (A_{11, A_{12, A_{12}}, A_{44, A_{44}}, A_{66}) = b \int_A E(z) \left( 1, z, z^2, z^3, z^4, z^6 \right) dz \]

\[ = b \sum_{k=1}^{3} \int_{z_l}^{z_{l+1}} E^{(k)}(z) \left( 1, z, z^2, z^3, z^4, z^6 \right) dz; \]  

\[ (B_{11, B_{22}, B_{44}) = b \int_{z_l}^{z_{l+1}} G(z) \left( 1, z^2, z^4 \right) dz = b \sum_{k=1}^{3} \int_{z_l}^{z_{l+1}} G^{(k)}(z) \left( 1, z^2, z^4 \right) dz \]

The kinetic energy of the beam resulted from Eq. (5) has the form

\[ T = \frac{1}{2} \int_A \rho^d(z) \left( \dot{u}^2 + \dot{w}^2 \right) dAdx \]

\[ = \frac{1}{2} \int_A \left[ I_{11} \dot{u}_0^2 + I_{12} \dot{u}_0 \dot{w}_0 + I_{22} \left( \frac{5}{4} \dot{\gamma}_0 - \dot{w}_{0,xx} \right)^2 + I_{22} \left( \frac{5}{4} \dot{\gamma}_0 - \dot{w}_{0,xx} \right)^2 \right] \]

\[ - \frac{10}{3h^2} I_{44} \dot{u}_0 \dot{\gamma}_0 - \frac{10}{3h^2} I_{44} \dot{\gamma}_0 \left( \frac{5}{4} \dot{\gamma}_0 - \dot{w}_{0,xx} \right) + \frac{25}{9h^4} I_{66} \dot{\gamma}_0^2 \]  

where the over dot denotes the derivative with respect to time variable; \( I_{11, I_{12}, \ldots, I_{66} \)} are the mass moments, defined as

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\[ (I_{11}, I_{12}, I_{22}, I_{34}, I_{44}, I_{66}) = b \int_0^h \rho(z)(1, z, z^2, z^3, z^4, z^5) dz \]
\[ = b \sum_{k=1}^3 h_k^{\alpha_k} \rho^{(\alpha_k)}(z)(1, z, z^2, z^3, z^4, z^5) dz \quad (k = 1, 2, 3) \quad (12) \]

The potential of the moving load \( V \) is simply given by
\[ V = - \int_0^L F_0 w_0(x, t) \delta(x - vt) \, dx \quad (13) \]

where \( \delta(\cdot) \) is the Dirac delta function, and \( x \) is the abscissa, measured from the left end of the beam.

4. FG-CNTRC SANDWICH BEAM ELEMENT

Consider a two-node beam element with length \( l \). The element vector of nodal displacements \( d \) contains eight components as
\[ d = \{ d_u, d_w, d_\gamma \}^T \quad (14) \]

where
\[ d_u = \{ u_{01}, u_{02} \}^T ; \quad d_w = \{ w_{01}, w_{01}, w_{02}, w_{02} \}^T ; \quad d_\gamma = \{ \gamma_{01}, \gamma_{02} \}^T \quad (15) \]
are, respectively, the vectors of values for \( u_0, w_0 \) and \( \gamma_0 \) at nodes 1 and 2. In the above equations and hereafter, a superscript \('T'\) denotes the transpose of a vector or a matrix.

Linear functions are used to interpolate the axial displacement \( u_0 \) and transverse shear rotation \( \gamma_0 \), while cubic Hermite polynomials are used for the transverse displacement \( w_0 \) as
\[ u_0 = N d_u ; \quad w_0 = H d_w ; \quad \gamma_0 = N d_\gamma \quad (16) \]
where \( N = \{ N_1, N_2 \} \) and \( H = \{ H_1, H_2, H_3, H_4 \} \) are the matrices of the linear and Hermite shape functions.

Using the above interpolations, one can write the strain energy in Eq. (9) in the form
\[ U = \frac{1}{2} \sum_{ne}^m d^T k d \quad \text{with} \quad k = k_{uu} + k_{uw} + k_{uy} + k_{wu} + k_{wy} + k_{yy} \quad (17) \]
where \( ne \) is the total number of elements, and \( k \) is the element stiffness matrix.

In the above equation, \( k_{uu}, k_{uw}, \ldots k_{yy} \) are the element stiffness matrices stemming from the axial, bending, shear deformations and their couplings with the following expressions
Similarly, the kinetic energy of the beam in Eq. (11) can be rewritten as

\[ T = \frac{1}{2} \sum \dot{\mathbf{d}}^T \mathbf{m} \mathbf{d} \] with \( \mathbf{m} = \mathbf{m}_{uu} + \mathbf{m}_{uw} + \mathbf{m}_{wy} + \mathbf{m}_{ww} + \mathbf{m}_{wy} + \mathbf{m}_{yy} \) (19)

where \( \mathbf{m} \) is the element mass matrix.

The expressions for the sub-matrices in the above equation are as follows:

\[
\begin{align*}
\mathbf{m}_{uu} &= \int_0^1 \mathbf{N}^T \mathbf{I}_1 \mathbf{N} \, dx; \\
\mathbf{m}_{uw} &= -\int_0^1 \mathbf{N}^T \mathbf{I}_{12} \mathbf{H}_{ss} \, dx; \\
\mathbf{m}_{wy} &= \int_0^1 \mathbf{N}^T \mathbf{I}_{12} \mathbf{H}_{ss} \, dx; \\
\mathbf{m}_{ww} &= \int_0^1 (\mathbf{H}_{ss} \mathbf{I}_{11} \mathbf{H} + \mathbf{H}_{ss} \mathbf{I}_{22} \mathbf{H}) \, dx; \\
\mathbf{m}_{wy} &= \int_0^1 \mathbf{N}^T \mathbf{I}_{12} \mathbf{N} \, dx; \\
\mathbf{m}_{yy} &= 25 \int_0^1 \mathbf{N}^T \left[ \frac{1}{16} \mathbf{I}_{22} - \frac{1}{2h^2} \mathbf{I}_{44} + \frac{1}{4h^2} \mathbf{I}_{66} \right] \, dx.
\end{align*}
\]

The discrete equation of motion for the dynamic analysis of the beam can be written in the form

\[ \mathbf{M} \ddot{\mathbf{d}} + \mathbf{K} \mathbf{d} = \mathbf{F}_{ex} \] (21)

where \( \mathbf{M} \) and \( \mathbf{K} \) are, respectively, the global mass and stiffness matrices, obtained by assembling the matrices \( \mathbf{m} \) and \( \mathbf{k} \) over the elements; \( \mathbf{D} \) and \( \dot{\mathbf{D}} \) are, respectively, the vectors of nodal displacements and accelerations; \( \mathbf{F}_{ex} \) is the vector of the nodal external force with the following form:

\[ \mathbf{F}_{ex} = \sum \mathbf{f}_{ex} \] (22)

where \( \mathbf{f}_{ex} = \{0 \ 0 \ F_1 \mathbf{H}_{1} \ F_2 \mathbf{H}_{2} \ F_3 \mathbf{H}_{3} \ F_4 \mathbf{H}_{4} \ 0 \ 0 \}^T \) is the element nodal force vector. Except for the element under the load \( F_0 \), the element nodal force vector \( \mathbf{f}_{ex} \) is zero for all other elements, and the interpolation functions \( \mathbf{H}_i \) (\( i = 1, \ldots, 4 \)) are evaluated at the current position of the force \( F_0 \).

The system of Eq. (21) can be solved by the Newmark method. The average acceleration method which ensures the numerically unconditional stability is adopted herein.

5. NUMERICAL RESULTS AND DISCUSSION

A FG-CNTRC sandwich beam with simply supported ends is considered in the numerical investigation in this section. Poly-methyl methacrylate (PMMA) with material properties as \( E = 2.5 \text{GPa}, \rho = 1190 \text{ kg/m}^3, \nu = 0.3 \) is chosen for the matrix phase; the armchair (10,10) SWCNTs with
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\[ E_{11}^{\mathrm{CNT}} = 5.6466 \, \mathrm{TPa}, \quad E_{22}^{\mathrm{CNT}} = 7.08 \, \mathrm{TPa}, \quad G_{12}^{\mathrm{CNT}} = 1.9445 \, \mathrm{TPa}, \quad \rho^{\mathrm{CNT}} = 1400 \, \mathrm{kg/m^3}, \]

and \( v^{\mathrm{CNT}} = 0.175 \) are selected as reinforcements. The CNT efficiency parameters \( \eta_i \) are determined by matching Young’s moduli \( E_{11}, E_{22} \) and shear modulus \( G_{12} \) of CNTRCs obtained from the rule of mixture given by Han and Elliott [18]. For example \(( \eta_1, \eta_2, \eta_3 ) = (0.137, 1.022, 0.715)\) for \( V_{\text{CNT}}' = 0.12; \), \(( \eta_1, \eta_2, \eta_3 ) = (0.142, 1.626, 1.138)\) for \( V_{\text{CNT}}' = 0.17; \), \(( \eta_1, \eta_2, \eta_3 ) = (0.141, 1.585, 1.109)\) for \( V_{\text{CNT}}' = 0.28.\)

Titanium alloy (Ti-6Al-4V) with \( E = 113.8 \, \mathrm{GPa}, \quad \rho = 4430 \, \mathrm{kg/m^3}, \quad v = 0.342 \) is chosen as the core material of the sandwich beam. The total thickness of the sandwich beam is 0.01 m. The ratio of homogeneous core thickness to face sheet thickness is defined by \( h_c/h_f \).

The fundamental frequency parameter is defined as

\[ \mu = \omega L \sqrt{I_{110}/A_{110}}, \quad (23) \]

where \( A_{110} \) and \( I_{110} \) are the values of \( A_i \) and \( I_i \) of a homogeneous beam made from pure core material, and \( \omega \) is the fundamental frequency. A uniform increment time step \( \Delta t = \Delta T/200 \) with \( \Delta T \) as the total time necessary for the load crossing the beam, is used for the Newmark procedure. For the convenience of discussion, the following dynamic magnification factor \( D_d \) is introduced

\[ D_d = \max \left( \frac{w(L/2,t)}{w_h} \right), \quad (24) \]

where \( w_h = LF_0/48EI \) is the static deflection of a pure Ti-6Al-4V beam under the load \( F_0 = 100 \, \text{kN}, \) acting at the mid-span.

5.1. Formulation verification

Before investigating the vibration characteristics of the beam, the accuracy of the finite element model is firstly confirmed.

**Table 2.** Comparison of frequency parameter of sandwich beams with \( h_c/h_f = 8, L/h = 20 \)

| Distribution | Source | \( V_{\text{CNT}}' = 0.12 \) | \( V_{\text{CNT}}' = 0.17 \) | \( V_{\text{CNT}}' = 0.28 \) |
|--------------|--------|-----------------|-----------------|-----------------|
| FG-V         | Wu and Kitipornchai [12] | 0.1453 | 0.1588 | 0.1825 |
|              | Present | 0.1406 | 0.1545 | 0.1790 |
| UD           | Wu and Kitipornchai [12] | 0.1432 | 0.1560 | 0.1785 |
|              | Present | 0.1384 | 0.1517 | 0.1749 |

**Table 3.** Comparison of frequency parameter of sandwich beams with \( L/h = 20, \ V_{\text{CNT}}' = 0.17 \)

| Distribution | Source | \( h_c/h_f = 8 \) | \( h_c/h_f = 6 \) | \( h_c/h_f = 4 \) |
|--------------|--------|-----------------|-----------------|-----------------|
| FG-V         | Wu and Kitipornchai [12] | 0.1588 | 0.1642 | 0.1743 |
|              | Present | 0.1545 | 0.1608 | 0.1720 |
| UD           | Wu and Kitipornchai [12] | 0.1560 | 0.1599 | 0.1668 |
|              | Present | 0.1517 | 0.1563 | 0.1643 |
Table 4. Comparison of frequency parameter of sandwich beams with $V_{CNT}^* = 0.17, h_i/h_f = 8$

| Distribution | Source | $L/h=10$ | $L/h=20$ | $L/h=30$ |
|--------------|--------|----------|----------|----------|
| FG-V         | Wu and Kitipornchai [12] Present | 0.3124 | 0.1588 | 0.1062 |
|              | Present | 0.3119 | 0.1580 | 0.1056 |
| UD           | Wu and Kitipornchai [12] Present | 0.3070 | 0.1560 | 0.1043 |
|              | Present | 0.2995 | 0.1517 | 0.1014 |

Tables 2-4 compare the fundamental frequency parameter of the FG-CNTRC sandwich beam with the results of Wu and Kitipornchai [12], where the differential quadrature method has been employed. The frequency parameter is received for sandwich beam with two types of CNT distribution named as FG-V and UD. Very good agreement between the frequency parameter of the present work with that reported in [12] is obtained from Tables 2-4, regardless of the total CNT volume fraction $V_{CNT}^*$, ratio $h_i/h_f$ and aspect ratio $L/h$, noting that a Timoshenko beam theory is used to formulate governing equations in [12].

5.2. Natural frequencies

Table 5 lists fundamental frequency parameter of FG-CNTRC sandwich beam for five different types of CNT distribution. As seen from the Table, the frequency parameter $\mu$ increases with increasing the total CNTs volume fraction $V_{CNT}^*$, especially $\mu$ more significantly increases for a smaller $h_i/h_f$ ratio.

Table 5. The fundamental frequency parameter of sandwich beam for different $h_i/h_f$ and $L/h$ ratios

| $L/h$ | Type | $h_i/h_f = 8$ | $h_i/h_f = 6$ | $h_i/h_f = 4$ |
|-------|------|---------------|---------------|---------------|
|       | $V_{CNT}^*$ | $V_{CNT}^*$ | $V_{CNT}^*$ | $V_{CNT}^*$ |
| 5     | 0.12  | 0.17          | 0.28          | 0.12  | 0.17          | 0.28          | 0.12  | 0.17          | 0.28          |
|       | 0.28  | 0.3616        | 0.3616        | 0.28  | 0.3616        | 0.3616        | 0.28  | 0.3616        | 0.3616        |
| 10    | 0.2376 | 0.2994        | 0.3443        | 0.2376 | 0.2994        | 0.3443        | 0.2376 | 0.2994        | 0.3443        |
|       | 0.3443 | 0.3616        | 0.3616        | 0.3443 | 0.3616        | 0.3616        | 0.3443 | 0.3616        | 0.3616        |
| 30    | 0.1091 | 0.0993        | 0.1045        | 0.1091 | 0.0993        | 0.1045        | 0.1091 | 0.0993        | 0.1045        |
|       | 0.1045 | 0.1187        | 0.1187        | 0.1045 | 0.1187        | 0.1187        | 0.1045 | 0.1187        | 0.1187        |
| 20    | 0.1384 | 0.1516        | 0.1749        | 0.1384 | 0.1516        | 0.1749        | 0.1384 | 0.1516        | 0.1749        |
|       | 0.1749 | 0.1838        | 0.1838        | 0.1749 | 0.1838        | 0.1838        | 0.1749 | 0.1838        | 0.1838        |
| 10    | 0.1236 | 0.1487        | 0.1707        | 0.1236 | 0.1487        | 0.1707        | 0.1236 | 0.1487        | 0.1707        |
|       | 0.1707 | 0.1775        | 0.1775        | 0.1707 | 0.1775        | 0.1775        | 0.1707 | 0.1775        | 0.1775        |
| 5     | 0.2736 | 0.2994        | 0.3443        | 0.2736 | 0.2994        | 0.3443        | 0.2736 | 0.2994        | 0.3443        |
|       | 0.3443 | 0.3616        | 0.3616        | 0.3443 | 0.3616        | 0.3616        | 0.3443 | 0.3616        | 0.3616        |
| 5     | 0.2736 | 0.2994        | 0.3443        | 0.2736 | 0.2994        | 0.3443        | 0.2736 | 0.2994        | 0.3443        |
|       | 0.3443 | 0.3616        | 0.3616        | 0.3443 | 0.3616        | 0.3616        | 0.3443 | 0.3616        | 0.3616        |
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| FG-V  | 0.5325 | 0.5816 | 0.6650 | 0.5445 | 0.5602 | 0.7027 | 0.5686 | 0.6442 | 0.7640 |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| FG-Å  | 0.5160 | 0.5597 | 0.6347 | 0.5188 | 0.5709 | 0.6577 | 0.5233 | 0.5871 | 0.6899 |

Of the five types of the CNT distribution, the FG-V sandwich beam has the highest frequency, while the FG-Å sandwich beam gives the smallest result. It is easy to see that the results obtained for beam with UD, FG-X, FG-O distributions are very close together. Moreover, the decrease of the ratio \( h_1/h_f \) leads to the increase in the frequency parameter, especially at smaller values of the aspect ratio \( L/h \). This is because the sandwich beam gets higher stiffness corresponding to a smaller ratio \( h_1/h_f \). Besides, the influence of the aspect ratio \( L/h \) on the frequency parameter \( \mu \) is also seen in the Table 5, where the frequency parameter increases with decreasing aspect ratio \( L/h \).

5.3. Forced vibration

Figure 3 shows the time histories for dimensionless mid-span deflection of the sandwich beam for three types of CNT distribution, namely FG-V, UD and FG-Å. The figure is plotted with an aspect ratio \( L/h = 20 \), a total CNTs volume fraction \( V_{CNT}^* = 0.17 \), two values of the ratio \( h_1/h_f = 4 \), \( h_1/h_f = 8 \) and two values of the moving load velocity, \( v = 20 \text{ m/s}, v = 100 \text{ m/s} \).

![Figure 3. Time histories for dimensionless mid-span transverse displacement for different types of CNT distribution (\( L/h = 20, V_{CNT}^* = 0.17 \))](image_url)
The difference of three types of CNT distribution is clearly seen from the figure, especially for the smaller values of \( \frac{h_i}{h_j} \) (Figure 3a, c). The mid-span deflection of the sandwich beams corresponding to the distribution FG-V of CNT is the smallest, while that of the sandwich beam with FG-\( \Lambda \) type distribution is the highest. For both values of the \( \frac{h_i}{h_j} \) ratio, the mid-span deflection increases with increasing moving load velocity. In addition, the sandwich beam is subjected to more vibration cycles when it is under load with lower moving velocities. This can be explained by the lower ratio of the moving load speed to the critical speed as in case of the isotropic beams [19]. As can be observed from Figure 3, the mid-span deflection of the FG-V beam insignificantly increases when increasing the \( \frac{h_i}{h_j} \) ratio, regardless of the moving velocity.

*Table 6. Dynamic magnification factors for \( L/h = 20 \) and at different moving load velocities and \( \frac{h_i}{h_j} \) ratios*

| \( v \) (m/s) | Type | \( \frac{h_i}{h_j} = 8 \) | \( \frac{h_i}{h_j} = 6 \) | \( \frac{h_i}{h_j} = 4 \) |
|-------------|------|-----------------|-----------------|-----------------|
|             |      | \( V_{\text{CNT}}^* \) | \( V_{\text{CNT}}^* \) | \( V_{\text{CNT}}^* \) |
| 1.12        | UD   | 0.17            | 0.28            | 0.28            |
| 1.1925      | 0.9419 | 0.6827          | 1.2102          | 0.9284          | 0.6216          |
| 1.1900      | 0.9401 | 0.6808          | 1.2040          | 0.9241          | 0.6177          |
| 1.1949      | 0.9436 | 0.6846          | 1.2162          | 0.9327          | 0.6256          |
| 1.1285      | 0.8958 | 0.6345          | 1.0993          | 0.8492          | 0.5669          |
| 1.2622      | 0.9916 | 0.7363          | 1.3396          | 1.0171          | 0.7070          |
| 1.2138      | 0.9913 | 0.7506          | 1.2162          | 0.9907          | 0.6965          |
| 1.2107      | 0.9897 | 0.7487          | 1.2089          | 0.9868          | 0.6925          |
| 1.2167      | 0.9928 | 0.7523          | 1.2234          | 0.9946          | 0.7005          |
| 1.1357      | 0.9502 | 0.7036          | 1.1327          | 0.9173          | 0.6247          |
| 1.3010      | 1.0335 | 0.8019          | 1.3753          | 1.0703          | 0.7810          |
| 1.2070      | 0.9905 | 0.7892          | 1.2162          | 0.9907          | 0.6965          |
| 1.2055      | 0.9898 | 0.7882          | 1.2107          | 0.9897          | 0.7487          |
| 1.2085      | 0.9912 | 0.7901          | 1.2167          | 0.9928          | 0.7523          |
| 1.1558      | 0.9649 | 0.7564          | 1.1357          | 0.9502          | 0.7036          |
| 1.2621      | 1.0164 | 0.8239          | 1.3010          | 1.0335          | 0.8019          |
| 1.7810      | 1.4380 | 1.0152          | 1.8025          | 1.3967          | 0.9341          |
| 1.7791      | 1.4361 | 1.0134          | 1.7987          | 1.3932          | 0.9311          |
| 1.7828      | 1.4399 | 1.0168          | 1.8061          | 1.4002          | 0.9370          |
| 1.7180      | 1.3743 | 0.9576          | 1.7053          | 1.3052          | 0.8587          |
| 1.8479      | 1.5068 | 1.0785          | 1.9093          | 1.4996          | 1.0211          |

*Table 7. Dynamic magnification factors for \( L/h = 20 \) and at different moving load velocities and \( \frac{h_i}{h_j} \) ratios*

| \( v \) (m/s) | Type | \( \frac{h_i}{h_j} = 8 \) | \( \frac{h_i}{h_j} = 6 \) | \( \frac{h_i}{h_j} = 4 \) |
|-------------|------|-----------------|-----------------|-----------------|
|             |      | \( V_{\text{CNT}}^* \) | \( V_{\text{CNT}}^* \) | \( V_{\text{CNT}}^* \) |
| 0.12        | UD   | 0.17            | 0.28            | 0.28            |
| 1.2506      | 1.0266 | 0.7675          | 1.2844          | 1.0090          | 0.7259          |
| 1.2479      | 1.0239 | 0.7647          | 1.2775          | 1.0024          | 0.7205          |
| 1.2533      | 1.0295 | 0.7700          | 1.2911          | 1.0159          | 0.7313          |
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|      | FG-V | 1.1933 | 1.0015 | 0.7680 | 1.1955 | 0.9253 | 0.6593 |
|------|------|--------|--------|--------|--------|--------|--------|
|      | FG-A | 1.2741 | 1.0839 | 0.8442 | 1.3166 | 1.3983 | 1.1137 | 0.8097 |
| 50   | UD   | 1.2683 | 1.0724 | 0.8245 | 1.2914 | 0.7873 | 1.3255 | 1.0224 | 0.7336 |
|      | FG-X | 1.2669 | 1.0707 | 0.8229 | 1.2886 | 1.0481 | 1.3183 | 1.0171 | 0.7274 |
|      | FG-O | 1.2697 | 1.0741 | 0.8259 | 1.2940 | 1.0546 | 0.7902 | 1.3325 | 1.0276 | 0.7400 |
|      | FG-V | 1.2323 | 1.0292 | 0.7887 | 1.2339 | 0.9871 | 1.7350 | 1.2215 | 0.9464 | 0.6731 |
|      | FG-A | 1.3052 | 1.1180 | 0.8616 | 1.3514 | 1.1216 | 0.8438 | 1.4395 | 1.1408 | 0.8285 |
| 100  | UD   | 1.2662 | 1.1049 | 0.8656 | 1.2923 | 1.0982 | 0.8178 | 1.3409 | 1.0884 | 0.7540 |
|      | FG-X | 1.2644 | 1.1035 | 0.8638 | 1.2903 | 1.0955 | 0.8145 | 1.3356 | 1.0817 | 0.7472 |
|      | FG-O | 1.2679 | 1.1063 | 0.8673 | 1.2944 | 1.1009 | 0.8209 | 1.3462 | 1.0951 | 0.7607 |
|      | FG-V | 1.2338 | 1.0697 | 0.8238 | 1.2467 | 1.0446 | 0.7590 | 1.2568 | 0.9959 | 0.6741 |
|      | FG-A | 1.3190 | 1.1416 | 0.9100 | 1.3599 | 1.1553 | 0.8830 | 1.4392 | 1.1908 | 0.8580 |

Figure 4. Variation of dynamic magnification factor with type of CNT distribution (L/h=20)

Tables 6-7 present the dynamic magnification factor $D_d$ of FG-CNTRC sandwich beam for five different types of CNT distribution and different moving load velocities. The aspect ratios $L/h = 20$ and $L/h = 5$ are respectively used in the computation in two tables. It is observed from the tables that the factor $D_d$ decreases with increasing the total CNTs volume fraction $V_{CNT}^*$, regardless of the CNT distribution. This decrease is observed more clearly when the moving load velocity is greater and the $h_i/h_f$ ratio is smaller. Similar to the frequency parameter, the factor $D_d$ obtained for three types of CNT distribution, UD, FG-X and FG-O, are
quite close together, while the factors $D_d$ received from FG-V and FG-Λ distributions are the smallest and highest, respectively.

In the Figure 4, the relation between the dynamic magnification factor $D_d$ and the moving load speed $v$ is illustrated for FG-V, UD and FG-Λ sandwich beams. Two values of the total CNTs volume fraction $V_{CNT}^*=0.12$, $V_{CNT}^*=0.28$, and two values of the ratio $h_i/h_f = 4$ and $h_i/h_f = 8$ are chosen to plot the figure. Once again, we can see that when $V_{CNT}$ is higher and the $h_i/h_f$ ratio is smaller, the factor $D_d$ received from the beams corresponding to the three types of CNT distribution is markedly different. Moreover, for the beams with $V_{CNT}^*=0.28$, the factor $D_d$ significantly increases with increasing the $h_i/h_f$ ratio and this is clearly observed for all FG-V, UD and FG-Λ sandwich beams. Especially, the dynamic magnification factor $D_d$ achieves the maximum value at a higher moving load speed for the beam having a lower $h_i/h_f$ ratio and higher $V_{CNT}^*$.

6. CONCLUSIONS

Vibration of FG-CNTRC sandwich beams under a moving load with five different types of CNT distribution has been presented. The sandwich beam consists of three layers, a homogeneous core and two face sheets of CNTRC materials. The effective properties of two CNTRC faces are determined by the extended rule of mixture. Based on the third-order shear deformation theory, a finite element formulation has been derived and employed to investigate vibration characteristics of the beam. The obtained numerical results show that CNT distribution has an important influence on the vibration behavior of the beam. An increase of the total volume fraction of CNTs leads to an increase in the frequency parameter and a decrease in the dynamic magnification factor, and the effect of CNT volume fraction is more significant for the beam with a smaller $h_i/h_f$ ratio. This study also shows that among the five types of the CNT distribution, the sandwich beam with FG-V type of CNT distribution has the lowest dynamic magnification factor while its fundamental frequency is the highest. Noting that although the numerical investigation presented here is only for simply supported beams, the formulation derived in the present work can also be used for FG-CNTRC sandwich beam analysis with other boundary conditions.

CRediT authorship contribution statement. Dr. Thi Thom Tran developed the computer code used in analysis and prepared the manuscript. Master Thi Hien Trinh performed the computations. Dr. Dinh Kien Nguyen set up the problem, checked the results and final manuscript.

Declaration of competing interest. We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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