Abstract

We give necessary and sufficient geometric conditions for a distribution (or a Pfaffian system) to be locally equivalent to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$. We study the geometry of that class of systems, in particular, the existence of corank one involutive subdistributions. We also distinguish regular points, at which the system is equivalent to the canonical contact system, and singular points, at which we propose a new normal form that generalizes the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$ in a way analogous to that how Kumpera-Ruiz normal form generalizes the canonical contact system on $J^n(\mathbb{R}, \mathbb{R})$, which is also called Goursat normal form.

Keywords: Contact systems, involutive subdistributions, Pfaffian systems, Kumpera-Ruiz normal forms, Goursat normal form, jet spaces.

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Introduction

Consider $J^n(R^k, R^m)$, the space of $n$-jets of smooth maps from $R^k$ into $R^m$, and denote by $(q^1, \ldots, q^k, u_1, \ldots, u_m, p^\sigma_i)$, for $1 \leq i \leq m$ and for $1 \leq |\sigma| \leq n$, the canonical coordinates, also called natural coordinates, on this space (see e.g. [3], [22], and [26]), where $q^j$, for $1 \leq j \leq k$, represent independent variables and $u_i$, for $1 \leq i \leq m$, represent dependent variables; the vector of non-negative integers $\sigma = (\sigma_1, \ldots, \sigma_k)$ is a multi-index such that $|\sigma| = \sigma_1 + \cdots + \sigma_k \leq n$; and $p^\sigma_i$, for $1 \leq i \leq m$, correspond to the partial derivatives $\partial^{(|\sigma|)}u_i/\partial q_\sigma$. Any smooth map $\varphi$ from $R^k$ into $R^m$ defines a submanifold in $J^n(R^k, R^m)$ by the relations

$$p^\sigma_i = \frac{\partial^{(|\sigma|)}\varphi_i}{\partial q_\sigma}(q^1, \ldots, q^k),$$

for $1 \leq i \leq m$ and $1 \leq |\sigma| \leq n$. This submanifold is called the $n$-graph of $\varphi$. It turns out that all $n$-graphs are integral submanifolds, of dimension $k$, of a distribution called the canonical contact system on $J^n(R^k, R^m)$ or the Cartan distribution on $J^n(R^k, R^m)$. The Pfaffian system that annihilates this distribution, which is also called the canonical contact system (see e.g. [3] and [22]), is given in the canonical coordinates of $J^n(R^k, R^m)$ by

$$dp^\sigma_i - \sum_{j=1}^k p^\sigma_i p^{\sigma+1}_j dq^j = 0,$$

for $1 \leq i \leq m$ and for $1 \leq |\sigma| \leq n - 1$, where $\sigma + 1_j = (\sigma_1, \ldots, \sigma_j + 1, \ldots, \sigma_k)$.

The above description explains the importance of contact systems in geometric theory of (partial) differential equations and in differential geometry. In the former, a (partial) differential equation is interpreted as a submanifold in $J^n(R^k, R^m)$ and thus it is natural to study the geometry of pairs consisting of a contact system and a submanifold, see e.g. [3], [17], and [26]. In the latter, contact systems describe, for instance, diffeomorphisms which preserve the $n$-graphs of applications (for example, $n$-graphs of curves in the case $k = 1$), see e.g. [3] and [22].

A natural problem which arises is to characterize those distributions which are (locally) equivalent to a canonical contact system. This problem was posed by
Pfaff [24] in 1815 and seems still to be open in its full generality although many important particular solutions have been obtained. In the case \( n = 1 \), with \( m = 1 \) and an arbitrary \( k \) the final solution has been obtained by Darboux [10] in his famous theorem generalizing earlier results of Pfaff [24] and Frobenius [12]. The case \( n = 2, m = 1 \) and \( k = 1 \) was solved by Engel in [11]. The case \( n \geq 2, m = 1 \) and \( k = 1 \) was solved by E. von Weber [27], Cartan [6] and Goursat [14] (at generic points) and by Libermann [16], Kumpera and Ruiz [14], and Murray [21] (at an arbitrary point). The case \( n = 1, k \) and \( m \) arbitrary has been studied and solved by Bryant [2] (see also [3]).

This paper is devoted to the problem of when a given distribution is locally equivalent to the canonical contact system in the case \( k = 1, n \) and \( m \) arbitrary, that is to the canonical contact system for curves. This problem has been studied by Gardner and Shadwick [13], Murray [21], and Tilbury and Sastry [25] (as a particular case of the the problem of equivalence to the so-called extended Goursat normal form). Their solutions are based on a result of [13] that assures the equivalence provided that a certain differential form satisfies precise congruence relations. The problem of how to verify the existence of such a form had apparently remained open. This difficulty was solved by Aranda-Bricaire and Pomet [1], who proposed an algorithm which determines the existence of such a form. Their solution, although being elegant and checkable, uses the formalism of infinite dimensional manifolds and thus goes away from classical results characterizing contact systems.

The aim of our paper is two-fold. Firstly, in Theorem 1.1, we give geometric checkable conditions, based on the classical notion of Engel’s rank, which characterize regular contact systems for curves, i.e., for \( k = 1 \) and arbitrary \( n \) and \( m \). Secondly, we extend our approach to singular points and we prove, in Theorem 3.2, that any singular contact system can be put to a normal form for which we propose the name extended Kumpera-Ruiz normal form. That form generalizes canonical contact systems for curves in a way analogous to that how Kumpera-Ruiz normal form (see [15]; compare [5], [1], [4], [19], [20], and [23]) generalizes Goursat normal form, that is the canonical contact system on \( J^n(\mathbb{R}, \mathbb{R}) \). When checking conditions of Theorem 1.1 we must determine whether a given distribution possesses a corank one involutive subdistribution. An elegant answer given to this problem by Bryant [2] implies that our conditions become checkable.
The paper is organized as follows. In Section 1 we define the canonical system for curves and we give the first main result of the paper, Theorem 1.1, characterizing distributions that are locally equivalent to a canonical contact system for curves. In Section 2 we discuss the problem of whether a given distribution possesses a corank one involutive subdistribution and we recall Bryant’s solution of this problem (see also Appendices A and B). In Section 3 we introduce extended Kumpe-Ruiz normal forms and give the second main result of the paper, Theorem 3.2, stating that any singular contact system is locally equivalent to an extended Kumpe-Ruiz normal form. Finally, Section 4 contains proofs of all our results.

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1 The Canonical Contact System for Curves

A rank $k$ distribution $\mathcal{D}$ on a smooth manifold $M$ is a map that assigns smoothly to each point $p$ in $M$ a linear subspace $\mathcal{D}(p) \subset T_p M$ of dimension $k$. In other words, a rank $k$ distribution is a smooth rank $k$ subbundle of the tangent bundle $TM$. Such a field of tangent $k$-planes is spanned locally by $k$ pointwise linearly independent smooth vector fields $f_1, \ldots, f_k$ on $M$, which will be denoted by $\mathcal{D} = (f_1, \ldots, f_k)$. Two distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ defined on two manifolds $M$ and $\tilde{M}$, respectively, are equivalent if there exists a smooth diffeomorphism $\varphi$ between $M$ and $\tilde{M}$ such that $(\varphi_*, \mathcal{D})(\tilde{p}) = \tilde{\mathcal{D}}(\tilde{p})$, for each point $\tilde{p}$ in $\tilde{M}$.

The derived flag of a distribution $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \cdots$ defined inductively by

$$\mathcal{D}^{(0)} = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}], \quad \text{for } i \geq 0. \quad (1)$$

The Lie flag is the sequence of modules of vector fields $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by

$$\mathcal{D}_0 = \mathcal{D} \quad \text{and} \quad \mathcal{D}_{i+1} = \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \quad \text{for } i \geq 0. \quad (2)$$

In general, the derived and Lie flags are different; though for any point $p$ in the underlying manifold the inclusion $\mathcal{D}_i(p) \subset \mathcal{D}^{(i)}(p)$ clearly holds, for $i \geq 0$. 

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For a given distribution $D$, defined on a manifold $M$, we will say that a point $p$ of $M$ is a regular point of $D$ if all the elements $D_i$ of its Lie flag have constant rank in a small enough neighborhood of $p$. A distribution $D$ is said to be completely nonholonomic if, for each point $p$ in $M$, there exists an integer $N(p)$ such that $D_{N(p)}(p) = T_p M$. A distribution $D$ is said to be involutive if its first derived system satisfies $D^{(1)} = D^{(0)}$.

An alternative description of the above defined objects can also be given using the dual language of differential forms. A Pfaffian system $I$ of rank $s$ on a smooth manifold $M$ is a map that assigns smoothly to each point $p$ in $M$ a linear subspace $I(p) \subset T^*_p M$ of dimension $s$. In other words, a Pfaffian system of rank $s$ is a smooth subbundle of rank $s$ of the cotangent bundle $T^* M$. Such a field of cotangent $s$-planes is spanned locally by $s$ pointwise linearly independent smooth differential 1-forms $\omega_1, \ldots, \omega_s$ on $M$, which will be denoted by $I = (\omega_1, \ldots, \omega_s)$. Two Pfaffian systems $I$ and $\tilde{I}$ defined on two manifolds $M$ and $\tilde{M}$, respectively, are equivalent if there exists a smooth diffeomorphism $\varphi$ between $M$ and $\tilde{M}$ such that $I(p) = (\varphi^* \tilde{I})(p)$, for each point $p$ in $M$.

For a Pfaffian system $I$, we can define its derived flag $I^{(0)} \supset I^{(1)} \supset \cdots$ by the relations $I^{(0)} = I$ and $I^{(i+1)} = \{ \alpha \in I^{(i)} : d\alpha \equiv 0 \mod I^{(i)} \}$, for $i \geq 0$, provided that each element $I^{(i)}$ of this sequence has constant rank. In this case, it is immediate to see that the derived flag of the distribution $D = I^\perp$ coincides with the sequence of distributions that anihilate the elements of the derived flag of $I$, that is

$$D^{(i)} = (I^{(i)})^\perp, \quad \text{for } i \geq 0.$$  

For a given Pfaffian system $I$, we will say that a point $p$ of $M$ is a regular point if $p$ is a regular point for the distribution $D = I^\perp$, that is if all elements $D_i$ of the Lie flag are of constant rank in a small enough neighborhood of $p$.

Consider the space $J^n(\mathbb{R}, \mathbb{R}^m)$ of jets of order $n \geq 1$ of functions from $\mathbb{R}$ into $\mathbb{R}^m$. This space is diffeomorphic to $\mathbb{R}^{(n+1)m+1}$. The canonical coordinates associated to $\mathbb{R}$ (denoted by $x_0^0$) and to $\mathbb{R}^m$ (denoted by $x_0^0, \ldots, x_m^0$) can be used to define the canonical coordinates on $J^n(\mathbb{R}, \mathbb{R}^m)$, which will be denoted by

$$x_0^0, x_1^0, \ldots, x_m^0, x_1^1, \ldots, x_m^1, \ldots, x_1^n, \ldots, x_m^n,$$

with obvious indentifications $x_0^0 = q$ and $x_i^0 = u_i$, for $1 \leq i \leq m$, and $x_i^j = p_i^j$, for
$1 \leq i \leq m$ and $1 \leq j \leq n$ (see the beginning of the Introduction). Observe that any smooth map $\varphi$ from $\mathbb{R}$ into $\mathbb{R}^m$ defines a curve in $J^n(\mathbb{R}, \mathbb{R}^m)$ by the relations $x^i_j = \varphi_j^{(i)}(x^0_0)$, for $0 \leq i \leq n$ and $1 \leq j \leq m$, where $\varphi_j^{(i)}$ denotes the $i$-th derivative with respect to $x^0_0$ of the $j$-th component of $\varphi$. This curve is called the $n$-graph of $\varphi$. It is clear that not all curves in $J^n(\mathbb{R}, \mathbb{R}^m)$ are $n$-graphs of maps. In order to distinguish the “good” curves from the “bad” ones, we should introduce a set of constraints on the velocities of curves in $J^n(\mathbb{R}, \mathbb{R}^m)$. In other words, we should endow $J^n(\mathbb{R}, \mathbb{R}^m)$ with a nonholonomic structure.

The canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$ is the completely nonholonomic distribution spanned by the following family of vector fields:

$$\left( \frac{\partial}{\partial x^1_0}, \ldots, \frac{\partial}{\partial x^m_0}, \sum_{i=0}^{n-1} \sum_{j=1}^{m} x^{i+1}_j \frac{\partial}{\partial x^i_j} + \frac{\partial}{\partial x^0_0} \right).$$

By definition, if a curve in $J^n(\mathbb{R}, \mathbb{R}^m)$ is the $n$-graph of some map then it is an integral curve of the canonical contact system. More precisely, a section $\sigma : \mathbb{R} \to J^n(\mathbb{R}, \mathbb{R}^m)$ is the $n$-graph of a curve $\varphi : \mathbb{R} \to \mathbb{R}^n$ if and only if it is an integral curve of the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$ (see e.g. [22]).

The aim of our paper is to give a complete answer to the question “Which distributions are locally equivalent to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$?”. The following result will be the starting point of our study.

**Theorem 1.1 (contact systems for curves)** A rank $m+1$ distribution $\mathcal{D}$ on a manifold $M$ of dimension $(n+1)m+1$ is equivalent, in a small enough neighborhood of any point $p$ in $M$, to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$ if and only if the two following conditions hold, for $0 \leq i \leq n$.

(i) Each element $\mathcal{D}^{(i)}$ of the derived flag has constant rank $(i+1)m+1$ and contains an involutive subdistribution $\mathcal{L}_i \subset \mathcal{D}^{(i)}$ that has constant corank one in $\mathcal{D}^{(i)}$.

(ii) Each element $\mathcal{D}_i$ of the Lie flag has constant rank $(i+1)m+1$.

This result yields a constructive test for the local equivalence to the canonical contact system for curves, provided that we know how to check whether or not a given distribution admits a corank one involutive subdistribution. We give in the next section a checkable necessary and sufficient condition for the existence of such a distribution. The proof of Theorem 1.1 will be given in Section 4.
2 Corank One Involutive Subdistributions

The aim of this Section is to give an answer to the following question: “When does a given constant rank distribution $D$ contain an involutive subdistribution $L \subset D$ that has constant corank one in $D$?” In fact, the answer to this question is an immediate consequence of a result contained in Bryant’s Ph.D. thesis [2]. Links between Bryant’s result and the characterization of the canonical contact system for curves have also been observed by Aranda-Bricaire and Pomet [1].

A characteristic vector field of a distribution $D$ is a vector field $f$ that belongs to $D$ and satisfies $[f, D] \subset D$. The characteristic distribution of $D$, which will be denoted by $C$, is the module spanned by all its characteristic vector fields. It follows directly from the Jacobi identity that the characteristic distribution is always involutive. For a constant rank Pfaffian system $I$, the characteristic distribution of $I^\perp$ is often called the Cartan system of $I$. We refer the reader to [3] for a definition of the Cartan system given in the language of Pfaffian systems.

The Engel rank [3] of a Pfaffian system $I$, at a point $p$, is the largest integer $\rho$ such that there exists a 1-form $\omega$ in $I$ for which we have $(d\omega)^\rho(p) \neq 0 \mod I$. The Engel rank of a constant rank distribution $D$ will be, by definition, the Engel rank of its anihilator $D^\perp$. Obviously, the Engel rank $\rho$ of a distribution equals zero at each point if and only if the distribution is involutive.

We will now give an equivalent definition of the Engel rank in the language of vector fields, in the particular case when $\rho = 1$, which will be important in the paper. Let $D$ be a distribution such that $D^{(0)}$ and $D^{(1)}$ have constant ranks $d_0$ and $d_1$, respectively, and denote $r_0(p) = d_1(p) - d_0(p)$. Assume that $d_0 \geq 2$ and $r_0 \geq 1$. Take a family of vector fields

$$(f_1, \ldots, f_{d_0}, g_1, \ldots, g_{r_0})$$

such that $D^{(0)} = (f_1, \ldots, f_{d_0})$ and $D^{(1)} = (f_1, \ldots, f_{d_0}, g_1, \ldots, g_{r_0})$. The structure functions $c_{ij}^k$ associated to those generators, for $1 \leq i < j \leq d_0$ and $1 \leq k \leq r_0$, are the smooth functions defined by the following relations:

$$[f_i, f_j] = \sum_{k=1}^{r_0} c_{ij}^k g_k \mod D^{(0)}, \quad \text{for } 1 \leq i < j \leq d_0.$$ 

It is important to point out that the structure functions are not invariantly related to the distribution $D$, since they depend on the choice of generators.
Assume that the Engel rank $\rho$ of $\mathcal{D}$ is constant and that $r_0 \geq 1$. It is easy to check that $\rho = 1$ if and only if either $d_0 = 2$, or $d_0 = 3$, or $d_0 \geq 4$ and the structure functions satisfy the relations

$$c_{ij}^p c_{kl}^q - c_{ik}^p c_{jl}^q + c_{jk}^p c_{il}^q - c_{jl}^p c_{ik}^q + c_{kl}^p c_{ij}^q = 0,$$

(4)

for each sextuple $(i,j,k,l,p,q)$ of integers such that $1 \leq i < j < k < l \leq d_0$ and $1 \leq p \leq r_0$ and $1 \leq q \leq r_0$.

The following result is a particular case of Bryant’s algebraic Lemma 2 (see also 3), which is the cornerstone of Bryant’s characterization of the canonical contact system on $J^1(\mathbb{R}^k, \mathbb{R}^m)$.

**Lemma 2.1 (Bryant)** Let $\mathcal{D}$ be a distribution such that $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ have constant ranks $d_0$ and $d_1$, respectively. Assume that $r_0 \geq 1$. Then the two following conditions are equivalent:

(i) The characteristic distribution $\mathcal{C}$ of $\mathcal{D}$ has constant rank $c_0 = d_0 - r_0 - 1$ and the Engel rank $\rho$ of $\mathcal{D}$ is constant and equals 1;

(ii) The distribution $\mathcal{D}$ contains a subdistribution $\mathcal{B} \subset \mathcal{D}$ that has constant corank one in $\mathcal{D}$ and satisfies $[\mathcal{B}, \mathcal{B}] \subset \mathcal{D}$.

Observe that if the first condition is satisfied then we must necessarily have $r_0 \leq d_0 - 1$. The following result is included in the proof of Bryant’s 2 normal form Theorem (see also 3). In Appendix A we give an alternative proof of its surprising Item (iii); our proof explains the role of the assumption $r_0 \geq 3$ by relating it to the Jacobi identity.

**Lemma 2.2 (Bryant)** Let $\mathcal{D}$ be a distribution such that $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ have constant ranks $d_0$ and $d_1$, respectively. Assume that the distribution $\mathcal{D}$ contains a subdistribution $\mathcal{B} \subset \mathcal{D}$ that has constant corank one in $\mathcal{D}$ and satisfies $[\mathcal{B}, \mathcal{B}] \subset \mathcal{D}$.

(i) If $r_0 = 1$ then the distribution $\mathcal{D}$ contains an involutive subdistribution $\mathcal{L} \subset \mathcal{D}$ that has constant corank one in $\mathcal{D}$;

(ii) If $r_0 \geq 2$ then $\mathcal{B}$ is unique;

(iii) If $r_0 \geq 3$ then $\mathcal{B}$ is involutive.

Observe that, in the first item of the above Lemma, the involutive subdistribution $\mathcal{L}$ can be different from $\mathcal{B}$, which is not necessarily involutive. The following result
is a direct consequence of Bryant’s work. In order to avoid the trivial case \( r_0 = 0 \), for which the existence of a corank one involutive subdistribution is obvious, we will assume that \( r_0 \geq 1 \).

**Corollary 2.3 (corank one involutive subdistributions)** Let \( \mathcal{D} \) be a distribution such that \( \mathcal{D}^{(0)} \) and \( \mathcal{D}^{(1)} \) have constant ranks \( d_0 \) and \( d_1 \), respectively. Assume that \( r_0 \geq 1 \). Then, the distribution \( \mathcal{D} \) contains an involutive subdistribution \( \mathcal{L} \subset \mathcal{D} \) that has constant corank one in \( \mathcal{D} \) if and only if the three following conditions hold:

(i) The characteristic distribution \( \mathcal{C} \) of \( \mathcal{D} \) has constant rank \( c_0 = d_0 - r_0 - 1 \);

(ii) The Engel rank \( \rho \) of \( \mathcal{D} \) is constant and equals 1;

(iii) If \( r_0 = 2 \) then, additionally, the unique corank one subdistribution \( \mathcal{B} \subset \mathcal{D} \) such that \([\mathcal{B}, \mathcal{B}] \subset \mathcal{D}\) must be involutive.

We would like to emphasize that the above conditions are easy to verify, as well as the conditions of Corollary 2.4 below. Indeed, for any distribution, or the corresponding Pfaffian system, we can compute the characteristic distribution \( \mathcal{C} \) and check whether or not the Engel rank equals 1 using, respectively, the formula (12) and the condition (13) of Appendix B. This gives the solution if \( r_0 \neq 2 \). If \( r_0 = 2 \) we have additionally to check the involutivness of the unique distribution \( \mathcal{B} \) satisfying \([\mathcal{B}, \mathcal{B}] \subset \mathcal{D}\), whose explicit construction is also given in Appendix B.

Combining Theorem 1.1 and Corollary 2.3 we get the following characterization of the canonical system on \( J^n(\mathbb{R}, \mathbb{R}^m) \), using the language of Pfaffian systems.

**Corollary 2.4 (contact systems for curves)** Let \( \mathcal{I} \) be a Pfaffian system of rank \( nm \), defined on a manifold \( M \) of dimension \((n + 1)m + 1\). If \( m \neq 2 \), the Pfaffian system \( \mathcal{I} \) is locally equivalent, at a given point \( p \) of \( M \), to the canonical contact system on \( J^n(\mathbb{R}, \mathbb{R}^m) \) if and only if

(i) The rank of each derived system \( \mathcal{I}^{(i)} \) is constant and equals \((n - i)m\), for \( 0 \leq i \leq n \);

(ii) The Engel rank of \( \mathcal{I}^{(i)} \) is constant and equals 1, for \( 0 \leq i \leq n \);

(iii) The rank of each Cartan system \( \mathcal{C}(\mathcal{I}^{(i)}) \) is constant and equals \((n + 1 - i)m + 1\), for \( 0 \leq i \leq n - 1 \);

(iv) The point \( p \) is a regular point for \( \mathcal{I} \).
In other words, if \( m \neq 2 \), the characterization of the canonical contact system on \( J^n(\mathbb{R}, \mathbb{R}^m) \) turns out to be a natural combination of that given for \( J^1(\mathbb{R}, \mathbb{R}^m) \) by Bryant (see [2] and [3]) and that given for \( J^n(\mathbb{R}, \mathbb{R}) \) by Murray [21].

### 3 Extended Kumpera-Ruiz Normal Forms

The aim of this Section is to study the class of distributions that satisfy condition (i) of Theorem 1.1 but fail to satisfy condition (ii) of that theorem. The fist condition describes the geometry of the canonical contact system while the second condition characterizes regular points. In this sense, systems that satisfy the former but fail to satisfy the latter can be considered as “singular” contact systems for curves. We will show that any such distribution can be brought to a normal form for which we propose the name extended Kumpera-Ruiz normal form. Those forms generalize the canonical contact system on \( J^1(\mathbb{R}, \mathbb{R}^m) \) in a way analogous to that how Kumpera-Ruiz normal forms (see e.g. [15], [19], [20], and [23]) generalize the canonical system on \( J^1(\mathbb{R}, \mathbb{R}) \), which is also called Goursat normal form.

Consider the family of vector fields \( \kappa^1_1 = (\kappa^1_1, \ldots, \kappa^1_m, \kappa^1_0) \) that span the canonical contact system on \( J^1(\mathbb{R}, \mathbb{R}^m) \), where
\[
\kappa^1_1 = \frac{\partial}{\partial x^1}, \ldots, \kappa^1_m = \frac{\partial}{\partial x^m}, \\
\kappa^1_0 = x^1 \frac{\partial}{\partial x^1} + \cdots + x^m \frac{\partial}{\partial x^m} + \frac{\partial}{\partial x^0},
\]
and the the family of vector fields \( \kappa^2_1 = (\kappa^2_1, \ldots, \kappa^2_m, \kappa^2_0) \) that spans the canonical contact system on \( J^2(\mathbb{R}, \mathbb{R}^m) \), where
\[
\kappa^2_1 = \frac{\partial}{\partial x^1}, \ldots, \kappa^2_m = \frac{\partial}{\partial x^m}, \\
\kappa^2_0 = x^2 \frac{\partial}{\partial x^1} + \cdots + x^m \frac{\partial}{\partial x^m} + x^1 \frac{\partial}{\partial x^1} + \cdots + x^m \frac{\partial}{\partial x^m} + \frac{\partial}{\partial x^0}.
\]
Loosely speaking, we can write
\[
\kappa^2_1 = \frac{\partial}{\partial x^1}, \ldots, \kappa^2_m = \frac{\partial}{\partial x^m}, \\
\kappa^2_0 = x^1 \kappa^1_1 + \cdots + x^m \kappa^1_m + \kappa^1_0.
\]
In order to make this precise we will adopt the following natural notation. Consider an arbitrary vector field \( f \) given on \( J^{n-1}(\mathbb{R}, \mathbb{R}^m) \) by
\[
f = \sum_{i=0}^{n-1} \sum_{j=1}^{m} f^i_j (x^{n-1}) \frac{\partial}{\partial x^j} + f^0_0 (x^{n-1}) \frac{\partial}{\partial x^0},
\]
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where $\mathbb{R}^{n-1}$ denotes the coordinates $x^0_0, x^0_1, \ldots, x^0_m; x^1_0, \ldots, x^1_m, \ldots, x^{n-1}_1, \ldots, x^{n-1}_m$ of $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. We can lift the vector field $f$ to a vector field on $J^n(\mathbb{R}, \mathbb{R}^m)$, which we also denote by $f$, by taking

$$f = \sum_{i=0}^{n-1} \sum_{j=1}^m f_j(x^{n-1}) \frac{\partial}{\partial x_j} + f_0(x^{n-1}) \frac{\partial}{\partial x_0} + 0 \cdot \frac{\partial}{\partial x_1} + \cdots + 0 \cdot \frac{\partial}{\partial x_m}.$$  

That is, we lift $f$ by translating it along the directions $\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_m}$.

**Notation 3.1 (lifts of vector fields)** From now on, in any expression of the form $c^i(x)\kappa^i$, the vector fields $\kappa^0, \ldots, \kappa^{n-1}$ should be considered as their above defined lifts.

Let $\kappa^{n-1} = (\kappa^n_0, \ldots, \kappa^n_m, \kappa^n_0)$ denote a family of vector fields defined on $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. A *regular prolongation*, with a parameter $c^n$, of $\kappa^{n-1}$, denoted by $\kappa^n = R_{c^n}(\kappa^{n-1})$, is a family of vector fields $\kappa^n = (\kappa^n_0, \ldots, \kappa^n_m, \kappa^n_0)$ defined on $J^n(\mathbb{R}, \mathbb{R}^m)$ by

$$\kappa^n_0 = (x^n_0 + c^n_0)\kappa^{n-1}_0 + \cdots + (x^n_m + c^n_m)\kappa^{n-1}_m + \kappa^{n-1}_0,$$

where $c^n = (c^n_0, \ldots, c^n_m)$ is a vector of $m$ real constants. A *singular prolongation*, with a parameter $c^n$, of $\kappa^{n-1}$, denoted by $\kappa^n = S_{c^n}(\kappa^{n-1})$, is a family of vector fields $\kappa^n = (\kappa^n_0, \ldots, \kappa^n_m, \kappa^n_0)$ defined on $J^n(\mathbb{R}, \mathbb{R}^m)$ by

$$\kappa^n_0 = (x^n_0 + c^n_0)\kappa^{n-1}_0 + \cdots + (x^n_{m-1} + c^n_{m-1})\kappa^{n-1}_{m-1} + \kappa^{n-1}_m + x^n_m\kappa^{n-1}_0,$$

where $c^n = (c^n_0, \ldots, c^n_{m-1}, 0)$ is a vector of $m$ real constants, the last one being zero.

A family of vector fields $\kappa^n$ on $J^n(\mathbb{R}, \mathbb{R}^m)$, for $n \geq 1$, will be called an *extended Kumpera-Ruiz normal form* if $\kappa^n = \sigma_n \circ \cdots \circ \sigma_2(\kappa^1)$, where for each $2 \leq i \leq n$ the map $\sigma_i$ equals either $R_{c^i}$ or $S_{c^i}$, for some vector parameters $c^i$. In other words, a Kumpera-Ruiz normal form is a family of vector fields obtained by successive prolongations from the family of vector fields that spans the canonical contact system on $J^1(\mathbb{R}, \mathbb{R}^m)$.

The above defined prolongations and prolongations-based definition of extended Kumpera-Ruiz normal forms generalizes for contact systems analogous operations introduced by the authors [23] for Goursat structures.
Let \( x : M \to \mathbb{R}^{(n+1)m+1} \cong J^n(\mathbb{R}, \mathbb{R}^m) \) be a local coordinate system on a manifold \( M \), in a neighborhood of a given point \( p \) in \( M \). We will say that an extended Kumera-Ruiz normal form on \( J^n(\mathbb{R}, \mathbb{R}^m) \), defined in \( x \)-coordinates, is centered at \( p \) if we have \( x(p) = 0 \). For example, on \( J^2(\mathbb{R}, \mathbb{R}^2) \), we have the two following extended Kumera-Ruiz normal forms

\[
\left( \frac{\partial}{\partial x_1^1} , \frac{\partial}{\partial x_2^1} , x^1_1 \frac{\partial}{\partial x_1^1} + x^2_1 \frac{\partial}{\partial x_2^1}, x^1_2 \frac{\partial}{\partial x_1^1} + x^2_2 \frac{\partial}{\partial x_2^1} + \frac{\partial}{\partial x_0^1} \right)
\]

\[
\left( \frac{\partial}{\partial x_1^1} , \frac{\partial}{\partial x_2^1} , x^1_1 \frac{\partial}{\partial x_1^1} + x^2_1 \frac{\partial}{\partial x_2^1} + \frac{\partial}{\partial x_0^1} \right),
\]

defined by \( R_{(0,0)}(\kappa^1) \) and \( S_{(0,0)}(\kappa^1) \), respectively. These two normal forms are obviously centered at zero.

The following theorem is the second main contribution of the paper. It asserts that extended Kumera-Ruiz normal forms serve as local normal forms for all singular contact systems for curves, that is for all distributions that satisfy condition (i) of Theorem 1.1 but fail to fulfill the regularity condition (ii) of that theorem.

**Theorem 3.2 (extended Kumera-Ruiz normal forms)** A distribution \( \mathcal{D} \) of rank \( m+1 \) on a manifold \( M \) of dimension \( (n+1)m+1 \) is equivalent, in a small enough neighborhood of any point \( p \) in \( M \), to a distribution spanned by an extended Kumera-Ruiz normal form, centered at \( p \) and defined on a suitably chosen neighborhood of zero, if and only if each element \( \mathcal{D}^{(i)} \) of its derived flag has constant rank \((i+1)m+1\) and contains an involutive subdistribution \( \mathcal{L}_i \subset \mathcal{D}^{(i)} \) that has constant corank one in \( \mathcal{D}^{(i)} \), for \( 0 \leq i \leq n \).

## 4 Proof of the Two Main Theorems

In this Section we provide a proof of Theorem 3.2. We start with several Lemmas — which will be used in the proof — that describe the geometry of incidence between characteristic distributions \( \mathcal{C}_i \) and involutive corank one subdistributions \( \mathcal{L}_i \) of \( \mathcal{D}^{(i)} \). Then we prove Theorem 3.2. Finally, we conclude Theorem 1.1 as a corollary of Theorem 3.2.

For \( i \geq 0 \), we will denote by \( \mathcal{C}_i \) the characteristic distribution of \( \mathcal{D}^{(i)} \). It follows directly from the Jacobi identity that \( \mathcal{C}_i \subset \mathcal{C}_{i+1} \). Define \( d_i(p) = \dim \mathcal{D}^{(i)}(p) \) and \( c_i(p) = \dim \mathcal{C}_i(p) \). Moreover, denote \( r_i(p) = d_{i+1}(p) - d_i(p) \).
Though the following result is a direct consequence of Bryant’s algebraic Lemma, we will give its proof as a warm up exercise. Indeed, the method used to prove the inclusion $C_0 \subset L_0$ is also used in the proofs of inclusions $L_0 \subset L_1$ and $L_0 \subset C_1$, which will be considered later.

**Lemma 4.1 ($C_0 \subset L_0$)** Let $\mathcal{D}$ be a distribution such that $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ have constant ranks $d_0$ and $d_1 \geq d_0 + 1$, respectively. If the distribution $\mathcal{D}$ contains an involutive subdistribution $L_0 \subset \mathcal{D}^{(0)}$ that has constant corank one in $\mathcal{D}^{(0)}$ then the ranks of $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(1)}$ satisfy $r_0 \leq d_0 - 1$. Moreover:

(i) The characteristic distribution $C_0$ satisfies $C_0 \subset L_0$;

(ii) The rank of $C_0$ is constant and equal to $d_0 - r_0 - 1$.

**Proof:** Assume that $\mathcal{D}$ contains an involutive subdistribution $L_0 \subset \mathcal{D}^{(0)}$ of constant corank one. The relation $r_0 \leq d_0 - 1$ is obvious. In order to prove by contradiction that $C_0 \subset L_0$, assume that for some point $p$ the vector space $C_0(p)$ is not contained in $L_0(p)$. Then, in a small enough neighborhood of $p$, we can assume that the distribution $\mathcal{D}^{(0)}$ is a direct sum $\mathcal{D}^{(0)} = (h) \oplus L_0$, where $h$ is a vector field that belongs to $C_0$ but that does not belong to $L_0$. Since $[L_0, L_0] \subset \mathcal{D}^{(0)}$ we have $\mathcal{D}^{(1)} = \mathcal{D}^{(0)} + [h, L_0]$. But since $[h, \mathcal{D}^{(0)}] \subset \mathcal{D}^{(0)}$ we have $\mathcal{D}^{(1)} = \mathcal{D}^{(0)}$, which is impossible because $r_0 \geq 1$.

Now, let us compute the rank of $C_0$. Since the corank of $L_0$ in $\mathcal{D}^{(0)}$ equals 1, we have a local decomposition $\mathcal{D}^{(0)} = (f_{d_0}) \oplus L_0$, where $f_{d_0}$ is an arbitrary vector field that belongs to $\mathcal{D}^{(0)}$ but that does not belong to $L_0$. Since $\mathcal{D}^{(1)} = \mathcal{D}^{(0)} + [f_{d_0}, L_0]$ we can find, locally, a family of vector fields

$$(f_1, \ldots, f_{r_0}, f_{r_0+1}, \ldots, f_{d_0-1})$$

that span $L_0$ and satisfies

$$\mathcal{D}^{(1)} = \mathcal{D}^{(0)} \oplus ([f_{d_0}, f_1], \ldots, [f_{d_0}, f_{r_0}]).$$

It follows that, for $r_0 + 1 \leq i \leq d_0 - 1$, we can find some smooth functions $\alpha_{ij}$ such that $[f_{d_0}, f_i] = \sum_{j=1}^{r_0} \alpha_{ij} [f_{d_0}, f_j] \bmod \mathcal{D}^{(0)}$. On the one hand, we have $\dim C_0(p) \geq d_0 - r_0 - 1$, at each point $p$, because the vector fields $h_i = f_i - \sum_{j=1}^{r_0} \alpha_{ij} f_j$, for $r_0 + 1 \leq i \leq d_0 - 1$, satisfy $[h_i, \mathcal{D}^{(0)}] \subset \mathcal{D}^{(0)}$ and are pointwise linearly independent. But, on the other hand, we have $\dim C_0(p) \leq d_0 - r_0 - 1$, at any point $p$, because
otherwise we would have \( \dim \mathcal{D}^{(1)}(p) \leq d_1 - 1 \). Hence \( \dim \mathcal{C}_0(p) = d_0 - r_0 - 1 \), for each point \( p \) in the underlying manifold. \( \square \)

**Lemma 4.2** (\( \mathcal{L}_0 \subset \mathcal{L}_1 \)) Let \( \mathcal{D} \) be a distribution such that \( \mathcal{D}^{(0)} \), \( \mathcal{D}^{(1)} \), and \( \mathcal{D}^{(2)} \) have constant ranks \( d_0 \), \( d_1 \geq d_0 + 2 \), and \( d_2 \geq d_1 + 2 \), respectively. Assume that each distribution \( \mathcal{D}^{(i)} \), for \( i = 0 \) and \( 1 \), contains an involutive subdistribution \( \mathcal{L}_i \subset \mathcal{D}^{(i)} \) that has constant corank one in \( \mathcal{D}^{(i)} \). Then \( \mathcal{L}_0 \subset \mathcal{L}_1 \).

**Proof:** Assume that there exists a point \( p \) such that the vector space \( \mathcal{L}_0(p) \) is not contained in \( \mathcal{L}_1(p) \). We will show that this assumption leads to \( d_2 \leq d_1 + 1 \). To start with, observe that on the one hand we have \( \dim \mathcal{D}^{(0)}(q) \cap \mathcal{L}_1(q) \leq d_0 - 1 \), for any point \( q \) in a small enough neighborhood of \( p \), because \( \mathcal{L}_0(p) \) is not contained in \( \mathcal{L}_1(p) \); but that on the other hand we have

\[
\dim \mathcal{D}^{(1)}(q) \geq \dim \left( \mathcal{D}^{(0)}(q) \cup \mathcal{L}_1(q) \right) = \dim \mathcal{D}^{(0)}(q) + \dim \mathcal{L}_1(q) - \dim \left( \mathcal{D}^{(0)}(q) \cap \mathcal{L}_1(q) \right),
\]

for any point \( q \), which implies that \( \dim \left( \mathcal{D}^{(0)}(q) \cap \mathcal{L}_1(q) \right) \geq d_0 - 1 \). Therefore, locally around \( p \), we have \( \dim \mathcal{D}^{(0)}(q) \cap \mathcal{L}_1(q) = d_0 - 1 \). An analogous argument can be applied to the intersection \( \mathcal{L}_0 \cap \mathcal{L}_1 \) in order to show that \( \dim \left( \mathcal{L}_0(q) \cap \mathcal{L}_1(q) \right) = d_0 - 2 \).

The above relations between the ranks of \( \mathcal{D}^{(0)} \), \( \mathcal{D}^{(0)} \cap \mathcal{L}_1 \), and \( \mathcal{L}_0 \cap \mathcal{L}_1 \) imply that we can find a local basis \( (f_1, \ldots, f_{d_0}) \) of \( \mathcal{D}^{(0)} \) such that \( \mathcal{L}_0 = (f_1, \ldots, f_{d_0-2}) \) and \( \mathcal{L}_0 \cap \mathcal{L}_1 = (f_1, \ldots, f_{d_0-2}, f_{d_0}) \). Moreover, we can assume that \( \mathcal{C}_0 = (f_1, \ldots, f_{c_0}) \), where \( c_0 < d_0 - 1 \). Indeed, by Lemma [4.1], we have \( \mathcal{C}_0 \subset \mathcal{L}_0 \) and \( \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{L}_1 \), which obviously implies \( \mathcal{C}_0 \subset \mathcal{L}_0 \cap \mathcal{L}_1 \).

Since the vector field \( f_{d_0} \) does not belong to \( \mathcal{L}_0 \), we have \( \mathcal{D}^{(1)} = \mathcal{D}^{(0)} + [f_{d_0}, \mathcal{L}_0] \). Therefore, the distribution \( \mathcal{D}^{(1)} \) admits the following local decomposition

\[
\mathcal{D}^{(1)} = \mathcal{D}^{(0)} \oplus ([f_{d_0}, f_{c_0+1}], \ldots, [f_{d_0}, f_{d_0-1}]).
\]

Denote \( g_i = [f_{d_0}, f_i] \), for \( c_0 + 1 \leq i \leq d_0 - 1 \). Since the vector field \( f_{d_0-1} \) does not belong to \( \mathcal{L}_1 \), we have \( \mathcal{D}^{(2)} = \mathcal{D}^{(1)} + [f_{d_0-1}, \mathcal{D}^{(1)}] \). Therefore, all vector fields in \( \mathcal{D}^{(2)} \) are linear combinations of those belonging to \( \mathcal{D}^{(1)} \) and of the vector fields \( [f_{d_0-1}, g_i] \), for \( c_0 + 1 \leq i \leq d_0 - 1 \).

Now, observe that \( \mathcal{L}_0 \cap \mathcal{L}_1 \subset \mathcal{C}_1 \). Indeed, the distribution \( \mathcal{D}^{(1)} \) admits a local decomposition \( \mathcal{D}^{(1)} = (f_{d_0-1}) \oplus \mathcal{L}_1 \). But \( [\mathcal{L}_0 \cap \mathcal{L}_1, \mathcal{L}_1] \subset \mathcal{L}_1 \) and \( [\mathcal{L}_0 \cap \mathcal{L}_1, f_{d_0-1}] \subset \mathcal{L}_0 \). Hence \([\mathcal{L}_0 \cap \mathcal{L}_1, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(1)} \).
We claim that each vector field \([f_{d_0-1}, g_i]\), for \(c_0 + 1 \leq i \leq d_0 - 2\), belongs to \(\mathcal{D}^{(1)}\). Indeed, the Jacobi identity gives

\[ [f_{d_0-1}, g_i] + [f_{d_0}, [f_i, f_{d_0-1}]] + [f_i, [f_{d_0-1}, f_{d_0}]] = 0. \]

On the one hand, the vector field \([f_i, [f_{d_0-1}, f_{d_0}]\] belongs to \(\mathcal{D}^{(1)}\) because the vector field \(f_i\) belongs to \(\mathcal{L}_0 \cap \mathcal{L}_1\), which is contained in \(\mathcal{L}_0\). But on the other hand, the vector field \([f_{d_0}, [f_i, f_{d_0-1}]\] also belongs to \(\mathcal{D}^{(1)}\) because \([f_i, f_{d_0-1}]\) belongs to \(\mathcal{L}_0\). It follows that the vector field \([f_{d_0-1}, g_i]\) belongs to \(\mathcal{D}^{(1)}\). Hence \(\mathcal{D} = \mathcal{D}^{(1)} + [f_{d_0-1}, g_{d_0-1}]\), which obviously implies \(d_2 \leq d_1 + 1\). It follows that we must have \(\mathcal{L}_0 \subset \mathcal{L}_1\). \(\square\)

**Lemma 4.3** (\(\mathcal{L}_0 \subset \mathcal{L}_1\)) Let \(\mathcal{D}\) be a distribution such that \(\mathcal{D}^{(0)}\), \(\mathcal{D}^{(1)}\), and \(\mathcal{D}^{(2)}\) have constant ranks \(d_0\), \(d_1 \geq d_0 + 2\), and \(d_2 \geq d_1 + 2\), respectively. Assume that each distribution \(\mathcal{D}^{(i)}\), for \(i = 0\) and \(1\), contains an involutive subdistribution \(\mathcal{L}_i \subset \mathcal{D}^{(i)}\) that has constant corank one in \(\mathcal{D}^{(i)}\). Then \(\mathcal{L}_0 \subset \mathcal{L}_1\).

**Proof:** Take local generators \((f_1, \ldots, f_{d_0-1}, f_{d_0})\) of \(\mathcal{D}^{(0)}\) such that

\[ \mathcal{L}_0 = (f_1, \ldots, f_{d_0-1}). \]

Since by Lemma 4.2 we have \(\mathcal{L}_0 \subset \mathcal{L}_1\), each vector field \(f_i\), for \(1 \leq i \leq d_0 - 1\), belongs to \(\mathcal{L}_1\). Now observe that, since the corank of \(\mathcal{L}_1\) in \(\mathcal{D}^{(1)}\) equals 1, we can take \(r_0\) vector fields \(g_1, \ldots, g_{r_0}\) in \(\mathcal{D}^{(1)}\) that are linearly independent mod \(\mathcal{D}^{(0)}\) and such that each vector field \(g_i\), for \(1 \leq i \leq r_0 - 1\), belongs to \(\mathcal{L}_1\). It follows that there exist two smooth functions \(\alpha\) and \(\beta\) such that

\[ \mathcal{L}_1 = (f_1, \ldots, f_{d_0-1}, g_1, \ldots, g_{r_0-1}, \alpha f_{d_0} + \beta g_{r_0}). \]

We want to prove that \(\mathcal{L}_0 \subset \mathcal{L}_1\), that is that \([f_i, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(1)}\), for \(1 \leq i \leq d_0 - 1\). By the definition of \(\mathcal{D}^{(1)}\) we have \([f_i, \mathcal{D}^{(0)}] \subset \mathcal{D}^{(1)}\), for \(1 \leq i \leq d_0 - 1\), and by the involutivity of \(\mathcal{L}_1\) we have \([f_i, g_j] \in \mathcal{D}^{(1)}\), for \(1 \leq j \leq r_0 - 1\). Therefore, what remains to prove is that \([\mathcal{L}_0, g_{r_0}] \subset \mathcal{D}^{(1)}\).

We will prove that \([\mathcal{L}_0, g_{r_0}] \subset \mathcal{D}^{(1)}\) by contradiction. Assume that, for some \(1 \leq i \leq d_0 - 1\), there exists a point \(p\) such that \([f_i, g_{r_0}] (p) \notin \mathcal{D}^{(1)} (p)\). This implies that \([f_i, g_{r_0}] (q) \notin \mathcal{D}^{(1)} (q)\), for each point \(q\) in a small neighborhood \(U\) of \(p\). But, since \(\mathcal{L}_1\) is involutive, the vector field \([f_i, \alpha f_{d_0} + \beta g_{r_0}]\) belongs to \(\mathcal{D}^{(1)}\), which clearly implies that \(\beta [f_i, g_{r_0}]\) also belongs to \(\mathcal{D}^{(1)}\). Therefore, we must have \(\beta (q) = 0\), for
each point \( q \) in \( U \). It follows that, in a small enough neighborhood of \( p \), we have \( \mathcal{D}^{(0)} \subset \mathcal{L}_1 \), which implies that \( \mathcal{D}^{(1)} \subset \mathcal{L}_1 \) because \( \mathcal{L}_1 \) is involutive. Since \( \mathcal{L}_1 \) has corank one in \( \mathcal{D}^{(1)} \), this is impossible. Therefore \( [\mathcal{L}_0, g_{ro}] \subset \mathcal{D}^{(1)} \), which implies that \( \mathcal{L}_0 \subset \mathcal{C}_1 \).

\[ \square \]

**Lemma 4.4 (canonical distribution)** Let \( \mathcal{D} \) be a distribution such that \( \mathcal{D}^{(0)} \), \( \mathcal{D}^{(1)} \), and \( \mathcal{D}^{(2)} \) have constant ranks \( d_0 = m + 1 \), \( d_1 = 2m + 1 \), and \( d_2 = 3m + 1 \), respectively. If \( m \geq 2 \) then assume, additionally, that each distribution \( \mathcal{D}^{(i)} \), for \( i = 0 \) and \( 1 \), contains an involutive subdistribution \( \mathcal{L}_i \subset \mathcal{D}^{(i)} \) that has constant corank one in \( \mathcal{D}^{(i)} \). Under these assumptions, we have \( \mathcal{L}_0 = \mathcal{C}_1 \), that is the distribution \( \mathcal{D}^{(0)} \) contains a unique involutive subdistribution \( \mathcal{L}_0 \) that has constant corank one in \( \mathcal{D}^{(0)} \) and satisfies \( [\mathcal{L}_0, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(1)} \).

**Proof:** For \( m = 1 \) the result is well known (see e.g. [1], [15], [18], and [19]; see also [23]). Now, if \( m \geq 2 \) then Item (ii) of Lemma 4.1 (applied to \( \mathcal{C}_1 \)) implies that \( \dim \mathcal{C}_1(p) = 2(2m + 1) - (3m + 1) - 1 = m \) and Lemma 4.3 that \( \mathcal{L}_0(p) \subset \mathcal{C}_1(p) \), for each point \( p \) in the underlying manifold. But \( \dim \mathcal{L}_0(p) = m \). Thus \( \mathcal{L}_0(p) = \mathcal{C}_1(p) \), which implies that \( \mathcal{L}_0 \) is uniquely characterized by \( [\mathcal{L}_0, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(1)} \). \[ \square \]

The following result is a natural generalization of a theorem used by E. von Weber [27] in his study of Goursat structures. In fact, the main idea we will use in our proof of Theorem 3.2 is quite close to Weber’s original idea. A good introduction to the work of E. von Weber is Cartan’s paper [3]. In our own paper [23], the two main results of [27] are given in a more modern language.

**Lemma 4.5 (extended Weber normal form)** Let \( \mathcal{D} \) be a distribution defined on a manifold \( M \) of dimension \( (n + 1)m + 1 \). Assume that \( \mathcal{D}^{(0)} \) and \( \mathcal{D}^{(1)} \) have constant ranks \( m + 1 \) and \( 2m + 1 \), respectively, and that \( \mathcal{D}^{(0)} \) contains an involutive subdistribution \( \mathcal{L}_0 \subset \mathcal{D}^{(0)} \) that has constant corank one in \( \mathcal{D}^{(0)} \) and satisfies \( \mathcal{L}_0 \subset \mathcal{C}_1 \). Then, in a small enough neighborhood of any point \( p \) in \( M \), the distribution \( \mathcal{D} \) is equivalent to a distribution spanned on \( J^n(\mathbb{R}, \mathbb{R}^m) \) by a family of vector fields that has the following form:

\[
\left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}, y_1^n \zeta_1^{n-1} + \cdots + y_m^n \zeta_m^{n-1} + \zeta_0^{n-1} \right),
\]

16
where \( \mathcal{L}_0 = \left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m} \right) \) and the vector fields \( \zeta_{n-1}^1, \ldots, \zeta_{n-1}^m, \zeta_{0}^{n-1} \) are lifts of vector fields on \( J^{n-1}(\mathbb{R}, \mathbb{R}^m) \), that is,

\[
\zeta_i^{n-1} = \zeta_i^{n-1}(y_0^0, y_1^0, \ldots, y_m^0, \ldots, y_i^{n-1}, \ldots, y_m^{n-1}), \text{ for } 0 \leq i \leq m.
\]

Moreover, the set of local coordinates \((y_0, y_1, \ldots, y_m, \ldots, y_i^{n-1}, \ldots, y_m^{n-1})\), from \( M \) into \( J^n(\mathbb{R}, \mathbb{R}^m) \) can be taken to be centered at \( p \).

**Proof:** It follows directly from Frobenius’ theorem, applied to the distribution \( \mathcal{L}_0 \), that the distribution \( \mathcal{D} \) is locally equivalent to a distribution spanned on \( \mathbb{R}^{(n+1)m+1} \) by a family of vector fields that has the following form:

\[
\left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}, \sum_{i=0}^{n-1} \sum_{j=1}^{m} \alpha_j^i(z) \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_0} \right),
\]

where \( \mathcal{L}_0 = \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m} \right) \) and the local coordinates \( z_0^0, z_1^0, \ldots, z_m^0, \ldots, z_1^n, \ldots, z_m^n \) are centered at \( p \).

Since \( \dim \mathcal{D}^{(1)}(p) = 2m + 1 \) we can assume, after a permutation of the \( z \)-coordinates, if necessary, that the real \( m \times m \) matrix

\[
T = \left( \alpha_{0}^{n-1} \right), \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq m,
\]

has full rank \( m \), in a small enough neighborhood of zero. We can assume, moreover, that \( \alpha_i^{n-1}(0) = 0 \), for \( 1 \leq i \leq m \). Otherwise, replace the coordinate \( z_i^{n-1} \) by \( z_i^{n-1} - z_0^0 \alpha_i^{n-1}(0) \). Now, we can define a new set of centered local coordinates

\[
(y_0^0, y_1^0, \ldots, y_m^0, \ldots, y_1^n, \ldots, y_m^n) = \psi(z_0^0, z_1^0, \ldots, z_m^0, \ldots, z_1^n, \ldots, z_m^n)
\]

by taking \( y_i^n = \alpha_i^{n-1}(z) \), for \( 1 \leq i \leq m \), and by taking \( y_j^i = z_j^i \), as the remaining coordinates. Since the matrix \( T \) has rank \( m \), this change of coordinates is indeed a local diffeomorphism. Hence, the distribution \( \mathcal{D} \) is locally equivalent to a distribution spanned, on a small enough neighborhood of zero, by a pair of vector fields that has the following form:

\[
\left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}, \sum_{j=1}^{m} y_j^n \frac{\partial}{\partial y_j} + \sum_{i=0}^{n-2} \sum_{j=1}^{m} \beta_{j}^i(y) \frac{\partial}{\partial y_j} + \frac{\partial}{\partial y_0} \right).
\]

Since \( \mathcal{L}_0 \subset \mathcal{C}_1 \) we have \([\mathcal{L}_0, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(1)} \). But this inclusion clearly implies that, for \( 1 \leq i \leq n-2 \) and \( 1 \leq j \leq m \), we have \( \partial^2 \beta_j^i / \partial y_k^n \partial y_l^n \equiv 0 \), for \( 1 \leq k \leq m \) and
1 ≤ l ≤ m. It follows that all functions β^i_j, for 1 ≤ i ≤ n − 2 and 1 ≤ j ≤ m, are affine with respect to the variables y_1^n, ..., y_m^n, that is
\[ β^i_j(y) = \sum_{k=1}^{m} a^i_{jk}(\overline{y}^{-1}) y_k^n + a^i_{j0}(\overline{y}^{-1}). \]
where \( \overline{y}^{-1} \) denotes the coordinates \( (y_{0}^{0}, y_{1}^{0}, \ldots, y_{m}^{1}, \ldots, y_{0}^{1}, \ldots, y_{n}^{-1}) \). Now, define
\[ \zeta^1_k = \frac{\partial}{\partial y_{1}^{k}} \sum_{j=1}^{n-2} \sum_{i=1}^{m} a^i_{jk}(\overline{y}^{-1}) \frac{\partial}{\partial y_{j}^{1}}, \quad \text{for} \ 1 \leq k \leq m, \]
and
\[ \zeta^0_0 = \frac{\partial}{\partial y_{0}^{0}} \sum_{j=1}^{n-2} \sum_{i=1}^{m} a^i_{j0}(\overline{y}^{-1}) \frac{\partial}{\partial y_{j}^{1}}. \]
This definition shows that \( \mathcal{D} \) is locally equivalent to
\[ \left( \frac{\partial}{\partial y_{1}^{1}}, \ldots, \frac{\partial}{\partial y_{m}^{1}}, y_{1}^{1} \zeta^1_1 + \cdots + y_{m}^{1} \zeta^1_m + \zeta^0_0 \right), \]
where the vector fields \( \zeta^1_1, \ldots, \zeta^1_m, \zeta^0_0 \) are lifts of vector fields on \( J^{n-1}(\mathbb{R}, \mathbb{R}^m) \).
It follows directly from our construction that \( \mathcal{L}_0 = (\frac{\partial}{\partial y_{1}^{1}}, \ldots, \frac{\partial}{\partial y_{m}^{1}}). \) \( \Box \)

**Proof of Theorem 3.2:** We will proceed by induction on the integer \( n \geq 1 \). For \( n = 1 \), the Theorem is a direct consequence of Lemma 4.3. Thus, assume that the Theorem is true for \( n - 1 \geq 1 \) and consider a rank \( m + 1 \) distribution \( \mathcal{D} \), defined on a manifold \( M \) of dimension \( (n+1)m+1 \), such that each element \( \mathcal{D}^{(i)} \) of its derived flag has constant rank \( (i + 1)m + 1 \) and contains an involutive subdistribution \( \mathcal{L}_i \subset \mathcal{D}^{(i)} \) that has constant corank one in \( \mathcal{D}^{(i)} \), for \( 0 \leq i \leq n \). Let \( p \) be an arbitrary point in \( M \).

By Lemma 4.4, the involutive distribution \( \mathcal{L}_0 \), which has corank one in \( \mathcal{D}^{(0)} \), satisfies \( \mathcal{L}_0 \subset \mathcal{C}_1 \). We can thus apply Lemma 4.5, which states that the distribution \( \mathcal{D} \) is equivalent, in a small enough neighborhood of \( p \), to a distribution spanned on \( J^n(\mathbb{R}, \mathbb{R}^m) \) by a family of vector fields \( (\zeta^n_1, \ldots, \zeta^n_m, \zeta^n_0) \) that has the following form:

\[ \zeta^n_1 = \frac{\partial}{\partial y_{1}^{1}}, \ldots, \zeta^n_m = \frac{\partial}{\partial y_{m}^{1}} \]
\[ \zeta^n_0 = y_{1}^{1} \zeta^n_1 + \cdots + y_{m}^{1} \zeta^n_m + \zeta^n_0, \]

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where the vector fields $\zeta_1^{n-1}, \ldots, \zeta_m^{n-1}, \zeta_0^{n-1}$ are lifts of vector fields on $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. In the rest of the proof we will assume that $D = (\zeta_1^n, \ldots, \zeta_m^n, \zeta_0^n)$. Note that the $y$-coordinates are centered at zero.

The aim of the proof will be to construct a local change of coordinates

$$(x_0^0, x_1^0, \ldots, x_m^0, \ldots, x_1^{n-1}, \ldots, x_m^{n-1}) = \phi^n(y_0^0, y_1^0, \ldots, y_m^0, \ldots, y_1^{n-1}, \ldots, y_m^{n-1}),$$

a Kumpera-Ruiz normal form $F$ on $J^n(\mathbb{R}, \mathbb{R}^m)$, and a smooth map $\mu^n : J^n(\mathbb{R}, \mathbb{R}^m) \to GL(m + 1, \mathbb{R})$, given by an $(m + 1) \times (m + 1)$ matrix $(\mu^n_{ij}(y))$, such that

$$\phi^n_*(\zeta_i^n) = \sum_{j=1}^m (\mu^n_{ij} \circ \psi^n) \kappa_j^n, \quad \text{for } 1 \leq i \leq m, \quad (7)$$

and

$$\phi^n_*(\zeta_0^n) = \sum_{j=0}^m (\mu^n_{0j} \circ \psi^n) \kappa_j^n, \quad (8)$$

where $\psi^n = (\phi^n)^{-1}$ denotes the inverse of the local diffeomorphism $\phi^n$. Moreover, we will ask the $x$-coordinates to be centered at zero, that is $\phi^n(0) = 0$. Observe that we take $\mu^n_{i0} = 0$, for $1 \leq i \leq m$, in order to transform the canonical distribution of our distribution $D$, which is given by $L_0 = (\zeta_1^n, \ldots, \zeta_m^n)$ (see Lemma 4.4), into the canonical distribution of the Kumpera-Ruiz normal form, which is given by $(\kappa_1^n, \ldots, \kappa_m^n)$.

Recall that the vector fields $\zeta_1^{n-1}, \ldots, \zeta_m^{n-1}, \zeta_0^{n-1}$ are lifts of vector fields on $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. If we take $F = (\zeta_1^{n-1}, \ldots, \zeta_m^{n-1}, \zeta_0^{n-1})$ then we will obtain a decomposition $D^{(1)} = L_0 \oplus F$. Since $D^{(1)}$ contains an involutive subdistribution $L_1$ that has constant corank one in $D^{(1)}$, it follows directly from the relation $L_0 \subset L_1$ (see Lemma 4.2) that $F$ contains an involutive subdistribution that has constant corank one in $F$. In fact, it is easy to prove (using the relations $\mathcal{C}_i \subset \mathcal{C}_{i+1}$ and $\mathcal{L}_i = \mathcal{C}_i$) that if $D$ satisfies the conditions of Theorem 3.2 on $J^n(\mathbb{R}, \mathbb{R}^m)$ then $F$ satisfies the conditions of this Theorem on $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. Now, recall that we have assumed that the Theorem is true on $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. The distribution $F$ is thus locally equivalent to a distribution spanned by a Kumpera-Ruiz normal form on $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$, centered at zero. It follows that there exists a local diffeomorphism

$$(x_0^0, x_1^0, \ldots, x_m^0, \ldots, x_1^{n-1}, \ldots, x_m^{n-1}) = \phi^{n-1}(y_0^0, y_1^0, \ldots, y_m^0, \ldots, y_1^{n-1}, \ldots, y_m^{n-1}),$$
a Kumpera-Ruiz normal form \((κ_1^{n-1}, \ldots, κ_m^{n-1}, κ_0^{n-1})\) on \(J^{n-1}(\mathbb{R}, \mathbb{R}^m)\), and a smooth map \(μ^{n-1}: J^{n-1}(\mathbb{R}, \mathbb{R}^m) \to GL(m+1, \mathbb{R})\) such that

\[
φ^n_i(ζ_i^{n-1}) = \sum_{j=0}^{m} (μ_{ij}^{n-1} \circ ψ^{n-1})κ_j^{n-1}, \quad \text{for} \ 0 \leq i \leq m,
\]

where \(ψ^{n-1} = (φ^{n-1})^{-1}\) denotes the inverse of the local diffeomorphism \(φ^{n-1}\). Note that we have \(φ^{n-1}(0) = 0\).

The following Lemma can be easily proved by a direct computation.

**Lemma 4.6 (triangular tangent maps)**  Let \(φ^n = (φ^{n-1}, φ_1^n, \ldots, φ_m^n)^T\) be a diffeomorphism of \(J^n(\mathbb{R}, \mathbb{R}^m)\) such that its first \(nm + 1\) components, which are given by \(φ^{n-1}\), depend on the first \(nm + 1\) coordinates only. Moreover, let \(f\) be a vector field on \(J^n(\mathbb{R}, \mathbb{R}^m)\) of the form \(f = αf^{n-1} + f_n\), where \(α\) is a smooth function on \(J^n(\mathbb{R}, \mathbb{R}^m)\), the vector field \(f^{n-1}\) is the lift of a vector field on \(J^{n-1}(\mathbb{R}, \mathbb{R}^m)\), and the only non-zero components of \(f_n\) are those that multiply \(\frac{∂}{∂y_1}, \ldots, \frac{∂}{∂y_m}\). Then, we have

\[
φ^n_i(f) = (α \circ ψ^n)φ^{n-1}_i(f^{n-1}) + \sum_{i=1}^{m} ((L_f φ_i^n) \circ ψ^n) \frac{∂}{∂y_j^n}.
\]  (9)

Note that the vector field \(φ^{n-1}_i(f^{n-1})\) is lifted to \(J^n(\mathbb{R}, \mathbb{R}^m)\) and that the coordinates \(x_1^n, \ldots, x_m^n\) are those given by \(φ_1^n, \ldots, φ_m^n\), respectively.

**Regular case:** If \(μ_0^{n-1}(0) \neq 0\) then we can complete \(φ^{n-1}\) to a zero preserving diffeomorphism \(φ^n\) of \(J^n(\mathbb{R}, \mathbb{R}^m)\) by taking \(φ^n = (φ^{n-1}, φ_1^n, \ldots, φ_m^n)^T\), where

\[
φ^n_j(y) = \frac{\sum_{i=1}^{m} μ_{ij}^{n-1} y_i^n + μ_0^{n-1}}{\sum_{i=1}^{m} μ_{ij}^{n-1} y_i^n + μ_0^{n-1}} - \frac{μ_0^{n-1}(0)}{μ_0^{n-1}(0)}, \quad \text{for} \ 1 \leq j \leq m.
\]

It is easy to check, using Lemma [4.7] below, that \(φ^n\) is a local diffeomorphism (because \(μ^{n-1}\) is invertible). In this case, we define \(c_i^n = (μ_0^{n-1}/μ_0^{n-1})(0)\) for \(1 \leq i \leq m\), \(μ_0^n = L_φ^{n-1}φ^n\) for \(0 \leq i \leq m\) and \(1 \leq j \leq m\), and \(μ_0^n = \sum_{i=1}^{m} μ_0^{n-1} y_i^n + μ_0^{n-1}\). Moreover, the Kumpera-Ruiz normal form \((κ_1^n, \ldots, κ_m^n, κ_0^n)\) is defined to be the regular prolongation, with parameter \(c^n = (c_1^n, \ldots, c_m^n)\), of \((κ_1^{n-1}, \ldots, κ_m^{n-1}, κ_0^{n-1})\).
Let us check that, in this case, relation (8) holds. By relation (9) we have:

\[
\phi^n_i(\zeta^n_0) = \sum_{i=1}^{m} (y^n_i \circ \psi^n_0) \phi^{n-1}_i(\zeta^{n-1}_0) + \phi^{n-1}_n(\zeta^{n-1}_0) + \sum_{i=1}^{m} \left( (L_{\zeta^n_0} \phi^n_i) \circ \psi^n_0 \right) \frac{\partial}{\partial x^n_i}
\]

\[
= \sum_{i=1}^{m} (y^n_i \circ \psi^n_0) \left( \sum_{j=0}^{m} (\mu^{n-1}_{ij} \circ \psi^{n-1}) \kappa^{n-1}_j \right) + \sum_{j=0}^{m} (\mu^{n-1}_{0j} \circ \psi^{n-1}) \kappa^{n-1}_j
\]

\[
+ \sum_{i=1}^{m} (\mu^n_{0i} \circ \psi^n) \kappa^n_i
\]

\[
= \sum_{j=0}^{m} \left( \left( \sum_{i=1}^{m} \mu^{n-1}_{ij} y^n_i + \mu^{n-1}_{0j} \right) \circ \psi^n_0 \right) \kappa^{n-1}_j + \sum_{i=1}^{m} (\mu^n_{0i} \circ \psi^n) \kappa^n_i
\]

\[
= \left( \sum_{j=1}^{m} \left( \sum_{i=1}^{m} \mu^{n-1}_{ij} y^n_i + \mu^{n-1}_{00} \right) \circ \psi^n_0 \right) \kappa^{n-1}_j + \sum_{i=1}^{m} (\mu^n_{0i} \circ \psi^n) \kappa^n_i
\]

\[
= (\mu^n_{00} \circ \psi^n_0) \left( \sum_{j=1}^{m} (x^n_j + c^n_j) \kappa^{n-1}_j + \kappa^{n-1}_0 \right) + \sum_{i=1}^{m} (\mu^n_{0i} \circ \psi^n) \kappa^n_i
\]

Moreover, for \(1 \leq i \leq m\), we have

\[
\phi^n_i(\zeta^n_0) = \sum_{j=1}^{m} \left( (L_{\zeta^n_0} \phi^n_j) \circ \psi^n_0 \right) \kappa^n_j = \sum_{j=1}^{m} (\mu^{n-1}_{ij} \circ \psi^n) \kappa^n_j.
\]

It follows that both (9) and (8) hold.

**Singular case:** Suppose now that \(\mu^{n-1}_{00}(0) = 0\). Since the matrix \(\mu^{n-1}\) is invertible in a small enough neighborhood of zero, we can assume that there exists an integer \(1 \leq i \leq m\) such that \(\mu^{n-1}_{0i}(0) \neq 0\). After a permutation of the coordinates \(y^n_1, \ldots, y^n_m\), if necessary, we can assume that \(\mu^{n-1}_{0m}(0) \neq 0\). Now, like in the regular case, we can complete \(\phi^{n-1}\) to a zero preserving diffeomorphism \(\phi^n\) of \(J^n(\mathbb{R}, \mathbb{R}^m)\) by taking

\[
\phi^n = (\phi^{n-1}, \phi^n_1, \ldots, \phi^n_m)^T,
\]

where

\[
\phi^n_j(y) = \frac{\sum_{i=1}^{m} \mu^{n-1}_{ij} y^n_i + \mu^{n-1}_{0j}}{\sum_{i=1}^{m} \mu^{n-1}_{im} y^n_i + \mu^{n-1}_{0m}} - \frac{\mu^{n-1}_{0j}(0)}{\mu^{n-1}_{0m}(0)}, \quad \text{for } 1 \leq j \leq m - 1,
\]

and

\[
\phi^n_m(y) = \frac{\sum_{i=1}^{m} \mu^{n-1}_{0i} y^n_i + \mu^{n-1}_{00}}{\sum_{i=1}^{m} \mu^{n-1}_{im} y^n_i + \mu^{n-1}_{0m}}.
\]
In this case, we define \( c_i^n = (\mu_{0i}^{n-1}/\mu_{0m}^{n-1})(0) \) for \( 1 \leq i \leq m - 1 \). Observe that we can take \( c_m^n = 0 \) because \( \mu_{0m}^{n-1}(0) = 0 \). We take \( \mu_{ij}^n = L_{\kappa_i^n}^{-1} \phi_i^n \), for \( 0 \leq i \leq m \) and \( 1 \leq j \leq m \), and \( \mu_{00}^n = \sum_{i=1}^m \mu_{0m}^{n-1} \phi_i^n + \mu_{0m}^{n-1} \). Moreover, the Kumpera-Ruiz normal form \( (\kappa_1^n, \ldots, \kappa_m^n, \kappa_0^n) \) is defined to be the singular prolongation, with parameter \( c^n = (c_1^n, \ldots, c_{m-1}^n, 0) \), of \( (\kappa_1^{n-1}, \ldots, \kappa_m^{n-1}, \kappa_0^{n-1}) \). Let us check that relation (8) holds. We have:

\[
\phi^n_*(\zeta^n_0) = \sum_{i=1}^m (y_i^n \circ \psi^n) \phi_i^{n-1}(\zeta_i^{n-1}) + \phi^n_*(\zeta^n_0) + \sum_{i=1}^m ((L_{\kappa_i^n}^{-1} \phi_i^n) \circ \psi^n) \frac{\partial}{\partial x_i^n} \\
= \sum_{i=0}^m \left( \sum_{i=1}^m \mu_{ij}^{n-1} \phi_i^n + \mu_{0j}^{n-1} \right) \circ \psi^n \kappa_j^{n-1} + \sum_{i=1}^m (\mu_{0i} \circ \psi^n) \kappa_i^n \\
= \left( \sum_{i=1}^m \left( \sum_{i=1}^m \mu_{ij}^{n-1} \phi_i^n + \mu_{0j}^{n-1} \right) \circ \psi^n \right) \kappa_j^{n-1} + \sum_{i=1}^m (\mu_{0i} \circ \psi^n) \kappa_i^n \\
= (\mu_{00} \circ \psi^n) \left( \sum_{j=1}^{m-1} \left( x_j^n + c_j^n \right) \kappa_j^{n-1} + \kappa_m^{n-1} + x_m^n \kappa_0^{n-1} \right) + \sum_{i=1}^m (\mu_{0i} \circ \psi^n) \kappa_i^n \\
= \sum_{i=0}^m (\mu_{0i} \circ \psi^n) \kappa_i^n.
\]

Moreover, like in the regular case, we have

\[
\phi^n_*(\zeta^n_0) = \sum_{j=1}^m ((L_{\kappa_i^n}^{-1} \phi_i^n) \circ \psi^n) \kappa_j^n = \sum_{j=1}^m (\mu_{ij} \circ \psi^n) \kappa_j^n,
\]

for \( 1 \leq i \leq m \). It follows that relations (7) and (8) hold in both cases.

We have thus proved that the conditions of Theorem 3.2 are sufficient for converting a distribution into extended Kumpera-Ruiz normal form. It is straightforward to check that these conditions are also necessary. \( \square \)

**Lemma 4.7 (M"obius transformations)** Consider a real \( n \times n \) matrix \( M \) that has the following form:

\[
M = \begin{pmatrix} A & b \\ c & d \end{pmatrix},
\]

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where $c$ is a row vector and $b$ a column vector, both of dimension $n - 1$, the real constant $d$ is non-zero, and $A$ is a real $(n - 1) \times (n - 1)$ matrix. The linear fractional transformation $\varphi$, from $\mathbb{R}^{n-1}$ into $\mathbb{R}^{n-1}$, defined in a small enough neighborhood of zero by $\varphi(x) = (Ax + b) / (cx + d)$ is a local diffeomorphism if and only if the matrix $M$ is invertible.

**Proof:** We have $\varphi_*(0) = (Ad - bc) / d^2$ and thus $\det \varphi_*(0) = (1/d^2) \det(Ad - bc)$. But $\det M = (1/d^{n-2}) \det(Ad - bc)$. Hence $\det \varphi_*(0) \neq 0$ if and only if $\det M \neq 0$. □

**Proof of Theorem 1.1:** Let $\mathcal{D}$ be a distribution of rank $m + 1$, defined on a manifold $M$ of dimension $(n+1)m+1$, that satisfies the conditions of Theorem 1.1. In particular, the distribution $\mathcal{D}$ satisfies the conditions of Theorem 3.2. Therefore there exists a Kumpeara-Ruiz normal form $(\kappa_1^n, \ldots, \kappa_m^n, \kappa_0^n)$ on $J^n(\mathbb{R}, \mathbb{R}^m)$, defined on a small enough neighborhood of zero, that is equivalent to the distribution $\mathcal{D}$ considered on a small enough neighborhood of any point $p$ in $M$. We will thus assume that $\mathcal{D} = (\kappa_1^n, \ldots, \kappa_m^n, \kappa_0^n)$.

Now, if we exclude Engel’s case ($m = 1$ and $n = 2$), for which Theorem 3.2 is well known to be true, it is straightforward to check that if the sequence of prolongations that defines our Kumpeara-Ruiz normal form contains a singular prolongation then there exists some integer $2 \leq i \leq n$ such that the Lie flag of $\mathcal{D}$ satisfies

$$\dim \mathcal{D}_i(0) < (i + 1)m + 1.$$  

It thus follows that $(\kappa_1^n, \ldots, \kappa_m^n, \kappa_0^n)$ has necessarily been obtained by a sequence of regular prolongations from the canonical contact system on $J^1(\mathbb{R}, \mathbb{R}^m)$. That is:

$$\mathcal{D} = \left( \frac{\partial}{\partial x_1^i}, \ldots, \frac{\partial}{\partial x_m^i}, \sum_{i=0}^{n-1} \sum_{j=1}^m \left( x_j^{i+1} + c_j^{i+1} \right) \frac{\partial}{\partial x_j^i} + \frac{\partial}{\partial x_0^i} \right).$$  

(10)

What remains to prove now is that the distribution (10), which is defined on a small enough neighborhood of zero, is locally equivalent to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$, also considered on a small enough neighborhood of zero. In other words, we have to normalize all constants $c_j^{i+1}$ by making them equal to zero. To this end, observe that the Lie algebra

$$\mathfrak{g} = \text{span}_\mathbb{R} \{ \kappa_1^n, \ldots, \kappa_m^n, \kappa_1^{n-1}, \ldots, \kappa_m^{n-1}, \ldots, \kappa_1^1, \ldots, \kappa_m^1, \kappa_0^n \},$$

is
of dimension \((n + 1)m + 1\), generated by the vector fields \((\kappa_1^a, \ldots, \kappa_m^a, \kappa_0^a)\) defining (10), has the same structure constants independently of the values of the parameters \(c^i_{j+1}\). Indeed, the only non-zero Lie brackets are those given by the relations 
\[ [\kappa_j^i, \kappa_0^a] = \kappa_j^{i-1}, \] 
for \(1 \leq i \leq n\) and \(1 \leq j \leq m\). By Cartan’s theorem [7] on equivalence of frames, our distribution is locally equivalent to the canonical contact system at zero (see e.g. [22] for a modern account on Cartan’s equivalence method). \(\Box\)

### Appendix

**Lemma A.1 (Bryant)** Let \(\mathcal{D}\) be a distribution such that \(\mathcal{D}^{(0)}\) and \(\mathcal{D}^{(1)}\) have constant ranks \(d_0\) and \(d_1\), respectively. Put \(r_0 = d_1 - d_0\). Assume that the distribution \(\mathcal{D}\) contains a subdistribution \(\mathcal{B} \subset \mathcal{D}\) that has constant corank one in \(\mathcal{D}\) and satisfies \([\mathcal{B}, \mathcal{B}] \subset \mathcal{D}\). If \(r_0 \geq 3\) then \(\mathcal{B}\) is involutive.

**Proof:** Assume that \(\mathcal{D}\) contains a subdistribution \(\mathcal{B} \subset \mathcal{D}\) that has constant corank one in \(\mathcal{D}\) and satisfies \([\mathcal{B}, \mathcal{B}] \subset \mathcal{D}\). By Lemma A.1, the rank of the characteristic distribution \(\mathcal{C}_0\) of \(\mathcal{D}^{(0)}\) is constant and \(\mathcal{C}_0 \subset \mathcal{B}\). Therefore, there exists a local basis \((f_1, \ldots, f_{d_0})\) of \(\mathcal{D}\) such that

\[ \mathcal{C}_0 = (f_1, \ldots, f_{c_0}) \quad \text{and} \quad \mathcal{B} = (f_1, \ldots, f_{d_0-1}), \]

Since \(\mathcal{B}\) satisfies \([\mathcal{B}, \mathcal{B}] \subset \mathcal{D}\) we have \(\mathcal{D}^{(1)} = \mathcal{D}^{(0)} + [f_{d_0}, \mathcal{B}]\) or, more precisely,

\[ \mathcal{D}^{(1)} = \mathcal{D}^{(0)} \oplus ([f_{d_0}, f_{c_0+1}], \ldots, [f_{d_0}, f_{d_0-1}] ). \quad (11) \]

Note that the relation \(d_1 \geq d_0 + 3\) implies that \(\text{card}\{c_0 + 1, \ldots, d_0 - 1\} \geq 3\).

We want to prove that \([\mathcal{B}, \mathcal{B}] \subset \mathcal{B}\). Let us first prove that \([f_i, f_j] \in \mathcal{B}\), for \(c_0 + 1 \leq i \leq d_0 - 1\) and \(c_0 + 1 \leq j \leq d_0 - 1\). Consider an arbitrary triple \(f_i, f_j, \) and \(f_k\) of vector fields such that the indices \(i, j,\) and \(k\) are pairwise different and contained in \(\{c_0 + 1, \ldots, d_0 - 1\}\) (it is important to stress that such a triple exists because \(d_1 \geq d_0 + 3\)). It follows from relation (11) that the three vector fields \([f_{d_0}, f_i], [f_{d_0}, f_j],\) and \([f_{d_0}, f_k]\) are linearly independent mod \(\mathcal{D}^{(0)}\). Moreover, since \([\mathcal{B}, \mathcal{B}] \subset \mathcal{D}^{(0)}\), there exist three smooth functions \(a, b, \) and \(c\) such that \([f_i, f_j] = af_{d_0} \mod \mathcal{B}, \)

\([f_j, f_k] = bf_{d_0} \mod \mathcal{B}, \) and \([f_k, f_i] = cf_{d_0} \mod \mathcal{B}.\) The Jacobi identity gives:

\[ [f_i, [f_j, f_k]] + [f_j, [f_k, f_i]] + [f_k, [f_i, f_j]] = 0, \]
which implies that \([f_i, bf_{d_0}] + [f_j, cf_{d_0}] + [f_k, af_{d_0}]\) belongs to \(D^{(0)}\), and thus that \(b[f_i, f_{d_0}] + c[f_j, f_{d_0}] + a[f_k, f_{d_0}]\) belongs to \(D^{(0)}\). The latter relation implies that \(a, b,\) and \(c\) are identically zero because \([f_{d_0}, f_i], [f_{d_0}, f_j],\) and \([f_{d_0}, f_k]\) are linearly independent mod \(D^{(0)}\). It follows that we have \([f_i, f_j] \in \mathcal{B}\), for \(c_0 + 1 \leq i \leq d_0 - 1\) and \(c_0 + 1 \leq j \leq d_0 - 1\).

Since \([C_0, C_0] = C_0\), what remain to prove is that \([C_0, \mathcal{B}] = \mathcal{B}\). The proof follows again from the Jacobi identity, applied to any triple \(f_i, f_j,\) and \(f_k\) of pairwise linearly independent vector fields such that \(f_i\) belongs to \(C_0\) and both \(f_j\) and \(f_k\) belong to \(\mathcal{B}\) but do not belong to \(C_0\).

\[ \square \]

### B Appendix

In this appendix we will provide, following Bryant [2], a way to check the conditions of Corollaries 2.3 and 2.4. Indeed, we will show how to verify whether or not the Engel rank equals 1, and how to construct explicitly the characteristic distribution of \(D\) and — when it exists — the unique corank one subdistribution \(\mathcal{B} \subset D\) satisfying \([\mathcal{B}, \mathcal{B}] \subset D\).

Consider a distribution \(D\) of constant rank \(d_0\), defined on a manifold of dimension \(N\). Let \(\omega_1, \ldots, \omega_{s_0}\), where \(s_0 = N - d_0\), be differential 1-forms locally spanning \(D^\perp\), the annihilator of \(D\), which we denote by

\[ D^\perp = (\omega_1, \ldots, \omega_{s_0}) . \]

We will denote by \(I\) the Pfaffian system generated by \(\omega_1, \ldots, \omega_{s_0}\).

For any form \(\omega \in D^\perp\), we put

\[ \mathcal{W}(\omega) = \{ f \in D : f \perp d\omega \in D^\perp \} . \]

Clearly, the characteristic distribution \(C\) of \(D\) is given by

\[ C = \bigcap_{i=1}^{s_0} \mathcal{W}(\omega_i) . \]

Now assume that \(D^{(1)}\) is of constant rank \(d_1 > d_0\), that is \(r_0 \geq 1\), or, equivalently, that the first derived system \(I^{(1)}\) is of constant rank smaller than \(s_0\). By a direct
calculation we can check (see e.g. [3]) that the Engel rank of the distribution $\mathcal{D}$, or of the corresponding Pfaffian system $\mathcal{I}$, equals 1 at $p$ if and only if
\[ (d\omega_i \wedge d\omega_j)(p) = 0 \mod \mathcal{I}, \] for any $1 \leq i \leq j \leq s_0$.

Now let us choose a family of differential 1-forms $\omega_1, \ldots, \omega_{r_0}, \omega_{r_0+1}, \ldots, \omega_{s_0}$ such that $(\mathcal{D}^{(0)})^\perp = (\omega_1, \ldots, \omega_{s_0})$ and $(\mathcal{D}^{(1)})^\perp = (\omega_{r_0+1}, \ldots, \omega_{s_0})$. Independently of the value of $r_0 \geq 2$, the unique distribution $\mathcal{B}$ satisfying $[\mathcal{B}, \mathcal{B}] \subset \mathcal{D}$ is given, as shown by Bryant [2], by
\[ \mathcal{B} = \sum_{i=1}^{r_0} W(\omega_i). \] (14)

In fact, Bryant has also proved that it is enough to take in the above sum only two terms corresponding to any $1 \leq i < j \leq r_0$. In order to verify, in the case $r_0 = 2$, the conditions of Corollary 2.3 we have additionally to check the involutivity of this explicitly calculable distribution.

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