On the Bargmann-Radon transform in the monogenic setting

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Abstract

In this paper we introduce and study a Bargmann-Radon transform on the real monogenic Bargmann module. This transform is defined as the projection of the real Bargmann module on the closed submodule of monogenic functions spanned by the monogenic plane waves. We prove that this projection can be written in integral form in terms the so-called Bargmann-Radon kernel. Moreover, we have a characterization formula for the Bargmann-Radon transform of a function in the real Bargmann module in terms of its complex extension and then its restriction to the nullcone in \(\mathbb{C}^m\). We also show that the formula holds for the Szegő-Radon transform that we introduced in [4]. Finally, we define the dual transform and we provide an inversion formula.

Key words: Monogenic functions, Bargmann modules, Bargmann-Radon transform.

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1 Introduction

In the paper [11] we considered an extension of the Segal-Bargmann transform, a unitary map from spaces of square-integrable functions to spaces of square-integrable holomorphic functions (see [4], [10], [13], [14], [17]). Specifically, we studied the higher dimensional extension based on monogenic functions with values in a Clifford algebra. This approach has been used, e.g., in [7], [9] to study quantum systems with internal, discrete degrees of freedom corresponding to nonzero spins.

In [11] we introduced a notion of Segal-Bargmann module (over the Clifford algebra) which is the set of entire functions, square integrable with respect to the Gaussian density and that are in the kernel of the Dirac operator. We also defined the Segal-Bargmann-Fock transform in this framework. The fact that monogenic functions admit a Fischer decomposition allows to prove a relation between the projection of the transform onto its monogenic part and the
Fourier-Borel kernel. It is also worthwhile to mention that this kernel, unlike what happens for Hardy or Bargmann spaces, is an exponential not a rational function.

In [4] we defined the so-called Szegő-Radon projection which may be abstractly defined as the orthogonal projection of a suitable Hilbert module of square integrable left monogenic functions onto the closed submodule of monogenic functions spanned by the monogenic plane waves \( \langle x, \tau \rangle^k \), where \( \tau = t + is \), \( t, s \) are orthogonal unit 1-vectors. This transformation does not exactly correspond to the Radon transform. However it is a canonical map from \( m \)-dimensional monogenic functions to 2-dimensional monogenic functions, like in the case of the Clifford-Radon transform, see [3], [15]. The Clifford-Radon transform and, more in general, the Radon transform are important tools with several applications for example in tomography.

In this paper we combine the approaches in [4] and [11]. We introduce and study a Bargmann-Radon transform on the real monogenic Bargmann module. Similarly to what we have done in [4] in the Szegő-Radon case, it is defined as the projection of the real Bargmann module on the closed submodule of monogenic functions spanned by the monogenic plane waves \( \langle x, \tau \rangle^k \), where \( \tau = t + is \), \( t, s \) are orthogonal unit 1-vectors. This transformation does not exactly correspond to the Radon transform. However it is a canonical map from \( m \)-dimensional monogenic functions to 2-dimensional monogenic functions, like in the case of the Clifford-Radon transform, see [3], [15]. The Clifford-Radon transform and, more in general, the Radon transform are important tools with several applications for example in tomography.

In this section we collect some preliminary results and notations used in the rest of the paper. For more information on the material in this section, we refer the reader to [2], [3].

By \( \mathbb{R}_m \) we denote the real Clifford algebra over \( m \) imaginary units \( \xi_1, \ldots, \xi_m \) which satisfy the relations \( \xi_i \xi_j + \xi_j \xi_i = -2 \delta_{ij} \). An element \( x \) in the Clifford algebra is denoted by \( x = \sum_A \xi_A x_A \) where \( x_A \in \mathbb{R}, A = i_1 \ldots i_r, i_l \in \{1, 2, \ldots, n\}, i_1 < \ldots < i_r \) is a multi–index, \( \xi_A = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_r} \), and \( \xi_0 = 1 \). Similarly, we denote by \( \mathbb{C}_m \) we denote the complex Clifford algebra over \( m \) imaginary units \( \xi_1, \ldots, \xi_m \).

The so called 1-vectors are elements in \( \mathbb{R}_m \) which are linear combinations with real coefficients of the elements \( \xi_i, i = 1, \ldots, m \). The sets of 1-vectors is denoted by \( \mathbb{R}_m^{(1)} \). The map from \( \mathbb{R}_m \) to \( \mathbb{R}_m^{(1)} \) is given by \( (x_1, x_2, \ldots, x_m) \mapsto x = x_1 \xi_1 + \ldots + x_m \xi_m \) and it is obviously one-to-one.
The norm of a 1-vector is defined as $|\underline{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and the scalar product of $\underline{x}$ and $\underline{y} = y_1 \underline{e}_1 + \cdots + y_m \underline{e}_m$ is

$$\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + \cdots + x_m y_m.$$ 

In $\mathbb{C}_m$ there are automorphisms which leave the multivector structure invariant. In this paper we will use the so-called Hermitian conjugation

$$\lambda \mu^\dagger = \mu^\dagger \lambda^\dagger, \quad (\lambda A \underline{e}_A)^\dagger = \lambda^\dagger A^\dagger \underline{e}_A^\dagger, \quad \underline{e}_j^\dagger = -\underline{e}_j, \quad j = 1, \ldots, n,$

where $\lambda^\dagger$ stands for the complex conjugate of the complex number $\lambda$.

In the sequel, we will denote by $B(0,1)$ the unit ball with center at the origin in $\mathbb{R}^m$ while the symbol $S^{m-1}$ will denote its boundary, that is the sphere of unit 1-vectors in $\mathbb{R}^m$:

$$S^{m-1} = \{\underline{x} = e_1 x_1 + \cdots + e_m x_m : x_1^2 + \cdots + x_m^2 = 1\},$$

whose area, denoted by $A_m$ is given by

$$A_m = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}.$$

**Definition 2.1.** A function $f : \Omega \subseteq \mathbb{R}^m \to \mathbb{C}_m$ defined and continuously differentiable in the open set $\Omega$ is said to be (left) monogenic if it satisfies

$$\partial_{\underline{x}} f(\underline{x}) = \sum_{j=1}^m e_j \partial x_j f(\underline{x}) = 0.$$ 

If $f : \Omega \subseteq \mathbb{C}_m \to \mathbb{C}_m$ is as above, we say that $f$ is (left) monogenic in $\Omega$ if it is holomorphic and in the kernel of the complexified Dirac operator $\sum_{j=1}^m e_j \partial x_j$. We denote by $\mathcal{M}(\Omega)$ the right $\mathbb{C}_m$-module of (left) monogenic functions in $\Omega$.

A classical tool in Clifford analysis is the so-called Fischer decomposition. It provides a unique decomposition of an arbitrary homogeneous polynomial in $\mathbb{R}^m$ as

$$R_k(\underline{x}) = M_k(\underline{x}) + \underline{x} R_{k-1}(\underline{x}),$$

where the subscripts denote the degree of homogeneity of the polynomial and $M_k \in \mathcal{M}(\mathbb{R}^m)$. The monogenic polynomial $M_k$ is called monogenic part of $R_k$ and is denoted by $M(R_k)$. The Fischer decomposition of the function $\frac{1}{\underline{x}} (\underline{x}, \underline{u})^k$ can be written in terms of the so-called zonal spherical monogenics which are defined by

$$Z_k(\underline{x}, \underline{u}) = \frac{\Gamma\left(\frac{m}{2} - 1\right)}{2^{k+1} \Gamma\left(\frac{m}{2} + k\right)} |\underline{u}| |\underline{x}|^k \left((k + m - 2)C_k^{m-1}(t) + (m - 2)\frac{\underline{u} \wedge \underline{x}}{|\underline{u}|^2} C_k^{m}(t)\right) \quad (1)$$

where $t := \frac{(\underline{x}, \underline{u})}{|\underline{u}| |\underline{x}|}$ and $C_k^\lambda(t)$ are the Gegenbauer polynomials. Let us define $Z_{k,0}(\underline{x}, \underline{u}) = Z_k(\underline{x}, \underline{u})$ and

$$Z_{k,s}(\underline{x}, \underline{u}) = \frac{Z_{k-s,0}(\underline{x}, \underline{u})}{\beta_{s-k-s} \cdots \beta_{1-k-s}}, \quad k \geq s$$

with $\beta_{2s,k} = -2s$, $\beta_{2s+1,k} = -(2s + 2k + m)$. Then we have:

$$\frac{1}{k!} (\underline{x}, \underline{u})^k = \sum_{s=0}^k \frac{\beta_s}{\beta_{s-k-s} \cdots \beta_{1-k-s}} Z_{k,s}(\underline{x}, \underline{u}) u^s. \quad (2)$$

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Using (2), we obtain the Fischer decomposition of \( \exp(\langle z, x \rangle) \), see [6], namely
\[
\exp(\langle x, u \rangle) = \sum_{k=0}^{\infty} \frac{\langle x, u \rangle^k}{k!}
= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} x^s Z_{k,s}(x, u) u^s
= \sum_{s=0}^{\infty} x^s (\sum_{k=s}^{\infty} Z_{k,s}(x, u)) u^s
= E(x, u) + \sum_{s=1}^{\infty} x^s E_s(x, u) u^s,
\]
where \( E_s(x, u) = \sum_{k=s}^{\infty} Z_{k,s}(x, u) \). The function \( E(x, u) \) is the monogenic part of \( \exp(\langle x, u \rangle) \) and it is the Fourier-Borel kernel, see [6], [12]. Note that it is hermitian, namely, \( E^\dagger(u, x) = E(x, u) \).

3 The Bargmann-Radon transform

In this section we introduce and study the Bargmann-Radon transform on the (real) monogenic Bargmann module. In particular, we introduce the Bargmann-Radon kernel and we use it to express the Bargmann-Radon transform in integral form. We also show that this transform gives rise to monogenic functions that can be expressed in an interesting way on the nullcone.

We begin by giving the definition of the so-called monogenic Bargmann module (see section 5 in [11]):

**Definition 3.1.** The monogenic Bargmann module \( MB(\mathbb{R}^m) \) consists of the functions \( f \in \mathcal{M}(\mathbb{R}^m) \) such that
\[
f(z) e^{-|\omega|^2/4} \in L^2(\mathbb{R}^m),
\]
and equipped with the inner product
\[
\langle f, g \rangle_{MB} = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|\omega|^2/2} f^\dagger(\omega) g(\omega) d\omega.
\]

Note that an analogous definition has been given in [11], section 4, for functions in the kernel of the complexified Dirac operator (or its powers). More precisely, we have

**Definition 3.2.** The Segal-Bargmann-Fock space \( B(\mathbb{C}^m) \) is the Hilbert space of entire functions in \( \mathbb{C}^m \) which are square-integrable with respect to the \( 2m \)-dimensional Gaussian density, i.e.
\[
\frac{1}{\pi^m} \int_{\mathbb{C}^m} \exp(-|\omega|^2) |f(\omega)|^2 d\omega < \infty,
\]
and equipped with the inner product
\[
\langle f, g \rangle_B = \frac{1}{\pi^m} \int_{\mathbb{C}^m} \exp(-|\omega|^2) f^\dagger(\omega) g(\omega) d\omega.
\]
The monogenic Bargmann module \( MB(\mathbb{C}^m) \) is defined as
\[
MB(\mathbb{C}^m) = \mathcal{M}(\mathbb{C}^m) \cap B(\mathbb{C}^m),
\]
and it is equipped with the inner product defined in \( B(\mathbb{C}^m) \).
**Definition 3.3.** We define the Bargmann transform of \( f \in \mathcal{MB}(\mathbb{R}^m) \) as

\[
\mathcal{B}[f(x)e^{-|x|^2/4}] = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \exp\left(-\frac{1}{2} \langle \bar{z}, z \rangle + \frac{1}{4} |z|^2\right) f(x)e^{-|x|^2/4} d\bar{x}.
\]

We note that

\[
\mathcal{B}[f(x)e^{-|x|^2/4}] = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \exp\left(-\frac{1}{2} \langle \bar{z}, z \rangle + \frac{1}{4} |z|^2\right) f(x)e^{-|x|^2/4} d\bar{x}
\]

\[
= f(z)
\]

\[
= (2\pi)^{-m/2} \int_{\mathbb{R}^m} E(\bar{z}, x)f(x)e^{-|x|^2/2} d\bar{x},
\]

where \( E(\bar{z}, x) \) is the monogenic part of \( \exp(\langle \bar{z}, x \rangle) \), see Section 2.

Let us denote by \([\cdot, \cdot]\) the Fischer inner product in \( \mathcal{M}(\mathbb{R}^m)\):

\[
[R, S] = R(\partial_x)^\dagger S(x)|_{x=0}.
\]

From this definition, we immediately obtain the formula

\[
f(u) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-|\bar{x}|^2/2} E(u, \bar{x})f(x) d\bar{x}
\]

\[
= [E^\dagger(u, x), f(x)].
\]

By taking the holomorphic extensions of \( f, g \) to \( \mathbb{C}^m \) and using the fact that \( \mathcal{MB}(\mathbb{R}^m) \) equipped with the Fischer inner product and \( \mathcal{MB}(\mathbb{C}^m) \) are isometric (see \[11\]) we obtain

\[
\langle f, g \rangle_{\mathcal{MB}} = \frac{1}{\pi^m} \int_{\mathbb{C}^m} e^{-|z|^2} f(z)^\dagger g(z) dz.
\]

Moreover, by the definition of Fischer product extended to functions defined over \( \mathbb{C}^m \), we deduce

\[
f(u) = \frac{1}{\pi^m} \int_{\mathbb{C}^m} e^{-|\bar{z}|^2} E(\bar{z}, u)^\dagger f(z) d\bar{z}.
\]

We now consider the following submodules of the module \( \mathcal{MB}(\mathbb{R}^m) \).

**Definition 3.4.** For any given \( \tau = t + is \), \( t, s \in \mathbb{R}^m \), where \( |t| = |s| = 1 \), \( t \perp s \), the closure of the right \( \mathbb{C}_m \)-module consisting of all finite linear combinations

\[
\sum_{\ell \in \mathbb{N}} \langle x, \tau \rangle^\ell \tau
\]

is denoted by \( \mathcal{MB}(\tau) \).

The following result from \[4\] is useful for the computations in the sequel:

**Proposition 3.5.** Let \( t, s \in \mathbb{R}^n \) be such that \( |t| = |s| = 1 \) and \( \langle t, s \rangle = 0 \) and let \( \tau = t + is \in \mathbb{C}^m \). Then

1. \( \tau^\dagger \tau = 4 \tau \),
2. \( \tau^2 = 0 \),
3. \( \tau^\dagger + \tau \tau^\dagger = 4 \).
We note that since $\tau^2 = 0$ then $\tau$ is an element in the nullcone in $\mathbb{C}^m$.

**Definition 3.6.** We define the Bargmann-Radon transform of $f \in \mathcal{M}B(\mathbb{R}^m)$ as

$$\mathcal{R}_\tau[f] = \text{Proj}_{\mathcal{M}B(\tau)} f$$

where $\text{Proj}_{\mathcal{M}B(\tau)}$ denotes the projection on $\mathcal{M}B(\tau)$.

The Bargmann–Radon kernel is of the same form as the Szegő-Radon kernel introduced in [4] but with different coefficients:

$$B_\tau(u, x) = \sum_{\ell=0}^{\infty} \lambda_\ell(u, \tau)^\ell \tau^\dagger(x, \tau) \ell.$$

**Remark 3.7.** The calculations of the coefficients $\lambda_\ell$ follow from the fact that the integral

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|u|^2/2} \tau^\dagger(x, \tau) \ell \langle x, \tau \rangle \ell' dx$$

is zero when $\ell \neq \ell'$, because of the definition of the Fischer inner product. If $\ell = \ell'$ the integral equals

$$\frac{(-1)^\ell}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|u|^2/2} \langle x, \tau \rangle^{2\ell} dx.$$

This last integral does not depend on $\tau$ so we can compute it for a specific choice of $\tau$, for example $\tau = e_1 + i e_2$. We have, by setting $y = (x_3, \ldots, x_m)$,

$$\int_{\mathbb{R}^m} e^{-|u|^2/2} (x_1^2 + x_2^2) dx = \int_{\mathbb{R}^{m-2}} e^{-|u|^2/2} dy \int_0^{+\infty} \int_0^{2\pi} e^{-r^2/2} r^{-2\ell} dr d\theta = (2\pi)^{m/2} \int_0^{+\infty} 2^\ell e^{-s^2} ds = (2\pi)^{m/2} 2^\ell \ell!.$$

**Lemma 3.8.** The formula

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|u|^2/2} B_\tau(u, x) \tau(x, \tau) \ell dx = \tau \langle u, \tau \rangle \ell$$

holds if and only if

$$\lambda_\ell = \frac{(-1)^\ell}{\ell! 4 \cdot 2^\ell}.$$

Moreover

$$B_\tau(u, x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! 4 \cdot 2^\ell} (u, \tau)^\ell \tau^\dagger(x, \tau) \ell = \frac{\tau \tau^\dagger}{4} \exp(-\frac{1}{2} (u, \tau)(x, \tau^\dagger)).$$

**Proof.** We compute

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|u|^2/2} B_\tau(u, x) \tau(x, \tau) \ell dx = \tau \tau^\dagger \tau \lambda_\ell (-1)^\ell 2^\ell \langle u, \tau \rangle \ell$$

using Remark 3.7. We have:

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|u|^2/2} B_\tau(u, x) \tau(x, \tau) \ell dx = \tau \tau^\dagger \tau \lambda_\ell (-1)^\ell 2^\ell \langle u, \tau \rangle \ell$$

$$= \tau \langle u, \tau \rangle \ell.$$
if and only if  
\[ \tau \tau^\dagger \lambda \ell(-1)^\ell \ell!2^\ell = \tau. \]

By Proposition 3.5 we have that  \( \tau \tau^\dagger \tau = 4\tau \) and so we obtain the statement. \( \square \)

Since  \( B_\tau \) is a reproducing kernel for the generators of  \( MB(\tau) \), and since  \( \tau \tau^\dagger \) commutes with  \( \langle u, \tau \rangle \) we immediately have:

**Corollary 3.9.** The function  \( B_\tau(u, x) \) is a reproducing kernel for the  \( C^m \)-module  \( MB(\tau) \).

The following result expresses the Bargmann-Radon transform of  \( f \in MB(\mathbb{R}^m) \) in terms of the Bargmann-Radon kernel:

**Theorem 3.10.** Let  \( f \in MB(\mathbb{R}^m) \). The following formula holds

\[ \mathcal{R}_\tau[f](\bar{u}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|x|^2/2} B_\tau(u, x) f(x) dx. \]

**Proof.** The assertion follows using standard arguments. First of all, we note that the operator  \( P \) defined by

\[ P[f](\bar{u}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|x|^2/2} B_\tau(u, x) f(x) dx \]

is idempotent on  \( MB(\mathbb{R}^m) \) and coincides with the identity on  \( MB(\tau) \) by virtue of Corollary 3.9. The fact that the kernel  \( B_\tau(u, x) \) is hermitian gives  \( \langle Pf, g \rangle = \langle f, Pg \rangle \). Thus  \( P \) is the orthogonal projection of  \( MB(\mathbb{R}^m) \) on  \( MB(\tau) \) and thus it coincides with  \( \mathcal{R}_\tau \) as stated. \( \square \)

Next result is interesting because it shows that the Bargmann-Radon transform of  \( f \in MB(\mathbb{R}^m) \) is a monogenic function which can be seen as a suitable multiple of the restriction to the nullcone of its extension to  \( \mathbb{C}^m \):

**Theorem 3.11 (Characterization formula).** The Bargmann-Radon transform of  \( f \in MB(\mathbb{R}^m) \) is a monogenic function that can be expressed as:

\[ \mathcal{R}_\tau[f](\bar{u}) = \frac{\tau \tau^\dagger}{4} f(-\frac{1}{2}\tau^\dagger \langle u, \tau \rangle). \]

**Proof.** First of all, any entire holomorphic function  \( h \) can be written as

\[ h(\bar{z}) = M[h](\bar{z}) + z h(\bar{z}) \]

where  \( M[h] \) denotes the monogenic part of  \( h \). Since  \( \tau^2 = (\tau^\dagger)^2 = 0 \) we have that

\[ \tau^\dagger h(\tau^\dagger) = \tau^\dagger M[h](\tau^\dagger). \]

In particular, if we take  \( h(\bar{z}) = \exp(-\frac{\lambda}{2} \langle z, \bar{z} \rangle) \) we obtain:

\[ \tau^\dagger \exp(-\frac{\lambda}{2} \langle z, \bar{z} \rangle) = \tau^\dagger (E(\tau^\dagger, -\frac{\lambda}{2}z) + \tau^\dagger \ldots) \]

\[ = \tau^\dagger E(\tau^\dagger, -\lambda/2z) \]

\[ = \tau^\dagger E(-\lambda/2\tau^\dagger, \bar{z}), \]

\[ (4) \]
thus, using (3), we have:

\[ f(-\frac{\lambda}{2} \tau^\dagger) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|\mathbf{u}|^2/2} E(\frac{-\lambda}{2} \tau^\dagger, \mathbf{x}) f(\mathbf{x}) \, d\mathbf{x}. \]

We now note that we can rewrite the formula in Theorem 3.10 as

\[ \mathcal{R}_\tau[f](\mathbf{u}) = \frac{1}{(2\pi)^{m/2}} \tau \tau^\dagger \int_{\mathbb{R}^m} e^{-|\mathbf{x}|^2/2} \exp\left(\frac{1}{2}\lambda \langle \mathbf{x}, \tau^\dagger \rangle \right) f(\mathbf{x}) \, d\mathbf{x} \]

Using (4) in the last formula, we get the statement.

\[ \square \]

4 The Szegő-Radon transform

In our paper [4] we considered instead of the ambient module \( M\mathcal{B}(\mathbb{R}^m) \) another \( \mathbb{C}_m \)-module that we recall below:

**Definition 4.1.** The monogenic Szegő module is defined as the right \( \mathbb{C}_m \)-module \( M\mathcal{L}^2(B(0,1)) \) of the monogenic functions \( f : B(0,1) \subset \mathbb{C}_m \to \mathbb{C}_m \) for which \( \lim_{r \to 1} f(r\omega) \in \mathcal{L}^2(S^{m-1}) \), equipped with the Hilbert inner product

\[ \langle f, g \rangle_{M\mathcal{L}^2} = \int_{S^{m-1}} f^\dagger(\omega)g(\omega) \, dS(\omega). \]

By the extension of the Cauchy formula, for \( \mathbf{x} \in B(0,1) \), we have

\[ f(\mathbf{x}) = \frac{1}{A_m} \int_{S^{m-1}} \frac{\mathbf{x} - \omega}{|\mathbf{x} - \omega|^m} \omega f(\omega) \, dS(\omega) \]

where \( A_m = \frac{2\pi^{m/2}}{\Gamma(m/2)} \). For the standard Szegő-Radon transform we again start from the plane waves

\[ f_{\tau,k}(\mathbf{x}) = \langle \mathbf{x}, \tau \rangle^k \tau \]

where \( \tau = t + is \) with \( |t| = |s| = 1 \) and \( t \perp s \) so that \( \tau \tau^\dagger + \tau^\dagger \tau = 4 \), see Proposition 3.5. Since

\[ f_{\tau,k}^\dagger(\mathbf{x}) = (-1)^k \langle \mathbf{x}, \tau^\dagger \rangle^k \tau^\dagger \]

we obtain (see [4]):

\[ \langle f_{\tau,k}, f_{\tau,k} \rangle = 2\pi^{m/2} \frac{\Gamma(k+1)}{\Gamma(m/2+1)}. \]

In the Szegő-module we can give the analogue of Definition 3.4:

**Definition 4.2.** We denote by \( M\mathcal{L}^2(\tau) \) the submodule of \( M\mathcal{L}^2(B(0,1)) \) which is the closure of the \( \mathbb{C}_m \)-module consisting of all finite linear combinations \( \sum_k f_{\tau,k}(\mathbf{x})a_k, a_k \in \mathbb{C}_m \) of monogenic plane waves \( f_{\tau,k}(\mathbf{x}) \).

As we have done in the previous section, we can consider the orthogonal projection on this submodule and we can describe its kernel, see [4]:

\[ \square \]
Definition 4.3. The Szegő-Radon transform \( R_{\tau}[f] \) of \( f \in M\mathcal{L}^2(B(0,1)) \) is defined as the orthogonal projection of \( f \) on the submodule \( M\mathcal{L}^2(\tau) \).

The kernel of this projection is given by

\[
K(x, y) = \frac{\tau \tau^\dagger}{4} \frac{\Gamma(m/2)}{2\pi^{m/2}} (1 + \langle x, \tau \rangle \langle y, \tau^\dagger \rangle)^{-m/2}
= \frac{\tau \tau^\dagger}{4} \sum_{k=0}^{\infty} \frac{\Gamma(m/2 + k)}{2^{m/2} \Gamma(k + 1)} \langle x, \tau \rangle^k \langle y, \tau^\dagger \rangle^k
\]

and so we have

\[
R_{\tau}[f] = \int_{S^{m-1}} K_{\tau}(x, \omega) f(\omega) dS(\omega).
\]

The kernel \( K_\tau \) can be directly related to the Szegő kernel

\[
S(x, \omega) = \frac{1}{A_m} \frac{1 + x^\dagger \omega}{1 + |x|^2 - 2 \langle x, \omega \rangle}^{m/2}
\]

via the formula proved in the following lemma:

Lemma 4.4. We have

\[
K_\tau(x, \omega) = \frac{\tau \tau^\dagger}{4} S(-\langle x, \tau \rangle \frac{\tau}{2}, \omega)
\]

Proof. By setting \( \lambda = \langle x, \tau \rangle \), we obtain that

\[
S(-\lambda \frac{\tau}{2}, \omega) = \frac{1}{A_m} \frac{1 - \frac{\lambda}{2} \tau^\dagger \omega}{(1 + \lambda \tau^\dagger \omega)^{m/2}}
\]

since \( \langle \tau^\dagger, \tau^\dagger \rangle = 0 \), so the term that contains \( |x|^2 \) disappear. So the formula follows from the fact that \( \tau \tau^\dagger (\tau^\dagger \omega) = 0 \).

The previous lemma allows to prove that the Szegő-Radon transform satisfies the same characterization formula that we have obtained in Section 3 in the case of the Bargmann-Radon transform. This fact motivates the use of the same symbol \( R_{\tau} \) for both. Indeed we have:

Theorem 4.5 (Characterization formula). Let \( f \in M\mathcal{L}^2(B(0,1)) \). Then the following formula holds:

\[
R_{\tau}[f](\tau) = \frac{\tau \tau^\dagger}{4} f(-\frac{1}{2} \tau^\dagger \langle u, \tau \rangle).
\]

Proof. Lemma 4.4 implies directly the equalities

\[
R_{\tau}[f](\tau) = \int_{S^{m-1}} K_{\tau}(x, \tau) f(\omega) dS(\omega)
= \frac{\tau \tau^\dagger}{4} \int_{S^{m-1}} S(-\frac{\tau}{2}, \omega) f(\omega) dS(\omega)
= \frac{\tau \tau^\dagger}{4} f(-\frac{1}{2} \langle u, \tau \rangle),
\]

and the statement follows.
Example 4.6. The following example is important because we consider the function
\[ g(x) = \sigma(x, a)^\ell, \quad a^2 = 0, \quad \ell \in \mathbb{N} \]
which generates the module of all spherical monogenics of degree \( \ell \). It can be verified with direct computations that
\[ R_\tau[g](\tau) = \frac{\tau\tau^\dagger}{4} a \left( -\frac{1}{2}(\tau^\dagger, a)(\tau, \tau) \right)^\ell. \]

5 The dual Bargmann-Radon transform and the inversion formula

Both the dual transform and the inversion formula will be the same for the Bargmann-Radon transform and for the Szegő-Radon transform. The main results for the Szegő-Radon transform were presented in [4]. Here we repeat the results from the Bargmann-Radon point of view.

For every inner spherical monogenic \( P_k(x) \) we have the formula
\[ P_k(u) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|\mathcal{E}|^2/2} Z_k(u, x) P_k(x) dx, \]
which is in accordance with the fact that the Fourier-Borel kernel
\[ E(u, x) = \sum_{k=0}^\infty Z_k(u, x) \]
is the reproducing kernel for the monogenic Bargmann module, see formula (3).

The Bargmann-Radon transform maps a monogenic function \( f \) into \( MB(\mathbb{R}^m) \) into \( MB(\tau) \) and it can be expressed as, see Theorem 3.11:
\[ R_\tau[f](\tau) = \frac{\tau\tau^\dagger}{4} f(-\frac{1}{2}(\tau^\dagger, x)(\tau, x)). \]

Definition 5.1. Let \( F(u, \tau) \) be a function in \( BM(\tau) \). The dual Bargmann-Radon transform of \( F \) is defined by
\[ \tilde{R}[F](u) = \frac{1}{A_m A_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} F(u, \ell + i\sigma)dS(\tau)dS(\sigma), \]
where for fixed \( \ell \in S^{m-1} \) the sphere \( S^{m-2} \) contains the elements \( \sigma \in S^{m-1} \) such that \( \sigma \perp \ell \).

Note that the dual Bargmann-Radon transform is in fact the average of a function over the Stiefel manifold of 2-frames.

Our main task in now to compute \( \tilde{R}[R_\tau[f]](u) \) and to relate this with \( f \). As \( f \) admits a monogenic Taylor series
\[ f(x) = \sum_{k=0}^\infty P_k(x) \tag{5} \]
where \( P_k(x) \) are inner spherical monogenics of degree \( k \), it will be sufficient to study \( \tilde{R}[R_\tau P_k](u) \), where
\[ R_\tau[P_k](u) = \frac{(-1)^k \tau\tau^\dagger}{2k!} \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|\mathcal{E}|^2/2}(\langle u, \tau \rangle (\langle x, \tau^\dagger \rangle)^k P_k(x) dx. \tag{6} \]

In our paper [4], see Theorem 5.4, we have proved a result which will be crucial in the sequel. We repeat it here for the sake of completeness and adapting the notation to the present setting:
Theorem 5.2. For $\tau = t + is$, there exists a constant $\lambda_k$ such that

$$\frac{1}{A_{m-1}A_m} \int_{S^{m-1}} dS(t) \int_{S^{m-2}} dS(s) \langle x, \tau \rangle^k \langle y, \tau \rangle^k \tau\tau^\dagger = (7)$$

where for fixed $t \in S^{m-1}$ the sphere $S^{m-2}$ contains the elements $s \in S^{m-1}$ such that $s \perp t$ and the constant $\lambda_k$ is given by

$$\lambda_k = \frac{2\pi(-1)^k A_{m-2}}{(k + m - 2)A_m C_{k-1}^{m/2-1}} \prod_{j=2} k \Gamma \frac{(m - 1) \Gamma (k + 1)}{\Gamma (m + k - 1)}$$

We are now ready to compute $\tilde{R}[R_{\tau P_k}](u)$:

Theorem 5.3. We have the relation

$$\tilde{R}[R_{\tau P_k}](u) = \frac{1}{2} \frac{\Gamma (m - 1) \Gamma (k + 1)}{\Gamma (m + k - 1)} P_k(u)$$

Proof. In order to compute the dual Bargmann-Radon transform of $R_{\tau P_k}$, we need to compute the integral

$$I_k := \frac{1}{A_{m-1}A_m} \int_{S^{m-1}} dS(t) \int_{S^{m-2}} dS(s) \langle x, \tau \rangle^k \langle y, \tau \rangle^k \tau\tau^\dagger dS(t)dS(s).$$

Using the fact that

$$C_{k-1}^{m/2-1}(1) = \frac{\Gamma (m - 2 + k)}{\Gamma (m - 2) \Gamma (k + 1)}$$

we thus obtain that in fact

$$I_k = \mu_k Z_k(u, x)$$

with

$$\mu_k = \frac{2\pi(-1)^k A_{m-2}}{A_m \Gamma (m - 2) \Gamma (k + 1)} 2^{k+1} \Gamma (k + 1)$$

and $Z_k$ is the zonal spherical monogenic function defined in [1]. Since

$$\frac{A_{m-2}}{A_m} = \frac{m - 2}{2\pi}$$

we have

$$\mu_k = 2^{k+1}(-1)^k \frac{\Gamma (m - 1) \Gamma (k + 1)}{\Gamma (m + k - 1)}.$$

Thus in order to compute the dual transform of $R_{\tau P_k}$ we in fact have to compute

$$\frac{(-1)^k}{2^{k+2}k!} I_k = \frac{1}{2} \frac{\Gamma (m - 1) \Gamma (k + 1)}{\Gamma (m + k - 1)} Z_k(u, x)$$

We now have that

$$\tilde{R}[R_{\tau P_k}](u) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-|x|^2/2} \tilde{R}[F](x) P_k(x) dx.$$
\[ \tilde{R}[F](x) = \frac{(-1)^k 1}{2^k k! \cdot 4} \tilde{R}[\bar{x} \bar{z}^\dagger (u, \bar{z})^k / (x^\dagger)^k] \]
\[ = \frac{1}{2} \frac{\Gamma(m - 1) \Gamma(k + 1)}{\Gamma(m + k - 1)} Z_k(u, x) \]

which, together with the reproducing property of \( Z_k \) leads to the result. \( \square \)

Now we prove the following:

**Theorem 5.4 (Inversion formula).** Let \( f \in \mathcal{MB}(\mathbb{R}^m) \), then

\[ f(u) = \frac{2}{(m - 2)!}(m - 1 - \Gamma) \ldots (1 - \Gamma) \tilde{R}[\mathcal{R}_x f](u), \]

where

\[ \Gamma := -\bar{x} \wedge \partial_x \]

is the \( \Gamma \) operator.

**Proof.** It is sufficient to decompose \( f \) in Taylor series \( f(x) = \sum_{k=0}^{\infty} P_k(x) \), to notice that \( \Gamma P_k = -kP_k \) and to apply the previous result. \( \square \)

**Remark 5.5.** We observe that:

(i) Monogenic functions satisfy the equation

\[ (E + \Gamma)f(x) = 0 \]

where \( E = \sum_{j=1}^{n} x_j \partial_{x_j} \) is the Euler operator. So for monogenic functions we also have that

\[ f(u) = \frac{2}{(m - 2)!}(m - 1 + E) \ldots (1 + E) \tilde{R}[\mathcal{R}_x f](u). \]

(ii) Notice that

\[ \tilde{R}[\mathcal{R}_x f](u) = \frac{1}{A_m A_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} \frac{\tau \tau^\dagger}{4} f \left( \frac{\tau^\dagger}{2} (u, \bar{z}) \right) dS(t) dS(s) \]

for which we have to use the complex extension \( f(z) \) of \( f(x) \) and put

\[ z = -\frac{\tau^\dagger}{2} (u, \bar{z}). \]

An interesting consequence of this theory is an implicit formula for the monogenic part \( M[h] \) of a holomorphic function belonging to the Bargmann module \( \mathcal{B}(\mathbb{C}^m) \). Any entire holomorphic function \( h \) admits the Fischer decomposition

\[ h(z) = \sum_{\ell=0}^{\infty} z^\ell h_\ell(z) \]

where \( h_\ell \) is complex monogenic, that is \( \partial_z h_\ell(z) = 0 \). Using this fact we can prove:
Theorem 5.6. The monogenic part $M[h]$ of an entire holomorphic function $h$ is given by (for $\mathbf{u} \in \mathbb{C}^m$ or in $\mathbb{R}^m$)

$$M[h](\mathbf{u}) = \Theta \frac{1}{A_mA_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} \frac{\tau^\dagger}{4} h \left(-\frac{\tau^\dagger}{2} (\mathbf{u}, \tau)\right) \, dS(t)dS(s)$$

where

$$\Theta := \frac{2}{(m-2)!} (m-1-\Gamma)(m-2-\Gamma) \cdots (1-\Gamma).$$

Proof. The result hold in the case when

$$f(\mathbf{u}) = M[h](\mathbf{u})$$

is the monogenic Bargmann module, i.e., when $h \in \mathcal{B}(\mathbb{C}^m)$ because indeed

$$f(\mathbf{u}) = \Theta \frac{1}{A_mA_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} \frac{\tau^\dagger}{4} f \left(-\frac{\tau^\dagger}{2} (\mathbf{u}, \tau)\right) \, dS(t)dS(s).$$

Now, using the Fischer decomposition $h(\bar{z}) = \sum_{\ell=0}^{\infty} z^{\ell} h_\ell(\bar{z})$ of $h$ we obtain for $\bar{z} = -\frac{\tau^\dagger}{2} (\mathbf{u}, \tau)$

$$\frac{\tau^\dagger}{4} h \left(-\frac{\tau^\dagger}{2} (\mathbf{u}, \tau)\right) = \frac{\tau^\dagger}{4} \sum_{\ell=0}^{\infty} \bar{z}^{\ell} h_\ell(\bar{z})$$

$$= \frac{\tau^\dagger}{4} h_0 \left(-\frac{\tau^\dagger}{2} (\mathbf{u}, \tau)\right)$$

and $h_0 = M[h] = f$. The result extends to holomorphic functions in a neighborhood of the origin. \hfill $\square$

To our knowledge this is the first time that the monogenic part of a function is given in integral form and, as a possible application, we provide two examples namely the monogenic part of the Fourier-Borel kernel and of the Szegő kernel.

Example 5.7. The Fourier-Borel kernel $E(\mathbf{u}, \mathbf{z})$.

As we explained at the end of Section 2, the Fourier-Borel kernel is the monogenic part of the function $\exp(\langle \bar{z}, \tau \rangle)$. Applying Theorem 5.6, where we set $h(\bar{z}) = \exp(\langle \bar{z}, \tau \rangle)$ so that

$$h \left(-\frac{\tau^\dagger}{2} (\mathbf{u}, \tau)\right) = \exp \left(-\frac{\tau^\dagger}{2} (\mathbf{u}, \tau), \langle \bar{z}, \tau \rangle \right) = \exp \left(-\frac{1}{2} \langle \mathbf{u}, \tau \rangle \langle \bar{z}, \tau \rangle \right) ,$$

we have

$$E(\mathbf{u}, \mathbf{z}) = \Theta \frac{1}{A_mA_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} \frac{\tau^\dagger}{4} \exp \left(-\frac{1}{2} \langle \mathbf{u}, \tau \rangle \langle \bar{z}, \tau \rangle \right) \, dS(t)dS(s).$$

This formula expresses the Fourier-Borel kernel in terms of the integral

$$H(\mathbf{u}, \mathbf{z}) = \frac{1}{A_mA_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} \frac{\tau^\dagger}{4} \exp \left(-\frac{1}{2} \langle \mathbf{u}, \tau \rangle \langle \bar{z}, \tau \rangle \right) \, dS(t)dS(s)$$

that is a zonal biregular function.
Example 5.8. The Szegő kernel $S(u, x)$.

We recall that

$$S(z, x) = \frac{1}{A_m} \frac{1 + \bar{z}x}{(1 + \langle \bar{z}, z \rangle |x|^2 - 2 \langle \bar{z}, x \rangle)^{m/2}}.$$ 

Thus, using Theorem 5.6, we obtain

$$S(u, x) = \Theta \frac{\Theta}{A_m^2 A_{m-1}} \int_{S^{m-1}} \int_{S^{m-2}} \frac{\tau \tau^\dagger}{2} (1 + \langle u, \tau \rangle \langle \bar{z}, \tau^\dagger \rangle)^{-m/2} dS(\tau) dS(\bar{z}),$$

which is also the monogenic part of the Cauchy-Hua kernel

$$(1 + \langle \bar{z}, z \rangle \langle w^\dagger, w^\dagger \rangle + 2 \langle \bar{z}, w^\dagger \rangle)^{-m/2}, \quad w^\dagger = -x,$$

see also [8], [10].

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