SHELAH'S PARTITION FUNCTIONS AND
THE HALES-JEWETT NUMBERS

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Abstract. In this paper we study several partition relations, defined by Saharon Shelah, and relate them to the Hales-Jewett numbers. In particular we give an upper bound for the Hales-Jewett numbers using the primitive recursive function $f^8$, which belongs to the class $E^5$ of the Grzegorczyk hierarchy and grows slower than the function $f^{13}$. This improves the recent result of the first author and Shelah.

1. Introduction

The Hales-Jewett theorem is the bread and butter of Ramsey theory. In the words of [5], “the Hales-Jewett theorem strips van der Waerden’s theorem of its unessential elements and reveals the heart of Ramsey theory. It provides a focal point from which many results can be derived and acts as a cornerstone for much of the more advanced work”.

Van der Waerden’s theorem [15] states that a finite coloring of $\mathbb{N}$ contains arbitrarily long monochromatic arithmetic progressions. In fact, in 1927, he proved a seemingly stronger statement (which is actually equivalent to the previous statement, by compactness) as follows.

Theorem 1.1 (Van der Waerden). For all positive integers $n$ and $r$ there exists the least positive integer $W(r,n)$ such that any $r$-coloring of $\{1,\ldots, W(r,n)\}$ contains a monochromatic $n$-term arithmetic progression.

In 1963, Hales and Jewett [6] made the crucial observation that van der Waerden’s famous “double-induction proof” can be adapted to higher-dimensions.
Theorem 1.2 (Hales and Jewett). For all positive integers $n$, $r$ there exists the least positive integer $\text{HJ}(r, n)$ such that any $r$-coloring of $n^{\text{HJ}(r, n)}$ contains a monochromatic line.

Roughly speaking, the $k$-dimensional Hales-Jewett theorem says that for a given finite set $\Lambda$ of alphabets and a finite set $C$ of colors, every coloring of the combinatorial space $U_{M, \Lambda} = \{ \eta : \eta : M \to \Lambda \}$ has, under some appropriate assumptions, a monochromatic $k$-dimensional subspace. How large is $\text{HJ}(r, n)$? This is a famous open problem. The best known upper bound on $\text{HJ}(r, n)$ was proved by Shelah [11] in 1988. It is a primitive recursive function which is far better than the original Hales-Jewett threshold. The original van der Waerden–Hales–Jewett proof led to the notorious Ackermann function. However, Shelah’s improved bound is still far too large for “tractable combinatorics”. To get an idea about the order of magnitude of Shelah’s bound, look at the following example:

$$\text{HJ}(2, 4) \leq 2^{2^{2^2}}$$

where the height of the tower is 24. In [7], Matet streamlines Shelah’s argument, in particular replacing a brute force argument with a clever use of Ramsey’s theorem, to give a proof that takes less than a page. We refer to [8] for a compact version of Shelah’s proof. In the other direction, finding lower bounds for Hales-Jewett numbers is important. In their original paper, Hales and Jewett [6] also proved a linear lower bound to the Hales-Jewett numbers by an explicit construction. In [12], Shelah introduced a long list of primitive recursive functions of the form $f^\omega(-, -)$ and $f^-\omega(-, -)$ and compared them with the Hales-Jewett numbers in order to approach a tight upper (lower) bound.

The partition function $f^{13}$ was established by Shelah and the first author in [3]. By using this function they give an upper bound for the Hales-Jewett numbers, and as an application they acquire an alternative proof of the fact that the Hales-Jewett numbers belong to the class $E^5$ of the Grzegorczyk hierarchy. Also a list of similar partition functions is provided in [14] to give a primitive recursive upper bound for the generalized version of the Hales-Jewett numbers.

In this paper we study some partition relations defined in [3] and [12] and relate them to the Hales-Jewett numbers. In particular, we give an upper bound for the Hales-Jewett numbers using the primitive recursive function $f^{8, *}$, which belongs to the class $E^5$ of the Grzegorczyk hierarchy and grows slower than $f^{13}$.

See Section 2 for undefined notions

In computer science, a primitive recursive function can be described by only do-loops.
The relationships between these partition functions can be summarized in the following diagrams, where by \( \rightarrow \) we mean \( \leq \), and dashed arrows (\( \dashrightarrow \)) indicate the new results in this paper. Also \( \text{HJ} \) and \( \text{WC} \) denote the Hales-Jewett number and the Gallai-Witt number respectively, and the partition functions \( f_8, f_{8*}, f_9, f_{9*} \) and \( f_{13} \) are defined in [12] and [3] respectively.

\[
\begin{align*}
\text{HJ}_C(m, \Lambda) & \quad f_{13}^\Lambda(m, C) \\
\uparrow & \quad \uparrow \\
 f_{8*}^\Lambda(m, C) & \quad f_{9*}^\Lambda(m, C) \quad \dashrightarrow \quad m \cdot \text{HJ}_C(1, |\Lambda|^m) \\
\uparrow & \quad \uparrow \\
f_8^\Lambda(m, C) & \quad f_9^\Lambda(m, C)
\end{align*}
\]

and

\[
\begin{align*}
f_{8*}^\Lambda(|\Lambda|^2 \cdot \text{WC}(|\Lambda|, 1), C) & \quad f_{9*}^\Lambda(|\Lambda|^2 \cdot \text{WC}(|\Lambda|, 1), C) \quad \dashrightarrow \quad f_{13}^\Lambda(|\Lambda|^2 \cdot \text{WC}(|\Lambda|, 1), C) \\
\uparrow & \\
\text{HJ}_C(1, \Lambda) & \quad f_{13}^\Lambda(|\Lambda| \cdot \text{WC}(|\Lambda|, 1), C)
\end{align*}
\]

The paper is organized as follows. In Section 2 we express some preliminaries which will be used for the rest of the paper and also some elementary results. Then in Section 3 we prove a main theorem that gives an upper bound for the Hales-Jewett numbers in terms of the function \( f_{8*}^\Lambda \in E^5 \).

2. SOME PRELIMINARIES AND BASIC RESULTS

This section is devoted to some preliminaries including notation and conventions and also basic definitions and results. We refer the reader to [3], [12], [13] and [14] for some related results.

2.1. Notation and conventions. Throughout this paper we use the following notation and conventions.

1. \( i, j, k, l, m, n \) denote natural numbers.

2. For a natural number \( n \), \([n]\) denotes the set \( \{1, \ldots, n\} \).

3. Given a natural number \( n \), we identify it with \( n = \{0, \ldots, n - 1\} \).

4. \( \Lambda \) denotes a finite non-empty set of alphabets, whose elements are usually denoted by \( \alpha, \beta, \ldots \).

5. \( C \) denotes a finite non-empty set; the set of colors.
(6) $M, N, \ldots$ denote finite non-empty linear orders.

(7) We use lowercase bold letters $d, e, \ldots$ to denote coloring functions.

(8) If $(M, \leq_M)$ is a linear order and $r > 0$, then $[M]^r$ denotes the set of all $r$-element sets $\{u_0, \ldots, u_{r-1}\}$ of $M$ such that $u_0 <_M \cdots <_M u_{r-1}$. Also, $[M]^{<l} = \bigcup_{0<r<l} [M]^r$.

(9) $U_{M,\Lambda} = \{\eta \mid \eta : M \rightarrow \Lambda$ is a function $\}$. 

2.2. Primitive recursive functions. By a number theoretic function we mean a function from $\mathbb{N}^k$ to $\mathbb{N}$. Let $g, h$ and $f$ be number theoretic functions of arities $k, k + 2$ and $k + 1$ respectively. Recall that $f$ is said to be defined by primitive recursion from $g$ and $h$ whenever for every $x \in \mathbb{N}^k$ and every $n \in \mathbb{N}$ we have:

$$
\begin{cases}
    f(0, x) = g(x), \\
    f(n + 1, x) = h(f(n, x), n, x).
\end{cases}
$$

There is a simpler kind of primitive recursion which is appropriate to define unary functions, namely

$$
\begin{cases}
    f(0) = m, \\
    f(n + 1) = h(f(n), n).
\end{cases}
$$

where $m \in \mathbb{N}$ and $h : \mathbb{N}^2 \rightarrow \mathbb{N}$.

Definition 2.1. The class of primitive recursive functions is the smallest set of number theoretic functions that contains the constant zero function, the successor function and the projection functions, and is closed under composition and primitive recursion.

The simplest function which is not primitive recursive was constructed by Ackermann in [1].

2.3. The Grzegorczyk hierarchy. For each $n \in \mathbb{N}$, we define the function $E_n$ as follows

$$
E_0(x, y) = x + y, \\
E_1(x) = x^2 + 2, \\
E_{n+2}(0) = 2,
$$

and

$$
E_{n+2}(x + 1) = E_{n+1}(E_{n+2}(x)).
$$

Observe that each $E_n$ is primitive recursive.

Definition 2.2. For each $n \in \mathbb{N}$ we define the $n^{th}$ Grzegorczyk’s class $E^n$ to be the smallest set of number theoretic functions that contains the functions $E_k$ for $k < n$, the constant zero function, the successor function and the projection functions, and is closed
under composition and limited primitive recursion (that is, if \(g, h, j \in \mathcal{E}^n\) and \(f\) is defined by primitive recursion from \(g\) and \(h\), has the same arity as \(j\) and is pointwise bounded by \(j\), then \(f\) belongs to \(\mathcal{E}^n\) as well).

**Fact 2.1.** The following properties hold.

1. A number theoretic function \(f\) is primitive recursive if and only if \(f \in \mathcal{E}^n\) for some \(n \in \mathbb{N}\).
2. For every \(n \in \mathbb{N}\) we have \(\mathcal{E}^n \subset \mathcal{E}^{n+1}\).
3. If \(g, h \in \mathcal{E}^n\) for some \(n \in \mathbb{N}\) and \(f\) is defined by primitive recursion from \(g\) and \(h\), then \(f \in \mathcal{E}^{n+1}\).
4. For every \(n \in \mathbb{N}\) and every \(f \in \mathcal{E}^n\) of arity \(k\) there exists \(F \in \mathcal{E}^n\) of arity \(k\) which dominates \(f\) pointwise and satisfies

\[
F(x_1, \ldots, x_k) \leq F(y_1, \ldots, y_k)
\]

for every \(x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{N}\) with \(x_i \leq y_i\) for all \(i \in [k]\).

See [2, Appendix A] and [10] for more details on the Grzegorczyk hierarchy.

2.4. Van der Waerden numbers [15]. Van der Waerden’s theorem states that for any positive integers \(r\) and \(m\) there exists a positive integer \(N\) such that if the integers \(\{1, 2, \ldots, N\}\) are colored, each with one of \(r\) different colors, then there are at least \(m\) integers in arithmetic progression all of the same color. The smallest such \(N\) is the van der Waerden number \(W(r, m)\).

Here we first define the multidimensional van der Waerden numbers (also called Gallai-Witt numbers) which is denoted by \(W_C(h, m)\).

**Definition 2.3.** The Gallai-Witt number \(W_C(h, m)\) is defined to be the minimal \(n\) such that if \(d : U_{h,n} \rightarrow C\) is a \(C\)-coloring of \(U_{h,n}\), then we can find \(d > 0\) and a sequence \(\langle m_e | e < h \rangle\) of natural numbers such that for each \(e < h\), \(m_e + d \cdot m < n\) and \(d\) is constant on

\[
\{\langle m_e + d \cdot i_e | e < h \rangle | i_0 \leq m, \ldots, i_{h-1} \leq m\}.
\]

The existence of the Gallai-Witt numbers is guaranteed by Gallai-Witt theorem, proved independently by Gallai (Grünwald), as quoted in [9] and Witt [16].

**Remark 2.4.** Given natural numbers \(m, r\), the van der Waerden number \(W(r, m)\) is defined to be the least \(n\) such that for any coloring \(d : n \rightarrow r\), there exists a \(d\)-monochromatic arithmetic progression of length \(m\). Then \(W(r, m + 1) = W_C(1, m)\), where \(|C| = r\).

\(^3\)There is no publication by Gallai that contains it; however, Rado in [9] proved it and credits Gallai.
By a celebrated result of Gowers [4] we have
\[ W(r, m) \leq 2^{2^{2^{2m+9}}}, \]
in particular, \( W(r, m) \in \mathcal{E}^3 \).

2.5. The Hales-Jewett Theorem. The notions of line and \( m \)-dimensional subspace play an important role in the Hales-Jewett theorem.

**Definition 2.5.** A set \( L \subseteq \mathcal{U}_{M, \Lambda} \) is a line of \( \mathcal{U}_{M, \Lambda} \) if there exist a non-empty subset \( M_0 \subseteq M \) and \( \rho : M \setminus M_0 \rightarrow \Lambda \) such that for every \( \eta \in \mathcal{U}_{M, \Lambda}, \eta \in L \) whenever
(a) \( \eta|M \setminus M_0 = \rho \).
(b) \( \eta|M_0 \) is constant.

**Definition 2.6.** A set \( S \subseteq \mathcal{U}_{M, \Lambda} \) is an \( m \)-dimensional subspace of \( \mathcal{U}_{M, \Lambda} \) if there are pairwise disjoint non-empty subsets \( M_\ell, \ell < m, \) of \( M \) and a map \( \rho : M \setminus \bigcup_{\ell<m} M_\ell \rightarrow \Lambda \) such that for every \( \nu \in \mathcal{U}_{M, \Lambda}, \nu \in S \) if and only if
1. \( \nu|M \setminus \bigcup_{\ell<m} M_\ell = \rho \).
2. for every \( \ell < m, \nu|M_\ell \) is constant.

It is evident that a line of \( \mathcal{U}_{M, \Lambda} \) is just a 1-dimensional subspace of \( \mathcal{U}_{M, \Lambda} \).

**Notation 2.7.** Given an \( m \)-dimensional subspace \( S \) of \( \mathcal{U}_{M, \Lambda} \) as in Definition 2.6, we denote it by
\[ S = S(\langle M_\ell : \ell < m \rangle, \rho). \]

**Definition 2.8.** The Hales-Jewett number, \( \text{HJ}_C(m, \Lambda) \), is defined to be the minimal \( k \) such that for any linear order \( M \) of size \( k \) and any \( C \)-coloring \( d : \mathcal{U}_{M, \Lambda} \rightarrow C, \) there exists a \( d \)-monochromatic \( m \)-dimensional subspace of \( \mathcal{U}_{M, \Lambda} \).

By Hales-Jewett [4], \( \text{HJ}_C(m, \Lambda) \) is finite. The original proof by Hales and Jewett was based on double induction and did not give any primitive recursive bounds for \( \text{HJ}_C(m, \Lambda) \). In [11], Shelah showed that the function \( \text{HJ}_C(m, \Lambda) \) is primitive recursive, and indeed \( \text{HJ}_C(m, \Lambda) \in \mathcal{E}^5 \).

Let us now borrow a long list of definitions from [12], which are essentially some variants of the Hales-Jewett numbers.

**Definition 2.9.** ([12])
1. \( \xi_{\Lambda}^d(m, C) \), where \( |\Lambda| \) divides \( m \), is defined to be the minimal \( k \) divisible by \( |\Lambda| \) such that for any linear order \( M \) of size \( k \) and any coloring \( d : \mathcal{U}_{M, \Lambda} \rightarrow C, \) there are pairwise disjoint subsets \( M_\ell, \ell < m, \) of \( M \) and a map \( \rho : M \setminus \bigcup_{\ell<m} M_\ell \rightarrow \Lambda \) such that:
Lemma 2.11. (1) for all \( l_0 < l_1 < m, |M_{l_0}| = |M_{l_1}| > 0, \\
(2) \( d \) is constant on \( T \), where \( T \) consists of those \( \nu \in S(\langle M_i : l < m \rangle, \rho) \) such that for every \( \alpha \in \Lambda, \)
\[
\left| \{ l < m : \nu|M_i \equiv \alpha \} \right| = \frac{m}{|\Lambda|}.
\]

(2) \( f^\Lambda_m(m, \Lambda) \) is defined similarly without clause (a).

We now define a modified version of the above partition functions.

Definition 2.10. (1) \( f^\Lambda_{\Lambda^*}(m, \Lambda) \), where \( |\Lambda| \) divides \( m \), is defined as in \( f^\Lambda_0(m, \Lambda) \), where clause 2.9(b) is replaced by:

(b\*): for \( \nu_1, \nu_2 \in S(\langle M_i : l < m \rangle, \rho), d(\nu_1) = d(\nu_2) \) provided that for all \( \alpha \in \Lambda \\
\left| \{ l < m : \nu_1|M_i \equiv \alpha \} \right| = \left| \{ l < m : \nu_2|M_i \equiv \alpha \} \right|.
\]

(2) \( f^\Lambda_{\Lambda^*}(m, \Lambda) \) is defined similarly, using \( f^\Lambda_m(m, \Lambda) \) by replacing clause 2.9(b) by (b\*).

The next lemma is evident from the above definitions.

Lemma 2.11. (1) \( f^\Lambda_{\Lambda^*}(m, \Lambda) \leq f^\Lambda_0(m, \Lambda) \).
(2) \( f^\Lambda_{\Lambda^*}(m, \Lambda) \leq f^\Lambda_{\Lambda^*}(m, \Lambda) \).
(3) Let \( \ell \in \{8, 9\} \). Then \( f^\Lambda_0(m, \Lambda) \leq f^\Lambda_{\Lambda^*}(m, \Lambda) \).
(4) \( f^\Lambda_{\Lambda^*}(m, \Lambda) \leq HJ(\Lambda, \Lambda) \).

In order to compare these partition functions with the Hales-Jewett numbers, we define \( HJ(\Lambda, \Lambda) \) which is above all of these functions and still not far from \( HJ(\Lambda, \Lambda) \).

Definition 2.12. The partition function \( HJ(\Lambda, \Lambda) \) is defined to be the minimal \( k \) such that for any linear order \( M \) of size \( k \) and any \( C \)-coloring \( d : U_{M,\Lambda} \to C \), there exists a \( m \)-dimensional subspace \( S = S(\langle M_i : l < m \rangle, \rho) \) such that

(1) \( d \) is constant on \( S \), and
(2) for every \( l, l' < m, |M_l| = |M_{l'}| \).

Note that \( HJ(\Lambda) \) is essentially HJ with the additional condition that \( M_l \)'s have the same size.

Lemma 2.13. \( HJ(\Lambda, \Lambda) \leq m \cdot HJ(\Lambda, |\Lambda|^m) \).

Proof. Let \( n = HJ(\Lambda, |\Lambda|^m) \), and set \( M = n \times m \). Let \( d : U_{M,\Lambda} \to C \) be a \( C \)-coloring. Define \( F : U_{n,\Lambda^m} \to U_{M,\Lambda} \) as follows: let \( \eta : n \to \Lambda^m \), then \( F(\eta) : M \to \Lambda \) is defined by
\[
F(\eta)(i, \ell) = \eta(i)(\ell).
\]
Let $e : U_{n,\Lambda^m} \to C$ be defined as $e(\eta) = d(F(\eta))$.

\[
\begin{array}{c}
U_{M,\Lambda} \\
F \uparrow \\
U_{n,\Lambda^m}
\end{array} \xrightarrow{d} C \xleftarrow{e}
\]

By our assumption, there are $N \subseteq n$ and $\rho : n \setminus N \to \Lambda^m$ which witness $HJ_C(1, |\Lambda|^m)$ with respect to $e$. Thus $e$ is constant on $\{\eta : n \to \Lambda^m : n \setminus N \text{ constant and } n \supseteq \rho\}$. For $\ell < m$ set

\[M_\ell = N \times \{\ell\} \subseteq M.\]

Define also $\varrho : M \setminus \bigcup_{\ell < m} M_\ell \to \Lambda$ by $\varrho(i, \ell) = \rho(i)(\ell)$. Note that for all $\ell, \ell' < m$ we have $|M_\ell| = |M_{\ell'}| > 0$. Now suppose that $v_1, v_2 \in S\big((M_\ell : \ell < m), \varrho\big)$. Define $\eta_1, \eta_2 : n \to \Lambda^m$ such that for all $i < n$ and $\ell < m$:

\[
\begin{align*}
\eta_1(i)(\ell) &= v_1(i, \ell), \\
\eta_2(i)(\ell) &= v_2(i, \ell).
\end{align*}
\]

Now it follows that

- $\eta_1, \eta_2 \supseteq \rho$ (as $v_1, v_2 \supseteq \varrho$)
- $\eta_1|N$ is constant; to see this suppose $i_1, i_2 \in N \subseteq n$. Then for all $\ell < m$

  \[
  \begin{align*}
  \eta_1(i_1)(\ell) &= v_1(i_1, \ell) \\
  &= v_1(i_2, \ell) \\
  &= \eta_1(i_2)(\ell).
  \end{align*}
  \]

  Thus $\eta_1(i_1) = \eta_1(i_2)$.
- similarly $\eta_2|N$ is constant.

Thus $e(\eta_1) = e(\eta_2)$ which implies that

\[d(v_1) = d(F(\eta_1)) = e(\eta_1) = e(\eta_2) = d(F(\eta_2)) = d(v_2).\]

The lemma follows.

\[\square\]

\textbf{Remark 2.14.} Note that since clearly $HJ_C(m, \Lambda) \leq HJ_C^v(m, \Lambda)$ and $\mathbf{f}^{\mathcal{A},*}_\Lambda(m, C) \leq HJ_C^v(m, \Lambda)$, Lemma 2.13 shows that

\begin{enumerate}
  \item $\mathbf{f}^{\mathcal{A},*}_\Lambda(m, C) \leq m \cdot HJ_C(1, |\Lambda|^m)$.
  \item $HJ_C(m, \Lambda) \leq m \cdot HJ_C(1, \Lambda^m)$ (see [2, Page 26]).
\end{enumerate}
Let $M$ be a finite linear order and let $\text{Sym}(M)$ be the set of all permutations of $M$. This induces an equivalence relation $E_M$ on $U_{M,\Lambda}$ defined by

$$\rho_1 E_M \rho_2 \iff \exists \pi \in \text{Sym}(M), \ \rho_2 = \rho_1 \circ \pi.$$ 

**Remark 2.15.** Suppose $\eta_1, \eta_2 \in U_{N,\Lambda}$. Then it is easily seen that

$$\eta_1 E_N \eta_2 \iff \text{for all } \alpha \in \Lambda, |\eta_1^{-1}\{\alpha\}| = |\eta_2^{-1}\{\alpha\}|.$$ 

We borrow the following definitions from [3].

**Definition 2.16.** $f_{\Lambda}^{13}(m, C)$ is defined to be the minimal $k$ such that for any linear order $M$ of size $k$ and any $C$-coloring $d : U_{M,\Lambda} \to C$, there are $N \subseteq M$ of size $m$ and a function $\rho : M \setminus N \to \Lambda$ such that for all $\eta_1, \eta_2 \in U_{N,\Lambda}$,

$$\eta_1 E_N \eta_2 \implies d(\eta_1 \cup \rho) = d(\eta_2 \cup \rho).$$

The next theorem gives the relation between $f_{\Lambda}^{13}(m, C)$ and $HJ_C(n, \Lambda)$.

**Theorem 2.17** (see [3]). The following statements hold.

1. $f_{\Lambda}^{13}(m, C) \in E^5$.
2. $HJ_C(n, \Lambda) \leq f_{\Lambda}^{13}\big(|\Lambda| \cdot W_C(|\Lambda|, m), C\big)$. 

The next lemma compares $f_{\Lambda}^{9,\ast}(m, C)$ and $f_{\Lambda}^{13}(m, C)$.

**Lemma 2.18.** $f_{\Lambda}^{9,\ast}(m, C) \leq f_{\Lambda}^{13}(m, C)$. In particular $f_{\Lambda}^{9,\ast}(m, C) \in E^5$.

**Proof.** Let $k = f_{\Lambda}^{13}(m, C)$. Suppose $M$ is a linear order of size $k$ and let $d : U_{M,\Lambda} \to C$ be a $C$-coloring. Let $N \subseteq M$ of size $m$ and $\rho : M \setminus N \to \Lambda$ witness $k = f_{\Lambda}^{13}(m, C)$. For each $l < m$ let $M_l = \{a_l\}$, where $N = \langle a_l : l < m \rangle$ is the increasing enumeration of $N$. Now suppose $\nu_1, \nu_2 \in S(M_l : l < m, \rho)$ and suppose they satisfy the requirement $2.10(c^*)$. It then immediately follows that $\nu_1 E_N \nu_2$ and hence $d(\nu_1 \cup \rho) = d(\nu_2 \cup \rho)$. \hfill $\Box$

In the above proof, the sets $M_l, l < m$ all have size 1. By a similar argument, we can prove the following lemma.

**Lemma 2.19.** Let $n > 0$ and $f_{\Lambda,n}^{9,\ast}(m, C) \leq f_{\Lambda}^{13}(m \cdot n, C)$, where $f_{\Lambda,n}^{9,\ast}(m, C)$ is defined as in $f_{\Lambda}^{9,\ast}(m, C)$ with the additional requirement that for all $\ell < m, |M_\ell| = n$.

3. **Bounding Hales-Jewett Numbers with $f_{\Lambda}^{8,\ast}$**

This section contains a main theorem which compares Hales-Jewett numbers with $f_{\Lambda}^{8,\ast}$.

**Main Theorem 1.** $HJ_C(1, \Lambda) \leq f_{\Lambda}^{8,\ast}\big(|\Lambda|^2 \cdot W_C(|\Lambda|, 1), C\big)$. 

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Lemma 3.1. HJ\(_C(1, \Lambda) \leq f^{8*}_\Lambda(m^*, C)\), where \(m^*\) satisfies the following:

\((\oplus)\): assume \(\Omega = \{\ell = \langle \ell_\alpha : \alpha \in \Lambda \rangle \in U_{\Lambda, m^*} : \sum_{\alpha \in \Lambda} \ell_\alpha = m^*\}\) and \(d : \Omega \to C\) is a \(C\)-coloring. Then there are \(\ell^* > 0\) and \(\langle \ell^*_\alpha : \alpha \in \Lambda \rangle\) such that letting,

\[
\bar{\ell}^\alpha(\beta) = \begin{cases} 
\ell^*_\beta & \beta \neq \alpha \\
\ell^* + \ell^*_\alpha & \beta = \alpha
\end{cases}
\]

then \(\bar{\ell}^\alpha \in \Omega\) for \(\alpha \in \Lambda\) and \(d|\{\bar{\ell}^\alpha : \alpha \in \Lambda\}\) is constant.

Proof. Let \(m^{**} = f^{8*}_\Lambda(m^*, C)\) and assume \(d : U_{m^{**}, \Lambda} \to C\) is a \(C\)-coloring.

Claim. There is a \(C\)-coloring \(e : U_{m^*, \Lambda} \to C\) such that

1. if \(\eta_1, \eta_2 \in U_{m^*, \Lambda}\) and for all \(\alpha \in \Lambda, |\eta_1^{-1}\{\alpha\}| = |\eta_2^{-1}\{\alpha\}|\), then \(e(\eta_1) = e(\eta_2)\), and
2. if there is an \(e\)-monochromatic line in \(U_{m^*, \Lambda}\), then there is a \(d\)-monochromatic line in \(U_{m^{**}, \Lambda}\).

Proof. We abuse notation to identify \(m^*\) and \(m^{**}\) with linear orders of size \(m^*\) and \(m^{**}\). By definition of \(f^{8*}_\Lambda(m^*, C)\), there are pairwise disjoint subsets \(\langle M_\ell : \ell < m^* \rangle\) of \(m^{**}\) and a map \(\rho : m^{**} \setminus \bigcup_{\ell < m^*} M_\ell \to \Lambda\) such that for every \(\nu_1, \nu_2 \in S(\langle M_\ell : \ell < m^* \rangle, \rho)\),

\(d(\nu_1) = d(\nu_2)\) provided that for each \(\alpha \in \Lambda\)

\[
|\{\ell < m^* : \nu_1[M_\ell \equiv \alpha]\}| = |\{\ell < m^* : \nu_2[M_\ell \equiv \alpha]\}|
\]

Define \(F : U_{m^*, \Lambda} \to U_{m^{**}, \Lambda}\) with \(\eta \mapsto F(\eta) : m^{**} \to \Lambda\) where

\[
F(\eta)(t) = \begin{cases} 
\eta(t) & t \in M_\ell, (\ell < m^*) \\
\rho(t) & t \in m^{**} \setminus \bigcup M_\ell
\end{cases}
\]

Now, define a \(C\)-coloring \(e : U_{m^*, \Lambda} \to C\) with \(e(\eta) = d(F(\eta))\). We show that it satisfies the conditions of the claim. First, let \(\eta_1, \eta_2 \in U_{m^*, \Lambda}\) and for all \(\alpha \in \Lambda, |\eta_1^{-1}\{\alpha\}| = |\eta_2^{-1}\{\alpha\}|\). Then for all \(\alpha \in \Lambda\),

\[
|\{\ell < m^* : F(\eta_1)[M_\ell \equiv \alpha]\}| = |\{\ell < m^* : F(\eta_2)[M_\ell \equiv \alpha]\}|
\]

It follows that \(e(\eta_1) = d(F(\eta_1)) = d(F(\eta_2)) = e(\eta_2)\).

Second, let \(N \subseteq m^*\) and \(\tau : m^* \setminus N \to \Lambda\) be a \(e\)-monochromatic line in \(U_{m^*, \Lambda}\). Let \(N' = \bigcup_{\ell \in N} M_\ell\) and \(\tau' : m^{**} \setminus N' \to \Lambda\) with \(\tau'|M_\ell = \tau(\ell)\) for \(\ell \in m^* \setminus N\), and \(\tau'|m^{**} \setminus \bigcup_{\ell < m^*} M_\ell = \rho\). We claim that \((N', \tau')\) is a \(d\)-monochromatic line in \(U_{m^{**}, \Lambda}\). Let \(\xi_1, \xi_2 \in U_{m^{**}, \Lambda}\) such that \(\xi_1|N' \equiv \alpha\) and \(\xi_2|N' \equiv \beta\). We show that \(d(\xi_1 \cup \tau') = d(\xi_2 \cup \tau')\).

It suffices to find \(\eta_1, \eta_2 : N \to \Lambda\) such that \(\xi_i \cup \tau' = F(\eta_i \cup \tau)\) for \(i = 1, 2\). Let \(\eta_i \equiv \alpha\)
and $\eta_2 \equiv \beta$. Then

\[ d(\xi_1 \cup \tau') = d(F(\eta_1 \cup \tau)) = e(\eta_1 \cup \tau) = e(\eta_2 \cup \tau) = d(F(\eta_2 \cup \tau)) = d(\xi_2 \cup \tau'). \]

Now, we show that there is an $e$-monochromatic line in $U_{m^*, \Lambda}$. Define a $C$-coloring $c : \Omega \to C$ with $\eta \mapsto e(G(\eta))$, where for every $\eta = \langle \ell_\alpha : \alpha \in \Lambda, \sum_{\alpha \in \Lambda} \ell_\alpha = m^* \rangle \in \Omega$, $G(\eta) : m^* \to \Lambda$ is a map such that if $\{P_\alpha : \alpha \in \Lambda\}$ is a partition of $m^*$ with $|P_\alpha| = \ell_\alpha$, then $G(\eta) \upharpoonright P_\alpha \equiv \alpha$. Notice that $c$ does not depend on the partition and so $c$ is well-defined. By using $(\oplus)$ for coloring $c$, there are $\ell^*$ and $\langle \ell^*_\alpha : \alpha \in \Lambda \rangle$ such that if

\[ \bar{\ell}^\alpha(\beta) = \begin{cases} \ell^*_\beta & \beta \neq \alpha \\ \ell^*_\alpha + \ell^* & \beta = \alpha \end{cases} \]

then for each $\alpha \in \Lambda$, $\bar{\ell}^\alpha \in \Omega$ and $c|\{\bar{\ell}^\alpha : \alpha \in \Lambda\}$ is constant. Now, let $\{P'*_{\alpha} : \alpha \in \Lambda\} \cup \{P'\}$ partition $m^*$ such that $|P'| = \ell^*$ and for every $\alpha \in \Lambda$, $|P'_\alpha| = \ell^*_\alpha$. Set $N'' = P'$ and let $\tau'' : m'' \setminus N'' \to \Lambda$ such that $\tau''|P''_\alpha \equiv \alpha$. Then $(N'', \tau'')$ is a witness of an $e$-monochromatic line in $U_{m^*, \Lambda}$, and the result follows. \hfill \Box

**Lemma 3.2.** Let $m^* = |\Lambda|^2 \cdot W_C(|\Lambda|, 1)$. Then $m^*$ satisfies $(\oplus)$ from Lemma 3.1.

**Proof.** Set $h = |\Lambda|$, $n = W_C(h, 1)$ and $k = h^2 \cdot n$. Let $d : U_{h, h} \to C^4$ be a $C$-coloring. Define $F : U_{h, n} \to U_{h, h}$ as follows. Fix $\eta \in U_{h, n}$, then for each $e < h$,

\[ F(\eta)(e) = h \cdot \eta(e) + h \cdot n - \sum_{i < h} \eta(i). \]

$F$ is easily seen to be well-defined and that

\[ \sum_{e < h} F(\eta)(e) = h \sum_{e < h} \eta(e) + h^2 \cdot n - \sum_{e < h} \sum_{i < h} \eta(i) = k. \]

Define the coloring $e : U_{h, n} \to C$ by $e(\eta) = d(F(\eta))$. By definition of $W_C$, we can find $\bar{m} = \langle m_e : e < h \rangle$ and $d > 0$ such that $e$ is constant on $\langle \vartheta_e = \bar{m} + d \cdot i_e : e < h \rangle$, where $\bar{m} + d \cdot i_e = \langle m_0, \ldots, m_{e-1}, m_e + d, m_{e+1}, \ldots, m_{h-1} \rangle$. Let $\ell = h \cdot n - (\sum_{e < h} m_e) - d$. This means that $d$ is constant on $\langle \vartheta^*_e : e < h \rangle$, where

\[ \vartheta^*_{e} = \langle h \cdot m_0 + \ell, \ldots, h \cdot m_{e-1} + \ell, h \cdot m_e + d + \ell, h \cdot m_{e+1} + \ell, \ldots, h \cdot m_{h-1} + \ell \rangle \]

Now let $\ell^*_e = h \cdot m_e + \ell$ and $\ell^* = d$. It is easily seen they are as required by $m^*$. \hfill \Box

To complete the proof of the main theorem, combine Lemma 3.1 and Lemma 3.2. \hfill \Box

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4Note that only members of $\Omega$ matter. In fact we apply $d$ to elements of the form $F(\eta)$, which are always in $\Omega$.\hfill \Box
Remark 3.3. Notice that Main Theorem 1 provides an alternative and apparently a better upper bound for $HJ_C(1,\Lambda)$ in comparison with Theorem 2.17(2).

For primitive recursive functions $f$ and $g$, set

$$f \leq E g \iff \min \{n : f \in E^n\} \leq \min \{n : g \in E^n\},$$

and $f = E g$ is defined similarly. Our results show that

$$f^8 = E f^8 \ast = E f^9 = E f^9 \ast = E HJ \leq E f^{13}.$$

Now we can ask the following natural question.

Question 3.4. Is $f^{13} \leq E HJ$?

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