Local Central Limit Theorem for a Random Walk Perturbed in One Point

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Received: 9 April 2018 / Accepted: 20 June 2019 / Published online: 9 July 2019
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Abstract
We consider a symmetric random walk on the $\nu$-dimensional lattice, whose exit probability from the origin is modified by an antisymmetric perturbation and prove the local central limit theorem for this process. A short-range correction to diffusive behaviour appears in any dimension along with a long-range correction in the one-dimensional case.

Keywords Local central limit theorems · Inhomogeneous random walks

Mathematics Subject Classification (2010) 60F05 · 60G50 · 60J10

1 Statement of the Result

Let $\{\xi_n\}_{n \in \mathbb{N}}$ be symmetric i.i.d. random variables taking values in $\mathbb{Z}^\nu$, such that

$\mathbb{E}[\xi_1] = 0; \quad B_{ij} := \mathbb{E}[\xi_{1,i}\xi_{1,j}]; \quad \mathbb{E}[\xi_1^{\alpha_1} \cdots \xi_1^{\alpha_\nu}] < \infty, \quad |\alpha| = 4,$ (1.1)

where $\xi_{1,i}$ is the $i$-th component of $\xi_1$ and we use the customary multi-index notation $\alpha := (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}_0^\nu$, $|\alpha| := \alpha_1 + \ldots + \alpha_\nu$. The positive definite matrix $B$ has eigenvalues $\{|\sigma_i^2|\}_{i=1,...,\nu}$ and in the one-dimensional case we simply write $\sigma$ rather than $\sigma_1$. We denote by $S_n$ the random walk with increments $\xi_n$ and initial condition $S_0 = x_0 \in \mathbb{Z}^\nu$ and we require $\{S_n\}_{n \in \mathbb{N}}$ to be aperiodic and irreducible [5]. Moreover let

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\( \{\eta_n\}_{n \in \mathbb{N}} \) be another sequence of i.i.d. random variables in \( \mathbb{Z}^\nu \) such that the matrix
\[
(\mathbb{E}[\eta_{1,i} \eta_{1,j}])_{i,j=1,\ldots,\nu}
\]
is positive definite and
\[
\mathbb{E}[\eta_1] =: d \neq 0; \quad \mathbb{E}[\eta_{1,i}^\alpha \eta_{1,j}^\nu] < \infty, \quad |\alpha| = 3. \tag{1.2}
\]

In this note we consider the aperiodic and irreducible Markov chain \( \{X_n\}_{n \in \mathbb{N}} \) on \( \mathbb{Z}^\nu \) defined by
\[
X_n = X_{n-1} + \xi_n + (\eta_n - \xi_n)\delta_{X_{n-1},0}, \quad X_0 = x_0 \in \mathbb{Z}^\nu,
\]
where \( \delta_{x,y} \) denotes as customary the Kronecker delta. Away from the origin this is just the symmetric random walk. Every time it hits zero, it exits with a different probability given by
\[
\mathbb{P}(X_n = x \mid X_{n-1} = 0) = \mathbb{P}(\eta_1 = x).
\]
We set
\[
p(x) := \mathbb{P}(\xi_1 = x), \quad q(x) := \mathbb{P}(\eta_1 = x),
\]
and
\[
a(x) := \mathbb{P}(\eta_1 = x) - \mathbb{P}(\xi_1 = x) = q(x) - p(x),
\]
so that we can conveniently represent the transition probability for \( X_n \) as
\[
\mathbb{P}(X_n = x \mid X_{n-1} = y) = p(x - y) + \delta_{y,0}a(x). \tag{1.4}
\]
Of course
\[
\sum_{x \in \mathbb{Z}^\nu} a(x) = 0, \quad \sum_{x \in \mathbb{Z}^\nu} xa(x) = d.
\]

We will assume antisymmetry of \( a(x) \):
\[
a(x) = -a(-x). \tag{1.5}
\]
Note that this entails
\[
a(x) = q_a(x), \quad p(x) = q_s(x),
\]
where \( q_a, q_s \) denotes respectively the antisymmetric and symmetric part of \( q(x) \).

The main result of this note follows. As customary, \( o\left(\frac{1}{n^p}\right) \) denotes a quantity approaching zero faster than \( n^{-p} \) uniformly in \( x \in \mathbb{Z}^\nu \) and we set \( |B| := \sqrt{\det B} \).

**Theorem 1.1** Let \( x \in \mathbb{Z}^\nu \), \( P_n(x) := \mathbb{P}(X_n = x \mid X_0 = 0) \). Then for any \( \nu \geq 1 \) we have
\[
P_n(0) = \frac{1}{(2\pi n)^{\nu/2} |B|} + o\left(\frac{1}{n^{\nu/2}}\right),
\]
while for \( x \neq 0 \) it is
- if \( \nu = 1 \)
\[
P_n(x) = \frac{e^{-\frac{x^2}{2\sigma^2n}}}{\sqrt{2\pi \sigma^2 n}} \left( 1 + \frac{d}{\sigma^2} \text{sign}(x) \right) + \frac{\psi_1(x)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),
\]
where \( \psi_1 \) is an odd function with \( \sup_{|x|} |\psi_1(x)| \leq C \) and \( |\psi_1(x)| \leq C \frac{n}{|x|} \) for all \( \varepsilon > 0 \).
• If \(\nu = 2\)

\[
P_n(x) = \frac{e^{-(x,B^{-1}x)}_n}{2\pi |B| n} + \frac{\psi_2(x)}{n} + o\left(\frac{1}{n}\right),
\]

where \(\psi_2(x)\) is a bounded odd function such that \(|\psi_2(x)| \leq \frac{C}{|x|}\);

• if \(\nu \geq 3\)

\[
P_n(x) = \frac{e^{-(x,B^{-1}x)}_n}{(2\pi n)^{\frac{\nu}{2}} |B|} + \frac{(q_\alpha G_\nu)(x)}{n^{\frac{\nu}{2}}} + o\left(\frac{1}{n^{\frac{\nu}{2}}}\right),
\]

where \(G_\nu\) is the \(\nu\)-dimensional Green function.

**Remark 1.2** The form of the one dimensional short range correction is typical under the assumption that only three moments are finite, see e.g. [5, Theorem 2.3.10]. In higher dimension the short range correction decays faster with \(|x|\). Indeed the \(\nu\)-dimensional \((\nu \geq 3)\) Green function satisfies \(\lim_{|x| \to \infty} |x|^{\nu-2} G_\nu(x) \leq C\), see e.g. [5, Theorem 4.3.1]).

Despite the simplicity of this problem, to the best of our knowledge the sole other works on it are [6] and [2]. In [6] Minlos and Zhizhina proved the local limit theorem for more general perturbation, acting in a finite neighbourhood of the origin, but a.s. bounded increments. Boldrighini and Pellegrinotti in [2] studied the particular case of single point perturbation in one dimension, with analytic increment distribution, giving a more precise information about the terms in the leading order asymptotics. Finally for the transient case \((\nu \geq 3)\) our theorem can be recovered by the more general result [4, Theorem 7].

Our approach is simply based on the inversion of the characteristic function, as in the usual local limit theorem for random walks (see e.g. [5], Chapter 1). We emphasise that since we use real methods, we can drop the analyticity assumptions of [2, 6]. Moreover, we do not compute sub-leading corrections to the diffusive scale, but our method in principle would allow this calculation (not without effort, though). In the more involved context of random walk in random environment a similar approach has been used in [1].

The perturbed walk we consider inherits recurrence or transience from the symmetric random walk. We remark that the antisymmetry assumption on \(a(x)\) simplifies the calculations and already captures all the interesting features of the problem. This appears evident for instance looking at the asymptotic formula for \(P_n(x)\) in [2] \((\nu = 1)\): the symmetric part of the perturbation affects only the form of the short-range correction and the coefficient of the long range correction (not precisely \(d/\sigma^2\), but just proportional to it).

Lastly, we briefly recall some notions on the Hermite polynomials we will use along the proof (we refer for instance to [3]). We denote by \(H_\alpha(x)\) the Hermite polynomials on \(\mathbb{R}\). The Rodrigues formula reads

\[
e^{-\frac{x^2}{2}} H_\alpha(x) = (-1)^\alpha \partial^\alpha \left(e^{-\frac{x^2}{2}}\right) = \int_\mathbb{R} \frac{d\lambda}{\sqrt{2\pi}} e^{-ix\lambda} e^{-\frac{\lambda^2}{2}} (i\lambda)^\alpha.
\]
The Hermite polynomials form a basis of $L^2 \left( e^{-\frac{x^2}{2}} dx \right)$ with

$$\int_{\mathbb{R}} H_n(x) H_m(x) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = n! \delta_{nm}.$$ 

Any $f \in L^2 \left( e^{-\frac{x^2}{2}} dx \right)$ can be written as $f = \sum_{n \geq 0} c_n H_n$, where the equality is in $L^2$ mean and $c_n$ are given by

$$c_n = \frac{1}{n!} \int_{\mathbb{R}} f(x) H_n(x) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx.$$ 

In particular, using $H_n(x) = (-1)^n H_n(-x)$, for $f = \text{sign}(x)$ we can compute

$$\frac{1}{2} \int_{-\infty}^{\infty} \text{sign}(x) H_{2n+1}(x) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \int_{0}^{\infty} H_{2n+1}(x) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = H_{2n}(0) = (-1)^n \frac{(2n)!}{2^n n!}.$$ 

The series of $\text{sign} x$ converges pointwise uniformly in each interval [3, Theorem 9.1.6]. Throughout we adopt the following notations: $\hat{f}(\lambda) := \sum_{x \in \mathbb{Z}^\nu} f(x) e^{i\lambda x}$ is the usual Fourier transform on $\mathbb{Z}^\nu$; $C$ denotes an absolute constant which may change line by line.

## 2 Proof

First we get a nice representation formula for antisymmetric $a$.

**Lemma 2.1** Let $a(x) = -a(-x)$ for any $x \in \mathbb{Z}^\nu$. Then

$$P_n(x) = p^{sn}(x) + \sum_{k=0}^{n-1} p^{sk}(0)(q_n * p^{*(n-k-1)}(x)), \quad (2.1)$$

where $p^{sn}$ is recursively defined by $p^{sn}(x) = (p * p^{*(n-1)})(x)$, $p^{s0}(x) = \delta_{x,0}$.

**Proof** We introduce the hitting times

$$\tau := \inf\{n \geq 1 : X_n = 0\}, \quad \tau' := \inf\{n \geq 1 : S_n = 0\}$$

and the first return probabilities ($n \geq 1$)

$$f_n(x) := \mathbb{P}(\tau = n | X_0 = x), \quad f'_n(x) := \mathbb{P}(\tau' = n | S_0 = x).$$

We also set

$$g_n(x, y) := \mathbb{P}(X_n = y, \tau \geq n | X_0 = x).$$

We note that for $x \neq 0$ one has

$$g_n(x, y) := \mathbb{P}(X_n = y, \tau \geq n | X_0 = x) = \mathbb{P}(S_n = y, \tau \geq n | X_0 = x). \quad (2.2)$$
The antisymmetry of $a(y)$ yields $f_1(0) = f'_1(0) = p(0)$ and for $n \geq 2$

$$f_n(0) = \sum_{y \neq 0} P_1(y) f_{n-1}(y) = \sum_{y \neq 0} (p(y) + a(y)) f'_{n-1}(y) = \sum_{y \neq 0} p(y) f'_{n-1}(y) + \sum_{y \neq 0} a(y) f'_{n-1}(y) = f'_n(0).$$

Therefore we obtain $f_n(0) = f'_n(0)$ for all $n \in \mathbb{N}$, whence $P_n(0) = \mathbb{P}(X_n = 0|X_0 = 0) = \mathbb{P}(S_n = 0|S_0 = 0) = p^{*n}(0)$. Then

$$P_n(x) = \sum_{k=0}^{n-1} P_k(0) g_{n-k}(0, x) = \sum_{k=0}^{n-1} p^{*k}(0) \sum_{y \in \mathbb{Z}^\nu/\{0\}} (p(y) + a(y)) g_{n-k-1}(y, x)$$

$$= p^{*n}(x) + \sum_{k=0}^{n-1} p^{*k}(0) \sum_{y \in \mathbb{Z}^\nu/\{0\}} a(y) \mathbb{P}(S_{n-k-1} = x, \tau' \geq n - k - 1|S_0 = y),$$

where the first identity is a decomposition with respect to the last visit of the walk to the origin, the second identity follows by the definition of $g_n$ and the last identity follows by (2.2). Now we use

$$\sum_{y \in \mathbb{Z}^\nu/\{0\}} a(y) \mathbb{P}(S_{n-k-1} = x, \tau' \geq n - k - 1|S_0 = y)$$

$$= \sum_{y \in \mathbb{Z}^\nu/\{0\}} a(y) \mathbb{P}(S_{n-k-1} = x|S_0 = y),$$

which is a consequence of the antisymmetry of $a$. Indeed we need to show that

$$\sum_{y \in \mathbb{Z}^\nu/\{0\}} a(y) \mathbb{P}(S_{n-k-1} = x, \tau' < n - k - 1|S_0 = y) = 0. \quad (2.4)$$

To prove (2.4) we use the symmetry of the walk $S_n$ and the antisymmetry of $a$. We have

$$\mathbb{P}(S_{n-k-1} = x, \tau' < n - k - 1|S_0 = y) = \mathbb{P}(S_{n-k-1} = x, \tau' < n - k - 1|S_0 = -y). \quad (2.5)$$

which follows by

$$\mathbb{P}(S_{n-k-1} = x, \tau' < n - k - 1|S_0 = y)$$

$$= \sum_{\ell=1}^{n-k-2} \mathbb{P}(S_\ell = 0, \tau' = \ell|S_0 = y) \mathbb{P}(S_{n-k-1} = x|S_\ell = 0)$$

$$= \sum_{\ell=1}^{n-k-2} \mathbb{P}(S_\ell = 0, \tau' = \ell|S_0 = -y) \mathbb{P}(S_{n-k-1} = x|S_\ell = 0)$$

$$= \mathbb{P}(S_{n-k-1} = x, \tau' < n - k - 1|S_0 = -y),$$

where we used

$$\mathbb{P}(S_\ell = 0, \tau' = \ell|S_0 = y) = \mathbb{P}(S_\ell = 0, \tau' = \ell|S_0 = -y),$$
which follows by the symmetry of $P$. Combining (2.5) with the antisymmetry of $a$, immediately gives (2.4). Thus we get

$$P_n(x) = p_n^*(x) + \sum_{k=0}^{n-1} p^k(0) \sum_{y \in \mathbb{Z}^v / \{0\}} a(y) \mathbb{P}(S_{n-k-1} = x | S_0 = y)$$

$$= \sum_{k=0}^{n-1} p^k(0)(a * p^{(n-k-1)})(x),$$

that is the (2.1).

Let now

$$\varphi_n(\lambda) := \sum_{x \in \mathbb{Z}^v} e^{i\lambda x} P_n(x), \quad (2.7)$$

be the characteristic function of $X_n$. Using (2.1) it can be expressed as

$$\varphi_n(\lambda) = \hat{p}_n(\lambda) + \phi_n(\lambda), \quad (2.8)$$

$$\phi_n(\lambda) := \sum_{k=0}^{n-1} p^k(0) q_a(\lambda) \hat{p}^{n-k-1}(\lambda). \quad (2.9)$$

The formula (2.9) will be the starting point of our proof. Note that we have already proved that $P_n(0)$ is unchanged by the perturbation, since the second summand on the r.h.s. of the (2.1) vanishes in $x = 0$. Henceforth we will assume then $x \neq 0$.

Now we set

$$I_n := \sum_{k=0}^{n-1} p^k(0) \int_{[-\pi, +\pi]^v \setminus [-\delta_n, \delta_n]^v} \frac{d^v \lambda}{(2\pi)^v} e^{-i\lambda \cdot x} \hat{p}^{n-k-1}(\lambda) \hat{q}_a(l), \quad (2.10)$$

$$II_n := \sum_{k=0}^{n-1} p^k(0) \int_{[-\delta_n, \delta_n]^v} \frac{d^v \lambda}{(2\pi)^v} e^{-i\lambda \cdot x} \hat{p}^{n-k-1}(\lambda) \hat{q}_a(\lambda), \quad (2.11)$$

where for $\nu = 1$ we set $\delta_n$ a decreasing sequence of positive numbers such that $\lim_{n \to \infty} \delta_n^2 \ln n = 0$ and $\delta_n := n^{\varepsilon - 2}$ with $\varepsilon > 0$ for $\nu \geq 2$. Obviously $P_n(x) = p_n^*(x) + I_n + II_n$. We evaluate separately $I_n$ and $II_n$ in the following lemmas.

**Lemma 2.2**

1. Let $\nu = 1$. Then

$$I_n = \frac{\psi_1(x)}{\sqrt{n}} + o \left( \frac{1}{n^{1/2}} \right), \quad (2.12)$$

where $\psi_1$ is an odd function that $\sup_{|x|} |\psi_1(x)| \leq C$ and $|\psi_1(x)| \leq C |x|^{\varepsilon}$ for all $\varepsilon > 0$.

2. Let $\nu = 2$. Then

$$I_n = \frac{\psi_2(x)}{n} + o \left( \frac{1}{n} \right), \quad (2.13)$$

where $\psi_2$ is a bounded odd function such that $\lim_{|x| \to \infty} |x||\psi_2(x)| \leq C$. 

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Let \( \nu \geq 3 \) and \( G_\nu \) denote the \( \nu \)-dimensional Green function. Then

\[
I_n = \left( q_a * G_\nu \right)(x) \left( \frac{1}{n^{\nu/2}} \right) + o \left( \frac{1}{n^{\nu/2}} \right).
\]

(2.14)

Lemma 2.3

1. Let \( \nu = 1 \). Then

\[
II_n = e^{-\frac{x^2}{2\sigma^2}} \frac{d}{\sqrt{2\pi\sigma^2}} \text{sign}(x) + o \left( \frac{1}{n^{1/2}} \right).
\]

(2.15)

2. Let \( \nu \geq 2 \). Then

\[
II_n = o \left( \frac{1}{n^{\nu/2}} \right).
\]

(2.16)

According to formulas (2.7) and (2.8), we can then prove the main theorem by Fourier inversion using the local central limit theorem for the homogeneous probability [5]. The proofs of the lemmas are presented below.

Proof of Lemma 2.2

We recall that for \( k \geq 1 \)

\[
p^*(0) = \frac{1}{(2\pi|B|k)^{\frac{\nu}{2}}} + o \left( \frac{1}{k^{\frac{\nu}{2}}} \right).
\]

(2.17)

and due to aperiodicity there exists \( b > 0 \) such that

\[
|\hat{p}(\lambda)| \leq 1 - b|\lambda|^2 \leq e^{-b|\lambda|^2}
\]

(2.18)

for all \( \lambda \in [-\pi, \pi]^\nu \) (see for instance Lemma 2.3.2 in [5]). We fix \( \theta \in (0, 1) \) and we split the sum over \( 0 \leq k \leq n - n^\theta \) and \( n - n^\theta < k \leq n - 1 \). By (2.17) and (2.18) we get

\[
\left| \sum_{0 \leq k \leq n - n^\theta} p^*(0) \hat{p}^{n-k-1}(\lambda) \right| \leq \sum_{n^\theta - 1 \leq h \leq n - 1} p^{*(n-h-1)}(0) |\hat{p}(\lambda)|^h \leq e^{-bn^\theta \lambda^2} \rho_n,
\]

(2.19)

where \( \rho_n := \sum_{n^\theta \leq h \leq n} p^*(h) \) that, recalling (2.17), is bounded by a constant (uniform in \( n \)) for \( \nu \geq 3 \), by \( C \ln n \) for \( \nu = 2 \) and by \( C \sqrt{n} \) for \( \nu = 1 \). Again by (2.17) we get

\[
\sum_{n - n^\theta < k \leq n - 1} p^*(0) \hat{p}^{n-k-1}(\lambda) = \sum_{0 \leq h < n^\theta - 1} p^{*(n-h-1)}(0) \hat{p}(\lambda)^h
\]

\[
= \frac{1 + o(1)}{|B|(2\pi n)^{\nu/2}} \sum_{h=0}^{h=[n^\theta-1]} \hat{p}(\lambda)^h
\]

\[
= \frac{1 + o(1)}{|B|(2\pi n)^{\nu/2}} \frac{1 - \hat{p}^{[n^\theta]}(\lambda)}{1 - \hat{p}(\lambda)}.
\]

(2.20)
Therefore

\[ I_n = R_1(\lambda) + \frac{1 + o(1)}{|B| (2\pi n)^{\nu / 2}} \int_{[-\pi, +\pi] \setminus [-\delta_n, \delta_n]^\nu} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{1 - \hat{p}^{(\nu)}(\lambda)}{1 - \hat{\rho}(\lambda)} \hat{q}_a(\lambda) \]

where

\[ |R_1(\lambda)| \leq e^{-bn^\theta \delta_n^2} \rho_n \sup_{|\lambda| > \delta_n} \hat{q}_a(\lambda) \leq C e^{-bn^\theta \delta_n^2} \rho_n. \]

We can further decompose

\[
\int_{[-\pi, +\pi] \setminus [-\delta_n, \delta_n]^\nu} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{1 - \hat{p}^{(\nu)}(\lambda)}{1 - \hat{\rho}(\lambda)} \hat{q}_a(\lambda) = \int_{[-\pi, +\pi] \setminus [-\delta_n, \delta_n]^\nu} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} + R_2(\lambda). \tag{2.21}
\]

where

\[ |R_2(\lambda)| \leq e^{-bn^\theta \delta_n^2} \left( \max_{|\lambda| > \delta_n} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} \right) \leq e^{-bn^\theta \delta_n^2} \left( \max_{|\lambda| > \delta_n} \hat{q}_a(\lambda) \right) \leq C e^{-bn^\theta \delta_n^2} \delta_n^{-2}. \]

Summarising

\[
I_n = \frac{1}{|B| (2\pi n)^{\nu / 2}} \int_{[-\pi, +\pi] \setminus [-\delta_n, \delta_n]^\nu} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} + O \left( e^{-n^\theta \delta_n^2} (\delta_n^{-2} + \rho_n) \right) + O \left( \frac{1}{n^{\nu / 2}} \right). \tag{2.22}
\]

\[ I_n \]

is an odd function of \( x \in \mathbb{Z}^\nu \) (due to the parity of \( \hat{p}, \hat{q}_a \)) plus a remainder \( o \left( \frac{1}{n^{\nu / 2}} \right) \).

We complete the proof treating separately the transient and the recurrent case.

**Proof of (2.14) (\( \nu \geq 3 \))** Let first \( \nu \geq 3 \) and recall that

\[
\int_{[-\pi, +\pi]^v} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} = G_v(x)
\]

is the Green function of the free walk. We can write

\[
\int_{[-\pi, +\pi] \setminus [-\delta_n, \delta_n]^v} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} = (q_a * G_v)(x)
\]

\[ + \int_{[-\delta_n, \delta_n]^v} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)}.
\]

Now we show that

\[
\int_{[-\delta_n, \delta_n]^v} \frac{d^v \lambda}{(2\pi)^v} e^{-ix \cdot \lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} = O \left( \delta_n^{\nu - 1} \right), \quad \nu \geq 2. \tag{2.23}
\]
so that the statement (3) follows letting \( \theta = 1/2 \) since \( \delta_n = n^{e-1/2} \) for some \( e > 0 \).

To prove (2.23) we use (2.18) and \( |\hat{q}_a(\lambda)| \leq C|\lambda| \) for \( \delta_n \) (and so \( \lambda \)) sufficiently small. This leads us to

\[
\left| \int_{[-\delta_n,\delta_n]^2} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} \right| \leq C_n \int_{[-\delta_n,\delta_n]^2} \frac{d^2\lambda}{(2\pi)^2} \frac{1}{|\lambda|} \leq C_n \delta^{\nu-1}, \quad \nu \geq 2. \tag{2.24}
\]

\( \square \)

In the recurrent cases we have to evaluate directly the integral in (2.22).

**Proof of (2.13) \( (\nu = 2) \)** Let us assume \( x_1 \geq x_2 \) (the case \( x_2 \geq x_1 \) is analogous). Noting that \( \hat{q}_a(\lambda) \) is everywhere regular but in the origin, we decompose

\[
\int_{[-\pi,\pi]^2 \setminus [-\delta_n,\delta_n]^2} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} = \int_{[-\pi,\pi]^2} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \frac{2id\cdot\lambda}{(B\lambda,\lambda)} + \int_{[-\pi,\pi]^2} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \left( \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} - \frac{2id\cdot\lambda}{(B\lambda,\lambda)} \right) + O(\delta_n), \tag{2.25}
\]

where we used again (2.23). Now we show that \( \partial_{\lambda_1} \left( \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} - \frac{2id\cdot\lambda}{(B\lambda,\lambda)} \right) \) is continuous on \([-\pi, \pi]^2\). Recalling the parity of \( \hat{q}_a(\lambda) \) and \( \hat{\rho}(\lambda) \) we can expand \( \hat{q}_a(\lambda) = id\cdot\lambda + r_1(\lambda) \), \( \hat{\rho}(\lambda) = 1 - \frac{1}{2}(B\lambda, \lambda) + r_2(\lambda) \), where \( r_1(\lambda) = o(\lambda^3) \) and \( r_2(\lambda) = o(\lambda^4) \). Then we can write

\[
\partial_{\lambda_1} \left( \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} - \frac{2id\cdot\lambda}{(B\lambda,\lambda)} \right) \bigg|_{x=0} \text{ as}
\]

\[
\lim_{\lambda \to 0} \frac{1}{\lambda_1} \left( \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} - \frac{2id\cdot\lambda}{(B\lambda,\lambda)} \right) = \lim_{\lambda \to 0} \frac{1}{\lambda_1} \left( \frac{2r_1(\lambda)(B\lambda,\lambda) + 4id\cdot\lambda r_2(\lambda))}{(B\lambda,\lambda)((B\lambda,\lambda) - 2r_2(\lambda))} \right),
\]

that is bounded. Thus, integrating by parts the function \( e^{-ix\cdot\lambda} = \frac{i}{x_1} \partial_{\lambda_1} e^{-ix\cdot\lambda} \) in the variable \( \lambda_1 \), we get

\[
\left| \int_{[-\pi,\pi]^2} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \left( \frac{\hat{q}_a(\lambda)}{1 - \hat{\rho}(\lambda)} - \frac{2id\cdot\lambda}{(B\lambda,\lambda)} \right) \right| \leq C_{x_1} \frac{C\sqrt{2}}{|x|}. \tag{2.26}
\]

To handle the first term on the r.h.s. of (2.25) we first notice that, again integrating by parts and using the continuity of \( \frac{d\lambda}{(B\lambda,\lambda)} \) on \([-\pi, \pi]^2 \setminus \{ |\lambda| \leq \pi \} \), we get

\[
\int_{[-\pi,\pi]^2} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \frac{d\cdot\lambda}{(B\lambda,\lambda)} = \frac{1}{n} \int_{|\lambda| \leq \pi} \frac{d^2\lambda}{(2\pi)^2} e^{-ix\cdot\lambda} \frac{d\cdot\lambda}{(B\lambda,\lambda)} + R_1(x). \tag{2.27}
\]
with $|R_1(x)| \lesssim \frac{C}{|x|}$. We change variables $\lambda = O \tilde{\lambda}$ and $x = \tilde{x}O^{-1}$, with $O^{-1} = OT$ and $O^{-1}BO = \text{diag}(\sigma_1^2, \sigma_2^2)$, so that
\[
\int_{|\lambda| \leq \pi} \frac{d^2 \lambda}{(2\pi)^2} e^{-i\tilde{x} \cdot \lambda} \frac{d \cdot \lambda}{(B \lambda, \lambda)} = \int_{|\lambda| \leq \pi} \frac{d^2 \tilde{\lambda}}{(2\pi)^2} e^{-i\tilde{x} \cdot \tilde{\lambda}} \frac{d \cdot \tilde{\lambda}}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2}, \tag{2.28}
\]
where $\tilde{d} = Od$. Since the function $\frac{d \tilde{\lambda}_1 + d \tilde{\lambda}_2}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2}$ is continuous on $\{ |\lambda| \leq \pi \} \setminus [-\pi/\sqrt{2}, \pi/\sqrt{2}]^2$, integrating by parts we get
\[
\int_{|\lambda| \leq \pi} \frac{d^2 \lambda}{(2\pi)^2} e^{-i\tilde{x} \cdot \lambda} \frac{d \cdot \lambda}{(B \lambda, \lambda)} = \int_{[-\pi/\sqrt{2}, \pi/\sqrt{2}]^2} \frac{d^2 \tilde{\lambda}}{(2\pi)^2} e^{-i\tilde{x} \cdot \tilde{\lambda}} \frac{d \tilde{\lambda}_1 + d \tilde{\lambda}_2}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2} + R_2(x), \tag{2.29}
\]
with $|R_2(x)| \lesssim \frac{C}{|x|}$. Then we will prove that also the first term on the r.h.s. of (2.29) satisfies the same bound
\[
\left| \int_{[-\pi/\sqrt{2}, \pi/\sqrt{2}]^2} \frac{d^2 \tilde{\lambda}}{(2\pi)^2} e^{-i\tilde{x} \cdot \tilde{\lambda}} \frac{d \tilde{\lambda}_1 + d \tilde{\lambda}_2}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2} \right| \leq \frac{C}{|x|}.
\]
By triangle inequality and symmetry we only need to prove
\[
\left| \int_{[-\pi/\sqrt{2}, \pi/\sqrt{2}]^2} \frac{d^2 \tilde{\lambda}}{(2\pi)^2} e^{-i\tilde{x} \cdot \tilde{\lambda}} \frac{d \tilde{\lambda}_1}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2} \right| \leq \frac{C}{|x|}.
\]
We use
\[
\frac{1}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2} = \int_0^\infty dy e^{-y(\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2)}
\]
and Fubini’s theorem to obtain
\[
\int_{[-\pi/\sqrt{2}, \pi/\sqrt{2}]^2} \frac{d^2 \tilde{\lambda}}{(2\pi)^2} e^{-i\tilde{x} \cdot \tilde{\lambda}} \frac{d \tilde{\lambda}_1}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2} = \tilde{d}_1 \int_0^\infty dy \left[ \int_{-\pi}^\pi \frac{d \tilde{\lambda}_2}{2\pi} e^{-i \tilde{x}_2 \tilde{\lambda}_2 - y \sigma_2^2 \tilde{\lambda}_2^2} \right] \left[ \int_{-\pi}^\pi \frac{d \tilde{\lambda}_1}{2\pi} e^{-i \tilde{x}_1 \tilde{\lambda}_1 - y \sigma_1^2 \tilde{\lambda}_1^2} \right] + \tilde{d}_1 \int_0^\infty dy \left[ \int_{-\infty}^\infty \frac{d \tilde{\lambda}_2}{2\pi} e^{-i \tilde{x}_2 \tilde{\lambda}_2 - y \sigma_2^2 \tilde{\lambda}_2^2} \right] \left[ \int_{-\infty}^\infty \frac{d \tilde{\lambda}_1}{2\pi} e^{-i \tilde{x}_1 \tilde{\lambda}_1 - y \sigma_1^2 \tilde{\lambda}_1^2} \right]
\]
and
\[
\int_{[-\pi/\sqrt{2}, \pi/\sqrt{2}]^2} \frac{d^2 \tilde{\lambda}}{(2\pi)^2} e^{-i\tilde{x} \cdot \tilde{\lambda}} \frac{d \tilde{\lambda}_1}{\sigma_1^2 \lambda_1^2 + \sigma_2^2 \lambda_2^2} = \tilde{d}_1 \int_0^\infty dy \left[ \int_{-\pi}^\pi \frac{d \tilde{\lambda}_2}{2\pi} e^{-i \tilde{x}_2 \tilde{\lambda}_2 - y \sigma_2^2 \tilde{\lambda}_2^2} \right] \left[ \int_{-\pi}^\pi \frac{d \tilde{\lambda}_1}{2\pi} e^{-i \tilde{x}_1 \tilde{\lambda}_1 - y \sigma_1^2 \tilde{\lambda}_1^2} \right] + \tilde{d}_1 \int_0^\infty dy \left[ \int_{-\infty}^\infty \frac{d \tilde{\lambda}_2}{2\pi} e^{-i \tilde{x}_2 \tilde{\lambda}_2 - y \sigma_2^2 \tilde{\lambda}_2^2} \right] \left[ \int_{-\infty}^\infty \frac{d \tilde{\lambda}_1}{2\pi} e^{-i \tilde{x}_1 \tilde{\lambda}_1 - y \sigma_1^2 \tilde{\lambda}_1^2} \right]
\]
The first term can be explicitly computed and its modulus equals
\[
\left| \frac{1}{4\sigma_1\sigma_2} \left( \frac{d_1\bar{x}_1}{\sigma_1^2} \right) \int_0^\infty \frac{dy}{y^2} e^{-\frac{1}{y^2} \left( \frac{\bar{x}_1^2}{\sigma_1^2} + \frac{\bar{x}_2^2}{\sigma_2^2} \right)} \right| \leq C \frac{1}{|x|}, \tag{2.30}
\]
where the last inequality follows by the change of variables \( y' = \left( \frac{\bar{x}_1^2}{\sigma_1^2} + \frac{\bar{x}_2^2}{\sigma_2^2} \right)^{-1} y \).

For the second term we will show that
\[
\int_0^\infty dy \left( \int_\pi^\infty d\bar{\lambda}_1 e^{-i\bar{\lambda}_1\bar{x}_1} \right) \left( \int_\pi^\infty d\bar{\lambda}_2 e^{-i\bar{\lambda}_2\bar{x}_2} \right) \leq C \frac{1}{|x_1||x_2|} \leq \frac{2C}{|x|}. \tag{2.31}
\]
Then, since the other contributions are analogous, the statement will follow by triangle inequality. Integrating by parts the functions \( e^{-i\bar{\lambda}_j\bar{x}_j} = \frac{i}{\bar{x}_j} \partial_{\bar{\lambda}_j} e^{-i\bar{\lambda}_j\bar{x}_j} \) we can bound
\[
\left| \int_\pi^\infty d\bar{\lambda}_2 e^{-i\bar{\lambda}_2\bar{x}_2 - y\sigma_2^2\bar{x}_2^2} \right| \leq \frac{1}{|x_2|} \left( e^{-y\sigma_2^2\pi^2} + \int_\pi^\infty d\bar{\lambda}_2 y\sigma_2^2\bar{\lambda}_2 e^{-y\sigma_2^2\bar{\lambda}_2^2} \right) \leq \frac{2}{|x_2|} e^{-y\sigma_2^2\pi^2} \tag{2.32}
\]
and similarly
\[
\left| \int_\pi^\infty d\bar{\lambda}_1 e^{-i\bar{\lambda}_1\bar{x}_1 - y\sigma_1^2\bar{x}_1^2\bar{\lambda}_1} \right| \leq \frac{1}{|x_1|} \left( e^{-y\sigma_1^2\pi^2} + \int_\pi^\infty d\bar{\lambda}_1 y\sigma_1^2\bar{\lambda}_1 e^{-y\sigma_1^2\bar{\lambda}_1^2} \right) \leq \frac{2}{|x_1|} e^{-y\sigma_1^2\pi^2} + \frac{C}{\sqrt{y}\sigma_1} \tag{2.33}
\]
where in the last lines we changed variables \( \bar{\eta}_1 = \sqrt{y}\sigma_1\bar{\lambda}_1 \) and used \( \int_0^\infty d\bar{\eta}_1 e^{-\bar{\eta}_1^2} \leq C/2 \) and \( \int_0^\infty d\bar{\eta}_1 e^{-\bar{\eta}_1^2} \leq C/2 \). Since the product of (2.32) and (2.34) is integrable over \( y \in (0, \infty) \) the (2.31) has been proved. For the third term we notice that it equals
\[
\left| \frac{1}{4\sigma_1\sigma_2} \int_0^\infty dy \frac{1}{y^2} e^{-\frac{1}{y^2} \left( \frac{\bar{x}_1^2}{\sigma_1^2} + \frac{\bar{x}_2^2}{\sigma_2^2} \right)} \left[ \int_{(-\infty, -\pi) \cup (\pi, \infty)} \frac{d\bar{\lambda}_1}{2\pi} e^{-i\bar{\lambda}_1\bar{x}_1 - y\sigma_1^2\bar{x}_1^2\bar{\lambda}_1} \right] \right|
\]
that is again bounded by \( C/|x| \) using the (2.30) and the integrability of the function \( e^{-y\sigma_1^2\bar{x}_1^2\bar{\lambda}_1} \). The same bound for the fourth term is analogous. \( \square \)
Proof of (2.12) (\(\nu = 1\)) The leading term in (2.22) reads

\[
\frac{1}{\sqrt{n}} \int_{[-\pi, \pi] \setminus [-\delta_n, \delta_n]} \frac{d\lambda}{2\pi} e^{ix\lambda} \frac{\hat{q}_a(\lambda)}{1 - \hat{p}(\lambda)}
\]

= \frac{1}{\sqrt{n}} \int_{[-\pi, \pi] \setminus [-\delta_n, \delta_n]} \frac{d\lambda}{2\pi} \left( \frac{d e^{ix\lambda}}{\sigma^2 \lambda} + e^{ix\lambda} \left( \frac{\hat{q}_a(\lambda)}{1 - \hat{p}(\lambda)} - \frac{id}{\sigma^2 \lambda} \right) \right).

The derivative of the integrand function in the second summand on the r.h.s. is bounded on \([-\pi, \pi]\) (the proof is as in the \(\nu = 2\) case), therefore the integral can be estimated by \(C/|x|\). The first integral gives the main contribution:

\[
\frac{d}{\sigma^2 \pi \sqrt{n}} \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} d\lambda \lambda \frac{\sin(x\lambda)}{\lambda} = \frac{d}{\sigma^2 \pi \sqrt{n}} (\sin(\delta_n x) - \sin(\pi x)),
\]

where \(\sin(t) = \int_0^\infty \frac{\sin(s)}{s} ds\) is the sine integral function. It stays bounded for small argument and decays as \(1/t\) for large \(t\). This gives (2.12).

This concludes the proof of the first lemma.

Proof of Lemma 2.3 The change of variables \(\lambda \rightarrow \lambda \sqrt{n}\) maps the cube \([-\delta_n, \delta_n]^{\nu}\) in \([-\delta_n \sqrt{n}, \delta_n \sqrt{n}]^{\nu}\). By the cumulant expansion of \(\hat{p}(\lambda)\) and the Taylor expansion of \(\hat{q}_a(l)\) we obtain

\[
II_n = \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} d\lambda \frac{\sin(x\lambda)}{\lambda} e^{-i\frac{x}{\sqrt{n}}\lambda} \sum_{k=0}^{n-1} p^*_k(0) e^{(Bl,l)_{2n}} \left[ i(l \cdot d) \sqrt{n} + r(l \sqrt{n}) \right],
\]

with \(\lim sup_{|\lambda| \rightarrow 0} |r(\lambda)|/|\lambda|^3 \leq C\). Now we separately handle the case \(\nu \geq 2\), and \(\nu = 1\).

Proof of (2.16) (\(\nu \geq 2\)) Because of our choice of \(\delta_n\) for \(\nu \geq 2\) we have

\[
|II_n| = \left| \int_{[-n^\nu, n^\nu]^{\nu}} d\lambda \frac{\sin(x\lambda)}{\lambda} e^{-i\frac{x}{\sqrt{n}}\lambda} \sum_{k=0}^{n-1} p^*_k(0) e^{(Bl,l)_{2n}} \left[ i(l \cdot d) \sqrt{n} + r(l \sqrt{n}) \right] \right| \leq C \int_{[-n^\nu, n^\nu]^{\nu}} d\lambda \frac{i(l \cdot d)}{\sqrt{n}^{\nu}} + r\left(\frac{l \sqrt{n}}{\sqrt{n}^{\nu}}\right) \leq C' n^{\nu+1} \sqrt{n}^{\nu-1} = o\left(\frac{1}{n^{\nu}}\right).
\]

Proof of (2.15) (\(\nu = 1\)) It is convenient to set for \(|\lambda| \leq \delta_n \sqrt{n}\)

\[
K_n(\lambda) := \frac{\sqrt{n}}{2\sigma^2 \lambda^2 \sqrt{\pi}} \int_{\frac{\sigma^2 \lambda^2 (n-1)}{2n}}^{\frac{\sigma^2 \lambda^2}{2n}} dt \frac{e^t}{\sqrt{t}}.
\]

We can write \(K_n(\lambda)\) as a series as follows

\[
K_n(\lambda) = \sum_{\ell \geq 0} \left(\frac{\sigma^2 \lambda^2}{2n}\right)^\ell \frac{1}{\ell!} \frac{(n-1)^{\ell+\frac{1}{2}} - 1}{\ell + \frac{1}{2}}.
\]

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The proof is done simply by expanding the exponential. Then
\[
\sum_{k=1}^{n-1} p^*(k) e^{\frac{\sigma^2}{2n}} = \frac{1}{\sqrt{2\pi \sigma^2 n}} \sum_{t=1/n}^{(n-1)/n} e^{\frac{\sigma^2}{2t}} + o\left(\frac{1}{\sqrt{n}}\right)
\]
and estimating the Riemann sum with the integral (through the second derivative) we obtain
\[
\left| \frac{1}{\sqrt{2\pi \sigma^2 n}} \sum_{t=1/n}^{(n-1)/n} e^{\frac{\sigma^2}{2t}} - K_n(\lambda) \right| \leq e^{\frac{\sigma^2}{2n}} \left(1 - \frac{1}{n}\right) - e^{\frac{\sigma^2}{2n}}. \quad (2.39)
\]
When we plug this last expression into (2.36), this can be readily estimated by
\[
\frac{1}{n^2} \int_{\mathbb{R}} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} \left[ \frac{i(\lambda d)}{\sqrt{n}} + r \left(\frac{\lambda}{\sqrt{n}}\right) \right] + \frac{d}{n} \int_{\mathbb{R}} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} \left(1 - \frac{1}{n}\right) \left[ \frac{|\lambda|}{\sqrt{n}} + r \left(\frac{\lambda}{\sqrt{n}}\right) \right] = o\left(\frac{1}{\sqrt{n}}\right).
\]
Therefore
\[
I_1 = \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} K_n(\lambda) \left[ \frac{i(\lambda d)}{\sqrt{n}} + r \left(\frac{\lambda}{\sqrt{n}}\right) \right] + o\left(\frac{1}{\sqrt{n}}\right). \quad (2.40)
\]
Using \(|r(\lambda)| \leq C|\lambda|^3\) for \(\lambda \in [-\delta_n, \delta_n]\) we have
\[
\int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} K_n(\lambda) r \left(\frac{\lambda}{\sqrt{n}}\right) \leq \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} K_n(\lambda) |\lambda| |\lambda|^2 n^{-1} = 2 \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} \sum_{\ell \geq 0} (\sigma |\lambda|)^{2\ell} |\lambda|^3. \quad (2.41)
\]
Now we split the sum with \(n\); the tail is easily bounded:
\[
2 \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} \sum_{\ell \geq n} (\sigma |\lambda|)^{2\ell+1} \frac{\lambda^2}{n} \leq \frac{2}{2n + 1} \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} \frac{|\lambda|^3}{n} = o\left(\frac{1}{\sqrt{n}}\right).
\]
For the remaining part of the sum we have
\[
2 \int_{[-\delta_n \sqrt{n}, \delta_n \sqrt{n}]} \frac{d\lambda}{2\pi \sqrt{n}} e^{-\frac{\sigma^2}{2n}} \sum_{\ell = 0}^{n} (\sigma |\lambda|)^{2\ell+1} \frac{\lambda^2}{n} \leq \sum_{\ell = 0}^{n} \frac{2\delta^2}{n! (2\ell + 1) 2^\ell} \int_{\mathbb{R}} \frac{d\lambda}{2\pi} e^{-\frac{\sigma^2}{2n}} (\sigma |\lambda|)^{2\ell+1} = \sum_{\ell = 0}^{n} \frac{1}{2\ell + 1} \frac{2\delta^2}{\sqrt{n}} = \frac{2\delta^2 \ln n}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right).
\]
due to our choice of $\delta_n$ such that $\limsup_{n \to \infty} \delta_n^2 \ln n = 0$. For the linear term in $\lambda$ we get

$$II_n = 2 \sum_{\ell \geq 0} \int_{-\sqrt{n}}^{\sqrt{n}} \frac{d\lambda}{2\pi \sqrt{n}} e^{-i \frac{\lambda^2}{2} \sigma^2} \left( \sum_{\ell=0}^{\ell} \frac{1}{(2\ell+1)!} \lambda^{2\ell} \right) \frac{\lambda d\lambda}{\sqrt{n}} + o\left( \frac{1}{\sqrt{n}} \right)$$

\[
= \frac{d}{\sigma^2} 2 \sum_{\ell \geq 0} \frac{1}{2\ell! (2\ell+1)} \int_{\mathbb{R}} \frac{d\lambda}{2\pi \sqrt{n}} e^{-i \frac{\lambda^2}{2} \sigma^2} e^{-\frac{\lambda^2}{2n}} \left( i\lambda \right) + o\left( \frac{1}{\sqrt{n}} \right)
\]

The leading term can be explicitly computed. Using (1.7) we obtain the expansion in Hermite polynomials of sign \(\frac{x}{\sigma \sqrt{n}} = \text{sign}(x)\):

\[
II_n = \frac{d}{\sigma^2} 2 \sum_{\ell \geq 0} \frac{H_{2\ell}(0)}{(2\ell+1)!} \frac{H_{2\ell+1}\left( \frac{x}{\sigma \sqrt{n}} \right)}{2\pi \sqrt{n}\sigma^2} e^{-\frac{x^2}{2n\sigma^2}} + o\left( \frac{1}{\sqrt{n}} \right)
\]

\[
= \frac{d}{\sigma^2} \sum_{\ell \geq 0} \left[ 2 \int_0^\infty dy \frac{H_{2\ell+1}(y)}{\sqrt{2\pi} (2\ell+1)!} e^{-\frac{y^2}{2}} \right] H_{2\ell+1}\left( \frac{x}{\sigma \sqrt{n}} \right) e^{-\frac{x^2}{2n\sigma^2}} + o\left( \frac{1}{\sqrt{n}} \right)
\]

\[
= \frac{d}{\sigma^2} \sum_{\ell \geq 0} \left[ \int_0^\infty dy \sqrt{2\pi} \text{sign}(y) \frac{H_{2\ell+1}(y)}{(2\ell+1)!} e^{-\frac{y^2}{2}} \right] H_{2\ell+1}\left( \frac{x}{\sigma \sqrt{n}} \right) e^{-\frac{x^2}{2n\sigma^2}}
\]

\[
+ o\left( \frac{1}{\sqrt{n}} \right)
\]

\[
= \frac{d}{\sigma^2} \text{sign}(x) e^{-\frac{x^2}{2n\sigma^2}} + o\left( \frac{1}{\sqrt{n}} \right)
\]

This concludes the proof of the second lemma and therefore of the main theorem.

\[\Box\]

**Acknowledgements** The authors thank C. Boldrighini for suggesting the problem. R. L. is supported by the ERC grant 676675 FLIRT.

**References**

1. Boldrighini, C., Marchesiello, A., Saffirio, C.: Weak dependence for a class of local functionals of Markov chains on $\mathbb{Z}^d$. Methods Funct. Anal. Topology 21, 302–314 (2015)
2. Boldrighini, C., Pellegrinotti, A.: Random walk on $\mathbb{Z}$ with one-point inhomogeneity. Markov Proc. Rel. Fields 18, 421–440 (2012)
3. Szegö, G.: Orthogonal Polynomials, vol. XXIII. AMS Colloquium Publications, Providence (1933)
4. Korshunov, D.A.: Limit theorems for general markov chains. Siberian Math. J. 42(2), 301–316 (2001)
5. Lawler, G.F., Limic, V.: Random Walk: A Modern Introduction. Cambridge University Press, Cambridge (2010)
6. Minlos, R.A., Zhizhina, E.A.: Local limit theorem for a non homogeneous random walk on the lattice. Theory Probab. Appl. 39, 513–529 (1994)

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