Coulomb screening correction to the $Q$ value of the triple alpha process in thermal plasmas

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Background: The triple alpha reaction is a key to $^{12}$C production and is expected to occur in weakly-coupled, thermal plasmas as encountered in normal stars.

Purpose: We investigate how Coulomb screening affects the structure of a system of three alpha particles in such a plasma environment by precise three-body calculations within the Debye-Hückel approximation.

Methods: A three-alpha model that has the Coulomb interaction modified in the Yukawa form is employed by including two kinds of the three-alpha potential in such a way as to reproduce the Hoyle state energy in vacuum. Precise three-body wave functions are obtained by a superposition of correlated Gaussian bases with the aid of the stochastic variational method.

Results: The energy shifts of the Hoyle state due to the Coulomb screening are obtained as a function of the Debye screening length. The results, which automatically incorporate the finite size effect of the Hoyle state, are consistent with the conventional result based on the Coulomb correction to the chemical potentials of ions that are regarded as point charges in a weakly-coupled, thermal plasma.

Conclusions: We have given a theoretical basis to the conventional point-charge approach to the Coulomb screening problem relevant for nuclear reactions in normal stars by providing the first evaluation of the Coulomb corrections to the $Q$ value of the triple alpha process that produces a finite size Hoyle state.

I. INTRODUCTION

In the past few decades, the structure of the $^{12}$C spectrum has been one of the most interesting phenomena in nuclear physics. An accurate description of the production process of the $^{12}$C element, which is one of the most abundant elements, is a key to understanding the stellar nucleosynthesis, where $^{12}$C is created in the fusion of three $^4$He nuclei (α particles) through the formation of the $^8$Be resonant state as an intermediate state. To explain the abundance of $^{12}$C, in the 1950s, Hoyle proposed the existence of a resonant state of $^{12}$C with $J^\pi = 0^+$, the same spin-parity as the ground state, at an energy just above the three-α threshold. This state, which is called the Hoyle state, was experimentally confirmed soon afterwards and has been believed to play an essential role in increasing the production rate of $^{12}$C.

From an astrophysical point of view, a dense and cold helium plasma appears in the outer layer of an X-ray bursting, accreting neutron star, in which the triple alpha reaction leads to unstable helium burning. In such a plasma, the Coulomb repulsion is screened off at large distances by the surrounding degenerate electrons in a manner that is dependent on the plasma density. This phenomenon can affect the triple alpha reaction rate as it shifts the energy of the Hoyle state.

In this paper, we study the Coulomb screening effect on the Hoyle state in such a plasma environment as encountered in a normal star that undergoes a stable burning of helium. To describe the structure of the Hoyle state of $^{12}$C, we perform precise three-body calculations in terms of the correlated Gaussian expansion with the aid of the stochastic variational method. In these calculations, three α particles are approximated as structureless point particles, while phenomenological two- and three-α potentials that reasonably reproduce the empirical $^8$Be and Hoyle state energies are employed. Finally, the Coulomb screening effect is incorporated into the Coulomb interaction in the Yukawa form. This form is relevant as long as the Debye-Hückel approximation is valid. This approximation gives a good description of the long-range Coulomb screening in a weakly-coupled, thermal plasma of interest here.

Since the screening acts to reduce the Coulomb interaction between α particles, this astrophysical environment would make it less repulsive than that in a free space. In fact, the screening effect in the three-α system was studied to search for the possible Efimov states. It was concluded that due to the nature of the Hoyle state that appears as the three-α first excited state, a series of the Efimov $0^+$ states might appear above the Hoyle state in possible astrophysical environments where the two-α ground state energy would become almost zero. According to Ref. [13], a full treatment of the three-body problem with short-range and Coulomb interactions could shed some light on the conjecture that the Hoyle state may emerge as an Efimov state.

The paper is organized as follows. In Sec. [11] we explain how to construct the wave function of the three-α system. In doing so, the variational method and model...
Hamiltonian are described. Section III is devoted to a description of the correction to the Hoyle state energy by the Coulomb screening in a weakly-coupled plasma in the zero-size limit of ions including a $^{12}$C nucleus in the Hoyle state. The validity of the Yukawa form of the screened Coulomb potential is discussed in this section. In Sec. IV, we show the calculated results for the energy shift of the Hoyle state due to the screening and compare them with those based on the Coulomb corrections to the chemical potential of point ions in weakly-coupled plasmas. Conclusions of this work are drawn in Sec. V.

II. THREE-Å DESCRIPTION OF THE HOYLE STATE

In this section, we describe how to obtain the three-Å wave function while allowing for the Coulomb screening.

A. Variational calculation with correlated Gaussian expansion

We begin by summarizing a variational approach, which will be adopted to obtain a precise solution of the three-body Schrödinger equation. The Hamiltonian for the three-Å system is specified as

$$H = \sum_{i=1}^{3} T_i - T_{cm} + \sum_{i<j} [V_{ij}^{2\alpha} + V_{ij}^{\text{Coul}}(C)] + V_{123\alpha},$$

where $T_i$ is the kinetic energy operator of the $i$th α particle, and the center-of-mass kinetic energy $T_{cm}$ is appropriately subtracted. Details of the two- and three-Å potentials, $V_{ij}^{2\alpha}$ and $V_{123\alpha}$, as well as the screened Coulomb potential $V_{ij}^{\text{Coul}}(C)$ with the screening factor $C$, will be given in the next section.

The wave function of the three-Å system can be expanded by a number $(K)$ of symmetrized $(S)$ correlated Gaussian basis functions $G$ as

$$\Psi^{(n)} = \sum_{k=1}^{K} c^{(n)}_k S G(A_k, x).$$

The set of the coefficients $(c_1^{(n)}, \ldots, c_K^{(n)})$, where $n$ denotes a label of the state $(n = 0, \ldots, K - 1)$ with $n = 0$ being the ground state, can be determined by solving the generalized eigenvalue equation

$$\sum_{j=1}^{K} H_{ij} c_j^{(n)} = E^{(n)} \sum_{j=1}^{K} B_{ij} c_j^{(n)},$$

where

$$H_{ij} = \langle SG(A_i, x) | H | SG(A_j, x) \rangle$$

and

$$B_{ij} = \langle SG(A_i, x) | SG(A_j, x) \rangle$$

are the Hamiltonian and overlap matrix elements, respectively.

Here, the coordinate set $\tilde{x} = (x_1, x_2)$, where the tilde denotes the transpose of the matrix, is taken as the Jacobi coordinates excluding the center of mass of the three-Å system $x_3$. These three coordinates are defined as

$$x_i = \sum_{j=1}^{3} U_{ij} r_j,$$

where $r_i$ denotes the $i$th single-Å coordinate, and

$$U = \begin{pmatrix} 1 & -1 & 0 \\ \frac{1}{3} & \frac{1}{3} & -1 \\ \frac{2}{3} & \frac{2}{3} & 1 \end{pmatrix}$$

is the transformation matrix. Finally, the correlated Gaussian basis function is defined by $[11, 12]$

$$G(A, x) = \exp \left( -\frac{1}{2} \tilde{x} A \tilde{x} \right)$$

$$= \exp \left( -\frac{1}{2} A_{11} x_1^2 - \frac{1}{2} A_{22} x_2^2 - A_{12} x_1 : x_2 \right).$$

Each correlated Gaussian is specified by a symmetric, positive-definite $2 \times 2$ matrix $A$. The diagonal elements of the matrix $A$ can be related to the Gaussian falloff parameters as $1/\sqrt{A_{ii}}$, while the off-diagonal element controls the correlations among the different relative coordinates.

The wave function of the system has to have a proper symmetry under interchange of identical particles. The symbol $S$ denotes a symmetrizer that assures the basis function being totally symmetric with respect to any exchange of particles. One of the major advantages of the correlated Gaussian is invariance of its functional form under any coordinate transformation, which allows us to easily manipulate exchange of the particles as needed for the symmetrization of the basis function. In fact, we superpose the six permutations among the three identical bosons, which in turn can be expressed by an appropriate choice of the transformation matrix $T_{P}$. The quadratic form $\tilde{y} A \tilde{y}$ can thus be rewritten as $\tilde{x} T_{P} A T_{P} \tilde{x}$ with the transformation of the coordinate set $\tilde{y} = T_{P} \tilde{x}$, which leaves the functional form of the correlated Gaussian unchanged. This convenient property makes the correlated Gaussian basis suitable for treating few-body systems accompanied by strong interparticle correlations $[16, 17]$. Most of the matrix elements, including $H_{ij}$ and $B_{ij}$, can be analytically obtained as functions of a number of variational parameters, i.e., the matrix elements $A_{ij}$ for each basis $[11, 12, 18]$, which are in turn optimized by the stochastic variational method $[11, 12]$. In practice, the diagonal matrix elements of the matrix $A$ are generated as random numbers in the ranges of $0 < 1/\sqrt{A_{11}} < 20$ fm and $0 < 2/\sqrt{3} A_{22} < 20$ fm in such a way that one can describe the asymptotic wave function due to the Coulomb
screening at large distances. The correlation among the particles is taken into account via the off-diagonal matrix element $A_{ij}$, which is determined by defining the two-dimensional rotation matrix $R(\theta)$ with randomly generated rotational angles $\theta$ and multiplying it to the diagonal matrix $D_{ij} = A_{ij}\delta_{ij}$ as $RDR$.

We remark in passing that the above-mentioned formalism holds also for description of the ground state structure of $^{8}$Be. In this case, one can omit $V_{\alpha \alpha}$ from the Hamiltonian \footnote{\label{footnote:11}Tsuchiya et al. (1967)} and set $A_{22} = A_{12} = 0$ in the Gaussian basis function \footnote{\label{footnote:8}Cameron et al. (1958)}.

### B. Potential terms in the three-\(\alpha\) Hamiltonian

In describing $V_{\alpha \alpha}$ in Eq. \footnote{\label{footnote:11}Tsuchiya et al. (1967)} not only in as simple a form as possible but also in such a way as to reasonably reproduce low energy $\alpha$-$\alpha$ scattering data, we assume the $\alpha$ particle to be an inert point boson. Several versions of the potential models constructed under such an assumption are known (see, e.g., Refs. \footnote{\label{footnote:12}Ali-Bodmer (1968), \footnote{\label{footnote:22}Ali-Bodmer (1968)}}). In this paper, we employ the modified Ali-Bodmer (AB) potential \footnote{\label{footnote:12}Ali-Bodmer (1968), \footnote{\label{footnote:22}Ali-Bodmer (1968)}}, which is designed to provide the S-wave $^{8}$Be ($0^+_1$) resonance position $E_r$ with 88.84 keV \footnote{\label{footnote:28}Cameron et al. (1958)}, a value close to the empirical one 91.8 keV \footnote{\label{footnote:28}Cameron et al. (1958)}. The explicit form of the potential is given by

$$V_{i,j}^{2\alpha} = 125 \exp \left( -\frac{r_{ij}^2}{1.53^2} \right) - 30.18 \exp \left( -\frac{r_{ij}^2}{2.85^2} \right),$$

where the energy and length are given in units of MeV and fm, respectively, and $r_{ij} \equiv |r_i - r_j|$.

To take the Debye screening in thermal plasmas into account, we replace the bare Coulomb potential between point charges located at $r_i$ and $r_j$ by the Yukawa form

$$V_{ij}^{\text{Coul}} = \frac{4e^2}{r_{ij}} \exp(-Cr_{ij}).$$

Here, the parameter $C$ acts as the inverse of the length of the Coulomb screening. The validity of this form of the screened potential and the relevant value of $C$ will be given in Sec. \footnote{\label{footnote:13}III}. We remark in passing that for more realistic calculations, the charge form factor of an $\alpha$ particle, $f$, can be incorporated into Eq. \footnote{\label{footnote:13}III} as

$$V_{ij}^{\text{Coul}} = \int d^3u_i \int d^3u_j f(u_i)f(u_j) \frac{4e^2}{|r_i + u_i - (r_j + u_j)|} \exp(-C(|r_i + u_i - (r_j + u_j)|),$\footnotetext{\label{footnote:11}Tsuchiya et al. (1967)}

where the integral of $f$ over the whole space is set to unity. To incorporate such a finite size effect in the three-$\alpha$ system, it is reasonable to assume the Gaussian charge form factor for the $\alpha$ particle, leading to the explicit form of the Coulomb potential,

$$V_{ij}^{\text{Coul}} = \frac{4e^2}{r_{ij}} \exp\left(\kappa r_{ij}\right)$$

with $\kappa = 0.60141 \text{ fm}^{-1}$ \footnote{\label{footnote:13}Cameron et al. (1958)}. We will use Eq. \footnote{\label{footnote:13}III} unless otherwise noted.

Finally, we consider the three-$\alpha$ potential, which naturally occurs due to the internal structure of each $\alpha$ particle. This potential has to be allowed for because it is known that the empirical energies of the states close to three-$\alpha$ threshold are not well reproduced only from the two-$\alpha$ potential \footnote{\label{footnote:24}Tsuchiya et al. (1967)}. As was done in Ref. \footnote{\label{footnote:3}Cameron et al. (1958)}, one can introduce a simplified potential among three point $\alpha$ particles as a correction to $V_{ij}^{2\alpha}$ in the form of

$$V_{123} = v_r \exp\left(-\frac{R^2}{b_r^2}\right) - v_a \exp\left(-\frac{R^2}{b_a^2}\right). \footnote{\label{footnote:12}Ali-Bodmer (1968), \footnote{\label{footnote:22}Ali-Bodmer (1968)}}$$

with $R^2 \equiv \sqrt{\sum_{i=1}^{N}(r_i - x_i)^2} = \sqrt{\frac{3}{4}x_1^2 + \frac{1}{3}x_2^2}$. In order to see the model dependence, we employ two kinds of the three-$\alpha$ potential. One is the potential that has an attractive term alone \footnote{\label{footnote:3}Cameron et al. (1958)}; the parameters are set to $v_r = 0, v_a = 152.2$ MeV, and $b_r = 2.58$ fm (Set 1) in such a way as to reproduce the empirical Hoyle state energy in vacuum. The other includes a repulsive term, which reasonably occurs given the Pauli principle among three $\alpha$ particles composed of nucleons \footnote{\label{footnote:25}Cameron et al. (1958), \footnote{\label{footnote:26}Cameron et al. (1958)}}. We have fixed the parameters as $v_r = 48.0$ MeV, $b_r = 1.20$ fm, $v_a = 134$ MeV, and $b_a = 2.66$ fm (Set 2), to obtain better reproduction of the $^{12}$C ground state energy at $C = 0$. Owing to difference in the structure of these two models for the three-$\alpha$ potential, we expect some difference in the spatial scale of the Hoyle state in vacuum, which in turn may lead to difference in the energy of the Hoyle state at nonzero $C$ in such a way that the larger scale, the stronger Coulomb screening.

The physical constants that we employ in this paper are $h^2/m_\alpha = 10.5254$ MeV fm$^2$ and $e^2 = 1.43996$ MeV fm, where $m_\alpha$ is the mass of an $\alpha$ particle in vacuum.

### III. SCREENING CORRECTION TO TRIPLE ALPHA REACTIONS IN THE POINT-CHARGE APPROXIMATION

Before exhibiting the numerical solutions to the three-body problem as described in the previous section, we follow conventional approaches to the Coulomb screening by assuming that all the ions involved, including a $^{12}$C nucleus in the Hoyle state, are point charges and then estimate how much carbon is produced via triple alpha reactions as encountered in normal stars that undergo a stable burning of helium. In such environments in which the temperature $T$ is of order or even higher than $10^8$ K and the mass density $\rho$ is typically $10^3-10^6$ g cm$^{-3}$, the Hoyle state ($C^+$) occurs via two successive resonant reactions ($\alpha + \alpha \rightarrow \text{Be}$ and $\text{Be} + \alpha \rightarrow C^+$), where Be denotes the $^8$Be ground state \footnote{\label{footnote:27}Beckwith et al. (1967)}. The Debye screening results in the Yukawa form of the Coulomb interaction among $\alpha$ particles, which in turn acts to enhance carbon production \footnote{\label{footnote:28}Cameron et al. (1958)}.

There are two ways of evaluating such enhancement in the carbon production. One is a direct one in which...
the difference in the $Q$ value between the screened and non-screened cases is obtained from the Coulomb energy of the point-like Hoyle state and then is incorporated into the Saha prediction of the carbon production dominated by the Boltzmann factor $e^{Q/k_BT}$ [27]. Another is an indirect one in which the Coulomb correction to the chemical potential of each component, which is regarded as a point particle even for a nucleus in the Hoyle state, is calculated in the Debye-Hückel approximation and then is incorporated into the chemical equilibrium condition between three α particles and a nucleus in the Hoyle state [28–29]. As far as the system is sufficiently hot to become a weakly coupled, non-degenerate plasma that is charge neutral, both approaches have to give a consistent result for the enhancement in the carbon production. We remark that corrections due to the electron Fermi degeneracy and/or the strong Coulomb coupling deviate from the simple Debye-screened form [30].

As for the first approach, all we have to do is to give the $C$ value appropriately. In the case of the Debye screening, $C^{-1}$ is the Debye screening length defined as

$$
\lambda_D = \left[\frac{k_BT}{4\pi e^2 (n_e + \sum_i n_i Z_i^2)}\right]^{1/2},
$$

where $n_i$ and $Z_i$ are the averaged number density and charge number of ions of species $i$, and $n_e$ is the averaged number density of electrons. Due to charge neutrality, $n_e = \sum_i Z_i n_i$ is satisfied. We can estimate the value of $\lambda_D$ by assuming that hydrogen is exhausted and that $\sum_i n_i Z_i^2$ is dominated by $\alpha$ particles ($i = \alpha$). The latter assumption is validated if one notes the fact that under chemical equilibrium, $n_{c\alpha}$ is proportional to $e^{Q/k_BT}$, which is generally negligible. In the range of $T$ and $\rho$ as considered here, $\lambda_D$ is of order $10^{-3}–10^{-4}$ fm.

Eventually, this $\lambda_D$ determines corrections to the Coulomb potential of a quantum system of three $\alpha$ particles as

$$
\Delta V_C = \sum_{j<k} \frac{4e^2}{r_{jk}} - \frac{4e^2}{\lambda_D},
$$

where $r_{jk}$ is the distance operator between the $j$th and $k$th $\alpha$ particles. The expectation value of $\Delta V_C$, which can be obtained in the present three-body calculations as $\Delta E_C = \langle \Delta V_C \rangle$, gives rise to decrease in the mass of the Hoyle state and hence increase in the $Q$ value. Since the distance between the fusing particles is generally far shorter than $\lambda_D$, we obtain, by using the Taylor expansion with respect to $r_{jk}$,

$$
\Delta E_C = -\frac{12e^2}{\lambda_D} + O(\langle r_{jk} \rangle).
$$

Equation (16) suggests that both in the weak screening limit and in the zero-size limit of the Hoyle state, the increase in the $Q$ value amounts to $12e^2/\lambda_D$. We remark in passing that the number of electrons remain unchanged by the triple alpha reaction and that the gamma decay of the Hoyle state (two-photon processes) is not considered here because the lower-lying carbon states are not always described in terms of three $\alpha$ particles.

As for the second approach, we first write down the Helmholtz free energy density of the system as

$$
f = f_0 + f_{DH},
$$

where

$$
\begin{align*}
  f_0 &= n_e m_e c^2 + \sum_i n_i m_i c^2 \\
  &\quad - n_e k_BT \left\{ \ln \left[ \frac{2}{n_e \left( \frac{m_e k_BT}{2\pi\hbar^2} \right)^{3/2}} \right] + 1 \right\} \\
  &\quad - \sum_i n_i k_BT \left\{ \ln \left[ \frac{g_i}{n_i \left( \frac{m_i k_BT}{2\pi\hbar^2} \right)^{3/2}} \right] + 1 \right\}.
\end{align*}
$$

with the electron ($i$ ion) rest mass $m_e$ ($m_i$) and the number, $g_i$, of internal degrees of freedom of $i$ ions, is the ideal gas part of the free energy density, and

$$
f_{DH} = - (n_e + \sum_i n_i Z_i^2) \frac{e^2}{3\lambda_D}
$$

is the lowest order Coulomb correction to $f_0$, i.e., the Debye-Hückel term appropriate for a multi-component classical plasma. From this free energy density, one can derive the chemical potential of electrons and of $i$ ions as

$$
\mu_e = m_e c^2 - k_BT \ln \left[ \frac{2}{n_e \left( \frac{m_e k_BT}{2\pi\hbar^2} \right)^{3/2}} - \frac{e^2}{2\lambda_D} \right]
$$

and

$$
\mu_i = m_i c^2 - k_BT \ln \left[ \frac{g_i}{n_i \left( \frac{m_i k_BT}{2\pi\hbar^2} \right)^{3/2}} - \frac{Z_i^2 e^2}{2\lambda_D} \right]
$$

respectively. The last term of the right side in Eq. (21) corresponds to the Coulomb correction, $\mu_i^{\text{Coul}}$, to the chemical potential of $i$ ions.

We then apply the chemical potentials given by Eqs. (20) and (21) to the chemical equilibrium condition for $3\alpha \leftrightarrow C^*$,

$$
3\mu_\alpha = \mu_{C^*}.
$$

Note that $\mu_e$ does not come in because the triple alpha process involves no beta process. We thus obtain

$$
n_{C^*} = n_\alpha^3 e^{Q_0/k_BT} \lambda_\alpha^3 \lambda_{C^*}^3 e^{12e^2/\lambda_D k_BT},
$$

where $Q_0 = (3m_\alpha - m_{C^*}) c^2$ is the $Q$ value in the ideal gas limit, $\lambda_\alpha = \sqrt{2\pi\hbar^2/(m_e k_BT)}$ is the thermal de Broglie wavelength, and $g_{C^*} = g_\alpha = 1$. In the absence of screening, Eq. (23) reduces to the Saha prediction of the carbon production. The Debye screening induces the factor $e^{12e^2/\lambda_D k_BT}$ via $3\mu_\alpha - \mu_{C^*} = 12e^2/\lambda_D$, which is consistent with the first approach that predicts increase in the $Q$ value by $12e^2/\lambda_D$ in the weak screening limit.
IV. RESULTS AND DISCUSSIONS

We now proceed to exhibit the numerical results for the energy and size of the Hoyle state in weakly-coupled, thermal plasmas. The former will then be compared with the point-charge prediction of the $Q$ value shift as given in the previous section.

A. Coulomb screening effects on the three-$\alpha$ system

![Graph](image)

**FIG. 1.** Energy of the screened Hoyle state of the three-$\alpha$ system calculated with respect to the three-$\alpha$ threshold as a function of the screening factor $C$. The result for the screened ground state of $^8\text{Be}$ is also plotted for comparison.

In the present framework based on the Hamiltonian $H$, the lowest energy state ($n = 0$) corresponds to the ground state of $^{12}\text{C}$, while the Hoyle state appears as a resonant/bound excited state. Since the decay width of the Hoyle state is small, such a resonant state can be essentially described as a bound state $[31–33]$. In fact, in the two-$\alpha$ system, the ground state energy of $^8\text{Be}$ is obtained as 88.8 keV. With the Set 1 Hamiltonian ($C = 0$), the energies of the $^{12}\text{C}$ ground state and the Hoyle state are $-9.40$ and $0.349$ MeV, respectively, which are consistent with the results of Ref. $[2]$, while the Set 2 Hamiltonian ($C = 0$) gives $-8.99$ and $0.475$ MeV for the ground and Hoyle states, respectively.

Figure 1 plots the energies of the screened Hoyle state and the ground state of $^8\text{Be}$ with respect to the three-$\alpha$ threshold, $E_{C^*}$ and $E_{\text{Be}}$, evaluated as a function of the Coulomb screening factor $C$. As expected, the Hoyle state energy decreases with increasing the Coulomb screening factor $C$ and eventually approaches the three-$\alpha$ potential calculated in the absence of the Coulomb term in the Hamiltonian. When $C = 0.0162$ fm$^{-1}$, $E_{\text{Be}}$ becomes $\sim 10^{-5}$ MeV, which suggest that the condition for appearance of the Efimov state is met. We nevertheless find that for both Hamiltonians the Hoyle state is still in a resonance state with $E_{C^*} = 0.082$ (0.210) MeV for Set 1 (Set 2), excluding that the hypothesis that the Hoyle state is bound by the Efimov attraction. Beyond $C \sim 0.05$ fm$^{-1}$, we observe that the Hoyle state appears as a bound state, that is, the energy becomes below the $^8\text{Be}$ one, and that the asymptotic energy ($C = \infty$) calculated with Set 2 is $-3.62$ MeV, being slightly higher than that with Set 1 ($-3.89$ MeV) because Set 2 includes the repulsive component in the three-$\alpha$ potential. In this situation, however, the screening is too strong for the Debye-Hückel approximation to be valid. We can also observe the behavior of the energy shift at small $C$ does not depend strongly on the choice of the three-$\alpha$ potential, which will be discussed quantitatively later in this section.

To examine more details of the correlated motion of the three-$\alpha$ system, we calculate the root-mean-square (rms) pair distance defined by $d(n) = \sqrt{\langle \Psi^{(n)} | x^2 | \Psi^{(n)} \rangle}$. Figure 2 plots the results for the rms pair distance of the Hoyle state as a function of $C$, together with those of the ground state of $^8\text{Be}$ for comparison. For Set 1, as long as $C$ is small, the rms pair distance of the Hoyle state is significantly shorter than that of the $^8\text{Be}$ ground state due to stronger binding in the three-$\alpha$ system. At a critical $C$ where the Hoyle state becomes bound, the rms pair distance of the $^8\text{Be}$ ground state becomes so short as to coincide with that of the Hoyle state. This is not the case with Set 2, which provides the Hoyle state with a pair distance that is longer than not only the same quantity calculated from Set 1, but also the $^8\text{Be}$ result for any positive $C$. This behavior comes from the repulsive component of the three-$\alpha$ potential in Set 2. In fact, the rms pair distance of the $^8\text{Be}$ ground state with $C = \infty$ becomes 4.75 fm, while that of $C^*$ is 4.75 fm for Set 1 and 4.92 fm for Set 2. Such repulsion also acts to enhance the rms radius of the Hoyle state, which can be measured from the center of mass of the system as 3.43 (2.74) fm for Set 1 and 3.71 (2.84) fm for Set 2, respectively, in the case of $C = 0$ ($C = \infty$).

B. Screening-induced enhancement of carbon production

Let us now consider a realistic situation in which the value of $C$ is set to the inverse of the Debye screening length $[14]$, $\lambda_D^{-1}$. The use of $\lambda_D$ is strictly applicable to a plasma in which all the components (ions and electrons) behave like a nearly ideal, thermal gas. For example, at the highest density of interest here, the Fermi degeneracy can play a role in modifying the present description of the screening correction to the Coulomb interaction as suggested in Sec. $[13]$

Next, we reestimate screening-induced enhancement of the carbon production in normal stars by allowing for the spatial structure of the Hoyle state as precisely evaluated in the previous subsection. To do so, we follow the same line of argument of the direct approach shown in Sec.
Instead of taking the zero-size limit as in Sec. III we just substitute $E_{C^*}(C)$ into $Q$ and thereby estimate the screening-induced $Q$ value shift and enhancement factor as $\Delta Q(C) = E_{C^*}(C = 0) - E_{C^*}(C)$ and $e^{\Delta Q(C)/k_B T}$, respectively.

Finally, we compare the resultant $Q$ value shift due to the screening, $\Delta Q(C)$, with the conventional prediction, $12e^2/\lambda_D$, obtained for point charges. In Fig. 3 we show such a comparison by regarding $C$ as $\lambda_D^{-1}$. The obtained $\Delta Q$ values are insensitive to the three-$\alpha$ potential and hence the size of the Hoyle state. Both for Set 1 and Set 2, the results agree with the conventional prediction in the limit of $C \to 0$, as they should. For typical $\lambda_D$, such an agreement seems to be intact.

Despite such a good agreement, there has to be a difference in the screening-induced $Q$ value shift between the finite-size and zero-size cases of the Hoyle state. This difference, as discussed in Sec. III, is expected to depend on the model adopted for the three-$\alpha$ potential because the two models give an appreciable difference in the prediction of the spatial scale of the Hoyle state as shown in Fig. 2. To estimate the difference in the $Q$ value shift, however, it is inappropriate to use the result for $\Delta Q(C)$ because even in the weak coupling and classical limit, the Debye-Hückel approximation holds only for a description of the long-range Coulomb screening. In fact, within this approximation, the radial distribution function, $g(r)$, for $\alpha$ particles, i.e., the probability of finding another $\alpha$ particle at a distance of $r$ from the origin at which an $\alpha$ particle is already located, is known to be negative near $r = 0$ and hence unphysical [34]. In the case of the triple alpha reactions, the fusing $\alpha$ particles are inevitably located in the immediate vicinity of the partner. It is thus necessary to properly take into account the short-range spatial correlation.

To do so, it is convenient to note that for thermal plasmas one can generally express $g(r)e^{\Delta Q(C)/k_B T}$ in a power series of $r^2$ near $r = 0$ [35]. Then it is reasonable to define the effective potential $w(r)$ between two $\alpha$ particles via $g(r) = e^{-w(r)/k_B T}$. According to Ref. [36], $w(r)$ can be expanded as

$$w(r) = \frac{4e^2}{r} - \frac{4e^2}{\lambda_D} + \frac{1}{4} \left(\frac{r}{a_\alpha}\right)^2 + O(r^4)$$

with $a_\alpha = (3/4\pi n_\alpha)^{1/3}$. The second term of the right side corresponds to $-(2\mu_\alpha^\text{Coul} - \mu_\text{Be}^\text{Coul})$, i.e., the conventional point-charge prediction of the screening-induced $Q$ value shift for $\alpha + \alpha \rightarrow \text{Be}$ under chemical equilibrium $2\alpha \leftrightarrow \text{Be}$, while the third term comes from two closely separated $\alpha$ particles in the uniform electron background. For typical separation as depicted in Fig. 2 as well as for typical $T$ and $\rho$, the ratio of the third to second term is only of order $10^{-7}$. The usage of $w(r)$ instead of the Yukawa potential in Eq. (15) would thus reproduce the screening-induced $Q$ value shift of the triple alpha process in the point-charge approximation, $12e^2/\lambda_D$, while adding an $O(10^{-7})$ correction due to the finite size effect of the Hoyle state. This implies that the difference in such a $Q$ value shift between the two models for the three-$\alpha$ potential would also be negligible. We remark in passing that in the present estimate of the terms beyond the second one in Eq. (24), possible corrections due to the electron screening, the strong force potential $V_{\alpha}^{2\alpha}$, and the quantum nature of fusing $\alpha$ particles are ignored.

**V. CONCLUSION**

In this paper, we have revisited the Coulomb screening correction to the $Q$ value of the triple alpha pro-
cess in weakly-coupled, thermal plasmas by newly obtaining the precise three-α wave function within the Debye-Hückel approximation. Through variational calculations that incorporate the two models for the three-α potential in such a way as to reproduce the empirical energy of the Hoyle state in vacuum, we find that the conventional point-charge analysis gives a very good estimate of the screening-induced $Q$ value shift in normal stars that undergo a stable burning of helium.

Many questions nevertheless remain. It would be straightforward to perform the same kind of three-body calculations by considering a more realistic situation, e.g., by using the effective Coulomb potential instead of the Yukawa potential. Estimates of the $Q$ value shift in different environments as may be encountered in X-ray bursting, accreting neutron stars would also be interesting.

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