SUPERCATEGORIZATION OF QUANTUM KAC-MOODY ALGEBRAS

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Abstract. We show that the quiver Hecke superalgebras and their cyclotomic quotients provide a supercategorification of quantum Kac-Moody algebras and their integrable highest weight modules.

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1. Introduction

The idea of categorification dates back to I. B. Frenkel. He proposed that one can construct a tensor category whose Grothendieck group is isomorphic to the quantum group $U_q(\mathfrak{sl}_2)$ (see, for example, [5, 7]). Since then, a construction of a tensor category or a 2-category whose Grothendieck group possesses a given algebraic structure has been referred to as categorification.

One of the most prominent examples of categorification is the Lascoux-Leclerc-Thibon-Ariki theory, which clarifies the mysterious connection between the representation theory of quantum affine algebras of type $A^{(1)}_{n-1}$ and the modular representation theory of Hecke algebras at roots of unity. In [25], Misra and Miwa constructed an integrable representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$, called the Fock space representation, on the space spanned by colored Young diagrams. They also showed that the affine crystal consisting of $n$-reduced colored Young diagrams is isomorphic to the highest weight crystal $B(\Lambda_0)$. In [23], using the Misra-Miwa construction, Lascoux, Leclerc and Thibon discovered an algorithm of computing Kashiwara’s lower global basis (=Lusztig’s canonical basis) elements corresponding to $n$-reduced Young diagrams and conjectured that the transition matrices evaluated at 1 coincide with the composition multiplicities of simple modules inside Specht modules over Hecke algebras.

The Lascoux-Leclerc-Thibon conjecture was proved by Ariki in a more general form [1]. Combining the geometric method of Kazhdan-Lusztig and Ginzburg with combinatorics of Young diagrams and Young tableaux, Ariki proved that the Grothendieck group of the category of finitely generated projective modules over cyclotomic Hecke
algebras give a categorification of integrable highest weight modules over affine Kac-Moody algebras of type $A_{n-1}^{(1)}$. Moreover, he showed that the Kashiwara-Lusztig global basis is mapped onto the isomorphism classes of projective indecomposable modules, from which the Lascoux-Leclerc-Thibon conjecture follows. Actually, he proved the conjecture in a more general form because his categorification theorem holds for highest weight modules of arbitrary level.

Next problem is to prove a quantum version or a graded version of Ariki’s categorification theorem. The key to this problem was discovered by Khovanov-Lauda and Rouquier. In [20, 21, 26], Khovanov-Lauda and Rouquier independently introduced a new family of graded algebras, called the Khovanov-Lauda-Rouquier algebras or quiver Hecke algebras, and proved that the quiver Hecke algebras provide a categorification of the negative half of quantum groups associated with all symmetrizable Cartan data. Furthermore, Khovanov and Lauda conjectured that the cyclotomic quiver Hecke algebras give a graded version of Ariki’s categorification theorem in much more generality. That is, the cyclotomic quiver Hecke algebras should give a categorification of integrable highest weight modules over all symmetrizable quantum Kac-Moody algebras. This conjecture was proved by Kang and Kashiwara [10]. Their proof was based on: (i) a detailed analysis of the structure of quiver Hecke algebras and their cyclotomic quotients, (ii) the proof of exactness of restriction and induction functors, (iii) the existence of natural isomorphisms, which is one of the axioms for the $\mathfrak{sl}_2$ categorification developed by Chuang and Rouquier [4].

On the other hand, in [3], Brundan and Kleshchev showed that, when the defining parameter is a primitive $(2l + 1)$-th root of unity, the representation theory of some blocks of affine Hecke-Clifford superalgebras and their cyclotomic quotients is controlled by the representation theory of quantum affine algebras of type $A_{2l}^{(2)}$ at the crystal level. In [27], Tsuchioka proved a similar statement for the affine and cyclotomic Hecke-Clifford superalgebras with the defining parameter at $2(l + 1)$-th root of unity and the quantum affine algebras of type $D_{l+1}^{(2)}$. On the other hand, in [28], Wang introduced spin affine Hecke algebras (resp. cyclotomic spin Hecke algebras) and showed that they are Morita superequivalent to affine Hecke-Clifford superalgebras (resp. cyclotomic Hecke-Clifford superalgebras).

In [12], motivated by the works of Brundan-Kleshchev [3], Kang, Kashiwara and Tsuchioka introduced a new family of graded superalgebras, called the quiver Hecke superalgebras, which is a super version of Khovanov-Lauda-Rouquier algebras. They
also defined the notion of quiver Hecke-Clifford superalgebras and showed that these superalgebras are weakly Morita superequivalent to the corresponding quiver Hecke superalgebras. Moreover, they showed that, after some completion, quiver Hecke-Clifford superalgebras are isomorphic to affine Hecke-Clifford superalgebras.

Now the natural question arises: What do these superalgebras categorify? The purpose of this paper is to provide answers to this question. We will show that the quiver Hecke superalgebras and their cyclotomic quotients give a supercategorification of quantum Kac-Moody algebras and their integrable highest weight modules. (In [6], Ellis, Khovanov and Lauda dealt with the case when \( I = \{ i \} \).)

Recall that a supercategory is a category with an endofunctor \( \Pi \) and a natural isomorphism \( \xi : \Pi^2 \to \text{id} \) such that \( \xi \circ \Pi = \Pi \circ \xi \). Thus a supercategorification means a construction of a supercategory whose Grothendieck group possesses a given algebraic structure.

To explain our main results more precisely, we fix some notations and conventions. Let \( A = A_0 \oplus A_1 \) be a superalgebra and let \( \phi_A \) be the involution defined by \( \phi_A(a) = (-1)^\epsilon a \) for \( a \in A_\epsilon \) with \( \epsilon = 0, 1 \). We denote by \( \text{Mod}(A) \) the category of left \( A \)-modules, which becomes a supercategory with the functor \( \Pi \) induced by \( \phi_A \). On the other hand, we denote by \( \text{Mod}_{\text{super}}(A) \) the category of left \( A \)-supermodules \( M = M_0 \oplus M_1 \) with \( \mathbb{Z}_2 \)-degree preserving homomorphisms as morphisms. The category \( \text{Mod}_{\text{super}}(A) \) is endowed with a structure of supercategory given by the parity shift functor \( \Pi \).

For each \( n \geq 0 \), let \( R(n) \) be the quiver Hecke superalgebra over a base field \( k \) generated by \( e(\nu)(\nu \in I^n) \), \( x_k \) (\( 1 \leq k \leq n \)), \( \tau_l \) (\( 1 \leq l < n \)) with the defining relations given in Definition 4.1. Set \( R(\beta) = e(\beta)R(n) \), where \( \beta \in \mathbb{Q}^+ \), \( e(\beta) = \sum_{\nu \in I^\beta} e(\nu) \). For a dominant integral weight \( \Lambda \), let \( R^\Lambda(\beta) \) denote the cyclotomic quiver Hecke superalgebra at \( \beta \) (see Definition 7.1). Let \( \text{Mod}(R(\beta)) \) (resp. \( \text{Proj}(R(\beta)) \)) be the category of \( \mathbb{Z} \)-graded (resp. finitely generated projective) left \( R(\beta) \)-modules. We also denote by \( \text{Rep}(R(\beta)) \) the category of \( \mathbb{Z} \)-graded \( R(\beta) \)-modules that are finite-dimensional over \( k \). We define the categories \( \text{Mod}(R^\Lambda(\beta)) \), \( \text{Proj}(R^\Lambda(\beta)) \) and \( \text{Rep}(R^\Lambda(\beta)) \) in a similar manner.

On the other hand, let \( U_q(\mathfrak{g}) \) be the quantum Kac-Moody algebra associated with a symmetrizable Cartan datum and let \( U^-_q(\mathfrak{g}) \) be the \( A \)-form of the negative half of \( U_q(\mathfrak{g}) \) with \( A = \mathbb{Z}[q, q^{-1}] \). For a dominant integral weight \( \Lambda \), we denote by \( V(\Lambda) \) the integrable highest weight module generated by the highest weight vector \( v_\Lambda \) with weight \( \Lambda \). We also denote by \( V_\Lambda(\Lambda) \) the \( A \)-form \( U^-_q(\mathfrak{g})v_\Lambda \) of \( V(\Lambda) \) and by \( V_\Lambda(\Lambda)^\vee \) its dual \( A \)-form.
Our goal in this paper is to prove the following isomorphisms

\[(1.1) \quad V_{\Lambda}(\Lambda) \sim \rightarrow \text{Proj}(R^\Lambda), \quad V_{\Lambda}(\Lambda)^\vee \sim \rightarrow \text{Rep}(R^\Lambda),\]

\[(1.2) \quad U_{\Lambda}^{-}(g) \sim \rightarrow \text{Proj}(R), \quad U_{\Lambda}^{-}(g)^\vee \sim \rightarrow \text{Proj}(R),\]

where

\[
\text{Proj}(R^\Lambda) = \bigoplus_{\beta \in \mathbb{Q}^+} \text{Proj}(R^\Lambda(\beta)), \quad \text{Rep}(R^\Lambda) = \bigoplus_{\beta \in \mathbb{Q}^+} \text{Rep}(R^\Lambda(\beta)),
\]

\[
\text{Proj}(R) = \bigoplus_{\beta \in \mathbb{Q}^+} \text{Proj}(R(\beta)), \quad \text{Rep}(R) = \bigoplus_{\beta \in \mathbb{Q}^+} \text{Proj}(R(\beta)),
\]

and \([\bullet]\) denotes the Grothendieck group.

For this purpose, we define the \(i\)-restriction and the \(i\)-induction superfunctors \((i \in I)\)

\[
E_i^\Lambda: \text{Mod}(R^\Lambda(\beta + \alpha_i)) \rightarrow \text{Mod}(R^\Lambda(\beta)), \quad F_i^\Lambda: \text{Mod}(R^\Lambda(\beta)) \rightarrow \text{Mod}(R^\Lambda(\beta + \alpha_i))
\]

by

\[
E_i^\Lambda(N) = e(\beta, i)N = e(\beta, i)R^\Lambda(\beta + \alpha_i) \otimes_{R^\Lambda(\beta + \alpha_i)} N, \quad F_i^\Lambda(M) = R^\Lambda(\beta + \alpha_i)e(\beta, i) \otimes_{R^\Lambda(\beta)} M,
\]

where \(M \in \text{Mod}(R^\Lambda(\beta))\) and \(N \in \text{Mod}(R^\Lambda(\beta + \alpha_i))\).

The first main result of this paper shows that the functors \(E_i^\Lambda\) and \(F_i^\Lambda\) are exact and hence they induce well-defined operators on \([\text{Proj}(R^\Lambda)]\) and \([\text{Rep}(R^\Lambda)]\). To prove this, we take a detailed analysis of the structure of \(R(\beta)\) and \(R^\Lambda(\beta)\), and show that \(R^\Lambda(\beta + \alpha_i)e(\beta, i)\) is a projective right \(R^\Lambda(\beta)\)-supermodule (Theorem 8.7).

Next, we prove a super version of \(\mathfrak{sl}_2\)-categorification. That is, for \(\lambda = \Lambda - \beta\), we show that there exist natural isomorphisms given below (Theorem 9.6):

(i) if \(\langle h_i, \lambda \rangle \geq 0\), then we have

\[
\Pi_iq_i^{-2}F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i, \lambda \rangle - 1} \Pi_k^k q_i^{2k} \sim \rightarrow E_i^\Lambda F_i^\Lambda,
\]

(ii) if \(\langle h_i, \lambda \rangle < 0\), then we have

\[
\Pi_iq_i^{-2}F_i^\Lambda E_i^\Lambda \sim \rightarrow E_i^\Lambda F_i^\Lambda \oplus \bigoplus_{k=0}^{\langle h_i, \lambda \rangle - 1} \Pi_i^{k+1}q_i^{-2k-2},
\]

where \(\Pi_i = \text{id}\) if \(i\) is even and \(\Pi_i = \Pi\) if \(i\) is odd.

We then show that the endomorphisms induced on \([\text{Proj}(R^\Lambda)]\) and \([\text{Rep}(R^\Lambda)]\) by the parity shift functor \(\Pi\) coincide with the identity. Therefore the above isomorphisms
(i) and (ii) of functors show that $[\text{Proj}(R^\Lambda)]$ and $[\text{Rep}(R^\Lambda)]$ have a structure of $U_\Lambda(\mathfrak{g})$-module.

Next, we show that the simple objects of $\text{Rep}(R^\Lambda)$ form a basis of $[\text{Rep}(R^\Lambda)]$, which shares a particular property of \textit{strong perfect bases} introduced by Berenstein-Kazhdan ([2]). By the general theory of strong perfect bases, we conclude that $[\text{Rep}(R^\Lambda)]$ is isomorphic to $V_\Lambda(\Lambda)^\vee$. Then by duality, $[\text{Proj}(R^\Lambda)]$ is isomorphic to $V_\Lambda(\Lambda)$. Finally, by taking the projective limit with respect to $\Lambda$, we conclude that there exist natural isomorphisms in (1.2) (Theorem 10.2, Corollary 10.3).

In principle, this paper follows the outline of [10]. However, our argument is substantially different from the one in [10] in the following sense:

(i) Our supercategorification theorem would give various applications. For instance, we can generalize the Brundan-Kleshchev categorification theorem ([3]) in the level of crystal to the quantum level.

(ii) We make use of the strong perfect basis theory to characterize the $A$-forms of $V(\Lambda)$ and obtain a supercategorification of $V(\Lambda)$.

(iii) By taking a projective limit, we obtain a categorification of $U_q^{-}(\mathfrak{g})$, which is opposite to the usual approach.

(iv) Since we deal with skew polynomials rather than polynomials, the calculation involved are much more subtle and complicated.

Note that any simple object $M$ of $\text{Rep}(R^\Lambda)$ is self-associate, i.e., $\Pi M \simeq M$. Hence the parity functor $\Pi$ induces the identity on the Grothendieck group $[\text{Rep}(R^\Lambda)]$. On the other hand, any simple object of the category $\text{Rep}_{\text{super}}(R^\Lambda)$ of finite-dimensional $R^\Lambda$-supermodules is never self-associate. Hence, $\Pi$ induces a non-trivial action on the Grothendieck group $[\text{Rep}_{\text{super}}(R^\Lambda)]$, and

$$[\text{Rep}(R^\Lambda)] \simeq \frac{[\text{Rep}_{\text{super}}(R^\Lambda)]}{(\Pi - 1)[\text{Rep}_{\text{super}}(R^\Lambda)]}.$$  

The study of $\text{Rep}_{\text{super}}(R^\Lambda)$ will be carried out in a forthcoming paper. Note that in [9], Hill and Wang studied the categories of supermodules over quiver Hecke superalgebras under an additional condition on the Cartan datum.

This paper is organized as follows. In the following two sections, we recall some of the basic properties of quantum Kac-Moody algebras, integrable highest weight modules and supercategories. In Section 4, we take a detailed analysis of the structure of
quiver Hecke superalgebras. In Section 5, we prove the existence of natural isomorphisms and short exact sequences which are necessary in proving our main results. In Section 6, we show that the Grothendieck group $\text{Rep}(R)$ has a strong perfect basis. In Section 7 and 8, we prove that the superfunctors $E_i^\Lambda$ and $F_i^\Lambda$ are exact and they send projectives to projectives. In Section 9, we prove that $E_i^\Lambda$ and $F_i^\Lambda$ satisfy certain commutation relations, which is a super version of $\mathfrak{sl}_2$-categorification. In the final section, we conclude that the quiver Hecke superalgebras and their cyclotomic quotients provide supercategorification of quantum Kac-Moody algebras and their integrable highest weight modules.

2. Quantum Kac-Moody algebras

Let $I$ be an index set. An integral square matrix $A = (a_{ij})_{i,j \in I}$ is called a Cartan matrix if it satisfies

(i) $a_{ii} = 2,$  \hspace{.2cm} (ii) $a_{ij} \leq 0$ for $i \neq j,$  \hspace{.2cm} (iii) $a_{ij} = 0$ if $a_{ji} = 0.$

In this paper, we assume that $A$ is symmetric; i.e., there is a diagonal matrix $D = \text{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $DA$ is symmetric.

A Cartan datum $(A,P,\Pi,\Pi^\vee)$ consists of

1. a Cartan matrix $A,$
2. a free abelian group $P,$ called the weight lattice,
3. $\Pi = \{\alpha_i \in P \mid i \in I\},$ called the set of simple roots,
4. $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \text{Hom}(P,\mathbb{Z}),$ called the set of simple coroots,

satisfying the following conditions:

(a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I,$
(b) $\Pi$ is linearly independent.

The weight lattice $P$ has a symmetric bilinear pairing $(\mid \rangle$ satisfying

$$(\alpha_i \mid \lambda) = s_i \langle h_i, \lambda \rangle \text{ for all } \lambda \in P.$$ 

Therefore, we have $(\alpha_i | \alpha_j) = s_i a_{ij}.$ We set $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0}, \text{ for all } i \in I\},$ which is called the set of dominant integral weights. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ is called the root lattice. Set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i.$ For $\beta = \sum k_i \alpha_i \in Q^+,$ $|\beta| := \sum_{i \in I} k_i$ is called the height of $\beta.$
For an indeterminate $q$, set $q_i = q^{h_i}$ and define the $q$-integers

$$[n]_i = \frac{q^n_i - q^{-n}_i}{q_i - q^{-i}_i}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \left[ \begin{array}{c} m \\ n \end{array} \right]_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

**Definition 2.1.** The quantum Kac-Moody algebra $U_q(\mathfrak{g})$ associated with a Cartan datum $(\mathcal{A}, \mathcal{P}, \Pi, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with $1$ generated by $e_i, f_i$ $(i \in I)$ and $q^h$ $(h \in \mathcal{P}^\vee)$ subject to the following defining relations:

1. $q^0 = 1, \quad q^h q^{h'} = q^{h+h'}$ for $h, h' \in \mathcal{P}^\vee$,
2. $q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i$ for $h \in \mathcal{P}^\vee, \ i \in I$,
3. $e_i f_j - f_j e_i = \delta_{ij} K_i - K_i^{-1} / q_i - q_i^{-1}$ where $K_i = q_i^h$,
4. $\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(1-a_{ij}+r)} e_j^{(r)} = \sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(1-a_{ij}-r)} f_j^{(r)} = 0$ if $i \neq j$,

where $e_i^{(k)} := e_i^k / [k]_i!$ and $f_i^{(k)} := f_i^k / [k]_i!$ for $k \in \mathbb{Z}_{\geq 0}$.

Let $U_q^{-}(\mathfrak{g})$ (resp. $U_q^{+}(\mathfrak{g})$) be the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $f_i$’s (resp. $e_i$’s) and let $U_q(\mathfrak{g})^0$ be the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $q^h$ $(h \in \mathcal{P}^\vee)$. Then we have the triangular decomposition

$$U_q(\mathfrak{g}) \simeq U_q^{-}(\mathfrak{g}) \otimes U_q(\mathfrak{g})^0 \otimes U_q^{+}(\mathfrak{g}),$$

and the root space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in \mathcal{Q}} U_q(\mathfrak{g})_{\alpha} \quad \text{where} \quad U_q(\mathfrak{g})_{\alpha} := \{ x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{(h,\alpha)} x \text{ for any } h \in \mathcal{P}^\vee \}.$$

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and denote by $U_{\mathcal{A}}^{-}(\mathfrak{g})$ (resp. $U_{\mathcal{A}}^{+}(\mathfrak{g})$) the $\mathcal{A}$-subalgebra of $U_q^{-}(\mathfrak{g})$ generated by $f_i^{(n)}$ (resp. $e_i^{(n)}$) and denote by $U_{\mathcal{A}}^0(\mathfrak{g})$ the $\mathcal{A}$-subalgebra generated by $q^h$ and $\prod_{k=1}^n (1 - q^k q^{-h})$ for all $m \in \mathbb{Z}_{\geq 0}$ and $h \in \mathcal{P}^\vee$. Let $U_{\mathcal{A}}(\mathfrak{g})$ be the $\mathcal{A}$-subalgebra generated by $U_{\mathcal{A}}^0(\mathfrak{g})$, $U_{\mathcal{A}}^{-}(\mathfrak{g})$ and $U_{\mathcal{A}}^{+}(\mathfrak{g})$. Then $U_{\mathcal{A}}(\mathfrak{g})$ also has the triangular decomposition

$$U_{\mathcal{A}}(\mathfrak{g}) \simeq U_{\mathcal{A}}^{-}(\mathfrak{g}) \otimes U_{\mathcal{A}}^0(\mathfrak{g}) \otimes U_{\mathcal{A}}^{+}(\mathfrak{g}).$$

Fix $i \in I$. For any $P \in U_q^{-}(\mathfrak{g})$, there exist unique $Q, R \in U_q^{-}(\mathfrak{g})$ such that

$$e_i P - P e_i = \frac{K_i Q - K_i^{-1} R}{q_i - q_i^{-1}}.$$
We define the endomorphisms $e'_i, e''_i : U_q^-(g) \rightarrow U_q^-(g)$ by
\[ e'_i(P) = R, \quad e''_i(P) = Q. \]
Regarding $f_i$ as the endomorphism of $U_q^-(g)$ defined by the left multiplication by $f_i,$ we obtain the $q$-boson commutation relations:
\begin{equation}
(2.1) \quad e'_i f_j = q^{-a_{ij}} f_j e'_i + \delta_{i,j}.
\end{equation}

**Definition 2.2 ([16]).** The quantum boson algebra $B_q(g)$ associated with a Cartan matrix $A$ is the associative algebra over $\mathbb{Q}(q)$ generated by $e'_i, f_i$ ($i \in I$) satisfying the following defining relations:
\begin{enumerate}
  \item $e'_i f_j = q^{-a_{ij}} f_j e'_i + \delta_{i,j},$
  \item $\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1}{r} \right] e'_i e'^{1-a_{ij} - r} f_j e'^r = \sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1}{r} \right] f_i^{1-a_{ij} - r} f_j f_i^r = 0$ if $i \neq j.$
\end{enumerate}

**Lemma 2.3 ([16, Lemma 3.4.7, Corollary 3.4.9]).**
\begin{enumerate}
  \item If $x \in U_q^-(g)$ and $e'_i x = 0$ for all $i \in I$, then $x$ is a constant multiple of 1.
  \item $U_q^-(g)$ is a simple $B_q(g)$-module.
\end{enumerate}

The $\mathbb{Q}(q)$-vector space $U_q^-(g)$ has a unique non-degenerate symmetric bilinear form $(\ , \ )$ satisfying the following properties ([16, Proposition 3.4.4]):
\[ (1, 1) = 1, \quad (e'_i u, v) = (u, f_i v) \quad \text{for } i \in I \text{ and } u, v \in U_q^-(g). \]

The dual of $U^-_{\mathbb{A}}(g)$ is defined to be
\[ U^-_{\mathbb{A}}(g)^\vee := \{ v \in U_q^-(g) \mid (u, v) \in \mathbb{A} \text{ for all } u \in U^-_{\mathbb{A}}(g) \}. \]

**Definition 2.4.** We denote by $\mathcal{O}_{\text{int}}$ the abelian category consisting of $U_q(g)$-modules $V$ satisfying the following conditions:
\begin{enumerate}
  \item $V$ has a weight decomposition with finite-dimensional weight spaces; i.e.,
  \[ V := \bigoplus_{\mu \in \mathbb{P}} V_\mu \quad \text{with} \quad \dim V_\mu < \infty, \]
  where $V_\mu := \{ v \in V \mid q^h v = q^{(h, \mu)} v \text{ for all } h \in \mathbb{P}^\vee \},$
  \item there are finitely many $\lambda_1, \ldots, \lambda_s \in \mathbb{P}$ such that
  \[ \text{wt}(V) := \{ \mu \in \mathbb{P} \mid V_\mu \neq 0 \} \subset \bigcup_{i=1}^{s} (\lambda_i - \mathbb{Q}^+), \]
\end{enumerate}
(3) for any \( i \in I \), the action of \( f_i \) on \( V \) is locally nilpotent.

For each \( \Lambda \in P^+ \), let \( V(\Lambda) \) denote the irreducible highest weight \( U_q(\mathfrak{g}) \)-module with highest weight \( \Lambda \). It is generated by a unique highest weight vector \( v_\Lambda \) with the defining relations:

\[
q^h v_\Lambda = q^{\langle h, \Lambda \rangle} v_\Lambda, \quad e_i v_\Lambda = 0, \quad f_i^{(\langle h, \Lambda \rangle + 1)} v_\Lambda = 0 \quad \text{for all } h \in P^\vee \text{ and } i \in I.
\]

It is known that the category \( \mathcal{O}_{\text{int}} \) is semisimple and that all the irreducible objects are isomorphic to \( V(\Lambda) \) for some \( \Lambda \in P^+ \). (See, for example, [8, Theorem 3.5.4].)

There exists a unique non-degenerate symmetric bilinear form \((,\rangle\) on \( V(\Lambda) \) (\( \Lambda \in P^+ \)) satisfying

\[
(v_\Lambda, v_\Lambda) = 1, \quad (e_i u, v) = (u, f_i v) \quad \text{for any } u, v \in V(\Lambda) \text{ and } i \in I.
\]

For a \( U_q(\mathfrak{g}) \)-module \( M \), an \( \mathbb{A} \)-form of \( M \) is a \( U_{\mathbb{A}}(\mathfrak{g}) \)-submodule \( M_{\mathbb{A}} \) such that \( M = \mathbb{Q}(q) \otimes_{\mathbb{A}} M_{\mathbb{A}} \) and the weight space \( (M_{\mathbb{A}})_\lambda := M_{\mathbb{A}} \cap M_{\lambda} \) is finitely generated over \( \mathbb{A} \) for any \( \lambda \in P \). We define an \( \mathbb{A} \)-form \( V_{\mathbb{A}}(\Lambda) \) of \( V(\Lambda) \) by

\[
V_{\mathbb{A}}(\Lambda) = U_{\mathbb{A}}(\mathfrak{g}) v_{\Lambda}.
\]

The dual of \( V_{\mathbb{A}}(\Lambda) \) is defined to be

\[
V_{\mathbb{A}}(\Lambda)^\vee = \{ v \in V(\Lambda) \mid (u, v) \in \mathbb{A} \text{ for all } u \in V_{\mathbb{A}}(\Lambda) \}.
\]

Then \( V_{\mathbb{A}}(\Lambda)^\vee \) is also an \( \mathbb{A} \)-form of \( V(\Lambda) \).

Now, we will give the definition of perfect bases and strong perfect bases, and prove their basic properties. These properties will play a crucial role in proving our main result (Theorem 10.2).

Let \( V = \bigoplus_{\lambda \in P} V_\lambda \) be a \( P \)-graded \( \mathbb{Q}(q) \)-vector space. We assume that there are finitely many \( \lambda_1, \ldots, \lambda_s \in P \) such that

\[
\text{wt}(V) := \{ \mu \in P \mid V_\mu \neq 0 \} \subseteq \bigcup_{i=1}^s (\lambda_i - \mathbb{Q}^+).
\]

Furthermore, assume that \( e_i \) (\( i \in I \)) acts on \( V \) in such a way that \( e_i V_\lambda \subset V_{\lambda+\alpha_i} \). For any \( v \in V \) and \( i \in I \), we define

\[
\varepsilon_i(v) := \min\{ n \in \mathbb{Z}_{\geq 0} \mid e_i^{n+1} v = 0 \}.
\]
If \( v = 0 \), then we will use the convention \( \varepsilon_i(0) = -\infty \). One can easily check that, for \( k \in \mathbb{Z}_{\geq 0} \),

\[
V_{i}^{<k} := \{ v \in V \mid \varepsilon_i(v) < k \} = \text{Ker} e_i^k.
\]

**Definition 2.5 ([2, 13]).**

(i) A \( \mathbb{Q}(q) \)-basis \( B \) of \( V \) is called a **perfect basis** if

(a) \( B = \bigcup_{\mu \in \text{wt}(V)} B_\mu \), where \( B_\mu := B \cap V_\mu \),

(b) for any \( b \in B \) and \( i \in I \) with \( e_i(b) \neq 0 \), there exists a unique element \( \tilde{e}_i(b) \in B \) such that

\[
e_i b - c_i(b) \tilde{e}_i(b) \in V_i^{<\varepsilon_i(b)-1}
\]

for some \( c_i(b) \in \mathbb{Q}(q)^\times \),

(c) if \( b, b' \in B \) and \( i \in I \) satisfy \( \varepsilon_i(b) = \varepsilon_i(b') > 0 \) and \( \tilde{e}_i(b) = \tilde{e}_i(b') \), then \( b = b' \).

(ii) We say that a perfect basis is **strong** if \( c_i(b) = [\varepsilon_i(b)]_i \) for any \( b \in B \) and \( i \in I \); i.e.,

\[
e_i b - [\varepsilon_i(b)]_i \tilde{e}_i(b) \in V_i^{<\varepsilon_i(b)-1}.
\]

For a perfect basis \( B \), we set \( \tilde{e}_i(b) = 0 \) if \( e_i b = 0 \). We can easily see that for a perfect basis \( B \)

\[
V_i^{<k} = \bigoplus_{b \in B, \varepsilon_i(b) < k} \mathbb{Q}(q)b.
\]

**Example 2.6 ([16, 17]).** Recall that the \( U_q(\mathfrak{g}) \)-module \( V(\Lambda) \) has the **upper global basis**

\[
G^\vee(\Lambda) := \{ G^\vee(b) \mid b \in B(\Lambda) \},
\]

which is parametrized by the crystal basis \( B(\Lambda) \) of \( V(\Lambda) \).

Then \( G^\vee(\Lambda) \) is a strong perfect basis of \( V(\Lambda) \). Moreover, the upper global basis \( G^\vee(\Lambda) \) is an \( A \)-basis of \( V_A(\Lambda)^\vee \).

For any sequence \( i = (i_1, \ldots, i_m) \in I^m \ (m \geq 1) \), we inductively define a binary relation \( \preceq_i \) on \( V \setminus \{0\} \) as follows:

if \( i = (i) \), \( v \preceq_i v' \Leftrightarrow \varepsilon_i(v) \leq \varepsilon_i(v') \),

if \( i = (i; i') \), \( v \preceq_i v' \Leftrightarrow \begin{cases} 
\varepsilon_i(v) < \varepsilon_i(v') \\
\text{or} \\
\varepsilon_i(v) = \varepsilon_i(v'), \ e^\varepsilon_i(v)(v) \preceq_i e^{\varepsilon_i(v')}(v').
\end{cases}
\]

We write
• \( v \equiv_i v' \) if \( v \preceq_i v' \) and \( v' \preceq_i v \),
• \( v' \prec_i v \) if \( v' \preceq_i v \) and \( v \not\equiv_i v' \).

We can easily verify:

1. for all \( v \in V \setminus \{0\}, V^{\prec_i v} := \{0\} \cup \{v' \in V \setminus \{0\} \mid v' \prec_i v\} \) forms a \( \mathbb{Q}(q) \)-linear subspace of \( V \); indeed, we have

\[
V^{\prec_i v} = \left\{ u \in V \mid e_{i_m}^{\ell_m} \cdots e_{i_1}^{\ell_1} u = 0, \ e_{i_k}^{1+\ell_k} e_{i_{k-1}}^{\ell_{k-1}} \cdots e_{i_1}^{\ell_1} u = 0 \text{ for } 1 \leq k < m \right\},
\]
where \( v_0 = v, \ell_k = \varepsilon_{i_k}(v_{k-1}) \) and \( v_k = e_{i_k}^{\ell_k} v_{k-1} \) \( (1 \leq k \leq m) \).

2. if \( v \not\equiv_i v' \), then

\[
v + v' \equiv_i \begin{cases} v & \text{if } v' \prec_i v, \\ v' & \text{if } v \prec_i v'. \end{cases}
\]

For \( i \) and \( v \in V \setminus \{0\} \), we set

\[
e_i^{\top}(v) := e_i^{(\varepsilon_i(v))} v.
\]

We can easily see the following:

\[\text{(2.4)}\]

If \( B \) is strong perfect, then \( e_i^{\top}(b) = \tilde{e}_i^{\varepsilon_i(b)} b \) for any \( b \in B \).

For each \( i = (i_1, \ldots, i_m) \in I^m \), define a map \( e_i^{\top} : V \setminus \{0\} \to V \setminus \{0\} \) by

\[
e_i^{\top} := e_{i_m}^{\top} \circ \cdots \circ e_{i_1}^{\top}.
\]

Then \( e_i^{\top} B \subset \mathbb{Q}(q)^* B \). If \( B \) is strong perfect, then we have \( e_i^{\top} B \subset B \).

Let \( V^H := \{ v \in V \mid e_i v = 0 \text{ for all } i \in I \} \) be the space of highest weight vectors in \( V \) and let \( B^H = V^H \cap B \) be the set of highest weight vectors in \( B \). Then, (2.3) implies that

\[
V^H = \bigoplus_{b \in B^H} \mathbb{Q}(q)b.
\]

In [2], Berenstein and Kazhdan proved the following version of uniqueness theorem for perfect bases.

**Theorem 2.7 ([2]).** Let \( B \) and \( B' \) be perfect bases of \( V \) such that \( B^H = (B')^H \). Then there exist a map \( \psi : B \to B' \) and a map \( \xi : B \to \mathbb{Q}(q)^* \) such that \( \psi(b) - \xi(b)b \in V^{\prec_i b} \) holds for any \( b \in B \) and any \( i = (i_1, \ldots, i_m) \) satisfying \( e_i^{\top}(b) \in V^H \). Moreover, such \( \psi \) and \( \xi \) are unique and \( \psi \) commutes with \( \tilde{e}_i \) and \( \varepsilon_i \) \( (i \in I) \).
From the axioms of strong perfect basis and Theorem 2.7, one can easily prove the following corollary.

**Corollary 2.8.** Let $B$ and $B'$ be strong perfect bases of $V$ such that $B^H = (B')^H$. Then the map $\xi : B \to \mathbb{Q}(q)^\times$ given in Theorem 2.7 is a trivial map; i.e.,

$$\psi(b) - b \in V^{\sim b} \quad \text{for all } b \in B.$$

**Lemma 2.9.** Let $B$ be a strong perfect basis of $V$.

(i) For any finite subset $S$ of $B$, there exists a finite sequence $i = (i_1, \ldots, i_m)$ of $I$ such that $e_1^{\top}(b) \in B^H$ for any $b \in S$.

(ii) Let $b_0 \in B^H$ and let $i = (i_1, \ldots, i_m)$ be a finite sequence in $I$. Then $S := \{b \in B \mid e_1^{\top}(b) = b_0\}$ is linearly ordered by $\preceq_i$.

**Proof.** (i) is almost evident. In order to see (ii), it is enough to show that if $b, b' \in S$ satisfy $b \equiv_i b'$, then $b = b'$. If we set $v_0 = b$, $\ell_k = \varepsilon_{i_k}(v_{k-1})$ and $v_k = e_{i_k}^{(\ell_k)}v_{k-1}$ ($1 \leq k \leq m$), then $v_m = b_0$. Similarly, if we set $v'_0 = b'$, $\ell'_k = \varepsilon_{i_k}(v'_{k-1})$ and $v'_k = e_{i_k}^{(\ell'_k)}v'_{k-1}$ ($1 \leq k \leq m$), then $\ell'_k = \ell_k$ and $v'_m = b_0$. Thus we have $v_k = e_{i_k}^{\ell_k}v_{k-1}$ and $v'_k = e_{i_k}^{\ell'_k}v'_{k-1}$. Hence Definition 2.5 (i) (c) shows that $v'_k = v_k$ for all $k$. □

The following proposition gives a characterization of $V_h(\Lambda)^\vee$ by using strong perfect bases.

**Proposition 2.10.** Let $M$ be a $U_q(g)$-module in $\mathcal{O}_{\text{int}}$ such that $\text{wt}(M) \subset \Lambda - \mathbb{Q}^+$. Let $M_h$ be an $A$-submodule of $M$. Assume that $e_i^{(a)}M_h \subset M_h$, $(M_h)_\Lambda = Av_\Lambda$ and $M$ has a strong perfect basis $B \subset M_h$ such that $B^H = \{v_\Lambda\}$. Then we have

(a) $M_h \cong V_h(\Lambda)^\vee$,

(b) $B$ is an $A$-basis of $M_h$.

**Proof.** Since $M$ has only one highest weight vector $v_\Lambda$, $M$ is isomorphic to $V(\Lambda)$.

Since $(M_h)_\Lambda = Av_\Lambda$ and

$$V_h(\Lambda)^\vee = \left\{ u \in V(\Lambda)_\Lambda \mid e_{i_1}^{(a_1)} \cdots e_{i_\ell}^{(a_\ell)}u \in Av_\Lambda \text{ for all } (i_1, \ldots, i_\ell) \right\},$$

it is obvious that $M_h \subset V_h(\Lambda)^\vee$.

Conversely, for $u \in V_h(\Lambda)^\vee$, write $u = \sum_{b \in B} c_b b$ with $c_b \in \mathbb{Q}(q)$. Let us show $c_b \in A$ for any $\lambda \in P$ and $b \in B_\Lambda$. We take $i = (i_1, \ldots, i_m)$ such that $e_1^{\top}b = v_\Lambda$ for any $b \in B_\Lambda$. We shall show $c_b \in A$ by the descending induction with respect to the linear order $\preceq_i$. 


By the induction hypothesis, we may assume that $c_{b'} \in A$ for any $b' \in B$ such that $b \prec_i b'$.

Then setting $v_0 = b$, $\ell_k = e_{ii}(v_{k-1})$ and $v_k = e_{ik}^{(\ell_k)} v_{k-1}$ $(1 \leq k \leq m)$, we have $e_{im}^{(\ell_m)} \cdots e_{i1}^{(\ell_1)} u = c_b v_{\Lambda} + \sum_{b \prec_i b'} c_{b'} e_{im}^{(\ell_m)} \cdots e_{i1}^{(\ell_1)} b' \in V_{\Lambda}(\Lambda)'$. Hence we obtain $c_b \in A$. □

3. Supercategories and superbimodules

In this section, we briefly review the notion of supercategory, superfunctor, superbimodule and their basic properties. (See [12, Section 2] for more details.)

3.1. Supercategories.

Definition 3.1.

(i) A supercategory is a category $\mathcal{C}$ equipped with an endofunctor $\Pi_{\mathcal{C}}$ of $\mathcal{C}$ and an isomorphism $\xi_{\mathcal{C}}: \Pi_{\mathcal{C}}^2 \sim \text{id}_{\mathcal{C}}$ such that $\xi_{\mathcal{C}} \circ \Pi_{\mathcal{C}} = \Pi_{\mathcal{C}} \circ \xi_{\mathcal{C}} \in \text{Hom}(\Pi_{\mathcal{C}}^2, \Pi_{\mathcal{C}})$.

(ii) For a pair of supercategories $(\mathcal{C}, \Pi, \xi)$ and $(\mathcal{C}', \Pi', \xi')$, a superfunctor from $(\mathcal{C}, \Pi, \xi)$ to $(\mathcal{C}', \Pi', \xi')$ is a pair $(F, \alpha_F)$ of a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ and an isomorphism $\alpha_F: F \circ \Pi \sim \Pi' \circ F$ such that the diagram

$$
\begin{array}{ccc}
F \circ \Pi^2 & \overset{\alpha_F \circ \Pi}{\longrightarrow} & F \circ \Pi \\
| & & | \\
F & \overset{F \circ \xi}{\longrightarrow} & F \\
\downarrow \text{id}_F & & \downarrow \xi' \circ F \\
F & \overset{\xi' \circ F}{\longrightarrow} & F
\end{array}
$$

commutes. If $F$ is an equivalence of categories, we say that $(F, \alpha_F)$ is an equivalence of supercategories.

(iii) Let $(F, \alpha_F)$ and $(F', \alpha_{F'})$ be superfunctors from a supercategory $(\mathcal{C}, \Pi, \xi)$ to $(\mathcal{C}', \Pi', \xi')$. A morphism from $(F, \alpha_F)$ to $(F', \alpha_{F'})$ is a morphism of functors $\varphi: F \rightarrow F'$ such that

$$
\begin{array}{ccc}
F \circ \Pi & \overset{\varphi \circ \Pi}{\longrightarrow} & F' \circ \Pi \\
\alpha_F & \downarrow \text{id}_F & \downarrow \alpha_{F'} \\
\Pi' \circ F & \overset{\Pi' \circ \varphi}{\longrightarrow} & \Pi' \circ F'
\end{array}
$$

commutes.

In this paper, a supercategory is assumed to be an additive category.
For superfunctors $F: C \to C'$ and $F': C' \to C''$, the composition $F' \circ F: C \to C''$ becomes a superfunctor by taking the composition
\[F' \circ F \circ \Pi_{\varepsilon} \xrightarrow{F' \circ \alpha_{\varepsilon}} F' \circ \Pi_{\varepsilon'} \circ F \xrightarrow{\alpha_{\varepsilon'} \circ F} F' \circ F \circ \Pi_{\varepsilon'} \circ F' \circ F\]
as $\alpha_{F' \circ F}$.

The functors $id_{\varepsilon}$ and $\Pi$ are superfunctors by taking $\alpha_{id_{\varepsilon}} = id_{\Pi}: id_{\varepsilon} \circ \Pi \to \Pi \circ id_{\varepsilon}$ and $\alpha_{\Pi} = -id_{\Pi'}: \Pi \circ \Pi \to \Pi \circ \Pi$. Note the sign. The morphism $\alpha_{F}: F \circ \Pi \to \Pi' \circ F$ is a morphism of superfunctors.

### 3.2. Superalgebras

A superalgebra is a $\mathbb{Z}_2$-graded algebra. Let $A = A_0 \oplus A_1$ be a superalgebra. We denote by $\phi_A$ the involution of $A$ given by $\phi_A(a) = (-1)^{p(a)}a$ for $a \in A_\epsilon$ with $\epsilon = 0, 1$. We call $\phi_A$ the parity involution.

The category of $A$-modules $\text{Mod}(A)$ is naturally endowed with a structure of supercategory. The functor $\Pi$ is induced by the parity involution $\phi_A$. Namely, for $M \in \text{Mod}(A)$, $\Pi M := \{\pi(x) \mid x \in M\}$ with $\pi(x) + \pi(x') = \pi(x + x')$ and $a\pi(x) = \pi(\phi_A(a)x)$ for $a \in A$. The morphism $\xi: \Pi^2 \to \text{id}$ is given by $\pi(\pi(x)) \mapsto x$.

An $A$-supermodule is an $A$-module with a decomposition $M = M_0 \oplus M_1$ such that $A_\epsilon M_{\epsilon'} \subset M_{\epsilon+\epsilon'}$ ($\epsilon, \epsilon' \in \mathbb{Z}_2$).

Let $\text{Mod}_{\text{super}}(A)$ be the category of $A$-supermodules. The morphisms in this category are $A$-linear homomorphisms which preserve the $\mathbb{Z}_2$-grading. Then $\text{Mod}_{\text{super}}(A)$ is also endowed with a structure of supercategory. The functor $\Pi$ is given by the parity change: namely, $(\Pi M)_\epsilon := \{\pi(x) \mid x \in M_{\epsilon-1}\}$ ($\epsilon = 0, 1$) and $a\pi(x) = \pi(\phi_A(a)x)$ for $a \in A$ and $x \in M$. The isomorphism $\xi_M: \Pi^2 M \to M$ is given by $\pi(\pi(x)) \mapsto x$ ($x \in M$). Then there is a canonical superfunctor $\text{Mod}_{\text{super}}(A) \to \text{Mod}(A)$. For an $A$-supermodule $M$, the parity involution $\phi_M: M \to M$ of $M$ is defined by $\phi_M|_{M_\epsilon} = (-1)^\epsilon \text{id}_{M_\epsilon}$. Then we have $\phi_M(ax) = \phi_A(a)\phi_M(x)$ for any $a \in A$ and $x \in M$.

Let $A$ and $B$ be superalgebras. We define the multiplication on the tensor product $A \otimes B$ by
\[(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\varepsilon_1\varepsilon_2}(a_1a_2) \otimes (b_1b_2)\]
for $a_i \in A_{\varepsilon_i}$, $b_i \in B_{\varepsilon_i}$ ($\varepsilon_i, \varepsilon'_i = 0, 1$). Then $A \otimes B$ is again a superalgebra. Note that we have $A \otimes B \cong B \otimes A$ as a superalgebra by the supertwist map
\[A \otimes B \cong B \otimes A, \quad a \otimes b \mapsto (-1)^{\varepsilon_1\varepsilon_2}b \otimes a \quad (a \in A_{\varepsilon_1}, b \in B_{\varepsilon_2}).\]
If $M$ and $N$ are an $A$-module and an $B$-module respectively, then $M \otimes N$ has a structure of $A \otimes B$-module by

$$(a \otimes b)(u \otimes v) = (-1)^{\varepsilon_0} (au) \otimes (bv)$$

for $a \in A$, $b \in B_{\varepsilon}$, $u \in M_{\varepsilon'}$, $v \in N$ ($\varepsilon, \varepsilon' = 0, 1$).

### 3.3. Superbimodules.

Let $A$ and $B$ be superalgebras. An $(A,B)$-superbimodule is an $(A,B)$-bimodule with a $\mathbb{Z}_2$-grading compatible with the left action of $A$ and the right action of $B$. For an $(A,B)$-superbimodule $L$, let $F_L: \text{Mod}(B) \rightarrow \text{Mod}(A)$ be the functor $N \mapsto L \otimes B N$. Then $F_L$ is a superfunctor, where

$$\alpha_{F_L}: F_L \Pi N = L \otimes_B \Pi N \rightarrow \Pi F_L N = \Pi (L \otimes_B N)$$

is given by $s \otimes \pi(x) \mapsto \pi(\phi_L(s) \otimes x)$ ($s \in L$, $x \in N$).

For an $(A,B)$-superbimodule $L$, its parity twist $\Pi L$ is $\Pi L$ as a left $A$-module and its right $B$-module structure is given by $a \pi(s)b = \pi(\phi_A(a)sb)$ ($s \in L$, $a \in A$, $b \in B$). Then there exists a canonical isomorphism of superfunctors $\eta: F_{\Pi L} \sim \Pi \circ F_L$. The isomorphism $\eta_N: (\Pi L) \otimes_B N \sim \Pi (L \otimes_B N)$ is given by $\pi(s) \otimes x \mapsto \pi(s \otimes x)$. It is an isomorphism of superfunctors since one can easily check the commutativity of the following diagram:

$$\begin{array}{ccc}
F_{\Pi L} \circ \Pi & \xrightarrow{\eta \circ \Pi} & \Pi \circ F_L \circ \Pi \\
\alpha_{F_{\Pi L}} & \downarrow & \alpha_{(\Pi \circ F_L)} \\
\Pi \circ F_{\Pi L} & \xrightarrow{\Pi \circ \eta} & \Pi \circ \Pi \circ F_L \\
\end{array}$$

by using $\phi_{\Pi L}(\pi(s)) = -\pi(\phi_L(s))$.

### 4. The quiver Hecke superalgebras

In this section we recall the construction of quiver Hecke superalgebras and investigate its basic properties. We take as a base ring a graded commutative ring $k = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} k_n$. We assume the following condition:

$$2 \text{ is invertible in } k_0.$$
4.1. Definition of quiver Hecke superalgebras. We assume that a decomposition
\( I = I_{\text{even}} \sqcup I_{\text{odd}} \) is given. We say that a Cartan matrix \( A = (a_{ij})_{i,j \in I} \) is colored by \( I_{\text{odd}} \) if
\[ a_{ij} \in 2\mathbb{Z} \quad \text{for all } i \in I_{\text{odd}} \text{ and } j \in I. \]
From now on, we assume that \( A \) is colored by \( I_{\text{odd}} \).

We define the parity function \( p: I \rightarrow \{0, 1\} \) by
\[ p(i) = 1 \quad \text{if } i \in I_{\text{odd}} \quad \text{and} \quad p(i) = 0 \quad \text{if } i \in I_{\text{even}}. \]
Then we naturally extend the parity function on \( I^n \) and \( \mathbb{Q}^+ \) as follows:
\[ p(\nu) := \sum_{k=1}^{n} p(\nu_k) \quad \text{for all } \nu \in I^n, \]
\[ p(\beta) := \sum_{k=1}^{r} p(i_k) \quad \text{for } \beta = \sum_{k=1}^{r} \alpha_{i_k} \in \mathbb{Q}^+. \]

For \( i \neq j \in I \) and \( r, s \in \mathbb{Z}_{\geq 0} \), we take \( t_{i,j:(r,s)} \in k^{-2(\alpha_i|\alpha_j) - r(\alpha_i|\alpha_i) - s(\alpha_j|\alpha_j)} \) such that
\[ t_{i,j:(-a_{ij},0)} \in k^s, \quad t_{i,j:(r,s)} = t_{j,i:(s,r)}, \]
\[ t_{i,j:(r,s)} = 0 \quad \text{if } i \in I_{\text{odd}} \text{ and } r \text{ is odd}. \]

Let \( P_{ij} := k\langle w, z \rangle / (zw - (-1)^{p(i)p(j)}wz) \) be the \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded \( k \)-algebra where \( w \) and \( z \) have the \( \mathbb{Z} \times \mathbb{Z}_2 \)-degree \( ((\alpha_i|\alpha_i), p(i)) \) and \( ((\alpha_j|\alpha_j), p(j)) \), respectively. Let \( Q_{i,j} \) be an element of \( P_{ij} \) which is of the form
\[
Q_{i,j}(w, z) = \begin{cases} 
\sum_{r,s \in \mathbb{Z}_{\geq 0}} t_{i,j:(r,s)} w^r z^s & \text{if } i \neq j, \\
0 & \text{if } i = j.
\end{cases}
\]

Then \( Q_{i,j}(w, z) \) is an even element and \( Q = (Q_{i,j})_{i,j \in I} \) satisfies
\[ Q_{i,j}(w, z) = Q_{j,i}(z, w) \quad \text{for } i, j \in I, \]
\[ Q_{i,j}(w, z) = Q_{i,j}(-w, z) \quad \text{for } i \in I_{\text{odd}} \text{ and } j \in I. \]

**Definition 4.1** ([12]). The quiver Hecke superalgebra \( R(n) \) of degree \( n \) associated with the Cartan datum \((A, P, \Pi, \Pi')\) and \( (Q_{i,j})_{i,j \in I} \) is the superalgebra over \( k \) generated by \( e(\nu) \ (\nu \in I^n) \), \( x_k \ (1 \leq k \leq n) \), \( \tau_a \ (1 \leq a \leq n - 1) \) with the parity
\[ p(e(\nu)) = 0, \quad p(x_k e(\nu)) = p(\nu_k), \quad p(\tau_a e(\nu)) = p(\nu_a)p(\nu_{a+1}) \]
subject to the following defining relations:
(R1) $e(\mu)e(\nu) = \delta_{\mu,\nu}e(\nu)$ for all $\mu, \nu \in I^n$, and $1 = \sum_{\nu \in I^n} e(\nu)$, 
(R2) $x_{p}x_{q}e(\nu) = (-1)^{p(\nu)p(\nu_q)}x_{q}x_{p}e(\nu)$ if $p \neq q$, 
(R3) $x_{p}e(\nu) = e(\nu)x_{p}$ and $\tau_{a}e(\nu) = e(s_{a}\nu)\tau_{a}$, where $s_{a} = (a, a+1)$ is the transposition on the set of sequences, 
(R4) $\tau_{a}x_{p}e(\nu) = (-1)^{p(\nu)p(\nu_{a+1})}x_{p}\tau_{a}e(\nu)$, if $p \neq a, a+1$, 
(R5) 
\[ (\tau_{a}x_{a+1} - (-1)^{p(\nu_{a})p(\nu_{a+1})}x_{a}\tau_{a})e(\nu) = (x_{a+1}\tau_{a} - (-1)^{p(\nu_{a})p(\nu_{a+1})}\tau_{a}x_{a})e(\nu) = \delta_{\nu_{a},\nu_{a+1}}e(\nu), \]
(R6) $\tau_{a}^{2}e(\nu) = Q_{\nu_{a},\nu_{a+1}}(x_{a}, x_{a+1})e(\nu)$, 
(R7) $\tau_{a}\tau_{b}e(\nu) = (-1)^{p(\nu_{a})p(\nu_{a+1})p(\nu_{b})p(\nu_{b+1})}\tau_{b}\tau_{a}e(\nu)$ if $|a - b| > 1$, 
(R8) 
\[ (\tau_{a+1}\tau_{a+1} - \tau_{a}\tau_{a+1})e(\nu) = \begin{cases} 
\dfrac{Q_{\nu_{a},\nu_{a+1}}(x_{a+2}, x_{a+2}) - Q_{\nu_{a},\nu_{a+1}}(x_{a}, x_{a+1})}{x_{a+2}^2 - x_{a}^2}e(\nu) & \text{if } \nu_{a} = \nu_{a+2} \in I_{even}, \\
(-1)^{p(\nu_{a+1})(x_{a+2} - x_{a})}\dfrac{Q_{\nu_{a},\nu_{a+1}}(x_{a+2}, x_{a+1}) - Q_{\nu_{a},\nu_{a+1}}(x_{a}, x_{a+1})}{x_{a+2}^2 - x_{a}^2}e(\nu) & \text{if } \nu_{a} = \nu_{a+2} \in I_{odd}, \\
0 & \text{otherwise}. 
\end{cases} \]

The $\mathbb{Z}$-grading on $R(n)$ is given by:
\[ \deg_{\mathbb{Z}}(e(\nu)) = 0, \quad \deg_{\mathbb{Z}}(x_{k}e(\nu)) = (\alpha_{\nu_{k}}|\alpha_{\nu_{k}}), \quad \deg_{\mathbb{Z}}(\tau_{a}e(\nu)) = -(\alpha_{\nu_{a}}|\alpha_{\nu_{a+1}}). \]

We understand $R(0) \cong k$, and $R(1)$ is isomorphic to $k^[x_{1}]$ where $k^[x_{1}] = \bigoplus_{i \in I} k e(i)$ is the direct sum of the copies $ke(i)$ of the algebra $k$.

For $\nu = (\nu_{1}, \ldots, \nu_{n}) \in I^{n}$ and $1 \leq m \leq n$, we set
\[ \nu_{<m} := (\nu_{1}, \ldots, \nu_{m-1}) \quad \text{and} \quad \nu_{>m} := (\nu_{m+1}, \ldots, \nu_{n}). \]

For $a, b \in \{1, \ldots, n\}$ with $a \neq b$, we define the elements of $R(n)$ by
\[ e_{a,b}^{ev} = \sum_{\nu_{a} = \nu_{b} \in I_{even}}\nu \in I^{n}, e(\nu), \quad e_{a,b}^{od} = \sum_{\nu_{a} = \nu_{b} \in I_{odd}}\nu \in I^{n}, e(\nu) \quad \text{and} \quad e_{a,b} = e_{a,b}^{ev} + e_{a,b}^{od}. \]

For any $\nu \in I^{n}$ ($n \geq 2$), let
\[ \mathcal{P}_{\nu} := k(x_{1}, \ldots, x_{n})/(x_{a}x_{b} - (-1)^{p(\nu_{a})p(\nu_{b})}x_{b}x_{a})_{1 \leq a < b \leq n} \]
be the superalgebra generated by $x_k$ ($1 \leq k \leq n$) with $\mathbb{Z} \times \mathbb{Z}_2$-degree $((\alpha_{\nu k} | \alpha_{\nu k}), p(\nu_k))$ ($k = 1, \ldots, n$). Let $\mathcal{P}_p^{ev}$ be the subalgebra of $\mathcal{P}_\nu$ generated by $x_k^{1+p(\nu_k)}$ ($1 \leq k \leq n$). Then $\mathcal{P}_p^{ev}$ is isomorphic to the polynomial ring $k[x_1^{1+p(\nu_1)}, \ldots, x_n^{1+p(\nu_n)}]$. Set

$$\mathcal{P}_n = \bigoplus_{\nu \in \mathbb{I}^n} \mathcal{P}_\nu e(\nu) \quad \text{and} \quad \mathcal{P}_n^{ev} = \bigoplus_{\nu \in \mathbb{I}^n} \mathcal{P}_\nu^{ev} e(\nu).$$

Then $\mathcal{P}_n^{ev}$ is contained in the center of $\mathcal{P}_n$ and

$$(4.8) \quad Q_{\nu a, \nu a+1}(x_a, x_{a+1}) e(\nu) \text{ belongs to } \mathcal{P}_n^{ev} \text{ for all } \nu \in \mathbb{I}^n \text{ and } 1 \leq a < n.$$  

For $1 \leq k < n$, we define the algebra endomorphism $\overline{s}_k$ of $\mathcal{P}_n$ by

$$(4.9) \quad \overline{s}_k(x_p e(\nu)) = (-1)^{p(\nu_k)p(n_\nu)} x_{s_k(p)} e((\nu_k)) \quad \text{for } 1 \leq p \leq n,$$

where $s_k = (k, k+1) \in S_n$ is the transposition which acts on $\mathbb{I}^n$ in a natural way. For $f \in \mathcal{P}_n$ and $1 \leq k < n$, define

$$\partial_k f = \frac{f - \overline{s}_k f}{x_{k+1} - x_k} e_{k,k+1}^{ev} + \frac{(x_{k+1} - x_k)f - \overline{s}_k f(x_{k+1} - x_k)}{x_{k+1}^2 - x_k^2} e_{k,k+1}^{od},$$

$$f^{\partial_k} = \frac{f - \overline{s}_k f}{x_{k+1} - x_k} e_{k,k+1}^{ev} + \frac{f(x_{k+1} - x_k) - (x_{k+1} - x_k)(\overline{s}_k f)}{x_{k+1}^2 - x_k^2} e_{k,k+1}^{od}.$$  

Then one can easily show that

$$(4.11) \quad \partial_k f, \quad f^{\partial_k} \in \mathcal{P}_n, \quad \tau_k f = (\overline{s}_k f)\tau_k + \partial_k f, \quad f\tau_k = \tau_k(\overline{s}_k f) + f^{\partial_k}$$

and

$$\partial_k(x_j) = (x_j)^{\partial_k} = \delta_{j,k+1} e_{k,k+1}^{ev} + \delta_{j,k} e_{k,k+1}^{od} + \delta_{j,k+1} e_{k,k+1}^{ev} + \delta_{j,k+1} e_{k,k+1}^{od},$$

$$\partial_k(fg) = (\partial_k f)g + (\overline{s}_k f)\partial_k g, \quad (fg)^{\partial_k} = f^{\partial_k} + (f^{\partial_k})\overline{s}_k g.$$  

For $\beta \in \mathbb{Q}^+$ with $|\beta| = n$, set

$$I^\beta = \{ \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{I}^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}.$$  

We define

$$R(m, n) = R(m) \otimes_k R(n) \subset R(m + n),$$

$$e(n) = \sum_{\nu \in \mathbb{I}^n} e(\nu), \quad e(\beta) = \sum_{\nu \in I^\beta} e(\nu), \quad e(\alpha, \beta) = \sum_{\mu \in \mathbb{I}^n, \nu \in I^\beta} e(\mu, \nu),$$

$$R(\beta) = e(\beta)R(n), \quad R(\alpha, \beta) = R(\alpha) \otimes_k R(\beta) \subset R(\alpha + \beta).$$
\[
e(n, i^k) = \sum_{\nu \in I^{n+k}, \nu_{n+1}=\cdots=\nu_{n+k}=i} e(\nu), \quad e(i^k, n) = \sum_{\nu \in I^{n+k}, \nu_1=\cdots=\nu_k=i} e(\nu),
\]
\[
e(\beta, i^k) = e(\beta, k\alpha_i) = e(\beta + k\alpha_i) e(n, i),
\]
\[
e(i^k, \beta) = e(k\alpha_i, \beta) = e(\beta + k\alpha_i) e(i^k, n)
\]
for \(\alpha, \beta \in \mathbb{Q}^+\).

**Proposition 4.2** ([12, Corollary 3.15]). For each \(w \in S_n\), we choose a reduced expression \(s_{i_1} \cdots s_{i_\ell}\) of \(w\) and write \(\tau_w = \tau_{i_1} \cdots \tau_{i_\ell}\). Then

\[
\{ x_{a_1} \cdots x_{a_n} \tau_w e(\nu) \mid a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n, w \in S_n, \nu \in I^n \}
\]
forms a basis of the \(k\)-module \(R(n)\).

**Remark 4.3.** In general, \(\tau_w\) depends on the choice of reduced expressions of \(w\). However, we still write \(\tau_w\) after choosing a reduced expression of \(w\). In \(I = \{i\}\) case, by the axioms in Definition 4.1, \(\pm \tau_w\) does not depend on the choice of reduced expressions of \(w \in S_n\); i.e., for any two reduced expressions \(w = s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}\), we have

\[
\tau_{i_1} \cdots \tau_{i_r} = \pm \tau_{j_1} \cdots \tau_{j_r}.
\]

By the proposition above we have:

**Lemma 4.4.** The algebra \(R(n+1)\) has a direct sum decomposition

\[
R(n+1) = \bigoplus_{a=1}^{n+1} R(n, 1) \tau_n \cdots \tau_a = \bigoplus_{a=1}^{n+1} R(n) \otimes k^I [x_{n+1}] \tau_n \cdots \tau_a.
\]

In particular, \(R(n+1)\) is a free \(R(n, 1)\)-module of rank \(n+1\).

Let \(\text{Mod}(R(\beta))\) (resp. \(\text{Proj}(R(\beta))\)) be the category of arbitrary (resp. finitely generated projective) \(\mathbb{Z}\)-graded \(R(\beta)\)-modules. We denote by \(\text{Rep}(R(\beta))\) the category of \(\mathbb{Z}\)-graded left \(R(\beta)\)-modules which are coherent over \(k_0\). Note that we assume modules to be only \(\mathbb{Z}\)-graded but not \(\mathbb{Z} \times \mathbb{Z}_2\)-graded. The morphisms in these categories are \(R(\beta)\)-linear homomorphisms which preserve \(\mathbb{Z}\)-grading. Then \(\text{Mod}(R(\beta))\) is an abelian category, and \(\text{Proj}(R(\beta)), \text{Rep}(R(\beta))\) are subcategories of \(\text{Mod}(R(\beta))\) stable under extensions. Hence they are exact categories.

Since \(R(\beta)\) is a superalgebra, the category \(\text{Mod}(R(\beta))\) has a supercategory structure induced by the parity involution \(\phi := \phi_{R(\beta)}\) as seen in §3.2.
For a \(Z\)-graded \(R(\beta)\)-module \(M = \bigoplus_{i \in Z} M_i\), let \(M(k)\) denote the \(Z\)-graded \(R(\beta)\)-module obtained from \(M\) by shifting the grading by \(k\); i.e., \(M(k)_i := \bigoplus_{i \in Z} M_{i+k}\). We also denote by \(q\) the grading shift functor

\[
(qM)_i = M_{i-1}.
\]

Let \(\pi\) be an odd element with the defining equation \(\pi^2 = 1\). For any superring \(R\), we define

\[
R^\pi := R \otimes Z[\pi] \simeq R \oplus R\pi.
\]

Thus \(A^\pi = \mathbb{Z}[q, q^{-1}, \pi]\) with \(\pi^2 = 1\). We denote by \([\text{Proj}(R(\beta))]\) and \([\text{Rep}(R(\beta))]\) the Grothendieck group of \(\text{Proj}(R(\beta))\) and \(\text{Rep}(R(\beta))\), respectively. Then \([\text{Proj}(R(\beta))]\) and \([\text{Rep}(R(\beta))]\) have the \(A^\pi\)-module structure given by \(q[M] = [qM]\) and \(\pi[M] = [\Pi M]\), where \([M]\) is the isomorphism classes of an \(R(\beta)\)-module \(M\).

For \(\alpha, \beta \in \mathbb{Q}^+\), consider the natural embedding

\[
e_{\alpha, \beta}: R(\alpha) \otimes R(\beta) \hookrightarrow R(\alpha + \beta),
\]

which maps \(e(\alpha) \otimes e(\beta)\) to \(e(\alpha, \beta)\). For \(M \in \text{Mod}(R(\alpha, \beta))\) and \(N \in \text{Mod}(R(\alpha + \beta))\), we define

\[
\text{Ind}_{\alpha, \beta} M = R(\alpha + \beta)e(\alpha, \beta) \otimes_{R(\alpha, \beta)} M \in \text{Mod}(R(\alpha + \beta)),
\]

\[
\text{Res}_{\alpha, \beta} N = e(\alpha, \beta)N \in \text{Mod}(R(\alpha, \beta)).
\]

Then the Frobenius reciprocity holds:

\[
\text{Hom}_{R(\alpha+\beta)}(\text{Ind}_{\alpha, \beta} M, N) \simeq \text{Hom}_{R(\alpha, \beta)}(M, \text{Res}_{\alpha, \beta} N).
\]

Given \(\alpha, \alpha', \beta, \beta' \in \mathbb{Q}^+\) with \(\alpha + \beta = \alpha' + \beta'\), let

\[
_{\alpha, \beta} R_{\alpha', \beta'} := e(\alpha, \beta)R(\alpha + \beta)e(\alpha', \beta').
\]

We also write \(_{\alpha+\beta} R_{\alpha, \beta} := R(\alpha + \beta)e(\alpha, \beta)\) and \(_{\alpha', \beta'} R_{\alpha+\beta'} := e(\alpha', \beta')R(\alpha' + \beta')\). Note that \(_{\alpha, \beta} R_{\alpha', \beta'}\) is a \(\mathbb{Z} \times \mathbb{Z}\)-graded \((R(\alpha), R(\alpha', \beta'))\)-superbimodule. Now we obtain Mackey’s Theorem for quiver Hecke superalgebras.

**Proposition 4.5.** The \(\mathbb{Z} \times \mathbb{Z}\)-graded \((R(\alpha) \otimes R(\beta), R(\alpha') \otimes R(\beta'))\)-superbimodule \(_{\alpha, \beta} R_{\alpha', \beta'}\) has a graded filtration with graded subquotients isomorphic to

\[
\Pi^p(\gamma \beta - \beta')(\alpha R_{\alpha - \gamma, \gamma}) \otimes (\beta R_{\beta + \gamma' - \beta', \gamma' - \gamma}) \otimes R'(\alpha - \gamma, \alpha' + \alpha R_{\alpha'}) \otimes (\gamma, \beta' - \gamma R_{\beta'})(-(\gamma | \beta + \gamma - \beta')),\]

where \(R' = R(\alpha - \gamma) \otimes R(\gamma) \otimes R(\beta + \gamma - \beta') \otimes R(\beta' - \gamma)\) and \(\gamma\) ranges over the set of \(\gamma \in \mathbb{Q}^+\) such that \(\alpha - \gamma, \beta' - \gamma\) and \(\beta + \gamma - \beta' = \alpha' + \gamma - \alpha\) belong to \(\mathbb{Q}^+\).
The proof is similar to that of [20, Proposition 2.18].

Hereafter, an $R(n)$-module always means a $\mathbb{Z}$-graded $R(n)$-module.

4.2. The algebra $R(n\alpha_i)$. In this subsection, we briefly review the results about $R(n\alpha_i)$ developed in [6, 9, 20, 26]. For the sake of simplicity, we assume that

\begin{equation}
\begin{aligned}
{k_0} \text{ is a field and the } k_i \text{'s are finite-dimensional over } k_0
\end{aligned}
\end{equation}

throughout this subsection.

Under the condition (4.15), the $\mathbb{Z}$-graded algebra $R(\beta)$ satisfies the conditions:

(a) its $\mathbb{Z}$-grading is bounded below and
(b) each homogeneous subspace $R(\beta)_t$ is finite-dimensional over $k_0$ ($t \in \mathbb{Z}$).

Hence we have

(i) $R(\beta)$ has the Krull-Schmidt direct sum property for finitely generated modules,

(ii) Any simple module in Mod($R(\beta)$) is finite-dimensional over $k_0$ and has

an indecomposable finitely generated projective cover (unique up to isomorphism),

(iii) there are finitely many simple modules in Rep($R(\beta)$) up to grade shifts and isomorphisms.

We now consider the case $\beta = n\alpha_i$. For $1 \leq k < n$, let $b_k := \tau_k x_{k+1} \in R(n\alpha_i)$. Then, by a direct computation, we have

\begin{equation}
\begin{aligned}
b_r b_s = b_s b_r & \quad \text{if } |r - s| > 1, \\
b_r b_{r+1} b_r = b_{r+1} b_r b_{r+1}, & \quad \text{for } 1 \leq r < n - 1.
\end{aligned}
\end{equation}

Thus, for $w \in S_n$, $b_w$ is well-defined.

We denote by $w[1, n]$ the longest element of the symmetric group $S_n$, and set

\begin{equation}
\begin{aligned}
b(i^n) := b_w[1, n].
\end{aligned}
\end{equation}

Then (4.17) implies

$$b(i^n)^2 = b(i^n) \quad \text{and} \quad b_k b(i^n) = b(i^n) b_k = b(i^n) \quad \text{for } 1 \leq k < n.$$
and define

\[(4.20) \quad [n]_i^\pi = \frac{(\pi_i q_i)^n - q_i^{-n}}{\pi_i q_i - q_i^{-1}}, \quad [n]_i^\pi! = \prod_{k=1}^n [k]_i^\pi.\]

Recall that \(q_i = q^{(\alpha_i | \alpha_i)/2}\). The algebra \(R(n\alpha_i)\) decomposes into the direct sum of indecomposable projective \(\mathbb{Z} \times \mathbb{Z}_2\)-graded modules as follows:

\[(4.21) \quad R(n\alpha_i) \simeq [n]_i^\pi! P(i^n) = \bigoplus_{k=0}^{n-1} q_i^{1-n+2k} \Pi_i^k P(i^n),\]

where

\[P(i^n) := \Pi_i^{\frac{n(n-1)}{2}} R(n\alpha_i) b(i^n) \left\langle \frac{n(n-1)}{4} (\alpha_i | \alpha_i) \right\rangle.\]

Note that \(P(i^n)\) is an indecomposable projective \(\mathbb{Z} \times \mathbb{Z}_2\)-graded module unique up to isomorphism and \(\mathbb{Z} \times \mathbb{Z}_2\)-grading shift. Note also that \(R(n\alpha_i) b(i^n) \simeq R(n\alpha_i) / \sum_{k=1}^{n-1} R(n\alpha_i) \tau_k\).

On the other hand, there exists an irreducible \(\mathbb{Z} \times \mathbb{Z}_2\)-graded \(R(n\alpha_i)\)-module \(L(i^n)\) which is unique up to isomorphism and \(\mathbb{Z} \times \mathbb{Z}_2\)-grading shift:

\[(4.22) \quad L(i^n) := \text{Ind}_{k[x_1] \otimes \cdots \otimes k[x_n]}^{R(n\alpha_i)} 1,\]

where \(1\) is the trivial \(k[x_1] \otimes \cdots \otimes k[x_n]\)-module which is isomorphic to \(k_0\). Hence \(L(i^n)\) is the \(R(n\alpha_i)\)-module generated by the element \(u(i^n)\) of \(\mathbb{Z} \times \mathbb{Z}_2\) degree \((0,0)\) with the defining relation

\[x_k u(i^n) = 0 \quad (1 \leq k \leq n), \quad k_s u(i^n) = 0 \quad (s > 0).\]

By Proposition 4.2, the \(R(n\alpha_i)\)-module \(L(i^n)\) has a \(k_0\)-basis

\[\{ \tau_w \cdot u(i^n) \mid w \in S_n \}.\]

Set

\[L_k := \{ v \in L(i^n) \mid x_n^k \cdot v = 0 \} \quad (k \geq 0).\]

Since \(x_n\) anticommutes with all \(x_i\) \((i = 1, \ldots, n-1)\) and \(\tau_j\) \((j = 1, \ldots, n-2)\), \(L_k\) has an \(R((n-1)\alpha_i)\)-module structure. Moreover \(L_k\) has a \(k_0\)-basis

\[\{ \tau_{w_1} \tau_{w_2} \cdots \tau_s u(i^n) \mid w \in S_{n-1}, n - k + 1 \leq s \leq n \},\]

and \(L_n = L(i^n)\). Thus we have a supermodule isomorphism

\[(4.23) \quad L_k / L_{k-1} \simeq \Pi_i^{k-1} L(i^{n-1}) \langle (1 - k)(\alpha_i | \alpha_i) \rangle \quad \text{for} \ 1 \leq k \leq n.\]
Here the $\mathbb{Z} \times \mathbb{Z}_2$-grading shift is caused by the $(\mathbb{Z} \times \mathbb{Z}_2)$-degree of $\tau_{n-1} \cdots \tau_{n-k+1} u(i^n)$. Note that $L_k = x_n^{n-k} L(i^n)$ for $0 \leq k \leq n$.

**Lemma 4.6.** Let $w[1,n]$ be the longest element of $S_n$. Then we have

$$R(n\alpha_i)\tau_{w[1,n]} R(n\alpha_i) = R(n\alpha_i).$$

**Proof.** We have

$$\tau_{w[1,n]} = \tau_{w[1,n-1]\tau_{n-1}\tau_{n-2} \cdots \tau_1}.$$ 

Hence by induction, it is enough to show that for $1 \leq a \leq n-1$

$$\tau_{w[1,n-1]\tau_{n-1}\tau_{n-2} \cdots \tau_{a+1}} \in R(n\alpha_i)\tau_{w[1,n-1]} R(n\alpha_i).$$

Note that

$$x_n \tau_{w[1,n-1]} \tau_{n-1} \tau_{n-2} \cdots \tau_a = \pm \tau_{w[1,n-1]} x_n \tau_{n-1} \tau_{n-2} \cdots \tau_a$$

$$= \pm \tau_{w[1,n-1]} (\pm \tau_{n-1} x_{n-1} + 1) \tau_{n-2} \cdots \tau_a$$

$$= \pm \tau_{w[1,n-1]} \tau_{n-1} \tau_{n-2} \cdots \tau_a x_{a+1}$$

Thus we have (4.24) and our assertion follows. □

5. The superfunctors $E_i$, $F_i$ and $\overline{F}_i$

In this section, we define the superfunctors $E_i$, $F_i$ and $\overline{F}_i$ on Mod($R(\beta)$) and investigate their relations among themselves. In the later sections, these relations will play an important role in proving the categorification theorem for cyclotomic quiver Hecke superalgebras.

Recall that an $R(n)$-module means a $\mathbb{Z}$-graded $R(n)$-module.

**5.1. Definitions of simple root superfunctors.** Let

$$E_i : \text{Mod}(R(\beta + \alpha_i)) \to \text{Mod}(R(\beta)),$$

$$F_i : \text{Mod}(R(\beta)) \to \text{Mod}(R(\beta + \alpha_i))$$

be the superfunctors given by

$$E_i(N) = e(\beta, i) N \simeq e(\beta, i) R(\beta + \alpha_i) \otimes_{R(\beta + \alpha_i)} N$$
for \( N \in \text{Mod}(R(\beta + \alpha_i)) \) and \( M \in \text{Mod}(R(\beta)) \).

Then \((F_i, E_i)\) is an adjoint pair; i.e., \(\text{Hom}_{R(\beta + \alpha_i)}(F_i M, N) \simeq \text{Hom}_{R(\beta)}(M, E_i N)\). Let \(n = |\beta|\). There are natural transformations:

\[
\begin{align*}
x_{E_i} & : E_i \to \Pi_i q_i^{-2} E_i, & x_F & : F_i \to \Pi_i q_i^{-2} F_i, \\
\tau_{E_{ij}} & : E_i E_j \to \Pi^{(p(j)p(i))} q^{(\alpha_i|\alpha_j)} E_j E_i, & \tau_{F_{ij}} & : F_i F_j \to \Pi^{(p(j)p(i))} q^{(\alpha_i|\alpha_j)} F_j F_i
\end{align*}
\]

induced by

(a) the left multiplication by \(x_{n+1}\) on \(e(\beta, i) N\) for \(N \in \text{Mod}(R(\beta + \alpha_i))\),
(b) the right multiplication by \(x_{n+1}\) on the kernel \(R(\beta + \alpha_i)e(\beta, i)\) of the functor \(F_i\),
(c) the left multiplication by \(\tau_{n+1}\) on \(e(\beta, i, j) N\) for \(N \in \text{Mod}(R(\beta + \alpha_i + \alpha_j))\),
(d) the right multiplication by \(\tau_{n+1}\) on the kernel \(R(\beta + \alpha_i + \alpha_j)e(\beta, j, i)\) of the functor \(F_iF_j\).

By the adjunction, \(\tau_{E_{ij}}\) induces a natural transformation

\[
F_j E_i \to \Pi^{(p(j)p(i))} q^{(\alpha_i|\alpha_j)} E_i F_j.
\]

(See Theorem 5.2 for more detail.)

Let \(\xi_n : R(n) \to R(n + 1)\) be the algebra homomorphism given by

\[
\xi_n(x_k) = x_{k+1}, \quad \xi_n(\tau_i) = \tau_{i+1}, \quad \xi_n(e(\nu)) = \sum_{i \in I^1(n)} e(i, \nu)
\]

for all \(1 \leq k \leq n, 1 \leq \ell \leq n\) and \(\nu \in I^n\). We denote by \(R^1(n)\) the image of \(\xi_n\).

For each \(i \in I\) and \(\beta \in \mathbb{Q}^+\), let \(F_{i,\beta} := R(\beta + \alpha_i)v(i, \beta)\) be the \((R(\beta + \alpha_i))\)-supermodule generated by \(v(i, \beta)\) of \(\mathbb{Z} \times \mathbb{Z}_2\)-degree \((0, 0)\) with the defining relation \(e(i, \beta)v(i, \beta) = v(i, \beta)\). The supermodule \(F_{i,\beta}\) has an \((R(\beta + \alpha_i), R(\beta))\)-superbimodule structure whose right \(R(\beta)\)-action is given by

\[
av(i, \beta) \cdot b = a \xi_n(b)v(i, \beta) \quad \text{for} \quad a \in R(\beta + \alpha_i) \text{ and } b \in R(\beta).
\]

In a similar way, we define the \((R(n + 1), R(n))\)-superbimodule structure on \(R(n + 1)v(1, n)\) by

\[
av(1, n) \cdot b = a \xi_n(b)v(1, n) \quad \text{for} \quad a \in R(n + 1) \text{ and } b \in R(n).
\]
Hence
\[ R(n + 1)v(1, n) \simeq \bigoplus_{i \in I, |\beta|=n} R(\beta + \alpha_i)v(i, \beta). \]

Now, for each \( i \in I \), we define the superfunctor
\[ \overline{F}_i : \text{Mod}(R(\beta)) \rightarrow \text{Mod}(R(\beta + \alpha_i)) \text{ by } N \mapsto \overline{F}_{i, \beta} \otimes_{R(\beta)} N. \]

5.2. Functorial relations. We shall investigate the commutation relations for the superfunctors \( E_i, F_i \) and \( \overline{F}_i \) \((i \in I)\).

Proposition 5.1. The homomorphism of \((R(n), R(n - 1))\)-superbimodules
\[ \tilde{\rho} : R(n)e(n - 1, j) \otimes_{R(n-1)} \bigcap_{i, \beta} q^{-(\alpha_i, \alpha_j)} \cap_{i, p(j)} e(n - 1, i)R(n) \longrightarrow e(n, i)R(n + 1)e(n, j) \]
given by
\[ x \otimes \pi^{p(i)p(j)} y \mapsto x\tau_n y \quad (x \in R(n)e(n - 1, j), \ y \in e(n - 1, i)R(n)) \]
does an isomorphism of \((R(n), R(n))\)-superbimodules
\[ \rho : R(n)e(n - 1, j) \otimes_{R(n-1)} \bigcap_{i, \beta} q^{-(\alpha_i, \alpha_j)} \cap_{i, p(j)} e(n - 1, i)R(n) \oplus e(n, i)R(n, 1)e(n, j) \]
\[ \simeq e(n, i)R(n + 1)e(n, j). \]

Proof. The homomorphism \( \tilde{\rho} \) is well-defined since we have
\[ a e(n - 1, j)\tau_n e(n - 1, i) = e(n - 1, j)\tau_n e(n - 1, i) \phi^{p(i)p(j)}(a) \] for any \( a \in R(n - 1) \),
where \( \phi \) is the parity involution defined in § 3.2. Thus it induces a homomorphism
\[ \rho' : R(n)e(n - 1, j) \otimes_{R(n-1)} \bigcap_{i, \beta} q^{-(\alpha_i, \alpha_j)} \cap_{i, p(j)} e(n - 1, i)R(n) \rightarrow \frac{e(n, i)R(n + 1)e(n, j)}{e(n, i)R(n, 1)e(n, j)}. \]

Thus it is enough to show that \( \rho' \) is an isomorphism. Since
\[ R(n) = \bigoplus_{a=1}^n \tau_a \cdots \tau_{n-1} k[x_n] \otimes_k R(n - 1), \]
we have
\[ R(n)e(n - 1, j) \otimes_{R(n-1)} e(n - 1, i)R(n) \]
\[ = \left( \bigoplus_{a=1}^n \tau_a \cdots \tau_{n-1} k[x_n e(j)] \otimes_k R(n - 1) \right) \otimes_{R(n-1)} e(n - 1, i)R(n) \]
\[ \simeq \bigoplus_{a=1}^n \tau_a \cdots \tau_{n-1} k[x_n e(j)] \otimes_k e(n - 1, i)R(n). \]
On the other hand,
\[
e(n, i)R(n + 1)e(n, j) \sim e(n, i)R(n, 1)e(n, j) = \bigoplus_{a=1}^{n+1} e(n, i)\tau_a \cdots \tau_n k[x_{n+1}e(j)] \otimes_k e(n - 1, i)R(n) \\
\sim \bigoplus_{a=1}^{n} e(n, i)\tau_a \cdots \tau_n k[x_{n+1}e(j)] \otimes_k e(n - 1, i)R(n).
\]

By (4.11), for \( f \in k[x_n e(j)] \), \( y \in e(n - 1, i)R(n) \) and \( 1 \leq a \leq n \), we have
\[
\tau_a \cdots \tau_{n-1} f \tau_n y = \tau_a \cdots \tau_{n-1} (\tau_n (\tau_{n-1}) + (f^{\beta_h})e_{n,n+1}) y \\
\equiv \tau_a \cdots \tau_{n-1} (\tau_{n-1} (\tau_{n-2}) + (f^{\beta_h})e_{n,n+1}) y \mod e(n, i)R(n, 1)e(n, j).
\]

Hence \( \rho' \) is right \( R(n) \)-linear and \( \rho'(\tau_a \cdots \tau_{n-1} f) = \tau_a \cdots \tau_n (\tau_{n} (f)) \). Since \( f \mapsto \tau_n (f) \) induces an isomorphism \( k[x_n e(j)] \cong k[x_{n+1} e(j)] \), our assertion follows. \( \square \)

**Theorem 5.2.** There exist natural isomorphisms
\[
E_i F_j \simeq \begin{cases} \quad q^{-(\alpha_i, \alpha_i)} F_j \Pi_{p(i)p(j)} E_i & \text{if } i \neq j, \\
\quad q^{-(\alpha_i, \alpha_i)} F_i \Pi_i E_i \oplus k[t_i] \otimes 1 & \text{if } i = j,
\end{cases}
\]
where \( t_i \) is an indeterminate of \( (\mathbb{Z} \times \mathbb{Z}_2) \)-degree \( (\langle \alpha_i, \alpha_i \rangle, p(i)) \) and
\[
k[t_i] \otimes 1 : \text{Mod}(R(\beta)) \to \text{Mod}(R(\beta))
\]
is the functor defined by \( M \mapsto k[t_i] \otimes M \).

**Proof.** Note that the kernels of \( F_j E_i \) and \( E_i F_j \) on \( \text{Mod}(R(\beta)) \) are given by
\[
R(\beta - \alpha_i + \alpha_j) e(\beta - \alpha_i, j) \otimes e(\beta - \alpha_i, \alpha_i) R(\beta) \quad \text{and}
\]
\[
e(\beta + \alpha_j - \alpha_i, i) R(\beta + \alpha_j) e(\beta, j),
\]
respectively. Since
\[
R(n)e(n - 1, j) \otimes_{R(n-1)} e(n - 1, i)R(n) e(\beta) \\
\simeq R(\beta - \alpha_i + \alpha_j) e(\beta - \alpha_i, j) \otimes e(\beta - \alpha_i, \alpha_i) R(\beta) \quad \text{and}
\]
\[
e(n, i)R(n + 1)e(n, j)e(\beta, j) = e(\beta + \alpha_j - \alpha_i, i) R(\beta + \alpha_j) e(\beta, j),
\]
our assertion is obtained by applying the exact functor \( \bullet e(\beta, j) \) on (5.1). \( \square \)
Remark 5.3. For an $R(\beta)$-module $M$, the $R(\beta)$-module structure on $k[t_i] \otimes M$ is given by
\[ a(t_i^k \otimes s) = t_i^k \otimes \phi^{k_\nu}(a)s \quad \text{for } a \in R(\beta), \ s \in M. \]

Thus we have an isomorphism of functors
\[ k[t_i] \otimes \text{Id} \simeq \bigoplus_{k \geq 0} (q^{(\alpha_i\alpha_i)} \Pi_i)^k. \]

Remark 5.4. The morphism $k[t_i] \otimes \text{Id} \rightarrow E_i F_i$ intertwines $t_i : k[t_i] \otimes \text{Id} \rightarrow q_i^{-2} \Pi_i k[t_i] \otimes \text{Id}$ and $E_i x_{F_i} : E_i F_i \rightarrow q_i^{-2} E_i \Pi_i F_i$.

Furthermore, the morphism $E_i F_i \rightarrow q_i^{-2} \Pi_i E_i$ intertwines $E_i x_{F_i}$ and $q_i^{-2} x_{F_i} \Pi_i E_i$.

Proposition 5.5. There exists an injective $(R(n), R(n))$-bimodule homomorphism
\[ \Phi : R(n)v(1, n-1) \otimes_{R(n)} R(n) \rightarrow R(n+1)v(1, n) \]
given by
\[ x v(1, n-1) \otimes y \mapsto x \xi_n(y) v(1, n). \]

Moreover, its image $R(n)R^1(n)$ has a decomposition
\[ R(n)R^1(n) = \bigoplus_{a=2}^{n+1} R(n, 1) \tau_n \cdots \tau_a = \bigoplus_{a=0}^{n-1} \tau_a \cdots \tau_1 R(1, n). \]

Proof. Since the proof is similar to that of [10, Proposition 3.7], we omit it. \qed

By a direct calculation, for $1 \leq k \leq n$, $1 \leq \ell \leq n-1$ and $\nu \in I^3$, we can easily see that
\[ x_k e(\nu, i) \tau_n \cdots \tau_1 e(i, \nu) \equiv (-1)^{p(\nu) p(\beta)} p(\nu_k) \tau_n \cdots \tau_1 x_{k+1} e(i, \nu), \]
\[ \tau_\ell e(\nu, i) \tau_n \cdots \tau_1 e(i, \nu) \equiv (-1)^{p(\nu) p(\beta)} p(\nu_\ell) p(\nu_{\ell+1}) \tau_n \cdots \tau_1 \tau_{\ell+1} e(i, \nu), \]
\[ x_{n+1} e(\nu, i) \tau_n \cdots \tau_1 e(i, \nu) \equiv (-1)^{p(\nu) p(\beta)} \tau_n \cdots \tau_1 x_1 e(i, \nu) \quad \text{mod } R(n)R^1(n). \]

Note that
\[ p(\tau_n \cdots \tau_1 e(i, \nu)) = p(i) p(\beta), \quad p(x_k e(\nu, i)) = p(\nu_k), \]
\[ p(\tau_\ell e(\nu, i)) = p(\nu_\ell) p(\nu_{\ell+1}), \quad p(x_{n+1} e(\nu, i)) = p(i). \]
Hence
\[ a\tau_n \cdots \tau_1 e(i, \beta) \equiv \tau_n \cdots \tau_1 e(i, \beta) \phi^{p(i)p(\beta)}(\xi_n(a)), \]
\[ x_{n+1} e(\beta, i) \tau_n \cdots \tau_1 e(i, \beta) \equiv (-1)^{p(i)p(\beta)} \tau_n \cdots \tau_1 x_1 e(i, \beta) \mod R(n)R^1(n) \quad \text{for any } a \in R(\beta). \]

By Proposition 5.5, there exists a right $R(n)$-linear map
\[ \varphi_1: R(n+1)v(1, n) \to R(n) \otimes k^f[x_{n+1}] \]
given by
\[ (5.3) \quad R(n+1)v(1, n) \to \text{Coker}(\Phi) \cong \bigoplus_{a=1}^{n+1} R(n, 1)\tau_a \cdots \tau_1 R(1, n) / \bigoplus_{a=2}^{n+1} R(n, 1)\tau_a \cdots \tau_1 R(1, n) \cong R(n, 1)\tau_n \cdots \tau_1 R(1, n) \cong R(n) \otimes k^f[x_{n+1}] \cong R(n) \otimes k^f[t]. \]

Similarly, there is another map \( \varphi_2: R(n+1)v(1, n) \to k^f[x_1] \otimes R(n) \) given by
\[ (5.4) \quad R(n+1)v(1, n) \to \text{Coker}(\Phi) \cong \bigoplus_{a=0}^{n} \tau_a \cdots \tau_1 R(1, n) / \bigoplus_{a=2}^{n} \tau_a \cdots \tau_1 R(1, n) \cong \tau_n \cdots \tau_1 R(1, n) \cong k^f[x_1] \otimes R(n) \cong k^f[t] \otimes R(n). \]

By restricting $\Phi$ to
\[ R(\beta + \alpha_j - \alpha_i) v(j, \beta - \alpha_i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i, i) R(\beta), \]
which is the kernel of $F_jE_i$ on Mod($R(\beta)$), (5.3) and (5.4) can be rewritten as
\[ e(\beta + \alpha_j - \alpha_i, i) R(\beta + \alpha_j) v(j, \beta) \]
\[ \xrightarrow{\varphi_1} \text{Coker}(\Phi) \cong \bigoplus_{a=1}^{n+1} R(\beta + \alpha_j - \alpha_i, i)\tau_a \cdots \tau_1 R(1, \beta) / \bigoplus_{a=2}^{n+1} R(\beta + \alpha_j - \alpha_i, i)\tau_a \cdots \tau_1 R(1, \beta) \cong \delta_{i,j} R(\beta, i) \cong \delta_{i,j} R(\beta, i) \cong \delta_{i,j} R(\beta) \otimes k[t_i] \]
and
\[ e(\beta + \alpha_j - \alpha_i, i) R(\beta + \alpha_j) v(j, \beta) \]
\[ \xrightarrow{\varphi_2} \text{Coker}(\Phi) \cong \bigoplus_{a=0}^{n} e(\beta + \alpha_j - \alpha_i, i)\tau_a \cdots \tau_1 R(1, \beta) / \bigoplus_{a=1}^{n} e(\beta + \alpha_j - \alpha_i, i)\tau_a \cdots \tau_1 R(1, \beta) \cong \delta_{i,j} \tau_n \cdots \tau_1 R(\beta, \beta) \cong \delta_{i,j} R(\beta, \beta) \cong \delta_{i,j} R(\beta, \beta) \cong \delta_{i,j} R(\beta) \otimes k[t_i] \]
\[ \simeq \delta_{i,j} k[x_1 e(i)] \otimes R(\beta) \simeq \delta_{i,j} k[t_i] \otimes R(\beta). \]

Therefore by (5.2), \( \varphi_1 \) and \( \varphi_2 \) coincide and we obtain:

**Theorem 5.6.** (i) There is a natural isomorphism
\[ F_j E_i \xrightarrow{\sim} E_i F_j \quad \text{for } i \neq j. \]

(ii) There is an exact sequence in \( \text{Mod}(R(\beta)) \)
\[ 0 \to F_i E_i M \to E_i F_i M \to \prod_{p(i) \neq p(i')} q^{-\langle \alpha_i | \beta \rangle} k[t_i] \otimes M \to 0, \]
which is functorial in \( M \in \text{Mod}(R(\beta)) \). Here \( t_i \) is an indeterminate of \((\mathbb{Z} \times \mathbb{Z}_2)\)-degree \((\langle \alpha_i | \alpha_i \rangle, p(i))\).

### 6. Crystal structure and strong perfect bases

In this section, we will show that we can choose a set of irreducible \( R(\beta) \)-modules \( (\beta \in \mathbb{Q}^+) \) which gives a strong perfect basis of \([\text{Rep}(R(\beta))]\). We will also show that irreducible modules are always isomorphic to their parity changes. To prove these facts, we need to employ the categorical crystal theories which are developed in [20, 22, 24]. However, in the quiver Hecke superalgebra case, we can use the arguments similar to those given in [20, Section 3.2] with slight modifications. Therefore we only state the required results (I1)–(I4) below, and we will focus on the basic properties of perfect bases.

In this section, we assume (4.15); i.e., \( k_0 \) is a field and the \( k_i \)'s are finite-dimensional over \( k_0 \).

For \( M \in \text{Rep}(R(\beta)) \) and \( i \in I \), define
\[
\begin{align*}
\Delta_{i,k} M &= e(\beta - k\alpha_i, i^k) M \in \text{Rep}(R(\beta - k\alpha_i, k\alpha_i)), \\
\varepsilon_i(M) &= \max\{k \geq 0 \mid \Delta_{i,k} M \neq 0\}, \\
E_i(M) &= e(\beta - \alpha_i, i) M \in \text{Rep}(R(\beta - \alpha_i)), \\
F'_i(M) &= \text{Ind}_{\beta,\alpha_i}(M \boxtimes L(i)) \in \text{Rep}(R(\beta + \alpha_i)), \\
\tilde{e}_i(M) &= \text{soc}(E_i(M)) \in \text{Rep}(R(\beta - \alpha_i)), \\
\tilde{f}_i(M) &= \text{hd}(F'_i M) \in \text{Rep}(R(\beta + \alpha_i)).
\end{align*}
\]
Here, $\text{soc}(M)$ means the socle of $M$, the largest semisimple subobject of $M$ and $\text{hd}(M)$ means the head of $M$, the largest semisimple quotient of $M$. We set $\varepsilon_i(M) = -\infty$ for $M = 0$.

Then we have the following statements.

(I1) If $M$ is an irreducible $R(\beta)$-module and $\varepsilon_i(M) > 0$, then $\tilde{e}_i M$ is irreducible.

(I2) If $M$ is an irreducible $R(\beta)$-module, then $\tilde{f}_i M$ is irreducible.

(I3) Let $M$ be an irreducible $R(\beta)$-module and $\varepsilon = \varepsilon_i(M)$. Then $\Delta_{i} M$ is isomorphic to $N \boxtimes L(i^\varepsilon)$ for some irreducible $R(\beta - \varepsilon\alpha)$-module $N$ with $\varepsilon_i(N) = 0$. Moreover, $N \simeq \tilde{e}_i^\varepsilon(M)$.

(I4) Let $M$ be an irreducible $R(\beta)$-module. Then $\tilde{f}_i \tilde{e}_i M \simeq M$. If $\varepsilon_i(M) > 0$, then we have $\tilde{f}_i \tilde{e}_i M \simeq M$.

Lemma 6.1. Let $z$ be an element of $[\text{Rep}(R(\beta))]$ such that $[E_i^k]z = 0$. Here $[E_i^k]$ is the map from $[\text{Rep}(R(\beta))]$ to $[\text{Rep}(R(\beta - k\alpha_i))]$ induced by the exact functor $E_i^k$. Then $z$ is a linear combination of $[M]$'s, where $M$ are irreducible $R(\beta)$-modules with $\varepsilon_i(M) < k$.

Proof. Write $z = \sum a_M[M]$, where $a_M \in \mathbb{Z}$ and $M$ ranges over the set of isomorphic classes of irreducible $R(\beta)$-modules. Let $\ell$ be the largest $\varepsilon_i(M)$ with $a_M \neq 0$. Then by (I3), $[E_i^\ell]z = \dim L(i^\ell) \sum_{\varepsilon_i(M) = \ell} a_M[\tilde{e}_i^\ell M]$. Hence if $\ell \geq k$, then $[E_i^k]z = 0$, which is a contradiction. Hence we obtain the desired result.

Proposition 6.2. Let $M$ be an irreducible module in $\text{Rep}(R(\beta))$. Assume that $\varepsilon := \varepsilon_i(M) > 0$. Then we have

$$[E_i M] = \pi_i^{1-\varepsilon} q_i^{1-\varepsilon} [\varepsilon]_i [\tilde{e}_i M] + \sum_k [N_k],$$

where $N_k$ are irreducible modules with $\varepsilon_i(N_k) < \varepsilon_i(\tilde{e}_i M) = \varepsilon - 1$.

Proof. By (I3), we have

$$\Delta_{i} M \simeq \tilde{e}_i^\varepsilon M \boxtimes L(i^\varepsilon).$$

Similarly, we have

$$\Delta_{i} \tilde{e}_i M \simeq \tilde{e}_i^\varepsilon M \boxtimes L(i^{\varepsilon-1}).$$

On the other hand, (4.23) implies that

$$[L(i^\varepsilon)] = \pi_i^{1-\varepsilon} q_i^{1-\varepsilon} [\varepsilon]_i [L(i^{\varepsilon-1})]$$
as an element of $[\text{Rep}(R(0))]$. Thus we obtain
\[ [E_i^{e-1}]([E_iM] - \pi_1^{1-\varepsilon} q_i^{1-\varepsilon} [\varepsilon_i] \pi_i [\tilde{e}_i M]) = 0. \]
Hence the desired result follows from the preceding lemma.

By a similar argument to the one in [20, Corollary 3.19], we have the following lemma.

**Lemma 6.3.** For any irreducible $R(\beta)$-module $M$,
\[ k_0 \simeq \text{End}_{R(\beta)}(M). \]

Now we are ready to prove the following fundamental result on irreducible modules over quiver Hecke superalgebras.

**Theorem 6.4.** For any irreducible $R(\beta)$-module $M$, we have
\[ \Pi M \simeq M. \]
In particular, $\Pi$ acts as the identity on $[\text{Rep}(R(\beta))]$ and $[\text{Proj}(R(\beta))]$.

**Proof.** We shall prove it by induction on $|\beta|$. If $|\beta| > 0$, there exists $i \in I$ such that $\varepsilon_i(M) > 0$. Since the endofunctor $\Pi$ commutes with the functor $E_i$,
\[ \tilde{e}_i(\Pi M) \simeq (\Pi \tilde{e}_i M). \]
By induction hypothesis, $\Pi \tilde{e}_i(M) \simeq \tilde{e}_i M$. Hence we obtain
\[ \tilde{e}_i M \simeq \tilde{e}_i \Pi M. \]
Then $\Pi M \simeq M$ follows from (I4).

By (4.16), our assertion also holds for $[\text{Proj}(R(\beta))]$. \qed

Together with Proposition 6.2, we obtain the following corollary.

**Corollary 6.5.** For any irreducible $R(\beta)$-module $M$, we have
\[ (6.2) \quad [E_i M] = q_i^{1-\varepsilon} [\varepsilon_i] [\tilde{e}_i M] + \sum_k [N_k], \]
where $N_k$'s are irreducible modules with $\varepsilon_i(N_k) < \varepsilon_i(M) - 1$. 


Let $\psi: R(\beta) \to R(\beta)$ be the involution given by
\begin{equation}
\psi(ab) = \psi(b)\psi(a), \quad \psi(e(\nu)) = e(\nu), \quad \psi(x_k) = x_k, \quad \psi(\tau_l) = \tau_l,
\end{equation}
for all $a, b \in R(\beta)$.

For any $M \in \text{Mod}(R(\beta))$, we denote by $M^* = \text{Hom}_{k_0}(M, k_0)$ the $k_0$-dual of $M$ whose left $R(\beta)$-module structure is induced by the involution $\psi$: namely, $(af)(s) = f(\psi(a)s)$ for $f \in \text{Hom}_{k_0}(M, k_0)$, $a \in R(\beta)$ and $s \in M$. We say that $M$ is self-dual if $M^* \simeq M$.

The following lemma tells that $\tilde{e}_i$ commutes with the duality up to a grade shift.

**Lemma 6.6.** For any irreducible $R(\beta)$-module $M$ such that $\varepsilon_i(M) > 0$, we have
\begin{equation}
(q_i^{1-\varepsilon_i(M)}\tilde{e}_iM)^* \simeq q_i^{1-\varepsilon_i(M)}\tilde{e}_i(M^*).
\end{equation}

**Proof.** Set $\varepsilon = \varepsilon_i(M)$. By (6.2), we have
\[ [E_i M] = [\varepsilon]_i[q_i^{1-\varepsilon_i} \tilde{e}_i M] + \sum_k [N_k]. \]
Here $N_k$'s are irreducible modules with $\varepsilon_i(N_k) < \varepsilon_i(M) - 1$. Since $E_i$ commutes with the duality functor, we have
\[ [E_i(M^*)] = [\varepsilon]_i[(q_i^{1-\varepsilon_i} \tilde{e}_i M)^*] + \sum_k [(N_k)^*]. \]
On the other hand, applying (6.2) to $M^*$, we obtain
\[ [E_i(M^*)] = [\varepsilon]_i[q_i^{1-\varepsilon_i} \tilde{e}_i(M^*)] + \sum_k [(N_k')^*] \]
with $\varepsilon_i(N_k') < \varepsilon_i(M) - 1$. Hence we obtain the desired result. \qed

**Proposition 6.7.** For any irreducible $R(\beta)$-module $M$, there exists $r \in \mathbb{Z}$ such that $q^r M$ is self-dual, that is,
\[ (q^r M)^* \simeq q^r M. \]

**Proof.** Using induction on $|\beta|$, we shall show that there exists $r \in \mathbb{Z}$ such that $q^r M$ is self-dual.

Assume $|\beta| > 0$ and take $i \in I$ such that $\varepsilon := \varepsilon_i(M) > 0$.

Then, by the induction hypothesis, there exists $r \in \mathbb{Z}$ such that $q^r q_i^{1-\varepsilon_i} \tilde{e}_i M$ is self-dual. Then the preceding lemma implies
\[ q_i^{1-\varepsilon_i} \tilde{e}_i(q^r M) \simeq (q_i^{1-\varepsilon_i} \tilde{e}_i(q^r M))^* \simeq q_i^{1-\varepsilon_i} \tilde{e}_i((q^r M)^*). \]
Hence by (14), we get \( q^r M \simeq (q^r M)^* \). \( \square \)

Finally we obtain the following theorem which shows the existence of strong perfect basis of \([\text{Rep}R(\beta)]\).

**Theorem 6.8.** For \( \beta \in \mathbb{Q}^+ \), let \( \text{Irr}_0 R(\beta) \) be the set of isomorphism classes of self-dual irreducible \( R(\beta) \)-modules. Then

\[
\{ [M] \mid M \in \text{Irr}_0 R(\beta) \}
\]

is an \( A \)-basis of \([\text{Rep}R(\beta)]\). Moreover, it is a strong perfect basis; i.e., it satisfies the property (2.2).

**Proof.** The proof is an immediate consequence of Proposition 6.7 and (6.2). \( \square \)

The following lemma is a categorification of the \( q \)-boson relation (2.1).

**Lemma 6.9.** For all \( \beta \in \mathbb{Q}^+ \) and \( M \in \text{Rep}(R(\beta)) \), we have isomorphisms and exact sequences.

\[
E_i F'_j M \simeq \Pi^{p(i)p(j)} q^{-(\alpha_i|\alpha_j)} F'_j E_i M \quad \text{for } i \neq j,
\]

\[
0 \to M \to E_i F'_j M \to \Pi_i q^{-2} q^{-(\alpha_i|\alpha_j)} F'_j E_i M \to 0.
\]

**Proof.** By Theorem 5.2 and Remark 5.4, all the columns and rows in the following commutative diagram are exact except the bottom row.

```
0 \to \delta_{ij} q^i \Pi_i k[t_i] \otimes M \to q^2 \Pi_i E_i F_j M \to q^2 \Pi_i \Pi^{p(i)p(j)} q^{-(\alpha_i|\alpha_j)} F'_j E_i M \to 0
```

Hence the bottom row is also exact. \( \square \)
7. Cyclotomic quiver Hecke superalgebras

In this section, we define the cyclotomic quiver Hecke superalgebra $R^\Lambda$ and study its elementary properties.

7.1. Definition of cyclotomic quotients. For $\Lambda \in \mathbb{P}^+$ and $i \in I$, we choose a monic polynomial of degree $\langle h_i, \Lambda \rangle$

$$a_i^\Lambda(u) = \sum_{k=0}^{\langle h_i, \Lambda \rangle} c_{i;k}u^{\langle h_i, \Lambda \rangle-k}$$

with $c_{i;k} \in k_{\mathcal{E}_{\langle \alpha_i | \alpha_i \rangle}}$ such that $c_{i;0} = 1$ and $c_{i;k} = 0$ if $i \in I_{\text{odd}}$ and $k$ is odd. Hence $a_i^\Lambda(x_1)e(i)$ has the $\mathbb{Z} \times \mathbb{Z}_2$-degree

$$\langle \langle h_i, \Lambda \rangle(\alpha_i | \alpha_i), p(i)\rangle(h_i, \Lambda) \rangle.$$

For $1 \leq k \leq n$, define

$$a^\Lambda(x_k) = \sum_{\nu \in \mathbb{I}^n} a^\Lambda_{\nu_k}(x_k)e(\nu) \in R(n).$$

**Definition 7.1.** Let $\beta \in \mathbb{Q}^+$ and $\Lambda \in \mathbb{P}^+$. The cyclotomic quiver Hecke superalgebra $R^\Lambda(\beta)$ at $\beta$ is the quotient algebra

$$R^\Lambda(\beta) = \frac{R(\beta)}{R(\beta)a^\Lambda(x_1)R(\beta)}.$$ 

7.2. Structure of cyclotomic quotients. We shall prove that the cyclotomic quotients are finitely generated over $k$. For the definition of $\varpi_a$ and $\partial_a$, see (4.9) and (4.10).

**Lemma 7.2.** Assume that $fe(\nu)M = 0$ for $M \in \text{Mod}(R(n))$, $f \in P_n$, $\nu \in \mathbb{I}^n$ and $1 \leq a < n$ such that $\nu_a = \nu_{a+1} = i$. Then we have

$$(\partial_a f)(x_a - x_{a+1})^p(i)e(\nu)M = 0 \quad \text{and}$$

$$(x_a^2 + x_{a+1}^2)^p(i)(\varpi_a f)e(\nu)M = (\varpi_a f)(x_a^2 + x_{a+1}^2)^p(i)e(\nu)M = 0.$$ 

**Proof.** By (4.10), we have

$$x_{a+1}^{1+p(i)} - x_a^{1+p(i)}\tau_a f \tau_a e(\nu) = ((x_{a+1} - x_a)^p(i)f - \varpi_a f(x_{a+1} - x_a)^p(i)) \tau_a e(\nu)$$

$$= ((x_{a+1} - x_a)^p(i)f \tau_a - (-1)^p(i)\varpi_a f \tau_a (x_{a+1} - x_a)^p(i)) e(\nu)$$

$$= ((x_{a+1} - x_a)^p(i)f \tau_a - (-1)^p(i)\tau_a f(x_{a+1} - x_a)^p(i) + (-1)^p(i)\partial_a f(x_{a+1} - x_a)^p(i)) e(\nu).$$
Hence we have $(\partial_a f)(x_a - x_{a+1})^{p(i)}e(\nu)M = 0$. It follows that
\[
0 = (x_a^{1+p(i)} - x_{a+1}^{1+p(i)})(\partial_a f)(x_a - x_{a+1})^{p(i)}e(\nu)M
\]
\[
= ((x_a - x_{a+1})^{p(i)}f(x_a - x_{a+1})^{p(i)} - \overline{s}_a f(x_a - x_{a+1})^{2p(i)}) e(\nu)M.
\]
Thus $(x_a - x_{a+1})^{2p(i)}(\overline{s}_a f)e(\nu)M = (x_a^2 + x_{a+1}^2)^{p(i)}(\overline{s}_a f)e(\nu)M = 0$. \hfill \Box

**Lemma 7.3.** There exists a monic polynomial $g(u)$ with coefficients in $k$ such that $g(x_a) = 0$ in $R^\Lambda(\beta)$ $(1 \leq a \leq n)$.

**Proof.** If $a = 1$, $g(x_1^2) = \prod_{i \in I} a_i^\Lambda(-x_1)a_i^\Lambda(x_1)$ satisfies the condition. Hence, by induction on $a$, it is enough to show the following statement:

For any monic polynomial $g(u) \in k[u]$ and $\nu \in I^n$, we can find

a monic polynomial $h(u) \in k[u]$ such that

$h(x_a^2)e(\nu)M = 0$ for any $R(\beta)$-module $M$ with $g(x_a^2)M = 0$.

(i) Suppose $\nu_a \neq \nu_{a+1}$. In this case, we have

$g(x_a^2)Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e(\nu)M = g(x_a^2)\tau_a^2e(\nu)M = \tau_\nu g(x_a^2)e(\nu)M = 0$.

Since $Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})$ is a monic polynomial in $x_a^{1+p(\nu_a)}$ with coefficients in $k[x_a^{1+p(\nu_a)}]$, there exists a monic polynomial $h(u)$ such that

$h(x_a^2) \in k[x_a^{1+p(\nu_a)}e(\nu), x_a^{1+p(\nu_{a+1})}e(\nu)]g(x_a^2) + k[x_a^{1+p(\nu_a)}e(\nu), x_a^{1+p(\nu_{a+1})}e(\nu)]g(x_a^2)Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})$.

Then $h(x_a^2)e(\nu)M = 0$.

(ii) Suppose $\nu_a = \nu_{a+1}$. Then Lemma 7.2 implies

$g(x_a^2)(x_a^2 + x_{a+1}^2)^{p(\nu_a)}e(\nu)M = 0$.

Then we can apply the same argument as (i). \hfill \Box

**Lemma 7.4.** Let $f \in \mathcal{P}_{n+1}$ be a monic polynomial of degree $m$ in $x_{n+1}$ whose coefficients are contained in $\mathcal{P}_n \otimes k^I$. Set $\overline{R} = e(n, i^{m+1})R(n + m + 1)e(n, i^{m+1})$. Then we have

$\overline{RfR} = \overline{R}$. 

Proof. We will prove the following statement by induction on $k$:

\begin{equation}
\partial_{k-1} \cdots \partial_{n+1} f e(n, i^{m+1}) \tau_{w[n+1,k]} \in \overline{RfR}
\end{equation}

for $n+1 \leq k \leq n+m+1$. Here $w[n+1,k]$ is the longest element of the subgroup $S_{[n+1,k]}$ generated by $s_a$ ($n+1 \leq a < k$). (See Remark 4.3.) Assuming (7.2), by multiplying $\tau_{w[n+1,k]}^{-1}w[n+1,k+1]$ from the right, we have

\[
\partial_{k-1} \cdots \partial_{n+1} f e(n, i^{m+1}) \tau_{w[n+1,k+1]} \in \overline{RfR}.
\]

By multiplying $\tau_k$ from the left, we have

\[
\tau_k (\partial_{k-1} \cdots \partial_{n+1} f) e(n, i^{m+1}) \tau_{w[n+1,k+1]} = (\tau_k (\partial_{k-1} \cdots \partial_{n+1} f) \tau_k + \partial_k \cdots \partial_{n+1} f) e(n, i^{m+1}) \tau_{w[n+1,k+1]} = \partial_k \cdots \partial_{n+1} f e(n, i^{m+1}) \tau_{w[n+1,k+1]} \in \overline{RfR}.
\]

Here we have used the fact that $\tau_k \tau_{w[n+1,k+1]} = 0$.

Thus the induction proceeds and we obtain (7.2) for any $k$. Since $\partial_{n+m} \cdots \partial_{n+1} f = 1$, our assertion follows from $\overline{R\tau_{w[n+1,n+m+1]}R} = R$ in Lemma 4.6. \hfill $\square$

**Corollary 7.5.** For $\beta \in Q^+$ with $|\beta| = n$ and $i \in I$, there exists $m$ such that

\[ R^\Lambda(\beta + k\alpha_i) = 0 \text{ for any } k \geq m. \]

**Proof.** By Lemma 7.3, there exists a monic polynomial $g(u)$ of degree $m$ such that

\[ g(x_n)R^\Lambda(\beta) = 0. \]

Lemma 7.4 implies $e(n, i^k)R^\Lambda(\beta + k\alpha_i) = 0$ for $k > m$. Now our assertion follows from similar arguments to [11, Lemma 4.3 (b)]. \hfill $\square$

## 8. The superfunctors $E_i^\Lambda$ and $F_i^\Lambda$

In this section, we define the superfunctors $E_i^\Lambda$ and $F_i^\Lambda$ on $\text{Mod}(R^\Lambda(\beta))$ and show that they induce well-defined exact functors on $\text{Proj}(R^\Lambda(\beta))$ and $\text{Rep}(R^\Lambda(\beta))$.

For each $i \in I$, we define the superfunctors

\[ E_i^\Lambda : \text{Mod}(R^\Lambda(\beta + \alpha_i)) \to \text{Mod}(R^\Lambda(\beta)), \]

\[ F_i^\Lambda : \text{Mod}(R^\Lambda(\beta)) \to \text{Mod}(R^\Lambda(\beta + \alpha_i)) \]
by

\[ E_i^A(N) = e(\beta, i)N = e(\beta, i)R^A(\beta + \alpha_i) \otimes_{R^A(\beta + \alpha_i)} N, \]
\[ F_i^A(M) = R^A(\beta + \alpha_i)e(\beta, i) \otimes_{R^A(\beta)} M \]

for \( M \in \text{Mod}(R^A(\beta)) \) and \( N \in \text{Mod}(R^A(\beta + \alpha_i)). \)

For each \( i \in I, \beta \in \mathbb{Q}^+ \) and \( m \in \mathbb{Z}, \) let

\[ K_{i,\beta}^m := R(\beta + \alpha_i)v(i, \beta)T_i^m \]

be the \( R(\beta + \alpha_i) \)-supermodule generated by \( v(i, \beta)T_i^m \) with the defining relation

\[ e(i, \beta)v(i, \beta)T_i^m = v(i, \beta)T_i^m. \]

We assign to \( v(i, \beta)T_i^m \) the \((\mathbb{Z} \times \mathbb{Z}_2)\)-degree \((0, 0)\). The supermodule \( K_{i,\beta}^m \) has an \((R(\beta + \alpha_i), k[t_i] \otimes R(\beta))\)-superbimodule structure whose right \( k[t_i] \otimes R(\beta) \)-action is given by

\[
\begin{align*}
av(i, \beta)T_i^m \cdot b &= a \xi_n(b)v(i, \beta)T_i^m, \\
v(i, \beta)T_i^m \cdot t_i &= a \phi_i^m(x_1)v(i, \beta)T_i^m = (-1)^{p(i)}ax_1v(i, \beta)T_i^m
\end{align*}
\]

for \( a \in R(\beta + \alpha_i) \) and \( b \in R(\beta). \) Here, \( \phi_i := \phi^{p(i)} \) and \( \phi \) is the parity involution (see § 3.2).

In the sequel, we sometimes omit the \( \mathbb{Z} \)-grading shift functor \( q \) when \( \mathbb{Z} \)-grading can be neglected.

Set \( \Lambda_i := \langle h_i, \Lambda \rangle. \) We introduce \((R(\beta + \alpha_i), R^A(\beta))\)-superbimodules

\[
\begin{align*}
F^A &:= R^A(\beta + \alpha_i)e(\beta, i), \\
K_0 &:= R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} R^A(\beta), \\
K_1 &:= K_{i,\beta}^\Lambda \otimes_{R(\beta)} \Pi_{i,\beta}^{\Lambda_i + p(\beta)} R^A(\beta) \\
&= R(\beta + \alpha_i)v(i, \beta)T_i^{\Lambda_i} \otimes_{R(\beta)} \Pi_{i,\beta}^{\Lambda_i + p(\beta)} R^A(\beta).
\end{align*}
\]

For \( i \in I, \) let \( t_i \) be an indeterminate of \( \mathbb{Z} \times \mathbb{Z}_2 \)-degree \(((\alpha_i|\alpha_i), p(i)). \) Then \( k[t_i] \) is a superalgebra. The superalgebra \( k[t_i] \) acts on \( K_1 \) from the right by the formula given in (8.1). Namely,

\[
\left( av(i, \beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i + p(\beta)} b \right) t_i = a \phi_i^{p(\beta)}(x_1)v(i, \beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i + p(\beta)} \phi_i(b)
\]

for \( a \in R(\beta + \alpha_i) \) and \( b \in R^A(\beta). \) Here \( \pi_i := \pi^{p(i)} \) and \( \phi_i = \phi^{p(i)}. \) On the other hand, \( k[t_i] \) acts on \( R(\beta + \alpha_i)e(\beta, i), F^A \) and \( K_0 \) by multiplying \( x_{n+1} \) from the right. Thus \( K_0, F^A \)and \( K_1 \) have a graded \((R(\beta + \alpha_i), k[t_i] \otimes R^A(\beta))\)-superbimodule structure.
By a similar argument to the one given in [10, Lemma 4.8, Lemma 4.16], we obtain the following two lemmas:

Lemma 8.1.

(i) Both $K_1$ and $K_0$ are finitely generated projective right $k[t_i] \otimes R^\Lambda(\beta)$-supermodules.

(ii) For any monic (skew)-polynomial $f(t_i) \in \mathcal{P}_n[t_i]$, the right multiplication by $f(t_i)$ on $K_1$ induces an injective endomorphism of $K_1$.

Lemma 8.2. For $i \in I$ and $\beta \in \mathbb{Q}^+$ with $|\beta| = n$, we have

(i) \( R(\beta + \alpha_i)a^{\Lambda}(x_1)R(\beta + \alpha_i) = \sum_{a=0}^{n} R(\beta + \alpha_i)a^{\Lambda}(x_1)\tau_1 \cdots \tau_a, \)

(ii) \( R(\beta + \alpha_i)a^{\Lambda}(x_1)R(\beta + \alpha_i)e(\beta, i) = R(\beta + \alpha_i)a^{\Lambda}(x_1)R(\beta)e(\beta, i) + R(\beta + \alpha_i)a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\beta, i). \)

For $1 \leq a \leq n$, $1 \leq \ell < n$ and $\nu \in I^n$, by applying a similar argument given in [11, Lemma 4.7], we have

\[
\begin{align*}
x_{a+1}a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\nu, i) &\equiv (-1)^{p(\nu_a)(\Lambda_i+p(\nu))}a^{\Lambda}(x_1)\tau_1 \cdots \tau_n x_a e(\nu, i), \\
\tau_{\ell+1}a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\nu, i) &\equiv (-1)^{p(\nu_{\ell+1})(\Lambda_i+p(\nu))}a^{\Lambda}(x_1)\tau_1 \cdots \tau_n \tau_\ell e(\nu, i), \\
x_1 a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\nu, i) &\equiv (-1)^{p(\nu)}a^{\Lambda}(x_1)\tau_1 \cdots \tau_n x_{n+1} e(\nu, i) \\
&\mod R(n + 1)a^{\Lambda}(x_1)R(n)e(n, i).
\end{align*}
\]

Hence we have

\[
a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\beta, i)c \equiv \phi_{1, a^{\Lambda}}^{\Lambda_i+p(\beta)}(\xi_1(\beta))a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\beta, i) \\
\mod R(n + 1)a^{\Lambda}(x_1)R(\beta)e(\beta, i)
\]

for any $\beta \in \mathbb{Q}^+$ with $|\beta| = n$ and $c \in R(n)$.

Let $P: K_1 \to K_0$ be the homomorphism defined by

\[
xv(i, \beta)T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i+p(\beta)}y \mapsto x a^{\Lambda}(x_1)\tau_1 \cdots \tau_n e(\beta, i) \otimes y
\]

for $x \in R(\beta + \alpha_i)$ and $y \in R^\Lambda(\beta)$. Then, by (8.3) and (8.4), $P$ becomes an $(R(\beta + \alpha_i), k[t_i] \otimes R(\beta))-superbimodule$ homomorphism.

Let $pr: K_0 \to F^\Lambda$ be the canonical projection. Using Lemma 8.2, one can easily see that

\[
\text{Im}(P) = \text{Ker}(pr) = \frac{R(\beta + \alpha_i)a^{\Lambda}(x_1)R(\beta + \alpha_i)e(\beta, i)}{R(\beta + \alpha_i)a^{\Lambda}(x_1)R(\beta)e(\beta, i)} \subset K_0.
\]
Hence we obtain an exact sequence of \((R(\beta + \alpha_i), k[t_i] \otimes R(\beta))\)-superbimodules

\[
K_1 \xrightarrow{P} K_0 \xrightarrow{\text{pr}} F^\Lambda \rightarrow 0.
\]

We will show that \(P\) is actually injective by constructing an \((R(\beta + \alpha_i), k[t_i] \otimes R(\beta))\)\-bilinear homomorphism \(Q\) such that \(Q \circ P\) is injective.

For \(1 \leq a \leq n\), we define the elements \(\varphi_a\) and \(g_a\) of \(R(\beta + \alpha_i)\) by

\[
\varphi_a = \sum_{\nu \in I^\beta + \alpha_i, \nu_a \neq \nu_{a+1}} \tau_a e(\nu) + \sum_{\nu \in I^\beta + \alpha_i, \nu_a = \nu_{a+1} \in I_{\text{even}}} (1 - (x_{a+1} - x_a)\tau_a) e(\nu)
\]

\[(8.7)\]

\[
+ \sum_{\nu \in I^\beta + \alpha_i, \nu_a = \nu_{a+1} \in I_{\text{odd}}} ((x_{a+1} - x_a) - (x_{a+1}^2 - x_a^2)\tau_a) e(\nu)
\]

and

\[
g_a = \sum_{\nu \in I^\beta + \alpha_i, \nu_a \neq \nu_{a+1}} \tau_a e(\nu) + \sum_{\nu \in I^\beta + \alpha_i, \nu_a = \nu_{a+1} \in I_{\text{even}}} (x_{a+1} - x_a)(1 - (x_{a+1} - x_a)\tau_a) e(\nu)
\]

\[(8.8)\]

\[
+ \sum_{\nu \in I^\beta + \alpha_i, \nu_a = \nu_{a+1} \in I_{\text{odd}}} (x_{a+1}^2 - x_a^2)((x_{a+1} - x_a) - (x_{a+1}^2 - x_a^2)\tau_a) e(\nu).
\]

The elements \(\varphi_a\)'s are called \(\textit{intertwiners}\), and \(g_a\)'s are their variants.

Note that if \(\nu_a = \nu_{a+1} \in I_{\text{even}},\)

\[
\varphi_a e(\nu) = (x_a\tau_a - \tau_a x_a) e(\nu) = (\tau_a x_{a+1} - x_{a+1}\tau_a) e(\nu)
\]

\[(8.9)\]

\[= (1 - \tau_a(x_a - x_{a+1})) e(\nu),\]

and if \(\nu_a = \nu_{a+1} \in I_{\text{odd}},\)

\[
\varphi_a e(\nu) = (x_a^2\tau_a - \tau_a x_a^2) e(\nu) = (\tau_a x_{a+1}^2 - x_{a+1}^2\tau_a) e(\nu)
\]

\[(8.10)\]

\[= ((x_a - x_{a+1}) - \tau_a(x_a^2 - x_{a+1}^2)) e(\nu).\]

\textbf{Lemma 8.3.} For \(1 \leq a \leq n\) and \(\nu \in I^{n+1}\), we have

\[
\varphi_a e(\nu) = e(s_a\nu)\varphi_a,
\]

\[(8.11)\]

\[
x_{s_a(b)}\varphi_a e(\nu) = (-1)^{p(\nu_a)p(\nu_{a+1})p(\nu_b)}\varphi_a x_b e(\nu) \quad (1 \leq b \leq n + 1),
\]

\[
\tau_b\varphi_a e(\nu) = (-1)^{p(\nu_a)p(\nu_{a+1})p(\nu_b)p(\nu_{b+1})}\varphi_a \tau_b e(\nu) \quad \text{if } |b - a| > 1,
\]

\[
\tau_a\varphi_{a+1}\varphi_a = \varphi_{a+1}\varphi_a\tau_{a+1},
\]
and
\[ g_a e(v) = e(s_a v) g_a, \]
\[ x_{s_a (b)} g_a e(v) = (-1)^{p(v_a) + p(v_{a+1})} g_a x_b e(v) \quad (1 \leq b \leq n + 1), \]
\[ \tau_b g_a e(v) = (-1)^{p(v_a) + p(v_{b+1})} g_a \tau_b e(v) \quad \text{if} \ |b - a| > 1, \]
\[ \tau_a g_{a+1} g_a = g_{a+1} g_a \tau_{a+1}. \]

Proof. By the defining relations of quiver Hecke superalgebras, the third equality can be verified immediately. If \( \nu_a \neq \nu_{a+1} \) or \( \nu_a = \nu_{a+1} \in I_{\text{even}} \), the first and second equalities were proved in [10, Lemma 4.12]. We will prove the second equality in (8.11) when \( \nu_a = \nu_{a+1} \in I_{\text{odd}} \). Let \( b = a \). Then

\[ x_{a+1} \varphi_a e(v) = x_{a+1}^2 - x_{a+1} x_a - (x_{a+1}^2 - x_a^2)(x_{a+1} \tau_a) e(v) \]
\[ = x_{a+1}^2 - x_{a+1} x_a - (x_{a+1}^2 - x_a^2)(-\tau_a x_a + 1) e(v), \]
and
\[ \varphi_a x_a e(v) = x_{a+1} x_a - x_a^2 - (x_{a+1}^2 - x_a^2)(\tau_a x_a) e(v). \]

Therefore we have
\[ x_{a+1} \varphi_a e(v) + \varphi_a x_a e(v) = 0. \]

Similarly, we can prove the equality when \( b = a + 1 \).

Let \( S = \tau_a \varphi_{a+1} \varphi_a - \varphi_a \varphi_{a+1} \tau_a + 1 \). Using the second equality, we have

\[ (\tau_a \varphi_{a+1} \varphi_a) x_a e(v) = (-1)^{p(v_a) + p(v_{a+1}) + p(v_a) + p(v_{a+2}) + p(v_{a+1}) + p(v_{a+2})} x_{a+2} (\tau_a \varphi_{a+1} \varphi_a) e(v), \]
\[ (\varphi_a \varphi_{a+1} \tau_{a+1}) x_a e(v) = (-1)^{p(v_a) + p(v_{a+1}) + p(v_a) + p(v_{a+2}) + p(v_a) + p(v_{a+1})} x_{a+2} (\varphi_a \varphi_{a+1} \tau_{a+1}) e(v), \]
\[ (\tau_a \varphi_{a+1} \varphi_a) x_{a+1} e(v) = (-1)^{p(v_{a+1}) + p(v_{a+1}) + p(v_a) + p(v_{a+2})} x_{a+1} \varphi_{a+1} \varphi_a e(v) \]
\[ = (-1)^{p(v_{a+1}) + p(v_{a+1}) + p(v_a) + p(v_{a+2})} x_{a+1} \varphi_{a+1} \varphi_a e(v), \]
\[ (\varphi_a \varphi_{a+1} \tau_{a+1}) x_{a+1} e(v) = (-1)^{p(v_{a+1})} \varphi_{a+1} \varphi_a (x_{a+2} \tau_{a+1} - \epsilon_{a+1,a+1}) \varphi_a e(v), \]
\[ = (-1)^{p(v_{a+1})} (p(v_{a+1}) + p(v_a) + p(v_{a+2}) + p(v_a) + p(v_{a+2}) + p(v_a) + p(v_{a+1})) x_{a+1} \varphi_{a+1} \varphi_a \tau_{a+1} e(v) \]
\[ - (-1)^{p(v_{a+1})} p(v_{a+2}) \varphi_{a+1} \varphi_a \epsilon_{a+1,a+2} e(v), \]
\[ (\tau_a \varphi_{a+1} \varphi_a) x_{a+2} e(v) = (-1)^{p(v_{a+2})} (p(v_{a+2}) + p(v_{a+1}) + p(v_{a+2}) + p(v_a) + p(v_{a+2}) + p(v_a) + p(v_{a+1})) x_{a+1} \varphi_{a+1} \varphi_a \tau_{a+1} e(v) \]
\[ = (-1)^{p(v_{a+2})} (p(v_{a+2}) + p(v_{a+1}) + p(v_{a+2}) + p(v_a) + p(v_{a+2}) + p(v_a) + p(v_{a+1})) \]
Lemma 4.12] we conclude that

\[\left((-1)^{b(\nu_{a+1})}p(\nu_{a+2})x_a\tau_a + e_{a,a+1}\varphi_{a+1}\varphi_a e(\nu)\right)\]

\[= (-1)^{b(\nu_{a+2})}(p(\nu_a)p(\nu_{a+1})+p(\nu_a)p(\nu_{a+2}))x_a\tau_a\varphi_{a+1}\varphi_a e(\nu)\]

\[+ (-1)^{b(\nu_{a+2})}(p(\nu_a)p(\nu_{a+1})+p(\nu_a)p(\nu_{a+2}))\varphi_{a+1}\varphi_a e_{a,a+1,a+2} e(\nu),\]

\[(\varphi_{a+1}\varphi_a \tau_{a+1})x_{a+2} e(\nu) = \varphi_{a+1}\varphi_a ((-1)^{b(\nu_{a+2})}x_{a+1}\tau_{a+1} + e_{a+1,a+2} e(\nu))\]

\[= (-1)^{b(\nu_{a+2})}(p(\nu_a)p(\nu_{a+1})+p(\nu_a)p(\nu_{a+2})+p(\nu_{a+1})p(\nu_{a+2}))x_a\tau_a\varphi_{a+1}\varphi_a e(\nu)\]

\[+ \varphi_{a+1}\varphi_a e_{a,a+1,a+2} e(\nu).\]

Hence we have \(Sx_b = \pm x_{s_{a,a+2}b}S\) for all \(b\). Using a similar argument given in [10, Lemma 4.12] we conclude that \(S = 0\).

The equalities in (8.12) easily follows from (8.11).

Hence we have

\[ag_n \cdots g_1 e(i, \beta) = g_n \cdots g_1 e(i, \beta) \phi_i^{p(\beta)}(\xi_n(a))\]

for any \(a \in R(\beta)\) and,

\[x_{n+1} g_n \cdots g_1 e(i, \beta) = (-1)^{p(\beta)p(i)} g_n \cdots g_1 e(i, \beta) x_1.\]

Using a similar method to the construction of \(P\), we obtain the following proposition:

**Proposition 8.4.** There is an \((R(\beta + \alpha_i), k[t_i] \otimes R^A(\beta))\)-bilinear homomorphism

\[Q: K_0 \rightarrow K'_1 := R(\beta + \alpha_i) v(i, \beta) \otimes_{R(\beta)} \Pi_i^{p(\beta)} R^A(\beta)\]

defined by

\[ae(\beta, i) \otimes b \mapsto a g_n \cdots g_1 v(i, \beta) \otimes \pi_i^{p(\beta)} b\]

for \(a \in R(\beta + \alpha_i)\) and \(b \in R^A(\beta)\). Here, the right action of \(t_i\) on \(K'_1\) is given by

\[av(i, \beta) \otimes \pi_i^{p(\beta)} b \mapsto (-1)^{p(i)p(\beta)} a x_1 v(i, \beta) \otimes \pi_i^{p(\beta)} \phi_i(b).\]

**Theorem 8.5.** For each \(\nu \in I^\beta\), set

\[A_{\nu}(t_i) = a_i^{\nu}(t_i) \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{i,\nu_a}(t_i, x_a) \prod_{1 \leq a \leq n, \nu_a = i \in I_{add}} (x_a - t_i)^2 e(\nu),\]

and define

\[A(t_i) := \sum_{\nu \in I^\beta} A_{\nu}(t_i) \in k[t_i] \otimes R^A(\beta).\]

Then the composition

\[Q \circ P: K_1 \rightarrow K'_1\]
Thus we may assume that \( \tau \) coincides with the right multiplication by \((-1)^{p(\Lambda_i) + p(\beta)} A(t_i); \) i.e.,
\[
av(i, \beta) T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i + p(\beta)} b \rightarrow (av(i, \beta) \otimes \pi_i^{p(\beta)} \phi_i^T(b)) (-1)^{p(\Lambda_i) p(\beta)} A(t_i)
\]
\[
= av(i, \beta) A(t_i) \otimes \pi_i^{p(\beta)} b.
\]

**Proof.** If \( i \in I_{\text{even}} \), our assertion was proved in [10, Theorem 4.15]. If \( i \in I_{\text{odd}} \), then it suffices to show that
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu) = a^\Lambda(x_1) \tau_1 \cdots \tau_n e(\nu, i) g_n \cdots g_1
\]
\[
\equiv A'_\nu \mod R(\beta + \alpha_i) a^\Lambda(x_2) R^1(\beta),
\]
where
\[
A'_\nu = a^\Lambda(x_1) \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{i, \nu_a}(x_1, x_{a+1}) \prod_{1 \leq a \leq n, \nu_a = i, \nu = I_{\text{odd}}} (x_{a+1} - x_1)^2 e(i, \nu).
\]

We will prove (8.14) by induction on \(|\beta| = n\). If \( n = 0 \), the assertion is obvious. Thus we may assume that \( n \geq 1 \).

Note that, by (8.10), we have
\[
\tau_n e(\nu, i) g_n = \begin{cases} 
\tau_n e(\nu, i) \tau_n = Q_{i, \nu_n}(x_n, x_{n+1}) e(\nu, i, \nu_n) & \text{if } \nu_n \neq i, \\
\tau_n (x_{n+1}^2 - x_n^2)(x_{n+1} - x_n) e(\nu, i) & \text{if } \nu_n = i.
\end{cases}
\]

(i) We first assume that \( \nu_n \neq i \). Then, by (4.8), we have
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu)
\]
\[
= a^\Lambda(x_1) \tau_1 \cdots \tau_n g_{n-1} \cdots g_1 e(i, \nu)
\]
\[
= a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} g_{n-1} \cdots g_1 e(i, \nu) Q_{i, \nu_n}(x_1, x_{n+1})
\]
\[
\equiv A'_{\nu_n} Q_{i, \nu_n}(x_1, x_{n+1}) = A'_\nu \mod R(\beta + \alpha_i) a^\Lambda(x_2) R^1(\beta) e(i, \beta).
\]

(ii) Assume that \( \nu_n = i \). Then we have
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_n g_n \cdots g_1 e(i, \nu)
\]
\[
= a^\Lambda(x_1) \tau_1 \cdots \tau_n (x_{n+1} - x_n)(x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu).
\]

Note that
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} g_n \cdots g_1 e(i, \nu) = \pm g_n \cdots g_1 a^\Lambda(x_2) \tau_2 \cdots \tau_n \equiv 0
\]
\[
\mod R(\beta + \alpha_i) a^\Lambda(x_2) R^1(\beta) e(i, \beta).
\]

By (8.10), the formula (8.17) can be written as
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} (\tau_n (x_{n+1}^2 - x_n^2) - (x_{n+1} - x_n)) (x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu) \equiv 0.
\]
Thus
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} \tau_n(x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu) \\
\equiv a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1}(x_{n+1} - x_n)(x_n^2 - x_{n+1}^2) g_{n-1} \cdots g_1 e(i, \nu) \\
\equiv (-1)^{p(\nu < n)} A'_{\nu < n}(x_{n+1} - x_1)(x_{n+1}^2 - x_1^2).
\]
Since the right multiplication by \((x_{n+1}^2 - x_1^2)\) and \((x_{n+1} - x_1)\) on \(K_1\) are injective by Lemma 8.1, we conclude that
\[
a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} \tau_n(x_{n+1}^2 - x_n^2) g_{n-1} \cdots g_1 e(i, \nu) \\
\equiv (-1)^{p(\nu < n)} A'_{\nu < n}(x_{n+1} - x_1),
\]
which implies
\[
(-1)^{p(\nu < n)} a^\Lambda(x_1) \tau_1 \cdots \tau_{n-1} \tau_n(x_{n+1}^2 - x_n^2)(x_{n+1} - x_n) g_{n-1} \cdots g_1 e(i, \nu) \\
\equiv (-1)^{p(\nu < n)} A'_{\nu < n}(x_{n+1} - x_1)^2.
\]
Then, (8.16), together with \(A'_\nu = A'_{\nu < n}(x_{n+1} - x_1)^2\), implies the desired result. \(\square\)

By applying the same argument given in [10, Lemma 4.19], we have the following lemma.

**Corollary 8.6.** Set
\[
K'_0 := R(\beta + \alpha_i)e(\beta, i)T_i^{\Lambda_i} \otimes_R \Pi_i^{\Lambda_i} R^\Lambda(\beta).
\]
Then the following diagram commutes

\[
\begin{array}{ccc}
K_1 & \xrightarrow{P} & K_0 \\
\downarrow{(-1)^{p(i)\Lambda_i p(\beta) A(t_i)}} & & \downarrow{(-1)^{p(i)\Lambda_i p(\beta) A(t_i)}} \\
K'_1 & \xrightarrow{P'} & K'_0.
\end{array}
\]

Here, \(P': K'_1 \to K'_0\) is given by
\[
av(i, \beta) \otimes \pi_i^{p(\beta)} b \mapsto a a^\Lambda(x_1) \tau_1 \cdots \tau_n e(\beta, i) T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i} b,
\]
and \((-1)^{p(i)\Lambda_i p(\beta)} A(t_i): K_0 \to K'_0\) is given by
\[
a \otimes b \mapsto (a T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i} \phi_i^{\Lambda_i}(b)) (-1)^{p(i)\Lambda_i p(\beta)} A(t_i) = (-1)^{p(i)\Lambda_i p(\beta)} a A(t_i) T_i^{\Lambda_i} \otimes \pi_i^{\Lambda_i} b.
\]
In particular, for any $\nu \in I^\beta$, we have
\[
g_n \cdots g_1 a^\Lambda(x_1) \tau_1 \cdots \tau_n e(\nu, i) \otimes e(\beta)
\]
\begin{equation}
(8.18)
= (-1)^{p(i) + p(\beta)} a^\Lambda_i(x_{n+1}) \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{\nu_a, i}(x_a, x_{n+1}) \prod_{1 \leq a \leq n, \nu_a = i} (x_a - x_{n+1})^2 e(\nu, i) \otimes e(\beta)
\end{equation}
in $R(\beta + \alpha_i) e(\beta, i) \otimes_{R(\beta)} \Lambda^\Lambda(\beta)$.

Since $K_1$ is a projective $R^\Lambda(\beta) \otimes k[t_i]$-supermodule by Lemma 8.1 and $A(t_i)$ is a monic (skew)-polynomial (up to a multiple of an invertible element) in $t_i$, by a similar argument to the one in [10, Lemma 4.17, Lemma 4.18], we conclude:

**Theorem 8.7.** The module $F^\Lambda$ is a projective right $R^\Lambda(\beta)$-supermodule and we have a short exact sequence consisting of right projective $R^\Lambda(\beta)$-supermodules:
\[
0 \rightarrow K_1 \xrightarrow{P} K_0 \rightarrow F^\Lambda \rightarrow 0.
\]

Since $K_1$, $K_0$, and $F^\Lambda$ are the kernels of functors $\Xi_i$, $F_i$ and $F_i^\Lambda$, respectively, we have:

**Corollary 8.8.** For any $i \in I$ and $\beta \in \mathbb{Q}^+$, there exists an exact sequence of $R(\beta + \alpha_i)$-modules
\[
0 \rightarrow \Pi_i^{\Lambda_i + p(\beta)} q^{(\alpha_i | 2\Lambda - \beta)} \Xi_i^\Lambda M \rightarrow F_i^\Lambda M \rightarrow F_i^\Lambda M \rightarrow 0,
\]
which is functorial in $M \in \text{Mod}(R^\Lambda(\beta))$.

For $\alpha \in \mathbb{Q}^+$, let $\text{Proj}(R^\Lambda(\alpha))$ denote the category of finitely generated projective $\mathbb{Z}$-graded $R^\Lambda(\alpha)$-modules, and by $\text{Rep}(R^\Lambda(\alpha))$ the category of $\mathbb{Z}$-graded $R^\Lambda(\alpha)$-modules coherent over $k_0$. Then we conclude that the functors $E_i^\Lambda$ and $F_i^\Lambda$ are well-defined on $\bigoplus_{\alpha \in \mathbb{Q}^+} \text{Proj}(R^\Lambda(\alpha))$ and $\bigoplus_{\alpha \in \mathbb{Q}^+} \text{Rep}(R^\Lambda(\alpha))$:

**Theorem 8.9.** Set
\[
\text{Proj}(R^\Lambda) = \bigoplus_{\alpha \in \mathbb{Q}^+} \text{Proj}(R^\Lambda(\alpha)), \quad \text{Rep}(R^\Lambda) = \bigoplus_{\alpha \in \mathbb{Q}^+} \text{Rep}(R^\Lambda(\alpha)).
\]
Then the functors $E_i^\Lambda$ and $F_i^\Lambda$ are well-defined exact functors on $\text{Proj}(R^\Lambda)$ and $\text{Rep}(R^\Lambda)$, and they induce endomorphisms of the Grothendieck groups $[\text{Proj}(R^\Lambda)]$ and $[\text{Rep}(R^\Lambda)]$. 
Proof. By Theorem 8.7, \( F^\Lambda \) is a finitely generated projective module as a right \( R^\Lambda(\beta) \)-supermodule and as a left \( R^\Lambda(\beta + \alpha_i) \)-supermodule. Similarly, \( e(\beta, i)R^\Lambda(\beta + \alpha_i) \) is a finitely generated projective module as a left \( R^\Lambda(\beta) \)-supermodule and as a right \( R^\Lambda(\beta + \alpha_i) \)-supermodule. Now our assertions follow from these facts immediately. \( \square \)

9. Commutation relations between \( E_i^\Lambda \) and \( F_i^\Lambda \)

The main goal of this section is to show that the superfunctors \( E_i^\Lambda \) and \( F_i^\Lambda \) satisfy certain commutation relations, from which we obtain a supercategorification of \( V(\Lambda) \).

**Theorem 9.1.** For \( i \neq j \in I \), there exists a natural isomorphism

\[
E_i^\Lambda F_j^\Lambda \cong q^{-\langle \alpha_i, \alpha_j \rangle} \Pi^{p(i)p(j)} F_j^\Lambda E_i^\Lambda.
\]

**Proof.** By Proposition 5.1, we already know

\[
e(n, i)R(n + 1)e(n, j) \cong q^{-\langle \alpha_i, \alpha_j \rangle} R(n)e(n - 1, j) \otimes_{R(n-1)} \Pi^{p(i)p(j)} e(n - 1, i)R(n).
\]

Applying the functor \( R^\Lambda(n) \otimes_{R(n)} \bullet \otimes_{R(n)} R^\Lambda(n)e(\beta) \) on (9.2), we obtain

\[
e(n, i)R(n + 1)e(\beta, j)
\]

\[
= q^{-\langle \alpha_i, \alpha_j \rangle} R^\Lambda(n)e(n - 1, j) \otimes_{R^\Lambda(n - 1)} \Pi^{p(i)p(j)} e(n - 1, i)R^\Lambda(n)e(\beta)
\]

\[
\cong q^{-\langle \alpha_i, \alpha_j \rangle} \Pi^{p(i)p(j)} F_j^\Lambda E_i^\Lambda R^\Lambda(\beta).
\]

Note that

\[
E_i^\Lambda F_j^\Lambda R^\Lambda(\beta) = \left( \frac{e(n, i)R(n + 1)e(n, j)}{e(n, i)R(n + 1)a^\Lambda(x_1)R(n + 1)e(n, j)} \right) e(\beta).
\]

Thus it suffices to show that

\[
e(n, i)R(n + 1)a^\Lambda(x_1)R(n + 1)e(n, j)
\]

\[
= e(n, i)R(n)a^\Lambda(x_1)R(n + 1)e(n, j) + e(n, i)R(n + 1)a^\Lambda(x_1)R(n)e(n, j).
\]

Since, by (7.1), \( a^\Lambda(x_1)\tau_k = \pm \tau_k a^\Lambda(x_1) \) for all \( k \geq 2 \), we have

\[
R(n + 1)a^\Lambda(x_1)R(n + 1) = \sum_{a=1}^{n+1} R(n + 1)a^\Lambda(x_1)\tau_a \cdots \tau_n R(n, 1)
\]

\[
= R(n + 1)a^\Lambda(x_1)R(n, 1) + R(n + 1)a^\Lambda(x_1)\tau_1 \cdots \tau_n R(n, 1)
\]
\[= R(n + 1) a^\Lambda(x_1) R(n, 1) + \sum_{a=1}^{n+1} R(n, 1) \tau_n \cdots \tau_n a^\Lambda(x_1) \tau_1 \cdots \tau_n R(n, 1)\]
\[= R(n + 1) a^\Lambda(x_1) R(n, 1) + R(n, 1) a^\Lambda(x_1) R(n + 1) + R(n, 1) \tau_n \cdots \tau_1 a^\Lambda(x_1) \tau_1 \cdots \tau_n R(n, 1).\]

For \(i \neq j\), we get
\[e(n, i) R(n, 1) \tau_n \cdots \tau_1 a^\Lambda(x_1) \tau_1 \cdots \tau_n R(n, 1) e(n, j) = 0,\]
and our assertion (9.3) follows. \(\square\)

Let us recall the following commutative diagram, a super-version of [10, (5.5)], derived from Theorem 5.2, Theorem 5.6 and Corollary 8.8:
(9.4)
The kernels of these functors provide the following commutative diagram of \((R(\beta), R^A(\beta))\)-superbimodules:

\[
\begin{array}{cccccc}
0 & \rightarrow & L'_1 & \rightarrow & L'_0 & \rightarrow & q_i^{-2}F_i^A\Pi_iE_i^A R^A(\beta) & \rightarrow & 0 \\
0 & \rightarrow & L_1 & \rightarrow & L_0 & \rightarrow & E_i^A F_i^A R^A(\beta) & \rightarrow & 0 \\
\kappa[t_i]T_i^A \otimes \Pi_i^A R^A(\beta) & \xrightarrow{A} & \kappa[t_i] \otimes R^A(\beta) & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

where

\[
\begin{align*}
L'_0 &= q_i^{-2}R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} \Pi_i e(\beta - \alpha_i, i) R^A(\beta), \\
L'_1 &= q_i^{(\alpha_i/2A-\beta)}R(\beta)v(i, \beta - \alpha_i)T_i^A \otimes_{R(\beta - \alpha_i)} \Pi_i^{A_i + p(\beta)} e(\beta - \alpha_i, i) R^A(\beta), \\
L_0 &= e(\beta, i)R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} R^A(\beta), \\
L_1 &= q_i^{(\alpha_i/2A-\beta)}e(\beta, i)R(\beta + \alpha_i)v(i, \beta)T_i^A \otimes_{R(\beta)} \Pi_i^{A_i + p(\beta)} R^A(\beta).
\end{align*}
\]

The homomorphisms in the diagram (9.5) can be described as follows (cf. [10, §5.2]):

- \(P\) is given by (8.5). It is \((R(\beta + \alpha_i), \kappa[t_i] \otimes R^A(\beta))\)-bilinear.
- \(A\) is defined by chasing the diagram. Note that it is \(R^A(\beta)\)-linear but not \(\kappa[t_i]\)-linear.
- \(B\) is given by taking the coefficient of \(\tau_n \cdots \tau_1\). It is \((R(\beta), \kappa[t_i] \otimes R(\beta))\)-linear (see the remark below).
- \(F\) is given by \(a \otimes \pi_i b \mapsto a\tau_n \otimes b\) for \(a \in R(\beta)e(\beta - \alpha_i, i)\) and \(b \in e(\beta - \alpha_i, i) R^A(\beta)\) (See Proposition 5.1).
- \(C\) is the cokernel map of \(F\). It is \((R(\beta), R^A(\beta))\)-bilinear but does not commute with \(t_i\).

Remark 9.2. The map \(B\) can be described as

\[
B(x^j_{n+1}a\tau_n \cdots \tau_k v(i, \beta)T_i^A \otimes \pi_i^{A_i + p(\beta)} b) = \delta_{k,1}t_i^j T_i^A \otimes \pi_i^{A_i} \phi_i^{A_i}(a)b
\]
for $a \in R(\beta)$ and $b \in R^\lambda(\beta)$. Then
\[
B \left( (x_{n+1}^i a \tau_n \cdots \tau_k v(i, \beta) T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} b) t_i \right) = B \left( \delta_{k,1} (-1)^{p(i)(\Lambda_i + p(\beta))} (x_{n+1}^i a \tau_n \cdots \tau_k v(i, \beta) T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} \phi_i(b) \right) = B \left( \delta_{k,1} (x_{n+1}^i a x_{n+1} \tau_n \cdots \tau_k v(i, \beta) T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} \phi_i(b) \right) = \delta_{k,1} (x_{n+1}^i + 1 T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} \phi_i^{\Lambda_i + 1}(a) \phi_i(b)).
\]
On the other hand,
\[
B \left( (x_{n+1}^i a \tau_n \cdots \tau_k v(i, \beta) T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} b) \right) t_i = \delta_{k,1} (t_i T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} \phi_i^{\Lambda_i + 1}(a) b) t_i = \delta_{k,1} (x_{n+1}^i + 1 T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} \phi_i^{\Lambda_i + 1}(a) \phi_i(b)).
\]

Thus $B$ is right $(k[t_i] \otimes R(\beta))$-linear.

Define
\[
T = T_i^\Lambda_i \otimes \pi_i^{\Lambda_i} 1 \in k[t_i] T_i^\Lambda_i \otimes \Pi_i^\Lambda_i R^\lambda(\beta), \quad T_1 = v(i, \beta) T_i^\Lambda_i \otimes \pi_i^{\Lambda_i + p(\beta)} 1 \in L_1.
\]
The element $T$ has $\mathbb{Z}_2$-degree $p(i) \Lambda_i$ and $T_1$ has $\mathbb{Z}_2$-degree $p(i)(\Lambda_i + p(\beta))$. Note that
\[
T t_i = t_i T \quad \text{and} \quad T_1 t_i = (-1)^{p(i)p(\beta)} t_i T_1.
\]

Let $p$ be the number of $\alpha_i$ appearing in $\beta$. Define an invertible element $\gamma \in k^\times$ by
\[
(-1)^{p(i)\Lambda_i p(\beta) + p} \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{i, \nu_a}(t_i, x_a) \prod_{1 \leq a \leq n, \nu_a = i \in I_{\text{odd}}} (x_a - t_i)^2
\]
\[
\gamma^{-1} t_i^{-(h_\nu, \beta) + 2(1 + p(i))p} + \text{(terms of degree } < -(h_\nu, \beta) + 2(1 + p(i))p \text{ in } t_i).
\]
Note that $\gamma$ does not depend on $\nu \in I^\beta$.

Set $\lambda = \Lambda - \beta$ and
\[
(9.7) \quad \varphi_k = A(T t_i^k) \in k[t_i] \otimes R^\lambda(\beta).
\]

From now on, we investigate the kernel and cokernel of the map $A$ which are the key ingredients of the proof of Theorem 9.6 below. For this purpose, the following proposition is crucial.
Proposition 9.3. The element $\gamma \varphi_k$ is a monic (skew)-polynomial in $t_i$ of degree $\langle h_i, \lambda \rangle + k$.

Here and in the sequel, for $m < 0$, we say that a (skew)-polynomial $\varphi$ is a monic polynomial of degree $m$ if $\varphi = 0$.

To prove Proposition 9.3, we need some preparation. Define a map $E : L'_0 \to R^A(\beta)$ by

$$\sum_{i} a \otimes \pi_i b \mapsto a \phi_i(b) \quad \text{for} \quad a \in R(\beta)e(\beta - \alpha_i, i) \quad \text{and} \quad b \in e(\beta - \alpha_i, i)R^A(\beta).$$

We define the endomorphism $\circ(x_n \otimes 1)$ of $L'_0$ by

$$(a \otimes \pi_i b)(x_n \otimes 1) = (-1)^{p(i)} ax_n \otimes \pi_i \phi_i(b).$$

Lemma 9.4. Let

$$L'_0 := R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} \Pi_i e(\beta - \alpha_i, i)R^A(\beta).$$

Then for any $z \in L'_0$, we have

$$F(z) t_i = F(z(x_n \otimes 1)) + e(\beta, i) \otimes E(z).$$

Proof. We may assume $z = a \otimes \pi_i b$. Note that

$$F(z) = a \tau_n e(\beta - \alpha_i, i^2) \otimes b, \quad E(z) = a \phi_i(b).$$

Thus

$$F(z) t_i = a \tau_n e(\beta - \alpha_i, i^2) x_{n+1} \otimes \phi_i(b)
= a((-1)^{p(i)} x_n \tau_n + 1) e(\beta - \alpha_i, i^2) \otimes \phi_i(b)
= (-1)^{p(i)} a x_n \tau_n e(\beta - \alpha_i, i^2) \otimes \phi_i(b) + ae(\beta - \alpha_i, i^2) \otimes \phi_i(b)
= (-1)^{p(i)} F(ax_n \otimes \pi_i \phi_i(b)) + e(\beta, i) \otimes E(z)
= F(z(x_n \otimes 1)) + e(\beta, i) \otimes E(z).$$

By Proposition 5.1, we have

$$e(\beta, i)R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} R^A(\beta)
= F((R(\beta)e(\beta - \alpha_i, i) \otimes_{R(\beta - \alpha_i)} e(\beta - \alpha_i, i)R^A(\beta)))
\oplus e(\beta, i)(k[t_i] \otimes R^A(\beta)).$$
Using the decomposition (9.10), we may write

\[(9.11)\]

\[P(e(\beta, i)\tau_n \cdots \tau_1 T_1 t_i^k) = F(\psi_k) + e(\beta, i)\varphi_k\]

for uniquely determined \(\psi_k \in L'_0\) and \(\varphi_k \in k[t_i] \otimes R^A(\beta)\). On the other hand, we have

\[A(T_i^k) = AB(e(\beta, i)\tau_n \cdots \tau_1 T_1 t_i^k) = CP(e(\beta, i)\tau_n \cdots \tau_1 T_1 t_i^k) = \varphi_k.\]

Hence the definition of \(\varphi_k\) coincides with the definition given in (9.7). Note that

\[F(\psi_{k+1}) + e(\beta, i)\varphi_{k+1} = P(e(\beta, i)\tau_n \cdots \tau_1 T_1 t_i^{k+1}) = P(e(\beta, i)\tau_n \cdots \tau_1 T_1 t_i^k) t_i\]

\[= (F(\psi_k) + e(\beta, i)\varphi_k) t_i = F(\psi_k(x_n \otimes 1)) + e(\beta, i)E(\psi_k) + e(\beta, i)\varphi_k t_i,\]

which yields

\[(9.12)\]

\[\psi_{k+1} = \psi_k(x_n \otimes 1), \quad \varphi_{k+1} = E(\psi_k) + \varphi_k t_i.\]

Now we will prove Proposition 9.3. By Corollary 8.6, the equality

\[g_n \cdots g_1 x_1^e(i, \nu)a^A(x_1)\tau_1 \cdots \tau_n\]

\[= (-1)^{(k + \Lambda_i)p(i)p(\beta)}a_i^A(x_{n+1}) \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{\nu_a,i}(x_a, x_{n+1}) \prod_{1 \leq a \leq n, \nu_a = i \in I_{\text{odd}}} (x_a - x_{n+1})^2 e(\nu, i)\]

holds in \(R(\beta + \alpha_i)e(\beta, i) \otimes R(\beta) R^A(\beta)\), which implies

\[AB(g_n \cdots g_1 x_1^e(i, \nu)T_1) = C\left((-1)^{(k + \Lambda_1)p(i)p(\beta)}x_{n+1}^A(x_{n+1}) \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{\nu_a,i}(x_a, x_{n+1}) \prod_{1 \leq a \leq n, \nu_a = i \in I_{\text{odd}}} (x_a - x_{n+1})^2 e(\nu, i)\right)\]

\[= (-1)^{(k + \Lambda_1)p(i)p(\beta)}t_i^k a_i^A(t_i) \prod_{1 \leq a \leq n, \nu_a \neq i} Q_{\nu_a,i}(x_a, t_i) \prod_{1 \leq a \leq n, \nu_a = i \in I_{\text{odd}}} (x_a - t_i)^2 e(\nu).\]

On the other hand, since \(B\) is the map taking the coefficient of \(\tau_n \cdots \tau_1\), we have

\[B(g_n \cdots g_1 x_1^e(i, \nu)T_1) = B\left((-1)^{k p(i)p(\beta)}x_{n+1}^e(i, \nu)_{\nu_a = i} - (x_{n+1}^{1+p(i)} - x_{a}^{1+p(i)})^2 e(\nu, i)\tau_n \cdots \tau_1 T_1\right)\]
Thus we have

\[ A(t_i^k \prod_{\nu_a = i} (t_i^{1+p(i)} - x_a^{1+p(i)})^2 T_e(\nu)). \]

(9.13) 

Set

\[ S_i = \sum_{\nu \in I^\beta} \prod_{\nu_a = i} (t_i^{1+p(i)} - x_a^{1+p(i)})^2 T_e(\nu) \in k[t_i] \otimes R^A(\beta), \]

\[ F_i = \gamma(-1)^{A, p(i)p(\beta) + p} a_i(t_i) \sum_{\nu \in I^\beta} \prod_{\nu_a = i} \mathcal{Q}_{\nu_a, i}(x_a, t_i) \prod_{1 \leq a \leq n, \nu_a \neq i} \prod_{1 \leq a \leq n, \nu_a = i} (x_a - t_i)^2 e(\nu) \]

\[ \in k[t_i] \otimes R^A(\beta). \]

Then they are monic (skew)-polynomials in \( t_i \) of degree \( 2(1 + p(i))p \) and \( \langle h_i, \lambda \rangle + 2(1 + p(i))p \), respectively. Note that \( S_i \) is contained in the center of \( k[t_i] \otimes R^A(\beta) \) and \( F_i \) commutes with \( t_i \). Hence (9.13) can be expressed in the following form:

(9.14) 

\[ \gamma A(t_i^k S_i T) = t_i^k F_i. \]

Lemma 9.5. For any \( k \geq 0 \), we have

(9.15) 

\[ t_i^k F_i = (\gamma \varphi_k) S_i + h_k, \]

where \( h_k \in k[t_i] \otimes R^A(\beta) \) is a polynomial in \( t_i \) of degree \( < 2(1 + p(i))p \). In particular, \( \gamma \varphi_k \) coincides with the quotient of \( t_i^k F_i \) by \( S_i \).

Proof. By (9.12),

(9.16) 

\[ A(at_i) - A(a)t_i \in k[t_i] \otimes R^A(\beta) \] is of degree \( \leq 0 \) in \( t_i \),

for any \( a \in k[t_i] T_i^A \otimes \Pi_i A^\beta \). We will show

(9.17) 

for any polynomial \( f \) in the center of \( k[t_i] \otimes R^A(\beta) \) in \( t_i \) of degree \( m \in \mathbb{Z}_{\geq 0} \) and \( a \in k[t_i] T_i^A \otimes R^A(\beta), A(af) - A(a)f \) is of degree \( < m \).
We will use induction on $m$. Since $A$ is right $R^\Lambda(\beta)$-linear, (9.17) holds for $m = 0$. Thus it suffices to show (9.17) when $f = t_ig$. By the induction hypothesis, (9.17) is true for $g$. Then we have

$$A(af) - A(a)f = (A(at_ig) - A(at_ig) + (A(at_i) - A(a)t_ig).$$

It follows that the first term is of degree $< \deg(g)$ in $t_i$ and the second term is of degree $< \deg(g) + 1$, which proves (9.17). Thus we have

$$t^k_i \gamma^{-1} F_i - \varphi_k S_i = t^k_i \gamma^{-1} F_i - A(t^k_i T_i)S_i = A(t^k_i S_iT_i) - A(t^k_i T_i)S_i$$

by (9.14) and it is of degree $< 2(1 + p(i)p$ by applying (9.17) for $f = S_i$. □

Therefore, by Lemma 9.5, we conclude $\gamma \varphi_k$ is a monic (skew)-polynomial in $t_i$ of degree $\langle h_i, \lambda \rangle + k$, which completes the proof of Proposition 9.3.

**Theorem 9.6.** Let $\lambda = \Lambda - \beta$. Then there exist natural isomorphisms of endofunctors on $\text{Mod}(R^\Lambda(\beta))$ given below.

(i) If $\langle h_i, \lambda \rangle \geq 0$, then we have

\begin{equation}
\Pi_i q_i^{-2} F_i^{(h_i, \lambda)^{-1}} \otimes R^\Lambda(\beta) \sim E_i F_i^{(h_i, \lambda)^{-1}}.
\end{equation}

(ii) If $\langle h_i, \lambda \rangle < 0$, then we have

\begin{equation}
\Pi_i q_i^{-2} F_i^{(h_i, \lambda)^{-1}} \otimes R^\Lambda(\beta) \sim E_i F_i^{(h_i, \lambda)^{-1}} \otimes \bigoplus_{k=0}^{\langle h_i, \lambda \rangle - 1} \Pi_i^{k+1} q_i^{-2(k+2)}.
\end{equation}

**Proof.** Due to Proposition 9.3 and (9.12), we can apply the arguments in [10, Theorem 5.2] with a slight modification. Hence we will give only a sketch of proof.

From the Snake Lemma, we get an exact sequences of $R^\Lambda(\beta)$-superbimodules:

$$0 \to \text{Ker} A \to \Pi_i q_i^{-2} F_i^{(h_i, \lambda)^{-1}} \Pi_i E_i^\Lambda R^\Lambda(\beta) \to E_i^\Lambda F_i^\Lambda R^\Lambda(\beta) \to \text{Coker} A \to 0.$$ 

If $a := \langle h_i, \lambda \rangle \geq 0$, then Proposition 9.3 yields

$$\text{Ker} A = 0, \bigoplus_{k=0}^{a-1} \Pi_i^{k+1} q_i^{-2(k+2)} \otimes R^\Lambda(\beta) \simeq \text{Coker} A$$

and our first assertion follows.
If \( a := \langle h_i, \lambda \rangle < 0 \), then Proposition 9.3 implies \( \text{Coker}A = 0 \). By (9.12), we can prove that there is an isomorphism

\[
\text{Ker} A \cong \bigoplus_{k=0}^{-a-1} k t_i^k \otimes R^A(\beta),
\]

which completes the proof. \( \square \)

10. Supercategorification

In this section, applying the results obtained in the previous sections, we will show that the quiver Hecke superalgebra \( R(\beta) \) and its cyclotomic quotient \( R^A(\beta) \) \((\beta \in \mathbb{Q}^+)\) give supercategorifications of \( U^-_\Lambda(\mathfrak{g}) \) and \( V^A(\Lambda) \), respectively.

From now on, we assume (4.15); i.e., \( k_0 \) is a field and the \( k_i \)'s are finite-dimensional over \( k_0 \). Set

\[
\text{Proj}(R^A) = \bigoplus_{\beta \in \mathbb{Q}^+} \text{Proj}(R^A(\beta)) \quad \text{and} \quad \text{Rep}(R^A) = \bigoplus_{\beta \in \mathbb{Q}^+} \text{Rep}(R^A(\beta)).
\]

Recall the anti-involution \( \psi : R^A(\beta) \to R^A(\beta) \) given by (6.3). For \( N \in \text{Mod}(R^A(\beta)) \), let \( N^\psi \) be the right \( R^A(\beta) \)-module obtained from \( N \) by the anti-involution \( \psi \) of \( R^A(\beta) \). By (4.16) and Theorem 6.4, we have the pairing

\[
([\text{Proj}(R^A)] \times [\text{Rep}(R^A)] \to \mathbb{A}
\]
given by

\[
([P], [M]) \mapsto \sum_{n \in \mathbb{Z}} q^n \dim_{k_0} (P^\psi \otimes_{R^A} M)_n.
\]

Lemma 6.3 implies

**Lemma 10.1.** The Grothendieck groups \([\text{Proj}(R^A)]\) and \([\text{Rep}(R^A)]\) are \( \mathbb{A} \)-dual to each other by this pairing.

From Theorem 8.9, we can define endomorphisms \( E_i \) and \( F_i \), induced by \( E^A_i \) and \( F^A_i \), on the Grothendieck groups \([\text{Proj}(R^A)]\) and \([\text{Rep}(R^A)]\) as follows:

\[
[\text{Proj}(R^A(\beta))] \xrightarrow{F_i := [F^A_i]} [\text{Proj}(R^A(\beta + \alpha_i))],
\]

\[
E_i := [q_i^{1-\langle h_i, \Lambda - \beta \rangle} E^A_i]
\]
\[ \begin{array}{c}
\text{[Rep}(R^\Lambda(\beta))]) \xrightarrow{F_i} \text{[Rep}(R^\Lambda(\beta + \alpha_i))].
\end{array} \]

Then, from the isomorphisms (9.1), (9.18), (9.19) and Theorem 6.4, we obtain the following identities in [Proj](R^\Lambda(\beta)) and [Rep](R^\Lambda(\beta)):

\[ E_i F_j = F_j E_i \quad \text{if } i \neq j, \]

\[ E_i F_i = F_i E_i + \frac{q^{(h_i, \Lambda - \beta) - (h_i, \Lambda - \beta)}}{q_i - q_i^{-1}} \quad \text{if } \langle h_i, \Lambda - \beta \rangle \geq 0, \]

\[ E_i F_i + \frac{q^{(h_i, \Lambda - \beta) - (h_i, \Lambda - \beta)}}{q_i - q_i^{-1}} = F_i E_i \quad \text{if } \langle h_i, \Lambda - \beta \rangle \leq 0. \]

Let \( K_i \) be an endomorphism on [Proj](R^\Lambda(\beta)) and [Rep](R^\Lambda(\beta)) given by

\[ K_i|_{\text{Proj}(R^\Lambda(\beta))} := q_i^{(h_i, \Lambda - \beta)}, \quad K_i|_{\text{Rep}(R^\Lambda(\beta))} := q_i^{(h_i, \Lambda - \beta)}. \]

Then (10.2) can be rewritten as the commutation relation (Q3) in Definition 2.1:

\[ [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}. \]

Define the superfunctors \( E_i^{\Lambda(n)} \) and \( F_i^{\Lambda(n)} \),

\[ \text{Mod}(R^\Lambda(\beta)) \xrightarrow{E_i^{\Lambda(n)}} \text{Mod}(R^\Lambda(\beta + n\alpha_i)) \]

by

\[ E_i^{\Lambda(n)} : N \mapsto (R^\Lambda(\beta) \otimes P(i^n))^\psi \otimes_{R^\Lambda(\beta) \otimes R(n\alpha_i)} e(\beta, i^n) N, \]

\[ F_i^{\Lambda(n)} : M \mapsto R^\Lambda(\beta + n\alpha_i) e(\beta, i^n) \otimes_{R^\Lambda(\beta) \otimes R(n\alpha_i)} (M \boxtimes P(i^n)) \]

for \( M \in \text{Mod}(R^\Lambda(\beta)) \) and \( N \in \text{Mod}(R^\Lambda(\beta + n\alpha_i)). \) Then they induce the endomorphism \( E_i^n/n_i! \) and \( F_i^n/n_i! \) by (4.21).

Note that

(i) the action of \( E_i \) on [Proj](R^\Lambda) and [Rep](R^\Lambda) is locally nilpotent,

(ii) if the module \([M]\) in [Rep](R^\Lambda(\beta)) satisfies \( E_i[M] = 0 \) for all \( i \in I, \)

then \( \beta = 0. \)

By Corollary 7.5, we see that
the action $F_i$ on $[\text{Proj}(R^A)]$ and $[\text{Rep}(R^A)]$ are locally nilpotent. Therefore, by (10.3) and [19, Proposition B.1], the endomorphisms $F_i$ and $E_i$ satisfy the quantum Serre relation (Q4) in Definition 2.1. Hence $[\text{Proj}(R^A)]$ and $[\text{Rep}(R^A)]$ are endowed with a $U_A(g)$-module structure.

Note that $[\text{Proj}(R)] := \bigoplus_{\beta \in Q^+} [\text{Proj}(R(\beta))]$ and $[\text{Rep}(R)] := \bigoplus_{\beta \in Q^+} [\text{Rep}(R(\beta))]$ are also $A$-dual to each other. The exact functors $E_i: \text{Rep}(R(\beta + \alpha_i)) \rightarrow \text{Rep}(R(\beta))$ and $F'_i: \text{Rep}(R(\beta)) \rightarrow \text{Rep}(R(\beta + \alpha_i))$ defined in (6.1) induce endomorphisms $E'_i$ and $F'_i$ on $[\text{Proj}(R)]$, respectively. Hence, (6.5) implies the following commutation relation in $[\text{Rep}(R)]$:

\begin{equation}
E'_i F'_j = q^{-(\alpha_i | \alpha_j)} F'_j E'_i + \delta_{i,j}.
\end{equation}

Similarly, we define

\[
\text{Proj}(R(\beta)) \xrightarrow{F_i} \text{Proj}(R(\beta + \alpha_i)) \quad \text{and} \quad \text{Proj}(R(\beta)) \xleftarrow{E'_i} \text{Proj}(R(\beta + \alpha_i))
\]

by

\[
F_i P := R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} P \quad \text{and} \quad E'_i Q := \frac{e(\beta, i)R(\beta + \alpha_i)}{e(\beta, i)x_{n+1}R(\beta + \alpha_i)} \otimes_{R(\beta + \alpha_i)} Q,
\]

where $|\beta| = n$. Then they are well-defined on $\text{Proj}(R)$, and we obtain an exact sequence

\[
0 \rightarrow \delta_{i,j} \text{ id} \rightarrow E'_i F_j \rightarrow \Pi^{(i)p(j)} q^{-(\alpha_i | \alpha_j)} F'_j E'_i \rightarrow 0.
\]

Thus the exact functors induce the endomorphisms $E'_i$ and $F'_i$ on $[\text{Proj}(R)]$ and satisfy the same equation in (10.5). (See [14, Lemma 5.1], for more details.)

Let $\text{Irr}_0(R^A(\beta))$ be the set of isomorphism classes of self-dual irreducible $R^A(\beta)$-modules, and $\text{Irr}_0(R^A) := \bigsqcup_{\beta \in Q^+} \text{Irr}_0(R^A(\beta))$. Then \{ $[S] \mid S \in \text{Irr}_0(R^A)$ \} is a strong perfect basis of $[\text{Rep}(R^A)]$ by Theorem 6.8. By Proposition 2.10, (10.4 ii) and Lemma 10.1, we conclude:

**Theorem 10.2.** For $\Lambda \in \text{P}^+$, we have

\begin{equation}
V_A(\Lambda)^\vee \simeq [\text{Rep}(R^A)] \quad \text{and} \quad V_A(\Lambda) \simeq [\text{Proj}(R^A)]
\end{equation}

as $U_A(g)$-modules.
The fully faithful exact functor $\text{Rep}(R^\Lambda(\beta)) \to \text{Rep}(R(\beta))$ induces an $A$-linear homomorphism $[\text{Rep}(R^\Lambda)] \to [\text{Rep}(R)]$. Hence $[\text{Rep}(R^\Lambda)] \to [\text{Proj}(R^\Lambda)]$ is surjective.

Denote by $B^\text{low}_A(g)$ (resp. $B^\text{up}_A(g)$) the $A$-subalgebra of $B_q(g)$ generated by $e'_i$ and $f^{(n)}_i$ (resp. by $e'_i/\lbrack n_i \rbrack!$ and $f_i$) for all $i \in I$ and $n \in \mathbb{Z}_{>0}$.

As a $U^-_A(g)$-module, $U^-_A(g)$ is the projective limit of $V^-_A(\Lambda)$. Hence, Theorem 10.2 implies the following corollary:

**Corollary 10.3.** There exist isomorphisms:

$$U^-_A(g)^\vee \simeq [\text{Rep}(R)] \quad \text{as a } B^\text{up}_A(g)-\text{module},$$

$$U^-_A(g) \simeq [\text{Proj}(R)] \quad \text{as a } B^\text{low}_A(g)-\text{module}.$$
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