Approximating the discrete and continuous median line segments in $d$ dimensions

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Abstract

Consider a set $P$ of $n$ points in $\mathbb{R}^d$. In the discrete median line segment problem, the objective is to find a line segment bounded by a pair of points in $P$ such that the sum of the Euclidean distances from $P$ to the line segment is minimized. In the continuous median line segment problem, a real number $\ell > 0$ is given, and the goal is to locate a line segment of length $\ell$ in $\mathbb{R}^d$ such that the sum of the Euclidean distances between $P$ and the line segment is minimized.

To begin with, we show how to compute $(1 + \epsilon \Delta)$- and $(1 + \epsilon)$-approximations to a discrete median line segment in time $O(n^{1+\epsilon} \log n)$ and $O(n^2 \epsilon^{-d})$, respectively, where $\Delta$ is the spread of line segments spanned by pairs of points. While developing our algorithms, by using the principle of pair decomposition, we derive new data structures that allow us to quickly approximate the sum of the distances from a set of points to a given line segment or point. To our knowledge, our utilization of pair decompositions for solving minsum facility location problems is the first of its kind – it is versatile and easily implementable.

Furthermore, we prove that it is impossible to construct a continuous median line segment for $n \geq 3$ non-collinear points in the plane by using only ruler and compass. In view of this, we present an $O(n^{2/3} \epsilon^{-d})$-time algorithm for approximating a continuous median line segment in $\mathbb{R}^d$ within a factor of $1 + \epsilon$. The algorithm is based upon generalizing the point-segment pair decomposition from the discrete to the continuous domain. Last but not least, we give an $(1 + \epsilon)$-approximation algorithm for solving the constrained median line segment problem in $\mathbb{R}^2$ where an endpoint or the slope of the median line segment is given at input. These algorithms are space-subdivision and prune-and-search approaches, and their time complexities are sub-quadratic in $n$.

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1 Introduction

In this paper, we consider the following two problems in facility location theory.

- **Discrete median line segment problem.** Given a set $P$ of $n$ points in $\mathbb{R}^d$, locate a line segment bounded by two points of $P$ such that the sum of the Euclidean distances from $P$ to the line segment is minimized.

- **Continuous median line segment problem.** Given a set $P$ of $n$ points in $\mathbb{R}^d$ and a positive real number $\ell$, locate a line segment of length $\ell$ such that the sum of the Euclidean distances from $P$ to the line segment is minimized.

For simplicity of discussion, the median line segment problem refers to the continuous form of the problem, unless indicated otherwise.

**Related work** The median line segment problem is closely related to one of the oldest non-trivial problems in facility location theory – the (generalized) Fermat-Torricelli problem,
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which asks to find a point with the minimal sum of distances to a given set $P$ of $n$ points. The optimal point is referred to as the Fermat-Toricelli point or simply the (geometric) median. For $n \geq 5$ points in general position, the Fermat-Toricelli point cannot be constructed by strict usage of ruler and compass [3, 16]. In other words, the Fermat-Toricelli problem is unsolvable by radicals over the field of rationals. Consequently, no exact algorithm exists under computational models with basic arithmetic operations and the extraction of $k$-th roots. This leaves us with only numerical or symbolic approximation methods for $n \geq 5$ points (e.g., see [3, 7, 8]). Furthermore, it remains unclear whether the problem is in $NP$.

In the discrete form of the Fermat-Toricelli problem, the median is commonly known as the medoid, which refers to the point in $P$ that has the minimal sum of distance to $P$. One can find the medoid of $P$ by simply computing all $\binom{n}{2}$ pairwise distances. However, it has been argued that an exact algorithm does not exist for solving the medoid problem in $o(n^2)$ time [18]. Many approaches have thus been developed to compute the medoid approximately (or exactly under statistical assumptions) in sub-quadratic time (e.g., see [2, 10, 18, 14]).

**Our results** In either of our problems, given an $\epsilon > 0$, a line segment $s$ is said to be a $(1 + \epsilon)$-approximate solution if the sum of the distances from $s$ to $P$ is at most $(1 + \epsilon)$ times the sum of the distances from the optimal line segment to $P$.

**Discrete median line segment problem.** For points in the plane ($d = 2$), it is possible to compute a discrete median segment exactly in $O(n^2)$ time using a rotational sweep algorithm as proposed by Daescu and Teo [9]. However, there seems to be no obvious generalization of the exact method to higher dimensions ($d \geq 3$). Ergo, we strive to obtain efficient geometric approximation algorithms for solving the discrete median line segment problem in $\mathbb{R}^d$.

We begin in Section 2.1 by proposing an algorithm for computing a $(1 + \epsilon)$-approximate discrete median line segment in time $O(n^2\epsilon^{-d}\log n)$. Our algorithm uses a new data structure named well separated subset decomposition (WSSD), which extends on the classical idea of the well separated pair decomposition (WSPD) by Callahan [6]. WSSD partitions $P$ into $O(\log n)$ clusters that preserve the distances of $P$ to each of the $\binom{n}{2}$ candidate line segments.

In Section 2.2 we improve upon our $O(n^2\epsilon^{-d}\log n)$-time algorithm by taking an additional step and representing the $\Theta(n^2)$ candidate line segments in a linear-sized WSPD. By utilizing both WSSD and WSPD, the resulting approach yields a $(1 + \epsilon\Delta)$-approximation in time $O(ne^{-2d}\log n)$, where $\Delta$ is the “spread” of the candidate line segments (see Preliminaries for detailed definition).

Finally, in Section 2.3 we directly encode the distances between $P$ and line segments spanned by pairs of points of $P$ in a single novel quadratic-sized data structure called well separated pair decomposition for point-segment pairs (WSPD-PS). WSPD-PS is a careful modification of the standard WSPD, and it allows us to compute a $(1 + \epsilon)$-approximate discrete median line segment in time $O(n^2\epsilon^{-d})$, provided that the pairwise distances associated with the well separated point-segment pairs are computed and summed in the right order. Moreover, by using WSPD-PS, we can compute the closest distance between $P$ and line segments spanned by pairs of points of $P$ in time $O(n^2)$, which is a poly-logarithmic factor improvement over the previously best-known result given by Goswami et al. [12].

As a byproduct of our approaches above, new deterministic $(1 + \epsilon)$-approximation algorithms are feasible for solving the medoid problem in time either $O(ne^{-d}\log n)$ or $O(n\log n + ne^{-d})$. We speculate that our approaches can be further generalized to solve any discrete median problem involving a facility of constant geometric complexity.
Continuous median line segment problem. We prove that it is impossible to construct a median line segment for \( n \geq 3 \) non-collinear points in \( \mathbb{R}^2 \) by using only ruler and compass (Section 3). Hence, we present an \( O(n^d \epsilon^{-d}) \)-time algorithm for computing a \((1 + \epsilon)\)-approximate median line segment in \( d \) dimensions (Section 4). This algorithm is based on generalizing WSPD-PS to represent the distances between \( P \) and line segments of length \( \ell \) in \( \mathbb{R}^d \).

In addition, we address the median line segment problem with different geometric constraints (Section 5). Particularly, a \((1 + \epsilon)\)-approximation algorithm is given for solving the point-anchored median line segment problem in the plane. In this constrained problem, an endpoint of the median line segment is given as part of the input. By essentially dividing the space around the anchor point into \( O(n) \) intervals with certain geometric properties, our algorithm finds an approximate solution in \( O(n \epsilon^{-2} \alpha_d^{-1}) \) time, where \( \alpha_d \) is a parameter dependent on the coordinates of \( P \). Furthermore, we provide an algorithm for computing a \((1 + \epsilon)\)-approximate constant-slope median line segment in \( \mathbb{R}^2 \), where the slope of the median line segment is fixed at input. Our algorithm is a tailored extension of the prune-and-search approach given by Bose et al. [5], and its running time is \( O(k n \log n) \), where \( k = \frac{2\pi}{\cos^{-1}(1+\epsilon)} \).

Preliminaries Before discussing our results further, we formally define some of the notations used in the paper (see Appendix A for additional definitions used only in the appendices).

For any two points \( a \) and \( b \) in \( \mathbb{R}^2 \), let \( ab \) denote the line segment bounded by \( a \) and \( b \), and let \( d(a, b) = ||ab|| \) be the Euclidean distance between \( a \) and \( b \). Furthermore, for any point \( c \) in \( \mathbb{R}^d \), we denote by \( d(c, ab) \) the shortest distance between point \( c \) and line segment \( ab \).

For a given point set \( P \), we denote by \( D \) the diameter of \( P \); that is, \( D = \max\{d(a, b) : a, b \in P\} \). Let \( E = \min\{d(c, ab) : a, b, c \in P \text{ and } a, b \neq c\} \). We define \( \Delta = D/E \) to be the “spread” of the line segments spanned by pairs of points in \( P \).

For simplicity of arguments, we assume that point set \( P \) has been scaled and translated so that \( P \) lies within a unit hypercube \([0, 1]^d\), unless stated otherwise.

2 Approximating the discrete median line segment

A set \( P \) of \( n \) points in Euclidean space defines a set of \( \binom{n}{2} = \Theta(n^2) \) distinct candidate line segments, for each of which we need, as it would seem at a minimum, to compute \( n - 2 \) distances, resulting in a total of \( \Omega(n^3) \) computations.

2.1 \( O(n^2 \epsilon^{-d} \log n) \)-time \((1 + \epsilon)\)-approximation

We propose an \( O(n^2 \epsilon^{-d} \log n) \)-time approximation algorithm that involves the following partitioning scheme. With respect to each candidate line segment, we partition \( P \) into \( O(\log n) \) point sets according to their distances to the candidate line segment such that each point set can be approximated by a single representative point. We call the resulting partition a well separated subset decomposition (WSSD).

Well separated subset decomposition (WSSD)

Let \( C \) denote a subset of \( P \). Define \( s > 0 \) to be a parameter called separation factor. With respect to a candidate line segment \( ab \), where \( a, b \in P \) and \( a \neq b \), if the points of \( C \) can be enclosed within a Euclidean ball of radius \( r \) such that the closest distance from this ball to \( ab \) is at least \( sr \), then \( C \) is said to be \( s \)-well separated from \( ab \) (Figure 1).

Definition 2.1 (WSSD). Given a set \( P \) of \( n \) points, a line segment \( ab \), and a separation factor \( s > 0 \), an \( s \)-WSSD with respect to \( ab \) is defined as a collection of subsets of \( P \), denoted
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**Figure 1** A well separated subset $C$ with respect to a line segment $ab$.

by $\{C_1, C_2, ..., C_m\}$, such that (I) $C_i \subseteq P$ for $1 \leq i \leq m$, (II) $C_i \cap C_j = \emptyset$ for $1 \leq i, j \leq m$ and $i \neq j$, (III) $\bigcup_{i=1}^{m} C_i = P$, and (IV) $C_i$ is $s$-well separated from $ab$ for $1 \leq i \leq m$.

An $s$-WSSD can be constructed from either a kd-tree [4] or a balanced box decomposition (BBD) tree [1]. Both of these data structures are based on the hierarchical subdivision of space into rectilinear regions called cells. For a set $P$ of $n$ points in $\mathbb{R}^d$, it is possible to build, in time $O(n \log n)$, an optimized kd-tree [11] or a BBD-tree with height $O(\log n)$ and space $O(n)$. In either tree, each non-terminal node has two children, and each leaf node contains a single point. Unlike a kd-tree, the cells of a BBD-tree have a bounded aspect ratio, and the sizes of the cells decrease by (at least) a factor of 1/2 with each descent of $2^d$ levels in the tree. For a brief overview of the two tree data structures, please refer to Appendix B.

**Theorem 2.2.** For a set $P$ of $n$ points and any $s > 0$, with respect to a line segment $ab$, one can construct an $s$-WSSD of size $O(s^d \log n)$ in time $O(n \log n + s^d \log n)$.

**Proof.** We begin by building a kd-tree or a BBD-tree for $P$. Each leaf node, which contains a single point, is treated as having an infinitesimally small cell containing its point.

The construction of an $s$-WSSD, with respect to a given line segment $ab$, is based on a recursive process. Throughout the construction, we maintain a collection of sets that satisfy properties (I), (II), and (III) as stated in Definition 2.1. When the procedure terminates, all the sets generated will fulfill property (IV). Each set of the $s$-WSSD will be encoded as a node in the kd-tree or BBD-tree.

Let $u$ denote a node in either tree. Consider the smallest Euclidean ball that encloses the cell of node $u$. If the ball is $s$-well separated from line segment $ab$, then we report node $u$ as an $s$-well separated subset. Otherwise, we apply the procedure recursively to each child node $u_i$ of $u$. A formal description of the procedure is given below as a pseudocode.

**Algorithm** $\text{WSSD}(u, ab, s)$
1. if ($u$ is $s$-well separated from $ab$) then return $u$;
2. else return $\text{WSSD}(u_i, ab, s)$ for each child $u_i$ of $u$;

To evaluate the number of $s$-well separated subsets generated in this recursive process, we use a packing argument, which slightly differs depending on either an optimized kd-tree or a BBD-tree is used as the basis for the construction of the $s$-WSSD. However, in both cases, the asymptotic upper bound on the number of $s$-well separated subsets generated is $O(s^d \log n)$, with the distinction that the upper bound applies to the worst case for the BBD-tree, whereas the upper bound is derived with respect to the average (expected) case for the kd-tree (see Appendix C for details). Together with $O(n \log n)$ time to build either tree, the overall running time is $O(n \log n + s^d \log n)$.

We now describe a technical lemma associated with an $s$-WSSD, which will be used later in approximating a discrete median line segment.
Lemma 2.3 (WSSD Utility Lemma). If subset $C$ is s-well separated from line segment $ab$, and $c, c' \in C$, then we have $d(c', ab) \leq (1 + \frac{4}{s})d(c, ab)$.

Proof. The above relationship between $d(c', ab)$ and $d(c, ab)$ can be obtained using a proof by exhaustion, which involves analyzing a finite number of mutually exclusive cases depending on the locations of $c$ and $c'$ with respect to $ab$. For details, please refer to Appendix D.

WSSD-based approximation

This section discusses the usage of a WSSD for approximating a discrete median line segment for $P$. We present the arguments only for the WSSD constructed from a BBD-tree, since the analysis is similar for the case of using a kd-tree, aside from that the resulting time complexity would be of the average case instead of the worst.

Theorem 2.4. Given a set $P$ of $n$ points in $\mathbb{R}^d$, for any $\epsilon > 0$, a $(1 + \epsilon)$-approximation to a discrete median line segment for $P$ can be computed in time $O(n^2 \epsilon^{-d} \log n)$.

Proof. We begin by constructing an s-WSSD with respect to each of the $\Theta(n^2)$ candidate line segments. The total construction time is bounded by $O(n \log n + n^2 s^d \log n) = O(n^2 s^d \log n)$.

We make a small augmentation to the construction of the WSSD as follows. When building a BBD-tree, we associate each node $u$ of the tree with a quantity $|u|$ indicating the number of points lying within its cell. When we output a node $u$ as an s-well separated subset with respect to a line segment $ab$ in the decomposition process, we report $|u|$ and the farthest point within the cell of node $u$ from $ab$ (which is not necessarily a point of $P$). Since the farthest point within a hypercube from $ab$ is one of the $2^d$ vertices of the hypercube, we can find the farthest point in $O(2^d)$ time, which is just $O(1)$ given that $d$ is treated as a constant. Thus, the overall running time for the construction of the WSSD remains the same as before.

Let $\{C_i : 1 \leq i \leq m_C\}$ be the collection of s-well separated subsets with respect to a line segment $ab$. Let $\alpha(C_i)$ denote the farthest point within the cell containing $C_i$ from line segment $ab$, and let $|C_i|$ be the number of points in $C_i$. With respect to each candidate line segments $ab$, where $a, b \in P$ and $a \neq b$, we compute the distance sum $\sum_i |C_i| \cdot d(\alpha(C_i), ab)$, and output the line segment $ab$ achieving the smallest distance sum.

Suppose that the aforementioned approach yields line segment $ab$ as an approximate discrete median line segment for $P$. Then, by Lemma 2.3, for each s-well separated subset $C_i$ with respect to $ab$, we have $|C_i| \cdot d(\alpha(C_i), ab) \leq \sum_{c \in C_i} (1 + \frac{4}{s})d(c, ab)$. Since $\alpha(C_i)$ is the farthest point within the cell containing $C_i$ from $ab$, by summing over all $i$, we obtain

$$\sum_{p \in P} d(p, ab) \leq \sum_i |C_i| \cdot d(\alpha(C_i), ab) \leq \left(1 + \frac{4}{s}\right)\sum_{p \in P} d(p, ab)$$

Let $xy$ denote an exact discrete median segment for $P$, where $x, y \in P$ and $x \neq y$. Let $\{Z_i : 1 \leq i \leq m_Z\}$ be the set of s-well separated subsets with respect to $xy$. We then have

$$\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq \sum_i |C_i| \cdot d(\alpha(C_i), ab) \leq \sum_i |Z_i| \cdot d(\alpha(Z_i), xy) \leq \left(1 + \frac{4}{s}\right)\sum_{p \in P} d(p, xy)$$

Given any $\epsilon > 0$, we set $s = 4/\epsilon$. Then, we obtain $\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq (1 + \epsilon)\sum_{p \in P} d(p, xy)$. This implies that the output line segment $ab$ is a $(1 + \epsilon)$-approximation to a discrete median line segment for $P$. The overall running time is bounded by $O(n^2 (4/\epsilon)^d \log n) = O(n^2 \epsilon^{-d} \log n)$, which is $O(n^2 \log n)$ if we treat $\epsilon$ as a constant.
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**Remark.** The approach described above could be used to produce a \((1 + \epsilon)\)-approximation to the medoid, in which case an \(s\)-WSSD is constructed with respect to each candidate point \(p \in P\) (instead of each candidate line segment). The running time would be \(O(n\epsilon^{-d}\log n)\).

### 2.2 \(O(n\epsilon^{-2d} \log n)\)-time \((1 + \epsilon\Delta)\)-approximation

In this section, we derive an \(O(n\epsilon^{-2d} \log n)\)-time approximation algorithm with a performance guarantee dependent on \(\Delta\). To accomplish that, we encode the \(\Theta(n^2)\) candidate line segments approximately using WSPD in addition to the WSSD described in the prior section.

#### Well separated pair decomposition (WSPD)

**Definition 2.5 (WSPD).** For a set \(P\) of \(n\) points and a separation factor \(s > 0\), an \(s\)-WSPD is a collection of pairs of subsets of \(P\), denoted as \(\{\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}\}\), such that (I) \(A_i, B_i \subseteq P\) for \(1 \leq i \leq m\), (II) \(A_i \cap B_i = \emptyset\) for \(1 \leq i \leq m\), (III) \(\bigcup_{i=1}^m A_i \otimes B_i = P\otimes P\), and (IV) \(A_i\) and \(B_i\) are \(s\)-well separated for \(1 \leq i \leq m\).

By using a (compressed) quadtree, an \(s\)-WSPD of size \(O(s^d n)\) can be constructed for \(P\) in \(O(n \log n + s^d n)\) time [6]. For the remainder of Section 2.2 we assume that the points of \(P\) are in general position – that is, no three points are collinear, and no two points coincide. In the lemma below, we describe a utility property of an \(s\)-WSPD with respect to line segments spanned by points in each well separated pair.

**Lemma 2.6 (WSPD Utility Lemma).** If pair \(\{A, B\}\) is \(s\)-well separated, \(a, a' \in A\), and \(b, b' \in B\), then for any point \(c \in P \setminus \{a, a', b, b'\}\), we have \(d(c, a'b') \leq (1 + \frac{2}{s} \Delta) d(c, ab)\).

**Proof.** As with Lemma 2.3 one can prove the current lemma using an exhaustive proof (see Appendix E for the case analysis).

#### WSPD+WSSD-based approximation

**Theorem 2.7.** Given a set \(P\) of \(n\) points in \(\mathbb{R}^d\), for any \(\epsilon > 0\), a \((1 + \epsilon\Delta)\)-approximation to a discrete median line segment for \(P\) can be computed in time \(O(n\epsilon^{-2d} \log n)\).

**Proof.** We construct a WSPD for \(P\). We augment the quadtree used for constructing the WSPD so that each node of the tree is associated with a representative point lying inside its cell. For each well separated pair \(\{A, B\}\) in the WSPD, let \(rep(A)\) and \(rep(B)\) denote the representative points of \(A\) and \(B\), respectively. For each well separated pair \(\{A, B\}\), we compute an \(s\)-WSSD with respect to line segment \(rep(A)rep(B)\).

For each well separated pair \(\{A, B\}\), let \(\{C_i : i = 1, 2, \ldots\}\) be the collection of \(s\)-well separated subsets with respect to \(rep(A)rep(B)\). Let \(\alpha(C_i)\) denote the farthest point within the cell containing \(C_i\) from \(rep(A)rep(B)\), and let \(|C_i|\) be the number of points in \(C_i\). For each well separated pair \(\{A, B\}\), we compute the distance sum \(\sum_{i} |C_i| \cdot d(\alpha(C_i), rep(A)rep(B))\), and output the line segment \(rep(A)rep(B)\) achieving the smallest distance sum.

Suppose that the above approach yields line segment \(ab\) as an approximate discrete median line segment for \(P\). Let \(\{A, B\}\) be the \(s\)-well separated pair of which \(a\) and \(b\) are the representative points. By Lemma 2.3, for each \(s\)-well separated subset \(C_i\) with respect \(ab\), we have \(|C_i| \cdot d(\alpha(C_i), ab) \leq \sum_{c \in C_i} (1 + \frac{4}{s}) d(c, ab)\). Since \(\alpha(C_i)\) is the farthest point within the cell containing \(C_i\) from \(ab\), by summing over all \(i\), we obtain

\[
\sum_{p \in P} d(p, ab) \leq \sum_{i} |C_i| \cdot d(\alpha(C_i), ab) \leq \left(1 + \frac{4}{s}\right) \sum_{p \in P} d(p, ab)
\]

(1)
Let \( xy \) denote an exact discrete median line segment for \( P \), where \( x, y \in P \) and \( x \neq y \). Let \( \{F, G\} \) be the well separated pair containing these points, and let \( f \) and \( g \) denote their respective representatives. Let \( \{H_i : i = 1, 2, \ldots\} \) be the set of \( s \)-well separated subsets with respect to \( fg \). According to Lemma 2.6, for any point \( p \in P \), we have \( d(p, fg) \leq (1 + \frac{2}{s} \Delta) d(p, xy) \). By summing over all points \( p \in P \), we obtain \( \sum_{p \in P} d(p, fg) \leq \sum_{p \in P} (1 + \frac{2}{s} \Delta) d(p, xy) \). Since \( xy \) has the minimal distance sum, we have

\[
\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, fg) \leq \left(1 + \frac{2}{s} \Delta\right) \sum_{p \in P} d(p, xy)
\]

(2)

Note that \( \sum_{i} |C_i| \cdot d(\alpha(C_i), ab) \leq \sum_{i} |H_i| \cdot d(\alpha(H_i), fg) \leq (1 + \frac{4}{s}) \sum_{p \in P} d(p, fg) \). So, together with the inequalities \( ▶ \) and \( ▶ \), we obtain

\[
\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq \left(1 + \frac{4}{s}\right) \left(1 + \frac{2}{s} \Delta\right) \sum_{p \in P} d(p, xy)
\]

Assume that \( s \geq 1 \); if \( 0 < s < 1 \), we replace \( s \) with \( \max(s, 1) \). Since \( \Delta = D/E \) and \( D \geq E \), the multiplicative term on the right-hand side of the second inequality above becomes

\[
1 + (2\Delta + 4)\frac{1}{s} + (8\Delta)\frac{1}{s^2} \leq 1 + (10\Delta + 4)\frac{1}{s} = 1 + \left(\frac{10D + 4E}{E}\right)\frac{1}{s} \leq 1 + \frac{14}{s} \Delta
\]

Given any \( \epsilon > 0 \), let \( s = 14/\epsilon \). We then have \( \sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq (1 + \epsilon \Delta) \sum_{p \in P} d(p, xy) \), implying that the output line segment \( ab \) is a \((1 + \epsilon \Delta)\)-approximation.

Recall that there are \( O(s^2n) \) well separated pairs. With respect to each pair, we have \( O(s^4 \log n) \) well separated subsets. Hence, the overall running time is \( O(s^2n \log n) = O((14/\epsilon)^2d \log n) = O(n \log n) \), which is \( O(n \log n) \) if we treat \( \epsilon \) as a constant. ☐

**Remark.** Given an \( \epsilon > 0 \), if we set \( s = \frac{8\Delta}{-(\Delta+2) + (\Delta+2)^2 + 8\epsilon \Delta} \) in the method above, the resulting solution would be a \((1 + \epsilon)\)-approximation (without a dependency on \( \Delta \)) to a discrete median line segment for \( P \). However, in order to set the value of \( s \), we would be required to determine \( \Delta \) in addition to the given \( \epsilon \). Let \( t(n) \) denote the time complexity for computing \( \Delta \). Then, the overall running time of the algorithm is \( O(t(n) + n \epsilon^{-2d} \log n) \).

### 2.3 \( O(n^2 \epsilon^{-d}) \)-time \((1 + \epsilon)\)-approximation

To compute a \((1 + \epsilon)\)-approximate discrete median line segment in time \( O(n^2 \epsilon^{-d}) \), we create a new data structure, which is named well separated pair decomposition for point-segment pairs (WSPD-PS), to efficiently represent the distances between points and line segments spanned by pairs of points.

**Well separated pair decomposition for point-segment pairs (WSPD-PS)**

Let \( A, B, \) and \( C \) be subsets of \( P \). As before, we define \( s > 0 \) to be the separation factor. We denote by \( r \) be the smallest radius of a Euclidean ball such that each of \( A, B, \) and \( C \) can be enclosed within such a ball. Let \( T \) denote the sphero-cylinder resulting from the Minkowski sum of a ball of radius \( r \) and the line segment bounded by the two centers of the balls enclosing \( A \) and \( B \). Set \( \{A, B\} \) is said to be \( s \)-well separated from \( C \) if the closest distance between \( T \) and the ball of radius \( r \) enclosing \( C \) is at least \( sr \) (Figure 2).
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Figure 2 A well separated pair $\{A, B, C\}$ with separation factor $s$.

Definition 2.8 (WSPD-PS). Given a set $P$ of $n$ points and a parameter $s > 0$, an $s$-WSPD-PS is defined as a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}, C_m\}$, such that (I) $A_i, B_i, C_i \subseteq P$ for $1 \leq i \leq m$, (II) $\{A_i, B_i\} \cap C_j = \emptyset$ for $1 \leq i \leq m$, (III) $\bigcup_{i=1}^{m} (A_i \times B_i) \times C_i = P \times P \times P$, and (IV) $A_i, B_i$ and $C_i$ are $s$-well separated for $1 \leq i \leq m$.

Theorem 2.9. For a set $P$ of $n$ points and any $s > 0$, one can construct an $s$-WSPD-PS of size $O(s^d n^2)$ in time $O(s^d n^2)$.

Proof. We begin by building a compressed quadtree for $P$. The quadtree can be built in $O(n \log n)$ time, and is of size $O(n)$. For simplicity of arguments, we assume that the quadtree is not compressed but of size $O(n)$. This allows us to assume that nodes of the same level in the quadtree have the same cell size. We construct a WSPD-PS for $P$ by recursively trying to separate each pair of nodes $\{u, v\}$ from each node $w$ in the quadtree, while keeping the invariant that $u, v$, and $w$ are nodes of the same level. The algorithm is presented as a pseudocode below (where the blue parts are augmentations only necessary for obtaining an approximate discrete median line segment, which will be discussed in the next section).

Algorithm WSPD-PS($u, v, w, s, \text{par}$)
1. if $(u = v = w = P)$ and $\{u, v\}$ and $w$ are $s$-well separated) then output $\{u, v\}, w, \text{par}$;
2. else let $u_1, \ldots, u_\alpha$ be the children of $u$;
3. let $v_1, \ldots, v_\beta$ be the children of $v$;
4. let $w_1, \ldots, w_\gamma$ be the children of $w$;
5. for $i \leftarrow 1$ to $\alpha$
6. do for $j \leftarrow 1$ to $\beta$
7. do $X \leftarrow Y \leftarrow \emptyset$;
8. newpar $\leftarrow$ par;
9. for $k \leftarrow 1$ to $\gamma$
10. do if ($u_i, v_j$, or $w_k$ is empty) then ignore $\{u_i, v_j, w_k\}$;
11. else if ($\{u_i, v_j\}$ and $w_k$ are $s$-well separated)
12. then add $\{u_i, v_j\}, w_k$ to $X$;
13. else add $\{u_i, v_j\}, w_k$ to $Y$;
14. if ($X \neq \emptyset$) then newpar $\leftarrow \{u_i, v_j\}$;
15. for each $\{u, v\}, w \in X$ do output $\{u, v\}, w, \text{par}$;
We claim that the total number of well separated pairs generated by the algorithm is $O(n^2 s^d)$ (see Appendix F for details). Then, together with $O(n \log n)$ time to build the quadtree, the overall running time of the algorithm is $O(n \log n + n^2 s^d) = O(n^{2} s^d)$.  

We now proceed to describe a utility property associated with an $s$-WSPD-PS, which will later be made use of in approximating a discrete median line segment for $P$.  

Lemma 2.10 (WSPD-PS Utility Lemma). For any pair $(\{A, B\}, C)$ in an $s$-WSPD-PS, if $a, a' \in A$, $b, b' \in B$, and $c, c' \in C$, then we have (I) $d(c, a'b') \leq (1 + \frac{3}{s})d(c, ab)$, and (II) $d(c', ab) \leq (1 + \frac{4}{s})d(c, ab)$.  

Proof. Part (I) of the lemma can be proven in the same way as Lemma 2.6, with the only difference that $s\epsilon$ is bounded from above by $d(c, ab)$ instead of $D$ (see Appendix F). Part (II) of the lemma is identical to Lemma 2.3.  

WSPD-PS-based approximation  
Recall that $E$ denotes the closest (non-zero) distance between points of $P$ and line segments spanned by pairs of points of $P$. One can find $E$ exactly using brute force in $O(n^3)$ time. For points in $\mathbb{R}^2$, $E$ can be determined in $O(n^2 \log n)$ time using the proximity query data structure given in [12]. Interestingly, we could use an $s$-WSPD-PS for $P$ to find $E$ exactly.  

Lemma 2.11. For $n$ points in $\mathbb{R}^d$ and $s > 2$, one can compute $E$ in $O(s^d n^2)$ time.  

Proof. See Appendix G.  

In Lemma 2.11 since the value of $s$ can be chosen arbitrarily close to 2, the running time is $O(2^d n^2) = O(n^2)$, where $d$ is assumed to be a constant. This result could be of independent interest, as it is an improvement over the previously best-known algorithm.  

Because of Lemma 2.11 we can determine an upper bound for $\Delta$ in $O(s^d n^2)$ time, and subsequently obtain a $(1 + \epsilon)$-approximation using the algorithm in Section 2.2, provided that $\Delta$ is bounded (i.e., $P$ is in general position). However, the resulting algorithm would have a time complexity dominated by the cost of computing $\Delta$, which is $O(s^d n^2)$, and would involve three different pair decomposition schemes (i.e., WSSD, WSPD, and WSPD-PS). In what follows, we show that one can derive a $(1 + \epsilon)$-approximation algorithm by using only WSPD-PS and without making any general assumption on $P$.  

Theorem 2.12. Given a set $P$ of $n$ points in $\mathbb{R}^d$, for any $\epsilon > 0$, one can compute a $(1 + \epsilon)$-approximation to a discrete median line segment of $P$ in $O(n^2 e^{-d})$ time.  

Proof. We begin by constructing an $s$-WSPD-PS for $P$. For ease of reference, for any well separated pair $(\{u, v\}, w)$ in the $s$-WSPD-PS, we call $\{u, v\}$ the segment set of the pair.  

Augmenting WSPD-PS construction  
In the quadtree used for constructing the WSPD-PS, we associate each node $u$ with a representative point $rep(u)$ and a quantity $|u|$ indicating the number of points located within its cell. In addition, we output each well separated pair $(\{u, v\}, w)$ along with a segment set $\{x, y\}$, if any, where $x$ and $y$ are the lowest ancestors of $u$ and $v$, respectively, such that i) $x$ and $y$ are nodes of the same level (in the non-compressed quadtree), and ii) $\{x, y\}$ is $s$-well separated from some node of the same level as $x$ and $y$. We call $\{x, y\}$ the parent segment set of $(\{u, v\}, w)$ (and of $\{u, v\}$), and $\{u, v\}$ is a child segment set of $\{x, y\}$. In the pseudocode (WSPD-PS), the blue parts are incorporated for keeping track of the parent segment set $\{x, y\}$ for each well separated pair $(\{u, v\}, w)$.  

for each $\{u, v\}, w \in Y$ do call WSPD-PS $u, v, w, s, newpar$;
Approximating the discrete and continuous median line segments in $d$ dimensions

**Finding an approximate median line segment**  Let $\Gamma$ denote the set of well separated pairs in an $s$-WSPD-PSS for $P$. We denote by $\mathcal{CS}$ a cell size, $\mathcal{PSS}$ a parent segment set, and $SS$ a segment set. Let $\Gamma[\mathcal{CS}, \mathcal{PSS}, SS]$ denote the set of well separated pairs with cell size $\mathcal{CS}$, parent segment set $\mathcal{PSS}$, and segment set $SS$. Let $\sigma[SS]$ denote the distance sum for segment set $SS$. We process $\Gamma$ in increasing $CS$ as described in the pseudocode below.

**Algorithm** ApproxMedSeg($\Gamma$)

1. $F \leftarrow \emptyset$;
2. for each $\mathcal{CS}$ in increasing order
3. do for each $\mathcal{PSS}$ (in any order)
4. do for each $SS$ (in any order)
5. do compute $\sigma[SS] = \sum_i(|w_i| \cdot d(rep(w_i), rep(u_i)rep(v_i)))$
   where $(\{u_i, v_i\}, w_i) \in \Gamma[\mathcal{CS}, \mathcal{PSS}, SS]$;
6. $SS_{\text{min}} \leftarrow \arg \min_{SS} \sigma[SS]$;
7. let $(u, v)$ be the $SS_{\text{min}}$, and let $\{x, y\}$ be the $\mathcal{PSS}$;
8. $\sigma[\mathcal{PSS}] \leftarrow \sigma[SS_{\text{min}}]$; $rep(x) \leftarrow rep(u)$; $rep(y) \leftarrow rep(v)$;
9. if ($\mathcal{PSS}$ has no parent segment set)
10. then add $(rep(x)rep(y), \sigma[\mathcal{PSS}])$ to $F$;
11. output $(ab, \sigma) \in F$ with the minimum $\sigma$;

Briefly, at a given $\mathcal{CS}$, we update $\sigma[SS]$ for each $SS$ sharing the same $\mathcal{PSS}$ by considering its associated well separated pairs (line 5 of the pseudocode). Let $SS_{\text{min}}$ be the $SS$ with the minimum distance sum after the update. We record the distance sum for $SS_{\text{min}}$ as that for its $\mathcal{PSS}$, and replace the representative points for the $\mathcal{PSS}$ of $SS_{\text{min}}$ with those for $SS_{\text{min}}$. When the algorithm terminates, of all segment sets without a parent segment set, we report the one with the minimum distance sum.

Since each well separated pair is processed only once, the total running time is $O(s^d n^2)$.

**Correctness of algorithm**  Let $xy$ be an exact discrete median line segment of $P$. Let $ab$ be the solution returned by the algorithm. We claim that $\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq (1 + \frac{\epsilon}{6} + \frac{s^2}{2}) \sum_{p \in P} d(p, xy)$. Given an $\epsilon > 0$, if we set $s = \frac{2 + \sqrt{25 + 16\epsilon^2}}{6}$, then we have $\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq (1 + \epsilon) \sum_{p \in P} d(p, xy)$ (see Appendix H for details).

**Remark.** The approach above could be applied to produce a $(1 + \epsilon)$-approximation to the medoid (using WSPD instead of WSPD-PS). The running time would be $O(n \log n + n\epsilon^{-d})$.

3 Inconstructibility of the continuous median line segment

**Theorem 3.1.**  The construction of a median line segment is, in general, impossible for $n = 3$ and more points in the plane by strict usage of ruler and compass.

**Proof.**  In order to prove the theorem, we require the following lemma.

**Lemma 3.2.**  Let $p^*$ denote the Fermat-Torricelli point for a point set $\{p_1, p_2, p_3\}$. Let $\beta = \arg \max_i ||p^*p_i||$. For $i \neq \beta$, let $\eta_i$ be the distance from $p_\beta$ to the foot of the altitude from $p_i$.

A. If $\ell \leq ||p^*p_\beta||$, then there exists a median line segment $s^* = a^*b^*$ such that its endpoint $a^*$ coincides with $p^*$, and $s^*$ lies in $p^*p_\beta$ (Figure 3A).

B. If $\ell > ||p^*p_\beta||$, then there is a median line segment $s^* = a^*b^*$ such that its endpoint $a^*$ coincides with $p_\beta$. 


1. $l \leq \min \{\eta_i : i \neq \beta\}$. For $i \neq \beta$, let $\phi_i$ be the acute angle formed by $b^* p_i$ and the line supporting $s^*$. Endpoint $b^*$ must be located such that $\phi_i = \phi_j$, where $i \neq j$ (Figure 3B).

2. $l > \min \{\eta_i : i \neq \beta\}$ For $i, j \neq \beta$ and $i \neq j$, endpoint $b^*$ must be located such that $|\|h_i\|/|d_i\| = |\|h_j\|/|d_j\|$ (i.e., Lemma I.2 holds).

![Figure 3](image)

Figure 3 Illustrations for Lemma 3.2. (A) Part A. (B) Case 1 of part B.

Proof. See Appendix I.

Part A of Lemma 3.2 essentially implies that if $l \leq |\|p^* p_\beta\||$, then a median line segment $s^*$ can be constructed by ruler and compass, since the exact Euclidean construction of the Fermat-Torricelli point for $n = 3$ points is possible. However, in part B of Lemma 3.2 ($\ell > |\|p^* p_\beta\||$) – case 1 in particular – in order to construct a median line segment $s^*$, we have to look for a point $b^*$ on the circumference of a circle of radius $\ell$ centered at $a^* = p_\beta$ such that the rays emanating from $p_i$ and $p_j$, where $i \neq j$ and $i, j \neq \beta$, meeting at $b^*$ make equal angles with the normal at $b^*$. This is known as the Alhazen’s billiard problem, to which the general solution has been proven to be inconstructible using only ruler and compass [17].

4 Approximating the continuous median line segment

In this section, we derive an $O(n \log n + n^d \epsilon^{-d})$-time algorithm for computing a $(1 + \epsilon)$-approximate median line segment of length $l$ for $P$ in $d$ dimensions. The algorithm entails generalizing WSPD-PS to represent the distances between $P$ and line segments of length $\ell$ in $\mathbb{R}^d$. The result is a novel pair decomposition in the continuous domain with possible other applications in computational geometry. We now describe said generalization in detail.

Well separated pair decomposition for point-segment pairs with segment length $\ell$ (WSPD-PS- $\ell$)

Recall that $P$ has been scaled and translated so that it is contained within $[0, 1]^d$. Assume that length $\ell$ has also been scaled along with $P$. Furthermore, we assume that the scaling of $P$ and $\ell$ is sufficient such that the set of all line segments of length $\ell$, which intersect the convex hull of $P$, is contained within $[0, 1]^d$. Note that $\ell \leq D \leq \sqrt{d}$. Since a median line segment of length $\ell$ must intersect the convex hull of $P$ (see Lemma I.3), it must also lie within $[0, 1]^d$. Define $\{Q, R\} = \{\alpha, \beta : \alpha, \beta \in [0, 1]^d \text{ and } d(\alpha, \beta) = \ell\}$.

Let $A$ and $B$ be compact (i.e., closed and bounded) subsets of $[0, 1]^d$. Let $C$ be a subset of $P$. Let $r$ denote the smallest radius of a Euclidean ball such that each of $A$, $B$, and $C$ can be enclosed within such a ball. Let $\Pi_A$ and $\Pi_B$ be the balls of radius $r$ enclosing $A$ and $B$,
Approximating the discrete and continuous median line segments in $d$ dimensions

respectively. Let $d(\Pi_A, \Pi_B)$ denote the closest distance between $\Pi_A$ and $\Pi_B$. Set $\{A, B\}$ is said to be an $\ell$-segment set if $\ell - 4r \leq d(\Pi_A, \Pi_B) \leq \ell$.

Let $T$ be the spherocylinder resulting from the Minkowski sum of a ball of radius $r$ and the line segment bounded by the two centers of balls $\Pi_A$ and $\Pi_B$. Define $s > 0$ to be a parameter called separation factor. Set $\{A, B\}$ is said to be $s$-well separated from $C$ if the closest distance between $T$ and the ball of radius $r$ enclosing $C$ is at least $sr$.

**Definition 4.1 (WSPD-PS-$\ell$).** Given a set $P$ of $n$ points, a real number $\ell > 0$, and a parameter $s > 0$, an $s$-WSPD-PS-$\ell$ is defined as a set of pairs $\{(A_1, B_1), C_1\}, \ldots, \{(A_m, B_m), C_m\}$, such that (I) $A_i, B_i \subseteq [0, 1]^d$ and $C_i \subseteq P$ for $1 \leq i \leq m$, (II) $A_i, B_i \cap C_j = \emptyset$ for $1 \leq i \leq m$, (III) $\cup_{i=1}^m A_i \times C_i \supseteq \{Q, R\} \times P$, (IV) $\{A_i, B_i\}$ and $C_i$ are $s$-well separated for $1 \leq i \leq m$, and (V) $\{A_i, B_i\}$ is an $\ell$-segment set for $1 \leq i \leq m$.

Note from the above that an $s$-WSPD-PS-$\ell$ is different from an $s$-WSPD-PS in two main aspects - i) $A_i$ and $B_i$ are subsets of $[0, 1]^d$ instead of $P$, and ii) $\{A_i, B_i\}$ is an $\ell$-segment set.

**Theorem 4.2.** For a set $P$ of $n$ points in $\mathbb{R}^d$, a positive real number $\ell$, and any $s > 0$, one can construct an $s$-WSPD-PS-$\ell$ of size $O(s^dn^d)$ in time $O(n\log n + s^dn^d)$.

**Proof.** At first, we build a BBD-tree for $P$ in $O(n\log n)$ time. The BBD-tree is of size $O(n)$ and height $O(\log n)$. We then construct a (region) quadtree for the unit hypercube, where the leaf nodes of the quadtree have the same cell size as the smallest size group in the BBD-tree. We claim that the quadtree is of size $O(n^d)$ and can be built in the same amount of time.

To construct an $s$-WSPD-PS-$\ell$, we follow a similar paradigm as for constructing an $s$-WSPD-PS. We recursively try to separate every pair of nodes $\{u, v\}$ in the quadtree from every node $w$ in the BBD-tree, all the while ensuring that i) $\{u, v\}$ is an $\ell$-segment set, and ii) $u, v,$ and $w$ have the same cell size. Using a packing argument, we can show that the number of well separated pairs in an $s$-WSPD-PS-$\ell$ is $O(s^dn^d)$, provided that $\ell \leq D \leq \sqrt{d}$.

**Remark.** If $\ell \geq D$, then we effectively have the median line problem in $\mathbb{R}^d$, which can be solved exactly in $O(n^2)$ time for $d = 2$. However, there is no known exact $O(n^3)$-time algorithm for solving the median line problem for $d > 2$. Since a problem instance with $\ell \geq D$ is the same as that with $\ell = D$, our approximation approach can be used to solve both the median line segment and line problems by simply setting $\ell = \min(D, \ell)$.

The total running time of the algorithm is $O(n\log n + n^d + n^d s^d) = O(n\log n + s^dn^d)$.

For a detailed proof, please refer to Appendix L.

**WSPD-PS-$\ell$-based approximation**

**Theorem 4.3.** Given a set $P$ of $n$ points in $\mathbb{R}^d$ and a real number $\ell > 0$, for any $\epsilon > 0$, one can compute a $(1 + \epsilon)$-approximate median line segment of length $\ell$ for $P$ in $O(n\log n + n^d\epsilon^{-d})$ time.

**Proof.** Note that the utility lemma for $s$-WSPD-PS (Lemma 2.10) can be directly applied to $s$-WSPD-PS-$\ell$. As a result, we can prove the current theorem using the same argument as described in the proof of Theorem 2.12, with the only difference that the representative points chosen for a given segment set must be of pairwise distance $\ell$.

**5 Approximating the constrained continuous median line segments**

In this section, we provide approximation algorithms for two median line segment problems in the plane under certain geometric restrictions.
Anchored-point median line segment problem. Given a set $P$ of $n$ points in $\mathbb{R}^2$, a point $q \in \mathbb{R}^2$, and a real number $\ell > 0$, find a line segment of length $\ell$ with an endpoint at $q$ such that the sum of the Euclidean distances from $P$ to the line segment is minimized.

Remark. It follows from the proof of Theorem 3.1 that the anchored point median line segment problem is, in general, not solvable by radicals over $\mathbb{Q}$ for $n \geq 2$ points.

Theorem 5.1. For the point-anchored median line segment problem in $\mathbb{R}^2$, given any $\epsilon > 0$, one can compute a $(1 + \epsilon)$-approximate solution in time $O(ne^{-2}\alpha_\theta^{-1})$, where $\alpha_\theta$ is a function dependent on the coordinates of $P$.

Proof. Let $s = ab$ denote any line segment of length $\ell$ with a fixed endpoint $a = q$. Let $\theta$ be the counterclockwise angle of $s$ with respect to the positive $x$-axis rooted at $a$. We divide the range of angle $\theta \in [0, 2\pi)$ into a set of small contiguous intervals such that the relative error in computing the sum of the distances from $P$ to a line segment $s$, whose angle $\theta$ is given by a boundary of an interval, does not exceed $\epsilon$. We claim that the number of intervals is bounded by $2\pi/(e^2\alpha_\theta)$ in the worst case, where $\alpha_\theta$ is some function dependent on the coordinates of $P$. Since it takes $O(n)$ time to compute the sum of distances for each candidate line segment (defined by the boundaries of the intervals), we can obtain a solution, whose sum of distances to $P$ is at most $(1 + \epsilon)$ times that of the optimal solution, in $O(ne^{-2}\alpha_\theta^{-1})$ time (please see Appendix M for details).

Constant-slope median line segment problem. Given a set $P$ of $n$ points in $\mathbb{R}^2$, an angle $\theta$, and a real number $\ell > 0$, find a line segment of length $\ell$ making angle $\theta$ with the abscissa axis such that the sum of the Euclidean distances from $P$ to the line segment is minimized.

Theorem 5.2. For the constant-slope median line segment problem in $\mathbb{R}^2$, given any $\epsilon > 0$, one can find a line segment, whose sum of distances to $P$ is at most $(1 + \epsilon)$ times that of the optimal line segment, in time $O(kn\log n)$, where $\epsilon = \frac{1}{\sqrt{\cos(2\pi/k)}} - 1$.

Proof. Given an $\epsilon > 0$, we begin by defining the so-called $k$-oriented distance function [13] to approximate the Euclidean distance between points within a factor of $(1 + \epsilon)$, where $k = \cos(\frac{2\pi}{1 + \epsilon})$. Since the resulting objective function, which is in terms of the $k$-oriented distances, is convex and piecewise linear, we can find the minimum of the objective function in $O(kn\log n)$ time using the prune-and-search approach described by Bose et al. [6] after careful modifications. For details of the proof, please refer to Appendix N.

Remark. Alternatively, the space-subdivision procedure previously used in approximating a point-anchored median line segment could be extended to address the constant-slope variant. The resulting $(1 + \epsilon)$-approximation algorithm would have a time complexity of $O(n^2 + ne^{-4}\alpha_{xy})$, where $\alpha_{xy}$ is a function dependent on the coordinates of $P$.

Final remark

When $d \approx n$, we could use a low-distortion embedding given by [15] to reduce an instance of our $d$-dimensional problem to an $m$-dimensional instance with a $(1 + \delta)$-approximation to the optimal solution, where $m = O(\delta^{-2}\log n)$. For example, in the median line segment problem in $\mathbb{R}^d$, if we map $P$ to $\mathbb{R}^m$ and solve the reduced problem in $\mathbb{R}^m$ using our $(1 + \epsilon)$-approximation approach, the resulting solution would be a $(1 + \lambda)$-approximation, where $\lambda = 3 \cdot \max(\epsilon, \delta)$. Our algorithm would take time $O(n^m\epsilon^{-m})$, where $m = O(\lambda^{-2}\log n)$.
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### A Additional definitions

The notations defined below are used only in the appendices.

For any line segment $ab$ in $\mathbb{R}^d$, let $H_a$ (resp. $H_b$) be the hyperplane containing $a$ (resp. $b$), orthogonal to $ab$, and not containing $ab$. Let $S_a$ (resp. $S_b$) be the closed half-space bounded by $H_a$ (resp. $H_b$) and not containing $ab$. Define $S_{ab} = \mathbb{R}^d \setminus (S_a \cup S_b)$.

For a line segment $ab$ in $\mathbb{R}^2$, let $L_{ab}$ be the line containing $ab$. Let $H^+$ denote a closed half-plane bounded by $L_{ab}$, and let $H^- = \mathbb{R}^2 \setminus H^+$. Define $S^+_a = S_a \cap H^+$, $S^-_a = S_a \cap H^-$, $S^+_b = S_b \cap H^+$, $S^-_b = S_b \cap H^-$, $S^+_{ab} = S_{ab} \cap H^+$, and $S^-_{ab} = S_{ab} \cap H^-$ (Figure 4).

![Figure 4 Regions defined with respect to a line segment ab.](image)

### B Tree data structures

**kd-tree** A kd-tree [4] is a simple generalization of the binary tree for partitioning points in $k$ dimensions.

Consider a kd-tree for a set $P$ of $n$ point. The root of the tree represents the entire point set $P$. Each non-terminal node in the tree represents a subset of $P$ and a partitioning of the subset. Each non-terminal node has two children, representing the two subsets resulting from the partitioning. For each non-terminal node, we subdivide its associated point set into two (disjoint) subsets using an axis-orthogonal hyperplane oriented in one of the $k$ directions. Specifically, each non-terminal node of the tree is associated with two quantities – i) cutting dimension and ii) cutting value. For a given node, if its cutting dimension is $i \in [1, k]$, then all the points whose $i$-th coordinate is less than or equal to the cutting value are assigned to the left child node, and the remaining points are assigned to the right child node. Each leaf node of the kd-tree contains a single point. Geometrically, each node of the kd-tree implicitly represents a hyper-rectangular region of space (which is called a cell).

An optimized kd-tree [11] can be built for point set $P$ as follows. We select the cutting dimension to be the one along which the points have the greatest spread (i.e., difference between the largest and smallest coordinates). We choose the cutting value to be the median coordinate along the cutting dimension. This guarantees that the kd-tree has height $O(\log n)$.
For a set $P$ of $n$ points in $\mathbb{R}^d$, it is possible to build a kd-tree of height $O(\log n)$ and space $O(n)$ in time $O(n \log n)$. The cell associated with each node of the kd-tree is a $d$-dimensional hyper-(rectangular) cuboid. As we descend from a parent node to its child node, the size (i.e., largest side length) of the cell decreases. Any two cells associated with a given level of the kd-tree are interior-disjoint.

**Balanced box decomposition (BBD) tree** A balanced box decomposition (BBD) tree [1] is a data structure based on partitioning space into rectilinear regions of bounded aspect ratio.

We define a box to be a rectangle (i.e., $d$-fold product of closed intervals on the coordinate axes in $\mathbb{R}^d$) such that the ratio of its longest to shortest side (i.e., aspect ratio) is bounded by some constant. The size of a rectangle is the length of its longest side.

Consider a BBD-tree for a set $P$ of $n$ points in $\mathbb{R}^d$. Each node of the BBD-tree is associated with a cell, which could be either a box or the set theoretic difference of two boxes (one enclosed within the other). Thus, each cell consists of an outer box and an optional inner box. Each cell is associated with the set of points lying within the cell. The size of the cell is the size of the outer box.

Each inner box satisfies a property called stickiness. Formally, an inner box is sticky if it is surrounded by $3d - 1$ identical boxes (in a grid-like fashion), and the interior of each of these boxes lies entirely either inside or outside the outer box.

For a set $P$ of $n$ points in $\mathbb{R}^d$, it is possible to construct a BBD-tree, in $O(n \log n)$ time, with the following properties: i) the tree has $O(n)$ nodes and height $O(\log n)$, ii) the cells have bounded aspect ratio, and the sizes of the cells decrease by at least a factor of $1/2$ with each descent of $2d$ levels in the tree, and iii) inner boxes are sticky relative to their outer boxes.

Note that the sizes of the cells decrease exponentially as we descend a constant number of levels in the tree. The tree is constructed through a recursive application of two partitioning operations – splits and shrinks. Briefly, a split partitions a cell by an axis-orthogonal hyperplane, whereas a shrink partitions a cell using a box that lies within the cell. Each non-terminal node in the resulting tree has two children. The details of the construction of the BBD-tree are presented in [1].

**WSSD construction analysis**

**kd-tree-based s-WSSD** Each of the nodes at a given level in an optimized kd-tree is associated with (nearly) the same number of points (which is a result of choosing the median as the cutting value). Consider the nodes at a given level $\lambda$ of the kd-tree, where the cell associated with each node contains $b_\lambda$ points. Let $V$ denote the volume of the cell associated with any node at level $\lambda$ in the kd-tree. As shown by Friedman et al. [11], the expected volume of each such cell is approximately

$$E[V] = \frac{b_\lambda}{n + 1} \frac{1}{P_\lambda}$$

where $P_\lambda$ is the probability density averaged over the cell (assuming that the probability distribution of the points within the cell is approximately constant). Let $x$ be the size of the cell. Then, the expected size $E[x]$ of the cell is simply the $d$-th root of the expected volume of the cell – that is,

$$E[x] = \left(\frac{b_\lambda}{n + 1} \frac{1}{P_\lambda}\right)^{1/d}$$
In the procedure \(WSSD()\), we divide a node \(u\) only if the call \(WSSD(u, ab, s)\) is non-terminal — that is, node \(u\) is not an \(s\)-well separated subset. The expected number of nodes at level \(\lambda\) that are divided in the procedure must be bounded from above by the expected number of cells at level \(\lambda\) that overlap the Minkowski sum of line segment \(ab\) and a ball of radius \(sr\) (i.e., a hyper-capsule; see Figure 5) — that is,

\[
\left(1 + \left[\frac{2sr}{E[x]}\right]\right)^d + ||ab|| \left(1 + \left[\frac{2sr}{E[x]}\right]\right)^{d-1}
\]

Given that \(r = E[x]\sqrt{d}/2\) and \(||ab|| \leq D \leq \sqrt{d}\), the upper bound becomes

\[
\left(1 + s\sqrt{d}\right)^d + \sqrt{d} \left(1 + s\sqrt{d}\right)^{d-1} = O(s^d) + O(s^{d-1})
\]

Assume that \(s \geq 1\); if \(0 < s < 1\), we replace \(s\) with \(\max(s, 1)\). We then have \(O(s^d) + O(s^{d-1}) \leq O(s^d)\). Since there are \(O(\log n)\) levels in the kd-tree, the expected number of non-terminal calls to the procedure \(WSSD()\) is \(O(s^d \log n)\). Hence, the expected number of \(s\)-well separated subsets is \(2 \cdot O(s^d \log n) = O(s^d \log n)\). Together with \(O(n \log n)\) time to build the kd-tree, the overall expected running time is \(O(n \log n + s^d \log n)\).

**BBD-tree-based \(s\)-WSSD** For the case of a BBD-tree, we will use a similar packing argument as that for a kd-tree. Recall that point set \(P\) has been scaled so that it is enclosed within a unit hypercube. This implies that the cells of the BBD-tree have sizes that are powers of 1/2.

For analysis purposes, we congregate the nodes in the BBD-tree into groups according to the sizes of their associated cells. Define size group \(i\) to be the set of nodes whose cell size is \(1/2^i\). Note that a node and its child may have the same size, and thus we cannot apply the packing argument directly to each size group. Define the base group for size group \(i\) to be the subset of nodes in the size group that are leaf nodes or whose children belong to the next smaller size group. The cells corresponding to the nodes in a base group are pairwise interior-disjoint. For each base group \(i\), the number of cells overlapping the Minkowski sum of line segment \(ab\) and a ball of radius \(sr\) is bounded from above by

\[
\left(1 + \left[\frac{2sr}{1/2^i}\right]\right)^d + ||ab|| \left(1 + \left[\frac{2sr}{1/2^i}\right]\right)^{d-1}
\]

Since \(r = (1/2^i)\sqrt{d}/2\), \(||ab|| \leq D \leq \sqrt{d}\), and \(s = \max(s, 1)\), the upper bound becomes \(O(s^d)\). Note that at most \(2d\) levels of ancestors above the nodes in the base group can be in the same size group. In addition, the BBD-tree is \(O(\log n)\) in height, which implies that the total number of base groups is bounded by \(O(\log n)\). As a result, the number of \(s\)-well separated subsets with respect to line segment \(ab\) is \(O(s^d \log n)\). Recall that it takes \(O(n \log n)\) time to build a BBD-tree. Hence, the overall running time is bounded by \(O(n \log n + s^d \log n)\).
Proof of Lemma 2.3

Depending on the locations of points \(c\) and \(c'\) with respect to line segment \(ab\), we have the following cases (refer to Figure 6 for illustrations).

**Case 1.** \(c \in S_a\) and \(c' \in S_a\)  Note that \(d(c, ab) = ||ca||\) and \(d(c', ab) = ||c'a||\). Due to the triangle inequality, we have \(||c'a|| \leq ||ca|| + ||cc'||\). Since \(C\) is enclosed within a ball of radius \(r\), and \(C\) is \(s\)-well separated from \(a\), we have \(||c'a|| \leq ||ca|| + 2r = ||ca|| + \frac{2s}{r}sr \leq ||ca|| + \frac{2}{s}||ca|| = (1 + \frac{2}{s})||ca||\). Therefore, \(d(c', ab) \leq (1 + \frac{2}{s})d(c, ab)\). Note that the case of \(c, c' \in S_b\) is similar to the above.

**Case 2.** \(c \in S_a\) and \(c' \in S_{ab}\)  Observe that \(d(c, ab) = ||ca||\) and \(d(c', ab) = ||c'x'||\), where \(x'\) is the orthogonal projection of \(c'\) onto \(ab\). Since \(||c'x'|| = \sqrt{||c'a||^2 - ||x'a||^2} \leq ||c'a||\), using the same reasoning as in the previous case, we have \(||c'x'|| \leq ||ca|| + ||cc'|| \leq (1 + \frac{2}{s})||ca||\), which implies that \(d(c', ab) \leq (1 + \frac{2}{s})d(c, ab)\). The above analysis similarly applies to the case where \(c \in S_b\) and \(c' \in S_{ab}\).

**Case 3.** \(c \in S_{ab}\) and \(c' \in S_a\)  Let \(x\) denote the orthogonal projection of \(c\) onto \(ab\). Notice that \(d(c, ab) = ||cx||\) and \(d(c', ab) = ||c'a||\). Through an application of the triangular inequality, we have \(||c'a|| \leq ||cx|| + ||ax|| + ||cc'||\). Since \(c \in S_{ab}\) and \(c' \in S_a\), we have \(||ax|| \leq ||cc'||\). Then, given that \(||cc'|| \leq 2r\) and \(||cx|| \geq sr\), we obtain \(||c'a|| \leq ||cx|| + 2||cc'|| \leq ||cx|| + 4r \leq (1 + \frac{4}{s})||cx||\). That is, \(d(c', ab) \leq (1 + \frac{4}{s})d(c, ab)\). Note that the same analysis applies to the case where \(c \in S_{ab}\) and \(c' \in S_b\).

**Case 4.** \(c \in S_{ab}\) and \(c' \in S_{ab}\)  Let \(x\) and \(x'\) be the orthogonal projections of \(c\) and \(c'\), respectively, onto \(ab\). We then have \(d(c, ab) = ||cx||\) and \(d(c', ab) = ||c'x'||\). Due to the triangle inequality, we obtain \(||c'x'|| \leq ||cx|| + ||cc'|| + ||xx'||\). Notice that \(||xx'|| \leq ||cc'||\). Thus, \(||c'x'|| \leq ||cx|| + 2||cc'|| \leq ||cx|| + 4r \leq (1 + \frac{4}{s})||cx||\), which implies that \(d(c', ab) \leq (1 + \frac{4}{s})d(c, ab)\).
Thus, we have Case 3. That plane determined by points \( \Pi \) be the plane (i.e., 2-flat) determined by points \( \Pi \). Let \( a \) be the orthogonal projection of point \( c \) onto line segment \( ab \). Then, we have \( d(c, ab) = ||ca|| \). Let \( x' \) be the orthogonal projection of point \( c \) onto line segment \( a'b' \). Then, we have \( d(c, a'b') = ||ca'|| \). Observe that \( ||cx'|| = \sqrt{||ca'||^2 - ||a'x'||^2} \leq ||ca'|| \). Then, by following the same argument as in the previous case, we have \( ||cx'|| \leq (1 + \frac{3}{2}\Delta)||ca|| \), which is equivalent to \( d(c, a'b') \leq (1 + \frac{3}{2}\Delta)d(c, ab) \). Due to symmetry, the case of \( c \in S_b \cap S_{a'} \) is similar to the above.

Case 1. \( c \in S_a \) and \( c \in S_{a'} \). Observe that \( d(c, ab) = ||ca|| \), and \( d(c, a'b') = ||ca'|| \). Due to the triangle inequality and the fact that \( A \) is enclosed within a ball of radius \( r \), we have \( ||ca'|| \leq ||ca|| + ||a'a'|| \leq ||ca|| + 2r = ||ca|| + \frac{2}{s}sr \). Since \( sr \), which is the minimum separation distance between \( A \) and \( B \), is bounded from above by \( D \), we have \( ||ca'|| \leq ||ca|| + \frac{2}{s}D \). Notice that \( ||ca|| \geq \mathcal{E} \). We then obtain \( ||ca'|| \leq ||ca|| + \frac{2\mathcal{E}}{s} = (1 + \frac{2\mathcal{E}}{4s})||ca|| = (1 + \frac{3}{2}\Delta)||ca|| \). Thus, \( d(c, a'b') \leq (1 + \frac{3}{2}\Delta)d(c, ab) \). Note that the case of \( c \in S_b \cap S_{a'} \) is similar to the above.

Case 2. \( c \in S_a \) and \( c \in S_{a'b'} \). Note that \( d(c, ab) = ||ca|| \). Let \( x' \) be the orthogonal projection of point \( c \) onto line segment \( a'b' \). Then, we have \( d(c, a'b') = ||ca'|| \). Observe that \( ||cx'|| = \sqrt{||ca'||^2 - ||a'x'||^2} \leq ||ca'|| \). Then, by following the same argument as in the previous case, we have \( ||cx'|| \leq (1 + \frac{3}{2}\Delta)||ca|| \), which is equivalent to \( d(c, a'b') \leq (1 + \frac{3}{2}\Delta)d(c, ab) \). Due to symmetry, the case of \( c \in S_b \cap S_{a'b'} \) is similar to the above.

Case 3. \( c \in S_{ab} \) and \( c \in S_{a'} \). Let \( x \) be the orthogonal projection of point \( c \) onto line segment \( ab \). Observe that \( d(c, ab) = ||cx|| \). In addition, note that \( d(c, a'b') = ||ca'|| \).

**Observation E.1.** \( ||ca'|| \leq ||cx|| + 2r \).

**Proof.** Let \( y \) denote the closest point on line segment \( a'b' \) from point \( x \). Because of the triangle inequality and the fact that \( a' \) is the nearest point on line segment \( a'b' \) from \( c \), we have \( ||ca'|| \leq ||cy|| \leq ||cx|| + ||xy|| \). We claim that \( ||xy|| \leq 2r \), which leads to \( ||ca'|| \leq ||cx|| + 2r \), as observed.

Now we prove the claim that \( ||xy|| \leq 2r \) by using contradiction. Assume that \( ||xy|| > 2r \). Let \( \Pi \) be the plane (i.e., 2-flat) determined by points \( a, b, \) and \( y \). Similarly, let \( \Gamma \) be the plane determined by points \( a', b', \) and \( x \). Observe that the two planes \( \Pi \) and \( \Gamma \) intersect in a straight line containing line segment \( xy \) (Figure 7).

Let \( a^* \) and \( b^* \) be the orthogonal projections of \( a \) and \( b \), respectively, onto plane \( \Gamma \). Note that \( ||a^*a'|| = \sqrt{||aa'||^2 - ||a^*a||^2} \leq ||a'a'|| \leq 2r \). Similarly, we have \( ||b^*b'|| \leq ||bb'|| \leq 2r \). Observe that \( a^*xb^* \) is a line segment in plane \( \Gamma \) specifically, an orthogonal projection of line segment \( axb \) onto plane \( \Gamma \).

Let \( a^{**} \) and \( b^{**} \) be the orthogonal projections of \( a^* \) and \( b^* \), respectively, onto the line containing line segment \( a'yb' \). Since \( a^{**} \) is the nearest point on the line containing line segment \( a'yb' \) from \( a^* \), if \( ||a^*a'|| \leq 2r \), then \( ||a^{**}|| \leq 2r \). Given that \( a^*a^{**} \) is parallel to \( xy \),
and \( a^{**}y \) is orthogonal to \( xy \) and \( a^*a^{**} \), if \( ||xy|| > 2r \), then we have \( ||a^*x|| > ||a^{**}y|| \), which implies that \( \angle a^*xy = \sin^{-1} \frac{||a^{**}y||}{||a^*x||} < \pi/2 \).

Due to symmetry, by using a similar argument as the above, we can also find that \( \angle b^*xy < \pi/2 \). Hence, we have \( \angle a^*xy + \angle b^*xy < \pi \); that is, \( a^*b^* \) is not a line segment, which in turn implies that \( axb \) is not a line segment. Ergo, by contradiction, we have proven that \( ||xy|| \) must be equal to or less than \( 2r \).

\[ \text{\( \Box \)} \]

Due to the observation above, we have \( ||ca'|| \leq ||cd|| + \frac{2r}{\pi}sr \). Since \( sr \leq D \) and \( ||cx|| \geq \mathcal{E} \), we obtain \( ||ca'|| \leq ||cx|| + \frac{2D}{\pi}||cx|| = (1 + \frac{2}{s}b) ||cx|| \), which implies that \( d(c, a'b') \leq (1 + \frac{2}{s}b)d(c, ab) \). Lastly, the case of \( c \in S_{ab} \cap S_{cb} \) is similar to the above because of symmetry.

**Case 4.** \( c \in S_{ab} \) and \( c \in S_{a'b'} \)  
Let \( x \) and \( x' \) be the orthogonal projections of point \( c \) onto line segment \( ab \) and \( a'b' \), respectively. Then, we have \( d(c, ab) = ||cx|| \) and \( d(c, a'b') = ||cx'|| \). Observe that \( ||cx'|| \leq ||cx|| + 2r \), which leads to \( ||cx'|| \leq (1 + \frac{2}{s}b)||cx|| \), as in the previous case. Thus, \( d(c, a'b') \leq (1 + \frac{2}{s}b)d(c, ab) \).

**Case 5.** \( c \in S_a \) and \( c \in S_b \)  
Notice that \( d(c, ab) = ||ca|| \) and \( d(c, a'b') = ||cb'|| \). Since \( b' \) is the closest point on \( a'b' \) from \( c \), we have \( ||cb'|| \leq ||ca'|| \). Using the triangle inequality and a similar argument as before, we obtain \( ||cb'|| \leq ||ca|| + ||ab'|| \leq ||ca|| + 2r \leq (1 + \frac{2}{s}b)||ca|| \). Thus, \( d(c, a'b') \leq (1 + \frac{2}{s}b)d(c, ab) \). Note that the case of \( c \in S_b \cap S_{a'} \) is similar to the above due to symmetry.

Putting all the cases together, we conclude that \( d(c, a'b') \leq (1 + \frac{2}{s}b)d(c, ab) \), regardless of the position of point \( c \) with respect to line segments \( ab \) and \( a'b' \).

### F. Number of well separated pairs in Theorem 2.9

A call to \( \text{WSPD-PS}() \) is said to be terminal if it does not make any call to \( \text{WSPD-PS}() \) (i.e., \( Y \) remains an empty set at all times throughout the call). Thus, a terminal call can generate at most \( 2^{3d} \) well separated pairs. To evaluate the total number of well separated pairs in the resulting \( \text{WSPD-PS} \), it suffices to count the number of terminal calls. Since a terminal call may only arise as a call to \( \text{WSPD-PS}() \) in a non-terminal call, we instead bound the number of calls to \( \text{WSPD-PS}() \) made by all the non-terminal calls. We claim that, in any non-terminal call, for every pair of nodes \( u_i \) and \( v_j \) (iterated in the first two outer loops of the algorithm), the number of calls to \( \text{WSPD-PS}() \) (as in the final for loop in the algorithm) is bounded by \( O(s^d) \). Since there are overall at most \( O(n^2) \) pairs of nodes in the (compressed) quadtree, the total number of calls to \( \text{WSPD-PS}() \) is \( O(n^2s^d) \).
We are now left to establish the claim that for any pair of nodes \( u_i \) and \( v_j \) (of the same level), the number of calls to \([\text{WSPD-PS}]\) is bounded by \( O(s^d) \). For a pair of nodes \( u_i \) and \( v_j \), a call to \([\text{WSPD-PS}]\) is made only if \( \{u_i, v_j\} \) is not \( s \)-well separated from some node \( w_k \). Let \( x \) denote the side length of the cells of nodes \( u_i, v_j, \) and \( w_k \). Let \( r \) be the radius of the Euclidean ball enclosing each of the cells of nodes \( u_i, v_j, \) and \( w_k \). Note that \( r = x\sqrt{d}/2 \).

Assume that \( s \geq 1 \) (if not, let \( s = \max(s, 1) \)). Let \( c_{u_i}, c_{v_j}, \) and \( c_{w_k} \) be the centers of the balls enclosing the cells of nodes \( u_i, v_j, \) and \( w_k \), respectively. Since \( \{u_i, v_j\} \) is not \( s \)-well separated from \( w_k \), the closest distance between line segment \( c_{u_i}c_{v_j} \) and point \( c_{w_k} \) must be at most \( 2r + sr \leq 3sr \). Let \( M \) denote the Minkowski sum of \( c_{u_i}c_{v_j} \) and a ball of radius \( 3sr \). The set of nodes \( w_k \) that are not \( s \)-well separated from \( \{u_i, v_j\} \) must correspond to the cells of side length \( x \) overlapping \( M \). Using a similar packing argument as in the proof of Theorem 2.2 for a pair of nodes \( u_i \) and \( v_j \), the number of such nodes \( w_k \) must be bounded by \( O(s^d) \).

Putting all these together, recall that there are \( O(n) \) nodes in the compressed quadtree, yielding at most \( O(n^2) \) pairs of nodes. Any pair of nodes (in a non-terminal call) invokes \( O(s^d) \) calls to \([\text{WSPD-PS}]\), and a terminal call generates at most \( 2^{3d} \) well-separated pairs. Therefore, the total number of well separated pairs is bounded by \( O(2^{3d}n^2s^d) = O(n^2s^d) \), since \( d \) is a constant.

**G Proof of Lemma 2.11**

Using Theorem 2.9, we first construct an \( s \)-WSPD-PS for \( P \) with \( s > 2 \). We make a small augmentation to the quadtree used for the construction of the WSPD-PS so that each node of the tree is associated with a representative point lying within its cell (if any).

For any well separated pair \( \{\{A, B\}, C\} \) in the \( s \)-WSPD-PS, let \( \text{rep}(A) \), \( \text{rep}(B) \), and \( \text{rep}(C) \) denote the representative points associated with \( A, B, \) and \( C \), respectively. Then, for each well separated pair \( \{\{A, B\}, C\} \) in the \( s \)-WSPD-PS, we compute the shortest distance between point \( \text{rep}(C) \) and the line segment bounded by points \( \text{rep}(A) \) and \( \text{rep}(B) \). We return the smallest such distance. The overall running time is \( O(s^dn^2) \).

We now argue that this approach yields the exact closest distance (not just an approximation). Let \( \{\{x, y\}, z\} \) be the actual closest pair, and let \( \{\{A, B\}, C\} \) be its corresponding well separated pair in the \( s \)-WSPD-PS. If \( x = \text{rep}(A) \), \( y = \text{rep}(B) \), and \( z = \text{rep}(C) \), then the representatives yield the exact closest distance. Suppose that \( x \neq \text{rep}(A) \). Since \( \text{rep}(A), x \in A \), where \( A \) is enclosed within a ball of radius \( r \), we have \( ||\text{rep}(A)x|| \leq 2r \). Recall that \( \{\{A, B\}, C\} \) is an \( s \)-well separated pair, which implies \( d(z, xy) \leq sr \). Given that \( s > 2 \), we have \( ||\text{rep}(A)x|| \leq 2r < sr \leq d(z, xy) \), which contradicts the fact that \( \{\{x, y\}, z\} \) is the closest pair. Thus, \( x = \text{rep}(A) \) must hold. By a symmetrical argument, \( y = \text{rep}(B) \) and \( z = \text{rep}(C) \).

Therefore, we could choose \( s \) to be greater than 2, and the resulting closest representative pair computed from an \( s \)-WSPD-PS would yield the exact closest distance between points of \( P \) and line segments spanned by pairs of points of \( P \).

**H Correctness of algorithm in Theorem 2.12**

Let \( xy \) be an exact discrete median line segment of \( P \). Let \( ab \) be the representative segment returned as the approximate solution by the algorithm \([\text{ApproxMedSeg}]\). To establish the correctness of the algorithm, we have to show that

\[
\sum_{p \in P} d(p, xy) \leq \sum_{p \in P} d(p, ab) \leq (1 + \epsilon) \sum_{p \in P} d(p, xy)
\]
The first inequality holds because no line segment bounded by any other two points can have a smaller distance sum than the exact discrete median line segment \( xy \). To prove the second inequality, consider the segment set \( \{U, V\} \) such that i) \( x, a \in U \) and \( y, b \in V \), and ii) \( x, a \notin U \) and \( y, b \notin V \) for all child segment sets \( \{U', V'\} \) of \( \{U, V\} \).

First, we examine the set of distance computations for the levels above that of \( \{U, V\} \). Let \( \Lambda \) denote the set of all well separated pairs such that for each pair \( \{(U''_i, V''_i), W_i\} \in \Lambda \), \( \{(U''_i, V''_i) \) is an ancestor segment set of \( \{U, V\} \). For each well separated pair \( \{(U''_i, V''_i), W_i\} \in \Lambda \), according to Lemma 2.10 we have

\[
|W_i| \cdot d(rep(W_i), ab) \leq \left( 1 + \frac{2}{s} \right) |W_i| d(rep(W_i), xy)
\]

If we sum over all the pairs in \( \Lambda \), we then have

\[
\sum_i |W_i| \cdot d(rep(W_i), ab) \leq \left( 1 + \frac{2}{s} \right) \sum_i |W_i| d(rep(W_i), xy)
\]

Secondly, we examine the distance computations for the levels below that of \( \{U, V\} \). For any segment set \( \{A, B\} \), let \( \sigma[\{A, B\}, ab] \) denote the distance sum computed for \( \{A, B\} \) in the algorithm, where \( a \in A \) and \( b \in B \) are the representative points used in computing the distance sum. Let \( X, Y \) be the child segment set of \( \{U, V\} \) such that \( x \in X \) and \( y \in Y \). Similarly, let \( \{A, B\} \) denote the child segment set of \( \{U, V\} \) such that \( a \in A \) and \( b \in B \). Note that \( X \cap A = \emptyset \) and \( Y \cap B = \emptyset \).

Let \( \{X', Y'\} \) be the lowest descendent segment set of \( \{X, Y\} \) such that \( x \in X' \) and \( y \in Y' \). Let \( x' \) and \( y' \) be the representative points associated with \( X' \) and \( Y' \). If \( x' = x \) and \( y' = y \), then the distance sum computed for \( \{X', Y'\} \) is \( \sigma[\{X', Y'\}, x'y'] = \sigma[\{X', Y\}, xy] \).

Otherwise, according to Lemma 2.10, we have \( \sigma[\{X', Y'\}, x'y'] \leq (1 + \frac{2}{3}) \sigma[\{X', Y\}, xy] \).

Let \( x^* \in X \) and \( y^* \in Y \) be the pair of representative points used (in the algorithm) to compute the distance sum for segment set \( \{X, Y\} \). Since \( a \) and \( b \) are the pair of representative points produced by the algorithm as the solution, we must have \( \sigma[\{A, B\}, ab] \leq \sigma[\{X, Y\}, x^*y^*] \). If \( x^* = x' \) and \( y^* = y' \), then \( \sigma[\{A, B\}, ab] \leq \sigma[\{X, Y\}, x^*y^*] = \sigma[\{X, Y\}, x'y'] \leq (1 + \frac{3}{s}) \sigma[\{X, Y\}, xy] \). Otherwise, at some level between that of \( \{X, Y\} \) and \( \{X', Y'\} \), \( x' \) and \( y' \) must be “overtaken” by some other pair \( x'' \) and \( y'' \) such that \( x'' \) and \( y'' \) are the representatives of some segment set \( \{X'', Y''\} \), and ii) \( \sigma[\{X'', Y''\}, x''y''] \leq \sigma[\{X'', Y''\}, x'y'] \), where \( \{X'', Y''\} \) is an ancestor segment set of \( \{X', Y'\} \) and of the same level as \( \{X'', Y''\} \) (see Figure 3 for an illustration). Clearly, this sort of “overtaking” could happen multiple times as we ascend the levels from that of \( \{X', Y'\} \) to \( \{X, Y\} \) in the algorithm. At the end of the ascension, \( x^* \) and \( y^* \) prevail, and we have \( \sigma[\{A, B\}, ab] \leq \sigma[\{X, Y\}, x^*y^*] \leq \sigma[\{X, Y\}, x'y'] \leq (1 + \frac{2}{3}) \sigma[\{X, Y\}, xy] \).

As the algorithm terminates, \( \{ab, \sigma\} \) is yielded as the approximate solution, where

\[
\sigma = \sum_i |W_i| \cdot d(rep(W_i), ab) + \sigma[\{A, B\}, ab]
\]

is the minimum distance sum reported along with line segment \( ab \). By applying Lemma 2.10 we have

\[
\sum_{p \in P} d(p, ab) \leq \left( 1 + \frac{4}{s} \right) \left( \sum_i |W_i| \cdot d(rep(W_i), ab) + \sigma[\{A, B\}, ab] \right)
\]

\[
\leq \left( 1 + \frac{4}{s} \right) \left( 1 + \frac{2}{s} \right) \left( \sum_i |W_i| \cdot d(rep(W_i), xy) + \sigma[\{X, Y\}, xy] \right)
\]
Figure 8 Pair \( \{x'', y''\} = \{x^*, y^*\} \) overtakes \( \{x', y'\} \) in the distance sum computation during the execution of the algorithm \( \text{ApproxMedSeg}() \) such that \( \sigma[\{X, Y\}, x^*y^*] \leq \sigma[\{X, Y\}, x'y'] \leq (1 + \frac{2}{s})\sigma[\{X, Y\}, xy] \). Note that the depiction of set \( Y \) is similar to that of set \( X \) and thus omitted.

\[
\begin{align*}
\sum_{p \in P} d(p, ab) &\leq \left( 1 + \frac{4}{s} \right) \left( 1 + \frac{2}{s} \right) \left( 1 + \frac{4}{s} \right) \sum_{p \in P} d(p, xy) \\
&= \left( 1 + \frac{10}{s} + \frac{32}{s^2} + \frac{32}{s^3} \right) \sum_{p \in P} d(p, xy)
\end{align*}
\]

Since \( s = \max(s, 1) \), we obtain

\[
\sum_{p \in P} d(p, ab) \leq \left( 1 + \frac{10}{s} + \frac{64}{s^2} \right) \sum_{p \in P} d(p, xy)
\]

Given an \( \epsilon > 0 \), if we set \( s = \frac{5 + \sqrt{25 + 64\epsilon^2}}{\epsilon} \), then we have

\[
\sum_{p \in P} d(p, ab) \leq (1 + \epsilon) \sum_{p \in P} d(p, xy)
\]

1 Proof of Lemma 3.2

We begin by restating the median line segment problem as follows. Given a set \( P \) of \( n \) points \( p_1, \ldots, p_n \in \mathbb{R}^d \) and a positive real number \( \ell \), locate a line segment \( s = ab \) of length \( \ell \) such that the objective function \( f(s) = \sum_{i=1}^{n} d(p_i, s) \) is minimized.

**Characterization of the median line segment** Given a line segment \( s \), for each point \( p_i \in P \), let \( q_i \) be the closest point on line segment \( s \) to point \( p_i \). Define a unit vector from \( q_i \) to point \( p_i \) as \( \bar{u}(s, p_i) = \frac{p_i - q_i}{||p_i - q_i||} \). Let \( x_i \) and \( y_i \) be the \( x \)- and \( y \)-coordinates of point \( p_i \), respectively. Consider the following characterization of a median line segment.

**Lemma 1.1.** Assume that \( p_1, \ldots, p_n \in \mathbb{R}^d \) are non-collinear points, where \( n \geq 3 \) and \( 2 \leq d \leq 3 \).

**A. Floating case.** When a line segment \( s \) does not contain any given point of \( P \) (i.e., \( s \subset \mathbb{R}^d \setminus P \)), if \( s \) is a median line segment, then

\[
\sum_{i=1}^{n} \bar{u}(s, p_i) = 0 \tag{3}
\]
B. Absorbed case. When a line segment \( s \) contains a given point \( p_j \in P \), if \( s \) is a median line segment, then

\[
\left\| \sum_{1 \leq i \leq n; i \neq j} \bar{u}(s, p_i) \right\| \leq 1
\]  \hspace{1cm} (4)

Proof. See Appendix J.

In proving the floating case of Lemma 1.1 we show that the gradient of the objective function \( f \) must be zero for a median line segment. This leads to the following additional characterization of a median line segment in \( \mathbb{R}^2 \).

Lemma 1.2. Assume that \( p_1, \ldots, p_n \in \mathbb{R}^2 \) are non-collinear points, where \( n \geq 3 \). Let \( a \) and \( b \) be the endpoints of a line segment \( s \). Denote by \( x_a \) and \( y_a \) the \( x \)- and \( y \)-coordinates of \( a \), respectively. Let \( \theta \) be the counterclockwise angle of \( s = ab \) with respect to the positive \( x \)-axis rooted at \( a \). If \( s \) is a median line segment, then we have the following.

A. Floating case. Line segment \( s \) does not contain any point of \( P \).

For \( p_j = a \):

\[
\sum_{1 \leq i \leq n \atop p_i \in S_{ab} \atop p_i \neq p_j} \frac{(x_i - x_a) (-\sin \theta) + (y_i - y_a) \cos \theta}{\sqrt{(x_i - x_a)^2 + (y_i - y_a)^2}} \left[ (x_i - x_a) (\cos \theta) - (y_i - y_a) \sin \theta \right] = 0
\]  \hspace{1cm} (5)

B. Absorbed case. Line segment \( s \) contains a point \( p_j \) of \( P \).

For \( p_j = a \):

\[
\sum_{1 \leq i \leq n \atop p_i \in S_{ab} \atop p_i \neq p_j} \frac{(x_i - x_a) (-\sin \theta) + (y_i - y_a) \cos \theta}{\sqrt{(x_i - x_a)^2 + (y_i - y_a)^2}} \left[ (x_i - x_a) (\cos \theta) - (y_i - y_a) \sin \theta \right] = 0
\]  \hspace{1cm} (6)

For \( p_j \neq a \):

\[
\left\| \sum_{1 \leq i \leq n \atop p_i \in S_{ab} \atop p_i \neq p_j} \frac{(x_i - x_a) (-\sin \theta) + (y_i - y_a) \cos \theta}{\sqrt{(x_i - x_a)^2 + (y_i - y_a)^2}} \left[ (x_i - x_a) (\cos \theta) - (y_i - y_a) \sin \theta \right] \right\| \leq 1
\]  \hspace{1cm} (7)

Proof. See Appendix K.
Geometric interpretation  The geometric characterization above, while being analytical in nature, has a very realistic appeal, especially through a mechanical or physical interpretation. If the unit vectors $\bar{u}(s^*, p_i)$ are seen as forces, then Equation 3 of Lemma I.1 essentially implies that the sum of forces on a median line segment $s^*$ is zero (so that the line segment does not translate). Similarly, according to Equation 4 or Equation 5 of Lemma I.2, the sum of torques (moments) on a median line segment $s^*$ is zero (so that the line segment does not rotate). In other words, a median line segment $s^*$ must be in “static” equilibrium — that is, i) the vectorial sum of the forces acting on a median line segment equals zero, and ii) the sum of the moments (about any arbitrary point) of all forces equals zero.

On the location of the median line segment  For a point set $P$, we denote the convex hull of $P$ by $[P]$, the boundary of $[P]$ by $bd[P]$, and the interior of $[P]$ by $int[P]$.

Lemma I.3. Given a set $P$ of $n$ points in $\mathbb{R}^d$, the intersection of a median line segment $s^*$ and $[P]$ is non-empty — that is, $s^* \cap [P] \neq \emptyset$.

Proof. It suffices to show that if $s^* \cap [P] = \emptyset$, then $\sum_{i=1}^{n} \bar{u}(s^*, p_i) \neq 0$ (because if $\sum_{i=1}^{n} \bar{u}(s^*, p_i) = 0$, then $s^*$ cannot be a median line segment). If $s^* \in \mathbb{R}^d \setminus int[P]$, then there exists a hyperplane $H$ passing through an endpoint of $s^*$ (say, $a^*$) supporting $[P]$ such that i) $P \not\subset H$ (since $int[P]$ is non-empty), and ii) one of the two closed half-spaces bounded by $H$ contains $s^*$ but not $P$, and the other contains $P$ but not $s^*$. Let $S_H$ denote the half-space that contains $s^*$, and $S_H'$ the half-space that contains $P$. All unit vectors $\bar{u}(s^*, p_i)$, where $1 \leq i \leq n$, emanate from $s^* \subset S_H$, passing through $H$, to $p_i \in P \subset S_H'$, and at least one of the unit vectors is not contained in $H$. Thus, the sum of the unit vectors, $\sum_{i=1}^{n} \bar{u}(s^*, p_i)$, has a non-zero component perpendicular to $H$, implying that the sum cannot be zero. ▶

Characterization of the classic Fermat-Torricelli point  Recall that $p^*$ denote the Fermat-Torricelli point for a given point set $\{p_1, p_2, p_3\}$. The characterization of the Fermat-Torricelli point is divided into two different cases.

FT1. If all three angles of the triangle $\triangle p_1p_2p_3$ are less than $2\pi/3$, then $p^*$ is the isogonal point of the triangle $\triangle p_1p_2p_3$.

FT2. If one of the angles of the triangle $\triangle p_1p_2p_3$ is greater than or equal to $2\pi/3$, then $p^*$ is the vertex of the obtuse angle of the triangle $\triangle p_1p_2p_3$.

Now we are in position to prove Lemma I.2.

Part A  Due to Lemma I.3, a median line segment $s^*$ cannot be located such that all three points $\{p_1, p_2, p_3\}$ belong to either $S_a^*$ or $S_b^*$. Furthermore, since $\ell < D$, all three points cannot be in $S_{a^*b^*}$, and one point is in either $S_a^*$ or $S_b^*$, then the sum of the unit vectors from the three given points to line segment $s^*$ has a non-zero component parallel to $s^*$. A similar imbalance occurs if two points lie within either $S_a^*$ or $S_b^*$, and one point is in $S_{a^*b^*}$. Thus, the only possible scenarios worth considering are the following: (I) one point is located in each of $S_a^*$, $S_{a^*b^*}$, and $S_b^*$, and (II) two points belong to either $S_a^*$ or $S_b^*$, and one point is in whichever of $S_a^*$ or $S_b^*$ does not contain the other two points.

We begin by examining scenario (I). As one shall see, scenario (II) is covered in the analysis of scenario (I). Suppose that a line segment $s$ is located such that neither of its endpoints coincides with a point of $\{p_1, p_2, p_3\}$, $p_1 \in S_a$, $p_2 \in S_{ab}$, and $p_3 \in S_b$. Assume, without loss of generality, that $\theta = 0$ (through affine transformation) and $p_3 = p_1$. 


For point $p_1$, let $\alpha_1$ denote the acute angle formed by line segment $ap_1$ and the line supporting $s$ (Figure 9). Similarly, for point $p_3$, define $\alpha_3$ as the acute angle formed by line segment $bp_3$ and the line containing $s$. If $s$ is a median line segment, then due to part A of Lemma 1.1, we must have $\alpha_1 = \alpha_3 = \pi/6$. Let $w_2 \in [0, 1]$ be the distance from $a$ to $q_2$, where $q_2$ is the closest point on $s$ to $p_2$. Denote by $||\bar{d}_2||$ the distance from $q_2$ (on $s$) to $p_2$, and let $||\bar{h}_2|| = ||\bar{d}_2||w_2$. Based on part A of Lemma 1.2, we obtain $\sin \alpha_1 + \sin \alpha_3 = ||\bar{h}_2||/||\bar{d}_2||$, which in turn implies that $||\bar{h}_2|| = ||\bar{d}_2||$; that is, the closest point on $s$ to $p_2$ is $b$ ($q_2 = b$) if $s$ is indeed a median line segment. Hence, we have $p_1 \in S_a$ and $p_2, p_3 \in S_b$, as in scenario (II).

Figure 9 Scenario (I) in proving part A of Lemma 3.2

For any point $p$, let $f(p)$ denote the sum of Euclidean distances from points $p_1$, $p_2$, and $p_3$ to $p$. We know that for any point $b \neq p^*$, $f(p^*) < f(b)$ because $p^*$ is the median point having the minimal sum of distances to the three given points (see Figures 10 and 11 for detailed arguments). Since $||bp_1|| < ||ab|| + ||ap_1||$ by the triangle inequality, we have $f(b) < f(s) + \ell$, which in turn implies that $f(p^*) - \ell < f(s)$; that is, a median line segment must lie in $p^*p_1$ and have an endpoint (in this case, $b$) coinciding with $p^*$. This concludes the proof for part A, and now we proceed to proving part B.

Figure 10 Case FT1. $f(p^*) < f(b)$ for any $b \neq p^*$. Note that $f(p^*) = ||p_2p^*|| + ||p^*Q^*|| + ||Q^*D||$, where $||p^*Q^*|| = ||p^*p_3||$, $||Q^*D|| = ||p^*p_1||$, and $p_2p^*Q^*D$ is a line segment. In addition, $f(b) = ||bp_2|| + ||bQ|| + ||QD||$, where $||bQ|| = ||bp_3||$ and $||QD|| = ||bp_1||$.

Part B When $\ell > ||p^*p_3||$, for $s = ab$ to be a median line segment, an endpoint of $s$ (say, $a$) must coincide with $p_\beta$ (i.e., the vertex of the smallest acute angle of $\triangle p_1p_2p_3$). Assume, without loss of generality, that $p_\beta = p_1$, and $p_1p_3$ lies in the positive $x$-axis rooted at $p_1$. 
In case 1 of part B, it suffices to show that when $\ell \leq \min\{\eta_i : i \neq 1\}$, points $p_2$ and $p_3$ belong to $S_b$ for $0 \leq \theta \leq \angle p_3 p_1 p_2$. Notice that when $p_2, p_3 \in S_b$, Lemma [12] requires that $||\overline{h_2}||/||\overline{d_2}|| = ||\overline{h_3}||/||\overline{d_3}||$. For $i \in \{2, 3\}$, let $\phi_i$ be the acute angle formed by line segment $b^p p_i$ and the line supporting $s^*$. Since $\sin \phi_i = ||\overline{h_i}||/||\overline{d_i}||$, we have $\phi_2 = \phi_3$.

Now we prove that when $\ell \leq \min\{\eta_i : i \neq 1\}$, both $p_2$ and $p_3$ are indeed located in $S_b$ for $0 \leq \theta \leq \angle p_3 p_1 p_2$. We present the argument only for point $p_2$, and a similar argument would apply to $p_3$. Let $\theta_2$ be the angle $\angle b p_1 p_2$ such that $\angle p_1 b p_2 = \pi/2$. Note that $\cos \theta_2 = \ell/||p_1 p_2||$. Moreover, the endpoint $b$ of line segment $s$, with its endpoint $a$ anchored at $p_1$, is the closest point on $s$ to $p_2$, as long as $\theta_2 = \cos^{-1}(\ell/||p_1 p_2||) \geq \angle p_3 p_1 p_2$. Given that

$$\angle p_3 p_1 p_2 = \cos^{-1}\left(\frac{|p_2 p_3|^2 - |p_1 p_2|^2 - |p_1 p_3|^2}{-2 \cdot |p_1 p_2| \cdot |p_1 p_3|}\right)$$

we have

$$\ell \leq \frac{|p_2 p_3|^2 - |p_1 p_2|^2 - |p_1 p_3|^2}{-2 \cdot |p_1 p_3|} = \eta_2$$

where $\eta_2$ is the distance from $p_1$ to the foot of the altitude of the triangle $\triangle p_3 p_1 p_2$ from vertex $p_2$. Hence, if $\ell \leq \min\{\eta_i : i = 1, 2\}$, then $b$ is the closest point on $s$ to both $p_2$ and $p_3$. This concludes the proof for case 1 of part B.

In case 2 of part B, while we have $\ell > \min\{\eta_i : i \neq \beta\}$, four possible scenarios could be encountered, depending on the length $\ell$ of line segment $s$ relative to $||p_1 p_2||$ and $||p_1 p_3||$. Each of the four scenarios consists of the range $0 \leq \theta \leq \angle p_3 p_1 p_2$ divided into at most three distinctive intervals, each of which corresponds to either i) $p_2, p_3 \in S_b$, ii) $p_2 \in S_b$ and $p_3 \in S_{ab}$, iii) $p_2 \in S_{ab}$ and $p_3 \in S_b$, or iv) $p_2, p_3 \in S_{ab}$. For ease of reference, the intervals just described are denoted as $I_{p_2, p_3 \in S_b}$, $I_{p_2 \in S_b, p_3 \in S_{ab}}$, $I_{p_2 \in S_{ab}, p_3 \in S_b}$, and $I_{p_2, p_3 \in S_{ab}}$. Assume, without loss of generality, that $||p_1 p_2|| \leq ||p_1 p_3||$. If $\ell \leq ||p_1 p_2|| = \min\{||p_3 p_i|| : i \neq \beta\}$, then we could have either of the two scenarios (each of which has three intervals): i) $I_{p_2 \in S_b, p_3 \in S_{ab}}$, ii) $I_{p_2 \in S_{ab}, p_3 \in S_b}$. Assume, without loss of generality, that $||p_1 p_2|| \leq ||p_1 p_3||$. If $||p_1 p_2|| < \ell \leq ||p_1 p_3||$, then we have the following scenario with two intervals: $I_{p_2, p_3 \in S_{ab}}$, $I_{p_2 \in S_{ab}, p_3 \in S_b}$. If $\ell > ||p_1 p_3||$, then we have the scenario consisting of a single interval: $I_{p_2, p_3 \in S_{ab}}$.

Recall, that, according to Lemma [12] for $s$ to be a median line segment, we must have $||\overline{h_2}||/||\overline{d_2}|| = ||\overline{h_3}||/||\overline{d_3}||$. Thus, in case 2 of part B, this required condition must be fulfilled in one of the four intervals. Particularly, for interval $I_{p_2, p_3 \in S_{ab}}$, the condition of $||\overline{h_2}||/||\overline{d_2}|| = ||\overline{h_3}||/||\overline{d_3}||$ is equivalent to $\phi_2 = \phi_3$ (as in case 1 of part B). This concludes the proof for part B of the lemma.
Approximating the discrete and continuous median line segments in $d$ dimensions

Remark. Let $\gamma = \arg \min_{i} ||p^* p_i||$. As a consequence of Lemma I.3 (part A), if $l \leq \|p^* p_i\|$, then there is a median line segment $s^*$ such that its endpoint $a^*$ coincides with $p^*$, and $s^*$ lies in either $p^* p_1$, $p^* p_2$, or $p^* p_3$. Hence, $s^*$ is not unique in this case.

J Proof of Lemma I.1

For ease of exposition, we first consider the two-dimensional case ($d = 2$).

Let $a$ and $b$ denote the two endpoints of line segment $s$. Let $x_a$ and $y_a$ be the $x$- and $y$-coordinates of $a$, respectively. The $x$- and $y$-coordinates of $b$ can be expressed as $x_b = x_a + \ell \cos \theta$ and $y_b = y_a + \ell \sin \theta$, respectively, where $\theta$ denotes the counterclockwise angle of $ab$ with respect to the positive $x$-axis rooted at $a$. Note that given a constant $\ell$, line segment $s$ is uniquely defined by three parameters $x_a$, $y_a$, and $\theta$.

Let $x_i$ and $y_i$ be the $x$- and $y$-coordinates, respectively, of any point $p_i \in P$. Assume, without loss of generality, that $\ell = 1$. The objective function $f$ can be written, in terms of the three parameters $x_a$, $y_a$, and $\theta$, as

$$f(x_a, y_a, \theta) = \sum_{1 \leq i \leq n, p_i \in S_a} \sqrt{(x_i - x_a)^2 + (y_i - y_a)^2} + \sum_{1 \leq i \leq n, p_i \in S_{ab}} |(x_i - x_a)(-\sin \theta) + (y_i - y_a)\cos \theta| + \sum_{1 \leq i \leq n, p_i \in S_b} \sqrt{(x_i - x_a - \cos \theta)^2 + (y_i - y_a - \sin \theta)^2}$$

Let $s^*$ denote a median line segment bounded by endpoints $a^*$ and $b^*$. Line segment $s^*$ is defined by $(x_{a^*}, y_{a^*}, \theta^*)$.

Floating case If $s^*$ does not contain any point of $P$ (i.e., $s^* \cap P = \emptyset$), then function $f$ is differentiable at $(x_{a^*}, y_{a^*}, \theta^*)$. Since $(x_{a^*}, y_{a^*}, \theta^*)$ is a minimum point of $f$, the gradient of $f$ must be zero at $(x_{a^*}, y_{a^*}, \theta^*)$; that is, the following must hold:

$$\nabla f(x_{a^*}, y_{a^*}, \theta^*) = \begin{bmatrix} \frac{\partial f}{\partial x} (x_{a^*}, y_{a^*}, \theta^*) \\ \frac{\partial f}{\partial y} (x_{a^*}, y_{a^*}, \theta^*) \\ \frac{\partial f}{\partial \theta} (x_{a^*}, y_{a^*}, \theta^*) \end{bmatrix} = 0$$

Taking the first partial derivative of $f$ with respect to $x_a$ yields

$$\frac{\partial f}{\partial x_a} = -\sum_{1 \leq i \leq n, p_i \in S_a} \frac{x_i - x_a}{\sqrt{(x_i - x_a)^2 + (y_i - y_a)^2}} - \sum_{1 \leq i \leq n, p_i \in S_{ab}} \frac{(x_i - x_a)(-\sin \theta) + (y_i - y_a)\cos \theta}{\sqrt{(x_i - x_a - \cos \theta)^2 + (y_i - y_a - \sin \theta)^2}} + \sum_{1 \leq i \leq n, p_i \in S_b} \frac{x_i - x_a - \cos \theta}{\sqrt{(x_i - x_a - \cos \theta)^2 + (y_i - y_a - \sin \theta)^2}}$$

The minimum of $f$ is obtained when $\frac{\partial f}{\partial x_a} = 0$ and $(x_a, y_a, \theta) = (x_{a^*}, y_{a^*}, \theta^*)$. Notice that each individual $i$-th term of the summation on the right-hand side of the equation above
represents the \(x\)-component of the unit vector \(\bar{u}(s, p_i)\). Hence, the sum of the \(x\)-components of the unit vectors \(\bar{u}(s, p_i)\), where \(1 \leq i \leq n\), must be equal to zero when \(s = s^*\).

Similarly, taking the first partial derivative of \(f\) with respect to \(y_a\) yields

\[
\frac{\partial f}{\partial y_a} = - \sum_{1 \leq i \leq n} \frac{y_i - y_a}{\sqrt{(x_i - x_a)^2 + (y_i - y_a)^2}} - \sum_{1 \leq i \leq n} \frac{(x_i - x_a)(-\sin \theta) + (y_i - y_a)\cos \theta}{[(x_i - x_a)(-\sin \theta) + (y_i - y_a)\cos \theta]} - \sum_{1 \leq i \leq n} \frac{y_i - y_a - \sin \theta}{\sqrt{(x_i - x_a - \cos \theta)^2 + (y_i - y_a - \sin \theta)^2}}
\]

Each \(i\)-th term of the summation on the right-hand side of the equation above represents the \(y\)-component of the unit vector \(\bar{u}(s, p_i)\). Thus, \(\frac{\partial f}{\partial y_a} = 0\) implies that the sum of the \(y\)-components of the unit vectors \(\bar{u}(s, p_i)\), where \(1 \leq i \leq n\), must be zero when \(s = s^*\). We conclude that Equation (3) in the floating case (part A) of the lemma must hold.

**Absorbed case** We now assume that a median line segment \(s^*\) contains a given point of \(P\), say \(p_1\). Suppose that \(a^* = p_1 \in P\). Since \((x_{a^*}, y_{a^*}, \theta^*) = (x_1, y_1, \theta^*)\) is a minimum point of the objective function \(f\), for every unit vector \(\bar{v} \in \mathbb{R}^2\), \(f\) has a directional derivative \(D_{\bar{v}}f(x_1, y_1, \theta^*) \geq 0\), even though \(f\) is not differentiable at \((x_1, y_1, \theta^*)\). Note that \(p_1 \in S_{a^*}\). Define

\[
f_{S_{a^*}}(p_1, a^*) = \|p_1 - a^*\| = \sqrt{(x_1 - x_{a^*})^2 + (y_1 - y_{a^*})^2}
\]

and

\[
f'(x_{a^*}, y_{a^*}, \theta^*) = f'(x_1, y_1, \theta^*) = f(x_1, y_1, \theta^*) - f_{S_{a^*}}(p_1, a^*)
\]

For every unit vector \(\bar{v} \in \mathbb{R}^2\), we have

\[
D_{\bar{v}}f(x_1, y_1, \theta^*) = D_{\bar{v}}f'(x_1, y_1, \theta^*) + D_{\bar{v}}f_{S_{a^*}}(p_1, a^*)
\]

Note that

\[
D_{\bar{v}}f'(x_1, y_1, \theta^*) = \nabla f(x_1, y_1, \theta^*) \cdot \bar{v}
\]

and

\[
D_{\bar{v}}f_{S_{a^*}}(p_1, a^*) \big|_{p_1 = a^*} = \lim_{t \to 0} \frac{f_{S_{a^*}}(p_1 + t\bar{v}, a^*) - f_{S_{a^*}}(p_1, a^*)}{t} \big|_{p_1 = a^*}
\]

\[
= \lim_{t \to 0} \frac{\|p_1 + t\bar{v} - a^*\| - \|p_1 - a^*\|}{t} \big|_{p_1 = a^*}
\]

\[
= \lim_{t \to 0} \frac{\|t\bar{v}\|}{t} = 1
\]

We thus have

\[
D_{\bar{v}}f(x_1, y_1, \theta^*) = D_{\bar{v}}f'(x_1, y_1, \theta^*) + D_{\bar{v}}f_{S_{a^*}}(p_1, a^*)
= \nabla f(x_1, y_1, \theta^*) \cdot \bar{v} + 1
\]
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Particularly, consider
\[ \vec{v} = -\frac{\nabla f(x_1, y_1, \theta^*)}{\|\nabla f(x_1, y_1, \theta^*)\|} \]

Then,
\[ D_v f(x_1, y_1, \theta^*) = \nabla f(x_1, y_1, \theta^*) \cdot \left( -\frac{\nabla f(x_1, y_1, \theta^*)}{\|\nabla f(x_1, y_1, \theta^*)\|} \right) + 1 \]
\[ = -\|\nabla f(x_1, y_1, \theta^*)\| + 1 \]

Since $D_v f(x_1, y_1, \theta^*) \geq 0$, we have
\[ -\|\nabla f(x_1, y_1, \theta^*)\| + 1 \geq 0 \]
\[ \|\nabla f(x_1, y_1, \theta^*)\| \leq 1 \]

We achieve the same conclusion when we apply the same argument above to the case of $p_1 \in S_{ab}$ and $p_1 \in S_b$. We have thus proven the absorbed case (part B) of the lemma.

The arguments just described for two dimensions ($d = 2$) can be easily applied to prove the lemma for the case of three-dimensional space ($d = 3$). The only difference is the initial formulation of function $f$, which is expressed in terms of five parameters – namely, $x_a$, $y_a$, $z_a$, $\theta$, and $\phi$, where $z_a$ is the $z$-coordinate of $a$, $\theta$ is the azimutal angle of $ab$ (with respect to the $x$-axis rooted at $a$), and $\phi$ is the polar angle of $ab$ (with respect to the $z$-axis rooted at $a$). This concludes the proof of the lemma.

**Remark.** Generally, the converse of Lemma I.1 is not true because the objective function $f$ is not convex. In fact, $f$ is a sum of convex and quasi-concave functions. While a minimum point of $f$ must satisfy Lemma I.1, a stationary point of $f$ satisfying Lemma I.1 may not be a minimum of $f$. In addition, since function $f$ may not be, in general, strictly convex, a minimum point of $f$ may not be unique – that is, there may exist multiple median line segments for an instance of the problem with $n \geq 3$ non-collinear points (see the remark in Appendix I for an example).

**K Proof of Lemma I.2**

In the floating case, we have previously shown that the gradient of $f$ must be zero for a median line segment $s^*$. Thus, the first partial derivative of $f$ with respect to $\theta$,
\[ \frac{\partial f}{\partial \theta} = \sum_{1 \leq i \leq n} \frac{(x_i - x_a)(-\sin \theta) + (y_i - y_a)\cos \theta}{|x_i - x_a|(-\sin \theta) + (y_i - y_a)\cos \theta} [(x_i - x_a)(-\cos \theta) - (y_i - y_a)\sin \theta] \]
\[ + \sum_{1 \leq i \leq n} \frac{(x_i - x_a - \cos \theta)\sin \theta - (y_i - y_a - \sin \theta)\cos \theta}{\sqrt{(x_i - x_a - \cos \theta)^2 + (y_i - y_a - \sin \theta)^2}} \]

must have a value of zero at $(x_{a^*}, y_{a^*}, \theta^*)$, proving the floating case of the lemma.

In the absorbed case, when a median line segment passes through a point $p_j$ of $P$ at $a$, we can think of it as the median line segment problem with a fixed endpoint $a$, where $a = p_j \in P$. Then, while $(x_a, y_a)$ is fixed at $(x_j, y_j)$, the associated objective function $f$ becomes one of a single variable $\theta$ and a summation of distances over $n - 1$ points of $P \setminus \{p_j\}$. Since $f(\theta)$ attains a minimum at $\theta^*$, the gradient of $f(\theta)$ must be zero at $\theta = \theta^*$, where the gradient of $f(\theta)$ is simply the first derivative of $f$ with respect to $\theta$, which is in turn simply
Equation (6) in part B of the lemma. If a median line segment passes through a point of \( P \) at a point in \( P \) other than \( a \), then we can use the same line of argument detailed in the proof of Lemma 1.1 to reach the conclusion that the magnitude of the gradient of \( f \), including that of \( \frac{\partial f}{\partial x} \), must be less than or equal to one in the absorbed case. This concludes the proof of the absorbed case of the lemma.

\[ \text{Remark.} \] Let \( L_a \) be the line supporting \( s^* \). Recall that the length \( \ell \) of \( s^* \) is assumed to be one without loss of generality. For each point \( p_i \in P \), let \( w_i \) denote the distance from \( a^* \) to \( q_i \), where \( q_i \) is the closest point on \( s^* \) to \( p_i \). Note that \( w_i \in [0, 1] \). Equation (5) (or Equation (6)) in Lemma 1.2 claims that for all points \( p_i \in S_{a^*b^*} \cup S_{b^*} \), the sum of the components of the unit vectors \( \bar{u}(s^*, p_i) \) normal to \( L_{s^*} \) multiplied (scaled) by \( w_i \) must be zero. For a point \( p_i \in S_{a^*b^*} \cup S_{b^*} \), let \( d_i \) denote the vector from \( q_i \) to \( p_i \), and let \( h_i \) be the component of \( d_i \) normal to \( L_{s^*} \) multiplied by \( w_i \). See Figure 12 for an illustration, in which \( p_1 \) is a point in \( S_{a^*b^*} \), and \( p_2 \) is a point in \( S_{b^*} \). For point \( p_1 \), the component of the unit vector \( \bar{u}(s^*, p_1) \) normal to \( L_{s^*} \) multiplied by \( w_1 \) is equal to \( \bar{h}_1/||d_1|| \), and \( \bar{h}_2/||d_2|| \) for point \( p_2 \).

![Figure 12 Geometric characterization of part A of Lemma 1.2](image)

\[ \text{L Proof of Theorem 4.2} \]

At first, we build a BBD-tree for \( P \) in \( O(n \log n) \) time. The BBD-tree is of size \( O(n) \) and height \( O(\log n) \). Recall that \( P \) has been scaled so that it is contained within a unit hypercube. Then, the cells of the BBD-tree have sizes that are powers of \( 1/2 \). Define size group \( i \) to be the set of nodes whose cell size is \( 1/2^i \). Define the base group for size group \( i \) to be the subset of nodes in the size group that are leaf nodes or whose children belong to the next smaller size group. Let \( i = \alpha \) be the smallest size group. Since at most \( 2d \) levels of ancestors above the nodes in each \( i \)-th base group can be in the same size group \( i \), and the height of the BBD-tree is \( O(\log n) \), we have \( \alpha = O(\log n) \).

We construct a (region) quadtree for the unit hypercube \([0, 1]^d\), where the leaf nodes of the quadtree have a cell size of \( 1/2^\alpha \). The quadtree is of size \( \sum_{i=0}^{\alpha} 2^d = \sum_{i=0}^{\log n} 2^d = O(2^{d \log n}) = O(n^d) \), and can be constructed in the same amount of time.

An \( s \)-WSPD-PS-\( \ell \) is constructed using a recursive procedure similar to that for an \( s \)-WSPD-PS (see Theorem 2.9) but with the following differences. Throughout the recursive algorithm, we maintain a collection of pairs that satisfy properties (I), (III), and (V) as stated in Definition 4.1. When the algorithm ends, all output pairs will be well separated and thus fulfilling properties (II) and (IV). Each pair \( \{(A_i, B_i), C_i\} \) in the WSPD-PS-\( \ell \) will be represented implicitly as a pair composed of three nodes \( \{u, v, w\} \), where \( i \) u, v, and w
have the same cell size, ii) $u$ and $v$ are nodes of the same level in the quadtree, and iii) $w$ is a node in the BBD-tree.

For any pair \((\{u,v\},w)\) in the decomposition process, if \(\{u,v\}\) is an $\ell$-segment set and $s$-well separated from $w$, then we output the pair. Otherwise, let $j$ be the size group to which node $w$ belongs. Let $W'$ be the set of descendants of node $w$ whose size group is $j + 1$ (i.e., next smaller size group). Note that it takes at most $2d$ levels of descent to reach $W'$ in the BBD-tree. Let $U'$ and $V'$ be the set of children of nodes $u$ and $v$, respectively, in the quadtree. For each \(\{u',v'\}\) where $u' \in U'$ and $v' \in V'$, if \(\{u',v'\}\) is an $\ell$-segment set, then we recursively try to separate \(\{u',v'\}\) from nodes $w' \in W'$. This recursive procedure ensures that all the pairs considered involve nodes having the same cell size (even though the nodes are in two different trees).

Let $\text{WSPD-PS-}\ell(u,v,w,s,\ell,\text{par})$ denote the resulting algorithm after adapting $\text{WSPD-PS}(u,v,w,s,\text{par})$ to construct an $s$-WSPD-PS-\ell. To prove that the number of well separated pairs in an $s$-WSPD-PS-\ell is bounded by $O(s^d n^d)$, we use an argument similar to that for an $s$-WSPD-PS. We evaluate the number of calls to $\text{WSPD-PS-}\ell()$ made by all the non-terminal calls. In any non-terminal call, for a pair of nodes $u_i$ and $v_j$ (at a given level in the quadtree), the number of calls to $\text{WSPD-PS-}\ell()$ is bounded by $O(s^d)$. Since there are $O(n \log n)$ levels in the quadtree, and the number of pairs of nodes at each level $i$ is bounded by $2di$ and $2di = 4di$, we have at most $\sum_{i=0}^{\log n} 4^di = O(4^d \log n) = O(2^d n^d) = O(n^d)$ pairs of nodes $\{u_i, v_j\}$. Thus, the total number of calls to $\text{WSPD-PS-}\ell()$ is $O(n^d s^d)$.

Now we are left to show that for any pair of nodes $u_i$ and $v_j$ (of the same level in the quadtree), the number of calls to $\text{WSPD-PS-}\ell()$ is bounded by $O(s^d)$. As with the procedure $\text{WSPD-PS}(u,v,w,s,\text{par})$, for any pair of nodes $\{u_i, v_j\}$, a call to $\text{WSPD-PS-}\ell()$ is made only if $\{u_i, v_j\}$ is not $s$-well separated from some node $w_k$. Using the same packing argument as given in the proof of Theorem 2.9, the number of such nodes $w_k$ can be shown to be bounded by $O(s^d)$, provided that $\ell \leq D \leq \sqrt{d}$.

Recall that each non-terminal node in the BBD-tree has two children, and for any node in the BBD-tree, it takes at most $2d$ levels of descent to reach its descendants with half of its cell size. Thus, for a given node in the BBD tree, the number of such descendants is bounded by $2^2d$. Consequently, a terminal call to $\text{WSPD-PS-}\ell()$ can generate at most $2^d \cdot 2^d \cdot 2^d = 2^{4d}$ well-separated pairs. Since a terminal call may occur only as a call to $\text{WSPD-PS-}\ell()$ in a non-terminal call, the total number of well separated pairs in an $s$-WSPD-PS-\ell is bounded by $O(2^{4d} s^d n^d) = O(s^d n^d)$, given that $d$ is treated as a constant. Finally, together with time $O(n \log n)$ and $O(n^d)$ to build the BBD-tree and quadtree, respectively, the total running time of the algorithm is $O(n \log n + n^d + n^d s^d) = O(n \log n + s^d n^d)$.

**M Proof of Theorem 5.1**

Let $s$ denote any line segment of length $\ell$ with an endpoint fixed at $q$. Assume, without loss of generality, that the fixed endpoint of line segment $s$ is $a = q = (0,0)$ (by a translation of $P$), and the length of line segment $s$ is $\ell = 1$ (through a uniform scaling of $P$). Let $\theta$ be the counterclockwise angle of line segment $s$ with respect to the positive $x$-axis rooted at $a$. The sum of the distances from $P = \{p_1, p_2, ..., p_n\}$ to line segment $s$ is given by the following objective function:

$$f(\theta) = \sum_{1 \leq i \leq n} \sqrt{x_i^2 + y_i^2}$$
where \( x_i \) and \( y_i \) are the \( x \)- and \( y \)-coordinates of \( p_i \in P \), respectively. We consider \( \theta \in [0, \pi/2) \) only, and each subsequent quadrant can be handled analogously. The quadrant \([0, \pi/2)\) can be divided into a set \( T \) of at most \( \Theta(n) \) contiguous intervals, in each of which the subsets of points of \( P \) in \( S_a, S_{+ab}, S_{-ab}, S_{+b}, \) and \( S_{-b} \), respectively, remain constant. We partition each interval of \( T \) into a number of small sub-intervals such that the relative error in computing the sum of the distances from \( P \) to a line segment \( s \), whose angle \( \theta \) is given by a boundary of a sub-interval, does not exceed \( \epsilon \). To evaluate the number of sub-intervals, we perform the following analysis.

Let \( I \) denote a sub-interval. Suppose that the optimal line segment \( s^* \) lies within \( I \). First, we note that the distance from any given point \( p_i \in S_a \) to endpoint \( a \) of line segment \( s \) remains constant within sub-interval \( I \). For simplicity of notation, the subscript \( i \) is dropped, and \( p \) is equivalent to \( p_i \) hereafter.

![Figure 13](image)

Figure 13 A point \( p \in P \) located in (A) \( S_{+ab}^+ \) or (B) \( S_b^+ \).

For a point \( p \in S_{+ab}^+ \), let \( d_p = d(p, s) \) denote its orthogonal distance to a line segment \( s \) whose location is defined by a boundary of interval \( I \) (Figure 13A). Suppose that \( d_p^* = d(p, s^*) \) is the distance from \( p \) to the optimal line segment \( s^* \). We rotate the coordinate system such that the positive \( x \)-axis contains \( s \), and the first quadrant of the defined \( xy \)-plane contains sub-interval \( I \) (and thus \( s^* \)). Specifically, consider the worst-case scenario where \( s \) and \( s^* \) are located at the two ends of sub-interval \( I \). Let \( \Delta \theta \) be the size of sub-interval \( I \). In addition, let \( x_p \) and \( y_p \) denote the \( x \)- and \( y \)-coordinates, respectively, of point \( p \). In order to have \( d_p \leq (1 + \epsilon) d_p^* \), the following must hold:

\[
\begin{align*}
    d_p &\leq (1 + \epsilon) d_p^* \\
    y_p &\leq (1 + \epsilon) (-x_p \sin \Delta \theta + y_p \cos \Delta \theta) \\
    \frac{1}{1 + \epsilon} &\leq \frac{x_p}{y_p} \sin \Delta \theta + \cos \Delta \theta \\
    &= \sqrt{1 + \left( \frac{x_p}{y_p} \right)^2} \cos \left( \Delta \theta + \tan^{-1} \frac{x_p}{y_p} \right)
\end{align*}
\]
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\[ \Delta \theta \leq \cos^{-1} \left( \frac{1}{(1 + \epsilon) \sqrt{1 + \left( \frac{1}{y_p} \right)^2}} \right) - \tan^{-1} \frac{x_p}{y_p} \]

Let $A_{ab,p}^+$ denote the right-hand term of the last inequality above. Given that

\[ A_{ab,p}^+ \geq \epsilon \left( \cos^{-1} \left( \frac{1}{2 \sqrt{1 + \left( \frac{x_p}{y_p} \right)^2}} \right) - \tan^{-1} \frac{x_p}{y_p} \right) = \epsilon \alpha_{ab,p}^+ \]

for $0 < \epsilon < 1$, if we have $\Delta \theta = \epsilon \alpha_{ab,p}^+$, then the desired condition $d_p \leq (1 + \epsilon)d_p^*$ is fulfilled.

Note that $\alpha_{ab,p}^+$ is a trigonometric function in terms of the coordinates of point $p$. We can satisfy $d_p \leq (1 + \epsilon)d_p^*$ for all points $p \in S_{ab}^+$ if we set $\Delta \theta = \epsilon \cdot \min \{ \alpha_{ab,p}^+ : p \in S_{ab}^+ \}$.

The analysis for $S_{ab}^+$ is similar to that for $S_{ab}^-$ due to symmetry, and we obtain $\{ \alpha_{ab,p}^- : p \in S_{ab}^- \}$ accordingly.

We can also perform a similar analysis for each point $p \in S_b^+$. Let $d_p = d(p,s)$ denote the distance from $p$ to endpoint $b$ of a line segment $s$ located at a boundary of sub-interval $I$ (Figure 13.3). Let $d_p^* = d(p, s^*)$ be the shortest distance from $p$ to the optimal line segment $s^*$. As before, we define a coordinate system on $s$ such that the positive $x$-axis contains $s$, and the first quadrant of the $xy$-plane contains sub-interval $I$, at whose boundaries $s$ and $s^*$ are positioned. Let $\Delta \theta$ be the size of sub-interval $I$. If $d_p \leq (1 + \epsilon)d_p^*$, then we have

\[ d_p \leq (1 + \epsilon)d_p^* \]

\[ \sqrt{(x_p - 1)^2 + y_p^2} \leq (1 + \epsilon) \sqrt{(x_p - \cos \Delta \theta)^2 + (y_p - \sin \Delta \theta)^2} \]

\[ \frac{(x_p - 1)^2 + y_p^2}{(1 + \epsilon)^2} \leq (x_p - \cos \Delta \theta)^2 + (y_p - \sin \Delta \theta)^2 \]

\[ = x_p^2 - 2x_p \cos \Delta \theta + y_p^2 - 2y_p \sin \Delta \theta + 1 \]

\[ \frac{1}{2} \left( \frac{(x_p - 1)^2 + y_p^2}{(1 + \epsilon)^2} - x_p^2 - y_p^2 - 1 \right) \geq x_p \cos \Delta \theta + y_p \sin \Delta \theta \]

\[ = \sqrt{x_p^2 + y_p^2} \cos \left( \Delta \theta + \tan^{-1} \left( \frac{-x_p}{y_p} \right) \right) \]

\[ \Delta \theta \leq \tan^{-1} \left( \frac{x_p}{y_p} \right) - \cos^{-1} \left( \frac{1}{2 \sqrt{x_p^2 + y_p^2}} \left( \frac{(x_p - 1)^2 + y_p^2}{(1 + \epsilon)^2} - x_p^2 - y_p^2 - 1 \right) \right) \]

Let $A_{b,p}^+$ denote the right-hand side of the last inequality above. Since

\[ A_{b,p}^+ \geq \epsilon^2 \left[ \tan^{-1} \left( \frac{x_p}{y_p} \right) - \cos^{-1} \left( \frac{1}{2 \sqrt{x_p^2 + y_p^2}} \left( \frac{(x_p - 1)^2 + y_p^2}{(1 + \epsilon)^2} - x_p^2 - y_p^2 - 1 \right) \right) \right] = \epsilon^2 \alpha_{b,p}^+ \]

where $\epsilon' = \min(1, \epsilon_p)$,

\[ \epsilon_p = \sqrt{\frac{(x_p - 1)^2 + y_p^2}{\left( x_p - \frac{x_p}{\sqrt{x_p^2 + y_p^2}} \right)^2 + \left( y_p - \frac{y_p}{\sqrt{x_p^2 + y_p^2}} \right)^2}} - 1 \]
We denote by $\Delta \theta = \epsilon^2 \alpha_{b,p}^+$, then $d_p \leq (1 + \epsilon)d_p^*$ is satisfied. Note that $\alpha_{b,p}^+$ is a trigonometric function dependent on the coordinates of point $p$. In order to uphold $d_p \leq (1 + \epsilon)d_p^*$ for all points $p \in S_b^+$, we can simply set $\Delta \theta = \epsilon^2 \cdot \min \{\alpha_{b,p}^+ : p \in S_b^+\}$.

Points $p \in S_b^+$ can be handled analogously as those in $S_b^-$, and we obtain $\{\alpha_{b,p}^- : p \in S_b^-\}$ as the result.

In summary, for each given interval $\tau \in T$, we compute $\alpha_{ab}^+ = \min \{\alpha_{ab,p}^+ : p \in S_{ab}^+\}$, $\alpha_{ab}^- = \min \{\alpha_{ab,p}^- : p \in S_{ab}^-\}$, $\alpha_{b}^+ = \min \{\alpha_{b,p}^+ : p \in S_b^+\}$, and $\alpha_{b}^- = \min \{\alpha_{b,p}^- : p \in S_b^-\}$. We then use $\Delta \theta = \min \{\epsilon \alpha_{ab}^+, \epsilon \alpha_{ab}^-, \epsilon^2 \alpha_b^+, \epsilon^2 \alpha_b^-\}$ for partitioning the given interval $\tau$ into sub-intervals of size at most $\Delta \theta$.

We now derive an upper bound on the number of sub-intervals as follows. Let $s(\tau)$ denote the set $\{\alpha_{ab}^+, \alpha_{ab}^-, \alpha_b^+, \alpha_b^-\}$ computed for each interval $\tau$ of $T$. Define $\alpha_{\theta} = \min \{\alpha \in s(\tau) : \tau \in T\}$. Then, we have a total of $2\pi/(\epsilon^2 \alpha_{\theta})$ sub-intervals in the worst case. Since it takes $O(n)$ algebraic operations to compute the sum of distances for each candidate line segment (defined by the boundaries of the sub-intervals), we can obtain a solution, whose sum of distances to $P$ is at most $(1 + \epsilon)$ times that of the optimal solution, in $2\pi n/(\epsilon^2 \alpha_{\theta}) = O(ne^{-2}\alpha_{\theta}^{-1})$ time.

**Proof of Theorem 5.2**

We denote by $s = ab$ any line segment of length $\ell$ making angle $\theta$ with the positive $x$-axis. Assume, without loss of generality, that $\theta = 0$ and $\ell = 1$. Let $x_a$ and $y_a$ be the $x$- and $y$-coordinates of the endpoint $a$ of line segment $s$, respectively. Then, the sum of the distances from $P = \{p_1, p_2, ..., p_n\}$ to line segment $s$ can be written as the following objective function:

$$f(x_a, y_a) = \sum_{1 \leq i \leq n} \sqrt{(x_i - x_a)^2 + (y_i - y_a)^2} + \sum_{1 \leq i \leq n} (y_i - y_a) + \sum_{1 \leq i \leq n} (y_a - y_i) + \sum_{1 \leq i \leq n} \sqrt{(x_i - x_a - 1)^2 + (y_i - y_a)^2}$$

where $x_i$ and $y_i$ are the $x$- and $y$-coordinates of $p_i \in P$, respectively.

**Remark.** $f$ is a piecewise convex function, where each piece consists of a sum of two convex functions and two linear functions, and the transition between any two consecutive pieces corresponds to a point of $P$ moving between $S_a$, $S_{ab}^+$, $S_{ab}^-$, and $S_b$. Since the number of such transitions is bounded by $\Theta(n)$, the minimum of function $f$ can be obtained by solving $\Theta(n)$ two-variable convex optimization problems.

We begin by defining the so-called $k$-oriented distance function $d_k$ [13, 21] to approximate the Euclidean distance $d$ as follows.

**$k$-oriented distance** A cone in $\mathbb{R}^2$ is defined as the intersection of two half-planes, each of whose supporting lines contains the origin $O$. A simplicial cone $c$ has a diameter bounded by an angle $\gamma$ if, for any two points $p$ and $q$ in $c$, we have $\angle pOq \leq \gamma$. Let $C = \{c_1, ..., c_k\}$ be a set of $k$ cones, each of which has a diameter bounded by $\gamma$, and $C$ forms a partition of $\mathbb{R}^2$. Note that $k$ is a function of $\gamma$. Thus, $C$ could be a set of cones defined by the rays originating at $O$ making angles $\{(i - 1)2\pi/k : 1 \leq i \leq k\}$ with respect to the abscissa axis. The two rays that bound a cone $c$ are called the axes of $c$. For a point $p \in \mathbb{R}^2$, let $t_i(p)$ denote
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$p$ represented in the coordinate system whose axes are those of $c_i$. For a point $p$ in a cone $c_i$, $d_k(O, p) = ||t_i(p)||$ is called the $k$-oriented distance from $O$ to $p$, and is defined as the length of the shortest path from $O$ to $p$ traveling only in the directions parallel to the axes of $c_i$. For any two points $p$ and $q$ in $c_i$, we have $d_k(p, q) = d_k(O, q - p)$. Notice that, if $\gamma = \pi/4$, then the corresponding $d_k$ is known as the rectilinear (Manhattan) distance function. For any two points $p, q \in \mathbb{R}^2$, $d(p, q) \leq d_k(p, q) \leq (1 + \epsilon)d(p, q)$, where $\epsilon$ is a positive constant that decreases as $k$ increases. Specifically, $\epsilon = \frac{1}{\sqrt{\cos(2\pi/k)}} - 1$.

Recall that the objective function $f(s)$ denotes the sum of the Euclidean distances from $P$ to $s$. We can approximate $f(s)$ using

$$f_k(s) = \sum_{1 \leq i \leq n} d_k(p_i, a) + \sum_{1 \leq i \leq n} d_k(p_i, b) + \sum_{1 \leq i \leq n} y_i - y_a + \sum_{1 \leq i \leq n} y_a - y_i$$

Observe that function $f_k(s)$ is convex and piecewise linear. Hence, we can find the minimum of $f_k(s)$ using the prune-and-search approach described by Bose et al. [5] after some careful modifications.

**Prune and search** Consider the set of cones $C$ used in evaluating $d_k$. Recall that each cone $c \in C$ is defined by two lines. Let $L$ be the set of lines defining $C$. For each point $p \in P$, we create a point at a distance $\ell$ to the right of $p$. Let $P'$ denote the newly created set of points. For each point $p \in P \cup P'$, we construct a copy of $L$ such that each of the lines in $L$ passes through $p$. The result is an arrangement of lines $A$. Observe that each cell of $A$ corresponds to a linear piece of the surface $f_k$. Consequently, $f_k$ reaches a minimum when the endpoint $a$ of line segment $s$ coincides with a vertex of $A$.

We now describe a prune-and-search algorithm to find the lowest point on the surface $f_k$. Note that $A$ consists of $k$ sets of parallel lines. Let $H_i$ denote a given set of parallel lines in $A$, where $1 \leq i \leq k$. We begin by finding a median line $h \in H_i$ that divides $H_i$ into two nearly equal sets. Line $h$ partitions $\mathbb{R}^2$ into two half-planes, $h_1$ and $h_2$, one of which contains a minimum of $f_k$. Suppose, without loss of generality, that $h_1$ contains the minimum. Then, we can simply ignore all the lines in $h_2$, and continue to recurse on $h_1$. This recursive process takes $O(\log n)$ rounds for each set $H_i$.

In each aforesaid round, we first find a point $p_h$ on $h$ that minimizes $f_k$. We can then, based on $p_h$, determine if the minimum lies in $h_1$ or $h_2$.

The problem of finding $p_h$ is a one-dimensional instance of our problem (i.e., constrained to line $h$). Since $f_h$ is piecewise linear, $p_h$ lies on an intersection of $h$ with some other line in $H = \{H_1, \ldots, H_k\} \setminus h$. Hence, we i) compute all the intersections of $h$ with $H$, ii) find the median intersection point $q_m$ and the two intersection points $q_1$ and $q_2$ that are adjacent to $q_m$ on $h$, and iii) determine if $p_h$ lies to the left of $q_m$, right of $q_m$, or is $q_m$ by evaluating $f_k(q_m), f_k(q_1)$, and $f_k(q_2)$.

Let $u$ be the size of $H$. The time complexity of finding $p_h$ is given by the recurrence relation $T(u) = T(u/2) + O(u + Q(n))$, where $Q(n)$ denotes the query time to evaluate $f_k$. This recurrence solves to $O(u + Q(n)\log u)$.

After finding $p_h$, we determine whether the minimum lies in $h_1$ or $h_2$ as follows. Consider two opposite rays $r_1$ and $r_2$, which are i) originating at $p_h$, ii) orthogonal to $h$, and iii) contained in $h_1$ and $h_2$, respectively. We identify the first lines $h_{r_1}$ and $h_{r_2}$ intersected by $r_1$ and $r_2$, respectively. Let $v_1$ (resp. $v_2$) be the intersection point of $h_{r_1}$ and $r_1$ (resp. $h_{r_2}$ and $r_2$). There are three possible cases to be considered: (1) If $f_k(v_1) \leq f_k(p_h) \leq f_k(v_2)$, then a minimum of $f_k$ lies in $h_1$. (2) If $f_k(v_1) \geq f_k(p_h) \geq f_k(v_2)$, then a minimum of $f_k$ lies
in $h_2$. (3) If $f_k(v_1) > f_k(p_h)$ and $f_k(v_2) > f_k(p_h)$, then $p_h$ is a minimum of $f_k$. Verifying these cases require the computation of all the intersections of $H$ with $v_1$ and $v_2$, and the evaluation of $f_k$ at $v_1$ and $v_2$. So, the time complexity of determining whether a minimum lies in $h_1$ or $h_2$ is $O(u + Q(n))$.

Observe that $u = O(kn)$. Thus, the time taken by the recursive procedure for each set $H_i$ is given by the recurrence relation $T(n) = T(kn/2) + O(kn + Q(n) \log kn)$, which solves to $O(kn + Q(n) \log kn)$. Given that we have $k$ sets $H_i$, the overall time taken by the prune-and-search algorithm to compute the point that minimizes $f_k$ is $O(P(n) + k(kn + Q(n) \log kn))$, where $P(n)$ is the preprocessing cost to construct the data structure for evaluating $f_k$, and $Q(n)$ is the query time to evaluate $f_k$.

**Computing $f_k$** Recall that the set of cones $C$ is defined by the rays originating at $O$ with directions $\{(j - 1)2\pi/k : 1 \leq j \leq k\}$. Let $C^a = \{c^a_j : 1 \leq j \leq k\}$ be the set of cones $C$ after being translated such that the apex of the cones is located at $a$. Similarly, let $C^b = \{c^b_j : 1 \leq j \leq k\}$ denote the set of cones $C$ translated such that the apex of the cones is at $b$. Without loss of generality, assume that the value of $k$ is chosen such that $(j - 1)2\pi/k$ is equal to $\pi/2$, $\pi$, and $3\pi/2$ for some values of $j$. Let $\beta = (j - 1)2\pi/k$, where $1 \leq j \leq k$. Let $C_I, C_{II}, C_{III}$, and $C_{IV}$ denote the subset of cones in $C$ contained within quadrant I ($0 \leq \beta \leq \pi/2$), quadrant II ($\pi/2 \leq \beta \leq \pi$), quadrant III ($\pi \leq \beta \leq 3\pi/2$), and quadrant IV ($3\pi/2 \leq \beta \leq 2\pi$), respectively. We define similar quadrant notations for $C^a$ and $C^b$. Let $[X]$ be an indicator function: $[X]$ equals to one if the predicate $X$ is true, and zero otherwise.

Then, our objective function can rewritten as

$$
f_k(s) = \sum_{i=1}^{n} d_k(p_i, a) \left[ p_i \in c^a : c^a \in C^a_{II} \cup C^a_{III} \right]$$

$$+ \sum_{i=1}^{n} d_k(p_i, b) \left[ p_i \in c^b : c^b \in C^b_{II} \cup C^b_{IV} \right]$$

$$+ \sum_{i=1}^{n} (y_i - y_a) \left[ p_i \in c^a \land p_i \in c^b : c^a \in C^a_I, c^b \in C^b_I \right]$$

$$+ \sum_{i=1}^{n} (y_a - y_i) \left[ p_i \in c^a \land p_i \in c^b : c^a \in C^a_{IV}, c^b \in C^b_{IV} \right]$$

(8)

In the plane, a point $p_i$ is said to dominate a point $p_j$, denoted as $p_i \geq p_j$, if and only if $x_i \geq x_j$ and $y_i \geq y_j$. Note that a point $p_i$ is in $c^a_j$ if and only if $t_j(p_i) \geq t_j(a)$. In addition, for a point $p_i$ located in $c^a_j$, we have $d_k(a, p_i) = d_k(O, p_i - a) = ||t_j(p_i)|| - ||t_j(a)||$. Hence, the first summation term in Equation (8) can be written as

$$\sum_{i=1}^{n} d_k(p_i, a) \left[ p_i \in c^a : c^a \in C^a_{II} \cup C^a_{III} \right]$$

$$= \sum_{j=k/4+1}^{3k/4} \sum_{i=1}^{n} d_k(p_i, a) \left[ p_i \in c^a_j \right]$$

$$= \sum_{j=k/4+1}^{3k/4} \sum_{i=1}^{n} (||t_j(p_i)|| - ||t_j(a)||) \left[ t_j(p_i) \geq t_j(a) \right]$$

$$= \sum_{j=k/4+1}^{3k/4} \left( \sum_{i=1}^{n} ||t_j(p_i)|| \left[ t_j(p_i) \geq t_j(a) \right] - ||t_j(a)|| \sum_{i=1}^{n} \left[ t_j(p_i) \geq t_j(a) \right] \right)$$
Similarly, the second summation term in Equation (8) can expressed as

\[
\sum_{i=1}^{n} d_k (p_i, b) \left[ p_i \in c^b : c^a \in C_I^b \cup C_{IV}^b \right] \\
= \sum_{j=1}^{k/4} \sum_{i=1}^{n} \left( \| t_j (p_i) \| - \| t_j (b) \| \right) \left[ t_j (p_i) \geq t_j (b) \right] \\
+ \sum_{j=3k/4+1}^{k/4} \sum_{i=1}^{n} \left( \| t_j (p_i) \| - \| t_j (b) \| \right) \left[ t_j (p_i) \geq t_j (b) \right]
\]

As for the third and fourth summations in Equation (8),

\[
\sum_{i=1}^{n} (y_i - y_a) \left[ p_i \in c^a \land p_i \in c^b : c^a \in C_I^a, c^b \in C_{IV}^b \right] \\
= \sum_{i=1}^{n} (y_i - y_a) \left[ p_i \in c^a : c^a \in C_I^a \right] - \sum_{i=1}^{n} (y_i - y_a) \left[ p_i \in c^b : c^b \in C_{IV}^b \right] \\
= \sum_{j=1}^{k/4} \sum_{i=1}^{n} (y_i - y_a) \left[ t_j (p_i) \geq t_j (a) \right] - \sum_{j=1}^{k/4} \sum_{i=1}^{n} (y_i - y_a) \left[ t_j (p_i) \geq t_j (b) \right] \\
= \sum_{j=1}^{k/4} \left( \sum_{i=1}^{n} y_i \left[ t_j (p_i) \geq t_j (a) \right] - y_a \sum_{i=1}^{n} \left[ t_j (p_i) \geq t_j (a) \right] \right) \\
- \sum_{j=1}^{k/4} \left( \sum_{i=1}^{n} y_i \left[ t_j (p_i) \geq t_j (b) \right] - y_a \sum_{i=1}^{n} \left[ t_j (p_i) \geq t_j (b) \right] \right)
\]

and

\[
\sum_{i=1}^{n} (y_a - y_i) \left[ p_i \in c^a \land p_i \in c^b : c^a \in C_{IV}^a, c^b \in C_{III}^b \right] \\
= \sum_{i=1}^{n} (y_a - y_i) \left[ p_i \in c^a : c^a \in C_{IV}^a \right] - \sum_{i=1}^{n} (y_a - y_i) \left[ p_i \in c^b : c^b \in C_{IV}^b \right] \\
= \sum_{j=3k/4+1}^{k} \sum_{i=1}^{n} (y_a - y_i) \left[ t_j (p_i) \geq t_j (a) \right] - \sum_{j=3k/4+1}^{k} \sum_{i=1}^{n} (y_a - y_i) \left[ t_j (p_i) \geq t_j (b) \right] \\
= \sum_{j=3k/4+1}^{k} \left( \sum_{i=1}^{n} y_i \left[ t_j (p_i) \geq t_j (a) \right] - y_a \sum_{i=1}^{n} \left[ t_j (p_i) \geq t_j (a) \right] \right) \\
- \sum_{j=3k/4+1}^{k} \left( \sum_{i=1}^{n} y_i \left[ t_j (p_i) \geq t_j (b) \right] - y_a \sum_{i=1}^{n} \left[ t_j (p_i) \geq t_j (b) \right] \right)
\]
Putting them together, we have

\[
 f_k(s) = \sum_{j=1}^{k/4} \left( \sum_{i=1}^{n} (t_j(p_i)) [t_j(p_i) \geq t_j(b)] - (t_j(b)) \sum_{i=1}^{n} [t_j(p_i) \geq t_j(b)] \right) \\
+ \sum_{j=1}^{k/4} \left( \sum_{i=1}^{n} y_i [t_j(p_i) \geq t_j(a)] - (t_j(a)) \sum_{i=1}^{n} [t_j(p_i) \geq t_j(a)] \right) \\
- \sum_{j=1}^{k/4} \left( \sum_{i=1}^{n} [t_j(p_i) \geq t_j(b)] - (t_j(b)) \sum_{i=1}^{n} [t_j(p_i) \geq t_j(b)] \right) \\
+ \sum_{j=k/4+1}^{3k/4} \left( \sum_{i=1}^{n} [t_j(p_i) \geq t_j(a)] - (t_j(a)) \sum_{i=1}^{n} [t_j(p_i) \geq t_j(a)] \right) \\
+ \sum_{j=3k/4+1}^{k} \left( y_a \sum_{i=1}^{n} [t_j(p_i) \geq t_j(a)] - \sum_{i=1}^{n} y_i [t_j(p_i) \geq t_j(a)] \right) \\
- \sum_{j=3k/4+1}^{k} \left( y_a \sum_{i=1}^{n} [t_j(p_i) \geq t_j(b)] - \sum_{i=1}^{n} y_i [t_j(p_i) \geq t_j(b)] \right) \\
\tag{9}
\]

Given a query point \( q \), \( \sum_{i=1}^{n} (t_j(p_i)) [t_j(p_i) \geq t_j(q)] \) or \( \sum_{i=1}^{n} y_i [t_j(p_i) \geq t_j(q)] \) can be computed by using a dominance query asking for the number of points in \( \{t_j(p_1), ..., t_j(p_n)\} \) that dominates \( t_j(q) \) multiplied by \( (t_j(p_i)) \) or \( y_i \). Similarly, \( (t_j(q)) \sum_{i=1}^{n} [t_j(p_i) \geq t_j(q)] \) or \( y_q \sum_{i=1}^{n} [t_j(p_i) \geq t_j(q)] \) is equivalent to the number of points in \( \{t_j(p_1), ..., t_j(p_n)\} \) that dominates \( t_j(q) \) multiplied by \( (t_j(q)) \) or \( y_q \). Range trees can be used to answer such dominance queries. Specifically, we use range trees to store \( \{t_j(p_1), ..., t_j(p_n)\} \) along with the sum of coordinates of the points (i.e., \( (t_j(p_i)) \)) and the y-coordinate of the points (i.e., \( y_i \)). For a set of \( n \) points in the plane, a two-dimensional range tree can be constructed in \( O(n \log n) \) time using \( O(n \log n) \) space, and a generalized dominance counting query can be answered in \( O(\log n) \) time [20]. Consider the following preprocessing procedure. For \( 1 \leq j \leq k \), we construct a range tree \( T_j \) containing \( \{t_j(p_1), ..., t_j(p_n)\} \). Then, using the range trees, Equation (9) can be evaluated in \( O(k \log n) \) time.

Recall that the total time to find a line segment \( s \) (of unit length and zero slope) that minimizes \( f_k \) is \( O(P(n) + k(n + Q(n) \log kn)) \). Since \( P(n) = O(kn \log n) \) and \( Q(n) = O(k \log n) \), the overall running time of our algorithm is \( O(kn \log n) \).