The Dekel-Zhao profile: a mass-dependent dark-matter density profile with flexible inner slope and analytic potential, velocity dispersion, and lensing properties

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ABSTRACT

We explore a function with two shape parameters for the dark-matter halo density profile subject to baryonic effects, which is a special case of the general Zhao family of models applied to simulated dark-matter haloes by Dekel et al. This profile has variable inner slope and concentration parameter, and analytic expressions for the gravitational potential, velocity dispersion, and lensing properties. Using the Numerical Investigation of a Hundred Astrophysical Objects cosmological simulations, we find that it provides better fits than the Einasto profile and the generalized NFW profile with variable inner slope, in particular towards the halo centres. We show that the profile parameters are correlated with the stellar-to-halo mass ratio $M_{\text{star}}/M_{\text{vir}}$. This defines a mass-dependent density profile describing the average dark-matter profiles in all galaxies, which can be directly applied to observed rotation curves of galaxies, gravitational lenses, and semi-analytic models of galaxy formation or satellite–galaxy evolution. The effect of baryons manifests itself by a significant flattening of the inner density slope and a 20 per cent decrease of the concentration parameter for $M_{\text{star}}/M_{\text{vir}} = 10^{-3.5}–10^{-2}$, corresponding to $M_{\text{star}} \sim 10^{7}–10^{8} \, M_{\odot}$. The accuracy by which this profile fits simulated galaxies is similar to certain multiparameter mass-dependent profiles, but its fewer parameters and analytic nature make it most desirable for many purposes.

Key words: galaxies: evolution – galaxies: haloes – dark matter.

1 INTRODUCTION

Dark-matter (DM) halo density profiles in DM-only cosmological simulations are well described by the ‘NFW’ parametrization (Navarro, Frenk & White 1996, 1997; Springel et al. 2008; Navarro et al. 2010) from dwarf haloes to large clusters, although with some systematic deviations (e.g. Navarro et al. 2004, 2010; Gao et al. 2008; Macciò, Dutton & van den Bosch 2008; Springel et al. 2008). This density profile scales with radius as

$$\rho_{\text{NFW}}(r) = \frac{\rho_{c}}{x(1+x)^{2}},$$

with $x = r/r_{s}$, $r_{s}$ being a characteristic scale radius at which the density logarithmic slope equals 2 in absolute value. This radius defines a concentration $c_{\text{DMO}} = R_{\text{vir}}/r_{s}$, which depends on the halo virial mass $M_{\text{vir}}$ and redshift (e.g. Bullock et al. 2001; Wechsler et al. 2002; Dutton & Macciò 2014) – both $M_{\text{vir}}$ and the virial radius $R_{\text{vir}}$ being set by cosmology. The inner $\rho \propto r^{-1}$ ‘cusp’ of the NFW parametrization is at odds with observations of DM dominated dwarf, low-surface-brightness, and dwarf satellite galaxies as well as clusters, which infer shallower ‘cores’ (e.g. Flores & Primack 1994; Moore 1994; McGaugh & de Blok 1998; van den Bosch & Swaters 2001; de Blok et al. 2008; de Blok 2010; Kuzio de Naray & Spekkens 2011; Oh et al. 2011, 2015; Newman et al. 2013a,b; Adams et al. 2014). The introduction of baryonic processes such as cooling, star formation and feedback resulting from star formation or active galactic nuclei (AGNs) in the simulations can alleviate this ‘cusp–core discrepancy’ by transforming cusps into cores (e.g. Governato et al. 2010, 2012; Macciò et al. 2012, 2020; Zolotov et al. 2012; Martizzi, Teyssier & Moore 2013; Teyssier et al. 2013; Di Cintio et al. 2014a; Chan et al. 2015; Tollet et al. 2016; Peirani et al. 2017).

Baryonic processes can affect DM haloes in different ways. When baryons cool slowly and accumulate at the centre of a DM halo, they steepen the potential well, leading to an adiabatic contraction of the DM distribution and even more severe cusps (Blumenthal et al. 1986; Gnedin et al. 2004; O’Hare et al. 2007). When a clump of gas or a satellite galaxy moves within the halo, it can transfer part of its orbital energy and angular momentum to the DM background through dynamical friction (Chandrasekhar 1943;
This latter process dynamically ‘heats’ the DM halo and has been shown to contribute to core formation (El-Zant, Shlosman & Hoffman 2001; El-Zant et al. 2004; Tonini, Lapi & Salucci 2006; Romano-Díaz et al. 2008; Del Popolo 2009; Goerdt et al. 2010; Cole, Dehnen & Wilkinson 2011; Nipoti & Binney 2015). When stellar winds, supernova explosions or AGNs generate outflows, they induce mass and potential fluctuations that can also dynamically heat the DM and form cores (Dekel & Silk 1986; Dekel, Devor & Hetzroni 2003a; Dekel et al. 2003b; Read & Gilmore 2005; Mashchenko, Couchman & Wadsley 2006; Mashchenko, Wadsley & Couchman 2008; Governato et al. 2012; Peñarrubia et al. 2012; Pontzen & Governato 2012; 2014; Zolotov et al. 2012; Martizzi et al. 2013; Teyssier et al. 2013; Madau, Shen & Governato 2014; El-Zant, Freundlich & Combes 2016; Dutton et al. 2016b; Peirani et al. 2017; Freundlich et al. 2020). Other processes such as galactic bars (Weinberg & Katz 2002) or tidal effects at the halo outskirts (More, Diemer & Kravtsov 2015) may also affect the DM distribution.

These different processes are reflected in hydrodynamical simulations, which display a variety of DM halo responses to the introduction of baryons, notably depending on stellar and halo masses. In particular, Di Cintio et al. (2014a), Chan et al. (2015), Tollet et al. (2016) and Dutton et al. (2016b) show that the inner slope of simulated DM haloes displays a minimum for stellar masses between $10^7$ and $10^{10} \, M_\odot$, while it rises above the NFW slope when the stellar mass exceeds $10^{10} \, M_\odot$. This behaviour can be interpreted in terms of a competition between outflows induced by feedback and the confinement imposed by halo gravity (e.g. Dekel & Silk 1986; Peñarrubia et al. 2012); for very low stellar masses, the inner slope follows that of DM-only NFW haloes; between $10^7$ and $10^{10} \, M_\odot$, outflows overcome halo gravity, leading to the expansion of the halo; above $10^{10} \, M_\odot$, the accumulation of baryons leads to adiabatic contraction, although the introduction of AGN feedback in simulations can partially counteract adiabatic contraction at high halo mass (Macciò et al. 2020). Hydrodynamical simulations of dwarf galaxies by Mashchenko et al. (2008), Madau et al. (2014), Verbeke, Vandenbroucke & De Rijcke (2015), Read, Agertz & Collins (2016), and Dutton et al. (2016b) further suggest that the main parameter driving the halo response is the stellar-to-halo mass ratio rather than the stellar or halo mass itself. The different responses of the DM halo as well as the potentially smooth transition between cusps and cores motivates a parametrization of DM halo density profiles that would reflect the different halo shapes induced by baryonic physics or environment. In particular, a parametrization with free inner slope in addition to a free concentration parameter would enable to follow the transition between cusps and cores.

Different parametrizations allowing some inner slope flexibility have been proposed (Einasto 1965; Jaffe 1983; Hernquist 1990; Dekhnen 1993; Evans 1994; Tremaine et al. 1994; Burkert 1995; Zhao 1996; Jing & Suto 2000; Navarro et al. 2004; Merritt et al. 2006; Stoehr 2006; An & Zhao 2013; Di Cintio et al. 2014a; Schaller et al. 2015; Oldham & Auger 2016; Dekel et al. 2017). Amongst them, the Einasto profile (Einasto 1965; Navarro et al. 2004; Mamon, Biviano & Murante 2010; Retana-Montenegro et al. 2012; An & Zhao 2013) with two free shape parameters provides excellent fits to DM cusps and analytic expressions for the mass and the gravitational potential (including incomplete gamma functions for the potential) as well as for the surface density, the deflection angle and the deflection potential relevant for lensing studies (involving Fox $H$ functions, cf. equation (31) below for their definition and Retana-Montenegro et al. 2012), but does not seem to fully recover the innermost part of shallower density profiles (Dekel et al. 2017; Section 3.2). Modified NFW and Einasto profiles allowing constant-density cores have been proposed by Read et al. (2016) and Lazar et al. (2020), but at the expense of analyticity (in particular, the analyticity of the concentration). The profile proposed by Dehnen (1993) and Tremaine et al. (1994) has the particularity to have analytic expressions for the mass, the gravitational potential, and the velocity dispersion (in terms of elementary functions) and, in certain cases, for the distribution function and the surface density (in terms of elementary functions for some of the cases), but its unique shape parameter does not allow to recover the diversity of DM haloes. More generally, Zhao (1996, hereafter Z96) shows that double-power-law density profiles of the form

$$
\rho(r) = \frac{\rho_c}{x^{a+1} \left(1 + bx^{1/g} \right)^{k-1}}
$$

where $x = r/r_c$, $r_c$ a characteristic radius, and $\rho_c$ a characteristic density, have analytic expressions for the gravitational potential, the enclosed mass, and the velocity dispersion (in terms of elementary functions) provided that $b = n$ and $g = 3 + kn$, where $n$ and $k$ can be any natural numbers. Within this general Zhao family of profiles with four shape parameters $(a, b, g, n)$, associated with the characteristic radius), Dekel et al. (2017, hereafter D17) show that the specific profile with $n = 2$ and $k = 1$, i.e. $b = 2$ and $g = 3.5$ in equation (2), provides excellent fits for DM haloes in simulations with and without baryons, ranging from steep cusps to flat cores. This specific profile with two remaining shape parameters $(a$ and $c)$, hereafter referred to as the Dekel-Zhao (DZ) profile, notably captures cores better than the Einasto profile. In Freundlich et al. (2020, hereafter F20), we accordingly used it to model the cusp–core transformation by outflow episodes induced by feedback, and further derived analytic expressions for the velocity dispersion in such DM haloes with additional fiducial baryonic mass distributions (in terms of incomplete beta functions). We note that Zhao (1997) provides analytic approximations for the distribution function and the projected line-of-sight velocity dispersion of this profile, while An & Zhao (2013, hereafter AZ13) offers a general parametrization of density profiles$^1$ that includes both double power-law profiles (including the NFW and other profiles) and the Einasto profile, with general analytic expressions for the gravitational potential, the enclosed mass, the velocity dispersion (in terms of incomplete beta and gamma functions), and the surface density (in terms of Fox $H$ functions).

Without being concerned by the non-analyticity of the potential and kinetic energy associated with most density profiles given by equation (2), Di Cintio et al. (2014a) analyse a suite of hydrodynamical simulations to obtain functional forms for the shape parameters $a$, $b$, $g$ and the concentration parameter associated with $r_c$ as a function of the stellar-to-halo mass ratio $M_{\text{star}}/M_{\text{vir}}$ at redshift $z = 0$. This enables them to define a mass-dependent density profile (hereafter Di Cintio+) for DM haloes, whose parameters are entirely set by the stellar and halo masses and which reflects the halo response to baryonic processes, since $M_{\text{star}}/M_{\text{vir}}$ represents an integrated star formation efficiency including the effects of feedback. The Di Cintio+ profile not only enables to fit simulated DM distributions, but it is widely used to model observed rotation curves (e.g. Allaert, Gentile & Baes 2017; van Dokkum et al. 2019; Wasserman et al.

$^1$The AZ13 parametrization is characterized by a logarithmic density slope

$$
\frac{d \ln \rho}{d \ln r} = a + x^{1/b}/\left(1 + xx^{1/g} \right),
$$

which leads to equation (2) with $g = s^{-1}$ for the density profile when $s > 0$ and to the Einasto density profile when $s = 0$ (cf. their equations 5a and 6a).
2019; Cautun et al. 2020) and at times to parametrize semi-analytical models of satellite evolution (e.g. Carleton et al. 2019). It however lacks analytic expressions for the gravitational potential, the velocity dispersion, and lensing properties such as the projected surface density and mass, the deflection angle and the magnification.

In this article, we review the analytic properties of the DZ parametrization of DM density profiles, as established in Z96, AZ13, D17, and F20, and further derive expressions for its lensing properties in terms of Fox H functions and series expansions. We systematically test this parametrization in a large suite of cosmological hydrodynamical zoom-in simulations, compare it both to the Einasto model and the generalized NFW model with variable inner slope, and obtain the dependences of its two shape parameters on stellar and halo mass. This enables us to establish it as a mass-dependent profile including the influence of baryons, whose accuracy is comparable to the Di Cintio+ profile but with the advantage of having analytic expressions for the gravitational potential and the velocity dispersion. We further give an integral expression for its associated isotropic distribution function. This model can be directly applied to model rotation curves for assessing halo masses, and also to gravitational lenses and semi-analytical models.

This article unfolds as follows: in Section 2, we recall the analytic properties of the spherically symmetric DZ profile, in particular its associated gravitational potential and velocity dispersion, and derive analytic expressions for its lensing properties; in Section 3, we systematically test the profile in the Numerical Investigation of a Hundred Astrophysical Objects (NIHAO) suite of hydrodynamical cosmological simulations (Wang et al. 2015) and quantify the mass dependence of its two free parameters, the inner logarithmic slope \(a\) and the concentration \(c_2\); in Section 4, we provide prescriptions to describe DM haloes given their stellar and halo masses and to model rotation curves with the DZ profile.

2 ANALYTICS

2.1 General case

2.1.1 Mean density profile

To describe the transition from cusps to cores and alterations of the DM distribution due to environmental effects while enabling straightforward analytic expressions of the density, mass and circular velocity profiles of DM haloes, D17 proposed a functional form similar to equation (2) for the mean density profile within a sphere of radius \(r\),

\[
\bar{\rho}(r) = \frac{\bar{\rho}_c}{1 + (r/r_c)^{b(\beta - a)}} \tag{4}
\]

where \(\bar{\rho}_c\) is a characteristic density, \(r = r r_c\) with \(r_c = R_{\text{vir}}/c\) an intermediate characteristic radius, \(a\) and \(\beta\) the inner and outer asymptotic slopes, \(b\) a middle shape parameter, and \(c\) a concentration parameter. The normalization factor \(\bar{\rho}_c\) can be expressed as \(\bar{\rho}_c = c^3 \bar{\rho}_{\text{vir}}\), with \(c = c^{c-3}(1 + c^{1/b})^{b/(\beta - a)}\) and \(\bar{\rho}_{\text{vir}} = 3M_{\text{vir}}/4\pi R_{\text{vir}}^3\) the mean mass density within \(R_{\text{vir}}\). As the virial radius \(R_{\text{vir}}\) is set by cosmology for a given halo mass through \(\bar{\rho}_{\text{vir}} = \Delta \bar{\rho}_{\text{crit}}\) with \(\Delta\) the overdensity, this functional form effectively depends on four shape parameters: \(a, b, \beta\) and \(c\).

2.1.2 Mass, velocity, force, and density profiles

The enclosed mass, circular velocity, and force profiles stemming from equation (4) can be expressed as

\[
M(r) = \frac{4\pi r^3}{3} \bar{\rho}(r) = \mu M_{\text{vir}} x^3 \bar{\rho}(r)/\bar{\rho}_c, \tag{5}
\]

\[
V^2(r) = \frac{GM(r)}{r} = c^2 V_{\text{vir}}^2 x^2 \frac{\bar{\rho}(r)}{\bar{\rho}_c}, \tag{6}
\]

and

\[
F(r) = -\frac{GM(r)}{r^2} = -c^2 c F_{\text{vir}} x^2 \bar{\rho}(r)/\bar{\rho}_c, \tag{7}
\]

where \(V_{\text{vir}}^2 = GM_{\text{vir}}/R_{\text{vir}}\) and \(F_{\text{vir}} = -GM_{\text{vir}}/R_{\text{vir}}^2\). In turn, the density profile is obtained by deriving the expression of the enclosed mass:

\[
\rho(r) = \frac{1}{4\pi r^2} \frac{dM}{dr} = \frac{3}{3} \left(1 + \frac{3 - \beta}{3 - \alpha} x^{1/b}\right) \frac{1}{1 + x^{1/b} \bar{\rho}(r)}. \tag{8}
\]

This expression reduces to equation (2) when \(\beta = 3\), with \(g = 3 + 1/b\) and \(\rho_c = (1 - a/3)\bar{\rho}_c\). More generally, each term of equation (8) is analogous to equation (2) with \(g = \beta + 1/b\) so the results of Z96 apply: this density profile allows analytic expressions for the gravitational potential and the velocity dispersion provided that \(b = n\) and \(\beta = 3 + k/n\), where \(n\) is a natural number and \(k\) a positive or null integer.

2.1.3 Inner slope and concentration

In the density profile derived from equation (4), the shape parameter \(a\) may not be the slope at the resolution limit \(0.01R_{\text{vir}}\) in the case of the NIHAO simulations, cf. Wang et al. 2015) and \(c\) does not necessarily reflect the actual concentration of the halo as for an NFW profile. The logarithmic slope of the density profile expressed in equation (8) is

\[
s(r) = \frac{\frac{\ln \rho}{dr}}{\frac{\ln r}{r}} = \frac{a + \beta b^{-1} x^{1/b}}{1 + x^{1/b}} - \frac{3 - \beta}{3 - \alpha} - \frac{3 - \beta}{3 - \alpha} x^{1/b}, \tag{9}
\]

so \(s_1 = s(0.01R_{\text{vir}})\) measures the inner logarithmic slope at the resolution limit in the NIHAO simulations. Equation (9) further enables to define a concentration parameter \(c_2\) similar to the NFW parameter, corresponding to the radius \(r_2\) at which the logarithmic slope \(s\) of the density profile equals 2. This radius is such that

\[
c_2 \equiv \frac{R_{\text{vir}}}{r_2} = c \left(\frac{\beta + b^{-1} - 2}{2 - a}\right)^b, \tag{10}
\]

which coincides with \(c\) when \(a + \beta + b^{-1} = 4\). Another concentration parameter, \(c_{\text{max}}\), can be defined from the radius \(r_{\text{max}}\) at which the circular velocity peaks (cf. Appendix A, Supporting Information). The logarithmic slope at the resolution limit \((s_1\) for the NIHAO simulations) and \(c_2\) (or \(c_{\text{max}}\)) can be used as effective inner slope and concentration when describing the density profile.

2.2 The DZ profile

Using three pairs of simulated haloes at different masses with and without baryons at \(z = 0\) from the NIHAO suite of simulations (Wang et al. 2015). D17 show that the functional form of equation (8) with \(b = 2\) and \(\beta = 3\) yields excellent fits for haloes ranging from steep cusps to flat cores. They notably show that this parametrization, here referred to as the DZ profile, matches simulated profiles better than the NFW and Einasto profiles, capturing cores better, in addition to providing fully analytic expressions for the density, the mass, the gravitational potential, and the velocity dispersion. We further
show in F20 that density profile fits using this parametrization enable to recover the simulated gravitational potentials and the velocity dispersions of simulated haloes. The upper left panel of Fig. 1 highlights the variety of density profiles from cusps to cores that can be described by the DZ profile, with four examples of different inner slope ($s_1 = 0$ and 1) and concentration ($c_2 = 5$ and 15). These fiducial examples correspond to different rotation curves, velocity dispersions, gravitational potentials and distribution functions.

In the following subsections, we recall the analytic expressions of the gravitational potential and velocity dispersion. In Section 2.3, we obtain analytic expressions for quantities relevant to gravitational lensing. In Appendix A (Supporting Information), we further express we obtain analytic expressions for quantities relevant to gravitational potential and velocity dispersion. In Section 2.3, we recall sum expressions for the velocity dispersion in haloes with fiducial baryonic components from F20. In Appendix D (Supporting Information), we give an integral expression of the distribution function. Finally, in Sections 3 and 4, we test the DZ profile over the whole NIHAO suite of simulations and establish it as a mass-dependant profile whose shape parameters $s_1$ and $c_2$ only depend on the stellar-to-halo mass ratio.

### 2.2.1 Shape parameters

Introducing $\bar{x} = 3$ and $b = 2$ in equation (8), the DZ density profile is

$$\rho(r) = \frac{\rho_c}{x^{a}(1 + x^{-1/2})^{3.5-a}}$$

with $x = r/r_c$, $\rho_c = (1 - a/3)\bar{x}\rho_c$ while $\bar{\rho}_c = c^3 \mu \bar{\rho}_c$, $\mu = c^{-a} (1 + e^{1/2}(b^3 - a)$ and $\bar{\rho}_c = 3M_{\text{vir}}/4\pi R_{\text{vir}}^3$, and two shape parameters $a$ and $c = R_{\text{vir}}/r_c$. The inner logarithmic slope $s_1$ at the resolution $r_1$ from equation (9) is

$$s_1 = a + 3.5c^{1/2}(r_1/R_{\text{vir}})^{1/2}$$

while the concentration parameters is

$$c_2 = c \left( \frac{1.5}{2-a} \right)^2.$$
are equivalent in describing the density profile. Indeed, $a$ and $c$ can be expressed as functions of $s_1$ and $c_2$,

$$a = \frac{1.5s_1 - 2(3.5 - s_1)(r_1/R_{\text{vir}})^{1/2}c_2^{1/2}}{1.5 - (3.5 - s_1)(r_1/R_{\text{vir}})^{1/2}c_2^{1/2}}$$

(14)

and

$$c = \left(\frac{s_1 - 2}{(3.5 - s_1)(r_1/R_{\text{vir}})^{1/2} - 1.5c_2^{1/2}}\right)^2.$$  

(15)

In the following, analytic expressions are expressed in terms of $(a, c)$ while numerical tests focus on $(s_1, c_2)$. Equations (12), (13), (14), and (15) enable to switch from the two couples of parameters at will.

It is further possible to define a core radius $r_{\text{core}}$ corresponding to a given value of the logarithmic slope, namely

$$r_{\text{core}} = \frac{R_{\text{vir}}}{\epsilon} \left(\frac{s_{\text{core}} - a}{3.5 - s_{\text{core}}}\right)^2$$

(16)

with $s_{\text{core}} = s(r_{\text{core}})$. Since the logarithmic slope $s$ is an increasing function of radius with $s(r = 0) = a$, this equation is only valid when $s_{\text{core}} \geq a$. We find that $s_{\text{core}} = 1$ enables to retrieve a radius close to what one’s eye identifies as a core (cf. Fig. 1). This value also corresponds to the slope at the core radius of a pseudo-isothermal halo. Moreover, we note from Fig. 8 below that $s_1 = 1$ lies right below the 1σ scatter of the inner slope $s_1$ at low mass and hence marks the threshold below which core formation occurs. By analogy with the Burkert (1995) and ‘Lucky13’ (Li et al. 2020) cored profiles, one could also choose $s_{\text{core}} = 1.5$. We point out that the slopes at the core radii of the ‘core-NFW’ (Read et al. 2016) and ‘core-Einasto’ (Lazar et al. 2020) profiles are not fixed to a specific value. At given $a$ and $c$, the core radii from equation (16) defined at different $s_{\text{core}}$ can be related to one another through constant factors depending only on $a$.

Equation (5) also enables to express the half-mass radius, or more generally the radius

$$r_f = \frac{R_{\text{vir}}}{\epsilon} \left(\frac{\mu}{f(\sigma/r)}\right)^{1/(6 - 2\epsilon)}.$$  

(17)

enclosing a DM mass $M(r_f) = fM_{\text{vir}}$. The half-mass radius of a DZ halo truncated at the virial radius corresponds to $f = 0.5$ in this equation. We stress that neither the cuspy NFW profile, nor the cored pseudo-isothermal, Burkert (1995), and ‘Lucky13’ (Li et al. 2020) profiles, nor the Einasto, ‘core-Einasto’ (Lazar et al. 2020), ‘core-NFW’ (Read et al. 2016), and generalized NFW profiles with flexible inner slope have analytic expressions for the half-mass radius and therefore $r_f$ (cf. also the table of Fig. 15).

### 2.2.2 Gravitational potential

The mass, circular velocity, force, and logarithmic slope profiles of the DZ profile can be expressed analytically from equations (5), (6), (7), and (9) with $b = 2$ and $\overline{f} = 3$. Its density (equation 11) follows the form of equation (2) with $b = 2$ and $\overline{f} = 3/1 + 1/2$ so the DZ profile also allows analytic expressions for the gravitational potential and the velocity dispersion (Z96).

Assuming that the gravitational potential vanishes at infinity and that the halo density profile is truncated at the virial radius yields the gravitational potential per unit mass$^2$

$$U(r) = -\frac{GM_{\text{vir}}}{R_{\text{vir}}} - \int_r^{R_{\text{vir}}} \frac{GM(r)}{y^2} \, dy$$

$$= -\frac{V_{\text{vir}}^2}{2} \left(1 + 2c\mu \int_x^{\zeta} (\zeta - 2a) \, d\zeta\right)$$

(18)

within the virial radius, with $V_{\text{vir}}^2 = GM_{\text{vir}}/R_{\text{vir}}$, $\chi = r\sigma/r_c$, $\zeta = \chi^{1/2}(1 + 1/2\epsilon)$, and $\zeta_c = c^{1/2}(1 + 1/2\epsilon)$. When $a \neq 2$ and $\epsilon \neq 5/2$, this yields

$$U(r) = -\frac{V_{\text{vir}}^2}{2} \left(1 + 2c\mu \left(\frac{\chi^{2(2-a)} - \chi^{2(2-a)}}{2(2-a)} - \frac{\chi_{c}^{2(2-a)+1} - \chi^{2(2-a)+1}}{2(2-a) + 1}\right)\right).$$  

(19)

As noted in Zhao (1997) and AZ13, equation (18) and hence equation (19) can be rewritten in terms of incomplete beta functions (cf. also Appendix C1, Supporting Information).

#### 2.2.3 Velocity dispersion

The equilibrium of a spherical collisionless system can be described by the spherical Jeans equation stemming from the Boltzmann equation (Binney & Tremaine 2008, equation 4.215), which yields the radial-velocity dispersion

$$\sigma_v^2(r) = \frac{G}{\rho(r)} \int_r^{R_{\text{vir}}} \frac{\rho(r')M(r')r'' \, dr'}{\rho(r)}$$

(20)

for a halo truncated at the virial radius when the anisotropy parameter $\beta \equiv 1 - \sigma_v^2/2\sigma_T^2$, where $\sigma_T$ is the tangential velocity dispersion, is null (isotropic case) and the boundary condition is $\lim_{r \rightarrow \infty} \sigma_v^2 = 0$. For a DZ density profile as in equation (11), this leads to

$$\sigma_v^2(r) = 2c\mu \frac{G}{R_{\text{vir}}} \frac{\rho_c}{\rho(r)} \int_x^{\zeta} \chi^{3-4a} - 2a \, d\zeta,$$  

(21)

or

$$\sigma_v^2(r) = 2c\mu \frac{G}{R_{\text{vir}}} \frac{\rho_c}{\rho(r)} \frac{B[4 - 2a, 9, \overline{f}]}{\chi_{c}^{2(2-a)+1}}$$

(22)

where $B(a, b, x) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt$ is the incomplete beta function and the brackets denote the difference of the enclosed function between 1 and $\chi$, i.e. $\int f(\chi) \, d\chi = \int f(\chi) - f(\chi) \, d\chi$. We extend here the definition of the incomplete beta function appearing inside the brackets to negative parameters since the integral of equation (21) is well-defined as long as $\overline{f} > 0$ such that the bracketed term is also well-defined. This equation is a specific case of equation (B6) of AZ13, and it can further be expressed in terms of finite sums (Z96, F20), as recalled in the present Appendix B (Supporting Information). The sum expressions enable to express the velocity dispersion in terms of elementary functions.

In Appendix C (Supporting Information), we further recall expressions from appendix B of F20 for the velocity dispersion in haloes with baryons (i) where the ratio between the DM and the total masses follows a power law, (ii) where the baryons are concentrated to a central point mass, (iii) where they constitute a uniform sphere,

$^2$ We use the variable change $\zeta = z^{1/2}(1 + z^{1/2})/2$ with $z = r/\sigma/r_c$, which is such that $z^{1/2} = \zeta(1 - \zeta), \ z = z^{1/2} = 1/(1 - \zeta)$, and $dz = 2\zeta(1 - \zeta)^{-3} \, d\zeta$.

$^3$ If $a = 2$, it instead yields $U(r) = -V_{\text{vir}}^2/(1 + 2\mu[\ln(\chi_c/\chi) + 1 - \chi_{c}])$ and if $a = 5/2$, $U(r) = -V_{\text{vir}}^2/(1 + 2\mu[\chi - \right\ln(\chi_c/\chi)])$ but such specific rational values of $a$ are unlikely to arise from fits.
(iv) where they constitute a singular isothermal sphere, and (v) where they themselves follow the DZ profile.

When the anisotropy parameter \( \beta \) is constant but not necessarily equal to zero, the Jeans equation corresponds to a differential equation in \( \rho \) whose solution is

\[
\sigma_s^2(r) = \frac{G}{r^{2\beta}} \rho(r) \int_0^R \rho'(r') M(r') r'^{-2\beta-2} \, dr'
\]

assuming that \( \lim_{r \to +\infty} \sigma_s^2(r) = 0 \) (Binney & Tremaine 2008, equation 4.216). Following similar steps as for equation (22), this leads to

\[
\sigma_s^2(r) = 2c \mu \frac{GM_\text{vir}}{R_{\text{vir}}} \frac{\rho_c}{\rho(r)} \frac{1}{r^{2\beta}} \left[ B(4 - 4\alpha + 4\beta, 9 + 4\beta, \xi) \right]^{\frac{1}{\beta}}.
\]

### 2.3 Lensing properties

#### 2.3.1 Surface density

The mass surface density of a spherically symmetric lens is obtained by integrating the three-dimensional density profile along the line of sight,

\[
\Sigma(R) = \int_{-\infty}^{+\infty} \rho(r) \, dz
\]

where \( R \) is the projected radius measured from the centre of the lens and \( r = \sqrt{R^2 + z^2} \) is the three-dimensional radius. This expression can be written as the Abel transform

\[
\Sigma(R) = 2 \int_{\frac{R}{\rho}}^{\infty} \frac{\rho(r) \, dr}{r^2 - R^2},
\]

which yields

\[
\Sigma(X) = 2 \rho_c r_c \int_X^{r_c} \frac{dx}{x^2(1 + x^{1/2})^{2(3.5-a)} \sqrt{x^2 - X^2}}
\]

with \( X = R r_c \) and \( c = R_{\text{vir}} / r_c \) for a DZ density profile truncated at the virial radius. This integral can be broken into two terms such that

\[
\Sigma(X) = \tilde{\Sigma}(X) - \tilde{\Sigma}(c)
\]

the surface density associated with an untruncated DZ profile. When \( a < 1 \), this expression yields at the centre

\[
\tilde{\Sigma}(0) = 4 \rho_c r_c B(2 - 2\alpha, 5)
\]

with the variable change used to obtain equations (19) and (22). However, the integral cannot be easily expressed in terms of elementary functions for all values of \( a \) when \( X \neq 0 \). Following Mazure & Capelato (2002), Baes & van Hese (2011), Baes & Gentile (2011), and Retana-Montenegro et al. (2012), who expressed similar integrals involving Sérsic and Einasto profiles in terms of the Meijer G and Fox H functions, we use the Mellin transform method (Marichev 1983; Adamchick 1996; Fikioris 2007) to evaluate it as the Mellin–Barnes integral

\[
\tilde{\Sigma}(R) = 4 \sqrt{\pi} \rho_c \frac{X}{2 \pi i} \int_{L} \Gamma(4y - 2\alpha) \Gamma(7 - 4y) \Gamma(y - \frac{3}{2}) \Gamma(y) \left[ X^2 \right]^{-y} \, dy
\]

where \( L \) is a vertical line in the complex plane (cf. Appendix E, Supporting Information). This integral can be recognized as a Fox H function (e.g. Fox 1961; Mathai & Saxena 1978; Srivastava, Gupta & Goyal 1982; Kilbas & Saigo 1999, 2004; Mathai, Saxena & Haubold 2009), which is generally defined as the inverse Mellin transform of a product of gamma functions,

\[
H_{p,q}^{m,n} \left( \begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) = \frac{1}{2\pi i} \int_{L} \prod_{j=1}^{p} \Gamma(b_j + y) \prod_{j=1}^{q} \Gamma(1 - a_j - A_j y) \prod_{j=m+1}^{p} \Gamma(1 - b_j - B_j y) \prod_{j=n+1}^{q} \Gamma(a_j + A_j y) \, dz
\]

where the couples \( \mathbf{a} \) and \( \mathbf{b} \) indicate the coefficients in the gamma functions with \( A_j, B_j > 0 \) and \( a_j, b_j \) complex numbers while \( 0 < m < q \) and \( 0 < n < p \) are integers. With this definition, the surface density associated with the untruncated DZ profile can be compactly written as

\[
\tilde{\Sigma}(X) = 4 \sqrt{\pi} \rho_c r_c X H_{2,1}^{1,2} \left( \frac{-6, 4}{-1, 1}, (0, 1) \right) \left( X^2 \right).
\]

This expression has explicit series expansions depending on the nature of the poles of the gamma functions at the denominator of the integrand of the Mellin–Barnes integral (e.g. Kilbas & Saigo 1999; Baes & Gentile 2011), which are given in Appendix F (Supporting Information). We note that equation (32) is a specific case of equation (C1) of AZ13, which includes both other double power-law profiles and the Einasto profile, and that AZ13 also provide analytic expressions for the limiting behaviours of this surface density when \( X \to 0 \) and \( X \to \infty \) in terms of elementary functions.

The cumulative mass contained within an infinite cylinder of radius \( R \) is

\[
\tilde{\mathcal{M}}(R) = 2\pi \int_{0}^{R} \tilde{\Sigma}(R') R' \, dR'
\]

for an untruncated DZ profile and \( \mathcal{M}(R) = \tilde{\mathcal{M}}(R) - \pi R^2 \tilde{\Sigma}(c) \) for a DZ profile truncated at the virial radius. Injecting equation (30) and inverting the two integrals involved yields

\[
\tilde{\mathcal{M}}(X) = 4 \pi^{3/2} \rho_c r_c X H_{3,3}^{2,2} \left( \frac{-6, 4}{-\frac{1}{2}, 1}, (0, 1) \right) \left( X^2 \right),
\]

which also has an explicit series expansion (Appendix F, Supporting Information).

#### 2.3.2 Deflection angle

A gravitational lens deflects light from background sources depending on their projected distance \( R \) in the lens plane. The deflection angle \( \hat{\alpha}(R) \) of a thin axially symmetric lens where the distances between the source, the lens, and the observer are much larger than the size of the lens is directly related to its cumulative mass \( \mathcal{M}(R) \) through

\[
\hat{\alpha}(R) = \frac{4G R \mathcal{M}(R)}{c^2 R}
\]

(Schneider, Ehlers & Falco 1992, equation 8.5), \( c \) being here the speed of light. Introducing \( D_L, D_S \), and \( D_L S \) the angular distances respectively between the observer and the lens, between the observer and the source, and between the lens and the source, one can express the scaled deflection angle

\[
\alpha(R) = \frac{D_L D_S}{r_c D_S} \hat{\alpha}(R)
\]

and the convergence

\[
\kappa(R) = \frac{\Sigma(R)}{\Sigma_{\text{crit}}}
\]
We give analytic expressions for the lensing potential in Appendix F

\[ \alpha \text{ profile truncated at the virial radius, } \tilde{\alpha} \text{ which has a series expansion analogous to that of } \gamma \text{ axially symmetric lens reads} \]

\[ \tilde{\alpha}(X) = \frac{\sqrt{\pi} \tilde{\kappa}_0}{\Gamma(2 - 2a)\Gamma(5)} X^2 H_{3/3}^2 \left[ (-6, 4), (-1/2, 1), (0, 1), \left( -\frac{1}{2}, 1, (-2a, 4), (-1/2, 1) \right) X^2 \right] , \] (38)

which has a series expansion analogous to that of \( \tilde{\lambda}(X) \). For a DZ profile truncated at the virial radius, \( \alpha(X) = \tilde{\alpha}(X) - X\Sigma(c) / \Sigma_{\text{crit}} \).

We give analytic expressions for the lensing potential in Appendix F (Supporting Information).

For an axially symmetric lens, multiple images occur if and only if the central convergence \( \kappa_0 = \Sigma(0) / \Sigma_{\text{crit}} > 1 \) when the surface density does not increases with \( X \), while there is only one image when \( \kappa_0 \leq 1 \) (Schneider et al. 1992, section 8). If \( a \geq 1 \), the DZ profile has a singular surface density at the centre and there can be multiple images for all masses. However, if \( a < 1 \), the surface density is not singular and there can be multiple images only if \( \kappa_0 > 1 \).

2.3.3 Shear and magnification

The Jacobian between the unlensed and lensed coordinate systems depends on the convergence \( \kappa \) and on the lensing shear, which for a axially symmetric lens reads

\[ \gamma(X) = \frac{\Sigma(X) - \Sigma(X)}{\Sigma_{\text{crit}}} \] (39)

with

\[ \Sigma(X) = \frac{2}{X^2} \int_0^X x \Sigma(x) dx, \] (40)

the average surface density within \( X \). The average surface density for an untruncated DZ profile can be expressed as

\[ \Sigma(X) = \frac{4\sqrt{\pi} \rho_0 r_c}{\Gamma(7 - 2a)} X H_{3/3}^2 \left[ (-6, 4), (-1/2, 1), (0, 1), \left( -\frac{1}{2}, 1, (-2a, 4), (-1/2, 1) \right) X^2 \right] \] (41)

in terms of a Fox \( H \) function while the average surface density of a DZ profile truncated at the virial radius is \( \Sigma(X) = \tilde{\Sigma}(X) - \tilde{\Sigma}(c) \).

Both have series expansions (Appendix F, Supporting Information). Equations (32), (41), and the definitions of the convergence \( \kappa \) (equation 37) and of the shear \( \gamma \) (equation 39) enable to determine the magnification factor \( \mu(X) = [1 - \gamma(X)] \) by which the source luminosity is amplified (Schneider et al. 1992, equations 5.21 and 5.25). This factor, which is the inverse of the determinant of the Jacobian between the unlensed and lensed coordinate systems, comprises of a term depending on the convergence \( \kappa \) that describes the isotropic focusing of the light rays in the lens plane and of a term depending on the shear \( \gamma \) that accounts for the anisotropic focusing due to the tangential stretching of the image.

Fig. 2 displays the radial profiles of some of the lensing properties of the four fiducial DZ haloes of different inner slope and concentration shown in Fig. 1, assumed to be truncated at the virial radius. We note that the shear \( \gamma \) mainly depends on the concentration away from the halo centre, with higher concentration leading to more shear, while steeper inner densities induce more shear near the centre. The quantities expressed in this section as well as those shown in Fig. 2 assume spherically symmetric haloes. Generalizations to elliptical DZ haloes can be obtained by substituting the projected radius \( R \) with an expression depending on the ellipticity of the lens (e.g. Schneider et al. 1992; Golse & Kneib 2002; Meneghetti, Bartelmann & Moscardini 2003).

3 THE DZ PROFILE IN SIMULATIONS

3.1 The NIHAO simulations

We systematically test the DZ profile on the simulated DM haloes at \( z = 0 \) of the NIHAO project (Wang et al. 2015), which provides a set of about 90 cosmological zoom-in hydrodynamical simulations run with the improved Smoothed Particle Hydrodynamics (SPH) code gasolinel2 (Wadsley, Keller & Quinn 2017). Each simulation is run at the same resolution with and without baryons, but we focus here on the hydrodynamical simulations including the effects of baryons. The simulations assume a flat \( \Lambda \)CDM cosmology with Planck Collaboration (2014) parameters, namely \( \Omega_h = 0.121, \Omega_b = 0.0490, H_0 = 67.1 \text{ km s}^{-1} \text{Mpc}^{-1}, \sigma_8 = 0.8344, \) and \( n = 0.6824 \).
They include a subgrid model describing the turbulent mixing of metals and thermal energy (Wadsley, Veeravalli & Couchman 2008), cooling via hydrogen, helium and other metal lines in a uniform ultraviolet ionizing and heating background (Shen, Wadsley & Stinson 2010) and star formation according to the Kennicutt–Schmidt relation when the temperature falls below 15 000 K and the density reaches 10.3 cm$^{-3}$ (Stinson et al. 2013). Stars inject energy back to their surrounding interstellar medium (ISM) through ionizing feedback from massive stars (Stinson et al. 2013) and supernovae (Stinson et al. 2006). During the pre-supernova feedback phase, 13 per cent of the total stellar luminosity – which is typically $2 \times 10^{50}$ erg M$_{\odot}^{-3}$ of the entire stellar population over the 4 Myr preceeding the explosion of high-mass stars – is ejected into the surrounding gas. During the supernova feedback phase, stars whose mass is comprised between 8 and 40 M$_{\odot}$ eject 4 Myr after their formation both an energy $E_{SN} = 10^{51}$ erg and metals into their surrounding ISM according to the blast-wave formalism described in Stinson et al. (2006). Cooling is delayed for 30 Myr inside the blast region to prevent the energy from supernova feedback to be radiated away. Without cooling, the added supernova energy heats the surrounding gas, which both prevents star formation and models the high pressure of the blast wave. AGN feedback is not included.

The NIHAO sample comprises isolated haloes chosen from dissipationless cosmological simulations (Dutton & Macciò 2014) with halo masses between $\log(M_{\text{vir}}/M_{\odot}) = 9.5$–12.3. Their merging histories, concentration and spin parameters were not taken into account in the selection. The virial radius $R_{\text{vir}}$ is defined as the radius within which the average total density is $\Delta$ times the critical density of the Universe, where $\Delta$ is defined according to Bryan & Norman (1998). The virial mass $M_{\text{vir}}$ is the total mass enclosed within $R_{\text{vir}}$. The particle masses and force softening lengths are chosen to resolve the DM mass profile below 1 per cent of the virial radius at all masses in order to resolve the half-light radius of the galaxies. Stellar masses, which are calculated within $0.15R_{\text{vir}}$, range from $5 \times 10^4$ to $2 \times 10^{11}$ M$_{\odot}$, i.e. from dwarfs to Milky Way sized galaxies, with morphologies, colours, and sizes that correspond well with observations (e.g. Stinson et al. 2015; Wang et al. 2015; Dutton et al. 2016a). As shown by Tollet et al. (2016), Dutton et al. (2016b), D17, F20, and Macciò et al. (2020), NIHAO DM haloes display a variety of inner slopes ranging from steep cusps to flat cores, cores being more prevalent at $z = 0$ for stellar masses comprised between $10^7$ and $10^{10}$ M$_{\odot}$.

### 3.2 Fitting procedure and results

#### 3.2.1 Density profile fits and rotation curves

We fit the logarithm of the density profile of each simulated halo at $z = 0$ according to the DZ parametrization (equation 11) through a least-squares minimization between 0.01 $R_{\text{vir}}$ (the resolution limit) and $R_{\text{vir}}$. Since $R_{\text{vir}}$ and $M_{\text{vir}}$ are set, $a$ and $c$ are the only free parameters. We impose the inner logarithmic slope at the resolution limit to be positive, namely $s_1 \geq 0$ with $s_1 = (0.01 R_{\text{vir}})$ expressed in equation (12). The profile radii $r$ are spaced logarithmically, with $N \sim 100$ radii $r_i$ between 0.01$R_{\text{vir}}$ and $R_{\text{vir}}$. The inner slope $s_1$ and the concentration parameter $c_2$ associated with the fit result can be derived from $a$ and $c$ with equations (12) and (13). The rms of the residuals between the simulated log $\rho$ and the model is used to evaluate the relative goodness of fit in the range 0.01 $R_{\text{vir}}$–$R_{\text{vir}}$, and we also define $\sigma_1$ the rms of the residuals in the central region of the halo between 0.01 $R_{\text{vir}}$ and 0.1 $R_{\text{vir}}$. The residuals themselves can be seen in Appendix H (Supporting Information). The absolute value of $\sigma_1$ (or $\sigma_2$) is sensitive to the smoothness of the simulated profile, in particular to the resolution of the simulations, the number of radii used, and the binning procedure for the profile. We thus mostly use it to compare the performance of different models in fitting a given target profile. We notably note that with profile radii spaced logarithmically, the effective weight assigned to the inner region of the halo is larger than it would have been with linearly-space radii.

Fig. 3 displays the DZ fit results to the DM density profile for eight fiducial $z = 0$ NIHAO haloes of different masses, simulated with baryons. This selection includes the two haloes studied more specifically in F20, g1.08e11 and g6.12e10, but is otherwise arbitrary in each mass range. The best-fitting profile parameters $a$ and $c$ as well as the corresponding inner slope $s_1$ and concentration $c_2$ are indicated. The mass dependence of the DM halo response to baryons described by Di Cintio et al. (2014a), Tollet et al. (2016), and Dutton et al. (2016b) is already visible in this figure, with the lowest-mass halo having a relatively steep cusp, haloes with stellar masses between $10^7$ and $10^{10}$ M$_{\odot}$ shallower cores, and the two most massive haloes steeper inner slopes.

The figure further compares the fits according to the DZ parametrization with fits according to the Einasto and the generalized NFW with free inner slope (gNFW) parametrizations. We recall that the Einasto density profile (Einasto 1965; Navarro et al. 2004, AZ13) can be expressed as

$$\rho_{\text{Einasto}}(r) = \rho_0 \exp\left(-\left(\frac{r}{r_a}\right)^{\nu} - 1\right)$$

with $r_a$ the radius where the logarithmic density slope equals 2, $\rho_0$ the corresponding density and $\nu$ a shape parameter. The gNFW profile refers to equation (2) with $b = 1$ and $g = 3$ (e.g. AZ13), i.e.

$$\rho_{\text{gNFW}}(r) = \frac{\rho_0}{x(1 + x)^{a}}$$

with $x = r/r_a$ and $a$ the innermost slope. These two profiles have two free shape parameters ($c_2 = R_{\text{vir}}/r_a$ and $\nu$ for the Einasto profile, $a$ and $c = R_{\text{vir}}/r_a$ for the gNFW profile) as is the case for the DZ parametrization. As notably indicated by the rms $\sigma_1$ and $\sigma_2$, Einasto fits are significantly worse for shallow inner density slopes than the other two, which seem to follow each other closely. This is particularly visible in the inner part of the density profile.

Fig. 4 compares the DM circular velocity profiles of the eight fiducial haloes of Fig. 3 with those resulting from the density profile fits. As for the density profile fits, we define $\sigma_V$ and $\sigma_{V_c}$ the rms of the residuals between the simulated circular velocity $V_c$, and the model, in the ranges 0.01 $R_{\text{vir}}$–$R_{\text{vir}}$ and 0.01 $R_{\text{vir}}$–0.1 $R_{\text{vir}}$, respectively. Although we note that there may be some $\lesssim 10$ per cent offset in the velocity prescription at high masses, the DM profile fares significantly better than the other two parametrizations in recovering the DM circular velocity profiles, as indicated by the systematically lower values of $\sigma_1$ and $\sigma_{V_c}$. The inadequacy of the Einasto and gNFW profiles is strikingly towards the innermost part of the rotation curve. Fig. 5 confirms the trends seen in Figs 3 and 4 over the whole NIHAO sample at $z = 0$ by systematically comparing the rms $\sigma_1$, $\sigma_2$, $\sigma_V$, and $\sigma_{V_c}$ distributions of the three two-parameter models. We point out that the circular velocities at small radii obtained for the DZ, Einasto, and gNFW profiles are significantly impacted by the behaviour of these profiles below the resolution limit of 0.01 $R_{\text{vir}}$. 

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Figure 3. Model versus simulated density profiles: the dark-matter density profiles at \( z = 0 \) of eight arbitrary NIHAO galaxies with baryons (plain red line) at different masses with their best-fitting DZ, Einasto, and gNFW profiles (dashed, dotted, and thin dashed black lines, respectively), for radii covering the range between \( 0.01R_{\text{vir}} \) and \( R_{\text{vir}} \). The rms errors \( \sigma \) and \( \sigma_c \) and the best-fitting parameters \( s_1 \) and \( c_2 \) of the different parametrizations are indicated, as well as \( M_{\text{star}}, M_{\text{vir}}, \) and \( M_{\text{star}}/M_{\text{vir}} \). The gNFW fits follow closely the DZ fits; the Einasto fits do not recover the inner density profiles as well as the others.
The Dekel-Zhao profile

Figure 4. Model versus simulated rotation curves: dark-matter circular velocity profiles, \( V_c(r) = \sqrt{GM(r)/r} \), of the eight \( z = 0 \) NIHAO galaxies shown in Fig. 3 (plain red line) together with those inferred from the DZ, Einasto, and gNFW fits to their density profiles (dashed, dotted, and thin dashed black lines, respectively). The velocity of each galaxy is normalized to its maximum value \( V_{\text{max}} \), which is an increasing function of mass. The rotation curves inferred from the DZ fits to the density profiles recover better the simulated curves than those inferred from the Einasto and gNFW fits. The residuals of \( V_c/V_{\text{max}} \) are shown in Appendix H (Supporting Information).
3.2.2 Model versus simulated parameters

To quantify further the adequation of the different profile parametrizations, we define an inner slope $s_1^*$ and a concentration $c_1^*$ directly measured from the simulated density and logarithmic slope profiles. The former is the average slope between 0.01 $R_{\text{vir}}$ and 0.02 $R_{\text{vir}}$, as notably used by Tollet et al. (2016); the latter corresponds to the radius where the logarithmic slope equals 2. Since the simulated slope profile can be relatively noisy, we smooth it using a Savitsky-Golay filter with maximum window size when measuring $c_1^*$. Fig. H1 in Appendix H (Supporting Information) illustrates how $s_1^*$ and $c_1^*$ are obtained from the simulated profiles. The definitions of these two quantities each have their own shortcomings, notably as $s_1^*$ may in principle be different than the innermost slope at 0.01 $R_{\text{vir}}$ and as $c_1^*$ may be affected by the smoothing, but do enable to capture reasonable inner slopes and concentrations. We use these quantities as references to describe the inner slope and concentration differences between model and simulation, $\Delta s = s_{\text{model}} - s_1^*$ and $\Delta c = c_{\text{model}} - c_1^*$.

Fig. 6 compares the inner slopes and concentrations derived from the DZ fits ($s_1$ and $c_2$) with those measured on the simulated profiles ($s_1^*$ and $c_1^*$), highlighting very strong correlations (with Pearson correlation coefficients $r > 0.85$) with some scatter (0.26 for $s_1$, 0.10 for $\log c_2$): the DZ fits enable to retrieve the inner slope and concentration measured from the simulated profiles. We further define $V_{\text{max}}^*$ and $R_{\text{max}}^*$ the maximum velocity and the corresponding radius on the simulated circular velocity profiles such as those shown in Fig. 4, as well as $\Delta V = (V_{\text{max, model}} - V_{\text{max}}^*)/V_{\text{max}}^*$ and $\Delta R = (R_{\text{max, model}} - R_{\text{max}}^*)/R_{\text{max}}^*$ the relative difference between the values derived from the density profile fits and those measured on the simulated profiles. Fig. 7 shows the distributions of $\Delta s$, $\Delta c$, $\Delta V$, and $\Delta R$ for the DZ, Einasto, and gNFW fits for all NIHAO galaxies with baryons at $z = 0$. The figure shows that the DZ parametrization provides inner slopes closest to $s_1^*$ on average while the other two, and in particular the Einasto parametrization, systematically overestimate the inner slope. This can already be seen in Fig. 3, where the Einasto fit is in most cases above the simulated density profile in the innermost part. The three parametrizations slightly tend to underestimate the concentration compared to that measured from the slope profile, but we recall that the latter may be affected by the smoothing. In this regard, the Einasto parametrization seems to yield a higher systematic offset than the other two, but the scatters are similar. The three parametrizations recover the maximum velocity with a relative error $\lesssim 5$ per cent, but with a systematic overestimation of $\sim 1$ per cent on average for DZ and Einasto, $\sim 3$ per cent for gNFW. The maximum radius is less well recovered by all parametrizations, with relative errors spread around a 10 per cent overestimate in the...
DZ case (6 per cent for gNFW, 20 per cent for Einasto) with a \(\sim0.05\) per cent standard deviation.

We conclude from this analysis that the DZ parametrization provides significantly better fits to DM density profiles than Einasto and marginally better than gNFW, and infer better fits than both to the circular velocity profile. It enables to recover the inner density slope \(s_1\) with a \(\pm0.27\) scatter but a negligible systematic error, the concentration \(c_2\) with a \(\pm5\) scatter and a limited \(-2\) systematic offset on average (0.1 dex scatter and \(-0.05\) dex offset in log \(c_2\)). It retrieves the maximum velocity \(V_{\text{max}}\) with a \(\pm2\) per cent scatter and a \(\pm1\) per cent systematic offset, the corresponding radius with a \(\pm30\) per cent scatter and a \(\pm13\) per cent offset (0.11 dex scatter and +0.05 dex offset in log \(R_{\text{max}}\)).

### 3.3 Mass dependence of the profile parameters

#### 3.3.1 Mass dependence of \(s_1\) and \(c_2\)

Fig. 8 shows the dependence of the inner slope \(s_1\) and the concentration \(c_2\) derived from the DZ density profile fits on the stellar mass \(M_{\text{star}}\), the halo mass \(M_{\text{vir}}\) and the stellar-to-halo mass ratio \(M_{\text{star}}/M_{\text{vir}}\). At low stellar mass, halo mass, and stellar-to-halo mass ratio, the halo is dominated by DM and hence follow the NFW slope \((s_1 \approx 1.25)\) and concentration \((c_2 \approx 10)\); at intermediate mass and stellar-to-halo mass ratio, stellar feedback is strong enough to overcome the gravitational potential and expand the halo; at high mass and stellar-to-halo mass ratio, there is adiabatic contraction of the halo due to the steepening of the gravitational potential. Halo expansion occurs for stellar masses between \(10^7\) to \(10^{10}\) \(M_\odot\), halo masses between \(10^{10.5}\) and \(10^{11.5}\) \(M_\odot\), and stellar-to-halo mass ratios between \(10^{-3.5}\) and \(10^{-2}\). As noted by Di Cintio et al. (2014a), the range in stellar-to-halo mass ratio where core formation occurs is in agreement with the analytic calculation of Peñarrubia et al. (2012) comparing the energy baryons must inject into a DM halo to remove its central cusp and the energy released by Type II supernovae explosions. We note that there is a hint of a small drop of the concentration \(c_2\) in the range where core formation happens: feedback not only affect the inner part of the DM distribution, but also puffs-up the halo at larger scales. We recall that the NIHAO simulations used here do not include AGN feedback. As a consequence, the most massive haloes of the sample are partially overcooled, with log \((M_{\text{star}}/M_{\text{vir}})\) close to \(-1\). When AGN feedback is included, the stellar mass of the most massive haloes is reduced, their DM distribution relaxes, and their inner slope slightly decreases (Blank et al. 2019; Macciò et al. 2020).

We try to capture the behaviour of the inner slope \(s_1\) as a function of \(M_{\text{star}}\), \(M_{\text{vir}}\) and \(M_{\text{star}}/M_{\text{vir}}\) using the function

\[
s_1(x) = s' + s'' \log \left( 1 + \frac{x}{x_0} \right)^{v} \tag{45}\]

where \(x_0\), \(s'\), \(s''\), and \(v\) are adjustable parameters and \(\log = \log_{10}\). We impose \(s'\) and \(s''\) to be similar for the three variables \(x\), which yields a unique asymptotical value \(s' = 1.25\) when \(x\) goes to zero. This value corresponds approximately to an NFW cusp in the absence of baryons. Fig. 8 further displays the fitting function obtained by Tollet et al. (2016) for the measured slope \(s_1\) between 1 per cent and 2 per cent of the virial radius \(R_{\text{vir}}\) in the same suite of cosmological zoom-in simulations with baryons. Motivated by Dutton & Macci`o (2014) and Di Cintio et al. (2014b), we try to capture the behavior of the concentration \(c_2\) as a function of \(M_{\text{star}}\), \(M_{\text{vir}}\), and \(M_{\text{star}}/M_{\text{vir}}\) using the function

\[
c_2(x) = c' + \left( 1 + \frac{x}{x_0} \right)^{v} \tag{46}\]

where \(x_0\), \(c'\), \(c''\), and \(v\) are adjustable parameters. We impose \(c'\) to be similar for the three variables \(x\), yielding \(c' = 11.5\). This asymptotical value when \(x\) goes to zero is in accordance with fitting functions for the NFW concentration (e.g. Dutton & Macci`o 2014). The values of the different fitting parameters are indicated in Table 1, together with the rms of the residuals (\(\sigma\)). This latter quantity is obtained through an iterative process excluding points beyond 3 \(\sigma\) : this process does not affect the rms values of \(s_1\), which are equal to the standard deviation of the residuals, but does affect those of \(c_2\) as it excludes some of the points at high mass or high mass ratio. The steep exponential rise of \(c_2\) indeed leads to artificially high residuals when taking only \(y\)-axis errors into account, which is reflected in the standard deviation of the residuals. The value of \(\sigma\) obtained by the iterative process and indicated in the figure and the table corresponds to a very good approximation to the standard deviation inferred from the difference between the 16 and 84 per cent quantiles of the residuals (which should be equal to 2\(\sigma\)).

Although the different panels highlight significant scatter, we note that the tightest relations are those as a function of the stellar-to-halo mass ratio: both the inner slope \(s_1\) and the concentration \(c_2\) react to the presence of baryons. The smaller scatter obtained for the stellar-
Figure 8. Mass dependence of the DZ parameters: the inner slope $s_1$ and the concentration $c_2$ derived from the density profile fits as a function of stellar mass $M_{\text{star}}$, halo mass $M_{\text{vir}}$, and stellar-to-halo mass ratio $M_{\text{star}}/M_{\text{vir}}$. The inner slope $s_1$ is fitted using the function proposed by Tollet et al. (2016) specified in equation (45): the best-fitting curve is shown as the plain black line, while the Tollet et al. (2016) fit to $s_1^{\star}$ is shown as the dashed black line. The concentration $c_2$ is fitted using the function specified in equation (49). The values of the best-fitting parameters are indicated in Table 1. The rms $\sigma$ of the residuals, which is highlighted in grey, is obtained through an iterative process excluding points beyond $3\sigma$: this process does not affect the rms for $s_1$ but does affect that of $c_2$ as it excludes some of the points at high masses. The mass dependence of $s_1$ is marked by the presence of cores for $M_{\text{star}}$ between $10^7$ and $10^{10}$ $M_\odot$, $M_{\text{vir}}$ between $10^{10.5}$ and $10^{11.5}$ $M_\odot$, and $log M_{\text{star}}/M_{\text{vir}}$ between $-3.5$ and $-2$, adiabatic contraction above. The mass dependence of $c_2$ also reflects adiabatic contraction at high masses. The tightest relations are those as a function of the stellar-to-halo mass ratio.

Table 1. Best-fitting parameters and relative errors for the slope and concentration relations shown in Figs 8 and 9. The fitting functions are specified in equations (45), (46), and (49). We impose $s$, $s^{\prime}$, and $c^{\prime}$ to be the same as a function of the three variables $M_{\text{star}}$, $M_{\text{vir}}$, and $M_{\text{star}}/M_{\text{vir}}$ for $s_1$ and $c_2$. The rms $\sigma$ of the residuals within $3\sigma$ is also indicated.

| Relation         | $x_0$         | $s^{\prime}$ | $s^{\prime\prime}$ | $v$  | $\mu$ | $\sigma$ |
|------------------|---------------|--------------|---------------------|------|-------|----------|
| $s_1(M_{\text{star}})$ | $5.18 \times 10^7$ | 1.25         | 0.37                | 1.51 | 0.35  |          |
| $s_1(M_{\text{vir}})$ | $3.99 \times 10^{10}$ | 1.25         | 0.37                | 3.00 | 0.38  |          |
| $s_1 \left( \frac{M_{\text{star}}}{M_{\text{vir}}} \right)$ | $1.30 \times 10^{-3}$ | 1.25         | 0.37                | 2.98 | 0.34  |          |
| $\left( \frac{s_1}{s_{DMO}} \right) \left( \frac{M_{\text{star}}}{M_{\text{vir}}} \right)$ | $1.30 \times 10^{-3}$ | 1            | 0.32                | 2.86 | 0.28  |          |

$C_0 (M_{\text{star}})$: $2.38 \times 10^{10}$

$C_0 (M_{\text{vir}})$: $1.05 \times 10^{12}$

$C_0 \left( \frac{M_{\text{star}}}{M_{\text{vir}}} \right)$: $3.04 \times 10^{-2}$

$C_0 \left( \frac{M_{\text{star}}}{M_{\text{vir}}} \right)$: $2.43 \times 10^{-2}$

To isolate the effect of the introduction of baryonic processes on the inner slope $s_1$ and the concentration $c_2$, we normalize these two quantities by their expected NFW values in DM-only simulations, $s_{DMO}$ and $c_{DMO}$. Namely, we use the best-fitting relation for the NFW concentration as a function of halo mass (measured using Bryan & Norman 1998) from Dutton & Macciò (2014),

$$c_{DMO} = 1.025 - 0.097 \log \left( \frac{h M_{\text{vir}}}{10^{12} M_\odot} \right)$$

3.3.2 Comparison with the DM-only parameters

To show that the stellar-to-halo mass ratio (the 'integrated star formation efficiency') is the best parameter to capture the effect of baryons on the DM distribution. This was also suggested by hydrodynamical simulations of dwarf galaxies (e.g. Mashchenko et al. 2008; Madau et al. 2014; Verbeke et al. 2015; Read et al. 2016), which showed that core formation occurs above a critical mass depending on the halo mass.
with $h = 0.671$ the dimensionless Hubble parameter (Planck Collaboration 2014), the corresponding NFW slope at $0.01\,\sigma_{\text{vir}}$ being

$$s_{\text{DMO}} = \frac{1 + 0.03c_{\text{DMO}}}{1 + 0.01c_{\text{DMO}}}.$$  \hfill (48)

Fig. 9 shows the slope and concentration ratios $s_1/s_{\text{DMO}}$ and $c_2/c_{\text{DMO}}$ as a function of the stellar-to-halo mass ratio $M_{\text{star}}/M_{\text{vir}}$, which is the variable leading to the lowest scatter in Fig. 8. Fig. 9 highlights the formation of shallow cores for log $M_{\text{star}}/M_{\text{vir}}$ between $-3.5$ and $-2$ and adiabatic contraction above. Both effects are visible not only in terms of $s_1/s_{\text{DMO}}$, but also in terms of $c_2/c_{\text{DMO}}$: while the slope ratio decreases from 1 to below 0.5 before increasing above 1.5 as $M_{\text{star}}/M_{\text{vir}}$ increases, the concentration ratio decreases to $\sim 0.8$ before sharply rising up to $\sim 3$. This drop in halo concentration for log $(M_{\text{star}}/M_{\text{vir}})$ between $-3.5$ and $-2$ had not been seen previously, as highlighted by the dashed line obtained by Di Cintio et al. (2014b), but was also recently reported by Lazar et al. (2020) using the FIRE-2 simulations.

We fit the slope ratio $s_1/s_{\text{DMO}}$ as a function of $M_{\text{star}}/M_{\text{vir}}$ with the function of equation (45) and $s = 1$ to impose an NFW slope when $M_{\text{star}}/M_{\text{vir}}$ goes to zero. The concentration ratio is fitted as a function of $M_{\text{star}}/M_{\text{vir}}$ with the function of equation (49) plus a second power-law term to account for the dip of concentration when log $M_{\text{star}}/M_{\text{vir}}$ is between $-3.5$ and $-2$, namely

$$\left(\frac{c_2}{c_{\text{DMO}}}\right) (x) = c' \left(1 + \left(\frac{x}{x_0}\right)^v\right) - x^b$$  \hfill (49)

with $x_0$, $c'$, $v$, and $b$ four adjustable parameters constrained to yield $c_2/c_{\text{DMO}} = 1$ at log $M_{\text{star}}/M_{\text{vir}} = -6$. Table 1 lists the best-fitting parameters of the functions describing the slope and concentration ratios and the rms of the residuals, which indicates the scatter of the two relations. A large part of this scatter has a physical origin related to the individual merger and star formation histories of the simulated galaxies. In particular, we note that the scatter in stellar mass at fixed halo mass is estimated to be between 0.16 and 0.2 dex at $z = 0$ (e.g. More et al. 2009; Behroozi, Wechsler & Conroy 2013; Reddick et al. 2013). The processes responsible for this scatter, such as mergers, star formation, and feedback, are expected to affect DM haloes as well (cf. Section 1) and hence the inner slope and the concentration parameter associated with the DZ fits. We further note from the colour scale on both panels that the inner slope and concentration ratios $s_1/s_{\text{DMO}}$ and $c_2/c_{\text{DMO}}$ are correlated.

## 4 A MASS-DEPENDENT PROFILE

### 4.1 Prescriptions

Section 3.3 establishes the DZ profile as a mass-dependent profile, whose shape parameters $s_1$ and $c_2$ (or equivalently, $a$ and $c$) are set by the stellar-to-halo mass ratio $M_{\text{star}}/M_{\text{vir}}$. It further provides fitting functions for the dependences of $s_1$ and $c_2$ on $M_{\text{star}}/M_{\text{vir}}$. As for the Di Cintio+$\gamma$ profile, it is thus possible to derive the shape of the DM distribution taking into account the effect of baryons for any halo given its stellar or halo mass. While the Di Cintio+$\gamma$ profile uses four shape parameters including the concentration, the DZ profile describes the DM distribution with only two parameters, with the advantage to have analytic expressions for the gravitational potential and the velocity dispersion (cf. Z96, D17), the resulting kinetic energy (cf. F20), and lensing properties (cf. Section 2). Inspired by the appendix of Di Cintio et al. (2014b), we provide here prescriptions to derive the DZ DM profile associated with any given halo.

(i) The inputs are the halo mass $M_{\text{vir}}$ and the stellar mass $M_{\text{star}}$. If only one of the two quantities is known, one can use an abundance-matching $M_{\text{star}}/M_{\text{vir}}$ relation to derive the other one (e.g. Behroozi et al. 2013, 2019; Moster, Naab & White 2013; Rodríguez-Puebla et al. 2017).

(ii) Determine the virial radius $R_{\text{vir}}$ using the overdensity criterion

$$M_{\text{vir}} = \frac{4\pi}{3} R_{\text{vir}}^3 \Delta \rho_{\text{crit}}$$  \hfill (50)

with $\Delta = 18\pi^2 + 82\chi - 39\chi^2$ at $z = 0$ for $x = \Omega_0 - 1$ from Bryan & Norman (1998) and $\rho_{\text{crit}} = 3\bar{H}^2/8\pi G$ the critical density of the
University. With the Planck Collaboration (2014) parameters, \( \Delta = 103.5 \) and \( \rho_{\text{crit}} = 124.9\, M_\odot \text{pc}^{-3} \).

(iii) Compute the inner slope and concentration ratios \( s_1/s_{\text{DMO}} \) and \( c_2/c_{\text{DMO}} \) from the stellar-to-halo mass ratio \( M_{\text{star}}/M_{\text{vir}} \) using the fitting functions from equations (45) and (49), whose best-fitting parameters are indicated in Table 1. These functions were obtained in the range \( -5 \leq \log(M_{\text{star}}/M_{\text{vir}}) \leq -1 \) and converge to 1 for smaller values of \( \log(M_{\text{star}}/M_{\text{vir}}) \).

(iv) Obtain the slope \( s_1 \) and the concentration \( c_2 \) from the corresponding ratio using the typical concentration \( c_{\text{DMO}} \) of a DM-only NFW halo from Dutton & Macciò (2014), recalled in equation (47), and the corresponding inner slope at \( 0.01R_{\text{vir}} \), \( s_{\text{DMO}} \), expressed in equation (48).

(v) Convert \( s_1 \) and \( c_2 \) into the DZ parameters \( a \) and \( c \) using equations (14) and (15). We recall that these latter parameters are not as physically meaningful as \( s_1 \) and \( c_2 \).

(vi) Obtain the scale radius \( r_c \) and the characteristic density \( \rho_c \) entering the expression of the density, \( r_c = R_{\text{vir}}\mu c \) and \( \rho_c = (1 - a/3)c^2\mu\rho_{\text{crit}} \), with \( \mu = c^4 - 2(1 + c^2)^{1/2} - a \) and \( \rho_{\text{crit}} = \frac{3M_{\odot}}{4\pi R_{\odot}^3} = \Delta\rho_{\text{crit}} \).

(vii) Determine the mass-dependent density profile using equation (11); the corresponding circular velocity profile using equations (4) and (6), with \( \rho_c = c^2\mu\rho_{\text{crit}}, b = 2, \) and \( \mathcal{R} = 3 \). The gravitational potential profile is obtained from equation (19), the velocity dispersion profile from equation (22), the projected surface density profile from equation (32) or its series expansion (equation F5), the scaled deflexion angle from equation (38) or its series expansion (deduced from equation F6), the lensing shear from the average projected surface density of equation (41) or its series expansion (equation F10).

In the following section, we show that these prescriptions for the DZ profile are in relatively good agreement with simulated density and circular velocity profiles and fare as good as the Di Cintio+ prescriptions given the stellar and halo masses. When fitting rotation curves of galaxies, we however advocate to release the mass-dependent prescription for the concentration \( c_2 \) and to leave this parameter free (as advocated for the Di Cintio+ profile by Di Cintio et al. 2014b). This enables to obtain extremely good fits to simulated density and circular velocity profiles (cf. Section 4.3).

4.2 Accuracy of the mass-dependent prescriptions

Fig. 10 compares the inner logarithmic slope and the concentration stemming from the mass-dependent prescriptions of Section 4.1 with \( s_1^* \) and \( c_2^* \) determined directly from the simulated profiles (cf. Section 3.2). Although the Pearson correlation coefficients are slightly lower than those of Fig. 6, the inner slope and concentration are well recovered. Overall, these mass-dependent prescriptions enable to recover the inner slope \( s_1 \) with \( \pm 0.31 \) scatter and a negligible systematic error and the concentration \( c_2 \) with \( \pm 9 \) scatter and a small \(-1.5 \) systematic offset (0.12 dex offset and \(-0.05 \) dex offset in \( \log c_2 \)). As further shown in Figs G3 and G4, these prescriptions recover the maximum velocity \( V_{\text{max}} \) with \( \pm 9 \) per cent scatter and a +3 per cent offset, and the corresponding radius \( R_{\text{max}} \) with \( \pm 31 \) per cent scatter and a +12 per cent offset (0.14 dex scatter and +0.05 dex offset in \( \log R_{\text{max}} \)). The scatters and offsets in \( \Delta s, \Delta c, \Delta V, \) and \( \Delta R \) are comparable to those described for the fits in Section 3.2 but the rms errors and the discrepancies between prescribed and simulated profiles are significantly higher (\( \sigma \) progressing on average from 0.046 to 0.080, \( \sigma_V \) from 0.027 to 0.072, and similar trends for \( \sigma_{s_1} \) and \( \sigma_V^{s_1} \), especially at high stellar-to-halo mass ratio: while the overall scatters and offsets are preserved, discrepancies arise on a case by case basis. The difference between the parametrized and the simulated rotation curves can be as high as 20 per cent. As discussed in the following section, releasing the constraint on the concentration when fitting rotation curves enables to significantly improve the fits.

In Appendix G (Supporting Information), we further compare the current mass-dependent prescriptions with those of Di Cintio et al. (2014b). For this other mass-dependent profile, the four shape parameters entering equation (2) – namely \( a, b, \) and \( c_2 \) – and the concentration parameter associated with the scale radius \( r_c \) are expressed as a function of the stellar-to-halo mass ratio while the scale density \( \rho_c \) is deduced from the halo mass, since the enclosed mass associated with the profile must verify \( M(R_{\text{vir}}) = M_{\text{vir}} \). Both for the current and Di Cintio+ prescriptions, all the parameters describing the profiles are set given the stellar and halo masses. Appendix G (Supporting Information) shows that the prescriptions of Section 4.1 provide equally good (or even marginally better) fits to the simulated density and velocity profiles than the Di Cintio+ prescriptions. We caution however that while the current prescriptions stem from the NIHAO sample itself, the Di Cintio+ prescriptions were obtained from a smaller sample of 10 simulated galaxies (the MaGICC sample; Brook et al. 2012; Stinson et al. 2013), such that the slightly better accuracy of the current prescriptions is most likely due to the different...
Figure 11. Prescribed versus simulated density profiles when fitting rotation curves (leaving the concentration free): the dark-matter density profiles at $z = 0$ of the 8 arbitrary NIHAO galaxies shown in Fig. 3 (plain red line) with their DZ (dashed) and Di Cintio+ (dotted) one-parameter fit to the rotation curves. For the DZ profile, the inner slope $s_1$ is set by the fitting function of Fig. 9 (cf. equation 45 and Table 1) but the concentration $c_2$ is allowed to vary; for the Di Cintio+ profile, the shape parameters $a, b, g$ of equation (2) are set by their mass-dependent prescriptions (Di Cintio et al. 2014b, equation 3), the scale radius $r_c$ is allowed to vary, and the characteristic density $\rho_c$ is constrained by the halo mass $M_{\text{vir}}$. The masses $M_{\text{star}}, M_{\text{vir}}, M_{\text{star}}/M_{\text{vir}}$ and the rms errors $\sigma$ and $\sigma_{\text{centre}}$ are indicated. Both the DZ and the Di Cintio+ parametrizations provide extremely good fits to the density profiles.
Figure 12. Prescribed versus simulated rotation curves when leaving the concentration free: dark-matter circular velocity profiles, $V_c(r) = \sqrt{GM(r)/r}$, of the eight $z = 0$ NIHAO galaxies shown in Fig. 3 (plain red line) together with those inferred from the DZ and Di Cintio+ one-parameter fit to the rotation curves (dashed and dotted lines, respectively). For the DZ profile, the inner slope $s_1$ is set by the fitting function of Fig. 9 (cf. equation 45 and Table 1) but the concentration $c_2$ is allowed to vary; for the Di Cintio+ profile, the shape parameters $a, b, g$ of equation (2) are set by their mass-dependent prescriptions (Di Cintio et al. 2014b, equation 3), the scale radius $r_\text{c}$ is allowed to vary, and the characteristic density $\rho_\text{c}$ is constrained by the halo mass $M_{\text{vir}}$. The velocity of each galaxy is normalized to its maximum value $V_{\text{max}}$, which is an increasing function of mass. Both the DZ and the Di Cintio+ parametrizations provide extremely good fits to the rotation curves, with differences below 10 per cent that are well within observational errors.
applied to simulated haloes whose mass $M$ is similar to what is advocated for the Di Cintio concentration $c$, and their scale density $\rho(\sigma) \propto e^{-\sigma^2/c^2}$. This motivates to release the mass constraint on the $M$ to-halo mass ratio parameters being $c$ galaxies simulated with baryons. The median values for the two prescriptions, which are highlighted by vertical lines above the $\Delta 1$ profile of the Di Cintio $\sigma_\text{vir}$–$R_\text{vir}$–$0.1$ subhalo is known, enforcing $M(R_{\text{vir}}) = M_{\text{vir}}$ effectively leaves one free parameter ($r_c$ or its associated concentration).

Figs 11 and 12 show the density and circular velocity profiles resulting from one-parameter fits to the rotation curves using the DZ profile and its current mass-dependence prescription for the inner logarithmic slope $s_1$ for the eight fiducial NIHAO haloes shown in Fig. 3, together with the corresponding Di Cintio+ one-parameter fits and the simulated profiles. The inner slope $s_1$ of the DZ profile is set by the fitting function of Fig. 9 (cf. equation 45 and Table 1) given the stellar and halo masses while its concentration $c_2$ is allowed to vary. The shape parameters $a$, $b$, $g$ of the Di Cintio+ profile (equation 2) are set by their mass-dependent prescriptions (Di Cintio et al. 2014b, equation 3), while the scale radius $r_c$ is allowed to vary. The DZ fit is characterized by density $\rho_1$ and its mass dependence in the domain where $\rho_{\text{DZ}}$ increases exponentially. This motivates to release the mass constraint on the concentration $c_2$ when modelling rotation curves of galaxies with the DZ profile, thus treating it as a two-parameter profile (the two parameters being $M_{\text{star}}$ and $c_2$ given the stellar mass $M_{\text{star}}$). This is similar to what is advocated for the Di Cintio$^+$ profile. When applied to simulated haloes whose mass $M_{\text{vir}}$ is known, enforcing $M(R_{\text{vir}}) = M_{\text{vir}}$ effectively leaves one free parameter ($r_c$ or its associated concentration).

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and −0.08 dex offset in log $c_2$), the maximum velocity $V_{\text{max}}$ with a 3 per cent scatter and a +0.6 per cent offset, and the corresponding radius $r_{\text{max}}$ with a ±30 per cent scatter and a +20 per cent offset (0.11 dex scatter and +0.08 dex offset in log $r_{\text{max}}$). These scatters and offsets are comparable to those obtained previously, except for Δ$V$ where they are significantly smaller. The differences between the parametrized and the simulated rotation curves are below 10 per cent at any radius and for any galaxy, i.e. well within observational errors.

In contrast, the NFW profile used for DM haloes is in contrast unable to describe such rotation curves in the presence of baryons, with differences as high as 50 per cent in the intermediate-mass range where core formation occurs (Di Cintio et al. 2014b).

5 CONCLUSION

Baryonic processes affect the DM haloes in which galaxies are embedded, their inner density profiles ranging from steep NFW-like cusps as in DM-only simulations (Navarro et al. 1996, 1997) at low stellar masses, flat cores in the stellar mass range between $10^7$ and $10^9$ $M_\odot$, and cusps steeper than NFW at higher stellar masses (e.g. Di Cintio et al. 2014b; Tollet et al. 2016; Dutton et al. 2016b).

In this article, we study a parametrization of DM haloes that enables to describe this variety of halo responses to baryonic processes with a variable inner logarithmic slope $s_1$ and a variable concentration parameter $c_2$. This parametrization, which we refer to here as the DZ profile, is a specific case of the Zhao family of double power-law models (equation 2, Z96) in which the outer logarithmic slope is set to $g = 3.5$ and the exponent describing the transition between the inner and outer regions to $b = 2$. As shown by Z96 and AZ13, it allows analytic expressions for the gravitational potential and the velocity dispersion, which we recall in Section 2.2.2 (equations 19 and 22).

Using three pairs of haloes at different masses with and without baryons at $z = 0$, taken from the NIHAO suite of hydrodynamical cosmological zoom-in simulations (Wang et al. 2015), D17 show that this parametrization yields excellent fits to the density and circular velocity profiles of DM haloes ranging from steep cusps to flat cores, notably capturing cores better than the NFW and Einasto (1965) profiles. In F20, we further derive the kinetic energy associated with this DZ profile and show that it fits well with the simulated quantity.

In this article, we extend the work done by Z96, AZ13, D17, and F20 by gathering most analytic expressions obtained for the DZ profile (Sections 2.1 and 2.2), by deriving additional analytic expressions for its lensing properties in terms of Fox $H$ functions (Section 2.3) and by testing this profile over the whole NIHAO suite of simulations at $z = 0$ (Section 3). We also provide analytic expressions in terms of the maximum circular velocity and radius $V_{\text{max}}$ and $r_{\text{max}}$ (Appendix A, Supporting Information), a second-order Taylor expansion of the distribution function (Appendix D, Supporting Information), expressions for the velocity dispersion and the kinetic energy in the presence of an additional baryonic component (Appendix C, Supporting Information), and series expansions of the lensing properties (Appendix F, Supporting Information). Table 2 summarizes the analytic expressions available for the DZ profile. The systematic test on the NIHAO simulations enables us to quantitatively show that the DZ profile provides better fits to the density and circular velocity profiles of DM haloes than the other two-parameter Einasto and generalized NFW with variable inner slope profiles, in particular in the innermost regions (Section 3.2).

But most importantly, this test enables us to describe the mass dependence of the inner slope $s_1$ and concentration parameters $c_2$ associated with the DZ profile (Section 3.3) and to establish it as a mass-dependent profile (Section 4) on par with the double power-law Di Cintio+ profile proposed by Di Cintio et al. (2014b) – with the advantage to have analytic expressions for many of its properties and only two shape parameters instead of four. We show that both $s_1$ and $c_2$ correlate with stellar and halo mass, especially with the stellar-to-halo mass ratio $M_{\text{halo}}/M_{\text{star}}$, and we provide fitting functions for the corresponding relations. The inner logarithmic slope $s_1$ corresponds to the NFW slope for log ($M_{\text{halo}}/M_{\text{star}}$) between −3.5 and −2, and to steepervan-NFW inner density profiles for log ($M_{\text{halo}}/M_{\text{star}}$) > −2 (Fig. 9, equation 45, and Table 1). The concentration $c_2$ similarly corresponds to the NFW concentration at low $M_{\text{halo}}/M_{\text{vir}}$, becomes slightly (~20 per cent) smaller than the NFW concentration for log ($M_{\text{halo}}/M_{\text{vir}}$) between −3.5 and −2, and increases exponentially compared to NFW for log ($M_{\text{halo}}/M_{\text{vir}}$) > −2 (Fig. 9, equation 49, and Table 1). In terms of stellar mass, the range for core formation and halo expansion corresponds to $10^7$ to $10^{10}$ $M_\odot$.

The DZ profile thus enables to follow the expansion of the halo due to baryons in the mass range with log ($M_{\text{halo}}/M_{\text{vir}}$) between −3.5 and −2 not only in terms of inner logarithmic slope as for the Di Cintio+ profile but also in terms of concentration—i.e. at larger radii than those concerned by the inner slope. With the fitting functions of $s_1$ and $c_2$ as functions of $M_{\text{halo}}/M_{\text{vir}}$, the DM distribution in haloes

### Table 2. Analytic expressions for the DZ profile, which depends on two shape parameters – $s_1$ and $c_2$, or equivalently, the inner slope $s_1$ and the concentration $c_2$.

| Quantity | Equation |
|----------|----------|
| Density | $\rho(r) = \frac{\rho_0}{x^s (1 + x^{1/2})^{3.5(s-a)}}$ with $\Delta V = 3\Delta\rho$ |
| Characteristic radius | $r_c = R_{\text{vir}}/c$ |
| Characteristic density | $\rho_0 = (1 - a/3)\rho_c$ |
| Characteristic av. density | $\rho_{\text{av}} = c^3(\rho_0/r_c^3)$ |
| Average virial density | $\rho_{\text{vir}} = 3M_{\text{vir}}/4\pi r_{\text{vir}}^3 = \Delta\rho_{\text{vir}}$ |
| Mass factor | $\mu = e^{-\gamma - (1 + zL^2 e^{c/23} - a)}$ |
| Inner slope $s_1$ from $a$, $c$ | Equation (12) |
| Concentration $c_2$ from $a$, $c$ | Equation (13) |
| Parameter $a$ from $s_1$, $c_2$ | Equation (14) |
| Parameter $c$ from $s_1$, $c_2$ | Equation (15) |
| Core radius $r_{\text{core}}$ | Equation (16) |
| Half-mass radius and $r_j$ | Equation (17) |
| Maximum velocity radius $r_{\text{max}}$ | Equation (A1) with $b = 2$ and $g = 3$ |
| Maximum velocity $V_{\text{max}}$ | Equation (A2) with $b = 2$ and $g = 3$ |
| Concentration $c_{\text{max}}$ from $a$, $c$ | Equation (A7) |
| Parameter $a$ from $s_1$, $c_{\text{max}}$ | Equation (A8) |
| Parameter $c$ from $s_1$, $c_{\text{max}}$ | Equation (A9) |
| Average density | Equation (4) with $b = 2$ and $g = 3$ |
| Enclosed mass | Equation (5) |
| Circular velocity | Equation (6) |
| Gravitational force | Equation (7) |
| Logarithmic slope | Equation (9) with $b = 2$ and $g = 3$ |
| Gravitational potential | Equation (19) |
| Velocity dispersion | Equations (22), (B1) and (B3) |
| Surface density | Equations (32) and (F5) |
| Average surface density | Equations (41) and (F10) |
| Projected mass | Equations (34) and (F6) |
| Deflection angle | Equation (38) and from equation (F6) |
| Lensing shear | Equation (39) |
| Lensing potential | Equations (F8) and (F9) |
| Distribution function | Equations (D5) and (D6) (integral forms) |
| $s_1(M_{\text{halo}}/M_{\text{vir}})$ | Equations (45) and (48), Table 1 |
| $c_2(M_{\text{halo}}/M_{\text{vir}})$ | Equations (49) and (47), Table 1 |
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| Profile |
|---------|
| NFW 1996 |
| superNFW 2018 |
| pISO 1995 |
| Burkt 1995 |
| Lucky13 2020 |
| Einasto 1965 |
| coreEinasto 2020 |
| aβγ/ Di Cintio 2014 |
| gNFW 2016 |
| coreNFW 2016 |
| Dekel-Zhao 2017 |  
| Dekel+2017 |  
| Freundlich+2020 |  

| Expression & shape parameters |
|-------------------------------|
| $\rho = \frac{\rho_c}{x(1+x^n)}$ |
| $\rho = \frac{x(1+x^2)}{b}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |
| $\rho = \frac{x(1+x^2)^{n/2}}{c}$ |

| Analytic expressions |
|----------------------|
| $c_2(M(r), V(r), \sigma(r)) \Phi(\Sigma(r), f(r))$ |

| Mass-dependence |
|-----------------|
| $c(M_{\text{halo}})$ |

| $x = r/r_g$, $c = R_{200}/r_g$, $x_M = M_{200}/M_{\text{halo}}$, $c_2 = R_{200}/r_{200}$ |

| variable inner slope |
|----------------------|

| cusp |
|------|

| cores |
|------|

| variable inner slope |
|----------------------|

| Figure 15. The DZ profile against other existing parametrizations of DM halo density profiles. For each parametrization, we indicate the analytic expression of the density (or mass), its shape parameters, whether analytic expressions for the concentration ($c_2$) where the logarithmic density slope equals 2 in absolute value, enclosed mass ($M_c$), circular velocity ($V_c$), radial velocity dispersion ($\sigma_z$), gravitational potential ($\Phi$), projected surface density ($\Sigma$), average surface density ($\overline{\Sigma}$), and distribution function ($f$) are available to the best of our knowledge, and whether the shape parameters have been expressed as functions of the stellar and halo masses ($M_{\text{star}}$ and $M_{\text{halo}}$) using numerical simulations. The projected surface densities $\Sigma$ and $\overline{\Sigma}$ enable to define lensing properties such as the convergence, the shear and the magnification. The parametrizations listed alongside the DZ profile include the NFW (e.g. Navarro et al. 1996, 1997; Łokas & Mamon 2001; Evans & An 2006; Elías-Díaz & Müller 2007, AZ13) and ‘superNFW’ (Lilley, Evans & Sanders 2018) cuspy profiles, the pseudo-isothermal (pISO), Burkt (1995), and ‘Lucky13’ (Li et al. 2020) cored profiles, and the Einasto (e.g. Einasto 1965; Retana-Montenegro et al. 2012; Dutton & Macciò 2014, AZ13), ‘core-Einasto’ (Lazar et al. 2020), double power-law $a\beta\gamma$ (e.g. Z96, AZ13, Di Cintio et al. 2014b), generalized NFW (gNFW, e.g. Umetsu et al. 2011; Mamon et al. 2019), and ‘core-NFW’ (Read et al. 2016) profiles with flexible inner slope, more suited to describe the diversity of DM halo shapes in the presence of baryons. The ‘core-Einasto’ profile can become a two-parameter profile by fixing its parameter $a$ (Lazar et al. 2020), but limited to fitting cored profiles in a certain mass range. For the double power-law $a\beta\gamma$ profile (equation 2), $M(r)$, $V(r)$, $\sigma_z(r)$, and $\Phi(r)$ can be expressed using elementary functions in certain cases, in particular within the family of profiles with $b = n$ and $g = 3 + kn$ where $k$, $n$ are natural integers (Z96, AZ13). The Di Cintio+ profile corresponds to a double power-law profile whose shape parameters are set by the stellar-to-halo mass ratio (Di Cintio et al. 2014b). In this case, only $c_2$, $M(r)$, and $V(r)$ have analytic expressions (using non-elementary functions for the latter two). The mass-dependent prescriptions as a function of halo mass for the NFW and Einasto profiles stem from dark-matter-only simulations (e.g. Dutton & Macciò 2014). The DZ profile is a double-law profile with $x = r/r_g$, $c = R_{200}/r_g$, $x_M = M_{200}/M_{\text{halo}}$, $c_2 = R_{200}/r_{200}$ available non-elementary functions not available only certain cases. |

| ranging from dwarfs to Milky Way like in stellar mass is set by the stellar and halo masses. We show that the mass-dependent DZ profile thus established is as accurate as the multiparameter Di Cintio+ profile to describe density and circular velocity profiles of DM halos (Section 4.2), in particular when the concentration parameter is left free (Section 4.3). In Fig. 15, we compare the DZ profile with existing parametrizations of DM halo density profiles, emphasizing on the number of parameters, the availability of analytic expressions, and the availability of mass-dependent prescriptions derived from simulations. Amongst the parametrizations with variable inner slope, the DZ profile stands out for its available analytic expressions and its mass-dependent prescriptions as a function of the stellar-to-halo mass ratio, taking into account the effect of baryons. |

| We caution that this study relies on a specific suite of hydrodynamical cosmological simulations (NIHAO; Wang et al. 2015), which is notably characterized by a strong stellar feedback implementation with a blast-wave formalism and delayed cooling and no AGN feedback. We note that the Di Cintio+ profile was proposed using a previous suite of simulations (MaGiCC; Brook et al. 2012; Stinson et al. 2013) with a similar implementation. Other simulation suites with different feedback schemes (e.g. Mashchenko et al. 2008; Teyssier et al. 2013; Madau et al. 2014; Verbeke et al. 2015; Read et al. 2016) suggest a similar behaviour of the inner density profile of DM haloes as a function of the stellar-to-halo mass ratio. This behaviour can be understood in theoretical terms as a competition between outflows induced by feedback and the confinement imposed by the halo gravity. |

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(e.g. Dekel & Silk 1986; Read & Gilmore 2005; Peñarrubia et al. 2012; Pontzen & Governato 2012; Dutton et al. 2016b; El-Zant et al. 2016, F20). As such, the halo response to baryonic processes may not necessarily depend on the details of the feedback implementation as long as outflows are well-reproduced in the simulations. These outflows are expected to affect the stellar and gaseous components of galaxies, such that the good agreement of NIHAO galaxies with observations in terms of morphologies, colour, sizes and rotation curves (Stinson et al. 2015; Wang et al. 2015; Dutton et al. 2016a, 2017; Obreja et al. 2019; Santos-Santos et al. 2020) may reflect outflows comparable to those of actual galaxies (Tollet et al. 2019) and of other simulation suites reproducing the aforementioned observables. We leave detailed tests of the DZ profile in other simulation suites with different feedback implementations for future work.

The accuracy of the DZ profile to describe the DM distributions of simulated haloes makes it a useful tool to study the evolution of DM density profiles, to model rotation curves of galaxies, to parametrize gravitational lenses, and to implement in semi-analytical models of galaxy formation and evolution. The analytic expressions for the gravitational potential, the velocity dispersion and the lensing properties can notably be used to model core formation in DM haloes from outflow episodes resulting from feedback, as in F20, to model gravitational lenses, to generate halo potentials or initial conditions for simulations, to compare different DM distributions in semi-analytical models (Jiang et al. 2020), and to quantify simulated and observed rotation curves of galaxies without numerical integrations.

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DATA AVAILABILITY

We provide codes to implement the DZ profile at https://github.com /JonathanFreundlich/Dekel_profile. The simulation data underlying this article will be shared on reasonable request to the corresponding author.

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SUPPORTING INFORMATION

Supplementary data are available at MNRAS online.

Appendix A. Profile parameters in terms of $r_{\text{max}}$ and $V_{\text{max}}$

Appendix B. The velocity dispersion as a sum of elementary functions

Appendix C. Potential and velocity dispersion with an additional mass

Appendix D. Notes on the distribution function

Appendix E. Analytical lensing properties with the Mellin transform method

Appendix F. Series expansion of the lensing properties

Appendix G. Mass-dependent prescriptions

Appendix H. Additional figures

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