Three loop calculations and inclusive $V_{cb}$

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We discuss the impact of the recent $O(\alpha_s^2)$ calculations of the semileptonic width of the $b$ quark and of the relation between pole and kinetic heavy quark masses by Fael et al. on the inclusive determination of $|V_{cb}|$. The most notable effect is a reduction of the uncertainty. Our final result is $|V_{cb}| = 42.16(51) 10^{-3}$.

INTRODUCTION

The purpose of this note is to study the implications of the recent $O(\alpha_s^2)$ calculations by Fael, Schönwald and Steinhauser on the determination of the Cabibbo-Kobayashi-Maskawa matrix element $|V_{cb}|$ from inclusive semileptonic $B$ decays, see Refs. [1] [3] for the most recent results. As is well-known, the values of $|V_{cb}|$ determined from inclusive semileptonic $B$ decays and from $B \to D^*\ell\bar{\nu}$ have differed for a long time and, despite significant experimental and theoretical efforts, the situation remains quite confusing. Recent accounts of the $V_{cb}$ puzzle can be found in Refs. [1] [2].

The latest lattice calculations [10–13], which for the first time explore the $\bar{B} \to D^*\ell\bar{\nu}$ form factors at non-zero recoil, have not clarified the issue and a preliminary comparison of these results shows a few discrepancies [14].

The third order perturbative correction to the $b \to c\ell\bar{\nu}$ decay width computed in Ref. [1] and partially checked in Ref. [13] represents a fundamental step to improve the precision in the extraction of $|V_{cb}|$ from inclusive $B$ decays. Indeed, perturbative corrections are sizeable – they reduce the semileptonic rate by over 10% – and provide the dominant theoretical uncertainty. In the following we will study the impact of the $O(\alpha_s^2)$ corrections on the central value and uncertainty of $|V_{cb}|$.

The other relevant three-loop calculation in our analysis is that of Refs. [2] [3], which concerns the relation between the pole (or $\overline{\text{MS}}$) and the kinetic masses of a heavy quark. This calculation allows us to convert recent high-precision determinations of the $b$ quark $\overline{\text{MS}}$ mass [16, 17] into the kinetic scheme [18] with an uncertainty of about 15 MeV, or $\overline{\text{MS}}$ scheme with 4 active quark flavours, $\overline{\text{MS}}$ charm mass at $\mu = 2 \text{ GeV}$.

THE TOTAL SEMILEPTONIC WIDTH

Our starting point is the Operator Product Expansion (OPE) for the total semileptonic width (see Ref. [22] for a complete list of references):

$$\Gamma_{sl} = \Gamma_0 f(\rho) \left[ 1 + a_1 a_s + a_2 a_s^2 + a_3 a_s^3 - \left( \frac{1}{2} - p_1 a_s \right) \frac{\mu_s^2}{m_b^2} + (g_0 + g_1 a_s) \frac{\mu_G^2(m_b)}{m_b^2} + d_0 \rho_D^2 - g_0 \rho_L S \frac{\rho_D^2}{2} + \ldots \right]$$

(1)

where $\Gamma_0 = A_{ew}[V_{cb}]^2 G_F^2 m_b \ln(m_b^2/\mu^2)/192\pi^3$, $f(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln \rho$, $a_s = (\alpha_s(m_b))/\pi$ is the strong coupling in the $\overline{\text{MS}}$ scheme with 4 active quark flavours, $\rho = (m_c(m_c)/m_b \ln(m_b^2/\mu^2))^2$ is the squared ratio of the $\overline{\text{MS}}$ charm mass at the scale $\mu_c = m_c(m_c)$, and of the $b$ quark kinetic mass with a cutoff $\mu \sim 1 \text{ GeV}$, $m_b^k(m_b)$. $A_{ew} \approx 0.104$ is the leading electroweak correction. The parameters $\mu_s^2, \rho_D^2, \ldots$ are nonperturbative expectation values of local operators in the $B$ meson defined in the kinetic scheme with cutoff $\mu$. They are generally extracted from a fit to central moments of $\mu$.
the lepton energy and of the hadronic invariant mass distributions in semileptonic $B$ decays \cite{ref1, ref2}, for which the same contributions as in Eq. (1) are included, with the exception of the $\mathcal O(\alpha_s^3)$ corrections which are available only for the width.

The coefficients in Eq. (1) depend on three unphysical scales: the scale of the $\overline{\text{MS}}$ strong coupling constant $\mu_b$, that of the $\overline{\text{MS}}$ charm mass $\mu_c$, and the wilsonian cutoff $\mu$ employed in the kinetic scheme definition of the $b$ mass and of the OPE matrix elements. We choose the $\overline{\text{MS}}$ scheme for the charm mass because all high-precision determinations of this mass are expressed in this scheme and we prefer to escape uncertainties related to the scheme conversion; in the following we choose $1.6 \text{ GeV} \lesssim \mu_c \lesssim 3 \text{ GeV}$, avoiding scales which are either too low or too high to provide a good convergence of the perturbative series. The kinetic scheme \cite{ref1, ref2, ref3} provides a short-distance, renormalon-free definition of $m_b$ and of the OPE parameters by introducing a hard cutoff $\mu$ to factor out the infrared contributions from the perturbative calculation. The cutoff $\mu$ should ideally satisfy $\Lambda_{\text{QCD}} \ll \mu \ll m_b$; in the following we will vary it in the range $0.7-1.3 \text{ GeV}$. Finally, the scale of the strong coupling constant will be varied in the range $2-8 \text{ GeV}$.

Table I of Ref. \cite{ref1} shows the size of the various coefficients in Eq. (1) for a couple of typical scale-settings. The third order coefficient $a_3$ is new and stems from the calculation of Ref. \cite{ref1}. We reproduce the numerical results of Ref. \cite{ref1} for the coefficients $a_i$.

As a first step in our analysis, we employ the results of the default fit of Ref. \cite{ref1} with $\mu_b = m_b^{\text{kin}}$, $\mu_c = 3 \text{ GeV}$, and $\mu = 1 \text{ GeV}$, to extract $|V_{cb}|$ from Eq. (1). To this end, we employ the total semileptonic branching fraction obtained in the same fit and $\tau_B = 1.579(5) \text{ ps}$ \cite{ref22}. Notice that the 2014 default fit included a constraint on $\bar m_c(3 \text{ GeV})$, but not on $m_b^{\text{kin}}$. Small shifts have to be applied to the values of $m_b^{\text{kin}}$, $\mu_c^2$ and $\rho_B^2$ extracted from the fit in order to account for missing two-loop charm mass effects in the kinetic scheme definition adopted in the 2014 fit. These effects have now been computed in Ref. \cite{ref3}, where it was found that they reduce to decoupling effects and that they can be taken into account by expressing the kinetic scheme definitions in terms of $\mu_b$. As noted in Ref. \cite{ref1}, the better convergence of the perturbative expansion with $\mu_c = 2 \text{ GeV}$, already observed in Ref. \cite{ref22}, carries on at the three loop level, but the cancellations appear somewhat accidental. Since the physical scale of the decay is actually lower than $m_b$, we believe a more appropriate choice for the scale of $\alpha_s^{(4)}$ is $\mu_b = m_b^{\text{kin}}/2$, which with $\mu_c = 2 \text{ GeV}$ leads to $|V_{cb}| = 42.59(44)_{fb}(33)_{\text{exp}} \times 10^{-3}$ and

\[
\Gamma_{\text{sl}} = \Gamma_0 f(\rho) \left[ 0.9255 - 0.1162 \alpha_s - 0.0350 \alpha_s^2 - 0.0097 \alpha_s^3 \right] = 0.5401 \Gamma_0, \tag{2}
\]

where the first term differs from 1 because of the power corrections. We can also repeat the same exercise evolving the value of $\bar m_c(3 \text{ GeV}) = 0.987(13) \text{ GeV}$ from the fit to $\mu_c = 2 \text{ GeV}$, which gives $\bar m_c(2 \text{ GeV}) = 1.091(14) \text{ GeV}$ and extract again $|V_{cb}|$ using $\mu_c = 2 \text{ GeV}$ in Eq. (1). We get $|V_{cb}| = 42.59(44)_{fb}(33)_{\text{exp}} \times 10^{-3}$ and

\[
\Gamma_{\text{sl}} = \Gamma_0 f(\rho) \left[ 0.9258 - 0.0878 \alpha_s - 0.0179 \alpha_s^2 - 0.0005 \alpha_s^3 \right] = 0.5374 \Gamma_0. \tag{3}
\]

As noted in Ref. \cite{ref1}, the better convergence of the perturbative expansion with $\mu_c = 2 \text{ GeV}$, already observed in Ref. \cite{ref22}, carries on at the three loop level, but the cancellations appear somewhat accidental. Since the physical scale of the decay is actually lower than $m_b$, we believe a more appropriate choice for the scale of $\alpha_s^{(4)}$ is $\mu_b = m_b^{\text{kin}}/2$, which with $\mu_c = 2 \text{ GeV}$ leads to $|V_{cb}| = 42.59(44)_{fb}(33)_{\text{exp}} \times 10^{-3}$ and

\[
\Gamma_{\text{sl}} = \Gamma_0 f(\rho) \left[ 0.9255 - 0.1140 \alpha_s - 0.0011 \alpha_s^2 + 0.0103 \alpha_s^3 \right] = 0.5381 \Gamma_0. \tag{4}
\]

We see from Eqs. (2)-(4) that the typical size of the three-loop corrections is $1\%$ and that the perturbative series converge well at different values of the scales. A conservative estimate of the residual perturbative uncertainty on $\Gamma_{\text{sl}}$ is therefore $0.5\%$, but it is worth studying the scale dependence of the width in more detail. In Fig. 1 we show the $\mu_b$ and $\mu_c$ dependence of Eq. (1) at two and three loops, using the inputs of the 2014 default fit. The scale dependence is reduced by the inclusion of the three loop contribution by over a factor 2, and the red curves appear to be flatter than the blue ones. The region of minimal scale dependence is the one around $\mu_b \sim \mu_c \approx 2-3 \text{ GeV}$. We also studied the dependence of the width on the kinetic scale in the range $0.7 < \mu < 1.3 \text{ GeV}$ for different values of $\mu_{b,c}$, finding similar results. Defining $\Delta \mu_{b,c}$ the maximum percentage deviation of the red solid lines in Fig. 1 and $\Delta \mu$ accordingly, with $\mu_b = m_b^{\text{kin}}/2$ and $\mu_c = 2 \text{ GeV}$, we get

\[
\Delta \mu_b = 0.44\%, \quad \Delta \mu_c = 0.44\%, \quad \Delta \mu = 0.67\%. \tag{5}
\]
Based on all this, we conservatively estimate a residual perturbative uncertainty of 0.7% in $\Gamma_{sl}$ and consequently of 0.35% in $|V_{cb}|$ for our new default scenario, corresponding to $\mu = 1$ GeV, $\mu_c = 2$ GeV and $\mu_b = m_b^{\text{kin}}/2 \approx 2.3$ GeV.

Beside the purely perturbative contributions, there are various other sources of uncertainty in the calculation of the semileptonic width [26], but the work done in the last few years has been fruitful. After the $O(\alpha_s/m_b^2)$ corrections [27, 28], the $O(\alpha_s\rho_D^3/m_b^3)$ corrections to $\Gamma_{sl}$ have been recently computed in Ref. [21] (the $O(\alpha_s\rho_{LS}^3)$ corrections to $\Gamma_{sl}$ follow from the $O(\alpha_s\mu_c^3/m_b^3)$ and are tiny). They are expressed in terms of $m_b$ in the on-shell scheme and of $m_c(m_b)$. After converting their result to the kinetic scheme and changing the scale of the semileptonic width [26], but the work done in the last few years has been fruitful. After the conversion to the kinetic scheme the $O(3, 4)$ terms generate new $O(\mu^3\alpha_s^2)$ and $O(\mu^3\alpha_s^3)$ contributions that tend to compensate their effect. The resulting final shift on $|V_{cb}|$ is $+0.05\%$ with $\mu_c = 3$ GeV, $\mu_b = m_b^{\text{kin}}$ and $+0.1\%$ for $\mu_c = 2$ GeV, $\mu_b = m_b^{\text{kin}}/2$, and we choose to neglect it in the following.

After the calculation of the $O(\alpha_s\rho_D^3)$ contribution, the main residual uncertainty in $\Gamma_{sl}$ is related to higher power corrections. The Wilson coefficients of the $O(1/m_b^4, 1/m^5)$ contributions have been computed at the tree level [29] — here the $O(1/m^5)$ effects include $O(1/m_b^3m_c^2)$, sometimes referred to as Intrinsic Charm — but little is known about the corresponding 27 matrix elements. The Lowest Lying State Approximation (LLSA) [29] has been employed to estimate them and to guide the extension [3] of Ref. [4] to $O(1/m_b^4, 1/m^5)$. In the LLSA, the $O(1/m_b^4, 1/m^5)$ contributions increase the width by about 1%, but there is an important interplay with the semileptonic fit: as shown in Ref. [5], the $O(1/m_b^3, 1/m^5)$ corrections to the moments and their uncertainties modify the results of the fit in a subtle way and the final change in $\Gamma_{sl}$ is about $+0.5\%$, a result stable under changes of the LLSA assumptions [5]. We therefore expect the $O(1/m_b^4, 1/m^5)$ corrections to decrease $|V_{cb}|$ by 0.25% with respect to the default fit. Although the uncertainty attached to this value is mostly included in the theoretical uncertainty of the 2014 fit results, we may consider an additional 0.3% uncertainty for the width. Further uncertainties stem from unknown $O(\alpha_s^2/m_b^2)$, and $O(\alpha_s^2\rho_D^3/m_b^3)$ corrections, but they are all likely to be at or below the 0.1% level. The so-called Intrinsic Charm contributions, related to soft charm, lead to the $O(1/m_b^3m_c^2)$ corrections mentioned above, but also to terms of $O(\alpha_s/m_b^2m_c)$ which may contribute up to 0.5% to the width [30]. Finally, one expects quark-hadron duality to break down at some point. Combining all the discussed sources of uncertainty, we estimate the total remaining uncertainty in $\Gamma_{sl}$ to be 1.2%.

In the end, using the inputs of the 2014 default fit and setting $\mu_c = 2$ GeV, $\mu_b = m_b^{\text{kin}}/2$ for the central value, we obtain

$$|V_{cb}|_{2014} = 42.49(44)_{\text{th}}(33)_{\text{exp}}(25)\times 10^{-3} = 42.48(60)\times 10^{-3}$$ (6)

where the uncertainty due to $\Gamma_{sl}$ has been reduced by a factor 2 with respect to Ref. [4].

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**FIG. 1.** Scale dependence of $\Gamma_{sl}$ at fixed values of the inputs and $\mu^{\text{kin}} = 1$ GeV. Dashed (solid) lines represent the two (three) loop calculation. In the left plot ($\mu_b$-dependence) the blue (red) curves are at $\mu_c = 3(2)$ GeV; in the right plot ($\mu_c$-dependence) the blue (red) curves $\mu_b = m_b^{\text{kin}}(m_b^{\text{kin}}/2)$. 
With respect to Ref. [4], in particular, we consider a 20%, instead of a 30%, shift in $\rho$ corrections discussed in the previous section, we reduce the theoretical uncertainties used in the fit to the moments that the fit to the moments is based on an $O$ reference value and in Table I we display the complete results of this fit.

The safety shift in $\chi_{\min}$ which correspond to $\overline{m}_c(2\text{GeV}) = 1.093(8)$ and $m_b^{\text{kin}}(1\text{GeV}) = 4.565(19)\text{GeV}$, where for the latter we have used option B of [3] for the definition of $m_b^{\text{kin}}$. We now repeat the 2014 default fit with both these constraints, slightly updating the theoretical uncertainty estimates. In view of the small impact of the $O(1/m_b^2, 1/m^3)$ and $O(\alpha_s\rho_D^3)$ corrections discussed in the previous section, we reduce the theoretical uncertainties used in the fit to the moments with respect to Ref. [4]. In particular, we consider a $20\%$, instead of a $30\%$, shift in $\rho_D^3$ and $\rho_{LS}^3$, and reduce to 4 MeV the safety shift in $m_{c,b}$. For all of the other settings and for the selection of experimental data we follow Ref. [4].

While the central values of the fit are close to those of 2014, the uncertainty on $m_b^{\text{kin}}$ ($\overline{m}_c(3\text{GeV})$ decreases from 20(12) to 12(7) MeV, and we get $|V_{cb}| = 42.39(32)_{\text{th}}(32)_{\text{exp}}(25)\Gamma 10^{-3}$ with $\chi^2_{\min}/dof = 0.46$. The very same fit performed with $\mu_c = 2\text{GeV}$ and $\mu_b = m_b^{\text{kin}}/2$ gives

$$|V_{cb}| = 42.16(30)_{\text{th}}(32)_{\text{exp}}(25)\Gamma 10^{-3}$$

with $\chi^2_{\min}/dof = 0.47$ and we neglect the very small shift due to the $O(\alpha_s\rho_D^3)$ correction to $\Gamma_{sl}$. This is our new reference value and in Table I we display the complete results of this fit.

Let us now comment on the interplay between the fit to the moments and the use of Eq. (1). First, we observe that the fit to the moments is based on an $O(\alpha_s^2)$ calculation without $O(\alpha_s\rho_D^3)$ contributions, and that the lower precision in the calculation of the moments with respect to the width inevitably affects the determination of $|V_{cb}|$. This is clearly visible in Eq. (6), where the theoretical component of the error is larger than the residual theory error associated with the width. However, only a small part of that uncertainty is related to the purely perturbative corrections, which are relatively suppressed in some semileptonic moments but sizeable in $\Gamma_{sl}$, as we have seen above. In other words, an $O(\alpha_s^2)$ calculation of the moments is unlikely to improve the precision of the fit significantly, and the inclusion of $O(\alpha_s^2)$ corrections only in $\Gamma_{sl}$ is perfectly justified. On the other hand, an $O(\alpha_s/m_b^3)$ calculation of the moments can have an important impact on the $|V_{cb}|$ determination. This is because the semileptonic moments, and the hadronic central moments, are highly sensitive to the OPE parameters. Since the power correction related to $\rho_D^3$ amounts to about 3% percent in Eq. (1), an $O(\alpha_s)$ shift on $\rho_D^3$ induced by perturbative corrections to the moments can have a significant impact in the determination of $|V_{cb}|$.

### Table I

| $m_b^{\text{kin}}$ | $\overline{m}_c(2\text{GeV})$ | $\mu^2_c$ | $\rho_D^3$ | $\mu^2_{3}(m_b)$ | $\rho^3_{LS}$ | $\text{BR}_{c\ell\nu}$ | $10^3|V_{cb}|$ |
|------------------|-----------------|--------|----------|-----------------|--------|------------------|------------------|
| 4.573 | 1.092 | 0.477 | 0.185 | 0.306 | -0.130 | 10.66 | 42.16 |
| 0.012 | 0.008 | 0.056 | 0.031 | 0.050 | 0.092 | 0.15 | 0.51 |
| 1 | 0.307 | -0.141 | 0.047 | 0.612 | -0.196 | -0.064 | -0.420 |
| 1 | 0.018 | -0.010 | -0.162 | 0.048 | 0.028 | 0.061 |
| 1 | 0.735 | -0.054 | 0.067 | 0.172 | 0.429 |
| 1 | -0.157 | -0.149 | 0.091 | 0.299 |
| 1 | 0.001 | 0.013 | -0.225 |
| 1 | -0.033 | -0.005 |
| 1 | 0.684 |

### Updating the Semileptonic Fit

Despite ongoing analyses of the $q^2$ and $M_X$-moments at Belle and Belle II [31, 32], no new experimental result on the semileptonic moments has been published since the 2014 fit [4]. On the other hand, new lattice determinations of $m_b$ and $m_c$ have been presented, improving their precision by roughly a factor 2. We use the FLAG 2019 averages [17] with $N_f = 2 + 1 + 1$ for $m_b$ and $m_c$,

$$\overline{m}_c(3\text{GeV}) = 0.988(7)\text{GeV},$$
$$m_b(m_b) = 4.198(12)\text{GeV},$$

which correspond to $\overline{m}_c(2\text{GeV}) = 1.093(8)$ and $m_b^{\text{kin}}(1\text{GeV}) = 4.565(19)\text{GeV}$, where for the latter we have used option B of [3] for the definition of $m_b^{\text{kin}}$. All parameters are in GeV at the appropriate power and all, except $m_c$, in the kinetic scheme at $\mu = 1\text{GeV}$. The first and second rows give central values and uncertainties, the correlation matrix follows.
take this into account. We also note that a fit without theoretical errors is a very poor fit ($\chi^2/dof \sim 2$) with $|V_{cb}|$ decreased by slightly less than 1 $\sigma$.

An important problem of the semileptonic fit is the sensitivity to the ansatz employed for the correlation among the theoretical uncertainties associated with the various observables [24]. We have studied the dependence of the result of Eq. (8) on the modelling of the theoretical correlations following Ref. [25] closely. Since the results shown above have been obtained using scenario D from Ref. [25] with $\Delta = 0.25$ GeV, we have repeated the fit with option B, with option C using various values of $\xi$, and with option D for $\Delta$ in the range $0.15 - 0.30$ GeV. The central values for $|V_{cb}|$ vary between $42.05 \pm 10^{-3}$ and $42.28 \pm 10^{-3}$. These results are very much in line with Fig. 1 and Table 3 of Ref. [25] and therefore we do not add any uncertainty related to the theoretical correlations in Eq. (8).

We have also performed a fit including $\mathcal{O}(1/m_Q^3, 1/m_b^3)$ corrections, in analogy with Ref. [5], to check the consistency with our main result of Eq. (8). We assign an error to the LLSA predictions and assume gaussian priors for all of the 27 dimension 7 and 8 matrix elements. The error is chosen as the maximum of either 60% of the parameters value in the LLSA or $\Lambda_{LL}^n/2$ ($n = 4, 5$), with $\Lambda_{LL} = 0.55$ GeV, see Ref. [5] for additional details. As already noticed in Ref. [5], higher power corrections tend to decrease the value of $|V_{cb}|$. A fit performed with the same theory errors of Ref. [5] and $\mu_c = 2$ GeV and $\mu_b = m_b^{kin}/2$ leads to $|V_{cb}| = 42.00(32)_{th}(32)_{exp}(25) \times 10^{-3} = 42.00(53) \times 10^{-3}$, which is consistent with Eq. (8). Following the discussion above, one could slightly reduce the theory uncertainties in this fit with the only consequence of a small reduction on the error of $|V_{cb}|$.

Finally, repeating the reference fit of Table I without a constraint on $m_b$ we obtain an independent determination of $m_b^{kin}(1$ GeV) = 4.579(16) GeV, which translates into $m_b^\text{exp}(m_b) = 4.210(22)$ GeV. This determination, which still relies on the lattice determination of $m_c$ reported in [7], is compatible with the FLAG $N_f = 2 + 1 + 1$ average for $m_b^\text{exp}(m_b)$ and competitive with other current determinations of $m_b$.

**DISCUSSION**

From a theoretical point of view, the reliability of the determination of $|V_{cb}|$ from inclusive semileptonic decays depends on our control of higher order effects. The new three-loop calculation of Ref. [1] shows that higher order perturbative effects are under control, and that they lie within the previously estimated uncertainties. This progress, together with the work done on higher power corrections [5, 29] and on perturbative corrections to the Wilson coefficients of power suppressed operators [21, 27, 28], led us to estimate a residual theoretical error on $\Gamma_{sl}$ of about 1.2%, and to slightly reduce the theoretical uncertainty in the fit to the moments.

Our final result is shown in Eq. (8). It is very close to previous determinations of $|V_{cb}|$ [4, 5, 38], but the total uncertainty is now 1.2%, one third smaller than in [4]. This reduction of the uncertainty reflects a better control of higher order effects, but it is also due to improved determinations of the heavy quark masses. The dominant single component of the uncertainty in Eq. (8) is now related to the experimental determination of the moments and of the semileptonic branching fraction, which are expected to be improved at Belle II. Future experimental analyses should also consider new observables beyond the traditional moments of the lepton energy and hadronic invariant mass distributions. For instance, the forward-backward asymmetry $27/29$ and the moments of the leptonic invariant mass $|\chi|^2$ distribution would enhance the sensitivity to the OPE matrix elements and reduce the uncertainty on $|V_{cb}|$. Because of reparametrisation invariance, the $q^2$-moments and $\Gamma_{sl}$ depend on a reduced number of OPE matrix elements [28], so that a fit to the $q^2$-moments at $\mathcal{O}(1/m_Q^3)$ involves only 8 parameters. This nice property allows for an independent check of the treatment of higher power corrections adopted in [5], but it is unlikely to lead to a competitive determination of $|V_{cb}|$. The $q^2$-moments will also constrain the soft charm effects considered in [30]. As far as the current experimental analyses are concerned, there are various aspects that require closer scrutiny. We refer in particular to the subtraction of QED corrections made with PHOTOS [29], to the impact of Coulomb interactions, to the contribution of $D^{**}$ states and to the correlations which play a crucial role in the fit.

Finally, turning to ways in which theory can improve the inclusive determination of $|V_{cb}|$, we have already argued that the most important missing higher order effects are probably the $\mathcal{O}(\alpha_s/m_Q^3)$ corrections to the moments. Lattice QCD calculations provide precise constraints on the heavy quark masses, see Eq. (7), which are going to improve in the future, but we now have methods to compute differential distributions and their moments directly on the lattice [40]. While it is still unclear whether a determination of $\Gamma_{sl}$ competitive with Eq. (1) can be achieved at the physical $b$ mass, these lattice calculations might be able to enhance the predictive power of the OPE by accessing quantities which are inaccurate or beyond the reach of current experiments and are highly sensitive to the non-perturbative parameters. The computation of meson masses at different quark mass values [16, 41] can also provide useful information when
the data are analysed in the heavy quark expansion. At the moment, however, the reduction of the uncertainty in Eq. (8) exacerbates the $V_{cb}$ puzzle, and calls for renewed efforts to solve an unwelcome anomaly, impervious to New Physics explanations [42, 43].

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