CLOSED-FORM SOLUTIONS FOR THE LUCAS-UZAWA GROWTH MODEL WITH LOGARITHMIC UTILITY PREFERENCES VIA THE PARTIAL HAMILTONIAN APPROACH

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Abstract. In this paper, we present a dynamic picture of the two sector Lucas-Uzawa model with logarithmic utility preferences and homogeneous technology as was proposed by Bethmann [3] for a Robinson Crusoe economy. We use a newly developed partial Hamiltonian approach to derive a new set of closed-form solutions for the model with logarithmic utility preferences and homogeneous technology. Unlike the previous literature, our model yields three distinct closed-form solutions to the model. We establish the growth rates of all the variables which fully describe the dynamics of the model. Even though the first closed-form solution provides static growth rates and the other two provide dynamic growth rates, in the long run all the closed-form solutions approach the same static balanced growth path.

1. Introduction. The Lucas-Uzawa model of endogenous growth [10, 19] is one of the fundamental models of economic growth theory, and has been extensively used to analyze the determinants of long run economic growth across countries. Over time, different quantitative and qualitative techniques have been employed to analyze the solutions of the Lucas-Uzawa model with or without externalities for the case of CES utility preferences. Benhabib and Perli [2], Caballe and Santos [5] solved the basic Lucas-Uzawa model by using dimension reduction techniques while Mulligan and Sala-i-Martin [12] utilized time elimination methods. At the same time, Xie [20], Boucekkine and Ruiz-Tamarit [4], Chilarescu [7] used methods like parameter restrictions or hypergeometric functions to find closed-form solutions. The recently developed partial Hamiltonian approach [13, 14] and partial Lagrangian approach [15] provide a unique way of establishing closed-form solutions for economic growth models as well as other fields of applied mathematics (see e.g [14]). Naz et al [16] derived completely new closed-form solutions for the Lucas-Uzawa model by using the partial Hamiltonian approach [13, 14] while Naz and Chaudhry [17] provided a comparison of closed-form solutions derived by the partial Hamiltonian approach [16] and the classical approach [4, 7] for the Lucas-Uzawa model.

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established closed-from solutions for the Lucas-Uzawa model with externalities in terms of hypergeometric functions and Chilarescu \[8\] derived one closed-form solution of the Lucas-Uzawa model with externalities.

Recently, Bethmann \[3\] developed a stylized version of the two sector Lucas-Uzawa model with logarithmic utility preferences and homogeneous technology for a Robinson Crusoe economy. He utilized a dynamic programming technique after employing a dimension reduction technique to study this model. Also, Chilarescu and Sipos \[9\] derived closed-form solutions for the variables in the model proposed by Bethmann in terms of numerically computable functions involving integrals.

In this paper, we utilize the partial Hamiltonian approach \[13, 14\] to derive the first integrals and closed-form solutions for the Lucas-Uzawa model with logarithmic preferences and homogeneous technology. In order to do so, we begin by deriving the first integrals for this model via the partial Hamiltonian approach \[13, 14\]. Then we utilize the derived first integrals to construct closed-form solutions. Using this methodology, we find multiple closed-form solutions to the model which is first time that this has been found in the economic growth literature. The multiple equilibria is a topic of particular interest to economists and an example of this is Chaudhry et al \[6\] in which multiple equilibria for an economic growth model with environmental quality was investigated.

We also calculate the growth rates of all the variables in the model for each of our solutions. Our first solution provides static growth rates while the other two provide dynamic growth rates, but in the long run all approach the same static value which is in line with economic fundamentals. Note that in the previous literature \[9\] only one solution was established and this was in terms of two numerically computable functions involving integrals.

The layout of the paper is as follows. In Section 2, we introduce the Lucas-Uzawa model with logarithmic preferences and homogeneous technology. We utilize the partial Hamiltonian approach to find the first integrals of the dynamical system of ordinary differential equations (ODEs). In Section 3, the closed-form solutions of the dynamical system of ODEs are constructed by utilizing two first integrals. In Section 4, the closed-form solutions of the dynamical system of ODEs are constructed by utilizing only one first integral. In Section 5, we analyze the growth rates of all the variables in the system and discuss the convergence of the closed-form solutions to the balanced growth path. Finally, our conclusions are presented in Section 6.

2. First integrals of the Lucas-Uzawa model. In this section we set up the Lucas-Uzawa growth model with logarithmic utility preferences which we will solve using our new methodology. Also, we apply our partial Hamiltonian methodology to obtain the first integrals that can be used to solve the model.

2.1. Model description. The representative agent’s utility function is defined as

$$\max_{c,u} \int_{0}^{\infty} e^{-\rho t} \ln(c) dt,$$

subject to the constraints of physical capital and human capital:

$$\dot{k}(t) = Ak^\alpha (uh)^{1-\alpha} - c, \ k_0 = k(0),$$

$$\dot{h}(t) = \delta (1-u)h, \ h_0 = h(0),$$

where $\rho > 0$ is the discount factor, $\alpha$ is the elasticity of output with respect to physical capital, $A > 0$ is the level of technology in the goods sector, $\delta > 0$ is
the level of technology in the education sector, \( k \) is physical capital, \( h \) is human capital, \( c \) is per capita consumption and \( u \) is the fraction of labor allocated to the production of physical capital.

The current value Hamiltonian function for this problem is defined as

\[
H(t, c, u, k, h, \lambda, \mu) = \ln(c) + \lambda [Ak^\alpha (uh)^{1-\alpha} - c] + \mu \delta (1 - u)h,
\]

where \( \lambda(t) \) and \( \mu(t) \) are costate variables. The transversality conditions are

\[
\lim_{t \to \infty} e^{-\rho t} \lambda(t)k(t) = 0, \quad \lim_{t \to \infty} e^{-\rho t} \mu(t)h(t) = 0.
\]

The necessary first order conditions for optimal control are

\[
c = \frac{1}{\lambda}, \quad (5)
\]

\[
\frac{\lambda A}{\mu \delta} (1 - \alpha) = \left( \frac{uh}{k} \right)^\alpha, \quad (6)
\]

\[
\dot{k}(t) = Ak^\alpha (uh)^{1-\alpha} - c, \quad (7)
\]

\[
\dot{h}(t) = \delta (1 - u)h, \quad (8)
\]

\[
\dot{\lambda} = \rho \lambda - A \alpha \lambda \left( \frac{uh}{k} \right)^{1-\alpha}, \quad (9)
\]

\[
\dot{\mu} = \mu(\rho - \delta). \quad (10)
\]

The growth rates of consumption, \( c \), and physical capital, \( k \), are given by

\[
\frac{\dot{c}}{c} = \alpha A \left( \frac{uh}{k} \right)^{1-\alpha} - \rho, \quad (11)
\]

\[
\frac{\dot{u}}{u} = \frac{1 - \alpha}{\alpha} \delta - \frac{c}{k} + \delta u. \quad (12)
\]

The Hamiltonian is non-concave in the control and state variables due to terms \( k^\alpha (uh)^{1-\alpha} \) and \( (1 - u)h \). Therefore, Mangasarian's sufficiency theorem [11] cannot be employed for this model and we establish sufficiency by utilizing Arrow's theorem [1]. The optimal value for \( u \) from (6) is

\[
u = \left( \frac{\lambda A}{\mu \delta} (1 - \alpha) \right)^{\frac{1}{\alpha}} \left( \frac{h}{k} \right). \quad (13)
\]

Using the optimal values of \( c \) and \( u \) from (5) and (13) the maximized Hamiltonian is defined as

\[
H^0(t, k, h, \lambda, \mu) = \ln \left( \frac{1}{\lambda} \right) + \lambda \left[ Ak \left( \frac{\lambda A(1 - \alpha)}{\mu \delta} \right)^{\frac{1-\alpha}{\alpha}} - \frac{1}{\lambda} \right] + \mu \delta \left[ h - \left( \frac{\lambda A(1 - \alpha)}{\mu \delta} \right)^{\frac{1}{\alpha}} \right]. \quad (14)
\]

The maximized Hamiltonian (14) is always concave in the state variables, \( k \) and \( h \). Therefore, we can conclude that the first-order conditions provided in (5)-(10) represent an optimal path for the problem solved by the representative consumer which is not necessarily unique.
We will derive closed-form solutions for the original variables \(c(t), u(t), k(t), h(t), \lambda(t)\) and \(\mu(t)\) of the dynamical system of ODEs (7)-(10) by utilizing the partial Hamiltonian approach.

2.2. Partial Hamiltonian operators and first integrals. The partial Hamiltonian determining (see [13, 14]) equation for the current-value Hamiltonian (3) is

\[
\lambda(\eta^1_\theta + \eta^1_h + \dot{\eta}^1_h) + \mu(\eta^2_\theta + \eta^2_h + \dot{\eta}^2_h) - \eta^1 \alpha \lambda Au^{1-\alpha}k^{\alpha-1}h^{1-\alpha} - \eta^2 \lambda A(1-\alpha)k^{\alpha}u^{1-\alpha}h^{-\alpha} - \left( \ln(c) + \lambda[A k^{\alpha} (uh)^{1-\alpha} - c] + \mu \delta(1-u)h \right)(\xi_t + \dot{k} \xi_k + \dot{h} \xi_h) = B_t + \dot{k}B_k + \dot{h}B_h
\]

\[
+ |\eta^1 - \xi(A k^{\alpha} u^{1-\alpha} h^{1-\alpha} - c)(\lambda \rho) + |\eta^2 - \xi \delta(1-u)h|(-\mu \rho),
\]

Equation (15), after expansion and separation with respect to powers of the control variables \(c\) and \(u\), yields an overdetermined system for \(\xi, \eta^1, \eta^2, B\). For the sake of brevity, the detailed calculations required to solve the system of equations (15) will not be reproduced here and we will only give the partial Hamiltonian operators, gauge terms and first integrals. The partial Hamiltonian operators, gauge terms and first integrals are given by

\[
\xi = 0, \eta^1 = 0, \eta^2 = e^{(\delta-\rho)t}, B(t) = 0,
\]

\[
I_1 = \lambda A(1-\alpha) \left( \frac{k}{uh} \right)^\alpha e^{(\delta-\rho)t},
\]

\[
\xi = 0, \eta^1 = e^{-\rho t}k, \eta^2 = e^{-\rho t}h, B = \frac{e^{-\rho t}}{\rho},
\]

\[
I_2 = \left( \lambda k - \frac{1}{\rho} + \frac{A(1-\alpha)}{\delta c} \left( \frac{k}{uh} \right)^\alpha h \right) e^{-\rho t}.
\]

The partial Hamiltonian approach yields two first integrals. In the next section, these first integrals will be utilized to derive the closed-form solutions for the dynamical system of ODEs associated with the Lucas-Uzawa model.

3. Closed-form solution for the Lucas-Uzawa model via \(I_1\) and \(I_2\). In this section, we utilize the first integrals \(I_1\) and \(I_2\) to derive two distinct sets of closed-form solutions for all the variables in the Lucas-Uzawa model. These first integrals hold for fairly general values of the parameters of the model.

By setting \(I_1 = a_1\), we have

\[
\lambda A(1-\alpha) \left( \frac{k}{uh} \right)^\alpha e^{(\delta-\rho)t} = a_1,
\]

where \(a_1\) is an arbitrary constant. Setting \(z = \frac{hu}{k}\), equation (17) can be re-written as

\[
z = \left( \frac{\lambda A(1-\alpha)}{\delta a_1} \right)^{\frac{1}{\alpha}} e^{\frac{(\delta-\rho)}{\alpha} t},
\]

where \(\lambda = 1/c\). Equation (9) with the aid of equation (18) results in

\[
\dot{\lambda} - \rho \lambda = -A \alpha \lambda^{\frac{1}{\alpha}} \left( \frac{A(1-\alpha)}{\delta a_1} \right)^{\frac{1-\alpha}{\alpha}} e^{\frac{(\delta-\rho)}{\alpha} t},
\]
which is a Bernoulli’s equation. The solution of equation (19) is
\[ \lambda = \frac{1}{c} = \left[ \frac{\alpha A}{\delta} \left( \frac{1 - \alpha}{a_1 \delta} \right)^{\frac{1}{\alpha}} e^{-\frac{(\rho - \delta)(1 - \alpha)}{\alpha} t} + a_2 e^{-\frac{(1 - \alpha)\rho}{\alpha} t} \right]^{\frac{1}{1 - \alpha}}, \]  
where \( a_2 \) is an arbitrary constant. Equations (18) and (20) provide the following expression for \( z \)
\[ z = \left[ \frac{\alpha A}{\delta} + a_2 \left( \frac{(1 - \alpha)A}{a_1 \delta} \right)^{\frac{1}{\alpha}} e^{-\frac{(1 - \alpha)\rho}{\alpha} t} \right]^{\frac{1}{1 - \alpha}}, \]  
and \( z^* = \left( \frac{aA}{\delta} \right)^{\frac{1}{1 - \alpha}} \) is the steady state solution. It is important to mention here that the system of differential equations (7)-(10) provides two sets of solutions depending on if \( a_2 = 0 \) and \( a_2 \neq 0 \). Thus we discuss two cases: Case I: \( a_2 = 0 \), and Case II: \( a_2 \neq 0 \).

3.1. Case I: \( a_2 = 0 \) and thus \( z = z^* \). For \( a_2 = 0 \), we have
\[ z(t) = \left( \frac{\alpha A}{\delta} \right)^{\frac{1}{1 - \alpha}} = z^*. \]  
Equation (18) gives
\[ \lambda(t) = \frac{a_1 \delta}{(1 - \alpha)A} e^{(\rho - \delta)t} z^*^{\alpha}, \]  
and \( \lambda = \frac{1}{c} \) provides
\[ c(t) = \frac{(1 - \alpha)A}{a_1 \delta z^*^{\alpha}} e^{(\rho - \delta)t}. \]  
Using the initial conditions \( c(0) = c_0 \) and \( k(0) = k_0 \), we have
\[ c_0 = \frac{(1 - \alpha)A}{a_1 \delta z^*^{\alpha}}. \]  
Equation (7) gives the following solution for \( k(t) \):
\[ k(t) = \frac{c_0 \alpha}{\alpha \rho - \alpha \delta + \delta} e^{(\rho - \delta)t} + a_3 e^{\frac{\delta}{\alpha} t}. \]  
The transversality condition (4) for \( k \) is satisfied provided that \( a_3 = 0 \) and the initial condition \( k(0) = k_0 \) which gives
\[ \frac{c_0}{k_0} = \rho - \delta + \frac{\delta}{\alpha}. \]  
Next, we make use of second first integral. Setting \( I_2 = a_4 \), we have
\[ \lambda k - \frac{1}{\rho} + \frac{A(1 - \alpha)}{\delta c} \left( \frac{k}{uh} \right)^{\alpha} h = a_4 e^{\rho t}, \]  
and this yields
\[ h(t) = \frac{c_0 \delta z^*^{\alpha}}{(1 - \alpha)A} \left( a_4 e^{\rho t} - \frac{k}{c_0} \frac{k_0}{\rho} + 1 \right) e^{-(\rho - \delta)t}. \]  
The transversality condition for \( h \) is satisfied provided \( a_4 = 0 \). Moreover, imposing the initial condition that \( h(0) = h_0 \) the expression for \( h \) simplifies to
\[ h(t) = h_0 e^{-(\rho - \delta)t}, \]  
where \( h_0 = \frac{\delta k_0 z^*}{\rho} \). Finally \( u = \frac{kz}{\alpha} \), which completes the solution.
We can summarize these closed-form solutions for all variables in the following simple forms:

\[
\begin{align*}
    c(t) &= c_0 e^{-(\rho-\delta)t}, \\
    k(t) &= k_0 e^{-(\rho-\delta)t}, \\
    u(t) &= \frac{\rho}{\delta} u^*, \\
    h(t) &= h_0 e^{-(\rho-\delta)t}, \\
    \lambda(t) &= \frac{1}{c_0} e^{(\rho-\delta)t}, \\
    \mu(t) &= \frac{(1-\alpha)A}{c_0 \delta z^*} e^{(\rho-\delta)t},
\end{align*}
\]  

(30)

where \( c_0 = \rho - \delta + \frac{\delta}{\alpha} \) and \( h_0 = \frac{\delta k_0 z^*}{\rho} \).

What makes this solution extremely interesting is that it is completely new to the economic growth literature. This newly derived closed-form solution (30) for all the variables of the Lucas-Uzawa model with logarithmic utility satisfies the transversality conditions as well as the initial conditions provided that the parameters and initial conditions satisfy

\[
\frac{c_0}{k_0} = \rho - \delta + \frac{\delta}{\alpha}, \quad h_0 = \frac{\delta k_0 z^*}{\rho}.
\]  

(31)

It is worth mentioning here that the closed-form solution (30) satisfies a linear version of the model.

3.2. Case II: \( a_2 \neq 0 \) and thus \( z \neq z^* \). The expression for \( \lambda \) from equation (18) can be alternatively given as

\[
\lambda(t) = \frac{a_1 \delta}{A(1-\alpha)} e^{(\rho-\delta)t} z(t)^\alpha,
\]  

(32)

where \( z \) is the same as given in (21). Using the initial condition \( z(0) = z_0 \), the expression for \( z(t) \) in (21) takes following form:

\[
z(t) = \frac{z^* z_0}{\left(z_0^{1-\alpha} + (z^* - z_0^{1-\alpha}) e^{-(1-\alpha)\frac{\delta}{\alpha} t}\right)^{1/\alpha}},
\]  

(33)

where

\[
a_2 = \frac{z^* - z_0^{1-\alpha}}{\delta c_0^{\frac{1}{\alpha}}}. \tag{34}
\]

The expression for \( c = \frac{1}{A} \), using the initial condition \( c(0) = c_0 \), is given by

\[
c(t) = c_0 z_0^{\alpha} e^{(\delta - \rho)t} z(t)^{-\alpha},
\]  

(35)

where \( a_1 = \frac{A(1-\alpha)}{\delta z_0^{\alpha}} \). Equation (10) provides the following expression of the costate variable \( \mu = a_1 e^{(\rho-\delta)t} \). The differential equation (7) for \( k \) results in the following integrable differential equation

\[
\dot{k} - A z^{1-\alpha} k = -c_0 z_0^{\alpha} e^{-(\rho-\delta)t} z^{-\alpha},
\]  

(36)
and it provides
\[ k(t) = c_0 z_0^\alpha z(t)^{-1} e^{\frac{\alpha}{1-\alpha}} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right), \]  
(37)
where
\[ F(t) = \int_0^t z(t)^{1-\alpha} e^{(\delta - \rho - \frac{\delta}{1-\alpha}) t} dt, \]  
(38)
and \( k(0) = k_0 \) is utilized. It is important to mention here that
\[ \lim_{t \to \infty} F(t) = m, \]  
(39)
is a non-zero finite number \( m \), since the integrand for \( F(t) \) is a positive function.

The transversality condition (4) for \( k \) becomes
\[ \lim_{t \to \infty} z(t)^{\alpha-1} \lim_{t \to \infty} \frac{\frac{k_0 z_0^{1-\alpha}}{c_0} - F(t)}{e^{-\frac{\alpha}{1-\alpha}(1-\alpha)t}} = 0. \]  
(40)
This is established by applying l’Hôpital provided that
\[ \lim_{t \to \infty} \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) = 0, \]  
(41)
and \( m = \frac{z_0^{1-\alpha} k_0}{c_0} \) where \( m \) is defined as in (39). Setting \( L_2 = a_4 \), we find
\[ \left( \lambda k - \frac{1}{\rho} + \frac{A(1-\alpha)}{\beta c} \left( \frac{k}{uh} \right) \alpha h \right) e^{-\rho t} = a_4, \]  
(42)
and this gives
\[ h(t) = \frac{a_4}{a_1} e^{\delta t} + \frac{1}{\rho a_1} e^{-(\rho - \delta) t} - \frac{1}{a_1} z(t)^{\alpha-1} e^{\frac{\alpha}{1-\alpha} t} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right). \]  
(43)
The transversality condition for \( h \) takes the following form:
\[ \lim_{t \to \infty} \left( \frac{a_4}{a_1} + \frac{1}{\rho a_1} e^{-\rho t} - \frac{1}{a_1} z(t)^{\alpha-1} e^{\frac{\alpha}{1-\alpha} t} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right) \right) = 0. \]  
which is satisfied provided \( a_4 = 0 \). The expression for the variable \( h \) takes the following form:
\[ h(t) = \frac{\rho c_0 h_0}{c_0 - \rho k_0} \left[ \frac{1}{\rho} e^{-(\rho - \delta) t} - z(t)^{\alpha-1} e^{\frac{\alpha}{1-\alpha} t} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right) \right]. \]  
(44)
The variable \( u \) can be determined from \( u = z k / h \) and is given by
\[ u(t) = \frac{u_0 z_0^{\alpha-1} (c_0 - \rho k_0) \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right)}{k_0 \left( e^{(\delta - \rho - \frac{\delta}{1-\alpha}) t} - \rho z(t)^{\alpha-1} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right) \right)}. \]  
(45)
This completes the solution.
We can summarize the closed-form solutions for all variables as follows:

\[ c(t) = c_0 z_0^\alpha e^{(\delta - \rho) t} z^{-\alpha}, \]

\[ k(t) = c_0 z_0^\alpha z(t)^{-1} e^{\frac{\rho}{c_0} t} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right), \]

\[ h(t) = \frac{\rho c_0 \delta}{c_0 - \rho k_0} \left[ \frac{1}{\rho} e^{-(\rho - \delta) t} - z(t)^{\alpha-1} e^{\frac{\rho}{c_0} t} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right) \right], \]

\[ u(t) = \frac{k_0 e^{(\delta - \rho - \frac{\rho}{c_0}) t} - \rho z(t)^{\alpha-1} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right)}{e^{(\delta - \rho - \frac{\rho}{c_0}) t} - \rho z(t)^{\alpha-1} \left( \frac{k_0 z_0^{1-\alpha}}{c_0} - F(t) \right)}, \]

\[ \mu(t) = \frac{A(1 - \alpha)}{\delta z_0^\alpha c_0} e^{(\rho - \delta) t}, \]

\[ \lambda(t) = \frac{1}{c_0 z_0^\alpha} e^{(\rho - \delta) t} z^\alpha, \]

where

\[ F(t) = \int_0^t z(t)^{\alpha-1} e^{(\delta - \rho - \frac{\rho}{c_0}) t} dt, \quad \lim_{t \to \infty} F(t) = \frac{k_0 z_0^{1-\alpha}}{c_0}, \]

\[ \frac{\delta u_0 z_0^\alpha}{A(1 - \alpha) k_0 z_0} = \frac{\rho}{c_0 - \rho k_0}, \]

\[ z(t) = \frac{z^* z_0}{\left( z_0^{1-\alpha} + (z_0^{1-\alpha} - z^*_0^{1-\alpha}) e^{-(1-\alpha) \frac{\rho}{c_0} t} \right)^{\frac{1}{1-\alpha}}}, \quad z^* = \left( \frac{\alpha A}{\delta} \right)^{\frac{1}{1-\alpha}}. \]

It is important to mention here that by utilizing l'Hôpital rule, \( \lim_{t \to \infty} u(t) = \frac{\rho}{c_0} = u^* \).

Chilarescu and Sipos [9] derived only one solution for this model. Their closed-form solutions for \( c(t) \) and \( k(t) \) were the same as we have derived in (46), but for the variables \( u(t) \) and \( k(t) \) their solution involved another unknown integral, while our solution contains only one integral expression. Moreover, our closed-form solution (46) satisfies the transversality as well as the initial conditions. In the next Section, we will show how one can utilize one first integral \( I_1 \) to derive the solution found by Chilarescu and Sipos [9].

4. Closed-form solution for the Lucas-Uzawa model via \( I_1 \). Now we show how one can utilize only one first integral \( I_1 \) to derive a closed-form solution. By setting \( I_1 = a_1 \), we will arrive at equations (17)-(21). As was the case above, the following two scenarios will arise: Case I: \( a_2 = 0 \), and Case II: \( a_2 \neq 0 \).

It is straightforward to show that for the case \( a_2 = 0 \), one can derive the same solution as given in equation (30).

For \( a_2 \neq 0 \), we will follow the same procedure as described in the previous section to derive equations (32)-(41). Substituting \( c(t) \) and \( k(t) \) from equations (35) and (37) into equation (12), we have

\[ \frac{\dot{u}}{u} = \frac{1 - \alpha}{\alpha} \delta - \frac{\rho c_0}{\delta k_0 z_0^{1-\alpha}} - F(t) + \rho u. \]
The solution of equation (48) with the initial condition $u(0) = u_0$ is given by

$$u(t) = \frac{\left(\frac{\delta}{\alpha} - \delta\right)u_0 \left[k_0 z_0^{1-\alpha} - F(t)\right]}{\left[\left(\frac{\delta}{\alpha} - \delta + \delta u_0\right)\frac{k_0 z_0^{1-\alpha}}{c_0} - \delta u_0 G(t)\right]e^{(\delta - \frac{\alpha}{\delta})t} - \delta u_0 \left[k_0 z_0^{1-\alpha} - F(t)\right]},$$  \hspace{1cm} (49)

where

$$G(t) = \int_0^t z(s)^{1-\alpha} e^{-\rho s} ds.$$  \hspace{1cm} (50)

It is not difficult to show that closed-form solution (53) which was derived by utilizing l'Hôpital rule the solution (49) attains equilibrium value in long run i.e. $\lim_{t \to \infty} u(t) = z^* = u^*$ provided that

$$\lim_{t \to \infty} \left[\left(\frac{\delta}{\alpha} - \delta + \delta u_0\right)\frac{k_0 z_0^{1-\alpha}}{c_0} - \delta u_0 G(t)\right] = 0.$$  \hspace{1cm} (51)

The variable $h$ can be determined from $h = zk/u$ and is given by

$$h(t) = \left[\left(\frac{\delta}{\alpha} - \delta + \delta u_0\right)\frac{k_0 z_0^{1-\alpha}}{c_0} - \delta u_0 G(t)\right]e^{(\delta - \frac{\alpha}{\delta})t} - \delta u_0 \left[k_0 z_0^{1-\alpha} - F(t)\right] \frac{h_0 c_0}{k_0 z_0^{1-\alpha} \left(\frac{\delta}{\alpha} - \delta\right)} e^{\frac{\alpha}{\delta} t},$$  \hspace{1cm} (52)

which satisfies the transversality condition (4) provided (51) holds.

We can summarize the closed-form solutions for all variables as follows:

$$c(t) = c_0 z_0^\alpha e^{(\delta - \rho) t} z^{-\alpha},$$

$$k(t) = k_0 z_0^\alpha (t - 1) e^{\frac{\alpha}{\delta} t} \left(\frac{k_0 z_0^{1-\alpha}}{c_0} - F(t)\right),$$

$$h(t) = \left[\left(\frac{\delta}{\alpha} - \delta + \delta u_0\right)\frac{k_0 z_0^{1-\alpha}}{c_0} - \delta u_0 G(t)\right]e^{(\delta - \frac{\alpha}{\delta})t} - \delta u_0 \left[k_0 z_0^{1-\alpha} - F(t)\right] \frac{h_0 c_0}{k_0 z_0^{1-\alpha} \left(\frac{\delta}{\alpha} - \delta\right)} e^{\frac{\alpha}{\delta} t},$$

$$u(t) = \frac{\left(\frac{\delta}{\alpha} - \delta\right)u_0 \left[k_0 z_0^{1-\alpha} - F(t)\right]}{\left[\left(\frac{\delta}{\alpha} - \delta + \delta u_0\right)\frac{k_0 z_0^{1-\alpha}}{c_0} - \delta u_0 G(t)\right]e^{(\delta - \frac{\alpha}{\delta})t} - \delta u_0 \left[k_0 z_0^{1-\alpha} - F(t)\right]},$$

$$\mu(t) = \frac{A(1 - \alpha)}{\delta z_0^\alpha} e^{(\rho - \delta) t},$$

$$\lambda(t) = \frac{1}{\delta z_0^\alpha} e^{(\rho - \delta) t} z^\alpha,$$  \hspace{1cm} (53)

where

$$F(t) = \int_0^t z(s)^{1-\alpha} e^{(\delta - \rho - \frac{\alpha}{\delta}) s} ds, \quad \lim_{t \to \infty} F(t) = \frac{k_0 z_0^{1-\alpha}}{c_0},$$

$$G(t) = \int_0^t z(s)^{1-\alpha} e^{-\rho s} ds, \quad \lim_{t \to \infty} G(t) = \frac{\left(\frac{\delta}{\alpha} - \delta + \delta u_0\right) k_0 z_0^{1-\alpha}}{\delta u_0},$$

$$z(t) = \frac{z^* z_0}{\left(z_0^{1-\alpha} + (z^{1-\alpha} - z_0^{1-\alpha}) e^{-\frac{(1 - \alpha)}{\delta} t}\right)^{1-\alpha}}, \quad z^* = \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{\alpha - 1}}.$$  \hspace{1cm} (54)

It is not difficult to show that closed-form solution (53) which was derived by utilizing $I_1$ is exactly the same as the solution found by Chilarescu and Sipos [9].
Thus, there exists three sets of closed-form solutions (30), (46) and (53) for the Lucas-Uzawa model with logarithmic time-preferences. The question that arises is whether the closed-form solutions (30), (46) and (53) converge to the same balanced growth path in the long run. We address this issue in the next section.

5. Convergence to the balanced growth path. In this section, we determine whether the closed-form solutions (30), (46) and (53) converge to the same balanced growth path in the long run. In order to do this, we derive the growth rates of per capita consumption, $c$, physical capital, $k$, human capital, $h$, the fraction of labor allocated to the production of physical capital, $u$, and the costate variables $\mu$ and $\lambda$ (30). The growth rates of all the variables for the closed-form solution (30) are

$$\frac{\dot{c}}{c} = \delta - \rho,$$
$$\frac{\dot{k}}{k} = \delta - \rho,$$
$$\frac{\dot{h}}{h} = \delta - \rho,$$
$$\frac{\dot{u}}{u} = 0,$$
$$\frac{\dot{\lambda}}{\lambda} = \rho - \delta,$$
$$\frac{\dot{\mu}}{\mu} = \rho - \delta.$$

This simple closed-form solution yields constant growth rates. The growth rates of consumption, $c$, the physical capital, $k$, and the growth rate of human capital are all the same and equal to $\delta - \rho$. The growth rate of the fraction of labor allocated to the production of physical capital, $u$, is zero. The growth rates of the costate variables $\lambda$ and $\mu$ are equal to $\rho - \delta$.

The growth rates of all the variables for the second closed-form solution (46) after simplifications are

$$\frac{\dot{c}}{c} = \delta - \rho - \frac{\dot{z}}{z},$$
$$\frac{\dot{k}}{k} = \delta - \rho - \frac{\dot{z}}{z} - \frac{z^{1-\alpha}e^{(\delta - \rho - \frac{\dot{z}}{z})t}}{\frac{k_0z_0^{1-\alpha}}{\gamma_0} - F(t)} - \frac{\dot{z}}{z},$$
$$\frac{\dot{h}}{h} = \delta - \rho + \frac{\dot{z}}{z} + \frac{z^{1-\alpha}e^{(\delta - \rho - \frac{\dot{z}}{z})t}}{\frac{k_0z_0^{1-\alpha}}{\gamma_0} - F(t)} - \frac{\dot{z}}{z},$$
$$\frac{\dot{u}}{u} = \frac{\dot{k}}{k} - \frac{\dot{h}}{h} + \frac{\dot{z}}{z},$$
$$\frac{\dot{\lambda}}{\lambda} = \rho - \delta - \frac{\dot{z}}{z},$$
$$\frac{\dot{\mu}}{\mu} = \rho - \delta.$$

(56)
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where

\[
\dot{z} = \frac{\delta(z^* - z_0^1 - z^1) e^{-\frac{(1-\alpha)\delta t}{\alpha}}}{\alpha(z^1 - z_0^1 - z^1) e^{-\frac{(1-\alpha)\delta t}{\alpha}}}.
\]

(57)

An extremely interesting and important result is that one of our closed-form solutions (46) provides a dynamic growth rate since this is the first time in the economic growth literature that a dynamic growth rate has been established in the context of a closed-form solution.

It is clear from equation (57) that \(\dot{z}\) approaches zero as \(t \to \infty\) which means that the rate of growth of \(z\) decreases asymptotically as we approach the steady state. The growth rates of consumption, \(c\), physical capital, \(k\), and human capital, \(h\), decrease over time and approach \(\delta - \rho\) as \(t \to \infty\). The growth rate of the fraction of labor allocated to the production of physical capital, \(u\), approaches zero as \(t \to \infty\). The growth rates of the costate variables \(\lambda\) and \(\mu\) approach the value \(\rho - \delta\) as \(t \to \infty\). Again this is an important result because it implies that even though the transition paths followed by the economic variables in the two possible solutions may differ in the short run, in the long run the economic variables in both of our solutions reach the same steady state. It is worth mentioning here that l'Hôpital rule is applied whenever it was necessary to establish the growth rates of variables as \(t \to \infty\).

For closed-form solution (53), the growth rates of the variables \(c(t)\), \(k(t)\), \(\lambda(t)\) and \(\mu(t)\) are the same as we have derived for solution (46). One can construct the growth rates of the variables \(h(t)\) and \(u(t)\) for the solution (53) as well. Also for (53), the growth rate of human capital, \(h\), approaches \(\delta - \rho\) as \(t \to \infty\) whereas the growth rate of the fraction of labor allocated to the production of physical capital, \(u\), approaches zero as \(t \to \infty\) [9].

Moreover, the closed-solutions (30), (46) and (53) satisfy

\[
\lim_{t \to \infty} u(t) = \frac{\rho}{\delta} = u^*, \quad \lim_{t \to \infty} \frac{c}{k} = \rho - \delta + \frac{\delta}{\alpha}.
\]

(58)

This shows that our newly derived solutions, (30) and (46), satisfy all properties stated in Proposition 1 of Chilarescu [9] for the closed-form solution (53).

6. Conclusions. We have derived closed-form solutions for a stylized version of the two sector Lucas-Uzawa model with logarithmic utility preferences and homogeneous technology in a Robinson Crusoe economy. The partial Hamiltonian approach provided two first-integrals for this model. These first integrals are then utilized to construct closed-form solutions for two different cases: \(z = z^*\) and \(z \neq z^*\). For the case where \(z = z^*\), the closed-form solutions for all variables are presented in (30) and they satisfy the appropriate transversality conditions. For the case where \(z \neq z^*\), two closed-form solutions, (46) and (53), are established whereas in the previous literature [9], only one solution (53) had been derived. For the case where \(z \neq z^*\), the expressions for the levels of consumption and capital stock in the newly obtained solution (46) are same as those found in [9], while the level of human capital and the amount of labor allocated to the production of physical capital are different. We also derive the growth rates of all the variables in the model. One solution provides static growth rates while the other two solutions provide dynamic growth rates, but in the long run all the closed-form solutions approach the same static value of balanced growth path which makes economic sense.

It is important to note that the closed-form solutions (for all variables in the model) presented in (30) and (46) are completely new to the literature and have
not been derived before. This discovery of multiplicity in terms of closed-form solutions for the Lucas-Uzawa model with logarithmic utility preferences is a significant contribution to the field of economic growth theory.

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