ARITHMETIC TOPOLOGY OF 4-MANIFOLDS

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Abstract. We construct a functor from the smooth 4-dimensional manifolds to the hyper-algebraic number fields, i.e. fields with non-commutative multiplication. It is proved that the simply connected 4-manifolds correspond to the abelian extensions. We recover the Rokhlin and Donaldson’s Theorems from the Galois theory of the non-commutative fields.

1. Introduction

The arithmetic topology studies an interplay between the 3-dimensional manifolds and the fields of algebraic numbers [Morishita 2012] [6]. The idea dates back to C. F. Gauss. Namely, there exists a map $F$ from the 3-dimensional manifolds $M^3$ to the algebraic number fields $K$. Such a map transforms links $\mathcal{Z} \subset M^3$ (knots $\mathcal{K} \subset M^3$, resp.) into the ideals (prime ideals, resp.) of the ring of integers $O_K$ of $K$. The map $F$ is a functor on the category of 3-dimensional manifolds $M^3$ with values in the category of algebraic number fields $K$ [7, Theorem 1.2].

The aim of our note is an extension of $F$ to the smooth 4-dimensional manifolds $M^4$. The range of $F$ is no longer the fields of algebraic numbers, but the fields $K$ with non-commutative multiplication, e.g. the quaternions or the cyclic division algebras. We refer to $K$ as a hyper-algebraic number field. The Galois theory of such fields was elaborated by [Cartan 1947] [2] and [Jacobson 1940] [4], see [Harchemko 1996] [14] for a detailed account.

To introduce the map $F$, let $N$ be a finite index subgroup of the mapping class group $\text{Mod} \, M^3$, where $M^3$ is a 3-dimensional manifold. The subgroup $N$ gives rise to a smooth branched cover $M^4_N$ of the 4-sphere $S^4$, see [Piergallini 1995] [12] and Section 2.1. On the other hand, we have $\hat{N}/\hat{\mathbb{Z}} \cong G_K$, where $\hat{N}$ ($\hat{\mathbb{Z}}$, resp.) is a profinite completion of $N$ ($\mathbb{Z}$, resp.) and $G_K$ is the absolute Galois group of $K$ [8, Theorem 1.2]. Recall that the field $K$ can be recovered up to an isomorphism from the group $G_K$, see [Uchida 1976] [13, Corollary 2] and remark 2.5. We define $F$ as a composition of maps:

$$M^4_N \mapsto \hat{N}/\hat{\mathbb{Z}} \cong G_K \mapsto K. \quad (1.1)$$

Remark 1.1. The map $F$ can be equivalently defined via a $C^*$-algebra $E_{M^4}$ generated by the diffeomorphisms of $M^4$ [10]. Namely, the K-theory of $E_{M^4}$ gives rise to a number field $K$ whose central simple algebra is isomorphic to $K$. In particular, the exotic smoothings of $M^4$ are classified by the Brauer group of $K$, see [10] for the details.

2010 Mathematics Subject Classification. Primary 16W20, 57N13; Secondary 11R32.

Key words and phrases. 4-dimensional manifolds, non-commutative Galois theory.
Let $O_K$ be the ring of integers of the field $K$. Unlike $O_K$, the ring $O_K$ is always simple, i.e. all two-sided ideals of $O_K$ are trivial. Hence we must use the dynamical ideals (dynamials), i.e. the crossed products $O_K \rtimes m\mathbb{Z}$ by an endomorphism of $O_K$ of degree $m \geq 1$, see [9] for the details and motivation. The dynamial $O_K \rtimes m\mathbb{Z}$ is minimal if and only if $m = p$ is a prime number [9, Theorem 1.7].

To formalize our results, recall that the groups $N \not\cong N'$ are a Grothendieck pair, if $\hat{N} \cong \hat{N}'$. We shall denote by $\mathcal{M}^4$ a category of all smooth 4-dimensional manifolds $\mathcal{M}^4$, such that: (i) the set $\mathcal{M}^4$ contains no $\mathcal{M}_N^4$ and $\mathcal{M}_{N'}^4$, such that $N$ and $N'$ are a Grothendieck pair and (ii) the arrows of $\mathcal{M}^4$ are differentiable maps between such manifolds. Denote by $\mathcal{K}$ a category of the hyper-algebraic number fields, such that the arrows of $\mathcal{K}$ are injective homomorphisms between these fields. The knotted surface is a transverse immersion $\iota : X_{g_1} \cup \ldots X_{g_n} \hookrightarrow \mathcal{M}^4$ of a collection of the 2-dimensional orientable surfaces $X_{g_i}$ of the genera $g_i \geq 0$. The $\iota(X_{g_1} \cup \ldots X_{g_n})$ is called a surface knot $\mathcal{X}$ if $n = 1$ and a surface link $\mathcal{L}$ otherwise. Our main result can be formulated as follows.

**Theorem 1.2.** The map $F : \mathcal{M}^4 \to \mathcal{K}$ is a covariant functor, such that:

(i) $F(S^4) \cong \mathbb{H}$ is the field of quaternions;

(ii) the dynamials $O_K \rtimes m\mathbb{Z}$ correspond to the surface links $\mathcal{L} \subset \mathcal{M}^4$;

(iii) the minimal dynamials $O_K \rtimes p\mathbb{Z}$ correspond to the surface knots $\mathcal{X} \subset \mathcal{M}^4$.

**Remark 1.3.** The functor $F$ is independent of the choice of manifold $\mathcal{M}^3$ used in the construction of $F$ according to the formula (1.1). Indeed, using the manifolds $\{\mathcal{M}_t^4 | \pi_1(\mathcal{M}_t^4) \cong \text{Mod } \mathcal{M}_t^3\}$ described in Section 2.1, one can always take a connected sum $\#\mathcal{M}_t^4$, so that $\text{Mod } \mathcal{M}^3 \cong \pi_1(\#\mathcal{M}_t^4)$.

Let $K$ be a Galois extension of $\mathbb{H}$, i.e. an extension having the Galois group $\text{Gal } K$ and obeying the fundamental correspondence of the Galois theory [Cartan 1947] [2], [Jacobson 1940] [4] and [Харчевоно 1996] [14]. Recall that $K$ is said to be an abelian extension, if $\text{Gal } K$ is an abelian group. Theorem 1.2 implies the following result.

**Corollary 1.4.** Let $\mathcal{M}^4 \in \mathcal{M}^4$ be a simply connected 4-manifold. Then $K = F(\mathcal{M}^4)$ is an abelian extension.

**Remark 1.5.** The converse of 1.4 is false.

The article is organized as follows. Section 2 contains a brief review of the 4-dimensional topology, the Galois theory of non-commutative fields and the arithmetic topology. Theorem 1.2, corollary 1.4 and remark 1.5 are proved in Section 3. In Section 4 we give a proof of the Rokhlin and Donaldson’s Theorems using the Galois theory of non-commutative fields.

2. Preliminaries

This section contains a brief review of the smooth 4-dimensional manifolds, the Galois theory of non-commutative fields and the arithmetic topology. We refer the reader to [Cartan 1947] [2], [Morishita 2012] [6] and [Piergallini 1995] [12] for a detailed account.
2.1. Topology of 4-dimensional manifolds. Let $\mathcal{M}^3$ be a closed 3-dimensional manifold. Denote by $\text{Mod} \, \mathcal{M}^3$ the mapping class group of $\mathcal{M}^3$, i.e. a group of isotopy classes of the orientation-preserving diffeomorphisms of $\mathcal{M}^3$. The $\text{Mod} \, \mathcal{M}^3$ is a finitely presented group [Hatcher & McCullough 1990] [3].

Since each finitely presented group can be realized as the fundamental group of a smooth 4-manifold, we denote by $\mathcal{W}^4 \in \mathcal{M}^3$ a 4-manifold with $\pi_1(\mathcal{W}^4) \cong \text{Mod} \, \mathcal{M}^3$. It will be useful for us to represent $\mathcal{W}^4$ as a PL (and, therefore, smooth) 4-fold (or 5-fold, if necessary) branched cover of the sphere $S^4$. Let $X_g$ be a closed surface, which we assume for the sake of brevity to be orientable of genus $g \geq 0$. Recall that there exists a transverse immersion $\iota : X_g \rightarrow S^4$, such that $\mathcal{W}^4$ is the 4-fold PL cover of $S^4$ branched at the points of $X_g$ [Piergallini 1995] [12]. In other words, the $\text{Mod} \, \mathcal{M}^3$ is a normal subgroup of index 4 of the fundamental group $\pi_1(S^4 - X_g)$.

Remark 2.1. The immersion $\iota$ and surface $X_g$ need not be unique for given $\mathcal{W}^4$. Yet one can always restrict to a canonical choice of $\iota$ and $X_g$, which we always assume to be the case.

Definition 2.2. By $\mathcal{M}_N^3$, we understand a cover of the manifold $\mathcal{W}^4$ corresponding to the finite index subgroup $N$ of the fundamental group $\pi_1(\mathcal{W}^4)$.

Remark 2.3. In view of the continuous map $\mathcal{W}^4 \rightarrow S^4$, the manifold $\mathcal{M}_N^3$ is a smooth 4d-fold branched cover of the sphere $S^4$, where $d$ is the index of $N$ in $\text{Mod} \, \mathcal{M}^3$. The manifold $\mathcal{M}_N^3$ is a regular (Galois) cover of $S^4$ if and only if $N$ is a normal subgroup of $\text{Mod} \, \mathcal{M}^3$.

2.2. Galois theory for non-commutative fields. Denote by $K$ a division ring, i.e. a ring such that the set $K^\times := K - \{0\}$ is a group under multiplication. If the group $K^\times$ is commutative, then $K$ is a field. For otherwise, we refer to $K$ as a non-commutative field.

Roughly speaking, the Galois theory for $K$ is a correspondence between the subfields of $K$ and the subgroups of a Galois group $\text{Gal} \, K$ of the field $K$. Namely, let $G$ be a group of automorphisms of the field $K$. It is easy to see, that the set

$$I_G = \{ x \in K \mid g(x) = x \text{ for all } g \in G\}$$

(2.1)

is a subfield of the field $K$.

Definition 2.4. An extension $K$ of the field $L$ is called Galois, if there exists a group $G$ of automorphisms of the field $K$, such that $L \cong I_G$. The Galois group of $K$ with respect to $L$ is defined as $\text{Gal} \, K \cong G$. The absolute Galois group $G_K$ is defined as a Galois group of the algebraic closure of $K$. The $G_K$ is a profinite group.

Remark 2.5. By an adaption of the argument for the case of fields [Uchida 1976] [13, Corollary 2], one can show that the group $G_K$ defines the underlying field $K$ up to an isomorphism, see lemma 3.3.

Unlike the case of fields, the inner automorphisms of $K$ are a non-trivial group. Indeed, consider an automorphism $h_g : K \rightarrow K$ given by the formula:

$$x \mapsto g^{-1}xg, \quad x \in K, \quad g \in K^\times.$$

(2.2)

By $\text{Inn} \, (K)$ we denote a group of such automorphisms under the composition $h_{g_1} \circ h_{g_2} = h_{g_1g_2}$, where $g_1, g_2 \in K^\times$. It follows from (2.2) that $\text{Inn} \, (K) \cong K^\times / C$, where $C$ is the center of $K^\times$. 


Let $G$ be a finite group of automorphisms of the field $K$. A normal subgroup 
$\Gamma := G \cap \text{Inn}(K)$ of $G$ consists of the inner automorphisms of the field $K$. Consider a group ring

$$\mathbb{B}(\Gamma) = \sum_{g_i \in \Gamma} \sum_{h_j \in C} g_i h_j.$$ (2.3)

It is easy to see, that $\mathbb{B}(\Gamma)$ is a finite-dimensional algebra over its center $C$. The following result has been established in [Cartan 1947] [2, Théorème 1] and [Харченко 1996] [14, p. 139].

**Theorem 2.6.** If $K$ is a Galois extension, then the corresponding Galois group $G$ satisfies the short exact sequence of groups:

$$1 \to \Gamma \to G \to G/\Gamma \to 1,$$ (2.4)

where $\iota$ is an inclusion and $|G/\Gamma| = \dim_C \mathbb{B}(\Gamma)$.

Recall that $\mathbb{B}(\Gamma)$ is a Frobenius algebra [Харченко 1996] [14, Theorem 3.5.1]. In other words, there exists a non-degenerate bilinear form

$$Q : \mathbb{B}(\Gamma) \times \mathbb{B}(\Gamma) \to C,$$ (2.5)

such that $Q(xy, z) = Q(x, yz) = Q(x, zy) = Q(xz, y) = Q(zx, y) = Q(z, xy)$.

**Remark 2.7.** The form (2.5) is symmetric if and only if $\mathbb{B}(\Gamma)$ is a commutative ring.

**Proof.**

$$Q(xy, z) = Q(x, yz) = Q(x, zy) = Q(xz, y) = Q(zx, y) = Q(z, xy).$$ $\square$

### 2.3. Arithmetic topology

The arithmetic topology studies an interplay between 3-dimensional manifolds and number fields [Morishita 2012] [6]. Let $\mathfrak{M}^3$ be a category of closed 3-dimensional manifolds, such that the arrows of $\mathfrak{M}^3$ are homeomorphisms between the manifolds. Likewise, let $\mathbf{K}$ be a category of the algebraic number fields, where the arrows of $\mathbf{K}$ are isomorphisms between such fields. Let $\mathfrak{M}^3 \in \mathfrak{M}$ be a 3-manifold, let $S^3 \in \mathfrak{M}$ be the 3-sphere and let $O_K$ be the ring of integers of $K \in \mathbf{K}$.

**Theorem 2.8.** ([7, Theorem 1.2]) The exists a covariant functor $F : \mathfrak{M}^3 \to \mathbf{K}$, such that:

(i) $F(S^3) = Q$;

(ii) each ideal $I \subseteq O_K = F(\mathfrak{M}^3)$ corresponds to a link $\mathcal{J} \subset \mathfrak{M}$;

(iii) each prime ideal $I \subseteq O_K = F(\mathfrak{M}^3)$ corresponds to a knot $\mathcal{K} \subset \mathfrak{M}$.

The following construction of $F$ is based on [Uchida 1976] [13]. Let $X_{g,n}$ be an orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components. Denote by $\text{Mod} X_{g,n}$ the mapping class group of $X_{g,n}$, i.e. a group of isotopy classes of the orientation and boundary-preserving diffeomorphisms of the surface $X_{g,n}$. Let $N$ be a finite index subgroup of $\text{Mod} X_{g,n}$. We omit the construction of a 3-manifold $\mathfrak{M}^3 \in \mathfrak{M}$ from $N$ referring the reader to [7, Remark 1.1]. As usual, let $G_K$ be the absolute Galois group of $K \in \mathbf{K}$, while $\hat{N}$ and $\hat{Z}$ denote the profinite completion of the groups $N$ and $Z$, respectively. Let $G_K \to K$ be an injective map defined in [Uchida 1976] [13, Corollary 2].
Theorem 2.9. ([8, Theorem 1.2]) The map:
\[ \mathcal{M}^3_N \to \tilde{N}/\tilde{\mathbb{Z}} \cong G_K \to K \]
(2.6)
coincides with the functor \( F \) of theorem 2.8. Such a functor is injective, unless \( N \) and \( N' \) are a Grothendieck pair. Moreover, for every normal finite index subgroup \( N' \subseteq N \), there exists a regular cover \( \mathcal{M}^3_{N'} \) of \( \mathcal{M}^3_N \) and an intermediate field \( K' = F(\mathcal{M}^3_{N'}) \), such that \( K \subseteq K' \) and \( \text{Gal}(K'/K) \cong N/N' \).

Remark 2.10. The map (2.6) extends to the 4-manifolds \( \mathcal{M}^4 \) and non-commutative fields \( K \), see (1.1). Such an extension is at the heart of our paper.

3. Proofs

3.1. Proof of theorem 1.2. For the sake brevity, we shall focus on the first part by proving that \( F: \mathcal{M}^4 \to \mathfrak{R} \) is a covariant functor, while referring the reader to [9, Section 4] for the proof of items (i)-(iii) of theorem 1.2. We shall prove \( F \) to be a functor with respect to homeomorphisms first, and then generalize to arbitrary differentiable mappings, see remark 3.4.

3.1.1. Let us show that \( F: \mathcal{M}^4 \to \mathfrak{R} \) is a covariant functor. For clarity, we split the proof in a series of lemmas.

Lemma 3.1. The manifolds \( \mathcal{M}^4_N, \mathcal{M}^4_{N'} \in \mathcal{M}^4 \) are homeomorphic, if and only if, the subgroups \( N, N' \subseteq \text{Mod} \mathcal{M}^3 \) are isomorphic.

Proof. (i) Let \( \mathcal{M}^3_N \) be homeomorphic to \( \mathcal{M}^3_{N'} \) by a homeomorphism \( h: \mathcal{M}^3_N \to \mathcal{M}^3_{N'} \). Since both \( \mathcal{M}^3_N \) and \( \mathcal{M}^3_{N'} \) cover the 4-sphere \( S^4 \) branched over a surface \( X_g \to S^4 \), one gets a commutative diagram in Figure 1. By definition, \( N \cong p_*(\pi_1(\mathcal{M}^3_N)) \subseteq \pi_1(S^4 - X_g) \). Likewise, \( N' \cong p'_*(\pi_1(\mathcal{M}^3_{N'})) \subseteq \pi_1(S^4 - X_g) \). Since \( \mathcal{M}^3_N \) is homeomorphic to \( \mathcal{M}^3_{N'} \), one gets an isomorphism of the fundamental groups \( \pi_1(\mathcal{M}^3_N) \cong \pi_1(\mathcal{M}^3_{N'}) \). Because the maps \( p_* \) and \( p'_* \) are injective, we conclude that \( N \) and \( N' \) are isomorphic subgroups of \( \pi_1(S^4 - X_g) \) and, therefore, of the group \( \text{Mod} \mathcal{M}^3 \). In view of the above and remark 2.1, the necessary condition of lemma 3.1 is proved.

(ii) Let \( N \cong N' \) be isomorphic subgroups of \( \text{Mod} \mathcal{M}^3 \). As explained in Section 2.1, the groups \( N \) and \( N' \) define a pair of branched covers \( \mathcal{M}^3_N \) and \( \mathcal{M}^3_{N'} \) of \( S^4 \). In view of the diagram in Figure 1, one obtains an inclusion of groups \( N, N' \subseteq \pi_1(S^4 - X_g) \). Since \( N \cong N' \), the subgroups \( N \) and \( N' \) are conjugate in the group \( \pi_1(S^4 - X_g) \). In other words, there exists an element \( g \in \pi_1(S^4 - X_g) \), such that \( N' = g^{-1}Ng \). It is well known, that the conjugate subgroups of \( \pi_1(S^4 -
Lemma 3.2. Up to the Grothendieck pairs, the subgroups \( N \) and \( N' \) of \( \text{Mod} \, \mathcal{M}^3 \) are isomorphic, if and only if, the groups \( \widehat{N}/\widehat{\mathbb{Z}} \cong G_K \) and \( \widehat{N'}/\widehat{\mathbb{Z}} \cong G_{K'} \) are isomorphic.

Proof. (i) Let \( N \cong N' \) be a pair of isomorphic subgroups of \( \text{Mod} \, \mathcal{M}^3 \). Recall that a profinite group \( \widehat{N} \) is a topological group defined by the inverse limit

\[
\widehat{N} := \lim_{\leftarrow} \frac{N}{N_k},
\]

where \( N_k \) runs through all open normal finite index subgroups of \( N \). It follows from (3.1), that if \( N \cong N' \), then \( \widehat{N} \cong \widehat{N'} \). Since \( G_K \cong \widehat{N}/\widehat{\mathbb{Z}} \) and \( G_{K'} \cong \widehat{N'}/\widehat{\mathbb{Z}} \), we conclude that \( G_K \cong G_{K'} \). The necessary condition of lemma 3.2 is proved.

(ii) Suppose that \( G_K \cong \widehat{N}/\widehat{\mathbb{Z}} \) and \( G_{K'} \cong \widehat{N'}/\widehat{\mathbb{Z}} \) are isomorphic groups. Let us show, that \( N \cong N' \) are isomorphic subgroups of \( \text{Mod} \, \mathcal{M}^3 \). To the contrary, let \( N \not\cong N' \). Since \( N \) and \( N' \) cannot be a Grothendieck pair, we conclude that \( \widehat{N} \not\cong \widehat{N'} \). But \( G_K \cong \widehat{N}/\widehat{\mathbb{Z}} \) and \( G_{K'} \cong \widehat{N'}/\widehat{\mathbb{Z}} \) and, therefore, \( G_K \not\cong G_{K'} \). One gets a contradiction proving the sufficient condition of lemma 3.2. \( \square \)

Lemma 3.3. The absolute Galois groups \( G_K \) and \( G_{K'} \) are isomorphic if and only if the underlying non-commutative fields \( \mathbb{K} \) and \( \mathbb{K}' \) are isomorphic.

Proof. The proof is an adaption of the argument of [Uchida 1976] [13, Corollary 2] to the case of non-commutative fields. Namely, we introduce a topology on \( G_K \) according to the formula (3.1). Let \( G_K \) and \( G_{K'} \) be open subgroups of an absolute Galois group \( G \), and let \( \sigma : G_K \to G_{K'} \) be a topological isomorphism. To prove lemma 3.3, it is enough to show that \( \sigma \) can be extended to an inner automorphism of \( G \), which corresponds an isomorphism \( \mathbb{K} \cong \mathbb{K'} \). Indeed, one takes an open normal subgroup \( N \) of \( G \) contained in \( G_K \) and \( G_{K'} \). The group \( N \) induces an isomorphism \( \sigma_N : G_K/N \to G_{K'}/N \). It can be shown that \( \sigma_N \) extends to an inner automorphism of the group \( G/N \). We repeat the construction over all open normal subgroups of \( G \) and obtain an explicit formula for the required inner automorphism of \( G \). Lemma 3.3 follows. \( \square \)

Remark 3.4. Lemmas 3.1-3.3 are true for the differentiable mappings. In this case one obtains the inclusion of the corresponding non-commutative fields.

Lemmas 3.1-3.3 and formula (1.1) imply that \( F : \mathcal{M}^4 \to \mathcal{S} \) is a covariant functor.

3.1.2. The detailed proof of items (i)-(iii) of theorem 1.2 is given in [9, Theorem 4.1]. We refer the reader to this paper for the proof and examples of the sphere knots.

Theorem 1.2 is proved.
\[ H_1(\mathcal{M}^4; \mathbb{Z}) \cong \text{Id} \]
\[ \begin{array}{ccc}
\mathcal{M}^4 & \xrightarrow{F} & \mathbb{K} \\
\downarrow & & \downarrow \\
S^4 & \xrightarrow{F} & L
\end{array} \]

\[ H_1(S^4 - X_g; \mathbb{Z}) \cong \pi_1(S^4 - X_g) / [\pi_1(S^4 - X_g), \pi_1(S^4 - X_g)] \] \hspace{1cm} (3.2)

is also a finite abelian group.

Consider a commutative diagram in Figure 2. An automorphism of \( H_1(S^4 - X_g; \mathbb{Z}) \) comes from a homeomorphism \( h : \mathcal{M}^4 \to \mathcal{M}^4 \) of the manifold \( \mathcal{M}^4 \). Theorem 1.2 says that \( h \) must correspond to an automorphism \( \sigma_h \) of the field \( \mathbb{K} \), such that \( \sigma_h(L) = L \) for a subfield \( L \subseteq K \). (For simplicity, the reader can think \( L \cong \mathbb{H} \) is the field of quaternions.) Since the automorphisms \( \sigma_h \) generate the Galois group of the extension \( \mathbb{K} \mid L \), we conclude that:

\[ \text{Gal } \mathbb{K} \cong \text{Aut } (H_1(S^4 - X_g; \mathbb{Z})). \] \hspace{1cm} (3.3)

Recall that the group of automorphisms of a finite abelian group is always an abelian group. Indeed, a finite abelian groups can be written in the form \( \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k\mathbb{Z} \) for some distinct primes \( p_i \). On the other hand, \( \text{Aut } (G \oplus H) \cong \text{Aut } (G) \oplus \text{Aut } (H) \), where \( G \) and \( H \) are finite abelian groups of the coprime order. Thus

\[ \text{Aut } (H_1(S^4 - X_g; \mathbb{Z})) \cong \text{Aut } (\mathbb{Z}/p_1\mathbb{Z}) \oplus \cdots \oplus \text{Aut } (\mathbb{Z}/p_k\mathbb{Z}). \] \hspace{1cm} (3.4)

Since the group of automorphism of a cyclic group is a cyclic group, using (3.4) we conclude, that the group \( \text{Aut } (H_1(S^4 - X_g; \mathbb{Z})) \) is abelian. In view of (3.3), the group \( \text{Gal } \mathbb{K} \) is also abelian. Corollary 1.4 is proved.

3.3. Proof of remark 1.5. Let \( \mathbb{K} \) be an abelian extension. Then \( \text{Gal } \mathbb{K} \) is a finite abelian group. Let us show, that the group \( H_1(S^4 - X_g; \mathbb{Z}) \) can be infinite. Indeed, one can always write \( H_1(S^4 - X_g; \mathbb{Z}) \cong \mathbb{Z}^k \oplus \text{Tors} \), where \( k \geq 0 \) and \( \text{Tors} \) is a finite abelian group. As explained, we have \( \text{Aut } (H_1(S^4 - X_g; \mathbb{Z})) \cong \text{Aut } (\mathbb{Z}^k) \oplus \text{Aut } (\text{Tors}) \). Recall that \( \text{Aut } (\mathbb{Z}^k) \cong \text{GL}_k(\mathbb{Z}) \). If \( k = 1 \), then the group \( \text{Aut } (\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \) is abelian and finite. Yet the group \( H_1(S^4 - X_g; \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Tors} \) is infinite. In other words, the corresponding 4-manifold \( \mathcal{M}^4 \in \mathcal{M}^4 \) cannot be simply connected.
4. ROKHLIN AND DONALDSON’S THEOREMS REVISITED

In this section we give a proof of the Rokhlin and Donaldson’s Theorems based on the Galois theory of non-commutative fields. To outline the proof, let \( \mathcal{M}^4 \in 2\mathbb{R}^4 \) be a simply connected smooth 4-manifold. The corollary 1.4 says that \( \mathbb{K} = F(\mathcal{M}^4) \) is an abelian extension. Consider a subgroup of the inner automorphisms \( \Gamma \) of the abelian group \( \text{Gal} \mathbb{K} \). Let \( \mathbb{B}(\Gamma) \) be the corresponding group ring, see (2.3). Denote by \( Q \) a symmetric bilinear form on the \( \mathbb{B}(\Gamma) \), see (2.5) and remark 2.7. For \( \Gamma \) a finite abelian group, we calculate both \( \mathbb{B}(\Gamma) \) and \( Q \), see lemmas 4.1 and 4.2. On the other hand, theorem 1.2 implies an isomorphism of the \( \mathbb{Z} \)-modules:

\[
\mathbb{B}(\Gamma) \cong H_2(\mathcal{M}^4; \mathbb{Z}),
\]

(4.1)

see lemma 4.4. From the map \( Q : \mathbb{B}(\Gamma) \times \mathbb{B}(\Gamma) \to \mathbb{Z} \), one gets a symmetric bilinear form on the homology group \( H_2(\mathcal{M}^4; \mathbb{Z}) \). As a corollary, we recover from the arithmetic of \( Q \) the Rokhlin and Donaldson’s Theorems for the simply connected smooth 4-manifolds. Let us pass to a detailed argument.

Lemma 4.1. The group ring of a finite abelian group \( \Gamma \cong \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k\mathbb{Z} \) is isomorphic to a direct sum of the cyclotomic fields, i.e.

\[
\mathbb{B}(\Gamma) \cong \mathbb{Z}(\zeta_{p_1}) \oplus \cdots \oplus \mathbb{Z}(\zeta_{p_k}),
\]

(4.2)

where \( \zeta_{p_i} \) is the \( p_i \)-th root of unity.

Proof. An elegant proof of this fact can be found in [Ayoub & Ayoub 1969] [1]. □

Thus to calculate \( Q \), we can restrict to the cyclotomic fields \( \mathbb{Q}(\zeta_{p_i}) \) and take the tensor product over \( p_i \). Recall that the trace form on \( \mathbb{Q}(\zeta_{p_i}) \) is a symmetric bilinear form:

\[
\text{Tr}_{\mathbb{Q}(\zeta_{p_i})} : \mathbb{Q}(\zeta_{p_i}) \times \mathbb{Q}(\zeta_{p_i}) \to \mathbb{Q}, \text{ such that } (x, y) \mapsto \text{tr}(xy),
\]

(4.3)

where \( \text{tr} \) is the trace of an algebraic number. The trace form (4.3) is equivalent to the form \( \text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(x, x) := \text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(x^2) \) via the formula:

\[
\text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(x, y) = \frac{1}{2} \left[ \text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(x + y)^2 - \text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(x^2) - \text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(y^2) \right].
\]

(4.4)

Lemma 4.2. The following classification is true:

\[
\text{Tr}_{\mathbb{Q}(\zeta_{p_i})}(x^2) = \begin{cases} 
p(1), & \text{if } p \text{ is odd prime} \\
2^n(1) \left( (1) \oplus (-1) \oplus (2^{n-1} - 1) \times H \right), & \text{if } p = 2^n, \ n \geq 4.
\end{cases}
\]

(4.5)

where we use the Witt ring notation \( p(1) \) for the diagonal quadratic form of dimension \( p \), \( 2^n(1) \) for the Pfister form of dimension \( 2^n \) and \( H \) for the hyperbolic (split) plane, see [Lam 2005] [5, Chapter X] for definitions.

Proof. We refer the reader to [Otake 2014] [11, Theorem 2.1] for the proof. □

Remark 4.3. The case \( n = 1 \) (\( n = 2; n = 3 \)) in formula (4.5) corresponds to the complex numbers (quaternions; octonions), respectively; see [Lam 2005] [5, Chapter X]. We omit these values of \( n \), since none of the fields is contained in \( \mathbb{K} \).
Lemma 4.4. If $H_2(\mathcal{M}^4;\mathbb{Z})$ is the second homology of a simply connected manifold $\mathcal{M}^4$, then $\mathbb{B}(\Gamma) \cong H_2(\mathcal{M}^4;\mathbb{Z})$, where $\cong$ is an isomorphism of the corresponding $\mathbb{Z}$-modules.

Proof. Recall that $\text{Gal} \mathbb{K}$ is acting on the field $\mathbb{K}$ by the automorphisms of $\mathbb{K}$. Theorem 1.2 says that such automorphisms correspond to the homeomorphisms of the manifold $\mathcal{M}^4$. Recall that each homeomorphism $h : \mathcal{M}^4 \to \mathcal{M}^4$ defines a linear map $h^* : H_2(\mathcal{M}^4;\mathbb{Z}) \to H_2(\mathcal{M}^4;\mathbb{Z})$. Since $\Gamma \subseteq \text{Gal} \mathbb{K}$, one gets a linear representation:

$$\rho : \Gamma \to \text{Aut}(H_2(\mathcal{M}^4;\mathbb{Z})).$$

(4.6)

Consider a regular representation of $\Gamma$ by the automorphisms of $H_2(\mathcal{M}^4;\mathbb{Z}) \cong \mathbb{Z}^k$. It is easy to see, that such a representation coincides with $\rho$ and, therefore, we conclude that $k = |\Gamma|$. Moreover, since $\mathbb{B}(\Gamma)$ is the group ring of $\Gamma$, one gets an isomorphism $\mathbb{B}(\Gamma) \cong H_2(\mathcal{M}^4;\mathbb{Z})$ between the corresponding $\mathbb{Z}$-modules. Lemma 4.4 is proved. \hfill \Box

Corollary 4.5. (Rokhlin and Donaldson)

(i) Definite intersection form of a simply connected smooth 4-manifold is diagonalizable;

(ii) Signature of the intersection form of a simply connected smooth 4-manifold is divisible by 16.

Proof. As explained, we identify the intersection form:

$$H_2(\mathcal{M}^4;\mathbb{Z}) \times H_2(\mathcal{M}^4;\mathbb{Z}) \to \mathbb{Z}$$

(4.7)

with the trace form $\text{Tr}_{\mathbb{Q}(\zeta_p)}(x,y)$ given by the formulas (4.3)-(4.5). Accordingly, we have to consider the following two cases.

(i) Let us consider the case $\text{Tr}_{\mathbb{Q}(\zeta_p)}(x^2) = p(1)$, where $p$ is an odd prime. Let $\{1, \zeta_p, \ldots, \zeta_p^{p-1}\}$ be the standard basis in the ring of integers $\mathbb{Z}[\zeta_p]$ of the cyclotomic field $\mathbb{Q}(\zeta_p)$. The corresponding formula (4.5) can be written as:

$$\text{Tr}_{\mathbb{Q}(\zeta_p)}(x^2) = \sum_{i=0}^{p-1} x_i^2, \quad x_i \in \mathbb{Z},$$

(4.8)

where $x = x_0 + x_1\zeta_p + \cdots + x_{p-1}\zeta_p^{p-1}$ for an $x \in \mathbb{Z}[\zeta_p]$. Using (4.4), one gets a symmetric bilinear form on $H_2(\mathcal{M}^4;\mathbb{Z})$:

$$\text{Tr}_{\mathbb{Q}(\zeta_p)}(x,y) = \sum_{i=0}^{p-1} x_iy_i, \quad x_i, y_i \in \mathbb{Z}.$$ 

(4.9)

It remains to notice, that $\mathcal{M}^4 \in \mathfrak{M}^4$ is a smooth manifold and (4.9) is a positive definite diagonalizable intersection form. Item (i) of corollary 4.5 follows.

(ii) Let us consider the case $\text{Tr}_{\mathbb{Q}(\zeta_p)}(x^2) = 2^n(1 \oplus (1 \oplus (-1) \oplus (2^{n-1}-1) \times H)$, where $p = 2^n$ and $n \geq 4$; see remark 4.3 explaining the restriction $n \geq 4$. Let
{1, ζ_{2^{n+1}}, \ldots, ζ_{2^{n+1}}^{-1}}$ be the standard basis in the $\mathbb{Z}[ζ_{2^{n+1}}]$. The corresponding formula $(4.5)$ can be written as:

$$Tr_{Q(ζ_{2^{n+1}})}(x^2) = \sum_{i=0}^{2^n-1} x_i^2 + (x_{2^n}^2 - x_{2^{n+1}}^2) + \left( \sum_{i=2^n+2}^{3 \times 2^{n-1}-1} x_i^2 - \sum_{i=3 \times 2^{n-1}}^{2^{n+1}-1} x_i^2 \right),$$

(4.10)

where $x = x_0 + \cdots + x_{2^{n+1}-1} ζ_{2^{n+1}}^{-1} \in \mathbb{Z}[ζ_{2^{n+1}}]$. Using $(4.4)$, one obtains a symmetric bilinear form:

$$Tr_{Q(ζ_{2^{n+1}})}(x, y) = \sum_{i=0}^{2^n-1} x_i y_i + \left( x_{2^n} y_{2^n} - x_{2^n+1} y_{2^{n+1}} \right) + \left( \sum_{i=2^n+2}^{3 \times 2^{n-1}-1} x_i y_i - \sum_{i=3 \times 2^{n-1}}^{2^{n+1}-1} x_i y_i \right).$$

(4.11)

It is easy to see, that $Tr_{Q(ζ_{2^{n+1}})}(x, y)$ is a diagonal bilinear form on $H_2(\mathcal{M}; \mathbb{Z})$. The signature of $(4.11)$ is equal to $2^n$, since the number of positive and negative 1’s for the terms in brackets is the same, while the signature of the first sum is $2^n$. It remains to notice, that $n \geq 4$ and, therefore, the signature of $Tr_{Q(ζ_{2^{n+1}})}(x, y)$ is divisible by 16. Since $\mathcal{M}$ is a simply connected smooth 4-manifold, one gets item (ii) of corollary 4.5. □

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