We analyze the Casalbuoni-Brink-Schwarz superparticle model on a 2-dimensional curved spacetime as a super Finsler metric defined on a (2,2)-dimensional supermanifold. We propose a nonlinear Finsler connection which preserves this Finsler metric and calculates it explicitly. The equations of motion of the superparticle are reconstructed in the form of auto-parallel equations expressed by the super nonlinear connection.
I. INTRODUCTION

Einstein constructed the theory of gravity by considering the geodesic equations of a point particle. Following his line of thought, a natural way to construct a supergravity model is by the super Riemannian formulation proposed by Arnowitt and Nath \[1\]. They extended the standard Riemannian manifold to a supermanifold that contains anticommuting Majorana spinors as coordinate functions. They defined a connection, a curvature and field equations on this supermanifold through almost the same procedure as in the Einstein’s gravity. However, in spite of high expectations, the solutions of the field equations did not include a superspace with global supersymmetry \[2\]. Nevertheless, we still think that it is an ideal path to supergravity and believe that some modifications to the connection can salvage this method.

The simplest superparticle model was first given by Casalbuoni \[3\], motivated by his study on the classical limit of fermion systems, which was analyzed by Brink and Schwarz later on \[4\]. The relation between the dynamics of this superparticle and the supergravity constraint equations via twistorial interpretation was suggested by Witten \[5\]. Though the relation between superparticle and supergravity seems quite natural, this is the only literature that clearly states it.

In this paper, we consider the Casalbuoni-Brink-Schwarz 2-dimensional superparticle Lagrangian as a super Finsler metric on a supermanifold (for the literature on supermanifold, we refer \[6, 7\]). We extend the nonlinear connection method on Finsler manifold invented by Kozma and Ootsuka \[8\] to a super Finsler manifold. Despite the fact that an explicit calculation of Finsler connection is in general difficult, their technique makes the calculation much easier. It is also applicable to a degenerate Finsler metric, which is a required property for our super Finsler metric. In the major literature on Finsler geometry \[9–13\], its connection defines parallel transports on the tangent bundle (line element space). Our parallel transports stay on the manifold (point space). This standpoint is called point-Finsler approach \[8, 14, 15\], and it is well suited for physical applications. We do not need the linear part of the standard Finsler connection. As for superconnection, our connection has a different definition discussed by Bejancu \[16\] and Vacaru and Vicol \[17\], but the same one as defined by DeWitt \[6\]. We further extend the latter definition to a nonlinear connection so that it can be applied to degenerate super metrics. Since this resembles Einstein’s theory of
general relativity, and taking the fact that the Casalbuoni-Brink-Schwarz model is a particle model with internal degrees of freedom (pseudoparticle) into account, we are certain that it leads a theory of gravity for a matter with internal degrees of freedom. We also believe it corresponds to a supergravity without the Rarita-Schwinger field.

In section 2, we give a quick review of the spinor structure and an analysis of the Casalbuoni-Brink-Schwarz model in terms of a super Finsler manifold. Section 3 is devoted to a definition of a nonlinear Finsler connection on a supermanifold. In section 4, we express the equations of motion of the superparticle as auto-parallel equations.

II. CASALBUONI-BRINK-SCHWARZ MODEL ON CURVED SUPER SPACETIME

Casalbuoni-Brink-Schwarz superparticle model was originally defined on a flat spacetime. We generalize this model to a curved spacetime as \[5\]. We consider 2-dimensional spacetime for simplicity and present it as a super Finsler metric \( L \) defined on a \((2,2)\)-dimensional super Finsler manifold \( M^{(2,2)} \). We take the even submanifold \( M^{(2,0)} \) of \( M^{(2,2)} \) as a Lorentzian manifold \((M^{(2,0)}, g)\) and assume the Lorentzian metric has signature \((+, -)\). The dynamical variables of this model are \( x^\mu (\mu = 0, 1) \), \( \xi^A (A = 1, 2) \), where \( x^\mu \) represent spacetime even coordinates and \( \xi^A \) are Grassmann odd coordinates which are components of Majorana spinors \( \xi = |\xi\rangle \xi^A \). The ket \( |\xi\rangle \) denotes spinor basis. There exists an inner product \( B_{AB} \) in the spinor space, \( B_{AB} = B_{BA}, B^2 = 1 \) \[18, 19\], which defines a cospinor of \( |\xi\rangle \) as \( \langle \xi | = \xi^AB_{AB} \) so that \( \langle \xi | \xi' \rangle = \xi^AB_{AB}c_{B}^{\xi'}. \) With cospinor basis \( \langle A | \), the component of the spinor can be extracted as \( \xi^A = \langle A | \xi \rangle \). The matrix \( B_{AB} \) satisfies \( B\gamma^aB^{-1} = t\gamma^a \), and \( \gamma^a \) are the gamma matrices that admit the property

\[
(\gamma^a)^A_C(\gamma^b)_C^B + (\gamma^b)^A_C(\gamma^a)_C^B = 2\eta^{ab}\delta^A_B. \tag{1}
\]

Here \( a, b = 0, 1 \) stand for the indices of flat spacetime whose metric is \( \eta_{ab} \). We use notations \( (\gamma^a)_{AB} := B_{AC}(\gamma^a)_C^B = (\gamma^a)_{BA} \). For example, we can take gamma matrices and spinor metric

\[
(\gamma^0)^A_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma^1)^A_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}
\]
and $B_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We start with the following Lagrangian

$$L(x, dx, \xi, d\xi) = \sqrt{g_{\mu\nu}(x)\Pi^\mu\Pi^\nu}, \quad \Pi^\mu = dx^\mu + \langle \xi|\gamma^\mu(x)|d\xi\rangle$$ \hspace{1cm} (3)

$$\gamma^\mu(x) := \gamma^a e^a_{\mu}(x), \quad g_{ab} = \eta^{ab} \theta_a \otimes \theta_b, \quad \theta^a = e^a_{\mu}(x)dx^\mu,$$ \hspace{1cm} (4)

where $e^a_{\mu}(x)$ are zweibeins. We regard the Lagrangian $L$ as a super Finsler metric because it satisfies the properties of super Finsler metric described below.

We set $z^I := (x^\mu, \xi^A)$, and capital Roman letters starting from $I, J, \cdots$ stands for both spacetime and spinor indices. $z^I$ and $dz^I$ satisfy the following commutation relations

$$z^I z^J = (-1)^{|I||J|} z^J z^I, \quad |I| = \begin{cases} 0 & (I = \mu), \\ 1 & (I = A). \end{cases}$$ \hspace{1cm} (5)

We also use the abbreviation $(z, dz)$ for $(z^I, dz^I) = (x^\mu, \xi^A, dx^\mu, d\xi^A)$. The symbol $d$ is called a total derivative, and $dz^I$ plays a role of a coordinate function of the tangent space. Namely, for a vector field $v = \frac{\partial}{\partial z^I} v^I$ on the supermanifold $M^{(2,2)}$, it gives

$$dz^I(v) = v^I.$$ \hspace{1cm} (6)

**Definition II.1.** Suppose we have a well-defined differentiable function $L : D(L) \subset TM^{(2,2)} \to \mathbb{R}$, where $D(L)$ is a subbundle of the tangent bundle $TM^{(2,2)}$. $L$ is called **super Finsler metric** when it admits the homogeneity condition

$$L(z, \lambda dz) = \lambda L(z, dz), \quad \lambda > 0.$$ \hspace{1cm} (7)

The set $(M^{(2,2)}, L)$ is called a **super Finsler manifold**.

We do not assume positivity: $L > 0$ and regularity: $\det (g_{IJ}(x, dx)) \neq 0$ for $g_{IJ}(z, dz) := \frac{1}{2} \frac{\partial^2 L^2}{\partial dz^I \partial dz^J}$, since these conditions are too strong for physical applications. Note that the homogeneity condition implies

$$\frac{\partial L}{\partial dz^I} dz^I = L.$$ \hspace{1cm} (8)

**Remark 1.** We call the super Finsler metric given by (3) and (4), **Casalbuoni-Brink-Schwarz metric**.
Firstly, we show symmetries of the system when the Lorentzian manifold \((M^{(2,0)}, g)\) is flat, \(L = \sqrt{\eta_{ab} \Pi^a \Pi^b}\). \(\Pi^a = dx^a + \langle \xi | \gamma^a | d\xi \rangle\), which has Poincaré symmetry and supersymmetry. These symmetries are written in terms of vector fields on the supermanifold \(M^{(2,2)}\).

**Definition II.2.** The Lie derivative of \(L\) along a vector field \(v = \frac{\partial}{\partial z^l} v^l\) on the supermanifold is defined by

\[
\mathcal{L}_v L := \frac{\partial L}{\partial z^l} v^l + \frac{\partial L}{\partial dz^l} dv^l. \tag{9}
\]

**Definition II.3.** A vector field \(v\) is said to be a Killing vector field, when it satisfies

\[
\mathcal{L}_v L = 0. \tag{10}
\]

The Killing vector field which corresponds to the Lorentz transformation is

\[
v = \frac{\partial}{\partial x^a} \varepsilon^a b_x b^c + \frac{\partial}{\partial d_x b} \varepsilon^b c_c c + \frac{\partial}{\partial \xi A} s^A B \xi B - \frac{\partial}{\partial d \xi A} s^A B d \xi B, \tag{11}
\]

where \(\varepsilon_{ab}\) and \(s_{AB}\) are arbitrary anti-symmetric tensors, \(\varepsilon_{ab} := \eta_{ac} \varepsilon^c_b = -\varepsilon_{ba}\) and \(s_{AB} := B_{AC} s^C B = -s_{BA}\). We can check that

\[
\mathcal{L}_v L = \frac{\Pi_a}{L} \left[ \frac{\partial \Pi^a}{\partial x^b} \varepsilon_{b} c_c c + \frac{\partial \Pi^a}{\partial d_x b} \varepsilon^b c_c c + \frac{\partial \Pi^a}{\partial \xi A} s^A B \xi B - \frac{\partial \Pi^a}{\partial d \xi A} s^A B d \xi B \right] \tag{12}
\]

\[
= \frac{\Pi_a}{L} \left[ \varepsilon^a b d_x b + (\gamma^a)_{AC} d \xi C s^A B \xi B - \xi^C (\gamma^a)_{C A} s^A B d \xi B \right] \tag{13}
\]

\[
= \frac{\Pi_a}{L} \left[ \varepsilon^a b d_x b + \varepsilon^a b \xi C (\gamma^b)_{C A} d \xi A \right] = \frac{1}{L} \varepsilon^a b \Pi_a \Pi^b = 0, \tag{14}
\]

where we used the identity \(s^A (\gamma^a)_{C B} - (\gamma^a)_{C A} s^C B = \varepsilon_a (\gamma^b)_{A B}\). For translation, we have

\[
v = \frac{\partial}{\partial x^a} \varepsilon^a, \tag{15}
\]

with an arbitrary constant \(\varepsilon^a\). Supersymmetry transformation is described by

\[
v = \frac{\partial}{\partial x^a} \langle \xi | \gamma^a | \varepsilon \rangle + \frac{\partial}{\partial \xi A} \varepsilon^A =: Q_A \varepsilon^A, \quad \frac{1}{2} \{Q_A, Q_B\} = \frac{\partial}{\partial x^a} (\gamma^a)_{AB}, \tag{16}
\]

with an arbitrary Grassmann number \(\varepsilon^A\).

Secondly, we derive the equations of motion of the model when \((M^{(2,0)}, g)\) is not flat. For convenience, we rewrite (3) using zweibeins.

\[
L = \sqrt{\eta_{ab} \Pi^a \Pi^b}, \quad \Pi^a = \theta^a + \langle \xi | \gamma^a | d\xi \rangle = e^a(x) \Pi^a. \tag{17}
\]
The action integral is given by the integration of the Finsler metric along an oriented curve $c$ on $M^{(2,2)}$,

$$A[c] := \int_c L = \int_{t_0}^{t_1} c^* L = \int_{t_0}^{t_1} L \left( c^* z^I, c^* dz^I \right) = \int_{t_0}^{t_1} L \left( z^I(t), \frac{dz^I(t)}{dt} \right) dt,$$

where a map $c : t \in \mathbb{R} \mapsto c(t) \in M^{(2,2)}$ is a parametrization of the curve $c$, and $c^* L$ represents the pullback of $L$ by the map $c$. The variation of the action is given by

$$\delta A[c] = \int_c \delta L,$$

where

$$\delta L = \frac{\eta_{ab} \Pi^a}{L} \delta \Pi^b = \frac{\Pi_a}{L} (\delta (e^a_\mu dx^\mu) + \langle \delta \xi | \gamma^a | d\xi \rangle) + \langle \xi | \gamma^a | d\delta \xi \rangle$$

$$= d \left\{ \frac{\Pi_a}{L} e^a_\mu \delta x^\mu - \frac{\Pi_a}{L} \langle \xi | \gamma^a | \delta \xi \rangle \right\} + \frac{\Pi_a a_\mu}{L} \partial_\mu e^a_\nu dx^\nu \delta x^\mu + \langle \delta \xi | \frac{\Pi_a}{L} \gamma^a | d\xi \rangle$$

$$- d \left( \frac{\Pi_a}{L} \right) \delta x^\mu - d \left( \frac{\Pi_a}{L} \right) \langle \xi | \gamma^a | \delta \xi \rangle - d \langle \xi | \frac{\Pi_a}{L} \gamma^a | \delta \xi \rangle.$$ (20)

The Euler-Lagrange equations are extracted from the $\delta x^\mu$ part and $\delta \xi$ part:

$$0 = c^* \left[ \frac{\Pi_a}{L} \partial_\mu e^a_\nu dx^\nu - d \left( \frac{\Pi_a}{L} \right) \right],$$

$$0 = c^* \left[ 2 \frac{\Pi_a}{L} \gamma^a | d\xi \rangle + d \left( \frac{\Pi_a}{L} \right) | \xi \rangle \right].$$ (21) (22)

These equations have terms including $d^2 z^I$, whose pullback by $c^*$ is given by

$$c^* d^2 z^I = d \left( c^* dz^I \right) = d \left( \frac{dz^I(t)}{dt} \right) = \frac{d^2 z^I(t)}{dt^2} (dt)^2.$$ (23)

Further on, we will omit the pullback symbol $c^*$ for notational simplicity. We can rewrite (21) as

$$0 = \frac{\Pi_b}{L} e^a_\mu \partial_\mu e^b_\nu dx^\nu - e^a_\mu d \left( \frac{\Pi_a}{L} \right) = \frac{\Pi_b}{L} e_a (e^b_\nu) dx^\nu - d \left( \frac{e^a_\mu \Pi_a}{L} \right) + \frac{\Pi_b}{L} d e_a^\mu$$

$$= \frac{\Pi_b}{L} L e a \theta^b - \frac{\Pi_b}{L} d L e a x^\nu + \frac{\Pi_b}{L} d e_a^\mu - d \left( \frac{\Pi_a}{L} \right) = \frac{\Pi_b}{L} L e a \theta^b - d \left( \frac{\Pi_a}{L} \right),$$ (24)

where $\text{div}$ is the exterior derivative of a 1-form, $\text{div} \theta^a := \frac{1}{2} (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) dx^\mu \wedge dx^\nu$, which gives a 2-form, and $e_{ca}$ the interior product. With the above equation, (22) becomes

$$\frac{1}{L} \Pi_a \gamma^a | d\xi \rangle = - \frac{1}{2L} \Pi_b e_a \text{div} \theta^b \gamma^a | \xi \rangle.$$ (25)
Since the above equation is a first-order differential equation of \( |\xi| \), it becomes a constraint on the supermanifold. Multiplying it by \( \frac{\Pi_c \gamma^c}{L} \), we have

\[
\frac{1}{L^2} \Pi_a \Pi_c \gamma^c \gamma^a |d\xi| = \frac{1}{L^2} \Pi_a \Pi_c (\gamma^{ca} + \eta^{ca}) |d\xi| = |d\xi| = -\frac{1}{2L^2} \Pi_b \Pi_c \gamma^c \gamma^a |\xi| \epsilon_{e_a} \text{div} \theta^b, \tag{26}
\]

where \( \gamma^{ca} := \frac{1}{2} (\gamma^c \gamma^a - \gamma^a \gamma^c) \). In the second equality, we used \( \Pi_a \Pi_c \eta^{ca} = L^2 \). From the identity

\[
\langle \xi | \gamma^n \gamma^c \gamma^a | \xi \rangle = \langle \xi | \gamma^n (\gamma^{ca} + \eta^{ca}) | \xi \rangle = \varepsilon^{ca} \langle \xi | \gamma^n \gamma^{01} | \xi \rangle = \varepsilon^{ca} \left( \eta^{n0} \langle \xi | \gamma^1 | \xi \rangle - \eta^{n1} \langle \xi | \gamma^0 | \xi \rangle \right), \tag{27}
\]

and \( \langle \xi | \gamma^n | \xi \rangle = 0 \), we have

\[
\langle \xi | \gamma^n | d\xi \rangle = -\frac{1}{2L^2} \Pi_b \Pi_c \langle \xi | \gamma^n \gamma^c \gamma^a | \xi \rangle \epsilon_{e_a} \text{div} \theta^b = 0. \tag{28}
\]

Using together the definition of torsion

\[
T^a = \text{div} \theta^a + \omega^a_b \wedge \theta^b = \frac{1}{2} T^a_{bc} \theta^b \wedge \theta^c, \tag{29}
\]

where \( \omega^a_b \) is the spin connection, equation (26) becomes

\[
|d\xi| = -\frac{\Pi_b \Pi_c}{2L^2} (\epsilon_{e_a} T^b - \omega^b_{na} \theta^n + \omega^b_{on} \theta^n) \gamma^c \gamma^a |\xi| = -\frac{\theta_b \theta_c}{2L^2} (\epsilon_{e_a} T^b - \omega^b_{na} \theta^n + \omega^b_{on} \theta^n) (\gamma^{ca} + \eta^{ca}) |\xi|. \tag{30}
\]

The right hand side is calculated with an anti-symmetric tensor \( \varepsilon^{ab} (\varepsilon^{01} = 1) \) as,

\[
\theta_b \theta_c \epsilon_{e_a} T^b (\gamma^{ca} + \eta^{ca}) = -L^2 \theta_b T^b_{01} \gamma^{01}, \tag{31}
\]

\[
- \varepsilon^{ca} \theta^b \theta_c \theta^o \omega_{bna} + \varepsilon^{ca} \theta^b \theta_c \omega_{ba} = \{(\theta^0)^2 - (\theta^1)^2\} \omega_{01} = \Pi_a \Pi^a \omega_{01} = L^2 \omega_{01}, \tag{32}
\]

\[
- \theta^b \theta^o \theta^a \omega_{bna} + \theta^b \theta^a \omega_{ba} = -\theta^b \theta^a \omega_n \omega_{bna} = 0, \tag{33}
\]

to result in

\[
|d\xi| = \frac{1}{2} (\theta^b T^b_{01} - \omega_{01}) \gamma^{01} |\xi| = -\frac{1}{2} (K_{01b} \theta^b + \omega_{01}) \gamma^{01} |\xi| = -\frac{1}{4} (K_{abc} + \omega_{abc}) \gamma^{ab} \theta^c |\xi|, \tag{34}
\]

where we denoted the contorsion

\[
K_{abc} := \frac{1}{2} (T_{abc} + T_{bca} - T_{cab}). \tag{35}
\]
Applying abbreviations such as

\[ \hat{\omega}_c := \frac{1}{2}(K_{abc} + \omega_{abc}) \gamma^{ab}, \quad \hat{\omega} := (K_c + \omega_c) \theta^c = \frac{1}{2}(K_{abc} + \omega_{abc}) \gamma^{ab} \theta^c, \]

we obtain a simple expression

\[ |d\xi| + \frac{1}{2} \hat{\omega}|\xi| = 0. \]  

(37)

With (28), the equation (21) becomes

\[ 0 = \frac{\theta^a}{L} \partial_\mu e^a \gamma^\nu - d \left( \frac{g_{\mu \nu} dx^\nu}{L} \right) \]

\[ = \frac{1}{2L} \partial_\mu g_{\alpha \beta} dx^\alpha dx^\beta - d \left( \frac{g_{\mu \nu} dx^\nu}{L} \right). \]  

(38)

After considering the pullback by \( c^* \), this is exactly the same as the equation of motion of a relativistic free particle on a Lorentzian manifold. The Casalbuoni-Brink-Schwarz superparticle can be identified as a relativistic particle with spin obeying equation (37) as the internal degree of freedom.

### III. NONLINEAR FINSLER CONNECTION ON A SUPERMANIFOLD

In this section we will define a connection on a supermanifold which expresses naturally the geodesics of a superparticle. For this purpose we follow the definition given by Kozma and Ootsuka [8]. In their formulation, the Berwald connection is redefined as a nonlinear connection directly on point-Finsler space, and extended also to comprise the singular case. Such definition is advantageous for our purpose to consider the generalization to a supermanifold. We define a nonlinear generalization of the cotangent bundle \( NT^*M^{(2,2)} \) as

\[ T^*M^{(2,2)} \subset NT^*M^{(2,2)} := \{ a(z, dz) | a(z, \lambda dz) = \lambda a(z, dz), \lambda > 0 \}. \]  

(39)

A nonlinear 1-form \( a \in NT^*M^{(2,2)} \) is a function of \( z^I \) and \( dz^I \) and defines a map \( a : \Gamma(TM^{(2,2)}) \to C^\infty(M^{(2,2)}), a(X) := a(z, dz(X)), X \in TM^{(2,2)} \) which satisfies a homogeneity condition \( a(\lambda X) = \lambda a(X) \). It is not linear because \( a(X + Y) \neq a(X) + a(Y) \).

**Definition III.1.** Let \( \Gamma(T^*M^{(2,2)}) \) be a section of the cotangent bundle \( T^*M^{(2,2)} \) on a supermanifold \( M^{(2,2)} \) and \( \nabla : \Gamma(T^*M^{(2,2)}) \to \Gamma(T^*M^{(2,2)} \otimes NT^*M^{(2,2)}) \) a map such that
satisfies
\[ \nabla dz^I &= -N^I_J \otimes dz^J, \]  
\[ N^I_J(z, \lambda dz) = \lambda N^I_J(z, dz), \]  
\[ \frac{\partial N^I_J}{\partial dz^K} = (-1)^{|J||K|} \frac{\partial N^I_K}{\partial dz^J}, \]  
\[ \frac{\partial L}{\partial z^I} = \frac{\partial L}{\partial dz^J} N^J_I. \]  
(40) \(\) (41) \(\) (42) \(\) (43)

Then, \(N^I_J\) is called a nonlinear super Finsler connection on \(M^{(2,2)}\).

Unlike the linear connection, in general, \(N^I_J\) is not linear in \(dz^I\); namely \(N^I_J(z, dz) \neq N^I_K(z)dz^K\). The condition (41) means that the connection \(N^I_J\) is degree 1 homogeneous. For a nonlinear connection, the condition (42) does not mean the torsion is zero, while for the linear case, it becomes a torsion-free condition. The last condition (43) implies that the connection preserves the super Finsler metric: \(\nabla L := \frac{\partial L}{\partial z^I} \nabla z^I + \frac{\partial L}{\partial dz^I} \nabla dz^I = 0\). We define the quantities \(G^I := \frac{1}{2} N^I_J dz^J\) and call them super Berwald functions. They are degree 2 homogeneity functions with respect to \(dz^I\): \(G^I(z, \lambda dz) = \lambda^2 G^I(z, dz)\). From this homogeneity condition, we have
\[ \frac{\partial G^I}{\partial dz^J} = \frac{1}{2} \left( N^I_J + (-1)^{|J||K|} \frac{\partial N^I_K}{\partial dz^J} dz^K \right) = N^I_J. \]  
(44)

**Remark 2.** The nonlinear connection defined above satisfies the linearity \(\nabla(\rho_1 + \rho_2) = \nabla \rho_1 + \nabla \rho_2\) for sections \(\rho_1\) and \(\rho_2\) of \(T^* M^{(2,2)}\), which fails for the sections of \(TM^{(2,2)}\). Moreover, for physical problems, covariant quantities appear more often than contravariant quantities do. For these reasons, we proposed the definition [III.1]. However, using such connection, we can also define the nonlinear connection for a vector field \(X = \frac{\partial}{\partial z^I} X^I\), by
\[ \nabla X := (dX^I + N^I_J(z, dz(X))dz^J) \otimes \frac{\partial}{\partial z^I}. \]  
(45)

Here \(N^I_J(z^K, dz^K(X)) = N^I_J(z^K, X^K)\). The connection above defines a map \(\nabla : \Gamma(TM^{(2,2)}) \rightarrow \Gamma(T^* M^{(2,2)} \otimes TM^{(2,2)})\) with \(\nabla(\lambda X) = \lambda \nabla X, \lambda > 0\) and \(\nabla(X + Y) \neq \nabla X + \nabla Y\).

For the superparticle model, we have the following results on a nonlinear connection.

**Theorem III.1.** Let \(L\) be the Casalbuoni-Brink-Schwarz metric, then the super Berwald
functions for $L$ and constraints are given by

$$G^\mu = \frac{1}{2} \Gamma^\mu_{\alpha\beta} dx^\alpha dx^\beta + \frac{1}{L} \Pi^\mu \langle \xi|d\xi \rangle - \frac{1}{2L^2} \Pi_a dx^\nu \langle \xi|\gamma^\mu|d\xi \rangle - \frac{1}{2} g^\mu\beta L_{\beta\nu} \partial^\alpha \langle \xi|\gamma_\alpha|d\xi \rangle + \langle \xi|\gamma^\mu|\lambda \rangle,$$

$$(46)$$

$$G^A = \frac{1}{2L^2} \Pi_a dx^\nu \langle d\xi|\gamma^A \rangle - \lambda^A,$$

$$(47)$$

$$C_A := M_A - M_\mu \langle \gamma^\mu \rangle_{AB} \xi^B = 0,$$

$$(48)$$

where $\lambda^A$ are arbitrary functions of $(z^I, dz^I)$ which are second order homogeneous with respect to $dz^I$, and

$$M_\mu := \frac{1}{2L} \left\{ -\Pi_a \partial_\mu e^a dx^\nu + \left( \eta_{ab} - \frac{1}{L^2} \Pi_a \Pi_b \right) e^b_\mu dx^\nu + \Pi_a e^a_\mu \right\},$$

$$(49)$$

$$M_A := \frac{1}{2L} \left\{ 2\Pi_a \langle d\xi|\gamma^A \rangle + \left( \eta_{ab} - \frac{1}{L^2} \Pi_a \Pi_b \right) \Pi^A e_\mu dx^\mu \langle \xi|\gamma^B |A \rangle \right\}.$$ 

$$(50)$$

Proof. Firstly, we multiply (43) by $dz^I$ from the right and obtain

$$\frac{\partial L}{\partial z^I} dz^I = \frac{\partial L}{\partial dz^J} N^J_I dz^I = 2 \frac{\partial L}{\partial dz^I} G^I.$$ 

$$(51)$$

Considering the homogeneity condition (5), we find a particular solution for $G^I$:

$$G^I = \frac{1}{2} \left( \frac{\partial L}{\partial dz^I} \right) \frac{dz^I}{L}.$$ 

$$(52)$$

Since we are considering $(2,2)$-dimensional supermanifold, we need 4 independent vectors to span the general solution. We choose vectors

$$l^I_\mu := \delta^I_\mu - \frac{\Pi_\mu}{L^2} dz^I, \quad l^I_B := \left( l^\mu_B \right) := \left( \xi|\gamma^\mu|B \right) - \langle A|B \rangle$$

$$(53)$$

for the basis. It is easy to check that these vectors vanish when they are contracted with $\frac{\partial L}{\partial dz^I}$ from the left. Thus, we can write the general solution as

$$G^I = \frac{1}{2} \left( \frac{\partial L}{\partial dz^I} dz^I \right) \frac{dz^I}{L} + l^I_\mu \lambda^\mu + l^I_A \lambda^A,$$

$$(54)$$

where $\lambda^\mu, \lambda^A$ are arbitrary functions of $(z^I, dz^I)$, and are second order homogeneous with respect to $dz^I$. Since there are 5 non-independent vectors $(dz^I, l^I_\mu, l^I_B)$ in the solution, we can choose one additional condition for the coefficients $\lambda^\mu$. For convenience, we set

$$\Pi_\mu \lambda^\mu = 0.$$ 

$$(55)$$
For further calculation, we define
\[ L_{IJ} := \frac{\partial^2 L}{\partial dz^I \partial dz^J}, \]
\[ L_{\mu\nu} = \frac{\partial^2 L}{\partial dx^\mu \partial dx^\nu} = \frac{1}{L} \left( g_{\mu\nu} - \frac{1}{L^2} \Pi_\mu \Pi_\nu \right) = L_{\nu\mu}, \]
\[ L_{A\nu} = \frac{\partial^2 L}{\partial \xi^A \partial dx^\nu} = \langle \xi | \gamma^\alpha L_{\alpha\nu} | A \rangle = L_{\nu A}, \]
\[ L_{AB} = \frac{\partial^2 L}{\partial \xi^A \partial \xi^B} = \langle \xi | \gamma^\alpha L_{\alpha\beta} | A \rangle \langle \xi | \gamma^\beta | B \rangle = -L_{BA}. \]

From (44), (53), and (54) we have
\[ N_I^J = \frac{\partial G^I}{\partial dz^J} = \frac{1}{2} \left( \frac{\partial L}{\partial z^K} \right) \left( \frac{1}{L} \delta^I_J - \frac{dz^I}{L^2} \frac{\partial L}{\partial dz^J} \right) + \frac{1}{2}(-1)^{|I||J|} \left( \frac{\partial L}{\partial dz^J} + \frac{\partial^2 L}{\partial dz^J \partial z^K} dz^K \right) L \]
\[ + \delta^I_J \frac{\partial \lambda^\mu}{\partial dz^J} + \frac{\partial \lambda^A}{\partial dz^J} \delta^I_J + \delta^I_J \frac{\partial \lambda^A}{\partial dz^J} \delta^I_J + (-1)^{|I|} \frac{\partial \lambda^A}{\partial dz^J} \delta^I_J A^A. \] (60)

The last term will vanish due to (53). When this is multiplied by \( \frac{\partial L}{\partial dz^J} \), only few terms remain:
\[ \frac{\partial L}{\partial dz^J} N_I^J = \frac{1}{2} \left( \frac{\partial L}{\partial z^K} \right) \left( \frac{1}{L} \delta^I_J - \frac{dz^I}{L^2} \frac{\partial L}{\partial dz^J} \right) - \lambda^\mu L_{\mu J}. \] (61)

The relation (43) says that the left hand side of (61) is equal to \( \frac{\partial L}{\partial z^J} \), which leads to
\[ \lambda^\mu L_{\mu J} = \frac{1}{2} \left( -\frac{\partial L}{\partial z^J} + \frac{\partial^2 L}{\partial dz^J \partial z^K} dz^K \right) =: M_J. \] (62)

We separate the above equation into two pieces. For \( J = A \), we leave it as a constraint \( C_A = M_A - \lambda^\mu L_{\mu A} = 0 \). For \( J = \mu \), we rewrite it into a matrix equation
\[ \begin{pmatrix} L_{\mu\nu} & \Pi_\nu \\ \Pi_\nu & L \end{pmatrix} \begin{pmatrix} \lambda^\mu \\ 0 \end{pmatrix} = \begin{pmatrix} M_\mu \\ 0 \end{pmatrix}. \] (63)

The second row is the condition (55). This matrix has the inverse matrix
\[ \begin{pmatrix} \tilde{L}_{\mu\nu} & \Pi_\nu \\ \Pi_\nu & L \end{pmatrix}, \quad \tilde{L}_{\mu\nu} := L g_{\mu\nu} - \frac{\Pi_\mu \Pi_\nu}{L}, \] (64)
and we have
\[ \begin{pmatrix} \lambda^\mu \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{L}_{\mu\nu} M_\nu \\ \frac{1}{L} \Pi_\nu M_\nu \end{pmatrix}. \] (65)
With this $\lambda^\mu$, we obtain
\[ G^I = \frac{1}{2} \left( \frac{\partial L}{\partial z^J} dz^J \right) \frac{dz^I}{L} + l^I_\mu \tilde{L}^{\mu\nu} M_\nu + l^I_\lambda A^\lambda. \] (66)

The second row of (65) automatically holds. This can be checked by considering the constraint
\[ 0 = C_A = M_A - \lambda^\mu \langle \xi | \gamma^\alpha L_{\alpha\mu} | A \rangle = M_A - M_\alpha \langle \xi | \gamma^\alpha | A \rangle, \] (67)

where (62) is used for the last equality. Taking the contraction with $d\xi^A$, we get
\[ 0 = \langle C | d\xi \rangle = M_A d\xi^A - M_\alpha (\Pi^\alpha - dx^\alpha) = M_I dz^I - M_\alpha \Pi^\alpha = -M_\alpha \Pi^\alpha. \] (68)

For the last equality, we used (8) to obtain
\[ M_I dz^I = \frac{1}{2} \left( -\frac{\partial L}{\partial z^I} dz^I + \frac{\partial^2 L}{\partial dz^I \partial z^K} dz^K dz^I \right) = 0. \] (69)

The explicit expressions of $M_I$ are calculated straightforward.

Using the relation between the Christoffel symbol and zweibeins,
\[ \Gamma_{\mu\alpha\beta} dx^\alpha dx^\beta = \frac{1}{2} (\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}) dx^\alpha dx^\beta \]
\[ = \eta_{ab} (e^b_\beta \partial_\alpha e^a_\mu + e^a_\mu \partial_\alpha e^b_\beta - e^a_\alpha \partial_\mu e^b_\beta) dx^\alpha dx^\beta, \] (70)

we obtain
\[ 2LM_\mu = \Gamma_{\mu\alpha\beta} dx^\alpha dx^\beta - \frac{1}{L^2} \Pi_\alpha \Pi_\beta e^\mu_\sigma \sigma^\nu_\alpha \Pi^\nu dx^\nu - \langle \xi | \gamma_\alpha | d\xi \rangle t_\alpha \partial_\mu \text{div} \theta^a. \] (71)

With the above relation and
\[ \frac{\partial L}{\partial z^I} dz^J = \frac{1}{L} \Pi_\alpha e^\alpha_\mu dx^\mu, \] (72)

the even part of the super Berwald function $G^\mu$ becomes
\[ G^\mu = \frac{1}{2} \left( \frac{\partial L}{\partial z^J} dz^J \right) \frac{dx^\mu}{L} + l^\mu_\alpha \tilde{L}^{\alpha\beta} M_\beta + l^\mu_\lambda A^\lambda \]
\[ = \frac{1}{2L^2} \Pi_\sigma e^\sigma_\nu dx^\nu dx^\mu + \left( \frac{1}{2} \Pi^\mu \Pi^\beta \right) M_\beta + \langle \xi | \gamma^\mu | \lambda \rangle \]
\[ = \frac{1}{2L^2} \Pi_\sigma e^\sigma_\nu dx^\nu dx^\mu + \frac{1}{2} \Gamma_{\alpha\beta\gamma} dx^\alpha dx^\beta - \frac{1}{2L^2} \Pi_\sigma e^\sigma_\nu dx^\nu \Pi^\mu - \frac{1}{2} g^{\mu\beta} t_\beta \partial_\mu \text{div} \theta^a \langle \xi | \gamma_\alpha | d\xi \rangle \]
\[ - \frac{1}{L} \Pi^\mu \Pi^\beta M_\beta + \langle \xi | \gamma^\mu | \lambda \rangle \]
\[ = \frac{1}{2} \Gamma_{\alpha\beta\gamma} dx^\alpha dx^\beta + \frac{1}{L} \langle C | d\xi \rangle - \frac{1}{2L^2} \Pi_\sigma e^\sigma_\nu dx^\nu \langle \xi | \gamma^\mu | d\xi \rangle - \frac{1}{2} g^{\mu\beta} t_\beta \partial_\mu \text{div} \theta^a \langle \xi | \gamma_\alpha | d\xi \rangle + \langle \xi | \gamma^\mu | \lambda \rangle. \] (73)
In the last line, we used the identity (68). For the odd part $G^A$, the relation
\[ l^A_\alpha \tilde{L}^\alpha_\beta = -\frac{1}{L^2} \Pi_\alpha \tilde{L}^\alpha_\beta d\xi^A = 0 \] (74)
assures
\[ G^A = \frac{1}{2} \left( \frac{\partial L}{\partial z^I} dz^I \right) \frac{d\xi^A}{L} + l^A_\alpha \tilde{L}^\alpha_\beta M_\beta + l^A_\beta \lambda^B \]
\[ = \frac{1}{2L^2} \Pi_\alpha de^a_\nu dx^\nu d\xi^A - \lambda^A. \] (75)

Note that the super Berwald functions (46) and (47) are nonlinear with respect to $dz^I$. This result cannot arise if linear connections are assumed from the start as in [1]. We think this is why they cannot construct the supergravity. Only nonlinear connection is allowed for the Casalbuoni-Brink-Schwarz model.

Without odd variables, that is $\xi^A = 0$, the connection $N^\mu_\nu$ becomes the usual Riemannian connection. Therefore, our formulation is a natural extension.

**Proposition III.1.** The constraint $C_A = 0$ is equivalent to the equation (25) and eventually leads to (28), $\langle \xi | \gamma^a | d\xi \rangle = 0$.

**Proof.** From the definition of $M_I$, we have
\[ 2C_A = 2M_A - 2M_\mu (\gamma^\mu)_{AB} \xi^B \]
\[ = -\frac{\partial L}{\partial \xi^A} + \frac{\partial L}{\partial d\xi^A \partial x^\mu} dx^\mu + \frac{\partial^2 L}{\partial d\xi^A \partial \xi^B} d\xi^B \]
\[ - \left( -\frac{\partial L}{\partial x^\mu} + \frac{\partial^2 L}{\partial dx^\mu \partial x^\nu} dx^\nu + \frac{\partial^2 L}{\partial dx^\mu \partial \xi^C} d\xi^C \right) (\gamma^\mu)_{AB} \xi^B. \] (76)

We put the results
\[ \frac{\partial L}{\partial \xi^A} = -\frac{\Pi_\mu}{L} (\gamma^\mu)_{AB} \xi^B, \] (77)
\[ \frac{\partial^2 L}{\partial d\xi^A \partial x^\mu} = \frac{\partial}{\partial x^\mu} \left( \frac{\Pi_\nu}{L} \right) (\gamma^\nu)_{AB} \xi^B + \frac{\Pi_\nu e_\nu^a (\gamma^a)_{AB} L \xi^B,} \] (78)
\[ \frac{\partial^2 L}{\partial d\xi^A \partial \xi^B} = \frac{\Pi_\mu}{L} (\gamma^\mu)_{AB} - \left( \frac{\Pi_\mu}{L} \right) \frac{\partial}{\partial \xi^B} (\gamma^\mu)_{AC} \xi^C, \] (79)
\[ \frac{\partial L}{\partial x^\mu} = \frac{\Pi_\mu}{L} e_\nu^a dx^\nu, \] (80)
\[ \frac{\partial^2 L}{\partial dx^\mu \partial x^\nu} = \frac{\partial}{\partial x^\nu} \left( \frac{\Pi_\mu}{L} \right), \] (81)
\[ \frac{\partial^2 L}{\partial dx^\mu \partial \xi^C} = \left( \frac{\Pi_\mu}{L} \right) \frac{\partial}{\partial \xi^C}. \] (82)
into equation (76), and obtain

\[ 2C_A = \frac{\Pi}{L} (\gamma^\mu)_{AB} d\xi^B + \frac{\partial}{\partial x^\mu} \left( \frac{\Pi^\nu}{L} \right) dx^\mu (\gamma^\nu)_{AB} \xi^B + \frac{\Pi^\nu}{L} \partial_{\mu} e_{\nu}^a dx^\mu (\gamma^a)_{AB} \xi^B \\
+ \frac{\Pi}{L} (\gamma^\mu)_{AB} d\xi^B - \left( \frac{\Pi}{L} \right) \frac{\partial}{\partial \xi^B} (\gamma^\mu)_{AC} \xi^C d\xi^B \\
+ \frac{\Pi^a}{L} \partial_{\mu} e_{\nu}^a d\xi^\nu (\gamma^\mu)_{AB} \xi^B - \frac{\partial}{\partial x^\mu} \left( \frac{\Pi^\mu}{L} \right) dx^\nu (\gamma^\mu)_{AB} \xi^B - \left( \frac{\Pi}{L} \right) \frac{\partial}{\partial \xi^B} d\xi^C (\gamma^\mu)_{AB} \xi^B \\
= \frac{2\Pi^\mu}{L} (\gamma^\mu)_{AB} d\xi^B + \frac{\Pi^b}{L} (e^b_{\nu} \partial_{\mu} e_{\nu}^a + e_{\nu}^a \partial_{\nu} e_{\mu}^b) dx^\mu (\gamma^a)_{AB} \xi^B \\
= \frac{2\Pi^a}{L} (\gamma^a)_{AB} d\xi^B + \frac{\Pi^b}{L} e_{\nu}^a \text{div}^\theta (\gamma^a)_{AB} \xi^B. \tag{83} \]

Thus, \( C_A = 0 \) means the equation (25). \( \square \)

**IV. AUTO-PARALLEL EQUATIONS**

To rewrite the Euler-Lagrange equations into auto-parallel equations, (42) is the key condition. With these nonlinear super Finsler connections, we have the following result.

**Theorem IV.1.** The Euler-Lagrange equations of the superparticle are expressed as the auto-parallel equations

\[ 0 = c^\ast \left[ d^2 x^\mu + \Gamma^\mu_{\alpha\beta} dx^\alpha dx^\beta + \frac{2}{L} \Pi^\alpha \langle \xi | d\xi \rangle - \frac{1}{L^2} \Pi_\alpha d^2 x^\alpha dx^\nu \langle \xi | \gamma^\mu | d\xi \rangle - g^{\mu\beta} \partial_{\beta} \text{div}^\theta \langle \xi | \gamma^a | d\xi \rangle \\
- \frac{\lambda}{L} dx^\mu - \langle \xi | \gamma^\mu | \lambda \rangle \right], \tag{84} \]

\[ 0 = c^\ast \left[ d^2 \xi^A + \frac{1}{L^2} \Pi_a d^2 x^\alpha dx^\nu d\xi^A - \frac{\lambda}{L} d\xi^A + \lambda^A \right], \tag{85} \]

with the constraint

\[ c^\ast (C_A) = 0. \tag{86} \]

Here, \( \lambda \) and \( \lambda^A \) are arbitrary functions of \( (z^I, dz^I) \), and are second order homogeneous with respect to \( dz^I \).
Proof. We start with the Euler-Lagrange equation. Making use of condition (42), we have

$$0 = \frac{\partial L}{\partial z^I} - d \left( \frac{\partial L}{\partial dz^I} \right)$$

$$= \frac{\partial L}{\partial z^I} - \frac{\partial^2 L}{\partial dz^I \partial z^J} dz^J - \frac{\partial^2 L}{\partial dz^I \partial dz^J} d^2 z^J$$

$$= \frac{\partial L}{\partial z^I} - (-1)^{|I||J|} \left( \frac{\partial L}{\partial z^J} \right) \frac{\partial}{\partial dz^I} d^2 z^J - L_{IJ} d^2 z^J$$

$$= \frac{\partial L}{\partial z^J} N^J_I - (-1)^{|I||J|} \left( \frac{\partial L}{\partial z^K} \frac{\partial N^K_J}{\partial z^I} \right) \frac{\partial}{\partial dz^J} d^2 z^J - L_{IJ} d^2 z^J$$

$$= \frac{\partial L}{\partial z^J} N^J_I - \left\{ (-1)^{|I||J|} \frac{\partial L}{\partial z^K} \frac{\partial N^K_J}{\partial dz^I} + (-1)^{|I||K|} L_{IK} N^K_J \right\} d^2 z^J - L_{IJ} d^2 z^J$$

$$= -L_{IJ} (d^2 z^J + 2G^J). \quad (87)$$

Since

$$L_{IJ} \frac{dz^J}{L} = 0, \quad L_{IJ} l^I_A = 0, \quad (88)$$

we can expand it as

$$d^2 z^J + 2G^J = \frac{\lambda}{L} \frac{dz^J}{L} + l^I_A \lambda^A, \quad \lambda$$

$$d^2 \xi^A = -\frac{1}{L^2} \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B - \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B$$

$$d^2 \xi^A = -\frac{1}{L^2} \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B - \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B$$

$$d^2 \xi^A = -\frac{1}{L^2} \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B - \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B$$

$$d^2 \xi^A = -\frac{1}{L^2} \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B - \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\alpha \xi^B + \sum_{\alpha \beta} \theta^\alpha \partial_\beta \xi^B.$$
To evaluate the parameter $|\lambda\rangle$, take the total derivative of (37),
\[
|d^2\xi\rangle = d\left\{-\frac{1}{2}\hat{\omega}_c\theta^c|\xi\rangle\right\} = -\frac{1}{2}\{d\hat{\omega}_c\theta^c|\xi\rangle + \hat{\omega}_cd\theta^c|\xi\rangle + \hat{\omega}|d\xi\rangle\}. \tag{94}
\]
For the part $d\theta^c$, we have
\[
d\theta^c = d(e^c_\mu dx^\mu) = de^c_\mu dx^\mu + e^c_\mu d^2x^\mu = de^c_\mu dx^\mu - e^c_\mu \Gamma^\mu_{\alpha\beta}dx^\alpha dx^\beta + \frac{\lambda}{L}\theta^c + \langle\xi|\gamma^c|\lambda\rangle,
\]
where we substituted (92) into $d^2x^\mu$ in the second line. Then we obtain
\[
|d^2\xi\rangle = -\frac{1}{2}\left\{d\hat{\omega}_c\theta^c|\xi\rangle - \hat{\omega}_c\omega^c_{ab}\theta^a\theta^b|\xi\rangle + \frac{\lambda}{L}\hat{\omega}_c\theta^c|\xi\rangle + \langle\xi|\gamma^c|\lambda\rangle\hat{\omega}_c|\xi\rangle + \hat{\omega}|d\xi\rangle\right\}. \tag{96}
\]
The third term becomes $\frac{\lambda}{L}|d\xi|$ due to the equation (37). Comparing this and the equation (93), we obtain
\[
|\lambda\rangle = -\frac{1}{L^2}\eta_{ab}\theta^b de^c_\mu dx^\mu|d\xi\rangle + \frac{1}{2}d\hat{\omega}_c\theta^c|\xi\rangle - \frac{1}{2}\hat{\omega}_c\omega^c_{ab}\theta^a\theta^b|\xi\rangle + \frac{1}{2}\hat{\omega}|d\xi\rangle. \tag{97}
\]
Put this $|\lambda\rangle$ back into (92) and (93), and the result follows.

**Remark 3.** With (37) and (95), the equation (93) becomes
\[
d^2\xi^A = -\frac{1}{2}d\hat{\omega}^A_B\xi^B + \frac{1}{4}\hat{\omega}^A_B\omega^B_C\xi^C, \tag{98}
\]
and this is equivalent to
\[
D(D|\xi\rangle) = 0, \quad D|\xi\rangle := |d\xi\rangle + \frac{1}{2}\hat{\omega}|\xi\rangle. \tag{99}
\]
By the terminology of constrained systems, we can say that $C_A = 0$ is a second-class constraint, since the Lagrange multiplier $\lambda^A$ is determined, as suggested in [3, 4] for the flat case.

**V. DISCUSSION**

In this paper, we have newly defined a nonlinear connection on a super Finsler manifold and calculate it in the case of the Casalbuoni-Brink-Schwarz metric. We have expressed how
the equations of motion of the superparticle are rewritten in the form of the auto-parallel equations. Our explicit calculation displays the nonlinear connection truly plays a critical role in this process, though the last corollary indicates that the connection would become linear after exposed the constraint $C_A = 0$. This setup is fundamentally different from the one in Arnowitt-Nath [1] where only a linear connection is used. Considering the fact that our procedure is similar to Einstein’s approach to a relativistic particle in his theory of general relativity, we are on the right track to construct a theory of supergravity form a superparticle. The Casalbuoni-Brink-Schwarz model leads a theory of gravity for a matter with internal degrees of freedom. To prove it, we are now working on the derivation of the induced connection, Finsler curvature, and torsion on the constraints. For supergravity, the system with an additional Rarita-Schwinger field is underway as well. We also note that this method is applicable to any higher dimensional systems, which is remarkable because an explicit calculation of Finsler connection is difficult even in a 2-dimensional case.

ACKNOWLEDGMENTS

We thank Prof. L. Kozma, Prof. M. Morikawa and Prof. A. Sugamoto for valuable discussions.

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