Formation and Evolution of the Dust in Galaxies. III.  
The Disk of the Milky Way

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ABSTRACT

Context. Models of chemical evolution of galaxies including the dust are nowadays required to decipher the high-z universe. In a series of three papers we have tackled the problem and set a modern chemical evolution model in which the interstellar medium (ISM) is made of gas and dust. In the first paper (Piovan et al., 2011a) we revised the condensation coefficients for the elements that typically are present in the dust. In the second paper (Piovan et al., 2011b) we have implemented the dust into the Padova model with infall and radial flows and tested it against the observational data for the Solar Neighbourhood (SoNe). In this paper (the third of the series) we extend it to the whole Disk of the Milky Way (MW).

Aims. The Disk is used as a laboratory to analyze the spatial and temporal behaviour of (i) several dust grain families with the aid of which we can describe the ISM dust in a simple way, (ii) the abundances in the gas, dust, and total ISM of the elements present in the dust emitted by stars and the dust grains grown by accretion in the ISM, and finally (iii) the depletion of the same elements.

Methods. The temporal evolution of the dust and gas across the Galactic Disk is calculated under the effect of radial flows and a Bar in the central regions. The gradients of the abundances of C, N, O, Mg, Si, S, Ca and Fe in gas and dust across the Disk are derived as a function of time. 

Results. The theoretical gradients nicely reproduce those derived from Cepheids, OB stars, Red Giants and HII regions. This provides the backbone for the companion processes of dust formation and evolution across the Galactic Disk. We examine in detail the contributions to dust by AGB stars, SN\ae and grain accretion in the ISM at different galactocentric distances across the Disk. Furthermore, we examine the variation of the ratio between silicates and carbonaceous grains with time and position in the Disk. Finally, some hints about the depletion of the elements in regions of high and low SFR (inner and outer Disk) are presented.

Conclusions. The results obtained for the Disk of the MW make it possible to extend the model we have developed to other astrophysical situations such as galaxies of different morphological types, like ellipticals or star-busters, or even to a different theoretical modelling, like the chemo-dynamical N-Body simulations.

Key words. Galaxies - Dust; Galaxies – Spirals; Galaxies – Milky Way

1. Introduction

New generation, powerful telescopes and space instrumentation have unveiled a high-z universe surprisingly obscured by large amounts of dust, and consequently spurred a new generation of chemical and spectral galaxy models in which the formation, presence and evolution of dust is taken into account (See for instance Zhukovska et al. 2003, Calura et al. 2008, Valiante et al. 2009, Pipino et al. 2011, Gall et al. 2011a,b, Mattsson 2011, Valiante et al. 2011, Acharyya et al. 2011, Kemper et al. 2011, Dwek & Cherchneff 2011; to mention the most recent ones). Given the interest in understanding the properties of the early universe, most of these models are tuned toward QSOs or star-burst galaxies. They try to get clues about the role of dust in the early stages and to estimate how much dust can be produced both by stars and other processes in the interstellar medium (ISM) on a given timescale in order to match the amount of dust observed in high-z objects (Gall et al. 2011a,b, Mattsson 2011; Dwek & Cherchneff 2011). However, because of the huge distances, the information on dust to our disposal for these extremely far objects comes from (i) the extinction curve, that suggests clues about dust composition by the comparison with the one for galaxies in the local such as the SMC, LMC, and M31; (ii) estimates of the dust mass from the fluxes observed in the pass-bands to our disposal. Other pieces of information are typically related to the upper limit to the age of the target, its metallicity and star formation rate (SFR) and, finally, the total masses in form of stars, molecular gas (if CO observations are available), and Dark Matter (Valiante et al. 2011). Since we are observing the integrated properties of these galaxies, the history of chemical enrichment for individual elements and different distances from the centre of the galaxy are hardly available. A detailed information about the way in which the various elements behave is however required if we want to cross compare data with theoretical modelling of dust formation/evolution, because it would correlate the processes affecting the dust with the gas composition existing in that part of the galaxy we intend to model...
This indeed would be the typical case in which the relative proportion of silicates/carbonaceous grains and iron dust develop with time. In particular, we analyze the radial behaviour of (i) the radial and time evolution of many dust-related quantities, including chemical abundances, and abundance ratios. Furthermore, we present and discuss data for the gradients in gas content, chemical abundances, and abundance ratios. In particular, we analyze the radial behaviour of (i) some typical grain families; (ii) the elements involved in the dust formation, and (iii) the depletion in the innermost and outermost regions. Finally, in Sect. 6 we discuss the results and draw some general conclusions.

2. The chemical model

The model we are going to develop stems from the original infall model proposed long ago by Chiosi (1980) in the latest multi-ring formulation by Portinari & Chiosi (2000) and Portinari et al. (2004) for disk galaxies. This latter includes the radial flows of matter and the galactic bar that were introduced to reproduce the radial gradients and the density profile. The adopted stellar yields are the original ones by Portinari et al. (1998) in its latest version (Portinari 2010 - private communication). The model is able to describe the evolution of the abundances of the different refractory elements trapped into dust, properly simulating the process of injection of the star-dust into the ISM (Piovan et al. 2011a) and dust accretion/evaporation and destruction processes whose net balance determines the amount of dust in the ISM. The model traces the evolution of the abundance of some typical grain families supposed to represent the dust in the ISM and single of elements trapped into dust (Piovan et al. 2011b).

The Disk of the MW is subdivided in N concentric circular rings where \( r \) is the galacto-centric distance. Each ring or shell is identified by the mid radius \( r_k \) with \( k = 1, \ldots, N \). In most cases, radial flows of interstellar gas and dust are neglected, so that each ring / shell evolves independently from the others, but in our model the exchange of matter between contiguous shell is allowed. The Disk is described by the surface mass density as a function of the radial coordinate \( r \) and time \( t \); \( \sigma(r_k, t) \). Depending on the case, \( \sigma \) can refer to the ISM (\( \sigma^G \)), in turn split into dust or gas (\( \sigma^D \) or \( \sigma^G \) respectively), to the stars (\( \sigma^* \)) or to the total mass (simply \( \sigma \)). At every radius \( r_k \), the surface mass density is supposed to slowly grow by infall of either primordial or already enriched gas and to fetch at the present age \( t_c \) the mass density profile across the Galactic Disk for which an exponential profile is best suited to represent the surface mass density distribution: \( \sigma(r_k, t_c) \propto \exp(-r_k/r_d) \), where \( r_d \) is the scale radius of the Galactic Disk, that is typically estimated of the order of 4 – 5 Kpc. Since the final density profile is a priori known, one may normalize to it the current total surface mass density of the ISM. The evolution of the generic elemental species \( i \)-th in the dust at the radial distance \( r_k \) from the centre of the galaxy and at the time \( t \), is described as:

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (\nabla n_i) = 0 \]

1 Thereinafter, by star-dust we refer to the dust directly produced and emitted into the ISM, by AGB stars and supernovae (SNe) both of type I and type II.
where $\chi^D = \chi^D_i (r_k, t)$. The first term at the r.h.s. of eqn. (3) is the depletion of dust because of the star formation that consumes both gas and dust (assumed uniformly mixed in the ISM). The second term is the contribution by stellar winds from low mass stars to the enrichment of the $i$-th component of the dust. With respect to standard equations for the sole gas component, the condensation coefficients $\delta^D_{c,i}$ determine the fraction of material in stellar winds that goes into dust with respect to that in gas (local condensation of dust in the stellar outflow of low-intermediate mass stars). The third term is the contribution by stars not belonging to binary systems and not going into type II SNe (the same coefficients $\delta^D_{c,i}$ are used). The fourth term is the contribution by stars not belonging to binary systems, but going into type II SNe. The condensation efficiency in the ejecta of type II SNe are named as $\delta_U$. The fifth term is the contribution of massive stars going into type II SNe. The sixth and seventh term represent the contribution by the primary star of a binary system, distinguishing between those becoming type II SNe from those failing this stage and using in each situation the correct coefficients. The eighth term is the contribution of type Ia SNe, where the condensation coefficients are named as $\delta^I_{D}$, to describe the mass fraction of the ejecta going into dust. For all the $\delta_{c,i}$ coefficients we adopt the prescriptions presented in [Piovan et al. 2011a]. The last four terms describe: (1) the outflow of dust due to galactic winds (in the case of disk galaxies this term can be set to zero); (2) the radial flows of matter between contiguous shells; (3) the accretion term describing the accretion of grain onto bigger particles in cold clouds; (4) the destruction term taking into account the effect of the shocks of SNe on grains, obviously giving a negative contribution. The infall term in the case of dust can be neglected because we can assume that the primordial material entering the galaxy is made by gas only without a solid dust component mixed to it. Equations similar to eqns. (3) must be written for the ISM and gas. They are identified by the suffix $M$ (ISM) and $G$ (gas). These equations are given in detail in [Piovan et al. 2011b] and are shortly summarized below.

Indicating the contribution to the yields by stellar winds and type Ia and II SNe with the symbols $W_{c,D} (r_k, t)$, (and similar expressions for gas and ISM as a whole) and neglecting the outflow term, eqns. (3), the equations for the the gas and ISM are

$$\frac{d}{dt} M_i (r_k, t) = -\chi^A_i (r_k, t) \psi (r_k, t) + W_{c,M} (r_k, t) + \frac{d}{dt} M_i (r_k, t)_{rf}$$

$$\frac{d}{dt} D_i (r_k, t) = -\chi^D_i (r_k, t) \psi (r_k, t) + W_{c,G} (r_k, t) + \frac{d}{dt} D_i (r_k, t)_{rf}$$

$$\frac{d}{dt} G_i (r_k, t) = -\chi^G_i (r_k, t) \psi (r_k, t) + W_{c,G} (r_k, t) + \frac{d}{dt} G_i (r_k, t)_{rf}$$

It is soon evident that the dust creation/destruction and the radial flows make the system of differential equations more complicated than the original one by [Tielbörker & Arnett 1975] for a one-zone closed-box model. As the ISM is given by the sum of gas and dust, only two of these equations are required, furthermore Eqn. in the case of radial flows eqn. (2) can be used only if the gas and dust flow with the same velocity.

In the model by [Piovan et al. 2011b], many prescriptions/values for key quantities are explored. In brief we consider (i) nine cases for the IMF and five cases for the SFR, and for each SFR different specific efficiencies can be used; (ii) two possible models of accretion of dust grains in the cold regions of the ISM; (iii) different cases of star-dust formation in AGB stars, type II and type Ia SNe; (4) the age at which the effects induced by the Galactic Bar are taken into account. The presence of the Bar is mimicked by a change in the pattern of the velocities for the matter flowing from one shell to another [Portinari & Chiosi 2000]. The main motivation for the Bar is to reproduce the molecular ring at 4-5 Kpc in the gas distribution across the Galactic Disk. The velocity pattern is taken from [Portinari & Chiosi 2000] and is suitably chosen for every SFR and age $t_{Bar}$ at which the effect of the Bar is supposed to begin (the reader should refer [Portinari & Chiosi 2000] for all the details). In this paper, we keep constant the prescription for the velocity pattern (Bar) assigned by [Portinari & Chiosi 2000] to each case of SFR and age $t_{Bar}$.

Table I summarizes the many combinations of the parameters and associated models that are possible. For a thorough discussion of the effects induced by changing one
at a time the various parameters on the formation, destruction and evolution of (the Solar Neighborhood, in particular) the reader should refer to Piovan et al. (2011b). High number of parameters implies a high number of possible models. So a code must be found to establish a easy to read correspondence between the particular combination of parameters and the associated model. Each model is identified by a string of nine letters (the number of parameters) in italic face whose position in the string and alphabetic order corresponds to a particular parameter and choice for it. The code is presented in Table 1. The sequence should be read from top to bottom: for instance, the string DRAABABAB corresponds to Kroupa 1998 IMF, Schmidt SFR, ANN model for the fraction of molecular clouds in the ISM, $\chi_{MC}$, Dwek (1998) accretion model, Zhukovska et al. (2008) recipe for the yields of dust from type Ia SNe, Dwek (1998) condensation efficiencies for type II SNe, Ferrarotti & Gall (2000) condensation efficiencies for AGB stars, no effect of the Bar on the inner regions, and high efficiency $\nu$ of the SFR. Radial flows are always be included by default. We do not explore here the effects of different IMFs and SFR that have been already examined by Piovan et al. (2011b).

We focus here on the radial behaviour of dust formation and evolution and seek to follow the following questions. What are the dominant types of dust at different ages of the Galaxy and radial distances? Are the predicted radial gradient of element abundances in agreement with the observations in such a way to allow a correct calculation of dust formation on consistent initial conditions? What are the differences between the inner and outer regions of the Galaxy? To highlight these and other related issues we set up the so-called “reference model” in which we vary the key parameters. This reference model is identified by the string DRAABABAB and it includes the most refined recipes for the production of dust by stars and the accretion of dust in the ISM. It is worth noticing here the combination of the parameters we have chosen does not necessarily yield the model best fitting the observations across the Galactic Disk, but it only meant to provide the gross features on the physical processes for dust formation across the Disk and clues on the general behaviour of the system. Clearly, the model must be in satisfactory agreement with the observational data, even though it does not perfectly match it.

### 3. Modelling radial flows

A key term in the description of the properties of gas and dust along the Disk is the one of radial flow in eqn. driving the exchange of gas/dust between contiguous shells. The radial flows are described according to the model developed by Portinari & Chiosi (2000) however extended to a two components fluid made of gas and dust each of which in principle moving with its own velocity. Recalling that each circular ring/shell is identified by the mid-point radius $r_\kappa$ and the inner and outer borders by the radii $r_\kappa - \frac{1}{2}$ and $r_\kappa + \frac{1}{2}$, the material can flow across the inner and outer borders with velocity $v_k - \frac{1}{2}$ and $v_k + \frac{1}{2}$, respectively. Flow velocities are taken positive outward and negative inward. Let denote with $F_D(r)$ the radial flow of dust and $F_G(r)$ that of gas, so that the total flux of ISM is $F_M(r) = F_D(r) + F_G(r)$. The motion of this two-components fluid alter the gas surface density in the $k$-th shell. The most convenient description of it is as follows: If there is no velocity drift between gas and dust, as we will later assume, it is more convenient to follow the ISM and the dust, whereas for different gas and dust velocities it is more convenient to follow dust and gas as two independent fluids. Therefore we have:

$$\frac{[d\Omega^2_x(k)]}{dt}_{rf} = \frac{1}{\pi} \left( \frac{r_{k+1}^2 - r_{k-1}^2}{r_k^2} \right) \Delta F_C(r_k)$$

where $C = M, G, D$ (C is for component) means that we have the same formal expression for ISM ($M$), gas ($G$) and dust ($D$). The term $\Delta$ is $F_C(r_k) = F_C(r_{k+\frac{1}{2}}) - F_C(r_{k-\frac{1}{2}})$. The flux across the external border of the ring/shell $r_{k+\frac{1}{2}}$ is:

$$F_C(r_{k+\frac{1}{2}}) = 2\pi r_{k+\frac{1}{2}} v_k [\chi(r_{k+\frac{1}{2}}) \sigma_k^2 + \chi(-v_k c^2_{k+\frac{1}{2}}) \sigma_{k+1}^2]$$

where $\chi(x)$ is the step function: $\chi(x) = 1$ or 0 for $x > 0$ or $\leq 0$, respectively. Equ. (6) are a sort of “upwind approximation” for the advection term to be included in

### Table 1. Parameters of the models. Column (1) is the parameter number, column (2) the associated physical quantity, and column (3) the source and the italic symbols are the identification code we have adopted. See the text for some further information and Piovan et al. (2011b) for a detailed description of each choice.

| n     | Parameter | Source and identification label |
|-------|-----------|----------------------------------|
| 1     | IMF       | Salpeter (A), Larson (B), Kennicutt (C) Kroupa orig. (D), Chabrier (E), Arimoto (F), Kroupa 2007 (G), Scalo (H), Larson SoNe (I) |
| 2     | SFR law   | Constant SFR (A), Schmidt (B), Talbot & Arnett (C), Dopita & Ryder (D), Wyse & Silk (E) |
| 3     | $\chi_{MC}$ model | Artificial Neural Networks model (A), Constant $\chi_{MC}$ as in the Solar Neigh. (B) |
| 4     | Accr. model | Modified Dwek (1998) and Calura et al. (2008) (A); adapted Zhukovska et al. (2008) model (B) |
| 5     | SNe Ia model | Dust injection adapted from: Dwek (1998), Calura et al. (2008) (A), Zhukovska et al. (2008) (B) |
| 6     | SNe II model | Dust injection adapted from: Dwek (1998) (A), Zhukovska et al. (2008) (B), Nozawa et al. (2003, 2004, 2007) (C) |
| 7     | AGB model | Dust injection adapted from: Dwek (1998) (A), Ferrarotti & Gall (2000) (B) |
| 8     | Galactic Bar | No onset (A), onset at $t_{G} - 4$ Gyr (B), onset at $t_{G} - 1$ Gyr (C) |
| 9     | Efficiency SFR | Low efficiency (A), medium efficiency (B), high efficiency (C) |

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the model equations, describing either inflow or outflow depending on the sign of the velocity $v_{k+\frac{1}{2}}$. Clearly, since there are two components, gas and dust, it may happen that $v_{k+\frac{1}{2}} \neq v_{d,k+\frac{1}{2}}$. The simplest case is for $v_{k+\frac{1}{2}} = v_{d,k+\frac{1}{2}} = v_{k+\frac{1}{2}}$, i.e., when gas and dust are coupled and move along the same direction. Analogous expression holds for the flux $F_{C}(r_{k-\frac{1}{2}})$ across the inner border of the ring/shell.

Let’s take the inner edge $r_{k-\frac{1}{2}}$ at the midpoint between $r_{k-1}$ and $r_{k}$, and similarly for the outer edge $r_{k+\frac{1}{2}}$. Writing Eqn. [3] separately for each chemical species $i$ and as function of $G_{i}$‘s, we may write the radial flow term in eqn. [3] and similar ones for other components as follows:

$$\frac{d}{dt}C_i(r_{k-1}, t) = \alpha_i C_i(r_{k-1}, t) - \beta_i C_i(r_{k-1}, t) + \gamma_i C_i(r_{k+1}, t) + \delta_i C_i(r_{k+1}, t)$$

where again $C = M/G/D$ and:

$$\alpha_i = \frac{2}{r_{k} + \frac{r_{k-1} + r_{k+1}}{2}} \left( \chi C - \frac{r_{k-1} + r_{k}}{r_{k+1} - r_{k-1}} \right)$$

$$\beta_i = \frac{2}{r_{k} + \frac{r_{k-1} + r_{k+1}}{2}} \left( \chi C - \frac{r_{k+1} + r_{k}}{r_{k+1} - r_{k-1}} \right)$$

$$\gamma_i = - \frac{2}{r_{k} + \frac{r_{k-1} + r_{k+1}}{2}} \left( \chi C - \frac{r_{k} + r_{k+1}}{r_{k+1} - r_{k-1}} \right)$$

The terms at the r.h.s. of the eqn. [7] describe the contribution of the 3 contiguous shells to driving the matter flows: the first term represents the gas being gained by shell $k$ from the shell $k-1$, the second term is the gas lost by the shell $k$ towards the shells $k-1$ and $k+1$, and the third term is the gas acquired by the shell $k$ from the shell $k+1$ [Portinari & Chiosi 2000]. The quantities $\sigma_{A(k-1)}$, $\sigma_{Ah}$, and $\sigma_{A(k+1)}$ are compact notations for the present-day surface mass density profile $\sigma(r_{j}, t_{C}) \equiv \sigma_{j}(r_{j})$ where $j = k-1, k, k+1$, and $A$ stands for accreted matter. The coefficients given by Eqns. [3] are all $\geq 0$ and depend only on the shell $k$, not on the chemical species $i$ considered. They will be different for gas and dust if a drift between gas and dust is considered. If there is no drift, then the $M$ equation can be used together with one between dust and gas equations. If the velocity pattern is constant in time, $\alpha_i$, $\beta_i$, and $\gamma_i$ are also constant in time. There is an important point to note concerning the present-day surface mass density profile. In the case of static models (no radial flows) the present-day surface mass density is known a \textit{a priori} and is determined by the mass accretion law one has assumed, i.e., $\sigma(r_{j}, t_{C}) \equiv \sigma_{j}(r_{j})$. In other words, in static models the radial profile for accretion can be directly chosen so as to match the observed present–day surface density in the Disk [Portinari et al. 1998; Portinari & Chiosi 1999].

The inclusion of radial gas flows changes the expected final density profile (as it is now set up by infall and radial flows). Therefore, $\sigma(r_{k}, t_{C}) \neq \sigma_{j}(r_{j})$ and $\sigma(r, t_{C})$ cannot be known in advance. It is known indeed only a \textit{posteriori} (see Portinari & Chiosi 2000, for more details). At the end of each simulation we need to check how much radial flows have altered the actual density profile $\sigma(r_{k}, t_{C})$ with respect to the pure accretion profile $\sigma_{A}(r_{k})$. In any case, if the flow speeds are small (of the order of $v \sim 1 \text{ km sec}^{-1}$), the two profiles are expected to be close each other.

### 3.1. Boundary conditions

Eqns. [7] need to be modified for the innermost and the outermost shells, since for these regions at shells $k-1$ or $k+1$ are not defined.

#### 3.1.1. The innermost shell

Our model is limited to the Disk of the MW and cannot follow the innermost regions of the Galaxy where the Bulge dominates. A correct description of this region would require a more complicated model: a spherical component simulating the Bulge combined with the Disk each of which with its own mass distribution and formation history. Therefore we truncate the Disk to a innermost boundary (ring) located where the Bulge stars dominating, i.e. at about $r_{1} = 2 \sim 2.5$ kpc. At this innermost boundary, the first shell is taken symmetric with respect to $r_{1}$ and its innermost radius is

$$r_{\frac{1}{2}} = \frac{3r_{1} - r_{2}}{2}$$

At this layer we impose that always $v_{1/2} \leq 0$, therefore $k=1$, and Eqn. [7] become

$$\frac{d}{dt}C_{i}(r_{1}, t) = -\beta_{i} C_{i}(r_{1}, t) + \gamma_{i} C_{i}(r_{2}, t)$$

with:

$$\beta_{i} = -\frac{1}{2r_{1}} \left[ v_{\frac{3}{2}} \frac{3r_{1} - r_{2}}{2} - \chi(v_{\frac{3}{2}}) v_{\frac{1}{2}} \frac{r_{1} + r_{2}}{2} \right]$$

$$\gamma_{i} = -\chi(v_{\frac{3}{2}}) v_{\frac{1}{2}} \frac{r_{1} + r_{2}}{2} \frac{r_{1} + r_{2}}{2} r_{1} r_{2} - \sigma_{A} \sigma_{B}$$

#### 3.1.2. Boundary conditions at the Disk edge

The Galactic Disk is conceived as made of a inner region, where star formation and chemical enrichment occurs and stars gas and dust exist, and a external region in which only gas is present, no stars can be formed and no chemical enrichment can take place. This picture is supported both by observational data showing that in external spirals HI disks are observed to extend much beyond the optical disk and theoretical considerations about stability preventing SF beyond a certain radius (Toomre 1964; Quirk 1972).

The part of the Disk we are interested in is the first one, which may extend up to about 20–22 kpc. Therefore the mid radius of the most external shell ($k = N$) can be located at $r_{N} = 20 – 22$ kpc. all the regions belonging to the Disk external to this limit can be considered as a large reservoir of gas with primordial composition from which gas can flow inside. At the present time when the gravitational settling of the proto-galactic cloud is over, the radial flow of gas from this external region of Disk can be even more efficient than the classical infall mechanism. If no star formation occurs, the evolution of the gas surface density in the
outer disk \( \forall r > r_{N+1/2} \) can be described by \( \text{(Lacey & Fall } 1985; \text{Portinari & Chiosi } 2000) \):

\[
\frac{\partial \rho [G(r,t)]}{\partial t} = A(r) e^{-\frac{r}{\tau(r)}} - \frac{1}{r} \frac{\partial}{\partial r} [r v \cdot G(r,t)] \tag{12}
\]

Without star formation, there is no chemical evolution and the pattern of abundances always remain the primordial one \( (X_{i,inf}) \). We assume that this primordial material is also dust-free, that is \( D(r,t) = 0 \) for \( r > r_{N+\frac{1}{2}} \). As a consequence of it, the normalized surface density of the outer Disk is given only by \( \mathcal{M}(r,t) = G(r,t) \) for \( r > r_{N+\frac{1}{2}} \) and this component is the only one to be considered in Eqn. (12). Other simplifying conditions for the the most external part of the Disk have been discussed by \( \text{Portinari & Chiosi } 2000 \). They all apply for \( r > r_{N+\frac{1}{2}} \). In brief, the infall time-scale is uniform \( \tau(r) = \tau(r_N) \); the inflow velocity is uniform and constant \( v(r,t) \equiv v_{N+\frac{1}{2}} \) and the infall profile is flat \( A(r) \equiv A_{ext} \) in agreement with the observed gas discs in spirals. With these assumptions, Eqn. (12) becomes:

\[
\frac{\partial \rho [G(r,t)]}{\partial t} + v \frac{\partial \rho [G(r,t)]}{\partial r} = A e^{-\frac{r}{\tau}} - \frac{v}{\tau} \tag{13}
\]

where \( \tau \equiv \tau(r_N) \), \( v \equiv v_{N+\frac{1}{2}} \), and \( A \equiv A_{ext} \) to simplify the notation. Eqn. (13) has a straightforward analytical solution \( \text{(Portinari & Chiosi } 2000) \):

\[
\sigma (r,t) = A \times \left( 1 - e^{-\frac{r}{\tau}} + \frac{v}{\tau} \left( e^{-\frac{r}{\tau} + e^{-\frac{r}{\tau}}} - (t-T_r) \right) \right) \tag{14}
\]

where \( T_r \geq 0 \) is the time when radial inflows are assumed to start. Eqn. (14) is our boundary condition at the outermost edge.

Notice that relation (14) is the solution of Eqn. (13) in the idealized case of an infinite, flat gas layer extending boundless to any \( r > r_N \) (see also Appendix B). Of course, this does not correspond to gaseous disks surrounding real spirals. However, as we are considering only slow inflow velocities \( (v \sim 1 \text{ km sec}^{-1}) \), with typical values of \( r_N = 20 - 22 \text{ kpc} \) and \( t_C = 12.8 \text{ Gyr} \), the gas actually drifting into the inner disk shells will be just the gas originally accreted from regions within \( r \sim 35 - 40 \text{ kpc} \). Therefore, the boundary condition (14) remains valid as long as the gas layer extends out to \( r \sim 35 - 40 \text{ kpc} \), a very plausible assumption since observed gaseous disks extend over a few tens or even ~ 100 kpc.

3.1.3. The outermost shell

We take a reference external radius \( r_{ext} > r_N \) in the outer disc where the (total and gas) surface density \( \sigma(r_{ext},t) \equiv \sigma_{ext}(t) \) is given by the boundary condition (14): typically, \( r_N = 22 \text{ kpc} \) and \( r_{ext} \sim 24 \text{ kpc} \). We take the outer edge of the shell at the midpoint:

\[
r_{N+\frac{1}{2}} = (r_N + r_{ext})/2 \tag{15}
\]

The radial flow term for the \( N \)-th shell is:

\[
\left[ \frac{d}{dt} C_i(r_N,t) \right]_{rf} = \alpha_N^C C_i(r_{N-1},t) - \beta_N^C C_i(r_N,t) + \omega_N^C(t) \tag{16}
\]

where:

\[
\omega_N^C(t) = -X_{i,inf} \chi(-v_{N+\frac{1}{2}}) \frac{v_N^C}{v_{N+\frac{1}{2}}} \times \frac{2}{r_{N-1} + 2N + 2 \tau} \sigma_{ext}(t) \sigma_N \tag{17}
\]

for \( \mathcal{M} \) and \( G \), while for dust \( \omega_N^C(t) = 0 \).

3.2. The numerical solution

Using Eqns. (7), (10) and (14), neglecting for the moment the terms for dust accretion and destruction, the basic set of Eqns. (2), (3) and (4) can be written as:

\[
\frac{d}{dt} C_i(r_N,t) = \alpha_N^C C_i(r_{N-1},t) + \beta_N^C C_i(r_N,t) + \gamma_N^C C_i(r_N,t) + W_i C_i(r_N,t) \tag{18}
\]

where we have introduced:

\[
\omega_C^C(t) = - (\eta_C^C(r_N,t) + \beta_N^C) \leq 0 \tag{19}
\]

\[
\eta_C^C(r_N,t) \equiv \frac{\Psi}{\tau(r_N,t)} \tag{20}
\]

The terms \( W_i C_i(r_N,t) \) are defined as in Eqns. (2), (3) and (4). We refer to \( \text{Portinari et al. } 1998 \) for further details on the quantity \( \eta \). Neglecting, for the time being, that \( \eta \)'s and \( W_i \)'s contain the \( C_i \)'s themselves, we are dealing with a linear, first order, non homogeneous system of differential equations with non constant coefficients, of the kind:

\[
\frac{d C_i}{dt} = A(t) C_i(t) + W_i(t) \tag{21}
\]

There is a system like (21) for each chemical species \( i \), but the matrix of the coefficients \( A(t) \) is independent of \( i \). The way by which this system is solved and then dust evolution is calculated depends on the specific hypotheses made for gas and dust.

(1) Radial flows are neglected but dust is included. Since the radial flows are not started from the beginning of the evolution of the galaxy, this case usually happens in the early stages of the evolution of the MW and it lasts some Gyrs \( \text{(Portinari & Chiosi } 2000) \). In this case the evolution of the \( \mathcal{M} \) and \( D \) components should be followed. The classical static system for the ISM in which the different rings evolve independently \( \text{(2) (without flowing term) is solved according to the classical Talbot & Arnett technique (Portinari et al. } 1998) \). Once this system is solved the accretion/destruction terms for dust can be taken into account (see \( \text{Piovan et al. } 2011b \)) is solved to find the evolution of the dust. The presence of the accretion/destruction terms does no longer allow to use the standard resolving method and the new system of ordinary differential equations must be integrated with a different technique. We adopt a combination of the ODEINT routine (4-th order Runge-Kutta) with controlled step-size \( \text{(Press et al. } 1992) \) and the powerful DOP853 Runge-Kutta routine (method of Dormand...
& Prince) of order 8 with a 5th order estimator and a 3rd order correction (Hairer et al. 2010a, b).

(2) Radial flows and dust are both included. The equations to be solved depend on whether or not some velocity drift exists between dust and gas. The simplest and most widely adopted hypothesis is that not only dust is homogeneously mixed to the gas, but both radially flow at the same velocity. In this case, the two systems of equations to be solved are (2) and (3). The system (2) is reduced to the form (18) by including the radial flows and solved according to the technique developed by Portinari & Chiosi (2000) to whom the reader should refer for all details. Once the evolution of the ISM is computed, Eqn. (3) for dust is setup by means of accretion/destruction terms (Piovan et al. 2011b). The system of ordinary differential equations is now solved with the ODEINT and DOP853 integrators. The reason to calculate first the evolution for the ISM is that the equations for the accretion/destruction processes regulating the dust evolution include the total amount of interstellar matter as an ingredient.

(3) Velocity drift between the ISM components. In such a case, the global system of differential equations, now including a gas and dust together, made by Eqns. (4) and (3), with the radial flow term as in Eqn. (3) and the accretion/destruction terms for dust must be solved. In such a case it is not possible to obtain a priori the evolution of the ISM. Again the ODEINT and DOP853 integrators are used to evolve the system as a function of time.

Some general considerations can be made: rather small time-steps are needed to keep stable the numerical model; the required time-steps get smaller and smaller at increasing the flow velocities and making thinner the shells. The integrators of ordinary differential equations self-regulate the internal time-step according to the required precision once the external time-step is chosen. To describe the gas flows in a disk with an exponential density profile, the shells are equi-spaced in the logarithmic, rather than linear, scale (so that they roughly have the same mass, rather than the same width). We model the Galactic Disk using 33 shells from 2 to 21 kpc, equally spaced in the logarithmic scale; their width ranges from \(\sim 0.15\) kpc for the inner shells to \(\sim 1-1.5\) kpc for the outermost ones. With such a grid spacing, and velocities up to \(\sim 1\) km sec\(^{-1}\), suitable time-steps are of \(10^{-4}\) Gyr. This means that roughly 1.5 x \(10^5\) time-steps, times 35 shells, are needed to complete each model, which would translate in excessive computational times. This drawback is avoided by separating the time-scales in the code following the suggestion by Portinari & Chiosi (2000).

4. The Radial variation of the fraction of MCs

The accretion of dust in the MW Disk can be simulated in our model by means of two possible choices: (1) a simple model that adopts typical accretion timescales to simply determine the amount of each element embedded into dust; (2) a more refined model with variable timescales that follows the accretion of dust grains in some typical grain families inside the cold regions of the Disk. This latter description is the one included in our reference model and is adopted here to examine the radial behaviour of dust formation and evolution. Since in our one-phase chemical model no description is available for a multi-phase ISM, it is unavoidable to estimate from time to time (possibly related to the physical properties of the system) the amount mass in cold molecular clouds (MCs) in each ring. The partition between MCs and diffuse ISM (the remaining part of the ISM) crucially enters the accretion term in eqns. (2), (3) and (4) and drives the evolution of the dust budget. In this formulation the fraction, \(\chi_{MC}\), of ISM locked up in MCs is a parameter. For this reason we have included two possibilities for \(\chi_{MC}\). In the first one, from the present data for the Solar Vicinity, \(\chi_{MC}\) is pre-
network in usage here is taken from Grassi et al. (2011) to whom the reader should refer for all details. It contains two layers of hidden neurons whose number is \( n_h = 5 \) and \( n_g = 25 \), respectively and has two input neurons, i.e. \((\text{SFR}/\text{SFR}_\odot, \text{SFR}/\text{SFR}_\odot, \text{SFR})\) (plus the bias input that is always set to a constant value) and one output neuron, i.e. \( \sigma_{H_2}(r_t, t_G) \). The evolution of the RMS is monotonic and decreases to approximately \( \text{RMS} = 10^{-3} \) after \( 10^6 \) iterations. In the left panel of Fig. 2 we show the fit found by the ANNs (solid line) compared with the data (crosses). We suppose now that the correlation between the three parameters established from the present-day data remains the same as a function of the Galaxy age and therefore we drop the dependence on time. Furthermore, adopting the same ANN weights, we extrapolate the ANN predictions to the whole space of the parameters \( \text{SFR}/\text{SFR}_\odot, \sigma^2(r_t, t) \) and \( \sigma_{H_2}(r, t) \). The contours and isolines describing the extrapolated regions are shown in left and middle panels of Fig. 2.

Some comments are worth being made here. First, we have no warranty that the behaviour in the extrapolated region is so regular as predicted by the ANNs. In the huge gaps uncovered by the experimental data for the MW, we could for example have valleys and peaks so that to cast light on this problem a complete coverage of the parameter space with observational data for spiral galaxies in general would be required. However, this goes beyond the aims of this study. It suffices here to check that extrapolating the MW data does not lead to un-physical results. The predictions seem to be reasonable: for low SFR or low gas amount, the \( H_2 \) mass is low. For high SFR and gas mass we have a reasonable amount of \( H_2 \). Even if we look only at the region where the SFR is high but the total amount of gas is low, the prediction for \( H_2 \) is also reasonable. Even if the correlation between \( H_2 \) and SFR is not clear and somewhat in contradictions with the expectations, stars are seen to form from the cold molecular gas (Kennicutt 1998) and a significant amount of \( H_2 \) is physically acceptable when SFR is significant. Second, the region covered by the ANN is bounded by the limits \([\text{SFR}/\text{SFR}_\odot, \text{SFR}/\text{SFR}_\odot, \text{SFR}]\) and \([\sigma^2(r_t, t)_{\text{min}}, \sigma^2(r_t, t)_{\text{max}}]\). This may be a problem in particular with high SFRs, because the limits for the ANN have been set basing upon data for the SFR in the innermost regions of the Disk where the uncertainty is high. To cope with this point of embarrassment, when the SFR is higher than about \( 3 \times \text{SFR}_\odot \), we replace the predictions based on the ANNs with a simpler fit based on \( \sigma_{H_2}(r, t_G) \) versus SFR data and we keep the fraction \( \chi_{MC} \) in the interval \( 0.4 - 0.5 \). All of this is shown in the right panel of Fig. 2.

5. Radial gradients in element abundances

Studying the properties of dust in the SoNe is by far easier than across the Disk of the MW: indeed the data to our disposal for the Disk are much fewer and of poorer quality than those for the SoNe. For instance, we cannot estimate the depletion of elements at different distances from the SoNe toward the Galactic Center and/or the outermost regions of the Disk. However, we can compare the radial gradients in element abundances, as indicated by Cepheid stars, Open Clusters, HII regions and O,B stars. In the
case of a satisfactory agreement between the observations and theoretical predictions for the element abundances, we may argue with some confidence that the predictions for radial variation of the dust properties are reasonably good. In relation to this see the case of the relative abundances of Mg and Si and how they would affect the formation of silicates in the dust and depletion of these elements as thoroughly discussed by Piovan et al. (2011b).

In the following, we examine the radial variations of the elemental abundances to check if our multi-shells model, in presence of radial flows of gas and inner bar, is able to reproduce in a satisfactory way at least the main radial properties of the MW. As already said we are not looking for the best model in absolute sense but only the a general agreement with observational data so that our predictions for dust formation and evolution stand on solid ground.

Two important observational constraints must be satisfied and reproduced by the models, namely: (i) the radial distribution of the gas and (ii) the radial gradients in those elements that are involved in the dust formation. In Fig. 3 starting from the reference model introduced in Sect. 2, we compare the density profiles for models calculated with different IMF, efficiency of the star formation $\nu$ and the infall timescale $\tau$. Twelve models are shown. This sample is taken from the wider grid of models calculated by Piovan et al. (2011b) to reproduce the depletion of the elements observed in the SoNe.

In Fig. 2, the efficiency is (0.3 and 0.7). Not all the combinations have been represented, but only the most significant ones. Data for the SoNe are taken from Dame (1993) and the hatched area represents the confidence region taking into account the errors in the observational estimates. Data for the SoNe gas mass density are from Dickey (1993); Dame (1993) and Olling & Merrifield (2001).

The whole sample contains model for four values of the infall timescale $\nu$ ($\nu=0.3$ and $\nu=0.7$), three IMFs (Salpeter, Kroupa multi-power law and Larson adapted to the SN), four values of the initial mass function  $\nu$ ($\nu=0.3$ and $\nu=0.7$), three IMFs (Salpeter, Kroupa multi-power law and Larson adapted to the SN), four values of the infall timescale $\tau$ ($\tau=3$, $\tau=6$ and $\tau=9$ Gyr), the efficiency $\nu$ (0.3 and 0.7). Not all the combinations have been represented, but only the most significant ones. Data for the SoNe are taken from Dame (1993) and the hatched area represents the confidence region taking into account the errors in the observational estimates. Data for the SoNe gas mass density are from Dickey (1993); Dame (1993) and Olling & Merrifield (2001).
twelve models are shown in Fig. 3. They are with original stellar yields by Portinari et al. [1998] (no correction to the Mg abundance is applied). Since we have no data for the element depletion as a function of the radial distance, the slight under-abundance of Mg is less of a problem here. Furthermore, selecting the twelve cases, very short infall timescales have been excluded because the models would not reproduce the element depletions observed in the SoNe. Finally, since we have no hints on the observational error associated to the mass density profile \( \sigma_g(r) \) as a function of the galacto-centric distance, at each distance we apply an error derived from the ratio \( \Delta \sigma_g/\sigma_g \) for the SoNe, and assumed to remain constant. Indeed, already in the SoNe, there is a significant uncertainty in the local gas density, the estimates of which go from 7-13 \( M_\odot \) pc\(^{-2}\) (Dickey 1993), to \( \sim 8 \) [Dame 1993] up to 13-14 [Olling & Merrifield 2001].

It is soon evident that while some models fairly agree with the observational data for \( \sigma_g \), others significantly deviate from it in particular for the innermost regions: models with long infall timescale and low star formation efficiency (such as the model with \( \tau = 9 \) Gyr and \( \nu = 0.3 \)) do not consume enough gas. In literature the problem is often solved either introducing a suitable radial dependence for \( \tau \) or changing the IMF in such a way that the low-mass long-lived stars are favored. However, for the purposes of the present study, we prefer to keep both \( \tau \) and IMF constant with the galacto-centric distance. Finally, it is worth noting the crucial contribution of the bar whose presence nicely reproduces the gas density distribution in the innermost regions, simulating the bump at about 5 kpc.

In Fig. 4 we show the gradients across the Galactic Disk in the abundance ratios \([\text{Mg}/\text{H}], [\text{Fe}/\text{H}], [\text{C}/\text{H}], [\text{Si}/\text{H}], [\text{O}/\text{H}], [\text{S}/\text{H}], [\text{Ca}/\text{H}] \) and \([\text{N}/\text{H}]\). All these elements are also involved in dust formation and evolution. For each element we group the observational data according to the source of the abundances, namely O and B stars, field Red Giants (RGs), HII regions, Cepheid variables and open clusters. The references for the observational data are listed in the footnote\(^7\). Only models with intermediate values of \( \tau (1 < \tau < 9) \) are compared with the data. Since most likely different sources trace the gradients at different galactic ages, we plot the gradients as they were at the time the Sun was born (about 4.5 Gyr ago, the thin blue and black lines) and at the present age (thick red and magenta lines). Most objects under examination should fall into the region spanned by the gradients during this time interval. In general, theoretical predictions and data fairly agree with minor discrepancies that can be, in some cases, attributed to the yields, such as the slight Mg under-abundance and/or C and S over-abundance [Portinari et al. 1998]. Other major discrepancies between the theoretical and observational abundance gradients that depend on the type of source under consideration are expected and can be taken to indicate the performance of the models. For instance, following Kovtyukh et al. [2002], in intermediate mass stars the C under-abundance (carbon is deficient at about \([\text{C}/\text{H}] = -0.4\)) and the N over-abundance (nitrogen is enhanced at about \([\text{N}/\text{H}] = +0.4\)) has been predicted by Luck [1978] and Luck & Lambert [1981] and confirmed in studies of Cepheids and non-variable super-giants [Andrievsky et al. 1996; Kovtyukh et al. 1996]. This is more or less what we observe when the evolution of the theoretical element abundances C and N along the Disk are superposed to the measurements of C and N abundances in Cepheids (intermediate mass stars of 3 to 9 M_\odot). Finally, comparing theory and data, one should keep in mind that models do not take star dynamics and kinematics into account and the stars always remain in the same region where they are born: the radial flows act only on gas and dust.

Radial gradients in dust. Assessed that the radial gradients, in the gaseous element abundances and total gaseous density, reasonably agree with the observational data, at least for some combinations of the input parameters, we examine the radial formation and evolution of dust in the MW Disk being confident that it is grounded on a realistic evolution of the global abundances in the ISM. In Fig. 5 we show for five selected ages, from the early stages to the current time, the time evolution of the radial abundance of four typical grain families, that is silicates, carbonaceous grains, iron grains and generic grains formed with Ca, N and S. We grouped these last three elements into a single group for the sake of simplicity because of their low abundance. The contributions to each one of the four groups have been split among the three sources of dust, that is star-dust from SNae and AGB stars [Piovan et al. 2011a] and accretion in the cold regions of the ISM [Piovan et al. 2011b]. The amount of dust in all the regions continuously grows following the enrichment in metals produced by the many generations of stars. In the early stages of the evolution, at 0.1 Gyr, star-dust from SNae is the main source of dust across the whole disk. At 0.5 Gyr the accretion of dust in the ISM starts to be significant in the innermost regions and slowly becomes the dominating source in the whole Disk, earlier for the innermost regions and later for the outermost ones. AGB stars contribute mostly with carbon-rich grains in the early evolution at low metallicity Z and only when a significant metallicity is reached they start enriching the ISM in silicates. It is worth noting here that the evolutionary scheme across the Disk we have just outlined is a consequence of the role played by the star-dust from SNae in the early stages. Reducing the yields of dust by SNae would obviously change the the above results for the early stages.

In Fig. 6 we present the evolution with time of the radial dependence of the logarithm of the dust abundance \( \sigma_d^D (r_k, t) \) for all the elements in our list taking part to dust formation, namely C, N, O, Mg, Si, S, Ca and Fe. Nine ages are represented from the early stages to the current time. All the elements more or less follow the same evolutionary trend, and eventually reach a typical profile with a bump at 4-5 kpc in agreement with the gas density profile (as expected). It is interesting to note that while for the innermost-central rings of the disk the enrichment in dust already reaches a maximum at about 5 Gyr, for the outermost regions the dust budget keeps growing until the present age. The reason for that can be easily
attributed to the delayed contribution by accretion in the ISM occurring in the outer regions with low star formation (see [Piovan et al. 2011b, for more details]. In brief, as accretion of dust grains in the ISM gets important only when and if some enrichment in metals has taken place, in the outer regions of low star formation this is inhibited by the low metal content that can be reached over there.

It is interesting to examine not only the abundance evolution of each element in the dust, but also the relative abundance of each element embedded into dust (or the relative fraction of each dust grain family) with respect to the total dust budget. For this reason in Fig. 7 we show the evolution of the radial mass fraction of each element embedded into dust \( \sigma_i(r_k, t) / \sigma_{D}(r_k, t) \), normalized to the total dust budget, for some of the elements in our list and/or some families of grains. In particular, we show the fraction of silicates (which means Mg, Si, Fe and O embedded into quartz/pyroxenes/olivines by accretion in the ISM or star-dust) and carbonaceous grains (this latter is nearly coincides with the amount of C, once subtracted the amount of C belonging to the silicon carbide). Let us examine the behaviour of silicates and carbonaceous grains in some detail. At the beginning of the evolution, where only the most massive SNe contribute, the mass of dust in silicates is a bit larger than in carbon based grains. This is in agreement with the recent observations of SN 1987A by the Herschel Space Observatory [Matsunaga et al. 2011]; the significant emission by a population of cold dust grains observed by PACS and SPIRE can be reproduced only with a mixture of both carbonaceous grains and silicates with these latter more abundant than the C-based dust grains. This is indeed what is shown in Fig. 5 at the age of 0.1 Gyr, when SNe play the major role. As soon as AGB stars start to contribute (at low/sub-solar metallicities a 5-6 solar masses has a lifetime of the order of 0.1 Gyr) the mixture of SNe and low-Z AGB stars slowly increases the fraction of carbonaceous grains and reduces that of silicates. Indeed, low-Z AGB stars are easily C-star and mainly inject carbon in the ISM. At ages in the range 0.5-1 Gyr, at least in the innermost regions, the formation of dust grains in the cold ISM starts to be a very efficient process and strongly influences the total budget. Our silicates are made by Mg, Si, O and Fe; the timescale for the formation of silicates in the early evolution, when SNe enriched the ISM with all of these elements, is therefore shorter than the one for carbonaceous grains (carbon is also partially locked in the unreactive CO).

Carbon is injected by SNe (plus a significant contribution by low-Z AGB stars), but in any case carbon is less than the sum of all the other refractory elements involved in silicates. Their total number density in the ISM, compared to the one of carbon atoms, favors the formation of silicates during the early evolution. This can be seen in the upper right panel of Fig. 5: the fraction of carbon locked up in CO also plays an important role. In the panels of Fig. 7 for the carbonaceous grains and silicates, one can note that, as a consequence of the behaviour described above, the fraction of silicates in the innermost regions grows until the age of about 1 Gyr is reached, while the fraction of C-rich dust decreases. Afterwards, as AGB stars join SNe to increase the number density of Carbon atoms in the ISM, the accretion of carbon in cold MCs becomes also efficient, and the silicates/carbonaceous grains ratio slowly decreases again. A different behaviour takes place in the outermost regions: at the beginning there is a ratio similar to the inner regions, but the fraction silicates/carbonaceous grains keeps decreasing because (i) the accretion in the ISM has long timescales and starts late due to the low densities and Z; (ii) the strong contribution of C-based dust, due to the low-Z AGB stars, dominates in a significant window between SNe and start of the ISM accretion. The final result is the high fraction of carbon dust with respect to silicates in the outer region. It is interesting now to compare our results with the evolution of the gradient in Carbon predicted by [Zhukovska & Gail 2009]. Since in their case SNe are very poor factories of silicates, but significant injectors of Carbon-rich dust, [Zhukovska & Gail 2009] adopt for the early evolutionary stages in the inner regions, a mixture almost totally made by Carbon-based dust. In our case, on the contrary, we obtain in the inner Disk that the partition between the two main dust types is more balanced and both play a comparable role. In any case, in the outer regions we find results very close to those by [Zhukovska & Gail 2009]. In brief, at the current age we all find that the final gradient in relative mass fraction of carbonaceous grains has a similar positive slope reaching in the outermost region a value as high as 70% of the dust mass embedded in carbonaceous grains.

The behaviour of iron dust has the opposite trend: it starts very flat and at the current time we have a negative gradient, with iron dust significantly contributing to the dust mass in the center of the MW. In the early stages, we have some contribution to the iron budget by SNe (however less than for silicates and carbonaceous grains, see top panels in Fig. 5 and because of the low metallicities a negligible contribution by low-Z AGB stars. As time goes on, the sum of the contributions from (i) CCSNe, (ii) accretion in the ISM that for iron dust starts to be significant at \( t > 1 - 2 \) Gyr, (iii) AGB stars that reach solar metallicities in the inner regions and start to inject iron dust, and, finally, (iv) type Ia SNe [Piovan et al. 2011b]. The sum of all of these contributions yields a gradient with negative slope.

Radial gradients in element depletion. Once examined the results for the radial gradients in element abundances both in the gas and dust, we discuss the corresponding gradient in element depletion due to dust formation. First, we concentrate on the innermost and outermost regions of the Disk. In Fig. 5 we show the results for the depletion of C, N, O, Mg, Si, Ca and Fe in the ISM in an inner ring of the MW Disk centered at 2.3 Kpc. This is a region of high density and high star formation in turn. The results are shown for both the original (upper panel) and corrected Mg yields (lower panel). Indeed, as described in [Piovan et al. 2011b], a small correction to the original yields for the Mg under-abundance leads to different (better) results for the dust mixture as the Mg to Si ratio affects the formation of silicates and the final results for the SoNe. The same group of models presented in [Piovan et al. 2011b] and used to study the depletion in the SoNe are shown for the inner and outer regions, but limited to those with the coefficient of star formation efficiency \( \nu = 0.3 \) and \( \nu = 0.7 \). The infall time scale is
varied between 1 and 9 Gyr (or between 3 and 9 Gyr in the Mg-corrected models) and three IMFs are examined, namely the Kroupa multi-power-law IMF, the Larson IMF adapted to the SoNe and, finally, the Salpeter one for the sake of comparison. We also show the un-depleted (no dust formation) abundances of the gas (i.e. the gas+dust total abundance). The abundances referring to the same models are connected by a line (dashed or continuous for $\nu = 0.3$ and $\nu = 0.7$ respectively) to better show how the elemental abundances fall down because of the dust formation. As expected, all the elements exhibit a lower abundance in the gas. Some common features are: (i) models with corrected yields of Mg (bottom panels) show a stronger depletion because they form more silicates, due to the more balanced ratio between Mg and Si; (ii) Mg is much more depleted than Si because of the higher abundance $\Delta[X_{\text{gas}}/H]$ of Si (actually $\Delta[X_{\text{gas}}/H]$ of Si and Mg are nearly the same but the total starting abundance of Si is higher); (iii) compared to the depletion in the SoNe examined by Piovan et al. (2011b), there is a less significant difference between the models with $\nu = 0.3$ and $\nu = 0.7$, with the exception of the case with $\tau = 1$ Gyr. The reason for this can be tracked in the fast onset of the ISM accretion phase in both cases. Very early on, accretion drives the dust budget and tends to smooth differences due to injections of stardust; (iv) the IMFs with higher number of SNae produce a stronger enrichment than the case with a smaller population of SNae, but the $\Delta[X_{\text{gas}}/H]$ caused by the depletion is quite similar.

The situation for a companion outermost region of low density and low star formation centered at 15 kpc is presented in Fig. 9. The meaning of the symbols is the same as before. In this Figure we applied a small to the right for the case $\nu = 0.7$ to avoid superposition of the models. Comparing the data displayed in Fig. 8 and Fig. 9 we note that: (i) in general the innermost regions form more dust with a higher reduction of the gas abundance $\Delta[X_{\text{gas}}/H]$. This is the result of forming dust in a medium richer in metals and denser in which the accretion process is favored; (ii) the effect of the different efficiencies of star formation (parameterized by $\nu$) is more evident in the low density/low SFR regions because it makes more evident the effect of metals and ISM enrichment on dust formation; (iii) in the outermost regions almost all the elements suffer a comparable depletion $\Delta[X_{\text{gas}}/H]$, simply because the accretion process is very weak over there.

Finally, in the six panels of Fig. 10 we show the abundances of [Mg/H], [Fe/H], [C/H], [Si/H], [S/H] and [Ca/H] in the gas as a function of the galacto-centric distance. To single out the difference between models with and without dust depletion, in all the diagrams we show the radial gradient at the current age for the ISM as a whole (that is gas+dust, left sub-panels in each plot) and for the gaseous component alone (right sub-panels in each plot) including the effect of depletion. Twelve combinations of the parameters IMF, $\nu$ and $\tau$ are shown for both the un-depleted (left panels) and depleted (right panels) cases. For each group of four panels we have therefore six un-depleted cases on the top left sub-panel and six on the bottom left sub-panel. The same six plus six cases are shown for the depleted gas in the top and bottom right sub-panels. As expected, dust globally lowers the amount of elements in the gas across the whole disk, but also in some cases alters significantly what would be the slope of the gradient in absence of dust. In particular, in the innermost regions, where a higher density of heavy atoms is available to accrete on the seeds, the regularity of the gradient breaks more strikingly than in the outermost regions where the gradient keeps roughly the original slope. The innermost regions, that are also affected by the bar ($\tau < 5$ kpc), present a even more irregular behaviour, in particular for the elements most depleted and participating to the complex process of silicates formation, like Magnesium and Silicon. Indeed, Carbon keeps a regular gradient as in our model no other heavy elements are supposed to participate to the accretion process of carbonaceous grains. Iron is also very irregular in the innermost regions, since it participates both to the formation of silicates and iron dust grains.

6. Discussion and conclusions

This is the last paper of a series of three (Piovan et al. 2011a,b) devoted to the study of star-dust and dust accretion in the ISM. We have analyzed in some detail the chemical gradients (gas and dust) across the Disk of the MW. The MW as a whole and its Disk and SoNe have been considered as ideal laboratories in which theories of chemical evolution and dust formation are compared with observational data. A suitable description of the Disk including radial flows of matter and the presence of a inner Bar has been adopted to simultaneously explain the bump on the inner gas density and the gradients in chemical abundances. This model provides the frame to the formation and evolution of the dust, one of the components of the ISM (the gas elemental species being the others).

The main conclusions of this study can be summarized as follows:

- For plausible combinations of the parameters, the same that in Piovan et al. (2011b) allowed to nicely reproduce the properties of the SoNe, both for gas and dust, we find a good general agreement with the mass gradients observed in the MW Disk, both for the total gas mass and the single element abundances. These latter have been compared with the abundances observed in different types of objects, like Cepheids, open clusters, HII regions, O and B stars and Red Giants. Some striking differences between models and observations for C and N can be easily explained by means of the stellar evolution theories.

- The evolutionary scheme for dust formation and evolution across the Disk, emerging from the simulations, is as follows: SNae dominate the dust budget in the early stages in the whole disk. As time goes on, AGB stars enter the game: they contribute to the C-based dust budget if the metallicity keeps low, switch to O-rich dust as soon as the metallicity is near solar or supersolar. For this reason AGB stars behave in a different way in the inner regions where the metallicity quickly increases and in the outer regions where the metallicity keeps low for long time. The ISM accretion is the main contributor to the dust budget across the whole disk. It starts very early on in the innermost regions due to the faster enrichment and the highest number densities of the elements involved in dust and much later in the outer regions where metallicity and density are lower and where AGB stars of low-Z can be very important.
for many Gyr. It is clear from this trend that a crucial role is played by the SNe: the faster they enrich the ISM in metals, the faster the ISM starts to play a decisive role. Furthermore, the higher the condensation efficiencies, the higher is the amount of dust that can be injected during the time interval before the ISM dominates. The uncertainties on the efficiency of dust condensation in SNe (Zukovska et al. 2008; Gall et al. 2011a,b) could clearly change the picture during the early evolutionary phases (Zhukovska et al. 2008; Gall et al. 2011a,b).

The radial distribution of the dust mass follows the behaviour of the gas showing a bump in the inner zone and a negative slope at the present time. It decreases moving outwards. The fractional radial mass of silicates has also a negative slope, whereas the carbonaceous grains have a positive slope. They also play a more and more important role as we move toward the outskirts. This is ultimately due to the combined effect of several causes: (i) the yield from SNe that enrich the ISM with elements involved in silicates and dust where star formation is more active; (ii) the long time interval during which low-Z AGB stars inject carbon in the outskirts of the Disk, and (iii) the delay in the onset of the ISM accretion in the outer regions.

The depletion of the elements is more significant in the inner regions, with the highest star formation and densities, while in the outskirts we have less depletion due to lower efficiency of dust accretion and star-dust injection.

As expected, the depletion significantly alters the radial gradients in the gas abundances, in particular for the elements involved in silicates, for which (at least in our simplified model) many elements enter the process and the key element driving the process can vary with time, making the whole thing more irregular.

We can conclude by saying that the picture drawn by the chemical model for the Galactic Disk in presence of dust is consistent with the way in which SNe, AGB stars of different Z and ISM contribute to the total dust budget. In particular, since the radial properties of the elements along the disk are satisfactorily reproduced, we are confident that the dust enrichment scenario described is based upon a consistent general enrichment in metals of the ISM.

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Fig. 4. Evolution of the Galactic abundances of $\text{[Mg/H]}$, $\text{[Fe/H]}$, $\text{[C/H]}$, $\text{[Si/H]}$, $\text{[O/H]}$, $\text{[S/H]}$, $\text{[Ca/H]}$ and $\text{[N/H]}$ as a function of the galacto-centric distance for different types of abundance indicators. In all the diagrams we present the evolution of the radial gradient from the age at which the Sun was formed (thin lines) to the current age (thick lines). The elemental abundances are measured in the following sources: O and B stars, Field Red Giants (RGs), HII regions, open clusters and Cepheid variables. The sources have been divided into two or three groups/panels, depending on the available ones for that element, in such a way to examine the gradients from different sources. The following models are considered: Kroupa IMF, $\tau = 3$ and $\nu = 0.3$ (continuous thin black and thick red lines); Kroupa IMF, $\tau = 6$ and $\nu = 0.7$ (continuous thin blue and thick magenta lines); Salpeter IMF, $\tau = 3$ and $\nu = 0.3$ (dotted thin black and thick red lines); Salpeter IMF, $\tau = 6$ and $\nu = 0.3$ (dotted thin blue and thick magenta lines); Larson SoNe IMF, $\tau = 3$ and $\nu = 0.3$ (dashed thin black and thick red lines), and finally Larson SoNe IMF, $\tau = 6$ and $\nu = 0.3$ (dashed thin blue and thick magenta lines).
Fig. 5. Temporal evolution of the radial contribution to the abundance of dust by the four types of grain among which we distributed the single elements (that is silicates, carbonaceous grains, iron dust grains and generic grains embedding Ca, N or S - see Zhukovska et al. (2008) and Piovan et al. (2011b) for more details) and the three sources of dust, namely SNæ, AGB stars and accretion in the ISM. All the contributions have been properly corrected for the destruction of dust. Five ages are represented from the early stages to the current time. Upper left panel: the radial contribution of the four types of dust grains to the total dust budget at 0.1 Gyr. We show: silicates (continuous lines), carbonaceous grains (dashed lines), iron dust (dotted lines) and, finally, generic grains containing S, Ca and N (dot-dashed lines). For each group we distinguish the net contributions from the ISM accretion, AGB and SNæ, that is: ISM-C, AGB-C and SNæ-C for carbonaceous grains, ISM-Sil, AGB-Sil and SNæ-Sil for silicates, ISM-Fe, AGB-Fe and SNæ-Fe for the iron dust and finally, ISM-Ca/S/N, AGB-Ca/S/N and SNæ-Ca/S/N for the other grains. In all cases the abundance of CO is fixed to $\xi_{CO} = 0.30$ taken as the reference value. For the last two ages we also show for the ISM the results for $\xi_{CO} = 0.15$ and $\xi_{CO} = 0.45$. Upper Central panel: the same as in the upper left panel but for 0.5 Gyr. Upper Right panel: the same as in the upper left panel but for 1.0 Gyr. Lower Left panel: the same as in the upper left panel but for 5.0 Gyr. Lower Right panel: the same as in the upper left panel but for 12.8 Gyr. Since the ratio $\sigma_D/\sigma_H$ grows at varying age, the scale of the y-axis is continuously shifted according to the value of the plotted data.
Fig. 6. Temporal evolution of the logarithmic radial abundance of dust $\sigma_r^D(r_k,t)$, in [M_⊙pc^{-2}], for all the elements belonging to our set and involved into the process of dust formation, that is C, N, O, Mg, Si, S, Ca and Fe (see also Piovan et al. (2011) for more details). All the contributions have been properly corrected for the dust destruction. Nine ages are represented from the early stages to the current time, that is 0.1, 0.2, 0.5, 1, 2.5, 5, 7, 10 and 12.8 Gyr, assuming that the formation of the MW started when the Universe was $\sim$0.9 Gyr old (Gail et al. 2009). **Upper panels:** from left to right the time evolution of the radial abundance $\sigma_r^D$ for Mg and Fe. **Central panels:** the same as in the upper panels but for C, N and Si, from left to right. **Lower panels:** the same as in the central panels, but for S, Ca and O. Since the abundance $\sigma_r^D$ changes at varying the age in a different range for each element, the scale of the y-axis is continuously shifted according to the represented range of values.
Fig. 7. Temporal evolution of the radial mass fraction of single elements or grain families embedded into dust $\sigma_i^D(r_k,t)/\sigma_i^D(r_k,t)$, normalized to the total dust budget, for some of the elements belonging to our set and involved in the process of dust formation (that is C, N, O, Mg, Si, S, Ca and Fe) and some grain types (silicates, carbonaceous grains, iron grains and Ca/N/S - see also Piovan et al. (2011b) for more details). All the contributions have been properly corrected for the dust destruction. Nine ages are represented as in Fig. 6 from the early stages to the current time, that is 0.1, 0.2, 0.5, 1, 2.5, 5, 7, 10 and 12.8 Gyr, assuming that the formation of the MW started when the Universe was $\sim$0.9 Gyr old (Gail et al. 2009). Upper panels: from left to right the time evolution of the radial mass fraction of Mg and Fe, normalized to the total dust mass. Central panels: the same as in the upper panels but for O and Si, from left to right. Lower panels: the same as in the central panels, but for S, N and Ca (humped together) and O, silicates (that is the mass fraction of Mg, Si, Fe and O involved into quartz/pyroxenes/olivines) and C (that in practice, once subtracted the carbon embedded in SiC, forms the carbonaceous grains).
Fig. 8. Depletion of the elements C, N, O, Mg, Si, S, Ca and Fe in the ISM for an inner ring of the MW centered at 2.3 Kpc. Top panels: the element depletions are obtained using the original yields, no correction for the Mg under-abundance is applied. The same combinations of the parameters adopted by Piovan et al. (2011b) to simulate the depletion in the SoNe are considered, that is three IMF (Salpeter, Kroupa multi power-law and Larson adapted to the SoNe) and four infall time scales (τ = 1, τ = 3, τ = 6 and τ = 9). In each panel the global ISM abundance (gas+dust) is shown. The values of the abundances are connected with continuous lines for the case ν = 0.7 (filled squares and circles), while for ν = 0.3 (open squares and circles) we use dashed lines. The squares represent the global ISM abundance, while the circles the depleted gas abundance. For each panel there is a label showing the combination IMF/infall timescale we have adopted. Bottom panels: the same as in the top panel but with the original yield corrected for the Mg under-abundance. Only three infall timescales are shown (τ = 3, τ = 6 and τ = 9).
Fig. 9. Depletion of the elements C, N, O, Mg, Si, S, Ca and Fe in the ISM for an outer ring of the MW at 15.0 Kpc. **Top panels:** the element depletions are obtained using the original yields, no correction for the Mg under-abundance is applied. The choice of the parameters and the meaning of the symbols are the same as for Fig. 8. For the sake of a better representation of the data, the case with \( \nu = 0.7 \) is slightly shifted to the right in such a way to avoid the superposition with \( \nu = 0.3 \). As in Fig. 8, the dashed lines connect the un-depleted and depleted gas abundances for the \( \nu = 0.3 \) case. **Bottom panels:** the same of the top panel, but with the original yield corrected for the Mg under-abundance. The same timescales of Fig. 8 are used.
as a whole (that is gas+dust, left panels in each plot) and the gaseous component are considered for both the undepleted (left panels) and depleted (right panels) cases.

Taking into account the depletion of the elements due to the presence of dust, 12 combinations of the parameters \( \tau, \nu \) (also examined in Piovan et al. (2011b) for the SoNe.

The Galactic abundances of \([\text{Mg}/\text{H}]\), \([\text{Fe}/\text{H}]\), \([\text{C}/\text{H}]\), \([\text{Si}/\text{H}]\), \([\text{S}/\text{H}]\) and \([\text{Ca}/\text{H}]\) in the gas as a function of the galacto-centric distance. In all diagrams we present the radial gradient at the current age for both the ISM component as a whole (that is gas+dust, left panels in each plot) and the gaseous component alone (right panels in each plot) taking into account the depletion of the elements due to the presence of dust. Twelve combinations of the parameters are considered for both the un-depleted (left panels) and depleted (right panels) cases. The legend has been split between left and right panels for the sake of clarity. The following cases are shown. Kroupa IMF: \((\tau, \nu) = (3, 0.3)\) (continuous green line), \((\tau, \nu) = (6, 0.7)\) (continuous blue line), \((\tau, \nu) = (9, 0.3)\) (continuous red line) and \((\tau, \nu) = (9, 0.7)\) (continuous blue line); Salpeter IMF: \((\tau, \nu) = (3, 0.3)\) (dotted green line), \((\tau, \nu) = (6, 0.3)\) (dotted blue line), \((\tau, \nu) = (9, 0.3)\) (dotted red line), and \((\tau, \nu) = (9, 0.3)\) (dotted blue line); Larson SoNe IMF: \((\tau, \nu) = (3, 0.3)\) (dashed green line), \((\tau, \nu) = (6, 0.3)\) (dashed blue line), \((\tau, \nu) = (9, 0.7)\) Gyr (dashed green line) and \((\tau, \nu) = (8, 0.3)\) (dashed blue line). All theses cases are also examined in Piovan et al. (2011b) for the SoNe.

**Fig. 10.** The Galactic abundances of \([\text{Mg}/\text{H}]\), \([\text{Fe}/\text{H}]\), \([\text{C}/\text{H}]\), \([\text{Si}/\text{H}]\), \([\text{S}/\text{H}]\) and \([\text{Ca}/\text{H}]\) in the gas as a function of the galacto-centric distance. In all diagrams we present the radial gradient at the current age for both the ISM component as a whole (that is gas+dust, left panels in each plot) and the gaseous component alone (right panels in each plot) taking into account the depletion of the elements due to the presence of dust. Twelve combinations of the parameters are considered for both the un-depleted (left panels) and depleted (right panels) cases. The legend has been split between left and right panels for the sake of clarity. The following cases are shown. Kroupa IMF: \((\tau, \nu) = (3, 0.3)\) (continuous green line), \((\tau, \nu) = (6, 0.7)\) (continuous blue line), \((\tau, \nu) = (9, 0.3)\) (continuous red line) and \((\tau, \nu) = (9, 0.7)\) (continuous blue line); Salpeter IMF: \((\tau, \nu) = (3, 0.3)\) (dotted green line), \((\tau, \nu) = (6, 0.3)\) (dotted blue line), \((\tau, \nu) = (9, 0.3)\) (dotted red line), and \((\tau, \nu) = (9, 0.3)\) (dotted blue line); Larson SoNe IMF: \((\tau, \nu) = (3, 0.3)\) (dashed green line), \((\tau, \nu) = (6, 0.3)\) (dashed blue line), \((\tau, \nu) = (9, 0.7)\) Gyr (dashed green line) and \((\tau, \nu) = (8, 0.3)\) (dashed blue line). All theses cases are also examined in Piovan et al. (2011b) for the SoNe.