Maximal Operator for the Higher Order Calderón Commutator

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Abstract. In this paper, we investigate the weighted multilinear boundedness properties of the maximal higher order Calderón commutator for the dimensions larger than two. We establish all weighted multilinear estimates on the product of the $L^p(\mathbb{R}^d, w)$ space, including some peculiar endpoint estimates of the higher dimensional Calderón commutator.

1 Introduction

In the recent work [21], the author studied the multilinear boundedness of the higher order Calderón commutator. The purpose of this paper is to further generalize those results to the weighted space for its maximal type operator. Before stating our main results, let us give some notation and background. Define the truncated higher ($n$-th) order Calderón commutator by

$$\mathcal{C}_\varepsilon[\nabla A_1, \ldots, \nabla A_n, f](x) = \int_{|x-y| \geq \varepsilon} K(x-y) \left( \prod_{i=1}^n \frac{A_i(x) - A_i(y)}{|x-y|} \right) \cdot f(y) \, dy,$$

where $n$ is a positive integer and $K$ is the Calderón–Zygmund convolution kernel on $\mathbb{R}^d \setminus \{0\}$ $(d \geq 2)$, which means that $K$ satisfies the following three conditions:

\begin{align}
(1.1) \quad & |K(x)| \leq |x|^{-d}, \\
(1.2) \quad & \int_{r < |x| < R} K(x) (x/|x|)^\alpha \, dx = 0,
\end{align}

for all $0 < r < R < \infty$ and for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| = n$,

\begin{align}
(1.3) \quad & |K(x-y) - K(x)| \leq |y|^{\delta}/|x|^{d+\delta}
\end{align}

for some $0 < \delta \leq 1$ if $|x| > 2|y|$. Then we define the higher order Calderón commutator and its maximal operator by

$$\mathcal{C}[\nabla A_1, \ldots, \nabla A_n, f](x) = \lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon[\nabla A_1, \ldots, \nabla A_n, f](x),$$

$$\mathcal{C}_*[\nabla A_1, \ldots, \nabla A_n, f](x) = \sup_{\varepsilon > 0} |\mathcal{C}_\varepsilon[\nabla A_1, \ldots, \nabla A_n, f](x)|.$$
It is the standard context to check that these functions $C[\nabla A_1, \ldots, \nabla A_n, f](x)$ and $C_+[\nabla A_1, \ldots, \nabla A_n, f](x)$ are well defined for $A_1, \ldots, A_n, f \in C^\infty_c(\mathbb{R}^d)$ (see e.g., [16]). This kind of commutator was first introduced by A. P. Calderón [2] when $n = 1$ and $K(x)$ is a homogeneous kernel and later [3,4] for the higher order one (see also [6,7]). One can easily see that the first order Calderón commutator $C[\nabla A, f](x)$ is a generalization of

$$[A, S]f(x) = A(x)S(f)(x) - S(Af)(x) = -p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{A(x) - A(y)}{x - y} f(y)dy,$$

where $S = \frac{d}{dx} \circ H$ and $H$ denotes the Hilbert transform. It is well known that the commutator $[A, S]$ and its generalization are elementary operators in harmonic analysis, which play an important role in the theory of the Cauchy integral along Lipschitz curve in $\mathbb{C}$, the boundary value problem of elliptic equation on non-smooth domain, the Kato square root problem on $\mathbb{R}$, and the mixing flow problem (see e.g., [2,4,8,10,13,17,19,23–25] for the details).

Many classical known results about the higher order Calderón commutator take place in the setting of the dimension $d = 1$. For example, the endpoint estimate that the $n$-th order Calderón commutator $C$ maps $L^1(\mathbb{R}) \times \cdots \times L^1(\mathbb{R})$ to $L^{1+\infty}(\mathbb{R})$ was proved by C. P. Calderón [5] when $n = 1$, Coifman and Meyer [6] when $n = 1, 2$ and Duong, Grafakos, and Yan [11] when $n \geq 1$. Here we point out that one important fact used by Coifman and Meyer [6], Duong, Grafakos, and Yan [11] is that the one dimensional higher order Calderón commutator can be reduced to the multilinear Calderón Zygmun operator (see the very nice exposition [17, Chapter 7] and the references therein). However, when the dimension $d \geq 2$, things become complicated, since Calderón commutator is a non-standard multilinear Calderón–Zygmund operator. If we consider the Calderón–Zygmund kernel $K(x) = |x|^{-d}$, then the sharp bilinear estimates (except some endpoint estimates) of the first order Calderón commutator in this case has been established by Fong [14] via the time-frequency analysis method. For the more general Calderón–Zygmund kernel or even rough homogeneous kernel, the author [21] established all multilinear boundedness of the higher order Calderón commutator for the higher dimensions, especially the endpoint estimate that the $n$-th order Calderón commutator $C$ maps the product of Lorentz space $L^{d,1}(\mathbb{R}^d) \times \cdots \times L^{d,1}(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ to $L^{d,\infty}(\mathbb{R}^d)$.

The weighted results related to the Calderón commutator is also only known for the case $d = 1$. Duong, Gong, Grafakos, Li, and Yan [12, Theorem 4.3] proved that $C_+ \text{ maps } L^{q_1}(\mathbb{R}, w) \times \cdots \times L^{q_n}(\mathbb{R}, w) \times L^p(\mathbb{R}^d, w)$ to $L^r(\mathbb{R}, w)$ if $\frac{1}{r} = (\sum_{i=1}^n \frac{1}{q_i}) + \frac{1}{p}$ with $\frac{1}{n+1} < r < \infty$, $1 < q_1, \ldots, q_n \leq \infty$, $1 < p < \infty$ and $w \in \cap_{i=1}^n A_{q_i}(\mathbb{R}) \cap A_p(\mathbb{R})$. For the endpoint estimate, Grafakos, Liu and Yang [18, Corollary 1.7] showed that $C_+ \text{ maps } L^1(\mathbb{R}, w) \times \cdots \times L^1(\mathbb{R}, w) \times L^1(\mathbb{R}, w)$ to $L^{1+\infty}(\mathbb{R}, w)$ under the assumption $w \in A_1(\mathbb{R})$. The method used in Duong et al. [12] and Grafakos et al. [18] is both that by establishing the weighted theory for a class of multilinear Calderón–Zygmund operators with a non-smooth kernel and then applying it to the Calderón commutator for the dimension $d = 1$. For the higher dimensional case of the Calderón commutator, no proper weighted multilinear Calderón–Zygmund theory can be applied directly.
In this paper, we are interested in the following weighted strong type multilinear estimate (or weighted weak type estimate) for the maximal operator of the higher order Calderón commutator

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^r(R^d, w)} \leq \left( \prod_{i=1}^{n} \| \nabla A_i \|_{L^q_i(R^d, w)} \right) \| f \|_{L^p(R^d, w)},
\]

where \( \frac{1}{r} = \left( \sum_{i=1}^{n} \frac{1}{q_i} \right) + \frac{1}{p} \) with \( 1 \leq q_i \leq \infty, \ (i = 1, \ldots, n), \) and \( 1 \leq p \leq \infty. \) However, it is unknown whether those kind of estimates hold for the maximal Calderón commutator \( \mathcal{C}_s, \) even in the unweighted case. In this paper, we will work directly on the weighted space and state our main results as follows.

**Theorem 1.1** Let \( d \geq 2 \) and \( n \) be a positive integer. Suppose \( K \) satisfies (1.1), (1.2), and (1.3). Assume that \( \frac{1}{r} = \left( \sum_{i=1}^{n} \frac{1}{q_i} \right) + \frac{1}{p} \) with \( 1 \leq q_i \leq \infty, \ (i = 1, \ldots, n), \) and \( 1 \leq p \leq \infty. \) Suppose \( w \in \left( \bigcap_{i=1}^{n} A_{\text{max}}(\frac{n}{q_i}, 1)(R^d) \right) \cap A_p(R^d). \) We have the following conclusions:

(i) If \( \frac{d}{d+n} < r < \infty, \ 1 < q_i \leq \infty \ (i = 1, \ldots, n) \) and \( 1 < p \leq \infty, \) then (1.4) holds.

(ii) If \( \frac{d}{d+n} \leq r < \infty \) with \( q_i = 1 \) for some \( i = 1, \ldots, n, \) or \( p = 1, \) or \( r = \frac{d}{d+n}, \) then the following multilinear estimate holds:

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{r, \infty}(R^d, w)} \leq \left( \prod_{i=1}^{n} \| \nabla A_i \|_{L^q_i(R^d, w)} \right) \| f \|_{L^p(R^d, w)},
\]

and in this case, if \( q_i = d \) for some \( i = 1, \ldots, n, \) \( L^{q_i}(R^d, w) \) in the above inequality should be replaced by \( L^{d,1}(R^d, w), \) the weighted Lorentz space. Specifically, we have the following endpoint estimate:

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{2^n, \infty}(R^d, w)} \leq \left( \prod_{i=1}^{n} \| \nabla A_i \|_{L^{q_i,1}(R^d, w)} \right) \| f \|_{L^1(R^d, w)}.
\]

**Remark 1.2**

(i) These results in Theorem 1.1 are new, even in the unweighted case when the dimension \( d \geq 2. \)

(ii) When \( 0 < r < \frac{d}{d+n}, \) these multilinear strong type estimates (1.4) (or weak type estimates (1.5)) do not hold for the maximal Calderón operator \( \mathcal{C}_s. \) In fact, some counterexamples has been constructed in [21, Theorem 1.1] to show that those multilinear strong type estimates (or weak type estimates) fail even for the operator \( \mathcal{C} \) in the case \( 0 < r < \frac{d}{d+n}. \) Thus, our results in Theorem 1.1 are optimal in this sense.

(iii) The condition of the weight \( w \in \left( \bigcap_{i=1}^{n} A_{\text{max}}(\frac{n}{q_i}, 1)(R^d) \right) \cap A_p(R^d) \) seems to be unnatural at the first sight, since it does not appear previously. However, this kind of condition is only appropriate for the higher dimensional Calderón commutator, as we will see in our later proof. In fact, \( w \in A_{\text{max}}(\frac{n}{q_i}, 1)(R^d) \) comes from \( \nabla A_i \in L^{q_i}(R^d, w) \) and \( w \in A_p(R^d) \) comes from \( f \in L^p(R^d, w). \) When the dimension \( d = 1, \) (1.4) turns out to be that \( \mathcal{C}_s \) maps \( L^{q_1}(R, w) \times \cdots \times L^{q_n}(R, w) \times L^p(R^d, w) \) to \( L^r(R, w) \) if \( \frac{1}{n+1} < r < \infty, \ 1 < q_i, \ldots, q_n \leq \infty, \ 1 < p \leq \infty, \) and \( w \in \bigcap_{i=1}^{n} A_{q_i}(R, w) \cap A_p(R), \) which has been proved by Duong, Gong, Grafakos, Li, and Yan [12, Theorem 4.3] except the endpoint case \( q_i = \infty \) for some \( i \) or \( p = \infty. \) Therefore, even in the one dimensional case (1.4) is new at the endpoint case \( q_i = \infty \) for some \( i \) or \( p = \infty. \) To the best knowledge of the author, (1.4) is new when \( d \geq 2. \)
(iv) Notice that $L^{1,1}(\mathbb{R}, w) = L^1(\mathbb{R}, w)$. Therefore, when the dimension $d = 1$, (1.6) is just that the maximal $n$-th order Calderón commutator maps $L^1(\mathbb{R}, w) \times \cdots \times L^1(\mathbb{R}, w)$ to $L^{1, \infty}(\mathbb{R}, w)$ under the assumption $w \in A_1(\mathbb{R})$, which has been proved by Grafakos, Liu, and Yang [18, Corollary 1.7]. To the best knowledge of the author, (1.6) is new when $d \geq 2$. Although we assume that $d \geq 2$ in our main results, the proof presented in this paper is also valid for $d = 1$. Therefore, even when $d = 1$, the proof of (1.4) and (1.6) here are quite different from the one given by Duong, Gong, Grafakos, Li, and Yan [12] and Grafakos, Liu, and Yang [18]; thus, we give new proofs of (1.4) and (1.6) for $d = 1$.

(v) Currently, there is extensive research on seeking the optimal quantitative weighted bound for singular integral. We do not pursue this topic in this paper but hope to work on it in the future.

Notice first that if $q_i = \infty$ with $i = 1, \ldots, n$, i.e., $A_i$ is a Lipschitz function, then $C[\nabla A_1, \ldots, \nabla A_n, \cdot]$ is a standard Calderón–Zygmund operator. By the standard weighted theory of the Calderón–Zygmund operator, we can easily get that $C_\ast$ maps $L^{\infty}(\mathbb{R}^d, w) \times \cdots \times L^{\infty}(\mathbb{R}^d, w) \times L^p(\mathbb{R}^d, w)$ to $L^p(\mathbb{R}^d, w)$ for $1 < p < \infty$ and $L^{\infty}(\mathbb{R}^d, w) \times \cdots \times L^{\infty}(\mathbb{R}^d, w) \times L^1(\mathbb{R}^d, w) \to L^{1, \infty}(\mathbb{R}^d, w)$. Recall the method used in [12] or [18]: by establishing the Cotlar inequality for the multilinear Calderón–Zygmund operator, the authors in [12] or [18] proved the weighted multilinear estimates for the Calderón–Zygmund operator and then applied them to the one-dimensional Calderón commutator. There are also variants of the Cotlar inequality for higher dimensional Calderón commutators, which is available only for the multilinear estimates (1.4) in the case that all $q_i > d, i = 1, \ldots, n, r > 1$ (see Proposition 3.3). To deal with the remaining case, our strategy is as follows. We straightforwardly establish the endpoint estimates in (ii) of Theorem 1.1, which means that we need to give some weak type estimates. Note that $A_i$ belongs to the Sobolev space $W^{1, q_i}(\mathbb{R}^d, w)$. We will construct an exceptional set that satisfies the required weighted weak type estimate. And on the complementary set of exceptional set, the function $A_i$ is a Lipschitz function with a bound $\lambda \lambda^\nu$. Then, roughly speaking, the strong type estimate and the weak type $L^{1, \infty}(\mathbb{R}^d, w)$ boundedness (with $q_i = \infty, i = 1, \ldots, n$) of $C_\ast[\nabla A_1, \ldots, \nabla A_n, f](x)$ could be applied on the complementary set of the exceptional set. To construct the exceptional set, we will make use of the Mary Weiss maximal operator and the weighted Sobolev inequality.

This paper is organized as follows. First, some preliminary lemmas are presented in Section 2. In Section 3, we give the proof of Theorem 1.1. The proof is divided into several cases. In Subsection 3.1, we prove some strong type estimates of Theorem 1.1(i). The proofs of Theorem 1.1(ii) are given in Subsections 3.2 and 3.3. In Subsection 3.4, we will use the linear Marcinkiewicz interpolation with some strong type estimates of (i) and full weak type estimates of (ii) to show the rest of (i) in Theorem 1.1.

**Notation** Throughout this paper, we only consider the dimension $d \geq 2$, and the letter $C$ stands for a positive finite constant that is independent of the essential variables, though not necessarily the same one in each occurrence. By $A \lesssim B$ we mean that $A \leq CB$ for some constant $C$. By the notation $C_\varepsilon$ means that the constant depends on the parameter $\varepsilon$. We have that $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$;
2 Some Preliminary Lemmas

In this section, we will introduce the weighted properties of some operators that are useful in the proof of Theorem 1.1. Those operators include the Hardy–Littlewood maximal operator with order $\delta$, the maximal sharp function operator, the Mary Weiss maximal operator, and many others. Also, a weighted Sobolev inequality is needed.

**Definition 2.1 ($$A_p(\mathbb{R}^d)$$ weight)** A nonnegative locally integrable function $w$ on $\mathbb{R}^d$ is said to be an $A_p(\mathbb{R}^d)$ weight if there exists a constant $C > 0$ such that

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{\frac{1}{p}} \, dx \right)^{p-1} \leq C < \infty,$$

where the supremum is taken to be all cube $Q$ in $\mathbb{R}^d$. The smallest constant $C$ for (2.1) holds is called the $A_p$ bound of $w$ and is denoted by $[w]_{A_p}$. We call $w$ an $A_1(\mathbb{R}^d)$ weight if there exists a constant $C$ independent of $Q$ such that

$$\frac{1}{|Q|} \int_Q w(z) \, dz \leq C w(y), \quad a.e. \quad y \in Q.$$

And we set the smallest constant $C$ in (2.2) as $[w]_{A_1}$, which is called the $A_1$ bound of $w$. We also set $A_\infty(\mathbb{R}^d) = \bigcup_{1 \leq q < \infty} A_q(\mathbb{R}^d)$.

It is easy to see that an equivalent definition of $A_1(\mathbb{R}^d)$ weight is that $M(w) \leq C w(x)$, where $M$ is the Hardy–Littlewood maximal operator. Recall the following basic fact about $A_p(\mathbb{R}^d)$ weight (see [16]):

$$A_p(\mathbb{R}^d) \nsubseteq A_q(\mathbb{R}^d), \text{ if } 1 \leq p < q \leq \infty.$$

**Lemma 2.2 (see [20] or [22])** Suppose that $w \in A_\infty(\mathbb{R}^d)$. Let $0 < \delta, q < \infty$. Then there exists a constant $C$ depends only on $w, \delta, q$ such that

$$\int_{\mathbb{R}^d} [M_\delta f(x)]^q w(x) \, dx \leq C \int_{\mathbb{R}^d} [M_\delta^q f(x)]^q w(x) \, dx.$$
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holds for any function provided that the left side integral is finite. Here, $M_\delta$ and $M_\delta^d$ are the Hardy–Littlewood maximal operator with order $\delta$ and the maximal sharp function operator, which are defined as

$$M_\delta(f)(x) = \sup_{r > 0} \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y)|^\delta \, dy \right)^{\frac{1}{\delta}},$$

$$M_\delta^d(f)(x) = \sup_{Q} \inf_{\epsilon} \left( \frac{1}{|Q|} \int_{Q} |f(z) - c|^\delta \, dz \right)^{\frac{1}{\delta}},$$

where $Q$ is a cube in $\mathbb{R}^d$ and $Q(x, r)$ is a cube with center $x$ and sidelength $r$.

Next we state some properties of a special maximal function introduced first by Mary Weiss (see [5]), which is defined as

$$M(\nabla A)(x) = \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{|A(x + h) - A(x)|}{|h|}.$$

**Lemma 2.3** Suppose that $w \in A_p/d(\mathbb{R}^d)$ with $p > d$. Let $\nabla A \in L^p(\mathbb{R}^d, w)$. Then $M$ is bounded on $L^p(\mathbb{R}^d, w)$; that is,

$$\|M(\nabla A)\|_{L^p(\mathbb{R}^d, w)} \leq \|\nabla A\|_{L^p(\mathbb{R}^d, w)}.$$

**Proof** By using the density argument, it is sufficient to consider $A$ as a $C^\infty$ function with compact support. Then by the result of [5, Lemma 1.4], we get that for any $q > d$,

$$\frac{|A(x) - A(y)|}{|x - y|^d} \lesssim \left( \frac{1}{|x - y|^d} \int_{|x - z| \leq 2|x - y|} |\nabla A(z)|^q \, dz \right)^{\frac{1}{q}}.$$

Since $w \in A_p/d(\mathbb{R}^d)$, by the reverse Hölder inequality of $A_p/d(\mathbb{R}^d)$ weight (see [16]) and its definition, there exist $\epsilon > 0$ such that $w \in A_{p/d-\epsilon}(\mathbb{R}^d)$ and $p/d - \epsilon \geq 1$. Therefore, we can choose $q$ in the above inequality such that $p/d - \epsilon = p/q$ and $d < q < p$. Applying the fact that the Hardy–Littlewood maximal operator $M$ maps $L^q(\mathbb{R}^d, w)$ to itself if $1 < s \leq \infty$ and $w \in A_s(\mathbb{R}^d)$, we can get

$$\|M(\nabla A)\|_{L^q(\mathbb{R}^d, w)} \leq \|M_q(\nabla A)\|_{L^q(\mathbb{R}^d, w)} = \|M(|\nabla A|^q)\|_{L^{p/q}(\mathbb{R}^d, w)} \leq \|\nabla A\|_{L^p(\mathbb{R}^d, w)},$$

which completes the proof. 

**Lemma 2.4** Let $A$ be a function such that $\nabla A \in L^{d,1}(\mathbb{R}^d, w)$, the Lorentz space with weight $w \in A_1(\mathbb{R}^d)$. Then for any $\lambda > 0$,

$$w(\{x \in \mathbb{R}^d : M(\nabla A)(x) > \lambda\}) \lesssim \lambda^{-d} \|\nabla A\|_{L^{d,1}(\mathbb{R}^d, w)}^d.$$

**Proof** By the density argument, it is sufficient to consider $A$ as a smooth function with compact support. Using the formula [26, p. 125, (17)], one can write

$$A(x) = C_d \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^d} \partial_j A(y) \, dy = \mathbb{K} * f(x),$$
where $\mathbb{K}(x) = 1/|x|^{d-1}$, $f = C_d \sum_{j=1}^{d} R_j(\partial_j A)$ with $R_j$s the Riesz transforms. Notice that $w \in A_1(\mathbb{R}^d) \subseteq A_p(\mathbb{R}^d)$ for all $1 < p \leq \infty$. By applying the general form of the Marcinkiewicz interpolation theorem (see [28, p. 197, Theorem 3.15]), we obtain that the Riesz transform $R_j$ maps $L^{d,1}(\mathbb{R}^d, w)$ to itself. Then it is easy to see that
\[
\|f\|_{L^{d,1}(\mathbb{R}^d, w)} \lesssim \|\nabla A\|_{L^{d,1}(\mathbb{R}^d, w)}.
\]
Hence to finish the proof, it is sufficient to prove that
\[
w\left(\{x \in \mathbb{R}^d : M(\nabla A)(x) > \lambda\}\right) \lesssim \lambda^{-d} \|f\|_{L^{d,1}(\mathbb{R}^d, w)}^d
\]
with $A = \mathbb{K} \ast f$. Below we will show that for any $x \in \mathbb{R}^d$, the estimate
\[
|A(x+h) - A(x)| \lesssim |h| T(f)(x)
\]
holds uniformly for $h \in \mathbb{R}^d \setminus \{0\}$ where $T$ is an operator maps $L^{d,1}(\mathbb{R}^d, w)$ to $L^{d,\infty}(\mathbb{R}^d, w)$. Once we show this, we get (2.3), and hence finish the proof of Lemma 2.4. We write
\[
A(x+h) - A(x) = \int_{|x-y| \leq 2|h|} |x+h-y|^{-d+1} f(y) dy - \int_{|x-y| \leq 2|h|} |x-y|^{-d+1} f(y) dy
\]
\[
+ \int_{|x-y| > 2|h|} \left(|x+h-y|^{-d+1} - |x-y|^{-d+1}\right) f(y) dy
\]
\[
= I + II + III.
\]
Consider $I$ first. Observe that $\mathbb{K} \in L^{d',\infty}(\mathbb{R}^d)$ where $d' = d/(d-1)$. Set $B(x, r) = \{y \in \mathbb{R}^d : |x-y| \leq r\}$. Applying the rearrangement inequality (see [16, p. 74, Exercise 1.4.1]), we obtain that
\[
|I| \leq \int_{\mathbb{R}^d} \mathbb{K}(x+h-y) |f| \chi_{B(x,2|h|)}(y) dy \leq \int_0^\infty \mathbb{K}^+(s) \left(f \chi_{B(x,2|h|)}\right)^+(s) ds
\]
\[
\leq \left( \int_0^\infty (f \chi_{B(x,2|h|)})^+(s) \frac{ds}{s} \right) \cdot \sup_{s>0} \left(\mathbb{K}^+(s) s^{\frac{1}{d'}}\right)
\]
\[
\lesssim \|f \chi_{B(x,2|h|)}\|_{L^{d,1}(\mathbb{R}^d)} \|\mathbb{K}\|_{L^{d',\infty}(\mathbb{R}^d)},
\]
where $f^+$ stands for the decreasing rearrangement of $f$. Applying the definition of Lorentz space, we can get that $\|\chi_E\|_{L^{d,1}(\mathbb{R}^d)} = \|\chi E\|_{L^{d,1}(\mathbb{R}^d)}$ holds for any characteristic function $\chi_E$ of set $E$ of finite measure; thus, $\|\chi_{B(x,2|h|)}\|_{L^{d,1}(\mathbb{R}^d)} = C_d |h|$. Then we obtain that
\[
|I| \lesssim |h| A(f)(x), \quad \text{where} \quad A(f)(x) = \sup_{r>0} \frac{\|f \chi_{B(x,r)}\|_{L^{d,1}(\mathbb{R}^d)}}{\|\chi_{B(x,r)}\|_{L^{d,1}(\mathbb{R}^d)}}.
\]
In the sequel it is sufficient to show that the operator $\Lambda$ maps $L^{d,1}(\mathbb{R}^d, w)$ to $L^{d,\infty}(\mathbb{R}^d, w)$. Note that $L^{d,1}(\mathbb{R}^d, w)$ is a Banach space (see e.g., [28, p. 204, Theorem 3.22]); it suffices to show that $\Lambda$ is restricted of type $(d, d)$, thus is $\|\Lambda(\chi_E)\|_{L^{d,\infty}(\mathbb{R}^d, w)} \lesssim w(\chi_E)^{\frac{1}{d}}$ (see e.g., [16, p. 62, Lemma 1.4.20]). However, in this case, the proof is equivalent to showing that
\[
w\left(\{x \in \mathbb{R}^d : M(\chi_E)(x) > \lambda\}\right) \lesssim \lambda^{-1} \|\chi_E\|_{L^1(\mathbb{R}^d, w)},
\]
where $M$ is the Hardy–Littlewood maximal operator. Since $M$ is weighted weak type (1,1) if $w \in A_1(\mathbb{R}^d)$, hence we prove that $\Lambda$ maps $L_{d,1}(\mathbb{R}^d, w)$ to $L_{d,\infty}(\mathbb{R}^d, w)$.

Next consider $II$. Observe that the kernel $k(x) := \varepsilon^{-1}|x|^{-d+1}\chi_{\{|x| \leq \varepsilon\}}$ is radial non-increasing and $L^1$ integrable in $\mathbb{R}^d$, we get

$$|II| \leq \|k\|_{L^1(\mathbb{R}^d)} |h| M(f)(x).$$

Notice that $L_{p,1}(\mathbb{R}^d, w) \subset L^p(\mathbb{R}^d, w)$ and $M$ maps $L^p(\mathbb{R}^d, w)$ to itself for $1 < p < \infty$. Hence we get that $M$ maps $L_{d,1}(\mathbb{R}^d, w)$ to $L_{d,\infty}(\mathbb{R}^d, w)$.

Finally consider $III$. Notice that it suffices to consider $|x - y| > 2|h|$. Then applying the Taylor expansion of $|x - y + h|^{-d+1}$, we get

$$\frac{1}{|x - y + h|^{d-1}} - \frac{1}{|x - y|^{d-1}} = (-d + 1) \sum_{j=1}^{d} h_j \frac{x_j - y_j}{|x - y|^{d+1}} + R(x, y, h),$$

where the remainder term $R(x, y, h)$ in the Taylor expansion satisfies

$$|R(x, y, h)| \leq C|h|^2|x - y|^{-d-1} \text{ if } |x - y| > 2|h|.$$

Inserting (2.4) into the term $III$ with the above estimate of $R(x, y, h)$, we conclude that

$$|III| \leq |h| \sum_{j=1}^{d} R_j^*(f)(x) + |h|^2 \int_{|x - y| > 2|h|} |x - y|^{-d-1}|f(y)|dy,$$

where $R_j^*$ is the maximal Riesz transform defined by

$$R_j^*(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x - y| > \varepsilon} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) dy \right|.$$

Since $R_j^*$ is bounded on $L^p(\mathbb{R}^d, w)$ for $1 < p < \infty$, we immediately obtain that $R_j^*$ maps $L_{d,1}(\mathbb{R}^d, w)$ to $L_{d,\infty}(\mathbb{R}^d, w)$. The second term that controls $III$ can be dealt similar to that of the estimate of $II$ once we observe that $\varepsilon|x|^{-d-1}\chi_{\{|x| > \varepsilon\}}$ is radial non-increasing and $L^1$ integrable.

In the following, we introduce a weighted Sobolev inequality and a key weighted weak type estimate for $\nabla A \in L^p(\mathbb{R}^d, w)$ with $1 \leq p < d$. Define the weighted Hardy–Littlewood maximal operator of order $p$ $M_{w,p}$ and the weighted maximal operator $\mathfrak{M}_{w,s}$ by

$$M_{w,p}(f)(x) = \sup_{r > 0} \left( \frac{1}{w(Q(x, r))} \int_{Q(x, r)} |f(y)|^p w(y) dy \right)^{1/p},$$

$$\mathfrak{M}_{w,s}(\nabla A)(x) = \sup_{r > 0} \left( \frac{1}{w(Q(x, r))} \int_{Q(x, r)} \frac{|A(x) - A(y)|^s}{r} w(y) dy \right)^{1/s},$$

where $Q(x, r)$ is a cube with center $x$ and sidelength $r$.

**Lemma 2.5** (see [9]) If $w$ is an $A_1(\mathbb{R}^d)$ weight, then the following weighted Sobolev inequality

$$\left( \int_{\mathbb{R}^d} |g(x)|^p w(y) dy \right)^{1/p} \leq C \left( \int_{\mathbb{R}^d} [w(x)^{-\frac{1}{2}}|\nabla g(x)|]^{p} w(x) dx \right)^{1/2}.$$
holds for $1 \leq p < d$ and $1/s = 1/p - 1/d$, where $g$ is a $C^1$ function with compact support and the constant $C$ does not depend on $g$.

**Lemma 2.6** Let $w \in A_1(\mathbb{R}^d)$ and $\nabla A \in L^p(\mathbb{R}^d, w)$ with $1 \leq p < d$. Set $1/s = 1/p - 1/d$. Then we have

$$w \left( \{ x \in \mathbb{R}^d : \mathfrak{M}_{w,s}(\nabla A)(x) > \lambda \} \right) \leq \lambda^{-p} \| \nabla A \|^p_{L^p(\mathbb{R}^d, w)}.$$  

**Proof** By using a standard limiting argument, we only need to consider $A$ as a $C^\infty$ function with compact support. Fix a cube $Q(x, r)$. Choose a $C^\infty$ function $\phi$ such that $\phi(y) \equiv 1$ if $y \in Q(x, r)$, $\text{supp} \phi \subset Q(x, 2r)$ and $\| \nabla \phi \|_{L^\infty(\mathbb{R}^d)} \leq r^{-1}$. Consider the auxiliary function $\phi(y)(A(x) - A(y))$ where $x$ is fixed and $y$ is the variable. Using the weighted Sobolev inequality in Lemma 2.5 and the property (2.2) of $A_1(\mathbb{R}^d)$ weight, one can get that

$$\left( \int_{Q(x, r)} |A(x) - A(y)|^s w(y) dy \right)^{\frac{1}{s}} \leq \left[ \int_{\mathbb{R}^d} \left( \frac{1}{r} \left( \int_{Q(x, 2r) \setminus Q(x, r)} |\nabla A(y)|^p w(y) dy \right)^{\frac{1}{p}} \right]^{\frac{1}{s}}$$

$$\leq \left[ \int_{Q(x, 2r)} \left( \frac{1}{r} \left( \int_{Q(x, 2r) \setminus Q(x, r)} |A(x) - A(y)|^p w(y) dy \right)^{\frac{1}{p}} \right]^{\frac{1}{s}}.$$  

The above estimate, via the doubling property of $w(x) dx$ (i.e., $w(2Q) \leq w(Q)$, see [16]) and $\frac{1}{s} = \frac{1}{p} - \frac{1}{d}$, yields that

$$\left( \frac{1}{w(Q(x, r))} \int_{Q(x, r)} \frac{|A(x) - A(y)|^s}{r} w(y) dy \right)^{\frac{1}{s}} \leq M_{w, p}(\nabla A)(x) + S_p(\nabla A)(x),$$  

where

$$S_p(\nabla A)(x) := \left[ \frac{1}{w(Q(x, 2r))} \int_{Q(x, 2r) \setminus Q(x, r)} \frac{|A(x) - A(y)|^p}{r} w(y) dy \right]^{\frac{1}{p}}.$$  

Again using the fact that $w(x) dx$ satisfies the doubling property, one can see that the Hardy–Littlewood maximal operator $M_{w,1}$ with the weight $w$ is of weak type (1,1), and thus so is $M_{w,1}$ maps $L^1(\mathbb{R}^d, w)$ to $L^{1,\infty}(\mathbb{R}^d, w)$. Then we get that $M_{w, p}$ maps $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$. Therefore to complete the proof, it is enough to show that $S_p(\nabla A)(x) \leq T(\nabla A)(x)$ with $T$ mapping $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$.

Below we give some explicit estimates of $A(x) - A(y)$ similar to that in the proof of Lemma 2.4. By the formula given in [26, p. 125, (17)], we can write

$$A(x) = C_d \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^d} \partial_j A(y) dy.$$
Split \( A(x) - A(y) \) into three terms as follows,

\[
(2.5) \quad A(x) - A(y) = C_d \sum_{j=1}^{d} \left[ \int_{|x-z| \leq 2|x-y|} \frac{x_j - z_j}{|x-z|^d} \partial_j A(z) \, dz \right. \\
- \int_{|x-z| \leq 2|x-y|} \frac{y_j - z_j}{|y-z|^d} \partial_j A(z) \, dz \\
+ \int_{|x-z| > 2|x-y|} \left( \frac{x_j - z_j}{|x-z|^d} - \frac{y_j - z_j}{|y-z|^d} \right) \partial_j A(z) \, dz \\
= I(x) + II(x) + III(x).
\]

Plug the above three terms back into \( S_p(\nabla A)(x) \) and define these three terms as \( S_{p,1}(\nabla A)(x) \), \( S_{p,2}(\nabla A)(x) \), and \( S_{p,3}(\nabla A)(x) \), respectively.

Let us first consider \( S_{p,1}(\nabla A)(x) \). By applying the Hölder inequality,

\[
|I(x)|^p \leq |x - y|^{p-1} \left( \int_{|x-z| \leq 2|x-y|} \frac{\left| \nabla A(z) \right|^p}{|x-z|^{d-1}} \, dz \right).
\]

Plugging this inequality into \( S_{p,1}(\nabla A)(x) \) with \(|x - y| = r\), and then using that the kernel \( \tilde{k}(x) = \varepsilon^{-1} |x|^{-d+1} \chi_{\{|x| \leq \varepsilon \}} \) is a radial non-increasing function and \( L^1 \) integrable in \( \mathbb{R}^d \), we get that

\[
S_{p,1}(\nabla A)(x) \lesssim \left( r^{-1} \int_{|x-z| \leq \varepsilon} \frac{\left| \nabla A(z) \right|^p}{|x-z|^{d-1}} \, dz \right)^{\frac{1}{p}} \lesssim M_p(\nabla A)(x).
\]

It is easy to see that \( M_p \) maps \( L^p(\mathbb{R}^d, w) \) to \( L^{p,\infty}(\mathbb{R}^d, w) \) with \( w \) an \( A_1(\mathbb{R}^d) \) weight is equivalent to that the Hardy–Littlewood maximal operator \( M \) maps \( L^1(\mathbb{R}^d, w) \) to \( L^{1,\infty}(\mathbb{R}^d, w) \), which is, however, well known.

Next we consider \( S_{p,2}(\nabla A)(x) \). By using the Hölder inequality to deal with \( II(x) \) as those of \( I(x) \), then applying \(|x - y| \approx r\) and the Fubini theorem, we get

\[
S_{p,2}(\nabla A)(x) \lesssim \left[ \frac{1}{\omega(Q(x,2r))} \int_{Q(x,2r) \setminus Q(x,r)} \frac{1}{r} \left( \int_{|x-z| \leq \varepsilon} \frac{\left| \nabla A(z) \right|^p}{|y-z|^{d-1}} \, dz \right) w(y) \, dy \right]^\frac{1}{p} \\
\lesssim \left[ \frac{1}{\omega(Q(x,2r))} \int_{|x-z| \leq \varepsilon} r^{-1} \left( \int_{|y-z| \leq \varepsilon} \frac{w(y)}{|y-z|^{d-1}} \, dy \right) \left| \nabla A(z) \right|^p \, dz \right]^\frac{1}{p} \\
\lesssim \left[ \frac{1}{\omega(Q(x,2r))} \int_{|x-z| \leq \varepsilon} M(w)(z) \left| \nabla A(z) \right|^p \, dz \right]^\frac{1}{p} \\
\lesssim \left[ \frac{1}{\omega(Q(x,2r))} \int_{|x-z| \leq \varepsilon} \left| \nabla A(z) \right|^p w(z) \, dz \right]^\frac{1}{p} \lesssim M_{w,p}(\nabla A)(x),
\]

where in the third inequality, we use again the fact that the kernel function \( \tilde{k}(x) = \varepsilon^{-1} |x|^{-d+1} \chi_{\{|x| \leq \varepsilon \}} \) is a radial non-increasing function and \( L^1 \) integrable in \( \mathbb{R}^d \); the last second inequality follows from (2.2). As previously shown, \( M_{w,p} \) maps \( L^p(\mathbb{R}^d, w) \) to \( L^{p,\infty}(\mathbb{R}^d, w) \).
We consider $S_{p,3}(\nabla A)(x)$. Set $\mathcal{K}_j(x) = \frac{x_j}{|x|^r}$. Notice that $|x - z| > 2|x - y|$. Applying the Taylor expansion of $\mathcal{K}_j(x - z)$, we get
\[
\mathcal{K}_j(x - z) - \mathcal{K}_j(y - z) = \sum_{i=1}^d (x_i - y_i) \partial_i \mathcal{K}_j(x - z) + R(x, y, z),
\]
where the Taylor expansion's remainder term $R(x, y, z)$ satisfies
\[
|R(x, y, z)| \lesssim |x - y|^2|x - z|^{-d - 1}, \quad \forall |x - z| > 2|x - y|.
\]
Plunge the Taylor expansion's main term and remainder term into $S_{p,3}(\nabla A)(x)$ and split $S_{p,3}(\nabla A)(x)$ as two terms $S_{p,3,m}(\nabla A)(x)$ (related to main term) and $S_{p,3,r}(\nabla A)(x)$ (related to remainder term), respectively. Then by $|x - y| \approx r$, we have the following estimate of $S_{p,3,m}(\nabla A)(x)$:
\[
S_{p,3,m}(\nabla A)(x) \lesssim \left[ \frac{1}{w(Q(x, 2r))} \int_{Q(x, 2r)} \left( \sum_{j=1}^d \sum_{i=1}^d \left| \int_{|x - z| > 2|x - y|} \partial_i \mathcal{K}_j(x - z) \partial_j A(z) dz \right| w(y) dy \right]^\frac{1}{p}
\]
\[
\lesssim \sum_{j=1}^d \sum_{i=1}^d T_{i,j}^*(\partial_i A)(x),
\]
where the maximal singular integral operator $T_{i,j}^*(f)(x)$ is defined by
\[
T_{i,j}^*(f)(x) = \sup_{\epsilon > 0} \left| \int_{|x - y| > \epsilon} \partial_i \mathcal{K}_j(x - y) f(y) dy \right|.
\]
One can easily check that the kernel $\partial_i \mathcal{K}_j(x - y)$ is a standard Calderón–Zygmund convolution kernel that satisfies $(1.1)$, $(1.3)$ and has mean value zero on $S^{d-1}$. Then by the standard weighted Calderón–Zygmund theory (see [16]), $T_{i,j}$ is bounded on $L^p(\mathbb{R}^d, w)$, so $T_{i,j}^*$ maps $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$.

Finally, one can apply the method similar to that of $I$ to handle the remainder term $S_{p,3,r}(\nabla A)(x)$. Indeed, by the Hölder inequality and $|x - y| \approx r$, we get
\[
S_{p,3,r}(\nabla A)(x) \lesssim \left[ \frac{1}{w(Q(x, 2r))} \int_{Q(x,2r) \setminus Q(x,r)} r \left( \int_{r \leq |x - z|} \left| \nabla A(z) \right|^p \frac{dz}{|x - z|^{d+1}} w(y) dy \right)^\frac{1}{p}
\]
\[
\lesssim M_p(\nabla A)(x),
\]
where in the last inequality we use that the function $\epsilon |x|^{-d-1} \chi_{\{|x| > \epsilon\}}$ is radial non-increasing and $L^1$ integrable. As shown in the estimate of $I$, we get that $M_p$ maps $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$. Hence, we complete the proof. 

**Remark 2.7** When giving an estimate in $(2.5)$, we in fact prove that the inequality
\[
\frac{|A(x) - A(y)|}{|x - y|} \lesssim M(\nabla A)(x) + M(\nabla A)(y) + \sum_{i=1}^d \sum_{j=1}^d T_{i,j}^*(\partial_j A)(x)
\]
holds for almost every $x, y \in \mathbb{R}^d$ if $A$ is a $C^\infty$ function, where $T_{i,j}^*$ is defined in $(2.6)$. 

**Lemma 2.8** Let \( \{ Q_k \}_k \) be the disjoint cubes in \( \mathbb{R}^d \). Denote by \( l(Q_k) \) the side length of \( Q_k \). Define the operator \( T_s \) as
\[
T_s(f)(x) = \sum_k \int_{Q_k} \frac{l(Q_k)^s}{[l(Q_k) + |x - y|]^{d+s}} |f(y)| dy.
\]
Suppose that \( 1 \leq q \leq \infty \), \( w \in A_q(\mathbb{R}^d) \) and \( f \in L^q(\mathbb{R}^d, w) \). Then for any \( s > 0 \), we get that
\[
\| T_s(f) \|_{L^q(\mathbb{R}^d, w)} \lesssim \| f \|_{L^q(\mathbb{R}^d, w)}.
\]

**Proof** If \( q = 1 \), Lemma 2.8 just follows from property (2.2) of \( A_1(\mathbb{R}^d) \) weight and the Fubini theorem. In fact, we have
\[
\| T_s(f) \|_{L^1(\mathbb{R}^d, w)} \leq \sum_k \int_{Q_k} \left[ \int_{\mathbb{R}^d} \frac{w(x) \cdot l(Q_k)^s}{[l(Q_k) + |x - y|]^{d+s}} dx \right] |f(y)| dy
\]
\[
\lesssim \sum_k \int_{Q_k} M(w)(y) \cdot |f(y)| dy
\]
\[
\lesssim [w]_{A_1} \sum_k \int_{Q_k} |f(y)|w(y) dy \lesssim [w]_{A_1} \| f \|_{L^1(\mathbb{R}^d, w)},
\]
where the second inequality follows from that splitting the kernel
\[
\frac{l(Q_k)^s}{[l(Q_k) + |x - y|]^{d+s}}
\]
into two parts according to whether \( |x - y| \leq l(Q_k) \) or \( |x - y| > l(Q_k) \); the third inequality follows from the property (2.2) and in the last inequality we use that \( Q_k \) is cubes disjoint to each other. After we establish that \( T_s \) is bounded on \( L^1(\mathbb{R}^d, w) \) with bound \([w]_{A_1}\), the proof of the case \( 1 < q < \infty \) just follows from the famous extrapolation theorem (see, e.g., [16, Theorem 7.5.3]). If \( q = \infty \), apply the Fubini theorem,
\[
|T_s(f)(x)| \lesssim \sum_{Q_k} \| f \|_{L^\infty(Q_k)} \sup_{x \in Q_k} \int_{Q_k} \frac{l(Q_k)^s}{[l(Q_k) + |x - y|]^{d+s}} dy \lesssim \| f \|_{L^\infty(\mathbb{R}^d)}.
\]
Then \( T_s \) is bounded on \( L^\infty(\mathbb{R}^d, w) \) is just a consequence of the chain of inequalities:
\[
\| T_s(f) \|_{L^\infty(\mathbb{R}^d, w)} \lesssim \| T_s(f) \|_{L^\infty(\mathbb{R}^d)} \lesssim \| f \|_{L^\infty(\mathbb{R}^d, w)} \lesssim \| f \|_{L^\infty(\mathbb{R}^d)},
\]
which can be proved as follows. Notice that we have the equivalent definition of \( L^\infty(\mathbb{R}^d, \mu) \): \( \| f \|_{L^\infty(\mathbb{R}^d, \mu)} = \sup \{ \alpha : \mu(\{ x \in \mathbb{R}^d : |f(x)| > \alpha \}) > 0 \}. \) The first inequality in (2.8) follows from the fact that \( w(E) > 0 \) implies \( |E| > 0 \). Likewise, the last inequality in (2.8) follows from the fact that \( |E| > 0 \) implies \( w(E) > 0 \), because \( w(x) = 0 \) only for the points in a set of measure zero by the definition of \( A_\infty(\mathbb{R}^d) \) weight. Hence, we complete the proof.

**Remark 2.9** By the last argument above, for any \( w \in A_\infty(\mathbb{R}^d) \), the equality
\[
\| f \|_{L^\infty(\mathbb{R}^d)} \approx \| f \|_{L^\infty(\mathbb{R}^d, w)}
\]
holds. We will apply this equivalence straightforwardly many times later.
3 Proof of Theorem 1.1

3.1 Some Basic Strong Type Multilinear Estimates

In the sequel, we begin to give the proof of Theorem 1.1. In this subsection, we will first show our theorem in the case where \( q_1 = \cdots = q_n = \infty, r = p \in [1, \infty) \) which is not complicated and the case \( d < q_1, \ldots, q_n \leq \infty, 1 < r < \infty, p = \infty \).

**Proposition 3.1** Let \( q_i = \infty \) with \( i = 1, \ldots, n \), \( 1 \leq r = p < \infty \). We have the following conclusions.

(i) If \( p = r \in (1, \infty), w \in A_p(\mathbb{R}^d) \), then

\[
\| \mathcal{C}_a[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^p(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^n \| \nabla A_i \|_{L^\infty(\mathbb{R}^d, w)} \right) \| f \|_{L^p(\mathbb{R}^d, w)}.
\]

(ii) If \( p = r = 1, w \in A_1(\mathbb{R}^d) \), then

\[
\| \mathcal{C}_a[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{1, \infty}(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^n \| \nabla A_i \|_{L^\infty(\mathbb{R}^d, w)} \right) \| f \|_{L^1(\mathbb{R}^d, w)}.
\]

**Proof** The proof of this lemma is quite standard, so we just give some key steps. When \( q_1 = \cdots = q_n = \infty, A_i \) is a Lipschitz function for \( i = 1, \ldots, n \). Fix all \( A_i \). Observe that the kernel

\[
\mathfrak{R}(x, y) := K(x - y) \left( \prod_{i=1}^n \frac{A_i(x) - A_i(y)}{|x - y|} \right)
\]

is a standard Calderón–Zygmund kernel satisfying the boundedness condition and regularity condition with bound \( \prod_{i=1}^n \| \nabla A_i \|_{L^\infty(\mathbb{R}^d)} \) (see e.g., [17, Definition 4.1.2]). Then we have the following \( L^2 \) boundedness

\[
\| \mathcal{C}[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^2(\mathbb{R}^d)} \leq \left( \prod_{i=1}^n \| \nabla A_i \|_{L^\infty(\mathbb{R}^d)} \right) \| f \|_{L^2(\mathbb{R}^d)}.
\]

This in fact can be seen by using the famous \( T1 \) theorem (see [17]) or by applying the mean value formula

\[
\frac{A_i(x) - A_i(y)}{|x - y|} = \int_0^1 \left( \frac{x - y}{|x - y|}, \nabla A_i(sx + (1 - s)y) \right) ds
\]

to reduce the operator \( \mathcal{C} \) to the following operator introduced by Christ and Journé [8]:

\[
\mathcal{C}_{\mathcal{C}}[a_1, \ldots, a_n, f](x) = \text{p.v.} \int_{\mathbb{R}^d} k(x - y) \left( \prod_{i=1}^n m_{x, y} a_i \right) f(y) dy,
\]

which maps \( L^\infty(\mathbb{R}^d) \times \cdots \times L^\infty(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d) \) (see [8]). Here in the above operator \( k(x - y) \) is a standard Calderón–Zygmund kernel and \( m_{x, y} a = \int_0^1 a(sx + (1 - s)y) dy \). Then the rest of the proof just follows from the standard weighted Calderón–Zygmund theory (see [16, Theorem 7.4.6]).

**Proposition 3.2** Suppose that \( 1 < r < \infty, d < q_1, \ldots, q_n \leq \infty \) and \( \frac{1}{r} = \sum_{i=1}^n q_i \). Let \( w \in \bigcap_{i=1}^n A_{\frac{q_i}{2}}(\mathbb{R}^d) \). Assume that \( \nabla A_i \in L^{q_i}(\mathbb{R}^d, w), i = 1, \ldots, n \) and \( f \in L^\infty(\mathbb{R}^d, w) \).
Then we get
\[ \|C[\nabla A_1, \ldots, \nabla A_n, f]\|_{L^r(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^n \|\nabla A_i\|_{L^q_i(\mathbb{R}^d, w)} \right) \|f\|_{L^\infty(\mathbb{R}^d, w)}. \]

**Proof.** By the standard limiting arguments, it is enough to consider that each \( A_i \) are \( C_c^\infty \) functions and \( f \) is bounded compact function. Then one can easily check that
\[ \int_{\mathbb{R}^d} [M_\delta (C[\nabla A_1, \ldots, \nabla A_n, f])(x)]^t w(x) \, dx \text{ is finite} \] (for example one may use the method in [22, p. 1248] to show this). Therefore, using the Fefferman–Stein inequality in Lemma 2.2, we can get that for any \( \delta > 0 \),
\[ \|C[\nabla A_1, \ldots, \nabla A_n, f]\|_{L^r(\mathbb{R}^d, w)} \lesssim \|M_\delta (C[\nabla A_1, \ldots, \nabla A_n, f])\|_{L^r(\mathbb{R}^d, w)} \lesssim \|M_\delta^1 (C[\nabla A_1, \ldots, \nabla A_n, f])\|_{L^r(\mathbb{R}^d, w)}. \]

In the following, we need to give an estimate of the maximal sharp function. Fix \( x \) and a cube \( Q \ni x \). Define \( f_1 = f \chi_{3Q} \) and \( f_2 = f - f_1 \). Then write
\[ C[\nabla A_1, \ldots, \nabla A_n, f](x) = C[\nabla A_1, \ldots, \nabla A_n, f_1](x) + C[\nabla A_1, \ldots, \nabla A_n, f_2](x). \]

Choose a constant \( c = C[\nabla A_1, \ldots, \nabla A_n, f_2](x) \) in the maximal sharp function. Then we see that this maximal sharp function \( M_1^1 (C[\nabla A_1, \ldots, \nabla A_n, f]) \) is bounded by the following two functions:
\[
I(x) + II(x) := \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |C[\nabla A_1, \ldots, \nabla A_n, f_1](z)|^\delta \, dz \right)^{\frac{1}{\delta}} + \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |C[\nabla A_1, \ldots, \nabla A_n, f_2](z) - C[\nabla A_1, \ldots, \nabla A_n, f_2](x)|^\delta \, dz \right)^{\frac{1}{\delta}}.
\]

We first consider the above first function \( I(x) \). Define \( \tilde{A}_i = A_i \chi_{3Q} \). Then for any \( z \in Q \), we can write
\[ C[\nabla A_1, \ldots, \nabla A_n, f_1](z) = C[\tilde{\nabla} A_1, \ldots, \tilde{\nabla} A_n, f_1](z). \]

Choose \( \delta \leq d/n \). Applying the Hölder inequality, strong type multilinear estimate (see [21, Theorem 1.1]), and the definition of \( \tilde{A}_i \), we can get
\[
\left( \frac{1}{|Q|} \int_{\tilde{\nabla} A_1, \ldots, \tilde{\nabla} A_n, f_1}(z)^\delta \, dz \right)^{\frac{1}{\delta}} \lesssim \|C[\tilde{\nabla} A_1, \ldots, \tilde{\nabla} A_n, f_1]\|_{L^\frac{\delta}{\delta - 1}(Q, \mathbb{R}^d)} \lesssim \|f_1\|_{L^\infty(\mathbb{R}^d)} \prod_{i=1}^n \|\tilde{\nabla} A_i\|_{L^q_i(\mathbb{R}^d, \mathbb{R}^d)} \lesssim \|f\|_{L^\infty(\mathbb{R}^d)} \prod_{i=1}^d M_d(\nabla A_i)(x),
\]
where \( M_d \) is the Hardy–Littlewood maximal operator of order \( d \). Notice that \( w \in \bigcap_{i=1}^n A_{q_i/d}(\mathbb{R}^d) \), by using the weighted boundedness of the Hardy–Littlewood maximal operator, one can easily get that \( \|M_d(\nabla A_i)\|_{L^{q_i}(\mathbb{R}^d, w)} \lesssim \|\nabla A_i\|_{L^{q_i}(\mathbb{R}^d, w)} \). Therefore,
\[ \|I\|_{L^r(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^n \|\nabla A_i\|_{L^{q_i}(\mathbb{R}^d, w)} \right) \|f\|_{L^\infty(\mathbb{R}^d, w)}. \]
Next we turn to $II(x)$. Write

$$
\mathcal{C} [\nabla A_1, \ldots, \nabla A_n, f_2](z) - \mathcal{C} [\nabla A_1, \ldots, \nabla A_n, f_2](x) = 
\int_{(3Q)^c} \left[ \mathcal{R}(z, y) - \mathcal{R}(x, y) \right] f(y) \, dy,
$$

where \( \mathcal{R}(x, y) := K(x - y) \prod_{i=1}^n \frac{A_i(z) - A_i(y)}{|x - y|^n} \). Then write

$$
\mathcal{R}(z, y) - \mathcal{R}(x, y)
= \left( \frac{K(z - y)}{|z - y|^n} - \frac{K(x - y)}{|x - y|^n} \right) \prod_{i=1}^n (A_i(z) - A_i(y)) 
+ \frac{K(x - y)}{|x - y|^n} \left( \prod_{i=1}^n (A_i(z) - A_i(y)) - \prod_{i=1}^n (A_i(x) - A_i(y)) \right) 
=: \mathcal{R}_1(z, x, y) + \mathcal{R}_2(z, x, y).
$$

We consider the term \( \mathcal{R}_1(z, x, y) \). Notice that \( x, z \in Q \) and \( y \in (3Q)^c \); then \( |z - y| \approx |x - y| \). By the regularity condition (1.3) and the formula (2.7), we get that

$$
|\mathcal{R}_1(z, x, y)| \lesssim \frac{(I(Q)^\delta}{|x - y|^{d+\delta}} \prod_{i=1}^d [M(\nabla A_i)(z) + T(\nabla A_i)(y)],
$$

where here and in the following, \( T \) is the sum of combination of the Hardy–Littlewood maximal operator and maximal singular integral \( T^*_1 \) defined in (2.6), which both map \( L^q(\mathbb{R}^d, w) \) to itself for \( 1 < q < \infty \).

Next we consider the term \( \mathcal{R}_2(z, x, y) \). We can split \( \mathcal{R}_2(z, x, y) \) into \( n \) terms and apply (2.7):

$$
\mathcal{R}_2(z, x, y) \lesssim \frac{1}{|x - y|^{d+\delta}} \left| \sum_{i=1}^n [A_i(z) - A_i(x)] \prod_{k=1}^{i-1} [A_k(x) - A_k(y)] \right| 
\times \prod_{k=i+1}^n [A_k(z) - A_k(y)] 
\lesssim \frac{I(Q)}{|x - y|^{d+\delta}} \prod_{i=1}^n [M(\nabla A_i)(z) + T(\nabla A_i)(x) + T(\nabla A_i)(y)].
$$

Combining these estimates of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), we get

$$
|\mathcal{R}(z, y) - \mathcal{R}(x, y)|
\lesssim \frac{(I(Q)^\delta}{|x - y|^{d+\delta}} \prod_{i=1}^n [M(\nabla A_i)(z) + T(\nabla A_i)(x) + T(\nabla A_i)(y)]
\lesssim \frac{(I(Q)^\delta}{|x - y|^{d+\delta}} \sum_{\mathbb{N}^{\oplus}_1} \prod_{i \in \mathbb{N}_1} M(\nabla A_i)(z) \prod_{i \in \mathbb{N}_2} T(\nabla A_i)(x) \prod_{i \in \mathbb{N}_3} T(\nabla A_i)(y),
$$

where in the last inequality we divide \( \mathbb{N}^{\oplus}_1 = N_1 \cup N_2 \cup N_3 \) with \( \mathbb{N}_1 = \{1, \ldots, n\} \) and \( N_1, N_2, N_3 \) non intersecting each other. Plugging the above estimates into \( II(x) \)
and applying the H"older inequality, we get that

\[
II(x) \leq \sum_{n=1}^{\infty} \left( \prod_{i \in \mathbb{N}_2} T(\nabla A_i)(x) \right) \left( \frac{1}{|Q|} \int_Q \int_{(3Q)^c} \frac{(I(Q))^{\delta}}{|x - y|^{d + \delta}} \right) \times \left( \prod_{i \in \mathbb{N}_3} T(\nabla A_i)(y) \right) f(y) dy \\
\times \left( \prod_{i \in \mathbb{N}_1} T(\nabla A_i)(z) \right) dz \\
\leq \|f\|_{L^q(\mathbb{R}^d, w)} \sum_{n=1}^{\infty} \left( \prod_{i \in \mathbb{N}_2} T(\nabla A_i)(x) \right) M \left( \prod_{i \in \mathbb{N}_3} T(\nabla A_i)(x) \right) \times M \left( \prod_{i \in \mathbb{N}_1} T(\nabla A_i)(x) \right).
\]

Now, using the H"older inequality and the fact that $M$ and $T$ are bounded on $L^q(\mathbb{R}^d, w)$ for $1 < q < \infty$, we get

\[
\|II\|_{L^r(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^{\infty} \|\nabla A_i\|_{L^q(\mathbb{R}^d, w)} \right) \|f\|_{L^\infty(\mathbb{R}^d, w)},
\]

which completes the proof.

**Proposition 3.3** Suppose that $1 < r < \infty$, $d < q_1, \ldots, q_n \leq \infty$, $p = \infty$ and $\frac{1}{\tau} = \sum_{i=1}^{\infty} \frac{1}{q_i}$. Let $w \in \cap_{i=1}^{\infty} A_{q_i/d}(\mathbb{R}^d)$. Then we get

\[
\|C_\epsilon[\nabla A_1, \ldots, \nabla A_n, f]\|_{L^r(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^{\infty} \|\nabla A_i\|_{L^{q_i}(\mathbb{R}^d, w)} \right) \|f\|_{L^\infty(\mathbb{R}^d)}.
\]

**Proof** Let $\varphi$ be a $C_c^\infty$ function that is supported in $\{x \in \mathbb{R}^d : |x| < 1/4\}$, $\varphi(x) = 1$ if $|x| < 1/8$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. Set $\varphi_\epsilon(x) = \epsilon^{-d} \varphi(\epsilon^{-1} x)$. It is easy to see that $\varphi_\epsilon \ast C[\nabla A_1, \ldots, \nabla A_n, f](x)$ is bounded by $M(C[\nabla A_1, \ldots, \nabla A_n, f])(x)$. By the weighted boundedness of the Hardy–Littlewood maximal operator $M$ and Proposition 3.2, we get that

\[
\|M(C[\nabla A_1, \ldots, \nabla A_n, f])\|_{L^r(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^{\infty} \|\nabla A_i\|_{L^{q_i}(\mathbb{R}^d, w)} \right) \|f\|_{L^\infty(\mathbb{R}^d, w)}.
\]

So, to complete the proof, it is sufficient to show that the difference

\[
C_\epsilon[\nabla A_1, \ldots, \nabla A_n, f](x) - \varphi_\epsilon \ast C[\nabla A_1, \ldots, \nabla A_n, f](x)
\]

is controlled uniformly in $\epsilon$ by a function that is bounded from $L^{q_1}(\mathbb{R}^d, w) \times \cdots \times L^{q_n}(\mathbb{R}^d, w) \times L^\infty(\mathbb{R}^d, w)$ to $L^r(\mathbb{R}^d, w)$. We write the difference in the above equality as follows:

\[
\int_{\mathbb{R}^d} \varphi_\epsilon(z) \left( \int_{|x - y| > \epsilon} \left( \tilde{R}(x, y) - \tilde{R}(x - z, y) \right) f(y) dy \right) dz \\
+ \int_{\mathbb{R}^d} \varphi_\epsilon(z) \left[ \text{p.v.} \int_{|x - y| < \epsilon} \tilde{R}(x - z, y) f(y) dy \right] dz = P_\epsilon(x) + Q_\epsilon(x),
\]
where \( \tilde{\mathcal{K}}(x, y) := K(x - y) \prod_{i=1}^{n} \frac{A_i(x) - A_i(y)}{|x - y|^n} \). Now we first give an estimate of \( Q_\varepsilon(x) \). By the Fubini theorem,

\[
Q_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x - z) \left[ \text{p.v.} \int_{|x - z| < \varepsilon} \tilde{\mathcal{K}}(z, y) f(y) \, dy \right] dz
\]

Notice that \(|x - y| < \varepsilon\) and \(|x - z| < \frac{\varepsilon}{4}\). For each \( i = 1, \ldots, n \), define \( \tilde{A}_i(\cdot) = A_i(\cdot) \chi_{\{|x - z| < \varepsilon\}} \). Then all \( A_i \) in the above inequality can be replaced by \( \tilde{A}_i \). Choose \( \tilde{r} = (\sum_{i=1}^{n} \frac{1}{q_i}) + \frac{1}{p} \) such that \( 1 < \tilde{r} < +\infty, 1 \leq \tilde{q}_i < q_i < \infty \) for all \( i = 1, \ldots, n \), \( 1 < p < \infty \). Then by the multilinear boundedness properties of \( C[\nabla A_1, \ldots, \nabla A_n, f](x) \), we may continue to give an estimate of \( Q_\varepsilon(x) \) as follows:

\[
|Q_\varepsilon(x)| \leq \| f \|_{L^\infty(\mathbb{R}^d)} \varepsilon^d \prod_{i=1}^{n} \| \nabla \tilde{A}_i \|_{L^{\tilde{q}_i}(\mathbb{R}^d)} \| \varphi_\varepsilon \|_{L^{\tilde{r}}(\mathbb{R}^d)}
\]

As we have done in the proof of Lemma 2.3, we see that \( M_{\tilde{q}_i} \) maps \( L^{\tilde{q}_i}(\mathbb{R}^d, w) \) to itself for \( w \in A_{\tilde{q}_i/d}(\mathbb{R}^d) \). Then by using the Hölder inequality, we get that

\[
\| \sup_{\varepsilon} |Q_\varepsilon| \|_{L^r(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^{n} \| \nabla A_i \|_{L^{\tilde{q}_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^\infty(\mathbb{R}^d, w)}.
\]

Next we turn to \( P_\varepsilon(x) \). Write

\[
\tilde{\mathcal{K}}(x, y) - \tilde{\mathcal{K}}(x - z, y) = \left( \frac{K(x - y)}{|x - y|^n} - \frac{K(x - z - y)}{|x - z - y|^n} \right) \prod_{i=1}^{n} (A_i(x) - A_i(y))
\]

\[
+ \frac{K(x - z - y)}{|x - z - y|^n} \left[ \prod_{i=1}^{n} (A_i(x) - A_i(y)) - \prod_{i=1}^{n} (A_i(x - z) - A_i(y)) \right]
\]

\[
=: I + II.
\]

We consider the term \( I \). Notice that \(|x - y| > \varepsilon\) and \(|z| < \frac{\varepsilon}{4}\); then \(|x - y| \approx |x - z - y| \).

By the regularity condition (1.3) and (2.7), we get that

\[
|I| \leq \frac{\varepsilon^d}{|x - y|^{d+\delta}} \prod_{i=1}^{d} [M(\nabla A_i)(x) + T(\nabla A_i)(y)].
\]

Consider the term \( II \). We can split \( II \) into \( n \) terms and use (2.7):

\[
|II| \leq \frac{1}{|x - y|^{d+n}} \sum_{i=1}^{n} \left| \prod_{k=1}^{i-1} [A_k(x - z) - A_k(y)] \prod_{k=i+1}^{n} [A_k(x) - A_k(y)] \right|
\]

\[
\leq \frac{\varepsilon}{|x - y|^{d+n}} \prod_{i=1}^{n} [M(\nabla A_i)(x) + T(\nabla A_i)(x - z) + T(\nabla A_i)(y)].
\]
Combining the estimates of $I$ and $II$, we get

$$| \mathcal{R}(x, y) - \mathcal{R}(x, z, y)|$$

$$\leq \frac{e^d}{|x - y|^{d + \delta}} \prod_{i=1}^{n} [M(\nabla A_i)(x) + T(\nabla A_i)(x - z) + T(\nabla A_i)(y)]$$

$$\leq \frac{e^d}{|x - y|^{d + \delta}} \sum_{i \in N_2} \prod_{i \in N_1} M(\nabla A_i)(x) \prod_{i \in N_2} T(\nabla A_i)(x - z) \prod_{i \in N_3} T(\nabla A_i)(y),$$

where in the last inequality we divide $\mathbb{N}_n^u = N_1 \cup N_2 \cup N_3$ with $N_1, N_2, N_3$, not intersecting each other. Plugging the above estimate into $P_\varepsilon(x)$, we get that

$$|P_\varepsilon(x)| \leq \sum_{i \in N_1} \left[ \prod_{i \in N_1} M(\nabla A_i)(x) \right] \int_{\mathbb{R}^d} \int_{|x - y| > \varepsilon} \frac{e^d}{|x - y|^{d + \delta}}$$

$$\times \left[ \prod_{i \in N_3} T(\nabla A_i)(y) \right] f(y) dy$$

$$\times q_\varepsilon(z) \left[ \prod_{i \in N_2} T(\nabla A_i)(x - z) \right] dz$$

$$\leq \|f\|_{L^q(\mathbb{R}^d)} \sum_{i \in N_1} \left[ \prod_{i \in N_1} M(\nabla A_i)(x) \right] M \left[ \prod_{i \in N_2} T(\nabla A_i) \right](x)$$

$$\cdot M \left[ \prod_{i \in N_3} T(\nabla A_i) \right](x).$$

Now using the Hölder inequality and the fact that $M, T$ are bounded on $L^q(\mathbb{R}^d, w)$ for $1 < q < \infty$, we get that

$$\|\sup_\varepsilon P_\varepsilon\|_{L^r(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^{n} \|\nabla A_i\|_{L^q(\mathbb{R}^d, w)} \right) \|f\|_{L^q(\mathbb{R}^d, w)},$$

which completes the proof.

### 3.2 Case: All $q$s are Larger than $d$

In this subsection, we consider the case where $d/(d + n) = r < \infty$ and $d \leq q_1, \ldots, q_n \leq \infty$. Without loss of generality, we assume that the first $q_1, \ldots, q_l > d$ and $q_{l+1}, \ldots, q_n = d$ with $0 \leq l \leq n$. Here and in the following, when $l = 0$, we mean all $q_1 = \cdots = q_n = d$. The proof of the case $p = \infty$ is slightly different from that of $1 \leq p < \infty$. So we shall give two propositions below. Let us see the case $1 \leq p < \infty$ first, and we emphasize in the proof where it does not work for $p = \infty$.

**Proposition 3.4** Let $\frac{1}{d} = \left( \sum_{i=1}^{n} \frac{1}{q_i} \right) + \frac{1}{p} \cdot \frac{d}{d+n} \leq r < \infty$, $d < q_1, \ldots, q_l \leq \infty$, and $q_{l+1} = \cdots = q_n = d$ with $0 \leq l \leq n, 1 \leq p < \infty$. Suppose that $w \in \left( \bigcap_{i=1}^{n} A_{\text{max}}(\frac{q_i}{d}, 1) (\mathbb{R}^d) \right) \cap A_p(\mathbb{R}^d)$. Then

$$\|\nabla_A f\|_{L^{r,\infty}(\mathbb{R}^d, w)} \approx$$

$$\left( \prod_{i=1}^{l} \|\nabla A_i\|_{L^{q_i}(\mathbb{R}^d, w)} \right) \left( \prod_{i=l+1}^{n} \|\nabla A_i\|_{L^{q_i}(\mathbb{R}^d, w)} \right) \|f\|_{L^p(\mathbb{R}^d, w)},$$

where $L^{d,1}(\mathbb{R}^d, w)$ is the weighted Lorentz space.
Proof By the density limiting argument and scaling argument, it is sufficient to prove that when \( A_i \) \((i = 1, \ldots, n)\) and \( f \) are \( C^\infty \) functions with compact supports,

\[
\|\nabla A_i\|_{L^{\infty}(\mathbb{R}^d, w)} = \|\nabla A_j\|_{L^{\infty}(\mathbb{R}^d, w)} = \|f\|_{L^p(\mathbb{R}^d, w)} = 1,
\]

for \( i = 1, \ldots, l \) and \( j = l + 1, \ldots, n \), the inequality

\[
w(\{x \in \mathbb{R}^d : \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f](x) > \lambda\}) \lesssim \lambda^{-r}
\]

holds for any \( \lambda > 0 \). Fix \( \lambda > 0 \). For convenience we set

\[
E_\lambda = \{x \in \mathbb{R}^d : \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f](x) > \lambda\}.
\]

Our goal is to show \( w(E_\lambda) \lesssim \lambda^{-r} \). First assume that all \( q_1, \ldots, q_l < \infty \). Once the proof in this situation is well understood, we can modify the proof to the other case that there exist some \( q_l = \infty \) for \( i = 1, \ldots, l \). We will show how to do this in the last part of the proof. Define the exceptional set

\[
J_{i, l} = \{x \in \mathbb{R}^d : \mathcal{M}(\nabla A_i)(x) > \lambda^{\frac{2}{l_i}}\}
\]

for \( i = 1, \ldots, n \). Here it should be pointed out that the above definition is meaningless if \( q_i = \infty \). Therefore, we need to assume all \( q_i < \infty \) first. By Lemmas 2.3 and 2.4, \( \mathcal{M} \) maps \( L^p(\mathbb{R}^d, w) \) to itself for \( p > d \) and maps \( L^{d,1}(\mathbb{R}^d, w) \) to \( L^{d,\infty}(\mathbb{R}^d, w) \); i.e.,

\[
\begin{align*}
\text{(3.1)} \quad & w(J_{i, l}) \lesssim \lambda^{-r} \|\nabla A_i\|_{L^{\infty}(\mathbb{R}^d, w)}^{\frac{l_i}{l}} = \lambda^{-r}, & \quad i = 1, \ldots, l, \\
& w(J_{j, l}) \lesssim \lambda^{-r} \|\nabla A_j\|_{L^{\infty}(\mathbb{R}^d, w)}^{\frac{l_j}{l}} = \lambda^{-r}, & \quad j = l + 1, \ldots, n.
\end{align*}
\]

Set \( J_k = \cup_{i=1}^n J_{i, k} \). Since \( w(x)dx \) satisfies the doubling property, we can choose an open set \( G_\lambda \) which satisfies the conditions \( J_\lambda \subset G_\lambda \) and \( w(G_\lambda) \lesssim w(J_\lambda) \). By property (3.1) of \( J_{i, k} \), we see that \( w(G_\lambda) \lesssim \lambda^{-r} \). Next making a Whitney decomposition of \( G_\lambda \) (see e.g., [16]), we can obtain a family of disjoint dyadic cubes \( \{Q_k\}_k \) such that

\[
\begin{align*}
\text{(i)} \quad & G_\lambda = \bigcup_{k=1}^\infty Q_k; \\
\text{(ii)} \quad & \sqrt{d} \cdot l(Q_k) \leq \text{dist}(Q_k, (G_\lambda)^c) \leq 4\sqrt{d} \cdot l(Q_k).
\end{align*}
\]

With those properties (i) and (ii), for each \( Q_k \), we can construct a larger cube \( Q_k^* \) so that \( Q_k \subset Q_k^* \), \( Q_k^* \) is centered at \( y_k \) and \( y_k \in (G_\lambda)^c \), \( l(Q_k^*) \approx l(Q_k) \). By the property (ii) above, the distance between \( Q_k \) and \( (G_\lambda)^c \) equals \( \text{Cl}(Q_k) \). Therefore, by the construction of \( Q_k^* \) and \( y_k \), one can get

\[
\text{(3.2)} \quad \text{dist}(y_k, Q_k) \approx l(Q_k), \quad w(Q_k^*) = w(Q_k).
\]

Now we come back to give an estimate of \( w(E_\lambda) \). Split \( f \) into two parts \( f = f_1 + f_2 \), where \( f_1(x) = f(x)\chi_{(G_1)^c}(x) \) and \( f_2(x) = f(x)\chi_{G_1}(x) \). By the definition of \( J_\lambda \), when restricted on \( (G_1)^c \), \( A_1 \) is a Lipschitz function with \( \|\nabla A_1\|_{L^\infty((G_1)^c)} \lesssim \lambda^{\frac{2}{l_1}} \) for \( i = 1, \ldots, n \). Let \( \widetilde{A}_1 \) represent the Lipschitz extension of \( A_1 \) from \( (G_1)^c \) to \( \mathbb{R}^d \) (see [26, p. 174, Theorem 3]) so that for each \( i = 1, \ldots, n \),

\[
\begin{align*}
\widetilde{A}_i(y) &= A_i(y) \quad & \text{if } y \in (G_1)^c, \\
|\widetilde{A}_i(x) - \widetilde{A}_i(y)| &\leq \lambda^{\frac{2}{l_i}}|x - y| \quad & \text{for all } x, y \in \mathbb{R}^d.
\end{align*}
\]
Since the operator $C_{a}[\cdots,\cdot]$ is sub-multilinear, we split $E_{\lambda}$ as three terms and give estimates as follows:

$$w(\{x \in \mathbb{R}^{d} : C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f](x) > \lambda\})$$
$$\leq w(10G_{A}) + w(\{x \in (10G_{A})^{c} : C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f_{1}](x) > \lambda/2\})$$
$$+ w(\{x \in (10G_{A})^{c} : C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f_{2}](x) > \lambda/2\}).$$

The above first term satisfies $w(10G_{A}) \leq \lambda^{-r}$, which is our required estimate. In the following, we only consider the second terms. Notice that we only need to consider $x \in (10G_{A})^{c}$. By the definition of $f_{i}$, it is not difficult to see that

$$C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f_{1}](x) = C_{a}[\nabla \widetilde{A}_{1}, \ldots, \nabla \widetilde{A}_{n}, f_{1}](x).$$

With this equality in hand, Proposition 3.1 $(1 \leq p < \infty)$ implies

$$(3.3) \quad w(\{x \in (10G_{A})^{c} : C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f_{1}](x) > \lambda/2\})$$
$$= w(\{x \in (10G_{A})^{c} : C_{a}[\nabla \widetilde{A}_{1}, \ldots, \nabla \widetilde{A}_{n}, f_{1}](x) > \lambda/2\})$$
$$\leq \lambda^{-p} \left( \prod_{i=1}^{n} \| \nabla \widetilde{A}_{i} \|_{L^{\infty}(\mathbb{R}^{d},w)} \right) \| f_{1} \|_{L^{p}(\mathbb{R}^{d},w)} \leq \lambda^{-p} + \frac{\lambda^{-r}}{\pi_{i=1}^{n} \frac{\lambda}{\pi_{i}}} = \lambda^{-r}.$$ 

If $p = \infty$, the above method does not work. We will show how to prove this kind of estimate in the next proposition.

Let us turn to $C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f_{2}](x)$. Recall $N_{j}^{i} = \{i, i+1, \ldots, j\}$ and our construction of $G_{A}, y_{k}, Q_{k}$ and $Q_{k}^{c}$ above (3.2). Then by property (i) of $\{Q_{k}\}_{k}$, we can write $f_{2} = \sum_{k} f_{k} \chi_{Q_{k}}$. Therefore, we can get

$$C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f_{2}](x) = \sum_{k} C_{a}[\nabla A_{1}, \ldots, \nabla A_{n}, f \chi_{Q_{k}}](x).$$

In the following we need to study carefully $\prod_{i=1}^{n} \frac{A_{i}(x) - A_{j}(y)}{|x - y|}$. We will separate it into several terms and then give an estimate for each term. Write

$$\prod_{i=1}^{n} \frac{A_{i}(x) - A_{j}(y)}{|x - y|}$$
$$= \prod_{i=1}^{n} \left( \frac{\widetilde{A}_{i}(x) - \widetilde{A}_{i}(y)}{|x - y|} + \frac{\widetilde{A}_{i}(y) - \widetilde{A}_{i}(y_{k})}{|x - y|} + \frac{A_{i}(y_{k}) - A_{i}(y)}{|x - y|} \right)$$
$$= \sum_{i \in N_{1}} \left( \frac{\widetilde{A}_{i}(x) - \widetilde{A}_{i}(y)}{|x - y|} \right) \left( \prod_{i \in N_{2}} \frac{\widetilde{A}_{i}(y) - \widetilde{A}_{i}(y_{k})}{|x - y|} \right) \left( \prod_{i \in N_{3}} \frac{A_{i}(y_{k}) - A_{i}(y)}{|x - y|} \right)$$
$$= I(x, y) + II(x, y, y_{k}),$$

where in the third equality we divide $N_{j}^{i} = N_{1} \cup N_{2} \cup N_{3}$ with $N_{1}, N_{2}, N_{3}$, not intersecting each other; and $I(x, y), II(x, y, y_{k})$, are defined as follows
(3.4) \[ I(x, y) = \prod_{i=1}^{n} \frac{\widetilde{A}_i(x) - \widetilde{A}_i(y)}{|x - y|}, \]

\[ II(x, y, y_k) = \sum_{N_1 \in \mathbb{N}_1^n} \left[ \prod_{i \in N_1} \frac{\widetilde{A}_i(x) - \widetilde{A}_i(y)}{|x - y|} \right] \times \left[ \prod_{i \in N_2} \frac{\widetilde{A}_i(y) - \widetilde{A}_i(y_k)}{|x - y|} \right] \left[ \prod_{i \in N_3} \frac{A_i(y_k) - A_i(y)}{|x - y|} \right]. \]

By the above decomposition, we in fact write \( \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f \chi_{Q_1}](x) \) into \( 3^n \) terms and separate these terms into two parts according \( I \) and \( II \).

**Weighted estimate of \( \mathcal{C}_s[\cdot, \cdot, \cdot] \) related to \( I \).** This estimate is similar to (3.3). In fact, in this case there is only one term \( \mathcal{C}_s[\nabla \widetilde{A}_1, \ldots, \nabla \widetilde{A}_n, f_2] \). Then by Proposition 3.1 \((1 \leq p < \infty)\), we get

\[ w(\{ x \in (10G_1) : \mathcal{C}_s[\nabla \widetilde{A}_1, \ldots, \nabla \widetilde{A}_n, f_2](x) > \lambda/2 \}) \leq \lambda^{-p} \left( \prod_{i=1}^{n} \| \nabla \widetilde{A}_i \|_{L^p(\mathbb{R}^d, w)} \right) f_2 \| f_2 \|_{L^p(\mathbb{R}^d, w)} \leq \lambda^{-p} \sum_{i=1}^{n} \lambda^{-r_i} = \lambda^{-r}. \]

If \( p = \infty \), the above argument may not work again.

**Weighted Estimate of \( \mathcal{C}_s[\cdot, \cdot, \cdot] \) Related to \( II \).** It is sufficient to consider one term \( \mathcal{C}_s[\cdot, \cdot, \cdot] \) related to \( II \) in which \( N_1 \) is a proper subset of \( \mathbb{N}_1^n \). In such a case, without loss of generality, we may suppose that \( N_1 = \{1, \ldots, v\}, N_2 = \{v + 1, \ldots, m\}, \) and \( N_3 = \{m + 1, \ldots, n\} \) with \( 0 \leq v \leq m \leq n \) and \( v < n \). Here if \( v = 0 \), it means that \( N_1 = \emptyset \); if \( v = m \), \( N_2 = \emptyset \); if \( m = n \), \( N_3 = \emptyset \). With this notation, it is easy to see that \( N_1 \) is a proper subset of \( \mathbb{N}_1^n \). By a slight abuse of notation, we still utilize \( II(x, y, y_k) \) to represent one term related to \( N_1, N_2 \) and \( N_3 \) in (3.4) and utilize \( H_{II}(x) \) to represent \( \mathcal{C}_s[\cdot, \cdot, \cdot] \) related to \( II(x, y, y_k) \); i.e.,

\[ H_{II}(x) = \sup_{\epsilon > 0} \left| \sum_{k} \int_{|x - y| > \epsilon} K(x - y)II(x, y, y_k)f(y)\chi_{Q_k}(y)dy \right|. \]

Notice that \( y_k \in (G_1)^c \), thus \( y_k \in (J_{i, \lambda})^c \). Therefore, we obtain that

\[ \mathcal{M}(\nabla A_i)(y_k) \leq \lambda^{\frac{r_i}{q_i}}, \quad \text{for } i = m + 1, \ldots, n. \]

With the above fact and \( \widetilde{A}_i \) being a Lipschitz function with bound \( \lambda^{r_i/q_i} \) for \( i = 1, \ldots, m \), we get

\[ |II(x, y, y_k)| \leq \lambda^{\sum_{i=1}^{m} \frac{r_i}{q_i}} \left| \frac{y - y_k}{|x - y|} \right|^{n - v} \prod_{i=m+1}^{n} \mathcal{M}(\nabla A_i)(y_k) \leq \lambda^{\sum_{i=1}^{m} \frac{r_i}{q_i}} \left| \frac{y - y_k}{|x - y|} \right|^{n - v}. \]

Since it is sufficient to consider \( x \in (10G_1)^c \), for \( y \in Q_k, |x - y| \geq 2l(Q_k) \approx |y - y_k| \) by (3.2). Combining the above discussion with (1.1), we obtain
\[ H_{II}(x) \leq \sum_{k} \int_{Q_k} |K(x-y)| \cdot |II(x, y, y_k)| \cdot |f(y)| \, dy \]
\[ \leq \lambda^{\sum_{i=1}^{n} \frac{r_i}{p_i}} \sum_{k} \int_{Q_k} \frac{l(Q_k)^{n-v}}{l(Q_k) + |x-y|^{d+n-v}} |f(y)| \, dy \]
\[ = \lambda^{\sum_{i=1}^{n} \frac{r_i}{p_i}} T_{n-v} f(x), \]
where \( T_{n-v} \) is defined as in Lemma 2.8. Applying the Chebyshev inequality with the above estimate, and using Lemma 2.8 (notice that \( n - v \geq 1 \) because \( N_i \) is a proper set of \( \mathbb{N}^n \)), we finally get
\[ w(\{x \in (10G_1)^c : H_{II}(x) > \lambda\}) \leq \lambda^{-p + \sum_{i=1}^{n} \frac{r_i}{p_i}} \| T_{n-v} f \|_{L^p(\mathbb{R}^d, w)} \]
\[ \leq \lambda^{-r} \| f \|_{L^p(\mathbb{R}^d, w)}. \]
Hence, we finish the proof of the term II. If \( p = \infty \), the above last argument may not work and some different discussion should be involved; see the proof in the next proposition.

Finally, we show how to modify our proof here to the case \( q_i = \infty \) for some \( i = 1, \ldots, l \). We can assume that only \( q_1 = \cdots = q_u = \infty \) with \( 1 \leq u \leq l \). Thus, \( A_1, \ldots, A_u \) are Lipschitz functions, which in fact are nice functions. Then we just fix \( A_1, \ldots, A_u \) in the rest of the proof. We only make a construction of exceptional set for \( A_{u+1}, \ldots, A_n \) and study \( \prod_{i=u+1}^{n} A_i(x) - A_i(y) \) the same way as we have done previously. After that, using that \( A_1, \ldots, A_u \) are Lipschitz functions to deal with all estimates involved with \( A_1, \ldots, A_u \), we could obtain our required bound.

**Proposition 3.5** Let \( \frac{1}{r} = (\sum_{i=1}^{n} \frac{1}{p_i}) + \frac{d}{d-n} \leq r < \infty, d < q_1, \ldots, q_l \leq \infty, \) and \( q_{l+1}, \ldots, q_n = d \) with \( 0 \leq l \leq n, p = \infty \). Suppose that \( w \in \cap_{i=1}^{n} A_{\max\{\frac{q_i}{p_i}, 1\}}(\mathbb{R}^d) \). Then
\[ \|c_{\ast} [\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{r, \infty}(\mathbb{R}^d, w)} \leq \left( \prod_{i=1}^{l} \| \nabla A_i \|_{L^{r_1}(\mathbb{R}^d, w)} \right) \left( \prod_{i=l+1}^{n} \| \nabla A_i \|_{L^{d,1}(\mathbb{R}^d, w)} \right) \| f \|_{L^{\infty}(\mathbb{R}^d, w)}, \]
where \( L^{d,1}(\mathbb{R}^d, w) \) is the weighted Lorentz space.

**Proof** The proof here is similar to that of Proposition 3.4, so we will be brief and only indicate necessary modifications here. Proceeding from the proof of Proposition 3.4, there are four different arguments.

The first one is that when we choose the set \( E_{\lambda} \), we set
\[ E_{\lambda} = \{x \in \mathbb{R}^d : c_{\ast} [\nabla A_1, \ldots, \nabla A_n, f](x) > C_0 \lambda\}, \]
where \( C_0 \) is a constant determined later. Our goal is to show \( w(E_{\lambda}) \lesssim \lambda^{-r} \). We split \( E_{\lambda} \) into several terms and give estimates as follows:
\[ w(\{x \in \mathbb{R}^d : c_{\ast} [\nabla A_1, \ldots, \nabla A_n, f](x) > C_0 \lambda\}) \]
\[ \leq w(10G_{\lambda}) + w(\{x \in (10G_1)^c : c_{\ast} [\nabla A_1, \ldots, \nabla A_n, f_1](x) > C_0 \lambda/2\}) \]
\[ + w(\{x \in (10G_1)^c : c_{\ast} [\nabla A_1, \ldots, \nabla A_n, f_2](x) > C_0 \lambda/2\}). \]
The first term above satisfies \( w(10G_1) \leq \lambda^{-r} \), so it is sufficient to consider the second and third terms. Thus, we only need consider \( x \in (10G_1)^c \).

The second difference is the estimate related to the second term in (3.6). Here we choose \( \tau, \tilde{q}_1, \ldots, \tilde{q}_n \), such that \( 1 < \tau < \infty, q_1 < \tilde{q}_1 < \infty, \ldots, q_n < \tilde{q}_n < \infty, d < \tilde{q}_1, \ldots, \tilde{q}_n \), and \( \frac{1}{\tau} = \sum_{i=1}^{n} \frac{1}{q_i} \). Using Proposition 3.3 with those above \( \tau, \tilde{q}_1, \ldots, \tilde{q}_n \) and the fact that \( A_i \) is a Lipschitz function on \((G_1)^c\) with Lipschitz bound \( \lambda \frac{2}{\tilde{q}_i} \) for \( i = 1, \ldots, n \), we can obtain

\[
\begin{align*}
\mathbb{P}(x \in (10G_1)^c : \mathbb{E}_x[\nabla A_1, \ldots, \nabla A_n, f_1](x) > C_0 \lambda / 2) \\
\leq \mathbb{P}(x \in (G_1)^c : \mathbb{E}_x[\nabla (A_1X_{(G_1)^c}), \ldots, \nabla (A_nX_{(G_1)^c}), f_1](x) > C_0 \lambda / 2) \\
\leq \lambda^{-r} \left( \prod_{i=1}^{n} \| \nabla (A_iX_{(G_1)^c}) \|_{L_{\infty}(\mathbb{R}^d, w)} \right) \| f_1 \|_{L_{\infty}(\mathbb{R}^d, w)} \\
\leq \lambda^{-r} \left( \prod_{i=1}^{n} \| \nabla A_i \|_{L_{\infty}(\mathbb{R}^d, w)} \right) \left( \prod_{i=1}^{n} \| \nabla \tilde{A}_i \|_{L_{\infty}(\mathbb{R}^d, w)} \right) \| f_1 \|_{L_{\infty}(\mathbb{R}^d, w)} \\
\leq \lambda^{-r} \left( \sum_{i=1}^{n} \frac{1}{q_i} \right)^{-r} \left( \sum_{i=1}^{n} \frac{1}{\tilde{q}_i} \right) = \lambda^{-r}.
\end{align*}
\]

Next consider the estimate related to the third term in (3.6). As done in the proof of Proposition 3.4, we divide \( \mathbb{E}_x[\nabla A_1, \ldots, \nabla A_n, f_2](x) \) into several terms and then separate these terms into two parts corresponding to I and II in (3.4). So we get

\[
\begin{align*}
\mathbb{P}(x \in (10G_1)^c : \mathbb{E}_x[\nabla A_1, \ldots, \nabla A_n, f_2](x) > C_0 \lambda / 2) \\
\leq \mathbb{P}(x \in (G_1)^c : \mathbb{E}_x[\nabla \tilde{A}_1, \ldots, \nabla \tilde{A}_n, f_2](x) > C_0 \lambda / 4) \] + \mathbb{P}(x \in (10G_1)^c : H_{II}(x) > C_0 \lambda / 4) \]

\[
\leq \lambda^{-1} \left( \prod_{i=1}^{n} \| \nabla \tilde{A}_i \|_{L_{\infty}(\mathbb{R}^d)} \right) \| f_2 \|_{L_{\infty}(\mathbb{R}^d, w)} \leq \lambda^{-1} \left( \sum_{i=1}^{n} \frac{1}{\tilde{q}_i} \right) = \lambda^{-r}.
\]

The third difference is the weighted estimate of \( \mathbb{E}_x[\cdots, \cdot] \) related to I. Here we use Lemma 3.1 and the estimate \( \| f_2 \|_{L_{\infty}(\mathbb{R}^d, w)} \leq \| f \|_{L_{\infty}(\mathbb{R}^d, w)} w(G_1) \leq \lambda^{-r} \) to get

\[
\begin{align*}
\mathbb{P}(x \in (10G_1)^c : \mathbb{E}_x[\nabla A_1, \ldots, \nabla A_n, f_2](x) > C_0 \lambda / 4) \\
\leq \lambda^{-1} \left( \prod_{i=1}^{n} \| \nabla A_i \|_{L_{\infty}(\mathbb{R}^d)} \right) \| f_2 \|_{L_{\infty}(\mathbb{R}^d, w)} \leq \lambda^{-1} \left( \sum_{i=1}^{n} \frac{1}{q_i} \right) = \lambda^{-r}.
\end{align*}
\]

The fourth difference is the weighted estimate of \( \mathbb{E}_x[\cdots, \cdot] \) related to II. We shall prove that

\[
(3.7) \quad \{ x \in (10G_1)^c : H_{II}(x) > C_0 \lambda / 4 \} = \emptyset.
\]

In fact, by (3.5) and Lemma 2.8 with \( q = \infty \), we get for any \( x \in (10G_1)^c \),

\[
H_{II}(x) \leq C_d \lambda \sum_{i=1}^{n} \frac{1}{\tilde{q}_i} \| f \|_{L_{\infty}(\mathbb{R}^d, w)} = C_d \lambda.
\]

If we choose \( C_0 > 4C_d \), we get (3.7). So we complete the proof.  

\[ \blacksquare \]

### 3.3 Case: Some \( q_i \)s are smaller than \( d \) and some are not.

In this subsection, we consider the case: \( d/(d + n) \leq r < \infty \) with at least one \( q_i < d \), \( 1 \leq p \leq \infty \). By our condition, the weight \( w \) satisfies \( w \in \left( \bigcap_{l=1}^{n} A_{\max(\mathbb{F}_{p,l}, 1)}(\mathbb{R}^d) \right) \cap A_p(\mathbb{R}^d) = A_1(\mathbb{R}^d) \). Without loss of generality, we can suppose that \( d \leq q_1, \ldots, q_l \leq \infty \) and \( 1 \leq q_{l+1}, \ldots, q_n < d \) with \( 0 \leq l < n \). If \( l = 0 \), it means that all \( q_1, \ldots, q_n \in [1, d) \).
Also, we suppose that \( q_1 = \cdots = q_k = d \) and \( d < q_{k+1}, \ldots, q_l \leq \infty \) with \( 0 \leq k \leq l \). If \( k = 0 \), we mean that there is no index in \( q_1, \ldots, q_l \) equal to \( d \), i.e., \( d < q_1, \ldots, q_l \leq \infty \). If \( k = l \), we mean that \( q_1 = \cdots = q_l = d \). Since the proof of \( p = \infty \) is a little different from that of \( 1 \leq p < \infty \), we shall give two propositions.

**Proposition 3.6** Suppose \( w \in A_1(\mathbb{R}^d) \). Let \( \frac{1}{r} = \left( \sum_{i=1}^{n} \frac{1}{q_i} \right) + \frac{1}{p} \cdot \frac{d}{d+n} \leq r < \infty \), \( q_1 = \cdots = q_k = d \), \( d < q_{k+1}, \ldots, q_l \leq \infty \) and \( 1 \leq q_{l+1}, \ldots, q_n < d \) with \( 0 \leq k \leq l \), and \( 0 \leq l < n, 1 \leq p < \infty \). Then

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{r,\infty}(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^{k} \| \nabla A_i \|_{L^{\infty,1}(\mathbb{R}^d, w)} \right) \left( \prod_{i=k+1}^{n} \| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^p(\mathbb{R}^d, w)},
\]

where \( L^{d,1}(\mathbb{R}^d, w) \) is the weighted Lorentz space.

**Proof** We need to prove that for any \( \lambda > 0 \), the following inequality holds:

\[
w(\{ x \in \mathbb{R}^d : \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f](x) > \lambda \}) \lesssim \lambda^{-r} \left( \prod_{i=1}^{k} \| \nabla A_i \|_{L^{\infty,1}(\mathbb{R}^d, w)} \right) \left( \prod_{i=k+1}^{n} \| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^p(\mathbb{R}^d, w)}.
\]

By the standard density and scaling argument, it is sufficient to consider that each \( A_i \) \((i = 1, \ldots, n)\) and \( f \) as smooth functions with compact supports and

\[
\| \nabla A_1 \|_{L^{d,1}(\mathbb{R}^d, w)} = \cdots = \| \nabla A_k \|_{L^{d,1}(\mathbb{R}^d, w)} = 1,
\]

\[
\| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} = 1 \text{ for } i = k+1, \ldots, n, \text{ and } \| f \|_{L^p(\mathbb{R}^d, w)} = 1.
\]

As done in the proof of Proposition 3.4, we first suppose that all \( q_{k+1}, \ldots, q_l \leq \infty \), since the other case is easy, and we will show lastly how to modify the proof to the case that there exist \( q_i = \infty \) for some \( i = k+1, \ldots, l \). Fix \( \lambda > 0 \) and set

\[
E_\lambda = \{ x \in \mathbb{R}^d : \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f](x) > \lambda \}.
\]

Our goal is to show \( w(E_\lambda) \lesssim \lambda^{-r} \). The main idea is to construct some exceptional set such that the \( w \) measure of exceptional set is bounded by \( \lambda^{-r} \), which is our required estimate. At the same time, on the complementary set of exceptional set these functions \( A_i \)s should be Lipschitz functions with bound \( \lambda^{\frac{1}{q_i}} \) for each \( i = 1, \ldots, n \). The constructions of exceptional sets are different between \( d \leq q_i < \infty \) and \( 1 \leq q_i < d \). Now we begin our constructions of some exceptional sets.

**Step 1: Exceptional Set related to \( q_1, \ldots, q_l \)**

Define the exceptional set for \( i = 1, \ldots, l \):

\[
J_{i,\lambda} = \{ x \in \mathbb{R}^d : \mathcal{M}(\nabla A_i)(x) > \lambda^{\frac{1}{q_i}} \}; \quad J_{\lambda} = \bigcup_{i=1}^{l} J_{i,\lambda}.
\]
Since \( w \in A_1(\mathbb{R}^d) \), by Lemmas 2.3 and 2.4, \( M \) maps \( L^p(\mathbb{R}^d, w) \) to itself for \( p > d \) and maps \( L^{d, \infty}(\mathbb{R}^d, w) \) to \( L^{d, \infty}(\mathbb{R}^d, w) \), i.e.,

\[
\begin{align*}
w(J_i, \lambda) & \lesssim \lambda^{-r} \| \nabla A_i \|_{L^{d, \infty}(\mathbb{R}^d, w)} = \lambda^{-r}, \quad i = 1, \ldots, k, \\
w(J_i, \lambda) & \lesssim \lambda^{-r} \| \nabla A_i \|_{L^{d, \infty}(\mathbb{R}^d, w)} = \lambda^{-r}, \quad i = k + 1, \ldots, l.
\end{align*}
\]

So we obtain that \( w(J_\lambda) \lesssim \lambda^{-r} \).

**Step 2: Calderón–Zygmund Decomposition**

By the formula given in [26, p. 125, (17)], for each \( A_i, i = l + 1, \ldots, n \), we can write

\[
A_i(x) = \sum_{j=1}^d C_d \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^d} \partial_j A_i(y) dy =: \sum_{j=1}^d A_{i,j}(x).
\]

Notice that \( w \in A_1(\mathbb{R}^d) \) in this case. For each \( |\partial_j A_i|^{q_j} \in L^1(\mathbb{R}^d, w) \) with \( j = 1, \ldots, d \) and \( i = l + 1, \ldots, n \), making a Calderón–Zygmund decomposition at level \( \lambda' \), we may have the following conclusions (see e.g., [15, p. 413, Theorem 3.5]):

1. **(cz-i)\( \partial_j A_i = g_{j,i} + b_{j,i}, \| g_{j,i} \|_{L^1(\mathbb{R}^d)} \lesssim \lambda \frac{\lambda'}{\lambda}, \| g_{j,i} \|_{L^{q_j}(\mathbb{R}^d, w)} \lesssim \| \partial_j A_i \|_{L^{q_j}(\mathbb{R}^d, w)};**
2. **(cz-ii)\( b_{j,i} = \sum_{Q \in Q_{j,i}} b_{j,i,Q}, \text{supp} \ b_{j,i,Q} \subset Q, \text{where } Q_{j,i} \text{ is a countable set of disjoint dyadic cubes;**
3. **(cz-iii)\( \text{Let } E_{j,i} = \bigcup_{Q \in Q_{j,i}} Q, \text{then } w(E_{j,i}) \lesssim \lambda^{-r} \| \partial_j A_i \|_{L^{q_j}(\mathbb{R}^d, w)};**
4. **(cz-iv)\( \int b_{j,i,Q}(y) dy = 0 \text{ for each } Q \in Q_{j,i}, \text{the unweighted estimate } \| b_{j,i,Q} \|_{L^1(\mathbb{R}^d)} \lesssim \lambda'|Q| \text{ and the weighted estimate } \| b_{j,i,Q} \|_{L^{q_j}(\mathbb{R}^d, w)} \lesssim \| \partial_j A_i \|_{L^{q_j}(\mathbb{R}^d, w)} \text{ hold.**

We shall split \( A_{i,j} \) into two parts according to the above Calderón–Zygmund decomposition (cz-i):

\[
A_{i,j}^g(x) = C_d \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^d} g_{j,i}(y) dy,
\]

\[
A_{i,j}^b(x) = C_d \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^d} b_{j,i}(y) dy.
\]

Define the **exceptional set** \( B_\lambda = \bigcup_{j=l+1}^n \bigcup_{i=1}^d E_{j,i} \). Then by (cz-iii), we obtain \( w(B_\lambda) \lesssim \lambda^{-r} \).

**Step 3: Exceptional Set \( D_\lambda \)**

Set \( \frac{1}{s_j} = \frac{1}{q_j} - \frac{1}{d} \) for \( i = l + 1, \ldots, n \). Notice that \( w \) belongs to \( A_1(\mathbb{R}^d) \). Define the following exceptional set

\[
D_{i,\lambda} = \left\{ x \in \mathbb{R}^d : M_{w, s_j}(\nabla A_i)(x) > \lambda \frac{\lambda'}{\lambda} \right\},
\]

where the maximal operator \( M_{w, s_j} \) is defined in the paragraph above Lemma 2.5. Denote \( D_\lambda = \bigcup_{i=l+1}^n D_{i,\lambda} \). Then by Lemma 2.6, we get that

\[
w(D_{i,\lambda}) \lesssim \lambda^{-r} \| \nabla A_i \|_{L^{q_j}(\mathbb{R}^d, w)} = \lambda^{-r}, \quad w(D_\lambda) \lesssim \lambda^{-r}.
\]
Step 4: Exceptional Set $F_\lambda$

For each $j = 1, \ldots, d$, $i = l + 1, \ldots, n$, define the functions
\[ \Delta_{j,i}^{l,i}(x) = \sum_{Q \in \mathcal{Q}_{j,i}} \frac{l(Q)}{l(Q) + |x - y_Q|^{d+1}} m(Q), \]
where $y_Q$ is the center of $Q$. Define another exceptional set
\[ F_{j,i,\lambda} = \{ x \in \mathbb{R}^d : \Delta_{j,i}^{l,i}(x) > 1 \}, \quad F_\lambda = \bigcup_{j=1}^d \bigcup_{i=l+1}^n F_{j,i,\lambda}. \]

Notice that we have $w \in A_1(\mathbb{R}^d)$. We claim the following property of $A_1$ weight: For any cube $Q$ and $\alpha > 1$, there exists a constant $C$ independent of $Q$ and $\alpha$ such that
\begin{equation}
(3.8) \quad w(\alpha Q) \leq C \alpha^d w(Q).
\end{equation}
In fact by (2.2) in Definition 2.1, there exist a constant $C$ independent of $\alpha$ and $Q$ such that
\[ \frac{1}{|\alpha Q|} \int_{\alpha Q} w(z) \, dz \leq C \inf_{y \in Q} w(y) \leq C \frac{1}{|Q|} \int_Q w(y) \, dy, \]
which immediately implies (3.8). In the following, by using the Chebyshev inequality in the first inequality, (3.8) with $\alpha = 2^k$ in the last second inequality, and (cz-iii) in the last inequality, we get
\[ w(F_{j,i,\lambda}) \leq \int_{\mathbb{R}^d} \Delta_{j,i,\lambda}(x) w(x) \, dx \leq \sum_{Q \in \mathcal{Q}_{j,i}} \left[ \int_{\mathbb{R}^d} \frac{l(Q)^{d+1}}{l(Q) + |x - y_Q|^{d+1}} w(x) \, dx \right] \]
\[ \leq \sum_{Q \in \mathcal{Q}_{j,i}} \left[ \int_Q w(x) \, dx + \sum_{k=1}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} \frac{l(Q)^{d+1}}{l(Q) + |x - y_Q|^{d+1}} w(x) \, dx \right] \]
\[ \leq \sum_{Q \in \mathcal{Q}_{j,i}} \left[ w(Q) + \sum_{k=1}^{\infty} w(2^k Q) \frac{2^k Q}{2^{k(d+1)}} \right] \leq \sum_{Q \in \mathcal{Q}_{j,i}} w(Q) \leq \lambda^{-r}. \]
Therefore, we obtain that $w(F_\lambda) \lesssim \lambda^{-r}$.

Step 5: Exceptional Set $H_\lambda$

Define the exceptional set for $i = l + 1, \ldots, n$, $j = 1, \ldots, d$,
\[ H_{i,j,\lambda} = \{ x \in \mathbb{R}^d : M(\nabla A_{i,j}^g)(x) > \lambda^{r/q_i} \}, \quad H_\lambda = \bigcup_{i=l+1}^n \bigcup_{j=1}^d H_{i,j,\lambda}. \]
Notice that by the definition of $A_{i,j}^g$, for each $s = 1, \ldots, d$, we get
\[ \mathcal{F}(\partial_s A_{i,j}^g)(\xi) = C \frac{\xi_s \xi_j}{|\xi|^2} \mathcal{F}(g_{j,i})(\xi) \Rightarrow \nabla A_{i,j}^g = CRR_j g_{j,i}, \]
where $\mathcal{F}$ is the Fourier transform, $R_j$ is the Riesz transform, and $R = (R_1, \ldots, R_d)$. Recall $w \in A_1(\mathbb{R}^d)$. Since $R_j$ is weighted strong type $(q,q)$ for $1 < q < \infty$, we get
\[ \|\nabla A_{i,j}^g\|_{L^q(\mathbb{R}^d, w)} \leq \|g_{j,i}\|_{L^q(\mathbb{R}^d, w)}. \]
Choose $d < q < \infty$; by the Chebyshev inequality,
Lemma 2.3 and (cz-i) in Step 2, we get
\[ w(H_{i,j,\lambda}) \leq \lambda^{-\frac{d}{4n}} \int_{\mathbb{R}^d} |M(\nabla A_{i,j}^\lambda)(x)|^q w(x) \, dx \leq \lambda^{-\frac{d}{4n}} \int_{\mathbb{R}^d} |\nabla A_{i,j}^\lambda(x)|^q w(x) \, dx \]
\[ \leq \lambda^{-\frac{d}{4n}} \int_{\mathbb{R}^d} |g_{j,i}(x)|^q w(x) \, dx \leq \lambda^{-r} \int_{\mathbb{R}^d} |g_{j,i}(x)|^q w(x) \, dx \leq \lambda^{-r}. \]
Therefore, we get \( w(H_{i}) \leq \lambda^{-r} \).

**Step 6: Final Exceptional Set \( G_\lambda \)**

Based on the construction of \( J_\lambda, B_\lambda, D_\lambda, F_\lambda, H_\lambda \) in Step 1–5 and the fact that \( w \) satisfies the doubling property, we choose an open set \( G_\lambda \) that satisfies the following conditions:

(a) \( \{ 10J_\lambda \cup 10B_\lambda \cup 10D_\lambda \cup 10F_\lambda \cup 10H_\lambda \} \subset G_\lambda; \)

(b) \( w(G_\lambda) \leq w(J_\lambda) + w(B_\lambda) + w(D_\lambda) + w(F_\lambda) + w(H_\lambda). \)

Applying the previous weighted estimates of \( J_\lambda, B_\lambda, D_\lambda, F_\lambda, \) and \( H_\lambda, \) we obtain that \( w(G_\lambda) \leq \lambda^{-r}. \) Next, making a Whitney decomposition of \( G_\lambda \) (see [16]), we may obtain a family of disjoint dyadic cubes \( \{ Q_k \} \) such that

(i) \( G_\lambda = \bigcup_{k=1}^{\infty} Q_k; \)

(ii) \( \sqrt{d} \cdot l(Q_k) \leq \text{dist}(Q_k, (G_\lambda)^c) \leq 4\sqrt{d} \cdot l(Q_k). \)

By property (ii) above, the distance between \( Q_k \) and \( (G_\lambda)^c \) equals \( Cl(Q_k). \) For each \( Q_k \) above, we could construct a larger cube \( Q_k^* \) so that \( Q_k \subset Q_k^*, Q_k^* \) is centered at \( y_k \) and \( y_k \in (G_\lambda)^c, \) \( l(Q_k^*) \approx l(Q_k). \) Therefore, by the construction of \( Q_k^* \) and \( y_k, \) we get that

\[ \text{dist}(y_k, Q_k) \approx l(Q_k). \]

Clearly, the exceptional set \( G_\lambda \) constructed in Step 6 satisfies that \( w(G_\lambda) \leq \lambda^{-r}. \)

Below we will prove that these functions \( A_{i,j} \)s are Lipschitz functions on \( (G_\lambda)^c. \)

**Step 7: Lipschitz Estimates of \( A_i \) on \( (G_\lambda)^c \)**

Choose any \( x, y \in (G_\lambda)^c. \) By the exceptional set \( J_\lambda \) constructed in Step 1, we see that for \( i = 1, \ldots, l, \)

\[ |A_i(x) - A_i(y)| \leq \lambda^{\frac{d}{4n}}|x - y|. \]

(3.10)

In the following, we only consider \( i = l + 1, \ldots, n. \) By the Calderón–Zygmund decomposition in Step 2, it is sufficient to prove that \( A_{i,j}^\lambda \) and \( A_{i,j}^\lambda \) satisfy Lipschitz estimates on \( (G_\lambda)^c \) for each \( i = l + 1, \ldots, n \) and \( j = 1, \ldots, d. \) First, it is easy to see that \( A_{i,j}^\lambda \) satisfies Lipschitz estimates by the construction of \( H_\lambda \) in Step 5. In fact, \( x, y \in (G_\lambda)^c \) implies that \( x, y \in H_\lambda^*; \) we obtain that for \( i = l + 1, \ldots, n, j = 1, \ldots, d, \)

\[ |A_{i,j}^\lambda(x) - A_{i,j}^\lambda(y)| \leq \lambda^{\frac{d}{4n}}|x - y|. \]

(3.11)

We now prove that \( A_{i,j}^\lambda \) is a Lipschitz function on \( (G_\lambda)^c. \) Recall the Calderón–Zygmund decomposition properties (cz-ii), (cz-iii), and (cz-iv) in Step 2. For each
\[ b_{j,i} = \sum_{Q \in \Omega_{j,i}} b_{j,i,Q}, \text{ supp } b_{j,i,Q} \subset Q, \text{ where } \Omega_{j,i} \text{ is a countable set of disjoint dyadic cubes. Then for each } Q \in \Omega_{j,i}, \text{ we define} \]
\[ A_{i,j}^{b_{0}}(x) = C_{d} \int_{\mathbb{R}^{d}} \frac{x_j - z_j}{|x - z|^{d}} b_{j,i,Q}(z) \, dz. \]

Now we fix a dyadic cube \( Q \in \Omega_{j,i}. \) We are going to give a straightforward Lipschitz estimate of \( A_{i,j}^{b_{0}}. \) By the construction of \( G_{\lambda}, \) we get that \( x, y \in (10B_{\lambda})^{c}, \text{ i.e., } x, y \in (10Q)^{c}; \) therefore, we obtain \( \text{dist}(x, Q) \geq \frac{9}{2} l(Q) \) and \( \text{dist}(y, Q) \geq \frac{9}{2} l(Q). \) Let \( z_{0} \) be the center of \( Q. \) Without loss of generality, assume that \( |x - z_{0}| \leq |y - z_{0}|. \) Choose a point \( Z \in \mathbb{R}^{d} \) such that
\[ |x - Z| < 100|x - y|; \quad |y - Z| \leq 100|x - y|; \quad |X - z_{0}| > \frac{2}{5}|x - z_{0}| \]
for any \( X \) belonging to the polygonal with vertex \( x, y, Z. \) We could draw a figure to show that such a point \( Z \) exists under the condition that \( \text{dist}(x, Q) > \frac{9}{2} l(Q) \) and \( \text{dist}(y, Q) > \frac{9}{2} l(Q). \) Now we write \( A_{i,j}^{b_{0}}(x) - A_{i,j}^{b_{0}}(y) = A_{i,j}^{b_{0}}(x) - A_{i,j}^{b_{0}}(Z) + A_{i,j}^{b_{0}}(Z) - A_{i,j}^{b_{0}}(y). \) By using the mean value formula, we have
\[ A_{i,j}^{b_{0}}(x) - A_{i,j}^{b_{0}}(Z) = \int_{0}^{1} (x - Z, \nabla(A_{i,j}^{b_{0}})(tx + (1 - t)Z)) \, dt, \]
\[ A_{i,j}^{b_{0}}(Z) - A_{i,j}^{b_{0}}(y) = \int_{0}^{1} (Z - y, \nabla(A_{i,j}^{b_{0}})(tZ + (1 - t)y)) \, dt. \]

For any \( 0 \leq t \leq 1, \) the points \( tx + (1 - t)Z \) and \( tZ + (1 - t)y \) lie in the polygonal with vertex \( x, y, Z. \) Notice that \( |x - z_{0}| \geq 5 l(Q). \) Then by our choice of \( Z, \) we get
\[ |tx + (1 - t)Z - z_{0}| > \frac{2}{5}|x - z_{0}| > 2 l(Q), \]
\[ |tZ + (1 - t)y - z_{0}| > \frac{2}{5}|x - z_{0}| > 2 l(Q). \]

We set \( Z(t) \) equal to \( tx + (1 - t)Z \) or \( tZ + (1 - t)y \) and \( K_{j}(x) = x_{j}/|x|^{d}. \) Using the cancelation condition of \( b_{j,i,Q}, (3.13) \) and the unweighted estimate in (cz-iv) of Step 2, we get that
\[ |\nabla(A_{i,j}^{b_{0}})(Z(t))| = \left| \int_{\mathbb{R}^{d}} \left[ (\nabla K_{j})(Z(t) - z) - (\nabla K_{j})(Z(t) - z_{0}) \right] b_{j,i,Q}(z) \, dz \right| \]
\[ \leq \frac{l(Q)}{[l(Q) + |x - z_{0}|]^{d+1}} \| b_{j,i,Q} \|_{L^{1}(Q)} \]
\[ \leq \lambda^{\frac{n}{n+1}} \frac{l(Q)}{[l(Q) + |x - z_{0}|]^{d+1}} |Q|. \]

Combining the above arguments with (3.12) and the construction of \( Z, \) we obtain
\[ |A_{i,j}^{b_{0}}(x) - A_{i,j}^{b_{0}}(y)| \leq \lambda^{\frac{n}{n+1}} \frac{l(Q)}{[l(Q) + |x - z_{0}|]^{d+1}} |Q|x - y|. \]
Notice that \( x \in (G_\lambda)^c \) implies that \( x \in (F_\lambda)^c \) in Step 3. Then we get that

\[
|A_{i,j}^b(x) - A_{i,j}^b(y)| \leq \lambda \hat{\n}\|x - y\| \sum_{Q \in \Omega_i} \frac{I(Q)}{I(Q) + |x - z_Q|^{d+1}} |Q| \\
\leq \lambda \hat{\n}\|x - y\|.
\]

Therefore, we conclude that the Lipschitz estimates in (3.10) for \( i = 1, \ldots, l \), good function (3.11), and bad function (3.14) for \( i = l + 1, \ldots, n \), to obtain that for any \( i = 1, \ldots, n, x, y \in (G_\lambda)^c \),

\[
|A_i(x) - A_i(y)| \leq \lambda \hat{\n}\|x - y\|. 
\]

**Step 8: Weighted Estimate of \( E_\lambda \)**

We come back to give an estimate of \( E_\lambda \). Split \( f \) into two parts \( f = f_1 + f_2 \) where \( f_1(x) = f(x) \chi_{(G_\lambda)^c}(x) \) and \( f_2(x) = f(x) \chi_{G_\lambda}(x) \). By the Lipschitz estimate in (3.15), when restricted on \((G_\lambda)^c\), \( A_i \) is a Lipschitz function with \( \|\n A_i\|_{L^\infty((G_\lambda)^c)} \leq \lambda \hat{\n} \) for \( i = 1, \ldots, n \). Let \( \tilde{A}_i \) represent the Lipschitz extension of \( A_i \) from \((G_\lambda)^c\) to \( \mathbb{R}^d \) (see [26, p. 174, Theorem 3]) so that for each \( i = 1, \ldots, n \),

\[
\tilde{A}_i(y) = A_i(y) \quad \text{if } y \in (G_\lambda)^c, \\
|\tilde{A}_i(x) - \tilde{A}_i(y)| \leq \lambda \hat{\n}\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d. 
\]

Since the operator \( C_*[\ldots, \cdot] \) is sub-multilinear, we split \( E_\lambda \) as three terms:

\[
w\{x \in \mathbb{R}^d : C_*[\n A_1, \ldots, \n A_n, f](x) > \lambda\} \\
\leq w(10G_\lambda) + w\{x \in (10G_\lambda)^c : C_*[\n A_1, \ldots, \n A_n, f_1](x) > \lambda/2\} \\
+ w\{x \in (10G_\lambda)^c : C_*[\n A_1, \ldots, \n A_n, f_2](x) > \lambda/2\}.
\]

The first term above satisfies \( w(10G_\lambda) \leq \lambda^{-r} \), which is our required estimate. Below we consider the second term. We only consider \( x \in (10G_\lambda)^c \). By the definition of \( f_1 \),

\[
C_*[\n A_1, \ldots, \n A_n, f_1](x) = C_*[\n \tilde{A}_1, \ldots, \n \tilde{A}_n, f_1](x).
\]

Notice that \( w \in A_1(\mathbb{R}^d) \ni A_1(\mathbb{R}^d) \). Applying the above equality and Proposition 3.1 (1 \( \leq p < \infty \)), we derive that

\[
w\{x \in (10G_\lambda)^c : C_*[\n \tilde{A}_1, \ldots, \n \tilde{A}_n, f_1](x) > \lambda/2\} \\
= w\{x \in (10G_\lambda)^c : C_*[\n \tilde{A}_1, \ldots, \n \tilde{A}_n, f_1](x) > \lambda/2\} \\
\leq \lambda^{-p} \left( \prod_{i=1}^n \|\n \tilde{A}_i\|_{L^\infty(\mathbb{R}^d, w)} \right) \|f_1\|_{L^p(\mathbb{R}^d, w)} \leq \lambda^{-p} \sum_{i=1}^n \frac{1}{\hat{\n}} \lambda^{-r}.
\]

If \( p = \infty \), the above argument does not work.
**Step 9: Weighted Estimate of** \( \mathcal{E}_s[\nabla A_1, \ldots, \nabla A_n, f_2](x) \)

Recall \( \mathbb{N}_j = \{ i, i + 1, \ldots, j \} \) and the construction of \( G_\lambda, y_k, Q_k \), and \( Q_k^* \) above (3.9). Then we can write \( f_2 = \sum_k f_\chi_{Q_k} \). So

\[
\mathcal{E}_s[\nabla A_1, \ldots, \nabla A_n, f_2](x) = \sum_k \mathcal{E}_s[\nabla A_1, \ldots, \nabla A_n, f_\chi_{Q_k}](x).
\]

In the following, we study \( \prod_{i=1}^n \frac{A_i(x)-A_i(y)}{|x-y|} \). We separate it into several terms and then give an estimate for each term. Write

\[
\prod_{i=1}^n \frac{A_i(x) - A_i(y)}{|x-y|} = \prod_{i=1}^n \left( \frac{\tilde{A}_i(x) - \tilde{A}_i(y)}{|x-y|} + \frac{\tilde{A}_i(y) - \tilde{A}_i(y_k)}{|x-y|} + \frac{A_i(y_k) - A_i(y)}{|x-y|} \right)
\]

\[
= \sum_{N_1 \in \mathbb{N}_1^*} \left( \prod_{i \in N_1} \frac{A_i(x) - A_i(y)}{|x-y|} \right) \left( \prod_{i \in N_2} \frac{\tilde{A}_i(y) - \tilde{A}_i(y_k)}{|x-y|} \right) \left( \prod_{i \in N_3} \frac{A_i(y_k) - A_i(y)}{|x-y|} \right)
\]

\[
= I(x, y) + II(x, y, y_k) + III(x, y, y_k) + IV(x, y, y_k),
\]

where in the third equality, we divide \( \mathbb{N}_1^* = N_1 \cup N_2 \cup N_3 \) with \( N_1, N_2, N_3 \), not intersecting each other. Then \( I(x, y), II(x, y, y_k), III(x, y, y_k), \) and \( IV(x, y, y_k) \) are defined as follows

(3.17) \[
I(x, y) = \prod_{i=1}^n \frac{\tilde{A}_i(x) - \tilde{A}_i(y)}{|x-y|},
\]

\[
II(x, y, y_k) = \sum_{N_1 \in \mathbb{N}_1^*} \left( \prod_{i \in N_1} \frac{A_i(x) - A_i(y)}{|x-y|} \right) \left( \prod_{i \in N_2} \frac{\tilde{A}_i(y) - \tilde{A}_i(y_k)}{|x-y|} \right),
\]

\[
III(x, y, y_k) = \sum_{N_3 \neq \emptyset, N_3 \subset \{1, \ldots, l\}} \left( \prod_{i \in N_1} \frac{A_i(x) - A_i(y)}{|x-y|} \right) \left( \prod_{i \in N_2} \frac{\tilde{A}_i(y) - \tilde{A}_i(y_k)}{|x-y|} \right) \times \left( \prod_{i \in N_3} \frac{A_i(y_k) - A_i(y)}{|x-y|} \right),
\]

\[
IV(x, y, y_k) = \sum_{N_3 \neq \emptyset, N_3 \cap \{1+1, \ldots, n\} \neq \emptyset} \left( \prod_{i \in N_1} \frac{A_i(x) - A_i(y)}{|x-y|} \right) \times \left( \prod_{i \in N_2} \frac{\tilde{A}_i(y) - \tilde{A}_i(y_k)}{|x-y|} \right) \left( \prod_{i \in N_3} \frac{A_i(y_k) - A_i(y)}{|x-y|} \right).
\]

In the above decomposition, we in fact divide \( \mathcal{E}_s[\nabla A_1, \ldots, \nabla A_n, f_\chi_{Q_k}](x) \) into \( 3^n \) terms and separate these terms into four parts according \( I, II, III, \) and \( IV \).
Step 10: Weighted Estimate of $C_n[\cdots, \cdot]$ Related to $I$.

In this case there is only one term; i.e., $C_n[\nabla \tilde{A}_1, \ldots, \nabla \tilde{A}_n, f_2]$. Then by Proposition 3.1 (1 ≤ $p < \infty$), we get

$$w(\{ x \in (10G_1)^c : C_n[\nabla \tilde{A}_1, \ldots, \nabla \tilde{A}_n, f_2](x) > \lambda/2 \}) \leq \lambda^{-p} \left( \prod_{i=1}^{n} \| \nabla \tilde{A}_i \| _{L^\infty(\mathbb{R}^d, w)} \right) \| f_2 \| _{L^p(\mathbb{R}^d, w)} \leq \lambda^{-p \sum_{i=1}^{n} \frac{\rho_i}{q_i}} = \lambda^{-r}.$$  

If $p = \infty$, the above argument may not work.

Step 11: Weighted Estimate of $C_n[\cdots, \cdot]$ Related to $II$.

It is sufficient to consider one term $C_n[\cdots, \cdot]$ related to $II$ in which $N_1$ is a proper subset of $\mathbb{N}_1^n$ and $N_3 = \emptyset$. In such a case, without loss of generality, we can suppose $N_1 = \{1, \ldots, \nu\}$, $N_2 = \{\nu + 1, \ldots, n\}$ with $0 \leq \nu < n$. Here if $\nu = 0$, it means that $N_1 = \emptyset$. With this notation, we see that $N_3$ is a proper subset of $\mathbb{N}_1^n$. By a slight abuse of notation, we still use $II(x, y, y_k)$ to represent one term related to $N_1$, $N_2$, and $N_3$ in (3.17) and use $H_{II}(x)$ to represent $C_n[\cdots, \cdot]$ related to $II(x, y, y_k)$, i.e.,

$$H_{II}(x) = \sup_{\varepsilon > 0} \left| \sum_{k} \int_{|x-y| > \varepsilon} K(x-y) II(x, y, y_k) f \chi_{Q_k}(y) dy \right|.$$  

Notice that $\tilde{A}_i$ is a Lipschitz function with bound $\lambda_j \tilde{n}_j$ for $i = 1, \ldots, n$ by (3.16). Then we obtain that

$$|II(x, y, y_k)| \leq \lambda^{\sum_{i=1}^{n} \frac{\rho_i}{q_i}} |y - y_k|^{n - \nu} |x - y|^{d - \nu - n}.$$  

Since we only need to consider $x \in (10G_1)^c$, then by (3.9), we obtain that

$$|x - y| \geq 2l(Q_k) \approx |y - y_k| \text{ for any } y \in Q_k.$$  

Now combining with (1.1), the above estimate of $II(x, y, y_k)$ and (3.18), we obtain that

$$H_{II}(x) \leq \sum_{k} \int_{Q_k} |K(x-y)| \cdot |II(x, y, y_k)| \cdot |f(y)| dy \leq \lambda^{\sum_{i=1}^{n} \frac{\rho_i}{q_i}} \sum_{k} \int_{Q_k} \frac{l(Q_k)^{n-v}}{|l(Q_k) + |x-y||^{d+n-\nu}} |f(y)| dy.$$  

Utilizing the Chebyshev inequality, the above estimate of $H_{II}$ and Lemma 2.8 (since $n - \nu \geq 1$), we finally obtain that

$$w(\{ x \in (10G_1)^c : H_{II}(x) > \lambda \}) \leq \lambda^{-p + \sum_{i=1}^{n} \frac{\rho_i}{q_i}} \| T_{n-\nu} f \| _{L^p(\mathbb{R}^d, w)} \leq \lambda^{-r} \| f \| _{L^p(\mathbb{R}^d, w)}.$$  

Hence, we finish the proof related to $II$.  

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Step 12: Weighted Estimate of $C_*[\cdots, \cdot]$ Related to $III$.

It is sufficient to consider one term $C_*[\cdots, \cdot]$ related to $III$ in which $N_1$ is a proper subset of $\mathbb{N}^n$ and $N_3$ is a nonempty subset of $\{1, \ldots, l\}$. By the condition in this proposition, for any $i \in N_3$, $d \leq q_i < \infty$. Thus $\nabla A_i \in L^q(\mathbb{R}^d, w)$ (or $L^{q_i,1}(\mathbb{R}^d, w)$ if $q_i = d$) with $d \leq q_i < \infty$. Then by using the fact that $y_k$ lies in the $(G_\lambda)^c$, i.e., $y_k \in (J_{i,\lambda})^c$, we give the estimates in $N_3$ as follows

$$\frac{|A_i(y_k) - A_i(y)|}{|y - y_k|} \leq M(\nabla A_i)(y_k) \leq \lambda^{\frac{q_i}{q_i - 1}}, \text{ for } i \in N_3,$$  

(3.19)

Define $v = \text{card}(N_1)$. Then we see that $0 \leq v < n$. By a slight abuse of notation, we still utilize $III(x, y, y_k)$ to stand for one term related to $N_1$, $N_2$, and $N_3$ in (3.17) and use $H_{III}(x)$ to represent $C_*[\cdots, \cdot]$ related to $III(x, y, y_k)$, i.e.,

$$H_{III}(x) = \sup_{\varepsilon > 0} \left| \sum_k \int_{|x-y| > \varepsilon} K(x - y) III(x, y, y_k) f \chi_{Q_k}(y) dy \right|.$$

From the fact that $\widetilde{A}_i$s are Lipschitz functions with bounds $\lambda v^{q_i}$ for $i \in N_1 \cup N_2$ and (3.19), we obtain that

$$|III(x, y, y_k)| \leq \lambda^\sum_{i \in N_1 \cup N_2} \frac{v^{q_i}}{\lambda^{v^{q_i}}} \frac{|y - y_k|^{n-v}}{|x - y|^{n-v}} \prod_{i \in N_3} M(\nabla A_i)(y_k) \leq \lambda^\sum_{i = 1}^n \frac{v^{q_i}}{\lambda^{v^{q_i}}} \frac{|y - y_k|^{n-v}}{|x - y|^{n-v}}.$$

Inserting this estimate of $III(x, y, y_k)$ into $H_{III}$, combining with (1.1) and (3.18), and next utilizing the Chebyshev inequality and Lemma 2.8 (since $n - v \geq 1$), we finally obtain that

$$w(\{x \in (10G_\lambda)^c : H_{III}(x) > \lambda\}) \leq \lambda^{-\rho} \|T_{n-v} f\|_{L^p(\mathbb{R}^d, w)}^p \leq \lambda^{-\rho} \|f\|_{L^p(\mathbb{R}^d, w)}^p.$$

Hence, we finish the proof of this part.

Step 13: Weighted Estimate of $C_*[\cdots, \cdot]$ Related to $IV$.

It is sufficient to consider one term $C_*[\cdots, \cdot]$ related to $IV$ in which $N_1 \subsetneq \mathbb{N}^n$ and $N_3 \neq \emptyset$ with $N_3 \cap \{l + 1, \ldots, n\} \neq \emptyset$. In such a case, without loss of generality, we can suppose $l + 1, \ldots, v \in N_3$ with $l + 1 \leq v \leq n$ and $v + 1, \ldots, n$ belongs to $N_1$ or $N_2$. So we can assume that $N_3 = \{i, \ldots, w, l + 1, \ldots, v\}$ with $0 \leq i \leq w \leq l$. Define $u = \text{card}(N_1)$. Then $n - u \geq 1$. With this notation, we can easily see that $N_3$ is a nonempty set with $N_3 \cap \{l + 1, \ldots, n\} \neq \emptyset$. Note that $w \in A_1(\mathbb{R}^d)$. By a slight abuse of notation, we still use $IV(x, y, y_k)$ to stand for one term related to $N_1$, $N_2$, and $N_3$ in (3.17) and use $H_{IV}(x)$ to stand for $C_*[\cdots, \cdot]$ related to $IV(x, y, y_k)$, i.e.,

$$H_{IV}(x) = \sup_{\varepsilon > 0} \left| \sum_k \int_{|x-y| > \varepsilon} K(x - y) IV(x, y, y_k) f \chi_{Q_k}(y) dy \right|.$$

Note that $d \leq q_1, \ldots, q_l \leq \infty$ and $1 \leq q_{l+1}, \ldots, q_n < d$. Recall that in Step 3, we set $\frac{1}{s_i} = \frac{1}{q_i} - \frac{1}{d}$ for $i = l + 1, \ldots, n$. We also set $\frac{1}{q} = (\sum_{i=l+1}^n \frac{1}{s_i}) + \frac{1}{p}$. Since $r \geq \frac{d}{d+n}$
and \( \frac{1}{r} = (\sum_{i=1}^{n} \frac{1}{q_i}) + \frac{1}{p} \), we could obtain \( 1 \leq q \leq \infty \), which will be crucial when we use Lemma 2.8. With (3.9) and \( \tilde{A}_i \) is a Lipschitz function with bound \( \lambda^{i/q_i} \) for \( i \in N_1 \cup N_2 \), we have

\[
|IV(x, y, y_k)|
\leq \lambda^{(\sum_{i \in N_1 \cup N_2} \frac{1}{q_i})} \frac{1}{\lambda^{(\sum_{i \in N_1 \cup N_2} \frac{1}{q_i})}} \left( \prod_{i=1}^{w} M(\nabla A_i)(y_k) \right) \sum_{i=1}^{L} \frac{|A_i(y_k) - A_i(y)|}{l(Q_k)}
\leq \lambda^{(\sum_{i \in \lambda} \frac{1}{q_i})} \frac{1}{\lambda^{(\sum_{i \in \lambda} \frac{1}{q_i})}} \left( \prod_{i=1}^{w} M(\nabla A_i)(y_k) \right) \sum_{i=1}^{L} \frac{|A_i(y_k) - A_i(y)|}{l(Q_k)}.
\]

Then, inserting the above estimate of \( IV \) into \( H_{IV} \) with (1.1) and (3.18), we get

\[
H_{IV}(x) \leq \sum_{k} \int_{Q_k} |K(x - y)| \cdot |IV(x, y, y_k)| \cdot |f(y)| dy
\leq \lambda^{(\sum_{i \in \lambda} \frac{1}{q_i})} \frac{1}{\lambda^{(\sum_{i \in \lambda} \frac{1}{q_i})}} \sum_{k} \int_{Q_k} \frac{l(Q_k)^{n-u}}{|x - y|^{d+n-u}} h_{1,v}(y) dy
= \lambda^{(\sum_{i \in \lambda} \frac{1}{q_i})} \frac{1}{\lambda^{(\sum_{i \in \lambda} \frac{1}{q_i})}} \sum_{k} T_{n-u}(h_{1,v})(x),
\]

where the operator \( T_{n-u} \) is defined in Lemma 2.8 and the function \( h_{1,v}(y) \) is defined as

\[
h_{1,v}(y) = \sum_{Q_k} \prod_{i=1}^{L} \left( \frac{|A_i(y_k) - A_i(y)|}{l(Q_k)} \right) \chi_{Q_k}[f](y).
\]

Using the Chebyshev inequality and the above estimate of \( H_{IV} \), applying Lemma 2.8 (note that \( 1 \leq q \leq \infty \) and \( n - u \geq 1 \)), we finally obtain that

\[
(3.20) \quad w(\{x \in (10G_1)^{c} : H_{IV}(x) > \lambda\}) \leq \lambda^{-q + (\sum_{i \in \lambda} \frac{1}{q_i}) \frac{q}{q_i}} \int_{(10G_1)^{c}} \left[ T_{n-u}(h_{1,v})(x) \right]^{q} w(x) dx
\leq \lambda^{-q + (\sum_{i \in \lambda} \frac{1}{q_i}) \frac{q}{q_i}} \| h_{1,v} \|_{L^{q}(\mathbb{R}^{d}, w)}^{q}.
\]

In the following we give an estimate of \( \| h_{1,v} \|_{L^{q}(\mathbb{R}^{d}, w)}^{q} \). We write

\[
\| h_{1,v} \|_{L^{q}(\mathbb{R}^{d}, w)}^{q} \leq \sum_{Q_k} \int_{Q_k} \prod_{i=1}^{L} \left( \frac{|A_i(y_k) - A_i(y)|}{l(Q_k)} \right)^{q} |f(y)|^{q} w(y) dy
\leq \sum_{Q_k} \prod_{i=1}^{L} \left[ \int_{Q_k} \left( \frac{|A_i(y_k) - A_i(y)|}{l(Q_k)} \right)^{s_i} w(y) dy \right]^{\frac{q}{s_i}} \left[ \int_{Q_k} |f(y)|^{p} w(y) dy \right]^{\frac{q}{p}}
\leq \sum_{Q_k} \prod_{i=1}^{L} \left[ \int_{Q_k} \left( \frac{|A_i(y_k) - A_i(y)|}{l(Q_k)} \right)^{s_i} w(y) dy \right]^{\frac{q}{s_i}} \left[ \int_{Q_k} |f(y)|^{p} w(y) dy \right]^{\frac{q}{p}}
\leq \sum_{Q_k} \prod_{i=1}^{L} \left( \frac{M_{w,s_i}(\nabla A_i)(y_k)}{l(Q_k)} \right)^{\frac{q}{s_i}} \left[ \int_{Q_k} |f(y)|^{p} w(y) dy \right]^{\frac{q}{p}},
\]

where in the second inequality we use the Hölder inequality, and the third inequality follows from the fact that \( Q_k \subset Q_k^* \), \( y_k \) is the center of \( Q_k^* \) and \( l(Q_k^*) = l(Q_k) \).
Notice that \( y_k \) lies in the \((G_A)^c\), i.e., \( y_k \in (D_{1,\lambda})^c \) (see Step 3). Then we obtain that
\[
\mathcal{M}_{w,i_l}((\nabla A_i)(y)) \leq \lambda \frac{\|f\|_p}{\|f\|_{L_p}} \quad \text{for } i = l + 1, \ldots, v.
\]
Using the above inequality, the Hölder inequality again, and (cz-iii) in Step 2, we get
\[
\|h_{1,v}\|_{L^q(\mathbb{R}^d,w)} \leq \lambda^{\frac{q}{q_1}} \left( \sum_{i_l=1}^{v} w(Q_{i_l}) \right)^{\frac{q}{q_1}} \|f\|_p \lambda^{\frac{q}{q_1}} \leq \lambda \left( \sum_{i_l=1}^{v} w(Q_{i_l}) \right)^{\frac{q}{q_1}} \|f\|_p \leq \lambda \left( \sum_{i_l=1}^{v} w(Q_{i_l}) \right)^{\frac{q}{q_1}} \|f\|_p.
\]
Plunge the above estimate into (3.20); with some elementary calculations, we finally obtain that
\[
w(\{x \in (10G_A)^c : H_{IV}(x) > \lambda\}) \lesssim \lambda^{-\frac{\theta}{q}} \left( \sum_{i_l=1}^{v} w(Q_{i_l}) \right)^{\frac{q}{q_1}} \|f\|_p \leq \lambda^{-\tau},
\]
hence, we finish the proof of the term IV.

Finally, we show how to modify the above argument to the case \( q_i = \infty \) for some \( i = k+1, \ldots, l \). Notice that only in Step 1 the construction of exceptional set is involved with \( A_{k+1}, \ldots, A_l \). We assume that only \( q_{k+1} = \cdots = q_u = \infty \) with \( k+1 \leq u \leq l \). Therefore, \( A_{k+1}, \ldots, A_u \) are Lipschitz functions. Then we just fix \( A_{k+1}, \ldots, A_u \) in the rest of the proof. In Step 1 we modify the argument that we only make a construction of exceptional set for \( A_1, \ldots, A_k \) and \( A_{u+1}, \ldots, A_l \). These proofs in Steps 2–8 are the same. Later when studying
\[
\left( \prod_{i_l=1}^{n} \frac{A_i(x) - A_i(y)}{|x - y|} \right) = \left( \prod_{i_l=k+1}^{u} \prod_{i_l=k+1}^{u+1} \prod_{i_l=k+1}^{u+1} \frac{A_i(x) - A_i(y)}{|x - y|} \right),
\]
we just use the same method as in Steps 9–13 to deal with the terms from \( \prod_{i_l=k+1}^{u} \prod_{i_l=k+1}^{u+1} \), since the term \( \prod_{i_l=k+1}^{u} \frac{A_i(x) - A_i(y)}{|x - y|} \) could be absorbed by the kernel \( K(x - y) \) if we observe that \( K(x - y) \prod_{i_l=k+1}^{u} \frac{A_i(x) - A_i(y)}{|x - y|} \) is a standard Calderón–Zygmund kernel. □

**Proposition 3.7** Let \( \frac{d - n}{p - n} \leq r \leq 1 \), \( q_1 = \cdots = q_k = d \), \( d < q_{k+1}, \ldots, q_l \leq \infty \), and \( l \leq q_{l+1}, \ldots, q_n \leq d \) with \( 0 \leq k \leq l \) and \( 1 \leq l < n \), \( p = \infty \). Suppose that \( w \in A_1(\mathbb{R}^d) \). Then
\[
\|C_\ast[\nabla A_1, \ldots, \nabla A_n, f]\|_{L^{r,\infty}(\mathbb{R}^d,w)} \lesssim \left( \frac{\|\nabla A_i\|_{L^{d,1}(\mathbb{R}^d,w)}}{\|\nabla A_i\|_{L^{d,1}(\mathbb{R}^d,w)}} \right) \|f\|_{L^\infty(\mathbb{R}^d,w)} \|
\]
where \( L^{d,1}(\mathbb{R}^d, w) \) is the standard Lorentz space.

**Proof** The proof is similar to that of Proposition 3.6, and one could follow the idea in the proof of Proposition 3.5, so the details of the proof is omitted. □

### 3.4 Interpolation

Notice that we have already proved all the cases in Theorem 1.1(ii) by Propositions 3.4–3.7. And only the part strong type multilinear estimates of (i) in Theorem 1.1 has been established by Propositions 3.1 and 3.3. The rest of part (i) in Theorem 1.1 just
follows from the linear Marcinkiewicz interpolation (see [28] or [1]). In the following we show how to do this.

Since the maximal Calderón commutator \( \mathcal{C}_s \) is \((n + 1)\)-th submultilinear, when using the Marcinkiewicz interpolation, our main strategy is to consider \( \mathcal{C}_s \) as a sublinear operator if we fix part of \( n \) variables.

Let \( \nabla A_i \in L^{q_i}(\mathbb{R}^d, w) \) and \( f \in L^p(\mathbb{R}^d, w) \) with \( \frac{1}{r} = \left( \sum_{i=1}^n \frac{1}{q_i} \right) + \frac{1}{p} \), \( \frac{d}{d+n} < r < \infty \), \( 1 < q_i \leq \infty \) \((i = 1, \ldots, n)\), and \( 1 < p \leq \infty \). Let \( w \in \left( \prod_{i=1}^n A_{\max(\frac{q_i}{p}, 1)}(\mathbb{R}^d) \right) \cap A_p(\mathbb{R}^d) \).

Our goal is to show the following strong type estimate:

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^r(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^n \| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^p(\mathbb{R}^d, w)}.
\]

(3.21)

We divide the proof into several cases. We first consider the case where all \( q_i \neq d \) for \( i = 1, \ldots, n \). Therefore, by Theorem 1.1(ii), the multilinear estimates (1.5) are not involved with \( L^{d,1}(\mathbb{R}^d, w) \) spaces. We further divide this case into two cases: \( 1 < p < \infty \) and \( p = \infty \). Consider first the case \( 1 < p < \infty \). We fix all \( \nabla A_i, q_i \) and \( w \in \left( \prod_{i=1}^n A_{\max(\frac{q_i}{p}, 1)}(\mathbb{R}^d) \right) \cap A_p(\mathbb{R}^d) \). By the basic property of \( A_p(\mathbb{R}^d) \) weight, \( w \in A_{p_1}(\mathbb{R}^d) \) for all \( p_1 > p \). If we choose \( p_1, r_1 \) such that \( p < p_1 < \infty \) and \( \frac{1}{r_1} = \left( \sum_{i=1}^n \frac{1}{q_i} \right) + \frac{1}{p_1} \), then by Theorem 1.1(ii),

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{r_1}(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^n \| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^{p_1}(\mathbb{R}^d, w)}.
\]

(3.22)

Since \( w \in A_p(\mathbb{R}^d) \), by the reverse Hölder inequality of \( A_p(\mathbb{R}^d) \) weight (see [16]) and its definition, there exist \( \varepsilon > 0 \) such that \( w \in A_{p-\varepsilon} \) and \( p - \varepsilon \geq 1 \). Then we may choose \( p_0, r_0 \) such that \( p - \varepsilon \leq p_0 < p \), \( \frac{d}{d+n} < r_0 < \infty \) and \( \frac{1}{r_0} = \left( \sum_{i=1}^n \frac{1}{q_i} \right) + \frac{1}{p_0} \). Hence, we obtain \( w \in \left( \prod_{i=1}^n A_{\max(\frac{q_i}{p_0}, 1)}(\mathbb{R}^d) \right) \cap A_{p_0}(\mathbb{R}^d) \). By using Theorem 1.1(ii), we get

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{r_0,\infty}(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^n \| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^{p_0}(\mathbb{R}^d, w)}.
\]

(3.23)

Applying the Marcinkiewicz interpolation with (3.22) and (3.23), we establish the strong type estimate (3.21) provided that all \( q_i \neq d \) and \( 1 < p < \infty \). Next we consider another case where all \( q_i \neq d \) and \( p = \infty \). By our condition \( r < \infty \), there is at least one \( q_i < \infty \). Without loss of generality, we may suppose that \( q_1 < \infty \). If \( d < q_1 < \infty \), then the rest of proof is similar to the case \( q_i \neq d \) and \( 1 < p < \infty \) once fixing \( \nabla A_i, q_i \) for \( i = 2, \ldots, n \), \( f \in L^\infty(\mathbb{R}^d, w) \), and \( w \in \prod_{i=2}^n A_{\max(\frac{q_i}{p}, 1)}(\mathbb{R}^d) \). If \( 1 < q_i < d \), then \( w \in A_1(\mathbb{R}^d) \) by our condition. Therefore it is easy to show (3.21) using Theorem 1.1(ii) once we fix \( \nabla A_i, q_i \) for \( i = 2, \ldots, n \), \( f \in L^\infty(\mathbb{R}^d, w) \), and \( w \in A_1(\mathbb{R}^d) \).

Second, let us consider the case where there is only one \( q_i \) that equals \( d \). Without loss of generality, we may suppose \( q_1 = d \). Then by our condition, \( w \in A_1(\mathbb{R}^d) \) in this case. Fix \( \nabla A_i, q_i \) for \( i = 2, \ldots, n \), \( f \in L^p(\mathbb{R}^d, w) \) and \( w \in A_1(\mathbb{R}^d) \). Then we can choose \( r_0, r_1, q_{1,0}, q_{1,1} \) such that \( \frac{d}{d+n} < r_0 < r_1 < \infty \), \( 1 < q_{1,0} < d < q_{1,1} \), \( 1 = 1 \), \( \frac{1}{r_0} = \frac{1}{q_{1,0}} + \left( \sum_{i=2}^n \frac{1}{q_i} \right) + \frac{1}{p} \). Then by Theorem 1.1(ii), we get

\[
\| \mathcal{C}_s[\nabla A_1, \ldots, \nabla A_n, f] \|_{L^{r_0,\infty}(\mathbb{R}^d, w)} \lesssim \left( \prod_{i=1}^n \| \nabla A_i \|_{L^{q_i}(\mathbb{R}^d, w)} \right) \| f \|_{L^{p_0}(\mathbb{R}^d, w)}.
\]

(3.24)

Using the Marcinkiewicz interpolation with the above two estimates, we get (3.21) in the case where \( q_1 = d \) and all \( q_2, \ldots, q_n \neq d \).
Maximal Operator of Calderón Commutator

Finally, we consider the general case where there are \( m \) numbers of \( q_i \)s that equal \( d \). We only need to show \( m = 2 \); the general case just follows from the induction. Without loss of generality, we suppose that \( q_1 = q_2 = d \). In this case, \( w \in A_1(\mathbb{R}^d) \). Fix \( \nabla A_i, q_i \) for \( i = 2, \ldots, n, f \in L^p(\mathbb{R}^d, w) \) and \( w \in A_1(\mathbb{R}^d) \). Then we can choose \( r_0, r_1, q_{1,0}, q_{1,1} \), such that
\[
\frac{d}{dx} < r_0 < \infty, \quad 1 < q_{1,0} < d < q_{1,1}, \quad \frac{1}{r_0} = \frac{1}{q_{1,0}} + \left( \sum_{i=2}^n \frac{1}{q_i} \right) + \frac{1}{p}, \quad \text{and} \quad \frac{1}{r_1} = \frac{1}{q_{1,1}} + \left( \sum_{i=2}^n \frac{1}{q_i} \right) + \frac{1}{p}.
\]
Since \( q_2 = d \), by the result of the case there is only one \( q_i = d \) we discussed above, so we get the strong type estimate
\[
C_s[L, \nabla A_2, \ldots, \nabla A_n, f] : L^{q_1}(\mathbb{R}^d, w) \longrightarrow L^{r_1}(\mathbb{R}^d, w) \quad j = 0, 1.
\]

Using the Marcinkiewicz interpolation with the above two estimates, we get (3.21) in the case \( q_1 = q_2 = d \) and all \( q_3, \ldots, q_n \neq d \). Applying the induction of \( m \), we finish the proof.

Remark 3.8 Instead of using the linear Marcinkiewicz interpolation in this proof, another possibly more straightforward method is the multilinear interpolation with change of measures. To the best knowledge of the author, this kind of multilinear interpolation with change of measures is currently unknown. Therefore, it will be interesting to establish the multilinear version of Stein–Weiss interpolation with change of measures (see [27]).

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