The conjugate profile of the circular teeth of a spur gear.  
Part I: Problem statement

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Abstract. The paper presents the manner of finding the conjugate profile of a circular tooth of a spur gear. A synthesis of the method of enveloping applied to the cam mechanisms is presented in order to relate it the profile of the spur gears. The main argument of employing cam mechanisms is the possibility of obtaining any follower law of motion using a minimum number of parts- the cam and the follower. In the case of the mechanisms with flat face follower, the cam is obtained as an envelope of successive positions of the follower. The gear mechanisms are a particular case of cam mechanisms. The major requirement imposed to this mechanism is to transmit the rotational motion between two shafts with a constant transmission ratio. From here it results that the profile a geared wheel can be completely identified when there are known the distance between the axes, the transmission ratio and the profile of one of the wheels. The most used curve as tooth flank is the involute of a circle, due to the fact that this curve has as conjugate curve an involute, too. Although the involute profiles are common in most of the technical appliances, there are cases when they cannot satisfy the functional constraints of certain devices. As example, in the mechanical watches technology, large transmission ratios are needed and the gears with small number of teeth are used as routine. But this necessity is better fulfilled by cycloidal profiles than the involute ones. The circular profiles for the spur gear are the oldest gears due to the simple profile. The exact conjugate profile of a circular tooth obtained by enveloping by means of dedicated software is presented.

1. Introduction

In the monographs on the theory of mechanisms the chapters concerning the synthesis of mechanisms emphasize the synthesis of mechanical function generators [1-2]. For the multi-mobile mechanisms which are widespread in practical application, it is required finding a mechanism capable to offer an output signal having the form:

\[ \theta^m_{p_1} = f(\theta_1, \theta_2, ..., \theta_p, \lambda_1, \lambda_2, ..., \lambda_q) , \]

where \( \theta_i \), \( i = 1 \div p \) are the characteristic kinematical parameters corresponding to the position of the \( p \) driving elements of the mechanism and \( \lambda_j \), \( j = 1 \div q \) are the constructive parameters of the mechanism. If the situation of a mono-mobile mechanism is considered, \( p = 1 \), the solution of the problem can be found by recalling a mechanism from one of the two main classes of mechanisms:
linkages and higher pair mechanisms. An imposed law of motion (1) to the driven element can be provided by different manners, the most widespread solutions are either using an electrical motor with constant rotational speed and ensuring the change of speed by a mechanical solution or by using an electrical motor coupled to a electronically signal generator, in this second case the motor actuating a mechanical system with constant transmission ratio [3-4]. For the linkages, the exact solution of the problem can be found for a reduced number of situations due to the small number of constructive parameters of the mechanism. It is looked for a mechanism for which the final element has a motion as close as possible to the theoretical imposed motion, of the form:

$$\theta_{\text{aprox}} = f(\theta_1, \lambda_1, \lambda_2, \ldots, \lambda_q)$$  \hspace{1cm} (2)

This stipulation is attained following an optimization process of the constructive parameters, expressed by the condition:

$$|\theta_{\text{aprox}}^n - \theta_{\text{ex}}^n| < \Delta$$  \hspace{1cm} (3)

Here $\Delta$ is the maximum value of the deviation between the theoretical and approximate signal for the final element. A foremost problem occurring in this case is indicated by Hunt [5], namely by the optimization mentioned process only the constructive dimensions of a mechanism with predefined structure are optimized. But the optimization method cannot answer the question: which linkage, from all possible structural solutions is best appropriate to the dimensional optimization in order to satisfy the condition (3). All these aspects can be overcome by accepting as synthesis solution a mechanism with higher pairs [6], that is in fact a cam mechanism [7-13]. A cam mechanism presents the property of transmitting the motion directly from the driving element to the driven element by means of a higher pair. By an accurate design of the driving element (the cam) one can obtain whichever intended law of motion of the driven element (the follower). The employment of cam mechanisms rise some problems: the high cost of the profiled element (the cam) and, from functional point of view, increased values of the contact stresses together to the risk of wear occurrence  in the higher pair. The two main types of planar cam mechanisms are the mechanisms with tip follower [14], presented in figure 1, and the mechanisms with curved face follower, shown in figure 2.

Figure 1. Cam mechanisms with tip follower.

The essential difference between the two types of mechanisms refers to the contact point. For the flat face follower, the contact point $C$ moves along the follower but for the tip follower, the contact point $V$ is immobile on the follower. This remark permits replacing the theoretical mechanism formed by the tip follower and the theoretic profile with a real mechanism, having a follower with a roller of $R$.
radius placed in the point $V$, joined to the follower at its center. This roller tracks the real profile \((PR)\) which is a curve equidistant to the theoretic profile. In this manner, the sliding friction between the tip of the follower and the theoretical profile is replaced by rolling friction, more advantageous, if thinking in terms of wear and efficiency of the mechanism. In the case of mechanisms with flat face follower, the displacement of the contact point on the follower doesn't permit the replacement of the follower with a roller follower and thus the existence of sliding friction in the higher pair must be accepted.

![Figure 2](image2.png)

**Figure 2.** Cam mechanisms with curved face follower.

There is though a situation when, the friction between the cam and the faced follower may be replaced by rolling friction, specifically for a circular cam profile.

![Figure 3](image3.png)

**Figure 3.** For curved face follower and circular cam, the sliding friction may be replaced by rolling friction.

An analysis based on figure 3 shows that for the mechanism in figure 3.a, the elements change the roles and the circular profile of the cam [14-17], allows for considering the mechanism from figure 3.b as a mechanism with rotating cam and roller oscillating follower, where the roller is the disc of the cam.

### 2. Application for the gear case

A particular yet common practical case occurs when for a mechanism with two mobile parts, both in rotating motion, it is required that the relation (1) takes the particular form:

$$\theta_2 = k \theta_1$$  \hspace{1cm} (4)

where $k$ is a constant. In this case the mechanism becomes a gear mechanism and the ratio:
\[ \theta_1 / \theta_2 = k = i_{12} \]  \hspace{1cm} (5)

represents the transmission ratio of the gear. The flanks of the gear are in fact the surfaces that figure the higher pair. The axis of relative motion is a straight line immobile in space \cite{18}. During gearing, the \( \Delta_{12} \) line describe, with respect to the axes of the teethed wheels, two revolute surfaces \( \Sigma_{12} \) one sheet hyperboloid, cones or cylinders), representing the axodes of relative motions. During the motion, the two axodes roll and slide simultaneously with respect to \( \Delta_{12} \) axis remaining tangent permanently, and reciprocate enveloping process of these takes place. This remark is the fundament of gears machining by enveloping. Therefore, with known relative position of the axes of the geared wheels, the transmission ratio and one of the surfaces, the other surface can be unequivocal determined.

![Diagram](image_url)

**Figure 4.** Principle of teeth generation by enveloping

When the axes \( \Delta_1 \) and \( \Delta_2 \) are parallel, the two hyperboloids become tangent cylinders \( C_{w1}, C_{w2}, \) rolling without slipping one over the other. The mechanism is currently a planar one and the entire gearing process can be studied in a section normal to the axes of the wheels. The intersections between the rolling cylinders and the plane of study are two tangent circles \( c_{w1} \) and \( c_{w2} \) known as pitch circles. The principle of obtaining the conjugate profile of a teethed wheel by enveloping is shown in figure 4. The gears 1 and 2 are represented by the pitch circles with the centers \( O_1, O_2, \) and radii \( r_{w1,2} \). The two circles are tangent in the point \( C \) where two points \( C_{1,2} \) from the two circles overlap. The pure rolling condition requires that:

\[ v_{C1} = v_{C2} \]  \hspace{1cm} (6)

with the direct consequence:
For external gearing, the wheels must rotate opposite sense. There are considered two planes $p_1, p_2$, attached to the two wheels. In the first plane, the curve $γ_1$ is considered, defined by the equation:

$$F(x_1, y_1) = 0$$  \hspace{5cm} (8)

The profile $γ_2$ of the second wheel in the plane $p_2$ is obtained applying to the plane $p_1$ a rotation of angle $θ_1$ and to the plane $p_2$ a contrary rotation of angle $θ_2$, and additionally obeying the eq. (7). The equation of the curve $γ_2$ from the plane $p_2$ has the form:

$$G[x_2, y_2, θ_1, θ_2(θ_1)] = 0$$  \hspace{5cm} (9)

Or, more generally:

$$H(x_2, y_2, θ_1) = 0$$  \hspace{5cm} (10)

The equation (10), represents a family of curves [19] depending on parameter $θ_1$, in the plane $p_2$. The equation of the flank of the profile is given by the system:

$$H[x_2, y_2, θ_1] = 0$$

$$\frac{∂H[x_2, y_2, θ_1]}{∂θ_1} = 0$$  \hspace{5cm} (11)

That represents the equation of envelope family of curves (10).

From figure 4 it results that there are two manners of obtaining the profile of the wheel 2:

a) one considers a fixed ground and the two wheels are mobile but obeying the condition (7)

b) one considers the wheel 2 immobile while the wheel 1 performs a planetary motion about the point $O_2$ in a manner that ensures pure rolling condition between the pitch circles.

The methodology presented above cannot emphasise another aspect related to the trajectory of the contact point between the flanks. Considering the conjugate flanks of the wheels as known, the mechanism resulting is a mechanism with rotating cam and curved face follower, with the contact point $P$ mobile on both curves and also mobile in the plane $Oxy$ of the ground. During gearing, the contact point $P$ describes in the plane of the ground a curve $A$ known as line of action.

To solve this aspect, the fundamental law of gearing should be used as it permits simultaneous defining of conjugate profile and of line of action. To exemplify these aspects, the manner of obtaining the conjugate profile of a wheel with elliptical tooth profile is presented next, figure 5. As expected, the modified shape of the curve and the changed relative position of the axes influence the shape of the profile of the conjugate wheel. Amid all types of curves employed as tooth flank the involute of the circle imposed itself from several reasons: the mathematical study due to Euler proved that the conjugate of an involute is an involute too; the involute is a technological curve because it can be obtained using tools with straight-line cutting edge; the transmission ratio is unmodified when the distance between axes changes; the line of action is a straight line. The machining process of involute teeth using rack type tools is presented in figure 6. As noticed from figure 6, for a reduced number of teeth, $z < 17$, the undercutting phenomenon occurs, with negative consequences upon the contact ratio and strength of the tooth. The demand of large transmission ratios requires pinions with small number of teeth. A solution for this requirement is offered by the cycloidal profile gears, figure 7, [20]. Although the technological process is more complex compared to the one for obtaining the involute gears, they are required by precision mechanics applications since they present the advantage of
running with no interference even for small number of teeth and they offer conditions for lubricant film formation. The geometry of wheel with cycloid profile is presented in Fig. 7.

Figure 5. Obtaining the conjugate of a gear with elliptic flank by enveloping

Figure 6. Involute teeth generation using a rack type tool

Figure 7. Cycloidal wheel with two teeth

The cycloidal profile of the tooth is realized by pairs of arcs of hypocycloid and epicycloid, generated with the same circle and a moving circle of diameter \( d_i / 4 \). The mentioned types of profiles are widely used in machine technology but the idea of employing geared wheels with circular flanks is strongly attractive. Using a gear with cylindrical teeth would allow for, as shown above, replacing the sliding friction with the rolling friction between the flanks. Additionally, the manufacturing costs for cylindrical teeth are reduced as the possibility of machining independent teeth and assembling them on the crown exists. Jacob Leupold, in the well-known _Teatrum machinarium_ [21] presents several solutions of gearing for cylindrical flanks. In literature [22-23] there are also presented constructive solutions where one of the gears has circular profile teeth. From the cost price point of view, to approximate the theoretical flanks of the conjugate wheel with flanks obtained via simpler technological curves would be especially useful but with the consequence of slight deviation of the constant transmission ration.
3. Conclusions
The paper presents the general principles concerning the structural synthesis of function generators mechanisms. An analysis upon the manner the two main categories- the lower pairs mechanisms and the higher pairs mechanisms, respond to these requirement, reaches the conclusion that theoretically, the higher pairs mechanisms permit providing any law of motion to the driven element while the lower pair mechanisms permit only an approximation of an imposed law of motion. For the lower pairs mechanisms the necessity of increased precision in providing the law of motion of the driven element requires an increased structural complexity.

The higher pair mechanisms have the great advantage of permitting the execution of any law of motion using only two elements: the profiled driving element-the cam, and the driven element-the follower. Regardless of spatial or planar mechanism, a higher pair must exist between the cam and the follower: a class 1 higher pair for spatial mechanisms and a class 4 higher pair for planar mechanisms. The main inconvenience of the cam mechanisms consist in the intricacy of obtaining the profile of the cam which is a complicated, non-technological curve. No matter which constructive type the follower is, roller of flat-face, the profile of the cam will result as an envelope of the successive positions occupied by the follower in the plane of the cam.

The gear mechanisms can be regarded as a special case of cam mechanisms with oscillating follower which satisfy the condition of constant ratio between the angular velocities of the cam and of the follower.

Some particularities, concerning the method of obtaining the conjugate profile of a gear when the profile of the teeth, the transmission ratio and the distance between the axes are known, are presented. Though the involute teeth gears are quasi-unanimously accepted as solutions for transmitting the rotational motion with constant ratio, there are fields where these gears don’t satisfy the functional or technological requirements. The gears with circular tooth profile have as main advantage a tool plain from constructive point of view and the possibility of replacing the sliding friction between the flanks – always present for the involute gears, with rolling friction.

Simulation software is used to obtain the conjugate profile for a gear with circular profile. The possibility of replacing the exact conjugate profile with an approximate profile but preferable technologically, but with the rebate to the requirement of rigorously constant transmission ratio would led to gears with lower costs and higher reliability.

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