Cohomology of the variational complex

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Abstract. Cohomology of the variational bicomplex in the calculus of variations in classical field theory are computed in the class of exterior forms of finite jet order. This provides a solution of the global inverse problem of the finite order calculus of variations.

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1 Introduction

Let $Y \rightarrow X$ be a smooth fibre bundle of a field model. We study cohomology of the variational bicomplex of exterior forms on the infinite order jet space $J^\infty Y$ of $Y \rightarrow X$. The exterior differential on $J^\infty Y$ splits into the sum of the vertical differential $d_V$ and the horizontal differential $d_H$. These differentials, together with the variational operator $\delta$, constitute the variational bicomplex of exterior forms on $J^\infty Y$.

Note that the two differential algebras of exterior forms $O^*_\infty$ and $Q^*_\infty$ are usually considered on $J^\infty Y$. The $O^*_\infty$ consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. Lagrangian field theory is phrased in terms of $O^*_\infty$. Its cohomology, except de Rham cohomology and a particular result of [23] on $\delta$-cohomology, remains unknown. The $Q^*_\infty$ is the structure algebra of the sheaf of germs of exterior forms on finite order jet manifolds. For short, one can say that it consists of exterior forms of locally finite jet order. The $d_H$- and $\delta$-cohomology of $Q^*_\infty$ has been investigated in [2, 21]. Due to Lemma 8 below, we simplify this investigation and complete it by the study of $d_V$-cohomology of $Q^*_\infty$. We prove that the differential algebra $O^*_\infty$ has the same $d_H$- and $\delta$-cohomology as $Q^*_\infty$ (see Theorem 16 below). This provides a solution of the global inverse problem of the calculus of variations in the class of finite order Lagrangians. The main point for applications is that the obstruction to the exactness of the calculus of variations is given by closed forms on the fibre bundle $Y$, and is of first order.
2 The differential calculus on $J^\infty Y$

Smooth manifolds throughout are assumed to be real, finite-dimensional, Hausdorff, paracompact, and connected. Put further $\text{dim}X = n \geq 1$.

Jet spaces provide the standard framework in theory of non-linear differential equations and the calculus of variations [6, 10, 19, 23]. Recall that the $r$-order jet space $J^rY$ consists of sections of $Y \rightarrow X$ identified by $r + 1$ terms of their Taylor series. The key point is that $J^rY$ is a smooth manifold. It is coordinated by $(x^\lambda, y^i, y^i_\Lambda)$, where $(x^\lambda, y^i)$ are bundle coordinates on $Y \rightarrow X$ and $\Lambda = (\lambda_k \ldots \lambda_1), |\Lambda| = k \leq r$, denotes a symmetric multi-index. The infinite order jet space $J^\infty Y$ is defined as a projective limit $(J^\infty Y, \pi^\infty_r)$ of the inverse system $X \pi^r \leftarrow Y \pi^r \leftarrow \cdots \leftarrow J^{r-1}Y \pi^r \leftarrow \cdots$ (1) of finite order jet manifolds $J^rY$ of $Y \rightarrow X$, where $\pi^r_{r-1}$ are affine bundles. The surjections $\pi^\infty_r: J^\infty Y \rightarrow J^rY$ (2) obey the composition condition $\pi^\infty_i = \pi^j_i \circ \pi^\infty_j, \forall j > i$. The set $J^\infty Y$ is provided with the coarsest topology such that all surjections (2) are continuous. The base of open sets of this topology consists of the inverse images of open subsets of finite order jet manifolds under the surjections (2), which thus are open maps. With this topology, $J^\infty Y$ is a paracompact Fréchet (but not Banach) manifold modelled on a locally convex vector space of formal series $\{x^\lambda, y^i, y^i_\Lambda, \ldots\}$ [1, 21]. Bearing in mind the well-known Borel theorem, one can say that $J^\infty Y$ consists of equivalence classes of sections of $Y \rightarrow X$ identified by their Taylor series at points $x \in X$. A bundle coordinate atlas $\{U_Y, (x^\lambda, y^i)\}$ of $Y \rightarrow X$ yields the manifold coordinate atlas $\{((\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)), 0 \leq |\Lambda|, of $J^\infty Y$, together with the transition functions $y^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x^\lambda} d_\mu y^i_\Lambda$, (3) where $\lambda + \Lambda$ is the multi-index $(\lambda \lambda_k \ldots \lambda_1)$ and $d_\lambda$ are the total derivatives $d_\lambda = \partial_\lambda + \sum_{|\Lambda| \geq 0} y^i_{\lambda+\Lambda} \partial_i^\Lambda$. Moreover, $Y$ is a strong deformation retract of $J^\infty Y$ (see Appendix A for an explicit form of a homotopy map).
Since $J^\infty Y$ is not a Banach manifold, the familiar geometric definition of differential objects on $J^\infty Y$ is not appropriate (see, e.g., [1], [20]). One uses the fact that $J^\infty Y$ is a projective limit of the inverse system of manifolds (1). Given this inverse system, we have the direct system

\[ O^* \to O^*_0 \to O^*_1 \to \cdots \to O^*_r \to \cdots \]  

of differential algebras $O^*_r$ of exterior forms on finite order jet manifolds, where $\pi^*_r$ are pull-back monomorphisms. This direct system admits a direct limit $(O^*_\infty, \pi^*_r)$ in the category of $\mathbb{R}$-modules. It consists of exterior forms on finite order jet manifolds modulo the pull-back identification, together with the $\mathbb{R}$-module monomorphisms $\pi^*_k: O^*_k \to O^*_\infty$ which obey the composition condition $\pi^*_i = \pi^*_j \circ \pi^*_k$, $\forall j > i$. Operations of the exterior product $\wedge$ and the exterior differentiation $d$ of exterior forms on finite order jet manifolds commute with the pull-back maps $\pi^*_r$ and, thus, constitute the direct systems of the order-preserving endomorphisms of the direct system (4). These direct systems have the direct limits which make $O^*_\infty$ a graded differential algebra. The $O^*_\infty$ is a differential calculus over the $\mathbb{R}$-ring $O_\infty^0$ of continuous real functions on $J^\infty Y$ which are the pull-back of smooth real functions on finite order jet manifolds by surjections (4). Passing to the direct limit of de Rham complexes on finite order jet manifolds, de Rham cohomology of the differential algebra $O^*_\infty$ has only been found [3, 7]. This coincides with de Rham cohomology of the fibre bundle $Y$ (see Section 4). However, this is not a way of studying other cohomology of the graded differential algebra $O^*_\infty$.

To solve this problem, let us enlarge $O^*_\infty$ to the $\mathbb{R}$-ring $Q^*_\infty$ of continuous real functions on $J^\infty Y$ such that, given $f \in Q^*_\infty$ and any point $q \in J^\infty Y$, there exists a neighborhood of $q$ where $f$ coincides with the pull-back of a smooth function on some finite order jet manifold. The reason lies in the fact that the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $Q^*_\infty$ [21]. Therefore, sheaves of $Q^*_\infty$-modules on $J^\infty Y$ are fine and, consequently, acyclic. Then, the abstract de Rham theorem on cohomology of a sheaf resolution can be called into play.

Remark 1. Throughout, we follow the terminology of [12] where by a sheaf $S$ over a topological space $Z$ is meant a sheaf bundle $S \to Z$. Accordingly, $\Gamma(S)$ denotes the canonical presheaf of sections of the sheaf $S$, and $\Gamma(S)$ is the group of global sections of $S$. All sheaves below are ringed spaces, but we omit this terminology if there is no danger of confusion.

Let us define a differential calculus over the ring $Q^*_\infty$. Let $O^*_r$ be a sheaf of germs of exterior forms on the $r$-order jet manifold $J^r Y$ and $\Gamma(O^*_r)$ its canonical presheaf. There
is the direct system of canonical presheaves
\[ \Gamma(O^*_X) \xrightarrow{\pi^*_1} \Gamma(O^*_0) \xrightarrow{\pi^*_0} \Gamma(O^*_1) \xrightarrow{\pi^*_1} \cdots \xrightarrow{\pi^*_r} \Gamma(O^*_r) \xrightarrow{\pi^*_r} \cdots, \]
where \( \pi^*_r \) are pull-back monomorphisms with respect to open surjections \( \pi^*_r \). Its direct limit \( O^*_\infty \) is a presheaf of graded differential \( \mathbb{R} \)-algebras on \( J^\infty Y \). The germs of elements of the presheaf \( O^*_\infty \) constitute a sheaf \( Q^*_\infty \) on \( J^\infty Y \). It means that, given a section \( \phi \in \Gamma(U, Q^*_\infty) \) over an open subset \( U \subset J^\infty Y \) and any point \( q \in U \), there exists a neighbourhood \( U_q \subset U \) of \( q \) such that \( \phi|_{U_q} \) is the pull-back of a local exterior form on some finite order jet manifold. However, \( Q^*_\infty \) does not coincide with the canonical presheaf \( \Gamma(Q^*_\infty) \) the sheaf \( Q^*_\infty \). The structure algebra \( Q^*_\infty = \Gamma(J^\infty Y, Q^*_\infty) \) of the sheaf \( Q^*_\infty \) is a desired differential calculus over the \( \mathbb{R} \)-ring \( Q^*_\infty \). There are obvious \( \mathbb{R} \)-algebra monomorphisms
\[ O^*_\infty \rightarrow Q^*_\infty, \quad O^*_\infty \rightarrow \Gamma(Q^*_\infty). \]

For short, we agree to call elements of \( Q^*_\infty \) the exterior forms on \( J^\infty Y \). Restricted to a coordinate chart \((\pi^*_0)^{-1}(U_Y)\) of \( J^\infty Y \), they can be written in a coordinate form, where horizontal forms \( \{dx^\lambda\} \) and contact 1-forms \( \{\theta^i_\lambda = dy^i_\lambda - y^i_{\lambda+\Lambda}dx^\lambda\} \) constitute the set of generators of the differential calculus \( Q^*_\infty \). There is the canonical splitting
\[ Q^*_\infty = \bigoplus_{k,s} Q^{k,s}_{\infty}, \quad 0 \leq k, \quad 0 \leq s \leq n, \]
of \( Q^*_\infty \) into \( Q^0_{\infty} \)-modules \( Q^{k,s}_{\infty} \) of \( k \)-contact and \( s \)-horizontal forms, together with the corresponding projections
\[ h_k : Q^*_\infty \rightarrow Q^{k,s}_{\infty}, \quad 0 \leq k, \quad h^s : Q^*_\infty \rightarrow Q^{*,s}_{\infty}, \quad 0 \leq s \leq n. \]

Accordingly, the exterior differential on \( Q^*_\infty \) is decomposed into the sum \( d = d_H + d_V \) of horizontal and vertical differentials such that
\[ d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi), \quad \phi \in Q^*_\infty, \]
\[ d_V \circ h^s = h^s \circ d \circ h^s, \quad d_V(\phi) = \theta^i_\lambda \wedge \partial^i_\lambda \phi. \]

They are nilpotent, i.e.,
\[ d_H \circ d_H = 0, \quad d_V \circ d_V = 0, \quad d_V \circ d_H + d_H \circ d_V = 0. \]

**Remark 2.** It should be emphasized that, in the class of exterior forms of locally finite order, all local operators are well-defined because these forms depend locally on a finite number of variables and all sums over these variables converge.

**Remark 3.** Traditionally, one attempts to introduce the differential algebra \( Q^*_\infty \) of locally pull-back forms on \( J^\infty Y \) in a standard geometric way [4, 7, 20, 21]. The difficulty lies in the geometric interpretation of derivations of the \( \mathbb{R} \)-ring \( Q^0_{\infty} \), as vector fields on the Fréchet manifold \( J^\infty Y \).
3 The variational bicomplex

Being nilpotent, the differentials \( d_V \) and \( d_H \) provide the natural bicomplex \( \{ Q^{k,m}_\infty \} \) of the sheaf \( \Omega^*_\infty \) on \( J^\infty Y \). To complete it to the variational bicomplex, one considers the projection \( \mathbb{R} \)-module endomorphism

\[
\tau = \sum_{k>0} \frac{1}{k} \tau \circ h_k \circ h^n,
\]

\[
\tau(\phi) = (-1)^{|\Lambda|} \theta^i \wedge [d_\Lambda (\partial_i^\Lambda] \phi], \quad 0 \leq |\Lambda|,
\]

\( \phi \in \Gamma(\Omega^\infty) \), of \( \Omega^*_\infty \) such that

\[
\tau \circ d_H = 0, \quad \tau \circ d \circ \tau - \tau \circ d = 0.
\]

Introduced on elements of the presheaf \( \Omega^*_\infty \) (see, e.g., [7, 10, 22]), this endomorphism is induced on the sheaf \( \Omega^*_\infty \) and its structure algebra \( \Omega^*_\infty \). Put

\[
E_k = \tau(\Omega^k,n_\infty), \quad E_k = \tau(\Omega^k,n_\infty), \quad k > 0.
\]

Since \( \tau \) is a projection operator, we have isomorphisms

\[
\Gamma(E_k) = \tau(\Gamma(\Omega^k,n_\infty)), \quad E_k = \tau(\Omega^k,n_\infty).
\]

The variational operator on \( \Omega^*_\infty \) is defined as the morphism \( \delta = \tau \circ d \). It is nilpotent, and obeys the relation

\[
\delta \circ \tau - \tau \circ d = 0. \quad (5)
\]

Let \( \mathbb{R} \) and \( \Omega^*_X \) denote the constant sheaf on \( J^\infty Y \) and the sheaf of exterior forms on \( X \), respectively. The operators \( d_V, d_H, \tau \) and \( \delta \) give the following variational bicomplex of sheaves of exterior forms on \( J^\infty Y \):

\[
\begin{array}{cccccccc}
0 & \to & \Omega^k,0_\infty & \overset{d_H}{\to} & \Omega^k,1_\infty & \overset{d_H}{\to} & \cdots & \overset{d_H}{\to} & \Omega^k,n_\infty \to E_k \to 0 \\
0 & \to & \Omega^1,0_\infty & \overset{d_H}{\to} & \Omega^1,1_\infty & \overset{d_H}{\to} & \cdots & \overset{d_H}{\to} & \Omega^1,n_\infty \to E_1 \to 0 \\
0 & \to & \mathbb{R} & \overset{d_H}{\to} & \Omega^0,0_\infty & \overset{d_H}{\to} & \cdots & \overset{d_H}{\to} & \Omega^0,n_\infty \equiv \Omega^0,n
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & \to & \Omega^0,0_X & \overset{d}{\to} & \Omega^0,1_X & \overset{d}{\to} & \cdots & \overset{d}{\to} & \Omega^0,n_X \to 0 \\
0 & \to & \mathbb{R} & \overset{d}{\to} & \Omega^1,0_X & \overset{d}{\to} & \cdots & \overset{d}{\to} & \Omega^1,n_X \to 0
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & \to & \Omega^1,0_X & \overset{d}{\to} & \Omega^1,1_X & \overset{d}{\to} & \cdots & \overset{d}{\to} & \Omega^1,n_X \to 0 \\
0 & \to & \mathbb{R} & \overset{d}{\to} & \Omega^0,0_X & \overset{d}{\to} & \cdots & \overset{d}{\to} & \Omega^0,n_X \to 0
\end{array}
\]
The second row and the last column of this bicomplex form the variational complex

\[
0 \rightarrow \Omega^0_\infty \rightarrow \Omega^1_\infty \rightarrow \cdots \rightarrow \Omega^n_\infty \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \cdots
\]  

(7)

The corresponding variational bicomplexes \( \{ \Omega^*_\infty, E_k \} \) and \( \{ O^*_\infty, E_k \} \) of the differential calculus \( \Omega^*_\infty \) and \( O^*_\infty \) take place.

There are the well-known statements summarized usually as the algebraic Poincaré lemma (see, e.g., [18, 22]).

Lemma 1. If \( Y \) is a contractible fibre bundle \( \mathbb{R}^{n+p} \rightarrow \mathbb{R}^n \), the variational bicomplex \( \{ O^*_\infty, E_k \} \) of the graded differential algebra \( O^*_\infty \) is exact.

It follows that the variational bicomplex of sheaves (6) is exact for any smooth fibre bundle \( Y \rightarrow X \). Moreover, all sheaves \( Q^{k,m}_\infty \) in this bicomplex are fine, and so are the sheaves \( \mathcal{E}_k \) in accordance with the following lemma.

Lemma 2. Sheaves \( \mathcal{E}_k \), \( k > 0 \), are fine.

Proof. Though \( \mathbb{R} \)-modules \( E_{k>1} \) fail to be \( \Omega^0_\infty \)-modules [22], one can use the fact that the sheaves \( \mathcal{E}_{k>0} \) are projections \( \tau(\Omega^k_\infty) \) of sheaves of \( \Omega^0_\infty \)-modules. Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be a locally finite open covering of \( J^\infty Y \) and \( \{ f_i \in \Omega^0_\infty \} \) the associated partition of unity. For any open subset \( U \subset J^\infty Y \) and any section \( \varphi \) of the sheaf \( \Omega^k_\infty \) over \( U \), let us put \( h_i(\varphi) = f_i \varphi \). Then, \( \{ h_i \} \) provide a family of endomorphisms of the sheaf \( \Omega^k_\infty \), required for \( \Omega^k_\infty \) to be fine. Endomorphisms \( h_i \) of \( \Omega^k_\infty \) also yield the \( \mathbb{R} \)-module endomorphisms \( \overline{h}_i = \tau \circ h_i : \mathcal{E}_k \rightarrow \mathcal{E}_k \) of the sheaves \( \mathcal{E}_k \). They possess the properties required for \( \mathcal{E}_k \) to be a fine sheaf. Indeed, for each \( i \in I \), there is a closed set \( \text{supp} f_i \subset U_i \) such that \( \overline{h}_i \) is zero outside this set, while the sum \( \sum_{i \in I} \overline{h}_i \) is the identity morphism. \( \square \)

This Lemma simplify essentially our cohomology computation of the variational bicomplex in comparison with that in [2, 21]. Since all sheaves except \( \mathbb{R} \) and \( \pi^{\infty,*} \Omega^*_X \) in the bicomplex (8) are fine, the abstract de Rham theorem ([12], Theorem 2.12.1) can be applied to columns and rows of this bicomplex in a straightforward way. We will quote the following variant of this theorem (see Appendix B for its proof).

Theorem 3. Let

\[
0 \rightarrow S \xrightarrow{h_0} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \quad p > 1,
\]  

(8)

be an exact sequence of sheaves on a paracompact topological space \( Z \), where the sheaves \( S_p \) and \( S_{p+1} \) are not necessarily acyclic, and let

\[
0 \rightarrow \Gamma(Z, S) \xrightarrow{h_0} \Gamma(Z, S_0) \xrightarrow{h^0} \Gamma(Z, S_1) \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} \Gamma(Z, S_p) \xrightarrow{h^p} \Gamma(Z, S_{p+1})
\]  

(9)
be the corresponding cochain complex of structure groups of these sheaves. The $q$-cohomology groups of the cochain complex for $0 \leq q \leq p$ are isomorphic to the cohomology groups $H^q(Z,S)$ of $Z$ with coefficients in the sheaf $S$.

4 De Rham cohomology of $J^{\infty}Y$

Let us start from de Rham cohomology of the graded differential algebra $\mathcal{O}^*_\infty$.

**Proposition 4.** There is an isomorphism

$$H^*(\mathcal{O}^*_\infty) = H^*(Y)$$

between de Rham cohomology $H^*(\mathcal{O}^*_\infty)$ of $\mathcal{O}^*_\infty$ and de Rham cohomology $H^*(Y)$ of the fibre bundle $Y$.

**Proof.** The proof is based on the fact that the de Rham complex

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \overset{d}{\to} \mathcal{O}^1_\infty \overset{d}{\to} \cdots$$

of $\mathcal{O}^*_\infty$ is the direct limit of de Rham complexes of exterior forms on finite order jet manifolds. Since the exterior differential $d$ commutes with the pull-back maps $\pi_{r-1}^*$, these complexes form a direct system. Then, in accordance with the well-known theorem $[8]$, the cohomology groups $H^*(\mathcal{O}^*_\infty)$ of the de Rham complex (10) are isomorphic to the direct limit of the direct system

$$0 \to H^*(X) \overset{\pi^*}{\to} H^*(Y) \overset{\pi_{1,r}^*}{\to} H^*(J^1Y) \to \cdots$$

of de Rham cohomology groups $H^*(J^rY)$ of finite order jet manifolds $J^rY$. The forthcoming Lemma 5 completes the proof. $\square$

**Lemma 5.** De Rham cohomology of any finite-order jet manifold $J^rY$ is equal to that of $Y$.

**Proof.** Since every fibre bundle $J^rY \to J^{r-1}Y$ is affine, $J^{r-1}Y$ is a strong deformation retract of $J^rY$, and so is $Y$ (see Appendix A). Then, in accordance with the Vietoris–Begle theorem $[4]$, cohomology $H(J^rY, \mathbb{R})$ of $J^rY$ with coefficients in the constant sheaf $\mathbb{R}$ coincides with that $H(Y, \mathbb{R})$ of $Y$. The well-known de Rham theorem completes the proof. $\square$

Turn now to de Rham cohomology of the graded differential algebra $\mathcal{Q}^*_\infty$. Let us consider the complex of sheaves

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \overset{d}{\to} \mathcal{Q}^1_\infty \overset{d}{\to} \cdots$$

(11)
on $J^\infty Y$ and the de Rham complex of their structure algebras

$$0 \to \mathbb{R} \to Q_0^\infty \xrightarrow{d} Q_1^\infty \xrightarrow{d} \cdots.$$  \hfill (12)

**Proposition 6.** There is an isomorphism

$$H^*(Q_\infty^*) = H^*(Y)$$

of de Rham cohomology $H^*(Q_\infty^*)$ of the graded differential algebra $Q_\infty^*$ to that $H^*(Y)$ of the fiber bundle $Y$.

**Proof.** The complex (11) is exact due to the Poincaré lemma, and is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$ since $Q_r^\infty$ are sheaves of $Q_0^\infty$-modules. Then, by virtue of Theorem \ref{thm:3}, we have the cohomology isomorphism

$$H^*(Q_\infty^*) = H^*(J^\infty Y, \mathbb{R}).$$ \hfill (13)

Lemma \ref{lem:7} below completes the proof. \hfill \Box

**Lemma 7.** There is an isomorphism

$$H^*(J^\infty Y, \mathbb{R}) = H^*(Y, \mathbb{R}) = H^*(Y)$$ \hfill (14)

between cohomology $H^*(J^\infty Y, \mathbb{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$, that $H^*(Y, \mathbb{R})$ of $Y$, and de Rham cohomology $H^*(Y)$ of $Y$.

**Proof.** Since $Y$ is a strong deformation retract of $J^\infty Y$, the first isomorphism in (14) follows from the above-mentioned Vietoris–Begle theorem \ref{thm:4}, while the second one is a consequence of the de Rham theorem. \hfill \Box

Since the graded differential algebras $O_\infty^*$ and $Q_\infty^*$ have the same de Rham cohomology, we agree to call

$$H^*(J^\infty Y) = H^*(Q_\infty^*) = H^*(O_\infty^*)$$

the de Rham cohomology of $J^\infty Y$.

Proposition \ref{prop:6} shows that every closed form $\phi \in Q_\infty^*$ splits into the sum

$$\phi = \varphi + d\xi, \quad \xi \in Q_\infty^*,$$ \hfill (15)

where $\varphi$ is a closed form on the fiber bundle $Y$. Accordingly, Proposition \ref{prop:6} states that, if $\phi$ in this splitting belongs to $O_\infty^*$, so is $\xi$. The decomposition (15) will play an important role in the sequel.
5 Cohomology of $d_V$

Let us consider the vertical exact sequence of sheaves

$$0 \to \mathcal{O}_X^m \xrightarrow{\pi^*} \mathcal{Q}_{\infty}^{0,m} \xrightarrow{d_V} \mathcal{Q}_{\infty}^{1,m} \xrightarrow{d_V} \cdots, \quad 0 \leq m \leq n,$$

in the variational bicomplex (6) and the corresponding complex of their structure algebras

$$0 \to \mathcal{O}^m(X) \xrightarrow{\pi^*} \mathcal{Q}_{\infty}^{0,m} \xrightarrow{d_V} \mathcal{Q}_{\infty}^{1,m} \xrightarrow{d_V} \cdots.$$ (17)

**Proposition 8.** There is an isomorphism

$$H^*(m, d_V) = H^*(Y, \pi^* \mathcal{O}_X^m)$$

(18)

of cohomology groups $H^*(m, d_V)$ of the complex (17) to cohomology groups $H^*(Y, \pi^* \mathcal{O}_X^m)$ of $Y$ with coefficients in the pull-back sheaf $\pi^* \mathcal{O}_X^m$ on $Y$.

**Proof.** The exact sequence (17) is a resolution of the pull-back sheaf $\pi^* \mathcal{O}_X^m$ on $J^\infty Y$. Then, by virtue of Theorem 3, we have a cohomology isomorphism

$$H^*(m, d_V) = H^*(J^\infty Y, \pi^* \mathcal{O}_X^m).$$

The isomorphism (18) follows from the facts that $Y$ is a strong deformation retract of $J^\infty Y$ and that $\pi^* \mathcal{O}_X^m$ is the pull-back onto $J^\infty Y$ of the sheaf $\pi^* \mathcal{O}_X^m$ on $Y$.  

**Corollary 9.** Cohomology groups $H^{\geq \dim Y}(m, d_V)$ vanish.

The cohomology groups $H^*(m, d_V)$ have a $C^\infty(X)$-module structure. For instance, let

$$Y \cong X \times V \to X$$

be a trivial fibre bundle with a typical fibre $V$. There is an obvious isomorphism of $\mathbb{R}$-modules

$$H^*(m, d_V) = \mathcal{O}_X^m \otimes H^*(V).$$

(19)

6 Cohomology of $d_H$

Turn now to the rows of the variational bicomplex (6). We have the exact sequence of sheaves

$$0 \to \mathcal{Q}_{\infty}^{k,0} \xrightarrow{d_H} \mathcal{Q}_{\infty}^{k,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}_{\infty}^{k,n} \xrightarrow{\tau} \mathcal{E}_k \to 0, \quad k > 0.$$

Since the sheaves $\mathcal{Q}_{\infty}^{k,0}$ and $\mathcal{E}_k$ are fine, this is a resolution of the fine sheaf $\mathcal{Q}_{\infty}^{k,0}$. It states immediately the following.
Proposition 10. The cohomology groups \( H^*(k, d_H) \) of the complex

\[
0 \rightarrow Q^k_\infty \xrightarrow{d_H} Q^{k,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q^{k,n}_\infty \xrightarrow{\tau} E_k \rightarrow 0, \quad k > 0,
\]

are trivial.

This result at terms \( Q^{k,n}_\infty \) recovers that of [21]. The exactness of the complex (20) at the term \( Q^{k,n}_\infty \) means that, if

\[ \tau(\phi) = 0, \quad \phi \in Q^{k,n}_\infty, \]

then

\[ \phi = d_H \xi, \quad \xi \in Q^{k,n-1}_\infty. \]

Since \( \tau \) is a projection operator, there is the \( \mathbb{R} \)-module decomposition

\[ Q^{k,n}_\infty = E_k \oplus d_H(Q^{k,n-1}_\infty). \]

Remark 4. One can derive Proposition 10 from Theorem 3, without appealing to that sheaves \( \mathfrak{E}_k \) are acyclic.

Let us consider the exact sequence of sheaves

\[ 0 \rightarrow \mathbb{R} \rightarrow \Omega^0_\infty \xrightarrow{d_H} \Omega^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{0,n}_\infty \]

where all sheaves except \( \mathbb{R} \) are fine. Then, from Theorem 3 and Lemma 7, we state the following.

Proposition 11. Cohomology groups \( H^r(d_H), r < n, \) of the complex

\[ 0 \rightarrow \mathbb{R} \rightarrow \Omega^0_\infty \xrightarrow{d_H} \Omega^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{0,n}_\infty \]

are isomorphic to de Rham cohomology groups \( H^r(Y) \) of \( Y. \)

This result recovers that of [21], but let us say something more.

Proposition 12. Any \( d_H \)-closed form \( \sigma \in Q^{*,<n}_\infty \) is represented by the sum

\[ \sigma = h_0 \varphi + d_H \xi, \quad \xi \in Q^*_\infty, \]

where \( \varphi \) is a closed form on the fibre bundle \( Y. \)

Proof. Due to the relation

\[ h_0 d = d_H h_0, \]
the horizontal projection $h_0$ provides a homomorphism of the de Rham complex (12) to the complex

$$0 \to \mathbb{R} \to Q^0_{\infty} \xrightarrow{d_H} Q^0_{\infty,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q^0_{\infty,n} \xrightarrow{d_H} 0. \tag{25}$$

Accordingly, there is a homomorphism

$$h^*_0 : H^r(J^\infty Y) \to H^r(d_H), \quad 0 \leq r \leq n, \tag{26}$$

of cohomology groups of these complexes. Proposition 8 and Proposition 11 show that, for $r < n$, the homomorphism (26) is an isomorphism (see the relation (33) below for the case $r = n$). It follows that a horizontal form $\psi \in Q^{0,<n}$ is $d_H$-closed (resp. $d_H$-exact) if and only if $\psi = h_0 \phi$ where $\phi$ is a closed (resp. exact) form. The decomposition (15) and Proposition 10 complete the proof.

Proposition 13. If $\phi \in Q^{0,<n}$ is a $d_H$-closed form, then $d_V \phi = d \phi$ is necessarily $d_H$-exact.

Proof. Being nilpotent, the vertical differential $d_V$ defines a homomorphism of the complex (25) to the complex

$$0 \to Q^{1,0}_{\infty} \xrightarrow{d_H} Q^{1,1}_{\infty} \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q^{1,n}_{\infty} \xrightarrow{d_H} 0$$

and, accordingly, a homomorphism of cohomology groups $H^*(d_H) \to H^*(1,d_H)$ of these complexes. Since $H^{<n}(1,d_H) = 0$, the result follows.

7 Cohomology of the variational complex

Let us prolong the complex (22) to the variational complex

$$0 \to \mathbb{R} \to Q^0_{\infty} \xrightarrow{d_H} Q^0_{\infty,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} Q^0_{\infty,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \to \cdots \tag{27}$$

of the graded differential algebra $Q^*_\infty$. In accordance with Lemma 4, the variational complex (4) is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$. Then, Theorem 3 and Lemma 7 give immediately the following.

Proposition 14. There is an isomorphism

$$H^*_\text{var} = H^*(Y) \tag{28}$$

between cohomology $H^*_\text{var}$ of the variational complex (27) and de Rham cohomology of the fibre bundle $Y$. 11
The isomorphism (28) recovers the result of [21] and that of [2] at terms \( Q_0, E_1 \), but let us say something more. The relation (5) for \( \tau \) and the relation (24) for \( h_0 \) define a homomorphisms of the de Rham complex (12) of the algebra \( Q^* \) to the variational complex (27). The corresponding homomorphism of their cohomology groups is an isomorphism. Then, in accordance with the splitting (15), we come to the following assertion which completes Proposition 12.

**Proposition 15.** Any \( \delta \)-closed form \( \sigma \in Q_{k,n}^*, k \geq 0 \), is represented by the sum

\[
\sigma = h_0 \varphi + d_H \xi, \quad k = 0, \quad \xi \in Q^{0,n-1}, \tag{29a}
\]
\[
\sigma = \tau(\varphi) + \delta(\xi), \quad k = 1, \quad \xi \in Q^{0,n}, \tag{29b}
\]
\[
\sigma = \tau(\varphi) + \delta(\xi), \quad k > 1, \quad \xi \in E_{k-1}, \tag{29c}
\]

where \( \varphi \) is a closed \((n+k)\)-form on \( Y \).

8 Cohomology of \( O^*_\infty \)

Thus, we have the whole cohomology of the graded differential algebra \( Q^*_\infty \). The following theorem provides us with \( d_H \)- and \( \delta \)-cohomology of the graded differential algebra \( O^*_\infty \).

**Theorem 16.** Graded differential algebra \( O^*_\infty \) has the same \( d_H \)- and \( \delta \)-cohomology as \( Q^*_\infty \).

**Proof.** Let the common symbol \( D \) stand for the coboundary operators \( d_H \) and \( \delta \) of the variational bicomplex. Bearing in mind the decompositions (23), (29a) – (29c), it suffices to show that, if an element \( \phi \in O^*_\infty \) is \( D \)-exact with respect to the algebra \( Q^*_\infty \) (i.e., \( \phi = D \varphi, \varphi \in Q^*_\infty \)), then it is \( D \)-exact in the algebra \( O^*_\infty \) (i.e., \( \phi = D \varphi', \varphi' \in O^*_\infty \)).

Lemma 1 states that, if \( Y \) is a contractible fibre bundle and a \( D \)-exact form \( \varphi \) on \( J^\infty Y \) is of finite jet order \([\varphi]\) (i.e., \( \varphi \in O^*_\infty \)), there exists an exterior form \( \varphi' \in O^*_\infty \) on \( J^\infty Y \) such that \( \phi = D \varphi \). Moreover, a glance at the homotopy operators for \( d_H \) and \( \delta \) [18] shows that the jet order \([\varphi]\) of \( \varphi \) is bounded for all exterior forms \( \phi \) of fixed jet order. Let us call this fact the finite exactness of the operator \( D \). Given an arbitrary fibre bundle \( Y \), the finite exactness takes place on \( J^\infty Y|_U \) over any open subset \( U \) of \( Y \) which is homeomorphic to a convex open subset of \( \mathbb{R}^{\dim Y} \). Now, we show the following.

(i) Suppose that the finite exactness of the operator \( D \) takes place on \( J^\infty Y \) over open subsets \( U, V \) of \( Y \) and their non-empty overlap \( U \cap V \). Then, it is also true on \( J^\infty Y|_{U \cup V} \).

(ii) Given a family \( \{U_\alpha\} \) of disjoint open subsets of \( Y \), let us suppose that the finite exactness takes place on \( J^\infty Y|_{U_\alpha} \) over every subset \( U_\alpha \) from this family. Then, it is true on \( J^\infty Y \) over the union \( \bigcup_\alpha U_\alpha \) of these subsets.
If the assertions (i) and (ii) hold, the finite exactness of $D$ on $J^\infty Y$ takes place since one can construct the corresponding covering of the manifold $Y$ ([3], Lemma 9.5).

Proof of (i). Let $\phi = D\varphi \in \mathcal{O}^*_\infty$ be a $D$-exact form on $J^\infty Y$. By assumption, it can be brought into the form $D\varphi_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi_V$ on $(\pi^\infty_0)^{-1}(V)$, where $\varphi_U$ and $\varphi_V$ are exterior forms of finite jet order. Due to the decompositions ([23], (29a)–(29c)), one can choose the forms $\varphi_U$, $\varphi_V$ such that $\varphi - \varphi_U$ on $(\pi^\infty_0)^{-1}(U)$ and $\varphi - \varphi_V$ on $(\pi^\infty_0)^{-1}(V)$ are $D$-exact forms. Let us consider their difference $\varphi_U - \varphi_V$ on $(\pi^\infty_0)^{-1}(U \cap V)$. It is a $D$-exact form of finite jet order which, by assumption, can be written as $\varphi_U - \varphi_V = D\sigma$ where an exterior form $\sigma$ is also of finite jet order. Lemma 17 below shows that $\sigma = \sigma_U + \sigma_V$ where $\sigma_U$ and $\sigma_V$ are exterior forms of finite jet order on $(\pi^\infty_0)^{-1}(U)$ and $(\pi^\infty_0)^{-1}(V)$, respectively. Then, putting

$$\varphi'_U = \varphi_U - D\sigma_U, \quad \varphi'_V = \varphi_V + D\sigma_V,$$

we have the form $\phi$ equal to $D\varphi'_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi'_V$ on $(\pi^\infty_0)^{-1}(V)$, respectively. Since the difference $\varphi'_U - \varphi'_V$ on $(\pi^\infty_0)^{-1}(U \cap V)$ vanishes, we obtain $\phi = D\varphi'$ on $(\pi^\infty_0)^{-1}(U \cup V)$ where

$$\varphi' \overset{\text{def}}{=} \begin{cases} \varphi'|_U = \varphi'_U, \\ \varphi'|_V = \varphi'_V \end{cases}$$

is of finite jet order.

Proof of (ii). Let $\phi \in \mathcal{O}^*_\infty$ be a $D$-exact form on $J^\infty Y$. The finite exactness on $(\pi^\infty_0)^{-1}(U \cup V)$ holds since $\phi = D\varphi$ on every $(\pi^\infty_0)^{-1}(U_\alpha)$ and, as was mentioned above, the jet order $[\varphi_\alpha]$ is bounded on the set of exterior forms $D\varphi_\alpha$ of fixed jet order $[\phi]$. \hfill $\square$

**Lemma 17.** Let $U$ and $V$ be open subsets of a fibre bundle $Y$ and $\sigma \in \mathcal{O}^*_\infty$ an exterior form of finite jet order on the non-empty overlap $(\pi^\infty_0)^{-1}(U \cap V) \subset J^\infty Y$. Then, $\sigma$ splits into a sum $\sigma_U + \sigma_V$ of exterior forms $\sigma_U$ and $\sigma_V$ of finite jet order on $(\pi^\infty_0)^{-1}(U)$ and $(\pi^\infty_0)^{-1}(V)$, respectively.

**Proof.** By taking a smooth partition of unity on $U \cup V$ subordinate to the cover $\{U, V\}$ and passing to the function with support in $V$, one gets a smooth real function $f$ on $U \cup V$ which is 0 on a neighborhood of $U - V$ and 1 on a neighborhood of $V - U$ in $U \cup V$. Let $(\pi^\infty_0)^*f$ be the pull-back of $f$ onto $(\pi^\infty_0)^{-1}(U \cup V)$. The exterior form $((\pi^\infty_0)^*f)\sigma$ is zero on a neighborhood of $(\pi^\infty_0)^{-1}(U)$ and, therefore, can be extended by 0 to $(\pi^\infty_0)^{-1}(U)$. Let us denote it $\sigma_U$. Accordingly, the exterior form $(1 - ((\pi^\infty_0)^*f))\sigma$ has an extension $\sigma_V$ by 0 to $(\pi^\infty_0)^{-1}(V)$. Then, $\sigma = \sigma_U + \sigma_V$ is a desired decomposition because $\sigma_U$ and $\sigma_V$ are of finite jet order which does not exceed that of $\sigma$. \hfill $\square$

It is readily observed that Theorem [10] is applied to de Rham cohomology of $\mathcal{O}^*_\infty$ whose isomorphism to that of $\mathcal{Q}^*_\infty$ has been stated by Proposition [3] and Proposition [6].
9 The global inverse problem in the calculus of variations

The variational complex (27) provides the algebraic approach to the calculus of variations on fiber bundles in the class of exterior forms of locally finite jet order [7, 10, 22]. For instance, the variational operator $\delta$ acting on $Q^{0,n}_{\infty}$ is the Euler–Lagrange map, while $\delta$ acting on $E_1$ is the Helmholtz–Sonin map. Let

$$L = \mathcal{L}\omega \in Q^{0,n}_{\infty}, \quad \omega = dx^1 \wedge \cdots dx^n,$$

be a horizontal density on $J^\infty Y$. One can think of $L$ as being a Lagrangian of locally finite order. Then, the canonical decomposition (21) leads to the first variational formula

$$dL = \tau(dL) + (\text{Id} - \tau)(dL) = \delta_1(L) + dH(\phi), \quad \phi \in Q^{1,n-1}_{\infty},$$

where the exterior form

$$\delta_1(L) = (-1)^{|\Lambda|}d\Lambda(\partial_{\Lambda}L)\theta^i \wedge \omega$$

is the Euler–Lagrange form associated with the Lagrangian $L$.

Let us relate the cohomology isomorphism (28) to the global inverse problem of the calculus in variations. As a particular repetition of Proposition 15, we come to its following solution in the class of Lagrangians of locally finite order.

**Theorem 18.** A Lagrangian $L \in Q^{0,n}_{\infty}$ is variationally trivial, i.e., $\delta(L) = 0$ if and only if

$$L = h_0\phi + dH\xi, \quad \xi \in Q^{0,n-1}_{\infty},$$

where $\phi$ is a closed $n$-form on $Y$ (see the expression (29a)).

**Theorem 19.** An Euler–Lagrange-type operator $\mathcal{E} \in E_1$ satisfies the Helmholtz condition $\delta(\mathcal{E}) = 0$ if and only if

$$\mathcal{E} = \delta(L) + \tau(\phi), \quad L \in Q^{0,n}_{\infty},$$

where $\phi$ is a closed $(n+1)$-form on $Y$ (see the expression (29b)).

**Theorem 19** recovers the result of [2, 21].

**Remark 5.** As a consequence of Theorem 18, one obtains that the cohomology group $H^n(d_H)$ of the complex (23) obeys the relation

$$H^n(d_H)/H^n(Y) = \delta(Q^{0,n}_{\infty}),$$

where $\delta(Q^{0,n}_{\infty})$ is the $\mathbb{R}$-module of Euler–Lagrange forms on $J^\infty Y$. 

14
Theorem 16 leads us to the similar solution of the global inverse problem in the class of finite order Lagrangians. This is the case of higher order Lagrangian field theory. Namely, the theses of Theorem 18 and Theorem 19 remain true if all exterior forms in expressions (31) and (32) belong to $O^\infty$. Thus, the obstruction to the exactness of the finite order calculus of variations is the same as for exterior forms of locally finite order, without minimizing the order of Lagrangians. In particular, we recover the result of [23].

Note that the local exactness of the calculus of variations has been proved in the class of exterior forms of finite order by use of homotopy operators which do not minimize the order of Lagrangians (see, e.g., [18, 22]). The infinite variational complex of such exterior forms on $J^\infty Y$ has been studied by many authors (see, e.g., [7, 10, 18, 22]). However, these forms on $J^\infty Y$ fail to constitute a sheaf. Therefore, the cohomology obstruction to the exactness of the calculus of variations has been obtained in the class of exterior forms of locally finite jet order which make up the differential algebra $Q^\infty$ [2, 21]. Several statements without proof were announced in [3]. A solution of the global inverse problem in the calculus of variations in the class of exterior forms of a fixed jet order has been suggested in [2] by a computation of cohomology of the fixed order variational sequence (see [15, 24] for another variant of such a variational sequence). The key point of this computation lies in the local exactness of the finite order variational sequence which however requires rather sophisticated ad hoc technique in order to be reproduced (see also [16]). Therefore, the results of [2] were not called into play. The first thesis of [2] agrees with Theorem 18 for finite order Lagrangians, but says that the jet order of the form $\xi$ in the expression (31) is $k - 1$ if $L$ is a $k$-order variationally trivial Lagrangian. The second one states that a $2k$-order Euler–Lagrange operator can be always associated with a $k$-order Lagrangian.

Theorem 18 and Theorem 19 for elements of $O^\infty$ provide a solution of the global inverse problem in time-dependent mechanics treated as a particular field theory on smooth fiber bundles over $X = \mathbb{R}$ [17]. Note that, in time-dependent mechanics, the inverse problem is more intricate than in field theory. Given a second order dynamic equation, one studies the existence of an associated Newtonian system and its equivalence to a Lagrangian one [17]. Since a fiber bundle $Y \to \mathbb{R}$ is trivial, de Rham cohomology of $Y$ is equal to that of its typical fiber $M$, and so is de Rham cohomology $H^*(J^\infty Y)$ of $J^\infty Y$. The $d_V$-cohomology groups of the differential algebra $O^\infty$ are given by the isomorphism (19) such that

$$H^*(0, d_V) = H^*(1, d_V) = C^\infty(\mathbb{R}) \otimes H^*(M).$$

The variational complex (27) in time-dependent mechanics takes the form

$$0 \to \mathbb{R} \to Q^0_\infty \xrightarrow{d_t} Q^0_\infty \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \xrightarrow{} \cdots.$$ 

Its cohomology coincides with de Rham cohomology of $M$. In particular, Theorem 18 states that a Lagrangian $L$ of time-dependent mechanics is variationally trivial if and
only if it takes the form

\[ L = (\varphi_t + \varphi_i y_i^t)dt + d_t \xi, \]

where \( \varphi = \varphi_t dt + \varphi_i dy^i \) is a closed 1-form on \( Y \) (see also [3]).

## 10 Cohomology of conservation laws

Let us concern briefly cohomology of conservation laws in Lagrangian formalism on \( J^\infty Y \), but everything below is also true for a finite order Lagrangian formalism. Let \( u \) be a vertical vector field on a fibre bundle \( Y \to X \), treated as a generator of a local 1-parameter group of gauge transformations of \( Y \). Its infinite order jet prolongation

\[ J^\infty u = d_\Lambda u \partial^\Lambda, \quad 0 \leq |\Lambda|, \]

is a derivation of the ring \( Q^0_\infty \), and also defines the contraction \( u \rfloor \phi \) and the Lie derivative

\[ L_{J^\infty u} \phi \overset{\text{def}}{=} J^\infty u \rfloor d\phi + d(J^\infty u \rfloor \phi) \]

of elements of the differential algebra \( Q^*_\infty \). It is easily justified that

\[ J^\infty u \rfloor d_H \phi = -d_H (J^\infty u \rfloor \phi), \quad \phi \in Q^*_\infty. \]

Let \( L \) be a Lagrangian on \( J^\infty Y \). By virtue of the first variational formula (30), the Lie derivative of the Lagrangian \( L \) along \( J^\infty u \) reads

\[ L_{J^\infty u} L = J^\infty u \rfloor dL = u \rfloor \delta L - d_H (J^\infty u \rfloor \phi), \quad (34) \]

where

\[ J_u = -J^\infty u \rfloor \phi \in Q^{0,n-1}_\infty \]

is called the symmetry current along the vector field \( u \). If \( L \) is an \( r \)-order Lagrangian, we come to the familiar expression for a symmetry current

\[ J_u = -J^\infty u \rfloor \phi = h_0 (J^{2r-1} u \rfloor \rho_L) + \varphi \]

where \( \rho_L \) is a \((2r - 1)\)-order Lepagean equivalent of the Lagrangian \( L \) [10, 14], and \( \varphi \) is a \( d_H \)-closed form. Of course, a symmetry current \( J_u \) in the expression (34) is not defined uniquely, but up to a \( d_H \)-closed form. In finite order Lagrangian formalism, one usually sets

\[ J_u = h_0 (J^{2r-1} u \rfloor \rho_L), \]
but the problem of a choice of a Lepagean equivalent \( \rho_L \) remains \([3, 10]\).

If the Lie derivative \((34)\) vanishes, we obtain the weak conservation law

\[
d_HJ_u = d\lambda J^\lambda \omega \approx 0
\]
on the shell \( \text{Ker}\delta(L) \), i.e., the global section \( d_HJ_u \) of the sheaf \( \Omega^0,\infty \) on \( J^\infty Y \) takes zero values at points of the subspace \( \text{Ker}\delta(L) \subset J^\infty Y \) given by the condition \( \delta(L) = 0 \). Then, one can say that the divergence \( d_HJ_u \) is a relative \( d_H \)-cocycle on the pair of topological spaces \((J^\infty Y, \text{Ker}\delta(L))\). Of course, it is a \( d_H \)-coboundary, but not necessarily a relative \( d_H \)-coboundary since \( J_u \not\approx 0 \). Therefore, the divergence \( d_HJ_u \) of a conserved current \( J_u \) can be characterized by elements of the relative \( d_H \)-cohomology group \( H^n_{\text{rel}}(J^\infty Y, \text{Ker}\delta(L)) \).

For instance, any conserved Noether current in the Yang–Mills gauge theory on a principal bundle \( P \) with a structure group \( G \) is well known to reduce to a superpotential, i.e., \( J_u = W + dH U \) where \( W \approx 0 \) \([10, 11]\). Its divergence \( d_HJ_u \) belongs to the trivial element of the relative cohomology group \( H^n_{\text{rel}}(J^2 Y, \text{Ker}\delta(L_{YM})) \), where \( Y = J^1 P/G \).

Let now \( N^n \subset X \) be an \( n \)-dimensional submanifold of \( X \) with a compact boundary \( \partial N^n \). Let \( s \) be a section of the fibre bundle \( Y \rightarrow X \) and \( \overline{s} = J^\infty s \) its infinite order jet prolongation, i.e., \( y^i_A \circ \overline{s} = d\lambda s^i, 0 < |\Lambda| \). Let us assume that \( \overline{s}(\partial N^n) \subset \text{Ker}\delta(L) \). Then, the quantity

\[
\int_{N^n} \overline{s} d_HJ_u = \int_{\partial N^n} \overline{s}^* J_u
\]

depends only on the relative cohomology class of the divergence \( d_HJ_u \). For instance, in the above mentioned case of gauge theory, the quantity \((35)\) vanishes.

Let \( N^{n-1} \) be a compact \((n - 1)\)-dimensional submanifold of \( X \) without boundary, and \( s \) a section of \( Y \rightarrow X \) such that \( \overline{s}(N^{n-1}) \subset \text{Ker}\delta(L) \). Let \( J_u \) and \( J'_u \) be two currents in the first variational formula \((34)\). They differ from each other in a \( d_H \)-closed form \( \varphi \). Then, the difference

\[
\int_{N^{n-1}} \overline{s} (J_u - J'_u)
\]
depends only on the homology class of \( N^{n-1} \) and the de Rham cohomology class of \( \overline{s}^* \varphi \). The latter is an image of the \( d_H \)-cohomology class of \( \varphi \) under the morphisms

\[
H^{n-1}(d_H) \xrightarrow{h_0} H^{n-1}(Y) \xrightarrow{s^*} H^{n-1}(X).
\]

In particular, if \( N^{n-1} = \partial N^n \) is a boundary, the quantity \((36)\) always vanishes.
11 Appendix A

If \( Q \to Z \) is an affine bundle coordinated by \((z^\lambda, q^i)\), the map

\[
[0, 1] \times Q \ni (t, z^\lambda, q^i) \mapsto (z^\lambda, tq^i + (1 - t)s^i(z)),
\]

where \( s \) is a global section of \( Q \to Z \), provides a homotopy from \( Q \) to \( Z \) identified with \( s(Z) \subset Q \). Similarly, a desired homotopy from \( J^\infty Y \to Y \) is constructed

Let \( \gamma(k), k \leq 1 \), be global sections of the affine jet bundles \( J^k Y \to J^{k-1} Y \). Then, we have a global section

\[
\gamma: Y \ni (x^\lambda, y^i) \mapsto (x^\lambda, y^i, y^i_{\Lambda} = \gamma(\Lambda, \gamma_{\Lambda} \circ \cdots \circ \gamma(1)) \in J^\infty Y. \tag{37}
\]

of the open surjection \( \pi_0^\infty: J^\infty Y \to Y \). Let us consider the map

\[
[0, 1] \times J^\infty Y \ni (t; x^\lambda, y^i, y^i_{\Lambda}) \mapsto (x^\lambda, y^i, y^i_{\Lambda}) \in J^\infty Y, \quad 0 < |\Lambda|,
\]

\[
y^i_{\Lambda} = f_k(t)y^i_{\Lambda} + (1 - f_k(t))\gamma(k)^{(i)}(x^\lambda, y^i, y^i_{\Sigma}), \quad |\Sigma| < k = |\Lambda|, \tag{38}
\]

where \( f_k(t) \) is a continuous monotone real function on \([0, 1]\) such that

\[
f_k(t) = \begin{cases} 0, & t \leq 1 - 2^{-k}, \\ 1, & t \geq 1 - 2^{-(k+1)}. \end{cases} \tag{39}
\]

A glance at the transition functions (3) shows that, although written in a coordinate form, this map is globally defined. It is continuous because, given an open subset \( U_k \subset J^k Y \), the inverse image of the open set \((\pi_k^\infty)^{-1}(U_k) \subset J^\infty Y\), is the open subset

\[
(t_k, 1] \times (\pi_k^\infty)^{-1}(U_k) \cup (t_{k-1}, 1] \times (\pi_{k-1}^\infty)^{-1}(\pi_{k-1}^k[U_k \cap \gamma(k)(J^{k-1} Y)]) \cup \cdots \cup [0, 1] \times (\pi_0^\infty)^{-1}(\pi_0^k[U_k \cap \gamma(k) \circ \cdots \circ \gamma(1)(Y)])
\]

of \([0, 1] \times J^\infty Y\), where \([t_r, t] = \text{supp } f_r\). Then, the map (38) is a desired homotopy from \( J^\infty Y \to Y \) which is identified with its image under the global section (37).

12 Appendix B

**Proof.** For \( q = 0 \), the manifested isomorphism follows from the fact that \( H^0(Z, S) = \Gamma(Z, S) \) for any sheaf \( S \) on \( Z \). To prove other ones, let us replace the exact sequence (8) with

\[
0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-2}} S_{p-1} \xrightarrow{h^{p-1}} \text{Ker } h^p \to 0
\]
and consider the short exact sequences

\[
0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} \text{Ker } h^1 \to 0,
\]
\[
0 \to \text{Ker } h^{r-1} \xrightarrow{\text{in}} S_{r-1} \xrightarrow{h^{r-1}} \text{Ker } h^r \to 0, \quad 1 < r \leq p.
\]

They give the corresponding exact cohomology sequences

\[
0 \to H^0(Z, S) \to H^0(Z, S_0) \to H^0(Z, \text{Ker } h^1) \to H^1(Z, S) \to \\
H^1(Z, S_0) \to \cdots,
\] (40)
\[
0 \to H^0(Z, \text{Ker } h^{r-1}) \to H^0(Z, S_{r-1}) \to H^0(Z, \text{Ker } h^r) \to \\
H^1(Z, \text{Ker } h^{r-1}) \to H^1(Z, S_{r-1}) \to \cdots.
\] (41)

Since sheaves $S_r$, $0 \leq r < p$, are acyclic, the exact sequence (40) falls into

\[
0 \to H^0(Z, S) \to H^0(Z, S_0) \to H^0(Z, \text{Ker } h^1) \to H^1(Z, S) \to 0,
\]
\[
H^k(Z, \text{Ker } h^1) = H^{k+1}(Z, \text{Ker } h^0), \quad 1 \leq k,
\] (42)

and, similarly, the exact sequence (41) does

\[
0 \to H^0(Z, \text{Ker } h^{r-1}) \to H^0(Z, S_{r-1}) \to H^0(Z, \text{Ker } h^r) \to \\
H^1(Z, \text{Ker } h^{r-1}) \to 0,
\]
\[
H^k(Z, \text{Ker } h^r) = H^{k+1}(Z, \text{Ker } h^{r-1}), \quad 1 \leq k.
\] (43)

The equalities (44) for the couples of numbers $(k = m, r = q - m)$, $1 \leq m \leq q - 2$, and the equality (42) for $k = q - 1$ lead to the chain of isomorphisms

\[
H^1(Z, \text{Ker } h^{q-1}) = H^2(Z, \text{Ker } h^{q-2}) = \cdots = H^q(Z, \text{Ker } h^0) = H^q(Z, S).
\] (45)

The exact sequence (43) for $r = q$ contains the exact sequence

\[
H^0(Z, S_{q-1}) \xrightarrow{h^{q-1}} H^0(Z, \text{Ker } h^q) \to H^1(Z, \text{Ker } h^{q-1}) \to 0.
\] (46)

Since $H^0(Z, S_{q-1}) = \Gamma(Z, S_{q-1})$ and $H^0(Z, \text{Ker } h^q) = \text{Ker } h^q$, the result follows from (45) and (46) for $0 < q \leq p$. 

\[\square\]
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