Model-Checking of Linear-Time Properties Based on Possibility Measure

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Abstract—Using possibility measure, we study model-checking of linear-time properties in possibilistic Kripke structures. First, the notion of possibilistic Kripke structures and the related possibility measure are introduced, then model-checking of reachability and repeated reachability linear-time properties in finite possibilistic Kripke structures are studied. Standard safety properties and \( \omega \)-regular properties in possibilistic Kripke structures are introduced, the verification of regular safety properties and \( \omega \)-regular properties using finite automata are thoroughly studied. It has been shown that the verification of regular safety properties and \( \omega \)-regular properties in a finite possibilistic Kripke structure can be transformed into the verification of reachability properties and repeated reachability properties in the product possibilistic Kripke structure introduced in this paper. Several examples are given to illustrate the methods presented in the paper.

Index Terms—Possibilistic Kripke structure, possibility measure, model checking, linear temporal logic, regular language.

I. INTRODUCTION

In the last four decades, computer scientists have systematically developed theories of correctness and safety in different aspects, such as methodologies, techniques and even automatic tools for correctness and safety verification of computer systems; see for examples [1], [5], [11], [20]. Of which, model checking has been established as one of the most effective automated techniques for analyzing correctness of software and hardware designs [2], [5], [18]. A model checker checks a finite-state system against a correctness property expressed in a propositional temporal logic such as Linear Temporal Logic (LTL) or Computation Tree Logic (CTL). These logics can express safety (e.g., No two processes can be in the critical section at the same time) and liveness (e.g., Every job sent to the printer will eventually print) properties [1], [10], [11], [12], [20], [24]. Model checking has been effectively applied to reasoning about correctness of hardware, communication protocols, software requirements, etc. Many industrial model checkers have been developed, including SPIN [8], SMV [18].

Whereas model-checking techniques focus on the absolute guarantee of correctness - “it is impossible that the system fails” - in practice such rigid notions are hard, or even impossible, to guarantee. Instead, systems are subject to various phenomena of an uncertainty nature, such as message incomplete or garbling and the like, and correctness - “with 99 percent chance the system will not fail” or “the system will not fail most often” - is becoming less absolute. To handle with the systematic verification which has something to do with uncertainties in probability, Hart and Sharir in 1986 [7] investigated the logic of timing sequence in probability propositions and applied probability theory to model checking in which the uncertainty is modeled by probability measure. In 2008, Baier and Katoen [2] systematically introduced the principle and method of model checking based on probability measure and related applications with Markov chain models for probability systems.

On the other hand, since Zadeh proposed the theory of fuzzy sets in 1965 [27], many scholars have been devoting themselves to the research in this theory and its applications. As a branch of the theory of fuzzy sets, possibility measure (28) (more general, fuzzy measure (22)) is a development of classical measure, which focuses on non-additive cases (c.f. [9], [25]) that is different from the probability measure which is additive. Most problems in real situations are complicated and non-additive. As a matter of fact, fuzziness seems to pervade most human perception and thinking processes as noted by Zadeh, especially, modeling human-centered systems, for example, biomedical systems (15), criminal trial systems, decision making systems (6), linguistic quantifiers (26). Therefore, it is necessary to do some research work in the theory and applications of model checking on non-deterministic systems of non-additive measure, especially, fuzzy measure. And this paper attempts to initiate an LTL model checking based on possibility measure.

In this paper, the notion of possibilistic Kripke structure is introduced by combining the system with fuzzy uncertainty, then a possibility measure is induced by the given possibilistic Kripke structure. Linear-time properties specify the traces that a possibilistic Kripke structure should exhibit. Informally speaking, one could say that a linear-time property specifies the admissible (or desired) behavior of the system under consideration. In the following we provide a formal definition of such properties. This definition is rather elementary, and gives an exam-

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ple of what a linear-time property is. In particular, the possibilities of model checking of reachability and repeat reachability are studied, which can be proceeded by solving certain fuzzy relation equations using the least fixed-point method or by constructing the transitive closure of transition possibility distribution. Safety properties and \( \omega \)-regular properties in possibilistic Kripke structures are introduced. Some calculation methods related to transition possibility distribution. Safety properties and \( \omega \)-regular properties in possibilistic Kripke structures can be calculated by the possibility of reachability and repeated reachability properties of the product possibilistic Kripke structure.

The rest of the paper is organized as follows. In Section II, we provide our main definition of possibilistic Kripke structure, and a possibility measure on its paths is introduced. In Section III, linear-time properties in a possibilistic Kripke structure, and a possibility measure on its paths are introduced. In Section IV, we require the labeling function \( L \) to be a function from the state set \( S \) to \([0,1]^{AP}\), where \([0,1]^{AP}\) denotes all fuzzy subsets of \( AP \), which contains the fuzzy uncertainty. In this case, we can transform it into a possibilistic Kripke structure \( M' = (S', AP, I', P', L') \) as in Definition 1 as follows, let \( D = im(L) = \{ L(s)(A) | s \in S, A \in AP \} \), \( S' = S \times D \), and \( I'(s, d) = I(s) \) for any \( (s, d) \in S \times D \), \( P'(s, d, (s', d')) = P(s, s') \), \( L'(s, d) = \{ A \in AP | L(s, A) \geq d \} \). Then the property of \( M' \) can be obtained by that of \( M' \).

(2) In Definition 1 we require the transition possibility distribution and initial distribution are normal, i.e., \( \bigwedge_{s \in S} P(s, s') = 1 \) and \( \bigwedge_{s \in S} I(s) = 1 \), where we use \( \bigwedge X \) or \( \bigvee X \) to represent the least upper bound (or supremum) or the largest lower bound (or infimum) of the subset \( X \subseteq [0,1] \), respectively. These conditions are corresponding to the transition probability distribution and probability initial distribution in probabilistic Kripke structure (2), where the supremum operation is replaced by the sum operation. They form the main differences between possibilistic Kripke structure and probabilistic Kripke structure. In fact, in fuzzy uncertainty, the order instead of the additivity is one of the most important factors to be considered.

The states \( s \) with \( I(s) > 0 \) are considered as the initial states. For state \( s \) and \( T \subseteq S \), let \( P(s, T) \) denote the possibility of moving from \( s \) to some state \( t \in T \) in a single step, that is,

\[
P(s, T) = \bigwedge_{t \in T} P(s, t).
\]

Paths in possibilistic Kripke structure \( M \) are infinite paths in the underlying digraph. They are defined as infinite state sequence \( \pi = s_0 s_1 \cdots s_n \cdots \in S^\omega \) such that \( P(s_i, s_{i+1}) > 0 \) for all \( i \in I \). Let Paths(M) denote the set of all paths in \( M \), and Paths_{fin}(M) denotes the set of finite path fragments \( s_0 s_1 \cdots s_n \) where \( n \geq 0 \) and \( P(s_i, s_{i+1}) > 0 \) for \( 0 \leq i \leq n - 1 \). Let Paths(s) denote the set of all paths in \( M \) that start in state \( s \). Similarly Paths_{fin}(s) denotes the set of finite path fragments \( s_0 s_1 \cdots s_n \) such that \( s_0 = s \).

For a state \( s \), the set of direct successors (written as Post(s) ) and direct predecessors (written Pre(s)) are defined as follows:

- \( \text{Post}(s) = \{ s' | s \in \text{Post}(s, s') \geq 0 \}; \)
- \( \text{Pre}(s) = \{ s' | s \in \text{Pre}(s', s) \geq 0 \}; \)
- \( \text{Post}^*(s) = \{ s' | s \in \text{Post}^*(s') \}; \)
- \( \text{Pre}^*(s) = \{ s' | s \in \text{Pre}^*(s') \}; \)

For \( B \subseteq S \), \( \text{Post}^*(B) = \bigcup_{s \in B} \text{Post}^*(s) \), \( \text{Pre}^*(B) = \bigcup_{s \in B} \text{Pre}^*(s) \).

Remark 2: For a possibilistic Kripke structure \( M = (S, P, I, AP, L) \), the transition possibility distribution \( P \) is a fuzzy relation on \( S \), its transitive closure \( P^* \), say, is also a transition distribution defined as follows, \( P^*(s, t) = \bigvee \{ P(s_0, s_1) \land \cdots \land P(s_{k-1}, s_k) | s_0 = s, s_1, \ldots, s_k, s_{k+1} \in S, k \geq 1 \} \) for \( s, t \in S \). Using \( P^* \), we can construct a new possibilistic Kripke structure \( M^* \), say, as \( M^* = (S, P^*, I, AP, L) \).

Furthermore, if \( M \) is finite, then for any \( s, t \) in \( S \), there exists a finite state sequence \( s_0 s_1 \cdots s_k \), which is written as \( m(s, t) \) in the following, i.e., \( m(s, t) = s_0 s_1 \cdots s_k \) such that \( P^*(s, t) = \bigwedge_{i=0}^{k-1} P(s_i, s_{i+1}) \). We also use \( P(m(s, t)) \) to represent \( P^*(s, t) = \bigwedge_{i=0}^{k-1} P(s_i, s_{i+1}) \).
A possibility measure induced by a possibilistic Kripke structure, the number of states is \( \Omega \) is a nonempty set and \( I \) is a distribution that can be replaced by minimum and maximum operations of multiplication and addition operations of real numbers. The vector \( I \) are given by

\[
I = \begin{align*}
1 & \quad 0 \\
0 & \quad 0 \\
0 & \quad 0 
\end{align*}
\]

Remark 4: A matrix is called a fuzzy matrix if all its elements are taken from the unit interval \([0,1]\). The composition operation of fuzzy matrices is similar to ordinary matrix multiplication operation, just let ordinary multiplication and addition operations of real numbers be replaced by minimum and maximum operations of real numbers, which is called \( \max - \min \) composition operation. We use the symbol \( \circ \) to represent the \( \max - \min \) composition operation. Then for a possibilistic distribution \( P \), its transitive closure is, \( P^+ = P \cup P^2 \cup \cdots = \bigvee_{i=1}^{\infty} P^i \), where \( P^{k+1} = P^k \circ P \) for any positive integer number \( k \). When \( M \) is a finite possibilistic Kripke structure, the number of states is \( n \), say, then \( P^+ = \bigvee_{i=1}^{\infty} P^i \)

B. Possibility measure induced by a possibilistic Kripke structure

Definition 2: A \((\sigma-)\)algebra is a pair \((X, \Omega)\) where \( X \) is a nonempty set and \( \Omega \) is a set consisting of subsets of \( X \) that contains the empty set and is closed under complementation and (countable) unions. Then \((X, \Omega)\) is called a measurable space.

Given a measurable space \((X, \Omega)\), recall that a possibility measure \((28)\) over the algebra \( \Omega \) is a mapping \( m : \Omega \to [0,1] \) satisfying the following conditions,

(i) \( m(\emptyset) = 0, m(X) = 1 \).

(ii) If \( A \subseteq B \) in \( \Omega \), then \( m(A) \leq m(B) \).

(iii) If \( A_i \in \Omega, i \in I \), then \( m(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} m(A_i) \).

Sugeno \((22)\) calls a mapping \( m : \Omega \to [0,1] \) a fuzzy measure over \( \Omega \) if it satisfies (1) and (ii). Therefore, a possibility measure is also a fuzzy measure.

If \( m \) is a possibility measure over the powerset \( 2^X \), then \( m \) is determined by its behaviors on singletons, i.e., if \( \Omega = \mathcal{P}(X) \), then

\[
m(A) = \max_{a \in A} m(a)
\]

for any subset \( A \) of \( X \).

Remark 5: (1) Possibility measure is a class of non-additive measure, it is close related to probability measure. Possibility measure is useful for design in the presence of uncertainties which involve modeling linguistic imprecision or uncertainties with little numerical data to develop a valid probabilistic model. It plays an important role in the human perception and thinking process, especially, modeling human-centered systems, biomedical systems \((16)\), criminal trial, decision making \((6)\), linguistic quantifiers \((24)\). As an example, let us see its application in the diagnosing an ill patient with incomplete data.

An ill patient has symptoms of pneumonia, bronchitis, emphysema and cold, but the data of information is incomplete. A doctor gives an estimation by his experience using possibility measure as follows,

\[
\begin{align*}
&0.5 \text{ pneumonia} \\
&0.3 \text{ bronchitis} \\
&0 \text{ emphysema} \\
&1 \text{ cold}
\end{align*}
\]

which can be extended onto the power-set \( \Omega = \mathcal{P}(X) \) using Eq. (1), where \( X = \{\text{pneumonia, bronchitis, emphysema, cold}\} \).

(2) Possibility measure is extensional while probability measure is intensional. This makes possibility measures easier to compute than probability measures \((4)\). This is also an advantage of using of possibility measures in model checking.

In the following, we give a possibility measure over a possibilistic Kripke structure \( M \).

Definition 3: \((2)\) Given a Kripke structure \( M \), the cylinder set of \( \hat{\tau} = s_0 \cdots s_n \in \text{Paths}_{fin}(M) \) is defined as,

\[
\text{Cyl}(\hat{\tau}) = \{ \tau \in \text{Paths}(M) | \hat{\tau} \in \text{Pref}(\tau) \},
\]

where \( \text{Pref}(\tau) = \{ \tau' \in \text{Paths}_{fin}(M) | \tau' \text{' is a finite prefix of } \tau \} \).

Remark 6: \((1)\) Assume that \( M \) is a possibilistic Kripke structure, then \( \Omega = \mathcal{P}(\text{Paths}(M)) \) is the algebra generated by \( \text{Cyl}(\hat{\tau})| \hat{\tau} \in \text{Paths}_{fin}(M) \) on \( \text{Paths}(M) \).

Proof: For any \( \tau \in \text{Paths}(M) \), let \( \tau = s_0s_1s_2 \cdots \). For all \( j \geq 0 \), write \( \tau_i = s_0s_1 \cdots s_j \). Obviously, \( \bigcup_{\tau \in \text{Paths}(M)} \text{Cyl}(\hat{\tau}) = \{ \tau \} \), so \( \{ \tau \} \in \Omega \) holds for all \( \tau \in \text{Paths}(M) \). For any \( A \subseteq \text{Paths}(M), A = \bigcup \{ \{ \tau \} | \tau \in A \} \), so \( \Omega = \mathcal{P}(\text{Paths}(M)) \).
(2) If $M$ has at least two elements, then $\text{Paths}(M)$ has size continuum, and the \(\sigma\)-algebra $\Omega_\sigma$ generated by $\{\text{Cyl}(\pi) | \pi \in \text{Paths}_{fin}(M)\}$ has size at most continuum. In this case, $2^{\text{Paths}(M)}$ has size larger than continuum, so the $\sigma$-algebra $\Omega_\sigma$ generated by $\{\text{Cyl}(\pi) | \pi \in \text{Paths}_{fin}(M)\}$ is not $2^{\text{Paths}(M)}$. Furthermore, by the observation of the proof of Remark 1, for any $\pi \in \text{Paths}(M)$, $[\pi] \in \Omega_\sigma$, and the following facts hold:

(2-1) For any two elements $\pi_1, \pi_2 \in \text{Paths}_{fin}(M)$,

\[\text{Cyl}(\pi_1) \cap \text{Cyl}(\pi_2) = \begin{cases} \text{Cyl}(\pi_2), & \pi_1 \in \text{Pref}(\pi_2); \\ \text{Cyl}(\pi_1), & \pi_2 \in \text{Pref}(\pi_1); \\ \emptyset, & \text{otherwise}. \end{cases}\]

(2-2) If $E = \text{Cyl}(s_0 \cdots s_k) \in \Omega$, then $E^c = \text{Paths}(M) - E = \bigcup_{i=1}^{k} \cup_{s \in S} \text{Cyl}(s_0 \cdots s_{i-1} s | s \in S) \in \Omega$.

For these reasons, we can define a possibility measure on $\Omega = 2^{\text{Paths}(M)}$.

**Definition 4:** For a possibilistic Kripke structure $M$, let $\text{Paths}(M) = \bigcup_{s \in S} \text{Paths}(s)$. A function $P^M : \text{Paths}(M) \rightarrow [0,1]$ is defined as follows: for any $\pi \in \text{Paths}(M)$, $\pi = s_0 \pi_1 \cdots$, $P^M(\pi) = \bigcup_{i=0}^{\infty} P(s_i, s_{i+1})$. Furthermore, for $A \subseteq \text{Paths}(M)$, define $P^M(A) = \bigvee \{P^M(\pi) | \pi \in A\}$, then we have a well-defined function $P^M : 2^{\text{Paths}(M)} \rightarrow [0,1]$. We call $P^M$ the possibility measure over $\Omega = 2^{\text{Paths}(M)}$ as it satisfies the conditions (i)-(iii) in Definition 2 as shown in Theorem 8 below. If $M$ is clear form the context, then $M$ is omitted and we simply write $P$ for $P^M$.

First, let us see how to calculate the possibility measure $P$ over the cylinder sets.

**Proposition 7:** Let $M$ be a possibilistic Kripke structure. Then the possibility measure of the cylinder sets are given by

\[P(\text{Cyl}(s_0 \cdots s_n)) = I(s_0) \bigwedge_{i=0}^{n-1} P(s_i, s_{i+1}),\]

where $P(s_0) = 1$, specially, $P(\text{Cyl}(s_0)) = I(s_0)$.

**Proof:** As $\text{Cyl}(s_0 \cdots s_n) = \bigcup_{\pi \in S^M s_0 \cdots s_n \in \text{Pref}(\pi)}$, then

\[P(\text{Cyl}(s_0 \cdots s_n)) = \bigvee \{I(s_0) \bigwedge_{i=0}^{\infty} P(s_i, s_{i+1}) | s_0 \cdots s_n \in \text{Pref}(\pi), s_{n+1}, \cdots \in S\} = I(s_0) \bigwedge_{i=0}^{\infty} P(s_i, s_{i+1}) \bigvee \bigwedge_{i=n}^{\infty} P(s_i, s_{i+1}) | s_i \in S, i > n \}.

Since $P$ is a possibilistic distribution, i.e., for all states $s$, $\forall s' \in \text{Paths}(s, s') = 1$ holds. Then, for any $\varepsilon > 0$, $s \in S$, there exists $t \in S$, such that $P(s, t) \geq 1 - \varepsilon$. It follows that for all non-negative integer $i$, if $i \geq n$, then there exists $s_i \in S$ such that $P(s_i, s_{i+1}) \geq 1 - \varepsilon$. This shows

\[\bigvee \bigwedge_{i=n}^{\infty} P(s_i, s_{i+1}) | s_i \in S \right) = 1.\]

So

\[P(\text{Cyl}(s_0 \cdots s_n)) = I(s_0) \bigwedge_{i=0}^{n-1} P(s_i, s_{i+1}).\]

In the following, we also use $P(s_0 \cdots s_n)$ to represent $\bigwedge_{i=0}^{n} P(s_i, s_{i+1})$.

**Theorem 8:** $P$ is a possibility measure on $\Omega = 2^{\text{Paths}(M)}$.

**Proof:** Let us show that $P$ satisfies conditions (i)-(iii) in Definition 2.

(i) (1) As $P(\emptyset) = \bigvee \{P(\pi) | \pi \in \emptyset\} = 0$, then $P(\emptyset) = 0$.

(2) As $\text{Paths}(M) = \bigcup_{s \in S} \text{Paths}(s) = \bigcup_{s \in S} \text{Cyl}(s)$, then $P(\text{Paths}(M)) = \bigvee_{s \in S} P(\text{Cyl}(s))$, according to Proposition 7 $P(\text{Paths}(M)) = \bigvee_{s \in S} I(s)$. Since $I$ is a possibility distribution, i.e., $\bigvee_{s \in S} I(s) = 1$. Hence, $P(\text{Paths}(M)) = 1$.

(ii) Since $P(\bigcup_{i \in A} A_i) = \bigvee \{P(\pi) | \pi \in \bigcup_{i \in A} A_i\} = \bigvee_{i \in A} P(\pi_i) \bigvee P(A_i)$. The condition (ii) holds.

(iii) holds trivially by the definition of $P$.

In Fig. 2, let $\pi_i = s_i s_1 f n$, $A_i = \{\pi_n, \pi_{n+1}, \cdots\}$, then $\bigcap_{i \in A} A_i = \emptyset$, and thus $P(\bigcap_{i \in A} A_i) = P(\emptyset) = 0$. On the other hand, $P(\pi_1) = P(s_1 s_2) = 1/2$, then we get $P(A_1) = \bigvee P(\pi_i) P(\pi_{i+1}) \cdots = 1/2$, it follows that $\bigwedge_{i \in A} P(\pi_i) = 1/2$. Hence, $P(\bigcap_{i \in A} A_i) \neq \bigwedge_{i \in A} P(\pi_i)$.

![Fig. 2. A possibilistic Kripke structure M with two states](image_url)

**Remark 9:** (1) For path starting in a certain (possibly noninitial) state $s$, the same construction is applied to the possibilistic Kripke structure $M$ that resulting from $M$ by letting $s$ as the unique initial state. Formally, for $M = (S, P, I, AP, L)$ and state $s$, $M_s$ is defined by $M_s = (S, P, s, AP, L)$, where $s$ denotes an initial distribution with only one initial state $s$.

(2) The possibility measure $P$ is defined on the algebra $\Omega = 2^{\text{Paths}(M)}$. Of course it can also be restricted to a $\sigma$-algebra $\Omega_\sigma$. The latter case corresponds with the probabilistic model checking (2). Clearly in possibilistic Kripke structure, the possibility measure is defined on $\Omega = 2^{\text{Paths}(M)}$, thus for any set $A \in 2^{\text{Paths}(M)}$, $A$ can be measured in the sense of $P$. And then we can make the discussion more widely compared with probability measure.

**Example 10:** Consider the possibilistic Kripke structure $M$ in Figure 1, let us give several calculations of possibility measure.

\[P(\text{Cyl}(s_0 s_1 s_2)) = I(s_0) \land P(s_0, s_1) \land P(s_1, s_2) = 1.\]

\[P(\text{Cyl}(s_0 s_1 s_2)') = P(\text{Cyl}(s_0 s_2) \cup \text{Cyl}(s_1 s_2)) = P(\text{Cyl}(s_0 s_2)) \lor P(\text{Cyl}(s_1 s_3)),\]
I(s₀) ∧ P(s₀, s₂) = 0.2, Po(Cyl(s₀s₁s₂)) = I(s₀) ∨ P(s₀, s₁) ∨ P(s₁, s₂) = 0.9, then Po(Cyl(s₀s₂)) = 0.2 ∨ 0.9 = 0.9. It follows that Po(Cyl(s₀s₁s₂)) + Po(Cyl(s₀s₁)) ≠ 1.

It follows that possibility measures are not additive, and hence are different from probability measures. Model-checking based on a possibility measure has different behavior compared with that based on a probability measure. We refer to [3] for further comparison between probability measures and possibility measures.

III. POSSIBILITY MEASURES OF REACHABILITY AND REPEATED REACHABILITY LINEAR-TIME PROPERTIES

The quantitative model-checking problem that we are confronted with is: given a possibilistic Kripke structure M and an LTL property P, compute the possibility measure for the set of paths in M for which P holds. We consider some special cases: properties of reachability, constraint reachability and repeated reachability, in this section. For this purpose, let us first present the notion of linear-time properties in a possibilistic Kripke structure.

A. Linear-time properties

Some of the relevant definition of LTL (short for linear-temporal logic) are presented as follows:

Definition 5: ([2]) Syntax of LTL. LTL formulae over the set AP of atomic propositions are formed according to the following grammar,

ϕ ::= true ∨ ϕ ∧ ϕ ∨ ¬ϕ ∨ ϕ ∨ ϕ ∨ ϕ

where a \in AP.

For example, ϕ = (a ∨ b ⊕ c) ∧ a is an LTL formula, but ϕ = ∨ a ∨ (b ∨ c) is not, where a, b, c \in AP.

Definition 6: ([2]) Semantics of LTL. Assume \pi = s₀s₁s₂\ldots is a path starting s₀ in a possibilistic Kripke structure M, \pi[ι] = s₀s₁s₂\ldots, a \in AP and set satisfaction relation ( |= ) as follows:

\pi |= true;
\pi |= a \iff a \in L(s₀);
\pi |= ϕ₁ ∧ ϕ₂ \iff \pi |= ϕ₁ and \pi |= ϕ₂;
\pi |= ¬ϕ \iff \pi \not |= ϕ;
\pi |= ϕ \iff \pi[ι] |= ϕ;
\pi |= ϕ₁ ∨ ϕ₂ \iff \exists k ≥ 0, \pi[k] |= ϕ₂ and \pi[ι] |= ϕ₁ for all 1 ≤ i ≤ k − 1.

The until operator allows to derive the temporal modalities ("eventually", sometimes in the future) and □ ("always", from now on forever) as follows:

ϕ = true ∨ ϕ, □ϕ = ¬ϕ ∧ ϕ.

As a result, the following intuitive meaning of ◊ and □ is obtained: ◊ϕ ensures that ϕ will be true eventually in the future. □ϕ is satisfied if and only if it is not the case that eventually ¬ϕ holds. This is equivalent to the fact that ϕ holds from now on forever.

Definition 7: Let M = (S, P, I, AP, L) be a possibilistic Kripke structure without terminal states, i.e., for any state s, there exists a state \tau such that P(s, \tau) = 0. The trace of the infinite path fragment \pi = s₀s₁\ldots is defined as trace(π) = L(s₀)L(s₁)\ldots. The trace of the finite path fragment \tilde{π} = s₀s₁\ldots sₙ is defined as trace(\tilde{π}) = L(s₀)L(s₁)\ldots L(sₙ).

The set of traces of a set Π of paths is defined in the usual way, trace(Π) = { trace(π) | \pi \in Π}. Let Traces(s) denote the set of traces initiated at s, and Traces(M) the set of traces of the possibilistic Kripke structure M, i.e., Traces(s) = trace(Paths(s)) and Traces(M) = ∪ s∈S Traces(s).

LTL formulae stand for properties of paths (or in fact traces). This means that a path can either fulfill an LTL formula or not. To precisely formulate when a path satisfies an LTL formula, we proceed as follows. First, the semantics of an LTL formula ϕ is defined as a language Words(ϕ) that contains all infinite words over the alphabet 2^AP which are traces of paths that satisfy ϕ. That is, Words(ϕ) = { trace(π) ∈ (2^AP)^ω | π |= ϕ }.

Definition 8: A linear-time property (LP property) over the set of atomic propositions AP is a subset of (2^AP)^ω.

Note that it suffices to consider infinite words only (and not finite words), as possibilistic Kripke structure without terminal states are considered.

Definition 9: Let P be an LP property over AP and M = (S, P, I, AP, L) be a possibilistic Kripke structure without terminal states. Then, M = (S, P, I, AP, L) satisfies P, denoted M |= P, iff Traces(M) ⊆ P. State s ∈ S satisfies P, notation s |= P, whenever Traces(s) ⊆ P.

B. POSSIBILITY MEASURE OF LP PROPERTY

For a countable set Σ, any subset of Σ^ω is called a language of infinite words, sometimes also called an ω-linguage. Languages will be denoted by the symbol L.

Definition 10: Let M be a possibilistic Kripke structure and P an LP property (both over AP). The possibility for M to exhibit a trace in P, denoted P^M(P), is defined by

P^M(P) = P^M(\{ π ∈ Paths(M) | trace(π) ∈ P \}).

Similarly, for the LTL formula ϕ we write Pω(ϕ) for Pω(Words(ϕ)), i.e., Pω(ϕ) = Pω(Paths(ϕ)) = Pω(π ∈ Paths(M) | π |= ϕ). For state s of M, we write Pω(s | |= ϕ) for Pω(π ∈ Paths(s) | π |= ϕ), i.e., Pω(s | |= ϕ) = Pω(π ∈ Paths(s) | π |= ϕ).

C. REACHABILITY POSSIBILITY

One of the elementary questions for the quantitative analysis of systems modeled by possibilistic Kripke structures is to compute the possibility of reaching a set B of states, where B may represent a set of certain bad states which should be visited only with some small possibility, or dually, a set of good states which should rather be visited frequently.

This subsection focuses on computing P(ϕB).
The possibility measure of eventually reaching B is given by:

\[
P_0(\partial B) = \bigvee_{s_0 \in \text{Paths}_{s_0}(M) \cap (S \cup B)} P_{0}(Cyl(s_0, \ldots, s_n))
\]

\[
= \bigvee_{s_0 \in \text{Paths}_{s_0}(M) \cap (S \cup B)} \bigvee_{i=1}^{n-1} P(s_i, s_{i+1})
\]

Using the transitive closure \( P^* \), \( P(\partial B) \) has a very simple form, i.e.,

\[
P_0(\partial B) = \bigvee_{s \in S} P(s) \cup P^*(s, t)
\]

In the following, we give another approach to calculate \( P(\partial B) \), which is adopted in the probabilistic model-checking of reachability (2).

Let variable \( x_s \) denote the possibility measure of reaching \( B \) from \( s \), i.e., \( x_s = P_0(s \models \partial B) \), for arbitrary \( s \in S \). The goal is to compute \( x_s \) for all state \( s \). There are three cases to consider.

1. \( B \) is not reachable from \( s \) in the underlying directed graph of \( M \), then \( x_s = 0 \).
2. \( B \) is reachable from \( s \), i.e., \( x_s > 0 \). Moreover, if \( s \in B \), then \( x_s = 1 \). For the state \( s \) is not in \( B \), it holds that

\[
x_s = \bigvee_{s \in S, \partial B} P(s, t) \cup \bigvee_{s \in S, \partial B} P(s, u)
\]

This equation states that either \( B \) is reached within one step, i.e., by a finite path fragment \( su \) with \( u \in B \) (second summand, \( \bigvee_{s \in S, \partial B} P(s, u) \)), or first a state \( t \in S \setminus B \) is reached from which \( B \) is reached - this corresponds to path fragments \( st \cdots u \) of length \( \geq 2 \) where all states (except the last one) do not belong to \( B \) (first summand, \( \bigvee_{s \in S, \partial B} P(s, t) \)).

Let \( S \times \preceq(B) \times B \) denote the set of states \( s \in S \setminus B \) such that there is a path fragment \( s_0 \cdots s_n(n > 0) \) with \( s_0 = s \). Then for the vector \( X = (x_s)_{s \in S} \), we have:

\[
X = A \times X \cup b,
\]

where \( A = (P(s, t))_{s \in S} \) and the vector \( b = (b_s)_{s \in S} \) contains the possibilities of reaching \( B \) from \( S \) within one step, i.e., \( b_s = P_0(s, B) \).

The above technique yields the following two phase algorithm to compute reachability possibility in finite Markov chains: first, perform a graph analysis to compute the set \( X \), and then generate the matrix \( A \) and the vector \( b \). Second, solve the fuzzy relation equation \( X = A \times X \cup b \). This problem is addressed below by characterizing the desired possibility vector as the least solution in \([0,1]^S\). This characterization enables us to compute the possibility measure by a finite iteration method. In fact, we present a characterization for a slightly more general problem, viz. constrained reachability (i.e., until properties).

\[D.\] Constrained reachability possibility

Let \( M = (S, P, I, AP, L) \) be a possibilistic Kripke structure and \( B, C \subseteq S \). Consider the event of reaching \( B \) via a finite path fragment which ends in a state \( s \in B \), and visits only states in \( C \) prior to reaching \( s \). Using LTL–like notations, this event is denoted by \( C \cup B \). The event \( \partial B \) considered above agrees with \( S \cup B \). For \( n \geq 0 \), the event \( C \cup^{\leq n} B \) has the same meaning as \( C \cup B \), except that it is required to reach \( B \) (via states in \( C \)) within \( n \) steps. Formally, \( C \cup^{\leq n} B \) is the union of the basic cylinders spanned by path fragments \( s_0 \cdots s_k \) such that \( k \leq n \) and \( s_j \in C \) for all \( 0 \leq i < k \) and \( s_k \in B \).

Let \( S_{a_1}, S_{a_2}, S \) be a partition of \( S \) such that,

1. \( B \subseteq S_{a_1} \subseteq \{s \in S | P(s \models C \cup B) = 1\} \)
2. \( S(C \cup B) \subseteq S_{a_2} \subseteq \{s \in S | P(s \models C \cup B) = 0\} \)
3. \( S \subseteq S(S_{a_1} \cup S_{a_2}) \)

For all state \( s \), if \( s \in S_{a_1} \), we have \( P(s \models C \cup B) = 1 \); if \( s \in S_{a_2} \), then \( P(s \models C \cup B) = 0 \); if \( s \in S \), then we get a fuzzy matrix \( A = (P(s), t, s) \), by omitting the rows and columns for the states \( s \in S_{a_1} \cup S_{a_2} \) from \( P \) and \( b = (b_s)_{s \in S} \).

**Theorem 11**: (Least Fixed Point Characterization)

The vector \( X = (P(s \models C \cup B))_{s \in S} \) is the least fixed point of the operator \( \Phi : [0,1]^S \to [0,1]^S \), which is given by \( \Phi(Y) = A \times Y \cup b \).

(iii) Furthermore, if \( X^{(0)} = 0 \) is the vector consisting of zeros only, and for \( n \geq 0 \), \( X^{(n+1)} = \Phi(X^{(n)}) \), then

\[
\Phi(Y) = (X^{(0)})_{s \in S},
\]

where \( X^{(0)} = P_0(s \models C \cup^{\leq n} S) \) for each state \( s \in S_{a_1} \).

(ii) We prove the well-definedness of \( \Phi \) as a function from \([0,1]^S\) to \([0,1]^S\).

\[\text{Proof:}\] (i) We prove the well-definedness of \( \Phi \) as a function from \([0,1]^S\) to \([0,1]^S\).

For \( Y = (y_s)_{s \in S} \), the vector \( \Phi(Y) = (y'_s)_{s \in S} \), and \( y'_s = \bigvee_{s \in S} P(s, t) \cup \bigvee_{s \in S} P(s, u) \).

Clearly, \( y'_s \in [0,1]^S \), therefore \( \Phi(Y) \in [0,1]^S \).

Next we prove the fixed point property, i.e., \( X = \Phi(X) \).

Since if \( s \in S_{a_1} \), then \( x_s = P_0(s \models C \cup B) = 1 \); if \( s \in S_{a_2} \), then \( x_s = P_0(s \models C \cup B) = 0 \). For all \( s \in S_{a_1} \), we derive that

\[
x_s = \bigvee_{s \in S} P_0(s, t) \cup \bigvee_{s \in S} P_0(s, u)
\]

which is the component for state \( s \) in the vector \( \Phi(X) \).

Hence \( X = \Phi(Y) \).

(ii) Let us first show that \( x_s^{(0)} = P_0(s \models C \cup^{\leq n} S) \) for each state \( s \in S_{a_1} \) by induction on \( n \).

If \( n = 0 \), \( x_s^{(0)} = 0 \), and \( P_0(s \models C \cup^{\leq n} S) = P_0(\emptyset) = 0 \), so \( x_s^{(0)} = 0 \).

If \( n = 1 \),

\[
x_s^{(1)} = \bigvee_{s \in S} (P(s, t) \cup x_s^{(0)}) \cup P(s, s) = \bigvee_{s \in S} P(s, u)
\]

and

\[
P_0(s \models C \cup^{\leq n} S) = \bigvee_{s \in S} P(Cyl(s)) = \bigvee_{s \in S} P(s)
\]

Hence, \( x_s^{(1)} = P_0(s \models C \cup^{\leq n} S) \).

Assume that for any \( n \geq 2 \), \( x_s^{(n)} = P_0(s \models C \cup^{\leq n} S) \), let
us calculate $x^{(n+1)}_s$ as follows:

$$x^{(n+1)}_s = \Phi^s(\chi^{(n)}_s)$$

$$= \forall_{t_0 \in S}(P(s, t_0) \land x^{(n)}_s) \lor P(s, S_{-1})$$

$$= [\forall_{t_0 \in S} P(s, t_0) \land \Phi P(t_0) = C \cup S_{-1}]) \lor P(s, S_{-1})$$

$$= \forall_{t_0 \in S} P(s, t_0) \land \Phi P(t_0) \lor \Phi (t_0, t_1) \lor \Phi (t_1, t_2) \lor \cdots$$

$$= \forall_{t_0, t_1, t_2, \cdots, t_{n-1} \in S} (\Phi (t_0) \lor \Phi (t_1, t_2) \lor \cdots)$$

$$= \forall_{t_0, t_1, t_2, \cdots, t_{n-1} \in S} \Phi x^{(n)}_s.$$

Hence,

$$x^{(n+1)}_s = \Phi^s(\chi^{(n)}_s).$$

This shows that $x^{(n)}_s = \Phi^s(\chi^{(n)}_s)$ for each state $s \in S_T$, and the condition (ii-1) holds.

Since $C \cup S_{-1}$ is the countable union of the events $C \cup S_{-1}$, for $s \in S_T$, let $x'_s = \Phi^s(\chi^{(n)}_s)$, then we have,

$$x'_s = \Phi^s(\chi^{(n)}_s).$$

Then $x'_s$ is the least fixed point characterization in Theorem 11.

Example 13: Consider the possibilistic Kripke structure $M$ in Example 8, the event of interest is $C \cup S_{-1}$ where $B = \{s_3\}, C = \{s_0, s_1, s_2\}$. We shall compute the bounded constrained reachability possibility $x_3 = \Phi^s(\chi^{(n)}_s) \lor \Phi (s, S_{-1})$, for all states $s \in S_T$, where we take $S_0 = \emptyset, S_{-1} = \{s_3\}, S_1 = \{s_0, s_1, s_2\}$.

Using the state order $s_0 < s_1 < s_2$, the possibility matrix $A$ and the vector $b$ are given by,

$$A = \begin{pmatrix} 0 & 1 & 0.2 \\ 0 & 0 & 1 \\ 0 & 0.7 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0.9 \\ 1 \end{pmatrix}.$$

The least fixed point characterization suggests the following iterative scheme,

$$X^{(0)} = 0, X^{(n+1)} = A \times X^{(n)} \lor b,$$

where $X^{(n)} = \Phi^s(\chi^{(n)}_s)$, for all states $s \in S_T$, where we take $S_0 = \emptyset, S_{-1} = \{s_3\}, S_1 = \{s_0, s_1, s_2\}$. Then we can obtain that

$$X^{(1)} = A \times X^{(0)} \lor b = b,$$

$$X^{(2)} = A \times X^{(1)} \lor b = \begin{pmatrix} 0 & 1 & 0.2 \\ 0 & 0 & 1 \\ 0 & 0.7 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ 0.9 \\ 1 \end{pmatrix} \lor \begin{pmatrix} 0 \\ 0.9 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.9 \\ 1 \\ 1 \end{pmatrix},$$

$$X^{(3)} = A \times X^{(2)} \lor b = \begin{pmatrix} 0 & 1 & 0.2 \\ 0 & 0 & 1 \\ 0 & 0.7 & 0 \end{pmatrix} \circ \begin{pmatrix} 0.9 \\ 1 \\ 1 \end{pmatrix} \lor \begin{pmatrix} 0.9 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It follows that $X^{(n)} = X^{(3)}$ for any $n \geq 3$. That is,

$$X = \Phi^s(\chi^{(n)}) \lor \Phi (s, S_{-1})$$

$$= \lim_{n \to \infty} X^{(n)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

On the other hand, clearly, $X = \Phi^s(\chi^{(n)}) \lor \Phi (s, S_{-1}) = \Phi^s(\chi^{(n)}) \lor \Phi (s, S_{-1})$, using Theorem 11 to the fuzzy relation equation

$$X = A \times X \lor b,$$

where $X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$, this equation can be rewritten as

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0.2 \\ 0 & 0 & 1 \\ 0 & 0.7 & 0 \end{pmatrix} \circ \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \lor \begin{pmatrix} 0 \\ 0.9 \\ 1 \end{pmatrix},$$

solving this fuzzy relation equation yields the solution

$$X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$
E. Repeated reachability possibility

This section focuses on quantitative properties of repeated reachability of finite possibilistic Kripke structure which can be verified using graph analysis, i.e., by just considering the underlying digraph of the finite possibilistic Kripke structure, combining the transition possibility distribution.

For a finite possibilistic Kripke structure $M$, let $B \subseteq S$ be a set of states in $M$, and $a$ a state in $M$. For the event $\Box B$, i.e., the set of all paths that visit $B$ infinitely, let us calculate $Po(s|\Box B)$. Since $Po(s|\Box B) = \forall_{a \in B} Po(s|\Box a)$, it suffices to calculate the possibility $Po(s|\Box a)$ for any $a \in B$. Let us give some analysis on how to calculate $Po(s|\Box a)$ in the following.

**Theorem 14:** For a finite possibilistic Kripke structure $M$, $a \in S$, we have

$$Po(s|\Box a) = \forall_{T} Po(D_T),$$

where $T$ ranges over all strongly connected subsets of the digraph of $M$ such that $a \in T$, and for $T = \{t_1, \ldots, t_k\}$,

$$D_T = \bigvee_{\pi \in S_k} \bigwedge_{i=0}^{k} Po(t_{\pi(0)}, t_{\pi(1)}).$$

where $S_k$ denotes the set of all permutations on the set $\{1, \ldots, k\}$.

Recall a strongly connected subset of the digraph $M$ is a subset $T$ of $S$ such that, for any $s, t \in T$, there are state sequence $t_1 \cdots t_k$ in $T$ such that $t_1 = s, t_k = t$ and $P(t_i, t_{i+1}) > 0$ for any $1 \leq i < k$.

**Proof:** Let $T = \{t_1, \ldots, t_k\}$ be a strongly connected subset of $M$ containing $a$. For $\pi \in S_k$, if

$$\bigwedge_{i=0}^{k} Po(t_{\pi(0)}, t_{\pi(1)}) \neq 0,$

then, by the observation in Remark 2 for any $i$, there exists a state sequence $m(t_{\pi(0)}, t_{\pi(1)})$ that leads from state $t_{\pi(0)}$ to state $t_{\pi(1)}$ such that $Po(t_{\pi(0)}, t_{\pi(1)}) = Po(m(t_{\pi(0)}, t_{\pi(1)})$. Let $\pi = m(s, t_{\pi(1)})m(t_{\pi(1)}, t_{\pi(2)}) \cdots m(t_{\pi(k-1)}, t_{\pi(k)})$. Since $a \in T$, it follows that $Po(|\Box a)$, and $Po(M^k(\pi)) = \bigwedge_{i=0}^{k} Po(t_{\pi(0)}, t_{\pi(1)})$. This shows that $D_T \leq Po(s|\Box a)$ for any strongly connected subset $T$ of $M$ such that $a \in T$.

Conversely, if $\pi \vdash |a$ and the first state of $\pi$ is $\pi(1) = s$, let $T = \inf(\pi)$, the set of states occurring infinitely often in $\pi$. Then $T$ is a strongly connected subset of $M$ such that $a \in T$. Assume that $T = \{t_1, \ldots, t_k\}$, let $t_j = \min\{n|\pi(n) = t_j\}$ for any $1 \leq j \leq k$, where $\pi(n)$ denotes the $n$-th state of $\pi$. We may assume that $t_1 < t_2 < \cdots < t_k$. Let $\pi' = s(\pi(t_1) \cdots \pi(t_k))$. Since $\pi$ has the form $s \cdots t_1 \cdots t_k \cdots t_1 \cdots$, by the definition of $P^+$, it follows that $\forall_{T} Po(D_T) \leq Po(M^k(\pi'))$, where $Po(M^k(\pi)) = \bigwedge_{i=0}^{k} Po(t_{\pi(0)}, t_{\pi(1)})$. Hence, $Po(M^k(\pi)) \leq D_T$. This shows that $Po(s|\Box a) \leq \forall_{T} D_T$, where $T$ ranges over all strongly connected subset of the digraph of $M$ such that $a \in T$.

Therefore, we have the required equality.

**Lemma 15:** For a finite possibilistic Kripke structure $M$, if $T'$ and $T$ both are strongly connected subset of $M$ containing $a$, and $T' \subseteq T$, then we have $D_{T'} \leq D_T$.

This is obvious by the definition of $D_T$ and $D_T$.

By this lemma, to calculate $Po(s|\Box a)$, it is sufficient to calculate $D_T$ for those minimal strongly connected subsets containing $a$ in $M$. Note that if $P^+(a, a) \neq 0$, the minimal strongly connected subset containing $a$ in $M$ is unique, i.e., $T = \{a\}$. In this case,

$$Po(s|\Box a) = D_T = P^+(s, a) \wedge P^+(a, a).$$

If $P^+(a, a) = 0$, then there is no strongly connected subset containing $a$ in $M$. In this case, $Po(s|\Box a) = 0$, and the equality $Po(s|\Box a) = P^+(s, a) \wedge P^+(a, a)$ also holds.

Therefore, in any case, $Po(s|\Box a) = P^+(s, a) \wedge P^+(a, a)$. Then we have the following theorem.

**Theorem 16:** Let $M$ be a finite possibilistic Kripke structure and $B \subseteq S$. Then we have,

$$Po(s|\Box B) = \forall_{a \in B} Po(s|\Box a).$$

Since the calculation of $P^+$ can be done by some simple graph-search algorithm combining with the minimum and maximum operations in the unit interval $[0,1]$ or some simple fuzzy matrix algorithms, then $Po(s|\Box B)$ can be effectively calculated.

In the probabilistic model checking of repeated reachability linear-time properties (see Ref. [2]), a different approach which is not appropriate to possibilistic model checking is adopted, which is more complex than our method for the possibilistic model checking of repeated reachability linear-time properties.

**Example 17:** Consider the possibilistic Kripke structure $M$ in Example 1. By a simple calculation, the corresponding possibilistic Kripke structure $M^+$ using the transitive closure $P^+$ as the transition possibility distribution is presented in Fig. 3. Then, by Theorem 16 we have $Po(s_0|\Box s_1) = P^+(s_0, s_1) \wedge P^+(s_1, s_1) = 0.7, Po(s_0|\Box s_2) = P^+(s_0, s_2) \wedge P^+(s_2, s_2) = 0.7, Po(s_0|\Box s_3) = P^+(s_0, s_3) \wedge P^+(s_3, s_3) = 1$.

![Fig.3. The corresponding $M^+$ of $M$ in Fig.1](image-url)
(S, P, I, AP, L), the goal is to compute $P^M(P)$. The LT property $P$ is represented by means of a finite automaton $\mathcal{A}$. The possibilistic Kripke structures in this section are assumed finite.

A. Finite automata over finite words and infinite words

First, let us recall the notion of finite automata theory [2], [21].

A (nondeterministic) finite automaton (NFA) is a tuple $\mathcal{A} = (Q, \Sigma, \delta, I, F)$, where $Q$ denotes a finite set of states, $\Sigma$ is a finite input alphabet, $\delta : Q \times \Sigma \rightarrow 2^Q$ is a transition function, $I \subseteq Q$ is a set of initial states, and $F \subseteq Q$ is a set of accept (or final) states.

The transition function $\delta$ can be identified with the relation $\longrightarrow \subseteq Q \times \Sigma \times Q$ given by $q \xrightarrow{u} q'$ iff $q' \in \delta(q, u)$. Thus, often the notion of transition relation (rather than transition function) is used for $\delta$. Intuitively, $q \xrightarrow{u} q'$ denotes that the automaton can move from state $q$ to state $q'$ when reading the input symbol $u$.

Next, we give the notion of a run for a finite automaton $\mathcal{A}$. Let $w = u_1 \cdots u_n$ be a finite word. A run for $w$ in $\mathcal{A}$ is a finite sequence of states $q_0 q_1 \cdots q_n$ such that $q_i \xrightarrow{u_{i+1}} q_{i+1}$ for all $0 \leq i \leq n - 1$.

A run $q_0 q_1 \cdots q_n$ is called successful if $q_0 \in I$ and $q_n \in F$. A finite word $w \in \Sigma^*$ is called accepted by $\mathcal{A}$ if there is a successful run for $w$. The accepted language of $\mathcal{A}$, denoted $L(\mathcal{A})$, is the set of finite word in $\Sigma^*$ accepted by $\mathcal{A}$, i.e., $L(\mathcal{A}) = \{w \in \Sigma^* \mid \exists$ a successful run in $\mathcal{A}$ for the word $w\}$. For a language $L \subseteq \Sigma^*$, if there is an NFA $\mathcal{A}$ such that $L = L(\mathcal{A})$, then $L$ is called a regular language over $\Sigma$.

When an NFA is an acceptor of infinite word, then we have the notion of (nondeterministic) Büchi automaton (NBA). An NBA has the same structure as an NFA, $\mathcal{A} = (Q, \Sigma, \delta, I, F)$, say, the difference is the run and the accepting language of $\mathcal{A}$.

Let $\sigma = u_1 u_2 \cdots \in \Sigma^\omega$ be an infinite word, a run for $\sigma$ in $\mathcal{A}$ is an infinite sequence of states $q_0 q_1 \cdots$ such that $q_i \xrightarrow{u_{i+1}} q_{i+1}$ for all $i \geq 0$. A run $q_0 q_1 \cdots$ is accepting if $q_0 \in I$ and $q_i \in F$ for infinite many indices $i$. The accepting (infinite) language of $\mathcal{A}$ is $L_\omega(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \exists$ a successful run in $\mathcal{A}$ for the word $\sigma\}$.

For a language $L \subseteq \Sigma^\omega$, if there is an NBA $\mathcal{A}$ such that $L = L_\omega(\mathcal{A})$, then $L$ is called an $\omega$-regular language over $\Sigma$.

B. Possibility measure of regular safety property $P_{safe}$

Safety properties are often characterized as “nothing bad should happen”. Formally, in classical case, safety property is defined as an LT property over AP such that any infinite word where $P$ does not hold contains a bad prefix. For convenience, we use the dual notion of good prefixes to define safety property here. Of course, they are equivalent.

**Definition 11:** For a property $P$, the good prefixes of $P$, say $\text{GPref}(P)$ is defined by

$$\text{GPref}(P) = \{\hat{\sigma} \in (2^AP)^* \mid \hat{\sigma} \in \text{Prefix}(\sigma), \sigma \in P\},$$

An LT property $P_{safe}$ is called a safety property provided that, for a $\sigma$, if for all $\hat{\sigma} \in \text{Prefix}(\sigma)$, $\hat{\sigma} \in \text{GPref}(P_{safe})$, then $\sigma \in P_{safe}$, i.e.,

$$\{\sigma \in (2^AP)^* \mid \forall \hat{\sigma} \in \text{Prefix}(\sigma), \hat{\sigma} \in \text{GPref}(P_{safe}) \} = P_{safe}.$$

**Definition 12:** A safety property $P_{safe}$ over $\mathcal{A}$ is regular if its set of good prefixes constitutes a regular language over $2^AP$.

For regular safety property $P_{safe}$, there is an automaton accepting the good prefixes $\text{GPref}(P_{safe})$.

Let $\mathcal{A} = (Q, 2^AP, \delta, I, F)$ be an NFA for the good prefixes of a regular safety property $P_{safe}$. That is, $P_{safe} = \{A_0A_1 \cdots (2^AP)^n \mid 0 \leq A_n \in L(\mathcal{A})\}$ where $A_0(q, A) = \{q\}$ for each $A \subseteq AP$ and each state $q \in Q$. Furthermore, let $M = (S, P, I, AP, L)$ be a finite possible Kripke structure.

**Definition 13:** Let $M = (S, P, I, AP, L)$ be a possibilistic Kripke structure and $\mathcal{A} = (Q, 2^AP, \delta, I, F)$ be an NFA. The product $M \otimes \mathcal{A}$ is a possibilistic Kripke structure $M \otimes \mathcal{A} = (S \times Q, P', I', \delta', AP', L')$, where

1. $AP' = S \times Q$, and $L'(\langle s, q \rangle) = \langle s, q \rangle$ for any $\langle s, q \rangle \in S \times Q$;
2. $I'((s, q)) = \begin{cases} I(s), & \text{if } q \in \delta(q_0, L(s)) \text{ for some } q_0 \in I; \\ 0, & \text{otherwise.} \end{cases}$
3. the transition possibility distribution of $M \otimes \mathcal{A}$ is

$$P'((\langle s, q \rangle), (\langle s', q' \rangle)) = \begin{cases} P(s, s'), & \text{if } q' \in \delta(q, L'(s')); \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 18:** (1) For the definition of LT properties we have assumed that the possibilistic Kripke structure $M$ has no terminal states. It is, however, not guaranteed that $M \otimes \mathcal{A}$ possesses this property, even if $M$ does. This stems from the fact that in NFA $\mathcal{A}$ there may be a state $q$ that has no direct successor states for some set $u$ of atomic propositions, i.e., with $\delta(q, u) = \emptyset$. As we know, this technical problem can be treated by requiring $\delta(q, u) \neq \emptyset$ for all states $q \in Q$ and input $u$, i.e., the finite automaton $\mathcal{A}$ is complete. Note that imposing the requirement $\delta(q, u) \neq \emptyset$ is not a severe restriction, as any NFA can be easily transformed into an equivalent one that satisfies this property by introducing a state $q_{trap}$ and
adding transition $q \xrightarrow{u} q_{\text{trap}}$ to $\mathcal{A}$ whenever $\delta(q, u) = 0$ or $q = q_{\text{trap}}$.

(2) For each path fragment $\pi = s_0 s_1 s_2 \ldots$ in $M$, since $\mathcal{A}$ is required complete, there exists at least a run $q_0 q_1 q_2 \ldots$ in $\mathcal{A}$ for trace$(\pi) = L(s_0) L(s_1) L(s_2) \ldots$ such that $q_0 \in I$, $q_{i+1} \in \delta(q_i, L(s_i))$ and $\pi^+ = \langle s_0, q_1 \rangle \langle s_1, q_2 \rangle \langle s_2, q_3 \rangle \ldots$ is a path fragment in $M \otimes \mathcal{A}$. The corresponding $\pi^+$ is not unique in general, let us denote the set of all such $\pi^+$ by the symbol $S(\pi)$. The definition of $I'$ and $P'(s, q, (s', q'))$ guarantees that $P_0(\pi) = P_0(S(\pi))$. Furthermore, we have the following equality for any $X \subseteq \text{Paths}(M)$:

$$P_0^M(\pi \pi \in X) = P_0^M(\mathcal{A}(\cup \subseteq S(\pi))).$$

(3) Every path fragment in $M \otimes \mathcal{A}$ which starts in state $(s, \delta(q_0, L(s)))$ arises from the combination of a path fragment in $M$ and a corresponding run in $\mathcal{A}$.

The following theorem shows that $P_0(s \models P_{\text{safe}})$ can be derived from the possibility measure of the event $\square \Box B$ in $M \otimes \mathcal{A}$, where $B = S \times F$.

**Theorem 19:** Let $P_{\text{safe}}$ be a regular safety property, $\mathcal{A}$ be an NFA for the set of good prefixes of $P_{\text{safe}}$, $M$ be a possibilistic Kripke structure, and $s$ be a state in $M$. Then,

$$P_0^M(s \models P_{\text{safe}}) = P_0^M(\mathcal{A}(s, \delta(q_0, L(s))) \models \Box B),$$

where $B = S \times F$.

**Proof:** Let $\Pi$ be the set of paths that start in $s$ and accept $P_{\text{safe}}$, i.e.,

$$\Pi = \{ \pi \in \text{Paths}(s) | \text{Pref}(\text{trace}(\pi)) \cap L(\mathcal{A}) \neq \emptyset \}.$$

The set $\Pi^+$ is the set of paths in $M \otimes \mathcal{A}$ that start in $(s, \delta(q_0, L(s)))$ and eventually reach an accept state of $\mathcal{A}$, i.e.,

$$\Pi^+ = \{ \pi^+ \in \text{Paths}(s, \delta(q_0, L(s))) | \pi^+ \models \Box B \}.$$

By the observation in Remark [18], for the measurable set $\Pi$ of paths in $M$ and state $s$,

$$P_0(\Pi) = P_0(\langle s, (q_0, L(s)) \rangle)(\{ \pi^+ | \pi \in \Pi \}) = P_0(\langle s, (q_0, L(s)) \rangle)(\Pi^+) = P_0^M(\langle s, \delta(q_0, L(s)) \rangle) \models \Box B).$$

**C. Possibility measure of $\omega$-regular property**

Let us now consider the wider class of LT properties, i.e., $\omega$-regular properties. An LT property $P$ is $\omega$-regular whenever $P$ defines an $\omega$-regular language.

For the $\omega$-regular property $P$, $\mathcal{A}$ is assumed to be a Büchi automaton accepting $P$. We use the symbol $P_0(s \models P)$ to represent $P_0(s \models P)$, i.e.,

$$P_0(s \models P) = P_0^M(\{ \pi \in \text{Paths}(s) | \text{trace}(\pi) \in L_\omega(\mathcal{A}) = P \}).$$

(9)

It can now be shown, using similar arguments as for regular safety properties, that the possibility measure of the event $\square \Box B$ in the product possibilistic Kripke structure $M \otimes \mathcal{A}$ coincides with the possibility measure of accepting $P$ by $\mathcal{A}$. The possibility measure of the event $\Box \Box B$ can be calculated in polynomial time as shown in Theorem [15], where $B = S \times F$.

**Theorem 20:** Let $\mathcal{A}$ be an NBA and $M$ a finite possibilistic Kripke structure. Then, for all states $s$ in $M$,

$$P_0^M(s \models \mathcal{A}) = P_0^M(\mathcal{A}(s, \delta(q_0, L(s))) \models \Box \Box B),$$

where $B = S \times F$.

**Proof:** The connection between a path $\pi$ in $M$ and the corresponding path $\pi^+$ in $M \otimes \mathcal{A}$ is as follows, trace$(\pi) \in L_\omega(\mathcal{A})$ iff $\pi^+ \models \Box \Box B$. By the observation in Remark [18], the possibility measure for path $\pi$ in $M$ with trace$(\pi) \in L_\omega(\mathcal{A})$ agrees with the possibility measure of generating a path $\pi^+$ in $M \otimes \mathcal{A}$ which arises by the lifting of path $\pi$ in $M$ where trace$(\pi) \in L_\omega(\mathcal{A})$. The latter agrees with the possibility measure for the paths $\pi^+$ in $M \otimes \mathcal{A}$ with $\pi^+ \models \Box \Box B$.

**V. An illustrative example**

We now give an example to illustrate the construction of this paper.

Suppose that there is an animal sicking for a new disease. For the new disease, the doctor has no complete knowledge about it, but he (or she) believes by experience that these drugs such as Ribavirin, Ofloxacin and Thymosin may be useful to the disease.

For simplicity, it is assumed that the doctor considers roughly the animal’s condition to be three states, say, “poor”, “fair” and “excellent”. It is vague when the animal’s condition is said to be “poor”, “fair” and “excellent”. Since the animal’s condition can simultaneously belong to “poor”, “fair” and “excellent” with respective memberships in the real life situation ([16], [3], [17]). Therefore, when a possibilistic Kripke structure is used to model the treatment processes of the animal, a fuzzy state is naturally denoted as a three-dimensional vector $[a_1, a_2, a_3]$, which is represented as the possibility distribution of the animal’s condition over states “poor”, “fair” and “excellent”.

Similarly, it is imprecise to say that at what point exactly the animal has changed from one state to another state after a drug treatment (i.e., event), because the drug event occurring may lead a state to multistates with respective membership. Therefore, the treatment process is modeled by a possibilistic Kripke structure, a transition possibility distribution is represented by a $3 \times 3$ matrix.

Suppose that the treatment process of the animal is modeled by the following possibilistic Kripke structure $M = (S, P, I, AP, L)$, where $S = AP = \{\text{poor, fair, excellent}\}$,

$$P = \begin{bmatrix} 0.5 & 1 & 0.5 \\ 0.2 & 0.5 & 1 \\ 0.2 & 1 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $L(s) = \{s\}$ for any $s \in S$.

The structure $M$ is presented in Fig. 4, and the corresponding $M^+$ is presented in Fig. 5, where we use the symbols $p, f, e$ to represent the states or the atomic propositions “poor”, “fair” and “excellent” respectively.

Let us do some model checking using the above structure $M$. 
First, let us calculate $\text{Po}(\diamond \{\text{excellent}\})$. Using Eq. (1), we have,

$$\text{Po}(\diamond \{\text{excellent}\}) = \bigvee_{s \in S} I(s) \wedge P^+(s, \text{excellent}) = 1.$$ 

Using Eq. (7), we have,

$$\text{Po}(\text{poor} \mid = \square \diamond \{\text{excellent}\}) = P^+(\text{poor}, \text{excellent}) \wedge P^+(\text{excellent}, \text{excellent}) = 1 \wedge 1 = 1.$$ 

Consider a regular safety property $P_{\text{safety}}$ with good prefixes accepted by an NFA $\mathcal{A}$ as shown in Fig. 6, here $G\text{Pref}(P_{\text{safety}}) = (2^{AP})^*\{\text{excellent}\}$, which represents the property “the drug will eventually be useful for the disease”.

Next, let us consider an $\omega$-regular property $L = \{\text{poor}\}\{\text{poor}\}^\omega$ which can be accepted by an NBA $\mathcal{B}$ as shown in Fig. 8. Here $L$ represents a property “the drug is useless for the disease”. The product possibilistic Kripke structure $M \otimes \mathcal{A}$ is shown in Fig. 9.

**VI. Conclusions**

$LTL$ model checking based on possibility measure is a fuzzy measure extension of classical model checking. Both the possibilistic and probabilistic model-checking solve certain uncertainty of error or other stochastic behavior occurring in various real world applications. In this paper, we studied several important possibility measures of $LT$ properties and $LTL$ formulae corresponding to them. Concretely, we introduced the notions of $LT$ properties; several particular $LT$ properties such
as reachability and repeatedly reachability were introduced. More generally, LT properties such as regular safety properties, \( \omega \)-regular properties using automata theory were studied. In fact, we introduced the product possibilistic Kripke structure of a possibilistic Kripke structure and a finite automaton. In which, the computation of possibility measure of possibilistic Kripke structure meeting LT property can be translated into reachability possibility or repeated reachability possibility of the product possibilistic Kripke structure. With these notions, we gave the quantitative verification methods of regular safety properties and \( \omega \)-regular properties.

This is an initial work on the model checking using possibility measure. There are many things to be done along this direction.

- We use max-min composition of fuzzy relations in this paper. There are other forms of composition of fuzzy relations, such as max-product composition. They may be more appropriate in some real world applications of fuzzy sets than max-min composition. Then the related work using other composition instead of max-min composition can be done in the future.

- We use the normal possibility distribution in this paper (see conditions (2) and (3) in the definition of possibilistic Kripke structure). These restrictions are too strict for some applications of the method proposed in this paper. It is natural to relax these restrictions and to use generalized possibility measure in the model-checking.

- The properties considered in this paper are classical, we can further consider the properties with fuzzy uncertainties. In this case, we can use fuzzy automata (see for example [14]) instead of classical automata to describe the related properties of systems.

- As we know, there has been many work on model checking of LT properties in multi-valued systems, see for example [15] and references therein. In our future work, we shall give some comparisons of our method with the methods in multi-valued model checking [15].

- Another direction is to extend the method used in this paper to the CTL model checking in possibilistic Kripke structure.

Of course, the most important thing is to give some case studies of the methods proposed in this paper.

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