Theoretical analysis of quantum dynamics in 1D lattices: Wannier-Stark description

Quentin Thonmann, Jean Claude Garreau and Véronique Zehnlé
Laboratoire de Physique des Lasers, Atomes et Molécules, UMR 8523,
and Centre d’Etudes et de Recherches Laser et Applications,
Université des Sciences et Technologies de Lille,
F-59655 Villeneuve d’Ascq Cedex, France

(March 31, 2022)

This paper presents a formalism describing the dynamics of a quantum particle in a one-dimensional tilted time-dependent lattice. The description uses the Wannier-Stark states, which are localized in each site of the lattice and provides a simple framework leading to fully-analytical developments. Particular attention is devoted to the case of a time-dependent potential, which results in a rich variety of quantum coherent dynamics.

Pacs number(s): 03.65.-w, 03.75.-b, 32.80.Lg, 32.80.Pj

I. INTRODUCTION

Quantum dynamics in a periodic lattice is one of the oldest problems of quantum mechanics, whose basis have been settled by Bloch and Zener [1,2]. In the 30’s. Aimed at the description of the electron motion in crystalline lattices, this problem has largely been considered, for about half a century, as an academic one, because dissipation effects forbid the observation of most quantum effects in the motion of a crystalline electron. Laser cooling of atoms has brought a revival of the interest on such problems, as it produces atoms whose de Broglie wavelength is comparable to the wavelength of the light interacting with the atoms, and whose resulting kinetic energy is comparable to the typical lightshift induced by the radiation. The latter feature means that the cold atoms can be trapped in light potentials (or dipole potentials). The former means that the atom dynamics in such a potential is, in absence of dissipation, essentially quantum. Moreover, the main source of dissipation is spontaneous emission, which can be arbitrarily reduced (if one disposes of a powerful enough laser), whereas keeping a constant light potential, just by an increase of the laser-atom detuning [3].

Light potentials are a consequence of the displacement of atomic levels resulting from the interaction with light, corresponding to a process in which a photon is absorbed transferring the atom to an (virtual) excited state from which the atoms de-excites back to the ground-state by stimulated emission. Such a process induces an energy shift of the atomic levels that can be deduced from second order perturbation theory and which is proportional to light intensity, to the square of the coupling (that is, to $|d_{eg}·\epsilon|^2$, where $d_{eg}$ is the dipole matrix element between the states $g$ and $e$, and $\epsilon$ is the polarization vector of the light), and to the inverse of the laser-atom detuning $\delta L = \omega L - \omega_{eg}$ ($\omega_{eg}$ is the Bohr frequency between states $g$ and $e$). Any spatial gradient of this energy shift produces a (conservative) force, and thus a potential. A simple example is that of a standing wave formed by two counter-propagating parallel-polarized beams. Placed in such a standing wave, an atom perceives a periodic one-dimensional potential whose strength varies sinusoidally in the space. Standing waves (with little variations) form the model potential considered in the present work.

Light potentials generated by standing waves have been used in many experimental studies of quantum dynamics. For example, Bloch oscillations have been observed both with single atoms [3] and with a Bose-Einstein condensate (BEC) [4] in an accelerated standing wave. Wannier-Stark ladders [6] and collective tunneling effects [7] have also been studied with such a system. Atoms placed in an intense, phase-modulated, or pulsed, standing wave realize a paradigmatic system for theoretical and experimental studies of quantum chaos, the so-called Quantum Kicked Rotor [8,9].

In this paper, we consider the quantum dynamics of an atom (of mass $M$) placed in a tilted sinusoidal potential whose phase (that is, the position of its nodes) can be modulated in an arbitrary way, corresponding to the Hamiltonian

$$H = \frac{p^2}{2M} + v_0 \cos\{2k_L [x - x_0(t)]\} + f(t)x$$

(1)

where $x_0(t)$ is a phase and $f(t)$ a force, both being (eventually) time-dependent, and $k_L = 2\pi/\lambda L$ is the wavenumber of the standing wave. Different temporal dependences of $x_0(t)$ can be considered. For instance, the accelerated case, $x_0(t) = (1/2)at^2$, which has been studied in [3,4,6], is equivalent to an inertial force $F = Ma$ in the frame of the potential (see appendix A).

A natural energy unit in such a context is the “recoil energy”, defined as the change in kinetic energy of the atom corresponding to the absorption a photon, given by

$$E_R = \frac{\hbar^2 k_L^2}{2M}$$

(2)

to which one can associate a recoil frequency $\omega_R = E_R/\hbar$ and a recoil momentum $p_R = \hbar k_L$, etc. It is also useful to re-scale the variables: $X \equiv x/(\lambda_L/2)$, where $\lambda L/2$ is the step of the periodic lattice, and $\tau \equiv \omega_R t$. With these definitions the above Hamiltonian takes the following form, which is retained in the rest of the paper:
\[ H = \frac{p^2}{2m^*} + V_0 \cos\{2\pi[X - X_0(\tau)]\} + F(\tau)X \]  

where \( V_0 \equiv v_0/E_r, F \equiv f\lambda_L/2E_R \) (note that in this system the momentum operator in the real space is \( P = -i(\partial/\partial X), \hbar = 1 \), the reduced mass is \( m^* = \pi^2/2 \) and \( d = 1 \) is the step of the lattice). For simplicity, in what follows, we shall write the rescaled variables \( x, p \) and \( t \).

In the next section, we briefly review the time-independent Hamiltonian case in Eq. (3), introduce the Wannier-Stark basis, and show that it leads to a very simple description of the Bloch oscillation. In the following sections, we shall discuss the more complicated dynamics that arises when a harmonic modulation of the lattice is applied.

II. THE BLOCH OSCILLATION IN THE WANNIER-STARK DESCRIPTION

The Bloch oscillation (BO) is a well known phenomenon discovered by Zener while studying the quantum properties of an electron in a (perfect) crystal submitted to a constant electric field \( \text{[4]} \). The BO arises when a small spatial tilt is added to the lattice: Quantum particles do not fall along the slope of the potential, but perform a periodic, space-limited, oscillation. BO is thus a strictly quantum behavior. We shall call “lattice” the untilted potential, and “tilted lattice” the sum of the lattice and the tilted potential.

The BO is usually described in the basis of the so-called Bloch states \( \text{[1]} \), i.e., the eigenstates of the lattice. In this paper, we use another framework corresponding to the eigenstates of the tilted potential. While the lattice is invariant under spatial translations by a multiple of the spatial period \( d \), the tilted lattice has a more complicated symmetry: it is invariant under simultaneous spatial translation by a lattice period \( d \) and energy translation by \( \omega_B = Fd \) (\( \omega_B \) is the “Bloch frequency”). One then expects the eigenenergies to form “ladder” structures separated by \( \omega_B \), the so-called Wannier-Stark ladders, introduced by Wannier in connection with the problem of electrons in a crystal submitted to a homogeneous electric field \( \text{[2]} \). Each element of the ladder corresponds to eigenfunctions (the Wannier-Stark states) centered at a given well, and thus separated by an integer multiple of \( d \). The form, and even the existence of these eigenstates has been the object of a long controversy, that has been settled only recently \( \text{[13]} \). In the present paper, we consider a spatially limited lattice, extending over many periods, and limited by an infinite-height box. This changes only very slightly the “bulk” properties of the system, and the eigenenergies and eigenstates obtained numerically display (to a very good approximation) the expected ladder structure described above. In this framework, Wannier-Stark states (WSS) are the eigenfunctions of the system. The relation to the case of an infinite tilted lattice makes no problem if the corresponding states (Wannier-Stark

The form, and even the existence of these eigenstates has been evidenced in 1988 in a semi-conductor superlattice \( \text{[14]} \), and 1996 with cold atoms in an optical lattice \( \text{[5]} \). Note that, being spatially localized, Wannier-Stark states provide an ideal tool for the description of the wave function of a cold atom (as produced by a “Sisyphus-boosted” MOT) placed in the potential, whose de Broglie wavelength (around \( \lambda_L/3 \)) is of the order of the lattice period \( \lambda_L/2 \).

Consider the properties of the time-independent Hamiltonian \( H_0 \)

\[ H_0 = \frac{p^2}{2m^*} + V_0 \cos(2\pi x) + Fx \]  

where \( F \) is a constant force. The eigenfunctions of \( H_0 \) are Wannier-Stark states forming an energy ladder whose separation is \( \omega_B = Fd, E_{nm} = E_m + nFd \).

![FIG. 1. (a) Periodic potential with a tilt. (b) The WSS state \( \varphi_n \) is localized in the well of index \( n = 1 \) and has appreciable overlap with neighbor lattice sites \( n = 0 \). This eigenfunction is obtained for \( V_0 = 2.5 \) and \( F = 0.5 \). The “numerical” box includes 64 lattice sites.](image)

The BO can be advantageously described by using the Wannier-Stark state (WSS) localized inside a given individual lattice well, corresponding to the lowest energy of the states associated to this well (see Fig. \( \text{[1]} \)). Note that considering only the ground state of each well is equivalent to the restriction to the first Bloch band in the description based on Bloch states. We also choose strong enough \( F \) and \( V_0 \) to produce well-localized WSS \( \text{[13]} \). The WSS associated to the lattice well labeled \( n \) is noted \( \varphi_n(x) \) (supposed real) and the corresponding eigenenergy is \( E_n \) (we drop the index \( m \)). The symmetries of the potential discussed above then imply:

\[ \varphi_{n+p}(x) = \varphi_n(x - pd) \]  

and

\[ E_{n+p} = E_n + pFd \]
this Bloch oscillation is proportional to \(X\) of the BO appears here clearly as an interference effect frequency along the slope, the atom performs an oscillation with the quantum motion in a tilted lattice: instead of falling [16]. This result evidences a counter-intuitive property of if the atom is localized in only one well. Eq. (8) also pre- quasi-momentum classical frame as the periodic evolution of the atom’s reference basis. The oscillation is described in a semi- states (eigenstates of the untilted lattice) are taken as the is evidenced in our approach. The developments are simpler if we use a unitary transformation that transforms the modulation in a time- dependent force. Physically, this is equivalent to move to an accelerated reference frame in which the lattice is at rest [22], adding thus an inertial force (Appendix A). In this frame, the new Hamiltonian is given by Eq. (8) plus a time-dependent force \(F'(t)\):

\[
x_0(t) = a \sin(\omega t).
\]

(10)

The developments are simpler if we use a unitary transformation that transforms the modulation in a time-dependent force. Physically, this is equivalent to move to an accelerated reference frame in which the lattice is at rest [22], adding thus an inertial force (Appendix A). In this frame, the new Hamiltonian is given by Eq. (8) plus a time-dependent force \(F'(t)\):

\[
F'(t) = m^* \frac{d^2 x_0(t)}{dt^2} = -m^* a \omega^2 \sin(\omega t) = -F_0 \sin(\omega t)
\]

(11)

where \(F_0 \equiv m^* a \omega^2\) is the amplitude of the inertial force. The harmonic time-dependence in Eq. (11) is the analog of an AC electric field for electrons in a (perfect) crystal. The dynamics in such a system can be described in a quite simple fashion by writing the state of the atom as a superposition of WSS, Eq. (7). The coefficients \(c_n(t)\) can be obtained by reporting Eq. (7) into the Schrödinger equation

\[
[H_0 - F_0 x \sin(\omega t)] \Psi(x, t) = i \frac{\partial \Psi(x, t)}{\partial t}
\]

(12)

where \(H_0\) is given by Eq. (3), with eigenstates \(\varphi_n(x)\). This produces the following set of coupled differential equations for the \(c_n(t)\):

\[
\dot{c}_n(t) = -i E_n c_n(t) + i F_0 \sin(\omega t) \sum_m X_{n,m} c_m(t)
\]

(13)

where \(\dot{c}_n \equiv dc_n/dt\). Neglecting temporarily the coupling between different WSS, (i.e putting \(X_{n,m} = X_{n,n} \delta_{m,n}\)), the amplitudes are obtained simply as \(c_n = \exp[i \phi_n(t)]\), with the time-dependent phase

\[
\phi_n(t) = -E_n t - \frac{F_0 X_{n,n}}{\omega} \cos(\omega t)
\]

(14)

where \(X_{n,n} = X_{0,0} + n d\), depends on the site index \(n\) [10]. We now write \(c_n(t) \equiv d_n(t) e^{i \phi_n(t)}\). The amplitudes \(d_n\) obey the following system of differential equations:

\[
\dot{d}_n = i F_0 \sum_{m \neq n} X_{n,m} d_m(t) \exp\{i [\phi_m(t) - \phi_n(t)]\} \sin(\omega t) .
\]

(15)

After Eq. (14), the phase difference \(\phi_m(t) - \phi_n(t)\) is:

III. THE MODULATED POTENTIAL IN THE WANNIER-STARK DESCRIPTION: RESONANT DYNAMICS

With the existence of the natural frequency \(\omega_B\) of the system in mind, one is tempted to investigate the quantum dynamics in presence of a harmonic external forcing. The WSS approach proves to be very efficient, since it allows a fully-analytical description. After some general considerations, we study in this section the case of resonant forcing, and show that it leads to a very rich and interesting dynamics. The general (non-resonant) case will be treated in the next section. Consider the Hamiltonian of Eq. (3) with a constant force \(F\) and a lattice phase modulation

\[
x_0(t) = a \sin(\omega t).
\]

(10)
\[ \phi_n(t) - \phi_n(0) = (n-n) \left[ \omega_B t + \frac{F_0 d}{\omega} \cos(\omega t) \right] \]  
(16)

where we used Eq. (13). Eq. (14) can be recast as
\[ \dot{d}_n = iF_0 \sum_{p \neq n} X_p d_{n+p} \left[ e^{-ip\omega_B t} e^{-i(\omega F_0 d/\omega) \cos(\omega t)} \right] \sin(\omega t) \]  
(17a)

\[ = \frac{F_0}{2} \sum_{p \neq n} X_p d_{n+p} \sum_l (-i)^l J_l \left( \frac{F_0 d}{\omega} \right) \left\{ e^{i(l+1)\omega t - p\omega_B t} - e^{-i(l-1)\omega t - p\omega_B t} \right\} \]  
(17b)

where, \( X_p \equiv X_{n+p} \). \( J_n(x) \) is the Bessel function of the first kind, and we have used the well-known property of Bessel functions:
\[ e^{-iz\cos(\omega t)} = \sum_{l=-\infty}^{+\infty} J_l(z) (-i)^l e^{ilt} \]  
(18)

Note that the sum over lattice sites (i.e., over \( p \)) extends only over a few neighbors sites, since the coupling coefficients \( X_p \) rapidly shrink to zero. If the modulation is smooth enough to avoid projections on other states of the system, the sum over the harmonics of the modulation (i.e., \( t \)) is also limited to a few terms close to \( t = 0 \) (typically, \( l_{\text{max}} \sim F_0 d/\omega = \pi n^* \omega d \sim O(1) \)). Therefore, only a finite number of terms are to be retained in the above expression. On the other hand, the evolution of \( d_n \) described by Eqs. (17b) is a sum of oscillations with frequencies \( (l \pm 1)\omega - p\omega_B \). In the following we keep only the so-called secular terms, that is, terms that oscillate slowly or do not oscillate at all. The resulting “close to resonance” dynamics is observed when \( (l \pm 1)\omega \approx p\omega_B \), i.e. when the forcing frequency \( \omega_B \) is commensurable (or almost) with the system’s natural frequency \( \omega_B \).

Let us consider the simpler resonant case, \( \omega = \omega_B \). Due to the relative strength of the factors \( X_p \) we can keep, to a good accuracy, only the contribution of the next-neighbor site (\( p = 1 \)), which leads to the following expression:
\[ \dot{d}_n(t) = \Omega_1 \left[ d_{n+1} - d_{n-1} \right] \]  
(19)

where
\[ \Omega_1 = \frac{F_0 X_1}{2} \left[ J_0 \left( \frac{F_0 d}{\omega_B} \right) + J_2 \left( \frac{F_0 d}{\omega_B} \right) \right] = \frac{\omega_B X_1}{d} J_1 \left( \frac{F_0 d}{\omega_B} \right) \]  
(20)

This equation is similar to a “dipole coupling” between sites \( n \) and \( n \pm 1 \) where \( \Omega_1 \) plays the role of a Rabi frequency. Note that, contrary to intuition, the coupling towards the left or towards the right neighbor is the same.

The meaning of Eq. (19) can be better appreciated by searching for the plane-wave solutions of the form:
\[ d_n(t) = e^{i(k_0 d n + \omega t)} \]  
(21)

The (dimensionless) wavenumber \( k_0 \) takes into account the phase difference between neighbor sites. Substitution into Eq. (13) leads to the dispersion relation
\[ \omega = 2\Omega_1 \sin(k_0 d) \]  
(22)

and to the group velocity \( v_g \equiv \frac{d\omega}{dk_0} \):
\[ v_g = 2\Omega_1 d \cos(k_0 d) \]  
(23)

This result shows that the dynamics is different depending on the wave number \( k_0 \). For instance, if \( k_0 d = \pm \pi/2 \) (phase-quadrature from site to site), \( v_g = 0 \) and there is no global motion. If \( k_0 d = \pi \), \( v_g = -2\Omega_1 d \) and the global motion is a fall along the slope of the potential with the maximum speed \( 2\Omega_1 d \). More interesting is the case \( k_0 = 0 \), where \( v_g = 2\Omega_1 d \): the atom then climbs up the slope of the potential with a constant maximum speed: there is, in this case, coherent transfer of energy from the modulation to atom, thanks to the particular phase relations between neighbor sites. Note also that, contrary to the motion of a classical particle, the speed \( |v_g| \) is independent of the sense of displacement: the wavepacket climbs the slope up or down at the same speed.

More detailed information on the wavepacket motion can be grabbed by writing the amplitude \( d_n \) in the more general form:
\[ d_n(t) = f_n(t) e^{i(k_0 d n + \omega t)} \]  
(24)

where \( f_n \) are complex amplitudes describing the envelope of the atomic wavepacket, assumed to vary slowly in time as compared to the frequency \( \omega_B \), and in space as compared to the lattice period \( d \). Reporting the above expression into Eq. (13), we get:
\[ \dot{f}_n + i\omega f_n = \Omega_1 \left[ \cos(k_0 d) \left( f_{n+1} - f_{n-1} \right) + i \sin(k_0 d) \left( f_{n+1} + f_{n-1} \right) \right] \]  
(25)

Using the dispersion relation Eq. (22) and keeping only slowly varying contributions, we obtain:
\[ \dot{f}_n = \Omega_1 \left[ \cos(k_0 d) \left( f_{n+1} - f_{n-1} \right) + i \sin(k_0 d) \left( f_{n+1} + f_{n-1} - 2f_n \right) \right] \]  
(26)

Since \( f_n \) vary slowly in space, we can take the continuous limit (with respect to the variable \( x = nd \)) and deduce the following equation for the wavepacket envelope:
\[ \dot{f}(x, t) = \Omega_1 \left( 2d \cos(k_0 d) \frac{\partial}{\partial x} + id^2 \sin(k_0 d) \frac{\partial^2}{\partial x^2} \right) f(x, t) \]  
(27)

This equation is an interesting piece of information. If \( k_0 d = \pm \pi/2 \), one has a diffraction equation of the form \( \dot{f} = \pm i\Omega_1 d^2 \frac{\partial^2}{\partial x^2} f \), which describes the spreading of the
atomic wavepacket (i.e. diffraction) with no global displacement. The case \( k_0d = 0, \pi \) gives the wave equation \( \hat{f} = v_g \partial_x f \) \((v_g = 2d\Omega_1)\), describing a wavepacket traveling with constant velocity \( v_g \) and no deformation: the wavepacket presents an ascending or descending coherent motion. Mixed behaviors, i.e. spreading at a diffusion rate \( \Omega_1d^2 \sin(k_0d) \) and uniform displacement with group velocity \( v_g = 2\Omega_1d\cos(k_0d) \), are found for other values of \( k_0 \). The general solution can be easily obtained by performing a spatial Fourier transform of Eq. (27).

In Fig. 3 the wavepacket, prepared with \( k_0 = 0 \) has a uniform displacement while preserving its shape. Its group velocity obtained from the numerical simulations is \( v_g = 0.030 \), in very good agreement with the theoretical value from Eq. (23) which is \( v_g = 0.032 \).

Other resonant behaviors are observed if \( \omega = q\omega_B \) (\( q \) integer). From the general expression of Eq. (17), one finds for instance a next-to-neighbor \((n \rightarrow n\pm 2)\) resonant interaction if \( \omega = 2\omega_B \), leading to:

\[
\ddot{d}_n(t) = \Omega_2 [d_{n+2} - d_{n-2}]
\]

with \( \Omega_2 = (\omega_B/d)X_2J_1(F_0d/\omega_B) \). Note that \( X_2 \ll X_1 \). We illustrate in Fig. 4 the temporal evolution of a wavepacket, obtained by numerical integration of the Schrödinger equation. The initial state is prepared as a superposition of two packets: one packet is constructed with in-phase amplitudes (that is, \( c_n \) and \( c_{n+2} \) have the same phase for \( n \) odd), and the other one is constructed with amplitudes in phase-opposition (that is, \( c_n \) and \( c_{n+2} \) have opposite sign for \( n \) even). The first packet moves with velocity \( v_g = 4\Omega_2d \), and the second with \( v_g = -4\Omega_2d \). The figure displays an original behavior showing each of these two initially inter-penetrated packets moving independently in opposite directions, creating a highly delocalized state.

**IV. The modulated potential: General case**

In this section we generalize the results of the preceding section to the case of a non-resonant modulation. We follow essentially the same steps as in Sec. III and we shall skip algebraic details of the calculations. Coming back to Eq. (17a) and looking for a solution of the form:

\[
d_n(t) = e^{i(k_0nt + \phi(t))}
\]

one gets the instantaneous frequency:
\[
\dot{\phi} = 2F_0 \sum_{p > 0} X_p \cos\{p[k_0d - \theta(t)]\} \sin(\omega t) \tag{30}
\]

where \( \theta(t) = \omega_B t + (F_0d/\omega) \cos(\omega t) \), and we used \( X_p = X_{-p} \). The group velocity \( v_g = d\phi/d\omega \) is thus

\[
v_g = 2F_0d \sum_{p > 0} pX_p \sin\{p[\theta(t) - k_0d]\} \sin(\omega t) \tag{31}
\]

As in the preceding section, more detailed behavior is obtained by putting

\[
d_a(t) = f_n(t)e^{ik_0dn + \phi(t)} \tag{32}
\]

where \( f_n \) are slowly-varying amplitudes. The generalization of Eq. (26) then is:

\[
\dot{f}_n = iF_0 \sum_{p \neq 0} X_p[f_{n+p} - f_n]e^{ip(k_0d - \theta)} \sin(\omega t) \tag{33}
\]

or,

\[
\dot{f}_n = F_0 \sum_{p > 0} X_p \sin(\omega t)
\]

\[
\{ - [f_{n+p} - f_{n-p}] \sin[p(k_0d - \theta)]
\]

\[
+i[(f_{n+p} + f_{n-p} - 2f_n)] \cos[p(k_0d - \theta)] \tag{34}
\]

Taking the continuous limit of the above expression then produces an equation describing both the propagation and the diffraction of the wavepacket:

\[
\dot{f}(x, t) = \left( v_g(t) \frac{\partial}{\partial x} + iD(t) \frac{\partial^2}{\partial x^2} \right) f(x, t)
\]

\[
+ 2iF_0 \sum_{p \geq 2} X_p \cos[p(\theta - k_0d)] \sin(\omega t)f(x, t) \tag{35}
\]

where

\[
D(t) = F_0d^2 \sum_{p > 0} p^2X_p \cos[p(\theta - k_0d)] \sin(\omega t) \tag{36}
\]

and the group velocity \( v_g \) is given by Eq. (31). Note that the last term in Eq. (35) is a phase term which is \( O(X_2) \ll 1 \) and does not contribute to the probability density \( |f(x, t)|^2 \). For the sake of lightness, it is not considered in the following.

The Fourier transform of Eq. (35) with respect to \( x \) produces an algebraic equation for the Fourier transform \( \tilde{f}(k, t) \) of \( f(x, t) \), whose solution is:

\[
\tilde{f}(k, t) = e^{ikx'(t)}e^{ik^2\Delta(t)} \tilde{f}(k, 0) \tag{37}
\]

and thus:

\[
f(x, t) = \frac{1}{\sqrt{2\pi}} \int e^{ik[x + x'(t)]}e^{ik^2\Delta(t)} \tilde{f}(k, 0) \tag{38}
\]

with

\[
x'(t) = \int_0^t v_g(\tau)d\tau \tag{39}
\]

\[
\Delta(t) = \int_0^t D(\tau)d\tau . \tag{40}
\]

This expression describes a coherent motion of the wavepacket formed of an oscillatory motion with the time-dependent group velocity Eq. (31), and a diffusive motion with a time-dependent diffusion coefficient \( D(t) \). For example, if one builds an initial gaussian packet of width \( a_0 \), \( f(x, 0) = \exp(-x^2/a_0^2) \) one finds, after some straightforward calculations:

\[
|f(x, t)|^2 = \frac{a_0}{a(t)} \exp\left( -\frac{2x^2}{a(t)^2} \right) , \tag{41}
\]

where

\[
a(t) = a_0 \left[ 1 + 16 \frac{\Delta(t)^2}{a_0^2} \right]^{1/2} . \tag{42}
\]

FIG. 5. Spatio-temporal behavior of \(|\Psi(x, t)|^2\) obtained numerically by integration of the Schrödinger equation (gray level convention: the maximum values of \(|\Psi(x, t)|^2\) are depicted in black), from \( t = 0 \) to \( t = 4\pi/\delta \) (that is, 2 periods of the beat frequency \( \delta = \omega - \omega_B = 0.02 \)). Other parameters are the same as in Fig. 3.

The physical meaning of our developments can be evidenced by considering the case where \( \omega \) differs from \( \omega_B \) by a small detuning \( \delta = \omega - \omega_B, |\delta| \ll \omega_B \), and keep only leading-order terms of order \( O(\delta^{-1}) \). The wavepacket then undergoes a harmonic oscillation at the beat frequency \( \delta \) with a group velocity given by Eq. (31)

\[
v_g(t) = 2\Omega_1d \cos(k_0d + \delta t) \tag{43}
\]

corresponding to a periodic mean position displacement...
\[ \langle x(t) \rangle = x(0) - \frac{2\Omega_1 d}{\delta} [\sin(k_0 d + \delta t) - \sin(k_0 d)] \] (44)

The width of the wavepacket oscillates in a breathing mode which is governed by:

\[ \Delta(t) = \frac{\Omega_1 d^2}{\delta} [\cos(k_0 d + \delta t) - \cos(k_0 d)] . \] (45)

The results of Sec. II for a resonant excitation are naturally recovered in the limit \( \delta \rightarrow 0 \).

---

**V. CONCLUSION**

We have studied, in a fully-analytical way, the dynamics of a wavepacket in a static and time-modulated tilted potential in the framework of the Wannier-Stark states. This basis is well suited to the description of the state of a cold atom (as produced by a Sisyphus-boosted MOT). Moreover, it provides a simple description the atomic dynamics, which is proved to be very rich: A variety of coherent motions are obtained depending on the preparation of the initial wavepacket and its site-to-site quantum coherence. We can note that the description introduced here is, in its principle, independent of the details of the lattice, provided it presents localized states in the lattice sites. It is therefore generalizable to other kinds of lattices.

Let us finally, mention that the present work distinguishes from the more usual “solid-state” approach which is based on Bloch functions. We postpone for a forthcoming work the detailed comparison between the Wannier-Stark and Bloch approaches.

---

**VI. ACKNOWLEDGMENTS**

Laboratoire de Physique des Lasers, Atomes et Molécules (PhLAM) is UMR 8523 du CNRS et de l’Université des Sciences et Technologies de Lille. Centre d’Etudes et Recherches Lasers et Applications (CERLA) is supported by Ministère de la Recherche, Région Nord-Pas de Calais and Fonds Européen de Développement Economique des Régions (FEDER).

**APPENDIX A: UNITARY TRANSFORMATION**

Eq. (11) is obtained if we perform a unitary transformation

\[ U(t) = e^{iX_0(t)}e^{-i\beta(t)}X e^{i\gamma(t)} \] (A1)

where we have included translation operators in space and momentum with \( \beta = 2m^*X_0(i.e \text{ moment of a particle of mass } m^*) \). In this framework, following [20], we obtain (with \( UXU^+ = X + X_0 \) and \( UPU^+ = P + \beta \))

\[ H' = UHU^+ + i (d_1U)U^+ = \frac{(P + \beta)^2}{2m^*} + V_0 \cos(2\pi X) + F(X + X_0(t)) - \dot{X}_0(t)P + \beta(t)(X + X_0) - \dot{\gamma} \]

with \( \dot{\gamma} = FX_0 + m^* \dot{X}_0 X_0 + \frac{(m^*/2)\ddot{X}_0^2}{} \):

\[ H' = \frac{P^2}{2m^*} + V_0 \cos(2\pi X) + (F + m^* \ddot{X}_0(t))X \] (A2)

Therefore, in the frame of the periodic potential, the Hamiltonian contains an inertial force proportional to \( \ddot{X}_0(t) \).
[9] H. Amman, R. Gray, I. Shvarchuck, and N. Christiensen, Phys. Rev. Lett. 80, 4111 (1998).
[10] J. Ringot, P. Szriftgiser, J. C. Garreau, and D. Delande, Phys. Rev. Lett. 85, 2741 (2000).
[11] M. B. D’Arcy, R. M. Godum, M. K. Oberthaler, M. K. Cassetari, G. S. Summy, Phys. Rev. Lett. 87, 074102 (2001).
[12] G. Wannier, Rev. Mod. Phys. 62, 645 (1962).
[13] For a complete review of the subject, see G. Nenciu, Rev. Mod. Phys. 63, 91 (1991).
[14] J. Bleuse, G. Bastard, and P. Voisin, Phys. Rev. Lett 60, 220 (1988).
[15] H. Fukuyama, R. A. Bari, and H. C. Fogdeby, Phys. Rev. B 8, 5579 (1973).
[16] By virtue of the translational properties of the WSS, Eqs. (1) and (2), $X_{n,n+p} (p \neq 0)$ does not depend on $n$: $X_{n,n+p} = \int \varphi_n(x)x(\varphi_{n+p}(x))dx = \int \varphi_0^*(x-na)x\varphi_p(x-na)dx = \int \varphi_0^*(x')(x'+na)\varphi_p(x')dx' = X_{0,0}$, where we used the orthogonality of the WSS. Note however that $X_{n,n} = X_{0,0} + na$ does depend on $n$.
[17] While the nearest neighbors interaction is proportional to $X_{01} = 0.13$ the next to neighbor interaction is $X_{02} = 7.8 \times 10^{-3}$ i.e the amplitude of the oscillation at $2\omega_B$ is roughly 20 times smaller than that at $\omega_B$ (the numerical values were obtained for $F = 0.5$ and $V_0 = 2.5$).
[18] N. W. Ashcroft N.W and N. D. Mermin, Solid State Physics, Holt Rinehart and Winston, 1976.
[19] M. Holthaus, J. Opt. B: Quantum Semiclass. Opt. 2, 589 (2000).
[20] M. Ben Dahan, Thèse de doctorat (unpublished) Paris, 1997.