STABILITY CRITERIA FOR MULTIPHASE PARTITIONING PROBLEMS WITH VOLUME CONSTRAINTS

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Dedicated to the memory of Paul Fife, a man of brilliance, extraordinary intuition, and very high morality.

Abstract. We study the stability of partitions involving two or more phases in convex domains under the assumption of at most two-phase contact, thus excluding in particular triple junctions. We present a detailed derivation of the second variation formula with particular attention to the boundary terms, and then study the sign of the principal eigenvalue of the Jacobi operator. We thus derive certain stability criteria, and in particular we recapture the Sternberg-Zumbrun result on the instability of the disconnected phases in the more general setting of several phases.

1. Introduction. The partitioning of a set into a number of subsets (the “phases”) so that the dividing hypersurface (the “interface”) has minimal area, is a problem of geometric analysis and calculus of variations. It is of high importance in the physical sciences and engineering because of its relation to surface tension. Examples include a variety of phenomena ranging from the annealing of metals (Mullins) to the segregation of biological species (Ei et al). Two phase systems formed by the mixing of two polymers or a polymer and a salt in water are used for the separation of cells, membranes, viruses, proteins, nucleic acids, and other biomolecules. The partitioning between the two phases is dependent on the surface properties of the materials. An overview of the physical aspects of the subject is offered in. Early studies of the mathematical problem of partitioning include Nitsche’s paper, and Almgren’s Memoir (see also White.)

Paul Fife was one of the top applied mathematicians of his time with significant and lasting contributions to Diffuse Waves, Diffuse Interfaces, Stefan problems and Phase Field Models. His monographs “Dynamics of Internal Layer and Diffuse Interfaces” and “Mathematical Aspects of Reactions and Diffusive Systems” and the IMA volume “Dynamical Issues in Combustion Theory” are classics. Fife with his collaborators studied extensively the dynamical problems related to the generation of partition and to their coarsening. For a sample see [4, 5, 6, 2].

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Sternberg-Zumbrun [21], treated the static problem and proved that disconnected two phase partitions of convex sets are always unstable. The S-Z formulas with little notation changes are given in the next theorem. For the reader’s convenience some details are given to make the exposition self-contained. Throughout this paper, we take $\Omega \subset \mathbb{R}^N$ to be a bounded domain with smooth boundary $\Sigma = \partial \Omega$.

**Definition 1.1.** Let $M$ be a $n$-dimensional $C^1$ submanifold of $\mathbb{R}^N$ with boundary and $V$ an open subset of $\mathbb{R}^N$ such that $V \cap M \neq \emptyset$. A variation of $M$ is a collection of diffeomorphisms $(\xi^t)_{t \in I}$, $I = ]-\delta, \delta[$, $\delta > 0$, $\xi^t : V \to V$ such that

1. The function $\xi(x,t) = \xi^t(x)$ is $C^2$
2. $\xi^0 = id_V$
3. $\xi^t|_{V \setminus K} = id_{V \setminus K}$ for some compact set $K \subset V$.

In place of the $\xi^t$ we often consider their extensions by identity to all of $\mathbb{R}^N$.

With each variation we associate the first and second variation fields

$$w(x) = \xi_x(x,0), \quad a(x) = \xi_{tt}(x,0)$$

known [20] also as velocity and acceleration fields, $\xi_t, \xi_{tt}$ denoting first and second partial derivative in $t$.

As $M \subset \Omega, \partial M \subset \Sigma$, admissible variations of $M$ should respect the rigidity of the boundary of $\Omega$. In this connection S-Z suggested that admissible variations of $M$ be obtained by solving the ODE

$$\frac{d\xi}{dt} = w(\xi), \quad \xi(0) = x \tag{1}$$

for any given first variation vector field $w$ and then setting $\xi^t(x) = \xi_x(t)$, where $\xi_x$ is the solution of (1) for the initial condition $\xi_x(0) = x$. The requirement for rigid container walls is satisfied by selecting $w$ so that $w(p) \in T_p\Sigma$ for all $p \in \Sigma$, $T_pX$ denoting as usual the tangent space of $X$. In this paper we consider only normal variations, i.e. those satisfying $w(p) \in N_pM$ for all $p \in M$; $N_pM$ is the normal space of $M$ at $p$.

A partitioning of $\Omega$ (see Figure 1) is associated with a set of interfaces $M = (M_i)_{i=1}^n$, which are assumed to be non-intersecting $C^2$-hypersurfaces with boundary $(C^2)$ in $\Sigma = \partial \Omega$. A minimal partitioning $M$ is a critical point of the area functional $A$ under the assumed volume constraints, i.e.

$$\delta A(M) := \frac{d}{dt} A(M^t) \bigg|_{t=0} = 0$$

for all variations preserving $\Sigma$ and the volume of the phases. In this equation $M^t = \xi^t(M)$, $\xi^t$ is a variation (Definition 1.1), and $A(M)$ is the area of $M$. A partition is disconnected when the space occupied by at least one phase is a disconnected set. For example, the partition of Figure 1 is disconnected, as phases 1, 2 occupy disconnected sets.

**Theorem 1.2** (Sternberg-Zumbrun). Let $M = (M_i)_{i=1}^n$ be a minimal 2-phase partitioning in $\Omega$. Then for any normal variation of $M$, which preserves $\Sigma$ and the volume of the phases, i.e.

$$\int_M f = 0,$$

the second variation of area of $M$ is given by

$$\delta^2 A(M) = \frac{d^2}{dt^2} A(M^t) \bigg|_{t=0} = \int_M (|\text{grad}_M f|^2 - |B_M|^2 f^2) - \int_{M \cap \Sigma} \sigma f^2$$
Figure 1. Three-phase partitioning of a set $\Omega$. Subsets painted with the same color contain material of the same phase. Interfaces $(M_1, \ldots, M_4)$ and subsets $(\Omega_1, \ldots, \Omega_5)$ are identified by successive indexing. Phases are enumerated in the same way: yellow is 1, green is 2, and cyan is 3. An alternative, more convenient for the calculations, identification scheme of subsets is shown in parentheses: the first index corresponds to a phase and the second enumerates the connected components of the subsets occupied by that phase; for example $\Omega_{12}(\equiv \Omega_4)$ is the second connected component of phase 1. For connected phases (i.e. those occupying a single connected subset) we omit, for brevity, the second index; e.g. $\Omega_{31} \equiv \Omega_3$.

The orientation of $M$ is selected so that $\sigma \geq 0$ on $\Sigma$ for convex $\Omega$.

Using this formula with $r = 2$, S-Z proved that for convex $\Omega$, when $\sigma \neq 0$ on $\Sigma$, every disconnected two phase partitioning is necessarily unstable. Recall that by definition a minimal partitioning is stable when $\delta^2 A(M) > 0$ for all variations $w \neq 0$ preserving $\Sigma$ and the volume of the phases.

In this paper we study the stability of general $m$-phase partitions in convex domains. The problem under consideration is shown in Figure 1. The notation is as in the caption of the same Figure. The notation (see [24]) $M_{ij}$, where $i, j$ are phases in contact, was found inconvenient for the calculations, as it required an additional third index; for example, with reference to Figure 1 the part of the interface between phases 1 and 2 which lies between $\Omega_1$ and $\Omega_2$ would have to be denoted as $M_{12}^{(1)}$, and that between $\Omega_4$ and $\Omega_5$ as $M_{12}^{(2)}$; now summation over all phases would not make sense, as not all possible phase contacts are present in the partition. The conditions of the problem is constancy of phase volume and the assumption of at most two-phase contact.

In the following section an extension of the S-Z formula to $m$-phase problems is given in Proposition 1. The instability of disconnected multiphase partitions follows as an application of this. In Section 3 we develop the spectral theory of the bilinear form expressing the second variation of area, which is the main tool for proving our stability/instability results. Proposition 5 states that, for normalized variations, the minimum of the second variation of area exists and is given by the principal eigenvalue. The difficulties in obtaining this result are (i) the boundary integral $\int_{\partial M} f^2$ cannot be bounded above by $\int_M f^2$, and (ii) admissible variations need not
satisfy the boundary condition \( (15) \) of the related eigenvalue problem. They were handled by developing an interpolation estimate for the boundary integral in Lemma 3.1. An extension of Proposition 5 to \( m \)-phase partitioning problems is immediate. Proposition 6 gives a characterization of all connected \( m \)-phase partitionings by reduction to the two phase case.

The last two sections are devoted to applications. In Section 4 we prove the existence of unstable partitionings in \( \mathbb{R}^N \) when \( \Omega \) satisfies hypothesis (H) (see Section 4), and that spherical partitionings are stable when they are not in contact with the boundary of \( \Omega \). As a byproduct, we also prove that spherical partitionings in bounded sets are never absolute minimizers of the area functional under volume constraint. Applications to 2-dimensional problems have been included in Section 5. The main results here are criteria for instability, Propositions 9 and 12. Proposition 13 shows that sufficiently small partitions are stable. For related work see \([15, 8, 13]\).

2. Multiphase partitioning problems. For more than two phases we consider the functional

\[
A(M) = \sum_{i=1}^{r} \gamma_i A(M_i).
\]

The coefficients \( \gamma_i > 0 \) have the physical meaning of surface energy density. The notation is as in the caption of Figure 1; properties and fields associated with a particular interface are identified by the same indexing method as the associated interface; for example the normal field and surface density of \( M_2 \) (see Figure 1) are \( N_2 \) and \( \gamma_2 \) (of course, \( \gamma_1 = \gamma_4 \)). The summation in (2) extends over all interfaces of the partition problem.

The partition \( M = (M_i)_{i=1}^{r} \) is considered oriented, its orientation being determined by the orientations of the \( M_i \). There are \( 2^r \) possible orientations for \( M \). Most of the following formulas depend on the orientation of \( M \). Admissible variations for the functional in (2) are those preserving phase volume. They can be directly obtained from general variations by rendering them volume preserving (see [21]). For convenience, we use the method of Lagrange multipliers, which involves the following modified (weighted) functional:

\[
A^\star(M) = A(M) - \sum_{j=1}^{m} \lambda_j \left( \sum_{k=1}^{P_j} |\Omega_{jk}| - V_j \right)
\]

In this formula \( |\cdot| \) denotes volume, \( m \) is the number of phases, \( P_j \) is the number of distinct sets phase \( j \) is split (indexed by \( k \)), \( V_j \) is the volume of phase \( j \) and \( \lambda_j \) is the Lagrange multiplier corresponding to the volume constraint for the \( j \)-th phase. Since \( \sum_{j=1}^{m} \sum_{k=1}^{P_j} |\Omega_{jk}| = |\Omega| \), there are only \( m - 1 \) linearly independent constraints and we could have used only \( m - 1 \) Lagrange multipliers. For convenience, we use \( m \) multipliers.

Example 2.1. In the case of a disconnected 3-phase partition the weighted area functional is given by (see Fig. 2)

\[
A^\star(M) = \sum_{i=1}^{3} \gamma_i A(M_i) - \lambda_1 (|\Omega_{11}| + |\Omega_{12}|) - \lambda_2 |\Omega_2| - \lambda_3 |\Omega_3|
\]

The volume constants \( V_j \) were dropped as they play no part in the variational process.
The following proposition extends Theorem 1.2 to more than two phases.

**Proposition 1.** Let $\{M_i\}_{i=1}^r$ be a minimal multiphase partitioning with volume constancy constraint for the phases. Further let $N_i$ be the unit normal field of $M_i$. Then

(i) The scalar mean curvature $\kappa_i = H_i \cdot N_i$ of each interface $M_i$ is constant.

(ii) The scalar mean curvatures satisfy the relation

$$\sum_{j=1}^r \gamma_j \kappa_j = 0 \quad (5)$$

(iii) Each $M_i$ is normal to $\Sigma$, i.e., on each $M_i \cap \Sigma$ we have $N_i \cdot N_\Sigma = 0$ or $N_i(p) \in T_p \Sigma$ for all $p \in \partial M_i$. $N_\Sigma$ is the normal field of $\Sigma$.

(iv) For any admissible variation of $M$, i.e., one preserving $\Sigma$ and the volume of the phases, the second variation of area of $M$ is given by

$$\delta^2 A(M) = \sum_{i=1}^r \gamma_i \int_{M_i} (|\nabla f_i|^2 - |B_{M_i}|^2 f_i^2) - \sum_{i=1}^r \gamma_i \int_{\partial M_i} \sigma_i f_i^2$$

where $\sigma_i = H_\Sigma(N_i, N_i)$.

**Proof.** For concreteness we consider the disconnected 3-phase partitioning of Fig. 2 with the indicated orientation. Let $w$ be any variation of $M$. By (4), the formula for the first variation of the area of a manifold [20] $\delta A(M_i) = \int_{M_i} \text{div}_M w$ and

$$\delta |\Omega_{jk}| = \int_{\partial \Omega_{jk}} w \cdot N_{\partial \Omega_{jk}},$$

$N_{\partial \Omega_{jk}}$ being the unit outward normal field of $\partial \Omega_{jk}$, which is easily established, we obtain:

$$\delta A^*(M) = \sum_{i=1}^3 \gamma_i \int_{M_i} \text{div}_M w - \lambda_1 \left( \int_{M_1} w \cdot N_1 - \int_{M_2} w \cdot N_3 \right)$$

$$- \lambda_2 \left( \int_{M_2} w \cdot N_2 - \int_{M_1} w \cdot N_1 \right)$$

$$- \lambda_3 \left( \int_{M_3} w \cdot N_3 - \int_{M_2} w \cdot N_2 \right).$$

**Figure 2.** Disconnected 3-phase partitioning. Differently shaded regions correspond to different phases. The notation is as in Figure 1. The phases are from left to right 1, 2, 3, and 1 (see first index of notation in parentheses). $N_i$ is the unit normal field of interface $M_i$ (in the indicated orientation).
Let \( H_i \) be the mean curvature vector field of \( M_i \), \( \nu_i \) the unit tangent field of \( M_i \) which is normal to \( \partial M_i \) (also known as “conormal” field) and \( f_i = w \cdot \nu_i \) the normal component of the variation field on \( M_i \) (tangential components are irrelevant and are disregarded from the outset). Let \( (\cdot)^\top \) denote projection on the tangent space of a manifold. Application of the identity [20]

\[
\text{div}_M w = \text{div}_M w^\top - H \cdot w^\perp,
\]

the divergence theorem for manifolds [20], \( \int_{\partial M_i} \text{div}_M w^\top = \int_{\partial M_i} w \cdot \nu_i \), and reordering of terms give:

\[
\delta A^*(M) = \sum_{i=1}^{3} \gamma_i \int_{\partial M_i} w \cdot \nu_i - \int_{M_i} (\gamma_1 \kappa_1 + \lambda_1 - \lambda_2) f_1 - \int_{M_2} (\gamma_2 \kappa_2 + \lambda_2 - \lambda_3) f_2 - \int_{M_3} (\gamma_3 \kappa_3 + \lambda_3 - \lambda_1) f_3.
\]

Standard arguments render

\[
\gamma_1 \kappa_1 = \lambda_2 - \lambda_1, \quad \gamma_2 \kappa_2 = \lambda_3 - \lambda_2, \quad \gamma_3 \kappa_3 = \lambda_1 - \lambda_3
\]

which prove (i) and (ii) and \( w \cdot \nu_i = 0 \), which proves (iii). By variation of the Lagrange multipliers we recover the volume constancy constraints:

\[
\int_{M_1} f_1 = \int_{M_2} f_2 = \int_{M_3} f_3
\]

For the proof of (iv) we start by Simon’s general formula for the second variation of area [20]

\[
\delta^2 A(M) = \int_{M_i} \left[ \text{div}_M a + (\text{div}_M w)^2 + g^{rs} \langle (D_E w)^\perp, (D_E w)^\perp \rangle - g^{rk} g^{sl} \langle D_E w, E_s \rangle \langle D_E w, E_k \rangle \right]
\]

In this formula, \( w, a \) are the first and second variation fields, \( (\cdot)^\top \) denotes projection on the tangent space of \( M_i \), \( (\cdot)^\perp \) denotes projection on the normal space of \( M_i \), \( E_1, \cdots, E_{N-1} \) are the basis vector fields in a chart, \( g_{rs} = E_r \cdot E_s \) are the covariant components of the metric tensor of \( M_i \) and \( g^{kl} \) its contravariant components. Summation convention applies throughout this paper. The notation \( \langle , \rangle \) is alternatively used to denote scalar product in lengthier expressions. Recalling that \( w \) is normal to \( M_i \), i.e. \( w^\perp = w = f_i N_i \), we have

\[
g^{rs} \langle (D_E w)^\perp, (D_E w)^\perp \rangle = g^{rs} (D_E f_i)(D_E f_i) = |\text{grad}_{M_i} f_i|^2
\]

and

\[
g^{rk} g^{sl} \langle D_E w, E_s \rangle \langle D_E w, E_k \rangle = g^{rk} g^{sl} \langle f_i D_E N_i, E_s \rangle \langle f_i D_E N_i, E_k \rangle = f_i^2 g^{ik} g^{jl} \langle D_E N_i, E_s \rangle \langle D_E N_i, E_k \rangle = f_i^2 g^{ik} g^{jl} \langle N_i, D_E E_s \rangle \langle N_i, D_E E_k \rangle = f_i^2 B^k B^l = f_i^2 |B_{M_i}|^2
\]

In this equation \( B_{rk} = \langle N, D_E E_k \rangle \) are the covariant components of the second fundamental form tensor \( II(u, v) = \langle N, D_u v \rangle \) (\( u, v \) are tangent vector fields) and...
\( B^*_r = g^{s_k} B_{r_k} \). By \([6]\) as \( w^\top = 0 \) we obtain
\[
\text{div}_{M_i} w = -\kappa_i f_i.
\] (9)

Combination of \([6]\) with the equality \( a = D_w w \), which is obtained by taking the time derivative of \([11]\), gives for \( \text{div}_{M_i} a \):
\[
\text{div}_{M_i} a = \text{div}_{M_i} a^\top - \kappa_i N_i \cdot D_w w
\]

By Lemma 2.3 we have for the second variation of the \(|\Omega_{jk}|\):
\[
\delta^2 |\Omega_{11}| = \int_{M_1} f_1 \text{div}_{\mathbb{R}^N} w, \quad \delta^2 |\Omega_{12}| = - \int_{M_3} f_3 \text{div}_{\mathbb{R}^N} w
\]
\[
\delta^2 |\Omega_2| = \int_{M_2} f_2 \text{div}_{\mathbb{R}^N} w - \int_{M_1} f_1 \text{div}_{\mathbb{R}^N} w
\]
\[
\delta^2 |\Omega_3| = \int_{M_3} f_3 \text{div}_{\mathbb{R}^N} w - \int_{M_2} f_2 \text{div}_{\mathbb{R}^N} w
\]

By \([4]\)
\[
\delta^2 A^*(M) = \sum_{i=1}^{3} \gamma_i \delta^2 A(M_i) - \lambda_1 (\delta^2 |\Omega_{11}| + \delta^2 |\Omega_{12}|) - \lambda_2 \delta^2 |\Omega_2| - \lambda_3 \delta^2 |\Omega_3|
\]

Replacing for \( \delta^2 A(M_i) \), \( \delta^2 |\Omega_{jk}| \) by the above equalities and rearranging give the following expression for \( \delta^2 A^*(M) \):
\[
\delta^2 A^*(M) = \sum_{i=1}^{3} \gamma_i \int_{M_i} (||\text{grad}_{M_i} f_i ||^2 - |B_{M_i}|^2 f_i^2) \\
+ \int_{M_1} [\gamma_1 \text{div}_{M_1} a^\top - \gamma_1 \kappa_1 N_1 \cdot D_w w + \gamma_1 \kappa_1^2 f_1^2 - (\lambda_1 - \lambda_2) f_1 \text{div}_{\mathbb{R}^N} w] \\
+ \int_{M_2} [\gamma_2 \text{div}_{M_2} a^\top - \gamma_2 \kappa_2 N_2 \cdot D_w w + \gamma_2 \kappa_2^2 f_2^2 - (\lambda_2 - \lambda_3) f_2 \text{div}_{\mathbb{R}^N} w] \\
+ \int_{M_3} [\gamma_3 \text{div}_{M_3} a^\top - \gamma_3 \kappa_3 N_3 \cdot D_w w + \gamma_3 \kappa_3^2 f_3^2 - (\lambda_3 - \lambda_1) f_3 \text{div}_{\mathbb{R}^N} w]
\]

By \([7]\) the integral on the second row assumes the form
\[
\int_{M_1} [\gamma_1 \text{div}_{M_1} a^\top - \gamma_1 \kappa_1 N_1 \cdot D_w w + \gamma_1 \kappa_1^2 f_1^2 + \gamma_1 \kappa_1^2 f_1 \text{div}_{\mathbb{R}^N} w] = \\
\int_{M_1} [\gamma_1 \text{div}_{M_1} a^\top - \gamma_1 \kappa_1 f_1 (N_1 \cdot D_{N_i} w - \text{div}_{\mathbb{R}^N} w) + \gamma_1 \kappa_1^2 f_1^2] = \\
\int_{M_1} [\gamma_1 \text{div}_{M_1} a^\top + \gamma_1 \kappa_1 f_1 \text{div}_{M_1} w + \gamma_1 \kappa_1^2 f_1^2] = \\
\int_{M_1} [\gamma_1 \text{div}_{M_1} a^\top - \gamma_1 \kappa_1^2 f_1^2 + \gamma_1 \kappa_1^2 f_1^2] = \int_{M_1} \gamma_1 \text{div}_{M_1} a^\top
\]

On the second and third equalities we have used the identity \([20]\)
\[
\text{div}_{\mathbb{R}^N} X = \text{div}_{M} X + N \cdot D_{N} X,
\]
and Equation (9): $X$ is any differentiable field on $M$. Application of the divergence theorem gives

$$
\int_{M_1} \gamma_1 \text{div}_M a^T = \gamma_1 \int_{\partial M_1} a \cdot \nu = \gamma_1 \int_{\partial M_1} \nu \cdot D_w w = \gamma_1 \int_{\partial M_1} \nu \cdot D N_1 N_1 = -\gamma_1 \int_{\partial M_1} f_2 \nu \cdot D N_1 N_1 = \gamma_1 \int_{\partial M_1} f_2 \nu \cdot D N_1 N_1
$$

In a similar fashion we reformulate the last two rows in the expression for $\delta A^*(M)$ and this completes the proof of (iv).

**Remark 1.** The convention for the second fundamental form of $\Sigma = \partial \Omega$ is $N_\Sigma = -\nu$ so that $\text{II}_\Sigma(N_i, N_i)$ is always non-negative for convex $\Omega$. $N_i$ is the normal field of $M_i$, which is tangent to $\Sigma$.

**Remark 2.** The equation of (ii) of the proposition depends on orientation choice. In the case of a disconnected 2-phase partition we have $\gamma_1 = \gamma_2 = \gamma_{12}$, the interfacial energy density of phases 1 and 2, and (ii) reduces to $\kappa_1 + \kappa_2 = 0$, which, in the 2-dimensional case, implies the interfaces are circular arcs of equal radii. This restricts considerably the number of possible realizations of minimal disconnected multiphase partitions.

**Example 2.2.** Let $\Omega$ be an ellipse centered at 0 with major and minor semiaxes $a$, $b$. For $a > x_0 > 0$ the tangents at $(x_0, \pm y_0)$ have equations

$$
\frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) = 0, \quad \frac{x_0}{a^2}(x - x_0) - \frac{y_0}{b^2}(y + y_0) = 0.
$$

These lines intersect at $x = x_0 + \frac{a^2 y_0}{b^2 x_0} = \frac{a^2}{x_0}$. Hence the radius of the circular arc, which intersects the ellipse at right angles, is given by

$$
R^2 = \left(\frac{a^2}{x_0} - x_0\right)^2 + y_0^2 = \left(\frac{a^4 y_0^2}{b^4 x_0^2} + 1\right) y_0^2
$$

This is a monotone decreasing function of $x_0$ and from this it follows, with simple geometric arguments, that all possible minimal, disconnected, 2-phase partitionings of an ellipse, are pairs of transversal circular arcs symmetric about the $x$ or $y$-axis.

As an application of Proposition 1 we prove the instability of disconnected 3-phase partitions in a convex set. Given any such partition, we choose a variation which is constant on each interface. The volume constraints are satisfied if we choose

$$
f_i = \frac{1}{A(M_i)}, \quad i = 1, 2, 3.
$$

By (iv) of Proposition 1 we obtain

$$
\delta^2 A(M) = -\sum_{i=1}^{3} \frac{\gamma_i}{A(M_i)^2} \int_{M_i} |B_{M_i}|^2 - \sum_{i=1}^{3} \frac{\gamma_i}{A(M_i)^2} \int_{\partial M_i} \sigma_i
$$

which is negative when $\sigma_i \neq 0$. Generalization to an arbitrary number of phases and phase splittings is immediate:

**Proposition 2.** Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and convex set with $C^{2,\alpha}$ boundary. Assume a stable $m$-phase partitioning of $\Omega$, with volume constraint, and let $M = (M_i)_{i=1}^m$ be the set of the interfaces. Further, assume that $\Omega$ is strictly convex,
in particular \( H_{\partial \Omega}(N_i, N_i) > 0 \) at all points of \( \partial M_i \cap \Sigma, \ i = 1, \cdots, r \). Then the partition is connected.

We close this section by proving the Lemma that was used in the proof of part (iv) of Proposition 1.

**Lemma 2.3.** In the setting of Proposition 1, the second variation of volume of any distinct phase \( \Omega_j \) is given by

\[
\delta^2 |\Omega_j| = \int_{M_j} (\text{div}_{\mathbb{R}^N} w) w \cdot N_{\partial \Omega_j} \tag{10}
\]

In this equation \( N_{\partial \Omega_j} \) is the unit outward normal field of \( \partial \Omega_j \) and \( M_j \) denotes collectively the interfacial part of \( \partial \Omega_j \), i.e. \( M_j = \partial \Omega_j \setminus \partial \Omega \).

**Proof.** Let \( (\xi^t)_{t \in I} \) be a variation of \( R^N \) and \( \Omega_j^t = \xi^t(\Omega_j) \). Then

\[
|\Omega_j^t| = \int_{\xi^t(\Omega_j)} dx = \int_{\Omega_j} J_{\xi^t}(y) dy
\]

where \( J_{\xi^t} \) is the Jacobian of \( \xi^t \). For the second variation of this functional we have

\[
\delta^2 |\Omega| = \frac{d^2}{dt^2} |\Omega|^t \bigg|_{t=0} = \int_{\Omega} \left. \frac{\partial^2}{\partial t^2} J_{\xi^t}(y) \right|_{t=0} dy.
\]

Application of the rule of differentiation of determinants and straight-forward manipulations give

\[
\frac{\partial^2}{\partial t^2} J_{\xi^t}(y) \bigg|_{t=0} = \text{div}_{\mathbb{R}^N} a + \frac{\partial w^\alpha}{\partial x^\alpha} \frac{\partial w^\beta}{\partial x^\beta} - \frac{\partial w^\alpha}{\partial x^\beta} \frac{\partial w^\beta}{\partial x^\alpha}.
\]

We are using Greek indices for vector components and coordinates in the surrounding space \( \mathbb{R}^N \), and Roman for submanifolds. Summation convention is applicable to Greek indices as well. Formula (10) follows from this equality, the identity

\[
\frac{\partial w^\alpha}{\partial x^\alpha} \frac{\partial w^\beta}{\partial x^\beta} - \frac{\partial w^\alpha}{\partial x^\beta} \frac{\partial w^\beta}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} \left( w^\alpha \frac{\partial w^\beta}{\partial x^\beta} - w^\beta \frac{\partial w^\alpha}{\partial x^\beta} \right)
\]

\[
= \text{div}_{\mathbb{R}^N} \left( (\text{div}_{\mathbb{R}^N} w) w - D_w w \right),
\]

Gauss’s theorem in \( \mathbb{R}^N \) and \( a = D_w w \). Since the variation preserves \( \partial \Omega \), the integral over \( \partial \Omega_j \setminus M_j \) drops. \qed

3. **Spectral analysis of the 2nd variation of area form.** To keep the length of formulas to a minimum and focus on the essence of the argument, we present the details for the two phase partitioning problem and then indicate how the results generalize to more phases.

3.1. **Two phase partitioning problem.** Let \( M \) be the interface of a two phase partitioning problem in \( \Omega \), which is assumed minimal, i.e. \( \delta A(M) = 0 \). For linearized stability we naturally study the minimal eigenvalue of the bilinear form

\[
J(f) = \int_{M} \left( |\nabla^M f|^2 - |B_M|^2 f^2 \right) - \int_{\partial M} \sigma f^2 \tag{11}
\]

For brevity we will write \( \nabla^M f \) in place of \( \nabla^M f \). Although \( J \) and \( \delta^2 A^*(M) \) are identical expressions, we consider them from a different point of view: for the purposes of spectral analysis \( M \) is fixed and \( J \) is a nonlinear functional on a properly
defined functional space on \( M \) containing the admissible variations of \( M \); thus its elements \( f \) satisfy the conditions of volume constancy
\[
\int_M f = 0 \tag{12}
\]
and the normalization condition
\[
\int_M f^2 = 1. \tag{13}
\]

As a matter of convenience, we introduce Lagrange multipliers and the corresponding modified functional
\[
J^*(f; \lambda, \mu) = \int_M \left( |\nabla_M f|^2 - |B_M|^2 f^2 \right) - \int_{\partial M} \sigma f^2 - \lambda \int_M f - \mu \int_M f^2
\]
and we are interested in the critical points of \( J^* \) or \( J \) with the conditions (12) and (13).

**Proposition 3.** A necessary and sufficient condition for a \( C^2 \) function \( f \) on \( M \) to be a critical point of \( J^* \), or equivalently \( J \) with the conditions (12) and (13), is that it satisfies the following inhomogeneous PDE with Neumann boundary condition:
\[
\Delta_M f + (\mu + |B_M|^2) f = -\frac{\lambda}{2} \tag{14}
\]
\[
D_{\nu} f = \sigma f \quad \text{on} \partial M \tag{15}
\]

**Remark 3.** In (14) \( \Delta_M \) is the Laplace-Beltrami operator on \( M \) defined by
\[
\Delta_M f = \text{div}_M (\nabla_M f) = g^{-1/2} (g^{ij} f_{,i})_{,j}
\]
in a local coordinate system \( q^1, \cdots, q^{N-1} \), where \( g = \det(g_{ij}) \), \( g_{ij} \) is the metric tensor and the comma operator denotes partial derivative in the respective coordinate, i.e. \( f_{,i} = \frac{\partial f}{\partial q^i} = D_{q^i} f \). The summation convention on pairs of identical indices is assumed throughout this paper. As \( M \) is fixed, \( g_{ij} \) is fixed and (14) is a linear equation.

**Remark 4.** The \( C^2 \) condition on \( f \) can be relaxed by considering the weak form of (14), (15).

**Proof.** The first variation of \( J^* \) is given by
\[
\delta J^*(f)\phi = \frac{d}{dt} J^*(f + t\phi) \bigg|_{t=0}
\]
\[
= 2 \int_M \left( |\nabla M f \cdot \nabla M \phi - |B_M|^2 f \phi \right) - 2 \int_{\partial M} \sigma f \phi - 2 \mu \int_M f \phi - \lambda \int_M \phi
\]
By Green’s formula for manifolds we obtain
\[
\delta J^*(f)\phi = -2 \int_M \left( \Delta_M f + |B_M|^2 f + \mu f + \frac{1}{2} \frac{\lambda}{2} \right) \phi + 2 \int_{\partial M} (\nabla_M f \cdot \nu - \sigma f) \phi.
\]
When \( \phi \) is a \( C^\infty \) function with compact support in the interior of \( M \), the second integral on the right side drops and by the fundamental lemma of the calculus of variations we obtain (14). When the support of \( \phi \) intersects the boundary of \( M \), \( \delta J^*(f)\phi = 2 \int_{\partial M} (\nabla_M f \cdot \nu - \sigma f)\phi \) and, again by the same argumentation, we obtain (15), in view of the identity \( D_{\nu} f = \nabla_M f \cdot \nu \). The converse is trivial.
Two are the relevant problems: (a) given a partitioning, to show that it is unstable, i.e. to find a particular admissible variation \( f \) such that \( J(f) < 0 \), and (b) to prove that a partitioning \( M \) is stable, i.e. for any admissible variation \( f \neq 0 \) we have \( J(f) > 0 \). The proposition next shows that for problems of the first category it suffices to find a negative eigenvalue \( \mu < 0 \) of problem (14).

**Proposition 4.** Let \( M \) be a minimal two phase partitioning and \( f \) an eigenfunction of problem (14) with corresponding eigenvalue \( \mu \). Then

\[
J(f) = \mu.
\]

In particular, if \( \mu < 0 \), \( M \) is unstable.

**Remark 5.** Proposition 4 implies that no lower bound is necessary for the functional \( J \), in order to conclude that a minimal partitioning is unstable, when a negative eigenvalue is at hand. This is in contrast with problems of category (b).

**Proof.** Multiplication of (14) by \( f \) and integration over \( M \) gives in view of (12) and (13):

\[
\int_M f \Delta_M f + \mu \int_M |B_M|^2 f^2 = 0
\]

Application of Green’s formula on the first integral gives

\[
- \int_M \nabla \cdot \nabla f^2 + \int_{\partial M} f \nu \cdot \nabla f + \mu \int_M |B_M|^2 f^2 = 0.
\]

By (11) and (15) we obtain \( J(f) = \mu \). The second assertion follows trivially from this.

In class (b) we need to know in advance that the functional \( J \) has a minimum under the conditions (12) and (13). The difficulty is that the boundary integral \( \int_{\partial M} f^2 \) cannot be bounded above by \( \int_M f^2 \). However, if \( f \in W^{1,2}(M) \equiv H^1(M) \), by the boundary trace embedding theorem ([1] 5.34-5.37, pp 163-166; [25] 8, pp 120-132) we have

\[
\int_{\partial M} f^2 \leq c_{BT}^2 \left( \int_M |\nabla f|^2 + \int_M f^2 \right)
\]

where \( c_{BT} \) is a constant depending on \( M \). Now using this estimate for the boundary integral and the estimates \( |B_M|^2 \leq b_0^2 \), \( \sigma \leq \sigma_0 \), \( b_0 \) and \( \sigma_0 \) being certain constants depending on \( M \) and \( \Sigma \), we obtain

\[
J(f) \geq \int_M |\nabla f|^2 - b_0^2 \int_M f^2 - \sigma_0 c_{BT}^2 \left( \int_M |\nabla f|^2 + \int_M f^2 \right)
\]

\[
= (1 - \sigma_0 c_{BT}^2) \int_M |\nabla f|^2 - (b_0^2 + \sigma_0 c_{BT}^2) \int_M f^2
\]

from which we can conclude coercivity of \( J \) if \( \sigma_0 < 1/c_{BT}^2 \). In this way we can prove (see [23], Theorem 1.2, p.4) that for sufficiently small principal curvatures of \( \Sigma \) in a neighborhood of \( \partial M \), \( J \) has a minimum, which is necessarily a critical point of \( J^* \), hence an eigenfunction of (14) with BC (15). As a consequence, by (16), if \( J^* \) has no non-positive eigenvalues, we can draw the conclusion that \( M \) is stable.

We can drop the hypothesis of sufficiently small principal curvatures of \( \Sigma \) in a neighborhood of \( \partial M \) by replacing (17) by an interpolation estimate. The standard notation for Sobolev spaces is used: \( |u|_{L^2(M)} = (\int_M u^2)^{1/2} \), \( |u|_{H^1(M)} = (\int_M |\nabla u|^2)^{1/2} \), \( |\nabla u|_{L^2(M)} = (\int_M |\nabla u|^2)^{1/2} \), \( |u|_{L^2(\partial M)} = (\int_{\partial M} u^2)^{1/2} \) are the standard norms of \( L^2(M), H^1(M) \) and \( L^2(\partial M) \).
Lemma 3.1. Let $M$ be a bounded $C^{2,\alpha}$ submanifold of $\mathbb{R}^N$ with boundary. Then for every $\epsilon > 0$ there is a constant $c_\epsilon$ such that for any $u \in H^1(M)$

$$|u|_{L^2(\partial M)} \leq \epsilon |u|_{H^1(M)} + c_\epsilon |u|_{L^2(M)} \quad (19)$$

Proof. We prove the estimate by contradiction. Assume there is a $\epsilon_0 > 0$ for which (19) does not hold. Then for each $n \in \mathbb{N}$ there is a $u_n \in H^1(M)$ such that $|u_n|_{H^1(M)} = 1$ and

$$|u_n|_{L^2(\partial M)} > \epsilon_0 + n |u_n|_{L^2(M)} \quad (20)$$

By (20) and (17) we obtain

$$\int_{\partial M} \nu \cdot \nabla u \, d\nu \leq c_\epsilon \int_M u^2 \quad (21)$$

Since $(u_n)$ is bounded in $H^1(M)$, we have, possibly by passing to a subsequence, $u_n \rightharpoonup u$ in $L^2(M)$. By $u_n \to 0$ in $L^2(M)$ we easily conclude $u = 0$. From the compactness of the embedding $H^1(M) \hookrightarrow L^2(\partial M)$ (see Adams [1] 5.34-5.37, pp 163-166 for the flat case) it follows that $u_n \to 0$ in $L^2(\partial M)$, which contradicts (20). \qed

As an example we prove (19) directly for a bounded hypersurface $M$ of $\mathbb{R}^N$ with boundary.

Example 3.2. Assume that there is a $x_0 \in \mathbb{R}^N$ such that $x_0 \cdot \nu(p) > 0$ for all $p \in \partial M$. Without loss of generality set $x_0 = 0$. For $u \in C^\infty(M)$, $x$ being the position vector in $\mathbb{R}^N$, by (6)

$$\int_M \text{div}_M xu^2 = - \int_M H \cdot xu^2 + \int_{\partial M} (x \cdot \nu) u^2$$

we obtain

$$\int_{\partial M} (x \cdot \nu) u^2 = \int_M H \cdot xu^2 + \int_M (u^2 \text{div}_M x + 2ux \cdot \nabla^M u) \leq \frac{1}{2} \int_M (H \cdot x)^2 u^2 + \frac{1}{2} \int_M u^2 + \int_M u^2 \text{div}_M x$$

$$+ \frac{1}{\epsilon^2} \int_M |x|^2 u^2 + \epsilon^2 \int_M |\nabla^M u|^2 \quad (21)$$

By the compactness of $\partial M$ and $\overline{M}$ there are positive constants $c_0, c_1, c_2$ such that $x \cdot \nu \geq c_0$, $|x|^2 \leq c_1$ and $|H| \leq c_2$. To compute $\text{div}_M x$ apply the definition of operator $\text{div}_M$ (Simon [20]) in a chart with basis vector fields $E_1, \cdots, E_{N-1}$:

$$\text{div}_M x = g^{ij} (DE_i x, E_j) = g^{ij} (E_i, E_j) = g^{ij} g_{ij} = N - 1$$

By (21) we obtain

$$c_0 |u|^2_{L^2(\partial M)} \leq \epsilon^2 |u|^2_{H^1(M)} + \left(N - \frac{1}{2} + \frac{1}{2} c_1 \epsilon^2 + \frac{1}{\epsilon^2} \right) |u|^2_{L^2(M)}$$

By a density argument we extend to $u \in H^1(M)$. \qed

Estimate (19) makes it possible to select $\epsilon > 0$ so small that the coefficient of $\int_M |\nabla^M f|^2$ in (18) becomes positive. The following inequality replaces (18):

$$J(f) \geq (1 - 2\sigma_0 \epsilon^2)|u|^2_{H^1(M)} - (b_0^2 + 1 + 2\sigma_0 c_2^2)|u|^2_{L^2(M)} \quad (22)$$

Now we can conveniently prove the main result for problems of class (b):
Proposition 5. Consider a minimal two phase partitioning in \( \Omega \subset \mathbb{R}^N \) with interfaces \( M \). Then for any \( f \in H^1(M) \) satisfying \([12], [13]\) we have

\[
J(f) \geq \mu_1
\]

where \( \mu_1 \) is the smallest eigenvalue of problem \([14], [15]\). In particular, if \( \mu_1 > 0 \), \( M \) is stable.

Proof. Let \( X = \{ u \in H^1(M) : |u|_{L^2(M)} \leq 1, \int_M u = 0 \} \). By the continuity of the \( L^2(M) \)-norm and \( u \mapsto \int_M u \) as mappings \( H^1(M) \to \mathbb{R} \), it follows that \( X \) is a closed subset of \( H^1(M) \). The convexity of \( X \) is clear. Hence \( X \) is a weakly closed subset of \( H^1(M) \). By \([22]\) it follows immediately that \( J \) is coercive on \( X \). The sequential weakly lower semicontinuity of \( J \) follows from the sequential weakly lower semicontinuity of the norm of \( H^1(M) \) and the compactness of the embedding \( H^1(M) \hookrightarrow L^2(M) \): \( u_n \overset{H^1}{\to} u \) implies \( u_n \overset{L^2}{\to} u \). The conditions in \([23]\) Theorem 1.2 are satisfied and by this we conclude that \( J \) attains its infimum in \( X \). The position of the infimum is a critical point of \( J^* \) and, as it was shown in Proposition \([4]\), it is a solution of equation \([14]\) with BC \([15]\). Inequality \([23]\) follows trivially form this. \( \square \)

3.2. Three and more phases. Since disconnected partitionings in strictly convex sets are always unstable according to Proposition \([2]\) we need only consider connected partitionings. The functional \( J \) for the \((m+1)\)-phase partitioning problem reads:

\[
J(f_1, \ldots, f_m) = \sum_{i=1}^m \gamma_i \int_{M_i} (|\nabla f_i|^2 - |B_{M_i}|^2 f_i^2) - \sum_{i=1}^m \gamma_i \int_{\partial M_i} f_i^2
\]

The volume constraints are

\[
\int_{M_i} f_i = 0, \quad i = 1, \ldots, m
\]

the normalization condition is

\[
\sum_{i=1}^m \int_{M_i} f_i^2 = 1
\]

and the corresponding modified functional is

\[
J^*(f_1, \ldots, f_m; \lambda_1, \ldots, \lambda_m, \mu) = \sum_{i=1}^m \gamma_i \left[ \int_{M_i} (|\nabla f_i|^2 - |B_{M_i}|^2 f_i^2) - \int_{\partial M_i} f_i^2 \right]
\]

\[
- \sum_{i=1}^m \lambda_i \int_{M_i} f_i - \mu \sum_{i=1}^m \int_{M_i} f_i^2
\]

Proposition \([3]\) extends without difficulty to connected multiphase problems, with \([14]\) holding on each interface in the following form:

\[
\gamma_i \Delta_{M_i} f_i + (\mu + \gamma_i |B_{M_i}|^2) f_i = -\frac{\lambda_i}{2}
\]

The boundary conditions retain the form of \([15]\) for each \( i \). Propositions \([4], [5]\) hold as they are. In the proof of Proposition \([5]\) the considered space is

\[
X = \{ (u_1, \ldots, u_m) \in H^1(M) : |u|_{L^2(M)} \leq 1, \int_{M_i} u_i = 0, \quad i = 1, \ldots, m \}
\]

Here we have the following definitions: \( H^1(M) = H^1(M_1) \times \cdots \times H^1(M_m) \), \( L^2(M) = L^2(M_1) \times \cdots \times L^2(M_m) \) and \( |u|^2_{L^2(M)} = \sum_{i=1}^m |u_i|^2_{L^2(M_i)} \) as usually.
The proposition next characterizes all connected \( m \)-phase partitionings by stating that each \( m \)-phase partitioning problem reduces to \( m - 1 \) independent 2-phase problems.

**Proposition 6.** Let \( \Omega \) be an open, bounded subset of \( \mathbb{R}^N \) with \( C^2 \) boundary. Assume a connected minimal \( m \)-phase partitioning of \( \Omega \) with volume constancy constraint, and let \( M = (M_i)_{i=1}^{m-1} \) be the interfaces of the partitioning. Then the partitioning is stable if and only if each \( M_i \) \( (i = 1, \cdots, m - 1) \) is stable. The partitioning is unstable if and only if there is at least one unstable \( M_i \) \( (i = 1, \cdots, m - 1) \).

The proof is straight-forward and follows by the fact that the \( M_i \) appear individually in the form of the second variation of area \( J \) and the constraints.

Examples for stable and unstable multiphase partitionings are easily constructed from 2-phase partitionings by means of Proposition 6. Therefore our applications focus on 2-phase partitionings.

### 4. Some applications to partitioning problems in \( \mathbb{R}^N \).

As applications of the spectral analysis of the 2nd variation of area, we derive in this section some general conclusions concerning partitionings in \( \mathbb{R}^N \).

#### 4.1. Stability of \( N \)-dimensional spherical partitionings.

The stability of partitionings in which one phase has the shape of a sphere, \( M = S^{N-1} \), has a direct physical meaning. It is the basis for modeling the stability of emulsions, i.e. suspensions of small liquid droplets or deformable solid particles in a surrounding fluid.

As \( \partial M = \emptyset \), all boundary integrals are absent, in particular the boundary condition (15). In 3-dimensions one may directly proceed to the solution of (14) in spherical polar coordinates using periodic conditions in place of (15). The spectral theory of the operator \( \Delta_M \) is well-known (see for example [7], Ch. V, 8, p. 314; Ch. VII, 5, pp 510-512 and [22] IV.2). Its eigenvalues are given by \( \lambda_l = -l(l + 1) \), \( l = 0, 1, \cdots \) and the corresponding eigenvectors are the spherical harmonics \( Y^m_l \), \( m = 0, \pm 1, \cdots, \pm l \), the first of which are

\[
Y^0_0(\theta, \phi) = \frac{1}{2\sqrt{\pi}}, \quad Y^0_1(\theta, \phi) = \frac{3}{2\sqrt{6\pi}} \cos \theta, \quad Y^\pm_1(\theta, \phi) = \pm \frac{3}{2\sqrt{6\pi}} \sin \theta e^{\pm i\phi}
\]

Since the eigenvectors of \( J \) satisfy the volume constancy condition (12), the first of these is not admissible. Integration of (14) in view of (12) gives \( \lambda = 0 \). Thus the minimal eigenvalue of \( J \), as obtained by \( l = 1 \), is \( \mu_1 = l(l + 1) - (N - 1) = 0 \), which would imply neutral stability. A more careful examination of these eigenvectors reveals that they are not true variations, but translations along the axes of the coordinate system. On discarding them and proceeding to the next available eigenvalue, \( l = 2 \), we have

\[
\mu_1 = l(l + 1) - (N - 1) = 4 > 0
\]

which implies stability. For \( N > 3 \) (see [22]) \( \lambda_l = -l(l + N - 2) \), \( l = 0, 1, \cdots \) and again \( \mu_1 = 0 \). Discarding translations again yields stability. In the next section we prove that the same situation pertains also to the 2-dimensional case. For the following proposition, no regularity and convexity conditions are necessary for \( \Omega \).

**Proposition 7.** Let \( \Omega \subset \mathbb{R}^N \) be an open set and \( \Omega_1 = B(x_0, R) \) a ball such that \( \overline{\Omega_1} \subset \Omega \). The two-phase partitioning of \( \Omega \) defined by \( M = \partial \Omega_1 \) is stable.
Note that balls are never absolute minimizers of partitioning problems in bounded sets. This is most conveniently seen by moving the ball until it comes into contact with the boundary, \( M \cap \partial \Omega \neq \emptyset \). Then it is clear that the translated ball is not even minimal, since at the contact point the normal of \( M \) is not tangent to \( \partial \Omega \). This implies the existence of a variation which decreases the ball’s area and this proves that the original ball is not an absolute minimizer.

4.2. Existence of unstable two phase partitionings. The existence of stable partitionings is guaranteed by the existence of absolute minimizers (see \[24\] Th. 1.4, p. 6). Here we give a proof of existence of unstable partitionings in \( \mathbb{R}^N \) under certain conditions on \( \Omega \). Let \( M \) be a minimal two phase partitioning of \( \Omega \). We assume \( \Omega \) is convex and has the following property: for all sufficiently large \( \sigma \) there is a convex set \( \Omega \) containing \( M \) and satisfying the condition

\[(H): \Omega \) contacts \( \Sigma = \partial \Omega \) along \( \partial M \), i.e. \( T_p\partial \Omega = T_p\Sigma \) at all \( p \in \partial M \), and \( II_{\partial \Omega, p}(N_p, N_p) \geq \sigma, \ p \in \partial M \)

\((N_p \) is the normal field of \( M \) at \( p \).

Example 4.1. (i) Let \( \Omega = B(0, \frac{1}{\sigma_0}) \subset \mathbb{R}^2 \), \( D \) a circle intersecting \( \partial \Omega \) at right angles, and \( M = D \cap \Omega \). If \( M \cap \partial \Omega = \{p_1, p_2\} \), and \( C_1^\sigma, C_2^\sigma \) are the circles of radius \( \frac{1}{\sigma} \) contained in \( \Omega \) and contacting \( \Sigma = \partial \Omega \) at \( p_1, p_2 \) respectively, then \( \Omega \) satisfies \( (H) \) with \( \Omega_{\sigma} = (M \cup C_1^\sigma \cup C_2^\sigma) \) (\( \{\cdot\} \) denotes convex hull).

(ii) Consider a rhombus and a circular arc with its center positioned at one of its vertices. Let \( \Omega \) be the solid obtained by the revolution of the rhombus about the axis of the rhombus that passes through the center of the circular arc. Let \( M \) be the surface obtained by the revolution of the arc. By elementary geometric arguments similar to (i) it is easily established that for all sufficiently large \( \sigma \) there is a convex set \( \Omega_{\sigma} \) containing \( M \) and satisfying condition \( (H) \).

Proposition 8. Let \( M \) be a minimal partitioning of \( \Omega \) in \( \mathbb{R}^N \). Assume that \( \Omega \) satisfies condition \( (H) \). Then there is a convex set \( \Omega^* \) for which \( M \) is an unstable partitioning.

Proof. Assume \( M \) is stable or neutrally stable, for otherwise there is nothing to prove. Let

\[ \mu_1 = \inf_{f_M, f = 0, |f|_{L^2(M)} = 1} J(f) \geq 0 \]

Let \( \epsilon > 0 \) small and \( f \) be a variation of \( M \) satisfying \([13], [12] \) and such that \( J(f) < \mu_1 + \epsilon \). We can assume that \( f \) is not identically vanishing on \( \partial M \), for if \( f = 0 \) on \( \partial M \), select \( f_0 \in \mathbb{R}, f_0 \neq 0 \) such that

\[ -\int_M |B_M|^2 - \int_{\partial M} II(N, N) < \frac{2}{f_0} \int_M |B_M|^2 f \]

and then we have \( f + f_0 = f_0 \neq 0 \) on \( \partial M \) and

\[ J(f + f_0) = J(f) - f_0^2 \int_M |B_M|^2 - 2f_0 \int_M |B_M|^2 f - f_0^2 \int_{\partial M} II(N, N) < J(f). \]

Since \( \Omega, M \) were assumed to satisfy \( (H) \) there is a \( \sigma > 0 \) and \( \Omega_{\sigma} =: \Omega^* \) such that

\[ \sigma > \frac{k}{|f|_{L^2(\partial M)}^2} + \sup_{p \in \partial M} II_{\partial \Omega, p}(N_p, N_p) \]
II_{\partial M}(N, N) \geq \sigma\]

on \(\partial M\). \(k\) is a positive number to be fixed later. We have

\[
J_{\Omega^*}(f) = \int_M \left( |\nabla_M f|^2 - |B_M|^2 f^2 \right) - \int_{\partial M} \Pi_{\partial M}(N, N) f^2 \leq \int_M \left( |\nabla_M f|^2 - |B_M|^2 f^2 \right) - \sigma \int_{\partial M} f^2 \leq \int_M \left( |\nabla_M f|^2 - |B_M|^2 f^2 \right) - \int_{\partial M} \Pi_{\partial M}(N, N) f^2 - k
\]

Choosing \(k > \mu_1 + \epsilon\) completes the proof. \(\square\)

5. Application to 2-dimensional partitioning problems. Two-dimensional partitionings are particularly simple, for in this case

\[
\Delta_M f = \frac{d^2 f}{ds^2}
\]

where \(s\) is the arc length of \(M\) and the integrals over \(\partial M\) reduce to numbers. The boundary condition (15) reduces to

\[
\pm \frac{df}{ds} = \sigma f, \text{ at } s = 0, L
\]

(28)

\(L = |M|\) being the length of \(M\). The plus sign applies at \(s = L\) and minus at \(s = 0\). We are using the values \(\sigma = \sigma_1\) at \(s = 0\) and \(\sigma = \sigma_2\) at \(s = L\). From part (i) of Proposition 1 the curvature \(\kappa\) is constant, thus the only possibilities for \(M\) are line segments and circular arcs. The orientation of \(M\) is selected so that \(\kappa \geq 0\). Further, \(|B_M| = \kappa = 1/R\), where \(R\) is the radius of the arc or \(\infty\) for line segments.

For equation (14) we have the following three types of solution, depending on the sign of \(\mu + |B_M|^2\):

\(I\) \[f(s) = -\frac{\lambda}{2k^2} + C \sin(ks) + D \cos(ks), \quad k^2 = \mu + \kappa^2 \Leftrightarrow \mu > -\kappa^2\]

\(II\) \[f(s) = \frac{\lambda}{2k^2} + Ce^{ks} + De^{-ks}, \quad k^2 = -(\mu + \kappa^2) \Leftrightarrow \mu < -\kappa^2\]

\(III\) \[f(s) = -\frac{\lambda}{4} s^2 + Cs + D \quad \mu = -\kappa^2\]

The constants \(\lambda, C, D\) in each case are determined by the two conditions of (28) and one of (12), which form a linear homogeneous system of three equations in these three variables. The condition for existence of solutions of this system is obtained by setting its determinant to 0, which gives a nonlinear equation for \(k\). With a solution for \(k\) at hand, we can determine the eigenvalue \(\mu\) by the last column in the above table of possible solutions for \(f\) and the eigenvectors \((\lambda, C, D)\), each determining an eigenfunction \(f\) of problem (14). Not all three cases (I)-(III) need to be considered, depending on the problem under study.

5.1. Case I: \(-\kappa^2 < \mu < 0 \Leftrightarrow 0 < k < 1\). By (28) \(s = 0, L\) and (12) we obtain the system

\[-\frac{\lambda}{2k^2} + \frac{k}{\sigma_1} C + D = 0\]
\[-\frac{\lambda}{2k^2} + \left[ \sin(kL) - \frac{k}{\sigma_2^2} \cos(kL) \right] C + \left[ \cos(kL) + \frac{k}{\sigma_2^2} \sin(kL) \right] D = 0 \tag{29}\]

\[-\frac{\lambda L}{2k} + [1 - \cos(kL)] C + \sin(kL) D = 0\]

The case of stable and neutrally stable eigenvalues \(\mu \geq 0\) is contained here.

5.2. **Case II**: \(\mu < -\kappa^2, k > 0\). By (28) \(s = 0, L\) and (12) we obtain the system

\[-\frac{\lambda}{2k^2} + \left(1 + \frac{k}{\sigma_1}\right) C + \left(1 - \frac{k}{\sigma_1}\right) D = 0 \]

\[-\frac{\lambda}{2k^2} + \left(1 - \frac{k}{\sigma_2}\right) e^{kL} C + \left(1 + \frac{k}{\sigma_2}\right) e^{-kL} D = 0 \tag{30}\]

\[-\frac{\lambda L}{2k} + (e^{kL} - 1) C - (e^{-kL} - 1) D = 0\]

By the first of equations (30), \(C = -\sigma_1 D\), and the remaining two equations give a system, the solvability of which is equivalent to the equation

\[\sigma_1 \sigma_2 L^2 - 4(\sigma_1 + \sigma_2) L + 12 = 0. \tag{31}\]

As a consequence, when it happens that the length of the interface has one of the following two values

\[L = 2 \frac{\sigma_1 + \sigma_2 \pm \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}}{\sigma_1 \sigma_2}\]

the partitioning is unstable. We will prove a more general result in Proposition 9.

5.3. **Case III**: \(\mu = -\kappa^2\). As previously we obtain the system

\[C + \sigma_1 D = 0\]

\[\frac{1}{4} L (2 - \sigma_2 L) \lambda + (\sigma_2 L - 1) C + \sigma_2 D = 0 \tag{32}\]

\[-\frac{1}{6} L^2 \lambda + LC + 2D = 0\]

By the first of equations (31), \(C = -\sigma_1 D\), and the remaining two equations give a system, the solvability of which is equivalent to the equation

\[\sigma_1 \sigma_2 L^2 - 4(\sigma_1 + \sigma_2) L + 12 = 0. \tag{33}\]

As a consequence, when it happens that the length of the interface has one of the following two values

\[L = 2 \frac{\sigma_1 + \sigma_2 \pm \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}}{\sigma_1 \sigma_2}\]

the partitioning is unstable. We will prove a more general result in Proposition 9.

5.4. **Stability and instability criteria.** The following proposition generalizes the previous result.

**Proposition 9.** Let \(\Omega\) be a bounded, convex, open subset of \(\mathbb{R}^2\) and \(M\) a minimal two phase partitioning of \(\Omega\) with length \(L\). Assume there is a neighborhood of the points \(\partial M\) in \(\Sigma = \partial \Omega\) which is a \(C^2\) curve and the curvatures of \(\Sigma\) at these points are \(\sigma_1, \sigma_2\). If \(L\) satisfies the condition

\[2 \frac{\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}}{\sigma_1 \sigma_2} \leq L \leq 2 \frac{\sigma_1 + \sigma_2 + \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}}{\sigma_1 \sigma_2} \tag{33}\]

then \(M\) is unstable.

**Proof.** We only have to prove the inequalities. Letting \(B = \frac{\lambda}{2k^2}, x = Lk, a = \sigma_1 L\) and \(b = \sigma_2 L\), the solvability condition for system (30) of case (II) (in the variables \(B, C, D\)) is

\[D(x) = \begin{vmatrix} 1 & 1 + \frac{x}{a} & 1 - \frac{x}{a} & \frac{x}{a} \\ 1 & (1 - \frac{x}{a}) e^x & (1 + \frac{x}{a}) e^{-x} & e^x - 1 \\ x & e^x - 1 & 1 - e^{-x} & 0 \end{vmatrix} = 0\]
Performing operations we obtain

\[
D(x) = 4 \left[ 1 + \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) x^2 \right] \cosh x - 2 \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{x^2}{ab} \right) x \sinh x - 4.
\]

For \( x \to +\infty \) it is easily established that \( D(x) \to -\infty \). At \( x = 0 \) we have \( D(0) = 0 \), so we expand \( D \) in a power series of \( x \) about 0:

\[
D(x) = -\frac{1}{6ab} (ab - 4a - 4b + 12)x^4 + O(x^5)
\]

We have \( D(x) > 0 \) in a neighborhood \( ]0, \delta[ \), \( \delta > 0 \), when \( ab - 4a - 4b + 12 < 0 \).

For \( \sigma_1 = \sigma_2 = \sigma \), by (33) we have instability when \( \frac{2}{\pi} \leq L \leq \frac{6}{\pi} \) and \( L \to \infty \) as \( \sigma \to 0 \), which suggests that for flat boundaries all minimal partitionings are stable.

This is conveniently proved by solving systems (29), (30) and (31):

**Proposition 10.** Let \( \Omega \) and \( M \) satisfy the conditions of Proposition 9 and \( \sigma_1 = \sigma = \sigma_2 \). Then \( M \) is stable.

**Proof.** When \( \sigma_1 = \sigma_2 = 0 \), by (32) it is clear that (31) has no solution and it is easily checked that (30) has only the trivial solution. System (29) reduces to

\[
\lambda = C = 0, \quad \sin(kL)D = 0
\]

which has nontrivial solutions only when \( \sin(kL) = 0 \), i.e. \( kL = n\pi, n \in \mathbb{N} \). By the restriction \( 0 < \frac{\pi}{L} < 1 \) for case (I) it follows that \( \kappa L > n\pi \). As \( \omega = \kappa L \) is the angle of the circular sector defined by \( M \) and the tangents to \( \partial \Omega \) at the extremities of \( M \), by the convexity of \( \Omega \) we have \( \omega < \pi \). The case \( k = \kappa \) corresponding to \( \mu = 0 \) gives also no eigenvalues. Thus we have only positive eigenvalues, and this according to Proposition 5 implies stability.

As a next application of equations (29)-(31) we give a proof of the stability of circles. This is essentially a variational proof of the well-known fact that among all 2-dimensional geometric shapes of equal area, circles have least perimeter.

**Proposition 11.** Let \( \Omega \subset \mathbb{R}^2 \) be an open set and \( \Omega_1 = B(x_0, R) \) such that \( \overline{\Omega_1} \subset \Omega \). Then \( M = \partial \Omega_1 \) is a stable two phase partitioning of \( \Omega \).

**Proof.** Integration of (14) in view of (12) gives \( \lambda = 0 \). From systems (29)-(31) only the third equation remains in place. Additionally we have the periodicity condition \( f(0) = f(\frac{2\pi}{\kappa}) \). It is easily checked that cases (II) and (III) have only trivial solutions, while for (I) we have

\[
C \sin \frac{2\pi k}{\kappa} + D \left( \cos \frac{2\pi k}{\kappa} - 1 \right) = 0
\]

\[
\left( 1 - \cos \frac{2\pi k}{\kappa} \right) C + D \sin \frac{2\pi k}{\kappa} = 0
\]

which has nontrivial solutions if and only if \( \cos \frac{2\pi k}{\kappa} = 1 \), i.e. \( k = \kappa \) which corresponds to neutral stability as in this case \( \mu = 0 \). The eigenvectors are

\[
f(s) = C \sin(\kappa s) + D \cos(\kappa s)
\]
i.e. linear combinations of \( f_1(s) = \sin(\kappa s) \) and \( f_2(s) = \cos(\kappa s) \) expressing translations along the \( y \) and \( x \) axis respectively. Since they are essentially not true variations of \( M \), we can discard them and thus there are only positive eigenvalues, which proves the assertion.

Partitionings with large interfaces are unstable. More precisely:

**Proposition 12.** Let \( \Omega \) and \( M \) satisfy the conditions of Proposition 9. If \( L \) satisfies the condition

\[
L > 2 \frac{\sigma_1 + \sigma_2 + \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}}{\sigma_1 \sigma_2}
\]

(34)

then \( M \) is unstable.

**Proof.** Integration of (14), taking into account (15) and (12), gives

\[
\int_{\partial M} \sigma f + \int_M |B_M|^2 f = -\frac{1}{2} \lambda |M|,
\]

which in two dimensions simplifies to the equation

\[
\sigma_1 f(0) + \sigma_2 f(L) = -\frac{1}{2} \lambda L
\]

(35)

By remarking that when \( f \) is an eigenfunction of (14) then also \( f^\ast \) defined by \( f^\ast(s) = f(L-s) \) is an eigenfunction for the same eigenvalue, we obtain the equation

\[
\sigma_2 f(0) + \sigma_1 f(L) = -\frac{1}{2} \lambda L
\]

(36)

The solution of the system (35), (36) is

\[
f(0) = f(L) = -\frac{\lambda L}{2(\sigma_1 + \sigma_2)}
\]

By (II) we have \( f(0) = \frac{\lambda}{2k^2} + C + D \) and \( f(L) = \frac{\lambda}{2k^2} + C e^{kL} + D e^{-kL} \), from which it follows that

\[
C = -\frac{\lambda}{2} \frac{1 - e^{-kL}}{e^{kL} - e^{-kL}} \left( \frac{1}{k^2} + \frac{L}{\sigma_1 + \sigma_2} \right), \quad D = -\frac{\lambda}{2} \frac{e^{kL} - 1}{e^{kL} - e^{-kL}} \left( \frac{1}{k^2} + \frac{L}{\sigma_1 + \sigma_2} \right).
\]

By the volume conservation equation as expressed by the third of (30) we obtain

\[
\frac{kL}{2} \coth \frac{kL}{2} = 1 + \frac{(kL)^2}{(\sigma_1 + \sigma_2)L}
\]

(37)

By considering the function

\[
f(x) = \frac{x e^x + 1}{2 e^x - 1} - 1 - \frac{x^2}{c}
\]

where \( x = kL \) and \( c = (\sigma_1 + \sigma_2)L \), with Taylor series expansion \( f(x) = \left( \frac{1}{12} - \frac{1}{2} \right) x^2 + \mathcal{O}(x^4) \) about 0, and remarking that \( f(x) > 0 \) in a neighborhood \( |0, \delta| \) if \( c > 12 \), while \( f(x) \) becomes negative for \( x \) sufficiently large, we conclude the existence of a positive root of \( f(x) = 0 \). When \( L \) satisfies the condition (34), from \( \sigma_1 \sigma_2 \leq \frac{1}{4} (\sigma_1 + \sigma_2)^2 \) we obtain \( (\sigma_1 + \sigma_2)L > 12 \), and this completes the proof.

We conclude by proving that sufficiently small partitions are stable.

**Proposition 13.** Let \( \Omega \) and \( M \) be as in Proposition 9. If \( L = |M| \) is sufficiently small, then \( M \) is stable.
Proof. If $L < L_0$, where $L_0 = 2^{\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}}$, case (III) has no solution. The proof that case (II) has no solution follows by rewriting $\lambda$ in the form

$$\frac{x}{2} (e^x + 1) = \left(1 + \frac{1}{2}x^2\right) (e^x - 1),$$

($x = kL$, $c = (\sigma_1 + \sigma_2)L$), expanding both sides in Taylor series, and comparing corresponding coefficients of powers of $x$.

We are looking for eigenvalues in the range $-\kappa^2 < \mu \leq 0$, or equivalently $0 < k \leq \kappa$. The determinant of the linear system (29) in the variables $B = -\frac{1}{x^2}$, $C$, $D$ is given by

$$D_1(x) = \begin{vmatrix} 1 & \frac{k}{\sigma_1} & \frac{k}{\sigma_2} \\ \sin x - \frac{k}{\sigma_1} \cos x & \cos x + \frac{k}{\sigma_2} \sin x \\ \cos x & \sin x \end{vmatrix}$$

$$= 2(1 - \cos x) - \left[x + k \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) - \frac{1}{\sigma_1\sigma_2} k^2 x\right] \sin x + k \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) x \cos x$$

with $x = kL$. We expand $\sin x$, $\cos x$ into a Taylor series about 0:

$$D_1(x) = \frac{1}{\sigma_1\sigma_2} k^2 x^2 - \frac{1}{3} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) x^3 + \frac{1}{12} \left(1 - 2 \frac{k^2}{\sigma_1\sigma_2}\right) x^4 + O(x^5).$$

Now assume $k$ is a root of $D_1(kL) = 0$ in $[0, \kappa]$ for $L \leq L_0$, i.e.

$$0 = \frac{1}{\sigma_1\sigma_2} k^2 - \frac{1}{3} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) L + \frac{1}{12} \left(1 - 2 \frac{k^2}{\sigma_1\sigma_2}\right) L^2 + O(L^3),$$

(38)

Substituting $x = kL$ and eliminating $k^2$ give

$$0 = \frac{1}{\sigma_1\sigma_2} - \frac{1}{3} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right) \sigma_3 L + \frac{1}{12} \left(1 - 2 \frac{k^2}{\sigma_1\sigma_2}\right) \sigma_3^2 L^2 + O(L^3),$$

(39)

from which in the limit $L \to 0$ we obtain

$$0 = \frac{1}{\sigma_1\sigma_2}$$

which is absurd. Hence, possibly for a lower value for $L_0$, the equation $D_1(kL) = 0$ has no roots in $[0, \kappa]$ for all $L < L_0$. \qed

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