On the Regret Minimization of Nonconvex Online Gradient Ascent for Online PCA

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Abstract
Non-convex optimization with global convergence guarantees is gaining significant interest in machine learning research in recent years. However, while most works consider either offline settings in which all data is given beforehand, or simple online stochastic i.i.d. settings, very little is known about non-convex optimization for adversarial online learning settings. In this paper we focus on the problem of Online Principal Component Analysis in the regret minimization framework. For this problem, all existing regret minimization algorithms are based on a positive semidefinite convex relaxation, and hence require quadratic memory and SVD computation (either thin of full) on each iteration, which amounts to at least quadratic runtime per iteration. This is in stark contrast to a corresponding stochastic i.i.d. variant of the problem, which was studied extensively lately, and admits very efficient gradient ascent algorithms that work directly on the natural non-convex formulation of the problem, and hence require only linear memory and linear runtime per iteration. This raises the question: can non-convex online gradient ascent algorithms be shown to minimize regret in online adversarial settings?

In this paper we take a step forward towards answering this question. We introduce an adversarially-perturbed spiked-covariance model in which, each data point is assumed to follow a fixed stochastic distribution with a non-zero spectral gap in the covariance matrix, but is then perturbed by adversarial noise. This model is a natural extension of a well studied standard stochastic setting that allows for non-stationary (adversarial) patterns to arise in the data and hence, might serve as a significantly better approximation for real-world data-streams. We show that in a certain regime of parameters, when the non-convex online gradient ascent algorithm is initialized with a “warm-start” vector, it provably minimizes the regret with high probability. We further discuss the possibility of computing such a “warm-start” vector. Our theoretical findings are supported by empirical experiments on both synthetic and real-world data.

1 Introduction
Nonconvex optimization is ubiquitous in contemporary machine learning, ranging from optimization over sparse vectors or low-rank matrices to training Deep Neural Networks. While traditional (yet still highly active) research on nonconvex optimization focuses mostly on efficient convergence to stationary points, which in general need not even be a local minima, let alone a global one, a more-recent line of work focuses on proving convergence to global minima, usually under certain simplifying assumptions that on one hand make the nonconvex problem tractable, and on the other hand, are sufficiently reasonable in some scenarios of interest. One of the most studied and well known nonconvex optimization problems in machine learning underlies the fundamental task of Principal
**Component Analysis** (PCA), in which, given a set of $N$ vectors in $\mathbb{R}^d$, one wishes to find a $k$-dimensional subspace for $k << d$, such that the projections of these vectors onto this subspace is closest in square-error to the original vectors. It is well known that the optimal subspace corresponds to the span of the top $k$ eigenvectors of the covariance matrix of the data-points. Henceforth, we focus our discussion to the case $k = 1$, i.e., extracting the top principal component. Quite remarkably, while this problem is non-convex (since extracting the top eigenvector amounts to maximizing a convex function over the unit Euclidean ball), a well known iterative algorithm known as Power Method (or Power Iterations), which simply starts with a random unit vector and repeatedly applies the covariance matrix to it (and then normalizes the result to have unit norm), converges to the global optimal solution rapidly. The convergence guarantee of the PM, can also be shown to imply that the nonconvex projected gradient ascent method with random initialization and a fixed step-size also converges to the top principal component.

In a recent line of work, the convergence of non-convex gradient methods for PCA was extended to a natural online stochastic i.i.d. setting of the problem, in which, given a stream of data points sampled i.i.d. from a fixed distribution, the goal is to converge to the top eigenvector of the covariance of the underlying distribution as the sample size increases, yielding algorithms that require only linear memory (i.e., do not need to store the entire sample or large portions of it at any time) and linear runtime to process each data point, see for instance [15, 3, 18, 10, 1, 14, 22].

In a second recent line of research, researchers have considered Online PCA as a sequential decision problem in the adversarial framework of regret minimization (aka online learning, see for instance the recent introductory texts [17, 8], e.g., [20, 21, 16, 4, 6, 2]). In this framework, for each data-point, the online algorithm is required to predict a unit vector (i.e., a subspace of dimension one, recall we are in the case $k = 1$) before observing the data-point, and the goal is to minimize regret which is the difference between the square-error of the predictions made and the square-error of the principal component of the entire sequence of data. Different from the i.i.d. stochastic setting, in this framework, the data may be arbitrary (though assumed to be bounded in norm), and need not follow a simple generative model. Formally, the regret is given by

$$
\text{regret} := \sum_{i=1}^{N} \| x_i - w_i w_i^T x_i \|_2^2 - \min_{\|w\|_2 = 1} \sum_{i=1}^{N} \| x_i - w w_i^T x_i \|_2^2,
$$

where $\{x_i\}_{i \in [N]} \subset \mathbb{R}^d$ is the sequence of data points, and $\{w_i\}_{i \in [N]}$ is the sequence of predictions made by the online algorithm. Using standard manipulations, it can be shown that

$$
\text{regret} = \max_{\|w\|_2 = 1} \sum_{i=1}^{N} (w_i^T x_i)^2 - \sum_{i=1}^{N} (w_i^T x_i)^2 = \lambda_1 \left( \sum_{i=1}^{N} x_i x_i^T \right) - \sum_{i=1}^{N} (w_i^T x_i)^2,
$$

where $\lambda_1(\cdot)$ denotes the largest (signed) eigenvalue of a real symmetric matrix.

Naturally, the arbitrary nature of the data in the online learning setting, makes the problem much more difficult than the stochastic i.i.d. setting. Notably, all current algorithms which minimize regret cannot directly tackle the natural nonconvex formulation of the problem, but consider a well known (tight) convex relaxation, which “lifts” the decision variable from the unit Euclidean ball in $\mathbb{R}^d$ to the set of all $d \times d$ positive semidefinite matrices of unit trace (aka the spectrahedron). While this reformulation allows to obtain regret-minimizing algorithms in the online adversarial settings (since the problem
becomes convex), they are dramatically less efficient than the standard nonconvex gradient methods. In particular, all such algorithms require quadratic memory (i.e., $O(d^2)$), and require either a thin or full-rank SVD computation of a full-rank matrix to process each data point, which amounts to at least quadratic runtime per data point (for non-trivially-sparse data), see [20, 21, 16, 4, 6, 2]. This phenomena naturally raises the question:

Can Nonconvex Online Gradient Ascent be shown to minimize regret for the Online PCA problem?

While in this paper we do not provide a general answer (either positive or negative), we do take a step forward towards understanding the applicability of nonconvex gradient methods to the Online PCA problem, and to online nonconvex optimization in general. We introduce a "semi-adversarial" setting, which we refer to as adversarially-perturbed spiked-covariance model, which assumes the data follows a standard i.i.d. stochastic distribution with a covariance matrix that admits a non-zero spectral gap, however, each data point is then perturbed by some arbitrary, possibly adversarial, vector of non-trivial magnitude. We view this model as a natural extension of the standard stochastic spiked covariance model (which was studied extensively in recent years) due to its ability to capture arbitrary (adversarial) patterns in the data. Hence, we believe the suggested model might provide a much better approximation for real-world data streams. We formally prove that in a certain regime of parameters, which concerns both the spectral properties of the distribution covariance and the magnitude of adversarial perturbations, given a "warm-start" initialization which is sufficiently correlated with the top principal component of the distribution covariance, the natural nonconvex online gradient ascent algorithm guarantees an $\tilde{O}(\sqrt{N})$ regret bound with high probability. In particular, the algorithm requires only $O(d)$ memory and $O(d)$ runtime per data point. We further discuss the possibilities of computing such a "warm-start" vector (i.e., initializing from a "cold-start"). Finally, we present empirical experiments with both synthetic and real-world datasets which complement our theoretical analysis.

2 Assumptions and Results

In this section we formally introduce our assumptions and main result. As discussed in the introduction, since our aim is make progress on a highly non-trivial problem of providing global convergence guarantees for a non-convex optimization algorithm in an online adversarial setting, our results do not hold for arbitrary (bounded) data, as is usually standard in convex online learning settings, but only on for a more restricted family of input streams, namely those which follow a model we refer to in this paper as the adversarially-perturbed spiked-covariance model. Next we formally introduce this model.

2.1 Adversarially-Perturbed Spiked-Covariance Model

Throughout the paper we assume the data, i.e., the vectors $\{x_t\}_{t \in [N]}$, satisfy the following assumption.

Assumption 1 (Perturbed Spiked Covariance Model). We say a sequence of $N$ vectors $\{x_t\}_{t \in [N]} \subset \mathbb{R}^d$ satisfies Assumption 1 if for all $t \in [N]$, $x_t$ can be written as $x_t = q_t + v_t$, where $\{q_t\}_{t \in [N]}$ are sampled i.i.d. from a distribution $D$ and $\{v_t\}_{t \in [N]}$ is a sequence of arbitrary bounded vectors such that the following conditions hold:
1. the vectors \( \{v_t\}_{t \in [N]} \) all lie in a Euclidean ball of radius \( V \) centered at the origin, i.e., \( \max_{t \in [N]} \|v_t\|_2 \leq V \)

2. the support of \( \mathcal{D} \) is contained in a Euclidean ball of radius \( R \) centered at the origin, i.e., \( \sup_{q \in \text{support}(\mathcal{D})} \|q\|_2 \leq R \)

3. \( \mathcal{D} \) has zero mean, i.e., \( \mathbb{E}_{q \sim \mathcal{D}}[q] = 0 \)

4. the covariance matrix \( Q := \mathbb{E}_{q \sim \mathcal{D}}[qq^\top] \), admits an eigengap \( \delta(Q) := \lambda_1(Q) - \lambda_2(Q) \) which satisfies \( \delta(Q) \geq V \sqrt{\lambda_1(Q)} + V^2 + \varepsilon \), for some \( \varepsilon > 0 \).

We now make a few remarks regarding Assumption 1. Item (1) assumes that the adversarial perturbations are bounded which is standard in the online learning literature, Item (2) is also a standard assumption, which is used to apply standard concentration arguments for sums of i.i.d random variables. Item (3), i.e., the assumption that the distribution as zero mean, while often standard, is not mandatory for our analysis to hold, however since it greatly simplifies the analysis and results in a much wider regime of parameters to which our result is applicable, we make it.

To better understand item (4), it helps to think of \( \delta(Q), V^2, \varepsilon \) as quantities proportional to \( \lambda_1(Q) \), i.e., consider \( \delta(Q) = c_3 \lambda_1(Q), V^2 = c_V \lambda_1(Q), \varepsilon = c_\varepsilon \lambda_1(Q) \), for some universal constants \( c_3, c_V, c_\varepsilon \in (0, 1) \). Now, item (4) in the assumption boils down the the condition \( c_3 \geq \sqrt{c_V^2 + c_\varepsilon^2 + \varepsilon^2} \) i.e., the eigengap in the covariance \( Q \) needs to dominate the adversarial perturbations in a certain way. We further discuss this assumption after presenting our main theorem - Theorem 1 in the following Subsection.

Connection with stochastic i.i.d. models: note that when setting \( V = 0 \) in Assumption 1 (i.e., there is no adversarial component), our setting reduces to the well studied standard stochastic i.i.d. setting. In particular, in this case item (4) in Assumption 1 simply reduces to the standard assumption in this model that the covariance admits an eigengap bounded away from zero (\( \delta(Q) \geq \varepsilon \)). Hence, the model introduced above can be seen as a natural, yet highly non-trivial, extension of the standard stochastic model to a “more expressive” online adversarial model, that might serve as a better approximation for real-world data-streams in online-computation environments.

2.2 Algorithm and Convergence Result

For simplicity of the analysis we consider the data as arriving in blocks of length \( \ell \), where \( \ell \) is a parameter to be determined later. Towards this end, we assume that \( N = T \ell \) for some integer \( T \) and we consider prediction in \( T \) rounds, such that on each round \( t \in [T] \), the algorithm predicts on all \( \ell \) vectors in the \( t \)th block, which we denote by \( x^{(1)}_t, \ldots, x^{(\ell)}_t \). It is important to emphasize that, while our algorithm considers the original data in blocks, it requires only \( O(d) \) memory and \( O(d) \) time to process each data point \( x^{(i)}_t \) for any \( t \in [T], i \in [\ell] \). Our algorithm, which we refer to as nonconvex online gradient ascent, is given below (see Algorithm 1).
The following theorem states our main result.

**Theorem 1.** Consider a sequence \( \{x_t\}_{t \in [N]} \) which follows Assumption \( \mathbb{A} \) and fix \( p \in (0, 1) \). For large enough \( N \), there exists an integer \( \ell = O \left( \frac{R^4 (\delta(Q)^2 - V^2 (\lambda_1(Q) + V^2))^2}{\lambda_1(Q)^2} \log \frac{d N}{p} \right) \), such that applying Algorithm \( \mathbb{A} \) with blocks of length \( \ell \) and initialization \( \tilde{w}_1 \) which satisfies \( \langle \tilde{w}_1, x \rangle^2 \geq 1 - \frac{4 \delta(Q)^2 V^2}{2 \lambda_1(Q)} \), where \( x \) is the leading eigenvector of \( Q \) (as defined in Assumption \( \mathbb{A} \)), and with learning rate \( \eta = \frac{1}{\sqrt{\ell (R + V)^2}} \), guarantees that with probability at least \( 1 - p \), the regret is upper-bounded by \( O \left( \sqrt{N \log \frac{d N}{p} \frac{\lambda_1(Q) R^2 (R + V)^2}{\delta(Q)^2 - V^2 (\lambda_1(Q) + V^2)}} \right) \).

Theorem 1 roughly says that when the distribution covariance has a large-enough eigengap with respect to the adversarial perturbations (item 4 in Assumption \( \mathbb{A} \)), then non-convex OGA converges from a “warm-start” with \( O(\sqrt{N}) \) regret. Intuitively, the condition on the eigengap implies that the best-in-hindsight eigenvector cannot be far from the leading eigenvector of the distribution covariance by more than a certain constant. Hence, Theorem 1 can be seen as an online “local” convergence result. Importantly, it is not hard to show that under the conditions of the theorem, the best-in-hindsight eigenvector can also be far from both the initial vector \( \tilde{w}_1 \) and from \( x \) by a constant (hence in particular both \( \tilde{w}_1 \) and \( x \) can incur linear regret). Hence, while our setting is strictly easier than the fully adversarial online learning setting, it still a highly non-trivial online learning setting. In particular, all previous algorithms that provably minimize the regret under the conditions of Theorem \( \mathbb{A} \) require quadratic memory and quadratic runtime per data-point.

### 2.2.1 Computing a ”warm-start” vector

We now discuss the possibility of satisfying the ”warm-start” requirement in Theorem \( \mathbb{A} \).

First, we note that given the possibility to sample i.i.d. points from the underlying distribution \( D \), it is straightforward to obtain a warm-start vector \( \tilde{w}_1 \), as required by Theorem \( \mathbb{A} \) by simply initializing \( \tilde{w}_1 \) to be the leading eigenvector of the empirical covariance of a size-\( n \) sample of such points. It is not difficult to show via standard tools such as the Davis-Kahan sin \( \theta \) theorem and a Matrix-Hoeffding concentration inequality (see for instance the proof of the following Lemma \( \mathbb{A} \)), that for any \( (\epsilon, p) \in (0, 1)^2 \), a sample of size \( n = O \left( \frac{(R+V)^4 \ln(d/p)}{\epsilon^2 \delta(Q)^2} \right) \) suffices, so the outcome \( \tilde{w}_1 \) satisfies: \( \langle \tilde{w}_1, x \rangle^2 \geq 1 - \epsilon \) with probability at least \( 1 - p \).

If sampling directly from \( D \) is not possible, the following lemma, whose proof is given in the sequel, shows that with a simple additional assumption on the parameters \( \delta(Q), \lambda_1(Q), V^2 \), it is possible to obtain the warm-start initialization directly using data that follows Assumption \( \mathbb{A} \). Moreover, the sample-size \( n \) required is independent of \( N \), and hence using for instance the first \( n \) vectors in the stream to compute such initialization deteriorates the overall regret bound in Theorem \( \mathbb{A} \) only by a lower-level term.

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**Algorithm 1** Nonconvex Online Gradient Ascent for Online PCA

1: input: unit vector \( \tilde{w}_1 \), learning rate \( \eta > 0 \)
2: for \( t = 1 \ldots T \) do
3: predict vector \( \tilde{w}_t \)
4: observe \( \ell \) vectors \( x_t^{(1)}, \ldots, x_t^{(\ell)} \) and payoff \( \sum_{i=1}^{\ell} \langle \tilde{w}_t, x_t^{(i)} \rangle^2 \)
5: \( \tilde{w}_{t+1} \leftarrow \frac{\tilde{w}_t + \eta \sum_{i=1}^{\ell} x_t^{(i)} x_t^{(i)\top} \tilde{w}_t}{\| \tilde{w}_t + \eta \sum_{i=1}^{\ell} x_t^{(i)} x_t^{(i)\top} \tilde{w}_t \|} \)
6: end for
Lemma 1. [warm-start] Suppose that in addition to Assumption 1 it also holds that
\[ \delta(Q) \geq (16\lambda_1(Q)V^2)^{1/3} \]
Then, for every \( p \in (0, 1) \) there exists a sample size \( n = O \left( \frac{(R+V)^\lambda_1\log(d/p)}{\delta^3} \right) \) such that initializing \( \hat{w}_1 \) to be the leading eigenvector of the empirical covariance \( \hat{X} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i^\top \), where \( \mathbf{x}_1, \ldots, \mathbf{x}_n \) follow Assumption 1, guarantees that with probability at least \( 1 - p \):
\[ (\hat{w}_1^\top \mathbf{x})^2 \geq 1 - \frac{\delta(Q) - V^2}{2\lambda_1(Q)}. \]

3 Analysis

At a high-level, the proof of Theorem 1 relies on the combination of the following three ideas:

1. We build on the fact that the Online PCA problem, when cast as online linear optimization over the spectrahedron (i.e., when the decision variable is lifted from a unit vector to a positive semidefinite matrix of unit trace), is online learnable via a standard application of online gradient ascent, which achieves an \( O(\sqrt{N}) \) regret bound (however requires a full SVD computation on each iteration to compute the projection onto the spectrahedron).

2. We prove, that under Assumption 1, the above “inefficient” algorithm, when initialized with a proper “warm-start” vector, guarantees that the projection onto the spectrahedron is always a rank-one matrix (hence, only a rank-one SVD computation per iteration is required).

3. Finally, we show that the nonconvex online gradient ascent algorithm, Algorithm 1, approximates sufficiently well the steps of the above algorithm (in case the projection is rank-one), avoiding SVD computations all together.

We introduce the following notation that will be used throughout the analysis. For all \( t \in [T] \), we define \( \mathbf{X}_t := \sum_{i=1}^{t} \mathbf{X}_i^{(i)} \mathbf{X}_i^{(i)^\top} \), \( \mathbf{Q}_t := \sum_{i=1}^{t} \mathbf{q}_i^{(i)} \mathbf{q}_i^{(i)^\top} \), \( \mathbf{V}_t := \sum_{i=1}^{t} \mathbf{v}_i^{(i)} \mathbf{v}_i^{(i)^\top} \), and \( \mathbf{M}_t := \sum_{i=1}^{t} \mathbf{q}_i^{(i)^\top} \mathbf{v}_i^{(i)^\top} + \mathbf{v}_i^{(i)} \mathbf{q}_i^{(i)^\top} \). Note that \( \mathbf{X}_t = \mathbf{Q}_t + \mathbf{M}_t + \mathbf{V}_t \). Recall that we let \( \mathbf{Q} \) denote the covariance matrix associated with the distribution \( \mathcal{D} \) (as detailed in Assumption 1), and we let \( \lambda_1(Q), \ldots, \lambda_d(Q) \) denote its eigenvalues in descending order. Also, we let \( \mathbf{x} \) denote the leading eigenvector of \( \mathbf{Q} \), which under Assumption 1 is unique. We also define \( \mathbf{D}_t := \mathbf{Q}_t - \ell \cdot \mathbf{Q} + \mathbf{M}_t \). Note that \( \mathbf{X}_t = \ell \cdot \mathbf{Q} + \mathbf{V}_t + \mathbf{D}_t \). Intuitively, under Assumption 1, \( 1/\ell \mathbf{D}_t \) converges to zero in probability as \( \ell \to \infty \).

We denote by \( \mathcal{S} \) the spectrahedron, i.e., \( \mathcal{S} := \{ \mathbf{W} \in \mathbb{R}^{d \times d} \mid \mathbf{W} \succeq 0, \text{Tr}(\mathbf{W}) = 1 \} \), and we let \( \Pi_S[\mathbf{W}] \) denote the Euclidean projection of a symmetric matrix \( \mathbf{W} \in \mathbb{R}^{d \times d} \) onto \( \mathcal{S} \).

Our main building block towards proving Theorem 1 is to analyze the regret of a different non-convex algorithm for Online PCA. The meta-algorithm, Algorithm 2, builds on the standard convexification scheme for Online PCA, i.e., “lifting” the decision set from the unit ball to the spectrahedron, however, instead of computing exact projections onto the spectrahedron, it follows a nonconvex approach of only approximating the projection via a rank-one solution. We refer to it as a meta-algorithm, since for a given approximation parameter \( \gamma \), it only requires on each iteration to find an approximate eigenvector of the matrix to be projected onto \( \mathcal{S} \).

Note that a straightforward implementation of Algorithm 2 with \( \gamma = 0 \) corresponds to updating \( \mathbf{w}_{t+1} \) via accurate SVD of the \( d \times (\ell + 1) \) matrix \( \left( \mathbf{w}_t, \mathbf{x}_t^{(1)}, \ldots, \mathbf{x}_t^{(\ell)} \right) \), which already yields an algorithm with \( O(\ell d) \) memory and \( O(\ell d) \) amortized runtime per data-point.
Lemma 3. Suppose that on some iteration

Lemma 2. Let \( w \in \mathbb{R}^d \) be a unit vector and let \( X \in \mathbb{R}^{d \times d} \) be positive semidefinite. Let \( w' \) be the leading eigenvector of the matrix \( W := \text{ww}^\top + \eta X \), for some \( \eta > 0 \). If \( w^\top Xw \geq \frac{\lambda_1(X) + \lambda_2(X)}{2} \), then it follows that \( w'w'^\top = \Pi_S[W] \).

Proof. Recall \( w' \) denotes the leading eigenvector of \( W \) and let \( y_2, \ldots, y_d \) denote the other eigenvectors in non-increasing order. It is well known that the projection of \( W \) onto \( S \) is given by

\[
\Pi_S[W] = (\lambda_1(W) - \lambda)w'w'^\top + \sum_{i=2}^{d} \max\{0, \lambda_i(W) - \lambda\} y_i y_i^\top ,
\]

where \( \lambda \) is a non-negative real such that \( \lambda_1(W) - \lambda + \sum_{i=2}^{d} \max\{\lambda_i(W) - \lambda, 0\} = 1 \). Thus, if we show that \( \lambda_1(W) \geq 1 + \lambda_2(W) \), then it in particular follows that \( \Pi_S[W] = w'w'^\top \).

Note that on one hand,

\[
\lambda_1(W) \geq w^\top Ww = 1 + \eta w^\top Xw . \tag{1}
\]

On the other-hand, using the last inequality, we can also write

\[
\lambda_2(W) \leq \lambda_1(W) + \lambda_2(W) - w^\top Ww = \lambda_1(W) + \lambda_2(W) - 1 - \eta w^\top Xw . \tag{2}
\]

Using Ky Fan’s eigenvalue inequality, we have that

\[
\lambda_1(W) + \lambda_2(W) = \lambda_1(ww^\top + \eta X) + \lambda_2(ww^\top + \eta X) \\
\leq \lambda_1(ww^\top + \lambda_2(ww^\top) + \lambda_1(\eta X) + \lambda_2(\eta X) \\
= 1 + \eta (\lambda_1(X) + \lambda_2(X)) . \tag{3}
\]

Thus, by combining Eq. (1), (2), (3), we arrive at the following sufficient condition so that \( w'w'^\top = \Pi_S[W] \):

\[
1 + \eta w^\top Xw \geq 1 + 1 + \eta (\lambda_1(X) + \lambda_2(X)) - \left(1 + \eta w^\top Xw\right),
\]

which is equivalent to the condition \( w^\top Xw \geq \frac{\lambda_1(X) + \lambda_2(X)}{2} \).

Lemma 3. Suppose that on some iteration \( t \) of Algorithm 2 it holds that

\[
(w_t^\top x)^2 \geq 1 - \frac{\delta(Q)}{2\lambda_1(Q)} + \frac{2\|D_t\|}{\ell \cdot \lambda_1(Q)} .
\]

Then, \( w_{t+1}w_{t+1}^\top = \Pi_S[W_{t+1}] \).
Proof. Recall that $X_t := \sum_{i=1}^{\ell} x_i^{(i)} x_i^{(i)\top}$, and denote $W_{t+1} = \hat{w}_t \hat{w}_t^\top + \eta X_t$.

Using Lemma 2 it suffices to show that

$$\hat{w}_t \top X_t \hat{w}_t \geq \frac{\lambda_1(X_t) + \lambda_2(X_t)}{2}. \tag{4}$$

On one hand we have

$$\hat{w}_t \top X_t \hat{w}_t = \hat{w}_t \top (\ell \cdot Q + V_t + D_t) \hat{w}_t \geq \ell \cdot \hat{w}_t \top Q \hat{w}_t - \|D_t\|$$

$$\geq \ell (\hat{w}_t \top x)^2 \cdot x \top Q x - \|D_t\| = \ell (\hat{w}_t \top x)^2 \cdot \lambda_1(Q) - \|D_t\|. \tag{5}$$

On the other hand,

$$\lambda_1(X_t) + \lambda_2(X_t) = \lambda_1(\ell \cdot Q + V_t + D_t) + \lambda_2(\ell \cdot Q + V_t + D_t)$$

$$\leq \lambda_1(\ell \cdot Q + V_t) + \lambda_2(\ell \cdot Q + V_t) + 2\|D_t\|$$

$$\leq \lambda_1(\ell \cdot Q) + \lambda_2(\ell \cdot Q) + \lambda_1(V_t) + \lambda_2(V_t) + 2\|D_t\|$$

$$\leq \ell (\lambda_1(Q) + \lambda_2(Q)) + \text{Tr}(V_t) + 2\|D_t\|$$

$$\leq \ell (\lambda_1(Q) + \lambda_2(Q) + V^2) + 2\|D_t\|, \tag{6}$$

where (a) follows from Ky Fan’s eigenvalue inequality.

Combining Eq. 4, 5, 6, we arrive at the following sufficient condition so that $w_{t+1} \hat{w}_{t+1} = \Pi_{S}[W_{t+1}]$:

$$(\hat{w}_t \top x)^2 \geq \frac{\lambda_1(Q) + \lambda_2(Q) + V^2 + 4\ell^{-1}\|D_t\|}{2 \cdot \lambda_1(Q)}$$

$$= \frac{2\lambda_1(Q) - \delta(Q) + V^2 + 4\ell^{-1}\|D_t\|}{2 \lambda_1(Q)}$$

$$= 1 - \frac{{\delta(Q) - V^2}}{2 \lambda_1(Q)} + \frac{2\|D_t\|}{\ell \cdot \lambda_1(Q)}. \square$$

Lemma 4. Suppose that on some iteration $t$ of Algorithm 2 it holds that $(x \top \hat{w}_t)^2 \geq \frac{1}{2}$. Then, for any learning rate $\eta > 0$, it holds that

$$(x \top w_{t+1})^2 \geq (x \top \hat{w}_t)^2 + \eta \ell \frac{(1 - (x \top \hat{w}_t)^2) \cdot \delta(Q) - (x \top \hat{w}_t)^2 V^2 - 4\ell^{-1}\|D_t\|}{\lambda_1(W_{t+1}) - \lambda_2(W_{t+1})}.$$

Proof. Fix some iteration $t$. We introduce the short notation $w = \hat{w}_t$, $w' = w_{t+1}$, $W = W_{t+1} = \hat{w}_t \hat{w}_t^\top + \eta X_t$, $\lambda_1 = \lambda_1(W_{t+1})$, $\lambda_2 = \lambda_2(W_{t+1})$, and for all $i \geq 2$, $y_i$ is the eigenvector of $W_{t+1}$ associated with eigenvalue $\lambda_i$.

It holds that

$$\lambda_1 w' w' + \sum_{i=2}^{d} \lambda_i y_i y_i^\top = W = w w^\top + \eta X_t.$$

Thus, we have that

$$(x \top w')^2 = \frac{x \top W x - \sum_{i=2}^{d} \lambda_i (x \top y_i)^2}{\lambda_1} = \frac{(x \top w)^2 + \eta x \top X_t x - \sum_{i=2}^{d} \lambda_i (x \top y_i)^2}{\lambda_1}.$$
Note that \( \sum_{i=2}^{d} \lambda_i (\mathbf{x}^\top \mathbf{y}_i)^2 \leq \lambda_2 (\|\mathbf{x}\|^2 - (\mathbf{x}^\top \mathbf{w})^2) = \lambda_2 (1 - (\mathbf{x}^\top \mathbf{w})^2) \). Thus, we have that

\[
(x^\top w')^2 \geq \frac{(x^\top w)^2 + \eta x^\top X_i x - \lambda_2 (1 - (x^\top w')^2)}{\lambda_1 - \lambda_2}.
\]

Rearranging we obtain,

\[
(x^\top w')^2 \geq \frac{(x^\top w)^2 + \eta x^\top X_i x - \lambda_2}{\lambda_1 - \lambda_2} = (x^\top w)^2 + \eta x^\top X_i x + (1 - \lambda_1 + \lambda_2)(x^\top w)^2 - \lambda_2 = (x^\top w)^2 + \eta x^\top X_i x + (1 - \lambda_1 + \lambda_2)(x^\top w)^2 + \lambda_2 (2(x^\top w)^2 - 1).
\]

Note that via Ky Fan’s inequality we have that

\[
\lambda_1 + \lambda_2 \leq \lambda_1 (\mathbf{w}^\top \mathbf{w}) + \lambda_2 (\mathbf{w}^\top \mathbf{w}) + \lambda_1 (\eta \mathbf{X}_t) + \lambda_2 (\eta \mathbf{X}_t) = 1 + \eta (\lambda_1 (\mathbf{X}_t) + \lambda_2 (\mathbf{X}_t)).
\]

Also, \( \lambda_2 (\mathbf{w}^\top \mathbf{w} + \eta \mathbf{X}_t) \geq \lambda_2 (\eta \mathbf{X}_t) = \lambda_2 (\mathbf{X}_t) \).

Thus, using our assumption that \((x^\top w)^2 \geq 1/2\), we have that

\[
(x^\top w')^2 \geq (x^\top w)^2 + \eta x^\top X_i x + (\lambda_1 (\mathbf{X}_t) + \lambda_2 (\mathbf{X}_t))(x^\top w)^2 + \lambda_2 (\mathbf{X}_t)(2(x^\top w)^2 - 1).
\]

Note that

\[
x^\top X_i x = x^\top (\ell \cdot \mathbf{Q} + \mathbf{V}_t + \mathbf{D}_t) x \geq x^\top (\ell \cdot \mathbf{Q} + \mathbf{D}_t) x \geq \ell \cdot x_1 (\mathbf{Q}) - \|\mathbf{D}_t\|,
\]

\[
\lambda_2 (\mathbf{X}_t) = \lambda_2 (\ell \cdot \mathbf{Q} + \mathbf{V}_t + \mathbf{D}_t) \geq \lambda_2 (\ell \cdot \mathbf{Q} + \mathbf{D}_t) \geq \ell \cdot \lambda_2 (\mathbf{Q}) - \|\mathbf{D}_t\|,
\]

\[
\lambda_1 (\mathbf{X}_t) + \lambda_2 (\mathbf{X}_t) \leq \lambda_1 (\ell \cdot \mathbf{Q}) + \lambda_2 (\ell \cdot \mathbf{Q}) + \lambda_1 (\mathbf{V}_t + \mathbf{D}_t) + \lambda_2 (\mathbf{V}_t + \mathbf{D}_t)
\]

\[
\leq \ell (\lambda_1 (\mathbf{Q}) + \lambda_2 (\mathbf{Q})) + \lambda_1 (\mathbf{V}_t) + \lambda_2 (\mathbf{V}_t) + 2\|\mathbf{D}_t\|
\]

\[
\leq \ell (\lambda_1 (\mathbf{Q}) + \lambda_2 (\mathbf{Q})) + \text{Tr}(\mathbf{V}_t) + 2\|\mathbf{D}_t\|
\]

\[
\leq \ell (\lambda_1 (\mathbf{Q}) + \lambda_2 (\mathbf{Q}) + V^2) + 2\|\mathbf{D}_t\|,
\]

where (a) follows again from Ky Fan’s inequality.

Plugging-in both the above inequalities, we have that

\[
(x^\top w')^2 \geq (x^\top w)^2 + \eta \ell (\lambda_1 (\mathbf{Q}) - (\lambda_1 (\mathbf{Q}) + \lambda_2 (\mathbf{Q}) + V^2) \cdot (x^\top w)^2 + \lambda_2 (\mathbf{Q}) \cdot (2(x^\top w)^2 - 1) - 4\ell^{-1}\|\mathbf{D}_t\|)
\]

\[
= (x^\top w)^2 + \eta \ell (1 - (\mathbf{w}^\top \mathbf{x})^2) \cdot (\lambda_1 (\mathbf{Q}) - \lambda_2 (\mathbf{Q})) - (x^\top w)^2 V^2 - 4\ell^{-1}\|\mathbf{D}_t\|
\]

\[
= (x^\top w)^2 + \eta \ell (1 - (x^\top w)^2) \cdot \delta(\mathbf{Q}) - (x^\top w)^2 V^2 - 4\ell^{-1}\|\mathbf{D}_t\|.
\]

\[\square\]
Lemma 5. Suppose that when applying Algorithm \[2\] the following conditions holds:

\[
\forall t \in [T] : \frac{1}{\ell} \|D_t\| \leq \epsilon \leq \frac{1}{18\lambda_1(Q)} (\delta(Q)^2 - V^4 - V^2\lambda_1(Q)),
\]

\[
\eta \leq \min \left\{ \frac{\epsilon}{4\ell\lambda_1(Q)} \cdot (V^2 + 4\epsilon), \frac{1}{\ell(R + V)^2} \right\},
\]

\[
\gamma \leq \min \left\{ \frac{\epsilon}{4\lambda_1(Q)}, \frac{9\eta\ell}{2} \right\},
\]

\[
(w_t^\top x)^2 \geq 1 - \frac{\delta(Q) - V^2 - 4\epsilon}{2\lambda_1(Q)}.
\]

Then, for all \(t \in [T]\), \(w_{t+1}w_{t+1}^\top = \Pi_S[W_{t+1}]\).

Proof. In light of Lemma \[3\] it suffices to show that on each iteration \(t\), it holds that

\[
(w_t^\top x)^2 \geq 1 - \frac{\delta(Q) - V^2 - 4\epsilon}{2\lambda_1(Q)}.
\]

We prove this inequality indeed holds for all \(t \in [T]\) via induction.

Note that for \(t = 1\), this clearly holds by our assumption on \(w_1\).

Suppose now the assumption holds for some \(t \geq 1\). In the following we let \(\lambda_i\) denote the \(i\)th largest eigenvalue of the matrix \(W_{t+1} := w_tw_t^\top + \eta X_t\). Note that under the induction hypothesis and Assumption \[4\], it in particular holds that \((w_t^\top x)^2 \geq 1/2\), and hence we can invoke Lemma \[4\].

We consider two cases.

If \((w_t^\top x)^2 \geq 1 - \frac{\delta(Q) - V^2 - 5\epsilon}{2\lambda_1(Q)}\), then using Lemma \[4\] we have that

\[
(w_{t+1}^\top x)^2 \geq 1 - \frac{\delta(Q) - V^2 - 5\epsilon}{2\lambda_1(Q)} - \eta\ell V^2 + 4\epsilon - \eta\ell \lambda_1 - \lambda_2
\]

\[
\geq (a) 1 - \frac{\delta(Q) - V^2 - 5\epsilon}{2\lambda_1(Q)} - \eta\ell (V^2 + 4\epsilon),
\]

were (a) follows since under the induction hypothesis, we in particular have that \(w_{t+1}w_{t+1}^\top = \Pi_S[W_{t+1}]\) (see Lemma \[3\]), which in turn implies that \(\lambda_1 \geq 1 + \lambda_2\) (see proof of Lemma \[2\]).

Thus, for any \(\eta \leq \frac{\epsilon}{32\lambda_1(Q)(V^2 + 4\epsilon)}\) we obtain

\[
(w_{t+1}^\top x)^2 \geq 1 - \frac{\delta(Q) - V^2 - \frac{9}{2}\epsilon}{2\lambda_1(Q)}.
\]

Moreover, we have that

\[
(w_{t+1}^\top x)^2 \geq (w_{t+1}^\top x)^2 - \|w_{t+1}w_{t+1}^\top - \hat{w}_{t+1}w_{t+1}\|_F \geq (w_{t+1}^\top x)^2 - \gamma
\]

\[
\geq 1 - \frac{\delta(Q) - V^2 - \frac{9}{2}\epsilon}{2\lambda_1(Q)} - \gamma.
\]

Thus, for any \(\gamma \leq \frac{\epsilon}{4\lambda_1(Q)}\) the claim indeed holds for the first case.

On the other hand, in case \((w_t^\top x)^2 < 1 - \frac{\delta(Q) - V^2 - 5\epsilon}{2\lambda_1(Q)}\), by an application of Lemma \[4\] we have that
\[
(\mathbf{w}_{t+1}^\top \mathbf{x})^2 \geq (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 + \eta \ell (1 - (\hat{\mathbf{w}}_t^\top \mathbf{x})^2) \cdot \delta(\mathbf{Q}) - (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 V^2 - 4\epsilon \geq (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 + \frac{\delta(\mathbf{Q}) - \delta(\mathbf{Q}) + V^2}{\lambda_1 - \lambda_2} (1 - \frac{\delta(\mathbf{Q}) - V^2 - 5\epsilon}{2\lambda_1(\mathbf{Q})}) - 4\epsilon.
\]

Moreover, as before, we have that
\[
(\hat{\mathbf{w}}_t^\top \mathbf{x})^2 \geq (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 + \frac{\delta(\mathbf{Q}) - V^2}{\lambda_1(\mathbf{Q})} \cdot \delta(\mathbf{Q}) - V^2 \cdot (1 - \frac{\delta(\mathbf{Q}) - V^2 - 5\epsilon}{\lambda_1(\mathbf{Q})}) - 9\epsilon.
\]

where (a) follows from our assumption on \((\hat{\mathbf{w}}_t^\top \mathbf{x})^2\) in this second case, and (b) follows, since Assumption 1 implies that \(\max_{i} \{\delta(\mathbf{Q}), V^2\} \leq \lambda_1(\mathbf{Q})\).

Thus, for any
\[
\epsilon \leq \frac{1}{18\lambda_1(\mathbf{Q})} (\delta(\mathbf{Q})^2 - V^4 - V^2\lambda_1(\mathbf{Q})),
\]

we have that
\[
(\mathbf{w}_{t+1}^\top \mathbf{x})^2 \geq (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 + \eta \frac{\delta(\mathbf{Q})^2 - V^4 - V^2\lambda_1(\mathbf{Q})}{2\lambda_1(\mathbf{Q})(\lambda_1 - \lambda_2)}.
\]

Moreover, as before, we have that
\[
(\hat{\mathbf{w}}_t^\top \mathbf{x})^2 \geq (\mathbf{w}_{t+1}^\top \mathbf{x})^2 - \gamma \geq (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 + \eta \frac{\delta(\mathbf{Q})^2 - V^4 - V^2\lambda_1(\mathbf{Q})}{2\lambda_1(\mathbf{Q})(\lambda_1 - \lambda_2)} - \gamma \geq (\hat{\mathbf{w}}_t^\top \mathbf{x})^2 + \frac{\eta \delta(\mathbf{Q})^2 - V^4 - V^2\lambda_1(\mathbf{Q})}{4\lambda_1(\mathbf{Q})} - \gamma,
\]

where (a) follows since Assumption 1 implies that \(\delta(\mathbf{Q})^2 - V^4 - V^2\lambda_1(\mathbf{Q}) \geq 0\), and since
\[
\lambda_1 - \lambda_2 \leq \lambda_1 \leq 1 + \eta \|\mathbf{x}_t\| \leq 1 + \eta \sum_{i=1}^{\ell} \|\mathbf{q}_t(i) + \mathbf{v}_t(i)\|^2 \leq 1 + \eta \ell(R + V)^2 \leq 2,
\]

where the last inequality follows from our assumption that \(\eta \leq \frac{1}{4\lambda_1(\mathbf{Q})}\). Thus, for any \(\gamma \leq \eta \frac{\delta(\mathbf{Q})^2 - V^4 - V^2\lambda_1(\mathbf{Q})}{4\lambda_1(\mathbf{Q})}\) (which in particular holds for \(\gamma \leq 9\eta\ell/2\)), we have that
\[
(\hat{\mathbf{w}}_t^\top \mathbf{x})^2 \geq (\mathbf{w}_{t+1}^\top \mathbf{x})^2 \geq 1 - \frac{\delta(\mathbf{Q}) - V^2 - 4\epsilon}{2\lambda_1(\mathbf{Q})},
\]

as needed.

**Lemma 6** (Convergence of Algorithm 2). Consider an application of Algorithm 2 to a sequence \(\{(\mathbf{x}_t^{(1)}, \ldots, \mathbf{x}_t^{(\ell)})\}_{t\in[T]}\) which follows Assumption 1, and suppose that all conditions stated in Lemma 5 hold. Then,
\[
\max_{\|\mathbf{w}\| = 1} \frac{T}{t=1} \frac{T}{i=1} (\mathbf{w}^\top \mathbf{x}_t^{(i)})^2 - \frac{T}{t=1} \frac{T}{i=1} (\hat{\mathbf{w}}_t^\top \mathbf{x}_t^{(i)})^2 \leq \frac{3\gamma T \sqrt{\ell}}{\sqrt{2\eta}} + \frac{\eta T^2 (R + V)^4}{2}.
\]
Proof. By an application of Lemma 5 it holds for all $t \in [T]$ that $w_{t+1}w_{t+1}^\top = \Pi_S[w_{t+1}]$.
Thus, using standard arguments, we have that for all $t \in [T]$ it holds that
\[
\|w_{t+1}w_{t+1}^\top - w^*w^*^\top\|^2_F \leq \|w_{t+1}^\top - w^*w^*^\top\|^2_F.
\]
\[= \|\hat{w}_t\hat{w}_t^\top - w^*w^*^\top\|^2_F + 2\eta(\hat{w}_t\hat{w}_t^\top - w^*w^*^\top) \cdot X_t + \eta^2\|X_t\|^2_F.
\]

Note that
\[
\|\hat{w}_{t+1}\hat{w}_{t+1}^\top - w^*w^*^\top\|^2_F = \|\hat{w}_{t+1}\hat{w}_{t+1}^\top + w_{t+1}w_{t+1}^\top - w_{t+1}w_{t+1}^\top - w^*w^*^\top\|^2_F
\]
\[\leq \|w_{t+1}w_{t+1}^\top - w^*w^*^\top\|^2_F + 3\sqrt{2}\|w_{t+1}w_{t+1}^\top - w_{t+1}w_{t+1}^\top\|_F
\]
\[\leq \|w_{t+1}w_{t+1}^\top - w^*w^*^\top\|^2_F + 3\sqrt{2}\gamma,
\]
where (a) follows since for any two unit vectors $y, z$ it holds that $\|yy^\top - zz^\top\|_F \leq \sqrt{2}$. Combining both of the above bounds, we obtain
\[
(w^*w^*^\top - \hat{w}_t\hat{w}_t^\top) \cdot X_t \leq \frac{1}{2\eta}(\|\hat{w}_t\hat{w}_t^\top - w^*w^*^\top\|^2_F - \|\hat{w}_{t+1}\hat{w}_{t+1}^\top - w^*w^*^\top\|^2_F) + \frac{3\sqrt{2}\gamma}{2\eta} + \frac{\eta}{2}\|X_t\|^2_F.
\]

Summing over all iterations we obtain the bound
\[
\sum_{t=1}^T w^*x_tw^\top - \sum_{t=1}^T \hat{w}_t\hat{w}_t^\top \leq \frac{1}{\eta} + \frac{3\gamma T}{2\eta} + \frac{\eta}{2}\sum_{t=1}^T \|X_t\|^2_F.
\]

Finally, note that under Assumption 1 it holds for all $t \in [T]$ that
\[
\|X_t\|_F = \|\sum_{i=1}^\ell x_t(i)x_t(i)^\top\|_F \leq \sum_{i=1}^\ell \|x_t(i)x_t(i)^\top\|_F = \sum_{i=1}^\ell \|x_t(i)\|^2
\]
\[= \sum_{i=1}^\ell \|q_t(i) + v_t(i)\|^2 \leq \ell(R + V)^2.
\]

3.1 Convergence of Algorithm 1

Lemma 7. Consider some iteration $t$ of Algorithm 1 and let $w_{t+1}$ denote the leading eigenvector of the matrix $W_{t+1} := \hat{w}_t\hat{w}_t^\top + \eta X_t$. If $\eta \leq \frac{1}{\sqrt{10}\eta(R + V)^2}$, then it holds that
\[
\|\hat{w}_{t+1}\hat{w}_{t+1}^\top - w_{t+1}w_{t+1}^\top\|_F \leq \sqrt{5}(\eta(R + V)^2)^2.
\]

Proof. Let us denote by $y_2, \ldots, y_d$ the $(d - 1)$ non-leading eigenvectors of the matrix $W_{t+1}$. Since both $w_{t+1}, \hat{w}_{t+1}$ are unit vectors, we have that
\[
\|\hat{w}_{t+1}\hat{w}_{t+1}^\top - w_{t+1}w_{t+1}^\top\|_F^2 = 2\left(1 - \langle \hat{w}_{t+1}w_{t+1}^\top, y_i \rangle^2 \right) = 2\sum_{i=2}^d (\hat{w}_{t+1}w_{t+1}^\top)^2.
\]

Note that by the update rule of Algorithm 1 the vector $\hat{w}_{t+1}$ could be written as
\[
\hat{w}_{t+1} = \frac{(I + \eta X_t)\hat{w}_t}{\|(I + \eta X_t)\hat{w}_t\|} = \frac{W_{t+1}\hat{w}_t}{\|W_{t+1}\hat{w}_t\|}.
\]
Hence, $\mathbf{w}_{t+1}$, is the result of applying a single iteration of the Power Method, initialized with the vector $\mathbf{w}_t$, to the matrix $\mathbf{W}_{t+1}$. Let us denote by $y_2, \ldots, y_d$ the $(d - 1)$ non-leading eigenvectors of $\mathbf{W}_{t+1}$. Using standard arguments, see for instance Eq. (18) in [5], we have that

$$\sum_{i=2}^{d} (\mathbf{w}_{t+1}^\top y_i)^2 \leq \sum_{i=2}^{d} (\mathbf{w}_i^\top y_i)^2 \left( \frac{\lambda_2(\mathbf{W}_{t+1})}{\lambda_1(\mathbf{W}_{t+1})} \right)^2.$$

Since $\mathbf{w}_t$ is a unit vector, we have that $\sum_{i=2}^{d} (\mathbf{w}_i^\top y_i)^2 = 1 - (\mathbf{w}_t^\top \mathbf{w}_{t+1})^2$. Moreover, we can bound

$$\frac{\lambda_2(\mathbf{W}_{t+1})}{\lambda_1(\mathbf{W}_{t+1})} = \frac{\lambda_2(\mathbf{w}_t \mathbf{w}_t^\top + \eta \mathbf{X}_t)}{\lambda_1(\mathbf{w}_t \mathbf{w}_t^\top + \eta \mathbf{X}_t)} \leq \frac{\lambda_2(\mathbf{w}_t \mathbf{w}_t^\top) + \lambda_1(\eta \mathbf{X}_t)}{\lambda_1(\mathbf{w}_t \mathbf{w}_t^\top)} \leq \eta \|\mathbf{X}_t\|,$$

where the first inequality follows from Weyl’s inequality for the eigenvalues. Thus, plugging-in into Eq. (9), we have that

$$\|\mathbf{w}_{t+1}\mathbf{w}_{t+1}^\top - \mathbf{w}_{t+1}^\top \mathbf{w}_{t+1}\|_F^2 \leq 2 \frac{1 - (\mathbf{w}_t^\top \mathbf{w}_{t+1})^2}{(\mathbf{w}_t^\top \mathbf{w}_{t+1})^2} \eta^2 \|\mathbf{X}_t\|^2.$$  \hspace{1cm} (10)

Using the Davis-Kahan sinθ theorem (see for instance Theorem 2 in [7]), we have that

$$1 - (\mathbf{w}_t^\top \mathbf{w}_{t+1})^2 = \frac{1}{2} ||\mathbf{w}_t \mathbf{w}_t^\top - \mathbf{w}_{t+1} \mathbf{w}_{t+1}^\top||_F^2 \leq \frac{2 ||\mathbf{w}_t \mathbf{w}_t^\top - \mathbf{W}_{t+1}||^2}{(\lambda_1(\mathbf{w}_t \mathbf{w}_t^\top) - \lambda_2(\mathbf{w}_t \mathbf{w}_t^\top))^2} = 2 \eta^2 ||\mathbf{X}_t||^2.$$

Plugging back into Eq. (10) and using the fact that $\eta \leq \frac{1}{\sqrt{2} ||\mathbf{X}_t||}$ (see bound on $||\mathbf{X}_t||$ in Eq. (5)), we can conclude that

$$\|\mathbf{w}_{t+1} \mathbf{w}_{t+1}^\top - \mathbf{w}_{t+1}^\top \mathbf{w}_{t+1}\|_F^2 \leq \frac{4(\eta ||\mathbf{X}_t||)^4}{1 - 2\eta^2 ||\mathbf{X}_t||^2} \leq 5\eta^4 ||\mathbf{X}_t||^4 \leq 5(\eta \ell (R + V))^4,$$

where (a) and (b) follow from our assumption on $\eta$ and the bound [8].

**Lemma 8** (Matrix Hoeffding). Under the conditions of Assumption [19], it holds for all $t \in [T]$ and for all $\epsilon > 0$ that

$$\Pr \left( \|\frac{1}{\ell} \mathbf{D}_t\| \geq \epsilon \right) \leq 2d \cdot \exp \left( - \frac{\epsilon^2 \ell}{128 R^4} \right).$$

**Proof.** By a straightforward application of the Matrix Hoeffding inequality (see for instance [19]), we have for any fixed $t \in [T]$ that

\[
\Pr (\|\frac{1}{\ell} (\mathbf{Q}_t - \ell \cdot \mathbf{Q})\| \geq \epsilon) \leq d \cdot \exp \left( - \frac{\epsilon^2 \ell}{32 R^2} \right), \\
\Pr (\|\frac{1}{\ell} \mathbf{M}_t\| \geq \epsilon) \leq d \cdot \exp \left( - \frac{\epsilon^2 \ell}{32 V^2 R^2} \right).
\]

Thus, the lemma follows from applying both of the above bounds with parameter $\epsilon/2$ and noting that $V^2 \leq R^2$.  \hspace{1cm} $\square$
We can now finally prove Theorem 1.

**Proof.** The proof follows from straightforward application of the tools we have developed thus-far. We assume for simplicity that $N = T \cdot \ell$ for our choice of $\ell$. Note this is without loss of generality, since the remainder $(N - \ell \cdot \lfloor N/\ell \rfloor)$ affects the bound in the theorem only via lower-order terms.

Let us define $\epsilon := \frac{\delta(Q)^2 \cdot 2 + \lambda_1(Q)}{18 \lambda_1(Q)}$, and note this choice corresponds to the RHS of Eq. (7). Thus, for a certain $\ell = O(\ell^d \log \frac{\ell}{p})$, we have by an application of Lemma 8 that with probability at least $1 - p$ it holds for all $t \in [T]$ that $\frac{1}{T} ||D_t|| \leq \epsilon$. Define $\gamma := \sqrt{5} (\eta \ell (R + V)^2)$, where $\eta$ is the chosen learning rate. Note that for a large enough $N$, all parameters $\epsilon, \eta, \gamma, \hat{w}_1$ satisfy the conditions of Lemma 6 with probability at least $1 - p$, and thus, by invoking Lemma 6 we have that with probability at least $1 - p$ that

$$\lambda_1 \left( \sum_{i=1}^N x_i x_i^\top \right) - \sum_{t=1}^\ell \sum_{i=1}^\ell (\hat{w}_t x_i)^2 \leq \frac{3\gamma T}{\sqrt{2\eta}} + \frac{\eta T \ell^2 (R + V)^4}{2} \leq O \left( \sqrt{T \ell (R + V)^2} \right) = O \left( \sqrt{N \ell (R + V)^2} \right),$$

where (a) follows from plugging the value of $\eta, \gamma$ and (b) follows since $N = T \cdot \ell$. The theorem now follows from plugging-in the bound on $\ell$. \hfill \Box

### 3.2 Proof of Lemma 1 (”warm-start”)

**Proof.** Let $\hat{w}_1$ be the leading eigenvector of the normalized covariance $\hat{X} = \frac{1}{n} \sum_{i=1}^n x_i x_i^\top$, where for all $i \in [n] \ x_i = q_i + \nu_i$. Clearly, $E[\hat{X}] = Q + \frac{1}{n} \sum_{i=1}^n \nu_i \nu_i^\top$, and thus $||Q - \hat{X}||^2 \leq 2 ||Q - E[\hat{X}]||^2 + 2 ||\hat{X} - E[\hat{X}]||^2 = 2 \frac{1}{n} \sum_{i=1}^n ||\nu_i \nu_i^\top||^2 + 2 \Delta^2$, where we use the notation $\Delta = ||\hat{X} - E[\hat{X}]||$. Via the Davis-Kahan $\sin \theta$ theorem (see for instance Theorem 2 in [7]) and using the short notation $\delta = \delta(Q)$, we have that

$$(\hat{w}_1 x)^2 = 1 - \frac{1}{2} ||\hat{w}_1 \hat{w}_1^\top - xx^\top||^2 \geq 1 - 2 \frac{||X - Q||^2}{\delta^2} \geq 1 - \frac{4V^4}{\delta^2} - 4 \Delta^2.$$

Now, using the short notation $\lambda_1 = \lambda_1(Q)$, the warm-start condition in Theorem 1 boils down to the condition

$$\frac{4V^4 + \Delta^2}{\delta^2} \leq \frac{\delta - V^2}{2 \lambda_1} \iff 8 \lambda_1 V^4 + \delta^2 V^2 + 8 \lambda_1 \Delta^2 - \delta^3 \leq 0.$$

Solving the above inequality for $V^2$ we obtain the solution interval:

$$0 \leq V^2 \leq \frac{-\delta^2 + \sqrt{\delta^4 + 32 \lambda_1 \delta^4 - 256 \lambda_1^2 \Delta^2}}{16 \lambda_1}.$$

In particular, for

$$\Delta \leq \frac{\sqrt{7} \delta^{3/2}}{16 \lambda_1},$$

(11)
we obtain the feasible sub-interval:

\[
0 \leq V^2 \leq \frac{\delta^3/2}{4\sqrt{\lambda_1}},
\]

which is equivalent to the requirement \(\delta(Q) \geq (16\lambda_1(Q)V^4)^{1/3}\). We conclude the proof with the simple observation that using a standard Matrix Hoeffding concentration bound (see for instance Lemma 8 in the sequel), it suffices to take \(n = O(\frac{(R+V)^3\lambda_1 \log(d/p)}{\delta^3})\) for the bound in (11) to hold with probability at least \(1 - p\).

\[\square\]

4 Experiments

We test the following algorithms. Algorithm 2 with block-size \(\ell = 1\), where \(\hat{\mathbf{w}}_{t+1}\) is computed via rank-one SVD (R1-OGA), a similar algorithm which uses non-unit block-size \(\ell > 1\) (BR1-OGA), the non-convex online gradient ascent, Algorithm 4 with unit block-size \(\ell = 1\) (Nonconvex-OGA), and the convex online gradient ascent (equivalent to Algorithm 4 but uses accurate Euclidean projections onto the spectrahedron) with unit block-size (Conv-OGA). Since computing that exact projection for Conv-OGA via a full SVD is highly time consuming, we approximate it by extracting only the five leading components. Finally, we record the regret of the initial “warm-start” vector \(\hat{\mathbf{w}}_1\) (BaseVec), which serves as the initialization for all algorithms. For all datasets we plot for each iteration \(t\) the average-regret against the leading eigenvector in hindsight (w.r.t. all data) up to time \(t\).

We consider the following three datasets.

**Synthetic:** a random dataset is constructed by generating Gaussian zero-mean data with a random covariance matrix \(\mathbf{Q}\) with eigenvalues \(\lambda_i = 15 \cdot 0.3^{i-1}\) for all \(i \in [d]\), and perturbing them using independent Gaussian zero-mean noise with random covariance matrix \(\mathbf{V}\) with eigenvalues \(\mu_i = 3 \cdot 0.3^{i-1}\) for all \(i \in [d]\), where we use \(d = 100\). We set the number of data points to \(N = 10000\), and we compute the initialization \(\hat{\mathbf{w}}_1\) for all algorithms by computing the leading eigenvector of a sample of size 100 (i.e., 1% of \(N\)) based on samples from the covariance \(\mathbf{Q}\). For the algorithm BR1-OGA we set \(\ell = 10\). We average the results of 30 i.i.d. experiments.

**MNIST:** we use the training set of the MNIST handwritten digit recognition dataset [13] which contains 60000 28x28 images, which we split into \(N = 59400\) images for testing, while 600 images (i.e., 1% of data) are used to compute the initialization \(\hat{\mathbf{w}}_1\). For the algorithm BR1-OGA we set \(\ell = 5\).

**CIFAR10:** we use the CIFAR10 tiny image dataset [12] which contains 50000 32x32 images in RGB format. We convert the images to grayscale and use \(N = 49900\) images for testing and 100 images (i.e., 0.2% of data) are used to compute the initialization. For BR1-OGA we set \(\ell = 5\).

The results for all three datasets are given in figure 1. It can be seen that indeed all algorithms improve significantly over the “warm-start” base vector. We also see that all algorithms indeed attain low average-regret, and in particular are competitive with OGA which follows a convex approach (up to the approximation of the projection via thin SVD).

To further examine the applicability of our theoretical approach, for all datasets, we recorded for algorithm BR1-OGA the fraction of projection errors, i.e., the percent of number of iterations \(t\) on which the projection of the matrix \(\mathbf{W}_{t+1} = \hat{\mathbf{w}}_t\hat{\mathbf{w}}_t^\top + \eta\mathbf{X}_t\) onto the
spectrahedron $S$ is not a rank-one matrix. The results are 6.24%, 0.26%, 0%, for synthetic, MNIST and CIFAR10, respectively. These low error rates indeed support our theoretical analysis which hinges on showing that under our data model (recall Assumption 1) and given a "warm-start" initialization, the projections of the matrices $W_t$ in Algorithm 2 are always rank-one.

5 Discussion

In this paper we took a step forward towards understanding the ability of highly-efficient non-convex online algorithms to minimize regret in adversarial online learning settings. We focused on the particular problem of online principal component analysis with $k = 1$, and showed that under a "semi-adversarial" model, in which the data follows a stochastic distribution with adversarial perturbations, and given a “warm-start” initialization, the natural nonconvex online gradient ascent indeed guarantees sublinear regret. Our theory is further supported by empirical evidence.

We hope this work will motivate further research on online nonconvex optimization with global convergence guarantees. Future directions of interest may include extending our analysis to an even wider regime of parameters and extracting $k$ principal components at once. Also, it is interesting if in the standard adversarial setting, it can be shown that online nonconvex gradient ascent achieves low-regret, or on the other-hand, to show that there exist instances on which it cannot guarantee non-trivial regret. Finally, moving beyond PCA, other online learning problems of interest that may benefit from a non-convex approach include online matrix completion [9, 11], and of course, provable online learning of deep networks.

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