Quantum Dot circuit-QED thermoelectric diodes and transistors

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Recent breakthroughs in quantum-dot circuit-quantum-electrodynamics (circuit-QED) systems are important both from a fundamental perspective and from the point of view of quantum photonic devices. However, understanding the applications of such setups as potential thermoelectric diodes and transistors has been missing. In this paper, via the Keldysh nonequilibrium Green’s function approach, we show that cavity-coupled double quantum-dots can serve as excellent quantum thermoelectric diodes and transistors. Using an enhanced perturbation approach based on polaron-transformations, we find non-monotonic dependences of thermoelectric transport properties on the electron-photon interaction. Strong light-matter interaction leads to pronounced rectification effects for both charge and heat, as well as thermal transistor effects in the linear transport regime, which opens up a cutting-edge frontier for quantum thermoelectric devices.

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Introduction.— Recently, there has been a flurry of activities and progress in probing and controlling hybrid light-matter systems which sit at the confluence of mesoscopic physics and quantum optics.1–7 Few examples of such hybrid light-matter systems include quantum-dot (QD) circuit-Quantum ElectroDynamics (c-QED) systems8–16, cold atoms coupled to light17, and optomechanical devices18–24. A rich setup is a class of recent experiments where QDs at finite voltage bias have been integrated with superconducting microwave resonators12,25–28, accomplishing sufficiently strong light-matter coupling. Such QD-c-QED systems offer a rich platform for studying nonequilibrium open quantum systems. Experiments are versatile, tunable (large windows of parameters) and scalable. These QD c-QED setups are important both from a fundamental perspective (investigating correlations, transport, entanglement, bosonic statistics) and from the point of view of device applications (quantum microwave amplifiers and lasers in microwave regime). From the perspective of devices, the focus and success until now has been on realizing photon emitters, microwave amplifiers and even single-atom lasers. However, there has been no work on investigating these systems as potential quantum diodes and rectifiers30–33 which is the aim of this paper.

Manipulation and separation of thermal and electrical currents at mesoscopic scales are of fundamental interest and have technological impact on high-performance thermoelectric devices.5,31,34–52. In this paper, we investigate the inelastic thermoelectric transport assisted by microwave photons residing in the cavity, as well as elastic tunneling transport. The strong light-matter interaction provides an excellent avenue for realizing quantum thermoelectric devices. By employing the non-equilibrium Green’s function approach,53–55 we show that due to the nonlinearity induced by the electron-photon interaction, significant charge and thermal rectification effects can be realized by properly tuning the QDs energy. We further show that these QD c-QED setups exhibit thermal transistor effects even in the linear transport regime, and thus provide a platform for unprecedented thermal control.

Model.— As schematically depicted in Fig. 1, we consider double QDs (DQD) that is connected to two electronic reservoirs and a photonic bath. The QDs are defined with tunable electronic energy levels $E_l$ and $E_r$ by local gate-voltages. $t$ is the tunneling between the QDs, $\Gamma_L$ and $\Gamma_R$ are the hybridization energies of the dots to the source and drain electrodes (labeled by $L$ and $R$, respectively), respectively. Charge current, electronic heat current, and photonic heat current are induced by applying a voltage bias between the terminals $L$ and $R$ and a temperature difference between the three terminals. The system is described by the Hamiltonian,

$$\hat{H} = \hat{H}_{c-DQD} + \hat{H}_{lead} + \hat{H}_{dot-lead},$$

with

$$\hat{H}_{c-DQD} = \hat{H}_{DQD} + \hat{H}_p + \hat{H}_{e-p}.$$  \hspace{1cm}(2)

Note that in the above equation, $H_{c-DQD}$ represents the composite particle comprised of double QDs ($H_{DQD}$), the single-mode cavity photon ($H_p$) and their coupling ($H_{e-p}$) as elucidated below,

$$\hat{H}_{DQD} = \sum_{i=L,R} E_i \hat{d}_i \hat{d}_i^\dagger + (t \hat{d}_L^\dagger \hat{d}_R + H.c.),$$

$$\hat{H}_{e-p} = g \omega_c (\hat{d}_i^\dagger \hat{d}_i + \hat{d}_r^\dagger \hat{d}_r)(\hat{a} + \hat{a}^\dagger),$$

$$\hat{H}_p = \omega_c \hat{a} \hat{a}^\dagger.$$  \hspace{1cm}(3c)

Here $\hat{d}_i$ creates an electron in the $i$th QD with an energy $E_{i}$, and $\hat{a}$ creates an electron in the photonic mode with frequency $\omega_c$. The $l(r)$ QD is located next to and strongly coupled
Drain

Source

Drain

Source

FIG. 1: (Color online) (a) A schematic representation of the model. The mesoscopic system is effectively housed in the microwave cavity. Wavy lines indicate the light-matter coupling $g$. Tunneling rates between the dots and the electron leads ($\Gamma_L, \Gamma_R$) and in between the dots ($t$) can be tuned via gate-controlled tunnel barriers. Electrons travel from source into the first QD (with energy $E_L$) and then hop to the second QD (with a different energy $E_r$) assisted by a photon from the photonic bath. (b) Illustration of possible photon-assisted inelastic transport processes. (c) Illustration of possible elastic transport processes. Here $E_d$ and $E_D$ are the QDs energy after hybridization. (d) Elastic and inelastic electric conductance $G$ as a function of $g$ for different $\mu$, where $E_l = 0$, $E_r = 1\omega_c$, $\Gamma_0 = 0.1\omega_c$, $k_B T_L = k_B T_R = k_B T_p = k_B T = 0.1\omega_c$, $t = 0.3\omega_c$ and $g = 0.1$.

with the left (right) lead. The tunneling elements from the $l^{th}$ QD to the right lead and that from the $r^{th}$ QD to the left lead are assumed negligible. $\hat{a}^\dagger$ and $\hat{a}$ create and annihilate a photon with energies $\omega_c$ (we set $\hbar = 1$).

The last term describes the light-matter interactions with dimensionless strength $g$. The Hamiltonians

$$\hat{H}_{\text{lead}} = \sum_{j=L,R} \sum_k \epsilon_{j,k} \hat{d}^\dagger_{j,k} \hat{d}_{j,k},$$

$$\hat{H}_{\text{dot-lead}} = \sum_k V_{L,k} \hat{d}^\dagger_{L,k} + \sum_k V_{R,k} \hat{d}^\dagger_{R,k} + \text{H.c.}$$

describe the electronic leads and the tunneling between the QDs and the leads, respectively.

We first diagonalize the Hamiltonian $\hat{H}_{-DQD} = \hat{H}_{DQD} + \hat{H}_p + \hat{H}_{e-p}$, and write it in terms of a new set of electronic operators $\hat{D} = \sin \theta \hat{d}^\dagger + \cos \theta \hat{d}$ and $\hat{d} = \cos \theta \hat{d}^\dagger - \sin \theta \hat{d}$, where $\theta = \frac{1}{2} \arctan \left( \frac{2E_c}{\epsilon} \right)$ and $\epsilon \equiv E_r - E_l$.

The corresponding levels are $E_D = \frac{E_c + E_l}{2} + \sqrt{\frac{\epsilon^2}{4} + t^2}$ and $E_d = \frac{E_c + E_l}{2} - \sqrt{\frac{\epsilon^2}{4} + t^2}$. Using these operators, the Hamiltonian can be written as $\hat{H}_{-DQD} = E_D \hat{D}^\dagger \hat{D} + E_d \hat{d}^\dagger \hat{d} + \omega_c \hat{b}^\dagger \hat{b} + g (\hat{D}^\dagger \hat{D} + \hat{d}^\dagger \hat{d})(\hat{b}^\dagger + \hat{b})$. By employing $\Gamma_\alpha(\omega) = 2\pi \sum_k |V_{\alpha k}|^2 \delta(\omega - \epsilon_{k\alpha})$, the couplings rates between leads and QDs (in local basis $d^\dagger_0|0\rangle$ and $d^\dagger_0|0\rangle$) become

$$\hat{\Gamma}^L = \left( \begin{array}{ccc} \Gamma_L & 0 & 0 \\ 0 & 0 & \Gamma_R \end{array} \right).$$

The unitary transformation matrix between the local basis and the diagonal basis is $U = \left( \begin{array}{cc} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{array} \right)$.

Hence, the tunnel coupling matrix between QDs and leads in the diagonal basis become

$$\hat{\Gamma}_r^{\text{rot}} = U \hat{\Gamma}^L U^\dagger = \Gamma_L \left( \begin{array}{cc} \sin^2 \theta & -\cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{array} \right),$$

$$\hat{\Gamma}_r^{\text{rot}} = U \hat{\Gamma}^R U^\dagger = \Gamma_R \left( \begin{array}{cc} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{array} \right).$$

To break the left-right mirror symmetry and induce efficient energy filtering, we set the tunnel coupling to be Lorentzian (7): $\Gamma_L = \Gamma_0 \frac{\Gamma_0^2}{(\omega - E_l)^2 + \Gamma_0^2}$, $\Gamma_R = \Gamma_0 \frac{\Gamma_0^2}{(\omega - E_r)^2 + \Gamma_0^2}$.

After obtaining the c-DQD Green’s functions (see Appendix A) the elastic and inelastic currents can be calculated (see Appendix B) as ($e < 0$ is the electronic charge)

$$I^L_e|_{\text{el}} = e \int \frac{d\omega}{2\pi} \text{Tr}(\hat{G}^L_r(\omega) \hat{G}_r \hat{P}_r(\omega) \hat{G}^r \hat{P}^r \hat{G}^L \hat{P}^L) \hat{G}^L \hat{P}^L),$$

$$I^L_e|_{\text{inel}} = e \int \frac{d\omega}{2\pi} \text{Tr}(\hat{G}^L_r(\omega) \hat{G}_r \hat{P}_r(\omega) \hat{G}^r \hat{P}^r \hat{G}^L \hat{P}^L),$$

The elastic and inelastic heat currents flowing into the $L$ lead is calculated as

$$I^L_Q|_{\text{el}} = \int \frac{d\omega}{2\pi} (\omega - \mu_L) \text{Tr}(\hat{G}^L_r(\omega) \hat{G}_r \hat{P}_r(\omega) \hat{G}^r \hat{P}^r \hat{G}^L \hat{P}^L),$$

$$I^L_Q|_{\text{inel}} = \int \frac{d\omega}{2\pi} (\omega - \mu_L) \text{Tr}(\hat{G}^L_r(\omega) \hat{G}_r \hat{P}_r(\omega) \hat{G}^r \hat{P}^r \hat{G}^L \hat{P}^L),$$

where $I_L = [\text{e}^{(\omega - \mu_L)/k_B T_L} + 1]^{-1}$ is the Fermi-Dirac distributions for $L$ reservoir. For the charge and heat currents flowing into the $R$ lead, the same expressions hold once $L \rightarrow R$.

Here, $\mu_L(R) = \mu \pm \Delta \mu/2$, $\mu$ is the equilibrium chemical potential and $\Delta \mu$ is the electrochemical potential bias. Charge conservation implies that $I^L_e + I^R_e = 0$, while energy conservation requires $I^L_Q + I^R_Q + \mu_L I^L_e/e + \mu_R I^R_e/e = 0$. The net charge
current flowing from the left reservoir to the right one is then
\[ I_e = \frac{1}{2}(I_e^R - I_e^L). \] (10)

The heat current flowing into the photonic bath is
\[ I_Q^R = -(I_Q^L + I_Q^R) + \frac{\mu_R}{e} I_e^L + \frac{\mu_R}{e} I_e^R, \] (11)
and the net heat current exchanged between the \( L \) and \( R \) leads is
\[ I_Q = \frac{1}{2}(I_Q^R - I_Q^L). \] (12)

Our results for c-DQD Green’s function and elastic currents in this paper are non-perturbative in electron-photon coupling \( g \) and valid for small dot-lead coupling \( \Gamma_{L(R)} \) (compared to all other energiescales) as proved in Ref. 35. We treat inelastic currents in a perturbative way, i.e., we include diagrams in Fig. 5. However, as shown in Ref. 35, the elastic currents inelastic electric conductance, the elastic electric conductance on \( g \), i.e., we include diagrams in Fig. 5. However, as shown in Ref. 35, the elastic currents inelastic electric conductance, the elastic electric conductance on \( g \), do modify the elastic electric conductance, mainly due to two factors (as shown in Ref. 35): the shift of the electronic energy due to polaron renormalization which is proportional to \( g^2 \) and the side-bands effect. These two effects are sensitive to the chemical potential which determines the distribution on the main peak and the side-bands, leading to various dependences of the elastic electric conductance on \( g \). The dependence of the inelastic electric conductance on \( g \) is proportional to \( g^2 \) for small \( g \). The dependence on \( g \) becomes stronger when \( g > 0.03 \), i.e., at the crossover between weak-coupling and strong-coupling regimes.

Thermoelectric rectification effects.— Beside the conventional charge and heat rectification effects, coupled charge and heat transport also allows cross-rectification effects such as, charge rectification induced by temperature differences (Here we term it as Seebeck rectifications, since it corresponds to the asymmetry of the Seebeck effect with respect to forward and backward temperature differences) and heat rectification induced by voltage biases (Peltier rectifications) \(^{31} \). To the best of our knowledge, there is no study on such cross-rectification effects in c-QED systems. The magnitude of the rectification effects is calibrated by
\[ R_{te} = \frac{I_e(\Delta \mu) + I_e(-\Delta \mu)}{|I_e(\Delta \mu)| + |I_e(-\Delta \mu)|}. \] (13)

for the charge rectification, and
\[ R_{te} = \frac{I_Q(\Delta \mu) + I_Q(-\Delta \mu)}{|I_Q(\Delta \mu)| + |I_Q(-\Delta \mu)|}, \] (14)
for the Peltier rectification. A typical electrochemical potential difference \( \Delta \mu \) for pronounced rectification effects is comparable with \( k_B T \).

In Fig. 2, we demonstrate and study the charge and Peltier rectifications. The asymmetric charge and heat transport with respect to the forward and backward voltage biases are shown in Figs. 2(a) and 2(b). We find that the asymmetry is induced by the inelastic transport processes. As shown in Ref. 35, the elastic currents
are anti-symmetric with respect to forward and backward voltage and temperature biases. Since the asymmetry only arises from the inelastic transport, the light-matter interaction plays the essential role for both charge and Peltier rectifications. Strong light-matter interaction leads to strong rectification effects. Figs. 2(c)-2(d) give the dependences of the rectification effects on the QDs energies. There are hot-spots for charge and Peltier rectifications which are distributed symmetrically around the line $E_l = E_r$ as has been shown in Ref. 31 for the weak-coupling regime. However, the symmetry around the $E_l = -E_r$ line is broken because of the polaron energy shift, e.g., $E_D \rightarrow E_D - g^2\omega_c^2$, and the side band effects. These figures also show that rectification effects can be enhanced (the peak values are increased and/or the area of the hot-spots are increased) by increasing the light-matter coupling.

Fig. 3 shows the dependences of the charge and Peltier rectifications on the light-matter coupling $g$ for various QDs energies. General trends can be observed: for small $g$, the dependences follow a power-law $\sim g^{\nu+\mu}$ with $\nu$ close to 0.8 but depends on specific temperatures and electrochemical potential differences; for large $g$, the power-law dependences are not valid any more, leading generally to non-monotonic dependences. The power-law exponents are seemingly independent of the QDs energies. These power-law dependences indicate that rectifications are induced by high-order electron-photon interaction effects which requires generally strong light-matter interaction.

**Thermal transistor effects in the linear transport regime.**— It was accepted for a long time that non-linear transport is the prerequisite for thermal transistor effects. It was first argued in Ref. 31 that thermal transistor effects can emerge in the linear-transport regime if phonon-assisted transport is considered. However, the rate equation method used in Ref. 31 is valid only for the weak-coupling regime. Here we show, using more rigorous Green’s function method that such linear thermal transistor effect also exists in c-QED systems. If we consider pure thermal conduction (i.e., the electrochemical potential difference is set to zero), the linear thermal transport properties of the system is given by

\[
\begin{pmatrix}
I_P^n \\
I_Q^n
\end{pmatrix} =
\begin{pmatrix}
K_{PP} & K_{PR} \\
K_{RP} & K_{RR}
\end{pmatrix}
\begin{pmatrix}
T_P - T_L \\
T_R - T_L
\end{pmatrix} =
\begin{pmatrix}
\partial T_P I_Q^R \\
\partial T_P I_Q^R
\end{pmatrix},
\]

where $K_{PP} = \partial I_P / \partial T_P$, $K_{PR} = \partial I_P / \partial T_R$, $K_{RP} = \partial I_R / \partial T_P$ and $K_{RR} = \partial I_R / \partial T_R$ in the limit $T_L, T_R, T_P \to T$. From the above, the heat current amplification factor is given by

\[
\alpha = \left| \frac{\partial T_P I_Q^R}{\partial T_P I_Q^R} \right| = \frac{K_{RP}}{K_{PP}}.
\]

As schematically illustrated in Figs. 4(a) and 4(b), the condition for thermal transistor is $\alpha > 1^{31}$. In Figs. 4(c) and 4(d), we find that the coefficient $\alpha$ is very sensitive to the QDs energies which can be controlled easily via gate-voltages in experiments. In particular, there are hot-spots for $\alpha$ to be considerably larger than 1. Detailed dependences of the heat currents and $\alpha$ on the QD energy $E_l$ is shown in Fig. 4(c). It is shown that pronounced thermal transistor effect can be achieved at considerably large heat currents if light-matter coupling $g$ is strong. In general, strong light-matter interaction helps the thermal transistor effect in the linear-transport regime. It also enhances the heat currents significantly.

**Concluding remarks.**— It is shown that QD systems placed at finite voltage bias and integrated with a superconducting c-QED architecture can serve as excellent charge and Peltier rectifiers (diodes). Thermal transistor effects in the linear transport regime is also found thanks to photon-assisted inelastic transport. Although the paper is primarily discussed for a QD c-QED architecture, our results are very applicable to molecular junctions as well where the role of photons is played by the molecular vibrations. However, we have not considered the role of nanowire and substrate phonons. Future work will involve understanding the role of phonons and studying the impact of onsite and...
inter-site Coulomb interactions \(^{72-74}\).

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Appendix A: Non-perturbative hybridized dot Green’s function

We start by analytically solving the eigenproblem for the DQD c-QED model. Following the relation \( G^r(t) = \Theta(t)(G^r(t) - G^c(t)) \), \( G^c(t) = -\Theta(-t)(G^r(t) - G^c(t)) \), and utilizing \( \Theta(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \), we have the retarded (advanced) Green’s function:

\[
G^{r,a}_{0D}(\omega) = \int \frac{d\omega}{2\pi} \int \frac{d\omega_1}{2\pi} \int dt e^{i\omega t} e^{-i(\omega_1-\omega)t} \times [G^{r,a}_{0D}(\omega_1) - G^{r,a}_{0D}(\omega_1)],
\]

(A1)

Following the method of Ref.[35], we first detail the calculation of the lesser Green’s function

\[
G^\prime_{0D}(\omega) = i \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle d^\dagger_0(t) d(t) \rangle
= i \int_{-\infty}^{+\infty} dt e^{i\omega t} \sum_\varphi \sum_\psi \langle \varphi | \rho d^\dagger_0(0) | \psi \rangle \
\times \langle \psi | e^{iH_{c-DQD}t} d_0(0) e^{-iH_{c-DQD}t} | \varphi \rangle,
\]

(A2)

where \( \rho = e^{-\beta H_{c-DQD}} / Z \) with \( Z = \text{Tr}(e^{-\beta H_{c-DQD}}) \). Here \( | \varphi \rangle \) and \( | \psi \rangle \) are the possible eigenstates.

We introduce a cavity photon basis with displacements shifted by different QD states through the e-p coupling \(^{35}\)

\[
|n\rangle_v = \frac{|\hat{A}_v^\dagger|^n}{\sqrt{n!}} \exp \left(-g_{c,\hat{a}}^2/2 - g_{c}\hat{a}\right)|0\rangle,
\]

(A3)

where \( \hat{A}_v = \hat{a}^\dagger + g_{c} \) denotes the creator that creates a photon displaced from the original position by a value \( g_{c} \) depending on the electronic state, that is, \( g_{0} = 0 \), \( g_{D} = g_{d} = g \), and \( g_{DD} = g_{D} + g_{d} = 2g \). Therefore, with the help of the cavity photon basis, the solution to the eigenvalue problem is

\[
\begin{align}
0 |0, n\rangle_H_{c-DQD} |0, n\rangle &= n\omega_{c}, \\
D_{0}(D, |H_{c-DQD}D, n\rangle D) &= \omega_{c} + \tilde{E}_{D}, \\
d_{d}(n, |H_{c-DQD}d, n\rangle d) &= \omega_{c} + \tilde{E}_{d}, \\
d_{d}(Dd, |H_{c-DQD}Dd, n\rangle Dd) &= \omega_{c} + \tilde{E}_{Dd},
\end{align}
\]

(A4)-(A7)

where \( \tilde{E}_{D} = \tilde{E}_{D} - \omega_{c}g_{D}, \tilde{E}_{d} = \tilde{E}_{d} - \omega_{c}g_{d}, \) and \( \tilde{E}_{Dd} = \tilde{E}_{D} + \tilde{E}_{d} - 2\omega_{c}g_{D}g_{d} \). Obviously, \(|0, n\rangle_0, \langle D, n\rangle_D, \langle d, n\rangle_d, \langle Dd, n\rangle_{Dd}\) are four possible eigenstates and \( \omega_{v}, \omega_{c} + \tilde{E}_{D}, \omega_{c} + \tilde{E}_{d}, \omega_{c} + \tilde{E}_{Dd} \) are the corresponding possible eigenvalues.

There are only two nonzero combinations for calculating \( G_{0D}^\prime(\omega) \): \( |D, n\rangle_D \) and \( |0, m\rangle_0 \), or \( |Dd, n\rangle_{Dd} \) and \( |d, m\rangle_d \), and we

\[
\begin{align}
G_{0D}^\prime(\omega) &= \frac{2\pi i}{Z} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \delta(\omega - (n - m)\omega_{c} - \tilde{E}_{D}) \times e^{-\beta(n\omega_{c} + \tilde{E}_{D})} |D,n\rangle_0 \langle m|_0 D + \delta(\omega - (n - m)\omega_{c} - (\tilde{E}_{Dd} - \tilde{E}_{d})) \times e^{-\beta(n\omega_{c} + \tilde{E}_{Dd})} |Dd,n\rangle_{Dd} \langle m|_d Dd \right].
\end{align}
\]

(A8)

The detailed expression of \( \langle v|n\rangle_{c,\hat{a}} \), denoting the inner product of modified phonon states with effective displacements \( g_{b} \) and \( g_{c} \), can be derived as follows:

\[
\langle v|n\rangle_{c,\hat{a}} = \left( \frac{\langle \hat{a} + g_{b}\rangle^n}{\sqrt{n!}} \exp\left(-g_{b}^2/2 - g_{c}\hat{a}\right) \langle \hat{a} + g_{b}\rangle^n \right) \left( \frac{\langle \hat{a} + g_{c}\rangle^m}{\sqrt{m!}} \exp\left(-g_{c}^2/2 - g_{c}\hat{a}\right) \langle \hat{a} + g_{c}\rangle^m \right) = \exp\left(-\langle \hat{a} + g_{c}\rangle^2/2 \right) \langle \hat{a} + g_{c}\rangle^m \langle \hat{a} + g_{b}\rangle^n \right) \left( \frac{\langle \hat{a} + g_{c}\rangle^m}{\sqrt{m!}} \exp\left(-g_{c}^2/2 - g_{c}\hat{a}\right) \langle \hat{a} + g_{c}\rangle^m \right) = \exp\left(-\langle \hat{a} + g_{c}\rangle^2/2 \right)
\]

where

\[
D_{nm} = \frac{(-1)^n D_{nm}(g_{b} - g_{c})}{\sqrt{n!m!}}
\]

is invariant under the exchange of indices \( n, m \). Note, to get the third equivalence, we utilized the relation \( \exp(c\hat{a}) f(\hat{a}^\dagger + c, \hat{a}) = f(\hat{a}^\dagger + c, \hat{a}) \).
Therefore, the lesser Green’s function can be further reduce to:

\[
G_{0D}^<(\omega) = \frac{2\pi i}{Z} \sum_{n,m=0}^{\infty} \left[ \delta(\omega - \Delta_{nm}^{(1)})e^{-\beta(n\omega_0 + \bar{E}_D)} + \delta(\omega - \Delta_{nm}^{(2)})e^{-\beta(n\omega_0 + \bar{E}_{Dd})} \right] D_{nm}^2(g_D), \tag{A10}
\]

where

\[
\Delta_{nm}^{(1)} = (n - m)\omega_c + \bar{E}_D, \\
\Delta_{nm}^{(2)} = (n - m)\omega_c + (\bar{E}_D - 2\omega_c g_D g_d), \\
D_{nm}(g_D) = e^{-\mathcal{g}_D^{2}/2} \sum_{k=0}^{\min(n,m)} \frac{(-1)^k \sqrt{n!m!}g_D^{n+m-2k}}{(n-k)!(m-k)k!}, \\
Z = (1 + N_P)(1 + e^{-\beta\bar{E}_D} + e^{-\beta\bar{E}_d} + e^{-\beta\bar{E}_{Dd}}).
\]

Here \(N_P = 1/(e^{\beta\omega_c} - 1)\) denotes the Bose distribution of the photon population with inverse temperature \(\beta \equiv 1/k_B T_p\) and \(\bar{E}_D = E_D - \omega_c g_D g_d\), \(\bar{E}_d = E_d - \omega_c g_d^2\) and \(\bar{E}_{Dd} = \bar{E}_D + \bar{E}_d - \omega_c g_D g_d\).

Similarly, for the greater Green’s function, we can obtain

\[
G_{0D}^>(\omega) = -i \int dt e^{i\omega t} \langle d_D(t) d_D^\dagger(0) \rangle = -\frac{2\pi i}{Z} \sum_{n,m=0}^{\infty} \left[ \delta(\omega - \Delta_{nm}^{(1)})e^{-\beta m\omega_c} + \delta(\omega - \Delta_{nm}^{(2)})e^{-\beta(n\omega_0 + \bar{E}_d)} \right] D_{nm}^2(g_D), \tag{A11}
\]

Substituting the expressions of the greater and lesser Green’s functions into Eq.(A10) we get the advanced and retarded Green’s functions of the c-DQD

\[
G_{0D}^{>/(a)}(\omega) = \frac{1}{Z} \sum_{n,m=0}^{\infty} \left[ \frac{e^{-\beta m\omega_c} + e^{-\beta(n\omega_0 + \bar{E}_D)}}{\omega - \Delta_{nm}^{(1)} \pm i0^+} + \frac{e^{-\beta(n\omega_0 + \bar{E}_D)} + e^{-\beta(n\omega_0 + \bar{E}_d)}}{\omega - \Delta_{nm}^{(2)} \pm i0^+} \right] D_{nm}^2(g_D), \tag{A12}
\]

**Appendix B: Inelastic and Elastic Currents**

With the Green’s functions, we can now study the quantum transport by calculating the charge current

\[
I^c_e = e \frac{d}{dt} \langle \sum_k \hat{d}_k \hat{d}_k^\dagger \rangle = e \int \frac{d\omega}{2\pi} I_L(\omega) \tag{B1}
\]

and heat current

\[
I^q_h = \frac{d}{dt} \langle \sum_k (\epsilon_k^L - \mu_L)\hat{d}_k \hat{d}_k^\dagger \rangle = \int \frac{d\omega}{2\pi} (\omega - \mu_L)I_L(\omega) \tag{B2}
\]

leaving electrode L, The Green’s function calculation yields

\[
I_L(\omega) = -i\text{Tr}(\tilde{G}_{tot}^>(\omega) \tilde{G}_{tot}^<(-\omega) - \tilde{G}_{tot}^>(\omega) \tilde{G}_{tot}^<(-\omega)) - i\int L(\omega) \langle [\tilde{G}_{tot}^>(\omega) - \tilde{G}_{tot}^<(-\omega)] \rangle, \tag{B3}
\]

which in terms of the total Green’s functions are \(\tilde{G}_{tot}^>(\omega)\), \(\tilde{G}_{tot}^<(\omega)\), and \(\tilde{G}_{tot}^0(\omega)\) are the lesser, advanced and retarded Green’s function, respectively. By using the

| Symbol | Meaning |
|--------|---------|
| \(\omega\) | \(\omega\) Zeroth order dot-lead coupling |
| \(\omega_{\text{est}}\) | \(\omega_{\text{est}}\) 2nd order dot-lead coupling (elastic current) |
| \(\omega_{\text{est}}\) | \(\omega_{\text{est}}\) 2nd order dot-lead coupling (inelastic current) |
| \(\omega\) | \(\omega\) Electron-photons coupling, coupling strength \(V\) |
| \(\omega\) | \(\omega\) Lead-dot coupling, coupling strength \(V\) |
| \(\omega\) | \(\omega\) Photon |
| \(\omega\) | \(\omega\) Bare dot |

FIG. 5: (color online). Symbol and Feynman diagram for the Green’s function \(G_{tot}\) which is the main ingredient in the transport calculations.

FIG. 6: (color online). Symbol and Feynman diagram for the Green’s function \(G_{tot}\) for the 4th order electron-photon coupling.
Dyson equation and the Keldysh formula, we have the total retarded (advanced) Green’s function,

\[ \hat{G}^\text{tot}_r (\omega) = \hat{G}^\text{tot}_r (\omega) \{ \hat{\Sigma}^> (\omega) + \hat{\Sigma}^< (\omega) \} \hat{G}^\text{tot}_r (\omega), \]  

(B4)

where

\[ \hat{G}^\text{tot}_r (\omega) = \left[ \left( \hat{G}^\text{r}_1 (\omega) \right)^{-1} - \hat{\Sigma}^> (\omega) \right]^{-1}, \]  

(B5)

here

\[ \hat{G}^\text{r}_1 (\omega) = \left[ \left( \hat{G}^\text{r}_0 (\omega) \right)^{-1} - \hat{\Sigma}^> (\omega) \right]^{-1}, \]  

(B6)

and

\[ \hat{\Sigma}^> (\omega) = \begin{pmatrix} G^{r>}_0 (\omega) & 0 \\ 0 & G^{a>}_0 (\omega) \end{pmatrix}, \]  

(B7)

As is seen from the above equations, the self-energy on the dot includes two contributions. The first, \( \Sigma^> \), is due to the coupling with the leads,

\[ \hat{\Sigma}^> = -i[\hat{G}^L (1-f_L) + \hat{G}^R (1-f_R)], \]  

(B8)

\[ \hat{\Sigma}^< = i[\hat{G}^L (f_L) + \hat{G}^R (f_R)], \]  

(B9)

\[ \hat{\Sigma}^{r>}_{\alpha} = \mp i[\hat{G}^L (f_L) + \hat{G}^R (f_R)]/2. \]  

(B10)

The second contribution to the self-energy results from the interaction with the photons, and in the second order of \( g \) reads

\[ \hat{\Sigma}^> (\omega) = i g^2 \int \frac{d\omega'}{2 \pi} \frac{1}{\omega - \omega_c - \omega' + i 0^+} \left[ (1 + N_P) \hat{G}^> (\omega') - N_P \hat{G}^< (\omega') \right] + \frac{N_P \hat{G}^< (\omega') - (1 + N_P) \hat{G}^> (\omega')}{\omega + \omega_c - \omega' + i 0^+}, \]  

(B11)

and

\[ \hat{\Sigma}^< (\omega) = g^2 \left[ N_P \hat{G}^< (\omega - \omega_c) + (1 + N_P) \hat{G}^< (\omega + \omega_c) \right]. \]  

(B12)

Inserting the expressions for the Green’s function \( \hat{G}^\text{tot}_r \) into the above equation, one finds that \( I_L \) can be written as a sum of two terms, one arising from the elastic transitions of the transport electrons and the other coming from the inelastic ones,

\[ I_L (\omega) = I_L^e (\omega) + I_L^{inel} (\omega), \]  

(B13)

The elastic-process contribution is

\[ I_L^e (\omega) = Tr(\hat{\Gamma}^L_\text{rot} (\omega) \hat{G}^r_\text{tot} (\omega) \hat{\Sigma}^> (\omega) + 2 f_L (\omega) \hat{\Sigma}^< (\omega) \hat{G}^a_\text{tot} (\omega)), \]  

(B14)

while the inelastic one is

\[ I_L^{inel} (\omega) = Tr(\hat{\Gamma}^L_\text{rot} (\omega) \hat{G}^r_1 (\omega) \hat{\Sigma}^> (\omega) + 2 f_L (\omega) \hat{\Sigma}^< (\omega) \hat{G}^a_1 (\omega)), \]  

(B15)

So we can get the elastic and inelastic currents,

\[ I_L^el |_{el} = e \int \frac{d\omega}{2 \pi} Tr(\hat{\Gamma}^L_\text{rot} (\omega) \hat{G}^r_\text{tot} (\omega) \hat{\Sigma}^> (\omega) + 2 f_L (\omega) \hat{\Sigma}^< (\omega) \hat{G}^a_\text{tot} (\omega)), \]  

(B16a)

\[ I_L^el |_{inel} = e \int \frac{d\omega}{2 \pi} Tr(\hat{\Gamma}^L_\text{rot} (\omega) \hat{G}^r_1 (\omega) \hat{\Sigma}^> (\omega) + 2 f_L (\omega) \hat{\Sigma}^< (\omega) \hat{G}^a_1 (\omega)). \]  

(B16b)

Meanwhile, we can get the heat current

\[ I_Q^el = \int \frac{d\omega}{2 \pi} (\omega - \mu \lambda) Tr(\hat{\Gamma}^L_\text{rot} (\omega) \hat{G}^r_\text{tot} (\omega) \hat{\Sigma}^> (\omega) + 2 f_L (\omega) \hat{\Sigma}^< (\omega) \hat{G}^a_\text{tot} (\omega)), \]  

(B17a)

\[ I_Q^{inel} = \int \frac{d\omega}{2 \pi} (\omega - \mu \lambda) Tr(\hat{\Gamma}^L_\text{rot} (\omega) \hat{G}^r_1 (\omega) \hat{\Sigma}^> (\omega) + 2 f_L (\omega) \hat{\Sigma}^< (\omega) \hat{G}^a_1 (\omega)). \]  

(B17b)

We have to point that we treat inelastic in a perturbative way, but as much as we can we include high order corrections in Fig. 5. However, as shown in diagram of Fig. 6, we missed the next order terms that they are 4th order and higher order.
