NON-AUTONOMOUS WEAKLY DAMPED PLATE MODEL ON TIME-DEPENDENT DOMAINS

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Abstract. We are concerned with dynamics of the weakly damped plate equation on a time-dependent domain. Under the assumption that the domain is time-like and expanding, we obtain the existence of time-dependent attractors, where the nonlinear term has a critical growth.

1. Introduction. In this paper, we study the long-time behaviors of a weakly damped plate equation on the time-dependent domain:

\[ \partial_t^2 u + \Delta^2 u + \partial_t u + \lambda u + f(u) = g(x, t), \quad \text{in} \quad Q_{\tau}, \]

\[ u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on} \quad \Sigma_{\tau}, \]

\[ u(x, \tau) = u_0^\tau(x), \quad \partial_t u(x, \tau)|_{t=\tau} = u_1^\tau(x), \quad x \in \Omega_{\tau}, \]

where the external forcing function \( g(x, t) \in L^2_{\text{loc}}(Q_{\tau}) \), \( u_0^\tau \) and \( u_1^\tau \) are the initial data and \( \nu = (\nu_x, \nu_t) \) is exterior normal of \( Q_{\tau} \). Here the space-time domain \( Q_{\tau} \) satisfies

\[ Q_{\tau} \subset \mathbb{R}^n \times (\tau, \infty), \quad \tau \in \mathbb{R}, \]

such that its intersection with the hyperplane \( \{ (x, s) \in \mathbb{R}^{n+1} | s = t \} \) is a bounded domain \( \Omega_t \in \mathbb{R}^n \) with the boundary \( \Gamma_t = \partial \Omega_t \). So the domain \( Q_{\tau} \) and its lateral boundary \( \Sigma_{\tau} \) can be defined by

\[ Q_{\tau} = \bigcup_{t > \tau} \{ \Omega_t \times \{ t \} \} \quad \text{and} \quad \Sigma_{\tau} = \bigcup_{t > \tau} \{ \Gamma_t \times \{ t \} \}, \]
respectively. In general, $Q_\tau$ is non-cylindrical along the $t$-axis. We say that a
domain $Q_\tau$ is expanding if $\Omega_s \subset \Omega_t$ whenever $s \leq t$ and is contracting in the reverse
case.

When the domain is cylindrical along the $t$-axis and $g$ does not depend on time, equation (1) becomes an autonomous plate equation. The plate equation arises in the nonlinear theory of oscillations, and has attracted considerable attention in the past decades. Khanmamedov [12] presented the existence of global attractor for the plate equation with the damping $a(x)u_t$, and considered the localized damping case [13]. Yang-Zhong [29] established the existence of global attractor for the plate equation with the nonlinear damping. Fatori et al [10] dealt with the vibrating plates with the nonlinear strain of the $p$-Laplacian type. Al-Gharabli-Messaoudi [1] considered the long-time behaviors of a plate equation with a nonlinear damping and a logarithmic source term.

If the domain is non-cylindrical along the $t$-axis, since the boundary of $\Omega_t$ is a function of time, we know that equation (1) is intrinsically non-autonomous on moving boundary domains. This is also true in the case where the classical theory generally fails to capture the dissipation mechanism, no matter whether the external forcing function depends on time or not [7, 23]. There are a great number of theoretical issues concerning the non-cylindrical domains [7, 11, 15–18, 20–23, 26, 32]. Main motivation of the study naturally comes from applications, such as the Stefan problems [25], the behavior of particles in time-dependent potential well [8], the model of tumor growth [4], and the control problems [24] etc. For the semi-linear heat equations on non-cylindrical domains, Kloeden et al [15] proved the existence of pullback attractors for the non-cylindrical domains with the monotonicity condition. Later, similar results were presented for the non-cylindrical domains with some spatially diffeomorphic transformation [16].

On the other hand, as for the semi-linear wave equations on non-cylindrical domains, the theory of pullback attractors was presented [23] and applied to the oscillation equation. The long-time behaviors of the wave equation with linear damping was studied and the existence of the time-dependent global attractor was explored [7]. Pullback attractors were investigated to the weakly damped wave equations on the moving boundary domains by using the compactness criterion [20]. Under certain assumptions on the non-cylindrical domains, the existence of pullback attractors for the weakly damped wave equations was established [32]. However, to the best of our knowledge, not much has been undertaken on the dynamics of the plate equation on such non-cylindrical domains.

In the present paper, we restrict our attention to the asymptotic dynamics of the problem (1) on time-dependent domains. Under the assumptions that the lateral boundary is time-like, the domains are expanding and the nonlinear team $f$ has a critical growth rate, we will establish the existence of a pullback attractor.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminary results on pullback attractors for non-autonomous dynamical systems. In Section 3, we prove the well-posedness of equation (1) and present some appropriate energy estimates. Section 4 is dedicated to the pullback asymptotic compactness and the existence of minimal pullback attractors.

2. Preliminaries. In this section, in order to present the proofs of our main results in a straightforward manner, we introduce the assumptions on system (1)-(3) and some related results on non-autonomous dynamical systems. For convenience, we
use $C$ or $C_i$ ($i = 1, 2,\ldots$) to denote a generic positive constant that may change from line to line.

2.1. Assumptions. (A1) Assumptions on the domain. Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\Gamma$ and $0 \in \Omega$. We consider the time-dependent domain:

$$\Omega_t = \{x \in \mathbb{R}^n | x_i = h_i(t)y_i, \ 1 \leq i \leq n, \ y_i \in \Omega, \ t \in \mathbb{R}\},$$  
where $h_i \in C^\alpha(\mathbb{R})$ with constants $h_m, h_M > 0$ such that

$$h_m \leq h_i(t) \leq h_M, \ t \in \mathbb{R}, \ 1 \leq i \leq n,$$
and there exists a constant $\gamma \geq 0$ such that

$$0 \leq h_i'(t) \leq \gamma, \ t \in \mathbb{R}, \ 1 \leq i \leq n.$$

Remark 1. Since $0 \in \Omega$, assumptions (5) and (6) imply that $\Omega_t$ is expanding, and there exist two bounded domains $\Omega_*^-, \Omega_*^+ \subset \mathbb{R}^n$ such that

$$\Omega_*^- \subset \Omega_t \subset \Omega_*^+.$$  

From (7) we have a uniform Poincaré type inequality:

$$\|u\|_{L^2(\Omega_t)}^2 \leq \frac{1}{\lambda_*^1} \|
abla u\|_{L^2(\Omega_t)}^2, \ t \in \mathbb{R},$$

where $\lambda_*^1$ depends on $\Omega_*^+$. Moreover, there exists a vector-valued function $r : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^n$ defined by

$$r(y, t) = (h_1(t)y_1, h_2(t)y_2, \ldots, h_n(t)y_n).$$

Its inverse $\phi(\cdot, t) = r^{-1}(\cdot, t)$ is $C^2$-diffeomorphic (see [16,18]) and can be written as

$$\phi(x, t) = (y, t), \quad y = \left(\frac{x_1}{h_1(t)}, \frac{x_2}{h_2(t)}, \ldots, \frac{x_n}{h_n(t)}\right).$$

(A2) Assumptions on the forcing function. For the nonlinear term $f$, we assume that $f \in C^1(\Omega)$ satisfies

$$|f'(u)| \leq C(1 + |u|^p)$$

for some positive constants $C, p > 0$ with $p(n - 4) \leq 4$. We also assume that there exist $\beta \in (0, \lambda_*^1)$ and $\rho > 0$ such that

$$F(u) \geq -\frac{\beta}{2} u^2 - \rho \quad \text{and} \quad f(u)u \geq F(u) - \frac{\beta}{2} u^2 - \rho, \ u \in \mathbb{R},$$

where

$$F(u) = \int_0^u f(s)ds.$$

For the external forcing function $g(x, t)$, we assume that

$$g \in L^2_{\text{loc}}(Q_\tau)$$

and there exists $\delta_0 > 0$ such that

$$\int_{-\infty}^0 e^{\delta_0 s} \|g(s)\|_{L^2(\Omega_s)}^2 ds < \infty,$$

with $\delta_0 \leq \delta_1$ and

$$\delta_1 = \frac{2}{3} \left\{ \frac{\lambda_*^1 - \beta}{2\lambda_*^1}, \frac{\lambda_*^1 - \beta}{2 + 3(\lambda_*^1 - \beta)} \right\}. $$
2.2. Pullback attractors. In this subsection we introduce some basic definitions and results on pullback attractors \([3, 7, 15, 16, 20]\).

**Definition 2.1.** Let \(\{X_t\}_{t \in \mathbb{R}}\) be a family of non-empty metric spaces. An evolutionary process is a two-parameter family of \(\{U(t, \tau) : X_\tau \to X_t, \; t \geq \tau, \; \tau \in \mathbb{R}\}\) if

(i) \(U(\tau, \tau) = I_\tau\) is the identity operator on \(X_\tau\), \(\tau \in \mathbb{R}\); and

(ii) \(U(t, s)U(s, \tau) = U(t, \tau), \; -\infty < s \leq t < \infty\).

In addition, the evolutionary process is said to be closed if a sequence \(x_n \to x\) in \(X_\tau\) and \(U(t, \tau)x_n \to y\) in \(X_t\), then

\[U(t, \tau)x = y.\]

We suppose that \(\mathcal{D}\) is a nonempty class of parameterized elements of the form:

\[\mathcal{D} = \{D(t) : D(t) \subset X_t \text{ and } D(t) \neq \emptyset, \; t \in \mathbb{R}\}.\]

**Definition 2.2.** A class \(\mathcal{D}\) is said to be inclusion closed if \(\hat{\mathcal{D}} \in \mathcal{D}\) and

\[C(t) \subset X_t, \; C(t) \subset D(t), \; t \in \mathbb{R},\]

then \(\hat{\mathcal{C}} \in \mathcal{D}\).

**Definition 2.3.** A family \(\hat{\mathcal{D}} \in \mathcal{D}\) with non-empty elements is said to be pullback \(\mathcal{D}\)-absorbing for a process \(U(t, \tau) : X_\tau \to X_t\), if for any \(t \in \mathbb{R}\) and \(\hat{\mathcal{D}} \in \mathcal{D}\), there exists \(\tau_0 = \tau_0(t, \hat{\mathcal{D}})\) such that

\[U(t, \tau)\mathcal{D}(\tau) \subset B(t), \; \tau \leq \tau_0.\]

**Definition 2.4.** A process \(u(t, \tau) : X_\tau \to X_t\) is said to be pullback \(\mathcal{D}\)-asymptotically compact if the sequence \(\{U(t, \tau_n)x_n\}\) is relatively compact in \(X_t\) for any \(t \in \mathbb{R}\), \(\hat{\mathcal{D}} \in \mathcal{D}\), and any sequences \(\{\tau_n\}\) and \(\{x_n\}\) with \(x_n \in D(\tau_n)\).

**Notation.** For each \(t \in \mathbb{R}\), we denote the Hausdorff semi-distance between nonempty subsets \(B\) and \(C\) of \(X_t\) by

\[\text{dist}_{X_t}(B, C) = \sup_{x \in B} \inf_{y \in C} \|x - y\|_{X_t}.\]

**Definition 2.5.** A family \(\hat{A} = \{A(t) : A(t) \subset X_t, \; A(t) \neq \emptyset, \; t \in \mathbb{R}\}\) is said to be a pullback \(\mathcal{D}\)-attractor for \(U(\cdot, \cdot)\), if

(i) \(A(t)\) is a compact subset of \(X_t\) for \(t \in \mathbb{R}\);

(ii) \(\hat{A}\) is pullback \(\mathcal{D}\)-attracting, i.e.

\[\lim_{\tau \to -\infty} \text{dist}_{X_t}(U(t, \tau)D(\tau), A(t)) = 0, \; \hat{\mathcal{D}} \in \mathcal{D} \text{ and } t \in \mathbb{R};\]

(iii) \(\hat{\mathcal{A}}\) is invariant, i.e.

\[U(t, \tau)A(\tau) = A(t), \; -\infty < \tau \leq t < \infty.\]

Moreover, a pullback \(\mathcal{D}\)-attractor is said to be minimal if \(\hat{\mathcal{C}}\) is a \(\mathcal{D}\)-absorbing family with non-empty element, then \(A(t) \subset C(t)\) for \(t \in \mathbb{R}\).

**Proposition 2.6** (\([15, 20]\)). Let \(U(t, \tau) : X_\tau \to X_t\) be a closed evolution process defined on a family of metric spaces \(\{X_t\}_{t \in \mathbb{R}}\). Suppose that the process \(U(t, \tau)\) is
pullback $\mathcal{D}$-asymptotically compact and $\hat{B} \in \mathcal{D}$ is a family of pullback $\mathcal{D}$-absorbing sets for $U(t, \tau)$. Then, the family $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is defined by

$$A(t) = \bigcup_{\hat{D} \in \mathcal{D}} \Lambda(\hat{D}, t)$$

where

$$\Lambda(\hat{D}, t) = \bigcap_{s \leq \tau \leq s} \bigcup_{t \in \mathbb{R}} U(t, \tau)\hat{D}(\tau)$$

is the minimal pullback $\mathcal{D}$-attractor for $U(t, \tau)$. If $\hat{B} \in \mathcal{D}$, then

$$A(t) = \Lambda(\hat{B}, t).$$

In addition, if $B(t)$ is closed for $t \in \mathbb{R}$ and the class $\mathcal{D}$ is inclusion closed, then the pullback $\mathcal{D}$-attractor $\mathcal{A}$ belongs to $\mathcal{D}$.

For the existence of pullback attractors in the autonomous systems [30, 31] and stochastic systems [27, 28], the compactness of the family of processes is usually needed. We now recall the method of contractive functions to verify the asymptotic compactness. This technique was used in [5, 6, 14] for autonomous systems, in [26] for non-autonomous systems, and in [20, 22, 32] for time-dependent spaces.

**Definition 2.7.** Let $X$ be a metric space. We say that a function $\phi : X \times X \to \mathbb{R}$ is contractive on a bounded subset $B$ of $X$ if for any sequence $\{x_n\}$ of $B$ there exists a subsequence $\{x_{nk}\}$ such that

$$\lim_{k \to \infty} \lim_{l \to \infty} \phi(x_{nk}, x_{nl}) = 0.$$

**Proposition 2.8 ( [20]).** Let $\{X_t\}_{t \in \mathbb{R}}$ be a family of Banach spaces and $U(t, \tau) : X_\tau \to X_t$ be an evolution process that possesses a pullback $\mathcal{D}$-absorbing family $\hat{B} = \{B(t)\}_{t \in \mathbb{R}}$. For any $t \in \mathbb{R}$ and $\varepsilon > 0$, there exists a time $\tau_\varepsilon \leq t$ and a contractive function $\phi : B(\tau_\varepsilon) \times B(\tau_\varepsilon) \to \mathbb{R}$ such that

$$\|U(t, \tau_\varepsilon)x - U(t, \tau_\varepsilon)y\|_X \leq \varepsilon + \phi(x, y), \quad x, y \in B(\tau_\varepsilon).$$

Then the process is pullback $\mathcal{D}$-asymptotically compact.

3. **Well-posedness.** In this section, we recall the functional spaces, present some estimates on the energy function, and prove the well-posedness of system (1)-(3). Hereinafter, we will focus on the dynamics of system (1) in the space $X_t = H^2_0(\Omega_t) \times L^2(\Omega_t)$, equipped with the inner-product:

$$\|(u, v)\|_{X_t}^2 = \|\Delta u\|_{L^2(\Omega_t)}^2 + \lambda\|u\|_{L^2(\Omega_t)}^2 + \|v\|_{L^2(\Omega_t)}^2,$$

and the resultant norm:

$$(u_1, v_1), (u_2, v_2) \in X_t \quad \text{if} \quad \int_{\Omega_t} \Delta u_1 \cdot \Delta u_2 dx + \lambda \int_{\Omega_t} u_1 u_2 dx + \int_{\Omega_t} v_1 v_2 dx.$$

3.1. **Function spaces of non-cylindrical domains.** We start with some definitions and properties of function spaces in the non-cylindrical domains:

$$Q_{\tau, T} = \bigcup_{t \in (\tau, T)} \{\Omega_t \times \{t\}\} \quad \text{and} \quad \Sigma_{\tau, T} = \bigcup_{t \in (\tau, T)} \{\Gamma_t \times \{t\}\}$$

with $\Omega_t$ satisfying (4)-(5) and $\tau < T$. 
**Definition 3.1.** Let $B_t$ be a Banach space contained in $L^1_{\text{loc}}(\Omega_t)$. Then we define the space:

$$L^q(\tau, T; B_t) = \left\{ u \in L^1_{\text{loc}}(Q_{\tau,T}) | u(t) \in B_t \text{ for a.e. } t \in (\tau, T) \text{ and } \int_\tau^T \| u(t) \|_{B_t}^q \, dt < \infty \right\},$$

where $q \geq 1$, and its norm denoted by

$$\|u\|_{L^q(\tau, T; B_t)} = \left( \int_\tau^T \| u(t) \|_{B_t}^q \, dt \right)^{\frac{1}{q}}.$$

Let $\hat{u}$ be the null extension of $u \in \{ L^q(\tau, T; L^p(\Omega_t)), \ p, q \geq 1 \}$ such that

$$\hat{u}(x, t) = \begin{cases} u(x, t), & x \in \Omega_t, \\ 0, & x \in \mathbb{R}^n \setminus \Omega_t. \end{cases}$$

Then

$$u \in L^q(\tau, T; L^p(\Omega_t)) \Rightarrow \hat{u} \in L^q(\tau, T; L^p(\mathbb{R}^n))$$

and

$$u \in L^q(\tau, T; H^2_0(\Omega_t)) \Rightarrow \hat{u} \in L^q(\tau, T; H^2(\mathbb{R}^n)),$$

with

$$\frac{\partial \hat{u}}{\partial x_i} = \frac{\partial u}{\partial x_i}, \quad 1 \leq i \leq n.$$

**Definition 3.2.** The weak time-derivative $u' = \partial_t u$ is defined by

$$\langle u', \phi \rangle = -\int_\tau^T \int_{\Omega_t} u(x, t) \phi'(x, t) \, dx \, dt, \quad \phi \in C_c^\infty(Q_{\tau,T}).$$

We can verify that

$$\hat{u}' = \hat{u}'.$$

**Definition 3.3.** We say that a function $u$ in $L^1_{\text{loc}}(Q_{\tau,T})$ belongs to $C([\tau, T]; L^2(\Omega_t))$ if its null extension $\hat{u}$ belongs to $C([\tau, T]; L^2(\mathbb{R}^n))$. We say that a sequence $\{ u_m \}$ converges to $u$ in $C([\tau, T]; L^2(\Omega_t))$ as $m \to \infty$, if the sequence $\{ \hat{u}_m \}$ converges to $\hat{u}$ in $C([\tau, T]; L^2(\mathbb{R}^n))$ as $m \to \infty$.

These two definitions can also be applied to the space $H^2(\cdot)$.

Denote

$$v(y, t) = u(r(y, t), t) = u(x, t), \quad t \in \mathbb{R}.$$ When condition (6) is satisfied, we have the following relationship between $u$ and $v$.

**Lemma 3.4.** ([16]) For any $1 \leq p \leq \infty$, we have $u \in L^p(\Omega_t)$ if and only if $v(t) \in L^p(\Omega)$. Moreover, there exist constants $C_1$ and $C_2$ dependent on $p$ such that

$$C_1 \| u(t) \|_{L^p(\Omega_t)} \leq \| v(t) \|_{L^p(\Omega)} \leq C_2 \| u(t) \|_{L^p(\Omega_t)}.$$

Similarly, there exist constants $C_3$ and $C_4$ dependent on $p$ such that

$$C_3 \| \Delta u(t) \|_{L^2(\Omega_t)} \leq \| \Delta v(t) \|_{L^2(\Omega)} \leq C_4 \| \Delta u(t) \|_{L^2(\Omega_t)}.$$

Combining these estimates with the embedding inequality [9], we have

$$\mu_q \| u \|_{L^q(\Omega_t)} \leq \| \Delta u(t) \|_{L^2(\Omega_t)}, \quad 1 \leq q \leq \frac{2n}{n - 4},$$

where $\mu_q$ does not depend on time $t$. 

Lemma 3.5. ([16]) For any $1 \leq p, q \leq \infty$, we have $u \in L^q(\tau, T; L^p(\Omega_t))$ if and only if $v \in L^q(\tau, T; L^p(\Omega))$. Moreover, there exist two positive constants $C_5 = C_5(p, q, \tau, T)$ and $C_6 = C_6(p, q, \tau, T)$ such that

$$C_5\|u\|_{L^q(\tau, T; L^p(\Omega_t))} \leq \|v\|_{L^q(\tau, T; L^p(\Omega))} \leq C_6\|u\|_{L^q(\tau, T; L^p(\Omega_t))}. \quad (14)$$

Analogously, we have $u \in L^2(\tau, T; H^2_0(\Omega_t))$ if and only if $v \in L^2(\tau, T; H^2_0(\Omega))$, and there exist $C_7 = C_7(\tau, T)$ and $C_8 = C_8(\tau, T)$ such that

$$C_7\|u\|_{L^2(\tau, T; H^2_0(\Omega_t))} \leq \|v\|_{L^2(\tau, T; H^2_0(\Omega))} \leq C_8\|u\|_{L^2(\tau, T; H^2_0(\Omega_t))}. \quad (15)$$

Lemma 3.6. If condition (A1) holds and the sequence $\{u_n\}$ is bounded in the space $L^2(\tau, T; H^2_0(\Omega_t))$ such that $\partial_t u_n$ is a bounded sequence in $L^2(\tau, T; L^2(\Omega_t))$, then for any $s \in \left[2, \frac{2n}{n-4}\right)$, there exists a subsequence $\{u_{nk}\}$ that converges strongly in $L^2(\tau, T; L^s(\Omega_t))$.

Proof. Let $v_n(\cdot, t) = u_n(\cdot, t)$. According to (14)-(15) we know that $\{v_n\}$ is bounded in $L^2(\tau, T; H^2_0(\Omega_t))$ and $\partial_t v_n$ is bounded in $L^2(\tau, T; L^2(\Omega_t))$. By virtue of Aubin-Lions’s theorem [19], there exists a subsequence $\{v_{nk}\}$ and a function $v \in L^2(\tau, T; L^s(\Omega_t))$ such that

$$v_{nk} \to v \text{ in } L^2(\tau, T; L^s(\Omega_t)), \quad s \in \left[2, \frac{2n}{n-4}\right).$$

From inequality (14), we obtain

$$\|u_{nk} - u\|_{L^2(\tau, T; L^s(\Omega_t))} \leq \frac{1}{C_5} \|v_{nk} - v\|_{L^2(\tau, T; L^s(\Omega_t))},$$

where $u(\cdot, t) = v (r^{-1}(\cdot), t)$.

3.2. Energy estimates. In this subsection, we establish some estimates on the energy function.

Definition 3.7. A function $u$ is said to be a weak solution of the problem (1)-(3) if for $\tau \leq T$ and $u \in C([\tau, T]; H^2_0(\Omega_t)) \cap C^1([\tau, T]; L^2(\Omega_t))$ with $u(\tau) = u^0$ and $\partial_t u(\tau) = u^1_t$, there holds

$$\int_\tau^T \int_{\Omega_t} \left\{ -\frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} + \Delta u \Delta \phi + \frac{\partial u}{\partial t} \phi + \lambda u \phi + f(u) \phi - g \phi \right\} \, dx \, dt = 0$$

for $\phi \in L^2(\tau, T; H^2_0(\Omega_t))$ satisfying $\partial_t \phi \in L^2(\tau, T; L^2(\Omega_t))$ and $\phi(\tau) = \phi(T) = 0$. The solution above is called a strong solution if there also holds

$u \in L^\infty(\tau, T; H^1(\Omega_t) \cap H^2_0(\Omega_t))$, $\partial_t u \in L^\infty(\tau, T; H^1_0(\Omega_t))$, $\partial_t^2 u \in L^\infty(\tau, T; H^2_0(\Omega_t))$.

Lemma 3.8 ([20]). Assume that $Q_t$ has a regular lateral boundary $\Sigma_t$ and $w \in C^1(\mathbb{R}; L^2(\Omega_t))$. Then we have

$$\frac{d}{dt} \int_{\Omega_t} w(x, t) \, dx = \int_{\Omega_t} \partial_t w \, dx - \int_{\Gamma_t} w(x, t) \nu_\Sigma \, d\sigma.$$ 

Define the energy function by

$$E(t) = \frac{1}{2} \int_{\Omega_t} (|\partial_t u|^2 + |\Delta u|^2 + \lambda |u|^2) \, dx + \int_{\Omega_t} F(u) \, dx.$$
Lemma 3.9. Under conditions \((A_1)-(A_2)\), the energy function \(E(t)\) and the strong solution \(u\) satisfy
\[
\frac{d}{dt}E(t) \leq -\int_{\Omega_t} |\partial_t u|^2 dx + \int_{\Omega_t} g(x,t)\partial_t u dx.
\] (16)

Proof. According to Lemma 3.8, multiplying both sides of equation (1) by \(\partial_t u\) and integrating it over \(\Omega_t\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |\partial_t u|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} |\Delta u|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} \lambda |u|^2 dx + \frac{d}{dt} \int_{\Omega_t} F(u) dx
\]
\[
= -\int_{\Omega_t} |\partial_t u|^2 dx + \int_{\Omega_t} g(x,t)\partial_t u dx + R,
\] (17)

where
\[
R = -\int_{\Gamma_t} F(u)\nu_t d\sigma - \frac{1}{2} \int_{\Gamma_t} |\Delta u|^2 \nu_t d\sigma + \int_{\Gamma_t} \Delta u \nabla (\partial_t u) \nu_x d\sigma.
\] (18)

Note that \(\nabla x,t = 0\). According to [2], it follows from (2) that
\[
\frac{\partial^2 u}{\partial x_i \partial t} = \partial_x (\partial_t u) \nu_t \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i^2} = \partial_x (\partial_x u) \nu_x.
\]

Substituting this into (18) leads to
\[
R = \frac{1}{2} \int_{\Gamma_t} \left( \sum_{i=1}^n \partial_x (\partial_t u) \nu_x \right)^2 \nu_t d\sigma.
\]

Since the domain \(\Omega_t\) is expanding, we know that \(\nu_t \leq 0\) and \(R \leq 0\). Using (17), we arrive at (16) immediately. \(\square\)

Lemma 3.10. Under conditions \((A_1)-(A_2)\), there exist positive constants \(\beta_0, C_f\) and \(C_F\) independent of the initial value and initial time such that
\[
\beta_0 \|(u(t), \partial_t u(t))\|^2_{X_t} - C_f \leq E(t) \leq C_F \left( 1 + \|(u(t), \partial_t u(t))\|_{X_t}^{p+2} \right).
\] (19)

Proof. Following [20, 32], from (10) and (8) we get
\[
\int_{\Omega_t} F(u) dx \geq -\frac{\beta}{2\lambda^*_t} \int_{\Omega_t} |\Delta u|^2 dx - \rho |\Omega^*|.
\]

Let
\[
\beta_0 = \frac{1}{2} \left( 1 - \frac{\beta}{\lambda^*_t} \right) \quad \text{and} \quad C_f = \rho |\Omega^*|.
\] (20)

We see that the left hand side of inequality (19) holds. From (9) there exists a constant \(C' > 0\) such that
\[
F(u) \leq C' (1 + |u|^{p+2}).
\]

Thus, we have
\[
\int_{\Omega_t} F(u) dx \leq C' \left( |\Omega^*| + \|u\|_{L_{p+2}(\Omega_t)}^{p+2} \right).
\]

From (13), the right hand side of inequality (19) holds if \(C_F = \max \{ |\Omega^*|, \mu_{p+2}^{-1} \} \). \(\square\)
3.3. Well-posedness. In the subsection, we state the well-posedness of the problem (1)-(3). We make the coordinate transformation to convert the space-time domain into the cylindrical domain. In this way, the problem is converted into an equivalent initial-boundary problem which has a more complicated structure defined on a cylindrical domain.

**Lemma 3.11.** There exist positive constants $C_9, C_{10}$ and $C_{11}$ independent of $\tau, t$ and $z \in X_\tau$ such that

$$
\|U(t, \tau)z\|_{Y_{X_\tau}}^2 \leq C_9 (1 + \|z\|_{X_{\tau}}^{r+2}) e^{-\delta_1 (t-\tau)} + C_{10} \int_{\tau}^{t} e^{-\delta_1 (t-s)} \|g(s)\|_{L^2(\Omega_\tau)}^2 ds + C_{11} f,
$$

where $\delta_1$ is defined in (12).

**Proof.** Let

$$
\Phi(t) = \int_{\Omega_\tau} u \partial_t u dx \quad \text{and} \quad E_\varepsilon(t) = E(t) + \varepsilon \Phi(t),
$$

where $\varepsilon > 0$ is a constant. It follows from (8), (19) and Young’s inequality that

$$
|\Phi(t)| \leq \frac{1}{2} \max \left\{1, \frac{1}{\lambda_1} \right\} \|u, \partial_t u\|_{X_\tau}^2 \leq \frac{1}{2\beta_0} \max \left\{1, \frac{1}{\lambda_1} \right\} (E(t) + C_f).
$$

Taking $\varepsilon_0 = \beta_0 \min\{1, \lambda_1^*\}$, we get

$$
\frac{1}{2} E(t) - \frac{1}{2} C_f \leq E_\varepsilon(t) \leq E(t) + \frac{1}{2} E(t) + \frac{1}{2} E(t), \quad \varepsilon \leq \varepsilon_0, \quad t \in \mathbb{R}.
$$

Since $u = 0$ on $\Gamma_t$, by Lemma 3.8 we have

$$
\frac{d}{dt} \Phi(t) = \int_{\Omega_\tau} |\partial_t u|^2 dx + \int_{\Omega_\tau} u \partial_t^2 u dx.
$$

Substituting this equality into equation (1) yields

$$
\frac{d}{dt} \Phi(t) = -E(t) + \frac{3}{2} \int_{\Omega_\tau} |\partial_t u|^2 dx - \frac{1}{2} \int_{\Omega_\tau} |\Delta u|^2 dx - \frac{1}{2} \lambda \int_{\Omega_\tau} |u|^2 dx - \Phi(t) + \int_{\Omega_\tau} g u dx + \int_{\Omega_\tau} |F(u) - f(u)u| dx.
$$

(23)

It follows from (8) and Young’s inequality that

$$
-\Phi(t) \leq \frac{\beta_0}{2} \int_{\Omega_\tau} |\Delta u|^2 dx + \frac{1}{2\beta_0 \lambda_1} \int_{\Omega_\tau} |\partial_t u|^2 dx
$$

and

$$
\int_{\Omega_\tau} g u dx \leq \frac{\beta_0}{2} \int_{\Omega_\tau} |\Delta u|^2 dx + \frac{1}{2\beta_0 \lambda_1} \|g(t)\|_{L^2(\Omega_\tau)}^2.
$$

From (8), (10) and (20), we have

$$
\int_{\Omega_t} |F(u) - f(u)u| dx \leq \frac{\beta}{2\lambda_1} \int_{\Omega_t} |\Delta u|^2 dx + C_f.
$$

(24)

Since $\beta_0 \leq \frac{1}{2}$, substituting inequality (24) into (23), we obtain

$$
\frac{d}{dt} \Phi(t) \leq -E(t) + \frac{1}{2} \left(3 + \frac{1}{\beta_0 \lambda_1} \right) \int_{\Omega_\tau} |\partial_t u|^2 dx + \frac{1}{2\beta_0 \lambda_1} \|g(t)\|_{L^2(\Omega_\tau)}^2 + C_f.
$$

(25)
Using (16) and Young’s inequality leads to
\[
\frac{d}{dt} E(t) \leq -\frac{1}{2} \int_{\Omega_t} |\partial_t u|^2 dx + \frac{1}{2} \|g(t)\|_{L^2(\Omega_t)}^2. \tag{26}
\]

Let
\[
\varepsilon = \min \left\{ \varepsilon_0, (3 + \frac{1}{\beta_0 \lambda_1})^{-1} \right\}.
\]

It follows from (25) and (26) that
\[
\frac{d}{dt} E_\varepsilon(t) \leq -\varepsilon E(t) + \|g(t)\|_{L^2(\Omega_t)}^2 + 4\varepsilon C_f.
\]

Combining the above inequality with (22) yields
\[
\frac{d}{dt} E_\varepsilon(t) \leq -\frac{2\varepsilon}{3} E_\varepsilon(t) + \|g(t)\|_{L^2(\Omega_t)}^2 + \frac{4\varepsilon}{3} C_f.
\]

Applying Gronwall’s lemma over \([\tau, t]\), we obtain
\[
E_\varepsilon(t) \leq E_\varepsilon(\tau) e^{-\frac{2\varepsilon}{3}(t-\tau)} + \int_\tau^t e^{-\frac{2\varepsilon}{3}(t-s)} \left( \|g(s)\|_{L^2(\Omega_s)}^2 + \frac{4\varepsilon}{3} C_f \right) ds.
\]

Since
\[
\int_\tau^t e^{-\frac{2\varepsilon}{3}(t-s)} ds \leq \frac{3}{2\varepsilon},
\]
we have
\[
E_\varepsilon(t) \leq E_\varepsilon(\tau) e^{-\frac{2\varepsilon}{3}(t-\tau)} + \int_\tau^t e^{-\frac{2\varepsilon}{3}(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + 2C_f.
\]

It follows from (22) that
\[
E(t) \leq 3E(\tau) e^{-\frac{2\varepsilon}{3}(t-\tau)} + 2 \int_\tau^t e^{-\frac{2\varepsilon}{3}(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + 6C_f. \tag{27}
\]

In view of the definition of \(\delta_1\), we have \(2\varepsilon/3 = \delta_1\). Then it follows from (27) and (19) that
\[
\|U(t, \tau)z\|_{X_t}^2 \leq C_9 (1 + \|z\|_{X_\tau}^{p+2}) e^{-\delta_1(t-\tau)} + C_{10} \int_\tau^t e^{-\delta_1(t-s)} \|g(s)\|_{L^2(\Omega_s)}^2 ds + C_{11} C_f,
\]
where \(C_9 = 3C_F \beta_0^{-1}\), \(C_{10} = 2\beta_0^{-1}\) and \(C_{11} = 7\beta_0^{-1}\).

\begin{theorem}
Under conditions \((A_1) - (A_2)\) and given an initial data \((u_0^\tau, u_1^\tau) \in H_0^1(\Omega_\tau) \times L^2(\Omega_\tau)\), on any interval \([\tau, t]\) with \(t > \tau\), the problem \((1)-(3)\) has a unique weak solution which continuously depends on the initial data. Moreover, this problem generates a continuous process:
\[
U(t, \tau) : X_\tau \to X_t, \ t > \tau \in \mathbb{R},
\]
where \(U(t, \tau)(u_0^\tau, u_1^\tau) = (u(t), \partial_t u(t))\).
\end{theorem}
the initial data, and then the uniqueness of solution follows. Let (1.13), applying the Galerkin method we can prove the existence of weak solutions of the problem (1) with the initial data $z_i = (u_{i0}^0, u_{i0}^1)$ $i = 1, 2$. We set

$$U(t, \tau)z_i = (u_i(t), \partial_t u_i(t)) \text{ and } w(t) = u_1(t) - u_2(t).$$

Then $w(t)$ satisfies

$$\begin{cases}
\partial_t w + \Delta^2 w + \partial_t w + \lambda w = f(u_2) - f(u_1), & x \in \Omega_t, \ t \geq \tau, \\
w = \frac{\partial w}{\partial \nu} = 0, & x \in \Sigma_t, \\
w(x, \tau) = u_{10}^0(x) - u_{20}^0(x), & \partial_t w(x, t)|_{t = \tau} = u_{11}^1(x) - u_{21}^1(x), & x \in \Omega_\tau.
\end{cases} \tag{29}$$

Multiplying both sides of the equation in (29) by $\partial_t w$ and integrating it over $\Omega_t$ leads to

$$\frac{d}{dt} \int_{\Omega_t} (|\partial_t w|^2 + |\Delta w|^2 + \lambda |w|^2) dx + 2 \int_{\Omega_t} |\partial_t w|^2 dx \leq 2 \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w dx. \tag{28}$$

From (28) we have

$$2 \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w dx \leq C \int_{\Omega_t} (1 + |u_1|^p + |u_2|^p) |w| |\partial_t w| dx. \tag{29}$$

By virtue of the embedding inequalities [9], we know that there exists a constant $C$ independent of time $t$ such that

$$\|u\|_{L^{p+2}(\Omega_t)} \leq C\|u\|_{H^2(\Omega_t)}.$$

From condition (A2) and by using Hölder’s inequality, there exists a constant $C > 0$ such that

$$2 \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w dx \leq C \left(1 + \|u_1\|_{L^{p+2}(\Omega_t)}^p + \|u_2\|_{L^{p+2}(\Omega_t)}^p\right) \|w\|_{L^{p+2}(\Omega_t)} \|\partial_t w\|_{L^2(\Omega_t)}$$

$$\leq C \left(1 + \|u_1\|_{L^{p+2}(\Omega_t)}^p + \|u_2\|_{L^{p+2}(\Omega_t)}^p\right) \left(\|\Delta w\|_{L^2(\Omega_t)}^2 + \|\partial_t w\|_{L^2(\Omega_t)}^2\right).$$
Fixing a time interval \([\tau, T]\), from Lemma 3.11, we have a positive constant \(C_0\) such that
\[
2 \int_{\Omega_t} (f(u_2) - f(u_1)) \partial_t w dx \\
\leq C_0 \left( \|\Delta w\|_{L^2(\Omega_t)}^2 + \|\partial_t w\|_{L^2(\Omega_t)}^2 \right), \quad \tau \leq t \leq T.
\]
Using Gronwall’s lemma, we obtain
\[
\|U(t, \tau)z_1(\tau) - U(t, \tau)z_2(\tau)\|_{X_i} \leq e^{C_0(t-\tau)} (\|z_1 - z_2\|_{X_i}) .
\]

\[\Box\]

4. Pullback \(\mathcal{D}\)-attractor for plate equation.

4.1. Pullback \(\mathcal{D}\)-absorbing set. In this subsection, we prove dissipativity of the problem (1)-(3) based on an analytical approach \([20, 22, 32]\). Given a function \(\rho : \mathbb{R} \to \mathbb{R}^+\), we can define a family of closed balls:
\[
\mathcal{B}_{X_i}(0, \rho(t)) = \{z \in X_i \mid \|z\|_{X_i} \leq \rho(t)\}, \quad t \in \mathbb{R},
\]
with the radius \(\rho(t)\) satisfying
\[
\lim_{\tau \to -\infty} |\rho(\tau)|^{p+2} e^{\delta_1 \tau} = 0, \quad (30)
\]
where \(\delta_1 > 0\) is a decay coefficient, depending on \(\beta\) and \(\lambda_1\), given in (12).

Define the universe \(\mathcal{D}\) by
\[
\mathcal{D} = \left\{ \hat{D} \mid D(t) \neq \emptyset \text{ and } D(t) \subset \overline{B_{X_i}(0, \rho_B(t))} \text{ with } \rho_B(t) \text{ satisfying } (30) \right\}. \quad (31)
\]

**Lemma 4.1.** Under conditions \((A_1)-(A_2)\), for any initial data \(z_\tau = (u_0(\tau), u_1(\tau)) \in D(\tau) \subset X_\tau\), there exists \(\rho_0 > 0\) such that the family \(\mathcal{B}_0 = \{B_0(t) = \overline{B_{X_i}(0, \rho_0(t))} \}_{t \in \mathbb{R}}\) is pullback \(\mathcal{D}\)-absorbing for the evolution process \(U(t, \tau)\) corresponding to equation (1).

**Proof.** Since \(\delta_0 \leq \delta_1\), from (21) we know
\[
\|U(t, \tau)z_\tau\|_{X_i}^2 \leq C_9 e^{-\delta_1 t} (1 + |\rho_B(\tau)|^{p+2}) e^{\delta_1 \tau} + C_{10} \int_\tau^t e^{-\delta_0(s-\tau)} \|g(s)\|_{L^2(\Omega_i)}^2 ds + C_{11} C_f . \quad (32)
\]
for \(z_\tau \in D(\tau)\). Define \(\rho_0(t)\) by
\[
|\rho_0(t)|^2 = C_{10} \int_{-\infty}^t e^{-\delta_0(s-\tau)} \|g(s)\|_{L^2(\Omega_i)}^2 ds + C_{11} C_f + 1. \quad (33)
\]
Then
\[
\|U(t, \tau)z_\tau\|_{X_i}^2 \leq C_9 e^{-\delta_1 t} (1 + |\rho_B(\tau)|^{p+2}) e^{\delta_1 \tau} + |\rho_0(t)|^2 .
\]

From (11), we know that the definition of \(\rho_0(t)\) is well-defined. According to (30), we have
\[
(1 + |\rho_B(\tau)|^{p+2}) e^{\delta_1 \tau} \to 0, \quad \text{as } \tau \to -\infty.
\]
So there exists \(\tau_0(t, \hat{D}) < t\) such that
\[
\|U(t, \tau)z_\tau\|_{X_i}^2 \leq |\rho_0(t)|^2, \quad \tau < \tau_0(t, \hat{D}), \quad z_\tau \in D(\tau) . \quad (34)
\]
That is,
\[
U(t, \tau)D(\tau) \subset B_0(t), \quad \tau < \tau_0(t, \hat{D}). \quad \Box
\]
4.2. Pullback \( \mathcal{D} \)-asymptotic compactness. In this subsection, we will verify the pullback \( \mathcal{D} \)-asymptotic compactness of the process by using the contractive function, and present the existence of minimal pullback attractors.

Let \((u_i(t), \partial_t u_i(t))\) be the solution of equation (1) with the initial data \(z_i = (u_i^0, u_i^1)\) \((i=1,2)\) and

\[
U(t, \tau)z_i = (u_i(t), \partial_t u_i(t)), \quad w(t) = u_1(t) - u_2(t).
\]

**Lemma 4.2.** Let \( \mathcal{B}_0 \) be the pullback \( \mathcal{D} \)-absorbing family as described in Lemma 4.1. Then there exist a constant \( \delta_2 > \delta_1 \) and a constant \( C_{\tau,t} > 0 \) depending on \( \tau \) and \( t \) such that

\[
\|U(t, \tau)z_1 - U(t, \tau)z_2\|_{\mathcal{H}_\tau}^2 \leq 3 |\rho_0(\tau)|^2 e^{-\delta_2(\tau-\tau)}
\]

\[
+ C_{\tau,t} \int_\tau^t \|u_1(s) - u_2(s)\|^2_{L^2(\tau,t)} ds
\]

\[
+ 4 \int_\tau^t \int_{\Omega_s} (f(u_2) - f(u_1))\partial_t w dx ds
\]

for \( t \geq \tau \), where \( z_i \in \mathcal{B}_0(\tau) \).

**Proof.** Define the energy function:

\[
G(t) = \frac{1}{2} \| (w(t), \partial_t w(t)) \|^2_{\mathcal{H}_\tau}.
\]

Multiplying both sides of equation (29) by \( \partial_t w \), integrating it over \( \Omega_t \) and taking into account Lemma 3.8, we obtain

\[
\frac{d}{dt} G(t) \leq - \int_{\Omega_t} |\partial_t w|^2 dx + \int_{\Omega_t} (f(w) - f(u_1))\partial_t w dx.
\]

Let

\[
\Psi(t) = \int_{\Omega_t} w\partial_t w dx \quad \text{and} \quad G_\alpha(t) = G(t) + \alpha \Psi(t), \quad \alpha > 0.
\]

It follows from Young’s inequality that

\[
|\Psi(t)| \leq \max \left\{ 1, \frac{1}{\lambda_1^*} \right\} G(t).
\]

Set

\[
\alpha_0 = \frac{1}{2} \min \{ 1, \lambda_1^* \}.
\]

We have

\[
\frac{1}{2} G(t) \leq G_\alpha(t) \leq \frac{3}{2} G(t), \quad 0 \leq \alpha \leq \alpha_0, \ t \in \mathbb{R}.
\]

In view of \( w = 0 \) on \( \Gamma_\lambda \), we get

\[
\Psi'(t) = \int_{\Omega_t} w\partial_t^2 w + |\partial_t w|^2 dx.
\]

Using (8), (29) and Young’s inequality, we deduce

\[
\Psi'(t) = \int_{\Omega_t} |\partial_t w|^2 dx - \int_{\Omega_t} w\Delta^2 w dx - \lambda \int_{\Omega_t} |w|^2 dx - \int_{\Omega_t} w\partial_t w dx
\]

\[
+ \int_{\Omega_t} (f(u_2) - f(u_1)) w dx
\]

\[
\leq - G(t) + \left( \frac{3}{2} + \frac{1}{2\lambda_1^*} \right) \int_{\Omega_t} |\partial_t w|^2 dx + \int_{\Omega_t} (f(u_2) - f(u_1)) w dx + M(t)
\]
Using the fact
\[
\alpha \text{ equation (1) Lemma 4.3.}
\]
Under conditions
where
\[
M(t) = \frac{1}{2} \int_{\Omega_t} \lambda |w|^2 dx + \frac{1}{2} \int_{\Omega_t} (\lambda_1^* |w|^2 - |\Delta w|^2) dx.
\]
According to the embedding inequality, there exists a constant \( C \) independent of time \( t \) such that
\[
\|u\|_{L^{p+2}(\Omega_t)} \leq C\|\Delta u\|_{L^2(\Omega_t)}.
\]
Using (9) and Hölder’s inequality, we derive
\[
\int_{\Omega_t} (f(u_2) - f(u_1))wdx \leq C \left( 1 + \|u_1\|_{L^{p+2}(\Omega_t)}^p + \|u_2\|_{L^{p+2}(\Omega_t)}^p \right) \|w(t)\|_{L^{p+2}(\Omega_t)}^2
\]
\[
\leq C \left( 1 + \|U(t, \tau)z_1\|_{L^p(\Omega_t)}^p + \|U(t, \tau)z_2\|_{L^p(\Omega_t)}^p \right) \|w(t)\|_{L^{p+2}(\Omega_t)}^2.
\]
It follows from (32) and (33) that
\[
\|U(t, \tau)z\|_{L^p(\Omega_t)}^2 \leq C_9(1 + |\rho(t)|^{p+2})e^{-\delta_{1}(t-\tau)} + |\rho_0(t)|^2.
\]
Then there exists a constant \( K(t, \tau) > 0 \) such that
\[
\Psi'(t) \leq -G(t) + \left( \frac{3}{2} + \frac{1}{2\lambda_1^*} \right) \int_{\Omega_t} |\partial_t w|^2 dx + K(t, \tau)\|w(t)\|_{L^{p+2}(\Omega_t)}^2.
\]
Set
\[
\alpha = \min \left\{ \alpha_0, \left( \frac{3}{2} + \frac{1}{2\lambda_1^*} \right)^{-1} \right\}.
\]
Using the fact \( \alpha \leq 1 \) and inequality (36), we get
\[
\frac{d}{dt} G_\alpha(t) \leq -\alpha G(t) + K(t, \tau)\|w(t)\|_{L^{p+2}(\Omega_t)}^2 + \int_{\Omega_t} (f(u_2) - f(u_1))\partial_t wdx.
\]
By virtue of (37) and Gronwall’s lemma over \([\tau, t]\), we have
\[
G_\alpha(t) \leq G_\alpha(\tau)e^{\frac{\alpha}{3}(t-\tau)} + \sup_{s \in [\tau, t]} K(t, s) \int_{\tau}^{t} e^{\frac{\alpha}{3}(t-s)}\|w(s)\|_{L^{p+2}(\Omega_s)}^2 ds
\]
\[
+ \int_{\tau}^{t} e^{\frac{\alpha}{3}(t-s)} \int_{\Omega_s} (f(u_2) - f(u_1))\partial_t wdx ds.
\]
Taking \( \delta_2 = 2\alpha/3 \geq \delta_1 \) and using (37) leads to
\[
G(t) \leq 3G(\tau)e^{-\delta_2(t-\tau)} + 2 \sup_{s \in [\tau, t]} K(t, s) \int_{\tau}^{t} \|w(s)\|_{L^{p+2}(\Omega_s)}^2 ds
\]
\[
+ 2 \int_{\tau}^{t} \int_{\Omega_s} (f(u_2) - f(u_1))\partial_t wdx ds.
\]
From (34), we obtain
\[
G(\tau) \leq \frac{1}{2} |\rho_0(\tau)|^2.
\]
Hence, inequality (35) follows by choosing \( C_{\tau,s} = 4\sup_{s \in [\tau, t]} K(t, s) \).

**Lemma 4.3.** Under conditions \((A_1) - (A_2)\), the evolution process corresponding to equation (1) is pullback \( \mathcal{D} \)-asymptotically compact.
Proof. We divide our discussions into two steps.

**Step 1.** In order to apply Proposition 2.8, we will construct a contractive function based on (35). Given \( t \in \mathbb{R} \) and \( \varepsilon > 0 \), from (33) we have

\[
|\rho_0(\tau)|^2 = C_{10} e^{-\delta_0 \tau} \int_{-\infty}^{\tau} e^{\delta_0 s} \|g(s)\|^2_{L^2(\Omega_s)} ds + C_{11} \mathbf{f} + 1, \quad \tau \leq t.
\]

Since \( \delta_0 < \delta_2 \) and the integral above is non-increasing as \( \tau \) decreases, we get

\[
\lim_{\tau \to -\infty} |\rho_0(\tau)|^2 e^{-\delta_2 (t-\tau)} = \lim_{\tau \to -\infty} e^{-(\delta_2-\delta_0) \tau} C_{10} \left( \int_{-\infty}^{\tau} e^{\delta_0 s} \|g(s)\|^2_{L^2(\Omega_s)} ds \right) e^{-\delta_2 t} = 0.
\]

There exists \( \tau_\varepsilon = \tau_\varepsilon(t, \varepsilon) \leq t \) such that

\[
3|\rho_0(\tau_\varepsilon)|^2 e^{-\delta_2 (t-\tau_\varepsilon)} < \varepsilon^2.
\]

Define \( \phi_\varepsilon : B_0(\tau_\varepsilon) \times B_0(\tau_\varepsilon) \to \mathbb{R} \) by

\[
\phi_\varepsilon^2(z_1, z_2) = C_{\tau_\varepsilon} \int_{\tau_\varepsilon}^{t} \left[ u_1(s) - u_2(s) \right]_{L^2(\Omega_s)}^2 ds + 4 \int_{\tau_\varepsilon}^{t} \int_{\Omega_s} (f(u_2) - f(u_1)) \partial_t u dx ds,
\]

where the positive constant \( C_{\tau_\varepsilon} \) is defined by (35). It follows from Lemma 4.2 that

\[
\|U(t, \tau_\varepsilon) z_1 - U(t, \tau_\varepsilon) z_2 \|_{L^2(\Omega_s)} \leq \varepsilon + \phi_\varepsilon(z_1, z_2), \quad z_1, z_2 \in B_0(\tau_\varepsilon).
\]

**Step 2.** Due to Proposition 2.8, we only need to verify that \( \phi_\varepsilon \) is contractive on \( B_0(\tau_\varepsilon) \). Given any sequence \( \{z_n\}_{n \in \mathbb{N}} \) of \( B_0(\tau_\varepsilon) \), let \( (u_n, \partial_t u_n) \) be the solution corresponding to the initial data \( z_n \). From (21) we find

\[
\|U(s, \tau_\varepsilon) z_n\|_{L^2(\Omega_s)} \leq M_{\tau_\varepsilon, t}, \quad s \in [\tau_\varepsilon, t].
\]

It follows that for some constant \( C_{12} > 0 \) there holds

\[
\|u_n\|_{L^2(\tau_\varepsilon, t; H_0^1(\Omega_s))} \leq C_{12} \text{ and } \|\partial_t u_n\|_{L^2(\tau_\varepsilon, t; L^2(\Omega_s))} \leq C_{12}.
\]

By Lemma 3.6, we see that there exist \( u \) and a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that

\[
u_{n_k} \to u \text{ strongly in } L^2(\tau_\varepsilon, t; L^{p+2}(\Omega_s)).
\]

From (38) we have

\[
\lim_{k \to \infty} \lim_{t \to \infty} C_{\tau_\varepsilon} \int_{\tau_\varepsilon}^{t} \|u_{n_k}(s) - u_n(s)\|^2_{L^2(\Omega_s)} ds = 0.
\]

Consider the nonlinear term \( f \) in \( \phi_\varepsilon \):

\[
\int_{\tau_\varepsilon}^{t} \int_{\Omega_s} (f(u_{n_k}) - f(u_n))(\partial_t u_{n_k} - \partial_t u_n) dx ds
\]

\[
= \int_{\Omega_s} [F(u_{n_k}(t)) + F(u_n(t))] dx - \int_{\Omega_s} [F(u_{n_k}(\tau_\varepsilon)) + F(u_n(\tau_\varepsilon))] dx
\]

\[
- \int_{\tau_\varepsilon}^{t} \int_{\Omega_s} [f(u_{n_k}) \partial_t u_{n_k} + f(u_n) \partial_t u_n] dx ds.
\]

It follows from (9) and (10) that

\[
|F(u)| \leq C'(1 + |u|^{p+2}).
\]
Then the corresponding Nemytskii operator $H_F : L^{p+2} (\Omega_\ast) \rightarrow L^1 (\Omega_\ast)$ is continuous for each fixed $s$. By (38) we have

$$\lim_{k \to \infty} \lim_{t \to \infty} \int_{\Omega_\ast} [F(u_{n_k} (t)) + F(u_n (t))] dx \rightarrow \int_{\Omega_\ast} [F(u_{n_k} (\tau_\varepsilon)) + F(u_n (\tau_\varepsilon))] dx$$

$$= 2 \int_{\Omega_\ast} F(u(t)) dx - 2 \int_{\Omega_\ast} F(u(\tau_\varepsilon)) dx.$$  (40)

Since the Nemytskii operator $N_F : H^2_0 (\Omega_\ast) \rightarrow L^2 (\Omega_\ast)$ maps the bounded sets into the bounded sets [9], for any fixed $s$ we have $f(u_n) \rightarrow f(u)$ weakly in $L^2 (\Omega_\ast)$. That is,

$${\lim_{k \to \infty}} \lim_{t \to \infty} \int_{\tau_\varepsilon}^t \int_{\Omega_\ast} [f(u_{n_k}) \partial_t u_{n_k} + f(u_n) \partial_t u_n] dx ds$$

$$= 2 \int_{\tau_\varepsilon}^t \int_{\Omega_\ast} f(u(s)) \partial_t u(s) dx ds$$

$$= 2 \int_{\Omega_\ast} F(u(t)) dx - 2 \int_{\Omega_\ast} F(u(\tau_\varepsilon)) dx.$$  (41)

Combining (40) with (41) leads to

$${\lim_{k \to \infty}} \lim_{t \to \infty} \int_{\tau_\varepsilon}^t \int_{\Omega_\ast} (f(u_{n_k}) - f(u_n)) (\partial_t u_{n_k} - \partial_t u_n) dx ds = 0.$$  

Using this limit and (39), we know that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$${\lim_{k \to \infty}} \lim_{t \to \infty} \phi (z_{n_k}, z_{n_l}) = 0.$$  

Thus, following Proposition 2.8 we arrive at the desired result. 

**Theorem 4.4.** Suppose that conditions (A1) – (A2) hold. Then the process associated to the problem (1)-(3) admits a minimal pullback $\mathcal{D}$-attractor $\mathcal{A}$, where $\mathcal{D}$ is defined by (31). In addition, if $\delta_0 \leq 2 \delta_1/(p + 2)$, then $\mathcal{A}$ belongs to $\mathcal{D}$.

**Proof.** Using Proposition 2.6, and Lemmas 4.1 and 4.3 implies the existence of the minimal pullback $\mathcal{D}$-attractor. It remains to verify that the pullback $\mathcal{D}$-attractor belongs to $\mathcal{D}$ with $\delta_0 < 2 \delta_1/(p + 2)$. Since the class $\mathcal{D}$ is inclusion closed, by Proposition 2.6 we only need to show that $\widehat{B}_0$ belongs to $\mathcal{D}$, i.e.

$$\lim_{\tau \to -\infty} |\rho_0 (\tau)|^{p+2} e^{\delta_1 \tau} = 0.$$

From (33), we have

$$|\rho_0 (\tau)|^{p+2} e^{\delta_1 \tau}$$

$$= \left( C_{10} e^{-\tau \delta_0} \int_{-\infty}^{\tau} e^{\delta_0 s} \|g(s)\|^2_{L^2 (\Omega_\ast)} ds + C_{11} C_f + 1 \right)^{\frac{p+2}{2}} e^{\delta_1 \tau}$$

$$\leq 2^{\frac{p+2}{2}} \left[ C_{10} \left( \int_{-\infty}^{\tau} e^{\delta_0 s} \|g(s)\|^2_{L^2 (\Omega_\ast)} ds \right)^{\frac{p+2}{2}} e^{-\delta_0 \tau \frac{p+2}{2} + \delta_1 \tau} + (C_{11} C_f + 1)^{\frac{p+2}{2}} e^{\delta_1 \tau} \right].$$

Since

$$\delta_1 - \delta_0 \frac{p+2}{2} > 0,$$
we obtain
\[ \lim_{\tau \to -\infty} |\rho_0(\tau)|^{p+2}e^{\delta_1 \tau} = 0. \]
Consequently, the pullback $\mathcal{D}$-attractor belongs to $\mathcal{D}$.

REFERENCES

[1] M. M. Al-Gharabli and S. A. Messaoudi, Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term, J. Evol. Equ., 18 (2018), 105–125.
[2] C. Bardos and G. Chen, Control and stabilization for the wave equation, Part III: Domain with moving boundary, SIAM J. Control Optim., 19 (1981), 123–138.
[3] A. N. Carvalho, J. A. Langa and J. C. Robinson, Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems, Springer, New York, 2013.
[4] X. Chen and A. Friedman, A free boundary problem for an elliptic-hyperbolic system: An application to tumor growth, SIAM J. Math. Anal., 35 (2003), 974–986.
[5] I. Chueshov and I. Lasiecka, Attractors for second-order evolution equations with a nonlinear damping, J. Dynam. Differential Equations, 16 (2004), 469–512.
[6] I. Chueshov and I. Lasiecka, Global attractors for von Karman evolutions with a nonlinear boundary dissipation, J. Differential Equations, 198 (2004), 196–231.
[7] M. Conti, V. Pata and R. Temam, Attractors for processes on time-dependent space. Application to Wave equation, J. Differential Equations, 255 (2013), 1254–1277.
[8] D. R. da Costa, C. P. Dettmann and E. D. Leonel, Escape of particles in a time-dependent potential well, Phys. Rev. E, 83 (2011), 066211.
[9] L. C. Evans, Partial Differential Equations, 2nd ed., vol. 19, American Mathematical Society, Providence, RI, 2010.
[10] L. H. Fatori, M. A. Jorge Silva, T. F. Ma and Z. Yang, Long-time behavior of a class of thermoelastic plates with nonlinear strain, J. Differential Equations, 259 (2015), 4831–4862.
[11] Z. Feng, Duffing-van der Pol-type oscillator systems, Discrete Contin. Dyn. Syst., 7 (2014), 1231–1257.
[12] A. Kh. Khanmamedov, Existence of a global attractor for the plate equation with a critical exponent in unbounded domain, Appl. Math. Lett., 18 (2005), 827–832.
[13] A. Kh. Khanmamedov, Global attractors for the plate equation with a localized damping and a critical exponent in an unbounded domain, J. Differential Equations, 225 (2006), 528–548.
[14] A. Kh. Khanmamedov, Global attractors for von Karman equations with nonlinear interior dissipation, J. Math. Anal. Appl., 318 (2006), 92–101.
[15] P. E. Kloeden, P. Marín-Rubio and J. Real, Pullback attractors for a semilinear heat equation in non-cylindrical domain, J. Differential Equations, 244 (2008), 2062–2090.
[16] P. E. Kloeden, J. Real and C. Sun, Pullback attractors for a semilinear heat equation on time-varying domains, J. Differential Equations, 246 (2009), 4702–4730.
[17] I. Lasiecka, T. F. Ma and R. N. Monteiro, Long-time dynamics of vectorial von Karman system with nonlinear thermal effects and free boundary conditions, Discrete Contin. Dyn. Syst. Ser. B, 23 (2018), 1037–1072.
[18] J. Limaco, L. A. Medeiros and E. Zuazua, Existence, uniqueness and controllability for parabolic equations in non-cylindrical domain, in: Seventh Workshop on Partial Differential Equations, Part II, Rio de Janeiro, 2001, in: Mat. Contemp., 23 (2002), 49–70.
[19] J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod; Gauthier-Villars, Paris, 1969.
[20] T. F. Ma, P. Marín-Rubio and C. M. Surco Chuño, Dynamics of wave equation with moving boundary, J. Differential Equations, 262 (2017), 3317–3342.
[21] T. F. Ma and T. M. Souza, Pullback dynamics of non-autonomous wave equations with acoustic boundary condition, Differential Integral Equations, 30 (2017), 443–462.
[22] F. Meng, M. Yang and C. Zhong, Attractors for wave equations with nonlinear damping on time-dependent space, Discrete Cont. Dyn. Syst. B, 21 (2016), 205–225.
[23] F. Di Plinio, G. S. Duane and R. Temam, Time dependent attractor for the oscillon equation, Discrete Cont. Dyn. Syst., 29 (2011), 141–167.
[24] H. M. Soner and S. E. Shreve, A free boundary problem related to singular stochastic control: Parabolic case, Comm. Partial Differential Equations, 16 (1991), 373–424.
[25] J. Stefan, Über die Theorie der Eishbildung, insbesondere über die Eishbildung im Polarmeere, *Ann. Phys.*, **278** (1891), 269–286.

[26] C. Sun and Y. Yuan, *L*-type pullback attractors for a semilinear heat equation on time-varying domains, *Proc. Roy. Soc. Edinburgh Sect. A*, **145** (2015), 1029–1052.

[27] Z. Wang and S. Zhou, Existence and upper semicontinuity of random attractors for non-autonomous stochastic strongly damped wave equation with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **37** (2017), 2787–2812.

[28] Z. Wang and S. Zhou, Random attractor and random exponential attractor for stochastic non-autonomous damped cubic wave equation with linear multiplicative white noise, *Discrete Contin. Dyn. Syst.*, **38** (2018), 4767–4817.

[29] L. Yang and C. Zhong, Global attractor for plate equation with nonlinear damping, *Nonlinear Anal.*, **69** (2008), 3802–3810.

[30] Z. Yang and Z. Liu, Longtime dynamics of the quasi-linear wave equations with structural damping and supercritical nonlinearities, *Nonlinearity*, **30** (2017), 1120–1145.

[31] Z. Yang, and Z. Liu, Stability of exponential attractors for a family of semilinear wave equations with gentle dissipation, *J. Differential Equations*, **264** (2018), 3976–4005.

[32] F. Zhou, C. Sun and X. Li, Dynamics for the damped wave equations on time-dependent domains, *Discrete Contin. Dyn. Syst. B*, **23** (2018), 1645–1674.

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