MULTI-DIMENSIONAL KURAMOTO-SIVASHINSKY-ZAKHAROV-KUZNETSOV EQUATION POSED ON ADMISSIBLE MULTI-DIMENSIONAL DOMAINS

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Abstract. An initial-boundary value problem for the n-dimensional (n is a natural number from the interval [2,7]) Kuramoto-Sivashinsky-Zakharov-Kuznetsov equation posed on smooth bounded domains in $\mathbb{R}^n$ was considered. The existence and uniqueness of global regular solutions as well as their exponential decay have been established. A connection between an order of the stationary part of the equation and admissible dimensions of a domain has been revealed.

1. Introduction

In this work, we study the existence, uniqueness, regularity and exponential decay of global solutions to an initial-boundary value problem for the $n$-dimensional Kuramoto-Sivashinsky-Zakharov-Kuznetsov equation (KS-ZK)

$$\phi_t + \Delta^2 \phi + \Delta \phi + \Delta \phi_{x_1} + \frac{1}{2} |\nabla \phi|^2 = 0 \quad (1.1)$$

with the purpose to reveal a connection between an order of the stationary part of the KS-ZK equation and admissible dimensions of domains involved. Here $n$ is a natural number from the interval [2,7], $\Delta$ and $\nabla$ are the Laplacian and the gradient in $\mathbb{R}^n$. In [15], Kuramoto studied the turbulent phase waves and Sivashinsky in [25] obtained an asymptotic equation which modeled the evolution of a disturbed plane flame front. See also [9]. Mathematical results on initial and initial-boundary value problems for one-dimensional (1.1) are presented in [3, 4, 7, 8, 12, 16, 23, 26, 31], see references there for more information. Multi-dimensional problems for equations of (1.1) type can be found in [2, 5, 7, 21, 22, 23, 24, 26], where some results on existence, regularity and nonlinear stability of solutions have been presented. In

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one-dimensional case, Kuramoto-Sivashinsky type equations which included the (KdV) term $\phi_{xxx}$ have been considered [3, 4, 12, 16, 31]. The aim of these studies were intentions to understand simultaneous influences of dispersion and dissipation on stability of systems modeled by these equations. In [16], an initial-boundary value problem for the KdV equation was regularized by initial-boundary value problems for the Kuramoto-Sivashinsky equation with a consequent passage to the limit with respect to regularization parameters.

In multi-dimensional cases, $\phi_{xxx}$ must be replaced by $\Delta \phi_{x_1}$, Zakharov-Kuznetsov term, see [30], where the propagation of nonlinear ion-sonic waves in a plasma submitted to a magnetic field directed along the $x_1$-axis was studied. This Kuramoto-Sivashinsky-Zakharov-Kuznetsov (KS-ZK) equation may be useful in studies of nonlinear stability of a three-dimensional viscous film flowing down an inclined surface; see [29], where it has been observed that the three-dimensional structure of waves can suppress an instability due to presence of the fourth-order derivative term.

There is an extensive literature on existence, uniqueness and stability of global solutions for both ZK equation, see [10, 11, 26] and for the Kuramoto-Sivashinsky equation, see [7, 12, 18, 20, 22, 31]. Recently, appeared published papers on solvability of initial-boundary value problems for two-dimensional and three-dimensional Kuramoto-Sivashinsky-Zakharov-Kuznetsov equation in bounded and unbounded domains, see [17, 19]. In [18], an initial-boundary value problem for the n-dimensional Kuramoto-Sivashinsky equation posed on smooth domains in $\mathbb{R}^n$, $n \in [2, 7]$ has been studied.

This motivated us to consider n-dimensional Kuramoto-Sivashinsky-Zakharov-Kuznetsov equation and to define natural $n$ which allow to prove the existence, uniqueness and exponential decay of global regular solutions. Consequently, we call (1.1) Kuramoto-Sivashinsky-Zakharov-Kuznetsov equation (KS-ZK).

For $n$ dimensions, (1.1) can be rewritten in the form of the following system:

\begin{align}
(u_j)_t + \Delta^2 u_j + \Delta u_j + (\Delta u_j)_{x_1} + \frac{1}{2} \sum_{i=1}^{n} (u_i)^2_{x_j} &= 0, \quad (1.2) \\
(u_i)_{x_j} &= (u_j)_{x_i}, \quad i \neq j, \quad i, j = 1, ..., n, \quad (1.3)
\end{align}

where $u_j = \phi_{x_j}$. First essential problem that arises while one studies stability either for (1.1) or for (1.2)-(1.3) is a destabilizing effect of $\Delta u_j$ that may be damped by a dissipative term $\Delta^2 u_j$ provided a domain has
some specific properties. Naturally, so called "thin domains" appear where some dimensions are small while the others may be large, see [14, 24].

Our approach is based on the Faedo-Galerkin method and simple, transparent conditions for the geometry of a domain which suppress instability. These conditions were obtained using Steklov's inequalities.

Second essential problem is presence of semi-linear terms in (1.2) which are interconnected. Differently from the one-dimensional case, this does not allow to obtain the first estimate independent of time and solutions and leads to a connection between geometric properties of a domain and initial data.

Our work has the following structure: Section I is Introduction. Section 2 contains notations and auxiliary facts. In Section 3, formulation of an initial-boundary value problem for (1.2)-(1.3) posed on bounded smooth domains in $\mathbb{R}^n$ with boundary conditions of a solution and the Laplace operator of the solution on boundaries of domains are given. The existence and uniqueness of a global regular solution, exponential decay of the $H^2$-norm have been established. Moreover, a "smoothing" effect has been observed here. Section 4 contains conclusions.

2. Notations and Auxiliary Facts

Let $D_n$ be a sufficiently smooth domain in $\mathbb{R}^n$ satisfying the Cone condition, [1], and $x = (x_1, ..., x_n) \in D_n$, where $n$ is a fixed natural number $n \in [2, 7]$. We use the standard notations of Sobolev spaces $W^{k,p}$, $L^p$ and $H^k$ for functions and the following notations for the norms [1] for scalar functions $f(x,t)$:

$$
\|f\| = \int_{D_n} |f|^2 dx, \quad \|f\|_{L^p(D_n)} = \int_{D_n} |f|^p dx,
$$

$$
\|f\|_{W^{k,p}(D_n)} = \sum_{0 \leq \alpha \leq k} \|D^\alpha f\|_{L^p(D_n)}, \quad \|f\|_{H^k(D_n)} = \|f\|_{W^{k,2}(D_n)}.
$$

When $p = 2$, $W^{k,p}(D_n) = H^k(D_n)$ is a Hilbert space with the scalar product

$$
((u, v))_{H^k(D_n)} = \sum_{|\beta| \leq k} (D^\beta u, D^\beta v), \quad \|u\|_{L^\infty(D_n)} = \text{ess sup}_{D_n} |u(x)|.
$$

We use a notation $H^k_0(D_n)$ to represent the closure of $C^\infty_0(D_n)$, the set of all $C^\infty$ functions with compact support in $D_n$, with respect to the norm of $H^k(D_n)$.
Lemma 2.1 (Steklov’s Inequality [27]). Let \( v \in H^1_0(0, L) \). Then
\[
\frac{\pi^2}{L^2} \|v\|^2 \leq \|v_x\|^2.
\] (2.1)

Lemma 2.2 (Differential form of the Gronwall Inequality). Let \( I = [t_0, t_1] \). Suppose that functions \( a, b : I \to \mathbb{R} \) are integrable and a function \( a(t) \) may be of any sign. Let \( u : I \to \mathbb{R} \) be a positive differentiable function satisfying
\[
u_t(t) \leq a(t)u(t) + b(t), \quad \text{for } t \in I \text{ and } u(t_0) = u_0, \] (2.2)
then
\[
u(t) \leq u_0 e^{\int_{t_0}^t a(\tau) d\tau} + \int_{t_0}^t e^{\int_{t_0}^s a(\tau) d\tau} b(s) ds.
\]

Proof. Multiply (2.2) by the integrating factor \( e^{\int_{t_0}^t a(\tau) d\tau} \) and integrate the result from \( t_0 \) to \( t \). \( \square \)

The next Lemmas will be used in estimates:

Lemma 2.3 (See: [13], Theorem 9.1). Let \( n \) be a natural number from the interval \([2, 7]\); \( D_n \) be a sufficiently smooth bounded domain in \( \mathbb{R}^n \) satisfying the Cone condition and \( v \in H^4(D_n) \cap H^1_0(D_n) \). then for all natural \( n \in [2, 7] \)
\[
\sup_{D_n} |v(x)| \leq C_n \|v\|_{H^4(D_n)}.
\] (2.3)
The constant \( C_n \) depends on \( n, D_n \).

Lemma 2.4. Let \( f(t) \) be a continuous positive function such that
\[
f'(t) + (\alpha - kf^n(t))f(t) \leq 0, \quad t > 0, \quad n \in \mathbb{N},
\] (2.4)
\[
\alpha - kf^n(0) > 0, \quad k > 0.
\] (2.5)
Then
\[
f(t) < f(0)
\] (2.6)
for all \( t > 0 \).

Proof. Obviously, \( f'(0) + (\alpha - kf^n(0))f^n(0) \leq 0 \). Since \( f \) is continuous, there exists \( T > 0 \) such that \( f(t) < f(0) \) for every \( t \in [0, T) \). Suppose that \( f(0) = f(T) \). Integrating (2.4), we find
\[
f(T) + \int_0^T (\alpha - kf^n(t))f(t) dt \leq f(0).
\]
Since
\[ \int_0^T (\alpha - k f^n(t)) f(t) \, dt > 0, \]
then \( f(T) < f(0) \). This contradicts that \( f(T) = f(0) \). Therefore, \( f(t) < f(0) \) for all \( t > 0 \).

The proof of Lemma 2.4 is complete. \( \square \)

### 3. KS-ZK System Posed on Bounded Smooth Domains in \( \mathbb{R}^n \)

#### Special Basis

Let \( \Omega_n \) be the minimal nD-parallelepiped containing a given bounded smooth domain \( \bar{D}_n \in \mathbb{R}^n \), \( n = 2, \ldots, 7 \):
\[ \Omega_n = \{ x \in \mathbb{R}^n; x_i \in (0, L_i) \}, \ i = 1, \ldots, n. \]

Fixing any natural \( n = 2, \ldots, 7 \), consider in \( Q_n = D_n \times (0, t) \) the following initial-boundary value problem:

\[
\begin{align*}
(u_j)_t + \Delta^2 u_j + \Delta u_j + (\Delta u_j)_{x_1} + \frac{1}{2} \sum_{i=1}^{n} (u_i)^2_{x_j} &= 0; \\
(u_i)_{x_j} = (u_j)_{x_i}, & i \neq j; \\
(u_j)_{x_j} &= 0, \quad t > 0; \\
(u_j)(x, 0) &= u_{j0}(x), \ j = 1, \ldots, n; \ x \in D_n. 
\end{align*}
\]  
(3.1)

\[
\begin{align*}
(a^n f)^2 &\leq \| \nabla f \|^2, \quad a^2 \| f \|^2 \leq \| \Delta f \|^2, \quad a \| \nabla f \|^2 \leq \| \Delta f \|^2, \quad (3.5) \\
a^2 \| \Delta f \|^2 &\leq \| \Delta^2 f \|^2, \quad a \| \Delta f_{x_1} \|^2 \leq \| \Delta^2 f \|^2, \quad \text{where } a = \sum_{i=1}^{n} \frac{\pi^2}{L_i^2}. \quad (3.6)
\end{align*}
\]

**Lemma 3.1.** Let \( f \in H^4(D_n) \cap H_0^1(D_n) \), \( \Delta f|_{\partial D_n} = 0 \). Then
\[ \| \nabla f \|^2 \geq \sum_{i=1}^{n} \| f_{x_i} \|^2. \]

**Proof.** By definition,
\[
\| \nabla f \|^2 = \sum_{i=1}^{n} \| f_{x_i} \|^2.
\]

Extending \( f \in H_0^1(D_n) \) by 0 into the minimal parallelepiped \( \Omega_n \) as \( \bar{f} \), making use of Steklov’s inequalities in \( \Omega_n \) and taking into account that \( \| \nabla \bar{f} \|_{\Omega_n} = \| \nabla f \|_{D_n} \), we get
\[
\| \nabla f \|^2 \geq \sum_{i=1}^{n} \frac{\pi^2}{L_i^2} \| f \|^2 = a \| f \|^2.
\]
On the other hand,
\[ a\|f\|^2 \leq \|\nabla f\|^2 = -\int_{D_n} f\Delta f \, dx \leq \|\Delta f\|\|f\|. \]
This implies
\[ a\|f\| \leq \|\Delta f\| \quad \text{and} \quad a^2\|f\|^2 \leq \|\Delta f\|^2. \]
Consequently, \( a\|\nabla f\|^2 \leq \|\Delta f\|^2. \) Similarly,
\[ \|\Delta f\|^2 = (\Delta^2 f, f) \leq \|\Delta^2 f\|\|f\| \leq \frac{1}{a}\|\Delta^2 f\|\|\Delta f\|, \]
\[ \|\Delta f_{x_1}\|^2 \leq \|\nabla \Delta f\|^2 = -(\Delta^2 f, \Delta f) \leq \|\Delta^2 f\|\|\Delta f\| \leq \frac{1}{a}\|\Delta^2 f\|^2. \]
Assertions of Lemma 3.1 follow from these inequalities. Proof of Lemma 3.1 is complete. □

Remark 3.1. Assertions of Lemma 3.1 are true if the function \( f \) is replaced respectively by \( u_j, j = 1, ..., n \).

Lemma 3.2. Let \( a > 1 \). In conditions of Lemma 3.1,
\[ \|f\|^2(t)_{H^2(D_n)} \leq 3\|\Delta f\|^2(t), \quad (3.7) \]
\[ \|f\|^2(t)_{H^1(D_n)} \leq 5\|\Delta^2 f\|^2(t), \quad (3.8) \]
\[ \sup_{D_n} \|f(x)\| \leq C_s\|\Delta^2 f\|, \quad \text{where} \ C_s = 5C_n. \quad (3.9) \]

Proof. To prove (3.8), making use of Lemma 3.1, we find
\[ \|f\|^2_{H^1(D_n)} = \|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 + \|\nabla \Delta f\|^2 + 2\|\Delta^2 f\|^2 \]
\[ \leq \left( \frac{1}{a^4} + \frac{1}{a^2} + \frac{1}{a} + 1 \right)\|\Delta^2 f\|^2. \]
Since \( a > 1 \), then (3.8) follows. Similarly, (3.7) can be proved. Moreover, taking into account Lemma 2.3, we get (3.9). □

Theorem 3.1. Let \( n \) be a natural number from the closed interval \([2,7]\), let \( D_n \in \mathbb{R}^n \) be a bounded smooth domain satisfying the Cone condition and \( \Omega_n \) be the minimal \( nD \)-parallelepiped containing \( \bar{D}_n \). Let
\[ 2a = 2\sum_{i=1}^{n} \frac{\pi^2}{L_i^2} > 3 + 5^{1/2}, \quad \theta = 1 - \frac{1}{a} - \frac{1}{a^{1/2}} > 0. \quad (3.10) \]
Given
\[ u_{j0}(D_n) \in H^2(D_n) \cap H^1_0(D_n), \quad j = 1, ..., n \]
such that
\[ \theta - \frac{2C^2\lambda^3}{a\theta} \left( \sum_{j=1}^{n} \|\Delta u_j\|^2(0) \right) > 0, \quad (3.11) \]

Then there exists a unique global regular solution to (3.1)-(3.4):

\[ u_j \in L^\infty(\mathbb{R}^+; H^2(D_n) \cap H^1_0(D_n)) \cap L^2(\mathbb{R}^+; H^4(D_n) \cap H^1_0(D_n)); \]

\[ u_{jt} \in L^2(\mathbb{R}^+; L^2(D_n)), \quad j = 1, \ldots, n. \]

Moreover,

\[ \sum_{j=1}^{n} \|\Delta u_j\|^2(t) \leq \left( \sum_{j=1}^{n} \|\Delta u_{j0}\|^2 \right) \exp\{-a^2t\theta/2\} \quad (3.12) \]

and

\[ \sum_{i=1}^{n} \|\Delta u_i\|^2(t) + \int_{0}^{t} \sum_{i=1}^{n} \|\Delta^2 u_i\|^2(\tau)d\tau \leq C \sum_{i=1}^{n} \|\Delta u_{i0}\|^2, \quad t > 0. \]

**Proof.** Let \( \{w_i(x)\} \) be eigenfunctions of the following problem:

\[ \Delta^2 w_i - \lambda_i w_i = 0, \quad x \in D_n; \quad w_i|_{\partial D_n} = \Delta w_i|_{\partial D_n} = 0. \]

We construct approximate solutions to (3.1)-(3.4) in the form

\[ u^N_j(x, t) = \sum_{i=1}^{N} g^j_i(t) w_i(x); \quad j = 1, \ldots, n. \]

Unknown functions \( g^j_i(t) \) satisfy the following initial problems:

\[ \frac{d}{dt} \langle u^N_j, w_j \rangle(t) + \langle \Delta u^N_j, \Delta w_j \rangle(t) - \langle \nabla u^N_j, \nabla w_j \rangle(t) \\
- \langle (\nabla u^N_j)_{x_j}, \nabla w_j \rangle(t) + \frac{1}{2} \sum_{i=1}^{n} \langle (u^N_i)^2, w_j \rangle(t) = 0, \quad (3.13) \]

\[ g^j_i(0) = g^j_{i0}, \quad j = 1, \ldots, n. \quad (3.14) \]

By Caratheodory’s existence theorem, [6], Theorems 1.2 of Chapter 1, there exist solutions to (3.13)-(3.14) at least locally in \( t \). All the estimates we will prove will be done on smooth solutions to (3.1)-(3.4). Naturally, the same estimates are true also for approximate solutions \( u^N_j \).

**Estimate I** Multiply (3.1) by \( 2\Delta^2 u_j \) to obtain

\[ \frac{d}{dt} \|\Delta u_j\|^2(t) + 2\|\Delta^2 u_j\|^2(t) - 2\|\Delta^2 u_j\|(t) \|\Delta u_j\|(t) \]
\[-2\| (\Delta u_j)_{x_1} \|(t) \| \Delta^2 u_j \|(t) \]
\[+2 \sum_{i=1}^{n} ((u_i)(u_i)_{x_j}, \Delta^2 u_j)(t) \leq 0, \ j = 1, ..., n.\]

By Lemma 3.1, \(a \| \Delta u_j \| \leq \| \Delta^2 u_j \|, \ a^{1/2}(\| \Delta u_j \|)_{x_1} \leq \| \Delta^2 u_j \|, \) hence the last inequality can be rewritten as

\[\frac{d}{dt} \| \Delta u_j \|^2(t) + 2 \left(1 - \frac{1}{a} - \frac{1}{a^{1/2}}\right) \| \Delta^2 u_j \|^2(t) \]
\[+2 \sum_{i=1}^{n} ((u_i)(u_i)_{x_j}, \Delta^2 u_j)(t) \leq 0, \ j = 1, ..., n.\]

By definition, \(\theta = 1 - \frac{1}{a} - \frac{1}{a^{1/2}}\) must be positive. Solving the inequality

\[1 - \frac{1}{a} - \frac{1}{a^{1/2}} > 0\]

with respect to \(a\), we find that \(2a > 3 + 5^{1/2}\), see (3.10). In [17, 19], it was defined \(a > 3\). This condition is more restrictive, but more explicit.

Making use of Lemma 3.1 and definition of \(\theta\), we get

\[\frac{d}{dt} \| \Delta u_j \|^2(t) + 2\theta \| \Delta^2 u_j \|^2(t) \]
\[\leq 2 \sum_{i=1}^{n} (\sup_{D} |u_i(t)| \| \nabla u_i \|(t)) \| \Delta^2 u_j \|(t), \ j = 1, ..., n. \] (3.15)

Taking into account Lemmas 2.5, 3.1, 3.2, we can rewrite (3.15) as

\[\frac{d}{dt} \| \Delta u_j \|^2(t) + 2\theta \| \Delta^2 u_j \|^2(t) \]
\[\leq 2 \left[ \sum_{i=1}^{n} \sup_{D_{a}} |u_i(x, t)| \| \nabla u_i \|(t) \right] \| \Delta^2 u_j \|(t) \]
\[\leq 2 \left[ C_s \sum_{i=1}^{n} \| \Delta^2 u_i \|(t) \| \nabla u_i \|(t) \right] \| \Delta^2 u_j \|(t); j = 1, ..., n. \] (3.16)

Summing over \(j = 1, ..., n\) and making use of Lemma 3.1, we rewrite (3.16) in the form:

\[\frac{d}{dt} \sum_{j=1}^{n} \| \Delta u_j \|^2(t) + 2\theta \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \]
\begin{align*}
&\leq 2C sn \left( \sum_{j=1}^{n} \| \nabla u_j \|(t) \right) \left[ \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \right] \\
&\leq \left[ \frac{\theta}{2} + \frac{2C^2 n^2}{\theta} \left( \sum_{j=1}^{n} \| \nabla u_j \|^2(t) \right) \right] \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \\
&\leq \left[ \frac{\theta}{2} + \frac{2C^2 n^3}{\theta \theta} \left( \sum_{j=1}^{n} \| \nabla u_j \|^2(t) \right) \right] \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \\
&\leq \left[ \frac{\theta}{2} + \frac{2C^2 n^3}{a\theta} \left( \sum_{j=1}^{n} \| \Delta u_j \|^2(t) \right) \right] \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t).
\end{align*}

Taking this into account, we transform (3.16) as follows:

\begin{align*}
&\frac{d}{dt} \sum_{j=1}^{n} \| \Delta u_j \|^2(t) + \frac{\theta}{2} \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \\
&+ \left[ \theta - \frac{2C^2 n^3}{a\theta} \left( \sum_{j=1}^{n} \| \Delta u_j \|^2(t) \right) \right] \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \leq 0. \quad (3.17)
\end{align*}

Condition (3.11) and Lemma 2.4 guarantee that

\[ \theta - \frac{2C^2 n^3}{a\theta} \left( \sum_{j=1}^{n} \| \Delta u_j \|^2(t) \right) > 0, \quad t > 0. \]

Hence, by Lemma 3.1, (3.17) can be rewritten as

\begin{align*}
&\frac{d}{dt} \sum_{j=1}^{n} \| \Delta u_j \|^2(t) + \frac{\theta}{2} \sum_{j=1}^{n} \| \Delta^2 u_j \|^2(t) \leq 0. \quad (3.18)
\end{align*}

Integrating, we get

\begin{align*}
\sum_{i=1}^{n} \| \Delta u_j \|^2(t) \leq \sum_{j=1}^{n} \| \Delta u_{j_0} \|^2 \exp \{ -a^2 \theta t/2 \}. \quad (3.19)
\end{align*}

Since

\[ \theta - \frac{2C^2 n^3}{a\theta} \left( \sum_{j=1}^{n} \| \Delta u_j \|^2(t) \right) > 0, \quad t > 0, \]

returning to (3.17), we find

\begin{align*}
\sum_{i=1}^{n} \| \Delta u_i \|^2(t) + \int_{0}^{t} \sum_{i=1}^{n} \| \Delta^2 u_i \|^2(\tau) d\tau \leq C \sum_{i=1}^{n} \| \Delta u_{i_0} \|^2. \quad (3.20)
\end{align*}
By Lemma 3.1, \( a^{1/2} \| \nabla \Delta u_j \| \leq \| \Delta^2 u_j \| \), then directly from (3.1),

\[
(u_j)_t \in L^2(\mathbb{R}^+; L^2(D_n)), \ j = 1, \ldots, n.
\]

Obviously, these inclusions and estimates (3.19), (3.20) do not depend on \( N \) that allow us to pass to the limit as \( n \to \infty \) in (3.13), (3.14) and to prove the existence part of Theorem 3.1.

**Lemma 3.3.** A regular solution of Theorem 3.1 is uniquely defined.

**Proof.** Let \( u_j \) and \( v_j, j = 1, \ldots, n \) be two distinct solutions to (3.1)-(3.4). Denoting \( w_j = u_j - v_j \), we come to the following system:

\[
(w_j)_t + \Delta^2 w_j + \Delta w_j + \Delta (u_j)_{x_1} + \frac{1}{2} \sum_{i=1}^n \left( u_i^2 - v_i^2 \right)_{x_j} = 0, \tag{3.21}
\]

\[
(w_j)_{x_i} = (w_i)_{x_j}, \ i \neq j, \tag{3.22}
\]

\[
w_j|_{\partial D_n} = \Delta w_j|_{\partial D_n} = 0, \ t > 0; \tag{3.23}
\]

\[
w_j(x, 0) = 0 \text{ in } D_n, \ j = 1, \ldots, n. \tag{3.24}
\]

Multiply (3.21) by \( 2w_j \) to obtain

\[
\frac{d}{dt} \| w_j \|^2(t) + 2 \| \Delta w_j \|^2(t) - 2 \| \nabla w_j \|^2(t) \leq 2 \frac{1}{a^{1/2}} \| \Delta w_j \|^2(t),
\]

\[
-2 \| \nabla (w_i)_{x_1} \| \| \nabla w_j \| \leq \sum_{i=1}^n (|\{ u_i + v_i \} w_i, (w_j)_{x_j}|)(t). \tag{3.25}
\]

Making use of Lemmas 2.3, 3.1, 3.2, we estimate

\[
I_1 = 2 \| \nabla (w_i)_{x_1} \| \| \nabla w_j \| \leq 2 \frac{1}{a^{1/2}} \| \Delta w_j \|^2(t),
\]

\[
2 \| w_j \|^2(t) \leq \frac{2}{a} \| \Delta w_j \|^2(t).
\]

\[
I_2 = \sum_{i=1}^n (|\{ u_i + v_i \} w_i, (w_j)_{x_j}|) \leq \sum_{i=1}^n (\sup_{D_n} |\{ u_i + v_i \}| \| w_i \| \| (w_j)_{x_j} \|)
\]

\[
\leq C_s \sum_{i=1}^n \| w_i \| \left( \| \Delta^2 (u_i + v_i) \| \right) \| \nabla (w_j) \|
\]

\[
\leq C_s \frac{1}{a^{1/2}} \sum_{i=1}^n \| w_i \| \left( \| \Delta^2 (u_i + v_i) \| \right) \| \Delta w_j \|
\]

\[
\leq \frac{c}{2} \| \Delta w_j \|^2 + \frac{nC^2}{2a \epsilon} \left( \sum_{i=1}^n \{ \| \Delta^2 u_i \|^2 + \| \Delta^2 v_i \|^2 \} \right) \sum_{j=1}^n \| w_j \|^2.
\]
Here $\epsilon$ is an arbitrary positive number. Substituting $I_1, I_2$ into (3.25), we get

$$\frac{d}{dt} \|w_j\|^2(t) + (2\theta - \frac{\epsilon}{2})\|\Delta w_j\|^2(t) \leq \frac{nC_2^2}{2a\epsilon} \left[ \sum_{i=1}^{n} \{ \|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2 \} \right] \sum_{j=1}^{n} \|w_j\|^2. \quad (3.26)$$

Taking $\epsilon = 4\theta$ and summing up over $j = 1, \ldots, n$, we transform (3.26) as follows:

$$\frac{d}{dt} \sum_{j=1}^{n} \|w_j\|^2(t) \leq C \left[ \sum_{i=1}^{n} \{ \|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2 \} \right] \sum_{j=1}^{n} \|w_j\|^2, \quad i = 1, \ldots, n.$$

Since by (3.20),

$$\|\Delta^2 u_i\|^2(t), \|\Delta^2 v_i\|^2(t) \in L^1(\mathbb{R}^+),$$

then by Lemma 2.2,

$$\|w_j\|^2(t) \equiv 0 \quad j = 1, \ldots, n, \text{ for all } t > 0.$$

Hence

$$u_j(x, t) \equiv v_j(x, t); \quad j = 1, \ldots, n.$$

The proof of Lemma 3.3, and consequently, Theorem 3.1 are complete.

4. Conclusions

In this work, we studied the initial-boundary value problem for the n-dimensional Kuramoto-Sivashinsky-Zakharov-Kuznetsov system (1.2), (1.3) posed on bounded smooth domains $D_n \in \mathbb{R}^n$. The problem on a bounded domains $D_n$ has homogeneous conditions for a solution and the Laplace operator of the solution on the boundaries of the domains. Making use of a special base, we established the existence, uniqueness and exponential decay of the $H^2$-norm as well as "smoothing effect": for initial data from $H^2(D_n) \cap H_0^1(D_n)$, we proved that for all natural $n \in [2, 7]$ solutions pertained to $L^2(\mathbb{R}^+; H^4(D_n))$. A set of admissible domains has been defined which eliminate destabilizing effects of terms $\Delta u_j$ by dissipation of $\Delta^2 u_j$.

Since these problems do not admit the first estimate independent of $t$ and solutions, in order to prove the existence of global solutions,
we put conditions connecting geometrical properties of domains with initial data, see (3.11).

For $n = 2, 3$, initial-boundary value problems have been studied in [17, 19], nevertheless, we included these cases because they are covered by the same scheme and are special cases of the general results.

Conflict of Interests

The author declares that there is no conflict of interest regarding the publication of this paper.

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