CURVES ON K3 SURFACES IN DIVISIBILITY TWO

YOUNGHAN BAE AND TIM-HENRIK BUELLES

ABSTRACT. We prove a conjecture of Maulik, Pandharipande, and Thomas expressing the Gromov–Witten invariants of K3 surfaces for divisibility two curve classes in all genus in terms of weakly holomorphic quasimodular forms of level two. Then, we establish the holomorphic anomaly equation in divisibility two in all genus. Our approach involves a refined boundary induction, relying on the top tautological group of the moduli space of smooth curves, together with a degeneration formula for reduced virtual fundamental class with imprimitive curve classes. We use the double ramification relations with target variety as a new tool to prove the initial condition. The relationship between the holomorphic anomaly equation for higher divisibility and the conjectural multiple cover formula of Oberdieck and Pandharipande is discussed in detail and illustrated with several examples.

CONTENTS

0. Introduction 2
1. Initial condition 7
2. Degeneration formula 12
3. Compatibilities 14
4. Proof of Theorem 1 and 3 21
5. Examples 23
6. Multiple cover formula and Hecke operators 25
Appendix A. A proof of degeneration formula 34
References 36

Date: June 1, 2020.
0. Introduction

Let $S$ be a complex nonsingular projective K3 surface and $\beta \in H_2(S, \mathbb{Z})$ an effective curve class. Gromov–Witten invariants of $S$ are defined via intersection theory on the moduli space $\overline{M}_{g,n}(S, \beta)$ of stable maps from $n$-pointed genus $g$ curves to $S$. This moduli space comes with a virtual fundamental class. However, the virtual class vanishes for $\beta \neq 0$ so, instead, we use the reduced class\(^1\)

$$\overline{M}_{g,n}(S, \beta)^{red} \in A_{g+n}(\overline{M}_{g,n}(S, \beta), \mathbb{Q}).$$

For integers $a_i \geq 0$ and cohomology classes $\gamma_i \in H^*(S, \mathbb{Q})$ we define

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{S}^g, \beta = \int_{\overline{M}_{g,n}(S, \beta)^{red}} \prod_{i=1}^{n} \psi_i^{a_i} \cup \text{ev}_i^*(\gamma_i),$$

where $\text{ev}_i: \overline{M}_{g,n}(S, \beta) \to S$ is the evaluation at $i$-th marking and $\psi_i$ is the cotangent class at the $i$-th marking. By the deformation invariance of the reduced class, the invariant only depends on the norm $\langle \beta, \beta \rangle$ and the divisibility of the curve class $\beta$.

0.1. Quasimodularity. Gromov–Witten invariants of K3 surfaces for primitive curve classes are well-understood since the seminal paper by Maulik, Pandharipande, and Thomas [24]. The invariants are coefficients of weakly holomorphic\(^2\) quasimodular forms with pole of order at most one [24, Theorem 4]. For imprimitive curve classes, the quasimodularity is conjectured with the level structure [24, Section 7.5].

The quasimodularity can be stated in a precise sense via elliptic K3 surfaces. Let

$$\pi: S \to \mathbb{P}^1$$

be an elliptic K3 surface with a section and denote by $B, F \in H_2(S, \mathbb{Z})$ the class of the section resp. a fiber. For any $m \geq 1$ one defines the descendent potential

$$F_{g,m}(\tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)) = \sum_{h \geq 0} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,mB+hF}^S q^{h-m}.$$  

Note that this generating series involves curve classes $mB + hF$ of different divisibilities, bounded by $m$.

\(^1\)We will identify this class with its image under the cycle class map $A_* \to H_2$.

\(^2\)Weakly holomorphic means holomorphic on the upper half plane with possible pole at the cusp $i\infty$. 
It is convenient to use the following homogenized insertions which will lead to quasimodular forms of pure weight. Let \( 1 \in H^0(S) \) and \( p \in H^4(S) \) be the identity resp. the point class. Denote

\[
W = B + F \in H^2(S)
\]

and let

\[
U = \mathbb{Q}\langle F, W \rangle \subset H^2(S)
\]

be the hyperbolic plane in \( H^2(S) \) and let \( U^\perp \subset H^2(S) \) be its orthogonal complement with respect to the intersection form. We only consider second cohomology classes which are pure with respect to the decomposition

\[
H^2(S, \mathbb{Q}) \cong \mathbb{Q}\langle F \rangle \oplus \mathbb{Q}\langle W \rangle \oplus U^\perp.
\]

Following [7, Section 4.6], define a modified degree function \( \deg \) by

\[
\deg(\gamma) = \begin{cases} 
2 & \text{if } \gamma = W \text{ or } p, \\
1 & \text{if } \gamma \in U^\perp, \\
0 & \text{if } \gamma = F \text{ or } 1.
\end{cases}
\]

For \( m \geq 1 \), consider the Hecke congruence subgroup of level \( m \)

\[
\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod m \right\}
\]

and let \( \text{QMod}(m) \) be the space of quasimodular forms for the congruence subgroup \( \Gamma_0(m) \subset \text{SL}(2, \mathbb{Z}) \). Let \( \Delta(q) \) be the modular discriminant

\[
\Delta(q) = q \prod_{n \geq 1} (1-q^n)^{24}.
\]

Our first main result proves level two quasimodularity of \( F_{g,2} \):

**Theorem 1.** Let \( \gamma_1, \ldots, \gamma_n \in H^*(S) \) be homogeneous on the modified degree function \( \deg \). Then \( F_{g,2} \) is the Fourier expansion of a quasimodular form

\[
F_{g,2}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n)) \in \frac{1}{\Delta(q)^2} \text{QMod}(2)
\]

of weight \( 2g - 12 + \sum_i \deg(\gamma_i) \) with pole at \( q = 0 \) of order at most 2.

0.2. **Holomorphic anomaly equation.** In the physics literature, the (conjectural) holomorphic anomaly equation [3, 4] predicts hidden structures of the Gromov–Witten partition function associated to Calabi–Yau varieties. For the past few years, there has been an extensive work to prove the holomorphic anomaly equation in many cases: local \( \mathbb{P}^2 \) [21], the quintic threefold [10, 13], K3 surface with primitive curve classes [27], elliptic fibration [28] and \( \mathbb{P}^2 \) relative to a smooth cubic [5].
In our context, the holomorphic anomaly equation fixes the non-holomorphic parameter of the Gromov–Witten partition function of K3 surfaces in terms of lower weight partition functions. Every quasimodular form for $\Gamma_0(m)$ can be written uniquely as a polynomial in $C_2$ with coefficients which are modular forms for $\Gamma_0(m)$ \cite[Proposition 1]{15}. Here,

$$C_2(q) = -\frac{1}{24}E_2(q)$$

is the renormalized second Eisenstein series. Assuming quasimodularity, the holomorphic anomaly equation computes the derivative of $F_{g,m}$ with respect to the $C_2$ variable. See \cite{27} for the proof of holomorphic anomaly equation for K3 surfaces with primitive curve classes and \cite{28} for the holomorphic anomaly equation associated to elliptic fibrations.

Define an endomorphism \cite[Section 0.6]{27}

$$\sigma : H^*(S^2) \rightarrow H^*(S^2)$$

by the following assignments:

$$\sigma(\gamma \boxtimes \gamma') = 0$$

if $\gamma$ or $\gamma' \in H^0(S) \oplus \mathbb{Q}\langle F \rangle \oplus H^4(S)$, and for $\alpha, \alpha' \in U^\perp$,

$$\sigma(W \boxtimes W) = \Delta_{U^\perp}, \quad \sigma(W \boxtimes \alpha) = -\alpha \boxtimes F,$$

$$\sigma(\alpha \boxtimes W) = -F \boxtimes \alpha, \quad \sigma(\alpha, \alpha') = \langle \alpha, \alpha' \rangle F \boxtimes F.$$ 

We denote by $\sigma = \sigma_1 \boxtimes \sigma_2$ the Künneth decomposition of $\sigma$.

Recall the virtual fundamental class for trivial curve classes which will play a role for the holomorphic anomaly equation. For $\beta = 0$ we have an isomorphism

$$\overline{M}_{g,n}(S, 0) \cong \overline{M}_{g,n} \times S$$

and the virtual class is given by

$$[\overline{M}_{g,n}(S, 0)]^{vir} = \begin{cases} [\overline{M}_{0,n} \times S] & \text{if } g = 0, \\ c_2(S) \cap [\overline{M}_{1,n} \times S] & \text{if } g = 1, \\ 0 & \text{if } g \geq 2. \end{cases}$$

Also, consider the pullback of the diagonal of $\mathbb{P}^1$

$$\Delta_{\mathbb{P}^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^{2} \delta_i \boxtimes \delta_i^\vee.$$
Define the generating series

$$H_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n)$$

(1) \[= F_{g-1,m}(\alpha; \gamma_1, \ldots, \gamma_n, \Delta_{p^i}) + 2 \sum_{\{g=g_1+g_2\} \atop {i \in \{1, 2\}}} F_{g_1,m}(\alpha_{I_1}; \gamma_{I_1}, \delta_i) F_{g_2}(\alpha_{I_2}; \gamma_{I_2}, \delta_i^\vee) - 2 \sum_{i=1}^n F_{g,m}(\alpha_{\psi_i}; \gamma_1, \ldots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \ldots, \gamma_n) + 20 \sum_{i<j}^n \langle \gamma_{i}, F \rangle F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_{i-1}, F, \gamma_{i+1}, \ldots, \gamma_n) - 2 \sum_{i,j}^n F_{g,m}(\alpha; \gamma_1, \ldots, \sigma_{\text{th}}(\gamma_i, \gamma_j), \ldots, \sigma_{\text{th}}(\gamma_i, \gamma_j), \ldots, \gamma_n),
\]

where $F_{\text{vir}}$ denotes the generating series for virtual fundamental class. In most cases this term vanishes. The equation takes almost the same form for arbitrary $m$, only the last two terms acquire a factor of $\frac{1}{m}$. The appearance of these factors is explained in Section 6.2, see also Example 14. We conjecture that the holomorphic anomaly equation has the following form:

**Conjecture 2.**

(2) \[\frac{d}{dC_2} F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = H_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) \cdot\]

For primitive curve classes, the holomorphic anomaly equation is proven in [27]. We prove Conjecture 2 when $m = 2$:

**Theorem 3.** For any $g \geq 0$,

(3) \[\frac{d}{dC_2} F_{g,2}(\alpha; \gamma_1, \ldots, \gamma_n) = H_{g,2}(\alpha; \gamma_1, \ldots, \gamma_n) \cdot\]

### 0.3. Multiple cover formula.

Motivated by the Katz–Klemm–Vafa (KKV) formula, Oberdieck and Pandharipande conjectured a formula which computes imprimitive invariants from the primitive invariants:

---

\[\text{Here, instead of descendent insertions we use a tautological class } \alpha \in R^*(\overline{M}_{g,n}), \text{ see the comment in Section 1.}\]
Conjecture 4. ([26, Conjecture C2]) For a primitive curve class \( \beta \),
\[
\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,m\beta} = \sum_{d \mid m} d^{2g-3+\deg}\langle \tau_{a_1}(\varphi_d(\gamma_1)) \cdots \tau_{a_n}(\varphi_d(\gamma_n)) \rangle_{g,\varphi_d(m\beta)}.
\]

The invariants on the right hand side are with respect to primitive curve classes\(^4\).

Our proof of Theorem 1 provides an algorithm to reduce divisibility two invariants to low genus invariants for which the multiple cover formula is known\(^5\). However, further idea seems to be necessary to prove the full conjecture. The degeneration to the normal cone of a smooth elliptic fiber \( E \subset S \) intertwines invariants of \( S \) with invariants of \( \mathbb{P}^1 \times E \) in a non-trivial way. This phenomenon is illustrated in Example 19 for the genus 2 invariants \( \langle \tau_0(p)^2 \rangle_{2,2\beta} \).

In Section 6 we apply Conjecture 4 to an elliptic K3 surface to deduce a conjectural multiple cover formula for the descendent potentials \( F_{g,m} \). The multiple cover formula for any divisibility \( m \) is then expressed in terms of Hecke operators and translation \( q \mapsto q^d \) acting on the primitive series \( F_{g,1} \). Together with the holomorphic anomaly equation for primitive curve classes [27] this naturally leads to the above conjecture for the holomorphic anomaly equation for higher divisibility.

**Proposition* 5.** Let \( m \geq 1 \). Assume the multiple cover formula (4) holds for all curve classes of divisibility \( d \mid m \) and all descendent insertions. Then the holomorphic anomaly equation (2) holds.

0.4. **Plan of the paper.** We prove the quasimodularity and the holomorphic anomaly equation by induction on the genus and the number of markings. In Section 1, we derive the multiple cover formula, which implies the holomorphic anomaly equation, for genus 0, genus 1 and some genus 2 decendents from the KKV formula. The genus 2 computation relies on double ramification relations with target variety. This result

\(^4\)Section 6 contains all relevant definitions.

\(^5\)The genus 0 and genus 1 cases are proved by Lee and Leung in [19, 20]. Their proof involves degeneration formula in symplectic geometry which is not possible in algebraic geometry. We present an algebro-geometric approach using the KKV formula.
serves as the initial condition for our induction. In Section 3.3, compatibility of the holomorphic anomaly equation with the degeneration formula is presented. The degeneration formula for the reduced class reduces arbitrary descendents to descendents with at most one point insertion. In Section 4, we use previous results to prove Theorem 1 and 3. The property of the top tautological group $R^{g-1}(M_{g,n})$ reduces higher genus cases to lower genus invariants discussed in Section 1. In Section 6 we discuss the multiple cover formula for descendent potentials and derive the candidate for the holomorphic anomaly equation in higher divisibility.

Acknowledgements. We are grateful to G. Oberdieck, R. Pandharipande, J. Shen, L. Wu and Q. Yin for many discussions on the Gromov–Witten theory of K3 surfaces. We want to thank D. Radchenko for useful suggestions on quasimodular forms.

Y. B. was supported by ERC-2017-AdG-786580-MACI and the Korea Foundation for Advanced Studies. T.-H. B. was supported by ERC-2017-AdG-786580-MACI.

The project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation program (grant agreement No. 786580).

1. Initial condition

This section contains a proof of the multiple cover formula in genus 0 and genus 1 for any divisibility $m$. It is a direct consequence of the KKV formula. However, as initial condition for our induction we also require a special case in genus 2, which cannot be easily deduced from the KKV formula. We treat this descendent potential separately, using double ramification relations [2] for K3 surfaces. This approach is likely to give relations in any genus and will be pursued in the future.

The multiple cover formula for divisibility $m$ implies the holomorphic anomaly equation for divisibility $m$ by Proposition 16. This section thus provides all initial conditions used in Section 4. We will find it convenient to use the descendent potential form Conjecture 22 of the multiple cover formula, which is equivalent to Conjecture 4 by Lemma 23. The compatibility with respect to restriction to boundary strata, as discussed in Section 3.1, will be used frequently.
For $2g - 2 + n > 0$, let

$$R^*(\mathcal{M}_{g,n}) \subseteq H^{2*}(\mathcal{M}_{g,n})$$

be the tautological ring of $\mathcal{M}_{g,n}$. For a tautological class $\alpha \in R^*(\mathcal{M}_{g,n})$, we consider the invariants

$$\langle \alpha; \gamma_1, \ldots, \gamma_n \rangle = \int_{[\mathcal{M}_{g,n}(S,\beta)]_{red}} \pi^* \alpha \prod_{i=1}^{n} \text{ev}_i^*(\gamma_i),$$

where $\pi: \mathcal{M}_{g,n}(S,\beta) \to \mathcal{M}_{g,n}$ is the stabilization morphism. We write

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = \sum_{h \geq 0} \langle \alpha; \gamma_1, \ldots, \gamma_n \rangle_{g,mB+hF} q^{h-m}$$

for the generating series in divisibility $m$. By the usual trading of cotangent line classes, these generating series are related to the ones defined via cotangent classes on $\mathcal{M}_{g,n}(S,\beta)$. Thus, they define equivalent data.

1.1. Revisiting KKV formula. The Katz–Klemm–Vafa (KKV) formula implies that the generating series of $\lambda_g$-integrals

$$F_{g,m}(\lambda_g; \cdot)$$

satisfy the multiple cover formula [30]. Here, $\lambda_g = c_g(E_g)$ is the top Chern class of the rank $g$ Hodge bundle $E_g$ on $\mathcal{M}(S,\beta)$. The KKV formula will be the starting point of our genus induction.

**Proposition 6.** The multiple cover formula holds in genus 0 and genus 1 for all $m \geq 1$.

**Proof.** When $g = 0,1$, the tautological ring $R^*(\mathcal{M}_{g,n})$ is spanned by boundary strata. Thus, one can replace descendents $\alpha \in R^*(\mathcal{M}_{g,n})$ by classes in $H^*(S)$. By the divisor equation and the dimension constraint, we can reduce to the case $F_{0,2}(\emptyset)$ and $F_{1,2}(\tau_0(p))$.

The genus 0 case is covered by the full Yau–Zaslow formula [17, 30]. The genus 1 case follows from the genus 2 KKV formula. Using the boundary expression of $\lambda_2$ on $\mathcal{M}_2$, we have

$$F_{2,m}(\lambda_2; \cdot) = \frac{1}{240} F_{1,m}(\psi_1; \Delta_s) + \frac{1}{1152} F_{0,m}(\cdot; \Delta_s, \Delta_s)$$

$$= \frac{1}{10} F_{1,m}(\tau_0(p)) + \frac{1}{60} D^2 q F_{0,m}(\emptyset),$$

where $\Delta_s \subset S \times S$ is the diagonal class. Therefore, $F_{1,m}(\tau_0(p))$ satisfies Conjecture 22. \qed
Similarly, we can use the boundary expression for
\( \lambda_3 \in R^3(\overline{M}_3) \)
and deduce the multiple cover formula for the genus 2 descendent potential
\[ F_{2,m}(\tau_1(p)) \]
In fact, we observe the following more general consequence of the boundary expression for \( \lambda_{g+1} \):

**Proposition 7.** Let \( m \geq 1 \) and \( g \geq 1 \). Assume the multiple cover formula Conjecture 22 holds for \( m \) and all descendents of genus \( < g \).

Then Conjecture 22 holds for \( F_{g,m}(\tau_{g-1}(p)) \).

**Proof.** Let \( \delta \in R^1(\overline{M}_g) \) be the boundary divisor corresponding to a curve with nonseparating node. Denote two half edges as \( h \) and \( h' \). Recall that \((−1)^g \lambda_g\) is equal to the double ramification cycle \( DR_g(\emptyset) \) with the empty condition. We use this formula for genus \( g+1 \). By [14, Theorem 1],
\[
(-1)^{g+1} \lambda_{g+1} = DR_{g+1}(\emptyset)
\]
\[
= \frac{1}{2} \left[ - \frac{1}{(g+1)!} \sum_{w=0}^{r-1} \left( \frac{w^2}{2} (\psi_h + \psi_{h'}) \right)^g \right] \delta + \text{lower genus},
\]
where \([\cdots],r\) is the coefficient of the linear part of a polynomial in \( r \).

The leading term is nonzero by Faulhaber’s formula.

By [18, Lemma 5.2], \( \psi \)-monomials in \( R^{\geq g}(\overline{M}_{g,n}) \) can be written as boundary strata with \( \psi \)-classes. Thus we can write\(^6\)
\[
(\psi_1 + \psi_2)^g = c \quad \frac{1}{2} + \text{lower genus}
\]
in \( R^g(\overline{M}_{g,2}) \) for some \( c \in \mathbb{Q} \). Therefore it suffices to prove that \( c \) is nonzero. Recall that \( \lambda_g \lambda_{g-1} \) vanishes on \( \overline{M}_{g,n} \setminus M^r_{g,n} \), so
\[
\int_{\overline{M}_{g,2}} (\psi_1 + \psi_2)^g \lambda_g \lambda_{g-1} = c \int_{\overline{M}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}.
\]
Both integrals are nonzero by [14, Lemma 8]. Therefore we get the result.

\(^6\)Number under each vertex is the genus and legs correspond to markings.
1.2. **Double ramification relations.** In this section we prove Conjecture 4 for the special genus 2 descendent potential

\[ F_{2,m}(\tau_1(\gamma)\tau_0(\rho)), \quad \gamma \in H^2(S), \]

using *double ramification relations with target variety* developed in [1, 2]. This completes our set of initial data.

Let \( \mathcal{Pic}_{g,n} \) be the Picard stack for the universal curve over the stack of prestable curves \( \mathcal{M}_{g,n} \). Let

\[ \pi : \mathcal{C} \to \mathcal{Pic}_{g,n}, \quad s_i : \mathcal{Pic}_{g,n} \to \mathcal{C}, \quad \mathcal{L} \to \mathcal{C} \]

be the universal curve, the \( i \)-th section and the universal line bundle. The following operational classes on \( \mathcal{Pic}_{g,n} \) are obtained from the universal structures:

- \( \psi_i = c_1(s_i^*\omega) \in A^1_{\text{op}}(\mathcal{Pic}_{g,n}) \),
- \( \xi_i = c_1(s_i^*\mathcal{L}) \in A^1_{\text{op}}(\mathcal{Pic}_{g,n}) \),
- \( \eta = \pi^*(c_1(\mathcal{L})^2) \in A^1_{\text{op}}(\mathcal{Pic}_{g,n}) \).

Let \( A = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) be a vector of integers satisfying

(5) \[ \sum_i a_i = d, \]

where \( d \) is the degree of the line bundle. We denote by \( P_{c,r}^{g,A} \) the codimension \( c \) component of the class

\[ \sum_{\Gamma \in \mathcal{G}_{g,n,d}} \frac{r^{-h_1(\Gamma)}}{\text{Aut}(\Gamma)} \sum_{w \in W_{\Gamma,r}} \prod_{v \in V(\Gamma)} \exp \left( \frac{1}{2} a_i^2 \psi_i + a_i \xi_i \right) \prod_{v \in V(\Gamma)} \exp \left( -\frac{1}{2} \eta(v) \right) \cdot \prod_{e = (h, h')} \frac{1 - \exp \left( -\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'}) \right)}{\psi_h + \psi_{h'}}. \]

We refer to [2] for details about the notations. This expression is polynomial in \( r \) when \( r \) is sufficiently large. Let \( P_{g,A}^c \) be the constant part of \( P_{g,A}^{c,r} \).

**Theorem 8.** ([2, Theorem 8]) \( P_{g,A,d}^c = 0 \) for all \( c > g \) in \( A^1_{\text{op}}(\mathcal{Pic}_{g,n}) \).

After restricting \( P_{g,A,d}^c \) to (5), this expression is a polynomial in \( a_1, \ldots, a_{n-1} \). The polynomiality will be used to get refined relations.

Let \( L \) be a line bundle on \( S \) with degree

\[ \int_{\beta} c_1(L) = d. \]
The choice of a line bundle $L$ induces a morphism
$$\varphi_L: \overline{M}_{g,n}(S, \beta) \to \Pic_{g,n}, \ [f: C \to S] \mapsto (C, f^*L).$$
Then Theorem 8 gives relations
$$P^c_{g,A,d}(L) = \varphi_L^*P^c_{g,A,d} \cap [\overline{M}_{g,n}(S, \beta)]_{\text{red}} = 0$$
in $A_{g+n-c}(\overline{M}_{g,n}(S, \beta))$. In Section 3.1 we check that these relations are compatible with the multiple cover formula.

We return to our problem in genus two. By the Getzler–Ionel vanishing on $\overline{M}_{2,n}$, the dimension constraint, and the divisor equation any descendent insertion reduces to the following three cases:
$$F_{2,m}(\tau_1(p)), \ F_{2,m}(\tau_0(p)^2), \ F_{2,m}(\tau_1(\gamma)\tau_0(p)) \quad \text{with } \gamma \in H^2(S).$$
The first case is treated in Proposition 7 and follows from the KKV formula in genus three and lower genus. The second case for $m = 2$ is treated as part of the proof of Theorem 1 in Section 4. We use (6) to prove the multiple cover formula for the third case. The point class $p$ is thought of as the product of $F$ and $W$.

**Proposition 9.** For $\gamma \in H^2(S)$, the generating series $F_{2,m}(\tau_1(\gamma)\tau_0(p))$ satisfies Conjecture 22.

**Proof.** We treat the case $\gamma = F$ first. We use the line bundle $\mathcal{O}_S(F)$ on $S$. The $\eta$-class vanishes in this case because $\langle F, F \rangle = 0$. Consider the $[a_1^4]$-coefficient of
$$P^3_{2,A,m}(F)|_{a_2 = m - a_1}.$$ Notice that there is only one boundary stratum with a genus 2 vertex
\[
\begin{array}{c}
1 \\
2 \\
0 \\
2
\end{array}
\]
and this term does not contribute to the $[a_1^4]$-coefficient. Therefore the $[a_1^4]$-coefficient is
$$-\frac{1}{2}\xi_1\xi_2\psi_1 - \frac{1}{2}\xi_1\xi_2\psi_2 - \frac{1}{2}m\xi_1\psi_1\psi_2 + \frac{3}{4}m\xi_2\psi_1\psi_2 + \text{lower genus}.$$ Integrating
$$\text{ev}_2^*(W)P^3_{2,A,m}(F)|_{a_2 = m - a_1}$$
against the reduced class gives
$$-\frac{1}{2}F_{2,m}(\tau_1(F)\tau_0(p)) - \frac{1}{2}F_{2,m}(\tau_0(F)\tau_1(p)) + \text{lower genus}.$$
The second term equals $-\frac{m}{2} F_{2,m}(\tau_1(p))$ by the divisor equation. Thus, $F_{2,m}(\tau_1(F) \tau_0(p))$ is a linear combination of terms which satisfy Conjecture 22. Switching the role of $F$ and $W$, we get the same result for $\gamma = W$.

Now consider $\gamma \in U^\perp$. We use a similar argument as above. This time, however, we consider the $[a_1^3]$-coefficient of

$$\text{ev}^*_1(\gamma) \text{ev}^*_2(W) F_{2,A,m}(F) |_{a_3 = m-a_1-a_2}.$$  

Since $\gamma \in U^\perp$, there is only one boundary stratum which can produce $F_{2,m}(\tau_1(\gamma) \tau_0(p))$:

$\begin{array}{ccc}
1 & \rightarrow & 2 \\
2 & \leftarrow & 0 & \leftarrow & 3
\end{array}$

This term does not contribute to the $[a_1^3]$-coefficient because the weight from the edge is a multiple of $a_2$. Therefore the $[a_1^3]$-coefficient of (7) gives

$$-\frac{1}{8} F_{2,m}(\tau_2(\gamma) \tau_0(W) \tau_0(F)) - \frac{1}{4} F_{2,m}(\tau_1(\gamma) \tau_0(W) \tau_1(F))$$

$$-\frac{1}{8} F_{2,m}(\tau_0(\gamma) \tau_0(W) \tau_2(F)) + \text{lower genus} = 0.$$  

We thus conclude by the divisor equation for $W$. \hfill \Box

**Remark 10.** In fact, for $\gamma \in U^\perp$ the above generating series vanishes (and thus trivially satisfies the multiple cover formula). A proof in the primitive case is given in [8, Lemma 4].

## 2. Degeneration Formula

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. For $m \geq 1$, let $\beta = mB + hF$ be a curve class. Choose a smooth fiber $E$ of $S \to \mathbb{P}^1$. Let $\epsilon : \mathcal{S} \to \mathbb{A}^1$ be the total space of the degeneration to the normal cone of $E$ in $S$. This space corresponds to the degeneration

$$S \sim S \cup_E \mathbb{P}^1 \times E.$$  

Over the center $t : 0 \hookrightarrow \mathbb{A}^1$, the fiber is $S \cup_E \mathbb{P}^1 \times E$ and over $t \neq 0$, the fiber is isomorphic to $S$. Let $\overline{M}_{g,n}(\epsilon, \beta)$ be the moduli space of stable maps to the degeneration $\mathcal{S}$. Over $t \neq 0$, this moduli space is isomorphic to $\overline{M}_{g,n}(S, \beta)$ and over $t = 0$, this moduli space parametrizes stable maps to the expanded target

$$\tilde{S}_0 = S \cup_E \mathbb{P}^1 \times E \cup_E \cdots \cup_E \mathbb{P}^1 \times E.$$
Let
\[ \nu = (g_1, g_2, n_1, n_2, h_1, h_2) \]
be a splitting of the discrete data \( g, n, h \) and let \( \beta_i = mB + h_iE \) be the splitting of the curve class. An ordered partition of \( m \)
\[ \mu = (\mu_1, \ldots, \mu_l) \]
specifies the contact order along the relative divisor \( E \).

Let \( L = \text{length}(\mu) \) and \( \overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu \) be the fiber product
\[ \overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu = \overline{M}_{g_1,n_1}(S/E, \beta_1)_\mu \times_{E^l} \overline{M}_{g_2,n_2}(\mathbb{P}^1 \times E/E, \beta_2)_\mu \]
of the boundary evaluations at relative markings\(^7\) and let
\[ \iota_{\nu\mu} : \overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu \rightarrow \overline{M}_{g,n}(\mathcal{S}_0, \beta) \]
be the finite morphism.

The degeneration formula for reduced class has a special feature in terms of evaluation at relative markings. Let \( x \) be the intersection of the section of the elliptic fibration and the fiber \( E \). We consider \((E, x)\) as an abelian variety. Let \( K \) be the kernel of the following morphism between abelian varieties
\[ E^l \rightarrow \text{Pic}^0(E), \ (x_i) \mapsto \mathcal{O}\left( \sum \mu_i(x_i - x) \right). \]
The subvariety \( K \) is nonsingular of pure codimension 1 and the number of connected components is equal to \( (\gcd(\mu_i)_{i=1}^l)^2 \). A lift \( \tilde{\beta} \) of the curve class \( \beta \) to the total family \( \mathcal{S} \) defines a line bundle \( \mathcal{O}(\tilde{\beta}) \). The evaluation at relative markings factors through \( K \times E^l \) because the flat family \( \epsilon \) gives a linear equivalence of line bundles on each fiber. Let \( \Delta_K = K \subset K \times E^l \) be the diagonal embedding.

**Theorem 11.** The reduced virtual class of maps to the degeneration (8) satisfies the following properties.

(i) For \( \iota_t : \{t\} \hookrightarrow \mathbb{A}^1 \), the Gysin pullback of reduced class is given by
\[ \iota_t^1[\overline{M}_{g,n}(\epsilon, \beta)]^{\text{red}} = [\overline{M}_{g,n}(\mathcal{S}_t, \beta)]^{\text{red}}. \]

(ii) For the special fiber,
\[ [\overline{M}_{g,n}(\mathcal{S}_0, \beta)]^{\text{red}} = \sum_{\nu, \mu} \frac{\prod \mu_i}{l!} \iota_{\nu\mu}^* [\overline{M}_{g,n}(\mathcal{S}_0, \nu)_\mu]^{\text{red}}. \]

\(^7\)We put \( \cdot \) to indicate (possibly) disconnected theory. Namely, for each connected component \( C \) of the domain curve, intersection of \( C \) with the relative divisor \( E \) is nontrivial.
(iii) On the special fiber, we have the factorization
\[ [\overline{M}_{g,n}(S_0,\nu)]_{\text{red}} = \Delta_K \left([\overline{M}_{g_1,n_1}(S/E,\beta_1)]_{\mu_{\text{red}}} \times [\overline{M}_{g_2,n_2}(\mathbb{P}^1 \times E,\beta_2)]_{\nu_{\text{vir}}} \right). \]

Proof. When \( m \geq 1 \), the reduced class of the disconnected moduli space \( \overline{M}_{g,n}(S/E,\beta) \) vanishes. Therefore disconnected theory can only appear on the bubble \( \mathbb{P}^1 \times E \). The proof is given in Appendix A. \( \square \)

Denote an ordered cohomology weighted partition by 
\[ \mu = ((\mu_1,\delta_1),\ldots,(\mu_l,\delta_l)) \text{, } \delta_i \in H^*(E) \]
and let \( \omega \in H^2(E) \) be the point class. The descendent potential for the pair \( (S,E) \) is defined analogously to the absolute case:
\[ F_{g,m}(\alpha;\gamma_1,\ldots,\gamma_n \mid \mu) = \sum_{h \geq 0} \langle \alpha;\gamma_1,\ldots,\gamma_n \mid \mu \rangle_{g,mB+hF} q^{h-m}. \]

The descendent potential for the pair \( (\mathbb{P}^1 \times E,E) \) is defined by
\[ G_{g,m}(\bullet;\gamma_1,\ldots,\gamma_n \mid \mu) = \sum_{h \geq 0} \langle \alpha;\gamma_1,\ldots,\gamma_n \mid \mu \rangle_{g,mB+hF} q^{h}. \]
As a corollary, we get the degeneration formula of reduced Gromov–Witten invariants.

Corollary 12. Let \( \gamma_1,\ldots,\gamma_n \in H^*(S) \) and choose a lift of these cohomology classes to the total space \( S \). Then
\[ F_{g,m}(\tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)) = \sum_{\mu} \frac{\prod_i H_i}{\Pi_i} F_{g,m}(\ldots \mid \mu) \cdot G_{g,m}(\ldots \mid (\mu \vee), \omega), \]
where
\[ \mu \vee = ((\mu_1,\delta_1^\vee),\ldots,(\mu_l,\delta_l^\vee)) \text{ and } \mu_\omega = ((\mu_1,\omega),\ldots,(\mu_l,\omega)). \]

3. Compatibilities

This section contains all relevant compatibilities used in the paper. First, we comment on tautological relations, used in Section 1 and Section 4. Then we discuss the compatibility of the holomorphic anomaly equation with the divisor equation, as well as the degeneration formula.
3.1. **Tautological relations.** We show that the relations among descendent potentials coming from tautological relations on $\bar{M}_{g,n}(S, \beta)$ are compatible with the multiple cover formula.

(i) A crucial point for the compatibility with boundary expressions for tautological classes on $\bar{M}_{g,n}(S, \beta)$ is the splitting behavior of the reduced class. A boundary stratum with a separating node corresponding to

$$g = g_1 + g_2, \quad m = m_1 + m_2$$

has a contribution

$$F_{g_1, m_1} \cdot F^{\text{vir}}_{g_2, m_2} + F^{\text{vir}}_{g_1, m_1} \cdot F_{g_2, m_2}.$$ 

Because the virtual class for non-zero curve classes vanishes, the contribution $F^{\text{vir}}$ is a number. Therefore, no non-trivial products of generating series appear when using boundary expressions. Then the compatibility follows from two facts. Firstly, restriction to a boundary divisor creates a diagonal class $\Delta_S$. Since

$$\left(\deg - \deg\right)(\Delta_S) = 0,$$

the factor $m^{\deg - \deg}$ in Conjecture 22 remains unchanged. Secondly, since $\deg(\Delta_S) = 2$, the formula for

$$\ell = 2g - 3 + \deg$$

is compatible with boundary restriction.

(ii) For $c > g, A \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, consider the series of relations

$$P_{g,bA,db}^c(L^{\otimes b}) = 0$$

obtained by tensoring the line bundle $L$ by $b$ times. For each coefficient of a monomial in $a_i$-variables, this expression is polynomial in $b$ and hence each coefficient of $b$-variable is a relation. As a consequence, each term of a relation $P_{g,A,m}^c(F)$ gives the same value of

$$m^{\deg - \deg},$$

where $\deg(\xi) = 1$ and $\deg(\xi) = 0$, as in Definition 0.1. The same holds true with the roles of $F$ and $W$ interchanged. Thus, the relations are compatible with the operator

$$m^{\deg - \deg} T[m]_{2g-3+\deg},$$

which gives the multiple cover formula in Conjecture 22.
3.2. Divisor equation. For primitive curve classes, it was pointed out in [27, Section 3.6, Case (i)] that the holomorphic anomaly equation in genus 0 is compatible with the divisor equation. For divisibility $m$, let

$$\frac{d}{d\gamma} = \langle \gamma, F \rangle D_q + m \langle \gamma, W \rangle, \quad \gamma \in H^2(S).$$

Then the compatibility with the divisor equation corresponds to

$$H_{g,m}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_{n-1}}(\gamma_{n-1}) \tau_0(\gamma_n))$$

$$= \frac{d}{d\gamma_n} H_{g,m}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_{n-1}}(\gamma_{n-1}))$$

$$- 2k F_{g,m}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_{n-1}}(\gamma_{n-1}))$$

$$+ \sum_{i=1}^{n-1} H_{g,m}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_{i-1}}(\gamma_i \cup \gamma_n) \ldots \tau_{a_{n-1}}(\gamma_{n-1})), $$

where $k$ is the weight of $F_{g,m}(\tau_{a_1}(\gamma_1) \ldots \tau_{a_{n-1}}(\gamma_{n-1}))$ and we have used the commutator relation

$$\left[ \frac{d}{dC_2}, D_q \right] = -2k.$$

The same check as in the primitive case works for arbitrary divisibility. This relies on the fact that divisor equation for $W$ is the same as applying the differential operator

$$D_q = q \frac{d}{dq}$$

to the generating series. Indeed, for the curve class $\beta = mB + hF$,

$$\langle \beta, W \rangle = -2m + h + m = h - m,$$

which matches the exponent of $q^{h-m}$ in the generating series $F_{g,m}$. The divisor equation for $F$ acts as multiplication by $m$ on the generating series.

In Section 4, the refined induction reduces any generating series ultimately to genus 0 and 1. We thus have to justify compatibility of the holomorphic anomaly equation for generating series of the form

$$F_{1,m}(\tau_0(p) \tau_0(\gamma_1) \ldots \tau_0(\gamma_n)), \quad \gamma_i \in H^2(S).$$

This compatibility however is true. By Proposition 6, the multiple cover formula, which is compatible with the divisor equation, holds in genus $\leq 1$. Thus, we also find compatibility for the holomorphic anomaly equation.
**Example 13.** We consider $F_{0,m}(\tau(W)^2)$ to illustrate the above compatibility. The corresponding series $H_{0,m}$ is

$$H_{0,m}(\tau(W)^2) = -4F_{0,m}(\tau(1)\tau(W)) + \frac{40}{m}F_{0,m}(\tau(F)\tau(W)).$$

In the above notation, $\gamma_n = W$ is the second $W$ and $k = -10$ is the weight of $F_{0,m}(\tau(W))$. We have to check that

$$H_{0,m}(\tau(W)^2) = D_qH_{0,m}(\tau(W)) + 20F_{0,m}(\tau(W)).$$

By the dilaton equation, we can verify

$$H_{0,m}(\tau(W)^2) - D_qH_{0,m}(\tau(W)) = -2D_qF_{0,m}(\tau(1)) - 4F_{0,m}(\tau(W)) + \frac{20}{m}F_{0,m}(\tau(F)\tau(W))$$

$$= 4D_qF_{0,m}(\emptyset) - 4D_qF_{0,m}(\emptyset) + 20F_{0,m}(\tau(W))$$

$$= 20F_{0,m}(\tau(W)).$$

**Example 14.** The above example in genus 0 illustrates how the second last term in the holomorphic anomaly equation (2) plays a role. We consider

$$F_{1,m}(\tau_1(W)\tau_0(W))$$

to show how the last term, i.e. the term involving $\sigma$, interacts non-trivially with the other terms. The corresponding series $H_{1,m}$ are

$$H_{1,m}(\tau_1(W)\tau_0(W)) = 2F_{0,m}(\tau_1(W)\tau_0(W)\tau_0(1)\tau_0(F))$$

$$- 2\left(F_{1,m}(\tau_2(1)\tau_0(W)) + F_{1,m}(\tau_1(W)\tau_1(1))\right)$$

$$+ \frac{20}{m}\left(F_{1,m}(\tau_1(F)\tau_0(W)) + F_{1,m}(\tau_1(W)\tau_0(F))\right)$$

$$- \frac{2}{m}F_{1,m}(\psi; \Delta_U^{-1}),$$

$$H_{1,m}(\tau_1(W)) = 2F_{0,m}(\tau_1(W)\tau_0(1)\tau_0(F))$$

$$- 2F_{1,m}(\tau_2(1))$$

$$+ \frac{20}{m}F_{1,m}(\tau_1(F)).$$

Let $k = -8$ be the weight of $F_{1,m}(\tau_1(W))$. Then (11) is equivalent to

$$H_{1,m}(\tau_1(W)\tau_0(W)) = D_qH_{1,m}(\tau_1(W)) - 2kF_{1,m}(\tau_1(W)).$$

The term $F_{1,m}(\psi; \Delta_U^{-1})$ can be computed using

$$\psi = \frac{1}{24}[\delta_0] \in A^1(M_{1,1}),$$
where \([\delta_0] \in A^1(\overline{M}_{1,1})\) is the class representing the nodal curve with a self node. The genus 0 contribution vanishes by the divisor equation. Since the rank of \(U^\perp\) is 20, we obtain the genus 1 contribution

\[
F_{1,m}(\psi_1; \Delta_{U^\perp}) = 20 F_{1,m}(\tau_0(p)) .
\]

The divisor equation for \(F\) implies that

\[
\frac{20}{m} F_{1,m}(\tau_1(W)\tau_0(F)) = 20 F_{1,m}(\tau_1(W)) + \frac{20}{m} F_{1,m}(\tau_0(p)) .
\]

We can now verify the compatibility by a direct computation using divisor and dilaton equation:

\[
H_{1,m}(\tau_1(W)\tau_0(W)) = D_q H_{1,m}(\tau_1(W)) - 2 F_{1,m}(\tau_1(W)) - 2 F_{1,m}(\tau_1(W)\tau_1(1)) + \frac{20}{m} F_{1,m}(\tau_0(p)) + \frac{20}{m} F_{1,m}(\tau_1(W)\tau_0(F)) - \frac{2}{m} F_{1,m}(\psi_1; \Delta_{U^\perp}) = D_q H_{1,m}(\tau_1(W)) - 4 F_{1,m}(\tau_1(W)) + \frac{20}{m} F_{1,m}(\tau_0(p)) + \frac{20}{m} F_{1,m}(\tau_1(W)\tau_0(F)) - \frac{40}{m} F_{1,m}(\tau_0(p)) = D_q H_{1,m}(\tau_1(W)) + 16 F_{1,m}(\tau_1(W)) ,
\]

3.3. **Relative holomorphic anomaly equations.** Assuming quasi-modularity, we have two ways to compute the derivative of \(F_{g,m}\) with respect to \(C_2\):

(i) Apply the degeneration formula Corollary 12, together with the holomorphic anomaly equations for \((S, E)\) and \((\mathbb{P}^1 \times E, E)\).

(ii) Apply the holomorphic anomaly equation (3) for \(S\), followed by the degeneration formula for each term.

We argue that both ways yield the same result. This compatibility is parallel to the compatibility proved in [28, Section 4.6]. We first state the holomorphic anomaly equations for the relevant relative geometries.

**Relative** \((\mathbb{P}^1 \times E, E)\). Consider \(\pi: \mathbb{P}^1 \times E \rightarrow \mathbb{P}^1\) as a trivial elliptic fibration over \(\mathbb{P}^1\). For the pair \((\mathbb{P}^1 \times E, E)\) the holomorphic anomaly equation holds for cycle-valued generating series [28]. The equation for descendent potentials can thus be obtained by integrating against tautological classes \(\alpha \in R^*(\overline{M}_{g,n})\). For insertions \(\gamma_i \in H^*(\mathbb{P}^1 \times E, \mathbb{Q})\)
we will simply write $\gamma$. Let $\mu = ((\mu_1, \delta_1), \ldots, (\mu_l, \delta_l))$ be an ordered cohomology weighted partitions. For ordered cohomology weighted partitions $\mu$ and $\mu'$, denote

$$G_{g,m}^{\bullet} (\mu \mid \alpha, \gamma \mid \mu') = \sum_{h \geq 0} \langle \mu \mid \alpha; \gamma \mid \mu' \rangle_{g,m, P^1 + hE} q^h$$

by the disconnected rubber generating series for $P^1 \times E$ relative to divisors at 0 and $\infty$. Let $\Delta_E \subset E \times E$ be the class of the diagonal. Define the generating series

$$P_{g,m}^{\bullet} (\alpha; \gamma \mid \mu) = G_{g-1,m}^{\bullet} (\alpha; \gamma, \Delta_{P^1} \mid \mu)$$

$$+ 2 \sum_{g=g_1+g_2 \geq 1} \prod_{i=1}^h \frac{b_i}{h!} G_{g_1,m}^{\bullet} (\alpha_{I_1}; \gamma_{I_1} \mid ((b, 1), (b_i, \Delta_{E,i})_{i=1}^h))$$

$$\times G_{g_2,m}^{\bullet} (((b, 1), (b_i, \Delta_{E,i})_{i=1}^h) \mid \alpha_{I_2}; \gamma_{I_2} \mid \mu)$$

$$- 2 \sum_{i=1}^n G_{g,m}^{\bullet} (\alpha; \psi_i \gamma_1, \ldots, \gamma_{i-1}, \pi_* \pi^* \gamma_i, \gamma_{i+1}, \ldots, \gamma_n \mid \mu)$$

$$- 2 \sum_{i=1}^l G_{g,m}^{\bullet} (\alpha; \gamma \mid (\mu_1, \delta_1), \ldots, (\mu_l, \psi_{\text{rel}}^i \pi_* \pi^* \delta_i), \ldots, (\mu_l, \delta_l))$$

where $\psi_i^{\text{rel}}$ is the cotangent line class at the $i$-th relative marking and $\Delta_E = \sum \Delta_{E,i} \otimes \Delta_{E,i}^\vee$ is the pullback of the K"unneth decomposition of $\Delta_E$ at the corresponding relative marking. The holomorphic anomaly equation takes the form:

**Proposition 15.** ([28, Proposition 20]) $G_{g,m}^{\bullet} (\alpha; \gamma \mid \mu)$ is a quasimodular form and

$$\frac{d}{dC_2} G_{g,m}^{\bullet} (\alpha; \gamma \mid \mu) = P_{g,m}^{\bullet} (\alpha; \gamma \mid \mu).$$

**Relative $(S, E)$.** Since the log canonical bundle of $(S, E)$ is nontrivial, relative moduli spaces in fiber direction has nontrivial virtual fundamental class. Define

$$F_{g,0}^{\text{vir-rel}} (\alpha; \gamma \mid \emptyset) = \sum_{h \geq 0} \langle \alpha; \gamma \mid \emptyset \rangle_{S/E, \text{vir}}^{g,h} q^h.$$

Recall that we denote the pullback of the diagonal of $P^1$ as

$$\Delta_{P^1} = 1 \boxtimes F + F \boxtimes 1 = \sum_{i=1}^2 \delta_i \boxtimes \delta_i^\vee.$$
Define a generating series

\[ H_{g,m}^{\text{rel}}(\alpha; \gamma | \mu) = F_{g-1,m}^{\text{rel}}(\alpha; \gamma, \Delta_{P_1} | \mu) + 2 \sum_{g=g_1+g_2} \sum_{\{1,\ldots,n\} = I_1 \sqcup I_2} \sum_{i \in \{1,2\}} \frac{\prod_{i=1}^h b_i}{h!} F_{g_1,m}^{\text{rel}}(\alpha_{I_1}; \gamma_{I_1} | \{(b, 1), (b_i, \Delta_{E,E_i}^\partial)_{i=1}^h\}) \]

\times G_{g_2,m}^{\text{vir}}((b, 1), (b_i, \Delta_{E,E_i}^\partial)_{i=1}^h | \alpha_{I_2}; \gamma_{I_2} | \mu)

- 2 \sum_{i=1}^n F_{g,m}^{\text{rel}}(\alpha \psi; \gamma_1, \ldots, \gamma_{i-1}, \pi^* \pi_* \gamma_i, \gamma_{i+1}, \ldots, \gamma_n | \mu)

- 2 \sum_{i=1}^l F_{g,m}^{\text{rel}}(\alpha; \gamma | (\mu_1, \delta_1), \ldots, (\mu_i, \psi_i^{\text{rel}} \pi^* \pi_* \delta_i), \ldots, (\mu_l, \delta_l)))

+ 20 \sum_{i=1}^n \langle \gamma_i, F \rangle F_{g,m}^{\text{rel}}(\alpha; \gamma_1, \ldots, \gamma_{i-1}, F, \gamma_{i+1}, \ldots, \gamma_n | \mu)

- 2 \sum_{i<j} F_{g,m}^{\text{rel}}(\alpha; \gamma_1, \ldots, \sigma_{i}^{\text{th}}(\gamma_i, \gamma_j), \ldots, \sigma_{j}^{\text{th}}(\gamma_i, \gamma_j), \ldots, \gamma_n | \mu).

The conjectural holomorphic anomaly equation for \((S, E)\) has the following form:

\[ F_{g,m}^{\text{rel}}(\alpha; \gamma | \mu) \in \frac{1}{\Delta(q)^m} \text{QMod}(m) \]

and

\[ \frac{d}{dC_2} F_{g,m}^{\text{rel}}(\alpha; \gamma | \mu) = H_{g,m}^{\text{rel}}(\alpha; \gamma | \mu). \]

**Proposition 16.** Let \( m \geq 1 \). Assuming quasimodularity for \( F_{g,m} \) and \( F_{g,m}^{\text{rel}} \), the holomorphic anomaly equations are compatible with the degeneration formula in the above sense.

**Proof of Proposition 16.** The proof given in [28, Proposition 21] treats virtual fundamental classes, not reduced classes. The splitting behavior of the reduced class with respect to restriction to boundary divisors [24, Section 7.3] calls for a slight adaptation of the proof. For this, we introduce a formal variable \( \varepsilon \) with \( \varepsilon^2 = 0 \). We can then interpret reduced Gromov–Witten invariants of the K3 surface as integrals against the
class
\[ [\mathcal{M}_{g,n}(S,\beta)]^{\text{vir}} + \varepsilon [\mathcal{M}_{g,n}(S,\beta)]^{\text{red}} \]
followed by taking the \([\varepsilon]\)-coefficient. We consider similar class for \(S/E\).
This class has the advantage of satisfying the usual splitting behavior of virtual fundamental classes. Thus, for this class one can follow the proof of compatibility given in [28, Proposition 21]. All the terms appearing in the computation (ii) also appear in computation (i). We are left with proving the cancellation of the remaining terms in (i). This follows from comparing \(\psi_{i}^{\text{rel}}\)-class and the \(\psi\)-class pulled-back from the stack of target degeneration [28, Lemma 22]. In particular, we match the following terms: the third term of \(H^{\text{rel}}\) times \(G^{\text{rel}}\) with the fourth term of \(F^{\text{rel}}\) times \(P^{\text{rel}}\), and analogously for the fifth term of \(H^{\text{rel}}\) times \(G^{\text{rel}}\) with the second term of \(F^{\text{rel}}\) times \(P^{\text{rel}}\). □

Main advantage of the holomorphic anomaly equation is that it is compatible with the degeneration formula. Thus, the genus reduction from the degeneration formula connects the low genus results with arbitrary genus predictions. On the other hand, it is not even clear to say what should be the compatibility of the multiple cover formula and the degeneration formula.

4. Proof of Theorem 1 and 3

4.1. Proof of Theorem 1. The proof proceeds via induction on the pair \((g, n)\) ordered by the lexicographic order: \((g', n') < (g, n)\) if

- \(g' < g\)
- \(g' = g\) and \(n' < n\).

Recall the dimension constraint of insertions:

\[ g + n = \deg(\alpha) + \sum_{i} \deg(\gamma_{i}). \]

We separate the proof into several steps.

Case 1. If all cohomology classes \(\gamma_{i}\) satisfy \(\deg(\gamma_{i}) \leq 1\), then \(\deg(\alpha) \geq g\) and by the strong form of Getzler–Ionel vanishing [12, Proposition 2] we have \(\alpha = \iota_{*}\alpha'\) with \(\alpha' \in R^{*}(\partial\mathcal{M}_{g,n})\) and \(\iota: \partial\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}\). We are thus reduced to lower \((g, n)\).

Case 2. Assume \(\deg(\alpha) \leq g - 2\) or equivalently, there exist at least two descendents of the point class. We use the degeneration to the

---

8We thank G. Oberdieck for pointing this out.
normal cone of a smooth elliptic fiber:

\[ S \sim S \cup E (\mathbb{P}^1 \times E). \]

We specialize the point class to the bubble \( \mathbb{P}^1 \times E \). Let \( C = C' \cup C'' \) be the splitting of a domain curve appearing in the degeneration formula in Theorem 11. Namely, \( C' \) is the component on \( S \) and \( C'' \) is the component on \( \mathbb{P}^1 \times E \). We argue that this splitting has non-trivial contribution only for \( g(C') < g \). If \( g(C') = g \), this forces \( C'' \) to be a disconnected union of two rational curves. Since the degree of the curve class along the divisor is \( \langle 2B + hF, F \rangle = 2 \), the two descendents of the point class then force the cohomology weighted partition to be \((1, 1)^2\) on the bubble or, equivalently, \((1, \omega)^2\) for \((S, E)\). This contribution vanishes because of the splitting behavior of the reduced class.

**Case 3.** Assume \( \deg(\alpha) = g - 1 \) or equivalently, there exists only one descendent of the point class. We may thus assume \( \gamma_1 = p \). If \( n = 1, g \geq 2 \), we can move \( \tau_{g-1}(p) \) to the bubble and the genus on \( S \) drops.

When \( n \geq 2 \), moving the point class to the bubble as in Case 2 may not reduce the genus. In particular, moving \( \tau_0(p) \) to the bubble has non-trivial contribution from rational curves on the bubble. On the other hand, if \( a \geq 1 \), moving \( \tau_a(p) \) to the bubble reduces the genus on \( S \) because of the dimension constraint.

We use Buryak, Shadrin and Zvonkine’s description of the top tautological group \( R^{g-1}(M_{g,n}) \) [9]. For any \( \alpha \in R^{g-1}(\overline{M}_{g,n}) \) the restriction of \( \alpha \) to \( M_{g,n} \) is a linear combination of

\[
R^{g-1}(M_{g,n}) = \mathbb{Q}\langle \psi_1^{g-1}, \psi_2^{g-1}, \ldots, \psi_n^{g-1} \rangle
\]

and the boundary term is also tautological class in \( R^{g-1}(\partial \overline{M}_{g,n}) \). By the divisor equation and subsequent use of (13), we can reduce to cases for \( \leq (g, 2) \). When \( g \geq 3 \), (13) has a different basis

\[
R^{g-1}(M_{g,2}) = \mathbb{Q}\langle \psi_1^{g-1}, \psi_1 \psi_2^{g-2} \rangle
\]

which is an easy consequence of generalized top intersection formula. Therefore, we may assume the descendent of the point class is of the form \( \tau_a(p) \) with \( a \geq 1 \). Now, specializing this insertion to the bubble \( \mathbb{P}^1 \times E \) reduces the genus and hence the same argument as in Case 2 applies. The genus 2 case is covered in Section 1.2.
Relative vs. absolute. We reduced to invariants for \((S, E)\) with genus \(g' < g\). As explained in [24, Lemma 31] (see also [23]), the degeneration formula provides an upper triangular relation between absolute and relative invariants for all pairs \(\leq (g', n')\). Thus, our induction applies.

4.2. Proof of Theorem 3. We argue by showing that each induction step in the proof of Theorem 1 is compatible with the holomorphic anomaly equation. Nontrivial step appears when the degeneration formula is used. From the compatibility result Proposition 16, we are reduced to proving the relative holomorphic anomaly equation for lower genus relative generating series \(F_{g', 2}^{rel}\) for \((S, E)\) and relative generating series for \((\mathbb{P}^1 \times E, E)\). The holomorphic anomaly equation for \((\mathbb{P}^1 \times E, E)\) is established in [27]. Because of the relative vs. absolute correspondence [23], we are reduced to proving the holomorphic anomaly equation for \(F_{g', 2}\) in genus 0, 1 and some genus 2 descendents. We proved the multiple cover formula for these cases in Section 1, which implies the holomorphic anomaly equation by Proposition 5.

Remark 17. Parallel argument shows that we can always reduce the proof for arbitrary descendent insertions to the case when the number of point insertions is less than or equal to \(m - 1\).

5. Examples

Example 18. We compute \(F_{1,2}(\tau_1(F))\) via topological recursion in genus one and illustrate Conjecture 22. Let \([\delta_0] \in A^1(\overline{M}_{1,1})\) be the class representing the nodal curve with a self node. Since

\[
\psi_1 = \frac{1}{24}[\delta_0] \in A^1(\overline{M}_{1,1}),
\]

we obtain

\[
F_{1,1}(\tau_1(F)) = \frac{1}{24}F_{0,1}(\tau_0(F)\tau_0(\Delta_S)) = \frac{1}{12}F_{0,1}(\tau_0(F)\tau_0(F \times W))
\]

\[= \frac{1}{12}D_q F_{0,1},\]

where \(\Delta_S \subset S \times S\) is the diagonal class. Analogously,

\[
F_{1,2}(\tau_1(F)) = \frac{1}{24}F_{0,2}(\tau_0(F)\tau_0(\Delta_S)) = \frac{1}{3}D_q F_{0,2}.
\]

Using the multiple cover formula in genus zero

\[
F_{0,2} = T_2 F_{0,1} + \frac{1023}{8192} F_{0,1}(q^2),
\]

...
Lemma 20. We obtain
\[ F_{1,2}(\tau_1(F)) = \frac{1}{3}D_qF_{0,2} = 2T_2\frac{1}{12}D_qF_{0,1} + \frac{1023}{1024}B_2\frac{1}{12}D_qF_{0,1} \]
\[ = 2T_2F_{1,1}(\tau_1(F)) + (2^9 - 2^{-10})B_2F_{1,1}(\tau_1(F)), \]
in perfect agreement with Conjecture 22.

Example 19. We compute \( F_{2,2}(\tau_0(p))^2 \) via degeneration formula and verify the multiple cover formula. The first two terms are computed by the classical geometry of K3 surface in [26]. For simplicity we write \( F_{1,2} = F_{1,2}(\tau_0(p)) \). The relative invariants for \((S, E)\) can be written in terms of absolute invariants:

Lemma 20. (i) \( F_{0,2}^{rel}(\emptyset \mid (1, 1)^2) = 2F_{0,2} \),
(ii) \( F_{1,2}^{rel}(\emptyset \mid (1, 1), (1, \omega)) = F_{1,2} - 2F_{0,2}D_qC_2 \),
(iii) \( F_{1,2}^{rel}(\emptyset \mid (2, 1)) = \frac{4}{3}D_qF_{0,2} - 4C_2F_{0,2} \).

Proof. It is a standard computation of the relative vs. absolute correspondence [23].

The relative invariants for \((\mathbb{P}^1 \times E, E)\) can be computed by the Gromov–Witten invariants of \( E \).

Lemma 21. (i) \( G_{0,1}^{rel}(\tau_0(p) \mid (1, 1)) = 1 \), \( G_{0,1}^{rel}(\emptyset \mid (1, \omega)) = 1 \),
(ii) \( G_{1,1}^{rel}(\tau_0(p) \mid (1, \omega)) = D_qC_2 \), \( G_{1,1}^{rel}(\tau_0(p)^2 \mid (1, 1)) = 2D_qC_2 \),
(iii) \( G_{2,1}^{rel}(\tau_0(p)^2 \mid (1, \omega)) = (D_qC_2)^2 \),
(iv) \( G_{1,2}^{rel}(\tau_0(p)^2 \mid (2, \omega)) = D_q^2C_2 \), \( G_{1,2}^{rel}(\tau_0(p)^2 \mid (1, \omega)^2) = D_q^3C_2 \).

Consider the degeneration where two point insertions move to the bubble \( \mathbb{P}^1 \times E \). By Theorem 11,
\[ F_{2,2}(\tau_0(p)^2) = (F_{1,2} - 2F_{0,2}D_qC_2)4D_qC_2 + \left( \frac{1}{3}D_qF_{0,2} - 4C_2F_{0,2} \right)2D_q^2C_2 \]
\[ + (2F_{0,2})\frac{1}{2}(D_q^3C_2 + 4(D_qC_2)^2) \]
\[ = 36q + 8760q^2 + 754992q^3 + 36694512q^4 + \cdots . \]

On the other hand, the primitive generating series
\[ F_{2,1}(\tau_0(p)^2) = \frac{(D_qC_2)^2}{\Delta(q)} \]
is computed in [6] and one can apply the multiple cover formula to obtain a candidate for \( F_{2,2}(\tau_0(p)^2) \). The first few terms of the two generating series match. It is enough to conclude that the two generating series are indeed equal because the space of quasimodular forms
with given weight is finite dimensional. However, it seems non-trivial to match the above formula from the degeneration with the formula provided by Conjecture 22.

6. Multiple cover formula and Hecke operators

This section contains a discussion of the multiple cover formula. We start by recalling the conjecture formulated in [26]. Then, we study the conjecture for the descendent potentials associated to elliptic K3 surfaces. The result is expressed in terms of Hecke operators. The discussion naturally leads to a candidate for the holomorphic anomaly equation in higher divisibility. We conclude with a proof of the multiple cover formula in fiber direction.

6.1. Multiple cover formula. Let $S$ be a nonsingular projective K3 surface, $\beta \in H_2(S, \mathbb{Z})$ be a primitive effective curve class, $m \in \mathbb{N}$ and $d \mid m$ be a divisor of $m$. The proposed formula by Oberdieck and Pandharipande involves a choice of a real isometry

$$\varphi_d : \left( H^2(S, \mathbb{R}), \langle \cdot, \cdot \rangle \right) \to \left( H^2(S_d, \mathbb{R}), \langle \cdot, \cdot \rangle \right)$$

between two K3 surfaces such that

$$\varphi_d \left( \frac{m}{d} \beta \right) \in H_2(S_d, \mathbb{Z})$$

is a primitive effective curve class. Such an isometry can always be found and Gromov–Witten invariants are in fact independent of the choice of isometry, see [8].

Consider integers $a_i \in \mathbb{N}$, cohomology classes $\gamma_i \in H^*(S, \mathbb{Q})$ and let $\deg = \sum \deg(\gamma_i)$. Then, the conjectured multiple cover formula [26, Conjecture C2], identical to Conjecture 4 in Section 0, is

$$\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \rangle_{g,m\beta} = \sum_{d \mid m} d^{2g-3+\deg} \langle \tau_{a_1}(\varphi_d(\gamma_1)) \ldots \tau_{a_n}(\varphi_d(\gamma_n)) \rangle_{g,\varphi_d(\frac{m}{d} \beta)}.$$

Let $S$ be an elliptic K3 surface with a section. For any $\ell \in \mathbb{Q}^*$ we define

$$\phi_\ell : H^*(S, \mathbb{Q}) \to H^*(S, \mathbb{Q})$$

---

9We view curve classes also as cohomology classes under the natural isomorphism $H_2(S, \mathbb{Z}) \cong H^2(S, \mathbb{Z})$.

10Notations here are as in Section 0. In particular, we use the modified degree function deg.
acting on $U = \mathbb{Q}(F,W)$ as
\[
\phi_\ell(F) = \ell F, \quad \phi_\ell(W) = \frac{1}{\ell} W
\]
and trivially on the orthogonal complement $U^\perp$. For $d \mid m$ and $d \mid h$ we have
\[
\phi_\frac{m}{d} \left( \frac{m}{d} B + \frac{h}{d} F \right) = B + \left( \frac{m(h - m)}{d^2} + 1 \right) F \quad \text{in } H_2(S,\mathbb{Z})
\]
which is a primitive curve class.

Altering the curve class via the isometry $\phi$ therefore results in additional factors of $\frac{d}{m}$ or $\frac{m}{d}$ while keeping the descendent insertions unchanged. This explains the change in exponents
\[
2g - 3 + \deg \leftrightarrow 2g - 3 + \underline{\deg}
\]
and the factor $m^{\deg - \underline{\deg}}$ in the multiple cover formula below for the descendent potential. We use the operator notation introduced in the next section, specifically Definition 25.

**Conjecture 22.** For deg-homogeneous classes $\gamma_i \in H^*(S,\mathbb{Q})$,
\[
F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = m^{\deg - \underline{\deg}} T[m]_{2g-3+\underline{\deg}} \left( F_{g,1}(\alpha; \gamma_1, \ldots, \gamma_n) \right),
\]
where $\deg = \sum \deg(\gamma_i)$ and $\underline{\deg} = \sum \underline{\deg}(\gamma_i)$.

Tautological classes play no role for the multiple cover behavior. Therefore the same formula is conjectured for the potential
\[
F_{g,m}(\tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)).
\]
Now we show our presentation of multiple cover formula is equivalent to the original formula.

**Lemma 23.** Conjectures 4 and 22 are equivalent.

**Proof.** By the deformation invariance of the reduced class, the Gromov–Witten invariants for arbitrary curve classes are fully captured by the elliptic K3 surface with a section. The primitive curves classes are $B + hF \in H_2(S,\mathbb{Z})$. Taking the coefficient of $q^{mh-m}$ in Conjecture 22 gives a multiple cover formula for the curve class $mB + mhF$ which matches the formula in Conjecture 4. It is the other implication which we have to justify.

The generating series $F_{g,m}$ involves curve classes $mB + hF$ of different divisibilities bounded by $m$. We apply Conjecture 4 to each invariant
and use the isometries $\phi$. Note that each appearance of $\gamma_i = F$ introduces a factor of $\frac{m}{d}$, while each appearance of $\gamma_i = W$ gives $\frac{d}{m}$.

Moreover, 

$$\left|\{i \mid \gamma_i = F\}\right| - \left|\{i \mid \gamma_i = W\}\right| = \deg - \overline{\deg},$$

and therefore

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_n) = \sum_{h \geq 0} \left< \alpha; \gamma_1, \ldots, \gamma_n \right>_{g,B+hF} q^{h-m}$$

$$= \sum_{h \geq 0} \sum_{d|m} d^{2g-3+\deg} \left( \frac{m}{d} \right)^{\deg - \overline{\deg}} \left< \alpha; \gamma_1, \ldots, \gamma_n \right>_{g,B+(\frac{mh-m}{d^2}+1)F} q^{h-m}$$

$$= m^{\deg - \overline{\deg}} \sum_{d|m} d^{2g-3+\deg} \left( \sum_{h \geq 0} \left< \alpha; \gamma_1, \ldots, \gamma_n \right>_{g,B+(\frac{mh-m}{d^2}+1)F} \left( q^d \right)^{h-m} \right)$$

$$= m^{\deg - \overline{\deg}} \sum_{d|m} d^{2g-3+\deg} \left( B_d U \sum_{h \geq 0} \left< \alpha; \gamma_1, \ldots, \gamma_n \right>_{g,B+hF} q^{h-1} \right)$$

$$= m^{\deg - \overline{\deg}} \sum_{d|m} d^{2g-3+\deg} B_d U \left( F_{g,1}(\alpha; \gamma_1, \ldots, \gamma_n) \right)$$

$$= m^{\deg - \overline{\deg}} T[m]^{2g-3+\deg} \left( F_{g,1}(\alpha; \gamma_1, \ldots, \gamma_n) \right).$$

□

**Remark 24.** One observation about the multiple cover formula is the following: Though the factor $m^{\deg - \overline{\deg}}$ depends on the cohomology insertions, the operator $T[m]^{2g-3+\deg}$ depends only on the weight of the primitive descendent potential $F_{g,1}$ because $2g - 3 + \overline{\deg}$ is recovered from the weight $2g - 12 + \overline{\deg}$.

### 6.2. Holomorphic anomaly equation in higher divisibility

We derive a candidate for the holomorphic anomaly equation for $m \geq 1$ from the conjectural multiple cover formula, such that both are compatible\(^{11}\). It turns out that the equation is almost identical to the one in the primitive case. Additional factors appear only in the last two terms, which are specific to K3 surfaces. We refer to [28, Section 7.3] for explanations on the appearance of these terms.

Let $\gamma_1, \ldots, \gamma_n \in H^*(S)$ with

$$\deg = \sum_i \deg(\gamma_i), \quad \overline{\deg} = \sum_i \overline{\deg}(\gamma_i) .$$

\(^{11}\)We should point out that this derivation should be lifted to cycle-valued. Tautological classes play no role here.
We will simply write $\gamma$ to denote $\gamma_1, \ldots, \gamma_n$. We apply the $\frac{d}{dC_2}$-derivative to Conjecture 22 and use the commutator relations Lemma 27 to obtain:

\[
\frac{d}{dC_2} F_{g,m}(\alpha; \gamma) = \frac{d}{dC_2} \left( m^{\deg - \deg} T[m]_{2g-3+\deg} F_{g,1}(\alpha; \gamma) \right) \\
= m^{\deg - \deg} \sum_{ad=m} a^{2g-3+\deg} \frac{d}{dC_2} B_u U_d F_{g,1}(\alpha; \gamma) \\
= m^{\deg - \deg} \sum_{ad=m} a^{2g-3+\deg} \frac{d}{dC_2} B_u U_d \frac{d}{dC_2} F_{g,1}(\alpha; \gamma) \\
= m^{\deg - \deg + 1} T[m]_{2g-5+\deg} \frac{d}{dC_2} F_{g,1}(\alpha; \gamma).
\]

We take the last row as the definition for $H_{g,m}$. Then, we apply the holomorphic anomaly equation for the primitive series

\[
\frac{d}{dC_2} F_{g,1}(\alpha; \gamma) = H_{g,1}(\alpha; \gamma)
\]

and go through the terms to see how they are effected:

(i) The degree $\deg$ of $F_{g-1,m}(\alpha; \gamma \Delta_{\psi})$ has increased by one. The genus, however, dropped by 1. Thus, the first term precisely matches the multiple cover formula, i.e.

\[
F_{g-1,m}(\alpha; \gamma \Delta_{\psi}) = m^{\deg - \deg + 1} T[m]_{2g-5+\deg} \left( F_{g-1,1}(\alpha; \gamma \Delta_{\psi}) \right).
\]

(ii) An analogous argument applies to the second term.

(iii) The modified degree $\deg$ of $F_{g,1}(\alpha \psi_1; \gamma_1, \ldots, \pi^* \pi_1 \gamma_1, \ldots, \gamma_n)$ has decreased by 2, whereas $\deg$ decreased by 1. Again we find that the term matches the multiple cover formula

\[
F_{g,m}(\alpha \psi_1; \gamma_1, \ldots, \pi^* \pi_1 \gamma_1, \ldots, \gamma_n) \\
= m^{\deg - \deg + 1} T[m]_{2g-5+\deg} \left( F_{g,1}(\alpha \psi_1; \gamma_1, \ldots, \pi^* \pi_1 \gamma_1, \ldots, \gamma_n) \right).
\]

(iv) The degree of $\langle \gamma_i, F \rangle F_{g,1}(\alpha; \gamma_1, \ldots, F, \ldots, \gamma_n)$ remains unchanged, whereas $\deg$ decreased by 2. An additional factor of $\frac{1}{m}$ therefore appears:

\[
\frac{1}{m} \langle \gamma_i, F \rangle F_{g,m}(\alpha; \gamma_1, \ldots, F, \ldots, \gamma_n) \\
= m^{\deg - \deg + 1} T[m]_{2g-5+\deg} \left( \langle \gamma_i, F \rangle F_{g,1}(\alpha; \gamma_1, \ldots, F, \ldots, \gamma_n) \right).
\]

(v) The term $F_{g,1}(\ldots, \sigma_1(\gamma_i, \gamma_j), \ldots, \sigma_2(\gamma_i, \gamma_j), \ldots)$ is similar to the previous case: $\deg$ remains unchanged, whereas $\deg$ decreases...
by 2, giving rise to an additional factor of \( \frac{1}{m} \):
\[
\frac{1}{m} \mathcal{F}_{g,m}(\gamma_1, \ldots, \sigma_1(\gamma_i, \gamma_j), \ldots, \sigma_2(\gamma_i, \gamma_j), \ldots, \gamma_n)
\]
\[
= m^{\text{deg} - \text{deg} + 1} T[m]_{2g-5+\text{deg}}(\mathcal{F}_{g,1}(\gamma_1, \ldots, \sigma_1(\gamma_i, \gamma_j), \ldots, \sigma_2(\gamma_i, \gamma_j), \ldots, \gamma_n))
\]

We arrive at the level \( m \) holomorphic anomaly equation (1) which appeared in Section 0.

**Proof of Proposition 5.** Assume that the multiple cover formula (4) holds for all divisors \( d \mid m \) and all descendent insertions. Using Lemma 23, also Conjecture 22 holds. By Proposition 29, the descendent potentials are quasimodular forms of level \( m \) and we can consider the \( \frac{d}{d\tau} \) derivative. When \( m = 1 \), the holomorphic anomaly equation is proven in [27, Theorem 4]. Therefore the commutator relation Lemma 27 (v) of \( \frac{d}{d\tau} \) and \( T[m]_{\ell} \) imply Conjecture 2. □

6.3. **Operators.** We recall basic properties of modular forms and Hecke operators. We choose operators to act from the left because it has an advantage to write simpler commutator relations. Since the material is standard, we will be brief. See [32, 33] for the basic theory of modular forms and Hecke operators.

For any (weakly holomorphic) modular function\(^{12}\) \( f : \mathbb{H} \to \mathbb{C} \) and \( d \in \mathbb{Z}_{>0} \) we define
\[
D_q f(\tau) = \frac{1}{2\pi i} \frac{\partial f}{\partial \tau}(\tau),
\]
\[
B_d f(\tau) = f(d\tau),
\]
\[
U_d f(\tau) = \frac{1}{d} \sum_{j=0}^{d-1} f\left(\frac{\tau + j}{d}\right).
\]

If \( f(\tau + 1) = f(\tau) \), let \( q = e^{2\pi i \tau} \) and express \( f \) as a Laurent series
\[
f(\tau) = \sum_{n=-s}^{\infty} a_n q^n.
\]

The action of the above operators is then (we let \( a_i = 0 \) for \( i < -s \))
\[
D_q f = q \frac{d}{dq} f, \quad B_d f = \sum_{n=-s}^{\infty} a_n q^n, \quad U_d f = \sum_{n=-s}^{\infty} a_d n q^n.
\]

For \( m \in \mathbb{N} \) and \( \ell \in \mathbb{Z} \), we define the operator \( T[m]_{\ell} \) acting on modular functions via

\(^{12}\)We will often simply say ‘modular form’.
Definition 25.
\[ T[m]_\ell = \sum_{d|m} d^\ell B_d U_{m/d}. \]

The above operators are related to classical Hecke operators \( T_d \) as follows. Let \( p \) be a prime number and let \( N \) be the level of the modular form. If \( p \nmid N \), the Hecke operator acting on modular forms of weight \( k \) is
\[ T_p = U_p + p^{k-1}B_p. \]
If \( p \mid N \), then \( U_p \) is the Hecke operator. However, we will only consider the action of Hecke operators on level 1 modular forms. In general, the Hecke operators satisfy
\[ T_{de} = T_d T_e, \quad \text{gcd}(d, e) = 1, \]
\[ T_{pr+1} = T_{pr}^r T - p^{k-1}T_{pr-1}, \quad r \in \mathbb{N}. \]

Remark 26. From the above relations, for any \( r \in \mathbb{N} \) one can write
\[ U_{pr} = T_{pr} - p^{k-1}B_p T_{pr-1}. \]

For later reference, we list the following basic commutator relations between the above operators:

Lemma 27. Let \( d, e \in \mathbb{N} \) and \( \ell \in \mathbb{Z} \). Then
(i) \( B_d B_e = B_{de} = B_e B_d \),
(ii) \( U_d U_e = U_{de} = U_e U_d \),
(iii) \( D_q B_d = d \cdot B_d D_q, \quad U_d D_q = d \cdot D_q U_d \),
(iv) \( \frac{d}{dC_2} U_d = d \cdot U_d \frac{d}{dC_2}, \quad B_d \frac{d}{dC_2} = d \cdot \frac{d}{dC_2} B_d \),
(v) \( \frac{d}{dC_2} T[m]_{\ell+2} = m \cdot T[m]_{\ell} \frac{d}{dC_2}, \quad T[m]_{\ell+2} D_q = m \cdot D_q T[m]_{\ell} \),
(vi) \( [\frac{d}{dC_2}, D_q] = -2k \).

Proof. The proof for (i)-(iii) follows directly from the definition. The commutator relation (vi) is well-known, see e.g. [33, Section 5.3]. For (iv) we use that \( d \cdot C_2(d \tau) - C_2(\tau) \) is modular for \( \Gamma_0(d) \) and therefore
\[ \frac{d}{dC_2} = \frac{1}{d} \cdot \frac{d}{dB_d C_2}. \]
It follows that for all \( n \geq 0 \)
\[ B_d \frac{d}{dC_2} C_2^n = n \cdot B_d C_2^{n-1} = d \cdot \frac{d}{dC_2} B_d C_2^n. \]
Let \( f \) be weakly holomorphic modular of weight \( k - 2n \). Then by induction on \( n \)

\[
B_d \frac{d}{dC_2} D_q^n f = B_d(D_q \frac{d}{dC_2}(D_q^{n-1} f) - (2k - 4)D_q^{n-1} f)
\]

\[
= \frac{1}{d} D_q B_d \frac{d}{dC_2}(D_q^{n-1} f) - (2k - 4)B_d(D_q^{n-1} f)
\]

\[
= D_q \frac{d}{dC_2} B_d(D_q^{n-1} f) - (2k - 4)B_d(D_q^{n-1} f)
\]

\[
= \frac{d}{dC_2} D_q B_d D_q^{n-1} f
\]

\[
= d \cdot \frac{d}{dC_2} B_d D_q^n f.
\]

The commutator of \( T_d \) and \( \frac{d}{dC_2} \) can be computed similarly. The result for \( U_d \) then follows from equation (14). The commutator relations in (v) follow directly from (iv).

We present an alternative formula for the operator \( T[m]_\ell \) in terms of Hecke operators below, which makes the conjectured level \( m \) quasi-modularity for \( F_{g,m} \) transparent.

### 6.4. Dirichlet convolution

Definitions in Section 6.3 are best explained in the language of Dirichlet convolutions. For arithmetic functions \( f, g: \mathbb{N} \to \mathbb{C} \), recall the convolution product

\[
(f \ast g)(n) = \sum_{ad=n} f(a)g(d).
\]

The point-wise multiplication defines another operation on arithmetic functions via

\[
(f \cdot g)(n) = f(n) \cdot g(n).
\]

If \( h \) is completely multiplicative, then

\[
(f \ast g) \cdot h = (f \cdot h) \ast (g \cdot h).
\]

We will need the following classical \( k \)-th power function resp. Möbius function:

\[
\text{Id}_\ell(n) = n^\ell
\]

\[
\mu(n) = \begin{cases} 
1, & \text{if } n \text{ square-free, even number of prime divisors,} \\
-1, & \text{if } n \text{ square-free, odd number of prime divisors,} \\
0, & \text{else.}
\end{cases}
\]
The definition of Dirichlet convolution extends formally to functions with values in algebras; denote
\[ B(n) = B_n, \quad T(n) = T_n, \quad U(n) = U_n. \]
Then Definition 25 can be rewritten as
\[ T[m]_\ell = \left( (\text{Id}_\ell \cdot B) \ast U \right)(m). \]
The equation (26) implies the well-known formula for the Hecke operators:
\[ T = (\text{Id}_{k-1} \cdot B) \ast U. \]
The Dirichlet inverse of \( \text{Id}_{k-1} \) is \( \mu \cdot \text{Id}_{k-1} \) and thus we may rewrite Definition 25 in terms of Hecke operators as
\[ T[m]_\ell = \left( (c_\ell \cdot B) \ast T \right)(m), \]
or in its explicit form as\(^{13}\)

**Lemma 28.**
\[ T[m]_\ell = \sum_{ad=m} c_\ell(a) B_a T_d, \]
where \( c_\ell = \text{Id}_\ell \ast (\mu \cdot \text{Id}_{k-1}) \), i.e.
\[ c_\ell(a) = \sum_{r|a} r^\ell \mu\left(\frac{a}{r}\right) \left(\frac{a}{r}\right)^{k-1}. \]

Let \( f \) be a weakly holomorphic quasimodular form of level 1 of weight \( k \) with pole of order at most 1 and let \( \Delta(q) \) be the modular discriminant. Note the following basic facts:

(i) The action of the Hecke operator \( T_{d}f \) is weakly holomorphic quasimodular of level 1 of the same weight \( k \) with pole of order at most \( d \),
(ii) the function \( B_{d}f \) is weakly holomorphic quasimodular for \( \Gamma_0(d) \) of the same weight \( k \), with pole of order at most \( d \),
(iii) the level \( m \) modular form
\[ \frac{\Delta(q)^m}{\Delta(q^m)} \]
is holomorphic.

As a direct consequence, the multiple cover formula implies level \( m \) quasimodularity.

\(^{13}\)Note that \( T[m]_\ell \) depends only on \( m \) and \( \ell \), not on the weight \( k \). The dependence on \( k \) in Lemma 28 thus cancels out. We omit \( k \) in the notation of \( c_\ell \).
Proposition* 29. If the generating series $F_{g,m}$ satisfies the multiple cover formula, it satisfies the quasimodularity conjecture. More precisely,

$$F_{g,m} \in \frac{1}{\Delta(q)^m} \text{QMod}(m).$$

6.5. Multiple cover formula in fiber direction. When the curve class is a multiple of the fiber class $F$, the multiple cover formula reduces to a property of the Gromov–Witten invariant of elliptic curves. Relevant properties are conjectured in [31].

Let $S \rightarrow \mathbb{P}^1$ be an elliptic K3 surface and $\beta = mF$. By Section 4, Case 1, we may assume at least one of insertions is the point class $\gamma_1 = p$ and $g \geq 1$. Move this point insertion to the bubble in the degeneration (8). Then the Gromov–Witten theory of $S$ localizes to the Gromov–Witten theory of $\mathbb{P}^1 \times E$ with the curve class $(0, mE)$. Let $\iota : E \hookrightarrow S$

be the inclusion of a fiber. The obstruction bundle computation shows that the invariant is of the form

$$\langle \tau_{a_1}(p)\tau_{a_2}(\gamma_2) \ldots \tau_{a_n}(\gamma_n) \rangle_{g,mF}^S = \langle \lambda_{g-1}; \tau_{a_1}(\omega)\tau_{a_2}(t^*\gamma_2) \ldots \tau_{a_n}(t^*\gamma_n) \rangle_{g,mE}^E,$$

where $\lambda_{g-1} = c_{g-1}(E_g)$. In particular, if $\gamma_i \in \mathbb{Q}(F) \oplus U^\perp \oplus \mathbb{Q}(p)$, the invariant vanishes. Consider the following generating series

$$F_g^E(\tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n)) = \sum_{m \geq 0} \langle \lambda_{g-1}; \tau_{a_1}(\gamma_1) \ldots \tau_{a_n}(\gamma_n) \rangle_{g,mE}^E q^m$$

where $\gamma_i = 1$ or $\omega$ and $\sum a_i + \sum \deg(\gamma_i) = g - 1 + n$.

The generating series $F_g^E$ has a simple description in terms of Eisenstein series. Let

$$C_{2k}(q) = -\frac{B_{2k}}{2k \cdot (2k)!} E_{2k}(q)$$

be the renormalized $2k$-th Eisenstein series. The following formula is conjectured in [31].

Lemma 30. For $g \geq 1$,

$$F_g^E(\tau_{g-1}(\omega)) = \frac{g!}{2^{g-1}} C_{2g}.$$

Proof. In [31, Proposition 4.4.7] this formula is given under assuming the Virasoro constraint for $\mathbb{P}^1 \times E$. The Virasoro constraint for any toric bundle over a nonsingular variety which satisfies the Virasoro constraint is proven in [11]. Combining this result with the Virasoro constraint for elliptic curves [29], the result follows. \qed
When \( \beta = mF \), Conjecture 4 is equivalent to the following proposition.

**Proposition 31.** There exists \( c \in \mathbb{Q} \) such that

\[
F_E^g (\tau_{a_1} \omega) \ldots \tau_{a_r} \omega \tau_{a_{r+1}} (1) \ldots \tau_{a_{r'}} (1)) = c D_q^{-1} F_E^g (\tau_{g-1} \omega).
\]

**Proof.** Boundary strata with a vertex of genus less than \( g \) do not contribute because the invariants involve \( \lambda_h \) vanishes on \( \overline{M}_{g,n}(E,m) \) when \( h \geq g \). If \( r' > r \), then \( \sum a_i \geq g \) and we can reduce to the case when \( r' = r \) by the topological recursion on the \( \psi \)-monomial in \( R^{2g} (\overline{M}_{g,n}) \) [18]. If \( r' = r \), then \( \sum a_i = g - 1 \) and similar argument as in Section 4, Case 3 can be applied. Therefore \( F_E^g \) is proportional to

\[
F_E^g (\tau_{g-1} \omega) \tau_0 (\omega)^{r-1}) = D_q^{-1} F_E^g (\tau_{g-1} \omega)
\]

where the equality comes from the divisor equation. \( \square \)

**Remark 32.** One can find a closed formula for the constant \( c \in \mathbb{Q} \) by integrating tautological classes on \( \overline{M}_{g,n} \).

**Appendix A. A proof of degeneration formula**

For self-contained exposition, we present a proof of the degeneration formula closely following [24, 25]. When \( m = 1, 2 \), a proof using symplectic geometry was presented in [19].

**Perfect obstruction theory.** For simplicity assume \( n = 0 \). General cases easily follow from this case. Let \( \epsilon : S \to \mathbb{A}^1 \) be the total family of the degeneration and \( \overline{M}_g (\epsilon, \beta) \to \mathbb{A}^1 \) be the moduli space of stable maps to the expanded target \( \widetilde{S} \). For the relative profile \( \nu \), the embedding

\[
\iota_{\nu} : \overline{M}_g (S_0, \nu) \to \overline{M}_g (\epsilon, \beta)
\]

can be realized as a Cartier pseudo-divisor \((L_{\nu}, s_{\nu})\).

Let \( E_\epsilon \to \mathbb{L}_{\overline{M}_g (\epsilon, \beta)} \) be the perfect obstruction theory constructed in [22]. Then the perfect obstruction theories \( E_0 \) and \( E_{\nu} \) of \( \overline{M}_g (S_0, \beta) \) and \( \overline{M}_g (S_0, \nu) \) sit in exact triangles

\[
L_{\nu} \to \iota_{\nu}^* E_\epsilon \to E_0 [1] \to L_0
\]

\[
L_{\nu} \to \iota_{\nu}^* E_\epsilon \to E_{\nu} [1].
\]
On $\overline{M}_g(S_0, \nu)$, the perfect obstruction theory splits as follows. Let $E_1$ and $E_2$ be the perfect obstruction theory of relative stable map spaces $\overline{M}_g(S/E, \beta_1)_\mu$ and $\overline{M}_g(\mathbb{P}^1 \times E/E, \beta_2)_\mu$ respectively. There exists an exact triangle

$$\bigoplus_{i=1}^l (N^\vee_{\Delta_E/E \times E})_i \to E_1 \boxplus E_2 \to E_\nu \to \cdots$$

where $(N^\vee_{\Delta_E/E \times E})_i$ is the pullback of the conormal bundle of the diagonal $\Delta_E \subset E \times E$ along the $i$-th relative marking.

**Reduced class.** Let $\rho: \tilde{S} \to S \times \mathbb{A}^1 \to S$ be the projection. By pulling back holomorphic symplectic form on $S$ via $\rho$, one can define a cosection of the obstruction sheaf of $E_\epsilon$

$$\text{Ob}_{\overline{M}_g(\epsilon, \beta)} \to \mathcal{O},$$

see [16, Section 5]. Dualizing the cosection gives a morphism

$$\gamma: \mathcal{O}[1] \to E_\epsilon.$$

Let $E_\epsilon^{\text{red}}$ be the cone of $\gamma$ which gives the reduced class on $\overline{M}_g(\epsilon, \beta)$. Similarly we can construct

$$\gamma_{\text{rel}}: \mathcal{O}[1] \to E_1$$

for the moduli space of relative stable maps $\overline{M}_g(S/E, \beta)$.

**Degeneration formula for reduced class.** Restricting $\gamma$ to $\overline{M}_g(S_0, \beta)$ and $\overline{M}_g(S_0, \nu)$, we get

$$\gamma_0: \mathcal{O}[1] \to t_0^* E_\epsilon \to E_0$$

$$\gamma_\nu: \mathcal{O}[1] \to t_\nu^* E_\epsilon \to E_\nu$$

where the compositions induce reduced classes. The exact triangles

$$L_0^\vee \to t_0^* E_\epsilon^{\text{red}} \to E_0^{\text{red}} \to \cdots$$

$$L_\nu^\vee \to t_\nu^* E_\epsilon^{\text{red}} \to E_\nu^{\text{red}} \to \cdots$$

still hold.

**Lemma 33.** We have an exact triangle

$$N^\vee_{\Delta_E/K \times E} \to E_1^{\text{red}} \boxplus E_2 \to E_\nu^{\text{red}} \to \cdots$$

on $\overline{M}_g(S_0, \nu)$ compatible with the structure maps to the cotangent complex.
Proof. Consider the diagram of complexes

\[
\begin{array}{ccc}
\mathcal{O}[1] \oplus 0 & \rightarrow & \mathcal{O}[1] \\
\gamma_{\text{rel}} \oplus 0 & \downarrow & \gamma_v \\
\bigoplus_{i=1}^{n(v)} (N_{\Delta_E/E \times E}^v) & \rightarrow & E_1 \oplus E_2 \rightarrow E_v, \\
\bigoplus_{i=1}^{n(v)} (N_{\Delta_E/E \times E}^v) & \downarrow & \bigoplus_{i=1}^{n(v)} (N_{\Delta_E/E \times E}^v) \rightarrow E_1^{\text{red}} \oplus E_2 \rightarrow E_v^{\text{red}},
\end{array}
\]

where middle horizontal morphisms are the exact triangle from (15). The square on the top commutes because the cosections for \(\tilde{S}\) and \((S, E)\) are both coming from the holomorphic symplectic form on \(S\). The vertical morphisms are exact triangles and hence induces a map between cones. Since the evaluation at relative markings factors through \(\Delta_K \subset K \times E\), hence the morphism

\[
\bigoplus_{i=1}^{n(v)} (N_{\Delta_E/E \times E}^v) \rightarrow E_1 \oplus E_2
\]

in (15) factors through \(N_{\Delta_K/K \times E}^v\). \(\square\)

Now Theorem 11 is a direct consequence of Lemma 33.

References

[1] Y. Bae, Tautological relations for stable maps to a target variety, 2019.
[2] Y. Bae, D. Holmes, R. Pandharipande, J. Schmitt, and R. Schwarz, Pixton’s formula and Abel-Jacobi theory on the Picard stack, 2020.
[3] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Holomorphic anomalies in topological field theories, Nuclear Phys. B 405 (1993), no. 2-3, 279–304.
[4] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Comm. Math. Phys. 165 (1994), no. 2, 311–427.
[5] P. Bousseau, H. Fan, S. Guo, and L. Wu, Holomorphic anomaly equation for (\(P^2, E\)) and the Nekrasov-Shatashvili limit of local \(P^2\), 2020.
[6] J. Bryan and N. C. Leung, The enumerative geometry of K3 surfaces and modular forms, J. Amer. Math. Soc. 13 (2000), no. 2, 371–410.
[7] J. Bryan, G. Oberdieck, R. Pandharipande, and Q. Yin, Curve counting on abelian surfaces and threefolds, Algebr. Geom. 5 (2018), no. 4, 398–463.
[8] T.-H. Buelles, Gromov–Witten classes of K3 surfaces, 2019.
[9] A. Buryak, S. Shadrin, and D. Zvonkine, Top tautological group of \(\mathcal{M}_{g,n}\), J. Eur. Math. Soc. (JEMS) 18 (2016), no. 12, 2925–2951.
[10] H.-L. Chang, S. Guo, and J. Li, BCOV’s Feynman rule of quintic 3-folds, 2018.
[11] T. Coates, A. Givental, and H.-H. Tseng, Virasoro Constraints for Toric Bundles, 2015.
[12] C. Faber and R. Pandharipande, *Relative maps and tautological classes*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 1, 13–49.
[13] S. Guo, F. Janda, and Y. Ruan, *Structure of Higher Genus Gromov-Witten Invariants of Quintic 3-folds*, 2018.
[14] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publ. Math. Inst. Hautes Études Sci. **125** (2017), 221–266.
[15] M. Kaneko and D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 165–172.
[16] Y.-H. Kiem and J. Li, *Low degree GW invariants of surfaces II*, Sci. China Math. **54** (2011), no. 8, 1679–1706.
[17] A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger, *Noether-Lefschetz theory and the Yau-Zaslow conjecture*, J. Amer. Math. Soc. **23** (2010), no. 4, 1013–1040.
[18] R. Kramer, F. Labib, D. Lewanski, and S. Shadrin, *The tautological ring of \(\mathcal{M}_{g,n}\) via Pandharipande-Pixton-Zvonkine r-spin relations*, Algebr. Geom. **5** (2018), no. 6, 703–727.
[19] J. Lee and N. C. Leung, *Yau-Zaslow formula on K3 surfaces for non-primitive classes*, Geom. Topol. **9** (2005), 1977–2012.
[20] J. Lee and N. C. Leung, *Counting elliptic curves in K3 surfaces*, J. Algebraic Geom. **15** (2006), no. 4, 591–601.
[21] H. Lho and R. Pandharipande, *Stable quotients and the holomorphic anomaly equation*, Adv. Math. **332** (2018), 349–402.
[22] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293.
[23] D. Maulik and R. Pandharipande, *A topological view of Gromov-Witten theory*, Topology **45** (2006), no. 5, 887–918.
[24] D. Maulik, R. Pandharipande, and R. P. Thomas, *Curves on K3 surfaces and modular forms*, J. Topol. **3** (2010), no. 4, 937–996, With an appendix by A. Pixton.
[25] D. Maulik and R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, A celebration of algebraic geometry, Clay Math. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 2013, pp. 469–507.
[26] G. Oberdieck and R. Pandharipande, *Curve counting on K3 × E, the Igusa cusp form \(\chi_{10}\), and descendent integration*, K3 surfaces and their moduli, Progr. Math., vol. 315, Birkhäuser/Springer, [Cham], 2016, pp. 245–278.
[27] G. Oberdieck and A. Pixton, *Holomorphic anomaly equations and the Igusa cusp form conjecture*, Invent. Math. **213** (2018), no. 2, 507–587.
[28] G. Oberdieck and A. Pixton, *Gromov-Witten theory of elliptic fibrations: Jacobi forms and holomorphic anomaly equations*, Geom. Topol. **23** (2019), no. 3, 1415–1489.
[29] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), no. 1, 47–108.
[30] R. Pandharipande and R. P. Thomas, *The Katz-Klemm-Vafa conjecture for K3 surfaces*, Forum Math. Pi **4** (2016), e4, 111.
[31] A. Pixton, *The Gromov-Witten theory of an elliptic curve and quasimodular forms*, Bachelor thesis, 2008.
[32] K. A. Ribet and W. A. Stein, *Lectures on modular forms and Hecke operators*, http://wstein.org/books/ribet-stein/main.pdf.
[33] D. Zagier, *Elliptic modular forms and their applications*, The 1-2-3 of modular forms, Universitext, Springer, Berlin, 2008, pp. 1–103.
ETH Zürich, Department of Mathematics
E-mail address: younghan.bae@math.ethz.ch

ETH Zürich, Department of Mathematics
E-mail address: buelles@math.ethz.ch