Abstract evolution equations with an operator function in the second term

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Abstract

In this paper, having introduced a convergence of a series on the root vectors in the Abel-Lidskii sense, we present a valuable application to the evolution equations. The main issue of the paper is an approach allowing us to principally broaden conditions imposed upon the right-hand side of the evolution equation in the abstract Hilbert space. In this way, we come to the definition of the function of an unbounded non-selfadjoint operator. Meanwhile, considering the main issue we involve an additional concept that is a generalization of the spectral theorem for a non-selfadjoint operator.

Keywords: Spectral theorem; Abel-Lidskii basis property; Schatten-von Neumann class; operator function; evolution equation.

MSC 47B28; 47A10; 47B12; 47B10; 34K30; 58D25.

1 Introduction

The application of results connected with the basis property in the Abell-Lidskii sense \[11\] covers many problems \[12\] in the framework of the theory of evolution equations. The central idea of this paper is devoted to an approach allowing us to principally broaden conditions imposed upon the right-hand side of the evolution equation in the abstract Hilbert space. In this way we can obtain abstract results covering many applied problems to say nothing on the far-reaching generalizations. We plan to implement the idea having involved a notion of an operator function. This is why one of the paper challenges is to find a harmonious way of reformulating the main principles of the spectral theorem having taken into account the peculiarities of the convergence in the Abel-Lidskii sense. However, our final goal is an existence and uniqueness theorem for an abstract evolution equation with an operator function at the right-hand side, where the derivative at the left-hand side is supposed to be of the integer order. The peculiar contribution is the obtained formula for the solution of the evolution equation. We should remind that involving a notion of the operator function, we broaden a great deal a class corresponding to the right-hand side. This gives us an opportunity to claim that the main issue of the paper is closely connected
with the spectral theorem for the non-selfadjoint unbounded operator. Here, we should make a 
brief digression and consider a theoretical background that allows us to obtain such exotic results.

We should recall that the concept of the spectral theorem for a selfadjoint operator is based 
on the notion of a spectral family or the decomposition of the identical operator. Constructing a 
spectral family, we can define a selfadjoint operator using a concept of the Riemann integral, it 
is the very statement of the spectral theorem for a selfadjoint operator. Using the same scheme, 
we come to a notion of the operator function of a selfadjoint operator. The idea can be clearly 
demonstrated if we consider the well-known representation of the compact selfadjoint operator as 
a series on its eigenvectors. The case corresponding to a non-selfadjoint operator is not so clear 
but we can adopt some notions and techniques to obtain similar results. Firstly, we should note 
that the question regarding decompositions of the operator on the series of eigenvectors (root 
vectors) is rather complicated and deserves to be considered itself. For this purpose, we need to 
involve some generalized notions of the series convergence, we are compelled to understand it in 
one or another sense, we mean Bari, Riesz, Abel (Abel-Lidskii) senses of the series convergence 
[1],[2]. The main disadvantage of the paper [1] is a sufficiently strong condition imposed upon 
the numerical range of values comparatively with the sectorial condition (see definition of the 
sectorial operator), there considered a domain of the parabolic type containing the spectrum of 
the operator. A reasonable question that appears is about minimal conditions that guaranty the 
desired result. At the same time the convergence in the Abel-Lidskii sense was established in the 
paper [10] for an operator class wider than the class of sectorial operators. A major contribution 
of the paper [11] to the theory is a sufficient condition for the Abel-Lidskii basis property of the 
root functions system for a sectorial non-selfadjoint operator of the special type. Considering 
such an operator class we strengthen a little the condition regarding the semi-angle of the sector, 
but weaken a great deal conditions regarding the involved parameters. Moreover, the central 
aim generates some prerequisites to consider technical peculiarities such as a newly constructed 
sequence of contours of the power type on the contrary to the Lidskii results [16], where a sequence 
of the contours of the exponential type was considered. In any case, we may say that a clarification 
of the results [16] devoted to the decomposition on the root vectors system of the non-selfadjoint 
operator has been obtained. In the paper [11], we used a technique of the entire function theory 
and introduced a so-called Schatten-von Neumann class of the convergence exponent. Using 
a sequence of contours of the power type, we invented a peculiar method how to calculate a 
contour integral, involved in the problem in its general statement, for strictly accretive operators 
satisfying special conditions formulated in terms of the norm. Here, we used an equivalence 
between operators with the discrete spectra and operators with the compact resolvent, for the 
results devoted to them can be easily reformulated from one to another realm.

In order to avoid the lack of information, consider a class of non-selfadjoint operators for which 
the above concept can be successfully applied. We ought to note that a peculiar scientific interest 
appears in the case when a senior term of the operator is not selfadjoint [26],[8], for in the contrary 
case there is a plenty of results devoted to the topic wherein the following papers are well-known 
[5],[14],[17],[18],[26]. The fact is that most of them deal with a decomposition of the operator on 
a sum, where the senior term must be either a selfadjoint or normal operator. In other cases, 
the methods of the papers [9],[8] become relevant and allow us to study spectral properties of 
operators whether we have the mentioned above representation or not. Here, we should remark 
that the methods [8] can be used in the natural way, if we deal with abstract constructions 
formulated in terms of the semigroup theory [10]. The central challenge of the latter paper is how 
to create a model representing a composition of fractional differential operators in terms of the
semigroup theory. We should note that motivation arises in connection with the fact that a second
order differential operator can be represented as a some kind of a transform of the infinitesimal
generator of a shift semigroup. Having been inspired by the novelty of the idea, we generalize a
differential operator with a fractional integro-differential composition in the final terms to some
transform of the corresponding infinitesimal generator of the shift semigroup. Having applied the
methods [8], we managed to study spectral properties of the infinitesimal generator transform
and obtain an outstanding result – asymptotic equivalence between the real component of the
resolvent and the resolvent of the real component of the operator. The relevance is based on the
fact that the asymptotic formula for the operator real component in most cases can be established
due to well-known results [24]. Thus, the existence and uniqueness theorems formulated in terms
of the operator order [12], subsequently generalized due to involving the notion of the operator
function, can cover a significantly large operator class.

Apparently, the application part of the paper appeals to the theory of differential equations.
In particular, the existence and uniqueness theorems for evolution equations with the right-hand
side – an operator function of a differential operator with a fractional derivative in final terms are
covered by the invented abstract method. In this regard such operators as the Riemann-Liouville
fractional differential operator, the Kipriyanov operator, the Riesz potential, the difference oper-
ator, the artificially constructed normal operator are involved [12].

2 Preliminaries

Let $C, C_i$, $i \in \mathbb{N}_0$ be real constants. We assume that a value of $C$ is positive and can be different
within a formula but values of $C_i$ are certain. Denote by $\text{int} M$, $\text{Fr} M$ the interior and the set of
boundary points of the set $M$ respectively. Everywhere further, if the contrary is not stated, we
consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote
by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $D(L)$, $R(L)$, $N(L)$ the domain of
definition, the range, and the kernel or null space of an operator $L$ respectively. Let $P(L)$ be the
resolvent set of an operator $L$. Denote by $\lambda_i(L)$, $i \in \mathbb{N}$ the eigenvalues of an operator $L$.
Suppose $L$ is a compact operator and $N := (L^*L)^{1/2}$, $r(N) := \dim R(N)$; then the eigenvalues
of the operator $N$ are called the singular numbers (s-numbers) of the operator $L$ and are denoted
by $s_i(L)$, $i = 1, 2, \ldots, r(N)$. If $r(N) < \infty$, then we put by definition $s_i = 0$, $i = r(N) + 1, 2, \ldots,$
According to the terminology of the monograph [2] the dimension of the root vectors subspace
corresponding to a certain eigenvalue $\lambda_k$ is called the algebraic multiplicity of the eigenvalue $\lambda_k$.
Denote by $n(r)$ a function equals to the number of the elements of the sequence $\{a_n\}_1^\infty$, $|a_n| \uparrow \infty$
within the circle $|z| < r$. Let $A$ be a compact operator, denote by $n_A(r)$ counting function a
function $n(r)$ corresponding to the sequence $\{s_i^{-1}(A)\}_1^\infty$. Let $\mathcal{S}_p(\mathfrak{H})$, $0 < p < \infty$ be a Schatten-von
Neumann class and $\mathcal{S}_\infty(\mathfrak{H})$ be the set of compact operators. In accordance with the terminology
of the monograph [1] the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_\mathfrak{H}, f \in D(L), \|f\|_\mathfrak{H} = 1\}$ is called the
numerical range of an operator $L$. An operator $L$ is called sectorial if its numerical range belongs
to a closed sector $S_\epsilon(\theta) := \{\zeta : |\arg(\zeta - i)| \leq \theta < \pi/2\}$, where $i$ is the vertex and $\theta$ is the
semi-angle of the sector $S_\epsilon(\theta)$. If we want to stress the correspondence between $\epsilon$ and $\theta$, then we
will write $\theta_\epsilon$. By the convergence exponent $p$ of the sequence $\{a_n\}_1^\infty \subset \mathbb{C}$, $a_n \neq 0$, $a_n \to \infty$ we
we mean the greatest lower bound for such numbers $\lambda$ that the following series converges

$$\sum_{n=1}^\infty \frac{1}{|a_n|^\lambda} < \infty.$$
More detailed information can be found in [15]. Denote by $\tilde{S}_p(\mathcal{H})$ the class of the operators such that

$$
\tilde{S}_p(\mathcal{H}) := \{ T \in \mathcal{S}_p, T \in \mathcal{S}_p, \forall \varepsilon > 0 \},
$$

we will call it *Schatten-von Neumann class of the convergence exponent*. We use the following notations

$$
M_f(r) := \max_{|z|=r} |f(z)|, \ m_f(r) := \min_{|z|=r} |f(z)|, \ z \in \mathbb{C}.
$$

Everywhere further, unless otherwise stated, we use notations of the papers [2], [4], [6], [7], [25].

Convergence in the Abel-Lidsky sense

In this subsection, we reformulate results obtained by Lidskii [16] in a more convenient form applicable to the reasonings of this paper. However, let us begin our narrative. In accordance with the Hilbert theorem (see [23], [2, p.32]) the spectrum of an arbitrary compact operator $B$ consists of the so-called normal eigenvalues, it gives us an opportunity to consider a decomposition to a direct sum of subspaces

$$
\mathcal{H} = \mathcal{H}_q \oplus \mathcal{M}_q,
$$

where both summands are invariant subspaces regarding the operator $B$, the first one is a finite dimensional root subspace corresponding to the eigenvalue $\mu_q$ and the second one is a subspace wherein the operator $B - \mu_q I$ is invertible. Let $n_q$ is a dimension of $\mathcal{H}_q$ and let $B_q$ is the operator induced in $\mathcal{H}_q$. We can choose a basis (Jordan basis) in $\mathcal{H}_q$ that consists of Jordan chains of eigenvectors and root vectors of the operator $B_q$. Each chain $e_{q_1}, e_{q_1+1}, ..., e_{q_k}, k \in \mathbb{N}_0$, where $e_{q_1}, \xi = 1, 2, ..., m$ are the eigenvectors corresponding to the eigenvalue $\mu_q$ and other terms are root vectors, can be transformed by the operator $B$ in accordance with the following formulas

$$
Be_{q_k} = \mu_q e_{q_k}, \ B e_{q_1} = \mu_q e_{q_1+1} + e_{q_1}, ..., Be_{q_k} + k = \mu_q e_{q_k} + k + e_{q_k-1}.
$$

(2)

Considering the sequence $\{\mu_q\}^\infty_1$ of the eigenvalues of the operator $B$ and choosing a Jordan basis in each corresponding space $\mathcal{H}_q$, we can arrange a system of vectors $\{e_i\}^\infty_1$ which we will call a system of the root vectors or following Lidskii a system of the major vectors of the operator $B$. Assume that $e_1, e_2, ..., e_{n_q}$ is the Jordan basis in the subspace $\mathcal{H}_q$. We can prove easily (see [16, p.14]) that there exists a corresponding biorthogonal basis $g_1, g_2, ..., g_{n_q}$ in the subspace $\mathcal{M}_q^\perp$.

Using the reasonings [11], we conclude that $\{g_i\}^{n_q}$ consists of the Jordan chains of the operator $B^*$ which correspond to the Jordan chains (2) due to the following formula

$$
B^* g_{q+k} = \mu_q g_{q+k}, \ B^* g_{q+k-1} = \mu_q g_{q+k-1} + g_{q+k}, ..., B^* g_{q} = \mu_q g_{q} + g_{q+1}.
$$

It is not hard to prove that the set $\{g_i\}^n$, $j \neq i$ is orthogonal to the set $\{e_i\}^n$ (see [11]). Gathering the sets $\{g_i\}^n$, $j = 1, 2, ..., n$, we can obviously create a biorthogonal system $\{g_i\}^\infty$ with respect to the system of the major vectors of the operator $B$. It is rather reasonable to call it as a system of the major vectors of the operator $B^*$. Note that if an element $f \in \mathcal{H}$ allows a decomposition in the strong sense

$$
f = \sum_{n=1}^{\infty} c_n e_n, \ c_n \in \mathbb{C},$$

4
then by virtue of the biorthogonal system existing, we can claim that such a representation is unique. Further, let us come to the previously made agreement that the vectors in each Jordan chain are arranged in the same order as in (2) i.e. at the first place there stands an eigenvector. It is clear that under such an assumption we have
\[ c_{q\xi + i} = \frac{(f, g_{q\xi + k - i})}{(e_{q\xi + i}, g_{q\xi + k - i})}, \quad 0 \leq i \leq k(q\xi), \]
where \( k(q\xi) + 1 \) is a number of elements in the \( q\xi \)-th Jordan chain. In particular, if the vector \( e_{q\xi} \) is included to the major system solo, there does not exist a root vector corresponding to the same eigenvalue, then
\[ c_{q\xi} = \frac{(f, g_{q\xi})}{(e_{q\xi}, g_{q\xi})}. \]

Note that in accordance with the property of the biorthogonal sequences, we can expect that the denominators equal to one in the previous two relations. Consider a formal series corresponding to a decomposition on the major vectors of the operator \( B \)
\[ f \sim \sum_{n=1}^{\infty} e_{n}c_{n}, \]
where each number \( n \) corresponds to a number \( q\xi + i \) (thus, the coefficients \( c_{n} \) are defined in accordance with the above and numerated in a simplest way). Consider a formal set of functions with respect to a real parameter \( t \)
\[ H_{m}(\varphi, z, t) := \frac{e^{\varphi(z)t}}{m!} \cdot \frac{d^{m}}{dz^{m}} \{ e^{-\varphi(z)t} \}, \quad m = 0, 1, 2, \ldots, . \]
Here we should note that if \( \varphi := z, z \in \mathbb{C} \), then we have a set of polynomials, what is in the origin of the concept, see [16]. Consider a series
\[ \sum_{n=1}^{\infty} c_{n}(t)e_{n}, \quad (3) \]
where the coefficients \( c_{n}(t) \) are defined in accordance with the correspondence between the indexes \( n \) and \( q\xi + i \) in the following way
\[ c_{q\xi + i}(t) = e^{-\varphi(\lambda q) t} \sum_{m=0}^{k(q\xi) - i} H_{m}(\varphi, \lambda q, t)c_{q\xi + i + m}, \quad i = 0, 1, 2, \ldots, k(q\xi), \quad (4) \]
here \( \lambda q = 1/\mu q \) is a characteristic number corresponding to \( e_{q\xi} \). It is clear that in any case, we have a limit \( c_{n}(t) \to \tilde{c}_{n}, t \to +0 \), where a value \( \tilde{c}_{n} \) can be calculated directly due to the formula (4). For instance in the case \( \varphi = z, z \in \mathbb{C} \), we have \( \tilde{c}_{n} = c_{n} \). Generalizing the definition given in [32, p.71], we will say that series (3) converges to the element \( f \) in the sense \( (A, \lambda, \varphi) \), if there exists a sequence of the natural numbers \( \{N_{j}\}_{j=1}^{\infty} \) such that
\[ f = \lim_{t \to +0} \lim_{j \to \infty} \sum_{n=1}^{N_{j}} c_{n}(t)e_{n}. \]
Note that sums of the latter relation forms a subsequence of the partial sums of the series \( B \).

We need the following lemmas \([10]\), in the adopted form also see \([11]\). In spite of the fact that the scheme of the Lemma 3 proof is the same we present it in the expanded form for the reader convenience. Further, considering an arbitrary compact operator \( B : \mathfrak{H} \rightarrow \mathfrak{H} \) such that \( \Theta(B) \subset Y_0(\theta), \theta < \pi \), we put the following contour in correspondence to the operator

\[ \vartheta(B) := \{ \lambda : |\lambda| = r > 0, |\arg\lambda| \leq \theta + \varsigma \} \cup \{ \lambda : |\lambda| > r, |\arg\lambda| = \theta + \varsigma \}, \]

where \( \varsigma > 0 \) is an arbitrary small number, the number \( r \) is chosen so that the operator \((I - \lambda B)^{-1}\) is regular within the corresponding closed circle. Here, we should note that the compactness property of \( B \) gives us the fact \((I - \lambda B)^{-1} \in \mathcal{B}(\mathfrak{H}), \lambda \in \mathbb{C} \setminus \text{int}B \). It can be proved easily if we note that in accordance with the Corollary 3.3 \([4, \text{p.268}]\), we have \( P(0) \subset \mathbb{C} \setminus \Theta(B) \).

**Lemma 1.** Assume that \( B \) is a compact operator, \( \Theta(B) \subset Y_0(\theta), \theta < \pi \), then on each ray \( \zeta \) containing the point zero and not belonging to the sector \( Y_0(\theta) \) as well as the real axis, we have

\[ \|(I - \lambda B)^{-1}\| \leq \frac{1}{\sin \psi_0}, \lambda \in \zeta, \]

where \( \psi_0 = \min\{|\arg \zeta - \theta|, |\arg \zeta + \theta|\} \).

**Lemma 2.** Assume that a compact operator \( B \) satisfies the condition \( B \in \tilde{\mathcal{H}}_\rho \), then for arbitrary numbers \( R, \kappa \) such that \( R > 0, 0 < \kappa < 1 \), there exists a circle \( |\lambda| = \tilde{R}, (1 - \kappa)R < \tilde{R} < R \), so that the following estimate holds

\[ \|(I - \lambda B)^{-1}\|_\Theta \leq e^{\gamma(|\lambda|)|\lambda|^\varrho} |\lambda|^m, |\lambda| = \tilde{R}, m = [\varrho], \varrho \geq \rho, \]

where

\[ \gamma(|\lambda|) = \beta(|\lambda|^{m+1}) + C\beta(|C\lambda|^{m+1}), \beta(r) = r^{-\frac{\varrho}{m+1}} \left( \int_0^r \frac{n_{Bm+1}(t)dt}{t} + r \int_r^\infty \frac{n_{Bm+1}(t)dt}{t^2} \right). \]

**Lemma 3.** Assume that \( B \) is a compact operator, \( \varphi \) is an analytical function inside \( \vartheta(B) \), then in the pole \( \lambda_q \) of the operator \((I - \lambda B)^{-1}\), the residue of the vector function \( e^{-\varphi(\lambda)t}B(I - \lambda B)^{-1}f, (f \in \mathfrak{H}) \), equals to

\[ \sum_{\zeta=1}^{m(q)} \sum_{i=0}^k \tilde{c}_{q\zeta+i}c_{q\zeta+i}(t), \]

where \( m(q) \) is a geometrical multiplicity of the \( q \)-th eigenvalue, \( k(q) + 1 \) is a number of elements in the \( q \)-th Jourdan chain,

\[ c_{q\zeta+j}(t) := e^{-\varphi(\lambda_q)t} \sum_{m=0}^{k(q)-j} \tilde{c}_{q\zeta+j+m}H_m(\varphi, \lambda_q, t). \]

**Proof.** Consider an integral

\[ J = \frac{1}{2\pi i} \oint_{\vartheta_q} e^{-\varphi(\lambda)t}B(I - \lambda B)^{-1}f d\lambda, f \in \text{R}(B), \]

where \( \text{R}(B) \) is a set of all \( f \) admitting the representation \( f = \sum_{n=0}^{\infty} f_n e^{-\theta_n(t)} \).

\[ J = \vartheta_q \left( \sum_{n=0}^{\infty} f_n e^{-\theta_n(t)} \right) \]

and from the residue Local Theorem

\[ J = \sum_{\zeta=1}^{m(q)} \sum_{i=0}^k \tilde{c}_{q\zeta+i}c_{q\zeta+i}(t). \]
where the interior of the contour \( \partial_q \) does not contain any poles of the operator \((I - \lambda B)^{-1}\), except of \( \lambda_q \). Assume that \( \mathfrak{N}_q \) is a root space corresponding to \( \lambda_q \) and consider a Jordan basis \( \{e_{q, j}\}, i = 0, 1, \ldots, k(q_e), \xi = 1, 2, \ldots, m(q) \) in \( \mathfrak{N}_q \). Using decomposition of the Hilbert space in the direct sum (1), we can represent an element

\[
f = f_1 + f_2,
\]

where \( f_1 \in \mathfrak{N}_q, \ f_2 \in \mathfrak{M}_q \). Note that the operator function \( e^{-\varphi(\lambda)B}(I - \lambda B)^{-1}f_2 \) is regular in the interior of the contour \( \partial_q \), it follows from the fact that \( \lambda_q \) ia a normal eigenvalue (see the supplementary information). Hence, we have

\[
\mathfrak{J} = \frac{1}{2\pi i} \oint_{\partial_q} e^{-\varphi(\lambda)B}(I - \lambda B)^{-1}f_1 d\lambda.
\]

Using the formula

\[
B(I - \lambda B)^{-1} = \frac{1}{\lambda} \{(I - \lambda B)^{-1} - I\} = \frac{1}{\lambda^2} \left\{ \left( \frac{1}{\lambda} I - B \right)^{-1} - \lambda I \right\},
\]

we obtain

\[
\mathfrak{J} = -\frac{1}{2\pi i} \oint_{\partial_q} e^{-\varphi(\zeta^{-1})B}(\zeta I - B)^{-1}f_1 d\zeta, \ \zeta = 1/\lambda.
\]

Now, let us decompose the element \( f_1 \) on the corresponding Jordan basis, we have

\[
f_1 = \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q)} e_{q, \xi+i} e_{q, \xi+i}.
\]

(5)

In accordance with the relation (2), we get

\[
Be_{q, \xi} = \mu q e_{q, \xi}, \ Be_{q, \xi+1} = \mu q e_{q, \xi+1} + e_{q, \xi}, \ldots, Be_{q, \xi+k} = \mu q e_{q, \xi+k} + e_{q, \xi+k-1}.
\]

Using this formula, we can prove the following relation

\[
(\zeta I - B)^{-1}e_{q, \xi+1} = \sum_{j=0}^{i} \frac{e_{q, \xi+j}}{(\zeta - \mu q)^{i-j+1}}.
\]

(6)

Note that the case \( i = 0 \) is trivial. Consider a case, when \( i > 0 \), we have

\[
\frac{(\zeta I - B)e_{q, \xi+j}}{(\zeta - \mu q)^{i-j+1}} = \frac{\zeta e_{q, \xi+j} - B e_{q, \xi+j}}{(\zeta - \mu q)^{i-j+1}} = \frac{e_{q, \xi+j}}{(\zeta - \mu q)^{i-j}} - \frac{e_{q, \xi+j-1}}{(\zeta - \mu q)^{i-j+1}}, \ j > 0,
\]

\[
\frac{(\zeta I - B)e_{q, \xi}}{(\zeta - \mu q)^{i+1}} = \frac{e_{q, \xi}}{(\zeta - \mu q)^{i}}.
\]

Using these formulas, we obtain

\[
\sum_{j=0}^{i} \frac{(\zeta I - B)e_{q, \xi+j}}{(\zeta - \mu q)^{i-j+1}} = \frac{e_{q, \xi}}{(\zeta - \mu q)^{i}} + \frac{e_{q, \xi+1}}{(\zeta - \mu q)^{i}} - \frac{e_{q, \xi}}{(\zeta - \mu q)^{i}} + \ldots
\]

(7)
where to concrete evolution equations. Finally, we discuss an approach that we can implement to apply the abstract theoretical results
and uniqueness theorem for evolution equation with the operator function at the right-hand side.

Using a notion of the convergence in the Abel-Lidskii sense. As a main result we prove an existence
the spectral family having involved the operators similar to Riesz projectors (see [2, p.20]) and
construction and formulate lemmas giving us a tool for further study. We consider an analogue of
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First of all, we consider statements with the necessary refinement caused by the involved functions,
operator function is supposed to be defined on the set of unbounded non-selfadjoint operators.

In this section, we have a challenge how to generalize results [11] in the way to make an efficient
tool for study abstract evolution equations with the operator function at the right-hand side. The

Note that using the reasonings of the last lemma, it is not hard to prove that
\[ c_{q+j}(t) \rightarrow \sum_{m=0}^{k(q)-j} c_{q+j+m} H_m(\varphi, \lambda, 0), \quad t \rightarrow +0. \]

### 3 Main results

In this section, we have a challenge how to generalize results [11] in the way to make an efficient
tool for study abstract evolution equations with the operator function at the right-hand side. The
operator function is supposed to be defined on the set of unbounded non-selfadjoint operators.
First of all, we consider statements with the necessary refinement caused by the involved functions,
here we should note that a particular case corresponding to a power function \( \varphi \) was considered
by Lidskii [16]. Secondly, we find conditions that guarantee convergence of the involved integral
construction and formulate lemmas giving us a tool for further study. We consider an analogue of
the spectral family having involved the operators similar to Riesz projectors (see [2, p.20]) and
using a notion of the convergence in the Abel-Lidskii sense. As a main result we prove an existence
and uniqueness theorem for evolution equation with the operator function at the right-hand side.
Finally, we discuss an approach that we can implement to apply the abstract theoretical results
to concrete evolution equations.
Lemma 4. Assume that the operator $B$ satisfies conditions of Lemma [1], the entire function $\varphi$ of the order less than a half maps the inside of the contour $\partial(B)$ into the sector $\Sigma_0(\varphi), \varphi < \pi/2$ for a sufficiently large value $|z|, z \in \text{int} \partial(B)$. Then the following relation holds

$$\lim_{t \to +0} \frac{1}{2\pi i} \int_{\partial(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = f, f \in \text{R}(B).$$  \hspace{1cm} (7)

Proof. Using the formula

$$B^2(I - \lambda B)^{-1} = \frac{1}{\lambda^2} \left\{ (I - \lambda B)^{-1} - (I + \lambda B) \right\},$$

we obtain

$$\int_{\partial(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \frac{1}{2\pi i} \int_{\partial(B)} e^{-\varphi(\lambda)t} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda -$$

$$- \frac{1}{2\pi i} \int_{\partial(B)} e^{-\varphi(\lambda)t} \lambda^{-2} (I + \lambda B) W f d\lambda = I_1(t) + I_2(t).$$

Consider $I_1(t)$. Since this improper integral is uniformly convergent regarding $t$, this fact can be established easily if we apply Lemma [1] then using the theorem on the connection with the simultaneous limit and the repeated limit, we get

$$\lim_{t \to +0} I_1(t) = \frac{1}{2\pi i} \int_{\partial(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda.$$

define a contour $\partial_R(B) := \text{Fr}\{\{\lambda : |\lambda| < R\} \setminus \text{int} \partial(B)\}$ and let us prove that

$$\frac{1}{2\pi i} \oint_{\partial_R(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda \to \frac{1}{2\pi i} \int_{\partial(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda, R \to \infty.  \hspace{1cm} (8)$$

Consider a decomposition of the contour $\partial_R(B)$ on terms $\partial_R(B) := \{\lambda : |\lambda| = R, \theta + \varsigma \leq \text{arg}\lambda \leq 2\pi - \theta - \varsigma\}$, $\partial_R := \{\lambda : |\lambda| = r, |\text{arg}\lambda| \leq \theta + \varsigma\} \cup \{\lambda : r < |\lambda| < R, \text{arg}\lambda = \theta + \varsigma\} \cup \{\lambda : r < |\lambda| < R, \text{arg}\lambda = -\theta - \varsigma\}$. It is clear that

$$\frac{1}{2\pi i} \oint_{\partial_R(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda = \frac{1}{2\pi i} \int_{\partial_R(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda +$$

$$+ \frac{1}{2\pi i} \int_{\partial_R} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda.$$ 

Let us show that the first summand tends to zero when $R \to \infty$, we have

$$\left\| \int_{\partial_R(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda \right\| \leq R^{-2} \int_{\theta+\varsigma}^{2\pi-\theta-\varsigma} \left\| (I\lambda^{-1} - B)^{-1} W f \right\|_{\partial} d\lambda.$$
Applying Corollary 3.3, Theorem 3.2 [4, p.268], we have
\[
\left\| (I\lambda^{-1} - B)^{-1} \right\| \leq R/\sin\varsigma, \lambda \in \tilde{\vartheta}_R(B).
\]

Substituting this estimate to the last integral, we obtain the desired result. Thus, taking into account the fact
\[
\frac{1}{2\pi i} \int_{\partial R} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda \to \frac{1}{2\pi i} \int_{\vartheta(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda, \quad R \to \infty,
\]
we obtain (8). Having noticed that the following integral can be calculated as a residue at the point zero, i.e.
\[
\frac{1}{2\pi i} \oint_{\vartheta_R(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda = \lim_{\lambda \to 0} \frac{d(I - \lambda B)^{-1}}{d\lambda} W f = f,
\]
we get
\[
\frac{1}{2\pi i} \int_{\vartheta(B)} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda = f.
\]

Hence \( I_1(t) \to f, \quad t \to +0 \). Let us show that \( I_2(t) = 0 \). For this purpose, let us consider a contour \( \vartheta_R(B) = \tilde{\vartheta}_R \cup \vartheta_R \), where \( \tilde{\vartheta}_R := \{ \lambda : |\lambda| = R, |\arg\lambda| \leq \theta + \varsigma \} \) and \( \vartheta_R \) is previously defined. It is clear that
\[
\frac{1}{2\pi i} \oint_{\vartheta_R(B)} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda = \frac{1}{2\pi i} \int_{\tilde{\vartheta}_R} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda +
\]
\[
\frac{1}{2\pi i} \int_{\vartheta_R} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda.
\]

Considering the second term having taken into account the definition of the improper integral, we conclude that if we show that there exists such a sequence \( \{R_n\}_n^\infty, \quad R_n \uparrow \infty \) that
\[
\frac{1}{2\pi i} \int_{\tilde{\vartheta}_{R_n}} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda \to 0, \quad n \to \infty,
\]
then we obtain
\[
\frac{1}{2\pi i} \oint_{\vartheta_{R_n}(B)} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda \to \frac{1}{2\pi i} \int_{\vartheta(B)} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda, \quad R \to \infty.
\]

Using the lemma conditions, we can accomplish the following estimation
\[
|e^{-\varphi(\lambda)t}| = e^{-Re \varphi(\lambda)t} \leq e^{-C|\varphi(\lambda)|t}, \quad \lambda \in \tilde{\vartheta}_R,
\]
where $R$ is sufficiently large. Using the condition imposed upon the order of the entire function and applying the Wieman theorem (Theorem 30 §18 Chapter I [15]), we can claim that there exists such a sequence $\{R_n\}_1^\infty, R_n \uparrow \infty$ that

$$\forall \varepsilon > 0, \exists N(\varepsilon) : e^{-C|\varphi(\lambda)|t} \leq e^{-Cm|\varphi(R_n)|t} \leq e^{-Ct|M\varphi(R_n)|\cos \pi \phi - \varepsilon}, \lambda \in \tilde{\vartheta}_{R_n}, n > N(\varepsilon),$$

where $\phi$ is the order of the entire function $\varphi$. Using this estimate, we get

$$\left\| \int_{\partial_{R_n}} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda \right\|_{\mathcal{B}} \leq Ce^{-Ct[M\varphi(R_n)]\cos \pi \phi - \varepsilon} \|Wf\|_{\mathcal{B}} \int_{-\tilde{\theta} - \varepsilon}^{\tilde{\theta} + \varepsilon} d\xi.$$  

It is clear that if the order $\phi$ less than a half then we obtain (9) and as a consequence (10). Since the operator function under the integral is analytic, then

$$\int_{\partial_{R_n}(B)} \lambda^{-2} e^{-\varphi(\lambda)t} (I + \lambda B) W f d\lambda = 0, n \in \mathbb{N}.$$ 

Combining this relation with (10), we obtain the fact $I_2(t) = 0$. The proof is complete.

**Remark 1.** Note that the statement of the lemma is not true if the order equals zero, in this case we cannot apply the Wieman theorem (more detailed see the proof of the Theorem 30 §18 Chapter I [15]). At the same time the proof can be easily transformed for the case corresponding to a polynomial function. Here, we should note that the reasonings are the same, we have to impose conditions upon the polynomial to satisfy the lemma conditions and establish an estimate analogous to (11). Now assume that $\varphi(z) = c_0 + c_1 z + \ldots + c_n z^n, z \in \mathbb{C}$, by easy calculations we see that the condition

$$\max_{k=0,1,\ldots,n} (|\arg c_k| + k\theta) < \pi/2,$$

gives us $|\arg \varphi(z)| < \pi/2, z \in \text{int}\tilde{\vartheta}(B).$ Thus, we have the fulfilment of the estimate (11). It can be established easily that $m\varphi(|z|) \to \infty, |z| \to \infty.$ Combining this fact with (11) and preserving the scheme of the reasonings presented in Lemma 4, we obtain (7).

Bellow, we consider an invertible operator $B$ and use a notation $W := B^{-1}$. This agreement is justified by the significance of the operator with a compact resolvent, the detailed information on which spectral properties can be found in the papers cited in the introduction section. Consider a function $\varphi$ that can be represented by a Laurent series about the point zero. Denote by

$$\varphi(W) := \sum_{n=-\infty}^{\infty} c_n W^n$$  

(12) 

a formal construction called by a function of the operator, where $c_n$ are coefficients corresponding to the function $\varphi$. The lemma given bellow are devoted to the study of the conditions under which being imposed the series of operators (12) converges on some elements of the Hilbert space $\mathcal{H}$, thus the operator $\varphi(W)$ is defined.
Lemma 5. Assume that $B$ is a compact operator, $\Theta(B) \subset \mathcal{L}_0(\theta)$, $\theta < \pi/2$, 
\[ \varphi(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad z \in \mathbb{C}, \quad s \in \mathbb{N}, \quad \max_{n=0,1,\ldots,s} (|\arg c_n| + n\theta) < \pi/2, \] (13)

then
\[ \frac{1}{2\pi i} \int_{\vartheta(B)} \varphi(\lambda) e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \varphi(W) u(t); \quad \lim_{t \to +0} \varphi(W) u(t) = \varphi(W) f, \] (14)

where
\[ u(t) := \frac{1}{2\pi i} \int_{\vartheta(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda, \quad f \in D(W^s). \]

Proof. Consider a decomposition of the Laurent series on two terms
\[ \varphi_1(z) = \sum_{n=0}^{s} c_n z^n; \quad \varphi_2(z) = \sum_{n=1}^{\infty} c_{-n} z^{-n}. \]

Consider an obvious relation
\[ \lambda^k B^k (E - \lambda B)^{-1} = (E - \lambda B)^{-1} - (E + \lambda B + \ldots + \lambda^{k-1} B^{k-1}), \quad k \in \mathbb{N}. \] (15)

It gives us the following representation
\[ \frac{1}{2\pi i} \int_{\vartheta(B)} \lambda^n e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = I_{1n}(t) + I_{2n}(t), \quad n \in \mathbb{Z}^- \cup \{0, 1, \ldots, s\}, \] (16)

where
\[ I_{1n} := \frac{1}{2\pi i} \int_{\vartheta(B)} e^{-\varphi(\lambda)t} (I - \lambda B)^{-1} W^{n-1} f d\lambda, \quad I_{2n}(t) := 0, \quad n = 0, \]
\[ I_{2n}(t) := \begin{cases} - \sum_{k=0}^{n-1} \beta_k(t) B^{k-n+1} f, & n > 0, \\ \sum_{k=0}^{n} \beta_k(t) B^{k-n+1} f, & n < 0 \end{cases}, \quad \beta_k(t) := \frac{1}{2\pi i} \int_{\vartheta(B)} e^{-\varphi(\lambda)t} \lambda^k d\lambda. \]

Let us show that $\beta_k(t) = 0$, define a contour $\vartheta_R(B) := \text{Fr} \{ \text{int} \vartheta(B) \cap \{ \lambda : r < |\lambda| < R \} \}$ and let us prove that
\[ I_{Rk}(t) := \frac{1}{2\pi i} \int_{\vartheta_R(B)} e^{-\varphi(\lambda)t} \lambda^k d\lambda \to \beta_k(t), \quad R \to \infty. \] (17)

Consider a decomposition of the contour $\vartheta_R(B)$ on terms $\tilde{\vartheta}_R := \{ \lambda : |\lambda| = R, \arg \lambda \leq \theta + \varsigma \}$ and $\check{\vartheta}_R := \{ \lambda : |\lambda| = r, \arg \lambda \leq \theta + \varsigma \} \cup \{ \lambda : r < |\lambda| < R, \arg \lambda = \theta + \varsigma \} \cup \{ \lambda : r < |\lambda| < R, \arg \lambda = -\theta - \varsigma \}$. We have
\[ \frac{1}{2\pi i} \int_{\vartheta_R(B)} e^{-\varphi(\lambda)t} \lambda^k d\lambda = \frac{1}{2\pi i} \int_{\tilde{\vartheta}_R} e^{-\varphi(\lambda)t} \lambda^k d\lambda + \frac{1}{2\pi i} \int_{\check{\vartheta}_R} e^{-\varphi(\lambda)t} \lambda^k d\lambda. \]
Having noticed that \( I_{Rk}(t) = 0 \), since the operator function under the integral is analytic inside the contour, we come to the conclusion that to obtain the desired result, we should show
\[
\frac{1}{2\pi i} \int_{\partial R} e^{-\varphi(\lambda)t} \lambda^k d\lambda \to 0, \ R \to \infty. \tag{18}
\]
We have
\[
\left| \int_{\partial R} e^{-\varphi(\lambda)t} \lambda^k d\lambda \right| \leq R^k \int_{\partial R} |e^{-\varphi(\lambda)t}| d\lambda \leq R^k+1 \int_{-\theta-\varsigma}^{\theta+\varsigma} e^{-i\text{Re} \varphi(\lambda)} d\arg\lambda.
\]
Consider a value \( \text{Re} \varphi(\lambda), \lambda \in \partial R \) for a sufficiently large value \( R \). Using the property of the principal part of the Laurent series in is not hard to prove that \( \forall \varepsilon > 0, \exists N(\varepsilon) : |\varphi_2(\lambda)| < \varepsilon, R > N(\varepsilon) \). It follows easily from the condition (13) that \( \text{Re} \varphi_1(\lambda) \geq C|\varphi_1(\lambda)|, \lambda \in \partial R \). It is clear that \( |\varphi_1(\lambda)| \sim |c_\lambda|R^s, \ R \to \infty. \) Thus, we have
\[
e^{-t\text{Re} \varphi(\lambda)} \leq e^{-C|\varphi(\lambda)|} \leq e^{-C|\lambda|^s t}, \lambda \in \partial R. \tag{19}
\]
Applying this estimate, we obtain
\[
\int_{-\theta-\varsigma}^{\theta+\varsigma} e^{-t\text{Re} \varphi(\lambda)} d\arg\lambda \leq \int_{-\theta-\varsigma}^{\theta+\varsigma} e^{-C|\varphi(\lambda)|} d\arg\lambda \leq e^{-CtR^s} \int_{-\theta-\varsigma}^{\theta+\varsigma} d\arg\lambda.
\]
The latter estimate gives us (18) from what follows (17). Therefore \( \beta_k(t) = 0 \) and we obtain the fact \( I_{2n}(t) = 0 \). Combining the fact of the operator \( W \) closedness (see [4, p.165] ) with the definition of the integral in the Riemann sense, we get easily
\[
W^n u(t) = \frac{1}{2\pi i} \int_{\partial(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} W^n f d\lambda, \ n = 0, 1, ..., s.
\]
Thus, using the formula (16), we obtain
\[
\frac{1}{2\pi i} \int_{\partial(B)} \varphi_1(\lambda) e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \varphi_1(W)u(t).
\]
Consider a principal part of the Laurent series. Using the formula (16), we get for values \( n \in \mathbb{N} \)
\[
\frac{1}{2\pi i} \int_{\partial(B)} \lambda^{-n} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = B^n u(t).
\]
Not that by virtue of a character of the convergence of the series principal part, we have
\[
\left\| \sum_{n=1}^{\infty} c_{-n} e^{-\varphi(\lambda)t} (I - \lambda B)^{-1} B^{n+1} f \right\|_{\mathcal{D}} \leq C \| f \|_{\mathcal{D}} \sum_{n=1}^{\infty} |c_{-n}| \cdot \| B \|_{\mathcal{D}}^{n+1} < \infty, \lambda \in \partial(B).
\]
Applying this estimate, we obtain
\[
\int_{\vartheta_j(B)} \varphi_2(\lambda) e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \sum_{n=1}^{\infty} c_{-n} \int_{\vartheta_j(B)} e^{-\varphi(\lambda)t} (I - \lambda B)^{-1} B^{n+1} f d\lambda, \quad j \in \mathbb{N},
\]
where
\[
\vartheta_j(B) := \{ \lambda : |\lambda| = r > 0, |\arg\lambda| \leq \theta + \varsigma \} \cup \{ \lambda : r < |\lambda| < r_j, r_j \uparrow \infty, |\arg\lambda| = \theta + \varsigma \}.
\]

Analogously to (19), we can easily get
\[
e^{-\text{Re}\varphi(\lambda)t} \leq e^{-C|\varphi(\lambda)|t} \leq e^{-C|\lambda|^t}, \quad \lambda \in \vartheta (B).
\]

Applying this estimate, we obtain
\[
\left\| \sum_{n=1}^{\infty} c_{-n} \int_{\vartheta_j(B)} e^{-\varphi(\lambda)t} (I - \lambda B)^{-1} B^{n+1} f d\lambda \right\| \leq C\|f\|_B \sum_{n=1}^{\infty} |c_{-n}| \cdot \|B^{n+1}\| \int_{\vartheta_j(B)} e^{-C|\lambda|^t} |d\lambda| \leq
\]
\[
\leq C\|f\|_B \sum_{n=1}^{\infty} |c_{-n}| \cdot \|B\|^{n+1} \int_{\vartheta(B)} e^{-C|\lambda|^t} |d\lambda| < \infty.
\]

Thus, we obtain the first relation (14). Let us establish the second relation (14). Using the formula (15), we obtain
\[
\frac{1}{2\pi i} \int_{\vartheta(B)} \varphi_2(\lambda) e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{-n} \int_{\vartheta(B)} e^{-\varphi(\lambda)t} (I - \lambda B)^{-1} B^{n+1} f d\lambda = \varphi_2(W)u(t).
\]

Note that the uniform convergence of the series at the left-hand side with respect to \( j \) follows from the latter estimate. Reformulating the well-known theorem of calculus on the absolutely convergent series in terms of the norm, we have
\[
\frac{1}{2\pi i} \int_{\vartheta(B)} \varphi_2(\lambda) e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{-n} \int_{\vartheta(B)} e^{-\varphi(\lambda)t} (I - \lambda B)^{-1} B^{n+1} f d\lambda = \varphi_2(W)u(t).
\]

Thus, we obtain the first relation (14). Let us establish the second relation (14). Using the formula (15), we obtain
\[
\frac{1}{2\pi i} \int_{\vartheta(B)} \lambda^n e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = I_{1n}(t) + I_{2n}(t), \quad n \in \mathbb{Z}^- \cup \{0, 1, \ldots, s\},
\]
where
\[
I_{1n}(t) := \frac{1}{2\pi i} \int_{\vartheta(B)} e^{-\varphi(\lambda)t} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f d\lambda, \quad I_{2n}(t) := 0, \quad n = -2,
\]
\[
I_{2n}(t) := \begin{cases} \sum_{k=-3}^{n} \beta_k(t) B^{k-n+1} f, & n \leq -3, \\ - \sum_{k=-2}^{n} \beta_k(t) B^{k-n+1} f, & n > -2, \end{cases}
\]
Using the proved above fact $\beta_k(t) = 0$, we have $I_{2n}(t) = 0$. Since in consequence of Lemma 1, inequality (20) for arbitrary $j \in \mathbb{N}, f \in D(W^*)$, we have
\[
e^{-\varphi(\lambda)t} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f \to \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f, \ t \to +0, \lambda \in \partial_j(B),
\]
where convergence is uniform with respect to the variable $\lambda$, the improper integral $I_{2n}(t)$ is uniformly convergent with respect to the variable $t$, then we get
\[
I_{2n}(t) \to \frac{1}{2\pi i} \int_{\partial(B)} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f d\lambda, \ t \to +0.
\]
Note that the last integral can be calculated as a residue, we have
\[
\frac{1}{2\pi i} \int_{\partial(B)} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f d\lambda = \lim_{\lambda \to 0} \frac{d(I - \lambda B)^{-1}}{d\lambda} W^{n+1} f = W^n f,
\]
\[
n \in \mathbb{Z}^{-} \cup \{0, 1, ..., s\}. \tag{22}
\]
It is obvious that using this formula, we obtain the following relation
\[
\lim_{t \to +0} \frac{1}{2\pi i} \int_{\partial(B)} \varphi_1(\lambda) e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} f d\lambda = \sum_{n=0}^{s} c_n W^n f, \ f \in D(W^*).
\]
Consider a principal part of the Laurent series. The following reasonings are analogous to the above, we get
\[
\left\| \sum_{n=1}^{\infty} c_{-n} \int_{\partial(B)} e^{-\varphi(\lambda)t} \lambda^{-2} (I - \lambda B)^{-1} B^{n-1} f d\lambda \right\| \leq C \|f\|_{\mathcal{D}} \sum_{n=1}^{\infty} |c_{-n}| \cdot \|B^{n-1}\| \int_{\partial(B)} |\lambda|^{-2} e^{-C|\lambda|t} |d\lambda| \leq
\]
\[
\leq C \|f\|_{\mathcal{D}} \sum_{n=1}^{\infty} |c_{-n}| \cdot \|B\|^{-n-1} \int_{\partial(B)} |\lambda|^{-2} |d\lambda| < \infty.
\]
It gives us the uniform convergence of the series with respect to $t$ at the left-hand side of the last relation. Using the analog of the well-known theorem of calculus on the absolutely convergent series, we have
\[
\sum_{n=1}^{\infty} c_{-n} \int_{\partial(B)} e^{-\varphi(\lambda)t} \lambda^{-2} (I - \lambda B)^{-1} B^{n-1} f d\lambda \to \sum_{n=1}^{\infty} c_{-n} \int_{\partial(B)} \lambda^{-2} (I - \lambda B)^{-1} B^{n-1} f d\lambda, \ t \to +0.
\]
Taking into account (21), (3), we get
\[
\lim_{t \to +0} \frac{1}{2\pi i} \int_{\partial(B)} \varphi_2(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \sum_{n=1}^{\infty} c_{-n} B^n f, \ f \in \mathcal{D}.
\]
It is clear that the second relation (14) holds. The proof is complete. \qed
Existence and uniqueness theorems

This paragraph is a climax of the paper, here we represent a theorem that put a beginning of a marvelous research based on the Abell-Lidskii method. The attempt to consider an operator function at the right-hand side was made in the paper [12], where we consider a case that is not so difficult since the corresponding function is of the power type. In contrast, in this paper we consider a more complicated case, a function that compels us to involve a principally different method of study. The existence and uniqueness theorem given below is based on the one of the number of theorems presented in [11].

Further, we will consider a Hilbert space $H$ consists of element-functions $u : \mathbb{R}_+ \to H$, $u := u(t), t \geq 0$ and we will assume that if $u$ belongs to $H$ then the fact holds for all values of the variable $t$. Notice that under such an assumption all standard topological properties as completeness, compactness e.t.c. remain correctly defined. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space $H$. The derivative is understood as the following limit

$$
\frac{u(t + \Delta t) - u(t)}{\Delta t} \xrightarrow{\delta} \frac{du}{dt}, \Delta t \to 0.
$$

Let $t \in \Omega := [a, b], 0 < a < b < \infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$
\sum_{i=0}^{n} u(\xi_i)\Delta t_i \xrightarrow{\theta} \int_{\Omega} u(t)dt, \lambda \to 0,
$$

where $(a = t_0 < t_1 < ... < t_n = b)$ is an arbitrary splitting of the segment $\Omega$, $\lambda := \max_i(t_{i+1} - t_i)$, $\xi_i$ is an arbitrary point belonging to $[t_i, t_{i+1}]$. The sufficient condition of the last integral existence is a continuous property (see[13, p.248]) i.e. $u(t) \xrightarrow{\delta} u(t_0), t \to t_0, \forall t_0 \in \Omega$. The improper integral is understood as a limit

$$
\int_{a}^{b} u(t)dt \xrightarrow{\delta} \int_{a}^{c} u(t)dt, b \to c, c \in [0, \infty].
$$

Let us study a Cauchy problem

$$
\frac{du}{dt} + \varphi(W)u = 0, \quad u(0) = h \in D(W), \quad (23)
$$

in the case when the operator $\varphi(W)$ is accretive we assume that $h \in \mathcal{H}$.

**Theorem 1.** Assume that $B$ is a compact operator, $\Theta(B) \subset L_0(\theta), \theta < \pi/2$, $B \in \tilde{S}_\rho$, moreover in the case $B \in \tilde{S}_\rho \setminus S_\rho$ the additional condition holds

$$
\frac{n_{B^{m+1}(r^{m+1})}}{r^\rho} \to 0, m = [\rho], \quad (24)
$$

the function $\varphi$ is satisfied the conditions of Lemma 5, the following condition holds $s > \rho$. Then a sequence of natural numbers $\{N_\rho\}_{\rho}^{\infty}$ can be chosen so that there exists a solution of the Cauchy
proof (23) in the form

\[ u(t) = \frac{1}{2\pi i} \int_{\partial(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda = \sum_{\nu=0}^{\infty} \sum_{q=N_{\nu}+1}^{\infty} \sum_{\xi=1}^{m(q) k(q_{\xi})} \sum_{i=0}^{\nu+1} e_{q_{\xi}+i} c_{q_{\xi}+i}(t), \]  

(25)

where

\[ \sum_{\nu=0}^{\infty} \left\| \sum_{q=N_{\nu}+1}^{\infty} \sum_{\xi=1}^{m(q) k(q_{\xi})} \sum_{i=0}^{\nu+1} e_{q_{\xi}+i} c_{q_{\xi}+i}(t) \right\| \leq \infty. \]  

(26)

Moreover, the existing solution is unique if the operator \( \varphi(W) \) is accretive.

**Proof.** Firstly, let us establish relation (26). Consider a contour \( \varphi \) where the function \( \gamma \), Note that in accordance with Lemma 3 [16] the following relation holds

\[ (I - \lambda B)^{-1} \| \leq e^{\gamma(|\lambda|)|\lambda|^m}, \quad m = [\rho], \quad |\lambda| = \tilde{R}_{\nu}, \quad R_{\nu} < \tilde{R}_{\nu} < R_{\nu+1}, \]

where the function \( \gamma(r) \) is defined in Lemma 2

\[ \beta(r) = r^{\frac{-\rho}{m+1}} \left( \int_0^r \frac{n_{B^{m+1}}(t)}{t} dt + r \int_r^{\infty} \frac{n_{B^{m+1}}(t)}{t^2} dt \right). \]

Note that in accordance with Lemma 3 [16] the following relation holds

\[ \sum_{i=1}^{\infty} \lambda_{i}^{m+1}(\tilde{B}) \leq \sum_{i=1}^{\infty} s_{i}^{m+1}(B) < \infty, \quad \varepsilon > 0, \]  

(27)

where \( \tilde{B} := (B^{m+1}A^{m+1})^{1/2} \). It is clear that \( \tilde{B} \in \tilde{\mathcal{S}}_\nu, \nu \leq \rho/(m+1) \). Denote by \( \vartheta_{\nu} \) a bound of the intersection of the ring \( \tilde{R}_{\nu} < |\lambda| < \tilde{R}_{\nu+1} \) with the interior of the contour \( \varphi(B) \), denote by \( N_{\nu} \) a number of poles being contained in the set \( \text{int} \varphi(B) \cap \{ \lambda : \nu < |\lambda| < \tilde{R}_{\nu} \} \). In accordance with Lemma 3 we get

\[ \frac{1}{2\pi i} \int_{\partial_{\vartheta_{\nu}}} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda = \sum_{q=N_{\nu}+1}^{\infty} \sum_{\xi=1}^{m(q) k(q_{\xi})} \sum_{i=0}^{\nu+1} e_{q_{\xi}+i} c_{q_{\xi}+i}(t), \quad h \in \mathcal{H}. \]  

(28)

Let us estimate the above integral, for this purpose split the contour \( \partial_{\vartheta_{\nu}} \) on terms \( \partial_{\vartheta_{\nu}} := \{ \lambda : |\lambda| = \tilde{R}_{\nu}, \arg \lambda \leq \theta + \varsigma \}, \partial_{\vartheta_{\nu+1}}, \partial_{\vartheta_{\nu+1}}, \partial_{\vartheta_{\nu+1}}, \arg \lambda = \theta + \varsigma \}, \partial_{\vartheta_{\nu+1}}, \arg \lambda = -\theta - \varsigma \}. \) Applying Lemma 2 relation (19), we get

\[ J_{\nu} := \left\| \int_{\partial_{\vartheta_{\nu}}} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda \right\| \leq \int_{\partial_{\vartheta_{\nu}}} e^{-t \text{Re} \varphi(\lambda)} \| B(I - \lambda B)^{-1} h \| \| d\lambda \| \leq \]

\[ \leq e^{\gamma(|\lambda|)|\lambda|^m} C e^{-C|\lambda|^\theta} \int_{-\theta-\varsigma}^{\theta+\varsigma} d\arg \lambda, \quad |\lambda| = \tilde{R}_{\nu}. \]
Thus, we get \( J_\nu \leq Ce^{\gamma(|\lambda|) |\lambda|^\rho - C|\lambda|^t} |\lambda|^{m+1} \), where \( m = [\rho] \), \(|\lambda| = \tilde{R}_\nu \). Let us show that for a fixed \( t \) and a sufficiently large \(|\lambda|\), we have \( \gamma(|\lambda|)|\lambda|^\rho - C|\lambda|^t < 0 \). It follows directly from Lemma 2 [14], we should consider \((27)\), in the case when \( B \in \mathcal{S}_p \) as well as in the case \( B \in \mathcal{S}_p \setminus \mathcal{S}_p \) but here we must involve the additional condition \((24)\). Therefore

\[
\sum_{\nu=0}^{\infty} J_\nu < \infty.
\]

Using the analogous estimates, applying Lemma [1] we get

\[
J_\nu^+ := \left\| \int_{\overline{\vartheta}_\nu} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda \right\|_{\mathcal{B}} \leq C\|h\|_{\mathcal{B}} \cdot C \int_{R_\nu} e^{-C(Re\varphi(\lambda))} |d\lambda| \leq C e^{-tCR^m} \int_{R_\nu} |d\lambda| = C e^{-tCR^m} \{R_{\nu+1} - R_\nu\}.
\]

\[
J_\nu^- := \left\| \int_{\overline{\vartheta}_\nu} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda \right\|_{\mathcal{B}} \leq C e^{-tCR^m} \int_{R_\nu} |d\lambda| = C e^{-tCR^m} \{R_{\nu+1} - R_\nu\}.
\]

The obtained results allow us to claim (the proof is left to the reader) that

\[
\sum_{\nu=0}^{\infty} J_\nu^+ < \infty, \quad \sum_{\nu=0}^{\infty} J_\nu^- < \infty.
\]

Using the formula \((28)\), the given above decomposition of the contour \( \overline{\vartheta}_\nu \), we obtain the relation \((25)\). Let us establish \((25)\), for this purpose, we should note that in accordance with relation \((28)\), the properties of the contour integral, we have

\[
\frac{1}{2\pi i} \oint_{\vartheta_{\tilde{R}_\nu}(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda = \sum_{\nu=0}^{p-1} \sum_{q=N_{\nu+1}}^{m(q)} \sum_{k(k)} \sum_{l=0}^{N(q) - C_{k+l} + i(t)} e_{N_{\nu+1}} \sum_{\xi=1}^{N_{\nu+1}} \sum_{\xi=1}^{N_{\nu+1}} e_{N_{\nu+1}}(t), \quad h \in \mathcal{B}, \quad p \in \mathbb{N},
\]

where the contour \( \vartheta_{\tilde{R}_\nu}(B) \) is defined in Lemma [5]. Using the proved above fact \( J_\nu \rightarrow 0, \nu \rightarrow \infty \), we can easily get

\[
\frac{1}{2\pi i} \oint_{\overline{\vartheta}_{\tilde{R}_\nu}(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda \rightarrow \frac{1}{2\pi i} \oint_{\vartheta(B)} e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda, \quad p \rightarrow \infty.
\]

The latter relation gives us the desired result \((25)\). Let us show that \( u(t) \) is a solution of the problem \((23)\). Applying Lemma [5] we get

\[
\varphi(W)u(t) = \frac{1}{2\pi i} \int_{\overline{\vartheta(B)}} \varphi(\lambda)e^{-\varphi(\lambda)t} B(I - \lambda B)^{-1} h d\lambda.
\]

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Now, we need establish the following relation

\[
\frac{du}{dt} = -\frac{1}{2\pi i} \int_{\partial(B)} \varphi(\lambda) e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h d\lambda, \quad h \in \mathcal{H},
\]

(29)
i.e. we can use a differentiation operation under the integral. For this purpose, let us prove that for an arbitrary \( \partial_j(B) \) (the definition is given in Lemma [5]) there exists a limit

\[
\frac{e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h - \varphi(\lambda) e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h}{\Delta t} \rightarrow -\varphi(\lambda) e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h, \quad \Delta t \rightarrow 0,
\]

(30)
where convergence is uniform with respect to \( \lambda \in \partial_j(B) \). Applying Lemma [1] we get

\[
\left\| \frac{e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h - \varphi(\lambda) e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h}{\Delta t} \right\|_{\mathcal{H}} \leq C \left\| \frac{e^{-\varphi(\lambda) t} - 1}{\Delta t} + \varphi(\lambda) \right\|_{\mathcal{H}} \max_{\lambda \in \partial_j(B)} e^{-\text{Re} \varphi(\lambda) t}.
\]

It is clear that

\[
\frac{e^{-\varphi(\lambda) t} - 1}{\Delta t} \rightarrow -\varphi(\lambda), \quad \Delta t \rightarrow 0,
\]

where convergence, in accordance with the Heine-Cantor theorem, is uniform with respect to \( \lambda \in \partial_j(B) \). Thus, we obtain (30). Using decomposition on the Taylor series, applying (20), we get

\[
\left\| \frac{e^{-\varphi(\lambda) t} - 1}{\Delta t} e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h \right\|_{\mathcal{H}} \leq |\varphi(\lambda)| e^{\|\varphi(\lambda)\|_{\mathcal{H}} t} e^{-\text{Re} \varphi(\lambda) t} \leq |\varphi(\lambda)| e^{(\Delta t - C t)|\varphi(\lambda)|} \leq |\varphi(\lambda)| e^{(\Delta t - C t)|\lambda|}, \quad \lambda \in \partial(B).
\]

Thus applying the latter estimate, Lemma [1] for a sufficiently small value \( \Delta t \), we get

\[
\left\| \int_{\partial(B)} \frac{e^{-\varphi(\lambda) t} B(I - \lambda B)^{-1} h d\lambda}{\Delta t} \right\|_{\mathcal{H}} \leq C \|h\|_{\mathcal{H}} \int_{\partial(B)} e^{-C|\lambda| t}|\lambda|^s d\lambda.
\]

The function under the integral at the right-hand side of the last relation guaranties that the improper integral at the left-hand side is uniformly convergent with respect to \( \Delta t \). These facts give us an opportinity to claim that the relation (29) holds. Here, we should explain that this conclusion is based upon the generalization of the well-known theorem of the calculus. In its own turn it follows easily from the theorem on the connection with the simultaneous limit and the repeated limit. We left a complete investigation of the matter to the reader having noted that the scheme of the reasonings is absolutely the same in comparison with the ordinary calculus. Thus, we obtain the fact that \( u \) is a solution of the equation (23).

Let us show that the initial condition holds in the sense \( u(t) \overset{\partial_j}{\rightarrow} h, \quad t \rightarrow +0 \). It becomes clear in the case \( h \in \text{D}(W) \), for in this case it suffices to apply Lemma [3] what gives us the desired result, i.e. we can put \( u(0) = h \). Consider a case when \( h \) is an arbitrary element of the Hilbert space \( \mathcal{H} \) and let us involve the accretive property of the operator \( \varphi(W) \). In accordance
with the above, for a fixed value of \( t \), we can understand a correspondence between \( u(t) \) and \( h \) as an operator \( S_t : \mathcal{H} \to \mathcal{H} \). Let us prove that \( \|S_t\|_{\mathcal{D} \to \mathcal{D}} \leq 1, \ t > 0 \). Firstly, assume that \( h \in D(W) \). Let us multiply the both sides of the relation (23) on \( u \) in the sense of the inner product, we get \((u', u)_{\mathcal{D}} + (\varphi(W)u, u)_{\mathcal{D}}\). Consider a real part of the last relation, we have

\[
\Re(u', u)_{\mathcal{D}} + \Re(\varphi(W)u, u)_{\mathcal{D}} = (u', u)_{\mathcal{D}}/2 + (u, u')_{\mathcal{D}}/2 + \Re(\varphi(W)u, u)_{\mathcal{D}}.
\]

Therefore \((\|u(t)\|^2_{\mathcal{D}})^{\prime} = -2\Re(\varphi(W)u, u)_{\mathcal{D}} \leq 0\). Integrating both sides from zero to \( \tau > 0\), we get \( \|u(\tau)\|^2_{\mathcal{D}} - \|u(0)\|^2_{\mathcal{D}} \leq 0 \). The last relation can be rewritten in the form \( \|S_t h\|^2_{\mathcal{D}} \leq \|h\|^2_{\mathcal{D}}, \ h \in D(W) \). Since \( D(W) \) is a dense set in \( \mathcal{H} \), we obviously obtain the desired result, i.e. \( \|S_t\|_{\mathcal{D} \to \mathcal{D}} \leq 1 \). Now, having assumed that \( h_n \xrightarrow{\mathcal{D}} h, n \to \infty, \ \{h_n\} \subset D(W), h \in \mathcal{H} \), consider the following reasonings \( \|u(t) - h\|^2_{\mathcal{D}} = \|S_t h - S_t h_n + S_t h_n - h_n + h_n - h\|_{\mathcal{D}} \leq \|S_t\| \cdot \|h - h_n\|_{\mathcal{D}} + \|S_t h_n - h_n\|_{\mathcal{D}} + \|h_n - h\|_{\mathcal{D}} \).

Note that \( S_t h_n \xrightarrow{\mathcal{D}} h_n, \ t \to +0 \). It is clear that if we choose \( n \) so that \( \|h - h_n\|_{\mathcal{D}} < \varepsilon/3 \) and after that choose \( t \) so that \( \|S_t h_n - h_n\|_{\mathcal{D}} < \varepsilon/3 \), then we obtain \( \forall \varepsilon > 0, \exists \delta(\varepsilon) : \|u(t) - h\|_{\mathcal{D}} < \varepsilon, \ t < \delta \). Thus, we can put \( u(0) = h \) and claim that the initial condition holds in the case \( h \in \mathcal{H} \). The uniqueness follows easily from the fact that \( \varphi(W) \) is accretive. In this case, repeating the previous reasonings, we come to \( \|\phi(\tau)\|^2_{\mathcal{D}} \leq \|\phi(0)\|^2_{\mathcal{D}}, \) where \( \phi \) is a sum of solutions \( u_1 \) and \( u_2 \). Notice that by virtue of the initial conditions, we have \( \phi(0) = 0 \). Therefore, the previous relation can hold only if \( \phi = 0 \). The proof is complete.

**Remark 2.** Note that generally the existence and uniqueness Theorem 2 [11] is based upon the Theorem 2 [11]. The corresponding analogs based upon the Theorems 3, 4 [11] can be obtained due to the same scheme and the proofs are not worth representing. At the same time the mentioned analogs can be useful because of special conditions imposed upon the operator \( B \) such as ones formulated in terms of the operator order [11]. Here we should also appeal to an artificially constructed normal operator presented in [12].

**Concrete operators**

It is remarkable that the made approach allows us to obtain a solution analytically for the right-hand side – a function of an operator belonging to a sufficiently wide class of operators. A plenty of examples are presented in the paper [12] where such well-known operators as the Riemann-Liouville fractional differential operator, the Kipriyanov operator, the Riesz potential, the difference operator are considered. An interesting example can be also found in the paper [8]. More general approach, implemented in the paper [10] allows us to build a transform of an operator belonging to the class of \( m \)-accretive operators. We should stress a significance of the last claim for the class contains the infinitesimal generator of a \( C_0 \) semigroup of contractions. In its own turn fractional differential operators of the real order can be expressed in terms of the infinitesimal generator of the corresponding semigroup, what makes the offered generalization relevant (more detailed see [10]). Bellow, we present a rather abstract example for which the paper results can be applied. Consider a transform of an \( m \)-accretive operator \( J \) acting in \( \mathcal{H} \)

\[
Z^\alpha_{G,F}(J) := J^\ast G J + F J^\alpha, \ \alpha \in [0,1),
\]

where symbols \( G, F \) denote operators acting in \( \mathcal{H} \). The Theorem 5 [10] gives us a tool to describe spectral properties of the transform \( Z^\alpha_{G,F}(J) \). Particularly, we can establish the order of the transform and its belonging to the Schatten-von Neumann class of the convergence exponent by virtue of the Theorem 3 [10]. Thus, having known the index of the Schatten-von Neumann class of the convergence exponent, we can apply Theorem 5 to the transform.
4 Conclusions

In this paper, we invented a technique to study evolution equations with the right-hand side a function of the non-selfadjoint unbounded operator. It is remarkable that, we may say that the main issue of the paper is an application of the spectral theorem to the special class of non-selfadjoint operators and in the natural way, we come to the definition of a function of the unbounded non-selfadjoint operator. Under this point of view, the main highlights of this paper are propositions analogous to the spectral theorem. We can perceive them as an introduction or a way of reformulating the main principles of the spectral theorem based upon the peculiarities of the convergence in the Abell-Lidsky sense. In this regard, the main obstacle that appears is how to define an analogue of a spectral family or decomposition of the identical operator.

However, as a main result we have obtained an approach allowing us to principally broaden conditions imposed upon the right-hand side of the evolution equation in the abstract Hilbert space. The application part of the paper appeals to the theory of differential equations. In particular, the existence and uniqueness theorems for evolution equations, with the right-hand side being presented by an operator function of a differential operator with a fractional derivative in final terms, are covered by the invented abstract method. In connection with this, such operators as a Riemann-Liouville fractional differential operator, Kipriyanov operator, Riesz potential, difference operator can be considered. Moreover, we can consider a class the artificially constructed normal operators for which the clarification of the Lidskii results relevantly works. Apparently, the further step in the theoretical study may be to consider an entire function that generates the operator function, note that a prerequisite of the prospective result is given by Lemma 4. In this case, some difficulties that may appear relate to the propositions analogous to Lemma 5 and Theorem 1. Having been inspired by the above ideas, we hope that the concept will have a further development.

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