Reduction of homomorphisms mod $p$ and algebraicity

Chandrashekhar Khare & Dipendra Prasad

Abstract

Let $K$ be a number field, and $A_1, A_2$ abelian varieties over $K$. Let $P$ (resp. $Q$) be a non-torsion point in $A_1(K)$ (resp. $A_2(K)$) such that for almost all places $v$ of $K$, the order of $Q$ mod $v$ divides the order of $P$ mod $v$. Then we prove (under some conditions on $A_i$, $i = 1, 2$) that there is a homomorphism $j$ from $A_1$ to $A_2$ such that $j(P) = Q$. We formulate and extend such a result for any subgroups of $A_i(K), i = 1, 2$. These results in particular extend the work of Corrales and Schoof from elliptic curves to abelian varieties.

1 Introduction

Let $A_1$ and $A_2$ be two abelian varieties over a number field $K$. There is a finite set $S$ of places of $K$ such that there are abelian schemes $A_i$ over $O_S$, the ring of $S$-integers of $K$, with generic fibres $A_i$. For $v$ not in $S$ (all the places $v$ we consider below are outside $S$ even if this is not mentioned explicitly) we can consider reduction mod $v$ of the schemes $A_i$ at $v$. We will abuse notation to denote these abelian varieties over finite fields also as $A_i$. Consider the specialisation map $sp_v: A_i(K) \to A_i(k_v)$, and denote the image of $A_i(K)$ by $A_i(v)$. We say that a homomorphism $\phi: A_1(K) \to A_2(K)$ specialises mod $v$ if there is a homomorphism $\phi_v: A_1(v) \to A_2(v)$ such that the diagram

$$
\begin{array}{ccc}
A_1(K) & \xrightarrow{\phi} & A_2(K) \\
sp_v \downarrow & & \downarrow sp_v \\
A_1(v) & \xrightarrow{\phi_v} & A_2(v)
\end{array}
$$

commutes. The aim of this paper is to answer the following question.
Question 1 Let $A_1$ and $A_2$ be abelian varieties over a number field $K$. Let $\phi : A_1(K) \to A_2(K)$ be a homomorphism of abelian groups that specialises mod $v$ for almost all places $v$ of $K$ to give a homomorphism $\phi_v$ as above. Is the restriction of $\phi$ to a subgroup of finite index of $A_1(K)$ induced by a homomorphism $\alpha_\phi \in \text{Hom}_K(A_1, A_2)$?

Remark: Although the question above is posed for a homomorphism from $A_1(K)$ to $A_2(K)$, the heart of the question is “pointwise”: namely given points $P$ in $A_1(K)$, and $Q$ in $A_2(K)$ such that for almost all places $v$ of $K$ the order of $Q$ in $A_2(k_v)$ divides the order of $P$ in $A_1(k_v)$, then is $Q$ related to $P$ by a homomorphism from $A_1$ to $A_2$? (Because of Lemma 1 in the next section we see that this is the essential content of the question.)

Here are the main theorems of this paper.

Theorem 1 Let $A_1$ and $A_2$ be two simple abelian varieties over a number field $K$ with the endomorphism rings over the algebraic closure of $K$ as $\mathbb{Z}$. Assume that $\dim (A_i) = g_i$ is either odd or $g_i = 2$ or 6. (We do not assume that $\dim A_1 = \dim A_2$.) Let $P$ (resp. $Q$) be a non-torsion point in $A_1(K)$ (resp. $A_2(K)$) such that for almost all places $v$ of $K$, the order of $Q$ mod $v$ divides the order of $P$ mod $v$. Then $A_1$ and $A_2$ are isogenous, and there is an isogeny $j$ from $A_1$ to $A_2$ such that $j(P) = nQ$ for some integer $n$. Furthermore, if one of the abelian varieties is an elliptic curve, we have the same conclusion without any restriction on its endomorphism ring.

Theorem 2 Let $A_1$ and $A_2$ be two simple abelian varieties over a number field $K$ with $\text{End}_K(A_i) = \mathbb{Z}$ and such that $\dim (A_i) = g_i$ is either odd or $g_i = 2$ or 6. In case one of the $A_i$’s is an elliptic curve, we allow the elliptic curve to have complex multiplication. Then we have the following.

1. For any abstract homomorphism

$$\phi : A_1(K) \to A_2(K)$$

that specialises mod $v$ for almost all places $v$ of $K$, an integral multiple of $\phi$ is given by a homomorphism from $A_1$ to $A_2$.

2. If $A_1(K)$ is infinite, $\phi$ itself is given by an isogeny.

The restrictions on dimension in the two theorems arise from those of Serre’s theorems in article 137 of [S-IV]. We prove theorem 1 in section 3. In
section 4 we prove a general lemma that answers Question [1] in the case of CM elliptic curves.

The main component of the proof given in this paper, besides the theorem of Serre on the image of Galois group on the \(\ell\)-adic Tate module of an Abelian variety, is the Kummer theory on Abelian varieties, initiated and studied by Ribet in several papers, and completed in his paper with Jacquinot, cf. [JR]. We recall these theorems from [JR] in section 2. We also make crucial use of a theorem of Bogomolov, cf. [B], that the image of Galois group on the \(\ell\)-adic Tate module of an Abelian variety contains an open subgroup of the homotheties.

The method we employ in the paper will be able to generalise theorem 1 to general abelian varieties if an analogue of Serre’s theorem was available for them.

We note that Theorems 1 and 2 in the special case of elliptic curves is due to Corrales and Schoof, see [CS]. However, our proof for elliptic curves is quite different from the proof of [CS].

In sections 6 and 7 we look at the analogue of these questions for the case of linear algebraic groups, and general rational varieties respectively.

2 Theorems due to Jacquinot and Ribet

In this section we recall two theorems due to Jacquinot and Ribet [JR] which play a fundamental role in this paper.

For any abelian variety \(A\) over a number field \(K\), we let \(A[\ell]\) denote the set of \(\ell\)-torsion points of \(A\) (defined over the algebraic closure \(\bar{K}\) of \(K\)). For any point \(P\) in \(A(K)\), \(P/\ell\) denotes any point \(Q\) in \(A(\bar{K})\) such that \(\ell Q = P\). The minimal field extension of \(K\) (inside \(\bar{K}\)) over which every point of \(A[\ell]\) and \(Q\) is defined is independent of the choice of \(Q\), and is denoted by \(K(A[\ell], P/\ell)\).

The following is theorem 3.1 of [JR].

**Theorem 3** Let \(A\) be an abelian variety over a number field \(K\), and let \(P \in A(K)\). Let \(B\) be the connected component of 0 in the Zariski closure of the subgroup of \(A(K)\) generated by \(P\). Then the Galois group of \(K(A[\ell], P/\ell)\) over \(K(A[\ell])\) is naturally isomorphic to \(B[\ell]\) for almost all primes \(\ell\).

From this theorem, [JR] deduce the following very important result for us.
Theorem 4 Let $A$ be an abelian variety over a number field $K$, and $P_1, P_2$ be points of $A(K)$. Then there exists a $K$-endomorphism $f : A \to A$, and an integer $n \geq 1$ such that $f(P_1) = nP_2$ if and only if,

$$K(A[\ell], P_2/\ell) \subseteq K(A[\ell], P_1/\ell),$$

for an infinite number of primes $\ell$.

3 Proof of Theorem 1

We will give the proof of theorem 1 when neither of the $A_i$’s is an elliptic curve with complex multiplication. The proof when one of the $A_i$’s is an elliptic curves with complex multiplication is very similar, and will be left to the reader.

Under our assumptions and by the theorems of Serre in [S-IV], for all $\ell$ large enough (see Théorème C on page 40 and the Corollaire on page 51 of the letter to Vignéras in [S-IV]), the image of the action of $G_K$ on $A_i[\ell]$ is $GSp_{2g_i}(\mathbb{Z}/\ell)$.

From theorem 3 (due to [JR]) recalled above, we know that for almost all primes $\ell$, the field extension $K(A_1[\ell], A_2[\ell], P/\ell)$ of $K$ obtained by attaching the $\ell$-torsion points $A_1[\ell]$ inside $A_1(\bar{K})$, and any $\ell$-division point $P/\ell$ of $P$, is a Galois extension of $K$ with Galois group $G_{P,\ell}$ which sits in the following split exact sequence,

$$0 \to A[\ell] \to G_{P,\ell} \to GSp_{2g_1}(\mathbb{Z}/\ell) \to 1.$$
will be at most $1/\ell$ of the density of the set of primes in $K(A_1[\ell], A_2[\ell])$ which split completely in $K(A_1[\ell], A_2[\ell], P/\ell)$. However, we will prove below that the set of primes in $K(A_1[\ell], A_2[\ell])$ which split in $K(A_1[\ell], A_2[\ell], P/\ell)$ and in $K(A_1[\ell], A_2[\ell], Q/\ell)$ is of much larger density, proving that $K(A_1[\ell], A_2[\ell], Q/\ell)$ is a subfield of $K(A_1[\ell], A_2[\ell], P/\ell)$.

A prime in $K(A_1[\ell], A_2[\ell])$ splits in $K(A_1[\ell], A_2[\ell], P/\ell)$ if $P$ has an $\ell$-division point in the corresponding residue field of $K(A_1[\ell], A_2[\ell])$. Now there are two possibilities under which $P$ will have an $\ell$-division point. One, in which the order of $P$ in the residue field is coprime to $P$, automatically will have an $\ell$-division point. Second, in which $P$ has order divisible by $\ell$ in which case if $P$ has an $\ell$-division point in the residue field of $K(A_1[\ell], A_2[\ell])$, then clearly there will exist an $\ell^2$-torsion point of $A_1$ which is also defined over this residue field of $K(A_1[\ell], A_2[\ell])$.

The field $K(A_1[\ell^2], A_2[\ell])$ is an abelian extension of $K(A_1[\ell], A_2[\ell])$ with Galois group isomorphic to $M_{p_{2g_1}}(\mathbb{Z}/\ell)$ which is the kernel of the natural map from $GSp_{2g_1}(\mathbb{Z}/\ell^2)$ to $GSp_{2g_1}(\mathbb{Z}/\ell)$. For any point $Z$ belonging to $A_1[\ell]$, the field $K(A_1[\ell], A_2[\ell], Z/\ell)$ is an abelian extension of $K(A_1[\ell], A_2[\ell])$ of degree $\ell^{2g_1}$. Each of these extensions correspond (by Galois theory) to subgroups $H_Z$ of $M_{p_{2g_1}}(\mathbb{Z}/\ell)$. The group $M_{p_{2g_1}}(\mathbb{Z}/\ell)$ operates on $(\mathbb{Z}/\ell)^{2g_1}$, and $H_Z$ is the stabilizer of the point $Z$ in $(\mathbb{Z}/\ell)^{2g_1}$. This implies in particular that every element of $H_Z$ has $1$ as an eigenvalue. Thus the determinant of $(1 - X)$ is $0$ for all $X$ belonging to $H_Z$, and for any choice of $Z$. Since the determinant $\det(1 - X)$ is a nonzero polynomial function on $M_{p_{2g_1}}(\mathbb{Z}/\ell)$, the set of zeros is a proper subset of cardinality bounded by a (fixed) polynomial $P_{d-1}(\ell)$ in $\ell$ of degree $d - 1$ where $d$ is the integer such that $\ell^d$ is the cardinality of $M_{p_{2g_1}}(\mathbb{Z}/\ell)$.

The set of primes of $K(A_1[\ell], A_2[\ell])$ for which $A_1$ has a point of order $\ell^2$ in its residue field correspond to primes $v$ of $K(A_1[\ell], A_2[\ell])$ such that $v$ splits in at least one of the extensions $K(A_1[\ell], A_2[\ell], Z/\ell) \subset K(A_1[\ell^2], A_2[\ell])$. Thus the corresponding Artin symbol for the field extension $K(A_1[\ell^2], A_2[\ell])$ of $K(A_1[\ell], A_2[\ell])$ must belong to one of the subgroups $H_Z$ of $M_{p_{2g_1}}(\mathbb{Z}/\ell)$.

The union of the subgroups $H_Z$ has order bounded by $P_{d-1}(\ell)$. Thus the density of the set of primes in $K(A_1[\ell], A_2[\ell])$ which are split in $K(A_1[\ell], A_2[\ell], P/\ell)$ but in none of $K(A_1[\ell], A_2[\ell], Z/\ell)$ is of density at least equal to $[\ell^d - P_{d-1}(\ell)]/\ell^{2g_1 + d}$ which is much larger than $1/\ell^{2g_1 + 1}$ (for large $\ell$). For these primes, the order of $P$ in the residue field is coprime to $\ell$, and since order of $Q$ divides the order of $P$, the order of $Q$ too is coprime to $\ell$ in the residue field. Thus these primes also split in $K(A_1[\ell], A_2[\ell], Q/\ell)$, and are of density greater than $1/\ell^{2g_1 + 1}$,
proving our earlier claim that $K(A_1[\ell], A_2[\ell], Q/\ell) \subset K(A_1[\ell], A_2[\ell], P/\ell)$. Hence by theorem 4 (due to [JR]) applied to the abelian variety $A = A_1 \times A_2$, and points $(P, 0), (0, Q)$, we get an endomorphism of $A$ taking $(P, 0)$ to an integral multiple of $(0, Q)$. This implies that there is an isogeny $j$ from $A_1$ to $A_2$ such that $j(P) = nQ$ for some integer $n$, proving the theorem.

4 Proof of Theorem 2 (part 1)

To avoid trivialities, we will assume in what follows that the image of $A_1(K)$ under the homomorphism being considered is infinite.

Given an abstract homomorphism

$$\phi : A_1(K) \rightarrow A_2(K)$$

that specialises mod $v$ for almost all places $v$ of $K$, it is clear that for any point $P$ of infinite order in $A_1(K)$, the order of $\phi(P) \ mod \ v$ which is a point in $A_2(k_v)$ divides the order of $P \ mod \ v$ which is a point in $A_1(k_v)$. Thus by theorem 1, there is an isogeny $j$ and an integer $n$ (both depending on the point $P$) such that $j(P) = n\phi(P)$. What we need to prove for the part 1 of theorem 2 is that $j$ and $n$ can be chosen independent of $P$, say on a torsion-free subgroup of $A_1(K)$. Note that if we can achieve $j(P) = n\phi(P_i)$ for the same $n$ (but perhaps different $j$) for a set of points $P_i$, we can assume that $n$ works for the subgroup of $A_1(K)$ generated by $P_i$. Since the equation $j(P) = n\phi(P)$ can be multiplied by any integer, we can thus assume that $n$ is independent of $P$. Thus for each torsion-free point $P$ of $A_1(K)$, we assume now that there is an isogeny $j$ dependent on $P$, and an integer $n \neq 0$ independent of $P$, such that $j(P) = n\phi(P)$. The following general lemma implies that $j$ is independent of $P$. (The proof of this fact, and of the lemma, for the case when $\text{End}(A_1) = \mathbb{Z}$ is rather trivial; the lemma has been formulated in this generality because of the possible application to the more general context.)

**Lemma 1** Let $A$ be a finitely generated free abelian group. Let $\mathcal{D}$ be a division algebra which contains $\mathbb{Q}$ and is finite dimensional over $\mathbb{Q}$. Let $\mathcal{O}$ be an order in $\mathcal{D}$. Suppose $\mathcal{O}$ acts on $A$ on the left making it into a left $\mathcal{O}$-module. Suppose that $f$ is an endomorphism of $A$ as an additive group such that for all $a \in A$, there exists $f_a \in \mathcal{O}$ such that $f(a) = f_a \cdot a$. Then $f$ is multiplication by an element of $\mathcal{O}$. 

6
Proof: Clearly $A \otimes \mathbb{Z} \mathbb{Q}$ is a vector space over $\mathbb{Q}$ on which $\mathcal{O} \otimes \mathbb{Z} \mathbb{Q} = \mathcal{D}$ acts, making it into a $\mathcal{D}$-vector space. From the hypothesis that $f(a) = f_a \cdot a$ for all $a \in A$, we find that any $\mathcal{D}$-subspace of $A \otimes \mathbb{Z} \mathbb{Q}$ is stable under $f$ (extended to $A \otimes \mathbb{Z} \mathbb{Q}$). Write $A \otimes \mathbb{Z} \mathbb{Q} = L_1 \oplus \cdots \oplus L_n$, as a direct sum of $\mathcal{D}$-subspaces $L_i$ of dimension 1 which as has been noted is invariant under $f$, i.e., $f(L_i) \subset L_i$. Write $M_i = L_i \cap A$; then $M_i$ is a lattice in $L_i$, and $\oplus M_i$ is a subgroup of finite index in $A$. Since both $L_i$ and $A$ are invariant under $f$, so is $M_i$. Also, each $L_i$ and $A$ being invariant under $\mathcal{O}$, so is $M_i$. We will now prove that the restriction of $f$ to $M_i$ is given by multiplication by an element $f_i$ in $\mathcal{O}$.

We can clearly assume that $M_i$ is a lattice in $\mathcal{D}$ which is invariant under $\mathcal{O}$. Also, after scaling by an element of $\mathcal{D}^*$ on the right, we can assume that the lattice $M_i$ in $\mathcal{D}$ contains 1.

Suppose that $f_i(1) = \alpha_i \in \mathcal{O}$. We can thus after replacing $f_i$ by $f_i - \alpha_i$, assume that $f_i(1) = 0$. We would like to prove that $f_i$ is identically zero. Assuming the contrary, let $x$ be an element in $M_i \cap \mathcal{O}$ such that $f_i(x) \neq 0$. After scaling $x$, we can moreover assume that $f_i(x)$ belongs to $\mathcal{O}$.

By hypothesis, for every element $m \in \mathbb{Z}$, there exists an element $\lambda_m \in \mathcal{O}$ such that $f_i(m + x) = \lambda_m \cdot (m + x)$. For an element $z \in \mathcal{D}$, let $\text{Norm}(z)$ denote the determinant of the left multiplication by $z$ on $\mathcal{D}$. Since $f_i(x) - f_i(m + x) = \lambda_m \cdot (x + m)$, it follows that $\text{Norm}(m + x)$ and $\text{Norm}(f_i(x))$ are integers in $\mathbb{Q}$, and $\text{Norm}(m + x)$ divides $\text{Norm}(f_i(x))$. Since $\text{Norm}(m + x)$ is a polynomial in $m$ with coefficients in $\mathbb{Z}$ of degree equal to the dimension of $\mathcal{D}$ over $\mathbb{Q}$ of leading term 1 and constant term $\text{Norm}(x)$, the polynomial $\text{Norm}(m + x)$ as $m$ varies takes arbitrary large values, hence cannot divide the fixed integer $\text{Norm}(f_i(x))$.

We have thus proved that $f$ restricted to any 1 dimensional $\mathcal{D}$ submodule of $A \otimes \mathbb{Z} \mathbb{Q}$ is multiplication by an element of $\mathcal{D}$. From this it is trivial to see that the action of $f$ on $A \otimes \mathbb{Z} \mathbb{Q}$ is multiplication by an element of $\mathcal{D}$, which must moreover lie in $\mathcal{O}$, completing the proof of the lemma.

Remark: The lemma above holds good only in the integral version stated above, and not for vector spaces, and hence is not totally trivial. We point out an example to illustrate that the analogue of the lemma is not true for vector spaces. For this, let $K$ be a finite extension of $\mathbb{Q}$ of degree $> 1$. There is an action of $K^*$ on $K$ via left or right multiplication. Let $f$ be an automorphism of $K$ (considered as a vector space over $\mathbb{Q}$) which does not arise from the action of an element of $K^*$. Such an automorphism $f$ satisfies the hypothesis of the previous lemma as for any $a \neq 0$, $f(a) \in K^*$, hence
\( f(a) = f_a \cdot a \) with \( f_a \in K^* \), but such an \( f \) does not satisfy the conclusion of the lemma.

## 5 Proof of theorem 2 (part 2)

The proof of the 2nd part of theorem 2 for CM elliptic curves follows from the theorem 2 of [CS] when combined with lemma 1. We will thus concentrate our efforts in proving this part under the assumption that \( \text{End}_K(A) = \mathbb{Z} \).

We begin with some preliminary results.

The following lemma is well known, cf. S.Lang’s book, Algebra, section 10 of the chapter on Galois theory for \( i = 1 \). It also follows from generalities on cohomology once we know it is true for \( i = 0 \), which is of course clear.

**Lemma 2** Let \( G \) be a group, and \( E \) a \( G \)-module. Let \( \tau \) be an element in the center of \( G \). Then \( H^i(G, E), i = 0, 1, \ldots \) is annihilated by the map on \( H^i(G, E) \) induced from the map \( x \mapsto \tau x - x \) from \( E \) to itself.

**Lemma 3** Let \( A \) be an abelian variety over a number field \( K \). Let \( K_{\ell^n} = K(A[\ell^n]) \), and \( G_{\ell^n} = \text{Gal}(K_{\ell^n}/K) \). Let \( G_{\ell^\infty} = \text{Gal}(K_{\ell^\infty}/K) \) with \( K_{\ell^\infty} = \bigcup_n K_{\ell^n} \). Then \( H^1(G_{\ell^n}, A[\ell^n]) (m \geq n) \) and \( H^1(G_{\ell^\infty}, A[\ell^n]) \) are of finite orders, bounded independent of \( m \) and \( n \).

**Proof:** We note that \( G_{\ell^\infty} \) being a compact \( \ell \)-adic Lie group, is topologically finitely generated, hence each finite quotient such as \( G_{\ell^m} \) is generated by a set of elements of cardinality independent of \( m \).

From the definition of \( H^1(G, A) \) in terms of maps \( \phi \) from \( G \) to \( A \) such that \( \phi(g_1g_2) = \phi(g_1) + g_1\phi(g_2) \), it follows that an element of \( H^1(G, A) \) is determined by a map on a set of generators of \( G \). Since \( A[\ell^n] \cong (\mathbb{Z}/\ell^n)^2 \) as abelian groups, it follows that \( H^1(G_{\ell^n}, A[\ell^n]) \) is a finitely generated abelian group which is generated by a set of elements of cardinality independent of \( m \).

It follows from a theorem of Bogomolov, cf. [B], that the \( \ell \)-adic Lie group \( G_{\ell^\infty} \) contains homotheties congruent to 1 modulo \( \ell^N \) for some integer \( N > 0 \). Therefore by Lemma 1, \( H^1(G_{\ell^m}, A[\ell^n]) \) is annihilated by \( \ell^N \). It follows that \( H^1(G_{\ell^m}, A[\ell^n]) \) is a finitely generated abelian group which is generated by a set of elements of cardinality independent of \( m, n \), and annihilated by \( \ell^N \), and thus is of finite order, bounded independent of \( m, n \).
The statement about $H^1(G_{\ell\infty}, A[\ell^n])$ follows either by noting that the cohomology $H^1(G_{\ell\infty}, A[\ell^n])$ can be calculated in terms of continuous cochains on $G_{\ell\infty}$, for which the earlier argument applies as well, or by noting that $H^1(G_{\ell\infty}, A[\ell^n])$ is the direct limit of $H^1(G_{\ell^m}, A[\ell^n])$ (direct limit over $m$), and a direct limit of finitely generated abelian groups each of which is generated by a set of elements of cardinality independent of $n$, and each annihilated by $\ell^N$, is of order bounded independent of $n$.

**Lemma 4** Given an abelian variety $A$ over $K$, a point $P$ of $A(K)$ of infinite order, and any prime $\ell$, there are infinitely many places $v$ of $K$ (in fact of positive density) such that the reduction of $P \mod v$ has order divisible by $\ell$.

**Proof:** We claim that there is a sufficiently large $n$ such that the extension $K_{P,\ell^n} = K(A[\ell^n], \frac{1}{\ell^n}.P)$ is a non-trivial extension of $K_{\ell^n} = K(A[\ell^n])$. If we grant the claim then there is a positive density of $v$ that split in $K(A[\ell^n])$ (we denote the Galois group $\text{Gal}(K_{\ell^n}/K)$ by $G_{\ell^n}$) but not in $K(A[\ell^n], \frac{1}{\ell^n}.P)$, and by inspection we see that for such $v$‘s the reduction of $P \mod v$ has order divisible by $\ell$.

From Lemma 2, it follows that $H^1(G_{\ell^n}, A[\ell^n])$ is bounded independently of $n$. Using maps between the Kummer sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & A(K)/\ell^n A(K) & \rightarrow & H^1(G_K, A[\ell^n]) & \rightarrow & H^1(G_K, A)[\ell^n] & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A(K_{\ell^n})/\ell^n A(K_{\ell^n}) & \rightarrow & H^1(G_{K_{\ell^n}}, A[\ell^n]) & \rightarrow & H^1(G_{K_{\ell^n}}, A)[\ell^n] & \rightarrow & 0
\end{array}
\]

the claim follows for large enough $n$, putting together the observation that the kernel of the restriction map $H^1(G_K, A[\ell^n]) \rightarrow H^1(G_{K_{\ell^n}}, A[\ell^n])$ is $H^1(G_{\ell^n}, A[\ell^n])$, and the fact that as $P$ is non-torsion the image of $P$ under the coboundary map (in the first exact sequence) in $H^1(G_K, A[\ell^n])$ has unbounded order as $n$ varies. From this the claim follows and hence the lemma.

We next have the following lemma.

**Lemma 5** Given an abelian variety $A$ over a number field $K$ satisfying the hypotheses of Theorem 2, a point $P$ of $A(K)$ of infinite order, and any prime $\ell$, there are infinitely many places $v$ of $K$ (in fact of positive density) such that the reduction of $P \mod v$ has order prime to $\ell$. 

9
Proof: From the proof of Lemma 3, it follows that $K_{P,\ell\infty}$ is an infinite extension of $K_{\ell\infty}$. Under the hypothesis of this lemma, $G_{\ell\infty}$ is an open subgroup of finite index of $GSp_{2g}(\mathbb{Z}_{\ell})$ by a theorem of Serre [S-IV]. (If the abelian variety is not principally polarised, the image of the Galois group does not necessarily sit inside a symplectic similitude group, and hence a slight modification to the argument given below will have to be made which we leave to the reader.) If we denote by $E_{\ell\infty}$ the Galois group of $K_{P,\ell\infty}$ over $K$, and $A_{\ell\infty}$, the Galois group of $K_{P,\ell\infty}$ over $K_{\ell\infty}$, we have an exact sequence of groups,

$$0 \rightarrow A_{\ell\infty} \rightarrow E_{\ell\infty} \rightarrow G_{\ell\infty} \rightarrow 1.$$ 

This exact sequence is a subsequence of the following (split) exact sequence of $\mathbb{Z}_{\ell}$-rational points of algebraic groups:

$$0 \rightarrow \mathbb{Z}_{\ell}^{2g} \rightarrow E(\mathbb{Z}_{\ell}) \rightarrow GSp(2g, \mathbb{Z}_{\ell}) \rightarrow 1,$$

(i.e., $A_{\ell\infty} \subset \mathbb{Z}_{\ell}^{2g}$, $E_{\ell\infty} \subset E(\mathbb{Z}_{\ell}), G_{\ell\infty} \subset GSp(2g, \mathbb{Z}_{\ell})$), in which $E(\mathbb{Z}_{\ell})$ is the semi-direct product $GSp(2g, \mathbb{Z}_{\ell})$ with $\mathbb{Z}_{\ell}^{2g}$. The embedding of $E_{\ell\infty}$ inside $E(\mathbb{Z}_{\ell})$ is obtained by choosing a sequence of points $P_n$ in $A$ with $\ell \cdot P_1 = P$, and $\ell \cdot P_{i+1} = P_i$. Since the action of $GSp(2g, \mathbb{Z}_{\ell})$ on $\mathbb{Z}_{\ell}^{2g}$ is irreducible, and $G_{\ell\infty}$ is an open subgroup of $GSp(2g, \mathbb{Z}_{\ell})$, it follows that $A_{\ell\infty}$ is an open subgroup of $\mathbb{Z}_{\ell}^{2g}$, and hence $E_{\ell\infty}$ is an open subgroup of the semi-direct product $GSp(2g, \mathbb{Z}_{\ell})$ with $\mathbb{Z}_{\ell}^{2g}$.

We will prove that the intersection of the fields $K_{P,\ell^n}$ and $K_{\ell^{n+1}}$ is $K_{\ell^n}$ for some $n$ large enough. This, together with the theorem of Bogomolov recalled earlier, will imply that there is a positive density of primes in $K$ which are split in $K_{P,\ell^n}$ and for which the Frobenius as an element of $Gal(K_{\ell^{n+1}}/K)$ is a non-trivial homothety in $GSp(2g, \mathbb{Z}/\ell^{n+1})$ which is congruent to 1 modulo $\ell^n$. For such primes $v$, it can be easily seen that the order of $P$ modulo $v$ is not divisible by $\ell$, completing the proof of the lemma.

It thus suffices to prove that the intersection of the fields $K_{P,\ell^n}$ and $K_{\ell^{n+1}}$ is $K_{\ell^n}$.

Let $E_{\ell^n}$ (resp. $G_{\ell^n}$) denote the Galois group of $K_{P,\ell^n}$ (resp. $K_{\ell^n}$) over $K$, and let $A_{\ell^n}$ denote the Galois group of $K_{P,\ell^n}$ over $K_{\ell^n}$. We have the exact sequence of groups,

$$0 \rightarrow A_{\ell^n} \rightarrow E_{\ell^n} \rightarrow G_{\ell^n} \rightarrow 1.$$ 

It is clear that the intersection of the fields $K_{P,\ell^n}$ and $K_{\ell^{n+1}}$ is $K_{\ell^n}$ if and only if inside the group $E_{\ell^n}$ which is a quotient of $E_{\ell^{n+1}}$, the image of $A_{\ell^{n+1}}$ is $A_{\ell^n}$.
Since $E_{\ell^\infty}$ is an open subgroup of the semi-direct product $GSp(2g, \mathbb{Z}_\ell)$ with $\mathbb{Z}_\ell^{2g}$, it follows that $E_{\ell^\infty}$ contains the natural congruence subgroup of level $\ell^n$ in this semi-direct product for some $n$, say $n = n_0$. From this it is clear that $E_{\ell^{n+1}}$ is the full inverse image of $E_{\ell^n}$ under the natural mapping from $E(\mathbb{Z}/\ell^{n+1})$ to $E(\mathbb{Z}/\ell^n)$, $n \geq n_0$, from which the surjectivity of the mapping from $A_{\ell^{n+1}}$ onto $A_{\ell^n}$ clearly follows, completing the proof of the lemma.

**Remark:** The above proof can be generalised to yield that for any abelian variety $A$ defined over $K$ and any point $P$ of $A(K)$ which does not project to a non-zero torsion point in any (geometric) subquotient of $A$, given a prime $\ell$ there are a positive density of places $v$ of $K$ such that $P \mod v$ has order prime to $\ell$.

**Remark:** Although we have given separate proofs of Lemmas 4 and 5, observe that Lemma 5 implies Lemma 4. This follows by applying Lemma 5 to the point of infinite order $P + R$ where $R$ is a nonzero $\ell$-torsion point. (We note that to prove lemma 4, we are allowed to go to a finite extension of $K$, and hence assume that $A$ has nonzero $\ell$-torsion point over $K$.) Lemma 5 implies that there are infinitely many places $v$ of $K$ for which the order of $P + R$ is coprime to $\ell$. It is easy to see that for such places, the order of $P$ must be divisible by $\ell$.

**Corollary 1** Under the hypothesis and notation of theorem 2, the image of any non-torsion point $P$ in $A_1$ is non-torsion in $A_2$.

**Proof:** Immediate from Lemma 5.

Lemma 5 allows us to strengthen theorem 1.

**Proposition 1** Let $A_1$ and $A_2$ be two simple abelian varieties over a number field $K$ with the endomorphism rings over the algebraic closure of $K$ as $\mathbb{Z}$. Assume that $\dim(A_1) = g_1$ is either odd or $g_1 = 2$ or 6. (We do not assume that $\dim A_1 = \dim A_2$.) Let $P$ (resp. $Q$) be a non-torsion point in $A_1(K)$ (resp. $A_2(K)$) such that for almost all places $v$ of $K$, the order of $Q \mod v$ divides the order of $P \mod v$. Then $A_1$ and $A_2$ are isogenous, and there is an isogeny $j_0$ from $A_1$ to $A_2$ such that $j_0(P) = Q$. Furthermore, if one of the abelian varieties is an elliptic curve, we have the same conclusion without any restriction on its endomorphism ring.
Proof: From theorem 1, there is an isogeny $j$ from $A_1$ to $A_2$ such that $j(P) = nQ$ for some integer $n$. We wish to prove that $n$ can be chosen to be 1. Let $\ell'$ be the highest power of a prime $\ell$ which divides $n$. We will prove that $\ell'$ torsion points of $A_1$ are contained in the kernel of $j$. If that were not the case, there would be an $\ell'$ torsion point of $A_1$, say $R$, which does not belong to the kernel of $j$. By lemma 5, there are infinitely many places $v$ of $K$ such that the order of $P + R$ is coprime to $\ell$ in the residue field $k_v$ of $K$, and hence the $\ell$-primary component of the orders of $P$ and $R$ are the same, of order $\ell'$. For such places $v$, the order of $j(P + R) = j(P) + j(R) = nQ + j(R)$ is also coprime to $\ell$. By choice, $j(R)$ is a nonzero torsion point on $A_2$ of order dividing $\ell'$, say $\ell^s$, $0 < s \leq r$. Thus since the order of $nQ + j(R)$ is coprime to $\ell$, the $\ell$-primary components of the order of $j(R)$ and $nQ$ are the same, hence the $\ell$-primary component of the order of $nQ$ is $\ell^s$, and therefore of $Q$, $\ell^{r+s}$, contradicting our hypothesis that order of $Q$ divides the order $P$ at each place. This proves that $\ell'$ torsion points of $A_1$ are contained in the kernel of $j$ where $\ell'$ is the highest power of $\ell$ dividing $n$. Thus all the $n$-torsion points of $A_1$ is contained in the kernel of $j$. Therefore the isogeny $j$ can be written as $n_0$ for an isogeny $j_0$ from $A_1$ to $A_2$. Thus we have $n(j_0(P) - Q) = 0$. This implies that $j_0(P) = Q + S$ for a certain torsion point $S$ on $A_2$.

If $S$ is nonzero, let the order of $S$ be divisible by a prime $\ell$. Choose a place $v$ of $K$ where the order of $P$, and hence of $Q$ and $j_0(P)$ are coprime to $\ell$. However, since $j_0(P) = Q + S$, its order is divisible by $\ell$, a contradiction to $S$ being nonzero.

End of the proof of part 2 of Theorem 2. Choose a torsion-free subgroup $B$ of $A_1(K)$ such that $A_1(K) = A_1(K)_{\text{tors}} \oplus B$. Then for each $P \in B$ we see from Corollary 4 that $\phi(P)$ is not torsion, and by Proposition 1 that $\phi(P) = j_0(P)$ for some isogeny $j_0$ that might a priori depend on $P$. Considering $P + Q$ for $P, Q \in B$ it follows that $j_0$ is independent of $P$. Now considering $\phi(P + P') = j_0(P + P')$ and $\phi(P) = j_0(P)$ where $P'$ is in $A(K)_{\text{tors}}$ and $P \in B$, we conclude that $\phi(P') - j_0'(P') = (j_0 - j_0')(P)$. The left hand side of this equation is a point of finite order, whereas right hand is of infinite order unless $j_0 = j_0'$, forcing $j_0 = j_0'$, and thus we are done with the proof of Theorem 2.
6 Rigidity for arithmetic groups

We begin with the following theorem for tori.

Proposition 2 Given homomorphism $\phi: \mathcal{O}_K^* \to \mathcal{O}_K^*$ that reduces mod $v$ for almost all places $v$ of a number field $K$, then $\phi$ is induced by the $m$th power map for some integer $m$.

Proof: The proof is a direct consequence of Theorem 1 of [RS] and the fact that any finite subgroup of $K^*$ is cyclic.

We next have the following theorem for arithmetic groups.

Theorem 5 Let $\Gamma$ be a subgroup of $SL(2,\mathbb{Z})$ of finite index. Let $\phi$ be a non-trivial homomorphism of $\Gamma$ into itself. Assume that for all primes $p$ in an infinite set $S$ of primes, $\phi$ factors to give a homomorphism $\phi_p : SL(2,\mathbb{Z}/p) \to SL(2,\mathbb{Z}/p)$

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
SL(2,\mathbb{Z}/p) & \longrightarrow & SL(2,\mathbb{Z}/p).
\end{array}
$$

Then $\phi$ is an automorphism which is the restriction to $\Gamma$ of the inner-conjugation action of an element in $GL(2,\mathbb{Q})$.

Proof: Let $A$ be the ring which is the direct product of $\mathbb{Z}/p$ for all $p$ in $S$. Clearly $\mathbb{Z}$ is a subring of $A$, and there is thus an injective homomorphism from $SL(2,\mathbb{Z})$ to $SL(2, A)$. Since there is an injective homomorphism from $SL(2,\mathbb{Z})$ to $SL(2, A)$ for $A$, the direct product of any infinite set of primes, it is clear that $\phi_p$ can be trivial for at most finitely many $p$ in $S$. After replacing $S$ by this slightly smaller set, we assume that $\phi_p$ is surjective for all $p$ in $S$, and hence the $\phi_p$ are given by the inner-conjugation action of an element $g_p$ in $GL(2,\mathbb{Z}/p)$. Here we are using the well-known facts:

1. any surjective homomorphism of $SL(2,\mathbb{Z}/p)$ into itself is given by the inner-conjugation of an element of $GL(2,\mathbb{Z}/p)$.

2. any homomorphism of $SL(2,\mathbb{Z}/p)$ into itself is either trivial or is surjective if $p > 3$. 

13
From this we see that the representations \( \phi : \Gamma \to GL_2(\mathbb{Q}) \) and the “identity” representation \( id : \Gamma \to GL_2(\mathbb{Q}) \) have the same trace. Further the second representation is irreducible. From this we conclude that \( \phi \) and \( id \) are conjugate by an element of \( GL_2(\mathbb{Q}) \) and further as \( \phi \) goes into \( \Gamma \) by comparing covolumes we see that \( \phi \) is an automorphism of \( \Gamma \).

**Remarks:**

1. The above proof is due to Serre: we had a different proof in an earlier version.

2. To deduce rigidity results for \( SL_2 \) one just needs that the abstract homomorphism specialises for any infinite set of primes rather than for almost all or even a positive density of primes which is crucial for abelian varieties. Further unlike abelian varieties the analog of Question 1 for \( SL_2 \) dealt with here is not a “pointwise question” (see remark following the question).

3. The proof works for \( SL(n, \mathbb{Z}) \) for any \( n \) to say that if \( \phi \) is a homomorphism of a subgroup of finite index of \( SL(n, \mathbb{Z}) \) onto another subgroup of finite index \( SL(n, \mathbb{Z}) \) which specialises for infinitely many primes \( p \) to give a homomorphism of \( SL(n, \mathbb{Z}/p) \) to itself (we recall that an automorphism of \( SL(n, \mathbb{Z}/p) \) is generated by inner automorphism from \( GL(n, \mathbb{Z}/p) \), and the automorphism \( A \to tA^{-1} \)), then \( \phi \) is algebraic.

4. Because of the strong rigidity theorem, the algebraicity of abstract homomorphisms of arithmetic lattices in semi-simple Lie groups is of interest only for arithmetic lattices in \( SL(2, \mathbb{R}) \) which are constructed using division algebras over totally real number fields. Our method clearly applies for such lattices too.

5. The following theorem when combined with the previous one, proves that for a subgroup \( \Gamma \) of finite index in \( SL(2, \mathbb{Z}) \), any homomorphism of \( \Gamma \) into itself which extends to \( SL(2, A) \) for \( A = \prod_{p \in T} \mathbb{Z}/p \), \( T \) an infinite set of primes, must be given by an inner-conjugation action by an element of \( GL(2, \mathbb{Q}) \), giving a different perspective to the earlier theorem.

**Theorem 6** Let \( G \) be any simply-connected, split, semisimple algebraic group over \( \mathbb{Q} \). Then any homomorphism of \( G(A) \) to itself where \( A \) is the product of \( \mathbb{Z}/p \) for primes \( p \) belonging to a set \( T \) that may be finite or infinite (and contains only sufficiently large primes) is factorisable, i.e., any homomorphism \( f \) from \( G(A) \) to itself is of the form \( \prod_{p \in A} f_p \) for certain homomorphisms \( f_p \) from \( G(\mathbb{Z}/p) \) to itself.

**Proof:** We will accomplish the proof of this theorem in several steps.
1. Any homomorphism from $G(\mathbb{Z}/p)$ to $G(\mathbb{Z}/q)$, $p$ not $q$ is trivial (for $p$ a sufficiently large prime).

Assume that the mapping is non-trivial. Then as $G(\mathbb{Z}/p)$ is a simple group modulo its center, any homomorphism from $G(\mathbb{Z}/p)$ to $G(\mathbb{Z}/q)$ must be injective when restricted to unipotent elements in $G(\mathbb{Z}/p)$. Because $p$ is not $q$, image of a unipotent element in $G(\mathbb{Z}/p)$ cannot have a unipotent component in the Jordan decomposition in $G(\mathbb{Z}/q)$. So image of any unipotent element in $G(\mathbb{Z}/p)$ is semi-simple in $G(\mathbb{Z}/q)$.

Note that a unipotent in $SL_2(\mathbb{Z}/p)$ has many powers that are conjugate to itself. By Jacobson-Morozov (which is applicable since we are looking only at large primes), the same holds good about non-trivial unipotents in $G(\mathbb{Z}/p)$. Hence the image of a non-trivial unipotent in $G(\mathbb{Z}/p)$ too will have many distinct powers that are conjugate to itself. But a semi-simple element in $G(\mathbb{Z}/q)$ has at most $|W|$ many powers that are conjugate to itself, where $|W|$ denotes the order of the Weyl group of $G$, completing the proof of this step.

2. Step 1 proves that any homomorphism from $G(A)$ to itself when restricted to direct sum is factorisable. However going from direct sum to direct product needs some more arguments and essentially the following step suffices.

3. Any homomorphism from $G(A^S)$ to $G(\mathbb{Z}/p)$ must be trivial for some finite set $S$ (depending on $p$) of primes in $A$ with $A^S = A - S$ where $S$ is the set of prime divisors in $A$ of the cardinality of $G(\mathbb{Z}/p)$, which we denote by $d$.

We first prove that a unipotent in $G(A^S)$ must go trivially to $G(\mathbb{Z}/p)$. Again by Jacobson-Morozov (applied to the ring $A^S$ which is a product of fields), this would follow if we can prove that under any homomorphism from $SL_2(A^S)$ to $G(\mathbb{Z}/p)$, any unipotent in $SL_2(A^S)$ must go trivially in $G(\mathbb{Z}/p)$. But this follows because multiplication by $d$ is an isomorphism on $A^S$ whereas it is trivial on $G(\mathbb{Z}/p)$.

We will be done if we can prove that unipotents in $G(A)$ for any set of primes $A$ generates $G(A)$. (This step is not true for an arbitrary ring $A$, but is true here as $A$ is a product of fields.)

4. Let $B$ be any Borel subgroup in $G$ defined over $\mathbb{Q}$. We will prove that any element of $B(A)$ belongs to the group generated by the unipotents.
We need to prove that for a torus $T$ contained in $B$, and defined over $\mathbb{Q}$, elements of $T(A)$ belong to the subgroup generated by the unipotents in $G(A)$.

For $SL_2(A)$, this follows from the matrix identity which is a product of 4 unipotent matrices taken from Deligne’s article in Modular Forms vol 2, Springer Lecture Notes in Mathematics, vol. 349.

\[
\begin{pmatrix}
  a^{-1} & 0 \\
  0 & a
\end{pmatrix} = \begin{pmatrix}
  1 & -a^{-1} \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  a-1 & 1
\end{pmatrix} \begin{pmatrix}
  1 & 1 \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  -(a-1)/a & 1
\end{pmatrix}
\]

As $G$ is simply-connected, any element of $T(A)$ is a product of elements of 1 dimensional tori in $T(A)$ arising out of the image of the diagonal torus in $SL_2$ under mappings of $SL_2$ to $G$ corresponding to the simple roots.

5. Unipotents in $G(A)$ for any set of primes $A$ generates $G(A)$.

This follows from step 4 combined with Bruhat decomposition (applied component-wise to write $g = (g_p)$ as $(u_p w_p b_p))$ and the fact that any element of the Weyl group is generated by the unipotent elements.

6. It follows from step 1 and 3 that any homomorphism from $G(A)$ to $G(\mathbb{Z}/p)$ is trivial on $G(A-p)$, hence we have proved that any homomorphism from $G(A)$ to itself is factorisable.

7  Rational Varieties

After having considered the case of Abelian varieties and semi-simple algebraic groups, it is tempting to consider arbitrary maps of algebraic varieties over number fields which reduce nicely under reduction modulo all finite primes, i.e. such that the following diagram

\[
\begin{array}{ccc}
V(K) & \xrightarrow{\phi} & V(K) \\
\downarrow_{sp_v} & & \downarrow_{sp_v} \\
V(k_v) & \xrightarrow{\phi_v} & V(k_v)
\end{array}
\]
commutes where $k_v$ is a residue field of the ring of integers of $K$, and to
ask whether such maps come from an algebraic one on $V$. The question
will naturally be more meaningful if $V$ is assured of many rational points,
if for instance, $V$ is a smooth projective rational variety. It seems specially
interesting to investigate it for flag variety $G/P$ for a parabolic $P$ in a semi-
simple split group $G$ over $K$. Here, we merely point out that the analogous
question for the affine line over $\mathbb{Z}$ is false, i.e. there exists a set-theoretic map
from $\mathbb{Z}$ to $\mathbb{Z}$ which is not polynomial but which makes the following diagram
commute.

\[
\begin{array}{c}
A^1(\mathbb{Z}) \xrightarrow{\phi} A^1(\mathbb{Z}) \\
\downarrow sp_p \downarrow sp_p \\
A^1(\mathbb{Z}/p) \xrightarrow{\phi_p} A^1(\mathbb{Z}/p).
\end{array}
\]

This is constructed for instance using

\[
\psi(n) = a_0 + a_1 n + a_2 n(n - 1) + a_3 n(n - 1)(n - 2) + \cdots,
\]
an infinite sum, which reduces to a finite sum for each $n$, where $a_i$ are integral,
and $n \geq 0$, and defining $\phi(n) = \psi(n^2)$. We find that since $\phi(n)$ is congruent
to $\phi(m)$ modulo any integer $N$ for which $m$ is congruent to $n$ modulo $N$, the
diagram above commutes for $\phi$.

Acknowledgement: We thank J.-P.Serre for helpful correspondence, and
K. Ribet for pointing out the preprint [BGK] which has some overlap with
the present work. We note, however, that the theorems of [BGK] are all
proved under the assumption that $A_1 = A_2$, and that our finer theorem 2.2
is not considered there.

8 References

[BGK] Banaszak, G., Gajda, W., Krasoń, P., A support problem for the
intermediate Jacobians of $\ell$-adic representations, preprint.

[CS] Corrales, C., and Schoof, R., The support problem and its elliptic ana-
logue, J. of Number Theory 64 (1997), 276–290.

[B] Bogomolov, F., Sur l’algébricité des représentations $l$-adiques, C. R. Acad.
Sci. Paris Sér. A-B 290 (1980), no. 15, 701–703.
[JR] Jacquinot, O., Ribet, K., *Deficient points on extensions of abelian varieties by* $\mathbb{G}_m$, Journal of Number Theory 25 (1987), 133–151.

[S] Serre, J-P., *Propriétés galoisiennes des points d’ordre finides courbes elliptiques*, Invent. Math. 15 (1972), 259–331.

[S-IV] Serre, J-P., *Oeuvres*, Vol. IV, articles 133–138, Springer-Verlag, 2000.

*Addresses of the authors:*

CK: Dept. of Math, University of Utah, 155 S 1400 E, Salt Lake City, UT 84112, USA: shekhar@math.utah.edu

School of Mathematics, Tata Institute of Fundamental Research, Colaba, Bombay-400005, INDIA. e-mail: shekhar@math.tifr.res.in

DP: Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad-211019, INDIA. e-mail: dprasad@mri.ernet.in