Abstract. We introduce three kinds of invariants of a virtual knot called the first, second, and third intersection polynomials. The definition is based on the intersection number of a pair of curves on a closed surface. The calculations of intersection polynomials are given up to crossing number four. We also study several properties of intersection polynomials.

1. Introduction

In classical knot theory, we study a circle embedded in a 3-dimensional sphere $S^3$ under an ambient isotopy. It is enough to consider the product $S^2 \times I$ of a 2-sphere $S^2$ and an interval $I$ instead of $S^3$. In this sense, it is natural to study a circle in the product $\Sigma_g \times I$ of a closed, connected, oriented surface $\Sigma_g$ of genus $g$ and $I$. Kauffman [9] leads the unification of such knot theories for all genera and introduces virtual knot theory. A virtual knot is described by a diagram on $\Sigma_g$ for some $g \geq 0$ under the projection $\Sigma_g \times I$ onto $\Sigma_g$. We are allowed to use three kinds of Reidemeister moves for diagrams and (de)stabilizations for surfaces. See Figure 1.

Some invariants of a virtual knot are natural generalizations of those of a classical knot such as knot groups and Jones polynomials [9], and some vanish for classical knots such as Sawollek polynomials [12] and writhe polynomials [11, 13, 10, 14]. In this paper, we will introduce three kinds of new invariants of the latter type.

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This paper is organized as follows. In Section 2, we define three kinds of Laurent polynomials $I_K(t)$, $II_K(t)$, and $III_K(t)$ of a virtual knot $K$ and prove the following.

**Theorem 1.1.** $I_K(t)$, $II_K(t)$, and $III_K(t)$ are invariants of $K$.

These invariants are called the first, second, and third intersection polynomials of $K$, respectively. We remark that they are trivial for classical knots (Lemma 2.11).

In Section 3, we explain how to calculate the intersection polynomials. We give the intersection polynomials for all the virtual knots up to crossing number four in Appendices A and B (Theorem 3.5). By observing these calculations, we have the following.

**Theorem 1.2.** $I_K(t)$, $II_K(t)$, and $III_K(t)$ are independent of each other.

Sections 4 and 5 are devoted to giving several applications of the intersection polynomials. In Section 4, we study the behaviors of intersection polynomials on symmetry of a virtual knot. Let $-K$, $K^\#$, and $K^*$ be the reverse, the vertical mirror image, and the horizontal mirror image of $K$, respectively. There are known several examples of virtual knots $K$ such that the eight knots derived from $K$ are mutually distinct (cf. [11]). In this paper, we prove the following.

**Theorem 1.3.** There are infinitely many virtual knot $K$ such that $K, -K, K^\#, K^*, -K^\#, -K^*, K^#*$, and $-K^#*$ are mutually distinct.

In Section 5, we give lower bounds of the crossing number $c(K)$ and the virtual crossing number $vc(K)$ of a virtual knot $K$ by intersection polynomials.

**Theorem 1.4.** For any virtual knot $K$, we have

(i) $c(K) \geq \deg X_K(t) + 1$ ($X = I, II, III$) and

(ii) $vc(K) \geq \deg X_K(t)$ ($X = I, II, III$).

In Appendix A, we give the table of $I_K(t)$ and $II_K(t)$ with $c(K) \leq 4$. In Appendix B, we give the table of $III_K(t)$ with $c(K) \leq 4$.

In forthcoming papers, we will study the behavior of intersection polynomials under a connected sum [5], a characterization of intersection polynomials [6], and a relationship with crossing changes [7].

2. Definitions

Let $\Sigma_g$ be a closed, connected, oriented surface of genus $g$, and $\alpha$ and $\beta$ closed, oriented curves on $\Sigma_g$. We often regard these curves as homology cycles of $H_1(\Sigma_g)$. The intersection number $\alpha \cdot \beta \in \mathbb{Z}$ is defined to be the homology intersection of the ordered pair $(\alpha, \beta)$. Geometrically it is calculated as follows. By perturbing $\alpha$ and $\beta$ if necessary, we may assume that $\alpha \cap \beta$ consists of $m$ transverse double points $p_1, \ldots, p_m$. At a double point $p_k$ ($1 \leq k \leq m$), if $\beta$ intersects $\alpha$ from the left or right as we walk along $\alpha$, we define $e_k = +1$ or $-1$, respectively. Then we have $\alpha \cdot \beta = \sum_{k=1}^{m} e_k$. See Figure 2. We remark that $\alpha \cdot \beta = -\beta \cdot \alpha$ and $\alpha \cdot \alpha = 0$ by definition.

We consider a circle embedded in $\Sigma_g \times [0, 1]$ for some $g \geq 0$. We identify two embedded circles up to ambient isotopies and (de)stabilizations. Such an equivalence class is called a virtual knot (cf. [2] [8] [9] [12]).
More precisely, a virtual knot is described by a diagram on $\Sigma_g$ which is a projection image under the projection $\Sigma_g \times [0, 1] \to \Sigma_g$ equipped with over/under-information at double points. A double point with over/under information is called a crossing. Two diagrams $(\Sigma_g, D)$ and $(\Sigma_g', D')$ present the same virtual knot if and only if there is a finite sequence of diagrams

$$(\Sigma_g, D) = (\Sigma_{g_1}, D_1), (\Sigma_{g_2}, D_2), \ldots, (\Sigma_{g_s}, D_s) = (\Sigma_{g'}, D')$$

such that for each $1 \leq i \leq s - 1$,

(i) $g_{i+1} = g_i$ holds and $D_{i+1}$ is obtained from $D_i$ by an orientation-preserving homeomorphism of $\Sigma_{g_i} = \Sigma_{g_{i+1}}$,

(ii) $g_{i+1} = g_i \pm 1$ holds and $\Sigma_{g_{i+1}}$ is obtained from $\Sigma_{g_i}$ by 1- or 2-handle surgery missing $D_i = D_{i+1}$. Such a deformation is called a stabilization or destabilization, respectively.

Throughout this paper, we assume that all virtual knots are oriented.

Let $D = (\Sigma, D)$ be a diagram of a virtual knot $K$, and $c_1, \ldots, c_n$ the crossings of $D$. We denote by $\gamma_D$ the closed, oriented curve on $\Sigma$ obtained from $D$ by ignoring over/under-information at $c_i$’s. Furthermore, we denote by $\gamma_i$ ($1 \leq i \leq n$) the closed, oriented curve as a part of $\gamma_D$ from the overcrossing to the undercrossing at $c_i$, and by $\overline{\gamma}_i$ the curve from the undercrossing to the overcrossing at $c_i$. See Figure 3. These curves satisfy $\gamma_i + \overline{\gamma}_i = \gamma_D$ and $\gamma_i \cdot \overline{\gamma}_i = \gamma_i \cdot (\gamma_D - \gamma_i) = \gamma_i \cdot \gamma_D$ as homology cycles on $\Sigma$. We call $\gamma_i$ and $\overline{\gamma}_i$ the cycles at $c_i$ on $\Sigma$.

**Definition 2.1** ([11][12][13][14]). The Laurent polynomial

$$W_D(t) = \sum_{i=1}^n \varepsilon_i (t^{\gamma_i} - 1) = \sum_{i=1}^n \varepsilon_i (t^{\gamma_i \cdot \gamma_D} - 1) \in \mathbb{Z}[t, t^{-1}]$$

is an invariant of $K$, where $\varepsilon_i$ is the sign of $c_i$. It is the writhe polynomial of $K$ and denoted by $W_K(t)$. The exponent $\gamma_i \cdot \overline{\gamma}_i$ is called the index of a crossing $c_i$.

Let $D^\#$ be the diagram obtained from $D$ by changing over/under-information at every crossing of $D$.

**Lemma 2.2** ([14]). For any diagram $D$ on $\Sigma$, we have $W_{D^\#}(t) = -W_D(t^{-1})$. 
Proof. Let $c_i^#$, $\ldots$, $c_n^#$ be the crossings of $D^#$ such that $c_i^#$ corresponds to $c_i$, $\varepsilon_i^#$ the sign of $c_i^#$, and $\gamma_i^#$ the cycle at $c_i^#$ on $\Sigma$ ($1 \leq i \leq n$). Then we have
\[
\varepsilon_i^# = -\varepsilon_i, \ \gamma_i^# = \overline{\gamma_i}, \ \text{and} \ \overline{\gamma_i}^# = \gamma_i.
\]
Therefore it holds that
\[
W_{D^#}(t) = \sum_{i=1}^{n} \varepsilon_i^#(t^{\gamma_i^#} - 1) = -\sum_{i=1}^{n} \varepsilon_i(t^{\overline{\gamma_i}} - 1) = -W_D(t^{-1}).
\]

Now we consider four kinds of Laurent polynomials as follows;
\[
\begin{align*}
\mathcal{f}_{01}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i^# \cdot \gamma_j^#} - 1), \\
\mathcal{f}_{10}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i} - 1), \\
\mathcal{f}_{00}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\gamma_i^#} - 1), \\
\mathcal{f}_{11}(D; t) &= \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (t^{\overline{\gamma_i}} - 1).
\end{align*}
\]

Lemma 2.3. For any diagram $D$ on $\Sigma$, we have the following.
(i) $\mathcal{f}_{10}(D; t) = \mathcal{f}_{01}(D^#; t)$.
(ii) $\mathcal{f}_{11}(D; t) = \mathcal{f}_{00}(D^#; t)$.

Proof. Let $c_i^#$, $\ldots$, $c_n^#$ be the crossings of $D^#$ such that $c_i^#$ corresponds to $c_i$, $\varepsilon_i^#$ the sign of $c_i^#$, and $\gamma_i^#$ the cycle at $c_i^#$ on $\Sigma$ ($1 \leq i \leq n$). Then we have
\[
\varepsilon_i^# = -\varepsilon_i, \ \gamma_i^# = \overline{\gamma_i}, \ \text{and} \ \overline{\gamma_i}^# = \gamma_i.
\]
Since it holds that
\[
\varepsilon_i \varepsilon_j = \varepsilon_i^# \varepsilon_j^#; \ \overline{\gamma_i} \cdot \gamma_j = \gamma_i^# \cdot \overline{\gamma_j}^#; \ \text{and} \ \overline{\gamma_i}^# \cdot \overline{\gamma_j}^# = \gamma_i^# \cdot \gamma_j^#,
\]
we have the conclusion by definition. \hfill $\Box$

In what follows, we often abbreviate $f_{pq}(D; t)$ to $f_{pq}(D)$ for $p, q \in \{0, 1\}$.

Lemma 2.4. If a diagram $D'$ is obtained from $D$ by a second or third Reidemeister move on $\Sigma$, then we have $f_{pq}(D) = f_{pq}(D')$ for any $p, q \in \{0, 1\}$.

Proof. Since $D'^#$ is obtained from $D^#$ by a second or third Reidemeister move on $\Sigma$, it is sufficient to prove the invariance of $\mathcal{f}_{01}(D)$ and $\mathcal{f}_{00}(D)$ by Lemma 2.3
A second Reidemeister move. Assume that $D'$ is obtained from $D$ by a second Reidemeister move removing a pair of crossings $c_1$ and $c_2$ of $D$. For $3 \leq i \leq n$, let $c_i'$ be the crossing of $D'$ corresponding to $c_i$, $\varepsilon_i'$ the sign of $c_i'$, and $\gamma_i'$ the cycle at $c_i'$ on $\Sigma$. Then it holds that
\[
\varepsilon_1 = -\varepsilon_2, \ \gamma_1 = \gamma_2, \ \text{and} \ \varepsilon_i' = \varepsilon_i, \ \gamma_i' = \gamma_i \ (3 \leq i \leq n).
\]
See Figure 4. A disk on $\Sigma$ is bounded by two arcs on $D$ connecting $c_1$ and $c_2$. Since $\gamma_{D'} = \gamma_D$ and $\gamma'_i = \gamma_i (3 \leq i \leq n)$, we have
\[
\begin{align*}
\epsilon_{01}(D) - \epsilon_{01}(D') &= \sum_{j=3}^{n} \epsilon_1 \epsilon_j (t^{\gamma_j \gamma_j} - 1 - t^{\gamma_j \gamma_j} - 1) \\
&+ \sum_{i=3}^{n} \epsilon_i \epsilon_1 (t^{\gamma_i \gamma_1} - 1) - \sum_{j=3}^{n} \epsilon_2 \epsilon_j (t^{\gamma_j \gamma_j} - 1) \\
&+ \sum_{i=3}^{n} \epsilon_i \epsilon_1 (t^{\gamma_i \gamma_1} - 1) + \sum_{i=3}^{n} \epsilon_i \epsilon_2 (t^{\gamma_i \gamma_2} - 1) \\
&= (t^{\gamma_1 \gamma_1} - 1 - t^{\gamma_1 \gamma_1} - 1) - (t^{\gamma_1 \gamma_1} - 1) + (t^{\gamma_1 \gamma_1} - 1) = 0.
\end{align*}
\]
Therefore $\epsilon_{01}(D)$ is invariant under a second Reidemeister move. On the other hand, the invariance of $\epsilon_{00}(D)$ is proved by
\[
\epsilon_{00}(D) - \epsilon_{00}(D') = (t^{\gamma_1 \gamma_1} - 1 - t^{\gamma_1 \gamma_1} - 1) + (t^{\gamma_1 \gamma_1} - 1) = 0.
\]

A third Reidemeister move. Assume that $D'$ is obtained from $D$ by a third Reidemeister move involving three crossings $c_1, c_2, c_3$ of $D$. For $1 \leq i \leq n$, let $c'_i$ be the crossing corresponding to $c_i$, $\epsilon'_i$ the sign of $c'_i$, and $\gamma'_i$ the cycle at $c'_i$ on $\Sigma$. Then it holds that
\[
\epsilon'_i = \epsilon_i \quad \text{and} \quad \gamma'_i = \gamma_i \quad (1 \leq i \leq n).
\]
Figure 5 shows $\gamma'_i = \gamma_i$ for $i = 1, 2, 3$. A disk on $\Sigma$ is bounded by three arcs of $D$ connecting $c_1$ and $c_2$, $c_1$ and $c_3$, and $c_2$ and $c_3$. Therefore both $\epsilon_{01}(D)$ and $\epsilon_{00}(D)$ are invariant under a third Reidemeister move. We remark that $\gamma_1 + \gamma_2 = \gamma_3$ holds in this case. \[\square\]
We use the notation $W_K(t) = W_D(t) = W_D(t) + W_D(t^{-1})$. It follows by Lemma 2.2 that

$$W_{D'}(t) = W_D(t) + W_D(t^{-1}) = W_D(t^{-1}) - W_D(t) = -W_D(t).$$

**Lemma 2.5.** If a diagram $D'$ is obtained from $D$ by a first Reidemeister move on $\Sigma$ as shown in Figure 6, then the difference $f_{pq}(D) - f_{pq}(D')$ is given as shown in Table 1.

![Figure 6](image)

|           | (a) $\varepsilon_1 = +1$ | (b) $\varepsilon_1 = -1$ | (c) $\varepsilon_1 = -1$ | (d) $\varepsilon_1 = +1$ |
|-----------|--------------------------|--------------------------|--------------------------|--------------------------|
| $f_{01}(D) - f_{01}(D')$ | $W_K(t)$ | $-W_K(t)$ | $-W_K(t)$ | $W_K(t)$ |
| $f_{10}(D) - f_{10}(D')$ | $W_K(t^{-1})$ | $-W_K(t^{-1})$ | $-W_K(t^{-1})$ | $W_K(t^{-1})$ |
| $f_{00}(D) - f_{00}(D')$ | 0 | 0 | $-W_K(t)$ | $W_K(t)$ |
| $f_{11}(D) - f_{11}(D')$ | $W_K(t)$ | $-W_K(t)$ | 0 | 0 |

**Table 1**

Proof. Assume that $c_1$ is removed from $D$ by a first Reidemeister move. For $2 \leq i \leq n$, let $c'_i$ be the crossing of $D'$ corresponding to $c_i$, $\varepsilon'_i$ the sign of $c'_i$, and $\gamma'_i$ the cycle at $c'_i$ on $\Sigma$. Then it holds that $\gamma'_i = \gamma_i$ and $\varepsilon'_i = \varepsilon_i$ ($2 \leq i \leq n$). By definition, we have $\gamma_1 = 0$ and $\tau_1 = \gamma_D$ for (a) and (b), and $\gamma_1 = \gamma_D$ and $\tau_1 = 0$ for (c) and (d).

$(p, q) = (0, 1)$. It holds that

$$f_{01}(D) - f_{01}(D') = \varepsilon_1^2(t^{\gamma_i \cdot \tau_i} - 1) + \sum_{j=2}^{n} \varepsilon_1 \varepsilon_j (t^{\gamma_j \cdot \tau_j} - 1) + \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_i \cdot \tau_i} - 1).$$

For (a) and (b), we have

$$f_{01}(D) - f_{01}(D') = \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_i \cdot \tau_i} - 1) = \varepsilon_1 \sum_{i=2}^{n} \varepsilon_i (t^{\gamma_i \cdot \tau_i} - 1) = \varepsilon_1 W_K(t).$$

For (c) and (d), by using the equation $\gamma_D \cdot \tau_j = (\gamma_j + \tau_j) \cdot \tau_j = \gamma_j \cdot \tau_j$, we have

$$f_{01}(D) - f_{01}(D') = \varepsilon_1 \sum_{j=2}^{n} \varepsilon_j (t^{\gamma_j \cdot \tau_j} - 1) = \varepsilon_1 \sum_{j=2}^{n} \varepsilon_j (t^{\gamma_j \cdot \tau_j} - 1) = \varepsilon_1 W_K(t).$$
\((p, q) = (1, 0)\). If \(D'\) is obtained from \(D\) by \((a), (b), (c), \) or \((d)\), then \(D^{\#}\) is obtained from \(D^{\#}\) by \((c), (d), (a), \) or \((b)\), respectively. By Lemmas 2.2, 2.3, and the equation for \((p, q) = (0, 1)\), it holds that

\[
f_{10}(D) - f_{10}(D') = f_{01}(D^{\#}) - f_{01}(D^{\#}) = (-\varepsilon_1) \cdot W_{D^{\#}}(t) = \varepsilon_1 W_D(t^{-1}).
\]

\((p, q) = (0, 0)\). It holds that

\[
f_{00}(D) - f_{00}(D') = \varepsilon_1^2 (t^{\gamma_1 - \gamma_1} - 1) + \sum_{j=2}^n \varepsilon_1 \varepsilon_j (t^{\gamma_1 - \gamma_j} - 1) + \sum_{i=2}^n \varepsilon_i \varepsilon_1 (t^{\gamma_i - \gamma_1} - 1).
\]

For \((a)\) and \((b)\), we have \(f_{00}(D) - f_{00}(D') = 0\) by \(\gamma_1 = 0\). For \((c)\) and \((d)\), by using \(\gamma_1 = \gamma_D\), it holds that

\[
f_{00}(D) - f_{00}(D') = \varepsilon_1 \sum_{j=2}^n \varepsilon_j (t^{\gamma_1 - \gamma_j} - 1) + \varepsilon_1 \sum_{i=2}^n \varepsilon_i (t^{\gamma_i - \gamma_1} - 1)
\]

\[
= \varepsilon_1 W_K(t^{-1}) + \varepsilon_1 W_K(t) = \varepsilon_1 W_K(t).
\]

\((p, q) = (1, 1)\). The proof is similar to the case \((p, q) = (1, 0)\). For \((a)\) and \((b)\), \(D^{\#}\) is obtained from \(D^{\#}\) by \((c)\) and \((d)\), respectively. By the equation for \((p, q) = (0, 0)\), we have

\[
f_{11}(D) - f_{11}(D') = f_{00}(D^{\#}) - f_{00}(D^{\#}) = (-\varepsilon_1) \cdot W_{D^{\#}}(t) = \varepsilon_1 W_D(t).
\]

For \((c)\) and \((d)\), since \(D^{\#}\) is obtained from \(D^{\#}\) by \((a)\) and \((b)\), respectively, it holds that \(f_{11}(D) - f_{11}(D') = f_{00}(D^{\#}) - f_{00}(D^{\#}) = 0\).

The writhe of a diagram \(D\) is the sum of the signs of crossings of \(D\), and denoted by \(\omega_D = \sum_{i=1}^n \varepsilon_i\). We consider two kinds of Laurent polynomials

\[
I_D(t) = f_{01}(D; t) - \omega_D W_K(t) \quad \text{and} \quad II_D(t) = f_{00}(D; t) + f_{11}(D; t) - \omega_D W_K(t).
\]

Theorem 2.6. The Laurent polynomials \(I_D(t)\) and \(II_D(t) \in \mathbb{Z}[t, t^{-1}]\) do not depend on a particular choice of a diagram \(D\) of a virtual knot \(K\).

\[
\]

Proof. Since the intersection numbers among \(\gamma_i\)’s and \(\overline{\tau}_i\)’s \((1 \leq i \leq n)\) do not change by a (de)stabilization, it is sufficient to consider the case that a diagram \(D'\) is obtained from \(D\) by a Reidemeister move on \(\Sigma\).

Assume that \(D'\) is obtained from \(D\) by a second or third Reidemeister move on \(\Sigma\). Since \(\omega_{D'} = \omega_D\) holds, we have \(I_{D'}(t) = I_D(t)\) and \(II_{D'}(t) = II_D(t)\) by Lemma 2.4.

Assume that \(D'\) is obtained from \(D\) by a first Reidemeister move such that a crossing \(c_1\) with the sign \(\varepsilon_1\) is removed from \(D\). Since it holds that \(\omega_D = \omega_{D'} + \varepsilon_1\) and \(f_{01}(D) - f_{01}(D') = \varepsilon_1 W_K(t)\) by Lemma 2.5, we have

\[
I_D(t) = f_{01}(D) - \omega_D W_K(t)
\]

\[
= f_{01}(D') + \varepsilon_1 W_K(t) - (\omega_{D'} + \varepsilon_1) W_K(t)
\]

\[
= f_{01}(D') - \omega_{D'} W_K(t) = I_{D'}(t).
\]
Similarly, since it holds that
\[(f_{00}(D) + f_{11}(D)) - (f_{00}(D') + f_{11}(D')) = \varepsilon_1 W_K(t)\]
by Lemma 2.5 we have $\Pi_D(t) = \Pi_{D'}(t)$. \hfill \Box

**Definition 2.7.** The Laurent polynomials $I_D(t)$ and $\Pi_D(t) \in \mathbb{Z}[t, t^{-1}]$ are called the first and second intersection polynomials of a virtual knot $K$, and denoted by $I_K(t)$ and $\Pi_K(t)$, respectively.

**Remark 2.8.** We can also consider the polynomial $f_{10}(D; t) - \omega_D W_K(t^{-1})$ which defines an invariant of $K$ by Lemmas 2.4 and 2.5. However, this is coincident with the first intersection polynomial $I_K(t^{-1})$. In fact, we have
\[f_{10}(D; t) - \omega_D W_K(t^{-1}) = \sum_{1 \leq i, j \leq n} \varepsilon_i (t^{\gamma_i} - 1) - \omega_D W_K(t^{-1}) = \sum_{1 \leq i, j \leq n} \varepsilon_i (t^{-\gamma_i} - 1) - \omega_D W_K(t^{-1}) = I_K(t^{-1}).\]

For a Laurent polynomial $h(t)$, we consider an equivalence relation in $\mathbb{Z}[t, t^{-1}]$ such that $f(t) \equiv g(t)$ (mod $h(t)$) holds if and only if $f(t) - g(t) = mh(t)$ for some $m \in \mathbb{Z}$. In the case of $h(t) = 0$, this equivalence relation gives $f(t) = g(t)$ only. For a diagram $D$ of a virtual knot $K$ on $\Sigma$, we consider the equivalence class $f_{00}(D)$ (mod $W_K(t)$). The invariance follows by Lemmas 2.4 and 2.5 immediately.

**Definition 2.9.** The equivalence class $f_{00}(D)$ (mod $W_K(t)$) is called the third intersection polynomial of a virtual knot $K$, and denoted by $III_K(t)$.

**Remark 2.10.** We can also consider the equivalence class $f_{11}(D)$ (mod $W_K(t)$) as an invariant of $K$. However, since $\Pi_K(t) = f_{00}(D) + f_{11}(D)$ (mod $W_K(t)$) holds by definition, we have
\[f_{11}(D) \equiv \Pi_K(t) - III_K(t) \mod W_K(t).\]

A virtual knot is classical if it is presented by a diagram on $S^2$. By definition, the writhe polynomial $W_K(t)$ vanishes for any classical knot $\Pi( \text{III} )$. The intersection polynomials satisfy the same property as follows.

**Lemma 2.11.** Any classical knot satisfies
\[I_K(t) = \Pi_K(t) = III_K(t) = 0.\]

**Proof.** All intersection numbers between two cycles on $S^2$ are zero. \hfill \Box

3. **Calculations**

Let $C$ be a closed, oriented curve on $\Sigma$ with a finite number of crossings. When we consider $C$ as the image of an immersion $S^1 \rightarrow \Sigma$, the curve $C$ is presented by a Gauss diagram $G$ consisting of the circle $S^1$ equipped with chords each of which connects the preimage of a crossing of $C$. The endpoints of chords admit signs with respect to the orientation of $C$ as shown in Figure 7.

The endpoints of a chord of $G$ divide the circle $S^1$ into two arcs. Let $\alpha \subset S^1$ be such an arc, and $P(\alpha)$ the set of endpoints of the chords of $G$ in the interior of $\alpha$. For an endpoint $x \in P(\alpha)$, we denote by $\text{sgn}(x)$ the sign of $x$, and by $\tau(x)$ the other endpoint of the chord incident to $x$. The arc $\alpha \subset S^1$ presents a cycle on $\Sigma$, which is also denoted by $\alpha \subset \Sigma$. See Figure 8.
Lemma 3.1. Let \( \overline{\alpha} \) be the complementary arc of \( \alpha \subset S^1 \). Then we have
\[
\alpha \cdot \overline{\alpha} = \sum_{x \in P(\alpha)} \operatorname{sgn}(x).
\]

Proof. Any chord whose endpoints both lie on \( \alpha \) does not contribute to the sum in the right hand side of the equation. Therefore it holds that
\[
\sum_{x \in P(\alpha)} \operatorname{sgn}(x) = \sum_{x \in P(\alpha), \tau(x) \in P(\overline{\alpha})} \operatorname{sgn}(x) = \alpha \cdot \overline{\alpha}.
\]

Let \( \alpha \) and \( \beta \subset S^1 \) be arcs for distinct chords \( a \) and \( b \) of \( G \), respectively. We consider an integer
\[
S(\alpha, \beta) = \sum_{x \in P(\alpha), \tau(x) \in P(\beta)} \operatorname{sgn}(x).
\]
It follows by definition that \( S(\alpha, \beta) = -S(\beta, \alpha) \). We say that the chords \( a \) and \( b \) of \( G \) are linked if their endpoints appear on \( S^1 \) alternately, and otherwise unlinked. The number \( S(\alpha, \beta) \) is equal to the sum of the signs of the endpoints indicated by dots as shown in Figure 9.

Then the intersection number \( \alpha \cdot \beta \) of the cycles \( \alpha \subset \Sigma \) is calculated as follows.

Lemma 3.2. (i) If \( a \) and \( b \) are unlinked, then \( \alpha \cdot \beta = S(\alpha, \beta) \).
(ii) Assume that $a$ and $b$ are linked as shown in Figure 10 where $\varepsilon, \delta \in \{\pm\}$. Then it holds that
\[
\alpha \cdot \beta = S(\alpha, \beta) + \frac{1}{2}(\varepsilon + \delta) \quad \text{and} \quad \beta \cdot \alpha = S(\beta, \alpha) - \frac{1}{2}(\varepsilon + \delta).
\]

**Proof.** We prove two cases $(\varepsilon, \delta) = (+, +)$ and $(+, -)$ in (ii) as shown in Figure 11. Other cases are similarly proved.

$(\varepsilon, \delta) = (+, +)$. We take a parallel copy of the curve $\alpha \subset \Sigma$ (or $\beta$) which lies on the left (or right) side of the original curve. See the left of Figure 11. Then the intersections between $\alpha$ and $\beta$ except one point near the crossing $b$ correspond to the endpoints $x \in P(\alpha)$ with $\tau(x) \in P(\beta)$. Since the sign of the exceptional intersection is equal to $+$, we obtain $\alpha \cdot \beta = S(\alpha, \beta) + 1$.

$(\varepsilon, \delta) = (+, -)$. Similarly to the above case, we consider parallel copies of $\alpha$ and $\beta$. In this case, there is no exceptional intersection near $b$. See the right of Figure 11. Therefore we have $\alpha \cdot \beta = S(\alpha, \beta)$.

**Example 3.3.** We consider three arcs $\alpha$, $\beta$, and $\gamma$ of the Gauss diagram as shown in Figure 12. We have
\[
S(\alpha, \beta) = -2, \quad S(\alpha, \gamma) = 1, \quad \text{and} \quad S(\beta, \gamma) = 0.
\]
Since $\alpha$ and $\beta$ are unlinked, it holds that $\alpha \cdot \beta = S(\alpha, \beta) = -2$ by Lemma 3.2(i). On the other hand, since $\alpha$ and $\gamma$, $\beta$ and $\gamma$ are linked, respectively, it holds that $\alpha \cdot \gamma = S(\alpha, \gamma) = 1$ and $\beta \cdot \gamma = S(\beta, \gamma) + 1 = 1$ by Lemma 3.2(ii).
Let $D \subset \Sigma$ be a diagram of a virtual knot $K$ with $n$ crossings $c_1, \ldots, c_n$, and $G$ the Gauss diagram of $D$. We also denote by $c_i$ the chord of $G$ corresponding to a crossing $c_i$ of $D$. Each chord of $G$ is oriented from the over-crossing to the under-crossing, and equipped with the same sign as that of the corresponding crossing of $D$. Then we see that if the sign of a chord is equal to $\varepsilon$, then the initial and terminal endpoints of the chord have the sign $-\varepsilon$ and $\varepsilon$, respectively. See Figure 13.

The endpoints of an oriented chord $c_i$ of $G$ divide the circle $S^1$ into two arcs. The arc from the initial endpoint to the terminal corresponds to the cycle $\gamma_i \subset \Sigma$ at the crossing $c_i$, and the other arc corresponds to $\overline{\gamma}_i$. Therefore we see that

$$
\gamma_i \cdot \overline{\gamma}_i = \gamma_i \cdot \gamma_D = \sum_{x \in P(\gamma_i)} \text{sgn}(x).
$$

If two chords $c_i$ and $c_j$ are unlinked, then it follows by Lemma 3.2(i) that

$$
\gamma_i \cdot \overline{\gamma}_j = S(\gamma_i, \overline{\gamma}_j), \quad \gamma_i \cdot \gamma_j = S(\gamma_i, \gamma_j), \quad \text{and} \quad \overline{\gamma}_i \cdot \overline{\gamma}_j = \overline{S}(\gamma_i, \gamma_j).
$$

We have similar equations in the case that $c_i$ and $c_j$ are linked by Lemma 3.2(ii). For example, we consider the case as shown in Figure 14. Then we have

$$
\gamma_i \cdot \overline{\gamma}_j = S(\gamma_i, \overline{\gamma}_j) + (\varepsilon_i - \varepsilon_j)/2, \\
\gamma_i \cdot \gamma_j = S(\gamma_i, \gamma_j) - (\varepsilon_i + \varepsilon_j)/2, \quad \text{and} \\
\overline{\gamma}_i \cdot \overline{\gamma}_j = S(\overline{\gamma}_i, \overline{\gamma}_j) + (\varepsilon_i + \varepsilon_j)/2.
$$

Let $c(K)$ denote the crossing number of $K$, which is the minimal number of crossings for all diagrams of $K$. The virtual knots up to crossing number four are given by Green [4]. In what follows, the labels of virtual knots are due to Green’s table.

**Example 3.4.** We consider the Gauss diagram $G$ of a virtual knot $K = 4.39$ as shown in Figure 15. Table 2 shows the intersection numbers $\gamma_i \cdot \overline{\gamma}_j$, $\gamma_i \cdot \gamma_j$, and
Figure 14

\[ \tau_i \cdot \tau_j \] for \( 1 \leq i, j \leq 4 \). Since we have \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = - \) and \( \varepsilon_4 = + \), it holds that

\[
W_K(t) = \sum_{i=1}^{n} \varepsilon_i (t^{\tau_i} - 1) = -t^3 + t^2 + 1 - t^{-1} \quad \text{and}
\]

\[
\overline{W}_K(t) = -t^3 + t^2 - t + 2 - t^{-1} + t^{-2} - t^{-3}.
\]

On the other hand, we have

\[
f_{01}(D) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\tau_i} - 1) = 2t^2 - 2t - 2 + 2t^{-1},
\]

\[
f_{00}(D) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\tau_i} - 1) = t^3 - t^2 - t^{-2} + t^{-3}, \quad \text{and}
\]

\[
f_{11}(D) = \sum_{1 \leq i, j \leq n} \varepsilon_i \varepsilon_j (t^{\tau_i} - 1) = t^2 - t - t^{-1} + t^{-2}.
\]

Since \( \omega_D = -2 \) holds, we obtain

\[
I_K(t) = -2t^3 + 4t^2 - 2t,
\]

\[
II_K(t) = -t^3 + 2t^2 - 3t + 4 - 3t^{-1} + 2t^{-2} - t^{-3}, \quad \text{and}
\]

\[
III_K(t) \equiv -t + 2 - t^{-1} \pmod{-t^3 + t^2 - t + 2 - t^{-1} + t^{-2} - t^{-3}}.
\]

Figure 15

**Theorem 3.5.** For the virtual knots \( K \) with \( c(K) \leq 4 \), the intersection polynomials \( I_K(t) \), \( II_K(t) \), and \( III_K(t) \) are given in Appendices [A] and [B].

By observing the calculations in Appendices [A] and [B], we see that the writhe polynomial and the intersection polynomials are independent of each other in the following sense.
Proposition 3.6. There are four pairs of virtual knots $K_i$ and $K'_i$ ($i = 1, 2, 3, 4$) which satisfy the following.

(i) For the virtual knots $K_1(t) \neq W_{K'_1}(t)$ and $X_{K_1}(t) = X_{K'_1}(t)$ ($X = I, II, III$).
(ii) $I_{K_2}(t) \neq I_{K'_2}(t)$ and $X_{K_2}(t) = X_{K'_2}(t)$ ($X = W, II, III$).
(iii) $II_{K_3}(t) \neq II_{K'_3}(t)$ and $X_{K_3}(t) = X_{K'_3}(t)$ ($X = W, I, III$).
(iv) $III_{K_4}(t) \neq III_{K'_4}(t)$ and $X_{K_4}(t) = X_{K'_4}(t)$ ($X = W, I, II$).

Proof. (i) For the virtual knots $K_1 = 4.36$ and $K'_1 = 4.65$, it holds that $W_{K_1}(t) = t^2 - 2 + t^{-2}$ and $W_{K'_1}(t) = -t^2 + 2 - t^{-2}$.

On the other hand, we have

\[
I_{K_1}(t) = I_{K'_1}(t) = -t^2 + 2 - t^{-2},
\]

\[
II_{K_1}(t) = II_{K'_1}(t) = -2t^2 + 4 - 2t^{-2}, \text{ and}
\]

\[
III_{K_1}(t) = III_{K'_1}(t) \equiv t^2 - 2 + t^{-2} \pmod{2t^2 - 4 + 2t^{-2}}.
\]

(ii) For the trivial knot $K_2 = O$ and the virtual knot $K'_2 = 4.16$, it holds that $I_{K_2}(t) = 0$ and $I_{K'_2}(t) = -t^2 + 3t - 3 + t^{-1}$.

On the other hand, we have

\[
W_{K_2}(t) = W_{K'_2}(t) = 0,
\]

\[
II_{K_2}(t) = II_{K'_2}(t) = 0, \text{ and}
\]

\[
III_{K_2}(t) = III_{K'_2}(t) = 0.
\]

(iii) For the virtual knots $K_3 = 3.2$ and $K'_3 = 4.33$, it holds that $II_{K_3}(t) = -2t + 4 - 2t^{-1}$ and $II_{K'_3}(t) = t^2 - 6t + 10 - 6t^{-1} + t^{-2}$.

On the other hand, we have

\[
W_{K_3}(t) = W_{K'_3}(t) = -t + 2 - t^{-1},
\]

\[
I_{K_3}(t) = I_{K'_3}(t) = -t + 2 - t^{-1}, \text{ and}
\]

\[
III_{K_3}(t) = III_{K'_3}(t) \equiv t - 2 + t^{-1} \pmod{2t - 4 + 2t^{-1}}.
\]

(iv) For the virtual knots $K_4 = O$ and $K'_4 = 4.13$, it holds that $III_{K_4}(t) = 0$ and $III_{K'_4}(t) = 2t - 4 + 2t^{-1}$.

On the other hand, we have

\[
W_{K_4}(t) = W_{K'_4}(t) = 0,
\]

\[
I_{K_4}(t) = I_{K'_4}(t) = 0, \text{ and}
\]

\[
II_{K_4}(t) = II_{K'_4}(t) = 0.
\]
The Gauss diagrams of the knots are illustrated in Figure 16.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gauss_diagrams.png}
\caption{Figure 16}
\end{figure}

**Remark 3.7.** For a virtual knot $K$, Silver and Williams [16] define a sequence of Alexander polynomials $\Delta_i(K)$ as an extension of the classical Alexander polynomial. The zero-th Alexander polynomial $\Delta_0(K)(u,v)$ is also defined by Sawollek [15]. Mellor [13] proves that the writhe polynomial is obtained from $\Delta_0(K)(u,v) = (1-uv)\Delta_0(K)(u,v)$ by the equation

$$W_K(t) = -\tilde{\Delta}_0(K)(t,t^{-1}).$$

It is natural to ask whether the intersection polynomials are also obtained from $\Delta_i(K)$. However this does not hold generally; in fact, for the virtual knot $K = 4.8$, we have $\Delta_0(K) = 0$ and $\Delta_i(K) = 1 (i \geq 1)$ which are coincident with those of the trivial knot. On the other hand, it holds that the intersection polynomials $I_K(t)$, $II_K(t)$, and $III_K(t)$ are all non-trivial.

4. Symmetries and crossing numbers

For a diagram $D$ on $\Sigma$ of a virtual knot $K$, let $-D$ be the diagram by reversing the orientation of $D$, $D\#$ the one by changing over/under-information at every crossing of $D$, and $D^*$ the one obtained by an orientation-reversing homeomorphism of $\Sigma$. The virtual knots presented by $-D$, $D\#$, and $D^*$ are called the reverse, the vertical mirror image, and the horizontal mirror image of $K$, and denoted by $-K$, $K\#$, and $K^*$, respectively. The writhe polynomials of these knots are given as follows.

**Lemma 4.1** ([1] [10] [14]). Any virtual knot $K$ satisfies

$$W_{-K}(t) = W_K(t^{-1}) \text{ and } W_{K\#}(t) = W_{K^*}(t) = -W_K(t^{-1}).$$

Therefore we have

$$\overline{W}_{-K}(t) = \overline{W}_K(t) \text{ and } \overline{W}_{K\#}(t) = \overline{W}_{K^*}(t) = -\overline{W}_K(t).$$

The intersection polynomials of $-K$, $K\#$, and $K^*$ are given as follows.

**Lemma 4.2.** For a virtual knot $K$, we have the following.

(i) $I_{-K}(t) = I_{K\#}(t) = I_{K^*}(t) = I_K(t^{-1}).$

(ii) $II_{-K}(t) = II_{K\#}(t) = II_{K^*}(t) = II_K(t).$

(iii) $III_{-K}(t) = III_{K\#}(t) = II_K(t) - III_K(t) \text{ and } III_{K^*}(t) = III_K(t).$
Proof. $I_{-K}(t), II_{-K}(t),$ and $III_{-K}(t)$. Let $c_i'$ be the crossing of $-D$ corresponding to $c_i, \gamma_i'$ the cycle at $c_i'$ on $\Sigma,$ and $\epsilon_i'$ the sign of $c_i'$ ($1 \leq i \leq n$). Then it holds that $\gamma_i' = -\overline{\gamma}_i$ and $\epsilon_i' = \epsilon_i$. By definition, we have

$$f_{01}(-D; t) = \sum_{1 \leq i, j \leq n} \epsilon_i' \epsilon_j'(t^{\gamma_i'} \overline{\gamma}_j - 1) = \sum_{1 \leq i, j \leq n} \epsilon_i \epsilon_j(t^{\gamma_i} \overline{\gamma}_j - 1) = f_{01}(D; t^{-1}),$$
$$f_{00}(-D; t) = \sum_{1 \leq i, j \leq n} \epsilon_i \epsilon_j(t^{\gamma_i} \overline{\gamma}_j - 1) = f_{11}(D; t),$$
$$f_{11}(-D; t) = \sum_{1 \leq i, j \leq n} \epsilon_i \epsilon_j(t^{\gamma_i} \overline{\gamma}_j - 1) = f_{00}(D; t).$$

Since $\omega_{-D} = \omega_D$ holds, we have

$$I_{-K}(t) = f_{01}(-D; t) - \omega_{-D} W_{-D}(t) = f_{01}(D; t^{-1}) - \omega_{-D} W_{-D}(t^{-1}) = I_K(t^{-1}),$$
$$II_{-K}(t) = f_{00}(D; t) - \omega_{-D} W_{-D}(t) + W_{-D}(t^{-1}) = f_{11}(D; t) + f_{00}(D; t) - \omega_{-D} (W_D(t^{-1}) + W_D(t)) = II_K(t)$$
and
$$III_{-K}(t) = f_{00}(-D; t) = f_{11}(D; t) = II_K(t) - f_{00}(D; t) = II_K(t) - II_K(t).$$

$I_K(t), II_K(t),$ and $III_K(t).$ We use the notations in the proof of Lemma 2.2. By the lemma, we have $f_{00}(D#; t) = f_{11}(D; t)$ and hence $f_{11}(D#; t) = f_{00}(D; t)$. Furthermore it holds that

$$f_{01}(D#; t) = \sum_{1 \leq i, j \leq n} (-\epsilon_i)(-\epsilon_j)(t^{\gamma_i} \overline{\gamma}_j - 1) = f_{01}(D; t^{-1}).$$

Since it holds that $\omega_{D#} = -\omega_D$ and $W_{D#}(t) = -W_D(t^{-1})$, we have

$$I_{K#}(t) = f_{01}(D#; t) - \omega_{D#} W_{D#}(t) = f_{01}(D; t^{-1}) - \omega_{D#} W_{D#}(t^{-1}) = I_K(t^{-1}),$$
$$II_{K#}(t) = f_{00}(D#; t) + f_{11}(D#; t) - \omega_{D#} (W_{D#}(t) + W_{D#}(t^{-1})) = f_{11}(D; t) + f_{00}(D; t) - \omega_{D#} (W_D(t^{-1}) + W_D(t)) = II_K(t)$$
and
$$III_{K#}(t) = f_{00}(D#; t) = f_{11}(D; t) = II_K(t) - f_{00}(D; t) = II_K(t) - III_K(t).$$

$I_K^*(t), II_K^*(t),$ and $III_K^*(t).$ Let $c_i^*$ be the crossing of $D^*$ corresponding to $c_i, \gamma_i^*$ the cycle at $c_i^*$ on $\Sigma,$ and $\epsilon_i^*$ the sign of $c_i^*$ ($1 \leq i \leq n$). Then it holds that $\gamma_i^* = -\overline{\gamma}_i, \gamma_i^* = -\gamma_i, \overline{\gamma}_j^* = -\overline{\gamma}_j,$ and $\epsilon_i^* = -\epsilon_i$. Since $f_{00}(D; t)$ and $f_{11}(D; t)$ are reciprocal, we have

$$f_{01}(D^*; t) = f_{01}(D; t^{-1})^{-1}, f_{00}(D^*; t) = f_{00}(D; t),$$
and
$$f_{11}(D^*; t) = f_{11}(D; t).$$

Since it holds that $\omega_{D^*} = -\omega_D$ and $W_{D^*}(t) = -W_D(t^{-1})$, we have

$$I_{K^*}(t) = f_{01}(D^*; t) - \omega_{D^*} W_{D^*}(t) = f_{01}(D; t^{-1}) - \omega_{D^*} W_{D^*}(t^{-1}) = I_K(t^{-1}),$$
$$II_{K^*}(t) = f_{00}(D^*; t) + f_{11}(D^*; t) - \omega_{D^*} (W_{D^*}(t) + W_{D^*}(t^{-1})) = f_{00}(D; t) + f_{11}(D; t) - \omega_{D^*} (W_D(t^{-1}) + W_D(t)) = II_K(t)$$
and
$$III_{K^*}(t) = f_{00}(D^*; t) = f_{00}(D; t) = III_K(t).$$
Proposition 4.3. If a virtual knot $K$ satisfies

(i) $2\text{III}_K(t) \neq \text{II}_K(t)$ (mod $\overline{W}_K(t)$),
(ii) $W_K(t) \neq W_K(t^{-1})$, and
(iii) $W_K(t) \neq -W_K(t^{-1})$ or $I_K(t) \neq I_K(t^{-1})$,

then the eight virtual knots

$K, -K, K^#, K^*, -K^#, -K^*, K^{##},$ and $-K^{##}$

are mutually distinct.

Proof. By Lemma 4.2(iii), the virtual knots $K, K^#, -K^#$, and $-K^{##}$ have the same third intersection polynomial $\text{III}_K(t)$, and $-K, -K^*, K^#$, and $K^{##}$ have $\text{II}_K(t) - \text{III}_K(t)$. Therefore, by the condition (i), it holds that

$$\{K, K^*, -K^#, -K^{##}\} \cap \{-K, -K^*, K^#, K^{##}\} = \emptyset.$$ 

Furthermore, by Lemma 4.1, the first four virtual knots $K, K^*, -K^#$, and $-K^{##}$ have the writhe polynomials $W_K(t)$, $-W_K(t^{-1})$, $-W_K(t)$, and $W_K(t^{-1})$, respectively. Since $W_K(t) \neq 0$ follows by the condition (ii), we have

$$\{K, K^*\} \cap \{-K^#, -K^{##}\} = \emptyset.$$ 

Finally, each of the pairs $K$ and $K^*$, and $-K^#$ and $-K^{##}$ can be distinguished by the condition (iii) and Lemma 4.2(i). We can prove that the latter four virtual knots $-K, -K^*, K^#$, and $K^{##}$ are mutually distinct similarly. □

Theorem 4.4. There are infinitely many virtual knots $K$ such that

$K, -K, K^#, K^*, -K^#, -K^*, K^{##},$ and $-K^{##}$

are mutually distinct.

Proof. Let $K_n$ ($n \geq 1$) be the virtual knot presented by the Gauss diagram $G_n$ as shown in Figure 17. It holds that

$$W_{K_n}(t) = t^{n+1} - (n + 1)t + (n + 1)t^{-1} - t^{-n-1},$$

$$I_{K_n}(t) = -t^{n+2} - nt^{n+1} + t - (2n - 1) - (n + 1)t^{-n-1} + 2 \sum_{i=1}^{n}(t^i + t^{-i}),$$

$$\text{II}_{K_n}(t) = -(t^{2n+2} + t^{-2n-2}) - (t^{n+1} + t^{-n-1}) - (2n + 1)(t^n + t^{-n}) - n(n + 1)(t^2 + t^{-2}) - (2n + 3)(t + t^{-1}) + 2(n^2 + n + 2) + 4 \sum_{i=1}^{n+1}(t^i + t^{-i})$$

and

$$\text{III}_{K_n}(t) = -(t^{n+1} + t^{-n-1}) - (n + 1)(t + t^{-1}) + 2 \sum_{i=1}^{n+1}(t^i + t^{-i}) - 2n,$$

where we have $\overline{W}_{K_n}(t) = 0$. Since these invariants of $K_n$ satisfy the conditions (i), (ii), and (iii) $I_{K_n}(t) \neq I_{K_n}(t^{-1})$ in Proposition 4.3 the eight kinds of virtual knots associated with $K_n$ are mutually distinct. Furthermore $K_n \neq K_m$ ($n \neq m$) holds by $\deg W_{K_n}(t) = n + 1$. We remark that $K_n$ satisfies $W_{K_n}(t) = -W_{K_n}(t^{-1})$. □
Example 4.5. We can construct an infinite family of virtual knots $K$ satisfying the conditions (i), (ii), and (iii) $W_K(t) \neq -W_K(t^{-1})$ in Proposition 4.3.

Let $K'_n (n \geq 3)$ be the virtual knot presented by the Gauss diagram $G'_n$ as shown in Figure 18. We have

$$W_{K'_n}(t) = -t^{n-1} + nt - n + t^{-1},$$

$$I_{K'_n}(t) = -t^n + (n-2)t^{n-1} - 2 \sum_{i=1}^{n-2} t^i + (2n-1) - nt^{-1},$$

$$II_{K'_n}(t) = -(t^n + t^{-n}) + (n-1)(t^{n-1} + t^{-n+1}) - 2 \sum_{i=1}^{n-2} (t^i + t^{-i}) - (n^2 - n + 1)(t + t^{-1}) + 2n^2 - 2,$$

and

$$III_{K'_n}(t) = -\sum_{i=1}^{n-2} (t^i + t^{-i}) - (t + t^{-1}) + 2n - 2 \pmod{W_{K'_n}(t)},$$

where $W_{K'_n}(t) = -(t^{n-1} + t^{-n+1}) + (n+1)(t + t^{-1}) - 2n$. Since these invariants of $K'_n$ satisfy the conditions (i), (ii), and (iii) $W_{K'_n}(t) \neq -W_{K'_n}(t^{-1})$ in Proposition 4.3, the eight kinds of virtual knots associated with $K'_n$ are mutually distinct. Furthermore $K'_n \neq K'_m (n \neq m)$ holds by $\deg W_{K'_n}(t) = n - 1$. We remark that $K'_n$ satisfies (iii) $I_{K'_n}(t) \neq I_{K'_n}(t^{-1})$ in Proposition 4.3. □
5. Real and virtual crossing numbers

For a Laurent polynomial $f(t)$, let $\deg f(t)$ denote the maximal degree of $f(t)$. The writhe polynomial $W_K(t)$ gives a lower bound of the crossing number $c(K)$ as follows.

Lemma 5.1 ([14]). Any non-trivial virtual knot $K$ satisfies $c(K) \geq \deg W_K(t) + 1$.

We remark that the minimal degree of $W_K(t)$ also gives a lower bound of $c(K)$, and denotes $\text{span } W_K(t)$.

Lemma 5.1 induces a weaker inequation

$$c(K) \geq \frac{1}{2} \text{span } W_K(t) + 1$$

immediately. The intersection polynomials also gives lower bounds of $c(K)$ as follows. Here, $\deg III_K(t)$ denotes the maximal number of $\deg f(t)$ for all $f(t)$ with $f(t) \equiv III_K(t)$ (mod $W_K(t)$).

Proposition 5.2. Let $K$ be a non-trivial virtual knot.

(i) $c(K) \geq \deg I_K(t) + 1$.
(ii) $c(K) \geq \deg II_K(t) + 1$.
(iii) $c(K) \geq \deg III_K(t) + 1$.

Proof. Assume that a diagram $D$ of $K$ satisfies $c(D) = c(K)$. Since $K$ is non-trivial, it holds that $c(D) \geq 2$. For $(\alpha, \beta) = (\gamma_i, \gamma_j)$, and $(\overline{\gamma}_i, \overline{\gamma}_j)$ with $i \neq j$, the intersection number $\alpha \cdot \beta$ is equal to $S(\alpha, \beta) - 1$, $S(\alpha, \beta)$, or $S(\alpha, \beta) + 1$ by Lemma [3,2] so that we obtain $\alpha \cdot \beta \leq S(\alpha, \beta) + 1$.

Since there are $c(D) - 2$ chords other than $c_{ij}$ and $c_{\overline{ij}}$ in the Gauss diagram of $D$, we have $S(\alpha, \beta) \leq c(D) - 2$. Therefore it holds that

$$\deg f_{pq}(D) \leq c(D) - 1 = c(K) - 1$$

for $(p, q) = (0, 1), (0, 0)$, and $(1, 1)$. Since $\deg W_K(t) \leq c(K) - 1$, we have the conclusion. \qed

As well as a diagram on $\Sigma$ or a Gauss diagram, a virtual knot is also presented by a virtual diagram in $\mathbb{R}^2$. It is an immersed circle in $\mathbb{R}^2$ with real and virtual crossings [9]. Here, the real crossings correspond to the crossings on $\Sigma$, and the virtual crossings are surrounded by small circles. The virtual crossing number of a virtual knot $K$ is the minimal number of virtual crossings for all virtual diagrams of $K$, and denoted by $\text{vc}(K)$.

The intersection polynomials are also calculated from a virtual diagram. For example, we consider the virtual diagram with three real crossings $c_1, c_2$, and $c_3$ and two virtual crossings as shown in the leftmost of Figure [19] which presents the virtual knot $K = 3.4$. To calculate $\gamma_1 \cdot \overline{\gamma}_2$, we draw the curves $\gamma_1$ and $\overline{\gamma}_2$ equipped with virtual crossings, and then take the sum of signs of two intersections with ignoring virtual crossings to obtain $\gamma_1 \cdot \overline{\gamma}_2 = 2$. See the second from the left in the figure. Similarly we obtain $\gamma_1 \cdot \gamma_2 = 0$ and $\overline{\gamma}_1 \cdot \overline{\gamma}_2 = -1$ as shown in the third and fourth from the left.

The writhe polynomial $W_K(t)$ gives a lower bound of the virtual crossing number $\text{vc}(K)$ as follows.

Lemma 5.3 ([14]). Any non-trivial virtual knot $K$ satisfies $\text{vc}(K) \geq \deg W_K(t)$.
We remark that Lemma 5.3 induces a weaker inequation
\[ \text{vc}(K) \geq \frac{1}{2} \text{span} W_K(t) \]
immediately, which is proved in [13]. The intersection polynomials also gives lower bounds of vc(K) as follows.

**Proposition 5.4.** Let K be a virtual knot.

(i) \( \text{vc}(K) \geq \deg I_K(t) \).

(ii) \( \text{vc}(K) \geq \deg II_K(t) \).

(iii) \( \text{vc}(K) \geq \deg III_K(t) \).

**Proof.** Let \( D \) be a virtual diagram of \( K \) in \( \mathbb{R}^2 \), and \( \alpha \) and \( \beta \) cycles on \( D \) with corners at (possibly the same) real crossings of \( D \). By a slight perturbation of \( \beta \) if necessary, we may assume that \( \alpha \) and \( \beta \) intersect in a finite number of double points near real and virtual crossings of \( D \) as explained as above. By Lemma 5.3 it is sufficient to prove that if the intersection number restricted to the real crossings between \( \alpha \) and \( \beta \) in \( \mathbb{R}^2 \) is equal to \( n \), then the number of virtual crossings of \( D \) is greater than or equal to \( |n| \).

Since the total intersection number between \( \alpha \) and \( \beta \) in \( \mathbb{R}^2 \) is equal to zero, the intersection number restricted to the virtual crossings between \( \alpha \) and \( \beta \) is equal to \(-n\).

Let \( v \) be a virtual crossing of \( D \) where two short paths \( \lambda \) and \( \lambda' \subset D \) intersect. If \( \alpha \) and \( \beta \) intersect in virtual crossings near \( v \), then there are four cases as follows.

(i) \( \alpha \supset \lambda, \alpha \not\supset \lambda', \) and \( \beta \not\supset \lambda, \beta \supset \lambda' \).

(ii) \( \alpha \supset \lambda, \lambda', \) and \( \beta \not\supset \lambda, \beta \supset \lambda' \).

(iii) \( \alpha \supset \lambda, \alpha \not\supset \lambda', \) and \( \beta \supset \lambda, \lambda' \).

(iv) \( \alpha \supset \lambda, \lambda' \) and \( \beta \supset \lambda, \lambda' \).

See Figure 20.

In the case (iv), the pair of virtual crossings between \( \alpha \) and \( \beta \) does not contribute to the intersection number; in fact, they have opposite signs. On the other hand, each case of (i)–(iii) contains a single virtual crossing. It follows that the number of virtual crossings of \( D \) in the cases (i)–(iii) is greater than or equal to \( |n| \). \( \square \)

**Example 5.5.** (i) Let \( K_n \) \((n \geq 1)\) be the virtual knot presented by the Gauss diagram and the virtual diagram as shown in Figure 21. We see that \( c(K_n) = n + 3 \).
and \( \text{vc}(K_n) = n + 2 \) can be detected by \( I_{K_n}(t) \) but not by \( W_{K_n}(t) \), \( II_{K_n}(t) \), and \( III_{K_n}(t) \). In fact, we have

\[
W_{K_n}(t) = t^{n+1} - t^2 - nt + n + 1 - t^{-1},
\]

\[
I_{K_n}(t) = -t^{n+2} + (n+1)t^{n+1} - t^n + (n+1)t - 2 \sum_{i=1}^{n} t^i,
\]

\[
II_{K_n}(t) = n(t^{n+1} + t^{-n-1}) - (n^2 + n + 2)(t + t^{-1}) + 2(n^2 + 2n + 2) - 2 \sum_{i=1}^{n} (t^i + t^{-i}), \text{ and}
\]

\[
III_{K_n}(t) \equiv n(t^2 + t^{-2}) - 2(t + t^{-1}) + 4 - \sum_{i=1}^{n} (t^i + t^{-i}) \pmod{W_{K_n}(t)},
\]

where \( W_{K_n}(t) = (t^{n+1} + t^{-n-1}) - (t^2 + t^{-2}) - (n+1)(t + t^{-1}) + 2n + 2 \).

(ii) Let \( K'_n \) \((n \geq 1)\) be the virtual knot presented by the Gauss diagram and the virtual diagram as shown in Figure 22. We see that \( c(K'_n) = n + 3 \) and \( \text{vc}(K'_n) = n + 2 \) can be detected by \( II_{K'_n}(t) \) but not by \( W_{K'_n}(t) \), \( I_{K'_n}(t) \), and \( III_{K'_n}(t) \).
In fact, we have

\[
\begin{align*}
W_{K_n}'(t) & = t^{n+1} - nt + n - 1 - t^{-1} + t^{-2}, \\
I_{K_n}'(t) & = (n-1)t^{n+1} + (2n+1)t - n - 1 - (n-2)t^{-1} + (n-1)t^{-2} - 2 \sum_{i=1}^{n} t^i, \\
II_{K_n}'(t) & = (t^{n+2} + t^{-n-2}) + (n-2)(t^{n+1} + t^{-n-1}) \\
& + (n-1)(t^2 + t^{-2}) - n(n+1)(t + t^{-1}) + 2(n^2 + n + 2) \\
& - \sum_{i=1}^{n} (t^i + t^{-i}) - \sum_{i=1}^{n} (t^{i-1} + t^{-i+1}), \text{ and} \\
III_{K_n}'(t) & \equiv -n(t + t^{-1}) + 4n - \sum_{i=1}^{n} (t^{i-1} + t^{-i+1}) \pmod{W_{K_n}'(t)},
\end{align*}
\]

where \(W_{K_n}'(t) = (t^{n+1} + t^{-n-1}) + (t^2 + t^{-2}) - (n-1)(t + t^{-1}) + 2n - 2\).

**Figure 22**

**References**

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Appendix A. Table of $W_K(t)$, $I_K(t)$, and $II_K(t)$.

Table 3 shows $W_K(t)$, $I_K(t)$, and $II_K(t)$ of a virtual knot $K$ up to crossing number four according to Green’s table [4] with a choice of orientations. We use the following notations:

\[
\begin{align*}
\{n\}(a_0 + a_1 + \cdots + a_m) &= a_0 t^n + a_1 t^{n+1} + \cdots + a_m t^{n+m} \\
b_0 + b_1 + b_2 + \cdots &= b_0 + b_1(t + t^{-1}) + b_2(t^2 + t^{-2}) + \cdots,
\end{align*}
\]

where $m \geq 1$ and $a_0 \neq 0$.

| $W_K(t)$ | $I_K(t)$ | $II_K(t)$ |
|---|---|---|
| 2.1 | 2 - 1 | 2 - 1 | 4 - 2 |
| 3.1 | -1(-1 + 1 + 1 - 1) | 0(-1 + 2 - 1) | 4 - 2 |
| 3.2 | -1(-2 + 3 + 0 - 1) | -1(-2 + 2 + 2 - 2) | 10 - 4 - 1 |
| 3.3 | 0(1 - 2 + 1) | 0 | 6 - 4 + 1 |
| 3.4 | 2 + 0 - 1 | 0 + 2 - 2 | 0 + 4 - 4 |
| 3.5 | 0 | 0 | 0 |
| 3.6 | 0 | 0 | 0 |
| 3.7 | 2 + 0 - 1 | 4 - 2 | 8 - 4 |
| 4.1 | 4 - 2 | 4 - 2 | 20 - 12 + 2 |
| 4.2 | 2 - 1 | 4 - 2 | 6 - 4 + 2 |
| 4.3 | 2 + 0 - 1 | -2 + 2 - 1 | 8 - 4 |
| 4.4 | 0 | 0 | 0 |
| 4.5 | -1(-1 + 1 + 1 - 1) | -1(-1 + 0 + 3 - 2) | 0 + 2 - 2 |
| 4.6 | -1(-2 + 3 + 0 - 1) | -1(-2 + 2 + 2 - 2) | 10 - 4 - 1 |
| 4.7 | 0 | 0 | 0 |
| 4.8 | 0 | 0 | 0 |
| 4.9 | -1(-1 + 1 + 1 - 1) | -1(-1 + 0 + 3 - 2) | 0 + 2 - 2 |
| 4.10 | -1(-2 + 3 + 0 - 1) | -1(-2 + 2 + 2 - 2) | 10 - 4 - 1 |
| 4.11 | 0 | 4 - 2 | 6 - 3 - 1 + 1 |
| 4.12 | 0 | 0 | 0 |
| 4.13 | 0 | 0 | 0 |
| 4.14 | 0 | 0 | 0 |
| 4.15 | -1(-2 + 3 + 0 - 1) | -1(-2 + 2 + 2 - 2) | 4 - 0 - 2 |
| 4.16 | 0 | 0 | 0 |
| 4.17 | -1(-1 + 1 + 1 - 1) | -1(-1 + 0 + 3 - 2) | 0 + 2 - 2 |
| 4.18 | 0 | 0 | 0 |
| 4.19 | 0 | 0 | 0 |
| 4.20 | 0 | 0 | 0 |
| 4.21 | 0 | 0 | 0 |
| 4.22 | 0 | 0 | 0 |
| 4.23 | $-1(-1 + 1 + 1 - 1)$ | $-1(-1 + 0 + 3 - 2)$ | $0 + 2 - 2$ |
| 4.24 | $-2(1 + 0 - 1 - 1 + 0 + 1)$ | $-2(-1 + 1 + 2 + 3 + 1 - 2)$ | $0 + 3 - 2 - 1$ |
| 4.25 | $4 - 2$ | $-1(-3 + 7 + 5 + 1)$ | $14 - 8 + 1$ |
| 4.26 | $-1(-1 + 0 + 2 + 0 - 1)$ | $-1(1 - 3 + 3 - 1)$ | $10 - 5 - 1 + 1$ |
| 4.27 | $2 - 1$ | $-1(-2 + 3 + 0 - 1)$ | $6 - 2 - 1$ |
| 4.28 | $-1(2 - 2 - 1 + 0 + 1)$ | $0(-3 + 4 + 1 - 2)$ | $10 - 5 + 1 - 1$ |
| 4.29 | $-1(-2 + 3 + 0 - 1)$ | $-1(-2 + 2 + 2 - 2)$ | $10 - 4 - 1$ |
| 4.30 | $2 - 1$ | $(1)(-1 + 2 - 1)$ | $10 - 6 + 1$ |
| 4.31 | 0 | 0 | 0 |
| 4.32 | $-1(-1 + 1 + 1 - 1)$ | $2 - 1$ | $4 - 2$ |
| 4.33 | $2 - 1$ | $2 - 1$ | $10 - 6 + 1$ |
| 4.34 | $0(1 - 2 + 1)$ | 0 | $6 - 4 + 1$ |
| 4.35 | $-1(-1 + 1 + 1 - 1)$ | $2 - 1$ | $4 - 2$ |
| 4.36 | $-2 + 0 + 1$ | $2 + 0 - 1$ | $4 + 0 - 2$ |
| 4.37 | $4 - 1 - 1$ | $6 - 1 - 2$ | $12 - 2 - 4$ |
| 4.38 | $0(1 - 2 + 1)$ | $1(2 - 4 + 2)$ | $4 - 4 + 2$ |
| 4.39 | $-1(-1 + 1 + 1 - 1)$ | $1(-2 + 4 - 2)$ | $4 - 3 + 2 - 1$ |
| 4.40 | $2 - 1$ | $2 - 1$ | $4 - 2$ |
| 4.41 | 0 | 0 | 0 |
| 4.42 | $0(1 - 1 - 1 + 1)$ | $1(-1 + 2 - 1)$ | $4 - 2$ |
| 4.43 | $4 - 2$ | $8 - 4$ | $16 - 8$ |
| 4.44 | $2 - 1$ | $0(1 - 2 + 1)$ | $2 - 2 + 1$ |
| 4.45 | $-1(-2 + 2 + 1 + 0 - 1)$ | $0(-2 + 3 + 0 - 1)$ | $12 - 6$ |
| 4.46 | 0 | $-1(-1 + 1 + 1 - 1)$ | $2 + 0 - 1$ |
| 4.47 | $-1(1 + 0 - 2 + 0 + 1)$ | $1(-1 + 4 + 4 + 0 - 1)$ | $8 - 4$ |
| 4.48 | $4 - 1 - 1$ | $2 + 1 - 2$ | $14 - 3 - 5 + 1$ |
| 4.49 | $0(1 - 2 + 1)$ | 0 | $6 - 4 + 1$ |
| 4.50 | $-1(-1 + 1 + 1 - 1)$ | $-1(-1 + 0 + 3 - 2)$ | $0 + 2 - 2$ |
| 4.51 | 0 | 0 | 0 |
| 4.52 | $2 - 1$ | $-2 + 1$ | $2 - 2 + 1$ |
| 4.53 | $4 - 2$ | $6 - 4 + 1$ | $12 - 8 + 2$ |
| 4.54 | $2 - 1$ | $-1(-2 + 3 + 0 - 1)$ | $6 - 2 - 1$ |
| 4.55 | 0 | $-4 + 2$ | $4 - 4 + 2$ |
| 4.56 | 0 | $0(2 - 4 + 2)$ | $-8 + 4$ |
| 4.57 | $-1(-1 + 1 + 1 - 1)$ | $0(-1 + 2 - 1)$ | $4 - 2$ |
| 4.58 | 0 | $-1(-1 + 1 + 1 - 1)$ | $8 - 4$ |
| 4.59 | 0 | $0(-1 + 1 + 1 - 1)$ | $8 - 4$ |
| 4.60 | $2 - 1$ | $2 - 1$ | $4 - 2$ |
| 4.61 | $2 - 1$ | $2 - 1$ | $4 - 2$ |
| 4.62 | $-2(-1 - 1 + 3 + 0 + 0 - 1)$ | $-1(-3 + 4 + 0 + 0 - 1)$ | $12 - 4 - 2$ |
| 4.63 | $-1(-2 + 3 + 0 - 1)$ | $-2(1 - 4 + 4 + 0 - 1)$ | $8 - 4$ |
| 4.64 | $0 - 1 + 1$ | $2 - 1$ | $4 - 2$ |
| 4.65 | $2 + 0 - 1$ | $2 + 0 - 1$ | $4 + 0 - 2$ |
| 4.66 | $-2(1 + 0 - 1 - 1 + 0 + 1)$ | $-2(1 - 3 + 2 + 1 - 1)$ | $8 - 3 - 2 + 1$ |
| 4.67 | $-1(1 - 1 - 1 + 1)$ | $2 - 1$ | $4 - 2$ |
| 4.68 | 0 | 0 | 0 |
|   |   |   |
|---|---|---|
| 4.69 | 2 | 2 |
| 4.70 | -1(-1 + 1 + 1 - 1) | 0(-1 + 2 - 1) |
| 4.71 | 0 | -2(-1 + 2 - 1 - 1 + 1 - 1) |
| 4.72 | 0 | -1(-1 + 1 + 1 - 1) |
| 4.73 | 4 | 2 |
| 4.74 | 2 | 1 |
| 4.75 | 0 | 2 + 0 - 1 |
| 4.76 | 0 | 4 + 0 - 2 |
| 4.77 | 0 | (-2 + 4 - 2) |
| 4.78 | -2(-1 + 1 + 1 - 1) | -2(-2 + 1 + 0 + 2 + 2 - 3) |
| 4.79 | 0(1 - 1 - 1 - 1) | 1(1 - 2 + 1) |
| 4.80 | -1(-3 + 4 + 0 + 0 - 1) | -1(-3 + 2 + 2 + 2 - 3) |
| 4.81 | 0(2 - 3 + 0 + 1) | 1(1 - 2 + 1) |
| 4.82 | 4 - 1 - 1 | 6 - 1 - 2 |
| 4.83 | -2(-1 + 0 + 2 - 1 + 1 - 1) | -2(-1 + 1 + 0 - 1 + 3 - 2) |
| 4.84 | 0 + 1 - 1 | 2 - 1 |
| 4.85 | 2 + 0 - 1 | -2(-1 + 1 + 0 + 0 + 1 - 1) |
| 4.86 | 2 + 0 - 1 | 4 - 2 |
| 4.87 | -2(-2 + 0 + 4 - 1 + 0 - 1) | -2(-4 + 2 + 4 - 1 + 2 - 3) |
| 4.88 | 1(1 - 2 + 1) | 1(-1 + 2 - 1) |
| 4.89 | 4 + 0 - 2 | 4 + 2 - 4 |
| 4.90 | 0 | 1(-2 + 4 - 2) |
| 4.91 | 4 - 1 + 0 - 1 | 4 - 1 + 2 - 3 |
| 4.92 | 2 + 0 + 0 - 1 | -2 + 2 + 2 - 3 |
| 4.93 | -1(-1 + 2 + 2 - 0 + 1) | -1(-1 + 2 + 0 - 2 + 1) |
| 4.94 | 2 | 2 |
| 4.95 | 2 + 0 + 0 - 1 | 2 + 0 + 0 - 1 |
| 4.96 | 2 + 0 - 1 | 4 + 0 - 2 |
| 4.97 | -2(1 - 1 + 1 + 0 - 0 + 1) | 0 |
| 4.98 | 0 | 0 - 1 + 2 - 1 |
| 4.99 | 0 | 8 - 4 |
| 4.100 | 4 - 2 | 8 - 4 |
| 4.101 | 2 + 0 + 0 - 1 | 2 + 0 + 0 - 1 |
| 4.102 | 0 - 1 + 0 + 1 | 4 - 1 - 2 + 1 |
| 4.103 | -2(-2 + 1 + 2 + 0 + 0 - 1) | -2(-2 + 1 + 2 + 0 + 0 - 1) |
| 4.104 | -2 + 0 + 0 + 1 | 6 - 2 - 2 + 1 |
| 4.105 | 0 | -8 + 4 |
| 4.106 | 2 + 0 - 1 | 0 + 2 - 2 |
| 4.107 | 0 | -4 + 2 |
| 4.108 | 0 | 0 |

Table 3
Appendix B. Table of $W_K(t)$, $f_{00}(D; t)$, $f_{11}(D; t)$, and $III_K(t)$.

Table 4 shows $W_K(t)$, $f_{00}(D; t)$, $f_{11}(D; t)$, and $III_K(t)$ of a virtual knot $K$ up to crossing number four with a choice of orientations. We remark that these polynomials are all reciprocal.

| $W_K(t)$ | $f_{00}(D; t)$ | $f_{11}(D; t)$ | $III_K(t)$ |
|----------|----------------|----------------|------------|
| 2.1      | 4 − 2          | −2 + 1         | −2 + 1     |
| 3.1      | 2 + 0 − 1      | 2 − 2 + 1      | 0          | 4 − 2     |
| 3.2      | 4 − 2          | 2 − 1          | −2 + 1     | 2 − 1     |
| 3.3      | 6 − 2 − 1      | −4 + 1 + 1     | −4 + 1 + 1 | 2 − 1     |
| 3.4      | 2 − 2 + 1      | 2 − 1          | 2 − 1      | 2 − 1     |
| 3.5      | 4 + 0 − 2      | −6 + 2 + 1     | −6 + 2 + 1 | −6 + 2 + 1|
| 3.6      | 0              | 0              | 0          | 0         |
| 3.7      | 4 + 0 − 2      | 2 − 2 + 1      | 2 − 2 + 1  | 2 − 2 + 1 |
| 4.1      | 8 − 4          | −8 + 2 + 2     | −4 + 1     | 0 − 2 + 2 |
| 4.2      | 2              | 0 + 2 − 2      | −4 + 2     | 0 − 2 − 2 |
| 4.3      | 8 − 4          | −8 + 4         | −10 + 4 + 1| 0          |
| 4.4      | 4 − 2          | 2 − 1          | 0 − 1 + 1  | 2 − 1     |
| 4.5      | 4 − 2          | −4 + 3 − 1     | −6 + 3     | 0 + 1 − 1 |
| 4.6      | 0              | 2 + 0 − 1      | 0          | 2 + 0 − 1 |
| 4.7      | 8 − 4          | −12 + 6        | −12 + 6    | −4 + 2    |
| 4.8      | 0              | 4 − 2          | 4 − 2      | −4 + 2    |
| 4.9      | 4 − 2          | −10 + 5        | −4 + 1 + 1 | −2 + 1    |
| 4.10     | 2 + 0 − 1      | 0              | −4 + 2     | 0         |
| 4.11     | 6 − 2 − 1      | 0 − 1 + 1      | −2 + 1     | 6 − 3     |
| 4.12     | 0              | 0              | 0          | 0         |
| 4.13     | 0              | −4 + 2         | 4 − 2      | −4 + 2    |
| 4.14     | 0 − 2 + 2      | 2 − 1          | 4 − 2 + 1  | 2 − 1     |
| 4.15     | 6 − 2 − 1      | −8 + 3 + 1     | −12 + 5 + 1| −2 + 1    |
| 4.16     | 0              | 0              | 0          | 0         |
| 4.17     | 2 + 0 − 1      | 0              | 4 − 2      | 0         |
| 4.18     | 4 − 2          | −2 + 1         | −2 + 1     | −2 + 1    |
| 4.19     | 2 + 0 − 1      | 0              | 4 − 2      | 0         |
| 4.20     | 2 − 2 + 1      | −2 + 1         | 2 − 1      | −2 + 1    |
| 4.21     | −4 + 0 + 2     | −2 + 0 + 1     | 2 − 1 + 1  | −2 + 0 + 1|
| 4.22     | 2 − 1 + 1 − 1  | 2 + 0 − 2 + 1  | 2 − 1      | 4 − 1 − 1 |
| 4.23     | 2 + 0 − 1      | −4 + 2         | 0          | −4 + 2    |
| 4.24     | −2 − 1 + 1 + 1 | −2 + 1 − 1 + 1 | −2 + 0 + 1 | 0 + 2 − 2 |
| 4.25     | 8 − 4          | −10 + 4 + 1    | −8 + 4     | −2 + 0 + 1|
| 4.26     | 0 + 1 + 0 − 1  | 6 − 3 − 1 + 1  | 4 − 2      | 6 − 2 − 1 |
| 4.27     | 4 − 2          | −2 + 1         | 0 + 1 − 1  | −2 + 1    |
| 4.28     | −4 + 1 + 0 + 1 | 2 − 2 + 0 + 1  | 0 − 1 + 1  | 6 − 3     |
| 4.29     | 6 − 2 − 1      | −6 + 1 + 2     | −8 + 3 + 1 | 6 − 3     |
| 4.30     | 4 − 2          | 0 − 1 + 1      | 2 − 1      | 0 − 1 + 1 |
| 4.31     | 0              | 4 − 2          | −4 + 2     | 4 − 2     |
| 4.32     | 2 + 0 − 1      | 4 − 2          | 0          | 4 − 2     |
\begin{tabular}{|c|c|c|c|}
\hline
4.33 & 4 - 2 & [-2 + 1] & [4 - 3 + 1] & [-2 + 1] \\
4.34 & 2 - 2 + 1 & 4 - 3 + 1 & [2 - 1] & 2 - 1 \\
4.35 & 2 + 0 - 1 & 0 & 4 - 2 & 0 \\
4.36 & -4 + 0 + 2 & -2 + 0 + 1 & [-2 + 0 + 1] & -2 + 0 + 1 \\
4.37 & 8 - 2 - 2 & -10 + 3 + 2 & [-10 + 3 + 2] & -2 + 1 \\
4.38 & 2 - 2 + 1 & -2 + 1 & 2 - 1 & -2 + 1 \\
4.39 & 2 - 1 + 1 - 1 & 0 + 0 - 1 + 1 & 0 - 1 + 1 & 2 - 1 \\
4.40 & 4 - 2 & 2 - 1 & 2 - 1 & 2 - 1 \\
4.41 & 0 & 0 & 0 & 0 \\
4.42 & 2 - 1 - 1 + 1 & 0 & 4 - 2 & 0 \\
4.43 & 8 - 4 & [-8 + 4] & [-8 + 4] & 0 \\
4.44 & 4 - 2 & [-4 + 1 + 1] & [-2 + 1] & 0 - 1 + 1 \\
4.45 & 4 - 1 + 0 - 1 & 2 - 2 + 0 + 1 & 2 - 2 + 0 + 1 & 6 - 1 - 2 \\
4.46 & 0 & 0 & 2 + 0 - 1 & 0 \\
4.47 & 0 - 1 + 0 + 1 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.48 & 8 - 2 - 2 & -6 + 2 + 0 + 1 & -12 + 3 + 3 & -6 + 2 + 0 + 1 \\
4.49 & 2 - 2 + 1 & 0 + 1 - 1 & 2 - 1 & 2 - 1 \\
4.50 & 2 + 0 - 1 & 0 & [-4 + 2] & 0 \\
4.51 & 0 & [-4 + 2] & 4 - 2 & [-4 + 2] \\
4.52 & 4 - 2 & [-2 + 1] & [4 - 3 + 1] & [-2 + 1] \\
4.53 & 8 - 4 & [-10 + 4 + 1] & [-10 + 4 + 1] & -2 + 0 + 1 \\
4.54 & 4 - 2 & 0 + 1 - 1 & [-2 + 1] & 0 + 1 - 1 \\
4.55 & 0 & 2 - 2 + 1 & 2 - 2 + 1 & 2 - 2 + 1 \\
4.56 & 0 & [-4 + 2] & [-4 + 2] & [-4 + 2] \\
4.57 & 2 + 0 - 1 & [-2 + 0 + 1] & 2 - 2 + 1 & 0 \\
4.58 & 0 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.59 & 0 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.60 & 4 - 2 & 2 - 1 & 2 - 1 & 2 - 1 \\
4.61 & 4 - 2 & [-6 + 3] & [-6 + 3] & -2 + 1 \\
4.62 & 6 - 1 - 1 - 1 & -2 + 0 + 0 + 1 & 2 - 2 + 0 + 1 & 4 - 1 - 1 \\
4.63 & 6 - 2 - 1 & 0 - 1 + 1 & [-4 + 1 + 1] & 6 - 3 \\
4.64 & 0 - 2 + 2 & 2 - 1 & 2 - 1 & 2 - 1 \\
4.65 & 4 + 0 - 2 & [-2 + 0 + 1] & [-2 + 0 + 1] & -2 + 0 + 1 \\
4.66 & -2 - 1 + 1 + 1 & 4 - 1 - 2 + 1 & 4 - 2 & 6 + 0 - 3 \\
4.67 & 2 + 0 + 1 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.68 & 0 & 0 & 0 & 0 \\
4.69 & 4 - 2 & [-6 + 3] & [-6 + 3] & -2 + 1 \\
4.70 & 2 + 0 - 1 & 2 - 2 + 1 & [-2 + 0 + 1] & 4 - 2 \\
4.71 & 0 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.72 & 0 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.73 & 8 - 4 & [-8 + 4] & [-8 + 4] & 0 \\
4.74 & 4 - 2 & [-4 + 1 + 1] & [-2 + 1] & 0 - 1 + 1 \\
4.75 & 0 & 2 + 0 - 1 & 2 + 0 - 1 & 2 + 0 - 1 \\
4.76 & 0 & [-2 + 2 - 1] & [-2 + 2 - 1] & [-2 + 2 - 1] \\
4.77 & 0 & 4 - 2 & 4 - 2 & 4 - 2 \\
4.78 & 6 - 1 - 1 - 1 & [-12 + 4 + 1 + 1] & [-8 + 2 + 1 + 1] & -6 + 3 \\
\hline
\end{tabular}
| 4.79 | $2 - 1 - 1 + 1$ | $-2 + 2 - 1$ | $2 + 0 - 1$ | $-2 + 2 - 1$ |
|------|----------------|----------------|--------------|--------------|
| 4.80 | $8 - 3 + 0 - 1$ | $-6 + 1 + 1 + 1$ | $-6 + 1 + 1 + 1$ | $2 - 2 + 1$ |
| 4.81 | $4 - 3 + 0 + 1$ | $4 - 1 - 1$ | $4 - 1 - 1$ | $4 - 1 - 1$ |
| 4.82 | $8 - 2 - 2$ | $-8 + 3 + 1$ | $-12 + 3 + 3$ | $-8 + 3 + 1$ |
| 4.83 | $4 - 1 + 0 - 1$ | $-2 + 0 + 0 + 1$ | $-2 + 1$ | $2 - 1$ |
| 4.84 | $0 + 2 - 2$ | $4 - 1 - 1$ | $0 - 1 + 1$ | $4 - 1 - 1$ |
| 4.85 | $4 + 0 + 2$ | $0 - 1 + 0 + 1$ | $0$ | $0 - 1 + 0 + 1$ |
| 4.86 | $4 + 0 - 2$ | $6 - 2 - 1$ | $2 - 2 + 1$ | $6 - 2 - 1$ |
| 4.87 | $8 - 1 - 2 - 1$ | $-10 + 2 + 2 + 1$ | $-10 + 2 + 2 + 1$ | $-2 + 1$ |
| 4.88 | $0 + 1 - 2 + 1$ | $2 - 1$ | $2 - 1$ | $2 - 1$ |
| 4.89 | $8 + 0 - 4$ | $-8 + 1 + 2 + 1$ | $-8 + 1 + 2 + 1$ | $0 + 1 - 2 + 1$ |
| 4.90 | $0$ | $4 - 2$ | $4 - 2$ | $4 - 2$ |
| 4.91 | $8 - 2 + 0 - 2$ | $-12 + 3 + 2 + 1$ | $-12 + 3 + 2 + 1$ | $-12 + 3 + 2 + 1$ |
| 4.92 | $4 + 0 + 0 - 2$ | $-10 + 2 + 2 + 1$ | $-10 + 2 + 2 + 1$ | $-10 + 2 + 2 + 1$ |
| 4.93 | $4 - 1 - 2 + 1$ | $0$ | $0$ | $0$ |
| 4.94 | $4 - 2$ | $-6 + 3$ | $-6 + 3$ | $-2 + 1$ |
| 4.95 | $4 + 0 + 0 - 2$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ |
| 4.96 | $4 + 0 - 2$ | $0$ | $-4 + 2$ | $0$ |
| 4.97 | $0 - 1 + 0 + 1$ | $2 + 0 - 1$ | $2 - 1 + 1$ | $2 + 0 - 1$ |
| 4.98 | $0$ | $4 - 2$ | $4 - 2$ | $4 - 2$ |
| 4.99 | $0$ | $8 - 4$ | $8 - 4$ | $8 - 4$ |
| 4.100 | $8 - 4$ | $-8 + 4$ | $-8 + 4$ | $0$ |
| 4.101 | $4 + 0 + 0 - 2$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ | $-2 + 0 + 0 + 1$ |
| 4.102 | $0 - 2 + 0 + 2$ | $4 - 1 + 2 + 1$ | $4 - 1 - 2 + 1$ | $4 - 1 - 2 + 1$ |
| 4.103 | $4 + 1 - 2 - 1$ | $0 - 1 + 0 + 1$ | $0 - 1 + 0 + 1$ | $4 + 0 - 2$ |
| 4.104 | $-4 + 0 + 0 + 2$ | $6 - 2 - 2 + 1$ | $6 - 2 - 2 + 1$ | $6 - 2 - 2 + 1$ |
| 4.105 | $0$ | $-8 + 4$ | $-8 + 4$ | $-8 + 4$ |
| 4.106 | $4 + 0 - 2$ | $-6 + 2 + 1$ | $-2 + 2 - 1$ | $-6 + 2 + 1$ |
| 4.107 | $0$ | $0 + 2 - 2$ | $0 + 0 - 2 + 2$ | $0 + 2 - 2$ |
| 4.108 | $0$ | $0$ | $0$ | $0$ |

**Table 4**

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