Dimensional Crossover in the Large N Limit

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Abstract: We consider dimensional crossover for an $O(N)$ Landau-Ginzburg-Wilson model on a $d$-dimensional film geometry of thickness $L$ in the large $N$-limit. We calculate the full universal crossover scaling forms for the free energy and the equation of state. We compare the results obtained using “environmentally friendly” renormalization with those found using a direct, non-renormalization group approach. A set of effective critical exponents are calculated and scaling laws for these exponents are shown to hold exactly, thereby yielding non-trivial relations between the various thermodynamic scaling functions.
§ 1. Introduction

Crossover behavior — the interpolation between qualitatively different effective degrees of freedom of a system as a function of scale — is an ubiquitous phenomenon in nature. Calculation of scaling functions associated with crossover behavior is, generally speaking, much more difficult than the calculation of critical exponents, the latter being calculable in an approximation scheme suitable for the asymptotic region around one critical point.

An important non-trivial and experimentally accessible example is seen in the context of dimensional crossover. As far as the fluctuations in a system are concerned there is a marked difference between an “environment” consisting of infinite three dimensional space and a three dimensional box of “size” $L$. By implementing a renormalization programme which is explicitly dependent on the relevant environmental parameters one obtains a globally defined RG which incorporates several fixed points at once and with which one can calculate the crossover scaling functions perturbatively using one uniform approximation scheme. Recently such a program which allows for the treatment of a wide class of crossovers has been developed [1-6] and dubbed with the epithet — “environmentally friendly” renormalization in recognition of the fact that a crossover is often induced by some relevant perturbation which can be associated with the effects of the “environment” on a system.

The gist of the approach is based on the intuition that a “good” coarse graining will be one that when effected to a length scale comparable to any set by the environment, will reflect the influence of the latter by continuously changing as a function of scale the type of effective degree of freedom being coarse grained. However, although the intuition is grounded in such Kadanoff/Wilson type coarse graining, such a procedure is notoriously difficult to implement. Instead the formalism we adopt is based on the field theoretic RG which represents the invariance under reparametrization of the couplings of the system, an idea which goes back to the original formulation of the RG in the ’50’s.

Just as there are good and bad coarse grainings so too there are good and bad reparametrizations. An environmentally friendly reparametrization is one that tracks the qualitatively changing nature of the effective degrees of freedom for a crossover system. As has been emphasized previously [6] a necessary condition for an environmentally friendly RG to satisfy is that the number of fixed points of the RG, defined globally on the space of parameters, be isomorphic to the number of points of scale invariance of the system. Unfortunately, for many crossovers, favorite forms of field theoretic renormalization, such as minimal subtraction, do not satisfy this criterion and therefore are of only limited use in describing crossover behavior.

Various new results, both formal and perturbative, have been obtained, predominantly in the context of dimensional crossover both above and below [2] the critical temperature.
The formalism has also been applied to several other crossover systems [3], and has been implemented in particle physics [4]. Two loop Padé resummed results have been obtained for the case of an $O(N)$ model on a layered geometry, and also for a quantal Ising model [5,6]. Effective exponents were calculated which asymptotically are in excellent agreement with known results [7]. Additionally, where available, there is good agreement between the results of environmentally friendly renormalization and numerical results from lattice simulations [8] and high temperature series [9] approximations.

Exactly solvable models provide a useful testing ground for ideas on phase transitions and quantum field theory. The growth of interest in conformal field theory in two dimensions, and the growing number of exactly solvable models in this same dimension, is evidence of the interest in such models. In this paper we wish to test environmentally friendly renormalization thoroughly in the context of an exact model. To date the only truly exactly solvable models, in the sense that all correlation functions are calculable, in any dimension, are the Gaussian and the Berlin-Kac spherical model [10], and variants of these. However, all higher vertex functions above two are identically zero for the Gaussian model and it has no ordered phase, pathologies absent in the spherical model. Stanley [11] established the equivalence for an infinite lattice of the partition functions of the $N = \infty$ limit of the $O(N)$ vector sigma model and the spherical model. The latter model was discussed from a field theoretic point of view by Wilson [12] whose analysis led to subsequent developments where the model served as the beginning point of a perturbative expansion in $1/N$ (see [13] for a detailed set of reprints on this topic). The original lattice spherical model was solved for both strictly finite geometries and geometries exhibiting a dimensional crossover by Barber and Fisher [14]. (See Rudnick [15] for a more recent discussion of the model in a purely finite geometry). While Allen and Pathria [16] have recently studied the models two point correlation function.

We choose as our testing ground the limit $N \to \infty$ of an $O(N)$ Landau-Ginzburg-Wilson model. This model is closely related to the spherical model and the $O(N)$ sigma model, but it contains an additional parameter $\lambda_B$ the $\varphi^4$ coupling, which away from the critical point governs the crossover to mean field behavior. It is the field theoretic formulation that we discuss in the following, our interest being the model in a film geometry. The purpose of the present paper is to present a solution of the Landau-Ginzburg-Wilson field theory variant of the spherical model from two points of view and analyze both the universal and non-universal aspects of the dimensional crossover both above and below the critical point.

The format of the paper is as follows: in section 2 we give a brief overview of the large-$N$ limit. We then derive, via a saddlepoint evaluation of the partition function, the scaling functions which incorporate the dimensional crossover and the crossover to mean field theory for the free energy and equation of state. We analyze in detail the universal scaling
limit of these functions and compare with known results for the spherical model. In section 3 we analyze the model with the environmentally friendly RG approach and demonstrate how the results of section 2 are recovered. In section 4 we calculate a set of effective exponents which are scaling functions that describe the crossovers between \(d\)-dimensional, \(d - 1\)-dimensional and mean field fixed points. In section 5 we show that the effective exponents satisfy natural analogs of the standard scaling laws, including hyperscaling. Finally we make our conclusions in section 6.

§ 2. The Large \(N\) limit.

The \(O(N)\) “microscopic” Landau-Ginzburg-Wilson Hamiltonian for a \(d\) dimensional film geometry of thickness \(L\) is given by

\[
\mathcal{H}[\varphi] = \int_0^L \int d^d x \left( \frac{1}{2} \nabla \varphi^a \nabla \varphi^a + \frac{1}{2} r_B(x) \varphi^a \varphi^a + \frac{\lambda_B}{4!} (\varphi^a \varphi^a)^2 - H^a(x) \varphi^a \right)
\]

where \(r_B(x) = m_B^2 + t_B(x)\). We will restrict our considerations to the case of a film which exhibits critical behavior \((d \geq 3)\) and make the standard assumptions that \(\lambda_B\) is temperature independent and that all the temperature dependence of the model is contained in the variable \(t_B\).

The partition function \(Z\) is obtained by performing the path integral over the order parameter fields, \(\varphi^a\), with the Hamiltonian (2.1). The generator of one particle irreducible vertex functions, \(G[\bar{\varphi}]\), where \(\bar{\varphi}^a\) is the induced magnetization, is the Legendre transform of \(W[H] = -\ln Z\). If the sources \(H^a\) and \(t_B\) are taken to be homogeneous then for a translationally invariant system \(\bar{\varphi}^a\) is also homogeneous and in the direction of \(H^a\) and \(G[\bar{\varphi}] = V \Gamma[\bar{\varphi}]\) where \(V\) is the volume. It is convenient however to retain the general case for the moment. The vertex functions \(\Gamma^{(a_1...a_N)}\) are the objects of primary interest to us as once these are known all the correlation functions of the theory can be reconstructed from them. To avoid over encumbering the notation we have dropped a conventional subscript \(B\), referring to “bare” quantities, from the fields and vertex functions.

In general for the \(O(N)\) model there are two types of modes: Those along the direction picked out by the field, \(H^a\), and those perpendicular to it. If we choose the direction of the field to be given by the unit vector \(n^a\) then using the two projectors

\[
P^{ab}_l = n^a n^b, \quad P^{ab}_t = \delta^{ab} - n^a n^b
\]

we can decompose a general vertex function into block diagonal form. We denote a generic vertex function by \(\Gamma^{(N)}_{l...t...t}\) where the number of \(l\) and \(t\) subscripts indicates whether a longitudinal or a transverse propagator is to be attached to the vertex at the corresponding
point. When all subscripts are either \( l \) or \( t \) we will use a single \( l \) or \( t \), for example \( \Gamma^{(N)}_l \) will be abbreviated \( \Gamma^{(N)}_t \).

Due to the Ward identities of the model it is sufficient to know only the \( \Gamma^{(N)}_l \) as all the other vertex functions can be reconstructed from these. Thus for example the equations of state, using the Ward identity \( \Gamma^{(1)}_l = \Gamma^{(2)}_t \bar{\phi} \), become

\[
\Gamma^{(2)}_l \bar{\phi} = H \quad \Gamma^{(1)}_t = 0 \quad (2.2)
\]

Decomposing \( \Gamma^{(ab)} \) yields \( \Gamma^{(2)}_l \), \( \Gamma^{(2)}_t \) and \( \Gamma^{(2)}_{lt} \). Ward identities imply that

\[
\Gamma^{(2)}_l = \Gamma^{(2)}_t + \frac{\Gamma^{(4)}_t}{3} \phi^2 \quad \text{and} \quad \Gamma^{(2)}_{lt} = 0 \quad (2.3)
\]

The large \( N \) limit is taken such that \( N \lambda_B \) is held fixed as \( N \to \infty \). In this setting it is possible to obtain exact expressions for the vertex functions of the theory. One can do this either by a direct resummation of the Feynman diagrams or via a saddle point approximation. In the latter approach the identity

\[
e^{-\frac{\lambda}{4} \int \phi^4} = A \int d\psi e^{\int \left(\frac{N}{2} \psi^2 - (\frac{\lambda N}{12}) \frac{1}{2} \psi \phi^2\right)} \quad (2.4)
\]

where \( A \) is a normalization constant, allows one to perform the now Gaussian \( \phi \) integral, which yields the effective Hamiltonian

\[
\mathcal{H}_{eff} \equiv \mathcal{H}_{eff}(\psi, H) = -\int \left\{ \frac{N}{2} \psi^2 + \frac{1}{2} \frac{H^2}{m_B^2 + t_B + \sqrt{N \lambda_B} \phi} \right\} + \frac{NV}{2} \bigcirc_{\Lambda} \quad (2.5)
\]

We use the diagrammatic notation of [6], where

\[
\bigcirc_{\Lambda} = \frac{T}{V} \ln \left[ -\nabla^2 + m_B^2 + t_B + \sqrt{N \lambda_B} \phi \right] \quad (2.6)
\]

and \((-1)^k / (k-1)!\) times the \( k \)'th derivative with respect to \( t_B \) will be represented by a circle with \( k \) dots, the dots representing the point at which each derivative acts. The subscript \( \Lambda \) represents the presence of an ultraviolet cutoff \( \Lambda \) that regulates the diagram. In the large \( N \) limit the integral over the auxiliary field \( \psi \) can be done in a saddle point approximation, the saddle point condition being

\[
\frac{\partial \mathcal{H}_{eff}}{\partial M^2} = 0 \quad (2.7)
\]
where $M^2 = m_B^2 + t_B + \psi \sqrt{\frac{N\lambda_B}{3}}$. The effective Hamiltonian $H_{eff}$ evaluated on the solution of (2.7) then yields the leading large $N$ behavior of $W[H, M^2]$ and a Legendre transform yields $\Gamma[\bar{\varphi}, M^2] + \Gamma_{reg}[t_B, \Lambda] = W + H\bar{\varphi}$ where
\[
\bar{\varphi} = -\frac{\partial W}{\partial H}.
\]
which we have split into a singular part $\Gamma[\bar{\varphi}, M^2]$, which vanishes at the bulk critical point, and a remaining regular part $\Gamma_{reg}[t_B, \Lambda]$. The function $\Gamma[\bar{\varphi}, M^2]$ determines the singular part of the free energy density to which it is related by $k_B T$.

For the translationally invariant case with homogeneous $H$ the saddle point constraint (2.7) together with (2.2) lead to
\[
M^2 \bar{\varphi} = H
\]
and
\[
M^2 = m_B^2 + t_B + \frac{\lambda_B}{6} \bar{\varphi}^2 + \frac{N\lambda_B}{6} \circ \Lambda
\]
The former is just the equation of state and the latter specifies the relationship between temperature and the transverse correlation length $M^{-1}$.

Using (2.10) the singular part of $\Gamma$ can be expressed in the form
\[
\Gamma = \frac{3M^4}{2\lambda_B} + \frac{N}{2} (\circ - M^2 \circ )
\]
while the regular part is given by
\[
\Gamma_{reg}[t_B, \Lambda] = -\frac{3(m_B^2 + t_B)^2}{2\lambda_B} + d_d \Lambda^d
\]
In (2.11) any cutoff dependence of the diagrams is subtracted so that they vanish at the bulk critical point. From equation (2.10) one finds
\[
\left. \frac{dM^2}{d\bar{\varphi}} \right|_{t_B} = \frac{\lambda_B}{1 + \frac{N\lambda_B}{6} \circ} \bar{\varphi}
\]
and
\[
\left. \frac{dM^2}{dt_B} \right|_{\bar{\varphi}} = \frac{1}{1 + \frac{N\lambda_B}{6} \circ}
\]
We have further the vertex functions
\[
\Gamma^{(0,1)} = \frac{3M^2}{\lambda_B}
\]
\[
\Gamma^{(0,2)} = \frac{3}{\lambda_B} \frac{1}{1 + \frac{N\lambda_B}{6} \circ}
\]
\[
\Gamma^{(0,3)} = \frac{N \Box}{(1 + \frac{N\lambda_B}{6} \Box)^2} \tag{2.17}
\]

and

\[
\Gamma_l^{(2)} = M^2 + \frac{dM^2}{d\varphi} \bar{\varphi} \tag{2.18}
\]

where we focus on the singular parts of these vertex functions only.

For later convenience, with \( H = 0 \), we define \( \mathcal{E} = \Gamma^{(0,1)} \) and \( \mathcal{C} = -\Gamma^{(0,2)} \). When the temperature dependence of \( t_B \) is taken to be \( t_B = \Lambda^2 \frac{t - T_c}{T} \), we have \( \mathcal{E} \) and \( \mathcal{C} \) proportional to the internal energy and specific heat respectively [17]. Given the Ward identities we need only specify the even transverse vertex functions \( \Gamma_l^{(N)} \) since the others can be obtained from these. Thus

\[
\Gamma_l^{(2)} = M^2 \tag{2.19}
\]

\[
\Gamma_l^{(4)} = \frac{\lambda_B}{1 + \frac{\lambda_B N}{6} \Box} \tag{2.20}
\]

\[
\Gamma_l^{(6)} = \frac{5}{9} \frac{\lambda_B^3 N \Box}{(1 + \frac{\lambda_B N}{6} \Box)^3} \tag{2.21}
\]

Note that the diagrammatic structure of all expressions is the same irrespective of whether we are treating the bulk problem, a film geometry, or the case of a completely finite geometry. However, the functions represented by the diagrams differ in the different cases. In the dimensional crossover problem each diagram depends explicitly on \( L \), the film thickness. For a \( d \)-dimensional layered geometry \((d < 4)\) with periodic boundary conditions in the infinite cutoff, \( \Lambda \rightarrow \infty \), limit

\[
\Box(M^2, L) = \frac{(d - 3)}{2} \sigma_d \sum_{n = -\infty}^{\infty} \frac{1}{[M^2 + (\frac{2\pi n}{L})^2]^{(5-d)/2}} \tag{2.22}
\]

where

\[
\sigma_d = -\frac{\Gamma(\frac{2-d}{2})}{(4\pi)^{d/2}} \tag{2.23}
\]

This diagram is well defined for \( d < 4 \) and \( M \neq 0 \) and can be used to specify the other diagrams from which it is derived by differentiation. The first of these is the tadpole, \( \Box \), which when required to vanish at the bulk critical point is given by

\[
\Box(M^2, L) = -\int_0^{\frac{L^2}{h^2}} \Box(x, L)dx + \frac{b_d}{L^{d-2}} \tag{2.24}
\]

where

\[
b_d = \frac{\Gamma(\frac{d-2}{2})\zeta(d-2)}{2\pi^{d/2}}. \tag{2.25}
\]
The other is the vacuum diagram which in the same limit and again required to vanish at the bulk critical point \((M = 0 \text{ and } L = \infty)\) is

\[
\Box(M^2, L) = - \int_0^M dx \int_0^x dy \Box(y, L) + \frac{b_d}{L^{d-2}} M^2 - \frac{a_d}{L^d}
\]  

(2.26)

where

\[
a_d = \frac{2\Gamma(d/2)\zeta(d)}{\pi^{d/2}}
\]

(2.27)

is a universal number associated with the critical theory defined by

\[
\frac{1}{2} a_d = L^d \left\{ \Gamma|T_c(\infty) - \Gamma|T_c(L) \right\}
\]

(2.28)

The critical temperature is that temperature for which \(M = 0\) and \(\bar{\varphi} = 0\). It is therefore determined by the zero of the right hand side of (2.10) with \(\bar{\varphi} = 0\). Now, since \(t_B\) by definition becomes zero at \(T_c(L)\), (2.10) determines \(m_B^2\) to be such as to cancel the last term in (2.24) and any cutoff dependence that would have given the diagram a non-zero value at the bulk critical point. This of course implies that \(m_B^2\) depends on \(L\) and can be written as \(m_B^2 = r_B - \Delta(L)\) where \(\Delta(L)\) vanishes for \(L \rightarrow \infty\) and \(r_B\) is \(L\) independent. From (2.24) we see that \(\Delta(L) = \frac{N\lambda_B}{6} \frac{b_d}{L^{d-2}}\). Likewise we can decompose \(t_B\) into an \(L\) independent part which would then vanish at \(T_c(\infty)\) and an \(L\) dependent part.

It is not difficult to see that the corresponding decomposition of \(t_B\) yields \(t_B(\infty) + \Delta(L)\). With the temperature dependence

\[
t_B(\infty) = \Lambda^2 \left( \frac{T - T_c(\infty)}{T} \right)
\]

(2.29)

discussed in [17], and the fact that \(t_B(\infty)\) vanishes at the bulk critical temperature we see that

\[
\Lambda^2 \frac{T_c(\infty) - T_c(L)}{T_c(L)} = \Delta(L) = \frac{N\lambda_B}{6} \frac{b_d}{L^{d-2}}
\]

(2.30)

and so \(\Delta(L)\) is proportional to the shift in critical temperature of the film from the bulk critical temperature. Since \(b_d\) is positive we see that the film critical temperature is suppressed relative to the bulk one and scales with the shift exponent \(d-2 = 1/\nu(d)\), \(\nu(d)\) being the bulk correlation exponent all of which is in agreement with the lattice results of Barber and Fisher [14]. Furthermore since \(b_d\) diverges at \(d = 3\) we see that for a three dimensional film the critical temperature \(T_c(L)\) is driven to zero and more careful analysis is appropriate.

In terms of the variable \(t_B\) (2.10) becomes

\[
M^2 = t_B + \frac{\lambda_B}{6} \bar{\varphi}^2 + \frac{N\lambda_B}{6} \Box'
\]

(2.31)
where $\bigcirc' = -\int_0^M dx \bigcirc(x, L)$. Normalizing so that $\Gamma$ vanishes at the bulk critical temperature $T = T_c(\infty)$ with zero external field $H = 0$ we have

$$\Gamma = \frac{3M^4}{2\lambda B} + \frac{N}{2} \left( \bigcirc' - M^2 \bigcirc' - \frac{a_d}{L^d} \right)$$

(2.32)

where $\bigcirc' = -\int_0^M dx \int_0^x dy \bigcirc(y, L, \infty)$.

Explicitly, one finds

$$\Gamma[M, L] = \frac{N}{2} \left\{ \frac{3M^d}{g} + \frac{G(d, z) - a_d}{L^d} \right\}$$

(2.33)

where $g = N\lambda BM^{d-4}$, $z = LM$, and the scaling function $G(d, z)$ for periodic boundary conditions is given by

$$G(d, z) = z^{d-1} \sigma_{d-1} \left\{ \frac{d-3}{d-1} - \frac{4}{d-1} \sum_{n=1}^\infty \left\{ \left[ 1 + \left( \frac{2\pi n}{z} \right)^2 \right]^{(d-1)/2} - \left( \frac{2\pi n}{z} \right)^{(d-1)/2} \right\} \right\}$$

(2.34)

which makes the small $z$ behavior manifest. A convenient integral representation of $G$ is

$$G(d, z) =\frac{d-2}{d} \sigma_d z^d + z^{d-1} \left\{ \frac{d-3}{d-1} \sigma_{d-1} \right.$$ 

$$\left. -\frac{4}{(4\pi)^{(d-1)/2} \Gamma(d-1)} \int_0^\infty dk k^{d-4} \left( k^2 + \frac{d-3}{2} \right) \ln \left[ \frac{1 - e^{-kz}}{k} \frac{\sqrt{1 + k^2}}{1 - e^{-\sqrt{1 + k^2}z}} \right] \right\}$$

For $d = 3$ we find the simpler result

$$G(3, z) = \frac{z^3}{12\pi} + \frac{1}{2\pi} \int_0^z dy \frac{y^2}{e^y - 1}$$

(2.35)

with $a_3 = \zeta(3)/\pi$.

The transverse correlation length $M^{-1}$ is determined in terms of $t_B$ and $\varphi$ by

$$\frac{3M^2}{\lambda_B} = \frac{3t_B}{\lambda_B} + \frac{1}{2} \varphi^2 - \frac{NM^{d-2}}{2} F(d, z)$$

(2.36)

where $F = -\frac{\bigcirc'}{M^{d-2}}$ and for periodic boundary conditions is given explicitly by

$$F(d, z) = \frac{\sigma_{d-1}}{z} \left\{ 1 + 2 \sum_{n=1}^\infty \left\{ \left[ 1 + \left( \frac{2\pi n}{z} \right)^2 \right]^{(d-3)/2} - \left( \frac{2\pi n}{z} \right)^{(d-3)} \right\} \right\}$$

$$= \sigma_d + \frac{1}{z} \left( \sigma_{d-1} - \frac{4}{(4\pi)^{(d-1)/2} \Gamma(d-3/2)} \int_0^\infty dk k^{d-4} \ln \left[ \frac{1 - e^{-kz}}{k} \frac{\sqrt{1 + k^2}}{1 - e^{-\sqrt{1 + k^2}z}} \right] \right)$$

(2.37)
Re-arranging the constraint equation (2.36) we can express it in terms of \( \tau = \frac{6t_B(\infty)}{N\lambda_B} \) with \( t_B(\infty) \) the \( L \) independent temperature parameter which vanishes at the bulk critical temperature and \( \tilde{\varphi} = \varphi/\sqrt{N} \). We then have (2.36) in the form

\[
w \equiv (\tau + \frac{b_d}{L^{d-2}} + \tilde{\varphi}^2)L^{d-2} = \frac{z^{d-2}}{g} + z^{d-2}F(d, z) \tag{2.38}
\]

and we see that \( \tau(L) = \frac{6t_B}{N\lambda_B} \) is given by

\[
\tau(L) = \tau + \Delta(L) \quad \text{with} \quad \Delta(L) = \frac{b_d}{L^{d-2}} \tag{2.39}
\]

If we consider the small \( \lambda_B \) limit we obtain mean field results, while the universal scaling regime is governed by the limit \( \lambda_B \to \infty \) in which case the universal form of the free-energy per component \( \tilde{\Gamma} \) is given by

\[
\tilde{\Gamma} = \frac{G(d, z) - a_d}{2L^d} \tag{2.40}
\]

The universal form of \( M[\tau, \tilde{\varphi}, L] \) is determined by

\[
w = Q^{-1}(z^2) = z^{d-2}F(d, z) \tag{2.41}
\]

In terms of the basic scaling variables

\[
\tilde{x} = (\tau + \frac{b_d}{L^{1/\nu(d)}})|\tilde{\varphi}|^{-1/\beta(d)} \quad \text{and} \quad \tilde{y} = L|\tilde{\varphi}|^{\nu(d)/\beta(d)} \tag{2.42}
\]

with \( \nu(d) = \frac{1}{d-2} \) and \( \beta(d) = \frac{1}{2} \) the bulk \( d \) dimensional exponents

\[
w = (1 + \tilde{x})\tilde{y}^{1/\nu(d)} \tag{2.43}
\]

Note that \( \tilde{x} \) could equally well be expressed in the form \( \tilde{x} = (b_d + \tau L^{1/\nu(d)})\tilde{y}^{-1/\nu(d)} \) so in general there is a choice of variables in which the scaling function can be expressed. For the large \( N \) limit, irrespective of whether we consider the universal limit or not, we see that in general only the combination \( \tau + |\tilde{\varphi}|^{1/\beta} \) plays a rôle. This has significant consequences for the effective exponents to be considered later, since there is a reduction from two variable scaling functions to scaling in terms of the single variable \( w = (|\tilde{\varphi}|^{1/\beta} + \tau)L^{1/\nu} + b_d \).

Equation (2.41) implies that\

\[
z^2 = Q(d, w) \tag{2.44}
\]

Substitution of (2.44) into (2.40) yields the scaling function \( G(d, w) \) for \( \tilde{\Gamma} \). Finally the equation of state is given by

\[
Q(d, w)\tilde{\varphi}L^{-1/\beta(d)} = \tilde{H} \tag{2.45}
\]

* More generally if we do not take the universal limit from (2.38) we have the more general two variable scaling form \( z^2 = Q(d, v, w) \) where \( v = N\lambda_B L^{d-4} \).
where $\tilde{H} = H/\sqrt{N}$. The asymptotic forms of the equation of state then become: For $L \to \infty$

\[
\sigma_d^{-\gamma(d)}(1 + \tilde{x})^{\gamma(d)}\varphi(d) = \tilde{H} \tag{2.46}
\]

and for $L \to 0$

\[
\left(\frac{L}{\sigma_d'}\right)^{\gamma(d')} (1 + \tilde{x})^{\gamma(d')}\varphi(d') = \tilde{H} \tag{2.47}
\]

where $\delta(d) = (d + 2)/(d - 2)$ and $d' = d - 1$. Both limiting forms (2.46) and (2.47) agree with the usual universal form of the equation of state [18] aside from the factors of $\sigma_d^{-\gamma(d)}$ and $(\frac{L}{\sigma_d'})^{\gamma(d')}$ which could be absorbed into a redefinition of $\varphi$ and $\tilde{H}$. We choose not to absorb dimension dependent or $L$ dependent factors into our variables as we are interested in a problem involving two dimensions at once with $L$ the interpolating physical variable.

Similarly $\tilde{\Gamma}$ interpolates between

\[
\tilde{\Gamma} = \rho_d(1 + \tilde{x})^{2-\alpha(d)}\varphi(2-\alpha(d))/\beta(d) \tag{2.48}
\]

for $L \to \infty$ and

\[
\tilde{\Gamma} = L^{1-\alpha(d')} \rho_{d'}(1 + \tilde{x})^{2-\alpha(d')}\varphi(2-\alpha(d'))/\beta(d') + \frac{a_d}{2L^d} \tag{2.49}
\]

for $z \to 0$ where

\[
\rho_d = \frac{\alpha(d)}{2} \sigma_d^{\alpha(d)-1} \quad \text{and} \quad \alpha(d) = \frac{d - 4}{d - 2}.
\]

For $d$ approaching three, we saw that the film critical temperature $T_c(L)$ is driven to zero as $b_d$ diverged with a simple pole at $d = 3$. However, $\sigma_{d-1}$ also diverges with a simple pole in this limit and in fact the scaling function $Q^{-1}(d, z)$ yields a simple pole divergence also. If we examine (2.24) more carefully we see that

\[
\bigcirc (M^2, L) = -\frac{M}{4\pi} - \frac{1}{2\pi L} \ln[1 - e^{-z}] \tag{2.50}
\]

and so the pole contributions cancel and we are left with the universal form of the constraint (2.38) in the form

\[
(\tau + \varphi^2)L = \frac{z}{4\pi} + \frac{1}{2\pi} \ln[1 - e^{-z}] \tag{2.51}
\]

in agreement with [14], who restricted their considerations to the zero field case, $H = 0$. From (2.51) we see

\[
Q(3, w) = \left\{2 \ln \left[\frac{e^{2\pi w} + \sqrt{e^{4\pi w} + 4}}{2}\right]\right\}^2 \tag{2.52}
\]
where we find it convenient to define \( w = (\tau + \varphi^2)L \). Then (2.52) together with (2.45) specifies the universal equation of state. Similarly (2.40) with (2.35) and (2.52) specifies \( \tilde{\Gamma} \).

For \( L \rightarrow \infty \) we have \( w \rightarrow \infty \) and we recover the three dimensional scaling function (2.46) discussed above, and for fixed \( L \) with \( \xi_L = M^{-1} \rightarrow \infty \) (2.51) gives

\[
w = \frac{1}{2\pi} \ln z \quad (2.53)
\]

so in the two dimensional critical regime which is governed by \( \tau \rightarrow -\infty \) we find

\[
\xi_L = L e^{-2\pi(\tau + \varphi^2)L}. \quad (2.54)
\]

The limiting form of the equation of state becomes

\[
e^{4\pi(\tau(L) + \varphi^2)} \varphi = \tilde{H} \quad \text{where} \quad \tau(L) = \tau - \frac{1}{2\pi L} \ln L \quad (2.55)
\]

in agreement with [14].

The other special dimension of interest is \( d = 4 \) where \( \circ \) has an ultraviolet divergence. In this case it is inappropriate to simply send \( \lambda_B \rightarrow \infty \) to recover the universal properties. For \( d = 4 \), when \( \lambda_B \) is sent to infinity in such a way as to cancel the divergent contribution from \( \circ(4, M, L, \Lambda) \) and render \( \Gamma^{(4)}_t \) finite we find the constraint retains the logarithmic corrections to scaling and becomes

\[
(\tau + \frac{1}{12L^2} + \varphi^2)L^2 = \frac{z^2}{(4\pi)^2} \ln(z^2) + \frac{z}{4\pi} \left\{ 1 - \frac{2}{\pi} \int_0^\infty dk \ln \left[ \frac{1 - e^{-kz}}{k} \right] \sqrt{1 + k^2} \right\} \quad (2.56)
\]

where \( z_0 = \kappa L \) with \( \kappa \) a remnant microscopic scale, such that \( M \ll \kappa \). Similarly the free energy scaling function is given by

\[
G(4, z) = \frac{z^4}{32\pi^2} (1 - \ln \frac{z^2}{z_0^2}) - \frac{1}{3\pi^2} \int_0^\infty dq q^2 \left\{ \frac{q^2 + 3}{\sqrt{q^2 + z^2}} \left( e^{q^2/2} - 1 \right) - \frac{1}{e^q - 1} \right\} \quad (2.57)
\]

with \( a_4 = -1/45 \). For \( M \rightarrow 0 \) with fixed \( L \) we recover the three dimensional results above, and for \( L \rightarrow \infty \) the constraint becomes

\[
\frac{(\tau + \varphi^2)}{\kappa^2} = \frac{1}{(4\pi)^2 \kappa^2} \ln \left( \frac{M^2}{\kappa^2} \right) \quad (2.58)
\]

while \( \tilde{\Gamma} \) becomes

\[
\tilde{\Gamma} = \frac{M^4}{64\pi^2} \left( 1 - \ln \frac{M^2}{\kappa^2} \right). \quad (2.59)
\]

To conclude, we have found that the universal limit \( \lambda_B \rightarrow \infty \) of the continuum Landau-Ginzburg-Wilson model recovers the results of Barber and Fisher for the spherical model.
Our results further interpolate between the integer dimensions they considered and we incorporate the presence of a homogeneous external field $H$ and the crossover to mean field theory. We found the scaling functions for the free energy and equation of state.

§ 3. Environmentally Friendly Renormalization

The purpose of this section is to use renormalization group techniques to recover the scaling functions in the large $N$ limit. We will treat the problem from the perspective of environmentally friendly renormalization. As before we assume that the finite system also exhibits critical behavior and that $3 \leq d \leq 4$. We will restrict our considerations to a film geometry with periodic boundary conditions, the geometry of interest being $S^1 \times \mathbb{R}^{d-1}$. Some of the analysis has appeared in other work [6], however, for the sake of completeness we will review the techniques and summarize the main results.

We indicate by $t_B(M)$ that temperature parameter which yields the transverse correlation length $\xi_L = M^{-1}$. The renormalization group method is to change from the original bare parameters to new renormalized ones. The renormalized parameters and vertex functions are related to the bare ones by

$$t(M, \kappa) = Z_{\varphi^2}^{-1} t_B(M) \quad \lambda(\kappa) = Z_\lambda(\kappa) \lambda_B \quad \varphi(\kappa) = Z_{\varphi^2}^{-1/2} \varphi_B$$

and

$$\Gamma^{(N,L)} = Z_{\varphi^2}^N Z_{\varphi^2}^L \lambda_B^{(N,L)} + \delta_{N0} \delta_{Ln} A^{(n)}(\kappa) \quad i = 0, 1, 2 \quad (3.1)$$

where $\kappa$ is an arbitrary renormalization scale. The renormalized vertex functions then obey the RG equation

$$\kappa \frac{d}{d\kappa} \Gamma^{(N,L)} + \left( L \gamma_{\varphi^2} - \frac{N}{2} \gamma_{\varphi^2} \right) \Gamma^{(N,L)} = \delta_{N0} \delta_{Ln} B^{(n)} \quad (3.2)$$

where the Wilson functions in the above are $\gamma_{\varphi} = \frac{d \ln Z_\varphi}{d \ln \kappa}$ and $\gamma_{\varphi^2} = -\frac{d \ln Z_{\varphi^2}}{d \ln \kappa}$. The final Wilson function is $\gamma_{\lambda} = \frac{d \ln Z_\lambda}{d \ln \kappa}$ and is related to the beta function, $\beta(\lambda)$ through the relation $\gamma_{\lambda} = \beta(\lambda)/\lambda$. Equation (3.2) is inhomogeneous for the three vertex functions $\Gamma$, $\Gamma^{(0,1)}$ and $\Gamma^{(0,2)}$; which are related to the free energy, internal energy and the specific heat respectively. By an appropriate set of normalization conditions the Wilson functions are related to the anomalous dimensions of the operators $\varphi$ and $\varphi^2$ respectively.

The normalization conditions which fix the particular parameterization we will use to describe physical quantities are,

$$\Gamma_t^{(2)} \bigg|_{NP} = \kappa^2 \quad (3.3)$$

$$\frac{\partial}{\partial k^2} \Gamma_t^{(2)} \bigg|_{NP} = 1 \quad (3.4)$$
where the subscript \( t \), introduced earlier, refers to the transverse components of the respective vertex functions, and \( NP \) refers to the normalization point. A simplifying feature in considering the large \( N \) limit is that the separate dependence on \( \varphi \) and \( t_B \) disappears into a dependence on the transverse correlation length. We choose our normalization point to be at zero momentum and transverse correlation length, \( \kappa^{-1} \). Away from the large \( N \) limit explicit account must be taken of the fact that one has a two scale rather than a one scale problem. The general \( O(N) \) case will be dealt with elsewhere.

In the normalization conditions (3.3–3.6) the film thickness is treated as a fixed passive variable. As has been emphasized on previous occasions [1,6] such “environmentally friendly” conditions are essential in order to obtain a perturbatively controllable description of the finite size crossover. The conditions (3.3–3.6) above specify the Wilson functions \( \gamma_{\varphi^2}, \gamma_{\varphi} \) and \( \gamma_{\lambda} \), which are explicitly \( L \) dependent and interpolate between those of a \( d \) and \( d-1 \) dimensional model as the ratio of film thickness to transverse correlation length ranges from infinity to zero.

The condition (3.3) is just the spherical constraint which we analyzed earlier and is therefore of a somewhat different status to the rest, serving to fix the relationship between the parameter \( \kappa \) and the physical variables \( t, \varphi \) and \( \lambda \). Essentially (3.4-3.6) determine the Wilson functions in terms of an arbitrary, fiducial transverse correlation length \( \kappa^{-1} \), while Eq. (3.3) determines the relationship between this correlation length and the temperature and magnetization.

In general we saw that it is sufficient to know the properties of the transverse vertex functions as the longitudinal ones could be generated directly from the Ward identities. The task now is to exhibit the relationship between the transverse correlation length, temperature and magnetization and thereby generate the equation of state for comparison with the results of section 2. To achieve this it is convenient to start with the differential statement

\[
d\Gamma^{(2)}_t = \Gamma^{(2,1)}_t(\varphi^2 dt + \frac{1}{6} \Gamma^{(4)}_t d\varphi^2
\]

(3.7)

If we integrate this relation along a contour of constant magnetization, \( \varphi \), from the critical isotherm \( t = 0 \) we obtain

\[
\Gamma^{(2)}_t(t, \varphi) = \Gamma^{(2)}_t(0, \varphi) + \int_0^t \Gamma^{(2,1)}_t(t', \varphi)dt'
\]

(3.8)
From the definitions of the Wilson functions implied by the normalization conditions (3.3–3.6) on inverting the relation (3.8) we find

\[
t + \int_0^{M(0, \bar{\varphi})} x dx (2 - \gamma_\varphi(x)) e^{-\int_x \gamma_\varphi(x') \frac{dx'}{2}} = \int_0^{M(t, \bar{\varphi})} x dx (2 - \gamma_\varphi(x)) e^{-\int_x \gamma_\varphi(x') \frac{dx'}{2}} \tag{3.9}
\]

Above we have the relation \( M = M(t, \bar{\varphi}) \), but parametrically in terms of \( M(0, \bar{\varphi}) \). We can determine the latter as an explicit function of \( \bar{\varphi} \) by integrating along the critical isotherm from the critical point. If we integrate (3.7) along the critical isotherm up from the critical point we find

\[
\Gamma^{(2)}(t, \bar{\varphi}) = \int_0^{\bar{\varphi}^2} \frac{\Gamma^{(4)}(x)}{6} dx = \int_0^{\bar{\varphi}^2} \frac{\Gamma^{(4)}(x)}{6} dx \tag{3.10}
\]

By choosing the normalization point \( NP \) in (3.3–3.6) on the critical isotherm such that the transverse correlation length is \( \kappa^{-1} \) and at zero momentum, we can express \( \Gamma^{(4)}_t \) in the form

\[
\Gamma^{(4)}_t = \lambda(\kappa) \exp \int_0^{M(0, \bar{\varphi})} (\gamma_\lambda(x) - 2 \gamma_{\varphi^2}(x)) \tag{3.11}
\]

where \( \gamma_\lambda(x) \) and \( \gamma_{\varphi^2}(x) \) are the resulting Wilson functions from this prescription. Inverting (3.10) one finds

\[
\bar{\varphi}^2 = \frac{6}{\lambda} \int_0^{M(0, \bar{\varphi})} (2 - \gamma_\varphi(x)) e^{-\int_0^x (\gamma_\lambda(x') - \gamma_{\varphi^2}(x')) \frac{dx'}{2}} dx \tag{3.12}
\]

The two equations (3.9) and (3.12) specify completely the relation between the transverse correlation length, temperature and magnetization. Finally, since the transverse correlation length is infinite on the co-existence curve, i.e. \( M(t_{\text{coex}}, \bar{\varphi}) = 0 \) the equation of the co-existence curve is given by

\[
t + \int_0^{M(0, \bar{\varphi})} x dx (2 - \gamma_\varphi(x)) e^{-\int_x \gamma_\varphi(x') \frac{dx'}{2}} = 0 \tag{3.13}
\]

where \( M(0, \bar{\varphi}) \) as a function of \( \bar{\varphi} \) is determined by equation (3.12).

The specific heat and the energy density are proportional to \( \Gamma^{(0,2)} \) and \( \Gamma^{(0,1)} \) respectively. We can treat these in a similar fashion to the above by beginning with the differential relation

\[
d\Gamma^{(0,2)} = \Gamma^{(0,3)} dt + \frac{1}{2} \Gamma^{(2,2)} d\bar{\varphi}^2 \tag{3.14}
\]

By integrating along a contour of constant \( \bar{\varphi} \) up from the co-existence curve we obtain

\[
\Gamma^{(0,2)} = \int_0^{M(t, \bar{\varphi})} (2 - \gamma_\varphi(x)) e^{2 \int_0^x \gamma_\varphi(x') \frac{dx'}{2}} dx^d - 5 dx \tag{3.15}
\]
and
\[ \Gamma^{(0,1)} = \int_0^x (2 - \gamma \varphi(x)) e^{\int_x^x \gamma \varphi^2(x') \, dx'} \int_0^x (2 - \gamma \varphi(x')) e^{2 \int_x^x \gamma \varphi^2(x') \, dx'} \Gamma^{(0,3)}(x') \, x'^{d-5} \, dx' \]
(3.16)

where
\[ \bar{\Gamma}^{(0,3)}(M) = \frac{\Gamma^{(0,3)}(\Gamma^{(2)})^3}{(\Gamma^{(2,1)})^3 M^d} \]
(3.17)

The advantage of integrating up from the co-existence curve is that we can extract the singular part by requiring that both \( \Gamma^{(0,1)} \) and \( \Gamma^{(0,2)} \) vanish there. We can impose such boundary conditions only at the critical point in the special case of when \( \alpha < 0 \) which is the case for the large \( N \) limit. A boundary condition at some other point is also possible but the formulae are more complicated.

For the large \( N \) limit we can evaluate the function \( \bar{\Gamma}^{(0,3)} \) exactly. We find diagrammatically that
\[ \bar{\Gamma}^{(0,3)} = N \frac{\mathcal{O}}{M^{d-6}} \]
(3.18)
which can be related to derivatives of the scaling function \( \mathcal{F} \). It is now a simple matter of performing the relevant integrations to recover the renormalization group results for these functions.

§§ 3.1 Equation of State

In the large \( N \)-limit one loop RG expressions become exact. Diagrammatically the renormalization constants \( Z_{\varphi}, Z_{\varphi^2} \) and \( Z_\lambda \) are
\[ Z_{\varphi^2} = Z_\lambda = 1 - \frac{\lambda_B N}{6} \mathcal{O}, \quad Z_{\varphi} = 0 \]
(3.19)

Note that \( \mathcal{O} \) is an explicit function of \( z = \kappa L \), with \( \kappa \) the fiducial finite size correlation length used as a running parameter.

In terms of the floating coupling \([1]\) \( h \), chosen so as to make the coefficient of the quadratic term in the resulting \( \beta \) function unity, one finds
\[ \beta(h, z) = -\varepsilon(z)h + h^2 \]
(3.20)

and
\[ \gamma_{\varphi^2}(h, z) = \gamma_\lambda(h, z) = h, \quad \gamma_{\varphi} = 0 \]
(3.21)

where the function \( \varepsilon(z) \) is
\[ \varepsilon(z) = \frac{6z^2 \mathcal{O}(z)}{\mathcal{O}(z)} - 2. \]
(3.22)
Similarly for the critical isotherm we have

$$\tilde{\gamma}_\varphi^2(h, z) = \tilde{\gamma}_\lambda(h, z) = h, \quad \tilde{\gamma}_\varphi = 0$$  \hspace{1cm} (3.23)$$

Equations (3.21) and (3.23) implies that \(\tilde{\gamma}_\lambda = \gamma_\varphi^2\) and hence with (3.12) we find (3.9) becomes

$$t_0 + \frac{\lambda_0 \varphi^2}{6} = \frac{2}{L^2} \int_0^z \frac{dx}{x} e^{\int_{z_0}^x (2-h(x')) \frac{dx'}{x}}$$  \hspace{1cm} (3.24)$$

where \(h\), the solution of (3.20), is

$$h(z, z_0, h_0) = \frac{e^{-\int_{z_0}^z \epsilon(x) \frac{dx}{x}}}{h_0^{-1} - \int_{z_0}^z \epsilon(x') \frac{dx'}{x'}} \hspace{1cm} (3.25)$$

with \(z_0 = \kappa_0 L\), \(\lambda_0 = \lambda(\kappa_0)\) and \(t_0 = t(M, \kappa_0)\). The steepest descent constraint of the large \(N\) limit has now been recovered in the form (3.24).

In (3.25) the initial coupling is specified at the “microscopic” scale \(\kappa_0\). For \(d < 4\) this microscopic scale can be sent to infinity while maintaining \(h_0\) finite. A universal floating coupling, \(h(z) = \frac{4z^2 \Theta(z)}{\Theta(z)}\), which is the separatrix solution of the differential equation is obtained. If this solution is used in (3.24) we obtain

$$t_0 + \frac{\lambda_0 \varphi^2}{6} = \frac{N \lambda_0}{6} M^{d-2} F(d, z)$$  \hspace{1cm} (3.26)$$

where \(\lambda_0 = \frac{6}{N \Theta(\kappa_0)}\) is the initial dimensional coupling corresponding to the separatrix solution. In the asymptotic regime (2.22) implies \(\lambda_0 = \frac{12 \kappa_0^{-d}}{N(d-2) \sigma_d}\). We see that we have recovered from our renormalization group arguments the universal form of the spherical constraint (2.41) where now \(\tau(L) = \frac{6 t_0}{\lambda_0}\). If one is interested in corrections to scaling, as is usually the case in comparing with experimental data, then \(\kappa_0\) should be left finite and fitted to the data. The cases of \(d = 3\) and \(d = 4\) require special care. For \(d = 4\) one can not ignore \(h_0\) but again when appropriate care is taken one recovers the results of the previous section.

For periodic boundary conditions

$$\epsilon(z) = 5 - d - (7 - d) \sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{z^2} \left( 1 + \frac{4\pi^2 n^2}{z^2} \right)^{\frac{d-2}{2}} \left( 1 + \frac{4\pi^2 n^2}{z^2} \right)^{\frac{d-2}{2}}$$  \hspace{1cm} (3.27)$$

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and the separatrix coupling

\[ h(z) = (5 - d) \sum_{n=-\infty}^{\infty} \frac{(1 + (\frac{2\pi n}{z})^2)^{\frac{(d-7)}{2}}}{\sum_{n=-\infty}^{\infty} (1 + (\frac{2\pi n}{z})^2)^{\frac{(d-5)}{2}}} \]  

(3.28)

For \( d = 3 \) the results are particularly simple

\[ h(z) = 1 + \frac{z}{\sinh z} \]  

(3.29)

\[ \varepsilon(z) = 1 + \frac{z^2 \coth(\frac{z}{2})}{\sinh z + z} \]  

(3.30)

With the explicit results above one recovers (2.37) for \( \mathcal{F} \) and hence \( Q \).

§ 4. Effective Exponents in the Large \( N \) limit

We will now put to use the formulae derived in the previous sections to calculate various scaling functions which describe the crossovers between \( d \) dimensional and \( d-1 \) dimensional critical behavior, and the crossover to mean field theory. A very useful way of representing a large class of scaling functions is in terms of effective critical exponents, which are functions that interpolate between the constant critical exponents associated with the different asymptotic regimes that characterize the crossover.

First of all we define, for \( T > T_c(L) \) and \( H = 0 \), an effective critical exponent \( \nu_{\text{eff}} \)

\[ \nu_{\text{eff}} = -\frac{d\ln \xi_L}{d\ln t_B} \bigg|_{H=0} \]  

(4.1)

where \( \xi_L \) is the correlation length associated with the transverse dimensions, i.e. the correlation length in the infinite dimensions (remember that \( t_B \sim T - T_c(L) \)). As \( \frac{1}{\xi_L} = M \) one finds from (2.13) that

\[ \nu_{\text{eff}} = \frac{1}{2} \left( \frac{1 - \frac{\lambda_6 N}{6M^2}}{1 + \frac{\lambda_6 N}{6}} \right) \]  

(4.2)

Explicitly

\[ \nu_{\text{eff}} = \left( \frac{1 + \frac{6}{g\mathcal{F}}}{d - 2 + \frac{d\ln \mathcal{F}}{d\ln z} + \frac{12}{g\mathcal{F}}} \right) \]  

(4.3)

where \( z = ML \) and the scaling function \( \mathcal{F} \), for periodic boundary conditions is given by (2.37).
As the coupling constant $\lambda_B$ is present in the scaling function it is not universal in the normal sense of the word. The coupling $g$ governs the corrections to scaling about the critical theory. In the limit $g \to 0$ one crosses over to mean field theory where $\nu_{\text{eff}} \to \frac{1}{2}$. In the critical regime, where $z \ll g$ and $1 \ll g$, the terms proportional to $g^{-1}$ may be neglected yielding

$$
\nu_{\text{eff}} = \left( d - 2 + \frac{d \ln \mathcal{F}}{d \ln z} \right)^{-1} \quad (4.4)
$$

which is now a true universal scaling function. We can also think of getting rid of the corrections to scaling by keeping a cutoff $\Lambda$ in the diagrams and making an appropriate choice of the non-universal coupling $\lambda_B$. To see this we divide the integration range $[0, \Lambda]$ into two parts $[0, \infty]$ and $[\infty, \Lambda]$. In the limit $\Lambda L \gg 1$ the integrand in the integration over the range $[\infty, \Lambda]$ can be approximated by the "bulk" expression to yield

$$
M^2 \frac{N \lambda_B}{3(4\pi)^{d/2}} \Gamma\left(\frac{d}{2}\right) \Lambda^{d-4} \quad (4.5)
$$

to corrections $O(\exp(-\Lambda L))$. Hence, the choice $\lambda_B = \frac{3(4-d)}{N} \frac{d}{2} \Gamma\left(\frac{d}{2}\right) \Lambda^{d-4}$ will eliminate the corrections to scaling. From (4.4) we see that as $z \to 0$ then $\nu_{\text{eff}} \to \frac{1}{(d-3)}$ whereas for $z \to \infty$, $\nu_{\text{eff}} \to \frac{1}{(d-2)}$. Thus $\nu_{\text{eff}}$ interpolates between the two exact asymptotic values associated with the spherical model in $d$ and $d-1$ dimensions.

We can also consider an effective exponent $\gamma_{\text{eff}} = \frac{-d \ln \chi}{d \ln |t_B|}$ where $\chi$ is the susceptibility for $H = 0$. As trivially $\chi = M^{-2}$, one obtains

$$
\gamma_{\text{eff}} = 2 \left( 1 + \frac{6}{g \mathcal{F}} \right)
$$

In the mean field limit $\gamma_{\text{eff}} \to 1$ whereas we have $\gamma_{\text{eff}} \to \frac{2}{(d-3)}$ as $z \to 0$ and $\gamma_{\text{eff}} \to \frac{2}{(d-2)}$ as $z \to \infty$. Thus once again we see the effective exponent interpolates between the known asymptotic fixed points of the model.

If we consider what happens for $T < T_c(L)$ and $H = 0$, the effective exponents corresponding to $\nu_{\text{eff}}$ and $\gamma_{\text{eff}}$ are ill defined, however, the effective exponent $\beta_{\text{eff}} = \frac{d \ln \bar{\varphi}}{d \ln |t_B|}$ is well defined. From the saddle point equation (2.10), as $\bigcirc'$ vanishes, due to the vanishing of the transverse mass on the coexistence curve, we see from (3.26) that

$$
\varphi^2 = \frac{6t}{\lambda} \quad (4.6)
$$

which implies that $\beta_{\text{eff}} = 1/2$, i.e. there is no crossover as one proceeds along the coexistence curve. This is in strong distinction to the Ising model where there is a crossover between the critical point and the strong coupling discontinuity fixed point at $T = 0$. The contrast is due to the fact that the coexistence curve is a line of first order transitions for $N = 1$ and a line of continuous transitions for $N > 1$. 

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Considering now the approach to the critical point as a function of field, $H$, on the critical isotherm, $T = T_c(L)$, we define an effective exponent $\delta_{eff} = \left. \frac{d \ln H}{d \ln \bar{\phi}} \right|_{t_B=0}$. This implies that

$$\delta_{eff} = 1 + \frac{\Gamma_t^{(4)} \bar{\phi}^2}{3 \Gamma_t^{(2)}} \quad (4.7)$$

From (2.9) at $T = T_c(L)$ one finds

$$\delta_{eff} = \left( \frac{3 + \frac{\lambda_B N}{6} \bigcirc - \frac{\lambda_B N}{3 M^2} \bigcirc'}{1 + \frac{\lambda_B N}{6} \bigcirc} \right)$$

or in terms of the universal scaling function $F$

$$\delta_{eff} = \left. \left( d + 2 + \frac{d \ln F}{d \ln z} \right) \right|_{t_B=0} + \left. \left( \frac{36}{g F} \right) \right|_{t_B=0}$$

$$\delta_{eff} = \left. \left( \frac{d + 2 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0} + \left. \left( \frac{12}{g F} \right) \right|_{t_B=0}$$

$$\delta_{eff} = \left. \left( \frac{d + 2 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0}$$

$$\delta_{eff} = \left. \left( \frac{d - 2 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0}$$

In the mean field limit $\delta_{eff} \to 3$ whilst in the universal limit, when we eliminate the corrections to scaling, we find

$$\delta_{eff} = \left. \left( \frac{d - 2 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0}$$

The specific heat and the energy density may also be discussed in terms of effective exponents. We may define, $\alpha_{cs}^e = \frac{d \ln C}{d \ln t_B}$ and $1 - \alpha_{cs}^e = \frac{d \ln E}{d \ln t_B}$, where $C$ is the specific heat and $E$ is the energy density for $H = 0$ respectively. From the definition of $\alpha_{cs}^e$ we have

$$\alpha_{cs}^e = 1 - \frac{t_B \Gamma_t^{(0,2)}}{\Gamma_t^{(0,1)}} \quad (4.11)$$

and using (2.15) and (2.16) we get diagrammatically

$$\alpha_{cs}^e = \lambda_B N \left( \bigcirc + \frac{\bigcirc'}{M^2} \right) \left( \frac{1}{6} \bigcirc \right)$$

$$\alpha_{cs}^e = \left. \left( \frac{d - 4 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0} + \frac{12}{g F}$$

$$\alpha_{cs}^e = \left. \left( \frac{d - 4 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0} + \frac{12}{g F}$$

In the mean field limit $\alpha_{cs}^e \to 0$ as expected, whilst eliminating the corrections to scaling yields the universal function

$$\alpha_{cs}^e = \left. \left( \frac{d - 4 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0} + \frac{12}{g F}$$

$$\alpha_{cs}^e = \left. \left( \frac{d - 4 + \frac{d \ln F}{d \ln z}}{d - 2 + \frac{d \ln F}{d \ln z}} \right) \right|_{t_B=0} + \frac{12}{g F}$$
Turning to the specific heat, diagrammatically

\[ \alpha_{\text{eff}}^s = -\frac{\lambda_B N}{3} M^2 \left( 1 - \frac{\lambda_B N}{6} \right) \left( 1 + \frac{\lambda_B N}{6} \right)^2 \]  

and in terms of the scaling function \( \mathcal{F} \)

\[ \alpha_{\text{eff}}^s = \left( 1 + \frac{6}{g\mathcal{F}} \right) \left( (d-2)(d-4) + 4(d-2) \frac{d\ln \mathcal{F}}{dz} + \frac{4z^4}{\mathcal{F} d(z)^2} \right) \left( d-2 + \frac{d\ln \mathcal{F}}{dz} + \frac{12}{g\mathcal{F}} \right)^2 \]  

This also vanishes in the mean field limit. The corresponding universal scaling function without corrections to scaling is

\[ \alpha_{\text{eff}}^s = \frac{((d-2)(d-4) + 4(d-2) \frac{d\ln \mathcal{F}}{dz} + \frac{4z^4}{\mathcal{F} d(z)^2})}{(d-2 + \frac{d\ln \mathcal{F}}{dz})^2} \]  

It is easy to verify that the two scaling functions interpolate between exactly the same asymptotic limits and that in particular the universal ones interpolate between \( \alpha(d') \) and \( \alpha(d) \) as \( z \) ranges from zero to infinity. For \( T < T_c \) on the coexistence curve the singular parts of the energy density and the specific heat are identical and thus either \( \alpha_{\text{eff}}^e \) or the amplitudes associated with these exponents are zero in this regime. In the current model it is the latter that is in fact the case.

§ 5. Effective Exponent Scaling Laws

In this section we wish to make some observations concerning certain algebraic relations between the effective exponents. First of all from equations (4.3), (4.5) and the fact that \( \eta_{\text{eff}} \equiv 0 \) due to the vanishing of \( \gamma_\phi \) in the large \( N \) limit, we see that the following relation is valid

\[ \gamma_{\text{eff}} = \nu_{\text{eff}} (2 - \eta_{\text{eff}}) \]  

Note that this is true even for the scaling functions that include the crossover to mean field theory as well as the universal crossover between the \( d \) and \( d - 1 \)-dimensional critical points. From the expressions for \( \gamma_{\text{eff}}, \alpha_{\text{eff}} \) and \( \beta_{\text{eff}} \) we see that

\[ \alpha_{\text{eff}}^e + 2\beta_{\text{eff}} + \gamma_{\text{eff}} = 2 \]  

also in the general case. The exponent \( \alpha_{\text{eff}}^s \) however, does not satisfy this relation. We see then that direct analogs of the normal scaling laws between exponents hold, however, here the relations are between entire scaling functions.

It is natural to ask if there are analogs of other scaling laws, in particular hyperscaling where \( 2 - \alpha = \nu d \). From equations (4.3), (4.13) and (4.16) it is clear that for fixed
dimensionality $d$, $2 - \alpha^i_{\text{eff}} \neq \nu_{\text{eff}} d$ where $i = e, s$. However, one can define the notion of an effective dimensionality $d^i_{\text{eff}}$ such that an effective hyperscaling law is valid. Defining $d^e_{\text{eff}} = (2 - \alpha^e_{\text{eff}})/\nu_{\text{eff}}$ one finds

$$d^e_{\text{eff}} = \left( \frac{d + \frac{d \ln \mathcal{F}}{d \ln z} + \frac{24}{g^2}}{1 + \frac{6}{g^2}} \right)$$

(5.3)

In the mean field limit $d^e_{\text{eff}} \to 4$, i.e. the upper critical dimension as one might expect. In the limit $z \to \infty$, one finds $d^e_{\text{eff}} \to d$, whilst in the limit $z \to 0$ for fixed $L$, $d^e_{\text{eff}} \to (d - 1)$. Of course one could also define an effective dimensionality via $\alpha^s_{\text{eff}}$ as $d^s_{\text{eff}} = (2 - \alpha^s_{\text{eff}})/\nu_{\text{eff}}$. The expression is unwieldy so we will not write it down. Again it interpolates between 4, $d$ and $d - 1$ in the appropriate asymptotic limits. However since the exponent $\alpha^s_{\text{eff}}$ did not satisfy the scaling law (5.2) we do not expect $d^s_{\text{eff}}$ to satisfy corresponding laws and in fact it doesn’t.

We might enquire now as to the validity of other scaling laws that involve the dimensionality explicitly. Noticing that $d^e_{\text{eff}} = 2 + \frac{1}{\nu_{\text{eff}}}$ (i.e. $\nu_{\text{eff}} = 1/(d^e_{\text{eff}} - 2)$) one arrives at the scaling relation

$$\beta_{\text{eff}} = \frac{\nu_{\text{eff}}}{2}(d^e_{\text{eff}} - 2 + \eta_{\text{eff}})$$

(5.4)

In like fashion one finds that

$$\delta_{\text{eff}} = \left( \frac{d^e_{\text{eff}} + 2 - \eta_{\text{eff}}}{d^e_{\text{eff}} - 2 + \eta_{\text{eff}}} \right)$$

(5.5)

We conclude here then that the effective exponents for the $N \to \infty$ limit of an $O(N)$ model satisfy natural analogs of the normal scaling laws, including hyperscaling, if we introduce the notion of an effective dimension $d^e_{\text{eff}}$. What are we to make of this? In the standard case of a single critical point the scaling laws tell us that out of all the critical exponents, which remember are numbers, only two are independent, hence a specification of two exponents, and the dimensionality of the system to account for hyperscaling, is sufficient. In the limit $N \to \infty$ we know that knowledge of only one exponent is necessary due to the triviality of $\eta$. From a field theoretic RG point of view we can rephrase this by saying that only one renormalization constant is needed; to renormalize the operator $\varphi^2$, no renormalization of $\varphi$ is required. In the crossover case, where one has to implement a global, non-linear RG as opposed to simply linearizing around a given fixed point, we are being told that of the six scaling functions $\alpha^e_{\text{eff}}, \nu_{\text{eff}}, \gamma_{\text{eff}}, \eta_{\text{eff}}, \delta_{\text{eff}}$ and $\beta_{\text{eff}}$ only one is independent, and that all relevant information can be encoded in the one function $d^e_{\text{eff}}$.

§ 6. Conclusions

In this paper we studied dimensional crossover for a $d$ dimensional film with periodic boundary conditions in an exactly solvable model — the large $N$ limit of an $O(N)$ Landau-Ginzburg-Wilson model, a model that also includes crossover to mean field behavior. We
obtained the scaling forms of the free energy and equation of state and extracted their universal limits where the crossover to mean field theory is eliminated. We studied the model using both direct methods and the techniques of environmentally friendly renormalization and were therefore able to compare this RG approach with an exact solution obtained by other means.

In the RG approach the equation of state was found by choosing normalization conditions at a fiducial value of the transverse correlation length and integrating it along particular contours in the \((t, \varphi)\) phase diagram; first along a contour of constant magnetization and second along the critical isotherm. The first was equivalent to a coordinate change \((t, \varphi) \to (M, \varphi)\). Both contours together yielded a complete description of the phase diagram of the model.

A natural set of effective exponents were defined and computed which also exhibited both dimensional crossover and the crossover to mean field exponents. By taking the universal limit we obtained effective exponents for the universal dimensional crossover. Finally we noticed that the effective exponents in the model obeyed natural analogs of the standard scaling laws. In the case of hyperscaling this was only true if one defined an effective dimensionality \(d_{eff}\) which interpolated between the two asymptotic limits \(d\) and \(d-1\). In this particular model it was found that all the effective exponents could be written as functions of \(d_{eff}\) and hence that there was really only one independent scaling function.

A natural question is whether or not the effective exponent laws extend to the case \(N \neq \infty\). The validity of effective exponent laws was discussed in the wider context of environmentally friendly renormalization in \([6]\), and using RG arguments it was shown there that certain scaling relations between thermodynamic functions exist which are a natural generalization of the scaling laws to crossover problems.

From a theoretical point of view the underlying object governing all effective exponents, whether derived from the energy \(E\) or the specific heat \(C\), is the singular part of the free energy, \(k_B T \tilde{\Gamma}\). Other thermodynamic quantities are derived from \(\tilde{\Gamma}\) by differentiation. Singular ones then have two asymptotic scaling forms, for example a thermodynamic function \(P\) has the form \(P \sim A_d^\pm(L)|T - T_c(L)|^{-\theta_d'} + S_d L^{-\varphi(d)}\) in the neighborhood* of \(T_c(L)\), and for \(L \to \infty, T \to T_c(\infty)\) it takes the form \(P \sim A_d^\pm|T - T_c(\infty)|^{-\theta_d}\). Quite generally one can decompose the scaling function \(P\) throughout the crossover into the form

\[
P = A^\pm(t, L) \exp\left(-\int_1^t \theta(x) \frac{dx}{x}\right) + S(t, L)
\]  

(6.1)

where at the respective asymptotic end points \(A^\pm\) gives the amplitude, \(\theta\) the exponent and \(S^\pm\) any shift that may have arisen in the thermodynamic function. However, in the

* Generally for continuous transitions one would expect \(S_d^+ = S_d^-\) and we assume this here.
crossover region this division is somewhat arbitrary. In the case of an effective exponent defined as the logarithmic derivative of \( P_s = P - S_d/L^{\varphi(d)} \) with respect to \( T - T_c(L) \) the decomposition is forced to take the form

\[
P = A_d^\pm(L) \exp \left( \int^t_1 \theta_{eff}(x) \frac{dx}{x} \right) + S_d L^{-\varphi(d)}
\]  

(6.2)

In the large \( N \) limit we have found that the above decomposition has the further property that for an appropriate definition of effective exponents, all the usual scaling relations are obeyed.

We see that the basic reason such scaling laws are obeyed in the large \( N \) limit is because the basic building function \( \tilde{\Gamma} \) involves \( t \) and \( \varphi \) only in the combination

\[
w = (\varphi^2 + \tau)L^{1/\nu} + b_d
\]

so that derivatives with respect to \( \tau \) and with respect to \( \varphi \) are intimately related. This explains why the energy rather than the specific heat provided the effective exponent that yielded the scaling laws. It is difficult to expect such a reduction to one scaling variable for more general models. Rather the two variables \( \tilde{x} \) and \( \tilde{y} \) should enter separately.

In general the decomposition (6.2) may not be possible, and is not expected to give effective exponents that obey scaling laws. However, we can choose a rather natural division into amplitude and floating effective exponent in the form (6.1) where the floating effective exponents do obey all the usual scaling relations including hyperscaling. A particularly convenient choice is that associated with separatrix exponents as advocated in [2] where the basic building functions for the exponents were \( \gamma_{\varphi}, \gamma_{\varphi^2} \) and \( \gamma_{\lambda} \) evaluated on the separatrix solution of the RG flow that connects the \( d \) and \( d' \) fixed points. Under such a division the amplitudes are non-singular functions of \( t \) and \( L \) that interpolate between the \( d \) and \( d' \)-dimensional amplitudes and the exponents capture all the singular behavior in the scaling functions. The shift is unaffected by this choice and retains the form \( S_d L^{-\varphi(d)} \).

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REFERENCES

[1] Denjoe. O’Connor and C. R. Stephens, Nucl. Phys. B360 (1991) 297; J.Phys. A25 (1992) 101.

[2] F. Freire, Denjoe O’Connor and C.R. Stephens, J. Stat. Phys. 74 (1994) 219.

[3] C.R. Stephens, Jou. Mag. Mag. Mat. 104-107 (1992) 297; Denjoe O’Connor and C. R. Stephens, Proc. Roy. Soc. 444 (1994) 287.
[4] Denjoe O’Connor, C.R. Stephens and F. Freire, *Mod. Phys. Lett.* **A8** (1993) 1779; M.A. van Eijck, Denjoe O’Connor and C.R. Stephens, *Int. J. Mod. Phys.* **A23** (1995) 3343.

[5] Denjoe O’Connor and C. R. Stephens, *Phys. Rev. Lett.* **72** (1994) 506.

[6] Denjoe O’Connor and C.R. Stephens, *Int. J. Mod. Phys.* **A9** (1994) 2805.

[7] G.A. Baker, B.G. Nickel and D.I. Meiron, *Phys. Rev.* **B17** (1978) 1365.

[8] K. Binder, *Phase Transitions and Critical Phenomena* Vol. **5B**, edited by Domb and Green (1976).

[9] T.W. Capehart and M.E. Fisher, *Phys. Rev.* **B13**, (1976) 5021.

[10] T.H. Berlin and M. Kac, *Phys. Rev.* **89** (1952) 821.

[11] H.E. Stanley, *Phys. Rev.* **176** (1968) 718.

[12] K.G. Wilson, *Phys. Rev.* **D7** (1973) 2911.

[13] *The Large N Limit in Quantum Field Theory and Statistical Physics* edited by E. Brézin and S.R. Wadia, World Scientific 1993.

[14] M. Barber and M.E. Fisher, *Ann. Phys.*, **77** (1973) 1.

[15] J. Rudnick, *Finite Size Scaling and Numerical Simulations of Statistical Systems* edited by V. Privman, World Scientific 1990.

[16] S. Allen and R.K. Pathria, *Phys. Rev.* **B50** (1994) 6765.

[17] F. Freire, Denjoe O’Connor and C.R. Stephens, *Phys. Rev.* **E53** (1996) 22.

[18] G.S. Joyce, *Phase Transitions and Critical Phenomena* Vol. **2** edited by C. Domb and M.S. Green, Academic Press 1972.