Locally EFX Allocations Over a Graph

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Abstract

The fairness notion of envy-free up to any good (EFX) has recently gained popularity in the fair allocation literature. However, few positive results about EFX allocations exist, with the existence of such allocations alone being an open question. In this work, we study an intuitive relaxation of EFX allocations: our allocations are only required to satisfy the EFX constraint for certain pre-defined pairs of agents. Since these problem instances can be represented using an undirected graph where the vertices correspond to the agents and each edge represents an EFX constraint, we call such allocations graph EFX (G-EFX) allocations. We show that G-EFX allocations exist for three different classes of graphs — two of them generalize the star $K_{1,n-1}$ and the third generalizes the three-edge path $P_3$. We also present and evaluate an algorithm using problem instances from Spliddit to show that G-EFX allocations are likely to exist for larger classes of graphs like long paths $P_n$.

1 Introduction

The problem of fairly allocating a set of indivisible goods among agents with preferences has been extensively studied by the EconCS community [Walsh, 2021]. Several notions of fairness have been proposed and analyzed in the last two decades; of all these notions, arguably the most compelling one is that of envy-freeness. In an envy-free allocation of goods, no agent prefers the set of goods allocated to any other agent to their own. Unfortunately, with indivisible goods, an envy-free allocation is not guaranteed to exist: consider an example with two agents and one indivisible good.

Two natural relaxations of envy-freeness have been explored in the literature — allocations that are envy-free up to one good (EF1) and allocations that are envy-free up to any good (EFX).

An allocation is EF1 if whenever an agent envies another agent, the envy can be eliminated by removing some item from the other agent’s allocated bundle. An allocation is EFX if whenever an agent envies another agent, the envy can be eliminated by removing any item from the other agent’s allocated bundle. EF1 allocations are guaranteed to exist for any instance of the fair allocation problem, and can in fact be computed in polynomial time [Lipton et al., 2004]. On the other hand, the existence of EFX allocations remains one of the biggest open questions in this subfield.

We introduce and study a relaxation of the EFX criterion where agents are represented by vertices on a fixed graph and allocations only need to satisfy the EFX constraint for all neighboring pairs of agents in the graph. This relaxation reduces to the usual notion of an EFX allocation when the underlying graph is complete.

In addition to being a generalization of EFX allocations, this model is also quite natural, as it captures envy under partial information. In the real world, agents typically do not envy other agents whose allocated bundles they are unaware of. In these cases, it suffices to only consider pairs of agents who are aware of each other and therefore can know each other’s allocated bundles. This problem corresponds precisely to a graph defined on the set of agents, with edges between agents who know each other.
1.1 Previous Work

EFX allocations were first introduced by Caragiannis et al. [2019b], who noted that EFX is likely the best relaxation of envy-freeness for indivisible goods.

EFX allocations are known to exist in certain special cases, either by restricting the number of agents or by placing constraints on the valuation functions of the agents. Plaut and Roughgarden [2020] present a number of useful results for EFX allocations, most notably EFX algorithms for \( n \) agents with identical general valuations, \( n \) agents with additive valuations where all agents rank items in the same order, and for two agents with arbitrary general valuations. Chaudhury et al. [2020] show that EFX allocations always exist for three agents with additive valuations, in a constructive proof. Their impressive analysis is highly involved and challenging to generalize to more than three agents. Mahara [2021] shows that EFX allocations exist for two types of agents with general valuations, again constructively, following similar techniques to Chaudhury et al. [2020]. Suksompong [2020] proves that for two agents with general valuations there are always at least two different EFX allocations, and this bound is tight. Amanatidis et al. [2021] prove that the maximum Nash welfare solution is EFX when valuations are additive and each good can have one of two possible values, and they provide a polynomial-time algorithm for EFX allocations in this special case. Babaioff et al. [2021] show that EFX allocations exist when agents have submodular valuations with binary marginal gains.

There have been a few variants of this problem which are centered on approximately EFX allocations. Plaut and Roughgarden [2020] define a \( c \)-EFX allocation as one in which for any pair of agents, either can drop any good from their allocated bundle, and be within a multiplicative factor of \( c \) from the other agent’s valuation for their bundle. They demonstrate an algorithm to achieve a \( 1/2 \)-EFX allocation. Amanatidis et al. [2020] subsequently show a single algorithm that achieves an allocation that is EFX, \((\phi - 1)\)-EFX, \( \frac{2}{\phi+2} \)-GMMS, and \( \frac{2}{3} \)-PMMS for additive valuations, where \( \phi \approx 1.618 \) is the golden ratio, and PMMS and GMMS are two other standard fairness criteria which formalize the well-known cut-and-choose protocol. Amanatidis et al. [2021] also present an alternative approximation rule for EFX allocations, called \( \alpha \)-vEFX, and show that the maximum Nash welfare solution is in fact a \( 1/2 \)-vEFX allocation.

Another line of work has sought incomplete EFX allocations, where the algorithm can decide not to allocate some of the goods. Caragiannis et al. [2019a] aim for high Nash welfare and EFX, showing that a possibly incomplete allocation exists that is EFX and achieves at least half of the maximum possible Nash welfare on all goods. Chaudhury et al. [2021b] give an algorithm such that no more than \( n \) goods go unallocated. Chaudhury et al. [2021a] bound the number of goods that remain unallocated using a graph theoretic function on the natural numbers called the rainbow cycle number. Berger et al. [2021] improve these results for four agents by constructing an allocation that leaves at most one good unallocated in that setting.

Another approach has been to study the probability of an EFX allocation existing under random valuations. Manurangsi and Suksompong [2021] show that when agents have additive valuations for individual items drawn independently at random, EFX allocations are likely to exist for any number of agents and goods.

A few other works have modeled fairness concepts using graphs. One line of work specifies a graph structure over divisible or indivisible goods and insists that agents’ bundles correspond to connected subgraphs [Bouveret et al., 2017, Bilò et al., 2022, Igarashi and Peters, 2019, Bei et al., 2021, Bei and Suksompong, 2021, Bei et al., 2020, Tucker-Foltz, 2021]. Other works have considered graph structures over agents. Beynier et al. [2019] investigate envy-free housing allocation over a graph, where agents receive one good each and must be envy-free. Bredereck et al. [2022] likewise consider envy-free allocations over a graph, with the goal of determining in polynomial or parametrized polynomial time if an envy-free allocation exists on the graph. Aziz et al. [2018] assume that agents can only view the allocations of adjacent agents in a graph. They seek allocations that are epistemically envy-free, where no vertex envies its neighbors, and furthermore, for any
vertex $x$, there is an allocation of the remaining goods (other than the ones allocated to $x$ and its neighbors) to the other agents so that $x$ does not envy any other agent. A few papers study scenarios which explicitly limit the operations of distributed algorithms to a graph structure, and aim to satisfy notions such as envy-freeness or maximum egalitarian welfare Beynier et al. [2018], Lange and Rothe [2019], Eiben et al. [2020], Kaczmarczyk and Niedermeier [2019], Varricchione [2021]. Although these works have similar motivation to ours, in that they consider a natural setting where agents only care about other agents if they are adjacent to them in some network, none of them to our knowledge addresses the question of finding EFX allocations on a graph. We believe this is a major step towards (dis)proving the general existence of EFX allocations.

1.2 Organization

This paper is arranged as follows. In Section 2, we set up the problem formally and define the notation we will use for the remainder of the paper. Our main theoretical results are stated in Section 3, with two similarly-structured subsections. In Section 3.1, we start with a warm-up result to show EFX allocations on stars (Proposition 3.1), and then go on to generalize it in two ways (Theorems 3.2 and 3.4). We also prove that these algorithms are highly efficient in the case of additive valuations in Theorem 3.6. Similarly, in Section 3.2, we start with a warm-up result to show EFX allocations on three-edge paths (Proposition 3.7), and then generalize this result (Theorem 3.8). In Section 4, we summarize some empirical approaches to generalizations of the problem, including an algorithm (Section 4.1) that seems to work for all instances of the problem if the underlying graph is a path, generated using real-world data from the Spliddit platform. We conclude with some discussions and directions for future research in Section 5.

2 Preliminaries and Notation

We have a set of $n$ agents $N = \{1, 2, \ldots, n\}$ and $m$ goods, $M = \{g_1, g_2, \ldots, g_m\}$. Each agent $i$ has a valuation function $v_i : 2^M \to \mathbb{R}_+ \cup \{0\}$ over the set of goods. We present results for two kinds of valuation functions. We call a valuation function general if the only constraint placed on it is monotonicity, i.e., for any $S \subseteq T \subseteq M$, $v_i(S) \leq v_i(T)$. We call a valuation function additive if the value of each subset $S \subseteq M$ is equal to the sum of the values of the goods in $S$, i.e. $v_i(S) = \sum_{g \in S} v_i(g)$. We abuse notation slightly and write $v_i(g)$ instead of $v_i(\{g\})$ for readability.

For agents $i, j \in N$ and goods $g_k, g_l \in M$, we sometimes write $g_k \triangleright_i g_l$ to mean $v_i(g_k) > v_i(g_l)$ (i.e. agent $i$ prefers good $g_k$ to $g_l$). We define $S \triangleright T$ analogously for subsets $S, T \subseteq M$. Agents $i$ and $j$ are said to have identical valuation functions iff for all $S \subseteq M$, $v_i(S) = v_j(S)$. Agents $i$ and $j$ with additive valuations are said to have consistent valuation functions iff for all $g_k, g_l \in M$, $g_k \triangleright_i g_l$ iff $g_k \triangleright_j g_l$ (i.e. the two agents have the same preference orders for the goods, but not necessarily the same valuations). Identical valuations are consistent, but the converse is not necessarily true.

An allocation is a partition of the set of goods $M$ to agents $N$, represented by a tuple $X = (X_1, X_2, \ldots, X_n)$ where $X_i$ is the subset of $M$ received by agent $i$. We typically refer to $X_i$ as the bundle that has been allocated to agent $i$. For a bundle $X_i$ and good $g$, we will sometimes write $X_i + g$ or $X_i - g$ to denote $X_i \cup \{g\}$ or $X_i \setminus \{g\}$ respectively. Given an allocation $X$, we say an agent $i$ envies an agent $j$ if $X_j \triangleright_i X_i$. An agent $i$ strongly envies an agent $j$ if there exists some good $g \in X_j$ such that $X_i - g \triangleright_i X_i$. We sometimes say the strong envy in this case equals $\max\{\max_{g \in X_i} (v_i(X_j - g) - v_i(X_i)), 0\}$. An allocation without any strong envy (i.e. where the strong envy is zero for all pairs $i, j \in N$) is an EFX allocation.

Let $G = (N, E)$ be an (undirected) graph on $n$ vertices $1, \ldots, n$, where the set of vertices corresponds to the set of agents $N$. Our goal is to output an allocation $X$ of the set of goods $M$ among the agents $N$ such that there is no edge $(i, j) \in E$ with agent $i$ strongly envious of agent $j$. Informally, we wish to allocate the set of goods $M$ among the agents, who correspond to vertices of $G$, but we
Figure 1: An example of a star graph and its generalizations. Each node $i$ in any of the graphs above is labeled by the valuation function $v_i$ of the corresponding agent. The valuation functions $v_{c_1}, v_{c_2}, v_{c_3}$ and $v_{c_4}$ are consistent.

only care about maintaining the EFX criterion along each of the edges. We call such an allocation a $G$-EFX allocation. In general, we will interchangeably talk about an agent $i$ and the vertex $i$ of $G$.

As a remark, note that this is a very natural generalization of the usual notion of EFX allocations, which corresponds to the case when the graph $G$ is the complete graph $K_n$.

We conclude this section with two simple observations:

- If $G$ has more than one connected component, it suffices to solve the allocation problem for any one of those components, say $C$, as that same allocation is trivially EFX on all of $G$. This is because the subgraph $G - C$ will consist of agents with no goods allocated to them, which is trivially EFX on $G - C$.

- If $G$ consists of at most three vertices, then an EFX allocation certainly exists on $G$, as in fact $K_2$ and $K_3$ are known to have EFX allocations [Plaut and Roughgarden, 2020, Chaudhury et al., 2020].

Therefore, without loss of generality, for the remainder of this paper, we restrict our attention to connected graphs with $n \geq 4$ vertices. We remark here that complete, exact EFX allocations are only known for these graphs when all agents have consistent valuations, when all agents have one of two different types of valuations, when each item can take one of two possible values, or when valuations are submodular with binary marginal gains [Plaut and Roughgarden, 2020, Mahara, 2021, Amanatidis et al., 2021, Babaioff et al., 2021].

3 Theoretical Results

In this section, we will prove that $G$-EFX allocations exist for three classes of graphs $G$. We will begin by showing $G$-EFX algorithms on two especially simple graphs – the star $K_{1,n-1}$, and the three-edge path $P_4$. Our three main results are generalizations of these two simple algorithms to more complex classes of graphs.

3.1 The Star and its Generalizations

A star consists of a “central” vertex and an arbitrary number of “outer” vertices, each with an edge only to the central vertex (see Figure 1a for the example of $K_{1,5}$). For uniformity, we consider stars with $n - 1$ outer vertices, to maintain consistency with the fact that there are $n$ agents in total. We start with a warm-up problem.
Proposition 3.1. For all $n \geq 1$, when $G$ is the star $K_{1,n-1}$, a G-EFX allocation exists for agents with arbitrary general (not necessarily additive) valuations.

Proof. Let the center of the star correspond to agent $c \in N$. If all the outer agents had the same valuation function $v_c$ as $c$, a complete EFX allocation would be guaranteed to exist [Plaut and Roughgarden, 2020]. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be one such EFX allocation where all the agents have the valuation function $v_c$.

We iterate through the outer vertices (in any order) and construct an allocation $X$ by allocating to each outer vertex its highest-valued bundle in $Y$ that has not already been allocated. We allocate the final bundle to the center of the star, i.e. to agent $c$.

We claim that the allocation $X$ is G-EFX. If $c'$ is an outer vertex, $c'$ does not envy $c$, as they picked their bundle $X_{c'}$ over the bundle $X_c$. The center, $c$, does not strongly envy any of the outer vertices, because $Y$ is EFX for agents with the valuation function $v_c$. Otherwise, we would have strong envy between the center and some outer vertex, say $c'$, implying that for some good $g \in X_{c'}$, we have $X_{c'} - g \succ c X_c$. However, this implies the existence of strong envy in the allocation $Y$, which contradicts the fact that $Y$ is EFX when all the agents have the valuation function $v_c$.

Proposition 3.1 can be immediately generalized to a larger class of graphs, consisting of a central group of vertices all having the same valuation function. The remaining vertices in this case can have edges to any subset of the central vertices but cannot have edges among themselves. We will refer to the central group of nodes as the core vertices (or agents), and designate them by $N' \subseteq N$, and the vertices in $N \setminus N'$ as the outer vertices. Note that the outer vertices induce an independent set $G[N \setminus N']$ in $G$. We give a few examples of these graphs in Figure 1. Let us formalize the result below.

Theorem 3.2. Suppose agents have general valuations, and $G = (N, E)$ consists of a core set of agents $N' \subseteq N$ with identical valuations, with $N \setminus N'$ inducing an independent set in $G$. Then a G-EFX allocation is guaranteed to exist.

Proof. The proof is similar to that of Proposition 3.1. Suppose the core vertices all have identical valuation function $v_c$, and let $Y = (Y_1, Y_2, \ldots, Y_n)$ be an EFX allocation for this problem instance assuming that all $n$ agents have this same valuation function $v_c$.

We construct an allocation $X$ as follows. We first iterate through the agents in $N \setminus N'$ (in any order) and allocate to each such agent its highest-valued bundle in $Y$ that has not already been allocated. Once this is done, we distribute the remaining bundles in $Y$ to the agents in $N'$ (in any order).

We claim that the allocation $X$ is G-EFX. If $i \in N \setminus N'$, then $i$ has neighbors only in $N'$. Clearly, $i$ does not envy any vertex in $N'$, because they picked their bundle $X_i$ over all bundles distributed among $N'$. If $i \in N'$, then $i$ has valuation function $v_c$ by definition. Of course, $i$ does not strongly envy any of the other vertices because $Y$ is EFX for agents with the valuation function $v_c$. Otherwise, we would have strong envy between $i$ and some other vertex, say $i'$, implying that for some good $g \in X_{i'}$, we have $X_{i'} - g \succ_i X_i$. However, this implies the existence of strong envy in the allocation $Y$, which is a contradiction.

Theorem 3.2 has an interesting corollary when there are only four agents with general valuations.

Corollary 3.3. Suppose there are four agents with general valuation functions, with two of them having identical valuations. Then, an allocation exists that satisfies the EFX criterion for all but possibly one pair of agents.

Proof. Theorem 3.2 implies the existence of a G-EFX allocation on the graph in Figure 1b. The EFX criterion is maintained between all edges in the graph, and therefore between all pairs of agents except possibly the one consisting of the top and bottom vertices.
Proposition 3.1 also admits a second generalization for additive valuations which mildly relaxes the requirement that all the core vertices $N'$ have the exact same valuation function, but places additional restrictions on the outer vertices. In this class of graphs, vertices in $N'$ only need to have consistent valuation functions, i.e., the same (weak) ranking over the set $M$ of goods. However, each outer vertex can only be connected to a neighborhood $\text{Nbd}_G(i) = \{j \in N : (i,j) \in E\}$ in $N'$ with identical valuation functions.

**Theorem 3.4.** Suppose agents have additive valuations, and $G = (N, E)$ consists of a core set of agents $N' \subseteq N$ with consistent valuations. Let $N' = N'_1 \sqcup \ldots \sqcup N'_k$, where all agents in $N'_k$ have the same valuation function $v_k$. Then a $G$-EFX allocation is guaranteed to exist as long as every vertex $i \in N \setminus N'$ has its neighborhood $\text{Nbd}_G(i) \subseteq N'_k$ for some $k$.

**Proof.** If all agents have consistent additive valuations, an EFX allocation is guaranteed to exist [Plaut and Roughgarden, 2020]. Let $Y = (Y_1, \ldots, Y_n)$ be an EFX allocation for this problem instance assuming all the core vertices in $N'$ have their respective valuation functions, and each vertex in $N \setminus N'$ has an identical valuation function to any of its neighbors (this is well-defined, by construction).

We construct an allocation $X$ as follows. We first iterate through the agents in $N \setminus N'$ (in any order). For each such agent $i$, suppose it is connected to agents in $N'_k$. We let $i$ choose its highest-valued bundle in $Y$ allocated to any agent with valuation $v_k$ and not yet chosen by any other agent. Once this is done for all $i \in N \setminus N'$, we distribute the remaining bundles in $Y$ to the agents in $N'$ while maintaining the invariant that any such agent $i$ receives a bundle allocated in $Y$ to an agent with valuation function $v_i$. We claim that this algorithm terminates with a bundle to each agent in $N$. There is no point in the algorithm where we wish to assign a bundle in $Y$ to an agent $i$, but all bundles in $Y$ allocated to agents with valuation $v_i$ have already been assigned. This follows by construction of $Y$, and the fact that each agent in $N \setminus N'$ only selects from bundles intended for neighbors of agents in $N'_k$.

We further claim that the allocation $X$ is $G$-EFX. If $i \in N \setminus N'$, with neighborhood $\text{Nbd}_G(i) \subseteq N'_k$, then $i$ is allocated their bundle from $Y$ before any node of $N'_k$, and they are all allocated from the same pool of bundles (corresponding to agents with valuation $v_k$). Therefore, $i$ does not envy any agent in $N'_k$. If $i \in N'$, they do not strongly envy their neighbors as that would violate the EFX property for the allocation $Y$ for similar reasons as in the proof of Theorem 3.2.

Note that Theorem 3.4 implies Theorem 3.2 when agents have additive valuations, simply by taking $K = 1$. While Theorem 3.4 certainly applies to several interesting and natural graphs (see Figures 1c and 1d, e.g.), a limiting factor is the constraint that each vertex in $N \setminus N'$ can only have neighbors in $N'$ that have identical valuations. One might be tempted to adapt our approach for Theorem 3.2 to prove the existence of $G$-EFX allocations in graphs where there is a core set of consistent agents, with the non-core agents having any arbitrary set of core neighbors. This would be a natural relaxation, and generalize both Theorems 3.2 and 3.4 in the additive case. Unfortunately, direct adaptation fails. The problem is that if a non-core agent $i$ has two core neighbors $j$ and $j'$, with consistent but non-identical evaluation functions $v_j$ and $v_{j'}$, then constructing an initial EFX allocation $Y$ seems to be hard, as it is not clear which of $v_j$ and $v_{j'}$ to assign to agent $i$. The following example illustrates this difficulty.

**Example 3.5.** Consider the allocation instance with three agents $\{1, 2, 3\}$ and six goods $\{g_1, \ldots, g_6\}$ defined over the two-edge path $P_3$ (see Figure 2a). Suppose the agents have additive valuations, given in Figure 2b.

Agents 1 and 3 have (weakly) consistent valuations, but they have a common neighbor 2 whose valuation is not consistent with either of them. If we try to use the same approach as in our proof for Theorem 3.4, we would first compute an EFX allocation for consistent valuations and then reallocate the bundles by letting 2 pick first. For this instance, however, we have to first choose which valuation profile to give agent 2 before computing the initial EFX allocation. If we give agent...
(a) The graph $P_3$

(b) The valuation functions

Figure 2: The allocation instance in Example 3.5

2 the same valuation as agent 3, we can easily find an EFX allocation for agents with consistent valuations $(v_1, v_3, v_5)$: e.g. $Y = (Y_1, Y_2, Y_3)$, with $Y_1 = \{g_1, g_2\}, Y_2 = \{g_3, g_4\}, Y_3 = \{g_5, g_6\}$. If we now give agent 2 their highest-valued bundle between $Y_2$ and $Y_3$, we get the final allocation $X$ as $X_1 = \{g_1, g_2\}, X_2 = \{g_5, g_6\}, X_3 = \{g_3, g_4\}$. However this is not G-EFX since agent 2 strongly envies agent 1.

The problem arose essentially because agent 2 was asked to pick their bundle from $\{Y_2, Y_3\}$, and so they were not allowed to choose agent 1’s bundle. If we modified the algorithm and let agent 2 pick agent 1’s bundle at the start, we would have $X_2 = \{g_1, g_2\}$. No matter how we distributed the remaining bundles, agent 1 always strongly envies agent 2.

The instance in Example 3.5 admits a G-EFX allocation; Proposition 3.1 implies this trivially, as does the main result in Chaudhury et al. [2020]. The obstacle, therefore, arises from the choices made in our algorithm. We relegate different ways of generalizing Theorem 3.4 to future work.

An important question when discussing EFX allocations is that of computational efficiency. This is known to be an intractable problem in general: Plaut and Roughgarden [2020] show that computing EFX allocations even when the valuations are identical has exponential query complexity. However, we show that when all agents have additive valuations, the $G$-EFX allocations from Theorems 3.2 and 3.4 can be computed in polynomial time.

**Theorem 3.6.** When agents have additive valuations, the $G$-EFX allocations in Theorems 3.2 and 3.4 can be computed in $O(mn^3)$ time, where $n$ and $m$ are the number of agents and number of goods respectively.

**Proof.** An EFX allocation can be computed for agents with consistent additive valuations in $O(mn^3)$ time [Plaut and Roughgarden, 2020]\(^1\). The initial EFX allocation $Y$ in Theorem 3.2 and 3.4 can be computed using this algorithm. The next step involves iterating through the agents and giving them their best unallocated bundle that satisfies certain additional conditions. For each agent, we can find this bundle in $O(nm)$ time since there are only at most $n$ bundles and computing the valuation of each bundle can be done in $O(m)$ time (using additivity). Therefore the second step of the algorithm takes $O(mn^2)$ time. This gives us a total time complexity of $O(mn^3)$.

\(\Box\)

### 3.2 The Three-Edge Path and its Generalization

We now move to another simple graph that is the starting point for our next set of results: the three-edge path graph $P_4$. Of course, Theorem 3.4 already proves the existence of a $G$-EFX allocation on a three-edge path in the particular case when the valuations of the two middle vertices are consistent with each other. In this subsection, as in the last one, we start with a warm-up problem, by proving the existence of a $G$-EFX allocation for a three-edge path whose valuations are unconstrained.

\(^1\)This assumes the goods are sorted in order of any of the consistent valuations, so the result is up to an additive factor of $O(m \log m)$. We will ignore this factor for our analysis, as it is a simple pre-processing step to sort our goods by the core vertex valuations.
This means there exists a good the valuation functions neighbor. Agents respectively since they were allocated a bundle that they (weakly) prefer to that of their unique

Suppose agents have general valuations, and Theorem 3.8. Let $Y = (Y_1, Y_2, Y_3, Y_4)$ be an EFX allocation of the set of goods $M$ on four agents with valuations $(v_2, v_2, v_3, v_3)$ respectively.

We construct an allocation $X$ from $Y$ as follows. We allocate to agent 1 their highest-valued bundle in the set $\{Y_1, Y_2\}$, and assign the other bundle in that set to agent 2. Similarly, we allocate to agent 4 their highest-valued bundle in the set $\{Y_3, Y_4\}$, and assign the other one to agent 3.

We claim the allocation $X$ is $G$-EFX on the path $P_4$. Agents 1 and 4 do not envy agents 2 and 3 respectively since they were allocated a bundle that they (weakly) prefer to that of their unique neighbor. Agents 2 and 3 do not strongly envy any other agent, because $Y$ is EFX for agents with the valuation functions $v_2$ or $v_3$. For instance, WLOG, suppose agent 2 strongly envies agent 1. This means there exists a good $g \in X_1$ such that $X_1 - g \succ_2 X_2$. But this would violate the fact that $X_2$, which is either $Y_1$ or $Y_2$, was a bundle given to an agent with valuation $v_2$ in $Y$, which is an EFX allocation.

Exactly as in Section 3.1, this result can be generalized to a larger class of graphs. Once again, we will have a central core set of vertices $N' \subseteq N$, corresponding to agents having one of two distinct valuation functions, say $v_k$ and $v_l$. All remaining vertices, i.e. the agents in $N \setminus N'$, induce an independent set in $G$ as before, and furthermore, they can have arbitrary neighborhoods among any one of the two types of agents in $N'$ (see Figure 3 for examples).

More formally, the core vertices $N'$ can now be partitioned as $N' = N'_k \cup N'_l$, with all agents in $N'_k$ with valuation $v_k$, and all agents in $N'_l$ with valuation $v_l$. The outer vertices in $N \setminus N'$ have arbitrary valuation functions, and for each $i \in N \setminus N'$, its neighborhood in $G$, $\text{Nbd}_G(i)$, satisfies either $\text{Nbd}_G(i) \subseteq N'_k$ or $\text{Nbd}_G(i) \subseteq N'_l$.

We now prove that when $G$ is of the form above, an EFX allocation always exists.

**Theorem 3.8.** Suppose agents have general valuations, and $G = (N, E)$ is of the form described above, i.e., consists of a core set of vertices $N' \subseteq N$ with two types of valuations, and all remaining agents in $N' \subseteq N$ with arbitrary valuations, but neighborhoods restricted to any of the two core groups of agents. Then a $G$-EFX allocation is guaranteed to exist.

Proof. Once again, we will use the fact that an EFX allocation is guaranteed to exist when each agent only has one of two types of valuations [Mahara, 2021]. Consider a modified instance of the

![Figure 3: A three-edge path and its generalizations. Each node $i$ in any of the graphs above is labeled by the valuation function $v_i$ of the corresponding agent.](image-url)
problem, on the same graph, but where all the outer vertices in $G$ have the same valuation function as their neighbors among the core vertices. Note that this is well-defined by construction, and furthermore, this instance has agents with only types $v_k$ and $v_{\ell}$. So let $Y = (Y_1, \ldots, Y_n)$ be an EFX allocation for this modified instance.

We first divide $Y$ into two pools of bundles based on the valuation function of the agent they were allocated to. Suppose $Y^k$ is the set of bundles allocated in $Y$ to agents with valuation $v_k$, and $Y^\ell$ is the set of bundles allocated to agents with valuation $v_{\ell}$.

We construct an allocation $X$ by allocating the bundles in $Y$ in a particular order. We start with the outer agents in $N \setminus N'$ whose neighborhood is contained in $N'_k$. We iterate through these agents (in any order), allocating to each such agent their highest-valued bundle in $Y^k$ that has not been allocated yet. Then, we assign the remaining bundles in $Y^k$ in any order to the agents in $N'_k$. We repeat this same procedure with the remaining agents and the set of bundles $Y^\ell$, starting with the outer agents with neighborhoods in $N'_\ell$, as before.

We claim that this algorithm terminates with a bundle to each agent in $N$. This follows by similar arguments as in the proof of Theorem 3.4. We also claim that the allocation $X$ is $G$-EFX. If $i \in N \setminus N'$, then $i$ is allocated their bundle from the same pool of bundles as all their neighbors, but before any of their neighbors are. So $i$ does not envy any of their neighbors. If $i \in N'$, they do not strongly envy their neighbors as that would violate the allocation $Y$ being EFX, by a similar argument as in the proof of Theorem 3.4.

We conclude this section by observing that Theorem 3.8 immediately shows the existence of $G$-EFX allocations for several interesting classes of small graphs (see Figure 3). Notably among these are the graph consisting of two arbitrary stars connected at their central vertices, and the four-edge path $P_5$, where any two of the degree-2 vertices have the same valuation function.

4 Empirical Results

In this section, we will discuss a more general setting that applies to a more robust class of graphs than in Section 3. One approach towards generalizing our techniques would certainly be to repeatedly relax the constraints in the assumptions of the theorems in the previous section, and for each such relaxation, fix our algorithm to maintain the $G$-EFX criterion at each step. However, in this section, we instead present a much simpler and more general algorithm that handles allocations on a large and natural class of graphs. Although we do not prove that this algorithm always terminates, it is the case that if it terminates, it does so with a $G$-EFX allocation on the input instance. We present empirical results using our algorithm in many (representative) real problem instances, and show that the algorithm terminates with a $G$-EFX allocation in all instances. The standard method of proving that an algorithm like ours terminates is to use a “potential function”, i.e. a loop variant bounded below that monotonically decreases at each round of our algorithm, indicating progress being made during each round [Benabbou et al., 2020, Chaudhury et al., 2020, Mahara, 2021]. We empirically investigate different natural candidates for such a potential function.

4.1 The “Sweeping” Algorithm

For simplicity, all our empirical results will use the path graph $P_n$ for the underlying graph $G$. We will assume that the vertices are $1, \ldots, n$ in that order along the path, and that agents have additive valuations. We will remark at the end on how to adapt the algorithm to trees. The algorithm proceeds as follows.

At the start of the algorithm, we assign all goods in $M$ to agent 1, who is first on the path. This allocation is nearly $G$-EFX already, except for the edge $(1, 2)$, where it likely violates the EFX criterion badly.
Table 1: Summary of the potential functions evaluated. $envy_X^i(i,j)$ denotes the amount by which $i$ envies $j$ in $X$. $strong-envy_X^i(i,j)$ denotes the amount by which $i$ strongly envies $j$ in $X$ i.e. the total envy $i$ has for $j$ after dropping $j$’s worst good from $i$’s perspective.

| Potential Function       | Closed-Form Expression                                                                 | Expectation   |
|--------------------------|----------------------------------------------------------------------------------------|---------------|
| Total Envy ($\phi_1$)   | $\sum_{i=1}^{n} envy_X^i(i,i+1) + envy_X^i(i+1,i)$                                    | Decreasing    |
| Total Strong Envy ($\phi_2$) | $\sum_{i=1}^{n} strong-envy_X^i(i,i+1) + strong-envy_X^i(i+1,i)$                     | Decreasing    |
| Minimum Valuation ($\phi_3$) | $\min v_i(X_i)$                                                                       | Increasing    |

Our algorithm consists of several rounds, each consisting of $2n - 3$ steps. Each round of the algorithm consists of “sweeping” the edges of $P_n$ step by step, from left to right in order (the forward sweep), and then back again (the reverse sweep). We only need to visit the edge $(n - 1, n)$ once in each round; all the other edges in $P_n$ are visited twice in every round, once during the forward sweep, and once during the reverse sweep.

This implies that on a particular step of any round, there is a well-defined edge of $P_n$ we are looking at, say the edge $(i, i+1)$, with current bundles $(X_i, X_{i+1})$. This step of the algorithm consists of “fixing” the allocation on the edge $(i, i+1)$ by re-allocating the goods in $X_i \cup X_{i+1}$ between the agents $i$ and $i + 1$ so that neither of them strongly envies the other. In other words, this particular edge is “fixed” in this step by moving around goods from either or both endpoints so that the EFX criterion is met along the edge. Recall that this is easy to do, as EFX allocations are known to exist for general valuations on two agents, due to Plaut and Roughgarden [2020], who use a cut-and-choose protocol, with one agent being the “cutter” and the other the “chooser”. Fixing a particular edge $(i, i+1)$ might negate the EFX criterion on a previously fixed edge, such as $(i + 1, i + 2)$; we then must fix that edge in a subsequent sweep or round. If an edge under consideration already meets the EFX criterion, then we do nothing and move on to the next step of the sweep.

Once a round is completed (i.e. we are back to the edge $(1, 2)$ at the end of a reverse sweep), we check if our current allocation is $G$-EFX on $P_n$. If it is, we are done; otherwise, we start the next round with the current allocation.

We conclude this subsection by remarking that this algorithm can be defined for other graphs, as well as with different initialization conditions. For instance, if $G$ is a tree, we could use an in-order traversal of its edges as our “sweeping” order. We could also start with a random allocation of the goods among the vertices of $G$ instead of assigning all of them to one particular vertex. We relegate the analysis of these variants to future work, and for the rest of this section, only discuss path graphs with the initialization condition as stated above. In addition, because the algorithm of Plaut and Roughgarden [2020] applies to two agents with general valuations, this algorithm could be applied to instances with general valuations. For the sake of simplicity, our experiments focus on additive valuation functions.

### 4.2 Performance and Potential Functions

We use the data from the website Spliddit²[Goldman and Procaccia, 2015] for our experiments, as this is the canonical repository of instances in this area of research. On Spliddit, users set up allocation problems with any number of goods and agents. To normalize the data, each agent is given 1,000 points to allocate across all of the goods, in integer amounts. Goods can be denoted as indivisible or divisible. We assume all goods are indivisible. There are 4,679 problem instances overall, 3,392 of which have three or more agents. Instances with two agents are trivial, so we exclude them from our analysis. We create a path on the $n$ agents in each problem instance in the obvious way, by numbering them 1 through $n$, and creating a path graph as stated in Section 4.1. All code

²See http://www.spliddit.org/.
used in these experiments is available at https://github.com/justinpayan/graph_efx.

Our first observation is that the algorithm successfully terminates with a $G$-EFX allocation in every single instance. In addition, the algorithm seems to be extremely fast, completing all instances in a matter of seconds on a laptop with an Intel Core i7 8th generation processor. The vast majority of instances require only a single round. 3,087 instances finished in one round, 296 required two rounds, 8 required three, and only a single instance required four rounds.

In order to analyze this algorithm, it suffices to show that it terminates, as the termination condition corresponds to an EFX allocation on the path. The standard technique for showing such a round-based algorithm terminates is to use a potential function $\phi$, a variable quantity that changes from round to round in a strictly monotonic way, indicating that the algorithm is making some progress. Typically, the potential function is bounded by a theoretical maximum or minimum, which implies that it cannot continue to increase indefinitely, and therefore the algorithm terminates. Several papers related to EFX allocations use potential functions in their arguments Benabbou et al. [2020], Chaudhury et al. [2020], Mahara [2021]. In our case, we wish to find a potential function that makes sufficient progress on each step or each round, to conclude that each round makes some progress towards termination. We only restrict our attention to potential function values in between each round, since any potential function which monotonically changes after each “fixing” step will automatically monotonically change after each round as well.

We investigate three reasonable candidates for potential functions in our trials, summarized in Table 1: $\phi_1$, which is the sum of the envy across all adjacent agents (in both directions); $\phi_2$, which is the sum of the strong envy over all adjacent agents; and $\phi_3$, which is the minimum value realized by any agent on its assigned bundle. We might expect $\phi_1$ or $\phi_2$ to be monotonically non-increasing over a run of our algorithm, as we repeatedly reassign goods to remove strong envy between pairs of agents. We also might expect $\phi_3$ to be monotonically non-decreasing, since at each step envious agents with relatively low value bundles receive higher-valued goods. Figure 4 shows the trajectories of these potential functions. Plots 4b and 4c show all examples with violations of monotonicity for $\phi_2$ and $\phi_3$, but there are 273 instances which violate monotonicity of $\phi_1$. While the instances violating monotonicity of $\phi_3$ appear to violate it by a small amount (potentially pointing the way to a modified potential function), the violations for $\phi_1$ have no discernible pattern. Total envy does not seem likely as a candidate for our potential function.

Because there is only a single instance violating monotonicity of strong envy, we present it here for further analysis in Table 2. Note that our algorithm always designates 1 as the cutter and 2 as the chooser when we are looking at the edge (1, 2), and 2 as the cutter and 3 as the chooser when we are looking at the edge (2, 3). However, if, during the first round, we reversed 2 and 3’s roles as cutter and chooser when we were looking at the edge (2, 3), we would immediately obtain a $G$-EFX
Table 2: Example instance where strong envy is not monotonically non-increasing. The original instance has \( v_3(g_1) = 995 \) and \( v_3(g_6) = 1 \), but we modify those valuations to remove ambiguity in execution. Initially, agent 2 has strong envy of 962 towards agent 1, and no other agents have any envy. After fixing the first edge, the allocation \( X \) is \( X_1 = \{g_1, g_2, g_4, g_6\}, X_2 = \{g_3, g_5\}, X_3 = \emptyset \).

We then fix the second edge to get \( X_1 = \{g_1, g_2, g_4, g_6\}, X_2 = \{g_3\}, X_3 = \{g_5\} \), and then the round finishes with \( X_1 = \{g_2, g_3\}, X_2 = \{g_1, g_4, g_6\}, X_3 = \{g_5\} \). Agent 3 now strongly envies 2 by 995, so the total strong envy has increased.

allocation. This means that for all Spliddit instances, there is some sequence of cutters and choosers in our single-edge protocol that causes the overall strong envy to be monotonically non-increasing. We omit the analysis in this paper, but this may be a promising direction for further work.

5 Discussion and Conclusion

Because the general EFX allocation problem seems intractably hard, there have been several major approaches in the literature on relaxations of the EFX criterion. As discussed in Section 1.1, notably among these are \( c \)-EFX allocations [Plaut and Roughgarden, 2020] and EFX partial allocations which burn or donate goods [Chaudhury et al., 2021b]. Our work in this paper introduces and studies a new relaxation motivated by the idea of agents having only partial information.

We believe that our proposed relaxation is particularly interesting for two reasons. First, we believe that unlike other existing EFX relaxations, our notion of fairness is natural, in the sense that most real life agents only care about other agents that they interact with. Our model seems to be a much more realistic representation of real-world settings, where stakeholders are typically concerned with only a subset of other stakeholders, a model corresponding precisely to a graph. We think this makes our model inherently more compelling to use in settings where agents have incomplete information about other agents’ bundles.

Second, studying EFX allocations on graphs gives us interesting insights into algorithmically exploiting the additional structure provided by the graph to make our allocations work, which we think is a key step towards proving the existence of EFX allocations in general. For instance, we show that when \( n = 4 \), and two out of the four agents have identical valuations, an allocation exists which satisfies the EFX criterion for all but one pair of agents (Corollary 3.3). It is not unlikely that there is a simple modification of our algorithm that relaxes any of these constraints, which would be a significant breakthrough in the major open problem of determining whether EFX allocations exist on four agents.

Our techniques and results provide a very promising direction for future work on EFX allocations. Our empirical results show that \( G \)-EFX allocations are likely to exist for several more general classes of graphs like paths \( P_n \). It would be interesting to find other natural classes of graphs that admit EFX allocations, on general valuations, or even additive ones.

There are also interesting questions to ask, that are purely on graph structures, and whether the existence of an EFX allocation on one graph can be used in some way to conclude the existence of an EFX allocation on another graph, for the same set of goods. For instance, consider the following question, which tries to use a \( G \)-EFX allocation to create an \( H \)-EFX allocation for a subgraph \( H \) of \( G \), where we view \( H \) as a new problem instance entirely.

**Problem 5.1.** Let \( G = (N, E) \) be a graph corresponding to some instance of the \( G \)-EFX allocation
problem on agents $N$ and goods $M$. Let $H = (N', E')$ be a subgraph of $G$, on the set of agents $N' \subseteq N$. Then, if there is a $G$-EFX allocation $X_G$ on $M$, can we use it to construct an $H$-EFX allocation $X_H$ on $M$ as well?

This is a natural question to ask, since we might wonder whether we can use this as a black box method to construct smaller allocations from larger ones, enabling us to start with an “easier” larger supergraph first. Of course, it suffices to only consider the case when $H$ is a connected induced subgraph of $G$. Furthermore, taking $G$ to be a complete graph, the conjecture asks: if an EFX allocation is guaranteed to exist for $n$ agents, can we conclude that an EFX allocation has to exist for strictly fewer agents as well?

Section 4 gives rise to several directions for future research as well. In our sweeping algorithm for paths, we use a classical subroutine from Plaut and Roughgarden [2020] to fix each edge. This subroutine involves a single choice — which agent cuts and which agent chooses. As discussed in Section 4.2, it seems that there is a choice of cutter and choosers for each edge for which our candidate potential function $\phi_2$ is, in fact, strictly non-increasing. This naturally raises the following question, for how much to hard-code the choice of cutter/chooser into our subroutines.

**Problem 5.2.** Are any of our potential functions, $\phi_1$, $\phi_2$, or $\phi_3$, strictly monotonic for some sequence of cutters and choosers during our sweeps in each round of the algorithm?

Of course, it is possible that we have not considered other important candidates for our potential function. It is feasible that the strong envy only moves in a single direction in the following sense: while the total strong envy can increase, as in Table 2, after every round there could be a lexicographic improvement in the vector of strong envy over all edges. If such a dynamic holds, then a vector of strong envies along the path might be the correct potential function to look at, that ensures progress in our algorithm. Alternatively, we might be able to prove that even if the strong envy increases during a particular round, it always goes down enough on subsequent rounds so that there is a net strict decrease over every few rounds. This amortized argument seems to hold true for the other candidates, $\phi_1$ and $\phi_3$, as well.

Finally, we also believe that the general study of fair allocation over graphs warrants further exploration. While we focus on EFX allocations in our work, several other notions of fairness like local proportionality and local max-min share might turn out to be very beautiful on graphs, and these offer scope for immediate future research.

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3For instance, if we create a supergraph of our graph with $m$ agents, where $m = |M|$ then it is easy to allocate a single item to each agent, which is a trivially EFX allocation.
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