EIGENVALUE STATISTICS IN QUANTUM IDEAL GASES

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Abstract

The eigenvalue statistics of quantum ideal gases with single particle energies $e_n = n^\alpha$ are studied. A recursion relation for the partition function allows to calculate the mean density of states from the asymptotic expansion for the single particle density. For integer $\alpha > 1$ one expects and finds number theoretic degeneracies and deviations from the Poissonian spacing distribution. By semiclassical arguments, the length spectrum of the classical system is shown to be related to sums of integers to the power $\alpha/(\alpha - 1)$. In particular, for $\alpha = 3/2$, the periodic orbits are related to sums of cubes, for which one again expects number theoretic degeneracies, with consequences for the two point correlation function.

1 Introduction

Most investigations of quantum chaos have focussed on the effects in single particle systems. The prime examples of frequently studied systems, such as hydrogen in a magnetic field, the standard map or small molecules all belong to this class (Eckhardt 1988, Casati and Chirikov 1995). Even electrons in a solid, a standard many body system, has until recently been reduced to a single (quasi)particle system. Yet the study of many-body quantum systems, even if they are integrable can be interesting for several reasons.

For once, the spin-statistics theorem, which requires quantum wave functions to be either totally symmetric (Bose-Einstein statistics) or totally anti-symmetric (Fermi-Dirac statistics) under exchange of particles changes the

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spectrum compared to the simple Maxwell-Boltzmann type superposition of the individual single particle density. This gives rise to changes in the total density of states, as is well known in statistical mechanics. The full program of implementing the permutation symmetry semiclassically in the mean density of states and in the trace formulas has recently been taken up by Weidenmüller et al. (1993a,b).

Experimentally, small clusters are examples of systems with many degrees of freedom for which one should use microcanonical averaging rather than canonical, since the number of atoms or electrons is rather fixed. The difference between microcanonical and canonical ensembles can be observed. The spectral statistics of small systems influences their thermodynamical behavior (Mühlenschlegel 1991).

Furthermore, symmetric systems can serve as a reference point for systems with weakly broken symmetries. In particular, it has been observed that weak symmetries, perhaps of dynamical origin, can give rise to strong degeneracies in spectra, the Shnirelman peak (Shnirelman 1993, Chirikov and Shepelyansky 1995).

In this contribution only integrable ideal gases will be analyzed. The energy levels are constrained to be a power of the quantum number, \( E_n = n^\alpha \), where \( n = 1, 2, 3, \ldots \) and \( \alpha > 1 \), except where noted. This family includes for instance the eigenenergies for a particle of mass \( m \) confined to a 1-d box of width \( L \), measured in units of \( 4 \pi^2 \hbar^2 / 2mL^2 \). It also describes the asymptotic eigenvalues for particles in homogeneous potentials of degree \( p \), for which \( \alpha = 2p / (p + 1) \) (Seligman et al. 1985, Seligman and Verbaarschot 1987). The harmonic case, \( \alpha = 1 \), causes problems for the stationary phase approximations used below and will only be considered occasionally. The deviations of its level spacing distribution from Poissonian have been studied at length previously (Berry and Tabor 1977a). The quantum ideal gases then have the eigenvalues

\[
E = \sum_{n_1, \ldots, n_D} n_i^\alpha \quad \text{with} \quad \begin{cases} 
\text{no constraint} & \text{Maxwell-Boltzmann (MB)} \\
n_1 \leq n_2 \leq \ldots \leq n_D & \text{Bose-Einstein (BE)} \\
n_1 < n_2 < \ldots < n_D & \text{Fermi-Dirac (FD)} 
\end{cases}
\]

where the different statistics have been indicated.

The questions addressed here concern the mean density of states (section 2), the level spacing distribution in particular for integer \( \alpha \) (section 3) and the behaviour of the pair correlation function (section 4). The results on the mean density of states are of a more general nature, whereas some ex-
amples are particular to the powers $\alpha$ considered. The final section contains some speculations on the relevance of number theory especially for the pair correlation function for some rational values of $\alpha$.

2 The partition function and the mean density of states

2.1 The connection

The partition function for a quantum mechanical system is defined as

$$Z(\beta) = \text{tr} e^{-\beta H} = \sum_j e^{-\beta E_j},$$

(2)

where the last sum extends over all eigenvalues $E_j$. The density of states $\rho(E)$ is related to $Z$ by a Laplace transform, so that the poles in an asymptotic expansion of $Z$ for small $\beta$ are related to the rate of divergence of $\rho(E)$ for large energy and thus to the mean density of states. In particular, if

$$Z(\beta) \sim \sum_j c_j \beta^{-\gamma_j}$$

(3)

then

$$\rho(E) \sim \sum_j \frac{c_j}{\Gamma(\gamma_j)} E^{\gamma_j - 1}.$$ 

(4)

As an example, in the harmonic case $\alpha = 1$, one has

$$Z(\beta) = \frac{1}{e^\beta - 1} \sim \frac{1}{\beta} - \frac{1}{2} + \frac{\beta}{12} - \frac{\beta^3}{720} + \ldots$$

(5)

and for the mean density of states

$$\overline{\rho}(E) = 1.$$ 

(6)

Thus, whereas the partition function contains contributions from positive powers of $\beta$ (which contain information on the shortest periodic orbits in the system, Berry and Howls 1994), these terms are cancelled in the density of states by the poles of the $\Gamma$-function in the denominator.
2.2 The recursion relation for $D$-particle partition functions

For the Maxwell-Boltzmann case with no restrictions on the integer sums, the partition function for $D$ particles can be written down explicitly,

$$
Z_{D}^{(MB)}(\beta) = (Z_{1}(\beta))^{D}.
$$

(7)

This is no longer possible for the symmetry reduced subspaces. However, there is a simple recursion relation involving the partition functions for all particle numbers up to $D$,

$$
Z_{D}(\beta) = \frac{1}{D} \sum_{k=1}^{D} (\pm 1)^{k+1} Z_{1}(k\beta) Z_{D-k}(\beta);
$$

(8)

the $+1$ applies in the Bose-Einstein subspace and the $-1$ in the Fermi-Dirac subspace. A combinatorial proof of this relation was given by Bormann and Franke (1993). A more direct analytical proof may be based on the grand canonical formalism (Reif 1965).

The grand canonical partition function is defined as

$$
\Omega(z, \beta) = \sum_{D=0}^{\infty} z^{D} Z_{D}(\beta)
$$

(9)

$$
= \sum_{D=0}^{\infty} \sum_{n_{1}, \ldots, n_{D}} z^{D} e^{-\beta \sum_{n_{i}}} ;
$$

(10)

the last sum on quantum numbers is restricted by the selection rules for the different statistics. Passing to an occupation number representation, where $g_{n}$ denotes the number of particles in quantum state $n$, one obtains

$$
\Omega(z, \beta) = \sum_{g_{1}, g_{2}, \ldots} z \sum_{n} g_{n} e^{-\beta \sum_{n} g_{n}} ;
$$

(11)

In the case of the Bose-Einstein statistics, the occupation numbers can take on all nonnegative integer values, and the summation on $g_{n}$ gives rise to geometric series. In the case of the Fermi-Dirac statistics, the occupation numbers can only take on the values 0 and 1. The result for both cases can be combined in a single expression,

$$
\Omega(z, \beta) = - \prod_{n=1}^{\infty} \left(1 - z e^{-\beta n} \right)^{-\theta},
$$

(12)
where \( \theta = +1 \) for Bose-Einstein and \( \theta = -1 \) for Fermi-Dirac statistics.

The \( D \)-particle partition function follows from taking an \( n \)-th derivative of \( \Omega \) with respect to \( z \) at \( z = 0 \). The first derivative becomes

\[
\frac{\partial \Omega(z, \beta)}{\partial z} = \sum_{n=1}^{\infty} \frac{e^{-\beta n_1 \alpha}}{1 - \theta z e^{-\beta n_2 \alpha}} \Omega(z, \beta)
\]  

(13)

\[
= S(z, \beta) \Omega(z, \beta).
\]  

(14)

The recursion relation (8) now follows from the formula for derivatives of a product and the observation that the \( k \)-th derivative of \( S \) at \( z = 0 \) is related to the single particle partition function,

\[
\frac{\partial^k S(z, \beta)}{\partial z^k} \bigg|_{z=0} = \theta^k Z_1((k + 1) \beta).
\]

(15)

This result holds for other forms of the single particle energies as well.

2.3 Examples

The first few partition functions when reduced to the single particle partition function become

\[
Z_2(\beta) = \frac{1}{2} \left( Z_1^2(\beta) \pm Z_1(2\beta) \right)
\]

\[
Z_3(\beta) = \frac{1}{6} \left( Z_1^3(\beta) \pm 3Z_1(\beta)Z_1(2\beta) + 2Z_1(3\beta) \right)
\]

\[
Z_4(\beta) = \frac{1}{24} \left( Z_1^4(\beta) \pm 6Z_1^2(\beta)Z_1(2\beta) + 8Z_1(\beta)Z_1(3\beta) + 3Z_1^2(2\beta) \pm 6Z_1(4\beta) \right)
\]

(16)

etc.

where the plus signs apply for the BE-statistics and the minus signs for the FD-statistics. The corresponding densities of states become

\[
\overline{\rho}_2(E) = \frac{1}{2} \sum_{n_1, n_2} \delta(E - n_1^\alpha - n_2^\alpha) \pm \frac{1}{2} \sum_n \delta(E - 2n^\alpha)
\]

\[
\overline{\rho}_3(E) = \frac{1}{6} \sum_{n_1, n_2, n_3} \delta(E - n_1^\alpha - n_2^\alpha - n_3^\alpha)
\]

\[
\pm \frac{1}{2} \sum_{n_1, n_2} \delta(E - n_1^\alpha - 2n_2^\alpha) \pm \frac{1}{3} \sum_n \delta(E - 3n^\alpha)
\]

(17)

etc.
The leading order contribution is $Z^{(MB)}/D!$, with the familiar permutation factor $D!$. Thus the mean density of states is to leading order the same for Bose-Einstein and Fermi-Dirac statistics and equal to $1/D!$ the density of states for the Maxwell-Boltzmann statistics. The next to leading order terms correct for the counting of states with two or more identical quantum numbers.

Further progress can be made if either the full single particle partition function or at least its asymptotic expansion are known. Such is the case for $\alpha = 1$, the harmonic oscillator, and for $\alpha = 2$ and $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, i.e. a particle on a ring.

The single particle partition function for the harmonic energies $\alpha = 1$ and its asymptotic expansion have been given above (3). The partition functions for two particles then become

$$Z_{2}^{BE}(\beta) = \frac{1}{2\beta^2} - \frac{1}{4\beta} - \frac{1}{24} + \frac{7}{1440} \beta^2 \pm \ldots$$

$$Z_{2}^{FD}(\beta) = \frac{1}{2\beta^2} - \frac{3}{4\beta} + \frac{11}{24} + \frac{7}{1440} \beta^2 \pm \ldots.$$ (18)

For the mean density of states, these expansions imply

$$\rho_{2}^{BE}(\beta) = \frac{1}{2} E - \frac{1}{4}$$

$$\rho_{2}^{FD}(\beta) = \frac{1}{2} E - \frac{3}{4}.$$ (20)

Here the correction terms have the same sign but different magnitude.

As a second example we take a particle on a ring (periodic boundary conditions on the interval), for which

$$Z_{1}(\beta) = \sum_{n=-\infty}^{+\infty} e^{-\beta n^2} \sim \left( \frac{\pi}{\beta} \right)^{1/2}.$$ (22)

From this one can derive that

$$Z_{2}(\beta) = \frac{1}{2} \left( \frac{\pi}{\beta} \right)^{-1} \pm \frac{\sqrt{2}}{4} \left( \frac{\pi}{\beta} \right)^{-1/2}$$

$$Z_{3}(\beta) = \frac{1}{6} \left( \frac{\pi}{\beta} \right)^{-3/2} \pm \frac{\sqrt{2}}{4} \left( \frac{\pi}{\beta} \right)^{-1} + \frac{\sqrt{3}}{9} \left( \frac{\pi}{\beta} \right)^{-1/2}$$

$$Z_{4}(\beta) = \frac{1}{24} \left( \frac{\pi}{\beta} \right)^{-2} \pm \frac{\sqrt{2}}{8} \left( \frac{\pi}{\beta} \right)^{-3/2}.$$ (23)
$+ \left( \frac{\sqrt{3}}{9} + \frac{1}{16} \right) \left( \frac{\pi}{\beta} \right)^{-1} \pm \frac{1}{8} \left( \frac{\pi}{\beta} \right)^{-1/2}$

and the densities

$\bar{\rho}_2(E) = \frac{\pi}{2} \pm \frac{\sqrt{2}}{4} E^{-1/2}$

$\bar{\rho}_3(E) = \frac{\pi}{3} E^{1/2} \pm \frac{\sqrt{2}}{4} \pi + \frac{\sqrt{3}}{9} E^{-1/2}$

$\bar{\rho}_4(E) = \frac{\pi^2}{24} E \pm \frac{\sqrt{2} \pi}{4} E^{1/2} + \left( \frac{\sqrt{3}}{9} + \frac{1}{16} \right) \pi \pm \frac{1}{8} E^{-1/2}$

Again, the plus sign refers to the Bose-Einstein statistics and the minus sign to the Fermi-Dirac statistics. Evidently, since the one particle partition function consists of the leading order term only, all lower order terms are due to symmetrization and the signs clearly reflect the statistics.

The size of the next to leading order corrections increase rapidly with the particle number $D$ and become very important for large $D$. As can easily been shown by induction,

$$Z_D(\beta) = \frac{\pi D/2}{D!} \beta^{-(D/2)} \left( 1 \pm c_D \beta^{1/2} \pm \cdots \right)$$

so that the mean density of states becomes

$$\bar{\rho}_D(\beta) = \frac{\pi D/2}{D! \Gamma(D/2)} E^{(D-2)/2} \left( 1 \pm c_D E^{-1/2} \pm \cdots \right).$$

with

$$c_D = \frac{\sqrt{2} D (D-1) \Gamma(D/2)}{4 \sqrt{\pi} \Gamma((D-1)/2)} \sim \frac{1}{4 \sqrt{\pi}} (D-1)^{5/2}.$$  

The energy $E_c$, where $c_D E_c^{-1/2} \sim 1$ thus increases like $(D-1)^5$. This may be compared to the groundstate energy of the Fermi system, $E_F \sim D^3$. Even when compared to this the importance of this term increases like $E_c/E_F \sim (D-1)^{5/3}$. The approach to the density of states of the classical ideal gas is thus very slow, and it will be difficult to estimate ground states accurately from the leading order terms (something that works surprisingly well in many cases in few degree of freedom systems). Some examples for the behaviour of the next to leading order corrections are shown in Fig. 1.

Similar behavior can be found in the harmonic oscillator case.
Figure 1: Mean density of states for $D$-particle systems in the Bose (top) and Fermi (bottom) subspaces for particles on a ring. The densities are normalized by the desymmetrized Maxwell-Boltzmann densities.

### 2.4 Asymptotic expansions of single particle partition functions

For integer $\alpha$ one can use the Euler-MacLaurin summation formula to derive an asymptotic expansion for the single particle partition function starting from the representation

$$
\sum_{n=1}^{\infty} e^{-\beta n^\alpha} = \int_0^\infty dx e^{-\beta x^{\alpha}} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{\partial^{2k-1}}{\partial n^{2k-1}} e^{-\beta n^\alpha} \bigg|_{n=0},
$$

(28)

where $B_{2k}$ are the Bernoulli numbers, $B_{2k} \sim (-1)^{k-1}(2k)!/(\pi^{2k}2^{2k-1})$. The leading order divergence comes from the integral, whereas the remaining
terms give a power series in $\beta$,

$$Z_1(\beta) \sim \Gamma \left( \frac{\alpha + 1}{\alpha} \right) \beta^{-1/\alpha} + \sum_j c_j \beta^{-\gamma_j}. \quad (29)$$

A power series expansion of $\exp (-\beta \nu^\alpha)$ shows that only the powers $\gamma = (2k - 1) \alpha$ carry non-zero weights. As noted before, these do not contribute to the density of states for the single particle but can be brought to live in the many particle system through combinations with sufficiently many powers of the first term.

For rational $\alpha$ this does not work, since the derivatives required in the Euler-MacLaurin formula do not terminate, giving diverging coefficients for the powers of $\beta$. Numerical results are consistent with the leading order behavior being given by the integral and the further terms in the desymmetrized version being a power series in $E^{1/\alpha}$.

For later reference, I note the mean density of states for $D$ particles that results from this asymptotic expansion,

$$\overline{\rho}_D(E) = \frac{\Gamma \left( \frac{D+1}{\alpha} \right)^D}{D! \Gamma \left( \frac{D}{\alpha} \right)} E^{(D-\alpha)/\alpha} + \cdots. \quad (30)$$

The mean density of states thus decreases for $D < \alpha$, is asymptotically constant for $D = \alpha$ and increases for $D > \alpha$.

### 3 Nearest neighbor spacings

#### 3.1 Numerical results

The Maxwell-Boltzmann gas with its huge average degeneracies because of permutation symmetry gives rise to a rather singular level spacing distribution,

$$P(s) = (1 - g) \delta(s) + \frac{1}{g^2} e^{-gs} \quad (31)$$

where the mean degeneracy $g = D!$. These massive degeneracies will obviously not survive perturbations. However, as Shnirelman’s observations show, the low lying spectra of a weakly perturbed system may still have a $\delta$-function contribution because of (almost) degeneracies between the perturbed symmetry related tori (shnirelman 1993, Casati and Shepelyansky 1995).
Within the symmetry reduced Bose-Einstein and Fermi-Dirac subspaces one does not expect this effect and generically finds a Poissonian spacing distribution (in accord with Berry and Tabor 1997a). However, for integer powers of $\alpha$, all the energy levels will be integer and there can be some number theoretic degeneracies (Casati et al 1985). In the case of $\alpha = 2$ it is known that the level spacing distribution collapses to a delta function. Thus although the density of states is constant, some large integers can be expressed in an increasing number of ways as sums of squares, and the gaps between them increase logarithmically.

As to the case of cubes, the density of two cubes decreases, so that the unfolded levels (with mean spacing one) are not simple multiples of integers. Thus a continuous range of unfolded spacings can be achieved and the level spacing distribution is essentially Poissonian, except for a small overshoot at the origin (Fig. 2a).

For three cubes, the density is constant, and the spacing distribution seems to converge to $\delta$-spikes with spacing $1/\rho(E)$. The deviations noticeable in Fig. 2b are due to finite $E$ effects: the Bose-Einstein spacings have a wing to lower values because of the decreasing density, the Fermi-Dirac spacings have one to higher values because of the increasing density. However, for four and more cubes, a strong $\delta$-function develops at the origin, eventually absorbing all the density (Fig. 2c,d).

3.2 Connection to Warings problem

In the case of integer $\alpha$, all eigenenergies are integer. Thus, if the density of states increases with energy, eventually one will reach a situation where the density exceeds one level per unit interval. This critical energy can be calculated to leading order from (30) to be

$$E_c(\alpha, D) = \left( \frac{\Gamma(\frac{\alpha+1}{\alpha})}{D! \Gamma\left(\frac{D}{\alpha}\right)} \right)^{-\alpha/(D-\alpha)}.$$  \hspace{1cm} (32)

For $E > E_c$ more and more levels have to fall onto the same integer, giving rise to a $\delta$-function at the origin in the spacings distribution. This will happen for $D > \alpha + 1$. The case $D = \alpha$ is marginal. For $\alpha = 2$ it is known that the spacings between numbers that can be represented as sums of squares increases logarithmically, giving rise to a logarithmically increasing degeneracy: the spacing distribution converges to a delta function at the
Table 1: Results on Warings numbers, taken from (Ribenboim 1991). The probabilistic lower estimate is often found to be too optimistic. The critical energies $E_c$ (computed using 32) in the last two columns increase rather rapidly. Some numerical consequences of this will be studied in section 4.3.

| $\alpha$ | $g(\alpha)$ | $G(\alpha)$ | $E_c(D = \alpha + 1)$ | $E_c(D = \alpha + 2)$ |
|--------|----------|----------|----------------|----------------|
| 2      | 4        | 4        | $3.8 \cdot 10^4$ | $2.6 \cdot 10^3$ |
| 3      | 9        | 4 $\leq G \leq 7$ | $1.0 \cdot 10^9$ | $1.3 \cdot 10^6$ |
| 4      | 19       | 16       | $1.6 \cdot 10^{15}$ | $6.0 \cdot 10^9$ |
| 5      | 37       | 6 $\leq G \leq 21$ | $2.4 \cdot 10^{23}$ | $2.8 \cdot 10^{14}$ |
| 6      | 73       | 9 $\leq G \leq 31$ | $4.6 \cdot 10^{33}$ | $1.6 \cdot 10^{20}$ |
| 7      | 143 $\leq g \leq 3806$ | 8 $\leq G \leq 45$ | $1.4 \cdot 10^{46}$ | $1.3 \cdot 10^{27}$ |
| 8      | 279 $\leq g \leq 36119$ | 32 $\leq G \leq 62$ | $9.0 \cdot 10^{60}$ | $1.6 \cdot 10^{35}$ |
| 9      | 548 $\leq g$ | 13 $\leq G \leq 82$ |  |  |

origin. For $\alpha = 3$ and higher, the distribution seems to converge to a stick diagram.

Since the density of states increases, one can ask whether all integers can in fact be represented as a sum of $D$ integers raised to the power $\alpha$. This is Warings problem (Ribenboim 1989). More precisely, define a number $g(\alpha)$ so that all integers can be represented if $D \geq g(\alpha)$. Since some small integers cause special problems, Waring considered another number $G(\alpha)$, such that if $D \geq G(\alpha)$ then all sufficiently large integers can be represented. Obviously $g(\alpha) \geq G(\alpha)$ and $G(\alpha) \geq \alpha + 1$ because of the above density argument. Some results are collected in table 1.

The considerations of the permutation symmetry give rise to a specialization. Since both the Maxwell-Boltzmann case (which agrees with the sums Waring considered) and the Bose-Einstein statistics allow for repetition of integers, Warings numbers remain unchanged. However, the Fermi-Dirac statistics poses the additional constraint that all the integers have to be different. Thus it is not possible to fill in 1’s if there is still a small gap to an integer. Therefore, the equivalent of $g(\alpha)$ makes no sense, some small integers will always be missed. However, the density of states still increases for $D \geq \alpha + 1$ without bound, so that the density of points is sufficient to reach all larger integers. Numerical tests suggest that at least for $\alpha = 3$ and $D = 4$ and 5 the density of points not represented among the lowest $10^6$ integers decreases, but that is insufficient since the numbers involved rapidly grow large. It might be interesting to study the existence and values of $\tilde{G}(\alpha)$,
such that for $D \geq \tilde{G}(\alpha)$ all sufficiently large integers can be represented as sums of $\alpha$’s powers of $D$ different integers. Obviously, $\tilde{G}(\alpha) \geq G(\alpha)$.

4 Form factor and pair correlations

4.1 Numerical Results

The level spacing distribution $P(s)$ is a complicated mixture of $n$-point correlation functions, $n = 2, 3, \ldots$ and thus not accessible to a complete semiclassical analysis. Some progress can be made for the two point correlation function

$$C(\epsilon) = \frac{\langle \rho(E + \epsilon/2)\rho(E - \epsilon/2) \rangle}{\langle \rho \rangle^2} \quad (33)$$

and the derived quantities spectral rigidity and number variance (Berry 1985, Seligman and Verbaarschot 1987, Verbaarschot 1987, Bohigas 1991). The Fourier transform of the correlation function (33), the form factor, is the absolute value square of the Fourier transform of the spectrum. Since the latter can be related to periodic orbits via the Berry-Tabor (1977b) semiclassical expansion or Poisson summation, a combination of classical results (Hannay and Ozorio de Almeida 1984) and quantum information (Berry 1985) can be used to estimate the form factor.

4.2 Berry-Tabor expansion and Poisson summation formula

The density of states in the Bose-Einstein and Fermi-Dirac subspaces is to leading order given by that for the Maxwell-Boltzmann case with corrections for energy levels where two or more quantum numbers coincide (cf. [18]). The Berry-Tabor (1977b) semiclassical expansion in terms of classical periodic orbits for the full density is thus a superposition of the ones for $D$ or fewer particles with Maxwell-Boltzmann statistics. Technically, the semiclassical expansion of Berry and Tabor reduces to a Poisson summation on the EBK quantized eigenvalues,

$$\sum_{n_i} \delta \left( E - \sum_{i=1}^{D} n_i^{\alpha_i} \right) = \sum_{m_i} \sum_{g(m_1, \ldots, m_D)}$$

$$\sum_{n_i} \delta \left( E - \sum_{i=1}^{D} n_i^{\alpha_i} \right) = \sum_{m_i} g(m_1, \ldots, m_D) \quad (34)$$

where

$$g(m_1, \ldots, m_D) = \int d^D x \prod_{i=1}^{D} \Theta(x_i) \delta \left( x - \sum_{i=1}^{D} n_i^{\alpha_i} \right) e^{2\pi x \cdot m}, \quad (35)$$
and
\[ \Theta(x) = \begin{cases} 
1 & x > 0 \\
1/2 & x = 0 \\
0 & x < 0 
\end{cases} \] (36)
is the Heaviside step function. The Fourier transform of the product of step functions gives rise to another set of corrections to the leading order term, which can be combined with the ones due to symmetrization.

Going through the algebra of expressing the delta function by its Fourier representation, doing the $x$-integrals and then the final $k$-integral in stationary phase, one ends up with an expression of the form (Seligman and Verbaarschot 1987)
\[ g(m_1, m_2, \ldots, m_D) \sim (\Pi m_i)^{-(\alpha-2)/(2(\alpha-1))} L^{(\alpha-1-D)/(2\alpha)} \cdot E^{-2(\alpha-D-1)/(2\alpha)} \exp \left(2\pi i L^{(\alpha-1)/\alpha} E^{1/\alpha} \right) \] (37)
where
\[ L = \sum m_i^{\alpha/(\alpha-1)}. \] (38)
Semiclassically, the $L(m_1, \ldots, m_D)$ are the periods of the classical motions. The lower order terms where one or two quantum numbers are equal are of similar form, but with coefficients multiplying the powers of $m_i$.

The main conclusion to be drawn from this is that the exponents $\alpha$ and $\tilde{\alpha} = \alpha/(\alpha - 1)$ are conjugate to one another. In particular, $\alpha = \frac{p+q}{p}$ corresponds to $\tilde{\alpha} = \frac{p+q}{q}$. Thus the periods for the eigenvalues with $\alpha = 3/2$ are sums of cubes. Since sums of four or more cubes show large degeneracies, the periodic orbit spectrum for this system is degenerate which should influence the pair correlation function.

### 4.3 Pair correlations

By the Berry-Tabor (1977b) expansion the density of states can be written
\[ \rho(E) = \sum_m g_m e^{2\pi S_m E^{1/\alpha}} \] (39)with the actions
\[ S_m = L(m)^{\alpha-1)/\alpha} \] (40)and the amplitudes given above. The Fourier transform in the scaling variable $x = E^{1/\alpha}$ (alternatively one can expand around a reference energy $E_0 \to \infty$ and consider a small interval around $E_0$) then consists of
δ-functions at actions $S_m$. When taking the absolute value square, there are two contributions, one from every orbit with itself and one from different orbits. If there are no degeneracies in orbit actions, the diagonal part gives the constant form factor expected for Poissonian distributed levels (Berry 1985). However, with degeneracies as in the case $\alpha = 3/2$, there are additional contributions from the cross terms and the form factor is higher by a factor $g$, the mean degeneracy of actions (Biswas et al 1991).

The diagonal approximation can be good at best up to a period $T_H \sim \hbar \rho(E)$, for then the individual eigenvalues can be resolved and a quantum sum rule predicts the saturation to a constant, the Fourier transform of a δ-function (Berry 1985). Since the degeneracy increases slowly with the period, its effect will only be noticeable if the density of states is sufficiently high so that for orbits of period near $T_H$ the degeneracies will become important. For the lowest 100000 states studied here and in the studies of Biswas et al (1991), no effect of periodic orbit degeneracies on pair correlation functions was found. This may have to do with the large numbers involved before the mean density of states is so high that degeneracies are enforced. This problem awaits further study.

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Figure 2: Level spacing distribution $P(s)$ for $\alpha = 3$ and various $D$. The dashed curves refer to the FD, the dotted curves to the $BE$ subspaces. (a) $D = 2$, (b) $D = 3$, (c) $D = 4$ and (d) $D = 5$. The number of levels included in each diagram was about 96000 for each symmetry subspace. In (c) and (d) there is a strong delta function at the origin. The remaining features seen are not stationary and disappear as the number of levels is increased.