A CLASS OF HESSIAN QUOTIENT EQUATIONS IN EUCLIDEAN SPACE

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Abstract. In this paper, we consider a class of Hessian quotient equations in Euclidean space. Under some sufficient condition, we obtain an existence result by the standard degree theory based on the prior estimates for the solutions to the Hessian quotient equations.

1. Introduction

Let $(M, g)$ be a smooth, compact Riemannian manifold of dimension $n \geq 3$. Define a $(0,2)$ tensor $\eta$ on $M$ by

$$\eta_{ij} = H g_{ij} - h_{ij},$$

where $g_{ij}$ and $h_{ij}$ are the first and second fundamental forms of $M$ respectively, $H(X)$ is the mean curvature at $X \in M$. In fact, $\eta$ is the first Newton transformation of $h$ with respect to $g$. The $\sigma_k$-curvature of $\eta$ is defined by

$$\sigma_k(\lambda(\eta)),$$

where $\sigma_k(\lambda(\eta))$ means $\sigma_k$ is applied to the eigenvalues of $g^{-1}\eta$ and the $k$-th elementary symmetric polynomial $\sigma_k$ is defined by:

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

In this paper, we study the problem of prescribed Weingarten curvature with $(k, l)$-Hessian quotient of $\lambda(\eta)$

$$\frac{\sigma_k(\lambda(\eta))}{\sigma_l(\lambda(\eta))} = f(X, \nu), \quad 2 \leq k \leq n, \quad 0 \leq l \leq k - 2,$$

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on a closed Riemannian manifold $M$, where $M$ is an embedded, closed manifold in $\mathbb{R}^{n+1}$, $f$ is given smooth functions in $\mathbb{R}^{n+1} \times \mathbb{S}^n$. $\nu(X)$ and $\kappa(X) = (\kappa_1(X), \cdots, \kappa_n(X))$ is the unit outer normal and the principal curvatures of hypersurface at $X$. Note that

$$\lambda_i(\eta) = H - \kappa_i = \sum_{j \neq i} \kappa_j, \quad \forall \ i = 1, \cdots, n.$$  

To ensure the ellipticity of (1.1), we have to restrict the class of hypersurfaces.

**Definition 1.1.** A smooth hypersurface $M \subset \mathbb{R}^{n+1}$ is called $(\eta, k)$-convex if $\lambda(\eta) \in \Gamma_k$ for any $X \in M$, where $\Gamma_k$ is the Garding’s cone

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall \ 1 \leq j \leq k \}.$$  

We mainly get the following theorem.

**Theorem 1.1.** Let $n \geq 3, k \geq 2, 0 \leq l < k-1$ and $f \in C^2((B_{r_2}\setminus B_{r_1}) \times \mathbb{S}^n)$ be a positive function. Assume that

$$f(X, \frac{X}{|X|}) \leq \frac{C^k_n}{C^l_n} (\frac{n-1}{r_2})^{k-l} \quad \text{for} \quad |X| = r_2,$$

$$f(X, \frac{X}{|X|}) \geq \frac{C^k_n}{C^l_n} (\frac{n-1}{r_1})^{k-l} \quad \text{for} \quad |X| = r_1,$$

and

$$\frac{\partial}{\partial \rho} \left[ \rho^{k-l} f(X, \nu) \right] \leq 0 \quad \text{for} \quad r_1 \leq |X| \leq r_2,$$

where $\rho = |X|$. Then there exists a $C^{4,\alpha}$, $(\eta, k)$-convex, star-shaped and closed hypersurface $M$ in $\{ r_1 \leq |X| \leq r_2 \}$ satisfies equation (1.1) for any $\alpha \in (0, 1)$.

**Remark 1.2.** The key to proving theorem 1.1 is to obtain the curvature estimate for this Hessian quotient type equation (1.1), which is established in Theorem 3.4. If we replace $\lambda(\eta)$ by $\kappa(X)$, Guan-Ren-Wang showed that the $C^2$ estimate fails for the quotient of curvature equation.

This kind of equations is motivated from the study of many important geometric problems. For example, when $k = n$, (1.1) becomes the following equation for $(\eta, n)$-convex hypersurface:

$$\det(\eta(X)) = f(X, \nu),$$  

(1.6)
which has been studied intensively by Sha [24, 25], Wu [28] and Harvey-Lawson [13]. It is interesting to consider the curvature equation (1.6) and its generalization. In [6], Chu-Jiao establish the curvature estimates for the equation, which replace the left hand of (1.6) by \( \sigma_k(\eta(X)) \). It’s worth noting that Theorem 1.1 recovers the existence results in [6]. In the complex setting, when \( k = n, l = 0 \), the equation (1.1) is called \((n-1)\) Monge-Ampère equation, which is related to the Gauduchon conjecture in complex geometry, more details see [8].

If we replace \( \lambda(\eta) \) by \( \kappa(X) \) and \( l = 0 \) in (1.1), the equation (1.1) becomes the classical prescribed curvature equation

\[
\sigma_k(\kappa(X)) = f(X, \nu),
\]

which has been widely studied in the past two decades. In fact, the curvature estimates are the key part for this prescribed curvature equation. When \( k = n \), the curvature estimates are established by Caffarelli-Nirenberg-Spruck [3]. When \( k = 2 \), the \( C^2 \) estimate for the equation (1.7) was obtained by Guan-Ren-Wang [12]. Spruck-Xiao [26] extended 2-convex case to space forms and give a simple proof for the Euclidean case. In [22, 23], Ren-Wang proved the \( C^2 \) estimate for \( k = n - 1 \) and \( n - 2 \) When \( 2 < k < n \), \( C^2 \) estimate was also proved for equation of prescribing curvature measures problem in [10, 11], where \( f(X, \nu) = \langle X, \nu \rangle \tilde{f}(X) \). Ivochkina [14, 15] considered the Dirichlet problem of the above equation on domains in \( \mathbb{R}^n \), and obtained \( C^2 \) estimates under some extra conditions on the dependence of \( f \) on \( \nu \). Caffarelli-Nirenberg-Spruck [4] and Guan-Guan [9] proved the \( C^2 \) estimate if \( f \) is independent of \( \nu \) and depends only on \( \nu \), respectively. Moreover, Some results have been obtained by Li-Oliker [20] on unit sphere, Barbosa-de Lira-Oliker [2] on space forms, Jin-Li [16] on hyperbolic space, Andrade-Barbosa-de Lira [11] on warped product manifolds.

The organization of the paper is as follows. In Sect. 2 we start with some preliminaries. \( C^0, C^1 \) and \( C^2 \) estimates are given in Sect. 3. In Sect. 4 we prove theorem 1.1

2. Preliminaries

2.1. Setting and General facts. For later convenience, we first state our conventions on Riemann Curvature tensor and derivative notation. Let \( M \) be a smooth manifold and \( g \) be a Riemannian metric on \( M \) with Levi-Civita connection \( \nabla \). For a \((s, r)\)-tensor
field $\alpha$ on $M$, its covariant derivative $\nabla \alpha$ is a $(s, r + 1)$-tensor field given by
\[
\nabla \alpha(Y^1, ..., Y^s, X_1, ..., X_r, X) = \nabla_X \alpha(Y^1, ..., Y^s, X_1, ..., X_r) = X(\alpha(Y^1, ..., Y^s, X_1, ..., X_r)) - \alpha(\nabla_X Y^1, ..., Y^s, X_1, ..., X_r) - \alpha(Y^1, ..., Y^s, X_1, ..., \nabla_X X_r).
\]
The coordinate expression of which is denoted by
\[
\nabla \alpha = (\alpha_{l_1 \cdots l_s}^{1 \cdots k_r}; j_1 \cdots k_r+1)\).
\]
We can continue to define the second covariant derivative of $\alpha$ as follows:
\[
\nabla^2 \alpha(Y^1, ..., Y^s, X_1, ..., X_r, X, Y) = (\nabla_Y (\nabla \alpha))(Y^1, ..., Y^s, X_1, ..., X_r, X).
\]
The coordinate expression of which is denoted by
\[
\nabla^2 \alpha = (\alpha_{l_1 \cdots l_s}^{1 \cdots k_r}; k_r+1).
\]
Similarly, we can also define the higher order covariant derivative of $\alpha$:
\[
\nabla^3 \alpha = \nabla(\nabla^2 \alpha), \nabla^4 \alpha = \nabla(\nabla^3 \alpha), ..., \]
and so on. For simplicity, the coordinate expression of the covariant differentiation will usually be denoted by indices without semicolons, e.g.,
\[
u_i, \quad u_{ij} \quad \text{or} \quad u_{ijk}
\]
for a function $u : M \to \mathbb{R}$.

Our convention for the Riemannian curvature $(3, 1)$-tensor $Rm$ is defined by
\[
Rm(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.
\]
Pick a local coordinate chart $\{x^i\}_{i=1}^n$ of $M$. The component of the $(3, 1)$-tensor $Rm$ is defined by
\[
Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R_{ijk^l} \frac{\partial}{\partial x^l}
\]
and $R_{ijkl} = g_{lm} R_{ijkm}$. Then, we have the standard commutation formulas (Ricci identities):
\[
\alpha_{i_1 \cdots i_s}^{l_1 \cdots l_s} - \alpha_{k_1 \cdots k_r; j i}^{l_1 \cdots l_s} = \sum_{a=1}^r R_{ijk^l} \alpha_{a_1 \cdots a_m}^{l_1 \cdots l_s} - \alpha_{k_1 \cdots k_r}^{l_1 \cdots l_s} - \sum_{b=1}^s R_{ijm} \alpha_{b_1 \cdots b_{l_b+1} \cdots l_r}^{l_1 \cdots l_s}.
\]
Let $M$ be an immersed hypersurface in $\mathbb{R}^{n+1}$. Denote $R_{ijkl}$ to be the Riemannian curvature of $M \subset \mathbb{R}^{n+1}$ with the induced metric $g$. Pick a local coordinate chart $\{x^i\}_{i=1}^n$.
on $M$. Let $\nu$ be a given unit normal and $h_{ij}$ be the second fundamental form $A$ of the hypersurface with respect to $\nu$, that is
\[ h_{ij} = -\left( \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right)_{\mathbb{R}^{n+1}}. \]

Recalling the following identities
\begin{align*}
(2.1) \quad \nabla_i \nabla_j X &= -h_{ij} \nu; \quad \text{Gauss formula} \\
(2.2) \quad \nabla_i \nu &= h_{ij} X^j; \quad \text{Weingarten formula} \\
(2.3) \quad \nabla_k h_{ij} &= \nabla_j h_{ik}; \quad \text{Codazzi equation}
\end{align*}

where $X^j = g^{jk} \nabla_k X$. Moreover, we have
\begin{align*}
(2.4) \quad \nabla_i \nabla_j h_{kl} &= \nabla_k \nabla_l h_{ij} + h_{jk}^m (h_{it} h_{km} - h_{im} h_{kt}) + h_{jl}^m (h_{tj} h_{km} - h_{im} h_{kj}).
\end{align*}

2.2. Star-shaped hypersurfaces in $\mathbb{R}^{n+1}$. Let $M$ be a star-shaped hypersurface in $\mathbb{R}^{n+1}$ which can represented by
\[ X(x) = \rho(x)x, \quad \text{for} \quad x \in S^n, \]
where $X$ is the position vector of the hypersurface $M$ in $\mathbb{R}^{n+1}$.

Let $\{e_1, \ldots, e_n\}$ be a smooth local orthonormal frame field on $S^n$ and $e_\rho$ be the radial vector field in $\mathbb{R}^{n+1}$. $D_i \rho = D_{e_i} \rho$, $D_i D_j \rho = D^2 \rho (e_i, e_j)$ denote the covariant derivatives of $u$ with respect to the round metric $\sigma$ of $S^n$. Then, the following formulas hold:

(i) The tangential vector on $M$ is
\[ X_i = \rho e_i + D_i \rho e_\rho \]
and the corresponding outward unit normal vector is given by
\begin{align*}
(2.5) \quad \nu &= \frac{1}{v} \left( e_\rho - \frac{1}{\rho^2} D^j \rho e_j \right),
\end{align*}
where $v = \sqrt{1 + \rho^{-2} |D \rho|^2}$ with $D^j \rho = \sigma^{ij} D_i \rho$. 

(ii) The induced metric $g$ on $M$ has the form
\[ g_{ij} = \rho^2 \sigma_{ij} + D_i \rho D_j \rho \]
and its inverse is given by
\[ g^{ij} = \frac{1}{\rho^2} \left( \sigma^{ij} - \frac{D^i \rho D^j \rho}{\rho^2 v^2} \right). \]

(iii) The second fundamental form of $M$ is given by
\[ h_{ij} = \frac{1}{v} \left( -D_i D_j \rho + \rho \sigma_{ij} + \frac{2}{\rho} D_i \rho D_j \rho \right) \]
and
\[ h^i_j = \frac{1}{\rho v} \left( \delta^i_j + \left[-\sigma^{ik} + \frac{D^i \rho D^k \rho}{\rho^2 v^2}\right] D_k D_j (\log \rho) \right). \]

The following Newton-Maclaurin inequality (see \[27,21\]) will be used frequently.

**Lemma 2.1.** Let $\lambda \in \mathbb{R}^n$. For $0 \leq l < k \leq n, r > s \geq 0, k \geq r, l \geq s$, the following is the Newton-Maclaurin inequality

\[ k(n - l + 1)\sigma_{l-1}(\lambda)\sigma_k(\lambda) \leq l(n - k + 1)\sigma_l(\lambda)\sigma_{k-1}(\lambda). \]

\[ \frac{[\sigma_k(\lambda)/C^n_k]}{[\sigma_l(\lambda)/C^n_l]} \leq \left[ \frac{\sigma_r(\lambda)/C^n_r}{\sigma_s(\lambda)/C^n_s} \right]^{\frac{r-s}{s}} \text{ for } \lambda \in \Gamma_k. \]

For convenience, we introduce the following notations:

\[ G(\eta) := \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{\frac{l}{k-l}}, \quad G^{ij} := \frac{\partial G}{\partial \eta_{ij}}, \quad G^{ij,rs} := \frac{\partial^2 G}{\partial \eta_{ij} \partial \eta_{rs}}, \quad F^{ii} := \sum_{k \neq i} G^{kk}. \]

Thus,
\[ G^{ii} = \frac{1}{k-l} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{\frac{1}{k-l} - 1} \frac{\sigma_{k-1}(\eta|i)\sigma_l(\eta) - \sigma_k(\eta)\sigma_{l-1}(\eta|i)}{\sigma_l^2(\eta)}. \]

If $\eta = \text{diag}(\mu_1, \mu_2, \cdots, \mu_n)$ with $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. It follows that
\[ G^{11} \geq G^{22} \geq \cdots \geq G^{nn}, \quad F^{11} \leq F^{22} \leq \cdots \leq F^{nn}. \]

To handle the ellipticity of the equation \[1.1\], we need the following important proposition and its proof is the same as Proposition 2.2.3 in \[5\].
Proposition 2.2. Let $M$ be a smooth $(\eta, k)$-convex closed hypersurface in $\mathbb{R}^{n+1}$ and $0 \leq l < k - 1$. Then the operator

$$(2.8) \quad G(\eta^i_j(X)) = \left(\frac{\sigma_k(\lambda(\eta))}{\sigma_l(\lambda(\eta))}\right)^{\frac{1}{k-l}}$$

is elliptic and concave with respect to $\eta^i_j(X)$. Moreover we have

$$(2.9) \quad \sum G^{ii} \geq \left(\frac{C^k_n}{C^n_l}\right)^{\frac{1}{k-l}}.$$

Proposition 2.3. Let $\eta$ be a diagonal matrix with $\lambda(\eta) \in \Gamma_k$, $0 \leq l \leq k - 2$ and $k \geq 3$. Then

$$(2.10) \quad -G^{li,i1}(\eta) = \frac{G^{11} - G^{ii}}{\eta_{ii} - \eta_{11}}$$

for $i \geq 2$.

Proof. We only need to proof the statement in case $l \geq 1$. According to the Proposition 2.1.4 in [5], we know that

$${\partial \sigma_k(\eta) \over \partial \eta^i_j} = \begin{cases} \sigma_{k-1}(\eta|i), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$${\partial^2 \sigma_k(\eta) \over \partial \eta^i_j \partial \eta^{kl}} = \begin{cases} \sigma_{k-2}(\eta|jk), & \text{if } i = j, k = l, i \neq k, \\ -\sigma_{k-2}(\eta|ik), & \text{if } i = l, j = k, i \neq j, \\ 0, & \text{otherwise}. \end{cases}$$

Thus

$$G^{li,i1}(\eta)(\eta_{ii} - \eta_{11}) = {1 \over k - l} \left(\sigma_k(\eta) \over \sigma_l(\eta)\right)^{\frac{1}{k-l} - 1} {1 \over \sigma^2_l(\eta)} \left(\partial^2 \sigma_k(\eta) \over \partial \eta^i_l \partial \eta^i_1\right) \left(\sigma_{k-2}(\eta|1i)\sigma_l(\eta) - \sigma_{l-2}(\eta|1i)\sigma_k(\eta)\right) (\eta_{ii} - \eta_{11})$$

Note that $\sigma_{k-1}(\eta|1) = \sigma_{k-1}(\eta|1i) + \eta_{ii} \sigma_{k-2}(\eta|1i)$, Then

$$G^{li,i1}(\eta)(\eta_{ii} - \eta_{11}) = -{1 \over k - l} \left(\sigma_k(\eta) \over \sigma_l(\eta)\right)^{\frac{1}{k-l} - 1} {1 \over \sigma^2_l(\eta)} \left(\sigma_{k-1}(\eta|1i)\sigma_l(\eta) - \sigma_{l-1}(\eta|1i)\sigma_k(\eta)\right)$$

$$= G^{ii} - G^{11}. \quad \square$$
In order to prove Theorem 1.1, we consider the family of equations as in [1, 18] for $0 \leq t \leq 1$

\[ \sigma_k(\lambda(\eta)) \sigma_l(\lambda(\eta)) = f^t(X, \nu), \]

where

\[ f^t(X, \nu) = tf(X, \nu) + (1 - t)\frac{C^k_n}{C^l_n}(n - 1)^{k-l}\left(\frac{1}{|X|^k} + \epsilon\left(1 - \frac{1}{|X|^k} - 1\right)\right), \]

where the constant $\epsilon$ is small sufficiently such that

\[ \min_{r_1 \leq \rho \leq r_2} \left(\frac{1}{\rho^{k-l}} + \epsilon\left(\frac{1}{\rho^{k-l}} - 1\right)\right) \geq c_0 > 0 \]

for some positive constant $c_0$.

### 3.1. $C^0$ Estimates

Now, we can prove the following proposition which asserts that the solution of the equation (1.1) have uniform $C^0$ bound.

**Theorem 3.1.** Assume $f \in C^2((B_{r_2}\setminus B_{r_1}) \times S^n)$ is a positive function. Under the assumptions (1.3) and (1.4) mentioned in Theorem 1.1, if $M \subset \mathbb{R}^{n+1}$ is a star-shaped, $(\eta, k)$-convex hypersurface satisfied the equation (3.1) for a given $t \in [0, 1]$, then

\[ r_1 < \rho(X) < r_2, \quad \forall X \in M. \]

**Proof.** Assume $\rho(x)$ attains its maximum at $x_0 \in S^n$ and $\rho(x_0) \geq r_2$, then recalling (2.6)

\[ h_j^i = \frac{1}{\rho^v} \left(\delta_j^i + [-\sigma^{im} + \frac{D^i\rho D^m\rho}{\rho^2\nu}]D_jD_m(\log \rho)\right), \]

which implies

\[ h_j^i(x_0) = \frac{1}{\rho}[\delta_j^i - \sigma^{im}D_jD_m(\log \rho)] \geq \frac{1}{\rho}\delta_j^i. \]

Then

\[ \eta_j^i(x_0) = H\delta_j^i - h_j^i \geq \frac{n-1}{\rho}\delta_j^i. \]

Note that $\frac{\sigma_k}{\sigma_l}$ for $0 \leq l \leq k - 2$ is concave in $\Gamma_k$. Thus

\[ \frac{\sigma_k(\lambda(\eta))}{\sigma_l(\lambda(\eta))} \geq \frac{\sigma_k(\frac{n-1}{\rho}\delta_j^i)}{\sigma_l(\frac{n-1}{\rho}\delta_j^i)} = \frac{C^k_n}{C^l_n}\left(\frac{n-1}{\rho}\right)^{k-l}. \]
On the other hand at $x_0$, the unit outer normal $\nu$ is parallel to $M$, i.e., $\nu = \frac{X}{|X|}$. If $\rho(x_0) = r_2$, then
\[ \frac{C_n^k (n-1)^{k-l}}{C_l^n (r_2)} > f'(X,\nu) = \frac{\sigma_k(\lambda(\eta))}{\sigma_l(\lambda(\eta))} \geq \frac{C_n^k (n-1)^{k-l}}{C_l^n (r_2)} \]
which is a contradiction. This shows $\sup \rho \leq r_2$. Similarly, we get $\inf_M \rho \geq r_1$ in view of (1.4).

Now, we prove the following uniqueness result.

**Proposition 3.2.** For $t = 0$, there exists an unique admissible solution of the equation (3.1), namely $M = S^n$.

**Proof.** Let $X$ be a solution of (3.1), for $t = 0$
\[ \frac{\sigma_k(\lambda(\eta))}{\sigma_l(\lambda(\eta))} = \frac{C_n^k (n-1)^{k-l}}{C_l^n (n-1)} \left( \frac{1}{|X|^{k-l}} + \epsilon \left( \frac{1}{|X|^{k-l}} \right) - 1 \right). \]
Assume $\rho(x)$ attains its maximum $\rho_{\text{max}}$ at $x_0 \in S^n$, then
\[ \frac{\sigma_k(\lambda(\eta))}{\sigma_l(\lambda(\eta))} \geq \frac{C_n^k (n-1)^{k-l}}{C_l^n (r_2)} \]
which implies
\[ \rho_{\text{max}} \leq 1. \]
Similarly,
\[ \rho_{\text{min}} \geq 1. \]
Thus, $\rho = 1$ is the unique solution of (3.1) for $t = 0$. \qed

3.2. $C^1$ Estimates. In this section, we establish the gradient estimate for the equation. The treatment of this section follows largely from Lemma 4.1 of [6].

Recalling that a star-shaped hypersurface $M$ in $\mathbb{R}^{n+1}$ can be represented by
\[ X(x) = \rho(x)x \quad \text{for} \quad x \in S^n, \]
where $X$ is the position vector of the hypersurface $M$ in $\mathbb{R}^{n+1}$.

In order to get the gradient estimate, we define a function $u = \langle X, \nu \rangle$. It is clear that
\[ u = \frac{\rho^2}{\sqrt{\rho^2 + |D\rho|^2}}. \]
Theorem 3.3. Under the assumption (1.5), if the closed star-shaped \((\eta, k)\)-convex hypersurface \(M\) satisfying the curvature equation (1.1) and the \(\rho\) has positive upper and lower bound. Then there exists a constant \(C\) depending only on \(n, k, l, \inf \rho, \sup \rho, \inf f\) and \(\|f\|_{C^1}\) such that
\[
|D\rho| \leq C.
\]

Proof. It is sufficient to obtain a positive lower bound of \(\langle X, \nu \rangle\). We consider
\[
\varphi = -\log u + \gamma(|X|^2),
\]
where \(\gamma(t)\) is a function which will be chosen later. Assume \(X_0\) is the maximum value point of \(\varphi\). If \(X\) is parallel to the normal direction \(\nu\) of at \(X_0\), we have \(\langle X, \nu \rangle = |X|^2\). Thus, our result holds. So, we assume \(X\) is not parallel to the normal direction \(\nu\) at \(X_0\), we may choose the local orthonormal frame \(\{e_1, \cdots, e_n\}\) on \(M\) satisfying
\[
\langle X, e_1 \rangle \neq 0, \quad \text{and} \quad \langle X, e_i \rangle = 0, \quad \forall \ i \geq 2.
\]
Using Weingarten equation, we obtain
\[
u_i = \sum_j h_{ij} \langle X, e_j \rangle = h_{i1} \langle X, e_1 \rangle.
\]
Then, we arrive at \(X_0\),
\[
0 = \varphi_i = -\frac{u_i}{u} + 2\gamma' \langle X, e_i \rangle = -\frac{h_{i1} \langle X, e_1 \rangle}{u} + 2\gamma' \langle X, e_i \rangle,
\]
which implies that
\[
h_{11} = 2u\gamma', \quad h_{i1} = 0, \quad \forall \ i \geq 2.
\]
Therefore, we can rotate the coordinate system such that \(\{e_i\}_{i=1}^n\) are the principal curvature directions of the second fundamental form \((h_{ij})\), i.e., \(h_{ij} = h_{ii}\delta_{ij}\). Thus,
\[
0 \geq F^{ii} \varphi_{ii}
\]
\[
= F^{ii} \left( -\frac{u_i}{u} + \frac{u_j^2}{u^2} + \gamma''(|X|^2)_i^2 + \gamma'(|X|^2)_i \right)
\]
\[
= -\frac{F^{ii} u_i}{u} + 4((\gamma')^2 + \gamma'')F^{11} \langle X, e_1 \rangle^2 + \gamma' F^{ii} (|X|^2)_i.
\]
Since \(\eta_{ii} = \sum_{j \neq i} h_{jj}\), we have
\[
\sum_i \eta_{ii} = (n - 1) \sum_i h_{ii}, \quad h_{ii} = \frac{1}{n - 1} \sum_k \eta_{kk} - \eta_{ii}.
\]
It follows that

\[ \sum_i F^{ii} h_{ii} = \sum_i \left( \sum_k G^{kk} - G^{ii} \right) \left( \frac{1}{n-1} \sum_l \eta_{ll} - \eta_{ii} \right) \]

\[ = \sum_i G^{ii} \eta_{ii} \]

\[ = \frac{1}{k-l} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{\frac{1}{k-l}} \sum_i \eta_{ii} \sigma_{k-1}(\eta) - \sigma_k(\eta) \sum_i \eta_{ii} \sigma_{l-1}(\eta) \]

\[ = \tilde{f}, \]

where \( \tilde{f} = f^{\frac{1}{k-l}}. \) Combining with (2.1), we have

\[ \gamma' F^{ii}(|X|^2)_{ii} = 2\gamma' \sum_i F^{ii} - 2\gamma' u \tilde{f}. \]  

Note that the curvature equation (1.1) can be written as

\[ G(\eta) = \tilde{f}, \]

Differentiating (3.7), we obtain

\[ G^{ii} \eta_{iik} = (d_X \tilde{f})(e_k) + h_{kk}(d_v \tilde{f})(e_k). \]

In fact

\[ F^{ii} h_{iik} = \sum_i \left( \sum_j G^{jj} - G^{ii} \right) h_{iik} \]

\[ = \left( \sum_j G^{jj} \right) H_k - \sum_i G^{ii} h_{iik} \]

\[ = \sum_i G^{ii} \eta_{iik} \]

\[ = (d_X \tilde{f})(e_k) + h_{kk}(d_v \tilde{f})(e_k). \]

Using Gauss formula (2.1), Weingarten formula (2.2) and Codazzi formula (2.3)

\[ u_i = h_{ii} \langle X, e_i \rangle, \quad u_{ii} = \sum_k h_{iik} \langle X, e_k \rangle - uh_{ii}^2 + h_{ii}. \]

Then

\[ -\frac{F^{ii} u_{ii}}{u} = -\frac{\langle X, e_1 \rangle}{u} \left( (d_X \tilde{f})(e_1) + h_{11}(d_v \tilde{f})(e_1) \right) + F^{ii} h_{ii}^2 - \tilde{f}. \]
Substituting (3.9) and (3.6) into (3.4),

\[
0 \geq -\frac{\langle X, e_1 \rangle}{u} \left( (d_X \bar{f})(e_1) + h_{11}(d_\nu \bar{f})(e_1) \right) + F_{ii} h_{ii}^2 - \frac{\bar{f}}{u} + 4((\gamma')^2 + \gamma'') F^{11} \langle X, e_1 \rangle^2 + 2\gamma' \sum_i F_{ii} - 2\gamma' u \bar{f}.
\]

Since $X = \langle X, e_1 \rangle e_1 + \langle X, \nu \rangle \nu$,

\[
(d_X \bar{f})(X) = \langle X, e_1 \rangle (d_X \bar{f})(e_1) + \langle X, \nu \rangle (d_\nu \bar{f})(\nu).
\]

From (1.5) and $X = \rho(x)x$, we see that

\[
0 \geq \frac{\partial}{\partial \rho} (\rho^{k-l} f) = \frac{\partial}{\partial \rho} \left( \rho^{k-l} \tilde{f}^{k-l} \right)
= (k-l)(\rho \tilde{f})^{k-l}(\tilde{f} + (d_X \bar{f})(X))
= (k-l)(\rho \tilde{f})^{k-l} \left( \tilde{f} + \langle X, e_1 \rangle (d_X \bar{f})(e_1) + \langle X, \nu \rangle (d_\nu \bar{f})(\nu) \right).
\]

It follows that

\[
- \left( \tilde{f} + \langle X, e_1 \rangle (d_X \bar{f})(e_1) \right) \geq u(d_\nu \bar{f})(\nu),
\]

which implies

\[
0 \geq (d_\nu \bar{f})(\nu) - 2\gamma' \langle X, e_1 \rangle (d_\nu \bar{f})(e_1) + F_{ii} h_{ii}^2 + 4((\gamma')^2 + \gamma'') F^{11} \langle X, e_1 \rangle^2 + 2\gamma' \sum_i F_{ii} - 2\gamma' u \bar{f}.
\]

Choosing

\[
\gamma(t) = \frac{\beta}{t},
\]

where $\beta$ is a constant to be determined later. Recalling that $h_{11} = 2\gamma' u < 0$ at $X_0$. From $H > 0$ we know that

\[
F^{11} = \sum_{j \neq 1} G^{ij} \geq \frac{1}{2} \sum_i G^{ii} = \frac{1}{2(n-1)} \sum_i F_{ii} \geq \frac{1}{2} \left( \frac{C^n_k}{C^n_l} \right)^{\frac{1}{k-l}}.
\]

Substituting these into (3.11),

\[
0 \geq -C \sum_i F_{ii} - 2C \gamma' \langle X, e_1 \rangle \sum_i F_{ii} + \frac{4}{n-1} (\gamma')^2 + \gamma' \langle X, e_1 \rangle^2 \sum_i F_{ii} + 2\gamma' \sum_i F_{ii},
\]
which implies

\[
0 \geq \frac{4}{n-1}(\beta^2 + \frac{4\beta^2}{\rho^2})(X,e_1)^2 - 2C\frac{\beta}{\rho^1}|\langle X, e_1 \rangle| - \frac{2\beta}{\rho^1} - C
\]

where \(\rho_1 = \inf_M \rho, \rho_2 = \sup_M \rho\). So we can choose \(\beta\) sufficiently large such that

\[|\langle X, e_1 \rangle| < \frac{1}{2}\inf_M \rho,\]

combining with the fact \(\rho^2 = u^2 + |\langle X, e_1 \rangle|^2\), we obtain

\[u(X_0) \geq C.\]

So our proof is completed. \(\square\)

3.3. \(C^2\) Estimates. Under the assumption \(1.3\), \(1.4\) and \(1.5\), from Theorem 3.1 and 3.3 we know that there exists a positive constant \(C\) depending on \(\inf_M \rho\) and \(\|\rho\|_{C^1}\) such that

\[
\frac{1}{C} \leq \inf_M u \leq u \leq \sup_M u \leq C.
\]

Theorem 3.4. Let \(M\) be a closed star-shaped \((\eta, k)\)-convex hypersurface satisfying the curvature equation \(1.1\) for some positive function \(f \in C^2(\Gamma)\), where \(\Gamma\) is an open neighborhood of the unit normal bundle of \(M\) in \(\mathbb{R}^{n+1} \times S^n\). Then, there exists a constant \(C\) depending only on \(n, k, \|\rho\|_{C^1}, \inf_M \rho, \inf f, \|f\|_{C^2}\) such that for \(1 \leq i \leq n\)

\[|\kappa_i(X)| \leq C, \quad \forall X \in M.\]

Proof. Since \(\eta \in \Gamma_k \subset \Gamma_1\), we see that the mean curvature is positive. It suffices to prove that the largest curvature \(\kappa_{\max}\) is uniformly bounded from above. Taking the auxiliary function

\[Q = \log \kappa_{\max} - \log(u - a) + \frac{A}{2}|X|^2,\]

where \(a = \frac{1}{2}\inf_M (u) > 0\) and \(A > 1\) is a constant to be determined later. Assume that \(X_0\) is the maximum point of \(Q\). We choose a local orthonormal frame \(\{e_1, e_2, \cdots, e_n\}\) near \(X_0\) such that

\[h_{ii} = \delta_{ij}h_{jj}, \quad h_{11} \geq h_{22} \geq \cdots \geq h_{nn}\]

at \(X_0\). Recalling that \(\eta_{ii} = \sum_k \eta_{kk} h_{kk}\), we have

\[\eta_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn}.\]
It can follow that
\[ G_{11} \geq G_{22} \geq \cdots \geq G_{nn}, \quad F_{11} \leq F_{22} \leq \cdots \leq F_{nn}. \]

We define a new function \( W \) by
\[
W = \log h_{11} - \log(u - a) + \frac{A}{2}|X|^2.
\]

Since \( h_{11}(X_0) = \kappa_{\text{max}}(X_0) \) and \( h_{11} \leq \kappa_{\text{max}} \) near \( X_0 \), \( W \) achieves a maximum at \( X_0 \).

Hence
\[
(3.13) \quad 0 = W_i = \frac{h_{11i}}{h_{11}} - \frac{u_i}{u - a} + A\langle X, e_i \rangle
\]
and
\[
(3.14) \quad 0 \geq F_{ii}W_i = F_{ii}(\log h_{11})_{ii} - F_{ii}(\log(u - a))_{ii} + \frac{A}{2}F_{ii}(|X|^2)_{ii}.
\]

We divide our proof in four steps.

**Step 1:** We show that
\[
0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} G_{i1i}h_{11i}^2 - \frac{F_{ii}h_{11i}^2}{h_{11}^2} + \frac{AF_{ii}h_{11i}^2}{u - a} + \frac{F_{ii}u_i^2}{(u - a)^2} + A \sum_i F_{ii} - C_0 h_{11} - C_0 \frac{1}{h_{11}} - AC_0,
\]
where \( C_0 \) depend on \( \inf_M \rho \), \( \|\rho\|_{C^2} \) and \( \|f\|_{C^2} \) and satisfy \( 1 + \sum_i \langle X, e_i \rangle^2 \leq C_0 \).

We apply the similar argument in (3.6),
\[
(3.16) \quad \frac{A}{2}F_{ii}(|X|^2)_{ii} = A \sum_i F_{ii}(1 - h_{ii}\langle X, \nu \rangle) = A \sum_i F_{ii} - Au f^\frac{1}{k_i - 1}.
\]

Using the similar argument in (3.8), we obtain
\[
F_{ii}h_{iik} = (d_X \tilde{f})(e_k) + h_{kk}(d_\nu \tilde{f})(e_k).
\]

By Gauss formula (2.1), Weingarten formula (2.2) and Codazzi formula (2.3)
\[
u_i = h_{ii}\langle X, e_i \rangle, \quad u_{ii} = \sum_k h_{iik}\langle X, e_k \rangle - uh_{ii}^2 + h_{ii}.
\]
It follows that
\begin{equation}
(3.17)
-F^{ii}(\log(u-a))_{ii} = -\frac{F^{ii}u_{ii}}{u-a} + \frac{F^{ii}u_{ii}^2}{(u-a)^2} \\
= -\frac{1}{u-a} \sum_{k} F^{ii}h_{iik}(X, e_k) + \frac{uF^{ii}h_{ii}^2}{u-a} - \frac{F^{ii}u_{ii}}{u-a} + \frac{F^{ii}u_{ii}^2}{(u-a)^2} \\
= -\frac{1}{u-a} \sum_{k} F^{ii}h_{iik}(X, e_k) + \frac{uF^{ii}h_{ii}^2}{u-a} - \frac{\tilde{f}}{u-a} + \frac{F^{ii}u_{ii}^2}{(u-a)^2} \\
\geq -\frac{1}{u-a} \sum_{k} h_{kk}(d_\nu \tilde{f})(e_k)(X, e_k) + \frac{uF^{ii}h_{ii}^2}{u-a} - \frac{\tilde{f}}{u-a} + \frac{F^{ii}u_{ii}^2}{(u-a)^2} - C_1,
\end{equation}
where $C_1$ depend on $\inf_M \rho$, $\|\rho\|_{C^1}$ and $\|f\|_{C^1}$.

Differentiating (3.7) twice, we obtain
\begin{equation}
F^{ii}h_{i1i1} = G^{ii}h_{i1i1} \geq -G^{ij,rs}h_{ij1}h_{rs1} + \sum_{k} h_{11k}(d_\nu \tilde{f})(e_k) - C_2h_{11}^2 - C_2,
\end{equation}
where $C_2$ depend on $\|f\|_{C^2}$. Applying the concavity of $G$ and Codazzi formula, we have
\begin{equation}
-G^{ij,rs}h_{ij1}h_{rs1} \geq -2 \sum_{i \geq 2} G^{1i,1i}h_{i1i1}^2 = -2 \sum_{i \geq 2} G^{1i,1i}h_{11}^2 = -2 \sum_{i \geq 2} G^{1i,1i}h_{i1i1}^2.
\end{equation}
Combining with (2.4), we have
\begin{equation}
F^{ii}(\log h_{11})_{ii} = \frac{F^{ii}h_{11i1}}{h_{11}} - \frac{F^{ii}h_{11i1}^2}{h_{11}^2} \\
= \frac{F^{ii}}{h_{11}} (h_{i1i1} + (h_{i1}^2 - h_{ii}h_{11})h_{ii} + (h_{ii}h_{11} - h_{i1}^2)h_{11}) - \frac{F^{ii}h_{11i1}^2}{h_{11}^2} \\
= \frac{F^{ii}}{h_{11}} h_{i1i1} - F^{ii}h_{ii11} - \frac{\tilde{f}}{h_{11}} - \frac{F^{ii}h_{11i1}^2}{h_{11}^2} \\
\geq -\frac{2}{h_{11}} \sum_{i \geq 2} G^{1i,1i}h_{11i1}^2 + \frac{1}{h_{11}} \sum_{k} h_{11k}(d_\nu \tilde{f})(e_k) - C_2\frac{1}{h_{11}} \\
- \frac{F^{ii}h_{11i1}^2}{h_{11}^2} - F^{ii}h_{ii11}^2 - C_2h_{11}.
Combining (3.14), (3.16), (3.17) and (3.18), we have

\[ 0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} G^{1,i,i} h_{11}^2 - \frac{F^{ii} h_{11}^2}{h_{11}} \]

\[ + \frac{1}{h_{11}} \sum_k h_{11k}(d_\nu \hat{f})(e_k) - \frac{1}{u-a} \sum_k h_{kk}(d_\nu \hat{f})(e_k) \langle X, e_k \rangle \]

\[ + \frac{a F^{ii} h_{ii}^2}{u-a} + \frac{F^{ii} u_i^2}{(u-a)^2} + A \sum_i F^{ii} - C_3 h_{11} - C_3 \frac{1}{h_{11}} - AC_3. \]

By (2.3) and (3.13),

\[ 0 \geq -\frac{2}{h_{11}} \sum_k h_{11k}(d_\nu \hat{f})(e_k) - \frac{1}{u-a} \sum_k h_{kk}(d_\nu \hat{f})(e_k) \langle X, e_k \rangle \]

\[ \geq - C_3 A, \]

which implies the inequality (3.15).

**Step 2:** There exists a positive constant \( \delta \) such that

\[ C_{n-1}^{k-1}[1 - (n - 2)\delta]^{k-1} - (n-1)\delta C_{n-2}^{k-2}[1 + (n - 2)\delta]^{k-2} > \frac{C_{n-1}^{k-1}}{2C_n^l}. \]

Let

\[ A = \left( \frac{\|f\|_{C_0}^{k-1} - \frac{2kC_n^l}{(n-k+1)C_n^{k-1}} + 1}{n-k+1} \right) C_0. \]

We show that there exist constants \( B_1 > 1 \) depending on \( n, k, l, \delta, \inf M \rho, \|\rho\|_{C^1} \) and \( \|f\|_{C^2} \), such that

\[ \frac{a F^{ii} h_{ii}^2}{2(u-a)} + \frac{A}{2} \sum_i F^{ii} \geq C_0 h_{11}, \]

if \( h_{11} \geq B_1 \).

**Case 1:** \( |h_{ii}| \leq \delta h_{11} \) for all \( i \geq 2 \).

In this case we have

\[ |\eta_{11}| \leq (n-1)\delta h_{11}, \quad [1 - (n - 2)\delta]h_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn} \leq [1 + (n - 2)\delta]h_{11}. \]
By the definition of $G^{ii}$ and $F^{ii}$, we obtain

\[
\sum F^{ii} = (n - 1) \sum G^{ii}
\]

\[
= \frac{n - 1}{k - l} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right) \left( \sum_{\sigma\leq\eta} \eta \right) - (n - l + 1) \sigma_k(\eta) \sigma_l(\eta) 
\]

\[
\geq \frac{C_n^k}{C_{n-1}^k} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right) \left( \sum_{\sigma\leq\eta} \eta \right) - (n - l + 1) \sigma_k(\eta) \sigma_l(\eta) 
\]

it implies that

\[
C_0 h_{11} \leq A \sum F^{ii}. 
\]

Case 2: $h_{22} > \delta h_{11}$ or $h_{nn} < -\delta h_{11}$.

In this case, we have

\[
a F^{ii} h_{ii}^2 
\]

\[
\geq \frac{a}{2 (\sup u - a)} \left( F^{22} h_{22}^2 + F^{nn} h_{nn}^2 \right) 
\]

\[
\geq \frac{a \delta^2}{2 (\sup u - a)} F^{22} h_{11}^2 
\]

\[
\geq \frac{a \delta^2}{2 n (\sup u - a)} \sum_i G^{ii} h_{11}^2 
\]

\[
\geq \left( \frac{C_n^k}{C_{n-1}^k} \right) \left( \frac{a \delta^2}{2 n (\sup u - a)} \right) h_{11}. 
\]

Then, we have

\[
\frac{a F^{ii} h_{ii}^2}{2 (u - a)} \geq C_0 h_{11} 
\]

if

\[
h_{11} \geq \left( \frac{C_n^k}{C_{n-1}^k} \right) \left( \frac{a \delta^2}{2 n (\sup u - a)} \right)^{-1} C_0. 
\]
**Step 3:** We show that

\[ |h_{ii}| \leq C_6 A, \]

if \( h_{11} \geq B_1 \), where \( C_6 \) is a constant depending on \( n, k, l, \inf \rho, \| \rho \|_{C^1} \) and \( \| f \|_{C^2} \).

Combining Step 1 and Step 2, we obtain

\[
0 \geq - \frac{2}{h_{11}} \sum_{i \geq 2} G_{i,i}^1 h_{11i}^2 - \frac{F_{i,i}^1 h_{ii}^2}{h_{11}^2} + \frac{a F_{i,i}^1 h_{ii}^2}{2(u - a)} + \frac{F_{i,i}^1 u_i^2}{(u - a)^2} + \frac{A}{2} \sum_i F_{i,i} - C_0 \frac{1}{h_{11}} - AC_0.
\]

(3.25)

Using (3.13), the Concavity of \( G \) and the Cauchy-Schwarz inequality,

\[
0 \geq - \frac{1 + \epsilon}{(u - a)^2} F_{i,i}^1 u_i^2 - (1 + \frac{1}{\epsilon}) A^2 F_{i,i}^1 (X, e_i)^2 \\
+ \frac{a F_{i,i}^1 h_{ii}^2}{2(u - a)} + \frac{F_{i,i}^1 u_i^2}{(u - a)^2} + \frac{A}{2} \sum_i F_{i,i} - C_0 \frac{1}{B_1} - AC_0 \\
\geq \left( \frac{a}{2(u - a)} - \frac{C_0 \epsilon}{(u - a)^2} \right) F_{i,i}^1 h_{ii}^2 - \left( 1 + \frac{1}{\epsilon} \right) A^2 C_0 - \frac{A}{2} \sum_i F_{i,i} - 2AC_0,
\]

where we used \( u_i = h_{ii} (X, e_i) \) in the second inequality. Choosing \( \epsilon = \frac{(u - a) a}{4C_0} \), then

\[
0 \geq \frac{a}{4(u - a)} F_{i,i}^1 h_{ii}^2 - \left( 1 + \frac{4C_0}{(u - a) a} \right) A^2 C_0 - \frac{A}{2} \sum_i F_{i,i} - 2AC_0
\]

(3.26)

\[
\geq \frac{a}{4(\sup u - a)} F_{i,i}^1 h_{ii}^2 - \left( 1 + \frac{4C_0}{a^2} \right) A^2 C_0 - \frac{A}{2} \sum_i F_{i,i} - 2AC_0.
\]

Note that \( \sum_i F_{i,i} = (n - 1) \sum_i G_{i,i} \geq (n - 1) \left( \frac{C}{C_n} \right)^{\frac{1}{k - l}} \) and

\[
F_{i,i} \geq F^{22} \geq \frac{1}{n(n - 1)} \sum_i F_{i,i}.
\]

Then (3.26) gives that

\[
0 \geq \frac{a}{4(\sup u - a) n(n - 1)} \left( \sum_{k \geq 2} h_{kk}^2 \right) \sum_i F_{i,i} - \left( 1 + \frac{4C_0}{a^2} \right) A^2 C_0 - \frac{A}{2} + \frac{2C_0}{n - 1} A \left( \frac{C}{C_n} \right)^{\frac{1}{k - l}} \sum_i F_{i,i},
\]

which implies that

\[
\sum_{k \geq 2} h_{kk}^2 \leq C_6 A^2.
\]
Step 4: We show that there exists a constant $C$ depending on $n, k, l, \inf_M \rho, \|\rho\|_{C^1}$ and $\|f\|_{C^2}$ such that

$$h_{11} \leq C.$$  

Without loss of generality, we assume that

$$h_{11} \geq \max \left\{ B_1, \left( \frac{32nC_0A^2(sup\ u - a)}{\epsilon a} \right)^{\frac{1}{2}}, \frac{C_0A}{\alpha} \right\},$$  

where $\alpha < 1$ will be determined later. Recalling $u_1 = h_{11}(X, e_1)$, by (3.13) and the Cauchy-Schwarz inequality, we have

$$\frac{F^{11}h_{11}^2}{h_{11}^2} \leq \frac{1 + \epsilon}{(u - a)^2} F^{11}u_1^2 + (1 + \epsilon)A^2F^{11}(X, e_1)^2$$  

and

$$\frac{F^{11}h_{11}^2}{h_{11}^2} \leq \frac{F^{11}u_1^2}{(u - a)^2} + C_0\epsilon F^{11}h_{11}^2 + \frac{(1 + \epsilon)C_0A^2F^{11}}{\epsilon}.$$  

We choose $\epsilon$ sufficiently small such that

$$\frac{F^{11}h_{11}^2}{h_{11}^2} \leq \frac{F^{11}u_1^2}{(u - a)^2} + \frac{aF^{ii}h_{ii}^2}{16(u - a)} + \frac{2C_0A^2F^{11}}{\epsilon}.$$  

Hence Combining with Step 3 and (3.27), we know that

$$\frac{F^{11}h_{11}^2}{h_{11}^2} \leq \frac{F^{11}u_1^2}{(u - a)^2} + \frac{aF^{ii}h_{ii}^2}{8(u - a)}$$  

and

$$|h_{ii}| \leq \alpha h_{11}, \quad \forall i \geq 2.$$  

Thus

$$\frac{1}{h_{11}} \leq \frac{1 + \alpha}{h_{11} - h_{ii}}.$$  

Combining with Proposition 2.3, we obtain

$$\sum_{i \geq 2} \frac{F^{ii}h_{11i}^2}{h_{11}^2} = \sum_{i \geq 2} \frac{F^{ii} - F^{11}h_{11i}^2}{h_{11}^2} + \sum_{i \geq 2} \frac{F^{11}h_{11i}^2}{h_{11}^2}$$  

$$\leq \frac{1 + \alpha}{h_{11}} \sum_{i \geq 2} \frac{F^{ii} - F^{11}h_{11i}^2}{h_{11}^2} + \sum_{i \geq 2} \frac{F^{11}h_{11i}^2}{h_{11}^2}$$  

$$= \frac{1 + \alpha}{h_{11}} \sum_{i \geq 2} \frac{G^{11} - G^{ii}h_{11}^2}{\eta_{ii} - \eta_{11}} + \sum_{i \geq 2} \frac{F^{11}h_{11i}^2}{h_{11}^2}$$  

$$= -\frac{1 + \alpha}{h_{11}} \sum_{i \geq 2} \frac{G^{1i,i}h_{11i}^2}{h_{11}^2} + \sum_{i \geq 2} \frac{F^{11}h_{11i}^2}{h_{11}^2}.$$  

Using (3.13), Cauchy-Schwarz inequality and the fact \( u_i = h_{ii} \langle X, e_i \rangle \), we have

\[
\sum_{i \geq 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} \leq 2 \sum_{i \geq 2} \frac{F^{11}u_i^2}{(u - a)^2} + 2A^2 \sum_{i \geq 2} F^{11} \langle X, e_i \rangle^2
\]

\[(3.31)\]

\[
\leq 2 \frac{C_0}{a^2} \sum_{i \geq 2} \frac{aF^{11}h_{11i}^2}{(u - a)} + 2nC_0A^2F^{11}
\]

\[
\leq \alpha^2 \frac{2nC_0aF^{11}h_{11i}^2}{a^2(u - a)} + \frac{\epsilon a}{16(\sup u - a)} F^{11}h_{11i}^2.
\]

We choose \( \alpha \) sufficiently small such that \( \alpha \leq \min \left\{ \sqrt{\frac{a^2}{32nC_0}}, 1 \right\} \), (3.30) and (3.31) implies that

\[
\sum_{i \geq 2} \frac{F^{11}h_{11i}^2}{h_{11}^2} \leq - \frac{2}{h_{11}} \sum_{i \geq 2} G^{11,i,i}h_{11i}^2 + \frac{aF^{11}h_{11i}^2}{8(u - a)}.
\]

(3.32)

Substituting (3.29) and (3.32) into (3.25), we obtain that

\[
0 \geq - \frac{aF^{11}h_{11i}^2}{4(u - a)} + \frac{A}{2} \sum_i F^{11} - C_0(A + 1)
\]

\[(3.33)\]

\[
\geq \frac{C_0}{2} h_{11} - C_0(A + 1),
\]

which implies that

\[
h_{11} \leq 2(A + 1).
\]

\[\square\]

4. The proof of Theorem 1.1

In this section, we use the degree theory for nonlinear elliptic equation developed in [19] to prove Theorem 1.1. The proof here is similar to [1, 16, 18]. So, only sketch will be given below.

After establishing the priori estimates in Theorem 3.1, Theorem 3.3, and Theorem 3.15, we know that the equation (1.1) is uniformly elliptic. From [7], [17], and Schauder estimates, we have

\[
|\rho|_{C^{4,\alpha}(\mathbb{S}^n)} \leq C
\]

for any \((\eta, k)\)-convex solution \(M\) to the equation (1.1), where the position vector of \(M\) is \(X = \rho(x)x\) for \(x \in \mathbb{S}^n\). We define

\[
C_0^{4,\alpha}(\mathbb{S}^n) = \{ \rho \in C^{4,\alpha}(\mathbb{S}^n) : M \text{ is } (\eta, k) - \text{convex}\}.
\]
Let us consider

\[ F(\cdot; t) : C^{4,\alpha}_0(S^n) \rightarrow C^{2,\alpha}(S^n), \]

which is defined by

\[ F(\rho, x; t) = \sigma_k(\lambda(\eta)) \sigma_l(\lambda(\eta)) - f^t(X, \nu), \]

where

\[ f^t(X, \nu) = tf(X, \nu) + (1 - t)C_k^n C_l^n (n - 1)^{k-l} \left( \frac{1}{|X|^{k-l}} + \epsilon \left( \frac{1}{|X|^{k-l}} - 1 \right) \right), \]

where the constant \( \epsilon \) is small sufficiently such that

\[ \min_{r_1 \leq \rho \leq r_2} \left( \frac{1}{\rho^{k-l}} + \epsilon \left( \frac{1}{\rho^{k-l}} - 1 \right) \right) \geq c_0 > 0 \]

for some positive constant \( c_0 \). Let

\[ \mathcal{O}_R = \{ \rho \in C^{4,\alpha}_0(S^n) : |\rho|_{C^{4,\alpha}(S^n)} < R \}, \]

which clearly is an open set of \( C^{4,\alpha}_0(S^n) \). Moreover, if \( R \) is sufficiently large, \( F(\rho, x; t) = 0 \) has no solution on \( \partial \mathcal{O}_R \) by the prior estimate established in (4.1). Therefore the degree \( \deg(F(\cdot; t), \mathcal{O}_R, 0) \) is well-defined for \( 0 \leq t \leq 1 \). Using the homotopic invariance of the degree, we have

\[ \deg(F(\cdot; 1), \mathcal{O}_R, 0) = \deg(F(\cdot; 0), \mathcal{O}_R, 0). \]

Theorem 3.2 shows that \( \rho_0 = 1 \) is the unique solution to the above equation for \( t = 0 \). Direct calculation show that

\[ F(s, x; 0) = -\epsilon C_k^n C_l^n (n - 1)^{k-l} \left( \frac{1}{s^{k-l}} - 1 \right). \]

Then

\[ \delta_{\rho_0} F(\rho_0, x; 0) = \frac{d}{ds} \big|_{s=1} F(s\rho_0, x; 0) = \epsilon C_k^n C_l^n (n - 1)^{k-l} (k-l) > 0, \]

where \( \delta F(\rho_0, x; 0) \) is the linearized operator of \( F \) at \( \rho_0 \). Clearly, \( \delta F(\rho_0, x; 0) \) takes the form

\[ \delta_w F(\rho_0, x; 0) = -a^{ij} w_{ij} + b^i w_i + \epsilon C_k^n C_l^n (n - 1)^{k-l} (k-l), \]

where \( a^{ij} \) is a positive definite matrix. Since \( \epsilon C_k^n C_l^n (n - 1)^{k-l} (k-l) > 0 \), thus \( \delta_{\rho_0} F(\rho_0, x; 0) \) is an invertible operator. Therefore,

\[ \deg(F(\cdot; 1), \mathcal{O}_R; 0) = \deg(F(\cdot; 0), \mathcal{O}_R, 0) = \pm 1. \]
So, we obtain a solution at $t = 1$. This completes the proof of Theorem \[\text{Theorem 1.1}\].

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