Conceptually Diverse Base Model Selection for Meta-Learners in Concept Drifting Data Streams

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Abstract

Meta-learners and ensembles aim to combine a set of relevant yet diverse base models to improve predictive performance. However, determining an appropriate set of base models is challenging, especially in online environments where the underlying distribution of data can change over time. In this paper, we present a novel approach for estimating the conceptual similarity of base models, which is calculated using the Principal Angles (PAs) between their underlying subspaces. We propose two methods that use conceptual similarity as a metric to obtain a relevant yet diverse subset of base models: (i) parameterised threshold culling and (ii) parameterless conceptual clustering. We evaluate these methods against thresholding using common ensemble pruning metrics, namely predictive performance and Mutual Information (MI), in the context of online Transfer Learning (TL), using both synthetic and real-world data. Our results show that conceptual similarity thresholding has a reduced computational overhead, and yet yields comparable predictive performance to thresholding using predictive performance and MI. Furthermore, conceptual clustering achieves similar predictive performances without requiring parameterisation, and achieves this with lower computational overhead than thresholding using predictive performance and MI when the number of base models becomes large.

Keywords— Base Model Selection, Concept Drift, Conceptual Similarity, Meta-Learners, Online Transfer Learning

1 Introduction

Learning in online data streams can be challenging as data availability may be limited if a rich history of observations cannot be retained [1]. Additionally, in dynamic, non-stationary environments, the underlying distribution of data, and the mapping from observations to response variables, referred to as a concept, can change over time [2], a phenomenon known as concept drift [3]. Meta-learners and ensembles can be used to improve predictive performance in online environments by combining models learnt historically throughout the data stream [4]. Historical models, which we refer to as base models, can be learnt to represent each concept
encountered as the data stream progresses. Base models can then be used as input to a meta-learner or ensemble to enhance predictive capabilities [5]. However, as the data stream progresses, and more base models are learnt, meta-learners and ensembles can become prone to overfitting, particularly when data availability is limited and the number of base models is large [6].

To improve generalisation, a relevant yet diverse subset of base models can be selected to be used as input to the meta-learner or ensemble [7]. Base model selection is often achieved within offline environments using metrics such as predictive performance to indicate relevancy, and Mutual Information (MI) between base model predictions to indicate pairwise diversity [8]. However, determining relevancy and diversity can be more challenging in online environments due to concept drift [9]. Encountering concept drift can impact the relevancy of a base model, and therefore requires the relevancy of base models to be continually re-evaluated as the data stream progresses. Many diversity metrics, such as MI, must also be recalculated due to their dependency on the current underlying distribution of data when identifying the covariance between model predictions. Repeatedly recalculating relevancy and diversity metrics may be computationally undesirable when using meta-learners or ensembles in data streams, particularly when drifts are encountered frequently or a large number of potential base models are available.

This challenge is not limited to learning in single online data streams. Meta-learners and ensembles can also be used in online Transfer Learning (TL) [10]. Online TL allows base models to be learnt from different sources of data and combined through the use of a meta-learner or ensemble to improve predictive performance [11]. The Bi-directional Online Transfer Learning (BOTL) framework allows knowledge transfer to be conducted bi-directionally across multiple online data streams [5, 6]. Base models are created to represent the concepts encountered within a data stream, and are transferred so that they can be used to improve the predictive performance in other data streams [5]. The number of base models transferred can quickly become large when the number of data streams in the framework is large, or when concept drifts are frequent. Therefore, selecting a relevant yet diverse subset of base models is required to prevent overfitting [6].

The contributions of this paper are:

(i) a novel approach for estimating the conceptual similarity of base models in concept drifting data streams,
(ii) the parameterised thresholding method for model culling, and the parameterless conceptual clustering method for obtaining a subset of relevant yet diverse base models using conceptual similarity, and
(iii) the use of parameterless conceptual clustering to reduce the number of models transferred in the BOTL framework.

Our conceptual similarity metric is determined using the Principal Angles (PAs) between the subspaces in which each base model was created. Thus, the similarity between pairs of base models remains static in the presence of concept drift and does not require recalculation. We present empirical results applying our methods in combination with BOTL using two synthetic data generators, namely a drifting hyperplane emulator and a smart home heating simulator, and real-world data predicting Time-To-Collision (TTC) from vehicle telemetry. Base models are obtained using three underlying concept drift detection strategies, specifically Reactive Proactive drift detection (RePro) [12], Adaptive Windowing (ADWIN) [4], and Adaptive Windowing with Proactive drift detection (AWPro) [6]. BOTL [6] is used to transfer models bi-directionally between online data streams. We use the thresholding and clustering approaches proposed in this paper to select a subset of base models from those learnt locally within a data stream, and received from other data streams for the Ordinary Least Squares (OLS) meta-learner in BOTL.

We show that conceptual similarity thresholding obtains predictive performances comparable to the existing approaches of performance and MI thresholding, while reducing the computational overhead associated with base models comparisons. Additionally, conceptual clustering achieves similar predictive performances without the need for a user defined culling parameter. Although conceptual clustering increases the computational overhead in comparison to conceptual similarity thresholding, it can be less computationally expensive than thresholding using diversity metrics that are not static, such as MI, when the number of base models is large. We also show that conceptual clustering can be used to reduce the number of models transferred in online TL.
The remainder of this paper is organised as follows. Section 2 outlines existing research relating to the creation of base models, combining base models, and selecting base models in online environments. Section 3 defines our novel approach of estimating the conceptual similarity between base models. In Section 4 we present our methods for selecting a subset of relevant yet diverse base models to be used as input to a meta-learner, namely (i) parameterised thresholding and (ii) parameterless conceptual clustering. This section also discusses how conceptual similarity clustering can be used to reduce the number of models transferred between online data streams in an online TL framework. In Section 5 we provide more insight into why the subset of base models used by a meta-learner should be relevant yet diverse. Section 6 provides details about the BOTL framework used to evaluate our techniques, and discusses the datasets used for empirical evaluation. Section 7 presents our empirical results of using parameterised thresholding and parameterless conceptual clustering as methods to select base models for the OLS-meta-learner in the BOTL framework. Results are also presented that show conceptual clustering can be used to reduce the number of models transferred between data streams in online TL. Finally, Section 8 concludes the paper, outlining our key findings and the possibilities for future work.

2 Related Work

Three important issues must be addressed when using meta-learners or ensembles in online environments. Firstly, base models must initially be created in online data streams. Secondly, base models must be combined to obtain an overarching prediction. Thirdly, if the number of base models becomes large in comparison to the available data, a subset of base models must be selected to be used by the meta-learner or ensemble to prevent overfitting.

2.1 Creating base models

When learning in online environments, predictive models must be updated or relearnt in order to adapt to concept drift [1]. Two common approaches to handling concept drift include incremental learners [2], and Concept Drift Detection strategies (CDDs) [1].

Incremental learners gradually update models parameters as new instances are observed in the data stream [1]. This allows incremental learners to react to concept drift. However, model parameters can be influenced by historical instances associated with a previously encountered concept [13]. Instance weighting and forgetting mechanisms can be used to prevent historical instances from negatively impacting predictive performance when concept drifts are encountered [2]. These techniques enable recently observed instances to have a greater influence on model parameters compared to historical instances, which may not be associated with the current concept [14]. Although incremental learners can be used in online environments, the effect of continually updating model parameters, without the ability to detect occurrences of concept drift, means that knowledge of previously encountered concepts can be lost as the data stream progresses [15]. This can be detrimental in environments that contain recurring concepts since the model parameters for a recurring concept must be re-learnt [16].

Alternatively, CDDs can be used in online learning environments [4, 5, 12]. CDDs typically learn a predictive model over a small window of recent observations. Model predictions are monitored over a sliding window of instances, with new instances added as they are observed [1]. Concept drifts are commonly detected by monitoring the predictive performance [12], or identifying changes to the distribution of predictive error [4]. When concept drifts are detected, alternative predictive models can be learnt or reused [6, 16].

CDDs typically only retain recent observations. This means that they are not influenced by historical instances that belong to a previously encountered concept when detecting drifts or learning model parameters [4]. Since model parameters are not usually incrementally updated when using CDDs, historical knowledge of previously encountered concepts can be retained, and in some approaches are reused when recurring concepts are encountered. For example, RePro [12] detects drifts by monitoring predictive performance. When the performance drops below a threshold, either a new model is learnt from the recent observations captured in the sliding window, or a historical model is reused in the presence of recurring concepts [17]. Other approaches
to detecting concept drifts estimate the precise point of drift by monitoring the distribution of predictive error. ADWIN [4] and AWPro [6] use this approach so that instances belonging to the previous concept can be discarded prior to learning a new model. ADWIN does not make use of historical models when encountering recurring concepts, while AWPro prioritises their reuse over learning new models [4,6].

To learn a model that generalises well when using a CDD, sufficient instances belonging to the new concept must be observed. During this period, ineffective predictions may be made as the previously learnt model continues to be used. This is known as the cold start problem, and introduces a trade-off between reacting quickly to drifts to prevent prolonged use of an ineffective model, and waiting for sufficient observations in order to learn a model that generalises well [13]. This means that predictive models may have to be learnt from small amounts of data. However, in many real-world data streams, historically encountered concepts may be similar to the newly encountered concept. To improve predictive performance, the models learnt to represent each of the concepts encountered throughout the data stream can be used as base models for a meta-learner or ensemble [19, 20, 21, 22]. This allows previously learnt models to be used to enhance predictive performance and reduce the impact of the cold start problem, particularly when concepts re-occur throughout the data stream [23, 24].

Some real-world applications that require predictions to be made in online environments consist of multiple devices, each learning similar tasks from independent data streams. For example, a low cost sensor may be used to make predictions via environmental monitoring in one location, while other low cost sensors conduct the same predictive task in different locations. Although each device learns from independent data streams, knowledge may have been learnt from one data stream that is beneficial to another. Sharing knowledge may improve predictive performance, and can be achieved through online TL techniques such as BOTL [6]. BOTL uses a CDD to create base models that represent the concepts encountered in an online data stream. Models are transferred to other data streams and used alongside locally learnt models as base models. Base model predictions are combined using an OLS meta-learner to improve predictive performance for unseen instances [5, 6]. Knowledge transfer is bi-directional such that knowledge learnt in one data stream is made available to others, and visa versa [6]. Other online TL frameworks learn base models in offline environments and transfer them to an online data stream [25, 26, 27, 11, 10]. However, regardless of the environment in which base models are learnt, meta-learners and ensembles can be used to combine model predictions to improve the predictive performance [7, 20, 8, 23, 28]. In this paper, we use RePro [12], ADWIN [4], and AWPro [6] as CDDs to create base models in the BOTL framework [5, 6].

2.2 Combining base models

In offline environments, the simplest approach to combining base models is to use ensembles with majority rules voting mechanisms for classification tasks, or prediction averaging for regression tasks [20]. However, these approaches give equal weightings to the predictions of each base model, which may not be appropriate in the presence of concept drifts. For example, when a drift is encountered, some base models may not make effective predictions for the new concept. Therefore, equally weighting models in the ensemble may hinder the overall predictive performance. To overcome this, weighted voting mechanisms can be used [23, 24, 22]. Weights can be determined by measuring the predictive performance of each base model over a recent window of instances in the data stream. However, weighted voting mechanisms are most beneficial when base models have been learnt to perform the same tasks, and have comparable success [29]. This may not be the case when learning in online data streams, since models may have been learnt to represent different concepts and may not perform comparably. Additionally, in a regression setting, weighted voting mechanisms rely upon the assumption that the response variable to be predicted will remain within a consistent range across all learning tasks. This may not be the case in online data streams that encounter concept drift, or when transferring knowledge between data streams in online TL frameworks.

Alternatively, since future concepts are unknown, a meta-learner can be used to combine the available base models. In comparison to weighting mechanisms, the use of a meta-learner typically allows for better generalisation by correcting for base model biases when some base models consistently perform poorly for segments of the data stream [29]. Correcting for such bias can greatly impact the overarching predictive performance since the base models obtained from historical concepts, or transferred from other data streams via online TL, are not
guaranteed to be relevant or useful when making predictions for the current concept. As the data stream progresses, the number of available base models may increase, and concept drifts may be encountered. Therefore, the weights associated with each base model must be updated [19,21,26].

Increasing the number of base models used by the meta-learner increases its representational capacity. This allows a better approximation of the underlying distribution of observable data to be made [30], which decreases the training error of the meta-learner, known as the empirical risk. However, when the number of base models becomes large, the meta-learner may overfit the window of observable data [31], which can increase the predictive error for unseen instances, known as the true risk [32]. This often occurs when the representational capacity of the meta-learner is too large to be able to effectively learn the meta-learner model parameters from the fixed sized window of observable data [33,34,30]. In such cases, the empirical risk can be a poor approximation of the true risk [33], which may lead to poor predictive performance for unseen instances. Therefore, to prevent overfitting, the representational capacity of the meta-learner can be reduced by selecting a subset of base models to be used as input. To ensure that the meta-learner makes effective predictions and generalises well for unseen instances, a relevant yet diverse subset of base models should be selected [7]. In this paper, we consider how relevancy and diversity metrics can be used to select a subset of base models to be used as input to the OLS meta-learner in the BOTL framework. Further discussion of how the empirical risk of a meta-learner can be minimised, while remaining a good approximation of the true risk, through the use of relevancy and diversity, is presented in Section 5.

2.3 Selecting base models

Identifying the best subset of base models is challenging in online environments since future concepts are unknown, a rich history of data often cannot be retained, and due to the presence of concept drift. Existing online ensemble techniques, such as Accuracy Weighted Ensemble (AWE) [28], Online Weighted Ensemble (OWE) [23] and Additive Expert ensemble (AddExp) [24], select base models using the recency of a model, the predictive performance, or a combination of the two, as indicators of relevancy. Using only the recency of a model as an indicator of relevancy may be undesirable in online data streams since a historical model may be more relevant than a recently learnt model due to recurring concepts [23,24]. The diversity among base models in online environments is considered less frequently. Five diversity metrics for regression ensembles were presented by Dutta, namely the correlation coefficient between base model predictions, the covariance between model predictions, the pairwise Chi-square of model predictions, the standard deviation of predictions as a disagreement bound, and the MI between model predictions [8]. MI is a frequently used regression ensemble diversity metric, where the pairwise diversity of base models is measured using the MI between predictions of base models over recent observations [21,6]. However, each of the measures of diversity proposed by Dutta [8] are dependent on the recent window of observations used to evaluate base models. This is problematic in concept drifting data streams and online TL environments since the diversity of base models that obtain similar predictive performance for some windows of observations, but have been learnt from different distributions, or learnt to represent different concepts, is not accounted for [21]. Additionally, due to the dependency on a window of recent observations, measuring diversity by the level of disagreement between base model predictions can be influenced by concept drift, and therefore must be recalculated as the data stream progresses.

To overcome this, we introduce a novel method for measuring the diversity of base models for meta-learners in online environments, which allows the pairwise conceptual similarity between base models to be estimated independently of the current distribution of observable data. This approach uses the similarity between the underlying subspaces in which each base model was learnt to obtain a measure of diversity that remains static as the data stream progresses, even in the presence of concept drift.

3 Conceptual Similarity

In this section, using the notation in Table 1, we present our novel method for estimating the conceptual similarity between pairs of base models in order to determine the diversity among base models to be used by meta-learners in online environments.
the unitary matrices and respectively. Using Singular Value Decomposition (SVD) we obtain SVD
variable function, mapping the predictions of base models, \( f \) associated response variable, and let \( X \)  
Definition 1 (Principal Angles (PAs) between subspaces) follows [35, 36, 37].
for two base models \( i \) and \( j \), let \( X_i \in \mathbb{R}^{N \times m} \) and \( X_j \in \mathbb{R}^{N \times m} \) denote the subspaces in which they were learnt, containing \( N \) instances, \( m-1 \) features, and the associated response variable, and let \( U_i \in \mathbb{R}^{N \times p} \) and \( U_j \in \mathbb{R}^{N \times q} \) represent orthonormal bases of \( X_i \) and \( X_j \) respectively. Using Singular Value Decomposition (SVD) we obtain \( \text{SVD}(U_i^\top U_j) = A \Sigma B \), where \( A \) and \( B \) are the unitary matrices and \( \Sigma \in \mathbb{R}^{p \times q} \) is the diagonal matrix of singular values, \( \Sigma = \text{diag}(s_1, \ldots, s_r) \), where \( r = \min(p, q) \) [37]. The PAs between subspaces are given by
\[
\Theta(X_i, X_j) = [\arccos(s_1), \ldots, \arccos(s_r)].
\]
3.1 Estimating conceptual distance

The conceptual similarity between the underlying concepts a pair of base models were learnt to represent can be estimated using PAs. In order to obtain the PAs in Definition 1, an orthonormal representation of the subspaces in which each base model was learnt is required. The Principal Components (PCs) of a subspace is an orthonormal representation, and can be obtained using Singular Value Decomposition (SVD) on the covariance matrix,

\[ \text{SVD}(X^T_i X_i) = U_i \Sigma V, \]

such that \( U_i \in \mathbb{R}^{N \times N} \) are the PCs, and \( \Sigma \) their singular values.

Therefore, to estimate the conceptual similarity of base models, both the model learnt to represent a concept, and an orthonormal representation of the training data must be stored in online learning frameworks, and transferred between data streams when using online TL. In these online learning frameworks, the memory and communication overhead required to store and transfer orthonormal representations of the subspace associated with each base model, \( i \), can be reduced by retaining only the first \( p \) PCs that capture 99.9\% of the variance of the original training data, \( X_i \). The diagonal matrix of singular values, \( \Sigma \), obtained through SVD in Equation 2, can be used to determine the number of PCs to retain.

Let \( \Sigma = \text{diag}([S_{11}, \ldots, S_{NN}]) \) represent the diagonal matrix of singular values in Equation 2. The number of PCs, \( p \), that capture 99.9\% of the variance can be identified using

\[ 1 - \frac{\sum_{i=1}^{p} S_{ii}}{\sum_{i=1}^{N} S_{ii}} \leq 0.001. \]

The percentage of variance captured by the \( p \) PCs can be decreased to reduce the impact of noise in the data stream. This allows the orthonormal representations, and the PAs between them, to be more robust to noise. However, alternative methods of obtaining orthonormal representations of each subspace, such as Laplacian PCA [38] and Robust PCA [39], can be used to improve robustness to noise and outliers in noisy data streams [40]. Once \( p \) has been identified we can reduce the matrix of \( N \) PCs, \( U_i \in \mathbb{R}^{N \times N} \), such that only the first \( p \) PCs are retained, \( \tilde{U}_i \in \mathbb{R}^{N \times p} \). As the reduced orthonormal representation, \( \tilde{U}_i \), captures 99.9\% of variance, for ease of notation throughout the remainder of this paper we use \( U_i \) to denote the reduced matrix of PCs. Therefore, from this point forward, \( U_i \in \mathbb{R}^{N \times p} \).

Using Definition 1, we can calculate the PAs between the subspaces in which two base models \( i \) and \( j \) were learnt, \( X_i \) and \( X_j \), using the reduced PCs, \( U_i \) and \( U_j \), as orthonormal bases. We define the conceptual distance between base models \( i \) and \( j \) as follows.

**Definition 2 (Conceptual distance between base models).** Let \( \Theta(X_i, X_j) \in \mathbb{R}^r \) be the vector of PAs, obtained using Definition 1 between the PCs \( U_i \) and \( U_j \) for \( X_i \) and \( X_j \) respectively. The conceptual distance between base models \( i \) and \( j \) is defined as

\[ d(i, j) = \frac{1}{r} \left( \sum_{h=1}^{r} (1 - \cos(\theta_h)) \right) = 1 - \frac{1}{r} \left( \sum_{h=1}^{r} \cos(\theta_h) \right), \]

where \( \theta_h \in \Theta(X_i, X_j) \) and \( r = \min(p, q) \). Therefore, \( d(i, j) \to 0 \) when base models \( i \) and \( j \) have been learnt from similar subspaces, and \( d(i, j) \to 1 \) when subspaces are dissimilar.

3.2 Estimating conceptual similarity

Finally, to estimate the conceptual similarity among base models, we create an affinity matrix.

**Definition 3 (Conceptual similarity).** The affinity matrix, \( \Delta \in \mathbb{R}^{|\mathcal{M}| \times |\mathcal{M}|} \), where \( |\mathcal{M}| \) is the number of available base models, is given by

\[ \Delta_{ij} = \begin{cases} \exp\left(\frac{-d(i,j)^2}{d_i d_j}\right) & \text{if } i \neq j \\ 0 & \text{otherwise,} \end{cases} \]

(5)
Algorithm 1: Concept similarity thresholding

Input: \( \mathcal{M}, f_i, U_i, \text{Dist}, \lambda_{CS} \)

for \( j \in \mathcal{M} \) do

\[ \text{Dist}_{i,j} = \text{getDistances}(U_i, \mathcal{M}.\text{getPCs}(j)) \text{ using } \text{Def 1} \text{ and Def 2} \]

\[ \text{Dist}_{j,i} = \text{Dist}_{i,j} \]

\[ \text{iAffinities} = \text{getAffinities}(\text{Dist}, i, \mathcal{M}) \text{ using } \text{Def 3} \]

if \( \forall j \in \text{iAffinities} < \lambda_{CS} \) then

Add \( \{f_i, U_i\} \) to \( \mathcal{M} \)

else

\( \forall j \in \mathcal{M} : \) Remove \( \text{Dist}_{i,j} \) and \( \text{Dist}_{j,i} \) from \( \text{Dist} \)

return \( \mathcal{M} \)

where \( \tilde{d}_i \) and \( \tilde{d}_j \) are local scaling parameters, which allows the conceptual difference between base models to be scaled by the surrounding neighbourhoods of \( i \) and \( j \) [41] such that \( \tilde{d}_i = d(i, k) \), where \( k \) is the \( k^{th} \) nearest neighbour of base model \( i \).

Zelnik-Manor and Perona [41] suggest that a value of \( k = 7 \) yields good results for local scaling, even for high-dimensional image segmentation and document classification tasks. However, since the dimensionality of the affinity matrix for base models is likely to be low in comparison to the affinity matrices used by Zelnik-Manor and Perona [41], we investigated the impact of using \( k \) values between 2 and 7. The results of using different local scaling parameters are briefly discussed in Section 7 and we show that these scaling parameter values obtain similar results, therefore in general we use \( k = 7 \). Local scaling is used to allow better affinities to be obtained when the density of conceptually similar base models varies [41]. Although this requires a parameter, \( k \), to be defined, it has been shown that parameter tuning is not typically needed for local scaling to perform well [40]. Alternatively, other scaling techniques could be used, such as density-aware kernels [40, 42], to amplify intra-cluster similarities in order to account for locally dense areas of conceptually similar base models [40].

Using Definition 3, we obtain the affinity matrix, \( \Delta \), where element \( \Delta_{ij} \rightarrow 1 \) when base models \( i \) and \( j \) are conceptually similar, and \( \Delta_{ij} \rightarrow 0 \) when they are dissimilar.

4 Base Model Selection

To select a relevant yet diverse subset of base models, we propose (i) parameterised thresholding and (ii) parameterless conceptual clustering, using the estimated conceptual similarity of base models. We also consider how conceptual clustering can be used to determine when a newly learnt base model should be transferred to other data streams in online TL settings.

4.1 Parameterised thresholding

As a data stream progresses, new models are made available when encountering concept drifts, or are received from other domains via online TL. The pairwise conceptual similarities between a new model, \( f_i \), and existing base models, \( f_j \in \mathcal{M} \), are calculated using the average PA between the two subspaces in which each model was learnt (Definitions 1 and 2), to obtain a locally scaled affinity metric (Definition 3). Given a user defined culling threshold, \( \lambda_{CS} \), models are added to \( \mathcal{M} \) if no existing base model in \( \mathcal{M} \) is considered conceptually similar. This means that in order for a new model, \( f_i \), to be used as input to the meta-learner, its affinity to all other available base models, \( \forall j \in \mathcal{M} : \Delta_{ij} \), is less than the conceptual similarity threshold, \( \lambda_{CS} \). Since conceptual similarity is calculated independently of the underlying distribution of data, a new model, \( f_i \), with an affinity to an existing base model greater than \( \lambda_{CS} \) can be discarded and does not need to be reconsidered as input to the meta-learner. This process, shown in Algorithm 1, obtains a diverse subset of base models.
Using a static diversity metric such as conceptual similarity prevents the need to re-evaluate base models since they do not need to be reconsidered once culled. However, the relevancy of remaining base models is dependent on the current concept. To ensure that a relevant subset of the remaining base models are used by the meta-learner, the predictive performance of each base model is evaluated to create the subset of base models, $M'$, that are used by the meta-learner to make predictions for unseen instances. The predictive performance of the models that remain after executing Algorithm 1, $M$, are evaluated over the current sliding window of data, $W$. Those that achieve an $R^2$ performance greater than a performance threshold, $\lambda_{\text{perf}}$, are included in $M'$. Base models that achieve an $R^2$ performance less than $\lambda_{\text{perf}}$ are temporarily excluded from the meta-learner until a concept drift is encountered. This ensures that the subset of base models used by the meta-learner remains relevant to the current concept.

Selecting culling parameters can be challenging as they are dependent on the presence of noise in the underlying data stream, the number of possible base models, and their separability for a given similarity metric [6]. Values that promote aggressive culling may result in discarding base models that are beneficial to the meta-learner, while values that are not aggressive enough may cause overfitting [6].

To overcome this, culling parameters could be updated as the data stream progresses using cross-validation. However, this would require the meta-learner to be validated for various threshold values over a small window of data. This increases computation, and the performance obtained by the meta-learner through cross-validation would likely be an overestimate since validation splits are unlikely to be independent due to the dependencies between consecutive instances within online data streams [43]. This means that considerable domain expertise is required to select the culling parameters for meta-learners in online environments.

### 4.2 Parameterless conceptual clustering

To prevent the need for domain expertise, we introduce parameterless conceptual clustering. As with conceptual similarity thresholding, when new models are learnt in a data stream or received via online TL, the affinity between existing base models, $f_j \in M$, and the new model, $f_i$, must be calculated using Definitions 1, 2 and 3. Once the affinity matrix, $\Delta$, has been obtained we can consider it as a fully connected graph, where nodes represent available base models, and edges represent their pairwise conceptual similarity. This allows graph clustering algorithms, such as Spectral Clustering (SC), to be used to identify groups of conceptually similar base models. SC algorithms typically have complexity $O(n^2)$ to create the similarity matrix, and $O(n^3)$ for spectral analysis [44], where $n$ is the number of available base models, $|M|$. Therefore, it is important to use a static similarity metric, such as conceptual similarity, so that updating the similarity matrix and clustering available base models is only required when a new base model is learnt, or received from another data stream. Using metrics such as MI would require both the similarity matrix and spectral analysis to be repeatedly updated due to the dependency on the current distribution of observable data. The computational complexity of this makes the use of metrics such as MI for clustering similar base models infeasible for most real-world applications.

We use Self-Tuning Spectral Clustering (STSC), a well known SC algorithm [41], which allows the number of clusters to be determined automatically. STSC uses the local scaling in Equation 5 and incrementally rotates the eigenvectors traditionally obtained from SC to estimate the number of clusters [41]. Automatically determining the number of clusters is advantageous when considering the diversity among base models in online environments, particularly in online TL, where it is not known how many concepts will be encountered in each data stream, or how similar the concepts learnt from different data streams will be.

Once clusters of conceptually similar base models have been identified using Algorithm 2 we create the subset of relevant yet diverse base models, $M'$, to be used as input to the meta-learner by selecting one base model from each cluster. To select a base model from each cluster, we evaluate their predictive performance on the current window of observable data, $W$, and select the model with the highest $R^2$ performance as an indicator of relevancy. In the case where the best performing model in a cluster is the local model, $f_i$, that has been learnt to represent the current concept, we also add the second best performing model in that cluster to $M'$. This ensures that the meta-learner can benefit from the additional support of a model learnt historically, or from another data stream, that is conceptually similar to the current concept.
Algorithm 2: Conceptual clustering base model selection

\textbf{Input:} $W$, $M$, $f_i$, $U_i$, $\text{Dist}$, $\Delta$

\begin{algorithmic}
  \FOR{$j \in M$}
    \STATE $\text{Dist}_{i,j} = \text{getDistances}(U_i, M.\text{getPCs}(j))$ using Def.1 and Def.2
    \STATE $\Delta_{i} = \text{getAffinityMatrix}(\text{Dist})$ using Def.3
  \ENDFOR
  \STATE $\text{clusterGroups} = \text{STSC}(\Delta)$ \[41]\n  \FOR{$c \in \text{clusterGroups}$}
    \STATE Add $\text{bestPerformingModel}(c, W)$ to $M'$
    \IF{$\text{bestPerformingModel}(c, W) = f_i$}
      \STATE Also add $\text{secondBestPerformingModel}(c, W)$ to $M'$
    \ENDIF
  \ENDFOR
  \RETURN $M'$
\end{algorithmic}

4.3 Reducing Transfer in Online TL

Parameterless conceptual clustering can also be used to reduce the number of models transferred in online TL frameworks such as BOTL. We achieve this by clustering locally learnt models prior to knowledge transfer. When a new model is learnt locally, it is only transferred to other data streams if it is assigned to a cluster that does not contain a model that has previously been transferred. Although this increases computation prior to transfer\[1\], it reduces the communication overhead by preventing two conceptually similar models, learnt from the same data stream, from being transferred. This also reduces computation in the data streams receiving transferred models by minimising the number of conceptually similar base models that must be evaluated in order to select a subset of base models to be used as input to the meta-learner.

5 Minimising Meta-Learner Risk using Relevancy and Diversity

Before presenting experimental results for our base model selection strategies, we consider how the empirical risk and true risk of a meta-learner is impacted by the number of available base models, and the importance of selecting a subset of relevant yet diverse base models as input to a meta-learner in an online environment.

5.1 Increasing the Number of Base Models

In this section we consider the effect of increasing the number of base models available to a meta-learner. We use the meta-learner in the BOTL framework to demonstrate this, where base models are learnt from each data stream using an underlying CDD. Further details on the BOTL framework and underlying CDDs are presented in Section\[6\].

Increasing the number of base models used by the meta-learner reduces empirical risk. This can be seen by considering the squared loss when using a single model, $f_i$, that has been locally learnt by the underlying CDD to represent the concept currently observed in the data stream without influence from historically learnt models, or models transferred from other data streams. This is equivalent to constraining the OLS meta-learner optimisation problem such that all base model weights are 0, except for $f_i$, which is given weight 1, such that

$$F_M^M(x_t^*) = w_0 + \sum_{j=1}^{j=k-1} w_j f_j(x_t) + w_i f_i(x_t), \text{ where } j \neq i,$$ \hspace{1cm} (6)
Figure 1: BOTL vs. CDDs: The difference in $R^2$ performance (a) between the OLS meta-learner in BOTL vs. the underlying CDDs of RePro, ADWIN, and AWPro, and the number of models used as base models (b) for a drifting hyperplane data stream with sudden drifts. Base models are learnt locally and transferred from 4 other sudden drifting hyperplane data streams.

\[
F_{M}^{*} (x^* t) = 0 + \sum_{j=1}^{j=k-1} 0 f_j (x^t) + 1 f_i (x^t),
\]

where $F_{M}$ is the unconstrained meta-learner, which learns weights $w_0, \ldots, w_k$ for each of the $k$ base models, and $F_{M}^{*}$ is the constrained meta-learner, where only the current model from the underlying CDD is used to make predictions \[6\]. To obtain the weights assigned to the $k$ base models in Equation \[6\] the OLS meta-learner must solve the optimisation problem,

\[
\min_{w_0, \ldots, w_k} \sum_{t=1}^{W} \left( y_t - \left( w_0 + \sum_{j=1}^{j=k-1} w_j f_j (x^t) + w_k f_i (x^t) \right) \right)^2.
\]

Since the optimisation problem in Equation \[8\] is convex, the empirical risk of the unconstrained meta-learner, $F_{M}$, which uses the predictions of all base models as inputs, is less than the empirical risk of the constrained meta-learner, $F_{M}^{*}$, which uses the locally learnt model alone. This occurs because the representational capacity of the meta-learner increases, allowing more complex underlying distributions in the data stream to be captured.

As the data stream progresses, the number of available base models increases as new models are learnt locally using the CDDs, or transferred via online TL. Since the window of available data remains fixed in size, the likelihood of overfitting increases as the number of base models becomes large in comparison to the window of available data \[31\]. This is illustrated in Figure \[1\] which shows the difference in predictive performance of the OLS meta-learner used by BOTL compared to using the model learnt via the underlying CDD alone in a sudden drifting data stream \[2\]. Combining base model predictions initially improves predictive performance. However, as the number of base models grows, the meta-learner becomes more susceptible to overfitting, causing a reduction in predictive performance, eventually resulting in worse performance than using the current locally learnt model alone. However, when using AWPro as the underlying CDD, we do not observe a decrease in performance as the data stream progresses. This is because AWPro prioritises the reuse of historical models in the presence of recurring concepts, reducing the number of base models available to the

\[2\] A description of the sudden drifting hyperplane dataset is provided in Section \[6\].
Using the principle of Empirical Risk Minimisation (ERM) [33], and the notation in Table 2, we can consider factors that impact the meta-learner’s ability to generalise well for unseen instances in a data stream between concept drifts.

### 5.2 ERM for Meta-Learners in Online Environments

In ERM, the overarching aim is to learn a function that maps the input space, $X$, to the response variable space, $Y$, drawn from some unknown distribution. If the distribution was known, we could find the function parameters, $\overrightarrow{w}$, that minimise the risk, $R(\overrightarrow{w})$, such that

$$ R(\overrightarrow{w}) = \int Q(z, \overrightarrow{w})dD $$

$$ = \sum_{\overrightarrow{w} \in W} Q(z, \overrightarrow{w}) $$

$$ R(\overrightarrow{w}_0) = \min_{\overrightarrow{w}} R(\overrightarrow{w}) , $$

where $z$ is an instance and response variable pair, $z = (x, y)$, and $Q(z, \overrightarrow{w})$ is the loss function, parameterised by $\overrightarrow{w}$, on instance $z$. For example, the squared loss of instance $x_t$ is $Q(z, \overrightarrow{w}) = (y_t - f(x_t))^2$ where model $f$ has parameters $\overrightarrow{w}$. Therefore, $R(\overrightarrow{w})$ is the risk, or loss, of a model parameterised by $\overrightarrow{w}$. We use $R(\overrightarrow{w}_0)$ to denote the risk of a model with optimal parameters, $\overrightarrow{w}_0$, which minimise the risk over the known distribution, $D$. However, since the distribution is unknown, ERM approximates the optimal risk by evaluating the loss over a window, $W$, of training samples, $Z_{|W|}$, to obtain the empirical risk, $R_{\text{emp}}(\overrightarrow{w}_{|W|})$. The empirical risk is defined as,

$$ R_{\text{emp}}(\overrightarrow{w}_{|W|}) = \frac{1}{|W|} \sum_{i=1}^{|W|} Q(z_i, \overrightarrow{w}_{|W|}) . $$

The empirical risk is inherently biased towards the training sample, $Z_{|W|}$ and therefore the function learnt often underestimates the risk of the same function when used on unseen instances of data belonging to the same concept. The risk over unseen instances is known as the true risk, and is denoted by $R(\overrightarrow{w}_{|W|})$. This means that for a given training sample

$$ R_{\text{emp}}(\overrightarrow{w}_{|W|}) < R(\overrightarrow{w}_{|W|}) . $$

---

**Table 2: ERM Notation**

| Definition | Notation |
|------------|----------|
| True risk of model with parameters $\overrightarrow{w}$ | $R(\overrightarrow{w})$ |
| Instance $x_t$ at time $t$ and respective response variable $y_t$ | $z_t = (x_t, y_t)$ |
| Loss function of a model parameterised by $\overrightarrow{w}$ for instance $z_t$ | $Q(z_t, \overrightarrow{w})$ |
| Optimal parameters for a given function | $\overrightarrow{w}_0$ |
| Training sample of size $|W|$ | $Z_{|W|}$ |
| Empirical risk of model with parameters $\overrightarrow{w}_{|W|}$ learnt over $Z_{|W|}$ | $R_{\text{emp}}(\overrightarrow{w}_{|W|})$ |
| Model parameters that minimise the empirical risk over $Z_{|W|}$ | $\overrightarrow{w}_{*|W|}$ |
Due to the randomness of \( Z_{|W|} \), the empirical risk and true risk, \( \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) \) and \( \mathcal{R} \left( \bar{w}_{|W|}^* \right) \), are also random sequences, and the law of large numbers states that the average converges to its expected value as the number of samples grows large [33, 34, 30]. Therefore, as \(|W| \to \infty\),

\[
\begin{align*}
\mathcal{R} \left( \bar{w}_{|W|}^* \right) & \to \mathcal{R} \left( \bar{w}_0^* \right), \\
\mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) & \to \mathcal{R} \left( \bar{w}_0^* \right).
\end{align*}
\]

As the number of training samples grows, a better estimate of the parameter values \( \bar{w}_{|W|}^* \) can be obtained for a fixed number of parameters \( k \), where \(| \bar{w}_{|W|}^* | = k \), such that the empirical risk tends to the true risk, \( \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) \to \mathcal{R} \left( \bar{w}_{|W|}^* \right) \). To obtain a set of parameters that are a good approximation of the optimal parameters, such that \( \bar{w}_{|W|}^* \to \bar{w}_0 \), the representational capacity of the model must be increased to encapsulate more complex underlying distributions in the data stream [33]. In turn, increasing the representational capacity of the model increases the number of parameters in \( \bar{w}_{|W|}^* \) which must be learnt. However, by increasing the complexity of the model, more training samples are required in order to ensure that the empirical risk, \( \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) \), is a good approximation of the true risk, \( \mathcal{R} \left( \bar{w}_{|W|}^* \right) \).

This means that to ensure a learnt model generalises well and approximates the model with optimal parameters, \( \bar{w}_0 \), the number of training samples and the model complexity must grow as a function of one another to guarantee convergence [34]. However, in online settings, it may not be feasible to retain a large number of historical instances. Additionally, due to the dynamic nature of learning in online environments, even when a large history of instances can be retained, a concept drift may be encountered prior to observing sufficient historical instances. Therefore, we consider the conditions under which the probability that the empirical risk, \( \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) \), is greater than the true risk, \( \mathcal{R} \left( \bar{w}_{|W|}^* \right) \), plus some \( \epsilon \), is bounded,

\[
P \left[ \left| \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) - \mathcal{R} \left( \bar{w}_{|W|}^* \right) \right| > \epsilon \right].
\]

Vapnik [30] showed that the bound on the empirical risk in Equation 12 can be written as

\[
\mathcal{R} \left( \bar{w}_{|W|}^* \right) \leq \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) + \varphi \left( \frac{n}{h_k} \right),
\]

where \( \varphi \left( \frac{n}{h_k} \right) \) is a confidence interval, dependent on the ratio between the number of training samples, \( n \), and the Vapnik-Chervonenkis (VC) dimension, \( h_k \), which is a measure of the complexity of a model. Therefore, although a complex model may have a low empirical risk, the confidence interval may be large when the ratio of training samples to model complexity is large [30]. In such cases, a model is said to overfit the training data. In order to reduce the right-hand side of Equation 13, a small confidence interval is required, which would indicate that a model generalises well on unseen instances. To minimise the confidence interval, \( \varphi \left( \frac{n}{h_k} \right) \), a model with a small VC dimension, \( h_k \), must be learnt. However, models with a small VC dimension have poorer representational capacity, thereby increasing the empirical risk, \( \mathcal{R}_{\text{emp}} \left( \bar{w}_{|W|}^* \right) \). This introduces a tradeoff between the confidence interval and representational capacity of the model. In order to obtain a model that generalises well, we must simultaneously find the VC dimension that minimises the confidence interval, and the parameter values that minimise the empirical risk [33, 34, 30].

To identify the number of base models that should be used, the meta-learner must repeatedly solve this optimisation problem every time a concept drift is encountered, or a model is received via online TL. For many online learning applications, solving this optimisation problem is not possible, or practical, due to its
computational complexity. Therefore, alternative approaches to decrease the complexity of the meta-learner must be considered in order to reduce the confidence interval, such as selecting a subset of the available base models to be used as input to the meta-learner.

5.3 Improving Generalisation for Meta-Learners in Online Environments

Overfitting caused by increasing the representational capacity of a meta-learner is known as the curse of dimensionality \[31\]. If future concepts were known, the curse of dimensionality could be avoided by discarding base models that are known not to be beneficial to the meta-learner for current and future concepts. However, in online environments future concepts are not known prior to learning. This means that subsets of base models must be selected by repeatedly evaluating the set of models learnt locally, and transferred when using online TL, as the data stream progresses.

In online ensembles, one approach to solving this challenge is to only use the \(k\) most recently learnt models as base models in the ensemble \[21, 24\]. However, in many real-world environments that encounter recurring concepts, recency may not be a good indicator of usefulness \[21\]. Instead, other feature selection and ensemble pruning approaches must be used to determine which models may be most beneficial. Ensemble pruning is based on the principle that combining the predictions of an appropriate subset of base models will provide improved predictive capabilities over combining all base models, as exemplified in bagging and boosting offline ensembles \[32\]. Therefore, ensemble pruning techniques can be used to select a subset of base models for meta-learners in online environments. Feature selection techniques can be applied by considering the predictions of each base model as input features to the meta-learner. These meta-features can be evaluated to consider how useful they are for predicting the current concept \[45\].

In an ensemble of regressors there is a bias-variance-covariance trade-off \[7, 46\], where the generalisation error of an ensemble of equally weighted regressors is:

\[
E \left[ (\hat{f}^M - y)^2 \right] = \text{bias}^2 + \frac{1}{M} \text{var} + \left( 1 - \frac{1}{|M|} \right) \text{covar},
\]

where

\[
\text{bias} = \frac{1}{|M|} \sum_i \left( E[f_i] - y \right),
\]

\[
\text{var} = \frac{1}{|M|} \sum_i \left( f_i - E[f_i] \right)^2 , \text{ and}
\]

\[
\text{covar} = \frac{1}{|M|(|M|-1)} \sum_i \sum_{i \neq j} \left( f_i - E[f_i] \right) \left( f_j - E[f_j] \right).
\]

This bias-variance-covariance decomposition also holds for non-uniformly weighted ensembles \[7\], and therefore the bias and variance of base models, and the covariance between them, can impact the generalisation ability of a meta-learner when base models are not equally weighted. Factors such as these can be considered by evaluating base models using metrics to determine which models should be used. Equation \[14\] indicates that base models should be relevant yet diverse in order to prevent the meta-learner overfitting.

5.4 Evaluating Base Models

When undertaking ensemble pruning in offline settings, it is desirable to obtain a relevant yet diverse subset of base models \[7\]. However, within non-stationary environments base model diversity is rarely accounted for. The performance and diversity of model predictions can be used to cull base models \[6\]. Using the \(R^2\) predictive performance of each base model on the current window of data available to the meta-learner allows models that make poor predictions to be removed. However, using performance alone as a metric to cull base models may not sufficiently reduce the number of models, allowing the meta-learner to overfit when the number of potential base models is high \[6\]. The covariance term, \(\text{covar}\), in Equation \[14\] accounts for
the pairwise difference of base models [47], which relates to their diversity. As the number of base models in the ensemble, \( |M| \), increases, the generalisation error decreases due to the variance term, \( \text{var} \). However, increasing the number of base models can cause the covariance term, \( \text{covar} \), to also increase. When simply using the performance of base models as a culling metric, the reduction of the covariance term in Equation 14 is not considered. To prevent the covariance term from significantly increasing, base models must be selected that have small, or negative covariance [47].

To account for the covariance term in Equation 14, the diversity of base models can be used to remove those that exhibit high covariance. To achieve this, the pairwise MI can be measured between the predictions of each of the base models on the current window of data available to the meta-learner. When a pair of base models have high MI, the base model with the lower predictive performance can be culled from the model set [6]. This reduces the number of redundant models used by the meta-learner, since including models with similar predictions provides no additional information and increases the covariance term in Equation 14.

Measuring diversity through the level of disagreement between base model predictions is a common approach in existing online ensemble pruning research [21]. However, this means that the diversity of base models must be recalculated as new instances are observed in the data stream. Although diversity can be estimated using these metrics [8], it does not guarantee that the underlying distributions of data from which each base model was created are diverse [8]. Additionally, disagreement in a regression setting can be highly skewed by a small number of differing predictions made by base models that have been learnt from conceptually similar distributions of data. Instead, diversity among the concepts learnt by each base model can be considered by estimating conceptual similarity.

Due to concept drift, base models must be re-evaluated to determine their relevancy to the current concept. Using metrics such as MI to indicate diversity, which are also dependent on the underlying distribution of observable data, may be undesirable as repeated pairwise comparisons between base models can become computationally expensive when the number of base models is large, or when drifts are encountered frequently. Therefore, the use of a diversity metric, such as conceptual similarity, that remains static in online environments, even in the presence of concept drift, is beneficial.

6 Experimental Set-Up

To evaluate the effectiveness of using conceptual similarity as a metric for base model selection techniques, we use parameterised thresholding and conceptual clustering to obtain a subset of base models for the OLS meta-learner in the BOTL framework. RePro [12], ADWIN [4] and AWPro [6] are used as the underlying CDDs to obtain Support Vector Regressors (SVRs) as base models from the concept drifting data streams. We have chosen SVRs to create base models since they have the representational capacity to model the underlying concepts encountered in each of the different data stream types used in this paper. However, other regression models can be used since both the BOTL framework and base model selection techniques are model agnostic.

In order to estimate the conceptual similarity between base models, when a model is selected for transfer in the BOTL framework, we transfer both the model, \( f_i \), which has been learnt to represent the current concept, and the reduced PCs, \( U_i \), obtained through Definition 1 and Equation 3. This means that the PCs associated with each base model are available in each data stream. We use the datasets provided in [5] and [6], which include two synthetic data stream generators, namely the drifting hyperplane data generator [6] and the smart home heating simulator [5], and real-world following distance datasets predicting Time To Collision (TTC) from vehicle telemetry [6]. BOTL is used to transfer base models between online data streams and CDDs of the same type, such that knowledge is shared between 5 drifting hyperplane, 5 heating simulator, and up to 17 following distance data streams for each CDD respectively.

6.1 Baseline Approaches

To empirically evaluate the effectiveness of estimating conceptual similarity as a diversity metric for selecting a subset of base models, we considered using existing ensemble pruning and meta-learner model selection techniques as baseline approaches. However, most online ensembles that can be used in regression settings, such
as AWE \[28\], OWE \[23\] and AddExp \[24\], combine base models using weighted averaging, where weights are bound between 0 and 1 \[32\]. The use of bounded weights introduces an assumption that the response variables each base model was learnt to predict have a consistent range of values. This assumption may not be valid when learning online in real-world environments. For example, when predicting the desired heating temperature for a smart home heating system, base models learnt over summer months may have different ranges of response variables in comparison to base models learnt over winter months. Since the future distribution of response variables is unknown in an online data stream, the response variable cannot be normalised to ensure all base models are learnt over consistent ranges of response variables. Therefore, bounding base model weights to be between 0 and 1 may lead to inaccurate predictions. Instead, we consider the underlying techniques employed by existing online ensembles to prune base models, and use an OLS meta-learner to combine base models since the weights are not bound between 0 and 1. For example AWE \[28\] and AddExp \[24\] prune base models using their predictive performance on the current window of observable data, therefore a similar technique, which evaluates the predictive performance of base models, can be used to obtain a subset of base models to be input to the OLS meta-learner.

We use the BOTL framework and two existing variants of BOTL that implement base model culling strategies as baseline approaches \[5, 6\]. We denote frameworks with no base model selection or culling as BOTL, frameworks with performance thresholding as P-Thresh \[3\] and frameworks with performance and MI thresholding as MI-Thresh \[4\]. BOTL uses all transferred models as base models for the OLS meta-learner, whereas P-Thresh and MI-Thresh use naïve culling thresholds that cull base models using similar techniques to pruning strategies implemented by existing online ensemble approaches. This allows a subset of base models to be selected by removing transferred models that are expected to be least beneficial. P-Thresh achieves this by culling transferred models that obtain \(R^2\) predictive performances below a threshold, \(\lambda_{\text{perf}}\), on the current window of observable data, and thus are deemed to be least relevant to the meta-learner. MI-Thresh culls the number of transferred models more aggressively, by considering the relevancy and diversity of the transferred models. This is achieved through culling redundant transferred models by considering the pairwise MI between model predictions. For a pair of models that obtain a MI above a threshold, \(\lambda_{\text{MI}}\), the poorer performing model of the two is culled. Of the remaining models, those that obtain an \(R^2\) performance below a performance threshold, \(\lambda_{\text{perf}}\), are also culled in order to obtain a relevant yet diverse subset of base models.

In addition to the BOTL variants, we also use the underlying CDD alone as a default baseline for what can be achieved without the use of a meta-learner to combine base model predictions. This allows the benefits and drawbacks of using meta-learners with differing base model selection techniques to be considered, and highlights the importance of base model selection strategies when the number of base models becomes large, which can be impacted by the choice of the underlying CDD.

### 6.2 CDDs

We use RePro \[12\], ADWIN \[4\] and AWPro \[6\] as underlying CDDs to detect concept drifts and create base models in each data stream. RePro and AWPro prioritise the reuse of historical models in the presence of recurring concepts \[6, 12\]. This is beneficial for online TL since it reduces the communicational overhead of transferring multiple models that represent the same concept, and also benefits meta-learners by reducing the number of base models that must be evaluated and compared in order to obtain a relevant yet diverse subset of base models. Unlike RePro, ADWIN and AWPro estimate the precise drift point within the window of recent observations so that instances belonging to the previous concept can be discarded, and do not influence the model learnt to represent a newly encountered concept \[4, 6\]. Discarding instances belonging to the previous concept is beneficial when evaluating base models since the retention of instances belonging to another concept may influence measures of diversity.

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| Dataset  | Characteristic                | Dataset Type | Drift Type | # Streams | Avg. $#x_t$ per Stream | Artificial Drifts per Stream |
|---------|-------------------------------|--------------|------------|-----------|------------------------|-----------------------------|
| SuddenA | Uniform Noise                 | Synthetic    | ✓/−        | 5         | 10,000                 | 20                          |
| SuddenB | Sensor Failure                | Synthetic    | ✓/−        | 5         | 10,000                 | 20                          |
| SuddenC | Intermittent Sensor Failure   | Synthetic    | ✓/−        | 5         | 10,000                 | 20                          |
| SuddenD | Sensor Deterioration          | Synthetic    | ✓/✓        | 5         | 10,000                 | 20                          |
| GradualA| Uniform Noise                 | Synthetic    | −/✓        | 5         | 11,900                 | 20                          |
| GradualB| Sensor Failure                | Synthetic    | ✓/✓        | 5         | 11,900                 | 20                          |
| GradualC| Intermittent Sensor Failure   | Synthetic    | ✓/✓        | 5         | 11,900                 | 20                          |
| GradualD| Sensor Deterioration          | Synthetic    | −/✓        | 5         | 11,900                 | 20                          |
| Heating | Weather Data                  | Hybrid       | ✓/✓        | 5         | 17,664                 | NA                          |
| Following| Vehicular Data               | Real-World   | ✓/✓        | 17        | 1909                   | NA                          |
6.3 Datasets

We evaluate base model selection using three dataset types presented in [5] and [6], namely drifting hyperplane, smart home heating simulator, and following distance datasets. The drifting hyperplane datasets are synthetic datasets that allow sudden, gradual and recurring drifts to be artificially introduced into the data streams. Common challenges of learning in online environments have been simulated in the drifting hyperplane datasets, including sensor failure, intermittent sensor failure, and sensor deterioration. The smart home heating simulator uses real-world weather data and a synthetic heating schedule to create a dataset of desired heating temperatures. Using real-world weather data to generate this dataset enables base model selection techniques to be evaluated on data streams containing drifts that are more representative of those seen in real-world environments. Finally, the real-world following distance dataset predicts Time To Collision (TTC) from vehicle to vehicle. An overview of the characteristics of these datasets is presented in Table 3.

6.3.1 Drifting Hyperplane

The drifting hyperplane datasets are modifications of a commonly used benchmark dataset [24], adapted for regression settings [6]. Each instance at time \( t \), \( x_t \), is a vector, \( x_t = \{ x_{t1}, x_{t2}, \ldots, x_{tn} \} \), containing \( n \) randomly generated, uniformly distributed, variables, \( x_{tn} \in [0, 1] \). For each instance, \( x_t \), a response variable, \( y_t \in [0, 1] \), is created using the function \( y_t = (x_{t0} + x_{t1} + x_{t2})/3 \), where \( p, q, \) and \( r \) reference three of the \( n \) variables of instance \( x_t \). This function represents the underlying concept, \( c_a \), to be learnt and predicted. Concept drifts are introduced by modifying which features are used to create \( y_t \). For example, an alternative concept, \( c_b \), may be represented by the function \( y_t = (x_{t1} + x_{t2} + x_{t3})/3 \), where \( \{ p, q, r \} \neq \{ u, v, w \} \) such that \( c_a \neq c_b \).

A variety of drift types have been synthesised in this generator, including sudden drifts, gradual drifts and recurring drifts. A sudden drift from concept \( c_a \) to concept \( c_b \) is created between time steps \( t \) and \( t + 1 \) by instantaneously changing the underlying function used to create \( y_t \) to an alternative function for \( y_{t+1} \). A gradual drift from concept \( c_a \) to \( c_b \) occurs between time steps \( t \) and \( t + m \), where \( m \) instances of data are observed during the drift. Instances of data created between \( t \) and \( t + m \) use one of the underlying concept functions, \( c_a \) or \( c_b \), to determine the response variable. The probability of an instance belonging to concept \( c_a \) decreases proportionally to the number of instances seen after time \( t \), while the probability of it belonging to \( c_b \) increases proportionally as we approach \( t + m \). Recurring drifts are created by introducing a concept \( c_c \) that reuses the underlying function defined by a previous concept, \( c_a \), such that we achieve conceptual equivalence where \( c_c = c_a \).

We create four variations of the drifting hyperplane datasets, introducing concept drifts that represent different problems that may be encountered when learning in real-world environments. The first variation simply introduces uniform noise, where \( y_t \pm 0.05 \) with probability 0.2. Datasets generated in this way are denoted as SuddenA and GradualA for sudden and gradual drifting data streams respectively. The second variant simulates sensor failure by setting a feature vector, \( i \), to 0 at time \( t \) for the remainder of the data stream with probability 0.001, such that \( x_{ti} = 0 \). In the scenario where feature \( i \) is used to create the response variable \( y_t \), we modify two other randomly selected feature vectors, \( j \) and \( k \), such that \( x_{tj} = x_{ti}/4 \) and \( x_{tk} = 3x_{ti}/4 \). This ensures that the underlying concept can still be learnt from the data. We denote datasets generated in this way as SuddenB and GradualB for sudden and gradual drifting data streams respectively.

The third variation simulates intermittent sensor failure by selecting a feature vector \( i \) to fail at time \( t - 1 \) with probability 0.001. Once selected to fail, the feature value at all subsequent time steps \( t \) is set to 0 such that \( x_{ti} = 0 \) with probability 0.3. Datasets generated using this scenario are denoted as SuddenC and GradualC.

The final variant emulates the deterioration of a sensor by including noise depending on the time step \( t \), such that \( x_{ti} = x_{ti} \pm (0.2(t/|X|)) \), where 0.2 is the maximum amount of noise added to \( x_{ti} \) and \( |X| \) is the number of instances in the dataset. This means that as the data stream progresses, more noise is added to an individual feature, simulating the gradual deterioration in accuracy of a sensor over time. Additionally, the probability of a sensor deteriorating increases as the data stream progresses, such that the probability of a

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1 Originally denoted as BOTL-C.I in [5] and [6].
2 Originally denoted as BOTL-C.II in [5] and [6].
feature being selected for deterioration at time \( t \) is \( 0.001(t/|X|) \). Datasets generated in this way are denoted as SuddenD and GradualD for sudden and gradual drifting data streams respectively.

### 6.3.2 Heating Simulator

This dataset is generated from a simulation of a smart home heating system, determining the desired room temperature for a user [5]. Heating temperatures were derived using weather data collected from a weather station in Birmingham, UK, from 2014 to 2016. This dataset contained rainfall, temperature and sunrise patterns, which were combined with a schedule, obtained from sampling an individual’s pattern of life, to determine when the heating system should be engaged. The schedule was synthesised to vary the desired temperatures based on time of day, day of week, and external weather conditions, creating concepts with more complex underlying distributions than those in the drifting hyperplane datasets. To create multiple data streams, weather data was sampled from overlapping time periods and used as input to the synthesised schedule to determine the desired heating temperatures. Due to the dependencies on weather data, each stream was subject to large amounts of noise. Concept drifts were introduced manually by changing the schedule, however, drifts also occurred naturally due to changing weather conditions. By sampling weather data from overlapping time periods, and due to seasonality, data streams follow similar trends, ensuring the predictive performance obtained in each data stream can benefit from knowledge transfer. By using concepts that have more complex underlying distributions, and are dependent on noisy data, the evaluation of BOTL and base model selection techniques on these data streams is more indicative of what is achievable when used in real-world environments.

### 6.3.3 Following Distance

Finally, the following distance dataset uses a vehicle’s following distance and speed to calculate Time To Collision (TTC) when following another vehicle [5]. Vehicle telemetry data such as speed, gear position, brake pressure, throttle position and indicator status, alongside sensor data that infer external conditions, such as temperature, headlight status, and windscreen wiper status, were recorded at a sample rate of 1Hz. Additionally, a selection of signals such as vehicle speed, brake pressure and throttle position were averaged over a window of 5 seconds to capture a recent history of vehicle state. Vehicle telemetry and environmental data can be used to make predictions that allow vehicle functionalities to be personalised and reflect current driving conditions. For example, Adaptive Cruise Control (ACC) can be personalised by predicting TTC to identify a driver’s preferred following distance. Data was collected from 4 drivers for 17 journeys which varied in duration, collection time and route. Each journey is considered to be an independent data stream, where 6 data streams were generated by 2 drivers driving 3 pre-defined routes [5] while the remaining 11 were generated by 2 drivers mostly commuting to and from the University of Warwick, UK. The maximum, minimum and average duration of these journeys were 83 minutes, 15 minutes and 43 minutes respectively. Each data stream is subject to concept drifts that occur naturally due to changes in the surrounding environment, such as road types and traffic conditions. BOTL enables knowledge to be learnt and transferred across journeys and between drivers.

### 7 Experimental Results

To evaluate the use of conceptual similarity to obtain a relevant yet diverse subset of base models, we use our base model selection techniques for the OLS meta-learner in BOTL and compare these model selection techniques to using the underlying CDD alone, and to BOTL without base model selection. We consider using performance thresholding (P-Thresh), MI thresholding (MI-Thresh), conceptual similarity thresholding (CS-Thresh), and conceptual similarity clustering (CS-Clust) as methods to obtain subsets of base models. We compare the predictive performances of each of these techniques and also consider the number of relevancy

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3This dataset is available at: https://github.com/hmckay/BOTL/blob/master/FollowingDistanceData.zip
and diversity metric calculations required to obtain the subsets, which includes the pairwise comparisons and evaluations of the predictive performances of base models.

We evaluate these approaches using the datasets originally introduced to evaluate the BOTL framework [5, 6], namely the sudden and gradual drifting hyperplane, smart home heating simulator, and following distance datasets, as detailed in Section 6.3. Window sizes of 30 instances for the sudden and gradual drifting hyperplane data streams, 480 instances capturing 10 days of heating and weather observations for smart home heating simulator, and 90 instances capturing 90 seconds of vehicle telemetry for following distance data streams, are used by the underlying CDDs and meta-learner. For CS-Thresh and MI-Thresh, we used a performance threshold, $\lambda_{perf} = 0.2$, to ensure that diverse base models are also relevant to the current concept. This performance threshold value has been chosen for MI-Thresh and CS-Thresh based on the results obtained by P-Thresh, presented in Figures 2–5, and is consistent with performance thresholds used in [6]. We also consider using conceptual clustering prior to knowledge transfer to reduce the number of base models transferred between data streams.

7.1 Parameterised Culling Thresholds

Figures 2–5 show the increase in performance of the BOTL meta-learner compared to using the underlying CDD alone, with increasingly aggressive culling parameter values for the sudden drifting hyperplane variants, gradual drifting hyperplane variants, smart home heating simulator and following data datasets respectively. Analysing the performance of the meta-learner with varying culling parameter values highlights that the selection of such parameter values is challenging, and can be dependent on many underlying factors, including the number of base models available, noise in the data stream, separability of base models for a given diversity metric, and the underlying CDD.

For sudden and gradual drifting hyperplane data streams with uniform noise (SuddenA, GradualA), in Figures 2 and 3, the meta-learner is unlikely to overfit, regardless of the number of base models, since the concepts to be learnt are simple and there is little noise in these variants of the synthetic data streams. This means that using aggressive culling parameters reduces the overall predictive performance of the meta-learner, since the meta-learner benefits from retaining more base models without suffering from the curse of dimensionality. However, in drifting hyperplane data streams with sensor failure (SuddenB and Gradual B), more aggressive culling parameters are required to prevent the meta-learner from overfitting when using ADWIN as the underlying CDD. This is necessary since the increase in noise increases the likelihood of the meta-learner overfitting, and ADWIN does not make use of historical models in the presence of recurring concepts, leading to large covariances between base models that have been learnt to represent the same concept. This indicates that aggressive culling techniques may be required in noisy data streams when using CDDs that do not reuse previously learnt models when concepts re-occur. Conversely, CDDs that reuse base models, such as RePro and AWPro, benefit from less aggressive culling parameters in these synthetic drifting hyperplane data streams.

When learning concepts with more complex underlying distributions, the meta-learner is more likely to overfit when the number of base models is large in comparison to the size of the window of available data. Data streams created using the smart home heating simulator are generated using real-world weather data, meaning that the concepts to be learnt are more complex, and contain more noise, in comparison to the drifting hyperplane data streams. These factors indicate that the meta-learner is likely to overfit when the number of base models becomes large, therefore requiring aggressive culling parameters. However, Figure 4 shows that less aggressive culling parameters can be chosen. This is observed since a large window size has been used to detect concept drifts, create base models, and train the meta-learner, in these data streams. This highlights the relationship between the meta-learners generalisation ability, representational capacity, and the amount of available training data.

In addition to this, a wider variety of culling parameters obtain similar predictive performances in the smart home heating simulator data streams. For example, there is little change in the predictive performance and

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6 For code and reproducibility notes see https://github.com/hmckay/BOTLv2
7 For Figures 2–5 all plots show increasingly aggressive culling parameter values on the x-axis with the least aggressive parameter value on the left of each plot, and most aggressive on the right.
Figure 2: Sudden Drifting Hyperplane: increase in $R^2$ performance compared to using the underlying CDD alone, and number of base models used by the BOTL meta-learner using increasingly aggressive culling threshold parameter values for performance, MI, and conceptual similarity thresholding, for variants of the sudden drifting hyperplane datasets when transferring base models between 5 data streams.
Figure 3: Gradual Drifting Hyperplane: increase in $R^2$ performance compared to using the underlying CDD alone, and number of base models used by the BOTL meta-learner using increasingly aggressive culling threshold parameter values for performance, MI, and conceptual similarity thresholding, for variants of the gradual drifting hyperplane datasets when transferring base models between 5 data streams.
Figure 4: Heating Simulator: increase in $R^2$ performance compared to using the underlying CDD alone, and number of base models used by the BOTL meta-learner using increasingly aggressive culling threshold parameter values for performance, MI, and conceptual similarity thresholding, for the smart home heating simulator datasets when transferring base models between 5 data streams.

Figure 5: Following Distance: increase in $R^2$ performance compared to using the underlying CDD alone, and number of base models used by the BOTL meta-learner using increasingly aggressive culling threshold parameter values for performance, MI, and conceptual similarity thresholding, for following distance datasets when transferring base models between 7 and 17 data streams.
number of base models selected for P-Thresh values ranging between 0.1 and 0.5, MI-Thresh values between 0.8 and 0.4, and CS-Thresh values between 0.8 and 0.5. A wider range of values can be used as culling thresholds for base model selection in the smart home heating simulator data streams because the number of base models available to the meta-learner does not change significantly over these ranges of culling parameter values. This indicates that the meta-learner’s sensitivity to culling parameter values is also dependent on how separable base models are for a given metric. For example, if large numbers of base models are equally diverse, aggressive culling parameters may be required to sufficiently reduce the number of base models available to the meta-learner to prevent overfitting.

Figure 5 shows the increase in predictive performance of the meta-learner compared to using the underlying CDD alone when transferring base models between 7 and 17 following distance data streams. We present the results for frameworks with differing numbers of data streams to highlight the difficulty in selecting culling parameter values when the number of data streams in the framework is large. The 7 data streams were chosen to be representative of the dataset containing 17 data streams, and contains vehicle telemetry data from one driver following two pre-defined routes, while the remaining 5 streams were obtained from two drivers commuting. The meta-learners that use ADWIN as the underlying CDD are most sensitive to the culling parameter values due to the increased covariance between base models in the presence of recurring concepts. However, when base models are transferred between 17 data streams, all meta-learners become sensitive to culling parameter values, regardless of the underlying CDD used. To prevent overfitting, aggressive culling parameters are required due to two factors. Firstly, using bi-directional knowledge transfer between 17 data streams increases the number of base models available to the meta-learner, and therefore aggressive culling parameters are required to sufficiently reduce the number of base models used as input to the meta-learner for the given window size of available data. Secondly, aggressive culling parameters are required since there may be high levels of covariance between predictions of base models from different data streams that encounter similar concepts. These challenges are highlighted when using P-Thresh as a culling mechanism. P-Thresh selects base models using performance alone, and when selecting base models from 17 following distance data streams the meta-learner overfits, even with aggressive culling parameters. This occurs because, after culling, the remaining base models obtain high $R^2$ performances, indicating similar predictions. This introduces high levels of covariance between inputs to the meta-learner. MI-Thresh and CS-Thresh also suffer from the increased number of base models and the covariance among the predictions of base models from different data streams. However, a wider range of culling parameters can be used to obtain a meta-learner with improved predictive performance over the underlying CDD. This reiterates the importance of using diversity when selecting base models.

The results presented in this section demonstrate that selecting appropriate culling threshold parameter values is challenging and may require domain expertise in order to prevent the meta-learner from overfitting. The aggressiveness of the culling threshold chosen can be dependent on the amount of training data available to the meta-learner, the number of base models available to select from, the complexity of the underlying distribution of the data stream, and the diversity between base models. Since each of these factors must be considered, selecting a culling parameter value may be difficult to determine in advance, and selecting a single threshold parameter for all online environments is not possible. However, the results presented in Figures 2-5 show that aggressive culling parameters are typically more beneficial since less aggressive culling parameters can be ineffective at reducing the number of base models sufficiently to prevent the meta-learner overfitting, as can be seen in the following distance datasets. Additionally, only a small reduction in predictive performance is observed when using an aggressive culling parameter in comparison to less aggressive culling parameters that are able to prevent the meta-learner from overfitting, as seen in the drifting hyperplane and heating simulator datasets.

Since the selection of effective culling threshold parameters is challenging, parameterless base model selection techniques are required. This can be achieved using conceptual similarity clustering, as introduced in Section 4.2. To compare our parameterless clustering approach to parameterised thresholding, we considered the results presented in Figures 2-5 and selected parameter values that showed an increase in predictive performance in comparison to using the underlying CDD alone across all dataset types. Therefore, in the remainder of this paper, we use MI-Thresh with $\lambda_{MI} = 0.2$ and CS-Thresh with $\lambda_{CS} = 0.4$ to compare parameterised culling thresholds with parameterless clustering. P-Thresh is not used as a baseline technique in the remainder.
of this paper since it is not able to effectively obtain a subset of base models to prevent overfitting in the BOTL framework with 17 following distance data streams.

### 7.2 Parameterless Clustering

Tables 4–7 show the $R^2$, PMCC$^2$ and RMSE predictive performances, the average number of base models used by the meta-learner, and the average number of relevancy and diversity metric calculations required to compare and evaluate base models for the various datasets and CDDs considered. These results show that, without base model selection, BOTL is more likely to overfit in noisy data streams, as seen in Tables 4b, 4d, 5b, 5d, and 7. However, using base model selection techniques that obtain a relevant yet diverse subset of base models reduces the likelihood of overfitting.

CS-Thresh obtains improved predictive performances in comparison to using the underlying CDD alone (with statistical significance $p < 0.01$). CS-Thresh also frequently outperforms BOTL without base model selection, and obtains comparable predictive performances to using MI-Thresh. For most datasets, CS-Clust also achieves this. However, CS-Clust obtains a poor $R^2$ predictive performance on SuddenB data streams when using RePro and ADWIN as underlying CDDs, as shown in Table 4b, and when using RePro for GradualB data streams, as shown in Table 5b. This indicates that CS-Clust is still susceptible to overfitting in data streams with large amounts of noise. Noise in these data streams prevents the clustering approach from effectively distinguishing between models learnt to represent different concepts. Conceptual clustering is more challenging in noisy environments because there is more variance in the underlying distribution of data used to create base models. This can affect the PCs created from the underlying distribution of data belonging to each concept, and can also increase the number of PCs required to capture 99.9% of the variance of the window of data. Capturing the increased noise within the PCs can reduce the PAs between these windows of synthetic data, making conceptual clustering more challenging due to a reduction in the separability of base models. To overcome this, fewer PCs could be used to calculate the PAs between the subspaces, thereby capturing less variance in the window of data used to create base models. Alternatively, other methods of obtaining orthonormal representations of data that are more robust to noise, such as Laplacian PCA [38] or Robust PCA [39], can be used [40].

Although CS-Clust overfits in these synthetic data streams, the PMCC$^2$ performances are statistically significantly ($p < 0.01$) greater than BOTL with no base model selection, and outperform the PMCC$^2$ performance obtained when using the underlying CDDs alone. This is observed since PMCC$^2$ is bound between 0 and 1, whereas $R^2$ is unbounded, $(-\infty, 1]$. This means that the $R^2$ predictive performance can be highly skewed if the meta-learner overfits for a small number of instances in the data stream. The observed improvement in PMCC$^2$ performance indicates that although CS-Clust can be susceptible to overfitting in these environments, the meta-learner overfits less frequently in comparison to the BOTL meta-learner that uses all base models. With the exception of these noisy synthetic data streams, CS-Clust obtains $R^2$ performances statistically significantly ($p < 0.01$) greater than the underlying CDDs, and outperforms the BOTL meta-learner with no base model selection techniques for the majority of CDDs, across all datasets except the heating simulator data streams, as shown in Table 6.

Table 6 shows that the BOTL framework with no base model selection strategy achieves the highest predictive performance for the smart home heating simulator data streams. Although the number of base models used by the meta-learner, $|M'|$, is greater than the number of base models used by meta-learners in following distance data streams, shown in Table 7, the BOTL meta-learner does not overfit in the heating simulator data streams. This is because the window of available data used to learn the meta-learner weights, $W$, is larger for the heating simulator data streams (480 instances) in comparison to following distance data streams (90 instances). This means that the meta-learner is able to use more base models before suffering from the curse of dimensionality [31]. This shows that base model selection techniques are not always necessary, however, the number of available base models cannot be known prior to learning. Although BOTL achieves the highest predictive performance, all base model selection techniques obtain $R^2$ performances that are statistically significantly ($p < 0.01$) greater than the underlying CDD, and obtain similar predictive performances to BOTL with no base model selection.
| (a) SuddenA: sudden drifting hyperplanes with uniform noise. | (b) SuddenB: sudden drifting hyperplanes with single sensor failure. | (c) SuddenC: sudden drifting hyperplanes with intermittent sensor failure. | (d) SuddenD: sudden drifting hyperplanes with gradual sensor deterioration. |
|---|---|---|---|
| **RePro** | **ADWIN** | **AWPro** |
| | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. |
| CDD | 0.886 | 0.887 | 0.062 | 1 | 0 | 0.851 | 0.854 | 0.071 | 1 | 0 | 0.830 | 0.835 | 0.076 | 1 | 0 |
| BOTL | 0.831 | 0.843 | 0.076 | 23 | 0 | 0.825 | 0.837 | 0.077 | 40 | 0 | **0.884** | 0.886 | 0.063 | 16 | 0 |
| MI-Thresh | 0.902 | 0.902 | 0.058 | 3 | 28507 | 0.887 | 0.888 | 0.062 | 2 | 51788 | 0.879 | 0.880 | 0.064 | 2 | 16699 |
| CS-Thresh | **0.891** | 0.891 | 0.061 | 1 | 2597 | **0.874** | 0.876 | 0.065 | 1 | 2480 | **0.861** | 0.863 | 0.069 | 1 | 1907 |
| CS-Clust | **0.903** | 0.904 | 0.058 | 4 | 5590 | **0.884** | 0.885 | 0.063 | 4 | 5174 | **0.876** | 0.878 | 0.065 | 4 | 1188 |
| CS-ClustRed | **0.901** | 0.901 | 0.058 | 3 | 3876 | **0.884** | 0.885 | 0.063 | 4 | 2135 | **0.875** | 0.877 | 0.065 | 3 | 809 |
| | | | | | | | | | | | | | | | |
| **RePro** | **ADWIN** | **AWPro** |
| | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. |
| CDD | 0.883 | 0.884 | 0.059 | 1 | 0 | 0.830 | 0.836 | 0.072 | 1 | 0 | 0.807 | 0.813 | 0.076 | 1 | 0 |
| BOTL | -1e+21 | 0.506 | +3e+9 | 25 | 0 | 5e+22 | 0.499 | +2e+10 | 40 | 0 | 2e+22 | 0.534 | +1e+10 | 18 | 0 |
| MI-Thresh | 0.904 | 0.905 | 0.054 | 3 | 27226 | 0.880 | 0.881 | 0.060 | 2 | 46413 | 0.873 | 0.875 | 0.062 | 2 | 16748 |
| CS-Thresh | **0.892** | 0.893 | 0.057 | 1 | 2403 | **0.862** | 0.864 | 0.065 | 1 | 3038 | **0.851** | 0.852 | 0.067 | 1 | 2398 |
| CS-Clust | -1e+17 | 0.899 | +5e+6 | 4 | 5134 | -1e+18 | 0.873 | +1e+7 | 4 | 5415 | **0.864** | 0.866 | 0.064 | 4 | 1369 |
| CS-ClustRed | **0.904** | 0.905 | 0.054 | 4 | 3366 | -8e+17 | 0.842 | +2e+7 | 4 | 2393 | **0.862** | 0.863 | 0.065 | 3 | 841 |
| | | | | | | | | | | | | | | | |

Table 4: Sudden Drifting Hyperplane: $R^2$, PMCC$^2$ and RMSE predictive performance, the average number of base models used by the meta-learner ($|\mathcal{M}'|$), and the average number of relevancy and diversity metric calculations to compare and evaluate base models (M.Calcs.) for variants of the sudden drifting hyperplane datasets when transferring models between 5 data streams in BOTL. Improved predictive performances with statistical t-test values $p < 0.01$ compared to the underlying CDD, while requiring fewer relevancy and diversity metric calculations than MI-Thresh are indicated with *. Of these, bold type indicates the approach with highest performance.
Table 5: Gradual Drifting Hyperplane: $R^2$, PMCC^2 and RMSE predictive performance, the average number of base models used by the meta-learner ($|\mathcal{M}'|$), and the average number of relevancy and diversity metric calculations to compare and evaluate base models (M.Calcs.) for variants of the gradual drifting hyperplane datasets when transferring models between 5 data streams in BOTL. Improved predictive performances with statistical t-test values $p < 0.01$ compared to the underlying CDD, while requiring fewer relevancy and diversity metric calculations than MI-Thresh are indicated with *. Of these, bold type indicates the approach with highest performance.

(a) GradualA: gradual drifting hyperplanes with uniform noise.

|       | RePro | ADWIN | AWPro |
|-------|-------|-------|-------|
|       | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ |
| CDD   | 0.849 | 0.849 | 0.068 | 1 | 0 | 0.796 | 0.803 | 0.079 | 1 | 0 | 0.798 | 0.804 | 0.078 | 1 | 0 |
| BOTL  | 0.771 | 0.797 | 0.084 | 1 | 30 | 0.795 | 0.813 | 0.079 | 1 | 38 | 0.807 | 0.852 | 0.078 | 1 | 0 |
| MI-Thresh | 0.892 | 0.892 | 0.058 | 3 | 56526 | 0.886 | 0.886 | 0.059 | 3 | 73607 | 0.887 | 0.888 | 0.059 | 3 | 31410 |
| CS-Thresh | *0.866 | 0.866 | 0.064 | 2 | 3246 | *0.854 | 0.854 | 0.067 | 1 | 2946 | *0.854 | 0.854 | 0.067 | 1 | 2404 |
| CS-Clust | *0.897 | 0.898 | 0.056 | 4 | 11979 | *0.877 | 0.878 | 0.061 | 4 | 4985 | *0.876 | 0.877 | 0.062 | 4 | 1448 |
| CS-ClustRed | *0.894 | 0.894 | 0.057 | 4 | 6912 | *0.872 | 0.873 | 0.063 | 3 | 1732 | *0.871 | 0.872 | 0.063 | 3 | 858 |

(b) GradualB: gradual drifting hyperplanes with single sensor failure.

|       | RePro | ADWIN | AWPro |
|-------|-------|-------|-------|
|       | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ |
| CDD   | 0.861 | 0.862 | 0.067 | 1 | 0 | 0.757 | 0.771 | 0.088 | 1 | 0 | 0.697 | 0.719 | 0.099 | 1 | 0 |
| BOTL  | +7e+20 | 0.322 | +3e+9 | 29 | 0 | +1e+21 | 0.322 | +3e+9 | 39 | 0 | +1e+21 | 0.348 | +3e+9 | 19 | 0 |
| MI-Thresh | 0.888 | 0.888 | 0.060 | 3 | 45556 | 0.863 | 0.863 | 0.067 | 2 | 58346 | 0.845 | 0.845 | 0.071 | 2 | 23961 |
| CS-Clust | *0.876 | 0.877 | 0.063 | 1 | 3497 | *0.834 | 0.835 | 0.073 | 1 | 3186 | *0.814 | 0.815 | 0.078 | 1 | 2554 |
| CS-ClustRed | *0.890 | 0.891 | 0.060 | 3 | 7501 | 4e+15 | 0.855 | +1e+6 | 4 | 2031 | *0.834 | 0.835 | 0.073 | 3 | 925 |

(c) GradualC: gradual drifting hyperplanes with intermittent single sensor failure.

|       | RePro | ADWIN | AWPro |
|-------|-------|-------|-------|
|       | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ |
| CDD   | 0.841 | 0.841 | 0.068 | 1 | 0 | 0.773 | 0.779 | 0.081 | 1 | 0 | 0.759 | 0.773 | 0.083 | 1 | 0 |
| BOTL  | 0.784 | 0.806 | 0.080 | 1 | 28 | 0.778 | 0.800 | 0.081 | 38 | 0 | *0.876 | 0.879 | 0.060 | 19 | 0 |
| MI-Thresh | 0.890 | 0.890 | 0.057 | 3 | 51948 | 0.878 | 0.878 | 0.060 | 3 | 69534 | 0.872 | 0.873 | 0.061 | 3 | 29224 |
| CS-Clust | *0.861 | 0.861 | 0.064 | 1 | 3132 | *0.829 | 0.830 | 0.071 | 1 | 2882 | *0.826 | 0.827 | 0.071 | 1 | 2482 |
| CS-ClustRed | *0.891 | 0.892 | 0.057 | 4 | 12289 | *0.856 | 0.856 | 0.065 | 4 | 4867 | *0.866 | 0.866 | 0.063 | 4 | 1427 |
| CS-ClustRed | *0.888 | 0.889 | 0.057 | 4 | 7151 | *0.854 | 0.855 | 0.065 | 3 | 1943 | *0.855 | 0.856 | 0.065 | 3 | 816 |

(d) GradualD: gradual drifting hyperplanes with gradual sensor deterioration.

|       | RePro | ADWIN | AWPro |
|-------|-------|-------|-------|
|       | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC^2 | RMSE | $|\mathcal{M}'|$ |
| CDD   | 0.845 | 0.846 | 0.080 | 1 | 0 | 0.484 | 0.593 | 0.136 | 1 | 0 | 0.513 | 0.623 | 0.130 | 1 | 0 |
| BOTL  | +2e+22 | 0.147 | +2e+10 | 31 | 0 | +1e+22 | 0.130 | +1e+10 | 25 | 0 | +2e+22 | 0.111 | +2e+10 | 16 | 0 |
| MI-Thresh | 0.898 | 0.898 | 0.065 | 4 | 47278 | 0.827 | 0.828 | 0.079 | 2 | 4119 | 0.745 | 0.746 | 0.097 | 1 | 2667 |
| CS-Clust | *0.881 | 0.881 | 0.070 | 1 | 4119 | *0.745 | 0.746 | 0.097 | 1 | 2667 | *0.758 | 0.759 | 0.094 | 1 | 2345 |
| CS-ClustRed | *0.898 | 0.899 | 0.065 | 5 | 15555 | *0.761 | 0.762 | 0.091 | 3 | 2452 | *0.766 | 0.767 | 0.089 | 3 | 1089 |
| CS-ClustRed | *0.894 | 0.895 | 0.066 | 4 | 9224 | *0.766 | 0.768 | 0.090 | 3 | 1212 | *0.764 | 0.765 | 0.090 | 3 | 704 |
Table 6: Heating Simulator: $R^2$, PMCC$^2$ and RMSE predictive performance, the average number of base models used by the meta-learner (|$\mathcal{M}'$|), and the average number of relevancy and diversity metric calculations to compare and evaluate base models (M.Calcs.) for smart home hearing simulator datasets when transferring models between 5 data streams in BOTL. Improved predictive performances with statistical t-test values $p < 0.01$ compared to the underlying CDD, while requiring fewer relevancy and diversity metric calculations than MI-Thresh are indicated with *. Of these, bold type indicates the approach with highest performance.

|                | RePro | ADWIN | AWPro |
|----------------|-------|-------|-------|
|                | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. |
| CDD            | 0.636 | 0.652  | 2.506 | 1     | 0      | 0.635 | 0.655  | 2.509 | 1     | 0      | 0.597 | 0.621  | 2.636 | 1     | 0      |
| BOTL           | 0.730 | 0.737  | 2.154 | 10    | 0      | 0.720 | 0.728  | 2.190 | 9     | 0      | 0.716 | 0.725  | 2.212 | 10    | 0      |
| MI-Thresh      | 0.709 | 0.716  | 2.238 | 3     | 3190   | 0.707 | 0.715  | 2.244 | 3     | 3790   | 0.690 | 0.701  | 2.307 | 3     | 3792   |
| CS-Thresh      | 0.705 | 0.712  | 2.254 | 3     | 490    | 0.701 | 0.709  | 2.270 | 3     | 594    | 0.686 | 0.694  | 2.326 | 3     | 512    |
| CS-Clust       | 0.709 | 0.715  | 2.241 | 3     | 794    | 0.709 | 0.717  | 2.240 | 3     | 877    | 0.689 | 0.699  | 2.313 | 3     | 1178   |
| CS-ClustRed    | 0.707 | 0.714  | 2.248 | 3     | 655    | 0.706 | 0.714  | 2.248 | 3     | 794    | 0.693 | 0.702  | 2.296 | 3     | 1015   |

Table 7: Following Distance: $R^2$, PMCC$^2$ and RMSE predictive performance, the average number of base models used by the meta-learner (|$\mathcal{M}'$|), and the average number of relevancy and diversity metric calculations to compare and evaluate base models (M.Calcs.) for following distance datasets when transferring models between 7 data streams in BOTL. The $R^2$ and PMCC$^2$ predictive performances and number of base model for other numbers of data streams is shown in Figure 6. Improved predictive performances with statistical t-test values $p < 0.01$ compared to the underlying CDD, while requiring fewer relevancy and diversity metric calculations than MI-Thresh are indicated with *. Of these, bold type indicates the approach with highest performance.

|                | RePro | ADWIN | AWPro |
|----------------|-------|-------|-------|
|                | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. | $R^2$ | PMCC$^2$ | RMSE | $|\mathcal{M}'|$ | M.Calcs. |
| CDD            | 0.545 | 0.564  | 0.603 | 1     | 0      | 0.441 | 0.502  | 0.676 | 1     | 0      | 0.516 | 0.554  | 0.628 | 1     | 0      |
| BOTL           | 0.653 | 0.685  | 0.533 | 5     | 0      | 0.419 | 0.400  | +2e+15| 10    | 0      | 0.456 | 0.671  | 0.586 | 7     | 0      |
| MI-Thresh      | 0.675 | 0.696  | 0.516 | 2     | 869    | 0.659 | 0.678  | 0.530 | 2     | 1302   | 0.684 | 0.695  | 0.512 | 2     | 768    |
| CS-Thresh      | 0.664 | 0.688  | 0.524 | 2     | 147    | 0.613 | 0.640  | 0.567 | 2     | 144    | 0.665 | 0.677  | 0.528 | 2     | 121    |
| CS-Clust       | 0.655 | 0.681  | 0.532 | 2     | 587    | 0.634 | 0.661  | 0.543 | 3     | 481    | 0.670 | 0.682  | 0.526 | 2     | 298    |
| CS-ClustRed    | 0.650 | 0.678  | 0.535 | 2     | 577    | 0.637 | 0.662  | 0.543 | 3     | 328    | 0.678 | 0.689  | 0.521 | 2     | 264    |
Figure 6 shows the predictive performance and number of base models used by the meta-learner with increasing numbers of following distance data streams. Without using base model selection techniques, BOTL can overfit, even when the number of data streams is small. However, CS-Thresh prevents the meta-learner overfitting in these real-world data streams, obtaining similar predictive performances to MI-Thresh, while CS-Clust achieves this without requiring a user defined threshold parameter.

Although CS-Clust requires additional computation for clustering, the use of conceptual similarity as a measure of diversity significantly reduces the number of pairwise comparisons between base models. Unlike MI-Thresh, the diversity metric remains static, independent of concept drifts, and therefore does not need to be recalculated as the data stream progresses. This is shown in Figure 7 which highlights the number of relevancy and diversity metric calculations required to compare and evaluate base models in CS-Clust compared to MI-Thresh, as the number of following distance data streams increases. A significant reduction in relevancy and diversity metric calculations is observed across all datasets, which can be seen in Tables 4–7.
Figure 7: Change in number of relevancy and diversity metric calculations required to compare and evaluate base models for CS-Clust vs. MI-Thresh for increasing numbers of following distance data streams.

7.3 Local Scaling in STSC

Parameterless conceptual clustering uses STSC [41] to create clusters of conceptually similar base models. Although the number of clusters of base models is determined by STSC, in order to perform SC the affinity matrix used by STSC undergoes local scaling, as shown in Section 3, using Definition 3. This allows better affinities to be obtained when the density of conceptually similar base models varies [41]. Throughout this paper a local scaling parameter of \( k = 7 \) was used, as suggested by Zelnik-Manor and Perona [41]. Although parameter tuning is not typically needed for local scaling to perform well [40], we consider the predictive performance of CS-Clust and the number of base models used by the OLS meta-learner with local scaling parameters varying between \( k = 2 \) and \( k = 7 \).

Figure 8 presents the predictive performance of CS-Clust and the number of base models selected as input to the OLS meta-learner using RePro, ADWIN and AWPro as CDDs, for sudden drifting hyperplane, gradual drifting hyperplane, heating simulator and following distance datasets. Figure 8 shows that varying the local scaling parameter, \( k \), has minimal effect on the predictive performance of CS-Clust. Additionally, the number of base models selected as input to the OLS meta-learner does not change significantly. Since CS-Clust selects a single model from each cluster, the number of base models used by the meta-learner is a good indicator of the number of clusters of conceptually similar base models identified.

These results support the suggestions in [41] and [40] that \( k = 7 \) is a good parameter for local scaling in STSC. However, accounting for locally dense areas in the affinity matrix can be challenging when clustering base models learnt from online environments, particularly if CDDs such as ADWIN are used, where base models are not reused in the presence of recurring concepts. Therefore, future work may consider alternative methods, such as density-aware kernels [40, 42] to overcome the challenges of clustering using affinity matrices with locally dense areas.

7.4 Reducing Redundant Transfer

To reduce knowledge transfer in BOTL, we used parameterless conceptual clustering on locally learnt base models, prior to transfer. This allowed conceptually similar models within a data stream to be identified. In BOTL-Red we apply this to BOTL (with no base model selection), and in CS-ClustRed to CS-Clust. Figure 9 shows that the number of models transferred can be reduced, without significant change in predictive performance. ADWIN does not reuse historically learnt models, and therefore a larger reduction in transferred base
Figure 8: $R^2$ and number of base models used by CS-Clust meta-learners for varying local scaling parameters in sudden drifting hyperplane, gradual drifting hyperplane, heating simulator, and following distance data streams, using RePro, ADWIN and AWPro as underlying CDDs.
Figure 9: $R^2$ performance, and number of base models used by meta-learners in BOTL, with reduced base model transfer (BOTL-Red and CS-ClustRed) compared to transferring every base model (BOTL and CS-Clust) for increasing numbers of following distance data streams.

models is observed since multiple models learnt from recurring concepts are no longer transferred. BOTL-Red does not prevent the meta-learner from overfitting since it only considers the diversity among locally learnt models, and does not use base model selection techniques to consider the conceptual similarity among models received from other data streams.

Since CS-ClustRed achieves similar performance to CS-Clust, it may be desirable to use CS-ClustRed when the communication overhead of knowledge transfer is high. However, there is a trade-off against additional computational overhead since conceptual clustering is needed both before knowledge transfer, to identify conceptually similar locally learnt models, and once base models have been received, to identify a relevant yet diverse subset of base models to be used by the meta-learner from local and transferred models.

8 Conclusion

CDDs can be used to make predictions in data streams that are subject to concept drift, where data availability may be limited. Meta-learners can improve predictive capabilities through the use of historically learnt knowledge, or transferred knowledge when online TL is used. However, as the number of available base models becomes large in comparison to the available data, a relevant yet diverse subset of base models must be selected to prevent the meta-learner overfitting and to improve generalisation.

We have presented parameterised thresholding and parameterless conceptual clustering methods that estimate conceptual similarity to identify diversity among base models, independent of the current distribution in the data stream. Comparable predictive performance to performance and MI thresholding have been obtained, while requiring fewer base model comparisons. Although conceptual clustering requires additional computation due to reforming clusters when new base models are made available, the reduction in base model comparisons and the avoidance of user defined parameters, makes it more applicable to real-world applications.
in which the number of base models and frequency of drifts are unknown and could become large.

Additionally, we have shown that conceptual clustering can be used in online TL to reduce the number of models transferred between concept drifting data streams. In future work, we will consider cross-domain conceptual similarity to further reduce the transfer of conceptually similar models. Cross-domain conceptual similarity will also allow locally learnt models to be replaced by conceptually similar models, learnt from other data streams, that prove to be more effective predictors. As both conceptual similarity and the BOTL framework are model agnostic, differing machine learning models can be used to create base models within each data stream, depending on the locally available computational resources. This may be beneficial in online TL frameworks since a domain with limited computational capacity can replace locally learnt base models with a conceptually similar base model that has been learnt in a domain that has greater computational resources. For example, if one domain in a BOTL framework is only able to utilise simple machine learning algorithms, such as Ridge Regression, to create base models, but other domains are capable of using more complex algorithms, such as SVRs, a locally learnt Ridge Regression model could be replaced by a conceptually similar SVR that has improved predictive capabilities.

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