Superselection sectors in the Ashtekar–Horowitz–Boulware model

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Abstract
We investigate refined algebraic quantization of the constrained Hamiltonian system introduced by Boulware as a simplified version of the Ashtekar–Horowitz model. The dimension of the physical Hilbert space is finite and asymptotes in the semiclassical limit to $(2\pi\hbar)^{-1}$ times the volume of the reduced phase space. The representation of the physical observable algebra is irreducible for generic potentials but decomposes into irreducible subrepresentations for certain special potentials. The superselection sectors are related to singularities in the reduced phase space and to the rate of divergence in the formal group averaging integral. There is no tunnelling into the classically forbidden region of the unreduced configuration space, but there can be tunnelling between disconnected components of the classically allowed region.

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1. Introduction

In this paper, we study quantization of the constrained Hamiltonian system introduced by Boulware [1] as a simplified version of the Ashtekar–Horowitz model [2]. Both systems have a four-dimensional unreduced phase space and a single constraint, quadratic in the momenta. The Ashtekar–Horowitz model was originally introduced in [2] to model the situation occurring in general relativity in which certain parts of the unreduced configuration space are not in the projection of the constraint hypersurface. These parts of the unreduced configuration space thus play no part in the classical theory, but they could give rise to tunnelling effects in Dirac-style quantizations [3, 4]. The quantization discussed in [2] indeed displayed such effects, containing physical states that have support in the classically forbidden region of the configuration space. Further developments using a variety of quantization schemes can be found in [1, 5–7]. In particular, there exists a path-integral quantization of Boulware’s system that exhibits no tunnelling into the classically forbidden region [1].
We shall investigate Boulware’s system within the refined algebraic quantization (RAQ) programme of [8–10] (for reviews, see [11, 12]). The main new issue of interest for us is that a RAQ quantum theory entails not just a physical Hilbert space \( \mathcal{H}_{\text{RAQ}} \) but also a precisely defined algebra \( \mathcal{A}_{\text{obs}} \) of physical observables. We wish to study this algebra and in particular ask whether its representation contains superselection sectors.

A major piece of technical input in RAQ is the rigging map, which maps a dense subspace of suitably well-behaved states in the unconstrained Hilbert space to distributional states that solve the constraints. In our system, the integral of matrix elements over the gauge group does not converge in absolute value, which complicates attempts to define a rigging map by group averaging. However, for generic potentials (in a sense that will be made precise), the formal group averaging expression nevertheless suggests a rigging map candidate. We show that this candidate is a genuine rigging map and the resulting representation of \( \mathcal{A}_{\text{obs}} \) is irreducible. For certain special potentials, the rigging map candidate becomes ill defined, owing to formally divergent terms, but we show that the candidate can then be replaced by a genuine rigging map by renormalizing the divergences. In this case, the representation of \( \mathcal{A}_{\text{obs}} \) decomposes into superselection sectors, labelled by the degrees of divergence in the formal rigging map candidate, and the representation within each superselection sector is irreducible. These results bear a qualitative similarity to the superselection sector results found in [13] in an \( SO(n, 1) \) gauge system but our sense of convergence in the group averaging is weaker and our superselection sector structure is richer.

The system also exhibits a striking connection between quantum superselection and classical singularities: superselection sectors exist precisely when some vectors in \( \mathcal{H}_{\text{RAQ}} \) are supported on the part of the unreduced configuration space that is associated with singular parts of the reduced phase space.

The physical Hilbert space is finite dimensional, and in the semiclassical limit its dimension asymptotes to \( (2\pi\hbar)^{-1} \) times the volume of the reduced phase space. The only superselection sector that remains significant in the semiclassical limit is the one whose rigging map requires no renormalization.

As in [1], there is no tunnelling of the kind found in [2] into the classically forbidden region of the unreduced configuration space. If the classically allowed region of the unreduced configuration space is not connected, there can be tunnelling between its components.

The rest of the paper is as follows. Section 2 introduces the classical system, and the quantization is carried out in section 3. Section 4 presents brief concluding remarks. The proofs of certain technical results are deferred to three appendices. We set \( \hbar = 1 \) except in the semiclassical limit discussion in section 3.

2. Classical system

The configuration space of the system is \( \mathcal{C} := T^2 \simeq S^1 \times S^1 \). We write the points in \( \mathcal{C} \) as \((x, y)\), where \( x \in S^1 \) and \( y \in S^1 \), and points in the phase space \( \Gamma := T^*\mathcal{C} \) as \((x, y, p_x, p_y)\), where \( p_x \in \mathbb{R} \) and \( p_y \in \mathbb{R} \).

The action reads

\[
S = \int dt \left( p_x \dot{x} + p_y \dot{y} - \lambda C \right),
\]

where the overdot denotes differentiation with respect to the parameter \( t \) and \( \lambda \) is a Lagrange multiplier associated with the constraint

\[
C := p_x^2 - R(y),
\]
where $R : S^1 \to \mathbb{R}$ is smooth. We assume $R$ to be positive at least somewhere. We also assume that $R$ has at most finitely many stationary points. It follows that $R$ has at least two stationary points and at most finitely many zeros. We further assume that each stationary point of $R$ has a nonvanishing derivative of $R$ of some order. To simplify the discussion of the classical system, we assume that no zero of $R$ is a stationary point. In section 3, we will introduce a further genericity condition on $R$ to control the quantum theory.

The constraint surface $\Gamma_1$ is the subset of $\Gamma_1$ where $C = 0$. By our assumptions about $R$, $\Gamma_1$ is the Cartesian product of $S^1 \times \mathbb{R} = \{(x, p_x)\}$ with finitely many disjoint circles in $S^1 \times \mathbb{R} = \{(y, p_y)\}$. We show in appendix A that each connected component of the reduced phase space $\Gamma_{\text{red}}$ is a two-dimensional symplectic manifold with certain one-dimensional singular subsets, and the symplectic volume of $\Gamma_{\text{red}}$ is finite and equal to $2\pi \int_{R > 0} |R'(y)|/\sqrt{R(y)} \, dy$ or $4\pi$ times the total variation of $\sqrt{R}$ over the subset of $S^1$ on which $R$ is positive. The singularities occur at the stationary points of $R$. This will become important on comparison to the quantum theory.

3. Quantization

In this section, we quantize the system, following refined algebraic quantization as reviewed in [12]. Subsection 3.1 fixes the structure in the auxiliary Hilbert space. The rigging map is constructed in subsection 3.2 under a certain genericity condition on $R$ and in subsection 3.3 under a weaker form of this condition.

3.1. Auxiliary structure

Our auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ is the space of square integrable functions on $C$ in the inner product

$$(\phi_1, \phi_2)_{\text{aux}} := \int \! dx \, dy \, \overline{\phi_1(x, y)} \phi_2(x, y),$$

(3.1)

where the overline denotes complex conjugation. To promote the classical constraint (2.2) into a quantum operator, we replace the momentum term $p_x^2$ by $-\partial^2/\partial x^2$, and in the potential term we replace the function $R(y)$ by the operator $\hat{R}$ that acts on $\phi \in \mathcal{H}_{\text{aux}}$ by $(\hat{R}\phi)(x, y) = R(y)\phi(x, y)$. The quantum constraint $\hat{C}$ thus reads

$$\hat{C} := -\frac{\partial^2}{\partial x^2} - \hat{R}.$$  

(3.2)

$\hat{C}$ is essentially self-adjoint on $\mathcal{H}_{\text{aux}}$ and exponentiates into the one-parameter family of unitary operators

$$U(t) := e^{-it\hat{C}}, \quad t \in \mathbb{R}.$$  

(3.3)

We next need to choose the test space $\Phi$, a linear subspace of sufficiently well-behaved states in $\mathcal{H}_{\text{aux}}$. Taking advantage of the Fourier decomposition in $x$, we take $\Phi$ to be the space of functions $f : C \to \mathbb{C}$ of the form $f(x, y) = \sum_{m \in \mathbb{Z}} e^{imx} f_m(y)$, where each $f_m : S^1 \to \mathbb{C}$ is smooth and only finitely many $f_m$ are different from zero for each $f$. $\Phi$ is clearly a dense linear subspace of $\mathcal{H}_{\text{aux}}$. If $f \in \Phi$, then

$$(U(t)f)(x, y) = \sum_m e^{-im[n^2-R(y)]} e^{imx} f_m(y),$$

(3.4)

which shows that $U(t)f \in \Phi$. $\Phi$ is thus invariant under $U(t)$. Note that if $f, g \in \Phi$, then

$$(f, g)_{\text{aux}} = 2\pi \sum_m \int \! dy \, f_m(y) g_m(y).$$

(3.5)
The above structure determines the RAQ observable algebra $A_{\text{obs}}$ as the algebra of operators $O$ on $H_{\text{aux}}$ such that the domains of $O$ and $O^\dagger$ include $\Phi$, $O$ and $O^\dagger$ map $\Phi$ to itself and $O$ commutes with $U(t)$ on $\Phi$ for all $t$. Note that if $O \in A_{\text{obs}}$, then also $O^\dagger \in A_{\text{obs}}$.

What remains is to specify the final ingredient in RAQ, an antilinear rigging map $\eta : \Phi \rightarrow \Phi^*$, where the asterisk denotes the algebraic dual, topologized by pointwise convergence. $\eta$ must be real and positive, states in its image must be invariant under the dual action of $U(t)$, and $\eta$ must intertwine with the representations of $A_{\text{obs}}$ on $\Phi$ and $\Phi^*$ in the sense that for all $O \in A_{\text{obs}}$ and $\phi \in \Phi$,

$$\eta(O\phi) = O(\eta\phi). \tag{3.6}$$

In terms of the matrix elements, (3.6) reads

$$\eta(O\phi_1)[\phi_2] = \eta(O\phi_1)[O^\dagger\phi_2], \tag{3.7}$$

where $\phi_1, \phi_2 \in \Phi$, the left-hand side denotes the dual action of $\eta(O\phi_1) \in \Phi^*$ on $\phi_2 \in \Phi$ and the right-hand side denotes the dual action of $\eta(O\phi_1) \in \Phi^*$ on $O^\dagger\phi_2 \in \Phi$. The rigging map then completely determines both the physical Hilbert space $\mathcal{H}_{\text{RAQ}}$ and the representation of $A_{\text{obs}}$ on it; $\mathcal{H}_{\text{RAQ}}$ is the Cauchy completion of the image of $\eta$ in the inner product

$$(\eta(\phi_1), \eta(\phi_2))_{\mathcal{H}_{\text{RAQ}}} := \eta(\phi_1)[\phi_2], \tag{3.8}$$

and the properties of $\eta$ and $A_{\text{obs}}$ imply that $\eta$ induces an antilinear representation of $A_{\text{obs}}$ on $\mathcal{H}_{\text{RAQ}}$, with the image of $\eta$ as the dense domain.

To find a rigging map, we shall place a genericity condition on $R$. In subsection 3.2, we work under a genericity condition that is relatively strong and will make the representation of $A_{\text{obs}}$ irreducible. In subsection 3.3, we weaken this condition in a way that will lead to superselection sectors.

### 3.2. Rigging map for generic $R$

Our construction of the rigging map will use the solutions to the equation

$$q^2 = R(y), \tag{3.9}$$

where the non-negative integer $q$ is a parameter and $y$ is regarded as the unknown. We assume that (3.9) has solutions for some $q$. From the assumptions on $R$, it follows that solutions only exist for finitely many $q$ and that for each $q$ there are at most finitely many solutions.

In this subsection, we assume that none of the solutions to (3.9) are stationary points of $R$. We write the solutions as $y_{qj}$, where the second index labels the solutions for given $q$.

Recall that the group averaging proposal seeks a rigging map as an implementation of the formal expression

$$\eta : \phi \mapsto \int_{-\infty}^{\infty} dt \, \phi^\dagger U(t). \tag{3.10}$$

A strategy proposed in [10] would be to try to define (3.10) in terms of integrated matrix elements as

$$\eta(\phi_1)[\phi_2] = \int_{-\infty}^{\infty} dt \, (\phi_1, U(t)\phi_2)_{\text{aux}}. \tag{3.11}$$

Examples in which this strategy can be successfully implemented are found in [13–16]. In our system, however, a saddle-point estimate shows that there are states for which the absolute value of the integrand on the right-hand side of (3.11) is asymptotically proportional to $|t|^{-1/2}$.
as $|t| \to \infty$, and for such states the integral is not absolutely convergent. While it may be possible to work with (3.11) in some appropriate weaker sense of conditional convergence, we shall not pursue this line here. Instead, we show that a formal reinterpretation of (3.11) leads to a map that can be directly proven to be a rigging map.

The integral expression (3.10) can be formally rewritten as (cf [8, 17, 18])

$$(\eta(f))(x, y) = 2\pi \sum_m \delta(m^2 - R(y)) e^{-imx} \frac{f_m(y)}{R(y)}$$

or equivalently as

$$(\eta(f))(x, y) = 2\pi \sum_{mj} e^{-imx} \frac{f_m(y)}{|R'(y_{|m|j})|} \delta(y, y_{|m|j}),$$

where the Dirac delta distributions in (3.12) and (3.13) are, respectively, those on $\mathbb{R}$ and $S^1$. While (3.10) remains formal, the right-hand sides of (3.12) and (3.13) are well-defined distributions on smooth functions on $C$. We now adopt (3.13) (or equivalently (3.12)) as the definition of our $\eta$ and proceed to show that this $\eta$ satisfies the rigging map axioms.

For $f, g \in \Phi$, (3.13) yields

$$\eta(f)[g] = (2\pi)^2 \sum_{mj} \frac{f_m(y_{|m|j}) g_m(y_{|m|j})}{|R'(y_{|m|j})|}.$$  

From (3.14) it is evident that $\eta$ is real and positive, and also that the image of $\eta$ is nontrivial and finite dimensional. From (3.4) it follows that vectors in the image of $\eta$ are invariant under the dual action of $U(t)$. We show in appendix B that $\eta$ intertwines with $A_{\text{obs}}$ in the sense of (3.7). $\eta$ is thus a rigging map and the physical Hilbert space $H_{\text{RAQ}}$ is the image of $\eta$ equipped with the inner product that can be read off from (3.8) and (3.14). Note that no Cauchy completion is needed since $H_{\text{RAQ}}$ is finite dimensional.

We show in appendix C that the representation of $A_{\text{obs}}$ on $H_{\text{RAQ}}$ is irreducible.

As all the states in $H_{\text{RAQ}}$ have their support in the classically allowed region of $C$, there is no tunnelling of the kind found in [2] into the classically forbidden region of $C$. If the classically allowed region of $C$ is not connected, there is however tunnelling between all its components that support states in $H_{\text{RAQ}}$.

Finally, consider the semiclassical limit. When $\hbar$ is reinstated, equation (3.9) becomes $\hbar^2 q^2 = R(y)$. In the limit $\hbar \to 0$, the dimension of $H_{\text{RAQ}}$ thus asymptotes to $2/\hbar$ times the total variation of $\sqrt{R}$ over the subset of $S^1$ on which $R$ is positive. From section 2, we see that this is $(2\pi\hbar)^{-1}$ times the volume of $\Gamma_{\text{red}}$. Although our $\Gamma_{\text{red}}$ is not compact, this is the semiclassical limit one might have expected on comparison with geometric quantization on compact phase spaces, such as quantization of angular momentum on the phase space $S^2$ [19, 20].

### 3.3. Rigging map with degenerate solutions to (3.9)

In this subsection, we allow stationary points of $R$ among the solutions to (3.9). As formulæ (3.13) and (3.14) then become ill defined, some modification is required.

We label the solutions to (3.9) as follows. Let $p$ denote the order of the lowest nonvanishing derivative of $R$ at a solution. For odd $p$, we write the solutions as $y_{pqj}$, where the last index enumerates the solutions with given $p$ and $q$. For even $p$, we write the solutions as $y_{p\epsilon qj}$, where $\epsilon \in \{1, -1\}$ is the sign of the $p$th derivative and the last index enumerates the solutions with given $p$, $\epsilon$ and $q$. 

Let $\mathcal{P}$ be the value set of the first index of the solutions $\{y_{pqj}\}$ and $\{y_{peqj}\}$. We assume that $\mathcal{P}$ has the following property:

If $p \in \mathcal{P}$, then $\mathcal{P}$ contains no factors of $p$ smaller than $p/2$.

The case of subsection 3.2 is recovered for $\mathcal{P} = \{1\}$.

For each odd $p \in \mathcal{P}$, we now define the map $\eta_p : \Phi \to \Phi^*$ by

$$
(\eta_p(f))(x, y) = 2\pi \sum_{mj} \frac{e^{-im_1 f_m(y)}}{|R(p)(y_{p|m})|^{1/p}} \delta(y, y_{p|m}).
$$

Similarly, for each even $p \in \mathcal{P}$ and each $\epsilon \in \{1, -1\}$ for which solutions to (3.9) exist, we define the map $\eta_{pe} : \Phi \to \Phi^*$ by

$$
(\eta_{pe}(f))(x, y) = 2\pi \sum_{mj} \frac{e^{-im_1 f_m(y)}}{|R(p)(y_{pe|m})|^{1/p}} \delta(y, y_{pe|m}).
$$

When $\mathcal{P} = \{1\}$, the only map defined by these formulae is $\eta_1$ from (3.15) with $p = 1$, and this map is identical to that in (3.13). We thus recover the results of subsection 3.2. When $\mathcal{P} \neq \{1\}$, the maps (3.15) and (3.16) with $p > 1$ receive contributions from precisely those solutions to (3.9) for which the corresponding terms in (3.13) diverge. We can therefore think of the maps (3.15) and (3.16) with $p > 1$ as appropriately renormalized versions of the respective ill-defined terms in (3.13). We show in appendix B that the coefficients in (3.15) and (3.16) are fixed by the requirement that the maps have the intertwining property (3.7).

If $f, g \in \Phi$, (3.15) and (3.16) give

$$
\eta_p(f)[g] = (2\pi)^2 \sum_{mj} \frac{\theta_m(y_{p|m})\theta_m(y_{p|m})}{|R(p)(y_{p|m})|^{1/p}},
$$

and

$$
\eta_{pe}(f)[g] = (2\pi)^2 \sum_{mj} \frac{\theta_m(y_{pe|m})\theta_m(y_{pe|m})}{|R(p)(y_{pe|m})|^{1/p}}.
$$

From (3.17) it is seen that each $\eta_p$ and $\eta_{pe}$ has a finite-dimensional, nontrivial image and satisfies the rigging map axioms, with the possible exception of the intertwining property (3.7). We show in appendix B that each $\eta_p$ and $\eta_{pe}$ also satisfies the intertwining property and hence provides a rigging map. Each of the images of these maps provides therefore a RAQ physical Hilbert space, denoted, respectively, by $\mathcal{H}^p_{\text{RAQ}}$ and $\mathcal{H}^{pe}_{\text{RAQ}}$, with the inner product given by (3.8) and (3.17). As all the spaces are finite dimensional, no Cauchy completion is needed.

As the images of any two of the rigging maps have trivial intersection in $\Phi^*$, we can regard $\mathcal{H}^p_{\text{RAQ}}$ and $\mathcal{H}^{pe}_{\text{RAQ}}$ as superselection sectors in the ‘total’ RAQ Hilbert space

$$
\mathcal{H}^\text{tot}_{\text{RAQ}} := \bigoplus_{p \text{ odd}} \mathcal{H}^p_{\text{RAQ}} \oplus \bigoplus_{p \text{ even}, \epsilon} \mathcal{H}^{pe}_{\text{RAQ}}.
$$

We show in appendix C that the representation of $A_{\text{obs}}$ on each $\mathcal{H}^p_{\text{RAQ}}$ and $\mathcal{H}^{pe}_{\text{RAQ}}$ is irreducible. This means that there are no further superselection sectors in $\mathcal{H}^\text{tot}_{\text{RAQ}}$.

There is again no tunnelling into the classically forbidden region of $\mathcal{C}$, but within each $\mathcal{H}^p_{\text{RAQ}}$ and $\mathcal{H}^{pe}_{\text{RAQ}}$ there can be tunnelling between the connected components of the classically allowed region of $\mathcal{C}$.

In the semiclassical limit, the dimension of $\mathcal{H}^1_{\text{RAQ}}$ asymptotes to $(2\pi\hbar)^{-1}$ times the volume of $\Gamma_{\text{red}}$, while the dimension of the orthogonal complement of $\mathcal{H}^1_{\text{RAQ}}$ in $\mathcal{H}^\text{tot}_{\text{RAQ}}$ remains...
4. Concluding remarks

In this paper, we have studied refined algebraic quantization (RAQ) of Boulware’s version of the Ashtekar–Horowitz model. Although the system did not allow a rigging map to be defined in terms of an absolutely convergent integral of matrix elements over the gauge group, the formal group averaging expressions nevertheless suggested a rigging map candidate, and we showed that for generic potential functions this candidate is a rigging map and the resulting representation of the RAQ observable algebra \( A_{\text{obs}} \) on the physical Hilbert space \( \mathcal{H}^{\text{tot}}_{\text{RAQ}} \) is irreducible. For certain special potentials, the rigging map candidate contained formally divergent terms, but a renormalization of these terms yielded a genuine rigging map, and in this case the representation of \( A_{\text{obs}} \) on \( \mathcal{H}^{\text{tot}}_{\text{RAQ}} \) decomposed into superselection sectors. The dimension of \( \mathcal{H}^{\text{tot}}_{\text{RAQ}} \) was in all cases finite and bore the expected semiclassical relation to the volume of the reduced phase space. The only superselection sector that remained significant in the semiclassical limit was the one whose rigging map required no renormalization.

The system exhibits a striking connection between the singular subsets in the reduced phase space \( \Gamma_{\text{red}} \) and the superselection sectors in the quantum theory. Because of the periodicity of the coordinate \( x \) on the unreduced configuration space \( \mathcal{C} \cong T^2 \), the conjugate momentum \( p_x \) gets quantized in integer values. For generic potentials, these integer values entirely miss the singular, measure zero subsets of \( \Gamma_{\text{red}} \), and in this case the quantum theory has no superselection sectors. However, when the potential is such that one or more of the quantized values of \( p_x \) hit some of the singular subsets of \( \Gamma_{\text{red}} \), superselection sectors arise in the quantum theory.

Although the compactness of \( \mathcal{C} \) simplified some aspects of the analysis, the compactness is as such not essential: the results remain qualitatively similar if the \( y \)-direction is unwrapped to the real axis, provided the range of \( y \) in which \( R \) takes positive values remains bounded. What is essential is the periodicity in the \( x \)-direction. As seen in appendix A, it is the \( x \)-periodicity that in the classical theory renders the volume of \( \Gamma_{\text{red}} \) finite and creates the singular subsets; in the quantum theory, it is the associated discreteness of \( p_x \) that makes the physical Hilbert space finite dimensional and allows the isolated stationary points of the potential to make nonzero contributions to the rigging map. If \( x \) takes values in \( \mathbb{R} \), these phenomena do not arise. The reduced phase space then has infinite volume and no singularities, the physical Hilbert space is infinite dimensional and the stationary points of \( R \) make a vanishing contribution to the rigging map.

The divergences in the rigging map appear to be related to the rate of divergence in the formal group averaging integral. It might be possible to investigate this issue in a more precise setting in systems where the variable \( y \in S^1 \) is replaced by a variable that takes values on a higher-dimensional space, say \( T^n \) with \( n > 1 \). For \( n \geq 3 \) and an \( R \) whose only stationary points are nondegenerate, it should then be possible to make the convergence of the averaging so strong that the uniqueness theorem of Giulini and Marolf [10] applies and implies in particular that there are no superselection sectors. Modifying the stationary point structure of \( R \) should then offer a range of options for weakening the sense of convergence and creating superselection sectors.

One would like to understand whether the connection between classical singularities and quantum superselection sectors extends from our specific system to more general classes of
constrained systems. On the classical side, the structure of the reduced phase space at the singularities can be described by the methods of singular Marsden–Weinstein reduction [21]. On the quantum side, the methods of RAQ can be regarded as a version of Rieffel induction [22], which has been argued [23, 24] to provide a natural quantum counterpart of the Marsden–Weinstein reduction. The basic language for discussing this connection appears therefore to be in place. It is likely that results in this direction would involve criteria on the RAQ test space. In the Rieffel induction language, such criteria can be described in terms of spectral continuity [24].

Our quantum theory appears physically reasonable, and when there are no superselection sectors, the theory can be regarded as a specification of operators on the physical Hilbert space constructed already in [1]. As the system was originally introduced in [1] as a simplified version of the Ashtekar–Horowitz (AH) model [2], one might expect our methods to produce a physically reasonable quantization also for the AH model. In the AH model, the variables \( x \) and \( y \) are interpreted as, respectively, the azimuthal and longitudinal angles on \( S^2 \), so that \( y \) has period \( 2\pi \) but \( 0 \leq x \leq \pi \), where the limits correspond to coordinate singularities at the north and south poles. The classical constraint can be promoted into a self-adjoint operator by introducing suitable boundary conditions at \( x = 0 \) and \( x = \pi \) [1], and we can then proceed essentially as in the Boulware system, recovering a finite-dimensional physical Hilbert space. Whether this quantization is physically reasonable seems now to hinge on one’s viewpoint on the classical system. On the one hand, if \( x = 0 \) and \( x = \pi \) are regarded as classically excluded, the reduced phase space has infinite volume and one would then expect a quantum theory with an infinite-dimensional Hilbert space [6, 7]. This viewpoint is adopted in the context of algebraic quantization in [6, 7]. On the other hand, the incompleteness of the Hamiltonian vector field of the constraint suggests that one might want to interpret the classical theory at \( x = 0 \) and \( x = \pi \) in terms of reflective boundary conditions of some sort, for example, as suggested by free motion at constant longitude on the round sphere. This viewpoint gives the reduced phase space finite volume, which leads one to expect a finite-dimensional Hilbert space in the quantum theory. Our quantization methods are thus compatible with the latter classical viewpoint. A refined algebraic quantization of the AH model that would be compatible with the former classical viewpoint remains an intriguing open problem.

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Appendix A. \( \Gamma_{\text{red}} \)

In this appendix, we verify the properties of \( \Gamma_{\text{red}} \) stated in the main text.

Each orbit generated by the constraint \( C \) on the constraint hypersurface \( \bar{\Gamma} \) has constant \( y \) and \( p_x \). We consider first the subset of \( \bar{\Gamma} \) where \( p_x \neq 0 \), which is always nonempty, and then include the subset (if nonempty) where \( p_x = 0 \).
A.I. \( p_s \neq 0 \)

Let \( I \subset S^1 \) be an open interval in which \( R \) takes positive values. The corresponding two subsets of \( \mathcal{R} \) are \( \mathcal{N} := \{(x, y, \sqrt{R(y)}, p_y) \mid x \in S^1, y \in I, p_s \in \mathbb{R}\} \) and a similar set with a minus sign in front of the square root. We consider \( \mathcal{N} \); the situation for the other set is similar.

The orbits that \( C \) generates in \( \mathcal{N} \) have constant \( y \), and they satisfy \( \dot{x} \neq 0 \) and \( \dot{p}_y = \frac{1}{2} \sqrt{R'} \sqrt{R} \). If \( x \) were not periodic, we could choose from each of the gauge orbits a unique point by the condition \( x = 0 \) and hence represent the projection of \( \mathcal{N} \) to \( \mathcal{R}_{\text{red}} \) as \( \mathcal{M} := \{(0, y, \sqrt{R(y)}, p_y) \mid y \in I, p_s \in \mathbb{R}\} \cong I \times \mathbb{R} \), with the symplectic form \( \Omega_{\text{red}} = dp_y \wedge dy \). As \( x \) is periodic, however, the projection of \( \mathcal{N} \) to \( \mathcal{R}_{\text{red}} \) is not \( \mathcal{M} \) but instead the quotient space \( \mathcal{M}/\mathbb{Z} \), where the \( \mathbb{Z} \) action is

\[
(y, p_y) \mapsto (y, p_y + \pi n \sqrt{R(y)})/\sqrt{R(y)}, \quad n \in \mathbb{Z}. \tag{A.1}
\]

The structure of \( \mathcal{M}/\mathbb{Z} \) now depends on whether \( I \) contains stationary points of \( R \).

A.I.1. No stationary points.

If \( I \) does not contain stationary points of \( R \), the \( \mathbb{Z} \) action (A.1) is properly discontinuous and \( \mathcal{M}/\mathbb{Z} \) is a symplectic manifold with topology \( I \times S^1 \). We can introduce on \( \mathcal{M} \) the adapted coordinates \((w, \theta)\) by \( w = \sqrt{R(y)} \) and \( \theta = 2p_s \sqrt{R(y)}/R(y) \), in which the symplectic form reads \( \Omega_{\text{red}} = dw \wedge d\theta \) and the \( \mathbb{Z} \) action (A.1) takes the form

\[
(w, \theta) \mapsto (w, \theta + 2\pi n), \quad n \in \mathbb{Z}, \tag{A.2}
\]

so that \( \theta \) becomes periodic with period \( 2\pi \) in \( \mathcal{M}/\mathbb{Z} \). The symplectic volume of \( \mathcal{M}/\mathbb{Z} \) is finite and equal to \( \pi \int_I |R'(y)|/\sqrt{R(y)} dy \) or \( 2\pi \) times the total variation of \( \sqrt{R} \) over \( I \).

A.I.2. Stationary points.

If \( I \) contains stationary points of \( R \), we may assume without loss of generality that \( y_0 \in I \) is the only such stationary point. We write \( I = (y_-, y_+), I_+ := (y_0, y_+) \) and \( I_- := (y_-, y_0) \), so that \( I \) is the disjoint union of \( I_+, I_- \) and \( \{y_0\} \). The \( \mathbb{Z} \) action (A.1) is then properly discontinuous for \( y \in I_+ \) and \( y \in I_- \) but not at \( y = y_0 \). \( \mathcal{M}/\mathbb{Z} \) consists thus of two open cylinders, coming, respectively, from \( I_+ \) and \( I_- \) and each being a symplectic manifold, joined together by a line at \( y = y_0 \). \( \mathcal{M}/\mathbb{Z} \) is clearly connected. It is not Hausdorff, since points on the line at \( y = y_0 \) do not have disjoint neighbourhoods.

We shall show that \( \mathcal{M}/\mathbb{Z} \) is not a manifold. We first construct a subset of \( \mathcal{M}/\mathbb{Z} \) that is homeomorphic to \( \mathbb{R}^2 \) and then show that the properties of this subset prevent \( \mathcal{M}/\mathbb{Z} \) from being a manifold.

To begin, let \( \mathcal{M}_s := \{(0, y, \sqrt{R(y)}, p_y) \in \mathcal{M} \mid y \in I_+ \} \) and \( q := (0, y_0, \sqrt{R(y_0)}, 0) \in \mathcal{M} \). The \( \mathbb{Z} \) action (A.1) restricts to \( \mathcal{M}_s \) and to \( \mathcal{M}_s \cup \{q\} \).

We introduce in \( \mathcal{M}_s \) the adapted coordinates \((w, \theta)\) as above. Writing \( w_0 := \sqrt{R(y_0)} \) and \( w_+ := \sqrt{R(y_+)} \), we then introduce in \( \mathcal{M}_s/\mathbb{Z} \) the coordinates \((u, v)\) by \( u = \sqrt{|w - w_0| \cos \theta} \) and \( v = \sqrt{|w - w_0| \sin \theta} \), where \( 0 < w^2 + v^2 < |w_+ - w_0| \).

Let \( S \cong \mathbb{R}^2 \) denote the space obtained by adding to \( \mathcal{M}_s/\mathbb{Z} \) in the chart \((u, v)\) the point \( u = 0 = v \). Given in \( \mathcal{M}_s/\mathbb{Z} \) a sequence of points that converges to the point \( u = 0 = v \) in \( S \), we can choose in \( \mathcal{M}_s \) a sequence of pre-images that converges to \( q \) in \( \mathcal{M}_s \cup \{q\} \). This shows that \( S \cong (\mathcal{M}_s \cup \{q\})/\mathbb{Z} \), where the homeomorphism holds in the sense of topological manifolds.

Two side remarks are in order. First, although we here only need \( S \) to be a topological manifold, we note that the symplectic form on \( \mathcal{M}_s/\mathbb{Z} \) continues into a symplectic form on \( S \) in the differentiable structure determined by the chart \((u, v)\). We have \( \Omega_{\text{red}} = 2 \text{sign}(w_+ - w_0) dv \wedge du \). Second, although the above construction made a specific choice for the point \( q \), a similar construction can be given if \( q \) is replaced by any point in \( \mathcal{M} \) at \( y = y_0 \).
Suppose now that $\mathcal{M}/\mathbb{Z}$ is a manifold. If so, it has to be two dimensional. Let $\tilde{q} \in \mathcal{M}/\mathbb{Z}$ be the projection of the point $q$ and let $U \simeq \mathbb{R}^2$ be a neighbourhood of $\tilde{q}$ in $\mathcal{M}/\mathbb{Z}$. As $\tilde{q} \in \mathcal{S}$, $U \cap \mathcal{S}$ is nonempty and open in $\mathcal{S}$. As $\mathcal{S}$ is a two-manifold, there exists a set $V \subset U \cap \mathcal{S}$ such that $\tilde{q} \in V$ and $V \simeq \mathbb{R}^2$. Since $V \subset U$, $V$ is open as a subset of $U$, and since $U$ is open in $\mathcal{M}/\mathbb{Z}$, $V$ is open in $\mathcal{M}/\mathbb{Z}$. But this is a contradiction since every neighbourhood of $\tilde{q}$ in $\mathcal{M}/\mathbb{Z}$ contains points that are not in $V$.

A.2. $p_s = 0$

Suppose that $\gamma_b \in S^1$ such that $R(\gamma_b) = 0$.

By our assumptions about $R$, the inverse function theorem [26] implies that there exists an open interval $J$, symmetric about 0, in which the equation $p_s^2 = R(y)$ can be solved for $y$ as $y = F(p_s)$, where $F : J \rightarrow \mathbb{R}$ is a smooth even function, $F(0) = \gamma_b$ and the only stationary point of $F$ is 0. The corresponding subset of $\Gamma$ is $Q := \{(x, F(p_s), p_s, p_s) \mid x \in S^1, p_s \in J, p_s \in \mathbb{R}\}$.

The orbits that $C$ generates in $Q$ have constant $p_s$, and they satisfy $p_s \neq 0$ and $\dot{x}/p_s = F'(p_s)$. Adopting the gauge $p_s = 0$, we find as above that the projection of $Q$ to $\Gamma_{\text{red}}$ can be represented as the set $\{(x, F(p_s), p_s, 0) \mid x \in S^1, p_s \in J\} \simeq S^1 \times \mathbb{R}$ and the symplectic form reads $\Omega_{\text{red}} = dp_s \wedge dx$.

Appendix B. The rigging maps are intertwiners

In this appendix, we show that the rigging maps defined in the main text have the intertwining property (3.7). We work under the assumptions of subsection 3.3, recovering in the special case $\mathcal{P} = \{1\}$ the situation of subsection 3.2.

Let $A \in \mathcal{A}_{\text{obs}}$. Let $n$ and $m$ be fixed integers and let $f, g \in \Phi$ such that $f(x, y) = e^{i n x} f_m(y)$ and $g(x, y) = e^{i m y} g_n(y)$. As $U(t)$ is unitary and commutes with $A^1$, we have $(U(-t)f, A^1 g)_\text{aux} = (f, U(t)A^1 g)_\text{aux} = (f, A^1 U(t) g)_\text{aux} = (Af, U(t) g)_\text{aux}$. Using (3.4) in the leftmost and rightmost expressions and performing the integration over $x$ in the inner products gives

$$\int \mathrm{d} y \, e^{i[R(y) - n^2]} f_m(y)(A^1 g)_m(y) = \int \mathrm{d} y \, e^{i[R(y) - n^2]} (Af)_n(y) g_n(y),$$

(B.1)

where for any $h \in \Phi$ and $B \in \mathcal{A}_{\text{obs}}$, we have introduced the notation $(B h)(x, y) =: \sum_k e^{i k x} (Bh)_k(y)$.

On each side of (B.1), we break the integral over $y \in S^1$ into a sum over integrals over open intervals $[I_a]$ whose end points are adjacent stationary points of $R$. Let $R_a$ be the restriction of $R$ to $I_a$ and let $R_a^{-1}$ be the inverse of $R_a$. Changing the integration variable in each of the integrals on the left-hand side to $s := R_a(y) - m^2$ and on the right-hand side to $s := R_a(y) - n^2$, we obtain

$$\int \mathrm{d} s \, e^{i\epsilon s} \sum_a \left[ \frac{f_m(A^1 g)_m}{|R'|} \right] (R_a^{-1}(s + m^2)) = \int \mathrm{d} s \, e^{i\epsilon s} \sum_a \left[ \frac{(Af)_n g_n}{|R'|} \right] (R_a^{-1}(s + n^2)),

(B.2)

where for given $s$ the sum on the left-hand side (right-hand side) is over the values of $\alpha$ for which $s + m^2$ (respectively, $s + n^2$) is in the image of $R_a$. The integrand on the left-hand side (right-hand side) is not defined at the stationary values of $R - m^2$ (respectively, $R - n^2$), which

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1 We are grateful to Nico Giulini for pointing out that in a similar argument given in [25], p 318, the space denoted therein by $\mathcal{M}$, should be replaced by the space in which the tilded holonomy parameters take arbitrary values, and this space is not the closure of $\mathcal{M}$.
are finitely many, but it is continuous in $s$ elsewhere and defines an $L^1$ function of the variable $s \in \mathbb{R}$.

Now, regarded as a function of $t \in \mathbb{R}$, each side of (B.2) is the Fourier transform of an $L^1$ function. (B.2) therefore implies the $L^1$ equality

$$\sum_a \left[ \int \frac{g_n(y_{p[n]} g_n(y_{p[n]} s)} |R_a^{-1}(s + m^2)|}{|R_a|} \right] (R_a(s + m^2)) = \sum_a \left[ \int \frac{(A f_n g_n(y_{p[n]} s)} |R_a|} {\left| R_a^{-1}(s + n^2) \right|} \right] (R_a(s + n^2)), \quad (B.3)$$

and the continuity observations above imply that the equality in (B.3) holds pointwise in $s$ except at the stationary values of $R - m^2$ and $R - n^2$.

Consider the right-hand side of (B.3) as a function of $s$. In a sufficiently small punctured neighbourhood of $s = 0$, this function is a sum of contributions in which a local inverse of $R$ takes $s + n^2$ close to some $y_{p[n]}$ and contributions in which two local inverses of $R$ take $s + n^2$ close to some $y_{p[n]}$. An elementary analysis shows that the contribution from near $y_{p[n]}$ has the asymptotic small $s$ expansion

$$\frac{(p)!^{1/p}}{p s} \left( \frac{(A f_n g_n(y_{p[n]} s)} |R(p)(y_{p[n]} s)|^{1/p} \right)^{n} + \sum_{k=2}^\infty \alpha_k(s^{1/p})^k, \quad (B.4)$$

where the fractional power $s^{1/p}$ stands for the branch that has the same sign as $s$. Similarly, the contribution from near $y_{p[n]}$ has the asymptotic small $s$ expansion

$$2\theta(s) \frac{(p)!^{1/p}}{p |s|} \left( \frac{(A f_n g_n(y_{p[n]} s)} |R(p)(y_{p[n]} s)|^{1/p} \right)^{n} + \sum_{k=3}^\infty \beta_k(s^{1/p})^k, \quad (B.5)$$

where $\theta$ is the Heaviside function. We have suppressed in (B.4) and (B.5) the various indices on the coefficients $\alpha_k$ and $\beta_k$. The factor $2$ in (B.5) arises because there are two contributing local inverses, and the sum in (B.5) lacks a $k = 2$ term because the $k = 2$ terms from the two contributing local inverses cancel.

Similar considerations apply to the left-hand side of (B.3).

By the property of the index set $\mathcal{P}$ stated in subsection 3.3, the leading contribution from each $p$ in the small $s$ expansion of (B.3) has a power distinct from that of sub-leading contributions from any higher $p$. Equating the coefficients order by order thus shows that for each odd $p$

$$\sum_{j} \frac{\int n_{\epsilon}(y_{p[n]})(A^{j}g_m(y_{p[n]} s))}{|R(p)(y_{p[n]} s)|^{1/p}} = \sum_{j} \frac{(A f_n g_n(y_{p[n]} s)} |R(p)(y_{p[n]} s)|^{1/p}}, \quad (B.6)$$

while for each even $p$ and each $\epsilon \in \{1, -1\}$

$$\sum_{j} \frac{\int n_{\epsilon}(y_{p[n]})(A^{j}g_m(y_{p[n]} s))}{|R(p)(y_{p[n]} s)|^{1/p}} = \sum_{j} \frac{(A f_n g_n(y_{p[n]} s)} |R(p)(y_{p[n]} s)|^{1/p}}. \quad (B.7)$$

In terms of the maps $\eta_p$ and $\eta_{pe}$ (3.17), (B.6) and (B.7) read

$$\eta_p(f[A^{j}g] = \eta_p(A f)[g], \quad (B.8a)$$

$$\eta_{pe}(f[A^{j}g] = \eta_{pe}(A f)[g]. \quad (B.8b)$$

By linearity, these arguments leading to (B.8) continue to hold when $f$ and $g$ are replaced by arbitrary vectors in $\Phi$. Hence, each $\eta_p$ and $\eta_{pe}$ have the intertwining property (3.7).

Finally, we note that the assumption about the index set $\mathcal{P}$ can be replaced by weaker assumptions that also involve $\epsilon$. For example, if only one sign of $\epsilon$ is known to occur, it suffices to assume that the even and odd subsets of $\mathcal{P}$ individually have the property stated in subsection 3.3.
Appendix C. Representation of $\mathcal{A}_{\text{obs}}$

In this appendix, we show that the representation of $\mathcal{A}_{\text{obs}}$ on each of the Hilbert spaces $\mathcal{H}_{\text{RAQ}}^p$ and $\mathcal{H}_{\text{RAQ}}^p$ of subsection 3.3 is irreducible. The irreducibility on the Hilbert space of subsection 3.2 follows as the special case $\mathcal{P} = \{1\}$.

We discuss the cases of odd and even $p$ separately.

C.1. $\mathcal{H}_{\text{RAQ}}^p$

Fix an odd $p \in \mathcal{P}$. To unclutter the notation, we will suppress $p$ in most of the formulae.

We first construct a set of tailored observables.

For each $y_{qj}$ and $y_{rk}$ (where the index $p$ is suppressed), we define a function $h_{qj;rk}$ from a neighbourhood of $y_{qj}$ to a neighbourhood of $y_{rk}$ by the formula

$$h_{qj;rk}(y) := R_{rk}^{-1}(R(y) - q^2 + r^2), \quad (C.1)$$

where $R_{rk}^{-1}$ is the inverse of the restriction of $R$ to a neighbourhood of $y_{rk}$. Raising both sides of the equation $R(y) - q^2 = R(h) - r^2$ to power $1/p$ and applying the implicit function theorem [26] shows that $h_{qj;rk}$ is well defined and smooth, and we can choose the domains to be pairwise disjoint and such that $(h_{qj;rk})^{-1} = h_{rk,qj}$.

For each $y_{qj}$, we choose a smooth function $\rho_{qj}$ on $S^1$, such that $\rho_{qj}(y_{qj}) = 1$ and the support of $\rho_{qj}$ is contained in the domain of $h_{qj;rk}$ for all $y_{rk}$.

We now define on $\Phi$ the operators $A_{mj;nk}$ (where the index $p$ is suppressed) such that if $f \in \Phi$, $f(x,y) = \sum e^{i\alpha_k} f_i(y)$, then

$$(A_{mj;nk} f)(x,y) = e^{imx} \rho_{|m|j}(y) f_m(h_{|m|j;|n|k}(y)). \quad (C.2)$$

In words, $A_{mj;nk}$ first annihilates from $f$ all components except the one whose $x$-dependence is $e^{imx}$, then modifies in this component the function of $y$ to the zero function everywhere except near $y = y_{mj}$ and finally maps to a vector whose $x$-dependence is $e^{imx}$, with $y$-dependence nonzero only near $y = y_{mj}$.

A direct computation shows that each $A_{mj;nk}$ commutes with $U(t)$. The adjoint of $A_{mj;nk}$ acts on $\Phi$ as

$$((A_{mj;nk})^\dagger f)(x,y) = e^{imx} \rho_{|m|j}((h_{|m|k;|m|j}(y))) f_m(h_{|m|k;|m|j}(y)), \quad (C.3)$$

and comparison of (C.2) and (C.3) shows that also the adjoint commutes with $U(t)$. Each $A_{mj;nk}$ is therefore in $\mathcal{A}_{\text{obs}}$.

With this preparation, we can prove the following proposition.

**Proposition 1.** Let $V \subset \mathcal{H}_{\text{RAQ}}^p$ be a linear subspace invariant under $\mathcal{A}_{\text{obs}}$, $V \neq \{0\}$. Then, $V = \mathcal{H}_{\text{RAQ}}^p$.

**Proof.** Let $v \in V$, $v \neq 0$. Let $u \in \Phi$ such that $v = \eta(u)$. We write $u(x,y) = \sum e^{i\alpha_k} u_i(y)$. From (3.17a) it follows that there exist $n$ and $k$ such that $u_n(y_{|n|k}) \neq 0$.

For each $m$ and $j$ such that $y_{|m|j}$ exists, we now define $w_{mj}^{mj} := A_{mj;nk} u$. It follows that $\eta(w_{mj}) \in V$, and from the construction of $A_{mj;nk}$ we see that for every $f \in \Phi$,

$$\eta(w_{mj}[f] = (2\pi)^2 \frac{u_n(y_{|n|k})}{|R^p(y_{|n|k})|^{1/p}} f_m(y_{|m|j}). \quad (C.4)$$

Comparison of (C.4) and (3.17a) shows that the set $\{\eta(w_{mj})\}$ spans $\mathcal{H}_{\text{RAQ}}^p$. \qed
C.2. $\mathcal{H}_{RAQ}^{p_0}$

Fix an even $p \in \mathbb{P}$ and fix $\epsilon$ so that solutions $y_{p(n)(j)}$ exist.

No restriction of $R$ to a neighbourhood of $y_{p(n)(j)}$ now has an inverse. However, we can give a meaning to (C.1) by raising both sides of the equation $\epsilon[R(y) - q^2] = \epsilon[R(h) - r^2]$ to power $1/p$, with the branches chosen so that (say) both sides are increasing, and applying the implicit function theorem. After this, the arguments go through as for odd $p$.

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