Vaidya-like exterior solution and formation of singularity in \( f(R, T) \) theory of gravity

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In this paper, we study the gravitational collapse of a fluid ball undergoing dissipation in the form of heat flux, in the framework of \( f(R, T) \) gravity. To apply the junction conditions in order to match the interior and the exterior spacetimes and analyze the dynamics of collapse, it is necessary to know the exterior metric for the radiating fireball, which is not yet available in the context of \( f(R, T) \) gravity. Therefore, at the outset, we have determined the exterior Vaidya-like radiating metric in \( f(R, T) \) gravity. Subsequently, we solved the \( f(R, T) \) field equations and used the \( f(R, T) \) junction conditions to match the exterior with the interior spacetime. The time of formation of the singularity and the time of formation of the apparent horizon have been determined. It is concluded that the final singularity is hidden behind the horizon.

I. INTRODUCTION

We know that the study of gravitational collapse leads to several important astrophysical results. When a massive star collapses under its own gravity, the end result has always been an area of interest for many years. The final phase of a collapse leads to the formation of spacetime singularities where normal laws of physics are no longer valid. Various aspects of collapse have been studied by researchers [1–13] over the years, which include dynamical instability, causal transport phenomenon, as well as the study of the progress of collapse in the case of different types of matter, which may or may not involve shear, heat flux, free-streaming radiation, and/or anisotropy of pressure. Different combinations of these factors, lead to the difference in the nature of the end state of collapse.

There have also been attempts to search for solutions to the Einstein Field Equations for various kinds of matter [10, 14], where, Vaidya’s metric [15] has been considered to describe the exterior spacetime outside the collapsing matter. Guha and Banerji found solutions for a cylindrical collapse of anisotropic charged fluid [16]. Sharif and Iftikhar [17] found numerical solutions to the Einstein-Maxwell field equations for a charged radiating matter fluid consisting of shear viscosity.

In recent times, observational data from the Type Ia supernova [18, 19] has shown that the universe is currently undergoing an accelerated expansion. This acceleration is explained by assuming the presence of an unknown component in its matter-energy sector, called the dark energy [20]. In addition to this, the observation of the galactic rotation curves also indicate the existence of an entity called “dark matter”, which seems to dominate the matter content of the universe. It seems that at extremely large scales, Einstein’s General Relativity (GR) theory is not a suitable option to explain the accelerating expansion of the universe. As an alternative to GR, new theories of modified gravity were developed [21–28] in order to explain astrophysical phenomena at large scales. One such theory was the \( f(R) \) theory in which the Einstein-Hilbert action was generalised by replacing the Ricci scalar \( R \) appearing in it, by a function of the Ricci scalar, \( f(R) \). This modification stemmed from the idea of including the higher order terms of curvature in the action integral, which give rise to the dark energy components. Moreover, the presence of these higher order curvature terms mimics the role of the cosmological constant \( \Lambda \) in GR at the current epoch of evolution of the universe (thereby representing the \( \Lambda \)CDM model), and integrates all the phases of evolution of the universe in a single model [29–31]. Barraco and Hamity [32] found spherically symmetric solutions in a first order approximation of \( f(R) \) theory of gravity, and showed that both the exterior and interior metrics satisfy the junction conditions. They also showed that at least one exterior solution is the Schwarzschild metric. Capozziello, Stable and Troisi [33] used the existence of Noether symmetry to find spherically symmetric case in \( f(R) \) theory. They also found classes of exact solutions for spherically symmetric case in \( f(R) \) theory, both for constant Ricci scalar \( R_0 \) as well as \( R(r) \) where \( r \) is the radial coordinate [34]. More spherically symmetric solutions in \( f(R) \) gravity were found using this Noether symmetry approach by Capozziello, Frusciante and Vernieri [35]. Multamäki and Vilja [36] examined static empty space solutions with spherical symmetry in \( f(R) \) theory of gravity. Chakrabarti and Banerjee [37] found apparent horizon formation time and singularity formation time for a perfect fluid collapse in \( f(R) \) gravity. Sharif and Kausar [38] found apparent horizons for spherically symmetric perfect fluid collapse in \( f(R) \) theory.

The \( f(R, T) \) theory, first proposed by Harko [39] is a further modification of GR, which can provide a suitable explanation for this accelerated expansion, without the need to invoke a cosmological constant, unlike in GR. This theory is a generalisation of the \( f(R) \) theory, where the function \( f(R) \) depended only on the geometry of the spacetime and not on the matter content. In the \( f(R, T) \) theory, the Ricci scalar \( R \) in the Einstein-Hilbert action is replaced by a function of \( R \) and \( T \), the latter being the trace of the energy momentum tensor appearing in the Einstein field equations. The \( f(R, T) \) function arises due to quantum effects or due to the existence of exotic imperfect matter fluids. In his paper [39], Harko also discussed a few cases for possible choices of the \( f(R, T) \) function. A suitable
choice is the linear form $R + 2\lambda T$ which gives rise to power-law type of scale factors. Sahoo et al. [40] showed that $f(R) + \lambda T$ gravity models act as alternatives to cosmic acceleration. Moraes and Sahoo constructed wormhole models [41], and also proposed a cosmological scenario from the simplest non-minimal matter-geometry coupling in the $f(R, T)$ theory of gravity [42]. Moraes, Correa and Lobato [43] found solutions for a static wormhole metric in the $f(R, T)$ framework with linearised form of the $f(R, T)$ function. Zaregonbadi, Farhoudi and Riazi [44] found solutions for a static spherically symmetric spacetime in $f(R, T)$ gravity, and extracted the expressions for the metric components in the case of the galactic halo, using minimal coupling. Sharif and Nawazish [45] found exact solutions to Bianchi I universe model using Noether symmetry, and used Noether Gauge symmetry to find conserved quantities for flat FRW universe and Bianchi I universe models. Amir and Sattar [46] investigated spherically symmetric collapse of a perfect fluid in $f(R, T)$ gravity, and determined the conditions for the formation of the apparent horizon. Abbas and Ahmed [47] studied charged perfect fluid collapse and apparent horizon formation in $f(R, T)$ theory. Yousaf et al. [48, 49] studied the influence of structure scalars obtained by orthogonal splitting of the Riemann tensor on various physical properties of the collapsing matter, such as energy density inhomogeneity, pressure anisotropy and shear viscosity in $f(R, T)$ theory, and provided an insight on how the evolution of the collapsing matter proceeds under the effect of tidal forces and these inhomogeneities. Yousaf [50] considered modelling of a gravastar in cylindrical symmetry in $f(R, T)$ theory, which could be a suitable alternative to a black hole as the final state of the collapse, and studied the effect of electromagnetic field on the mass-energy content of the middle thin shell of the gravastar, and also drew comparisons between its density and pressure, and its proper length and thickness. In a previous work [51], we studied the dynamical stability and heat transport for a collapsing dissipative fluid in $f(R, T)$ gravity.

A solution to the gravitational field equations outside a radiating star in the setting of $f(R, T)$ gravity is not yet available. Such a solution is necessary to apply the junction conditions for the smooth matching of the interior and the exterior spacetimes. In this paper, we aim to find an exterior solution to the collapsing matter, which is then used to apply the $f(R, T)$ junction conditions [52] in order to solve the $f(R, T)$ field equations corresponding to the interior spacetime. We assume that the matter distribution is spherically symmetric, and involves pressure isotropy and heat flux. It is known that the field equations get simplified in the spherically symmetric case, and further, that there is no generation of gravitational waves in such situations. The spherical symmetry can also be used to model realistic gravitational collapse cases with small deviations. The region outside the collapsing matter is considered to be filled with radiation energy density, similar to what Vaidya had considered in [53]. The separability of metric coefficients into spatial and temporal parts is considered, similar to the works of Sharif and Abbas [54], and Guha and Banerji [16]. The nature of the resulting singularity after calculating the time of formation of the singularity, and that of the apparent horizon, is studied in consideration with the work of Joshi, Goswami and Dadhich [55]. Sample parameter values are also considered and the results analysed.

In the works of Noureen and Zubair [56–58], the exterior metric considered was the Vaidya metric, while the analysis was done in $f(R, T)$ gravity. Keeping in mind the fact that the Vaidya exterior line-element is actually a solution of the exterior field equations for a radiating sphere in GR, in this work of ours, we have proceeded in a similar manner to find a likewise solution for the exterior spacetime in the $f(R, T)$ theory, and then utilise this metric for the $f(R, T)$ junction conditions, instead of the Darmois-Israel junction conditions, and see the nature of solutions that we arrive at. The time of formation of apparent horizon for different types of collapse have been studied previously in GR [59, 60]. Here, a study of the same is carried out in $f(R, T)$ theory.

The structure of this paper is as follows: Section II deals with the formalism of the $f(R, T)$ theory. In section III, an exterior Vaidya-like solution is derived for the collapsing matter cloud. Section IV consists of the description of the interior spacetime and the junction conditions using the exterior solution derived in section III. In section V, we determine the field equations for the interior spacetime. In section VI, the solution of the field equations are derived along with the time of formation of singularity. Section VII deals with the formation of apparent horizon of the collapsing system, and two sample cases with suitably chosen parameter values, which is followed by a summarization and discussion in section VIII.

II. FIELD EQUATIONS OF $f(R, T)$ GRAVITY

The modified Einstein-Hilbert action in $f(R, T)$ theory is given by

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R, T)}{16\pi G} + \mathcal{L}_m \right),$$

where $R$ is the Ricciscalar, and $T$ is the trace of the energy-momentum tensor of the distribution. Starting from this action, and choosing the matter Lagrangian $\mathcal{L}_m$ in the above equation to be given by $-\rho$ [51], where $\rho$ is the energy density, we have the field equations for $f(R, T)$ theory of gravitation [39] as:
\[ R_{\mu\nu} f_R - \frac{1}{2} g_{\mu\nu} f + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R = (1 + f_T) T^{(m)}_{\mu\nu} + \rho g_{\mu\nu} f_T, \]  

which when rearranged, gives

\[ R_{\mu\nu} = \frac{1}{f_R} \left[ (1 + f_T) T^{(m)}_{\mu\nu} + \rho g_{\mu\nu} f_T + \frac{1}{2} g_{\mu\nu} f + D_{\mu\nu} \right]. \]

Here, \( R_{\mu\nu} \) is the Ricci tensor, \( f_R \) and \( f_T \) are the derivatives of the \( f(R, T) \) function with respect to \( R \) and \( T \) respectively, \( g_{\mu\nu} \) is the metric of the corresponding four-dimensional spacetime, \( T^{(m)}_{\mu\nu} \) is the matter energy-momentum tensor, and \( D_{\mu\nu} \equiv (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R \) includes the higher order curvature terms, which acts as the source of dark energy.

Let us choose a simple form of the function given by \( f(R, T) = R + 2\lambda T \), so that \( f_R = 1 \), and \( f_T = 2\lambda \), which implies that \( D_{\mu\nu} = 0 \). Putting these back into the field equations we get,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda T g_{\mu\nu} = (1 + 2\lambda) T^{(m)}_{\mu\nu} + 2\lambda \rho g_{\mu\nu}. \]  

The trace of Eq. (4) is obtained as

\[ -R = (1 + 6\lambda)T + 8\lambda\rho. \]

### III. A VAIDYA-LIKE EXTERIOR SOLUTION IN \( f(R, T) \) GRAVITY

For the spacetime in the exterior region of the collapsing matter, let us consider a general spherically symmetric metric which will represent a Vaidya-like line element of the form:

\[ ds^2 = -X(r, t)^2 dt^2 - 2W(r, t) dt dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( X \) and \( W \) are arbitrary functions of the coordinates \( r \) and \( t \).

The energy-momentum tensor in the region outside the boundary of the collapsing system consists of radiation energy density denoted by \( \rho \), similar to the consideration of Vaidya \[53\] in the case of GR, i.e.,

\[ T^{(m)}_{\mu\nu} = \rho u_\mu u_\nu, \]  

where \( u^\mu \) is the four-velocity in the comoving frame, so that

\[ T = -\rho. \]

In view of this relation, the trace of the field equation given in (5) reduces to the form

\[ R = (1 - 2\lambda) \rho. \]

The field equations for the exterior metric (6) are given by

\[ R_{00} = \frac{X}{W^2 r} \left( X^2 XW'r - X' W' X^2 r + X^2 W X'' r + 2X' X^2 W - XW' W'r + XW W' r - 2XW' W + 2XW^2 \right) = \frac{1}{f_R} \left[ X^2 \left( \rho - \frac{f}{2} \right) + D_{00} \right], \]  

\[ R_{11} = \frac{2W'}{W r} = \frac{1}{f_R} \left[ (1 + f_T) \frac{W^2}{X^2} + D_{11} \right], \]  

\[ R_{22} = -\frac{(2X X' W'r - X^2 W' r + X^2 W - W^3)}{W^3} = \frac{1}{f_R} \left[ \left( \rho f_T + \frac{1}{2} f \right) r^2 + D_{22} \right]. \]
So that the Ricci scalar acquires the following form:

\[ R_{01} = \frac{(X^2 W_r - X' W' X_r + X W X'' r + 2 X W' - W' r + W W' r)}{W^2 r} = \frac{1}{f_R} \left[ W \left( \rho - \frac{f}{2} \right) + D_{01} \right], \quad (13) \]

where,

\[ D_{00} = \frac{f' R X^4 W'}{W^3} + \frac{f' R X^3 X'}{W^2} + \frac{f'' R X^4}{W} + \frac{2 f' R X^2 W'}{W^2} + \frac{f'' R X^2 X}{W} - \frac{f' R}{W} - \frac{2 f' R}{r}, \quad (14) \]

\[ D_{01} = -\frac{f' R X^2 W'}{W^2} + \frac{f' R X' X}{W} + \frac{X^2 f'' R}{W} + \frac{f'' R X^2}{W r} - f' R - \frac{2 f' R}{r}, \quad (15) \]

\[ D_{11} = -\frac{f'' R W'}{W} + f'' R, \quad (16) \]

and

\[ D_{22} = -\frac{r}{W^3} \left( -f' R X^2 W' r + 2 f' R X' W' r + X^2 W f'' r + f'' R X^2 W - 2 W^2 f' R r - W' r \right). \quad (17) \]

The Ricci scalar for this spacetime is given by

\[ R = \frac{2}{r^2 W^3} \left( -W X^2 r^2 - W X X'' r^2 + X' X W' r^2 - 4 X X' W r + W' W' r^2 + 2 X^2 r W' - W' W r^2 + W^4 - X^2 W \right). \quad (18) \]

In order to derive the exterior solution in \( f(R, T) \) gravity let us assume that

\[ X(r, t) = a(r)k(t), \quad (19) \]

and \( W(r, t) = b(r)k(t), \quad (20) \)

so that the Ricci scalar acquires the following form:

\[ R = -\frac{2}{r^2 b^3} \left( -a a' b' r^2 + a b a'' r^2 + a^2 b r^2 - 2 a r b' + 4 a a' b r + a^2 b - b^3 \right). \quad (21) \]

The above expression (21) for the Ricci scalar is only a function of \( r \). Comparing (21) with (9) we conclude that \( \rho \equiv \rho(r) \) as well for this specific choice.

In view of (19) and (20) we can write,

\[ \frac{X}{W} = \frac{a}{b} = k_1(r), \quad (say). \quad (22) \]

The chosen \( f(R, T) \) function \( f(R, T) = R + 2 \lambda T \) gives rise to a power-law type scale factor as shown by Harko [39]. For this choice of the \( f(R, T) \) function, we get from equation (12) after using equations (19), (20) and (22), the following result

\[ k_2^2 + k_2' r - \frac{b'}{b} (1 - k_2^2) = \frac{R r^2}{2} + 2 \lambda \left( \rho + \frac{T}{2} \right) r^2, \quad (23) \]

where, \( k_2^2 = 1 - k_1^2 \). Since \( T = -\rho \) according to (8), hence we have

\[ \frac{\rho + \frac{T}{2}}{2} = \frac{\rho}{2}. \]

Consequently, from equation (23) we obtain

\[ 1 - k_1^2 = k_2^2 = 1 - \frac{a^2}{b^2} = \frac{\lambda}{br} \int b r^2 dr + \frac{1}{2br} \int b R r^2 dr + \frac{1}{br} \int b' r dr. \quad (24) \]
Utilising the forms of $X(r, t)$ and $W(r, t)$ and the above equation (24), we finally arrive at the following expression for the line element

$$ds^2 = \left[ \frac{\lambda}{br} \int bho v \, dv + \frac{1}{2br} \int bR r^2 \, dr + \frac{1}{br} \int b'r \, dr - 1 \right] b^2 k^2 dt^2 - 2bkdtdr + r^2 d\Omega^2. \quad (25)$$

Let us denote the time coordinate for the exterior spacetime by $v$ and the radial coordinate by $Y$. Thus the exterior metric assumes the form

$$ds^2_+ = \left[ -1 + \frac{\lambda}{bY} \int bY^2 \, dY + \frac{1}{2bY} \int bRY \, dY + \frac{1}{bY} \int b'Y \, dY \right] k^2 b^2 dv^2 - 2bkdvdY + Y^2 d\Omega^2. \quad (26)$$

Equation (26) represents the Vaidya-like exterior solution in $f(R, T)$ theory which we were attempting to find. This exterior solution can now be utilised in the junction conditions necessary to analyse the collapse formalism in $f(R, T)$ theory.

For the sake of convenience of notation, we can write

$$ds^2_+ = [-1 + g_1(Y)] k^2 b^2 dv^2 - 2bkdvdY + Y^2 d\Omega^2, \quad (27)$$

where $g_1(Y) = \frac{\lambda}{bY} \int bY^2 \, dY + \frac{1}{2bY} \int bRY \, dY + \frac{1}{bY} \int b'Y \, dY$.

We now proceed to extract the line element in a form in which the integrals appearing in (27) are evaluated to yield exact expressions.

### A. An exact form of the Vaidya-like Exterior Metric in $f(R, T)$ gravity

Let us assume that

$$a(Y) = 1 - \frac{Q}{Y}, \quad (28)$$
$$b(Y) = \frac{Q}{Y}, \quad (29)$$

with $Q$ as a constant. This is a reasonable assumption, since we see that as $Y \to \infty$, we have $a \to 1$, and $b \to 0$, and our exterior spacetime reduces to the flat spacetime metric, provided the temporal part also changes accordingly. At the end of the radiation zone (say, $Y = Y_1$), we have empty space again, and so $\rho(Y) = 0$. Hence, let us consider a functional form of the type $\rho(Y) = \rho_0(Y_1 - Y)$, $\rho_0$ being a constant, where we see that as $Y$ increases, $\rho$ decreases and finally vanishes at $Y = Y_1$.

We choose these functional forms since these may be compared to the coefficients of the Vaidya metric in the GR case. In case of the Vaidya metric in GR, the time dependence of the temporal coefficient is not separable from the radial dependence, unlike in the $f(R, T)$ case shown here, where $k(v)$ contributes the temporal part.

With the above assumptions, we obtain

$$g_1(Y) = \frac{3Y}{Q} \left( 1 - \frac{Y}{2Q} \right) - \ln Y + \frac{\lambda \rho_0 Y^2}{6} \left( 3Y_1 - 2Y \right). \quad (30)$$

Hence the Vaidya-like exterior solution in $f(R, T)$ gravity is now given by

$$ds^2_+ = \left[ -1 + \frac{3Y}{Q} \left( 1 - \frac{Y}{2Q} \right) - \ln Y + \frac{\lambda \rho_0 Y^2}{6} \left( 3Y_1 - 2Y \right) \right] k^2 b^2 dv^2 - 2bkdvdY + Y^2 d\Omega^2. \quad (31)$$

### IV. The Interior Spacetime and the Junction Conditions

The timelike 3D-hypersurface $\Sigma$ which separates the interior and the exterior spacetime, and serves as the boundary of the collapsing matter, is given by

$$ds^2_\Sigma = -d\tau^2 + R(\tau)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad (32)$$

while the interior spacetime is assumed to be of the most general spherically symmetric form, given by

$$ds^2 = -A(r, t)^2 dt^2 + B(r, t)^2 dr^2 + C(r, t)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (33)$$
The junction conditions for \( f(R, T) \) theory of gravity \[ \text{[52]} \] are given by:

\[
\begin{align*}
\left[g_{ij}\right]_+ ^- &= 0 \quad \text{(34)} \\
\left[K\right]_+ ^- &= 0 \quad \text{(35)} \\
\left[R\right]_+ ^- &= 0 \quad \text{(36)} \\
f, R \left[\tilde{K}_{ij}\right]_+ ^- &= 0 \quad \text{(37)} \\
\left[\partial_a T\right]_+ ^- &= 0 \quad \text{(38)} \\
\left[T\right]_+ ^- &= 0 \quad \text{(39)}
\end{align*}
\]

Here, \( R \) is the Ricci scalar, \( K_{ij} \) is the extrinsic curvature tensor, with \( K \) being its trace-part, and \( \tilde{K}_{ij} \) being its trace-free part. The extrinsic curvature describes how the normal to the boundary \( \Sigma \) varies along the hypersurface.

Unlike the case in GR, where only the Darmois-Israel junction conditions \[ \text{[61, 62]} \] are necessary, which requires the matching of the spacetime metric and extrinsic curvature components across the hypersurface, in \( f(R, T) \) theory, we need to separately match the trace part and traceless parts of the extrinsic curvature, the Ricci scalar, as well as the trace of the energy momentum tensor and its derivative across the boundary. The extrinsic curvature components for the interior metric given by \( \text{(33)} \) are as follows:

\[
K_{\tau\tau}^+ = -\left(\frac{A'}{AB}\right)_\Sigma, \quad \text{(40)}
\]

\[
K_{\theta\theta}^- = \left(\frac{CC'}{B}\right)_\Sigma, \quad \text{(41)}
\]

and

\[
K_{\phi\phi}^- = K_{\theta\theta}^- \sin^2 \theta. \quad \text{(42)}
\]

The extrinsic curvature components for the exterior metric given by \( \text{(27)} \) are as follows:

\[
K_{\tau\tau}^+ = \left[\left(\frac{dv}{d\tau}\right)^{-1} \frac{d^2v}{d\tau^2} + 2k \left(\frac{dv}{d\tau}\right) \frac{db}{dY} \left(\frac{dY}{d\tau}\right)^2 + \frac{1}{k} \frac{db}{dv} \left(\frac{dv}{d\tau}\right)^2 - \frac{1}{b} \frac{db}{dY} \frac{dY}{d\tau}\right]
- \frac{k^2b^2}{2} \left(\frac{dv}{d\tau}\right)^2 \frac{dg_1}{dY} \frac{dY}{d\tau} - k^3b^2 \left(\frac{dv}{d\tau}\right)^3 \frac{db}{dY} (1-g_1)^2 + \frac{kb}{2} \left(\frac{dv}{d\tau}\right) \frac{dY}{dY} \right]_\Sigma, \quad \text{(43)}
\]

\[
K_{\theta\theta}^+ = [Y \left(\frac{dY}{d\tau} + bk \left(\frac{dv}{d\tau}\right) (1-g_1)\right)]_\Sigma, \quad \text{(44)}
\]

\[
K_{\phi\phi}^+ = K_{\theta\theta}^+ \sin^2 \theta. \quad \text{(45)}
\]

The first condition is identical to the first Darmois-Israel junction condition \[ \text{[61, 62]} \], and is given by

\[
ds_+^2 = ds_-^2 = ds_\Sigma^2. \quad \text{(46)}
\]

The second and fourth junction conditions give us

\[
\left[ -K_{\tau\tau}^+ + \frac{2}{R} K_{\theta\theta}^+ \right]_\Sigma = \left[ -K_{\tau\tau}^- + \frac{2}{R} K_{\theta\theta}^- \right]_\Sigma. \quad \text{(47)}
\]

and

\[
\left[ K_{\tau\tau}^+ + \frac{1}{R^2} K_{\theta\theta}^+ \right]_\Sigma = \left[ K_{\tau\tau}^- + \frac{1}{R^2} K_{\theta\theta}^- \right]_\Sigma. \quad \text{(48)}
\]
which when combined, gives us the second Darmois-Israel junction condition, given by

\[(K^+_i)_\Sigma = (K^-_i)_\Sigma\]  \hspace{2cm} (49)

From the first condition, using equations (27), (32) and (33), we have

\[
\frac{dt}{d\tau} = A(t, r_\Sigma)^{-1}, \quad Y_\Sigma(\tau) = C(t, r_\Sigma) = R_\Sigma(\tau). \hspace{2cm} (50)
\]

Equating equations (40) and (41) with equations (43) and (44) respectively, the second condition gives us

\[
\begin{align*}
- \left( \frac{A'}{AB} \right)_\Sigma &= \left[ \frac{dv}{d\tau} \right]^{-1} \frac{d^2v}{d\tau^2} + 2k \left( \frac{dv}{d\tau} \right) \left( \frac{db}{dY} \right) \left( \frac{dY}{d\tau} \right)^2 + \frac{1}{k} \left( \frac{dv}{d\tau} \right) - \frac{1}{b} \frac{db}{dY} \frac{dY}{d\tau} \\
- \frac{k^2b^2}{2} \left( \frac{dv}{d\tau} \right)^2 \frac{dg_1}{dY} \frac{dY}{d\tau} - k^3b^2 \left( \frac{dv}{d\tau} \right)^3 \frac{db}{dY} \left( 1 - g_1 \right)^2 + \frac{kb}{2} \left( \frac{dv}{d\tau} \right) \frac{dg_1}{dY} \right]_\Sigma.
\end{align*}
\]

\[
\left( \frac{CC'}{B} \right)_\Sigma = \left[ Y \left( \frac{dY}{d\tau} + bk \left( \frac{dv}{d\tau} \right) \left( 1 - g_1 \right) \right) \right]_\Sigma. \hspace{2cm} (51)
\]

Using (46) and (50) in equation (52), we find that

\[
\left( \frac{1}{bk} \left( \frac{dv}{d\tau} \right)^{-1} \right)_\Sigma = \left( \frac{C'}{B} + \frac{C}{A} \right)_\Sigma. \hspace{2cm} (53)
\]

Consequently, at the boundary \( \Sigma \) we have

\[
\frac{Cg_1}{2} = C \left[ 1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right]. \hspace{2cm} (54)
\]

From the form of our exterior metric, we see that \( g_1 \) is a function of \( Y \). At the boundary \( \Sigma \), the second equation of (50) gives us that \( Y_\Sigma(\tau) = C(t, r_\Sigma) \). Hence \( g_1 \) can be considered to be a function of \( C \) at the boundary. We introduce a function \( e(r, t) \), which we call the “form function” since it is completely determined by the metric coefficients of the interior spacetime, such that

\[
\frac{C}{2} \left[ 1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right] = e(r, t). \hspace{2cm} (55)
\]

This assumption is justified since the interior metric coefficients are all functions of \( r \) and \( t \). Temporal and radial variations of this function can be expressed in terms of the matter-energy variables using the field equations. It is interesting to note the resemblance between the form function \( e(r, t) \) and the Misner-Sharp mass function \( M^\Sigma \) in general relativity.

Defining \( U = \frac{C}{A} \), which is the collapse velocity, and \( H = \frac{C'}{A} \), which is the radial variation of the physical radius of the collapsing matter, we have the temporal and radial variations of the function \( e(r, t) \) from the field equations,

\[
D_T e = \frac{1}{A} \frac{\partial m}{\partial t} = -\frac{C^2}{2} \left[ U \left\{ (1 + 2\lambda) p + \lambda (3p + \rho) \right\} + (1 + 2\lambda) H q \right] \hspace{2cm} (56)
\]

and,

\[
D_C e = \frac{1}{C'} \frac{\partial m}{\partial r} = \frac{C^2}{2} \left[ (1 + \lambda) \rho - 3\lambda p + \frac{U}{H} (1 + 2\lambda) q \right] \hspace{2cm} (57)
\]

Integration of these equations over suitable limits can give us the form of the function \( e(r, t) \), that describe the geometry of the collapsing configuration. For the equation representing the temporal variation, we see that the right-hand side has a negative sign, which signifies that the function \( e(r, t) \) decreases with time, as the collapse proceeds. However the radial variation cannot be interpreted so easily since all the terms on the right-hand side are not positive.
Using (55), we obtain the following relation representing the third junction condition

\[
\left[ \frac{1}{b} \left( 2 \frac{d^2b}{dY^2} (g_1 - 1) + 3 \frac{dg_1}{dY} \frac{db}{dY} \right) + \frac{d^2g_1}{dY^2} + \frac{4}{Y} \left( \frac{1}{b} (g_1 - 1) \frac{db}{dY} + \frac{dg_1}{dY} \right) \right]_{\Sigma} = \left[ -2 \left( \frac{2C''}{C} + \frac{A''}{A} + \frac{2A'C'}{CA} \right) \right]_{\Sigma} 
\]

The term \( g_1(Y) \) encodes the effect of the Vaidya-like exterior spacetime on the interior of the collapsing ball.

V. FIELD EQUATIONS FOR THE INTERIOR SPACETIME

For the matter distribution in the interior spacetime, we consider a fluid with isotropic pressure and dissipation in the form of heat flux. Hence, the energy momentum tensor for the collapsing cloud is given by

\[
T_{ab} = (\rho + p) u_a u_b + pg_{ab} + q(u_a u_b),
\]

where, the four-velocity is given by \( u^a = A^{-1} \delta^a_0 \). We define \( \chi^a = B^{-1} \delta^a_1 \), where \( \chi^a \) is a unit vector in the radial direction. Then the heat flux vector is defined as \( q^a = q \chi^a \) where \( q = q(r, t) \) is the heat flux, and the four-velocity satisfies \( u^a u_a = -1 \).

For the \( f(R, T) \) function given by \( f(R, T) = R + 2\lambda T \), with the matter Lagrangian as \(-\rho\), the field equations take the following form:

\[
G_{\mu\nu} = (8\pi + 2\lambda) T_{ab} + \lambda g_{\mu\nu} (T + 2\rho).
\]

The non-zero components of the field equations are

\[
G_{00} = A^2 \left[ \frac{2}{A^2} \frac{\dot{B} \dot{C}}{BC} + \frac{\dot{C}^2}{AC} - \frac{1}{C^2} - \frac{1}{B^2} \left( \frac{2C''}{C} + \frac{A''}{A} - \frac{2A'C'}{AC} \right) \right] = (8\pi + 2\lambda) A^2 \rho - \lambda A^2 (T + 2\rho),
\]

\[
G_{11} = B^2 \left[ \frac{1}{A^2} \left( -\frac{2\dot{C}}{C} - \frac{\dot{C}^2}{AC} + \frac{2\dot{A}\dot{C}}{AC} + \frac{\ddot{A}}{AC} - \frac{\dot{C}}{C} \right) + \frac{1}{B^2} \left( \frac{C''}{C} + \frac{2A'C'}{AC} - \frac{1}{C^2} \right) \right] = (8\pi + 2\lambda) B^2 \rho + \lambda B^2 (T + 2\rho),
\]

\[
G_{22} = C^2 \left[ -\frac{1}{A^2} \left( \frac{\dot{B}}{B} - \frac{\dot{A}B}{AB} + \frac{\dot{B}C}{BC} - \frac{\ddot{A}C}{AC} + \frac{\dot{C}}{C} \right) + \frac{1}{B^2} \left( \frac{A''}{A} - \frac{A'B'}{AB} + \frac{A'C'}{AC} - \frac{B'C'}{BC} + \frac{C''}{C} \right) \right] = (8\pi + 2\lambda) C^2 \rho + \lambda C^2 (T + 2\rho),
\]

and

\[
G_{01} = -2 \left( \frac{\dot{C}' C'}{AC} - \frac{A'C''}{BC} \right) = -AB(8\pi + 2\lambda)q.
\]

Using the field equations for \( G_{11} \) and \( G_{01} \), and the junction conditions for \( f(R, T) \), we get

\[
(q)_{\Sigma} = \left[ p + \frac{1}{(8\pi + 2\lambda)} \left[ \lambda (T + 2\rho) + \frac{g_1}{C^2} + \frac{2}{C} Z_1 \right] \right]_{\Sigma},
\]

where we have

\[
Z_1 = 2k \frac{db}{dY} \left( \frac{dY}{dr} \right)^2 + \frac{1}{k} \frac{dk}{dr} - \frac{1}{b} \frac{db}{dY} \frac{dY}{dr} \left( \frac{dv}{dr} \right)^{-1} - k^2 b^2 \frac{dv}{dr} \frac{dg_1}{dY} \frac{dY}{dr} - k^2 b^2 \left( \frac{dv}{dr} \right)^2 \frac{db}{dY} \left( 1 - g_1 \right)^2 + \frac{kb}{2} \frac{dg_1}{dY}.
\]

All the above results are obtained at the boundary of the collapsing matter. If the fluid stops dissipating, we have \( q = 0 \), but the pressure \( p \) does not vanish at the boundary in that case, unlike the similar case in GR investigated by Santos, Herrera and others [2, 3], where the pressure at the boundary becomes zero in the absence of dissipation in the form of heat flux.
VI. SOLUTION OF THE INTERIOR FIELD EQUATIONS

The interior field equations are obtained in the form

\[
\frac{A''}{A B^2} - \frac{1}{A^2} \left( \frac{\dot{B}}{B} + 2 \frac{\dot{C}'}{C} \right) + \frac{\dot{A}}{A^2} \left( \frac{\dot{B}}{B} + 2 \frac{\dot{C}'}{C} \right) - \frac{A'B'}{A B^2} + 2 \frac{A'C''}{A B^2 C} = 8\pi p - \frac{R}{2} - \lambda T, \tag{67}
\]

\[
- \frac{A''}{A B^2} - \frac{2 \pi''}{B^2 C} + \frac{\dot{B}}{A B^2} + \frac{\dot{A}'}{A^3 B} + \frac{2 \dot{B} C'}{A^2 B C} + \frac{2 \dot{A}' C'}{A B^2} = 8\pi p + 2\lambda (p + \rho) + \frac{R}{2} + \lambda T, \tag{68}
\]

\[
\frac{1}{C^2} \left( 1 + \frac{\dot{C}^2}{A^2} - \frac{C''^2}{B^2} \right) + \frac{C}{A^2} \left( \ddot{C} - \frac{\dot{C} B}{C^2 B} - \frac{\dot{C} A}{C^2 A} \right) - \frac{C''}{B^2 C} - \frac{B'C'}{B^2 C} - \frac{A'C''}{A B^2 C} = 8\pi p + 2\lambda (p + \rho) + \frac{R}{2} + \lambda T, \tag{69}
\]

\[
\frac{\dot{C}'}{C} - \frac{A' \dot{C}}{A C} - \frac{B C'}{B C} = (4\pi + \lambda) q A B. \tag{70}
\]

On account of isotropy of pressure, equations (68) and (69) yield us

\[
- \frac{1}{A B^2} \left( A'' - \frac{A'B'}{B} - \frac{A'C''}{C} \right) + \frac{1}{A^2 B} \left( \ddot{B} - \frac{\dot{A} B}{A} + \frac{\dot{B} C}{C} \right) - \frac{1}{B^2 C} \left( C'' - \frac{B'C''}{B} \right) - \frac{1}{A^2 C} \left( \ddot{C} - \frac{\dot{A} C}{A} \right) - \frac{1}{C^2} \left( 1 + \frac{\dot{C}^2}{A^2} - \frac{C''^2}{B^2} \right) = 0. \tag{71}
\]

Let us consider that at the onset of collapse, the collapse profile is given by

\[
A(r, t) = A_0(r) s_1(t), \tag{72}
\]

\[
B(r, t) = B_0(r) s_2(t), \tag{73}
\]

\[
C(r, t) = r B_0(r) s_3(t). \tag{74}
\]

Here, \(A_0(r)\) and \(B_0(r)\) are static perfect fluid configurations just before the collapse has started. With the onset of collapse from \(t = 0\), the fluid starts behaving differently from a perfect fluid, in other words, it becomes a non-adiabatic fluid undergoing dissipation in the form of thermal radiation, and this gradual change in behaviour involving the dissipative effects is dependent on the time coordinate. This dependence is manifested in the temporal functions \(s_1(t)\), \(s_2(t)\) and \(s_3(t)\), which, when subjected to rescaling, can be expressed in the form of a single temporal function \(f_1(t)\) associated with the physical radius of the collapsing fluid. Metric coefficients separable in spatial and temporal parts, have been previously considered for charged cylindrical perfect fluid collapse in GR by Sharif and Abbas [54], dissipative collapse of charged cylindrical anisotropic fluid in GR by Guha and Banerjee [10].

By a rescaling of the coordinate time, as in [10], we can write

\[
A(r, t) = A_0(r), \tag{75}
\]

\[
B(r, t) = B_0(r), \tag{76}
\]

\[
C(r, t) = r B_0(r) f_1(t). \tag{77}
\]

This rescaling of time-coordinates ensures that the differential equation will consist of a single temporal function instead of three different functions of time, which would have been much more complicated for an analytical solution. At this initial stage, \(f_1(t) = 1\), and \(\dot{f}_1(t) \rightarrow 0\). These forms of the metric coefficients are the same as that considered by Chan [10, 14]. The field equations for the static perfect fluid configuration can be retrieved by using these forms of the metric coefficients and putting \(f_1(t) = 1\) in the equations (67) to (70). It may be mentioned here that considering the same time dependence for all three of the metric coefficients will result in a time independent equation of pressure isotropy, which cannot be used to find the form of the temporal function \(f_1\). Hence such an assumption is not useful in this case.

Using these forms of the metric coefficients in (71), we get

\[
\frac{1}{A_0} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_1^2}{f_1^2} \right) + \frac{1}{r^2 B_0^2 f_1^2} - D(r) = 0, \tag{78}
\]

where

\[
D(r) = -\frac{1}{A_0 B_0^2} \left( A_0'' - \frac{2 A_0' B_0'}{B_0} - \frac{A_0'}{r^2} \right) - \frac{B_0''}{B_0} - \frac{3 B_0'}{r B_0^2} - \frac{1}{r^2 B_0^2}.
\]
Now, as \( t \to -\infty \), we have \( f_1(t) = 1 \) and \( \dot{f}_1(t) \to 0 \). Utilising these conditions, and solving the above differential equation for \( f_1 \), we obtain

\[
f_1 = \pm \frac{1}{\sqrt{2Dk_3^3}} (C_1 Hck_3^2 - 2dk_3 - C_2 Hc)^{1/2},
\]

(79)

where \( C_1 \) and \( C_2 \) are integration constants, \( c(r) = \frac{1}{A_0^2} \), \( d(r) = \frac{1}{rB_0^2} \), \( H(r) = \sqrt{\frac{2D}{c}} \), and \( k_3 = \exp(ht) \). Since \( f_1 \) denotes the temporal dependence of physical radius which cannot be negative, we consider only the positive solution. For collapse to progress, we require \( \dot{f}_1 \leq 0 \).

From (79) we also have,

\[
\dot{f}_1 = \frac{1}{2f_1} \left( -\frac{C_1 H^2 c}{2Dk_3} + \frac{2Hd}{Dk_3^2} + \frac{3C_2 H^2 c}{2Dk_3^3} \right).
\]

(80)

Using the limiting case of the inequality for \( \dot{f}_1 \), we obtain

\[
t = \frac{1}{H} \ln \left( \frac{2d \pm \sqrt{4d^2 + 3C_1 C_2 (Hc)^2}}{C_1 Hc} \right).
\]

(81)

For real values of the discriminant, we get the condition

\[
-\frac{4d^2}{3(Hc)^2} \leq C_1 C_2,
\]

(82)

and additionally, the argument of the logarithm needs to be positive.

We also observe that since \( f_1 \) represents only the time dependence of the physical radius, it is independent of the radial coordinate. Hence its expression must have no dependence on \( r \). So \( k_3 \) must be independent of \( r \), implying that \( h \) must be a constant. So, \( \frac{2D}{c} = H^2 = \text{constant} \). Also, the terms \( \frac{2H}{c} \), and \( \frac{H}{c} \) are constants. We assign the following names to these constants:

\[
\frac{d}{D} = C_3, \quad H = C_4, \quad \frac{c}{D} = C_5, \quad \frac{d}{c} = \frac{C_3}{C_5} = C_6.
\]

(83-86)

Hence our expression for the time of singularity formation now becomes

\[
t_s = \frac{1}{C_4} \ln \left( \frac{C_6 \pm \sqrt{(C_6^2 + C_1 C_2 C_4^2)}}{C_1 C_4} \right),
\]

(87)

which leads to the condition

\[
-\frac{4C_6^2}{3C_4^2} \leq C_1 C_2,
\]

(88)

for real values of the logarithm. Also, the logarithm needs to have a positive argument. To get a feel of the physical conditions pertaining to the collapse, we now consider the formation of the apparent horizon.

VII. FORMATION OF APPARENT HORIZON

The apparent horizon is a boundary between outward directed light rays which bend inwards, and those which move outwards. Several authors have determined the time of formation of apparent horizon for various types of collapse in GR \([59, 60]\). Chakrabarti and Banerjee \([37]\) found apparent horizon formation time for a perfect fluid collapse in \( f(R) \) gravity. Sharif and Kausar \([38]\) found apparent horizons for spherically symmetric perfect fluid collapse in
$f(R)$ theory. Amir and Sattar [46] found apparent horizons for the collapse of perfect fluid with spherical symmetry in $f(R,T)$ gravity. Abbas and Ahmed [47] studied apparent horizon formation for charged perfect fluid collapse in $f(R,T)$ theory. For the formation of the apparent horizon, we have the requirement that any outward normal on its boundary must be null. This condition can be expressed as

$$g^{\mu\nu}C_{,\mu}C_{,\nu} = 0,$$  \hspace{1cm} (89)

which gives us the following condition:

$$\frac{\dot{C}^2}{A^2} = \frac{C^2}{B^2}.$$  \hspace{1cm} (90)

Using the expressions for $A$, $B$ and $C$, we arrive at the relation

$$\frac{\dot{f}_1^2}{f_1^2} = \frac{(B_0 + rB_0')^2 A_0^2}{r^2 B_0^4} = \delta^2.$$  \hspace{1cm} (91)

Since, the left hand side is a function of $t$ only, and the right hand side is a function of $r$ only, so $\delta^2$ must be a constant. We can see that the solution will be of the form $e^{\delta t}$. Replacing $\dot{f}_1^2$ by $\delta^2 f_1^2$ in the square of the equation (80), we obtain

$$f_1^4 = \frac{1}{4\delta^2} \left( -\frac{C_1 H^2 c}{2 Dk_3} + \frac{2 H d}{Dk_3^2} + \frac{3 C_2 H^2 c}{2 Dk_3^3} \right)^2.$$  \hspace{1cm} (92)

Taking the positive root, we get

$$f_1^2 = \frac{1}{2\delta} \left( -\frac{C_1 H^2 c}{2 Dk_3} + \frac{2 H d}{Dk_3^2} + \frac{3 C_2 H^2 c}{2 Dk_3^3} \right).$$  \hspace{1cm} (93)

Again, squaring the expression for $f_1$ in (79), we get

$$f_1^2 = \frac{C_1 H c}{2 Dk_3} - \frac{d}{Dk_3^2} - \frac{C_2 H c}{2 Dk_3^3}.$$  \hspace{1cm} (94)

Equating these two expressions and solving the resulting equation for $t$, we get the time of formation of the apparent horizon as follows

$$t_{ah} = \frac{1}{C_4} \ln \left( \frac{2 C_6 (\delta + C_4) \pm \sqrt{\left( 4 C_8^2 (\delta + C_4)^2 + C_1 C_2 C_4^2 (2\delta - C_4) (2\delta + 3 C_4) \right)}}{C_1 C_4 (2\delta - C_4)} \right).$$  \hspace{1cm} (95)

For real solutions, expressions within the square root have to be positive definite. Also, the argument of the logarithm needs to be positive. A similar procedure for obtaining the horizon formation time and the singularity formation time was adopted by Chakrabarti and Banerjee [37]. The difference between the time of singularity formation, and the time of formation of apparent horizon is given by

$$t_s - t_{ah} = \frac{1}{C_4} \ln \left( \frac{(2\delta - C_4) \left( C_6 \pm \sqrt{C_8^2 + C_1 C_2 C_4^2} \right)}{2 C_6 (\delta + C_4) \left( C_6 \pm \sqrt{C_8^2 + C_1 C_2 C_4^2} \right)} \right).$$  \hspace{1cm} (96)

From the expression for the difference between $t_s$ and $t_{ah}$, we see that the nature of the singularity depends on whether the argument of the logarithm is less than or greater than unity. If the argument is less than 1, then the overall expression is negative, assuming that $C_4$ is positive, which implies that the singularity forms before the apparent horizon does and hence is visible to the external observer, in other words, a naked singularity. On the other hand, if the argument is greater than 1, then the singularity forms at a later time than the apparent horizon, and hence, is hidden from the observer.

Joshi, Goswami and Dadhich [55] showed that in a case like above, where the singularities are not dependent on the radial coordinates, no naked singularity can result. The singularity is not central, as it occurs at every $r$ at the same time.
Using the previous expressions for \( d, c, \) and \( C_6 \) in (91) which follows from the condition for apparent horizon formation, we can write

\[
\frac{\delta^2}{C_6} = \frac{(B_0 + rB_0')^2}{B_0^2} = C_7^2. \tag{97}
\]

Integrating with respect to \( r \), we get \( B_0 = C_8r^n \). Putting it back in the expression for \( \delta^2 \), we obtain \( A_0 = \frac{C_8n^{n+1}\delta}{n+1} \).

It is to be noted that these power series solutions are valid only for the case where there is a formation of apparent horizon. We also see that \( D(r) \) vanishes with these expressions for \( A_0 \) and \( B_0 \). In order to interpret these results physically, we need to choose acceptable values of parameters necessary to compute the time of formation of the singularity, and compare with observational results so as to determine the credibility of our analysis.

**Sample Cases with Chosen Values of the Parameters**

**Case I**

As a sample case, considering the values of \( C_1, C_2, C_4 \) and \( C_6 \) to be unity each, which satisfy the inequality (88), from (87), the value of \( t_s \) comes out to be 0.881374s, which is a reasonably acceptable value according to [64], which gives the range of time taken for a massive star to collapse into a black hole with or without a supernova explosion to be about 0.1s - 0.5s.

In case of apparent horizon formation, we see that \( \delta \) has to be negative, since the physical radius of the matter decreases with time as the collapse occurs. Hence \( f_1(t) \) has to be a monotonically decreasing function of time \( t \).

Utilising the same parameter values for \( t_{ah} \), for the value of the logarithm to be defined, apart from the fact that \( \delta \) has to be negative, we see by examining the quantity under the square root in equation (95), that for the quantity to be positive definite, the restriction on the value of \( \delta \) take the following form : \( \delta \leq -1.411 \) and \( \delta \geq -0.08 \)

Choosing a value of \( \delta \) in this range, such as \( \delta = -3 \), and the positive of the square root so that the argument of the logarithm is positive, \( t_{ah} \) comes out to be 0.364917s. Hence \( t_s - t_{ah} > 0 \), which indicates that the apparent horizon forms before the singularity, resulting in the latter being hidden from the external observer.

**Case II**

Considering \( C_1, C_2, C_6 \) to be unity each, and \( C_4 = 2 \), and checking that these values satisfy (88), from (87), the value of \( t_s \) comes out to be 0.2406s, which lies between the range specified in [64]. Considering that \( \delta \) has to be negative for collapse to take place, and \( t_{ah} \) cannot be negative, we find that \( \delta \leq -6.274 \). Taking \( \delta = -7 \), we have from (95), \( t_{ah} = 0.041s \), which is less than \( t_s \). Once again the apparent horizon forms earlier than the singularity which is therefore hidden from the external observer.

**VIII. SUMMARY AND DISCUSSIONS**

To summarize, we have found a Vaidya-like ansatz for an exterior metric in the framework of \( f(R, T) \) gravity, which was not available earlier in literature, considering the spacetime exterior to the collapsing matter to be filled with radiation energy density. Accordingly, from the field equations in the \( f(R, T) \) gravity, we arrived at the expressions for the exterior metric coefficients, and thus determined the complete form of the exterior line element that we intended to use while applying the \( f(R, T) \) junction conditions, as we investigate the collapse of an isotropic matter fluid involving heat flux. Proceeding with the collapse formalism, the field equations are determined, the junction conditions are applied with the exterior solution derived by us, and the form function \( c(r, t) \) is identified, which resembles the Misner-Sharp mass of general relativity. Finally, to solve the field equations, the metric coefficients for the interior spacetime, \( A(r, t) \) and \( B(r, t) \) are assumed to be functions of the radial coordinate, and, in case of \( C(r, t) \) which represents the physical radius of the collapsing fluid, to be separable in space and time. Using these forms of the metric parameters, a differential equation in the temporal function \( f_1 \) is obtained by utilising the pressure isotropy condition. From that the time of the singularity formation \( t_s \) is obtained with the necessary restrictions imposed on the integration constants. It is seen that the resulting singularity is hidden behind an apparent horizon. The time of formation of apparent horizon, \( t_{ah} \) is also obtained, and the difference between \( t_s \) and \( t_{ah} \) is calculated. A power series solution in the radial coordinates is obtained for \( A_0(r) \) and \( B_0(r) \), which is valid strictly under the condition for the formation
of apparent horizon. Finally, the results are checked with some sample values of the parameters. It is seen that for our chosen values, the singularity formed at the end of the collapse is hidden behind an apparent horizon.

The following observations can be made in this context:

1. In the absence of heat flux (perfect fluid case), utilising equations (75), (76), (77) we would arrive at a relation between \( A_0(r) \) and \( B_0(r) \) from equation (64). Their ratio would be linear in the radial coordinate.

2. If the pressure was anisotropic, then the pressure in equation (65) would be the radial pressure, since the field equation for \( G_{11} \) was used to arrive at it.

3. In case of pressure anisotropy, we would have to find a different way to arrive at the solution for the temporal function \( f_1 \). In that case, equating the expression for \( q \) from equation (65) and equation (64) and using the equations (75), (76), (77) would have provided us with a differential equation for \( f_1 \) where the first term in the bracket in the RHS of equation (65) and the terms of equation (64) would have given additional terms involving \( f_1 \) and its time derivative. This would have resulted in a more complicated differential equation.

4. Presence of shear would have further added terms involving \( f_1 \) and its time derivative to the differential equation, since it would have appeared in the field equation for \( G_{11} \) and consequently in equation (65). Hence the differential equation would become more complex if effects like anisotropy and shear are involved in the collapse.

There is scope of further investigation of the effect of shear viscosity and pressure anisotropy in the collapsing star in \( f(R,T) \) gravity.

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