Interactions Between Solitons and Other Nonlinear Schrödinger Waves

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Abstract: The Nonlinear Schrödinger (NLS) equation is widely used in everywhere of natural science. Various nonlinear excitations of the NLS equation have been found by many methods. However, except for the soliton-soliton interactions, it is very difficult to find interaction solutions among different types of nonlinear excitations. In this Letter, two equivalent very simple methods, the truncated Painlevé analysis and the generalized tanh function expansion approaches, are developed to find interaction solutions between solitons and any other types of NLS waves. Especially, the soliton-cnoidal wave interaction solutions are explicitly studied by means of Jacobi elliptic functions and the third type of incomplete elliptic integrals. The static and moving optical soliton lattices appear naturally in all mediums described the NLS systems. Every peak of the cnoidal wave (or soliton lattice) elastically interacts with usual solitons except for phase shifts.

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The soliton and/or solitary wave equations connect rich histories of exactly solvable systems constructed in mathematical, statistical and many-body physics, and powerfully demonstrate the unity of nonlinear concepts across disciplines and scales from micro-physics and biology to cosmology [1]. Among these equations, the nonlinear Schrödinger (NLS) equation is the most ubiquitous [2]. Originally, the NLS equation is derived to describe the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium when dissipative processes are negligible (see for instance [3]). The NLS equation finds an important application in plasma physics, where it describes electron (Langmuir) waves [4]. The NLS equation in nonlinear optics is also well known to describe self-modulation and self-focusing of light in a Kerr-type nonlinear medium [5]. The great current interest in the NLS application was initiated by the prediction of the solitons in nonlinear optical fibers [6] and the concept of the

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soliton laser \[7\]. Furthermore, the NLS equation is widely used in ferromagnets with easy-axis anisotropy, molecular chains, nonideal Bose gas, nuclear matter, solid state medium, gravity waves, optical lattice, Bose-Einstein condensations, and so on \[8\].

The multiple soliton solutions of the NLS equation have been given by many authors via different methods, say, the Hirota’s bilinear method \[9\], the Darboux transformations (DT) and/or the Bäcklund transformations (BT) \[10\]. Using the DT and BT, in principle, one can obtain a new solution from a known one. However, in practice, one can only find multiple soliton solutions stemming from simple constant solutions. It is rather difficult to find new explicit solutions starting from nonconstant nonlinear waves such as the cnoidal waves and Painlevé waves via DT and BT. In Refs. \[11, 12\], the mutisoliton complexes on a cnoidal wave background have been studied by the DTs for the multi-component NLS equations, the sine-Gordon (SG) equation and the Toda lattice. Recently, it is found that combining the symmetry reduction method and the DT or BT related nonlocal symmetries \[13\], one can readily find the interaction solutions among solitons and other nonlinear excitations including the cnoidal waves for the KdV \[14\] and KP \[15\] equations. In this Letter, much more simple and effective methods are proposed for the NLS system while they are valid for any integrable systems.

As an illustration of our new methods, we consider the NLS equation as the special case of the Ablowitz-Kaup-Newell-Segur (AKNS) system

\[
p_t + \frac{1}{2} ib p_{xx} - ip^2 q = 0, \quad i \equiv \sqrt{-1},
\]

\[
q_t - \frac{1}{2} ib p_{xx} + iq^2 p = 0, \quad b \equiv \pm 1,
\]

with \(q\) being a conjugate of \(p\), i.e., \(q = p^*\).

To find significant interaction solutions of the AKNS system among solitons and other types of nonlinear excitations, we directly write down the following theorem:

**Theorem.** If \(w\) is a solution of

\[
\left( \frac{w_t}{w_x} \right)_t = \left( \frac{3w_t^2}{2w_x^2} + \frac{1}{2} w_x^2 - \frac{1}{4} \{w; x\} + 2\lambda b \frac{w_t}{w_x} \right)_x, \quad \{w; x\} \equiv \frac{w_{xxx}}{w_x} - \frac{3}{2} \frac{w_{xx}^2}{w_x^3},
\]

then

\[
p = \sqrt{b} \left[ w_x \tanh(w) - ib \frac{w_t}{w_x} - i\lambda - \frac{1}{2} \frac{w_{xx}}{w_x} \right] e^{iu},
\]

\[
q = \sqrt{b} \left[ w_x \tanh(w) + ib \frac{w_t}{w_x} + i\lambda - \frac{1}{2} \frac{w_{xx}}{w_x} \right] e^{-iu},
\]

is a solution of the AKNS system \(\text{[1]}\) with the consistent conditions for the ‘phase’ \(u\)

\[
u_x = 2b \frac{w_t}{w_x} + \lambda,
\]

\[
u_t = 3b \frac{w_t^2}{w_x^2} + 4\lambda \frac{w_t}{w_x} + bw_x^2 + 3\frac{b}{2} \lambda^2 - \frac{b}{2} \{w; x\}.
\]

The consistent condition \(u_{xt} = u_{tx}\) of \(\text{[4]}\) is nothing but \(\text{[2]}\).

To prove the theorem, we can use two very simple ways, the truncated Painlevé analysis and the tanh function expansion method. The Painlevé analysis is one of the best approaches to study and solve
special solutions for nonlinear physical systems. The function (especially the tanh function) expansion method is usually used to find traveling wave solutions of the nonlinear partial differential equations. Actually, the latter can be considered as the special case of the truncated Painlevé expansion approach. Using the standard Painlevé analysis, it is straightforward to find the following Lemma:

**Lemma.** If \( \phi \) is a solution of

\[
\left( \frac{\phi_t}{\phi_x} \right)_t = \left( \frac{3\phi_t^2}{2\phi_x^2} - \frac{1}{4} \{ \phi; x \} + 2\lambda b \frac{\phi_t}{\phi_x} \right)_x, \tag{5}
\]

then

\[
p = \sqrt{b} \left[ \frac{\phi_x}{\phi} - i b \frac{\phi_t}{\phi_x} - i \lambda - \frac{1}{2} \frac{\phi_{xx}}{\phi_x} \right] e^{iu}, \tag{6a}
\]

\[
q = \sqrt{b} \left[ \frac{\phi_x}{\phi} + i b \frac{\phi_t}{\phi_x} + i \lambda - \frac{1}{2} \frac{\phi_{xx}}{\phi_x} \right] e^{-iu}, \tag{6b}
\]

is a solution of the AKNS system \(^{(1)}\) with the consistent conditions for \( u \)

\[
u_x = 2b \frac{\phi_t}{\phi_x} + \lambda, \tag{7a}
\]

\[
u_t = 3b \frac{\phi_t^2}{\phi_x^2} + 4\lambda b \frac{\phi_t}{\phi_x} + \frac{3}{2} b \lambda^2 - \frac{b}{2} \{ \phi; x \}. \tag{7b}
\]

Because both the quantities \( \phi_t/\phi_x \) and \( \{ \phi; x \} \) are Möbius transformation

\[
\phi \rightarrow \frac{a\phi + b}{c\phi + d}, (ad - bc \neq 0) \tag{8}
\]

invariants, the Schwarzian AKNS (and NLS) equation \(^{(5)}\) is invariant under the Möbius transformation \(^{(8)}\).

Using the Lemma and the following straightening transformation,

\[
\phi = \frac{2}{\tanh(w) - 1}, \tag{9}
\]

the theorem is proved.

We call the transformation \(^{(9)}\) as the straightening transformation, because it transforms the single soliton solution of the Schwarzian AKNS \(^{(5)}\) and then the AKNS \(^{(1)}\) to a straight line solution \( w = k_0 x + \omega_0 t \). It is obvious that the solution expression \(^{(3)}\) is the generalization of the usual tanh function expansion method. Actually, the transformation \(^{(9)}\) converts the usual truncated Painlevé expansion approach to the most general extension of the special tanh function expansion method.

Now, the only left thing is to find the interaction solutions of the Eq. \(^{(2)}\) with respect to \( w \) or equivalently the Eq. \(^{(5)}\) regarding \( \phi \). The theorem reveals that the single soliton (or solitary wave) solution of the AKNS system \(^{(1)}\) is only a straightened solution \( w = kx + \omega t \) of Eq. \(^{(2)}\). This fact hints us that to find the interaction solutions between solitons and other nonlinear excitations, we need only to find solutions in the form of \( w = k_0 x + \omega_0 t + v \) where \( v \) is a function of \( x \) and \( t \). Here, we just write down the soliton-cnoidal wave interaction solution for \{ \( w, u \) \} equations \(^{(2)}\) and \(^{(4)}\)

\[
w = k_0 x + \omega_0 t + c E_n (\text{sn}(kx + \omega t, m), n + 1, m), \tag{10a}
\]
\[
    u = u_0 + \left(\lambda + 2b\frac{\omega_0}{k_0}\right) x + b \left(\frac{m^2c}{k_0(n+1)} + k_0^2 + \frac{3\lambda^2}{2} + 4\lambda \omega_0 + 3\frac{\omega_0^2}{k_0^2}\right) t + \frac{2bc(k\omega_0 - k_0\omega)}{k_0(k_0 + k)} E_x \left(\text{sn}(kx + \omega t, m), \frac{k_0(n+1)}{k_0 + k}, m\right),
\]

(10b)

where \(\{u_0, m, n, k_1, k_2\}\) are arbitrary constants, other two arbitrary constants related to the initial center positions of the soliton and the cnoidal wave have been removed, and

\[
    c^2 = n - \frac{nm^2}{1 + n}, \quad \delta^2 = 1,
\]

\[
    \omega = -bk\lambda - \frac{bk\delta \left[c^2k^3(c^2 - n) - 2ck_0k^2(n + nc^2 - n^2 - 2c^2) - 3kk_0^2(2nc^2 - c^2 - n^2) - 4nk_0^2\right]}{2\sqrt{nck_0(kc + k)(kc - nk_0)(ck_0 + kc^2 - kn)}},
\]

\[
    \omega_0 = -bk_0\lambda + \frac{bk_0k \left[c^2k^3(c^2 - n) + kk_0^2(2nc^2 - c^2 - n^2) + 2nk_0^3\right]}{2\sqrt{nck_0(kc + k)(kc - nk_0)(ck_0 + kc^2 - kn)}},
\]

(11)

In the solution (10), \(\text{sn}(z, m)\) is the usual Jacobi elliptic sine function and \(E_x(\zeta, n, m)\) is the third type of incomplete elliptic integral.

Correspondingly, for the physical quantity, the strength of the AKNS fields \(\{p, q\}\), \(I \equiv pq\) (i.e., \(I = |p|^2\) for NLS), we obtain

\[
    I = b\frac{[kc - k_0(S^2 - 1)]^2}{(1 - S^2)^2} \tanh^2(w) - \frac{2bc^2S\text{CD}}{(1 - S^2)^2} \tanh(w) + b\lambda^2 + \left(\frac{\omega_0}{k_0} + \frac{\omega}{k}\right) \lambda + \frac{bk^2c^2}{(1 - S^2)^2} + \frac{\omega\omega_0}{k_0} + b\frac{c^2k^3(c^2 - n) + k_0k(2nc^2 - c^2 - n^2) + 2nk_0^3}{2nk_0} + \frac{bk_0k(2n - 1) + n(2kc^2 - kn)}{n(S^2 - 1)},
\]

(12)

where \(w\) is given by (10), \(S \equiv \sqrt{n + 1}\text{sn}(kx + \omega t, m), C \equiv \sqrt{n + 1}\text{cn}(kx + \omega t, m),\) and \(D \equiv \text{dn}(kx + \omega t, m)\). The soliton-cnoidal wave interaction structure expressed by (12) with (10) is abundant. In the following, four types of special soliton-cnoidal waves are presented to show a variety of properties of the dark (gray) solitons under the background full of bright periodic cnoidal wave.

Fig. 1a exhibits the first type of special soliton-cnoidal wave structure of \(I\) determined by Eq. (12) for the NLS equation at \(t = 0\) with the parameter selected as

\[
    \{b, k_1, k_2, \delta, \lambda, m, n, c, \omega_0, \omega\} = \{1, 1, -1, -1, 0, 0.999, -0.01, 0.00898989, -0.0919418, 0.202157\},
\]

(13)

while Fig. 1b displays its time evolution. From Fig. 1b, two important features are observed: (i) The cnoidal wave looks like a moving optical soliton lattice. (ii) The interaction between soliton and everyone of the lattice is elastic except for the phase shifts.

Fig. 2a displays the second type of soliton-cnoidal wave structure with the parameter fixed as

\[
    \{b, k_1, k_2, \delta, \lambda, m, n, c, \omega_0, \omega\} = \{1, 1, 1, -1, 0, 0.9, -0.2, -0.05, 0.779897, 2.2540497\},
\]

(14)

at the initial time, and its time evolution is explored in Fig. 2b. It is seen from Fig. 2a that for small \(m\) (here \(m = 0.9\)) the cnoidal wave is different from the soliton lattice. However, the interaction between the soliton and every peak of the cnoidal periodic wave is also elastic with nonzero phase shifts.

Fig. 3a, Fig. 3b and Fig. 3c are the time evolutional plots for the third type of soliton-cnoidal wave interaction structure with the parameter determined as,

\[
    \{b, k_1, k_2, \delta, \lambda, m, n, c, \omega_0, \omega\} = \{1, 1, 1, -1, 1, 0.999, -0.875, -2.4720451, -3.0730997, -0.858140\},
\]

(15)
Fig 1: The first type of special soliton-cnoidal wave interaction solution for the NLS system given by (12) with the parameter selections (13): (a) The profile of the special structure at $t = 0$; (b) The density plot for time evolution.

at times $t = -10$, 0 and 10, respectively. It is remarkable that if the parameter $0 < n + 1 < 1$ is fixed smaller and smaller in Figs. 1-3, then the relative depth of the gray soliton lattice is shallower and shallower compared with that of the single non-lattice soliton.

Fig. 4 shows the fourth type of special soliton-cnoidal wave interaction solution with the parameters set as

$$\{b, k_1, k_2, \delta, \lambda, m, n, c, \omega_0, \omega\} = \{1, 1, 1, 1, 0.999, -0.00273650, -0.00142257, -0.665981, 0\}. \quad (16)$$

It is clear that the optical soliton lattice (cnoidal wave) can also be static. In Fig. 4b, the moving soliton elastically interacts with every static lattice with the same shift phase.

It is not surprising that when $m \to 1$ in Eq. (10), the soliton-cnoidal wave interaction solution reduces back to the two-soliton solution, whose interaction behaviors are displayed in Fig. 4 with the parameter selections

$$\{b, k_1, k_2, \delta, \lambda, m, n, c, \omega_0, \omega\} = \{1, 1, 1, -1, 0, 1, -0.95, -4.248259, -5.894616, 0.363449\}. \quad (17)$$

To sum up, the NLS system is studied by establishing two quite simple but equivalent methods, the truncated Painlevé expansion approach and the generalized tanh function expansion method, to find interactions among different nonlinear excitations. Especially, the soliton-cnoidal wave interaction solutions
are explicitly expressed by Jacobi elliptic functions and the third type of incomplete elliptic integral. To extend the known method to discover new phenomena in nonlinear physics, the key point is the straightening transformation \( (8) \) which straightens the single soliton to a straight line solution and transforms the truncated Painlevé expansion to an extended tanh function expansion method. The methods are valid for all integrable systems and or even nonintegrable models because both the truncated Painlevé analysis and the tanh expansion method can be used to find exact solutions of the partially solvable nonlinear models. The theorem (and the Lemma) can be used to find interaction solutions between solitons and any other NLS waves. However, for simplicity, only the soliton-cnoidal wave interaction solutions are discussed in detail. It is found that the soliton-cnoidal interaction solutions possess abundant structures. The nonlinear mediums described by NLS system themselves possess optical static or moving soliton lattice excitations, the cnoidal wave solutions with the modula very close to 1. Every soliton on the soliton lattice elastically interacts with traditional solitons with the only phase shift effect. In general, the cnoidal waves with smaller modula are periodic solutions which do not have localized structures. However, every peak of the cnoidal waves also elastically interacts with solitons with phase shifts.

The methods and what obtained in this Letter are valid for all the integrable models. The details on the methods for other nonlinear systems, other types of interaction solutions, other methods to solve
Fig. 3: Evolution of the third type of special soliton-cnoidal structure of the NLS system with the parameter selections at (a) $t = -10$, (b) $t = 0$ and $t = 10$ respectively.

Interaction solutions among different types of nonlinear excitations and so on will be reported in our future research work.

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Fig. 5. The special case of soliton-cnoidal wave interaction, two soliton case [12] with [14].

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