SUBMERSIONS, HAMILTONIAN SYSTEMS AND OPTIMAL SOLUTIONS TO THE ROLLING MANIFOLDS PROBLEM

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Abstract. Given a submersion \( \pi : Q \to M \) with an Ehresmann connection \( \mathcal{H} \), we describe how to solve Hamiltonian systems on \( M \) by lifting our problem to \( Q \) and doing our computations here. We also show how solutions of lifted Hamiltonian systems on \( Q \) can be viewed as particles in \( M \) under the influence of a generalization of the Lorentz force.

We investigate what this means in terms of normal and abnormal extremals in optimal control problems on respectively \( M \) and \( Q \). We give a demonstration of our theory by considering the system of one Riemannian manifold rolling on another without twisting or slipping. We show that the curves of minimal length connecting two given configurations can be considered as trajectories of objects with a non-constant gauge-charge under the influence of the Lorentz force. The Yang-Mills field is in this case given by the difference of the Riemannian curvatures of the manifolds involved. In the case of two surfaces rolling on each other, length minimizers are determined by the equation of a pendulum whose length is inverse proportional to the difference in Gaussian curvature of the two surfaces.

1. Introduction

Consider a (surjective) submersion \( \pi : Q \to M \) between two Riemannian manifolds \( Q \) and \( M \). Associated to this submersion we have splitting \( TQ = \mathcal{H} \oplus \mathcal{V} \) of the tangent bundle of \( Q \) into the subbundle \( \mathcal{V} := \ker \pi^* \) and its orthogonal complement \( \mathcal{H} \). Here, \( \pi^* \) denotes the differential of \( \pi \). The map \( \pi \) is called a Riemannian submersion if \( \pi^* \) restricted to the bundle \( \mathcal{H} \) is an isometry on every fiber. Several papers exist connecting the geometry of \( Q \) with \( M \) in this case, see e.g. \cite{8, 9, 10}. The important observation for us, is that the Riemannian geodesics of \( M \) are exactly the projections of geodesics in \( Q \) which are horizontal to \( \mathcal{H} \) at one point (and hence all points). If it is simpler to work out the computations of the geodesics on \( Q \) than on \( M \), this can be a great advantage. A simple example is the case when \( Q \) is a Lie group and \( M \) is a symmetric space.

This idea of solving the problem of geodesics “upstairs” was also used in \cite{12} for the case when the top space \( Q \) is a sub-Riemannian manifold.

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Specifically, for any Ehresmann connection $\mathcal{H}$ on a submersion $\pi: Q \rightarrow M$ into a Riemannian manifold $M$, we can obtain a sub-Riemannian structure on $Q$ by lifting the metric of $M$ to $\mathcal{H}$. In this case Riemannian geodesics in $M$ lift to sub-Riemannian normal geodesics in $Q$. For the definition of sub-Riemannian manifolds and normal geodesics, see Section 3.3.

Both Riemannian geodesics and normal sub-Riemannian geodesics may be considered as solutions of a Hamiltonian system. We will show that given any Hamiltonian function $H$ on $M$, we can solve the corresponding Hamiltonian system “upstairs”. We do this by making a choice of Ehresmann connection $\mathcal{H}$ on the submersion $\pi: Q \rightarrow M$, and using this connection, we can define a lifted Hamiltonian function $\tilde{H}$ on $Q$, from which we can obtain the solutions corresponding to $H$ in $M$.

We also show a converse result inspired by the following example. Consider a principle $G$-bundle $\pi: Q \rightarrow M$, where the base manifold $M$ is a Riemannian manifold. We again let $\mathcal{H}$ be an Ehresmann connection on $\pi$, required now to be invariant under the action of $G$. By lifting the metric of $M$ to $\mathcal{H}$, we give $Q$ the structure of a sub-Riemannian manifold. In this particular case, projections of the normal geodesics in $Q$ are the trajectories of gauge-charged particles under the influence of the Lorentz force [13]. Here, the gauge is an element in the dual Lie algebra of $G$ and the Yang Mills field is represented by the curvature of $\mathcal{H}$. See Section 3.4 for details.

Generalizing this idea, we described how solutions of lifted Hamiltonian systems on $Q$ look when viewed from the base space $M$. These projected curves are directed by a combination of the Hamiltonian vector field of the original Hamiltonian and a “force” depending on the curvature of $\mathcal{H}$. The concept of gauge is replaced by an introduction of parallel transport of forms which vanish on $\mathcal{H}$. In this way, we get a precise description of how the lifted Hamiltonian system in influenced by the original Hamiltonian and the Ehresmann connection $\mathcal{H}$, respectively.

The paper is structured as follows. In Section 2 we describe how to lift a Hamiltonian system from the base space to the top space of a submersion. Then we present our main theorem, connecting solutions of the two Hamiltonian system. In Section 3 we apply this result to optimal control problems, in particular sub-Riemannian manifolds. Given a submersion $\pi: Q \rightarrow M$, we both describe how to solve an optimal control problem given on $M$ by doing computations on $Q$, and how solutions of lifted optimal control problems on $Q$ look when viewed from $M$.

In Section 4 we give a specific example to show the effectiveness of our approach. We look at the problem of finding the optimal curves of the kinematic system of two Riemannian manifolds $M$ and $\tilde{M}$ rolling on each other under the constraint of high friction, preventing slipping and twisting. Here, results has, up until now, only been presented in some specific cases [10, 11], and even the case of surfaces with non constant curvature rolling on each other was previously unknown. We present the equations solving this optimal control problem using exactly the techniques described above. First, we
lift the problem from our original configuration space to the product of the orthogonal frame bundles \( F(M) \times F(\hat{M}) \), where it is simpler to do computations. Then, we describe the solutions by looking at the curve projected to one of the manifolds. We find that these projected curves behave as objects with an \( \mathfrak{so}(n) \)-gauge under the influence of the Lorentz force, where this gauge changes according to a given vector field. The Yang-Mills field is in this case given by the differences in the Riemannian curvatures. In particular, we show in the case of two surfaces rolling on each other without twisting or slipping, the solution of this problem are in many cases found by solving the equation of a pendulum, whose length is inverse proportional to the differences in Gaussian curvature of \( M \) and \( \hat{M} \) along the curve.

Some of the proofs, including the proof of the main theorem, are left to Section 5.

2. Lifting Hamiltonian systems

2.1. Notation and conventions. If \( \Pi^E : E \to M \) is a vector bundle over \( M \), we will denote the space of all its sections by \( \Gamma(E) \). For any section \( X : M \to E \), we will use \( X|_m \) rather than \( X(m) \) for the value of \( X \) at \( m \in M \). We let \( 0_E \) denote the zero-section of \( E \). For elements \( e_1, e_2 \in E_m \), both in the same fiber over \( m \in M \), define the vector \( \text{vl}_{e_1} e_2 \in T_{e_1} E \) by

\[
\text{vl}_{e_1} e_2 = \frac{d}{dt}\bigg|_{t=0} (e_1 + te_2).
\]

This is called the vertical lift of \( e_2 \) to \( e_1 \). Similarly, for any \( X \in \Gamma(E) \), we can define the vertical lift \( \text{vl} X \in \Gamma(T E) \) of \( X \) by \( \text{vl} X|_e = \text{vl}_{e_1} X|_{\Pi^E(e)} \).

If \( E \) is a subbundle of the tangent bundle, we say that a curve \( \gamma \) in \( M \) is \( E \)-horizontal if it is both absolutely continuous and satisfy \( \dot{\gamma}(t) \in E_{\gamma(t)} \) for almost every \( t \).

Rather than introducing a special notation for the interior product, we will denote the interior product of a form \( \eta \) with a vector field \( X \) by \( \eta(X, \cdot) \).

2.2. Submersions and Ehresmann connections. Let \( \pi : Q \to M \) be a submersion between two connected manifolds \( Q \) and \( M \). By a submersion, we will always mean a surjective map such that the differential map \( \pi_* : TQ \to TM \) is surjective as well. For each \( m \in M \), we write \( Q_m = \pi^{-1}(m) \) for the preimage of \( m \). Each \( Q_m \) is an embedded submanifold of \( M \) by the implicit function theorem. In what follows, objects related to \( Q \) will typically be marked by a tilde (\( \tilde{\cdot} \)). We call the subbundle \( \mathcal{V} = \ker \pi_* \) the vertical bundle. An Ehresmann connection \( \mathcal{H} \) on \( \pi : Q \to M \) is a subbundle of \( TQ \) satisfying \( TQ = \mathcal{H} \oplus \mathcal{V} \). Since for every \( q \in Q \), the map \( \pi_*|_{\mathcal{H}_q} : \mathcal{H}_q \to T\pi(q)M \) is an invertible linear map, a choice of Ehresmann connection allows us to define horizontal lifts of vectors, vector fields and curves.

To be more specific, for every vector \( v \in T_m M \) and \( q \in Q_m \), we define the horizontal lift \( h_q v \) of \( v \) to \( q \) as the unique element in \( \mathcal{H}_q \) which is sent
to \( v \) by \( \pi_v \). Similarly, any vector field \( X \in \Gamma(TM) \) has a horizontal lift \( hX \in \Gamma(TQ) \) given by formula
\[
hX|_q := h_q X|_{\pi(q)}.
\]
Finally, a curve \( \tilde{\gamma} \) in \( Q \) is a horizontal lift of an absolutely continuous curve \( \gamma \) in \( M \) if \( \tilde{\gamma} \) is \( H \)-horizontal and satisfy \( \pi(\tilde{\gamma}) = \gamma \). In other words, \( \tilde{\gamma} \) is a solution of the equation \( \dot{\tilde{\gamma}} = h_{\tilde{\gamma}(t)} \dot{\gamma}(t) \) for almost every \( t \). Clearly this implies that \( \tilde{\gamma} \) is uniquely determined by \( \gamma \) up to initial condition. For a general submersion \( \pi : Q \to M \) and curve \( \gamma \) in \( M \), we only know that a horizontal lift exists for short time. If any absolutely continuous curve in \( M \) has a horizontal lift that extends to all time, we say that the Ehresmann connection \( H \) is complete.

The curvature of an Ehresmann connection \( H \) is a vector valued 2-form on \( Q \), defined by
\[
\mathcal{R}(\tilde{X}, \tilde{Y}) = \text{pr}_V \left[ \text{pr}_H \tilde{X}, \text{pr}_H \tilde{Y} \right] \quad \text{for any } \tilde{X}, \tilde{Y} \in \Gamma(TQ).
\]
Here, \( \text{pr}_V \) and \( \text{pr}_H \) are the respective projections to \( H \) and \( V \) having the other bundle as their kernel. It is simple to verify that \( \mathcal{R} \) is anti-symmetric and \( C^\infty(Q) \)-linear in both arguments, hence making it a vector valued 2-form with values in \( V \).

By slightly abusing notation, we can write \( \mathcal{R}(X, Y) := \mathcal{R}(hX, hY) \) for any \( X, Y \in TQ \). Note that then \( \mathcal{R}(X, Y) = [hX, hY] - h[X,Y] \) since \( hX \) and \( hY \) are \( \pi \)-related to respectively \( X \) and \( Y \). With this convention, it is clear \( \mathcal{R} \) is also \( C^\infty(M) \)-linear in both arguments, however, \( \mathcal{R}(v_1, v_2) \) does not make sense as a vector for any \( v_1, v_2 \in T_m M \). Rather, we consider \( \mathcal{R}(v_1, v_2) \) as a vector field on \( Q_m \), given by formula \( q \mapsto \mathcal{R}(h_q v_1, h_q v_2) \).

2.3. Parallel transport of vertical vectors with respect to \( H \). Horizontal lifts are often considered as a generalization of parallel transport with respect to an affine connection. The reason why is the following. Let \( \nabla \) be an affine connection of \( M \) and let \( \mathcal{E}^\nabla \) be the subbundle of \( TT M \) defined as the collection of all tangent vectors of curves \( X(t) \) which are \( \nabla \)-parallel along their projection \( \gamma(t) \) in \( M \). Then \( \mathcal{E}^\nabla \) is an Ehresmann connection on the submersion \( \Pi^TM : TM \to M \) and horizontal lifts of curves in \( M \) to \( TM \) with respect to \( \mathcal{E}^\nabla \) exactly corresponds to parallel transport of vector fields. However, we can also define parallel transport by using the identity \( [hX, v_1 Y] = v_1 \nabla_X Y \), for any \( X, Y \in \Gamma(TM) \), where horizontal lifts are defined with respect to \( \mathcal{E}^\nabla \). We would like to generalize this latter approach to general submersions.

We define an operator
\[
\nabla : \Gamma(TM) \times \Gamma(V) \to \Gamma(V) \quad (X, \Upsilon) \mapsto \nabla_X \Upsilon := [hX, \Upsilon].
\]
The bracket \( [hX, \Upsilon] \) is indeed a vertical vector field, since \( hX \) and \( \Upsilon \) are \( \pi \)-related to respectively \( X \) and \( \Theta^TM \). This operator is \( \mathbb{R} \)-linear in the second
Lemma 1. Let $\gamma : [0, t_1] \to M$ be an absolutely continuous curve with $\gamma(0) = m_0$. Let $q_0$ be a given point in $Q_{m_0}$, and assume that there is a horizontal lift $\tilde{\gamma}$ of $\gamma$ with $\tilde{\gamma}(0) = q_0$. Then for any $\Upsilon_0 \in \mathcal{V}_{q_0}$, there is a unique vertical vector field $\Upsilon(t)$ along $\tilde{\gamma}$ with

$$\nabla_{\tilde{\gamma}(t)} \Upsilon(t) = 0 \text{ and } \Upsilon(0) = \Upsilon_0.$$ 

Definition 1. For any $\Upsilon_0 \in \mathcal{V}_{q_0}$, we say that $\Upsilon(t)$ is the $\nabla$-parallel transport of $\Upsilon_0$ along $\gamma$ if it satisfies $\nabla_{\tilde{\gamma}(t)} \Upsilon(t) = 0$, $\Upsilon(0) = \Upsilon_0$.

Proof of Lemma 1. Let $M$ and $Q$ be of respective dimensions $n$ and $n + \nu$. Let $x = (x_1, \ldots, x_n)$ be a chart defined on $U$ in $M$ around $m_0$, and choose a neighborhood $\tilde{U}$ in $\pi^{-1}(U)$ around $q_0$ sufficiently small such that $\mathcal{V}|_{\tilde{U}}$ is trivial. Let $\Upsilon_1, \ldots, \Upsilon_\nu$ be sections of $\mathcal{V}$, forming a basis on $\tilde{U}$. Since the image of $\tilde{\gamma}$ is compact, we can assume that it is contained in a neighborhood such as $\tilde{U}$, by taking a finite division of the curve, Write

$$\nabla_{\partial_{x_i}} \Upsilon_\kappa = \sum_{\mu=1}^{\nu} \Gamma_{i\kappa}^\mu \Upsilon_\mu,$$

and $\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{x}_i(t) \partial_{x_i}$. Then $\Upsilon(t) = \sum_{\kappa=1}^{\nu} b_\kappa(t) \Upsilon_\kappa|_{\tilde{\gamma}(t)}$ is a solution to

$$\dot{\tilde{\gamma}} = \sum_{i=1}^{n} \dot{x}_i h \partial_{x_i}, \quad 0 = \nabla_{\tilde{\gamma}} \Upsilon = \sum_{\kappa=1}^{\nu} \left( b_\kappa + \sum_{i=1}^{n} \sum_{\mu=1}^{\nu} x_i \Gamma_{i\kappa}^{\mu} \right) \Upsilon_\kappa|_{\tilde{\gamma}}.$$ 

These equations along with the initial conditions uniquely determine $\Upsilon$, and since it is a linear system, we also have existence. \qed

Remark 1. (a) Note that $\nabla$-parallel transport of an element $\Upsilon_0 \in \mathcal{V}_{q_0}$ along a curve $\gamma$ in $M$ is not guaranteed to exist for all time, only for all time for which the horizontal lift of $\gamma$ starting at $q_0$ exist. As a result, if $\mathcal{H}$ is complete, the $\nabla$-parallel transport exist for all time.

(b) Let $\mathcal{E}^\gamma$ be the subbundle of $T\mathcal{V}$, defined as the collection of all tangent vectors of vector fields $\Upsilon$ along $\mathcal{H}$-horizontal curves $\tilde{\gamma}$ in $Q$ such that $\beta$ is $\nabla$-parallel. Then $\mathcal{E}^\gamma$ is an Ehresmann connection on the submersion $\pi \circ \Pi^\gamma : \mathcal{V} \to M$.

In other words, the process of first finding a curve $\tilde{\gamma}$ which is a horizontal lift of a curve $\gamma$ in $M$, and after that consider parallel transport of a vector along $\tilde{\gamma}$ is really just a horizontal lift with respect to $\mathcal{E}^\gamma$.

(c) From the Jacobi-identity, we have

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} ) \Upsilon = [\mathcal{R}(X,Y), \Upsilon],$$

where $\mathcal{R}(X,Y)$ is the vector field on $M$ defined by $\nabla_Y X - \nabla_X Y - [X,Y]$. 


for any \( X, Y \in \Gamma(TM) \) and \( Y \in \Gamma(V) \).

Given a choice of splitting \( TQ = H \oplus V \) of \( TQ \), we will also have a corresponding splitting of the cotangent bundle \( T^*Q = \text{Ann}(V) \oplus \text{Ann}(H) \). By \( \text{Ann}(V) \) and \( \text{Ann}(H) \) we mean elements of \( T^*Q \) which vanish on respectively \( V \) or \( H \). Just as with elements of \( V \), we have a well defined \( \nabla \)-parallel transport of elements in \( \text{Ann}(H) \) along horizontal curves (or curves in \( M \), depending on your point of view).

For any \( \beta \in \Gamma(\text{Ann}(H)) \), we define \( \nabla_X \beta = \text{pr}_V \mathcal{L}_{hX} \beta \) for a vector field \( X \) on \( M \). In other words, \( \nabla_X \alpha \) is determined by the relation that for any vector field \( \tilde{Y} \) on \( Q \), we have

\[
hX(\text{pr}_V \tilde{Y}) = hX(\tilde{Y}) = (\nabla_X \beta)(\tilde{Y}) + \beta(\nabla_X \text{pr}_V \tilde{Y}).
\]

By similar arguments as in Lemma 1 we obtain that there is a unique solution to the equation

\[
\nabla_{\dot{\gamma}} \beta = 0, \quad \beta(0) = \beta_0 \in \text{Ann}(H)_{q_0}, q_0 \in Q_{\gamma(0)},
\]

which exist for all time that the horizontal lift of \( \gamma \) starting at \( q_0 \) exists.

2.4. **Induced submersion of the cotangent bundle and lifted Hamiltonian systems.** The choice of an Ehresmann connection gives us a way of fiber-wise defining a right inverse of the map \( \pi_* \), namely the horizontal lift. We want to use the Ehresmann connection to also define a fiber-wise left inverse for the map \( \pi^* \). More precisely, for any given \( q \in Q \), we have a map \( T^*_{\pi^*}(q)M \to T^*_qM \) given by \( p \to \pi^*(p)|_{T^*_qM} \). We remark that the image of this map is \( \text{Ann}(V)_q \). We introduce a vector bundle morphism

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{\pi^2} & T^*M \\
\Pi^*Q \downarrow & & \downarrow \Pi^*M \\
Q & \xrightarrow{\pi} & M
\end{array}
\]

determined by the following requirements

(i) \( \pi^2(\pi^*p) = p \) for any \( p \in T^*M \), \quad (ii) \( \ker \pi^2 = \text{Ann}(H) \).

The requirement (i) states that \( \pi^2 \) is a left inverse of \( \pi^* \). Note that from requirement (i) and (ii) we know that \( \pi^2(\tilde{p})(v) = \tilde{p}(h_qv) \) for any \( \tilde{p} \in T_q^*Q, v \in T^*_{\pi(q)}M \). Just as any choice of Ehresmann connection determines a left inverse of \( \pi^* \), we have that for any vector bundle morphism \( \pi^*: T^*Q \to T^*M \) over \( \pi \) satisfying (i), there is a unique corresponding Ehresmann connection

\[
\mathcal{H} = H^{\pi^2} = \left\{ \tilde{v} \in T_qQ, q \in Q : \alpha_0(v) = 0 \text{ for any } \alpha_0 \in \ker \pi^2|_{T^*_qQ} \right\}.
\]

With all of this formalism in place, we are now ready to present our main theorem. Let \( H \in C^\infty(T^*M) \) be a Hamiltonian function, and use \( \tilde{H} \) for its Hamiltonian vector field on \( M \). By abusing terminology slightly, we will often refer to an integral curve \( \lambda(t) \) of \( \tilde{H} \) as simply an *integral curve of \( H \).*
We lift this Hamiltonian to $Q$ using $\pi^2$. The following theorem connects the solutions of the lifted Hamiltonian system $\tilde{H} = H \circ \pi^2$ to the properties of the Hamiltonian $H$ and the Ehresmann connection $\mathcal{H}$.

**Theorem 1.** Let $H \in C^\infty(T^*M)$ and define $\tilde{H} \in C^\infty(T^*Q)$ as $\tilde{H} = H \circ \pi^2$.

(a) A curve in $\lambda(t)$ is an integral curve of $H$ if and only if every sufficiently short segment is the projection of an integral curve $\tilde{\lambda}(t)$ of $\tilde{H}$ which is contained in $\text{Ann}(\mathcal{V})$ (it is sufficient to require this in only one point).

(b) Let $\lambda : [0, t_1] \to T^*Q$ be an integral curve of $\tilde{H}$ with $\tilde{\gamma}(t) := \Pi_{T^*Q}^*(\tilde{\lambda}(t))$ and $\beta_0 := \text{pr}_V^*\tilde{\lambda}(0)$. Let $\pi^2(\tilde{\lambda}(t)) = \lambda(t)$ with $\gamma(t) = \Pi_{TM}^M(\lambda(t))$.

Then $\lambda$ is a solution to

\begin{equation}
\dot{\lambda}(t) = \tilde{H}|_{\lambda(t)} + \text{vl}_{\lambda(t)} \beta \mathcal{R}(\dot{\gamma}(t), \bullet),
\end{equation}

where the curve $\beta(t)$ in $\text{Ann}(\mathcal{H})$ determined by equations

\begin{equation}
\nabla_{\dot{\gamma}(t)} \beta(t) = 0, \quad \beta(0) = \beta_0.
\end{equation}

The curve $\tilde{\gamma}$ will be an $\mathcal{H}$-horizontal lift of $\gamma$.

In the above theorem, $\beta(t) \mathcal{R}(\dot{\gamma}(t), \bullet) \in T_{\gamma(t)}^*M$ is the element sending any $v \in T_{\gamma(t)}M$ to the value $\beta(t) \mathcal{R}(\dot{\gamma}(t), v)$. Note that $\mathcal{R}(\dot{\gamma}(t), v)$ is a vector field on and tangent to $Q_{\gamma(t)}$ and we consider this vector field evaluated by $\beta(t)$ at $\tilde{\gamma}(t)$.

The proof of this theorem is left to Section 5.1

2.5. Special case: Hamiltonian of a Riemannian manifold. We now consider the following special case. Let $(M, g)$ be a Riemannian manifold with metric tensor $g$. Associated to this metric, let $\mathbb{B} : TM \to T^*M$ be the vector bundle isomorphism $v \mapsto g(v, \cdot)$ with inverse $\mathbb{F}$. Use $\nabla$ for the Levi-Civita connection corresponding to $g$. Associated to the Riemannian structure, we also have a Hamiltonian function $H(p) = \mathbb{F}p(2p)$. The projection of integral curves of $H$ to $M$ gives us the Riemannian geodesics. If we apply Theorem 1(b) to this Hamiltonian, we get the following result.

**Corollary 1.** Let $\pi : Q \to M$ be a submersion into a Riemannian manifold $(M, g)$, with associated Hamiltonian $H$. Let $\mathcal{H} \subseteq TQ$ be a chosen Ehresmann connection, and define $\tilde{H}$ as in Section 2.4.

Let $\tilde{\lambda} : [0, t_1] \to T^*Q$ be an integral curve of $\tilde{H}$ with $\text{pr}_{V}^*\tilde{\lambda}(0) = \beta_0$. Use $\tilde{\gamma}$ for its projection to $Q$. Then $\tilde{\gamma}$ is a horizontal lift of a curve $\gamma$ in $M$, the latter being a solution of equation

\[ \nabla_{\gamma(t)} \dot{\gamma}(t) = \mathbb{F}\beta(t) \mathcal{R}(\dot{\gamma}(t), \bullet), \quad \gamma(0) = \pi(\tilde{\gamma}(0)), \]

where $\nabla_{\gamma} \beta(t) = 0$ and $\beta(0) = \beta_0$.

**Proof.** Pick a local orthonormal basis $e_1, \ldots, e_n$ of vector fields, and use these vector field to give the cotangent bundle local coordinates $p_i = p(e_i)$. It is
standard result that the Hamiltonian vector field in these local coordinates is given by
\[ \vec{H} = \sum_{i=1}^{n} p_i e_i - \sum_{i,j,k=1}^{n} p_j p_k \Gamma^i_{jk} \partial_{p_i}, \quad \Gamma^k_{ij} = g(e_k, \nabla_{e_i} e_j). \]

Let \( \lambda(t) = \pi^2(\tilde{\lambda}(t)) \). If \( \lambda(t) = \sum_{i=1}^{n} p_i(t) e_i |_{\gamma(t)} \) is a solution to
\[ \dot{\lambda}(t) = \vec{H}|_{\lambda(t)} + v_1(\lambda(t)) \beta(t) R(\dot{\gamma}(t), \bullet), \]
then \( e_i(\dot{\gamma}(t)) = p_i(t) \) and \( \dot{p}_i(t) = -\sum_{j,k=1}^{n} p_j(t)p_k(t) \Gamma^i_{jk} |_{\gamma(t)} + \beta(t) R(\dot{\gamma}(t), e_i) \).
The corollary then follows from the computation
\[
\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = \sum_{i=1}^{n} \dot{p}_i(t)e_i + \sum_{i,j,k=1}^{n} p_ip_j \Gamma^k_{ij} e_k
\]
\[
= \sum_{i=1}^{n} \left( -\sum_{j,k=1}^{n} p_j p_k \Gamma^i_{jk} + \beta(t) R(\dot{\gamma}(t), e_i) \right) e_i + \sum_{i,j,k=1}^{n} p_ip_j \Gamma^k_{ij} e_k
\]
\[
= \#\beta(t) R(\dot{\gamma}(t), \bullet).
\]

\[ \square \]

3. Lifted optimal control problems

3.1. Optimal control systems. Then are many different definitions and generalizations of an optimal control problem. We will use the definition found in [3, Section 2.1] and [2].

A smooth control system consists of a fiber bundle \( \xi : U \to M \) with fiber \( U \), along with a bundle morphism
\[
\begin{array}{c}
U \\
\downarrow f \\
\downarrow \xi \\
\downarrow \Pi^{TM} \\
M
\end{array} \quad \xrightarrow{T M} \quad .
\]

A measurable essentially bound curve \( \kappa(t) \) in \( U \) is called an admissible control if its projection \( \gamma(t) \) in \( M \) is an \( L^\infty \) curve satisfying \( \dot{\gamma} = f(\kappa(t)) \). We want to consider the following optimal control problem: For a smooth function \( \varphi : C^\infty(U) \to \mathbb{R} \) and two points \( m_0, m_1 \in M \), find the admissible control \( \kappa(t) \) which satisfies
\[ \gamma(0) = m_0, \quad \gamma(t_1) = m_1, \quad \text{with} \quad \xi(\kappa(t)) = \gamma(t), \]
and minimize the functional
\[ \kappa \mapsto \int_0^{t_1} \varphi(\kappa(t)) \, dt. \]

The latter functional is called the cost functional and \( \varphi \) is called the cost function.
Definition 2. We will denote the above optimal control problem as the optimal control problem associated to \((\xi, f, \varphi)\).

A sufficient condition for an admissible control to be a solution to this optimal control problem is given by the Pontryagin Maximum Principle (PMP). In order to present this result in a simpler way, we will just write the formulation assuming that \(U\) can be written as a trivial fiber bundle \(U = M \times U\), which can be considered a local version of the general case. See Remark 2 for how this statement can be reformulated for the case when \(U\) cannot be trivialized. For proof of this theorem, we refer to [1, Theorem 12.3].

Theorem 2. PMP for Optimal Control Problem with fixed time \(t_1\)

Let \(\hat{\kappa}(t) = (\hat{\gamma}(t), \hat{u}(t))\) be a solution to the optimal control problem associated to \((\xi, f, \varphi)\), where \(\hat{\gamma}(t)\) is a curve in \(M\) and \(\hat{u}(t)\) is a curve in \(U\). For each \(r \in \mathbb{R}, u \in U\), consider a Hamiltonian function

\[
H_{r,u}(p) = p(f(m, u)) + r \varphi(m, u), \quad p \in T^*_m M.
\]

Then there exists a curve \(\lambda : [0, t_1] \rightarrow T^* M\), and a number \(r \leq 0\) such that

(i) \(\Pi T^* M(\lambda(t)) = \hat{\gamma}(t)\).

(ii) \(\dot{\lambda}(t) = \hat{H}_r,\hat{u}(t)|_{\lambda(t)}\) for almost every \(t\),

(iii) \(H_{r,u}(\lambda(t)) = \max_{u \in U} H_{r,u}(\lambda(t))\) for almost every \(t\).

Moreover, if \(r = 0\), then \(\lambda\) never intersects the zero section of \(T^* M\).

Let us use the name extremals for solutions of PMP. They are called normal if \(r \neq 0\) (it is sufficient to consider \(r = -1\)) and abnormal if \(r = 0\). Remark that abnormal extremals do not depend on the function \(\varphi\), only the control system \((\xi, f)\).

Remark 2. If the fiber bundle cannot be trivialized, PMP can be reformulated in the following way. For any \(r \in \mathbb{R}\), define \(\mathcal{H}_r : U \times_M T^* M \rightarrow \mathbb{R}\) by \(\mathcal{H}_r(\kappa, p) = p(f(\kappa)) + r \varphi(\kappa)\). which takes the place of the Hamiltonian in (3). Requirement (ii) is then replaced with the identity

\[
\varsigma_{|\hat{\kappa}(t),\lambda(t)}(\dot{\lambda}, \text{pr}_2 \bullet) = (\mathcal{H}_r)_*|_{\hat{\kappa}(t),\lambda(t)}.
\]

where \(\text{pr}_2 : U \times_M T^* M \rightarrow T^* M\) is the projection and \(\varsigma\) is the canonical symplectic form on \(T^* M\). In the requirement (iii), the maximum needs to hold over all elements in \(U_{\hat{\kappa}(t)}\).

We will call \(\mathcal{H}_r\) the PMP-Hamiltonian of the optimal control problem associated to \((\xi, f, \varphi)\).

3.2. Submersions and a lifted optimal control problem. Consider the optimal control problem on \(M\) associated to a triple \((\xi, f, \varphi)\). Let \(\pi : Q \rightarrow M\) be a submersion into \(M\), equipped with a chosen Ehresmann connection \(\mathcal{H}\) and let \(\pi^2 : T^* Q \rightarrow TM\) be the corresponding left inverse of \(\pi^*\). Then we have a lifted optimal problem on \(Q\) associated to a triple \((\xi, \tilde{f}, \tilde{\varphi})\). This triple is defined by the points (a)-(c).
Let the PMP-Hamiltonian of \( \pi U \to \pi M \). Recall that this is defined as

\[
\pi U = \{(q, \kappa) \in Q \times U : \pi(q) = \xi(\kappa)\}.
\]

The map \( \tilde{\xi} \) is just the projection on the first coordinate. Let \( \tilde{\pi} \) denote the projection on the second coordinate. Then we have the following commutative diagram

\[
\begin{array}{ccc}
\pi^* U & \xrightarrow{\tilde{\pi}} & U \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\pi} & M
\end{array}
\]

(a) Define the fiber bundle \( \tilde{\xi} : \pi^* U \to Q \) as the pull-back bundle of \( \xi : U \to M \). We only need that show that it holds locally, hence we may assume

\[
\tilde{\pi}T
\]

For this system, we have the following result.

(b) Define a bundle morphism \( \tilde{f} : \pi^* U \to TQ \) by

\[
\tilde{f}(\tilde{\kappa}) = h_{\tilde{\xi}(\tilde{\kappa})} f(\tilde{\pi}(\tilde{\kappa})) \quad \text{for any } \tilde{\kappa} \in \pi^* U.
\]

In other words, if \( \tilde{\kappa} \) represents the pair \( (q, \kappa) \), then \( \tilde{f}(\tilde{\kappa}) = h_q f(\kappa) \).

(c) Finally, we define \( \tilde{\varphi} \in C^\infty(\pi^* U) \) by \( \tilde{\varphi} = \varphi \circ \tilde{\pi} \).

For this system, we have the following result.

**Proposition 1.** Let \( \mathcal{H}_r : U \times M T^* M, r \in \mathbb{R} \) be the PMP-Hamiltonian of the optimal control problem associated to \( (\xi, f, \varphi) \). Then \( \mathcal{H}_r \circ (\tilde{\pi} \times Q \pi^2) \) is the PMP-Hamiltonian of \( (\tilde{\xi}, \tilde{f}, \tilde{\varphi}) \). As a consequence,

(a) A curve \( \lambda \) in \( T^* M \) normal (resp. abnormal) extremal if and only if \( \lambda \), at least for short time, is the projection of a normal (resp. abnormal) extremal in \( T^* Q \) contained in \( \text{Ann}(V) \) (it is sufficient to require this in only one point).

(b) If \( \tilde{\kappa}(t) \) is a solution to the optimal control problem associated to \( (\tilde{\xi}, \tilde{f}, \tilde{\varphi}) \), then there is a number \( r \leq 0 \) and a curve \( \lambda \) in \( T^* M \) (in \( T^* M \setminus 0 \) if \( r = 0 \) ) such that

(i) \( \Pi T^* M(\lambda(t)) = \tilde{\xi}(\pi(\tilde{\kappa})) =: \gamma(t) \),

(ii) If \( \tilde{\pi}(\tilde{\kappa}(t)) = \kappa(t) \), then for almost every \( t \),

\[
\zeta|_{\kappa(t),\lambda(t)}(\hat{\lambda}, pr_2 \bullet) = (\mathcal{H}_r)_*|_{\kappa(t),\lambda(t)} - \beta(t) \mathcal{R}(\hat{\gamma}, \Pi T^* M pr_2 \bullet).
\]

(iii) For almost every \( t \),

\[
\mathcal{H}_r(\kappa(t), \lambda(t)) \leq \max_{\kappa \in \mathcal{U}_{\mu}(t)} \mathcal{H}(k, \lambda(t)).
\]

**Proof.** We only need that show that it holds locally, hence we may assume that \( U = M \times U \), and hence \( \pi^* U = Q \times U \). We then verify that for any \( \tilde{p} \in T^* Q \) and \( u \in U \), we have

\[
\tilde{H}_{r,u} : \tilde{p}(\tilde{f}(q, u)) + r\tilde{\varphi}(q, u),
\]

\[
= \tilde{p}(h_q f(\pi(q), u)) + r\varphi(\pi(q), u)
\]

\[
= \pi^2(\tilde{p})(f(\pi(q), u)) + r\varphi(\pi(q), u) = H_{r,u}(\pi^2(\tilde{p})).
\]
The rest follows from Theorem 1 and the fact that $\zeta(\nu \alpha, \bullet) = -\alpha(\Pi^*_TM \bullet)$ for any $\alpha \in \Gamma(T^*M)$. □

This relation can be a powerful and useful tool in many optimal control problems. We will explore the case of sub-Riemannian manifolds in detail.

3.3. Submersions and sub-Riemannian manifolds. A sub-Riemannian manifold is a triple $(M, D, g)$, where $M$ is a connected manifold, $D$ is a sub-bundle of $TM$ and $g$ is a metric tensor on $D$. Use $AC_D$ for the Hilbert manifolds of all $D$-horizontal curves defined on the interval $[0, 1]$ with an $L^2$-derivative. Let $AC_D(m_0, m_1)$ be the subset of $AC_D$ consisting of curves $\gamma$ satisfying $\gamma(0) = m_0$ and $\gamma(1) = m_1$. The distance function $d_{cc}$ in this geometry is called the Carnot-Carathéodory distance and is defined as

$$d_{cc}(m_0, m_1) = \inf \left\{ \int_0^1 g(\dot{\gamma}, \dot{\gamma})^{1/2} dt : \gamma \in AC_D(m_0, m_1) \right\}.$$

A sufficient condition for this distance to be finite, i.e. that any pair of points can be connected by a $D$-horizontal curve, is that $D$ is bracket generating. This means that $D$ along with the iterated brackets of its sections span the entire tangent bundle. To look for minimizers with respect to this distance function, is equivalent to look for curves in $AC_D(m_0, m_1)$ which minimize the energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) dt.$$

This can be viewed as the control system with data $(\Pi^D, \text{inc})$, where we try to minimize $E(\gamma)$. Here, $\Pi^D : D \to M$ is the projection and $\text{inc} : D \to TM$ is just the inclusion, The only difference is that we allow the derivatives to be in $L^2$ rather than $L^\infty$.

Normal extremals can be described in the following way. The metric tensor $g$ defines a bundle map $\sharp_g : T^*M \to D$ by the identity $g(\sharp_g p, v) = p(v)$ for any $v \in D$. Normal extremals are then the solution of the Hamiltonian system with Hamiltonian $H_{sR}(p) = \frac{1}{2}p(\sharp_g p)$. Projections to $M$ of normal extremals are always smooth and are length minimizers locally. We therefore call such curves in $M$ normal geodesics. Remark that if $\lambda$ is a normal extremal, and its projection to $M$ is the normal geodesic $\gamma$, then we have the relation

$$\dot{\gamma}(t) = \sharp_g \lambda(t) \quad (4)$$

Abnormal extremals $\lambda$ also have an alternate description. Let $\zeta$ be the canonical symplectic form on $T^*M$ and let $\lambda$ be a curve in $T^*M$. Then $\lambda$ is an abnormal extremal if and only if it is an absolutely continuous curve in Ann$(D)$ with an $L^2$ derivative, never meeting the zero section, such that $\zeta(\dot{\lambda}(t), \cdot)_{\text{Ann}(D)} = 0$. A curve fulfilling the latter requirement, is often referred to as a characteristic of $D$. We will use the term an abnormal curve

1Some authors prefer to work with sub-Riemannian geometry, having horizontal curves that have $L^\infty$ derivatives.
\[ \gamma(t) \] if the curve is the projection to \( M \) of an abnormal extremal \( \lambda(t) \). Abnormal curves are singular points of the map from the Hilbert manifold \( AC_D(m_0) \) of curves in \( AC_D \) starting at \( m_0 \) to the manifold \( M \) given by
\[
\text{end} : AC_D(m_0) \to M, \quad \text{end}(\gamma) = \gamma(1).
\]

For more details on sub-Riemannian manifolds, see [14].

Let \( \pi : Q \to M \) be a submersion, with vertical bundle \( \mathcal{V} \) and a chosen Ehresmann connection \( \mathcal{H} \). We lift the sub-Riemannian structure in the same way that we do with a more general optimal control problems. The lifted structure can be described as the sub-Riemannian manifold \((Q, \tilde{D}, \tilde{g})\), where
\[
\tilde{D} = \{ h_q v : v \in D_{m}, Q \in Q_{m}, m \in M \} = (\pi_*)^{-1}(D) \cap \mathcal{H},
\]
and \( \tilde{g} = \pi^* g \). We will say that this sub-Riemannian structure on \( Q \) is lifted from \( M \). We present the following corollaries of Theorem 1.

**Corollary 2.** A curve \( \lambda(t) \) in \( T^*M \) is a normal extremal if and only if it is a projection of a normal extremal in \( T^*Q \) contained in Ann(\( \mathcal{V} \)). Conversely, projections of normal extremals in \( T^*Q \) satisfy equation (1) with \( H = H_{SR} \).

**Remark 3.** In the special case when \( D = TM \), making the base space a Riemannian manifold, projections of normal geodesics are given by Corollary 1. The top space is then a sub-Riemannian manifold \((Q, H, \pi^* g)\). Also, \( H \)-horizontal lifts of Riemannian geodesics are normal geodesics. The latter fact was first observed in [12, Th. 6.2, Cor 6.5].

**Theorem 3.** A curve \( \gamma \) in \( M \) is abnormal if and only if any horizontal lift of any sufficiently short segment of the curve is abnormal.

Conversely, an \( L^2 \)-curve \( \tilde{\lambda}(t) \) in Ann(\( D \)) is an abnormal extremal if and only if
\[
\Pi T^*Q(\tilde{\lambda}(t)) = \tilde{\gamma}(t), \quad \pi^2(\tilde{\lambda}(t)) = \lambda(t),
\]
\[
\Pi T^*M(\lambda(t)) = \gamma(t), \quad \beta(t) = \pi^* \nabla_{\tilde{\gamma}}(t),
\]
satisfy
- \( \tilde{\gamma}(t) \) is \( \tilde{D} \)-horizontal (and hence \( \gamma \) is \( D \)-horizontal).
- \( \nabla_{\tilde{\gamma}} \beta = 0 \),
- \( \nabla_{\tilde{\gamma}}(\tilde{\lambda}, \bullet)|_{\text{Ann}(D)} = -\beta(t) \mathcal{R}(\dot{\gamma}, \Pi T^*M \bullet) \),

and \( \lambda \) and \( \beta \) does not vanish simultaneously.

The proof of this theorem uses elements of the proof of Theorem 1 and is therefore left to Section 5.2.

If we consider the special case when \( D = TM \), then Ann(\( D \)) contains just the zero-section. Hence the following is obvious.

**Corollary 3.** Let \( \pi : Q \to M \) be a submersion, let \( \mathcal{H} \) be an Ehresmann connection on \( \pi \). Consider the sub-Riemannian manifold \((Q, \mathcal{H}, \tilde{g})\) for some metric tensor \( \tilde{g} \) on \( \mathcal{H} \). Then a curve \( \tilde{\gamma}(t) \) in \( Q \) is an abnormal curve if and only if there is a non-zero curve \( \beta(t) \in \mathcal{V}_{\tilde{\gamma}(t)} \) such that
\[
\nabla_{\tilde{\gamma}} \beta = 0, \quad \beta(\mathcal{R}(\dot{\gamma}, \bullet)) = 0, \quad \text{where } \pi(\tilde{\gamma}(t)) = \gamma(t).
\]
3.4. Special case: Charged particles in a magnetic field or a Yang-Mills field. Consider the special case when the submersion \( \pi : Q \to M \) is a principal \( G \)-bundle, where \( G \) acts by a right action. Corresponding to this action, we have vector fields \( \sigma(A) \) defined such that for any \( A \) in the Lie algebra \( g \) of \( G \),

\[
\sigma(A)|_q = \left. \frac{d}{dt} \right|_{t=0} q \cdot \exp_G(tA).
\]

For every \( q \), the map \( g \mapsto V|_q \) defined by \( A \mapsto \sigma(A)|_q \) is a linear isomorphism.

Let \( \mathcal{H} \) be an Ehresmann connection which is invariant under the group action, that is, \( \mathcal{H}_p \cdot a = \mathcal{H}_p \cdot a \) for any \( a \in G \). Such an Ehresmann connection is called principal, and corresponding to it is a principal connection form \( \omega \), which is a \( g \)-valued one-form, defined by having \( \mathcal{H} \) as its kernel and satisfying \( \omega(\sigma(A)|_q) = A \). It is simple to verify that if \( \Upsilon \) is a vertical vector field along an \( \mathcal{H} \)-horizontal curve \( \tilde{\gamma} \) in \( Q \), and \( \pi(\tilde{\gamma}) = \gamma \), then

\[
\nabla_{\dot{\gamma}} \Upsilon = 0, \quad \text{if and only if} \quad \Upsilon(t) = \sigma(A)|_{\tilde{\gamma}(t)} \quad \text{for some} \quad A \in g.
\]

Introduce the curvature two-form \( \Omega \) by \( \Omega(\tilde{X}, \tilde{Y}) = d\omega(\tilde{X}, \tilde{Y}) + \omega([\tilde{X}, \tilde{Y}]) = -\omega(R(\tilde{X}, \tilde{Y})) \). Then we can rewrite (1) as

\[
\dot{\lambda} = \tilde{H}|_\lambda - \nabla_\lambda L\Omega(\dot{\gamma}, \cdot),
\]

for some constant \( L \in g^* \). If \( M \) is a Riemannian manifold, and \( H \) is the corresponding Hamiltonian, then we can write Corollary (11) as

\[
\nabla_{\dot{\gamma}} \dot{\gamma} = -\sharp L\Omega(\dot{\gamma}, \cdot).
\]

These are the equations for a particle with gauge \( L \) under the influence of the Yang-Mills field \( \Omega \). If \( G = U(1) \) or \( \mathbb{R} \), then \( L \) represents the charge and \( \Omega \) is a magnetic field. For more details, see [13] and [14, Chapter 12].

4. Optimal control of rolling manifolds

4.1. Definition of manifolds rolling without twisting or slipping. In this section, unless otherwise stated, \( M \) and \( \widehat{M} \) will denote connected oriented Riemannian manifolds, both of dimension \( n \). The Riemannian metrics of the respective manifolds will be denoted by \( g \) and \( \widehat{g} \). To avoid trivial considerations, we will always assume that \( n \geq 2 \).

We consider the kinematic system of \( M \) rolling on \( \widehat{M} \). The configuration space for this motion can be described as follows. Define the fiber bundle \( Q \) over \( M \times \widehat{M} \) as

\[
Q = \left\{ SO(T_m M, T_{\widehat{m}} \widehat{M}) : m \in M, \widehat{m} \in \widehat{M} \right\}.
\]

Here, \( SO(V, \widehat{V}) \) denotes the space of all linear, orientation preserving isometries between two oriented inner product spaces \( V \) and \( \widehat{V} \). Clearly each fiber is diffeomorphic to \( SO(n) \). An element \( q : T_m M \to T_{\widehat{m}} \widehat{M} \) in \( Q \) represents a configuration where \( M \) lie tangent to \( \widehat{M} \) at the points \( m \) and \( \widehat{m} \). Two
vectors, one in each tangent space, lie adjacent if one is mapped to the other by \( q \). A rolling is then a curve in this configuration space.

We will assume that we have high friction, giving us the constraints that we cannot slip or twist. By slipping, we mean moving one of the manifolds while remaining connected in the same point on the other. By twisting, we mean spinning in place, that is, we move in the same fiber over \( M \times \tilde{M} \), changing configuration, but not connecting points. Mathematically, we can write these constraints as follows.

Let \( \pi : Q \to M \) and \( \tilde{\pi} : Q \to \tilde{M} \) be the respective natural projections.

**Definition 3.** An absolutely continuous curve \( q(t) \) in \( Q \), with \( \gamma(t) = \pi(q(t)) \) and \( \tilde{\gamma}(t) = \tilde{\pi}(q(t)) \), is a rolling without slipping or twisting if it satisfies the following conditions for almost every \( t \),

- (No slipping condition:) \( q(t)\dot{\gamma}(t) = \dot{\tilde{\gamma}}(t) \),
- (No twisting condition:) Any vector field \( X(t) \) along \( \gamma(t) \) is parallel if and only if \( q(t)X(t) \) is parallel along \( \tilde{\gamma}(t) \).

In what follows, we will typically drop the phrase “without twisting or slipping” and just refer to a rolling, with the constraints being implicit. For more details on these types of systems, see e.g. [15, 6, 5].

Rolling without slipping or twisting can be described as horizontal curve relative to a subbundle \( D \) of \( TQ \) of rank \( n \). We describe this subbundle locally. On a sufficiently small neighborhood \( U \) on \( M \), choose an orthonormal basis \( e_1, \ldots, e_n \) of vector fields on \( U \). Define \( \tilde{U} \) and \( \tilde{e}_1, \ldots, \tilde{e}_n \) similarly on \( \tilde{M} \). We can then trivialize the \( Q \) over \( U \times \tilde{U} \) by

\[
Q|_{U \times \tilde{U}} \to U \times \tilde{U} \times SO(n)
\]

\( q \in SO(T_M M, T_{\tilde{M}} \tilde{M}) \mapsto (m, \tilde{m}, (q)_{ij}) \),

\( q_{ij} = \tilde{g} (\tilde{e}_i|_{\tilde{m}}, q(t)e_j|m) \).

Relative to this trivializations, define vector fields

\[
W_{\alpha\beta} = \sum_{s=1}^{n} (q_{sa} \partial_{q_{\alpha\beta}} - q_{sb} \partial_{q_{\alpha\beta}}).
\]

Then \( D \) on \( Q|_{U \times \tilde{U}} \) is spanned by vector fields

\[
\tilde{e}_j = e_j + qe_j + \sum_{1 \leq \alpha < \beta \leq n} (g(e_{\alpha}, \nabla_{e_j} e_{\beta}) - \tilde{g}(qe_{\alpha}, \nabla_{e_j} qe_{\beta})) W_{\alpha\beta}.
\]

Here, \( qe_j \) denotes the vector field \( q \mapsto qe_j|_{\pi(q)} \).

Notice that \( D \) is an Ehresmann connection for both \( \pi : Q \to M \) and \( \tilde{\pi} : Q \to \tilde{M} \). We can lift the Riemannian metric \( g \) on \( M \) to a metric \( h \) on \( D \) by

\[
h(v_1, v_2) = g(\pi_* v_1, \pi_* v_1), \quad v_1, v_2 \in D.
\]

In this metric, the vector fields in (7) form a local orthonormal basis. Notice that from the definition of \( D \), the metric \( h \) also satisfies \( h(v_1, v_2) = \tilde{g}(\tilde{\pi}_* v_1, \tilde{\pi}_* v_2) \). Hence, the sub-Riemannian structure \((D, h)\) can be considered lifted from both \( M \) and \( \tilde{M} \).
To find a sub-Riemannian length minimizer on $Q$ relative to $h$ with initial point $q_0$ and final point $q_1$, is to find the rolling $q(t)$ from $q_0$ to $q_1$ such that $\gamma(t) = \pi(q(t))$ (or equivalently $\hat{\gamma}(t) = \hat{\pi}(q(t))$) have minimal length. This problem can be quite complicated to attack, and few results exists. The first results was given for a sphere rolling on a plane $[10]$. It was shown that in this case, projection of geodesics to either the sphere or the plane, are curves with constant speed whose direction is a solution of the pendulum equation. Later results for more general manifolds of constant curvature are found in $[11]$. In addition, it was shown in $[7]$ that if the Riemannian curvatures $R$ satisfy the condition that the map

$$v_1 \wedge v_2 \mapsto R(\pi_* v_1, \pi_* v_2, \pi_* \cdot, \pi_* \cdot) - \widehat{R}(\widehat{\pi}_* v_1, \widehat{\pi}_* v_2, \widehat{\pi}_* \cdot, \widehat{\pi}_* \cdot), \quad v_1, v_2 \in D,$$

from $\Lambda^2 D$ to $\Lambda^2 D^*$ is invertible, then we both have that $D$ is bracket generating and there are no abnormal minimizers which are not also normal minimizers. Hence, in this case, we can focus on normal geodesics.

We would like to describe the geodesics in the general case. Unfortunately, computations in $Q$ defined as in $[9]$ has proved to be quite difficult. We therefore use Theorem $[11](a)$ to lift our problem to a space where computations can be done more easily. We start with some definitions.

### 4.2. Frame bundles.

We make the convention that whenever we mention $\mathbb{R}^n$, it will always come furnished with the standard orientation and the Euclidean inner product. We will write $GL(V, \hat{V})$ for the space of all linear isomorphisms of vector spaces $V$ to $\hat{V}$. A frame at the point $m \in M$, is a map $f \in GL(\mathbb{R}^n, T_m M)$. This can be considered as a basis $\{f_0, \ldots, f_n\}$ of $T_m M$, by identifying $f$ with the vectors

$$f_j := f(0, \ldots, 0, 1, 0, \ldots 0).$$

Conversely, any choice of basis determines a map $f$. If we denote by $\mathcal{F}_m(M) = GL(\mathbb{R}^n, T_m M)$, we can define a frame bundle $\pi : \mathcal{F}(M) \to M$ as the principal $GL(n)$-bundle with fibers $\mathcal{F}_m(M)$.

For a given affine connection $\nabla$ on $M$, we define a corresponding Ehresmann $\mathcal{E}$ on $\pi$ consisting of all tangent vectors of curves $f(t)$ such that each $f_1(t), \ldots, f_n(t)$ is parallel vector along $\pi f(t)$. This is sub-bundle is invariant under the right action of $GL(n)$.

For an oriented Riemannian manifold, define the oriented orthonormal frame bundle $\tau : F(M) \to M$ as the principle $SO(n)$-bundle, such that the fiber over $m$ is $F_m(M) := SO(\mathbb{R}^n, T_m M)$, that can be identified with the space of all positively orthonormal frames of $T_m M$.

Let us consider the case where $\nabla$ as the Levi-Civita connection on $M$. Since parallel translation along a curve preserves orthogonality, we can induce a principal Ehresmann connection $\mathcal{E}$ on $\tau : F(M) \to M$ instead of $\mathcal{F}(M)$. This Ehresmann connection again corresponds to an $so(n)$-valued principal connection one-form $\omega = (\omega_{\alpha\beta})$ on $F(M)$ defined as in Section 5.3.
Let $♭ : TM \to T^*M$ be the vector bundle isomorphism $v \mapsto \mathbf{g}(v, \cdot)$ and write $:\hat{\mathbf{g}} = \mathbf{g}^{-1}$. Define an $\mathbb{R}^n$ valued one-form $\theta = (\theta_i)$ on $F(M)$, by

$$\theta_i|_f = \tau^* \mathbf{g}(v, \cdot)$$

The differential of $\theta$ and $\omega$ are connected through the Cartan equations,

$$d\theta_i = -\sum_{k=1}^{n} \omega_{ik} \wedge \theta_k,$$

$$d\omega_{\alpha\beta} = -\sum_{k=1}^{n} \omega_{\alpha k} \wedge \omega_{k\beta} + \Omega_{\alpha\beta}.$$  

where $\Omega = (\Omega_{\alpha\beta})$ is the $\mathfrak{so}(n)$-valued curvature two-form. In this case, this is connected to the Riemannian curvature by

$$\Omega(f)(X, Y) = (\mathbf{g}(R^{\nabla}(X, Y)f_{\alpha}, f_{\alpha}))_{\alpha, \beta}.$$  

We can rewrite the Cartan equations in terms of vector fields rather than forms. For any $A = (A_{\alpha\beta}) \in \mathfrak{so}(n)$, let $\sigma(A)$ be defined as in (5). These satisfy $\theta_i(\sigma(A)) = 0$ and $\omega_{\alpha\beta}(\sigma(A)) = A_{\alpha\beta}$. We also have the vector fields $X_i$ with values in $\mathcal{E}$, defined by $X_i|_f = h_ff_i$, where the horizontal lift is with respect to $\mathcal{E}$. These satisfy $\theta_i(X_j) = \delta_i,j$ and $\omega_{\alpha\beta}(X_j) = 0$. Using this information along with the Cartan equations, we obtain

$$[X_i, X_j] = -\sigma(\Omega(X_i, X_j)).$$  

$$[X_j, \sigma(A)] = -\sum_{s=1}^{n} A_{sj} X_s.$$  

$$[\sigma(A), \sigma(B)] = \sigma([A, B]).$$

4.3. Lifting of the rolling problem. To simplify our computations, we want to lift our problem to $F(M) \times F(\hat{M})$. Define a principal $\text{SO}(n)$-bundle by

$$p : F(M) \times F(\hat{M}) \to \hat{Q}.$$

We will use the following result.

**Lemma 2.**[[7, Cor 1]] Let $q(t)$ be a curve in $Q$ which projects to the curve $(\gamma(t), \hat{\gamma}(t))$ in $M \times \hat{M}$. Then $q(t)$ is a rolling without twisting or slipping if and only if it is a projection of a curve $(f(t), \hat{f}(t))$ in $\hat{Q}$ such that

(a) $f_1(t), \ldots, f_n(t)$ are parallel along $\gamma(t),$
(b) $\hat{f}_1(t), \ldots, \hat{f}_n(t)$ are parallel along $\hat{\gamma}(t),$
(c) $f(t)^{-1}(\dot{\gamma}(t)) = \hat{f}(t)^{-1}(\dot{\hat{\gamma}}(t)),$
Remark 4.\hspace{1cm} So far, we have considered the curvature form when $\mathit{M}$ is a sub-Riemannian manifold. It appears in the geodesic equations in Theorem 4. However, its appearance is just the result of our choice of inner product in (8) to give $\mathit{S}^n$ of convenient coordinates. Let $\mathbb{H}$ be an Ehresmann connection. Theorem 4.4. Normal geodesics and optimal solutions. We will use Theorem \textbf{b) to describe the normal geodesics. In order to present our results, let us introduce the $\mathfrak{so}(n)$-valued two-form $\Omega^{S^n}$ on $\mathit{F}(\mathit{M})$ defined by

$$\Omega^{S^n}(\sigma(A), \cdot) = 0, \quad \Omega^{S^n}(\mathit{X}_a, \mathit{X}_s) = - (\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj})_{i,j}.$$ 

This notation is inspired by the fact that this form coincides with the curvature form when $\mathit{M} = S^n$. We will also identify $\mathfrak{so}(n)$ with $\mathfrak{so}(n)^*$ through the inner product

$$\langle A, B \rangle = \frac{1}{2} \text{tr}(A^\top B).$$

With slight abuse of notation, we also will use $\flat$ for this identification $\flat : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$, with inverse $\sharp$.

\textbf{Theorem 4.} Let $q(t)$ be a horizontal curve in $(\mathit{Q}, \mathit{D}, \mathit{h})$ with a projection $\gamma(t)$ in $\mathit{M}$. Then $q(t)$ is a normal geodesic if and only if there is vector fields $\mathit{V}(t)$ along $\gamma(t)$ and a curve $\mathit{L}(t)$ in $\mathfrak{so}(n)^*$ satisfying

$$\nabla_{\gamma} \dot{\gamma} = -\sharp L \left(\Omega(\gamma, \cdot) - \tilde{\Omega}(\dot{\gamma}, q \cdot)\right)$$

$$\sharp \dot{L} = -\Omega^{S^n}(\gamma, \mathit{V}), \quad \nabla_{\gamma} \mathit{V} = -\sharp L \tilde{\Omega}(\dot{\gamma}, q \cdot).$$

\textbf{Remark 4.} At first glance, it may seem striking that the curvature tensor of $S^n$ appears in the geodesic equations in Theorem 4. However, its appearance is just the result of our choice of inner product in (8) to give $\mathfrak{so}(n)^*$ convenient coordinates. There is also an apparent asymmetry between $\mathit{M}$ and $\tilde{\mathit{M}}$, since $\nabla_{\gamma} \mathit{V}$ only depends one the curvature of $\tilde{\mathit{M}}$, however, this is a bit of an illusion. Define
\( \tilde{V}(t) = V(t) + \dot{\gamma} \). Then it is still true that \( \sharp \dot{L} = -\Omega^{S^n}(\dot{\gamma}, \tilde{V}) \), but now we have \( \nabla_{\dot{\gamma}} \tilde{V} = -\sharp L \Omega(\dot{\gamma}, \cdot) \).

**Proof.** Let \((f(t), \tilde{f}(t))\) be some \(\tilde{D}\)-horizontal lift of \(\gamma(t)\) and let \(q(t) = \tilde{f}(t) \circ f(t)^{-1} \). We will use Corollary \(\text{II} \) to find the geodesics, recalling that \(\tilde{D}\) is an Ehresmann connection on \(F(M) \times F(\hat{M}) \to M \).

We start by finding the curvature \(\hat{R}\) of \(\tilde{D}\). Since elements \(X_i + \hat{X}_i\) form a basis, the curvature is \(\hat{R}(X_i + \hat{X}_i, X_j + \hat{X}_j) = -\sigma(\Omega(X_i, X_j)) - \hat{\sigma}(\Omega(X_i, X_j)) \).

Next, we determine the equations for parallel transport. Write \(\dot{\gamma}(t) = \sum_{i=1}^{n} \gamma_i(t) e_i \) for some local orthonormal basis \(e_1, \ldots, e_n \). Introduce local coordinates \(f_j = \sum_{i=1}^{n} f_{ij} e_i \). Using these coordinates

\[
\sigma(A) = \sum_{i,j,s=1}^{n} f_{is} A_{sj} \partial f_{ij},
\]

Since the \(\tilde{D}\)-horizontal lift of \(e_i\) is \(he_i = \sum_{s=1}^{n} f_{is} (X_s + \hat{X}_s)\), we have

\[
\nabla_{e_i} \hat{X}_j = -\hat{\sigma}(\hat{\Omega}(he_i, \hat{X}_j)).
\]

\[
\nabla_{e_i} \sigma(A) = -\nabla_{e_i} \hat{\sigma}(A) = \sum_{r,s=1}^{n} f_{ir} A_{sr} \hat{X}_s.
\]

If \(\beta(t)\) is a curve in \(\text{Ann}(\mathcal{V}) \cap \text{Ann}(\tilde{D})\), then it is on the form \(\sum_{j=1}^{n} v_j(t)(\theta_j - \hat{\theta}_j) + \sum_{i,j=1}^{n} L_{ij}(t)(\omega_{ij} - \hat{\omega}_{ij})\) where \(\sharp L = (L_{ij})\) is a curve in \(\mathfrak{so}(n)\). In order for \(\beta\) to be parallel, it must satisfy

\[
\langle \dot{\gamma}, A \rangle = \partial_t \beta(\hat{X}_j) = -\beta(\nabla_{\dot{\gamma}} \hat{X}_j) = -L \hat{\Omega}(h \dot{\gamma}, \hat{X}_j),
\]

\[
\langle \dot{\gamma}, A \rangle = \partial_t \beta(\sigma(A)) = \beta(\nabla_{\dot{\gamma}} \sigma(A)) = -\sum_{i,r,s=1}^{n} \gamma_i f_{ir} A_{sr} v_s
\]

\[
= -\left( \sum_{i=1}^{n} \left( \gamma_i f_{ir} v_s - \gamma_i f_{is} v_r \right) , A \right) = -\langle \Omega^{S^n}(\dot{\gamma}, V), A \rangle,
\]

where \(V\) is the vector field \(V(t) = \sum_{i=1}^{n} v_i(t) f_i(t)\).

We insert these results into formula Corollary \(\text{II} \) and we find

\[
\nabla_{\dot{\gamma}} \dot{\gamma} = \sharp \beta(t) \mathcal{R}(\dot{\gamma}, \cdot)
\]

\[
= -\sharp \left( \sum_{j=1}^{n} v_j(t)(\theta_j - \hat{\theta}_j) + \frac{1}{2} \sum_{i,j=1}^{n} L_{ij}(t)(\omega_{ij} - \hat{\omega}_{ij}) \right) \left( \sigma(\Omega(\dot{\gamma}, \cdot)) + \hat{\sigma}(\hat{\Omega}(q \dot{\gamma}, \cdot)) \right)
\]

\[
= -\sharp L \left( \Omega(\dot{\gamma}, \cdot) - \hat{\Omega}(q \dot{\gamma}, \cdot) \right)
\]

where \(\sharp \dot{L} = -\Omega^{S^n}(\dot{\gamma}, V)\) and \(\nabla_{\dot{\gamma}} V = -\sharp L \hat{\Omega}(q \dot{\gamma}, q \cdot)\). \(\square\)
4.5. The two dimensional case. Let us consider the case when the Riemannian manifolds $M$ and $\hat{M}$ are two-dimensional. Let $f_1(t)$ and $f_2(t)$ be a basis of orthogonal parallel vector fields along a curve $\gamma(t)$ in $M$. Write

$$\dot{\gamma} = a(\cos \theta f_1 + \sin \theta f_2).$$

(9)

Use respectively $\kappa$ and $\hat{\kappa}$ for the Gaussian curvatures of $M$ and $\hat{M}$. We will make the addition assumption that for any pair of points $(m, \hat{m}) \in M \times \hat{M}$ we have that $\kappa|_m - \hat{\kappa}|_{\hat{m}}$ never vanishes. The rolling distribution is bracket generating if and only if this condition holds \cite{4,1}. Under this assumption, we have the following result.

**Proposition 2.** Let $q(t)$ be an absolutely continuous curve in $Q$, with projection $\gamma(t)$. Write $\kappa(t) = \kappa|_{\gamma(t)}$ and $\hat{\kappa}(t) = \hat{\kappa}|_{q(t)\gamma(t)}$, and introduce the notation $\rho(t) = \frac{1}{\kappa(t) - \hat{\kappa}(t)}$. Represent $\gamma$ as in (9). Then $q(t)$ is a normal geodesic if and only if $a$ is constant, and $\theta$ is a solution to the equation

$$\ddot{\theta} + \frac{\dot{\rho}}{\rho} \dot{\theta} = \frac{A}{\rho} \sin(\theta - \phi_0) + \frac{a^2}{\rho} F,$$

(10)

where $F(t) = \int_0^t \rho(s)^2 \sin(\theta(t) - \theta(s))(\kappa(s)\dot{\kappa}(s) - \hat{\kappa}(s)\dot{\hat{\kappa}}(s))ds$ and $A$ and $\phi_0$ are arbitrary constants.

**Remark 5.** If $\kappa(t)\dot{\hat{\kappa}}(t) = \dot{\kappa}(t)\dot{\hat{\kappa}}(t)$, i.e. if one of the Gaussian curvatures is a constant times the other, the geodesic equations reduces to the equation a pendulum. More specifically, when $F(t) = 0$, (10) is the equation for the pendulum whose length varies by $\rho(t)$ under the influence of the force of gravity with magnitude $-\kappa$ and direction $\phi_0$ and a driving force of magnitude $\dot{\rho}\dot{\theta}$ resulting from the changing length. See Section 5.3 for details.

In particular, this happens if both of the manifolds have constant Gaussian curvature or if one of the manifolds is flat.

**Proof.** It is simple to verify from Theorem 4 that $|\dot{\gamma}|$ is always a first integral. In the notation of the same theorem, let $V = b_1(\cos \theta f_1 + \sin \theta f_2) + b_2(-\sin \theta f_1 + \cos \theta f_2)$. Also write $\Omega L$ and the curvature forms $\Omega$ and $\hat{\Omega}$ as

$$\left( \begin{array}{cc} 0 & L \\ -L & 0 \end{array} \right), \quad \left( \begin{array}{cc} 0 & -\kappa \\ \kappa & 0 \end{array} \right), \quad \text{and} \quad \left( \begin{array}{cc} 0 & -\hat{\kappa} \\ \hat{\kappa} & 0 \end{array} \right).$$

We can rewrite equations in Theorem 4 in these variables.

$$\dot{\theta} = L(\kappa - \hat{\kappa}),$$

$$\dot{L} = ab_2, \quad \dot{b}_1 = \theta b_2, \quad \dot{b}_2 = aL\kappa - \dot{\theta}b_1.$$  

This can be reduced to three equations by the identity $L = \frac{\dot{\theta}}{\kappa - \hat{\kappa}} = \rho \dot{\theta}$, which give us the formulation

$$b_1 = \dot{\theta} b_2 \quad \dot{b}_2 = \dot{\theta} \left( a\frac{\kappa}{\kappa - \hat{\kappa}} - b_1 \right), \quad \rho \ddot{\theta} + \dot{\rho} \theta = ab_2.$$  

(11)
Proof. Let \(\Omega = B \operatorname{vol} \), where \(\operatorname{vol} \) is the Riemannian volume form induced by the metric and orientation on \(M\). \(\Omega\) represents a magnetic field with magnitude \(B\), always normal to the manifold.

Let \(\gamma(t)\) be the trajectory of a charged object moving in \(M\). The initial charge is \(L_0\), however this charge changes according to a parallel vector field \(V\). Let \(X\) be a vector field on \(M\) extending \(V\) locally. Then the charge at time \(t\) is given by \(L_0 + \int_{\gamma|_{[0,t]}} \ast \phi X\), where \(\ast\) is the Hodge star operator. In other words, according to the positive orientation, movement in a direction rotated 90° from the direction of \(X\) will increase the charge, while movement in the opposite direction decreases the charge.

For some constants \(A\) and \(\phi_0\).

4.6. Rolling on \(\mathbb{R}^n\). Assume that \(\hat{M} = \mathbb{R}^n\) with the Euclidean metric and standard orientation. Then we can identify \(Q\) with \(\mathbb{R}^n \times F(M)\), by associating \(q : T_m M \to T_y \mathbb{R}^n\) with the pair \((y, q^{-1})\), where we have used the canonical identification of \(T_y \mathbb{R}^n\) with \(\mathbb{R}^n\). We then have the following reformulation of Theorem 4.

Corollary 4. Let \(q(t)\) be a rolling of \(M\) on \(\mathbb{R}^n\).

(a) Let \(\gamma(t)\) be its projection to \(M\). Then \(q(t)\) is a normal geodesic if and only if
\[
\nabla_{\dot{\gamma}} \dot{\gamma} = -\sharp L_0 \Omega(\dot{\gamma}, \cdot) + \sharp R(V, W, \dot{\gamma}, \bullet),
\]
where \(\nabla_{\dot{\gamma}} V = 0\) and \(\nabla_{\dot{\gamma}} W = \dot{\gamma}\), and \(L_0\) is a constant element in \(\mathfrak{so}(n)^*\).

(b) Identify \(q(t)\) with the curve \((y(t), f(t))\) in \(\mathbb{R}^n \times F(M)\). Then this is a normal geodesic if and only if
\[
\ddot{y} = -\sharp L_0 \Omega(f(t) \dot{y}(t), \bullet) + \sharp R(f(t) v, f(t)(y(t) - y(0)), f(t) \dot{y}(t), \bullet),
\]
for some \(v \in \mathbb{R}^n\) and \(L_0 \in \mathfrak{so}(n)^*\).

Proof. Here we have used that \(V\) becomes parallel when \(\hat{\Omega} = 0\). Also, since \(V\), now is parallel, if \(W\) is any other vector field along \(\gamma\), then \(\partial_t \Omega^{\mathfrak{so}}(W, V) = \Omega^{\mathfrak{so}}(\nabla_{\dot{\gamma}} W, V)\). The result then follows.
5. Proofs

5.1. Proof of Theorem 1 and necessary theory. Before we are ready to present our proof, which will come in Section 5.1.3, we need to introduce some concepts, such as lifted vertical and horizontal bundles and the $\pi^2$-symplectic lift of vectors. These concepts and some necessary lemmas are described in Sections 5.1.1 and 5.1.2 with a representation in local coordinates included in Section 5.1.4.

5.1.1. Lifted horizontal and vertical bundle. Let $\pi^2 : T^*Q \to T^*M$ be the submersion defined in Section 2.4, relative to some Ehresmann connection $\mathcal{H}$ on $\pi : Q \to M$. From the first mentioned submersion, we have vertical bundle $V^2 = \ker \pi^2_*$. We will show that this subbundle of $TT^*Q$ is symplectic with respect to the canonical symplectic form $\tilde{\omega}$ on $T^*Q$. By symplectic, we mean that the restriction of $\tilde{\omega}$ to $V^2$ is non-degenerate. We can then use this to define an Ehresmann connection $\mathcal{H}^2$ on $\pi^2$ as its symplectic complement.

Elements in the cotangent bundles of $Q$ and $M$ are denoted by respectively $\tilde{p}$ and $p$. We will use $\vartheta$ for the Louiville one-form on $T^*M$ defined by $\vartheta(w) = p(\Pi^{T^*M}_*w), w \in T_p T^*M$ and $\varsigma = -d\vartheta$ for the canonical symplectic form. Use $I$ for the bundle morphism $I : TT^*M \to T^*T^*M, I(w) = \varsigma(w, \bullet)$. Define $\tilde{\vartheta}, \tilde{\varsigma}$ and $\tilde{I}$ similarly on $Q$. Relative these structures, we will introduce a notation for what we will call the $\pi^2$-symplectic lift. For any $w \in T_p T^*M$ and $\tilde{p} \in (\pi^2)^{-1}(p)$, we will use $\ell_{\tilde{p}} w$ for the element

$$\ell_{\tilde{p}} w = \tilde{I}^{-1}\pi^2|_{\tilde{p}} I(w).$$

Here, by abusing notation, we have used $\pi^2|_{\tilde{p}}$ for the map $T_{\tilde{p}} T^*M \to T_{\tilde{p}} T^*Q, \alpha_0 \mapsto \alpha_0(\pi^2|_{\tilde{p}} \bullet)$. In other words, $\ell_{\tilde{p}} w$ is the unique element in $T_{\tilde{p}} T^*Q$ satisfying

$$\tilde{\varsigma}(\ell_{\tilde{p}} w, \tilde{u}) = \varsigma(w, \pi^2_* \tilde{u}), \quad \text{for any } \tilde{u} \in T_{\tilde{p}} T^*Q.$$

We use this notation in the following lemma.

**Lemma 3.** Let $\mathcal{H}^2$ be the symplectic complement of $V^2$, that is $\mathcal{H}^2 = \tilde{I}^{-1} \Ann(V^2)$.

(a) For any $\tilde{p} \in T^*Q$, the fiber of $\mathcal{H}^2$ is given by

$$\mathcal{H}^2_{\tilde{p}} = \{ \ell_{\tilde{p}} w : w \in T_{\pi^2(\tilde{p})} T^*M \}.$$

(b) For any $\tilde{p} \in T^*Q$ and $w \in T_{\pi^2(\tilde{p})} T^*M$, we have

$$\pi_* \Pi^{T^*Q}_* \ell_{\tilde{p}} w = \Pi^{T^*M}_* w.$$

(c) $V^2$ and $\mathcal{H}^2$ are symplectic, i.e. $\mathcal{H}^2 \oplus V^2 = TT^*Q$. Furthermore,

$$\Pi^{T^*Q}_* V^2 = \mathcal{V} \quad \text{and} \quad \Pi^{T^*Q}_* \mathcal{H}^2 = \mathcal{H}.$$

**Proof.** (a) This is an easy consequence of the fact that for any $\tilde{p} \in T^*Q$, the map $T_{\pi^2(\tilde{p})} T^*M \to \Ann(V^2)_{\tilde{p}}, \alpha_0 \mapsto \pi^2_* \alpha_0$ is a linear isomorphism.
Lemma 4. The following lemma describes the relation of these two notions.

\( h \)

5.1.2. Horizontal lifts with respect to symplectic lift and the horizontal lift. We will denote the latter by \( T \) have two different ways of lifting elements in \( T^*M \), by definition \( \pi^2 \), \( v \). Hence, for any vector field \( W \) on \( T^*M \), we have

\[ \alpha(T^*M_w) = \psi(W, vl) = \psi(\ell W, vl) = \alpha(T^*Q_\ell W) \]

Here, the vector field \( \ell W \) on \( T^*Q \) is defined in the obvious way. Since \( \alpha \) and \( W \) where arbitrary, the result follows.

(c) Clearly, \( \Pi^2, \psi = \Pi^2, \psi \ker \pi^2 \subseteq \ker \pi = \psi \). This map has to be surjective since it is a right inverse to the injective map \( \theta = \Pi^2, \psi \): \( TQ \to TQ^2 \) which maps \( \psi \) into \( \psi^2 \).

To prove the analogous result for \( T^2 \), observe that for any \( \tilde{p} \in TQ^2 \) and \( \tilde{\alpha} \in \text{Ann}(\mathcal{H})_q \), we have \( vl \tilde{p}, \tilde{\alpha} \in \psi \). By definition \( -\tilde{\psi}(vl \tilde{p}, \tilde{\alpha}) = \tilde{\alpha}(\Pi^2, \psi \ell W) = 0 \) for any \( \tilde{w} \in \psi^2 \). Hence, \( \Pi^2, \psi \mathcal{H}^2 \subseteq \psi \) and this map is surjective by (b).

Finally, assume that \( \tilde{w} \in (\mathcal{H}^2, \psi^2)_q \). From our previous results, this intersection is contained in the kernel of \( \Pi^2, \psi \). Hence, \( \tilde{w} = \psi \tilde{p}, \tilde{\alpha} \in T^*Q \) which is in the same fiber as \( \tilde{p} \). However, since \( \tilde{\psi}(vl \tilde{p}, \tilde{\alpha}) \) must annihilate both \( \mathcal{H}^2 \) and \( \psi^2 \) and since \( \mathcal{H}^2 \cup \psi^2 \) is mapped surjectively onto \( TM \), the identity \( -\tilde{\psi}(vl \tilde{p}, \tilde{u}) = \tilde{\alpha}(\Pi^2, \psi \tilde{u}) \)

implied that \( \tilde{\alpha} = 0 \). This shows that \( \mathcal{H}^2 \) and \( \psi^2 \) are transversal.

\( \Box \)

5.1.2. Horizontal lifts with respect to \( \mathcal{H}^2 \). A consequence of Lemma 3 (c), is that \( \mathcal{H}^2 \) is an Ehresmann connection on \( \pi^2 : T^*Q \to T^*M \). Hence, we have two different ways of lifting elements in \( TTM \) to \( \mathcal{H}^2 \), namely the \( \pi^2 \)-symplectic lift and the horizontal lift. We will denote the latter by \( h^2, \psi \). The following lemma describes the relation of these two notions.

Lemma 4. If \( \tilde{p} \in T^*Q \), the following relations hold.

(a)

\[ \Pi^2, \psi h^2, \psi w = \Pi^2, \psi \ell \tilde{p}w = h_q(\Pi^2, \psi w) \]

(b)

\[ h^2, \psi w = \ell \tilde{p}w - vl \tilde{p} R(h_q \Pi^2, \psi w, \bullet). \]

Here, \( \tilde{p} R(h_q \Pi^2, \psi w, \bullet) \in T^*Q \) is the covector \( \tilde{v} \to \tilde{p} R(h_q \Pi^2, \psi w, v) \).

As a consequence

\[ \pi^2, \psi \ell \tilde{p}w = w + vl \tilde{p} R(h_q \Pi^2, \psi w, \bullet) \]

\( p = \pi^2(\tilde{p}) \).
Remark 6. Written as a commutative diagram, let \( \tilde{p} \) in \( T^*Q \) satisfy

\[
\begin{array}{c}
\pi^2(p) \\
\Pi^*Q \\
\downarrow \quad \downarrow \\
\Pi^*M \\
\pi(p) \\
\end{array}
\]

Lemma 4 then states that for any that for any \( w \in T_p \Pi^*M \) with \( v = \Pi^*M \cdot w \), we have the relations

\[
\begin{array}{c}
\Pi^*Q \quad \Pi^*M \\
\downarrow \quad \downarrow \\
h_{\Pi^*Q} \quad h_{\Pi^*M} \\
\end{array}
\]

Observe that \( v_1 \tilde{p} \) only depends on the projection of \( \tilde{p} \) to \( \text{Ann}(\mathcal{H}) \). Hence, it is simple to see that \( h^2_{\tilde{p}}w = \ell_{\tilde{p}}w \) whenever \( \tilde{p} \) is inn \( \text{Ann}(\mathcal{V}) \).

Proof of Lemma 4

(a) This is immediate from the definition of horizontal lift and Lemma 3 (b) and (c).

(b) Let us split the Liouville form on \( T^*Q \) into two parts \( \tilde{\vartheta} = \vartheta^H + \vartheta^V \)

where

\[
\vartheta^H(w) = \tilde{p}(pr_{\mathcal{H}} \Pi^*Q \tilde{w}), \quad \vartheta^V(w) = \tilde{p}(pr_{\mathcal{V}} \Pi^*Q \tilde{w}), \quad \tilde{w} \in T_{\tilde{p}}T^*Q.
\]

We observe that for any \( \tilde{w} \in T_{\tilde{p}}T^*Q \),

\[
\vartheta^H(\tilde{w}) = \tilde{p}(pr_{\mathcal{H}} \Pi^*Q \tilde{w}) = \pi^2(\tilde{p})(\pi_* \Pi^*Q \tilde{w})
\]

\[
= \pi^2(\tilde{p})(\Pi^*M \cdot \pi^2 \tilde{w}) = \pi^2(\vartheta)(\tilde{w})
\]

which in turn imply \( \tilde{\vartheta} = \pi^2 \vartheta - d\vartheta^V \).

Since \( \mathcal{H}^2 \) is symplectic, \( h^2_{\tilde{p}}w \) is completely determined by the values of \( \tilde{I}(h^2_{\tilde{p}}w) \) on vectors \( h^2_{\tilde{p}}u \), where \( u \in T_{\pi^2(\tilde{p})}TM \). Define \( R^2 \) as the curvature of \( \mathcal{H}^2 \). From (a), we know that \( \Pi^*Q \cdot R^2(h^2_{\tilde{p}}w, h^2_{\tilde{p}}u) = R(h_q \Pi^*M \cdot w, h_q \Pi^*M \cdot u) \). We calculate

\[
\overline{\vartheta}(h^2_{\tilde{p}}w, h^2_{\tilde{p}}u) = \vartheta(w, u) - d\vartheta^V(h^2_{\tilde{p}}w, h^2_{\tilde{p}}u)
\]

\[
= \vartheta(w, u) + \vartheta(R^2(h^2_{\tilde{p}}w, h^2_{\tilde{p}}u))
\]

\[
= \vartheta(w, u) + \tilde{p} R(h_q \Pi^*M \cdot w, h_q \Pi^*M \cdot u)
\]

\[
= \overline{\vartheta}(\ell_{\tilde{p}}w, h^2_{\tilde{p}}u) - \overline{\vartheta}(v_1 \tilde{p} \ R(h_q \Pi^*M \cdot w, \cdot), h^2_{\tilde{p}}u)
\]

which shows \( \Box \)

We will state one of the observations made in the proof of Lemma 4 as a separate result, since we will need it later.
Lemma 5. Let $\vartheta^V$ be the one-form $\vartheta^V|_p = \tilde{p}(p_{\vartheta^V} \Pi^*T^*Q, \bullet)$ on $T^*Q$. Then
$$\zeta = \pi^2 \zeta - d\vartheta^V.$$

Lemma 6. Let $\lambda(t)$ be a curve in $T^*M$ with $\Pi^*T^*M(\lambda(t)) = \gamma(t)$. Let $\tilde{\lambda}(t)$ be an $\mathcal{H}^2$-horizontal lift of $\lambda(t)$ with $\Pi^*T^*Q(\tilde{\lambda}(t)) = \tilde{\gamma}(t)$. Then $\tilde{\lambda}(t) = \lambda^H(t) + \lambda^V(t)$, where $\lambda^H$ is the curve in $\text{Ann}(\mathcal{V})$ determined by
$$\lambda^H(t)(\tilde{v}) = \lambda(t)(\pi_* \tilde{v}) \text{ for any } \tilde{v} \in T_{\tilde{\gamma}(t)}Q,$$
and $\lambda^V$ is a curve in $\text{Ann}(\mathcal{H})$ satisfying
$$\nabla_{\tilde{\gamma}(t)} \lambda^V(t) = 0.$$

Proof. First observe that since $\pi^2$ commutes with addition on respectively $T^*Q$ and $T^*M$, both $\mathcal{V}^2$ and $\mathcal{H}^2$ are preserved under addition. In particular, the sum of two $\mathcal{H}^2$-horizontal curves are again $\mathcal{H}^2$-horizontal.

Let $\lambda^H(t)$ be defined as in (13). We will show that this is an $\mathcal{H}^2$-horizontal lift. Clearly $\pi^2(\lambda^H(t)) = \lambda(t)$, so all we need to show is that $\lambda^H(t)$ is $\mathcal{H}^2$-horizontal. By using the fact that we, at least for short time can write $\lambda^H$ as $t \mapsto \pi^* \alpha|_{\tilde{\gamma}(t)}$ for some one-form $\alpha$ on $M$, it is simple to verify that $\zeta(\lambda^H, \bullet)$ vanishes on $\mathcal{V}^2$.

Using the conclusions from the previous two paragraphs, we obtain that
$$\lambda^V(t) = \tilde{\lambda}(t) - \lambda^H(t)$$
is an $\mathcal{H}^2$-horizontal lift of $0_{\gamma(t)}$. Let $\tilde{\alpha}$ be a section of $\text{Ann}(\mathcal{H})$, which, at least for short time, satisfy $\tilde{\alpha}|_{\gamma(t)} = \tilde{\lambda}(t)$. Let $\Upsilon$ be any section of $\mathcal{V}$, and let $\tilde{W}$ be a section of $\mathcal{V}^2$ which projects to $\Upsilon$. Inserting this into the symplectic form on $T^*Q$, we have
$$\zeta(\lambda^V, \tilde{W}) = \partial_t \alpha(\Upsilon)|_{\tilde{\gamma}(t)} - \tilde{W} \alpha(\tilde{\gamma}(t)) - \alpha(\nabla_{\tilde{\gamma}(t)} \Upsilon) = (\nabla_{\tilde{\gamma}(t)} \alpha)(\Upsilon) = 0.$$

Here we have used that $\tilde{\gamma}$ is $\mathcal{H}$-horizontal. \hfill $\square$

5.1.3. Proof of Theorem 4. Let $H$ be any smooth function on $T^*M$ and let $\tilde{H} = H \circ \pi^2$. By definition, it is clear that $\ell \tilde{H}$ is the Hamiltonian vector field of $\tilde{H}$.

To prove (a), assume that $\lambda(t)$ is a curve in $T^*M$ along $\gamma(t)$. Let $\tilde{\gamma}$ be a horizontal lift and let $\lambda^H$ be defined as in (13). By Lemma 4 (b), we know that $\ell \tilde{H}$ and $h^2 \tilde{H}$ coincides on $\text{Ann}(\mathcal{V})$. The result then follows from Lemma 6.

For the proof of (b), introduce the map $\Phi^V : T^*Q \rightarrow T^*Q$ defined by $\Phi^V(\tilde{p}) = p_{\vartheta^V} \tilde{p}$. Obviously
$$\Phi^V_{\*}(h^2 w - \ell \vartheta w) = 0,$$
from Lemma 4 (b), since $\tilde{p} \mathcal{R}(h_{\vartheta^V} \Pi^*T^*M, \bullet)$ vanishes on vertical vectors. As such, if $\tilde{\lambda}$ is an integral curve of $\tilde{H}$ with $\Phi^V(\tilde{\lambda}) = \beta(t)$ and $\pi^2(\tilde{\lambda}) = \lambda$, then clearly $\nabla_{\tilde{\gamma}} \beta = 0$ by Lemma 6 where $\tilde{\gamma}$ is the projection of $\lambda$ to $M$. We
again use Lemma 4 (b), for the conclusion

\[ \dot{\lambda} = \pi^2 \lambda H|_{\dot{\lambda}} = H|_{\dot{\lambda}} + v_l \lambda \mathcal{R}(\dot{\gamma}, \cdot) = H|_{\dot{\lambda}} + v_l \beta \mathcal{R}(\dot{\gamma}, \cdot) \]

Finally, we know that \( \tilde{\gamma} = \Pi^{T^* Q}(\tilde{\lambda}) \) is an \( \mathcal{H} \)-horizontal lift of \( \gamma = \Pi^{T^* M}(\lambda) \) from Lemma 4 (a).

5.1.4. Representation in local coordinates. Recall that \( M \) is \( n \)-dimensional, while \( Q \) has dimension \( n + \nu \). Indices running from 1 to \( n \) will be in latin letters \( i, j, k \), while indices running from 1 to \( \nu \), will be in greek letters \( \kappa, \lambda, \mu \).

Let \((x, U)\) be a local coordinate system on \( M \), and let \( \bar{U} \) be a neighborhood of \( \pi^{-1}(U) \) such that \( \mathcal{V} \) trivialize over this neighborhood. Let \( \Upsilon_1, \ldots, \Upsilon_\nu \) be vector fields on \( \bar{U} \) forming a basis for \( \mathcal{V} \). We give \( T^* \bar{U} \) coordinates \( p = \sum_{i=1}^{n} p_i dx_i \) and \( T^* \bar{U} \) coordinates \( \tilde{p} = \sum_{i=1}^{n} a_i \pi^* dx_i + \sum_{\kappa=1}^{\nu} b_\kappa \Upsilon^*_\kappa \). Here, \( \Upsilon^*_\kappa \) is defined by formula

\[ \Upsilon^*_\kappa(h \partial_{x_i}) = 0, \quad \Upsilon^*_\kappa(\Upsilon_\mu) = \delta_{\kappa \mu}. \]

Relative to these coordinates,

\[ \pi^2 : \sum_{i=1}^{n} a_i \pi^* dx_i|_q + \sum_{\kappa=1}^{\nu} b_\kappa \Upsilon^*_\kappa|_q \mapsto \sum_{i=1}^{n} a_i dx_i|_{\pi^*(m)}. \]

After using the mentioned coordinates to identify \( T^* \bar{U} \) with \( \bar{U} \times \mathbb{R}^{n+\nu} \), the bundle \( \mathcal{V}^2 \) is consequently spanned by \( \Upsilon_\kappa \) and \( \partial_{b_\kappa} \). Write

\[ \mathcal{R}^\kappa_{ij} = \Upsilon^*_\kappa([h \partial_{x_i}, h \partial_{x_j}]), \quad \Gamma^\kappa_{i\mu} = \Upsilon^*_\kappa([h \partial_{x_i}, \Upsilon_\mu]), \quad c^\kappa_{\mu} = \Upsilon^*_\kappa([\Upsilon_{\lambda}, \Upsilon_\mu]). \]

Then \( \zeta = \sum_{j=1}^{n} dx_i \wedge dp_i \), while

\[ \tilde{\zeta} = \sum_{i=1}^{n} \pi^* dx_i \wedge da_i + \sum_{\kappa=1}^{\nu} \Upsilon^*_\kappa \wedge db_\kappa + \sum_{i=1}^{n} \sum_{\kappa, \mu=1}^{\nu} b_\kappa \Gamma^\kappa_{i\mu} \pi^* dx_i \wedge \Upsilon^*_\mu \]

\[ + \sum_{\kappa=1}^{\nu} \frac{b_\kappa}{2} \left( \sum_{i, j=1}^{n} \mathcal{R}^\kappa_{ij} \pi^* dx_i \wedge \pi^* dx_j + \sum_{\lambda, \mu=1}^{\nu} c^\kappa_{\lambda \mu} \Upsilon^*_\lambda \wedge \Upsilon^*_\mu \right). \]

It is then simple to verify that \( \mathcal{H}^2 \) is spanned by

\[ h^2 \partial_{x_i} = h \partial_{x_i} + \sum_{\kappa, \mu=1}^{\nu} b_\kappa \Gamma^\kappa_{i\mu} \partial_{b_\kappa} \quad \text{and} \quad h^2 \partial_{p_\kappa} = \partial_{a_\kappa}, \]

or

\[ \ell \partial_{x_i} = h \partial_{x_i} + \sum_{\kappa=1}^{\nu} \sum_{j=1}^{n} b_\kappa \mathcal{R}^\kappa_{ij} \partial_{a_j} + \sum_{\kappa, \mu=1}^{\nu} b_\kappa \Gamma^\kappa_{i\mu} \partial_{b_\kappa} \quad \text{and} \quad \ell \partial_{p_\kappa} = \partial_{a_\kappa}. \]
5.2. Proof of Theorem. We will use the notation of Section 5.1. First, consider any curve $t \mapsto \alpha(t)$ in $\text{Ann}(D)$ which has a projection $\gamma$ in $M$. For sufficiently short time, define $\tilde{\alpha}$ as the $H^2$-horizontal lift. Then by Lemma 6, this is in $\text{Ann}(\tilde{D})$. Hence, if $w \in T\text{Ann}(D) \subset TT M$, then $h^2_p w \in T\text{Ann}(\tilde{D})$.

Let $t \mapsto \tilde{\lambda}(t)$ be a characteristic of $\tilde{D}$ with $\pi^2(\tilde{\lambda}(t)) = \lambda(t)$. We also write $\Pi T^* Q(\tilde{\lambda}) = \tilde{\gamma}$ and $\Pi T^* M(\lambda) = \gamma$. Finally, let $\beta(t) = pr^*_\gamma \tilde{\lambda}(t)$. We then have the following identities that $\tilde{\lambda}(t)$ must satisfy.

For any $\tilde{\alpha}_0 \in \text{Ann}(\tilde{D})\tilde{\gamma}(t)$, we need to have

$$\tilde{\varsigma}(\dot{\tilde{\lambda}}, \nu\alpha|_{\tilde{\lambda}(t)} \alpha_0) = \alpha_0(\dot{\gamma}) = 0,$$

which is the horizontally requirement. Furthermore, let $\tilde{\alpha}$ be a one-form on $Q$ such that $\tilde{\alpha}|_{\tilde{\gamma}(t)} = \tilde{\lambda}(t)$ for some subsegment of $\lambda$. Without loss of generality, we may assume that $\pi^2(\tilde{\alpha}) = \alpha$ is also a one-form on $M$. Then, for any vector field $X$ on $M$,

$$\tilde{\varsigma}(\dot{\lambda}, h^2(\alpha_\star X)) = \varsigma(\dot{\lambda}, \alpha_\star X) + (\rho \tilde{\lambda}(t) R(\tilde{\dot{\gamma}}, X) = 0.$$

Finally, for any $\Upsilon \in \Gamma(\mathcal{V})$, we have

$$\tilde{\varsigma}(\dot{\tilde{\lambda}}, \alpha_\star \Upsilon) = -d\theta Y(\dot{\tilde{\lambda}}, \alpha_\star \Upsilon) = -\nabla_\gamma \beta(t)(\Upsilon) = 0.$$

5.3. Equations for a pendulum with varying length. Consider a pendulum in $\mathbb{R}^2$ with mass $m$, under the influence of a force $-mA(\cos \phi_0, \sin \phi_0)$. The pendulum is connected to a massless rod or string whose length varies according to the function $\rho(t)$. This change in length is done by an external force, which gives an extra driving force of magnitude $\dot{\rho} \theta$ in the tangential direction. We give the pendulum coordinates $\rho(t)(\cos \theta(t), \sin \theta(t))$. Newtons equation for this problem is then

$$m(\rho - \rho^2)(\cos \theta, \sin \theta) + m(\rho \dot{\theta} + 2 \rho \dot{\rho})(-\sin \theta, \cos \theta)$$

$$= -mA(\cos \theta - \phi_0)(\cos \theta, \sin \theta) + mA \sin(\theta - \phi_0)(-\sin \theta, \cos \theta)$$

$$+ \rho \dot{\theta}(-\sin \theta, \cos \theta).$$

The radial part of this equation is just the equation describing the constraint, while the tangential part is equal to

$$\dot{\theta} + 2 \frac{\rho \dot{\theta}}{\rho} = \frac{A}{\rho} \sin(\theta - \phi_0) + \frac{\rho}{\rho} \dot{\theta},$$

and which equals (10).

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