The Sato-Tate conjecture for a Picard curve with Complex Multiplication

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(with an appendix by Francesc Fité)

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Abstract

Let \( C / \mathbb{Q} \) be the genus 3 Picard curve given by the affine model \( y^3 = x^4 - x \). In this paper we compute its Sato-Tate group, show the generalized Sato-Tate conjecture for \( C \), and compute the statistical moments for the limiting distribution of the normalized local factors of \( C \).

1 Introduction

Serre [Ser12] provides a vast generalization of the Sato-Tate conjecture, which is known to be true for varieties with complex multiplication [Joh13]. As a down-to-earth example, in this paper we consider the Picard curve defined over \( \mathbb{Q} \) given by the affine model

\[ C: y^3 = x^4 - x. \]

One easily checks that \([0 : 1 : 0]\) is the unique point of \( C \) at infinite, and that \( C \) has good reduction at all primes different from 3. The Jacobian variety of \( C \) is absolutely simple and it has complex multiplication by the cyclotomic field \( K = \mathbb{Q}(\zeta) \) where \( \zeta \) is a primitive 9th root of unity.

With the help of Sage, we compile information on the number of points that the reduction of \( C \) has over finite fields of small characteristic:

| \( p \) | \( |C(\mathbb{F}_p)| \) | \( |C(\mathbb{F}_{p^2})| \) | \( |C(\mathbb{F}_{p^3})| \) |
|------|----------------|----------------|----------------|
| 2    | 3              | 5              | 9              |
| 5    | 6              | 26             | 126            |
| 7    | 8              | 50             | 365            |
| 11   | 12             | 122            | 1332           |
| 13   | 14             | 170            | 2003           |
| 17   | 18             | 392            | 4914           |
| 19   | 14             | 302            | 6935           |

Table 1: Number of points \( |C(\mathbb{F}_p)| \).

For every primer \( p \) of good reduction, we consider the local zeta function

\[ \zeta(C / \mathbb{F}_p; s) = \exp \left( \sum_{k \geq 1} \frac{|C(\mathbb{F}_{p^k})|}{p^{ks}} \right). \]
It follows from Weil’s conjectures [Wei49] that the zeta function is a rational function of \( T = p^{-s} \). That is,

\[
\zeta(C/F_p; T) = \exp \left( \sum_{k \geq 1} \frac{|C(F_p)|^k}{k} \right) = \frac{L_p(C, T)}{(1-T)(1-pT)}
\]

where the so-called local factor of \( C \) at \( p \)

\[
L_p(C, T) = \sum_{i=0}^{6} b_i T^i = \prod_{i=1}^{6} (1 - \alpha_i T)
\]

is a polynomial of degree 6 with integral coefficients and the complex numbers \( \alpha_i \) satisfy \( |\alpha_i| = \sqrt{p} \). In particular, it is determined by the three numbers \( |C(F_p)|, |C(F_{p^2})|, |C(F_{p^3})| \) according to:

\[
\begin{align*}
  b_0 &= 1 \\
  b_1 &= |C(F_p)| - (p + 1) \\
  b_2 &= (|C(F_{p^2})| - (p^2 + 1) + b_1^2)/2 \\
  b_3 &= (|C(F_{p^3})| - (p^3 + 1) - b_1^3 + 3b_2b_1)/3 \\
  b_4 &= pb_2 \\
  b_5 &= p^2b_1 \\
  b_6 &= p^3.
\end{align*}
\]

For all \( m \geq 1 \) it holds

\[
|C(F_{p^m})| = 1 + p^m - \sum_{i=1}^{6} \alpha_i^m.
\]

The local factors for small good primes are:

| \( p \) | \( L_p(C, T) \) |
|---|---|
| 2 | \((1 + 2T^2)(1 - 2T^2 - 4T^4)\) |
| 3 | \((1 + 3T^2)(1 - 3T^2 + 9T^4)\) |
| 5 | \(1 + 7T^3 + 343T^6\) |
| 7 | \((1 + 11T^2)(1 - 11T^2 + 121T^4)\) |
| 11 | \(1 - 65T^3 + 2197T^6\) |
| 13 | \((1 + 17T^2)^3\) |
| 17 | \(1 - 6T - 12T^2 + 169T^3 - 228T^4 - 2166T^5 + 6859T^6\) |

Table 2: Local factors \( L_p(C, T) \).

Even if for every such prime \( p \) all terms of the sequence

\[
|C(F_p)|, |C(F_{p^2})|, |C(F_{p^3})|, \ldots, |C(F_{p^m})|, \ldots \quad (m \geq 1)
\]

are determined by the first three, the obtention of these first three can be a hard computational task as soon as the prime \( p \) gets large. However, the presence of complex multiplication enables the fast computation of the local factors \( L_p(C, T) \) (see Section 4.2).

For future use, we introduce some notation. The ring of integers of \( K \) will be denoted by \( \mathcal{O} = \mathbb{Z}[\xi] \), and the unit group \( \mathcal{O}^* \simeq \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) has generators \( \epsilon_0 = -\xi^2, \epsilon_1 = \xi^4 - \xi^3 + \xi, \epsilon_2 = \xi^5 + \xi^2 - \xi \). Let \( \sigma_i \) denote the automorphism of \( \text{Gal}(K/\mathbb{Q}) \) determined by \( \sigma_i(\xi) = \xi^i \); one has that \( \sigma_2 \) generates
the Galois group Gal(K/ℚ) ∼ (ℤ/9ℤ)*. The unique ramified prime in K/ℚ is
3θ = (1 + ζ + ζ^4)^6.
Since the Jacobian variety Jac(C) has complex multiplication, the work of
Shimura and Taniyama [ST61] ensures the existence of an ideal m of the ring of
integers τ and a Grössencharakter ψ: I_K(m) → C*, where I_K(m) stands for the
group of fractional ideals coprime with m,

ψ(ατ) = \prod_{σ∈Φ^o} σ^{α} \quad \text{if } α \equiv 1 \pmod{m},

such that L(ψ,s) = L(C,s). The infinite type Φ is the reflex of the CM-type Φ
of Jac(C). Up to a finite number of Euler factors, one has

L(ψ,s) = \prod_p (1 - ψ(p)N(p)^{-s})^{-1} \quad \text{and } L(C,s) = \prod_p L_p(C,p^{-s})^{-1}.

Hence, the local factor L_p(C,T) can be obtained from the (monic) irreducible
polynomial of ψ(p) over ℚ according to

L_p(C,T) = T^6 \text{Irr}(ψ(p),1/T^f; ℚ)^{6/(fd)},

where f is the residual class degree of p in K, and d = [ℚ(ψ(p)): ℚ].

**Lemma 1.1.** There exists a Grössencharakter ψ: I_K(m) → C* of conductor
m = (1 + ζ + ζ^4)^6 and infinite type Φ = \{σ_1, σ_5, σ_7\} = \{σ_2, σ_2', σ_3'\}.

**Proof.** The following holds

ε_0^{18} \equiv 1 \pmod{m}, \quad ε_1^{6} \equiv 1 \pmod{m}, \quad ε_2^{3} \equiv 1 \pmod{m}.

Moreover, one readily checks that ε_0^{a}ε_1^{b}ε_2^{c} \equiv 1 \pmod{m} if and only if

(a, b, c) \equiv (0,0,0), \quad (2,1,2), \quad (4,2,1),
\quad (6,3,0), \quad (8,4,2), \quad (10,5,1),
\quad (12,6,0), \quad (14,7,2), \quad (16,8,1),

mod (18,9,3), respectively. Now an easy computation case-by-case shows that
if ε_0^{a}ε_1^{b}ε_2^{c} \equiv 1 \pmod{m}, then

\prod_{σ∈Φ^o} σ(ε_0^{a}ε_1^{b}ε_2^{c}) = 1.

By using that K has class number one, we define ψ(p) over prime ideals p
of τ coprime with m as follows. First we find a generator of p = (α), and then
search for

ε_0^{a}ε_1^{b}ε_2^{c}α \equiv 1 \pmod{m}

with 0 < a < 18, 0 < b < 9, and 0 < c < 3. The existence of such triple (a,b,c)
is guaranteed by the fact that (α,m) = 1 and the classes of the 486 possible
products ε_0^{a}ε_1^{b}ε_2^{c} exhaust all the elements in (τ/m)*. It follows that

ψ(p) = \prod_{σ∈Φ^o} σ(ε_0^{a}ε_1^{b}ε_2^{c}α)

is well-defined. Finally, one extends ψ over all ideals prime to m multiplicatively.
An argument along the same lines shows the non existence of a Grössencharakter of K of modulus (1 + ζ + ζ^4)^i for i < 4. Thus, ψ has conductor m. □
where \( \sigma \) is uniquely determined by the three properties:

\begin{enumerate}
\item [(i)] \(|J(p)| = \sqrt{N(p)}|; \\
\item [(ii)] \(J(p) \equiv 1 \pmod{m}; \\
\item [(iii)] \(J(p) \mathcal{O} = (p \cdot p^{\sigma_3} \cdot p^{\sigma_2}).
\end{enumerate}

One the one hand, it is easy to check that \( \psi(p) \) satisfies (i), (ii), and (iii). On the other hand, Holzapfel and Nicolae [HN02] show that for a primer power \( q \neq 1 \pmod{9} \) one has \(|C(\mathbb{F}_q)| = q + 1\), while for \( q = 1 \pmod{9} \) it follows

\[ |C(\mathbb{F}_p)| = N(p) + 1 - \text{Tr}_K/Q(J(p)), \]

where \( p \) is any prime ideal of the factorization of \( q\mathcal{O} \). The claim follows. \( \square \)

**Remark 1.1.** The proof of the last equalities takes 4 pages in the referenced article [HN02]. We are grateful to Francesc Fité for a more concise proof included in the appendix of the present paper.

The Grössencharakter \( \psi \) satisfies \( \sigma \psi(p) = \psi(\sigma p) \) for every prime ideal \( p \) and \( \sigma \in \text{Gal}(K/Q). \) The \( L \)-function of the curve \( C \) over \( K \) satisfies

\[ L(C_K, s) = \prod_{\sigma \in \text{Gal}(K/Q)} L(\sigma \psi, s) = L(C, s)^6. \]

The CM-type of \( \text{Jac}(C) \) is \( \Phi = \{ \sigma_2, \sigma_4, \sigma_8 \}, \) i.e. the reflex of \( \Phi^* \).
The Sato-Tate group \( \text{ST}(C) \)

For every prime \( p \neq 3 \), let us normalize the polynomials

\[ L_p^{\text{ST}}(C, T) = L_p(C, T^{\sqrt{p}}) \]

and call them normalized local factors of \( C \). Since they are monic, palindromic with real coefficients, roots lying in the unit circle and Galois stable, one can think of them as the characteristic polynomials of (conjugacy classes of) matrices in the unitary symplectic group

\[ \text{USp}(6, \mathbb{C}) = \{ M \in \text{GL}(6, \mathbb{C}) : M^{-1} = J^{-1} M^t J = M^* \} \]

where \( M^* \) denotes the complex conjugate transpose of \( M \), and \( J \) denotes the skew-symmetric matrix

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}.
\]

Roughly, the Sato-Tate group attached to \( C \) is defined to be a compact subgroup \( \text{ST}(C) \subseteq \text{USp}(6, \mathbb{C}) \) such that the characteristic polynomials of the matrices in \( \text{ST}(C) \) fit well with the normalized local factors \( L_p^{\text{ST}}(C, T) \), in the sense that the normalized local factors \( L_p^{\text{ST}}(C, T) \), as \( p \) varies, are equidistributed with respect to the Haar measure of \( \text{ST}(C) \) projected on the set of its conjugacy classes.

In analogy with Galois theory, the presence of some extra structure on \( C \) gives rise to proper subgroups of the symplectic group; moreover, the distribution of \( L_p^{\text{ST}}(C, T) \) can be viewed as a generalization of the classical Chebotarev distribution. Serre [Ser12] proposes a vast generalization of the Sato-Tate conjecture (born for elliptic curves) giving a precise recipe for \( \text{ST}(C) \). In this section, we calculate the Sato-Tate group \( \text{ST}(C) \) for our Picard curve \( C \).

**Proposition 2.1.** Up to conjugation in \( \text{USp}(6, \mathbb{C}) \), the Sato-Tate group of \( C \) is

\[
\text{ST}(C) = \left\langle \begin{pmatrix} u_1 \\ \tilde{u}_1 \\ u_2 \\ \tilde{u}_2 \\ u_3 \\ \tilde{u}_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right| |u_i| = 1 \right\}
\]

In particular, there is an isomorphism \( \text{ST}(C) \cong U(1)^3 \rtimes (\mathbb{Z}/9\mathbb{Z})^* \).

**Proof.** The recipe of Serre in [Ser12] is as follows. Fix an auxiliary prime \( \ell \) of good reduction (say \( \ell > 3 \)), and fix an embedding \( \iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C} \). Let

\[ \rho_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(V_\ell(\text{Jac}(C))) \cong \text{GL}(6, \mathbb{Q}_\ell) \]

be the \( \ell \)-adic Galois representation attached to the \( \ell \)-adic Tate module of the Jacobian variety of \( C \). Denote by \( G \) the Zariski closure of the image \( \rho_\ell(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \), and let \( G_1 \) be the Zariski closure of \( G \cap \text{Sp}_6(\mathbb{Q}_\ell) \), where \( \text{Sp}_6 \) denotes the symplectic group. By definition, the Sato-Tate group \( \text{ST}(C) \) is a maximal compact
subgroup of $G_1 \otimes_\mathbb{Q} \mathbb{C}$. In general, one hopes that this construction does not depend on $\ell$ and $t$, and this is the case for our Picard curve $C$. Indeed, since the CM-type of $\text{Jac}(C)$ is non-degenerate then the twisted Lefschetz group $\text{TL}(C)$ satisfies $G_1 = \text{TL}(C) \otimes \mathbb{Q}_\ell$ for all primes $\ell$ (see [FGL14] Lemma 3.5]). Recall that the twisted Lefschetz group is defined as

$$\text{TL}(C) = \bigcup_{\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \text{L}(C)(\tau),$$

where $\text{L}(C)(\tau) = \{ \gamma \in \text{Sp}_6(\mathbb{Q}) : \gamma \alpha \gamma^{-1} = \tau(\alpha) \text{ for all } \alpha \in \text{End}(\text{Jac}(C)) \otimes \mathbb{Q} \}$, where $\text{Jac}(C)$ denotes the base change to $\overline{\mathbb{Q}}$. Here, $\alpha$ is seen as an endomorphism of $H_1(\text{Jac}(C)_C, \mathbb{Q})$. The reason why the CM-type of $\text{Jac}(C)$ is non-degenerate is due to the fact that $\Phi^*$ is simple and $\dim \text{Jac}(C) = 3$ (see [Kub65] [Rib81]); alternatively, one checks that the $\mathbb{Z}$-linear map:

$$\mathbb{Z}[\text{Gal}(K/\mathbb{Q})] \to \mathbb{Z}[\text{Gal}(K/\mathbb{Q})], \quad \sigma_a \mapsto \sum_{\sigma_b \in \Phi} \sigma_b^{-1} \sigma_a$$

has maximal rank $1 + \dim(\text{Jac}(C)) = 4$. Then, by combining [BGK03] and [FKRS12] Thm.2.16(a)], it follows that the connected component of the identity $\text{TL}(C)^0$ satisfies

$$G_1^0 = \text{TL}(C)^0 \otimes \mathbb{Q}_\ell = \{ \text{diag}(x_1, y_1, x_2, y_2, x_3, y_3) \mid x_i, y_i \in \mathbb{Q}_\ell, x_i y_i = 1 \}.$$ 

Thus, the connected component of the Sato-Tate group for $C$ is equal to

$$\text{ST}(C)^0 = \{ \text{diag}(u_1, \overline{u}_1, u_2, \overline{u}_2, u_3, \overline{u}_3) : u_i \in U(1) \} \simeq U(1)^3.$$

According to [FKRS12] Prop. 2.17, it also follows that the group of components of $\text{ST}(C)$ is isomorphic to $\text{Gal}(K/\mathbb{Q})$. We claim that $\text{ST}(C) = \text{ST}(C)^0 \rtimes (\gamma)$, where

$$\gamma = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

To this end, we consider the automorphism of the Picard curve $C$ determined by $\alpha(x, y) = (\zeta^6 x, \zeta^2 y)$. We still denote by $\alpha$ the induced endomorphism of $\text{Jac}(C)$. Under the basis of regular differentials of $\Omega^1(C)$:

$$\omega_1 = \frac{dx}{y^2}, \quad \omega_2 = \frac{dx}{y}, \quad \omega_3 = \frac{xdx}{y^2},$$

the action induced is given by $\alpha^*(\omega_1) = \zeta^2 \omega_1$, $\alpha^*(\omega_2) = \zeta^4 \omega_2$, $\alpha^*(\omega_3) = \zeta^8 \omega_3$. By taking the symplectic basis of $H_1(\text{Jac}(C)_C, \mathbb{C})$ corresponding to the above basis (with respect to the skew-symmetric matrix $J$), we get the matrix

$$\alpha = \begin{pmatrix}
\zeta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & \zeta^4 & 0 & 0 & 0 \\
0 & 0 & 0 & \zeta^4 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta^8 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta^8
\end{pmatrix}.$$
As a consequence of [FKRS12, Prop. 2.17], we also obtain that, 

Remark 2.2. For every non-trivial irreducible representation \( \pi \) of \( \GL_2(\mathbb{F}_p) \), one checks that the matrix \( \gamma = (\gamma_1, \gamma_2) \) satisfies

\[
\gamma \alpha \gamma^{-1} = \alpha^2 \alpha,
\]

which implies that \( \gamma \in TL(\sigma_2) \). Hence, \( \gamma \) belongs to \( ST(C) \); finally, a short computations shows that \( \gamma^6 = -\Id \in ST(C)^0 \), but \( \gamma \) is not in \( ST(C)^0 \) for \( 1 \leq i < 6 \).

Remark 2.1. For future use, we compute the shape of the characteristic polynomials in each component of the Sato-Tate group. To this end, we take a random matrix \( \text{diag}(u_1, u_2, u_3, u_4) \) in the connected component \( ST^0(C) \), and we get:

\[
\begin{align*}
ST^0(C) \cdot \Id & : \prod_{i=1}^3 (T - u_i)(T - \overline{u}_i) \\
ST^0(C) \cdot \gamma & : T^6 - (u_1 u_2 u_3 + u_1 u_2 u_3) T^3 + 1 \\
ST^0(C) \cdot \gamma^2 & : (T^2 + 1)^3 \\
ST^0(C) \cdot \gamma^3 & : T^6 + (u_1 u_2 u_3 + u_1 u_2 u_3) T^3 + 1 \\
ST^0(C) \cdot \gamma^4 & : 6
\end{align*}
\]

Remark 2.2. As a consequence of [FKRS12 Prop. 2.17], we also obtain that, for every subextension \( K/K'/\mathbb{Q} \), one has \( ST(C_{K'}) = ST(C)^0 \times \langle \gamma \rangle \), where \( C_{K'} \) denotes the base change \( C \times \mathbb{Q} K' \).

3 Sato-Tate distribution

A general strategy to prove the expected distribution is due to Serre [Ser98]. For every non-trivial irreducible representation \( \phi : ST(C) \to \GL_m(\mathbb{C}) \), one needs to consider the \( L \)-function

\[
L(\phi, s) = \prod_{p \neq 3} \det(1 - \phi(x_p)p^{-s})^{-1},
\]

where \( x_p = \frac{1}{\sqrt{p}} \rho_c(\text{Frob}_p) \in ST(C) \), and then show that \( L(\phi, s) \) is invertible, in the sense that it has meromorphic continuation to \( \text{Re}(s) \geq 1 \) and it holds

\[
L(\phi, 1) \neq 0.
\]

Proposition 3.1. The Picard curve \( C : y^3 = x^4 - x \) satisfies the generalized Sato-Tate conjecture. More explicitly, the sequence

\[
\left\{ \left( \frac{\alpha_1 \psi(p)}{\sqrt{N(p)}}, \frac{\alpha_2 \psi(p)}{\sqrt{N(p)}}, \frac{\alpha_3 \psi(p)}{\sqrt{N(p)}} \right) \right\}_{p \neq 3} \subseteq U(1)^3 \times (\mathbb{Z}/9\mathbb{Z})^* \simeq ST(C),
\]

where \( p \) is any prime ideal of the factorization of \( p\ell \), is equidistributed over \( U(1)^3 \times (\mathbb{Z}/9\mathbb{Z})^* \) with respect to the Haar measure.

Proof. The irreducible representations of \( ST(C) \simeq U(1)^3 \times (\mathbb{Z}/9\mathbb{Z})^* \) can be described as follows (see [Ser77 §8.2]). For every triple \( b = (b_1, b_2, b_3) \) in \( \mathbb{Z}^3 \), we consider the irreducible character of \( U(1)^3 \) given by

\[
\phi_b : U(1)^3 \to \mathbb{C}^*, \quad \phi_b(u_1, u_2, u_3) = \prod_{i=1}^3 \frac{u_i^{b_i} - 1}{u_i^{b_i}},
\]
and let \( H_b = \{ h \in (\mathbb{Z}/9\mathbb{Z})^*: \phi_b(u_1, u_2, u_3) = \phi_b(h(u_1, u_2, u_3)) \} \).

The action of \((\mathbb{Z}/9\mathbb{Z})^*\) on \(U(1)^3\) is given by conjugation through powers of the matrix \( \gamma \); more precisely, for the generator \( g = 2 \) of \((\mathbb{Z}/9\mathbb{Z})^*\) we have \( \gamma^i(u_1, u_2, u_3) = (u_2, u_3, \bar{u}_1) \) since

\[
\begin{pmatrix}
  u_1 \\
  \tilde{u}_1 \\
  u_2 \\
  \tilde{u}_2 \\
  u_3 \\
  \bar{u}_3
\end{pmatrix}
\gamma^{-1} =
\begin{pmatrix}
  u_2 \\
  \tilde{u}_2 \\
  u_3 \\
  \tilde{u}_3 \\
  \bar{u}_1 \\
  u_1
\end{pmatrix}.
\]

An easy computation shows that \( H_b = \langle 2 \rangle \) or \( \langle 2^3 \rangle \) if and only if \( b = (0, 0, 0) \), while \( H_b = \langle 2^2 \rangle \) for \( b = (b_1, -b_1, b_1) \) with \( b_1 \neq 0 \), and \( H_b \) is trivial otherwise. Then, one has that

\[
\phi_b(u_1, u_2, u_3, h) = \prod_{i=1}^{3} u_i^{b_i}
\]

is a character of \( H := U(1)^3 \rtimes H_b \). By [Ser77] Prop. 25] every irreducible representation of \( G := U(1)^3 \rtimes (\mathbb{Z}/9\mathbb{Z})^* \) is of the form \( \theta := \text{Ind}^G_H(\phi_b \otimes \chi) \), where \( \chi \) is a character of \( H_b \) that may be viewed as a character of \( H \) by composing with the projection \( H \to H_b \).

Let \( \theta = \text{Ind}^G_H(\phi_b \otimes \chi) \) be an irreducible representation of \( U(1)^3 \rtimes (\mathbb{Z}/9\mathbb{Z})^* \) as above. If we denote the sequence by

\[
x_p = \left( \frac{\sigma_2 \psi(p)}{\sqrt{N(p)}}, \frac{\sigma_4 \psi(p)}{\sqrt{N(p)}}, \frac{\sigma_6 \psi(p)}{\sqrt{N(p)}} \right) \in U(1)^3 \rtimes (\mathbb{Z}/9\mathbb{Z})^*
\]

where \( p \) is any prime ideal of the factorization of \( p \mathcal{O} \), our claim is equivalent to show that the corresponding \( L \)-function

\[
L(\theta, s) = \prod_{p \neq 3} (1 - \det(\theta(x_p))p^{-s})^{-1}
\]

is invertible provided that \( (b_1, b_2, b_3) \neq (0, 0, 0) \). Assume first that \( H_b \) is trivial. Then, also \( \chi \) is trivial and one has

\[
L(\theta, s) = L(\phi_b, s) = \prod_{p \neq 3} \left( 1 - \frac{\sigma_2 \psi(p)^{b_1} \sigma_4 \psi(p)^{b_2} \sigma_6 \psi(p)^{b_3}}{\sqrt{N(p)}^{b_1 + b_2 + b_3}} \right).
\]

This can be seen as the \( L \)-function of the unitarized Grössencharakter

\[
\Psi := \frac{\sigma_2 \psi(\cdot)^{b_1} \sigma_4 \psi(\cdot)^{b_2} \sigma_6 \psi(\cdot)^{b_3}}{\sqrt{N(\cdot)^{(b_1 + b_2 + b_3)/2}}}
\]

Under our assumption \( (b_1, b_2, b_3) \neq (0, 0, 0) \) and by using the factorization of \( \psi(p) \mathcal{O} \) into prime ideals (see property (iii) in the proof of Proposition [1]), an easy computation shows that \( \Psi \) is non-trivial. Hecke showed [Hec20] that the \( L \)-function of a non-trivial unitarized Grössencharakter is holomorphic and non-vanishing for \( \text{Re}(s) \geq 1 \). In the remaining case, that is for \( H_b \) of order 3, one gets \( L(\Psi, s) = L(\theta, s)^{3} \) and the claim also follows by the same argument. \( \square \)
4 The moment sequences

In this section we will compute the moment sequences in two independent ways, one (exact) from the Sato-Tate group and the other one (numerically) by computing the local factors of our curve up to some bound.

Let \( \mu \) be a positive measure on \( I = [-d, d] \). Then, on the one hand, for every integer \( n \geq 0 \), the \( n \)th moment \( M_n[\mu] \) is by definition \( \mu(\phi_n) \), where \( \phi_n \) is the function \( z \mapsto z^n \). That is, we have

\[
M_n[\mu] = \int_I z^n \mu(z)
\]

The measure \( \mu \) is uniquely determined by its moment sequence \( M_n[\mu] \).

On the other hand, if a sequence \( \{a(p)\}_p \) is \( \mu \)-equidistributed, then the following equality holds:

\[
M_n[\mu] = \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} a_p^n.
\]

From now on, we shall denote by \( a_1(p) \), \( a_2(p) \), \( a_3(p) \) the higher traces according to

\[
T_p^{ST}(C, T) = 1 + a_1(p)T + a_2(p)T^2 + a_3(p)T^3 + a_2(p)T^4 + a_1(p)T^5 + T^6.
\]

Recall that due to the Weil’s conjectures, we know that

\[
a_1(p) \in [-6, 6], \quad a_2(p) \in [-15, 15], \quad a_3(p) \in [-20, 20].
\]

4.1 The distribution of \( ST(C) \)

For each \( i \) in \( \{1, 2, 3\} \), let \( \mu_i \) denote the projection on the interval \( I_i = [-\binom{6}{i}, +\binom{6}{i}] \) obtained from the Haar measure of the Sato-Tate group \( ST(C) \simeq U(1)^3 \times (\mathbb{Z}/9\mathbb{Z})^* \).

In general it is difficult to obtain the explicit distribution function, but because of the isomorphism stated in Proposition 2.1, we can easily compute the moment sequence of the Sato-Tate measure.

Similarly as in [FGL14] we shall split each measure as a sum of its restrictions to each component of \( ST(C)^0 \cdot \mathbb{A}^k \), where \( 0 \leq k \leq 5 \).

Therefore one has

\[
\mu_i = \frac{1}{6} \sum_{0 \leq k \leq 5} k \mu_i, \quad M_n[\mu_i] = \frac{1}{6} \sum_{0 \leq k \leq 5} M_n[k \mu_i]
\]

so we can compute the moments \( M_n[k \mu_i] \) separately for every \( k \) and then get the total moments \( M_n[\mu_i] \). To ease notation, we shall denote the moment sequences by

\[
M[\mu_i] := (M_0[\mu_i], M_1[\mu_i], M_1[\mu_i], \ldots, M_n[\mu_i], \ldots),
\]

and similarly for every \( M[k \mu_i] \).

In what follows, the characteristic polynomial of a matrix in \( USp(6) \) we will be denoted by

\[
P(T) = 1 + a_1 T + a_2 T^2 + a_3 T^3 + a_2 T^4 + a_1 T^5 + T^6.
\]

Case \( k = 1, 5 \): In these components, according to Remark 2.1 one has that \( P(T) = T^6 + 1 \), so that

\[
a_1 = a_2 = a_3 = 0.
\]
Hence,
\[ M_n[k\mu_1] = M_n[k\mu_2] = M_n[k\mu_3] = 0 \text{ for all } n \geq 1. \]

Case \( k = 2, 4 \): In these components, we have
\[ P(T) = T^6 \pm (u_1\bar{u}_2u_3 + \bar{u}_1u_2\bar{u}_3)T^3 + 1. \]

So that \( a_1 = a_2 = 0 \). Hence, it follows that
\[ M_n[k\mu_1] = M_n[k\mu_2] = 0 \text{ for all } n \geq 1. \]

To get the distribution of the third trace, since \( u_1, u_2, \) and \( u_3 \) are independent elements of \( U(1) \), the distribution of \( a_3(p) \) will correspond to the distribution of \( \alpha := u + \bar{u} \) for \( u \in U(1) \), and hence its associated moment sequence is
\[ M[k\mu_3] = (1, 0, 2, 0, 6, 0, 20, 0, \ldots). \]

Case \( k = 3 \): In this case one has that \( P(T) = (1 + T^2)^3 \), so that we have \( a_1 = a_3 = 0 \), while \( a_2 = 3 \). Hence, we obtain
\[ M_n[k\mu_1] = M_n[k\mu_3] = 0, \quad M_n[3\mu_2] = 3^n \text{ for all } n \geq 1. \]

Case \( k = 0 \): In this case one has that \( P(T) = \prod_{i=1}^3(T - u_i)(T - \bar{u}_i) \). If we develop this expression we get the following coefficients, where as above \( \alpha_i \) stands for the sum of \( u_i \) and its complex conjugate:
\[ a_1 = \alpha_1 + \alpha_2 + \alpha_3, \]
\[ a_2 = 3 + \alpha_1\alpha_2 + 2\alpha_2\alpha_3 + \alpha_1\alpha_3, \]
\[ a_3 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_1\alpha_2\alpha_3. \]

To get the sequences we proceed as follows. Recall that if \( X \) and \( Y \) denote independent random variables, then \( M_n[X] = E(X^n), E(X + Y) = E(X) + E(Y) \), and \( E(XY) = E(X)E(Y) \). Hence, one has
\[ M_n[X + Y] = E((X + Y)^n) = E \left( \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k} \right) = \sum_{k=0}^{n} \binom{n}{k} E(X^k)E(Y^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} M_k[X]M_{n-k}[Y]. \]

Since we know that \( M[\alpha] := M[\alpha_i] = (1, 0, 2, 0, 6, 0, 20, 0, \ldots) \) for \( i = 1, 2, 3 \), one gets:
\[ M_n[0\mu_1] = \sum_{a+b+c=n} \binom{n}{a,b,c} M_a[\alpha]M_b[\alpha]M_c[\alpha], \]
\[ M_n[0\mu_2] = \sum_{a+b+c+d=n} \binom{n}{a,b,c,d} 3^n M_{b+d}[\alpha]M_{b+c}[\alpha]M_{c+d}[\alpha], \]
\[ M_n[0\mu_3] = \sum_{a+b+c+d=n} \binom{n}{a,b,c,d} 2^{a+b+c} M_{a+d}[\alpha]M_{b+d}[\alpha]M_{c+d}[\alpha]. \]

Therefore we obtain the sequences:
\[ M[0\mu_1] = (1, 0, 6, 0, 90, 0, 1860, \ldots), \]
\[ M[0\mu_2] = (1, 3, 21, 183, 1845, \ldots), \]
\[ M[0\mu_3] = (1, 0, 32, 0, 4920, 0, 1109120, \ldots). \]

We can summarize the above results in the following proposition.
Proposition 4.1. With the above notations, the first moments of the measures of \(k\mu_i\) and \(\mu_i\) are as follows:

(i) The moments of the first trace are:

\[
M[\mu_1] = \begin{cases} 
(1, 0, 0, \ldots) & \text{if } k = 1, \ldots, 5; \\
(1, 0, 6, 0, 90, 0, 1860, \ldots) & \text{if } k = 0.
\end{cases}
\]

Hence, \(M[\mu_1] = (1, 0, 1, 0, 15, 0, 310, \ldots)\).

(ii) The moments of the second trace are:

\[
M[\mu_2] = \begin{cases} 
(1, 0, 0, \ldots) & \text{if } k = 1, 2, 4, 5; \\
(1, 3, 9, 27, \ldots) & \text{if } k = 3; \\
(1, 3, 21, 183, 1845, \ldots) & \text{if } k = 0.
\end{cases}
\]

Hence, \(M[\mu_2] = (1, 1, 5, 35, 321, \ldots)\).

(iii) The moments of the third trace are:

\[
M[\mu_3] = \begin{cases} 
(1, 0, 0, \ldots) & \text{if } k = 1, 3, 5; \\
(1, 0, 2, 0, 6, 0, 20, 0, \ldots) & \text{if } k = 2, 4; \\
(1, 0, 32, 0, 4920, 0, 1109120, \ldots) & \text{if } k = 0.
\end{cases}
\]

Hence, \(M[\mu_3] = (1, 0, 6, 0, 822, 0, 184860, 0\ldots)\).

4.2 The numerical sequences for \(C\)

Once we have computed the theoretical moment sequences from the Sato-Tate group \(\text{ST}(C)\), we wish to compute for every prime (up to some bound) its associated normalized local factor \(L_{ST}(A,T)\) to get the corresponding traces \(a_1(p)\), \(a_2(p)\) and \(a_3(p)\) and do the experimental equidistribution matching.

The Grössencharakter \(\psi\) attached to the Picard curve \(C\) permits us to perform this numerical experimentation within a reasonable time, in this case \(p \leq 2^{26}\) (about two hours of a standard laptop). We display the data obtained:

| \(n\) | \(M_n[\mu_1]\) | \(a_1\) | \(M_n[\mu_1] < 2^{26}\) | \(a_2\) | \(M_n[\mu_2] < 2^{26}\) | \(a_3\) | \(M_n[\mu_3] < 2^{26}\) |
|-------|----------------|-------|----------------|-------|----------------|-------|----------------|
| 0     | 1              | 1     | 1              | 1     | 1              | 1     |
| 1     | 0              | 0.000 | 1              | 0.999 | 0              | 0     |
| 2     | 1              | 0.998 | 5              | 4.991 | 6              | 5.984 |
| 3     | 0              | 0.005 | 35             | 34.868| 0              | −0.147|
| 4     | 15             | 14.946| 321            | 319.058| 822           | 815.937|
| 5     | 0              | 0     | −0.151         | 0     |
| 6     | 310            | 308.160|                |

Table 4: Numerical moment sequences computed for \(p\) up to \(2^{26}\).

We include graphics to display the histograms (for primes up to \(p \leq 2^{26}\)) showing the nondiscrete components of the three distributions \(\mu_i\).
Figure 1: Histogram of the first trace for primes $p \equiv 1 \pmod{9}$.

Figure 2: Histogram of the second trace for primes $p \equiv 1 \pmod{9}$.

Figure 3: Histogram of the third trace for primes $p \equiv 1 \pmod{9}$. 
Figure 4: Histogram of the third trace for primes $p \equiv 4, 7 \pmod{9}$.

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Appendix (by F. Fité)

We keep the notation of the article. Let $C: y^3 = x^4 - x$. Let $K$ denote the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta$ is a 9th root of unity. For every prime $p$ of $K$ coprime to 3, consider the character $\chi_p: \mathbb{F}_p^\ast \to \mathbb{C}^\ast$ such that $\chi_p(x)$ is the only 9th root of unity satisfying

$$\chi_p(x) \equiv x^{(N(p)-1)/9} \pmod{p}.$$

For $a, b \in \mathbb{Z}/9\mathbb{Z}$, define

$$J_{(a,b)}(p) := \sum_{x \in \mathbb{F}_p} \chi_p^a(x)\chi_p^b(1-x).$$

Proposition A.1. The number of points of $C$ defined over the finite field $\mathbb{F}_p$ is

$$|C(\mathbb{F}_p)| = \begin{cases} 1 + N(p) & \text{if } N(p) \not \equiv 1 \pmod{9}, \ (i) \\ 1 + N(p) + \text{Tr}_{K/\mathbb{Q}}(J_{(6,1)}(p)) & \text{if } N(p) \equiv 1 \pmod{9}, \ (ii) \end{cases}$$

Proof. Case (i) is considered in Proposition 1 and Proposition 2 of [HN02]. We now show case (ii), by giving an alternative and shorter proof of Proposition 3 of [HN02]. Let $C': v^9 = u(u+1)^6$. There is an isomorphism between $C$ and $C'$ given by

$$\phi: C \to C', \quad \phi(x,y) = \left( -\frac{1}{x^3}, \frac{y^2}{x^3} \right).$$

One easily sees that the inverse of $\phi$ is given by

$$\phi^{-1}: C' \to C, \quad \phi^{-1}(u,v) = \left( -\frac{(u+1)^2}{v^3}, -\frac{(u+1)^3}{v^4} \right).$$

Note that if $N(p) \not \equiv 1 \pmod{3}$, then exponentiation by 9 is an isomorphism of $\mathbb{F}_p$. Thus $C'$ has $N(p)$ affine points plus one point at infinity. Assume now that $N(p) \equiv 1 \pmod{9}$. By [IR90] Prop. 8.1.5, we have that

$$|C'(\mathbb{F}_p)| = 1 + \sum_{u \in \mathbb{F}_p} \sum_{v \in \mathbb{Z}/9\mathbb{Z}} \chi_p^a(u)\chi_p^b(u+1)$$

$$= 1 + N(p) + \sum_{a \in (\mathbb{Z}/9\mathbb{Z})^\ast} \sum_{u \in \mathbb{F}_p} \chi_p^a(u)\chi_p^b(u+1),$$

where for the second equality we have used [IR90] Thm. 1 (b), p. 93]. But writing $x = u+1$, we obtain

$$\sum_{u \in \mathbb{F}_p} \chi_p(u)\chi_p^6(u+1) = \chi_p(-1) \sum_{x \in \mathbb{F}_p} \chi_p^6(x)\chi_p(1-x).$$

Case (ii) of the proposition is a consequence of the equality $\chi_p(-1) = 1$ (this follows from the fact that the order of $\chi_p$ is odd).

To show that our result agrees with Proposition 3 of [HN02] it remains to show that $\text{Tr}_{K/\mathbb{Q}}(J_{(6,1)}(p)) = \text{Tr}_{K/\mathbb{Q}}(J_{(3,1)}(p))$. Indeed, by [BEW98] Thm. 2.1.5, one has

$$J_{(6,1)}(p) = J_{(2,1)}(p), \quad J_{(3,1)}(p) = J_{(5,1)}(p).$$

Since $5 \cdot 2 \equiv 1 \pmod{9}$, we deduce that

$$\text{Tr}_{K/\mathbb{Q}}(J_{(2,1)}(p)) = \text{Tr}_{K/\mathbb{Q}}(J_{(5,1)}(p)).$$

Finally, note that $J(p) = -J_{(3,1)}(p)$ in the notation of the article.
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