Smooth Optimal Control for a Class of Switched Systems Based on Fuzzy Theory and PSO

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Abstract. This paper proposes an approach to smooth optimal control design of two-point boundary value problem of switched systems with fuzzy operating regions. The switched system is modeled via a T-S fuzzy system, and then the fuzzy inference method of T-S fuzzy system is used to transform the switched system to a nonlinear system. For this nonlinear system, an optimal controller is designed. The control signal is a fuzzy control with weighted average defuzzifier. The fuzzy sets are used to partition the time space and a Particle Swarm Optimization (PSO) algorithm is proposed to optimize the weights. Simulations results demonstrate the applicability and effectiveness of the proposed method.

1. Introduction
Since many practical dynamical systems are of switching in nature, recently the control and analysis of these systems have been the focus of studies [1]. In this paper, we focus on the state dependent switched systems. In a state dependent switched system the continuous state space is partitioned into a finite or infinite number of operating regions by means of a family of switching surfaces, or guards. Whenever the system trajectory hits a switching surface, the continuous state jumps instantaneously to a new value, specified by a mapping. While some papers have been published on the analysis of such systems [2, 3], there are few works on optimal control of these systems.

In many practical applications, the switching surfaces are not known exactly and they are determined by linguistic variables based on human knowledge. Thus, in practice, we may have systems with fuzzy operating regions.

This paper proposes a novel approach for optimal control of a class of state dependent switched systems with fuzzy operating regions. Here, the switched system is firstly expressed via Takagi-Sugeno (T-S) fuzzy rules [4], and then by following the fuzzy inference method of T-S fuzzy systems, a nonlinear system is derived from the T-S fuzzy rules. Now, we have a nonlinear optimal control two-point boundary value problem. An optimal fuzzy control with weighted average defuzzifier is used, where the fuzzy sets are selected to partition the time space and a Particle Swarm Optimization (PSO) algorithm is proposed to optimize the weights. To demonstrate the applicability of the proposed method, it is applied to a switched two-point boundary value problem. Simulation results show the effectiveness of the proposed method.

This paper is organized as follows. Section 2 states problem formulation, assumptions, and preliminaries. Transformation based on T-S fuzzy systems is presented in Section 3. A brief description of the Particle Swarm Optimization algorithm is presented in Section 4. Section 5 describes and proposes the optimal controller, while in Section 6 a simulation example is demonstrated and finally, Section 7 concludes the main advantages of the proposed method.
2. Problem Formulation and Preliminaries

Let \( x \) be a vector in \( n \)-space \( \mathbb{R}^n \), \( u \) be a vector in \( m \)-space \( \mathbb{R}^m \), and \( t \) be a real variable. \( J = [t_a, t_b] \), with \( t_a < t_b \) is a real closed interval. The interior of this interval will be denoted by \( J^* \), \( J^* = (t_a, t_b) \).

This is the time interval in which the controlled system will evolve. The set \( A \) in \( \mathbb{R}^n \) is bounded, closed, pathwise-connected, which the trajectory of the system is constrained to stay in it for \( t \in J \). Two elements of \( A \), \( x_a \) and \( x_b \), are the initial and final states of the trajectory of the controlled system. \( U \) is a bounded closed subset of \( \mathbb{R}^m \). This is the set in which the control function are to take values. Let \( g_i, i = 1, \ldots, k \), be \( k \) continuous functions from \( \Omega = J \times A \times U \) to \( \mathbb{R}^n \). Consider the following state dependent switched system with uncertainty in the operating regions where they are described by human knowledge as follows:

\[
\dot{x} = g_k(t, x, u), \quad f_{ij}(x) \text{ is } M_{ij} \wedge \ldots \wedge f_{ik}(x) \text{ is } M_{ij}
\]

where, each \( M_{ij} \), are fuzzy sets in \( A \), \( k \) is the number of switched systems, \( f_{ij}(x) \) are known nonlinear functions which are functions of states and \( \wedge \) is fuzzy intersection.

**Assumption 2.1**

The trajectory functions \( t \in J \rightarrow x(t) \in A \) are absolutely continuous and the control function \( t \in J \rightarrow u(t) \in U \) is Lebesgue measurable.

**Assumption 2.2**

Without loss of generality it is assumed that the membership functions are normalized, thus we have

\[
\sum_{i=1}^{k} z_i(f_i(x)) = 1,
\]

where

\[
f_i(x) = [f_{i1}(x), f_{i2}(x), \ldots, f_{ik}(x)]
\]

\[
w_i(f_i(x)) = \prod_{j=1}^{p} \mu_{M_{ij}}(f_{ij}(x))
\]

\[
z_i(f_i(x)) = \frac{w_i(f_i(x))}{\sum_{i=1}^{k} w_i(f_i(x))}
\]

and \( \mu_{M_{ij}}: x \rightarrow [0, 1] \) is the membership function of fuzzy set \( M_{ij} \).

The objective is to minimize the following functional:

\[
\int_{J} f_{0}(t, x, u) \, dt
\]

such that, \( f_{0}: \Omega \rightarrow \mathbb{R} \) \(( \Omega = J \times A \times U \) \) be a continuous function, the control function \( u(.) \) takes values in the set \( U \), and the boundary conditions are \( x(t_a) = x_a, x(t_b) = x_b \). The cost functional (3) is written in a general form, and there is not any mathematical restrictions on the integrand function. However, in practice, the integrand is usually a positive semi-definite function, for every possible state and input combination.

3. Transformation based on T-S Fuzzy Systems
The fuzzy model proposed by Takagi-Sugeno (T-S) [5] is described by fuzzy IF-THEN rules which represent local input-output relations of a nonlinear system. The main feature of a T-S fuzzy model is to express the local dynamics of each fuzzy implication (rule) by a linear/nonlinear system model. The overall fuzzy model of the system is achieved by fuzzy “blending” of the linear/nonlinear system models. It is proved that many nonlinear dynamic systems can be represented by T-S fuzzy models (i.e., fuzzy models are universal approximators) [6]. The switched system (1) can be modeled via a T-S fuzzy logic system with the following IF-THEN rules:

Rule \( i \), where \( i = 1, 2, \ldots, k \):

\[
\text{IF } f_{i1}(x) \text{ is } M_{i1} \text{ and } \ldots \text{ and } f_{ip}(x) \text{ is } M_{ip} \\
\text{THEN } \dot{x}(t) = g_i(t, x, u)
\] (4)

Given the pair of \((x(t), u(t))\), the output of the fuzzy system derived from the original switched state dependent system is inferred as the following nonlinear system:

\[
\dot{x}(t) = \sum_{i=1}^{k} z_i(f_i(x))g_i(t, x, u) = h(t, x, u)
\] (5)

for all \( t \), where \( z_i(f_i(x)) \) is defined in Assumption 2.2 and based on this assumption we have \( \sum_{i=1}^{k} z_i(f_i(x)) = 1 \). Thus, the main problem is converted to the following optimal control problem:

\[
\int_{t_0}^{t_f} f_b(t, x, u) \, dt \\
\text{s.t.} \\
\dot{x} = h(t, x, u) \quad t \in J, \ x \in A, \ u \in U \\
x(t_0) = x_a \\
x(t_f) = x_b
\] (6)

4. Particle Swarm Optimization

Kennedy and Eberhart [7] introduced the Particle Swarm Optimization (PSO) algorithm, which is inspired by social behaviors observed in fish schools and bird flocks. PSO is mainly based on Swarm Intelligence paradigm, and also has many common properties with Evolutionary Computation methods. In PSO, particles are individuals, representing solutions of an optimization problem, and moving in the search space, obeying some permanent rules, to find the global optimum. Since its inception, PSO has been used for solving diverse range of optimization problems in the area of engineering and science [8, 9, 10] and different variants of the algorithm are proposed [11, 12].

A particle memorizes its best experience, namely personal best, and uses its own best performance and best performance of the whole swarm of particles, known as global best, to find a better point in the search space. Particles have five basic properties: Current Position, Velocity, Current Objective Value, Best Position ever found (or Personal Best), and Objective Value corresponding to the Best Position.

Consider a minimization problem, with cost function \( f : \mathbb{R}^n \to \mathbb{R} \). Hence solutions of this problem are real valued \( n \)-dimensional points. Assume that position of particle \( i \), at iteration \( t \), is represented by \( x_{i,[t]} \), and \( x_{i,j,[t]} \) stands for the value of \( j \)-th element of the position. Similarly, the velocity of particle \( i \), at iteration \( t \), and the value of its \( j \)-th element, are shown by \( v_{i,[t]} \) and \( v_{i,j,[t]} \), respectively. The symbols \( p_{i,[t]} \), \( p_{i,j,[t]} \), \( g_{[t]} \) and \( g_{j,[t]} \), stand for best position of particle \( i \), \( j \)-th element of best
position of particle \( i \), global best position, and \( j \)-th element of global best position, respectively. The update rules for PSO are defined by:

\[
\begin{align*}
\mathbf{v}_{i,j}(t+1) &= w\mathbf{v}_{i,j}(t) + c_1 r_1 (\mathbf{p}_{i,j}(t) - \mathbf{x}_{i,j}(t)) + c_2 r_2 (\mathbf{g}_{j}(t) - \mathbf{x}_{i,j}(t)) \\
\mathbf{x}_{i,j}(t+1) &= \mathbf{x}_{i,j}(t) + \mathbf{v}_{i,j}(t+1)
\end{align*}
\]

(7)

and

\[
\begin{align*}
\mathbf{v}_{i,j}(t+1) &= \mathbf{v}_{i,j}(t) + \mathbf{v}_{i,j}(t+1) \\
\mathbf{x}_{i,j}(t+1) &= \mathbf{x}_{i,j}(t) + \mathbf{v}_{i,j}(t+1)
\end{align*}
\]

(8)

where \( w \) is the inertia factor, \( c_1 \) is personal learning factor, \( c_2 \) is global learning factor, and \( r_1 \) and \( r_2 \) are random numbers uniformly distributed in the range \([0,1]\).

In this paper, the values of inertia factor and learning coefficients are chosen according to [13], as follows:

\[
w = \frac{2}{\phi - 2 + \sqrt{\phi^2 - 4\phi}}
\]

(9)

\[
c_1 = \phi_1 w
\]

\[
c_2 = \phi_2 w
\]

where \( \phi_1 \) and \( \phi_2 \) are positive real constants, and \( \phi = \phi_1 + \phi_2 \). A good choice for \( \phi_1 \) and \( \phi_2 \), is \( \phi_1 = \phi_2 = 2.05 \). Therefore according to (8), the values of PSO parameters are set as follows: \( w = 0.7298 \), and \( c_1 = c_2 = 1.4962 \). These values are used in the simulations of this paper.

5. Proposed Method

In this paper the optimal controller is the output of a Mamdani type fuzzy system. The fuzzy system contains fuzzifier, fuzzy rule base, inference engine and the defuzzifier. The input of the fuzzy system is selected as and the fuzzy rule base is built by the following fuzzy IF-THEN rules

\[
\text{IF } t \text{ is } A_l \text{ THEN } u \text{ is } B_l, \quad l = 1, \ldots, M
\]

(10)

where, \( A_l \) and \( B_l \) are Gaussian fuzzy sets such that \( A_l, l = 1, \ldots, M \) are used to partition the compact set \( J \), and \( B_l, l = 1, \ldots, M \) are linguistic description of control signal \( u \). Using singleton fuzzifier, product inference engine and center average defuzzifier the output of the fuzzy system which is also the control signal is obtained as

\[
u(t) = \frac{\sum_{i=1}^{M} w_i \mu_{A_i}(t)}{\sum_{i=1}^{M} \mu_{A_i}(t)}
\]

(11)

where, \( \mu_{A_i} \) is the membership function corresponding to fuzzy set \( A_i \) and \( w_i \) is the center of the membership function corresponding to fuzzy set \( B_i \). Thus we have a smooth control signal. On the other hand, the control signal \( u(t) \) based on (6) should be designed such that the cost functional

\[
\int_J f_x(t, x, u) \, dt
\]

(12)

is minimized and also the following boundary condition is satisfied:

\[
x(t_0) = x_a
\]

(13)

Therefore, the center of the output membership functions of the fuzzy system \( w_j \) is assumed free and tuned based on a PSO algorithm such that the following cost functional is minimized:
\[
\alpha \left[ \frac{x(f_\alpha) - x_i}{x_o} \right] + \int_0^T f_\alpha(t, x, u) \, dt
\]  

where, it contains (12) and also condition (13) is added to (14) as a penalty term. The penalty weight parameter \( \alpha \) is selected as a big positive constant. Initial state of the system is also a constraint in (6) and it is considered and satisfied in the simulation and optimization process.

6. Simulation Example
Consider the following nonlinear switched optimal control problem:

\[
\min I(u(.)) = \int_0^1 u^2(t) \, dt \\
\dot{x} = \frac{1}{2} x^2 \sin x + u, \quad x \text{ is Positive} \\
\dot{x} = x^3 + u, \quad x \text{ is Negative} \\
x(0) = -0.5 \\
x(1) = 0.8
\]  

where, \( x \in A = [-1,1] \), \( J = [0,1] \) and \( u \in U = [-1,8] \), \( M_1 = \text{Positive} \), \( M_2 = \text{Negative} \) and

\[
\mu_{\mu_1}(x) = \frac{1}{1 + \exp\left(-\frac{x}{0.01}\right)}, \\
\mu_{\mu_1}(x) = \frac{\exp\left(-\frac{x}{0.01}\right)}{1 + \exp\left(-\frac{x}{0.01}\right)}
\]  

Now, we use the following T-S fuzzy rules:

\[
\text{IF } x \text{ is } M_1 \text{ THEN } \dot{x} = \frac{1}{2} x^2 \sin(x) + u \\
\text{IF } x \text{ is } M_2 \text{ THEN } \dot{x} = x^3 + u
\]  

Now, by following the fuzzy inference method of T-S fuzzy systems the fuzzy blending of the rules in (17) becomes the following nonlinear system:

\[
\dot{x} = \frac{1}{1 + \exp\left(-\frac{x}{0.01}\right)} \left[ \frac{1}{2} x^2 \sin x + u + \exp\left(-\frac{x}{0.01}\right) \right] (x^3 + u) + \frac{\exp\left(-\frac{x}{0.01}\right)}{1 + \exp\left(-\frac{x}{0.01}\right)} (x^3 + u)
\]  

\[
\dot{x} = \frac{x^2}{1 + \exp\left(-\frac{x}{0.01}\right)} \left[ \frac{1}{2} \sin x + x \exp\left(-\frac{x}{0.01}\right) \right] + u
\]  

Therefore, we have the following optimal control problem:
Based on the cost functional (14), the nonlinear optimal control (19) can be written as

\[ \min I(u(.)) = \int_0^T u^2(t) \, dt \]

\[ x = \frac{x^2}{1 + \exp\left(-\frac{x}{0.01}\right)} \left[ \frac{1}{2} \sin x + x \exp\left(-\frac{x}{0.01}\right) \right] + u \]

(19)

\[ x(0) = -0.5 \]

\[ x(1) = 0.8 \]

Using the proposed method the smooth optimal control and the corresponding optimal trajectory is depicted in Figure 1 and 2, respectively. The smoothness of the fuzzy switching behaviour of the system itself and the smooth fuzzy control input results in the smooth response of the system, shown in Figure 1. In contrast, for normal non-fuzzy switching systems, due to discontinuous and crisp switching rules, sudden changes and jumps in the response can be observed.

**Figure 1.** Smooth Optimal Control Input resulted from PSO

**Figure 2.** Optimal State Trajectory corresponding to the Smooth Optimal Control Input shown in Figure 1.
7. Conclusion
In this paper we proposed a new approach for optimal control of a class of state dependent switched systems with fuzzy based operating regions. First the switched system was modeled via T-S fuzzy systems, and then by following the fuzzy inference method of T-S fuzzy system, the switched system was transformed to a nonlinear system. In the next step, a smooth control built by a fuzzy system with weighted average defuzzifier was proposed. To optimize the weights such that a desired cost functional is minimized a PSO algorithm was used. To show the effectiveness of the proposed method a simulation example was illustrated. In future works, we intend to propose a closed-form for the optimal control signal while the system is under modelling errors and unknown external disturbances.

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