Weak approximation for stochastic differential equations with jumps by iteration and hard bounds

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Abstract

We establish a novel theoretical framework in which weak approximation can be conducted in an iterative and convergent manner for a large class of multivariate inhomogeneous stochastic differential equations with jumps of general time-state dependent intensity. The proposed iteration scheme is built on a sequence of approximate solutions, each of which makes use of a jump time of the underlying dynamics as an information relay point in passing the past on to a previous iteration step to fill in the missing information on the unobserved trajectory ahead. We prove that the proposed iteration scheme is convergent and can be represented in a similar form to Picard iterates under the probability measure with its jump component suppressed. On the basis of the approximate solution at each iteration step, we construct upper and lower bounding functions that are convergent towards the true solution as the iterations proceed. We provide illustrative examples so as to examine our theoretical findings and demonstrate the effectiveness of the proposed theoretical framework and the resulting iterative weak approximation scheme.

Keywords: jump-diffusion processes; time-state dependent jump rate; Picard iteration; partial integro-differential equations; first exit times.

2020 Mathematics Subject Classification: 91B30, 60G51, 65M15, 65N15.

1 Introduction

Stochastic processes with jumps have long attracted a great deal of attention in a wide variety of applied areas since the distant past to the present day, ranging from social sciences [2, 8] and natural sciences [7, 15] to information and computer sciences [17], to name just a few. By introducing jumps, the underlying dynamics cannot only be made richer with possibly heavy-tailed marginals but also literally model characteristic jumps at the cost of analytical tractability. For instance, it is essential in actuarial science that the underlying dynamics contains jumps to represent insurance claims [4]. Most of the existing literature has focused on exploring analytical or, at least, semi-analytical solutions, whereas, due to the inherent complexity of the dynamics with jumps, the analytical solution can only be found in a limited number of rather classical models. In practice, insurance companies are more concerned with the risks in a finite time horizon, where the availability of analytical solutions is even more significantly limited.

The computation of expectation on jump processes has thus traditionally been tackled by exploring other avenues through approximation. For instance, in inversion-based approaches, computationally intensive numerical methods are often necessary even when the characteristic functions of the transition distribution are available in closed form [2], where the inversion-based density evaluation suffers from a prohibitive load for each parameter set in numerical procedure. Others include numerical methods for sample path generation [11, 5, 8, 13], for solving the relevant partial integro-differential equations [4, 14] and for constructing hard bounding functions [9, 10]. As is often the case, each has weaknesses in terms of, for instance, problem dimension, problem constraints, regularity requirements, implementability and complexity. As such, it is undoubtedly valuable to have many methodologies available so as to address different circumstances.

The aim of the present work is to establish a novel theoretical framework in which an iterative weak approximation scheme can be constructed for a large class of stochastic differential equations with jumps. The underlying class is large enough to accommodate the multivariate time-inhomogeneous dynamics, even in the absence of uniform ellipticity. Equally important is that its jump component can be governed by a general time-state dependent unbounded rate [8, 13]. In the proposed framework, a sequence (Theorem 3.2), in fact, two sequences (Theorem 3.4) of approximate solutions are set up with a jump time of the underlying dynamics as an information relay point in passing the past information on to a previous iteration step to fill in the missing information on the unobserved trajectory ahead. We prove that the resulting iteration scheme is, as it should be, convergent (Theorem 3.1) and, moreover, can even be monotonically convergent (Theorem 3.6) under suitable assumptions imposed on the input data. In particular, if approximation solutions are smooth enough, then, by using the approximate solution at each iteration step, upper and lower bounding functions can be constructed (Theorem 3.6) in the same spirit as [10], which form, so to speak, a 100% confidence interval for the true solution.

It turns out that those approximate solutions can be represented under the probability measure with its jump component suppressed (Theorem 3.5), that is, the underlying dynamics with jumps is then measured as though the dynamics contained no jumps. The iterative nature of those approximate solutions as a sequence in the original form does not look like a Picard iteration, whereas they, after this “de-jump” transform, may well do. This result is not only effective from a numerical point of view (for the obvious reason of the absence of prohibitive load for each parameter set in numerical procedure. Others include numerical methods for sample path generation [11, 5, 8, 13], for solving the relevant partial integro-differential equations [4, 14] and for constructing hard bounding functions [9, 10]. As is often the case, each has weaknesses in terms of, for instance, problem dimension, problem constraints, regularity requirements, implementability and complexity. As such, it is undoubtedly valuable to have many methodologies available so as to address different circumstances.

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of cumbersome random jumps), but also insightful from a broader perspective as the approximate solution can then be put in contrast with the Feynman-Kac representation of a suitable initial boundary value problem for partial differential equations (with the corresponding cumbersome integral term dropped), provided that an additional set of sufficient conditions is imposed on the problem parameter (Theorem 3.9), for which a variety of numerical methods, such as finite element methods, are available in the literature.

As is always the case in the context of time-state dependent jump rates, the proposed framework can benefit a lot from the uniform boundedness of the jump rate \([1]\). For instance, the proposed iteration scheme can then be made more implementable via the Poisson thinning (Theorem 3.7). In addition, the modes of convergence of the sequences of approximate solutions and associated hard bounding functions can be strengthened (Theorem 3.8). Remarkably, all the results are valid irrespective of problem dimension, whereas most existing numerical methods have mainly been focused on low, or even one, dimensions.

The rest of this paper is set out as follows. In Section 2 we summarize and formulate the background materials. We begin Section 3 with the construction of approximate solutions and their convergence towards the true solution as a sequence (Theorem 3.1). We then present theoretical findings in turn (Theorems 3.2, 3.4, and 3.5) respectively, in Sections 3.1, 3.2, and 3.3, without additional technical conditions, and construct hard bounding functions (Theorem 3.6) in Section 3.4. We close the main section by developing the Poisson thinning (Theorem 3.7) and strengthening the modes of convergence (Theorem 3.8), provided that the jump rate is uniformly bounded. In Section 4 we examine two illustrative examples so as to demonstrate the effectiveness of the proposed theoretical framework and the resulting iterative weak approximation scheme. To maintain the flow of the paper, we collect all derivations and proofs in the Appendix.

2 Problem formulation

We first develop the notation that will be used throughout the paper and introduce the underlying stochastic process. We denote the set of real numbers by \(\mathbb{R}\), the set of natural numbers excluding 0 by \(\mathbb{N} := \{1, 2, \ldots\}\), and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). We denote by \(\mathbb{R}^d\) the \(d\)-dimensional Euclidean space with \(\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}\). We fix a non-empty open convex subset \(D\) of \(\mathbb{R}^d\) with Lipschitz boundary \(\partial D\), and denote by \(\overline{D} := D \cup \partial D\) its closure.

Consider the stochastic process \(\{X_t : t \geq 0\}\) in \(\mathbb{R}^d\) defined by the stochastic differential equation:

\[
dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t + \int_{\mathbb{R}^d} z\mu(dx,dt;X_t-), \quad X_0 \in \overline{D},
\]

where \(\{W_t : t \geq 0\}\) is the standard Brownian motion in \(\mathbb{R}^d\), and the coefficients \(b\) and \(\sigma\) are Lipschitz continuous in space and locally bounded in time. For every \(x \in \overline{D}\), \(\mu(dx,dt;x)\) denotes a Poisson counting measure on \(\mathbb{R}^d_0 \times [0, \infty)\), whose compensator is \(\lambda(t,x)v(dx;x)dt\), where \(\lambda(t,x)\) is a non-negative function mapping from \([0, \infty) \times \overline{D}\) to \([0, \infty)\), and \(v(dx;x)\) is a finite Levy measure on \(\mathbb{R}^d_0\). As usual, we assume that the Brownian motion \(\{W_t : t \geq 0\}\) and the Poisson counting measure \(\mu(dx,dt;x)\) are mutually independent for \(x \in \overline{D}\).

Since the Lévy measure \(v(dx;x)\) is assumed to be finite \([12]\), it loses no generality to impose the standardization condition \(\int_{\mathbb{R}^d} v(dx;x) = 1\) for all \(x \in \overline{D}\). Moreover, since the singularity of jump sizes can be well captured by the Lévy measure \(v(dx;x)\) (for instance, a finite convolution of Poisson processes with different jump sizes), it is not restrictive to assume that the rate function \(\lambda\) is locally Lipschitz. In particular, as the rate function \(\lambda\) and the Lévy measure \(v\) depend on the state, the jump component in \((2.1)\) affects itself in a non-trivial way, that is, each jump can have an impact, not only on the current and future jump intensity and magnitude, but also on the drift and diffusion components, and, in fact, vice versa. Note that the rate function \(\lambda\) does not even need to be bounded \([8, 13]\) until we impose uniform boundedness in Section 3.5.

We denote by \(P^X_A(\cdot) := P_X(\cdot | X_0 = x)\) the probability measure under which \(X_t = x\) almost surely and the rate of jumps is governed by the function \(\lambda\), where \(\lim_{\delta \to 0^+} \delta^{-1} E^X_A(X_{t+\delta} \neq X_t) = \lambda(t,x)\) for almost every \((t,x) \in [0,T] \times \overline{D}\), due to the standardized assumption \(\int_{\mathbb{R}^d} v(dx;x) = 1\) for all \(x \in \overline{D}\). As usual, we denote by \(E^X_A(\cdot)\) the expectation taken under the probability measure \(P^X_A\). By formally setting \(\lambda = 0\) in the probability measure \(P^X_A\) and the expectation \(E^X_A(\cdot)\), one may switch on and off the jump component in the underlying process \((2.1)\).

**Remark 2.1.** (i) The underlying process \((2.1)\) can be thought of as an approximation of that with an infinite Lévy measure \(\int_{\mathbb{R}^d} v(dx;x) = +\infty\). For instance, the jump component can be parameterized as \(\int_{\mathbb{R}^d} z\mu_n(dx,dt;X_t-), \) where \(\mu_n(dx,dt;x)\) denotes a Poisson counting measure on \(\mathbb{R}^d_0 \times [0, \infty)\) for \(x \in \overline{D}\), whose compensator is now \(\lambda(t,x)v_n(dx;x)dt\), with \(\int_{\mathbb{R}^d} v_n(dx;x) = n\) for \(n \in \mathbb{N}\). Truncation by a sequence of Lévy measures of increasing intensities (that is, \(\{v_n(dx;x)\}_{n \in \mathbb{N}}\)) has already been well studied for major Lévy measures. See \([16]\) for a comprehensive survey on this topic.

(ii) The jump component can be generalized to, for instance, \(\int_{\mathbb{R}^d} m(t,X_t-;z)\mu(dx,dt;X_t-)\) as in \([10]\), so as to modulate each jump at \((t,X_t-)\) by a mapping \(m(t,X_t-;z)\). In the present paper, we do not adopt this generalization, as it does not seem to add essential value in the presence of the time-state dependent rate \(\lambda\) and the state dependent Poisson counting measure \(\mu(dx,dt;x)\).

Throughout, we reserve \(T\) for the terminal time of the interval of interest \([0,T]\). Define the first exit time of the underlying process \(\{X_t : t \geq 0\}\) from the closure of the domain \(\overline{D}\) later than a fixed time \(t\), as well as its capped one by the terminal time \(T\), by

\[
\eta_t := \inf\{s > t : X_s \notin \overline{D}\}, \quad \eta^T_t := \eta_t \land T.
\]
The aim of the present work is to construct a theoretical framework for iterative weak approximation schemes for the function given in the form of expectation:
\[
 u(t, x) := \mathbb{E}_{t}^{x} \left[ \mathbb{I}(\eta_{t} > T) \Theta_{t,T} g(X_{T}) + \mathbb{I}(\eta_{t} \leq T) \Theta_{t,\eta_{t}} \Psi(\eta_{t}, X_{\eta_{t}}, X_{\eta_{t} -}) - \int_{t}^{\eta_{t} \wedge T} \Theta_{t,s} \phi(s, X_{s}) ds \right], \quad (t, x) \in [0, T] \times \mathcal{D},
\]
(2.3)
with \( g : \mathcal{D} \to \mathbb{R}, \Psi : [0, T] \times (\mathbb{R}^{d} \setminus \mathcal{D}) \times \mathcal{D} \to \mathbb{R}, \phi : [0, T] \times \mathcal{D} \to \mathbb{R} \)
and
\[
 \Theta_{t,j} := \exp \left[ - \int_{t_{j}}^{t_{j+1}} r(s, X_{s}) ds \right], \quad 0 \leq t_{j} \leq t_{j+1} \leq T.
\]
(2.4)
For instance, the (finite-time version of) expected discounting penalty function \( \mathcal{H} \) is a special case, given by
\[
 u(t, x) = \mathbb{E}_{t}^{x} \left[ e^{-r(T-t)} w(X_{\eta_{t} \wedge T}, |X_{\eta_{t} \wedge T}|) \right], \quad (t, x) \in [0, T] \times (0, +\infty),
\]
with a spectrally negative Lévy measure. Here, on the event \( \{ \eta_{t} > T \} \), we have \( X_{\eta_{t} \wedge T} = X_{T} = X_{T} \), that is, \( w(X_{\eta_{t} \wedge T}, |X_{\eta_{t} \wedge T}|) = w(X_{T}, X_{T}) \), corresponding to the terminal condition \( g \) in (2.3). On the event \( \{ \eta_{t} \leq T \} \), the term \( w(X_{\eta_{t} \wedge T}, |X_{\eta_{t} \wedge T}|) \) can be represented by the boundary condition \( \Psi \) in (2.3) with its time argument suppressed.

Before moving to the main section, we summarize the basic assumptions that stand throughout.

**Assumption 2.2.** (a) The domain \( D \) is non-empty convex subset of \( \mathbb{R}^{d} \) with Lipschitz boundary \( \partial D \).
(b) For every \( x \in \mathcal{D} \), the intensity measure \( \nu(\{dz \mid x \}) \) is a Lévy and probability measure on \( \mathbb{R}^{d}_{0} \).
(c) The input data \( g, \Psi \) and \( \phi \) are uniformly bounded on their respective domains.
(d) The coefficients \( b \) and \( \sigma \) are locally bounded in time, and are Lipschitz continuous and grow at most linearly on \( \partial D \).

## 3 Main results

We are ready to start our main section. First, define the time of the first jump occurring after time \( t \) by
\[
 \tau_{1}^{(1)} := \inf \{ s > t : X_{s} \neq X_{t} \},
\]
(3.1)
and then, recursively, the time of the \( m \)-th jump occurring after time \( t \) by
\[
 \tau_{m}^{(m)} := \inf \{ s > \tau_{m}^{(m-1)} : X_{s} \neq X_{t} \},
\]
(3.2)
for \( m \in \mathbb{N} \), with the time of the zero-th jump after time \( t \) defined formally by \( \tau_{0}^{(0)} := t \). Next, define the function \( w_{0} : [0, T] \times \mathcal{D} \to \mathbb{R} \) by
\[
 w_{0}(t, x) := \mathbb{E}_{t}^{x} \left[ \mathbb{I}(\eta_{t} > T) \Theta_{t,T} g(X_{T}) + \mathbb{I}(\eta_{t} \leq T) \Theta_{t,\eta_{t}} \Psi(\eta_{t}, X_{\eta_{t}}, X_{\eta_{t} -}) - \int_{t}^{\eta_{t} \wedge T} \Theta_{t,s} \phi(s, X_{s}) ds \right], \quad (t, x) \in [0, T] \times \mathcal{D},
\]
(3.3)
which we let play a role as the initial approximate solution for the proposed iteration scheme. It is worth noting that the expectation (3.3) is taken under the probability measure \( \mathbb{P}_{t}^{x} \), namely, the jump component is suppressed with zero rate \( \lambda_{t} \equiv 0 \). In this case, thus, we observe no jumps, to say nothing of overshooting, that is, \( \Psi(\eta_{t}^{T}, X_{\eta_{t}^{T}}, X_{\eta_{t}^{T} -}) = \Psi(\eta_{t}^{T}, X_{\eta_{t}^{T}}, X_{\eta_{t}^{T} -}), \mathbb{P}_{t}^{x} \)-a.s. On the basis of the initial approximate solution (3.3), define the function \( w_{m} \) by
\[
 w_{m}(t, x) := \mathbb{E}_{t}^{x} \left[ \mathbb{I}(X_{\eta_{t} \wedge \tau_{m}^{(m)}} \in \mathcal{D}) \Theta_{t,\eta_{t} \wedge \tau_{m}^{(m)}} \Psi(\eta_{t}^{T} \wedge \tau_{m}^{(m)}, X_{\eta_{t} \wedge \tau_{m}^{(m)}}) \right.
\[
 + \mathbb{I}(X_{\eta_{t} \wedge \tau_{m}^{(m)}} \notin \mathcal{D}) \Theta_{t,\eta_{t} \wedge \tau_{m}^{(m)}} \Psi(\eta_{t}^{T} \wedge \tau_{m}^{(m)}, X_{\eta_{t} \wedge \tau_{m}^{(m)}}) - \int_{t}^{\eta_{t} \wedge \tau_{m}^{(m)}} \Theta_{t,s} \phi(s, X_{s}) ds \right],
\]
(3.4)
for \( m \in \mathbb{N} \) and \( (t, x) \in [0, T] \times \mathcal{D} \). We first claim that the sequence \( \{ w_{m} \}_{m \in \mathbb{N}_{0}} \) is convergent to \( u \), that is, the function of interest (2.3).

**Theorem 3.1.** **It holds, as \( m \to +\infty \), that \( w_{m}(t, x) \to u(t, x) \) for all \( (t, x) \in [0, T] \times \mathcal{D} \).**
3.1 Iterative approximation by paths until the first few jumps

On the basis of Theorem 3.1, the function \( w_m \), with a sufficiently large \( m \), can be employed as an approximation for \( u \), while a large \( m \) is nothing but observing the sample path until the \( m \)-th jump time \( \tau_{m}^{(m)} \). According to the representation (3.4), there would then be no essential difference from computation required for the original form (2.3) anymore. In particular, when the underlying stochastic process (2.1) is employed as a finite-intensity approximation of an infinite-intensity counterpart (Remark 2.1 (i)), an extremely large number of jumps would be required for reasonable accuracy of weak approximation.

This issue can be addressed in part with the aid of the following result, which leads to the construction of an iterative weak approximation scheme.

**Theorem 3.2.** Let \( m \in \mathbb{N} \) and \( n \in \{0, 1, 2, \ldots, m\} \). It holds that

\[
w_m(t, x) = \mathbb{P}_x^t \left( \mathbb{I}(X_{\tau_{m}^{(n)} \wedge T} \notin \Omega) \Theta_{t, \eta_T \wedge \tau_{m}^{(n)}} w_{m-n}(\eta_T \wedge \tau_{m}^{(n)}, x_{\tau_{m}^{(n)}}) \right.
\]

\[
+ \mathbb{I}(X_{\tau_{m}^{(n)} \wedge T} \notin \Omega) \Theta_{t, \eta_T \wedge \tau_{m}^{(n)}} \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -}), \int_{t}^{\eta_T \wedge \tau_{m}^{(n)}} \theta_{s, \tau}(x, X_s)ds \right), \tag{3.5}
\]

for all \((t, x) \in [0, T] \times \Omega\).

In comparison with the representation (3.4) of the approximate solutions \( \{w_m\}_{m \in \mathbb{N}_0} \), a few evident yet interesting features of the equivalent representation (3.5) can be summarized as follows. First of all, the representation (3.5) can obviously recover the original one (3.4) by setting \( n = m \). The opposite extreme is the case \( n = 0 \), which yields the trivial identity \( w_m = w_m \), due to \( \tau_0 = t \) by definition. When \( n \in \{1, \ldots, m-1\} \) in the middle, the representation (3.5) requires one to make an observation of the sample path until, at furthest, its \( n \)-th jump time \( \tau_{m}^{(n)} \) (no longer all the way up to the \( m \)-th jump as for the previous representation (3.4)), on the basis of the previous approximate solution \( w_{m-n} \) at \( n \) steps ago.

Here, the \( n \)-th jump time \( \tau_{m}^{(n)} \) can be thought of as an information relay point of passing the sample path behind to the previous step \( w_{m-n} \), which fills in the missing information on the unobserved trajectory ahead. From a computational point of view, a smaller error can be expected with a smaller number \( n \) of jumps required, since then the sample path for observation is generally shorter up to the \( n \)-th jump time, whereas the approximation of the previous step \( w_{m-n} \) needs to be sufficiently accurate to ensure so.

Before moving on to further theoretical developments, we provide an illustrative insight into the approximate solution \( w_m \) with the aid of the representation (3.5) in the context of the finite-time ruin probability, which is of significant interest in insurance mathematics. We remark that the additional conditions on the drift and diffusion coefficients are imposed so as to ensure that an exit can only occur by a jump.

**Proposition 3.3.** Suppose that, for every \( t \in [0, T] \), the drift \( b(t, \cdot) \) is not outward pointing at the boundary \( \partial D \), assume that there exists \( \varepsilon > 0 \) such that \( \int_{[0, T] \times \partial D} \sigma(t, x)dx = 0 \) where \( \partial D_\varepsilon := \{x \in \partial D : \text{dist}(x, \partial D) \leq \varepsilon\} \), and let \( r = 0, g = 0, \Psi = 1 \) and \( \phi = 0 \) in (2.3). It then holds that for \( m \in \mathbb{N} \) and \((t, x) \in [0, T] \times \Omega\),

\[
w_m(t, x) = \mathbb{P}_x^t \left( \mathbb{I}(X_{\tau_{m}^{(1)} \wedge T} \notin \Omega) + \mathbb{I}(\{X_{\tau_{m}^{(1)} \wedge T} \notin \Omega\} \cap \{X_{\tau_{m}^{(2)} \wedge T} \notin \Omega\}) \right.
\]

\[
+ \cdots + \mathbb{I}(\{X_{\tau_{m}^{(1)} \wedge T} \notin \Omega\} \cap \cdots \cap \{X_{\tau_{m}^{(m-1)} \wedge T} \notin \Omega\} \cap \{X_{\tau_{m}^{(m)} \wedge T} \notin \Omega\}) \right),
\]

and \( w_m(t, x) \geq w_{m-1}(t, x) \), with \( w_0 \equiv 0 \).

In words, in the context of the finite-time ruin probability, the approximate solution \( w_m \) represents the probability of ruin earlier than or right by the \( m \)-th jump. Hence, the convergence towards the true ruin probability \( u \) as well as its monotonicity are quite convincing. The monotonicity here is revisited in a general setting later in Theorem 3.4.

3.2 Iterative approximation along two routes with monotonicity

Apart from the iteration scheme based on the representation (3.5), we may construct, so to speak, a side road, which allows the iteration to proceed along an alternative route, involving the sequence \( \{v_m\}_{m \in \mathbb{N}} \), defined by

\[
v_m(t, x) := \mathbb{P}_x^t \left( \mathbb{I}(\tau_{m}^{(m)} \geq \eta_T \wedge T) \mathbb{I}(X_{\eta_T \wedge \tau_{m}^{(m)}} \notin \Omega) \Theta_{t, \eta_T \wedge \tau_{m}^{(m)}} \right.
\]

\[
+ \mathbb{I}(X_{\eta_T \wedge \tau_{m}^{(m)}} \notin \Omega) \Theta_{t, \eta_T \wedge \tau_{m}^{(m)}} \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -}), \int_{t}^{\eta_T \wedge \tau_{m}^{(m)}} \theta_{s, \tau}(x, X_s)ds \right), \tag{3.6}
\]
for \((t, x) \in [0, T] \times \overline{D}\), which initiates itself with \(v_1\), unlike the sequence \(\{w_m\}_{m \in \mathbb{N}_0}\) begins with \(w_0\). Other than that, the sequence \(\{v_m\}_{m \in \mathbb{N}}\) here differs only slightly from the original one \((3.4)\) by the event \(\{\tau_i^{(m)} < \eta_i^T\} \cap \{X_{\tau_i^{(m)}} \in \overline{D}\}\) in the first term of \((3.4)\), on which the sample path makes the \(m\)-th jump before the terminal time \(T\) or the first exit time \(\eta_i\), and does not exit due to the \(m\)-th jump. The sequence \(\{v_m\}_{m \in \mathbb{N}}\) is convergent to the true solution \(u\), along with \(\{w_m\}_{m \in \mathbb{N}_0}\), as follows.

**Theorem 3.4.** (a) It holds that for \((t, x) \in [0, T] \times \overline{D}\), \(m \in \mathbb{N}\) and \(n \in \{0, 1, \ldots, m - 1\}\),

\[
v_{m+1}(t, x) = v_{m-n}(t, x) + \mathbb{E}^x_0 \left[ \mathbb{I}(\eta_i^T < \eta_i^T) \Theta_i \mathbb{I}(T_i^{(m-n)} < \eta_i^T) X_{T_i^{(m-n)}} \right],
\]

and that \(v_m(t, x) \to u(t, x)\) as \(m \to +\infty\) for all \((t, x) \in [0, T] \times \overline{D}\).

(b) If \(g \geq 0\), \(\Psi \geq 0\) and \(\Phi \geq 0\) (or, if \(g \leq 0\), \(\Psi \leq 0\) and \(\Phi \geq 0\)), then the convergence in (a) is monotonic.

Hence, as long as the iteration begins with \(w_0\), one may stick to a single route (either \(\{w_m\}_{m \in \mathbb{N}_0}\) by Theorem 3.2, or \(\{v_m\}_{m \in \mathbb{N}}\) based on \((3.7)\)), or can even go back and forth between these two routes thanks to the interaction \((3.8)\), and even further, may skip a few steps (\(n\) in \((3.7)\) and \((3.8)\)) in exchange for making longer observation to get more jumps accordingly.

As has already been seen in Proposition 3.3, the input data \(r \equiv 0\), \(g \equiv 0\), \(\Psi \equiv 1\) and \(\Phi \equiv 0\) correspond to the finite-time ruin probability and yield a monotonically increasing sequence of approximate solutions \(w_m \geq w_{m-1}\) with \(w_0 \equiv 0\). Indeed, Theorem 3.4(b) is fully consistent with those, since the input data for the finite-time ruin probability satisfy the condition in (b), and moreover, since \(w_0 \equiv 0\) yields \(w_m = v_m\) due to \((3.4)\) and \((3.6)\) in comparison.

### 3.3 Iterative approximation by suppressing jumps

The recursive relation \((3.5)\) allows one to represent the function \(w_m\) on the basis of a previous one \(w_{m-n}\) (at \(n\) steps ago) and sample paths only up to their \(n\)-th jump. In this section, we further reduce the representation \((3.5)\) to an alternative one on the probability measure \(\mathbb{P}^x_0\), that is, under which the jump component is suppressed. To this end, we prepare the following notation:

\[
G_m(t, x) := \int_{\{z \in \mathbb{R}^n : x + z \in \overline{D}\}} w_{m-1}(t, x + z) \lambda(t, x) v(dz, x),
\]

\[
H(t, x) := \int_{\{z \in \mathbb{R}^n : x + z \in \overline{D}\}} \Psi(t, x + z, x) \lambda(t, x) v(dz, x),
\]

for \((t, x) \in [0, T] \times \overline{D}\) and \(m \in \mathbb{N}\), as well as

\[
\Lambda_{t_1, t_2} := \exp \left[ - \int_{t_1}^{t_2} \lambda(s, X_s) ds \right], \quad 0 \leq t_1 \leq t_2 \leq T.
\]

**Theorem 3.5.** It holds that

\[
w_m(t, x) = \mathbb{E}^x_0 \left[ \mathbb{I}(\eta_i^T > T) \Theta_i, T \Lambda_i, T g(X_T) + \mathbb{I}(\eta_i^T \leq T) \Theta_i, \eta_i, \Lambda_i, \eta_i, \Psi(\eta_i, X_{\eta_i}, X_{\eta_i}) \right.
\]

\[
\left. - \int_{t}^{T} \Theta_i, s \Lambda_i, s (\Psi(s, X_s) - G_{m-1}(s, X_s) - H(s, X_s)) ds \right],
\]

for \(m \in \mathbb{N}\) and \((t, x) \in [0, T] \times \overline{D}\).

Let us stress again that the approximate solution \((3.9)\) is written on the probability measure \(\mathbb{P}^x_0\), that is, with the jump component suppressed. In a sense, the representation \((3.9)\) can be thought of as a further step back from the shortest observation in \((3.5)\) (shortest, as with \(n = 1\)), whereas not identical to its trivial identity \(w_m = w_m\) with \(n = 0\). In short, the sample path strictly before the first jump is indistinguishable under the two probability measures \(\mathbb{P}^x_0\) and \(\mathbb{P}^x\), while the missing information caused by ceasing observation strictly before the first jump can still be compensated by the two terms \(G_{m-1}\) and \(H\) inside the integral, where the first term \(G_{m-1}\) carries over the information from the previous step \(w_{m-1}\).

It is also worth stressing that no ellipticity condition has been imposed on the underlying stochastic process \((2.1)\) whatsoever. As an extreme example, if the diffusion component is fully suppressed \((\sigma \equiv 0)\), then the underlying stochastic process \((2.1)\) is fully degenerate under the probability measure \(\mathbb{P}^x\), that is, the sample path is a deterministic function under \(\mathbb{P}^x\). The representation \((3.9)\) can then reduce to

\[
w_m(t, x) = \mathbb{I}(\eta_i^T > T) \Theta_i, T \Lambda_i, T g(x(T)) + \mathbb{I}(\eta_i^T \leq T) \Theta_i, \eta_i, \Lambda_i, \eta_i, \Psi(\eta_i, x(\eta_i), x(\eta_i))
\]

\[
- \int_{t}^{T} \Theta_i, s \Lambda_i, s (\Psi(s, x(s)) - G_{m-1}(s, x(s)) - H(s, x(s))) ds, \quad (t, x) \in [0, T] \times \overline{D},
\]

(3.10)
where the function $x$ here denotes the unique solution to the deterministic ordinary differential equation $x'(s) = b(s, x(s))$ for $s \in [t, T]$ with initial condition $x(t) = x$. Here, we have $\eta_t = \inf\{s \geq t : x(s) \notin D\}$, $\Theta_t = \exp[-\int_t^\eta r(s, x(s))ds]$, and $A_{\eta t} = \exp[-\int_t^\eta \lambda(s, x(s))ds]$, where the first exit time $\eta_t$ is also deterministic in this context. In brief, if the diffusion component is suppressed, then the iterative approximation can be conducted based on the recursive relation (3.10) in a fully deterministic manner. From a broader perspective, the deterministic identity (3.10) reveals the fact that the proposed iterative scheme based on the representation (3.9) can thus be thought of as a (probabilistic) Picard iteration process. In Section [4.1] we provide a concrete example for comprehensive demonstration of this fully deterministic iteration.

### 3.4 Hard bounding functions

So far, we have built a theoretical framework (Theorems 3.2, 3.4 and 3.5), based on which iterative weak approximation schemes can be constructed. It would make those approximation schemes more comprehensive and definitive to provide hard bounds for the true solution in a similar spirit to [10]. To this end, we prepare some notation. For $(t, x) \in [0, T] \times \mathbb{R}^d$, define

$$
\xi(t, x; \lambda) := \mathbb{E}_{t}^{x}[\int_{t}^{\eta_t} \Theta_t ds], \quad M_r(t, x) := \int_{[z \in \mathbb{R}^d : z \notin D]} \xi(t, x + z; 0)\lambda(t, x)\nu(dz, x) - \lambda(t, x)\xi(t, x; 0),
$$

(3.11)

with $M^U(t) := \text{esssup}_{(s,x) \in [t,T] \times D}(M_r(s, x))$ and $M^L(t) := \text{essinf}_{(s,x) \in [t,T] \times D}(M_r(s, x))$. Moreover,

$$
N_m(t, x) := \begin{cases} 
G_0(t, x) - \lambda(t, x)w_0(t, x) + H(t, x), & \text{if } m = 0, \\
G_m(t, x) - G_{m-1}(t, x), & \text{if } m \in \mathbb{N}, 
\end{cases}
$$

(3.12)

with the essential supremum $\{N^U_m(t)\}_{m \in \mathbb{N}_0}$ and infimum $\{N^L_m(t)\}_{m \in \mathbb{N}_0}$, defined as follows:

$$
N^U_m(t) := \text{esssup}_{(s,x) \in [t,T] \times D}(N_m(s, x)), \quad N^L_m(t) := \text{essinf}_{(s,x) \in [t,T] \times D}(N_m(s, x)), \quad t \in [0, T].
$$

(3.13)

Clearly, $N^U_m(\cdot)$ and $N^L_m(\cdot)$ are monotone for all $m \in \mathbb{N}_0$. Also, define the differential operator $\mathcal{L}$ by

$$
\mathcal{L}_t f(t, x) := \langle b(t, x), \nabla f(t, x) \rangle + \frac{1}{2} \text{tr} ((\sigma(t, x))^{\otimes 2}\text{Hess}_x f(t, x)), \quad (t, x) \in [0, \infty) \times D,
$$

for sufficiently smooth $f$. We have defined this differential operator on the boundary $\partial D$ as well, because the diffusion component can be degenerate there. We are now ready to state the result.

**Theorem 3.6.** Let $m \in \mathbb{N}$. If $w_{m-1}$ and $w_m$ are smooth enough so that $\partial_t$ and $\mathcal{L}$ exist almost everywhere on $[0, T] \times D$, then the functions $w_m$ and $w_{m-1}$ together satisfy

$$
\partial_t w_m(t, x) + \mathcal{L} w_m(t, x) = (r(t, x) + \lambda(t, x))w_m(t, x) + \phi(t, x) - G_{m-1}(t, x) - H(t, x), \quad a.e. (t, x) \in [0, T] \times D,
$$

(3.14)

with $w_m(T, \cdot) = g(\cdot)$. If, moreover, $M^L(0) < 1$ and $\xi(\cdot, t; 0)$ is smooth enough so that $\partial_t$ and $\mathcal{L}$ exist almost everywhere on $[0, T] \times D$, then it holds that for $(t, x) \in [0, T] \times D$,

$$
\frac{N^L_m(t)}{1 - M^L(t)} \xi(t, x; 0) \leq u(t, x) - w_m(t, x) \leq \frac{N^U_m(t)}{1 - M^U(t)} \xi(t, x; 0).
$$

(3.15)

The inequalities (3.15) can be employed as hard bounding functions for the true solution $u(t, x)$ at an arbitrary step $m$ on the basis of the two consecutive approximate solutions $w_m$ and $w_{m-1}$ required for obtaining the function $N_m$ via the formula (3.12). We remark that the smoothness conditions on the functions $w_{m-1}, w_m$ and $\xi(\cdot, t; 0)$ above do not require the uniform ellipticity or regularity of the input data. Indeed, shortly in Section [4.1] we derive hard bounding functions (3.15) in a problem setting without diffusion component ($\sigma \equiv 0$), yet within the scope of Theorem 3.6.

Let us stress that the partial differential equation (3.14) here is not a consequence of the Feynman-Kac formula, but holds true, simply based on the definition (3.4) without imposing a variety of additional sufficient conditions. It may still not be of use on its own in the absence of concrete information about the boundary $\partial D$, while we still present it here because it plays a crucial role not only in deriving the hard bounding functions (3.15), but also in proving the upcoming Theorem 3.8 as well as reappears in Theorem 3.9 with more sufficient conditions imposed.

### 3.5 Uniformly bounded jump rate

We have now been equipped with a comprehensive theoretical framework for convergent iterative approximation schemes (Sections 3.1, 3.2 and 3.3), along with hard bounding functions (Section 3.4), which altogether can already be considered well developed for implementation purposes. Before moving on to numerical illustrations in Section [4.1] we show that some implementation aspects may be eased in exchange...
for imposing additional restrictions on the problem setting. In particular, hereafter, the jump rate \( \lambda \) is assumed to be uniformly bounded over the domain, that is, there exists a constant \( \bar{\lambda} \) such that \( \lambda(t,x) \leq \bar{\lambda} \) for all \( (t,x) \in [0,T] \times \mathcal{D} \).

The first benefit from the jump rate function being uniformly bounded is the Poisson thinning, as is standard, for instance, in the context of exact simulation of jump-diffusion processes [1]. In our context, by introducing the Poisson thinning, one may simplify the computation of each approximate solution, particularly when Monte Carlo methods are employed, at the sacrifice of the iteration progress.

For its formulation, we prepare some notation. For every \( (t,x) \in [0,T] \times \mathcal{D} \), define the jump intensity measure \( \tilde{\nu}(dz,t,x) \) on \( \mathbb{R}^d \) (here, including the origin) by

\[
\tilde{\nu}(dz,t,x) := \frac{\lambda(t,x)}{\bar{\lambda}} \nu(dz;x), \quad z \in \mathbb{R}^d,
\]

with a mass at the origin \( \nu(\{0\};t,x) := 1 - \lambda(t,x)/\bar{\lambda} \). Note that the measure \( \tilde{\nu}(dz,t,x) \) remains finite and standardized, that is, \( \int_{\mathbb{R}^d} \tilde{\nu}(dz,t,x) = 1 \), just as the base Lévy measure \( v(dz;x) \). In view of the identity \( \int_{\mathbb{R}^d} \bar{\lambda} \tilde{\nu}(dz,t,x) = \int_{\mathbb{R}^d} \lambda(t,x) \nu(dz;x) \), one may then formulate a thinning method. That is, when the underlying process is at \( (t,x) \), a jump is generated with the largest rate \( \bar{\lambda} \) and then accepted with time-state dependent probability \( \lambda(t,x)/\bar{\lambda} \), whereas its rejection with the remaining probability \( 1 - \lambda(t,x)/\bar{\lambda} \) is formally represented as a zero-sized jump.

Next, for every \( (t,x) \in [0,T] \times \mathcal{D} \), denote by \( \overline{P}_x^\lambda \) and \( \overline{P}_x^{\lambda,x} \), respectively, the probability measure and the associated expectation under which the underlying process \( \{X_t : s \geq 0\} \) is governed, not by (2.1) but, by the following stochastic differential equation

\[
dX_t = b(s,X_t)ds + \sigma(s,X_t)dW_s + \int_{\mathbb{R}^d} \tilde{\mu}(dz,dx)ds, \quad X_0 = x,
\]

with initial state \( X_0 = x \), where \( \tilde{\mu}(dz,dt,x) \) denotes an extended Poisson counting measure on \( \mathbb{R}^d \times [0,\infty) \), whose compensator is given by \( \bar{\lambda} \tilde{\nu}(dz,t,x)dt \). The random counting measure here should be said to be “extended” in the sense that it permits zero-sized jumps with time-state dependent probability \( \tilde{\nu}(\{0\};t,x) \) out of all jumps generated with constant rate \( \bar{\lambda} \) wherever the underlying process is. The redefined underlying processes (3.16) under the probability measure \( \overline{P}_x^\lambda \) is indistinguishable in law to the original process (2.1) under \( \overline{P}_x^{\lambda,x} \). As such, the first exit time \( \eta \) and the discount factor \( \Theta_{t_1,t_2} \) can remain unaltered, respectively, as (2.2) and (2.3), between the two probability measures \( \overline{P}_x^\lambda \) and \( \overline{P}_x^{\lambda,x} \).

The real relevance of preparing the notation \( \overline{P}_x^\lambda \) and \( \overline{P}_x^{\lambda,x} \) lies in the way of counting jumps. The redefined underlying process (3.16) has been designed to make more jumps (due to the higher jump rate \( \bar{\lambda} \)), some of which are however invisible because of zero size. Namely, the definition of the jump times (3.1) and (3.2) is not capable of detecting such zero-sized jumps. Hence, out of necessity of counting in zero-sized jumps, we redefine a sequence of jump times by \( \{\tilde{\tau}^{(m)}_t \}_{m \in \mathbb{N}} \) on the redefined underlying process (3.16), where \( \tilde{\tau}^{(m)}_t \) denotes its \( m \)-th jump time among all jumps, some of which can be zero-sized. Accordingly, we redefine the discount function as \( \overline{\Lambda}_{t_1,t_2} := \exp[\int_{t_1}^{t_2} \lambda ds] = e^{-\lambda(t_2-t_1)} \) for \( 0 \leq t_1 \leq t_2 \), which is evidently no larger than the original one \( \Lambda_{t_1,t_2} \), due to the uniform dominance \( \lambda(t,x) \leq \bar{\lambda} \).

Now, in light of the representations (3.4) and (3.6), we define the sequences \( \{\tilde{w}_m\}_{m \in \mathbb{N}_0} \) and \( \{\tilde{v}_m\}_{m \in \mathbb{N}} \) of approximate solutions with the common initial approximative solution \( \tilde{w}_0 := w_0 \), respectively, by

\[
\tilde{w}_m(t,x) := \overline{P}_x^{\lambda,x} \left[ \mathbb{I}(X_{\eta_T}^{T(m)} \in \mathcal{D}) \tilde{\Theta}_{t,T(m)} \tilde{\Psi}(\eta_T^{T(m)},X_{\eta_T^{T(m)}}) \right.
\]

\[
+ \mathbb{I}(X_{\eta_T^{T(m)}} \notin \mathcal{D}) \tilde{\Theta}_{t,T(m)} \tilde{\Psi}(\eta_T^{T(m)},X_{\eta_T^{T(m)}}) - \int_{t}^{T(m)} \tilde{\Theta}_{s,T(m)} \tilde{\phi}(s,X_s)ds \right] \quad (3.17)
\]

\[
\tilde{v}_m(t,x) := \overline{P}_x^{\lambda,x} \left[ \mathbb{I}(\tilde{\tau}^{(m)}_T \geq \eta_T^T) \mathbb{I}(X_{\eta_T^{T(m)}} \in \mathcal{D}) \tilde{\Theta}_{t,T(m)} \tilde{\Psi}(\eta_T^{T(m)},X_{\eta_T^{T(m)}}) \right.
\]

\[
+ \mathbb{I}(X_{\eta_T^{T(m)}} \notin \mathcal{D}) \tilde{\Theta}_{t,T(m)} \tilde{\Psi}(\eta_T^{T(m)},X_{\eta_T^{T(m)}}) - \int_{t}^{T(m)} \tilde{\Theta}_{s,T(m)} \tilde{\phi}(s,X_s)ds \right] \quad (3.18)
\]

for \( (t,x) \in [0,T] \times \mathcal{D} \). We also prepare the notation

\[
\tilde{G}_{m-1}(t,x) := \int_{\{z \in \mathbb{R}^d : x+z \notin \mathcal{D}\}} \tilde{w}_{m-1}(t,x+z) \tilde{\lambda} \tilde{\nu}(dz,t,x) = (\tilde{\lambda} - \bar{\lambda}(t,x)) \tilde{w}_{m-1}(t,x) + \int_{\{z \in \mathbb{R}^d : x+z \notin \mathcal{D}\}} \tilde{w}_{m-1}(t,x+z) \bar{\lambda}(t,x) \nu(dz,t,x),
\]

while it turns out unnecessary to introduce the notation \( \tilde{H} \) in a similar manner, due to

\[
\tilde{H}(t,x) := \int_{\{z \in \mathbb{R}^d : x+z \notin \mathcal{D}\}} \tilde{\Psi}(t,x+z,x) \lambda(t,x) \nu(dz,t,x) = \int_{\{z \in \mathbb{R}^d : x+z \notin \mathcal{D}\}} \tilde{\Psi}(t,x+z,x,\lambda(t,x) \nu(dz,t,x) = H(t,x),
\]
Theorem 3.7. Suppose there exists a constant $\tilde{\lambda}$ such that $\lambda(t,x) \leq \tilde{\lambda}$ for all $(t,x) \in [0,T] \times D$.

(a) It holds that for $(t,x) \in [0,T] \times D$, $m \in \mathbb{N}$ and $n \in \{0,1,\ldots,m-1\}$,

$$\tilde{w}_m(t,x) = \mathbb{E}_x^t \left[ \right]$$

and

$$(b) It holds, as m \to +\infty, that \tilde{w}_m(t,x) \to u(t,x) and \tilde{v}_m(t,x) \to u(t,x) for all (t,x) \in [0,T] \times D. In particular, if g \geq 0, \Psi \geq 0 and \phi \leq 0 (or, if g \leq 0, \Psi \leq 0 and \phi \geq 0), then the latter convergence is monotonic.

(c) Let $m \in \mathbb{N}$. If $M_0(t) < 1$, and if $\tilde{w}_{m-1}$, $\tilde{w}_m$ and $\tilde{\xi}(\cdot,0)$ are smooth enough so that $\partial_1$ and $L$ exist almost everywhere on $[0,T] \times D$, then it holds that

$$\tilde{N}_m(t,x) \geq u(t,x) - \tilde{w}_m(t,x) \leq \tilde{N}_m(t,x) \geq u(t,x), \quad (t,x) \in [0,T] \times D, \quad (3.22)$$

where $\tilde{N}_m(t,x)$ and $\tilde{N}_m(t,x)$ are, respectively, the essential infimum and supremum of $\tilde{N}_m(s,x) := \tilde{G}_m(s,x) - \tilde{G}_{m-1}(s,x)$ on $[t,T] \times D$.

Another benefit from the jump rate function being uniformly bounded is the uniform convergence of the sequences $\{w_m\}_{m \in \mathbb{N}}$ and $\{v_m\}_{m \in \mathbb{N}}$ towards the true solution $u$, and as a consequence, the convergence of the upper and lower bounding functions $(3.15)$ to each other. We stress that the latter is a benefit from the former uniform convergence, because the supremum and infimum $(3.13)$ in the hard bounding functions are taken over the entire domain. In addition, the results here provide an important insight into a trade-off whether or not the Poisson thinning should be applied in terms of iteration progress per step relative to the computational complexity.

Theorem 3.8. Suppose there exists a constant $\tilde{\lambda}$ such that $\lambda(t,x) \leq \tilde{\lambda}$ for all $(t,x) \in [0,T] \times D$, and suppose that the function $w_0$ is smooth enough so that $\partial_1$ and $L$ exist almost everywhere on $[0,T] \times D$. If $|N_0(t)| + |N_0(t)| < +\infty$, then it holds, as $m \to +\infty$, that

$$\sup_{(t,x) \in [0,T] \times D} \left| w_m(t,x) - u(t,x) \right| = O(\lambda_m^m/m!), \quad \sup_{(t,x) \in [0,T] \times D} \left| v_m(t,x) - u(t,x) \right| = O(\lambda_m^m/m!),$$

with $\lambda_0 := \sup_{(t,x) \in [0,T] \times D} \{\lambda(s,x)(T-t)\}$, and that

$$\sup_{(t,x) \in [0,T] \times D} \left| \tilde{w}_m(t,x) - u(t,x) \right| = O(\tilde{\lambda}_m^m/m!), \quad \sup_{(t,x) \in [0,T] \times D} \left| \tilde{v}_m(t,x) - u(t,x) \right| = O(\tilde{\lambda}_m^m/m!).$$

Moreover, all the bounding functions of $(3.13)$ and $(3.22)$ tend pointwise to zero as $m \to +\infty$.

Due to the dominance $\lambda_0 \leq \tilde{\lambda}$, the difference in convergence rate above, between $O(\lambda_0^m/m!) and O(\tilde{\lambda}_m^m/m!)$, seems to imply that the original pair $\{w_m\}_{m \in \mathbb{N}}$ and $\{v_m\}_{m \in \mathbb{N}}$ outperforms the pair with thinning $\{\tilde{w}_m\}_{m \in \mathbb{N}}$ and $\{\tilde{v}_m\}_{m \in \mathbb{N}}$ as an iteration scheme, whereas this is a typical trade-off as the ones with thinning are generally easier to compute. Also, the difference in convergence rate indicates, just as is standard in the context of Poisson thinning, that the constant rate $\tilde{\lambda}$ should be chosen as small as possible with the most ideal value $\sup_{(t,x) \in [0,T] \times D} \lambda(t,x)$ whenever it is available. For further illustration, one can show, under the regularity condition on $w_0$ of Theorem 3.8 that, for all $(t,x) \in [0,T] \times D$,

$$w_m(t,x) = w_m(t,x) + \mathbb{E}_x^t \left[ \right] , \quad \tilde{w}_m(t,x) = \tilde{w}_m(t,x) + \mathbb{E}_x^t \left[ \right] ,$$
as a direct consequence of the proof of Theorem 3.8 in the Appendix. Looking at those two expressions together, a single step via the
thinning versio consists of the integration (of the common integrand \( \Theta_{\,t} \, N_0(s, X_s) \)) on an earlier and shorter interval, again due to its higher
jump rate based on the dominance \( \lambda (t, x) \leq \hat{\lambda} \). For the most direct comparison, by setting \( m = 1 \), we get

\[
\begin{aligned}
 w_1(t, x) &= w_0(t, x) + E_{\hat{\lambda}}^{x, t} \left[ \int_t^T \theta^{(1)} \Theta_{\,t} \, N_0(s, X_s) \, ds \right], \\
\tilde{w}_1(t, x) &= \tilde{w}_0(t, x) + E_{\hat{\lambda}}^{x, t} \left[ \int_t^T \theta^{(1)} \Theta_{\,t} \, N_0(s, X_s) \, ds \right],
\end{aligned}
\]

which implies, with \( w_0 = \tilde{w}_0 \) by definition, that the original one \( \{ w_m \}_m \in \mathbb{N}_0 \) is naturally expected to make more progress by one step, in
consistency with the difference in convergence rate in Theorem 3.8.

Additionally, under the uniform ellipticity on the diffusion component, the recursive relation (3.5) under probability measure \( P_{0}^{x} \) can
further be put into the context of the Feynman-Kac formula. To this end, we prepare some notation. For \( a = l + \alpha \) with \( l \in \mathbb{N}_0 \) and \( \alpha \in (0, 1] \)
and an open set \( B \in \mathbb{R} \times \mathbb{R}^d \), we denote by \( \mathcal{H}_{\alpha}(B) \) the Banach space of functions in \( \mathcal{C}^{(l+\alpha)}(B) \) having \( l \)-th space derivatives uniformly
\( \alpha \)-Hölder continuous and \( \lceil l/2 \rceil \)-time derivatives uniformly \( (a/2 - \lceil l/2 \rceil) \)-Hölder continuous. Then, the following is a standard result on the existence of the solution to the partial differential equation (3.14) and its equivalence to the probabilistic representation (3.9). We refer the reader to, for instance, [11 Chapter 5].

**Theorem 3.9.** Let \( m \in \mathbb{N} \) and assume that there exists \( \alpha \in (0, 1) \) for the following conditions hold:

(a) The domain \( D \) is non-empty convex subset of \( \mathbb{R}^d \) with Lipschitz boundary \( \partial D \).
(b) The coefficients \( b \) and \( \sigma \) are in \( \mathcal{H}_{l+\alpha}((0, T) \times D) \);
(c) The functions \( g(x) \) and \( \Psi(t, x, x) \) are bounded and continuous in \( (t, x) \) with \( \Psi(T, x, x) = g(x) \) for all \( x \in \partial D \);
(d) The functions \( r(t, x) \), \( \lambda(t, x) \), \( \phi(t, x) \) and \( G_{m-1}(t, x) + H(t, x) \) are bounded in \( \mathcal{H}_{\alpha}(0, T) \times D \);
(e) There exists a strictly positive constant \( c > 0 \) such that \( \| \xi \| \cdot (\alpha/2 + 2) \xi \) \( \geq c \| \xi \|^2 \) for all \( (t, x, \xi) \in (0, T) \times \mathbb{R}^d \).

Then, there exists a unique solution in \( \mathcal{C}^{1,2}((0, T) \times \mathbb{R}^d) \cap \mathcal{C}^{0,0}(0, T) \times \partial D \) to the partial differential equation (3.14) with initial and boundary conditions \( w_m(T, x) = g(x) \) for \( x \in \overline{D} \), and \( w_m(t, x) = \Psi(t, x, x) \) for \( (t, x) \in [0, T) \times \partial D \).

In fact, one can deduce, under the conditions of Theorem 3.9 that the true solution (2.3) (under the probability measure \( P_{0}^{x} \) in the
present of jump component) solves the partial integro-differential equation

\[
\partial_t u(t, x) + \mathcal{L}_t u(t, x) = (r(t, x) + \lambda(t, x))u(t, x) + \phi(t, x) - \int_{\mathbb{R}^d} u(t, x + z)\lambda(t, x)\nu(dz; x) - H(t, x),
\]

with \( u(T, x) = g(x) \) for \( x \in \overline{D} \) and \( u(t, x) = \Psi(t, x, x) \) for \( (t, x) \in [0, T) \times \partial D \). This boundary value problem does not lend itself well to
practical use, as partial integro-differential equations are generally not easy to deal with, whereas this fact provides an interesting insight
into the proposed framework of iterative nature. That is, the representation (3.9) can be thought of as a consequence of replacing the
unknown true solution \( u \) inside the integral by an approximate solution, say \( w_{m-1} \), so as to get a better approximate solution \( w_m \) by iteration.
The key point here is that the partial integro-differential equation loses its integral term, that is, then a partial differential equation, once this
replacement has taken place, corresponding to the probability measure \( P_{0}^{x} \) with jump component suppressed in the representation (3.9).

Before closing this section, let us stress again that the proposed convergent iterative weak approximation scheme has been constructed for
quite a large class of multidimensional stochastic differential equations with jumps, based on the previous theoretical developments (Sections
2.1, 2.2, 2.3 and 3.4) alone, without coming into the present subsection (Section 3.5), that is, without imposing uniform boundedness on jump rate or a variety of regularity conditions of Theorem 3.9 to say nothing of differentiability on the drift and diffusion coefficients as
in the exact simulation of one-dimensional sample paths [11 5 8 13].

4 Numerical illustrations

So far in this work, we have thoroughly constructed a theoretical framework in which weak approximation can be conducted in iterative
and convergent manners for quite a large class of multivariate inhomogeneous stochastic differential equations with jumps of time-state
dependent rate, rather than proposing concrete numerical procedures, such as discretization schemes and exact simulation methods (see, for instance, [11 5 8 12 13]). We devote the present section to two examples with a particular focus on effective illustration of the relevance of the proposed theoretical framework in the context of weak approximation, rather than assessing competitiveness of the resulting numerical
methodology in terms of complexity relative to the existing methods in the literature. As such, the two examples in what follows have been
chosen in such a way that sophisticated numerical approximation is not required.

4.1 Pure-jump processes

The first example we consider is based on the pure-jump process:

\[
X_s = x + b(s-t) + (N_s - N_t), \quad s \in [t, +\infty), \quad X_t = x(>0).
\] (4.1)
where \( b > 0 \) and \( \{N_t : s \in [t, +\infty)\} \) is a Poisson process with constant rate \( \lambda(>0) \) and a single jump size \( c(<0) \), certainly in the framework \((2.1)\) with \( b(\cdot, \cdot) \equiv b \), \( \sigma \equiv 0 \) and \( v(dz; \cdot) = \delta_{\{c\}}(dz) \). With the diffusion component suppressed \((\sigma \equiv 0)\), we intend to demonstrate the fully deterministic iteration \((3.10)\). We have adopted this example, where the jump rate is fully independent of time and state, so as to demonstrate most of our theoretical developments in the most illustrative manner. Hence, the next example in Section \( 4.2 \) concerns jump-diffusion processes with a time-state dependent jump rate.

Now, with the domain \( D = (0, +\infty) \), we investigate the finite-time ruin probability before terminal time \( T \):

\[
  u(t, x) = E^\lambda_\xi \left[ \eta_t \leq T \right] = E^\lambda_\xi \left[ \mathbb{I} \left( \eta_t \leq T \right) \right], \quad (t, x) \in [0, T] \times D,
\]

within the framework of Proposition \( 3.3 \) with \( r(\cdot, \cdot) \equiv 0 \), \( g(\cdot) \equiv 0 \), \( \Psi(\cdot, \cdot) \equiv 1 \) and \( \phi(\cdot, \cdot) \equiv 0 \). Hence, the sequence of approximate solutions \( \{w_m\}_{m \in \mathbb{N}_0} \) is monotonically increasing. For comparison purposes, we first derive the closed-form solutions of the first few approximate solutions with the aid of the expression for \( w_m \) in Proposition \( 3.3 \). Since \( b > 0 \) and \( x > 0 \), it is immediate that \( w_0(t, x) \equiv 0 \) for \( (t, x) \in [0, T] \times \bar{D} \). For the case \( m = 1 \), it holds that for \( (t, x) \in [0, T] \times [0, +\infty) \),

\[
  w_1(t, x) = E^\lambda_\xi \left[ \mathbb{I} \left( X_{\eta_2 \wedge T} \notin [0, +\infty) \right) \right] = E^\lambda_\xi \left[ \mathbb{I} \left( \tau^{(1)}_t \leq T \right) \mathbb{I} \left( X_{\tau^{(1)}_t} \notin [0, +\infty) \right) \right] = E^\lambda_\xi \left( E \leq \frac{c-x}{b} \wedge (T-t) \right),
\]

and thus

\[
  w_1(t, x) = \begin{cases} 
  1 - e^{-\lambda((c-x)/b) \wedge (T-t)}, & \text{if } (c-x)/b \wedge (T-t) \geq 0, \\
  0, & \text{otherwise},
  \end{cases}
\]

\[(4.2)\]

due to \( \tau^{(1)}_t \sim \text{Exp}(\lambda) \). For \( m = 2 \), observe that

\[
  \mathbb{I} \left( \{X_{\tau^{(2)}_t} \in [0, 0]\} \cap \{X_{\tau^{(2)}_t} \notin [0, +\infty)\} \right) = \mathbb{I} \left( \{t+E_1+E_2 \leq T\} \cap \{x+bE_1+c > 0\} \cap \{x+b(E_1+E_2)+2c < 0\} \right),
\]

where \( E_1 \) and \( E_2 \) are iid exponential random variables with rate \( \lambda \). The three events indicate, respectively, that the second jump occurs before the terminal time \( (t+E_1+E_2 \leq T) \), the location right after the first jump remains inside the domain \( (x+bE_1+c > 0) \), and the location right after the second jump is strictly outside the domain \( (x+b(E_1+E_2)+2c < 0) \). Hence, we obtain that for \( (t, x) \in [0, T] \times [0, +\infty) \),

\[
  w_2(t, x) = w_1(t, x) + \begin{cases} 
  e^{-\lambda t_1}(x) - e^{-\lambda t_2}(x) - \lambda e^{-\lambda t_3}(x)(\theta_2(t, x) - \theta_1(t, x)), & \text{if } \theta_2(t, x) \geq \theta_1(t, x), \\
  0, & \text{otherwise},
  \end{cases}
\]

\[(4.3)\]

where \( \theta_1(t, x) := \max\{0, (c-x)/b\} \) and \( \theta_2(t, x) := \max\{0, (c-x)/b \wedge (T-t)\} \). Note that \( w_2(t, x) \geq w_1(t, x) \) for all \( (t, x) \in [0, T] \times [0, +\infty) \), since the function \( e^{-\lambda t_1}(x) - e^{-\lambda t_2}(x) - \lambda e^{-\lambda t_3}(x)(\theta_2(t, x) - \theta_1(t, x)) \) is non-negative for \( (x, y) \in [0, +\infty)^2 \) with \( x \leq y \). It is evidently cumbersome to go beyond \( m = 2 \), whereas it is natural to conjecture, based on those closed-form expressions \( w_0 \equiv 0 \), \( (4.2) \) and \( (4.3) \), that the function \( w_m \) is continuous with only a countable number of non-differentiable points for all \( m \in \mathbb{N}_0 \).

Alternatively, we may derive an recursion formula based on the identity \((3.10)\). In the absence of the jump component under the probability measure \( P_0^X \), the underlying process \((4.1)\) reduces to a deterministic linear function \( x(s) = x + b(s-t) \). The deterministic function \( x(s) \) here never exits the domain \( (0, +\infty) \) due to \( b > 0 \), that is, the first exit time \( \eta_t \) is understood to be infinite. With constant rate \( \lambda \), \( v(dz; \cdot) = \delta_{\{c\}}(dz) \) and \( \Psi \equiv 1 \), we get

\[
  G_{m-1}(s, x(s)) = \lambda w_{m-1}(s, x(s) + b(s-t) + c) \mathbb{I}(x(s) + b(s-t) + c \geq 0), \quad H(s, x(s)) = \lambda \mathbb{I}(x(s) + b(s-t) + c < 0),
\]

for \( s \in [t, T] \). Therefore, with \( r \equiv 0, g \equiv 0, \phi \equiv 0 \) and \( \Psi \equiv 1 \), we get \( w_0 \equiv 0 \), and for each \( m \in \mathbb{N} \),

\[
  w_m(t, x) = \int_t^T \lambda e^{-\lambda(t-s)} (w_{m-1}(s, x(s) + b(s-t) + c) \mathbb{I}(x(s) + b(s-t) + c \geq 0) + \mathbb{I}(x(s) + b(s-t) + c < 0)) ds,
\]

\[(4.4)\]

according to the result \((3.10)\). Note that the closed-form expressions \((4.2) \) and \( (4.3) \) are consistent with the recursion formula \((4.4) \). Recalling \( w_m = v_m \) due to \( w_0 \equiv 0 \), as we have pointed out in Section \( 3.2 \), there is nothing really to illustrate Theorem \( 3.4 \) here.

As conjectured earlier, one can now ensure, by induction on the recursion \((4.4) \) starting with \( w_0 \equiv 0 \), that for every \( m \in \mathbb{N}_0 \), the function \( w_m \) is continuous with an increasing yet only countable number of non-differentiable points on \( [0, T) \times (0, +\infty) \). Moreover, we have \( \xi(t, x; 0) = T - t, M^L(t) = 0 \) and \( M^R(t) = -\lambda(T-t) \), due to \( \eta_t = +\infty, r \equiv 0 \) and \( \Phi(t, x) = \lambda(T-t)(1 + c > 0) \). Next, observe that

\[
  N_m(t, x) = G_m(t, x) - G_{m-1}(t, x) = \lambda \left[ w_m(t, x + c) - w_{m-1}(t, x + c) \right] \mathbb{I}(x + c > 0),
\]

which is non-negative, that is, \( N^L_m(t) \equiv 0 \) for all \( t \in [0, T] \) and \( m \in \mathbb{N} \), since the sequence \( \{N_m\}_{m \in \mathbb{N}_0} \) is monotonically increasing with \( w_0 \equiv 0 \) by Proposition \( 3.3 \). One can also deduce \( N^R_m(t) = \lambda w_m(t, -(m-1)c) + |N^L_m(0)| + |N^R_m(0)| < +\infty \) for all \( m \in \mathbb{N}_0 \) since \( \{w_m\}_{m \in \mathbb{N}} \) is bounded. That is, this problem setting lies in the scope of Theorems \( 3.6 \) and \( 3.8 \) with deterministic upper and lower bounds given, through \( (3.15) \), by

\[
  w_m(t, x) \leq u(t, x) \leq w_m(t, x) + \lambda(T-t)w_m(t, -(m-1)c).
\]

\[(4.5)\]
In what follows, with the unit drift $b = 1$ fixed, we examine two distinctive situations: large yet rare jumps and small yet intense jumps. For the first case of large yet rare jumps, with jump rate $\lambda = 1$ and the drawdown jump $c = -1$ of unit length, we plot in Figure 1 the first five approximate solutions and the associated upper bounding functions. To avoid overloading the figures, the upper bounding functions are only presented at the third, fourth and fifth iterations. The values of the true solution $u$ indicated by unfilled circles at six points in each figure are estimates by Monte Carlo methods using $10^7$ iid replications, which are large enough to narrow down 99% confidence intervals to almost a singleton.

As the iteration proceeds, the approximate solutions are supporting the ruin probability gradually from the lower initial state. This is not very surprising, as the present problem setting is built on a singular structure (positive linear drift with fatal drawdown jumps). Even after the approximate solution has captured the true solution from below (around $m = 500$ and (b) $m = 5$), the upper bound is still approaching downwards to the true solution and specifies the true solution, almost exactly and uniformly when $m = 5$, just as expected in Theorem 3.8.

Next, for the case of small yet intense jumps, we plot in Figure 2 the approximate solutions with intense jump rate $\lambda = 100$ and small drawdown jumps of (a) $c = -1/500$ and (b) $c = -1/1000$, in such a way that each jump is nearly invisible relative to the ascending drift $b = 1$. In this situation, the associated upper bounding functions are not useful (and thus not plotted in the figures) because the jump rate makes direct impact as is clear in (3.5). Still, it is encouraging that the proposed framework provides a fast convergent sequence of the approximate solutions in both distinctive situations of large yet rare jumps and small yet intense jumps.

### 4.2 Survival probability with time-state dependent jump rate

The previous example (Section 4.1) has concerned pure-jump processes on the basis of, in fact, already most of the presented theoretical results (Sections 3.1, 3.2, 3.3 and 3.4). We next turn to the case of jump-diffusion processes with a time-state dependent jump rate, so as to illustrate the effectiveness of the remaining developments, particularly, the Poisson thinning of Section 3.5. Note that this example is borrowed from [10] Example 2.2.1 for better demonstration purposes.

Consider a bounded space domain $D = (x_L, x_U)$ with $-\infty < x_L < x_U < \infty$. The underlying stochastic process is the Brownian motion with drift and jumps:

$$dX_s = bds + \sigma dW_s + \int_{\mathbb{R}_0} z \mu(dz; ds; X_s), \quad s \in [t, +\infty), \quad X_t = x,$$

where $b \in \mathbb{R}$, $\sigma > 0$ and $\{W_s : s \in [t, +\infty)\}$ is the standard Brownian motion in $\mathbb{R}$. We let the jump rate time-state dependent with its jump size distributed Gaussian as follows:

$$\lambda(t, x) = 5t(T - t)(x_U - x)(x - x_L), \quad \nu(dz; x) = \frac{1}{\sqrt{2\pi\rho}} \exp\left(-\frac{z^2}{2\rho}\right) dz, \quad (t, x, z) \in [0, T] \times (x_L, x_U) \times \mathbb{R}_0,$$

for some $\rho > 0$. We now examine the survival probability until the terminal time $T$, that is, $u(t, x) = \mathbb{P}_x^t (\eta_t > T)$, corresponding to the input data $(r, g, \Psi, \phi) = (0, 1, 0, 0)$ in the representation (2.3). The initial approximate solution is available in semi-analytical form [10]...
In order to apply the Poisson thinning for further iterations, we may find and thus set the smallest upper bound in this case, that is, \( \lambda = \sup_{(t,x) \in [0,T] \times [x_L,x_U]} \hat{\lambda}(t,x) = (5/16)T^2(x_U - x_L)^2 \). With the input data \((r,g,\Psi,\phi) = (0,1,0,0)\), the identity \((3.19)\) under the probability measure \(\mathbb{P}^x_t\) reduces to

\[
\tilde{w}_m(t,x) = \mathbb{P}^x_t \left[ \mathbb{1}(\eta_t > T)\tilde{N}_{L,T} + \int_t^{\eta_{T} \land T} \tilde{N}_{t,s} \tilde{G}_{m-1}(s,x)ds \right],
\]

for all \((t,x) \in [0,T] \times [x_L,x_U]\) and \(m \in \mathbb{N}\), where

\[
\tilde{G}_{m-1}(t,x) = (\tilde{\lambda} - \lambda(t,x))\tilde{w}_{m-1}(t,x) + \lambda(t,x) \int_{x_L - x}^{x_U - x} \tilde{w}_{m-1}(t,x+z) \frac{1}{\sqrt{2\pi \rho}} \exp \left( -\frac{z^2}{2\rho} \right) dz.
\]

Its semi-analytical solution is available as

\[
\tilde{w}_m(t,x) = e^{-\tilde{\lambda}(T-t)}w_0(t,x) + e^{-\frac{b}{\sigma^2}x} \sum_{k \in \mathbb{N}} \sin \left( k\pi \frac{x-x_L}{x_U-x_L} \right) \int_t^T \exp \left[ -\left( \frac{1}{2} \left( \frac{k\pi\sigma}{x_U-x_L} \right)^2 + \frac{b^2}{2\sigma^2} + \tilde{\lambda} \right)(s-t) \right] \beta_{m,k}(s)ds,
\]

for all \((t,x) \in [0,T] \times [x_L,x_U]\) and \(m \in \mathbb{N}\), where

\[
\beta_{m,k}(s) := \frac{2}{x_U-x_L} \int_{x_L}^{x_U} e^{\frac{b}{\sigma^2}y} \tilde{G}_{m-1}(s,y) \sin \left( k\pi \frac{y-x_L}{x_U-x_L} \right) dy;
\]

for \(s \in [0,T]\) and \(m \in \mathbb{N}\). Finally, for constructing the hard bounding functions \((3.22)\), the required component \(\xi(t,x,0) = \mathbb{E}^x_0[\eta_T] - t\) is also available in semi-analytical form [10] Example 2.2.1, as

\[
\xi(t,x,0) = h_0(x) - e^{-\frac{b}{\sigma^2}x} \sum_{k \in \mathbb{N}} c_k \sin \left( k\pi \frac{x-x_L}{x_U-x_L} \right) e^{-\left( \frac{b^2}{2\sigma^2} + \frac{k\pi\sigma}{x_U-x_L} \right)^2(T-t)}/2,
\]

for all \((t,x) \in [0,T] \times [x_L,x_U]\), where

\[
h_0(x) := \frac{(x_U - x)(e^{-2b(x_U - x)/\sigma^2} - 1) + (x_L - x)(1 - e^{-2b(x_U - x)/\sigma^2})}{b(e^{-2b(x_U - x)/\sigma^2} - e^{-2b(x_L - x)/\sigma^2})}, \quad c_k := \frac{2}{x_U-x_L} \int_{x_L}^{x_U} h_{0}(y)e^{\frac{b}{\sigma^2}y} \sin \left( k\pi \frac{y-x_L}{x_U-x_L} \right) dy,
\]

Figure 2: Plots of the approximate solutions \(\{w_m(0,x)\}_{m \in \{1,2,3,4,5\}}\), with \(m = 1\) (orange dash-dot), \(m = 2\) (green dash-dot), \(m = 3\) (blue dash), \(m = 4\) (purple dash) and \(m = 5\) (red solid). The six black unfilled circles in each figure indicate (very accurate Monte Carlo estimates of) the true values \(u(0,x)\) at six states.
for $x \in [x_L, x_U]$ and $k \in \mathbb{N}$.

In consistency with [10] Example 2.2.1, we set $T = 1$, $(x_L, x_U) = (0, 2)$, $b = 2$, $\sigma = 1$ and $\rho = 0.1$, and truncate the infinite series (4.6), (4.8) and (4.9) by 500 summands to ensure adequate convergences on computer. When constructing hard bounding functions according to the inequalities (3.22), we find the supremum and infimum of $\tilde{N}_m(\cdot, \cdot)$ and $M_\xi(\cdot, \cdot)$ approximately by discretization of the time-state domain $[0, T] \times [x_L, x_U]$ into equisized grids, each of $0.0005 \times 0.0005$.

Figure 3: The approximate solutions $\{\tilde{w}_m(0, x)\}_{m \in \{0,1,2,3\}}$ and the associated upper and lower bounding functions at two timepoints $t \in \{0, 0.5\}$ for $x \in [x_L, x_U]$ with $m = 0$ (pink dash-dot), $m = 1$ (orange dash-dot), $m = 2$ (green dash) and $m = 3$ (blue dash). The unfilled circles in (a) and (c) and the vertical whisker plots in (b) and (d) are, respectively, Monte Carlo estimates and 99% confidence intervals, constructed based on $10^5$ iid sample paths by the exact simulation method for jump-diffusion processes [1].

In Figure 3 we plot the approximate solutions $\{\tilde{w}_m(t, x)\}_{m \in \{0,1,2,3\}}$ as well as the associated upper and lower hard bounding functions at $t \in \{0, 0.5\}$ for $x \in [x_L, x_U]$. For justification and comparison purposes, we add Monte Carlo estimates (as unfilled circles) in (a) and (c), while 99% confidence intervals (as vertical whisker plots) in (b) and (d), at $x \in \{0.4, 0.8, 1.2, 1.6\}$, all constructed based on $10^5$ iid sample paths by the exact simulation method for jump-diffusion processes [1]. Those estimates and confidence intervals can certainly be improved by, for instance, increasing the sample path and/or employing relevant variance reduction methods, whereas we do not go in such directions here as we are not particularly concerned with a direct competition with existing methodologies in computing time. Instead, given that hard bounding functions (so to speak, 100% confidence intervals) are sufficiently competitive with those 99% confidence intervals already at the second ($m = 2$) or third ($m = 3$) iteration, it seems safe to claim that our theoretical framework has strong potential to provide an effective weak approximation method by iteration, particularly if the approximate solution can be computed efficiently, for instance, by numerical methods for partial differential equations. Let us, last not least, remind that the proposed framework is general enough to accommodate quite a large class of multidimensional inhomogeneous stochastic differential equations with jumps and, particularly, does not require the drift and diffusion coefficients to be as smooth as in existing univariate exact simulation methods.
Proof of Theorem 3.1. We collect all proofs here in the Appendix. To avoid overloading the paper, we skip nonessential details of somewhat routine nature in some instance, particularly when the

\[ \text{Proof of Theorem 3.1.} \]

We use the identities

\[ \{ \eta_t > T \} \cap \{ X_t \not\in \mathcal{D} \} = \emptyset, \quad \{ \eta_t > T \} \cap \{ X_t \in \mathcal{D} \} = \{ \eta_t > T \}. \]  

(A.1)

for \( t > 0 \). In words, on the event \( \{ \eta_t > T \} \), that is, no exit occurs before or at the terminal time \( T \), the location \( X_T \) cannot be strictly outside the closure \( \mathcal{D} \). The second identity is clearly the complement of the first one.

Now, if the first exit is caused by a jump, that is, \( X_{t_n} \neq X_{t_{n-1}} \), then the location \( X_{t_n} \) right after the jump is strictly outside the closure \( \mathcal{D} \) almost surely. If the first exit is caused by the diffusion component of infinite variation, then the exit must occur in the boundary \( \partial \mathcal{D} \) due to infinite variation, that is, \( X_{t_n} = X_{t_{n-1}} \in \partial \mathcal{D} \). Since those two cases are disjoint, we obtain

\[ \{ X_{t_n} = X_{t_{n-1}} \} = \{ X_{t_n} = X_{t_{n-1}} \} = \{ X_{t_n} \in \partial \mathcal{D} \}. \]  

(A.2)

With the aid of (A.1) and (A.2), we get

\[ \{ X_{t_n} \not\in \partial \mathcal{D} \} \cap \{ X_{t_n} = X_{t_{n-1}} \} = 1 \{ \{ \eta_t > T \} \cap \{ X_{t_n} = X_{t_{n-1}} \} \}, \]

and

\[ 1 \{ (X_{t_n}^{\tau^{(m)}} \in \partial \mathcal{D}) \cap \{ X_{t_n} = X_{t_{n-1}} \} = 1 \{ (\{ \eta_t > T \} \cap \{ X_{t_n} = X_{t_{n-1}} \} \} + 1 \{ \{ \eta_t \leq T \} \cap \{ X_{t_n} \in \partial \mathcal{D} \} \}, \]

\[ 1 \{ (X_{t_n}^{\tau^{(m)}} \not\in \partial \mathcal{D}) \cap \{ X_{t_n} = X_{t_{n-1}} \} = 1 \{ \{ \eta_t \leq T \} \cap \{ X_{t_n} \not\in \partial \mathcal{D} \} \}. \]

Combining those altogether, we obtain the identities

\[ 1 \{ X_{t_n}^{\tau^{(m)}} \in \partial \mathcal{D} \} = 1 \{ \{ \eta_t > T \} \} + 1 \{ \{ \eta_t \leq T \} \cap \{ X_{t_n} \in \partial \mathcal{D} \} \}, \]

\[ 1 \{ X_{t_n}^{\tau^{(m)}} \not\in \partial \mathcal{D} \} = 1 \{ \{ \eta_t \leq T \} \cap \{ X_{t_n} \not\in \partial \mathcal{D} \} \}. \]  

(A.3)

Since the jump rate is positive and finite, it holds that \( \tau_{\max}^{(m)} > T \) almost surely for sufficiently large \( m \) under the probability measure \( \mathbb{P}^{X_0,m} \). Hence, it holds, as \( m \to +\infty \), that for each \( (t, x) \in [0, T] \times \mathcal{D} \),

\[ \begin{align*}
 w_m(t, x) & \to E^{X_0}_m \left[ 1 \{ X_t^{\tau^{(m)}} \in \partial \mathcal{D} \} \Theta_{t, t_{n}} w_0(\eta_{t_{n}}, X_{t_{n}}^{\tau^{(m)}}) + 1 \{ X_t^{\tau^{(m)}} \not\in \partial \mathcal{D} \} \Theta_{t, t_{n}} \Psi(\eta_{t_{n}}, X_{t_{n}}^{\tau^{(m)}}, X_{t_{n}}^{\tau^{(m)}}) - \int_{t_{n}}^{T} \Theta_{t, s} \phi(s, X_s) \, ds \right] \\
 & = E^{X_0}_m \left[ 1 \{ \eta_t > T \} \Theta_{t, t_{n}} w_0(T, X_T) + 1 \{ \eta_t \leq T \} \Theta_{t, t_{n}} w_0(\eta_t, X_t) \right. \\
 & \quad + \left. 1 \{ \{ \eta_t \leq T \} \cap \{ X_t \in \partial \mathcal{D} \} \} \Theta_{t, t_{n}} \Psi(\eta_{t_{n}}, X_{t_{n}}^{\tau^{(m)}}, X_{t_{n}}^{\tau^{(m)}}) - \int_{t_{n}}^{T} \Theta_{t, s} \phi(s, X_s) \, ds \right],
\end{align*} \]  

(A.4)

which yields (A.3), due to \( w_0(t, x) = g(x) \) for \( x \in \mathcal{D} \) and \( w_0(\eta_t, X_t) = \Psi(\eta_t, X_t, X_{t_{n}}) = \Psi(\eta_t, X_t, X_{t_{n}}) \) on the event \( \{ X_{t_n} \in \partial \mathcal{D} \} \). Since \( g, \Psi, \phi \) and \( \Theta \) are bounded and \( \eta_{t_{n}} \in [0, T] \), the function \( w_0 \) is bounded due to the representation (A.1). Hence, for every \( m \in \mathbb{N} \), the random element in the expectation (A.4) is almost surely bounded. The passage to the limit in the convergence (A.4) can thus be justified by the bounded convergence theorem. \( \square \)
Proof of Theorem 3.2. First, by the definition (3.2), we have
\[ \eta_T = \inf \{ s > \eta : X_s \notin \mathcal{D} \} = \eta_T, \]
and since the random variable \( \tau \)

(6.6)

because the trajectory has already hit the boundary \( \partial \mathcal{D} \) by the diffusion component, or any, at time \( \eta_T \) and has already exited from the closure \( D \) by a jump at time \( \eta_T \). Next, it is straightforward that

(6.6)

we have applied (3.5) for the second equality. Noting \( \tau_{\eta_T} \geq \tau_{\eta_T} \) for all \( m \in \mathbb{N} \) and \( t > 0 \), due to (3.1) and (3.2), we obtain \( \eta_T \geq \tau_{\eta_T} \) and \( X_{\tau_{\eta_T}} \in \mathcal{D} \) on the event \( \{ \eta_T > \tau_{\eta_T} \} \), whereas \( \eta_T \geq \tau_{\eta_T} \) on the event \( \{ \eta_T < \tau_{\eta_T} \} \). The \( m \)-th jump time \( \tau_{\eta_T} \) can be introduced into the first term as follows:

(6.7)

where we have applied (3.7). In a similar manner, the second term can be rewritten as

(6.8)

where the last equality holds by \( \{ \eta_T > \tau_{\eta_T} \} \cap \{ X_{\tau_{\eta_T}} \notin \mathcal{D} \} = \emptyset \), and thus

(6.9)

Also, the third term can be rewritten as

(6.10)

By combining the first terms of (3.5), (3.7) and (3.9) and taking the conditional expectation on the stopped \( \sigma \)-field \( \mathcal{F}_{\eta_T} \), it holds by the strong Markov property of the underlying process [6 Chapter III] that

(3.5)

since the random variable \( 1 ( \eta_T > \tau_{\eta_T} ) \) is \( \mathcal{F}_{\eta_T} \)-measurable. On the whole, by substituting (3.8), (3.7), (3.9) and (3.11) into the representation (3.4), we obtain

(3.5)

which reduces to the desired expression (3.5) by combining the first two terms in the expectation.
Proof of Proposition 3.3 Observe that \( \{X_{nT}, \tau \} \in D \) as well as \( \{X_{nT}, \tau \} \notin D \) under \( \mathbb{P}^X \), since the drift is not outward pointing at the boundary as well as there is no diffusion component in the neighborhood \( \partial D \) of the boundary. Hence, one can rewrite the representation (3.3) as

\[
\mathbb{w}_m(t, x) = E^X_{\lambda} \left[ 1(X_{nT}, \tau) \notin D \right] + 1(X_{nT}, \tau) \in D] w_{m-1}(T \wedge \tau, X_{nT}, \tau) \right].
\]

Moreover, it holds true that \( \eta(t) < \eta(2) \) and \( \tau \wedge \tau(t) = \tau \wedge \tau(2) \), due to

\[
\tau \wedge \tau(1) = 1(\eta(1) \geq T)(T \wedge \tau(1)) + 1(\eta(1) < T)(T \wedge \tau(1)) = 1(\eta(1) \geq T) T + 1(\eta(1) < T)(T \wedge \tau(2))
\]

\[
= 1((\eta(1) \geq T) \wedge \{\eta(2) \geq T\}) T + 1((\eta(1) < T) \wedge \{\eta(2) \geq T\}) T + 1((\eta(1) < T) \wedge \{\eta(2) < T\}) \tau(2)
\]

\[
= 1(\eta(2) \geq T) T + 1(\eta(2) < T) \tau(2) = T \wedge \tau(2).
\]

Hence, we obtain

\[
\mathbb{w}_m(t, x) = E^X_{\lambda} \left[ 1(X_{nT}, \tau(1) \notin D) + 1(X_{nT}, \tau(1) \in D) \right] w_{m-1}(\eta(t) \wedge \tau(1), X_{nT}, \tau(1)) + E^X_{\lambda} \left[ 1(X_{nT}, \tau(2) \notin D) + 1(X_{nT}, \tau(2) \in D) \right] w_{m-2}(T \wedge \tau(2), X_{nT}, \tau(2)),
\]

where we have applied the tower property and the strong Markov property. This yields the desired identity by induction with the aid of \( \mathbb{w}_0(t, x) = E^X_{\lambda} \left[ 1(\eta \leq T) \right] \equiv 0 \) due to the degeneracy of the jump component. The dominance is evident from the increasing number of indicator functions in m.

Proof of Theorem 3.4 (a) Since overshooting can only be caused by a jump, the event \( \{X_{nT} \neq \tau(m+1) \notin D\} \) means that an exit occurs before the terminal time and, moreover, at latest at the \((m + 1)\)-st jump timing. Hence, it holds that

\[
1(X_{nT} \neq \tau(m+1) \notin D) = 1(\eta \leq \tau(m+1)) \wedge T) 1(X_{nT} \notin D) + (1(\eta \geq \tau(m+1)) 1(X_{nT} \notin D)
\]

\[
= 1(X_{nT} \neq \tau(m+1) \notin D) + 1(\tau(m) < \eta) 1(X_{nT} \neq \tau(m+1) \notin D).
\]

With this identity, the expression (3.6) (with \( m + 1 \) can be rewritten as follows:

\[
\mathbb{v}_{m+1}(t, x) = E^X_{\lambda} \left[ 1(\tau(m+1) \geq \eta) 1(X_{nT} \in D) \Theta_{x, \tau(m+1)} w_0(\eta, X_{nT}) + 1(X_{nT} \neq \tau(m+1) \notin D) \Theta_{x, \tau(m+1)} \Psi(\eta, x, X_{nT}) - \int_{t}^{\tau(m+1)} ds \Theta_{x, \tau(m+1)} \Psi(\eta, x, X_{nT}))
\]

\[
+ E^X_{\lambda} \left[ 1(\tau(m) < \eta) 1(X_{nT} \in D) \Theta_{x, \tau(m)} w_0(\eta, X_{nT}) + 1(\tau(m) < \eta) 1(X_{nT} \neq \tau(m+1) \notin D) \Theta_{x, \tau(m)} \Psi(\eta, x, X_{nT}) - \int_{t}^{\tau(m)} ds \Theta_{x, \tau(m)} \Psi(\eta, x, X_{nT}))
\]

\[
= \mathbb{v}_m(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} v_1(\tau(m), X_{nT}))
\]

\[
= \mathbb{v}_m(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} w_0(\eta, X_{nT})
\]

By applying the identity (A.12) recursively, we further obtain that

\[
\mathbb{v}_{m+1}(t, x) = \mathbb{v}_m(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} v_1(\tau(m), X_{nT}))
\]

\[
= \mathbb{v}_{m+1}(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} v_1(\tau(m), X_{nT}))
\]

\[
= \mathbb{v}_{m+1}(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} v_1(\tau(m), X_{nT}))
\]

which yields the first identity (3.7) for general \( n \in \{0, 1, \ldots, m-1\} \) by induction.

For the second identity (3.8), it is straightforward in light of (3.4) and (3.6) to obtain that

\[
\mathbb{w}_{m}(t, x) = \mathbb{v}_m(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} w_0(\eta, X_{nT})
\]

In a similar manner to the derivation of the identity (3.7), it holds by the identity (A.13) that

\[
\mathbb{w}_{m}(t, x) = \mathbb{v}_{m+1}(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} v_1(\tau(m), X_{nT}))
\]

\[
= \mathbb{w}_{m}(t, x) + E^X_{\lambda} \left[ 1(\tau(m) < \eta) \Theta_{x, \tau(m)} v_1(\tau(m), X_{nT}))
\]

which yields the first identity (3.8) for general \( n \in \{0, 1, \ldots, m-1\} \) by induction.

To prove the pointwise convergence, observe first that \( \tau(m) \geq \eta \geq \tau \), \( \mathbb{P}^X \)-a.s. eventually as \( m \to +\infty \), due to a finite jump intensity. Hence, it holds as \( m \to +\infty \) that

\[
\mathbb{v}_m(t, x) \to E^X_{\lambda} \left[ 1(X_{nT} \in D) \Theta_{x, \tau(m)} w_0(\eta, X_{nT}) + 1(X_{nT} \notin D) \Theta_{x, \tau(m)} \Psi(\eta, x, X_{nT}) - \int_{t}^{m} ds \Theta_{x, \tau(m)} \Psi(\eta, x, X_{nT}))
\]

which yields (3.9), due to \( 1(\eta > T) w_0(T, x) = 1(\eta > T) \phi(x, X_{nT}) \) and \( 1(X_{nT} \in D) w_0(\eta, X_{nT}) = 1(X_{nT} \in D) \Psi(\eta, x, X_{nT}) \). The passage to the limit can be justified by the bounded convergence theorem in a similar manner to Theorem 3.1.

(b) If \( g \geq 0 \), \( \Psi \geq 0 \) and \( \phi \leq 0 \) (respectively, if \( g \leq 0 \), \( \Psi \leq 0 \) and \( \phi \geq 0 \)), then \( w_0 \) and then \( v_1 \) are non-negative (respectively, non-positive) due to (3.3) and (3.6) over the domain. Therefore, the desired monotonicity holds true in light of the identity (3.7).
and let the probability measure $P^X_t$ represent the condition $X_t = Y_t = x$. We define discount functions on $\{Y_t : t \geq 0\}$ by $\Theta^Y_t := \exp[-\int_t^\infty r(s,Y_s)ds]$ and $\Lambda^Y_t := \exp[-\int_t^\infty \lambda(s,Y_s)ds]$ for $0 \leq t \leq T$, as well as the first exit time of the coupled process $\{Y_t : t \geq 0\}$ and its truncation, respectively, $\xi^Y_t := \inf\{s \geq t : Y_s \notin \mathcal{B}\}$ and $\xi^Y_T := \xi^Y_t \wedge T$, in a similar manner to (A.2) for the underlying process $\{X_t : t \geq 0\}$. It then holds $P^X_t$-a.s. that $X_t = Y_t$ for all $s \in [\xi^Y_t, \xi^Y_T)$, as well as $\Theta^Y_{t+} = \Theta^Y_t$ and $\Lambda^Y_{t+} = \Lambda^Y_t$ for all $0 \leq t \leq T$. Moreover, we have $\{\eta_t^Y < \xi^Y_t\} = \{\xi^Y_T < \xi^Y_t\}; P^X_t$-a.s., and the left-hand side indicates that the first exit occurs by the drift or diffusion component strictly before the first jump of $\{X_t : t \geq 0\}$, and vice versa. Hence, we get $\eta^Y_t \wedge \xi^Y_t = \xi^Y_T \wedge \xi^Y_t$, $P^X_t$-a.s., with which the identity (3.5) with $n = 1$ can be rewritten as

\[
w_m(t,x) = E^X_t \left[ 1(X_{\xi^Y_t \wedge \xi^Y_T} \in \mathcal{D}) \Theta^X_{t+} \Lambda^X_{t+} \xi^Y_t \wedge \xi^Y_T \1 \left( \xi^Y_T < \xi^Y_t \right) \right] = E^X_t \left[ 1(X_{\xi^Y_t \wedge \xi^Y_T} \notin \mathcal{D}) \xi^Y_t \wedge \xi^Y_T \1 \left( \xi^Y_T < \xi^Y_t \right) \right] \Phi_x(t,Y_t)ds,
\]

where we have applied the identity $Y_{\xi^Y_T} = Y_{\xi^Y_t}$. Now, fix $(t,x) \in [0,T] \times \mathcal{D}$ and denote by $\mathcal{F}$ the $\sigma$-field $\sigma(\{W_s : s \in [t,T]\})$. Note that the first exit time $\xi^Y_t$ is $\mathcal{F}$-measurable and, on the $\sigma$-field $\mathcal{G}$, the random variable $\xi^Y_t$ can be treated as the first jump time of an inhomogeneous Poisson process with rate function $\lambda(s,Y_s)$ for $s \in [t,T]$. For the first term of (A.15), it holds that

\[
E^X_t \left[ 1(\xi^Y_t > \xi^Y_T) \Theta^X_{\xi^Y_T \wedge \xi^Y_T} \1 \left( \xi^Y_T < \xi^Y_t \right) \right] = E^X_t \left[ E^X_{\xi^Y_T} \left[ 1(\xi^Y_t > \xi^Y_T) \Theta^X_{\xi^Y_T \wedge \xi^Y_T} \1 \left( \xi^Y_T < \xi^Y_t \right) \right] \right] = E^X_t \left[ \Lambda^X_{\xi^Y_T} \Theta^X_{\xi^Y_T \wedge \xi^Y_T} \1 \left( \xi^Y_T < \xi^Y_t \right) \right],
\]

where the last equality holds because (A.14) contains no jump component, and moreover, $1(\xi^Y_T > T)(Y_{\xi^Y_T} = 1(\xi^Y_T > T)g(Y_T) = 1(\xi^Y_T > T)g(Y_T)$, and $1(\xi^Y_T \leq T)w_m(Y_{\xi^Y_T}, Y_{\xi^Y_T}) = 1(\xi^Y_T \leq T)w_m(Y_{\xi^Y_T}, Y_{\xi^Y_T})$, due to (A.3) and (A.4). For the second term of (A.15), observe that

\[
E^X_t \left[ 1(\xi^Y_t \leq \xi^Y_T) \Theta^X_{\xi^Y_T \wedge \xi^Y_T} \1 \left( \xi^Y_T < \xi^Y_t \right) \right] = E^X_t \left[ \Lambda^X_{\xi^Y_T} \Theta^X_{\xi^Y_T \wedge \xi^Y_T} \1 \left( \xi^Y_T < \xi^Y_t \right) \right].
\]

In a similar manner, for the third term of (A.15), we obtain

\[
E^X_t \left[ 1(\xi^Y_t \leq \xi^Y_T) \1 \left( Y_{\xi^Y_T} \neq Y_{\xi^Y_T} \cap \mathcal{D} \right) \right] = E^X_t \left[ \int_{\mathcal{D}} \Theta^X_{t+} \Lambda^X_{t+} \Phi_x(t,Y_t)ds \right] = E^X_t \left[ \int_{\mathcal{D}} \Lambda^X_{t+} \Theta^X_{t+} \Phi_x(t,Y_t)ds \right],
\]

since the random variable $\Theta^X_{t+} \Phi_x(t,Y_t)$ is $\mathcal{G}$-measurable for all $s \in [t,T]$. On the whole, the representation (A.15) can be rewritten as

\[
w_m(t,x) = E^X_t \left[ 1(\xi^Y_T > T) \Lambda^X_{\xi^Y_T} \Theta^X_{\xi^Y_T \wedge \xi^Y_T} \1 \left( \xi^Y_T < \xi^Y_t \right) \right] = E^X_t \left[ \int_{\mathcal{D}} \Lambda^X_{t+} \Theta^X_{t+} \Phi_x(t,Y_t)ds \right],
\]

since the expectation $E^X_t$ has been replaced by $E^X_t$ since the integrand is written on the coupled process $\{Y_s : s \in [t,T]\}$ with no jumps involved. Finally, we get the desired result (3.9) since, under the probability measure $Y_0^X$, the two processes $\xi^Y_t$ and (A.14) on $[t,T]$ are indistinguishable.
Lemma A.1. Let $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be bounded and such that $\partial_t f$ and $\partial^2 f$ exist almost everywhere on $[0,\infty) \times \mathbb{D}$. It holds that for $(t, x) \in [0, T] \times \mathbb{D}$ and $\delta > 0$,

$$
E^x_A \left[ 1 \left( (X_{t+\delta})_{t+\delta} \in \mathbb{D} \right) T \right] [ f(t + \delta) \wedge \lambda, X_{t+\delta})_{t+\delta} ] - 1(x \in \mathbb{D})f(t, x)
$$

$$
= E^x_A \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \left( \partial_t f(s, X_s) + \mathcal{L} f(s, X_s) - \lambda(s, X_s) f(s, X_s) + \int_{\mathbb{R}^d} f(s, X_s + z) \lambda(s, X_s) v(dz; x_s) \right) ds,
$$

and

$$
E^x_A \left[ 1 \left( (X_{t+\delta})_{t+\delta} \notin \mathbb{D} \right) \Theta_{\eta} \right] [ f(\eta, X_\eta) ] = E^x_A \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \left( \partial_t f(s, X_s) + \mathcal{L} f(s, X_s) - \lambda(s, X_s) f(s, X_s) + \int_{\mathbb{R}^d} f(s, X_s + z) \lambda(s, X_s) v(dz; x_s) \right) ds.
$$

Proof of Lemma. It holds by the Ito formula that for $(t, x) \in [0, T] \times \mathbb{D}$ and $\delta > 0$,

$$
E^x_A \left[ \Theta_{\eta} f((t + \delta) \wedge \lambda, X_{t+\delta})_{t+\delta} \right] - f(t, x)
$$

$$
= E^x_A \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \left( \partial_t f(s, X_s) + \mathcal{L} f(s, X_s) - \lambda(s, X_s) f(s, X_s) + \int_{\mathbb{R}^d} f(s, X_s + z) \lambda(s, X_s) v(dz; x_s) \right) ds,
$$

and

$$
E^x_A \left[ 1 \left( (X_{t+\delta})_{t+\delta} \notin \mathbb{D} \right) \Theta_{\eta} \right] [ f((t + \delta) \wedge \lambda, X_{t+\delta})_{t+\delta} ] - 1(x \in \mathbb{D})f(t, x)
$$

$$
= E^x_A \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \left( \partial_t f(s, X_s) + \mathcal{L} f(s, X_s) - \lambda(s, X_s) f(s, X_s) + \int_{\mathbb{R}^d} f(s, X_s + z) \lambda(s, X_s) v(dz; x_s) \right) ds,
$$

where we have kept the left limits in time to be precise in the location of the trajectory. By the above two identities, we get

$$
E^x_A \left[ 1 \left( (X_{t+\delta})_{t+\delta} \notin \mathbb{D} \right) \Theta_{\eta} \right] [ f((t + \delta) \wedge \lambda, X_{t+\delta})_{t+\delta} ] = [E^x_A \left[ \Theta_{\eta} f((t + \delta) \wedge \lambda, X_{t+\delta})_{t+\delta} \right] - f(t, x)]
$$

$$
- [E^x_A \left[ 1 \left( (X_{t+\delta})_{t+\delta} \notin \mathbb{D} \right) \Theta_{\eta} \right] [ f((t + \delta) \wedge \lambda, X_{t+\delta})_{t+\delta} ] - 1(x \in \mathbb{D})f(t, x)]
$$

$$
= E^x_A \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \left( \partial_t f(s, X_s) + \mathcal{L} f(s, X_s) - \lambda(s, X_s) f(s, X_s) + \int_{\mathbb{R}^d} f(s, X_s + z) \lambda(s, X_s) v(dz; x_s) \right) ds,
$$

which concludes due to $\eta_t \leq \delta + \tau$ on the event $(X_{t+\delta})_{t+\delta} \notin \mathbb{D})$ and the integrand is absolutely continuous with respect to the Lebesgue measure $ds$. \hfill \square

Proof of Theorem 3.6 (a) Let $(t, x) \in [0, T) \times \mathbb{D}$ and $\delta \in (0, T - t)$. We rewrite the function $w_m$ in the representation (3.5) in a local form on the time interval $(t, t + \delta)$ as $\delta \to 0+$ by considering two cases separately as to whether or not there is at least one jump on the interval.

First, consider the case where there is no jump on the interval $(t, t + \delta) \ni \eta^T_\delta$; that is, the first jump time $\eta^T_\delta$ after $t$ is strictly later than the end time $\eta^T_\delta = (t + \delta) \wedge \eta^T$. Hence, we get $\eta^T_\delta = \eta^T_\delta \wedge \eta^T_\delta$. Moreover, observe that $\eta^T_\delta \wedge \eta^T_\delta = 1 (\eta^T_\delta \leq \delta + \tau)$$\eta^T_\delta \wedge (\eta^T_\delta \wedge \eta^T_\delta) = \eta^T_\delta \wedge \eta^T_\delta \wedge \eta^T_\delta$ due to (6.9). By applying those identities, it holds that

$$
E^x_A \left[ 1 \left( \eta^T_\delta > \eta^T_\delta \right) \left( 1 \left( \eta^T_\delta \wedge \eta^T_\delta = 1 \left( \eta^T_\delta \leq \delta + \tau \right) \eta^T_\delta \left( \eta^T_\delta \wedge \eta^T_\delta = \eta^T_\delta \wedge \eta^T_\delta \right) \right) \right] - \int_t^{\eta^T_\delta} \Theta_{s-} \phi(s, x) ds]
$$

where we have applied the Markov property, the tower rule and the representation (3.5).

Next, consider the case where there is at least one jump on the interval $(t, \eta^T_\delta)$, that is, the first jump time $\tau_\delta$ after $t$ is before or at latest at the end time $\eta^T_\delta$. It then holds that

$$
E^x_A \left[ 1 \left( \eta^T_\delta \leq \eta^T_\delta \right) \left( 1 \left( \eta^T_\delta \wedge \tau_\delta = 1 \left( \eta^T_\delta \leq \delta + \tau \right) \eta^T_\delta \left( \eta^T_\delta \wedge \tau_\delta = \eta^T_\delta \wedge \tau_\delta \right) \right) \right] - \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \phi(s, x) ds]
$$

due to $\eta^T_\delta = \tau_\delta$ on the event $(\tau^T_\delta \leq \eta^T_\delta) \cap \{ \eta^T_\delta \notin \mathbb{D} \}$. By combining the two identities (A.16) and (A.17), we obtain

$$
w_m(t, x) = E^x_A \left[ 1 \left( \eta^T_\delta > \eta^T_\delta \right) \Theta_{t+\delta} w_m(\eta^T_\delta \wedge \eta^T_\delta)
$$

$$
+ \left( \eta^T_\delta \leq \eta^T_\delta \right) \Theta_{t+\delta} \left( 1 \left( \eta^T_\delta \wedge \tau_\delta = 1 \left( \eta^T_\delta \leq \delta + \tau \right) \eta^T_\delta \left( \eta^T_\delta \wedge \tau_\delta = \eta^T_\delta \wedge \tau_\delta \right) \right) \right] - \int_t^{(t+\delta)\wedge\eta} \Theta_{s-} \phi(s, x) ds \right].
$$

Here, we first focus on the first term, that is, on the event $(\tau^T_\delta > \eta^T_\delta)$, where the underlying process starting at time $t$ has not made a jump until the endpoint $\eta^T_\delta \wedge \tau_\delta$ of the interval of interest. Hence, even when it exits from the domain, that is, $\{ \eta^T_\delta \leq \delta + \tau \}$, the exit is necessarily caused by the diffusion term in the boundary $\partial D$. With the aid of the given regularity of $w_m$, it holds $E^x_A$-a.s. on the event $(\tau^T_\delta > \eta^T_\delta)$ that

$$\Theta_{t+\delta} w_m(\eta^T_\delta \wedge \eta^T_\delta) = w_m(t, x) + \int_t^{\eta^T_\delta} \Theta_{s-} \left( -r(s, X_s) w_m(s, x_s) + \partial_t w_m(s, x_s) + \mathcal{L} w_m(s, x_s) ds + \int_t^s \Theta_{s-} \left( \mathcal{L} w_m(s, x_s), \sigma(s, x_s) dW_s \right) \right) ds.$$
that is, the jump component is degenerate on the interval \((t, t + \delta]\). Hence, it holds by taking the conditional expectation \(\mathbb{E}^x_\lambda\) that
\[
\mathbb{E}^x_\lambda \left\{ 1 \left( t^{(1)}_\lambda > \eta^{+\delta}_1 \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda) = w_m(t, x).
\]
By dividing this identity by \(\delta\) and taking a limit \(\delta \to 0^+\), we get
\[
\frac{1}{\delta} \mathbb{E}^x_\lambda \left\{ 1 \left( t^{(1)}_\lambda > \eta^{+\delta}_1 \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda) \to -\Theta(t, x) \mathbb{R}(t, x),
\]
where we have applied \(\mathbb{P}^{x}_\lambda \left( t^{(1)}_\lambda \leq \eta^{+\delta}_1 \right) - \delta \) \( t, x \) and \( 1 \left( t^{(1)}_\lambda > \eta^{+\delta}_1 \right) \to 1, \mathbb{P}^{x}_\lambda\)-a.s., due to \( \eta^{+\delta}_1 = t + \delta \) eventually \( \mathbb{P}^{x}_\lambda\)-a.s. By the same reasoning, we obtain
\[
\frac{1}{\delta} \mathbb{E}^x_\lambda \left[ \Theta_{t^{(1)}_\lambda} \phi(x, t) ds \right] \to \phi(x, t).
\]

Next, consider the event \( \{ t^{(1)}_\lambda \leq \eta^{+\delta}_1 \} \cap \{ X_{t^{(1)}_\lambda} \notin \mathcal{D} \} \), that is, the first jump occurs before (or right at) \( \eta^{+\delta}_1 \) and does not bring the trajectory strictly out of the closure \( \mathcal{D} \). In the other words, it holds true that the trajectory remains in the closure \( \mathcal{D} \) over the interval \( [t, t^{(1)}_\lambda \wedge \eta^{+\delta}_1] \), that is, \( X_t \in \mathcal{D} \) for all \( x \in [t, t^{(1)}_\lambda \wedge \eta^{+\delta}_1] \), \( \mathbb{P}^{x}_\lambda\)-a.s. Thus, this indicator function does not break down smoothness for the Ito formula. If \( w_{m-1} \) is as smooth as imposed, we thus have
\[
\mathbb{E}^x_\lambda \left\{ 1 \left( X_{t^{(1)}_\lambda} \wedge \eta^{+\delta}_1 \in \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda) = w_{m-1}(t, x),
\]
which we have applied Lemma \( \text{[A.1]} \). This, with the aid of the convergence \( \text{[A.19]} \) with \( m \) replaced by \( m - 1 \), yields the desired result
\[
\mathbb{E}^x_\lambda \left\{ 1 \left( t^{(1)}_\lambda \leq \eta^{+\delta}_1 \right) 1 \left( X_{t^{(1)}_\lambda} \wedge \eta^{+\delta}_1 \in \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda) = w_{m-1}(t, x)
\]
\[
\mathbb{E}^x_\lambda \left[ \Theta_{t^{(1)}_\lambda} \phi(x, t) ds \right] \to \phi(x, t),
\]
where \( \delta \to 0^+ \).

Finally, consider the event \( \{ t^{(1)}_\lambda \leq \eta^{+\delta}_1 \} \cap \{ X_{t^{(1)}_\lambda} \notin \mathcal{D} \} \), that is, the first jump occurs before or at \( \eta^{+\delta}_1 \) and brings the trajectory strictly out of the closure \( \mathcal{D} \). Hence, the first jump time \( t^{(1)}_\lambda \) is nothing but the first exit time before, or at latest, the endpoint of the interval \( [t, \eta^{+\delta}_1] \) of current interest, that is, \( t^{(1)}_\lambda = \eta^{+\delta}_1 \). We thus have
\[
1(X_{t^{(1)}_\lambda} \notin \mathcal{D}) = 1(X_{t^{(1)}_\lambda} \notin \mathcal{D}) = 0 \text{ on } [t, \eta^{+\delta}_1) \setminus \{ t^{(1)}_\lambda \} \text{ Conversely, if } 1(X_{t^{(1)}_\lambda} \notin \mathcal{D}) = 0 \text{ for all } x \in [t, \eta^{+\delta}_1], \text{ then no jump is allowed to occur before or at } \eta^{+\delta}_1 \text{ to bring the trajectory strictly out of the closure } \mathcal{D}. \text{ Therefore, it holds that}
\]
\[
\mathbb{E}^x_\lambda \left\{ 1 \left( t^{(1)}_\lambda \leq \eta^{+\delta}_1 \right) 1 \left( X_{t^{(1)}_\lambda} \notin \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda)
\]
\[
= \mathbb{E}^x_\lambda \left[ \Theta_{t^{(1)}_\lambda} \phi(x, t) ds \right] \to \phi(x, t),
\]
with the aid of Lemma \( \text{[A.1]} \). By dividing the identity by \( \delta \) and taking \( \delta \to 0^+ \), we get
\[
\mathbb{E}^x_\lambda \left\{ 1 \left( t^{(1)}_\lambda \leq \eta^{+\delta}_1 \right) 1 \left( X_{t^{(1)}_\lambda} \notin \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda) \to \mathbb{E}^x_\lambda \left[ \Theta_{t^{(1)}_\lambda} \phi(x, t) ds \right] = \mathbb{E}^x_\lambda \left[ \Theta_{t^{(1)}_\lambda} \phi(x, t) ds \right] = H(t, x),
\]
(3.22) since \( t^{(1)}_\lambda \leq \eta^{+\delta}_1 = t + \delta \) eventually \( \mathbb{P}^{x}_\lambda\)-a.s., due to a finite jump intensity. By rearranging \( \text{[A.18]} \) and combining the results \( \text{[A.19], [A.20], [A.21]} \) and \( \text{[A.22]} \), we obtain the desired partial differential equation \( \text{[3.14]} \).

To derive the initial condition, observe that for every \( x \in D \),
\[
w_m(t, x) = \mathbb{E}^x_\lambda \left\{ 1 \left( X_{t^{(1)}_\lambda} \notin \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} w_{m-1}(T, X_{t^{(1)}_\lambda}) + 1 \left( X_{t^{(1)}_\lambda} \in \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} \right\} \omega_{t^{(1)}_\lambda} \mathbb{R}(t^{(1)}_\lambda, X_{t^{(1)}_\lambda}; \lambda) = w_{m-1}(t, x),
\]
the based on the representation \( \text{[3.5]} \), due to \( \eta^{+\delta}_1 > T \) \( t^{(1)}_\lambda > T \) and \( X_{t^{(1)}_\lambda} = x \), \( \mathbb{P}^{x}_\lambda\)-a.s. Hence, by induction, we get \( w_m(t, x) = w_0(t, x) = g(x) \) for all \( x \in D \), again due to \( \eta^{+\delta}_1 > T \), \( \mathbb{P}^{x}_\lambda\)-a.s. in \( \text{[3.3]} \).

(b) Due to \( \text{[3.3]} \), we have \( w_0(T, x) = g(x) \) and the identity
\[
1 \left( X_{t^{(1)}_\lambda} \notin \mathcal{D} \right) \Theta_{t^{(1)}_\lambda} w_0(\eta^{+\delta}_1, X_{t^{(1)}_\lambda}) = 1 \left( \eta^{+\delta}_1 > T \right) \Theta_{t^{(1)}_\lambda} g(X_{t^{(1)}_\lambda}) + 1 \left( \eta^{+\delta}_1 \leq T \right) 1 \left( X_{t^{(1)}_\lambda} \in \partial D \right) \Theta_{t^{(1)}_\lambda} \Psi(\eta^{+\delta}_1, X_{t^{(1)}_\lambda}),
\]
almost surely under the probability measure \( \mathbb{P}^{x}_\lambda \). Thus, the representation \( \text{[3.5]} \) can be rewritten as
\[
\mathbb{E}^x_\lambda \left[ \Theta_{t^{(1)}_\lambda} w_0(\eta^{+\delta}_1, X_{t^{(1)}_\lambda}) \right] = w_0(t, x) = \mathbb{E}^x_\lambda \left[ \int_0^\eta^{+\delta}_1 \Theta_{t^{(1)}_\lambda} \phi(s, X_{t^{(1)}_\lambda}) ds \right].
\]
Equating this with the Dynkin formula under the smoothness conditions yields the partial differential equation

$$
\partial_t w(t,x) + \mathcal{L} w(t,x) = r(t,x) w(t,x) + \phi(t,x),
$$

for almost every \((t,x) \in [0,T] \times \mathbb{D}\). By rearranging \((A.24)\) and then applying it to the representation \((2.3)\), with the aid of Lemma \(A.1\) with \(f = w_0\) and \(\delta = T - t\), we get the following identity

$$
\begin{align*}
    u(t,x) &= \mathbb{E}^x \left[ 1 \{ \eta_T > t \} \Theta_T g(X_T) + 1 \{ \eta_T \leq t \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right].
\end{align*}
$$

Thus, by taking essential supremum and infimum over the time-state domain, the functions \(N_0^d(t)\) and \(N_0^d(t)\) can be taken out of the expectation to yield the inequalities

$$
N_0^d(t) \xi(t,x,\lambda) \leq u(t,x) - w_0(t,x) \leq N_0^d(t) \xi(t,x,\lambda), \quad (t,x) \in [0,T] \times \mathbb{D}.
$$

By setting \(g \equiv 0\), \(\Psi \equiv 0\) and \(\phi \equiv -1\) in \((2.3)\) and \((2.3)\), we get \(H \equiv 0\), \(M^d = N_0\), \(w_0(t,x) = \xi(t,x,\lambda_0)\), and \(u(t,x) = \xi(t,x,\lambda)\). Hence, the last inequalities reduce to \(M^d(t) \xi(t,x,\lambda) \leq \xi(t,x,\lambda) - \xi(t,x,\lambda_0) \leq M^d(t) \xi(t,x,\lambda)\).

By rearranging this inequality under the assumption \(M^d(t) \leq M^d(t) < 1\), we get

$$
\begin{align*}
    N_0^d(t) \xi(t,x,\lambda) &\leq \xi(t,x,\lambda) - \xi(t,x,\lambda_0) \\
    &\leq N_0^d(t) \xi(t,x,\lambda),
\end{align*}
$$

for almost every \((t,x) \in [0,T] \times \mathbb{D}\), with which the representation \((2.3)\) yields \(u(t,x) = w_0(t,x) + \mathbb{E}^x \left[ \int_0^T \Theta_s \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -}, w_0(s,x) + G_\xi(s,x) + H(s,x)) ds \right]

\text{for every} \quad (t,x) \in [0,T] \times \mathbb{D}, \quad \text{and Lemma A.1). Thus, by taking the essential supremum and infimum of the given inequality, we obtain the desired result.}

\(\Box\)

For the proofs of Theorems 3.7 and 3.8, we recall the notation \(\mathbb{E}^x\) and \(\mathbb{E}^x\), which respectively denote the probability measure and the associated expectation under which the underlying process is governed by the stochastic differential equation process \((3.10)\) with \(X_t = X\) almost surely. Moreover, we have denoted by \(\ell^{(m)}\) a sequence of jump times of the redefined underlying process \((3.10)\), where each \(\ell^{(m)}\) indicates an \(m\)-th jump time, which may be of a zero-sized jump.

\text{Sketch of proof of Theorem 3.7: To avoid overloading the paper with simple repetition, we do not go through the proofs of Theorems 3.1, 3.2, 3.4, 3.5 and 3.6 by highlighting required amendments. Instead, we summarize a few key points to be taken care of. Under the probability measure \(\mathbb{P}^x\), the underlying process \((3.16)\) makes jumps based on the constant rate \(\lambda\), irrespective of the time and state. Each jump is then rejected (by setting its size to zero) by the time-state dependent probability \(\tilde{v}(t,s)\). The jump times, denoted by \(\{\ell^{(m)}\}_{m \in \mathbb{N}}\), including those of zero-sized jumps, can thus be represented as a cumulative sum \(\ell^{(m)} \geq t + \sum_{i=1}^{\infty} E_i\), where \(\{E_i\}_{i \in \mathbb{N}}\) is a sequence of iid exponential random variables with rate \(\lambda\), that is, \(\mathbb{P}(\ell^{(1)} \geq s) = 0\), for \(s \geq t\). With those in mind, the result can be derived in a similar manner to the previous results.} \(\Box\)

\text{Proof of Theorem 3.8: By applying Lemma A.1 and the identity \((2.4)\), we obtain}

$$
\begin{align*}
    \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] &= \mathbb{E}^x \left[ \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds \right].
\end{align*}
$$

\text{By combining those identities along with \((2.3)\) and \((3.4)\), we get}

$$
\begin{align*}
    u(t,x) - w_0(t,x) &= \mathbb{E}^x \left[ 1 \{ \eta_T \not\in \mathbb{D} \} \Theta_T \mathbb{P}(\eta_T, X_{\eta_T}, X_{\eta_T -} - \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds) \right] \\
    &= \mathbb{E}^x \left[ \int_0^{\eta_T} \Theta_s \phi(s,X_s) ds \right], \quad (t,x) \in [0,T] \times \mathbb{D}.
\end{align*}
$$

(A.26)
where we have applied \(3.23\) and \(3.12\). Hence, due to \(\Theta \in (0, 1)\) and \([\eta^{(m)}_r, \eta^{(r)}] \subseteq [t, T]\), it holds that for \((t, x) \in [0, T] \times \mathcal{D}\),

\[
|u(t, x) - w_m(t, x)| \leq (T - t) \max \left\{ |N'(t)|, |N''(t)| \right\} P^{\lambda}(\tau^{(m)}_0) \leq T.
\] (A.27)

As for the rightmost probability, with \(\lambda_0(t) := \sup_{(t, x) \in [0, T] \times \mathcal{D}} \lambda(s, x)\), we have

\[
P^\lambda(\tau^{(m)}_0) \leq \sum_{k=m}^{\infty} e^{-\lambda_0(t)(T-t)} \frac{\lambda_0^k}{k!},
\]

for all \((t, x) \in [0, T] \times \mathcal{D}\), which is independent of the initial state \(x\). Recalling \(\lambda_0 = \sup_{(t, x) \in [0, T] \lambda_0(t)}(T - t)\), there exists a strictly positive constant \(c_m\) such that

\[
\sup_{(t, x) \in [0, T] \times \mathcal{D}} P^\lambda(\tau^{(m)}_0) \leq \sup_{(t, x) \in [0, T] \times \mathcal{D}} \sum_{k=m}^{\infty} e^{-\lambda_0(t)(T-t)} \frac{\lambda_0^k}{k!} \leq \sum_{k=m}^{\infty} e^{-\lambda_0} \frac{k^k}{k!} = e^{-\lambda_0} \frac{e^m}{m!} \rightarrow 0,
\] (A.28)

as \(m \rightarrow +\infty\). Here, the asymptotic inequality holds since \(\lambda_0\) is finite and \(e^{-\lambda_0 t}\) is increasing on \((0, k)\). The last equality holds

\[
\frac{m!}{\lambda_0^m} \sum_{k=m}^{\infty} e^{-\lambda_0} \frac{\lambda_0^k}{k!} = e^{-\lambda_0} \sum_{k=m}^{\infty} \frac{\lambda_0^k}{k!} \leq e^{-\lambda_0} \sum_{k=m}^{\infty} \frac{\lambda_0^{k-m}}{(k-m)!} = 1,
\]
due to \(m! \geq 1\) with equality only when \(m = 0\) or \(k = m\). Combining the inequalities (A.27) and (A.28) yields the desired uniform convergence. Therefore, by the identity (3.8), the triangle inequality yields

\[
|u(t, x) - w_m(t, x)| \leq |u(t, x) - w_0(t, x)| + \mathbb{E}^\lambda \left[ \Theta \left( \tau^{(m)}_0 < \eta^{(m)} \right) w_0(\tau^{(m)}_0, X^{(m)}_0) \right],
\]

where the last term has the same decay rate as (A.25) at most, due to the boundedness of \(w_0\). Finally, for each fixed \(t \in [0, T]\), it holds by (3.7) that for every \((s, x) \in [t, T] \times \mathcal{D}\),

\[
|N_m(s, x)| = |G_m(s, x) - G_{m-1}(s, x)| \leq \int_{[0, T] \times \mathcal{D}} |w_m(s, x + z) - w_{m-1}(s, x + z)| \lambda(s, x) \nu(dz, x),
\]

which yields \(N'_m(t) \rightarrow 0\) and \(N''_m(t) \rightarrow 0\), as \(m \rightarrow +\infty\), after taking supremum of \((s, x)\) over \([t, T] \times \mathcal{D}\), due to Theorem (3.8) and \(\int_{[0, T] \times \mathcal{D}} \nu(dz, x) = 1\) for all \(x\). The results for \(\tilde{w}_m\) can be derived in a similar manner to (A.27) under the probability measure \(\mathbb{P}^\lambda\), yet with the condition \(|N'_m(0)| + |N''_m(0)| < +\infty\) in common. \(\square\)