A Numerical Approach for Designing Unitary Space Time Codes with Large Diversity Product and Diversity Sum

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Abstract — We define the diversity function and analyze its limiting behavior which results in two important design criteria: the diversity product and the diversity sum. Numerical methods are derived which allows one to construct codes with excellent diversity function and excellent diversity product and sum.

I. INTRODUCTION
A multiple antenna communication system with \( M \) transmit antennas and \( N \) receive antennas operating in a non-coherent Rayleigh flat-fading channel can be characterized by the following equation:

\[
R = \sqrt{\frac{\rho T}{M}} \Phi H + W,
\]

where \( \Phi \) is an element from the unitary signal constellation \( \mathcal{V} := \{ \Phi_1, \ldots, \Phi_L \} \), \( T \) is the quasi-static period.

It is a basic design objective to construct constellations \( \mathcal{V} = \{ \Phi_1, \ldots, \Phi_L \} \) such that the pairwise probabilities \( P_{\Phi_i, \Phi_j} \) are as small as possible. The main purpose of this paper is to develop numerical procedures which allow one to construct unitary constellations with excellent diversity for any set of parameters \( M, N, T, L \) and for any signal to noise ratio \( \rho \).

II. DIVERSITY CRITERIA
We define a simplified function called the diversity function through:

\[
\mathcal{D}(\mathcal{V}, \rho) := \max_{l \neq l'} \frac{1}{2} \prod_{m=1}^{M} \left( 1 + \frac{(\rho T/M)^2}{4(1 + \rho T/M)}(1 - \delta_m(\Phi_l^* \Phi_{l'})) \right)^{-N}.
\]

As a result of the limiting behavior analysis of the diversity function, we derive the following design criteria: diversity product (for high SNR) and diversity sum (for low SNR).

The diversity product of a unitary constellation \( \mathcal{V} \) is defined as:

\[
\prod_{l=1}^{L} \frac{\min_{l \neq l'} \left| \prod_{m=1}^{M} (1 - \delta_m(\Phi_l^* \Phi_{l'})) \right|^2}{M}.
\]

The diversity sum is defined as follows: The diversity sum of a unitary constellation \( \mathcal{V} \) is defined as:

\[
\sum_{l=1}^{L} \sqrt{\frac{1 - \|\Phi_l^* \Phi_{l'}\|^2}{M}}.
\]

In general it seems to be difficult to come up with algebraic constructions of constellations with good diversity sum/product or more general good diversity function. In the next section we outline how numerical techniques can lead to near optimal constellations. More details on this can be found in [2]. Essential in the numerical algorithms is the Cayley transform [3].

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III. NUMERICAL DESIGN OF UNITARY CONSTELLATIONS WITH GOOD DIVERSITY
We used Simulated Annealing Algorithm (SA) and Genetic Algorithm (GA) (see e.g. [1]) to search the optimized constellation. The following constellations were all obtained in less than 3 minutes on an Intel Pentium 800MHz PC.

Results on Diversity Sum (GA)

| size | dim=2 | optimal DS | dim=3 | dim=4 | dim=5 |
|------|-------|------------|-------|-------|-------|
| 3    | 0.860 | 0.8860     | 0.833 | 0.811 | 0.779 |
| 4    | 0.802 | 0.8165     | 0.780 | 0.775 | 0.749 |
| 6    | 0.744 | 0.7746     | 0.750 | 0.729 | 0.717 |
| 10   | 0.683 | 0.7071     | 0.698 | 0.692 | 0.681 |

IV. CONSTELLATIONS OF DIFFERENT DIMENSIONS
We used Genetic Algorithm to optimize the diversity product and diversity sum at the same time to derive two constellations of dimension 3 and 4 respectively. We have chosen the size in such that the rate is comparable in each case.

| size | rate | DP | DS |
|------|------|----|----|
| 2 dim| 3    | 0.7925 | 0.8660 | 0.8660 |
| 3 dim| 5    | 0.7740 | 0.7183 | 0.7454 |
| 4 dim| 9    | 0.7925 | 0.5904 | 0.6403 |

The first graph below illustrates the performance of the three different constellations. One can see how the 4-dimensional constellation really performs well at high SNR.

REFERENCES
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