Principal Chiral Field at Large N

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Abstract

We present the exact and explicit solution of the principal chiral field in two dimensions for an infinitely large rank group manifold. The energy of the ground state is explicitly found for the external Noether’s fields of an arbitrary magnitude. The exact Gell-Mann -Low function exhibits the asymptotic freedom behaviour at large value of the field in agreement with perturbative calculations. Coefficients of the perturbative expansion in the renormalized charge are calculated. They grow factorially with the order showing the presence of renormalons. At small field we found an inverse logarithmic singularity in the ground state energy at the mass gap which indicates that at $N = \infty$ the spectrum of the theory contains extended objects rather then pointlike particles.

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1 Introduction

Recent progress in understanding of lower dimensional string theories is partially due to the advantage of discrete methods. It happened that the matrix quantum mechanics, despite its simplicity and solvability describes nonperturbative aspects of string theories at $d=1$ target space, i.e. $c = 1$ matter coupled to the two dimensional gravity.

The standard combinatorial methods of matrix models, as well as the continuous approaches have appeared so far to be ineffective for higher dimensional target space.

On the other hand, it has been known for a long time that certain matrix field theories are completely integrable in $2d$ for an arbitrary size of the matrix field. One of the most representative integrable matrix field theories is the principal chiral field (PCF) which describes a free field on a principal, say $SU(N)$, manifold:

$$S = \frac{N}{2\lambda_0} \int d^2x \, \text{tr} \left[ \partial_\mu g^\dagger \partial_\mu g \right] \quad (1)$$

where $g$ is an $N\times N$ unitary matrix. Its large $N$ solution has been anticipated for a long time to follow from its finite $N$ solution [1, 2, 3, 4].

Apart from being a model of Goldstone bosons whose strong interaction is entirely determined by the geometry of the manifold, this model has a long history of exploring its analogy to QCD [5] and contour geometry of gauge fields [6].

The large $N$ limit of the model is of particular interest. It is conceivable that it describes a string theory in two physical dimensions due to the analogy between planar Feynman graphs and the world sheets of a string. Of course one should not take this analogy literally: in asymptotically free theory neither the coupling constant $\lambda_0$ nor a renormalized coupling is a cosmological constant of a string [7]: due to renormalons the contribution of even planar graphs grows factorially with the order. However, some signs of the stringy behaviour will be seen in the nonperturbative regime.

In this paper we present the exact and explicit large $N$ solution for the chiral field in two dimensions.

The exact solution of the PCF (both $S$-matrix and Bethe Ansatz equations) was found in Ref.[1, 2]. It turns out that the spectrum of the, say,
$SU(N)$ model contains $N-1$ kinds of massive particles. They form multiplets belonging to the diagonal of the $SU(N) \otimes SU(N)$ associated with all fundamental representations of the $SU(N)$ algebra, namely, the vector representation and all antisymmetric tensors according to the Dynkin diagram. The spectrum of masses is

$$m_l = m \frac{\sin(\frac{n_l}{N})}{\sin(\frac{m}{N})}$$

where $l = 1, ..., N-1$ is the rank of a fundamental representation and $m = m_1$ is the mass of the vector particle. In the two-loop approximation it is

$$m = \Lambda \frac{1}{\sqrt{\lambda_0}} exp\left(-\frac{4\pi}{\lambda_0}\right)$$

where $\Lambda$ is a cutoff. All particles are bound states of the vector particles.

At large $N$ we must distinguish two physically different situations: $N \to \infty$ but $m = m_1 = \text{fixed}$. This means that $m_l = l m_1$, so that the $l$-th particle is not a bound state any more. It is decomposed into $l$ vector particles. This suggests that the interaction vanishes in this limit and we end up with a free massive field. This limit is not of a particular interest.

Below we consider another and the only physically interesting limit: $N \to \infty$ but the heaviest mass $m_{N/2} = \mu$ of the largest antisymmetric tensor remains fixed. In this case the masses fuse so that the mass spectrum becomes continuous. The label running along the Dynkin diagram becomes a continuous parameter. The energy of the massive excitation with a momentum $p$ will be therefore $\sqrt{p^2 + m_l^2} \approx \sqrt{p^2 + q^2}$ where we introduced $q = \mu \frac{\pi}{N} l$. We observe that an extra dimension emerges from the matrix structure of the field. This means that at $N = \infty$ particles do not form a discrete spectrum, so the theory ceases to be a theory of point like particles.

We shall find the energy of the ground state as a function of “the Noether’s” field by adding a term $tr(H_L Q_L + H_R Q_R)/2$ to the hamiltonian of the theory corresponding to the lagrangian (1). Here $Q_L = \int d^2 x g \partial_0 g^{-1}$ and $Q_R = \int d^2 x g^{-1} \partial_0 g$ are the Noether’s left and right charges and $H_{R(L)} = diag(h_1, h_2, ..., h_{N-1}, -h_{N-2}, -h_{N-1})$ is an element of the Cartan subalgebra. It amounts to introducing in eq.(1) the covariant derivative

$$D_\mu g = \partial_\mu g - i\delta_\mu (H_L g + g H_R)/2$$

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instead of the usual derivative \( \partial_\mu g \). In what follows, we shall consider only the case \( H_L = H_R = H \).

Parameters \( h_i \) play the role of chemical potentials for elementary particles of the model, so that the energy \( \mathcal{E}(h) \) is the energy of the ground state with a symmetry of the Young tableau \([1^{N-N_1}, 2^{N_1-N_2}, \ldots, N^{N_{N-1}}]\) where \( N_i = -d/dh_i \mathcal{E}(h) \). The field \( H \) introduces an energy scale into the theory and gives a valuable physical information.

The large \( N \) solution is explicit. To ease the references we state it now.

We use a special direction of the field which excites all types of the particles on equal footing:

\[
h_l = h \frac{\sin(\pi l)}{\sin(\pi/2)}
\]  

(5)

We show that the energy of the ground state is expressed in terms of modified Bessel functions:

\[
f(h) \equiv \frac{1}{N^2} \left( \mathcal{E}(h) - \mathcal{E}(0) \right) = -\frac{h^2}{8\pi} B^2 I_1(B)K_1(B)
\]  

(6)

where the parameter \( B \) is defined through

\[
\frac{m}{h} = BK_1(B)
\]  

(7)

The distribution of rapidities of physical particles will obey the simple semi-circle law with the support \( B \). The parameter \( B \) defines the value of rapidity corresponding to the Fermi momentum of the fused particles. We shall see, that \( B \) gives the most natural definition of the renormalized (running) coupling constant :

\[
\bar{\lambda}(h) = \frac{4\pi}{B}
\]  

(8)

The reader may find some similarity between \( B \) and the Fermi level of eigenvalues of the matrix field in the \( c=1 \) string theory [8].

The paper is organised as follows:

In section 2 we review the derivation of the spectral Bethe ansatz integral equations for rapidities.

In section 3 we solve the spectral equations in the large \( N \) limit.

In section 4 we obtain a singular behaviour on the threshold \( h \sim m \) of the spectrum.
In section 5 we show that our solution agrees with perturbative two loops calculations at large $h/m$ and find exact value of the mass (namely the ratio $m/\Lambda_{MS}$).

In section 6 we calculate all terms of perturbation theory in running coupling constant $\bar{\lambda}$ and show their factorial growth.

2 $S$-matrix and Bethe-ansatz equations for any $N$

Perhaps the most economical way to obtain the Bethe-ansatz equations for the chiral field is the factorized bootstrap method [3], rather than direct diagonalization of the hamiltonian of the model [1, 2]. The point is that the $S$-matrix can be easily found on the basis of some heuristic hypothesis. Let us give a sketch of this approach.

1) $S$-matrix. The chiral field is renormalizable and asymptotically free [9, 10]. It is invariant under the left-hand and right-hand group transformations $G \otimes G$ and the action (1) is defined only by the Lie algebra of $G$, i.e. does not depend on the representation of the group $G$. Therefore it is natural to assume that the elementary particles are massive and belong to some fundamental representations of the diagonal of the $G \otimes G$, whereas the antiparticles form conjugated representations. The model is integrable [11], therefore the scattering is factorized. Under these assumptions the minimal $S$-matrix (factorized scattering matrix with a minimal set of singularities) can be determined unambiguously. The complete proof of the minimal $S$-matrix being the scattering matrix of the chiral field requires a more sophisticated technique (see e.g. [1, 2]).

It turns out that once we assume that there is a particle in some, say, $l$-th, fundamental representation, the factorized bootstrap tells us that there are particles in all $N-1$ fundamental representations. In fact, they are bound states of an arbitrary chosen representation. Therefore it is convenient to start from the vector particle. The factorized $SU(N) \otimes SU(N)$ scattering matrix for vector particles is the tensor product of the $SU(N)$ factorized vector $S$-matrices $S = X(\theta)S(\theta) \otimes S(\theta)$. Here $\theta$ is a rapidity of a massive relativistic particle ($p^0 = m \cosh \theta, p^1 = m \sinh \theta$) and $X(\theta)$ is the CDD-ambiguity factor which cannot be determined by the factorization, unitary
and crossing symmetry conditions. The $SU(N)$ unitary, crossing invariant, factorized $S$-matrix of vector particles is well known \[12\]. It is

$$S(\theta) = u(\theta)(P^+ + \frac{\theta + i2\pi/N}{\theta - i2\pi/N} P^-)$$  \hspace{1cm} (9)$$

where $P^\pm$ is the projection operator onto symmetric (antisymmetric) states.

The amplitude in the symmetric channel $u(\theta)$ and the amplitude in the cross channel (particle- antiparticle scattering) $t(\theta) = \frac{1/2-\theta/(2\pi)}{1/2-1/N-\theta/(2\pi)} u(i\pi - \theta)$ obey the unitarity conditions $t(\theta)t(-\theta) = u(\theta)u(-\theta) = 1$. The minimal solution of these equations is

$$u(\theta) = \frac{\Gamma(1 - \frac{\theta}{2\pi i})\Gamma(\frac{1}{N} + \frac{\theta}{2\pi i})}{\Gamma(1 + \frac{\theta}{2\pi i})\Gamma(\frac{1}{N} - \frac{\theta}{2\pi i})}$$  \hspace{1cm} (10)$$

Finally the CDD-factor is chosen to cancel all double zeros and double poles on the physical sheet $0 < \text{Im}\theta < \pi$:

$$X(\theta) = \frac{\sinh(\frac{\theta}{2} + i\pi)}{\sinh(\frac{\theta}{2} - i\pi)}$$  \hspace{1cm} (11)$$

This is the $S$-matrix of the vector particles. It has a pole on the physical sheet at $\theta_b = 2\pi i/N$ in the antisymmetric channel. It corresponds to the first bound state (the second rank antisymmetric tensor) with a mass $m_2 = m \sin(2\pi/N)/\sin(\pi/N)$. The $S$-matrix of these particles can be also found by tensoring the vector $S$-matrix (the fusion procedure). It also has a pole in the antisymmetric channel, and so on. In this way the whole mass spectrum \[4\] can be generated.

2) Bethe-Ansatz Equations. The thermodynamical properties of the model can be obtained by imposing the periodic boundary conditions. For an integrable problem they imply the balance of two particle scattering phases and a phase of a free motion between collisions. For the $i$-th particle with the momentum $m \sinh \theta_i$, the periodic boundary conditions lead to the problem of the diagonalization of a product of two-particle S-matrices. Say, for a state with $N$ vector particles in the box $L$ we have

$$\exp(imL \sinh \theta_\alpha) = \prod_{\beta=1,\beta \neq \alpha}^{N} S_{\alpha \beta}(\theta_\alpha - \theta_\beta)$$  \hspace{1cm} (12)$$
The eigenvalues of the operator in the r.h.s of the eq.(12) can be found by the projection onto the symmetric subspace

$$\exp(imL \sinh \theta_\alpha) = \prod_{\beta=1, \alpha \neq \beta}^{N} \exp(i\phi(\theta_\alpha - \theta_\beta))$$ (13)

where $\exp(i\phi(\theta)) = u^2(\theta)X(\theta)$. To obtain the Bethe-Ansatz equation for the state which contains all kinds of particles, say, for the state with the Young tableau $[1^{N_1-N_2}, 2^{N_1-N_2}, ..., N^{N_N-1}]$ one has to consider complex rapidities of the bound states’ “strings” $\theta \rightarrow \theta + 2r\pi i/N$, where $\theta^l$ is a rapidity of the $l$-th particle and $r$ is an integer running between $-l/2$ and $l/2$. Substituting this into eq.(13) and multiplying equations over $r$ we shall obtain the equations for the rapidities of the state which contains $N_l$ particles of the kind $l$:

$$\exp(iLm_l \sinh \theta^{(l)}_\alpha) = \prod_{n=1}^{N-1} \prod_{\alpha, \beta}^{N_n} \exp(i\phi_{ln}(\theta^{(l)}_\alpha - \theta^{(n)}_\beta))$$ (14)

where

$$\phi_{ln}(\theta) = \sum_{-l/2<r<l/2,-n/2<r<n/2} \phi(\theta + 2r\pi/N + 2r'\pi/N)$$ (15)

After tedious calculations [2] we obtain

$$\frac{d\phi_{ln}(\theta)}{d\theta} = -2 \int_{0}^{\infty} d\omega [R_{ln}(\omega) - \delta_{ln}] \cos \omega \theta$$ (16)

where

$$R_{ln}(\omega) = 2 \frac{\sinh \left(\pi \omega \left(1 - \frac{1}{N} \max(l, n)\right)\right) \sinh \left(\pi \omega \frac{1}{N} \min(l, n)\right)}{\sinh \pi \omega}$$ (17)

Taking logarithm of the both sides of the Eqs.(14) we obtain the Bethe-Ansatz equations

$$m_l \sinh \theta^{(l)}_\alpha = \frac{2\pi}{L} J^{(l)}_\alpha + \frac{1}{L} \sum_{n=1}^{N-1} \sum_{\alpha, \beta}^{N_n} \phi_{ln}(\theta^{(l)}_\alpha - \theta^{(n)}_\beta)$$ (18)

where integers $J$ are the quantum numbers of the states. For the generalization of these equations to an arbitrary Bethe states see [2, 4]. The energy of this state is obviously

$$E = \frac{1}{L} \sum_{l=1}^{N-1} m_l \sum_{\alpha=1}^{N_1} \cosh \theta^{(l)}_\alpha$$ (19)
3) Spectral Equations. The next step is to find rapidities to minimize the energy (19) in the thermodynamic limit \( \mathcal{N}/L = n \), while \( L \to \infty \). We assume that the minimum of the energy corresponds to a dense smooth set of \( \theta \)'s, so one can describe them by the distribution function of rapidities of particles \( \rho_l(\theta) \) and the distribution of holes \( \tilde{\rho}_l(\theta) \). They are related by the equation

\[
\frac{1}{2\pi} m_l \cosh \theta = \tilde{\rho}_l(\theta) + \sum_n \int d\theta' R_{ln}(\theta - \theta') \rho_n(\theta')
\]  

where

\[
R_{ln}(\theta) = \frac{1}{\pi} \int_0^\infty d\omega R_{ln}(\omega) \cos \omega \theta
\]

Let us now turn to the energy of the state at the given fields \( h_l \)

\[
\mathcal{E} = \min_{n_l} (E - \sum h_l n_l) = \sum_l \int d\theta [m_l \cosh \theta - h_l \rho_l]
\]

Consider a small variation of \( \rho_l(\theta) \) and \( \tilde{\rho}_l(\theta) \) and introduce the (pseudo)energy \( \varepsilon^+_l(\theta) > 0 \) of a hole and the (pseudo)energy \( \varepsilon^-_l(\theta) < 0 \) of a particle [2], such as

\[
\delta \mathcal{E} = \sum_l \int d\theta [\varepsilon^+_l(\theta) \delta \tilde{\rho}_l(\theta) - \varepsilon^-_l(\theta) \delta \rho_l(\theta)].
\]

According to this definition at the ground state \( \varepsilon^+_l(\theta) = 0 \) if \( \tilde{\rho}_l(\theta) \neq 0 \) and \( \varepsilon^-_l(\theta) = 0 \) if \( \rho_l(\theta) \neq 0 \), i.e. \( \varepsilon^+_l(\varepsilon^-_l) \) and \( \tilde{\rho}_l(\rho_l) \) have nonoverlapping supports. Comparing with (22) and using (20) we find the spectral equations of the model [2]

\[
h_l - m_l \cosh \theta = \varepsilon^-_l(\theta) + \sum_n \int R_{ln}(\theta - \theta') \varepsilon^+_n(\theta'),
\]

\[
\mathcal{E} = -\frac{1}{2\pi} \sum_l \int \varepsilon^+_l(\theta) m_l \cosh \theta,
\]

where \( \varepsilon^+_l(\theta) > 0, \varepsilon^-_l(\theta) < 0, \varepsilon^+_l(\theta) \varepsilon^-_l(\theta) = 0 \).

4) Diagonalization of the Spectral Equations. To prepare the spectral equation (23) for the large \( N \) limit, we diagonalize the kernel matrix \( R_{ln} \). This is easy to do since it reflects the structure of the Dynkin diagram [4]. Indeed, its inverse is

\[
R_{ln}^{-1}(\omega) = \sinh^{-1} \frac{\pi|\omega|}{N} \left( \delta_{ln} \cosh \frac{\pi \omega}{N} - 1/2(\delta_{l,n+1} + \delta_{l+1,n}) \right)
\]

\[7\]
Let us introduce orthonormal eigenfunctions of the \((N-1)x(N-1)\) Cartan matrix \(C_{jk} = 2\delta_{jk} - \delta_{j,k+1} - \delta_{j+1,k}\):

\[
\chi_j^{(p)} = \sqrt{2/N} \sin \frac{\pi pj}{N}, \quad p = 1, 2, ..., N-1
\]

They are also the eigenvectors of \(R_{ij}

\[
R_{jk}(\omega) = \sum_{p=1}^{N-1} \chi_j^{(p)} \chi_k^{(p)} R^{(p)}(\omega)
\]

where

\[
R^{(p)}(\omega) = \frac{\sinh \frac{\pi|\omega|}{N}}{\cosh \frac{\pi|\omega|}{N} - \cos \frac{2\pi p}{N}} = \frac{2N}{\pi} \sum_{r=-\infty}^{\infty} \frac{|\omega|}{\omega^2 + (p + rN)^2}
\]

Then the spectral equations can be diagonalized with respect to the particle indices:

\[
\varepsilon^{(p)}(\theta) + \int R^{(p)}(\theta - \theta') \varepsilon^{(p)}(\theta') d\theta' = h^{(p)} - \delta_{p,1} M \cosh \theta,
\]

where

\[
M = \sum_{k=1}^{N-1} \chi_k^{(1)} m_k = \frac{m\sqrt{N}}{\sqrt{2} \sin(\pi/N)} \rightarrow \frac{mN^{3/2}}{\sqrt{2\pi}}
\]

and

\[
\varepsilon^{(p)}_{\pm} \equiv \sum_{k=1}^{N-1} \chi_k^{(p)} \varepsilon_k^{(\pm)}; \quad h^{(p)} \equiv \sum_{k=1}^{N-1} \chi_k^{(p)} h_k
\]

It is important to note that this transformation is valid and eq.(29) holds, providing that linear combination of particle’s (hole’s) (pseudo)energies remain positive (negative): \(\varepsilon^{(p)}_{(+) > 0} \varepsilon^{(p)}_{(-)} < 0\).
3 Large $N$ solution

One has to be careful using the last definition of the fourier transform of $\varepsilon_{\pm}(\omega) = \int_{-\infty}^{\infty} d\theta \varepsilon_{\pm}(\theta) \cos(\omega \theta)$: different $\varepsilon_{\pm}(\theta)$ entering the sum in (32) have different supports $(-B_k, B_k)$. It has to be taken into account when solving the integral equations.

However, if we take $h_k$ in the form (3), so that

$$h^{(p)} = h \delta_{p,1} \frac{\sqrt{N}}{\sqrt{2 \sin(\pi/N)}} \rightarrow_{N \rightarrow \infty} \frac{N^{3/2}}{\sqrt{2 \pi}} h \delta_{p,1}$$

(33)

it turns out that the ansatz

$$\varepsilon_{\pm}(\omega) = \varepsilon_{\pm}(\omega) \delta_{p,1} \frac{\pi}{\sqrt{8 N \sin(\pi/N)}} \rightarrow_{N \rightarrow \infty} \frac{N^{1/2}}{\sqrt{8}} \varepsilon_{\pm}(\omega) \delta_{p,1}$$

(34)

satisfies all the equations (29) for $p = 2, 3, ..., N-1$ and the equation with $p = 1$ (the Perron-Frobenius mode) gives an integral equation for the definition of $\varepsilon_{\pm}(\omega)$. Since $\varepsilon_{+}(\theta)$ is strictly positive it can be viewed as a newly defined density, with all appropriate analytical properties.

It implies that we look for a solution of (23) with equal supports $B_1 = B_2 = ... = B_{N-1} = B$. In what follows we consider eq.(29) only inside the interval $(-B, B)$ where $\varepsilon_{-}(\theta) = 0$, $\varepsilon_{+}(\theta) \equiv \varepsilon(\theta) > 0$. It is the only function which contributes to the ground state energy (30). It obeys eq.(29) with $p = 1$.

Further simplifications occur in the large $N$ limit. The kernels $R^{(p)}(\omega)$ look as:

$$R^{(p)}(\omega) = \frac{2N}{\pi} \frac{|\omega|}{\omega^2 + p^2}, \quad N \rightarrow \infty$$

(35)

Finally, for the choice (33) of the field we obtain the integral equation:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cos(\theta \omega) \frac{|\omega|}{\omega^2 + 1} \varepsilon(\omega) = h - m \cosh \theta, \quad |\theta| \leq B$$

(36)

where

$$\varepsilon(\omega) = \int_{-B}^{B} d\theta \varepsilon(\theta) \cos(\theta \omega)$$

Note that, in the large $N$ limit $R^{(p)}(\omega)$ vanishes at large $\omega$, whereas at finite $N$ it approaches 1 (see eq.(28)). This implies that $\varepsilon(\theta)$ is no longer differentiable.
at $\theta = \pm B$, but has a cusp. As a result the physics on the threshold $h \sim m$ will be changed drastically.

Equation (36) can be solved exactly. Note that if we act by the operator $(-\frac{\partial^2}{\partial \theta^2} + 1)$ on both sides of it we obtain a simple integral equation with the singular kernel:
\[
\frac{1}{\pi} P \int_{-B}^{B} \frac{d\theta' \varepsilon(\theta')}{(\theta - \theta')^2} = -h
\]
(37)

Its solution is the famous semi-circle law of Dyson:
\[
\varepsilon(\theta) = h \sqrt{B^2 - \theta^2}
\]
(38)

Its Fourier transform can be expressed through the Bessel function
\[
\varepsilon(\omega) = \pi h B J_1(B\omega)/\omega
\]
(39)

Plugging it into (38) and doing the exact integration over $\omega$ (see [13]) we obtain the relation (7) between the $m/h$ and $B$.

Now we can calculate the free energy as a function of $B$ using the relation (30) and our solution (38):
\[
f(h) \equiv \frac{1}{N^2} \left( \mathcal{E}(h) - \mathcal{E}(0) \right) = -\frac{1}{8\pi^2} mh \int_{-B}^{B} d\theta \cosh \theta \sqrt{B^2 - \theta^2}
\]
(40)

The integral (40) together with the relation (7) gives the result (3) for the energy.

Let us note that in the large $N$ limit any virtual and real processes involve all particles, since minimal energies are greater than a minimal separation between masses. A reasonable external field (5) excites all of them on equal footing and leads to the collective effects. That is why, even though the S-matrix of two, say, vector particles tends to unity at $N \to \infty$, the ground state energy is not that of a free field theory.

4 Singular behaviour on threshold

Here we will show that the theory exhibits qualitatively new features on the threshold $h \to m$ which corresponds to $B \to 0$. 
At small $B$ asymptotics of the Bessel functions $I_1(B) \to 1/\pi (B/2 + B^3/16 + ...)$ and $K_1(B) \to 1/B + \frac{B}{2} \ln(B/2) + ...$. Then, from (3) we obtain:

$$B^2 \simeq 4 \frac{\Delta}{|\ln \Delta|}, \quad \Delta \to 0$$

(41)

where we introduced $\Delta = h/m - 1$. This gives a singular behaviour on the threshold:

$$f(h) \simeq -(m/2\pi)^2 \frac{\Delta}{|\ln \Delta|}, \quad \Delta \to 0$$

(42)

It differs drastically from the threshold behaviour for a finite $N$ theory of massive particles, where we would have $3/2$ law (see e.g. [2, 14]):

$$f_N(h) \sim -m^2 (\Delta)^{3/2}$$

(43)

As we already mentioned, the reason for the singular behaviour is the emergency of an extra dimension in the large $N$ limit - the masses of physical particles are distanced by each other by the quantity of the order $1/N$, which is less than any energy scale left in the system. Therefore any external field excites a bundle of particles - a new extended object, which is characterized by an extra "momentum" $q$ in addition to the usual momentum $p$. Let us note that a similar behaviour has been found in the quasiclassical limit of the Sine-Gordon model [15], [16].

It is interesting to compare the results (41) and (42) with the expression for the ground state energy of the $c = 1$ matrix model [8]. The mechanism by which the inverse logarithmic behaviour with respect to the cosmological constant occurs also requires a parametrization through the fermi level of the corresponding fermions (whose coordinates are the eigenvalues of a hermitean matrix field). The Fermi level plays the role of a "hidden" parameter of the problem and the eigenvalues give rise to an extra (Liouville) degree of freedom of the theory [17].

5 Perturbative regime and the ratio $m/\Lambda_{\overline{MS}}$

The field $h$ fixes the scale of energies (with respect to the mass gap $m$). Large $h/m$ regime is the subject of the perturbative theory. From eq. (4) we conclude that it corresponds to $B \to \infty$. It follows from the large $B$
asymptotics of McDonald function \( K_1(B) = \sqrt{\frac{\pi}{2B}} e^{-B} \left( 1 + \frac{3}{8B} + O(1/B^2) \right) \)

that

\[
\frac{h}{m} = \sqrt{\frac{2}{\pi}} \frac{e^B}{\sqrt{B}} \left( 1 - \frac{3}{8B} + O(1/B^2) \right)
\] (44)

Solving for \( B \) we obtain:

\[
B = \ln \frac{h}{m} + \frac{1}{2} \ln \ln \frac{h}{m} + \frac{1}{2} \ln \frac{\pi}{2} + O\left( \frac{1}{\ln \frac{h}{m}} \right)
\] (45)

Using the large \( B \) asymptotics \( I_1(B) = \sqrt{\frac{1}{2\pi B}} e^B \left( 1 - \frac{3}{8B} + O(1/B^2) \right) \) we finally find from (44) and (45):

\[
16\pi f(h) = -h^2 B + O\left( \frac{h^2}{B} \right) = -h^2 \left( \ln \frac{h}{m} + \frac{1}{2} \ln \ln \frac{h}{m} + \frac{1}{2} \ln \frac{\pi}{2} + O\left( \frac{1}{\ln \frac{h}{m}} \right) \right)
\] (46)

This result reproduces correctly one- and two-loop terms of the perturbation theory as well as the universal non-perturbative constant \( \frac{1}{2} \ln \frac{\pi}{2} \) (first calculation of a similar constant was given in [19] for the Kondo problem).

Let us compare it with the perturbative result [20] following directly from (44). In our normalizations we have

\[
16\pi f_{\text{pert}}(h) = -8\pi \frac{h^2}{N\lambda(h)} \sum_{k=1}^{N} q_k^2 \frac{2h^2}{N^2} \sum_{k>j} (q_k - q_j)^2 [\ln |q_k - q_j| - \frac{1}{2}] + O\left( \frac{h^2}{\ln(h/m)} \right)
\] (47)

where

\[
q_k = \frac{h_k - h_{k-1}}{h}, \quad k = 1, 2, \ldots, N; \quad h_0 = h_N = 0
\] (48)

and the renormalized coupling in the minimal subtraction (\( \overline{MS} \)) scheme is:

\[
\overline{\lambda}^{-1}(h) = \frac{1}{4\pi} \left( \ln \frac{h}{2\Lambda_{\overline{MS}}} + \frac{1}{2} \ln \ln \frac{h}{\Lambda_{\overline{MS}}} + O\left( 1/\ln \frac{h}{\Lambda_{\overline{MS}}} \right) \right)
\] (49)

Or,

\[
h \frac{\partial}{\partial h} \overline{\lambda}(h) \equiv \beta(\overline{\lambda}) = -\frac{1}{4\pi} \overline{\lambda}^2 - \frac{1}{32\pi^2} \overline{\lambda}^3 + O(\overline{\lambda}^4)
\] (50)
Using an approximate solution of the BA equations at finite $N$ for a particular choice of the field which excites only the vector particles with the lowest mass $m_1$, the authors of [20] found the following ratio:

$$\frac{m}{\Lambda_{\overline{MS}}} = \sqrt{\frac{8\pi e}{\sin(\pi/N)}} \rightarrow_{N \to \infty} \sqrt{\frac{8\pi e}{N}}$$ \hspace{1cm} (51)

To compare it with our result (46) we have to calculate all the sums in (47) for our choice of fields (5):

$$q_k = \cos \frac{\pi}{N}(k - \frac{1}{2}) \rightarrow_{N \to \infty} \cos x, \hspace{1cm} 0 \leq x \leq \pi$$ \hspace{1cm} (52)

The calculation of corresponding sums in the large $N$ limit gives:

$$\frac{1}{N} \sum_k q_k^2 \rightarrow N \int_0^\pi \frac{dx}{\pi} \cos^2 x = 1/2$$ \hspace{1cm} (53)

$$\frac{1}{N^2} \sum_{k>j} (q_k - q_j)^2 [\ln |q_k - q_j| - 1/2] \rightarrow$$

$$1/2 \int_0^\pi \frac{dx}{\pi} \int_0^\pi \frac{dy}{\pi} \frac{1}{\pi} (\cos x - \cos y)^2 [\ln |\cos x - \cos y| - 1/2] = (1 - 2 \ln 2)/4$$ \hspace{1cm} (54)

Now matching our result (46) with the perturbation theory (47) we obtain the same ratio (51). So, two different choices of the field (that of [20] and of our’s) give the same mass gap in the $\overline{MS}$ scheme.

Let us note in the conclusion to this section that we can continue comparing further terms of the expansion (49) of our exact ground state energy in the inverse logarithms with the planar part of the Feynman graph expansion (47) for the ground state energy using relation (51). They must coincide in all orders.

### 6 Weak coupling expansion to any order and exact beta-function

The $\bar{\lambda}(h)$ defined eq.(49) is equal to $4\pi/B$ in the two-loop approximation (45). Since all next corrections are nonuniversal we may take $4\pi/B$ as the most natural definition of the renormalized coupling:

$$\bar{\lambda}(h) = 4\pi/B$$ \hspace{1cm} (55)
Having in hands the explicit expression (6) for the free energy we can find all orders of the renormalized perturbation theory. Let us notice that the function:

\[ y(B) = BK_1(B)I_1(B) \]

satisfies the 3-d order differential equation:

\[ y''' - \left(\frac{3}{B^2} + 4\right)y' + \frac{3}{B^3} y = 0 \]

If we look for the series expansion of \( y(B) \) we find from (57) the recurrence relations on the coefficients, which can be solved. Finally, by taking (55) into account we obtain:

\[ f(h)/h^2 = -\frac{1}{4\lambda} \left( 1 - \sum_{n=1}^{\infty} C_{2n} \left( \frac{\lambda}{4\pi} \right)^{2n} \right) \]

where

\[ C_{2n} = \frac{2n + 1}{2n - 1} \frac{(2n)!}{n!} \sim n \rightarrow \infty 2/\sqrt{\pi n} \left( \frac{n}{e} \right)^{2n} \]

Note that the expansion goes only in even powers.

As we see, in spite of the fact that every coefficient represents a sum over renormalized planar graphs, it grows factorially with the order. Most probably this happens because of the renormalons (some subsequence of logarithmically divergent graphs) giving the main factorial contribution in each order noticed long time ago by 'tHooft [7]. This means that we have an exponential number of graphs in each order but some of them give \((2n)!\) contribution after the momenta integration. More than that: the series is a non-signchanging one and thus non-Borel summable. Nevertheless, the free energy perfectly exists for any finite \( \lambda \). These phenomena seem to be imminent for any asymptotically free field theory.

One can still give a prescription how to sum up the series (58) and restore the result (6). Notice that:

\[ g(t) = 1 - \sum_{n=1}^{\infty} \frac{C_{2n}}{(2n)!} t^{2n} = F(-1/2, 3/2, 1, t^2/4) \]

where \( F \) is the hypergeometric function. It has the cut starting from \( t^2 = 4 \). We can continue \( g(t) \) analytically to the whole complex plane and make the
inverse Borel transform by integrating from 0 to $\infty$ with the exponential factor. The correct prescription restoring (3) is (see (55)): 

$$16\pi f(h)/h^2 = -B^2Re\int_0^\infty dt e^{-tB} g(t)$$

(61)

As we see from here even the Borel nonsummable series in the asymptotically free theories can be summed up by some special prescription. This prescription which uses a non-trivial procedure of analytical continuation might work well in other asymptotically free theories. Note that the integral in (61) taken along the real axes possesses also an exponentially small (for $B \to \infty$, $\bar{\lambda} \to 0$) imaginary part equal to $-K_1^2(B)B^2/\pi$. The similar properties of Borel integrals in 4d gauge theories were noticed in [18].

With the definition (8) of the running charge one can find from eq. (7) the exact beta-function:

$$\beta(\bar{\lambda}) = h \frac{\partial}{\partial h} \bar{\lambda} = -\frac{4\pi}{B^2} \frac{\partial \ln h/m}{\partial \ln h/m} = -4\pi \frac{K_1(B)}{B^2 K_0(B)}$$

(62)

or

$$\beta(\bar{\lambda}) = -\frac{1}{4\pi} \bar{\lambda}^2 \frac{K_1(\frac{4\pi}{\bar{\lambda}})}{K_0(\frac{4\pi}{\bar{\lambda}})} = -\frac{1}{4\pi} \bar{\lambda}^2 \sum_{n=2}^\infty b_n (\frac{\bar{\lambda}}{32\pi})^n$$

(63)

where

$$b_0 = 1, \ b_1 = 4, \ b_2 = -8, \ b_3 = 64, \ b_4 = -5^2 \cdot 2^5,$$

$$b_5 = 13 \cdot 2^{10}, \ b_6 = -1073 \cdot 2^8, \ b_7 = 103 \cdot 2^{16}, \ ...$$

(64)

$$b_n \sim -(-1)^n \sqrt{8/(\pi n)(4n/e)^n}$$

(65)

7 Discussion

Our results could be interesting from several physical points of view:

1. The model considered here is an example of an exactly solvable matrix model in 1+1 physical dimensions. A temptation would be to interpret it as a new string theory in 1+1 dimensional target space. Recall that a natural interpretation of big planar graphs could be given in terms of a sum of world sheets of a string (see [8] and references therein). An important condition for it is that the typical graphs should be big and dense, to describe a smooth
surface. For this one usually tunes the coupling to its (finite) critical value, corresponding to the "explosion" of the size of graphs. However, in the asymptotically free theory the critical coupling is equal to zero. It is the consequence of the fact that the theory has exponential corrections to the perturbative expansion, leading to the factorial divergency of its coefficients. In planar theories it can happen because of the presence of renormalons: some small portion of all the graphs has a factorially big weight (with respect to the order). These graphs seem to be not very much suitable for the interpretation in terms of random surfaces. Even the expansion in terms of the renormalized coupling contains the same factorial contributions, as we see from our results. The same scenario should be available to the multicolour QCD, invalidating this naive relation between planar graphs and world sheets of a hypothetic QCD string.

2. However, for both PCF and QCD a less trivial string scenario could be possible. It might exist a (nonperturbative) reformulation in terms of a new master field, with a new perturbation theory, already suitable for a string interpretation. The result (12) suspiciously reminds the similar result for the 1D bosonic string (from the matrix quantum mechanics) [8], if we take \((h - m)^2\) instead of the cosmological constant \(\lambda - \lambda_{\text{crit}}\). The similarity is even more striking if one remembers that for both models this asymptotics is related to the behaviour of the fermions on the top of the fermi-sea (1, 2), and the Fermi level is the most natural physical parameter in both cases. The semi-circle law for the distribution of rapidities (37) also suggests that at some value of field \(h\) we deal with an extended object.

3. Another interesting feature of the model (also observed in the \(c = 1\) matrix model) is the emergency of an extra dimension following from the matrix structure of the theories. This dimension is related to the random walk along \(A_{N-1}\) Dynkin diagram. The structure of this extra (third) dimension is well seen from the fact that the kernel of the problem (see eqs.(28) or (28)) looks as a propagator of a periodic motion in the space of rapidities versus Dynkin diagram. It reminds an extra (eigenvalue) dimension in the collective field formulation of the 1D matrix model usually attributed to the Liouville mode [17].

4. One of the immediate problems is to take into account the effects of finite \(N\). The most interesting effects of finite \(N\)'s correspond to the \(e^{-cN}\) corrections. To demonstrate it let us look at the expansion of the kernel (28) in the pole terms (see (28)). The main order for the large \(N\) corresponds to
the neglection of all poles but the first (on the finite distance from the origin on the imaginary axes of $\omega$). Next terms will give the exponentially small contributions (as well as the $1/N$ corrections) if we treat them as singularities in the corresponding integral equation.

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