ON THE LOGICAL AND COMPUTATIONAL PROPERTIES OF THE VITALI COVERING THEOREM

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Abstract. We study a version of the Vitali covering theorem, which we call WHBU and which is a direct weakening of the Heine-Borel theorem for uncountable coverings, called HBU. We show that WHBU is central to measure theory by deriving it from various central approximation results related to Littlewood’s three principles. A natural question is then how hard it is to prove WHBU (in the sense of Kohlenbach’s higher-order Reverse Mathematics), and how hard it is to compute the objects claimed to exist by WHBU (in the sense of Kleene’s schemes S1-S9). The answer to both questions is ‘extremely hard’, as follows: on one hand, in terms of the usual scale of (conventional) comprehension axioms, WHBU is only provable using Kleene’s $\exists^3$, which implies full second-order arithmetic. On the other hand, realisers (aka witnessing functionals) for WHBU, so-called $A$-functionals, are computable from Kleene’s $\exists^3$, but not from weaker comprehension functionals. Despite this hardness, we show that WHBU, and certain $A$-functionals, behave much better than HBU and the associated class of realisers, called $\Theta$-functionals. In particular, we identify a specific $A$-functional called $A_\S$ which adds no computational power to the Suslin functional, in contrast to $\Theta$-functionals. Finally, we introduce a hierarchy involving $\Theta$-functionals and HBU.

1. Introduction

The most apt counterpart in mathematical logic of the commonplace one cannot fit a square peg into a round hole is perhaps the following: a Turing machine cannot directly access third-order objects, like e.g. measurable functions. Thus, the development of measure theory in any framework based on Turing computability must proceed via second-order stand-ins for higher-order objects. In particular, the following frameworks, (somehow) based on Turing computability, proceed by studying the computational properties of certain countable representations of measurable objects: Reverse Mathematics ([76 X.1]), constructive analysis ([6 I.13] for an overview), predicative analysis ([19]), and computable analysis ([87]).

The existence of the aforementioned countable representations is guaranteed by various well-known approximation results. Perhaps the most basic and best-known among these results go by the name of Littlewood’s three principles. The latter are found throughout the literature, including Tao’s introduction to measure theory (see [29,60,78,81,89]), and were originally formulated by Littlewood as:

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1Note that Bishop’s constructive analysis is not based on Turing computability directly, but one of its ‘intended models’ is however (constructive) recursive mathematics (see [29]). One aim of Feferman’s predicative analysis is to capture constructive reasoning in the sense of Bishop.
There are three principles, roughly expressible in the following terms: Every (measurable) set is nearly a finite sum of intervals; every function (of class $L^p$) is nearly continuous; every convergent sequence of functions is nearly uniformly convergent. (37 p. 26)

The second and third principle are heuristic descriptions of the Lusin and Egorov theorems. In light of their fundamental role for measure theory, it is then a natural question how hard it is to prove these theorems, in the sense of Kohlenbach’s higher-order Reverse Mathematics (RM hereafter; see Section 2.1), and how hard it is to compute the countable approximations therein, in the sense of Kleene’s schemes S1–S9 (see Section 2.2). The aim of this paper is to answer these connected questions. As it turns out, the answer to both questions is ‘extremely hard’, as follows.

In Section 3 we show that the aforementioned approximation theorems (and related results) imply WHBU as in Principle 3.2, which is a version of the Vitali covering theorem that is a direct weakening of HBU; the latter is the Heine-Borel theorem for uncountable coverings as in Principle 2.9. In terms of standard comprehension axioms, WHBU is only provable using Kleene’s $\exists^3$, which implies full second-order arithmetic (see Section 2.1.4). Our approach to measure theory is akin to that of second-order RM (see Remark 3.1), but we shall also study the framework from [32], namely in Section 3.4. We show in Section 3.5 that our results pertaining to WHBU are robust, in that they do not depend on the framework at hand.

In Section 4 we will study the computational properties of realisers of WHBU, called weak fan functionals or $\Lambda$-functionals (see [46, 47]). Any $\Lambda$-functional is computable from $\exists^3$, but not from weaker comprehension functionals like $S^2_k$ that decide $\Pi^1_k$-formulas (see Section 2.1.4). Despite this observed hardness, we show that WHBU and $\Lambda$-functionals behave much better than (Heine-Borel) compactness and the associated class of realisers, called special fan functionals or $\Theta$-functionals. In particular, we identify a specific $\Lambda$-functional, called $\Lambda_S$, which adds no computational power to the Suslin functional, in contrast to $\Theta$-functionals. As an application, we show that higher-order $\Pi^1_k$-CA$_0$ plus WHBU cannot prove HBU. We also show that $\Theta$-functionals and (Heine-Borel) compactness yield new hierarchies akin to second-order arithmetic in Section 4.4.

In Section 5 we formulate the conclusion to this paper as follows: we discuss a conjecture and a template related to our results in Section 5.1.1 while an interesting ‘dichotomy’ phenomenon is observed in Section 5.1.2. In Section 5.2 we discuss some foundational musings related to the coding practise of Reverse Mathematics. Finally, some new insights regarding ‘normal’ and ‘non-normal’ mathematics have recently come to the fore in e.g. [52, 68], providing a more ‘grand scheme of things’ view of the results in this paper, as discussed in Remark 2.12. In a nutshell, the above results should be viewed as motivation for the development and study of a new scale not based on comprehension or discontinuous functionals.

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2Like for Heine-Borel compactness, there is no unique realiser for WHBU as in Principle 3.2 as we can always add dummy elements to the sub-cover at hand.

3It is shown in [44, 48] that $\Theta$-functionals yields realisers for ATR$_0$ when combined with the Turing jump functional $\exists^3$ from Section 2.2. $\Theta$-functionals also yield Gandy’s Superjump $\exists^4$, and even fixed points of non-monotone inductive definitions, when combined with the Suslin functional.
2. Preliminaries

We introduce Reverse Mathematics in Section 2.1 as well as its generalisation to higher-order arithmetic. In particular, since we shall study measure theory, we discuss the representation of sets in Section 2.1. As our main results are proved using techniques from computability theory, we discuss the latter in Section 2.2.

2.1. Reverse Mathematics.

2.1.1. Introduction. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([21,22]) and developed extensively by Simpson ([76]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics. We refer to [79] for a basic introduction to RM and to [75, 76] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach’s higher-order RM ([31]) essential to this paper, including the ‘base theory’ $\text{RCA}_0^\omega$ in Section 2.1.2. Since we shall study measure theory, we need to represent sets in $\text{RCA}_0^\omega$, as discussed in Definition 2.4.(vi) and (in more detail) Section 3.4.2.

Now, ‘classical’ RM is based on $L_2$, the language of second-order arithmetic $Z_2$. By contrast, higher-order RM makes use of the richer language of higher-order arithmetic. Indeed, while $L_2$ is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of all finite types $T$, defined by the two clauses:

(i) $0 \in T$ and (ii) If $\sigma, \tau \in T$ then $(\sigma \to \tau) \in T$,

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and where $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that $Z_2$ only includes objects of type 0 and 1.

The language $L_\omega$ includes variables $x^\rho, y^\rho, z^\rho, \ldots$ of any finite type $\rho \in T$. Types may be omitted when they can be inferred from context. The constants of $L_\omega$ includes the type 0 objects 0, 1 and $<_0, +_0, \times_0, =_0$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of ‘$=_0$’ as follows: for any objects $x^\tau, y^\tau$, we have

$$[x =_\tau y] \equiv (\forall z_1^\tau \ldots z_k^\tau)(xz_1 \ldots z_k =_0 yz_1 \ldots z_k),$$

(2.1)

if the type $\tau$ is composed as $\tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0)$. Furthermore, $L_\omega$ also includes the recursor constant $R_\sigma$ for any $\sigma \in T$, which allows for iteration on type $\sigma$-objects as in the special case (2.2). Formulas and terms are defined as usual.

2.1.2. The base theory of higher-order Reverse Mathematics. We introduce the base theory $\text{RCA}_0^\omega$ of higher-order RM and discuss its connection to $\text{RCA}_0$, the base theory of second-order RM.

**Definition 2.1.** The base theory $\text{RCA}_0^\omega$ consists of the following axioms.

1. Basic axioms expressing that 0, 1, $<_0, +_0, \times_0, =_0$ form an ordered semi-ring with equality $=_0$.

2. Basic axioms defining the well-known $\Pi$ and $\Sigma$ combinators (aka $K$ and $S$ in [3]), which allow for the definition of $\lambda$-abstraction.
The defining axiom of the recursor constant $R_0$: For $m^0$ and $f^1$:

$$R_0(f, m, 0) := m \quad \text{and} \quad R_0(f, m, n + 1) := f(n, R_0(f, m, n)). \quad (2.2)$$

(4) The axiom of extensionality: for all $\rho, \tau \in T$, we have:

$$(\forall x^\rho, y^\rho, \varphi^\rho)\left[ x =_\rho y \Rightarrow \varphi(x) =_\tau \varphi(y) \right]. \quad (E_{\rho, \tau})$$

(5) The induction axiom for quantifier-free\(^4\) formulas of $L_\omega$.

(6) QF-AC:\(^1,\omega\): The quantifier-free Axiom of Choice as in Definition 2.2.

**Definition 2.2.** The axiom QF-AC consists of the following for all $\sigma, \tau \in T$:

$$(\forall x^\sigma)(\exists y^\tau)A(x, y) \rightarrow (\exists Y^{\sigma \rightarrow \tau})(\forall x^\sigma)A(x, Y(x)), \quad \text{(QF-AC}^{\sigma, \tau})$$

for any quantifier-free formula $A$ in the language of $L_\omega$.

Recursion as in (2.2) is called primitive recursion; the class of functionals obtained from $R_\rho$ for all $\rho \in T$ is called Gödel’s system $T$ of all (higher-order) primitive recursive functionals.

Finally, as discussed in [31 §2], RCA$_0$ and RCA$_0^\omega$ prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. This is proved via the highly useful ECF-interpretation, discussed next.

**Remark 2.3** (The ECF-interpretation). The technical definition of ECF may be found in [31 p. 138, §2.6]. Intuitively speaking, the ECF-interpretation $[A]_{ECF}$ of a formula $A \in L_\omega$ is just $A$ with all variables of type two and higher replaced by countable representations of continuous functionals. Such representations are also (equivalently) called ‘associates’ or ‘codes’ (see [30 §4]). The ECF-interpretation connects RCA$_0^\omega$ and RCA$_0$ (see [31 Prop. 3.1]) in that if RCA$_0^\omega$ proves $A$, then RCA$_0$ proves $[A]_{ECF}$, again ‘up to language’, as RCA$_0$ is formulated using sets, and $[A]_{ECF}$ is formulated using types, namely only using type zero and one objects.

2.1.3. Basic definitions. We list some basic definitions and notations needed below.

Firstly, we use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in [31 p. 288-289].

**Definition 2.4** (Real numbers and related notions in RCA$_0^\omega$).

(i) Natural numbers correspond to type zero objects, and we use ‘$n^0$’ and ‘$n \in \mathbb{N}$’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘$q \in \mathbb{Q}$’ and ‘$<_{\mathbb{Q}}$’ have their usual meaning.

(ii) Real numbers are represented by fast-converging Cauchy sequences $q_{(\cdot)} : \mathbb{N} \rightarrow \mathbb{Q}$, i.e. such that $(\forall n^0, i^0)((|q_n - q_{n+i}|) <_{\mathbb{Q}} \frac{1}{2^n})$. We use the ‘hat function’ from [31 p. 289] to guarantee that every $f^1$ defines a real number.

(iii) We write ‘$x \in \mathbb{R}$’ to express that $x^1 := (q_{(\cdot)}^1)$ represents a real as in the previous item and write $[x](k) := q_k$ for the $k$-th approximation of $x$.

(iv) Two reals $x, y$ represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are equal, denoted $x =_R y$, if $(\forall n^0)((|q_n - r_n|) \leq 2^{-n+1})$. Inequality ‘$<_R$’ is defined similarly. We sometimes omit the subscript ‘$R$’ if it is clear from context.

(v) Functions $F : \mathbb{R} \rightarrow \mathbb{R}$ are represented by $\Phi^{1+1}$ mapping equal reals to equal reals, i.e. $(\forall x, y \in \mathbb{R})(x =_R y \Rightarrow \Phi(x) =_R \Phi(y))$.  

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\(^{4}\)To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language $L_\omega$; only quantifiers are banned.
(vi) The relation \( x \leq_\tau y \) is defined as in (2.1) but with \( \leq_0 \) instead of \( =_0 \).

Binary sequences are denoted \( f^1, g^1 \leq_1 1 \), but also \( f, g \in C \) or \( f, g \in 2^{\mathbb{N}} \).

We now discuss the issue of representations of real numbers.

**Remark 2.5.** First of all, introductory analysis courses often provide an explicit construction of \( \mathbb{R} \) (perhaps in an appendix), while in practise one generally makes use of the axiomatic properties of \( \mathbb{R} \), and not the explicit construction. Now, there are a number of different\(^5\) such constructions: Tao uses Cauchy sequences in his text [82] and discusses decimal expansions in the Appendix [82 §B]. Hewitt-Stromberg also use Cauchy sequences in [25 §5] and discuss Dedekind cuts in the exercises ([28 p. 46]). Rudin uses Dedekind cuts in [57] and mentions that Cauchy sequences yield the same result.

Secondly, Definition 2.4 is based on (fast-converging) Cauchy sequences, but Hirst has shown that over \( \text{RCA}_0 \), individual real numbers can be converted between various representations ([27]). Thus, the choice of representation in Definition 2.4 does not really matter, even over \( \text{RCA}_0 \). Moreover, the latter proves ([76 II.4.5]) that the real number system satisfies all the axioms of an Archimedean ordered field, i.e. we generally work with the latter axiomatic properties in \( \text{RM} \), rather than with the representations (whatever they are).

Thirdly, converting sequences of real numbers between representations cannot always be done over \( \text{RCA}_0 \), and \( \text{WKL}_0 \) or \( \text{ACA}_0 \) are sometimes needed, as also studied in [27]. By the results in the latter, (fast-converging) Cauchy sequences are the ‘best’ representation for the development of \( \text{RM} \).

The previous remark deals with weak systems: \( \exists^2 \) from Section 2.1.4 provides a uniform conversion facility between the various representations studied in [27].

Secondly, sets are represented by characteristic functions in Definition 2.6. Given \( (\exists^2) \) from Section 2.1.4, sets as in Definition 2.6 become ‘proper’ characteristic function, only taking values ‘0’ and ‘1’. For this and other reasons, we often assume the former axiom when dealing with sets.

**Definition 2.6.** [Sets in \( \text{RCA}_0^\omega \)] We let \( Y : \mathbb{R} \to \mathbb{R} \) represent subsets of \( \mathbb{R} \) as follows:

we write \( x \in Y \) for \( Y(x) >_R 0 \). A set \( Y \subseteq \mathbb{R} \) is called ‘open’ if for every \( x \in Y \), there is an open ball \( B(x, r) \subset Y \) with \( r \geq 0 \). A set \( Y \) is called ‘closed’ if the complement, denoted \( Y^c = \{ x \in \mathbb{R} : x \notin Y \} \), is open.

Hereafter, an ‘open set’ refers to Definition 2.6 while ‘RM-open set’ refers to the RM-definition of open set as in [76 II.5.6]. Now, one can effectively convert between RM-open sets and (RM-codes for) continuous characteristic functions (see [76 II.7.1]), i.e. our definition of (open) set is a generalisation of the RM-concept. We define countable sets as follows (see e.g. [33]).

**Definition 2.7.** A set \( A \subset \mathbb{R} \) is countable if there exists \( Y : \mathbb{R} \to \mathbb{N} \) such that

\[
(\forall x, y \in A)(Y(x) =_0 Y(y) \rightarrow x =_\mathbb{R} y). \tag{2.3}
\]

We say that \( Y \) as in (2.3) is injective on \( A \) or an injection from \( A \) to \( \mathbb{N} \).

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\(^5\)The ‘early’ constructions due to Dedekind (see e.g. [17]; using cuts) and Cantor (see e.g. [15]; using Cauchy sequences) were both originally published in 1872.
In case $Y$ as in (2.3) is also surjective, we say that $A$ is strongly countable, although we will not study this concept in this paper.

Finally, for completeness, we list our notational conventions on finite sequences.

**Notation 2.8** (Finite sequences). For $\rho = 0, 1$, we assume a dedicated type for 'finite sequences of objects of type $\rho$', namely $\rho^*$. Since the usual coding of pairs of numbers goes through in $\text{RCA}_0^{\omega}$, we shall not always distinguish between 0 and 0$^*$. Similarly, we do not always distinguish between 's$^\rho$' and 't$^\rho$' where the former is 'the object $s$ of type $\rho$', and the latter is 'the sequence of type $\rho^*$ with only element $s^\rho$'. The empty sequence for the type $\rho^*$ is '⟨⟩', usually with the typing omitted.

Furthermore, we denote by '|$s| = n'$ the length of the finite sequence $s^\rho = \langle s_0, s_1, \ldots, s_{n-1} \rangle$, where $|\langle \rangle| = 0$, i.e. the empty sequence has length zero. For sequences $s^\rho, t^\rho$, we denote by 's∗t' the concatenation of $s$ and $t$, i.e. $(s∗t)(i) = s(i)$ for $i < |s|$ and $(s∗t)(j) = t(|s|−j)$ for $|s| ≤ j < |s|+|t|$. For a sequence $s^\rho^*$, we define $\mathfrak{N}_N := \langle s(0), s(1), \ldots, s(N − 1) \rangle$ for $N^0 < |s|$. For a sequence $\alpha^0|^\rho$, we also write $\mathfrak{N}_N = \langle \alpha(0), \alpha(1), \ldots, \alpha(N − 1) \rangle$ for any $N^0$. By way of shorthand, $(\forall \varphi^\rho \in Q^\rho^*).A(q)$ abbreviates $(\forall \varphi^0 < |Q|).A(Q(i))$, which is (equivalent to) quantifier-free if $A$ is.

2.1.4. Some higher-order systems and functionals. We introduce some functionals and axioms which constitute the counterparts of second-order arithmetic

First of all, $\text{ACA}_0$ is readily derived from:

$$(\exists \mu^2)(\forall f^1).[(\exists n)(f(n) = 0) \rightarrow [f(\mu(f)) = 0 \land (\forall i < \mu(f))(f(i) \neq 0) \land ([\forall n](f(n) \neq 0) \rightarrow \mu(f) = 0)],$$

and $\text{ACA}_0^\omega \equiv \text{RCA}_0^{\omega} + (\mu^2)$ proves the same sentences as $\text{ACA}_0$ by [31, Theorem 2.5].

The (unique) functional $\mu^2$ in $\mu^1$ is also called Feferman's $\mu$ ([3]), and is clearly discontinuous at $f = 11\ldots$; in fact, $(\mu^2)$ is equivalent to the existence of $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $F(x) = 1$ if $x \geq 0$, and 0 otherwise ([31, §3]), and to

$$(\exists \varphi^2 \leq_2 1)(\forall f^1).[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0].$$

$$(\exists S^2 \leq_2 1)(\forall f^1).[(\exists g^1)(\forall n^0)(f(\mathfrak{N}_0) = 0) \leftrightarrow S(f) = 0],$$

and $\Pi_1^1-\text{CA}_0$ is readily derived from the following sentence:

$$(\exists S^2 \leq_2 1)(\forall f^1).[(\exists g^1)(\forall n^0)(f(\mathfrak{N}_0) = 0) \leftrightarrow S(f) = 0],$$

and $\Pi_1^1-\text{CA}_0^\omega \equiv \text{RCA}_0^{\omega} + (S^2)$ proves the same $\Pi_1^1$-sentences as $\Pi_1^1-\text{CA}_0$ by [62, Theorem 2.2]. The (unique) functional $S^2$ in $S^2$ is also called the Suslin functional ([31]). By definition, the Suslin functional $S^2$ can decide whether a $\Sigma_1^1$-formula in normal form, i.e. as in the left-hand side of $S^2$, is true or false. We similarly define the functional $S_k^2$ which decides the truth or falsity of $\Sigma_k^1$-formulas in normal form; we also define the system $\Pi_1^1-\text{CA}_0^\omega$ as $\text{RCA}_0^{\omega} + (S^2_k)$, where $(S^2_k)$ expresses that $S^2_k$ exists. Note that we allow formulas with function parameters, but not functionals here. In fact, Gandy's Superjump ([23]) constitutes a way of extending $\Pi_1^1-\text{CA}_0^\omega$ to parameters of type 2; see the discussion in [50, §2.3].

Thirdly, full second-order arithmetic $\text{Z}_2$ is readily derived from $\bigcup_k \Pi_1^k-\text{CA}_0^\omega$, or from:

$$(\exists E^3 \leq_3 1)(\forall Y^2).[(\exists f^1)Y(f) = 0 \leftrightarrow E(Y) = 0],$$

$$(\exists E^3 \leq_3 1)(\forall Y^2).[(\exists f^1)Y(f) = 0 \leftrightarrow E(Y) = 0]$$
and we therefore define \( Z^2_0 \equiv \text{RCA}_0^d \equiv \exists^3 \) and \( Z^2_2 \equiv \exists_k \Pi^1_k \text{-CA}_0^d \), which are conservative over \( Z_2 \) by [28] Cor. 2.6]. Despite this close connection, \( Z^2_2 \) and \( Z^2_0 \) can behave quite differently, as discussed in e.g. [48] §2.2. The functional from \((3^3)\) is also called \( \exists^3 \), and we use the same convention for other functionals.

Fourth, recall that the Heine-Borel theorem (aka Cousin’s lemma [16] p. 22]) states the existence of a finite sub-cover for an open cover of certain spaces. Now, a functional \( \Psi : \mathbb{R} \to \mathbb{R}^+ \) gives rise to the canonical cover \( \cup_{x \in I} I_x^\Psi \) for \( I \equiv [0, 1] \), where \( I_x^\Psi \) is the open interval \( x - \Psi(x), x + \Psi(x) \). Hence, the uncountable cover \( \cup_{x \in I} I_x^\Psi \) has a finite sub-cover by the Heine-Borel theorem; in symbols:

**Principle 2.9 (HBU).** \( (\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists y_1, \ldots, y_k \in I)(\forall x \in I)(\exists i \leq k)(x \in I_{y_i}^\Psi). \)

By the results in [18, 50], \( Z^2_2 \) proves HBU but \( Z^2_2 + \text{QF-AC}^{0,1} \) cannot, and many basic properties of the gauge integral (\( [18, 80] \)) are equivalent to HBU. We have also studied the Lindelöf lemma for \( \mathbb{R} \) in [18].

**Principle 2.10 (LIN).** \( (\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists (y_n)_{n \in \mathbb{N}})(\forall x \in \mathbb{R})(\exists k \in \mathbb{N})(x \in I_{y_k}^\Psi). \)

Furthermore, since Cantor space (denoted \( C \) or \( 2^\mathbb{N} \)) is homeomorphic to a closed subset of \([0, 1]\), the former inherits the same property. In particular, for any \( G^2 \), the corresponding ‘canonical cover’ of \( 2^\mathbb{N} \) is \( \cup_{f \in 2^\mathbb{N}} \overline{\mathcal{L}(f)} \) where \([\sigma^0] \) is the set of all binary extensions of \( \sigma \). By compactness, there are \( f_0, \ldots, f_n \in 2^\mathbb{N} \) such that \( \cup_{i \leq n}[\overline{f_i \mathcal{L}(f_i)}] \) still covers \( 2^\mathbb{N} \). By [18, Theorem 3.3], HBU is equivalent to the same compactness property for \( C \), as follows:

\[
(\forall G^2)(\exists f_1, \ldots, f_k \in 2^\mathbb{N})(\forall f \in 2^\mathbb{N})(\exists i \leq k)(f \in [\overline{f_i \mathcal{L}(f_i)})].
\]

On a technical note, when we say ‘finite sub-cover’, we mean the set of the associated neighbourhoods, not ‘just’ their union. We now introduce the specification \( \text{SFF}(\Theta) \) for a functional \( \Theta^2 \) which computes a finite sequence as in \( \text{HBU} \). We refer to such a functional \( \Theta \) as a realiser for the compactness of Cantor space, and simplify its type to ‘3’.

\[
(\forall G^2)(\forall f^1 \leq 1)(\exists g \in \Theta(G))(f \in [\mathcal{L}(g)]).
\]

Any functional \( \Theta \) satisfying \( \text{SFF}(\Theta) \) is called a special fan functional or simply a \( \Theta \)-functional. As to its provenance, \( \Theta \)-functionals were introduced as part of the study of the Gandy-Hyland functional in [44, §2] via a different definition. These are identical up to a term of Gödel’s \( T \) of low complexity by [17, Theorem 2.6].

Finally, we have studied countable sets in [52, 54], in the guise of the following.

**Principle 2.11 (cocode\(_0\)).** For any countable \( A \subset [0, 1] \), there is a sequence \((x_n)_{n \in \mathbb{N}} \) that contains all elements of \( A \).

This principle is ‘explosive’ in the sense that \( \Pi^1_1 \text{-CA}_0^d + \text{cocode}_0 \) proves \( \Pi^1_2 \text{-CA}_0 \), while \( \Pi^1_1 \text{-CA}_0^d \) is \( \Pi^1_3 \)-conservative over \( \Pi^1_1 \text{-CA}_0 \) (see [52, 54]).

2.2. **Higher-order computability.** As some of our main results are part of computability theory, we make our notion of ‘computability’ precise as follows.

(I) We adopt ZFC, i.e. Zermelo-Fraenkel set theory with the Axiom of Choice, as the official metatheory for all results, unless explicitly stated otherwise.

(II) We adopt Kleene’s notion of higher-order computation as given by his nine clauses S1-S9 (see [35], [61]) as our official notion of ‘computable’.
A thorough introduction to Kleene computability theory may be found in \cite{38}. We do recall an important notion from the latter.

**Remark 2.12** (Normal and non-normal mathematics). The distinction between ‘normal’ and ‘non-normal’ mathematics is based on the following definition.

For $n \geq 2$, a functional of type $n$ is called *normal* if it computes Kleene’s $\exists^n$ following S1-S9, and *non-normal* otherwise. (\cite{35 §5.4})

Similarly, we call a statement about type $n$ objects ($n \geq 2$) *normal* if it implies the existence of $\exists^n$ over Kohlenbach’s base theory from Section 2.1, and *non-normal* otherwise. We also use ‘*strongly* non-normal’ for type 3 functionals that do not compute $\exists^3$ relative to $\exists^2$. Note that by \cite{31 §3}, (($\exists^2$)) is equivalent to the existence of a discontinuous function on $\mathbb{R}$.

Historically, higher-order computability theory and higher-order RM have mostly been focused on the normal world. Recently, the authors have identified HBU and $\Theta$-functionals as interesting parts of the *non-normal* world (\cite{46–48}). Since HBU can be formulated in third-order arithmetic, ‘HBU is non-normal’ means that HBU does not prove ($\exists^2$) in this case. The associated $\Theta$-functionals are fourth-order and ‘a given $\Theta$-functional is non-normal’ thus means that it does not compute $\exists^3$. The same holds for WHBU (see Definition 3.2) and the associated $\Lambda$-functionals (see Definition 4.1). The uncountability of $\mathbb{R}$, when formulated using injections or bijections to $\mathbb{N}$, is similarly non-normal (see \cite{52–54}).

However, it is an empirical observation that the above non-normal theorems and functionals, which are intuitively ‘weak’, are classified as ‘hard to prove’ and the associated functionals as ‘hard to compute’ *relative to the normal scale based on comprehension and discontinuous functionals*: in each case $Z^2_2$ cannot prove the theorem and no $S^k_2$ can compute a realiser, while $Z^3_2$ and $\exists^3$ suffice. In this way, the normal scale gives intuitively *weak* non-normal theorems and functionals *the same* classification, namely *rather strong*. In this light, the normal scale seems unsuitable for analysing non-normal theorems and functionals. Thus, the need for the development of the non-normal scale arises, which is the topic of this paper, and also of \cite{16–54}. Here, Theorem 4.17 is a ‘milestone’ result from computability theory while Theorem 4.21 is a milestone in RM, where the non-normal nature of WHBU follows from that of HBU.

Finally, the importance of the of ‘normal versus non-normal’ distinction was only really understood by the authors after the completion of \cite{52–68}.

### 3. Reverse Mathematics and WHBU

#### 3.1. Introduction.** In this section, we study the RM of measure theory, WHBU in particular, as summarised by the following list.

- In Section 3.2 we introduce WHBU, a version of the Vitali covering theorem that is a direct weakening of HBU from Section 2.1.4. We investigate generalisations of WHBU akin to those studied in second-order RM.
- We show that various instances of Littlewood’s three principles (including Lusin’s and Egorov’s theorems) imply WHBU (Section 3.3).
- We study Kreuzer’s measure theory \cite{32} (Section 3.4). We derive Egorov’s theorem but show that the Heine-Borel theorem cannot be proved.
We show that WHBU also occurs in an alternative (very different) approach to the Lebesgue integral (Section 3.5), namely the gauge integral. A similar result for the Riemann integral is obtained.

Thus, WHBU is shown to arise naturally in different approaches to measure theory, i.e. our results can be said to be independent of the particular framework. Regarding the third item, we could obtain equivalences involving WHBU, but this would require a base theory beyond the scope of this paper.

Finally, we discuss an important convention as to the meaning of the Lebesgue measure. In a nutshell, except in Section 3.4, we interpret the Lebesgue measure in a ‘virtual’ or ‘comparative’ sense similar to the approach in second-order RM.

Remark 3.1 (A measure by any other name). First of all, Lebesgue measure theory can be developed in second-order RM (see e.g. [76, X.1]). However, the Lebesgue measure is defined via a supremum (see Definition [76, X.1.2]) that need not always exist in weak systems like RCA_0. Nonetheless, L_2-formulas like e.g.

\[
\lambda(E) = \inf \{ \sum_{k} |I_k| : (I_k)_{k \in \mathbb{N}} \text{ is a sequence of intervals and } E \subset \bigcup_{k} I_k \},
\]

(3.2)
in case this infimum exists. Due to the quantification over sequences, we observe that \( \mathbb{Z}_+^\mathbb{N} \) can always define \( \lambda(E) \), assuming this infimum exists. However, statements like ‘\( \lambda(E) > 1 \)’ can be interpreted in the aforementioned comparative sense, for which the associated infimum need not exist. To be absolutely clear, the formula ‘\( \lambda(E) \geq \frac{1}{2} \)’ is purely symbolic and short for

\[
\lambda(E) = \inf \{ \sum_{k} |I_k| : (I_k)_{k \in \mathbb{N}} \text{ is a sequence of intervals and } E \subset \bigcup_{k} I_k \},
\]

(3.2)
in this way, comparative statements about the Lebesgue measure of arbitrary sets make sense in RCA_0, even if the infimum in (3.2) does not always exist. It is important to note that this convention essentially ‘hard-wires’ the first Littlewood principle into the definition of the Lebesgue measure. Based on the above convention, we say that a set \( E \subseteq [0, 1] \) is measurable if

\[
\lambda(E) + \lambda([0, 1] \setminus E) \leq 1.
\]

Similarly, a measurable function \( f : \mathbb{R} \to \mathbb{R} \) is defined as usual, namely as saying that for all \( t \in \mathbb{R} \), the set \( \{ x \in \mathbb{R} : f(x) >_R t \} \) is measurable. Since the latter set or a union \( \bigcup_{n \in \mathbb{N}} A_n \) only exists as a set given \( \exists^2 \), we will often work over ACA_0^\omega.

Finally, the Lebesgue measure for open sets (in the sense of second-order RM) exists in ACA_0, while the latter system suffices for a general treatment of the subject (see [76, X.1]). We let \( \lambda_{\text{open}} \) be the non-normal statement that there exists \( \lambda : (\mathbb{R} \to [0, 1]) \to \mathbb{R} \) such that for open \( E \subset [0, 1], \lambda(E) \) equals the infimum as in (3.2).

Below, we show that the fragment \( \lambda_{\text{open}} \) of the Lebesgue integral allows for a generalisation of WHBU to general coverings of arbitrary closed sets.
3.2. The Vitali covering theorem and WHBU.

3.2.1. Introduction. In this section, we introduce WHBU, a version of the Vitali covering theorem that is a direct weakening of HBU from Section 2.1.4. We also establish some basic properties of WHBU in Section 3.2.2.

As to notation, recall that $(x - \Psi(x), x + \Psi(x))$ is denoted as $I^\Psi_x$ or $B(x, \Psi(x))$ for $\Psi : [0, 1] \to \mathbb{R}^+$, while $\cup_{x \in [0, 1]} B(x, \Psi(x))$ is called the canonical cover of the unit interval generated by $\Psi$. Also recall the convention concerning the Lebesgue measure from Remark 3.1. Now consider the following:

**Principle 3.2 (WHBU).** For $\Psi : [0, 1] \to \mathbb{R}^+$ and $\varepsilon > 0$, there are $y_0, \ldots, y_n \in [0, 1]$ such that the measure of $\cup_{i \leq n} I^\Psi_{y_i}$ is at least $1 - \varepsilon$.

As suggested by its name, WHBU is a weakening of HBU. In fact, HBU is to WHBU what WKŁ is to WWKL. The latter is weak weak König’s lemma and may be found in [76, X.1], while the ECF-translation converts the former two into the latter two. Now, WHBU constitutes the essence of Vitali’s covering theorem as follows, a version of which was introduced in 1907 ([86]).

If $I$ is a Vitali cover of $E \subseteq I$, then there is a sequence of disjoint intervals $I_n$ in $I$ such that $E \setminus \cup_{n \in \mathbb{N}} I_n$ has measure zero.

Indeed, a Vitali cover of a set is an open cover in which every element of the set can be covered by an open set of arbitrary small size ([50, Ch. 5.1]). Vitali’s covering theorem for countable coverings is equivalent, over RCA$_0$, to WWKL by [76, X.1.13]. Working in ACA$_0^+$ + QF-AC$^{0,1}$, we may apply the latter to WHBU to obtain a countable Vitali sub-cover of a set of measure 1 of a given Vitali cover. In this way Vitali’s covering theorem is provable from WHBU, while proving the latter from the former is an easy exercise. As discussed in [10, Note, p. 50-51], Borel actually proves the (countable) Heine-Borel theorem to justify his use of the following lemma:

If $1 > \varepsilon \sum_{n=0}^{+\infty} |a_n - b_n|$, then $\cup_{n \in \mathbb{N}} (a_n, b_n)$ does not cover $[0, 1]$.

However, the latter is equivalent to WWKL by [76, X.1.9], which provides some historical motivation and context.

Like for HBU, $\mathbb{Z}^\Omega_2$ proves WHBU, but $\mathbb{Z}^\Omega_2$ cannot. Hence, WHBU is quite hard to prove (in terms of conventional comprehension), and the finite sequence of reals in WHBU is similarly hard to compute: a $\Lambda$-functional is (equivalently) defined as (equivalently) defined in Section 3 by saying that $\Lambda(\Psi, \varepsilon)$ computes $\langle y_1, \ldots, y_n \rangle$ as in WHBU. Like for $\Theta$-functionals, there is no unique such $\Lambda$-functional. Moreover, no type two functional can compute a $\Lambda$-functional (see [46, 37]), which includes the comprehension functionals $S^\omega_2$ from Section 2.1.4. These hardness properties do not disappear if we restrict $\Psi : [0, 1] \to \mathbb{R}^+$ in WHBU to e.g. Borel or semi-continuous functions.

---

6For $\Psi : [0, 1] \to \mathbb{R}^+$, one defines $\Psi_k : ([0, 1] \times \mathbb{N}) \to \mathbb{R}^+$ as $\Psi_k(x) := \frac{\Psi(x)}{2^k}$, which yields a ‘canonical’ Vitali cover $\cup_{k \in \mathbb{N}} \cup_{x \in [0, 1]} B(x, \Psi_k(x))$ of $[0, 1]$ generated by $\Psi$.

7The model $\mathcal{M}$ from [53, §4.1] satisfies $\Pi^0_1$-CA$_0^+$ + QF-AC$^{0,1}$, but not WHBU. This model is obtained from the proof that a realiser for WHBU is not computable in any type two functional.

8The proof of [21, Theorem 16 and Cor. 17] goes through with trivial modification, i.e. WHBU restricted to Baire 2 or semi-continuous functions implies that there is no injection from $2^\mathbb{N}$ to $\mathbb{N}$. The latter statement is however not provable in $\mathbb{Z}^\Omega_2 + QF$-AC$^{0,1}$ by [53, Theorem 3.2].
3.2.2. Generalisations. We study the following rather straightforward generalisations of WHBU, the analogues of which have been studied in second-order RM, namely [76 IV.1.6] and [12 Lemma 3.13].

(a) We replace the ‘interval’ covering $\bigcup_{x \in [0,1]} B(x, \Psi(x))$ by a ‘general’ covering $\bigcup_{x \in [0,1]} O_x$, only assuming that the open $O_x$ contains $x$ for any $x \in [0,1]$.
(b) We replace the covering $\bigcup_{x \in [0,1]} B(x, \Psi(x))$ of the unit interval $[0,1]$ by a covering $\bigcup_{x \in E} B(x, \Psi(x))$ of arbitrary $E \subset [0,1]$.

As will become clear, these generalisations follow from WHBU and basic properties of the Lebesgue measure. We show that the same generalisations for HBU and the Lindelöf lemma are much stronger than the original principles, highlighting a fundamental difference. This kind of behaviour is also discussed in [70].

First of all, motivated by item (a) right above, we define the following notion.

**Definition 3.3.** A general open covering of $X \subset \mathbb{R}$ is a mapping $\lambda x. O_x : \mathbb{R} \to (\mathbb{R} \to \mathbb{R})$ such that $O_x$ is an open set containing $x$, for any $x \in X$.

The following result shows that a fragment of the Lebesgue measure already significantly generalises WHBU. We note that Kreuzer’s measure theory from Section 3.3 proves $\lambda x. O_x$ (see Remark 3.1) and WHBU (see Theorems 3.18 and 3.27).

**Theorem 3.4 (ACA$_0^\omega$ + $\lambda x. O_x$).** The following are equivalent.

- WHBU
- For a general open covering $\lambda x. O_x$ and $\varepsilon > 0$, there are $x_0, \ldots, x_k \in [0,1]$ such that the measure of $\cup_{i \leq k} O_{x_i}$ is $> 1 - \varepsilon$.

**Proof.** Let $\lambda y. O_y$ be as in the theorem and note that the following follows by definition for any $x \in [0,1]$:

$$\left(\exists n \in \mathbb{N}\right)\left(\lambda x. O_x \cap B(x, \frac{1}{2^m}) = \frac{1}{2^n}\right),$$

(3.3)

where we note that the set in (3.3) is open. Let $B(x)$ be the least $n$ as in (3.3). Then $\Psi(x) := 2^{-(B(x)+2)}$ yields a canonical covering of $[0,1]$. For $\varepsilon > 0$, WHBU yields $x_0, \ldots, x_k \in [0,1]$ such that $\cup_{i \leq k} B(x_i, \Psi(x_i))$ has measure at least $1 - \varepsilon$. Clearly, $x_0, \ldots, x_k \in [0,1]$ is also such that the measure of $\cup_{i \leq k} O_{x_i}$ is at least $1 - \varepsilon$. Indeed, while $B(x, \Psi(x))$ may not be a subset of $O_x$, we do have that the measure of $O_x$ is at least that of $B(x, \Psi(x))$. \hfill $\Box$

In short, WHBU ‘bootstraps’ itself to general open coverings, thanks to (a fragment of the) Lebesgue measure. By contrast, the Lindelöf lemma for general open coverings implies the ‘explosive’ principle cocode$_0$ from Section 2.1.4.

**Theorem 3.5 (ACA$_0^\omega$).** The Lindelöf lemma for general open coverings of $\mathbb{R}$ implies cocode$_0$.

**Proof.** Fix $A \subset [0,1]$ and $Y : [0,1] \to \mathbb{N}$ injective on $A$. Note that $\mu^2$ can enumerate all rationals in $A$, i.e. we may assume $A \cap \mathbb{Q} = \emptyset$. Define the set $B$ as follows:

$$y \in B \leftrightarrow (\exists n \in \mathbb{N}) \left[ y \in [2n+1, 2n+2] \land (y - (2n+1)) \in A \land Y(y - (2n+1)) = n \right].$$

Clearly, if we can enumerate the reals in $B$, we can enumerate the reals in $A$. Now define the following general open covering on $[n, n+1]$ (for any $n \in \mathbb{N}$):

$$O_x := \begin{cases} 
\text{the set } (n-1, n+3) \ \setminus \ B & \text{if } x \notin B \land x \in [n, n+1] \\
\text{the interval } B(x, d_n(x)) & \text{if } x \in B \land x \in (n, n+1),
\end{cases}$$

(3.4)
where \( d_n(x) = \min \left( \frac{|x-n|}{2}, \frac{|x-(n+1)|}{2} \right) \) in case \( x \in (n, n+1) \). Clearly, \( O_x \) is an open set such that \( x \in O_x \) for any \( x \geq 0 \) and the extension to \( \mathbb{R} \) is trivial. Now let \( (x_n)_{n \in \mathbb{N}} \) be such that \( \bigcup_{n \in \mathbb{N}} O_{x_n} \) covers \( \mathbb{R} \). Use \( \mu^2 \) to remove all elements in the sequence not in \( B \). The resulting sequence lists all reals in \( B \) as ‘by definition’ reals in \( B \) are not covered by the set in the first case of (3.4). Thus, we are done. \( \square \)

The proof of Theorem 3.4 also shows that any A-functional is readily generalised to general open coverings if we have access to the Lebesgue measure (for open sets). By Theorem 3.7, \( \Theta \)-functionals for general open coverings are quite explosive.

**Theorem 3.7.** Together with \( 3^2 \), a \( Z \)-functional computes \( \Omega_b \).

**Proof.** Let \( X \subset [0,1] \) be finite and define the following general open covering:

\[
O_x := \begin{cases}
\text{the set } (-2,2) \setminus X & \text{if } x \notin X \\
\text{the interval } (-2,2) & \text{otherwise},
\end{cases}
\tag{3.5}
\]

Clearly, \( O_x \) is an open set such that \( x \in O_x \) for any \( x \in [0,1] \). Now consider \( Z(\lambda x.O_x) = (x_0, \ldots, x_k) \) and note that in case for all \( i \leq k \), we have \( x_i \notin X \), the set \( X \) must be empty. Hence, define \( \Omega_b(X) \) as 0 in this case, and 1 otherwise. \( \square \)

Secondly, motivated by item (b) from the beginning of this section, we study coverings of arbitrary sets, as in the following principle.

**Principle 3.8 (WHBU\(^+\)).** For \( \Psi : [0,1] \to \mathbb{R}^+ \), \( \varepsilon > 0 \), and non-empty \( E \subset [0,1] \) with measure \( e \in [0,1] \), there are \( x_0, \ldots, x_k \in E \) such that the measure of \( \bigcup_{i \leq k} B(x_i, \Psi(x_i)) \) is at least \( e - \varepsilon \).

We let \( \text{WHBU}_{\text{RM}}^+ \) be \( \text{WHBU}^+ \) restricted to \( \text{RM} \)-closed sets \( E \). We now show that \( \text{WHBU} \) can be ‘bootstrapped’ as follows.

**Theorem 3.9 (ACA\(^0\))**. We have \( \text{WHBU} \leftrightarrow \text{WHBU}_{\text{RM}}^+ \leftrightarrow \text{WHBU}^+ \).

**Proof.** We first prove \( \text{WHBU} \to \text{WHBU}_{\text{RM}}^+ \). Fix \( \Psi : [0,1] \to \mathbb{R}^+ \), \( \varepsilon > 0 \), and \( \text{RM} \)-closed \( E \subset [0,1] \), where the latter is represented by the complement of \( \bigcup_{n \in \mathbb{N}} (a_n, b_n) \). Define \( \Phi : [0,1] \to \mathbb{R}^+ \) as follows:

\[
\Phi(x) := \begin{cases}
\Psi(x) & \text{in case } x \in E \\
D(x) & \text{in case } x \notin E,
\end{cases}
\tag{3.6}
\]

where \( D(x) \) is \( r \in \mathbb{Q}^+ \) such that \( B(x, r) \subset (a_m, b_m) \) in case \( m \in \mathbb{N} \) is the least natural such that \( x \in (a_m, b_m) \), and 0 otherwise. Apply \( \text{WHBU} \) to \( \bigcup_{x \in [0,1]} B(x, \Phi(x)) \) to obtain \( x_0, \ldots, x_k \in [0,1] \) such that the measure of \( \bigcup_{i \leq k} B(x_i, \Phi(x_i)) \) is at least \( 1 - \varepsilon \). Let \( y_0, \ldots, y_m \) be those \( x_i \) for \( i \leq k \) that are in \( E \), and let \( z_0, \ldots, z_n \) be
the remaining ones. Since all sets involved are intervals, we may use the usual
(second-order) Lebesgue measure \( \lambda \) and obtain:

\[
1 - \varepsilon < \lambda(\bigcup_{i \leq k} B(x_i, \Phi(x_i))) \leq \lambda(\bigcup_{j \leq m} B(y_j, \Phi(y_j))) + \lambda(\bigcup_{i \leq n} B(z_i, \Phi(z_i)))
\]

\[
= \lambda(\bigcup_{j \leq m} B(y_j, \Psi(y_j))) + \lambda(\bigcup_{i \leq n} B(z_i, \Phi(z_i))) \leq \lambda(\bigcup_{j \leq m} B(y_j, \Psi(y_j))) + (1 - \varepsilon),
\]

which shows that \( y_0, \ldots, y_m \in E \) are as required by \( \text{WHBU}^+_{\text{RM}} \).

To show that \( \text{WHBU}^+_{\text{RM}} \rightarrow \text{WHBU}^+ \), fix \( \Psi : [0, 1] \rightarrow \mathbb{R}^+ \), \( \varepsilon > 0 \), and \( E \subset [0, 1] \) with measure \( \lambda(E) = \varepsilon \). Then \( [0, 1] \setminus E \) has measure \( 1 - \varepsilon \) and by definition there is a sequence of open intervals \( (I_n)_{n \in \mathbb{N}} \) such that \( \bigcup_{n \in \mathbb{N}} I_n \) covers \( [0, 1] \setminus E \) and has measure at most \((1 - \varepsilon) + \varepsilon/2\). Then \( C := [0, 1] \setminus \bigcup_{n \in \mathbb{N}} I_n \) is RM-closed, satisfies \( C \subset E \), and has measure at least \( \varepsilon - \varepsilon/2 \). Now apply \( \text{WHBU}^+_{\text{RM}} \) for \( C \) and \( \varepsilon/2 \). \( \square \)

We observe that the second part of the proof only goes through because our
comparative interpretation of the Lebesgue measure essentially hard-codes Little-
wood’s first principle. The first part of the proof of the theorem is ‘effective’ and
shows that a \( \Lambda \)-functional is readily generalised to RM-closed sets. To generalise \( \Lambda \)-fun-
tionals to coverings of closed sets, let \( \text{WHBU}^+_{\text{closed}} \) be the restriction of \( \text{WHBU}^+ \)
to closed sets (Definition 2.6).

**Corollary 3.10 (\( \text{ACA}^0_0 + (\text{\lambda}_{\text{open}}) \)).** We have \( \text{WHBU} \rightarrow \text{WHBU}^+_{\text{closed}} \).

**Proof.** Recall \( B : \mathbb{R} \rightarrow \mathbb{N} \) as defined from (3.3). In the proof of Theorem 3.10 modify the functional \( \Phi \) from (3.4) as follows: \( \Phi(x) = \Psi(x) \) in case \( x \in E \), and \( \frac{1}{2\pi n^2} \) otherwise. The first part of the proof of the theorem now goes through. \( \square \)

The proof of the corollary shows that a \( \Lambda \)-functional is readily generalised to any
closed set, assuming the Lebesgue measure \( \lambda \) as in \( (\lambda_{\text{open}}) \). However, if we consider
Vitali covers as in Footnote \( \text{V} \) then we can shrink the measure of \( \bigcup_{i \leq k} B(x_i, \Psi(x_i)) \) from Theorem (3.9) below \( \varepsilon + \varepsilon \) if necessary. In this way, a \( \Lambda \)-functional generalised
to general open coverings of closed sets is computationally equivalent to the combi-
nation of: a ‘standard’ \( \Lambda \)-functional and the Lebesgue measure \( \lambda \) as in \( (\lambda_{\text{open}}) \). The
general case, involving coverings of arbitrary sets, goes through \textit{mutatis mutandis}.

Next, we show that the same generalisations for the Lindelöf lemma are quite
powerful. A ‘fixed radius interval covering’ of \( E \) is \( \bigcup_{x \in E} B(x, \varepsilon) \) for some \( \varepsilon > 0 \).

**Theorem 3.11 (\( \text{ACA}^0_0 \)).** The principle \( \text{code}_0 \) follows from the Lindelöf lemma
for fixed radius interval coverings of closed sets in \( \mathbb{R} \).

**Proof.** Consider the sets \( A, B \) from the proof of Theorem 3.10. Now apply the
Lindelöf lemma from the theorem to \( \bigcup_{x \in B}(x - \frac{1}{2}, x + \frac{1}{2}) \), which readily yields of enumeration on \( B \), and hence of \( A \). \( \square \)

On a historical note, Lindelöf in [36] formulates his lemma for general open
coverings of any set, while the Heine-Borel theorem for open coverings of closed sets may be found in e.g. [35].

Finally, Theorem 3.11 deals with coverings consisting of open intervals with a
fixed radius. Trivial as such coverings many seem, they play an important role in
Section 3.3 in the form of the following principle, which follows from \( \text{WHBU} \).

**Principle 3.12 (\( \text{WHBU}^+_{\text{RM}} \)).** For \( \varepsilon, \delta > 0 \) and non-empty RM-closed \( E \subset [0, 1] \),
there are \( x_0, \ldots, x_k \in E \) such that \( \bigcup_{i \leq k} B(x_i, \varepsilon) \) has measure at least \( \lambda(E) - \delta \).
We note that, $\text{WHBU}_{\text{RM}}$ is provable in $\text{WKL}_0$ by [60]. With the gift of hindsight, $\text{WHBU}_{\text{RM}}$ also readily follows from $\text{HBU}$, and $\text{ECF}$ yields the result from [60].

### 3.3. Littlewood’s three principles

In this section, we derive $\text{WHBU}_m$ from a number of theorems that embody Littlewood’s principles, like Lusin’s theorem and Egorov’s theorem (Section 3.3.1), Tao’s ‘Littlewood-like’ principles (Section 3.3.2), and convergence theorems (Remark 3.23). The restriction of $\text{WHBU}$ to measurable functions is $\text{WHBU}_m$.

We recall that $\text{WHBU}_{\text{RM}}$ from Section 3.2.2 is provable in $\text{RCA}_0 + \text{WKL}$ by [60].

We also recall the conventions regarding the Lebesgue measure from Remark 3.1. On a conceptual note, we will observe that the (comparative or virtual) definition of the Lebesgue measure as in Remark 3.1 essentially ‘hard-codes’ Lusin’s first principle, yielding substantial generalisations/robust variations of the Lusin and Egorov theorems (and related results).

#### 3.3.1. Theorems by Lusin and Egorov

In this section, we derive $\text{WHBU}_m$ from the well-known theorems due to Lusin and Egorov.

First of all, Lusin’s theorem expresses that any measurable function is a continuous function on nearly all of its domain. This theorem constitutes the second of Littlewood’s principles, and Lusin proved this theorem for real intervals in [39] in 1912, but it had been established previously by Borel ([11]), Lebesgue ([34]), and Vitali ([85]). We note that Lusin’s theorem is often proved via a straightforward application of Egorov’s theorem.

Secondly, there are multiple formulations of Lusin’s theorem ([29, 37, 56, 58, 88]) and we first study the one found in e.g. [88], as follows.

**Principle 3.13 (LUS).** For measurable $f : I \to \mathbb{R}$ and $\varepsilon > 0$, there exists measurable $E \subset [0, 1]$ with measure at least $1 - \varepsilon$ and such that $f$ restricted to $E$ is continuous.

We let $\text{LUS}_{\text{RM}}$ be LUS where the set $E$ is additionally assumed to be RM-closed. We note that Lusin’s original formulation from [39] involves a perfect set $E$, i.e. closed and no isolated points.

**Theorem 3.14 (ACA$_0^\omega$).** We have $\text{LUS} \leftrightarrow \text{LUS}_{\text{RM}}$ and $\text{LUS} \rightarrow \text{WHBU}_m$.

**Proof.** For the equivalence, let $f : I \to \mathbb{R}$ be measurable and $\varepsilon > 0$. Now apply LUS for $\varepsilon/2$ to obtain measurable $E \subset [0, 1]$ with measure at least $1 - \varepsilon/2$ and such that $f$ restricted to $E$ is continuous. Since $[0, 1] \setminus E$ has measure at most $\varepsilon/2$, there is a sequence of open intervals $(I_n)_{n \in \mathbb{N}}$ such that $\cup_{n \in \mathbb{N}} I_n$ has measure at most $\varepsilon$ and covers $[0, 1] \setminus E$. By definition, $C := [0, 1] \setminus \cup_{n \in \mathbb{N}} I_n$ is RM-closed, has measure at least $1 - \varepsilon$, and is contained in $E$. Thus, the set $C$ is as required for $\text{LUS}_{\text{RM}}$ and the equivalence follows.

For the second part, fix measurable $\Psi : [0, 1] \to \mathbb{R}^+$ and $\varepsilon > 0$, and let $E$ be as in $\text{LUS}_{\text{RM}}$. Since $E$ has an RM-code, it is separably closed by [13, Theorem 3.0]. Recall that [69, Theorem 2.4] provides the Tietze extension theorem for (third-order) functions that are (epsilon-delta) continuous on a separably closed set in $[0, 1]$. Hence, there is continuous $\Phi : [0, 1] \to \mathbb{R}$ which equals $\Psi$ on $E$. Following [30, §4],

---

9The proofs of Lusin’s theorem in e.g. [56, p. 74], [29, p. 64], and [37, p. 29], are basic applications of Egorov’s theorem.
\( \Phi \) has an RM-code, and applying \cite[IV.2.11]{WHBU}, we know \( \Phi \) attains its minimum on \( E \), which must be non-zero. We therefore have \((\exists \delta_0^\epsilon)(\forall x \in E)((|\Psi(x)| \geq \frac{1}{2\delta_0^\epsilon})\) and applying WHBU\textsubscript{RM}^{-} for \( \epsilon = \frac{1}{2\delta_0} \) finishes the proof. \( \square \)

Thirdly, consider the following alternative formulation of Lusin’s theorem.

**Principle 3.15 (LUS').** For measurable \( f : I \to \mathbb{R} \) and \( \epsilon > 0 \), there is a continuous \( g : I \to \mathbb{R} \) such that \( \{ x \in I : f(x) \neq g(x) \} \) has measure below \( \epsilon \).

**Theorem 3.16 (ACA\textsubscript{0}').** We have LUS' \( \to \) WHBU\textsubscript{m}.

**Proof.** For \( \epsilon > 0 \) and measurable \( \Psi : I \to \mathbb{R}^+ \), let \( g : I \to \mathbb{R} \) be the continuous function provided by LUS' for \( \epsilon/2 \). Define \( E = \{ x \in [0,1] : \Psi(x) = \sup_{y \in \mathbb{R}} g(y) \} \), using \( (\exists^2) \) and note that is has measure at least \( 1 - \epsilon/2 \). Since \([0,1] \setminus E \) has measure as most \( \epsilon/2 \), there is by definition a sequence \((I_n)_{n \in \mathbb{N}} \) of open intervals such that the union \( \bigcup_{n \in \mathbb{N}} I_n \) has measure at most \( \epsilon \) and covers \([0,1] \setminus E \). Define the RM-closed set \( F = [0,1] \setminus \bigcup_{n \in \mathbb{N}} I_n \) and note that \( F \subseteq E \) and that \( F \) has measure at least \( 1 - \epsilon \). Following \cite[§4]{WHBU}, one readily shows that \( g \) has an RM-code, and applying \cite[IV.2.11]{WHBU}, we know \( g \) attains its minimum on \( F \), which must be non-zero. We now have \((\exists \delta_0^\epsilon)(\forall x \in F)((|\Psi(x)| \geq \frac{1}{2\delta_0^\epsilon})\). Apply WHBU\textsubscript{RM} to obtain WHBU\textsubscript{m}. \( \square \)

A basic fact of measure theory is that measurable functions can be expressed as the (pointwise) limit of simple functions (\cite[Theorem 1.3.20]{WHBU}). By converting the latter in continuous piecewise linear functions, one readily derives Lusin’s theorem as in LUS, i.e. one also obtains WHBU\textsubscript{m}.

Fourth, we derive WHBU\textsubscript{m} from Egorov’s theorem, which was published around 1900 (\cite{WHBU}, \cite{WHBU}, \cite{WHBU}) and expresses that a convergent sequence of measurable functions is uniformly convergent outside an arbitrarily small set (see e.g. \cite[§1.3]{WHBU} and \cite[Ch. 3]{WHBU}). Thus, Egorov’s theorem corresponds to Littlewood’s third principle and we use the formulation from \cite{WHBU} as in EGO below. In \cite{WHBU}, one finds a weaker version EGO\textsuperscript{-}, involving convergence everywhere in the antecedent.

**Principle 3.17 (EGO).** Let \( f_n : (I \times \mathbb{N}) \to \mathbb{R} \) be a sequence of measurable functions converging almost everywhere to measurable \( f : I \to \mathbb{R} \), and let \( \epsilon > 0 \). Then there is measurable \( E \subseteq I \) of measure \( 1 - \epsilon \) with \( f_n \) converging uniformly to \( f \) on \( E \).

Let EGO\textsubscript{RM}^{-} be EGO\textsuperscript{-} with the extra assumption that \( E \) is RM-closed.

**Theorem 3.18 (ACA\textsubscript{0}').** We have EGO\textsuperscript{-} \( \iff \) EGO\textsubscript{RM}^{-} and EGO\textsuperscript{-} \( \to \) WHBU\textsubscript{m}.

**Proof.** For the equivalence, proceed as in the first part of Theorem \cite[§1.4]{WHBU}.

For the second part, fix \( \epsilon > 0 \) and measurable \( \Psi : I \to \mathbb{R}^+ \). Apply QF-\( \text{AC}^{1,0} \) to \((\forall x \in I)(\exists n \in \mathbb{N})(\Psi(x) > \frac{1}{2\epsilon})\) and let the resulting functional be named \( \Phi \). Using \( \exists^2 \), one readily guarantees that \( \Phi \) satisfies \( x = \sup_{y \in \mathbb{R}} y \to \Phi(x)=0 \Phi(y) \) for \( x,y \in \mathbb{R} \). Now define \( \Phi_n(x) := \min(\Phi(x),n) \). By definition, we have that \( \Phi_n \) converges to \( \Phi \) on \( I \). Let \( E \) be as in EGO\textsubscript{RM}^{-}, i.e.

\[
(\forall \epsilon' > 0)(\exists N \in \mathbb{N})(\forall x \in E)(\forall n \geq N)((|\Phi_n(x) - \Phi(x)| < \epsilon'))
\]

For \( \epsilon' = 1 \), take \( N_0 \) as in \cite[377]{WHBU}, and note that for \( x \in E \), we have \( \Psi(x) \geq \frac{1}{2N_0} \), by the definition of \( \Phi \). Applying WHBU\textsubscript{RM}^{-} for \( \epsilon = \frac{1}{2N_0} \) finishes the proof. \( \square \)
In conclusion, the Lusin and Egorov theorems imply \textsc{whbu}_m while the exact formulation of the former does not matter that much. In particular, the (comparative or virtual) definition of the Lebesgue measure as in Remark 3.11 essentially ‘hard-codes’ Lusin’s first principle, allowing us to ‘bootstrap’ e.g. \textsc{lus} to \textsc{lusrm}.

3.3.2. Littlewood-like principles. The literature contains a number of ‘Littlewood-like’ principles, i.e. statements similar to Littlewood’s three principles. We study two examples and sketch how they also imply \textsc{whbu}_m.

First of all, we study \textsc{llp}, which is the (part of the) ‘fourth’ Littlewood principle from [40] and one of Tao’s ‘Littlewood-like principles’ from [81], Ex. 1.3.25.

**Principle 3.19** (\textsc{llp}). For measurable \( f : \mathbb{R} \to \mathbb{R} \) and \( \varepsilon > 0 \), there exists a measurable set \( K \subseteq I \) such that \( \lambda(I \setminus K) < \varepsilon \) and \( f \) is bounded on \( K \).

Let \textsc{llprm} be \textsc{llp} where \( K \) is additionally \textsc{rm}-closed. One proves the following theorem in the way as in the previous section.

**Theorem 3.20** (\textsc{aca}^\omega_0). We have \textsc{llp} \iff \textsc{llprm} and \textsc{llp} \imp \textsc{whbu}_m.

**Proof.** Consider the function \( \Phi \) from the proof of Theorem 3.18. Then apply \textsc{whbu}^\omega for the resulting set. \( \square \)

Secondly, we say that \( f_n \) converges to \( f \) in measure on \( X \), if for every \( \varepsilon > 0 \) and \( k \in \mathbb{N} \), there is \( N \in \mathbb{N} \) such that for \( n \geq N \), the measure of \( \{ x \in X : |f_n(x) - f(x)| \geq \varepsilon \} \) is at most \( \frac{1}{k} \). The following theorem is called slightly weaker than Egoroff’s theorem by Royden in [56] p. 72], and connects the previous notion to pointwise convergence.

**Principle 3.21** (\textsc{wte}). Let \( f, f_n \) be measurable and such that \( (f_n(x))_{n \in \mathbb{N}} \) converges to \( f(x) \) for almost all \( x \in I \). Then \( (f_n)_{n \in \mathbb{N}} \) converges to \( f \) in measure.

**Theorem 3.22** (\textsc{aca}^\omega_0). We have \textsc{wte} \imp \textsc{whbu}_m.

**Proof.** Fix \( 1 > \varepsilon_0 > 0 \) and measurable \( \Psi : I \to \mathbb{R}^+ \). Let \( \Phi \) be as in the proof of Theorem 3.18 and define \( \Phi_n(x) := \min(\Phi(x), n) \). By definition, we have that \( \Phi_n \) converges to \( \Phi \) on \( I \). Use (3.2) to define \( E_n := \{ x \in I : |\Phi_n(x) - \Phi(x)| \geq \varepsilon_0 \} \). By \textsc{wte}, for any \( k \in \mathbb{N} \), there is \( N \in \mathbb{N} \) such that the measure of \( E_n \) is at most \( 1/2^k \) for \( n \geq N_0 \). For \( x \in (I \setminus E_{N_0}) \), \( |\Phi_{N_0}(x) - \Phi(x)| < 1 \) implies \( \Psi(x) \geq \frac{1}{2^{N_0+1}} \). As for Theorem 3.14, \textsc{whbu}_m follows, and we are done. Note that we need to use the first part of the proof of Theorem 3.14 to obtain \textsc{rm}-closed sets. \( \square \)

Finally, we discuss how the well-known convergence theorems associated to the Lebesgue integral imply \textsc{whbu}_m. Unfortunately, establishing this result requires a non-trivial integration theory, as provided by e.g. the system \textsc{aca}^\omega_0 + (\lambda_0) from Section 3.4 and also lots of technical details. Hence, we merely sketch these results.

**Remark 3.23.** First of all, the Lebesgue integral constitutes a generalisation of the Riemann integral; one of the advantages of the former is the superior treatment of limits of integrals. In particular, the dominated (resp. monotone) convergence theorem implies that pointwise convergence (ae) implies convergence of the associated integrals, assuming the sequence is dominated by an integrable function (resp. the sequence is non-negative and monotone).

Secondly, assuming the dominated convergence theorem, one derives \textsc{whbu}_m as in the following proof sketch. We implicitly use a number of properties of the Lebesgue measure and integral, provable in \textsc{aca}^\omega_0 + (\lambda_0) from Section 3.4, it seems.
In Section 3.3.1, we have derived

we list the main properties of Kreuzer’s approach as follows.

From [32] as in the axiom (\(3.4.2\)), a measure of measure theory.

About and around Kreuzer’s measure theory.

3.4.1. A measure of motivation. The system of (Lebesgue) measure theory from [32] is introduced in Section 3.4.2. This system is non-classical, i.e., a type theory. Now, as noted in item (vi) of Definition 2.4, we code sets as characteristic functions.

First of all, the system from [32] defines the Lebesgue measure on subsets of Cantor space, i.e. we need to represent such sets in \(\text{RCA}_0\), as the latter is officially a type theory. Now, as noted in item (ii) of Definition 2.4, we code sets as characteristic functions.

To be absolutely clear, a set \(Y \subseteq \mathbb{N}\) is given by a function \(f^Y_N\), and we write ‘\(n^0 \in Y\)’ for \(f_Y(n) = 0\). There are a number of ways of representing subsets of

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3.4.2. A measure of measure theory. We introduce the system of measure theory from [32] as in the axiom (\(\lambda\)) below.

First of all, the system from [32] defines the Lebesgue measure on subsets of Cantor space, i.e. we need to represent such sets in \(\text{RCA}_0\), as the latter is officially a type theory. Now, as noted in item (ii) of Definition 2.4, we code sets as characteristic functions.

Proof. (Sketch) Fix \(\Psi : I \rightarrow \mathbb{R}^+\) and \(\varepsilon > 0\) as in WHBU\(_m\). Use (3.2) to define \(f_n : I \rightarrow \mathbb{R}\) as 1 if \(\Psi(x) < \frac{1}{2^n}\), and 0 otherwise. Let \(A_n\) be the subset of \([0,1]\) represented by \(f_n\). Since \((\forall n \in \mathbb{N})(\forall x \in I)(f_n(x) \leq 1\mathbb{R})\) and \(f_n\) converges to \(f := 1\mathbb{R}\) pointwise on \(I\). We now have \(\lim_{n \to \infty} \int_I |f_n - f| d\lambda = 0\). By definition, we obtain \(\lim_{n \to \infty} \lambda(A_n) = 1\). Now let \(N_0\) be such that \(|1 - \lambda(A_n)| < \varepsilon\) for \(n \geq N_0\) and note that we may assume \(\varepsilon > \frac{1}{2^{N_0}}\) (just take a larger number if necessary). Then for \(x \in I \setminus A_{N_0}\), we have \(\Psi(x) \geq \frac{1}{2^{N_0}}\). As for Theorem 3.18, WHBU\(_m\) now follows.

Thirdly, assuming the monotone convergence theorem, one derives WHBU\(_m\) as in the following proof sketch. We implicitly use a number of properties of the Lebesgue measure and integral, provable in \(\text{ACA}_0\) from Section 3.4. It seems.

Proof. (Sketch) Fix \(\Psi : I \rightarrow \mathbb{R}^+\) and \(\varepsilon > 0\) as in WHBU\(_m\). Use (3.2) to define \(f_n : I \rightarrow \mathbb{R}\) as 1 if \(\Psi(x) < \frac{1}{2^n}\), and 0 otherwise. Let \(A_n\) be the subset of \([0,1]\) represented by \(f_n\). Since \((\forall n \in \mathbb{N})(\forall x \in I)(0 \leq f_n(x) \leq f_{n+1}(x) \leq 1\mathbb{R})\) and \(f_n \rightarrow f := 1\mathbb{R}\) pointwise everywhere, we have \(\lim_{n \to \infty} \int_I |f_n - f| d\lambda = 0\). By definition, we obtain \(\lim_{n \to \infty} \lambda(A_n) = 1\). Now let \(N_0\) be such that \(|1 - \lambda(A_n)| < \varepsilon\) for \(n \geq N_0\) and note that we may assume \(\varepsilon > \frac{1}{2^{N_0}}\) (just take a larger number if necessary). Now proceed as in the previous proof sketch.

In conclusion, numerous fundamental (approximation) results from measure theory imply WHBU\(_m\), sometimes over a non-trivial base theory. The development of measure theory even seems to go ‘hand in hand’ with the approximation theory.

3.4. About and around Kreuzer’s measure theory.

3.4.1. A measure of motivation. The system of (Lebesgue) measure theory from [32] is introduced in Section 3.3.2. This system is \(\text{ACA}_0\) extended with the axiom (\(\lambda\)) introducing an extension of the Lebesgue measure. This system is non-classical, as (\(\lambda\)) implies that all subsets of the Cantor space are measurable, like in e.g. [174]. We list the main properties of Kreuzer’s approach as follows.

(i) By [32] Theorem 3, the axiom (\(\lambda\)) gives rise to a \(\Pi^0_1\)-conservative extension of \(\text{ACA}_0\), i.e. the non-classical consequences of (\(\lambda\)) are limited.

(ii) By Theorem 3.21, the Heine-Borel theorem HBU\(_c\) for Cantor space is equivalent to its restriction to measurable functionals. In this light, the assumption that all subsets of Cantor space are measurable seems innocent if we are interested in the study of compactness (and related notions).

(iii) The system \(\text{ACA}_0\left(\lambda\right)\) does not prove the Heine-Borel theorem for \(2^\mathbb{N}\) (Corollary 3.26), but does prove Egorov’s theorem (Theorem 3.27).

(iv) No functional \(\Theta\) as in SFF(\(\Theta\)) is computable in Kreuzer’s Lebesgue measure \(\lambda\) and Feferman’s \(\mu\) (Theorem 3.28).

In Section 3.3.1, we have derived WHBU\(_m\) from Egorov’s theorem.

3.4.2. A measure of measure theory. We introduce the system of measure theory from [32] as in the axiom (\(\lambda\)) below.

First of all, the system from [32] defines the Lebesgue measure on subsets of Cantor space, i.e. we need to represent such sets in \(\text{RCA}_0\), as the latter is officially a type theory. Now, as noted in item (ii) of Definition 2.4, we code sets as characteristic functions.

To be absolutely clear, a set \(Y \subseteq \mathbb{N}\) is given by a function \(f^Y_N\), and we write ‘\(n^0 \in Y\)’ for \(f_Y(n) = 0\). There are a number of ways of representing subsets of
2\^N, and we follow Kreuzer’s approach from [32] §2. Define \(sg^1\) as \(sg(0) = 0\) and \(sg(k) = 1\) for \(k > 0\). \(sg^{k+1}(g)(n) := sg(g(n))\), which we also denote as \(sg\) when there can be no confusion, maps \(\mathbb{N}^N\) to \(2^{\mathbb{N}}\), and dispenses with a lot of notation. Indeed, a set \(X \subseteq 2^{\mathbb{N}}\) is then given by a functional \(F^X\), and we write \(g^1 \in X\) in case \(F^X(\text{sg}(g)) = 0\), i.e. we quantify over Baire space but always work ‘modulo \(sg\)’, as also expressed by the last line of the axiom \((\lambda)\) just below. For a sequence \(X^{0 \rightarrow \omega}\), we use \(X_n\) to denote the \(n\)-th element of that sequence, as usual.

Secondly, the system from [32] is\(^{13}\) then \(\mathcal{ACA}^0_\omega\) extended with the axiom \((\lambda)\):

\[
(\exists \lambda^3) \left( (\forall X^2)((\lambda)(X) \geq 0) \land (\forall f \in C)((X(f) \neq 0) \rightarrow (\lambda(X) = 0)) \land (\forall \lambda^0 \rightarrow 2)(\lambda(\cup_{j \in \mathbb{N}} X'_j) \equiv \sum_{i=0}^{\infty} \lambda(X'_i)) \land (\forall \lambda^0) (|\lambda|_{sg} = 2^{-|\lambda|}) \land (\forall X^2)(\lambda(X) = \lambda(\lambda f^1.\text{sg}(X(f))) = \lambda(X(\lambda n^0.\text{sg}(f(n))))))
\]

where \(X'_i := X_i \setminus \cup_{j<i} X'_j\). The last line of \((\lambda)\) indicates that \(\lambda\) is compatible with our coding of subsets of \(2^{\mathbb{N}}\). By [32] Theorem 3, \(\mathcal{ACA}^0_\omega + (\lambda)\) is a \(\Pi^1_2\)-conservative extension of \(\mathcal{ACA}^0_\omega\). Similar to [32], we will make use of a Skolem constant \(\lambda\) added to the language. As is well-known, \(2^{\mathbb{N}}\) and \([0, 1]\) are measure-theoretically equivalent. Indeed, \(\rho(\alpha) := \sum_{n=0}^{\infty} \frac{\alpha(n)}{2^n}\) is a measure-preserving surjection, and \((\lambda)\) thus defines a measure on \([0, 1]\). In a strong system like ZFC, we can prove that this measure will be identical to the Lebesgue measure on the class of Lebesgue-measurable sets.

Thirdly, \((\lambda)\) is ‘non-classical’ in nature as it implies that all subsets of \(2^{\mathbb{N}}\) are measurable. Indeed, the Axiom of Choice \(\mathcal{AC}\) is known to yield non-measurable sets (see e.g. [31] §1.2.3) if we require the measure to be countably additive and translation-invariant. Due to the absence of the latter requirement, \((\lambda)\) taken as a statement in ZFC does not violate \(\mathcal{AC}\), but it implies large cardinal\(^{14}\) axioms.

3.4.3. Some results. We establish some results in and about Kreuzer’s framework.

First of all, whether or not all subsets of \(2^{\mathbb{N}}\) are measurable turns out not to have an influence on open-cover compactness. Indeed, by Theorem 3.2.4 Heine-Borel compactness as in \(\mathcal{HBU}_c\) does not really change if we restrict to measurable functionals. In this light, assuming that all subsets of \(2^{\mathbb{N}}\) are measurable does not really change the strength of \(\mathcal{HBU}_c\), i.e. one should feel free to use this assumption when studying compactness, like in the form of \((\lambda)\).

\textbf{Theorem 3.24.} Given \((\exists 2)\), the theorem \(\mathcal{HBU}_c\) follows from the restriction of \(\mathcal{HBU}_c\) to measurable functionals.

\textit{Proof.} Fix \(f, g \in C\) and define \(h = \langle f, g \rangle\) by \(h(2n) = f(n)\) and \(h(2n + 1) = g(n)\). Let \(C^0\) be the set of \(\lambda x^0.0, f\) such that \(f \in C\). Then \(C^0\) is a compact subset of \(C\) of measure 0, so every total function that is continuous outside \(C^0\) will be measurable. Moreover \(C^0\) is homeomorphic to \(C\). Let \(F : C \rightarrow \mathbb{N}\) be arbitrary. Define \(F_0(\lambda x.0, f)) = 2F(f) + 1\), and define \(F_0(h) = n\) for the least \(n\) such that \(C_h(n)\) is disjoint from \(C^0\) if \(h\) is not in \(C^0\). Then \(F_0\) is measurable, and if we apply \(\mathcal{HBU}_c\) to \(F_0\), \(C^0\) can only be covered by \(C_{h(F_0(h))}\) for \(h \in C^0\), and by the homeomorphism, we obtain a finite sub-cover of the cover of \(C\) induced by \(F\). \(\square\)

\(^{10}\)Note that Kreuzer’s definition of \(\mathcal{ACA}^0_\omega\) in [32] is unfortunately different from ours. In this paper, we exclusively use the definition \(\mathcal{ACA}^0_\omega = \mathcal{RCA}^0_\omega + (\mu^0)\) from Section 2.2.

\(^{13}\)The axiom \((\lambda)\) implies there is a weakly inaccessible cardinal below the continuum, namely a weakly Mahlo cardinal. In particular, the assumption violates the axiom \(V = L\).
In hindsight, the previous theorem is not that surprising: the Axiom of Choice is not needed to prove $\text{HBU}_c$ (see [58, §4.1]): the latter is provable in $\mathbb{Z}_2^c$. Since the existence of non-measurable sets is intimately connected to the Axiom of Choice, it stands to reason the latter has no influence on $\text{HBU}_c$. Since special fan functionals compute realisers for $\text{HBU}_c$, we expect the following complimentary result, where $\text{LMC}(\lambda)$ is $(\lambda)$ without the leading existential quantifier.

**Theorem 3.25.** No functional $\Theta$ as in $\text{SFF}(\Theta)$ is computable in any $\lambda$ as in $\text{LMC}(\lambda)$ and Feferman’s $\mu$.

**Proof.** There is a partial functional of type $1 \to 1$ computable in $\mu$ which to a code for a Borel-subset $B$ of $C$ computes a binary representation of $\lambda(B)$, see [61, Section IV.1] or [47, Prop. 3.23]. If $\Theta$ were computable in $\lambda$, we could use this and the recursion theorem (for $\Sigma_1\Sigma_9$) to show that whenever we have an index $e$ for computing a functional $F$ from $\mu$ and some $f^1$, we can find a value of $\Theta(F)$ computable uniformly from $e$ and $f$. Since there is arithmetical $F$ such that $\Theta(F)$ cannot be hyperarithmetical (see [16][47]), this is impossible. Hence, no $\Theta$ is computable in $\lambda$ and $\mu$. $\square$

Secondly, we now show that Heine-Borel compactness as in $\text{HBU}_c$ is not provable from in Kreuzer’s framework.

**Corollary 3.26.** The system $\text{ACA}_0^\omega + (\lambda)$ cannot prove $\text{HBU}_c$.

**Proof.** Similar to the proof of the theorem, there is arithmetical $F_0$ such that for any finite sub-cover $\cup_{i \leq k} [\overline{f}F_0(f_i)]$ of the canonical cover $\cup_{f \in C} [\overline{f}F_0(f)]$, the finite sequence $\langle f_1, \ldots, f_k \rangle$ is not hyperarithmetic (see [16][47]). As in [32], denote by $T_0$ the sub-system of Gödel’s system $T$, where primitive recursion is restricted to the recursor $R_0$, and let $T_0[F]$ be $T_0$ extended with the functional $F$. Then $T_0[F]$ satisfies $(\lambda)$ by [32, Lemma 7]. Hence, there is a model of $\text{ACA}_0^\omega + (\lambda)$ in which $\text{HBU}_c$ is false, as it contains $F_0$ but the finite sub-cover for the associated canonical cover is lacking.

Alternatively, the proof of [32, Theorem 3] establishes the following term extraction procedure: if for arithmetical $A$, $\text{ACA}_0^\omega + (\lambda)$ proves $(\forall f^1)(\exists g^1)A(f, g)$, then a term $t$ can be extracted from this proof such that $\text{ACA}_0^\omega$ proves $(\forall f^1)A(f, t(f, \mu))$. Note the essential role of $\mu^2$ in the conclusion. If $\text{ACA}_0^\omega + (\lambda)$ proves $\text{HBU}_c$, we consider the latter restricted to the arithmetical functional $F_0$ from the previous paragraph. Note that we may replace $\mu^2$ the innermost universal quantifier over $C$ by a numerical quantifier. Hence, the resulting sentence (modulo some applications of $\mu^2$) has the right format for applying the previous term extraction result. However, this means we obtain a hyperarithmetical finite sub-cover for the canonical cover corresponding to $F_0$, a contradiction. $\square$

By [32, Remark 13], $\Pi_1^1-\text{ACA}_0^\omega + (\lambda)$ similarly cannot prove $\text{HBU}_c$, and the same for stronger systems. A lot of details need to be worked out to establish this result, however. The crucial part of the previous theorem is that $\text{HBU}_c$ restricted to *arithmetically defined* covers already yields non-hyperarithmetical functions (and in fact $\text{ATR}_0$; see [16][47]). In Section 4 we do establish that $\Pi_1^1-\text{ACA}_0^\omega + \text{WHBU}$ cannot prove $\text{HBU}_c$.\[12\]

---

12 In the case of the unit interval, this means that if we have a finite sub-cover for $[0, 1] \cap \mathbb{Q}$, we also have a finite sub-cover for $[0, 1]$ by adding the end-points of the intervals of the first covering.
Finally, Egorov’s theorem can be established in Kreuzer’s framework.

**Theorem 3.27.** The system \( \text{ACA}_0^\omega + (\lambda) \) proves Egorov’s theorem as follows: let \( f_n : (I \times \mathbb{N}) \to \mathbb{R} \) be a sequence converging almost everywhere to \( f : I \to \mathbb{R} \), and let \( \varepsilon > 0 \). Then there is \( E \subseteq I \) such that \( \lambda(E) > 1 - \varepsilon \) and \( f_n \) converges uniformly to \( f \) on \( E \).

**Proof.** The ‘usual’ proof (see e.g. [56, Ch. 3.6]) goes through as follows. First of all, the Lebesgue measure as in \( (\lambda) \) is ‘continuous from above’, i.e. we have

\[
[(\forall n \in \mathbb{N})(E \subseteq E_{n+1} \subseteq E_n \subseteq [0, 1]) \land E = \cap_{n \in \mathbb{N}}E_n] \to \lambda(E) = \lim_{n \to \infty} \lambda(E_n). \quad (3.8)
\]

Indeed, the proof of \((3.8)\) in e.g. [56, p. 63] amounts to nothing more than defining countable additivity, included in \( (\lambda) \), to obtain the consequent of \((3.8)\).

Secondly, define \( E_{n,k} := \cup_{m \geq n}\{x \in I : |f_m(x) - f(x)| \geq \frac{1}{k}\} \) and note that \( \lambda(\cap_{n \in \mathbb{N}}E_{n,k}) = 0 \) by the assumption that \( f_n \to f \) almost everywhere. Applying \((3.8)\), we obtain \( \lim_{n \to \infty} \lambda(E_{n,k}) = 0 \), which implies the following:

\[
(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall n \geq N)(\lambda(E_{n,k}) < \frac{1}{2^k}). \quad (3.9)
\]

Fix \( \varepsilon > 0 \) and use \((\mu^2)\) in \((3.9)\) to find \( g^1 \) such that \((\forall k \in \mathbb{N})(\lambda(E_{g(k),k}) < \frac{\varepsilon}{2^k})\) and define \( E := \cup_{k \in \mathbb{N}}E_{g(k),k} \). It is now straightforward to show that the set \( E \) is as required for EGO. Indeed, countable additivity implies that \( \lambda(E) \leq \sum_{n=0}^{\infty} \lambda(E_{g(k),k}) \leq \varepsilon \) while for \( x \in I \setminus E \), the rate of uniform convergence is \( g \). \( \square \)

In conclusion, while Kreuzer’s framework can establish fundamental results in measure theory like Egorov’s theorem, we cannot hope to prove any theorem based on the (uncountable) Heine-Borel theorem by Corollary 3.26. As shown in [48], many basic results in third-order arithmetic imply HBU, including the development of the gauge integral (see Section 3.3.1). However, the latter for absolutely integrable functions is exactly the Lebesgue integral ([5]).

3.5. **Alternative approaches.** The above results suggest that WHBU is essential to the development of the measure theory, the Lebesgue measure and integral in particular. To assuage any fears that these results depend on our choice of framework, we now consider a very different framework for the Lebesgue integral, and show that WHBU is essential there too.

One alternative framework is the **gauge integral** ([43, 80]) restricted to bounded functions (Sections 3.5.1 and 3.5.2). Basic properties of the (general) gauge integral were shown in [43, §3.3] to be equivalent to HBU. We discuss further applications of our alternative approach in Remark 3.3.1 including topological entropy

Finally, to drive home the point that WHBU emerges everywhere in integration theory, we establish the following result in Section 3.5.3 the monotone convergence theorem for nets of functions and the Riemann integral implies WHBU.

3.5.1. **Restricting the gauge integral.** The **gauge integral** is a generalisation of the Lebesgue and improper Riemann integral; it was introduced by Denjoy (in a different from) around 1912 and developed further by Lusin, Perron, Henstock, and Kurzweil ([80]). The definition of the gauge integral in Definition 3.28 is highly similar to the Riemann integral (and simpler than Lebesgue’s integral), but boasts a maximal ‘closure under improper integrals’, known as Hake’s theorem ([5, p. 195]).
The aforementioned scope and versatility of the gauge integral comes at a non-trivial ‘logical’ cost: as established in [48 §3], HBU is equivalent to many basic properties of the gauge integral, including uniqueness. The additivity of the gauge integral also requires discontinuous functions on \( \mathbb{R} \), and the resulting system is at the level of ATR\(_0\) by [46 Cor. 6.7] and [48 Theorem 3.3]. It is then a natural question if for natural sub-classes of functions, a weaker system, e.g. at the level of ACA\(_0\), suffices to develop the associated restricted gauge integral.

The positive answer to this question starts with a fundamental result, namely that for bounded \( f \) on bounded intervals, the following are equivalent: \( f \) is measurable, \( f \) is gauge integrable, and \( f \) is Lebesgue integrable ([9 p. 94]). Thus, the bounded functions on \([0, 1]\) constitute a sub-class with natural properties. Furthermore, the Riemann sum of bounded functions is ‘well-behaved’: the former sum does not vary much\(^{13}\) if we change the function on a small sub-interval. Hence, we may weaken HBU to only apply to ‘most’ of \( I \), which is exactly WHBU: the latter expresses that we have a finite sub-cover of any canonical cover, for ‘most’ of \( I \), i.e. a subset of measure \( 1 - \varepsilon \) for any \( \varepsilon > 0 \).

The previous discussion leads to the following definition. For brevity, we assume bounded functions on \( I \) to be bounded by 1. The crucial and (to the best of our knowledge) new concepts are ‘\( \varepsilon, \delta \)-fine’ and the \( L \)-integral in items \( \text{vii} \) and \( \text{viii} \). All other notions are part of the (standard) gauge integral literature (see e.g. [13]).

**Definition 3.28.** [Integrals]

(i) A gauge on \( I \equiv [0, 1] \) is any function \( \delta : \mathbb{R} \to \mathbb{R}^+ \).

(ii) A sequence \( P := (t_0, I_0, \ldots, t_k, I_k) \) is a tagged partition of \( I \), written ‘\( P \in \text{tp} \)’, if the ‘tag’ \( t_i \in \mathbb{R} \) is in the interval \( I_i \) for \( i \leq k \), and the \( I_i \) partition \( I \).

(iii) If \( \delta \) is a gauge on \( I \) and \( P = (t_i, I_i)_{i \leq k} \) is a tagged partition of \( I \), then \( P \) is \( \delta \)-fine if \( I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \) for \( i \leq k \).

(iv) If \( \delta \) is a gauge on \( I \) and \( P = (t_i, I_i)_{i \leq k} \) is a tagged partition of \( I \) and \( \varepsilon > 0 \), then \( P \) is \( \varepsilon, \delta \)-fine if \( \cup_{i=0}^k \tilde{I}_i \) has measure at least \( 1 - \varepsilon \), where \( \tilde{I}_i \) is \( I_i \) if \( I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \), and empty otherwise.

(v) For a tagged partition \( P = (t_i, I_i)_{i \leq k} \) of \( I \) and any \( f \), the Riemann sum \( S(f, P) = \sum_{i=0}^{n} f(t_i)I_i \) of \( f \) with \( P \), while the mesh \( \|P\| \) is \( \max_{i \leq n} |I_i| \).

(vi) A function \( f : I \to \mathbb{R} \) is Riemann integrable on \( I \) if there is \( A \in \mathbb{R} \) such that

\[
(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \text{tp})(\|P\| \leq \varepsilon \delta \to |S(f, P) - A| <_{\mathbb{R}} \varepsilon).
\]

(vii) A function \( f : I \to \mathbb{R} \) is gauge integrable on \( I \) if there is \( A \in \mathbb{R} \) such that

\[
(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \text{tp})(\|P\| \leq \varepsilon \delta \to |S(f, P) - A| <_{\mathbb{R}} \varepsilon).
\]

(viii) A bounded \( f : I \to \mathbb{R} \) is \( L \)-integrable on \( I \) if there is \( A \in \mathbb{R} \) such that

\[
(\forall \varepsilon > 0)(\exists \delta : \mathbb{R} \to \mathbb{R}^+)(\forall P \in \text{tp})(P \text{ is } \varepsilon, \delta \text{-fine} \to |S(f, P) - A| <_{\mathbb{R}} \varepsilon).
\]

(ix) A gauge modulus for \( f \) is a function \( \Phi : \mathbb{R} \to (\mathbb{R} \to \mathbb{R}^+) \) such that \( \Phi(\varepsilon) \) is a gauge as in items \( \text{vii} \) and \( \text{viii} \) for all \( \varepsilon > 0 \).

The real \( A \) from items \( \text{vii} \) and \( \text{viii} \) in Definition 3.28 is resp. called the Riemann and gauge integral. We always interpret \( \int_{a}^{b} f \) as a gauge integral, unless explicitly stated. We abbreviate ‘Riemann integration’ to ‘R-integration’, and the same for related notions. The real \( A \) in item \( \text{viii} \) is called the Lebesgue (or \( L \)) integral or restricted gauge integral due to the extra condition that \( f \) be bounded on \( I \).

\(^{13}\) Using the notions defined in Definition 3.28 if \( (\forall x \in I)(|f(x)| \leq d) \) for \( d \in \mathbb{N} \), then \( S(f, P) \) only varies \( \varepsilon/d \) if we change \( f \) in an interval \((a, b) \subseteq I \) such that \(|b - a| < \varepsilon|.|
Finally, using the Axiom of Choice, a gauge integrable function always has a

gauge modulus, but this is not the case in weak systems like $\text{RCA}_0^\omega$. However, to

establish the Cauchy criterion for gauge integrals as in [45 §3.3], a gauge modulus is essential.

For this reason, we sometimes assume a gauge modulus when studying

the RM of the gauge integral in Section 3.5.2. Similar ‘constructive enrichments’ exist in second-order

RM, as established by Kohlenbach in [30 §4].

3.5.2. Reverse Mathematics of the restricted gauge integral. We show that basic

properties of the L-integral imply (or are equivalent to) WHBU as follows. We have

based this development on Bartle’s introductory monograph [5] and [48, §3.5.2].

First of all, we show that WHBU is equivalent to the uniqueness of the L-integral,

to the fact that the latter extends the R-integral. Note that the names of the

two items in the theorem are from [5 p. 13-14]. Also note that a Riemann integrable

function is bounded, even in $\text{RCA}_0^\omega$.

**Theorem 3.29.** Over $\text{ACA}_0^\omega$, the following are equivalent to WHBU:

(i) **Uniqueness:** If a bounded function is L-integrable on $[0,1]$, then the L-

integral is unique.

(ii) **Consistency:** If a function is R-integrable on $[0,1]$, then it is L-integrable

there, and the two integrals are equal.

**Proof.** We prove WHBU $\implies$ (i) $\implies$ (ii) $\implies$ WHBU. To prove that WHBU implies

Uniqueness, assume the former, let $f$ be bounded and gauge integrable on $I$ and

suppose $f$ satisfies for $i = 1,2$ (where $A_i \in \mathbb{R}$) that:

$$(\forall \varepsilon > 0)(\exists \delta_i : \mathbb{R} \rightarrow \mathbb{R}^+) (\forall P \in \text{tp})(P \text{ is } \varepsilon-\delta_i\text{-fine } \implies |S(f, P) - A_i| < \varepsilon). \tag{3.10}$$

Fix $\varepsilon > 0$ and the associated $\delta_i : \mathbb{R} \rightarrow \mathbb{R}^+$ in (3.10) for $i = 1,2$. We define the

gauge $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ as $\delta(x) := \min(\delta_1(x), \delta_2(x))$. By definition, a partition which

is $\varepsilon-\delta\text{-fine}$, is also $\varepsilon-\delta_i\text{-finite}$ for $i = 1,2$. Now assume there is $P_0 \in \text{tp}$ which is

$\varepsilon-\delta\text{-finite}$, and note that we obtain the following by applying (3.10):

$$|A_1 - A_2| = \mathbb{R} |A_1 - S(f, P_0) + S(f, P_0) - A_2| \leq \mathbb{R} |A_1 - S(f, P_0)| + |S(f, P_0) - A_2| \leq \varepsilon.$$ 

Hence, we must have $A_1 = A_2$, and Uniqueness follows. What remains is to prove that for every gauge $\delta$ there exists a $\varepsilon-\delta\text{-fine}$ tagged partition. We emphasise the crucial nature of this existence: (3.10) is vacuously true if there is no $\varepsilon-\delta\text{-finite}$ tagged partition; in other words: we can only make meaningful use of the conclusion of (3.10), if we show the existence of a $\varepsilon-\delta\text{-finite}$ tagged partition.

Thus, fix $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ and apply WHBU to $\bigcup_{x \in I} (x - \delta(x), x + \delta(x))$ to obtain $w = \langle y_0, \ldots, y_k \rangle$ in $I$ such that the measure of $\bigcup_{n=0}^\kappa I_{y_n}^\delta$ is at least $1 - \varepsilon$. This finite sequence is readily converted into a tagged partition $P_0 := (z_j, I_j)_{j \leq l}$ (with $l \leq k$ and $z_j \in w$ for $j \leq l$) by removing overlapping segments and omitting redundant intervals ‘from left to right’. By definition, $z_j \in I_j \subset (z_j - \delta(z_j), z_j + \delta(z_j))$ for $j \leq l$, i.e. $P_0$ is $\varepsilon-\delta\text{-fine}$. While the previous two steps are straightforward, it should be noted that (i) WHBU is essential by the equivalences in the theorem, and (ii) to convert $w$ into a tagged partition, we need to compare real numbers (in the sense of deciding whether $x > \mathbb{R} 0$ or not) and this operation is only available in $\text{ACA}_0^\omega$.

To prove that Uniqueness implies Consistency, note that ‘$P$ is $\varepsilon-\delta\text{-fine}$’ follows from ‘$|P| \leq \delta$’ for the gauge $d_\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ which is constant $\delta > 0$, and any $\varepsilon > 0$. Rewriting the definition of Riemann integration with the first condition, we observe that an R-integrable function $f$ is also L-integrable (with a constant gauge $d_\delta$ for
every choice of \( \varepsilon > 0 \). The assumption *Uniqueness* then guarantees that \( A \) is the only possible L-integral for \( f \) on \( I \), i.e. the two integrals are equal.

To prove that *Consistency* implies WHBU, suppose the latter is false, i.e. there is \( \Psi_0 : \mathbb{R} \to \mathbb{R}^+ \) and \( \varepsilon_0 > 0 \) such that for all \( y_1, \ldots, y_k \in I \), the measure of \( \bigcup_{n=0}^{k} y_n \) is below \( 1 - \varepsilon_0 \). Note that the same property holds for all \( \varepsilon \leq \varepsilon_0 \). Now let \( f : I \to \mathbb{R} \) be R-integrable with R-integral \( A \in \mathbb{R} \). Define the gauge \( \delta_0 \) as \( \delta_0(x) := \Psi_0(x) \) and note that for any \( P \in \text{tp} \) and \( \varepsilon \leq \varepsilon_0 \), we have that \( P \) is not \( \varepsilon \)-\( \delta_0 \)-fine, as the tags of \( P \) would otherwise provided the reals \( y_i \) from WHBU. Hence, (3.11) below is vacuously true, as the underlined part is false:

\[
(\forall \varepsilon \in (0, \varepsilon_0])(\forall P \in \text{tp})(P \text{ is } \varepsilon\delta_0\text{-fine} \implies |S(f, P) - (A + 1)| < \varepsilon). \tag{3.11}
\]

However, (3.11) implies that \( f \) is L-integrable with L-integral \( A + 1 \), i.e. *Consistency* is false as the R and L-integrals of \( f \) differ. \( \square \)

The previous proof is similar to the related equivalence for HBU and *uniqueness* and *consistency* for the (unrestricted) gauge integral from [48 §3.3]. Other results in the latter section can be developed along the same lines with similar proofs. For this reason, we only mention these results without proof.

**Theorem 3.30.** Over ACA\(^0\) + QF-AC\(^{2,1}\), the following are equivalent to WHBU:

(i) There exists a bounded function which is not L-integrable with a modulus.

(ii) (Hake) If a bounded function \( f \) is L-integrable on \( I \) with modulus and R-integrable on \( [x, 1] \) for \( x > 0 \), the limit of R-integrals \( \lim_{x \to 0^+} \int_1^x f \) exists.

(iii) (weak Hake) If a bounded function \( f \) is L-integrable on \( I \) with modulus and R-integrable on \( [x, 1] \) for \( x > 0 \), the limit of R-integrals \( \lim_{x \to 0^+} \int_1^x f \) exists.

We point out that the function \( \kappa : I \to \mathbb{R} \) from [48 §3.3] is *unbounded*, i.e. the previous theorems do not apply. This function \( \kappa \) is used to show that HBU is equivalent to the existence of a gauge integrable function that is not Lebesgue integrable, i.e. for which the absolute value is not gauge integrable.

Finally, we discuss other possible applications of WHBU. We emphasise the speculative nature of the following remark.

**Remark 3.31** (Topological entropy). The notion of topological entropy \( h(\varphi) \) is introduced in [1] for a continuous function \( \varphi : X \to X \) and compact space \( X \). The number \( h(\varphi) \) is non-negative (possibly \(+\infty\) by [1 Ex. 3]) and crucially depends on open-cover compactness as follows: \( h(\varphi) \) is the supremum of \( h(\varphi, \mathfrak{A}) \) over all open covers \( \mathfrak{A} \) of \( X \). In turn, \( h(\varphi, \mathfrak{A}) = \lim_{n \to +\infty} \frac{1}{n} H(\mathfrak{A} \cup \varphi^{-1} \mathfrak{A} \cup \cdots \cup \varphi^{-n+1} \mathfrak{A}) \), where \( \mathfrak{A} \cup \mathfrak{B} = \{ A \cap B : A \in \mathfrak{A} \land B \in \mathfrak{B} \} \). Finally, \( H(\mathfrak{A}) = \log(N(\mathfrak{A})) \) is the entropy of the cover \( \mathfrak{A} \), where \( N(\mathfrak{A}) \) is the minimum number of sets in \( \mathfrak{A} \) that still cover \( X \). The similar notion of metric entropy is based on *partitions and distance* rather than the size of sub-covers. Hence, basic properties of metric entropy can be established in relatively weak systems (compared to say the hardness of HBU).

Moreover, it is not a leap of the imagination that basic properties of \( h(\varphi) \) imply HBU, even if \( X = [0, 1] \). The same holds for the *variational principle* that connects topological entropy to metric entropy (see e.g. [24]). To avoid the use of HBU, and the associated ‘explosion’\(^{14}\), one works with WHBU instead as follows: one defines \( H_0(\mathfrak{A}, \varepsilon) = \log(N_0(\mathfrak{A}, \varepsilon)) \) where \( N_0(\mathfrak{A}, \varepsilon) \) is the minimum number of sets

\(^{14}\)By [46 §6], the combination of HBU and (3.2) implies ATR\(_0\) over ACA\(^0\) + QF-AC\(^{2,1}\).
in $\mathfrak{A}$ such that the union has measure at least $1 - \varepsilon$. We then put $h_0(\varphi, \mathfrak{A}, \varepsilon) := \lim_{n \to +\infty} \frac{1}{n} H_0(\mathfrak{A} \cup \varphi^{-1}\mathfrak{A} \cup \cdots \cup \varphi^{-n+1}\mathfrak{A}, \varepsilon)$ and $h_0(\varphi, \varepsilon)$ is the supremum over covers $\mathfrak{A}$ of $X$. Assuming the supremum (involving the metric entropy) from the aforementioned variational principle is finite, $h_0(\varphi, \frac{1}{m})$ is a bounded increasing sequence, and hence $\lim_{n \to +\infty} h_0(\varphi, \frac{1}{m})$ exists. This limit seems a ‘worthy’ stand-in for $h(\varphi)$ when the latter is not well-defined (due to the absence of HBU).

3.5.3. Nets and the Riemann integral. Lest there be any doubt that WHBU is to be found everywhere in integration theory, we show in this section that the monotone convergence theorem for nets and the Riemann integral implies WHBU.

First of all, the notion of net is the generalisation of the concept of sequence to (possibly) uncountable index sets and any topological space. Nets were introduced about a century ago by Moore-Smith ([42]), who also proved e.g. the Bolzano-Weierstrass, Dini and Arzelà theorems for nets. The RM-study of these theorems may be found in [65–67], and each of them implies HBU. Moreover, only nets indexed by subsets of Baire space are used for these results, i.e. a ‘step up’ from sequences gives rise to HBU, and the same for this paper by Definition 3.33.

**Definition 3.32.** [Nets] A set $D \neq \emptyset$ with a binary relation ‘$\leq$’ is directed if

(a) The relation $\leq$ is transitive, i.e. $(\forall x, y, z \in D)((x \leq y \land y \leq z) \rightarrow x \leq z)$.

(b) The relation $\leq$ is reflexive, i.e. $(\forall x \in D)(x \leq x)$.

(c) For $x, y \in D$, there is $z \in D$ such that $x \leq z \land y \leq z$.

For such $(D, \leq)$ and topological space $X$, any mapping $x : D \to X$ is a net in $X$.

Since nets are the generalisation of sequences, we write $x_d$ for $x(d)$ to emphasise this connection. The relation ‘$\leq$’ is often not explicitly mentioned; we write ‘$d_1, \ldots, d_k \succeq d'$ for $(\forall i \leq k)(d_i \succeq d)$. We shall only consider nets indexed by subsets of $\mathbb{N}^\omega$, as follows.

**Definition 3.33.** [Directed sets and nets in RCA$_0^\omega$] A ‘subset $D$ of $\mathbb{N}^\omega$ is given by its characteristic function $F_D^2 \leq_2 1$, i.e. we write ‘$f \in D$’ for $F_D(f) = 1$ for any $f \in \mathbb{N}^\omega$. A ‘binary relation $\preceq$ on a subset $D$ of $\mathbb{N}^\omega$ is given by the associated characteristic function $G_{\preceq}^{(\mathbb{N} \times \mathbb{N}) \to \mathbb{N}}$, i.e. we write ‘$f \preceq g$’ for $G_{\preceq}(f, g) = 1$ and any $f, g \in D$. Assuming extensionality on the reals as in item (v) of Definition 2.1, we obtain characteristic functions that represent subsets of $\mathbb{R}$ and relations thereon. Using pairing functions, it is clear we can also represent sets of finite sequences (of real numbers), and relations thereon.

Thus, a net $x_d : D \to \mathbb{R}$ in RCA$_0^\omega$ is nothing more than a type $1 \to 1$-functional with extra structure on its domain $D \subseteq \mathbb{N}^\omega$ provided by $\preceq$. The definitions of convergence and increasing net are as follows, and now make sense in RCA$_0^\omega$.

**Definition 3.34.** [Convergence of nets] If $x_d$ is a net in $X$, we say that $x_d$ converges to the limit $\lim_d x_d = y \in X$ if for every neighbourhood $U$ of $y$, there is $d_U \in D$ such that for all $e \succeq d_U$, $x_e \in U$.

It goes without saying that for nets of functions $f_d : (D \times [0, 1]) \to \mathbb{R}$, properties of $f_d(x)$ like continuity pertain to the variable $x$, while the net is indexed by $d \in D$.

**Definition 3.35.** [Increasing net] A net $f_d : (D \times I) \to \mathbb{R}$ is increasing if $a \leq b$ implies $f_a(x) \leq_R f_b(x)$ for all $x \in I$ and $a, b \in D$. 

We formulate the monotone convergence theorem $\text{MCT}_\text{net}$ without measure theory, i.e. the Riemann integral is used. As it happens, $\text{MCT}_\text{net}$ is a special case of [2, 19.36] where the limit function is not assumed to be continuous. Bourbaki proves a stronger version in [7, IV.1, Theorem 1, p. 107].

**Principle 3.36 ($\text{MCT}_\text{net}$).** For continuous $f_d : (D \times I) \rightarrow \mathbb{R}$ forming an increasing net such that (the limit) $f = \lim_d f_d$ exists pointwise and is bounded and continuous, we have that $\lim_d \int_0^1 |f(x) - f_d(x)| \, dx = 0$.

Note that we need $\text{WWKL}$ to guarantee that the integral in $\text{MCT}_\text{net}$ exists, in light of [63, Theorem 10]. Arzelà already studied the monotone convergence theorem (involving sequences) for the Riemann integral in 1885, and this theorem is moreover proved in e.g. [83] using HBU.

**Theorem 3.37.** The system $\text{RCA}_0^\omega + \text{WWKL} + \text{MCT}_\text{net}$ proves WHBU.

**Proof.** In case $\neg (\exists^2)$, all $F : \mathbb{R} \rightarrow \mathbb{R}$ are continuous by [31, Prop. 3.12]. Hence, $\mathbb{Q} \cap [0,1]$ provides a countable sub-cover for the canonical cover corresponding to $\Psi : E \rightarrow \mathbb{R}^+$. By [70, X.1], $\text{WWKL}$ yields the sub-cover required for $\text{WHBU}$.

In case $(\exists^2)$, suppose $\neg \text{WHBU}$, i.e. there is some $\Psi : [0,1] \rightarrow \mathbb{R}^+$, and $\varepsilon_0 > \mathbb{R}$, such that for all $y_0, \ldots, y_n \in I$, the measure of $\bigcup_{i=0}^n f_{y_i}^\Psi$ is always below $1 - \varepsilon_0$. Now let $D$ be the set of finite sequences of reals in $I$ (without repetition) and define `$v \preceq u$’ for $w, v \in D$ if $(\forall i < |v|) (v(i) \in w)$. Clearly, $\preceq$ is transitive and reflexive, and also satisfies item (c) in Definition 3.32.

Now define $f_w : I \rightarrow \mathbb{R}$ as follows: if $w = \langle x \rangle$ for some $x \in I$, then $f_w$ is 0 outside of $I_x^\Psi$, while inside the latter, $f_w(x)$ is the piecewise linear function that is 1 at $x$, and 0 in $x + \Psi(x)$. If $w$ is not a singleton, then $f_w(x) = \max_{i < |w|} f_{\langle w(i) \rangle}(x)$.

Then $f_w$ is increasing (in the sense of Definition 3.35) and converges to the constant one function (in the sense of Definition 3.34), as for any $v \succeq \langle x \rangle$, we have $f_v(x) = 1$. Now, $\lim_w \int_0^1 f_w(x) \, dx = 1$ by $\text{MCT}_\text{net}$ and consider $1_w(x) \geq_R f_w(x)$, where the erstwhile is the indicator function for $\bigcup_{i < |w|} I_{\Psi(w(i))}$ (and Riemann integrable on $I$). Hence, there is $v_0$ such that $\int_0^1 1_{v_0}(x) \, dx > 1 - \varepsilon_0$, and as the left-hand side is the measure of $\bigcup_{i=0}^{|v_0|-1} f_{\Psi(v_0(i))}$, we obtain a contradiction. Hence $\text{WHBU}$ also follows in case $(\exists^2)$, and we are done.

Since the ECF-translation of $\text{MCT}_\text{net}$ readily follows from $\text{WWKL}$, we cannot obtain HBU from this convergence theorem. An equivalence $\text{MCT}_\text{net} \iff \text{WHBU}$ seems desirable, but we do not know a proof at this point. The absence of almost any structure on the index sets in $\text{MCT}_\text{net}$ is perhaps the cause of all difficulties.

### 4. Computability theory and measure theory

In this section, we study realisers for $\text{WHBU}$ in computability theory. In particular, we construct such a realiser, denoted $A_S$, that does not add any extra power to the Suslin functional as in $(\exists^2)$, in contrast to the Heine-Borel theorem and the Lindelöf lemma. We recall the definition of the Suslin functional:

\[ (\exists S^2 \leq_2 1)(\forall f^1) \left( \left( \exists g^1 \right)(\forall a^0) (f(\exists^0 n) = 0) \right) \iff S(f) = 0. \]  

\[ (S^2) \]

\[ \text{Suslin functional} \]
We introduce realisers for WHBU and some definitions in Sections 4.1 and 4.2. The construction of $\Lambda_S$ may be found in Section 4.3, as well as a proof that $\Lambda_S + S$ computes the same functions as the Suslin functional $S$. As an application, we show that $\Pi^1_1 \text{CA}_0 + \text{WHBU}$ does not prove HBU. We also introduce a new hierarchy for second-order arithmetic involving $\Theta$ and HBU in Section 4.4.

4.1. Introduction: WHBU and its realisers. We discuss the brief history of realisers for WHBU, list the associated definitions, and formulate the associated aim of this section in detail.

Now, the class of weak fan functionals, or simply $\Lambda$-functionals, was introduced in [46] and investigated further in [47]. This class arose in the study of a version of weak weak König’s lemma from Nonstandard Analysis, but minor variations of $\Lambda$-functionals also provide us with realisers of some classical theorems (not involving Nonstandard analysis) such as Vitali’s covering theorem for uncountable covers; see Section 3.2.1. We shall make use of the following definition. We recall Notation 2.8, in particular that for a finite sequence $\sigma \in \mathbb{N}^k$, ‘$f \in [\sigma]$’ means $\sigma = f^k$.

**Definition 4.1 (\Lambda-functional).** A functional $\Xi$ of type $2 \to (0 \to 1)$ is a $\Lambda$-functional if whenever $F : C \to \mathbb{N}$ we have that $\Xi(F) = \{f_i\}_{i \in \mathbb{N}}$ is a sequence in $C$ such that $\bigcup_{i \in \mathbb{N}} [\bar{f}_i F(f_i)]$ has measure 1.

Here $C$ is the Cantor space, identified with $\{0, 1\}^\mathbb{N} \subseteq \mathbb{N}^\mathbb{N}$. If $s$ is a finite binary sequence, we let $[s]$ be the set of extensions of $s$ in $C$, as before.

In [46] we proved the existence of a $\Lambda$-functional $\Lambda_0$ without using the Axiom of Choice. In [47] we showed that there is a $\Lambda$-functional $\Lambda_1$, called $\Lambda_\exists^2$ below, such that all elements in $C$ computable in $\exists^2 + \exists_2$ are also computable in $\exists^2$.

For $\Theta$ satisfying $SFF(\Theta)$ from Section 2.2, i.e. a realiser for the Heine-Borel theorem for uncountable covers, no such $\Theta$ is computable in $\Lambda_3^2$ and $\exists^2$ ([46],[48]). The aim of this section is to show that there is another $\Lambda$-functional, called $\Lambda_S$ and defined in (4.1), such that every function computable in $\Lambda_S$ and $S$ is computable from the Suslin functional $S$. Since the Superjump is computable in $S$ and any instance of $\Theta$ ([47] §4), it follows that no instance of $\Theta$ is computable in $\Lambda_S$ and $S$.

4.2. Background definitions and lemmas. In this section, we will introduce lemmas and concepts, mainly from [47], that are needed in Section 4.3.

**Definition 4.2.** We let $m$ be the standard product measure on $C = \{0, 1\}^\mathbb{N}$.

Since $C$ is trivially homeomorphic to any countable product of itself, we take the liberty to use $m$ as the measure of any further product of $C$ as well. We will use $A$, $B$ for such products and $X$, $Y$ and $Z$ for subsets of such products. All sets we (have to) deal with below are measurable, so we tacitly assume all sets are measurable. The following basic results of measure theory are used without reference.

**Proposition 4.3 (Basic measure theory).**

(a) If $X_n \subseteq A$ and $m(X_n) = 1$ for each $n \in \mathbb{N}$, then $m(\bigcap_{n \in \mathbb{N}} X_n) = 1$.

(b) If $X \subseteq A \times B$ has measure 1, then $m(\{x \in A \mid m(\{y \in B \mid (x, y) \in X\}) = 1\}) = 1$.

We shall make use of the general machinery on measure-theoretic uniformity for $S$ from [43], summarised in Definition 4.4 and Proposition 4.5. Our construction
of ΛS in Section 4.3 will be an adjustment of the construction of Λ_{32} from [47] to
the computability theory of S. The technical details of the constructions of Λ_{32}
and ΛS are quite similar, and we refer the interested reader to [47]; we shall rather
focus on the underlying intuition.

**Definition 4.4 (C-sets).** Let seq be the set of finite sequences of integers.

(a) A Suslin scheme on a set X is a map s ↦ P_s sending s ∈ seq to P_s ⊆ X.

(b) If P = \{P_s\}_{s ∈ \text{seq}} is a Suslin scheme, then define A(P) = \bigcup_{f ∈ \mathbb{N}^f} \bigcap_{n ∈ \mathbb{N}} P_{f_n}.

The functional A is known as the Suslin operator.

(c) The C-sets in \mathbb{N}^N and related spaces are the elements of the least set algebra
containing the open sets and being closed under the Suslin operator.

The notion of C-set was first introduced in [22] and is also studied in e.g. [13, 20, 74], a fact unknown to the authors before the beginning of 2020. Indeed, in
previous versions of this paper and [45], C-sets were called ‘Suslin sets’. All C-sets
have codes in \mathbb{N}^N in analogy with the coding of Borel sets and the set of such codes
has \Pi^1_1\text{-complexity. The next proposition is proved in detail in [45].}

**Proposition 4.5.**

(a) If A ⊆ \mathbb{N}^N is a C-set, then A is computable in S uniformly in any code for A.

(b) If A ⊆ \mathbb{N}^N is computable in S and f with computation-ranks bounded by the
countable ordinal \alpha, then A is a C-set and there is a code for A computable
in S, f and any \mathbb{N}_0\text{-code for } \alpha.

(c) If A ⊆ C is a C-set, then m(A) is computable in S and a code for A.

(d) If A ⊆ C is computable in S and m(A) > 0, then A contains an element computable in S. This basis theorem can be relativised to any f ∈ \mathbb{N}^N.

(e) The algebra of C-sets is a \sigma\text{-algebra, i.e. closed under countable unions and complements, and thus contains the Borel sets.}

We let \omega_1^{S, g} be the first ordinal that is not computable in S and g, while \omega_1^S is
\omega_1^{S, 0}. We also need the following result from [45].

**Proposition 4.6.** The set \{g ∈ C : \omega_1^S = \omega_1^{S, g}\} has measure 1.

Unless specified otherwise, the sets X, Y, Z considered below are computable in
S, possibly from parameters and at a countable level, i.e. they are C-sets. Without
pointing this out every time, we make use of the following result from [45].

**Proposition 4.7.** If A and B are measure-spaces as above and X ⊆ A × B is
a C-set, then \{x ∈ A | m(\{y ∈ B | (x, y) ∈ X\}) = 1\} is a C-set with a code
computable from S and any code for X.

**Notation 4.8.** The following notational conventions are used below:

(a) a, b, c are numerical arguments or values in computations, while \vec{a}, \vec{b}, \vec{c} are
finite sequences of such.

(b) i, j, n, m are integers for other purposes, such as indexing.

(c) f, g, h are elements of C, with finite sequences denoted \vec{f}, \vec{g}, \vec{h}.

(d) (f) denotes an infinite sequence (f) = \{f_n\}_{n ∈ \mathbb{N}} from C.

(e) Given (f) and ε = (c_0, . . . , c_{n-1}), we define (f)_ε := (f_{c_0}, . . . , f_{c_{n-1}}).

**Definition 4.9.** Let F : Y → \mathbb{N} where Y ⊆ C, and let (f) be as above.
Proof. We let \( S = \text{semi-computable in} \ S \). There is a well-ordering \( \prec \) when we have fixed \( f \). We let \( || \cdot || \) with the norm. This ordering has the desired properties. □

Note that we use commas to denote concatenations of finite sequences from \( 4.3 \). The construction of a weak \( \Lambda \)-functional. We construct the \( \Lambda \)-functional \( \Lambda S \) and show that \( \Lambda S + S \) computes the same functions as \( S \).

The following partial ordering is crucial to our construction of \( \Lambda S \).

**Lemma 4.10.** Let \( A \) be a finite product of \( C \) and let \( \bar{g} \) range over the elements of \( A \). Let \( c \) be a non-repeating sequence of integers of length \( k \), and let \( F : Z \to \mathbb{N} \) where \( Z \subseteq C \times C^k \times A \) is a measurable set. If \( Y \subseteq C^k \times A \) has measure 1, then

(a) the set of \( \langle f, \bar{g} \rangle \) such that \( \langle f, \bar{g} \rangle \in Y \) has measure 1,

(b) the measure of the following set is 1: the set of \( \langle f, \bar{g} \rangle \) such that either

\[
\text{m}(\{ f \mid f, \langle f, \bar{g} \rangle \in Z \}) = 1 \quad \text{and} \quad (f) \text{ is sufficient for } \lambda f.F(f, \langle f, \bar{g} \rangle),
\]

or

\[
\text{m}(\{ f \mid f, \langle f, \bar{g} \rangle \in Z \}) < 1 \quad \text{and} \quad (f) \text{ fails } \lambda f.F(f, \langle f, \bar{g} \rangle).
\]

The conclusions of the lemma do not change if we restrict \( f \) to sequences from a subset \( X \) of \( C \) of measure 1. The requirement that \( c \) is non-repeating is essential, since otherwise the set of \( \langle f, c \rangle \) will have measure 0.

**4.3. The construction of a weak \( \Lambda \)-functional.** We construct the \( \Lambda \)-functional \( \Lambda S \) and show that \( \Lambda S + S \) computes the same functions as \( S \).

The following partial ordering is crucial to our construction of \( \Lambda S \).

**Lemma 4.11.** There is a well-ordering \( \langle A, \prec \rangle \) of a subset of \( \mathbb{N} \) of order type \( \omega_1^S \), semi-computable in \( S \), such that for each \( a \in A \), \( \{ \langle b, c \rangle \mid b \prec c \prec a \} \) is computable in \( S \), uniformly in \( a \).

*Proof.* We let \( A \) be the set of computation tuples \( a = \langle e, \bar{a}, b \rangle \) such that \( \langle e \rangle(S, \bar{a}) = b \) with the norm \( || \cdot || \), and we let \( a \prec a' \) if \( ||a|| < ||a'|| \), or if \( ||a|| = ||a'|| \) and \( a < a' \). This ordering has the desired properties. □

We introduce some more notation.

**Notation 4.12.** We let \( [f] \) denote a family \( [f] = \{ (f_a) \}_{a \in A} = \{ \{ f_{a,i} \}_{i \in N} \}_{a \in A} \). When we have fixed \( [f] \), and \( a \in A \), we let \( [f]_a \) be \( [f] \) restricted to \( \{ b \in A \mid b \preceq a \} \) and we let \( [f]_{< a} \) be \( [f] \) restricted to \( \{ b \in A \mid b \prec a \} \).

**Definition 4.13.** Let \( [f] \) be as above. We define \( \Lambda_{[f]} \) as the partial functional, accepting partial functionals of type 2 as inputs, as follows: \( \Lambda_{[f]}(F) := \{ f_{a,i} \}_{i \in N} \) if

(a) \( (f_a) \) is sufficient for \( F \),
Similarly, \( \Lambda_{[f]_a} \) and \( \Lambda_{[f]_{<a}} \) are defined by replacing \([f]\) with \([f]_a\) or \([f]_{<a}\).

Since the specification for a \( \Lambda \)-functional only specifies the connection between \( F \) and \( \Lambda(F) \), and does not relate \( \Lambda(F) \) and \( \Lambda(G) \) for different \( F \) and \( G \), and since there is at least one \( \Lambda \)-functional \( \Lambda_0 \) (see [46]), functionals of the form \( \Lambda_{[f]} \) can be extended to total \( \Lambda \)-functionals. We trivially have the following.

**Lemma 4.14.** Let \( [f] \) be as above and also partially computable in \( S \). Then every function \( g \) computable in \( \Lambda_{[f]} \) is also computable in \( S \).

To obtain our main results, we must construct \([f]\) such that any function \( g \) computable in any total extension of \( \Lambda_{[f]} \) is still computable in \( S \). By convention, we let \( C^0 \) consist of a singleton, the empty sequence, with measure 1.

The following two lemmas are closely related to resp. [47] Lemmas 3.29 and 3.30. Indeed, the proof of Lemma 4.15 proceeds via a line-to-line translation from the \( \omega^C \)-recursion in the proof of [47] Lemma 3.29 to an \( \omega^S \)-recursion. Since the proof is long and technical with no new additions, we restrict ourselves to an outline of the proof here. Whenever the proof makes use of [47] Proposition 3.23, the proof of Lemma 4.15 makes use of Proposition 4.5 instead. Since the proof of Lemma 4.16 is based on details in the proof of Lemma 4.15 we refer to [47] Lemma 3.30 for the exact argument.

In the formulation of the latter lemma, we have taken the (notational) liberty to ignore other ways of listing the inputs. There is no harm in this since we may always use Kleene's S6 to permute inputs. Our motivation is that stating and proving the general result will be much more cumbersome, but all genuine mathematical obstacles are however gone.

**Lemma 4.15.** By \( S \)-recursion on \( a \in A \), we can construct \([f]\) = \( \{ (f_a) \}_{a \in A} \) and sets \( X_{a,k} \subseteq C^k \) of measure 1 (for each \( k \in \mathbb{N} \) and \( a \in A \)) such that an alleged computation \( \{e\} (\Lambda_{[f]_a}, S, \vec{a}, \vec{h}, \vec{g}) \) will terminate whenever the parameters satisfy:

(a) \( a \in A \) has norm \( \alpha = ||a|| \), \( e \) is a Kleene-index, \( \vec{a} \in \text{seq} \), and \( \vec{g} \in X_{a,k} \),

(b) \( \vec{h} \) is a sequence from \( \{f_{b,i} \mid i \in \mathbb{N} \land b \leq a \} \),

(c) there is some extension \([f']\) of \([f]_a\) such that \( \{e\} (\Lambda_{[f']}, S, \vec{a}, \vec{h}, \vec{g}) \downarrow \) with a computation of ordinal rank at most \( \alpha \).

**Proof.** (Outline)

- Given \( a \in A \), we assume that \([f]_{<a}\) is constructed, that all sets \( X_{b,k} \) are constructed for \( b < a \), that the induction hypothesis holds and that what is constructed so far, is computable in \( S \).
- The main step is, for each \( k \), to construct a \( C \)-set \( Z_{a,k} \subseteq C^\omega \times C^k \) of measure 1 such that for all \( \langle (f), \vec{g} \rangle \in Z_k \), the property stated in the Lemma will hold for \( \vec{g} \) and \( a \), except the requirement that \( (f) \) is computable in \( S \), if we extend \([f]_{<a}\) with \((f)\).
- We can then use Proposition 4.3, Proposition 4.5 and Gandy selection to find an \((f_a)\) computable in \( S \) such that the following set has measure one:

\[
X_{a,k} = \{ g \in C^k \mid \langle (f_a), \vec{g} \rangle \in Z_{a,k} \}. 
\]
• The set $Z_{a,k}$ is the intersection of countably many sets needed to handle each case given by the index $e$ and by how the sequence $\vec{h}$ is selected from $[f]_{<a}$ and $(f_a)$. We use Lemma 4.11 to handle the cases corresponding to Kleene’s $S\bar{S}$, ensuring that we can use $(f_a)$ as the value of $\Lambda_S(F)$ for all $F$ semi-computable in $S$ and $\Lambda_S$ and total on a set of measure 1 exactly at stage $|[a]|$.
• The whole construction is tied together using the recursion theorem for computing relative to $S$

This finishes the proof of the lemma.

For each $k \in \mathbb{N}$, define $X_k = \bigcap_{a \in A} X_{a,k}$. These sets are $C$-sets, but not with a code computable in $S$. They are complements of sets semi-computable in $S$. For a proof of the following lemma, see (the proof of) [45 Lemma 3.27].

**Lemma 4.16.** For each $k$ and $\vec{g} \in X_k$ we have $\mathbf{m}(\{ g \mid g, \vec{g} \in X_{k+1} \}) = 1$.

Let $\Lambda_0$ be the $\Lambda$-functional constructed in [46], and let $[f]$ be as constructed in the proof of Lemma 4.15. We define $\Lambda_S$ as follows:

$$
\Lambda_S(F) = \begin{cases} 
\Lambda_{f_i}(F) & \text{if defined} \\
\Lambda_0(F) & \text{otherwise}
\end{cases},
$$

and prove our main theorem as follows.

**Theorem 4.17.** If $f : \mathbb{N} \to \mathbb{N}$ is computable in $\Lambda_S + S$, then it is computable in $S$.

**Proof.** We shall prove the stronger claim (4.2) below by induction on the length of the computation. We need some notation as follows. Let $e$ be a Kleene index, let $a$ be a sequence from $\mathbb{N}$, let $\vec{h}$ be a sequence from $[f]$, and let $g, \vec{g}$ of length $k$ be a sequence from $\bigcap_{a \in A} X_{a,k}$ such that $\omega^S_e = \omega^S_{e, \vec{g}, \vec{g}}$. By Proposition 4.6, the final restriction does not alter the measure of the set. Now consider the following claim:

$$
\{ e \}(\Lambda_S, S, a, \vec{h}, \vec{g}) = \Lambda_S(\{ e \}(\Lambda, S, a, \vec{h}, \vec{g})) = b \rightarrow (\exists a \in A)(\{ e \}(\Lambda_{[f]_a}, S, a, \vec{h}, \vec{g}) = b)
$$

(4.2)

The theorem follows from the claim (4.2) and the total instances $\lambda c.\{ e \}(\Lambda_S, c)$.

We now prove the claim (4.2) by induction on the ordinal rank of the computation $\{ e \}(\Lambda_S, S, a, \vec{h}, \vec{g}) = b$. The proof is split into cases according to which Kleene-scheme $e$ represents, and all cases except those for application of $S$ or $\Lambda_S$ are trivial. Thus, we (only) consider the two cases (4.3) and (4.4). First, we consider

$$
\{ e \}(\Lambda_S, S, a, \vec{h}, \vec{g}) = S(\lambda s.\{ e_1 \}(\Lambda_S, S, s, \vec{a}, \vec{h}, \vec{g})),
$$

(4.3)

which yields the following by the induction hypothesis:

$$
(\forall s \in \text{seq})(\exists a \in A)\{(e_1)(\Lambda_{[f]_a}, S, s, \vec{a}, \vec{h}, \vec{g})\}]
$$

Since $\omega^S_e$ is $\vec{g}$-admissible, there is a bound on how far out in $A$ we need to go, i.e. $(\exists a \in A)(\forall s \in \text{seq})\{(e_1)(\Lambda_{[f]_a}, S, s, \vec{a}, \vec{h}, \vec{g})\}]]$, and $\{ e \}(\Lambda_{[f]_a}, S, \vec{a}, \vec{h}, \vec{g})$ follows.

For the second case, consider

$$
\{ e \}(\Lambda_S, S, a, \vec{h}, \vec{g}) = \Lambda_S(\lambda g.\{ e_1 \}(\Lambda_S, S, a, \vec{h}, g, \vec{g})).
$$

(4.4)

By Lemma 4.10 and the induction hypothesis, for almost all $g$ there is an $a_g \in A$ such that $\{ e_1 \}(\Lambda_{[f]_{a_g}}, S, \vec{a}, \vec{h}, g, \vec{g})]$. Now consider the sequence

$$
a \rightarrow \mathbf{m}(\{ g \mid e_1 \}_|[a]|(\Lambda_{[f]_a}, S, \vec{a}, \vec{h}, g, \vec{g})),
$$
that all terms in Gödel’s Tµ extra constants for total functionals. Recall that maximal type-structure of total functions, and that this is also the case when we use $\|a\|$. We need the following definition.

Moreover, we may assume that $\vec{h}$ is in $[f]_a$. Hence, our construction guarantees that $(f_b)$ is sufficient for $\lambda g.\{e_1\} (\Lambda_S, \vec{a}, \vec{h}, g, \vec{g})$, unless some $(f_b)$ already does the job for $b \prec a$. We may conclude that

$$\{e\} (\Lambda_S, \vec{a}, \vec{h}, \vec{g}) = (f_b) = \{e\} (\Lambda_{[f]_a}, \vec{a}, \vec{h}, \vec{g})$$

for some $b \leq a$. This ends the induction step, and we are done. □

As an application, we now use the functional $\Lambda_S$ to construct a model for $\Pi^1_1$-CA$^\omega_0 + \text{WHBU}$ in which $\neg \text{HBU}$ holds. We make use of the well-known fact that all terms in Gödel’s T have set-theoretical interpretations as elements of the maximal type-structure of total functions, and that this is also the case when we use extra constants for total functionals. Recall that $\mu^2$ is Feferman’s search operator from Section 2.2. We need the following definition.

**Definition 4.18.**

(a) Let $\text{SUS}$ be the set of functions computable in $S$.

(b) For any type $\sigma$, let $\text{SUS}_\sigma$ be the set of functions of type $\sigma$ in the maximal type-structure definable via a term in Gödel’s T using constants for $\mu^2$, $S$, $\Lambda_S$ and elements in SUS.

(c) We define a partial equivalence relation $\sim_\sigma$ on $\text{SUS}_\sigma$ by recursion on $\sigma$:

(i) The relation $\sim_0$ is the identity relation on $N$

(ii) If $\sigma = \tau \rightarrow \delta$ and $F_1, F_2 \in \text{SUS}_\sigma$, we let $F_1 \sim_\sigma F_2$ if $\phi_1 \sim_\tau \phi_2 \rightarrow F_1 \phi_1 \sim_\delta F_2 \phi_2$ for all $\phi_1, \phi_2 \in \text{SUS}_\tau$.

By the properties of $\Lambda_S$, we observe that $\text{SUS}_{0 \rightarrow 0} = \text{SUS}$. We also observe that $\sim_{0 \rightarrow 0}$ is the identity relation on SUS. It is then easy to see that all $\sim_\sigma$ are partial equivalence relations, i.e. transitive and symmetric, and thus equivalence relations on the set of self-equivalent functionals $F$. We may then form the Hereditarily Extentional Collapse (HEC), intimately related to the Mostovski collapse from set theory. For numerous applications of the HEC, we refer to [38].

We state the following well-known fact without proof.

**Lemma 4.19.** Seen as a model for $\mathbb{Z}_2$, SUS satisfies $\Pi^1_1$-comprehension, and all $\Pi^1_1$-formulas with parameters from SUS are absolute for $(\text{SUS}, \mathbb{N}^\mathbb{N})$.

Our next lemma will be important for showing that applying the HEC indeed produces a model of RCA$^\omega_0$.

**Lemma 4.20.** If $F \in \text{SUS}_\sigma$, then $F \sim_\sigma F$.

**Proof.** The proof is by induction on the term defining $F$. More precisely, we have to prove that if $t$ is a term of type $\sigma$, with free variables among $x_1^1, \ldots, x_n^\sigma$, then the interpretation $[[t]]$, seen as an element of $\tau_1 \rightarrow \cdots \rightarrow \tau_n \rightarrow \sigma$, will be self equivalent. Due to the possible use of $\lambda$-closure, $[[t]]$ is in our SUS-hierarchy.

The induction base deals with the variables and constants, including all the recursor constants $R_\sigma$. Of these, it is only the constant for $\Lambda_S$ that is nontrivial (and
new in this case). The induction steps consist of terms formed by λ-abstraction and by application, and are trivial (and well-known). So, let \( F \) and \( G \) be in \( \text{SUS}_{(0^{\sim_0})} \) be \( \sim \)-equivalent, meaning, in this case, that they are equal on \( \text{SUS} \). Since they are \( T \)-definable from \( \Lambda_S \), \( S \), \( \mu^2 \) and elements from \( \text{SUS} \), they are in particular Kleene computable from \( \Lambda_S \) and \( S \). Hence, we have
\[
\Lambda_S(F) = \Lambda_{[f]}(F) = \Lambda_{[f]}(G) = \Lambda_S(G), \tag{4.5}
\]
where \([f]\) and \( \Lambda_{[f]} \) are as constructed in the proof of Lemma 4.15. In light of (4.5), we may conclude \( F \sim_{(0^{\sim_0})} G \) and the proof is finished. \( \Box \)

Finally, the following theorem readily follows.

**Theorem 4.21.** The system \( \Pi^1_1-\text{CA}_0^\omega + \text{WHBU} \) cannot prove \( \text{HBU} \).

**Proof.** We construct a model \( \mathcal{M} \) of \( \Pi^1_1-\text{CA}_0^\omega + \text{WHBU} + \neg \text{HBU} \). This model \( \mathcal{M} \) is the aforementioned *hereditarily extensional collapse* \( \{ \text{SUS}_\sigma / \sim_\sigma \}_\sigma \) type, seen as a type structure. Clearly, \( \mathcal{M} \) satisfies \( (S^2) \) as \( \text{SUS} \) is unchanged under the collapse; \( \mathcal{M} \) satisfies \( \text{RCA}_0^\omega \) since \( \mathcal{M} \) is a model of Gödel’s \( T \) and we use \( \mu^2 \) to ensure that \( \text{QF-AC}^{1,0} \) is satisfied. The model \( \mathcal{M} \) satisfies \( \text{WHBU} \) since the collapse of \( \Lambda_S \) is a realiser for \( \text{WHBU} \) within \( \mathcal{M} \).

Next, we show that \( \text{HBU} \) fails in \( \mathcal{M} \). To this end, we consider \([44 \text{ Theorem } 1.c]) \) which establishes that the functional \( \Gamma \), the realiser for non-monotone inductive definitions, is computable in any \( \Theta \)-functional and \( S^2 \). Now, Richter proves in \([59]\) that even inductive definitions given by arithmetical functionals \( F : C \to C \) (actually \( \Pi^0_3 \) suffices) can have closure ordinals beyond the first recursively Mahlo ordinal; these can therefore be used to construct \( f \in C \) not computable in \( S^2 \).

Let \( F : C \to C \) be one such arithmetical functional. In the proof of \([44 \text{ Theorem } 1.c] \), a functional \( G : C \to C \) is defined in terms of \( F, \mu \), and \( S \) via a term of Gödel’s \( T \). This functional defines an open covering of \( C \) and its key property is that whenever \( f_1, \ldots, f_n \) defines a finite sub-covering, then the set inductively defined from \( F \) is computable in \( f_1, \ldots, f_n, S \), and in fact definable from \( f_1, \ldots, f_n, S, \mu^2 \) by a term of Gödel’s \( T \). Since \( F \) is chosen as an arithmetical functional such that the fixed point of the associated inductive definition is not in \( \mathcal{M} \), while \( G \) is in \( \mathcal{M} \), we must have that one of \( f_1, \ldots, f_n \) is outside \( \mathcal{M} \), and \( \text{HBU} \) fails in the latter. \( \Box \)

We conjecture that \( Z^2_\omega + \text{WHBU} \) cannot prove \( \text{HBU} \), but have no (idea of a) proof.

### 4.4. Two new hierarchies relating to second-order arithmetic

We have previously shown that the combination \( \mu^2 + \Theta^3 \) computes a realiser for \( \text{ATR}_0 \), while the latter schema (not involving a realiser) is provable in \( \text{ACA}_0^\omega + \text{HBU} \) \([40, 48, 50]\).

We refer to this phenomenon as an ‘explosion’ as both components are weak (in isolation) compared to the combination. The aim of this section is to exhibit a number of similar explosions that provide a sweeping generalisation of the aforementioned results, yielding new hierarchies parallel to the usual hierarchy of second-order arithmetic based on comprehension. A similar parallel hierarchy is described in \([11]\), but based on the *axiom of determinacy* from set theory, while we work with the -more natural in our opinion- theorem \( \text{HBU} \) and its realiser \( \Theta \).

First of all, as expected, a central role is played by transfinite recursion, which we now define. Let \( \text{WO}(X) \) express that \( X \) is a countable well-ordering as in \([70 \ V.1.1]\). The following definition may be found in \([70 \ VI.7]\). To be absolutely clear, \( \theta \) below is part of \( L_2 \); no type two parameters are allowed.
Definition 4.22. \([\Pi^1_k\text{-}TR_0]\) For any \(\theta \in \Pi^1_k\) and \(X \subseteq \mathbb{N}\), we have

\[
\text{WO}(X) \rightarrow (\exists Y \subseteq \mathbb{N})H_\theta(X, Y),
\]

where \(H_\theta(X, Y)\) states that \(X\) is a linear ordering and that \(Y = \{(m, j) : j \in \text{field}(X) \land \theta(n, Y^j)\}\) for \(Y^j := \{(m, i) : i <_X j \land (m, i) \in Y\}\).

We note the unfortunate use of ‘\(\theta\)' for an \(L_2\)-formula, and ‘\(\Theta\)' for special fan functionals. By [76 Table 4], \(\Pi^1_k\text{-}CA_0\) is strictly between \(\Pi^1_k\text{-}CA_0\) and \(\Pi^1_{k+1}\text{-}CA_0\).

Secondly, we prove the following two theorems.

Theorem 4.23. Uniformly for each instance of \(\Theta^3\), there is a type three functional \(\text{TR}(F, A, <)\), where \(F : 2^\mathbb{N} \rightarrow 2^\mathbb{N}\) and \(<_A\) is a binary relation on \(A \subseteq \mathbb{N}\), such that if \((A, <_A)\) is a well-ordering and \(a \in A\) then

\[
\text{TR}(F, A, <_A)(a) = F\{\{(b, c) \in A \times A : b <_A a \land c \in \text{TR}(F, A, <_A)(b)\}\},
\]

i.e. \(\text{TR}(F, A, <_A)(a)\) is the result of iterating \(F\) along \((A, <_A)\) up to \(a\).

Proof. Immediate by [47 Cor. 3.16].

A realiser for \(\Pi^1_k\text{-}TR_0\) is a functional that takes as input \(X \subseteq \mathbb{N}\), a binary relation \(<_X\), and \(f : \mathbb{N} \rightarrow \mathbb{N}\), and outputs \(Y \subseteq \mathbb{N}\) such that \(H_\theta(X, Y)\) if \(\text{WO}(X)\) and \(\theta\) is the \(\Pi^1_k\)-formula in Kleene normal form with \(f(x_1, \ldots, x_k, n) = 0\) providing the innermost quantifier-free part.

Corollary 4.24. The combination \(S^2_k + \Theta\) computes a realiser for \(\Pi^1_k\text{-}TR_0\).

Proof. This follows from the theorem by recalling that \(S^2_k\) decides \(\Pi^1_k\)-formulas.

Finally, we have the following theorem.

Theorem 4.25. The system \(\Pi^1_k\text{-}CA_0^\omega + \text{HBU}\) proves \(\Pi^1_k\text{-}TR_0\).

Proof. Our proof proceeds via contradiction: fix \(X \subseteq \mathbb{N}\) such that \(\text{WO}(X)\) and suppose we have \((\forall Y \in \mathbb{N})(\exists k^0)\neg H_\theta(k, X, Y)\) for some \(L_2\)-formula \(\theta \in \Pi^1_k\). Here, \(H_\theta(i, X, Y)\) is just \(H_\theta(X, Y)\) with the additional restriction \(j <_X k\), as can be found in [76 V.2.2]. Clearly, \(H_\theta(X, Y)\) is decidable given \(S^2_k\) and \(\text{QF-AC}^{1,0}\) applied to \((\forall Y \subseteq \mathbb{N})(\exists k^0)\neg H_\theta(k, X, Y)\) yields some \(G : C \rightarrow \mathbb{N}\). Use \(\text{HBU}\) to obtain \(f_1, \ldots, f_{k_0} \in C\) such that \(\cap_{i \leq k_0} [\neg G(f_i)]\) still covers \(C\). We now note that given \(S^2_k\), we can always apply transfinite recursion ‘once more’, i.e. given \(k\) and \(Y\) such that \(H(k, X, Y)\), we can define \(Z \supseteq \mathbb{N}\) such that \(H(k', X, Z)\), where \(k'\) is the least number above \(k\) according to \(<_X\). The same holds for a finite number of iterations via \(\text{IND}\). Now consider \(k_1 := \max_{1 \leq k_0} G(f_i)\) and note that we have obtained a contradiction; \(\text{IND}\) can be avoided by letting \(G^2\) be the least number as above. \(\square\)

5. Discussion and Conclusion

We discuss two observations (Section 5.1) and some foundational musings (Section 5.2) pertaining to our results.

5.1. Two observations. We discuss the possibility for a template based on our results, and an interesting observation dubbed dichotomy phenomenon.
5.1.1. Towards a template. The proof of Theorem 4.17 is similar to a proof of the existence of $\Lambda^{\exists_2}$, to such an extent that we just gave reference to that proof for many of the technicalities. Based on the theories on measure-theoretic uniformity for recursion in $\exists^2$ and in $\mathbf{S}$, the main constructions follow the same pattern. The question is how much further this kind of construction could lead.

The results on measure-theoretic uniformity turned out to be quite similar for computability in $\exists^2$ and in $\mathbf{S}$. In each case, the measure theory of subsets of the continuum computable in $\exists^2$ or $\mathbf{S}$ can be handled within the class of functions computable in $\exists^2$ and $\mathbf{S}$ via suitable coding. It seems unlikely that something like this can be relativised to all functionals of type 2. In this light, we offer the following open problem.

**Problem 1.** Is there a functional $F$ of type 2 such that for all $\Lambda$-functionals $\Xi$ there is a $\Theta$-functional computable in $F$ and $\Xi$?

We conjecture the answer to be negative, but see no way to establish this.

5.1.2. A dichotomy phenomenon. The main result of this section is another example of a ‘dichotomy’ phenomenon that we have observed during the study of functionals arising from classical theorems, namely as follows.

On one hand, positive results about relative dependence are of the form that elements in one class of functionals can uniformly be defined from elements in another class of interest via a term in a small fragment of Gödel’s $T$. On the other hand, negative results are of the form that there is one element $\Phi$ in one class such that no element $\Psi$ in the other class is computable in $\Phi$ in the sense of Kleene, often even not relative to any object of lower type.

We find this to be an interesting observation, and a source for classification of the (computational) strength of theorems.

5.2. Foundational musings. We discuss the foundational implications of our results, which we believe to be rather significant and different in nature from [48, 50].

As noted above, the development of measure theory in ‘computational’ frameworks like e.g. Reverse Mathematics, constructive mathematics, and computable analysis, proceeds by studying the computational properties of countable approximations of measurable objects. To be absolutely clear, theorems in these fields are generally not about objects themselves, but about representations of objects. Of course, this observation is of little concern in general as there are ‘representation theorems’ that express that ‘nice’ representations always exist. Nonetheless, there are two conceptual problems that arise from our results, as follows.

First of all, in the particular case of RM, there is a potential problem with using representations: the aim of RM is to find the minimal axioms required to prove theorems of ordinary mathematics ‘as they stand’ (see [76, I.8.9.5] for this exact wording). Thus, the logical strength/hardness of a theorem should not change upon the introduction of representations, lest this distort the RM-picture! However, we have identified interesting theorems, i.e. the Vitali covering theorem and WHBU, for which the hardness changes quite dramatically upon introducing codes. Indeed, in terms of (conventional) comprehension, WHBU is not provable in $\mathbb{Z}_2^4$ but provable $\mathbb{Z}_2^5$: the same holds if we restrict to Baire 2 or semi-continuous functions (see Footnote 8). However, $\text{ATR}_0 + \Delta^1_4$-induction proves HBU (and hence WHBU) formulated via codes for Borel functions by [4, Prop. 3.2].
Secondly, there is another, more subtle, aspect to our results, namely pertaining to the formalisation of mathematics in second-order arithmetic. Simpson (and many others) claims that the latter can accommodate large parts of mathematics:

\[ \ldots \] focusing on the language of second order arithmetic, the weakest language rich enough to express and develop the bulk of mathematics. (76 Preface)

Let us first discuss a concept for which the previous quote is undeniably correct: continuous functions, which are represented by codes in RM (see 76 II.6.1)). Now, Kohlenbach has shown in 30 §4] that WKL suffices to prove that every continuous function on Cantor space has a code. Hence, assuming WKL, a theorem in L_ω about (higher-order) continuous functions (on Cantor space) does not really change if we introduce codes, i.e. there is a perfect match between the theorem expressed in L_2 and L_ω. In other words, second-order WKL (working in RCA_0) proves that the second-order formalisation has the same scope as the original. In conclusion, L_2 can talk about certain continuous functions via codes, and WKL (working in RCA_0) guarantees that the approach-via-codes actually is talking about all (higher-order) continuous functions. In this light, Simpson’s quote seems justified and correct.

Our above results paint a different picture when it comes to measure theory: on one hand, a version of measure theory can be expressed and developed in L_2 using representations of measurable objects, as sketched in 76 X.1]. On the other hand, if one wants the guarantee that the development in L_2 using representations has the same scope or generality as the original theory involving measurable objects, one needs to know that each measurable object has a representation. To this end, one of course points to well-known approximation theorems like Lusin’s. However, the latter implies WHBU and is hard to prove in terms of conventional comprehension. In conclusion, L_2 can talk about certain measurable functions via codes, but to know that the approach-via-codes actually is talking about all measurable functions requires WHBU and hence Z_2^4, both of which are not part of second-order arithmetic.

In conclusion, second-order arithmetic uses codes to talk about certain objects of a given (higher-order) class, like continuous or measurable functions. However, to know that the L_2-development based on codes has the same scope or generality as the original theory, one needs the guarantee that every (higher-order) object has a code. Second-order arithmetic can apparently provide this guarantee in the case of continuous functions, but not in the case of measurable functions. Put another way, proving that second-order arithmetic can fully express measure theory, seriously transcends second-order arithmetic.

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