Archimedean-based Marshall-Olkin Distributions and Related Copula Functions

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Abstract

A new class of bivariate distributions is introduced that extends the Generalized Marshall-Olkin distributions of Li and Pellerey (2011). Their dependence structure is studied through the analysis of the copula functions that they induce. These copulas, that include as special cases the Generalized Marshall-Olkin copulas and the Scale Mixture of Marshall-Olkin copulas (see Li, 2009), are obtained through suitable distortions of bivariate Archimedean copulas: this induces asymmetry, and the corresponding Kendall’s tau as well as the tail dependence parameters are studied.

Keywords: Marshall-Olkin distribution, Marshall-Olkin copula, Kendall’s function, Kendall’s tau, tail dependence parameters

1 Introduction

In their seminal paper (see Marshall and Olkin, 1967), the authors introduce the Marshall-Olkin distribution whose survival version is

$$\bar{F}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)\}$$

$x_1, x_2 \geq 0, \lambda_1, \lambda_2, \lambda_3 > 0$. This is the distribution of a bivariate random vector $(M_1, M_2)$ with $M_1 = \min(X_1, X_3)$ and $M_2 = \min(X_2, X_3)$, where $X_1, X_2, X_3$ are three independent and exponentially distributed random variables.

For an interpretation of the Marshall-Olkin distribution, consider a system with two components $C_1$ and $C_2$ (electronic elements, engines, credit obligors, life-insured married couples, etc.). The random variables $X_1$ and $X_2$ represent the arrival times of a shock causing the failure or default or death (depending on the case) of $C_1$ and $C_2$, respectively (idosyncratic shocks), while $X_3$ is the arrival time of a shock causing the simultaneous end of the lifetimes of both $C_1$ and $C_2$ (systemic shock).

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In order to generalize this model, the case in which the underlying random variables $X_1, X_2, X_3$ are assumed to have marginal distributions different from the exponential one has been considered in the existing literature. The most general results, in this direction, are those obtained in Li and Pellerly (2011), where no restriction is made on the marginal distributions of the random variables $X_1, X_2, X_3$ while joint independence is again assumed. More precisely, the authors assume for $(X_1, X_2, X_3)$ a joint survival distribution of type

\[ \bar{F}(x, y, z) = \exp\left(-\left(H_1(x) + H_2(y) + H_3(z)\right)\right) \]  

(1)

where the $H_i$ are the cumulative hazard functions of the random variables $X_i$. The corresponding joint survival distribution of the random variables $M_1$ and $M_2$ is

\[ \bar{F}(x_1, x_2) = \exp\left(-H_1(x_1) - H_2(x_2) - H_3(\max(x_1, x_2))\right) \]

that is called Generalized Marshall-Olkin distribution. These distributions incorporate, as special cases, the Marshall-Olkin type distributions introduced in Muliere and Scarsini (1987), the bivariate Weibull distributions introduced in Lu (1989) and the bivariate Pareto distributions introduced in Asimit et al. (2010). The authors analyze the dependence structure implied by these distributions introducing the corresponding copula functions, that they call Generalized Marshall-Olkin copulas.

However, because of the independence among the original shocks (systemic and idiosyncratic), the dependence between $C_1$ and $C_2$ lifetimes is only given by the occurrence of the systemic shock. None the less, one could imagine concrete situations in which there is some additional dependence between $C_1$ and $C_2$ lifetimes given, for example, by some unobserved factors affecting all the original shocks (idiosyncratic and systemic) arrival times.

Consider for example the case of a firm interested in protecting itself against the failure of a production chain composed by a primary machine (or electronic component) in serial connection with two secondary machines in such a way that the failure of the primary machine causes the failure of all the productive system, while the failure of a secondary machine causes only the failure of the corresponding production line. Clearly the disease of all the three machines (representing the systemic and the idiosyncratic components of the system) is influenced by factors such as maintenance, good electric supply, and so on, and this fact induces dependence among them so that the disease of a production line can induce a change in the probability of failure of the primary machine. An insurance policy written in order to protect against lack in the production caused by the failure of all, or part, of the system, has to take into account the probability of failure of each machine (systemic and idiosyncratic effects) and the dependence among them.

The above is a lucky case, in which the systemic and the idiosyncratic components are well identified and observable. This is not in general the case. Consider, for example, a life insurance contract written by a married couple
with a final payment or goods transfer made when both spouses have died: such contracts are thought in order to transfer money or goods to heirs or to guarantee some annuity until death, using, for example, the house property as collateral security (this is the case of reverse mortgages contracts). In such cases the contract expires when both spouses have died and their death can occur separately or simultaneously. Their simultaneous death can be caused by the occurrence of some systemic event (for example some “catastrophic” event like a car accident, an earthquake and so on) but also by something more related to some dependence among their lifetimes idiosyncratic components: it is in fact well known that in very old people the death of one of the two spouses can induce in a very short period (so that the two events can be considered simultaneous) the death of the other. However, notice that, some catastrophic events causing the simultaneous death of the married couple, cannot be considered as fully external and independent: think of the case of a simultaneous death caused by a car accident in consequence of the fact that the one of the two that was driving had an heart attack. Clearly there are factors affecting both the simultaneous and the separated deaths: health care, social and government assistance, affective relationships and so on.

The idea of a common factor affecting all the components of a random vector is at the basis of the dependence structure represented by Archimedean copulas. Such a dependence can be in fact obtained starting from a vector of i.i.d. exponentially distributed components and dividing all of them by a positive random variable representing a common stochastic intensity (see Marshall and Olkin, 1988). This is the case of Archimedean copulas with completely monotone generator. The case of k-dimensional Archimedean copulas with k-monotone generator can be similarly constructed starting from a vector uniformly distributed on the unit k-dimensional simplex and multiplying each component by a non-negative random variable, representing again a common factor affecting all the random vector (see McNeil and Neslehová, 2009). In any case, all the random variables involved are modified by the common factor in the same way and, as it is well known, the induced dependence is symmetric.

In this paper we generalize the Li and Pellerey (2011) setting allowing for a dependence structure of Archimedean type among the original shocks arrival times. The restrictive fact of considering an exchangeable dependence structure represents a first step in the perspective to include dependence: this is a compromise between mathematical tractability and realism of the assumptions.

More precisely, we assume for \((X_1, X_2, X_3)\) a joint survival distribution function more general than that in (1), that is

\[
\tilde{F}(x, y, z) = G(H_1(x) + H_2(y) + H_3(z))
\]

where \(G : [0, +\infty) \rightarrow [0, 1]\) is the generator of a three-variate Archimedean copula: in this case the associated copula is Archimedean with generator \(G\) and
\[ H_i(x) = G^{-1}(F_i(x)) \] where \( F_i \) the marginal distribution of \( X_i \). The corresponding survival distribution of the random vector \((M_1, M_2)\) is

\[ \bar{F}(x_1, x_2) = G(H_1(x_1) + H_2(x_2) + H_3(\max(x_1, x_2))) \]

and we call it *Archimedean-based Marshall-Olkin distribution* and the copula that it induces *Archimedean-based Marshall-Olkin copula*.

As we will see, the family of the copulas so generated contains the two-dimensional Archimedean copulas, the Generalized Marshall-Olkin copulas and other well known families as specific cases. More precisely, we will show that any Archimedean-based Marshall-Olkin copula can be obtained through a suitable, in general asymmetric, distortion of a bivariate Archimedean copula with generator \( G \). In the case of symmetric distortions, we recover, even if under more restrictive assumptions, the generalization of Archimedean copulas introduced in Durante et al. (2007). In the case of linear distortions, we recover a proper subset of the Archimax copulas of Capéraà et al. (2000) that, when \( G \) is the Laplace transform of a positive random variable, coincide with the bivariate Scale Mixture of Marshall-Olkin copulas studied and applied in Li (2009), Bernhart et al. (2013) and Mai et al. (2013).

The impact of the distortion on the dependence structure induced by an Archimedean-based Marshall-Olkin copula is analyzed by studying its Kendall’s function, its Kendall’s tau and its tail dependence parameters.

The paper is organized as follows. In Section 2 we introduce the Archimedean-based Marshall-Olkin distribution. In Section 3 we derive the Archimedean-based Marshall-Olkin copula. Section 4 is devoted to the analysis of some dependence properties of the copulas introduced: in particular, the expression of the Kendall’s function and of the Kendall’s tau are derived and, in some particular cases, the tail dependence parameters are calculated.

## 2 The Archimedean-based Marshall-Olkin Distribution

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((X_1, X_2, X_3)\) be a random vector such that \(\mathbb{P}(X_i > 0) = 1\) for every \(i = 1, 2, 3\). We assume that their joint survival distribution is of type

\[ \bar{F}(x, y, z) = G(H_1(x) + H_2(y) + H_3(z)) \]

where

- \( G : [0, +\infty) \to [0, 1] \), with \( G(0) = 1 \) and \( G' \) exists on \((0, +\infty)\), it is non-positive, non-decreasing and concave; if \( x_G = \inf\{x \geq 0 : G(x) = 0\} \), we have that \( G \) is strictly decreasing on \([0, x_G)\) and so its inverse function \( G^{-1} : (0, 1) \to [0, x_G) \) is well defined: we can extend \( G^{-1} \) to the whole interval \([0, 1]\) by setting \( G^{-1}(0) = x_G \).
• for every $i = 1, 2, 3$, $H_i$ is continuous and strictly increasing for $x > 0$, $H_i(x) = 0$ for $x \leq 0$ and $\lim_{x \to +\infty} H_i(x) = x_G$.

Under these assumptions, the joint survival distribution can be rewritten as

$$ \bar{F}(x, y, z) = G(G^{-1}(\bar{F}_1(x)) + G^{-1}(\bar{F}_2(y)) + G^{-1}(\bar{F}_3(z))) $$

(2)

with $\bar{F}_i = G \circ H_i$. Thanks to Sklar’s theorem and Theorem 2 in McNeil and Nešlehová (2009), the $\bar{F}_i$, $i = 1, 2, 3$, are the marginal survival distributions of the random variables $X_i$ and the dependence structure of the random vector $(X_1, X_2, X_3)$ is of Archimedean type with generator $G$.

Clearly, this setting includes, as a specific case ($G(x) = e^{-x}$), the family of the Generalized Marshall-Olkin distributions introduced in Li and Pellerey (2011).

In what follows, with abuse of notation, we will denote with $f(+\infty)$ the $\lim_{x \to +\infty} f(x)$, when it exists.

**Remark 2.1.** In Muliere and Scarsini (1987), the particular case with $G(x) = e^{-x}$ and $H_i(x) = \lambda_i H(x)$, $\lambda_i > 0$, is studied and justified.

The choice of functions of type $H_i(x) = \lambda_i H(x)$ is allowed for any $G$ provided that $x_G = +\infty$. If $x_G < +\infty$, the requirement $H_i(+\infty) = x_G$ is satisfied if and only if $\lambda_1 = \lambda_2 = \lambda_3$ which corresponds to the case in which the vector $(X_1, X_2, X_3)$ is exchangeable.

Let us now consider the random vector $(M_1, M_2)$ defined by $M_1 = \min(X_1, X_3)$ and $M_2 = \min(X_2, X_3)$. The corresponding survival distribution function, for $t_1, t_2 > 0$, is

$$ \bar{F}_{M_1, M_2}(t_1, t_2) = \mathbb{P}(M_1 > t_1, M_2 > t_2) = \mathbb{P}(X_1 > t_1, X_2 > t_2, X_3 > \max(t_1, t_2)) = G(H_1(t_1) + H_2(t_2) + H_3(\max(t_1, t_2))). $$

(3)

We call the above distribution *Archimedean-based Marshall-Olkin distribution*.

Setting $K_i(t_i) = H_i(t_i) + H_3(t_i)$, the marginal survival distributions are

$$ \bar{F}_{M_i}(t_i) = G(K_i(t_i)), \ i = 1, 2. $$

(4)

**Remark 2.2.** Notice that if $x_G < +\infty$, then $\bar{F}_{M_1, M_2}(t_1, t_2) = 0$ on $\{(t_1, t_2) : H_1(t_1) + H_2(t_2) + H_3(\max(t_1, t_2)) \geq x_G\}$ and $\bar{F}_{M_i}(t_i) = 0$ for all $t_i \geq K_i^{-1}(x_G)$.
2.1 The singular component

Considering insurance contracts based on the lifetimes \( M_1 \) and \( M_2 \), it is of some relevance to separate the impact on the joint distribution (and, equivalently, on the price) of simultaneous failure or default or death (depending on the application in analysis) from the separated ones.

The simultaneous failure is clearly identified by the singular component of the distribution. It is in fact well known that Marshall-Olkin distributions (as well as the Generalized Marshall-Olkin distributions introduced in Li and Pellerey, 2011) admit a singularity: clearly this fact continues to hold in our extended setting.

To simplify the notation, we set

\[
\hat{H}(t) = \sum_{i=1}^{3} H_i(t).
\]

If \( M = \min(X_1, X_2, X_3) \), its survival distribution function is given, for \( t > 0 \), by

\[
P(M > t) = P(X_1 > t, X_2 > t, X_3 > t) = G(\hat{H}(t)).
\]

**Proposition 2.1.** Assume that \( G \) is twice differentiable and that each \( H_i \) is differentiable on \((0, +\infty)\). Then

\[
F_{M_1, M_2}(t) = P(M_1 \leq t, M_2 \leq t) = F_{M_1, M_2}^a(t) + F_{M_1, M_2}^s(t)
\]

where

\[
F_{M_1, M_2}(t) = 1 + G(\hat{H}(t)) - G(K_1(t)) - G(K_2(t))
\]  \hspace{1cm} (5)

and

\[
F_{M_1, M_2}^s(t) = -\int_0^t H'_3(x)G'(\hat{H}(x))dx.
\]

**Proof.** (5) is trivial.

Since

\[
\frac{\partial^2}{\partial t_2 \partial t_1} F_{AMO}(t_1,t_2) = \begin{cases} G''(H_1(t_1) + K_2(t_2)) H'_1(t_1) K'_2(t_2) & \text{if } t_1 < t_2 \\ G''(K_1(t_1) + H_2(t_2)) K'_1(t_1) H'_2(t_2) & \text{if } t_1 > t_2 \end{cases}
\]

it is a straightforward computation to show that

\[
P(M_1 \leq t, M_2 \leq t, M_2 > M_1) = G(\hat{H}(t)) - G(K_2(t)) - \int_0^t H'_1(x)G'(\hat{H}(x)) dt
\]

and

\[
P(M_1 \leq t, M_2 \leq t, M_2 < M_1) = 1 - G(K_1(t)) + \int_0^t K'_1(x)G'(\hat{H}(x)) dt
\]
and

\[ F_{M_1, M_2}^s(t) = \mathbb{P}(M_1 \leq t, M_2 \leq t, M_1 = M_2) = - \int_0^t H'_3(x)G' \left( \hat{H}(x) \right) dt. \]

As a consequence

\[
\mathbb{P}(M_1 = M_2) = - \int_0^{+\infty} H'_3(t)G' \left( \hat{H}(t) \right) dt
\]

\[
= - \int_0^{H^{-1}(x_G)} H'_3(t)G' \left( \hat{H}(t) \right) dt
\]

\[
= - \int_0^{H^{-1}(x_G)} \frac{H'_3(t)}{H'(t)} \hat{H}'(t)G' \left( \hat{H}(t) \right) dt
\]

\[
= \mathbb{E} \left[ \frac{H'_3(M)}{H'(M)} \right].
\]

**Example 2.1.**  

1. If \( G(x) = e^{-x} \), then

\[
F_{M_1, M_2}^s(t) = \int_0^t H'_3(t) \exp \left( -\hat{H}(t) \right) dt
\]

and

\[
\mathbb{P}(M_1 = M_2) = \int_0^{+\infty} H'_3(t) \exp \left( -\hat{H}(t) \right) dt
\]

and this is the case of the Generalized Marshall-Olkin distributions (see Li and Pellerey, 2011).

2. If \( x_G = +\infty \) and \( H_i = \lambda_i H \), then, if \( \hat{\lambda} = \sum_{i=1}^3 \lambda_i \),

\[
F_{M_1, M_2}^s(t) = \frac{\lambda_3}{\lambda} (1 - G(\hat{\lambda}H(t)))
\]

\[
= \frac{\lambda_3}{\lambda} \mathbb{P}(M \leq t)
\]

and

\[
\mathbb{P}(M_1 = M_2) = \frac{\lambda_3}{\lambda}.
\]

independently of \( G \). In particular, this case includes, when \( G(x) = e^{-x} \), the Marshall-Olkin type distribution introduced by Muliere and Scarsini (1987), and, if \( H_i(x) = \lambda_i x \), the standard Marshall-Olkin case distribution.
3. More in general, let us assume that $H_3$ is proportional to $H_1 + H_2$, that is $H_3 = cH_1 + H_2$, which is equivalent to: $F_{\min(M_1, M_2)}(t) = G\left(\frac{t}{c}H_3(t)\right)$. If $x_G < +\infty$, necessarily $c = \frac{1}{2}$, while any choice of $c > 0$ is allowed if $x_G = +\infty$. In any case, we have

$$F_{M_1, M_2}^s(t) = \frac{c}{c+1} \left(1 - G\left(\frac{c}{c+1}H_3(t)\right)\right)$$

and

$$\mathbb{P}(M \leq t)$$

that doesn’t depend on $G$.

4. Both the facts that $F_{M_1, M_2}^s$ is proportional to the cumulative distribution function of $M$ and that the singular component is independent of $G$ constitute a peculiarity of the above case. If $G(x) = (x+1)^{-1/\theta}$ (Clayton generator) with $\theta > 0$ and $H_1(x) + H_2(x) = H_3^2(x) + H_3(x)$ then

$$F_{M_1, M_2}^s(t) = \frac{1}{2 + \theta} \left(1 - (1 + H_3(t))^{-\frac{\theta}{\theta-1}}\right)$$

but

$$\mathbb{P}(M \leq t) = 1 - (1 + H_3(t))^{-\frac{\theta}{\theta-1}}.$$

Moreover, this is equivalent to

$$\mathbb{P}(M_1 = M_2) = \frac{1}{2 + \theta}$$

that depends on $\theta$.

Similarly, if $H_1(x) + H_2(x) = e^{H_3(x)} - H_3(x) - 1$ then, again,

$$F_{M_1, M_2}^s(t) = \frac{1}{\theta+1} \left(1 - e^{-\frac{\theta+1}{\theta}H_3(t)}\right)$$

with

$$\mathbb{P}(M \leq t) = 1 - e^{-\frac{1}{\theta}H_3(t)}$$

and

$$\mathbb{P}(M_1 = M_2) = \frac{1}{1 + \theta}.$$

3. The Archimedean-based Marshall-Olkin Copula Function

If $u_i = G(K_i(t_i))$, from (1), we get $t_i = K_i^{-1}(G^{-1}(u_i))$. Substituting it in (2), thanks to Sklar’s theorem, we obtain the copula associated to the vector
We have

\[ \bar{C}_{M_1,M_2}(u_1,u_2) = \begin{cases} 
  G(H_1 \circ K_1^{-1}(G^{-1}(u_1)) + G^{-1}(u_2)) & \text{if } K_1^{-1}(G^{-1}(u_1)) \leq K_2^{-1}(G^{-1}(u_2)) \\
  G^{-1}(u_1) + H_2 \circ K_2^{-1}(G^{-1}(u_2)) & \text{if } K_1^{-1}(G^{-1}(u_1)) > K_2^{-1}(G^{-1}(u_2)) 
\end{cases} \]

Setting \( D_i(x) = H_i \circ K_i^{-1}(x) \) for \( x \in [0, x_G] \) and \( i = 1, 2 \) we have

\[ \bar{C}_{M_1,M_2}(u_1,u_2) = \begin{cases} 
  G(D_1(G^{-1}(u_1)) + G^{-1}(u_2)) & \text{if } K_1^{-1}(G^{-1}(u_1)) \leq K_2^{-1}(G^{-1}(u_2)) \\
  G(G^{-1}(u_1) + D_2(G^{-1}(u_2))) & \text{if } K_1^{-1}(G^{-1}(u_1)) > K_2^{-1}(G^{-1}(u_2)) 
\end{cases} \]

Let us analyze the set

\[ F = \{(u_1,u_2) \in [0,1]^2 : K_1^{-1}(G^{-1}(u_1)) = K_2^{-1}(G^{-1}(u_2))\} = \{(u_1,u_2) \in [0,1]^2 : D_1(G^{-1}(u_1)) + G^{-1}(u_2) = G^{-1}(u_1) + D_2(G^{-1}(u_2))\}. \]

We have

- if \( D_1(x_G) > D_2(x_G) \), then \( [0,K_1^{-1}(x_G)] \subsetneq [0,K_2^{-1}(x_G)] \) and the set \( F \) is the graph of the function \( u_2 = h(u_1) = G \circ K_2 \circ K_1^{-1} \circ G^{-1}(u_1) \) for \( u_1 \in [0,1] \), where \( h \) is strictly increasing, \( h(1) = 1 \) and \( h(0) = \mathbb{P}(M_2 > K_1^{-1}(x_G)) > 0 \);
- if \( D_1(x_G) < D_2(x_G) \), then \( [0,K_1^{-1}(x_G)] \supsetneq [0,K_2^{-1}(x_G)] \) and the set \( F \) is the graph of the function \( u_1 = \hat{h}(u_2) = G \circ K_1 \circ K_2^{-1} \circ G^{-1}(u_2) \) for \( u_2 \in [0,1] \), where \( \hat{h} \) is strictly increasing, \( \hat{h}(1) = 1 \) and \( \hat{h}(0) = \mathbb{P}(M_1 > K_2^{-1}(x_G)) > 0 \);
- if \( D_1(x_G) = D_2(x_G) \), then the set \( F \) can be represented through both \( h \) and \( \hat{h} \) that are strictly increasing, \( h(1) = \hat{h}(1) = 1 \) and \( h(0) = \hat{h}(0) = 0 \).

It follows that

- if \( D_1(x_G) \geq D_2(x_G) \),

\[ \bar{C}_{M_1,M_2}(u_1,u_2) = \begin{cases} 
  G(D_1(G^{-1}(u_1)) + G^{-1}(u_2)) & \text{if } u_2 \leq h(u_1) \\
  G(G^{-1}(u_1) + D_2(G^{-1}(u_2))) & \text{if } u_2 > h(u_1) 
\end{cases} \]

- if \( D_1(x_G) < D_2(x_G) \),

\[ \bar{C}_{M_1,M_2}(u_1,u_2) = \begin{cases} 
  G(D_1(G^{-1}(u_1)) + G^{-1}(u_2)) & \text{if } u_1 \geq \hat{h}(u_2) \\
  G(G^{-1}(u_1) + D_2(G^{-1}(u_2))) & \text{if } u_1 < \hat{h}(u_2) 
\end{cases} \]

Clearly, we recover a copula of type (9) from a copula of type (8) by inverting \( D_1 \) with \( D_2 \) and viceversa.

**Remark 3.1.** Notice that \( D_i(x) \) and \( x - D_i(x) \) are continuous and strictly increasing on \([0, x_G]\). Moreover

\[ D_i(x) < x, \text{ for } x > 0 \]

and, if \( x_G = +\infty \), \( \lim_{x \to x_G} D_i(x) = +\infty \) and \( \lim_{x \to x_G} x - D_i(x) = +\infty \).
In next result we will prove that for any suitable pair of functions \(D_1\) and \(D_2\), the copula in (8) and (9) can be obtained starting from a three-variate distribution function of type (2) and following the construction presented above.

In what follows we set \(\hat{D}_i(x) = x - D_i(x)\). The following result holds:

**Lemma 3.1.** Let \(D_1, D_2\) be two strictly increasing and continuous functions on \([0, x_G]\) with \(D_1(0) = 0, D_1(x)\) strictly increasing and such that, if \(x_G = +\infty\), \(\lim_{x \to x_G} D_1(x) = D_1(x_G) = +\infty\) and \(\lim_{x \to x_G} D_i(x) = D_i(x_G) = +\infty\). Then, for every \(H_3\) defined on \([0, +\infty)\), continuous, strictly increasing and such that, \(H_3(0) = 0\) and \(H_3(+\infty) = x_G\), the function \(H_i(x) = D_i^{-1}(H_3(x)) - H_3(x)\), for \(x \in [0, H_3^{-1}(\hat{D}_i(x_G))]\), is the unique solutions of

\[
D_i(x) = H_i(x) + H_3^{-1}(x) \quad \text{for all} \ x \in [0, x_G].
\]

**Proof.** Notice that \(H_i(x)\) is strictly increasing on \([0, H_3^{-1}(\hat{D}_i(x_G))]\) and satisfies \(H_i(0) = 0\), \(H_i(H_3^{-1}(\hat{D}_i(x_G))) = D_i(x_G)\) and it is the only solution of

\[
\hat{D}_i(H_i(z) + H_3(z)) = H_3(z) \quad z \in [0, H_3^{-1}(\hat{D}_i(x_G))]
\]

which is equivalent to

\[
D_i(H_i(z) + H_3(z)) = H_i(z) \quad z \in [0, H_3^{-1}(\hat{D}_i(x_G))]
\]

Setting \(x = H_i(z) + H_3(z) \in [0, x_G]\) in (12) we get (11).

If \(x_G < +\infty\), \(H_i\) can be clearly extended so to be defined on the set \([0, +\infty)\), to be strictly increasing and with \(H_i(+\infty) = x_G\).

**Remark 3.2.** Notice that \(D_i\) and \(\hat{D}_i\) satisfy the same assumptions and if, given \(H_3, H_i\) solves (11), then \(H_i(x) = D_i^{-1}(H_3(x)) - H_3(x) = (\hat{D}_i^{-1} - Id)^{-1}(H_3(x))\) (where \(Id\) is the identity function) solves \(\hat{D}_i(x) = H_i(x) + H_3^{-1}(x)\) for all \(x \in [0, x_G]\).

**Example 3.1.**

1. If \(D_i(x) = \alpha_i x, \ \alpha_i \in (0, 1), i = 1, 2\), then \(H_i(x) = \frac{\alpha_i}{1 - \alpha_i} H_3(x), \ \text{for} \ i = 1, 2 \ \text{and} \ x \in [0, H_3^{-1}(1 - \alpha_i x_G)].\)

2. If \(D_i(x) = x - \frac{1}{\alpha_i} \log(\alpha_i x + 1), \ \alpha_i > 0, i = 1, 2, \ \text{then} \ H_i(x) = \frac{1}{\alpha_i} (e^{\alpha_i H_3(x)} - 1) - H_3(x), \ \text{for} \ i = 1, 2, \ \text{and} \ x \in [0, H_3^{-1}(\frac{1}{\alpha_i} \log(\alpha_i x_G + 1))].\)

3. If \(D_i(x) = x - \frac{1}{\alpha_i} (\sqrt{\alpha_i x + 1} - 1), \ \alpha_i > 0, i = 1, 2, \ \text{then} \ H_i(x) = \alpha_i H_3^2(x) + H_3(x), \ \text{for} \ i = 1, 2 \ \text{and} \ x \in [0, H_3^{-1}(\frac{1}{\alpha_i} (\sqrt{\alpha_i x_G + 1} - 1))].\)

4. If \(D_i(x) = \frac{1}{\alpha_i} (\sqrt{\alpha_i x + 1} - 1), \ \alpha_i > 0, i = 1, 2, \ \text{then} \ H_i(x) = \frac{1}{2 \alpha_i} \left( \sqrt{1 + 4 \alpha_i H_3(x)} - 1 \right), \ \text{for} \ i = 1, 2 \ \text{and} \ x \in [0, H_3^{-1}(x_G - \frac{1}{\alpha_i} (\sqrt{\alpha_i x_G + 1} - 1))]\).
Definition 3.1. Let $G : [0, +\infty) \to [0, 1]$, with $G(0) = 1$ and such that $G'$ exists on $(0, +\infty)$, it is non-positive, non-decreasing and concave. If $x_G = \inf\{x \geq 0 : G(x) = 0\}$, let $D_1, D_2$ be two strictly increasing and continuous functions on $[0, x_G]$ with $D_i(0) = 0$, $D_i(x)$ strictly increasing and such that if $x_G = +\infty$, $\lim_{x \to x_G} D_i(x) = D_i(x_G) = +\infty$ and $\lim_{x \to x_G} \hat{D}_i(x) = \hat{D}_i(x_G) = +\infty$. If we set $f(x) = \hat{D}_2^{-1} \circ \hat{D}_1(x)$ (when this is well defined), we call Archimedean-based Marshall-Olkin copula with generator $G$ and distortions $D_1, D_2$ the copula function $C_{AMO}$ so defined:

- if $D_1(x_G) \geq D_2(x_G)$,
  $$C_{AMO}(u, v) = \begin{cases} G( D_1(G^{-1}(u)) + G^{-1}(v) ) & \text{if } v \leq h(u) \\ G( G^{-1}(u) + D_2(G^{-1}(v)) ) & \text{if } v > h(u) \end{cases}$$

where $h(u) = G \circ f \circ G^{-1}(u)$;

- if $D_1(x_G) < D_2(x_G)$,
  $$C_{AMO}(u, v) = \begin{cases} G( D_1(G^{-1}(u)) + G^{-1}(v) ) & \text{if } u \geq \hat{h}(v) \\ G( G^{-1}(u) + D_2(G^{-1}(v)) ) & \text{if } u < \hat{h}(v) \end{cases}$$

where $\hat{h}(v) = G \circ f^{-1} \circ G^{-1}(v)$.

It is evident from the definition that the $C_{AMO}$ copula is a distortion of the Archimedean copula with generator $G$ through the functions $D_i$. More precisely, if $D_1 \neq D_2$ (that is if $H_1 \neq H_2$), the Archimedean copula is differently modified above and below the curve $F$ given in (7) and the obtained copula is obviously asymmetric.

Conversely, if $D_1 = D_2 = D$ (that is if $H_1 = H_2$) we get

$$C_{AMO}(u, v) = \begin{cases} G( D(G^{-1}(u)) + G^{-1}(v) ) & \text{if } v \leq u \\ G( G^{-1}(u) + D(G^{-1}(v)) ) & \text{if } v > u \end{cases}$$

and the obtained copula remains exchangeable. The family defined in (15) is contained in the class of copulas introduced in Durante et al. (2007), defined as

$$C_{\phi, \psi}(u, v) = \phi^{v-1}(\phi(u \land v) + \psi(u \lor v))$$

with $\phi : [0, 1] \to [0, +\infty]$, continuous, convex and strictly decreasing, $\psi : [0, 1] \to [0, +\infty]$, continuous, decreasing and such that $\psi(1) = 0$ and $\psi - \phi$ increasing in $[0, 1]$: (15) is trivially recovered from (16) when $\phi(1) = G^{-1}(1) = 0$ and $\psi(t) = D(G^{-1}(t))$.

Example 3.2. If $G(x) = e^{-x}$

$$C_{AMO}(u, v) = \begin{cases} ve^{-D_1(-\ln u)} & \text{if } v \leq \exp(-f(-\ln u)) \\ ue^{-D_2(-\ln v)} & \text{if } v > \exp(-f(-\ln u)) \end{cases}$$

and we recover the Generalized Marshall-Olkin copula introduced in Li and Pellerey (2011). In particular, if, for $i = 1, 2$, $D_i(x) = \alpha_i x$ with $\alpha_i \in (0, 1)$, we recover the classical Marshall-Olkin copula.
Example 3.3. If $D_i(x) = \alpha_i x$ for $i = 1, 2$ and $\alpha_i \in (0, 1)$, then

$$C_{AMO}(u,v) = \begin{cases} G\left(\alpha_1 G^{-1}(u) + G^{-1}(v)\right) & \text{if } v \leq G\left(\frac{1-\alpha_2}{1-\alpha_1} G^{-1}(u)\right) \\ G\left(G^{-1}(u) + \alpha_2 G^{-1}(v)\right) & \text{if } v > G\left(\frac{1-\alpha_1}{1-\alpha_2} G^{-1}(u)\right) \end{cases} \quad (17)$$

This particular specification constitutes a subclass of the family of copulas called Archimax copulas introduced in Capéraà et al. (2000). Archimax copulas are defined as

$$C_{G,A}(u,v) = G\left(\left(\frac{G^{-1}(u)}{G^{-1}(u) + G^{-1}(v)}\right) A\left(\frac{G^{-1}(u)}{G^{-1}(u) + G^{-1}(v)}\right)\right) \quad (18)$$

where $A : [0,1] \to [1/2,1]$ is a convex function such that $\max(t,1-t) \leq A(t) \leq 1$ for all $t \in [0,1]$ and copulas in (17) can be obtained choosing

$$A(t) = \begin{cases} (\alpha_1 - 1)t + 1 & \text{if } t \leq \frac{1-\alpha_2}{2-\alpha_1\alpha_2} \\ (1-\alpha_2)t + \alpha_2 & \text{if } t > \frac{1-\alpha_1}{2-\alpha_1\alpha_2} \end{cases}$$

The same fact holds if one considers piecewise linear distortions $D_i$, to which correspond piecewise affine functions $A$ in representation (18).

Moreover, the subset of copulas of type (17) when $G$ is the Laplace transform of a positive random variable, coincides with the family of bivariate Scale Mixture of Marshall-Olkin copulas (see Li (2009), Bernhart et al. (2013) and Mai et al. (2013)). These copulas are obtained as the copulas associated to the random vector $(\hat{X}, \hat{Y})$ where $(Z_1, Z_2)$ has a bivariate Marshall-Olkin distribution and $Y$ is an independent positive random variable. In the exchangeable case, May et al. (2013) propose an alternative construction: let $\epsilon_1, \epsilon_2$ be i.i.d. unit exponentially distributed random variables, $M > 0$ be a random variable with Laplace transform $G(x)$ and $\Lambda_t \neq 0$ be a Lévy subordinator with Laplace exponent $\Psi$; assuming that they are mutually independent, define

$$\tau_k = \inf\{t \geq 0 : M_t \geq \epsilon_k\}$$

where $M_t = M_{\Lambda^{-1}(1-p(t))/\Psi(1)}$ with $p(t)$ a given distribution function. The copula associated to $(\tau_1, \tau_2)$ is

$$C(u,v) = G\left(G^{-1}(\min(u,v)) + G^{-1}(\max(u,v))\frac{\Psi(2) - \Psi(1)}{\Psi(1)}\right)$$

which is obviously of type (17) with $\alpha_1 = \alpha_2 = \frac{\Psi(2) - \Psi(1)}{\Psi(1)}$.

Under the assumptions of Proposition 2.11

$$-\int_0^{F_{M_2}^{-1}(u)} H_3'(t)G''\left(H(t)\right) dt = \mathbb{P}(M_1 = M_2) =$$

$$= \mathbb{P}(F_{M_1}^{-1}(U) = F_{M_2}^{-1}(V)) =$$

$$= \mathbb{P}(K_1^{-1} \circ G(U) = K_2^{-1} \circ G(V)) =$$

$$= \mathbb{P}(F).$$
Hence, if \( G \) is twice differentiable and each \( D_i \) is differentiable on \([0, x_G]\), the copula function \( C_{AMO}(u, v) \) has a singularity on the curve \( F \) (see (17)), whose mass, expressed in terms of the distortions, is

\[
P(F) = - \int_0^{T^{-1}(x_G)} G'(T(x))dx
\]

where

\[
T(x) = \hat{D}_1^{-1}(x) + \hat{D}_2^{-1}(x) - x
\] (19)

It is a known fact that two Archimedean copulas with generators \( G_A \) and \( G_B \) coincide if and only if there exists \( \alpha > 0 \) such that \( G_A(x) = G_B(\alpha x) \) (see Corollary 2.2.6 in Alsina et al., 2006). In what follows we will present the analogous result for the Archimedean-based Marshall-Olkin copula functions.

**Theorem 3.2.** Let \( C^A \) and \( C^B \) be two Archimedean-based Marshall-Olkin copula functions with generators \( G_A \) and \( G_B \) and distortions \( A_1, A_2 \) and \( B_1, B_2 \), respectively.

\[
C^A \equiv C^B
\]

if and only if there exists \( m > 0 \) such that

\[
G_B(z) = G_A(mz) \text{ for } z \in [0, +\infty)
\] (20)

and

\[
B_i(z) = \frac{1}{m} A_i(mz) \text{ for } z \in [0, x_{G_B}] , \ i = 1, 2.
\] (21)

**Proof.** Since we are looking for \( C^A \equiv C^B \), necessarily the corresponding singularity sets \( F \) in (23) must coincide. Hence \( C^A \) and \( C^B \) must be of the same type (13) or (14). We show the proof in case (13) being the other perfectly identical. Hence, we assume \( A_1(x_{G_A}) \geq A_2(x_{G_A}) \), \( B_1(x_{G_B}) \geq B_2(x_{G_B}) \) and

\[
C^A(u, v) = \begin{cases} 
G_A \left( A_1(G_A^{-1}(u)) + G_A^{-1}(v) \right) & \text{if } v \leq h_A(u) \\
G_A \left( A_1(G_A^{-1}(u)) + A_2(G_A^{-1}(v)) \right) & \text{if } v > h_A(u)
\end{cases}
\]

and

\[
C^B(u, v) = \begin{cases} 
G_B \left( B_1(G_B^{-1}(u)) + G_B^{-1}(v) \right) & \text{if } v \leq h_B(u) \\
G_B \left( B_1(G_B^{-1}(u)) + B_2(G_B^{-1}(v)) \right) & \text{if } v > h_B(u)
\end{cases}
\]

Let \( h(u) = \min(h_A(u), h_B(u)) \). For \( C^A \equiv C^B \), necessarily

\[
G_A \left( A_1(G_A^{-1}(u)) + G_A^{-1}(v) \right) = G_B \left( B_1(G_B^{-1}(u)) + B_2(G_B^{-1}(v)) \right)
\] (22)

for \( u \in [0, 1] \) and \( v \leq h(u) \). We set \( \psi = G_A^{-1} \circ G_B : [0, x_{G_B}] \rightarrow [0, x_{G_A}] \), \( x = G_B^{-1}(u) \in [0, x_{G_B}] \) and \( y = G_B^{-1}(v) \in [0, x_{G_B}] \). So (22) can be rewritten as

\[
A_1(\psi(x)) + \psi(y) = \psi(B_1(x) + y)
\] (23)
with \( y \in [0, x_{G_B}] \) and \( x \leq G_B^{-1}(h^{-1}(G_B(y))) \). Since \( G_A \) and \( G_B \) are differentiable on \((0, x_{G_A})\) and \((0, x_{G_B})\), respectively, and their derivatives are strictly negative, \( \psi \) is differentiable on \((0, x_{G_B})\) as well. After differentiating \( \psi \) with respect to \( y \), we get
\[
\psi'(y) = \psi'(B_1(x) + y).
\]
This holds whenever \( 0 < x < G_B^{-1}(h^{-1}(G_B(y))) \); hence \( \psi' \), being locally constant, is also globally constant: then there exists \( m \) such that \( \psi'(y) = m \) for all \( y \in (0, x_{G_B}) \). It follows that \( \psi(y) = my + q \), for \( y \in [0, x_{G_B}] \), and since \( \psi \) is strictly increasing, necessarily, \( m > 0 \). Moreover, since \( \psi(0) = 0 \), \( q = 0 \).

Hence
\[
G_B(x) = G_A(mx), \quad x \in [0, x_{G_B}],
\]
and \( x_{G_B} = \frac{x_{G_A}}{m} \).
Substituting this relation in (23) we get
\[
A_1(G_A^{-1}(u)) = mB_1 \left( \frac{G_A^{-1}(u)}{m} \right)
\]
and, setting \( z = \frac{G_A^{-1}(u)}{m} \in [0, \frac{x_{G_A}}{m}] \), we obtain
\[
B_1(z) = \frac{1}{m}A_1(mz).
\]
In order to guarantee that \( C_A = C_B \)
\[
G_A \left( G_A^{-1}(u) + A_2(G_A^{-1}(v)) \right) = G_B \left( G_B^{-1}(u) + B_2(G_B^{-1}(v)) \right)
\]
must hold for \( v \in [0, 1] \) and \( u < \min(h_A^{-1}(v), h_B^{-1}(v)) \), as well. Using (24) we obtain
\[
A_2(G_A^{-1}(v)) = mB_2 \left( \frac{G_A^{-1}(v)}{m} \right)
\]
and, setting \( z = \frac{G_A^{-1}(v)}{m} \in [0, \frac{x_{G_A}}{m}] \), we obtain
\[
B_2(z) = \frac{1}{m}A_2(mz).
\]

The converse is trivial. \( \square \)

4 Dependence Properties

In this Section we will consider the concordance measure induced by the \( C_{AMO} \) copula by calculating the corresponding Kendall’s function and Kendall’s tau. Since, as we noticed, in general, the \( C_{AMO} \) copula is an asymmetric distortion of an Archimedean copula, we investigate the effect of such a distortion on the tail dependence parameters.

Let us start by observing that, thanks to (10),
\[
C_{AMO}(u, v) \geq G(G^{-1}(u) + G^{-1}(v)), \quad (u, v) \in [0, 1]^2.
\]
4.1 The Kendall’s function

We remind that the Kendall’s function of a copula $C$, is the cumulative distribution function of the random variable $C(U, V)$ with respect to the probability induced by $C$ (see Nelsen, 2006), that is the $C$-measure of the set

$$A_t = \{(u, v) \in [0, 1]^2 : C_{AMO}(u, v) \leq t\}.$$ 

Let $K_G(t)$ be the Kendall’s function of the Archimedean copula with generator $G$ and $K_{AMO}$ be the Kendall’s function of the copula $C_{AMO}$. Clearly, from (25)

$$K_{AMO}(t) \leq K_G(t), \quad t \in [0, 1].$$

More precisely,

**Theorem 4.1.**

$$K_{AMO}(t) = K_G(t) + G'(G^{-1}(t)) \cdot T^{-1}(G^{-1}(t))$$

where $T$ is defined in [14].

**Proof.** We follow the same ideas and spirit of the proof of the analogous result in the Archimedean case (see, for example, Theorem 4.3.4 in Nelsen, 2006).

We will prove the result in the case $D_1(x_G) \geq D_2(x_G)$, being the alternative case perfectly analogous.

We start with looking for the intersection of the $t$-level curve $\{(u, v) \in [0, 1]^2 : C_{AMO}(u, v) = t\}$, for $t > 0$, and the graph of the function $h$ of Definition 3.1, that is we solve for $u$ the equation

$$G(D_1(G^{-1}(u)) + G^{-1}(h(u))) = t$$

from which

$$D_1(G^{-1}(u)) + f(G^{-1}(u)) = G^{-1}(t)$$

or, setting $g(x) = D_1(x) + f(x),

$$g(G^{-1}(u)) = G^{-1}(t).$$

Since, thanks to the assumptions, $g$ is invertible, the unique solution is

$$u_t = G \circ g^{-1} \left(G^{-1}(t)\right) \in [t, 1]$$

and, if

$$v_t = h(u_t) = G \circ f \circ g^{-1} \left(G^{-1}(t)\right),$$

$(u_t, v_t)$ is the unique intersection of the level curve and the graph of $h$.

We split the set $A_t$ into three regions: the rectangle $R_t = [0, u_t] \times [0, v_t]$ and the sets

$$B_1 = \{(u, v) \in [0, 1]^2 : u \in [u_t, 1], v \leq G(G^{-1}(t) - D_1(G^{-1}(u)))\}$$
Let us compute now $P$ and $D$ exactly as done for $B$, continuous on $[0,1]$, if

$$P = \sum_{k=1}^{n} C(u_k, v_k).$$

Notice that $S_k = \sum_{k=1}^{n} R_k$, the $t$-level curve in $B_1$ is

$$v(u) = G(G^{-1}(u) - D_1(G^{-1}(u))).$$

For $k = 1, \ldots, n$, let $R_k = [t_{k-1}, t_k] \times [0, v(t_{k-1})]$. Clearly

$$P(R_k) = C(t_k, v(t_k)) - C(t_{k-1}, v(t_{k-1})) = C(t_k, v(t_k)) - t$$

and

$$C(t_k, v(t_k)) = G(D_1(G^{-1}(t_k)) + G^{-1}(v(t_k))) = G(G^{-1}(t) - \frac{D_1(G^{-1}(u_k))}{n}).$$

If $S_n = \sum_{k=1}^{n} R_k$,

$$P(S_n) = n \left[ G \left( G^{-1}(t) - \frac{D_1(G^{-1}(u_k))}{n} \right) - t \right] = G \left( G^{-1}(t) - \frac{D_1(G^{-1}(u_k))}{n} \right) D_1(G^{-1}(u_k))$$

and

$$P(B_1) = \lim_{n \to +\infty} P(S_n) = -G'(G^{-1}(t))D_1(G^{-1}(u_k)).$$

For the set $B_2$, considering the partition of the interval $[v_1, 1]$ given by the points

$$s_k = G \left( D_2^{-1} \left( D_2 \left( G^{-1}(v_k) \right) \left( 1 - \frac{k}{n} \right) \right) \right),$$

exactly as done for $B_1$, we get

$$P(B_2) = -G'(G^{-1}(t))D_2(G^{-1}(v_k)).$$

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It follows that, using (26) and (27),

\[ P(A_t) = P(R_t) + P(B_t) + P(B_2) = \]

\[ = t - G'(G^{-1}(t)) \left[ D_1(G^{-1}(u_t)) + D_2(G^{-1}(v_t)) \right] = \]

\[ = t - G'(G^{-1}(t)) \left[ D_1(g^{-1}(G^{-1}(t))) + D_2 \circ f(g^{-1}(G^{-1}(t))) \right] = \]

\[ = K_G(t) + G'(G^{-1}(t)) \left[ G^{-1}(t) - D_1(g^{-1}(G^{-1}(t))) - D_2 \circ f(g^{-1}(G^{-1}(t))) \right] \]

\[ = K_G(t) + G'(G^{-1}(t)) \left[ \text{Id} - D_1 \circ g^{-1} - D_2 \circ f \circ g^{-1} \right] \circ G^{-1}(t) \]

where \( \text{Id} \) is the identity function and \( K_G(t) = t - G'(G^{-1}(t))G^{-1}(t) \) is the Kendall’s function of the Archimedean copula with generator \( G \).

But, since \( g(x) = T \circ \hat{D}_1(x) \), where \( T \) is defined in (19), we have

\[ \text{Id} - D_1 \circ g^{-1} - D_2 \circ f \circ g^{-1} = (g - D_1 - D_2 \circ f) \circ g^{-1} = \]

\[ = (f - D_2 \circ f) \circ g^{-1} = \]

\[ = \hat{D}_2 \circ f \circ g^{-1} = \]

\[ = \hat{D}_1 \circ g^{-1} = \]

\[ = T^{-1} \]

and so

\[ K_{AMO}(t) = K_G(t) + G'(G^{-1}(t))T^{-1}(G^{-1}(t)). \]

\[ \square \]

In terms of the generating functions \( H_1, H_2, H_3 \) the Kendall's function can be rewritten as

\[ K_{AMO}(t) = K_G(t) + G'(G^{-1}(t)) \cdot H_3 \circ \hat{H}^{-1}(G^{-1}(t)). \]

In particular, if \( H_3 = c(H_1 + H_2) \), with \( c > 0 \) (see Example 24) we get

\[ K_{AMO}(t) = K_G(t) + \frac{c}{c+1} G'(G^{-1}(t))G^{-1}(t) \]  \hspace{1cm} (28)

and this case includes the case of linear distortions of Example 3.1.

Notice that (28) can be rewritten as \( K_{AMO}(t) = t - \frac{1}{c+1}G'(G^{-1}(t))G^{-1}(t) \), and if we consider two different generators \( G_A \) and \( G_B \) so that \( K_{G_A}(t) \leq K_{G_B}(t) \), but identical proportional parameter \( c \), then \( K_{G_{AMO}}(t) \leq K_{G_{B_{MO}}}(t) \), where with \( K_{G_{MO}} \) we denote the Kendall’s function of the Archimedean-based Marshall-Olkin copula with generator \( G_j \), for \( j = A, B \).

This fact doesn’t continue to hold in general as shown in next example.
Example 4.1. Let \( D_1(x) = D_2(x) = 1 + x - \sqrt{1 + 2x} \). It follows that
\[
K_{AMO}(t) = t - \frac{1}{4} G'(G^{-1}(t)) \left( \sqrt{1 + 4 \cdot G^{-1}(t)} - 1 \right)^2
\]
If we consider the Frank generator \( G_F(x) = -\frac{1}{\theta} \ln \left( 1 + e^{-x} (e^{-\theta} - 1) \right) \) with parameter \( \theta = 4 \) and the Gumbel generator \( G_G(x) = \exp \left( -\gamma^{1/\gamma} \right) \) with parameter \( \gamma = 2 \) we have
\[
K_{GF}(0.3) = 0.497 > 0.480 = K_{GG}(0.3)
\]
while
\[
K_{GFMO}(0.3) = 0.341 < 0.380 = K_{GGM}(0.3).
\]

Example 4.2. Let us consider \( D_i(x) = x^{\alpha_i} (\sqrt{\alpha_i x} + 1) - 1 \) (see Example 3.1). Then
\[
K_{AMO}(t) = K_G(t) + \frac{3G'(G^{-1}(t))}{2(\alpha_1 + \alpha_2)} \cdot \left( \sqrt{1 + \frac{4}{9} (\alpha_1 + \alpha_2) G^{-1}(t) - 1} \right)
\]

4.2 Kendall’s tau

It is known that the Kendall’s tau of a bivariate copula \( C \) is given by
\[
\tau_C = 4E[C(U,V)] - 1 = 3 - \frac{1}{2} \int_0^1 K(t)dt.
\]

Hence, we get
\[
\tau_{AMO} = 3 - \frac{1}{2} \int_0^1 K_{AMO}(t)dt =
\]
\[
= 3 - \frac{1}{2} \int_0^1 (K_G(t) + G'(G^{-1}(t)) \cdot T^{-1}(G^{-1}(t)))dt =
\]
\[
= \tau_G - \frac{4}{9} \int_0^1 G'(G^{-1}(t)) \cdot T^{-1}(G^{-1}(t))dt =
\]
\[
= \tau_G + 4 \int_0^{x_G} (G'(x))^2 \cdot T^{-1}(x)dx
\]
where \( \tau_G \) is the Kendall’s tau of the Archimedean copula with generator \( G \). Obviously, as expected from (25),
\[
\tau_{AMO} \geq \tau_G.
\]

In terms of the generating functions \( H_1, H_2, H_3 \) the Kendall’s tau can be rewritten as
\[
\tau_{AMO} = \tau_G + 4 \int_0^{x_G} (G'(x))^2 \cdot H_3 \circ \hat{H}^{-1}(x)dx.
\]
Example 4.3. If $G(x) = e^{-x}$ we recover the case studied in Li and Pellerey (2011) and we get

$$
\tau_{AMO} = 4 \int_{0}^{+\infty} e^{-2xT^{-1}(x)}dx.
$$

Example 4.4. Let us consider the Clayton case, that is $G(x) = (x+1)^{-1/\theta}$, with $\theta > 0$.

1. If $H_3 = c(H_1 + H_2)$, $c > 0$, then

$$
\tau_{AMO} = \tau_G + \frac{2}{c + 1} \frac{2}{\theta + 2} = \tau_G + \tau_{IMO} \frac{2}{\theta + 2}
$$

where $\tau_{IMO} = \frac{c}{c+1}$ is the Kendall’s tau when $G = e^{-x}$. Notice that this result continues to hold for $\theta \in (-\frac{1}{2}, 0)$ and $c = \frac{1}{2}$.

2. If $\hat{D}_1^{-1}(x) + \hat{D}_2^{-1}(x) = e^{\frac{x}{\theta}} - 1 + x$, with $1 \geq \gamma > 0$ (this corresponds to $H_1(z) + H_2(z) = e^{\frac{\theta(z)}{\theta_1}} - H_3(z) - 1$), we get $(\hat{D}_1^{-1}(x) + \hat{D}_2^{-1}(x) - x)^{-1} = \gamma \ln(x+1)$. Then

$$
\tau_{AMO} = \tau_G + \frac{4\gamma}{(2 + \theta)^2} = \frac{\theta^2 + 2\theta + 4\gamma}{(2 + \theta)^2},
$$

3. If $\hat{D}_1^{-1}(x) + \hat{D}_2^{-1}(x) = \left(\frac{x}{\gamma} + 1\right) - 1 + x$ with $\alpha > 1$, $1 \geq \gamma > 0$ (this corresponds to $H_1(z) + H_2(z) = \left(\frac{H_3(z)}{\gamma} + 1\right)^{\alpha} - 1 - H_3(z)$), we get

$$
(\hat{D}_1^{-1}(x) + \hat{D}_2^{-1}(x) - x)^{-1} = \gamma \left(\frac{x+1}{\theta} - 1\right). Then

$$
\tau_{AMO} = \tau_G + \frac{4\gamma}{(2 + \theta)(\alpha(2 + \theta) - \theta)} = \frac{1}{\theta + 2} \left(\theta + \frac{4\gamma}{\alpha(2 + \theta) - \theta}\right).
$$

4. If $D_1(x) = x - \frac{1}{\alpha_1} \sqrt{\alpha_1 x + 1} - 1$ (see Example 4.2), $(\hat{D}_1^{-1}(x) + \hat{D}_2^{-1}(x) - x)^{-1} = \frac{3}{2(\alpha_1 + \alpha_2)} \left(\frac{1}{\sqrt{\alpha_1 + \alpha_2}} x - 1\right)$. If $\alpha_1 + \alpha_2 = \frac{3}{2}$, we recover the previous case with $\gamma = \frac{3}{4}$ and $\alpha = 2$. Hence

$$
\tau_{AMO} = \frac{3\theta^2 + 12\theta + 8}{3(2 + \theta)(4 + \theta)}.
$$

Exactly as for the Kendall’s function, the presence of distortion functions strongly influences the Kendall’s tau. In Table II the values of the Kendall’s tau of some Archimedean copulas and the corresponding values of the Archimedean-based Marshall-Olkin copulas with same generator but distortions $D_1(x) = D_2(x) = 1 + x - \sqrt{1 + 2x}$ are reported in order to illustrate how the presence of the distortions can induce an inversion in the order of the concordance measure.
Table 1: Comparison of Kendall’s tau value with and without distortions.

| Generator | Parameter | $\tau_G$ | $\tau_{AMO}$ |
|-----------|-----------|----------|--------------|
| Clayton   | $\theta = 2$ | 0.5      | 0.78539      |
| Gumbel    | $\theta = 1.8$ | 0.375    | 0.81636      |
| Frank     | $\theta = 4$ | 0.388    | 0.906        |

### 4.3 Tail dependence

We recall that the upper and lower tail dependence parameters of a copula $C$ are given, respectively, by

$$\lambda_U = 2 - \lim_{u \uparrow 1} \frac{1 - C(u, u)}{1 - u} \quad \text{and} \quad \lambda_L = \lim_{u \downarrow 0} \frac{C(u, u)}{u}$$

when these limits exist and that if $\lambda_U \in (0, 1]$ ($\lambda_L \in (0, 1]$) we have that $C$ has upper(lower)-tail dependence (we refer to Section 5.4 in Nelsen (2006), for more details).

Let $\lambda_{L}^{AMO}$ and $\lambda_{U}^{AMO}$ be the lower and upper tail dependence parameters of the copula $C_{AMO}$ and $\lambda_{L}^{G}$ and $\lambda_{U}^{G}$ be the tail dependence parameters of the corresponding bivariate Archimedean copula with generator $G$.

Let $\mathcal{A}_1 = \{ x \in [0, x_G) : D_1(x) \geq D_2(x) \}$ and $\mathcal{A}_2 = \mathcal{A}_1^c$. Notice that $G^{-1}(u) \in \mathcal{A}_1$ if and only if $(u, u)$ lies on or below the curve $F$ defined in (7).

#### 4.3.1 $\lambda_{L}^{AMO}$

If $x_G < +\infty$, obviously $\lambda_{L}^{AMO} = 0$.

Let us now consider the case $x_G = +\infty$. We have

$$C_{AMO}(u, u) = G \left( G^{-1}(u) + D_1(G^{-1}(u)) 1_{\mathcal{A}_1}(G^{-1}(u)) + D_2(G^{-1}(u)) 1_{\mathcal{A}_2}(G^{-1}(u)) \right) .$$

Setting $x = G^{-1}(u)$,

$$\lambda_{L}^{AMO} = \lim_{u \downarrow 0} \frac{C_{AMO}(u, u)}{u} = \lim_{x \to +\infty} \frac{G(x + D_1(x)) 1_{\mathcal{A}_1}(x) + D_2(x) 1_{\mathcal{A}_2}(x)}{G(x)} .$$

If there exists $\bar{x} > 0$ such that $(\bar{x}, +\infty) \subset \mathcal{A}_i$ then

$$\lambda_{L}^{AMO} = \lim_{x \to +\infty} \frac{G(x + D_1(x))}{G(x)} .$$

The value of the above limit clearly depends on the type of decay of $G$ to zero. If the decay is of polynomial type, that is there exist $c, \gamma > 0$ such that
$G(x) \sim_{x \rightarrow +\infty} cx^{-\gamma}$, then

$$\lim_{x \rightarrow +\infty} \frac{G(x + D_i(x))}{G(x)} = \lim_{x \rightarrow +\infty} \left( 1 + \frac{D_i(x)}{x} \right)^{-\gamma}$$

Hence, if $\lim_{x \rightarrow +\infty} \frac{D_i(x)}{x} = \beta \in [0, 1]$ exists, we have that

$$\lambda_{AMO}^L = (1 + \beta)^{-\gamma}.$$ 

If the decay is exponential of type $G(x) \sim_{x \rightarrow +\infty} ce^{-ax\gamma}$, with $a, c, \gamma > 0$, we have

$$\lim_{x \rightarrow +\infty} \frac{G(x + D_i(x))}{G(x)} = \lim_{x \rightarrow +\infty} e^{-a((x + D(x))\gamma - x\gamma)} = \lim_{x \rightarrow +\infty} e^{-ax\gamma((1 + \frac{D(x)}{x})\gamma - 1)}.$$ 

If $\lim_{x \rightarrow +\infty} \frac{D_i(x)}{x} = \beta \in (0, 1]$, then $\lambda_{AMO}^L = 0$. 

If $\lim_{x \rightarrow +\infty} \frac{D_i(x)}{x} = 0$, then

$$\lambda_{AMO}^L = \lim_{x \rightarrow +\infty} e^{-ax\gamma((1 + \frac{D(x)}{x})\gamma - 1)} = \lim_{x \rightarrow +\infty} e^{-a\gamma \frac{D(x)}{x}}$$

and, if $\gamma \geq 1$, then $\lambda_{AMO}^L = 0$ while, if $\gamma < 1$, then

$$\lambda_{AMO}^L = e^{-a\gamma \lim_{x \rightarrow +\infty} \frac{D(x)}{x}}.$$ 

**Example 4.5.** Assume that there exists $\bar{x} > 0$ such that $(\bar{x}, +\infty) \subset A_i$ with $\lim_{x \rightarrow +\infty} \frac{D_i(x)}{x} = \beta_i$. We have

- in the Clayton case $G(x) = (x + 1)^{-1/\theta}$, $\theta > 0$, $\lambda_{AMO}^L = (1 + \beta_i)^{-1/\theta} \geq 2^{-1/\theta} = \lambda_{L}^G$;
- in the Gumbel case $G(x) = e^{-x^{1/\theta}}$, $\theta \geq 1$, with $\beta_i \in (0, 1]$, $\lambda_{AMO}^L = 0 = \lambda_{L}^G$;
- in the Frank case, $G(x) = -\frac{1}{\theta} \ln \left( 1 + e^{-x(e^{-\theta} - 1)} \right)$, $\theta \in \mathbb{R}$, $\lambda_{AMO}^L = 0 = \lambda_{L}^G$.

**4.3.2 $\lambda_{AMO}^U$**

Similarly as done for $\lambda_{AMO}^L$ we can calculate $\lambda_{AMO}^U$ in some specific cases.

If there exists $\check{x} > 0$ such that $(0, \check{x}) \subset A_i$, then

$$\lambda_{AMO}^U = 2 - \lim_{x \rightarrow 0} \frac{1 - G(x + D_i(x))}{1 - G(x)}.$$
If 
\[ 1 - G(x) \sim cx^\gamma \]
for some \( c, \gamma > 0 \), then 
\[ \lim_{x \to 0} \frac{1 - G(x + D_i(x))}{1 - G(x)} = \lim_{x \to 0} \left( 1 + \frac{D_i(x)}{x} \right)^\gamma \]
Hence, if \( \lim_{x \to 0} \frac{D_i(x)}{x} = \beta_i \in [0, 1] \), then 
\[ \lambda_{AMO}^U = 2 - (1 + \beta_i)^\gamma. \]

**Example 4.6.** Assume that there exists \( \bar{x} > 0 \) such that \( (0, \bar{x}) \subset A_i \) with 
\[ \lim_{x \to 0} \frac{D_i(x)}{x} = \beta_i \in [0, 1]. \]
We have 
- in the Clayton case \( G(x) = (x + 1)^{-1/\theta}, \theta > 0, \lambda_{AMO}^U = 1 - \beta_i \geq 0 = \lambda_{U}^G; \)
- in the Gumbel case \( G(x) = e^{-x^{1/\theta}}, \theta \geq 1, \lambda_{AMO}^U = 2 - (1 + \beta_i)^{1/\theta} \geq 2 - 2^{1/\theta} = \lambda_{U}^G; \)
- in the Frank case, \( G(x) = -\frac{1}{\theta} \ln \left( 1 + e^{-x^\theta} - 1 \right), \theta \neq 0, \lambda_{AMO}^U = 1 - \beta_i \geq 0 = \lambda_{U}^G. \)

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