COMMUTATOR THEORY FOR COMPATIBLE UNIFORMITIES

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Abstract. We investigate commutator operations on compatible uniformities. We define a commutator operation for uniformities in the congruence-modular case which extends the commutator on congruences, and explore its properties.

INTRODUCTION

The purpose of this paper is to generalize the commutator of congruences to a commutator of compatible uniformities. Commutator theory (on congruences) works best for congruences of algebras in congruence-modular varieties. The same is true of the commutator of uniformities described here. In fact, the commutator of congruences \( \alpha \) and \( \beta \) becomes a special case of that of uniformities, when we view \( \alpha \) and \( \beta \) as the uniformities \( \Ug\{\alpha\} \) and \( \Ug\{\beta\} \) that they generate, because we have \( \Ug\{[\alpha, \beta]\} = [\Ug\{\alpha\}, \Ug\{\beta\}] \).

We follow the development of Commutator Theory in [4] fairly closely. The main thesis of [10], where compatible uniformities were first studied systematically in the context of Universal Algebra, is that compatible uniformities can be considered a generalization of congruences. Often, there is a reasonably direct translation of congruence-theoretic arguments into uniformity-theoretic ones. Following this philosophy, we are able to generalize (in Sections 4 and 5) the concept of \( C(\alpha; \beta) \) (\( \alpha \) centralizes \( \beta \) modulo \( \delta \)) to compatible uniformities, and in the congruence-modular case, to define \([U, V]\) to be the least uniformity \( W \) such that \( C(U, V; W) \).

Another approach to the commutator \([\alpha, \beta]\), for congruences \( \alpha \) and \( \beta \), as discussed in [4], is to study congruences of the algebra \( A(\alpha) \). The congruence \( \beta \) is pushed out along the homomorphism \( \Delta_\alpha : A \to A(\alpha) \) that sends \( a \in A \) to \( \langle a, a \rangle \), yielding a congruence \( \Delta_{\alpha, \beta} \) which gives rise to \([\alpha, \beta]\). In the case of uniformities, we can replace \( \beta \) by a uniformity \( U \), and push it out along \( \Delta_\alpha \), yielding a compatible uniformity \( \Delta_{\alpha, U} \) on \( A(\alpha) \) which we then show gives rise to \([\Ug\{\alpha\}, U]\) in the important special case of algebras having term operations comprising a group structure. (This includes many familiar varieties of algebras, such as groups, rings, and varieties of nonassociative algebras.) This is done in Section 6. It is natural to ask whether the theory can be extended to give an interpretation of \([U, V]\) in terms of compatible uniformities on some algebra \( A(U) \). Unfortunately, the necessary definition of \( A(U) \) is not yet available.

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In the congruence-modular case, the properties of the commutator on compatible uniformities described in Section 7 duplicate those of the commutator of congruences, with some regrettable gaps in what we have been able to prove. In particular, because of the difficulties we encounter in working with joins of uniformities, the additivity of the commutator is not settled in general, although we prove it is true for some important special cases.

We also describe a natural commutator operation for congruential uniformities, which are uniformities that, as a filter of relations, have a base of congruences, and are thus of the form $Ug F$ for a filter $F$ in $\text{Con } A$. The commutator $[Ug F, Ug F']$ is defined in terms of the commutators of elements of $F$ and $F'$; see Section 8.

Section 9 is devoted to miscellaneous matters, including commutators of congruential uniformities on commutative rings. We prove that in that case, the two definitions of the commutator on congruential uniformities, the one given by the general definition of Section 5 and the other given by the formula of Section 8, coincide. This appears to be a special property of commutative rings; in general, we do not know even whether the commutator operation of Section 5, applied to congruential uniformities, always gives a congruential uniformity. We also discuss in this section the case of varieties which are congruence-distributive, where we show that as in the case of the commutator of congruences, the commutator of two compatible uniformities is simply their meet.

In the last section, we discuss the current state of some questions about compatible uniformities, uniformity lattices, and commutators of uniformities.

Preliminaries

**Category theory.** We follow [6] in terminology and notation. In particular, $1_a$ will stand for the identity arrow on an object $a$ in a category $C$.

**Lattice theory.** The reader should be familiar with lattices. We use $\top$ and $\bot$ to denote the greatest and least elements of a lattice, assuming they exist, and $\wedge$ and $\vee$ for the meet and join operations.

**Filters.** If $L$ is a lattice, then a nonempty subset $F \subseteq L$ is called a filter if $y \geq x \in F$ implies $y \in F$ and $x, y \in F$ imply $x \wedge y \in F$.

If $S \subseteq L$ is a nonempty set, then the filter generated by $S$, denoted by $\text{Fg}^L S$ or simply $\text{Fg} S$, is the subset of elements of $L$ that are greater than or equal to a finite meet of elements of $S$. An important special case, given $x \in L$, is $\text{Fg}\{x\}$, the principal filter generated by $x$, which is just the set of elements of $L$ greater than or equal to $x$.

Filters are ordered by reverse inclusion and the filters in a nonempty lattice form a complete lattice. The meet of a tuple of filters $F_i$ is given by $\bigwedge_i F_i = \text{Fg}(\bigcup_i F_i)$. The join of the tuple is the intersection: $\bigvee_i F_i = \bigcap_i F_i$.

If $F$ is a filter, a base for $F$ is a subset $B \subseteq F$ such that $x \in F$ implies $b \leq x$ for some $b \in B$. If $L$ is a lattice, then a subset $B \subseteq L$ is a base for a filter of $L$, or filter base, iff given any $x, y \in B$, there is a $z \in B$ such that $z \leq x \wedge y$. 
Universal algebra. We assume familiarity with universal algebra, as explained for example in [2]. We prefer to allow an algebra to have an empty underlying set, however. We denote the underlying set of an algebra \( A \) by \( |A| \).

If \( R \) is a binary relation on (the underlying set of) an algebra \( A \), then we denote by \( Cg R \) the smallest congruence \( \alpha \in \text{Con} A \) such that \( R \subseteq \alpha \).

Commutator theory. The commutator is a binary operation on congruences in the congruence lattice \( \text{Con} A \) of an algebra \( A \) in a congruence-modular variety, and which is sometimes defined for more general varieties.

If \( A \) is an algebra in a congruence-modular variety, and \( \alpha, \beta \in \text{Con} A \), we can define the commutator \( [\alpha, \beta] \) as the least \( \delta \in \text{Con} A \) such that \( \alpha \) centralizes \( \beta \) modulo \( \delta \), or in other words, such that \( \delta \) satisfies the \( \alpha, \beta \)-term condition. See the first part of Section 4 for detailed definitions. Whereas Section 4 gives these definitions in the general case, and doesn’t really define \( [\alpha, \beta] \) or its generalization to uniformities, Section 5 makes the assumption of congruence-modularity and proves the simplifications that make the definitions of \( [\alpha, \beta] \) and \([U, V]\) (for \( U, V \) compatible uniformities) so reasonable.
The commutator is so named because it generalizes the notion of the commutator of normal subgroups of a group. (Of course, the variety of groups is congruence-modular.) As a further example, the variety of commutative rings is congruence-modular, and for a commutative ring \( A \), the commutator is simply the product of ideals. That is, if \( I_\alpha \) denotes the ideal corresponding to \( \alpha \in \text{Con} A \), then we have \( I_{[\alpha,\beta]} = I_\alpha I_\beta \).

**Uniform universal algebra.** Universal algebra over the base category \( \text{Unif} \) of uniform spaces, as opposed to the category of sets, was first studied systematically in [10]. This paper develops commutator theory as a part of that subject. It will be best if the reader has access to [10] while reading this paper, but we will also devote most of the next two sections to summarizing some of the basic definitions and results that we need.

1. **Uniformities**

We denote the set of binary relations on a set \( S \) by \( \text{Rel} S \). \( \text{Rel} S \), ordered by inclusion, is a complete lattice.

If \( R \in \text{Rel} S \), then by \( R^{-1} \) we mean the relation \( \{ \langle x, y \rangle \mid \langle y, x \rangle \in R \} \), and by \( R^n \), for \( n > 0 \), we mean the \( n \)-fold relational product of \( n \) copies of \( R \).

Consider the following five conditions on a set \( U \subseteq \text{Rel} S \):

(U1) if \( U \in U \), and \( U \subseteq V \), then \( V \in U \)
(U2) if \( U, V \in U \), then \( U \cap V \in U \)
(U3) if \( U \in U \), then \( \Delta_S \defeq \{ \langle s, s \rangle \mid s \in S \} \subseteq U \)
(U4) if \( U \in U \), then \( U^{-1} \in U \)
(U5) if \( U \in U \), then \( V^2 \subseteq U \) for some \( V \in U \).

Then we say that \( U \) is a semiuniformity if \( U \) satisfies conditions (U1) through (U4), and a uniformity if it satisfies (U1) through (U5). Note that conditions (U1) and (U2) simply state that \( U \) is a filter of binary relations

**Proposition 1.1.** A filter \( U \) satisfies (U3) iff \( F_{\text{gRel} S} \{ \Delta_S \} \leq U \).

**Proposition 1.2.** A filter \( U \) satisfies (U4) iff \( U = U^{-1} \), where \( U^{-1} = \{ U^{-1} \mid U \in U \} \).

**Proposition 1.3.** If \( U, V \) are filters in \( \text{Rel} S \), then \( \{ U \circ V \mid U \in U, V \in V \} \) is a base for a filter \( U \circ V \) in \( \text{Rel} S \).

Note that for filters \( U \) and \( V \) of reflexive relations, \( U \cap V \leq U \circ V \). Thus, \( U \leq U \circ U \) if \( U \) satisfies (U3).

**Proposition 1.4.** A filter \( U \) satisfies (U5) iff \( U \circ U \leq U \).

**Notation.** If \( U \in U \), where \( U \) satisfies (U5), then by induction we can show that there is a \( V \in U \) such that \( V^n \subseteq U \). We denote such a \( V \) by \( ^nU \). This notation must be used with care, particularly in relation to quantifiers; we do not mean that \( ^nU \) is a function of \( U \); it is simply a shorthand for the statement that there exists such a \( V \) and that we will denote it by \( ^nU \).
The lattice operations. We denote the set of uniformities on a set $S$ by $\text{Unif} S$, and the set of semiuniformities by $\text{SemiUnif} S$. We order these sets by reverse inclusion, i.e., the ordering inherited from $\text{FilRel} S$.

The meet of an arbitrary tuple of uniformities on $S$, in the lattice $\text{FilRel} S$, is a uniformity. Thus, $\text{Unif} S$ admits arbitrary meets, and is a complete lattice. The same is true for $\text{SemiUnif} S$. The join of a tuple of semiuniformities is simply the intersection, and $\text{SemiUnif} S$ is a distributive lattice. The theory of joins of uniformities is more difficult.

Permutability. Permutability of congruences is an important condition in Universal Algebra, and the condition can be generalized to uniformities. Note that this subject was first discussed in [10], but the discussion there is not entirely correct; in particular, Theorem 6.1 is wrong.

If $U$ and $V$ are uniformities on a set $S$, then we say that $U$ and $V$ permute if $U \circ V = V \circ U$, and that $U$ and $V$ semipermute if either $U \circ V \leq V \circ U$, or $V \circ U \leq U \circ V$.

As we mentioned, the join operation in the lattice of uniformities can be difficult to deal with in the general case, but the case where $U$ and $V$ semipermute is an easy and important special case; the following theorem is a revised and corrected version of [10, Theorem 6.1]:

**Theorem 1.5.** Let $U, V \in \text{Unif} S$. Then $U \lor V = U \circ V$ iff $V \circ U \leq U \circ V$.

**Proof.** ($\Leftarrow$): It is trivial that $U \circ V$ satisfies (U1), (U2), and (U3). If $U \in U$ and $V \in V$, then since $V \circ U \leq U \circ V$, there are $\bar{U} \in U$ and $\bar{V} \in V$ such that $\bar{V} \circ \bar{U} \subseteq U \circ V$. But, $\bar{V} \circ \bar{U} = ((\bar{U}^{-1}) \circ (\bar{V}^{-1}))^{-1}$. Thus, $U \circ V$ also satisfies (U4). Finally, $(U \circ V) \circ (U \circ V) \leq (U \circ U) \circ (V \circ V) \leq U \circ V$, verifying (U5). Thus, $U \circ V$ is a uniformity. Since $U \leq U \circ V$ and $V \leq U \circ V$, we have $U \lor V \leq U \circ V$. However, $U \circ V \leq U \lor V = U \circ V$ by (U5).

($\Rightarrow$): If $U \lor V = U \circ V$, then $V \circ U \leq U \lor V$ by (U5).

The results in [10] that use permutability as a hypothesis, except for Theorem 6.1, are correct, and remain true if the hypothesis is weakened to semipermutability.

2. Compatible Uniformities

Compatibility. If $R$ is a relation on an algebra $A$, we say that $R$ is compatible (with the operations of $A$) if $a R a'$ implies $\omega^A(a) R \omega^A(a')$ for each operation symbol $\omega$, where $a R a'$ means that $a_i \land a'_i$ for all $i$.

We say that a filter $U$ of reflexive relations on an algebra $A$ is compatible if for each $U \in U$, and each basic operation symbol $\omega$, there is a $\bar{U} \in U$ such that $\omega(\bar{U}) = \{ (\omega(x), \omega(y)) \mid x, \bar{U} \\ y_i \text{ for all } i \} \subseteq U$. In this case, for any term $t$, given $U \in U$, there is a $\bar{U} \in U$ such that $t(\bar{U}) \subseteq U$.

We say that $U$ is singly compatible if for each $n$-ary term $t$ for $n \geq 1$, given $U \in U$, there is a $\bar{U} \in U$ such that $t(\bar{U}, a) \subseteq U$ for every $a \in A^{n-1}$.

**Lemma 2.1.** If $U$ is a uniformity on $A$, which is singly compatible, then $U$ is compatible.

**Proof.** This follows easily from (U5).
As a result of the Lemma, single compatibility will be of interest to us only for semiuniformities.

If $A$ is an algebra, we denote by $\text{SemiUnif } A$ the set of compatible semiuniformities on $A$, and by $\text{Unif } A$ the set of compatible uniformities.

Remark. Since a set $S$ can be seen as an algebra with no operations, the theory of $\text{Unif } S$ is subsumed by the theory of $\text{Unif } A$ for an algebra $A$. Parts of this section are therefore pertinent the study of $\text{Unif } S$ where $S$ is just a set.

If filters $U_i$ are compatible, so is $\bigwedge_i U_i$. It follows that the meet of an arbitrary tuple of compatible uniformities or semiuniformities is also compatible, so the sets of compatible uniformities and compatible semiuniformities are complete lattices. Similarly, if $U_i$ is a tuple of singly compatible semiuniformities, then $\bigwedge_i U_i$ is a singly compatible semiuniformity.

$Ug R$. If $R$ is a filter of relations on an algebra $A$, then $Ug R$ will denote the smallest compatible uniformity $U$ such that $R \subseteq U$. If we mean instead the smallest not-necessarily compatible uniformity, we will write $Ug^{|A|} R$.

Joins.

Proposition 2.2. If $U_i$, $i \in I$ are elements of $\text{Unif } A$, then $\bigvee_i U_i = Ug(\bigcap_i U_i)$.

Theorem 2.3. Let $A$ be an algebra. We have

1. The join (in the lattice $\text{Unif } |A|$) of a tuple of compatible uniformities is compatible.
2. The join (in the lattice $\text{SemiUnif } |A|$) of a tuple of singly compatible semiuniformities is singly compatible.

Proof. (1): See [10, Theorem 5.3].

(2): If $U_i$ are a tuple of singly compatible semiuniformities on $A$, then their join (in the lattice of semiuniformities on $|A|$) is $U = \bigcap_i U_i$. Let $t$ be an $n$-ary term operation. If $U \subseteq U_i$, then for each $i$, there is $U_i \subseteq U_i$ with $U_i \subseteq U$. Since each $U_i$ is singly compatible, $t(U_i, \Delta_A, \ldots, \Delta_A) \subseteq U$ for some $U_i \subseteq U_i$. Then $\bigcup_i U_i \subseteq U$, and $t(\bigcup_i U_i, \Delta_A, \ldots, \Delta_A) \subseteq U$. □

Examples of compatible uniformities. An important special case of a compatible uniformity is given by choosing $R = \{ \alpha \}$ where $\alpha \in \text{Con } A$. Then $Ug R = Fg^{\text{Rel } A} \{ \alpha \}$.

More generally, we can consider $Ug\{ \rho \}$ where $\rho \in \text{Rel } A$. However, we have

Theorem 2.4. $Ug\{ \rho \} = Ug\{ Cg \rho \}$, and is compatible if $\rho$ is.

Proof. It suffices to show that $Ug\{ Cg \rho \} \subseteq Ug \rho$, or in other words that if $U \subseteq Ug\{ \rho \}$, then $Cg\{ \rho \} \subseteq U$.

Let $U \subseteq Ug\{ \rho \}$. We have $\rho \subseteq U$, so $\rho \cup \Delta \subseteq U$ by (U3). Thus, we can reduce to the case where $\rho$ is reflexive by replacing $\rho$ with $\rho \cup \Delta$.

If $U \subseteq Ug\{ \rho \}$, then $\rho \subseteq U^{-1}$, so $\rho^{-1} \subseteq U$. Thus, we can further reduce to the case where $\rho$ is symmetric, by replacing $\rho$ by $\rho \cup \rho^{-1}$.

Finally, if $U \subseteq Ug\{ \rho \}$, then $\rho \subseteq \rho^n U$ for all $n \in \mathbb{N}$, which implies $\rho \subseteq U$, showing that $Cg \rho = \bigcup_n \rho^n \subseteq U$. 

As regards compatibility, it is easy to prove that if \( \rho \) is compatible, then so is \( \text{Cg}\rho \). It is obvious that \( \text{Ug}\{\alpha\} \) is compatible if \( \alpha \) is a congruence.

Another important special case is \( \mathcal{R} = F \), where \( F \) is a filter in \( \text{Con}\,\mathcal{A} \). In this case, \( \text{Ug}\mathcal{R} = \text{Fg}_{\text{Rel}\,\mathcal{A}}\,F \). A uniformity of this form called a congruential uniformity.

### Uniformities and congruences.

If \( \mathcal{U} \in \text{Unif}\,\mathcal{A} \), then \( \bigcap \mathcal{U} \in \text{Con}\,\mathcal{A} \). We may consider this as a mapping from \( \text{Unif}\,\mathcal{A} \) to \( \text{Unif}\,\mathcal{A} \), where we map \( \mathcal{U} \) to \( \text{Ug}\{\bigcap \mathcal{U}\} \); more generally, we can map \( \mathcal{U} \) to the filter of \( \kappa \)-fold intersections of relations in \( \mathcal{U} \), for \( \kappa \) some given infinite cardinal. The result will be a compatible uniformity \( \mathcal{V} \) such that \( \mathcal{V} \) admits \( \kappa \)-fold intersections of its elements. We say that \( \mathcal{V} \) satisfies the \( \kappa \)-fold intersection property.

#### Uniformities and homomorphisms.

If \( \mathcal{U} \) is a relation on an algebra \( \mathcal{A} \), and \( f : \mathcal{B} \to \mathcal{A} \) is a homomorphism from another algebra \( \mathcal{B} \), then we denote by \( f^{-1}(\mathcal{U}) \) the relation \( \{(b, b') \mid f(b) \in \mathcal{U} \} \). If \( \mathcal{U} \) is a filter of relations on \( \mathcal{A} \), then we denote by \( f^{-1}(\mathcal{U}) \) the filter \( \text{Fg}\{f^{-1}(\mathcal{U}) \mid \mathcal{U} \in \mathcal{U}\} \).

**Proposition 2.5.** Let \( \mathcal{A}, \mathcal{B} \) be algebras, and \( f : \mathcal{B} \to \mathcal{A} \) a homomorphism. We have

1. If \( \mathcal{U} \) is a compatible uniformity (compatible semiuniformity, singly compatible semiuniformity) on \( \mathcal{A} \), then \( f^{-1}(\mathcal{U}) \) is a compatible uniformity (respectively, compatible semiuniformity, singly compatible semiuniformity) on \( \mathcal{B} \).
2. The mapping \( \mathcal{U} \mapsto f^{-1}(\mathcal{U}) \) preserves arbitrary meets.

Now, suppose that we have \( \mathcal{U} \in \text{Fil}\,\text{Rel}\,\mathcal{A} \), and a homomorphism \( f : \mathcal{A} \to \mathcal{B} \). If \( \mathcal{U} \in \text{Unif}\,\mathcal{A} \), we define \( f_{sc}(\mathcal{U}) \) to be the meet of all \( \mathcal{V} \in \text{Unif}\,\mathcal{B} \) such that \( \mathcal{U} \leq f^{-1}(\mathcal{V}) \).

**Proposition 2.6.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be algebras, and \( f : \mathcal{A} \to \mathcal{B} \) a homomorphism. Then the mapping \( \mathcal{U} \mapsto f_{sc}(\mathcal{U}) \) preserves arbitrary joins.

Let \( \mathcal{A}, \mathcal{B} \) be algebras, and \( f : \mathcal{A} \to \mathcal{B} \) a homomorphism, and \( \mathcal{U} \) a relation on \( \mathcal{A} \). If \( n, n' \geq 0 \) and \( t \) is an \( (n + n') \)-ary term, we denote by \( L_{f,n,n',t}(\mathcal{U}) \) the set of pairs \( (t(b, f(a)), t(b, f(a'))) \) where \( b \in B^n \) and \( a, a' \in A^{n'} \) are such that \( a \mathcal{U} a' \).

**Theorem 2.7.** Given \( \mathcal{A}, \mathcal{B} \), \( f : \mathcal{A} \to \mathcal{B} \), and a uniformity \( \mathcal{U} \) on \( \mathcal{A} \), we have

1. For \( n, n', t \), the set of relations \( L_{f,n,n',t}(\mathcal{U}) \) on \( \mathcal{B} \), for \( U \in \mathcal{U} \), is a base for a filter \( \mathcal{L}_{f,n,n',t}(\mathcal{U}) \) of relations on \( \mathcal{B} \).
2. Given \( n, n', t \), if \( \mathcal{U} \) is compatible, then \( \mathcal{L}_{f,n,n',t}(\mathcal{U}) \leq f_{sc}(\mathcal{U}) \).
3. \( \mathcal{L}_f(\mathcal{U}) \) is a singly compatible semuniformity.
4. \( f_{sc}(\mathcal{U}) = \text{Ug}(\mathcal{L}_f(\mathcal{U})) \).

**Proof.** (1) is clear.

(2): Let \( V \in f_{sc}(\mathcal{U}) \). There is a \( \bar{V} \in f_{sc}(\mathcal{U}) \) such that \( \bar{V} \subseteq V \). There is a \( U \in \mathcal{U} \) such that \( U \subseteq f^{-1}(\bar{V}) \), because \( \mathcal{U} \leq f^{-1}(f_{sc}(\mathcal{U})) \). Then \( f(U) \subseteq V \) and \( L_{f,n,n',t}(U) \subseteq V \). Thus, \( \mathcal{L}_{f,n,n',t}(U) \leq f_{sc}(\mathcal{U}) \).

(3): (U1) and (U2) are clear. If \( t \) is the unary term \( t(x) = x \), then \( \mathcal{L}_{f,1,0,t}(\mathcal{U}) = \{\Delta_B\} \), proving (U3). \( L_{f,n,n',t}(U^{-1}) = L_{f,n,n',t}(U)^{-1} \), so if \( \mathcal{U} \) is a semuniformity, then so is \( \mathcal{L}_{f,n,n',t}(\mathcal{U}) \).
for all \( n, n', \) and \( t, \) proving (U4). To show single compatibility, we must show that given \( V \in \mathcal{L}_f(\mathcal{U}) \) and an \((\bar{n} + 1)\)-ary term \( \bar{t} \) for \( \bar{n} \geq 0 \), there is a \( \bar{V} \in \mathcal{L}_f(\mathcal{U}) \) such that \( \bar{t}(\bar{V}, \mathfrak{c}) \subseteq \bar{V} \) for any \( \mathfrak{c} \in B^{\bar{n}} \). It suffices to show that, given \( n, n' \), and \( t \), there is a \( U_{n,n',t} \in \mathcal{U} \) such that \( \bar{t}(L_{f,n,n',t}(U_{n,n',t}), \mathfrak{c}) \subseteq V \) for \( \mathfrak{c} \in B^n \). For, then \( \bar{U} = \bigcup_{n,n',t} L_{f,n,n',t}(U_{n,n',t}) \in \mathcal{L}_f(\mathcal{U}) \) and \( \bar{t}(\bar{U}, \mathfrak{c}) \subseteq V \) for all \( \mathfrak{c} \). Now, \( \bar{t}(L_{f,n,n',t}(U), \mathfrak{c}) \subseteq L_{f,n+n',t}(U) \) where \( \bar{t}(x, y, z) = \bar{t}(x, z, y) \). Thus, if we pick \( U_{n,n',t} \) such that \( L_{f,n+n',t}(U_{n,n',t}) \subseteq V \), which we can do because \( L_{f,n+n',t}(U) \leq f_{\ast c}(U) \) by (2), then we have \( \bar{t}(L_{f,n,n',t}(U_{n,n',t}), \mathfrak{c}) \subseteq V \).

(4): We have \( \mathcal{L}_f(\mathcal{U}) \leq f_{\ast c}(\mathcal{U}) \) by (2). Thus, \( Ug\mathcal{L}_f(\mathcal{U}) \leq f_{\ast c}(\mathcal{U}) \). To show the opposite inequality, it suffices to note that \( \mathcal{U} \leq f^{-1}(\mathcal{L}_f(\mathcal{U})) \). For, \( \mathcal{U} \leq f^{-1}(\mathcal{L}_{1,0,\bar{x}}(\mathcal{U})) \).

Remark. In [10, Section 11], there is another incorrect discussion about the procedure for finding the colimit of a diagram \( F : \mathcal{D} \rightarrow \mathbf{V}^{(\text{Unif})} \) in the category \( \mathbf{V}^{(\text{Unif})} \). The uniformity of the colimit is the smallest compatible uniformity greater than or equal to all of the \( \iota_d(\mathcal{U}(d)) \), where \( \iota_d \) is the insertion of \( F(d) \) into the colimit and \( \mathcal{U}(d) \) is the uniformity on \( F(d) \).

Completion. The completion of an algebra \( A \) with respect to a compatible uniformity \( \mathcal{U} \) is defined as the set of equivalence classes of Cauchy nets in \( A \) with respect to \( \mathcal{U} \), and we denote it by \( A/\mathcal{U} \). \( A/\mathcal{U} \) has a natural structure of uniform universal algebra. The mapping from \( A \) to \( A/\mathcal{U} \) taking \( a \in A \) to the equivalence class containing the constant nets at \( a \) is a uniform homomorphism onto a dense subset of \( A/\mathcal{U} \), and we denote this mapping by \( \text{nat} \mathcal{U} \).

Note that in [10], we used the notation \( \eta_d \) for \( \text{nat} \mathcal{U} \).

We denote the natural uniformity on \( A/\mathcal{U} \) by \( \mathcal{U}/\mathcal{U} \); it has a base of relations \( R(\mathcal{U}, \mathcal{U}) \) for \( \mathcal{U} \in \mathcal{U} \), where \( R(\mathcal{U}, \mathcal{U}) \) relates two equivalence classes \( k, k' \) of Cauchy nets iff there exist nets \( n \in k, n' \in k' \) such that \( n(d) \mathcal{U} n'(d') \) for large enough \( d \) and \( d' \).

More generally, if \( \mathcal{V} \) is another compatible uniformity on \( A \) such that \( \mathcal{U} \leq \mathcal{V} \), then there is a uniformity \( \mathcal{V}/\mathcal{U} \) on \( A/\mathcal{U} \) having a base of relations \( R(\mathcal{U}, \mathcal{V}) \) for \( \mathcal{V} \in \mathcal{V} \).

Remark. It follows from [10, Theorem 9.9(2)] that \( \mathcal{V}/\mathcal{U} = (\text{nat} \mathcal{U})_{\ast c}(\mathcal{V}) = (\text{nat} \mathcal{U})_{\ast}(\mathcal{V}) \).

Formation of the completion plays the same role in uniform universal algebra that formation of quotient algebras plays in standard universal algebra, and satisfies many of the same properties. See [10, Section 9]. As an example of the close relationship between these two constructions, if \( \alpha \in \text{Con} A \) then \( A/\mathcal{U}g\{ \alpha \} \cong \langle A/\alpha, \mathcal{U}_d \rangle \), where \( \mathcal{U}_d = Fg\{ \Delta \} \) is the discrete uniformity.

Joins of compatible uniformities. Joins in the lattice of compatible uniformities are the same as in the lattice of uniformities on the underlying set [10, Theorem 5.3].

Compatible uniformities on algebras in congruence-permutable algebras. We recall ([10, Theorems 6.4 and 6.2]) that if \( A \) is an algebra in a congruence-permutable variety, and \( \mathcal{U}, \mathcal{V} \in \text{Unif} A \), then \( \mathcal{U} \) and \( \mathcal{V} \) permute, and that \( \text{Unif} A \) is modular.

3. Topological Groups and Uniform Groups

A topological group is a group object in the category of topological spaces and continuous functions. Such an object is determined by a group \( G \) and a topology \( \mathcal{T} \) on \( G \) such that
the group operations are continuous functions. Thus, it is different from a uniform group, or group with a compatible uniformity \( \langle G, \mathcal{U} \rangle \), where the operations are required to be not only continuous but uniformly continuous.

**Axioms for topological groups.** If \( \langle G, \mathcal{T} \rangle \) is a topological group, then the set \( \mathcal{N} \) of neighborhoods of the identity \( e \) satisfies the following axioms:

\[
\begin{align*}
(G1) \text{ If } N \in \mathcal{N} \text{ and } N \subseteq N', \text{ then } N' \in \mathcal{N}. \\
(G2) \text{ If } N, N' \in \mathcal{N}, \text{ then } N \cap N' \in \mathcal{N}. \\
(G3) \text{ For any } N \in \mathcal{N}, \text{ there exists a neighborhood } \tilde{N} \in \mathcal{N} \text{ such that } \tilde{N}N = \{ xy \mid x, y \in \tilde{N} \} \subseteq N. \\
(G4) \text{ If } N \in \mathcal{N}, \text{ then } N^{-1} \in \mathcal{N}. \\
(G5) \text{ If } N \in \mathcal{N} \text{ and } a \in G, \text{ then } a^{-1}Na \in \mathcal{N}.
\end{align*}
\]

A stronger version of (G5) which will be useful to us is

\[
(G5') \text{ If } N \in \mathcal{N}, \text{ then } \bigcap_a a^{-1}Na \in \mathcal{N}.
\]

**Uniform groups.** A uniform group, or, group with a compatible uniformity, is a pair \( \langle G, \mathcal{U} \rangle \) where \( G \) is a group and \( \mathcal{U} \) is a compatible uniformity. If \( \langle G, \mathcal{U} \rangle \) is a uniform group, then the topology \( \mathcal{T} \) underlying the uniformity \( \mathcal{U} \) is compatible with the group operations. The neighborhood system \( \mathcal{N} \) of this topology is given by \( N \in \mathcal{N} \) iff \( N = \{ x \mid x U e \} \) for some \( U \in \mathcal{U} \). The conditions (G1) through (G5) can easily be verified.

**Translation invariance.** If \( A \) is a group, we say that a relation \( U \subseteq A^2 \) is left translation invariant (right translation invariant) if \( a \in A \) and \( b U b' \) imply \( ab U ab' \) (respectively, \( ba U b'a \)). If \( A \) is abelian, then left translation invariance and right translation invariance coincide and we simply say that a relation is translation invariant.

**Lemma 3.1.** Every compatible uniformity \( \mathcal{U} \) on a group \( A \) has a base of left translation invariant relations and a base of right translation invariant relations.

**Proof.** Given \( U \in \mathcal{U} \), let \( U' \in \mathcal{U} \) be such that if \( b U' c \), then \( ab U ac \) for any \( a \). Define the relation \( V \) by \( x V y \) iff there exist \( b, c \) such that \( x^{-1}y = b^{-1}c \) and \( b U' c \). Then clearly \( U' \subseteq V \), so that \( V \in \mathcal{U} \), and also \( V \subseteq U \) because \( b U' c \) and \( x^{-1}y = b^{-1}c \) imply \( x = (xb^{-1})b U (xb^{-1})c = xx^{-1}y = y \). But \( V \) is left translation invariant. This proves that \( \mathcal{U} \) has a base of left translation invariant relations; the proof that \( \mathcal{U} \) has a base of right translation invariant relations is similar. \( \square \)

**The left uniformity and right uniformity of a compatible topology.** Let \( \mathcal{N} \) be a neighborhood system for a compatible topology on \( G \). If \( N \in \mathcal{N} \), we define \( N_l = \{ \langle x, y \rangle \in G^2 \mid y \in xN \} \). The set of relations \( \{ N_l \mid N \in \mathcal{N} \} \) is a base for a compatible uniformity \( \mathcal{U}_{\mathcal{T}, l} \) on \( G \), called the left uniformity. Similarly, if \( N \in \mathcal{N} \), we define \( N_r = \{ \langle x, y \rangle \mid y \in Nx \} \), and the \( N_r \) form a base for the right uniformity, denoted by \( \mathcal{U}_{\mathcal{T}, r} \). The inverse operation is a uniform isomorphism (of the uniform structure) when viewed as a function from \( \langle G, \mathcal{U}_{\mathcal{T}, l} \rangle \) to \( \langle G, \mathcal{U}_{\mathcal{T}, r} \rangle \). Note that \( \mathcal{U}_{\mathcal{T}, l} \) has a base of left translation invariant relations, and \( \mathcal{U}_{\mathcal{T}, r} \) has a base of right translation invariant relations.
Theorem 3.2. Let \( \langle G, T \rangle \) be a topological group. There is at most one compatible uniformity \( U \) on \( G \) such that \( T \) is the topology underlying \( U \), and in this case, \( U = U_{T,1} = U_{T,r} \).

Proof. If \( U \) exists, then by Lemma 3.1, \( U \) has a base of left-invariant relations. It follows that if \( T \) is the underlying topology of a compatible uniformity \( U \), then \( T \) determines \( U \) as \( U_{T,1} \). Similar arguments apply to the right uniformity. \( \square \)

Theorem 3.3. Let \( \langle A, T \rangle \) be a topological group. The following are equivalent:

(1) \( U_{T,r} \leq U_{T,1} \)
(2) \( U_{T,1} \leq U_{T,r} \)
(3) \( U_{T,1} = U_{T,r} \)
(4) \( U_{T,1} \) is compatible
(5) \( U_{T,r} \) is compatible
(6) The neighborhood system \( N \) corresponding to \( T \) satisfies \((G6')\).

Proof. Clearly, (3) \( \implies \) (1) and (3) \( \implies \) (2). We have (4) \( \implies \) (3) and (5) \( \implies \) (3) by Theorem 3.2.

(1) \( \implies \) (6): If \( U_{T,r} \leq U_{T,1} \), then given \( N \in \mathcal{N} \), there is an \( \tilde{N} \in \mathcal{N} \) such that \( \tilde{N} \subseteq N \). That is, \( y \in \tilde{N} \implies y \in xN \), or \( xy^{-1} \in \tilde{N} \implies x^{-1}y \in N \). This implies by a change of variables that \( x \in \tilde{N} \implies a^{-1}xa \in N \) for all \( x \) and \( a \), i.e., (6).

To prove (6) \( \implies \) (4), we must show that \( U_{T,1} \) is compatible with respect to group multiplication and the inverse operation.

To show \( U_{T,1} \) is compatible with respect to group multiplication, it suffices to show that given \( N \in \mathcal{N} \), there is an \( \tilde{N} \in \mathcal{N} \) such that \( \langle x, x' \rangle, \langle y, y' \rangle \in \tilde{N} \implies \langle xy, x'y' \rangle \in N \), or in other words, \( x^{-1}x', y^{-1}y' \in \tilde{N} \implies y^{-1}x^{-1}x'y' \in N \). Given \( N \), there is an \( \tilde{N} \in \mathcal{N} \) such that \( a^{-1}\tilde{N}a \subseteq \tilde{N} \) for all \( a \). There is an \( \tilde{N} \in \mathcal{N} \) such that \( \tilde{N}N \subseteq \tilde{N} \), by \((G5')\). Finally, there is an \( \tilde{N} \in \mathcal{N} \) such that \( a^{-1}\tilde{N}a \subseteq \tilde{N} \) for all \( a \). Then

\[
x^{-1}x', y^{-1}y' \in \tilde{N} \implies x^{-1}x', y'y^{-1} \in \tilde{N} \\
\implies x^{-1}x'y'y^{-1} \in \tilde{N} \\
\implies y^{-1}x^{-1}x' \in N.
\]

To show that \( U_{T,1} \) is compatible with respect to the inverse operation, it suffices to show that for each \( N \), there is an \( \tilde{N} \) such that \( \langle x, y \rangle \in \tilde{N} \implies \langle x^{-1}, y^{-1} \rangle \in N \), or in other words, \( x^{-1}y \in \tilde{N} \implies xy^{-1} \in N \). Given \( N \), let \( \tilde{N} \) be such that \( \tilde{N}^{-1} \subseteq N \), and let \( \tilde{N} \) be such that \( xy \in \tilde{N} \implies yx \in \tilde{N} \) (true by \((G5')\)). Then \( x^{-1}y \in \tilde{N} \implies yx^{-1} \in \tilde{N} \implies xy^{-1} \in N \). Thus, (6) \( \implies \) (4).

The proof that (6) \( \implies \) (5) is similar. \( \square \)
4. Term Conditions, Centralization, and Related Commutator Operations

In this section, we will discuss various conditions we call term conditions, and define notions of centralization and commutator operations based on them. First, more or less following [4], we review the $\alpha, \beta$-term condition. Then, we generalize this to uniformities and give the $U, V$-term condition. We also give two weaker conditions, which we call the weak $\alpha, \beta$-term condition and the weak $U, V$-term condition. As we state these four conditions, we give corresponding notions of centralization. Then we define notions of commutator (binary operations on Con $A$ and Unif $A$) derived from the four types of centralization, and finally, we prove some relationships between the various notions, showing that centralization for congruences can be considered a special case of centralization for uniformities.

The $\alpha, \beta$-term condition. We begin with the $\alpha, \beta$-term condition. We consider it as coming in two equivalent forms:

If $\alpha, \beta, \delta \in \text{Con} A$, for some algebra $A$, then we say that $\delta$ satisfies the first form of the $\alpha, \beta$-term condition if for all $a, a' \in A$ such that $a \alpha a'$, for all $b, c \in A^n$, $n > 0$, such that $b \beta c$ (i.e., $b_i \beta c_i$ for all $i$), and all $(n + 1)$-ary terms $t, t(a, b) \delta t(a, c)$ implies $t(a', b) \delta t(a', c)$.

To give the second form of the $\alpha, \beta$-term condition, we first define, given $n \geq 0$, $n' \geq 0$, and $t$, an $(n + n')$-ary term, and binary relations $U, V$ on $A$, the set of $2 \times 2$ matrices

$$M_{n,n,t}(U, V) = \left\{ \left( \begin{array}{cc} t(a, b) & t(a, c) \\ t(a', b) & t(a', c) \end{array} \right) : a, a' \in A^n, b, c \in A^{n'}, \text{ with } a U a' \text{ and } b V c \right\}.$$  

Then we say that $\delta$ satisfies the second form of the $\alpha, \beta$-term condition if for all $n, n'$, and $t$,\( \left( \begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array} \right) \in M_{n,n',t}(\alpha, \beta) \) and $u_{11} \delta u_{12}$ imply $u_{21} \delta u_{22}$.

**Proposition 4.1.** The two forms of the $\alpha, \beta$-term condition are equivalent.

**Proof.** Clearly if $\delta$ satisfies the second form, it satisfies the first. Given $\delta$ satisfying the first form, $n$, $n'$, and $t$, and \( \left( \begin{array}{cc} t(a, b) & t(a, c) \\ t(a', b) & t(a', c) \end{array} \right) \in M_{n,n',t}(\alpha, \beta) \), such that $t(a, b) \delta t(a, c)$, we apply the first form $n$ times, changing one component of $a$ at a time, to obtain the conclusion that $t(a', b) \delta t(a', c)$. \hfill $\square$

In view of this equivalence, we simply say that $\delta$ satisfies the $\alpha, \beta$-term condition. When this is so, we also say that $\alpha$ centralizes $\beta$ modulo $\delta$, or $C(\alpha, \beta; \delta)$.

**Proposition 4.2.** We have

1. $C(\alpha, \beta; \alpha)$
2. $C(\alpha, \beta; \beta)$
3. $C(\alpha, \beta; \delta)$ and $\alpha' \leq \alpha, \beta' \leq \beta$ imply $C(\alpha', \beta'; \delta)$
4. If $C(\alpha, \beta; \delta_i)$ for $i \in I$, then $C(\alpha, \beta; \bigwedge_i \delta_i)$. 

The $U, V$-term condition. We now generalize the term condition to compatible uniformities. Let $A$ be an algebra, and let $U, V, W \in \text{Unif } A$.

We say that $W$ satisfies the first form of the $U, V$-term condition if for all $n > 0$, all $(n + 1)$-ary terms $t$, and all $W \in W$, there are $U \in U$, $V \in V$, $W \in W$ such that for all $a, a' \in A$ such that $a U a'$, and all $b, c \in A^n$ such that $b V c$ for all $i$, $t(a, b) W t(a', c)$ implies $t(a', b) W t(a', c)$.

We say that $W$ satisfies the second form of the $U, V$-term condition if for all $n, n'$, all $(n + n')$-ary $t$, and all $W \in W$, there are $U \in U$, $V \in V$, and $W \in W$ such that $\left( \begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array} \right) \in M_{n,n',t}(U, V)$ and $u_{11} W u_{12}$ imply $u_{21} W u_{22}$.

Theorem 4.3. The two forms of the $U, V$-term condition are equivalent.

Proof. Clearly, the second form of the condition implies the first form. In the other direction, given $n, n'$, and $t, t(a, b)$ and $t(a, c)$ can be changed to $t(a', b)$ and $t(a', c)$ one component of $a$ at a time. For the $\alpha, \beta$-term condition, the result is immediate, but in the uniformity-theoretic case, given $W \in W$, we must choose in reverse order $W_i$, $i = 1, \ldots, n$ such that the changes are valid, starting by choosing $U_n \in U$, $V_n \in V$, and $W_n \in W$ such that $a_n U_n$, $b V_n c$, and $t(a'_1, \ldots, a'_{n-1}, a_n), b) W_n t(a'_1, \ldots, a'_{n-1}, a_n), c)$ imply $t(a, b) W t(a', c)$, and ending by choosing $U_1, V_1$, and $W_1$ such that $a_1 U_1 a'_1$, $b V_1 c$, and $t(a, b) W_1 t(a, c)$ imply $t(a'_1, a_2, \ldots, a_n), b) W_2 t(a'_1, a_2, \ldots, a_n), c)$. We then let $U = \bigcap_i U_i$, $V = \bigcap_i V_i$, and $W = W_1$. $\square$

If $W$ satisfies the two equivalent forms of the $U, V$-term condition, then we say that $U$ centralizes $V$ modulo $W$, or $C(U, V; W)$.

Proposition 4.4. We have

1. $C(U, V; U)$
2. $C(U, V; V)$
3. $C(U, Y; W)$ and $U' \subseteq U$, $V' \subseteq V$ imply $C(U', V'; W)$
4. If $C(U, V; W_i)$ for $i \in I$, then $C(U, V; \bigcap_i W_i)$
5. $C(Ug\{ \alpha \}, Ug\{ \beta \}; Ug\{ \delta \})$ iff $C(\alpha, \beta, \delta)$.

Proof. To show (1), let $U \in U$ and $t$ be given. There is a symmetric $\bar{U} \in U$ such that $a \bar{U} a'$ implies $t(a, b) \bar{U} t(a', b)$ and $t(a, c) \bar{U} t(a', c)$. Then for any $V \in V$, $a \bar{U} a'$ and $t(a, b) \bar{U} t(a, c)$ imply $t(a', b) \bar{U} t(a', c)$.

To show (2), let $V \in V$ and $t$ be given. Then for some $\bar{V} \in V$, $b V c$ implies $t(a', b) \bar{V} t(a', c)$ for any $a'$, by uniform continuity of $t$, regardless of any consideration of $t(a, b)$ and $t(a, c)$.

(3) is obvious.

(4): Suppose the $W_i$ satisfy the $U, V$-term condition, and that $n > 0$, an $(n + 1)$-ary term $t$, and $W \in \bigcap_i W_i$ are given. Then $\bigcap_{j=1}^k W_j \subseteq W$ for some uniform neighborhoods $W_j \in W_{i_j}$, $i_j$ being selected values of the index $i$. It suffices to show that $\bigcap_{j=1}^k W_{i_j}$ satisfies the $U, V$-term condition.


Let \( U_j \in \mathcal{U}, V_j \in \mathcal{V}, W_j \in \mathcal{W}_j \) be relations as promised by the \( \mathcal{U}, \mathcal{V} \)-term condition for \( \mathcal{W}_j \), and let \( U = \bigwedge_j U_j, V = \bigwedge_j V_j, \) and \( W = \bigwedge_j W_j \). Then if \( a U a', b V c, \) and \( t(a, b) W t(a, c) \), we have \( t(a', b) W_j t(a', c) \) for each \( j \), whence \( t(a', b) W t(a', c) \).

(5) follows easily from the definitions.

\[ \square \]

**The weak term conditions.** We say that \( \delta \) satisfies the **weak \( \alpha, \beta \)-term condition** if for all \( n, n', \) and \( t, \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n,n',t}(\alpha, \beta) \) and \( u_{11} = u_{12} \) imply \( u_{21} \delta u_{22} \). In case \( \delta \) satisfies the weak \( \alpha, \beta \)-term condition, we say that \( \alpha \) **weakly centralizes** \( \beta \) **modulo** \( \delta \), or \( \tilde{C}(\alpha, \beta; \delta) \).

**Proposition 4.5.** We have

1. \( \tilde{C}(\alpha, \beta; \alpha) \)
2. \( \tilde{C}(\alpha, \beta; \beta) \)
3. \( \tilde{C}(\alpha, \beta; \delta) \) and \( \alpha' \leq \alpha, \beta' \leq \beta \) imply \( C(\alpha', \beta'; \delta) \)
4. If \( \tilde{C}(\alpha, \beta; \delta_i) \) for \( i \in I \), then \( \tilde{C}(\alpha, \beta; \bigwedge_i \delta_i) \)
5. \( \tilde{C}(\alpha, \beta; \delta) \) and \( \delta \leq \delta' \) imply \( \tilde{C}(\alpha, \beta; \delta') \)
6. \( C(\alpha, \beta; \delta) \) \( \implies \) \( \tilde{C}(\alpha, \beta; \delta) \).

We say that \( \mathcal{W} \) satisfies the **weak \( \mathcal{U}, \mathcal{V} \)-term condition** if for all \( n, n', \) and \( t, \) and all \( W \in \mathcal{W} \), there exist \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \) such that \( \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n,n',t}(U, V) \) and \( u_{11} = u_{12} \) imply \( u_{21} W u_{22} \). If \( \mathcal{W} \) satisfies the weak \( \mathcal{U}, \mathcal{V} \)-term condition, we say that \( \mathcal{W} \) **weakly centralizes** \( \mathcal{U} \) **modulo** \( \mathcal{V} \), or \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}) \).

**Proposition 4.6.** We have

1. \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{U}) \)
2. \( \tilde{C}(\mathcal{U}'; \mathcal{V}; \mathcal{V}) \)
3. \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}) \) and \( \mathcal{U}' \leq \mathcal{U}, \mathcal{V}' \leq \mathcal{V} \) imply \( \tilde{C}(\mathcal{U}', \mathcal{V}'; \mathcal{W}) \)
4. If \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}_i) \) for \( i \in I \), then \( \tilde{C}(\mathcal{U}, \mathcal{V}; \bigwedge_i \mathcal{W}_i) \)
5. \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}) \) and \( \mathcal{W} \leq \mathcal{W}' \) imply \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}') \)
6. \( C(\mathcal{U}, \mathcal{V}; \mathcal{W}) \) \( \implies \) \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}) \)
7. \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{W}) \) \( \iff \) \( \tilde{C}(\alpha, \beta; \delta) \).

**Proof.** (1): Given \( n, n', \) and \( t, \) and \( U \in \mathcal{U} \), there exists a symmetric \( U' \in \mathcal{U} \) such that \( a U' a' \) implies \( t(a, b) U t(a', b) \) for all \( b \). It follows that for any \( V \in \mathcal{V}, \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n,n',t}(U', V) \) and \( u_{11} = u_{12} \) imply \( u_{21} 2U u_{22} \).

(2) The proof of \( \tilde{C}(\mathcal{U}, \mathcal{V}; \mathcal{V}) \) is the same as the proof of \( C(\mathcal{U}, \mathcal{V}; \mathcal{V}) \).

(3) is obvious.

(4): Again we can reduce to the case \( I \) finite. For \( W \in \bigwedge_i \mathcal{W}_i, W = \bigcap_i \mathcal{W}_i \) for some \( \mathcal{W}_i \in \mathcal{W}_i \). Given \( n, n', \) and \( t, \) there exist \( U_i \in \mathcal{U}, V_i \in \mathcal{V} \) such that \( \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n,n',t}(U_i, V_i) \)
and \( u_{11} = u_{12} \) imply \( u_{21} W u_{22} \). Then \( \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n,n',t}(\bigcap_i U_i, \bigcap_i V_i) \) and \( u_{11} = u_{12} \) imply \( u_{21} W u_{22} \).

(5), (6), and (7) are clear. \( \square \)

Related commutator operations. These notions of centralization lead to definitions for binary operations on the lattice of compatible uniformities. Recall [4] that if \( A \) is an algebra, and \( \alpha, \beta \in \text{Con} A \), then \( C(\alpha, \beta) \) is defined as the least congruence on \( A \) satisfying the \( \alpha, \beta \)-term condition. Similarly, we denote by \( C(\mathcal{U}, \mathcal{V}) \) the least uniformity satisfying the \( \mathcal{U}, \mathcal{V} \)-term condition, by \( \tilde{C}(\alpha, \beta) \) the least congruence satisfying the weak \( \alpha, \beta \)-term condition, and by \( \tilde{C}(\mathcal{U}, \mathcal{V}) \) the least uniformity satisfying the weak \( \mathcal{U}, \mathcal{V} \)-term condition. These uniformities exist by Proposition 4.4(4) and Proposition 4.6(4).

These commutator operations have some common properties:

**Theorem 4.7.** Let \( \tilde{C}(x, y) \) stand for \( C(\alpha, \beta), C(\mathcal{U}, \mathcal{V}), \tilde{C}(\alpha, \beta), \) or \( \tilde{C}(\mathcal{U}, \mathcal{V}) \). Then

1. \( \tilde{C}(x, y) \leq x \land y \)
2. \( x' \leq x, y' \leq y \) imply \( \tilde{C}(x', y') \leq \tilde{C}(x, y) \).

Here are some explicit formulas for \( C(\alpha, \beta) \) and \( \tilde{C}(\alpha, \beta) \):

**Proposition 4.8.** Let \( A \) be an algebra, and \( \alpha, \beta \in \text{Con} A \). We have

1. \( C(\alpha, \beta) = \bigcup_{\nu} R_{\nu} \), where the relations \( R_{\nu} \) are defined for all ordinal numbers \( \nu \), as follows:

\[
R_{\nu} = \begin{cases} \Delta^A, & \nu = 0 \\
Cg \{ \langle t(a', b), t(a', c) \rangle \mid t \text{ is } (n+1)-\text{ary}, a \alpha a', b \beta c, \\
\text{and } t(a, b) R_{\nu'} t(a, c) \}, & \nu = \nu' + 1 \\
\bigcup_{\nu' < \nu} R_{\nu'}, & \text{for } \nu \text{ a limit ordinal} \end{cases}
\]

2. \( \tilde{C}(\alpha, \beta) = Cg \tilde{R} \) where

\[
\tilde{R} = \{ \langle u_{21}, u_{22} \rangle \mid \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n,n',t}(\alpha, \beta) \text{ for some } n, n', \text{ and } t, \text{ and } u_{11} = u_{12} \}.
\]

**Theorem 4.9.** If \( \alpha, \beta \in \text{Con} A \), then we have

1. \( C(\text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \}) = \text{Ug}\{ C(\alpha, \beta) \} \)
2. \( \tilde{C}(\text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \}) = \text{Ug}\{ \tilde{C}(\alpha, \beta) \} \).

**Proof.** (1): By Proposition 4.4(5), since \( C(\alpha, \beta) \) satisfies the \( \alpha, \beta \)-term condition, the uniformity \( \text{Ug}\{ C(\alpha, \beta) \} \) satisfies the \( \text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \} \)-term condition. Thus, \( C(\text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \}) \leq \text{Ug}\{ C(\alpha, \beta) \} \). To show the opposite inequality, we must show that \( W \in C(\text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \}) \) implies \( C(\alpha, \beta) \subseteq W \). By Proposition 4.8(1), \( C(\alpha, \beta) = \bigcup_{\nu} R_{\nu} \). However, by transfinite induction, and the \( \text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \} \)-term condition, we also have \( R_{\nu} \subseteq W \) for all \( \nu \) and \( W \). Thus, \( C(\alpha, \beta) \subseteq W \) and \( \text{Ug}\{ C(\alpha, \beta) \} \leq C(\text{Ug}\{ \alpha \}, \text{Ug}\{ \beta \}) \).
(2): The proof that \( \tilde{C}(\{\alpha\}, \{\beta\}) \leq \{\tilde{C}(\alpha, \beta)\} \) follows the same pattern as for the operation \( C(-, -) \). To show that \( \{\tilde{C}(\alpha, \beta)\} \leq \tilde{C}(\alpha, \beta) \subseteq W \), we must show that for every \( W \in \tilde{C}(\{\alpha\}, \{\beta\}) \), \( \tilde{C}(\alpha, \beta) \subseteq W \). However, it is clear that \( \tilde{R} \in W \), where \( \tilde{R} \) is the relation defined in the statement of Proposition 4.8(2). Thus, \( \tilde{R} \subseteq \bigcap \tilde{C}(\{\alpha\}, \{\beta\}) \). But this intersection is a congruence. Thus, \( \tilde{C}(\alpha, \beta) = Cg \tilde{R} \subseteq W \) by Proposition 4.8(2).

By this theorem, the commutator operations \( C(-, -) \) and \( \tilde{C}(-, -) \) on uniformities extend the corresponding commutator operations on congruences, and we can compute the commutators on congruences by computing the corresponding commutators of uniformities. The rule is to promote both arguments to uniformities, and then take the chosen commutator. The resulting uniformity then determines the desired congruence.

5. The Commutator on Uniformities in Congruence-Modular Varieties

As described in the previous section, the situation for a general variety is that we have defined two possibly different, possibly noncommutative commutator operations on uniformities, \( C(-, -) \) and \( \tilde{C}(-, -) \). We will show in this section that as it is with congruences [4], the situation is much simplified for congruence-modular varieties: these operations coincide and are commutative.

\[ \mathcal{M}(U, V), x_m(M), \text{and } X_m(U, V). \]

**Proposition 5.1.** Let \( U, V \in \text{Unif } A \). The set of sets of \( 2 \times 2 \) matrices \( M_{n,n',t}(U, V), U \in U, V \in V \) is a base for a filter \( \mathcal{M}_{n,n',t}(U, V) \) of sets of \( 2 \times 2 \) matrices of elements of \( A \).

If \( U, V \in \text{Unif } A \), then we define

\[ \mathcal{M}(U, V) = \bigvee_{n,n',t} \mathcal{M}_{n,n',t}(U, V) = \bigcap_{n,n',t} \mathcal{M}_{n,n',t}(U, V). \]

Let \( m_0, m_1, \ldots, m_k \) be a sequence of quaternary terms for \( A \). If \( M \) is a set of \( 2 \times 2 \)-matrices of elements of \( A \), we denote by \( x_m(M) \) the set of pairs

\[ \langle m_i(a, b, d, c), m_i(a, a, c, c) \rangle \]

such that \( i \leq k \) and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M \).

**Proposition 5.2.** Given a filter \( \mathcal{M} \) of sets of \( 2 \times 2 \)-matrices of elements of \( A \), the set of \( x_m(M) \), \( M \in \mathcal{M} \) forms a base for a filter \( x_m(\mathcal{M}) \) of relations on \( A \).

If \( n, n', \) and \( t \) are given, then we denote by \( X_{m,n,n',t}(U, V) \) the filter \( x_m(\mathcal{M}_{n,n',t}(U, V)) \), and by \( X_m(U, V) \) the filter \( x_m(\mathcal{M}(U, V)) \).
The commutator in modular varieties. The following theorem can be compared to \([4, \text{Proposition 4.2}(1)]\), which establishes the analogous result for commutators of congruences.

**Theorem 5.3.** If \(U, V, W\) are compatible uniformities on \(A\), an algebra in a congruence-modular variety with sequence of Day terms \(m\), then the following are equivalent:

\[
\begin{align*}
(1) \quad & X_m(U, V) \leq W \\
(2) \quad & X_m(V, U) \leq W \\
(3) \quad & C(U, V; W) \\
(4) \quad & C(V, U; W) \\
(5) \quad & \tilde{C}(U, V; W) \\
(6) \quad & \tilde{C}(V, U; W).
\end{align*}
\]

**Proof.** Obviously \((3) \Rightarrow (5)\) and \((4) \Rightarrow (6)\). We will show that also \((5) \Rightarrow (1) \Rightarrow (4)\). Exchanging \(U\) and \(V\) will then give \((6) \Rightarrow (2) \Rightarrow (3)\), completing the proof of equivalence.

\((5) \Rightarrow (1)\): Assume \(\tilde{C}(U, V; W)\). It suffices to show that if \(n, n', t, \) and \(W \in W\) are given, then there are \(U_{n, n', t} \in U\) and \(V_{n, n', t} \in V\) such that for all \(a, a' \in A^u\) with \(a U_{n, n', t} a'\), and all \(b, c \in A^{n'}\) such that \(b V_{n, n', t} c\), and for all \(i\), we have \(x W y\), where

\[
x = x(a, a', b, c) = m_i(t(a, b), t(a, c), t(a', c), t(a', b)),
\]

and

\[
y = y(a, a', b, c) = m_i(t(a, b), t(a, b), t(a', b), t(a', b)).
\]

For, this implies that \(x_m(M_{n, n', t}(U_{n, n', t}, V_{n, n', t})) \subseteq W\), and we then have

\[
x_m\left(\bigcup_{n, n', t} M_{n, n', t}(U_{n, n', t}, V_{n, n', t})\right) \subseteq W;
\]

but \(\bigcup_{n, n', t} M_{n, n', t}(U_{n, n', t}, V_{n, n', t}) \in \mathcal{M}(U, V)\). Thus, it follows that \(\mathcal{M}(U, V) \leq W\).

If we replace \(a\) by \(a'\) at its second occurrence in the right-hand expressions for \(x\) and \(y\), and \(a'\) by \(a\) in its second occurrence, then we obtain expressions for the same element \(z = m_i(t(a, b), t(a', c), t(a', c), t(a, b)) = m_i(t(a, b), t(a, b), t(a', b), t(a, b)) = t(a, b)\). Let \(s_i\) be the \((4n + 4n')\)-ary term given by

\[
s(g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4) = m_i(t(g_1, h_1), t(g_2, h_2), t(g_3, h_3), t(g_4, h_4)).
\]

Since \(\tilde{C}(U, V; W)\), there are \(U_i \in U\) and \(V_i \in V\) such that for all \(u_{11} u_{12} u_{21} u_{22} \in M_{4n, 4n', s_i}(U_i, V_i)\),

\[
\begin{pmatrix}
z & z \\
x & y
\end{pmatrix} = \begin{pmatrix}
s_i(a, a', a', a, b, b, b, b) & s_i(a, a', a', a, b, b, b, b) \\
s_i(a, a, a', a', b, c, c, b) & s_i(a, a, a', a', b, b, b, b)
\end{pmatrix} \in M_{4n, 4n', s_i}(U_i, V_i).
\]

Thus, \(x W y\). Letting \(U = \bigwedge U_i\) and \(V = \bigwedge V_i\), we have \(x_m(M_{n, n', t}(U, V)) \subseteq W\), proving that \((5) \Rightarrow (1)\).

To prove \((1) \Rightarrow (4)\), let \(W \in W\) and let \(n, n', \) and \(t\) be given. By \([10, \text{Lemma 7.1}]\) (a uniformity-theoretic generalization of \([4, \text{Lemma 2.3}]\)) there exists \(W \in W\) such that \(a, b, c\),
c, d ∈ A with b W d and m_i(a, a, c, c) W m_i(a, b, d, c) for all i imply a W c. By (1), there are U and V such that x_m(M_{n,n,t}(U^{-1}, V^{-1})) ⊆ W^{-1}. If \((\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}) \in M_{n,n,t}(V, U), \) then
\[
\begin{pmatrix} u_{21} & u_{11} \\ u_{22} & u_{12} \end{pmatrix} \in M_{n,n,t}(U^{-1}, V^{-1})
\]
for all i. It follows that if u_{11} W u_{12}, then u_{21} W u_{22}. A W, U, and V exists for each n, n', and t, implying (4).

If U, V ∈ Unif A, we define [U, V] to be the least W such that the six equivalent statements in the theorem hold. Of course, we then have [U, V] = C(U, V) = C(V, U) = \(\bar{C}(U, V) = \bar{C}(V, U) = Ug X_m(U, V)\).

**Corollary 5.4.** Let A be an algebra in a congruence-modular variety with Day terms m, and let U, V, W, and W' ∈ Unif A. If C(U, V; W) and W < W', then C(U, V; W').

6. Unif A(α) and the Commutator [Ug\{ α \}, U]

Recall that if A is an algebra, and α ∈ Con A, then A(α) is the subalgebra of A^2 of pairs ⟨a, b⟩ such that a α b. We will denote by π, π' : A(α) → A and \(\Delta_\alpha : A → A(\alpha)\) the homomorphisms defined respectively by ⟨a, b⟩ → a, ⟨a, b⟩ → b, and a → ⟨a, a⟩. (Note that in [4], the notation \(\Delta_A\) is used for \(\Delta_\alpha\), whereas we use \(\Delta_A\) to denote the diagonal set in A^2.)

If α, β ∈ Con A, where A is an algebra, then we can construct a congruence \(\Delta_{α,β} ∈ Con A(α)\) by extending β along \(\Delta_\alpha\). That is, \(\Delta_{α,β} = Cg\{⟨⟨a, a⟩, ⟨b, b⟩⟩ | a β b⟩\}.\)

**Theorem 6.1.** If A is an algebra in a congruence-modular variety, then \((π')^{-1}[α, β] = (\Delta_{α,β} \land \ker π) \lor \ker π'\).

**Proof.** See [4, Theorem 4.9 and Exercise 4.4].

The Theorem gives one way to compute the commutator [α, β]. It may help to illuminate the relationship between [α, β] and Con A(α) to note that the interval sublattice I_{Con A(α)}[\bot, \ker π] transposes up to the interval sublattice I_{Con A(α)}[\ker π', \top], which is of course isomorphic to Con A.

In this section, we will try to duplicate this result with β replaced by a compatible uniformity U, and Con A(α) replaced by Unif A(α), the lattice of compatible uniformities of A(α).

For this section, we will write [α, U], M(α, U), etc. for [Ug{ α }, U], M(Ug{ α }, U), etc.

**M(α, U) and \(\Delta_{α,U}\).** We define \(\Delta_{α,U}\) as the analog of \(\Delta_{α,β}\); that is, the compatible extension \((\Delta_α)_c(U)\) along \(\Delta_α\) of U ∈ Unif A to A(α). We will construct \(\Delta_{α,U}\) using the filter M(α, U) defined in Section 5.
Now, $\mathcal{M}(\alpha, \mathcal{U})$ is a filter of subsets of $2 \times 2$ matrices. We view each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a pair $\langle \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \rangle$ of elements of $A(\alpha)$ written as column vectors. Thus, $\mathcal{M}(\alpha, \mathcal{U})$ can be viewed as a filter of binary relations on $A(\alpha)$.

**Theorem 6.2.** Viewed in this way, $\mathcal{M}(\alpha, \mathcal{U})$ is a singly compatible semiuniformity on $A(\alpha)$, and $\Delta_{\alpha, \mathcal{U}} = \text{Ug} \mathcal{M}(\alpha, \mathcal{U})$.

**Proof.** We have $\mathcal{M}(\alpha, \mathcal{U}) = \mathcal{L}_{\Delta_{\alpha}}(\mathcal{U})$. Thus, by Theorem 2.7, $\mathcal{M}(\alpha, \mathcal{U})$ is a singly compatible semiuniformity and $\Delta_{\alpha, \mathcal{U}} = (\Delta_{\alpha})_{\mathcal{U}} = \text{Ug}(\mathcal{L}_{\Delta_{\alpha}}(\mathcal{U})) = \text{Ug} \mathcal{M}(\alpha, \mathcal{U})$. \(\square\)

For, the fact that ker $\pi$ is a congruence causes the modular law to be true in this special case, by [10, Theorem 7.5].

Thus, the interval $I_{\text{Unif } A(\alpha)}[\bot, \ker \pi]$ can be embedded into Unif $A$ via a mapping $\phi$ such that $(\pi')^{-1}(\phi(\mathcal{U})) = \mathcal{U} \vee \ker \pi'$. We shall see that if $\mathcal{U} \in \text{Unif } A$, where $A$ is an algebra with an underlying group structure, then $[\alpha, \mathcal{U}]$ does belong to the image of $\phi$, and $[\alpha, \mathcal{U}] = \phi(\Delta_{\alpha, \mathcal{U}} \land \ker \pi)$. (Note that in this case, the modular law also holds in Unif $A$, by [10, Theorem 6.4].)

Let us define $[\alpha, \mathcal{U}]' = \phi(\Delta_{\alpha, \mathcal{U}} \land \ker \pi)$ or, in other words, let $[\alpha, \mathcal{U}]'$ be the unique compatible uniformity on $A$ such that $(\pi')^{-1}([\alpha, \mathcal{U}]') = (\Delta_{\alpha, \mathcal{U}} \land \ker \pi) \lor \ker \pi'$.

**Theorem 6.3.** Let $A$ be an algebra, and let $\alpha \in \text{Con } A$ and $\mathcal{U} \in \text{Unif } A$. Then $[\alpha, \mathcal{U}] \leq [\alpha, \mathcal{U}]'$, with equality if $A$ has an underlying group structure.

**Proof.** To show that $[\alpha, \mathcal{U}] \leq [\alpha, \mathcal{U}]'$, it suffices to prove that $\widetilde{C}(\alpha, \mathcal{U}; [\alpha, \mathcal{U}])$. We have

$$(\pi')^{-1}[\alpha, \mathcal{U}]' = (\Delta_{\alpha, \mathcal{U}} \land \ker \pi) \lor \ker \pi'$$

$$\geq \ker \pi' \circ (\Delta_{\alpha, \mathcal{U}} \land \ker \pi) \circ \ker \pi'$$

$$\geq \ker \pi' \circ (\mathcal{M}(\alpha, \mathcal{U}) \land \ker \pi) \circ \ker \pi',$$ which implies that for all $W \in [\alpha, \mathcal{U}]'$, there is a $Q \in \mathcal{M}(\alpha, \mathcal{U})$ such that $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in Q$ and $u_{11} = u_{12}$ imply $u_{21} W u_{22}$. Given in addition $n$, $n'$, and $t$, there is a $U_{n, n', t} \in \mathcal{U}$ such that $M_{n, n', t}(\alpha, U_{n, n', t}) \subseteq Q$, by the definition of $\mathcal{M}(\alpha, \mathcal{U})$. It follows that $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in M_{n, n', t}(\alpha, U_{n, n', t})$ and $u_{11} = u_{12}$ imply $u_{21} W u_{22}$, proving that $\widetilde{C}(\alpha, \mathcal{U}; [\alpha, \mathcal{U}])$.

Now assume $A$ has an underlying group structure. (In particular, this implies that $A$ belongs to a congruence-permutable variety and Unif $A(\alpha)$ is modular, by [10, Theorem 6.2].)
To show $[\alpha, \mathcal{U}]' \leq [\alpha, \mathcal{U}]$, it suffices to show that $\Delta_{\alpha, \mathcal{U}} \wedge \ker \pi \leq (\pi')^{-1}([\alpha, \mathcal{U}])$. For, this implies

\[
(\pi')^{-1}([\alpha, \mathcal{U}]') = (\Delta_{\alpha, \mathcal{U}} \wedge \ker \pi) \vee \ker \pi' \\
\leq ((\pi')^{-1}([\alpha, \mathcal{U}]) \wedge \ker \pi) \vee \ker \pi' \\
= (\pi')^{-1}([\alpha, \mathcal{U}]).
\]

To show $\Delta_{\alpha, \mathcal{U}} \wedge \ker \pi \leq (\pi')^{-1}([\alpha, \mathcal{U}])$, it suffices to show that $x_{m}(\Delta_{\alpha, \mathcal{U}}) \leq [\alpha, \mathcal{U}]$. For, in that case, by [10, Lemma 7.1], if $W \in [\alpha, \mathcal{U}]$, and $\bar{W} \in [\alpha, \mathcal{U}]$ is such that $\bar{W}^{-1} \subseteq W$, then $\bar{W} \in [\alpha, \mathcal{U}]$ can be chosen, and $V \in \Delta_{\alpha, \mathcal{U}}$, such that $x_{m}(V) \subseteq \bar{W}$, so that $(y, z) \in V \implies \bar{W}, x \implies x, W, y$. We have $\ker \pi \wedge \ker \pi' = \bot \leq \Delta_{\alpha, \mathcal{U}}$, and if $\left( \begin{array}{cc} b & b \\ x & y \end{array} \right) \in \Delta_{\alpha, \mathcal{U}}$, we also have $\left( \begin{array}{cc} y & y \\ x & x \end{array} \right) \ker \pi' \left( \begin{array}{cc} b & b \\ x & x \end{array} \right) \ker \pi \left( \begin{array}{cc} y & y \\ x & x \end{array} \right)$. It follows by the Shifting Lemma ([10, Lemma 7.4]) that there is a $\bar{V} \in \Delta_{\alpha, \mathcal{U}}$ such that if $\left( \begin{array}{cc} b & b \\ x & y \end{array} \right) \in \bar{V}$, then $\left( \begin{array}{cc} y & y \\ x & x \end{array} \right) \in \bar{V}$, implying $x, W, y$. It follows that $\bar{V} \cap \ker \pi \leq (\pi')^{-1}(W) \cap \ker \pi$, proving that $\Delta_{\alpha, \mathcal{U}} \wedge \ker \pi \leq (\pi')^{-1}([\alpha, \mathcal{U}])$.

Now, $x_{m}(\mathcal{M}(\alpha, \mathcal{U})) = x_{m}(\alpha, \mathcal{U}) \leq [\alpha, \mathcal{U}]$ by Theorem 5.3. For ordinal numbers $\nu$, we define inductively

\[
\mathcal{R}_{\nu} = \begin{cases} 
\mathcal{M}(\alpha, \mathcal{U}), & \nu = 0 \\
\mathcal{R}_{\nu'} \circ \mathcal{R}_{\nu}, & \nu = \nu' + 1 \\
\bigcap_{\nu' < \nu} \mathcal{R}_{\nu'}, & \nu \text{ a limit ordinal.}
\end{cases}
\]

We claim that $x_{m}(\mathcal{R}_{\nu}) \leq [\alpha, \mathcal{U}]$ for every ordinal $\nu$. It is easy to see that $\mathcal{R}_{\nu}$ is a singular compatible semiuniformity for every $\nu$, and that the sequence becomes stationary at $\Delta_{\alpha, \mathcal{U}} = \Ug \mathcal{M}(\alpha, \mathcal{U})$. If there is a first ordinal $\nu$ such that $x_{m}(\mathcal{R}_{\nu}) \notin [\alpha, \mathcal{U}]$, then clearly, $\nu \neq 0$ and $\nu$ is not a limit ordinal. Thus, to prove the claim, and that $x_{m}(\Delta_{\alpha, \mathcal{U}}) \leq [\alpha, \mathcal{U}]$, it suffices to show that if $x_{m}(\mathcal{R}_{\nu}) \leq [\alpha, \mathcal{U}]$, then $x_{m}(\mathcal{R}_{\nu+1}) \in [\alpha, \mathcal{U}]$. We must show that, given $W \in [\alpha, \mathcal{U}]$, there is an $R \in \mathcal{R}_{\nu+1}$ such that $x_{m}(R) \subseteq W$.

We have the facts that the terms $m_{0}(x, y, z, w) = w$, $m_{1}(x, y, z, w) = xz^{-1}yxy^{-1}w$, and $m_{2}(x, y, z, w) = w$ are a sequence of $\Delta$-terms for any variety $\mathcal{V}$ of algebras with group structures. Without loss of generality, by the discussion of Section 3, we can also assume that $W$ is left translation invariant. Let $\bar{W} \in [\alpha, \mathcal{U}]$ be left translation invariant and such that $a \bar{W} a'$ and $b \bar{W} a' \Rightarrow ab W a'b'$. Since $x_{m}(\mathcal{R}_{\nu}) \leq [\alpha, \mathcal{U}]$ by the induction hypothesis, there is an $R \in \mathcal{R}_{\nu}$ such that $x_{m}(R) \subseteq W$. We have

\[
x_{m}(\bar{R}) = \left\{ \langle xw^{-1}zv^{-1}u, x \rangle \mid \left( \begin{array}{cc} x & y \\ u & v \end{array} \right), \left( \begin{array}{cc} y & z \\ v & w \end{array} \right) \in \bar{R} \right\} \cup \left\{ \langle x, x \rangle \mid x \in A \right\},
\]

where the second term in the union takes care of the contributions to $x_{m}(\mathcal{R}_{\nu+1})$ coming from the terms $m_{0}$ and $m_{2}$. Since $x_{m}(\bar{R}) \subseteq \bar{W}$, we have $\langle xw^{-1}yv^{-1}u, x \rangle, \langle yw^{-1}zv^{-1}y, y \rangle \in \bar{W}$. 


This implies that \( \langle v^{-1}yx^{-1}u, e \rangle, \langle w^{-1}zy^{-1}v, e \rangle \in W \), which implies that 
\[
\langle (w^{-1}zy^{-1}v)(v^{-1}yx^{-1}u), e \rangle = \langle w^{-1}zx^{-1}u, e \rangle \in W.
\]
But \( W \) was assumed left translation invariant, so this implies that \( \langle xw^{-1}zx^{-1}v, x \rangle \in W \), proving that \( x_m(R \circ R) \subseteq W \), and by induction that \( x_m(\Delta_{\alpha, I}) \leq [\alpha, U] \). \( \square \)

7. Properties of the Commutator

In this section, we discuss general properties of the commutator on uniformities, for a congruence-modular variety.

**Elementary properties.**

**Theorem 7.1.** Let \( A \) be an algebra in a congruence-modular variety \( V \). We have

1. If \( U, V \in \text{Unif} A \), then \([U, V] = [V, U]\).
2. The commutator is monotone, i.e., if \( U \leq U' \), then \([U, V] \leq [U', V]\).
3. \([U, V] \leq U \land V\).
4. If \( B \in V \), \( f : B \to A \) is a homomorphism, and \( U, V \in \text{Unif} A \), then \([f^{-1}(U), f^{-1}(V)] \leq f^{-1}([U, V])\).

**Proof.** (1): This is just a restatement of part of Theorem 5.3.
(2): If \( W \) satisfies the \( U', V \)-term condition, then it also satisfies the \( U, V \)-term condition. Or, use the obvious fact that \( X_m(U, V) \leq X_m(U', V) \).
(3): Follows from Theorem 4.7.
(4): We have
\[
X_m(f^{-1}(U), f^{-1}(V)) \leq f^{-1}(X_m(U, V))
\]
because \( f \) is a homomorphism. \( \square \)

**Additivity.**

**Theorem 7.2.** Let \( A \) be an algebra in a congruence-modular variety with sequence of Day terms \( m \), and let \( U_i \), for \( i \in I \), and \( V \) belong to \( \text{Unif} A \). Then \( \bigvee_{i \in I} [U_i, V] \leq [\bigvee_{i \in I} U_i, V] \), with equality if \( V \) is of the form \( Ug\{\alpha\} \) and \( A \) has an underlying group structure, or if \( I \) is finite and the \( U_i \) permute pairwise.

**Proof.** \( \bigvee_{i \in I} [U_i, V] \leq [\bigvee_{i \in I} U_i, V] \) by Theorem 7.1(2).

Let us denote \( \bigvee_{i \in I} [U_i, V] \) by \( W \) for the remainder of the proof.

Suppose \( V = Ug\{\alpha\} \) for some congruence \( \alpha \) and \( A \) has an underlying group structure. The commutator is commutative so we can switch the arguments on each side. Since \( \alpha \) centralizes \( U_i \) modulo \( [\alpha, V_i U_i] \), we have \( \Delta_{\alpha, U_i} \land \ker \pi \leq Z \), where \( Z = (\pi')^{-1}([\alpha, V_i U_i]) \). Because compatible extension preserves joins, we have \( \Delta_{\alpha, V \cdot U_i} = \bigvee_i (\Delta_{\alpha, U_i}) = Ug(\bigcap_i \Delta_{\alpha, U_i}) \), where the singly compatible semiuniformity \( R = \bigcap_i \Delta_{\alpha, U_i} \) satisfies \( R \land \ker \pi \leq Z \). Then by the same argument used in the proof of Theorem 6.3, \( Ug(R) \land \ker \pi \leq Z \). It follows that \([\alpha, \bigvee_{i \in I} U_i] \leq W \).

Now suppose that \( I = \{1, \ldots, k\} \) and that the \( U_i \) permute pairwise. We will show that \( W \) satisfies the \((\bigvee_{i \in I} U_i), V\)-term condition. Given \( W \in W \) and \( t \), we define \( W_0 = W \) and
for each $i \in I$, inductively, define $U_i \in \mathcal{U}$, $V_i \in \mathcal{V}$, and $W_i \in \mathcal{W}$ to be such that $a U_i a'$, $b V_i c$, and $t(a,b) W_i t(a,c)$ imply $t(a',b) W_i t(a',c)$, using the fact that $C(U_i, V; W)$. Then $a U_k \circ \ldots \circ U_1 a'$, $b (\bigcap_i V_i) c$, and $t(a,b) W_k t(a,c)$ imply $t(a',b) W t(a',c)$. However, $U_k \circ \ldots \circ U_1 \in \bigcap_i \mathcal{U}_i$, because the $\mathcal{U}_i$ permute pairwise; it follows that $\mathcal{W}$ satisfies the $(\bigcup_i \mathcal{U}_i), \text{V-term condition, proving that } \mathcal{W} = \bigcup_i [\mathcal{U}_i, \mathcal{V}] = [\bigcup_i \mathcal{U}_i, \mathcal{V}].$ \hfill $\square$

**Corollary 7.3.** The commutator of compatible uniformities on an algebra in a congruence-permutable variety is finitely additive.

**Proof.** By [10, Theorem 6.4], compatible uniformities on an algebra in a congruence-permutable variety permute pairwise. \hfill $\square$

**The homomorphism property.** Recall the definition of $R(\mathcal{W}, U)$ from Section 2.

**Lemma 7.4.** Let $V$ be a congruence-modular variety with sequence of Day terms $m$. If $A \in \mathcal{V}$, $\mathcal{W} \in \text{Unif} A$, $U$ and $V$ are relations on $A$, and $n$, $n'$, and $t$ are given, then $$x_m(M_{n,n',t}(R(\mathcal{W}, U), R(\mathcal{W}, V))) \subseteq R(\mathcal{W}, x_m(M_{n,n',t}(U, V))).$$

**Proof.** An element of $R(\mathcal{W}, U)$ is a pair of equivalence classes of Cauchy nets with respect to $\mathcal{W}$, having representatives such that for large enough indices, the values taken by the representatives are related by $U$. $R(\mathcal{W}, V)$ is defined similarly. Then an element of $x_m(M_{n,n',t}(R(\mathcal{W}, U), R(\mathcal{W}, V)))$ is a pair of equivalence classes having representatives such that for large enough indices, the values define pairs in $x_m(M_{n,n',t}(U, V))$. That is, such a pair belongs to $R(\mathcal{W}, x_m(M_{n,n',t}(U, V)))$. \hfill $\square$

**Theorem 7.5.** Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W} \in \text{Unif} A$, where $A \in \mathcal{V}$, a congruence-modular variety with Day terms $m$. Then

1. If $\mathcal{U} \geq \mathcal{W}$ and $\mathcal{V} \geq \mathcal{W}$, $[\mathcal{U}, \mathcal{V}] \vee \mathcal{W} = (\text{nat } \mathcal{W})^{-1}[\mathcal{U}/\mathcal{W}, \mathcal{V}/\mathcal{W}]$.
2. If $\mathcal{U}$ and $\mathcal{V}$ permute with $\mathcal{W}$, then $[\mathcal{U}, \mathcal{V}] \vee \mathcal{W} = (\text{nat } \mathcal{W})^{-1}[(\mathcal{U} \vee \mathcal{W})/\mathcal{W}, (\mathcal{V} \vee \mathcal{W})/\mathcal{W}]$.

**Proof.** (2) follows from (1) because if $\mathcal{U}$ and $\mathcal{V}$ permute with $\mathcal{W}$, then $[\mathcal{U}, \mathcal{V}] \vee \mathcal{W} = [\mathcal{U} \vee \mathcal{W}, \mathcal{V} \vee \mathcal{W}] \vee \mathcal{W}$, and we can apply (1) since $\mathcal{U} \vee \mathcal{W} \geq \mathcal{W}$ and $\mathcal{V} \vee \mathcal{W} \geq \mathcal{W}$.

As for (1), first we have by Theorem 7.1(4), $$[\mathcal{U}, \mathcal{V}] = [(\text{nat } \mathcal{W})^{-1}(\mathcal{U}/\mathcal{W}), (\text{nat } \mathcal{W})^{-1}(\mathcal{V}/\mathcal{W})]$$ $$\leq (\text{nat } \mathcal{W})^{-1}[\mathcal{U}/\mathcal{W}, \mathcal{V}/\mathcal{W}]$$ and also $\mathcal{W} = (\text{nat } \mathcal{W})^{-1}(\mathcal{W}/\mathcal{W}) \leq (\text{nat } \mathcal{W})^{-1}[\mathcal{U}/\mathcal{W}, \mathcal{V}/\mathcal{W}]$, because $\mathcal{W}/\mathcal{W}$ is the least element of the lattice Unif $A/\mathcal{W}$, whence $$[\mathcal{U}, \mathcal{V}] \vee \mathcal{W} \leq (\text{nat } \mathcal{W})^{-1}(\mathcal{U}/\mathcal{W}, \mathcal{V}/\mathcal{W})$$

To prove the opposite inequality, it suffices to show that $X_m(\mathcal{U}/\mathcal{W}, \mathcal{V}/\mathcal{W}) \leq ([\mathcal{U}, \mathcal{V}] \vee \mathcal{W})/\mathcal{W}$. If $Q \in [\mathcal{U}, \mathcal{V}] \vee \mathcal{W}$, then in particular, $Q \in [\mathcal{U}, \mathcal{V}]$. Then by the definition of $[\mathcal{U}, \mathcal{V}]$ and $X_m(\mathcal{U}, \mathcal{V})$, given $n$, $n'$, and $t$, there exist $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that $x_m(M_{n,n',t}(U, V)) \subseteq Q$. Then by Lemma 7.4, $x_m(M_{n,n',t}(R(\mathcal{W}, U), R(\mathcal{W}, V))) \subseteq R(\mathcal{W}, Q)$. This proves that $X_m(\mathcal{U}/\mathcal{W}, \mathcal{V}/\mathcal{W}) \leq ([\mathcal{U}, \mathcal{V}] \vee \mathcal{W})/\mathcal{W}$, because $R(\mathcal{W}, U) \in \mathcal{U}/\mathcal{W}$ and $R(\mathcal{W}, V) \in \mathcal{V}/\mathcal{W}$, and relations of the form $R(\mathcal{W}, Q)$ for $Q \in [\mathcal{U}, \mathcal{V}] \vee \mathcal{W}$ form a base for $([\mathcal{U}, \mathcal{V}] \vee \mathcal{W})/\mathcal{W}$. \hfill $\square$
8. Commutator Operations and Congruential Uniformities

Recall that a uniformity $U \in \text{Unif } A$ is congruential if it has a base of congruences. Given a filter of congruences $F$, it determines a congruential uniformity $Ug F$, of which $F$ is a base and which determines $F$.

For example, consider filters in $\text{Con } Z$, where $Z$ is the ring of integers. In addition to the principal filters $Fg\{ (n) \}$ for $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, there are many other filters such as $(m^\infty) \overset{\text{def}}{=} Fg\{ (p^n) \mid n \in \mathbb{N} \}$ for $m$ a nonzero natural number. (Of course, the mapping $m \mapsto (m^\infty)$ is not one-one.)

The mapping $F \mapsto Ug F$, from the lattice of congruential uniformities of $A$ into the lattice of uniformities, preserves arbitrary meets, and by [10, Theorem 6.3], if the algebra $A$ has permuting uniformities, it preserves finite joins.

For the time being, we will assume that the algebras we are discussing belong to a congruence-modular variety.

**Proposition 8.1.** Let $\alpha \in \text{Con } A$, and $F, F' \in \text{Fil } \text{Con } A$. Then the sets $\{ [\alpha, \phi] \mid \phi \in F \}$, $\{ [\phi, \alpha] \mid \phi \in F \}$, and $\{ [\phi, \phi'] \mid \phi, \phi' \in F \}$ are bases for filters $[\alpha, F]$, $[F, \alpha]$, and $[F, F']$ in $\text{Con } A$.

**Proposition 8.2.** Let $\alpha, \beta \in \text{Con } A$, and $F \in \text{Fil } \text{Con } A$. Then

1. $[Fg_{\text{Con } A}\{ \alpha \}, F_{\text{Con } A}\{ \beta \}] = F_{\text{Con } A}\{ [\alpha, \beta] \}$
2. $[\alpha, F] = [Fg_{\text{Con } A}\{ \alpha \}, F]$
3. $[F, \alpha] = [F, Fg_{\text{Con } A}\{ \alpha \}]$.

Thus, we have defined a binary operation on $\text{Fil } \text{Con } A$ which extends the commutator on $\text{Con } A$. Clearly this operation satisfies the elementary properties of the commutator as given in Theorem 7.1; we leave the statement of the theorem to the reader. Let us also prove that this commutator on $\text{Fil } \text{Con } A$ is finitely additive:

**Theorem 8.3.** If $F_i, i = 1, 2, \ldots, n$ and $F' \in \text{Fil } \text{Con } A$, then $[\bigvee_i F_i, F'] = \bigvee_i [F_i, F']$.

**Proof.** Clearly we have $\bigvee_i [F_i, F'] \leq [\bigvee_i F_i, F']$ by monotonicity. To prove the opposite inequality, let $\chi \in \bigvee_i [F_i, F']$ be of the form $\chi = \bigvee_i [\alpha_i, \beta_i]$ for $\alpha_i \in F_i$ and $\beta_i \in F'$. Then by the monotonicity and additivity of the commutator on congruences, we have $\chi \geq [\bigvee_i \alpha_i, \bigwedge_i \beta_i] \in [\bigvee_i F_i, F']$. This proves that $[\bigvee_i F_i, F'] \leq \bigvee_i [F_i, F']$. \hfill $\Box$

Now, let us relate this commutator operation on $\text{Fil } \text{Con } A$ to that on $\text{Unif } A$:

**Theorem 8.4.** Let $A$ be an algebra. We have

1. If $\alpha \in \text{Con } A$, and $F \in \text{Fil } \text{Con } A$, then $[Ug\{ \alpha \}, Ug F] \leq Ug[\alpha, F]$
2. If $\alpha \in \text{Con } A$, and $F \in \text{Fil } \text{Con } A$, then $[Ug F, Ug\{ \alpha \}] \leq Ug[F, \alpha]$.
3. If $F, F' \in \text{Fil } \text{Con } A$, then $[Ug F, Ug F'] \leq Ug[F, F']$.

**Proof.** (3): If $W \in Ug[F, F']$, then $[\alpha, \beta] \subseteq W$ for some $\alpha \in F$ and $\beta \in F'$, because such congruences form a base of $Ug[F, F']$. $[\alpha, \beta]$ satisfies the $\alpha, \beta$-term condition, so for all $t, a$ and $a'$ such that $a \alpha a'$, and $b$ and $c$ with $b \beta c$, $t(a, b) [\alpha, \beta] t(a, c)$ implies $t(a', b) [\alpha, \beta] t(a', c)$. 

\[ \begin{align*} \end{align*} \]
But $\alpha \in \text{Ug} F$ and $\beta \in \text{Ug} F'$. Thus, $\text{Ug}[F, F']$ satisfies the $\text{Ug} F, \text{Ug} F'$-term condition. If follows that $[\text{Ug} F, \text{Ug} F'] \leq \text{Ug}[F, F']$.

(1): By (3), $[\text{Ug}\{\alpha\}, \text{Ug} F] = [\text{Ug} Fg^{\text{Con} A}\{\alpha\}, \text{Ug} F] \leq \text{Ug}[Fg^{\text{Con} A}\{\alpha\}, F] = \text{Ug}[\alpha, F]$.

(2): similar to proof of (1).

Remark. Propositions 8.1 and 8.2, and Theorem 8.4 hold in non-congruence-modular varieties, if we replace $[\alpha, \beta]$ by $C(\alpha, \beta)$ and define $C(\alpha, F)$, $C(F, \alpha)$, and $C(F, F')$ or similarly if we replace $[\alpha, \beta]$ by $\bar{C}(\alpha, \beta)$ and define $\bar{C}(\alpha, F)$, $\bar{C}(F, \alpha)$, and $\bar{C}(F, F')$. We omit the details.

9. Miscellany

Congruential uniformities on commutative rings. For $A$ a commutative ring, Theorem 8.4 can be improved. For a translation invariant relation $U$ on $A$, we denote by $\delta(U)$ the set of differences $a - b$ for $a, b \in A$ such that $a U b$.

**Theorem 9.1.** If $A$ is a commutative ring, then the inequalities in the conclusion of Theorem 8.4 are equalities.

**Proof.** It suffices to show that $\text{Ug}[F, F'] \leq [\text{Ug} F, \text{Ug} F']$. We use the fact that $[\text{Ug} F, \text{Ug} F']$ satisfies the $\text{Ug} F, \text{Ug} F'$-term condition for the term $t(x, y) = xy$. Let $U \in [\text{Ug} F, \text{Ug} F']$. By Lemma 3.1, we may assume $U$ is translation-invariant without loss of generality. There are $\bar{U} \in [\text{Ug} F, \text{Ug} F']$, $\alpha \in F$, and $\beta \in F'$ such that $a \alpha a'$, $b \beta c$, and $ab \bar{U} ac$ imply $a' b U a' c$. Then if $b \beta c$ and $a \in I_\alpha$ (the ideal corresponding to $\alpha$), $0b = 0c$, implying $ab U ac$. It follows that $I_{[\alpha, \beta]} = I_\alpha I_\beta \subseteq \delta(U)$, and this implies that $[\alpha, \beta] \subseteq U$. \qed

It follows from this theorem that, for commutative rings, the commutators $[\alpha, F]$, $[F, \alpha]$ and $[F, F']$ can be considered as commutators of uniformities.

**Example.** In $\text{Unif} Z$, we have $[\text{Ug}(p^\infty), \text{Ug}(q^\infty)] = \text{Ug}((pq)^\infty)$, for prime numbers $p \neq q$, showing that the commutator of two compatible uniformities is not always equal to $\text{Ug}\{\alpha\}$ for some congruence $\alpha$.

Algebras in congruence-distributive varieties.

**Theorem 9.2.** Let $A$ be an algebra in a congruence-distributive variety, with Jónsson terms $d_0$, $\ldots$, $d_k$, and let $\mathcal{U}$, $\mathcal{V} \in \text{Unif} A$. Then $[\mathcal{U}, \mathcal{V}] = \mathcal{U} \wedge \mathcal{V}$.

**Proof.** We already know that $[\mathcal{U}, \mathcal{V}] \leq \mathcal{U} \wedge \mathcal{V}$. To prove the opposite inequality, we must show that if $W \in [\mathcal{U}, \mathcal{V}]$, then there are $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that $U \cap V \subseteq W$. We will use the Jónsson terms to prove this.

Let $W \in [\mathcal{U}, \mathcal{V}]$. We define $W_k, W_{k-1}, \ldots, W_0 \in [\mathcal{U}, \mathcal{V}]$ successively, as follows: Set $W_k = W$. If $n$ is odd, then there exist $W_{n-1} \in [\mathcal{U}, \mathcal{V}], U_{n-1} \in \mathcal{U}, V_{n-1} \in \mathcal{V}$ such that $d_n(a, b, a) W_{n-1} W_n(a, b, b) \alpha U_{n-1} \cap V_{n-1} \beta b$ imply $d_n(a, b, a) W_n d_n(a, b, b)$. If $n > 0$ is even, then there exist $W_{n-1}, U_{n-1}, V_{n-1}$ such that $d_n(a, a, a) W_{n-1} W_{n-1} d_n(a, a, b)$ and $U_{n-1} \cap V_{n-1} b$ imply $d_n(a, a, a) W_n d_n(a, a, b)$. Now let $U = \bigcap U_n$, $V = \bigcap V_n$, and $a U \cap V b$. We have
\[ d_0(a, a, a) = a = d_0(a, a, b) \text{ so } d_0(a, a, a) W_0 d_0(a, a, b). \] We further have

\[ d_2(a, b, a) = a = d_1(a, b, a) W_1 d_1(a, b, b) = d_2(a, b, b), \]

\[ d_3(a, a, a) = a = d_2(a, a, a) W_2 d_2(a, a, b) = d_3(a, a, b), \]

and so on, ending with \( a = d_k(a, b, a) W_k d_k(a, b, b) = b \) if \( k \) is even and with \( a = d_k(a, a, a) W_k d_k(a, a, b) = b \) if \( k \) is odd. In either case, we have shown that \( U \cap V \subseteq W_k = W \).

\[ \square \]

**Abelian algebras.** Recall that an algebra \( A \) in a congruence-modular variety is *abelian* if \([\top_A, \top_A] = \bot_A\). In this case, from Theorem 7.1 and Theorem 4.9, we have \([U, V] = U \{ \bot_A \}\) for any \( U, V \in \text{Unif } A \).

For example, abelian groups are abelian algebras, so we might consider the group of real numbers and the commutator \([U, U]\), where \( U \) is the unique compatible uniformity (compatible, that is, with respect to the abelian group operations) that gives rise to the usual topology on the group of real numbers. \( U \) is noncongruential and abelian. This example shows that noncongruential uniformities can have a commutator that is congruential, indeed of the form \( U \{ \alpha \} \) for \( \alpha \) a congruence.

10. Conclusions

In this final section, we will review some of the questions still open regarding the commutator of uniformities, and uniform universal algebra generally.

The most important question is the possible additivity and even complete additivity of the commutator, as holds for congruences and as we have proved for some special cases in Theorem 7.2 and Theorem 8.3. Many applications of commutator theory rely on this. An obstacle here is the difficulty of dealing with joins of compatible uniformities. A more specific question, which might be easier to settle, is complete additivity for compatible uniformities of an algebra in a congruence-permutable variety. We proved finite additivity in this case, using the fact that compatible uniformities permute pairwise.

The proof that, in the case of an algebra \( A \) with underlying group, formation of commutators with a uniformity of the form \( U \{ \alpha \} \) is completely additive, utilizing the theory of \( \text{Unif } A(\alpha) \), suggests that an appropriate definition for \( A(U) \) may help settle the additivity question.

A uniformity \( U \) on an algebra \( A \) in a congruence-modular variety \( V \) can be defined as *abelian* if \([U, U] = U \{ \bot_A \} = F \{ \Delta_A \}\). In the case of a congruence \( \alpha \), abelianness leads to a structure of abelian group object on the algebra \( A(\alpha) \), viewed as an algebra over \( A \) (that is, as an object in the comma category of algebras of \( V \) over \( A \)). The abelian group operations on this abelian group object can be obtained from any difference term. The problem of generalizing this theory to uniformities again depends upon the proper definition of \( A(U) \).

In the theory of uniform universal algebra, an important open question is the possible modularity of the uniformity lattice of an algebra in a congruence-modular variety. This has only been proved for algebras in congruence-permutable varieties and not more generally, although there is a partial result [10, Theorem 7.5].
A similar question is the possible distributivity of the uniformity lattice of an algebra in a congruence-distributive variety. This has been proved only for arithmetic algebras [10, Theorem 6.5]. Note that because $[\mathcal{U}, \mathcal{V}] = \mathcal{U} \wedge \mathcal{V}$ for congruence distributive algebras (Theorem 9.2) additivity of the commutator of compatible uniformities in this case is equivalent to distributivity of $\text{Unif} A$, and complete additivity is equivalent to the distributivity of meet over an arbitrary join.

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