Homogeneity of the pure state space for the separable nuclear $C^\ast$-algebras

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Abstract

We prove that the pure state space is homogeneous under the action of the group of asymptotically inner automorphisms for all the separable simple nuclear $C^\ast$-algebras. If simplicity is not assumed for the $C^\ast$-algebras, the set of pure states whose GNS representations are faithful is homogeneous for the above action.

1 Introduction

If $A$ is a $C^\ast$-algebra, an automorphism $\alpha$ of $A$ is asymptotically inner if there is a continuous family $(u_t)_{t \in [0, \infty)}$ in the group $U(A)$ of unitaries in $A$ (or $A + C1$ if $A$ is non-unital) such that $\alpha = \lim_{t \to \infty} \text{Ad} u_t$; we denote by $A_{\text{Inn}}(A)$ the group of asymptotically inner automorphisms of $A$, which is a normal subgroup of the group of approximately inner automorphisms. Note that each $\alpha \in A_{\text{Inn}}(A)$ leaves each (closed two-sided) ideal of $A$ invariant. It is shown, in [13, 1, 11], for a large class of separable nuclear $C^\ast$-algebras that if $\omega_1$ and $\omega_2$ are pure states of $A$ such that the GNS representations associated with $\omega_1$ and $\omega_2$ have the same kernel, then there is an $\alpha \in A_{\text{Inn}}(A)$ such that $\omega_1 = \omega_2 \alpha$. We shall show in this paper that this is the case for all separable nuclear $C^\ast$-algebras; in particular the pure state space of a separable simple nuclear $C^\ast$-algebra $A$ is homogeneous under the action of $A_{\text{Inn}}(A)$. We do not know of a single example of a separable $C^\ast$-algebra which does not have this property. See [8] for some problems on this and see 2.4 and 2.5 for remarks on the non-separable case.

Choi and Effros [5] have shown that $A$ is nuclear if and only if there is a net of pairs $(\sigma_\nu, \tau_\nu)$ of completely positive (CP) contractons such that $\lim \tau_\nu \sigma_\nu (x) = x$, $x \in A$, where $A \xrightarrow{\sigma_\nu} N_\nu \xrightarrow{\tau_\nu} A$ and $N_\nu$ is a finite-dimensional $C^\ast$-algebra. When $A$ is a non-unital $C^\ast$-algebra, $A$ is nuclear if and only if $A + C1$ is nuclear [7]. If $A$ is unital, we may assume that both $\sigma_\nu$ and $\tau_\nu$ are unit-preserving. We refer to [3, 11] for some other facts on nuclear $C^\ast$-algebras. We also quote [13] for a review on the subject.
Our proof of the homogeneity is a combination of the techniques leading up to the above result from [5] and the techniques from [11]. In section 2 we shall show how the homogeneity follows from inductive use of Lemma 2.1 (or 2.2), whose conclusion is very similar to the properties already used in [11]; this part follows closely [11] and so the proof will be sketchy. In section 3 we shall prove Lemma 2.1 from another technical lemma, Lemma 3.1, which shows some amenability of the nuclear $C^*$-algebras; this is the arguments often used for individual examples treated in [11] and so the proof will be again sketchy. Then we will give a proof of Lemma 3.1, which constitutes the main body of this paper and uses the results and techniques from [5].

We will conclude this paper, following [11], by generalizing Lemma 3.1 and then extend the main result, Theorem 2.3, to show that $\text{AInn}(A)$ acts on the pure state space of $A$ strongly transitively. See Theorem 3.8 for details.

2 Homogeneity

We first give a main technical lemma, whose conclusion is a slightly weaker version of Property 2.6 in [11]. We will give a proof in the next section.

**Lemma 2.1** Let $A$ be a nuclear $C^*$-algebra. Then for any finite subset $F$ of $A$, any pure state $\omega$ of $A$ with $\pi_\omega(A) \cap K(H_\omega) = (0)$, and $\epsilon > 0$, there exist a finite subset $G$ of $A$ and $\delta > 0$ satisfying: If $\varphi$ is a pure state of $A$ such that $\varphi \sim \omega$, and

$$|\varphi(x) - \omega(x)| < \delta, \quad x \in G,$$

then there is a continuous path $(u_t)_{t \in [0, 1]}$ in $U(A)$ such that $u_0 = 1$, $\varphi = \omega \text{Ad} u_1$, and

$$\|\text{Ad} u_t(x) - x\| < \epsilon, \quad x \in F, \quad t \in [0, 1].$$

In the above statement, $\pi_\omega$ is the GNS representation of $A$ associated with the state $\omega$; $H_\omega$ is the Hilbert space for this representation; $K(H_\omega)$ is the $C^*$-algebra of compact operators on $H_\omega$; $\varphi \sim \omega$ means that $\pi_\varphi$ is equivalent to $\pi_\omega$. We could also impose the extra condition that the length of $(u_t)$ is smaller than $\pi + \epsilon$ for the choice of the path $(u_t)$; see Property 8.1 in [11].

The following is an easy consequence:

**Lemma 2.2** Let $A$ be a nuclear $C^*$-algebra. Then for any finite subset $F$ of $A$, any pure state $\omega$ of $A$ with $\pi_\omega(A) \cap K(H_\omega) = (0)$, and $\epsilon > 0$, there exist a finite subset $G$ of $A$ and $\delta > 0$ satisfying: If $\varphi$ is a pure state of $A$ such that $\ker \pi_\varphi = \ker \pi_\omega$, and

$$|\varphi(x) - \omega(x)| < \delta, \quad x \in G,$$

then for any finite subset $F'$ of $A$ and $\epsilon' > 0$ there is a continuous path $(u_t)_{t \in [0, 1]}$ in $U(A)$ such that $u_0 = 1$, and

$$|\varphi(x) - \omega \text{Ad} u_1(x)| < \epsilon', \quad x \in F',$$

$$\|\text{Ad} u_t(x) - x\| < \epsilon, \quad x \in F.$$
Proof. Given \((F, \omega, \epsilon)\), choose \((G, \delta)\) as in the previous lemma. Let \(\varphi\) be a pure state of \(A\) such that \(\ker \pi_\varphi = \ker \pi_\omega\) and

\[
|\varphi(x) - \omega(x)| < \delta/2, \quad x \in \mathcal{G}.
\]

Let \(F'\) be a finite subset of \(A\) and \(\epsilon' > 0\) with \(\epsilon' < \delta/2\). We can mimic \(\varphi\) as a vector state through \(\pi_\omega\); by Kadison’s transitivity there is a \(v \in \mathcal{U}(A)\) such that

\[
|\varphi(x) - \omega \Ad v(x)| < \epsilon', \quad x \in F' \cup \mathcal{G},
\]

(see 2.3 of [17]). Since \(|\omega \Ad v(x) - \omega(x)| < \delta, \quad x \in \mathcal{G}\), we have, by applying Lemma 2.1 to the pair \(\omega\) and \(\omega \Ad v\), a continuous path \((u_t)\) in \(\mathcal{U}(A)\) such that \(u_0 = 1\), and

\[
\omega \Ad v = \omega \Ad u_1, \quad \|\Ad u_t(x) - x\| < \epsilon, \quad x \in F.
\]

Since \(|\varphi(x) - \omega \Ad u_1(x)| < \epsilon', \quad x \in F'\), this completes the proof. \(\square\)

We shall now turn to the main result stated in the introduction. We denote by \(\text{Almn}_0(A)\) the set of \(\alpha \in \text{Almn}(A)\) which has a continuous family \((u_t)_{t \in [0, \infty)}\) in \(\mathcal{U}(A)\) with \(u_0 = 1\) and \(\alpha = \lim \Ad u_t\); \(\text{Almn}_0(A)\) can be smaller than \(\text{Almn}(A)\) (e.g., \(\text{Almn}_0(A)\) may not contain \(\text{Inn}(A)\); see [10]).

**Theorem 2.3** Let \(A\) be a separable nuclear \(C^*\)-algebra. If \(\omega_1\) and \(\omega_2\) are pure states of \(A\) such that \(\ker \pi_{\omega_1} = \ker \pi_{\omega_2}\), then there is an \(\alpha \in \text{Almn}_0(A)\) such that \(\omega_1 = \omega_2 \alpha\).

Proof. Once we have Lemma 2.2, we can prove this in the same way as 2.5 of [11]. We shall only give an outline here.

Let \(\omega_1\) and \(\omega_2\) be pure states of \(A\) such that \(\ker \pi_{\omega_1} = \ker \pi_{\omega_2}\).

If \(\pi_{\omega_1}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_1}) \neq (0)\), then \(\pi_{\omega_1}(A) \supset \mathcal{K}(\mathcal{H}_{\omega_1})\) and \(\pi_{\omega_1}\) is equivalent to \(\pi_{\omega_2}\). Then by Kadison’s transitivity (see, e.g., 1.21.16 of [17]), there is a continuous path \((u_t)\) in \(\mathcal{U}(A)\) such that \(u_0 = 1\) and \(\omega_1 = \omega_2 \Ad u_1\).

Suppose that \(\pi_{\omega_1}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_1}) = (0)\), which also implies that \(\pi_{\omega_2}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_2}) = (0)\).

Let \((x_n)\) be a dense sequence in \(A\).

Let \(\mathcal{F}_1 = \{x_1\}\) and \(\epsilon > 0\) (or \(\epsilon = 1\)). Let \((\mathcal{G}_1, \delta_1)\) be the \((\mathcal{G}, \delta)\) for \((\mathcal{F}_1, \omega_1, \epsilon/2)\) as in Lemma 2.2 such that \(\mathcal{G}_1 \supset \mathcal{F}_1\). For this \((\mathcal{G}_1, \delta_1)\) we choose a continuous path \((u_{1t})\) in \(\mathcal{U}(A)\) such that \(u_{1,0} = 1\) and

\[
|\omega_1(x) - \omega_2 \Ad u_{1,1}(x)| < \delta_1, \quad x \in \mathcal{G}_1.
\]

Let \(\mathcal{F}_2 = \{x_i, \Ad u_{1,1}^*(x_i) \mid i = 1, 2\}\) and let \((\mathcal{G}_2, \delta_2)\) be the \((\mathcal{G}, \delta)\) for \((\mathcal{F}_2, \omega_2 \Ad u_{1,1}, 2^{-2} \epsilon)\) as in Lemma 2.2 such that \(\mathcal{G}_2 \supset \mathcal{G}_1 \cup \mathcal{F}_2\) and \(\delta_2 < \delta_1\). By 2.2 there is a continuous path \((u_{2t})\) in \(\mathcal{U}(A)\) such that \(u_{2,0} = 1\) and

\[
\|\Ad u_{2t}(x) - x\| < 2^{-1} \epsilon, \quad x \in \mathcal{F}_1,
\]

\[
|\omega_2 \Ad u_{1,1}(x) - \omega_1 \Ad u_{2,1}(x)| < \delta_2, \quad x \in \mathcal{G}_2.
\]
Let $\mathcal{F}_3 = \{x_i, \text{Ad} u_{2,1}^*(x_i) \mid i = 1, 2, 3\}$ and let $(\mathcal{G}_3, \delta_3)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_3, \omega_1 \text{Ad} u_{2,1}, 2^{-3} \epsilon)$ as in 2.2 such that $\mathcal{G}_3 \supset \mathcal{G}_2 \cup \mathcal{F}_3$ and $\delta_3 < \delta_2$. By 2.2 there is a continuous path $(u_{3,t})$ in $\mathcal{U}(A)$ such that $u_{3,0} = 1$ and

$$\|\text{Ad} u_{3,t}(x) - x\| < 2^{-2} \epsilon, \quad x \in \mathcal{F}_2,$$

$$(\omega_1 \text{Ad} u_{2,1}(x) - \omega_2 \text{Ad}(u_{1,1} u_{3,1})(x)) < \delta_3, \quad x \in \mathcal{G}_3.$$

We shall repeat this process.

Assume that we have constructed $\mathcal{F}_n, \mathcal{G}_n, \delta_n$, and $(u_{n,t})$ inductively. In particular if $n$ is even,

$$\mathcal{F}_n = \{x_i, \text{Ad}(u_{n-1,1}^* u_{n-3,1}^* \cdots u_{1,1}^*)(x_i) \mid i = 1, 2, \ldots, n\}$$

and $(\mathcal{G}_n, \delta_n)$ is the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_n, 2^{n-1} \epsilon\text{Ad}(u_{1,1} u_{3,1} \cdots u_{n-1,1}), 2^{-n} \epsilon)$ as in 2.2 such that $\mathcal{G}_n \supset \mathcal{G}_{n-1} \cup \mathcal{F}_n$ and $\delta_n < \delta_{n-1}$. And $(u_{n,t})$ is given by 2.2 for $(\mathcal{F}_{n-1}, \omega_1 \text{Ad}(u_{2,1} \cdots u_{n-2,1}), 2^{-n+1} \epsilon)$ and for $\mathcal{F}' = \mathcal{G}_n$ and $\epsilon' = \delta_n$ and it satisfies

$$(\omega_1 \text{Ad}(u_{2,1} u_{4,1} \cdots u_{n,1})(x) - \omega_2 \text{Ad}(u_{1,1} \cdots u_{n-1,1})(x)) < \delta_n, \quad x \in \mathcal{G}_n.$$

We define continuous paths $(v_t)$ and $(w_t)$ in $\mathcal{U}(A)$ with $t \in [0, \infty)$ by: For $t \in [n, n+1]$

$$u_t = u_{1,1} u_{3,1} \cdots u_{2n-1,1} u_{2n+1,t-n},$$

$$w_t = u_{1,1} u_{4,1} \cdots u_{2n-2,1} u_{2n+2,t-n}.$$

Then, since $\|\text{Ad} u_{nt}(x) - x\| < 2^{-n+1} \epsilon, \ x \in \mathcal{F}_{n-1}$, we can show that $\text{Ad} v_t$ (resp. $\text{Ad} w_t$) converges to an automorphism $\alpha$ (resp. $\beta$) as $t \to \infty$ and that $\omega_1 \beta = \omega_2 \alpha$. Since $\alpha, \beta \in \text{Aln}(A)$ and $\text{Aln}(A)$ is a group, this will complete the proof. See the proofs of 2.5 and 2.8 of [11] for details.

The notion of asymptotical innerness for automorphisms may be appropriate only for separable C*-algebras. Because any $\alpha \in \text{Aln}(A)$ can be obtained as the limit of a sequence in $\text{Im}(A)$, not just as the limit of a net there. Hence the following remark will not be a surprise; it may only suggest that we should take $\overline{\text{Im}}(A)$ or something bigger than $\text{Aln}(A)$ in place of $\text{Aln}(A)$, in formulating 2.3 for non-separable C*-algebras.

**Remark 2.4** There is a unital simple non-separable nuclear C*-algebra $A$ such that the pure states space of $A$ is not homogeneous under the action of $\text{Aln}(A)$.

We can construct such an example as follows. Let $A$ be a unital simple separable nuclear C*-algebra and $\Lambda$ an uncountable set. For each finite subset $F$ of $\Lambda$ we set $A_F = \otimes_{i \in A} A_i$ with $A_i \equiv A$ and take the natural inductive limit $A_\Lambda$ of the net $(A_F)$. Since $A_F$ is nuclear, it follows that $A_\Lambda$ is nuclear.

For each $X \subset \Lambda$ we define $A_X$ to be the C*-subalgebra of $A_\Lambda$ generated by $A_F$ with finite $F \subset X$. Note that for each $x \in A_\Lambda$ there is a countable $X \subset \Lambda$ such that $x \in A_X$.

Let $(u_n)$ be a sequence in $\mathcal{U}(A_\Lambda)$ such that $\text{Ad} u_n$ converges to $\alpha \in \text{Aut}(A_\Lambda)$ in the point-norm topology. Since there is a countable subset $X_n \subset \Lambda$ such that $u_n \in A_{X_n}, \alpha$ is
non-trivial only on $A_X$, where $X = \cup_n X_n$ is countable. Thus any $\alpha \in \text{Alnn}(A_\Lambda)$ has the above property of countable support.

For each $i \in \Lambda$ let $\omega_i$ and $\varphi_i$ be pure states of $A_i = A$ such that $\omega_i \neq \varphi_i$ and let $\omega = \bigotimes_{i \in \Lambda} \omega_i$ and $\varphi = \bigotimes_{i \in \Lambda} \varphi_i$. Then it follows that $\omega$ and $\varphi$ are pure states of $A_\Lambda$ and that $\omega \neq \varphi \alpha$ for any $\alpha \in \text{Alnn}(A_\Lambda)$. Hence $A_\Lambda$ serves as an example for the above remark.

In this case, however, we have an $\alpha \in \text{Inn}(A_\Lambda)$ such that $\omega = \varphi \alpha$ (since this is the case for each pair $\omega_i, \varphi_i$ from [2,3]) and it may be the case that the pure state space of $A_\Lambda$ is homogeneous under the action of $\text{Inn}(A_\Lambda)$.

**Remark 2.5** There is a unital simple non-separable non-nuclear $C^*$-algebra $A$ such that the pure state space of $A$ is not homogeneous under the action of $\text{Aut}(A)$.

There are plenty of such $C^*$-algebras at hand. Let $A$ be a factor of type II$_1$ or type III with separable predual $A_*$. Then $A$ is a unital simple non-separable non-nuclear $C^*$-algebra (see, e.g., [13] for non-nuclearity). Since $A$ contains a $C^*$-subalgebra isomorphic to $C_b(N) \cong C(\beta N)$ and $\beta N$ has cardinality $2^c$, the pure state space of $A$ has cardinality (at least) $2^c$, where $c$ denotes the cardinality of the continuum. (We owe this argument to J. Anderson.) On the other hand any $\alpha \in \text{Aut}(A)$ corresponds to an isometry on the predual $A_*$, a separable Banach space. Thus, since the set of bounded operators on a separable Banach space has cardinality $c$, $\text{Aut}(A)$ has cardinality (at most) $c$. Hence the pure state space of $A$ cannot be homogeneous under the action of $\text{Aut}(A)$.

We note in passing that $\text{Alnn}(A) = \text{Inn}(A)$ for any factor $A$ (or any quotient of a factor), since any convergent sequence in $\text{Aut}(A)$ with the point-norm topology converges in norm [9]. We also note that $\text{Inn}(A) = \text{Inn}(A)$ for any full factor [8, 16], since then $\text{Inn}(A)$ is closed in $\text{Aut}(A)$ with the topology of point-norm convergence in $A_*$ and so is closed in $\text{Aut}(A)$ with the topology of point-norm convergence in $A$.

### 3 Proof of Lemma 2.1

If $A$ is a non-unital $C^*$-algebra, $A$ is nuclear if and only if the $C^*$-algebra $A + C1$ obtained by adjoining a unit is nuclear. Hence to prove Lemma 2.1 we may suppose that $A$ is unital. In the following $U_0(A)$ denotes the connected component of 1 in the unitary group $U(A)$ of $A$.

**Lemma 3.1** Let $A$ be a unital nuclear $C^*$-algebra. Let $F$ be a finite subset of $U_0(A)$, $\pi$ an irreducible representation of $A$ on a Hilbert space $\mathcal{H}$, $E$ a finite-dimensional projection on $\mathcal{H}$, and $\epsilon > 0$. Then there exist an $n \in \mathbb{N}$ and a finite subset $G$ of $M_1(A)$ such that $xx^* \leq 1$ and $\pi(xx^*)E = E$ for $x \in G$, and for any $u \in F$ there is a bijection $f$ of $G$ onto $G$ with

$$\|ux - f(x)\| < \epsilon.$$
In the above statement, $M_{1n}(A)$ denotes the 1 by $n$ matrices over $A$; if $u \in A$ and $x = (x_1, x_2, \ldots, x_n) \in M_{1n}(A)$,

$$xx^* = \sum_{i=1}^{n} x_i x_i^* \in A,$$

$$ux = (ux_1, ux_2, \ldots, ux_n) \in M_{1n}(A).$$

We shall first show that Lemma 3.1 implies Lemma 2.1.

Let $F$ be a finite subset of $A$, $\omega$ a pure state of $A$ with $\pi_{\omega}(A) \cap K(H_{\omega}) = (0)$, and $\epsilon > 0$. Since $\mathcal{U}_0(A)$ linearly spans $A$, we may suppose that $F$ is a finite subset of $\mathcal{U}_0(A)$. For $\pi = \pi_{\omega}$ and the projection $E$ onto the subspace $C_{\Omega_{\omega}}$, we choose an $n \in \mathbb{N}$ and a finite subset $G$ of $M_{1n}(A)$ as in Lemma 3.1.

We take the finite subset

$$\{x_i x_j^* \mid x \in G; \ i, j = 1, 2, \ldots, n\}$$

for the subset $G$ required in Lemma 2.1. We will choose $\delta > 0$ sufficiently small later. Suppose that we are given a unit vector $\eta \in H_{\omega}$ satisfying

$$|\langle \pi(x_i^* \eta, \pi(x_j^* \eta) - \langle \pi(x_i^*) \Omega, \pi(x_j^*) \Omega \rangle| < \delta$$

for any $x \in G$ and $i, j = 1, 2, \ldots, n$, where $\Omega = \Omega_{\omega}$. Note that

$$\sum_{j=1}^{n} \|\pi(x_j^*) \Omega\|^2 = \langle \pi(xx^*) \Omega, \Omega \rangle = 1,$$

which implies that $|\langle \pi(xx^*) \eta, \eta \rangle - 1| < n\delta$. Thus the two finite sets of vectors $S_\Omega = \{\pi(x_i^*) \Omega \mid i = 1, \ldots, n, \ x \in G\}$ and $S_\eta = \{\pi(x_i^*) \eta \mid i = 1, \ldots, n, \ x \in G\}$ have similar geometric properties in $H_{\omega}$ if $\delta$ is sufficiently small. Hence we are in a situation where we can apply 3.3 of [11].

Let us describe how we proceed from here in a simplified case. Suppose that the linear span $L_{\Omega}$ of $S_\Omega$ is orthogonal to the linear span $L_{\eta}$ of $S_\eta$ and that the map $\pi(x_i^*) \Omega \mapsto \pi(x_i^*) \eta$ and $\pi(x_i^*) \eta \mapsto \pi(x_i^*) \Omega$ extends to a unitary on $L_{\Omega} + L_{\eta}$; in particular we have assumed that $\langle \pi(x_i^*) \eta, \pi(x_j^*) \eta \rangle = \langle \pi(x_i^*) \Omega, \pi(x_j^*) \Omega \rangle$ for all $i, j$. Since $U$ is a self-adjoint unitary, $F \equiv (1 - U)/2$ is a projection and satisfies that $e^{i\pi F} = U$ on the finite-dimensional subspace $L_{\Omega} + L_{\eta}$. By Kadison’s transitivity we choose an $h \in A$ such that $0 \leq h \leq 1$ and $\pi(h) |L_{\Omega} + L_{\eta} = F$. We set

$$\overline{h} = |G|^{-1} \sum_{x \in G} xhx^*,$$

where

$$xhx^* = \sum_{i=1}^{n} x_i hx_i^*.$$
Since
\[
\pi(xhx^*)(\Omega - \eta) = \sum \pi(x_i) F \pi(x_i^*)(\Omega - \eta),
\]
\[
= \sum \pi(x_i) \pi(x_i^*)(\Omega - \eta)
\]
and \( \pi(xhx^*)(\Omega + \eta) = 0 \), it follows that
\[
\pi(\hbar)(\Omega - \eta) = \Omega - \eta, \quad \pi(\hbar)(\Omega + \eta) = 0.
\]
Hence we have that \( e^{i\pi(\hbar)} \) switches \( \Omega \) and \( \eta \).

On the other hand for \( u \in \mathcal{F} \) there is a bijection \( f \) of \( \mathcal{G} \) onto \( \mathcal{G} \) such that \( \|ux - f(x)\| < \epsilon, \ x \in \mathcal{G} \). Since
\[
u h u^* - \hbar = |\mathcal{G}|^{-1} \sum_{x \in \mathcal{G}} \{(ux - f(x))hx^*u^* + f(x)h(x^*u^* - f(x)^*)\},
\]
it follows that \( \|u\hbar u^* - \hbar\| < 2\epsilon \). Thus the path \( (e^{it\pi(\hbar)})_{t \in [0,1]} \) almost commutes with \( \mathcal{F} \) and is what is desired. (Since what is required is \( \omega \eta = \omega \text{Ad} e^{i\pi(\hbar)} \), we may take the path \( (e^{it\pi(\hbar-1/2)}) \), whose length is \( \pi/2 \).)

If \( \mathcal{L}_\eta \) is not orthogonal to \( \mathcal{L}_\Omega \), we still find a unit vector \( \zeta \in \mathcal{H}_\omega \) such that
\[
|\langle \pi(x_i^*)\zeta, \pi(x_j^*)\zeta \rangle - \langle \pi(x_i^*)\Omega, \pi(x_j^*)\Omega \rangle| < \delta
\]
and such that \( \mathcal{L}_\zeta \) is orthogonal to both \( \mathcal{L}_\Omega \) and \( \mathcal{L}_\eta \). Here we use the assumption that \( \pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0) \). Then we combine the path of unitaries sending \( \eta \) to \( \zeta \) and then the path sending \( \zeta \) to \( \Omega \) to obtain the desired path.

The above arguments can be made rigorous in the general case; see [11] for details. □

We will now turn to the proof of Lemma 3.1, by first giving a series of lemmas. The following is an easy version of 3.4 of [2].

**Lemma 3.2** Let \( \pi \) be a non-degenerate representation of a \( C^* \)-algebra \( A \) on a Hilbert space \( \mathcal{H} \), \( E \) a finite-dimensional projection on \( \mathcal{H} \), \( \mathcal{F} \) a finite subset of \( A \), and \( \epsilon > 0 \). Then there is a finite-rank self-adjoint operator \( H \) on \( \mathcal{H} \) such that \( E \leq H \leq 1 \) and
\[
\|[[\pi(x), H]]\| < \epsilon, \ x \in \mathcal{F}.
\]

**Proof.** We define finite-dimensional subspaces \( V_k, k = 1, 2, \ldots, \) of \( \mathcal{H} \) as follows: \( V_1 = E\mathcal{H} \) and if \( V_k \) is defined then \( V_{k+1} \) is the linear span of \( V_k \) and \( xV_k, x^*V_k, \ x \in \mathcal{F} \), where we have omitted \( \pi \). Then \( (V_k) \) is increasing and
\[
x(V_{k+1} \ominus V_k) \subset V_{k+2} \ominus V_{k-1}, \ x \in \mathcal{F},
\]

with $V_0 = 0$. Denoting by $E_k$ the projection onto $V_k$ we define

$$H_n = \frac{1}{n} \sum_{k=1}^{n} E_k.$$ 

Then $E \leq H_n \leq E_n$. If $x \in \mathcal{F}$, we have, for $\xi \in V_{k+1} \ominus V_k$, that

$$(H_n x - x H_n) \xi = (H_n - \frac{n-k}{n})x \xi \in V_{k+2} \ominus V_{k-1}.$$ 

Hence for $\xi \in \mathcal{H}$,

$$(H_n x - x H_n) \xi = \sum_{k=0}^{n+1} (H_n x - x H_n)(E_{k+1} - E_k) \xi = \sum_{k=0}^{n+1} (H_n - \frac{n-k}{n})x (E_{k+1} - E_k) \xi,$$

and thus, by splitting the above sum into three terms, each of which is the sum over $k \mod 3 = i$ for $i = 0, 1, 2$, and estimating each, we reach

$$\|(H_n x - x H_n) \xi\| \leq \frac{3}{n} \|x\| \|\xi\|.$$ 

This implies that $\|[H_n, x]\| \leq 3/n$ for $x \in \mathcal{F}$. \qed

If $\pi$ is a representation of $A$ on a Hilbert space $\mathcal{H}$, we denote by $\pi_n$ the representation of $M_n \otimes A = M_n(A)$, the $n \times n$ matrix algebra over $A$, on the Hilbert space $\mathcal{C}^n \otimes \mathcal{H}$. If $x_i \in A$, then $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is naturally a diagonal element of $M_n(A)$.

**Lemma 3.3** Let $\pi$ be a non-degenerate representation of a unital $C^*$-algebra $A$ on a Hilbert space $\mathcal{H}$, $E$ a finite-rank projection on $\mathcal{H}$, $\mathcal{F}$ a finite subset of $\mathcal{U}_0(A)$, and $\epsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that each $u \in \mathcal{F}$ has a diagonal element $\hat{u} = u_1 \oplus u_2 \oplus \cdots \oplus u_n$ in $\mathcal{U}_0(M_n(A))$ satisfying $u_1 = u$, $u_n = 1$, and

$$\|u_i - u_{i+1}\| < \epsilon/2, \quad i = 1, 2, \ldots, n-1.$$ 

Furthermore there exists a finite-rank projection $F$ on $\mathcal{C}^n \otimes \mathcal{H}$ such that $F \geq E \oplus 0 \oplus \cdots \oplus 0$ and

$$\||\pi_n(\hat{u}), F|| < \epsilon, \quad u \in \mathcal{F}.$$ 

**Proof.** Since $\mathcal{U}_0(A)$ is path-wise connected, the first part is immediate.

Let $\delta > 0$, which will be specified sufficiently small later. By the previous lemma we choose a finite-rank self-adjoint operator $H_1$ on $\mathcal{H}$ such that $E \leq H_1 \leq 1$ and

$$\|[H_1, u_i]\| < \delta, \quad i = 1, 2, \quad u \in \mathcal{F}$$ 

where we have omitted $\pi$. Let $E_1$ be the support projection of $H_1$ and let $H_2$ be a finite-rank self-adjoint operator on $\mathcal{H}$ such that $E_1 \leq H_2 \leq 1$, and

$$\|[H_2, u_i]\| < \delta, \quad i = 2, 3, \quad u \in \mathcal{F}.$$
In this way we define $H_3, H_4, \ldots, H_{n-1}$ and set $H_n = E_{n-1}$, the support projection of $H_{n-1}$. We define an operator $F$ on $\mathbb{C}^n \otimes \mathcal{H}$ as a tri-diagonal matrix as follows:

$$F_{i,i} = H_i - H_{i-1}, \quad i = 1, \ldots, n,$$
$$F_{i,i+1} = F_{i+1,i} = \sqrt{H_i(1-H_i)}, \quad i = 1, \ldots, n-1,$$

where $H_0 = 0$. Noting that $H_i H_{i-1} = H_{i-1}$ and $H_1 \geq E$, it is easy to check that $F$ is a finite-rank projection and $F$ dominates $E \oplus 0 \oplus \cdots \oplus 0$. For $u \in \mathcal{F}$, we have that

$$\langle \hat{u} F - F \hat{u} \rangle_{i,i} = [u_i, H_i] - [u_i, H_{i-1}],$$
$$\langle \hat{u} F - F \hat{u} \rangle_{i,i+1} = [u_i, \sqrt{H_i(1-H_i)}] + \sqrt{H_i(1-H_i)}(u_i - u_{i+1}).$$

Thus, since $\|\sqrt{H_i(1-H_i)}\| \leq 1/2$, the norm of $[\hat{u}, F]$ is smaller than

$$\epsilon/2 + 2\delta + 2\max_i \|[u_i, \sqrt{H_i(1-H_i)}]\|,$$

which can be made smaller than $\epsilon$ for all $u \in \mathcal{F}$ by choosing $\delta$ small. \qed

When $E$ is a projection on a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(E\mathcal{H})$ the bounded operators on the subspace $E\mathcal{H}$.

**Lemma 3.4** Let $A$ be a unital nuclear $C^*$-algebra, $\pi$ an irreducible representation of $A$ on a Hilbert space $\mathcal{H}$, and $E$ a finite-rank projection on $\mathcal{H}$. Then the identity map on $A$ can be approximated by a net of compositions of CP maps

$$A \xleftarrow{\sigma\nu=\sigma'\nu \oplus \sigma''\nu} N\nu \oplus \mathcal{B}(E\nu \mathcal{H}) \xrightarrow{\tau\nu=\tau'\nu + \tau''\nu} A,$$

where $N\nu$ is a finite-dimensional $C^*$-algebra, $(E\nu)$ is an increasing net of finite-rank projections on $\mathcal{H}$ such that $E \leq E\nu$ and $\lim E\nu = 1$, $\sigma'\nu$ and $\sigma''\nu$ are unital CP maps such that $\sigma''\nu(x) = E\nu \pi(x) E\nu$, $x \in A$, and $\tau'\nu$ is a unital CP map such that

$$\pi \tau'\nu(a) E = 0, \quad a \in N\nu,$$
$$E \pi \tau''\nu(b) E = E b E, \quad b \in \mathcal{B}(E\nu \mathcal{H}).$$

**Proof.** There is a non-degenerate representation $\rho$ of $A$ such that $\rho$ is disjoint from $\pi$ and $\rho \oplus \pi$ is a universal representation, i.e., $\rho \oplus \pi$ extends to a faithful representation of $A^{**}$. Note that $(\rho \oplus \pi)(A^{**}) = \rho(A)^{''} \oplus \pi(A)^{''}$.

If the nuclear $C^*$-algebra $A$ is separable, $A^{**}$ is semidiscrete [3], which in turn implies that $\mathcal{R} = \rho(A)^{''}$ is semidiscrete. Hence the identity map on $\mathcal{R}$ can be approximated, in the point-weak* topology, by a net $(\tau'\nu, \sigma''\nu)$ of CP maps on $\mathcal{R}$, where $\sigma''\nu$ (resp. $\tau''\nu$) is a weak*-continuous unital CP map of $\mathcal{R}$ into a finite-dimensional $C^*$-algebra $N\nu$ (resp. of $N\nu$ into $\mathcal{R}$). By denoting $\sigma''\nu \rho$ by $\sigma''\nu$ again, we obtain a net of diagrams

$$A \xrightarrow{\sigma''\nu} N\nu \xrightarrow{\tau''\nu} \mathcal{R}.$$
such that $\tau'_\nu \sigma'_\nu(x)$ converges to $\rho(x)$ in the weak* topology for any $x \in A$.

If $A$ is separable or not, we have the characterization of nuclearity in terms of CP maps [3]; there is a net of diagrams of unital CP maps:

$$A \xrightarrow{\sigma'_\nu} N_\nu \xrightarrow{\tau'_\nu} A$$

such that $N_\nu$ is finite-dimensional and $\tau'_\nu \sigma'_\nu(x)$ converges to $x$ in norm for any $x \in A$. By denoting $\rho \tau'_\nu$ by $\tau'_\nu$ again, we obtain a net of diagrams:

$$A \xrightarrow{\sigma'_\nu} N_\nu \xrightarrow{\tau'_\nu} \mathcal{R}$$
as above; actually $\tau'_\nu \sigma'_\nu(x)$ converges to $\rho(x)$ in norm for any $x \in A$.

Since $\pi(A)'' = \mathcal{B}(\mathcal{H})$ is semidiscrete, there is such a net of CP maps on $\pi(A)''$ as for $\mathcal{R}$ as well. But we shall construct one in a specific way.

Let $(E_\nu)$ be an increasing net of finite-rank projections on $\mathcal{H}$ such that $E \leq E_\nu$, and $\lim E_\nu = 1$. We define $\sigma''_\nu : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(E_\nu \mathcal{H})$ by $\sigma''_\nu(x) = E_\nu x E_\nu$ and $\tau''_\nu : \mathcal{B}(E_\nu \mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by $\tau''_\nu(a) = a + \omega(a)(1 - E_\nu)$, where $\omega$ is a vector state, defined through a fixed unit vector in $E \mathcal{H}$. Then it is immediate that $(\sigma''_\nu, \tau''_\nu)$ has the desired properties. By denoting $\sigma''_\nu \pi$ by $\sigma''_\nu$ again, we obtain a net of diagrams:

$$A \xrightarrow{\sigma''_\nu} \mathcal{B}(E_\nu \mathcal{H}) \xrightarrow{\tau''_\nu} \pi(A)''$$
such that $\tau''_\nu \sigma''_\nu(x)$ converges to $\pi(x)$ in the weak* topology for any $x \in A$.

We may suppose that we use the same directed set $\{\nu\}$ for both $(\sigma'_\nu, \tau'_\nu)$ and $(\sigma''_\nu, \tau''_\nu)$. We set $\sigma_\nu = \sigma'_\nu \oplus \sigma''_\nu$, $M_\nu = N_\nu \oplus \mathcal{B}(E_\nu \mathcal{H})$, and $\tau_\nu = \tau'_\nu + \tau''_\nu$. By identifying $A^{**}$ with $\mathcal{R} \oplus \pi(A)''$, we have that

$$A \xrightarrow{\sigma_\nu} M_\nu \xrightarrow{\tau_\nu} A^{**}$$
approximate the identity map on $A$ (in the point-weak* topology), i.e., $\tau_\nu \sigma_\nu(x)$ converges to $x$ in the weak* topology for any $x \in A$.

Following [3] we approximate $\tau_\nu$ by unital CP maps of $M_\nu$ into $A$. This is done as follows. If $(e^k_{ij})$ denotes a family of matrix units of $M_\nu$, $\tau_\nu$ is uniquely determined by the positive element $\Lambda_\nu = (\tau_\nu(e^k_{ij}))$ in $M_\nu \otimes A^{**}$ (2.1 of [3]). Since $M_\nu \otimes A$ is dense in $M_\nu \otimes A^{**}$ in the weak* topology, we can, by general theory, approximate $\Lambda_\nu$ by positive elements in $M_\nu \otimes A$, in the weak* topology, which then determine CP maps of $M_\nu$ into $A$ (see the proof of 3.1 of [3]). In particular we approximate $\tau'_\nu : N_\nu \rightarrow A^{**}$ by CP maps $\psi' : N_\nu \rightarrow A$ satisfying

$$\pi \psi'(a)E = 0, \ a \in N_\nu,$$

and $\tau''_\nu : \mathcal{B}(E_\nu \mathcal{H}) \rightarrow A^{**}$ by CP maps $\psi'' : \mathcal{B}(E_\nu \mathcal{H}) \rightarrow A$ satisfying

$$E \pi \psi''(a)E = EaE, \ a \in \mathcal{B}(E_\nu \mathcal{H}).$$

This is indeed possible as shown by using Kadison’s transitivity. Moreover, by taking convex combinations of $\psi' + \psi''$, we may assume that $h = \psi'(1) + \psi''(1)$ is close to $1 \in A$.
in norm. By replacing \( \psi' \) by \( h^{-1/2}\psi'(\cdot)h^{-1/2} \) etc. we may suppose that \( \psi = \psi' + \psi'' \) is a unital CP map. Since \( hE = E = Eh \), this does not destroy the above properties imposed on \( \psi' \) and \( \psi'' \).

Restricting \( \sigma_\nu \) to \( A \) and retaining the same symbol \( \tau \) for the CP maps into \( A \) (instead of \( \psi \)), we now have a net of the compositions of unital CP maps:

\[
A \xrightarrow{\sigma_\nu} M_\nu \xrightarrow{\tau_\nu} A,
\]

which approximates the identity map in the point-weak topology.

By taking convex combinations of the above CP maps, we will obtain such a net which now approximates the identity map in the point-norm topology. For example, if \( (\lambda_\nu) \) is such that \( \lambda_\nu \geq 0 \), \( S = \{ \nu \mid \lambda_\nu > 0 \} \) is finite, and \( \sum_\nu \lambda_\nu = 1 \), then we define

\[
A \xrightarrow{\phi} \left( \bigoplus_{\nu \in S} N_\nu \right) \oplus \mathcal{B}(E_{\nu_0} \mathcal{H}) \xrightarrow{\psi} A,
\]

where \( \nu_0 \) is such that \( \nu_0 \geq \nu \), \( \nu \in S \), and

\[
\phi = \left( \bigoplus_{\nu \in S} \sigma'_\nu \right) \oplus \sigma''_{\nu_0}, \quad \psi = \sum_{\nu \in S} \lambda_\nu \tau'_\nu + \sum_{\nu \in S} \lambda_\nu \tau''_\nu p_\nu,
\]

with \( p_\nu : \mathcal{B}(E_{\nu_0} \mathcal{H}) \to \mathcal{B}(E_\nu \mathcal{H}) \) defined by the multiplication of \( E_\nu \) on both sides. By doing so, the properties \( \pi \psi'(a)E = 0 \) and \( E \pi \psi''(a)E = EaE \) are still retained, where \( \psi' \) is the first component of \( \psi \) etc. See [5] for technical details. \( \square \)

**Lemma 3.5** Let \( \sigma_\nu, \tau_\nu, M_\nu = N_\nu \oplus \mathcal{B}(E_\nu \mathcal{H}) \) be as in [3.4]. For any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( u \in \mathcal{U}(A) \) satisfies that \( \| u - \tau_\nu \sigma_\nu(u) \| < \delta \), there is a \( \nu \in \mathcal{U}(M_\nu) \) with \( \| u - \tau_\nu(v) \| < \epsilon \).

**Proof.** Suppose that \( A \) is represented on a Hilbert space \( H \). Since \( \tau = \tau_\nu \) is a unital CP map, by Steinspring’s theorem there is a representation \( \phi \) of \( M = M_\nu \) on a Hilbert space \( K \) which contains \( H \) such that \( \tau(a) = P\phi(a)P, \ a \in M, \) where \( P \) is the projection onto \( H \).

If \( u \in \mathcal{U}(A) \) satisfies that \( \| u - \tau \sigma(u) \| < \delta \), where \( \sigma = \sigma_\nu \) etc., it follows that

\[
\tau(\sigma(u)\sigma(u)^*) = P\phi\sigma(u)\phi\sigma(u)^*P \geq P\phi\sigma(u)P\phi\sigma(u)^*P \geq (1 - 2\delta)P.
\]

Let \( b \) denote \( \sigma(u)\sigma(u)^* \). Since \( P\phi(b)(1 - P)\phi(b)P = P\phi(b^2)P - (P\phi(b)P)^2 \leq P - (1 - 2\delta)^2P \), we have that \( \| P\phi(b)(1 - P) \| \leq 2\delta^{1/2} \). Since \( [P, \phi(b)] = P\phi(b)(1 - P) - (1 - P)\phi(b)P \), we also have that \( \|[P, \phi(b)]\| \leq 2\delta^{1/2} \). For any \( a \in M \) it follows that \( \| \tau(ba) - \tau(b)\tau(a) \| \leq 2\delta^{1/2}\|a\| \) and \( \| \tau(ba) - \tau(a) \| \leq 2(\delta^{1/2} + \delta)\|a\| \).
If \( e \) is the spectral projection of \( b \) corresponding to \( [\lambda, 1] \) for some \( \lambda \in (0, 1) \), then \( b \leq \lambda(1-e) + be \) and

\[
(1-2\delta)P \leq P\phi(b)P \leq \lambda P - \lambda P\phi(e)P + P\phi(be)P \leq \lambda P - \lambda P\phi(e)P + P\phi(e)P + 2(\delta + \delta^{1/2})P.
\]

Let \( \lambda = 1 - 4\delta - 2\delta^{1/2} - \delta^{1/4} \). Then the above inequality implies that

\[
\delta^{1/4}P \leq (4\delta + 2\delta^{1/2} + \delta^{1/4})P\phi(e)P,
\]

or \( \|P - P\phi(e)P\| \leq 4\delta^{3/4} + 2\delta^{1/4} \). Hence we have that \( \|\tau(e) - 1\| < 3\delta^{1/4} \) and \( \|\tau(be) - 1\| < 3\delta^{1/4} \) for a sufficiently small \( \delta > 0 \). Since \( be \leq (be)^{1/2} \leq e \), \( \tau((be)^{1/2}) \) is also close to 1.

Since \( \|\tau(e) - \tau((be)^{1/2})\| = \|P\phi((be)^{1/2})(1-P)\| < 3\delta^{1/8} \), \( \tau((be)^{1/2}) \) is also close to 1 (up to the order of \( \delta^{1/8} \) in this rough estimate); here \( (be)^{-1/2} \) is the inverse of \( (be)^{1/2} \) in \( eM_e \).

We now define a unitary \( v \) in \( M \) by \( v = (be)^{-1/2}\sigma(u) + y \), where \( y \) satisfies that \( yy^* = 1 - e \) and \( y^*y = 1 - \sigma(u)^*\sigma(u)^{-1} \sigma(u) \). Since \( (be)^{-1/2} \sigma(u) \sigma(u)^* \sigma(u)^{-1/2} = e \), \( v \) is indeed a unitary. Since \( \tau(y)\tau(y^*) \leq \tau(yy^*) = \tau(1-e) \leq 3\delta^{1/4} \), \( \|y\| \) is of the order of \( \delta^{1/8} \). Since \( \tau((be)^{-1/2}\sigma(u)) \) is close to \( \tau((be)^{-1/2})\tau(\sigma(u)) \) up to the order of \( \delta^{1/16} \), we can conclude that \( \|\tau(v) - \tau(\sigma(u))\| \) is close to zero up to the order of \( \delta^{1/16} \).

When \( (X, d) \) is a metric space, \( S \subset X \), and \( \varepsilon > 0 \), we call \( S \) an \( \varepsilon \)-net if \( \cup_{x \in S} B(x, \varepsilon) = X \), where \( B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\} \). When \( X \) has a finite \( \varepsilon \)-net, we denote by \( N(X, \varepsilon) \) the minimum of orders over all the finite \( \varepsilon \)-nets. If \( X \) is compact, then \( N(X, \varepsilon) \) is well-defined for any \( \varepsilon > 0 \).

**Lemma 3.6** Let \( (X, d) \) be a compact metric space. If \( S_1 \) and \( S_2 \) are \( \varepsilon \)-nets consisting \( N(X, \varepsilon) \) points, then there is a bijection \( f \) of \( S_1 \) onto \( S_2 \) such that \( d(x, f(x)) < 2\varepsilon \), \( x \in S_1 \).

**Proof.** Let \( \mathcal{F} \) be a non-empty subset of \( S_1 \) and set

\[
\mathcal{G} = \{ y \in S_2 \mid B(y, \varepsilon) \cap \cup_{x \in \mathcal{F}} B(x, \varepsilon) \neq \emptyset \}.
\]

Since \( \cup_{x \in \mathcal{F}} B(x, \varepsilon) \subset \cup_{x \in \mathcal{G}} B(x, \varepsilon) \), it follows that \( \mathcal{G} \cup S_1 \setminus \mathcal{F} \) is an \( \varepsilon \)-net and that the order of \( \mathcal{G} \) is greater than or equal to that of \( \mathcal{F} \). Then by the matching theorem we can find a bijection \( f \) of \( S_1 \) onto \( S_2 \) such that \( f(x) \in \{ y \in S_2 \mid B(x, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset \} \).

**Proof of Lemma 3.6** Let \( \pi \) be an irreducible representation of the unital nuclear \( C^* \)-algebra \( A \) on a Hilbert space \( \mathcal{H} \), \( E \) a finite-rank projection on \( \mathcal{H} \), \( \mathcal{F} \) a finite subset of \( \mathcal{U}_0(A) \), and \( \varepsilon > 0 \).

We apply Lemma 3.3 to this situation. Thus there exist an \( n \in \mathbb{N} \) and a finite-rank projection \( F \) on \( C^n \otimes \mathcal{H} \) such that

\[
F \geq E \oplus 0 \oplus \cdots \oplus 0,
\]

\[
\|[F, \pi_n(\hat{u})]\| < \varepsilon, \quad u \in \mathcal{F},
\]

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where \( \pi_n \) denotes the natural extension of \( \pi \) to a representation of \( M_n \otimes A \) on \( \mathbb{C}^n \otimes \mathcal{H} \); hereafter we shall simply denote \( \pi_n \) by \( \pi \). Let \( F_0 \) be a finite-rank projection on \( \mathcal{H} \) such that \( F \leq 1 \otimes F_0 \).

By Lemma 3.4 we find a net of diagrams

\[
A \xrightarrow{\sigma_v=\sigma_v'\otimes\sigma_v''} N_\nu \oplus \mathcal{B}(E_\nu \mathcal{H}) \xrightarrow{\tau_v=\tau_v''+\tau_v'} M_n \otimes A
\]

with \( F_0 \) in place of \( E \) as described there; in particular \( F_0 \leq E_\nu \). We take tensor product of these diagrams with \( M_n \); denoting \( \text{id}_n \otimes \sigma_v \) by the same symbol \( \sigma_v \) etc., we obtain

\[
M_n \otimes A \xrightarrow{\sigma_v=\sigma_v'\otimes\sigma_v''} M_n \otimes N_\nu \oplus M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) \xrightarrow{\tau_v=\tau_v''+\tau_v'} M_n \otimes A.
\]

Noting that \( F \in M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) = \mathcal{B}(\mathbb{C}^n \otimes E_\nu \mathcal{H}) \), we denote

\[
V_\nu = \mathcal{U}(M_n \otimes N_\nu \oplus M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) \cap \{ F \}')
\]

which is a compact group. Since \( (1 \otimes F_0)\pi \tau'_\nu(v) = 0 \) and \( (1 \otimes F_0)\pi \tau''_\nu(v)(1 \otimes F_0) = (1 \otimes F_0)v(1 \otimes F_0) \), we have that for each \( v \in V_\nu \)

\[
F\pi(\tau_\nu(v)\tau_\nu(v^*))F = F(1 \otimes F_0)\pi(\tau_\nu(v)\tau_\nu(v^*))(1 \otimes F_0)F,
\]

\[
= F(1 \otimes F_0)\pi(\tau''_\nu(v)\tau''_\nu(v^*))(1 \otimes F_0)F,
\]

\[
= F(1 \otimes F_0)v(1 \otimes F_0)v^*(1 \otimes F_0)F + F(1 \otimes F_0)\pi(\tau''_\nu(v))(1 \otimes (1 - F_0))\pi(\tau''_\nu(v^*))(1 \otimes F_0)F.
\]

Since the first term is \( F \) as \( [F, v] = 0 \), the second term must be zero. Hence it follows that

\[
F\pi(\tau_\nu(v)\tau_\nu(v^*))F = F,
\]

which implies that

\[
\pi(\tau_\nu(v)\tau_\nu(v^*))F = F.
\]

By multiplying \( E \oplus 0 \oplus \cdots \oplus 0 \) from the right we have that

\[
\sum_{j,k} \pi(\tau_\nu(v_{1j})\tau_\nu(v_{kj}^*))F_{k1}E = E.
\]

Since \( F \geq E \oplus 0 \oplus \cdots \oplus 0 \), we have that \( F_{k1}E = 0 \) for \( k \neq 1 \). Thus it follows that for \( v \in V_\nu \),

\[
\sum_{j=1}^{n} \pi(\tau_\nu(v_{1j})\tau_\nu(v_{1j}^*))E = E.
\]

By Lemma 3.3 (applied to \( M_n \otimes A \) instead of \( A \)) we choose \( \nu \) such that each \( u \in \mathcal{F} \) has a unitary \( \hat{u}' \in M_n \otimes N_\nu \oplus M_n \otimes \mathcal{B}(E_\nu \mathcal{H}) \) such that

\[
\|\tau_\nu(\hat{u}') - \hat{u}\| \approx 0
\]
as well as
\[ \| \tau_\nu \sigma_\nu (\hat{u}) - \hat{u} \| \approx 0. \]
Since
\[ (1 \otimes F_0) \hat{u}' (1 \otimes F_0) = (1 \otimes F_0) \pi (\tau''_\nu (\hat{u}')) (1 \otimes F_0) \]
\[ \approx (1 \otimes F_0) \pi (\tau_\nu (\hat{u}')) (1 \otimes F_0) \approx (1 \otimes F_0) \pi (\hat{u}) (1 \otimes F_0), \]
we have that
\[ \pi (\hat{u}) F \approx F \pi (\hat{u}) F \approx F \hat{u}' F \approx \hat{u}' F. \]
By choosing \( \nu \) sufficiently large, we may assume that
\[ \| \tau_\nu (\hat{u}') - \hat{u} \| < \epsilon, \ u \in \mathcal{F}. \]
By taking the unitary part of the polar decomposition of \( w = F \hat{u}' F + (1 - F) \hat{u}' (1 - F) \), we may assume that
\[ [\hat{u}', F] = 0, \ u \in \mathcal{F}. \]
Since \( \| w - \hat{u}' \| < 2 \epsilon \), we can estimate that
\[ \| \tau_\nu (\hat{u}') - \hat{u} \| < 3 \epsilon, \ u \in \mathcal{F}. \]
Since \( \| \tau_\nu (\hat{u}') \tau_\nu (\hat{u}')^* - 1 \| < 6 \epsilon \), we have that for any \( v \in V_\nu \),
\[ \| \tau_\nu (\hat{u}' v) - \tau_\nu (\hat{u}') \tau_\nu (v) \| < (12 \epsilon)^{1/2} < 4 \epsilon^{1/2}. \]
(See the proof of 3.3.) Hence for \( v \in V_\nu \)
\[ \| \hat{u} \tau_\nu (v) - \tau_\nu (\hat{u}' v) \| < 3 \epsilon + 4 \epsilon^{1/2}, \ u \in \mathcal{F}. \]
We choose an \( \epsilon \)-net \( \mathcal{G}' \) of \( V_\nu \) consisting of \( N(V_\nu, \epsilon) \) points and set
\[ \mathcal{G} = \{ (\tau_\nu (v_{11}), \tau_\nu (v_{12}), \ldots, \tau_\nu (v_{1n})) \mid v \in \mathcal{G}' \}. \]
Since \( \hat{u}' \mathcal{G}' \) is also an \( \epsilon \)-net of \( V_\nu \) for \( u \in \mathcal{F} \), Lemma 3.6 gives a bijection \( f \) of \( \mathcal{G}' \) onto \( \mathcal{G}' \) such that
\[ \| \hat{u}' v - f (v) \| < 2 \epsilon, \ v \in \mathcal{G}'. \]
Hence for each \( u \in \mathcal{F} \) there is a bijection \( f \) of \( \mathcal{G}' \) onto \( \mathcal{G}' \) such that
\[ \| \hat{u} \tau_\nu (v) - \tau_\nu (f (v)) \| < 5 \epsilon + 4 \epsilon^{1/2}, \]
which implies that regarding \( f \) as a map of \( \mathcal{G} \) onto \( \mathcal{G} \),
\[ \| u x - f (x) \| < 5 \epsilon + 4 \epsilon^{1/2}, \ x \in \mathcal{G}. \]
This completes the proof. \( \square \)

In Lemma 3.4 we could handle a mutually disjoint finite family of irreducible representations instead of just one. By doing so we can derive:
Lemma 3.7 Let $A$ be a unital nuclear $C^*$-algebra. Let $F$ be a finite subset of $U_0(A)$, $\pi$ a representation of $A$ on a Hilbert space $\mathcal{H}$ such that $\pi = \bigoplus_{i=1}^{k}\pi_k$ with $(\pi_i)_{i=1}^{k}$ a mutually disjoint family of irreducible representations of $A$, $E$ a finite-dimensional projection on $\mathcal{H}$, and $\epsilon > 0$. Then there exist an $n \in \mathbb{N}$ and a finite subset $G$ of $M_{1n}(A)$ such that $xx^* \leq 1$ and $\pi(xx^*)E = E$ for $x \in G$, and for any $u \in F$ there is a bijection $f$ of $G$ onto $G$ with
\[
\|ux - f(x)\| < \epsilon.
\]

A straightforward generalization of Lemma 3.4 would require that $E \in \pi(A)''$ in the above statement. But, since any finite-rank projection on $\mathcal{H}$ is dominated by such a one in $\pi(A)''$, we did not need it.

By having this at hand we can derive a stronger version of Lemma 2.1 and then strengthen Theorem 2.3. For example we will obtain:

Theorem 3.8 Let $A$ be a separable nuclear $C^*$-algebra. If $(\omega_i)_{1 \leq i \leq n}$ and $(\varphi_i)_{1 \leq i \leq n}$ are finite sequences of pure states of $A$ such that $(\omega_i)$ (resp. $(\varphi_i)$) are mutually disjoint and $\ker \omega_i = \ker \varphi_i$ for all $i$, then there is an $\alpha \in \text{Alm}_{10}(A)$ such that $\omega_i = \varphi_i\alpha$ for all $i$.

We will have to use a general form of Kadison’s transitivity for the proofs of the above results as in [7]. See Section 7 of [11] for details and for other consequences.

We do not know whether we could take an arbitrary non-degenerate representation of $A$ for $\pi$ in Lemma 3.7 (perhaps by weakening the requirement $\pi(xx^*)E = E$ by $\|\pi(xx^*)E - E\| < \epsilon$). If this were the case, we would obtain a new characterization of nuclearity which manifests a close connection with amenability of $A$ (cf. [7, 12, 14]).

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