Classification of quantum relativistic orientable objects

D M Gitman\(^1\) and A L Shelepin\(^2\)

\(^1\) Instituto de Física, Universidade de São Paulo, Caixa Postal 66318-CEP, 05315-970, São Paulo, SP, Brazil
\(^2\) Moscow Institute of Radio Engineering, Electronics and Automation, Prospect Vernadskogo, 78, 117454 Moscow, Russia
E-mail: gitman@dfn.if.usp.br and alex@shelepin.msk.ru

Received 4 October 2010
Accepted for publication 24 November 2010
Published 10 January 2011
Online at stacks.iop.org/PhysScr/83/015103

Abstract

Extending our previous work ‘Fields on the Poincaré group and quantum description of orientable objects’ (Gitman and Shelepin 2009 Eur. Phys. J. C 61 111–39), we consider here a classification of orientable relativistic quantum objects in 3 + 1 dimensions. In such a classification, one uses a maximal set of ten commuting operators (generators of left and right transformations) in the space of functions on the Poincaré group. In addition to the usual six quantum numbers related to external symmetries (given by left generators), there appear additional quantum numbers related to internal symmetries (given by right generators). Spectra of internal and external symmetry operators are interrelated, which, however, does not contradict the Coleman–Mandula no-go theorem. We believe that the proposed approach can be useful for the description of elementary spinning particles considered as orientable objects. In particular, it gives a group-theoretical interpretation of some facts of the existing phenomenological classification of spinning particles.

PACS numbers: 11.30.Cp, 11.30.Ly, 02.20.Qs

1. Introduction

In our previous work [1], we discussed a new approach for the description of orientable quantum relativistic objects in 3 + 1 dimensions. In such an approach, the orientable object is associated with a scalar field \( f(h) \) on the Poincaré group \( M(3, 1) \). The field depends on a 4-vector \( x^\mu \), which gives a position of the object, and on a 6-parameter matrix \( Z \in \text{Spin}(3, 1) \), which describes the object orientation.

The field \( f(h) \) admits two kinds of transformations, corresponding to a change of the laboratory or space-fixed reference frame (s.r.f.), as well as to a change of the local or body-fixed reference frame (b.r.f.),

\[
T(g_l, g_r) f(h) = f(g_l^{-1} h g_r). \tag{1}
\]

Here, left multiplication by \( g_l^{-1} \) corresponds to a change of the s.r.f. (Lorentz transformations), whereas right multiplication by \( g_r \) corresponds to a change of the b.r.f. There are two sets of transformation generators, namely right and left ones, and they are used to construct a maximal set of commuting operators in the space of functions \( f(h) \).

The set of all the transformations (1) form the direct product \( M(3, 1) \times M(3, 1) \). Possible external symmetries correspond to the left transformations, whereas some of the possible internal symmetries correspond to the right transformations. Indeed, external symmetries are usually defined as symmetries of the enclosing space, i.e. symmetries with respect to a change of the s.r.f., whereas internal ones are defined as symmetries of the body itself, in particular, symmetries with respect to a change of the b.r.f.

A classification of orientable objects is natural to define with the help of a maximal set of commuting operators, which are constructed from the generators of transformations (1). This set contains ten commuting operators (according to the number of group parameters) and consists of four operator functions of the left generators, four operator functions of the right generators and two Casimir operators, which can be constructed from the left generators, as well as from the right generators. Such a classification attributes ten quantum numbers to an orientable quantum object.
On the other hand, in relativistic quantum theory of point-like objects there exists Wigner’s classification [2], based on the left generators (generators of external symmetry transformations). Two Casimir operators determine the representation (mass and spin), while the remaining four operators determine, for instance, helicity and momentum. Thus, Wigner’s classification attributes only six quantum numbers to a relativistic quantum point-like object.

We believe that the proposed approach can be useful for the description of known spinning particles. In particular, their classification within this approach (considering spinning particles as orientable objects) attributes four additional quantum numbers.

One ought to mention that in the 1960s, attempts were made to unite internal and external symmetries in the framework of one group. Soon, however, the so-called no-go theorem [3] was proved (under some very general assumptions), stating that the symmetry group of the S-matrix is locally isomorphic to a direct product of the Poincaré group and the group of internal symmetries. However, on the basis of this, one often draws the too strong conclusion that a non-trivial relation between internal and external symmetries is impossible.

As was already mentioned, the transformations (1) of a field \( f(h) \) form the direct product of groups of internal and external symmetries, in agreement with the mentioned no-go theorem. Nevertheless, as will be demonstrated below, a non-trivial relation between internal and external symmetries is possible. Both transformation groups, corresponding to a change of the s.r.f. and b.r.f., act in the same space of 10-parameter functions \( f(h) \) and have the same Casimir operators that define the mass and the spin. By fixing eigenvalues of the Casimir operators and therefore fixing the representation, we obviously impose some conditions on the spectra of both left and right operators that enter the maximal representation (mass and spin), while the remaining four quantum numbers related to the orientation; these numbers, corresponding to right generators, are internal ones.

The paper is organized as follows. In section 1, we present a brief summary of the field on the Poincaré group (details can be found in [1]). In sections 2 and 3, we examine two sets of commuting operators in the space of functions on the Poincaré group that correspond to states with a fixed parity and chirality. We then consider the properties of right generators from these sets. In section 4, we consider a possible physical interpretation of a given classification of orientable objects. In sections 5 and 6, some sets of commuting operators are applied to a classification of orientable objects with spin-1/2 and -1. In section 7, we consider the classification of the fields on the homogeneous spaces of the Poincaré group.

We emphasize the fact that we examine only non-unitary finite-dimensional representations of the group \( \text{Spin}(3,1) \) and, accordingly, those of the group \( M(3,1) = T(4) \rtimes \text{Spin}(3,1) \), which corresponds to finite-component relativistic wave equations (‘relativistic quantum mechanics’ or ‘one-particle sector’). The consideration of unitary representations goes beyond the scope of this paper.

2. Orientable objects: right and left transformations

As was already mentioned, for a description of orientable objects it is convenient to use two reference frames: the laboratory (or s.r.f., related to the observer), with an orthonbasis \( e_\mu \), and the local (or b.r.f., related to the body), with the orthonbasis \( \xi_k, \zeta_k = v_k^\mu e_\mu \). For Euclidean spaces, \( \{e_i, e_j\} = \delta_{ij} \), and thus the elements of the matrix \( V = [v_k^\mu] \) satisfy the condition \( \sum_v v_i^v v_j^v = \delta_{ij} \), that is, the matrix \( V \) is orthogonal, \( V^{-1} = V^T \). For pseudo-Euclidean spaces (in particular, the 4D Minkowski space) the matrix \( V \) is pseudo-orthogonal, \( V^{-1} = \eta V^T \eta = \eta \operatorname{diag}(1, -1, \ldots, -1) \).

In Minkowski space, by using the homomorphism \( SL(2,C) \sim SO_0(3,1) \), one can describe the orientation by the matrix

\[
Z = \begin{pmatrix} z_1^1 & z_1^2 \\ z_2^1 & z_2^2 \end{pmatrix} \in SL(2,C),
\]

where \( Z = E \), \( E = \sigma^\mu e_\mu \) and \( \Xi = \sigma^\alpha \xi^\alpha \).

The quantities \( v_\alpha^\mu \in SO_0(3,1) \) are expressed in terms of \( z \) [1],

\[
v_\alpha^\mu = \frac{1}{2} (\sigma^\mu)_\beta^\alpha (\sigma_\beta)_{\alpha \beta} \xi^\alpha \zeta^\beta.
\]

We note that \( v_\alpha^\mu \) are tetrads, i.e. objects transformed as vectors (with respect to the first index, \( \mu \)) under the change of the s.r.f., being, at the same time, objects transformed as vectors (with respect to the second index, \( \alpha \)) under the change of the b.r.f.\(^1\).

The position of an orientable object in Minkowski space is therefore given by a 4-vector \( x \) (being coordinates of the

\(^1\) We underline ‘right’ indices in order to avoid confusion, since we shall consider quantities at fixed values of the indices (for instance, spinors \( z^2 \) and \( z^2 \)).
origin of the b.r.f. in the s.r.f.) and by the matrix of orientation of the Lorentz group, while \( z \) carries the spinor representation of this group. If one restricts the consideration to functions independent of \( z \), then (12) reduces to transformations of the left quasiregular representation, corresponding to the case of a usual scalar field \( f'(x') = f(x) \). If one restricts the consideration to functions independent of \( x \), then (12) reduces to transformations of the left generalized regular representation (GRR) of the Lorentz group.

Generators that correspond to translations and rotations have the form

\[
\hat{p}_\mu = i\partial/\partial x^\mu, \quad \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu},
\]

where \( \hat{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \) are the operators of orbital momentum projections and \( \hat{S}_{\mu\nu} \) are the operators of spin projections. The operators of right translations can also be presented in the form \( \hat{P}_R = -Z\hat{P}Z' \); the operators \( \hat{S}_{\mu\nu} \) and \( \hat{S}_{\mu\nu}^R \) depend only on \( z \) and \( \partial/\partial z \).

\[
\hat{S}_{\mu\nu} = i\left( (\sigma_{\mu\nu})_a \vec{c}^{a\alpha} \vec{c}^{a\beta} + (\sigma_{\mu\nu})^a \vec{c}^{a\alpha} \vec{c}^{a\beta} \right),
\]

where

\[
(\sigma_{\mu\nu})_a^\beta = \frac{1}{2}(\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu)^a_\beta, \quad (\sigma_{\mu\nu})^a_\beta = \frac{1}{4}(\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu)^a_\beta.
\]

In addition, it is convenient to present the spin operators in terms of 3D vector notation, \( \hat{S}_k = \frac{1}{2}\epsilon_{ijk}\hat{S}^{ij} \); \( \hat{B}_k = \hat{S}_{ijk} \); see formulae (A.1)–(A.3) of the appendix.

Below, we also consider phase transformations of \( Z \) (being symmetry transformations for a field on the Poincaré group [1]).

\[
Z' = Ze^{i\phi},
\]

with the generator (chirality operator)

\[
\hat{\Gamma}^5 = -i\partial/\partial \phi = \frac{1}{2} \left( \vec{z}^{a} \hat{b}^{a} \hat{a}^{b} - \vec{z}^{a} \hat{b}^{a} \hat{a}^{b} \right).
\]

In fact, this means a transition to an analysis of the group \( M(3, 1)_{\text{ext}} \times M(3, 1)_{\text{int}} \times U(1) \).

### 3. Maximal sets of commuting operators

A maximal set of commuting operators in the space of the functions on a group contains Casimir operators and an equal number of operators that are some functions of left and right generators [5].

Casimir operators that label irreps can be composed of both left and right generators, so that the ‘left’ and ‘right’ mass and spin are the same. For the Poincaré group \( M(3, 1) \), we have

\[
\hat{p}_m^2 = \eta^{\mu\nu}\hat{p}_\mu\hat{p}_\nu = \eta^m_\nu \hat{p}_m^R \hat{p}_n^R.
\]
\[
\hat{W}^2 = \eta_{\mu\nu} \hat{W}^\mu \hat{W}^\nu = \eta_{\mu\nu} \hat{W}_R^\mu \hat{W}_R^\nu, \tag{22}
\]
where
\[
\hat{W}^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \hat{p}_\nu \hat{J}_\rho \hat{p}_\sigma, \quad \hat{W}_R^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \hat{p}_R^\nu \hat{S}_R^\rho \hat{S}_R^\sigma.
\tag{23}
\]

As four operators, composed of left generators (generators of the group \(M(3,1)_{\text{ext}}\)), one can choose the Casimir operator \(\hat{p} \hat{S} = \hat{p}_R \hat{S}_R^3\) and the generators \(\hat{p}_3\) of the subgroup \(M(3)_{\text{ext}}\). The latter correspond to additive quantum numbers (if the quantum number is additive, then this quantum number of the composite system is the sum of the corresponding quantum numbers of subsystems). Eigenfunctions of these operators correspond to definite values of the helicity and the momentum.

The functions of right generators can be chosen in different ways, for instance, the set \(\hat{p}_R \hat{S}_R^3, \hat{p}_R^3\) (analogous to the set \(\hat{p} \hat{S}, \hat{p}_3\) for \(M(3,1)_{\text{ext}}\)), corresponding to the reduction \(M(3,1)_{\text{int}} \supset M(3)_{\text{int}} \supset T(3)_{\text{int}}\). However, as it follows from the explicit form of the operator \(\hat{p}_R^3\), its eigenfunctions \(f(x,z)\) contain arbitrarily large powers of \(z\) such that the corresponding representation of the Lorentz group is infinite dimensional.

We choose sets corresponding to the reduction \(M(3,1)_{\text{int}} \supset SL(2,C)_{\text{int}}\). Two Casimir operators of the ‘right’ Lorentz group \(SL(2,C)_{\text{int}}\) have the form
\[
\hat{S}^2 - \hat{B}^2 = \frac{1}{2} \hat{S}_R^3 \hat{S}_R^3 = \frac{1}{2} \hat{S}_R^{\mu\nu} \hat{S}_R^{\mu\nu},
\]
\[
\hat{S} \hat{B} = \frac{1}{16} \epsilon^{\mu\nu\rho\sigma} \hat{S}_R^\mu \hat{S}_R^\nu \hat{S}_R^\rho \hat{S}_R^\sigma, \tag{24}
\]
where \(\hat{S} = \hat{B}^\dagger\) are operators of spin and boost projections, see the appendix. In contrast to \(\hat{S}_R^3\), being the generators of the group \(M(3,1)_{\text{ext}}\), the operators of spin projections \(\hat{S}_R^{\mu\nu}\) are not generators of the group \(M(3,1)_{\text{ext}}\), see (14) and (15), and therefore operators (24) are functions of right (but not left) generators of the Poincaré group.

For \(SL(2,C)_{\text{int}}\), we use two sets of commuting operators corresponding to the reduction schemes
\[
SL(2,C)_{\text{int}} \supset U(1) \times U(1),
\]
\[
SL(2,C)_{\text{int}} \supset SU(2) \supset U(1). \tag{25}
\]

In the first case, two generators \(\hat{S}_R^3\) and \(\hat{B}_R^3\) of the maximal commutative (Cartan) subgroup of \(SL(2,C)_{\text{int}}\) correspond to additive quantum numbers. In the second case, the set includes the Casimir operator \(\hat{S}_R^3\) of \(SU(2)_{\text{int}}\) and \(\hat{S}_R^3\).

Therefore, we shall consider two sets of ten commuting operators on the group \(M(3,1)\):
\[
\hat{W}^3, \quad \hat{p}_\mu, \quad \hat{p} \hat{S} (\hat{S}_3 \text{ in the rest frame}), \tag{25}
\]
\[
\hat{S}^2 - \hat{B}^2, \quad \hat{S} \hat{B}, \quad \hat{S}_R^3, \quad \hat{B}_R^3,
\]
\[
\hat{W}^2, \quad \hat{p}_\mu, \quad \hat{p} \hat{S} (\hat{S}_3 \text{ in the rest frame}), \tag{26}
\]
\[
\hat{S}^2 - \hat{B}^2, \quad \hat{S} \hat{B}, \quad \hat{S}_R^3, \quad \hat{S}_R^3
\]

including the Lubanski–Pauli operator \(\hat{W}^2\), four left generators \(\hat{p}_\mu\) (the eigenvalue of the Casimir operator \(\hat{p}^2\) is evidently expressed through their eigenvalues), and helicity \(\hat{p} \hat{S}\), expressed through the left generators. The Casimir operators \(\hat{S}^2 - \hat{B}^2\) and \(\hat{S} \hat{B}\) of the subgroup \(SL(2,C)_{\text{int}}\) determine the characteristics \(j_1, j_2\) of the irreps \(T_{j_1,j_2}\) of the Lorentz group (see the appendix).

Eigenfunctions of the maximal sets of operators (25) are, at the same time, eigenfunctions of the chirality operator \(\hat{I}^3\) (20). Indeed, in the irreps of the Lorentz group \(T_{j_1,j_2}\), an eigenvalue of the chirality operator is \(\hat{I}^3 = j_1 - j_2\).

In addition to the states of definite chirality, the states of definite internal parity are, obviously, also of interest. The operator of space reflection \(\hat{P}\) anticommutes with the chirality operator \(\hat{I}^3\) and also with the operators \(\hat{S} \hat{B}\) and \(\hat{B}_R^3\). Therefore, eigenfunctions of the latter three operators change their sign under the action of \(\hat{P}\). The set (26) is more convenient to describe states of definite internal parity, because only one operator \(\hat{S} \hat{B}\) from this set does not commute with \(\hat{P}\). Eigenvalues of \(\hat{S} \hat{B}\) are proportional to \((j_1 - j_2)(j_1 + j_2 + 1)\), see (A.5). Thus, one can use quantum numbers, corresponding to the set (26), to characterize eigenstates of \(\hat{P}\), with only one change (replacement of the sign of \(j_1 - j_2\) by \(j_1 + j_2\))

4. Right generators and charges. Spectra

Despite the commutativity of the left and right generators, the corresponding quantum numbers are not independent. Since the same Casimir operators are constructed from left and right generators uniformly, within the framework of a representation (fixing their eigenvalues), the spectra of the corresponding left and right generators are the same.

The relation between left and right generators can be easily seen on the example of the group of 3D rotations \(SU(2)\), describing a 3D non-relativistic rotator [6–8]. Until recently, it has been the only example of a well-developed (by Wigner, Casimir and Eckart, back in the 1930s) theory, based on the use of both left and right transformations.

The concept of two coordinate systems is always present in the problem of rotation of a solid body, independently of whether it is described classically or quantum mechanically. One coordinate system (laboratory, or s.r.f.) is associated with the surrounding objects, while the other one (molecular, or b.r.f.) is associated with the body. Accordingly, there are two sets of operators of angular momentum— in the s.r.f. (left generators of the rotation group \(J_\lambda\)) and in the b.r.f. (right generators of the rotation group \(J_\lambda\)).

The maximal set of commuting operators in the space of functions on the group \(SU(2) \sim SO(3)\) consists of total angular momentum \(\hat{J}^2 = \hat{I}^2\) and two projections: \(J_3\) in the s.r.f. and \(J_3\) in the b.r.f. A classification of rotator states \(|j m k\rangle\) is made with the help of this set,
\[
\hat{J}^2 |J m k\rangle = j(j + 1) |J m k\rangle, \quad \hat{J}_3 |J m k\rangle = m |J m k\rangle, \quad \hat{I}_3 |J m k\rangle = k |J m k\rangle. \tag{27}
\]

By virtue of the relation \(\hat{J}_3 = \hat{I}_3^2\) the left and right irreps are labelled by the same \(j\) and the operators \(\hat{J}_3\) and \(\hat{I}_3\) have the same spectrum, namely their eigenvalues \(m\)
and \( k \) belong to the set \(-j, -j+1, \ldots, -1, j\). The operator \( \hat{I}_3 \), distinguishing equivalent representations in the decomposition of the left GRR of the rotation group into irreps, corresponds to an additive quantum number, independent of a choice of the s.r.f. This number plays an important role in the theory of molecular spectra [8, 9].

Let us now consider the cases of motions \( M(D) \) and \( M(D, 1) \), including rotations and translations. The generators of left rotations in this case consist of two summands—the orbital and spin momenta, \( \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu} \), whereas the generators of right rotations depend only on \( z \), \( \hat{J}^R = \hat{S}^R_{\nu} \).

The simplest example is a three-parameter group \( M(2) \). Here, we deal only with one operator of projection of the angular momentum, \( \hat{J} = \hat{L} + \hat{S} \); the operator of right projection coincides with the operator of spin projection, \( \hat{J}^R = -\hat{S} \). The maximal set of commuting operators can be composed of the operators of momentum and spin: \( \hat{p}_k, \hat{S}_k \). For the group \( M(3) \), the maximal set contains six commuting operators, which can be chosen as (reduction \( M(3) \rightarrow T(3) \), \( M(3) \rightarrow \text{Spin}(3) \rightarrow U(1) \))

\[
\hat{p}_k \cdot \hat{p}_l \hat{J}^k = \hat{p}_k \hat{S}^k, \quad \hat{S}_k \hat{S}^k = \hat{S}_k \hat{S}^k = 0, \quad \hat{S}_3, \quad S_3. \tag{28}
\]

Functions on the group \( M(3, 1) \) depend on ten parameters and, accordingly, there are ten commuting operators (two Casimir operators and two sets of four operators, constructed from left (14) and right (15) generators), see (25).

For a fixed mass and spin, i.e. in the framework of a representation determined by the Casimir operators of the Poincaré group, the spectra of left and right generators prove to be the same. In a similar way, the Casimir operators of the Lorentz group (24) are constructed from \( \hat{S}^\mu_{\nu} \) or \( \hat{S}^\mu_{\nu} \) uniformly and, therefore, the spectrum of operators of left and right spin projections for fixed values of \( j_1 \) and \( j_2 \) is the same. In particular, Casimir operators \( \hat{S}^2 = \sum (S_j)^2 \) and \( \hat{S}^2_R = \sum (S^2_R) \) of compact subgroups \( SU(2) \), \( \text{Spin}(3, 1), \text{SU}(2) \), \( \text{Spin}(3, 1) \) with eigenvalues \( S(S+1) \) and \( S(S+1) \) have the same spectrum; \( S \) and \( S_0 \) belong to the set \([j_1 - j_2, |j_1 - j_2| + 1, \ldots, j_1 + j_2] \), see (A.6). Note that ‘right’ quantum numbers \( S^R, S^R_k, B^R_k \) can be only integer for particles of integer spin and half-integer for particles of half-integer spin.

Left generators of the Poincaré group \( \hat{p}_\mu \) and \( \hat{J}_{\mu\nu} \) are associated with additive quantum numbers—the momentum and total angular momentum projections. Right generators \( \hat{p}^R_\mu \) and \( \hat{J}^R_{\mu\nu} \) and, in particular, the operators \( \hat{B}_3^R \) and \( \hat{S}^R_3 \) entering the maximal set are also associated with additive quantum numbers.

Right generators commute with left ones and, consequently, with the corresponding finite transformations. Therefore, under finite left transformations (changes of the s.r.f) the eigenfunctions of right generators remain eigenfunctions with the same eigenvalues. In other words, right generators determine internal quantum numbers that do not change under changes of the s.r.f. They can be identified with charges. Indeed, charges are usually understood as additive numbers that do not change under changes of the s.r.f. (Lorentz transformations). Right generators from the commutative (Cartan) subgroup satisfy such a definition.

Therefore, an orientable object is characterized by ten quantum numbers—six numbers (momentum \( p_\mu \), spin, helicity) are determined by the left generators and Casimir operators, whereas four numbers are determined by the right generators (Lorentz characteristics \( j_1, j_2 \) and two charges). In comparison with the usual description of fields in Minkowski space (4-momentum, spin, spin projection, representation of the Lorentz group), there appear two additional quantum numbers.

However, there is an essential difference between left and right generators: whilst left generators correspond to external, exact, symmetries, right generators correspond to internal symmetries that may be broken.

Let us turn to the example of a non-relativistic rotator, described by the quadratic Hamiltonian

\[
\hat{H} = \sum A_k (\hat{I}_k)^2, \tag{29}
\]

where \( A_k \) are the moments of inertia. For a completely symmetric rotator \( A_1 = A_2 = A_3 = A \), not only left transformations but also right ones are symmetry transformations of Hamiltonian (29), the symmetry group is \( \text{SO}(3) \times \text{SO}(3) \). In the axially symmetric case, only the right transformations (with the generator \( I_3 \)) about the axis \( \xi_3 \) are a symmetry of the body; the symmetry group is \( \text{SO}(3) \times \text{SO}(2) \). This symmetry corresponds to the additive quantum number \( k \) (see (27)). Finally, in the case of three different momenta of inertia the body is asymmetric, and therefore the left transformations with the generators \( I_k \) are not its symmetries; the symmetry group is \( \text{SO}(3) \).

Returning to the \( 3 + 1 \)-dimensional case, we consider the operator \( \hat{p}_\mu \hat{\Gamma}^{\mu\nu} \), where \( \Gamma^{\mu\nu} \) are linear differential operators in \( z \); this operator is invariant under the transformations \( M(3, 1) \times \text{Spin}(3, 1) \). The equations for the eigenvalues of this operator

\[
\hat{p}_\mu \hat{\Gamma}^{\mu\nu} f(x, z) = \varepsilon m s f(x, z), \tag{30}
\]

where \( \varepsilon = \pm 1 \), in the subspaces \( f(x, z, \zeta) \), \( f(x, \zeta, z) \), after a separation of the spatial and orientation variables for spin-1/2 and \(-1\) (corresponding to polynomials of first and second degree with respect to \( z \)), turn into the equations of Dirac and Duffin–Kemmer (see [1] for details).

The chiralit operator \( \hat{\Gamma}^5 \) and also the operators \( \hat{S} \hat{B} \) and \( \hat{B}^R_k \) from the set (25) do not commute with both space reflections \( \tilde{P} \) and \( \hat{p}_\mu \hat{\Gamma}^{\mu\nu} \). Thus the operator \( \hat{B}^R_3 \), as well as \( \Gamma^5 \), corresponds to a broken symmetry. In the subspace of functions \( f(x, z, \zeta) \), we have \( \hat{\Gamma}^5 = i \hat{B}^R_3 \), whereas in the subspace \( f(x, \zeta, z) \) we have \( \hat{\Gamma}^5 = -i \hat{B}^R_3 \). (Here, the multiplier \( i \) originates from the fact that we consider non-unitary finite-dimensional representation of the Lorentz group, for which the boost generators are anti-Hermitian.) Like chirality, \( \hat{B}^R_3 \) is an additive conserved number only for massless particles. Massive particles correspond to eigenfunctions of the operator of space parity \( \tilde{P} = \hat{I}_3 \), and the operator \( \hat{p}_\mu \hat{\Gamma}^{\mu\nu} \) which do not commute with the operators \( \hat{\Gamma}^5 \) and \( \hat{B}^R_3 \). Therefore, massive particles, described by equations of the form (30), cannot have a definite charge \( \hat{B}^R_3 \) and chirality \( \Gamma^5 \).

The quantum number corresponding to the generator \( \hat{S}^R_3 \) must be conserved also for massive Dirac particles, since \( \hat{S}^R_3 \) commutes not only with all left generators of the Poincaré group, but also with the operators \( \hat{p}_\mu \hat{\Gamma}^{\mu\nu} \) and \( \tilde{P} \).
The question arises as to whether the known elementary particles possess a conserved quantum number corresponding to the operator \( S_3^R \). The answer to this question, in fact, will also answer another question, namely whether one can consider usual particles as ‘orientable objects’ in the sense of the above definition.

5. \( S_3^R \) charge

One can attempt to associate the charge corresponding to the right generator \( S_3^R \) of the Poincaré group with the observable characteristics of physical particles. Note that although the ‘internal’ quantum numbers corresponding to right generators do not change under left transformations, the discrete transformations (automorphisms of the Poincaré group) act on both right and left generators (see [1] for details). The known behaviour under discrete transformations helps one to identify right characteristics with the properties of physical particles.

As to the operator \( S_3^R \):

1. it corresponds to an additive (conserved) quantum number;
2. it has integer eigenvalues for particles of integer spin and half-integer eigenvalues for particles of half-integer spin;
3. it does not change sign under space reflection;
4. it changes sign under charge conjugation.

Then it follows that this number equals zero for a real neutral particle (i.e. a particle that coincides with its own antiparticle); at the same time, real neutral particles with definite \( S_3^R \) have to have integer spin.

If one considers the \( S_3^R \) charge of particles described by finite-dimensional representations of the Lorentz group \( T_{[j_1, j_2]} \), then for particles of spin-1/2 two values are possible: \( 1/2 \) and \( -1/2 \); for particles of spin-1 three values are possible: \( 1 \), \( 0 \), and \( -1 \). In particular, for a photon and \( Z^0 \)-boson, as real neutral particles, we have \( S_3^R = 0 \).

Let us see which values of \( S_3^R \) can be associated with particles with wave functions \( f(x, z) \) that are eigenfunctions of the operator \( S_3^R \). \( S_3^R f(x, z) = S_3^R f(x, z) \). Since the sign of \( S_3^R \) changes under the charge conjugation, particles and antiparticles must have \( S_3^R \)-charges of opposite signs. For definiteness, let an electron \( e^- \) has the \( S_3^R \)-charge \( -\frac{1}{2} \), then a positron \( e^+ \) have \( S_3^R = \frac{1}{2} \). For \( W^- \), as a charged particle, it is natural to expect \( S_3^R = \pm 1 \) (the value \( S_3^R = 0 \) is excluded by a more detailed analysis, see below). Next, since \( \bar{\nu}_e \) only admits the values \( \pm \frac{1}{2} \), the reaction \( W^- \to e^- + \bar{\nu}_e \) implies \( S_3^R = -\frac{1}{2} \) for \( \bar{\nu}_e \) and \( S_3^R = 1 \) for \( W^- \). Therefore, we have

\[
\begin{align*}
\frac{1}{2} : & \quad v_e \quad e^+ \quad 1 : \quad W^+ \quad 0 : \quad \gamma, \quad Z^0, \\
-\frac{1}{2} : & \quad \bar{\nu}_e \quad e^- \quad -1 : \quad W^-.
\end{align*}
\]

Applying the same consideration to other families of fundamental fermions, we find the following classification with respect to the sign of \( S_3^R \):

\[
\begin{align*}
\frac{1}{2} : & \quad v_e \quad v_\mu \quad v_\tau \quad u \quad c \quad t, \\
-\frac{1}{2} : & \quad e^- \quad \mu \quad \tau \quad d \quad s \quad b.
\end{align*}
\]

Therefore, the \( S_3^R \)-charge, whose sign changes under both \( \hat{C} \hat{P} \hat{T} \) transformation and charge conjugation \( \hat{C} \), distinguishes not only particles and antiparticles but also the ‘up-down’ components in doublets of elementary fermions. This charge is conserved in any interactions, since the carriers of electromagnetic and strong interactions are characterized by \( S_3^R = 0 \), whereas we have already examined weak charged currents.

As a consequence of (31) and (32), we find the following empirical expression for \( S_3^R \) in terms of other charges:

\[
S_3^R = \frac{L - B}{2} + Q.
\]

where \( LB \) and \( Q \) are the lepton, baryon and electric charges, respectively. This formula relates the ‘right’ charge \( S_3^R \) with observable characteristics of particles.

For the above-mentioned fundamental particles of spin-1 the charge \( S_3^R \) coincides with the electric charge. Taking into account that \( (B - L)/2 = Q \), where \( Q \) is the average electric charge of the corresponding lepton or quark doublet, one can rewrite (33) as \( S_3^R = Q - \tilde{Q} \).

We also note that \( S_3^R \) coincides with the projection of the weak isospin \( T_3 \) for left components of particles and right of antiparticles (but not vice versa). For a massive Dirac particle \( T_3 \) is uncertain because the wave function contains left and right components with different \( T_3 = \pm 1/2 \) and \( T_3 = 0 \). On the other hand, the operator \( S_3^R \) commutes with the Dirac operator \( \gamma^0 \), and its eigenvalues for left and right components are the same. In the next section, we give an explicit form of the corresponding functions \( f(x, z) \).

Let us now consider the relation between ‘right’ quantum number \( S_3^R \) and spin. We have already noted that spectra of right and left spin operators are not independent, in particular, \( S_3^R \) can be only integer for particles of integer spin and half-integer for particles of half-integer spin \( s \),

\[
(-1)^s S_3^R = (-1)^s,
\]

and due to (33) we have

\[
(-1)^{L-B+2Q} = (-1)^{2s}.
\]

In 1961, Lurçat and Michel [10] noted that for all the known particles with integer \( B \) there holds the relation

\[
(-1)^{B+L} = (-1)^{2s},
\]

in other words, \( B + L + 2s \) is always even. Later, this observation resulted in the concept of \( R \)-parity, being positive for all the known particles, and defined as

\[
R = (-1)^{3(B-L)+2s},
\]

or \( R = (-1)^{3B+L+2s} \), where the multiplier 3 is introduced for the inclusion of quarks into the analysis.

On the condition that the electric charge is an integer, relation (35) is a consequence of (33). Indeed, since the right projection \( S_3^R \) and spin \( s \) must take integer and half-integer values simultaneously, both \( (B \pm L)/2 \) and spin \( s \) are integer and half-integer simultaneously. Furthermore, for fractional 1/3-multiple charges (quarks), we have \((-1)^{2s} = (-1)^3 S_3^R = (-1)^3 S_3^R = (-1)^3(L-B)\), since \( 6Q \) is even-valued. As a consequence, for all particles with integer charge \( Q \) or with 1/3-multiple charge \( Q \), \( R \)-parity is positive.
6. Spin-1/2: fermionic quadruplets

Consider irreps of $SL(2, C_{\text{int}})$, whose weight diagrams are determined by eigenvalues of generators of the right projections of spin $\mathcal{S}^R_3$ and $\mathcal{B}^R_3$.

Any finite-dimensional irrep occurs in the decomposition of left (or right) GRR with the multiplicity equal to its dimension. For instance, in the case of $SL(2, C)$ the irrep $T_{(1/2,0)}$ occurs in the decomposition of the left GRR twice: columns $(z^1 z^2)$ and $(z^3 z^2)$ carry this representation; analogously, the irrep $T_{(1/2,0)}$ also occurs in the decomposition of the right GRR twice (rows $(z^1 z^3)$ and $(z^2 z^3)$).

Linear functions of coordinates $z$ on the Lorentz group describe spin-1/2 particles. There are four linearly independent functions that are transformed differently under a change of a b.f.f. These are $z, z^2, z^3$ (we have omitted the usual ‘laboratory’ indices, since $z^1$ and $z^2$ are transformed in the same way under the action of $SL(2, C_{\text{int}})$). These functions can be arranged into linear combinations being eigenfunctions of various operators. The states of spin 1/2 with a definite charge $S^R_3$ and a chirality correspond to the functions

$$\mathcal{S}^R_3 = -1/2 \quad \delta^R_3 = 1/2 \quad R(\Gamma^5 = 1/2): \quad e^{i p x} \xi \quad e^{i p x} \eta$$

$$\mathcal{L}(\Gamma^5 = -1/2): \quad e^{i p x} \xi \quad e^{i p x} \eta.$$  \hspace{1cm} (37)

The states with a definite parity $\eta$ are eigenfunctions of the operator $\hat{P}$, $\hat{P} f(x, z) = \eta f(x, z)$, in the rest frame at $p_0 = \pm m$, we have

$$\mathcal{S}^R_3 = -1/2 \quad \delta^R_3 = 1/2 \quad \eta = 1: \quad e^{i m x} (\xi - \eta \hat{z}) \quad e^{i m x} (\xi + \eta \hat{z}),$$

$$\eta = -1: \quad e^{i m x} (\xi + \eta \hat{z}) \quad e^{i m x} (\xi - \eta \hat{z}).$$  \hspace{1cm} (38)

We note that functions from (38) are eigenfunctions for $\Gamma^{00}$ and $p_0$ and, therefore, they are solutions of the left-invariant equation (30).

Assuming a symmetry violation under right transformations of the Poincaré group and retaining only the particle–antiparticle symmetry, we find that a mixed picture becomes possible:

$$\mathcal{S}^R_3 = -1/2 \quad \delta^R_3 = 1/2 \quad \eta = 1: \quad e^{i m x} (\xi - \eta \hat{z}) \quad e^{i m x} (\xi + \eta \hat{z}),$$

$$\eta = -1: \quad e^{i m x} (\xi + \eta \hat{z}) \quad e^{i m x} (\xi - \eta \hat{z}).$$  \hspace{1cm} (39)

States from (37) correspond to the massless or ultra-relativistic case when one has two chiral left fermions (for example, $\tilde{e}_L$ and $\tilde{\nu}_L$) and two chiral right antifermions ($\tilde{e}_R$ and $\tilde{\nu}_R$), see figure 1(a).

The massive electron and neutrino correspond to states from (38). In this case, the electron $e^-$ and the neutrino $\nu$ have a positive internal parity; the positron $e^+$ and the antineutrino $\tilde{\nu}$ have a negative internal parity, see figure 1(b).

Finally, states from (39) can be considered as a good approximation within the range of energies much larger than the neutrino mass. In this approximation, the quadruplet of leptons $e^-, \tilde{e}^+, \nu_L, \tilde{\nu}_R$ contains an electron with a positive internal parity, a positron with a negative internal parity and a massless neutrino can be only a left one (chirality is negative) and the antineutrino can be only a right one (chirality is positive). In addition, a change of signs both of $\mathcal{S}^R_3$ and of chirality or parity corresponds to a transition to an antiparticle, whereas a change of sign alone leads to a transition to a state that is not present in (39).

The quark quadruplet $u, \tilde{u}, d, \tilde{d}$, where $u$ and $d$ are characterized by the same internal parity $\eta = 1$ and, according to (32), by opposite signs of $\mathcal{S}^R_3$, corresponds to (38). In addition, as should be expected, particles and corresponding antiparticles are characterized by opposite parity. However, in this case we also encounter a violation of symmetry with respect to right transformations of the Poincaré group—components of $SL(2, C_{\text{int}})$-doublet have different mass.

This classification can be visualized in the form of a weight diagram of the representation $T_{(1/2,0)} \oplus T_{(0,1/2)}$ of the group $SL(2, C_{\text{int}})$.

Note that in the framework of the suggested approach, in the massless case, we have left particles and right antiparticles only, see figure 1(a) and (37). If $m \neq 0$, we have both (left and right, $\Gamma^5 = \pm 1/2$, see figure 1(b) and (38)) components.

In the massless case $\mathcal{S}^R_3$ and $\delta^R_3$ coincide with weak isospin $T$ and its projection $T_3$. A more detailed analysis and, in particular, comparison with the Standard model requires the consideration of both interactions in terms of functions...
\( f(x, z) \) and the mechanism leading to symmetry breaking. We hope to present such an analysis in our next paper.

In addition to the eigenfunctions of \( \hat{S}_3^R \), one can also construct states with definite charge parity, \( \hat{C} f(x, z) = \eta f(x, z) \) (which describe the Majorana neutrino), or with \( \hat{C} \hat{P} \hat{T} \)-parity (the so-called ‘physical Majorana neutrino’, see [11, 12]).

Let us now consider functions corresponding to a massive particle, moving along the axis \( x^3 \). They can be obtained from functions in the rest frame (38), which are characterized by a certain internal parity, with the help of a Lorentz transformation

\[
P = U P_0 U^T, \quad Z = U Z_0, \quad \text{where} \quad P_0 = \pm \pm \text{diag} [m, m],
\]

\[
U = \text{diag} [e^{i\alpha}, e^{-i\alpha}] \in SL(2, \mathbb{C})_{\text{ext}},
\]

and the sign of \( P_0 \) corresponds to the sign of \( p_0 \),

\[
p_\mu = k_\mu \text{sign} p_0, \quad k_0 = m \cosh 2\alpha, \quad k_3 = m \sinh 2\alpha, \quad e^{i\alpha} = \sqrt{(k_0 \pm k_3)/m}.
\]

(40)

By applying these transformations to the state with \( \hat{S}_3^S = -1/2, \eta = 1 \) at \( p_0 > 0 \), we find

\[
f'_{m,1/2}(x, z) = e^{i(k_0 x + k_3 z)} [C_1 (\gamma^0 e^\alpha - z \gamma^3 e^{-\alpha}) + C_2 (\gamma^0 e^{-\alpha} + z \gamma^3 e^\alpha)],
\]

where the first term in the square brackets corresponds to \( s_3 = 1/2 \), and the second term corresponds to \( s_3 = -1/2 \). In the ultra-relativistic case with a positive \( c \) (i.e. with \( k_3 > 0 \)), there remain only two components,

\[
f'_{m,1/2}(x, z) \approx e^{i(k_0 x + k_3 z)} (C_1 \gamma^0 e^\alpha - C_2 \gamma^3 e^{-\alpha}),
\]

which are eigenfunctions of the helicity operator \( \hat{p} \hat{S} \) with the eigenvalues \( p_3 s_3 = \frac{1}{2} k_3 \text{sign} s_3 \); these components are also eigenfunctions of the operator \( \hat{\Gamma}^3 \) with the same sign. In a similar way, considering the case \( a < 0 \) and other states from (38), we conclude that in the ultra-relativistic limit with \( p_0 > 0 \), signs of chirality \( \hat{\Gamma}^3 \) and helicity \( \hat{p} \hat{S} \) are the same. We stress that the above conclusions derived for the ultra-relativistic case coincide with the results obtained from the Dirac equation.

Consider now the states corresponding to spin-1/2 particles from the viewpoint of solutions of left-invariant RWE of first order in more detail. Equation (30) for functions linear in \( z \) splits into a pair of Dirac equations for functions from subspaces \( f(x, z, \bar{z}) \) (\( S_3^S = -1/2 \)) and \( f(x, z, \bar{z}) \) (\( S_3^S = 1/2 \)). The sign of the mass term in these equations is \( \epsilon = \eta \text{ sign} p_0 \text{ sign} S_3^R \) (see [1, 13, 14] for details).

For spin \( s = 1/2 \), eigenfunctions of the operator \( \hat{S}_3^R \) and space parity \( \hat{P} = \hat{I}_r \) are \( \alpha^0 \pm \alpha^3, \alpha^0 \pm \alpha^3 \). In the rest frame, solutions of the above two mentioned Dirac equations with \( \epsilon = 1 \) have the form

\[
f_1(x, z) = e^{i\alpha x} C_\alpha (z^0 - \bar{z} \alpha^3) + e^{-i\alpha x} C_\alpha (z^0 + \bar{z} \alpha^3), \quad S_3^R = -1/2,
\]

(41)

\[
f_2(x, z) = e^{i\alpha x} D_\alpha (z^0 - \bar{z} \alpha^3) + e^{-i\alpha x} D_\alpha (z^0 + \bar{z} \alpha^3), \quad S_3^R = 1/2.
\]

(42)

As is known, the free Dirac equations have solutions corresponding to a pair of non-equivalent irreps of the improper Poincaré group with opposite signs of \( \eta \) and \( p_0 \). Consider the solution (41) of the first equation. Assuming, as usual, that the wave function of an antiparticle is a bispinor, being charge-conjugated to a certain negative-frequency solution of the Dirac equation [15], we find that the antiparticle is associated with the function \( e^{i\alpha x} C_\alpha (z^0 - \bar{z} \alpha^3) \), being complex-conjugated to the negative-frequency part of solution (41) \( \hat{C} \hat{P} \hat{T} \)-conjugation yields \( e^{i\alpha x} C_\alpha (z^0 + \bar{z} \alpha^3) \). This function is a solution of equation (30) with the same \( \epsilon = 1 \), see the first term of (42); however, it is characterized by opposite signs of \( \eta \) and \( S_3^R \) (\( \eta = -1, S_3^R = 1/2 \)). Thus, a particle (electron) and an antiparticle (positron) are described by positive-frequency solutions of equation (30) with \( \epsilon = 1 \) and opposite signs of \( S_3^R \) and \( \eta \).

There remain two unused functions, \( e^{i\alpha x} (\alpha^0 + \alpha^3) \) and \( e^{i\alpha x} (\alpha^0 - \alpha^3) \), that correspond to a particle with a negative parity and an antiparticle with a positive parity, which provides solutions of the Dirac equations with \( \epsilon = -1 \),

\[
f_3(x, z) = e^{i\alpha x} C_\alpha (z^0 + \bar{z} \alpha^3) + e^{-i\alpha x} C_\alpha (z^0 - \bar{z} \alpha^3),
\]

\[
S_3^R = -1/2,
\]

(43)

\[
f_4(x, z) = e^{i\alpha x} D_\alpha (z^0 + \bar{z} \alpha^3) + e^{-i\alpha x} D_\alpha (z^0 - \bar{z} \alpha^3),
\]

\[
S_3^R = 1/2.
\]

(44)

However, these functions with \( p_0 > 0 \) cannot describe the electron or positron: there does not exist an electron with negative parity and a positron with positive parity.

Consequently, it is natural to associate the remaining functions with other particles, as was done above.

7. Spin-1

Spin-1 particles are described by quadratic combinations of \( \gamma \), which are transformed with respect to the representations of the Lorentz group \( T_{[i_1j_2]b} \), \( j_1 + j_2 = 1 \). These are a 6D adjoint representation \( T_{[10]} \oplus T_{[01]} \) (two complex-conjugate matrices from \( SO(3, C) \)) and a 4D vector representation \( T_{[1\pm 1]} \) (matrix from \( SO(3, 1) \)).

One can see that if the spin part of a wave function is transformed according to the representation \( T_{[10]} \oplus T_{[01]} \), then the eigenvalue of the Casimir operator \( W^2 \) is equal to \(-m^2(s + 1), i.e. s \) is spin. Therefore, all the states carrying the representation \( T_{[10]} \oplus T_{[01]} \) have spin-1.

For this representation all six states are characterized by chirality \( \Gamma^3 = j_1 - j_2 = \pm 1 \). The multiple weight \( S_3^R \) is \( B^R_k = 0 \), being in the center of the diagram (figure 2(a)), corresponds to a pair of states. Note that states with a definite parity \( V_{\gamma_\epsilon}^{(\gamma_\eta)} \), corresponding to \( S_3^R = 0 \) (figure 2(b)), are also characterized by a definite charge parity, and therefore they can describe real neutral particles.

By restoring laboratory indices, one can easily see that each point of the weight diagram (figure 2) corresponds to three states (according to the number of possible spin
projections), which transform equally under \( SL(2, C)_{\text{int}} \), but differently under \( SL(2, C)_{\text{ext}} \). In particular, for states with \( S_3^\pm = 0 \) among four pairwise products, three functions correspond to spin-1, namely

\[
z^1 z^2 + \frac{1}{2} z^1 z^2 + \frac{1}{2} z^1 z^2 + \frac{1}{2} z^1 z^2,
\]

since, due to unimodularity, \( \det Z = z^1 z^2 + \frac{1}{2} z^1 z^2 = 1 \) is a Lorentz scalar.

Making a reduction to the compact group \( SU(2) \), we obtain two triplets: left and right, corresponding to the diagonals on figure 2(a), or triplets with a fixed parity, figure 2(b). Consider now the representation \( T_{\{1\}} \) (figure 3).

For this representation, all states have the chirality \( \Gamma^\pm = 0 \). The reduction to the compact subgroup \( SU(2) \) gives a triplet and singlet, which can be seen in figure 3(b). Besides, in contrast to \( T_{\{10\}} \oplus T_{\{01\}} \), for \( T_{\{1\}} \) possible values of spin are 1 and 0. Thus, to describe spin-1-particles, functions \( f(x, z) \), carrying representation \( T_{\{1\}} \), must satisfy certain subsidiary conditions. In particular, in the subspace of functions \( f(x, z, \bar{z}) \) at \( p^2 = m^2 \), we have [13]

\[
\hat{W}^2 = -m^2 (j_1 + j_2)(j_1 + j_2 + 1) + 4 \hat{p}_\mu q^\mu \hat{p}_\nu \hat{V}^\nu,
\]

where \( q^\mu = \frac{1}{2} \Gamma^\mu_{\alpha \beta} z^{\alpha} \bar{z}^{\beta} \) and \( \hat{V}^\mu = \frac{1}{2} \Gamma^\mu_{\alpha \beta} \partial^\alpha \bar{\partial}^\beta \). Consequently, in the case \( s = j_1 + j_2 \), a necessary and sufficient condition for spinorial irreducibility is given by

\[
\hat{p}_\mu q^\mu \hat{p}_\nu \hat{V}^\nu f(x, z, \bar{z}) = 0. \tag{46}
\]

For the representations \( T_{\{10\}} \) and \( T_{\{01\}} \), this condition is fulfilled identically, since in this case \( \hat{V}^\mu f(x, z) = 0 \). In the general case, taking into account that in the momentum representation, the action of the operator \( q^\mu \hat{p}_\mu \) is reduced to multiplication by a number, we arrive at alternative conditions,

\[
p_\mu q^\mu = 0, \tag{47}
\]

\[
\hat{p}_\nu \hat{V}^\nu f(x, z, \bar{z}) = 0. \tag{48}
\]

In the first case, we have a space of functions of two 4-vectors \( p_\mu, q_\mu \), which are subject to invariant constraints,

\[
p^2 = m^2, \quad p_\mu q^\mu = 0, \quad q^2 = 0. \tag{49}
\]

In the rest frame, according to (49), we have \( z^1 \bar{z}^1 + \frac{1}{2} z^2 \bar{z}^2 = 0 \). Such an approach to the construction of wave functions, describing elementary particles, was suggested by Wigner in the paper [16], where the discussion was restricted to particles of integer spin and to real-valued \( q_\mu \) with the constraints \( p^2 = m^2, \quad p_\mu q^\mu = 0, \quad q^2 = -1 \). Different generalizations of Wigner’s approach were considered in [17–21]. The second requirement (48) is only a condition on the functions \( \phi(x) \) and it does not concern the spinorial variables.

**Figure 2.** The weight diagrams of the representation \( T_{\{10\}} \oplus T_{\{01\}} \) of \( SL(2, C)_{\text{int}} \). (a) \( \Gamma^\pm = \pm 1 \), (b) \( \eta = \pm 1 \). The dotted lines join states related by the transformations of \( SU(2) \) related to \( SL(2, C)_{\text{int}} \).

**Figure 3.** The weight diagrams of the representation \( T_{\{1\}} \) of \( SL(2, C)_{\text{int}} \). \( \Gamma^\pm = 0 \). In (a), the dotted line joins states related by the transformations of \( SL(2, C)_{\text{int}} \). In (b), the dotted line joins states related by the transformations of \( SU(2) \subset SL(2, C)_{\text{int}} \).
Indeed, representing the function \( f(x, z) \) in the form

\[
 f(x, z) = \phi_\rho^a(x)z^{a\rho} = \Phi_\mu(x)q^\mu,
\]

\[
 \Phi_\mu(x) = -\tilde{\rho}_\mu^\alpha \phi_\alpha(x),
\]

we find [13] that \( \Phi_\mu(x) \) obey equations for spin-1-particles in the Proca form

\[
 (\tilde{\rho}^2 - m^2)\Phi_\mu(x) = 0, \quad \tilde{\rho}^\mu \Phi_\mu(x) = 0.
\]

Above we, considering possible values of \( S_3^R \) for \( W^\pm \) bosons, have excluded \( S_3^R = 0 \). Here we see that all states of spin-1 with \( j_1 + j_2 = 1 \) and \( S_3^R = 0 \) are related to real neutral particles and cannot be related with \( W^\pm \).

Therefore, in the framework of the present theory, we arrive to two families of particles of spin-1: two triplets, corresponding to \( T_{110} \oplus T_{011} \) and a quadruplet corresponding to \( T_{11\frac{1}{2}, \frac{1}{2}} \), which (since a reduction to a compact subgroup was done) is decomposed into a triplet with \( \eta = 1 \) and a singlet with \( \eta = -1 \). They are candidates to describe well-known particles of spin-1—the triplet of intermediate vector bosons and the photon.

To give an exact answer, let us consider the weight diagram of 6D adjoint irrep of the Lorentz group \( T_{110} \oplus T_{011} \) (figure 2). The multiple weight \( S_3^R = 0 \) can be related with real neutral particles—the photon \( \gamma \) and the \( Z^0 \)-boson, the weights with \( S_3^R = 1 \) and \( S_3^R = -1 \) can be related with \( W^+ \) and \( W^- \) bosons. In addition, each of the latter appears twice as \( W^\pm_2 \) and \( W^\pm_5 \) (linear combinations correspond to states with a definite parity \( W^\pm_\eta \) for \( \eta = \pm 1 \)).

As far as the 4D representation \( T_{11\frac{1}{2}, \frac{1}{2}} \) is concerned, the \( W^+ \) and \( W^- \) bosons can be associated with states having \( S_3^R = \pm 1 \), with the parity \( \eta = 1 \), whereas the photon and \( Z^0 \)-boson can be associated with two states having the zero charge \( S_3^R = 0 \), see figure 3(b).

However, more detailed considerations exclude the case \( T_{110} \oplus T_{011} \). In the massless limit not only \( S_3^R \), but also \( B_3^R \) and \( F_3^R \) are conserved quantum numbers. Then, \( e_{\eta \gamma}^+ \) and \( \psi_{\eta \gamma}^\pm \) are characterized by \( \gamma^+ = 1/2 \) and \( \gamma^\pm = -1/2 \) (see figure 1 and (37)), so for \( W^- \) we have \( \gamma^\pm = 0 \). The latter is fulfilled for \( T_{11\frac{1}{2}, \frac{1}{2}} \), but not for \( T_{110} \oplus T_{011} \). Analogously, it is easy to see (figure 1) that \( e_{\eta \gamma}^\mp \) and \( \psi_{\eta \gamma}^\pm \) are characterized by opposite values of \( B_3^R \), and therefore the charged \( W^- \) \( (S_3^R = -1) \) must have \( B_3^R = 0 \), which holds true for states with \( S_3^R = -1 \), described by the representation \( T_{11\frac{1}{2}, \frac{1}{2}} \) (figure 3), but not by \( T_{110} \oplus T_{011} \) (figure 2).

8. Quasiregular representations and spin description (geometrical models of spinning particles)

The consideration of GRR of the Poincaré group ensures the possibility of consistent description of particles with arbitrary spin by means of scalar functions on \( M \times \text{Spin}(3, 1) \), where \( M \) is Minkowski space. At the same time, for a description of spinning particles it is possible to use the spaces \( M \times L \), where \( L \) is some homogeneous space of the Lorentz group (one- or two-sheeted hyperboloid, cone, projective space and so on); see, for example, [16–27]. In some papers, fields on homogeneous spaces are considered; in other papers such spaces are treated as phase spaces of some classic mechanics, and the latter are treated as models of spinning relativistic particles.

These spaces appear in the framework of the next group-theoretical scheme. Let us consider the left quasiregular representation of the Poincaré group

\[
 T(g)f(g_0 K) = f(g^{-1}g_0 K), \quad K \in \text{Spin}(3, 1),
\]

and since \( x \) is invariant under right rotations (see (13))

\[
 g_0 \leftrightarrow (X, Z), \quad g_0 K \leftrightarrow (X, Z K).
\]

Therefore, the relation (52) defines the representation of the Poincaré group in the space of functions \( f(x, z K) \) on \( M \times \text{Spin}(3, 1)/K \).

Generally speaking, in the space of scalar functions on \( \text{Spin}(3, 1)/K \), one can realize only a part of irreps of the Lorentz group, and in the space of scalar functions on \( M \times \text{Spin}(3, 1)/K \) one can realize only a part of irreps of the Poincaré group. In particular, the case \( K = \text{Spin}(3, 1) \) corresponds to a scalar field on Minkowski space.

Thus the consideration of left quasiregular representations allows one to construct a number of spin models classified by subgroups \( K \subseteq \text{Spin}(3, 1) \). However, the dimension of the space \( M(3, 1)/K \) is reduced in comparison with \( M(3, 1) \) by the number of generators of the group \( K \), and respectively the number of commuting operators and the number of quantum numbers are reduced as well.

There exist 13 homogeneous spaces \( M(3, 1)/K \), containing Minkowski space [22, 23, 28]. We will consider five such spaces that were used in constructing different geometrical models, see table 1.

Here \( z_1, z_2, z_3, z_4 \) are elements of the first and second columns of the matrix \( Z \in \text{SL}(2, C) \); the 2×2 matrix \( Q \) corresponds to 4-vector \( q^\mu \).

\[
 Q = \begin{pmatrix} q^0 + q^3 & q^1 - iq^2 \\ q^1 + iq^2 & q^0 - q^3 \end{pmatrix}, \quad U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det U = 1.
\]

The latter vectors have different expressions in terms of \( z \) in different cases. For a cone, \( q^\mu = \sigma^\mu_{\rho\beta} z^{a\rho} z^{b\beta} \) [29]; in two other cases \( q^\mu \) is expressed via the tetrads \( v^\mu_{a\beta} = \sigma^\mu_{\rho\beta} c^a_{\rho\mu} v^\rho_{\alpha\beta} \) [1], then we have \( q^\mu = v^\mu_0 \) for the subgroup \( K_3 = SU(2) \); different \( SU(1, 1) \)-subgroups correspond to \( v^\mu_{1, 2}, v^\mu_{2}, v^\mu_{2} \) or to their linear combinations.

Let us discuss quantum numbers that can label quantum states corresponding to scalar functions

\[
 f'(y') = f(y), \quad f'(y') = T(g)f(y) = f(g^{-1}y), \quad y' = gy,
\]

\[
 y \in M(3, 1)/K
\]

defined on the above-listed spaces.

1. The scalar field \( f(x^\mu, z^a, z_0) \) on the 8D space \( M(3, 1)/K_1 \) (spinning space \( \text{SL}(2, C)/K_1 \)—a complex affine plane) depends on elements \( z^a \) of the first column of the matrix \( Z \) and complex conjugates \( z_0 \). Such a field was studied in [22, 23, 26]. It possesses four characteristics related to the orientation variables \( z \) (the
Let us consider a projective model. The 6D space $\mathbb{CP}^1 \sim S^3$.

| Name and dimension of space | Elements and transformations | Subgroup $K_i$ | Internal numbers |
|-----------------------------|-----------------------------|----------------|-----------------|
| 1 Complex affine plane      | $(z^1, z^2) \to (\alpha z^1 + \gamma z^2, \beta z^1 + \delta z^2)$ | $1$   | $j_1, j_2$ |
| 2 Complex projective line $\mathbb{CP}^1 \sim S^3$ | $z = \frac{z^1}{z^2}, z \to \frac{\alpha z^1 + \gamma}{\beta z^1 + \delta}$ | $(\alpha \beta)$ | - |
| 3 Lobachevskian 3-space (positive sheet of $H^1_{1,3}$ hyperboloid) | $Q \to UQU^\dagger$, det $Q = 1$ | $(\alpha \beta)$ | $j_1 = j_2, \eta$ |
| 4 Imaginary Lobachevskian 3-space (positive sheet of $H^1_{1,3}$ hyperboloid) | $Q \to UQU^\dagger$, det $Q = -1$ | $(\alpha \beta)$ | $j_1 = j_2, \eta$ |
| 5 The cone $H^1_{1,3}$ | $Q \to UQU^\dagger$, det $Q = 0$ | $(e^\nu, 0)$ | $j_1 = j_2$ |

In contrast to the case of functions on $M(3,1)$, the space $M(3,1)/K_1$ is not invariant under space reflection, $Z \to -Z^\dagger$, or $z^\mu \to -\bar{z}^\mu$, $\bar{z}^\mu \to z^\mu$ [1], and functions $f(x^\mu, z^\mu, \bar{z}^\mu)$ of elements of the first column convert to functions $f(x^\mu, -\bar{z}^\mu, z^\mu)$ of elements of the second column. Thus, states with a given parity cannot be described by scalar functions on $M(3,1)/K_1$.

2. Let us consider a projective model. The 6D space $M(3,1)/K_2$ (the spinning space $SL(2, C)/K_2$ is a 2D sphere) is a space of the least dimensions which can provide a spin description by one-component functions. Particle models in this space were studied in detail in [20, 21]. A relation to the previous (spinor) model is given by the relations

$$S^3_0 = j_2 - j_1, \quad iB_0^3 = (j_2 + j_1) .$$

In particular, this means that the functions $\phi(x^\mu, z, \bar{z})$ do not contain any information about a Lorentz group representation. We note that transformations (55) of the functions $\phi(x^\mu, z, \bar{z})$ are not reduced to an argument change, and such functions are not scalar ones with respect to the definition (54).

Space reflection transforms $z$ into $\bar{z} = z^1/z^2$, which means, as in the previous case, that states with an inner parity cannot be described by functions $\phi(x^\mu, z, \bar{z})$.

3. Vector models use functions $f(p_\mu, q_\mu)$ of 4-momentum $p_\mu$ and a spinning variable $q_\mu$.

$$\tilde{S}_{\mu \nu} = i(q_\mu \partial_q q_\nu - q_\nu \partial_q q_\mu), \quad \tilde{\mathbf{S}} \tilde{\mathbf{F}} = 0 .$$

Since the Casimir operator $\tilde{\mathbf{S}} \tilde{\mathbf{F}}$ of the Lorentz group is zero, we have $j_1 = j_2$. A reduction of an irrep $T_{(j_1, j_2)}$ of the Lorentz group to a compact rotation subgroup is given by the equation

$$T_{(j_1, j_2)} = \sum_{j=0}^{2j} T_j .$$

Thus, the models correspond to particles with integer spins that are described by the representation $T_{(j_1, j_2)}$ of the Lorentz group.

A 4-vector $q_\mu$ is given by a point on the hyperboloid $H^{1,3}$ or on the cone $H^1_{1,3}$ (see table 1), respectively $q_\mu q_\mu = 0, \pm 1$. The condition $p_\mu q_\mu = 0$ can be used to select states with a spin $s = 2j_1$ maximal for a given irrep $T_{(j_1, j_2)}$.

Thus, we have a family of models with scalar functions $f(p_\mu, q_\mu)$ and constraints

$$p_\mu p_\mu = m^2, \quad p_\mu q_\mu = 0, \quad q_\mu q_\mu = 0, \pm 1 .$$

The spaces of functions $f(p_\mu, q_\mu)$ on the hyperboloids are invariant under space reflection, $q_\mu = w_\mu \to -(-1)^{\delta_{0,1}} q_\mu$, $q_\mu = \nu_\mu \to (-1)^{\delta_{0,1}} q_\mu$, and therefore, such spaces can serve to describe states with a definite inner parity $\eta$. Functions on the cone depending on $q_\mu^\mu = \sigma^{\mu \nu} \partial_\nu z^\mu z^\nu$ under space reflection are converted to functions of $q_\mu^\mu = \sigma^{\mu \nu} \partial_\nu \bar{z}^\mu \bar{z}^\nu$. That is why one cannot construct such scalar functions corresponding to states with a definite parity $\eta$. 

\begin{table}[h]
\centering
\caption{Some homogeneous spaces related to the group $SL(2, C)$.}
\begin{tabular}{|c|c|c|c|}
\hline
Name and dimension of space & Elements and transformations & Subgroup $K_i$ & Internal numbers \\
\hline
1 Complex affine plane & $(z^1, z^2) \to (\alpha z^1 + \gamma z^2, \beta z^1 + \delta z^2)$ & $1$ & $j_1, j_2$
\hline
2 Complex projective line $\mathbb{CP}^1 \sim S^3$ & $z = \frac{z^1}{z^2}, z \to \frac{\alpha z^1 + \gamma}{\beta z^1 + \delta}$ & $(\alpha \beta)$ & -
\hline
3 Lobachevskian 3-space (positive sheet of $H^1_{1,3}$ hyperboloid) & $Q \to UQU^\dagger$, det $Q = 1$ & $(\alpha \beta)$ & $j_1 = j_2, \eta$
\hline
4 Imaginary Lobachevskian 3-space (positive sheet of $H^1_{1,3}$ hyperboloid) & $Q \to UQU^\dagger$, det $Q = -1$ & $(\alpha \beta)$ & $j_1 = j_2, \eta$
\hline
5 The cone $H^1_{1,3}$ & $Q \to UQU^\dagger$, det $Q = 0$ & $(e^\nu, 0)$ & $j_1 = j_2$
\hline
\end{tabular}
\end{table}
Thus, scalar functions on the homogeneous spaces $M \times (\text{Spin}(3, 1)/K)$, $K \subset \text{Spin}(3, 1)$, describe spinning particles; however, they correspond to states where a part of the inner (right) quantum numbers is fixed (i.e., they are expressed via other quantum numbers) or are not defined at all. In this case, the number of commuting operators and the number of quantum numbers that characterize the field are reduced by the number of generators of the subgroup $K$.

9. Concluding remarks

Orientable objects are described by the field $f(x, z)$ on the Poincaré group. Functions $f(x, z)$ depend on ten parameters and admit two kinds of transformations—left (change of space-fixed reference frame, or Lorentz transformations) and right (change of body-fixed reference frame). These transformations form the direct product $M(3, 1)_{\text{ext}} \times M(3, 1)_{\text{int}}$.

An orientable object is characterized by ten quantum numbers. Eight of them have a standard interpretation (these are the 4-momentum $\mu^i$, spin $s$, helicity and representation $(j_1, j_2)$ of the Lorentz group).

Two additional quantum numbers $S_3^R$ and $B_3^R$ that correspond to the generators $\hat{S}_3^R$ and $\hat{B}_3^R$ of the group $SL(2, C)_{\text{int}} \subset M(3, 1)_{\text{int}}$ can be interpreted as some charges. Indeed, the charges are additive quantum numbers, being independent of the choice of the laboratory reference frame. Generators of $SL(2, C)_{\text{int}}$ commute with the generators $M(3, 1)_{\text{ext}}$ (i.e., with the generators of the Lorentz transformations), and therefore 'right' quantum numbers $S_3^R$ and $B_3^R$ do not change under a change of the laboratory reference frame.

The two additional 'right' quantum numbers $S_3^R$ and $B_3^R$ characterizing orientable objects possess some properties with respect to discrete transformations, and their possible values are related to the spin value. It was noted that 'right' quantum numbers can be (in fact, uniquely) ascribed to all known elementary particles. Thus, we believe that the complete and, therefore, more adequate description of elementary (spinning) particles is achieved if one considers them as orientable objects and uses the corresponding relativistic classification theory developed in this paper.

In spite of the fact that left and right transformations commute, the spectra of left and right generators $\hat{S}_3^R$ and $\hat{B}_3^R$ are not independent. In particular, the 'right' charges $S_3^R$ and $B_3^R$ must be integer for particles with integer spin and half-integer for particles with half-integer spin. Note that, if $S_3^R$ is supposed to be a conserved quantum number, then $B_3^R$ has a definite value only for states with a definite chirality (but not parity, since $\hat{B}_3^R$ does not commute with the parity operator), i.e. $B_3^R$ can be conserved only for massless particles.

The classification of orientable objects yields the following properties in the one-particle sector. For fermions of spin-1/2, there are four states (quadruplet, realized by the up/down components of a weak doublet and by their antiparticles), distinguished by right generators of the Poincaré group: the sign of $B_3^R$-charge (instead of which one can choose the sign of chirality or internal parity) and by the sign of $S_3^R$-charge. Particles of spin-1 also form a quadruplet, whose quantum numbers coincide with those of $W^+, W^-, W^0, A^0$. In addition, potentially there are six other states corresponding to the representations $T_{100} \oplus T_{01}$ of the group $SL(2, C)_{\text{int}}$.

Note, once again, that in contrast with the left (external) symmetries, the right (internal) symmetries can be generally broken, and respectively states related by these symmetries can have different characteristics (including mass).

Obviously, the concrete realization of the idea to consider spinning particles as orientable objects is not rich enough to be able to reproduce all the quantum numbers of known elementary particles. We believe that further development of the idea can be related to an enlargement of the symmetry groups. In particular, to get quantum numbers, connected with colour $SU(3)$-symmetry, one can try to consider representations on functions $f(x, y, z)$, where $y = \{y_1, y_2, y_3\}$ carry fundamental irreps of $SU(3)$.

We have considered a description on the basis of finite-dimensional representations of the Lorentz group (and the related finite-dimensional RWE). In our forthcoming work, we hope to consider unitary infinite-dimensional representations of the Lorentz group and the related equations of the Majorana type, as well as the case of interaction.

Acknowledgment

DMG acknowledges continuous support from FAPESP and CNPq.

Appendix A. Generators and weight diagrams of the Lorentz group

In addition to the 4D vector notation for spin operators (see (16) and (17)), it is also convenient to use a 3D notation: $\hat{S}_k = \frac{1}{2} \epsilon_{ijk} \hat{S}^j$, $\hat{B}_k = \hat{S}_k$. In the space of functions on the group $f(z^a, \bar{z}^a)$ (functions of the four elements of a matrix $SL(2, C)$ (2)), a direct calculation yields for left and right generators$^3$ [13]

$$
\hat{S}_k = \frac{i}{2} (\partial \bar{z}^\tau \hat{\sigma}_k \partial z - \bar{z}^\tau \hat{\sigma}_k \partial z),
$$

$$
\hat{B}_k = \frac{i}{2} (\bar{z}^{\tau} \hat{\sigma}_k \partial z + z^{\tau} \hat{\sigma}_k \partial \bar{z}^\tau), \quad (A.1)
$$

$$
\gamma = (z^{\tau} \bar{z}^\tau), \quad \partial \bar{z}^\tau = (\partial/\partial z^1 \partial/\partial z^2)^T;
$$

$$
\hat{S}_k^\bar{z} = -\frac{i}{2} (\hat{\sigma}_k \partial \bar{z} - \bar{z} \hat{\sigma}_k \partial \bar{z}), \quad (A.2)
$$

$$
\hat{B}_k^\bar{z} = -\frac{i}{2} (\bar{z} \hat{\sigma}_k \partial z + \hat{\sigma}_k \partial \bar{z}), \quad \partial z^\tau = (\partial/\partial z^1 \partial/\partial z^2)^T.
$$

$^3$ For the sake of brevity, we have used the notation that we applied in [1, 13].

$z^a = (z^1, z^2, z^3)$, $\bar{z}^a = (\bar{z}^1, \bar{z}^2, \bar{z}^3)$, $\gamma = z^a \bar{z}^a = \bar{z}^a z^a = (\bar{z} \partial z + \partial \bar{z})^T = \bar{z} \partial z$. 

12
The terms... stand for analogous expressions obtained by the change \( z \to \xi' = (\xi_1 + \xi_2^2) \). In particular,
\[
\hat{S}_k^R = \frac{1}{2} \left( -z \partial_z + \xi \partial_\xi + \xi \partial_{\xi'} - \frac{z}{2} \partial_z \right),
\]
\[
\hat{B}_k^R = \frac{1}{2} \left( -z \partial_z - \xi \partial_\xi - \xi \partial_{\xi'} + \frac{z}{2} \partial_z \right).
\] (A.3)

It is known that from \( \hat{S}_k \) and \( \hat{B}_k \) one can construct linear combinations \( \hat{M}_k \) and \( \hat{\bar{M}}_k \),
\[
\hat{M}_k = \frac{1}{2} (\hat{S}_k - i \hat{B}_k) = z \sigma_k \partial_z + \xi \sigma_k \partial_\xi,
\]
\[
\hat{\bar{M}}_k = \frac{1}{2} (\hat{S}_k + i \hat{B}_k) = z \sigma_k \partial_z - \xi \sigma_k \partial_\xi,
\] (A.4)

such that \([\hat{M}_k, \hat{\bar{M}}_k] = 0\); in addition, for unitary representations of the Lorentz group, as it follows from the condition \( \hat{S}_k^R = \hat{S}_k, \hat{B}_k^R = \hat{B}_k \), the relation \( \hat{M}_k^+ = \hat{\bar{M}}_k \) must be fulfilled (for finite-dimensional non-unitary representations \( \hat{S}_k^R \neq \hat{S}_k, \hat{B}_k^R \neq \hat{B}_k \)).

Taking into account the fact that the operators \( \hat{M}_k \) and \( \hat{\bar{M}}_k \) satisfy commutation relations of the algebra \( su(2) \), we find the following relations for the spectra of the Casimir operators of the Lorentz subgroup:
\[
\hat{S}^2 - \hat{B}^2 = 2 (\hat{M}^2 + \hat{\bar{M}}^2) = 2 j_1 (j_1 + 1) + 2 j_2 (j_2 + 1) = -\frac{1}{4} (k^2 - \rho^2 - 4),
\]
\[
\hat{S} \hat{B} = -i (\hat{M}^2 - \hat{\bar{M}}^2) = -i j_1 (j_1 + 1) - j_2 (j_2 + 1) = k \rho,
\] (A.5)

where \( \rho = -i (j_1 + j_2 + 1), \ k = j_1 - j_2 \).

That is, the irreps of the Lorentz group \( SL(2, C) \) are labeled by a pair of numbers \( [j_1, j_2] \). It is convenient to label unitary infinite-dimensional irreps by pairs of numbers \( (k, \rho) \); in addition, the irreps \( (k, \rho) \) and \( (-k, -\rho) \) are equivalent [5, 29].

For finite-dimensional and unitary infinite-dimensional irreps of the group \( SL(2, C) \), the formulae for reduction to the compact \( SU(2) \)-subgroup have the respective forms
\[
T_{[j_1, j_2]}(k, \rho) = \sum_{j = j_1}^{j_2} T_j, \quad T_{(k, \rho)} = \sum_{j = k}^{\infty} T_j,
\] (A.6)

see [29].

The difference \( j_1 - j_2 \) (the difference between the number of dotted and undotted indices) can also be obtained as an eigenvalue of the chirality operator \( \hat{T}^5 \) (20).

Representations of low dimensions have a simple realization. The 2D irreps \( T_{[1/2, 0]} \) and \( T_{[0, 1/2]} \), which induce the transformations of spinors, are complex-conjugate matrices from \( SL(2, C) \); the 3D irreps \( T_{[1, 0]} \) and \( T_{[0, 1]} \) are complex-conjugate matrices from \( SO(3, C) \); and 4D matrices \( T_{[1/2, 1/2]} \), which induce the transformations of 4-vectors, are a representation by real-valued matrices from \( SO(3, 1) \).

The weight diagrams of the representations \( T_{[1/2, 0]} \oplus T_{[0, 1/2]} \) and \( T_{[1, 0]} \oplus T_{[0, 1]} \) are given by the figure. In addition, the axes \( S_3 \) and \(-iB_3 \) on which we indicate the eigenvalues of the corresponding operators are rotated by an angle of 45° with respect to the axes \( m_1 \) and \( m_2 \), on which we indicate eigenvalues of the operators \( \hat{M}_3 \) and \( \hat{\bar{M}}_k \).

---

**References**

[1] Gitman D M and Shelepin A L 2009 Fields on the Poincaré group and quantum description of orientable objects Eur. Phys. J. C 61 111–39

[2] Wigner E P 1939 On unitary representations of the inhomogeneous Lorentz group Ann. Math. 40 149–204

[3] Coleman S and Mandula J 1967 All possible symmetries of the S-matrix Phys. Rev. 159 1251–6

[4] Haag R, Lopuszański J T and Sohnius M 1975 All possible generators of supersymmetries of the S-matrix Nucl. Phys. B 88 257–74

[5] Barut A O and Raczka R 1977 Theory of Group Representations and Applications (Warszawa: PWN)

[6] Wigner E P 1959 Group Theory and its Application to the Quantum Mechanics of Atomic Spectra (New York: Academic)

[7] Landau L D and Lifschitz E M 1977 Quantum Mechanics (Course of Theoretical Physics vol 3) (Oxford: Pergamon)

[8] Biedenharn L S and Louck J D 1981 Angular Momentum in Quantum Physics (Reading, MA: Addison-Wesley)

[9] Zare R N 1988 Angular Momentum. Understanding Spatial Aspects in Chemistry and Physics (New York: Wiley)

[10] Lučet F and Michel L 1961 Sur les relations entre charges et spin Nuovo Cimento 21 574–6

[11] Kayser B and Goldhaber A S 1983 CPT and CP properties of Majorana particles and the consequences Phys. Rev. D 28 2341–4
[12] Kayser B, Gibrat-Debu F and Perrier F 1989 The Physics of Massive Neutrinos (Singapore: World Scientific)
[13] Gitman D M and Shelepin A L 2001 Fields on the Poincaré group: arbitrary spin description and relativistic wave equations Int. J. Theor. Phys. 40 603–84
[14] Buchbinder I L, Gitman D M and Shelepin A L 2002 Discrete symmetries as automorphisms of the proper Poincaré group Int. J. Theor. Phys. 41 753–90
[15] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1971 Relativistic Quantum Theory (New York: Pergamon)
[16] Wigner E P 1963 Theoretical Physics ed A Salam (Trieste, Vienna: IAEA) p 59
[17] Kim Y S and Wigner E P 1987 Cylindrical group and massless particles J. Math. Phys. 28 1175–9
[18] Biedenharn L C, Braden H W, Truini P and Dam H Van 1988 Relativistic wavefunctions on spinor spaces J. Phys. A 21 3593–610
[19] Hasiewicz Z and Siemion P 1992 A bosonic model for particles with arbitrary spin Int. J. Mod. Phys. A 7 3979–96
[20] Kuzenko S M, Lyakhovich S L and Segal A Yu 1995 A geometric model of the arbitrary spin massive particle Int. J. Mod. Phys. A 10 1529–52
[21] Lyakhovich S L, Segal A Yu and Sharapov A A 1996 Universal model of a $D = 4$ spinning particle Phys. Rev. D 54 5223–38
[22] Bacry H and Kihlberg A 1969 Wavefunctions on homogeneous spaces J. Math. Phys. 10 2132–41
[23] Kihlberg A 1970 Fields on a homogeneous space of the Poincaré group Ann. Inst. Henri Poincaré 13 57–76
[24] Boyer C P and Fleming G N 1974 Quantum field theory on a seven-dimensional homogeneous space of the Poincaré group J. Math. Phys. 15 1007–24
[25] Deriglazov A A and Gitman D M 1999 Classical description of spinning degrees of freedom of relativistic particles by means of commuting spinors Mod. Phys. Lett. A 14 709–20
[26] Drechsler W 1997 Geometro-stochastically quantized fields with internal spin variables J. Math. Phys. 38 5531–58
[27] Varlamov V V 2005 Maxwell field on the Poincaré group Int. J. Mod. Phys. A 20 4095–112
[28] Finkelstein D 1955 Internal structure of spinning particles Phys. Rev. 100 924
[29] Gel’fand I M, Graev M I and Vilenkin N Ya Generalized Functions vol 5 (New York: Academic)