Interacting fermions in one dimension: The Tomonaga-Luttinger model

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The theoretical description of interacting fermions in one spatial dimension is simplified by the fact that the low energy spectrum of noninteracting fermions is identical to the one for a harmonic chain. This fermion-boson transmutation allows to describe interacting fermions as a system of coupled oscillators. Tomonaga’s model of interacting nonrelativistic fermions on a ring is presented and first discussed in low order perturbation theory. After introducing the concept of two independent species of right and left moving fermions the exact solution of the Tomonaga-Luttinger model is discussed in detail. The momentum distribution and spectral functions are calculated using the method of the bosonization of the field operator. The general Luttinger liquid phenomenology is shortly discussed.

I. INTRODUCTION

The effects of the Coulomb interaction of electrons in metals can only be described approximately. With the progress in producing artificial low dimensional structures the theoretical work on one-dimensional interacting fermions has gained importance. Special features of the spectrum of low energy excitations in one dimension allow exact solutions of models of interacting fermions. This fermion-boson transmutation (FBT) was discussed in a recent publication on different levels of sophistication. In the following this paper is referred to as I. The basic idea can be understood on the level of elementary quantum statistical mechanics. Only in chapter V. of I the method of second quantization was introduced to calculate the momentum distribution of noninteracting fermions in the canonical ensemble. It was pointed out that the technique presented can also be used in the description of interacting fermions. From the response we obtained to I it is clear that many readers would have liked to see more explicitly the application to this problem. The purpose of this paper is to fill this gap on a level which requires only a basic knowledge of the method of second quantization but none of relativistic quantum field theory.

II. INTERACTING FERMIONS: THE TOMONAGA MODEL

The method of bosonization which was presented in I in the context of noninteracting electrons in one dimension is the key concept to understand ground state properties and the spectrum of excited states with low excitation energy.
also for interacting fermions. In a seminal paper Tomonaga studied nonrelativistic fermions in the high density limit, in which the spatial range of the two-body interaction is much larger than the interparticle distance. He studied spinless fermions on a ring, i.e. using periodic boundary conditions. The Hamiltonian \(H\) is a sum of the kinetic energy \(T = \sum_{i=1}^{N} \frac{p_i^2}{2m}\) and a two-body interaction \(V\)

\[
\hat{V} = \frac{1}{2} \sum_{i \neq j} V(\hat{x}_i - \hat{x}_j)
\]

\[
= \frac{1}{2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \psi^\dagger(x)\psi^\dagger(x')V(x - x')\psi(x')\psi(x),
\]

where in the second line we have switched to the language of second quantization, with \(\psi(x)\) the field operator which annihilates a fermion at the position \(x\). It can be expressed in terms of the annihilation operators \(c_n\) of momentum states in the standard way:

\[
\psi(x) = \frac{1}{\sqrt{L}} \sum_n e^{ik_nx} c_n,
\]

where the sum runs over momenta \(k_n = 2\pi n / L, n \in \mathbb{Z}\) compatible with the periodic boundary condition. The \(c_n(c^\dagger_n)\) obey canonical anticommutation relations

\[
[c_m, c_n]_+ = 0, [c_m, c^\dagger_n]_+ = \delta_{mn} \hat{1}.
\]

In order to be consistent with the periodic boundary condition \(\psi(x) = \psi(x + L)\) it is useful to expand the two body potential \(V(x - y)\) in a Fourier series

\[
V(x - y) = \frac{1}{L} \sum_n e^{ik_n(x-y)} v(k_n),
\]

\[
v(k_n) = \int_{-L/2}^{L/2} V(x)e^{-ik_nx}dx.
\]

Special choices of the Fourier transformed potential \(v(k)\) will be discussed later. In case \(V(0)\) exists, we can express \(\hat{V}\) in terms of the operator of the particle density

\[
\hat{\rho}(x) = \sum_{i=1}^{N} \delta(x - \hat{x}_i) = \psi^\dagger(x)\psi(x),
\]

where the \(\hat{x}_i\) are the position operators of the particles. Using \([\psi(x), \psi^\dagger(y)]_+ = \delta(x - y)\) for \(x, y \in [-L/2, L/2]\) one obtains

\[
V = \frac{1}{2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' V(x - x')\hat{\rho}(x)\hat{\rho}(x') - \frac{1}{2} V(0)\mathcal{N},
\]

where we have introduced the particle number operator \(\mathcal{N} \equiv \int_{-L/2}^{L/2} \rho(x)dx\). As in Tomonaga’s paper we present the main results for the spinless model and discuss the modifications due to the electron spin later. It is useful to Fourier decompose the density operators

\[
\hat{\rho}(x) = \frac{1}{L} \sum_n \hat{\rho}_n e^{ik_nx},
\]

\[
\hat{\rho}_n = \int_{-L/2}^{L/2} \hat{\rho}(x)e^{-ik_nx}dx.
\]

If we use the second quantized version \(\hat{\rho}(x) = \psi^\dagger(x)\psi(x)\) the \(\hat{\rho}_n\) can be expressed as follows

\[
\hat{\rho}_n = \sum_{n'} c_{n'}^\dagger c_{n'+n},
\]
In terms of the \( \hat{\rho}_n \) the interaction term is given by

\[
\hat{V} = \frac{1}{L} \sum_{n>0} v(k_n)\hat{\rho}_n\hat{\rho}_{-n} + \frac{1}{2L}N^2v(0) - \frac{1}{2}V(0)N. \tag{9}
\]

We have used \( v(k_n) = v(-k_n) \) as \( V(x) \) is real and \([\hat{\rho}_n, \hat{\rho}_m] = 0\), which follows most easily from the fact that only the commuting position operators \( \hat{x}_i \) enter in the definition of \( \hat{\rho}(x) \) in Eq. (5).

Now following Tomonaga we assume that the spatial range of the two-body interaction is much larger than the mean particle separation which is inversely proportional to the Fermi momentum \( k_F \). Two examples for \( v(k) \) with a finite value of \( v(0) \) which will be used later in the paper are shown in Fig. 1.

![Figure 1](attachment:image.png)

FIG. 1. Two forms of the dimensionless Fourier transform \( F(k) \equiv v(k)/(\pi v_F) \) of the two body potential.

The ground state of the noninteracting system with \( N \) fermions (\( N \) odd) is the Slater determinant |\( F > \) ("Fermi sea") in which the momentum states from \( -k_F = -2\pi n_F/L \) to \( k_F \) are occupied, where \( n_F = (N - 1)/2 \) and \( k_F \) is the Fermi momentum. A first insight into the nature of the interacting ground state can be obtained using first order perturbation theory in \( \hat{V} \)

\[
|\phi_0 \rangle = |F \rangle + \frac{\hat{Q}}{E_0^{(0)} - T} \hat{V}|F \rangle + \ldots \tag{10}
\]

where \( \hat{Q} = 1 - |F > < F| \) and \( E_0^{(0)} \) is the kinetic energy of the Fermi sea. Due to the projector \( \hat{Q} \) only the first term on the right hand side (rhs) of Eq. (9) contributes to the first order correction in the above equation. As we assume \( v(k_n) \) to be zero for \( k_n \) (much) larger than a cut-off \( k_c \ll k_F \) only states \( \hat{\rho}_n\hat{\rho}_{-n}|F \rangle \) with \( n \) of the order \( n_c = k_cL/2\pi \) or smaller occur in \( \hat{V}|F \rangle \). The second quantized form of \( \hat{\rho}_n \) Eq. (8) shows that \( \hat{\rho}_n\hat{\rho}_{-n}|F \rangle \) is a two particle-two hole state. For \( n_c > n > 0 \) the density operator \( \hat{\rho}_{-n} \) creates a particle-hole pair around the right Fermi point \( k = k_F \), while \( \hat{\rho}_n \) creates a particle-hole pair with opposite momentum around the left Fermi point \( k = -k_F \)

\[
\hat{Q}\hat{V}|F \rangle = \frac{1}{L} \sum_{n>0} v(k_n) \sum_{n'=-n_F-n}^{n_F+n} \sum_{n''=n_F+1}^{n_F+n} \epsilon_{n''}^{c\dagger} c_{n''+n} c_{n'-n} c_{n'-n}^\dagger c_{n''+n} |F \rangle \tag{11}
\]

The kinetic energy of the two-particle-two hole state \( \epsilon_{n''}^{c\dagger} c_{n''+n} c_{n'}^\dagger c_{n'-n} |F \rangle \) is given by \( E_0^{(0)} + \epsilon_{n''} - \epsilon_{n''+n} + \epsilon_{n'} - \epsilon_{n'-n} \) where \( \epsilon_n = (2\pi n/L)^2/2m_c \). As the particles and the holes are in the vicinity of the Fermi points one can linearize the energy dispersion around the two Fermi points in order to calculate \( \epsilon_{n'} - \epsilon_{n'-n} \) and \( \epsilon_{n''} - \epsilon_{n''+n} \). With the Fermi energy \( \epsilon_F = k_F^2/2m_c \) and the Fermi velocity \( v_F = k_F/m_c \) one uses instead of the one parabolic dispersion two linear branches

\[
\begin{align*}
\epsilon_{n,+} &\equiv \epsilon_F + v_F(n-n_F)2\pi/L \quad n > 0 \\
\epsilon_{n,-} &\equiv \epsilon_F - v_F(n+n_F)2\pi/L \quad n < 0
\end{align*} \tag{12}
\]
As will be discussed later the use of the linear dispersions, also shown in Fig. 2, leads to an error which vanishes in the high density limit \( k_F/k_c \to \infty \). It simplifies the expression for the kinetic energy of the two particle-two hole state to \( E_0^{(0)} + 2k_nv_F \) and leads to

\[
|\phi_0 \rangle \equiv |F \rangle = - \frac{1}{L} \sum_{n>0} \frac{v(k_n)}{2k_nv_F} \sum_{n''=-n_F-n}^{-n_F+1} \sum_{n'=n_F+1}^{n_F+n} c_{n''}^\dagger c_{n'}^\dagger c_{n'} c_{n''-n} |F \rangle + \ldots
\]

Higher order perturbation theory yields additional terms with multiple particle-hole pairs.

An important way to characterize the interacting ground state is to calculate its momentum distribution 
\( <\phi_0|\hat{n}_m|\phi_0 \rangle \equiv <\phi_0|c_m^\dagger c_m|\phi_0 \rangle \), i.e. to study how the Fermi step distribution of the noninteracting electrons is modified by the interaction. It follows from Eq. (13) that the leading correction to the Fermi function is quadratic in the two-body potential. For \( m > n_F \) one obtains e.g.

\[
<\hat{n}_m>^{(2)} = \frac{1}{L^2} \sum_{n \geq m-n_F}^{-n_F+1} \sum_{n'=-n_F-n}^{n_F+n} \frac{v(k_n)}{2k_nv_F}^2,
\]

where we have used that the two particle-two hole states are eigenstates of \( \hat{n}_m \). The sum over \( n'' \) can be trivially performed and yields a factor \( n \). The sum over \( n \) is especially simple for the step potential \( v(k) = v(0)\Theta(k_F^2 - k^2) \) where \( \Theta(\cdot) \) is the unit step function. If the sum over \( 1/n \) is approximated by an integral one finally obtains

\[
k_m > k_F : \quad <\hat{n}_m>^{(2)} = -\left(\frac{v(0)}{4\pi v_F}\right)^2 \ln\left(\frac{k_m-k_F}{k_c}\right) \cdot \Theta(k_c - (k_m-k_F)).
\]

The logarithmic singularity for \((k_m-k_F)/k_c \to 0\) is an artefact of the perturbation expansion, as the occupancies have to be smaller than one. The exact solution for the momentum distribution \(<\hat{n}_m>\) will be derived in detail later. In the thermodynamic limit \( L \to \infty, N/L = const \) one obtains a continuous function \( n(k) \) which in the neighbourhood of \( k_F \) is given by

\[
n(k) = \frac{1}{2} + c \left| \frac{k-k_F}{k_c} \right|^\alpha \text{ sign } (k_F-k).
\]

The constant \( \alpha \) is called the **anomalous dimension**. For weak two-body potentials \( \alpha \) is small and one can use \( x^\alpha = \exp(\alpha \ln x) = 1 + \alpha \ln x + O(\alpha^2) \) for fixed \( x \). By comparison with the perturbational result Eq. (14) one finds \( c = 1/2 \) and

\[
\alpha^{(2)} = \frac{1}{2} \left(\frac{v(0)}{2\pi v_F}\right)^2
\]

as the lowest order perturbational result for the anomalous dimension. A quantitative comparison of \(<\hat{n}_m>^{(2)}\) with the exact result will be given later.

Perturbation theory shows that there are three types of electrons: Those with momenta \( k \approx k_F \), which move with velocities \( \approx v_F \) which will be called "right movers", those with momenta \( k \approx -k_F \) and velocities \( \approx -v_F \), "left movers" and the "inert" electrons deep in the Fermi sea, which play no role in the "low energy physics". This basic insight of Tomonaga was used by Luttinger to further modify the model. In addition to the linearization Eq. (12) he extended the range of the possible \( k \)-values for both the right and left movers to \( 2\pi n/L \) with \( n \) positive and negative integers. This further simplifies the dispersion of the electrons, but introduces infinitely many negative energy electron states, which have to be filled in the ground state. As this infinite "Dirac sea" leads to various mathematical subtleties we prefer to work with a finite band cut-off. We introduce the momentum \( k_B = 2\pi n_0/L \geq 0 \) and allow states with momenta \( \geq -k_B \) for right movers and \( \leq k_B \) for left movers. In Fig.2 we have chosen \( k_B = 1.5k_F \) and show the states added as the dot-dashed lines.
FIG. 2. Energy dispersion as a function of momentum. The dashed curve shows the usual “nonrelativistic” dispersion used in the original Tomonaga model and the full curve the linearized version Eq. (12). The dot-dashed parts are the additional states for \( k_B = 1.5k_F \). The model discussed by Luttinger corresponds to \( k_B \to \infty \).

Later we also consider the limit \( k_B \to \infty \). The kinetic energy of the model solved exactly in the next section reads

\[
\hat{T}_{TL} = \sum_{n \in I_+} v_F k_n c_{n,+}^{\dagger} c_{n,+} + \sum_{n \in I_-} (-v_F k_n) c_{n,-}^{\dagger} c_{n,-},
\]

(18)

where \( I_+ = \{-m_0, -m_0 + 1, \ldots, \infty\} \) and \( I_- = \{-\infty, \ldots, m_0 - 1, m_0\} \). The creation and annihilation operators for the right \((\alpha = +)\) and left \((\alpha = -)\) movers obey the fermion anticommutation relations \((m \in I_\alpha, n \in I_\beta)\)

\[
[c_{m,\alpha}, c_{n,\beta}]_+ = 0 \quad [c_{n,\alpha}, c_{m,\beta}]_+ = \delta_{n,m} \delta_{\alpha,\beta} \hat{1}.
\]

(19)

In the interaction term \( \hat{V} \) Eq. (9) we replace \( \hat{\rho}_n \hat{\rho}_{-n} \) by \((\hat{\rho}_{n,+} + \hat{\rho}_{n,-})(\hat{\rho}_{-n,+} + \hat{\rho}_{-n,-})\) where

\[
\hat{\rho}_{n,\alpha} = \sum_{n'} w_{n',n'\alpha + n}^{\alpha} c_{n',\alpha}^{\dagger} c_{n',\alpha},
\]

(20)

with

\[
w_{n',n'\alpha + n}^{\alpha} = \begin{cases} 1 & \text{for } n', n' + n \in I_\alpha \\ 0 & \text{else} \end{cases}
\]

are the "density operators" of the right and left movers. Because \( w_{n,m}^{\alpha} \) is symmetric in \( n \) and \( m \) the relation \( \hat{\rho}_{m,\alpha} = \hat{\rho}_{-m,\alpha} \) holds.

It is not more difficult to work with the following generalization

\[
\hat{V}_{TL} = \frac{1}{L} \sum_{n > 0} \left\{ g_4(k_n) (\hat{\rho}_{n,+} + \hat{\rho}_{n,-} + \hat{\rho}_{-n,+} + \hat{\rho}_{-n,-}) + g_2(k_n) (\hat{\rho}_{n,+} + \hat{\rho}_{n,-} + \hat{\rho}_{-n,+} + \hat{\rho}_{-n,-}) \right\}

+ \frac{1}{2L} g_4(0) (N_+^2 + N_-^2) + \frac{1}{2L} g_2(0) \cdot 2N_+N_-
\]

(21)

where the \( N_\alpha \) are the particle number operators

\[
N_\alpha = \sum_{m \in I_\alpha} c_{m,\alpha}^{\dagger} c_{m,\alpha}
\]

(22)

and we have dropped the terms proportional to the particle number operators as they can be later included in the chemical potential. The second line in Eq. (21) can also be expressed in terms of the total particle number operator.
\( \mathcal{N} \equiv \mathcal{N}_+ + \mathcal{N}_- \) and the "current operator" \( \hat{J} \equiv \mathcal{N}_+ - \mathcal{N}_- \). For the interaction functions in Eq. (21) we have used the standard notation. If we put \( g_1(k) = g_2(k) = v(k) \) the Tomonaga-Luttinger Hamiltonian \( H_{TL} \equiv \hat{V}_{TL} + \hat{V}_{TL} \) yields in perturbation theory in \( \hat{V}_{TL} \) the same result for \( < \hat{n}_{m,+} > \) as in Eq. (15). In \( \hat{Q} \hat{V}_{TL} \hat{F}_{TL} \) only the terms proportional to \( \hat{\rho}_{n,-} - \hat{\rho}_{n,+} \) \( |F_{TL} > \) contribute for \( n > 0 \). Concerning the low energy physics the original model with the quadratic dispersion is equivalent to the Tomonaga-Luttinger (TL) model described by the Hamiltonian \( H_{TL} \) in the high density limit \( k_F/k_c \to \infty \).

There is an essential difference in the commutation relations of the \( \hat{\rho}_{m,\alpha} \) and the \( \hat{\rho}_m \). While as discussed previously \([\hat{\rho}_m, \hat{\rho}_n] = 0 \), which can be seen most easily in the first quantized version, the \( \hat{\rho}_{m,\alpha} \) do not all commute. From the anticommutation relations for the fermions on different branches it follows that \([\hat{\rho}_{m,+}, \hat{\rho}_{n,-}] = 0 \). The commutators \([\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\beta}] \) on the other hand are nontrivial

\[
[\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\alpha}] = \sum_{m',n'} [c^\dagger_{m',\alpha} c_{m'+m,\alpha}, c^\dagger_{n',\alpha} c_{n'+n,\alpha}] w^\alpha_{m', m'+m} w^\alpha_{n', n'+n} \tag{23}
\]

In going to the last equation the summation index was changed from \( n' \) to \( m' \) in the second term. It is easy to see that the rhs of Eq. (23) vanishes if \( m \) and \( n \) have the same sign. Take, for example, right movers and \( m \) and \( n \) positive. Then due to the factor \( w^\alpha_{m', m'+m} \) the rhs of Eq. (21) can only be nonzero for \( m' \geq -m_0 \) and \( m' + m + n \geq -m_0 \). But then also \( m' + m \geq -m_0 \) and \( m' + n \geq -m_0 \) and the commutator vanishes because of the factor \( w^\alpha_{m', m'+m} - w^\alpha_{n', n'+n} = 0 \). Next we consider the case \( n = -m \) shortly discussed in section V. in I. For \( m > 0 \) one obtains for the right movers

\[
[\hat{\rho}_{m,+}, \hat{\rho}_{m,-}] = \sum_{m' = -m_0}^{-m_0 - m - 1} c^\dagger_{m'+,+} c_{m'+,+} \equiv \mathcal{N}_+(m). \tag{24}
\]

We see that the commutator is given by the particle number of the lowest \( m \) right moving one-particle states. As explained earlier an interaction \( v(k) \) which drops off rapidly for \(|k| > k_c \) does not produce holes deep in the Fermi sea in the interacting ground state and eigenstates with low excitation energy. In this subspace of the total Hilbert space it is an excellent approximation which becomes asymptotically exact for \( m_0 \to \infty \) to replace \( \mathcal{N}_+(m) \) by \( m \) on the rhs of Eq. (24). In the interaction term \( \hat{V}_{TL} \) Eq. (20) only \( \hat{\rho}_{n,\alpha} \) for \(|n| \ll n_F \) contribute. Therefore it is sufficient to discuss the commutation relations Eq. (23) for \( m \) and \( n \) with different sign for \(|m|, |n| \ll n_F \). For \( m > -n > 0 \) one obtains for right movers

\[
[\hat{\rho}_{m,+}, \hat{\rho}_{m,-}] = \sum_{m' = -m_0}^{-m_0 - m - 1} c^\dagger_{m'+,+} c_{m'+,+} = \sum_{m' = -m_0}^{-m_0 - n - 1} c^\dagger_{m'+,+} c_{m'+,+.} \tag{25}
\]

The operator \( c^\dagger_{m'+,+} \) tries to create an electron deep in the Fermi-Dirac sea, which is not possible in the space of low lying excited states, as all states at the bottom of the band are occupied. Therefore the commutator vanishes in this subspace. Analogous arguments apply to the left movers and we finally obtain

\[
[\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\beta}] = \alpha \gamma \delta_{\alpha\beta} \delta_{m,-n}. \tag{26}
\]

The importance of this relation was first realized by Tomonaga who worked with \( k_B = 0 \). Luttinger made an error with the commutation relation for \( k_B = \infty \) later corrected by Mattis and Liebl.

### III. EXACT SOLUTION OF THE TL-MODEL USING BOSONIZATION

The commutation relations Eq. (26) for the operators \( \hat{\rho}_{m,\alpha} \) apart from a proper normalization factor, look like boson commutation relations. If we define the operators

\[
b_n = \frac{1}{|n|} \begin{cases} \hat{\rho}_{n,+} & \text{for } n > 0 \\ \hat{\rho}_{n,-} & \text{for } n < 0 \end{cases}
\]

(27)
and the corresponding adjoint operators \( b_n^\dagger \) using \( \hat{\rho}_{n,\alpha}^+ = \hat{\rho}_{-n,\alpha} \) Eq. (26) implies the usual bosonic commutation relations
\[
\{ b_n, b_m \} = 0, \quad \{ b_n, b_m^\dagger \} = \delta_{mn} \hat{1}.
\] (28)

The kinetic energy \( \hat{T}_{TL} \) in Eq. (18) consists of two commuting terms, which both are of the form discussed for fixed boundary condition in I. Therefore they both can be expressed in terms of the boson operators with the help of the Kronig identity \( [\rho_\alpha, \hat{N}^\dagger] = 0 \) which was stated in I without giving an explicit proof. This is therefore presented in appendix A. It yields
\[
\hat{T}_{TL} = \sum_{n \neq 0} v_F \left( \frac{2\pi}{L} \right) |n| b_n^\dagger b_n + v_F \left( \frac{\pi}{L} \right) (\hat{N}_{\dagger}^2 + \hat{N}^2) \] (29)

As we dropped terms proportional to \( \hat{N} \) in \( \hat{V}_{TL} \) which only lead to a renormalization of the chemical potential we also drop the term proportional to \( \hat{N} \) in the kinetic energy in the following. Neglecting also a constant term in the total Hamiltonian it reads
\[
H_{TL} = \sum_{n > 0} \left\{ k_n \left( v_F + \frac{g_1(k_n)}{2\pi} \right) \left( b_n^\dagger b_n + b_{-n}^\dagger b_{-n} \right) + \frac{k_n g_2(k_n)}{2\pi} \left( b_n^\dagger b_{-n}^\dagger + b_{-n} b_n \right) \right\} + \frac{\pi}{2L} [v_N \hat{N}^2 + v_J \hat{J}^2] = H_B + H_{N,L,J}
\] (30)

with the velocities
\[
v_N = v_F + \frac{g_1(0) + g_2(0)}{2\pi},
\]
\[
v_J = v_F + \frac{g_1(0) - g_2(0)}{2\pi}.
\] (31)

Here \( v_N \) determines the energy change for adding particles while \( v_J \) enters the energy change when the difference in the number of right and left movers is changed. For our original model with \( g_1(q) = g_2(q) = v(q) \) the velocity \( v_J \) equals the Fermi velocity \( v_F \) of the noninteracting system. As the particle number operators \( \hat{N}_\alpha \) commute with the boson operators \( b_n(b_n^\dagger) \) the two terms \( H_B \) and \( H_{N,L,J} \) in the Hamiltonian commute and can be treated separately.

The boson part \( H_B \) of the TL-Hamiltonian is bilinear in the boson creation and annihilation operators. As shown below it can be brought into the form
\[
H_B = \sum_{n \neq 0} \omega_n \alpha_n^\dagger \alpha_n + \text{const.}
\] (32)

by solving an eigenvalue problem, and the \( \alpha_n^\dagger (\alpha_n) \) are also boson creation (annihilation) operators. As \( H_B \) in Eq. (30) is a sum of commuting terms \( H_{B,n} \) only the coupling of the four operators \( b_n, b_{-n}, b_n^\dagger, b_{-n}^\dagger \) has to be considered at a time. Momentum conservation further reduces the \( 4 \times 4 \) eigenvalue problem to a \( 2 \times 2 \) problem. The eigenvalue problem is obtained by looking at the Heisenberg equations of motion for \( b_n(t) = e^{iH_B t} b_n e^{-iH_B t} \). This involves the commutators
\[
\left( \begin{array}{c} b_n, H_B \end{array} \right) = \left( \begin{array}{cc} k_n v_F + g_1(k_n)/(2\pi) & k_n g_2(k_n)/(2\pi) \\ -k_n g_2(k_n)/(2\pi) & -k_n v_F + g_1(k_n)/(2\pi) \end{array} \right) \left( \begin{array}{c} b_n \\ b_n^\dagger \end{array} \right)
\] (33)

The matrix \( \mathbf{M}^{(n)} \) on the rhs of Eq. (33) has the eigenvalues
\[
\omega_n^\pm = \pm v_F k_n \sqrt{\left( 1 + \frac{g_2(k_n)}{2\pi v_F} \right)^2 - \frac{g_1(k_n)^2}{4\pi^2 v_F^2}} \equiv \pm \omega_n
\] (34)

As \( \mathbf{M}^{(n)} \) is not symmetric one has to distinguish left and right eigenvectors. We denote the left eigenvector to \( \omega_n^+ \) as \( (c_n, s_n) \) where \( c_n \) and \( s_n \) can be chosen as real quantities as \( \mathbf{M}^{(n)} \) itself is real. The formal solution of the Heisenberg equations of motion \( i \dot{b}_n = [b_n, H] \) etc. reads
\[
\left( \begin{array}{c} b_n(t) \\ b_{-n}^\dagger(t) \end{array} \right) = e^{-i\mathbf{M}^{(n)} t} \left( \begin{array}{c} b_n \\ b_{-n}^\dagger \end{array} \right).
\] (35)
If we multiply this equation form the left with \((c_n, -s_n)\) we can replace the matrix by its eigenvalue \((c_n, -s_n)e^{-iM_n t} = (c_n, -s_n)e^{-i\omega_n t}\) which shows that the operator

\[
\alpha_n \equiv c_n b_n - s_n b_n^\dagger
\]

has the simple time evolution \(\alpha_n(t) = e^{-i\omega_n t} \alpha_n\). With a proper normalization of the left eigenvector the \(\alpha_n\) are therefore the boson operators which "diagonalize" \(H_B\) to the form Eq. (32). The \(\alpha_n\) should obey the boson commutation relation \([\alpha_n, \alpha_n^\dagger] = 1\). As Eq. (36) yields \([\alpha_n, \alpha_n^\dagger] = c_n^2 - s_n^2\) the left eigenvector \((c_n, -s_n)\) has to be normalized by \(c_n^2 - s_n^2 = 1\). Elementary calculation of the eigenvector then yields

\[
s_n^2 = \frac{1}{2} \left[ \frac{1 + g_4(k_n)/(2\pi v_F)}{\sqrt{(1 + g_4(k_n)/(2\pi v_F))^2 - (g_2(k_n)/(2\pi v_F))^2}} - 1 \right] = s_n^2
\]

In the noninteracting limit \(g_\nu \to 0\) the weights \(s_n\) vanish and the "Bogoljubov transformation" in Eq. (36) reduces to \(\alpha_n = b_n\). The dispersion relation \(\omega_n^{(0)} = v_F|k_n|\) of the bosons in the noninteracting case (see I) is in the interacting case replaced by \(\omega_n = v_c(k_n)|k_n|\) where the \(k\)-dependent "sound velocity" usually called the "charge velocity" follows from Eq. (34) as

\[
v_c(k_n) = v_F \sqrt{\left(1 + \frac{g_4(k_n) + g_2(k_n)}{2\pi v_F}\right) \left(1 + \frac{g_4(k_n) - g_2(k_n)}{2\pi v_F}\right)}.
\]

For our original model \(g_4(k) \equiv g_2(k) \equiv v(k)\) it is given by \(v_c(k_n) = v_F(1 + v(k_n)/(\pi v_F))^{1/2}\). The corresponding dispersion is shown in Fig. 3 for the two potentials \(v\) presented in Fig. 1.

![FIG. 3. Dispersion of the bosonic modes in Eq. (34) for the two potentials shown in Fig. 1. For the potential which is constant up to \(k_c\) the dispersion is discontinuous while for the smooth potential the approach to the noninteracting limit for large \(k\) is gradual.](image)

For \(|k_n|/k_c \ll 1\) the boson dispersion is approximately linear \(\omega_n \approx v_c|k_n|\) with \(v_c \equiv v_c(0)\). From Eqs. (31) and (38) we obtain the relation

\[
v_c = \sqrt{v_N v_F}
\]

first pointed out by Haldane.\footnote{For large momenta \(|k_n| \gg k_c\) the dispersion of the noninteracting case is approached. In the figure we have used a purely repulsive interaction \((v(k) \geq 0)\) which leads to \(v_c(k) \geq v_F\). For attractive interaction there is an instability if \(-v(q)\) gets larger than \(\pi v_F\)\footnote{leading to phase separation. In the following we concentrate on repulsive interactions.}

The interacting ground state \(|\{0\}_a, \tilde{N}_+, \tilde{N}_- >\) with the integers \(\tilde{N}_a \equiv N_a - m_0 - 1\) is determined by \(\alpha_n|\{0\}_a, \tilde{N}_+, \tilde{N}_- >= 0\). It can be constructed by a unitary transformation out of the noninteracting ground state}
\[ |\{0\}_b, \tilde{N}_+, \tilde{N}_- > := \left( \prod_{n=-m_0}^{N_-} c^\dagger_{n,-} \right) \left( \prod_{l=-m_0}^{N_+} c^\dagger_{l,+} \right) |\text{Vac} >. \] 

(40)

Here we have allowed the number of right and left movers to be different. Our original model corresponds to \( \tilde{N}_+ = \tilde{N}_- = n_F \). In the following we calculate expectation values in the interacting ground state by algebraic methods using \( \alpha_n |\{0\}_\alpha, \tilde{N}_+, \tilde{N}_- > = 0 \). Therefore the explicit form of this state is not needed. The excited states for given particle numbers \( \tilde{N}_+, \tilde{N}_- \) are constructed as for harmonic oscillators.

\[ |\{m_l\}_\alpha, \tilde{N}_+, \tilde{N}_- > := \prod_{l \neq 0} (\alpha_l^\dagger m_l) |\{0\}_\alpha, \tilde{N}_+, \tilde{N}_- >. \] 

(41)

Their energy is given by \( E_0(\tilde{N}_+, \tilde{N}_-) + \sum_{l \neq 0} m_l \omega_l \).

The nature of the interacting ground state shows up clearly in the momentum distribution \( \langle \hat{n}_{k,n,+} \rangle \). In the following we calculate \( \langle \hat{n}_{k,n,+} \rangle \) and show how the Fermi step of the noninteracting limit is modified near the right Fermi point \( k_F \). As in I it can be obtained by first calculating \( \langle \tilde{\psi}^\dagger_+(u) \tilde{\psi}_+(v) \rangle \), where \( \tilde{\psi}_+(v) \) is the auxiliary field operator defined in (B.6) and then using the inversion formula (B.24) to calculate the momentum distribution.

With the use of the bosonization formula (B.25) the calculation is quite simple. As we want to use \( \alpha_n |\{0\}_\alpha, \tilde{N}_+, \tilde{N}_- > = 0 \) we express in the operator function \( \phi_+ \) defined in (B.9) the \( b_n \) in terms of the \( \alpha_n, \alpha_n^\dagger \). From \( \alpha_n = c_n b_n - s_n b_n^\dagger \) in Eq. (36) one obtains the inversion

\[ b_n = c_n \alpha_n + s_n \alpha_n^\dagger. \] 

(42)

This yields

\[ i(\phi_+(u) - \phi_+(v)) = \sum_{n=1}^{\infty} e^{i n u u} e^{i n v v} (c_n \alpha_n + s_n \alpha_n^\dagger). \] 

(43)

As the \( \alpha_n \) and \( \alpha_n^\dagger \) commute, \( \exp[-i(\phi_+(u) - \phi_+(v))] \) can be written as a product of two exponentials with the annihilation part to the right. If applied to the interacting ground state this factor yields unity, i.e.

\[ e^{-i(\phi_+(u) - \phi_+(v))} |\{0\}\alpha, \tilde{N}_+, \tilde{N}_- > = \sum_{n=1}^{\infty} \left( \frac{\exp(it - k_F)}{\sqrt{\omega}} \right) s_n \alpha_n^\dagger |\{0\}_\alpha, \tilde{N}_+, \tilde{N}_- >. \] 

(44)

The expectation value of \( \tilde{\psi}^\dagger_+(u) \tilde{\psi}_+(v) \) as seen from Eq. (B.25) involves the overlap of the above state with itself. Using the Baker Hausdorff formula \( e^{-i[A,B]} = e^{-F} e^{-i[A,B]} \) we obtain

\[ < \tilde{\psi}^\dagger_+(u) \tilde{\psi}_+(v) > = \frac{\exp\left\{ - \sum_{n=1}^{\infty} \frac{2s_n^2}{n}(1 - \cos(\theta n)) \right\}}{1 - e^{it(\theta u - \theta v)}}. \] 

(45)

This is exactly of the form as the finite temperature canonical expectation value for noninteracting fermions (Eq.(50) in I) but with the Bose function \( b(n\Delta) \) replaced by \( s_n^2 \) which depends on the interaction as shown in Eq.(37). For arbitrary potentials therefore \( < \hat{n}_{k,n,+} > \) can be calculated by the recursive method outlined in the appendix of I. The sum in the exponent on the rhs of Eq. (45) is especially simple, if \( s_n^2 \) decays exponentially

\[ s_n^2 = s^2(0) e^{-2|k'|/k}. \] 

(46)

The corresponding potential \( v(k_n) \) can be obtained using Eq. (37) by solving a quadratic equation. The factor 2 in the exponent on the rhs of Eq. (46) was introduced because \( v(k_n) \) is proportional to \( s_n \) for large momenta. It is shown as the full line in Fig. 1 for \( s(0)^2 = 0.125 \). If the cosine is written as a sum of two exponentials sums of the type \( \sum_{n=1}^{\infty} q^{n} n = -\ln(1 - q) \) appear in the exponent on the rhs of Eq. (45). This yields

\[ < \tilde{\psi}^\dagger_+(u) \tilde{\psi}_+(v) > = \frac{\exp\left\{ - i(\tilde{N}_+ - \tilde{N}_-)(u - v) \right\}}{1 - e^{it(\theta u - \theta v)}} \left( \frac{1 - e^{-2/\pi c}}{1 - e^{it(\theta u - \theta v)2/n_c}} \right) \] 

(47)

The calculation of \( < \hat{n}_{m,+} > \) simplifies if we go to the thermodynamic limit \( L \to \infty, \tilde{N}_+/L = \text{const} \). Then it is appropriate to switch to the "physical" field operators \( \psi_\alpha(x) \) defined as in Eq. (2) but with \( c_n \) replaced by \( c_{n,\alpha} \). They are related to the auxiliary field operators defined in (B.6) by
\[ \psi_\alpha(x) = \frac{1}{\sqrt{L}} e^{i \frac{2\pi x}{L}}. \] (48)

The momentum distribution \( < \hat{n}_{k_m,+} > \) follows from the inversion formula (B.22) and the fact that \( < \tilde{\psi}_+^\dagger(u)\tilde{\psi}_+(v) > \) is a function of \( u - v \) only as

\[ < \hat{n}_{k_m,+} >= \int_{-L/2}^{L/2} dx e^{ik_n x} < \psi_+^\dagger(x)\psi_+(0) >. \] (49)

If one now takes the limit \( L \to \infty, 2\pi \tilde{N}_+/L \equiv k_F \) one obtains for fixed \( x \) from Eq. (47)

\[ < \psi_+^\dagger(x)\psi_+(0) >= \frac{i}{2\pi} \frac{e^{-ik_F x}}{(x+i0)} \left( \frac{(2/k_c)^2}{x^2 + (2/k_c)^2} \right)^{s^2(0)}. \] (50)

In contrast to the noninteracting case this zero temperature time-independent "Green's function" for large \( x \) does not decay as \( (1/x)^d \), where \( d = 1 \) is the spatial dimension, but as \( (1/x)^1 + 2s^2(0) \). Therefore the quantity

\[ \alpha = 2s^2(0) \] (51)

is called the "anomalous dimension". The momentum distribution in the thermodynamic limit will be denoted \( n_\alpha(k) \). The behaviour of \( n_+(k) \) near the right Fermi point is most easily studied by looking at its derivative

\[ \frac{dn_+(k)}{dk} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{(2/k_c)^2}{u^2 + (2/k_c)^2} \right)^{s^2(0)} e^{ikx} dx, \] (52)

where \( \tilde{k} \equiv k - k_F \). In the noninteracting case \( s^2(0) = 0 \) and \( (dn_+(k)/dk)^{(0)} = -\delta(\tilde{k}) \), where the negative delta function is due to the step at \( k = k_F \). For finite values of \( s^2(0) \) this singularity for \( \tilde{k} = 0 \) is weakened. For \( 2s^2(0) > 1 \) the function of \( x \) Fourier transformed in Eq. (52) decays faster than \( 1/|x| \) and \( (dn_+/dk)|_{\tilde{k}=0} \) is finite. For \( 0 < 2s^2(0) < 1 \) one obtains the behaviour mentioned in section II. As \( dn_+/dk \) is an even function of \( k \), it is sufficient to consider \( \tilde{k} > 0 \). With the substitution \( u = \tilde{k}x \) one obtains

\[ \frac{dn_+}{dk} = (2\tilde{k}/k_c)^{\alpha-1}(-\frac{1}{\pi}) \int_{0}^{\infty} \left( \frac{1}{u^2 + (2\tilde{k}/k_c)^2} \right)^{s^2(0)} \cos u du. \] (53)

For \( 0 < 2s^2(0) < 1 \) the limit \( \tilde{k} \to 0 \) in the integrand can be performed and the integral is finite, but \( dn_+/dk \) diverges at \( \tilde{k} = 0 \) because of the prefactor of the integral. Integration with respect to \( \tilde{k} \) finally yields Eq. (16). At finite temperatures the derivative of \( n_+ \) at \( k_F \) diverges like \( T^{\alpha-1} \) for \( T \to 0 \). The special case \( \alpha = 1 \) requires a separate treatment not given here.

The power law behaviour Eq. (16) of \( n_+(k) \) has been derived here for the special potential shown in Fig. 1 leading to the exponential decay Eq. (45) of \( s_\alpha^2 \). Different potentials with a different decay of \( s^2(k_n) \) lead for \( |\tilde{k}|/k_c \ll 1 \) to the same qualitative behaviour as for our special potential if \( v(k) \) goes to constant value for \( k \to 0 \). This implies a constant value \( s^2(0) \) and the anomalous dimension is given by Eq. (51). The anomalous dimension can be expressed in terms of the velocities \( v_N \) and \( v_F \) using Eqs. (31) and (37)

\[ 2s^2(0) = \frac{(v_F + v_N)^2}{v_N v_F} - 1 \equiv \frac{1}{2}(K + \frac{1}{K} - 2), \] (54)

where \( K \equiv \sqrt{v_F/v_N} > 0 \) is less than one for repulsive interactions and larger than one for attractive interactions.

The power law behaviour of \( n_+(k) \) for interactions \( v(k) \) with a finite limit \( v(0) \) in one spatial dimension is different from the "Fermi liquid" type behaviour of a usual three dimensional conductors in which the momentum distribution of the interacting ground states has a finite discontinuity of size \( 0 < Z < 1 \) at the Fermi momentum \( \frac{1}{2\pi} \). We note that for a finite one-dimensional system and small values of \( \alpha \) the distinction between the power-law behaviour and a finite discontinuity is not straightforward. In Fig. 4 we show the momentum distribution for a step potential (see Fig. 1) with \( v(0)/(2\pi v_F) = 1.5 \) and 0.3 corresponding to \( \alpha = 0.25 \) and \( \alpha = 0.0277 \) for different system sizes.
FIG. 4. Momentum distribution for the step potential of Fig. 1 with \( v(0)/(\pi v_F) = 3 \) and 0.6 corresponding to values of the anomalous dimension \( \alpha = 0.25 \) and 0.0554. For the larger interaction the distribution is shown for different system sizes (stars: \( n_c = 100 \), circles: \( n_c = 200 \)). For the smaller interaction the exact result for \( n_c = 20 \) (squares) is compared to the result from perturbation theory (triangles).

The size of the system is specified by the dimensionless ratio \( n_c = (k_c L)/2\pi \equiv (L/R)/2\pi \) where \( R \) is the spatial range of the interaction. For the smaller interaction we also compare the exact result with the lowest order perturbation theory result Eq. (15) in the range \( 0 < k - k_F < k_c \). As the dimensionless coupling constant \( v(0)/(2\pi v_F) = 0.3 \) is not really much smaller than one the deviations are already well visible.

Power laws occur for the TL-model also in other physical quantities. In the next section we present a short discussion of one-particle spectral functions which largely determine the current in a photoemission experiment.

IV. SPECTRAL PROPERTIES AND PHOTOEMISSION

For noninteracting electrons the "density of one-particle states per unit volume" (DOS) \( \rho_0(\epsilon) \) plays an important role for the calculation of thermodynamic properties. In one dimension it is defined as

\[
\rho_0(\epsilon) \equiv \frac{1}{T} \sum_{k_n} \delta(\epsilon - \epsilon_{k_n}).
\] (55)

In the thermodynamic limit \( \sum_{k} \rightarrow \frac{L}{2\pi} f(\epsilon) \, dk \) and for a system of noninteracting right movers (\( \epsilon_k = v_F k \)) one obtains \( \rho_0(\epsilon) = 1/(2\pi v_F) \), i.e. an energy independent DOS. For photoemission only the occupied states matter and one defines the "occupied DOS" \( \rho^< \) by multiplying the delta function in Eq. (55) with the Fermi function \( f(\epsilon_{k_n}) \), which at \( T = 0 \) is the step function at the Fermi energy \( \epsilon_F \). This simple DOS concept has to be generalized properly when the fermions interact. For a translational invariant system as discussed in the previous sections one defines

\[
\rho^<(\epsilon) \equiv \frac{1}{T} \sum_n < \phi_0(N)|c^+_n \delta(\epsilon + H - E_0(N))c_n|\phi_0(N)>.
\] (56)

as the generalization of the occupied DOS relevant for photoemission. In the noninteracting limit \( c_n|\phi_0(N) \rangle \) is the normalized Slater determinant where the state with momentum \( k_n \) has been removed relative to the N-electron ground state. As its energy is given by \( E_0(N) - \epsilon_{k_n} \) the spectral function \( \rho^<(\epsilon) \) reduces to the sum of delta functions as in Eq. (55) restricted to the occupied momenta. The individual terms in the sum of the rhs of Eq. (56) enter
resolved photoemission\cite{2,3}, while \(\rho^<(\epsilon)\) is needed for the description of angular integrated photoemission\cite{3}. For translational invariant systems expectation values with \(c_n^\dagger\) replaced by \(c_n^\dagger\) with \(n' \neq n\) in Eq. (56) vanish because of momentum conservation and one can write \(\rho^<(\epsilon)\) in the form

\[
\rho^<(\epsilon) = \langle \phi_0(N) | \psi^\dagger(x = 0) \left( \frac{1}{2\pi} \int e^{i(\epsilon + H - E_0(N))t} dt \right) \psi(x = 0) | \phi_0(N) \rangle >
\]

\[
= \int < \phi_0(N) | \psi^\dagger(x = 0) \psi(x = 0, t) | \phi_0(N) > e^{i\epsilon t} \frac{dt}{2\pi} = \int iG^<(t)e^{i\epsilon t} \frac{dt}{2\pi},
\]

where \(\psi(x, t) = e^{iHt}\psi(x)e^{-iHt}\) is the field operator in the Heisenberg representation. If we switch from the original nonrelativistic model to the TL-model the spectral function acquires an additional \(a\) label. In the following we sketch the calculation of \(\rho^<(\epsilon)\). For that purpose it is convenient to express the field operator \(\psi_+(v)\) Eq. (B.8) in terms of the \(\alpha_n, \alpha_n^\dagger\) instead of the \(b_n, b_n^\dagger\) because the former have a simple time dependence. In \(b_m = c_m\alpha_m + s_m\alpha_m^\dagger\) the operators \(\alpha_m\) and \(\alpha_m^\dagger\) commute. Therefore \(e^{i\phi_+(v)}\) (and \(e^{i\phi_+^\dagger(v)}\)) in Eq. (B.8) can be written as a product of two exponentials with the annihilation operators to the right. After using the Baker-Hausdorff formula once in order to complete the process of "normal ordering" (bringing all annihilation operators to the right of the creation operators) one obtains

\[
\psi_+(x) = \frac{A(n_c)}{\sqrt{L}} \hat{O}_+ \left( \frac{2\pi x}{L} \right) e^{i\chi_+(x)} e^{i\chi_+(x)}
\]

with \(i\chi_+(x) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left( c_m e^{ik_m x} \alpha_m - s_m e^{-ik_m x} \alpha_m \right) \)

and \(A(n_c) = e^{-\sum_{n=-\infty}^{\infty} s_n^2/n} \).

This is one of the most important formulas for the physics of one-dimensional interacting fermions. The time dependent operator \(\psi_+(x, t)\) follows from Eq. (58) by replacing \(\alpha_m\) and \(\alpha_{-m}\) by \(\alpha_m e^{-i\omega_m t}\) and \(\alpha_{-m} e^{-i\omega_m t}\) and \(U_a\) in \(\hat{O}_+\) (see Eq. (23)) by \(U_a(t)\). Various kinds of time dependent correlation functions can quite simply be calculated using this result\cite{4,5,6,7}. Here we restrict ourselves to \(G^<_1(t)\) in order to obtain \(\rho^<_1(\epsilon)\).

As \(U_+\) commutes with the bosonic operators we first show that the particle number changing operators \(U_+^\dagger\) and \(U_+\) lead to a simple time dependent factor. Using (B.17) one obtains

\[
U_+^\dagger e^{iHt}U_+ e^{-iHt}|0\rangle |\alpha, \hat{N}_+, \hat{N}_-\rangle = e^{-i(E_0(\hat{N}_+, \hat{N}_-) - E_0(\hat{N}_+ - 1, \hat{N}_-))t}|0\rangle |\alpha, \hat{N}_+, \hat{N}_-\rangle. \tag{59}
\]

As \(\psi_+(x)\) in Eq. (58) is normal ordered in the \(a\)'s one has to use the Baker-Hausdorff formula only once to normal order \(\psi_+(x = 0)\psi_+(x = 0, t)\). This yields

\[
ie^{i\mu t}G^<_1(t) = \frac{A^2(n_c)}{L} \int_0^L [\chi_+(x = 0, t = 0), \chi_+(x = 0, t)]
\]

\[
= \frac{A^2(n_c)}{L} \left( \sum_{n=1}^{\infty} \frac{1}{n} e^{i\omega_n t} \right)
\]

where \(\mu = E_0(\hat{N}_+, \hat{N}_-) - E_0(\hat{N}_+ - 1, \hat{N}_-)\) is the chemical potential, which we “eliminate” by discussing \(\rho^<_1\) as a function of \(\epsilon = \mu - \mu\). The spectral function can be calculated analytically in the “universal” low energy regime \(-k_{\text{min}}(v_F, v_c) \leq \epsilon \leq 0\) for the step potential which has a strictly linear boson dispersion \(\omega_n = v_c|k_n|\) for \(|k_n| \leq k_c\) (see Fig. 2) and a different linear dispersion \(\omega_n = v_F|k_n|\) for \(|k_n| > k_c\). This implies that we can write \(G^<_1(t)\) as a product of two power series in \(e^{i2\pi v_F t/L}\) and \(e^{i2\pi v_c t/L}\)

\[
iLe^{i\mu t}G^<_1(t) = A^2(n_c) \left( 1 + \sum_{m=n_c+1}^{\infty} a_m^{(n_c)} e^{i\omega_n t} \right) \left( \sum_{n=0}^{\infty} a_n^{(n_c)} e^{i\omega_n t} \right).
\]

Now the Fourier integral can be performed analytically. For \(v_F k_c \leq \epsilon \leq 0\) the \(a_m^{(n_c)}\) and \(v_c k_c \leq \epsilon \leq 0\) the \(a_m^{(n)}\) with \(m > n_c\) do not contribute and one obtains in the universal regime \(-k_{\text{min}}(v_F, v_c) \leq \epsilon \leq 0\)
The $d_n^{(n_e)}$ for $n \leq n_e$ can be given analytically as the sum in the exponent in Eq. (60) is of the form $(1+\alpha) \sum_{n=1}^{n_e} z^n/n$ with $z = \exp(i2\pi v_c t/L)$ and $\alpha = 2s^2(0)$. For the calculation of the $d_n^{(n_e)}$ for $n \leq n_e$ one makes no error to extend the sum to infinity, which yields $\exp[(1+\alpha) \sum_{n} z^n/n] = (1-z)^{-1+\alpha}$. Now one can use the well known power expansion of this function. Alternatively one can show that the recursion relation in (A.4) of I can simply be solved explicitly. If we also use $\sum_{n=1}^{n_e} 1/n \to \ln n_e + C$, where $C = 0.57721 \ldots$ is Euler’s constant, we find the exact weights of the delta peaks in Eq. (62)

$$A^2(n_e)d_n^{(n_e)} = \frac{e^{-\alpha C}}{n_e^\alpha} \prod_{\ell=1}^{n} \left(1 + \frac{\alpha}{\ell}\right)$$

for $n_e \geq n \geq 1$ and $d_0^{(n_e)} = 1$. For $n_e \geq n \gg 1$ one can use the product representation of Euler’s Gamma function to write the weights as

$$A^2(n_e)d_n^{(n_e)} = \frac{e^{-\alpha C}}{\Gamma(1+\alpha)} \left(\frac{n}{n_e}\right)^\alpha \left(1 + O(1/n_e)\right).$$

Now the thermodynamic limit can be performed in Eq. (62) and one obtains with $\epsilon_c \equiv v_c k_c$ for $\epsilon \leq \mu$ in the universal regime

$$\rho_+^<(\epsilon) = \frac{e^{-\alpha C}}{\Gamma(1+\alpha)} \cdot \frac{1}{2\pi v_c} \left(\frac{\mu - \epsilon}{\epsilon_c}\right)^\alpha.$$

The spectral weight near (below) the chemical potential $\mu$ is suppressed by the same power law behaviour as in $n_+(k) - 1/2$. This type of power law occurs also for potentials with a finite $v(0)$ but a different decay for momenta large compared to $k_c$. But then the prefactor is different and the power law behaviour does not hold exactly in a finite range as Eq. (65) but only asymptotically in the limit $(-\tilde{\epsilon})/\epsilon_c \to 0$.[3] This power law suppression of spectral weight was claimed to be experimentally verified in photoemission experiments with an organic compound which consists of weakly coupled chains.[3]

Reliable results for more complicated models of interacting fermions can often only be obtained using numerical methods for systems of rather small size. As our result Eq. (63) is exact also for finite systems we shortly discuss how one can extract the anomalous dimension even for small values of $n_e$. As Eq. (64) only holds for $n \gg 1$ it is not useful to fit the weights to the power law form. It is much easier to look at the scaling behaviour of the peak at $\epsilon = \mu$ with weight $A^2(n_e)d_n^{(n_e)} = e^{-\alpha C}/(n_e)^\alpha$ with system size. From a log-log-plot one can directly infer the value of $\alpha$.

For $-\tilde{\epsilon} \gg \epsilon_c$ the spectral function $\rho_+^<(\tilde{\epsilon})$ approaches the noninteracting value $1/(2\pi v_F)$. The crossover from the power law behaviour to this high energy limit is discussed in [3].

Finally we want to point out that the low energy power law behaviour of $\rho_+^<(\epsilon)$ given by Eq. (65) holds for arbitrary values of $\alpha$. No additional linear contribution occurs for $\alpha > 1$ as in the case of the momentum distribution $n_+(k) - 1/2$. This difference is easily lost without a proper treatment of the cut-offs.

V. SPIN-CHARGE SEPARATION IN THE MODEL INCLUDING SPIN

Electrons are spin one-half particles and for their description it is necessary to include the spin degree of freedom in the model. We choose a fixed quantization axis and denote the two spin states by $\sigma = \uparrow, \downarrow$. Then the field operator $\psi(x)$ acquires a spin index as the $c_n$. The density operator of spin projection $\sigma$ is defined as $\rho_\sigma(x) \equiv \psi_\sigma^\dagger(x)\psi_\sigma(x)$ and in the two-body interaction in Eq. (6) $V(x-x')\rho(x)\rho(x)$ is replaced by $\sum_{\sigma,\sigma'} V_{\sigma\sigma'}(x-x')\rho_\sigma(x)\rho_{\sigma'}(x')$. Usually the interaction $V_{\sigma\sigma'}(x-x')$ is independent of the spin indices.[8] After the linearization and introduction of the right and left movers the $\tilde{\rho}_{n,\sigma}$ in Eq. (20) obtain an additional spin index, as well as the boson operators in Eq. (27). The TL-Hamiltonian for spin 1/2 then reads

$$H^{(1/2)}_{TL} = \sum_{\sigma,\sigma'} \left\{ \sum_{n>0} k_n \left( v_F \delta_{\sigma\sigma'} + \frac{g\sigma\sigma'(k_n)}{2\pi} \right) \left( b_{n,\sigma}^\dagger b_{n,\sigma} + b_{-n,\sigma}^\dagger b_{-n,\sigma'} \right) \right\}.$$
\[
\frac{k_n g_2^{\sigma\sigma'}(k_n)}{2\pi} \left( b_{n,\sigma}^\dagger b_{-n,\sigma'} + b_{-n,\sigma'} b_{n\sigma} \right) + \frac{\pi}{L} \left( v_F \delta_{\sigma\sigma'} + \frac{g_4^{\sigma\sigma'}(0)}{2\pi} \sum_{\alpha} N_{\alpha,\sigma} N_{\alpha,\sigma'} + \frac{g_2^{\sigma\sigma'}(0)}{\pi} N_{+,\sigma} N_{-,\sigma'} \right).
\]

The interaction matrix elements \( g_2^{\sigma\sigma'}(k) \) are taken as depending on the relative spin only, i.e. \( g_2^{\sigma\sigma'} \) and \( g_2^{-\sigma'-\sigma} \) are independent of the spin label \( \sigma \). The interaction term in Eq. (66) couples the electrons with different spin. With the assumption on the \( g_2^{\sigma\sigma'} \) it is possible to switch to new boson operators \( b_{n,a} \) with \( a = c,s \)

\[
b_{n,c} = \frac{1}{\sqrt{2}} (b_{n,\uparrow} + b_{n,\downarrow})
\]

\[
b_{n,s} = \frac{1}{\sqrt{2}} (b_{n,\uparrow} - b_{n,\downarrow}),
\]

which obey \([b_{a,n}, b_{a',n'}] = 0\); \([b_{a,n}, b_{a',n'}^\dagger] = \delta_{aa'} \delta_{nn'} \mathbb{1}\). The kinetic energy can also be expressed in terms of “charge” \((c)\) and “spin” \((s)\) Bose operators using \( b_{n,\sigma}^\dagger b_{n,\sigma} + b_{n,\sigma} b_{n,\sigma}^\dagger = b_{n,c}^\dagger b_{n,c} + b_{n,c} b_{n,c}^\dagger \). If we define the interaction matrix elements \( g_{\nu,a}(q) \) via

\[
g_{\nu,c}(q) = g_2^{\sigma\sigma'}(q) + g_2^{-\sigma'-\sigma}(q)
\]

\[
g_{\nu,s}(q) = g_2^{\sigma\sigma'}(q) - g_2^{-\sigma'-\sigma}(q),
\]

and define \( N_{c,s}(\nu) = (N_\uparrow \pm N_\downarrow)/\sqrt{2} \) we can write \( H_{TL}^{(1/2)} \) as

\[
H_{TL}^{(1/2)} = H_{TL,c} + H_{TL,s},
\]

where the \( H_{TL,a} \) are of the form Eq. (30,31) but the interaction matrix elements have the additional label \( a \). The two terms on the rhs of Eq. (69) commute i.e. the “charge” and “spin” excitation are completely independent. This is usually called “spin-charge separation”\(\mathbb{1}\). The “diagonalization” of the two separate parts proceeds exactly as in section III and the low energy excitations are “massless bosons” \( \omega_{n,a} = v_\nu |k_n| \) with the charge velocity \( v_c \) and the spin velocity \( v_s \) usually different. For a spin independent interaction \( g_{\nu,s} \) in Eq. (68) vanishes and \( v_s \) equals the Fermi velocity \( v_F \). As discussed in the next section the Hamiltonian \( H_{TL}^{(1/2)} \) with properly chosen interaction matrix elements describes correctly the low energy physics of more complicated 1d models e.g. on a lattice. Then \( v_s \) and \( v_F \) are generally different even if the interaction on the lattice is spin independent.

The bosonization of the field operator \( \psi_{\alpha,\sigma}(x) \) proceeds exactly as in appendix B. In order to calculate quantities like the momentum distribution \( \langle \hat{n}_{k,+,\sigma} \rangle \) it is then useful to express the \( b_{n,\sigma} \) in \( i\phi_{+,\sigma}(v) \) like in (B.9) in terms of the charge and spin bosons

\[
i\phi_{+,\sigma}(v) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \sqrt{\nu} (b_{n,c} \pm b_{n,s}) \equiv i\phi_{+,c}(v) \pm i\phi_{+,s}(v),
\]

where the plus (minus) sign holds for the up (down) spin. Therefore e.g. the function \( \langle \psi_{+,+}^\dagger(x) \psi_{+,+}^\dagger (0) \rangle \) is a product of a charge and a spin part. The separate factors are calculated as in the spinless case. The only difference is a factor 1/2 in the exponent due to the additional factor \( 1/\sqrt{2} \) in Eq. (70). The final result is exactly of the form Eq. (45) but with \( 2s_{n,c}^2 \) replaced by \( s_{n,c}^2 + s_{n,s}^2 \). This leads to the power law behaviour presented in Eq. (16) as in the spinless case but the anomalous dimension given by

\[
\alpha = s_{n,c}^2(0) + s_{n,s}^2(0) \equiv \alpha_c + \alpha_s.
\]

The individual contributions can be expressed in terms of the \( K_\alpha \equiv (v_{L,a}/v_{N,a})^{1/2} \) as \( \alpha_a = (K_a - 1)^2/(4K_a) \) similar to Eq. (54). Also the result for \( G_{+,\sigma}(t) \) needed in the calculation of the integrated spectral function \( \rho_{+,\sigma}(\epsilon) \) takes the form of a product of a “spin” and a “charge” factor. In each factor the weight \( 1 + 2s_{n,c}^2 \) in the exponent of Eq. (60) is replaced by \( 1/2 + s_{n,c}^2 \). Therefore the Fourier transform of the \textit{individual factors} are in the low energy regime of the power law form Eq. (65) with \( \alpha \) replaced by \(-\frac{1}{2} + \alpha_a\). The convolution of the two power laws finally for \( \epsilon \leq \mu \) yields \( \rho_{+,\sigma}^{\phi}(\epsilon) \sim (\mu - \epsilon)^\alpha \) in the asymptotic regime as in the spinless model. Alternatively this follows more directly from the fact that \( G_{+,\sigma}^{\phi}(t) \) decays as \( t^{-(1+\alpha)} \) for large times.

In order to see a drastic difference between the model with and without spin, one has to calculate the individual terms in the sum in Eq. (56). The delta peaks of the noninteracting model are broadened into one “infrared” power law threshold in the model without spin \([3,4]\) and two power law singularities in the model including spin \([3,4]\).
VI. THE LUTTINGER LIQUID PHENOMENOLOGY

We have seen that the low energy physics of the TL-model is determined by two parameters \(v_c\) and \(K\) or \(v_F\) and \(v_J\) in the spinless model and four parameters \((v_c, K_c; v_s, K_s)\) or \((v_F, v_J, v_N, v_J')\) in the model including spin. They also determine the low temperature thermodynamics of the model. We shortly discuss the model including spin. The low temperature specific heat of a linear chain is given by \(\gamma T\), where the constant \(\gamma\) is inversely proportional to the sound velocity. This is the 1d version of Debye’s law. The ratio of \(\gamma\) and the corresponding value \(\gamma_0\) for the noninteracting fermions is therefore

\[
\frac{\gamma}{\gamma_0} = \frac{1}{2} \left( \frac{v_F}{v_c} + \frac{v_F}{v_s} \right). \tag{72}
\]

At zero temperature we can express the pressure \(p\) as \(p = -(\partial E_0/\partial L)_N\) and the compressibility \(\kappa \equiv - (\partial L/\partial p)_N/L = (L(\partial^2 E_0/\partial L^2)_N)^{-1} = ((N^2/L)(\partial^2 E_0/\partial N^2)_L)^{-1}.\) Because of \(N_c \equiv (N_{\uparrow} + N_{\downarrow})/\sqrt{2}\) the \(N\)-dependence of the ground state energy of the TL-model using Eq. (30) is given by \(v_{N_c}(\pi/4L)N^2\) leading to

\[
L \left( \frac{\partial^2 E_0}{\partial N^2} \right)_L = \frac{\pi}{2} v_{N_c} = \frac{\pi}{2} \frac{v_c}{K_c}. \tag{73}
\]

The compressibility ratio is therefore given by

\[
\frac{\kappa}{\kappa_0} = \frac{v_F}{v_c} K_c. \tag{74}
\]

The factor \(1/2\) in Eq. (73) is absent in the spinless case. We leave it as an exercise to show that one obtains similarly for the spin susceptibility

\[
\chi/\chi_0 = \frac{v_F}{v_s} K_s. \tag{75}
\]

These relations show that the \(K_s\) which determine the contributions \(\alpha_a\) to the anomalous dimension \(\alpha\) and therefore the power law behaviour of the Green’s functions also enter the low temperature thermodynamics.

It was an important insight by Haldane\(^4\) to realize that the low energy physics of the TL-model is generic for interacting fermions in one dimension. In the language of the renormalization group\(^5\) the TL-Hamiltonian is the fixed point Hamiltonian for 1d fermions with a repulsive interaction. This is the main reason for the importance of the TL-model for the physics in one dimension.

Usually the LL parameters \(K_a, v_a\) of the effective TL-model which describes the low energy physics of a given model can only be calculated approximately. An example is the original Tomonaga model with the quadratic energy dispersion (with or without spin) where it is not assumed that the interaction cutoff is much smaller than the Fermi momentum, i.e. an arbitrary repulsive interaction is used. An exception is the extreme short range limit \(V(x) = V_0 \delta(x)\). For the spinless model \(\tilde{V}\) then vanishes because the Pauli principle forbids two fermions at the same space point \((\psi(x)\psi(x) = 0\) in Eq. (1)) . For the corresponding model including spin ground state properties can be calculated exactly using the Bethe-ansatz method.\(^6\) The ratios \(K_c/v_c\) and \(K_s/v_s\) of the effective TL-model which describes the low energy physics of the model can be inferred by calculating the compressibility and spin susceptibility using Eqs. (74) and (75). The charge and spin velocities can be obtained as \((E_0^{c(s)}(2\pi/L) - E_0)/(2\pi/L)\), where \(E_0^{c(s)}(2\pi/L)\) is the lowest energy for charge (spin) excited states with total momentum \(2\pi/L\). From this information one can obtain the anomalous dimension, which determines the decay of the Green’s function without the need to calculate this function. For 1d models on a lattice, which cannot be solved exactly by the Bethe-ansatz method one can determine the LL parameters approximately by numerically calculating the quantities discussed above for small systems. In the case of weak interactions perturbation theory can be used. We show this to leading order for the spinless Tomonaga model with a short range interaction. The ground state energy up to first order in the interaction is given by the expectation value of \(H\) in the Fermi sea \(|\tilde{F}\rangle\:

\[
E_0^{(1)} = \sum_{n=-n_F}^{n_F} \frac{1}{2m_F} \frac{2\pi n}{L}^2 + \frac{1}{2L} \sum_{n=-n_F}^{n_F} \sum_{n'=-n_F}^{n_F} (v(0) - v(k_n - k_n')). \tag{76}
\]

The interaction contributes the Hartree and the Fock term. In the large system limit the sums can be replaced by integrals in the standard way and one obtains
\[ L \left( \frac{\partial^2 E_0}{\partial N^2} \right)_L = \pi v_F + v(0) - v(2k_F) \equiv \pi(v_c/K)^{(1)} . \]  

(77)

In order to obtain the charge velocity to linear order in the interaction we have to calculate the expectation value of \( H \) in the state \( \frac{2\pi}{L} c_{n_F+1}^\dagger c_{n_F} | F > \). Taking the difference with \( E_0 \) and dividing by \( 2\pi/L \) yields

\[ (v_c)^{(1)} = v_F + (v(0) - v(2k_F))/(2\pi). \]

(78)

The contribution of the interaction is due to the different Fock energies in the states \( \frac{2\pi}{L} > \) and \( | F > \). Combining the results from Eqs. (77) and (78) gives \( K = 1 - (v(0) - v(2k_F))/(2\pi v_F) + ... \) which leads to

\[ \alpha^{(2)} = \frac{1}{2} \left( \frac{v(0) - v(2k_F)}{2\pi v_F} \right)^2 . \]

(79)

If one generalizes the calculation of \( \langle \hat{n}_m \rangle^{(2)} \) in section II to the case of a short range interaction one obtains the same result for the anomalous dimension. This shows to leading order perturbation theory that the low energy physics of the Tomonaga model with an arbitrary range interaction is that of the TL model with \( g_2(0) = g_4(0) = v(0) - v(2k_F) \).

Because of the limited length of the paper the rapidly expanding field of electronic transport in one-dimensional interacting Fermi systems was not discussed. Important insights can rather quickly be obtained from the simple exercise of normal ordering of \( \psi_+^\dagger(x)\psi_-(x) \) with the use of Eq.(58) and the corresponding expression for the left movers. The interested reader should look up the relevant references in the review article cited, especially the important work by Kane and Fisher.

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APPENDIX A: THE KRONIG IDENTITY

In this appendix we generalize an identity found by Kronig in the context of the (now long forgotten) “neutrino theory of light”\[4\] to the case of a finite cut-off $k_B$. For the right movers it reads
\[
\sum_{l=1}^{\infty} lb_l^nb_l = \sum_{m=-m_0}^{\infty} (m + m_0)c_{m,+}^l + c_{m,+} - \frac{1}{2} \left(N_{+}^2 - N_{+} \right) .
\] (A.1)

For the proof we simply evaluate the lhs using Eqs. (20) and (27)
\[
\sum_{l=1}^{\infty} lb_l^nb_l = \sum_{l=1}^{\infty} \sum_{n=-m_0}^{\infty} c_{n+l,+}^l + c_{n+l,+}^l + \sum_{l=1}^{\infty} \sum_{n\neq n'} c_{n+l,+}^l + c_{n+l,+}^l + c_{n',+} + c_{n',+}
\]
\[
= \hat{A}_1 + \hat{A}_2 ,
\] (A.2)

where we have split the double sum over $n$ and $n'$ in the contributions from $n = n'$ and $n \neq n'$. We will show that the operator $\hat{A}_1$ yields the rhs of Eq. (A.1) while $\hat{A}_2$ vanishes
\[
\hat{A}_1 = \sum_{l=1}^{\infty} \sum_{n=-m_0}^{\infty} \hat{n}_{n+l,+} - \sum_{l=1}^{\infty} \sum_{n=-m_0}^{\infty} \hat{n}_{n+l,+} \hat{n}_{n,+}
\]
\[
= \sum_{m=-m_0}^{\infty} (m + m_0)\hat{n}_{m,+} - \frac{1}{2} \left(N_{+}^2 - N_{+} \right) .
\] (A.3)

The second equality follows from a change of summation variables and using $\hat{n}_{m,+}^2 = \hat{n}_{m,+}$. For the discussion of $\hat{A}_2$ we split the double sum into the contributions from $n > n'$ and $n < n'$
\[
\hat{A}_2 = \sum_{l=1}^{\infty} \sum_{n\neq n'} \left\{ \sum_{n=n'}^{\infty} c_{n+l,+}^l + c_{n,+}^l + c_{n',+} + c_{n'+l,+} + \sum_{n<n'} c_{n+l,+} + c_{n'+l,+} \right\} .
\] (A.4)

In the first term we write $n = n' + l'$ and use $l' \geq 1$ as the new summation variable and in the second term we write $n = n + l$. This yields
\[
\hat{A}_2 = \sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} \left\{ \sum_{n'=n}^{\infty} c_{n'+l+l',+}^l + c_{n'+l',+}^l + c_{n'+l',+}^l + c_{n'+l',+} + \sum_{n=-m_0}^{\infty} c_{n+l,+} + c_{n'+l'+n'+l,+} + c_{n,+} \right\} .
\] (A.5)

As the products of the four operators are both antisymmetric in $l$ and $l'$ the operator $\hat{A}_2$ vanishes after performing the summations, which completes the proof. If we multiply Eq. (A.1) with $v_F(2\pi/L)$ and use the analogous relation to (A.1) for the left movers we obtain Eq. (29) for the kinetic energy $T_{F\ell}$. We note that in the proof no use is made of the commutation relations of the $b_l$ and $b_l^\dagger$. The Kronig identity holds as an exact operator identity without performing the limit $k_B \to \infty$.

APPENDIX B: BOSONIZATION OF THE FIELD OPERATOR

In this appendix we generalize chapter V. of I on the bosonization of fermionic operators to the case of a single fermion operator. A new straightforward construction of the particle number changing part is given which simplifies earlier presentations\[5\].

We start with a simple relation which holds for operators $b, b^\dagger$ obeying $[b, b^\dagger] = \hat{1}$. It reads
\[
[b, e^{\lambda b^\dagger}] = \lambda e^{\lambda b^\dagger} .
\] (B.1)

To prove it one differentiates $b(\lambda) = e^{-\lambda b^\dagger}b e^{\lambda b^\dagger}$ with respect to $\lambda$ and obtains $db(\lambda)/d\lambda = \hat{1}$. Therefore $b(\lambda) = b + \lambda \hat{1} = e^{-\lambda b^\dagger}b e^{\lambda b^\dagger}$. Multiplying from the left with $e^{\lambda b^\dagger}$ yields (B.1). Next we define the following operators linear in the boson operators $b_n, b_n^\dagger$

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\[ A_+ = \sum_{n \neq 0} \lambda_n \hat{b}_n^\dagger + \mu_n \hat{b}_n \quad B_- = \sum_{n \neq 0} \mu_n \hat{b}_n \]  

with arbitrary constants \( \lambda_n \) and \( \mu_n \). Using (B.1) and elementary generalizations one can prove the commutation relations
\[
[b_m, e^{B_- e^{A_+}}] = \lambda_m e^{B_- e^{A_+}} \\
[b_m^\dagger, e^{B_- e^{A_+}}] = -\mu_m e^{B_- e^{A_+}}.
\]

In the following we consider fermionic operators \( \hat{S} \) which obey \( [b_m, \hat{S}] = -\lambda_m \hat{S} \) and \( [b_m^\dagger, \hat{S}] = \mu_m \hat{S} \). Then the operator \( \hat{O} \equiv \hat{S} e^{B_- e^{A_+}} \) commutes with all \( b_m \) and \( b_m^\dagger \). We therefore write
\[
\hat{S} = \hat{O} e^{-A_+ e^{-B_-}},
\]

and subsequently construct \( \hat{O} \) such that both sides of Eq. (B.4) yield identical matrix elements.

In the following we present the bosonization of the field operator for the right movers described by the \( c_{l,+} \) and just mention the corresponding result for the left movers. The \( c_{l,+} \) obey for \( m > 0 \) the following commutation relations
\[
[b_m, c_{l,+}] = -\frac{1}{\sqrt{m}} c_{l+m,+}, \quad [b_m^\dagger, c_{l,+}] = -\frac{1}{\sqrt{m}} c_{l-m,+}.
\]

The auxiliary field operator \( \tilde{\psi}_+(v) \) defined as
\[
\tilde{\psi}_+(v) = \lim_{m_0 \to \infty} \sum_{l \in \mathbb{Z}_+} e^{i l v} c_{l,+}
\]
obeys commutation relations as the operator \( \hat{S} \) discussed above
\[
[b_m, \tilde{\psi}_+(v)] = -\frac{1}{\sqrt{m}} e^{-i m v} \tilde{\psi}_+(v); \quad [b_m^\dagger, \tilde{\psi}_+(v)] = -\frac{1}{\sqrt{m}} e^{i m v} \tilde{\psi}_+(v).
\]

Therefore it is of the form
\[
\tilde{\psi}_+(v) = \hat{O}_+(v) e^{i \phi_+(v)} e^{i \tilde{\psi}_+(v)},
\]

where the operator \( i \phi_+(v) \) is given by
\[
i \phi_+(v) = \sum_{n=1}^{\infty} \frac{e^{i n v}}{\sqrt{n}} b_n.
\]

As \( \tilde{\psi}_+(v) \) reduces the number of right movers by one, the operator \( \hat{O}_+(v) \), which commutes with all \( b_m, b_m^\dagger \), also must have this property. In order to determine \( \hat{O}_+(v) \) we work with the eigenstates of the noninteracting system (compare Eq. (41))
\[
\{|m_l\}_b, \tilde{N}_+, \tilde{N}_- \} \equiv \prod_i \frac{\langle b_i^\dagger \rangle_m}{\sqrt{m!}} |\{0\}_b, \tilde{N}_+, \tilde{N}_- >.
\]

After the limit \( m_0 \to \infty \) the \( \tilde{N}_a \) can run from minus to plus infinity. It is easy to see that \( \hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- > \) has no overlap to excited states
\[
< \{m_l\}_b, \tilde{N}_+ - 1, \tilde{N}_- | \hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- > = < \{0\}_b, \tilde{N}_+ - 1, \tilde{N}_-| \prod_i \frac{\langle b_i \rangle_m}{\sqrt{m!}} \hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- >.
\]

As \( \hat{O}(v) \) commutes with the \( b_i \) the rhs of Eq. (B.11) vanishes unless all \( m_l \) are zero. This implies
\[ \hat{O}_+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- > = c_{\tilde{N}_+,\tilde{N}_-}^+(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- > , \quad (B.12) \]

where \( c_{\tilde{N}_+,\tilde{N}_-}^+(v) \) is a c-number. In order to determine the \( c_{\tilde{N}_+,\tilde{N}_-}^+(v) \) we calculate \( \langle \{0\}_b, \tilde{N}_+, \tilde{N}_-|\tilde{\psi}(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- \rangle \) using (B.6) as well as (B.8). In the calculation of the matrix element with the fermionic form (B.6) we use Eq. (40) which yields

\[ \langle \{0\}_b, \tilde{N}_+, \tilde{N}_-|\tilde{\psi}(v)|\{0\}_b, \tilde{N}_+, \tilde{N}_- \rangle = (-1)^{\tilde{N}_-} \delta_{l,l_0}. \quad (B.13) \]

The factor \((-1)^{\tilde{N}_-}\) occurs because we have to commute \( c_{l,l_0} \) through the product of \( N_- \) fermionic operators of the left movers if we assume \( m_0 \) to be odd. We note that no such factor occurs for the corresponding matrix element of the left movers. The calculation of the ground state to ground state matrix element of \( \tilde{\psi}_+(v) \) using (B.8) is simple as both exponentials involving the boson operators can be replaced by the unit operator and the matrix element is just \( c_{\tilde{N}_+,\tilde{N}_-}^+ \). The comparison therefore yields

\[ c_{\tilde{N}_+,\tilde{N}_-}^+(v) = (-1)^{\tilde{N}_-} e^{iv\tilde{N}_+} \quad (B.14) \]

and \( c_{\tilde{N}_+,\tilde{N}_-}^-(v) = e^{-iv\tilde{N}_-} \). Together with (B.12) and the definition \( \tilde{N}_3 \equiv N_3 - (m_0 + 1)\hat{l} \) this implies

\[ \hat{O}_+(v)e^{-i\tilde{N}_+ v(-1)^{\tilde{N}_-}|\{0\}_b, \tilde{N}_+, \tilde{N}_- > = |\{0\}_b, \tilde{N}_+, \tilde{N}_- > . \quad (B.15) \]

If we apply the operator \( \hat{O}_+(v)e^{-i\tilde{N}_+ v(-1)^{\tilde{N}_-} \rangle \) to the states in Eq. (B.10) and use again that \( \hat{O}_+(v) \) commutes with the boson operators we obtain

\[ \hat{O}_+(v)e^{-i\tilde{N}_+ v(-1)^{\tilde{N}_-}|\{m\}_b, \tilde{N}_+, \tilde{N}_- > = |\{m\}_b, \tilde{N}_+, \tilde{N}_- > . \quad (B.16) \]

This shows that the operator \( U_+ \equiv \hat{O}_+(v)e^{-i\tilde{N}_+ v(-1)^{\tilde{N}_-} \rangle \) is independent of \( v \) and given by

\[ U_+ = \sum_{\tilde{N}_+,\tilde{N}_-} \sum_{\{m\}} |\{m\}_b, \tilde{N}_+, \tilde{N}_- > = \langle \{m\}_b, \tilde{N}_+, \tilde{N}_- > . \quad (B.17) \]

It follows immediately that \( U_+ \) is unitary, i.e. \( U_+ U_+^\dagger = U_+^\dagger U_+ = \hat{1} \). From (B.15) one can infer that for arbitrary functions \( f \) of the number operator \( N_+ \) one has \( U_+ f(N_+) = f(N_+) U_+ \). Concerning the \( \tilde{N}_+ \)-dependence the operator \( U_+ \) resembles a one-dimensional tight-binding nearest neighbour hopping operator. Therefore the eigenstates of \( U_+ \) are just the corresponding “Blochstates”

\[ |\{m\}_b, k, N_- > = \frac{1}{\sqrt{2\pi}} \sum_{\tilde{N}_+=-\infty}^{\infty} e^{ik\tilde{N}_+} |\{m\}_b, \tilde{N}_+, \tilde{N}_- > \quad (B.18) \]

and the eigenvalue of \( U_+ \) is given by \( e^{ik} \). If one defines the operator

\[ \hat{k}_+ = \sum_{\{m\}} \sum_{\tilde{N}_-} \int_0^{\pi} |\{m\}_b, k, \tilde{N}_- > k(\{m\}_b, k, \tilde{N}_- > dk \quad (B.19) \]

the operator \( U_+ \) can be written as

\[ U_+ = e^{i\hat{k}_+} \quad (B.20) \]

In the “k-representation” the particle number operator \( N_+ \) acts like the differential operator \( i\partial/\partial k \), which seems to imply the commutation relation \( [\hat{N}_+, \hat{k}_+] = i\hat{1} \). This is not correct as an operator identity in the full Hilbert space. In the k-representation the operator \( \hat{k}_+ \) acts as a multiplication operator which destroys the 2\pi-periodicity in \( k \) unless the state \( |\psi > \) fulfills \( \langle \{m\}_b, k = \pi, N_- |\psi > = 0 = \langle \{m\}_b, k = -\pi, N_- |\psi > \). The direct calculation of the commutator using

\[ \tilde{N}_+ = \sum_{\tilde{N}_+,\tilde{N}_-} \sum_{\{m\}} |\{m\}_b, \tilde{N}_+, \tilde{N}_- > \tilde{N}_+ \langle \{m\}_b, \tilde{N}_+, \tilde{N}_- > . \quad (B.21) \]
yields

$$[\hat{N}_+, \hat{k}_+] = i \left( 1 - 2\pi \sum_{(m_l)} \sum_{\hat{N}_-} |\{m_l\}_b, k = \pi, \hat{N}_-\rangle \langle \{m_l\}_b, k = \pi, \hat{N}_-| \right)$$  \hspace{1cm} (B.22)

in contrast to statements in the literature\[8\].

To summarize we have shown that

$$\hat{O}_+(v) = U_+ e^{i\hat{N}_+ v} (-1)^{\hat{N}_-} = e^{ik_+} e^{i\hat{N}_+ v} (-1)^{\hat{N}_-}$$  \hspace{1cm} (B.23)

In the corresponding expression \(\hat{O}_-(u) = e^{ik_-} e^{-i\hat{N}_- u}\) no factor \((-1)^{\hat{N}_+}\) appears and therefore \(\hat{O}_+(v)\) and \(\hat{O}_-(u)\) anticommute, which is necessary to enforce anticommutation relations between \(\hat{\psi}_+(v)\) and \(\hat{\psi}_-(u)\). The bosonization of \(c_{l,+}\) itself is obtained by the inversion formula to (B.6)

$$c_{l,+} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\nu \hat{\psi}_+(v)} dv \hspace{1cm} (B.24)$$

We finally show how the bosonization of a single fermion operator described in this appendix relates to the bosonization of the product \(\hat{\psi}_+(u)\hat{\psi}_+(v)\) described in I. Using (B.8) and (B.21) we can write the product as

$$\hat{\psi}_+(u)\hat{\psi}_+(v) = e^{-i\phi_+(u)} e^{-i\phi_+(v)} e^{i\phi_+(u)} e^{-i\phi_+(v)} e^{-i\hat{N}_+ (u-v)}$$

$$= e^{-i(\phi_+(u) - \phi_+(v))} e^{-i(\phi_+(u) - \phi_+(v))} [\phi_+(u), \phi_+(v)] e^{-i\hat{N}_+ (u-v)} \hspace{1cm} (B.25)$$

Here we have used the Baker-Hausdorff formula \(e^A e^B = e^{B e^A e^{[A,B]}}\), valid if \([A,B]\) commutes with A and B. The commutator in the exponent is given by

$$[\phi_+(u), \phi_+(v)] = \lim_{\eta \to 0} \sum_{n,n' = 1}^{\infty} \frac{e^{inu}}{\sqrt{n}} \frac{e^{-in'v}}{\sqrt{n'}} \left[ b_n, b_n^\dagger \right] = \lim_{\eta \to 0} \sum_{n=1}^{\infty} e^{in(u-v+in)/n} \hspace{1cm} (B.26)$$

In order to make the infinite series convergent we have added a convergence factor \(e^{-\eta n}\) with \(\eta\) going to zero in the end of the calculation. This is necessary as we had to perform the limit \(m_0 \to \infty\) in Eq. (B.6), i.e. we work with an infinite Dirac see as in Luttinger’s paper\[8\]. Combining Eqs. (B.23) and (B.24) gives

$$\hat{\psi}_+(u)\hat{\psi}_+(v) = \frac{e^{-i\hat{N}_+ (u-v)}}{1 - e^{i(u-v+0)}} e^{-i(\phi_+(u) - \phi_+(v))} e^{-i(\phi_+(u) - \phi_+(v))} \hspace{1cm} (B.27)$$

which corresponds to Eqs. (47) and (49) of I.

A convergence factor also has to be introduced explicitly in the bosonization formula for the field operator (B.8) if no normal ordering is used. If one uses the Baker-Hausdorff formula and (incorrectly) assumes the commutation relation \([\hat{N}_+, \hat{k}_+] = i\hat{1}\) one obtains

$$\hat{\psi}_\alpha(u) = \lim_{\eta \to 0} \frac{1}{\sqrt{\eta}} e^{ib_\alpha(u)} \hspace{1cm} (B.28)$$

where the “phase field” operator

$$\tilde{b}_\alpha(u) \equiv \tilde{k}_\alpha + \alpha(\tilde{N}_\alpha - \frac{1}{2})u + \phi_\alpha(u) + (\phi_\alpha(u))^\dagger + \frac{\pi}{2}(1 + \alpha)\tilde{N}_-\alpha\hspace{1cm} (B.29)$$

with

$$\phi_\alpha(u) = -i \sum_{n \neq 0} \Theta(\alpha n) \frac{e^{inu}}{\sqrt{|n|}} b_n e^{-|n|\eta/2} \hspace{1cm} (B.30)$$

contains the boson as well as the particle number changing contribution. It is used in various applications\[8\] e.g. the discussion of transport. It is obviously more rigorous to use of the normal ordered form (B.8), as always done in the present paper.
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16 We should express a warning: Many authors in the present field, e.g., use a different sign convention in the exponent \( e^{i k_x x} \rightarrow e^{-i k_x x} \). This is in contradiction to the usual sign convention for a plane wave in quantum mechanics.
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