Commutant of $\hat{L}_{\mathfrak{sl}_2}(4, 0)$ in the cyclic permutation orbifold of $\hat{L}_{\mathfrak{sl}_2}(1, 0)^\otimes 4$

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Abstract

We study the commutant of the vertex operator algebra $\hat{L}_{\mathfrak{sl}_2}(4, 0)$ in the cyclic permutation orbifold model $(\hat{L}_{\mathfrak{sl}_2}(1, 0)^\otimes 4)^\tau$ with $\tau = (1\,2\,3\,4)$. It is shown that the commutant is isomorphic to a $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbifold model of a tensor product of two lattice type vertex operator algebras of rank one.

1 Introduction

Let $A_1 = \mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = 2$ be a root lattice of type $A_1$. It is well-known that the vertex operator algebra $V_{A_1}$ associated to the lattice $A_1$ is isomorphic to a simple affine vertex operator algebra $\hat{L}_{\mathfrak{sl}_2}(1, 0)$ of type $\mathfrak{sl}_2$ with level 1. For an integer $k \geq 2$, the cyclic sums of the weight one vectors in $V_{A_1}$ in the tensor product $V_{A_1}^\otimes k$ of $k$ copies of $V_{A_1}$ generate a vertex operator subalgebra isomorphic to $\hat{L}_{\mathfrak{sl}_2}(k, 0)$. The commutant $M$ of $\hat{L}_{\mathfrak{sl}_2}(k, 0)$ in $V_{A_1}^\otimes k$ has been studied well (see for example [JL], [LS], [LY]). Among other things the classification of irreducible modules for $M$ and the rationality of $M$ were established in [JL].

Let $\tau$ be a cyclic permutation on the tensor components of $V_{A_1}^\otimes k$ of length $k$. Then $\tau$ is an automorphism of the vertex operator algebra $V_{A_1}^\otimes k$ and every element of $\hat{L}_{\mathfrak{sl}_2}(k, 0)$ is fixed by $\tau$. Thus $\tau$ induces an automorphism of $M$. Our main concern is the orbifold model $M^\tau$ of $M$ by $\tau$, that is, the set of fixed points of $\tau$ in $M$, which is the commutant of $\hat{L}_{\mathfrak{sl}_2}(k, 0)$ in the orbifold model $(V_{A_1}^\otimes k)^\tau$.

The vertex operator algebra $\hat{L}_{\mathfrak{sl}_2}(k, 0)$ contains a subalgebra $T$ isomorphic to a vertex operator algebra associated to a rank one lattice generated by a square norm $2k$ element, which corresponds to a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. The commutant of $T$ in $V_{A_1}^\otimes k$ is a lattice type vertex operator algebra $V_{\mathbb{Z}A_{k-1}}$.

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On the other hand the commutant $K(\mathfrak{sl}_2, k)$ of $T$ in $\mathcal{L}_{\mathfrak{sl}_2}(k, 0)$ is called a parafermion vertex operator algebra of type $\mathfrak{sl}_2$. The parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ has been studied both in mathematics and in physics from various points of view (see for example [ALY], [DLY3], [DLWY], [DL]). We note that $M \otimes \mathcal{L}_{\mathfrak{sl}_2}(k, 0) \subset V_{A_1}^{sk}$ and $M \otimes K(\mathfrak{sl}_2, k) \subset V_{\sqrt{3}A_{k-1}}$. In fact, $M$ is the commutant of $K(\mathfrak{sl}_2, k)$ in $V_{\sqrt{3}A_{k-1}}$.

If $k = 2$, then $M$ is isomorphic to the simple Virasoro vertex operator algebra $L(\frac{3}{2}, 0)$ of central charge $\frac{1}{2}$. In this case $\tau$ acts trivially on $M$ and $M^\tau$ coincides with $M$. The first nontrivial case, that is, the orbifold model $M^\tau$ for the case $k = 3$ was studied in [DLTYY]. It was shown that $M^\tau$ is a $W_3$-algebra of central charge $\frac{6}{5}$. Furthermore, the classification of irreducible modules for $M^\tau$ was obtained and their properties were discussed in detail. Those results were used for the study of the vertex operator algebra $(V_{\sqrt{3}A_3})^\tau$ in [TY].

In this paper we consider the orbifold model $M^\tau$ for the case $k = 4$. The study of $M^\tau$ should lead to a better understanding of the structure of $(V_{\sqrt{3}A_3})^\tau$, for $M^\tau \otimes K(\mathfrak{sl}_2, 4)$ is contained in $(V_{\sqrt{3}A_3})^\tau$.

Let $L = \mathbb{Z}\alpha + \mathbb{Z}\alpha + \mathbb{Z}\alpha$ be an orthogonal sum of three copies of $\mathbb{Z}\alpha$, where $\langle \alpha, \alpha \rangle = 2$. The main idea is the use of an automorphism $\rho$ of the vertex operator algebra $V_L$ studied in [DLY2], which maps $V_N$ onto $V_L^\tau$. Here $N$ is a sublattice of $L$ isomorphic to the sublattice $\sqrt{2}A_3$ of $A_4^1$. It was shown in [DLY2] that $\rho(M) = \text{Com}_{V_L^\tau}(V_{Z\gamma})$; the commutant of $V_{Z\gamma}$ in $V_L^\tau$, where $\gamma = (\alpha, \alpha, \alpha) \in L$.

The cyclic permutation $\tau$ on the tensor components of $V_{A_4}^{\otimes 4} = V_{A_4}^1$ is a lift of an isometry of the underlying lattice $A_4^1$. We denote the isometry of $A_4^1$ by the same symbol $\tau$. The sublattice $\sqrt{2}A_3$ of $A_4^1$ is invariant under $\tau$. Hence we can discuss an isometry $\tilde{\tau}$ of $N$ corresponding to the isometry $\tau$ of $\sqrt{2}A_3$ by the isomorphism $N \cong \sqrt{2}A_3$. We extend $\tilde{\tau}$ to an isometry of $L$ and consider its lift to an automorphism of the vertex operator algebra $V_L$. We denote the automorphism by the same symbol $\tilde{\tau}$. Let $\tau' = \rho\tilde{\tau}\rho^{-1}$ be the conjugate of $\tilde{\tau}$ by $\rho$ so that $\rho(M^\tau) = \text{Com}_{V_L^\tau}(V_{Z\gamma})^\tau$. It turns out that $\text{Com}_{V_L^\tau}(V_{Z\gamma})^\tau$ can be expressed as $(V_{\gamma_1} \otimes V_{\gamma_2})^G\tilde{\tau}$, where $\gamma_1$ and $\gamma_2$ are elements of $L$ of square norm 12 and 4, respectively and $G$ is a group of automorphisms of $V_{\gamma_1} \otimes V_{\gamma_2}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is known that the vertex operator algebra $M$ is generated by the set of conformal vectors $\omega_\alpha$ of central charge $\frac{1}{2}$ associated to the positive roots $\alpha$ of type $A_3$ (see [JL], [LS]). However, it is difficult to describe the properties of the orbifold model $M^\tau$ in terms of those generators of $M$. By the result in this paper we can discuss $(V_{\gamma_1} \otimes V_{\gamma_2})^G$ instead of $M^\tau$, which seems to be easy to treat.

This paper is organized as follows. In Section 2 we review basic materials of vertex operator algebras such as conformal vectors and the commutant of a vertex operator subalgebra. In Section 3 we discuss two kinds of automor-
phisms of a vertex operator algebra $V_L$ associated to a positive definite even lattice $L$, one is a lift of the $-1$-isometry of the lattice $L$ and the other is an exponential of the operator $h_{(0)}$ for $h \in \mathbb{C} \otimes \mathbb{Z} L$. We also recall three automorphisms of a rank one lattice type vertex operator algebra $V_{Z\alpha}$ studied in [DLY1], [DLY2] for $\langle \alpha, \alpha \rangle = 2$. In Section 4 we introduce the commutant $M$ of $L_{\hat{sl}_2}(4, 0)$ in $L_{\hat{sl}_2}(1, 0)\otimes^4$ and its orbifold model $M^\tau$ by $\tau$. Finally, in Section 5 we prove that $M^\tau$ is isomorphic to a $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbifold model of a tensor product of two rank one lattice type vertex operator algebras.

The automorphism $\rho$ of the vertex operator algebra $V_L$ plays a key role in our argument. The use of $\rho$ was suggested by Ching Hung Lam. The authors are grateful to him for the important advice.

2 Preliminaries

In this section we review some basic notions and notations for vertex operator algebras (see [MN], [K], [LL]). Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra with the vacuum vector $1$ and the Virasoro vector $\omega$. We denote $\omega$ by $\omega^V$ also. The $n$-th product of $u, v \in V$ will be written as $u_{(n)}v$ for $n \in \mathbb{Z}$. We often regard $u_{(n)}$ as a $\mathbb{C}$-linear endomorphism of $V$. Two vectors $u$ and $v$ in $V$ are said to be mutually commutative if $u_{(n)}v = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. The eigenspace $V_n$ for $L_0 = \omega^{V}_{(1)}$ of eigenvalue $n \in \mathbb{Z}$ is finite dimensional. A vector in $V_n$ is said to be of weight $n$.

A vertex operator subalgebra of $V$ is a vertex subalgebra $U$ equipped with a Virasoro vector $\omega^U$. When $\omega^V = \omega^U$, $U$ is said to be full. For a pair of a vertex operator algebra $V$ and its subalgebra $U$, the subspace

$$\text{Com}_V(U) = \{v \in V \mid u_{(n)}v = 0 \text{ for } n \in \mathbb{Z}_{\geq 0} \text{ and } u \in U\}$$

becomes a vertex operator algebra with Virasoro vector $\omega^{\text{Com}_V(U)} = \omega^V - \omega^U$. We call it the commutant of $U$ in $V$. Actually, it is known that

$$\text{Com}_V(U) = \{v \in V \mid \omega^U_{(0)}v = 0\} \quad (2.1)$$

(see [FZ, Theorem 5.2]). Hence the commutant of $U$ in $V$ depends only on the Virasoro vector of $U$.

A vector $e \in V_2$ is called a conformal vector if $L^e_n = e_{(n+1)}$, $n \in \mathbb{Z}$ give a representation for the Virasoro algebra on $V$ of certain central charge. The Virasoro vector of a vertex operator subalgebra $U$ of $V$ is a conformal vector of $V$. Let $e$ be a conformal vector in $V$. For any vertex operator subalgebra $U$ with $\omega^U = e$, the commutant $\text{Com}_V(U)$ does not depend on $U$ by (2.1). In such a case we may write $\text{Com}_V(e) = \text{Com}_V(U)$.

Proposition 2.1. Let $V$ be a vertex operator algebra and $e^1, e^2$ mutually commutative conformal vectors in $V$. Then $\text{Com}_V(e^1 + e^2) = \text{Com}_{\text{Com}_V(e^1)}(e^2)$. 

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Proof. Let $U$ be a vertex subalgebra generated by $e^1$ and $e^2$. Since $e^1$ and $e^2$ are mutually commutative, $e_1 + e_2$ is the Virasoro vector of $U$. Thus we have

$$\text{Com}_V(e^1 + e^2) = \text{Com}_V(U)$$
$$= \{v \in V | e^{1}_{(n)}v = e^{2}_{(n)}v = 0 \text{ for } n \in \mathbb{Z}_{\geq 0}\}$$
$$= \{v \in \text{Com}_V(e^1) | e^{2}_{(n)}v = 0 \text{ for } n \in \mathbb{Z}_{\geq 0}\}$$
$$= \text{Com}_{\text{Com}_V(e^1)}(e^2).$$

\(\square\)

An automorphism of a vertex operator algebra $V$ is a linear isomorphism of $V$ preserving all $n$-th product, and fixing the vacuum vector $1$ and the Virasoro vector $\omega^V$. For a group $G$ consisting of automorphisms of $V$, the subset

$$V^G = \{v \in V | g(v) = v \text{ for } g \in G\}$$

is a full vertex operator subalgebra of $V$, which is called the orbifold model of $V$ by $G$. When $G = \langle \tau \rangle$ is a cyclic group, we denote $V^G$ by $V^\tau$ simply.

Let $V$ be a vertex operator algebra and $G$ an automorphism group of $V$. Let $U$ a vertex operator subalgebra of $V$ and assume that $g(\omega^U) = \omega^U$ for any $g \in G$. Then the restriction $g'$ of $g \in G$ to $\text{Com}_V(U)$ gives rise to an automorphism of $\text{Com}_V(U)$. In fact, for the automorphism group $H = \{g' | g \in G\}$ of $\text{Com}_V(U)$, we have

$$\text{Com}_V(U)^H = \text{Com}_{V^G}(\omega^U). \quad (2.2)$$

3 Lattice type vertex operator algebras and their automorphisms

In this section we discuss certain automorphisms of lattice type vertex operator algebras. Let $V_L$ be the vertex operator algebra constructed in [FLM] for a positive definite even lattice $(L, \langle , \rangle)$ of rank $d$. As a vector space $V_L$ is isomorphic to a tensor product of the symmetric algebra $S(\mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}])$ and the twisted group algebra $\mathbb{C}\{L\}$ of $L$, where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. In this paper we only consider the case where $\langle \alpha, \alpha \rangle \in 4\mathbb{Z}$ for any $\alpha \in L$ or $L$ is an orthogonal sum of rank one lattices. In such a case the central extension $\hat{L}$ of $L$ studied in [FLM] splits and $\mathbb{C}\{L\}$ is canonically isomorphic to the ordinary group algebra $\mathbb{C}[L]$. Thus we take $\mathbb{C}[L]$ in place of $\mathbb{C}\{L\}$ here. A standard basis of $\mathbb{C}[L]$ is denoted by $\{e^\alpha | \alpha \in L\}$ with multiplication $e^\alpha e^\beta = e^{\alpha + \beta}$. The vacuum vector of $V_L$ is $1 = 1 \otimes e^0$, and the Virasoro vector is given by

$$\omega^{V_L} = \frac{1}{2} \sum_{i=1}^{d} (h_i \otimes t^{-1})^2 \otimes e^0,$$
where \( \{h_1, \ldots, h_d\} \) is an orthonormal basis of \( \mathfrak{h} \). Every eigenvalue for \( L_0 = \omega^{V_L}_{(1)} \) on \( V_L \) is a nonnegative integer and the eigenspace \((V_L)_n\) with eigenvalue \( n \) is finite dimensional.

For simplicity, we regard the sets \( L, \mathfrak{h} \) and \( \{e^\alpha | \alpha \in L\} \) as subsets of \( V_L \), respectively, under the identification \( \alpha = (\alpha \otimes 1)(-1)1 \), \( h = h(-1)1 \), \( e^\alpha = 1 \otimes e^\alpha \) for \( \alpha \in L \) and \( h \in \mathfrak{h} \). Then we have \((V_L)_0 = \mathbb{C}1\) and

\[
(V_L)_1 = \mathfrak{h} \oplus \bigoplus_{\langle \alpha, \alpha \rangle = 2} \mathbb{C}e^\alpha.
\]

In fact, the weight of \( e^\alpha \) is \( \frac{1}{2}\langle \alpha, \alpha \rangle \) for \( \alpha \in L \).

We will need two kinds of automorphisms of \( V_L \). One is an involution \( \theta_L \) given by a lift of the \(-1\)-isometry of the lattice \( L \). We have

\[
\theta_L(\beta) = -\beta, \quad \theta_L(e^\beta) = e^{-\beta}
\]

for \( \beta \in L \). The set \((V_L)^{\theta_L}\) of fixed points of \( \theta_L \) is also denoted by \( V_L^+ \). The other is an inner automorphism \( I_h = \exp(2\pi\sqrt{-1}h(0)) \) for \( h \in \mathfrak{h} \). We have

\[
I_h(\beta) = \beta, \quad I_h(e^\beta) = e^{2\pi\sqrt{-1}\langle h, \beta \rangle}e^\beta
\]

for \( \beta \in L \). In particular,

\[
I_{\alpha/2\langle \alpha, \alpha \rangle}(e^{m\alpha}) = (-1)^m e^{m\alpha}
\]

for \( m \in \mathbb{Z} \) and \( 0 \neq \alpha \in L \). Since \( I_{h_1}I_{h_2} = I_{h_1+h_2} \) for \( h_1, h_2 \in \mathfrak{h} \), the automorphism \( I_h \) is of finite order if and only if \( h \in \mathbb{T}L \) for some \( T \in \mathbb{Z}_{>0} \). By the definition of \( \theta_L \) and \( I_h \), we see that

\[
\theta_L I_h \theta_L = I_{-h}
\]

for any \( h \in \mathfrak{h} \). Therefore,

\[
I_{-\frac{\alpha}{2\langle \alpha, \alpha \rangle}}(I_h \theta_L I_{\frac{\alpha}{2\langle \alpha, \alpha \rangle}}) = \theta_L.
\]

That is, \( I_h \theta_L \) is conjugate to \( \theta_L \) in \( \text{Aut}(V_L) \).

Let \( \mathbb{Z}\alpha \) be a rank one lattice with \( \langle \alpha, \alpha \rangle = 2 \), which is the root lattice of type \( A_1 \). In [DLY1, Section 2], three involutions \( \theta_1, \theta_2 \) and \( \sigma \) of \( V_{\mathbb{Z}\alpha} \) are considered. The involutions \( \theta_1 \) and \( \theta_2 \) are expressed as

\[
\theta_1 = I_{\frac{\alpha}{2}}, \quad \theta_2 = \theta_{\alpha},
\]

and \( \sigma \) is a unique extension of the involution of the Lie algebra \((V_{\mathbb{Z}\alpha})_1 \cong \mathfrak{sl}_2(\mathbb{C})\) given by

\[
\sigma(\alpha) = E^\alpha, \quad \sigma(E^\alpha) = \alpha, \quad \sigma(F^\alpha) = -F^\alpha,
\]

where \( E^\alpha = e^\alpha + e^{-\alpha} \) and \( F^\alpha = e^\alpha - e^{-\alpha} \). As automorphisms of \( V_{\mathbb{Z}\alpha} \), we have

\[
\sigma \theta_{\mathbb{Z}\alpha} \sigma = I_{\frac{\alpha}{4}}.
\]
4 Orbifold model $M^\tau$

In this section we introduce an orbifold model $M^\tau$. We use the notation $X_N$ to denote the root lattice of type $X_N$. We also write $X^i_N$ for an orthogonal sum of $i$ copies of the root lattice $X_N$.

For simplicity, we write $\mathcal{L}(k, 0)$ for the simple affine vertex operator algebra $\hat{\mathfrak{sl}}_2(k, 0)$ associated to $\mathfrak{sl}_2(\mathbb{C})$ of positive integer level $k$. It is well-known that $\mathcal{L}(1, 0)$ is isomorphic to the lattice type vertex operator algebra $V_{A_1}$ associated to the root lattice of type $A_1$. Thus we have natural isomorphisms

$$\mathcal{L}(1, 0)^{\otimes 4} \cong V_{A_1}^{\otimes 4} \cong V_{A_1}$$

of vertex operator algebras. Let $\tau$ be a cyclic permutation on $\mathcal{L}(1, 0)^{\otimes 4}$ defined by

$$\tau(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_4 \otimes a_1 \otimes a_2 \otimes a_3$$

for $a_i \in \mathcal{L}(1, 0)$. In fact, $\tau$ is a lift of an isometry

$$\tau : \alpha_i \mapsto \alpha_{i+1}, \quad i \in \mathbb{Z}/4\mathbb{Z}$$

of the lattice

$$A_4^4 = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$$

with $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$.

Set

$$H = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$

and consider a sublattice $\sqrt{2}A_3 = \sum_{i,j=1}^{4} \mathbb{Z}(\alpha_i - \alpha_j)$ of $A_4^4$. We note that $\mathbb{Z}H$ and $\sqrt{2}A_3$ are mutually orthogonal and that $H \equiv 4\alpha_1$ modulo $\sqrt{2}A_3$. Hence $A_4^4 = \bigoplus_{i=0}^{3} (i\alpha_1 + \mathbb{Z}H + \sqrt{2}A_3)$ and

$$V_{A_4^4} = \bigoplus_{i=0}^{3} V_{i\alpha_1 + \mathbb{Z}H + \sqrt{2}A_3}$$

as $V_{\mathbb{Z}H + \sqrt{2}A_3}$-modules. It follows that

$$\text{Com}_{V_{A_4^4}}(V_{\mathbb{Z}H}) = V_{\sqrt{2}A_3}.$$

Since $\tau$ leaves $V_{\mathbb{Z}H}$ invariant, $\tau$ induces an automorphism of the vertex operator algebra $V_{\sqrt{2}A_3}$, which will be also denoted by $\tau$. This automorphism is a lift of the restriction of the isometry $\tau$ (4.2) of $A_4^4$ to its sublattice $\sqrt{2}A_3$:

$$\tau(\alpha_i - \alpha_{i+1}) = \alpha_{i+1} - \alpha_{i+2}, \quad i \in \mathbb{Z}/4\mathbb{Z}. \quad (4.4)$$

Since $V_{\mathbb{Z}H}$ is contained in $(V_{A_4^4})^\tau$, it follows from (2.2) that

$$(V_{\sqrt{2}A_3})^\tau = \text{Com}_{(V_{A_4^4})^\tau}(V_{\mathbb{Z}H})$$
We next take two vectors
\[ E = e^{\alpha_1} + e^{\alpha_2} + e^{\alpha_3} + e^{\alpha_4}, \quad F = e^{-\alpha_1} + e^{-\alpha_2} + e^{-\alpha_3} + e^{-\alpha_4} \]
in \((V_{A_1^4})^\tau\). Then the set \( \{E, H, F\} \) generates a vertex operator subalgebra \( U \) of \((V_{A_1^4})^\tau\) isomorphic to \( L(4, 0) \). We consider the commutant
\[ M = \text{Com}_{V_{A_1^4}}(U) \cong \text{Com}_{L(1, 0)^4}(L(4, 0)). \]

We note that \( U \) contains \( V_{ZH} \) as a vertex operator subalgebra. The commutant
\[ K_0 = \text{Com}_U(V_{ZH}) \]
has been studied in [ALY], [DLY3] and [DLWy]. The Virasoro vector of \( K_0 \) is \( \omega_{K_0} = \omega_U - \omega_{VZH} \). Since \( \omega_{K_0} \) and \( \omega_{VZH} \) are mutually commutative conformal vectors and since \( \text{Com}_{V_{A_1^4}}(\omega_{VZH}) = V_{\sqrt{2}A_3} \), we see that
\[ M = \text{Com}_{V_{A_1^4}}(\omega_U) = \text{Com}_{V_{\sqrt{2}A_3}}(\omega_{K_0}) \]
by Proposition 2.1. Since \( U \) is contained in \((V_{A_1^4})^\tau\), we have
\[ M^\tau = \text{Com}(V_{\sqrt{2}A_3})^\tau(\omega_{K_0}) = \left(\text{Com}_{V_{\sqrt{2}A_3}}(\omega_{K_0})\right)^\tau. \]

5 Main results

In this section we show that \( M^\tau \) is isomorphic to a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-orbifold model of a tensor product of two lattice type vertex operator algebras of rank one. Following [DLY2], we study an isomorphism between the vertex operator algebras \( V_{\sqrt{2}A_3} \) and \( V_{A_1^4}^+ \). We remark that another isomorphism was considered in [DLY1] (see [DLY2, Remark 3.2]).

Throughout this section, let
\[ L = \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha \]
be an orthogonal sum of three copies of \( \mathbb{Z}\alpha \), where \( \langle \alpha, \alpha \rangle = 2 \). Set
\[ \alpha^{(1)} = (\alpha, 0, 0), \quad \alpha^{(2)} = (0, \alpha, 0), \quad \alpha^{(3)} = (0, 0, \alpha), \]
so that \( L = \mathbb{Z}\alpha^{(1)} + \mathbb{Z}\alpha^{(2)} + \mathbb{Z}\alpha^{(3)} \) and \( \langle \alpha^{(i)}, \alpha^{(j)} \rangle = 2\delta_{ij} \). The vertex operator algebra \( V_L \) is isomorphic to the tensor product \( V_{A_1^4}^{\otimes 3} \) of three copies of \( V_{A_1} = V_{Z\alpha} \). Hence the involution \( \sigma \) of \( V_{Z\alpha} \) defined in (3.2) induces naturally an involution \( \sigma \otimes \sigma \otimes \sigma \) of \( V_L \). Let
\[ \rho = I_4^{(\alpha^{(2)}+\alpha^{(3)})}(\sigma \otimes \sigma \otimes \sigma) \in \text{Aut}(V_L) \]
(5.1)
be a composite of $\sigma \otimes \sigma \otimes \sigma$ and the inner automorphism $I_{\frac{1}{4}(\alpha^{(2)}+\alpha^{(3)})}$ of $V_L$ with respect to $\frac{1}{4}(\alpha^{(2)}+\alpha^{(3)}) \in \frac{1}{4}L$ (see [DLY2, Section 3]). We note that $I_{\frac{1}{4}(\alpha^{(2)}+\alpha^{(3)})} = 1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha}$. By the definition of $\rho$, we have

$$\rho(\alpha^{(1)}) = E\alpha^{(1)}, \quad \rho(\alpha^{(i)}) = -E\alpha^{(i)} \quad (i = 2, 3),$$

$$\rho(E\alpha^{(i)}) = \alpha_i \quad (i = 1, 2, 3),$$

$$\rho(F\alpha^{(i)}) = -F\alpha^{(i)}, \quad \rho(F\alpha^{(i)}) = F\alpha_i \quad (i = 2, 3).$$

(5.2) (5.3) (5.4)

Set

$$\beta_1 = \alpha^{(1)} + \alpha^{(2)}, \quad \beta_2 = -\alpha^{(2)} + \alpha^{(3)}, \quad \beta_3 = -\alpha^{(1)} + \alpha^{(2)},$$

$$\gamma = \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}.$$

Then the set $\{\sqrt{2}\beta_1, \sqrt{2}\beta_2, \sqrt{2}\beta_3\}$ forms the set of simple roots of type $A_3$. We consider the sublattice $N = \sum_{i,j=1}^3 \mathbb{Z}(\alpha^{(i)} \pm \alpha^{(j)}) = \langle \beta_1, \beta_2, \beta_3 \rangle \mathbb{Z}$

of $L$. It is known that $N \cong \sqrt{2}D_3$ is isomorphic to the sublattice $\sqrt{2}A_3$ of the lattice $A_4^4$ (4.3) discussed in Section 4 by the correspondence

$$\beta_1 \leftrightarrow \alpha_1 - \alpha_2, \quad \beta_2 \leftrightarrow \alpha_2 - \alpha_3, \quad \beta_3 \leftrightarrow \alpha_3 - \alpha_4.$$ (5.5)

This isomorphism between the lattices $N$ and $\sqrt{2}A_3$ induces an isomorphism between the vertex operator algebras $V_N$ and $V_{\sqrt{2}A_3}$. Thus we can think of the vertex operator subalgebras $M$ and $K_0$ of $V_{\sqrt{2}A_3}$ discussed in Section 4 as vertex operator subalgebras of $V_N$.

We need the following facts in [DLY2].

**Theorem 5.1.** (1) $\rho(V_N) = V_L^+$. 
(2) $\rho(\omega^{K_0}) = \omega^{V_{\sqrt{2}A_3}}$. 
(3) $\rho(M) = \text{Com}_{V_L^+}(\omega^{V_{\sqrt{2}A_3}})$.

**Proof.** The assertions (1) and (2) follow from [DLY2, Lemmas 3.4 (3) and 3.3 (3)]. Then the assertion (3) follows from (4.5). \qed

Under the isomorphism (5.5) between $N$ and $\sqrt{2}A_3$, the isometry $\tau$ (4.4) of the lattice $\sqrt{2}A_3$ corresponds to an isometry

$$\beta_1 \mapsto \beta_2 \mapsto \beta_3 \mapsto -\alpha^{(2)} - \alpha^{(3)} \mapsto \beta_1$$

of the lattice $N$. This isometry of $N$ is the restriction of an isometry $\tilde{\tau}$ of the lattice $L$ of order 4 given by

$$\tilde{\tau} : \alpha^{(1)} \mapsto \alpha^{(3)} \mapsto -\alpha^{(1)} \mapsto -\alpha^{(3)} \mapsto \alpha^{(1)}, \quad \alpha^{(2)} \mapsto -\alpha^{(2)}.$$
The isometry $\tilde{\tau}$ of $L$ lifts to an automorphism of the vertex operator algebra $V_L$ of order 4, which is also denoted by $\tilde{\tau}$.

Actually,

$$\tilde{\tau} = (\theta_{Z\alpha} \otimes \theta_{Z\alpha} \otimes 1)t_{13}$$

is a composite of $t_{13}$ and $\theta_{Z\alpha} \otimes \theta_{Z\alpha} \otimes 1$, where $t_{13}$ denotes the transposition of the first component and the third one of the tensor product $V_L = V_{A_1}^{\otimes 3}$. By direct calculations, we have

$$\tilde{\tau}(\alpha^{(1)}) = \alpha^{(3)}, \quad \tilde{\tau}(\alpha^{(2)}) = -\alpha^{(2)}, \quad \tilde{\tau}(\alpha^{(3)}) = -\alpha^{(1)}, \quad (5.6)$$

$$\tilde{\tau}(e^{\pm\alpha^{(1)})} = e^{\pm\alpha^{(3)}}, \quad \tilde{\tau}(e^{\pm\alpha^{(2)})} = e^{\mp\alpha^{(2)}}, \quad \tilde{\tau}(e^{\pm\alpha^{(3)})} = e^{\mp\alpha^{(1)}}. \quad (5.7)$$

Now we consider the conjugate $\tau'$ of $\tilde{\tau}$ by $\rho$ (5.1).

$$\tau' = \rho\tilde{\tau}\rho^{-1} \in \text{Aut}(V_L).$$

**Lemma 5.2.** As automorphisms of $V_L$, we have

$$\tau' = (1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13}. \quad (5.8)$$

**Proof.** Using (3.3), we can calculate as follows.

$$\tau' = \rho\tilde{\tau}\rho^{-1}$$

$$= I_{\frac{1}{4}(a_1 + a_3)}\sigma\tilde{\tau}\sigma I_{\frac{1}{4}(a_1 + a_3)}$$

$$= (1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})(\sigma \otimes \sigma \otimes \sigma)(\theta_{Z\alpha} \otimes \theta_{Z\alpha} \otimes 1)t_{13}$$

$$\circ (\sigma \otimes \sigma \otimes \sigma)(1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})$$

$$= (\sigma\theta_{Z\alpha} \otimes I_{\frac{1}{4}\alpha}\sigma\theta_{Z\alpha} \otimes I_{\frac{1}{4}\alpha}\sigma)(1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13}$$

$$= (\sigma\theta_{Z\alpha}\sigma I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha}\sigma\theta_{Z\alpha}\sigma I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13}$$

$$= (1 \otimes I_{\frac{1}{4}\alpha} \otimes I_{\frac{1}{4}\alpha})t_{13}. \quad \square$$

Recall that $\gamma = \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}$. We set

$$\gamma_1 = \beta_2 - \beta_3 = \alpha^{(1)} - 2\alpha^{(2)} + \alpha^{(3)}, \quad \gamma_2 = -\beta_2 - \beta_3 = \alpha^{(1)} - \alpha^{(3)},$$

and consider $P = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 + \mathbb{Z}\gamma$. Since $\gamma_1, \gamma_2$ and $\gamma$ are mutually orthogonal, we have $V_P = V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2} \otimes V_{\mathbb{Z}\gamma}$. One can easily see that

$$P = (\alpha^{(1)} - \alpha^{(3)}, \alpha^{(2)} + 2\alpha^{(3)}, 6\alpha^{(3)})\mathbb{Z}. \quad (5.9)$$

Hence we have a coset decomposition

$$L = \cup_{i=0}^{5}(ia^{(3)} + P).$$
Since $\alpha^{(3)} = \frac{1}{6}\gamma_1 - \frac{1}{2}\gamma_2 + \frac{1}{3}\gamma$, we have
\[
3\alpha^{(3)} + P = \left(\frac{1}{2}\gamma_1 + \mathbb{Z}\gamma_1\right) + \left(\frac{1}{2}\gamma_2 + \mathbb{Z}\gamma_2\right) + \mathbb{Z}\gamma.
\]
Therefore,
\[
\text{Com}_V(V_{\mathbb{Z}\gamma}) = \text{Com}_{V_P \oplus V_{\alpha^{(3)} + P}}(V_{\mathbb{Z}\gamma}) = (V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}) \oplus (V_{\frac{1}{3}\gamma_1 + \mathbb{Z}\gamma_1} \otimes V_{\frac{1}{3}\gamma_2 + \mathbb{Z}\gamma_2}).
\]

**Lemma 5.3.**
1. The eigenvalues for $\tau'$ on $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ are $\pm 1$.
2. The eigenvalues for $\tau'$ on $V_{\frac{1}{3}\gamma_1 + \mathbb{Z}\gamma_1} \otimes V_{\frac{1}{3}\gamma_2 + \mathbb{Z}\gamma_2}$ are $\pm \sqrt{-1}$.
3. $(V_P \oplus V_{\alpha^{(3)} + P})' = (V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2})' \otimes V_{\mathbb{Z}\gamma}$.

**Proof.** Recall that we regard $L$ as a subset of $V_L$. Under the canonical identification between $V_L$ and $V_{\mathbb{A}_1}^\otimes 3$, we have
\[
\begin{align*}
\gamma_1 &= \alpha \otimes 1 \otimes 1 - 2(1 \otimes \alpha \otimes 1) + 1 \otimes 1 \otimes \alpha, \\
\gamma_2 &= \alpha \otimes 1 \otimes 1 - 1 \otimes 1 \otimes \alpha, \\
\gamma &= \alpha \otimes 1 \otimes 1 + 1 \otimes \alpha \otimes 1 + 1 \otimes 1 \otimes \alpha, \\
e^{\pm \gamma_1} &= e^{\pm \alpha} \otimes e^{\mp 2\alpha} \otimes e^{\pm \alpha}, \\
e^{\pm \gamma_2} &= e^{\pm \alpha} \otimes 1 \otimes e^{\mp \alpha}, \\
e^{\pm \gamma} &= e^{\pm \alpha} \otimes e^{\pm \alpha} \otimes e^{\pm \alpha},
\end{align*}
\]
respectively. Then by Lemma 5.2, we have
\[
\begin{align*}
\tau'(\gamma_1) &= \gamma_1, & \tau'(\pm \gamma_1) &= -e^{\pm \gamma_1}, \\
\tau'(\gamma_2) &= -\gamma_2, & \tau'(\pm \gamma_2) &= -e^{\mp \gamma_2}, \\
\tau'(\gamma) &= \gamma, & \tau'(\pm \gamma) &= e^{\pm \gamma}.
\end{align*}
\]
Since $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ is generated by the set $\{\gamma_1, \gamma_2, e^{\pm \gamma_1}, E^{\gamma_2}, F^{\gamma_2}\}$ consisting of eigenvectors for $\tau'$ whose eigenvalues are $\pm 1$, we have the assertion (1).

We note that $V_{\frac{1}{3}\gamma_1 + \mathbb{Z}\gamma_1} \otimes V_{\frac{1}{3}\gamma_2 + \mathbb{Z}\gamma_2}$ is an irreducible $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$-module and so it is generated by a nonzero vector $u = e^{\frac{1}{3}(\gamma_1 + \gamma_2)} + \sqrt{-1}e^{\frac{1}{3}(\gamma_1 - \gamma_2)}$. Since
\[
\begin{align*}
\tau'(e^{\frac{1}{3}(\gamma_1 + \gamma_2)}) &= \tau'(e^{\alpha^{(1)} + \alpha^{(2)}}) = e^{-\alpha^{(2)} + \alpha^{(3)}} = e^{\frac{1}{3}(\gamma_1 - \gamma_2)}, \\
\tau'(e^{\frac{1}{3}(\gamma_1 - \gamma_2)}) &= \tau'(e^{-\alpha^{(2)} + \alpha^{(3)}}) = -e^{\alpha^{(1)} - \alpha^{(2)}} = -e^{\frac{1}{3}(\gamma_1 + \gamma_2)}
\end{align*}
\]
we have $\tau'(u) = \sqrt{-1}u$. Hence by (1), the assertion (2) holds.

The assertion (3) follows from (1), (2) and (5.13).

Here we note that $\langle \gamma_1, \gamma_1 \rangle = 12$ and $\langle \gamma_2, \gamma_2 \rangle = 4$. Let
\[
g = I_{\frac{1}{3}\gamma_1} \otimes (I_{\frac{1}{3}\gamma_2} \theta_{\mathbb{Z}\gamma_2}) \in \text{Aut}(V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}).
\]
Then by (5.11) and (5.12), the restriction of $\tau'$ to $V_{\mathbb{Z}\gamma_1} \otimes V_{\mathbb{Z}\gamma_2}$ coincides with the automorphism $g$. Hence (5.10) and Lemma 5.3 imply the following proposition.
Proposition 5.4. Let $L$ and $\tau'$ be as above. Then
\[
\text{Com}_{V_L}(V_{Z\gamma})^{\tau'} = (V_{Z\gamma_1} \otimes V_{Z\gamma_2})^g.
\]

Let $M$ and $\tau$ be as in Section 4. Then we see that
\[
\rho(M^\tau) = \text{Com}_{V_{L^+}}(V_{Z\gamma})^{\tau'}
= \text{Com}_{V_L}(V_{Z\gamma})^{(\tau', \theta')}
= (V_{Z\gamma_1} \otimes V_{Z\gamma_2})^{(g, \theta')}
\]
by the definition of $\tau'$, Theorem 5.1 and Proposition 5.4, where
\[
\theta' = \theta_{Z\gamma_1} \otimes \theta_{Z\gamma_2} \in \text{Aut } (V_{Z\gamma_1} \otimes V_{Z\gamma_2}).
\]

Let $G = \langle g, \theta' \rangle$ be a subgroup of $\text{Aut } (V_{Z\gamma_1} \otimes V_{Z\gamma_2})$ generated by $g$ and $\theta'$. We note that
\[
G = \langle (I_{\frac{1}{2}}_{\gamma_1} \theta_{Z\gamma_1}) \otimes I_{\frac{1}{2}}_{\gamma_2}, I_{\frac{1}{2}}_{\gamma_1} \otimes (I_{\frac{1}{2}}_{\gamma_2} \theta_{Z\gamma_2}) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\]

Theorem 5.5. Let $M = \text{Com}_{V_{A_4^4}}(L(4,0))$ and $\tau$ a cyclic permutation of $V_{A_4^4}$ of length 4 as in Section 4. Let $G$ be as above. Then $M^\tau$ is isomorphic to the orbifold model $(V_{Z\gamma_1} \otimes V_{Z\gamma_2})^G$.

Let
\[
f = I_{\frac{1}{2}}_{\gamma_1} \otimes I_{\frac{1}{2}}_{\gamma_2} \in \text{Aut } (V_{Z\gamma_1} \otimes V_{Z\gamma_2}).
\]
By (3.1), we have
\[
f^{-1}((I_{\frac{1}{2}}_{\gamma_1} \theta_{Z\gamma_1}) \otimes I_{\frac{1}{2}}_{\gamma_2})f = \theta_{Z\gamma_1} \otimes I_{\frac{1}{2}}_{\gamma_2},
\]
\[
f^{-1}(I_{\frac{1}{2}}_{\gamma_1} \otimes (I_{\frac{1}{2}}_{\gamma_2} \theta_{Z\gamma_2}))f = I_{\frac{1}{2}}_{\gamma_1} \otimes \theta_{Z\gamma_2}.
\]

Let $g_1 = \theta_{Z\gamma_1} \otimes I_{\frac{1}{2}}_{\gamma_2}$ and $g_2 = I_{\frac{1}{2}}_{\gamma_1} \otimes \theta_{Z\gamma_2}$. Then $\langle g_1, g_2 \rangle = f^{-1}Gf \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and the following corollary holds.

Corollary 5.6. The vertex operator algebra $M^\tau$ is isomorphic to the orbifold model $(V_{Z\gamma_1} \otimes V_{Z\gamma_2})^{(g_1, g_2)}$.

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