Transverse Josephson effect due to spin-orbit coupling: Generation of transverse current without time-reversal symmetry breaking

Takehito Yokoyama
Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
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We investigate transverse Josephson current in superconductor/normal metal/superconductor junctions where the normal metal has Rashba type spin-orbit coupling. It is shown that transverse current arises from the spin-orbit coupling in the normal metal. This effect is specific to superconducting current and the transverse current vanishes in the normal state. In addition, this transverse Josephson effect is purely stationary and applied magnetic field is unnecessary to realize this effect, in contrast to the Hall effect in the normal state. We also discuss physical interpretation of this effect, comparing with the spin Hall effect.

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When magnetic field is applied to electrons, the Lorentz force acts on the electrons and a voltage drop appears in the direction perpendicular to the applied current. This is the celebrated (classical) Hall effect. About 100 years later, the quantum Hall effect has been discovered. When a strong magnetic field is applied to 2D electron gas perpendicularly, the longitudinal resistance vanishes while the Hall conductance is quantized to a rational multiple of $e^2/h$. The classical and quantum Hall effects arise by applying magnetic field. Therefore, the Hall effect occurs together with the time-reversal symmetry breaking. Several authors have proposed the models which exhibit a nonzero quantization of the Hall conductance by a spatially inhomogeneous magnetic flux with zero average and hence without Landau levels. In these models, time-reversal symmetry is also broken due to the inhomogeneous magnetic flux. On the other hand, there have been a few efforts to realize the Hall effect without any magnetic field (flux). It has been shown that a circularly polarized light radiation can induce Hall current in Rashba spin-orbit coupled metal and graphene. Since circularly polarized light is described by ac electric field, the Hall effect predicted in these works also requires the time-reversal symmetry breaking.

In this paper, we study transverse Josephson current in superconductor/normal metal/superconductor junctions where the normal metal has Rashba type spin-orbit coupling. It is shown that transverse current arises from the spin-orbit coupling in the normal metal under phase gradient, and the transverse current vanishes in the normal state. In addition, this transverse Josephson effect is purely stationary and applied magnetic field is unnecessary to realize this effect, in contrast to the Hall effect in the normal state. We also discuss physical interpretation of this effect, comparing with the spin Hall effect.

We consider a superconductor/normal metal/superconductor junction. The Hamiltonian of the superconductor and the normal metal are given by $H_S = H_0 + H_N$ and $H_N = H_0 + H_{so} + H_E + H_{so-bedroom}$, respectively. The superconductor and the normal metal are coupled via the tunneling Hamiltonian $H_T$ and the junction is described by a two barrier model. The total Hamiltonian of the system is thus given by $H_S + H_N + H_T$. The $H_0$, $H_{\Delta}$ and $H_{so}$ represent the kinetic energy, the superconducting order, and the Rashba type spin-orbit coupling, respectively:

$$H_0 = \sum_k \phi^\dagger_k \xi_0 \otimes \tau_3 \phi_k,$$  \hspace{1cm} (1)

$$H_{\Delta} = \sum_k \phi^\dagger_k \Delta \otimes \tau_1 \phi_k,$$  \hspace{1cm} (2)

$$H_{so} = -\sum_k E_{so} \cdot \phi^\dagger_k (k \times \sigma) \otimes \tau_3 \phi_k,$$  \hspace{1cm} (3)

with $\xi = \varepsilon_k - \varepsilon_F = \frac{\hbar^2 k^2}{2m} - \varepsilon_F$ and $\phi^\dagger_k = (c^\dagger_{k \uparrow}, c^\dagger_{k \downarrow}, ic_{-k \downarrow}, -ic_{-k \uparrow})$ where $\sigma$ and $\tau$ are Pauli matrices in spin and Nambu spaces, respectively. $\varepsilon_F$, $\Delta$, and $E_{so}$ are the Fermi energy, the gap function, and the vector pointing in the direction of the inversion symmetry breaking which characterizes the Rashba type spin-orbit coupling, respectively. Note that we adopt the basis in Ref. such that singlet pairing is proportional to the unit matrix in spin space. We consider Josephson current induced by phase gradient. The phase gradient along $j$ direction, $\nabla_j \varphi$, enters the Hamiltonian as follows

$$H_{\varphi} = \sum_k \phi^\dagger_k \frac{\hbar^2}{m} k_j \nabla_j \varphi \sigma_0 \otimes \tau_0 \phi_k,$$  \hspace{1cm} (4)

$$H_{so-\varphi} = -\sum_k E_{so} \cdot \phi^\dagger_k \nabla_j \varphi (e_j \times \sigma) \otimes \tau_0 \phi_k$$  \hspace{1cm} (5)

where $\varphi$ is half the phase of superconducting correlation and $\nabla_j \varphi$ is assumed to be spatially constant. We will treat $H_{so}$, $H_{\varphi}$, and $H_{so-\varphi}$ perturbatively. We schematically show the model in Fig. 1.

With the above Hamiltonians, the charge current op-
We consider the unper-reversal symmetry. The charge current \( j_c \) flows perpendicularly to the phase gradient \( \nabla \varphi \).

\[
j_{c,i} = -\frac{e^2}{m} \rho_i \cdot \mathbf{A}
\]

where \( e \) is the electron charge.

Before proceeding to the explicit calculation, let us discuss transverse current qualitatively based on the time-reversal symmetry. The charge current \( j_c \) is time-reversal odd while the electric field is even under time-reversal. Since the Ohm’s law relates quantities of different symmetries under time-reversal, the charge conductivity \( \sigma \) breaks the time-reversal symmetry and describes the inevitable joule heating and dissipation. A transverse current can flow under the applied electric field, which is hence dissipationless, but the time-reversal symmetry is compensated by the external magnetic field. Therefore, the Hall effect in the normal state occurs, inevitably accompanied by the time-reversal symmetry breaking.

On the other hand, let us consider the London equation, i.e., the response equation of supercurrent,

\[
j_c = \sigma \cdot \mathbf{E}
\]

where \( j_c \) is the charge current and \( \mathbf{E} \) is the applied electric field. The charge current is time-reversal odd while the electric field is even under time-reversal. Since the Ohm’s law relates quantities of different symmetries under time-reversal, the charge conductivity \( \sigma \) breaks the time-reversal symmetry and describes the inevitable joule heating and dissipation. A transverse current can flow under the applied electric field, which is hence dissipationless, but the time-reversal symmetry is compensated by the external magnetic field. Therefore, the Hall effect in the normal state occurs, inevitably accompanied by the time-reversal symmetry breaking.

We now calculate transverse Josephson current and give the analytical expression. We consider the unperturbed advanced Green’s functions in the normal metal of the form

\[
g_{k,\omega}^a = g_{0,k,\omega}^a \sigma_0 \otimes \tau_0 + g_{3,k,\omega}^a \sigma_0 \otimes \tau_3 + j_{k,\omega}^a \sigma_0 \otimes \tau_1
\]

where \( g_{0,k,\omega}^a \) and \( g_{3,k,\omega}^a \) are normal Green’s functions, while \( j_{k,\omega}^a \) is anomalous Green’s function. The anomalous Green’s function is in general nonzero in the normal metal due to the proximity effect. We perform perturbative calculation with respect to \( \mathbf{E}_\omega \) and \( \nabla \varphi \) up to a second and a first order, respectively. Diagrammatic representation of the transverse current is shown in Fig. 2. The transverse Josephson current can be expressed as

\[
j_{c,i} = \frac{i h^2 c}{m V} \sum_k \text{Tr} \mathbf{k}_i \sigma_0 \otimes \tau_0 G_{k,k}^{\omega}(t,t)
\]

where \( V \) is the total volume and \( \text{Tr} \) is taken over spin and Nambu spaces. \( G_{k,k}^{\omega}(t,t) \) is the lesser Green’s function of the total Hamiltonian. Performing perturbation with respect to \( H_{cl}, H_\omega \) and \( H_{so-\varphi} \), we expand the lesser component using the advanced Green’s functions by the Lan-greth theorem.\textsuperscript{11} Noting that \( g_{k,\omega}^{\omega} = f_\omega \left[ g_{k,\omega}^a - (g_{k,\omega}^a)^\dagger \right] \) with the lesser Green’s function \( g_{k,\omega}^a \) and the Fermi distribution function \( f_\omega \), and \( \delta_{ij} = \frac{\partial}{\partial \delta_{ij}} \), we can compute the transverse Josephson current. Then, the leading term of

![Diagram](image1.png)

**FIG. 1:** (Color online) Schematic picture of the model of superconductor/normal metal/superconductor junctions where the normal metal has the Rashba type spin-orbit coupling characterized by the vector \( \mathbf{E}_{so} \). The charge current \( j_c \) flows perpendicularly to the phase gradient \( \nabla \varphi \).

![Diagram](image2.png)

**FIG. 2:** Diagrammatic representations of the current densities with second-order contributions of the spin-orbit coupling \( \mathbf{E}_{so} \) and the first-order contributions of the phase gradient \( \nabla \varphi \). The wavy lines denote the spin-orbit interaction and dotted lines represent the phase gradient. The wavy and dotted lines intersecting at one point correspond to \( H_{so-\varphi} \).
The transverse current \((i \neq j)\) is given by the second order expansion with respect to \(E_{so}\) (the first order term vanishes), which results in the form

\[
 j_{c,i} \approx \frac{2i\hbar v^2}{V m^2} \nabla j \cdot \sum_{k,\omega} \text{Tr}_r k_i k_j (E_{so} \times k)^2 \left[ (g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 + \tau_3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3)^2 \right] < \\
+ \delta_{ij} \frac{2i\hbar v^2}{V m} \nabla j \cdot \sum_{k,\omega} \text{Tr}_r \tau_3 (E_{so} \times k)^2 \left[ g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 \right] < \\
- \frac{2i\hbar v^2}{V m} \nabla j \cdot \sum_{k,\omega} \text{Tr}_r k_j^2 \left[ (g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3 + g_{\tau,k,\omega}^3 g_{\tau,k,\omega}^3)^2 \right] < \\
= \frac{256e}{9V^2} \nabla j \cdot \sum_{k,\omega} \omega \varepsilon_k \text{Im} \left[ \varepsilon_k (f_{\kappa,k,\omega}^n)^2 \left\{ (g_{\kappa,\omega}^3)^2 - (f_{\kappa,k,\omega}^n)^2 \right\} + \frac{3}{2} g_{\kappa,\omega}^3 (f_{\kappa,k,\omega}^n)^2 \right] (11)
\]

where \(\text{Tr}_r\) means the trace in Nambu space (here we have already taken the trace in spin space). This is a general expression which is applicable to any Green’s function of the form of Eq. (9). We see that in the absence of the superconductivity \(f_{\kappa,k,\omega}^n \rightarrow 0\), the transverse current vanishes \(j_{c,i} \rightarrow 0\). From Eq. (11), we find that the direction of the transverse Josephson current is determined by the vector \(E_{so}\), which characterizes the spin-orbit coupling, as

\[
 j_{c,i} \approx [E_{so} \times (E_{so} \times \nabla \phi)]_i (12)
\]

for \(\vec{j}_{\perp} = \hat{\nabla} \phi\). Longitudinal Josephson current can flow under phase gradient without spin-orbit coupling. Thus, in the leading order, it is proportional to the phase gradient and does not depend on spin-orbit coupling. Then, Eq. (12) can be rewritten as

\[
 j_{c,i} \approx [E_{so} \times (E_{so} \times \vec{j}_j)]_i (13)
\]

where \(\vec{j}_j\) is the longitudinal Josephson current parallel to \(\hat{\nabla} \phi\). We also see when the phase gradient is along \(x\)-axis, to obtain finite \(j_{c,y}\) the vector \(E_{so}\) should have both \(x\) and \(y\) components.

When the normal metal is sufficiently thin and the interfaces between the normal metal and the superconductors are transparent, proximity effect becomes very strong such that the Green’s functions in the normal metal have the same form as those in the bulk superconductor:

\[
g_{k,\omega}^l = \frac{(\omega - i\gamma)\sigma_0 \otimes \tau_l + \xi\sigma_0 \otimes \tau_3 + \Delta_0 \otimes \tau_l}{(\omega - i\gamma)^2 - \xi^2 - \Delta^2}. \quad (14)
\]

The transverse current is then given by

\[
 j_{c,i} \approx \frac{64\pi e\nu}{9V} \nabla j \cdot \sum_{\omega} f_\omega \text{Re} \left[ \frac{\Delta^2}{(\omega - i\gamma)^2 - \Delta^2} \right] \\
= \frac{32e\nu}{9V\hbar} \nabla j \cdot \sum_{\omega} f_\omega \text{Re} \left[ \frac{\gamma}{\sqrt{\gamma^2 + \Delta^2}} - 1 \right] (15)
\]

at zero temperature where \(\gamma\) is the inelastic scattering rate by impurities and \(\nu\) is the density of states at the Fermi energy.

Let us discuss the mechanism of the transverse Josephson effect predicted in this paper, comparing with the spin Hall effect.\(^{13,14}\) Consider the spin-orbit coupling \(\vec{j}_s\) as a spin dependent potential. The spin Hall effect occurs since electron with spin-up and that with spin-down feel potential with the opposite sign and hence are scattered in the opposite direction as shown in Fig. 3. On the other hand, the Josephson current is carried by Cooper pair which consists of electron pair with opposite momentum and spin, \((k, \uparrow)\) and \((-k, \downarrow)\). As seen from Eq. (2), the Rashba type spin-orbit coupling is invariant under the sign change of both momentum and spin (since this term is time-reversal even). Thus, the two electrons which constitute the Cooper pair feel exactly the same potential and hence move in the same direction, resulting in the net transverse current.

As seen from the London equation Eq. (5), since both charge current and vector potential are odd under spatial inversion, the inversion symmetry breaking is unnecessary to obtain the charge current. In fact, the coef-
sufficient of the phase gradient is given by even (second) order with respect to the Rashba type spin-orbit coupling (see Eq. (11)), and hence does not break the inversion symmetry. This implies that when the system respects the inversion symmetry, the transverse Josephson effect predicted in this paper would also emerge due to spin-orbit coupling. Further investigation of the transverse Josephson effect based on such model as the Luttinger Hamiltonian, the spin-orbit coupling of which preserves the inversion symmetry, will be an interesting future work.

The present formalism assumes the translational symmetry of the system and hence the length of the normal metal does not appear in the results. To investigate the effect of the length of the normal metal, investigation based on, for example, the Usadel equation including spin-orbit coupling is necessary. To test the transverse Josephson effect predicted in this paper experimentally, one may use BiTeI, a recently discovered 3D Rashba system, as a spin-orbit coupled normal metal. For this material, we have covered 3D Rashba system, and hence does not break the inversion symmetry. Further investigation of the transverse Josephson effect based on such model as the Luttinger Hamiltonian, the spin-orbit coupling of which preserves the inversion symmetry, will be an interesting future work.

To test the transverse Josephson effect predicted in this paper experimentally, one may use BiTeI, a recently discovered 3D Rashba system, as a spin-orbit coupled normal metal. For this material, we have \( \nu \sim 5 \times 10^{-2} \) states/eV/unit cell, the lattice constant \( \sim 5 \AA \), \( k_F E_{so} \sim 100 \) meV, and the Fermi wavevector \( k_F \sim 0.1 \AA^{-1} \). Taking \( \Delta \sim 100 \) meV and \( \Delta \sim 1 \) meV, we estimate the current in Eq. (10) as \( j_{c,i} \sim 2 \times 10^{-4} \) A/cm². Our prediction could be confirmed by Josephson junctions with this material in four-terminal geometry by injecting longitudinal current. Since transverse Josephson current gives a phase shift along the transverse direction, one can detect the predicted effect by using superconducting quantum interference device attached to the normal metal transversely. The transverse Josephson effect is reflected as a shift in the interference pattern. In summary, we have investigated transverse Josephson current in superconductor/normal metal/superconductor junctions where the normal metal has Rashba type spin-orbit coupling. It has been shown that transverse current arises from the spin-orbit coupling in the normal metal. This effect is specific to superconducting current and the transverse current vanishes in the normal state. In addition, this transverse Josephson effect is purely stationary and applied magnetic field is unnecessary to realize this effect, in contrast to the Hall effect in the normal state. We have also presented physical interpretation of this effect, comparing with the spin Hall effect.

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Appendix. Here, we will derive Eq. (6). The velocity operator in \( \hat{\rho} \)-direction, \( v_{\hat{\rho}} \), is given by the derivative of the Hamiltonian with respect to the momentum:

\[
v_{\hat{\rho}} = \frac{\partial H_N}{\partial \delta k_{\hat{\rho}}} = \frac{\hbar}{m} k_{\hat{\rho}} \sigma_0 \otimes \tau_3 + \delta_{\hat{\rho}} k_{\hat{\phi}} \nabla_{\hat{\rho}} \varphi \sigma_0 \otimes \tau_0 - \frac{1}{\hbar} (\sigma \times E_{so})_i \otimes \tau_3.
\]

One obtains the current operator by multiplying the velocity operator by the charge operator: \(-e\sigma_0 \otimes \tau_3\):

\[
j_{c,i} = -e\sigma_0 \otimes \tau_3 v_{\hat{\rho}} = -\frac{e \hbar}{m} k_{\hat{\rho}} \sigma_0 \otimes \tau_0 - \delta_{\hat{\rho}} \frac{e \hbar}{m} \nabla_{\hat{\rho}} \varphi \sigma_0 \otimes \tau_3 + \frac{e}{\hbar} (\sigma \times E_{so})_i \otimes \tau_0.
\]

Note that the charge is opposite for electron and hole.

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20. Since the magnitude of the spin-orbit coupling of this material is large, the perturbative calculation with respect to the spin-orbit coupling may not be a good approximation. However, the qualitative result, the appearance of the transverse current, should remain.