Abstract

The n-sum graph Negami’s splitting formula for the Tutte polynomial is not valid in the region \((x-1)(y-1) = q\) for \(q = 1, 2, \ldots, n-1\) with the additional region \(y = 1\) if \(n > 3\). This region corresponds to (up to prefactors and change of variables) the Ising model, the \(q\)-state Potts model, the number of spanning forest generator and particularizations of these. We show splitting formulas for these specializations.

Keywords: Tutte polynomial, Graph Theory, Splitting formula

1. Introduction

The Tutte polynomial of a graph \(G\), also known as dichromate or Tutte-Whitney polynomial, is defined as the following subgraph generating function \([Tu]\):

\[
T(G; x, y) = \sum_{\substack{A \subseteq G \\ V(A) = V(G)}} (x - 1)^{\omega(A) - \omega(G)} (y - 1)^{\omega(A) + |E(A)| - |V(G)|}
\]

where \(A \subseteq G\) indicates that \(A\) is a subgraph of \(G\) and \(\omega(G)\) denotes the number of connected components of \(G\). It is the most general graph invariant that can be defined by the deletion-contraction algorithm:

\[
T(G; x, y) = T(G/e; x, y) + T(G - e, x, y)
\]

where \(e\) is neither a loop (and edge with coincident endpoints) nor a bridge (an edge whose deletion increases the number of connected components), with
\[ T(G; x, y) = x^i y^j \] if the edge set of \( G \) only has \( i \) bridges and \( j \) loops. Here \( G/e \) and \( G - e \) denote the contraction and deletion of the edge \( e \) respectively.

Computing the Tutte polynomial is in general an NP-hard problem [JVW].

Different specializations with respective prefactors and change of variables of the Tutte polynomial, naturally appear as classical invariants in several branches of mathematics, physics and engineering ([Ai], [Bo], [Bi], [BO]). For example, the Jones polynomial in knot theory [Jo], the reliability polynomial in network engineering, the Ising and Potts model in statistical mechanics ([Is], [On], [Po]), the random cluster model [FK], etc. (see Table 1).

Following [Ne], assume that the graph \( G \) splits in subgraphs \( K \) and \( H \) only sharing \( n \) common vertices \( U = V(K) \cap V(H) \). Let \( \Gamma(U) \) denote the partition lattice over \( U \) and let \( \mathcal{A} = \{U_1, U_2, \ldots U_k\} \) be one of these partitions. Denote by \( K/\mathcal{A} \) and \( H/\mathcal{A} \) the graphs obtained by identifying all vertices in each \( U_i \) of \( K \) and \( H \) respectively, see Figure 1. The following is Negami’s splitting formula for the Tutte polynomial (Corollary 4.7, iv, [Ne]):

\[
T(G; x, y) = \sum_{\mathcal{A}, \mathcal{B} \in \Gamma(U)} c_{\mathcal{A}\mathcal{B}}(x, y) \ T(K/\mathcal{A}; x, y) \ T(H/\mathcal{B}; x, y) \tag{1}
\]

where \( c_{\mathcal{A}\mathcal{B}}(x, y) \) are rational functions of \( x \) and \( y \) on the field of rational numbers. A colored version of this formula was developed in [Te] and the case of Tutte polynomials of generalized parallel connections of general matroids.

| Specialization | Invariant |
|----------------|-----------|
| \( xy = 1 \)   | Jones polynomial |
| \( y = 0 \)    | Chromatic polynomial |
| \( x = 1, y \neq 1 \) | Reliability polynomial |
| \( x = 0 \)    | Flow polynomial |
| \( (x - 1)(y - 1) = 2 \) | Ising model |
| \( (x - 1)(y - 1) = q \) | \( q \)-state Potts model |
| \( y \neq 1 \) | Random cluster model |
| \( y = 1 \) | Number of spanning forest generator |
| \( (1, 1) \) | Number of spanning tree |
| \( (2, 1) \) | Number of spanning forest |
| \( (1, 2) \) | Number of spanning subgraph |

Table 1: Specializations of the Tutte polynomial up to prefactors and change of variables.
Explicit splitting formulas were also given in [No] and [An]. As an application of the Feferman-Vaught Theorem, the existence of splitting formulas for a wide class of graph polynomials which includes the Tutte polynomial is proved in [Ma]. The above result is an existential theorem, it is not explicit like the others.

In view that some denominators of the coefficients $c_{AB}$ could annihilate restricted to certain regions, we wonder whether the formula holds for different specializations. For example, for the 2-sum such that $H$ and $K$ are connected we have the Brylawski sum ([Br, Corollary 6.14])

$$T(G; x, y) = \frac{1}{(x-1)(y-1)-1} \left[ (y-1) T(K; x, y) T(H; x, y) - T(K; x, y) T(H/\mathcal{A}; x, y) - T(K/\mathcal{A}; x, y) T(H; x, y) + (x-1) T(K/\mathcal{A}; x, y) T(H/\mathcal{A}; x, y) \right]$$

where $\mathcal{A}$ is the trivial or minimal partition of the common two vertices between $H$ and $K$. Figure 2 shows this factorization. Is clear that this formula does not hold in the region $(x-1)(y-1) = 1$.

In the next section we prove the following: For the $n$-sum graph, Negami’s formula [1] holds only over the region $(x-1)(y-1) \neq q$ such that $q = 1, 2, \ldots n-1$ with the additional constraint $y \neq 1$ if $n > 3$. The region where Negami’s formula doesn’t hold will be called the singular region.

In this paper, in the case of the $n$-sum graph such that $H$ and $K$ are connected, we show explicit splitting formulas for the Tutte polynomial over

---

1 The author is grateful to Prof. Lorenzo Traldi and Prof. Anna de Mier for these references and valuable comments.

2 As far as the author knows, this was the second known splitting formula for the Tutte polynomial after the well known factorization through an articulation point.
the singular region. We also show the interesting fact that these formulas are in general not unique; i.e. The Tutte polynomial over this region splits in several different ways. For example, for \( n \geq 4 \), the number of spanning forest generator \( T(G; x, 1) \) splits according to several different formulas. These are the main results of the paper.

2. Preliminaries

The Negami polynomial \( f(G, t, x, y) \) for a graph \( G \) is defined as follows [Ne]:

1. \( f(K_n, t, x, y) = t^n \)

2. \( f(G, t, x, y) = xf(G/e, t, x, y) + yf(G - e, t, x, y) \)

where \( e \) is an edge of \( G \) and \( K_n \) is the complement of the complete graph \( K_n \); i.e. \( n \) isolated vertices. The relationship between Negami and Tutte polynomial is the following:

\[
f(G; (x - 1)(y - 1), 1, y - 1) = (y - 1)^p(x - 1)^\omega(G)T(G; x, y)
\]

We define \( \Gamma(U) \) as the set of partitions of \( U \) with the following partial order: \( \gamma \leq \gamma' \) if \( \gamma' = \{U'_1, \ldots, U'_l\} \) is a refinement of \( \gamma = \{U_1, \ldots, U_m\} \); i.e. For every \( U'_i \) there is \( U_j \) such that \( U'_i \subseteq U_j \). The pair (\( \Gamma(U), \leq \)) is a partition lattice. We denote by \( \gamma \wedge \gamma' \) the infimum of \( \gamma \) and \( \gamma' \). Similarly, we denote by \( \gamma \vee \gamma' \) the supremum. Consider a total order on \( \Gamma(U) \) such that \( \gamma_i \leq \gamma_j \) imply \( i \leq j \). Define the \( |\Gamma(U)| \times |\Gamma(U)| \) matrix \( T_n \) such that its \((i, j)\)-entry is \( t^{\gamma_i \wedge \gamma_j} \) where \( |\gamma| \) denotes the number of blocks of the partition.
A closer look at Negami’s proof of his splitting formula\(^3\) (Theorem 4.2, \[Ne\]) shows that he actually proves the following more slightly general version:

**Theorem 2.1.** Let \(G\) be a graph obtained as a union of two graphs \(K\) and \(H\) sharing only the vertices \(U = \{u_1, \ldots, u_n\}\). Let \(t\) be a real number. Then,

\[
f(G) = \sum_{A,B \in \Gamma(U)} b_{AB}(t) f(K/A) f(H/B)
\]  

(4)

such that \(B_n(t) = (b_{AB}(t))\) is a matrix verifying the relation:

\[
T_n(t)B_n(t)T_n(t) = T_n(t)
\]

(5)

If the matrix \(T_n(t)\) is invertible for a specific value of \(t\), then \(B_n(t) = T_n(t)^{-1}\) and Negami’s original formula is reproduced. Because of the determinant formula \[Kr\]:

\[
det(T_n(t)) = \prod_{A \in \Gamma(U)} (|A|-1) \prod_{q=0}^{n-1} (t-q)
\]

(6)

the matrix \(T_n(t)\) is non invertible for \(t = 0, 1, \ldots, n-1\) and Negami’s formula does not hold. However, our generalized formulation solves this problem. For example, consider \(n > 1\) and the non invertible matrix \(T_n(1)\). The matrix \(B_n\) with one in the upper left corner and zero elsewhere is a solution of (5) and gives a well defined splitting formula. Moreover, as is shown in lemma 5.1 for every parameter \(t\) there is a solution of equation (5) hence a splitting formula (4) for the Negami polynomial.

Recall equation (3). Translating Negami’s splitting formula (4) to the Tutte polynomial, we get the corresponding version of Corollary 4.7, iv, \[Ne\]:

**Corollary 2.2.** Let \(G\) be a graph obtained as a union of two graphs \(K\) and \(H\) sharing only the vertices \(U = \{u_1, \ldots, u_n\}\). Then, for \((x,y)\) in the region \((x-1)(y-1) \neq 0\) we have\(^4\):

\[
T(G; x,y) = \sum_{A,B \in \Gamma(U)} c_{AB}(x,y) T(K/A; x, y) T(H/B; x, y)
\]

\(^3\)This will be done later in section 4. It is immediate from equations (10) and (11).

\(^4\)Recall that we are studying specializations. If we were studying the Tutte polynomial in the polynomial ring \(\mathbb{Z}[x, y]\), the pathological region \((x-1)(y-1) = 0\) wouldn’t appear in the analysis.
such that:

\[ c_{AB}(x,y) = b_{AB} ((x-1)(y-1)) (x-1)^{\omega(K/A) + \omega(H/B) - \omega(G)} (y-1)^{|A| + |B| - n} \quad (7) \]

where \( B_n(t) = (b_{AB}(t)) \) is a solution of equation (5).

See that in the original formulation by Negami, because of relation (7) and the determinant formula (6), the splitting (1) for the n-sum graph would hold only in the region \( (x-1)(y-1) \neq q \) such that \( q = 0, 1 \ldots n - 1 \); i.e. our formulation is an improvement.

As an example, consider the Brylawski sum (2). Along the curve \( (x-1)(y-1) = 1 \) this splitting formula is not defined. However, Corollary 2.2 provides the following splitting: The matrix with one in the bottom right corner and zero elsewhere is a solution of equation (5) with \( t = 1 \) hence by formula (7) we have the splitting:

\[ T(G; x,y) = (y-1) \ T(H; x,y) \ T(K; x,y) \]

Analogously, the matrix with one in the upper left corner and zero elsewhere is another solution and provides the splitting:

\[ T(G; x,y) = (x-1) \ T(K/A; x,y) \ T(H/A; x,y) \]

where \( A \) is the trivial or minimal partition. These splittings hold only in the region \( (x-1)(y-1) = 1 \) where the Brylawski sum (2) is not even defined. These are illustrated in Figure 3.

Recall that we are studying specializations. If we were studying the Tutte polynomial in the polynomial ring \( \mathbb{Z}[x, y] \), the singular region wouldn’t appear in the analysis for the matrix \( T_n(t) \) is invertible in the polynomial ring\(^5\).

\(^5\)This is because the diagonal term is the higher degree term of the determinant polynomial and cannot be canceled by linear combination of the other terms.
What happens in the region \((x - 1)(y - 1) = 0\)? Because of equation (3), there is no relationship between Negami and Tutte polynomials when \(x = 1\) or \(y = 1\) hence we cannot assure a priori the existence of splitting formulas in this region and they have to be calculated separately. Assuming that \(\omega(H) = \omega(K) = 1\), we will derive splitting formulas for this region in the following sections.

3. Splitting formula on \(x = 1\), \(y \neq 1\)

In the following sections we assume that \(\omega(H) = \omega(K) = 1\). With this condition we have that \(\omega(G) = \omega(K/A) = \omega(H/B) = 1\) for every pair of partitions \(A\) and \(B\) in \(\Gamma(U)\) hence \(\omega(K/A) + \omega(H/B) - \omega(G) = 1\).

The reader may be tempted to make the following mistake: Because \(\omega(K/A) + \omega(H/B) - \omega(G) = 1\), there is a priori no singularity and therefore evaluating \(x = 1\) in (7) gives a trivial splitting. However, recall that the coefficients \(b_{AB}(t)\) are rational functions and may have singularities at \(x = 1\).

Because the Tutte polynomial is continuous, we can just take the limit in the coefficients (7) and see if it defines a splitting formula; i.e. We define:

\[
c_{AB}(1, y) = \lim_{\substack{x \to 1 \\ y \neq 1}} c_{AB}(x, y) = \lim_{\substack{x \to 1 \\ y \neq 1}} (y - 1)^{|A|+|B|-n-1}b'_{AB} ((x - 1)(y - 1))
\]

where \(B'_n(t) = (b'_{AB}(t))\) is the inverse matrix of \(t^{-1}T_n(t)\). This is well defined for we are taking the limit and it is enough to consider a small enough reduced neighborhood of \(t = 1\) where the matrix is invertible. Because the inverse operation is continuous and the limit \(\lim_{t \to 0} t^{-1}T_n(t)\) is invertible ([Kn], [Bu]), we have:

\[
c_{AB}(1, y) = (y - 1)^{|A|+|B|-n-1}b'_{AB} \tag{8}
\]

where \(B'_n = (b'_{AB})\) is the inverse matrix of the matrix \(A_n = (a_{AB})\) such that \(a_{AB} = 1\) if \(|A \cap B| = 1\) and \(a_{AB} = 0\) otherwise. As it was expected, up to a prefactor this is the same splitting formula as the one for the Reliability polynomial studied in [BR].

\[
R(G; p) = \sum_{A, B \in \Gamma(U)} b'_{AB} R(K/A; p) R(H/B; p)
\]

\footnote{In that paper, the matrix \(A\) was called the connectivity matrix.}
where $R(G;p)$ is the reliability polynomial of $G$.

Formula (8) shows that the singularity at $x = 1$ is removable if $y \neq 1$; i.e. We have proved the following: There is an analytic continuation of the splitting coefficients $c_{AB}(x,y)$ to the region $x = 1, y \neq 1$ such that the splitting formula (1) holds.

4. Splitting formula on $y = 1$

Direct inspection on the cases $n = 1, 2, 3$ shows that there are no singularities at $y = 1$ in the coefficients (7) hence the splitting formula (1) holds in these cases.

However, the situation is different for $n > 3$. Here, the strategy of the previous section does not work: For $n > 3$ some pairs of partitions $(A, B)$ verify that $|A| + |B| - n - 1 < 0$ and $b'_{AB} \neq 0$ hence the splitting coefficients (7) corresponding to these pairs when $y$ tends to one diverge:

$$\lim_{y \to 1} c_{AB}(x,y) = \lim_{y \to 1} (y - 1)^{|A| + |B| - n - 1} b'_{AB} = \infty$$

and we have no splitting formula in this region; i.e. The singularity at $y = 1$ is not removable. The existence of these pairs of partitions is because otherwise the matrix $B'_n$ wouldn’t be invertible.

We will follow Negami’s strategy; i.e. Define auxiliary polynomials and derive relations between these and the contractions of the subgraphs $H$ and $K$.

In what follows, every definition or result valid for $K$ will be valid also for $H$ and the corresponding proof is verbatim. Hence we will work only with $K$. Consider the Negami polynomial expansion (Theorem 1.4 [Ne]):

$$f(G; t, x, y) = \sum_{A \subseteq G, V(A) = V(G)} t^{\omega(A)} x^{|E(A)|} y^{|E(G)| - |E(A)|}$$

Every spanning subgraph $Y \subseteq K$ defines a partition $\mathcal{P}(Y) \in \Gamma(U)$ via the following equivalence relation: $u_i$ is equivalent to $u_j$ if they belong to the

\footnote{The proof of this fact is verbatim to the proof of the second item in Lemma 5.1 and we will not reproduce it here.}
same connected component of $Y$. Define the auxiliary polynomial:

$$f_A(K; t, x, y) = \sum_{Y \subseteq K \atop \mathcal{V}(Y) = V(K) \atop \mathcal{P}(Y) = A} t^{\omega(Y) - |A|} x^{|E(Y)|} y^{|E(K)| - |E(Y)|}$$  \hspace{1cm} (9)

These polynomials verify:

$$f(G) = \sum_{A \in \Gamma(U)} f_A(K) f(H/A)$$  \hspace{1cm} (10)

$$f(K/A) = \sum_{B \in \Gamma(U)} t^{|A \wedge B|} f_B(K)$$  \hspace{1cm} (11)

As a consequence, we have Negami’s splitting Theorem 2.1 Define the auxiliary polynomial:

$$T_A(K; x, y) = \sum_{Y \subseteq K \atop \mathcal{V}(Y) = V(G) \atop \mathcal{P}(Y) = A} (x - 1)^{\omega(Y) - |A|} (y - 1)^{\omega(Y) + |E(Y)| - |V(K)|}$$

Lemma 4.1.

$$(x - 1) T(G; x, 1) = \lim_{t, \zeta \to 0 \atop t/\zeta \to x - 1} f(G; t, \zeta, 1) \zeta^{-|V(G)|}$$

$$T_A(K; x, 1) = \lim_{t, \zeta \to 0 \atop t/\zeta \to x - 1} f_A(K; t, \zeta, 1) \zeta^{-|V(K)| + |A|}$$

Proof: We prove only the first identity, the other is similar. Every spanning subgraph $A \subseteq G$ verifies:

$$\omega(A) + |E(A)| \geq |V(G)|$$

and the equality holds if and only if $A$ is a spanning forest with $\omega(A)$ trees. Then,

$$(x - 1) T(G; x, 1) = \sum_{i=1}^{\lfloor V(G) \rfloor} (x - 1)^i S_i$$  \hspace{1cm} (12)

\text{--- Footnotes ---}

\footnote{This is the polynomial $A(\gamma_i)$ in Negami’s proof (Theorem 4.2 \cite{Ne}).}

\footnote{These are equations (2) and (3) in Negami’s proof (Theorem 4.2 \cite{Ne}). They can also be derived by direct calculation in similar manner to \cite{BR}. These identities do not need the hypothesis $\omega(H) = \omega(K) = 1$.}
where $S_i$ is the number of spanning forests with $i$ trees. On the other hand we have:

$$f(G; t, \zeta, 1) = \sum_{A \subseteq G \atop V(A) = V(G)} t^{\omega(A)} \zeta^{|E(A)|} = \sum_{i=1}^{|V(G)|} t^i \left( S_i \zeta^{|V(G)|-i} + O(\zeta^{|V(G)|-i+1}) \right)$$

$$= \zeta^{|V(G)|} \sum_{i=1}^{|V(G)|} \left( \frac{t}{\zeta} \right)^i (S_i + O(\zeta))$$

Taking the limit the result follows. \hfill \Box

Taking the limits of equations (10) and (11) as in lemma 4.1 we have

$$T(G; x, 1) = \sum_{\mathcal{A} \in \Gamma(U)} T_A(K; x, 1) T(H/\mathcal{A}; x, 1) \quad (13)$$

$$T(K/\mathcal{A}; x, 1) = \sum_{B \in \Gamma(U)} \delta^0_{|\mathcal{A} \wedge B|+n-|\mathcal{A}|-|B|} (x-1)^{|\mathcal{A} \wedge B|-1} T_B(K; x, 1) \quad (14)$$

where $\delta^i$ is the Kronecker delta. Define the matrix $L_n(x) = (l_{AB}(x))$ whose entries are:

$$l_{AB}(x) = \delta^0_{|\mathcal{A} \wedge B|+n-|\mathcal{A}|-|B|} (x-1)^{|\mathcal{A} \wedge B|-1}$$

For example:

$$L_2(x) = \begin{pmatrix} 0 & 1 \\ 1 & x-1 \end{pmatrix}$$

$$L_3(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & x-1 \\ 0 & 1 & 0 & 1 & x-1 \\ 0 & 1 & 1 & 0 & x-1 \\ 1 & x-1 & x-1 & x-1 & (x-1)^2 \end{pmatrix}$$

We have proved the following splitting formula:

\footnote{To derive equation (14) from equation (11) we need the following fact: For every pair of partitions $\mathcal{A}, \mathcal{B} \in \Gamma(U)$ we have $|\mathcal{A} \wedge \mathcal{B}| + n - |\mathcal{A}| - |\mathcal{B}| \geq 0$. This follows from the identity:

$$|\mathcal{A} \wedge \mathcal{B}| + |\mathcal{A} \vee \mathcal{B}| \geq |\mathcal{A}| + |\mathcal{B}|$$}
Theorem 4.2. Let $G$ be a graph obtained as a union of two graphs $K$ and $H$ sharing only the vertices $U = \{u_1, \ldots, u_n\}$. Let $x$ be a real number. Then,

$$ T(G; x, 1) = \sum_{A, B \in \Gamma(U)} d_{AB}(x) T(K/A; x, 1) T(H/B; x, 1) $$

such that $D_n(x) = (d_{AB}(x))$ is a solution of the equation:

$$ L_n(x) D_n(x) L_n(x) = L_n(x) $$

(15)

In lemma 5.1 we show that there is always a solution of equation (15). For example, in the cases $n = 2, 3$, the unique solution $D_n(x)$ to equation (15) is:

$$ D_2(x) = \begin{pmatrix} 1 - x & 1 \\ 1 & 0 \end{pmatrix} $$

$$ D_3(x) = \frac{1}{2} \begin{pmatrix} (1 - x)^2 & 1 - x & 1 - x & 1 - x & 2 \\ 1 - x & -1 & 1 & 1 & 0 \\ 1 - x & 1 & -1 & 1 & 0 \\ 1 - x & 1 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} $$

In section 5 it will be shown that there is no unique solution in the case $n \geq 4$.

4.1. Example

Consider the case $n = 4$ and the point $(x, y) = (1, 1)$. Consider the following total order in the partition set $\Gamma(U)$:

$$ \{\{1, 2, 3, 4\}\} < \{\{1, 2, 3\}, \{4\}\} < \{\{1, 2, 4\}, \{3\}\} < \{\{1, 3, 4\}, \{2\}\} < \{\{2, 3, 4\}, \{1\}\} $$

$$ < \{\{1, 2\}, \{3, 4\}\} < \{\{1, 4\}, \{2, 3\}\} < \{\{1, 3\}, \{2, 4\}\} $$

$$ < \{\{1, 2\}, \{3\}, \{4\}\} < \{\{1, 3\}, \{2\}, \{4\}\} < \{\{1, 4\}, \{2\}, \{3\}\} $$

$$ < \{\{2, 4\}, \{1\}, \{3\}\} < \{\{2, 3\}, \{1\}, \{4\}\} < \{\{2, 4\}, \{1\}, \{2\}\} < \{\{1\}, \{2\}, \{3\}, \{4\}\} $$

Recall the matrix $A_4 = (a_{AB})$ such that $a_{AB} = 1$ if $|A \setminus \Gamma(U)$ | and $a_{AB} = 0$ otherwise:
In particular, from formula (8), the analytic continuation of formula (7)
to the region $x = 1$, we have the splitting formula:

$$T(G; 1, y) = -\frac{1}{6} \frac{1}{y-1} T(H/\{1, 2\}, \{3, 4\}; 1, y) T(K/\{1, 2\}, \{3, 4\}; 1, y) + \ldots$$

(16)

This expression is illustrated in Figure 4.

Formula (16) is not defined at $y = 1$; i.e. Negami’s formula (1) doesn’t hold at the point $(x, y) = (1, 1)$ in the case $n = 4$. However, we still have a splitting as follows: Recall the matrix $L_n(x) = (l_{AB}(x))$ whose entries are:

$$l_{AB}(x) = \delta_{|A\wedge B|+n-|A|-|B|} (x-1)^{|A\wedge B|-1}$$

In particular, at $x = 1$ we have $l_{AB}(1) = \delta_{1+n-|A|-|B|}$ and the matrix is the following:

$$L_4(1) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
A solution \( D_4 = (d_{AB}) \) of the equation \( L_4(1) D_4 L_4(1) = L_4(1) \) is the following:

\[
D_4 = \frac{1}{14} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & 4 & 4 & -3 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 4 & -3 & -3 & 4 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 & -3 & 4 & 4 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & -3 & -3 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 3 & 3 & 3 & 3 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & -4 & 3 & -4 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -4 & 3 & -4 & 3 & 3 & 0 \\
0 & -3 & -3 & 4 & 4 & -4 & 4 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 4 & -3 & 4 & 3 & 3 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -3 & -3 & 4 & 4 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 4 & -3 & 4 & 3 & -4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 4 & -3 & -3 & 4 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

By Theorem 4.2 the following is a splitting formula for the spanning tree number:

\[
T(G; 1, 1) = \sum_{A, B \in \mathcal{F}(U)} d_{AB} T(K/A; 1, 1) T(H/B; 1, 1)
\]

5. Degeneracy

In the following, \( \begin{pmatrix} n \\ i \end{pmatrix} \) denotes the Stirling number of the second kind whose value is the number of ways to partition a set of \( n \) elements into \( i \) disjoint nonempty subsets. We define \( \begin{pmatrix} n \\ 0 \end{pmatrix} = 0 \) for \( n \geq 1 \).

**Lemma 5.1.** Let \( t(x) \) be a real number. The equation \((5) \) \((15)\) has solution, and the solution is unique if and only if \( T_n(t) \) \((A(x))\) is invertible. Moreover,
1. The affine matrix space $\mathcal{M}_{n,t} \subseteq M_{|\Gamma(U)|}(\mathbb{R})$ of solutions of the equation (5) has the following dimension:

$$\dim_{\mathbb{R}} \mathcal{M}_{n,t} = \begin{cases} |\Gamma(U)|^2 - \left( \sum_{i=0}^{t} \binom{n}{i} \right)^2 & t = 0, 1, 2, \ldots n - 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Equation (15) has unique solution if and only if $n \leq 3$.

Proof: The matrix $T_n(t)$ is real and symmetric then there is an orthogonal (in particular real) matrix $O$ such that $O \, T_n(t) \, O^t$ is the diagonal matrix whose entries are the eigenvalues of $T_n(t)$. We can choose $O$ such that:

$$OT_n(t)O^t = \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix}$$

where $D_k$ is a $k \times k$ diagonal invertible matrix. Then, all of the solutions of equation (5) are the following:

$$B_n(t) = O^t \begin{pmatrix} M_1 & M_2 \\ M_3 & D_k^{-1} \end{pmatrix} O$$

such that $M_1, M_2, M_3$ are arbitrary real matrices and we have the result for equation (5). See that $T_n(t)$ is invertible if and only if $k = |\Gamma(U)|$ and in this case, $B_n(t) = T_n(t)^{-1}$. A verbatim argument proves the result for equation (15).

1. Instead of an orthogonal matrix, we can choose a non orthogonal $\Lambda \in M_{|\Gamma(U)|}(\mathbb{Z})$ such that

$$\Lambda \, T_n(t) \, \Lambda^t = \begin{pmatrix} (t - n + 1) \ldots (t - 1)t & 0 & \ldots & 0 & 0 \\ 0 & (t - n + 2) \ldots (t - 1)t & I_{\{n\}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & (t - 1)t & I_{\{2\}} \\ 0 & 0 & \ldots & 0 & t \end{pmatrix}$$

and the result follows verbatim.

\[11\] This result follows as an adaptation of the proof of the determinant formula in [Bu].
2. By direct calculation, the matrix $A_n(x)$ is invertible for $n = 1, 2, 3$ and every $x \in \mathbb{R}$. It rest to show that $A_n(x)$ is non invertible if $n \geq 4$. Define an ordering of $\Gamma(U)$ such that $A_i \leq A_j$ implies $i \leq j$. With respect to this ordering, consider the upper right submatrix $A'$ of the $AB$ elements of $A_n(x)$ such that $|A| = 2$ and $|B| = n - 1$. For these entries we have:

$$|A \wedge B| + n - |A| - |B| = |A \wedge B| - 1$$

hence all the entries not annihilated by the Kronecker delta must be one; i.e. $A'$ entries are zero or one. All of the entries to the left of $A'$ are zero for:

$$|A \wedge B| + n - |A| - |B| \geq |A \wedge B| \geq 1$$

and the Kronecker delta annihilates these entries. Taking the unit entry of the upper right corner of $A_n(x)$ as a pivot, Gauss elimination turns every entry to the right of $A'$ to zero. Thus we have a block whose entries are zero or one of dimension $\begin{bmatrix} \binom{n}{2} \times \binom{n}{n-1} \end{bmatrix}$. Because for $n \geq 4$, Gauss elimination on the block $A'$ gives a zero row and the proof is complete.

6. Acknowledgments

The author is deeply grateful to the anonymous referees for their great work, the final form of the paper is mainly due to them.

The author is grateful to Consejo Nacional de Ciencia y Tecnología (CONACYT), for its Cátedras CONACYT program.

References

[Ai] M. Aigner, A course in enumeration, Graduate Texts in Mathematics, Springer, 2007.

\footnote{These follows from the relations $\binom{n}{2} = 2^{n-1} - 1$ and $\binom{n}{n-1} = \frac{n(n-1)}{2}$.}
[An] A. Andrzejak, *Splitting formulas for Tutte polynomials*, J. Combin. Theory Ser. B, 70 (1997), 346-366.

[Bi] N.L. Biggs, *Algebraic Graph Theory*, Cambridge, Cambridge University Press, 1993.

[Bo] B. Bollobás, *Modern Graph Theory*, New York, Springer-Verlag, 1998.

[Br] T. H. Brylawski, *A combinatorial model for series-parallel networks*, Transactions of the American Mathematical Society, vol. 154, 1-22, 1971.

[BdM] J.E. Bonin, A. de Mier, *Tutte polynomials of generalized parallel connections*, Adv. in Appl. Math., 32 (2004), 31-43.

[BO] T.H. Brylawski, J.G. Oxley, *The Tutte polynomial and its applications*, Matroid Applications (N. White ed.), Cambridge Univ. Press, Cambridge, 1992, pp.123-225.

[BR] J.M. Burgos, F. Robledo *Factorization of network reliability with perfect nodes I: Introduction and statements*, Discrete Applied Mathematics, Vol.198, 2016.

[Bu] J.M. Burgos, *Factorization of network reliability with perfect nodes II: Connectivity matrix*, Discrete Applied Mathematics, Vol.198, 2016.

[FK] M.C. Fortuin, W.P. Kasteleyn, *On the random-cluster model: I. Introduction and relation to other models*, Physica (Elsevier) 57 (4), 536-564, (1972).

[Is] E. Ising, *Beitrag zur Theorie des Ferromagnetismus*, Z. Phys. 31, 253-258, (1925).

[Jo] V.F.R. Jones, *A polynomial invariant for knots via von Neumann algebra*, Bull. Amer. Math. Soc. (N.S.) 12, 103-111, (1985).

[JWV] F. Jaeger, D. Vertigan, D.J.A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Math. Proc. Cambridge Philos. Soc., 108 (1990), 35-53.
[Kr] C.Krattenthaler, *Advanced determinant calculus*, Séminaire Lotharingien de Combinatoire, issue 42, paper B42q.

[Ma] J.A.Makowsky, *Algorithmic uses of the Feferman-Vaught Theorem*, Annals of Pure and Applied Logic, 126, 159-213, (2004).

[Ne] S. Negami, *Polynomial invariants of graphs*, Trans. Amer. Math. Soc., 299 (1987), 601-622.

[No] S.D.Noble, *Complexity of graph polynomials*, PhD theses, New College, Oxford University, England, 1997.

[On] L.Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition*, Physical Review, Series II 65 (3-4), 117-149, (1944).

[Po] R.B.Potts, *Some Generalized Order-Disorder Transformations* Mathematical Proceedings 48 (1), 106-109, (1952).

[Tr] L.Traldi, *On the colored Tutte polynomial of a graph of bounded treewidth*, Discrete Applied Mathematics 154 (2006), 1032-1036.

[Tu] W.T.Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80-91.