NECESSARY OPTIMALITY CONDITION FOR TRILEVEL OPTIMIZATION PROBLEM

GAOXI LI AND ZHONGPING WAN
School of Mathematics and Statistics, Wuhan University
Wuhan 430072, China

JIA-WEI CHEN
School of Mathematics and Statistics, Southwest University
Chongqing, 400715, China

XIAOKE ZHAO
School of Mathematics and Statistics, Wuhan University
Wuhan 430072, China

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ABSTRACT. This paper mainly studies the optimality conditions for a class of trilevel optimization problem, of which all levels are nonlinear programs. We firstly transform this problem into an auxiliary bilevel optimization problem by applying KKT approach to the lower-level problem. Then we obtain a necessary optimality condition via the differential calculus of Mordukhovich. Finally, a theorem for existence of optimal solution is derived via Weierstrass Theorem.

1. Introduction. In the recent years, the multi-level optimization problems specially the bilevel and trilevel optimization problems are studied by many researchers [6, 7, 10, 12, 14, 18, 19, 20, 22, 23, 24, 26, 27]. Many organizational decisions have trilevel nested structures, they can be described as trilevel optimization problems (TOPs). For TOP, all decision makers attempt to optimize their objectives, but their decisions are affected by optimal objective values presented at other levels [24]. It has the following operation process: The top-level decision maker selects an action within a specified constraint set, then the middle-level decision maker selects an action within a constraint set determined by the action of the top-level maker, and finally the lower-level decision maker selects an action within a constraint set determined by the actions of top-level and middle-level decision makers [20]. This model can be used to solve cloud market pricing problem [21]. The cloud market pricing problem is that, all decision makers of the Software-as-a-Service (SaaS), the Platform-as-a-Service (PaaS) and the Infrastructure-as-a-Service (IaaS) wish to optimization their profit, but every levels’ price impact the setting of other levels’ price, so it is a trilevel optimization problem. This model can be applied to electric
system [1], transportation [5], supply chain management [25] and so on. In this paper, we consider the following trilevel optimization problem:

\[(TOP) \quad \min_{x,y,z} f_1(x,y,z) \quad s.t. \quad g_1(x,y,z) \leq 0, (y,z) \in \psi(x), \]

where \(\psi(x)\) is the solution set of the following problem

\[
\min_{y,z} f_2(x,y,z) \\
\text{s.t.} \quad g_2(x,y,z) \leq 0, z \in \omega(x,y),
\]

where \(\omega(x,y)\) is the solution set of the following problem

\[
\min_z f_3(x,y,z) \\
\text{s.t.} \quad g_3(x,y,z) \leq 0,
\]

here \(f_i : R^n \times R^m \times R^p \rightarrow R \quad i = 1, 2, 3; g_i : R^n \times R^m \times R^p \rightarrow R^p \quad i = 1, 2, 3\) \((g_i = (g_{i1}, \ldots, g_{ip})^\top)\); \(f_1, f_2\) and \(f_3\) are the top-level, middle-level and lower-level objective functions, respectively.

Sometimes \(\omega(x,y) (\psi(x))\) is not a singleton, so the middle-level (top-level) decision maker may not persuade the lower-level (middle-level) decision maker to select an optimal solution \(z \in \omega(x,y) ((z,y) \in \psi(x))\) that the middle (top) decision maker most want to get. But problem (2) (problem (1)) implies that the middle-level (top-level) decision maker is able to persuade the lower-level (middle-level) decision maker to select an optimal solution which is the best one from the middle-level (top-level) maker’s point of view. In fact this TOP (1)-(3) can be called optimistic trilevel optimization problem. For describing this model better, we introduce some definitions and hypotheses:

1. Feasible set for the lower-level for each fixed \((x,y)\):
\[K(x,y) = \{z \in R^p : g_3(x,y,z) \leq 0\}.\]

2. Inequality constraint set for the upper-level:
\[X = \{x \in R^n : \exists (y,z) \text{ s.t. } g_1(x,y,z) \leq 0\}.\]

3. Inequality constraint set for the middle-level:
\[Y = \{y \in R^m : \exists x \text{ s.t. } g_2(x,y,z) \leq 0\}.\]

4. The upper and middle level’s decision space:
\[Q(X,Y) = \{(x,y) \in R^{n+m} : \exists z \text{ s.t. } g_1(x,y,z) \leq 0, i = 1, 2\}.\]

5. The solution set of the lower-level optimization problem (3) for fixed \((x,y) \in Q(X,Y)\):
\[\omega(x,y) = \{z \in R^p : z \in \arg \min \{f_3(x,y,z) : g_3(x,y,z) \leq 0\}\}.\]

6. The solution set of the middle-level optimization problem for fixed \(x \in X\):
\[\psi(x) = \{(y,z) \in R^{m+p} : (y,z) \in \arg \min \{f_2(x,y,z) : g_2(x,y,z) \leq 0, z \in \omega(x,y)\}\}.\]

7. Inducible region of the trilevel optimization problem:
\[IR = \{(x,y,z) \in R^{n+m+p} : g_1(x,y,z) \leq 0, (y,z) \in \psi(x)\}.\]

From the definition of \(IR\), we know that TOP (1)-(3) is equal to the following problem

\[\min_{x,y,z} \{f_1(x,y,z) : (x,y,z) \in IR\}.\]
In the past 30 years, scholars mainly focused on linear TOP. For example, White [20] proposed a penalty approach to linear TOP. Alguacil et al. [1] proposed a novel two-stage solution approach that attains optimality with moderate computational effort by translating the linear TOP into an equivalent bilevel optimization problem. Zhang et al. [24] proposed a Kth-best algorithm for linear TOP. A sufficient optimality condition for this problem was obtained in this article. Huang, et al. [13] proposed an interactive intuitionistic fuzzy method for multilevel linear programming problems.

To the best of our knowledge, there are few optimality conditions for TOP (1)-(3). Since optimality conditions are essential for the design of algorithm and the proof of convergence, in this paper, we will discuss the optimality conditions for TOP (1)-(3).

The rest of this paper is organized as follows. In Section 2, we recall some important results about variational analysis. In Section 3, firstly we transform TOP (1)-(3) into an auxiliary bilevel optimization problem by KKT approach, and discuss the relationship between the two problems. Then we get a necessary optimality condition for the TOP (1)-(3) via the auxiliary bilevel optimization problem. In Section 4, we consider the existence of optimal solution via the auxiliary problem.

2. Preliminaries. In this section, we mainly recall some basic definitions and results about variational analysis which are needed in our main results.

Definition 2.1. [16, 17] Given a point $\bar{z}$, \(\limsup_{z \to \bar{z}} \Xi(z)\) is said to be the Kuratowski-Painlevé outer upper limit of a set-valued mapping $\Xi : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ at $\bar{z}$, if
\[
\limsup_{z \to \bar{z}} \Xi(z) := \{v \in \mathbb{R}^m : \exists z_k \to \bar{z}, v_k \to v \text{ with } v_k \in \Xi(z_k) \text{ as } k \to \infty\}.
\]
Its graph $\text{gph}\Xi$ is denoted as follows:
\[
\text{gph}\Xi := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : v \in \Xi(u)\}.
\]

Definition 2.2. [9] Given a set-valued mapping $\Xi : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ and a point $\bar{z}$ with $\Xi(\bar{z}) \neq \emptyset$, we say that $\Xi$ is inner semicompact at $\bar{z}$, if and only if, for every sequence $z_k \to \bar{z}$ with $\Xi(z_k) \neq \emptyset$ there is a sequence of $v_k \in \Xi(z_k)$ that contains a convergent subsequence as $k \to \infty$. It follows that the inner semicompactness holds whenever $\Xi$ is uniformly bounded around $\bar{z}$, that is, there exists a neighborhood $U$ of $\bar{z}$ and a bounded set $A \subset \mathbb{R}^m$ such that $\Xi(z) \subseteq A, \forall z \in U$.

Definition 2.3. [2] Let $P$ be a nonempty subset of $\mathbb{R}^n$. A set-valued mapping $\Xi : P \to 2^{\mathbb{R}^m}$ is said to be
(i) upper semicontinuous (shortly, usc) at $\bar{z} \in P$ if, for each open set $V \subset \mathbb{R}^m$ with $\Xi(\bar{z}) \subset V$, there exists $\delta > 0$ such that
\[
\Xi(z) \subset V, \quad \forall z \in B(\bar{z}, \delta).
\]
$\Xi$ is upper semicontinuous if it is upper semicontinuous at all $\bar{z} \in P$.

(ii) lower semicontinuous (shortly, lsc) at $\bar{z} \in P$ if, for each open set $V \subset \mathbb{R}^m$ with $\Xi(\bar{z}) \cap V \neq \emptyset$, there exists $\delta > 0$ such that
\[
\Xi(z) \cap V \neq \emptyset, \quad \forall z \in B(\bar{z}, \delta).
\]
$\Xi$ is lower semicontinuous if it is lower semicontinuous at all $\bar{z} \in P$.

(iii) compact-valued if, the images $\Xi(\nu)$ of all points $\nu \in P$ are compact.
Definition 2.4. [16, 17] For an extended real-valued function \( \psi : \mathbb{R}^n \to \mathbb{R} \), \( \partial \psi(z) \) is said to be the Fréchet subdifferential of \( \psi \) at a point \( z \) of its domain, if
\[
\partial \psi(z) = \left\{ v \in \mathbb{R}^n : \liminf_{z \to \hat{z}} \frac{\psi(z) - \psi(\hat{z}) - \langle v, z - \hat{z} \rangle}{\|z - \hat{z}\|} \geq 0 \right\},
\]
given a point \( \hat{z} \), \( \partial \psi(z) \) is said to be the basic/Mordukhovich subdifferential of \( \psi \) at \( \hat{z} \), if
\[
\partial \psi(z) = \limsup_{z \to \hat{z}} \partial \psi(z).
\]

If \( \psi \) is convex, \( \psi(\hat{z}) \neq \emptyset \), then \( \partial \psi(\hat{z}) \) reduce to the subdifferential in the sense of convex analysis:
\[
\partial \psi(\hat{z}) = \left\{ v \in \mathbb{R}^n : \psi(z) - \psi(\hat{z}) \geq \langle v, z - \hat{z} \rangle, \forall z \in \mathbb{R}^n \right\}.
\]
\( \partial \psi(\hat{z}) \) is nonempty and compact when \( \psi \) is local Lipschitz continuous function, its convex hull is the Clarke subdifferential \( \partial \psi(\hat{z}) \):
\[
\hat{\partial} \psi(z) = \co \partial \psi(z),
\]
here, “co” stands for the convex hull of the set in question. Via this link between the basic and Clarke subdifferential, we have the following convex hull property which plays an important role in this paper:
\[
\co \partial (-\psi)(\hat{z}) = -\co \partial \psi(\hat{z}).
\]

Definition 2.5. [17] Let \( \Omega \) be a nonempty subset of a finite dimensional space \( Z \), given \( z \in \Omega \), the cone
\[
\hat{N}(z; \Omega) = \left\{ \xi : \langle \xi, z' - z \rangle \leq o(\|z' - z\|) \ \forall z' \in \Omega \right\},
\]
is called regularity normal cone. The cone
\[
N(z; \Omega) = \left\{ \xi : \exists \xi_k \to \xi, z_k \to z(z_k \in \Omega) : \xi_k \in \hat{N}(z_k; \Omega) \right\},
\]
is called the limiting (Mordukhovich) normal cone to \( \Omega \) at point \( z \).

Proposition 1. [17] Let \( X \subset \mathbb{R}^n \) and \( D \subset \mathbb{R}^m \) be two closed sets, \( F : \mathbb{R}^n \to \mathbb{R}^m \) be a continuously differentiable mapping. Here \( F(x) = (f_1(x), \ldots, f_m(x)) \). Let \( C = \{ x \in X : F(x) \in D \} \), at any \( \bar{x} \in C \) one has
\[
\hat{N}(\bar{x}; C) \supset \left\{ \sum_{i=1}^{m} y_i \nabla f_i(\bar{x}) + z : y \in \hat{N}(F(\bar{x}); D), z \in \hat{N}(\bar{x}; X) \right\},
\]
where \( y = (y_1, y_2, \ldots, y_m) \). On the other hand, one has
\[
N(\bar{x}; C) \subset \left\{ \sum_{i=1}^{m} y_i \nabla f_i(\bar{x}) + z : y \in N(F(\bar{x}); D), z \in N(\bar{x}; X) \right\},
\]
at any \( \bar{x} \) satisfying the constraint qualification that, the only vector \( y \in N(F(\bar{x}); D) \) for which
\[
- \sum_{i=1}^{m} y_i \nabla f_i(\bar{x}) \in N(\bar{x}; X),
\]
is \( y = (0, \ldots, 0) \).
3. **Necessary optimality condition.** While designing algorithm for liner trilevel optimization problem (LTOP), Bard [4] translated LTOP into a bilevel optimization problem via replacing the lower level problem by its KKT conditions. The numerical results show that this is an effective method. In this part, we will firstly translate TOP (1)-(3) into a bilevel optimization problem by applying KKT approach to the lower-level problem. Then we will discuss the relationship between the two problems. And we hope that those results may be useful for designing algorithm for TOP. First of all, we give the definition of optimal solution of TOP (1)-(3) based on problem (4).

**Definition 3.1.** A point \((\bar{x}, \bar{y}, \bar{z})\) is called a local optimal solution of TOP (1)-(3), if there exists an open neighborhood \(U((\bar{x}, \bar{y}, \bar{z}), \delta)\) of \((\bar{x}, \bar{y}, \bar{z})\), \(\delta > 0\) such that \(f_1(x, y, z) \geq f_1(\bar{x}, \bar{y}, \bar{z}), \) for all \((x, y, z) \in U((\bar{x}, \bar{y}, \bar{z}), \delta) \cap IR\). \((\bar{x}, \bar{y}, \bar{z})\) is called a global optimal solution, if \(\delta = \infty\) can be selected.

Since we need the global optimal solutions of parametric optimization problem (2) and (3), we assume that problem (2) has global solution for every parameter, and we let the lower-level problem to be convex for every parameter. If the lower level problem is not convex for fixed parameter, the set of feasible solution is enlarged by adding local optimal as well as stationary solutions of the lower-level problem to it. Since we know that, the KKT conditions of lower-level problem is not always sufficient and necessary, so we suppose that the following Slater’s constraint qualification holds.

**Definition 3.2.** We say that \(K(x, y)\) satisfies Slater’s constraint qualification (Slater’s CQ) at \((x, y)\) \(\in \mathbb{R}^n \times \mathbb{R}^m\), if there exists \(\bar{z}(x, y) \in \mathbb{R}^p\) such that \(g_j(x, y, \bar{z}(x, y)) < 0, j = 1, \cdots, q_3\).

Replacing the lower level problem by its KKT conditions, then TOP (1)-(3) is reformulated the following auxiliary bilevel optimization problem

\[
\begin{align*}
\min_{x, y, z, \lambda} & \quad f_1(x, y, z), \\
\text{s.t.} & \quad g_1(x, y, z) \leq 0, \\
& \quad (y, z, \lambda) \in \psi_{kkt}(x), \tag{7}
\end{align*}
\]

where \(\lambda \in \mathbb{R}^{q_3}\), \(\psi_{kkt}(x)\) is the solution set of the following parametric MPEC problem

\[
\begin{align*}
\min_{y, z, \lambda} & \quad f_2(x, y, z) \\
\text{s.t.} & \quad g_2(x, y, z) \leq 0, \\
& \quad \nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^T \lambda = 0, \\
& \quad g_3(x, y, z)^T \lambda = 0, \\
& \quad \lambda \geq 0, g_3(x, y, z) \leq 0. \tag{8}
\end{align*}
\]

Dempe and Dutta have discussed the relationship between bilevel programming problem and MPEC problem in [7]. In what follows we will consider the relationship between bilevel programs (7)-(8) and TOP (1)-(3). For convenience, we define the following set

\[
\Lambda(x, y, z) := \left\{ \lambda \in \mathbb{R}^{q_3} : \quad \nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^T \lambda = 0, \lambda \geq 0, g_3(x, y, z)^T \lambda = 0, \right\}.
\]

**Theorem 3.3.** Assume the following conditions \((i)\) and \((ii)\) hold:

\[(i) \quad f_3(x, y, \cdot), g_3(x, y, \cdot), j = 1, 2, \cdots, q_3 \text{ are convex continuously differentiable functions on } K(x, y), \text{ and the inequality constraint set for the middle-level } Y \text{ is closed.}\]
(ii) The lower-level problem satisfies Slater’s CQ at any \((x, y) \in X \times Y\).

Then we have the following two results:

(a) If \((\bar{x}, \bar{y}, \bar{z})\) is a local optimal solution of TOP (1)-(3), then for each \(\lambda \in \Lambda(x, y, z)\), \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) is a local optimal solution of problem (7)-(8).

(b) For \(\lambda \in \Lambda(x, y, z)\), if \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) is a local optimal solution of problem (7)-(8), then \((\bar{x}, \bar{y}, \bar{z})\) is a local optimal solution of TOP (1)-(3).

Proof. (a) Let \((\bar{x}, \bar{y}, \bar{z})\) be a local optimal solution of TOP (1)-(3) restricting on \(U((\bar{x}, \bar{y}, \bar{z}), \delta)\), \(\delta > 0\) then \((\bar{y}, \bar{z})\) must be a global optimal solution of parametric problem (2) for fixed point \(\bar{x}\). Since the lower-level problem is a convex optimization problem, \(Y\) is a closed set, and Slater’s CQ is satisfied at \((\bar{x}, \bar{y})\), according to Theorem 2.1 of \([7]\), it follows that for each \(\bar{\lambda} \in \Lambda(\bar{x}, \bar{y}, \bar{z})\), \((\bar{y}, \bar{z}, \bar{\lambda})\) is a global optimal solution of parametric MPEC problem (8) for fixed point \(\bar{x}\). Further, \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) is a feasible point of problem (7).

To show that \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) is a local optimal solution of problem (7), it suffices to show that, there is no other feasible point \((x, y, z, \lambda)\) of problem (7) on \(U((\bar{x}, \bar{y}, \bar{z}), \delta) \times \mathbb{R}^p\) such that

\[
f_1(x, y, z) < f_1(\bar{x}, \bar{y}, \bar{z}).
\]

On the contrary, suppose that there is a feasible point \((x, y, z, \lambda)\) of problem (7) such that

\[
f_1(x, y, z) < f_1(\bar{x}, \bar{y}, \bar{z}).
\]

It is obvious that \((y, z, \lambda)\) is an optimal solution of parametric MPEC problem (8). Because the Slater’s CQ is satisfied on \(X \times Y\), from Theorem 2.3 in \([7]\), we know that \((y, z)\) is a global optimal solution of parametric bilevel optimization problem(2). Further \((x, y, z)\) is a feasible point of TOP (1)-(3). Combining this with (9) we known that this contradict with the fact that \((\bar{x}, \bar{y}, \bar{z})\) is an optimal solution of TOP (1)-(3) on \(U((\bar{x}, \bar{y}, \bar{z}), \delta)\),

(b) Suppose that \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) is an optimal solution of problem (7) on \(U((\bar{x}, \bar{y}, \bar{z}), \delta) \times \mathbb{R}^p\), \(\delta > 0\). Then there is no feasible point \((x, y, z, \lambda)\) \(\in U((\bar{x}, \bar{y}, \bar{z}), \delta) \times \mathbb{R}^p\) such that

\[
f_1(x, y, z) < f_1(\bar{x}, \bar{y}, \bar{z}).
\]

Combining the condition (1) and (ii) with Theorem 2.3 in \([7]\), it is easy to see that \((\bar{y}, \bar{z})\) is a global solution of parametric bilevel problem (2), further \((\bar{x}, \bar{y}, \bar{z})\) is a feasible point of TOP (1)-(3). We now prove that \((\bar{x}, \bar{y}, \bar{z})\) is an optimal solution of TOP (1)-(3) on \(U((\bar{x}, \bar{y}, \bar{z}), \delta, \delta > 0\). On the contrary, suppose that \((\bar{x}, \bar{y}, \bar{z})\) is not an optimal solution, then there exists another \((x, y, z) \in U((\bar{x}, \bar{y}, \bar{z}), \delta)\) feasible to TOP (1)-(3), such that

\[
f_1(x, y, z) < f_1(\bar{x}, \bar{y}, \bar{z}).
\]

From Theorem 2.1 in \([7]\), it follows that for each \(\lambda \in \Lambda(x, y, z)\), \((y, z, \lambda)\) is a global solution of parametric MPEC problem (8), furthermore \((x, y, z, \lambda)\) is a feasible point of problem (7). This contradicts the fact that \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) is a local optimal solution.

\[\square\]

Remark 1. Under the conditions of Theorem 3.3, it is easy to verify that the result (b) also holds in the case of global optimal solution.
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The following Example 1 shows that the assumptions of Theorem 3.3 are reasonable. Moreover, we illustrate that the Slater’s CQ is important by Example 2.

Example 1. We consider the following TOP
\[
\min_{x, y, z} \quad x + y + z \\
\text{s.t.} \quad 1 - x \leq 0, \quad x - 5 \leq 0, \quad (y, z) \in \psi(x),
\]
where \(\psi(x)\) is the solution set of following parametric problem
\[
\min_{y, z} \quad y^2 - 2z + x^2 \\
\text{s.t.} \quad 1 - y \leq 0, \quad y - 5 \leq 0, \quad z \in \omega(x, y),
\]
where \(\omega(x, y)\) is the solution set of the following parametric problem
\[
\min_{z} \quad -z \\
\text{s.t.} \quad z - xy \leq 0.
\]
Through some calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}) = (1, 1, 1)\). And it is easy to verify that problem (12) satisfies Slater’s CQ on \(X \times Y = [1, 5] \times [1, 5]\). KKT transformation can be shown as the following bilevel optimization problem.

\[
\min_{x, y, z, \lambda} \quad x + y + z \\
\text{s.t.} \quad 1 - x \leq 0, \quad x - 5 \leq 0, \quad (y, z, \lambda) \in \psi_{kkt}(x),
\]
where \(\psi_{kkt}(x)\) is the solution set of the following parametric problem
\[
\min_{y, z, \lambda} \quad y^2 - 2z + x^2 \\
\text{s.t.} \quad \lambda \geq 0, \quad z - xy \leq 0, \quad 1 - y \leq 0, \quad y - 5 \leq 0, \quad \lambda(z - xy) = 0, \quad -1 + \lambda = 0.
\]
Through a series of calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = (1, 1, 1, 1)\). That is we can obtain the solution of TOP (10)-(12) by solving bilevel problem (13)-(14).

Example 2. We consider the following TOP
\[
\min_{x, y, z} \quad x + y + z \\
\text{s.t.} \quad 1 - x \leq 0, \quad x - 8 \leq 0, \quad (y, z) \in \psi(x),
\]
where \(\psi(x)\) is the solution set of following parametric problem
\[
\min_{y, z} \quad (y - 1)^2 + xz^2 \\
\text{s.t.} \quad -y \leq 0, \quad y - 3 \leq 0, \quad z \in \omega(x, y),
\]
where \(\omega(x, y)\) is the solution set of the following parametric problem
\[
\min_{z} \quad x^2y^2z \\
\text{s.t.} \quad z^2 \leq 0.
\]
Through some calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}) = (0, 1, 1)\). And it is easy to verify that the lower level problem satisfies Slater’s CQ on \(X \times Y = [1, 8] \times [0, 3]\). KKT transformation can be shown as the following bilevel optimization problem.

\[
\min_{x, y, z, \lambda} \quad x + y + z \\
\text{s.t.} \quad 1 - x \leq 0, \quad x - 8 \leq 0, \quad (y, z, \lambda) \in \psi_{kkt}(x),
\]
where \(\psi_{kkt}(x)\) is the solution set of the following parametric problem
\[
\min_{y,z,\lambda} \quad (y - 1)^2 + x^2 \\
\text{s.t.} \quad \lambda \geq 0, \quad -y \leq 0, \\
y - 3 \leq 0, \quad z^2 \leq 0, \\
\lambda z^2 = 0, \quad x^2 y^2 + 2\lambda z = 0.
\] (19)

Through a series of calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = (1, 0, 0, \lambda)\), here, \(\lambda \geq 0\). That is, we can not obtain the solution of TOP (15)-(17) by bilevel problem (18)-(19).

**Remark 2.** From Theorem 3.3 we can see that the Slater’s CQ need to be satisfied on \(X \times Y\). Since the Slater’s CQ is used to guarantee the existence of KKT conditions, we can replace the Slater’s CQ by more weaker CQ such as Cottle’s CQ.

For parametric MPEC problem (8), we denote it’s feasible set as
\[
S(x) := \left\{ (y, z, \lambda) \in R^{m+p+q_3} : g_2(x, y, z) \leq 0, g_3(x, y, z) \leq 0, g_3(x, y, z)^\top \lambda = 0, \lambda \geq 0, \nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^\top \lambda = 0 \right\},
\]
then problem (8) is equal to
\[
\varphi_{kkt}(x) := \min_{y, z, \lambda} \{ f_2(x, y, z) : (y, z, \lambda) \in S(x) \}. \tag{20}
\]

Then auxiliary bilevel optimization problem (7)-(8) is a bridge between trilevel optimization problem and bilevel optimization problem. In order to get the necessary optimality condition of TOP (1)-(3), we should firstly consider the optimality condition of problem (7)-(8). As we know that, in general \(\varphi_{kkt}\) is non-differentiable, so we need to consider its subdifferential. Fortunately, Guo et al. [11] Dempe et al. [8] gave the calculating method of subdifferential of optimal-valued function for parametric MPEC problem (see Theorem 3.2 in [8]). We can obtain the subdifferential of \(\varphi_{kkt}\) by the way the same as Theorem 3.2 in [8]. First we need to define Lagrange functions and some constraint criterions.

For a fixed point \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \in gph S\), we have the following partition of the indices for the complementarity functions of \(S(\bar{x})\)
\[
\alpha = \alpha(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) := \{ i : \bar{\lambda}_i = 0, g_3^i(\bar{x}, \bar{y}, \bar{z}) < 0 \},
\]
\[
\beta = \beta(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) := \{ i : \bar{\lambda}_i = 0, g_3^i(\bar{x}, \bar{y}, \bar{z}) = 0 \},
\]
\[
\gamma = \gamma(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) := \{ i : \bar{\lambda}_i > 0, g_3^i(\bar{x}, \bar{y}, \bar{z}) = 0 \}.
\]

We define
\[
\alpha_k = \alpha_k(\bar{x}, y_k, z_k, \lambda_k) := \{ i : \lambda_{ki} = 0, g_3^i(\bar{x}, y_k, z_k) < 0 \},
\]
\[
\beta_k = \beta_k(\bar{x}, y_k, z_k, \lambda_k) := \{ i : \lambda_{ki} = 0, g_3^i(\bar{x}, y_k, z_k) = 0 \},
\]
\[
\gamma_k = \gamma_k(\bar{x}, y_k, z_k, \lambda_k) := \{ i : \lambda_{ki} > 0, g_3^i(\bar{x}, y_k, z_k) = 0 \}.
\]

The Lagrange-type function, associated with the parametric problem in (20) is
\[
L(x, y, z, \lambda, \eta^{g_2}, \eta^{g_3}, \eta^\lambda) := f_2(x, y, z) + g_2(x, y, z)^\top \eta^{g_2}
+ (\nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^\top \lambda)^\top \eta^{g_3} + \lambda^\top \eta^\lambda - g_4(x, y, z)^\top \eta^g,
\]
where \(\eta^{g_2} \in R^{g_2}, \eta^{g_3} \in R^{g_3}, \eta^\lambda \in R^\lambda\). The singular Lagrange-type function associated with the parametric problem in (20) is
\[
L_0(x, y, z, \lambda, \eta^{g_2}, \eta^{g_3}, \eta^\lambda) := g_2(x, y, z)^\top \eta^{g_2} + \lambda^\top \eta^\lambda
- g_3(x, y, z)^\top \eta^{g_3} + (\nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^\top \lambda)^\top \eta^{g_3},
\]
For simplicity, we denote

\[ L(x, y, z, \lambda) = L(x, y, z, \lambda, \eta^{g_2}, \eta^{f_{g_3}}, \eta^{g_3}, \eta^{\lambda}), \]

\[ L_0(x, y, z, \lambda) = L_0(x, y, z, \lambda, \eta^{g_2}, \eta^{f_{g_3}}, \eta^{g_3}, \eta^{\lambda}). \]

The derivative of \( L(x, y, z, \lambda) \) and \( L_0(x, y, z, \lambda) \) with respect to \((x, y, z, \lambda)\) at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) are denoted as \( \nabla L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \), \( \nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \). Here “\( \nabla \)” stand for “\( \nabla_{x,y,z,\lambda} \)” similarly hereinafter. The partial derivative of \( L(x, y, z, \lambda) \) with respect to \( x, y, z \) and \( \lambda \) at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) are denoted as \( \nabla_x L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}), \nabla_y L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}), \nabla_z L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \) and \( \nabla_\lambda L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \). Similarly, the partial derivative of \( L_0(x, y, z, \lambda) \) can be denoted as \( \nabla_x L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}), \nabla_y L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}), \nabla_z L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \) and \( \nabla_\lambda L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) \).

We define the set of M-type multipliers associated with problem (8) by

\[ \Lambda^{cm}(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = \Lambda^{cm}, \]

\[ \Lambda^{cm} = \left\{ (\eta^{g_2}, \eta^{f_{g_3}}, \eta^{g_3}, \eta^{\lambda}) : \begin{array}{l}
\eta^{g_2} \geq 0, \quad \eta^{g_3} = 0, \quad \eta^{\lambda} = 0, \\
\eta^{g_3} < 0 \wedge \eta^{\lambda} < 0 \Leftrightarrow \eta^{g_2} \eta^{g_3} = 0, \quad \eta^{g_3} = 0, \quad \eta^{\lambda} = 0,
\end{array} \right\}. \]

We define the set \( \Lambda^{cm, \lambda}_{\bar{y}, \bar{z}, \bar{\lambda}} \) which is obtained by replacing the gradients of \( f_2, g_2, \lambda, g_3, \nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z) \) in equality \( \nabla L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0 \) by their partial derivatives with respect to \( y, z, \lambda \), by

\[ \Lambda^{cm, \lambda}_{\bar{y}, \bar{z}, \bar{\lambda}} = \left\{ (\eta^{g_2}, \eta^{f_{g_3}}, \eta^{g_3}, \eta^{\lambda}) : \begin{array}{l}
\eta^{g_2} \geq 0, \quad \eta^{g_3} = 0, \quad \eta^{\lambda} = 0, \\
\eta^{g_3} < 0 \wedge \eta^{\lambda} < 0 \Leftrightarrow \eta^{g_2} \eta^{g_3} = 0, \quad \eta^{g_3} = 0, \quad \eta^{\lambda} = 0,
\end{array} \right\}. \]

The following solution mapping of problem (8), which plays a significant role in the next theorem, given by

\[ \psi_{kkl}(x) := \left\{ (y, z, \lambda) \in S(x) : f_2(x, y, z) \leq \varphi_{kkl}(x) \right\}. \]

To proceed in this part, we introduce the following two regularity conditions at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\). They are firstly defined in [8].

\[ \begin{align*}
\eta^{g_2} & \geq 0, \quad \eta^{g_3} = 0, \quad \eta^{\lambda} = 0, \\
\eta^{g_3} & < 0 \wedge \eta^{\lambda} < 0 \Leftrightarrow \eta^{g_2} \eta^{g_3} = 0, \quad \eta^{g_3} = 0, \quad \eta^{\lambda} = 0,
\end{align*} \]

\[ \Rightarrow \nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \quad \nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \]

\[ \nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \quad \nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0. \]

The upper-level regularity condition is

\[ \begin{align*}
\nabla g_1(\bar{x}, \bar{y}, \bar{z})^\top \epsilon = 0, \\
\epsilon \geq 0, \quad g_1(\bar{x}, \bar{y}, \bar{z})^\top \epsilon = 0,
\end{align*} \]

\[ \Rightarrow \epsilon = 0, \]
where, $\varepsilon \in \mathbb{R}^{q_1}$. This regularity condition will be used to ensure Proposition 1 correct. And they are useful in the proof process of the main theorem. Next we will give the subdifferential of the optimal value function $\varphi_{kkt}$ at $\bar{x}$ based on Theorem 3.2 in [8].

**Theorem 3.4.** Assume that $\psi_{kkt}$ is inner semicompact at $\bar{x}$, and regularity condition (21) holds at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ for all $(\bar{y}, \bar{z}, \bar{\lambda}) \in \psi_{kkt}(\bar{x})$. Then we have the subdifferential upper estimate

$$
\partial \varphi_{kkt}(\bar{x}) \subset \bigcup_{(\bar{y}, \bar{z}, \bar{\lambda}) \in \psi_{kkt}(\bar{x})} \left\{ \nabla_x f_2(\bar{x}, \bar{y}, \bar{z}) + \nabla_x g_2(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_2} + \nabla_{xx} f_3(\bar{x}, \bar{y}, \bar{z})^\top \eta^{f_3} - \nabla_{x \bar{y}} g_3(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_3} \right\}. 
$$

(24)

If in addition regularity condition (22) is satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, for all $(\bar{y}, \bar{z}, \bar{\lambda}) \in \psi_{kkt}(\bar{x})$, then the value function $\varphi_{kkt}$ is Lipschitz continuous around $\bar{x}$.

**Proof.** Since the proof process is similar to Theorem 3.2 in [8] we omit it here. \qed

We now consider bilevel optimization problem (7)-(8) in the optimal value reformulation. Let

$$
\Omega := \left\{ (x, y, z, \lambda) : x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}^p, \lambda \in \mathbb{R}^{q_3}, g_1(x, y, z) \leq 0 \right\}.
$$

Next we will define an abstract constraint for the problem and from here on, we define

$$
\varphi(x, y, z, \lambda) := f_2(x, y, z) - \varphi_{kkt}(x).
$$

Then problem (7) has the form

$$
\min_{x, y, z, \lambda} f_1(x, y, z) \ s.t. \ (x, y, z, \lambda) \in \Omega, \ \varphi(x, y, z, \lambda) \leq 0.
$$

(25)

We consider the following weak form of the well known basic constraint qualification

$$
\partial \varphi(x, \bar{y}, \bar{z}, \bar{\lambda}) \cap \{-bdN((x, \bar{y}, \bar{z}, \bar{\lambda}); \Omega)\} = \emptyset,
$$

(26)

here “bd” stands for the topological boundary of the set in question. We denote the feasible set of problem (25) as

$$
C := \{ (x, y, z, \lambda) \in \Omega : \varphi(x, y, z, \lambda) \leq 0 \}.
$$

From Theorem 3.3 in [9] we have the following Lemma which provides convenience for the calculation of normal cone. This Lemma needs the assumption that the constraint qualification (26) holds.

**Lemma 3.5.** Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ be a feasible point of problem (25). Assume that $\Omega$ be convex and $\varphi$ be Lipschitz continuous around $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$. Then

$$
N((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}); C) \subseteq \bigcup_{r \geq 0} r \partial \varphi(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) + N((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}); \Omega)
$$

provided that constraint qualification (26) holds at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$.

Next we will state one of the main results of this paper, which is a necessary optimality condition for problem (7)-(8).

**Theorem 3.6.** Let $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ be a local optimal solution of problem (7)-(8). Assume that the following conditions hold:

(i) $f_2$ is local Lipschitz continuous and differentiable around $(\bar{x}, \bar{y}, \bar{z})$, $f_1, g_1, g_2$ are differentiable around $(\bar{x}, \bar{y}, \bar{z})$, and $f_3, g_3$ are twice continuously differentiable around $(\bar{x}, \bar{y}, \bar{z})$;
(ii) The set-valued mapping \( \psi_{\text{kkt}} \) is inner semicompact at \( \bar{x}, \Omega \) is convex and constraint qualification (26) holds at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\);

(iii) The regularity condition (21) and (22) hold at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) for all \((\bar{y}, \bar{z}, \bar{\lambda}) \in \psi_{\text{kkt}}(\bar{x})\). And the upper-level regular condition (23) is satisfied at \((\bar{x}, \bar{y}, \bar{z})\) the following conditions hold:

\[
\nabla_x f_1(\bar{x}, \bar{y}, \bar{z}) + r \nabla_x f_2(\bar{x}, \bar{y}, \bar{z}) + \nabla_x g_1(\bar{x}, \bar{y}, \bar{z})^\top \varepsilon - r \Sigma_{k=1}^{n+1} \xi_k \left\{ \nabla_x f_2(\bar{x}, y_k, z_k) \right\} + \nabla_x g_2(\bar{x}, y_k, z_k)\nabla_{z-y} g_2(\bar{x}, y_k, z_k)^\top \eta_{k_2}^{g_2} + \nabla_{zz} g_3(\bar{x}, y_k, z_k)^\top \eta_{k_3} = 0, \quad (27)
\]

\[
\nabla_y f_2(\bar{x}, y_k, z_k) + \nabla_y g_2(\bar{x}, y_k, z_k)^\top \eta_{k_2}^{g_2} + \nabla_{yz} g_2(\bar{x}, y_k, z_k)^\top \eta_{k_3}^{g_2} + \nabla_{zz} g_3(\bar{x}, y_k, z_k)^\top \eta_{k_3} = 0, \quad (28)
\]

\[
\nabla_z f_2(\bar{x}, y_k, z_k) + \nabla_z g_2(\bar{x}, y_k, z_k)^\top \eta_{k_2}^{g_2} + \nabla_{zz} g_3(\bar{x}, y_k, z_k)^\top \eta_{k_3} = 0, \quad (29)
\]

\[
\nabla_z g_3(\bar{x}, y_k, z_k)^\top \eta_{k_3} + \eta_{k_3}^{g_2} = 0, \quad (30)
\]

\[
\nabla_y f_1(\bar{x}, \bar{y}, \bar{z}) + r \nabla_y f_2(\bar{x}, \bar{y}, \bar{z}) + \nabla_{yz} g_1(\bar{x}, \bar{y}, \bar{z})^\top \varepsilon = 0, \quad (31)
\]

\[
\nabla_z f_1(\bar{x}, \bar{y}, \bar{z}) + r \nabla_z f_2(\bar{x}, \bar{y}, \bar{z}) + \nabla_{zz} g_1(\bar{x}, \bar{y}, \bar{z})^\top \varepsilon = 0, \quad (32)
\]

\[
\varepsilon \geq 0, \quad g_1(\bar{x}, \bar{y}, \bar{z})^\top \varepsilon = 0, \quad (33)
\]

\[
\eta_{k_2}^{g_2} \geq 0, \quad g_2(\bar{x}, y_k, z_k)^\top \eta_{k_2}^{g_2} = 0, \quad (34)
\]

\[
\eta_{k_3}^{g_2} = 0, \quad \eta_{k_3}^{g_2} = 0, \quad i \in \alpha_k, \quad \eta_{k_3}^{g_2} = 0, \quad i \in \gamma_k, \quad (35)
\]

\[
\eta_{k_3}^{g_2} < 0 \land \eta_{k_3}^{g_2} < 0 \lor \eta_{k_3}^{g_2} \eta_{k_3}^{g_2} = 0, \quad i \in \beta_k. \quad (36)
\]

Proof. Let \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) be a local optimal solution of problem (25). Since \(f_1\) is continuously differentiable at \((\bar{x}, \bar{y}, \bar{z})\), it follows that

\[
0 \in \nabla f_1(\bar{x}, \bar{y}, \bar{z})^\top + N((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}); C).
\]

Due to \(\psi_{\text{kkt}}\) is inner semicompact at \(\bar{x}\), and regularity condition (21) and (22) hold at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\) for all \((\bar{y}, \bar{z}, \bar{\lambda}) \in \psi_{\text{kkt}}(\bar{x})\), from Theorem 3.4 it follows that optimal value function \(\varphi_{\text{kkt}}\) is Lipschitz continuous around \(\bar{x}\). Since \(f_2\) is local Lipschitz continuous at \((\bar{x}, \bar{y}, \bar{z})\), \(\varphi\) is Lipschitz continuous around \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\). Combining this with the convexity of \(\Omega\), from Lemma 3.5 it follows that there exists \(r \geq 0\) such that

\[
0 \in \nabla f_1(\bar{x}, \bar{y}, \bar{z})^\top + \nabla f_2(\bar{x}, \bar{y}, \bar{z}) + v \in r \partial(\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})) \times \{0\}^{m+p+q_3}.
\]

Furthermore, we have \(v \in N((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}); \Omega)\) such that

\[
-(\nabla f_1(\bar{x}, \bar{y}, \bar{z}) + r \nabla f_2(\bar{x}, \bar{y}, \bar{z}) + v) \in r \partial(-\varphi_{\text{kkt}})(\bar{x}) \times \{0\}^{m+p+q_3}.
\]

According to (5)-(6), it follows that

\[
\nabla f_1(\bar{x}, \bar{y}, \bar{z}) + r \nabla f_2(\bar{x}, \bar{y}, \bar{z}) + v \in r \co \partial \varphi_{\text{kkt}}(\bar{x}) \times \{0\}^{m+p+q_3}, \quad (37)
\]
Next, we will evaluate the basic normal cone $N((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}); \Omega)$. Since the upper-level regular condition (23) is satisfied at $(\bar{x}, \bar{y}, \bar{z})$, applying Proposition 1 and through some calculations one obtains

$$
N((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}); \Omega) \in \left\{ \begin{array}{l}
\nabla_x g_1(\bar{x}, \bar{y}, \bar{z})^T \varepsilon \\
\nabla_y g_1(\bar{x}, \bar{y}, \bar{z})^T \varepsilon \\
\nabla_z g_1(\bar{x}, \bar{y}, \bar{z})^T \varepsilon \\
0
\end{array} : \varepsilon \geq 0, \ g_1(\bar{x}, \bar{y}, \bar{z})^T \varepsilon = 0 \right\},
$$

provided the upper-level regular condition (23) holds. From this formula it follows that (33) holds.

Taking $\nu \in \text{co} \partial \varphi_{kkt}(\bar{x})$ and applying Carathéodory’s theorem [15], we find $\xi_k \in \mathbb{R}$, and $\nu_k \in \mathbb{R}^m$ with $k = 1, \ldots, n + 1$ such that

$$
\nu = \sum_{k=1}^{n+1} \xi_k \nu_k, \quad \sum_{k=1}^{n+1} \xi_k = 1, \quad \xi_k \geq 0, \quad \nu_k \in \partial \varphi_{kkt}(\bar{x}), \quad \text{for } k = 1, \ldots, n + 1.
$$

From Theorem 3.4 it is easy to see that for $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$, one has

$$
\nu_k = \nabla_x f_2(\bar{x}, y_k, z_k) + \nabla_x g_2(\bar{x}, y_k, z_k) \sum_{n} \sum_{n} \eta_k f_g \lambda + \nabla_x g_3(\bar{x}, y_k, z_k) \sum_{n} \sum_{n} \eta_k f_g \lambda
$$

$$
+ (\nabla_x g_3(\bar{x}, y_k, z_k) \sum_{n} \sum_{n} \eta_k f_g \lambda) \sum_{n} \sum_{n} \eta_k f_g \lambda - \nabla_x g_3(\bar{x}, y_k, z_k) \sum_{n} \sum_{n} \eta_k f_g \lambda,
$$

here, $(\eta_k^g, \eta_k^f, \eta_k^g) \in \Lambda_{\text{mm}} y_k, z_k, \lambda_k$. Combining (37)-(39), we can obtain conditions (27), (31) and (32) easily. According to $(\eta_k^g, \eta_k^f, \eta_k^g) \in \Lambda_{\text{mm}} y_k, z_k, \lambda_k$, we can obtain conditions (28)-(30) and (34)-(36).

**Remark 3.** In Theorem 3.6, the solution set mapping $\psi_{kkt}$ of parametric MPEC problem (8) need to be inner semicompact at $\bar{x}$. From Definition 2.2, we can assert that, if $\Pi = \{(x, x, z, \lambda) : g_1(x, y, z) \leq 0, (y, z, \lambda) \in S(x)\}$ is bounded, $\psi_{kkt}$ will be inner semicompact at $\bar{x}$.

The next examples are used to show that Theorem 3.6 is reasonable. Since it is a very tedious verification process, we only do simple describe.

**Example 3.** We consider the following TOP

$$
\begin{align*}
\min_{x, y, z} & \quad -x - y - z \\
\text{s.t.} & \quad x + y + z \leq 0, \ (y, z) \in \psi(x),
\end{align*}
$$

where $\psi(x)$ is the solution set of following parametric problem

$$
\begin{align*}
\min_{y, z} & \quad -y - z + x \\
\text{s.t.} & \quad y + z - 2z \leq 0, \ z \in \omega(x, y),
\end{align*}
$$

where $\omega(x, y)$ is the solution set of the following parametric problem

$$
\begin{align*}
\min_{z} & \quad -z \\
\text{s.t.} & \quad z - x - y \leq 0.
\end{align*}
$$

Through some calculation, we can get the optimal solution $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0)$. KKT transformation can be shown as the following bilevel optimization problem.

$$
\begin{align*}
\min_{x, y, z, \lambda} & \quad -x - y - z \\
\text{s.t.} & \quad x + y + z \leq 0, \ (y, z, \lambda) \in \psi_{kkt}(x),
\end{align*}
$$

where $\psi_{kkt}(x)$ is the solution set of the following parametric problem

$$
\begin{align*}
\min_{y, z, \lambda} & \quad -y - z + x \\
\text{s.t.} & \quad \lambda \geq 0, \\
& \quad y + z - 2x \leq 0, \ z - x - y \leq 0, \\
& \quad \lambda - 1 = 0, \ \lambda(z - x - y) = 0.
\end{align*}
$$
Through a series of calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = (0, 0, 0, 1)\). We can verify that problem (43) satisfies all the assumptions of Theorem 3.6, and there exist \(r = 1, n = 0, \varepsilon = 2, \xi = 1, \eta^f g k = 1, \eta^g z = 4, \eta^g z = 3, \eta^f = -1\), such that conditions (27)-(36) hold.

**Example 4.** We consider the following TOP

\[
\begin{align*}
\min_{x,y,z} & \quad x^2 \\
\text{s.t.} & \quad 1 - x \leq 0, \quad (y, z) \in \psi(x),
\end{align*}
\]

where \(\psi(x)\) is the solution set of following parametric problem

\[
\begin{align*}
\min_{y,z} & \quad y^2 \\
\text{s.t.} & \quad z + x - y \leq 0, \\
& \quad y - 80 \leq 0, \quad z \in \omega(x, y),
\end{align*}
\]

where \(\omega(x, y)\) is the solution set of the following parametric problem

\[
\begin{align*}
\min_z & \quad z - xy \\
\text{s.t.} & \quad 2 - z \leq 0.
\end{align*}
\]

It is obvious that parametric problem (47) is a convex optimization problem for fixed point \((x, y)\). Through a series of calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}) = (1, 3, 2)\). KKT transformation can be shown as the following bilevel optimization problem.

\[
\begin{align*}
\min_{x,y,z,\lambda} & \quad x^2 \\
\text{s.t.} & \quad 1 - x \leq 0, \quad (y, z) \in \psi_{kkt}(\bar{x}),
\end{align*}
\]

where \(\psi_{kkt}(x)\) is the solution set of the following parametric problem

\[
\begin{align*}
\min_{y,z,\lambda} & \quad y^2 \\
\text{s.t.} & \quad \lambda \geq 0, \quad y - 80 \leq 0, \\
& \quad 2 - z \leq 0, \quad z + x - y \leq 0, \\
& \quad 1 - \lambda = 0, \quad \lambda(2 - z) = 0.
\end{align*}
\]

Through a series of calculation, we can get the optimal solution \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = (1, 3, 2, 1)\). We can verify that all regularity conditions which are needed in Theorem 3.6 hold at \((\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})\), \(\varphi_{kkt}\) is inner semicompact at \(\bar{x}\), and there exist \(r = 0, n = 0, \xi = 1, \varepsilon = 6, (\eta^f_1, \eta^g_2) = (1, 0), \eta^g z = -6, \eta^f g k = 0, \eta^f = 0, \eta^f = 0\), such that conditions (27)-(36) hold.

Next, we will discuss the necessary condition for TOP (1)-(3).

**Theorem 3.7.** Let \((\bar{x}, \bar{y}, \bar{z})\) be a local optimal solution of TOP (1)-(3). Assume that all the conditions of Theorem 3.3 hold, and for \(\bar{\lambda} \in \Lambda(\bar{x}, \bar{y}, \bar{z})\) all conditions of Theorem 3.6 hold. Then we can find \(\xi_k \in R\), with \(k = 1, \ldots, n + 1\) such that \(\Sigma_{k=1}^{n+1} \xi_k = 1, \xi_k \geq 0, \r \geq 0, \) and \(\eta^g z \in R^v, \eta^f g k \in R^p, \eta^g z \in R^v, \eta^f k \in R^p, \varepsilon \in R\), \((\bar{y}_k, \bar{z}_k, \bar{\lambda}_k) \in \psi_{kkt}(\bar{x})\) such that the conditions (27)-(36) hold.

**Proof.** According to result (a) of Theorem 3.3 and Theorem 3.6, we can obtain this theorem easily, so we omit it here.
4. Existence of optimal solution. In this section we will consider the existence theorem of optimal solution for TOP (1)-(3). We know that inequality constraint set for the upper-level is

\[ X = \{ x : \exists (y, z) \text{ s.t. } g_1(x, y, z) \leq 0 \}. \]

The inducible region of TOP (1)-(3) is

\[ IR = \{ (x, y, z) \in R^{n+m+p} : (x, y, z) \in X \times \psi(x) \}. \]

Since the constraint functions \( g_1(x, y, z) \) have something to do with \( (y, z) \), it is difficult to guarantee the compactness of \( IR \) even if we assume that \( \psi(x) \) is compact. Then in this part we consider the reduced model of TOP (1)-(3) that constraint function \( g_1 \) contained only variable \( x \). That is

\[
\begin{align*}
\min_{x,y,z} & \quad f_1(x, y, z) \\
\text{s.t.} & \quad g_1(x) \leq 0, (y, z) \in \psi(x),
\end{align*}
\]

(50)

where \( \psi(x) \) is the solution set of the following problem

\[
\min_{y,z} f_2(x, y, z) \\
\text{s.t.} & \quad g_2(x, y, z) \leq 0, z \in \omega(x, y),
\]

(51)

where \( \omega(x, y) \) is the solution set of the following problem

\[
\min_{y,z} f_3(x, y, z) \\
\text{s.t.} & \quad g_3(x, y, z) \leq 0,
\]

(52)

here \( x \in R^n, y \in R^m \) and \( z \in R^p; f_i : R^n \times R^m \times R^p \to R, i = 1, 2, 3; g_1 : R^n \to R^n; g_i : R^n \times R^m \times R^p \to R^p, i = 2, 3. \)

Similarly to problem (1)-(3), the KKT transformation can be shown as follows

\[
\begin{align*}
\min_{x,y,z,\lambda} & \quad f_1(x, y, z), \\
\text{s.t.} & \quad g_1(x) \leq 0, (y, z, \lambda) \in \psi_{kkt}(x),
\end{align*}
\]

(53)

where \( \psi_{kkt}(x) \) is the solution set of the following parametric MPEC problem

\[
\begin{align*}
\min_{y,z,\lambda} & \quad f_2(x, y, z) \\
\text{s.t.} & \quad g_2(x, y, z) \leq 0, \\
\nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)\lambda & = 0, \\
\nabla_z g_3(x, y, z)\lambda & = 0, \\
\lambda & \geq 0, g_3(x, y, z) \leq 0.
\end{align*}
\]

(54)

We denote the inequality constraint set for the upper-level optimization program as

\[ X' := \{ x \in R^n : g_1(x) \leq 0 \}. \]

The feasible set of this problem is

\[ IR' = \{ (x, y, z, \lambda) \in R^{n+m+p+q_1} : g_1(x) \leq 0, (y, z, \lambda) \in \psi_{kkt}(x) \}. \]

(55)

The solution set mapping \( \psi_{kkt} \) of the parametric programming problem (54) is generally discontinuity. Therefore, set-valued mapping \( \psi_{kkt}(\cdot) \) don’t have continuous selection function in general. If \( \psi_{kkt}(x) \) is a singleton, we have the following existence theorem of optimal solution for problem (53)-(54).

Theorem 4.1. Let \( X' \) be a non-empty compact set, \( S(x) \) is nonempty and compact for every \( x \in X' \). Suppose that, for any \( x \in X' \), \( \psi_{kkt}(x) \) is singleton, \( \psi_{kkt}(\cdot) \) is upper semicontinuous, and \( f_1 \) is a continuous function. Then problem (53)-(54) has a global optimal solution provided it has a feasible solution.
Proof. Since $\psi_{kkt}(\cdot)$ is upper semicontinuous, $S(x)$ is nonempty and compact for every $x \in X'$, then $ghp\psi_{kkt} := \{(x, y, z, \lambda) : (y, z, \lambda) \in \psi_{kkt}(x)\}$ is a non-empty closed set. Because $X'$ is a non-empty compact set, and the intersection of $ghp\psi_{kkt}$ with $X' \times \mathbb{R}^{m+p+p}$ is compact, we can derive that $IR'$ is compact. According to Weierstrass Theorem, it follows that the optimal solution of problem (53)-(54) is obtained.

Next we will give the existence theorem of optimal solution for TOP (50)-(52).

**Theorem 4.2.** Assume that $f_3$, $g_3$ are continuously differentiable around $(\bar{x}, \bar{y}, \bar{z})$, $f_3(x, y, \cdot), g_3(x, y, \cdot)$ are convex functions on $K(x, y)$, and the inequality constraint set for the middle-level $Y$ is closed, and Slater’s CQ for the lower-level problem is satisfied at any $(x, y) \in X \times Y$. Moreover we assume that $X'$ is a non-empty compact set, $\psi_{kkt}(x)$ is singleton for every $x \in X$, $\psi_{kkt}(\cdot)$ is upper semicontinuous, and $f_1$ is a continuous function. Then TOP (50)-(52) has a global optimal solution provided it has a feasible solution.

Proof. According to Remark 1 and Theorem 4.1, we can obtain this theorem easily.

5. **Conclusions.** In this paper, we mainly studied the optimality conditions for TOP. We supposed that the upper-level decision maker is able to influence the middle-level decision maker’s choice, and the middle-level decision maker is able to influence the lower-level decision maker’s choice. Because it is a hard work to discuss this problem directly, we translate TOP into an auxiliary bilevel optimization problem, whose lower-level is MPEC problem. This transformation is reasonable. Then we obtained a necessary optimality condition and an existence theorem of optimal solution via this auxiliary bilevel optimization problem. The sufficient optimality condition is also very important for this problem, we will consider it in the future.

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**REFERENCES**

[1] N. Alguacil, A. Delgadillo and J. M. Arroyo, A trilevel programming approach for electric grid defense planning, *Computers and Operations Research*, 41 (2014), 282–290.
[2] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, A Wiley-Interscience Publication, New York, 1984.
[3] B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Non-linear Parametric Optimization*, Birkhauser Verlag, Basel-Boston, Mass., 1983.
[4] J. F. Bard, An investigation of the linear three level programming problem, *IEEE Transactions on Systems, Man and Cybernetics*, 5 (1984), 711–717.
[5] B. Si and Z. Gao, Optimal model for passenger transport pricing under the condition of market competition, *Journal of Transportation Systems Engineering and Information Technology*, 1 (2007), 72–78.
[6] X. Chi, Z. Wan and Z. Hao, Second order sufficient conditions for a class of bilevel programs with lower level second-order cone programming problem, *Journal of Industrial and Management Optimization*, 11 (2015), 1111–1125.
[7] S. Dempe and J. Dutta, Is bilevel programming a special case of a mathematical program with complementarity constraints?, *Mathematical Programming*, 131 (2012), 37–48.
[8] S. Dempe, B. S. Mordukhovich and A. B. Zemkoho, Sensitivity analysis for two-level value functions with applications to bilevel programming, *SIAM Journal on Optimization*, 22 (2012), 1309–1343.
[9] S. Dempe and A. B. Zemkoho, The generalized mangasarian-fromowitz constraint qualification and optimality conditions for bilevel programs, *Journal of Optimization Theory and Applications*, 148 (2011), 46–68.

[10] S. Dempe and A. B. Zemkoho, The bilevel programming problem: Reformulations, constraint qualifications and optimality conditions, *Mathematical Programming*, 138 (2013), 447–473.

[11] L. Guo, G. H. Lin, J. J. Ye and J. Zhang, Sensitivity analysis of the value function for parametric mathematical programs with equilibrium constraints, *SIAM Journal on Optimization*, 24 (2014), 1206–1237.

[12] J. Han, J. Lu, Y. Hu and G. Zhang, Tri-level decision-making with multiple followers: Model, algorithm and case study, *Information Sciences*, 311 (2015), 182–204.

[13] C. Huang, D. Fang and Z. Wan, An interactive intuitionistic fuzzy method for multilevel linear programming problems, *Wuhan University Journal of Natural Sciences*, 20 (2015), 113–118.

[14] G. Li, Z. Wan and X. Zhao, Optimality conditions for bilevel optimization problem with both levels programs being multiobjective, *Pacific journal of optimization*, 13 (2017), 421–441.

[15] O. L. Mangasarian, *Nonlinear Programming*, SIAM Classics in Applied Mathematics, volume 10, 1999.

[16] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I: Basic Theory*, Springer Science and Business Media, 2006.

[17] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, volume 317. Springer Science and Business Media, 2009.

[18] Z. Wan, L. Mao and G. Wang, Estimation of distribution algorithm for a class of nonlinear bilevel programming problems, *Information Sciences*, 256 (2014), 184–196.

[19] Z. Wan, G. Wang and B. Sun, A hybrid intelligent algorithm by combining particle swarm optimization with chaos searching technique for solving nonlinear bilevel programming problems, *Swarm and Evolutionary Computation*, 8 (2013), 26–32.

[20] D. White, Penalty function approach to linear trilevel programming, *Journal of Optimization Theory and Applications*, 93 (1997), 183–197.

[21] H. Xu and B. Li, Dynamic cloud pricing for revenue maximization, *IEEE Transactions on Cloud Computing*, 1 (2013), 158–171.

[22] J. J. Ye, Necessary optimality conditions for multiobjective bilevel programs, *Mathematics of Operations Research*, 36 (2011), 165–184.

[23] G. Zhang, J. Lu and Y. Gao, *Multi-level Decision Making*, Springer-Verlag Berlin Heidelberg, 2015.

[24] G. Zhang, J. Lu, J. Montero and Y. Zeng, Model, solution concept, and kth-best algorithm for linear trilevel programming, *Information Sciences*, 180 (2010), 481–492.

[25] Z. Zhang, G. Zhang, J. Lu and C. Guo, A fuzzy tri-level decision making algorithm and its application in supply chain, The 8th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT2013), Milan, Italy, 2013, 154–160.

[26] Y. Zheng, J. Liu and Z. Wan, Interactive fuzzy decision making method for solving bilevel programming problem, *Applied Mathematical Modelling*, 38 (2014), 3136–3141.

[27] Y. Zheng, Z. Wan, S. Jia and G. Wang, A new method for strong-weak linear bilevel programming problem, *Journal of Industrial and Management Optimization*, 11 (2015), 529–547.

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*E-mail address:* gaoxili@whu.edu.cn

*E-mail address:* mathwanzp@whu.edu.cn

*E-mail address:* J.W.chen713@163.com

*E-mail address:* zhaoxiaokehenan@126.com