The Least-squares Algorithm for Interferograms with Random Phase Shifts

Zhongsheng Zhai¹, Jinsong Li¹, Yanhong Zhang²*, Qinghua Lv² and Xuanze Wang¹

¹School of Mechanical Engineering, Hubei University of Technology, Wuhan, Hubei, 430068, China
²School of Science, Hubei University of Technology, Wuhan, Hubei, 430068, China
*Corresponding author’s e-mail: 1415481735@qq.com

Abstract. In surface shape measurement with phase-shifting interferometry, the extraction accuracy of initial phase is significant and it will influence the correct of measurement result. A least-squares algorithm is proposed to solve the parameters of interferograms with random and known phase shifts. It can rapidly extract the initial phase, modulation amplitude and background intensity without any iteration. Numerical simulations and optical experiments are implemented to demonstrate the performance of the proposed method. The simulation and experiment results show that the presented method is effective, rapid and accurate.

1. Introduction
Phase-shifting interferometry (PSI) is a useful technique in surface shape measurements with the advantages of non-contact, non-destruction, high accuracy and high resolution. The measurement correctness is relied on the extraction accuracy of initial phase. In traditional PSI, the phase shift between each interferogram is expected to keep constant. However, because of the influence of nonlinear phase shifter, mechanical vibration or air turbulence, there are some unavoidable phase shift errors[1]. Therefore, the accuracy of the initial phase and phase-shifts is important but often difficult to be guaranteed.

Many phase shifting algorithms have been developed to deal with the phase shift errors. F. Liu used Lissajous figure and ellipse fitting technology to correct the phase extraction error caused by inaccurate phase shifts in PSI, and the proposed method can correct the dynamic random phase-shift errors[2]. Some iterative algorithms and self-calibrating algorithms have been introduced to correct the phase-shift errors[3,4]. The main problem of these algorithms is that they will spend much time to find the convergence solution, and the results are highly dependent on the number of samples and especially on the initial estimated values. Some researchers used Spectral analysis[5], windowed Fourier transform[6] and wavelet transform to retrieve the phase, and Huang et al. compared the difference of these algorithms in phase extraction from a single fringe pattern in fringe projection profilometry[7].

The least-squares approach is a popular algorithm to estimate the initial phase and the phase shift. In 1991, Okada et al. first proposed a least-squares-based iterative algorithm to solve the approximate linear equations iteratively to determine phase-shift amounts and phase distributions simultaneously[8]. For coping with the limitation of number of frames (≥ 4) in the conventional iterative algorithms, Wang et al. proposed an advanced iterative algorithm (AIA) based on the least-squares iterative procedure. The proposed method can extract both the initial phase and phase shifts with three
randomly shifted interferograms[9]. The algorithm extracting the phase information proposed by Wang is not solve the initial phase directed by the intensity equation, but by the expansion of sine function. Even three frames can deduce the parameters, however, the fewer frames used in calculation, the bigger error it produced, and the parameters were estimated by expanding the sine function. In fact the equation can be solved directly with LS method.

In this paper, we propose a general least-squares algorithm to extract the parameters of interferograms: the initial phase, the background intensity and the modulation amplitude. It is a non-iterative algorithm, and it is also suitable for non-uniform phase shifts conditions. The proposed method can accurately and rapidly present the solution of parameters of interferograms. Moreover, it offers a way to solve the parameters of signal with sine form. The details of the proposed algorithm are described below.

2. Principles
Phase-shift fringe patterns are always expressed with two dimensions: one is frame index, and the other is pixel index[10]. In this paper, for calculating simple, we just consider the intensity variation of one pixel. Under that condition, the intensity of the pixel can be expressed as:

\[ Y_i = A \cos(\theta + X_i) + C + \epsilon_i \]  

(1)

where the subscript \( i = 1, 2, \ldots, N \) is used as a frame index and \( N \) is the total frame number, \( Y_i \), \( X_i \) are the acquired intensity and phase-shift of the \( i \)th frame, respectively, \( A \) is the modulation amplitude, \( \theta \) is initial phase, \( C \) is the background intensity, and \( \epsilon_i \) is described as the modeling error or noise, and is assumed to be a zero-mean white Gaussian stochastic process. In equation (1), \( Y_i \) and \( X_i \) are known, and parameters \( A, \theta, C \) are unknown and need to be solved.

According to the LS theory, the minimum LS error depends on the noise present in the acquired intensity and on its distortion. For solving easily, a function \( f(A, \theta, C) \) is used to express the LS error:

\[ f(\theta, A, C) = \sum_{i=1}^{N} \epsilon_i^2 = \sum_{i=1}^{N} [Y_i - A \cos(\theta + X_i) - C]^2 = \min \]  

(2)

For the known \( Y_i, X_i \), the LS principle require that

\[ \frac{\partial f}{\partial C} = \sum_{i=1}^{N} 2[Y_i - A \cos(\theta + X_i) - C] \times (-1) = 0 \]

\[ \frac{\partial f}{\partial A} = \sum_{i=1}^{N} 2[Y_i - A \cos(\theta + X_i) - C] \times \cos(\theta + X_i) = 0 \]  

(3)

\[ \frac{\partial f}{\partial \theta} = \sum_{i=1}^{N} -2A[Y_i - A \cos(\theta + X_i) - C] \times \sin(\theta + X_i) = 0 \]

Seemingly, it is hard to solve the equation (3) for they are nonlinear, and iterative approaches are suggested to find the values of \( A, \theta, C \). In fact, the equation (3) exist analytical solution, and the derived process is presented in follow. Equation (3) can be rewritten as:

\[ \sum_{i=1}^{N} (Y_i - C) = A \sum_{i=1}^{N} \cos(\theta + X_i) \]

\[ \sum_{i=1}^{N} (Y_i - C) \cos(\theta + X_i) = A \sum_{i=1}^{N} \cos(2\theta + 2X_i) + \frac{A}{2} N \]  

(4)

\[ \sum_{i=1}^{N} (Y_i - C) \sin(\theta + X_i) = A \sum_{i=1}^{N} \sin(2\theta + 2X_i) \]

From the first equation in (4), we can obtain

\[ C = \frac{1}{N} \sum_{i=1}^{N} Y_i - \frac{A}{N} \cos \sum_{i=1}^{N} \cos X_i + \frac{A}{N} \sin \sum_{i=1}^{N} \sin X_i \]  

(5)
Substituting equation (5) into the second and third equations of equation (4),
we define some parameters as
\[
\begin{align*}
    k_1 &= \sum_{i=1}^{N} Y_i, &
    k_2 &= \sum_{i=1}^{N} Y_i \cos X_i, &
    k_3 &= \sum_{i=1}^{N} Y_i \sin X_i, &
    k_4 &= \sum_{i=1}^{N} \sin X_i, \\
    k_5 &= \sum_{i=1}^{N} \cos X_i, &
    k_6 &= \sum_{i=1}^{N} \sin 2X_i, &
    k_7 &= \sum_{i=1}^{N} \cos 2X_i,
\end{align*}
\]

In equation (6), the values of \( N, \ Y_i, \) and \( X_i \) are known, then \( k_1, k_2, k_3, k_4, k_5, k_6 \) and \( k_7 \) are easily to be obtained and can be looked as known. Based on equation (4) -equation (6), it can be written as:
\[
\begin{align*}
    k_2 \cos \theta - k_3 \sin \theta - k_4 \cos \theta + k_5 \sin \theta
    &= \left[\frac{A}{2} k_1 - \frac{A}{2N} (k_2^2 - k_4^2) \right] \cos \theta
    + \left[\frac{A}{2} k_2 + \frac{A}{N} k_4 \right] \sin \theta
    + \frac{A}{2N} \left( k_2^2 + k_4^2 \right)
    - \frac{A}{2N} \left( k_2^2 - k_4^2 \right) \sin 2\theta \\
    k_5 \sin \theta + k_3 \cos \theta - k_4 \sin \theta - k_4 \cos \theta
    &= \left[\frac{A}{2} k_6 - \frac{A}{N} k_4 k_5 \right] \cos \theta
    + \left[\frac{A}{2} k_1 - \frac{A}{2N} (k_2^2 - k_4^2) \right] \sin 2\theta
\end{align*}
\]

In order to simplify the equation (7), we assume that
\[
\begin{align*}
    a &= k_2 - k_1k_5 &
    b &= -k_3 + k_1k_4 &
    c &= \frac{k_2 - (k_2^2 - k_4^2)}{2} \\
    d &= -k_6 + k_4k_5 &
    e &= \frac{N}{2} + \frac{(k_2^2 + k_4^2)}{2N}
\end{align*}
\]

Substituting equation (8) into equation (7), the equation (7) can be rewritten as
\[
\begin{align*}
    a \cos \theta + b \sin \theta &= A(c \cos 2\theta + d \sin 2\theta + e) \\
    -b \cos \theta + a \sin \theta &= A(-d \cos 2\theta + c \sin 2\theta)
\end{align*}
\]

After eliminating the parameter \( A \) in equation (9), and considering the relations of \( \tan \theta \), \( \sin \theta \) and \( \cos \theta \), we can obtain
\[
(-ad + bc + be) \cos \theta + (ac + bd - ae) \sin \theta = 0
\]

Therefore, we can obtain:
\[
\tan \theta = \frac{ad - bc - be}{ac + bd - ae}
\]

According to equation (11), the value of \( \theta \) calculated from \( \tan \theta \) is limited in \((-\pi/2, \pi/2)\). However, the initial phase \( \theta \) should be in the range \((-\pi, \pi)\). The problem is that the sign of the amplitude \( A \), which should over 0, is not taking into account. For getting the \( \theta \) in the proper range \((-\pi, \pi)\), using the result \( \tan \theta \) in equation (11) to express \( \cos 2\theta \) and \( \sin 2\theta \) in equation (9), and equation (9) can be rewritten as
\[
\begin{align*}
    \tan \theta
    &= \frac{(a + b \tan \theta)}{1 + \tan^2 \theta}
    &= A \sin \theta \left[ e \left( \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right)
    + d \left( \frac{2 \tan^2 \theta}{1 + \tan^2 \theta} \right)
    + c \left( \frac{2 \tan^2 \theta}{1 + \tan^2 \theta} \right) \right]
    + e\right]
    &+ d \left( \frac{2 \tan^2 \theta}{1 + \tan^2 \theta} \right)
    + c \left( \frac{2 \tan^2 \theta}{1 + \tan^2 \theta} \right)
\end{align*}
\]

According to equation (10), we define
\[\begin{align*}
    y &= ad - bc - be \\
    x &= ac + bd - ae
\end{align*}\]  

After substituting equation (13) into equation (12), we can get

\[\begin{align*}
    V &= A \sin \theta = \frac{axy + by^2}{c(x^2 - y^2) + e(x^2 + y^2) + 2dxy} \\
    H &= A \cos \theta = \frac{axy - by^2}{2cxy - d(x^2 - y^2)}
\end{align*}\]  

(14)

The parameters \(a, b, c, d, e, x\) and \(y\) in the both right sides of the equation (14) can be calculated with the equations above, and they can be looked as known. Considering the modulation amplitude \(A\) in equation (14) should be over zero, then we can obtain the solutions of initial phase \(\theta\) as:

\[\theta = \arctan2(V, H)\]  

(15)

Where \(\arctan2\) is the four-quadrant inverse tangent function, and the return value is within the range \((-\pi, \pi)\).

With equations (14) and (5), the modulation amplitude \(A\) and background intensity \(C\) can be solved as:

\[\begin{align*}
    A &= \left(H^2 + V^2\right)^{1/2} \\
    C &= k + \frac{kV - kH}{N}
\end{align*}\]  

(16)

Therefore, the three parameters \(A, \theta\) and \(C\) can be directly calculated by the equations (15) and (16). Moreover, these solutions for equation (1) is analytical and without iterative.

3. Experiments

An Optical experiment for measuring the surface shape of a mirror has also been carried out to prove the performance of the proposed method, and 34 interferograms with size of pixels 1292×964 are captured by a CCD. Figure 1 (a) presented the first interferogram. In order to obtain the initial phase to evaluate the surface morphology, we used the LS algorithm to extract the parameters with 34 interferograms. For proving the effectiveness of the proposed LS method, we choose two pixels P1 and P2 to analyze, their positions are (300, 300) and (400, 400) respectively, as showing in figure 1(a). The known phase shifts were non-uniform, and the intensity distribution of the P1 and P2 were shown in figure 1(b). We can find that the intensity distribution is not regular, and some big errors are existed in some phase shift. Figure 1(c) and 1(d) showed the fitting results of the two intensity sets respectively with LS method. With the equations (15) and (16), the initial phase of P1 and P2 were obtained as \(\theta_1 = -0.4292\) rad, \(\theta_2 = -0.9026\) rad, and then the surface testing can be fulfilled easily and quickly.
Figure 1. The simulation results of intensity distribution and error with LS for two cases: (a) real interferogram, (b) intensity distributions of two pixels with error, (c) and (d) fitting results with LS method.

From the results of simulation and optical experiment, we can conclude that our proposed algorithm can extract initial phase, background intensity and modulation amplitude effectively.

4. Conclusions

In summary, a least-squares algorithm for calculation the parameters of phase-shifted interferograms without iterative process method are proposed. With the rule of LS, we deduced the solution progress by extending the partial derivatives. With a single calculation, the parameter of the initial phase, the background intensity and the modulation amplitude can be extracted, and the solutions for the parameters are analytical. Numerical simulations and optical experiments are given to demonstrate the performance of the presented method, and the results show that the proposed method is rapid, effective and accurate. The proposed method also is suitable to solve signal with sine feature in other domain.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Nos. 51575164, 51405143, 51275157), the Open Fund of State Key Laboratory of Precision Measurement and Instrument (No PIL1209), and the Fund of Hubei University of Technology (Nos. HBSKFZD2014007, BSQD13048).

References

[1] Deck L L. (2014) Model-based phase shifting interferometry. Applied optics, 53: 4628-4636.
[2] Liu F, Wu Y, Wu F. (2015) Correction of phase extraction error in phase-shifting interferometry based on Lissajous figure and ellipse fitting technology. Optics express, 23: 10794-10807.
[3] Larkin K G. (2001) A self-calibrating phase-shifting algorithm based on the natural demodulation of two-dimensional fringe patterns. Optics Express, 9: 236-253.
[4] Xu J, Xu Q, Chai L. (2008) Iterative algorithm for phase extraction from interferograms with random and spatially nonuniform phase shifts. Applied optics, 47: 480-485.
[5] Servin M, Estrada J C, Quiroga J A. (2009) Spectral analysis of phase shifting algorithms. Optics Express, 17: 16423-16428.
[6] Kemao Q, Wang H, Gao W. (2008) Windowed Fourier transform for fringe pattern analysis: theoretical analyses. Applied optics, 47: 5408-5419.
[7] Huang L, Kemao Q, Pan B, et al. (2010) Comparison of Fourier transform, windowed Fourier transform, and wavelet transform methods for phase extraction from a single fringe pattern in fringe projection profilometry. Optics and Lasers in Engineering, 48: 141-148.
[8] Okada K, Sato A, Tsujiuchi J. (1991) Simultaneous calculation of phase distribution and scanning phase shift in phase shifting interferometry. Optics communications, 84: 118-124.
[9] Wang Z, Han B. (2004) Advanced iterative algorithm for phase extraction of randomly phase-shifted interferograms. Optics letters, 29: 1671-1673.