THE BICANONICAL MAP OF THE CARTWRIGHT-STEGER SURFACE

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ABSTRACT. We prove that the bicanonical map of the Cartwright-Steger surface is an embedding. We also discuss two minimal surfaces of general type, both covered by the Cartwright-Steger surface. One has \( K^2 = 2, p_g = 1, \pi_1 = \{1\} \) and the other has \( K^2 = 1, p_g = 0, \pi_1 = \mathbb{Z}/2\mathbb{Z} \).

The Cartwright-Steger surface is a minimal surface of general type with \( p_g = q = 1 \) and \( K^2 = 3c_2 = 9 \). It is an arithmetic ball quotient found by Cartwright and Steger [3], [4]. Reider’s theorem [10] implies that the bicanonical system of a ball quotient \( X \) is base point free, thus defines a morphism.

Let \( X \) be the Cartwright-Steger surface and

\[ \Phi_{2,X} : X \to \mathbb{P}^9 \]

be the bicanonical map. This map may fail to separate points only on certain curves, as specified in the criterion of Reider’s theorem [10]. In this note we prove that such curves do exist on the Cartwright-Steger surface.

Theorem 1. On the Cartwright-Steger surface \( X \) there exists no effective curve \( B \) such that either \( 3B \) is numerically equivalent to the canonical class \( K_X \) or \( K_XB = 2 \) and \( B^2 = 0 \). In particular, the bicanonical map \( \Phi_{2,X} \) is an embedding into \( \mathbb{P}^9 \).

It is known that Cartwright-Steger surface \( X \) has automorphism group \( \text{Aut}(X) = \mathbb{Z}/3\mathbb{Z} \) and the quotient is simply connected [4], and the action has only isolated fixed points, three of type \( 1/3(1,1) \) and six of type \( 1/3(1,2) \) [2]. The latter was also obtained by geometric arguments by F. Catanese, T. Domingo, M. Stover and the author, and by I. Dolgachev. It follows that the minimal resolution \( Y \) of the quotient \( X/\text{Aut}(X) \) is a minimal surface of general type with

\[ K_Y^2 = 2, \quad p_g(Y) := h^{2,0}(Y) = 1, \quad \pi_1(Y) = \{1\}. \]

L. Borisov found an involution \( \alpha \) on \( Y \), as a bi-product of his computation of the equation of \( Y \). This automorphism does not come from the ball. It turns out to be fixed point free and the quotient \( Z = Y/\langle \alpha \rangle \) is a minimal surface of general type with

\[ K_Z^2 = 1, \quad p_g(Z) = 0, \quad \pi_1(Z) = \mathbb{Z}/2\mathbb{Z}. \]
1. Known Facts on the Cartwright-Steger Surface

We collect known facts on the Cartwright-Steger surface [3], [4], [2], [5], [6]. Let 

$$
\pi : X = \mathbb{B}/\Pi \to \mathbb{P}(1, 2, 3) = \mathbb{B}/\bar{\Gamma}
$$

be the natural projection map of degree \([\bar{\Gamma} : \Pi] = 3.288\). The Deligne-Mostow quotient \(\mathbb{P}(1, 2, 3) = \mathbb{B}/\bar{\Gamma}\) contains a line \(D_B\) and a cuspidal curve \(D_A\). The curve \(D_B\) (resp. \(D_A\)) is the image of the set of mirrors of complex reflections of order 4 (resp. 3) in \(\bar{\Gamma}\). See p.111, [5]. Let \(P_1\) be the cusp of \(D_A\), \(P_2\) the tangential intersection point of \(D_A\) and \(D_B\), \(P_3\) the transversal intersection point of \(D_A\) and \(D_B\), and \(P_4, P_5\) be the singular points of type \(1/2(1, 1)\) and \(1/3(1, 2)\) respectively. We know that 

$$
\pi^{-1}(D_B) = E_1 + E_2 + E_3,
$$

where \(E_1, E_2, E_3\) are irreducible curves of geometric genus 4. As a reducible curve, \(\pi^{-1}(D_B)\) has 6 transversal branches at each \(O_i\), and is smooth elsewhere. Thus \(E_1, E_2, E_3\) intersect each other only at \(O_1, O_2, O_3\). Their multiplicities at \(O_i\) are given in Table 1. The intersection number \(E_i E_j\) is equal to the dot product of their multiplicity vectors if \(i \neq j\), and

$$
E_j^2 = 1 - g(E_j) + \sum_i m(E_j, O_i)[m(E_j, O_i) - 1].
$$

See Table 2. Here we use the fact that \(K_X E = 3g(E) - 3\) for a totally geodesic curve \(E\) of geometric genus \(g(E)\).

|   | \(O_1\) | \(O_2\) | \(O_3\) | \(g\) | \(p_a\) |
|---|--------|--------|--------|------|------|
| \(E_1\) | 3      | 1      | 2      | 4    | 8    |
| \(E_2\) | 2      | 1      | 3      | 4    | 8    |
| \(E_3\) | 1      | 4      | 1      | 4    | 10   |
| \(C_1, C_2\) | 0     | 1      | 2      | 4    | 5    |
| \(C_3, C_4\) | 4      | 3      | 2      |      | 10   |

Table 1.

There are two curves \(C_1, C_2\) of geometric genus 4, and two curves \(C_3, C_4\) of geometric genus 10 such that

$$
\pi^{-1}(D_A) = C_1 + C_2 + C_3 + C_4.
$$

As a reducible curve, \(\pi^{-1}(D_A)\) has 8 transversal branches at each \(O_i\). The preimage \(\pi^{-1}(P_1)\) consists of 36 points and \(\pi^{-1}(P_3)\) 72 points. The curves \(C_i\) intersect each other at \(O_i\), but also intersect transversally at points in \(\pi^{-1}(P_1)\), and nowhere else. Table 1 contains the multiplicities at \(O_1, O_2, O_3\) of the curves \(C_i\).

The information on the curves \(E_i, C_j\) are obtained from [5] and [6].
1.1. **Intersection numbers of the geodesic curves.** Through discussion with F. Catanese, M. Stover, D. Toledo, we have obtained the intersection numbers of curves $E_i, C_j$, as given in Table 2. This table confirms the computation of [2], where the entries involving $C_3, C_4$ are not given explicitly.

|     | $E_1$ | $E_2$ | $E_3$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $E_1$ | 5     | 13    | 9     | 11    | 11    | 25    | 25    |
| $E_2$ | 13    | 5     | 9     | 7     | 7     | 29    | 29    |
| $E_3$ | 9     | 9     | 9     | 9     | 9     | 27    | 27    |
| $C_1$ | 11    | 7     | 9     | $-1$  | 17    | 37    | 19    |
| $C_2$ | 11    | 7     | 9     | 17    | $-1$  | 19    | 37    |
| $C_3$ | 25    | 29    | 27    | 37    | 19    | 71    | 89    |
| $C_4$ | 25    | 29    | 27    | 37    | 19    | 89    | 71    |

**Table 2.**

In particular $p_a(C_3) = p_a(C_4) = 50$, filling up Table 1.

1.2. **The Néron-Severi group $\text{NS}(X)$.** By Cartwright and Steger [3],

$$H_1(X, \mathbb{Z}) = \mathbb{Z}^2.$$  

So by the universal coefficient theorem, $H^2(X, \mathbb{Z}) = \mathbb{Z}^5$, torsion free. The three curves, $E_1, E_3, C_1$, are numerically independent, so $\text{NS}(X)$ is a free group of rank 3. Note that $\text{Pic}(X)$ contains torsions, namely the torsion elements of the Picard variety $\text{Pic}^0(X)$, an elliptic curve in this case.

1.3. **Fix($\sigma$).** It is known that $\text{Aut}(X) \cong \mathbb{Z}/3\mathbb{Z}$ [4]. The order 3 automorphism $\sigma$ of $X$ fixes 9 points,

$$\text{Fix}(\sigma) = \{O_1, O_2, O_3, Q_1, \ldots, Q_6\},$$

where $O_i$ is of type $1/3(1, 1)$ and $Q_i$ of type $1/3(1, 2)$. The points $Q_1, \ldots, Q_6$ lie over the the singular point of type $1/3(1, 2)$ of the Deligne-Mostow quotient $\mathbb{P}(1, 2, 3)$, hence for any $i$

$$Q_i \notin \pi^{-1}(D_B) \cup \pi^{-1}(D_A).$$

The induced action of $\sigma$ on the elliptic curve $\text{Alb}(X)$ fixes 3 points. Let $F_0, F_1, F_2$ be the 3 $\sigma$-invariant Albanese fibres. One may assume that

$$O_1, O_2, O_3 \in F_0, \quad Q_1, Q_2, Q_3 \in F_1, \quad Q_4, Q_5, Q_6 \in F_2.$$  

1.4. **The action of $\sigma$ on $\pi^{-1}(D_B)$ and $\pi^{-1}(D_A)$.

- Each $E_i, C_j$ is $\sigma$-invariant, i.e. $\sigma(E_i) = E_i, \ i = 1, 2, 3$ and $\sigma(C_j) = C_j, \ j = 1, 2, 3, 4$.

This follows from the fact that both $\pi^{-1}(D_A)$ and $\pi^{-1}(D_B)$ have transversal branches at $O_i$. Indeed, locally at $O_i$, $\sigma(x, y) = (\zeta x, \zeta y)$, $\zeta$ is a third root of 1, hence preserves the tangent line of every branch.
Proposition 1. \( \text{Aut}(X) \) is cohomologically trivial, i.e., acts as the identity on \( H^2(X, \mathbb{Z}) = \mathbb{Z}^5 \).

**Proof.** Since the classes of the three curves, \( E_1, E_3, C_1 \), generate \( \text{NS}(X) \otimes \mathbb{Q} \), \( \text{Aut}(X) \) acts as the identity on \( \text{NS}(X) \otimes \mathbb{Q} \). By the holomorphic Lefschetz fixed point formula (cf. [7]), from the information on the fixed locus of \( \text{Aut}(X) \) one sees that \( \sigma^* \) acts on \( H^2(X, \mathcal{O}_X) \) as the identity, and on \( H^1(X, \mathcal{O}_X) \) as the multiplication by a third root of unity. It follows that \( \text{Aut}(X) \) acts as the identity on \( H^2(X, \mathbb{Q}) \), hence on \( H^2(X, \mathbb{Z}) \) since the latter has no torsion element. \( \square \)

**Remark 1.** \( \sigma^* \) acts on \( H^1(X, \mathbb{Z}) = \mathbb{Z}^2 \) nontrivially, namely as \( \sigma^*(v_1) = v_2 \), \( \sigma^*(v_2) = -v_1 - v_2 \) with respect to a suitable basis \( v_1, v_2 \). Thus in the theorem of [9] the condition that \( |K_X| \) is base point free is necessary.

1.5. **Linear equivalences among \( \text{Aut}(X) \)-invariant curves.** Since the quotient \( X/\text{Aut}(X) \) is simply connected [4], any numerical equivalence among \( \text{Aut}(X) \)-invariant curves is indeed a linear equivalence modulo an \( \text{Aut}(X) \)-invariant divisor class \( \in \text{Pic}^0(X) \) (note that the \( \text{Aut}(X) \)-action on \( \text{Pic}^0(X) \) fixes 3 elements, that form a subgroup of order 3.) Modulo \( \text{Pic}^0(X)^{\text{Aut}(X)} \),

\[
K_X \equiv E_3,
E_1 + E_2 \equiv 2E_3,
4E_3 \equiv C_1 + C_3 \equiv C_2 + C_4,
3E_3 \equiv E_1 + C_1 + C_2,
E_2 + E_3 \equiv C_1 + C_2.
\]

In particular, each of the above equivalences is a linear equivalence, once multiplied by 3, e.g., \( 3(E_1 + E_2) \equiv 6E_3 \).

**Remark 2.** It is not clear if the above equivalences are indeed linear equivalences, without being multiplied by 3. (Remark 5.7 in [2] needs proof or corrections. On the simply connected quotient \( X/\text{Aut}(X) \) a numerical equivalence between Cartier divisors is a linear equivalence, so is its pull back to \( X \), but this may not hold for \( \mathbb{Q} \)-Cartier divisors.) In general, if a compact complex surface \( V \) admits a \( \mathbb{Z}_m \)-action with only isolated fixed points such that \( V/\mathbb{Z}_m \) is simply connected, and if two \( \mathbb{Z}_m \)-invariant effective curves \( A \) and \( B \) on \( V \) are numerically equivalent, then \( mA \) and \( mB \) are linearly equivalent. But \( A \) and \( B \) may not be linearly equivalent, as there are examples, e.g., consider a product of two elliptic curves and its Kummer surface.

1.6. **The Albanese map.** Every fibre of the Albanese map \( \alpha : X \rightarrow \text{Alb}(X) \) is irreducible and reduced. Let \( F \) be a general smooth fibre. By [2]

\[
g(F) = 19
\]

and \( F \) is numerically equivalent to \( -E_1 + 5E_2 \).
First we recall Reider’s theorem [10] by stating the expanded version given in Theorem 11.4 of [1].

**Theorem 2.** [10] Let \( L \) be nef divisor on a smooth projective surface \( X \).

1. Assume that \( L^2 \geq 5 \). If \( P \) is a base point of the linear system \( |K_X + L| \), then \( P \) lies on an effective divisor \( B \) such that
   
   (a) \( BL = 0, \ B^2 = -1, \) or
   
   (b) \( BL = 1, \ B^2 = 0. \)

2. Assume that \( L^2 \geq 9 \). If \( P \) and \( Q \), possibly infinitely near, are not base points of \( |K_X + L| \) and fail to be separated by \( |K_X + L| \), then they lie on an effective curve \( B \), depending on \( P, Q \), satisfying one of the following:
   
   (a) \( BL = 0, \ B^2 = -2 \) or \(-1; \)
   
   (b) \( BL = 1, \ B^2 = -1 \) or \(0; \)
   
   (c) \( BL = 2, \ B^2 = 0; \)
   
   (d) \( L^2 = 9 \) and \( L \) is numerically equivalent to \( 3B \).

A ball quotient cannot contain a curve of geometric genus 0 or 1. Applying Reider’s theorem to \( L = K_X \), one sees that the bicanonical system of a ball quotient is base point free, thus defines a morphism. Moreover the bicanonical system is very ample unless the surface contains an effective divisor with the property (2c) or (2d).

Consider the case (2d). Suppose that \( K_X \) is numerically equivalent to \( 3B \). Since \( H^2(X, \mathbb{Z}) \) is torsion-free,

\[
K_X \equiv 3B + t
\]

for some \( t \in Pic^0(X) \), where "\( \equiv \)" is linear equivalence. Since \( Pic^0(X) \) is a divisible group, one can write \( t = 3t' \) in \( Pic^0(X) \), thus see that \( K_X \) is divisible by 3 in \( Pic(X) \). The 3-divisibility of \( K_X \) is equivalent to the liftability of the fundamental group \( \pi_1(X) \) to \( SU(2, 1) \) [8]. But the explicit computation of the fundamental group by [4] shows that \( \pi_1(X) \) does not lift to \( SU(2, 1) \). This rules out the possibility (2d).

It remains to consider the case (2c).

**Lemma 1.** Suppose that there is an effective divisor \( B \) on \( X \) with \( BK_X = 2, \ B^2 = 0 \). Then \( B \) is an irreducible smooth curve of genus 2.

**Proof.** Suppose that there is such an effective divisor \( B \). If \( B \) is reducible, then since \( K_X \) is ample, there is an irreducible component \( B_1 \) of \( B \) with \( B_1^2 \leq 0, \ B_1K_X = 1 \), impossible. If \( B \) is irreducible and singular, then \( B \) has geometric genus \( \leq 1 \), again impossible, since a ball quotient cannot contain a curve of geometric genus 0 or 1.

3. **Key Lemma**

The following lemma will play a key role in our proof of Theorem 1.
Lemma 2. Suppose that the Cartwright-Steger surface $X$ contains a smooth curve $B$ with $K_XB = 2, B^2 = 0$. Then the following hold.

1. All such curves $B$ define the same class in the Néron-Severi group $\text{NS}(X) \subset H^2(X, \mathbb{Z})$. In other words, the difference of two such curves is an element of the Picard variety $\text{Pic}^0(X)$ which is an elliptic curve.

2. The image $\sigma(B)$ under an automorphism $\sigma$ is another such curve, and is disjoint from $B$, where $\sigma$ is a generator of $\text{Aut}(X) \cong \mathbb{Z}_3$. The three curves $B, \sigma(B), \sigma^2(B)$ are mutually disjoint.

Remark 3. It can be shown that $mB$ does not move in an algebraic family for $m < 9$. Thus $\sigma(B) - B$, being an element of $\text{Pic}^0(X)$, has order at least 9, may have infinite order. In particular, $\sigma(B) - B$ is not a 3-torsion. The non-existence of such a curve $B$ is a subtle problem. The rest of this paper will be devoted to its proof.

On the Cartwright-Steger surface $X$ the three curves $E_1, E_3, C_1$ form a $\mathbb{Q}$-basis for $\text{NS}(X)$ with intersection matrix

$$\begin{pmatrix}
5 & 9 & 11 \\
9 & 9 & 9 \\
11 & 9 & -1
\end{pmatrix}$$

with determinant $18^2$. Any divisor $D$ on $X$ with

$$DE_1 = a, \quad DE_3 = b, \quad DC_1 = c$$

must be expressed as

$$(5) \quad D \sim xE_1 + yE_3 + zC_1$$

where

$$x = \frac{-5a + 6b - c}{18},$$

$$y = \frac{6a - 7b + 3c}{18},$$

$$z = \frac{-a + 3b - 2c}{18}.$$ 

The equality

$$(6) \quad D^2 = D(xE_1 + yE_3 + zC_1) = xa + yb + zc$$

becomes

$$2c^2 - 2(3b - a)c + 5a^2 + 7b^2 - 12ab + 18D^2 = 0.$$ 

The discriminant of this quadratic equation for $c$

$$\frac{\Delta}{4} = -(3a - 3b)^2 + 4b^2 - 36D^2$$

must be a square number, since $a, b, c$ are integers. In particular

$$(7) \quad 4b^2 - 36D^2 = (3a - 3b)^2 + s^2.$$
for some integer $s \geq 0$.

**Step I.** Suppose that $X$ contains a divisor $D$, not necessarily effective, with $K_X D = 2, D^2 = 0$. Then $D$ is numerically equivalent to either

$$D_I = 1/9(E_1 - E_3 + 2C_1) \sim 1/9(2E_3 + C_1 - C_2)$$

or

$$D_{II} = 1/9(-E_1 + 5E_3 - 2C_1) \sim 1/9(2E_3 - C_1 + C_2).$$

**Proof.** In this case, $b = DE_3 = DK = 2$ and $D^2 = 0$, thus (7) becomes

$$4b^2 - 36D^2 = 16 = (3a - 6)^2 + s^2$$

for some integer $s$. Note that $16 = 4^2 + 0^2$ is the unique expression as a sum of two squares. Since $3a - 6 \neq \pm 4$ for any integer $a$, we have $a = 2, \ s = 4$.

The quadratic equation (6) yields solutions for $c$

$$c = \left((3b - a) \pm \sqrt{\frac{\Delta}{4}}\right)/2 = ((3b - a) \pm s)/2 = 0 \text{ or } 4.$$

Thus we have two solutions: $(a, b, c) = (2, 2, 0)$ or $(2, 2, 4)$, yielding the two solutions $D_I$ and $D_{II}$. Finally the numerical equivalences follow from the linear equivalence

$$9E_3 \equiv 3(E_1 + C_1 + C_2)$$

(see the subsection 1.5.)

**Step II.** For any $i = 1, 2, 3$, $D_iE_i = D_{II}E_i = 2$.

$$D_IC_1 = 0, \ D_IC_2 = 4, \ D_IC_3 = 8, \ D_IC_4 = 4, \ D_IF = 8; \ D_{II}C_1 = 4, \ D_{II}C_2 = 0, \ D_{II}C_3 = 4, \ D_{II}C_4 = 8, \ D_{II}F = 8.$$

The $\mathbb{Q}$-divisors $D_I$ and $D_{II}$ cannot represent simultaneously integral divisors, i.e., cannot exist simultaneously in $\text{NS}(X)$.

**Proof.** The intersection numbers can be obtained by using Table 2. Another computation shows that

$$D_ID_{II} = 8/9,$$

not an integer, thus $D_I$ and $D_{II}$ cannot represent simultaneously integral divisors.

Once the order of the 3 points $O_1, O_2, O_3$ were fixed, the curves $E_1, E_2$ are distinguished from each other as they have different multiplicities at $O_1$. But $C_1$ and $C_2$ (resp. $C_3$ and $C_4$) cannot from each other, as they have the same multiplicity at each $O_i$. One may switch the names of $C_1$ and $C_2$, and simultaneously switch the names of $C_3$ and $C_4$. Under this switch, $D_I$ and $D_{II}$ are interchanged. Thus one may assume that only $D_I$ can be represented by an integral divisor.
Step III. All divisors $D$, not necessarily effective, with $K_X D = 2$, $D^2 = 0$ are numerically equivalent to
\[ D_I = 1/9(E_1 - E_3 + 2C_1) \sim 1/9(2E_3 + C_1 - C_2). \]
Here among the two curves in $\pi^{-1}(D_A)$ with multiplicities 0, 1, 2 at $O_1$, $O_2$, $O_3$, respectively the choice of $C_1$ is such that $DC_1 = 0$.

This completes the proof of the first statement of Lemma 2. The following proves the second statement.

Step IV. For any smooth curve $B$ on $X$ with $K_X B = 2$, $B^2 = 0$ the image $\sigma(B)$ under an automorphism $\sigma$ is another such curve, and is disjoint from $B$.

Proof. First note that $O_i \notin B$ for any $i$ (if $O_i \in B$, then the intersection number $BE_j = 2$ is greater than or equal to the product of the multiplicities of $B$ and $E_j$ at $O_i$ for all $j$, impossible.)

Suppose that $\sigma(B) = B$ as curves. Then, since $E_1$ is Aut($X$)-invariant, we have $\sigma(B \cap E_1) = B \cap E_1$. Since $BE_1 = 2$, the set $B \cap E_1$ is non-empty and has at most two points. Since $\sigma$ is of order 3, it fixes a point in $B \cap E_1$, which must be $Q_i$ for some $i$. Then $Q_i \in E_1$, impossible since none of the six points $Q_1, ..., Q_6$ is contained in the union of the 7 geodesic curves $E_1, E_2, E_3, C_1, ..., C_4$. This proves that $\sigma(B) \neq B$. They are numerically equivalent to each other by Step III, so $\sigma(B)B = B^2 = 0$, hence they are disjoint from each other. $\square$

4. The Quotient of the Cartwright-Steger Surface

The quotient $X/\text{Aut}(X)$ has
\[ K_{X/\text{Aut}(X)}^2 = \frac{1}{3}K_X^2 = 3 \]
and has 3 singular points of type $1/3(1,1)$ at the images $\bar{O_i}$ of $O_i$ and 6 singular points of type $1/3(1,2)$ at the images $\bar{Q_j}$ of $Q_j$. Let
\[ \nu: Y \to X/\text{Aut}(X) \]
be the minimal resolution, $R_i$ be the $(-3)$-curve lying over the singular point $O_i$, and $R_{j1} - R_{j2}$ the $A_2$-configuration of $(-2)$-curves lying over $Q_j$. Since $X/\text{Aut}(X)$ is simply connected by [4], so is $Y$. Thus a numerical equivalence between integral divisors on $Y$ is a linear equivalence. From the information in Section 1 one gets
\[ p_g(Y) := h^{2,0}(Y) = 1, \quad \pi_1(Y) = \{1\}, \quad h^{1,0}(Y) = 0. \]
Computing the adjunction, one gets
\[ K_Y = \nu^* K_{X/\text{Aut}(X)} - \frac{1}{3}(R_1 + R_2 + R_3), \]
which implies that
\[ K_Y^2 = 2. \]
This, together with Nöther formula, determines the remaining Hodge number
\[ h^{1,1}(Y) = 18. \]

**Notation.** For a curve \( D \) on \( X \), the image on \( X/\text{Aut}(X) \) will be denoted by \( \bar{D} \) and the proper transform of \( \bar{D} \) on \( Y \) by \( D' \).

**Proposition 2.**
1. \( K_Y = E'_3 + R_2 \). In particular, \( K_Y \) is nef.
2. \( Y \) is a simply connected minimal surface of general type with \( K^2_Y = 2, \, p_g(Y) = 1, \, b_2(Y) = 20, \, \text{rk} \text{Pic}(Y) = h^{1,1}(Y) = 18 \).
3. \( E_i' \) is a \((-3)\)-curve for \( i = 1, 2, 3 \).
4. \( C_1' \) and \( C_2' \) are (smooth) elliptic curves.
5. \( C_3' \) and \( C_4' \) are curves with geometric genus 1 and arithmetic genus 11.
6. The 12 \((-2)\)-curves \( R_{ij} \) are disjoint from the 10 curves \( E_i', R_j, C_k' \).
7. The intersection matrix of the 10 curves is given as follows:

|   | \( E_1' \) | \( E_2' \) | \( E_3' \) | \( R_1 \) | \( R_2 \) | \( R_3 \) | \( C_1' \) | \( C_2' \) | \( C_3' \) | \( C_4' \) |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( E_1' \) | -3 | 0 | 0 | 3 | 1 | 2 | 2 | 2 | 2 |
| \( E_2' \) | 0 | -3 | 0 | 2 | 1 | 3 | 0 | 0 | 4 | 4 |
| \( E_3' \) | 0 | 0 | -3 | 1 | 4 | 1 | 1 | 1 | 3 | 3 |
| \( R_1 \) | 3 | 2 | 1 | -3 | 0 | 0 | 0 | 0 | 4 | 4 |
| \( R_2 \) | 1 | 1 | 4 | 0 | -3 | 0 | 1 | 1 | 3 | 3 |
| \( R_3 \) | 2 | 3 | 1 | 0 | 0 | -3 | 2 | 2 | 2 |
| \( C_1' \) | 2 | 0 | 1 | 0 | 1 | 2 | -2 | 4 | 10 | 4 |
| \( C_2' \) | 2 | 0 | 1 | 0 | 1 | 2 | 4 | -2 | 4 | 10 |
| \( C_3' \) | 2 | 4 | 3 | 4 | 3 | 2 | 10 | 4 | 14 | 20 |
| \( C_4' \) | 2 | 4 | 3 | 4 | 3 | 2 | 4 | 10 | 20 | 14 |

**Table 3.**

8. \( \text{Pic}(Y) = \text{NS}(Y) \) is generated up to finite index by the 18 curves \( E_1', E_3', R_1, R_2, R_3, C_1', R_{ij} \).

9. There are many linear equivalences on \( Y \), e.g.,
   \[ E_1' + E_2' + R_1 + R_3 \equiv 2E_3' + 2R_2 \equiv 2K_Y, \]
   \[ C_1' + C_3' \equiv C_2' + C_4' \equiv 4E_3' + 4R_2 \equiv 4K_Y, \]
   \[ E_1' + C_1' + C_2' + R_3 \equiv 3E_3' + 3R_2 \equiv 3K_Y, \]
   \[ E_2' + E_3' + R_1 + R_2 \equiv C_1' + C_2'. \]

10. The image \( F' \) on \( Y \) of an Albanese fibre \( F \sim -E_1 + 5E_2 \) on \( X \)
    \[ F' \equiv \nu^*(-3E_1 + 15E_2) = -3E_1' + 15E_2' + 7R_1 + 4R_2 + 13R_3. \]
(11) The genus 19 fibration $|F'|$ on $Y$ over $\mathbb{P}^1$ has 3 reducible fibres,
\[ 3F'_0 + R_1 + R_2 + R_3, \]
\[ 3F'_1 + 2R_{12} + R_{11} + 2R_{22} + R_{21} + 2R_{32} + R_{31}, \]
\[ 3F'_2 + 2R_{42} + R_{41} + 2R_{52} + R_{51} + 2R_{62} + R_{61}. \]

5. THE QUOTIENT OF $Y$ BY BORISOV INVOLUTION

The canonical ring of $Y$ is generated by 1 element in degree 1, 3 elements in degree 2, 4 elements in degree 3.

L. Borisov has informed me that he found an octic equation for $Y$ in $\mathbb{P}^4$, by first obtaining equations of a ball quotient which is a $\mathbb{Z}_7 : \mathbb{Z}_3$ Galois cover of $X/\text{Aut}(X)$ (this cover does not factor through $X$), then getting the octic as the equation of the invariant functions. As a by-product he found an involution $\alpha$ of $Y$, which switches the six curves
\[ E'_1 \leftrightarrow R_3, \quad E'_2 \leftrightarrow R_1, \quad E'_3 \leftrightarrow R_2, \]
and permutes the six $A_2$-configurations $R_{i1} - R_{i2}$ into 3 orbits.

In this section we prove the following:

**Proposition 3.**

1. $\alpha(C'_i) = C'_i$ for $i = 1, 2, 3, 4$.
2. The involution $\alpha$ is fixed point free.
3. The quotient $Z := Y/\alpha$ is a minimal surface of general type with $K^2_Z = 1$, $p_g(Z) = 0$, $\pi_1(Z) = \mathbb{Z}/2\mathbb{Z}$.

**Lemma 3.** $\alpha(C'_1) = C'_1$ or $-R_3 - E'_1 + 3(E'_3 + R_2) - C'_1$.

**Proof.** Write $\alpha(C_1) = xE'_1 + yE'_3 + a_1R_1 + a_2R_2 + a_3R_3 + bC'_1 + \Sigma$ with rational coefficients, where $\Sigma$ is supported on $\cup R_{ij}$. Intersecting with the 6 curves $E'_i, R_j$, we get five independent equations, hence the reduced form
\[ \alpha(C_1) = x(R_3 + E'_1) - 3x(R_2 + E'_3) + (1 + 2x)C'_1 + \Sigma. \]
From $\alpha(C'_1)^2 = -2$, we get
\[ 12x^2 + 12x = \Sigma^2 \]
The right hand side is non-positive and the left is non-negative, so $\Sigma = 0$ and $x = 0$ or $-1$. $\square$

**Lemma 4.** The second possibility in Lemma 3 cannot occur.

**Proof.** Suppose that $\alpha(C'_1) = -R_3 - E'_1 + 3(E'_3 + R_2) - C'_1$.
Then $\text{Tr} \alpha |\text{NS}(Y) = -2$.

**Case 1.** $\alpha = -1$ on $H^2(\mathcal{O}_Y)$.
In this case, $p_g(Z) = q(Z) = 0$. By the topological Lefschetz fixed formula,
\[ e(Y^\alpha) = 2 + \text{Tr} \alpha |H^2(Y, \mathbb{Z}) = 2 + \text{Tr} \alpha |\text{NS}(Y) + 2\text{Tr} \alpha |H^2(\mathcal{O}_Y) = -2. \]
Let the fixed locus $Y^\alpha$ consist of $2m$ points $P_1, \ldots, P_{2m}$ and curves $A_1, \ldots, A_t$. Then $e(Y^\alpha) = 2m + \sum (2 - 2g(A_i)) = -2$, and by Hurwitz $e(Z) = -2$.

...
$2m + 10$ and $K_Z^2 = 2 - 2m$. Note that $H^2(Z, \mathbb{Z}) = \text{NS}(Z)$ is unimodular of rank $2m + 8$. On the other hand, $Z$ contains $2m$ $(-2)$-curves, three $A_2$-configurations, and two curves $\bar{R}_1, \bar{R}_3$ with 

$$\bar{R}_1^2 = \bar{R}_3^2 = -1, \quad \bar{R}_1 \bar{R}_3 = 3.$$ 

Thus $Z$ contains $2m + 8$ curves whose intersection matrix has $|\det| = 2^{2m} \cdot 3^3 \cdot 8$, not a square, a contradiction!

**Case 2.** $\alpha = 1$ on $H^2(\mathcal{O}_Y)$.

In this case, $p_g(Z) = 1, q(Z) = 0$. By the topological Lefschetz, $e(Y^\alpha) = 2$. Let $Y^\alpha$ consist of $2m$ points $P_1, \ldots, P_{2m}$ and curves $A_1, \ldots, A_t$. Then $e(Y^\alpha) = 2m + \sum (2 - 2g(A_i)) = 2$. By the holomorphic Lefschetz fixed point formula (cf. [7]),

$$1 - 0 + 1 = 2m/4 + \sum \left( (1 - g(A_i))/2 + A_i^2/4 \right) = e(Y^\alpha)/4 + \sum A_i^2/4,$$

so

$$\sum A_i^2 = 6, \quad \sum K_Y A_i = 2m - 8.$$ 

For surfaces with $p_g > 0$, $|\det \text{NS}|$ may not be a square. We argue in a different way. Every $\alpha$-invariant divisor such as $A = \sum A_i$, if never intersects the $6 A_2$-configurations, can be written as

$$A = x(E'_1 + R_3) + y(E'_3 + R_2)$$

for some rational numbers $x, y$. In our case

$$6 = A^2 = -2x^2 + 2y^2 + 4xy = 2(x + y)^2 - 4x^2,$$

$$2m - 8 = K_Y A = 2x + 2y.$$ 

Eliminating $y$, we get

$$2x^2 = (m - 4)^2 - 3.$$ 

Since $m$ is an integer, so is $x$, but then it is elementary to check that this Diophantine equation has no integer solution. □

This, together with the linear equivalences from Proposition 2, implies the first assertion of Proposition 3.

Now since $\alpha(C'_1) = C'_1$,

$$\text{Tr}_\alpha|\text{NS}(Y) = 0.$$ 

We will prove the last two assertions of Proposition 3.

**Case I.** $\alpha = -1$ on $H^2(\mathcal{O}_Y)$.

In this case, $p_g(Z) = q(Z) = 0$. By the topological Lefschetz fixed formula,

$$e(Y^\alpha) = 2 + \text{Tr}_\alpha|H^2(Y, \mathbb{Z}) = 2 + \text{Tr}_\alpha|\text{NS}(Y) + 2\text{Tr}_\alpha|H^2(\mathcal{O}_Y) = 0.$$ 

Let the fixed locus $Y^\alpha$ consist of $2m$ points $P_1, \ldots, P_{2m}$ and curves $A_1, \ldots, A_t$. Then

$$e(Y^\alpha) = 2m + \sum (2 - 2g(A_i)) = 0$$

for surfaces with $p_g > 0$, $|\det \text{NS}|$ may not be a square. We argue in a different way. Every $\alpha$-invariant divisor such as $A = \sum A_i$, if never intersects the $6 A_2$-configurations, can be written as

$$A = x(E'_1 + R_3) + y(E'_3 + R_2)$$

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$$2x^2 = (m - 4)^2 - 3.$$ 

Since $m$ is an integer, so is $x$, but then it is elementary to check that this Diophantine equation has no integer solution. □

This, together with the linear equivalences from Proposition 2, implies the first assertion of Proposition 3.

Now since $\alpha(C'_1) = C'_1$,

$$\text{Tr}_\alpha|\text{NS}(Y) = 0.$$ 

We will prove the last two assertions of Proposition 3.
and by Hurwitz,
\[ e(Z) = 2m + 11, \quad K_Z^2 = 1 - 2m. \]
By the holomorphic Lefschetz fixed point formula (cf. [7]),
\[ 1 - 0 - 1 = 2m/4 + \sum \left( (1 - g(A_i))/2 + A_i^2/4 \right) = e(Y^\alpha)/4 + \sum A_i^2/4, \]
so
\[ \sum A_i^2 = 0, \quad \sum K_Y A_i = 2m. \]
Every \( \alpha \)-invariant divisor such as \( A = \sum A_i \), if never intersects the 6 \( A_2 \)-configurations, can be written as
\[ A = x(E_1' + R_3) + y(E_3' + R_2) + bC_1' \]
for some rational numbers \( x, y, b \). In our case
\[ 0 = A^2 = -2x^2 + 2y^2 - 2b^2 + 4xy + 8bx + 4by. \]
Note that \( E_2^2 A = \alpha(E_3') \alpha(A) = R_2 A, E_2^2 A = R_1 A \) and \( E_1' A = R_3 A \). Since \( (E_1' + R_3) + (E_2' + R_1) \equiv 2E_3' + 2R_2 \), these imply that
\[ E_1' A + E_2' A = 2E_3' A. \]
By Lemma 5 the intersection number \( E_1' A \) is an even integer not exceeding \( E_1' \alpha(E_1') \). From these, we infer that
\[ E_1' A = E_2' A = E_3' A = 0 \quad \text{or} \quad E_1' A = E_2' A = E_3' A = 2. \]
In the latter case, \(-x + y + 2b = 3x + y = x + y + b = 2\) which together with the quadratic equation has no rational solution. In the former case, \(-x + y + 2b = 3x + y = x + y + b = 0\) which together with the quadratic equation has one solution \( x = y = b = 0 \). This implies that \( A = 0 \), hence \( Y^\alpha = \emptyset \). This completes the proof of Proposition 3 in this case.

**Case II.** \( \alpha = 1 \) on \( H^2(O_Y) \).

By the topological Lefschetz,
\[ e(Y^\alpha) = 4. \]
Let \( Y^\alpha \) consist of \( 2m \) points \( P_1, \ldots, P_{2m} \) and curves \( A_1, \ldots, A_t \). Then \( e(Y^\alpha) = 2m + \sum (2 - 2g(A_i)) = 4 \). By the holomorphic Lefschetz fixed point formula (cf. [7]),
\[ 1 - 0 + 1 = 2m/4 + \sum \left( (1 - g(A_i))/2 + A_i^2/4 \right) = e(Y^\alpha)/4 + \sum A_i^2/4, \]
so
\[ \sum A_i^2 = 4, \quad \sum K_Y A_i = 2m - 8. \]
As in the previous case, \( A = \sum A_i = x(E_1' + R_3) + y(E_3' + R_2) + bC_1' \) for some rational numbers \( x, y, b \) and
\[ 4 = A^2 = -2x^2 + 2y^2 - 2b^2 + 4xy + 8bx + 4by. \]
As in the previous case, \( E_1' A = E_2' A = E_3' A = 0 \) or \( 2 \), thus
\[ -x + y + 2b = 3x + y = x + y + b = 0 \]
or

\[-x + y + 2b = 3x + y = x + y + b = 2.\]

Either together with the quadratic equation has no rational solution.

The following Lemma completes the proof of Proposition 3.

**Lemma 5.** Let $\alpha$ be an involution on a smooth surface $V$. Let $D$ be an irreducible curve on $V$ such that $\alpha(D) \neq D$. If $P \in \alpha(D) \cap D$ is an isolated fixed point of $\alpha$, then every branch $D'$ of $D$ at $P$ is tangent to $\alpha(D')$. In particular, if $P \in \alpha(D) \cap D$ is a transversal intersection point, then either $P \neq \alpha(P)$ or $P$ lies on a point-wise fixed curve of $\alpha$.

**Proof.** At an isolated fixed point, $\alpha(x, y) = (-x, -y)$ in a suitable local coordinates $x, y$. So $\alpha$ preserves all tangential directions. \(\Box\)

6. PROOF OF THEOREM

Suppose that the Cartwright-Steger surface $X$ contains a smooth curve $B$ with $K_XB = 2, B^2 = 0$. Then By Step III,

\[B \sim 1/9(2E_3 + C_1 - C_2).\]

Let

\[p : X \to X/\langle \sigma \rangle\]

be the quotient map and

\[\nu : Y \to X/\langle \sigma \rangle\]

be the minimal resolution.

By Lemma 2, $p_*B$ is a smooth curve away from the singular points of $X/\langle \sigma \rangle$,

\[p^*p_*B = B + \sigma(B) + \sigma^2(B).\]

Since $p_*E = 3\bar{E}$ for any $\sigma$-invariant curve $E$, we see that

\[p_*B \sim \frac{1}{9}p_*(2E_3 + C_1 - C_2) = \frac{1}{3}(2\bar{E}_3 + \bar{C}_1 - \bar{C}_2),\]

thus

\[\nu^*p_*B \sim \frac{1}{3}\nu^*(2\bar{E}_3 + \bar{C}_1 - \bar{C}_2) = \frac{1}{3}(2E'_3 + C'_1 - C'_2 + \frac{2R_1 + 8R_2 + 2R_3}{3}).\]

By Proposition 3

\[\alpha\nu^*p_*B \sim \frac{1}{3}(2R_2 + C'_1 - C'_2 + \frac{2E'_2 + 8E'_3 + 2E'_1}{3}).\]

Then a direct computation using Table 3 in Proposition 2 gives

\[\langle \nu^*p_*B \rangle(\alpha\nu^*p_*B) = \frac{4}{3},\]

not an integer, a contradiction.
7. Further Discussion on the Surface $Z = Y/\langle \alpha \rangle$

The images of $E'_i$, $R_j$, $C'_k$, on $Z = Y/\langle \alpha \rangle$ will be denoted by $e_i$, $r_j$, $c_k$ respectively. Then

$$e_1 = r_3, \quad e_2 = r_1, \quad e_3 = r_2.$$  

**Proposition 4.**

(1) $K_Z = r_2 + t$ for the unique 2-torsion divisor $t$.

(2) $Z$ is a minimal surface of general type with $K^2_Z = 1$, $p_g(Z) = q(Z) = 0$, $\pi_1(Z) = \mathbb{Z}/2\mathbb{Z}$, $\text{rk} \text{Pic}(Z) = b_2(Z) = 9$.

(3) $r_1$ and $r_3$ are rational curves with one node, arithmetic genus 1.

(4) $r_2$ is a rational curve with two nodes, arithmetic genus 2.

(5) $c_1$ and $c_2$ are (smooth) elliptic curves.

(6) $c_3$ and $c_4$ are curves with geometric genus 1, arithmetic genus 6 and 5 nodes.

(7) The intersection matrix of the 7 curves is given as follows:

|     | $r_1$ | $r_2$ | $r_3$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $r_1$ | -1    | 1     | 3     | 0     | 4     | 4     |       |
| $r_2$ | 1     | 1     | 1     | 1     | 3     | 3     |       |
| $r_3$ | 3     | 1     | -1    | 2     | 2     | 2     |       |
| $c_1$ | 0     | 1     | 2     | -1    | 2     | 5     | 2     |
| $c_2$ | 0     | 1     | 2     | 2     | -1    | 2     | 5     |
| $c_3$ | 4     | 3     | 2     | 5     | 2     | 7     | 10    |
| $c_4$ | 4     | 3     | 2     | 2     | 5     | 10    | 7     |

**Table 4.**

(8) The three $A_2$-configurations $r_{i1} - r_{i2}$ are disjoint from the 7 curves $r_i, c_j$.

(9) $\text{Pic}(Z) = \text{NS}(Z) = H^2(Z, \mathbb{Z})$ is generated up to finite index by the 9 curves $r_1, r_2, c_1, r_{ij}$, whose intersection matrix has determinant $3^4$.

(10) There are many numerical equivalences on $Z$, e.g.,

$$r_1 + r_3 \sim 2r_2 \equiv 2K_Z,$$

$$c_1 + c_3 \sim c_2 + c_4 \sim 4r_2 \equiv 4K_Z,$$

$$c_1 + c_2 + r_3 \sim 3r_2 \sim 3K_Z,$$

$$r_1 + r_2 \sim c_1 + c_2.$$

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