A QFT approach to $W_{1+\infty}$

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Abstract

$W_{1+\infty}$ is defined as an infinite dimensional Lie algebra spanned by the unit operator and the Laurent modes of a series of local quasiprimary chiral fields $V^l(z)$ of dimension $l+1$ ($l = 0, 1, 2, \ldots$). These fields are neutral with respect to the $u(1)$ current $J(z) = V^0(z)$; as a result the $(l+2)$-fold commutator of $J$ with $V^l$ vanishes. We outline a construction of rational conformal field theories with stress energy tensor $T(z) = V^1(z)$ whose chiral algebras include all $V^l$'s. It is pointed out that earlier work on local extensions of the $u(1)$ current algebra solves the problem of classifying all such theories for Virasoro central charge $c = 1$.
Introduction

The infinite dimensional Lie algebra \( W_{1+\infty} \) has emerged in various contexts. It was first introduced as the non-trivial central extension \( \hat{D} \) of the Lie algebra of differential operators on the circle \([KP]\). The theory of its quasifinite highest weight representations was developed by Kac, Radul et al. \([KR, PKRW]\) (see also \([AFMO1, AFMO2]\), the latter reference containing a review and a bibliography on the subject). Among the many applications we should like to single those to the quantum Hall effect \([CTZ1, CTZ2]\) and to the bispectral problem (for a recent development and further references see \([BHY]\)).

The present paper starts with a quantum field theoretic (QFT) characterization of a larger algebra, \( \hat{gl} \), in terms of a basic bilocal field \( V(z_1, z_2) \). \( W_{1+\infty} \) is spanned by the local quasiprimary fields \( V \) entering the expansion of \( V \) around \( z_1 = z_2 \). We outline a program for constructing rational conformal field theory (RCFT) extensions of the chiral \( W_{1+\infty} \) algebra and point out that this program has been carried out in \([BMT]\) and \([PT]\) for Virasoro central charge \( c = 1 \) within the study of local extensions of the \( u(1) \) current algebra.

1 Chiral algebra generated by a bilocal “dipole field”

We shall begin by describing the QFT counterpart of \( \hat{gl} \), the central extension of the Lie algebra of infinite matrices with finitely many non-zero diagonals – see \([PKRW]\), Sec. 2. Our starting point will be a chiral algebra \( A_c = A_c(1, -1) \) generated by a bilocal field \( V(z_1, z_2) \) satisfying the commutation relations (CR)

\[
\left[ V(z_1, z_2), V(z_3, z_4) \right] = \left\{ V(z_1, z_4) + \frac{c}{z_{14}} \right\} \delta(z_{23}) - \left\{ V(z_3, z_2) + \frac{c}{z_{32}} \right\} \delta(z_{14})
\]

where \( z_{ij} = z_i - z_j \) and the complex variable \( \delta \)-function can be defined through its Laurent expansion (cf. \([K]\))

\[
\delta(z_{12}) = \frac{1}{z_{12}} + \frac{1}{z_{21}}, \quad \delta(z_{ij}) = \sum_{n=0}^{\infty} \frac{z_{n+1}}{z_i^n}
\]

and is characterized by the properties

\[
\int_{|z_1|=|z_2|} f(z_1) \delta(z_{12}) \frac{dz_1}{2\pi i} = f(z_2).
\]
The central charge $c \left( [c, V(z_1, z_2)] = 0 \right)$ will be regarded as a real number that labels our chiral algebra (along with the pair $\{1, -1\}$ of opposite charges to be identified shortly). The inhomogeneous term

$$\Psi(z_1, z_2; z_3, z_4) = \frac{c}{z_{14}} \delta(z_{23}) - \frac{c}{z_{32}} \delta(z_{14})$$

in (1.1) satisfies the 2-cocycle condition

$$\delta(z_{23})\Psi(z_1, z_4; z_5, z_6) - \delta(z_{14})\Psi(z_3, z_2; z_5, z_6) + \delta(z_{25})\Psi(z_3, z_4; z_1, z_6) - \delta(z_{16})\Psi(z_3, z_4; z_1, z_2) + \delta(z_{30})\Psi(z_5, z_4; z_1, z_2) - \delta(z_{45})\Psi(z_1, z_2; z_3, z_6) = 0$$

which guarantees the Jacobi identity for double commutators. In order to display the connection of the above Lie algebra structure with $\hat{gl}$ it suffices to insert the double Laurent expansion of $V(z_1, z_2) = \sum_{i, j \in \mathbb{Z}} E_{ij} z_1^i z_2^{-j - 1}$ in (1.1) and compute the resulting CR for the Weyl matrices $E_{ij}$:

$$\left[ E_{ij}, E_{kl} \right] = \delta_{jk} E_{il} - \delta_{il} E_{kj} + c \left\{ \theta(j) - \theta(i) \right\} \delta_{il} \delta_{jk},$$

$$\theta(i) = 1 \text{ for } i > 0, \quad \theta(i) + \theta(-i) = 1. \quad (1.5)$$

The CR (1.1) have a well defined limit when we set, after a possible differentiation, the arguments of $V$ equal to each other. The resulting local fields will be viewed as elements of the same chiral algebra $\mathcal{A}_c$. The first two of them, the $u(1)$ current

$$J(z) = V(z, z) = \sum_{n \in \mathbb{Z}} J_n z^{-n - 1} \quad (1.6)$$

and the (chiral) stress energy tensor

$$T(z) = \frac{1}{2} \left\{ (\partial_1 - \partial_2) V(z_1, z_2) \right\} \bigg|_{z_1 = z_2 = z} = \sum_{n \in \mathbb{Z}} L_n z^{-n - 2} \quad (1.7)$$

($\partial_i = \frac{\partial}{\partial z_i}$) generate a Lie subalgebra of $\hat{gl}$. Inserting (1.6) into (1.1) we find

$$\left[ V(z_1, z_2), J(z_3) \right] = \left\{ V(z_1, z_2) + \frac{c}{z_{12}} \right\} \left( \delta(z_{23}) - \delta(z_{13}) \right) \quad (1.8)$$

$$\left[ V(z_1, z_2), J_n \right] = \left\{ V(z_1, z_2) + \frac{c}{z_{12}} \right\} \left( z_2^{n} - z_1^{n} \right). \quad (1.9)$$
These relations justify the \{1, -1\} charge assignment for \(V\) (and the term “dipole field”). In the limit \(z_2 \to z_1\) we recover the \(u(1)\) current algebra:
\[
[J(z_1), J(z_2)] = -c \delta'(z_{12}) \iff [J_m, J_n] = cm \delta_{m,-n}.
\] (1.10)

Similarly, (1.1) and (1.7) imply
\[
[V(z_1, z_2), T(z_3)] = \left\{ \frac{c}{z_{12}} - \partial_1 V(z_1, z_2) \right\} \delta(z_{13}) - \left\{ \frac{c}{z_{12}} + \partial_2 V(z_1, z_2) \right\} \delta(z_{23})
\]
\[
\quad - \frac{1}{2} \left\{ V(z_1, z_2) + \frac{c}{z_{12}} \right\} \left\{ \delta'(z_{13}) + \delta'(z_{23}) \right\};
\] (1.11)

\[
[L_n, V(z_1, z_2)] = \left\{ z_1^n \left( D_1 + \frac{n+1}{2} \right) + z_2^n \left( D_2 + \frac{n+1}{2} \right) \right\} V(z_1, z_2)
\]
\[
\quad + cd_n(z_1, z_2), \quad D_i = z_i \partial_i;
\] (1.12)

the cocycle \(d_n(z_1, z_2) = -d_n(z_2, z_1)\) is homogeneous of degree \(n - 1\) in its arguments:
\[
d_n(z_1, z_2) = \frac{1}{2z_{12}} \sum_{i=1}^{n-1} (z_1^i - z_2^i)(n^{i-1} - z_2^{n-i}) \quad \text{for } n \geq 2,
\] (1.13)
\[
d_{-n}(z_1, z_2) = \frac{1}{z_1z_2} d_n \left( \frac{1}{z_1}, \frac{1}{z_2} \right) \quad (n \geq 2), \quad d_n = 0 \quad \text{for } n = 0, \pm 1.
\] (1.14)

The \(L_n\) generate the Virasoro algebra:
\[
[L_n, L_m] = (n-m)L_{n+m} + c \frac{n^3 - n}{12} \delta_{n,-m}.
\] (1.15)

The vanishing of the cocycle \(d_n\) for \(n = 0, \pm 1\) can be interpreted by saying that \(V\) is a quasi-primary bilocal field. We recall that \(V^l(z)\) is a local quasi-primary field of (conformal) dimension \(l + 1\) provided
\[
[L_n, V^l(z)] = z^n \left( D + (n+1)(l+1) \right) V^l(z) \quad \text{for } n = 0, \pm 1.
\] (1.16)

\((V^l\) is called primary if (1.16) holds for all \(n \in \mathbb{Z}\).)

The importance of the concept of a quasi-primary field stems from the fact that we shall be dealing, to start with, with the vacuum representation of the chiral field algebra in which the lowest weight (vacuum) vector \(|0\rangle\), defined by
\[
E_{ij}|0\rangle = 0 \quad \text{for } i \leq j
\]
\[
\quad \Rightarrow J_n|0\rangle = 0 \quad \text{for } n \geq 0, \quad L_n|0\rangle = 0 \quad \text{for } n \geq -1
\] (1.17)
is Möbius or su(1,1) invariant (it is annihilated by the conformal energy operator $L_0$ and by $L_{\pm 1}$). Möbius invariance determines local (2- and) 3-point functions of quasiprimary fields (up to a normalization) – see e.g. [FST]. Equations (1.1), (1.4) and (1.17) allow to write down the correlation functions of the basic bilocal field $V$. We have, in particular,

$$
\langle 0|V(z_1, z_2)V(z_3, z_4)|0\rangle = \frac{c}{z_{14}z_{23}}
$$

$$
\langle 0|V(z_1, z_2)V(z_3, z_4)V(z_5, z_6)|0\rangle = \frac{c}{z_{16}z_{23}z_{45}} - \frac{c}{z_{14}z_{25}z_{36}}.
$$

A realization of the above algebra for a positive integer $c$ is provided by the composite field (cf. [BPRSS])

$$
V_{\psi^*\psi}(z_1, z_2) = \sum_{i=1}^{c} :\psi_{i}^*(z_1)\psi_{i}(z_2):.
$$

where $\psi_i$ and their conjugates are free Fermi fields satisfying the canonical anticommutation relations

$$
\left[\psi_i(z_1), \psi_j(z_2)\right]_+ = 0 = \left[\psi_i^*(z_1), \psi_j^*(z_2)\right]_+
$$

$$
\left[\psi_i(z_1), \psi_j^*(z_2)\right]_+ = \delta_{ij}\delta(z_{12})
$$

or, in Fourier modes,

$$
\left[\psi_{i\mu}, \psi_{j\nu}^*\right]_+ = \delta_{\mu,-\nu}\delta_{ij}.
$$

In the vacuum (Neveu–Schwarz) sector $\psi^{(s)}(z)$ is expanded in integer powers of $z$:

$$
\psi^{(s)}(z) = \sum_{n\in\mathbb{Z}} \psi_{-n}^{(s)} \frac{1}{z^n}.
$$

The normal product $::$ in (1.20) is defined by either subtracting the vacuum expectation value,

$$
::\psi^*(z_1)\psi(z_2):: = \psi^*(z_1)\psi(z_2) - \frac{1}{z_{12}},
$$

or, equivalently, by ordering the Fourier modes:

$$
::\psi_{\mu}^*\psi_{\nu}:: = \begin{cases} 
\psi_{\mu}^*\psi_{\nu} & \text{for } \nu > 0 \\
-\psi_{\nu}\psi_{\mu}^* & \text{for } \nu < 0.
\end{cases}
$$

The main goal of this section is to establish an expansion formula for $V(z_1, z_2)$ in terms of a basis $V^l(z)$ of local quasiprimary fields (of conformal dimension $l + 1$) that is used in [CTZ1, CTZ2] for characterizing the $W_{1+\infty}$ algebra.
Theorem 1.1 Under the above assumptions the bilocal field $V$ admits the following expansion in local quasiprimary fields

$$V(z_1, z_2) = \sum_{l=0}^{\infty} \frac{(2l + 1)!}{(l!)^2} \int_{z_2}^{z_1} \frac{(z_1 - z)^l(z - z_2)^l}{z_{12}^{l+1}} V^l(z)dz$$  (1.23)

where $V^l$ satisfy (1.16) and the orthonormalization condition

$$\langle 0 | V^l(z_1) V^l'(z_2) | 0 \rangle = c \frac{(l!)^4}{(2l)!} z_{12}^{-2l-2} \delta_{ll'}.$$  (1.24)

As a consequence of (1.24) the 3-point function of $V$ and $V^l$ is computed to be

$$\langle 0 | V(z_1, z_2) V^l(z_3) | 0 \rangle = l! \frac{(2l)!}{(z_{13} z_{23})^{l+1}}.$$  (1.25)

Conversely, $V^l$ is computed in terms of $V$:

$$V^l(z) = \left(\frac{2l}{l}\right)^{-1} \lim_{z_{12} \to z} \sum_{k=0}^{l} \binom{l}{k}^2 \partial_k (-\partial_2)^{l-k} V(z_1, z_2)$$

$$= \frac{l!}{(2l)!} \lim_{z_{12} \to z} \partial_l (-\partial_2)^l \left\{ z_{12}^l V(z_1, z_2) \right\}.$$  (1.26)

Defining the Fourier modes of $V^l$ by the expansion formula

$$V^l(z) = \sum_{n \in \mathbb{Z}} V^l_n z^{-n-l-1}$$  (1.28)

one deduces that $V^l_n$ satisfy CR of the form

$$\left[ V^k_m, V^l_n \right] = (lm - kn) V^l_{m+n} + \sum_{\nu=1}^{\left\lfloor \frac{l+k+1}{2} \right\rfloor} p_\nu(k, m, l, n) V^l_{m+n-2\nu}$$

$$+ c \frac{(l!)^4}{(2l)!} \binom{l + m}{m - l - 1} \delta_{m-n} \delta_{kl}.$$  (1.29)

where $p_\nu$ are polynomials in their arguments.

**Proof.** We first check the consistency among (1.23), (1.24) and (1.25). To this end we take the matrix element of both sides of (1.23) between $V^l(z_3) | 0 \rangle$ and $\langle 0 |$ using (1.24) and the integral formula

$$\frac{(2l + 1)!}{(l!)^2} \int_{z_2}^{z_1} \frac{(z_1 - z)^l(z - z_2)^l}{z_{12}^{l+1}} dz = \frac{z_{12}^l}{(z_{13} z_{23})^{l+1}}.$$  (1.30)
This expression coincides with the unique up to a normalization Möbius invariant 3-point function. Conversely, using the uniqueness of the solution of the corresponding momenta problem we deduce the expression for the kernel multiplying $V^l(z)$ in the integrand of (1.23). The numerical coefficients are then computed from the requirement that the expansion (1.23) with $V^l$ satisfying (1.25) reproduces the 4-point function (1.18). The latter property is a consequence of the identity

$$\frac{1}{1 - \eta} = 1 + \sum_{l=1}^{\infty} \left(\frac{2l - 2}{l - 1}\right)^{-1} \eta^l F(l, l; 2l; \eta)$$

where the Gauss hypergeometric function is identified from the integral representation

$$F(l, l; 2l; \eta) = \frac{(2l - 1)!}{[(l - 1)!]^2} \int_{z_4}^{z_3} \frac{(z_3 - z)^{l-1}(z - z_4)^{l-1}}{(z_1 - z)(z_2 - z)^l} dz$$

$$= \frac{1}{B(l, l)} \int_0^1 \frac{t^{l-1}(1 - t)^{l-1}}{(1 - \eta t)^l} dt, \quad \left( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \right),$$

$t$ and $\eta$ being the cross-ratios

$$t = \frac{(z - z_4)z_{13}}{(z_1 - z)z_{34}}, \quad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}.$$ (1.33)

The expression (1.26) for $V^l$ (apart from the normalization factor) can be computed in two ways yielding the same answer. First, it follows from $L_1$ covariance (being indeed a special case of eqs. (3.68–71) of [FST]). Indeed, setting

$$V^l(z) = \lim_{z_{1,2} \to z} D_l(\partial_1, \partial_2) V(z_1, z_2),$$

where $D_l(\alpha, \beta)$ is an yet unknown homogeneous polynomial of degree $l$ of its arguments, we find from (1.12)

$$[L_1, V^l(z)] = \lim_{z_{1,2} \to z} D_l(\partial_1, \partial_2) \left\{ z_1(z_1\partial_1 + 1) + z_2(z_2\partial_2 + 1) \right\} V(z_1, z_2)$$

which agrees with (1.16) for $n = 1$ iff $D_l$ satisfies the differential equation

$$\left( \alpha \frac{\partial^2}{\partial \alpha^2} + \beta \frac{\partial^2}{\partial \beta^2} + \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \right) D_l(\alpha, \beta) = 0$$

(1.35)
which together with (homogenity and) the normalization condition
\[ \mathcal{D}_l(1,-1) = 1 \] (1.36)
yields the operator applied to \( V(z_1, z_2) \) in the right hand side of (1.26).

Another way to derive (1.26) is to use the equivalence of (1.26) and (1.27), to apply the operator \((\partial_1^n(-\partial_2)\partial_1^n)\) to both sides of (1.23) and to prove that only the term with \( l = n \) does not vanish in the limit \( z_1 = z_2 \).

We shall use a roundabout road to compute the coefficient to the first term in the right hand side of (1.29) introducing on the way the Taylor expansion of \( V(z_1, z_2) \) around the second point
\[ V(z_1, z_2) = \sum_{l=0}^{\infty} \frac{z_{12}^l}{l!} J^l(z_2), \quad J^l(z) = \partial_1^l V(z_1, z_2) \bigg|_{z_1 = z_2 = z}. \] (1.37)

**Lemma 1.2** The fields \( J^l \) satisfy the local CR
\[
\left[ J^k(z_1), J^l(z_2) \right] = \sum_{\nu \geq 1} \left\{ (-1)^\nu \left( \begin{array}{c} l \\ \nu \end{array} \right) J^{k+l-\nu}(z_2) - \left( \begin{array}{c} k \\ \nu \end{array} \right) J^{k+l-\nu}(z_1) \right\} \delta^{(\nu)}(z_{12}) \\
+ c(-1)^{l+1} \frac{k!l!}{(k+l+1)!} \delta^{(k+l+1)}(z_{12}) \] (1.38)
or, in Fourier–Laurent modes, for
\[ J^l(z) = \sum_{n \in \mathbb{Z}} J^l_n z^{l-n-1}, \]
\[
\left[ J^k_m, J^l_n \right] = (lm - kn) J^{l+k-1}_{m+n} \\
+ \left\{ l(l-1)m(m+2k-1) - k(k-1)n(n+2l-1) \right\} J^{l+k-2}_{m+n} \\
+ \cdots + (-1)^k c l! \left( \begin{array}{c} k + m \\ m - l - 1 \end{array} \right) \delta_{m,-n}. \] (1.39)

The proof of the lemma consists in a straightforward computation starting from (1.1) and (1.37) and using identities of the type
\[
\left[ \partial_2^m \delta(z_{12}) \right] \frac{z_{12}^l}{(l-n)!} = (-1)^n \left( \begin{array}{c} l \\ n \end{array} \right) \delta^{(n)}(z_{12}), \quad \delta^{(m)}(z_{12}) z_{12}^n = 0 \quad \text{for } n > m. \] (1.40)
Lemma 1.3 The quasiprimary fields \( V^l \) are expressed in terms of \( J^l \) and their derivatives by

\[
V^l(z) = \left(\frac{2l}{l}\right)^{-1} \sum_{k=0}^{l} (-1)^k \binom{l}{k} \left(\frac{2l - k}{l}\right) \partial^k J^{l-k}(z). \quad (1.41)
\]

To prove (1.41) we express \(-\partial_2\) in (1.27) as \(\partial_1 - (\partial_1 + \partial_2)\).

The term \((lm - kn)V^{l+k-1}_{m+n}\) in the right hand side of (1.29) is obtained from the corresponding term in the commutator of \(J\) modes (1.39) by noting that the coefficient to \(J^l\) in the expansion (1.41) is 1. This completes the proof of Theorem 1.1.

There are two obvious advantages in computing correlation functions in terms of quasiprimary fields: the orthogonality relation (1.24) (which has no match for \(J^l\) fields) and the possibility to use Möbius invariance which determines the 3-point functions. The spectrum of the Cartan subalgebra is, however, easier to write down in terms of representatives of the differential operators \(D^l\) (see [KR] and Sec. 2 below).

2 The subalgebra \(W^{(c)}\) of \(A_c\) and its QFT representations

Let \(W^{(c)}\) be the associative subalgebra of \(A_c\) spanned by (the vacuum representation of) finite linear combinations of the quasiprimary fields \(V^l\). It can be viewed as a covering of the (vacuum representation of) \(W_{1+\infty}\). The CR (1.29) have an important corollary which characterizes \(W^{(c)}\).

Proposition 2.1 The \((l+1)\)-fold commutator of current components \(J_m = V^0_m\) with \(V^l(z)\) is a c-number:

\[
\left[ J_{m_0}, \left[ J_{m_1}, \ldots, J_{m_l}, V^l(z) \right] \ldots \right] = cl! \prod_{\nu=0}^{l} (m_\nu z^{m_\nu-1}). \quad (2.1)
\]

It follows that the derivative (commutator) action of the Heisenberg algebra generated by \(J_m\) is nilpotent on finite linear combinations of \(V^l\). This motivates the following definition. An element \(A \in A_c\) is said to belong to \(W^{(c)}\) iff there exists a positive integer \(N = N(A)\) such that for any choice of the indices \(m_1, \ldots, m_N \in \mathbb{Z}\) the \(N\)-fold commutator \([J_{m_1}, \ldots, [J_{m_N}, A], \ldots]\) vanishes.

It follows that the associative local chiral algebra corresponding to \(W_{1+\infty}\) is generated by the quasiprimary fields \(V^l\). We proceed to summarizing the
results of [KR] on the unitary lowest weight modules $L_c$ of $\hat{D}$. Denote by $W(z^m f(D))$ the image of the differential operator $z^m f(D)$ in $\text{End} L_c$. We shall also use this notation for quasipolynomials $f$ (i.e., for polynomials of $D$ and of $e^{\lambda D}$). The basic CR can be written in a compact form [KR]

$$\left[ W(z^m e^{xD}), W(z^n e^{yD}) \right] = (e^{nx} - e^{my})W(z^{m+n} e^{(x+y)D}) - c \frac{e^{nx} - e^{my}}{e^{x+y} - 1} \delta_{m,-n}. \tag{2.2}$$

The positive energy unitary irreducible representations of $\hat{D} = W_{1+\infty}$ correspond to positive integers $c$ and are labeled by $c$ real charges $r = (r_1,\ldots,r_c)$. Let $|r\rangle$ be the corresponding minimal energy state satisfying

$$\{ W(-D^j) - \lambda_j \} |r\rangle = 0 = W(-z^m D^j) |r\rangle \quad \text{for } m \geq 0. \tag{2.3}$$

The eigenvalues $\lambda_l$ of Cartan operators are then recovered from the generating function [KR]

$$\Delta_r(x) := \sum_{l=0}^{\infty} \lambda_l \frac{x^l}{l!} = \sum_{i=1}^{c} \frac{e^{iR} - 1}{e^{x} - 1}. \tag{2.4}$$

We have, in particular,

$$\lambda_0 = \sum_{i=1}^{c} r_i, \quad \lambda_1 = \frac{1}{2} \sum_{i=1}^{c} (v_i^2 - r_i), \quad \lambda_2 = \sum_{i=1}^{c} \left( \frac{r_i^3}{3} - \frac{r_i^2}{2} + \frac{r_i}{3} \right).$$

We should like, to begin with, to reexpress these results in terms of the quasiprimary fields $V^l$ and their Laurent modes. This is done by using the modes $J^l_n$ (appearing in (1.39)) as intermediary. We have, according to [KR],

$$J^l_n = -W(z^n [D]_l), \quad [D]_0 = 1, \quad [D]_l = D(D-1)\ldots(D-l+1). \tag{2.5}$$

Using the relation (1.41) between $V^l$ and $\{J^{l-k}\}_{k=0,\ldots,l}$ we find

$$V^l_n = W(z^nf_{ln}(D)) \tag{2.6}$$

where $f_{ln}$ are polynomials of degree $l$:

$$f_{ln}(D) = -\binom{2l}{l}^{-1} \sum_{\nu=0}^{l} (-1)^{l-\nu} \binom{l}{\nu}^2 [D]_\nu [-D-n-1]_{l-\nu} = -D^l + \cdots \tag{2.7}$$

(we have noted that $\sum (\binom{l}{\nu})^2 = \binom{2l}{l}$).
The spectrum of the zero modes \( V_0^l \) can be computed independently starting from the expectation value of the dipole field \( V \) in the ground state \(|r\rangle\):

\[
\langle r|V(z, w)|r\rangle = \frac{1}{z - w} \sum_{i=1}^{c} \left\{ \left( \frac{z}{w} \right)^{r_i} - 1 \right\}.
\]  

(2.8)

Inserting in it the expansion (1.23) we find

\[
\sum_{i=1}^{c} \left\{ \left( \frac{z}{w} \right)^{r_i} - 1 \right\} = \sum_{l=0}^{\infty} \frac{1}{l!} V_l(r) \left\{ \left( 1 - \frac{w}{z} \right)^{l+1} F\left(l + 1, l + 1; 2l + 2; 1 - \frac{w}{z} \right) \right\}
\]  

(2.9)

(\( F \) being the hypergeometric function (1.32)). The eigenvalues of the zero modes

\[
V_l(r) = \langle r|V_0^l|r\rangle \quad (r = (r_1, \ldots, r_c))
\]  

(2.10)

are thus computed from the recurrence relation

\[
\sum_{i=1}^{c} r_i(r_i + 1) \ldots (r_i + n) \frac{1}{(n + 1)!} = \sum_{l=0}^{n} \binom{n}{l} \frac{n!(2l + 1)!}{(l!)^2(l + n + 1)!} V_l(r).
\]  

(2.11)

Another way to compute \( V_l(r) \) is to apply the operator \( (\partial_1^l - \partial_2^l) z_{12}^l \) to both sides of (2.8) and to use (1.27). The result is

\[
V_0(r) = \sum r_i \quad (= \lambda_0),
\]

\[
V_l(r) = B(l, l + 1) \sum_{k=0}^{l-1} \binom{l}{k} \frac{l}{k + 1} \sum_{i=1}^{c} r_i \frac{\Gamma(r_i + l - k)}{\Gamma(r_i - k)}, \quad l \geq 1.
\]  

(2.12)

We find, in particular,

\[
V_1(r) = \frac{1}{2} \sum r_i^2 \quad (= \lambda_1 + \frac{1}{2} \lambda_0), \quad V_2(r) = \frac{1}{3} \sum r_i^3 \quad (= \lambda_2 - \lambda_1 - \frac{1}{6} \lambda_0),
\]

\[
V_3(r) = \frac{1}{4} \sum \left( r_i^4 + \frac{1}{5} r_i^2 \right), \quad V_4(r) = \sum \left( \frac{1}{5} r_i^5 + \frac{1}{7} r_i^3 \right), \ldots
\]

\[
V_l(-r) = (-1)^{l+1} V_l(r).
\]  

(2.13)

The first two eigenvalues giving the total charge and the energy of the ground state justify the use of the term “charges” for the representation labels \( r_i \). For \( c \geq 1 \) a representation of lowest weight \( r \) is degenerate provided some of the differences \( r_i - r_j \) (for \( i \neq j \)) are integers. It is maximally degenerate if all such differences are integers. We shall say that a (reducible) positive energy \( W^{(c)} \) module \( \mathcal{H}_c \) gives rise to a QFT representation of \( W_{1+\infty} \) if all its
irreducible positive energy submodules \( L_c(r) \) are generated from the vacuum sector by \( W(c) \)-primary chiral vertex operators \( \phi_r(z) \) such that

\[
\phi_r(z)|0\rangle = e^{zL-1}r|0\rangle
\]

and if the resulting “field algebra” is closed under fusion. (For non-degenerate representations this means that the lowest weights form an abelian group: if \( |r\rangle, |s\rangle \in \mathcal{H}_c \) then also \( |r+s\rangle \in \mathcal{H}_c \). The fusion rules for degenerate representations are also written down in [FKRW].)

### 3 RCFT extensions of \( W(c) \). The case \( c = 1 \).

We set the problem of classifying the local extensions of the chiral algebra \( W(c) \) (with the same central charge \( c \)) which admit a finite number of QFT representations \( \pi_\nu \) such that their specialized characters

\[
\text{ch}_\nu(\tau) = \text{tr}_{\pi_\nu} q^{L_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}, \quad \text{Im} \tau > 0
\]

span a modular (i.e. \( \text{SL}(2, \mathbb{Z}) \)) invariant space.

We shall present the solution to this problem for \( c = 1 \) and shall end up with some remarks concerning the general case. We note that the algebra \( W(c) \) with \( c > 1 \) can be written as the tensor product of a \( W(1) \) factor and a chiral algebra of central charge \( c - 1 \) that involves no \( \text{U}(1) \) current in the degenerate case (see [CTZ2], Sec. 3). Thus the \( c = 1 \) theory is a necessary ingredient for the solution of the general problem.

The Bose–Fermi (\( \mathbb{Z}_2 \)-graded local) chiral algebras of \( c = 1 \) containing \( W(1) \) coincide with the (local or Fermi local) extensions of the \( \text{u}(1) \) current algebra classified in [BMT] and [PT]. Each such extension is generated by a pair of oppositely charged \( \hat{\text{u}}(1) \) primary fields \( \psi(z, \pm g) \) with \( g^2 \in \mathbb{N} \) satisfying the \( \mathbb{Z}_2 \)-graded (Bose–Fermi) locality

\[
z_{12}^g \left\{ \psi(z_1, g)\psi(z_2, \pm g) - (-1)^g \psi(z_2, \pm g)\psi(z_1, g) \right\} = 0.
\]

The condition that \( \psi \) is \( \hat{\text{u}}(1) \) primary means that it satisfies the operator Ward identities

\[
\left[ J(z_1), \psi(z_2, g) \right] = g\delta(z_{12})\psi(z_2, g)
\]

\[
\left[ T(z_1), \psi(z_2, g) \right] = -\Delta_g \delta'(z_{12})\psi(z_2, g) + \delta(z_{12})\psi'(z_2, g).
\]

The compatibility of these relations, together with the Sugawara formula

\[
T(z) = \frac{1}{2} :J(z)^2:,
\]

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which is a consequence of the $W_{1+\infty}$ CR \cite{125} for $c = 1$, yield the relation between charge and conformal dimension

$$
\Delta_g = \frac{1}{2} g^2 \ (\in \frac{1}{2} \mathbb{N})
$$

(3.6)

as well as the $\hat{u}(1)$ counterpart of the Knizhnik–Zamolodchikov equation \cite{1}:

$$
\psi'(z, g) = g J(z) \psi(z, g) = g \left\{ J_+(z) \psi(z, g) + \psi(z, g) J_-(z) \right\}
$$

(3.7)

where

$$
J_+(z) = \sum_{n=1}^{\infty} J_{-n} z^{n-1} = J(z) - J_-(z).
$$

(3.8)

The QFT representations of the resulting extended algebra $A(g^2) \supset W^{(1)}$ which respect the $\mathbb{Z}_2$-graded locality of $\psi(z, \pm g)$ are generated by a set of $2g^2$ (multivalued, fractional spin) charge fields $\phi(z, e_l)$ (including the unit operator) with charges

$$
e_l = \frac{l}{2g}, \quad 1 - g^2 \leq l \leq g^2.
$$

(3.9)

The relation

$$
\psi(e^{2\pi i} z, g) = \psi(z, g) e^{2\pi i g J_0}
$$

(3.10)

implies that $\psi(z, \pm g)$ are single or double valued in the sector $H_l$ with lowest weight vector $|e_l\rangle$ depending on the parity of $l$ (since $e^{2\pi i g e_l} = (-1)^l$). If $\psi(z, \pm g)$ are Bose fields (i.e. if $g^2$ is even) then $H_l$ with odd $l$ define $\mathbb{Z}_2$-twisted sectors. The subset of even $l$’s gives rise to a modular invariant local theory in that case. The specialized characters (3.1) in the sectors $H_l$ are given by \cite{PT}

$$
K_l(\tau, g^2) = \text{tr}_{H_l} q^{L_0 - \frac{c}{24}} = \frac{1}{\eta(\tau)} \sum_n q^{\frac{1}{2} (ng + \frac{1}{2})^2}
$$

(3.11)

where $\eta(\tau)$ is the Dedekind function

$$
\eta(\tau) = q^{\frac{1}{24}} \prod_{\nu=1}^{\infty} (1 - q^{2\nu})
$$

(3.12)

and the sum in (3.11) runs over all integers ($n \in \mathbb{Z}$). They span an $SL(2, \mathbb{Z})$ invariant space \cite{PT} (for a review – see \cite{FST}, Sec. 7). We find as special cases the extensions by a pair of free Fermi fields (satisfying (1.21)) for $g^2 = 1$, \cite{224}
and the level 1 $\text{su}(2)$ current algebra for $g^2 = 2$. In both cases the resulting RCFT has two (untwisted) sectors.

If $g^2$ is not square free, $g^2 = k^2g_0^2$ ($k, g_0 \in \mathbb{N}$) then the chiral algebra can be further extended, $\mathcal{A}(g^2) \subset \mathcal{A}(g_0^2)$. If we define a $\mathbb{Z}_k$-automorphism group of $\mathcal{A}(g_0^2)$ generated by

$$
\zeta^{\frac{\pm k}{g_0}} \psi(z, \pm g_0) \zeta^{-\frac{\pm k}{g_0}} = \zeta^{\pm 1} \psi(z, \pm g_0), \quad \zeta^k = 1 \quad (3.13)
$$

then $\mathcal{A}(g^2)$ appears as the subalgebra of $\mathbb{Z}_k$-invariant elements of $\mathcal{A}(g_0^2)$ and the corresponding RCFT is named a $\mathbb{Z}_k$-orbifold theory of $\mathcal{A}(g_0^2)$.

There are known local extensions of $\mathcal{W}(c)$ for arbitrary positive integer $c$. These are:

1. The $\mathbb{Z}_2$-graded algebra $\mathcal{A}_c(1,-1)$ generated by $c$ pairs of free Fermi fields (see (1.20), (1.21)) or the associated bosonic subalgebra generated by a pair of charge $\pm 2$ fields and by the level 1 $\text{so}(2c)$ currents.

2. Tensor products of $\hat{\text{su}}(c)_1$ theories with any of the above $\mathcal{A}(g^2)$ algebras.

The (most degenerate) minimal $W_{1+\infty}$ models can be embedded in a number of rational orbifold theories corresponding to finite subgroups of $(\text{SU}(c) \subset \text{SO}(2c))$ (cf. [DV]) that are currently under study [KT].

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References

[AFMO1] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, Phys. Lett. B332 (1994) 336–344; Commun. Math. Phys. 170 (1995) 151–179; see also H. Awata et al., Lett. Math. Phys. 31 (1994) 289–298.

[AFMO2] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, Representation theory of the $W_{1+\infty}$ algebra, Kyoto preprint, hep-th/9408158.

[BHY] B. Bakalov, E. Horozov, M. Yakimov, Tau-functions as highest weight vectors for $W_{1+\infty}$ algebra, Sofia preprint, hep-th/9510211.

[BMT] D. Buchholz, G. Mack, I. T. Todorov, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 20–56.
[BPRSS] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, X. Shen, Phys. Lett. B245 (1990) 447–452.

[CTZ1] A. Cappelli, C. A. Trugenberger, G. R. Zemba, Nucl. Phys. B396 (1993) 465–490; Phys. Rev. Lett. 72 (1994) 1902–1905; Nucl. Phys. B448 [FS] (1995) 470–504, hep-th/9502021.

[CTZ2] A. Cappelli, C. A. Trugenberger, G. R. Zemba, W_{1+\infty} dynamics of edge excitations, Torino preprint, cond-mat/9407095.

[DV³] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde, Commun. Math. Phys. 123 (1989) 485–526.

[FKRW] E. Frenkel, V. Kac, A. Radul, W. Wang, Commun. Math. Phys. 170 (1995) 337–357, hep-th/9405121.

[FST] P. Furlan, G. M. Sotkov, I. T. Todorov, Riv. Nuovo Cim. 12:6 (1989) 1–202.

[K] V. G. Kac, Vertex algebras, Lecture at this Workshop.

[KP] V. G. Kac, D. H. Peterson, Proc. Natl. Acad. Sci. USA 78 (1981) 3308–3312.

[KR] V. Kac, A. Radul, Commun. Math. Phys. 157 (1993) 429–457, hep-th/9308153.

[KT] V. G. Kac, I. T. Todorov, Affine orbifolds and rational conformal field theory extensions of W_{1+\infty}, (to appear).

[PT] R. R. Paunov, I. T. Todorov, Phys. Lett. B196 (1987) 519–526.

[T] I. T. Todorov, Phys. Lett. B153 (1985) 77–81.