THE HEAT FLOW CONJECTURE FOR POLYNOMIALS AND RANDOM MATRICES

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ABSTRACT. We study the evolution of the roots of a polynomial of degree \( N \), when the polynomial itself is evolving according to the heat flow. We propose a general conjecture for the large-\( N \) limit of this evolution. Specifically, we propose (1) that the log potential of the limiting root distribution should evolve according to a certain first-order, nonlinear PDE, and (2) that the limiting root distribution at a general time should be the push-forward of the initial distribution under a certain explicit transport map. These results should hold for sufficiently small times, that is, until singularities begin to form.

We offer three lines of reasoning in support of our conjecture. First, from a random matrix perspective, the conjecture is supported by a deformation theorem for the second moment of the characteristic polynomial of certain random matrix models. Second, from a dynamical systems perspective, the conjecture is supported by the computation of the second derivative of the roots with respect to time, which is formally small before singularities form. Third, from a PDE perspective, the conjecture is supported by the exact PDE satisfied by the log potential of the empirical root distribution of the polynomial, which formally converges to the desired PDE as \( N \to \infty \). We also present a “multiplicative” version of the the conjecture, supported by similar arguments.

Finally, we verify rigorously that the conjectures hold at the level of the holomorphic moments.

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1. Introduction

In this paper, we will look at the evolution of roots of polynomials as the polynomial evolves under the heat flow. Specifically, for each \( N = 1, 2, 3, \ldots \) and each \( \tau \in \mathbb{C} \), we will define the time-\( \tau \) heat flow operator \( \exp \left( \frac{\tau}{2N} \frac{d^2}{dz^2} \right) \) on polynomials \( p_N \) of degree \( N \) as

\[
\exp \left( \frac{\tau}{2N} \frac{d^2}{dz^2} \right) p_N(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\tau}{2N} \right)^k \left( \frac{d}{dz} \right)^{2k} p_N(z), \quad \tau \in \mathbb{C},
\]

where the right-hand side of (1.1) is a terminating series for all \( \tau \in \mathbb{C} \). Note that by convention we put a factor of \( N \) (the degree of \( p_N \)) into the denominator on the left-hand side of (1.1). This convention allows the evolution of the zeros to have a reasonable limiting behavior as \( N \) tends to infinity. We will also consider the evolution of zeros of polynomials under a “multiplicative” version of the heat flow. See also the supplemental document which provides several animations illustrating the ideas of the paper.

We note briefly that the behavior of real-rooted polynomials under the backward heat flow (taking \( \tau \) to be real and negative in (1.1)) is well understood. The Pólya–Benz theorem (e.g., [1, Theorem 1.2]) ensures that the roots remain real under the backward heat flow. The large-\( N \) behavior of the (real) roots can then be understood in terms of the notion of free additive convolution as follows. If the limiting root distribution of \( p_0 \) is a compactly supported measure \( \mu_0 \) on the real line, then for all \( t > 0 \), the limiting root distribution of \( p_{\lambda t} \) is the free additive convolution of \( \mu_0 \) with the semicircular distribution of variance \( t \). (See, for example, [3, Section 5.3.3] for information on the free additive convolution.) This result is due to Kabluchko [28, Theorem 2.11] and its proof uses convergence results of Arizmendi and Perales [4] for finite free convolution introduced by Marcus, Spielman, and Srivastava [33, 32]. See also the discussion of the heat flow on polynomials in the blog posts [38, 39] of Tao.

The present paper, however, is focused on polynomials with complex roots having a two-dimensional limiting distribution. In particular, the behavior predicted in Point 2 of Conjecture 1.3 below definitely does not hold in the real-rooted case.

The study of roots of polynomials under the heat flow is closely connected to the study of roots of polynomials under repeated differentiation, as in (to mention just a very few of the most directly relevant examples) work of O’Rourke and Steinerberger [35], Feng and Yao [16], Hoskins and Kabluchko [27], and our own paper with Jalowy and Kabluchko [21].
Definition 1.1. If $p^N$ is a polynomial of degree $N$, we define the empirical root distribution of $p^N$ as the probability measure on $\mathbb{C}$ given by

$$
\frac{1}{N} \sum_{j=1}^{N} \delta_{z_j},
$$

where $z_1, \ldots, z_N$ are the roots of $p^N$, listed with their multiplicity. If $\{p^N\}_{N=1}^\infty$ is a sequence of polynomials of degree $N$, we say that a measure $\mu$ is the limiting root distribution of the sequence if the empirical root distribution of $p^N$ converges weakly to $\mu$.

Definition 1.2. Suppose $\mu$ is a compactly supported probability measure on $\mathbb{C}$ with a bounded density with respect to the Lebesgue measure. Then we define the log potential $S: \mathbb{C} \to \mathbb{R}$ and the Cauchy transform $m: \mathbb{C} \to \mathbb{C}$ of $\mu$ by

$$
S(z) = \int_{\mathbb{C}} \log(|z - w|^2) \, d\mu(w);
$$

$$
m(z) = \int_{\mathbb{C}} \frac{1}{z - w} \, d\mu(w).
$$

We begin by considering the ordinary heat flow in (1.1). We refer to this flow as the “additive” (as opposed to “multiplicative”) case, because it is connected to Gaussian random matrix models, which can be constructed as sums (as opposed to products) of large numbers of independent random matrices.

We now present a general conjecture for the behavior of high-degree polynomials under the heat flow.

Conjecture 1.3. Suppose $p^N_0$ is a sequence of polynomials of degree $N$ having a compactly supported limiting root distribution $\mu_0$ whose Cauchy transform is globally Lipschitz. Then for all $\tau \in \mathbb{C}$, the empirical root measure of

$$
p^N_\tau(z) := \exp\left(\frac{\tau}{2N} \frac{d^2}{dz^2}\right) p^N(z) \quad (1.2)
$$

also converges weakly to a compactly supported probability measure $\mu_\tau$. Furthermore, there exists a constant $C > 0$ such that the following results hold.

1. The log potential $S(z, \tau)$ of $\mu_\tau$ is a $C^{1,1}$ solution of the PDE

$$
\frac{\partial S}{\partial \tau} = \frac{1}{2} \left( \frac{\partial S}{\partial z} \right)^2 \quad (1.3)
$$

in the set $\{z \in \mathbb{C}, |\tau| < C\}$. Here $\partial/\partial \tau$ and $\partial/\partial z$ denote the Cauchy–Riemann operators, given by

$$
\frac{\partial}{\partial \tau} = \frac{1}{2} \left( \frac{\partial}{\partial \tau_1} - i \frac{\partial}{\partial \tau_2} \right), \quad \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),
$$

where $\tau = \tau_1 + i\tau_2$ and $z = x + iy$.

2. Let $m_0$ be the Cauchy transform of $\mu_0$ and define a transport map $T_\tau: \mathbb{C} \to \mathbb{C}$ by

$$
T_\tau(z) = z - \tau m_0(z) \quad (1.4)
$$

Then for all $\tau$ with $|\tau| < C$, the measure $\mu_\tau$ is the push-forward of $\mu_0$ by $T_\tau$:

$$
\mu_\tau = (T_\tau)_*(\mu_0) \quad (1.5)
$$

where $(f)_*(\mu)$ denotes the push-forward of a measure $\mu$ by a map $f$. 


In Point 1, the condition $C^{1,1}$ means, more precisely, that the partial derivatives of $S$ with respect to the real and imaginary parts of $z$ are globally Lipschitz.

Suppose the roots of $p^N$ are distinct for $\tau$ in some domain $U \subset \mathbb{C}$, so that we can parameterize the roots holomorphically in $\tau \in U$ as $\{z_j(\tau)\}_{j=1}^N$. Then in light of (1.4) and (1.5), it is natural to expect that the roots will evolve as

$$z_j(\tau) \approx z_j(0) - \tau m_{s,\tau_0}(z_j(0)).$$

Although such a result does not follow from (1.5), the arguments in Section 3 will support this behavior.

The simplest case of Conjecture 1.3 is the one in which $\mu_0$ is uniform on the unit disk, in which case $m_0(z) = \bar{z}$ for all $z$ in the disk. Then $T_\tau(z) = z - \tau \bar{z}$ and for real $\tau$ between $-1$ and 1, the measure $\mu_\tau$ in (1.5) is uniform on an ellipse with semi-axes $1 - \tau$ and $1 + \tau$. See Figure 1 and see also Slide 3 in the supplemental document for an animation.

**Remark 1.4.** If we take $\tau$ to equal a real number $t$ in (1.3) and then take twice the real part of both sides, we obtain the following real-variables PDE:

$$\frac{\partial S}{\partial \tau} = \frac{1}{4} \left( \frac{\partial S}{\partial x} \right)^2 - \left( \frac{\partial S}{\partial y} \right)^2, \quad t \in \mathbb{R}. \quad (1.7)$$

Here $\partial/\partial \tau$ is the ordinary partial derivative in the real variable $t$.

The assumption that $m_0(z)$ is globally Lipschitz guarantees that the map $T_\tau$ is a homeomorphism of $\mathbb{C}$ to $\mathbb{C}$, when $|\tau|$ is small enough; see Lemma 4.2. This assumption will be satisfied if $\mu_0$ is absolutely continuous with respect to Lebesgue measure on the plane, with a sufficiently smooth density. But if $\mu_0$ is supported on the real line, $m_0$ will not even be continuous.

As explained in Sections 4.1.2 and 4.1.3 if we have enough regularity in the solution to (1.3), we can hope to deduce Point 2 of Conjecture 1 from Point 1. To do this, we will take the Laplacian of both sides of (1.3) to obtain a continuity equation satisfied by the density of the measure $\mu_\tau$, from which a push-forward theorem will hold, under suitable technical assumptions. But in Section 3 we will give a direct argument for Point 2 of the conjecture, which does not rely on Point 1.
Remark 1.5. The polynomials in Conjecture 1.3 can be deterministic or random. In the random case, one might imagine proving the result only “almost surely.” That is, one might prove that almost surely, both the hypothesis (convergence of the empirical root measure of $p^N_0$) and the conclusions (Points 1 and 2) hold. But this would be weaker than proving that every time the hypothesis hold, the conclusion also holds.

Remark 1.6. In a separate work with Jalowy and Kabluchko [20], we prove an almost sure version of Conjecture 1.3 for random polynomials with independent coefficients. One needs mild regularity assumptions on the coefficients to guarantee that the constant $C$ (denoted $t_{\text{sing}}$ in [20]) is positive. See Theorem 5.2 (PDE) and Theorem 3.3 (push-forward theorem) in [20].

In another work with the same authors [19], we establish rigorous results in [19] for the zeros of the Gaussian analytic function under the heat flow. See also [21] for similar results on the zeros of random polynomials under repeated differentiation.

We present three distinct lines of reasoning for Conjecture 1.3.

- An argument from the perspective of random matrix theory (Section 2). This line of reasoning is specific to the case in which $p^N_0$ is the characteristic polynomial of a certain class of random matrix models, and we would hope for an almost sure version of the conjecture.

- An argument from the perspective of dynamical systems (Section 3). We will argue that for large $N$, the roots move in approximately straight lines with constant velocity given by the negative of initial Cauchy transform. This argument supports Point 2 of the conjecture.

- An argument from the perspective of partial differential equations (Section 4). We will derive an exact PDE satisfied by the log potential of the empirical root measure of a polynomial evolving by the heat equation. This PDE formally converges to the PDE (1.3) in the limit $N \to \infty$. We then show how the PDE (1.3) implies a continuity equation for the density of the measure. With enough regularity, we can then show that Point 1 of Conjecture 1.3 implies Point 2.

We now present the “multiplicative” version of Conjecture 1.3. The heat flow in (1.8) arises naturally from certain multiplicative random matrix models. See Theorem 2.14 and Lemma 2.15. This flow is almost the same as the one discussed by Terry Tao in [39].

Conjecture 1.7. A “multiplicative” version of Conjecture 1.3 also holds, with the heat flow in (1.1) replaced by the operator

$$\exp \left\{ -\frac{(\tau - \tau_0)}{2N} \left( z^2 \frac{\partial^2}{\partial z^2} - (N - 2)z \frac{\partial}{\partial z} - N \right) \right\},$$

the PDE (1.3) replaced by

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2} \left( z^2 \left( \frac{\partial S}{\partial z} \right)^2 - z \frac{\partial S}{\partial z} \right)$$

and the transport map $T_\tau$ in (1.4) replaced by

$$\tilde{T}_\tau(z) = z \exp \left\{ \tau z m_0(z) - \frac{1}{2} \right\},$$
where \( m_0 \) is the Cauchy transform of the limiting root distribution of the initial polynomials.

Since the operator in the exponent in (1.8) maps \( z^j \) to a multiple of \( z^j \) for all \( j \), the exponentiated operator will also have this property. Conjecture 1.7 will be supported by appropriate versions of the three lines of reasoning described above supporting Conjecture 1.3.

**Rigorous results.** None of the perspectives discussed above leads to a rigorous argument for Conjecture 1.3 or Conjecture 1.7. We nevertheless obtain some rigorous results, as follows. First, we establish in Sections 2.4 and 2.6 a deformation theorem for the second moment of the characteristic polynomial of certain random matrix models, in both the “additive” and “multiplicative” settings. This result provides a conceptual link between (on the one hand) the deformation of certain random matrix models with respect to a parameter and (on the other hand) the deformation of the associated characteristic polynomials by an appropriate heat flow. We believe that generalizations of this result will give rise to other connections between random matrix models and differential flows on polynomials. Second, we show in Section 5 that our two main conjectures hold at the level of the holomorphic moments. Although these moments do not uniquely determine the limiting root distributions, the results of Section 5 provide substantial confirmation of our conjectures. We also remind the reader of rigorous results obtained for random polynomials, in a separate paper with Jalowy and Kabluchko. See Remark 1.6.

2. A RANDOM MATRIX PERSPECTIVE

2.1. The model deformation phenomenon. The two most basic results in random matrix theory are the circular and semicircular laws, for Ginibre matrices and GUE matrices, respectively. There is a curious connection between them: if \( \lambda \) has a uniform distribution over the unit disk in the plane (circular law), then \( 2 \text{Re} \lambda \) has a semicircular distribution on \([-2, 2]\) (semicircular law). Although this claim as stated is obvious, it is not obvious at the level of the random matrices. Why should twice the real part of the eigenvalues in the Ginibre ensemble have the same limiting distribution as the eigenvalues of the GUE?

Recent results have shown that this connection between the Ginibre ensemble and the GUE is not simply a coincidence, but is part of a broader phenomenon, which we call the model deformation phenomenon. First, results of [13] showed a similar result for of the free multiplicative Brownian motion \( b_t \) (large-\( N \) limit of Brownian motion in the general linear group \( GL(N; \mathbb{C}) \)) and the free unitary Brownian motion \( u_t \) (large-\( N \) limit of Brownian motion in the unitary group \( U(N) \)). According to Proposition 2.6 in [13], there is a canonical map \( \Phi_t \) from the support of the Brown measure of \( b_t \) to the unit circle such that push-forward by \( \Phi_t \) turns the Brown measure of \( b_t \) into the Brown measure (or law) of \( u_t \). This result should be viewed as a deformation of the relationship between the circular and semicircular laws.

Second, Ho and Zhong [25] extended the connection between the Ginibre ensemble and the GUE to the case of Ginibre or GUE matrices with an independent Hermitian matrix added. They also extended the connection between the free multiplicative Brownian motion \( b_t \) and the free unitary Brownian motion \( u_t \) to models of the form \( ub_t \) and \( uu_t \), where \( u \) is a unitary element freely independent of \( b_t \) and \( u_t \). Hall and Ho [17] [23] [24] then established a similar result connecting the
Brown measure of $x + iy$ to the law of $x + y$, where $x$ and $y$ are freely independent self-adjoint elements and $y$ is semicircular.

Third, Hall and Ho [18] extended the results of [13] and [25] to a continuous family of models of the form $ub_{s,τ}$. Here $b_{s,τ}$ is a family of free multiplicative Brownian motions (Section 2.6) labeled by a real variance parameter $s$ and a complex covariance parameter $τ$ and $u$ is unitary and freely independent of $b_{s,τ}$. In this case, there is a push-forward theorem relating the Brown measure of $ub_{s,τ}$ to the Brown measure of $ub_{s,τ}$; that is, the Brown measures for different values of $τ$ are related with $u$ and $s$ fixed. This paper also establishes a PDE for the log potential of the Brown measure with respect to $τ$, except possibly on the boundary of the support of the Brown measure.

Fourth, Zhong [41] established results similar to those of [18] in the additive case, considering the Brown measure of an element of the form $x_0 + zs,τ$, where $z_{s,τ}$ is an elliptic element (see Section 2.2) and $x_0$ is freely independent of $z_{s,τ}$. A major technical advance in [41] is that the element $x_0$ is completely arbitrary (other than being freely independent of $z_{s,τ}$). Zhong establishes a push-forward result in this generality, and certain more refined results under special assumptions on $x_0$.

Finally, the model deformation phenomenon has been observed in the setting of $R$-diagonal operators in the paper [21] by the present authors with Jalowy and Kabluchko. The paper considers operators of the form $AR_t$, where $A$ is an arbitrary $R$-diagonal operator, $R_t$ is an $R$-diagonal “transport operator” of a certain special form, and $R_t$ is assumed to be freely independent of $A$. One example would $R_t = (up_t)^2$, where $u$ is a Haar unitary and $p_t$ is a freely independent projection with trace $1 − t$. All the examples in [21] are related to the evolution of roots of polynomials under repeated applications of differential operators, with the case $R_t = (up_t)^2$ being related specifically to repeated differentiation of polynomials.

The purpose of this section is to propose a “polynomial” counterpart of the continuous push-forward results in [18] and [41]. Specifically, we will consider random matrix versions of the elements $x_0 + zs,τ$ and $ub_{s,τ}$. Then we will propose that an appropriate version of the heat flow, when applied to the characteristic polynomial of the random matrix model, will replicate the deformation of the random matrix model with respect to $τ$. Details of this idea are presented in Section 2.2 (additive case) and Section 2.6 (multiplicative case).

### 2.2. The additive heat flow conjecture for random matrices

Let $X^N$ and $Y^N$ be independent $N \times N$ GUE matrices and consider a matrix of the form

$$Z^N = e^{iθ}(aX^N + ibY^N),$$

where $a$ and $b$ are real numbers, assumed not both zero. We call such a matrix a (rotated) elliptic random matrix (with the parameter $θ$ giving the rotation). It is convenient to parameterize such matrices by a real, positive variance parameter $s$ and a complex covariance parameter $τ$, given by

$$s = \mathbb{E}\left\{\frac{1}{N} \text{Trace}((Z^N)^*Z^N)\right\},$$

$$τ = \mathbb{E}\left\{\frac{1}{N} \text{Trace}((Z^N)^*Z^N)\right\} − \mathbb{E}\left\{\frac{1}{N} \text{Trace}((Z^N)^2)\right\}. \quad (2.2)$$
This parameterization is motivated by [12], where (in the case $K_C = \mathbb{C}^N$) the law of $Z_N$ is denoted as $\mu_{s,\tau}$ and where the parameter $\tau$ plays the role of the complex time variable in the Segal–Bargmann transform.

The parameters $s$ and $\tau$ completely determine the distribution of the matrix $Z_N$; see [18, Section 2.1] for details. Using the Cauchy–Schwarz inequality, we can verify that

$$|\tau - s| \leq s. \quad (2.3)$$

We label such a matrix as $Z_{s,\tau}^N$ and we then consider a random matrix of the form

$$X_0^N + Z_{s,\tau}^N, \quad (2.4)$$

where $X_0^N$ is independent of $Z_{s,\tau}^N$.

Suppose now that $X_0^N$ converges almost surely in $*$-distribution to an element $x_0$ in a tracial von Neumann algebra. Define

$$z_{s,\tau} = e^{i\theta}(ax + iby) \quad (2.5)$$

where $x$ and $y$ are freely independent semicircular elements and the parameters $s$ and $\tau$ are defined by analogy to (2.2). When $|\tau - s| < s$, we may apply a result of Śniady [37] to conclude that the eigenvalue distribution of $X_0^N + X_{s,\tau}^N$ converges almost surely to the Brown measure of $x_0 + z_{s,\tau}$, where $x_0$ and $z_{s,\tau}$ are taken to be freely independent. (When $|\tau - s| < s$, the element $Z_N$ in (2.1) is equal in distribution to the sum of a multiple of a Ginibre matrix and an independent elliptic matrix, so that Theorem 6 in [37] applies.)

**Conjecture 2.1** (Arbitrary plus elliptic model). Fix $s > 0$ and two complex numbers $\tau_0$ and $\tau$ with $|\tau_0 - s| < s$ and $|\tau - s| \leq s$. Let $\mu_{s,\tau_0}$ be the Brown measure of $x_0 + z_{s,\tau_0}$, where $z_{s,\tau_0}$ is defined as in (2.5) and where $x_0$ is freely independent of $z_{s,\tau_0}$, and let $m_{s,\tau_0}$ be its Cauchy transform. Then $m_{s,\tau_0}$ is Lipschitz continuous and the Brown measure $\mu_{s,\tau}$ of $x_0 + z_{s,\tau}$ can be computed by the model deformation formula

$$\mu_{s,\tau} = (\Phi_{s,\tau_0,\tau} \ast \mu_{s,\tau_0}), \quad (2.6)$$

where the map $\Phi_{s,\tau_0,\tau} : \mathbb{C} \to \mathbb{C}$ is defined as

$$\Phi_{s,\tau_0,\tau}(z) = z - (\tau - \tau_0)m_{s,\tau_0}(z). \quad (2.7)$$

Furthermore, as $\tau$ varies over the set $|\tau - s| < s$ (with $x$ and $s$ fixed), the log potential of $S_{s,\tau}$ is a $C^1$ solution of the PDE

$$\frac{\partial S_{s,\tau}}{\partial \tau} = \frac{1}{2} \left( \frac{\partial S_{s,\tau}}{\partial z} \right)^2. \quad (2.8)$$

**Remark 2.2.** Substantial partial results in the direction of Conjecture 2.1 have been obtained. First, when $x_0$ is self-adjoint, the “additive” counterpart of the results of Hall–Ho [18] apply. (The proofs from [18] — which addresses the “multiplicative” case — apply with minor modifications in the easier additive case.) In this case, the push-forward result (2.6) holds and the PDE (2.8) holds at least away from the boundary of the support of $\mu_{s,\tau}$. Second, for general $x_0$ (assumed only to be freely independent of $z_{s,\tau_0}$), the results of Zhong [41] hold. A push-forward result holds in full generality [41, Theorem D] but since Zhong does not use PDE methods in [41], he does not address whether the PDE (2.8) holds.
We emphasize that Conjecture 2.1 is not the main focus of this paper. We present it to explain how Conjecture 2.3 below relates to the general proposal given in Conjecture 1.3. The idea is that the change from $\tau_0$ to $\tau$ in Conjecture 2.1 can be carried out by applying the heat flow to the characteristic polynomial of the random matrix with the first value of $\tau$.

**Conjecture 2.3.** Fix $s > 0$ and complex numbers $\tau_0$ and $\tau$ with $|\tau_0 - s| \leq s$ and $|\tau - s| \leq s$. Let $Z^N_{s,\tau_0}$ and $Z^N_{s,\tau}$ be elliptic random matrices with parameters defined as in (2.2). Let $X^N_0$ be independent of $Z_{s,\tau}$ and $Z_{s,\tau_0}$ and assume $X^N_0$ converges almost surely in distribution to an element $x_0$ in a tracial von Neumann algebra. Let $p_{s,\tau_0}$ be the random characteristic polynomial of $X^N_0 + Z^N_{s,\tau_0}$ and define a new random polynomial $q_{s,\tau_0,\tau}$ by

$$q_{s,\tau_0,\tau}(z) = \exp \left\{ \frac{(\tau - \tau_0)}{2N} \frac{\partial^2}{\partial z^2} \right\} p_{s,\tau_0}(z), \quad z \in \mathbb{C}.$$  

Then the empirical root measure of $q_{s,\tau_0,\tau}$ converges weakly almost surely to the limiting eigenvalue distribution of $X^N_0 + Z^N_{s,\tau}$.

The conjecture says that applying the heat operator for time $(\tau - \tau_0)/N$ to the polynomial $p_{s,\tau_0}$ gives a new polynomial $q_{s,\tau_0,\tau}$ whose roots resemble those of $p_{s,\tau}$—namely the eigenvalues of $X^N_0 + Z^N_{s,\tau}$. Thus, the heat flow effectively changes the value of $\tau$. We emphasize, however, that the conjecture is not claiming that the joint distribution of the roots of $q_{s,\tau_0,\tau}$ is the same as the joint distribution of the eigenvalues of $X^N_0 + Z^N_{s,\tau}$. Rather, the conjecture merely asserts that the limiting bulk distribution of the two collections of points is the same.

**Remark 2.4.** If Conjectures 2.1 and 2.3 hold, then Conjecture 1.3 also holds—that is, the expected PDE and push-forward results will hold for the limiting root distribution of $q_{s,\tau_0,\tau}$. But the argument we will present for Conjecture 2.3 in Section 2.4 is independent of the validity of Conjecture 2.1.

2.3. Examples. In this subsection, we present three examples of the general proposal in Conjecture 2.3. As our first example, we consider the case in which
$X^N_0$ is zero, and the parameters in (2.2) are $s = 1$ and $\tau_0 = 1$. In that case, $X^N_0 + Z^N_{s,\tau_0} = Z^N_{1,1}$ is a Ginibre matrix. We then take $\tau = 1 - t$, where $t$ is a real number between $-1$ and $1$, so that

$$(\tau - \tau_0) = -t, \quad -1 \leq t \leq 1.$$  

**Example 2.5** (Circular to elliptic and circular to semicircular). Let $Z^N$ be an $N \times N$ random matrix chosen from the Ginibre ensemble and let $p$ be its random characteristic polynomial. Fix a real number $t$ with $-1 \leq t \leq 1$ and define a new random polynomial $q_t$ by

$$q_t(z) = \exp\left\{ -\frac{t}{2N} \frac{\partial^2}{\partial z^2} \right\} p(z), \quad z \in \mathbb{C}. \quad (2.9)$$

For $-1 < t < 1$, Conjecture 2.3 says the empirical root measure of $q_t$ converges weakly almost surely to the uniform probability measure on the ellipse centered at the origin with semi-axes $1 + t$ and $1 - t$. For $t = 1$, Conjecture 2.3 says that the empirical root measure of $q_1$ converges weakly almost surely to the semicircular probability measure on $[-2, 2] \subset \mathbb{R}$.

In this setting, the initial measure $\mu_{s,\tau_0} = \mu_{1,1}$ is the uniform measure on the unit disk, so that $m_{s,\tau_0}(z) = \bar{z}$ for all $z$ in the unit disk. Thus, (1.6) says, in this case, that the roots $\{z_j(t)\}_{j=1}^N$ should evolve as

$$z_j(t) \approx z_j(0) + tj(0), \quad -1 \leq t \leq 1,$$  

(2.10)

where $\{z_j(0)\}_{j=1}^N$ are the eigenvalues of $Z^N$. In particular, we should have

$$z_j(1) \approx 2 \text{Re}[z_j(0)].$$

Figure 2 shows the similarity between the roots of $q_t$ and the eigenvalues of the corresponding elliptic random matrix model. See also Slide 4 in the supplemental document for an animation. Figure 3 meanwhile, illustrates the approximate straight-line motion of the roots predicted in (2.10). See also Slide 5 in the supplemental document for a dynamical version of Figure 3. We note that a rigorous version of Example 2.5 has been established in [21, Theorem 2.1]—but with the characteristic polynomial of a Ginibre matrix replaced by a Weyl polynomial. (A Weyl polynomial is a certain random polynomial whose limiting root distribution is uniform on the unit disk.)

As our second example of Conjecture 2.3, we consider the case in which $X^N_0$ is zero, $s = 1$, and $\tau_0 = 0$. In that case, $X^N_0 + Z^N_{s,\tau_0}$ is a GUE matrix.

**Example 2.6** (Semicircular to elliptic and semicircular to circular). Let $X^N$ be a GUE matrix and let $p$ be its random characteristic polynomial. Fix a real number $t$ with $0 \leq t \leq 2$ and define a new random polynomial $q_t$ by

$$q_t(z) = \exp\left\{ \frac{t}{2N} \frac{\partial^2}{\partial z^2} \right\} p(z), \quad z \in \mathbb{C}. \quad (2.10)$$

For $0 < t < 2$, Conjecture 2.3 says the empirical root measure of $q_t$ converges weakly almost surely to the uniform probability measure on the ellipse centered at the origin with semi-axes $2 - t$ and $t$. For $t = 2$, Conjecture 2.3 says that the empirical root measure of $q_2$ converges weakly almost surely to the semicircular distribution on the interval $[-2, 2]$ on the imaginary axis.
Figure 3. Plots of 100 of the curves $z_j(t)$, $0 \leq t \leq 1$, in Example 2.5 starting from the eigenvalues of a $1,000 \times 1,000$ Ginibre matrix. Each curve changes color from blue to red as $t$ increases.

We note that the eigenvalues of a GUE matrix are distinct with probability 1. (The joint distribution of the eigenvalues is absolutely continuous on $\mathbb{R}^N$—e.g., [8, Theorem 2.5.2]—so the measure of any hyperplane where two eigenvalues are equal is zero.) Furthermore, inside the collection of polynomials of degree $N$ with real coefficients, the set of polynomials with distinct real roots is open. Since the heat evolution is continuous, we can therefore conclude that the roots of $q_t(z)$ remain real and distinct for sufficiently small positive values of $t$. We believe, however, that when $N$ is large, the roots will very quickly begin to collide and move off the real axis. (This behavior is in the spirit of how the Riemann xi function—derived from the Riemann zeta function—behaves under the heat flow; see [40].) There is therefore no contradiction in conjecturing that the $N \to \infty$ root distribution of $q_t$ will be uniform on an ellipse in the plane, even for small positive $t$.

The top part of Figure 4 shows the characteristic polynomial $p$ of a GUE matrix with $N = 60$, multiplied by a suitable Gaussian to make the values of manageable size. (Specifically, it is convenient to multiply $p(x)$ by $2^{N/2}e^{-Nx^2/4}$, which of course does not change the zeros.) The bottom part of the figure then shows the polynomial $q_t$ with $t = 0.05$, multiplied by the same Gaussian. Already by the time $t = 0.05$, the number of real roots has dropped from 60 to 32. Figure 5 then shows the complex roots of the heat-evolved polynomial at time $t = 1$. See also Slide 6 in the supplemental document for an animation.

The anticipated trajectories of the roots in Example 2.6 is more complicated than in Example 2.5. After all, we are effectively trying to run time backward from $t = 1$ in the map (2.10), even though the $t = 1$ map $z \mapsto 2 \text{Re } z$ is not invertible. To put it a different way, Example 2.6 asserts that we can deform a one-dimensional distribution of points along the real axis into a two-dimensional uniform distribution on an ellipse or disk. This deformation cannot be achieved by applying a smooth map of the sort we have on the right-hand side of (2.10).

Rather, the behavior we expect is the following. If we write the roots $\{z_j(t)\}_{j=1}^N$ of $q_t$ in the form $z_j(t) = x_j(t) + iy_j(t)$, then we expect to have the approximate
Figure 4. The characteristic polynomial of a GUE matrix and the heat-evolution of the same polynomial at $t = 0.05$, both multiplied by a Gaussian. Shown for $N = 60$. The number of real roots is 60 (top) and 32 (bottom).

Figure 5. The points $\{z_j(1)\}_{j=1}^N$ in Example 2.6, with $N = 1,000$. The points display an obvious banding structure in the vertical direction but still approximate a uniform distribution on the unit disk.
equalities

\[ x_j(t) \approx x_j(0) - \frac{t}{2} x_j(0) \]

\[ y_j(t) \approx c_j t \sqrt{1 - \frac{1}{4} x_j(0)^2}, \]

where \( c_j \) is a random constant uniformly distributed between \(-1\) and \(1\). This predicted behavior can be understood as the limiting case of Conjecture 2.3 with \( X_0^N = 0 \), \( s = 1 \), and \( \tau_0 \) tending to zero. See Figure 6 along with Slide 7 in the supplemental document.

As our third example of Conjecture 2.3, we consider the case in which \( X_0^N \) is Hermitian and \( s = \tau_0 = 1 \). In that case, the limiting eigenvalue distribution of \( X_0^N + Z_1^N \) is computed in [25, Section 3] and the limiting eigenvalue distribution of \( X_0^N + Z_1^N \) is computed in [41, Section 6] (or the additive version of [18]). We then further specialize to the case in which the limiting eigenvalue distribution of \( X_0^N \) has half of its mass at \(1\) and half of its mass at \(-1\). Figure 7 shows the similarity between the eigenvalues of \( X_0^N + Z_1^N \) and the roots of the heat evolution of the characteristic polynomial of \( X_0^N + Z_1^N \). See also Slide 8 in the supplemental document for an animation.

2.4. Rigorous support for Conjecture 2.3: a deformation result for second moments. Suppose \( p^N \) is a family of random polynomials of degree \( N \). In our applications, we will take \( p^N \) to be either the characteristic polynomial of a random matrix or the heat evolution of such a characteristic polynomial. We then consider the second moment \( D^N : \mathbb{C} \to [0, \infty) \) of \( p^N \), defined as

\[ D^N(z) = \mathbb{E}[|p^N(z)|^2]. \]

(2.13)
We then define
\[ T^N(z) = \frac{1}{N} \log D^N(z) = \frac{1}{N} \log \mathbb{E}\{|P^N(z)|^2\} \]

We expect to be able to recover the limiting root distribution of \( P^N \) from the function \( D^N \) as follows:

\[
\text{limiting root distribution of } P^N = \frac{1}{4\pi} \Delta_z \left( \lim_{N \to \infty} T^N(z) \right). \quad (2.14)
\]

That is to say, we expect that the large-\( N \) limit of \( T^N \) will be the log potential of the limiting root distribution of \( P^N \). (This sort of claim is taken for granted in the physics literature.)

To understand the claim in (2.14), consider another function obtained by interchanging the expectation value with the logarithm in the formula for \( T^N \), namely

\[
S^N(z) = \frac{1}{N} \mathbb{E}\{\log |p^N(z)|^2\} = \mathbb{E}\left\{ \frac{1}{N} \sum_{j=1}^{N} \log |z - z_j^N|^2 \right\},
\]

where \( \{z_j^N\}_{j=1}^{N} \) is the collection of roots of \( P^N \), listed with their multiplicity. Then \( \frac{1}{4\pi} \Delta_z S^N \) is easily seen to be the expected empirical root distribution of \( Z^N \). (Put the Laplacian inside the expectation value and use that \( \frac{1}{4\pi} \log |z|^2 \) is the Green’s function for the Laplacian on the plane.)

Suppose now that we have, as usual, a concentration phenomenon, in which the root distribution of \( p^N \) is approaching a deterministic limit as \( N \) goes to infinity. (In the case of characteristic polynomials of random matrices, see, for example, Sections 2.3 and 4.4 in [3].) In that case, the large-\( N \) limit of the expected empirical root distribution should be the almost sure limit of the eigenvalue distribution itself.
In that case,

\[
\text{limiting root distribution of } p^N = \frac{1}{4\pi} \Delta_z \left( \lim_{N \to \infty} S^N(z) \right).
\]

But the same concentration phenomenon suggests that interchanging the expectation value with the logarithm should not have much effect, so that \( S^N \) and \( T^N \) should be almost equal. That is to say, if the empirical measure of the set \( \{ z_j \}_{j=1}^N \) is, with high probability, close to a deterministic measure \( \mu \), then both \( S^N \) and \( T^N \) should be close to the log potential of \( \mu \), and we should have

\[
\text{limiting root distribution of } p^N = \frac{1}{4\pi} \Delta_z \left( \lim_{N \to \infty} S^N(z) \right) = \frac{1}{4\pi} \Delta_z \left( \lim_{N \to \infty} T^N(z) \right),
\]

confirming (2.14).

We consider the second moments (as in (2.13)) of the characteristic polynomials of the random matrix models introduced in the previous section, starting from the additive case. Consider an \( N \times N \) “elliptic” random matrix \( Z_{s,\tau}^N \) with parameters \( s \) and \( \tau \), as in (2.1) and (2.2). Take another random matrix \( X_0^N \) that is independent of \( Z_{s,\tau}^N \) and define a function \( D^N \) by

\[
D^N(z, s, \tau) = \mathbb{E} \left\{ \left| \det(zI - (X_0^N + Z_{s,\tau}^N)) \right|^2 \right\}. \tag{2.15}
\]

Here \( z \) ranges over \( \mathbb{C} \), \( s \) ranges over \((0, \infty)\), and \( \tau \) ranges over the set of complex numbers with \( |\tau - s| < s \). Of course, this function depends also on the distribution of the random matrix \( X_0^N \) but we suppress this dependence in the notation.

We now come to the main theorem supporting the additive heat flow conjecture (Conjecture 2.3).

**Theorem 2.7** (Deformation theorem for second moment). Suppose \( \tau_0 \) and \( \tau \) are complex numbers satisfying \( |\tau_0 - s| \leq s \) and \( |\tau - s| \leq s \), in accordance with (2.3). Consider random matrices \( X_0^N + Z_{s,\tau_0}^N \) and \( X_0^N + Z_{s,\tau}^N \), respectively, where \( X_0^N \) is independent of \( Z_{s,\tau_0}^N \) and \( Z_{s,\tau}^N \) but not necessarily Hermitian. Then the function \( D^N \) in (2.15) can also be computed as

\[
D^N(z, s, \tau) = \mathbb{E} \left\{ \exp \left( \frac{(\tau - \tau_0)}{N} \frac{\partial^2}{\partial z^2} \right) \det(zI - (X_0^N + Z_{s,\tau_0}^N)) \right\}^2. \tag{2.16}
\]

The proposition says that one can compute \( D^N \) in two different ways: first, according to the definition (2.15), by taking the expectation of the magnitude-squared of the characteristic polynomial of the \((s, \tau)\)-model; or, second, by applying the heat operator \( \exp \left( \frac{(\tau - \tau_0)}{N} \frac{\partial^2}{\partial z^2} \right) \) to the characteristic polynomial of the \((s, \tau_0)\)-model and then taking the expectation of magnitude-square of this new polynomial.

In the notation of Conjecture 2.3, Theorem 2.7 says that the random polynomials \( q_{s,\tau_0,\tau} \) and \( p_{s,\tau} \) have the same second moments. Since we expect to recover the limiting root distributions of \( q_{s,\tau_0,\tau} \) and \( p_{s,\tau} \) from their second moments, as in (2.14), Theorem 2.7 provides a strong motivation for Conjecture 2.3.

Let us use the notation \( \{ z_j^{s,\tau_0} \}_{j=1}^N \) for the roots of \( p_{s,\tau} \) and the notation \( \{ z_j^{s,\tau_0}(\tau) \}_{j=1}^N \) for the roots of \( q_{s,\tau_0,\tau} \) (i.e., the roots of \( p_{s,\tau_0} \), evolved for time \( \tau \) by the heat flow).
Then Theorem 2.7 can be restated as
\[ E \left( \prod_{j=1}^{N} (z - z_j^{s,\tau})^2 \right) = E \left( \prod_{j=1}^{N} (z - z_j^{s,\tau_0}(\tau))^2 \right). \]

2.5. Proof of Theorem 2.7 The theorem will following easily from the following result, which is of independent interest.

Proposition 2.8. The function \( D_N \) in (2.15) satisfies the second-order linear PDEs
\[
\begin{align*}
\frac{\partial D_N}{\partial \tau} &= \frac{1}{2N} \frac{\partial^2 D_N}{\partial z^2}, \\
\frac{\partial D_N}{\partial \bar{\tau}} &= \frac{1}{2N} \frac{\partial^2 D_N}{\partial \bar{z}^2}
\end{align*}
\]
for \( \tau \) in the set \( |\tau - s| < s \).

In the proposition, it is understood that the distribution of the random matrix \( X_0 \) is fixed.

We establish two lemmas that will be used to prove the proposition. The first gives the PDE satisfied by the probability density of the random matrix \( Z_{s,\tau} \).

The second relates derivatives of a characteristic polynomial in the matrix variable to derivatives in the complex variable.

Consider \( M_N(\mathbb{C}) \) as a real vector space of dimension \( 2N^2 \), equipped with the real-valued inner product \( \langle \cdot, \cdot \rangle_N \) given by the scaled Hilbert–Schmidt inner product:
\[ \langle Z, W \rangle_N = N \text{Re}[\text{Trace}(Z^\dagger W)]. \]

We choose an orthonormal basis \( \{ X_j \}_{j=1}^{N^2} \cup \{ Y_j \}_{j=1}^{N^2} \) such that \( X_j \) is Hermitian and \( Y_j = iX_j \).

We then form the translation-invariant differential operators \( \tilde{X}_j, \tilde{Y}_j, Z_j, \bar{Z}_j \) on \( M_N(\mathbb{C}) \) as
\[
\begin{align*}
\tilde{X}_j f(A) &= \frac{d}{du} f(A + uX_j) \bigg|_{u=0}, \\
\tilde{Y}_j f(A) &= \frac{d}{du} f(A + uY_j) \bigg|_{u=0}, \\
Z_j &= \frac{1}{2} (\tilde{X}_j - i\tilde{Y}_j), \\
\bar{Z}_j &= \frac{1}{2} (\tilde{X}_j + i\tilde{Y}_j).
\end{align*}
\]
(2.19)

We then introduce operators \( \Delta, \partial^2, \) and \( \bar{\partial}^2 \) given by
\[
\Delta = \sum_{j=1}^{N^2} \tilde{X}_j^2; \quad \partial^2 = \sum_{j=1}^{N^2} Z_j^2; \quad \bar{\partial}^2 = \sum_{j=1}^{N^2} \bar{Z}_j.
\]
(2.20)

Lemma 2.9. Let \( \gamma_{s,\tau}^N \) be the density of the law of the random matrix \( Z_{s,\tau}^N \) defined in Section 2.2 Then \( \gamma_{s,\tau}^N \) satisfies the PDE
\[
\frac{\partial \gamma_{s,\tau}^N}{\partial \tau} = -\frac{1}{2} \partial^2 \gamma_{s,\tau}^N.
\]
(2.21)

Furthermore, if \( f \) is a (not necessarily holomorphic) polynomial function on the space \( M_N(\mathbb{C}) \) of all \( N \times N \) matrices with complex entries, we have
\[
\frac{\partial}{\partial \tau} E \{ f(X_0^N + Z_{s,\tau}^N) \} = -\frac{1}{2} E \{ \partial^2 f(X_0^N + Z_{s,\tau}^N) \}.
\]
(2.22)
Proof. Let $\Gamma_{s,\tau}^N$ be the Gaussian measure on $M_N(\mathbb{C})$ describing the law of $Z_{s,\tau}^N$. It is given by

$$\Gamma_{s,\tau}^N = \exp \left\{ \frac{1}{2} \Delta_{s,\tau} \right\} (\delta_0)$$

where $\delta_0$ is a $\delta$-function at the origin and where $\Delta_{s,\tau}$ is defined as

$$(2.23)$$

(The formula (2.23) is equivalent to Eq. (1.7) in [12]; see also the equation between Eqs. (1.13) and (1.14) in [12].) This operator is elliptic precisely when $|\tau - s| < s$ and semi-elliptic in the borderline case $|\tau - s| = s$. The PDE (2.21) follows formally from the commutativity of the operators on the right-hand side of (2.23). Rigorously, we can compute using the explicit formula in [12, Eq. (1.15)].

Meanwhile, since $X_0^N$ is independent of $Z_{s,\tau}^N$, we have

$$E\{f(X_0^N + Z_{s,\tau}^N)\} = \int \int f(a + b)\gamma_{s,\tau}^N(b) \, db \, d\nu(a),$$

where $db$ is the Lebesgue measure on $M_N(\mathbb{C})$ and $\nu$ is the law of $X_0^N$. We may then differentiate with respect to $\tau$ under the integral sign and integrate by parts to obtain (2.22). Alternatively, we may appeal to Proposition 4.7 in [12]. □

Lemma 2.10. Let $A$ be a variable ranging over $M_N(\mathbb{C})$, let $z$ be a variable ranging over $\mathbb{C}$, and let $B$ be a fixed matrix. Then

$$\partial^2 \det(zI - (B + A)) = -\frac{1}{N} \frac{d^2}{dz^2} \det(zI - (B + A)),$$

(2.24)

where $\partial^2$ is the operator defined in (2.20).

Proof. Both sides of (2.24) are easily seen to be polynomials in $z$ and the entries of $A$. It therefore suffices to establish the identity on the nonempty open set of pairs $(A, z)$ where $zI - A$ is invertible. We will use the notation

$$Q = (zI - (B + A))$$

$$R = Q^{-1}$$

and we can verify the following basic rules for computing:

$$Z_jA = X_j,$$

(2.25)

where $Z_j$ is as in (2.19). We then use Jacobi’s formula [31, Section 8.3] for the derivative of the determinant (invertible case):

$$\frac{d}{du} \det(A(u)) = \det(A(u))\text{Trace} \left( A^{-1} \frac{dA}{du} \right),$$

where Trace is the ordinary trace; and the formula [31, Section 8.4] for the derivative of the inverse:

$$\frac{d}{du} A(u)^{-1} = -A(u)^{-1} \frac{dA}{du} A(u)^{-1}.$$

In what follows, we use the normalized trace, $\text{tr}[\cdot] = \text{Trace}[\cdot]/N$. Using the tools in the previous paragraph, we can easily compute

$$Z_j \det(Q) = -N \det(Q)\text{tr}[RX_j]$$

$$Z_j^2 \det(Q) = N^2 \det(Q)\text{tr}[RX_j]^2 - N \det(Q)\text{tr}[(RX_jRX_j)].$$
We then use the “magic formulas” for sums involving an orthonormal basis \( \{X_j\}_{j=1}^{N^2} \) as in the proof of Lemma 2.9
\[
\sum_j X_jAX_j = \text{tr}[A]I \quad \text{(2.26)}
\]
\[
\sum_j \text{tr}[X_jA] \text{tr}[X_jB] = \frac{1}{N^2} \text{tr}[AB]. \quad \text{(2.27)}
\]
(See, for example, [11, Proposition 3.1], with a change of sign because the \( X_j \)'s here are Hermitian rather than skew-Hermitian.) We therefore obtain:
\[
\partial^2 \det(Q) = \det(Q)\text{tr}[R^2] - N \det(Q)\text{tr}[R]^2. \quad \text{(2.28)}
\]
Meanwhile, a similar computation shows
\[
\frac{d}{dz} \det(Q) = N \det(Q)\text{tr}[R] \quad \text{(2.29)}
\]
\[
\frac{d^2}{dz^2} \det(Q) = N^2 \det(Q)\text{tr}[R]^2 - N \det(Q)\text{tr}[R^2]. \quad \text{(2.30)}
\]
Comparing (2.28) to (2.30) gives the claimed result.

We these lemmas in hand, we are ready for the proof of Proposition 2.8.

**Proof of Proposition 2.8.** We apply Lemma 2.9 to the function \( f \) given by
\[
f(A) = |\det(zI - (B + A))|^2.
\]
We note that \( \partial^2 \) treats antiholomorphic functions as constants. We thus obtain
\[
\frac{\partial D^N}{\partial \tau} = -\frac{1}{2} \mathbb{E} \{ \partial^2 |\det(zI - (X^N_0 + Z^N_{s,\tau})|^2 \} \]
\[
= -\frac{1}{2} \mathbb{E} \{ \det(zI - (X^N_0 + Z^N_{s,\tau})) \ \partial^2 \det(zI - (X^N_0 + Z^N_{s,\tau})) \} . \quad \text{(2.31)}
\]
Using Lemma 2.10, (2.31) becomes
\[
\frac{\partial D^N}{\partial \tau} = \frac{1}{2N} \mathbb{E} \left\{ \frac{\partial^2}{\partial z^2} \det(zI - A) \right\} \frac{\partial^2}{\partial \tau^2} D^N, \]
as claimed in (2.17). Since \( D^N \) is real-valued, (2.18) follows.

Now for the proof of Theorem 2.7.

**Proof of Theorem 2.7.** Let \( \tilde{D}^N_{\tau_0}(s, \tau, z) \) denote the function on the right-hand side of (2.16), so that when \( \tau = \tau_0 \), we have \( \tilde{D}^N_{\tau_0}(s, \tau_0, z) = D^N(s, \tau_0, z) \). Our goal is to show that \( \tilde{D}^N_{\tau_0} = D^N \). The function \( \tilde{D}^N_{\tau_0} \) can be computed as
\[
\tilde{D}^N_{\tau_0}(s, \tau, z)
= \mathbb{E} \left\{ \exp \left( \frac{1}{2N} \left( (\tau - \tau_0) \frac{\partial^2}{\partial z^2} + (\tau - \tau_0) \frac{\partial^2}{\partial \tau^2} \right) \right) \right\} \text{tr}[\det(zI - (X^N_0 + Z^N_{s,\tau_0}))^2]
= \exp \left( \frac{1}{2N} \left( (\tau - \tau_0) \frac{\partial^2}{\partial z^2} + (\tau - \tau_0) \frac{\partial^2}{\partial \tau^2} \right) \right) \mathbb{E} \left\{ \text{tr}[\det(zI - (X^N_0 + Z^N_{s,\tau_0}))^2] \right\} . \quad \text{(2.32)}
\]
From the last expression in (2.32), we can see that $\hat{D}_0^N$ satisfies the same PDEs \((2.17)\) and \((2.18)\) as $D^N(s, \tau, z)$, as a function of $\tau$ and $z$. Thus,
\[
\hat{D}_0^N(s, \tau_0 + t(\tau - \tau_0), \tau_0, z) \text{ and } D^N(s, \tau_0 + t(\tau - \tau_0), z)
\] \hspace{1cm} (2.33)
will satisfy the same PDE in $t$ and $z$ for $0 \leq t \leq 1$, with equality at $t = 0$. Since both functions are, for all values of the other variables, polynomials in $z$ and $\bar{z}$ of degree $2N$, the PDE in $t$ and $z$ is actually an ODE with values in a finite-dimensional vector space. Thus, by uniqueness of solutions of ODEs, we conclude that the two functions in (2.33) are equal for all $t$; setting $t = 1$ gives the claimed result. \(\square\)

2.6. The multiplicative heat flow conjecture for random matrices. We now define a “multiplicative” version $B_{s,\tau}^N$ of the elliptic random matrix model $Z_{s,\tau}^N$. To do this, let $X_r^N$ and $Y_r^N$ be independent Brownian motions in the space of $N \times N$ Hermitian matrices, normalized so that $X_r^N$ and $Y_r^N$ are GUEs. Then, imitating [2.1], we define an elliptic Brownian motion by
\[
Z_{s,\tau}^N(r) = e^{i\theta}(aX_r^N + ibY_r^N),
\] \hspace{1cm} (2.34)
where the parameters $a$, $b$, and $\theta$ are chosen to give the desired values of $s$ and $\tau$ in (2.2) at $r = 1$. Then we introduce a Brownian motion $B_{s,\tau}^N(r)$ as the solution of the following stochastic differential equation
\[
dB_{s,\tau}^N(r) = B_{s,\tau}^N(r) \left( i \, dZ_{s,\tau}^N(r) - \frac{1}{2}(s - \tau) \, dr \right)
\] \hspace{1cm} (2.35)
\[B_{s,\tau}^N(0) = I \] \hspace{1cm} (2.36)
Here, the $dr$ term on the right-hand side of (2.35) is an Itô correction. The process $B_{s,\tau}^N(r)$ is a left-invariant Brownian motion living in the general linear group $GL(N; \mathbb{C})$. We typically take $r = 1$, since $B_{s,\tau}^N(r)$ has the same distribution as $B_{s,s,\tau}^N(1)$. We thus use the notation
\[
B_{s,\tau}^N = B_{s,\tau}^N(r)\big|_{r=1}.
\]

The law of $B_{s,\tau}^N$ is the measure denoted as $\mu_{s,\tau}$ in [12], where it serves to define norm on the target Hilbert space in the complex-time Segal–Bargmann transform. When $\tau = 0$, the distribution of $B_{s,0}^N$ is that of a Brownian motion in the unitary group at time $s$. We refer to $B_{s,\tau}^N$ as a “multiplicative” random matrix model because the solution to (2.35) and (2.36) can be approximated by multiplying together independent random matrices:
\[
B_{s,\tau}^N \approx \prod_{j=1}^k \left( I + \frac{i}{\sqrt{k}} Z_j - \frac{1}{2k} (s - \tau) I \right)
\]
for some large positive integer $k$, where $Z_1, \ldots, Z_k$ are independent random matrices with the same distribution as $Z_{s,\tau}^N$.

We let $b_{s,\tau}(r)$ be the “free” version of $B_{s,\tau}^N(r)$, obtained by replacing $X_r^N$ and $Y_r^N$ by their free counterparts and then solving the free version of (2.35) and (2.36). We again take $r = 1$ and use the notation $b_{s,\tau}$ for $b_{s,\tau}(1)$. When $\tau = s$, the process $b_{s,s}$ is Biane’s free multiplicative Brownian motion (denoted $\Lambda_s$ in [6, Section 4.2.1]). When $\tau = 0$, the process $b_{s,0}$ is Biane’s free unitary Brownian motion [5, Section 2.3]. When $\tau = t$ is real, Kemp [29] shows that the large-$N$ limit of $B_{s,\tau}^N$, in the sense of $*$-distribution, is $b_{s,t}$.
In the special case $\tau = s$, Driver, Hall, and Kemp [13], building on results of Hall and Kemp [22], compute the Brown measure of $b_{s,s}$ using a novel PDE method. Ho and Zhong [25] then compute the Brown measure of $ub_{s,s}$, where $u$ is a unitary element freely independent of $b_{s,s}$. Hall and Ho [18] then compute the Brown measure of $ub_{s,\tau}$ for general values of $\tau$. In all cases, we believe that the Brown measure of $ab_{s,\tau}$ coincides with the large-$N$ limit of the empirical eigenvalue distribution of $U_0^N B_{s,\tau}^N$, where $U_0^N$ is independent of $B_{s,\tau}^N$ and the limiting eigenvalue distribution of $U_0^N$ equals the law of $u$.

The following conjecture describes what we believe about the Brown measure of $ab_{s,\tau}$, where $a$ is freely independent of $b_{s,\tau}$ but not necessarily unitary.

**Conjecture 2.11.** Fix $s > 0$ and two complex numbers $\tau_0$ and $\tau$ with $|\tau_0 - s| < s$ and $|\tau - s| \leq s$. Let $\mu_{s,\tau_0}$ be the Brown measure of $ab_{s,\tau}$, where $a$ is invertible and freely independent of $b_{s,\tau}$, let $m_{s,\tau_0}$ be its Cauchy transform. Then $m_{s,\tau_0}$ is Lipschitz continuous and the Brown measure $\mu_{s,\tau}$ of $ab_{s,\tau}$ can be computed by the model deformation formula

$$
\mu_{s,\tau} = (\Phi_{s,\tau_0,\tau})*\mu_{s,\tau_0},
$$

where the map $\Phi_{s,\tau_0,\tau} : \mathbb{C} \to \mathbb{C}$ is defined as

$$
\Phi_{s,\tau_0,\tau}(z) = z \exp\left\{ \frac{\tau - \tau_0}{2} (2zm_{s,\tau_0}(z) - 1) \right\}.
$$

Furthermore, as $\tau$ varies over the set $|\tau - s| < s$ (with $a$ and $s$ fixed), the log potential of $S_{s,\tau}$ is a $C^1$ solution of the PDE

$$
\frac{\partial S_{s,\tau}}{\partial \tau} = -\frac{1}{2} \left( z^2 \left( \frac{\partial S}{\partial z} \right)^2 - z \frac{\partial S}{\partial z} \right). \tag{2.39}
$$

As in the additive case, the preceding conjecture is not the main focus of the present paper, but serves to connect Conjecture 2.13 below to the general multiplicative heat flow conjecture (Conjecture 1.7).

**Remark 2.12.** If $a$ is unitary, the push-forward result in (2.37) is known from Theorem 8.2 and Proposition 8.4 of [18]. In the unitary case, the PDE (2.39) holds except possibly on the boundary of the support of $\mu_{s,\tau}$. (By Proposition 6.2 and Theorem 7.5 in [18], we can set $\varepsilon = 0$ in [18] Theorem 4.2, except possibly on the boundary of the support.) Proving the conjecture for general $a$ is more difficult than in the corresponding additive case.

We now present the multiplicative version of Conjecture 2.3.

**Conjecture 2.13.** Fix $s > 0$ and complex numbers $\tau_0$ and $\tau$ such that $|\tau_0 - s| \leq s$ and $|\tau - s| \leq s$. Let $B_{s,\tau_0}^N$ and $B_{s,\tau}^N$ be Brownian motions as in (2.35) and (2.36). Suppose $A_N$ a random matrix, independent of $B_{s,\tau}^N$, that is converging almost surely in the sense of *-distribution to an element $a$ in a tracial von Neumann algebra. Let $p_{s,\tau_0}$ be the random characteristic polynomial of $A_N B_{s,\tau_0}^N$ and define a new random polynomial $q_{s,\tau_0,\tau}$ by

$$
q_{s,\tau_0,\tau}(z) = \exp\left\{ -\frac{(\tau - \tau_0)}{2N} \left( z^2 \frac{\partial^2}{\partial z^2} - (N - 2) z \frac{\partial}{\partial z} - N \right) \right\} p_{s,\tau_0}(z). \tag{2.40}
$$

Then the empirical root measure of $q_{s,\tau_0,\tau}$ converges weakly almost surely to the limiting eigenvalue distribution of $A_N B_{s,\tau}^N$. 

are complex numbers satisfying $\tau_0 = 1$, and $\tau = 1 + i$.

See Figure 8. See also Slide 9 in the supplemental document for an animation in the multiplicative case.

We now define a multiplicative version $\tilde{D}$ of the second moment function, given by

$$\tilde{D}^N(s, \tau, z) = \mathbb{E}\left\{ \left| \det(z - A_0^n B_{s, \tau}^N) \right|^2 \right\}.$$  \hfill (2.41)

**Theorem 2.14** (Deformation theorem for second moment). Suppose $\tau_0$ and $\tau$ are complex numbers satisfying $|\tau_0 - s| \leq s$ and $|\tau - s| \leq s$, in accordance with (2.3). Consider random matrices $A_0^n B_{s, \tau_0}^N$ and $A_0^n B_{s, \tau}^N$, respectively, where $A_0^n$ is independent of $B_{s, \tau_0}^N$ and $B_{s, \tau}^N$. Then the function $\tilde{D}^N$ in (2.15) can also be computed as

$$\tilde{D}^N(s, \tau, z) = \mathbb{E}\left\{ \exp \left\{ \frac{(\tau - \tau_0)}{2N} \left( z^2 \frac{\partial^2}{\partial z^2} - (N - 2)z \frac{\partial}{\partial z} - N \right) \right\} \det(z - A_0^n B_{s, \tau}^N) \right\}.$$  \hfill (2.42)

As in the additive case, we can rewrite this result as

$$\mathbb{E}\left\{ \prod_{j=1}^{N} (z - z_j^{s, \tau}) \right\}^2 = \mathbb{E}\left\{ \prod_{j=1}^{N} (z - z_j^{s, \tau_0}(\tau)) \right\}^2,$$

where $\{z_j^{s, \tau_0}\}_{j=1}^N$ are the roots of the characteristic polynomial of $A_0^n B_{s, \tau}^N$, and $\{z_j^{s, \tau_0}(\tau)\}_{j=1}^N$ are the roots of the characteristic polynomial of $A_0^n B_{s, \tau_0}$, after evolving for time $\tau - \tau_0$ by the multiplicative heat flow.
The proof of Theorem 2.14 is similar to the proof of Theorem 2.7 in the additive case. We briefly indicate the differences. We construct elements $X_j$ and $Y_j$ as in the additive case, but with $X_j$ taken to be skew-Hermitian, so that $Y_j = iX_j$ is Hermitian. Then we define the associated left-invariant vector fields by

$$\hat{X}_j f(Z) = \frac{d}{du} f(Ae^{uX_j}) \bigg|_{u=0}; \quad \hat{Y}_j f(Z) = \frac{d}{du} f(Ae^{uY_j}) \bigg|_{u=0}. $$

Then we construct “multiplicative” operators $\Delta^m, \partial^2 m$, and $\bar{\partial}^2 m$ by replacing $\tilde{X}_j$ and $\tilde{Y}_j$ by $\hat{X}_j$ and $\hat{Y}_j$ in (2.20). Then the density of the law of $B^N_{s,\tau}$ is a heat kernel measure with Laplacian equal to $s\Delta^m - \tau\partial^2 m - \bar{\tau}\bar{\partial}^2 m$, and these three operators commute [12, Lemma 5.9], and these three operators commute [12, Corollary 5.7]. The multiplicative version of (2.22) then follows from Proposition 4.7 in [12].

We then state and prove the multiplicative version of Lemma 2.10. Once this result is established, the proof of Theorem 2.14 proceeds as in the additive case.

**Lemma 2.15.** Let $B$ be a variable ranging over $M_N(\mathbb{C})$, let $z$ be a variable ranging over $\mathbb{C}$, and let $A$ be a fixed matrix. Then

$$\partial^2 m \det(zI - AB) = \frac{1}{N} \left\{ z^2 \frac{\partial^2}{\partial z^2} - z(N - 2) z \frac{\partial}{\partial z} - N \right\} \det(zI - AB),$$

where $\partial^2 m$ acts in the $B$ variable.

**Proof.** We use the notation

$$Q = zI - AB$$

$$R = Q^{-1}.$$ The multiplicative version of the operators $Z_j$ satisfy the basic identity

$$Z_j B = BX_j.$$ Then we compute

$$Z_j \det(Q) = -N \det(Q) \text{tr}[RABX_j]$$

$$Z_j^2 \det(Q) = N^2 \det(Q) \text{tr}[RABX_j] \text{tr}[RABX_j]$$

$$- N \det(Q) \text{tr}[RABX_j RABX_j]$$

$$- N \det(Q) \text{tr}[RABX^2_j]$$

Using the magic formulas (2.26) and (2.27), but this time with the sign reversed because the $X_j$’s and $Y_j$’s are skew Hermitian, we get

$$\partial^2 m \det(Q) = - \det(Q) \text{tr}[RABRAB] + N \det(Q) \text{tr}[RAB] \text{tr}[RAB]$$

$$+ N \det(Q) \text{tr}[RAB]$$

If we write

$$AB = zI - (zI - AB) = zI - Q,$$ we get

$$\text{tr}[RAB] = z \text{tr}[RAB] - 1$$

and

$$\text{tr}[RABRAB] = 1 - 2z \text{tr}[R] + z^2 \text{tr}[R^2]$$
and
\[ \text{tr}[RAB]\text{tr}[RAB] = 1 - 2z\text{tr}[R] + z^2\text{tr}[R]^2. \]
Thus,
\[
\partial_m^2 \det(Q) = \det(Q)\{(N - 1) - 2z(N - 1)\text{tr}[R]\}
+ \det(Q)z^2\{N\text{tr}[R]^2 - \text{tr}[R^2]\}
+ N\det(Q)(-1 + z\text{tr}[R]),
\]
which simplifies to
\[
\partial_m^2 \det(Q) = \det(Q)\{-1 - z(N - 2)\text{tr}[R]\}
+ \det(Q)z^2\{N\text{tr}[R]^2 - \text{tr}[R^2]\}.
\]
Using (2.29) and (2.30), we obtain the claimed result. □

3. A dynamical systems perspective

In this section, we record the system of differential equations satisfied by the roots of a polynomial as the polynomial itself evolves according to the heat equation. We then use this result to argue for the push-forward result in Point 2 of Conjecture 1.3. A similar argument holds in the multiplicative case.

3.1. The additive case. We now record a well-known (and elementary) result about the evolution of zeros of a polynomial undergoing the heat flow.

**Proposition 3.1.** Let \( p_0^N \) be a polynomial of degree \( N \) and let \( p_\tau^N \) be the heat evolution of \( p_0^N \), as defined in (1.2), so that
\[
\frac{\partial p_\tau^N}{\partial \tau} = 1 - \frac{2}{N^2} \sum_{k \neq j} \frac{1}{(z_j(\tau) - z_k(\tau))^3}.
\]
Suppose that, for some \( \sigma \in \mathbb{C} \), the zeros of \( p_\sigma^N \) are distinct. Then for all \( \tau \) in a neighborhood of \( \sigma \), it is possible to order the zeros of \( p_\tau^N \) as \( z_1(\tau), \ldots, z_N(\tau) \) so that each \( z_j(\tau) \) depends holomorphically on \( \tau \) and so that the collection \( \{z_j(\tau)\}_{j=1}^N \) satisfies the following system of holomorphic differential equations:
\[
\frac{dz_j(\tau)}{d\tau} = - \frac{1}{N} \sum_{k \neq j} \frac{1}{z_j(\tau) - z_k(\tau)}.
\]
The paths \( z_j(\tau) \) then satisfy
\[
\frac{d^2 z_j(\tau)}{d\tau^2} = - \frac{2}{N^2} \sum_{k \neq j} \frac{1}{(z_j(\tau) - z_k(\tau))^3}.
\]
The sums on the right-hand side of (3.2) and (3.3) are over all \( k \) different from \( j \), with \( j \) fixed. The result in (3.2) is discussed on Terry Tao’s blog [18] and dates back at least to the work of Csordas, Smith, and Varga [10, Lemma 2.4]. The result that (3.2) implies (3.3) dates back at least to work of Choodnovsky and Choodnovsky [8, Corollary 7]. We emphasize that the polynomial \( p_\tau^N \) in (1.2) is well-defined for all \( \tau \in \mathbb{C} \) by (1.2), whether the roots are distinct or not.

**Remark 3.2.** The second-order equations in (3.3) are the equations of motion for the rational Calogero–Moser system. (Take \( \omega = 0 \) and \( g^2 = -1/N \) in the notation of [18, Eq. (3)].) It follows that solutions to (3.2) are special cases of solutions to the rational Calogero–Moser system, in which the initial velocities are chosen to satisfy (3.2) at \( \tau = 0 \).
Remark 3.2 (together with Conjecture 2.3) indicates a novel connection between integrable systems and random matrix theory. The negative value of $g$ in the remark means that $\tau$ is real and all the $z_j$’s are initially real, the system is attractive and collisions can occur—allowing the points to move off the real line.

For completeness, we supply a proof of Proposition 3.1.

**Proof of Proposition 3.1.** The local holomorphic dependence of the roots on $\tau$ is an elementary consequence of the holomorphic version of the implicit function theorem, with the assumption that the roots of $p_\sigma$ are distinct guaranteeing that $dp_\sigma/dz$ is nonzero at each root.

If $z_j$ is a simple zero of a polynomial $p$, an easy power-series argument shows that

$$
\frac{p''(z_j)}{p'(z_j)} = 2 \left( \frac{p'(z)}{p(z)} - \frac{1}{z - z_j} \right) \bigg|_{z = z_j},
$$

from which we easily obtain

$$
\frac{p''(z_j)}{p'(z_j)} = 2 \sum_{k \neq j} \frac{1}{z_j - z_k},
$$

where the sum is over all $k$ different from $j$, with $j$ fixed. We may then differentiate the identity $p_\sigma^N(z_j(\tau)) = 0$ to obtain

$$
\frac{\partial p_\sigma^N}{\partial \tau}(z_j(\tau)) + (p_\sigma^N)'(z_j(\tau)) \frac{dz_j}{d\tau} = 0. \tag{3.5}
$$

Using (3.1), (3.5) gives

$$
\frac{dz_j}{d\tau} = -\frac{\partial p_\sigma^N}{\partial \tau}(z_j(\tau)) \frac{1}{(p_\sigma^N)'(z_j(\tau))} = -\frac{1}{2N} \frac{(p_\sigma^N)''(z_j(\tau))}{(p_\sigma^N)'(z_j(\tau))}.
$$

Applying (3.4) then gives (3.2).

For the second derivative, we suppress the dependence of $z_j$ on $\tau$ and we use (3.2) to make a preliminary calculation for each pair $j$ and $k$ with $j \neq k$:

$$
\frac{d}{d\tau}(z_j - z_k) = -\frac{1}{N} \sum_{l \neq j} \frac{1}{z_j - z_l} + \frac{1}{N} \sum_{l \neq k} \frac{1}{z_k - z_l}. \tag{3.6}
$$

We then split each sum over $l$ on the right-hand side of (3.6) into a sum over $l \notin \{j, k\}$ plus an additional term:

$$
\frac{d}{d\tau}(z_j - z_k) = -\frac{1}{N} \left( \frac{1}{z_j - z_k} + \sum_{l \notin \{j, k\}} \frac{1}{z_j - z_l} \right) + \frac{1}{N} \left( \frac{1}{z_k - z_j} + \sum_{l \notin \{j, k\}} \frac{1}{z_k - z_l} \right),
$$

which simplifies to

$$
\frac{d}{d\tau}(z_j - z_k) = -\frac{2}{N} \frac{1}{z_j - z_k} + \frac{1}{N} \sum_{l \notin \{j, k\}} \frac{z_j - z_k}{(z_j - z_l)(z_k - z_l)}. \tag{3.7}
$$
We then differentiate (3.2) using (3.7) to get
\[
\frac{d^2 z_j}{d\tau^2} = \frac{1}{N} \sum_{k \neq j} \frac{1}{(z_j - z_k)^2} \left( -\frac{2}{N} \frac{1}{z_j - z_k} + \frac{1}{N} \sum_{l \notin \{j,k\}} \frac{z_j - z_k}{(z_j - z_l)(z_k - z_l)} \right)
\]
\[
= -\frac{2}{N^2} \sum_{k \neq j} \frac{1}{(z_j - z_k)^3} + \frac{1}{N^2} \sum_{k,l: (j,k,l) \text{ distinct}} \frac{1}{(z_j - z_k)(z_j - z_l)(z_k - z_l)}.
\]
The last sum over \(k\) and \(l\) is zero because the range of the sum is invariant under interchange of \(k\) and \(l\), but the summand changes sign under interchange of \(k\) and \(l\), leaving us with the claimed result. \(\square\)

We now explain how Proposition 3.1 should imply push-forward result in Conjecture 1.3.

**Idea 3.3.** Suppose \(p_0^N\) is a sequence of polynomials of degree \(N\) whose limiting root distribution is a sufficiently regular probability measure \(\mu_0\) with compact support. Then we expect that the following results will hold.

1. The initial velocity of the roots will be approximately the Cauchy transform of \(\mu_0\) evaluated at the initial root:
   \[z_j'(0) \approx -m_0(z_j(0)).\]
2. The second derivative of the roots will be small at any time \(\tau\) for which the distribution of roots approximates a smooth two-dimensional distribution in the plane:
   \[z_j''(\tau) \approx 0.\]

If these results hold, then as in (1.6),
\[z_j(\tau) \approx z_j(0) - \tau m_0(z_j(0)),\]
in which case the push-forward result in Point 2 of Conjecture 1.3 will hold.

**Argument for Idea 3.3.** Point 1 is obtained by approximating the sum over \(k\) in (3.2), with the factor of \(1/N\), by an integral with respect to \(\mu_0\). Meanwhile, the right-hand side of (3.3) is formally of order \(1/N\), since it is a sum over \(N - 1\) values but we are dividing by \(N^2\). In reality, the sum is not as small as this naive calculation would suggest, because \(1/z^3\) is not a locally integrable function on the plane. Thus, there will be a substantial contribution in (3.3) from the values of \(z_k\) closest to \(z_j\). But as long as these nearest neighbors are of order \(1/\sqrt{N}\)—which we expect if the distribution of roots is two-dimensional—we still expect that the right-hand side of (3.3) will be of order \(1/\sqrt{N}\). \(\square\)

### 3.2. Solving the equations using the Calogero–Moser systems.

In this subsection, we explain how the theory of Calogero–Moser systems can be used to give an “explicit” formula for the roots of a heat-evolved polynomial. Of course, the formula still requires a calculation: finding the eigenvalues of a matrix. Nevertheless, it is notable that we can find the roots of the heat-evolved polynomial, for any one fixed time \(\tau\), directly in a comparatively simple way.
Let $p_{N}^{0}$ be a polynomial of degree $N$ with roots $z_1, \ldots, z_N$, listed with their multiplicities. Let $X_0$ be the diagonal matrix with the $z_j$’s on the diagonal. Let $Y$ with diagonal entries given by
\[ Y_{jj} = \frac{1}{N} \sum_{k \neq j} \frac{1}{z_j - z_k} \]  
and off-diagonal entries given by
\[ Y_{jk} = \frac{1}{N} \frac{1}{z_j - z_k}, \quad j \neq k. \]  
(Note that (3.8) has a sum but (3.9) does not.)

**Proposition 3.4.** For any polynomial $p_{N}^{0}$ of degree $N$, let $p_{N}^{\tau}$ be the heat-evolved polynomial, as in (1.2). Then for all $\tau \in \mathbb{C}$, the roots of $p_{N}^{\tau}$ are the eigenvalues of the matrix
\[ X_0 - \tau Y. \]  
Specifically, if $p_{N}^{0}$ is assumed to be monic—so that $p_{N}^{\tau}$ is also monic—then the characteristic polynomial of $X_0 - \tau Y$ is equal to $p_{N}^{\tau}$.

**Proof.** The claimed result is a direct application of the formulas for the Calogero–Moser system (3.3), with the initial velocities given by the first-order system in (3.2). See, for example, Section 2.7, just above the remark on p. 18, of the notes [15] by Etingof (with a different scaling). □

We now discuss how Proposition 3.4 could lead to the expected large-$N$ behavior of the roots of $p_{N}^{\tau}$. We would expect from (3.8) that the diagonal entries $Y_{jj}$ of $Y$ can be approximated for large $N$ by the initial Cauchy transform $m_0$:
\[ Y_{jj} \approx m_0(z_j). \]  
(3.10)

In particular, the diagonal entries of $Y$—and therefore of $X_0 - \tau Y$—should be of order 1 as $N$ tends to infinity. By contrast, the off-diagonal entries of $Y$—and therefore the off-diagonal entries of $X_0 - \tau Y$—should be small. Indeed, these entries are typically of order $1/N$, with some small fraction of them being larger (order $1/\sqrt{N}$ if the nearest-neighbor spacings are of order $1/\sqrt{N}$).

We then might plausibly hope that the eigenvalues of $X_0 - \tau Y$, especially if $|\tau|$ is small, might be well approximated simply by the diagonal entries of $X_0 - \tau Y$. In that case, Proposition 3.4 together with (3.10), would tell us that
\[ z_j(\tau) \approx z_j - \tau m_0(z_j), \]  
which is consistent with Idea 3.3 and with Point 2 of Conjecture 1.3. We emphasize, however, that we do not currently know how to make this line of reasoning rigorous. In particular, the simplest estimate for the eigenvalues of almost-diagonal matrices, namely the Gershgorin circle theorem (e.g., [26, Theorem 6.1.1]), is not sharp enough to be useful here. Specifically, the estimate in Gershgorin’s theorem involves the sum of the absolute values of the off-diagonal entries of the matrix over the rows of the matrix. Such sums are of at least of order 1 in our setting.

Nevertheless, Proposition 3.4 gives an intriguing alternative way of attacking the problem. If nothing else, the proposition seems to give a numerically efficient way to compute the roots of the heat-evolved polynomial $p_{N}^{\tau}$—for one fixed $\tau$—at least in the regime where Conjecture 1.3 holds. Proposition 3.4 can, for example,
accurately replicate Figure 5 in the “semicircular to circular” case but with much less computational effort than numerically solving the system of ODEs in (3.2).

3.3. Comparison to first-order Coulomb systems. It is instructive to compare the system of equations in (3.2) to first-order Coulomb systems in two dimensions, such as Eqs. (1.9) or (1.10) in the ICM lecture of Serfaty [36]. These equations read as, respectively,

\[ \frac{dz_j}{d\tau} = \frac{1}{N} \sum_{k \neq j} \frac{1}{\bar{z}_j - \bar{z}_k}; \quad (3.12) \]

\[ \frac{dz_j}{d\tau} = i \frac{1}{N} \sum_{k \neq j} \frac{1}{\bar{z}_j - \bar{z}_k}. \quad (3.13) \]

These equations are the gradient flow and rotated gradient flow, respectively, for the Coulomb energy for a collection of charged particles in the plane. (One can also study the second-order Newtonian dynamics of charged particles in the plane, as in Eq. (1.11) in [36].) Equation (3.12) also describes the dynamics of Ginzburg–Landau vortices, as in Eqs. (2.46) and (2.47) in [14]. Equations (3.12) and (3.13) are very similar to (3.2), except for the conjugates on \( z_j \) and \( z_k \) on the right-hand sides.

One might then suppose that the methods used to rigorously analyze the large-\( N \) behavior of the systems (3.12) and (3.13)—described in [36]—could be adapted to analyze (3.2) as well. This idea, however, turns out not to be correct. In an important technical sense, (3.2) is different from (3.12) and (3.13), namely that the system (3.2) is attractive in the \( x \)-direction and repulsive in the \( y \)-direction. That is, two nearby points \( z_j \) and \( z_k \) will attract each other if they are arranged horizontally from each other and will repel each other if they are arranged vertically from each other. This behavior is in contrast with (3.12), which is repulsive in both directions (nearby points repel each other), and with the (3.13), which is neither attractive nor repulsive (nearby points circle around each other). The mix of attractive and repulsive behavior in (3.2) means that the methods used to study (3.12) and (3.13) cannot be used without substantial modifications.

3.4. The multiplicative case. We now describe the multiplicative version of the results of Section 3.1.

**Proposition 3.5.** Let \( p_0^N \) be a polynomial of degree \( N \) and define \( p^N_\tau \) by

\[ p^N_\tau(z) = \exp\left\{ -\frac{\tau}{2N} \left( z^2 \frac{\partial^2}{\partial z^2} - (N - 2)z \frac{\partial}{\partial z} - N \right) \right\} p_0^N(z), \]

where the exponential, as applied to \( p_0^N \), is defined as a convergent power series. Suppose that the zeros of \( p_\sigma \) are nonzero and distinct for some \( \sigma \in \mathbb{C} \). Then for all \( \tau \) in a neighborhood of \( \sigma \), it is possible to order the zeros of \( p_\tau \) as \( z_1(\tau), \ldots, z_N(\tau) \) so that each \( z_j(\tau) \) is nonzero and depends holomorphically on \( \tau \), and so that the collection \( \{z_j(\tau)\}_{j=1}^N \) satisfies the following system of holomorphic differential equations:

\[ \frac{1}{z_j(\tau)} \frac{dz_j(\tau)}{d\tau} = \frac{1}{2N} \left[ 1 + \sum_{k \neq j} \frac{z_j(\tau) + z_k(\tau)}{z_j(\tau) - z_k(\tau)} \right]. \quad (3.14) \]
Furthermore, if we write each $z_j(\tau)$ as $z_j = e^{i w_j(\tau)}$ (but where we do not assume $w_j(\tau)$ is real), we have the second-derivative formula

\[
\frac{d^2 w_j}{d\tau^2} = \frac{1}{4N^2} \sum_{k \neq j} \frac{\cos((w_j - w_k)/2)}{\sin^3((w_j - w_k)/2)}.
\]

(3.15)

The system of ODEs in (3.14) is almost the same as the one discussed in Terry Tao’s second blog post [39] on the evolution of zeros of polynomials under heat flows, differing by a minus sign, the factor of $1/N$, and the “+1” on the right-hand side. Note that if $z$ and $w$ are in the unit circle, then $(z + w)/(z - w)$ is pure imaginary. Using this observation, and interpreting the left-hand side of (3.14) as the derivative of $\log z_j(\tau)$, we can verify the following result: If $\tau_0 = 0$ and the points $\{z_j(0)\}_{j=1}^N$ are all in the unit circle, then the points $\tilde{z}_j(t) = e^{-t/2} z_j(t)$ will be remain in the unit circle for $t \in \mathbb{R}$, for as long as the points $z_j(t)$ remain distinct. In this case, however, we are interested in going well past the time when the points collide.

As for the corresponding result (3.3) in the additive case, we expect that the right-hand side of (3.15) will be small when $N$ is large. Thus, we expect that the trajectories $w_j(\tau)$ will be approximately linear in $\tau$.

**Remark 3.6.** The equation (3.15) is the equation of motion for the trigonometric Calogero–Moser system, introduced by Sullivan. (Take $a = 1/2$ and $g^2 = -1/(2N^2)$ in Eq. (9) of [7].)

**Proof of Proposition 3.5.** The verification of (3.14) is very similar to the verification of (3.2) in Proposition 3.1 and is omitted. For (3.15), we first compute that

\[
\frac{d}{d\tau} \left( \frac{z_j + z_k}{z_j - z_k} \right) = 2 \frac{z_j \frac{dz_k}{d\tau} - z_k \frac{dz_j}{d\tau}}{(z_j - z_k)^2}.
\]

(3.16)

Then since $w_j = \frac{1}{t} \log z_j$, we use (3.16) and (3.14) to compute

\[
\frac{d^2 w_j}{d\tau^2} = \frac{1}{i} \frac{d}{d\tau} \left( \frac{1}{z_j} \frac{dz_j}{d\tau} \right) = \frac{1}{2N^2} \sum_{k \neq j} \frac{1}{(z_j - z_k)^2} \frac{1}{z_j z_k} \cdot \left( \sum_{l \neq k} z_k + z_l \frac{z_l}{z_k - z_l} - \sum_{l \neq j} z_j + z_l \frac{z_l}{z_j - z_l} \right).
\]

(3.17)

Then, as in the proof of Proposition 3.1, we write each sum over $l$ in (3.17) as a sum over $l \notin \{j, k\}$ plus one extra term, giving

\[
\frac{d^2 w_j}{d\tau^2} = \frac{1}{2iN^2} \sum_{k \neq j} \frac{1}{(z_j - z_k)^2} \frac{z_j z_k}{(z_k - z_l)(z_j - z_l)} \left( -2 \frac{z_j + z_k}{z_k - z_l} + \sum_{l \neq j, k} \left( \frac{z_k + z_l}{z_k - z_l} - \frac{z_j + z_l}{z_j - z_l} \right) \right).
\]
This result simplifies to
\[
\frac{d^2 w_j}{d\tau^2} = -\frac{1}{i N^2} \sum_{k \neq j} \frac{z_j z_k (z_j + z_k)}{(z_j - z_k)^3} + \frac{1}{2 i N^2} z_j \sum_{k,l; (j,k,l) \text{ distinct}} \frac{z_k z_l}{(z_k - z_l)(z_j - z_k)(z_j - z_l)}. \tag{3.18}
\]

Now, the second term on the right-hand side of (3.18) is zero because the summand changes sign under interchange of \(k\) and \(l\). For the first term, we recall that \(z_j = e^{ix_j}\) and compute that
\[
z_j z_k (z_j + z_k) = i \cos((x_j - x_k)/2) \sin^3((x_j - x_k)/2). \tag{3.19}
\]

Dropping the second term on the right-hand side of (3.18) and using (3.19) in the first term gives (3.15).

4. A PDE perspective

In this section, we offer arguments from a PDE perspective for Conjecture 1.3 and for a multiplicative analog of that conjecture.

4.1. PDE perspective: additive case. We consider a sequence \(P^N\) of polynomials of degree \(N\) whose empirical root distribution converges to a compactly supported probability measure \(\mu_0\). As in the statement of Conjecture 1.3, we further assume that the Cauchy transform \(m_0\) of \(\mu_0\) (as in Definition 1.2) is globally Lipschitz. We then let \(L_{m_0}\) denote the Lipschitz constant of \(m_0\):
\[
L_{m_0} = \sup_{z_1 \neq z_2} \left| \frac{m_0(z_1) - m_0(z_2)}{z_1 - z_2} \right| < \infty. \tag{4.1}
\]

We now let \(P^N_\tau\) denote the associated heat-evolved polynomials, as in (1.2). We now offer arguments for Points 1 and 2 of Conjecture 1.3 from a PDE perspective.

4.1.1. Point 1 of Conjecture 1.3. Point 1 of the conjecture—that the log potential of the limiting root distribution satisfies the PDE (1.3)—is supported by the following proposition.

**Proposition 4.1.** The logarithmic potential \(S^N(z, \tau) = \frac{1}{N} \log |P^N(z, \tau)|^2\) satisfies the following PDE
\[
\frac{\partial}{\partial \tau} S^N(z, \tau) = \frac{1}{2} \left( \frac{\partial S^N}{\partial z} \right)^2 + \frac{1}{2N} \frac{\partial^2 S^N}{\partial z^2} \tag{4.2}
\]
whenever \(z\) is not a root of \(P^N(z, \tau)\). Alternatively, the second term in (4.2) can be written in terms of the roots \(\{z_j(\tau)\}_{j=1}^N\) of \(P^N(z, \tau)\) so that
\[
\frac{\partial}{\partial \tau} S^N(z, \tau) = \frac{1}{2} \left( \frac{\partial S^N}{\partial z} \right)^2 - \frac{1}{2N^2} \sum_{j=1}^N \frac{1}{(z - z_j(\tau))^2}.
\]

**Proof.** Notice that \(P^N(z, \tau)\) is holomorphic in \(\tau\) and satisfies the equation
\[
\frac{\partial}{\partial \tau} P^N(z, \tau) = \frac{1}{2N} \frac{\partial^2}{\partial z^2} P^N(z, \tau).
\]
Around \((z, \tau)\) where \(P^N(z, \tau) \neq 0\), \(P^N\) has a branch of log, so that

\[
S^N = \frac{1}{N} \log P^N + \frac{1}{N} \log P^N.
\]

Since \(\log P^N\) is holomorphic in \(z\) and \(\tau\), \(\log P^N\) is anti-holomorphic in \(z\) and \(\tau\). The \(\tau\)-derivative of \(S^N\) is given by

\[
\frac{\partial S^N}{\partial \tau} = \frac{1}{2N^2} \left( \frac{\partial^2}{\partial z^2} P^N(z, \tau) \right)
= \frac{1}{2N^2} \left( \frac{\partial^2}{\partial z^2} e^{NS^N} \right)
= \frac{1}{2N^2} \left[ N^2 \left( \frac{\partial S^N}{\partial z} \right)^2 + N \frac{\partial^2 S^N}{\partial z^2} \right],
\]

which simplifies to (4.2).

Finally, by writing \(P^N(z, \tau)\) into a product of linear factors, the first and second \(z\)-derivatives of \(S^N\) are given by

\[
\frac{\partial S^N}{\partial z} = \frac{1}{N} \frac{\partial}{\partial z} P^N(z, \tau) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z - z_j(\tau)}
\]

and

\[
\frac{\partial^2 S^N}{\partial z^2} = -\frac{1}{N} \sum_{j=1}^{N} \frac{1}{(z - z_j(\tau))^2}.
\]

This proves the last assertion.

The PDE in (4.2) formally converges to the desired PDE (1.3) as \(N \to \infty\). We now look more carefully at the question of whether the error term (the second term on the right-hand side of (4.2)) is actually small for large \(N\).

Recall that

\[
\frac{\partial^2 S^N}{\partial z^2} = -\frac{1}{N} \sum_{i=1}^{N} \frac{1}{(z - z_i(\tau))^2}.
\]

We expect the estimate

\[
\frac{\partial^2 S^N}{\partial z^2} = -\frac{1}{N} \sum_{i=1}^{N} \frac{1}{(z - z_i(\tau))^2} \approx -\text{P.V.} \int \frac{1}{(z - w)^2} d\mu_\tau(w).
\]

Thus, the principal value integral should be finite as long as \(|\tau|\) is small enough so that \(\mu_\tau\) remains absolutely continuous on \(\mathbb{C}\). If this is true, the term \(\frac{1}{2N} \frac{\partial^2 S^N}{\partial z^2} \to 0\) as \(N \to \infty\), and the limit of \(S^N\) in Theorem 4.1 becomes the solution of the desired PDE (1.3).

Furthermore, the argument in the “Details of Step 1” portion of Section 4.1.3 (based on Theorem 1.1 and Remark 1.1 in [30]) indicates that if the solution is \(C^{1,1}\), it will automatically by \(C^{1,1}\), as claimed in Point 1 of Conjecture 1.3.
4.1.2. Brief argument for Point 2 of Conjecture 1.3. In this section, we give a brief argument in three steps why we believe the push-forward relation in Point 2 of Conjecture 1.3. Briefly, the argument is that Point 2 should follow from Point 1, modulo technical issues having to do with the regularity of the log potential. A separate argument for Point 2 of the conjecture was provided in Section 3; see Idea 3.3.

Step 1: We expect the PDE (1.3) has a unique solution for small time; that is, the logarithmic potential $S$ of the limiting root distribution in Conjecture 1.3 is the unique solution of the PDE (1.3) for small $|\tau|$. The PDE (1.3) is of Hamilton–Jacobi type and (sufficiently regular) solutions can be analyzed using the method of characteristics. The global Lipschitz assumption on the Cauchy transform $m_0$ of $\mu_0$ will guarantee that, for small enough $\tau$, each point in the plane can be reached by a unique characteristic curve, giving uniqueness.

Step 2: The density $W$ of the limiting measure $\mu_\tau$ can be calculated from the logarithmic potential $S$ by

$$W = \frac{1}{\pi} \frac{\partial^2 S}{\partial z \partial \bar{z}}.$$  

If $S$ is smooth enough, we may apply $\frac{1}{\pi} \frac{\partial}{\partial z}$ and then $\frac{\partial}{\partial \bar{z}}$ on both sides of the PDE (1.3), giving

$$\frac{\partial}{\partial \tau} \left( \frac{1}{\pi} \frac{\partial^2 S}{\partial z \partial \bar{z}} \right) = \frac{\partial}{\partial z} \left( \frac{\partial S}{\partial z} \right) \cdot \frac{1}{\pi} \left( \frac{\partial^2 S}{\partial \bar{z} \partial z} \right).$$

Hence, the density $W$ satisfies a (complex-time) continuity equation

$$\frac{\partial W}{\partial \tau} + \frac{\partial}{\partial z} \left[ -\frac{\partial S}{\partial z} W \right] = 0, \quad (4.5)$$

where we treat $S$ as a known quantity—the unique solution of the PDE (1.3).

We can use general results of continuity equations by converting the complex-time continuity equation into a real-time continuity equation. Equation (4.5) can be interpreted as saying that the measure $\mu_\tau = W \, dx \, dy$ flows along the vector field $-\partial S/\partial z$. (Here we interpret the complex number $-\partial S/\partial z$ as a vector in the plane.) As long as $\partial S/\partial z$ is Lipschitz in $z$, general results for continuity equations (e.g., [2]) will tell us that the measure $\mu_\tau$ is the push-forward of $\mu_0$ by the map obtained by flowing along $-\partial S/\partial z$.

Step 3: Step 2 in the argument says that $\mu_\tau$ is the push-forward of $\mu_0$ under the flow of $-\partial S/\partial z$. We now show that this map can be computed in a simple, explicit form as in Point 2 of Conjecture 1.3.

The Hamilton–Jacobi equation (1.3) can be analyzed using the Hamilton’s equations associated to the (complex-variable) Hamiltonian

$$H(z, p) = -\frac{1}{2} p^2, \quad (4.6)$$

where $p$ is the (complex-valued) momentum variable. The Hamiltonian is holomorphic in $z$ and $p$, and independent of $z$. While we are not aware of a general theory on complex-time Hamilton–Jacobi equations, we can do an analysis similar to [18, Section 5] by converting the equation into a real-time Hamilton–Jacobi equation. As we will see in Section 4.3, the result will be Hamilton–Jacobi formulas similar to the usual real-time formulas, as in the next paragraph.
The Hamilton’s equations associated to the Hamiltonian (4.6) are
\[
\frac{dz}{d\tau} = \frac{\partial H}{\partial p} = -p; \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial z} = 0,
\]
where we look for solutions that are holomorphic functions of \(\tau\). The initial condition \(z_0\) for \(z(\tau)\) can be chosen arbitrarily on \(\mathbb{C}\), but the initial condition \(p_0\) for \(p(\tau)\) is chosen to be
\[
p_0 = \frac{\partial}{\partial z_0} S(z_0, 0) = m_0(z_0).
\] (4.7)
The solution of the Hamilton’s equations are
\[
p(\tau) = p_0; \quad z(\tau) = z_0 - \tau p_0.
\] (4.8)
The solutions (4.8) are called the characteristic curves of the PDE (1.3). The first and second Hamilton–Jacobi formulas then hold:
\[
S(z(\tau), \tau) = S(z_0, 0) - \text{Re}[\tau p_0^2] \quad (4.9)
\]
\[
\frac{\partial S}{\partial z}(z(\tau), \tau) = p(\tau). \quad (4.10)
\]
These two formulas are derived in Section 4.3, Proposition 4.6.

If we combine the Hamilton’s equations and the second Hamilton–Jacobi formula, we have
\[
\frac{dz}{d\tau} = -p(\tau) = -\frac{\partial S}{\partial z}(z(\tau), \tau).
\]
The right-hand side of the above equation is exactly the vector field occurring in the continuity equation (4.5). Thus, the integral curves for the vector field \(-\partial S/\partial z\) are exactly the characteristic curves of the Hamilton–Jacobi equation (1.3). It is notable that the integral curves are holomorphic in \(\tau\).

The integral curves of \(-\partial S/\partial z\) are the characteristic curves \(z_0 - \tau p_0\), and \(p_0\) is \(m_0(z_0)\); the map \(T_\tau\) defined in Point 2 of Conjecture 1.3 is just the characteristic curves evaluated at \(\tau\). Recall that the general results of the continuity equation in Step 2 tells us that the density \(W\) of the limiting root distribution flows along the characteristic curves. This gives an argument for Point 2 of Conjecture 1.3.

\[4.1.3. \text{Details of Point 2 of Conjecture 1.3}\]

In this section, we fill in the missing details in Section 4.1.2. We also note certain places where the argument is not rigorous, for reasons having to do with the regularity of the log potential \(S\).

\textbf{Details of Step 1:} The PDE (1.3) is of Hamilton–Jacobi type with complex time and space variables. In Section 4.3, we transform the PDE into another Hamilton–Jacobi type PDE with a real time and two real space variables. Now, under quite general conditions (e.g., [9]), Hamilton–Jacobi equations have unique viscosity solutions. Furthermore, any \(C^1\) solution is automatically a viscosity solution, so there can be at most one \(C^1\) solution, giving the desired uniqueness result.

We also note that by Theorem 1.1 and Remark 1.1 in [30], one can construct a \(C^{1,1}\) solution to a Hamilton–Jacobi equation by the method of characteristics, \textit{provided} that the initial data are \(C^{1,1}\) and that the flow along the characteristic curves is a bi-Lipschitz homeomorphism. Thus, under these assumptions, any \(C^1\) solution must be given by the method of characteristics—and this \(C^1\) solution will actually be \(C^{1,1}\).

We now show that, in fact, every point in the plane is hit by a unique characteristic curve, provided that \(\tau\) is small enough. We first note that, in light of (4.7)
and (1.8), the characteristic curves can be computed in terms of the transport map as
\[ z(\tau) = T_\tau(z_0). \]
We then establish nice behavior for \( T_\tau \), for small \( \tau \).

**Lemma 4.2.** For all \( |\tau| < L_{m_0}^{-1} \) (where \( L_{m_0} \) is the Lipschitz constant of \( m_0 \)), \( T_\tau \) is a bi-Lipschitz homeomorphism of \( \mathbb{C} \) to \( \mathbb{C} \). Indeed, \( T_\tau \) is uniformly bi-Lipschitz for \( |\tau| \leq c \), for any \( c < L_{m_0} \).

**Proof.** Suppose that \( T_\tau(z_1) - T_\tau(z_2) = 0 \) for some \( z_1 \neq z_2 \). Then after an algebraic calculation, we get
\[
\frac{m_0(z_1) - m_0(z_2)}{z_1 - z_2} = -\frac{1}{\tau},
\]
which is impossible if \( |\tau| < L_{m_0}^{-1} \). This proves injectivity of \( T_\tau \). To prove surjectivity of \( T_\tau \), we note that \( T_\tau(z) \approx z \) for large \( |z| \). Thus, we can extend \( T_\tau \) to a continuous map on \( S^2 \) by defining \( T_\tau(\infty) = \infty \). Suppose now that, on the contrary, there exists \( z_0 \in \mathbb{C} \) such that the image \( T_\tau(S^2) \) does not contain \( z_0 \). Then this extended \( T_\tau \) is a continuous map from \( S^2 \) into \( S \setminus \{z_0\} \cong \mathbb{R}^2 \). By the Borsuk–Ulam theorem, \( T_\tau \) cannot be a injective map, contradicting the previously established injectivity of \( T_\tau \). (Alternatively, surjectivity may be proved using the fundamental group of the punctured plane, following a well-known proof of the fundamental theorem of algebra. See, for example, the proof of Theorem 56.1 in [34].)

Finally, we address the claimed bi-Lipschitz property of \( T_\tau \). We compute that
\[
\frac{T_\tau(z_1) - T_\tau(z_2)}{z_1 - z_2} = 1 + \tau \frac{m_0(z_1) - m_0(z_2)}{z_1 - z_2}.
\]
Thus, for any \( c < L_{m_0}^{-1} \),
\[
0 < 1 - cL_{m_0} \leq \left| \frac{T_\tau(z_1) - T_\tau(z_2)}{z_1 - z_2} \right| \leq 1 + cL_{m_0}
\]
if \( |\tau| \leq c \).

Before we move to the details of Step 2, we first define the time \( \tau_{\text{max}} \) where the PDE and the transport map \( T_\tau \) have nice behaviors.

**Notation 4.3.** Let \( \tau_{\text{max}} = \min(C, L_{m_0}^{-1}) \), where \( C \) is the constant in Conjecture L.3 and \( L_{m_0} \) is the Lipschitz constant of \( m_0 \), as in (4.1).

**Details of Step 2:** Step 2 of the argument in Section 4.1.2 concerns the deformation of the measure \( \mu_\tau \). Let \( W(z, \tau) \) be the density of \( \mu_\tau \). If \( S \) is \( C^3 \), then we can take \( \frac{1}{2} \partial^2 W/\partial z \partial \bar{z} \) on both sides of (1.3) to get the continuity equation. We expect that if the solution is only \( C^{1,1} \), the resulting continuity equation will still hold in the weak sense.

In the following result, which is proven in Section 4.3, we transform the complex-time continuity equation (4.5) into a (real-time) continuity equation for \( W \). Then in Step 3, we derive that \( \mu_\tau \) is the push-forward measure of \( \mu_0 \) by the map \( T_\tau \).

**Proposition 4.4.** Suppose that \( W \) satisfies the complex-time continuity equation (4.5). Let
\[
W^\tau(z, t) = W(z, t\tau).
\]
Then $W^\tau$ satisfies the real-time continuity equation
\[
\frac{\partial W^\tau}{\partial t} + \nabla \cdot (bW^\tau) = 0
\] (4.11)
where $b$ is the vector field
\[
b(z, \tau) = b(z, \tau) = -\left(\frac{\tau_1}{\tau_2}, \frac{\tau_2}{\tau_1}\right) = -\text{vect}\left(\tau \frac{\partial S}{\partial z}(z, \tau)\right)
\] (4.12)
where the operator vect takes a complex number $x + iy$ to the vector $(x, y)$ in $\mathbb{R}^2$.

By general results of the continuity equation (see, for example, Proposition 3.1 and Theorem 4.1 of [2]), if the initial condition of the continuity equation is the density of a measure, then the solution of a continuity equation is given by the push-forward of the measure by the flow of the vector field. We will show in details of Step 3 that the ODE describing the flow along the vector field (4.12) has a unique solution for every initial point on $C$. Theorem 4.1 of [2] then guarantees us the solution of the continuity equation is unique.

**Details of Step 3:** We consider the real-time continuity equation in (4.11) and we consider the integral curves of the associated vector field $-\text{vect}(\tau \partial S/\partial z)$. That is, we look for solutions $w^\tau(t)$ to
\[
\frac{dw^\tau(t)}{dt} = -\tau \frac{\partial S}{\partial z}(w^\tau(t), t) = -\tau \frac{\partial S}{\partial z}(z^\tau(t), t).
\] (4.13)
Now, assuming that the Hamilton–Jacobi formulas are applicable to the (real-variables version of) the equation for $S$, Proposition 4.7 will tell us that the right-hand side of (4.13) will be Lipschitz. Thus, by Picard–Lindelöf theorem, the ODE has a unique solutions. Then by Theorems 3.1 and 4.1 of [2], the continuity equation (4.11) has a unique solution, given by push-forward along the curves in (4.13).

We then solve (4.13) by a real-variables version of the argument we gave for Step 3 in Section 4.1.2. Specifically, we consider the function $z^\tau(t) = z_0 - t\tau p_0$ and $p^\tau(t) = p_0$ satisfy
\[
\frac{dz^\tau}{dt} = -\tau p^\tau(t) = -\tau \frac{\partial S^\tau}{\partial z}(z^\tau(t), t) = -\tau \frac{\partial S}{\partial z}(z^\tau(t), t).
\]
Thus, the curves $t \mapsto z^\tau(t)$ are the unique solutions of (4.13) with the initial condition $z_0$. Taking $t = 1$, we conclude that the measure $W(z, \tau)\, dx\, dy$ is the push-forward of $W(z, 0)\, dx\, dy$ under the map $z \mapsto z - \tau p_0 = z - \tau m_0(z)$.

### 4.2. PDE perspective: multiplicative case.
In the multiplicative case, we again consider a sequence of polynomial $P^N$ with distinct roots, whose empirical root distribution converges to a compactly supported probability measure $\nu_0$ on $\mathbb{C}$ whose Cauchy transform is globally Lipschitz.

In this section, we propose the conjecture of the heat evolution of $P^N$ using a PDE perspective. We call this the multiplicative case because the PDE is the same PDE as the one derived from multiplicative Brownian motion. The argument why we believe the conjecture holds is very similar to the additive case in Section 4.1.

**Conjecture 4.5.** Consider the new polynomial defined by
\[
Q^N(z, \tau) = \exp\left\{-\frac{\tau}{2N} \left(\frac{\partial^2}{\partial z^2} - (N - 2)z \frac{\partial}{\partial z} - N\right)\right\} P^N(z).
\]
Let $\nu^N_\tau$ be the empirical root distribution of $P^N(\cdot, \tau)$. Then the following holds.

1. There exists a constant $C > 0$ such that $\nu^N_\tau$ converges weakly to a probability measure $\nu_\tau$ on $\mathbb{C}$ whose logarithmic potential $S(z, \tau) = \int \log |z - w|^2 d\nu_\tau(w)$ is a $C^1$-solution of the PDE

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2} \left( z^2 \left( \frac{\partial S}{\partial z} \right)^2 - z \frac{\partial S}{\partial z} \right).$$

for $|\tau| \leq C$.

2. There exists $0 < \tau_{\text{max}} < C$ such that the probability measure $\nu_\tau$ on $\mathbb{C}$ is the push-forward of $\nu_0$ by

$$T^\text{mult}_\tau(z) = z \exp \left( \tau m_0(z) - \frac{1}{2} \right)$$

whenever $|\tau| \leq \tau_{\text{max}}$, where $m_0$ denotes the Cauchy transform of $\nu_0$.

4.3. Auxiliary calculations for the PDE (1.3). We will use the ordinary Hamilton–Jacobi method to prove that the solution of the PDE (1.3) satisfies the Hamilton–Jacobi formulas (4.9) and (4.10). The proof requires the solution $S$ to be $C^2$, although we expect that the result will continue to hold if the solution is only $C^1$.

Proposition 4.6. Suppose $S$ is a $C^2$ solution of (1.3) for $|\tau| \leq C$. Then the values of $S$ and $\partial S/\partial z$ can be calculated using the characteristic curves (4.8) as in (4.9) and (4.10).

Proof. We first turn the PDE (1.3) into a PDE with real time, and apply the first Hamilton–Jacobi formulas. (See, for example, Proposition 5.3 of [13].) For each $|\tau| \leq C$, define

$$S^\tau(z, t) = S(z, t\tau).$$

The $z$-derivative of $S^\tau$ and $S$ coincides with the same $(z, \tau)$, so that

$$\frac{\partial S^\tau}{\partial t} = \frac{\partial S}{\partial \tau} \frac{\partial \tau}{\partial t} + \frac{\partial S}{\partial \tau} \frac{\partial \tau}{\partial t} + \frac{\partial S^\tau}{\partial \tau} \frac{\partial \tau}{\partial t}$$

$$= 2\text{Re} \left[ \frac{\tau}{2} \left( \frac{\partial S^\tau}{\partial z} \right)^2 \right]$$

$$= \frac{\text{Re} \tau}{4} \left[ \left( \frac{\partial S^\tau}{\partial x} \right)^2 - \left( \frac{\partial S^\tau}{\partial y} \right)^2 \right] + \frac{\text{Im} \tau}{2} \frac{\partial S^\tau}{\partial x} \frac{\partial S^\tau}{\partial y},$$

where we have written $z$ into real variables by $z = x + iy$. Thus, $S^\tau(z, t)$ satisfies a Hamilton–Jacobi PDE, with Hamiltonian

$$H^\tau(x, y, p_x, p_y) = -\frac{1}{4} \text{Re}[\tau (p_x - ip_y)^2].$$

The Hamilton’s equations for this Hamiltonian are given as

$$\dot{x} = \frac{\partial H^\tau}{\partial p_x}; \quad \dot{y} = \frac{\partial H^\tau}{\partial p_y}; \quad \dot{p}_x = -\frac{\partial H^\tau}{\partial x}; \quad \dot{p}_y = -\frac{\partial H^\tau}{\partial y}.$$ (4.15)

Once the initial condition $z_0 = x_0 + iy_0$ is chosen, the initial conditions $p_{x,0}$ and $p_{y,0}$ of the Hamilton’s equations are taken to be

$$p_{x,0} = \frac{\partial S^\tau}{\partial x}(z_0, 0); \quad p_{y,0} = \frac{\partial S^\tau}{\partial y}(z_0, 0).$$
We then easily obtain
\[ \begin{align*}
p_x(t) &= p_{x,0}; \quad p_y(t) = p_{y,0} \\
x(t) &= x_0 - \frac{t}{2} \text{Re}[\tau(p_{x,0} - ip_{y,0})] \\
y(t) &= y_0 - \frac{t}{2} \text{Im}[\tau(p_{x,0} - ip_{y,0})].
\end{align*} \]

We then rewrite our results using the complex-valued functions \( z^\tau \) and \( p^\tau \) given by
\[ \begin{align*}
z^\tau(t) &= x(t) + iy(t) = z_0 - t\tau p_0 \\
p^\tau(t) &= \frac{1}{2}(p_x(t) - ip_y(t)).
\end{align*} \]

We can now apply Eqs. (5.11) and (5.12) of [13] to get the formulas
\[ \begin{align*}
S^\tau(z^\tau(t), t) &= S^\tau(z_0, 0) - t\text{Re}[\tau p_0^2] \\
\frac{\partial S^\tau}{\partial z^\tau}(z^\tau(t), t) &= p^\tau(t).
\end{align*} \]

Evaluating (4.16) and (4.17) at \( t = 1 \) gives the claimed formulas. \( \square \)

**Proposition 4.7.** Assume that the conclusion of Proposition 4.6 holds; that is, the characteristic curves (4.8) describe the values of \( S \) and \( \frac{\partial S}{\partial z} \) by the formulas (4.9) and (4.10). Then given any \( c < L_{m_0}^{-1} \), for any \( |\tau| \leq c \), \( \frac{\partial S}{\partial z}(z, \tau) \) is uniformly Lipschitz.

**Proof.** By the second Hamilton–Jacobi formula (4.10), the global Lipschitz assumption on \( m_0 \), and Lemma 4.2, we have
\[ \frac{\partial S}{\partial z}(z_1, \tau) - \frac{\partial S}{\partial z}(z_1, \tau) = |m_0(T_{-1}^{-1}(z_1)) - m_0(T_{-1}^{-1}(z_2))| \]
\[ \leq L_{m_0}|T_{-1}^{-1}(z_1) - T_{-1}^{-1}(z_2)| \]
\[ \leq L_{m_0}(1 + cL_{m_0})|z_1 - z_2| \]
for all \( |\tau| \leq c < L_{m_0}^{-1} \). \( \square \)

We now supply the proof of Proposition 4.4.

**Proof of Proposition 4.4.** Using the chain rule, we compute
\[ \frac{\partial W^\tau}{\partial t} = 2 \text{Re} \left( \tau \frac{\partial W}{\partial \tau} \right) = \text{Re} \left[ 2 \frac{\partial}{\partial z} \left( \tau \frac{\partial S}{\partial z} W^\tau \right) \right]. \]

Expanding the right-hand side, we get
\[ \frac{\partial W^\tau}{\partial t} = \frac{\partial}{\partial x} \left[ \left( \frac{\tau_1}{2} \frac{\partial S}{\partial x} + \frac{\tau_2}{2} \frac{\partial S}{\partial y} \right) W^\tau \right] + \frac{\partial}{\partial y} \left[ \left( \frac{\tau_2}{2} \frac{\partial S}{\partial x} - \frac{\tau_1}{2} \frac{\partial S}{\partial y} \right) W^\tau \right]. \]

This means we can write the above equation into the standard form of a continuity equation
\[ \frac{\partial W^\tau}{\partial t} + \nabla \cdot (bW^\tau) \]
where \( b \) is the vector field in the statement of the proposition.

\[ \square \]

5. Evolution of the holomorphic moments

If \( \mu \) is a compactly supported probability measure on the plane, we let \( M_a \) be the \( a \)-th holomorphic moment of \( \mu \), given as

\[
M_a = \int_{\mathbb{C}} z^a \, d\mu(z), \quad a = 0, 1, 2, \ldots,
\]

so that \( M_0 = 1 \). We emphasize that these moments do not uniquely determine the measure. (Every rotationally invariant choice of \( \mu \), for example, gives \( M_a = 0 \) for all \( a \geq 1 \).) Nevertheless, the holomorphic moments contain a lot of information about the measure.

5.1. The additive case. Now let \( p_0^N \) be a polynomial of degree \( N \) and let \( p_\tau^N \) be the heat evolution of \( p_0^N \), as in (1.1). We then let \( M_{a,N}(\tau) \) be the \( a \)-th holomorphic moment of the empirical root measure of \( p_\tau^N \), that is,

\[
M_{a,N}(\tau) = \frac{1}{N} \sum_{j=1}^N (z_{j,N}(\tau))^a,
\]

where \( \{z_{j,N}(\tau)\}_{j=1}^N \) are the roots of \( p_\tau^N \).

The following theorem gives mechanism for computing these moments inductively as a function of \( a = 0, 1, \ldots \), allows us to compute the large-\( N \) limit of the moments, and shows that the large-\( N \) moments are consistent with the prediction of the PDE in Point 1 in Conjecture 1.3. We may therefore say that Point 1 of Conjecture 1.3 holds “at the level of the holomorphic moments.”

**Theorem 5.1.** We have the following results.

1. For any polynomial \( p_0^N \) of degree \( N \), the moments \( M_{a,N}(\tau) \) in (5.1) satisfy

\[
\frac{dM_{0,N}}{d\tau} = \frac{dM_{1,N}}{d\tau} = 0
\]

and for \( a \geq 2 \),

\[
\frac{dM_{a,N}}{d\tau} = \frac{1}{2N} a(a-1)M_{a-2,N}(\tau) - \frac{a}{2} \sum_{b=0}^{a-2} M_{b,N}(\tau)M_{a-b-2,N}(\tau).
\]

2. Suppose now that \( p_0^N \) is a sequence of polynomials of degree \( N \) such that the limit

\[
M_a(0) := \lim_{N \to \infty} M_{a,N}(0)
\]

exists for all \( a \). Then the limit

\[
M_a(\tau) := \lim_{N \to \infty} M_{a,N}(\tau)
\]

exists for all \( \tau \) and these limiting moments satisfy the limiting version of (5.3), namely

\[
\frac{dM_a}{d\tau} = -\frac{a}{2} \sum_{b=0}^{a-2} M_b(\tau)M_{a-b-2}(\tau).
\]
Suppose \( \mu_\tau \) is a family of compactly supported probability measures whose log potential \( S(z, \tau) \) satisfies the PDE \((1.3)\) for \( z \) near infinity and \( \tau \) in some open set \( U \subset \mathbb{C} \). Let \( \hat{M}_a(\tau) \) be the \( a \)-th holomorphic moment of \( \mu_\tau \). Then these moments also satisfy \((5.4)\):

\[
\frac{d\hat{M}_a}{d\tau} = -\frac{a}{2} \sum_{k=0}^{a-2} \hat{M}_b(\tau) \hat{M}_{a-b-2}(\tau) \tag{5.5}
\]

for \( \tau \in U \).

Although we will establish \((5.3)\) using the PDE methods from Section 4, it is also verify the result using the ODE system \((3.2)\) in Section 3.1.

**Proof.** By \((4.2)\), the Cauchy transform \( m^N(z, \tau) = \frac{\partial S}{\partial \tau}(z, \tau) \) of the empirical root distribution of \( p^N_\tau \) satisfies the PDE

\[
\frac{\partial m^N}{\partial \tau} = \frac{\partial}{\partial z} \left[ \frac{1}{2}(m^N)^2 + \frac{1}{2N} \frac{\partial m^N}{\partial \tau} \right]. \tag{5.6}
\]

Meanwhile, the Cauchy transform \( m \) of any compactly supported probability measure \( \mu \) can be computed near infinity as

\[
m(z) := \int_C \frac{1}{z-w} \, d\mu(w) = \frac{1}{z} \int_C \left( 1 + \frac{w^2}{z} + \cdots \right) \, d\mu(w) = \frac{1}{z} \left( 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \right),
\]

where \( \{a_j\}_{j=0}^\infty \) are the holomorphic moments of \( \mu \).

We then have

\[
\frac{\partial}{\partial \tau} m^N(z, \tau) = \frac{1}{z} \left( \frac{a^1_{1,N}(\tau)}{z} + \frac{a^2_{2,N}(\tau)}{z^2} + \frac{a^3_{3,N}(\tau)}{z^3} + \cdots \right) \tag{5.7}
\]

and, after a short calculation,

\[
\frac{\partial}{\partial z} \left[ \frac{1}{2}(m^N)^2 + \frac{1}{2N} \frac{\partial m^N}{\partial \tau} \right] = \sum_{j=2}^\infty \frac{1}{z^{j+1}} \left( \frac{1}{2N} j(j-1)a_{j-2,N}(\tau) - \frac{j}{2} \sum_{k=0}^{j-2} a_{j,N}(\tau)a_{j-k,N}(\tau) \right). \tag{5.8}
\]

Equating coefficients between \((5.7)\) and \((5.8)\) gives the \( dM_1/N/d\tau = 0 \) (for the coefficient of \( 1/z^2 \)) and gives \((5.3)\) (for the coefficient of \( 1/z^{j+1} \), \( j \geq 2 \)).

Meanwhile, we can solve for \( M_{a,N}(\tau) \) inductively in \( a \), simply by integrating \((5.2)\) or \((5.3)\) with respect to \( \tau \), with a fixed value of \( M_{a,N}(0) \). The resulting expression will depend continuously on the parameter \( C = 1/N \), up to \( C = 0 \). If we integrate, let \( C \to 0 \), and then differentiate in \( \tau \), we obtain \((5.4)\).

Finally, if the log potential of \( \mu_\tau \) satisfies \((1.3)\) near infinity, then the Cauchy transform of \( \mu_\tau \) satisfies the PDE \((5.6)\), but without the \( 1/N \) term on the right-hand side. We may therefore repeat the above calculation without the \( 1/N \) term to get \((5.5)\). \( \Box \)
5.2. A “counterexample”. Example 2.6 describes what we expect to happen if we evolve the characteristic polynomial of a GUE matrix by the forward heat flow \( \exp \left\{ t \frac{d^2}{dz^2} \right\} \). The roots start out at \( t = 0 \) with an asymptotically semicircular distribution on \([-2, 2]\). Then for \( 0 < t < 2 \), the roots should be asymptotically uniform on an ellipse with semi-axes \( 2 - t \) and \( t \).

We might then expect that if \( p_N^0 \) is any sequence of polynomials of degree \( N \) having an asymptotically semicircular distribution of roots on \([-2, 2]\), the heat-evolved polynomials would have the same uniform limiting root distribution. But this claim is false. A counterexample is provided by the (scaled) Hermite polynomials

\[
H_N(z) = \exp \left\{ -\frac{1}{2N} \frac{d^2}{dz^2} \right\} z^N.
\]

These polynomials are known to have an asymptotically semicircular distribution of roots on \([-2, 2]\). But if we apply the forward heat equation, we get

\[
\exp \left\{ -\frac{1}{2N} \frac{d^2}{dz^2} \right\} H_N(z) = \exp \left\{ -\frac{(1-t)}{2N} \frac{d^2}{dz^2} \right\} z^N.
\]

For \( 0 < t < 1 \), the above polynomials are just rescaled Hermite polynomials and have an asymptotically semicircular distribution on \([-2(1-t), 2(1-t)]\). (The difference in behavior of the two types of polynomials is attributed to the extremely evenly spaced nature of the roots of the Hermite polynomial.)

Of course, the result in the preceding paragraph does not actually contradict the conjecture in Example 2.6, which is only about the characteristic polynomial of GUE matrices. Nor does it contradict Conjecture 1.3, since the Cauchy transform of the semicircular law is discontinuous on the interval \([-2, 2]\).

On the other hand, the discrepancy between the behavior of the Hermite polynomials and the (conjural) behavior of the characteristic polynomial of a GUE must be consistent with Theorem 5.1, which assumes only that the initial polynomials have well-defined limiting moments. (This assumption is certainly satisfied for both the Hermite polynomials and the characteristic polynomial of a GUE.) There is no contradiction, however: the holomorphic moments of the uniform measure on an ellipse with semi-axes \( 2 - t \) and \( t \) are the same as the moments of the semicircular distribution on \([-2(1-t), 2(1-t)]\). See [20, Proposition 6.3]. In particular, both sets of moments tend to zero as \( t \) tends to 1.

5.3. The multiplicative case. We now briefly indicate how the results of Section 5.1 need to be modified in the multiplicative case.

**Theorem 5.2.** The holomorphic moments in the multiplicative case satisfy Theorem 5.1 with the following changes. First, (5.2) is replaced by

\[
\frac{dM_{0,N}}{d\tau} = 0; \quad \frac{dM_{1,N}}{d\tau} = \frac{1}{2} M_{1,N}.
\]

(5.9)

Second, (5.3) is replaced by

\[
\frac{dM_{a,N}}{d\tau} = \frac{a}{2} M_{a,N} + \frac{a}{2} \sum_{b=1}^{a-1} M_{b,N} M_{a-b,N} - \frac{a}{2N} M_{a,N}.
\]

(5.10)

Third, (5.4) and (5.5) are replaced by the formal large-\( N \) limit of (5.10) (dropping the \( 1/N \) term). Last, in Point 3 of the theorem, the PDE (1.3) is replaced by (4.14).
Note that the right-hand side of (5.10) involves the moment $M_{a,N}$. But we can still easily integrate (5.10) to obtain $M_{a,N}$ by using an exponential integrating factor. The proof of Theorem 5.2 is very similar to the proof of Theorem 5.1 and is omitted.

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