LINEAR AND NONLINEAR CONVOLUTION OPERATOR EQUATIONS ON THE INFINITE STRIP

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Abstract. In the present paper we characterize the existence and uniqueness of maximal $L^p$-regular solutions of high order convolution operator equations. Particularly, we get coercive uniform estimates with respect to spectral parameter and we show that corresponding realization operator is $R$-positive and generates analytic semigroup in $L^p$. Then we apply these results to various problems of nonlinear integro-differential equations.

1. Introduction

The aim of this study is to obtain global existence results for wide class of convolution operator equations (COE)

$$\frac{\partial u}{\partial t} + \sum_{k=0}^{l} \left( a_k \ast \frac{\partial^k u}{\partial x^k} + b_k \frac{\partial^k u}{\partial x^k} \right) + \mu \ast Au + \nu Au = F \left( u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^i u}{\partial x^i} \right),$$

(1.1)

and

$$-\frac{\partial^2 u}{\partial t^2} + \sum_{k=0}^{l} \left( a_k \ast \frac{\partial^k u}{\partial x^k} + b_k \frac{\partial^k u}{\partial x^k} \right) + \mu \ast Au + \nu Au = F \left( u, \frac{\partial u}{\partial t} \right),$$

(1.2)

$$\alpha_1 u(0, x) + \beta_1 \frac{\partial u}{\partial t}(0, x) = f_1(x), \quad \alpha_2 u(T, x) + \beta_2 \frac{\partial u}{\partial t}(T, x) = f_2(x),$$

where $A$ is a linear operator in a UMD space $E$, $i < l$, $b_k, \nu \in \mathbb{C}$ and $a_k = a_k(x)$, $\mu = \mu(x) \in S'(R, \mathbb{C})$. Equations of above type arise, for instance in the study of heat flow in materials of fading memory type as well as some equations of population dynamics. For detailed information on this subject see [1-5].

First we investigate corresponding linear convolution equation with spectral parameter i.e.

$$(L + \lambda) u = \sum_{k=0}^{l} \left( a_k \ast \frac{\partial^k u}{\partial x^k} + b_k \frac{\partial^k u}{\partial x^k} \right) + \mu \ast Au + \nu Au + \lambda u = f,$$

(1.3)

in $L^p(R; E)$. Particularly, we get coercive uniform $L^p$-estimates with respect to $\lambda$, which in its turn implies $R$-positivity of $L$. The main tool we implement here is an operator-valued Fourier multiplier theorem (FMT) in $L^p(R; E)$. The exposition of FMT, their applications and some related references can be found in [6-9]. For the references concerning FMT in periodic function spaces, optimal regularity

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results for convolution operator equations (COE) and delay differential operator
equations see e.g. [10-14] and reference therein.
In the last section we utilize abstract global and local existence results in [15-16]
to handle nonlinear problems (1.1) and (1.2).
Let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \), where \( \alpha_i \) are integers. An \( E \)-valued generalized function \( D^\alpha f \) is called a generalized derivative in the sense of Schwartz distributions, if the equality
\[
< D^\alpha f, \varphi > = (-1)^{|\alpha|} < f, D^\alpha \varphi >
\]
holds for all \( \varphi \in S \). We indicate mixed derivative as follows
\[
D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}, \quad D_k^i = \left( \frac{\partial}{\partial x_k} \right)^i.
\]
The Fourier transform \( F : S(X) \to S(X) \) defined by
\[
(F f)(t) \equiv \hat{f}(t) = \int_{\mathbb{R}^N} \exp(-its) f(s) ds
\]
is an isomorphism whose inverse is given by
\[
(F^{-1} f)(t) \equiv \hat{f}(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(its) f(s) ds,
\]
where \( f \in S(X) \) and \( t \in \mathbb{R}^N \). It is clear that
\[
F(D^\alpha_x f) = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n} \hat{f}, \quad D^\alpha_x (F f) = F((-ix_1)^{\alpha_1} \cdots (-ix_n)^{\alpha_n} f)
\]
for all \( f \in S^i(R^n; E) \).
Suppose \( \Omega \subset \mathbb{R} \) and \( E_1 \to E_0 \), where \( \to \) denotes continuous and dense injection. \( W^1_p(\Omega; E_1, E_0) \) is a space of all functions \( u \in L_p(\Omega; E_0) \) such that \( u^{(k)} \in L_p(\Omega; E) \), \( k = 0, \cdots, l \) and
\[
\|u\|_{W^1_p(\Omega; E_1, E_0)} = \|u\|_{L_p(\Omega; E_1)} + \sum_{k=0}^l \|u^{(k)}\|_{L_p(\Omega; E_0)} < \infty.
\]
For \( E_0 = E_1 \) the space \( W^1_p(\Omega; E_1, E_0) \) will be denoted by \( W^1_p(\Omega; E_0) \).
Suppose \( E_1 \) and \( E_2 \) are two Banach spaces. \( B(E_1, E_2) \) will denote the space of all bounded linear operators from \( E_1 \) to \( E_2 \).
A linear operator \( A \) is said to be \( \varphi \)-positive in a Banach space \( E \), with bound \( M \) if \( D(A) \) is dense in \( E \) and
\[
\|(A + \lambda I)^{-1}\|_{B(E)} \leq M (1 + |\lambda|)^{-1}
\]
for all \( \lambda \in S_\varphi \), with \( \varphi \in [0, \pi) \), where \( M \) is a positive constant and \( I \) is identity operator in \( E \).
\( E(A^\theta) \) denotes the space \( D(A^\theta) \) with graphical norm
\[
\|u\|_{E(A^\theta)} = \left( \|u\|_E^p + \|A^\theta u\|_E^p \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \quad -\infty < \theta < \infty.
\]
2. Basic notations and parameter depended FMT

In the next section, we will study COE with spectral parameter via FMT in \(L_p\) spaces. Since we will deal with family of uniformly bounded functions, we need to formulate slightly extended version of FMT due to Strkalj and Weis [10]. Let us first introduce some basic definitions and facts.

**Definition 2.0.** A Banach space \(X\) is called UMD space if \(X\)-valued martingale difference sequences are unconditional in \(L_p\) \((R^d; X)\) for \(p \in (1, \infty)\). i.e. there exists a positive constant \(C_p\) such that for any martingale \(\{\varepsilon_k, k \in N\} \subset \{-1, 1\}\) and \(N \in N\)
\[
\|f_0 + \sum_{k=1}^{N} \varepsilon_k (f_k - f_{k-1})\|_{L_p(\Omega, \Sigma, \mu, X)} \leq C_p \|f_N\|_{L_p(\Omega, \Sigma, \mu, X)}.
\]

It is shown in [17] and [18] that a Hilbert operator
\[
(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{|x-y|} dy
\]
is bounded in the space \(L_p(R, X), p \in (1, \infty)\) for only those spaces \(X\), which possess the UMD property. UMD spaces include e.g. \(L_p\), \(l_p\) spaces and Lorentz spaces \(L_{pq}\), \(p, q \in (1, \infty)\).

**Definition 2.1.** Let \(X\) and \(Y\) be Banach spaces. A family of operators \(\tau \subset B(X, Y)\) is called \(R\)-bounded (see e.g. [6]) if there is a positive constant \(C\) and \(p \in [1, \infty)\) such that for each \(N \in N, T_j \in \tau, x_j \in X\) and for all independent, symmetric, \(\{-1, 1\}\)-valued random variables \(r_j\) on a probability space \((\Omega, \Sigma, \mu)\) the inequality

\[
\left\| \sum_{j=1}^{N} r_j T_j x_j \right\|_{L_p(\Omega, Y)} \leq C \left\| \sum_{j=1}^{N} r_j x_j \right\|_{L_p(\Omega, X)},
\]
is valid. The smallest such \(C\) is called \(R\)-bound of \(\tau\), we denote it by \(R_p(\tau)\).

The basic properties of \(R\)-boundedness are collected in the recent monograph of Denk et al. [6]. For the reader’s convenience, we present some results from [6].

(a) The definition of \(R\)-boundedness is independent of \(p \in [1, \infty)\).
(b) If \(\tau \subset B(X, Y)\) is \(R\)-bounded then it is uniformly bounded with

\[\sup \{\|T\| : T \in \tau\} \leq R_p(\tau).\]

(c) If \(X\) and \(Y\) are Hilbert spaces, \(\tau \subset B(X, Y)\) is \(R\)-bounded \(\iff\) \(\tau\) is uniformly bounded.

(d) Let \(X, Y\) be Banach spaces and \(\tau_1, \tau_2 \subset B(X, Y)\) be \(R\)-bounded. Then
\[
\tau_1 + \tau_2 = \{T + S : T \in \tau_1, S \in \tau_2\}
\]
is \(R\)-bounded as well, and \(R_p(\tau_1 + \tau_2) \leq R_p(\tau_1) + R_p(\tau_2)\).

(e) Let \(X, Y, Z\) be Banach spaces and \(\tau_1 \subset B(X, Y)\) and \(\tau_2 \subset B(Y, Z)\) be \(R\)-bounded. Then
\[
\tau_1 \tau_2 = \{ST : T \in \tau_1, S \in \tau_2\}
\]
is \(R\)-bounded as well, and \(R_p(\tau_1 \tau_2) \leq R_p(\tau_1)R_p(\tau_2)\).
One of the most important tools in $R$-boundedness is the contraction principle of Kahane. We shall frequently apply it in the next sections.

[6, Lemma 3.5.] Let $X$ be a Banach spaces, $n \in N$, $x_j \in X$, $r_j$ independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \Sigma, \mu)$ and $\alpha_j, \beta_j \in \mathbb{C}$ such that $|\alpha_j| \leq |\beta_j|$, for each $j = 1, \ldots, N$. Then

$$\left\| \sum_{j=1}^{N} \alpha_j r_j x_j \right\|_{L_p(\Omega, X)} \leq 2 \left\| \sum_{j=1}^{N} \beta_j r_j x_j \right\|_{L_p(\Omega, X)}.$$  

The constant $C$ can be omitted in case where $\alpha_j$ and $\beta_j$ are real.

**Definition 2.2.** A family of uniformly bounded functions $m_h : R^d \to \mathbb{C}$ is called a uniform collection of Fourier multipliers (UFM) if there exists a positive constant $C > 0$, independent of parameter $h \in Q$, such that

$$\left\| F^{-1} \left[ m_h \hat{f} \right] \right\|_{L_p(R^d; X)} \leq C \left\| f \right\|_{L_p(R^d, X)}$$

for all $f \in S(R^d; X)$.

The set of all $L_p(X)$-UFM will be denoted by $M_p(X)$ and the smallest constant $C$ satisfying (2.1) by $\|m_h\|_{M_p(X)}$.

**Theorem 2.3.** Let $X$ be a $UMD$ space. Then for any $p \in (1, \infty)$ there is a constant $C < \infty$ such that for all family of uniformly bounded functions $m_h : R^d \to \mathbb{C}$ whose distributional derivatives $D^\alpha m_h$ of order $\alpha \leq (1, \ldots, 1)$ are represented by functions, we have

$$\|m_h\|_{M_p(X)} \leq C \sup_h \sup \left\{ |\xi|^{[\alpha]} |D^\alpha m_h(\xi)|, \xi \in R^d \setminus \{0\}, \alpha \leq (1, \ldots, 1) \right\}.$$  

It is easy to see that Theorem 2.3 is a trivial consequence of [9, Proposition 2]. Since in [9, Proposition 2] author finds a constant $C$ independent of the multiplier functions, we can take supremum over parameters to obtain above theorem.

The following result is a parameter depended version of operator valued Miklin’s theorem [10, Theorem 4.4].

**Theorem 2.4.** Let $X$ and $Y$ be $UMD$ spaces and $1 < p \leq \infty$. If the family of operator-valued function $M_h : R^d \setminus \{0\} \to B(X,Y)$ has the property that their distributional derivatives $D^\alpha M_h$ of order $\alpha \leq (1, \ldots, 1)$ are represented by functions and

$$\sup_h R \left\{ |x|^{[\alpha]} D^\alpha M_h(x); x \in R^d \setminus \{0\}, (\alpha \leq 1, \ldots, 1) \right\} < \infty$$

holds, then $M_h$ is UFM.

Taking into consideration the estimate (4.4) of [10, Theorem 4.4] and by using the similar reasoning as in Theorem 2.3 we get the assertion of Theorem 2.4.

3. Maximal regularity for (1.3)

Now let us consider the high order integro-differential equation (1.3) in $L_p(R; E)$. Here we derive a sufficient condition which guarantee the maximal regularity of (1.3).
Definition 3.0. Let $E$ be a Banach space and $D(A)$ dense in $E$. A $\varphi$-positive operator $A$ is said to be $R$-positive if the following set
\[ \{ (1 + \xi)(A + \xi)^{-1} : \xi \in S_{\varphi} \} \]
is $R$-bounded.

In what follows $\hat{R}$ will denote the set of real numbers excluding zero i.e. $\hat{R} = R/\{0\}$.

Condition 3.1. Let $E$ be a Banach space and $A$ be a $R$-positive operator in $E$. Suppose the following are satisfied:

1. $b_k, \nu \in \mathbb{C}$, $a_k, \mu \in S'(R, \mathbb{C})$, $\hat{a}_k, \hat{\mu} \in C^1(\hat{R}, \mathbb{C})$ and
   \[ C_\mu = \inf_{\xi \in \hat{R}} |\hat{\mu}(\xi) + \nu| > 0 ; \]

2. There exist a constant $C_N$ such that
   \[ |N(\xi)| = \left| \sum_{k=0}^{l} (b_k + \hat{a}_k(\xi))(i\xi)^k \right| \geq C_N |\xi|^l ; \]

3. $\eta(\xi) = \frac{N(\xi)}{\hat{\mu}(\xi) + \nu} \in S_{\varphi_1}, \varphi_1 \in [0, \pi)$ and $\lambda \in S_{\varphi_2}, \varphi_2 \in [0, \pi)$ are so that $\varphi_1 + \varphi_2 < \pi$ and
   \[ \lambda + \eta(\xi) \in S_{\varphi}, \]

4. There are constants $C_1$ and $C_2$ so that
   \[ \left| \xi^m \frac{d^m}{d \xi^m} \hat{a}_k(\xi) \right| \leq C_1, \text{ for all } k = 0, 1, \cdots, l, \]
   \[ \left| \xi^m \frac{d^m}{d \xi^m} \hat{\mu}(\xi) \right| \leq C_2, \]
   for $m = 0, 1$ and $\xi \in \hat{R}$.

Taking into account Condition 3.1 and by using the Kahane’s contraction principle we shall estimate $R$-bounds of the following sets:
\[ \{ m_i(\xi, \lambda) : \xi \in \hat{R} \} \]
and
\[ \{ \xi \frac{d}{d \xi} m_i(\xi, \lambda) : \xi \in \hat{R} \}, \ i = 0, \cdots, 4, \]

where
\[ m_0(\xi, \lambda) = (\hat{\mu}(\xi) + \nu)^{-1} [A + \eta(\xi) + \lambda]^{-1}, \]
\[ m_1(\xi, \lambda) = \sum_{k=0}^{l} |\lambda|^{1-k} (i\xi)^k m_0(\xi, \lambda), \ m_2(\xi, \lambda) = A m_0(\xi, \lambda), \]
\[ m_3(\xi, \lambda) = \sum_{k=0}^{l} |\lambda|^{1-k} \hat{a}_k(\xi) (i\xi)^k m_0(\xi, \lambda) \] and $m_4(\xi, \lambda) = \hat{\mu}(\xi) A m_0(\xi, \lambda)$

The main theorem of this section is based on the following lemmas.
Lemma 3.2. Assume there are some constants $C_1$ and $C_2$ so that
\[ |\hat{a}_k(\xi)| \leq C_1 \text{ for all } k = 0, \ldots, l \]
and
\[ |\hat{\mu}(\xi)| \leq C_2 \]
for all $\xi \in \hat{R}$. If (1)-(3) of Condition 3.1 hold then \( \{m_i(\xi, \lambda) : \xi \in \hat{R}\}, i = 0, \ldots, 4 \) are $R$-bounded.

Proof. Since $\eta(\xi) \in S_{\varphi_1}$, $\varphi_1 \in [0, \pi)$ and $\lambda \in S_{\varphi_2}$ for $\varphi_2 \in [0, \pi)$, by [19, Lemma 2.3] there exist $K > 0$ independent of $\xi$ so that
\[ |1 + \eta(\xi) + \lambda|^{-1} \leq K (1 + |\eta(\xi)| + |\lambda|)^{-1}. \]
Therefore, for all $\xi \in R$ we have uniform estimate
\[
\frac{1}{|\hat{\mu}(\xi) + \nu|} \times \frac{1}{|1 + \eta(\xi)| + |\lambda|} \leq K \frac{1}{|\hat{\mu}(\xi) + \nu| + |\lambda| |\hat{\mu}(\xi) + \nu| + |N(\xi)|} \leq K \frac{1}{C_\mu + |\lambda| C_\mu + C N \|\xi\|}. \]
Now let us define families of operators
\[ \tau = \left\{ T_j = (1 + \eta(\xi_j) + \lambda) (A + \eta(\xi_j) + \lambda)^{-1} : \xi_j \in \hat{R} \right\}, \]
and
\[ \tau_i = \left\{ T_{j}^i = m_i(\xi_j, \lambda) : \xi_j \in \hat{R} \right\}, i = 0, \ldots, 4. \]
Taking into consideration $R$-positivity of $A$, applying the assumptions of Condition 3.1 and the Kahane’s contraction principle we get desired result:
\[
\left\| \sum_{j=1}^{N} r_j T_j^i x_j \right\|_X = \left\| \sum_{j=1}^{N} r_j \left( \frac{1}{(\hat{\mu}(\xi_j) + \nu)} (1 + \eta(\xi_j) + \lambda) T_j x_j \right) \right\|_X \leq \frac{2K}{C_\mu} \left\| \sum_{j=1}^{N} r_j T_j x_j \right\|_X \leq \frac{2K}{C_\mu} R_p(\tau) \left\| \sum_{j=1}^{N} r_j x_j \right\|_X, \]
where $X = L_\mu((0, 1), E)$. Hence
\[ R_p(\tau_0) \leq \frac{2K}{C_\mu} R_p(\tau). \]
It is clear that
\[ \sum_{k=0}^{l} |y|^k \leq l(1 + |y|^l) \]
for any $y \in C$. Thus making use of the above inequality we get
\[
\left\| \sum_{k=0}^{l} |\lambda|^{-\frac{k}{2}} (i\xi)^k \right\| \leq |\lambda| \left\| \sum_{k=0}^{l} \left( |\lambda|^{-\frac{k}{2}} |\xi|^k \right) \leq l \left( |\lambda| + |\xi|^l \right). \]
and
\[
\frac{1}{|\bar{\mu}(\xi) + \nu|} \left| \sum_{k=0}^{l} |\lambda|^{1-\frac{d}{d}} (\xi^{k}) \right| \leq Kl \frac{|\lambda| + |\xi|}{|\bar{\mu}(\xi) + \nu| + |\lambda| |\bar{\mu}(\xi) + \nu| + |N(\xi)|} 
\]
\[
\leq Kl \frac{|\lambda| + |\xi|}{C_{\mu} + |\lambda| C_{\mu} + C_{N} |\xi|} \leq KlM
\]
and
\[
\frac{1}{|\bar{\mu}(\xi) + \nu|} \left| \sum_{k=0}^{l} |\lambda|^{1-\frac{d}{d}} \hat{a}_{k}(\xi) (i\xi)^{k} \right| \leq KlMC_{1},
\]
where \(M^{-1} = \min \{C_{\mu}, C_{N}\}\). Again by the Kahane’s contraction principle we have
\[
\left\| \sum_{j=1}^{N} r_{j} T_{j}^{3} x_{j} \right\|_{X} = \left\| \sum_{j=1}^{N} \sum_{k=0}^{l} \frac{r_{j}}{\bar{\mu}(\xi) + \nu} (1 + \eta(\xi) + \lambda) T_{j} x_{j} \right\|_{X}
\]
\[
\leq 2KlM \left\| \sum_{j=1}^{N} r_{j} T_{j} x_{j} \right\|_{X} \leq 2KlMR_{p}(\tau) \left\| \sum_{j=1}^{N} r_{j} x_{j} \right\|_{X},
\]
and
\[
\left\| \sum_{j=1}^{N} r_{j} T_{j}^{3} x_{j} \right\|_{X} = \left\| \sum_{j=1}^{N} \sum_{k=0}^{l} \frac{r_{j}}{\bar{\mu}(\xi) + \nu} (1 + \eta(\xi) + \lambda) T_{j}^{3} x_{j} \right\|_{X}
\]
\[
\leq 2KlMC_{1}R_{p}(\tau) \left\| \sum_{j=1}^{N} r_{j} x_{j} \right\|_{X},
\]
which implies
\[
R_{p}(\tau_{1}) \leq 2KlMR_{p}(\tau)
\]
and
\[
R_{p}(\tau_{3}) \leq 2KlMC_{1}R_{p}(\tau).
\]
Finally, by virtue of the resolvent property we get
\[
R_{p}(\tau_{2}) \leq R_{p} \left\{ I \frac{1}{\bar{\mu}(\xi) + \nu} ; \xi_{j} \in \hat{R} \right\}
\]
\[
+ R_{p} \left\{ (\eta(\xi) + \lambda) (A + \eta(\xi) + \lambda)^{-1} ; \xi_{j} \in \hat{R} \right\}
\]
\[
\leq 2 \left( \frac{1}{C_{\mu}} + R_{p}(\tau) \right)
\]
and
\[
R_{p}(\tau_{4}) \leq 2C_{2} \left( \frac{1}{C_{\mu}} + R_{p}(\tau) \right)
\]
Next we estimate the derivatives of operator valued functions \(m_{i}(\xi)\).
Lemma 3.3. If the Condition 3.1 holds then \( \left\{ \xi \frac{d}{d \xi} m_1 (\xi, \lambda) : \xi \in \hat{R} \right\}, \ i = 0, \ldots, 4 \) are \( R \)-bounded.

**Proof.** For the sake of simplicity we only estimate \( \left\{ \xi \frac{d}{d \xi} m_1 (\xi, \lambda) : \xi \in \hat{R} \right\} \) and \( \left\{ \xi \frac{d}{d \xi} m_2 (\xi, \lambda) : \xi \in \hat{R} \right\} \). Computing the first derivative of \( m_1 \) we obtain

\[
R_p \left( \left\{ \xi \frac{d}{d \xi} m_1 (\xi, \lambda) : \xi \in \hat{R} \right\} \right) \leq \sum_{i=1}^{5} R_p (\tau_i),
\]

where

\[
\tau_1 = \left\{ \sum_{k=1}^{l} |\lambda|^{1-\frac{4}{\mu}} k (i \xi)^{k-1} m_0 (\xi, \lambda) : \xi \in \hat{R} \right\},
\]

\[
\tau_2 = \left\{ \frac{d \hat{u}}{d \xi} \frac{\hat{\mu}(\xi) + \nu}{m_1 (\xi, \lambda)} : \xi \in \hat{R} \right\},
\]

\[
\tau_3 = \left\{ m_1 (\xi, \lambda) \sum_{k=0}^{l} \frac{d \hat{u}}{d \xi} (i \xi)^{k} m_0 (\xi, \lambda) : \xi \in \hat{R} \right\},
\]

\[
\tau_4 = \left\{ m_1 (\xi, \lambda) \sum_{k=1}^{l} (b_k + \hat{a}_k (\xi)) k (i \xi)^{k-1} m_0 (\xi, \lambda) : \xi \in \hat{R} \right\}
\]

and

\[
\tau_5 = \left\{ m_1 (\xi, \lambda) \frac{N(\xi)}{\hat{\mu}(\xi) + \nu} m_0 (\xi, \lambda) : \xi \in \hat{R} \right\}.
\]

Making use of similar arguments as in Lemma 3.2 we get

\[
\frac{1}{|\hat{\mu}(\xi) + \nu|} \left| \sum_{k=1}^{l} |\lambda|^{1-\frac{4}{\mu}} k (i \xi)^{k-1} \right| \leq K l \left| \sum_{k=1}^{l} |\lambda|^{1-\frac{4}{\mu}} \right| \leq K l M_1,
\]

\[
\frac{1}{|\hat{\mu}(\xi) + \nu|} \left| \sum_{k=1}^{l} \frac{d \hat{u}}{d \xi} (i \xi)^{k} \right| \leq C_2 \frac{C_1}{\hat{\mu}},
\]

\[
\frac{1}{|\hat{\mu}(\xi) + \nu|} \left| \sum_{k=1}^{l} (b_k + \hat{a}_k (\xi)) k (i \xi)^{k-1} \right| \leq K l M_1 (C_b + C_1),
\]

and

\[
\frac{1}{|\hat{\mu}(\xi) + \nu|} \left| \sum_{k=1}^{l} \frac{N(\xi)}{\hat{\mu}(\xi) + \nu} \right| \leq K \frac{C_2}{\hat{\mu}} (C_b + C_1).
\]
Thus by the Kahane’s contraction principle we have
\[ R_p(\tau_1) \leq 2KlM_1R_p(\tau), \]
\[ R_p(\tau_2) \leq 2C_\mu KlMR_p(\tau), \]
\[ R_p(\tau_3) \leq 4C_\mu M (KlR_p(\tau))^2, \]
\[ R_p(\tau_4) \leq 4M M_1(C_b + C_1)(KlR_p(\tau))^2, \]
and
\[ R_p(\tau_5) \leq 4C_\mu^2 (C_b + C_1)lM (KR_p(\tau))^2. \]

Similarly, to show \( \left\{ \xi \frac{d}{d\xi} m_2(\xi, \lambda) : \xi \in R \setminus \{0\} \right\} \) is R-bounded, it suffices to estimate the following terms
\[
\left\{ \sum_{k=0}^{l} \frac{d}{d\xi}^k (i\xi)^k}{\hat{\mu}(\xi) + \nu}m_0(\xi, \lambda) : \xi \in \hat{R} \right\},
\]
\[
\left\{ \sum_{k=1}^{l} (b_k + \hat{a}_k(\xi)) k (i\xi)^{k-1}}{\hat{\mu}(\xi) + \nu}m_0(\xi, \lambda) : \xi \in \hat{R} \right\},
\]
and
\[
\left\{ \frac{N(\xi) \frac{d}{d\xi}^2}{(\hat{\mu}(\xi) + \nu)^2}m_0(\xi, \lambda) : \xi \in \hat{R} \right\}.
\]

By virtue of above techniques and Lemma 3.2 one can easily find R-bounds for these sets. The other cases can be proven analogously. Hence proof is completed.

**Corollary 3.4.** Let \( E \) be a UMD space and \( A \) be a R-positive operator in \( E \). If the Condition 3.1 holds and \( 1 < p < \infty \) then operator-valued functions \( m_i(\xi, \lambda) \) are UFM in \( L^p(R; E) \).

Now, making use of Corollary 3.4 we obtain our main result for this section.

**Theorem 3.5.** Let \( E \) be a UMD space and \( A \) be a R-positive operator in \( E \). If the Condition 3.1 is satisfied then for each \( f \in Y = L^p(R; E) \), \( (1.3) \) has a unique solution \( u \in W^{l}(R; E(A); E) \) and the coercive uniform estimate
\[
\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{p}} \left( \left\| a_k * \frac{d^k u}{dt^k} \right\|_Y + \left\| \frac{d^k u}{dt^k} \right\|_Y \right) + \| \mu * Au \|_Y + \| Au \|_Y \leq C \| f \|_Y \quad (3.1)
\]
holds.

**Proof.** Applying the Fourier transform to equation \( (1.3) \) we obtain
\[
(\hat{\mu}(\xi) + \nu) [A + \eta(\xi) + \lambda] \hat{u}(\xi) = \hat{f}(\xi).
\]
Due to positivity of $A$ and Condition 3.1, $A + \eta (\xi) + \lambda$ is invertible in $E$. Thus, solutions of the equation (1.3) can be represented in the form
\[
u (x) = F^{-1} \left( m_0 \hat{f} \right) \tag{3.2} \]
where
\[m_0(\xi, \lambda) = (\hat{\mu}(\xi) + \nu)^{-1} R(\xi, \lambda) \quad \text{and} \quad R(\xi, \lambda) = [A + \eta (\xi) + \lambda]^{-1}.
\]
Thus, by using (3.2) we obtain
\[
\sum_{k=0}^{l} |\lambda|^{-\frac{1}{k}} \left\| \frac{d^k u}{dt^k} \right\|_Y = \left\| F^{-1} \left[ m_1 (\xi, \lambda) \hat{f} \right] \right\|_Y,
\]
\[
\|Au\|_Y = \left\| F^{-1} \left[ m_2 (\xi, \lambda) \hat{f} \right] \right\|_Y,
\]
\[
\sum_{k=0}^{l} |\lambda|^{-\frac{1}{k}} \left\| a_k * \frac{d^k u}{dt^k} \right\|_Y = \left\| F^{-1} \left[ m_3 (\xi, \lambda) \hat{f} \right] \right\|_Y,
\]
and
\[
\|\mu * Au\|_Y = \left\| F^{-1} \left[ m_4 (\xi, \lambda) \hat{f} \right] \right\|_Y,
\]
where
\[m_1 (\xi, \lambda) = \sum_{k=0}^{l} |\lambda|^{-\frac{1}{k}} (i\xi)^k m_0(\xi, \lambda), \quad m_2 (\xi, \lambda) = Am_0(\xi, \lambda), \]
\[m_3 (\xi, \lambda) = \sum_{k=0}^{l} |\lambda|^{-\frac{1}{k}} \hat{a}_k(i\xi)^k m_0(\xi, \lambda) \quad \text{and} \quad m_4 (\xi, \lambda) = \hat{\mu}(\xi)Am_0(\xi, \lambda).
\]
By the Corollary 3.4 $m_i, \ i = 1, \cdots, 4$ are UFM in $L_p (R; E)$. Hence proof is completed.

Let $L$ be an operator generated by the problem (1.3) i. e.
\[
Lu = \sum_{k=0}^{l} a_k * \frac{d^k u}{dt^k} + \sum_{k=0}^{l} b_k \frac{d^k u}{dt^k} + \mu * Au + \nu Au,
\]
where
\[D (L) = W^l_p (R; E (A), E) = W^l_p (R; E) \cap L_p (R; E (A)).\]
The estimate (3.1) implies that $L$ is bijective and
\[
\|u\|_{W^l_p (R; E (A), E)} \leq C \|f\|_Y = C \|Lu\|_Y.
\]
Therefore $L$ has a continuous inverse
\[
L^{-1} : L_p (R, E) \rightarrow W^l_p (R; E (A), E). \tag{3.3}
\]
Under some restrictions on coefficients $a_k$ and $\mu$, it is also possible to show $L$ is continuous from $W^l_p (R; E (A), E)$ to $L_p (R, E)$. Really,
\[
\|Lu\|_Y \leq \sum_{k=0}^{l} \left\| a_k * \frac{d^k u}{dt^k} \right\|_Y + \sum_{k=0}^{l} \left\| b_k \frac{d^k u}{dt^k} \right\|_Y + \|\mu * Au\|_Y + \|\nu Au\|_Y.
\]
Since $\hat{a}_k$ and $\hat{\mu}$ are Fourier multipliers, we have
\[
\sum_{k=0}^{l} \left\| a_k \ast \frac{d^k u}{dt^k} \right\|_Y = \sum_{k=0}^{l} \left\| F^{-1} \left[ \hat{a}_k F \left( \frac{d^k u}{dt^k} \right) \right] \right\|_Y \leq C \sum_{k=0}^{l} \left\| \frac{d^k u}{dt^k} \right\|_Y
\]
and
\[
\| \mu \ast Au \|_Y \leq C \| Au \|_Y.
\]
Therefore
\[
\| Lu \|_Y \leq C \| u \|_{W^l_p(R; E(A), E)}.
\] (3.4)
Hence by (3.3) and (3.4) the operator $L$ is an isomorphism from $W^l_p(R; E(A), E)$ to $L_p(R, E)$ and
\[
\| Lu \|_{L_p(R, E)} \approx \| u \|_{W^l_p(R; E(A), E)}
\]
for all $u \in W^l_p(R; E(A), E)$.

Now let us write the above observation as a separate corollary.

**Result 3.6.** Let $E$ be a UMD space and $A$ be a $R$-positive operator in $E$. If Condition 3.1 holds then $L$ is an isomorphism from $W^l_p(R; E(A), E)$ to $L_p(R, E)$ i.e.
\[
\| Lu \|_{L_p(R, E)} \approx \| u \|_{W^l_p(R; E(A), E)}
\]
for all $u \in W^l_p(R; E(A), E)$.

**Result 3.7.** From (3.1) it follows that
\[
|\lambda| \| u \|_X \leq C \| f \|_X \tag{3.5}
\]
for all $\lambda \in S_{\varphi_2}$ (for some $\varphi_2 \in [0, \pi]$). Moreover, it is also easy to deduce from Lemma 3.2 and Lemma 3.3 that
\[
\| u \|_X \leq C \| f \|_X \tag{3.6}
\]
Thus, combining (3.5) and (3.6) we obtain the following resolvent estimate
\[
(1 + |\lambda|) \left\| (L + \lambda)^{-1} \right\|_{B(X)} \leq C.
\]

**Remark 3.8.** It is well known that
\[
W^l_p(R; E(A), E) \hookrightarrow L_p(R; E)
\]
whenever
\[
E(A) \hookrightarrow E.
\]
Hence, from Result 3.7 it follows that $L$ is a $\varphi$-positive operator in $L_p(R; E)$.

As an application of Theorem 2.3 consider the BVP for integro-differential equations
\[
\sum_{k=0}^{l} a_k \ast \frac{\partial^k u}{\partial x^k} + \sum_{k=0}^l b_k \frac{\partial^k u}{\partial x^k} + \int_{-\infty}^{\infty} \mu(x - z) \frac{\partial^2 u}{\partial y^2}(z, y) dt = f(x, y), \tag{3.7}
\]
u(x, 0) = u(x, 1) and $u_y(x, 0) = u_y(x, 1)$, $x \in (-\infty, \infty)$, $y \in (0, 1)$
in a mixed norm space $X = L_{p,q}(R \times [0,1])$ where $1 < p, q < \infty$. It is well-known that the differential expression
\[
Au = -u'' + bu
\]
defines a positive operator \( A \) acting on \( L_q([0,1]) \) with domain and satisfying \( u(0) = u(1) \) and \( u'(0) = u'(1) \). Thus by Theorem 3.5 the BVP (3.7) has a unique solution \( u \in W^{(1,2)}_{p,q}(R \times [0,1]) \) (anisotropic Sobolev spaces) and the following coercive inequality is valid:

\[
\sum_{k=0}^{l} \left( \left\| a_k \ast \frac{\partial^k u}{\partial x^k} \right\|_X + \left\| \frac{\partial^k u}{\partial y^k} \right\|_X \right) + \left\| \mu \ast \frac{\partial^2 u}{\partial y^2} \right\|_X \leq C \| f \|_X.
\]

4. Application to nonlinear equations

Here we will apply the main results of previous section to Cauchy problem for semilinear convolution equation (1.1) and the BVP for nonlinear elliptic equation (1.2). The main tool we use here is abstract existence and uniqueness theorem of Amann [15] and Shakhmurov [16]. For the exposition of abstract quasilinear equations see e.g. [15] and the references therein.

First we show that \( L \) is \( R \)-positive operator.

**Theorem 4.0.** Let \( E \) be a \( UMD \) space and \( A \) be a \( R \)-positive operator in \( E \). If Condition 3.1 holds and \( 1 < p < \infty \) then \( L \) is \( R \)-positive operator in \( L_p(R;E) \).

**Proof.** The Remark 3.8 ensures us \( \phi \)-positivity of \( L \) in \( L_p(R;E) \). Therefore it suffices to to show that the following set

\[
S = \left\{ (1+\lambda)(L+\lambda)^{-1} : \lambda \in S_\phi \right\}
\]

is \( R \)-bounded. From the proof of Theorem 3.5 and Result 3.6 we know

\[
(1+\lambda)(L+\lambda)^{-1} f = F^{-1} \left[ \sigma(\xi,\lambda) \hat{f} \right],
\]

for all \( f \in L_p(R;E) \), where

\[
\sigma(\xi,\lambda) = (1+\lambda)(\mu(\xi) + \nu)^{-1} [A + \eta(\xi) + \lambda]\]

By virtue of Lemma 3.2 and Lemma 3.3 one can easily prove

\[
\{ \sigma(\xi,\lambda) : \xi \in R \backslash \{0\} \} \text{ and } \{ \xi \sigma(\xi,\lambda) : \xi \in R \}
\]

are \( R \)-bounded sets and that \( \sigma(\xi,\lambda) \) is UFM in \( X = L_p(R;E) \). Thus

\[
\int_0^1 \left\| \sum_{j=1}^{m} r_j(y)(1+\lambda_j)(L+\lambda_j)^{-1} f_j \right\|_X dy = \int_0^1 \left\| \sum_{j=1}^{m} r_j(y) F^{-1} \left[ \sigma(\xi,\lambda_j) \hat{f_j} \right] \right\|_X dy
\]

\[
= \int_0^1 \left\| F^{-1} \left[ \sigma(\xi,\lambda_j) \sum_{j=1}^{m} r_j(y) \hat{f} \right] \right\|_X dy \leq C \int_0^1 \left\| \sum_{j=1}^{m} r_j(y) f_j \right\|_X dy,
\]

for each \( m \in \mathbb{N} \), \( (1+\lambda_j)(L+\lambda_j)^{-1} S, f_j \in L_p(R;E) \) and for all independent, symmetric, \( \{-1,1\} \)-valued random variables \( r_j \) on \([0,1]\). Hence we get the assertion.

Let \( E \) be a Banach space, \( 1 < p,q < \infty \) and \( p = (p,q) \). In what follows, \( L_p([0,T] \times R;E) \) will denote the space of all \( p \)-summable \( E \)-valued functions with
F urthermore,

for the definition of anisotropic Besov spaces see e.g. [23] and [8].

useful in the proof of our main theorems and their applications.

\[ \|f\|_{L_p([0,T]\times R;E)} = \left( \int_0^T \left( \int_{-\infty}^{\infty} \|f(t,x)\|_E^p \, dx \right)^{\frac{1}{p}} \, dt \right)^{\frac{1}{q}} < \infty. \]

Furthermore, \( W_p^{(1,l)}(J \times R; E(A), E) \) denotes anisotropic Sobolev spaces i.e. the space of all functions \( u \in L_p(J \times R; E(A)) \) such that \( \frac{\partial u}{\partial t} \) and \( \frac{\partial^k u}{\partial x_k} \in L_p(J \times R; E) \), \( k = 0, \ldots, l \) and

\[
\left\| \frac{\partial u}{\partial t} \right\|_{L_p(J \times R; E)} + \sum_{k=0}^l \left\| \frac{\partial^k u}{\partial x_k} \right\|_{L_p(J \times R; E)} + \| Au \|_{L_p(J \times R; E)} < \infty.
\]

It is well known that the Besov spaces has significant embedding properties. Let us recall some of them:

\[
W_{q,l+1}^s(X) \hookrightarrow B_{q,r}^s(X) \hookrightarrow W_q^l(X) \hookrightarrow L_q(X) \text{ where } l < s < l + 1,
\]

\[
B_{\infty,1}^s(X) \hookrightarrow C^s(X) \hookrightarrow B_{\infty,\infty}^s(X) \text{ for } s \in \mathbb{Z},
\]

and

\[
B_{p,1}^s(R^N, X) \hookrightarrow L_{\infty}(R^N, X) \text{ for } s \in \mathbb{Z}.
\]

For the definition of anisotropic Besov spaces see e.g. [23] and [8].

Now let us establish a lemma for representation of iterated spaces, that will be useful in the proof of our main theorems and their applications.

**Proposition 4.1.** Suppose \( \Omega_1, \Omega_2 \subseteq R^n \), \( s \in R \) and \( 1 \leq p, q, r \leq \infty \). Then:

(i) \[
B_{p,r}^s(\Omega_1, L_q(\Omega_2)) = B_{p,r}^{(s,0)}(\Omega_1 \times \Omega_2),
\]

(ii) \[
L_p(\Omega_1, B_{q,r}^s(\Omega_2)) = B_{p,r}^{(0,s)}(\Omega_1 \times \Omega_2)
\]

and

(iii) \[
W_p^l(\Omega_1, B_{q,r}^s(\Omega_2)) = B_{p,r}^{(l,s)}(\Omega_1 \times \Omega_2)
\]

**Proof.** (i) Taking into consideration [23] and the interpolation definition of Besov spaces i.e.

\[
B_{q,s}^s(R^N, X) = (L_q(R^N, X), W_q^m(R^N, X))_{\frac{s}{m},r},
\]

we get the first assertion:

\[
B_{p,r}^s(\Omega_1, L_q(\Omega_2)) = (L_p(\Omega_1, L_q(\Omega_2)), W_p^m(\Omega_1, L_q(\Omega_2)))_{\frac{s}{m},r}
\]

\[
= \left( L_p(\Omega_1 \times \Omega_2), W_p^{(m,0)}(\Omega_1 \times \Omega_2) \right)_{\frac{s}{m},r}
\]

\[
= B_{p,r}^{(s,0)}(\Omega_1 \times \Omega_2).
\]
(ii) Next we apply [22, Theorem 5.1.2] along with definition of Besov spaces, [23] and (i) we get desired result:

\[ L_p \left( \Omega_1, B_{q,r}^s(\Omega_2) \right) = \left( L_p \left( \Omega_1, L_q(\Omega_2) \right), L_p \left( \Omega_1, W^m_q(\Omega_2) \right) \right) \]

\[ \left| \frac{1}{m},r \right| \]

\[ = \left( L_p(\Omega_1 \times \Omega_2), W^{0,m}_p(\Omega_1 \times \Omega_2) \right) \]

\[ = B^{(0,s)}_{p,r}(\Omega_1 \times \Omega_2), \]

and

\[ W_p^l(\Omega_1, B^s_{q,r}(\Omega_2)) = \left( W_p^l(\Omega_1, L_q(\Omega_2)), W_p^l(\Omega_1, W^m_q(\Omega_2)) \right) \]

\[ \left| \frac{1}{m},r \right| \]

\[ = \left( W_p^{l(0)}(\Omega_1 \times \Omega_2), W_p^{l,m}(\Omega_1 \times \Omega_2) \right) \]

\[ = B^{(l,s)}_{p,r}(\Omega_1 \times \Omega_2). \]

(iii) Finally, with the help of (ii) and [23] we obtain

\[ B_{p,r}^s(\Omega_1, B^s_{q,r}(\Omega_2)) = \left( L_p \left( \Omega_1, B^s_{q,r}(\Omega_2) \right), L_p \left( \Omega_1, B^s_{q,r}(\Omega_2) \right) \right) \]

\[ \left| \frac{1}{m},r \right| \]

\[ = \left( B^{(0,\sigma)}_{p,r}(\Omega_1 \times \Omega_2), B^{(m,\sigma)}_{p,r}(\Omega_1 \times \Omega_2) \right) \]

\[ = B^{(s,\sigma)}_{p,r}(\Omega_1 \times \Omega_2). \]

Let \( X \) and \( Y \) be nonempty sets and \( J_T = [0, T) \). A nonlocal map \( F : X^J \rightarrow Y^J \) is said to have Volterra property (or Volterra map) if for every \( S \in J_T \) and \( u \in X \)

\[ F(u)|_{J_S} = F(u_{|J_S}). \]

It is clear that local maps satisfy Volterra property automatically.

By a solution on \( J_S \) of (1.1) we mean \( u \in W^{(1,l)}_{p,loc}(J \times R; E(A), E) \) such that \( u_{|J_S} \) belongs to \( W^{(1,l)}_{p}(J_S \times R; E(A), E) \) for each \( S \in J_T \) and satisfies a.e. the semilinear convolution operator equation (1.1).

It is known that if \( u \in W^{(1,l)}_{p}(J_S \times R; E(A), E) \) then

\[ \frac{\partial^j u}{\partial x^j} \in W^{(1, l-j)}_{p} \left( J \times R; (E(A), E) \frac{1}{1} \right), \quad 1 \leq j \leq l - 1. \]

Let \( B_0 \) be a cartesian product of above spaces, namely

\[ B_0 = \prod_{j=0}^{l-1} W^{(1, l-j)}_{p} \left( J \times R; (E(A), E) \frac{1}{1} \right). \]

Theorem 4.2. Let \( E \) be a UMD space, \( A \) be a \( R \)-positive operator in \( E \) with \( \varphi \in \left( \frac{n}{2}, \pi \right) \) and \( J = [0, T) \). Assume Condition 3.1 holds, \( 1 < p, q < \infty \) and \( u^0 \in B^{q,p}_{q,p}(R; E) \cap L_q \left( R; (E, E(A))_{1/p', p'} \right). \)

(i) \( F \) is a Volterra map from \( B_0 \) to \( L_p(J \times R, E) \), and
(ii) there exists \( r \in (p, \infty] \) such that \( F - F(0) \) is uniformly Lipschitz continuous on bounded subsets of \( B_0 \) with values in \( L_r(J \times R, E) \),
then (1.1) has a unique maximal solution. The maximal interval of existence, \( J_{\text{max}} \) is open in \( J \). Moreover, if for the unique maximal solution \( u \) of (1.1) \( F(u) \in L_p(J_{\text{max}} \times R, E) \), then \( J_{\text{max}} = J \).

**Proof.** Let \( E_1 = W^1_q(R; E(A), E) \) and \( E_0 = L_q(R; E) \) where
\[
E(A) \hookrightarrow E.
\]
By the intersection property of interpolation,
\[
(E_0, E_1)_{1/p',p} = B^{l/p'}_{q,p}(R; E) \cap L_q(R; E(E(A))_{1/p',p})
\]
It is clear that
\[
W^l_q(R; E(A), E) \hookrightarrow (E_0, E_1)_{1/p',p} \hookrightarrow L_q(R; E)
\]
and
\[
W^l_p(J; E_1, E_0) = W^l_{p(1,0)}(J \times R, E) \cap W^l_{p(0,1)}(J \times R, E(A), E) = W^l_{p(1,1)}(J \times R; E(A), E).
\]
Now consider the linear form of (1.1),
\[
\frac{\partial u}{\partial t} + \sum_{k=0}^l a_k \ast \frac{\partial^k u}{\partial x^k} + \sum_{k=0}^l b_k \frac{\partial^k u}{\partial x^k} + \mu \ast Au + \nu Au = f(t,x), \tag{4.1}
\]
\[
u(0,x) = u_0, \quad t \in (0,T), \quad x \in (-\infty, \infty)
\]
in \( L_p(J \times R; E) = L_p(J; E_0) \). By the Theorem 4.1 \( L \) is \( R \)-positive operator in \( L_q(R; E) \) with \( \varphi \in \left( \frac{\pi}{4}, \pi \right) \) and thus a generator of analytic semigroup. Since (4.1) can be written in an abstract form
\[
\frac{du}{dt} + Lu = f(t,x), \quad u(0) = u^0, \quad t \in J, \tag{4.2}
\]
[20, Theorem 4.2] and Theorem 3.5 ensures that, for each \( f \in Y = L_p(J \times R; E) \), (4.1) has a unique solution \( u \in W^l_{p(1,1)}(J \times R; E(A), E) \) and the following coercive estimate
\[
\left\| \frac{\partial u}{\partial t} \right\|_Y + \sum_{k=0}^l \left( \left\| a_k \ast \frac{\partial^k u}{\partial x^k} \right\|_Y + \left\| b_k \frac{\partial^k u}{\partial x^k} \right\|_Y \right) + \left\| \mu \ast Au \right\|_Y + \left\| Au \right\|_Y \leq C \left( \left\| f \right\|_Y + \left\| u_0 \right\|_E \right)
\]
holds. It is known that if \( A \) is a constant map, the maximal regularity property of problem (4.1) is independent of bounded intervals \( J \) and of \( p \) (see [20, Remarks 6.1(d) and (e)]).
In a similar manner nonlinear equation (1.1) reduces to the following abstract form
\[
\frac{du}{dt} + Lu = \tilde{F}(u), \quad u(0) = u^0, \quad t \in J.
\]
From assumptions (i) and (ii) it follows that \( \tilde{F} \) is a Volterra map from \( W^l_{p(1,0)}(J; E_1, E_0) \) to \( L_p(J; E_0) \), and \( F - \tilde{F}(0) \) is uniformly Lipschitz continuous on bounded subsets of \( W^l_{p(1,1)}(J; E_1, E_0) \) with values in \( L_q(J; E_0) \). Taking into account the fact that \( L \) has maximal regularity property independent of intervals \( J \) by [15, Theorem 2.1], (1.1) has a unique maximal solution \( u \) and this solution is globally defined whenever \( u \in W^l_{p(1,1)}(J_{\text{max}}) \).

Now let us study (4.1) and (1.1) in concrete settings.
Example 4.3. Suppose $A$ is an $n \times n$ matrix whose eigenvalues have positive real parts. Let $a = e^{-k|x|}$, $k > 0$, $b \in \mathbb{R}$ and $1 < p, q < \infty$. It is clear that

$$F(e^{-k|x|}) = \hat{a}(\xi) = \frac{2i\xi}{k^2 + \xi^2}$$

and

$$\eta(\xi) = -\xi^2 (\hat{a}(\xi) + b) \in S_\varphi, \ \varphi \in [0, \pi).$$

Moreover, $\hat{a}(\xi)$ satisfy all assumptions of Condition 3.1. Now consider the system of integro-differential equations

$$u_t + a * u_{xx} + bu_{xx} + Au = f(t, x), \ t \in (0, T), \ -\infty < x < \infty \quad (4.3)$$

$$u(0, x) = u_0 \in B_{q,p}'(R, R^n), \ x \in (-\infty, \infty).$$

Since all assumptions of the Theorem 4.2 are satisfied, for each $f \in Y = L_p(J \times R, R^n)$, (4.3) has a unique solution $u \in W_{p,2}^{(1,2)}(J \times R, R^n)$ and the coercive inequality holds

$$\|u_t\|_Y + \|a * u_{xx}\|_Y + \|u_{xx}\|_Y + \|u\|_Y \leq C (\|f\|_Y + \|u_0\|_{p/p'}) .$$

Example 4.4. Assume $a_k, b_k, \mu$ and $\nu$ satisfy assumptions of Condition 3.1, $l > 2$ and $f$ is a nonlocal map as in the Theorem 4.2. Let $A = -\Delta + c$ and $E = L_q(\Omega)$ where $\Omega \subset \mathbb{R}^2$ has sufficiently smooth boundary. From Proposition 4.1 we have

$$B_{q,p}'(R, L_q(\Omega)) \cap L_q \left( R, [L_q(\Omega), W^2(\Omega)]_{1/p}, p' \right) = B_{q,p}'(R \times \Omega) \cap B_{q,p}'(R \times \Omega) = B_{q,p}'(R \times \Omega) .$$

Now consider an initial-boundary value problem for 3D semilinear integro-differential equation

$$\frac{\partial u}{\partial t} + \sum_{k=0}^{l} \left( a_k * \frac{\partial^k u}{\partial x^k} + b_k \frac{\partial^k u}{\partial x^k} \right) - \mu * \Delta u - \nu \Delta u = f \left( u, u_x, \ldots, \frac{\partial^{l-1} u}{\partial x^{l-1}} \right), \ t \in (0, T) \quad (4.4)$$

$$u(0, x, y, z) = u_0, \ x \in (-\infty, \infty), \ (y, z) \in \Omega$$

$$u(t, x, y, z)|_{\partial \Omega} = 0 .$$

Therefore by Theorem 4.2, if $u_0 \in B_{q,p}'(R \times \Omega)$ then (4.4) has a unique maximal solution $u \in W_{p,2}^{(1,2)}(J \times R \times \Omega)$ where $s = (1, l, 2)$ and $p = (p, q).$

Let $1 < p, q < \infty$, $J_0 = [0, T_0]$ and $J = [0, T]$. Suppose boundary condition of (1.2) is nondegenerate i.e.

$$E_1 = W_q^1(R; E(A), E), \ E_0 = L_q(R; E) \text{ and } Y = W^2_p((0, T); E_1, E_0).$$

From the well known trace theorem [24] we know that if

$$u \in W^2_p((0, T); E_1, E_0)$$

then

$$u(0) \in X_0 = (E_1, E_0) \frac{1}{p-1} \in B_{q,p}'(R; E) \cap L_q \left( R; (E(A), E) \frac{1}{p-1} \right)$$

and

$$u'(0) \in X_1 = (E_1, E_0) \frac{p-1}{p} \in B_{q,p}'(R; E) \cap L_q \left( R; (E(A), E) \frac{p-1}{p} \right) ,$$
Let \( \bar{u} = (u, \frac{\partial u}{\partial t}) \) and \( \Omega_0, \Omega \) denote infinite strips \([0, T) \times \mathbb{R} \) and \([0, T_0) \times \mathbb{R} \) respectively and \( B_0 \) be a cartesian product of above spaces i.e.

\[
B_0 = X_0 \times X_1.
\]

**Theorem 4.5.** Let \( E \) be a UMD space and \( A \) be a \( R \)-positive operator in \( E \). Assume Condition 3.1 holds and \( F : \Omega_0 \times B_0 \to L_q (R; E) \) satisfies Caratheodory and Lipschitz continuity conditions i.e.

(i) \( F(\cdot, \bar{u}) \) is measurable for each \( \bar{u} \in B_0 \), \( F(t, \cdot) \) is continuous almost all \( t \in J_0 = [0, T_0) \),

(ii) \( f(t) = F(t, 0) \in L_p (\Omega_0, E) \), for each \( R > 0 \) there is a function \( \phi_R \in L_p (J_0) \) such that

\[
|F(t, \bar{u}) - F(t, \bar{v})|_{L_q(R;E)} \leq \phi_R(t) \left( |\bar{u} - \bar{v}|_{B_0} \right), \text{ a.a } t \in J, \bar{u}, \bar{v} \in B_0, |\bar{u}|_{B_0}, |\bar{v}|_{B_0} \leq R.
\]

If \( f_1, f_2 \in X_1 \), then there exist \( T \in J_0 \) such that (1.2) admits a unique solution \( u \in W^{2,1}_p (\Omega; E(A), E) \).

**Proof.** First we study linear form of (1.2) in the moving boundary \((0, b(\xi)) \) i.e.

\[
\frac{\partial^2 u}{\partial t^2} + \sum_{k=0}^{l} a_k \frac{\partial^k u}{\partial x^k} + \sum_{k=0}^{l} b_k \frac{\partial^k u}{\partial x^k} + \mu u + \nu Au = f(t, x),
\]

(4.5)

\[
\alpha_1 u(0, x) + \beta_1 \frac{\partial u(0, x)}{\partial t} = f_1(x), \quad \alpha_2 u(b(\xi), x) + \beta_2 \frac{\partial u(b(\xi), x)}{\partial t} = f_2(x), \quad x \in (-\infty, \infty).
\]

By the Theorem 4.1, \( L \) is \( R \)-positive operator in \( L_q (R; E) \). Since (4.5) can be written in abstract form

\[
-a''(t) + Lu(t) = f(t)
\]

\[
\alpha_1 u(0) + \beta_1 u'(0) = f_1, \quad \alpha_2 u(b(\xi)) + \beta_2 u'(b(\xi)) = f_2,
\]

[24. Theorem 4.1] and Theorem 3.5 ensures us that for each \( f \in Y_b = L_p ((0, b(\xi)) \times R; E) \), (4.1) has a unique solution \( u \in W^{2,1}_p ((0, b(\xi)) \times R; E(A), E) \). Similarly, nonlinear BVP (1.2) reduces to abstract form

\[
-a''(t) + Lu(t) = f(t, u, u')
\]

\[
\alpha_1 u(0) + \beta_1 u'(0) = f_1, \quad \alpha_1 u(T) + \beta_2 u'(T) = f_2.
\]

Since \( L \) is \( R \)-positive operator in \( E_0 = L_q (R; E) \), by [16, Theorem 4] there exist \( T \in [0, T_0) \) such that (1.2) admits a unique solution \( u \in W^2_p (J; E_1, E_0) = W^{2,1}_p (\Omega; E(A), E) \).

Finally, choosing \( p = q = 2 \) and \( l = 4 \) in (1.2) we consider BVP for the system of nonlinear equations:

\[
-a''(t) + \sum a \frac{\partial^4 u}{\partial x^4} + b \frac{\partial^4 u}{\partial x^4} + \sum A \frac{\partial^3 u}{\partial x^3} = f(t, u, \nabla u, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}), \quad t \in (0, T_0), \quad -\infty < x < \infty
\]

(4.6)

\[
\alpha_1 u(0, x) + \beta_1 \frac{\partial u(0, x)}{\partial t} = f_1(x), \quad \alpha_2 u(T, x) + \beta_2 \frac{\partial u(T, x)}{\partial t} = f_2(x), \quad x \in (-\infty, \infty),
\]

where \( A, a \) and \( b \) are as in Example 4.3. If \( f_1, f_2 \in B^{2,2}_p (R, R^n) \) then by Theorem 4.5, there exist \( T \in (0, T_0) \) such that (4.6) admits a unique solution \( u \in W^{2,4}_2 ([0, T] \times R, R^n) \).
There are a lot of positive operators in concrete Banach spaces. Therefore, choosing concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of $A$, we can obtain global existence results for various nonlinear convolution equations.

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