Dynamical Gauge Boson of Hidden Local Symmetry within the Standard Model

Koichi Yamawaki

1 Kobayashi-Maskawa Institute for the Origin of Particles and the Universe (KMI), Nagoya University, Nagoya 464-8602, Japan.

(Dated: July 18, 2018)

The Standard Model (SM) Higgs Lagrangian is straightforwardly rewritten into the scale-invariant nonlinear sigma model $G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V \simeq O(4)/O(3)$, with the (approximate) scale symmetry realized nonlinearly by the (pseudo) dilaton (= SM Higgs). It is further gauge equivalent to that having the symmetry $O(4)_{global} \times O(3)_{local}$, with $O(3)_{local}$ being the Hidden Local Symmetry (HLS). In the large $N$ limit of the scale-invariant version of the Grassmannian model $G/H = O(N)/[O(N-3) \times O(3)] \simeq O(N)_{global} \times [O(N-3) \times O(3)]_{local}$, identical to the SM for $N \to 4$, we show that the kinetic term of the HLS gauge bosons (“SM rho”) $\rho_\mu$ of the $O(3)_{local} \simeq [SU(2)_V]_{local}$ are dynamically generated by the nonperturbative dynamics of the SM itself. The dynamical SM rho stabilizes the skyrmion (“SM skyrmion”) $X_\pm$ as a dark matter candidate within the SM: The mass $M_{X_\pm} = O(10 \text{GeV})$ consistent with the direct search experiments implies the induced HLS gauge coupling $g_{HLS} = O(10^3)$, which realizes the relic abundance, $\Omega_{X_\pm} h^2 = O(0.1)$. If instead $g_{HLS} \lesssim 3.5$ ($M_\rho \lesssim 1.2 \text{ TeV}$), the SM rho could be detected with “narrow width” $\lesssim 100 \text{ GeV}$ at LHC, having all the “a = 2 results” of the generic HLS Lagrangian $L_A + aL_V$, i.e., $\rho$-universality, KSRF relations and the vector meson dominance, independently of “a”. There exists the second order phase transition to the unbroken phase having massless $\rho_\mu$ and massive $\pi$ (no longer NG bosons), both becoming massless free particles just on the transition point (scale-invariant ultraviolet fixed point). The results readily apply to the 2-flavored QCD as well.

I. INTRODUCTION

The Standard Model (SM) Higgs Lagrangian is customarily written in the linear sigma model which is convenient for the perturbation theory. Although the perturbative SM (pSM) has been very successful phenomenologically, there still remains the mystery of the origin of mass and the related Higgs particle itself within the pSM. Moreover, there is some concrete tension between the conventional (perturbative) understanding of the SM and the reality: apparent absence of the dark matter candidate, $\theta$ vacuum parameters due to instantons (strong CP problem, etc.), and absence of the first order phase transition for finite temperature and of large enough CP violation, both required by the baryogenesis, and so on. Although these problems might hint the possible new physics beyond the SM (BSM), they may simply indicate our ignorance of the full SM including the nonperturbative dynamics.

A possibility was in fact pointed out [1] that the dark matter candidate already exists in the SM as a soliton (“SM skyrmion”): The SM Higgs Lagrangian is rewritten [2] (see also [3, 4]) into the equivalent nonlinear sigma model having the hidden local symmetry (HLS) [6–10] as well as (approximately) scale-invariance, which then generates the kinetic term of the HLS gauge boson (“SM rho”) $\rho_\mu$ through nonperturbative physics at quantum level, thereby stabilizing a soliton as a skyrmion in the SM itself, similarly to the hadronic skyrmion (nucleon) stabilized by the $\rho$ meson as an HLS gauge boson within the same nonlinear sigma model (except for the scale symmetry). It was found [3] that the predicted skyrmion dark matter in the SM, $X_\pm$, with the mass $M_{X_\pm} = O(10 \text{GeV})$ consistent with the direct detection experiments, gives the relic abundance $\Omega_{X_\pm} h^2 \simeq 0.1$ in rough agreement with the observation.

In this paper, as a follow-up of Ref.[1], we establish the quantum dynamical generation of the SM rho, $\rho_\mu$, the gauge boson of the HLS hidden within the SM itself as a notable nonperturbative dynamics within the SM Higgs Lagrangian, based on an approach more reliable than that in Ref.[1].

In fact, pSM is not a whole story of the SM: Even within the perturbation the SM Higgs self-coupling $\lambda$ grows indefinitely to hit the Landau pole in the ultraviolet region thus eventually invalidating the perturbation itself, resulting
in some nonperturbative effects such as the bound state within the SM. Even if the perturbative Landau pole happens to be removed or pushed far above the weak scale so that the pSM would be logically consistent all the way up to the high scale, there arises ironically a notorious naturalness problem in turn. An immediate solution to this would be the nonperturbative quantum dynamics within the SM to show up with the “cutoff” A acting as the nonperturbative Landau pole not far from the weak scale without affecting the successful pSM at lower energy. Such a nonperturbative physics within the SM at lower scale may be a signal of the BSM having such a scale as a dual theory, similarly to the hadron-quark duality: With “cutoff” or “Landau pole” $O(\Lambda) = O(4\pi f_\pi)$, the nonlinear sigma model with HLS $\rho$ meson and skyrmion nucleon is dual to the underlying QCD.

Note also that the nonperturbative effects may not necessarily require the “strong coupling”: Sphaleron and instanton are well-known nonperturbative objects not to be described by the pSM but certainly exist in the SM as nonperturbative objects even for the weakest coupling. Also in the Georgi-Glashow model which is perturbatively renormalizable similarly to the SM Higgs Lagrangian, there exists a nonperturbative object, the ‘t Hooft-Polyakov monopole, even in the vanishing quartic coupling, known as a “Bogomol’nyi-Prasad-Sommerfield (BPS) limit”, similarly to the SUSY flat direction limit \([11]\). So even for the region of a small Higgs self-coupling, nonperturbative physics could be operative.

The nonperturbative quantum physics can often be better described by a different parameterization of the same Lagrangian at the classical level. In fact it was shown \([2]\) that the SM Higgs Lagrangian written in the linear sigma model on the broken vacuum can be straightforwardly cast through the polar decomposition into an (approximately) scale-invariant version of the nonlinear sigma model based on the manifold \(G/H = SU(2)_L \times SU(2)_R / SU(2)_V \simeq O(4)/O(3)\). Namely, both the (approximate) scale symmetry and internal symmetry \(G\) are realized nonlinearly, with the SM Higgs being nothing but a (pseudo) dilaton, \(\varphi\), a (pseudo) Nambu-Goldstone (NG) boson of the spontaneously broken scale symmetry, in addition to the NG bosons \(\pi\) living on \(G/H\). (Since the (pseudo) dilaton parts in such a parameterization are \(G\)-invariant, the discussions hereafter are confined, unless otherwise mentioned, to the nonlinear realization of the internal symmetry \(G\).) Once written in the form of nonlinear sigma model, one readily sees \([2]\) that it has the HLS, since it is known \([8, 9]\) that any nonlinear sigma model based on \(G/H\) à la Callan-Coleman-Wess-Zumino (CCWZ) \([12, 13]\) is gauge equivalent to another model (HLS Lagrangian) having a symmetry \(G_{\text{global}} \times H_{\text{local}}\), with the Lagrangian consisting of two invariants: \(L = L_A + aL_V\), \(a\) being a free parameter (See Appendix \([X]\)). While \(L_A\) is reduced after gauge fixing to the original \(G/H\) model, \(L_V\) term yields the mass of the HLS gauge boson \(\rho_0\), in such a way that the gauge symmetry (HLS) \(H_{\text{local}}\) and \(G_{\text{global}}\) are both spontaneously broken down to the diagonal group \(H = H_{\text{local}} \oplus H_{\text{global}} (H_{\text{global}} \subset G_{\text{global}})\) through the Higgs mechanism. In the case at hand, the SM Higgs Lagrangian rewritten into the (approximately) scale-invariant version of the \(G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V \simeq O(4)/O(3)\) nonlinear sigma model has the HLS \(H_{\text{local}} = [SU(2)_V]_{\text{local}} \simeq O(3)_{\text{local}}\). Here the HLS gauge boson is an auxiliary field (without kinetic term) as a static massive composite of the NG bosons and can be solved/gauged away at classical level: \(L_V = 0\). \(^2\)

However, it is well known (see Appendix \([13]\)) that the HLS gauge bosons in many nonlinear sigma models, such as the \(CP^{N-1}\) model with \(G/H = U(N)/[U(N-1) \times U(1)] \simeq SU(N)/[SU(N-1) \times U(1)] \simeq SU(N)_{\text{global}} \times U(1)_{\text{local}}\), do acquire kinetic term at quantum level through nonperturbative dynamics like the large \(N\) limit \([3, 10, 21, 29]\). \(^3\)

\(^1\) In the case of QCD, the perturbation in the linear sigma model already breaks down at physical point with \(\lambda \gg 1\), i.e., the “perturbative” Landau pole is very close to the nonperturbative one \(O(\Lambda)\). This is due to absence of the scale symmetry \([2]\), in sharp contrast to the SM Higgs Lagrangian \(\lambda \ll 1\), having the perturbative Landau pole far away from the weak scale (physical point).

\(^2\) A similar s-HLS model was studied \([4, 14, 16]\) as the effective theory of the walking technicolor \([17, 18]\) having the (approximate) scale symmetry and a pseudo-dilaton (“technidilaton”) as a light composite Higgs. The s-HLS model was also discussed in a different context, the ordinary QCD in medium \([19]\). Note that the pseudo-dilaton \(\varphi\) in the present paper is of course the SM Higgs itself (see a simple re-parameterization in Eqs. \([13]\) and \([22]\)), with decay constant \(F_\varphi = F_\rho = 246 \text{ GeV}\), which should not be confused with the technidilaton having a different decay constant and hence the SM particles different from those of the SM Higgs \([18]\) (see \([23]\) for a recent review of the technidilaton consistent with the LHC experiments in spite of the different couplings).

\(^3\) The \(CP^{N-1}\) model minimally written in terms of \((2N-2)\) NG bosons is usually parameterized having the symmetry \(SU(N)_{\text{global}} \times U(1)_{\text{local}}\), including redundancy: one constraint with Lagrange multiplier and the \(U(1)_{\text{local}}\) as an HLS whose gauge boson (having mass by the Higgs mechanism) is an auxiliary field to be solved away at classical level. It is well established \([3, 10, 21, 24]\) that in the large \(N\) limit, there exists a phase transition from the perturbative (broken) phase to nonperturbative (unbroken) phase and the HLS gauge boson in both phases necessarily acquires the kinetic term, becoming the propagating gauge boson, massive (broken phase) or massless (unbroken phase). In the unbroken phase the NG bosons at classical level are no longer the NG bosons but have a mass given by the Lagrange multiplier. The model in \(D = 4\) dimensions has a cutoff, an extra free parameter to define the quantum theory, acting as a Landau pole where the induced kinetic term of HLS gauge boson vanishes (“compositeness condition”) to return to the auxiliary field as at classical level \([9, 11]\). (See also \([24]\) for different formulation (renormalizable in the sense of effective theory) in \(D = 4\) dimensions,
In the unbroken phase a minimal parameterization of the classical Lagrangian without gauge symmetry redundancy is ill-defined at quantum level [9], thus the HLS parameterization is crucial to the nonperturbative quantum physics (Exactly the same applies to the present case as we shall explain later in details). This is in sharp contrast to the perturbation, the pSM, which is known [32] to be independent of the parameterization for a generic metric for the CCWZ nonlinear realization not just the original linear sigma model parameterization.

The same dynamical generation of the HLS gauge bosons in the large $N$ limit is also known in the nonlinear sigma model on the Grassmannian manifold, $G/H = U(N)/[U(N-p) \times U(p)]$ [31, 32] as an extension of $CP^{N-1}$ model ($p = 1$), and also on $G/H = O(N)/[O(N-p) \times O(p)]$ [31]. Similarly, in the Nambu-Jona-Lasinio (NJL) model [32], such a dynamical generation at quantum level of the kinetic term of the auxiliary field is in the large $N$ limit is very well known [33-35], see Appendix C for details. Thus the dynamical generation of the kinetic term of the auxiliary fields is a very common nonperturbative phenomenon.

Further in the case of the SM written in the form of a scale-invariant nonlinear sigma model, it was shown [1] that the massive (Higgsed) $[SU(2)^L]_{\text{local}}$ HLS gauge boson in the SM, SM rho, acquires kinetic term by the nonperturbative dynamics of the SM itself at order $O(p^4)$ in the systematic derivative expansion within the framework of the chiral perturbation theory [31, 32] in a version extended to include the HLS [10, 37, 38].

Here we show the key dynamical issue of Ref. [1], the quantum dynamical generation of the SM rho, the gauge boson of the HLS hidden within the SM itself, in a more transparent and well-established nonperturbative method than that in Ref. [1] #4, namely the conventional large $N$ expansion widely used for many nonperturbative dynamics. As an $N$ expansion of the SM with $G/H = SU(2)_L \times SU(2)_R/SU(2)_V \simeq O(4)/O(3)$, we take a Grassmannian manifold $G/H = O(N)/[O(N-p) \times O(p)]$, with $p = 3$ fixed, which, combined with the pseudo-dilaton parts, is reduced to precisely the SM Higgs Lagrangian for $N \to 4$ and $p = 3$.

According to the generic arguments [8, 9], this Grassmannian model is gauge equivalent to that having an HLS with the symmetry $G_{\text{global}} \times H_{\text{local}} \simeq O(N)_{\text{global}} \times [O(N-p) \times O(p)]_{\text{local}}$; this time the Lagrangian besides the dilatonic parts consists of three invariants: $\mathcal{L} = \mathcal{L}_A + a^{(p)} \mathcal{L}_V^{(p)} + a^{(N-p)} \mathcal{L}_V^{(N-p)}$, with $a^{(p)}$ and $a^{(N-p)}$ being free parameters corresponding to the $O(p)$ and $O(N-p)$ local gauge boson mass terms, respectively.

Similarly to the $CP^{N-1}$ model (see Appendix B), a popular parameterization to study the large $N$ limit of this model [31] is to use $p \times N$ real scalar field $\phi_{i,\alpha}$ $(i = 1, \ldots, p; \alpha = 1, \ldots, N)$ and introduce the HLS gauge boson, SM rho, $\rho_{\mu}$ by the covariant derivative $D_{\mu} \phi = (\partial_{\mu} - i p_{\mu}) \phi$. The $\phi$ consists of the $p(N-p) \times O(p)$ NG bosons $\rho$ and the $p(p+1)/2$ HLS gauge degrees of freedom (would-be NG bosons $\tilde{\rho}$ absorbed into $\rho_{\mu}$)#5, plus $p(p+1)/2$ redundant massive components corresponding to the constraints (through the Lagrange multipliers $n_{i,j}(x)$), besides the (pseudo-)dilaton (SM Higgs) $\varphi$ to make the theory equivalent to the SM in the $N \to 4$ limit with $p = 3$.

Curiously, this parameterization is equivalent to a specific “golden point” $a(= a^{(p)}) = 2$ realizing all the successful results in the QCD for the generic HLS Lagrangian, $\mathcal{L}_A + a \mathcal{L}_V$; when the kinetic term is simply summed [10, 11]. The $p-$universality, Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation II and the vector meson dominance (VMD) for $\pi$ form factor (See Eqs. (A34), (A35) and (A40) in Appendix A)

We first discuss the phase structure of this model in the large $N$ limit, which is essentially the same as that of the $CP^{N-1}$ model: (i) the broken phase $\langle \phi_{i,\alpha}(x) \rangle = \delta_{i,j} \sqrt{N} v \neq 0$, $\langle n_{i,j}(x) \rangle = 0$, with the NG bosons $\rho$ (with the decay constant $F_{\rho} = \sqrt{N} v$) living on the coset $G/H = O(N)/[O(N-p) \times O(p)]$ as in the classical theory, and (ii) the unbroken phase $\langle \phi_{i,\alpha}(x) \rangle = 0$, $\langle n_{i,j}(x) \rangle = \delta_{i,j} v; n \neq 0$, which exists only at the quantum level, with $\pi$ being no longer the NG bosons but massive.

In generic $D$ dimensions ($2 \leq D \leq 4$), the gap equation takes a form very similar to that of the $CP^{N-1}$ model, and also to that of the NJL model in $D$ dimensions though with opposite direction (weak coupling for the broken, and strong coupling for the unbroken). The SM Higgs boson $\varphi$, sitting in the theory both through the Higgs potential and the dilatonic factor, plays only a minor role for the phase structure.

Then we discuss the dynamical generation of the HLS gauge boson $\rho_{\mu}$. It was in fact already shown [31] in (a conventional non-scale invariant version of) this model that the kinetic term of the HLS gauge boson is dynamically
generated in the large $N$ limit in both phases. Although the pseudo-dilaton $\varphi (= \text{SM Higgs})$ in our case additionally is coupled to the HLS gauge boson, it does not affect the large $N$ counting of the 2-point function of the HLS gauge boson and hence is irrelevant to the kinetic term generation.

For concrete case $p = 3$ relevant to the SM (extension to $p \neq 3$ is trivial), we show that the $O(p)_{\text{local}}$ gauge boson $\rho_{\mu}$ becomes dynamical in the large $N$ limit, in both phases: massive (broken phase) or massless (unbroken phase), while not the $O(N - p)_{\text{local}}$ gauge boson, carrying index running $1, \cdots, N - p$ thus subject to all the planar diagrams contributions in the large $N$ limit, which stays as an auxiliary field (i.e., $C_{\chi}^{(N-p)} = 0$) in either phase. This is similar to the $SU(N - 1)_{\text{local}}$ gauge boson in the $CP^{N-1}$ model with $G/H = SU(N)/[SU(N - 1) \times U(1)]$, which, carrying the index running through $1, \cdots, N - 1$, is not dynamically generated in the large $N$ limit, in contrast to the dynamical generation of the $U(1)_{\text{local}}$ part. This is also contrasted to a popular $N$ extension $G/H = O(N)/O(N - 1) \simeq O(N)_{\text{global}} \times O(N - 1)_{\text{local}}$ whose $O(N - 1)_{\text{local}}$ HLS will not be dynamical in the large $N$ limit for the same reason as for the $O(N - p)_{\text{local}}$ gauge boson in our case. (In the limit to the SM, with $N \to 4$ ($p = 3$), $O(N - p)$ does not exist, anyway.)

We thus find that the kinetic term of the $O(3)_{\text{local}}$ HLS gauge boson $\rho_{\mu}$ in the large $N$ limit is indeed generated, with the $N$–independent induced HLS coupling ('t Hooft coupling) $\lambda_{\text{HLS}} = N g_{\text{HLS}}^2$ as given as:

$$\frac{1}{\lambda_{\text{HLS}}(\mu^2)} = N g_{\text{HLS}}^2(\mu^2) = \frac{1}{3(4\pi)^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) \to 0 \quad (\mu \to \Lambda), \quad (1)$$

where $\Lambda = e^{4/3} \Lambda \simeq 3.8 \Lambda$ (broken phase) (= $\Lambda$ (unbroken phase)) is identified with the Landau pole and $\mu^2$ the “renormalization scale” traded for $q^2$, the (momentum)² of $\rho_{\mu}$. The cutoff $\Lambda$ is needed to define the nonperturbative quantum theory by regularizing the divergence of the kinetic term which is absent as a counter term in the tree-level SM Lagrangian and thus cannot be renormalized in the ordinary sense, in sharp contrast to the pSM producing no such an extra kinetic term. Then the HLS gauge coupling $g_{\text{HLS}}$ (or $\Lambda$) is an extra free parameter of the nonperturbative dynamics within the SM, similarly to the $CP^{N-1}$ model in $D = 4$ (see footnote #3) and other nonperturbative dynamics. In the broken phase, we have the “on-shell” $\rho_{\mu}$ mass $M_{\rho}^2 = M_{\rho}^2(\mu^2 = M_{\rho}^2)$ for the mass function $M_{\rho}(\mu^2) = g_{\text{HLS}}^2(\mu^2) \cdot F_{\rho}^2$ as:

$$M_{\rho}^2 = g_{\text{HLS}}^2(M_{\rho}^2) \cdot F_{\rho}^2 = \lambda_{\text{HLS}}(M_{\rho}^2) \cdot 2\pi^2, \quad F_{\rho}^2 \equiv 2 \cdot N v^2 \simeq 2 \cdot (246 \text{ GeV})^2 \simeq (350 \text{ GeV})^2, \quad (2)$$

where $v$ is the $N$–independent VEV $v^2 = F_{\pi}^2/N$, and the factor 2 for $F_{\rho}^2/F_{\pi}^2$ reflects the covariant derivative parameterization which corresponds to $a = 2$.

Thus we find the dynamical gauge boson, SM rho $\rho_{\mu}$, of $O(3)_{\text{local}} \simeq [SU(2)_{\text{V}}]_{\text{local}}$ in the SM as the extrapolation $N \to 4$ with $p = 3$ (fixed) of the large $N$ limit.

Note that explicit $N$–dependence enters only in the relation Eq.(1) between $g_{\text{HLS}} = g_{\text{HLS}}(M_{\rho}^2)$ and the Landau pole $\Lambda$, while the phenomenologically relevant relation Eq.(2) has no explicit $N$–dependence, giving the SM value $F_{\rho} \simeq 246 \text{ GeV}$ ($F_{\rho} \simeq 350 \text{ GeV}$).

Needless to say, the SM fermions carries no $H_{\text{local}}$ charges, so that the dynamically generated new gauge symmetry HLS is trivially anomaly-free by construction.

Although the above results are obtained in a specific parameterization corresponding to $a = 2$, we further show that the large $N$ dynamics yields, independently of “$a$”, not just Eqs.(1) and (2) but all the on-shell relations, analogues of the successful results of $a = 2$ choice in the conventional HLS treatment (simply assuming the kinetic term) in QCD, i.e., the $\rho$–universality ($g_{\rho \pi \pi} = g_{\text{HLS}}$) and KSRF II, and some of the off-shell physics like analogue of the VMD of the $\pi$ form factor ($W/\gamma_{\pi} \rightarrow W_{L} W_{L}/Z_{L}$ dominated by $W/\gamma_{\pi} \rightarrow \rho_{\mu} \rightarrow W_{L} W_{L}/Z_{L}$ in the present case).

On the other hand, the dynamically generated $\rho_{\mu}$ propagator does depend on $a$ in the large $N$ limit, in such a way that the equation of motion of $\rho_{\mu}$ at classical level (in the absence of the kinetic term) is violated at quantum level by the term depending on $a$ as proportional to $1/a$. Thus the off-shell physics depends on $a$ in principle, with a curious exception of the vector meson dominance for $\pi$ form factor as mentioned above. As a result, at $a \to \infty$ the equation of motion of $\rho_{\mu}$ at classical level (in the absence of the kinetic term) remains intact at quantum level, so that the

---

#6 It is known that the large $N$ results remain qualitatively true even for the smallest value of $N = 2$ in the $CP^{N-1}$ model, which is checked by the equivalent O(3) model exactly solvable in 2 dimension. See e.g., Ref.[22]. Not to mention that the large $N_c$ QCD also well describes the reality with a small $N_c = 3$.

#7 Even in the generic case (not the present SM case) having gauged- $G_{\text{global}}$ anomaly (Wess-Zumino-Witten anomaly) as in QCD, the dynamical HLS such as the one associated with $\rho/\omega$ mesons in QCD is still anomaly-free, see Ref.[8, 12].
\[ \rho_\mu \text{ kinetic term is totally replaced by the Skyrme term, which was the case explicitly shown by Ref.} [1] \text{ that the SM skyrme as a dark matter is stabilized by the dynamical HLS gauge boson } \rho_\mu. \text{ We thus establish the key ingredient of Ref.} [1]. \]

Phenomenological implications of our result for the SM rho would be two different scenarios depending on the possible value of a single free parameter existing in the theory, \( M_\rho = g_{\rho\pi\pi} \cdot F_\rho \) or \( g_{\rho\pi\pi} = g_{\text{HLS}} \equiv g_{\text{HLS}}(M_\rho^2) = M_\rho/F_\rho = M_\rho/(350 \text{ GeV}) \) or the cutoff \( \Lambda \) (or the Landau pole \( \tilde{\Lambda} \)) in view of Eqs. (2) and (1). The cutoff \( \Lambda = e^{-4/3} \cdot M_\rho \cdot \exp \left[ \left( \frac{4}{3} \cdot \frac{1}{g_{\rho\pi\pi}^2} \right) \left( \frac{M_\rho^2}{M_\rho^2} \right) \right] \). We define \( \Lambda < M_\rho \), where \( g_{\rho\pi\pi} > 3.7 \text{ TeV} \) for instance, we would have \( g_{\rho\pi\pi} \approx 3.7 \text{ TeV} \times 4 / (4 F_\pi) \text{ (simple scale-up of the QCD } \rho \text{ meson). Thus this yields the width } \Gamma_\rho \approx \Gamma_{\rho \rightarrow W W} \approx \frac{g_{\rho\pi\pi}^2 M_\rho}{(48 \pi)} \approx 433 \text{ GeV} \), so broad as barely detectable at LHC. For larger (smaller) \( M_\rho \), the width gets larger (smaller) as \( \sim M_\rho^3 \), and the production cross section gets smaller (larger) as \( \sim 1/M_\rho^2 \), thus more difficult for \( M_\rho > 2 \text{ TeV} \) to be seen at LHC. The SM rho with narrow resonance \( \Gamma_\rho \lesssim 100 \text{ GeV} \) could be detected at LHC for \( M_\rho \gtrsim 1.2 \text{ TeV} \), which corresponds to \( g_{\text{HLS}} \lesssim 5.5 \) and \( \Lambda \gtrsim 50 \text{ TeV} \).

1) “Low \( M_\rho \) scenario” \( (M_\rho < 2.3 \text{ TeV}, \Lambda > M_\rho) \):

- The SM rho \( \rho_\mu \) at the collider experiments may be produced through Drell-Yang processes \( q\bar{q} \rightarrow W/Z/\gamma \rightarrow \rho_\mu \) with the coupling \( \rho_\mu = \alpha_{\text{em}} F_\rho / M_\rho = \alpha_{\text{em}} / g_{\rho\pi\pi}. \)

- Given a reference value \( M_\rho = 2 \text{ TeV} \) for instance, we would have \( g_{\rho\pi\pi} \approx 5.7 \text{ TeV} \times 4 / (4 F_\pi) \) (simple scale-up of the QCD \( \rho \) meson), which yields the width \( \Gamma_\rho \approx \Gamma_{\rho \rightarrow W W} \approx \frac{(g_{\rho\pi\pi}^2 M_\rho)}{(48 \pi)} \approx 433 \text{ GeV} \), so broad as barely detectable at LHC. For larger (smaller) \( M_\rho \), the width gets larger (smaller) as \( \sim M_\rho^3 \), and the production cross section gets smaller (larger) as \( \sim 1/M_\rho^2 \), thus more difficult for \( M_\rho > 2 \text{ TeV} \) to be seen at LHC. The SM rho with narrow resonance \( \Gamma_\rho \lesssim 100 \text{ GeV} \) could be detected at LHC for \( M_\rho \gtrsim 1.2 \text{ TeV} \), which corresponds to \( g_{\text{HLS}} \lesssim 5.5 \) and \( \Lambda \gtrsim 50 \text{ TeV} \).

2) “High \( M_\rho \) scenario” \( (M_\rho = O(10^2 - 10^3) \text{ TeV}, \Lambda < M_\rho) \), as a stabilizer of the skyrmeon dark matter \( X_\mu \):[1]

Even if no direct evidence were seen at the collider experiments, physical effects of the dynamical \( \rho_\mu \) are still observable through the skyrmeon dark matter \( X_\mu \) in the SM. In fact the SM skyrmeon is stabilized by the off-shell \( \rho_\mu \) in the short distance physics as shown in Ref. [1], the result of which corresponds to \( a \rightarrow \infty \) calculation, while the results are numerically similar even for \( a \sim 2 \)[2]. The HLS coupling is extremely large \( g_{\text{HLS}} = O(10^3) \) to realize \( M_{X_\mu} \lesssim O(10) \) GeV consistent with the direct detection of the dark matter, in rough agreement with the relic abundance of the dark matter: \( \Omega_{X_\mu} h^2 \approx 0.1 \) [1,2,3]. The cutoff is \( \Lambda = e^{-4/3} \Lambda \approx e^{-4/3} \cdot M_\rho = O(10^2 \text{ TeV}) \), where \( M_\rho = g_{\text{HLS}} \cdot F_\rho \) is a typical mass scale (no longer the “on-shell” mass, since the SM rho is deeply off-shell). In either scenario, the phenomenologically interesting nonperturbative SM physics has typical strong SM rho gauge coupling \( g_{\text{HLS}} \approx 1/3 - 10^3 \), which will have the cutoff \( \Lambda = O(10^6 - 10^2) \text{ TeV} \) close to the weak scale in sharp contrast to the pSM, thus resolving the naturalness problem within the full SM including the nonperturbative effects, even without recourse to the BSM.[8]

We further show that the theory of this type has a salient phase transition, through at this moment it is purely formal discussion. As in many nonlinear sigma models such as the \( CP^{N-1} \) model, the dynamical HLS \( O(p)_\text{local} \) gauge bosons \( \rho_\mu \) in the genuine nonperturbative unbroken phase of the \( O(N)/[O(N - p) \times O(p)] \) Grassmannian nonlinear sigma model in the large \( N \) limit are massless [31]. In this respect, we demonstrate that the gauge symmetry, HLS, is mandatory to keep the theory well-defined at quantum level not just in the broken phase but also in the unbroken phase, similarly to the \( CP^{N-1} \) model. In particular, in 2 dimensions the model having vanishing critical coupling has only the unbroken phase in accord with the Mermin-Wagner-Coleman theorem, thus the well-defined quantum theory exists only at presence of the HLS.

The (zero temperature) phase transition between the broken and the unbroken phase takes place independently of \( a \) as the second order phase transition at a critical point (nontrivial ultraviolet fixed point) of a dimensionless “coupling” related to the condensate (decay constant). The phase change goes through, with the (induced) HLS gauge coupling tending to zero continuously from both sides of the phases (second order phase transition), where the massless and massive spectrum interchanged between the HLS gauge boson \( \rho_\mu \) and the \( \pi \) modes (corresponding to NG bosons in the broken phase). This \( a \)-independent phase transition is compared with a similar symmetry restoration “Vector Manifestation” [10, 43] proposed in the non-scale-invariant nonlinear sigma model at \( a = 1 \) (a fixed point) at one loop (of \( O(p^4) \) in the sense of the chiral perturbation theory).

Finally, the results in this paper are of direct relevance to the 2-flavored QCD, not just the SM Higgs Lagrangian. In fact, the dynamical results obtained here for the SM in the large \( N \) limit are quite independent of the presence of the (pseudo) dilaton \( \varphi = \text{SM Higgs} \). Thus they apply most directly to the 2-flavored QCD, which is described by the same nonlinear sigma model (without scale invariance/pseudo-dilaton) having the \( \rho \) meson as the dynamical gauge boson of HLS, thereby proving all the otherwise mysterious “\( a = 2 \) relations” as the reality of QCD, such as the \( \rho \)-universality, KSRF relation II, and the VMD, be realized \( a \)- independently, together with the skyrmeon (nucleon).

---

#8 This indicates that the quadratic divergence corrections to the weak scale \( \delta F_\rho^2 \sim 4 \cdot \Lambda^2/(4 \pi)^2 \sim (0.1 \text{ TeV})^2 - (10 \text{ TeV})^2 \) (see the gap equation Eq. (35)). This also suggests a possibility that the SM in the full nonperturbative formulation eventually reveals itself as a “dual” to a possible BSM underlying theory with such a scale, similarly to the hadron-quark duality (nonlinear sigma model/chiral Lagrangian vs QCD).
stabilized by the dynamical \( \rho \) meson simply as nonperturbative dynamical effects in the large \( N \) limit, without recourse to the underlying QCD. The dynamically generated kinetic term has a new free parameter, the \( \rho \) coupling \( g_{\rho\pi\pi} = g_{\text{HLS}} \simeq 5.9 \) corresponding to \( m_{\rho} \approx 770 \text{ MeV} \) and through Eq.(11) we have \( \Lambda \simeq 1.1 \text{ GeV} (\simeq 4\pi f_\rho) \). Then this establish that the nonperturbative dynamics of the nonlinear sigma model having dynamical \( \rho \) meson is certainly dual to the underlying QCD, matched each other at \( \Lambda \sim \Lambda_\text{QCD} \). The Grassmannian manifold, thus being the right macroscopic theory dual to the underlying theory, QCD, gives us an unmistakable evidence for the observation [10, 38, 44] that the HLS is a “magnetic gauge symmetry” à la Seiberg duality [45] even in the non-SUSY QCD.

The paper is organized as follows:

In the next section we recapitulate the re-parameterization of the SM Higgs Lagrangian in terms of the (approximately) scale-invariant version of the nonlinear sigma model \( G/H = SU(2)_L \times SU(2)_R/SU(2)_V \simeq O(4)/O(3) \).

In Section III, as an \( N \) extension of the model \( G/H = O(4)/O(3) \simeq O(4)_{\text{global}} \times O(3)_{\text{local}} \), we introduce the HLS model \( G_{\text{global}} \times H_{\text{local}} = O(N)_{\text{global}} \times [O(N - p) \times O(p)]_{\text{local}} \) which is gauge equivalent to the “CCWZ representation” of the Grassmannian manifold \( G/H = O(N)/(O(N - p) \times O(p)) \).

In Section IV we first introduce an alternative parameterization of the model in terms of the covariant derivative of the \( p \times N \) component real field \( \phi \) with constraints through Lagrange multiplier. Solving the constraints we show that this parameterization is equivalent to a specific parameter choice \( a = 2 \) in the generic HLS Lagrangian. We then show that the effective action and gap equation in the large \( N \) limit and identify the two phases, broken and unbroken \( O(N) \) symmetry, with the scale symmetry spontaneously (and explicitly) broken in each phase by a different order parameter.

Section V is the main part of the paper where we demonstrate the dynamical generation of the \( O(p) \) local HLS gauge boson \( \rho_\mu \), while not of the \( O(N - p) \) local in the large \( N \) limit for a concrete case \( p = 3 \), and hence the dynamical generation of \( \rho_\mu \) in the SM. The second order phase transition with the vanishing induced HLS gauge coupling at the transition point is also discussed.

In section VI, the \( a \)—independence of the physical quantities of the \( \rho_\mu \) is shown, while the \( a \)—dependent part of the \( \rho_\mu \) propagator. Its physical implications are further discussed. In particular, the \( p \)—universality, KSRF relations, \( \text{I,II}, \) and the VMD are shown to be realized independently of the parameter \( a \), while the skyrmion dynamics is shown to depend on \( a \) in such a way that \( a \rightarrow \infty \) limit realizes the pure Skyrmie term. Phenomenological implications of the SM \( \rho \) for the collider physics and the skyrmion dark matter are further discussed, both acting as solution to the naturalness problem within the SM.

Section VII is devoted to Summary and Discussions.

In Appendix A we summarize the basic formalism of the CCWZ nonlinear sigma model based on the manifold \( G/H \), and its gauge-equivalent HLS model in the scale-invariant version. In Appendix B we recapitulate the well-known dynamical generation of the auxiliary field (HLS gauge boson) in \( CP^{N-1} \) model which is the same dynamical phenomenon in the large \( N \) limit as the SM discussed in the present paper. Appendix C is for a review of the dynamical generation of the auxiliary fields in the NJL model, which is also the same large \( N \) dynamical phenomenon as that in the present paper. Appendix D is for a direct calculation to prove the \( p \)—universality in the large \( N \) limit.

II. SM HIGGS LAGRANGIAN AS A NONLINEAR SIGMA MODEL

Let us first recapitulate the fact [2] that the SM Higgs Lagrangian is re-parameterized into a scale-invariant nonlinear sigma model.

A. \( G/H = SU(2)_L \times SU(2)_R/SU(2)_V \) parameterization

As is well-known, the SM Higgs Lagrangian takes the form of the \( SU(2)_L \times SU(2)_R \) linear sigma model:

\[
\mathcal{L}_{\text{SM}} = |\partial_\mu h|^2 - m^2 |h|^2 - \lambda |h|^4 \\
= \frac{1}{2} \left[ (\partial_\mu \hat{\sigma})^2 + (\partial_\mu \hat{\sigma}_a)^2 \right] - \frac{m^2}{2} \left[ \hat{\sigma}^2 + \hat{\sigma}_a^2 \right] - \frac{\lambda}{4} \left[ \hat{\sigma}^2 + \hat{\sigma}_a^2 \right]^2 \\
= \frac{1}{2} \text{tr} \left( \partial_\mu M \partial^\mu M^\dagger \right) - \frac{m^2}{2} \text{tr} \left( M M^\dagger \right) - \frac{\lambda}{4} \left( \text{tr} \left( M M^\dagger \right) \right)^2 ,
\]

(3)
The scale (dilatation) transformations for these fields are
\[ h = \left( \phi^+ \phi^0 \right) = \frac{1}{\sqrt{2}} \left( \frac{i\hat{\sigma} + \hat{\pi}}{\hat{\sigma} - i\hat{\pi}} \right), \]
and the 2 x 2 matrix \( M \) reads:
\[ M = (i\tau_2 h^*, h) = \frac{1}{\sqrt{2}} (\hat{\sigma} \cdot 1_{2x2} + 2i\hat{\pi}), \quad \left( \hat{\pi} \equiv \frac{\hat{\pi}_a}{2} \right), \]
which transforms under \( SU(2)_L \times SU(2)_R \) as
\[ M \rightarrow g_L M g_R^+, \quad (g_{R,L} \in SU(2)_{R,L}). \]

The potential term,
\[ V(\hat{\pi}, \hat{\sigma}) = m^2|h|^2 + \lambda|h|^4 \]
\[ = \frac{m^2}{2} [\hat{\sigma}^2 + \hat{\pi}_a^2] + \frac{\lambda}{4} [\hat{\sigma}^2 + \hat{\pi}_a^2]^2 = \frac{m^2}{2} \text{tr} (MM^\dagger) + \frac{\lambda}{4} \left( \text{tr} (MM^\dagger) \right)^2 \]
\[ = \frac{m^2}{2} \sigma^2 + \frac{\lambda}{4} \sigma^4 = V(\sigma), \]
with \( \sigma(x) \equiv \sqrt{\hat{\pi}^2(x) + \hat{\pi}_a^2(x)} \), has a minimum for \( m^2 < 0 \) at the chiral-invariant circle:
\[ \langle \sigma(x) \rangle = \sqrt{-\frac{m^2}{\lambda}} \equiv v = 246 \text{ GeV}. \]

On this vacuum the complex matrix \( M \) can be decomposed into a positive Hermitian (diagonalizable) matrix \( H \) and a unitary matrix \( U \) as \( M = H U \) ("polar decomposition") [9]:
\[ M(x) = H(x) \cdot U(x), \quad H(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} \sigma(x) & 0 \\ 0 & \sigma(x) \end{array} \right), \quad U(x) = \exp \left( i \frac{2\pi(x)}{F_\pi} \right), \quad F_\pi = v = \langle \sigma(x) \rangle, \]
with \( \pi(x) \equiv \pi^a(x) \tau^a / 2 (a = 1, 2, 3) \) and the \( \pi \) decay constant \( F_\pi \). The chiral transformation of \( M \) is carried by \( U \), while \( H \) is a chiral singlet such that:
\[ U \rightarrow g_L U g_R^+, \quad H \rightarrow H, \]
where \( g_{L/R} \in SU(2)_{L/R} \). Note that the physical Higgs is the radial mode \( \sigma \) which is a chiral-singlet (electroweak gauge singlet when electroweak coupling switched on), while \( \hat{\sigma} \) is a chiral non-singlet (electroweak gauge non-singlet) transforming to the chiral partner \( \hat{\pi} \) by the chiral rotation, both being tachyons with mass \( m^2 = m^2 < 0 \), in contrast to the physical modes \( \sigma \) and \( \pi \) (angular/phase modes or gauge parameters totally absorbed into \( W/Z \) in the unitary gauge). In fact \( \sigma \) is a chiral singlet and thus \( \langle \sigma(x) \rangle = v \neq 0 \) breaks spontaneously the scale symmetry, but not the chiral symmetry which is actually spontaneously broken by \( \langle U(x) \rangle = 1 \neq 0 \left( \langle \pi(x) \rangle = 0 \right) \).

We thus may parametrize \( \sigma(x) \) as the nonlinear base of the scale transformation:
\[ \sigma(x) = v \cdot \chi(\varphi), \quad \chi(\varphi) = \exp \left( \frac{\varphi(x)}{F_\varphi} \right), \quad F_\varphi = v, \]
such that \( \langle \chi(\varphi) \rangle = 1 \neq 0 \left( \langle \varphi(x) \rangle = 0 \right) \): Now the physical Higgs is \( \varphi(x) \) which is a dilaton, NG boson of the spontaneously broken scale symmetry, with the decay constant \( F_\varphi = v \#^9 \). Eq. 43 is then straightforwardly rewritten

---

#^9 The scale (dilatation) transformations for these fields are
\[ \delta_D \sigma = (1 + x^\mu \partial_\mu) \sigma, \quad \delta_D \chi = (1 + x^\mu \partial_\mu) \chi, \quad \delta_D \varphi = v + x^\mu \partial_\mu \varphi. \]
Although \( \chi \) is a dimensionless field, it transforms as that of dimension 1, while \( \varphi \) having dimension 1 transforms as the dimension 0, instead.
The opposite limit, $\lambda \rightarrow \infty$ with $v$ fixed, leads to the ordinary nonlinear sigma model, where we also have $V(\varphi) \rightarrow 0$ but with $\chi(x) \equiv 1$, so that the scale symmetry compensated by $\chi^2$ factor in Eq. (12) is completely lost. See Refs. [2, 3]. Either limit has no $\lambda$ coupling, but has derivative couplings instead, which are “weak” in the low energy $p^2/(4\pi v)^2 \ll 1$, so that the perturbation according to the derivative expansion ("chiral perturbation theory") makes sense.

Even if we take such a conformal/BPS limit, the theory is still an interacting theory with derivative coupling as in the usual chiral Lagrangian, and thus the quantum corrections will produce the trace anomaly of dimension 4, $\sim v^4 \chi^4 \ln \chi$, as a new source of the SM Higgs mass as a pseudo-dilaton, which, however, would do not affect the dynamical generation of the HLS gauge boson discussed here, similarly to the tiny explicit scale-symmetry breaking in the tree-level potential $V(\varphi)$ with $\lambda \approx 1/8 \ll 1$. 

\[ L_{SM} = \frac{1}{2} (\partial_{\mu} \sigma)^2 + \frac{\sigma^2}{4} \text{tr} (\partial_{\mu} U \partial^{\mu} U^\dagger) - V(\sigma) \]
\[ = \chi^2(\varphi) \cdot \left[ \frac{1}{2} (\partial_{\mu} \varphi)^2 + \frac{v^2}{4} \text{tr} (\partial_{\mu} U \partial^{\mu} U^\dagger) \right] - V(\varphi), \quad (12) \]
\[ V(\varphi) = V(\sigma) = \frac{\lambda}{4} v^4 \left[ (\chi^2(\varphi) - 1)^2 - 1 \right]. \quad (13) \]

Thus the SM Higgs Lagrangian Eq. (3) is trivially identical to Eq. (12). Note that the kinetic term of the latter, $\chi^2(\varphi) \cdot \left[ \frac{1}{2} (\partial_{\mu} \varphi)^2 + \frac{v^2}{4} \text{tr} (\partial_{\mu} U \partial^{\mu} U^\dagger) \right]$, contains the usual nonlinear sigma model $\frac{v^2}{4} \text{tr} (\partial_{\mu} U \partial^{\mu} U^\dagger)$ which transforms as dimension 2 to make the action not scale-invariant. However, the extra dilaton factor $\chi^2(\varphi) = e^{2\varphi(x)/v}$, transforming as dimension 2, makes the whole kinetic term to be dimension 4. Hence the action becomes scale-invariant as it should, since it is just a rewriting of the original kinetic term in Eq. (3) which is scale-invariant (dimension 4).

Actually the kinetic term coincides with the scale-invariant nonlinear chiral Lagrangian based on the coset $G/H = SU(2)_L \times SU(2)_R/SU(2)_L + R$, with the scale symmetry as well as the chiral symmetry being realized nonlinearly. The scale $v$ is in fact a measure of the spontaneous breaking but not the explicit breaking of the scale symmetry. Thus the SM Higgs $\varphi$ is nothing but a pseudo dilaton, with the explicit breaking of the scale symmetry from the potential term $V(\varphi)$ characterized by the dimensionless parameter:

\[ \lambda = \frac{M_\varphi^2}{2v^2} \approx 125 \text{ GeV}^2 / 2 \times (246 \text{ GeV})^2 \approx 1/8 \ll 1, \quad (14) \]

which is very close to the “conformal limit”,

\[ \lambda \rightarrow 0 \text{ with } v = \text{fixed}, \]

where $V(\varphi) \propto \lambda v^4 \rightarrow 0$ for any $\chi(x) = e^{\varphi(x)/v} \#^{10}$. The limit actually corresponds to the “Bogomol’nyi-Prasad-Sommerfield (BPS) limit” of ’t Hooft-Polyakov monopole in the Georgi-Glashow model [11], similarly to the SUSY flat direction #^{11}.

By the electro-weak gauging as usual: $\partial_{\mu} U = D_{\mu} U = \partial_{\mu} U - ig_2 W_{\mu} U + ig_1 U B_{\mu}$ in Eq. (12), we see the followings:

first, the physical Higgs field $\varphi$ as a pseudo-dilaton is a gauge singlet and hence manifestly gauge invariant object, in sharp contrast to the conventional shifting $h_0 = h_0' + v/\sqrt{2}$ or $\hat{\sigma} = \hat{\sigma}' + v$, where $h_0'$ or $\hat{\sigma}'$ is gauge variant. Second, the mass term of $W/Z$ is scale-invariant thanks to the dilaton factor $\chi$, and so is the mass term of the SM fermions $f$: $g_Y \tilde{f} f = (g_Y v/\sqrt{2})(\chi \tilde{f} f)$, all with the scale dimension 4. This implies that the couplings of the SM Higgs as a pseudo dilaton to all the SM particles are written in the scale-invariant form and thus obey the low energy theorem of the scale symmetry in perfect agreement with the experiments.

The low energy theorem for the pseudo dilaton $\varphi(q_\mu)$ coupling to the canonical matter filed $X$ at $q_\mu \rightarrow 0$ reads:

\[ g_{\varphi X^\dagger X} = \frac{2 M_X^2}{F_\varphi}, \quad g_{\varphi \bar{X} X} = \frac{M_X}{F_\varphi} \quad (F_\varphi = v), \quad (16) \]

for complex scalar and spin 1/2 fermion, respectively [40], which can also be read from the scale invariance of the mass term;

\[ M_X^2 \cdot \chi^2 X^\dagger X = M_X^2 X^\dagger X + \frac{2M_X^2}{v} \varphi X^\dagger X + \cdots, \]
\[ M_X \cdot \chi \bar{X} X = M_X \bar{X} X + \frac{M_X}{v} \varphi \bar{X} X + \cdots, \quad (17) \]
for the respective canonical field with the canonical dimension. #12

B. $G/H = O(4)/O(3)$ Parameterization

Since the nonlinear sigma model on the manifold $G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V$ is equivalent to another model $G/H = O(4)/O(3)$, we here present an explicit form of the scale-invariant form of the latter, based on the generic CCWZ formalism reviewed in Appendix A.1, where that for $G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V$ is given, see Eq. (A12). The result form is equivalent to Eq. (12) and hence to the SM Higgs Lagrangian Eq. (3). This is further regarded as an extrapolation $N \rightarrow 4$ with $p = 3$ of the Grassmannian manifold $G/H = O(N)/[O(N - p) \times O(p)]$ ($p = 3$) as will be discussed in the next section.

Here we take a vector representation of $O(4)$ with different normalization $\text{tr}(T_AT_B) = 2\delta_{AB}$, $T_A^t = -T_A$, with $\pi = \pi_a X_a$, $\text{tr}(\pi^2) = 2\pi_a^2$, we have

$$
\xi^{(4)}(\pi)_{\alpha\beta} = \left( \phi(\pi)_{i\beta} \Phi(\pi)_{4\beta} \right) = \exp \left( \frac{\pi_a \cdot X_a}{F_\pi} \right) = \exp \left[ \frac{1}{F_\pi} \begin{pmatrix} 0 & \pi_1 \\ -\pi_1 & -\pi_2 - \pi_3 \end{pmatrix} \right],
$$

$$
\left[ \xi^{(4)}(\pi) \right]^t \cdot \xi^{(4)}(\pi) = \phi^t(\pi) \phi(\pi) + \Phi^t(\pi) \Phi(\pi) = 1 = \xi^{(4)}(\pi) \cdot \left[ \xi^{(4)}(\pi) \right]^t,
$$

$$
\alpha, \beta = i, 4; \quad i = 1, 2, 3,
$$

(18)

which transforms under $G$ as $\xi^{(4)}(\pi) \rightarrow h(g, \pi) \cdot \xi^{(4)}(\pi) \cdot g^t$ ($g \in G$).

The Maurer-Cartan one-form reads:

$$
\alpha^{(4)}_{\mu} = \frac{1}{i} \partial_\mu \xi^{(4)}(\pi) \cdot \left( \xi^{(4)}(\pi) \right)^t = \frac{1}{i} \left( \left[ \frac{i}{F_\pi} \partial_\mu \pi + \frac{1}{2!} \left( \frac{i}{F_\pi} \right)^2 \partial_\mu [\pi, \partial_\mu \pi] + \frac{1}{3!} \left( \frac{i}{F_\pi} \right)^3 [\pi, [\pi, \partial_\mu \pi]] + \cdots \right] \right),
$$

$$
\alpha^{(4)}_{\mu \perp} = \frac{1}{2} \text{tr} \left( \alpha^{(4)}_{\mu}(\pi) X_a \right) \cdot X_a = \frac{1}{i} \left( \partial_\mu \Phi(\pi) \cdot \phi^t(\pi) 0_{3 \times 3} \right) \partial_\mu \phi(\pi) \cdot \Phi(\pi) \cdot \Phi(\pi) = \frac{1}{2F^2_\pi} \partial_\mu \pi + \cdots,
$$

$$
\alpha^{(4)}_{\mu ||} = \frac{1}{2} \text{tr} \left( \alpha^{(4)}_{\mu}(\pi) S_a \right) \cdot S_a = \frac{1}{i} \left( \partial_\mu \phi(\pi) \cdot \phi^t(\pi) 0_{1 \times 3} \right) \partial_\mu \Phi(\pi) \cdot \Phi(\pi) = \frac{i}{2F^2_\pi} [\pi, \partial_\mu \pi] + \cdots,
$$

and the CCWZ Lagrangian reads:

$$
\mathcal{L}_{\text{CCWZ}} = \frac{F^2_\pi}{4} \text{tr} \left( \left( \alpha^{(4)}_{\mu \perp}(\pi) \right)^2 \right) = -\frac{F^2_\pi}{2} \text{tr} \left( \partial_\mu \phi(\pi) \Phi^t(\pi) \partial^\mu \Phi(\pi) \phi^t(\pi) \right) + \frac{F^2_\pi}{2} \text{tr} \left( \partial_\mu \phi(\pi) \cdot \Phi^t(\pi) \Phi(\pi) \cdot \partial^\mu \phi^t(\pi) \right) + \frac{F^2_\pi}{2} \text{tr} \left( \partial_\mu \phi(\pi) \cdot \partial^\mu \phi^t(\pi) + \phi(\pi) \partial_\mu \phi^t(\pi) \cdot \phi(\pi) \partial^\mu \phi(\pi) \right) + \frac{1}{2} \left( \partial_\mu \pi_a \right)^2 + \cdots,
$$

(20)

where we have used $\Phi \phi^t = 0$ and $\Phi^t \Phi = 1 - \phi \phi^t$.

Then the SM Lagrangian Eq. (12) with $F_\pi = v$ is further rewritten in the form of the CCWZ Lagrangian similarly to Eq. (A11) as:

$$
\mathcal{L}_{\text{SM}} = \lambda^2(\varphi) \cdot \left[ \frac{1}{2} \left( \partial_\mu \varphi \right)^2 + \frac{v^2}{4} \text{tr} \left( \alpha^{(4)}_{\mu \perp}(\pi) \right)^2 \right] - V(\varphi).
$$

(21)

#12 For the general form of the low energy theorem of the scale symmetry including the anomalous dimension such as in the walking technicolor with large anomalous $\gamma_m = 1$ [17], see Ref. [18, 47].
It is now straightforward to introduce the HLS in the SM written in the form of the nonlinear realization $G/H$ besides the nonlinear realization of the scale symmetry. In the Appendix A2 the generic HLS formalism \[8, 9\] for the CCWZ representation $G/H$ is reviewed and the explicit (scale-invariant) HLS form of the SM is given for $G_{\text{global}} \times H_{\text{local}} = [SU(2)_L \times SU(2)_R]_{\text{global}} \times [SU(2)_Y]_{\text{local}}$, and in particular for $G_{\text{global}} \times H_{\text{local}} = O(4)_{\text{global}} \times O(3)_{\text{local}}$ in Eq. (A2) which is the base for the Grassmannian $N-$extension to be discussed in the followings. For reader’s convenience the physical implications of the standard HLS formalism (with the HLS kinetic term put by hand) are also reviewed in Appendix A3 which are for comparison with the genuine dynamical generation of the HLS kinetic term in the large $N$ limit to be given in the present paper.

III. GRASSMANNIAN $N$-EXTENSION OF THE SM

In order to discuss the dynamical generation of the HLS gauge bosons by the nonperturbative dynamics within the SM in terms of the $1/N$ expansion, in this section\#13 we here write down the scale-invariant HLS model with $G_{\text{global}} \times H_{\text{local}} = O(N)_{\text{global}} \times [O(N-p) \times O(p)]_{\text{local}}$. This is gauge equivalent to a scale-invariant version of the Grassmannian model on the manifold $G/H = O(N)/[O(N-p) \times O(p)]$, according to the generic HLS formalism \[8, 9\] (see Appendix A2). By taking $N = 4, p = 3$ the model is reduced into the scale-invariant HLS Lagrangian of the $G_{\text{global}} \times H_{\text{local}} = O(4)_{\text{global}} \times O(3)_{\text{local}}$, Eq. (A3), which is gauge equivalent to that of the $G/H = O(4)/O(3)$ model, Eq. (B1), and thus is equivalent to the SM Higgs Lagrangian Eq. (12) and Eq. (B)\#14.

Following the generic HLS formalism in Appendix A2 let us define the HLS version of the $N-$extension of the CCWZ base, an $N \times N$ real matrix field $\xi(x)$ which transforms under $G_{\text{global}} \times H_{\text{local}}$ as $\xi(x) \rightarrow h(x) \cdot \xi(x) \cdot g^{-1}$ with $h(x) \in H_{\text{local}}, g \in G_{\text{global}}$,

$$
\xi(x)_{\alpha\beta} = \left[ \exp \left( \frac{1}{2} \theta_{\gamma \delta} T_{\gamma \delta} \right) \right]_{\alpha\beta} = \exp \left[ \left( \begin{array}{c} (\theta_{ij})_{p \times p} \\ (\theta_{ik})_{N-p \times N-p} \\ (\theta_{jk})_{N-p \times N-p} \end{array} \right) \right] \cdot \exp \left( \frac{1}{F_p} \beta \cdot S_a \cdot \frac{\pi(x)}{F_{\pi}} \right) \cdot \exp \left( \frac{1}{F_p} \beta \cdot S_a \cdot \frac{\pi(x)}{F_{\pi}} \right)
$$

Following the generic HLS formalism in Appendix A2 let us define the HLS version of the $N-$extension of the CCWZ base, an $N \times N$ real matrix field $\xi(x)$ which transforms under $G_{\text{global}} \times H_{\text{local}}$ as $\xi(x) \rightarrow h(x) \cdot \xi(x) \cdot g^{-1}$ with $h(x) \in H_{\text{local}}, g \in G_{\text{global}}$,

$$
\xi(x)_{\alpha\beta} = \left[ \exp \left( \frac{1}{2} \theta_{\gamma \delta} T_{\gamma \delta} \right) \right]_{\alpha\beta} = \exp \left[ \left( \begin{array}{c} (\theta_{ij})_{p \times p} \\ (\theta_{ik})_{N-p \times N-p} \\ (\theta_{jk})_{N-p \times N-p} \end{array} \right) \right] \cdot \exp \left( \frac{1}{F_p} \beta \cdot S_a \cdot \frac{\pi(x)}{F_{\pi}} \right) \cdot \exp \left( \frac{1}{F_p} \beta \cdot S_a \cdot \frac{\pi(x)}{F_{\pi}} \right)
$$

$$
T_{\alpha\beta} = -T_{\beta\alpha}, T_{\gamma\delta}, \theta_{\alpha\beta} = -i (\delta_{\alpha\gamma} \theta_{\beta\delta} - \delta_{\alpha\delta} \theta_{\beta\gamma}) , \quad (\alpha, \beta = 1, 2, \cdots N), \quad \text{tr} (T_A T_B) = 2 \delta_{AB},
$$

$$
\theta_{\alpha\beta} = -\theta_{\beta\alpha}, \theta_{ij} = (\theta \cdot p) / F_p, \theta_{ik} = \rho_{ik} / F_p, \theta_{jk} = \rho_{jk} / F_p, \theta_{ik} = \pi_{ik} / F_{\pi}, \quad (i, j = 1 \cdots p, k, l = p + 1 \cdots N),
$$

$$
\xi' (x) \cdot \xi (x) = \phi(x) \phi(x) + \Phi(x) \Phi(x) = 1,
$$

$$
\xi (x) \cdot \xi' (x) = \left( \begin{array}{c} 1_{p \times p} \\ 0 \end{array} \right) \cdot \left( \begin{array}{c} (N-p)_{N \times N} \end{array} \right),
$$

\(\#13\) This section largely depends on the explicit Grassmannian coset parameterization given by Taichiro Kugo (private communication). We thank him for his generous offer.

\(\#14\) Extension to $G/H = [SU(N) \times SU(N)]/SU(N)$ and scale symmetry (not $[SU(N) \times SU(N)]$ linear sigma model!) is precisely the same form as Eq. (A2) with the $N \times N$ matrix

$$
U(x) = \exp \left( \frac{2i \pi(x)}{F_{\pi}}, \quad \pi(x) = \pi_a(x) T^a, \quad \chi(x) = \exp \left( \frac{\varphi}{F_{\varphi}}, \quad F_{\varphi}/\sqrt{N/2} = F_a = \nu, \quad (T^a)^2 = \frac{1}{2} \delta_{ab} T^a \right) \right)
$$

$T^a$ being the generator of $SU(N)$. In contrast, the $SU(N)_L \times SU(N)_R$ linear sigma model has two independent quartic couplings, \(\text{tr}(M M')^2\) and \(\text{tr}(M M')^2\), and $M$ has $N^2 - 1$ scalars in addition to $\sigma$. However this extension is not suitable for the study of the nonperturbative dynamics based on the large $N$ limit, because all planar graphs of the induced HLS gauge bosons loops are equally on the leading order of $1/N$ expansion. On the other hand, a popular $N$ extension of the $O(4)/O(3)$ model for the large $N$ limit is $G/H = O(N)/O(N-1)$. One might consider a gauge equivalent HLS model $O(N)_{\text{global}} \times O(N - 1)_{\text{local}}$, which however does not give the dynamical gauge boson of $O(N - 1)$, since again all planar graphs do contribute at the leading order.
The degrees of freedom of $\phi$ is $N \times p$ which is divided into $(N-p) \times p$ (NG bosons $\pi$) plus $p \times (p-1)/2$ (would-be NG bosons $\rho_\alpha^{(p)}$), plus $p \times (p+1)/2$ modes (i.e., constraints $\phi \phi^t = \mathbb{1}_{p \times p}$ corresponding to “massive” scalars in the broken phase, while the diagonal component out of them is a pseudo-dilaton in the unbroken phase),

$$N \times p \big|_\phi = (N-p) \times p \big|_\pi + \frac{p \times (p-1)}{2} \big|_\rho + \frac{p \times (p+1)}{2} \big|_{\text{constraint}},$$

and similarly for $\Phi$.

In the case of $N = 4, p = 3$ which corresponds to the SM, there exists an HLS only for the $O(p)_{\text{local}}$, since $O(N-p)$ does not exist. Moreover, as we shall see later, dynamical generation of the $O(N-p)_{\text{local}}$ does not take place in the large $N$ limit. Needless to say, when the gauge fixed $\delta^{(p)}_\rho = \rho_\mu^{(N-p)} = 0$, Eq. (23) is reduced to the CCWZ base in the original $O(N)/O(p) \times O(N-p)$ model, which for $N = 4, p = 3$ is nothing but the $O(4)/O(3)$ parameterization Eq. (18) in the SM Lagrangian Eq. (21).

The Maurer-Cartan one-form reads:

$$\alpha_\mu(x) = \frac{1}{i} \partial_\mu \xi(x) \cdot \xi^t(x) = \alpha_{\mu,\perp}(x) + \alpha_{\mu,\parallel}(x),$$

and its covariantized one:

$$\hat{\alpha}_\mu(x) = \frac{1}{i} D_\mu \xi(x) \cdot \xi^t(x) = \frac{1}{i} \left( \begin{array}{c} \alpha_{\mu,\perp}(x) \\ \alpha_{\mu,\parallel}(x) \end{array} \right),$$

where $\rho_\mu^{(p)}$ and $\rho_\mu^{(N-p)}$ are the HLS gauge bosons of $O(p)_{\text{local}}$ and $O(N-p)_{\text{local}}$, respectively. $\hat{\alpha}_{\mu,\perp}(x)$ and $\hat{\alpha}_{\mu,\parallel}(x)$ transform as

$$\hat{\alpha}_{\mu,\perp}(x) \rightarrow h(x) \cdot \hat{\alpha}_{\mu}(x) \cdot h^{-1}(x),$$

$$\hat{\alpha}_{\mu,\parallel}(x) \rightarrow h(x) \cdot \hat{\alpha}_{\mu}(x) \cdot h^{-1}(x),$$

where $h(x) \in O(p)_{\text{local}}, h(x) \in O(N-p)_{\text{local}}$.

Thus we arrive at an s-HLS model with $G_{\text{global}} \times H_{\text{local}} = O(N)_{\text{global}} \times [O(N-p) \times O(p)]_{\text{local}}$, as an $N-$ extension of the SM with HLS (see Eq. (A22)), which consists of three independent invariants at the lowest derivative

$$\mathcal{L}_{\text{SM-HLS}}^{(N,p)} = \chi^2(\varphi) \cdot \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + \mathcal{L}_A + a^{(p)} \mathcal{L}_V^{(p)} + a^{(N-p)} \mathcal{L}_V^{(N-p)} \right] - V(\varphi),$$

where

$$\mathcal{L}_A = \frac{F_p^2}{4} \text{tr} \left( \hat{\alpha}_{\mu,\perp}^2(x) \right) = \frac{F_p^2}{4} \text{tr} \left( \alpha_{\mu,\perp}^2(x) \right) = - \frac{F_p^2}{4} \text{tr} \left( \begin{array}{c} \partial_\mu \phi \cdot \phi \partial_\mu \Phi^t \phi^t \\ 0 \\ 0 \end{array} \right),$$

$$= \frac{F_p^2}{2} \text{tr} (\phi^t \partial_\mu \Phi \phi \partial_\mu \Phi^t) = \frac{F_p^2}{2} \text{tr} (\partial_\mu \phi \partial_\mu \phi + \phi \partial_\mu \phi)^2 = \frac{F_p^2}{2} \text{tr} (\partial_\mu \phi \partial_\mu \Phi^t + (\Phi \partial_\mu \Phi^t)^2),$$

$$= \frac{F_p^2}{4} \text{tr} \left( \alpha_{\mu,\parallel}^2(x) \right) = \frac{F_p^2}{4} \left( \frac{1}{F_p} \partial_\mu \pi + \cdots \right)^2 = \frac{1}{2} (\partial_\mu \pi a)^2 + \cdots,$$

(see Eq. (24) for the second line), and

$$a^{(p)} \mathcal{L}_V^{(p)} = \frac{(F_p^p)^2}{4} \text{tr} \left( \hat{\alpha}_{\mu,\parallel}^2(x) \right) = \frac{(F_p^p)^2}{4} \text{tr} \left[ \left( \frac{1}{i} \partial_\mu \phi \cdot \phi^t - \rho_\mu^{(p)} \right)^2 \right]$$

$$= \frac{(F_p^p)^2}{4} \text{tr} \left[ \left( \rho_\mu^{(p)} - \frac{1}{2F_p^p} \partial_\mu \rho^{(p)} \right) - \frac{i}{2F_p^p} \partial_\mu \rho^{(p)} - \frac{i}{2F_p^p} \partial_\mu \rho^{(p)} \right],$$

(31)
\[ a^{(N-p)} \mathcal{L}_{V}^{(N-p)} = \left( \frac{F_{p}^{(N-p)}}{4} \right)^{2} \text{tr}_{N-p \times N-p} \left( \left( \partial_{\mu} \phi \right) \right)^{2} = \left( \frac{F_{p}^{(N-p)}}{4} \right)^{2} \text{tr}_{N-p \times N-p} \left( \left( \frac{i}{\lambda} \partial_{\mu} \Phi \cdot \Phi^{t} - \rho^{(N-p)}_{\mu} \right) \right)^{2} \]

\[ = \left( \frac{F_{p}^{(N-p)}}{4} \right)^{2} \times \text{tr}_{N-p \times N-p} \left[ \left( \rho^{(N-p)}_{\mu} - \frac{\partial_{\mu} \rho^{(N-p)}}{F_{p}^{(N-p)}} \right) \right] - i \left[ \partial_{\mu} \rho^{(N-p)}, \rho^{(N-p)} \right] - i \frac{2 F_{p}^{(N-p)}}{2 F_{p}^{(N-p)}} \left[ \partial_{\mu} \pi, \pi \right] \right]^{2} \quad . \]  

(32)

Here we have two arbitrary parameters \( a^{(p)} \) and \( a^{(N-p)} \) instead of a single one, \( a \), in the standard HLS model (Eq. (A22)), with \( \left[ F_{p}^{(p)} \right]^{2} = a^{(p)} F_{p}^{2} = a^{(p)} v^{2} \), \( \left[ F_{p}^{(N-p)} \right]^{2} = a^{(N-p)} F_{p}^{2} = a^{(N-p)} v^{2} \), as the normalization of the kinetic term of \( \rho^{(p)} \) and \( \rho^{(N-p)} \), respectively. Note that the third line of Eq. (30) is a gauge-fixed form (unitary gauge \( \xi(x) = \xi(\pi) \), or \( \rho^{(N-p)} = 0 \)) as a consequence of the parameterization of the second line of Eq. (23) where \( \xi(\rho^{(p)}) \) and \( \xi(\rho^{(N-p)}) \) are automatically dropped in the trace, as noted for the generic case in Appendix A 2. At the classical level without the kinetic term of the HLS gauge bosons, \( \rho^{(p)} \) and \( \rho^{(N-p)} \), we can solve away these auxiliary fields through the respective equation of motion, which give \( \mathcal{L}_{V}^{(N-p)} = \mathcal{L}_{V}^{(p)} = 0 \), and we are left with \( \mathcal{L}_{A} \) which is identical to the genuine nonlinear sigma model based on \( G/H = O(N) \), \( O(N - p) \times O(p) \) as an \( N \)-extension of \( G/H = O(N) \) in Eq. (A22). When we take \( N = 4 \) with \( p = 3 \), the model is reduced to that of \( O(4) \) global \( \times O(3) \) local having a single parameter \( a = a^{(p=3)} \) for \( \mathcal{L}_{V} = \mathcal{L}_{V}^{(p=3)} \) and without \( \mathcal{L}_{(N-p)} \) term, which is identical to the standard s-HLS model (see Eq. (A22)). Thus the model is gauge equivalent to the scale-invariant version of the \( O(4) \) local model Eq. (21) and hence to the SM itself in the form of Eq. (12) and eventually to the standard SM Higgs Lagrangian Eq. (3):

\[ \lim_{N \rightarrow 4} \mathcal{L}_{SM-HLS}^{(N=4, p=3)} = \mathcal{L}_{SM-HLS} \simeq \mathcal{L}_{SM} \cdot \quad (33) \]

The model Eq. (29) is our starting Lagrangian.

IV. PHASE STRUCTURE AND PHASE TRANSITION

Now we expect nonperturbative dynamics of SM can be realized in the large \( N \) limit of our Lagrangian Eq. (29). We disregard the part \( a^{(N-p)} \mathcal{L}_{(N-p)} \), since it does not give rise to the dynamical generation of the kinetic term of \( \rho^{(N-p)} \) in the large \( N \) limit: Namely, in order to generate its kinetic term, all the planar graphs having the index \( \alpha \) running to \( 1 - N \) are involved, which not controllable even in that limit. The situation is the same as that in the \( CP^{N-1} = U(N)/[U(N-1) \times U(1)] \) model where only the \( U(1) \) local part among the whole HLS \( [U(N-1) \times U(1)] \) can generate the dynamical gauge boson in the large \( N \) limit (see Appendix B).

A. Covariant derivative parameterization and multiplier

The most convenient parametrization to study the large \( N \) dynamics is the \( p \times N \) matrix \( \phi \) in Eq. (33). To calculate the effective action in the large \( N \) limit we use rescaling the quantities and new abbreviations in notation:

\[ \phi \rightarrow F_{\pi} \phi, \quad \alpha^{(p)}_{\mu,||} = \alpha_{\mu,||} = \frac{1}{i F_{\pi}} \partial_{\mu} \phi \phi^{t} = \frac{G}{N} \phi \partial_{\mu} \phi^{t} , \quad \rho^{(p)}_{\mu} \rightarrow \rho_{\mu} , \quad \text{s.t.} \quad \phi \phi^{t} = F_{\pi}^{2} \cdot 1 \equiv \frac{N}{G} \cdot 1 , \]

(34)

in terms of which the Lagrangian consisting of the terms in Eq. (30) and (31) takes the form:

\[ \mathcal{L} = \mathcal{L}_{A} + a \mathcal{L}_{V} , \]

\[ \mathcal{L}_{A} = \frac{1}{2} \text{tr}_{p \times p} \left( \partial_{\mu} \phi \partial^{\mu} \phi^{t} + \frac{1}{F_{\pi}^{2}} \left( \phi \partial_{\mu} \phi^{t} \right)^{2} \right) = \frac{1}{2} \text{tr}_{p \times p} \left( \partial_{\mu} \phi \partial^{\mu} \phi^{t} + \frac{G}{N} \left( \phi \partial_{\mu} \phi^{t} \right)^{2} \right) , \]

\[ a \mathcal{L}_{V} = \frac{F_{\pi}^{2}}{4} \text{tr}_{p \times p} \left( \rho_{\mu} - \alpha_{\mu,||} \right)^{2} = \frac{F_{\pi}^{2}}{4} \text{tr}_{p \times p} \left( \rho_{\mu} - \frac{G}{N} \phi \partial_{\mu} \phi^{t} \right)^{2} = \frac{a}{2} \cdot \frac{N}{G} \frac{1}{2} \text{tr}_{p \times p} \left( \rho_{\mu} - \frac{G}{N} \phi \partial_{\mu} \phi^{t} \right)^{2} . \quad (35) \]
This suggests that $G = N/F_2^2$ may be regarded as the coupling and the large $N$ limit is taken as $N \to \infty$ with $G = N/F_2^2$ fixed (See Appendix B and C for a similar definition of the coupling in $CP^{N-1}$ model, and Ref. [10] for the chiral Lagrangian). Then our $N-$ extension of the SM takes the form

$$L_{\text{SM-HLS}}^{(N,p)} = \chi^2(\phi) \cdot \left[ \frac{1}{2} (\partial_\mu \phi)^2 + L_A + aL_V \right] - V(\phi),$$

which appears to depend on the arbitrary parameter $a$.

However, as far as $\rho_\mu$ is the auxiliary field without kinetic term, we may use the equation of motion $\rho_\mu = iG\phi \partial_\mu \phi^t$ (or simply add $aL_V(\equiv 0)$ with arbitrary weight $a$) and always rewrite the Lagrangian independently of the parameter $a$ as: #15:

$$L_A + aL_V = \frac{1}{2} \text{tr}_{p \times p} \left( D_\mu \phi \cdot (D^\mu \phi)^t \right), \quad D_\mu \phi = (\partial_\mu - i\rho_\mu) \phi, \quad \rho_\mu^t = -\rho_\mu. \quad (37)$$

This simply reflects the trivial fact that the classical theory without kinetic term of the $\rho_\mu$ is independent of $a$, although the quantum theory does in general depend on $a$ as we shall discuss later.

**B. Gap equation in the large $N$ limit**

We now consider the large $N$ limit nonperturbative dynamics of the $N$-extension of the SM as:

$$L_{\text{SM-HLS}}^{(N,p)} = \chi^2(\phi) \cdot \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \text{tr}_{p \times p} \left( D_\mu \phi \cdot (D^\mu \phi)^t \right) \right] - V(\phi). \quad (38)$$

As we repeatedly mentioned, for $N = 4, p = 3$ this is equivalent to Eq.[A22] and hence gauge equivalent to the SM Lagrangian Eq.[3] in the form of Eq.[12].

The Lagrangian can be further rewritten in terms of the rescaled fields:

$$\phi(x) = \chi(\phi) \cdot \phi(x), \quad \sigma(x) = \frac{1}{G^{1/2}} \cdot \chi(\phi) = \frac{1}{\sqrt{N}} \sigma(x), \quad (39)$$

having a canonical scale dimension $d_\phi = d_\sigma = 1$ for $D = 4$ ($d_\phi = d_\sigma = D/2 - 1$ for $2 \leq D \leq 4$ dimensions):

$$L_{\text{SM-HLS}}^{(N,p)} = \frac{1}{2} \text{tr}_{p \times p} \left[ D_\mu \phi \cdot (D^\mu \phi)^t - \eta(x) \left( \phi \phi^t - N\sigma^2 \mathbb{1} \right) \right] - V(\sigma), \quad \eta(x) = 1, \quad \sigma(x) = \frac{1}{\sqrt{N}} \sigma(x), \quad (40)$$

with $\lambda = N\lambda = \text{constant } (\text{"t Hooft coupling})$ in the $N \to \infty$ limit as usual. We have used Lagrange multiplier $p \times p$ matrix $\eta(x)$ (with the scale dimension $d_\eta = 2$) instead of the constraint $\phi \phi^t = F^2 \mathbb{1}$ ($\text{dimensionful field } \phi$ has the scale dimension $d_\phi = 0$), and also rescaled the dilaton decay constant $F_\phi = v \to p^{1/2} v$ such that $\chi(\phi) = e^{\phi/v} \to e^{\phi/(p^{1/2} v)}$. Note that Eq.[39] is equivalent to Eq.[38] which is scale-invariant except for $V(\varphi)$.

Indeed the constraint originates from the potential term $V(\varphi)$ itself, which breaks spontaneously the internal symmetry and the scale symmetry by the very existence of the explicit scale breaking term (SM Higgs mass $m^2$ in Eq.[3]). Thus once we imposed the constraint (or, Lagrange multiplier), the role of the potential $V(\varphi)$ is only limited, irrelevant to the phase structure (determined by the derivative couplings even for $V(\varphi) = 0$), thereby giving only the perturbative self interactions of $\phi$ itself, if any. Indeed, in the form of Eq.[33] the conformal limit (BPS limit), Eq.[31], i.e., $\lambda \to 0$ ($V(\varphi) \to 0$) with $\langle \sigma(x) \rangle (\neq 0)$ fixed, can be realized automatically by simply taking

---

#15 It is amusing that for a special value $a = 2$, Eq.[37] still remains true as a simple identity, even without using the equation of motion for $\rho_\mu$, namely the term $G^2 (\phi \partial_\mu \phi)^2$ gets automatically cancelled between those from $L_A$ and $aL_V = 2L_V$ to yields the form Eq.[37], even when the $\rho_\mu$ acquires the kinetic term. This is also the case in the auxiliary field formulation of the NJL model, see Appendix C. We shall return to this point later, Section V4.
The effective action at leading order of $1/N$ Lagrangian Eq. (40) should be regarded as the bare quantity and receives quantum corrections in the large $N$ limit. The effective action at leading order of $1/N$ expansion reads:

$$
\Gamma_{\text{eff}}[\phi, \eta, \sigma, \rho_\mu] = \int d^D x \frac{1}{2} \text{tr}_{\mu \nu} [D_\mu \phi(D^{\mu} \phi) - \eta(x) (\phi \phi^\dagger - N \sigma^2 1)] - V(\sigma)
+ \frac{i}{2} N \text{TrLn} (-D_\mu D^\mu - \eta) , \quad (2 \leq D \leq 4)
,$$

where in $D$ dimensions $\phi(x)$ and $\sigma(x)$ and $\eta(x)$ have a canonical dimension $d_\phi/\sigma = D/2 - 1$, and $d_\eta = 2$, respectively, while $\rho_\mu$ scales in the same way as the derivative in the covariant derivative, $d_{\rho_\mu} = 1$.

The effective potential for $\langle \phi_{i,\beta} \rangle = \sqrt{N} v(\delta_{i,j}, 0)$ and $\langle \eta_{i,j} \rangle = \eta \delta_{i,j}$, $\langle \sigma(x) \rangle = \sigma$ takes the form:

$$
\frac{1}{N_p} V_{\text{eff}}(v, \eta, \sigma) = \eta (v^2 - \sigma^2) + \frac{1}{N_p} V(\sigma) + \int \frac{d^D k}{i(2\pi)^4} \ln (k^2 - \eta) .
$$

This yields the gap equation:

$$
\frac{1}{N_p} \frac{\partial V_{\text{eff}}}{\partial v} = 2\eta v = 0 , \tag{43}
$$

$$
\frac{1}{N_p} \frac{\partial V_{\text{eff}}}{\partial \sigma} = -2\eta \sigma + \frac{\lambda}{\mu} \sigma (\sigma^2 - \frac{1}{G}) = 0 , \tag{44}
$$

$$
\frac{1}{N_p} \frac{\partial V_{\text{eff}}}{\partial \eta} = v^2 - \sigma^2 + \int \frac{d^D k}{i(2\pi)^4} \frac{1}{\eta - k^2} = 0 . \tag{45}
$$

Eq. (45) together with (43) is the same form as that of $CP^{N-1}$ in $D$ dimensions (see e.g., [7, 10]), and implies either of the two cases:

$$
\begin{cases}
\eta = 0 , \quad v \neq 0 ; \quad \text{case (i)} \\
v = 0 , \quad \eta \neq 0 ; \quad \text{case (ii)} .
\end{cases}
\tag{46}
$$

Eq. (44) yields two cases:

$$
\sigma = 0 ,
$$

$$
\sigma \neq 0 , \quad -2\eta + \frac{\lambda}{\mu} (\sigma^2 - \frac{1}{G}) = 0 . \tag{47}
$$

where the first solution $\sigma = 0$ in Eq. (47) contradicts Eqs. (45) and (43), and hence we are left with the second one, which implies $\eta = 0$ for $\lambda \to 0$, the BPS limit in the broken phase, case (i), while for $\lambda \neq 0$ we have:

$$
\sigma^2 = \frac{1}{G} + \frac{2p \eta}{\lambda} . \tag{48}
$$

The stationary condition in Eq. (45) gives a relation between $\eta$ and $v$. By putting $\eta = v = 0$ in Eq. (45), the critical point $G(\equiv G(\Lambda)) = G_{\text{crit}}(\equiv G_{\text{crit}}(\Lambda))$ separating the two phases in Eq. (46) is determined as

$$
\frac{1}{G_{\text{crit}}} = \int \frac{d^D k}{i(2\pi)^D} \frac{1}{-k^2} = \frac{1}{(\frac{D}{2} - 1) \Gamma(\frac{D}{2}) (4\pi)^{D/2} } . \tag{49}
$$

by which the integral in Eq. (46) reads:

$$
\int \frac{d^D k}{i(2\pi)^D} \frac{1}{\eta - k^2} = \frac{1}{G_{\text{crit}}} - \frac{\Gamma(2 - D/2)}{(D/2 - 1)} \cdot \eta^{D/2 - 1} / (4\pi)^{D/2} . \tag{50}
$$

---

#16 As we noted in the footnote to Eq. (15), even in the conformal limit, to define the quantum corrections we need the regularization scale $\Lambda$, and hence the scale symmetry is explicitly broken so as to give rise to the trace anomaly as the induced potential for $\phi$, such as $\sim v^4 \chi^4 \ln \chi$ for $D = 4$. This also yields a similar (but numerically different) self couplings of $\phi$. Direct test of the SM Higgs self couplings in future experiments will test the precise form of the $V(\langle \phi \rangle)$. 


Hence the gap equation takes the form:

\[ v^2 - \left( \frac{1}{G} - \frac{1}{G_{\text{crit}}} \right) - \frac{\eta^{D/2-1}}{(4\pi)^{D/2}} + \frac{2p\eta}{\lambda} = v^2. \] (51)

We may define the renormalized coupling at renormalization point \( \mu^2 \) as:

\[
\frac{1}{G} - \frac{1}{G_{\text{crit}}} = \frac{1}{G^{(R)}(\mu)} - \frac{1}{G^{(R)}_{\text{crit}(\mu)}},
\]

\[
\frac{1}{G^{(R)}_{\text{crit}}} = \frac{1}{G^{(R)}_{\text{crit}(\mu)}} - \frac{1}{G} - \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{\mu^2 - k^2},
\]

\[
\frac{1}{G^{(R)}(\mu)} = \frac{1}{G^{(R)}_{\text{crit}(\mu)}} - \int \frac{d^Dk}{i(2\pi)^D} \left( \frac{1}{-k^2} - \frac{1}{\mu^2 - k^2} \right)
= \Gamma(2-D/2) \frac{\mu^{D-2}}{(D/2-1)} \frac{1}{(4\pi)^{D/2}}.
\]

Now the gap equation Eq. (51) takes the form, depending on the phase (i) and (ii) as:

(i) \( G < G_{\text{cr}} : v \neq 0, \eta = 0 \)

\[
\frac{1}{G(\Lambda)} - \frac{1}{G_{\text{crit}}(\Lambda)} = \frac{1}{G^{(R)}_{\text{crit}(\mu)}} - \frac{1}{G^{(R)}(\mu)} = v^2 > 0,
\]

(ii) \( G > G_{\text{cr}} : v = 0, \eta \neq 0 \)

\[
\frac{1}{G(\Lambda)} - \frac{1}{G_{\text{crit}}(\Lambda)} = \frac{1}{G^{(R)}_{\text{crit}(\mu)}} - \frac{1}{G^{(R)}(\mu)} = -v_\eta^2 < 0.
\]

C. Phase Structure

The form of the gap equation, Eq. (51) and (52), is the same as that in NJL model up to the opposite sign, i.e., opposite phase for strong coupling \( G > G_{\text{crit}} \) vs weak coupling \( G < G_{\text{crit}} \), since the classical (bare) theory is formulated in the opposite phase (Wigner realization for the NJL model versus NG (nonlinear) realization for the present case). See e.g., Eq. (C.3) in Appendix C.

The case (i) is the perturbative phase where the classical theory structure remains. Eq. (51) is the gap equation for the spontaneous breaking of the symmetry \( G_{\text{global}} \times H_{\text{local}} \), with the Higgs mechanism of \( H_{\text{local}} \) yielding the “mass” of \( \rho_\mu \):

\[
(M_\rho)^2 = 2 \cdot N v^2 \neq 0,
\]

(with mass dimension \( D-2 \), as read from Eq. (11), with \( \langle \phi_{i,j}(x) \rangle = \sqrt{N} v (\delta_{i,j}, 0) \). \#17 This mass already differs from the “bare mass” \( 2N/G \) at classical level by the power divergent corrections \( 1/G_{\text{crit}} \) as seen in Eq. (51), but still receives additional quantum effects arising from the kinetic term after rescaled to the canonical form, see later discussions. The scale symmetry is also spontaneously broken by the same \( \langle \phi_{i,j}(x) \rangle = \langle \chi(\varphi) \cdot \phi_{i,j} \rangle = \sqrt{N} v \delta_{i,j}, v \neq 0 \), with \( \varphi \) in the \( \chi(\varphi) = e^{\varphi/F_\varphi} \) being the (pseudo-)dilaton with the decay constant \( F_\varphi = \sqrt{N} v \) at the quantum level (not the value at classical level \( \sqrt{pN/G} \)).

The case (ii) is a genuine nonperturbative phase in strong coupling \( G > G_{\text{crit}} \). It implies that the quantum theory is actually in the unbroken phase of \( G_{\text{global}} \times H_{\text{local}} \), although the theory at classical level is written in terms of the NG

\#17 The factor 2 is an “accidental value” \( a = 2 \) in the particular representation in Eq. (55), which can actually be shifted to arbitrary value as far as the equation of motion of \( \rho_\mu \) as the auxiliary field is used. See discussions for Eq. (27) and the related footnote.
boson variables as parts of $\phi$ living in the coset $G/H$ as if it were in the broken phase. The HLS gauge symmetry $H_{\text{local}}$ is thus never spontaneously broken and the gauge boson if exists as a particle should be massless. In fact, the originally the NG bosons $\pi$ in the $\phi$ (and would-be NG bosons $\tilde{\phi}$) at classical level (see Eq. (25) are no longer the NG bosons (would-be NG bosons) at quantum level by the nonperturbative dynamics (at large $N$) and acquire dynamically the mass

$$M_\pi^2 = M_\phi^2 = \eta \neq 0, \quad (G > G_{\text{crit}}), \quad v = 0,$$

as readily seen from Eq. (40). Note that $\langle \eta(x) \rangle = \eta = \mathbf{1} \neq 0$ breaks no internal symmetry but the scale symmetry due to VEV of the field $\eta(x)$ carrying the scale dimension 2. Writing the $p$-flavor-singlet (trace part) $\eta(x) = \eta \bar{\phi}_0(x)/\eta \cdot \mathbf{1}$, we may regard $\bar{\phi}_0(x)$ as a pseudo-dilaton in this phase (its mass from the trace anomaly due to the regularization with $\Lambda$, or the renormalization with $\mu$).

Eq. (54) and Eq. (55) imply that the dynamical phase transition from the case (i) ($\eta = 0, v^2 = 1/G - 1/G_{\text{crit}} > 0$) to the case (ii) ($v = 0, -v_\eta = 1/G - 1/G_{\text{crit}} < 0$) is induced continuously at the transition point $v = v_\eta = 0$ by the power divergent (quadratic divergent for $D = 4$) loop contributions $1/G_{\text{crit}}$ to the classical $(F_\pi^2)_0 = N/G$. In $D = 4$ it reads from the broken side ($v \to 0$)

$$v^2 = \frac{1}{G} - \frac{1}{G_{\text{crit}}} = \frac{F_\pi^2}{N} = \left(\frac{F_\pi^2}{N}\right)_{\text{crit}} - \frac{\Lambda^2}{(4\pi)^2} \to 0, \quad (G \to G_{\text{crit}} - 0),$$

(see, e.g., Ref. [10]), while from the side of the unbroken phase ($\eta \to 0$)

$$-v_\eta^2 = \frac{1}{G} - \frac{1}{G_{\text{crit}}} = \left(\frac{F_\pi^2}{N}\right)_{\text{crit}} - \frac{\Lambda^2}{(4\pi)^2} \to 0, \quad (G \to G_{\text{crit}} + 0).$$

The phase transition is the second order, analogously to the well-known gap equation in the NJL model (See Appendix [10]. In $D = 4$ this is the same as the pSM tuning the weak scale $F_\pi^2 = (246 \text{ GeV})^2$ against the quadratically divergent corrections to the (Higgs mass)$^2$, except that the cutoff $\Lambda$ (to be shown as related to the Landau pole) in the present case will be shown to be close to the weak scale, in contrast to the pSM whose Landau pole is much higher to be plagued with the naturalness problem.

Note also that the gap equations Eq. (54) and Eq. (55) are finite relations for $2 \leq D < 4$ as they should, since the theory is renormalizable. Indeed, the gap equations take the same form as that of the $D$-dimensional NJL model which is also renormalizable for $2 \leq D < 4$ [48] [49]. Then the full quantum theory has a beta function with a nontrivial ultraviolet fixed point for the dimensionless coupling $g = G \Lambda^{D-2}$:

$$\beta(g) = \Lambda \frac{\partial g}{\partial \Lambda} \bigg|_{v, \rho = \text{fixed}} = -(D - 2) \frac{g}{g_{\text{crit}}} (g - g_{\text{crit}}), \quad g_{\text{crit}} = G_{\text{crit}} \Lambda^{D-2} = (4\pi)^2 \left(\frac{D}{2} - 1\right) \Gamma\left(\frac{D}{2}\right),$$

(60)

(the same form of $\beta(g^{(R)})$ for $g^{(R)}(\mu) = G^{(R)}(\mu) \cdot \mu^{D-2}$ with $g^{(R)}_{\text{crit}} = G^{(R)}_{\text{crit}} \mu^{D-2})$, where $g_{\text{crit}}$ is the nontrivial (non-Gaussian) ultraviolet fixed point at which the interacting quantum theory is defined. This is in fact similar to that of the $D$-dimensional NJL model [48] [49].

Special attention should be paid to $D = 2$ dimensions, where $g_{\text{crit}} = 0$ and hence the case (i) (the classical/perturbative phase, broken phase with $v \neq 0$) does not exist at all, in accord with the Mermin-Wagner-Coleman theorem on absence of the spontaneous symmetry breaking in $D = 2$ dimensions. On the other hand, the gap equation Eq. (55) with $D = 2$ takes the form $\frac{1}{g} = -\frac{1}{\pi} \ln \frac{g}{g_{\text{crit}}}$, or:

$$\langle \eta(x) \rangle = \frac{\Lambda^2}{g(\Lambda)} \cdot \exp \left(\frac{4\pi}{g(\Lambda)}\right) = \mu^2 \cdot \exp \left(\frac{4\pi}{g^{(R)}(\mu)}\right),$$

(61)

where the scale symmetry appears to be spontaneously broken by $\langle \eta(x) \rangle \neq 0$ in the same sense as $D > 2$ (up to explicit breaking due to the trace anomaly), but actually undergoes the Berezinskii-Kosterlitz-Thouless (BKT) phase transition similarly to the $D = 2$ NJL model (Gross-Neveu model).\(^{18}\)

---

\(^{18}\) The $D = 2$ NJL model (Gross-Neveu model) also has $G_{\text{crit}} = 0$, which would imply the broken phase of the chiral symmetry for all $G > G_{\text{crit}} = 0$, opposite to the present model. This would be in apparent contradiction to the Mermin-Wagner-Coleman theorem, but actually undergoes the BKT phase transition at $G = G_{\text{crit}} = 0$, which is a typical example of the “conformal phase transition” [54].
For $D = 4$, on the other hand, the logarithmic divergence remains in $g^{(R)}(\mu) = G^{(R)}(\mu) \cdot \mu^2 = (4\pi)^{-2} \ln(\Lambda^2/\mu^2)$ due to the factor $\Gamma(2-D/2)$ in Eq.\([11]\) and Eq.\([12]\) in the large $N$ theory, in such a way that in the vicinity of the phase transition point $v/\Lambda, v_\eta/\Lambda \simeq 0$, the renormalized coupling behaves as

$$g^{(R)}(\mu) = \left[ \frac{1}{(4\pi)^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) \right]^{-1} \to 0 \quad (\Lambda^2 \to \infty), \quad (\text{triviality}), \quad (62)$$

$$\to \infty \quad (\mu^2 \to \Lambda^2), \quad (\text{Landau pole}), \quad (63)$$

even though the bare coupling $g(\Lambda)$ has a nontrivial UV fixed point $g_{\text{crit}} = (4\pi)^2$ (Gaussian fixed point). Namely the “renormalized” coupling $g^{(R)}(\mu)$ for $D = 4$ is infrared (IR) zero, having a trivial IR fixed point:

$$\beta(g^{(R)}(\mu)) = \mu \frac{\partial g^{(R)}(\mu)}{\partial \mu} = \frac{2}{(4\pi)^2} \left( g^{(R)}(\mu) \right)^2 > 0,$$  

and hence $g^{(R)}(\mu) \to 0$ as $\mu^2 \to 0$. This is the same as in the NJL model in four dimensions.

This appears different from the original SM which is perturbatively renormalizable. However, the perturbative SM in the original parameterization is also plagued with the Landau pole at a certain scale $\Lambda$ indicating that the theory is a trivial theory; the coupling $\lambda(\mu) \to 0$ ($\mu < \Lambda$) for the limit $\Lambda \to \infty$. So the situation is essentially the same. We here define the full quantum theory as a cutoff theory for $D = 4$ where the cutoff $\Lambda$ is regarded as the Landau pole, which will be explicitly related to the dynamically generated kinetic term of the HLS gauge boson, having no counter term and thus characterized by a new extra parameter, the HLS gauge coupling $g_{\text{HLS}}$. For $\Lambda \to \infty$ limit there still remains derivative coupling, which however vanishes at the infrared (low energy) limit, corresponding to the triviality of the conventional formulation of the SM.

V. DYNAMICAL GENERATION OF THE HLS GAUGE BOSONS IN THE SM (SM RHO)

A. Large $N$ result of the Grassmannian model

We now discuss the dynamical generation of the HLS gauge boson $\rho^a_i = \rho^a_i(S^a)^ij$. We are interested in the large $N$ limit of the SM as a scale-invariant Grassmannian model $O(N)_{\text{global}} \times O(p)_{\text{local}}$ with $N = 4$ and $p = 3$. Hereafter we confine ourselves to $O(N)_{\text{global}} \times O(3)_{\text{local}}$ model, in which case $O(3)$ generator takes the form $S^a_{ij} = i\epsilon_{iaj}$ such that:

$$\sum_{ij} \rho^a_{\mu} \rho^b_{\mu} \cdot (i\epsilon_{iaj}i\epsilon_{bjk}) = \sum_{ij} \rho^a_{\mu} \rho^b_{\mu} \cdot 2\delta^{ab} = 2 \cdot \sum_{a} \rho^a_{\mu} \rho^a_{\mu}, \quad (p = 3, S^a_{ij} = i\epsilon_{iaj}). \quad (65)$$

From the effective action Eq.\([11]\) the (amputated) two-point function of $\rho_\mu$ reads:

$$\frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \sum_{ij} \rho^a_{\mu}(-q) \cdot \tilde{\Gamma}^{(\rho)}(q) \cdot \rho^a_{\mu}(q) = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \sum_{a} \rho^a_{\mu}(-q) \cdot \Gamma^{(\rho)}(q) \cdot \rho^a_{\mu}(q) \quad (66)$$

where

$$\Gamma^{(\rho)}(q) = 2 \cdot \tilde{\Gamma}^{(\rho)}(q) = \frac{N}{2} \cdot 2 \cdot \left[ g_{\mu\nu} \cdot 2v^2 + \int \frac{d^D k}{i(2\pi)^D} \left[ \frac{(q + 2k)_{\mu} \cdot (q + 2k)_{\nu}}{((q + 2k)^2 - \eta)} \frac{(q - 2k)_{\mu} \cdot (q - 2k)_{\nu}}{(k^2 - \eta)} - 2 \frac{g_{\mu\nu}}{k^2 - \eta} \right] \right]$$

$$= 2N \left[ g_{\mu\nu} \cdot 2v^2 + \left( g_{\mu\nu} - \frac{g_{\rho\rho}q_{\mu}}{q^2} \right) \cdot q^2 \cdot f(q^2, \eta) \right], \quad (67)$$

$$f(q^2, \eta) = \frac{1}{2} \cdot \frac{\Gamma(2 - \frac{D}{2})}{\Gamma(2)} \int_0^1 dx \frac{(1-2x)^2}{|x(1-x)q^2 + \eta^2 - \frac{D^2}{4}|}. \quad (68)$$

#19 This corresponds to the extra free parameters in the “renormalized formulation” in the effective field theory approach for the nonlinear sigma model such as $\mathbb{CP}^{N-1}$ model in $D = 4$. Such an extra free parameter also appears in the $D = 4$ NJL model, i.e., dynamically generated kinetic term and quartic coupling of the auxiliary scalar field (analogue of the kinetic term of the HLS gauge boson in the present case), see Appendix and also in the chiral perturbation theory at loop orders (extra counter terms).
It reads (see e.g., \cite{51} for massless integral with $\eta = 0$):

\[
\begin{align*}
    f(q^2, 0) &= -\frac{1}{D - 1} \frac{\Gamma \left( 2 - \frac{D}{2} \right) \Gamma \left( D/2 - 1 \right)^2}{2(4\pi)^{D/2} \Gamma(D - 2)} (q^2)^{D/2 - 2} \\
    &= -\frac{1}{2} \cdot \frac{1}{3(4\pi)^2} \left[ \ln \left( \frac{\Lambda^2}{q^2} \right) + \frac{8}{3} \right] (D \to 4), \quad \text{(case (i)) : } v \neq 0, \eta = 0, \\
    f(0, \eta) &= -\frac{1}{3} \frac{\Gamma(2 - D/2)}{2(4\pi)^{D/2} \Gamma(2)} \eta^{D-4} \\
    &= -\frac{1}{2} \cdot \frac{1}{3(4\pi)^2} \left[ \ln \left( \frac{\Lambda^2}{\eta} \right) \right] (D \to 4), \quad \text{(case (ii)) : } v = 0, \eta \neq 0,
\end{align*}
\]

where for $D = 4$ we have retained the explicit $\ln q^2$-dependence in the broken phase (case (i)) which is relevant in the later discussions on the VMD.\footnote{21} while only took the pole residue as a divergent part in the unbroken phase (case (ii)) where the $\ln q^2$-dependence will not be discussed in this paper. Similar results have been given in Ref.\cite{31}.\footnote{22}

It then yields the propagator of the $O(3)_{\text{local HLS}}$ gauge boson $\rho_\mu$:

\[
\langle \rho_\mu \rho_\nu \rangle(q) = \mathcal{T} \cdot \mathcal{T} \langle T \{ \rho_\mu(x) \rho_\nu(0) \} \rangle = -\tilde{\Gamma}^{(\rho)}_{\mu\nu}(q)^{-1} = -2\Gamma^{(\rho)}_{\mu\nu}(q)^{-1} = 2 \cdot \langle \rho_\mu \rho_\nu \rangle(q) = 1 \cdot \left[ g_{\mu\nu} - \frac{g_\mu g_\nu}{v^2(-f^{-1}(q^2, \eta))} \right].
\]

In the $\rho_\mu$ basis (see Eq.\cite{55}) this takes the form in each phase in $D = 4$:

\[
\begin{align*}
    \text{(case(i) : } v \neq 0, \quad M_\rho^2 = \eta = 0):
    \langle \rho_\mu^a \rho_\nu^b \rangle(q) &= -\delta_{ab} \Gamma^{(\rho)}_{\mu\nu}(q)^{-1} = -\delta_{ab} \frac{1}{2} \tilde{\Gamma}^{(\rho)}_{\mu\nu}(q)^{-1} \\
    &\approx \delta_{ab} \frac{1}{N} \left( -2f(M_\rho^2, 0) \right)^{-1} \left[ g_{\mu\nu} - \frac{g_\mu g_\nu}{M_\rho^2} \right] \quad \text{near } q^2 = M_\rho^2, \\
    M_\rho^2 &= -f^{-1}(M_\rho^2, 0) \cdot v^2 = 2\lambda_{\text{HLS}} v^2 = 2 \cdot g_{\text{HLS}}^2 \cdot N v^2 = 2 \cdot g_{\text{HLS}}^2 F_\pi^2, \\
    \frac{1}{\lambda_{\text{HLS}}} &= \frac{1}{N g_{\text{HLS}}^2} = -2f(M_\rho^2, 0) = \frac{1}{3(4\pi)^2} \ln \left( \frac{\Lambda^2}{M_\rho^2} \right),
\end{align*}
\]

\[
\begin{align*}
    \text{(case(ii) : } v = 0, \quad M_\pi^2 = \eta \neq 0):
    \langle \rho_\mu^a \rho_\nu^b \rangle(q) &= \delta_{ab} \frac{1}{N} \left( -2f(0, \eta) \right)^{-1} \cdot g_{\mu\nu} + \text{gauge terms}, \\
    M_\rho^2 &= 0, \quad \frac{1}{\lambda_{\text{HLS}}} = \frac{1}{N g_{\text{HLS}}^2} = -2f(0, M_\pi^2) = \frac{1}{3(4\pi)^2} \ln \left( \frac{\Lambda^2}{M_\pi^2} \right).
\end{align*}
\]

where “gauge terms” depend on the gauge fixing as usual, and $\lambda_{\text{HLS}} = N g_{\text{HLS}}^2$ is the ’t Hooft coupling to be fixed in the large $N$ limit. In either phase we may introduce “running” coupling $g_{\text{HLS}}^2(\mu^2)$ with $\mu$ representing a typical mass.

\footnotetext{20}{We thank Taichiro Kugo (private communication) for giving us explicit calculations in the broken phase for $D = 4$.}

\footnotetext{21}{The finite part $8/3$ may be absorbed into the redefinition of the cutoff $\Lambda^2 \Rightarrow \tilde{\Lambda}^2 = (e^{4/3} \cdot \Lambda)^2$ (Landau pole) and ignored hereafter unless otherwise mentioned.}

\footnotetext{22}{There is a caveat about the coefficient $N$ for the loop integral part, which is from the loop of the $p \times (N - p)$ components $\pi$’s out of the full $p \times N$ components of $\Phi_{\mu, \alpha}$ ($i = 1, \cdots, p = 3$) for $(\rho_\mu)_{ij}$ is fixed, while $\alpha = 1, 2, \cdots, N$ is running in the loop) and thus the coefficient factor $N$ might be replaced by $N - p$. However, difference between $N$ and $N - p$ is only the ambiguity of the $1/N$ sub-leading effects which are on the same order of other (uncontrollable) sub-leading effects. There is no justification for keeping only a part of the sub-leading effects, ignoring others. In the standard sense of the $1/N$ expansion we here took only the leading order.}
scale $\mu = M_\rho$ for the case (i) and $\mu = M_\pi$ for (ii).

Note the transversality of the loop contribution (second term of Eq.(67)), which as a consequence of the gauge symmetry of HLS does imply the masslessness of the HLS gauge boson when $v = 0$, i.e., the unbroken phase (case (ii)). Were it not for the gauge symmetry, namely, the HLS, we could not have taken the inverse of $\Gamma^{(\rho)}_{\mu\nu}(q)$, so that the theory in the unbroken phase would be inconsistent, the same situation as the $CP^{N-1}$ model in the parameterization without gauge symmetry (see Appendix 3). (This would be a serious problem particularly for $D = 2$ where there exists only the unbroken phase in both the present case and the $CP^{N-1}$ model.) Thanks to the HLS as a gauge symmetry in the present case, we have a freedom to fix the gauge by adding the gauge-fixing term as usual and can take the inverse.

In the broken phase $v \neq 0$ (case (i)), on the other hand, the first term of Eq.(67) is from the $\rho_\mu$ mass term, also receiving the loop contributions via the gap equation Eq.(54), and plays a role of the gauge-fixing term (unitary gauge), since the gap equation solution $\langle \phi_{i,\beta}(x) \rangle = \sqrt{N}v(\delta_{i,j},0)$ with $v \neq 0$ already fixed the gauge. These results are precisely the same as those of the $CP^{N-1}$ model [8, 11, 21–27, 29].

In the case (ii), unbroken phase ($v = 0, \eta \neq 0$), the $\pi$ fields (and $\tilde{\rho}$) in $\phi$ are no longer the NG bosons (would-be NG bosons) and are massive with mass

$$M_\rho^2 = M_\pi^2 = \eta \neq 0,$$

(78)

and hence the only singularity of $f(q^2, M_\rho^2)$ arises from the two-particle threshold $q^2 = 4M_\rho^2$ and beyond. Thus $f(q^2, M_\rho^2)$ has no singularity at $q^2 = 0$ in the $q^2$ plane, $-f(0, M_\rho^2) \neq 0$, as seen from the explicit calculation in Eq.(75), and we see that the two-point Green function develops a genuine massless pole.

We then find the massless HLS gauge boson $\rho_\mu$ acquires the kinetic term:

$$\mathcal{L}_\rho = -\frac{1}{4g_{HLS}^2} \frac{1}{2} \text{tr}(\rho_{\mu\nu}^2) = -\frac{1}{4g_{HLS}^2} (\rho_{\mu\nu}^a)^2,$$

(79)

with $g_{HLS}^2$ given in Eq.(77). Hence the kinetic term of the massless HLS gauge boson indeed has been dynamically generated by the nonperturbative dynamics at 1/N leading order!!

As already noted [8, 28] for $CP^{N-1}$ model (see Appendix 3), the result is in perfect conformity with the Weinberg-Witten theorem [52] which forbids the dynamical generation of the massless particle with spin $J \geq 1$. The theorem was proved in the Hilbert space with positive definite metric and hence without gauge symmetry. This is in sharp contrast to our HLS Lagrangian Eq.(40) which does have a gauge symmetry thus is quantized with indefinite metric Hilbert space, and hence generates a massless composite gauge boson without conflict to the Weinberg-Witten theorem.

As to the case (i), broken phase ($v \neq 0, \eta = 0$), the $\rho_\mu$ have a mass and thereby decay into massless NG bosons $\pi$ in $\phi$ and has no pole in the physical (time-like) momentum plane. Still the kinetic term can be generated as in Eq.(72):

$$\mathcal{L}_\rho = -\frac{1}{4g_{HLS}^2} \frac{1}{2} \text{tr}(\rho_{\mu\nu}^2) = -\frac{1}{4g_{HLS}^2} (\rho_{\mu\nu}^a)^2,$$

(80)

with $g_{HLS}^2$ given in Eq.(44).

The kinetic term may be rescaled into the canonical form in

$$-\frac{1}{4g_{HLS}^2} (\rho_{\mu\nu}^a)^2 \rightarrow \frac{1}{4} (\rho_{\mu\nu}^a)^2,$$

(81)

in such a way that the $\rho_\mu$ mass reads the typical type of the Higgs mechanism in Eq.(73), which is independent of $N$.

We may define $F_\rho^2 \equiv M_\rho^2/g_{HLS}^2$ at quantum level, which then coincides with the $a = 2$ relation at classical level

$$F_\rho^2 = 2 \cdot F_\pi^2.$$  

(82)

Then the kinetic term is also dynamically generated in the $N \rightarrow \infty$ limit, in exactly the same way as the $CP^{N-1}$ model in $D = 4$ including the factor 2 in Eqs.(70) and (82) [8, 28].

In $D = 4$ the kinetic term operator has scale dimension 4 (hence scale-invariant) but the coefficient of the kinetic term $1/g_{HLS}^2 (\mu^2)$ necessarily depends on the cutoff, thus also violating the scale symmetry in a sense different from
2 ≤ D < 4.#23 Nevertheless the HLS gauge coupling at D = 4 also vanishes on the fixed point where the scale symmetry is realized in a trivial sense:

\[ g_{\text{HLS}}^2(\mu^2) \rightarrow 0 \quad \text{as} \quad \frac{\mu^2}{\Lambda^2} \rightarrow 0 \quad (g \rightarrow g_{\text{crit}}) \]  

(83)

This implies that in D = 4 dynamically generated HLS gauge coupling \( \alpha_{\text{HLS}}(\mu^2) = g_{\text{HLS}}^2(\mu^2)/(4\pi) \) has an trivial IR fixed point \( \alpha_{\text{HLS}}(\mu^2) \rightarrow 0 \ (\mu^2 \rightarrow 0) \), i.e., asymptotically non-free.

\[ \beta(\alpha_{\text{HLS}}(\mu^2)) = \mu^2 \frac{\partial \alpha_{\text{HLS}}(\mu^2)}{\partial \mu^2} = \frac{1}{3\pi} \alpha^2(\mu^2) > 0 . \]  

(84)

Thus the phase transition point \( g = g_{\text{crit}} \) is identified with the trivial IR fixed point for both \( g^{(R)}(\mu^2) \) and \( g_{\text{HLS}}^2(\mu^2) \) which vanish just on the IR fixed point.

In fact, as we noted before (Eq.(82)), although the original (bare) coupling \( g = GA^2 \) of the model has a nontrivial UV fixed point, the “renormalized” one \( g^{(R)}(\mu^2) = G^{(R)}(\mu^2)\mu^2 \) has an IR zero (trivial IR fixed point) \( g^{(R)}(\mu^2) \rightarrow 0 \ (\mu^2 \rightarrow 0) \) corresponding to the Landau pole \( g^{(R)}(\mu^2) \rightarrow \infty \ (\mu^2 \rightarrow \Lambda^2) \). Hence the HLS gauge bosons simply get decoupled at the critical point, both \( \rho_\mu \) and \( \pi \) becoming free massless particles, in accordance with the scale symmetry at the ultraviolet fixed point \( g = g_{\text{crit}} \) in Eq.(80).

On the same token, the \( D = 4 \) case in Eq.(73) and (74) indicates the Landau pole \( g_{\text{HLS}}^2(\mu^2) \rightarrow 0 \) as \( \Lambda^2 \rightarrow \infty \), in the same way as the original coupling \( g^{(R)}(\mu^2) = G^{(R)}(\mu^2) \) in Eq.(82). Another view of this result is

\[ \frac{1}{g_{\text{HLS}}^2(\mu^2)} \rightarrow 0 \quad (\mu^2 \rightarrow \Lambda^2) , \]  

(85)

that is, the kinetic term vanishes at the Landau pole in such a way that the HLS gauge boson \( \rho_\mu \) returned to the auxiliary filed as a static composite of \( \pi \), the situation sometimes referred to as “compositeness condition” #30 advocated in a reformulation of the top quark condensate model #32 for the composite Higgs.

In this viewpoint the HLS gauge bosons as bound states of \( \pi \)’s develop the kinetic term as we integrate the higher frequency modes in the large \( N \) limit from \( \Lambda^2 \) down to the scale \( \mu^2 \) in the sense of the Wilsonian renormalization group #11.

Thus either from the unbroken phase \( \eta \rightarrow 0 \) in Eq.(79) or broken phase \( \nu \rightarrow 0 \) in Eq.(80), the HLS gauge coupling and hence \( M_\rho^2 \) in Eq.(73) do vanish continuously across the phase transition point. The phase transition is the second order, similarly to the \( CP^{N-1} \) model #29. Then the HLS gauge boson gets degenerate with the massless \( \pi \) but actually decoupled, both \( \rho_\mu \) and \( \pi \) becoming massless free particles exactly on the phase transition point \( g = g_{\text{crit}} \) which is the UV fixed point of the original coupling \( g = GA^2 = NA^2/F_\pi^2 \) where the scale symmetry is realized. This looks similar to the “vector manifestation” #11 #33 of the Wigner realization of symmetry, with \( g_{\text{HLS}}^2 \rightarrow 0 \) and \( M_\rho^2 \rightarrow 0 \), \( a \rightarrow 1 \), while in the present case we shall show later that these quantities of \( \rho_\mu \) are independent of \( a \) and hence the phase transition is also independent of \( a \), accordingly.

### B. SM rho \( \rho_\mu \) in the extrapolation \( N \rightarrow 4 \) with \( p = 3 \)

Finally, the SM Higgs Lagrangian is equivalent to this model in an extrapolation \( N \rightarrow 4 \) with \( p = 3 \), Eq.(83), and hence Eqs.(83) and (79) clearly indicate the dynamical generation of the SM rho, the HLS gauge boson \( \rho_\mu \) in the SM, with \( N \rightarrow 4 \).

---

#23 In \( D \neq 4 \) dimensions the scale symmetry existing at classical level (in the conformal/BPS limit \( V(\varphi) \rightarrow 0 \)) has been broken by the dynamical generation of the kinetic term of the HLS gauge boson, having the scale dimension 4 (not \( D \)), which is traced back to the spontaneous scale-symmetry breaking due to \( \eta = \langle \eta(x) \rangle \neq 0 \) (unbroken phase) and \( \nu \neq 0 \) (broken phase). However, the HLS gauge coupling vanishes just on the ultraviolet fixed point \( g = g_{\text{crit}} \) (see Eq.(80)) approaching from both sides of the phases:

\[ g_{\text{HLS}}^2(\mu^2) \rightarrow 0 , \quad M_\rho \equiv 0 , \quad \mu^2 = M_\rho^2 = (M_\rho)^2 = \eta \rightarrow 0 , \quad (G \rightarrow G_{\text{crit}} + \cdot) , \]

\[ g_{\text{HLS}}^2(\mu^2) \rightarrow 0 , \quad M_\rho^2 = \eta \equiv 0 , \quad \mu^2 = M_\rho^2 \rightarrow 0 , \quad (G \rightarrow G_{\text{crit}} - \cdot) . \]

and hence the dynamically generated massless HLS gauge bosons get decoupled to be free particles, in conformity with the exact scale invariance at the fixed point.
We then conclude that as a simple extrapolation $N \to 4$ of the large $N$ result, Eq. (71), or Eqs. (72) - (77), the kinetic term of the HLS gauge boson $\rho_\mu$ in the SM is generated with a mass as

$$
\frac{1}{\lambda_{\text{HLS}}(\mu^2)} = \frac{1}{N g_{\text{HLS}}^2(\mu^2)} \bigg|_{N \to 4} = \frac{1}{3} \frac{1}{(4\pi)^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right),
$$

(86)

$$
M_\rho^2(\mu^2) = -\nu^2 f^{-1}(\mu^2, 0) = 2 \cdot \lambda_{\text{HLS}}(\mu^2) \nu^2 = g_{\text{HLS}}^2(\mu^2) \cdot F_\rho^2,
$$

(87)

$$
F_\rho^2 = 2 \cdot F_\pi^2 = 2 \cdot (246 \text{ GeV})^2 \simeq (350 \text{ GeV})^2,
$$

(88)

where $\mu^2 = M_\rho^2$ in the broken phase and $\mu^2 = M_\pi^2 = \eta$ in the unbroken phase is understood, respectively. The Landau pole in the broken (unbroken) phase is related to the cutoff $\Lambda$ as $\tilde{\Lambda} = e^{4/3} \cdot \Lambda$ ($= \Lambda$) (footnote #21). Note that the resultant expression in Eqs. (87) and (88) relevant to the phenomenology remains the same as that in $N \to \infty$, having no explicit dependence on $N$.

We shall later discuss possible phenomenological consequences of the dynamical generation of the SM rho, which will indicate that the cutoff or the Landau pole $\Lambda = \mathcal{O}(\text{TeV}) - \mathcal{O}(10^2 \text{ TeV})$ for the coupling strength $g_{\text{HLS}} = \mathcal{O}(10^5 - 10^6)$, depending on the collider physics or dark matter physics, respectively, either case being much close to the weak scale as a solution to the naturalness problem of the SM through the nonperturbative physics within the SM.

If the large $N$ results persist at least qualitatively for $N = 4$, then the (zero temperature) phase transition is of the second order, both $\rho_\mu$ and $\pi$ becoming degenerate as free massless particles just on the transition point $g = g_{\text{crit}}$ which is the scale-invariant ultraviolet fixed point. The unbroken phase has massless HLS gauge bosons $\rho_\mu$ and massive $\pi$ and $\rho$, while broken phase does massive $\rho_\mu$ (absorbing $\rho$) and massless $\pi$ (NG bosons). This phase transition is quite different from the second order phase transition in the conventional view based on the linear sigma model parameterization of the SM, where the unbroken phase is realized by the degenerate massive scalars, $\tilde{\sigma}$ and $\tilde{\sigma}$ in Eq. (38), which are interacting even at the transition point.

VI. $a-$ DEPENDENCE

So far we have discussed nonperturbative dynamics in the large $N$ limit in the particular form of the Lagrangian Eq. (38), which corresponds to $a = 2$ of the $aL_V$ term in the classical Lagrangian Eq. (36). At classical level the Lagrangian Eq. (38) is independent of the parameter $a$, since it is equivalent to Eq. (36) for any $a$, as far as we use the classical equation of motion for $\rho_\mu$ (by which $aL_V \equiv 0$). However, as noted below Eq. (38), the value $a = 2$ is special in that this equivalence holds without using the classical equation of motion. It will be seen more explicitly in Eq. (39) where the only different term between them is $(1 - \frac{a}{2}) \frac{G}{N} (\phi \partial_\mu \phi^I)^2$ which identically vanishes for $a = 2$.

Here we discuss $a-$dependence at the quantum level. For $a \neq 2$, not only the one-loop diagram, an infinite set of the bubble sum due to the additional term should be included in the large $N$ limit. We shall show that on-shell quantities, such as the $\rho_\mu$ pole position and the residue are independent of the parameter $a$ in a peculiar way. On the other hand, classical equation of motion for $\rho_\mu$ is violated at quantum level in the large $N$ limit in an $a-$dependent way: The (non-propagating) contact term proportional to $1 - \frac{a}{2}$ appears in the two-point Green function of the $\rho_\mu$ besides the propagating part of the unitary gauge form. Thus the off-shell physics can in principle depend on $a$.

In the $a \to \infty$ limit, however, such an $a-$dependence vanishes so that the classical equation of motion is recovered, thus the SM rho $\rho_\mu$ is totally replaced by the composite operator $\alpha_{\mu, I} \equiv i \frac{G}{N} \phi \partial_\mu \phi^I$ even at quantum level. This implies the dynamically generated $\rho_\mu$ kinetic term is fully replaced by the Skyrme term.

We here return to the original form of the HLS model, Eq. (38) with Eq. (35), where for the present purpose to see the $a-$dependence, we may disregard parts related to the dilaton (SM Higgs) $\phi(x)$ which is irrelevant to the dynamical generation of $\rho_\mu$ in the large $N$ dynamics and $a$ dependence, in which case the constraint is $\phi \phi^I = N \frac{\eta}{\Lambda} \mathbf{I}$ instead of $\phi \phi^I = N \sigma^2 \mathbf{I}$ (namely, disregarding $2p \eta / \Lambda = 6 \eta / \Lambda$ in the gap equation Eq. (71)), and thus the scale symmetry is no longer relevant.

Note also that the result exactly applies to the $\rho$ meson in the $2$-flavored QCD described by the same $G/H \simeq G_{\text{global}} \times H_{\text{local}}$ (without the scale symmetry nor dilaton) on the same footing as the SM Higgs Lagrangian.

We then discuss the action $S[\phi] = \int d^Dx L$ with the two terms of the Lagrangian Eq. (38) (we focus on $p = 3$ case
Introducing multiplier into the CCWZ parameterization may be heretical, since \( \mathcal{L} + a \mathcal{L}_V \) is written already based on the constraint \( \phi \phi^t = N \frac{1}{G} \mathbb{1} \). As far as the discussions on the broken phase are concerned, we do not need it at all. Here we included it for the unbroken phase discussions as well. The results are the same in either way, anyway, as we demonstrate equivalence between Eq. (38) and Eq. (39).

which as mentioned before is reduced to the action for Eq. (37) for arbitrary \( a \), as far as we use the classical equation of motion:

\[
\rho_\mu = (\rho_\mu)_{ij} = \rho_\mu^a S^a_{ij} = \alpha_{\mu,||} = i \frac{G}{N} \phi \partial_\mu \phi^t = i \frac{G}{N} \phi_{i\alpha} \partial_\mu \left( \phi^t \right)_{\alpha j}, \quad ((i, j) = 1, \ldots, p = 3; \alpha = 1, \ldots, N),
\]

\[
\frac{1}{2} \text{tr}_{p \times p} (S^a S^b) = \delta^{ab}.
\]

A. Problem with one-loop for \( a \neq 2 \)

Let us first consider the one-loop for Eq. (89) with arbitrary \( a \) (with extra assumption \( N \gg p = 3 \), which coincides with the large \( N \) limit only for \( a = 2 \)). We would have

\[
\Gamma^{(\rho)}(p) = 2 \tilde{\Gamma}^{(\rho)}(q) = \left[ \left( \frac{a}{2} \right) \left( \frac{2N}{G} \right) g_{\mu \nu} + \left( \frac{a}{2} \right)^2 2 B_{\mu \nu}(q) \right],
\]

where the bubble function \( B_{\mu \nu}(p) \) is given as:

\[
B_{\mu \nu}(q) = \frac{N}{2} \int \frac{d^D k}{(2\pi)^D} \frac{(2k + q)_{\mu}(2k + q)_{\nu}}{(k^2 - \eta)((k + q)^2 - \eta)}
= N \left[ \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) q^2 f(q^2, \eta) - g_{\mu \nu} \left( \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - \eta} \right) \right]
= N \left[ \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) q^2 f(q^2, \eta) - g_{\mu \nu} \left( \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - \eta} + \int \frac{d^D k}{(2\pi)^D} \left( \frac{1}{-k^2 + \eta} - \frac{1}{-k^2} \right) \right) \right]
= N \left[ \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) q^2 f(q^2, \eta) - \frac{1}{G_{\text{crit}}} g_{\mu \nu} \right],
\]

with

\[
\frac{1}{G_{\text{crit}}} \equiv \frac{1}{G_{\text{crit}}} - \frac{\Gamma(2 - \frac{D}{2})}{D/2 - 1} \cdot \frac{\eta^{D/2-1}}{(4\pi)^{D/2}} = \frac{1}{G_{\text{crit}}} - v_\eta^2.
\]

where \( v_\eta^2 \) is defined in the gap equation Eq. (31) (up to the term, \( 2\eta \lambda = 6\eta \bar{\lambda} \), as already noted), which now reads:

\[
v^2 = \frac{1}{G} - \frac{1}{G_{\text{crit}}}.
\]

\#24 Introducing multiplier into the CCWZ parameterization may be heretical, since \( \mathcal{L} + a \mathcal{L}_V \) is written already based on the constraint \( \phi \phi^t = N \frac{1}{G} \mathbb{1} \). As far as the discussions on the broken phase are concerned, we do not need it at all. Here we included it for the unbroken phase discussions as well. The results are the same in either way, anyway, as we demonstrate equivalence between Eq. (38) and Eq. (39).
Let us first check the case \( a = 2 \), where the additional four-\( \phi \) vertex in Eq. (39) is absent and the one loop dominance is literally valid in the large \( N \) limit. In fact Eq. (11) with Eq. (29) yields

\[
\Gamma^{(\rho)}_{\mu\nu}(q) = \left[ \frac{2N}{G} g_{\mu\nu} + 2B_{\mu\nu}(q) \right] \\
= N \left[ \frac{2}{G} \left( 1 - \frac{1}{G_{\text{crit}}} \right) g_{\mu\nu} + 2f(q^2, \eta) q^2 \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) \right],
\]

\[
\text{case (i)} = N \left[ 2\nu^2 g_{\mu\nu} + 2f(q^2, 0) q^2 \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{p^2} \right) \right], \quad (\nu \neq 0; \eta = M_\pi^2 = 0),
\]

\[
\text{case (ii)} = N \cdot 2f(q^2, \eta) q^2 \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right), \quad (\nu = 0; \eta = M_\pi^2 \neq 0), \quad (95)
\]

where use has been made of the gap equation Eq. (94).

The result Eq. (94) of course coincides with Eq. (67) based on Eq. (57) which is equivalent to Eq. (55) for \( a = 2 \) without use of the equation of motion. Note that in Eq. (67) making full use of the Lagrange multiplier (no tree-level \( \rho_{\mu} \) mass term), the contact term \( g_{\mu\nu} \) in the loop integral of \( \phi \) arising from the \( \rho_{\mu} \rho^\mu \phi \phi \) coupling makes the loop contribution to be transverse. On the other hand, such a loop graph contact term does not exist in the present CCWZ parameterization, Eq. (33), while the \( \rho_{\mu} \) tree mass term exists instead of \( \rho_{\mu} \rho^\mu \phi \phi \) coupling, which is combined with the tree contact term, resulting in the same answer.

Hence the one-loop \( \rho_{\mu} \) propagator at \( a = 2 \) coincides with Eq. (11) as its should, where \( F_\rho^2 = M_\rho^2 / g_{\text{HLS}, \rho}^2 = N 2v^2 = 2F_\pi^2 \). The result indicates that the quantum correction for \( F_\rho^2 \) and \( F_\pi^2 \) keeps the relation \( F_\rho^2 / F_\pi^2 = (F_\rho^2 / F_\pi^2)_0 = 2 = a \).

On the other hand, for \( a \neq 2 \), the one-loop result is depending on \( a \), obviously in disagreement with the large \( N \) limit result, Eq. (71):

\[
\Gamma^{(\rho)}_{\mu\nu}(q) = 2\Gamma^{(\rho)}_{\mu\nu}(q) = \left[ \frac{a}{2} \left( \frac{2N}{G} \right) g_{\mu\nu} + \frac{a^2}{2} 2B_{\mu\nu}(q) \right] \\
= N \left[ \frac{a}{2} \cdot \frac{2}{G} \left( 1 - \frac{a}{2G_{\text{crit}}} \right) g_{\mu\nu} + \frac{a^2}{2} 2f(q^2, \eta) q^2 \cdot \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) \right], \quad (96)
\]

which yields the \( \rho_{\mu} \) propagator \( \langle \rho_{\mu} \rho_{\nu} \rangle_{ab}^\rho(q) = \delta_{ab} (-\Gamma^{(\rho)}_{\mu\nu}(q))^{-1} = 1/2 \langle \rho_{\mu} \rho_{\nu} \rangle(q):

\[
\langle \rho_{\mu}^a \rho_{\nu}^b \rangle(q) = \frac{1}{2N} \delta_{ab} \left[ -f^{-1}(q^2, \eta) \left( \frac{2}{a} \right)^2 \cdot q^2 - (\frac{2}{a})^2 \cdot (-f^{-1}(q^2, \eta)(v')^2) \right] \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right). \quad (97)
\]

This reads in the broken phase:

\[
\langle \rho_{\mu}^a \rho_{\nu}^b \rangle(q) \approx \frac{1}{2N} \delta_{ab} \left[ -f^{-1}(M_\rho^2, 0) \left( \frac{2}{a} \right)^2 \cdot q^2 - (\frac{2}{a})^2 \cdot (-f^{-1}(M_\rho^2, 0)(v')^2) \right] \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{M_\rho^2} \right), \quad (98)
\]

where

\[
M_\rho^2 = \left( \frac{2}{a} \right)^2 \cdot \frac{1}{-2f(M_\rho^2, 0)} \cdot 2(v')^2, \quad (v')^2 = \frac{a}{2} \left( \frac{1}{G} - \frac{a}{2G_{\text{crit}}} \right),
\]

\[
g_{\text{HLS}, \rho}^2 = N \left( \frac{a}{2} \right)^2 \cdot (-2f(M_\rho^2, 0)). \quad (99)
\]

In the unbroken phase it takes the form:

\[
\langle \rho_{\mu}^a \rho_{\nu}^b \rangle(q) = \frac{1}{2N} \delta_{ab} f^{-1}(0, \eta) \left( \frac{2}{a} \right)^2 \cdot q^2 \cdot g_{\mu\nu} + \text{gauge terms},
\]

\[
g_{\text{HLS}, \rho}^2 = N \left( \frac{a}{2} \right)^2 \cdot (-2f(0, \eta)). \quad (100)
\]

Then both the pole position \( M_\rho^2 = g_{\text{HLS}, \rho}^2 \cdot F_\rho^2 \) and the residue \( g_{\text{HLS}, \rho}^2 \) are dependent on \( a \). Also the VMD is violated, since the direct coupling in Eq. (59) \( (1 - \frac{a}{2}) \cdot \frac{2N}{N} \left( \phi \partial_{\nu} \phi \right)^2 \) does exist. Note that \( (F_\rho/F_\pi)^2 = 2(v'/v)^2 \neq a \), which differs
from the classical relation \((F_\rho/F_\pi)^2 = a\) except for \(a = 2\).

However, all these results for \(a \neq 2\) are artifacts of the one-loop calculations (can be identified with the large \(N\) limit only for \(a = 2\)), in fact we shall next show \(a\)-independence of the on-shell quantities and \(F_\rho^2 = M_\rho^2/g_{\text{HLS}}^2 = 2F_\pi^2\) (\(a = 2\) relation!!) for arbitrary \(a\) in the genuine large \(N\) limit.

## B. Large \(N\) limit calculation

We now show that the on-shell quantities are independent of the parameter \(a\) in the large \(N\) limit, irrespectively of the phases.

Indeed, in the large \(N\) limit the dominant diagrams are not just the one-loop but do include an infinite sum of the bubble diagrams \(B_{\mu\nu}\) due to the additional four-\(\phi\) vertex proportional \((1 - a/2)^2\)\footnote{This is based on the observation by Taichiro Kugo (private communication), who showed the same result as Eq.\textbf{(107)} by more simplified calculation in the broken phase ignoring quantum corrections \(1/\xi\) in \(B_{\mu\nu}\), Eq.\textbf{(25)}, which implies identifying \(v^2 = 1/G\) through the gap equation Eq.\textbf{(14)}. We thank him for his very illuminating discussions.}:

\[
\Gamma^{(\rho)}_{\mu\nu}(q) = 2 \cdot \tilde{\Gamma}^{(\rho)}_{\mu\nu}(q) = \left(\frac{a}{2}\right) \left(\frac{2N}{G}\right) g_{\mu\nu} + 2 \left(\frac{a}{2}\right)^2 \left\{ B_{\mu\nu}(q) + B_{\mu\lambda}(q) \cdot \left(\frac{a}{2} - 1\right) \frac{G}{N} \cdot B_{\nu}^\lambda(q) + \ldots \right\}
\]

\[
= 2 \left[ \left(\frac{a}{2}\right) \left(\frac{N}{G}\right) g_{\mu\nu} + \left(\frac{a}{2}\right)^2 B_{\mu\lambda}(q) \cdot C_{\nu}^\lambda(q) \right],
\]

\[
C_{\mu\nu}(q) = g_{\mu\nu} + \left(\frac{a}{2} - 1\right) \frac{G}{N} B_{\mu\lambda}(q) \cdot C_{\nu}^\lambda(q),
\]

where \(B_{\mu\nu}(q)\) is given in Eq.\textbf{(92)}. Now \(C_{\mu\nu}(q)\) is given as

\[
C_{\mu\nu}(q) = g_{\mu\nu} + \left(\frac{1 - a}{2}\right) \frac{G}{N} B_{\mu\nu}(q) = \left[ 1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \right] \frac{g_{\mu\nu}}{q^2} \cdot q^2 + \left[ \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \right] f(q^2, \eta) \cdot q^2 \left( g_{\mu\nu} - \frac{g_{\mu\nu}}{q^2} \right),
\]

or

\[
C_{\mu\nu}(q) = \frac{1}{1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \cdot q^2} + \left[ 1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \right] f(q^2, \eta) \cdot q^2 \left( g_{\mu\nu} - \frac{g_{\mu\nu}}{q^2} \right).
\]

Then we get:

\[
\Gamma^{(\rho)}_{\mu\nu}(q) = 2N \left[ \left(\frac{a}{2}\right) \left(\frac{1}{G} \cdot \frac{1}{1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}}} \right) g_{\mu\nu} \right.
\]

\[
+ 2N \left( \left(\frac{a}{2}\right)^2 \frac{f(q^2, \eta) \cdot q^2}{G_{\text{crit}}} \right) \left[ \left(1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \right) \right] f(q^2, \eta) \cdot q^2 \left( g_{\mu\nu} - \frac{g_{\mu\nu}}{q^2} \right)
\]

\[
= \frac{N}{1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}}} \left[ a \left(\frac{1}{G} \cdot \frac{1}{G_{\text{crit}}} \right) g_{\mu\nu} + \frac{2 \left(\frac{a}{2}\right)^2 f(q^2, \eta) \cdot q^2}{\left(1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \right) \left(1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}} \right)} \left( g_{\mu\nu} - \frac{g_{\mu\nu}}{q^2} \right) \right]
\]

\[
= N A \left[ a \left(\frac{1}{G} \cdot \frac{1}{G_{\text{crit}}} \right) g_{\mu\nu} + \left(\frac{1}{\beta + \gamma \cdot q^2} \right) g_{\mu\nu} \right],
\]

where we defined (using the gap equation Eq.\textbf{(14)}):

\[
A = \beta^{-1} \left(\frac{1}{G} \cdot \frac{1}{G_{\text{crit}}} \right) = \beta^{-1} a v^2,
\]

\[
\alpha = 2 A^{-1} \beta \left(\frac{a}{2}\right)^2 f(q^2, \eta) = \frac{a}{2 \varepsilon^2} f(q^2, \eta), \quad \beta = 1 - \left(1 - \frac{a}{2}\right) \frac{G}{G_{\text{crit}}}, \quad \gamma = \left(1 - \frac{a}{2}\right) G f(q^2, \eta).
\]
Noting that
\[
\alpha + \gamma = \frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right) \frac{G}{\nu^2} \left(\frac{1}{\xi^2} - \frac{1}{G_{\text{NN}}}ight) f(q^2, \eta) = \frac{\beta}{\nu^2} f(q^2, \eta),
\]
we have the $\rho_\mu$ propagator by inverting $\Gamma_{\mu\nu}^{(\rho)}(q)$ (of course only for $a \neq 0$, since $\rho_\mu$ does not exist and $\Gamma_{\mu\nu}^{(\rho)}(q) \equiv 0$ for $a = 0$):
\[
\langle \rho_\mu^a \rho_\nu^b \rangle(q) = -\delta^{ab} \Gamma_{\mu\nu}^{(\rho)}(q)^{-1} = -\frac{1}{2} \delta^{ab} \Gamma_{\mu\nu}^{(\rho)}(q)^{-1}
\]
\[
= \delta^{ab} \frac{1}{N} A^{-1} \left[ \left(\frac{\alpha}{\alpha + \gamma}\right) g_{\mu\nu} + \left(-\alpha\beta(\alpha + \gamma)\right) g_{\mu\nu} \right] \left[ g_{\mu\nu} - \frac{g_{\mu\nu}}{\beta(\alpha + \gamma)} \right]
\]
\[
= \delta^{ab} \frac{1}{N} g_{\mu\nu} + \left(-\frac{1}{2} f^{-1}(q^2, \eta)\right) g_{\mu\nu} \left( g_{\mu\nu} - \frac{g_{\mu\nu}}{\xi^2} \right)
\]
\[
\approx \delta^{ab} \left( -\frac{1}{2} g_{\mu\nu} + \left(-\frac{1}{2} f^{-1}(q^2, \eta)\right) g_{\mu\nu} \left( g_{\mu\nu} - \frac{g_{\mu\nu}}{\xi^2} \right) \right)
\]
where
\[
M_\rho(q^2, \eta) = -v^2 \cdot f^{-1}(q^2, \eta) = g_{\text{HLS}}^2(q^2, \eta) \cdot (N \cdot 2v^2) = g_{\text{HLS}}^2(q^2, \eta) \cdot F^2 : \ a \neq 0 \text{ independent,}
\]
\[
M_\rho^2 = M_\rho(M_\rho^2, 0) = -v^2 \cdot f^{-1}(M_\rho^2, 0) = g_{\text{HLS}}^2 \cdot F^2, \quad M_\rho^2(q^2, M_\rho^2 \neq 0) \equiv 0,
\]
\[
g_{\text{HLS}}^2 = g_{\text{HLS}}^2(M_\rho^2, 0) = \frac{1}{2N} f^{-1}(M_\rho^2, 0), \quad (M_\rho^2, v^2 \neq 0, M_\rho^2 = \eta = 0),
\]
\[
= g_{\text{HLS}}^2(0, M_\rho^2) = -\frac{1}{2N} f^{-1}(0, M_\rho^2), \quad (M_\rho^2 = v^2 = 0, M_\rho^2 = \eta = 0),
\]
which for $a = 2$ indeed agrees with Eq. (111), with the concrete expression of $f(q^2, \eta)$ given in Eqs. (112) and (110).

In the unbroken phase $v = 0$, we have $N A g_{\mu\nu} = 0$ and $\Gamma_{\mu\nu}^{(\rho)}(q)$ becomes purely transverse and as it stands cannot be inverted, but thanks to the gauge symmetry, HLS, which exists for $a \neq 0$, we have a freedom to fix the gauge to take the inversion as usual, the same situation as Eq. (75) when inverting Eq. (67).

Note that in the large $N$ limit we have a universal ratio $F_\rho^2/F_\pi^2 = 2$ corresponding to “$a = 2$” independently of the classical parameter $a$:
\[
F_\rho^2 = \frac{M_\rho^2}{g_{\text{HLS}}^2} = 2 \cdot N v^2 = 2 \cdot F_\pi^2 \simeq 2 \cdot (246 \text{ GeV})^2 \simeq (350 \text{ GeV})^2,
\]
which is compared with the simple one-loop result Eq. (75) where $F_\rho^2/F_\pi^2$ is a complicated dependence on $a$ unless $a = 2$.

Thus we establish that the on-shell quantities, such as the pole position $M_\rho^2 = -f^{-1}(M_\rho^2, 0) \cdot v^2$ and the pole residue $-f^{-1}(M_\rho^2, 0) = 2N g_{\text{HLS}}^2$ are independent of the parameter $a$, and become identical to the values at $a = 2$ (the tree-level $a$-dependence of $(F_\rho^2) = a(F_\rho^2)$ disappears as $F_\rho^2 \to 2F_\pi^2$ at quantum level!!), so is the kinetic term of the dynamical gauge boson of HLS: We may say that the choice $a = 2$ good for the reality in QCD is not a mysterious parameter choice but the dynamical consequence of the quantum theory in the large $N$ limit!!

For all these $a$-independence of the on-shell quantities, however, the resultant $\rho_\mu$ propagator has an “unusual” contact term $\frac{2}{\nu^2} \frac{G}{\xi^2}$ which is cancelled for $a = 2$ and only for $a = 2$, in which case we are left with the standard massive vector meson propagator of the unitary gauge in the broken phase and the massless propagator in the unbroken phase in perfect agreement with Eq. (71) as it should be. This is compared with the one-loop result Eq. (75) having no such a contact term. Although this term by itself vanishes in the continuum limit $G \sim G_{\text{crit}} = C \left(\frac{14\pi^2}{\Lambda^2}\right) \to 0$ as $\Lambda \to \infty$ through the gap equation Eq. (113), discussing its origin might be useful to understand the theory at quantum level.
To see the origin of the $a-$dependent contact term, we now look into the relation: 

$$\langle \rho_\mu \rho_\nu \rangle(q) = \langle \alpha_{\mu,|a|} \alpha_{\nu,|a|} \rangle(q) - \frac{2G}{aN} g_{\mu\nu}, \quad \alpha_{\mu,|a|} = i \frac{G}{N} \phi \partial_\mu \phi^i, \quad (113)$$

which follows from the combined use of the Ward-Takahashi identities (for $a \neq 0$):

$$0 = \int \mathcal{D}\phi \frac{\delta}{\delta \rho_\nu(y)} \left( \rho_\mu(x) e^{iS[\phi]} \right) = \int \mathcal{D}\phi \left[ \delta^{(4)}(x-y) \cdot g_{\mu\nu} + \rho_\mu(x) \cdot \left( \frac{aN}{2G} \right) \left( \rho_\nu(y) - i \frac{G}{N} \phi \partial_\nu \phi^i(y) \right) \right] \cdot e^{iS[\phi]}, \quad (114)$$

where $S[\phi]$ is given in Eq. (112). The $a-$dependent contact term corresponds to the "tree $\rho_\mu$ propagator" (not propagating) with tree mass $a/(2G)$: $\langle \rho_\mu \rho_\nu \rangle^{(\text{tree})} = \frac{1}{N} \cdot (q^2 - (\frac{a}{2G}))^{-1} |q^2=0 \cdot g_{\mu\nu} = - \frac{2G}{aN} \cdot g_{\mu\nu}$. Thus the $\rho_\mu$ propagator at quantum level should depend on $a$ through the "unusual" contact term.

C. $\alpha_{\mu,|a|}$ as an $a-$independent genuine vector bound state

Here $\langle \alpha_{\mu,|a|} \alpha_{\nu,|a|} \rangle(q)$ should be $a-$independent, since it is independent of the presence of the auxiliary field $\rho_\mu$ in a manner similar to Eq. (113) taking an infinite sum of the bubble diagram. Now we show that it is indeed the case (in the genuine $a = 0$ case, where no HLS exists, we have a problem in the unbroken phase as noted before and to be repeated in the below).

Let us sum up the bubble diagrams in the large $N$ limit:

$$\langle \alpha_{\mu,|a|} \alpha_{\nu,|a|} \rangle(q) = \left( \frac{iG}{N} \right)^2 \langle \phi \phi \cdot \partial \phi \cdot \partial \phi \rangle(q) = \left( \frac{iG}{N} \right)^2 \left[ B_{\mu\nu}(q) + B_{\mu\lambda}(q) \cdot \left( - \frac{G}{N} \right) B_{\nu\lambda}(q) + \cdots \right]$$

$$= \left( \frac{G}{N} \right)^2 B_{\mu\lambda}(q) \cdot C_{\nu\lambda}(q),$$

$$C_{\mu\nu}(q) = g_{\mu\nu} + \frac{G}{N} B_{\mu\nu}(q) \cdot C_{\nu\lambda}(q),$$

$$C_{\mu\nu}(q) = \frac{G}{N} B_{\mu\nu}(q) \cdot C_{\nu\lambda}(q),$$

$$C_{\mu\nu}^{-1}(q) = g_{\mu\nu} + \frac{G}{N} B_{\mu\nu}(q) = \left( 1 - \frac{G}{G_{\text{crit}}} \right) g_{\mu\nu} + Gq^2 f(q^2, \eta) \left( g_{\mu\nu} - \frac{g_{\mu\nu} q^2}{q^2} \right)$$

$$= G \left[ v^2 g_{\mu\nu} + q^2 f(p^2, \eta) \left( g_{\mu\nu} - \frac{g_{\mu\nu} q^2}{q^2} \right) \right]. \quad (115)$$

Here we note that the "direct $4-\phi$ vertex" at the each end of the bubble graph, there are two relevant contributions, one is the genuine direct $4-\phi$ coupling $(\frac{a}{2} - 1) \frac{G}{N}$ and the other from the "tree $\rho_\mu$ propagator" $(-i \frac{a}{2}) \cdot \frac{G}{2G} (-i \frac{a}{2}) = - \frac{aG}{2N}$ such that

$$\left[ \left( \frac{a}{2} - 1 \right) \frac{G}{N} \right] + \left[ \frac{aG}{2N} \right] = - \frac{G}{N},$$

as in the above calculations, which makes the each vertex attached to the bubble $B_{\mu\nu}(q)$ in the sum to be independent of $a$.

Then we have

$$C_{\mu\nu}(q) = \frac{1}{1 - \frac{G}{G_{\text{crit}}} g_{\mu\nu} - \frac{Gp^2 f(q^2, \eta)}{1 - \frac{G}{G_{\text{crit}}} + Gq^2 f(q^2, \eta)} \left( g_{\mu\nu} - \frac{g_{\mu\nu} q^2}{q^2} \right)}$$

$$= \frac{1}{Gv^2 g_{\mu\nu} - \frac{q^2 f(q^2, \eta)}{v^2 + q^2 f(p^2, \eta)}}. \quad (117)$$

#26 This is due to Taichiro Kugo, private communications.
and hence

$$\langle \alpha_{\mu,||} \alpha_{\nu,||}\rangle(q) = \left( \frac{G}{N} \right)^2 B_{\mu \lambda}(q) \cdot C_{\lambda}^{\nu}(q) = \frac{G}{N} \frac{1}{v^2} \left[ -\frac{1}{G_{\text{crit}}} g_{\mu \nu} + \frac{1}{G} \frac{p^2 f(q^2, \eta)}{v^2 + q^2 f(q^2, \eta)} \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) \right]$$

$$= \frac{1}{N} \left[ G \cdot g_{\mu \nu} + \frac{-f^{-1}(q^2, \eta)}{q^2 - (-v^2 f^{-1}(q^2, \eta))} \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{-v^2 f^{-1}(q^2, \eta)} \right) \right].$$

(118)

The result is independent of \(a\), as it should be. We thus have a massive composite vector bound state \(\alpha_{\mu,||}\) in the broken phase, even without use of the auxiliary field \(\rho_{\mu}\), i.e., \(a = 0\), or the concept of HLS at all.

Now, Eq. (118) is consistent with Eq. (113) and \(\langle \rho_{\mu} \rho_{\nu}\rangle(q)\) in Eq. (107) (in the basis change \(\langle \rho_{\mu} \rho_{\nu}\rangle(q) = 2 \cdot \langle \rho_{\mu} \rho_{\nu}\rangle(q)\) (no sum) as noted in Eq. (67)):

$$\langle \rho_{\mu} \rho_{\nu}\rangle(q) = \frac{1}{N} \left[ \left( -\frac{2}{a} + 1 \right) G \cdot g_{\mu \nu} + \frac{-f^{-1}(q^2, \eta)}{q^2 - (-v^2 f^{-1}(q^2, \eta))} \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{-v^2 f^{-1}(q^2, \eta)} \right) \right].$$

(119)

Note again that \(a = 2\) is special in the sense that the \(\rho_{\mu}\) propagator takes the standard unitary gauge form without extra contact term as in Eq. (71).

Thus we have established that both \(M_{\gamma}^2\) and \(g_{\text{HLS}}^2\) are independent of the parameter \(a\), while the \(\rho_{\mu}\) propagator does have an \(a\)-dependent contact term, which implies the violation of the classical equation of motion. The physical meaning of this contact term will be discussed in the next subsection.

Further comments:

One might be puzzled by the contact term in Eq. (118), though. It is actually nothing peculiar. To see the bound states in the vector channel in the large \(N\) limit, we need the full amplitude containing the tree-level direct 4-\(\phi\) vertex in Eq. (116), as is done in the NJL model (see e.g., [36]):

$$T_{\mu \nu}(q) = -\frac{G}{N} g_{\mu \nu} + \left( \frac{G}{N} \right)^2 \left[ B_{\mu \lambda}(q) + B_{\lambda \mu}(q) \right] = -\frac{G}{N} g_{\mu \nu} + \langle \alpha_{\mu,||} \alpha_{\nu,||}\rangle(q)$$

$$= \frac{-f^{-1}(q^2, 0)}{p^2 - (-v^2 f(q^2, 0))} \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{-v^2 f(q^2, 0)} \right),$$

(120)

which is in fact the standard massive vector bound state propagator of the unitary gauge form in the broken phase. Here the tree contact term is precisely cancelled by the contact term arising from the infinite sum of the bubbles. So the existence of the contact term in Eq. (118) is of no peculiarity, or rather welcome.

This result in fact implies the “vector meson dominance (VMD)” for the \(\pi\pi\) scattering in the broken phase even without the auxiliary field \(\rho_{\mu}\), in sharp contrast to the conventional HLS formalism which is realized at \(a = 4/3\) for the \(\pi\pi\) scattering.

We shall further show later that the same kind of cancellation takes place for 4-\(\phi\) form factor (analogue of the pion electromagnetic form factor), with the tree contact term precisely cancelled by that arising from an infinite sum of the bubble diagrams, realizing the VMD not by the parameter choice but by the nonperturbative dynamics in the large \(N\) limit. Moreover, similar cancellation of the contact terms always take place not just for \(\alpha_{\mu,||}\) but also for \(\rho_{\mu}\) even including the “peculiar” \(a\)-dependent contact term, a phenomenon like a “miracle” of the large \(N\) dynamics.

In the unbroken phase \(v = 0\), on the other hand, \(C_{\mu \nu}^{-1}(q)\) has no \(g_{\mu \nu}\) term and hence cannot be inverted into \(C_{\mu \nu}(q)\). This is no problem, however, since we have a gauge symmetry, HLS, for \(a \neq 0\). It only means that we need to fix the gauge to take inversion for getting the massless propagator, the same situation as \(\langle \rho_{\mu} \rho_{\nu}\rangle(q)\) in Eq. (75) as repeatedly mentioned:

$$T_{\mu \nu}(q) = \frac{-f^{-1}(0, \eta)}{q^2} g_{\mu \nu} + \text{gauge terms} \quad \text{(symmetric phase).}$$

(121)

The independence of the on-shell quantities from the auxiliary field parameter \(a\) is also true in the NJL model (see Appendix C) where the auxiliary fields \(\tilde{a}\) and \(\tilde{\sigma}\) can be introduced with arbitrary weight, say \(\alpha\), as \(\frac{\alpha}{2G}(\pi^a + G\tilde{\alpha}\gamma_5 \tau^a \psi / \sqrt{2})^2 + (\tilde{\sigma} + G\tilde{\psi}/\sqrt{2})^2\), which may or may not cancel completely the effects of the four-fermion operators, \(G/4((\psi\gamma_5 \tau^a \psi)^2 + (\psi\psi)^2)\) except for \(\alpha = -1\) but the on-shell physics are the same for arbitrary \(\alpha\) in the large \(N\) limit.
There is a caveat, however: A distinctive difference between the present model and the NJL model is the unbroken phase, where our model is a consistent quantum theory only when the auxiliary field is introduced (i.e., \( a \neq 0 \)) to make the gauge symmetry HLS explicit as we noted repeatedly, while in the NJL model no such a problem exists in the unbroken phase even without the auxiliary field.

Incidentally, we may note that once the vector bound state \( \alpha_{\mu\lambda} \) in the broken phase is generated in the form of the standard unitary gauge propagator, Eq.\((120)\), it is well-known to give the Skyrme term in the region \( p^2 \ll M_\rho^2 \): \n
\[
L_{(\rho^\mu)}^{(p^\mu)} = \frac{F_\rho^2}{32M_\rho^2} \text{tr} ([L_\mu, L_\nu])^2 = \frac{1}{32g_{\text{HLS}}^2} ([L_\mu, L_\nu])^2 , \quad (q^2 \ll M_\rho^2) , \quad (L_\mu = \partial_\mu U \cdot U^\dagger, \quad U(x) = e^{2i\pi(x)/F_\rho}) \tag{122}
\]

with identification \( \epsilon^2 = g_{\text{HLS}}^2 \), and no non-Skyrme term \( \text{tr} ([L_\mu, L_\nu])^2 \). Here use has been made of Eq.\((112)\) and the notation and the normalization is the one for the standard \( SU(2)_L \times SU(2)_R / SU(2)_V \) (see Appendix A 2).

D. Physical implications of \( a \)-(in)dependence in the large \( N \) limit

We have shown that in the large \( N \) limit of the SM Higgs Lagrangian the on-shell quantities of the dynamically generated HLS gauge boson \( \rho_\mu \) are \( a \)-independence as in Eq.\((110)\), while \( a \)-dependence does exist in the off-shell quantities in the \( \rho_\mu \) propagator through the contact term as in Eq.\((107)\).

We here discuss what are independent of \( a \) and what else are not, and their physical implications, by focussing on the broken phase in the SM Higgs Lagrangian and the 2-flavor QCD on the same footing, since the dilatonic (SM Higgs boson) contributions are irrelevant to the large \( N \) limit physics of the HLS gauge bosons as we have already mentioned, although they are described by the scale-invariant (with dilaton) and the non-scale-invariant (without dilaton) version, respectively of the same Grassmannian nonlinear sigma model.

It turns out that all the “successful results of \( a = 2 \)” in QCD, namely \( \rho \)-universality and KSRF I, II, and the VMD of the form factor of the NG boson \( \pi \), and their analogues in the SM are realized independently of \( a \). We have already noted that the “VMD of \( \pi \pi \) scattering” is realized even without auxiliary field \( \rho_\mu \) (in the broken phase). On the other hand, the off-shell physics depend on \( a \) in principle: the skyrmion physics are dependent on \( a \) in such a way that at the \( a \to \infty \) where the classical equation of motion for \( \rho_\mu \) is recovered also at quantum level, whence the dynamically generated \( \rho_\mu \) kinetic term is totally replaced by the Skyrme term (not just low energy limit but also the high energy limit crucial to the stabilization of the skyrmion).

1. \( a \)-independent results

Let us first discuss the \( a \)-independent properties, which are relevant to the possible signatures of \( \rho_\mu \) resonances as the SM rho to be detectable at the collider experiments as well as the \( \rho \) meson properties in QCD.

Eq.\((110)\) and Eq.\((112)\) read:

\[
M_\rho(q^2) \equiv M_\rho^2(q^2, 0) = g_{\text{HLS}}^2(q^2)F_\rho^2 = 2g_{\text{HLS}}^2(q^2)F_\pi^2 , \quad (g_{\text{HLS}}^2(q^2) \equiv g_{\text{HLS}}^2(q^2, \eta = 0) , \tag{123}
\]

which is independent of \( a \). This implies the KSRF II (Eq.\((A35)\)), if the \( \rho \)-universality \( g_{\rho \pi \pi} = g_{\text{HLS}} \) (Eqs.\((A34)\)) is satisfied also independently of \( a \). Now we show that this is indeed the case.

Note that \( g_\rho \) is defined as the matrix element of the current \( \gamma_\mu^\rho \) of the gauged \( H_{\text{global}}(\subset G_{\text{global}}) \) for the \( \rho_\mu \) state as \( \langle 0|\gamma_\mu^\rho|\rho_\mu(q)\rangle \equiv \delta^{ab} \cdot g_\rho(q^2) \cdot \epsilon_\mu(q) = \delta^{ab} \cdot M_\rho(q^2) \cdot F_\rho \cdot \epsilon_\mu(q) \) and hence we have

\[
g_\rho(q^2) = M_\rho(q^2) \cdot F_\rho = g_{\text{HLS}}^2(q^2) \cdot F_\pi^2 = 2g_{\text{HLS}}^2(q^2) \cdot F_\pi^2 , \tag{124}
\]

which is also \( a \)-independent, where use has been made of Eq.\((123)\).

On the other hand, the KSRF I, Eq.\((A31)\) is of course independent of \( a \) even in the conventional HLS approach, and actually a low energy theorem of HLS [8, 9] proved to all orders of loop expansion [55]:

\[
g_\rho(q^2) = 2g_{\rho \pi \pi}(q^2) \cdot F_\pi^2 , \quad (\text{KSRF I}) . \tag{125}
\]
Comparing Eq. (124) with Eq. (123), we have the \( \rho \)-universality independently of \( a \):

\[
g^2_{\rho \pi \pi}(q^2) = g^2_{\text{HLS}}(q^2) = -\frac{1}{2N} f^{-1}(q^2, 0) = \frac{3 (4\pi)^2}{N \ln \left( \frac{4\pi}{q^2} \right)}, \quad (\rho - \text{universality}),
\]

where \( N \to 4 \) is understood for the SM as before (see Eq. (80) through Eq. (83)) and for the 2-flavor QCD as well. Then Eq. (123) reads the (generalized) KSRF II:

\[
M^2_{\rho}(q^2) = g^2_{\rho \pi \pi}(q^2) \cdot F^2_{\rho} = 2g^2_{\rho \pi \pi}(q^2) \cdot F^2_{\pi} \quad (\text{KSRF II}).
\]

The result is further confirmed by a direct computation without recourse to the KSRF I, which is given in Appendix D.

Thus the dynamically generated \( \rho_\mu \) in the large \( N \) limit reproduces the celebrated “\( a = 2 \) relations” Eqs. (A32) and (A34) in QCD even for arbitrary value of \( a \). In other words, Nature’s mysterious choice \( a = 2 \) in QCD is nothing but a dynamical consequence of the nonperturbative dynamics in the large \( N \) limit!! In the case of the SM Higgs Lagrangian the same result should also be checked at collider experiments if the \( \rho_\mu \) has a mass in the detectable region, as will be discussed later.

Next we discuss the VMD which is realized in the reality of the 2-flavored QCD, and should also be in the SM Higgs Lagrangian if the \( \rho_\mu \) mass is within the detection range of the collider experiments (see later discussions). Since it is off-shell physics, there is no a priori reason to believe it be realized \( a \)-independently. Nevertheless, it turns out to be the case. In contrast, it is well-known that the VMD for the \( \pi \) form factor is valid only for \( a = 2 \) in the conventional HLS formalism, at tree level with the kinetic term of \( \rho_\mu \) (simply assumed to be dynamically generated) [6–10]. Furthermore, at the one-loop order \( \mathcal{O}(p^4) \) in the chiral perturbation theory, it is even violated badly in general, particularly near the phase transition point [10, 56].

The NG boson form factor with the external gauge boson such as the \( \gamma \pi \pi \) in the hadron physics is given as usual by gauging \( H_{\text{global}}(\subset G_{\text{global}}) \) in the Lagrangian Eq. (59) as \( D_\mu \phi = (\partial_\mu - i \rho_\mu) \phi \Rightarrow D_\mu \phi = (D_\mu \phi + i \phi B_\mu) \) as in Eq. (A37).

For \( a = 2 \), there is no \( B_\mu \pi \), a direct coupling to the NG boson \( \pi \) (contained in \( \phi \), recall our parameterization Eq. (23)), as easily read from Eq. (37) by gauging \( H_{\text{global}}(\subset G_{\text{global}}) \) corresponding to the photon \( \gamma_\mu \) in hadrons physics. Therefore the vector meson dominance is trivially realized with the unitary gauge propagator in Eq. (119), which has no contact term for \( a = 2 \) but does have a nontrivial log \( q^2 \) dependence through \( g^2_{\text{HLS}} = (-2N f(q^2, 0))^{-1} \).

For arbitrary \( a \) we use the Ward-Takahashi identity Eq. (D4) for the Green function \( \langle \rho^{(R)}_\mu \pi \pi \rangle = \langle \alpha^{(R)}_{\mu, \pi} \pi \pi \rangle \) and the explicit computation Eq. (D2) in the Appendix D (with base change \( \rho_\mu^a \to \rho_\mu^a \)):

\[
\langle \rho^{(R)}_\mu(a) \pi(k) \pi(q+k) \rangle \big|^{k^2=(q+k)^2=0}_{\phi-\text{amputated}} = \langle \alpha^{(R)}_{\mu, \pi} (a) \pi(k) \pi(q+k) \rangle \big|^{k^2=(q+k)^2=0}_{\phi-\text{amputated}} = \frac{g_{\rho \pi \pi}(q^2)}{q^2 - g^2_{\text{HLS}}(q^2) \cdot F^2_{\rho}} \cdot \left( g_{\mu \nu} - \frac{q_\mu q_\nu}{g^2_{\text{HLS}}(q^2) \cdot F^2_{\rho}} \right) \cdot (q + 2k)^\nu,
\]

both of which have no \( a \)-dependence and no contact term. Here the \( \alpha^{(R)}_{\mu, \pi} \) is “renormalized” (rescaled to the canonical kinetic term) as \( \alpha^{(R)}_{\mu, \pi}(a) = g_{\text{HLS}}(q^2) \cdot \alpha_{\mu, \pi} \) and similarly for \( \rho_\mu \) (see Appendix D). In the generic case the form factor \( F_{\pi \pi}(q^2) \) for the gauged \( H_{\text{global}} \) current \( \gamma^a_\mu \) is just a linear combination of these two of the identical form:

\[
F_{\pi \pi}(q^2)(q + 2k)_\mu = \langle V^a_\mu(q) \pi(k) \pi(q + k) \rangle \big|^{k^2=(q+k)^2=0}_{\phi-\text{amputated}} = g_{\rho_\mu}(q^2) \cdot \left( \frac{a}{2} \langle \rho^{(R)}_\mu(a) \pi(k) \pi(q + k) \rangle + \left( 1 - \frac{a}{2} \right) \langle \alpha^{(R)}_{\mu, \pi}(a) \pi(k) \pi(q + k) \rangle \right) \big|^{k^2=(q+k)^2=0}_{\phi-\text{amputated}} \cdot (q + 2k)^\nu \]

\[
= \frac{g_{\rho_\mu}(q^2) \cdot g_{\rho \pi \pi}(q^2)}{g^2_{\text{HLS}}(q^2) \cdot F^2_{\rho} - q^2} \cdot \left( g_{\mu \nu} - \frac{q_\mu q_\nu}{g^2_{\text{HLS}}(q^2) \cdot F^2_{\rho}} \right) \cdot (q + 2k)^\nu,
\]

namely, the VMD is realized independently of \( a \). The contact terms are cancelled within each of \( \langle \rho^{(R)}_\mu \pi \pi \rangle \) and \( \langle \alpha^{(R)}_{\mu, \pi} \pi \pi \rangle \), separately, and hence the VDM follows independently of any combination of those.
The result has the log $q^2$ dependence through $g_{\text{HLS}}^2 = (-2Nf(q^2, 0))^{-1}$ as in Eq. (126),

$$F_{\pi\pi}(q^2) = \frac{g_{\rho}(q^2) \cdot g_{\rho\pi\pi}(q^2)}{g_{\text{HLS}}^2(q^2) \cdot F_\rho^2} - q^2 = \frac{g_{\pi\pi}^2(q^2) \cdot F_\rho^2}{g_{\text{HLS}}^2(q^2) \cdot F_\rho^2 - q^2} = \frac{g_{\rho\pi\pi}^2(q^2) \cdot F_\rho^2}{g_{\text{HLS}}^2(q^2) \cdot F_\rho^2 - q^2} \quad \text{(VMD)},$$

with a correct normalization $F_{\pi\pi}(0) = 1$ even including the log $q^2$ dependence. This differs from the naive VMD without such a log $q^2$ dependence, although it takes the same form near the on-shell $q^2 \approx M_\rho^2 = g_{\text{HLS}}^2(q^2 = M_\rho^2) \cdot F_\rho^2 = g_{\text{HLS}}^2 \cdot F_\rho^2$:

$$F_{\pi\pi}(q^2) \approx \frac{M_\rho^2}{F_\rho^2 - q^2} \quad (q^2 \approx M_\rho^2).$$

However, such a $q^2$ dependence is actually necessary for the modern version of the VMD to describe the correct $q^2$ behavior of the $\pi$ form factor and the related quantities in low energy QCD, in both the space-like and the time-like momentum regions, see e.g., [55, 59]. Then the VMD in a modern version is naturally realized in large $N$ limit of the present theory even without the auxiliary field $\rho_\mu$

It is compared with the standard HLS formulation which satisfies VMD only for $a = 2$, while $F_{\pi\pi}(0) = 1$ for any $a$ in a different way: [10]

$$F_{\pi\pi}(q^2) = \left(1 - \frac{a}{2}\right)_{\text{direct}} \left[+ \frac{g_{\rho\pi\pi}}{M_\rho^2 - q^2}\right]$$

$$= \left(1 - \frac{a}{2}\right)_{\text{direct}} \left[+ \frac{M_\rho^2}{2(M_\rho^2 - q^2)}\right],$$

where $g_{\rho\pi\pi} = \frac{a}{2}M_\rho^2$ is $a$–dependent in contrast to the above our corresponding relation which is $a$–independent.

2. $a$–dependent results

Another interesting off-shell physics is the skyrmion. First note that $\langle \rho_\mu \rho_\nu \rangle(q)$ in Eq. (119) for $a = 2$ and the pole in $T_{\mu\nu}(q)$ in Eq. (120) is identical. Then the $a = 2$ case gives the same Skyrmion term for $q^2 \ll M_\rho^2$:

$$\mathcal{L}_{\rho^3} = \frac{F_\rho^2}{32M_\rho^2} \text{tr} \left(\left[L_\mu, L_\nu\right]^2\right) = \frac{1}{32g_{\text{HLS}}^2} \left(\left[L_\mu, L_\nu\right]^2\right), \quad (a = 2; \ q^2 \ll M_\rho^2).$$

For $a \neq 2$, the above discrepancy between the two propagators, Eq. (119) and Eq. (118), disappears in the limit $a \to \infty$ and hence the equation of motion at classical level is recovered even at the quantum level as can be seen from Eq. (133). It then implies that the field strength of the dynamically generated HLS gauge boson $\rho_\mu$ reads [60]:

$$\rho_\mu \bigg|_{\rho_\mu = \alpha_\mu ||} = \partial_\nu \alpha_\nu || - \partial_\nu \alpha_\mu || - i \left[\alpha_\mu ||, \alpha_\nu ||\right] = i \left[\alpha_\mu, \alpha_\nu \right],$$

which for $N = 4$ and $p = 3$ yields the kinetic term into the form:

$$\mathcal{L}_{\text{kinetic}} = -\frac{1}{2g_{\text{HLS}}^2} \text{tr} \rho_\mu^2 \to \frac{1}{2g_{\text{HLS}}^2} \text{tr} \left(\left[\alpha_\mu, \alpha_\nu \right]\right)^2 = \frac{1}{32g_{\text{HLS}}^2} \text{tr} \left(\left[L_\mu, L_\nu\right]^2\right)^2, \quad (a \to \infty), \quad (c^2 = g_{\text{HLS}}^2),$$

where for comparison with the standard Skyrmion term expression, we have used notation/normalization of the equivalent $G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V$, with $\mathcal{L}_{\rho^3} = \partial_\mu U \cdot U^\dagger = 2i \cdot \xi_\mu \cdot \alpha_\mu || \cdot \xi_\mu$ in Eq. (A28). Hence the $a \to \infty$ limit in the large $N$ dynamics of the SM does generate the Skyrmion term. This is compared with Eq. (122) and also with Eq. (133) at $a = 2$, where the dynamically generated $\rho_\mu$ effects are replaced by the Skyrmion term only in the low energy limit $q^2 \ll M_\rho^2$.

In the case at hand, $a \to \infty$ limit, the $\rho_\mu$ kinetic term effects are completely replaced by the Skyrmion term for entire energy region not just in the low energy limit. This is crucial for the $SM$ skyrmion which is stabilized by the short distance off-shell dynamics. Thus in contrast to the on-shell physics independent of the parameter $a$, the off-shell effects such as the skyrmion do depend on the parameter $a$. 
One might suspect that a Higgs-like scalar bound state is also generated in the large $N$ limit, which would produce the non-Skyrme term $\left((L_\rho, L_\rho)^2\right)^2$ destabilizing the skyrmion in the low energy limit. Were it not for the elementary scalar, pseudo dilaton $\varphi$ as in the present SM case, then the model would be simply a usual Grassmannian nonlinear sigma model without dilaton field, which in fact would generate an $O(N)$-singlet dynamical Higgs-like particle in the large $N$ limit, in the same as the $CP^N$ model acting like a pseudo dilaton (see Appendix [1]). In the case at hand, having the elementary Higgs already, however, such a nonperturbative dynamics would not generate new particle but only gives quantum corrections of the already existing particle. So once such an elementary scale is included from the onset in the soliton equation, as done in [1, 42], the skyrmion physics based on the dynamical gauge boson of the HLS would not be affected further by the large $N$ dynamics acting on the dilatonic scalar sector.

Thus if the SM is close to the “Skyrme limit” $a \to \infty$, then the skyrmion in the SM having the elementary scalar is well described by $1/a$ expansion near the Skyrme limit. Actually, the $a$–dependence of Skyrmion as the dark matter in the SM will be rather weak all the way down to $a = O(1)$, as will be shown in the forthcoming paper [42].

3. Phenomenological implications of SM rho

Phenomenological implications of our result for the SM rho would be divided into two different scenarios depending on the possible value of a single extra free parameter existing in the nonperturbative theory, $M_\rho = g_{\rho \pi \pi} \cdot F_\rho$ (or $g_{\rho \pi \pi} = g_{\text{HLS}} = g_{\text{HLS}}(M_\rho^2) = M_\rho / F_\rho = M_\rho / (350 \text{GeV})$ or the cutoff $\Lambda$ (or the Landau pole $\Lambda$) in view of Eqs. (16)-(18):

$$\Lambda = e^{-4/3} \cdot \Lambda = e^{-4/3} \cdot M_\rho \cdot \exp \left[ \frac{4}{3} \left( 4 \pi F_\rho \right) \right] \cdot \Lambda_{\text{exp}}(136),$$

which implies that $\Lambda < M_\rho (g_{\text{HLS}} > 6.7, M_\rho > 23 \text{ TeV})$ and $\Lambda > M_\rho (g_{\text{HLS}} < 6.7, M_\rho < 23 \text{ TeV})$.

1) “Low $M_\rho$ scenario” ($M_\rho < 2.3 \text{ TeV}, \Lambda > M_\rho$):

Signatures of the SM rho $\rho_\mu$ at the collider experiments are qualitatively similar to the technirho in the walking technicolor based on the same type of the s-HLS effective theory [2, 14, 16], but more definite prediction due to a single parameter in the present case, arising from the $a$–dependence of the resonance parameters: Typical LHC signatures would be in the diboson channel through the Drell-Yan process with the $W/\gamma$ mixing: $q \bar{q} \to W/\gamma \to \rho_\mu \to W_L W_L / W_L Z_L$, characterized by the VMD, with the coupling, $\sim \alpha_{\text{em}} g_{\rho \pi \pi} M_\rho^2 = \alpha_{\text{em}} F_\rho / M_\rho = \alpha_{\text{em}} / g_{\text{HLS}}$. The production cross section depends on the parameter as $\sim 1 / M_\rho^2 \sim 1 / g_{\text{HLS}}^2$. Decays (branching ratios) are dominated by the diboson processes $\rho_\mu \to W_L W_L / W_L Z_L$, with the coupling $g_{\rho \pi \pi} = g_{\text{HLS}}$, characterized by the absence of the processes of $\rho_\mu \to W_L / Z_L + \varphi (\varphi = \text{SM Higgs} (= \text{pseudo-dilaton})$, in contrast to the “equivalence theorem results” in other models, because of the “conformal barrier” arising from the scale symmetry of the $\rho_\mu$ mass term in the s-HLS parameterization, similarly to the walking technirho [2, 16] (see also Appendix A3).

Given a reference value $M_\rho = 2 \text{ TeV}$ for instance, we would have $g_{\rho \pi \pi} \simeq 5.7$ and $\Lambda \simeq 3.3 \text{ TeV} \simeq 4 \pi F_\pi$ (simple scale-up of the QCD $\rho$ meson), perfectly natural scale with respect to the weak scale. This yields the width $\Gamma_\rho \simeq \Gamma_{\rho \to WW} \simeq g_{\rho \pi \pi}^2 M_\rho / (4 \pi) \simeq M_\rho / (4 \pi F_\rho^2) \simeq 433 \text{ GeV}$ [27], so broad as barely detectable at LHC. For larger (smaller) $M_\rho$, the width gets larger (smaller) as $\sim M_\rho^3 / g_\rho^2$, and the production cross section gets smaller (larger) as $\sim 1 / M_\rho^2$, thus more difficult for $M_\rho > 2 \text{ TeV}$ to be seen at LHC. The SM rho with narrow resonance $\Gamma_\rho \lesssim 100 \text{ GeV}$ could be detected at LHC for $M_\rho \lesssim 1.2 \text{ TeV}$, which corresponds to $g_{\text{HLS}} \lesssim 3.5$ and $\Lambda \gtrsim 50 \text{ TeV}$.

2) “High $M_\rho$ scenario” ($M_\rho \gg 2.3 \text{ TeV}, \Lambda < M_\rho$, as a stabilizer of the skyrmion dark matter $X_s$ [1]):

Even if no direct evidence were seen at the collider experiments, physical effects of the dynamical $\rho_\mu$ are still observable through the skyrmion dark matter $X_s$ in the SM. In fact the SM skyrmion is stabilized by the off-shell $\rho_\mu$ in the short distance physics as shown in Ref. [1], the result of which corresponds to $a \to \infty$ calculation. It was shown [1, 42] that the (complex scalar) skyrmion $X_s$, whose coupling to the SM Higgs as a pseudo-dilaton is given by the low energy theorem of the scale symmetry Eq. (10). Then the current direct detection experiments, XENON1T and PandaX-II [61, 63], give a constraint on the skyrmion mass (and simultaneously the skyrmion coupling to the SM Higgs which is directly related to the mass):

$$M_{X_s} \lesssim 11 \text{ GeV}, \quad \text{or equivalently,} \quad \lambda_{\varphi X_s X_s} = \frac{g_{\varphi X_s X_s}}{2 F_\varphi} = \frac{M_{X_s}^2}{F_\varphi} \lesssim 0.002, \quad \left( F_\varphi = F_\pi = \sqrt{N_v} = 246 \text{ GeV} \right), \quad (137)$$

#27 This is in contrast to the walking technirho [2, 14] of the typical one-family model ($N_D = 4$ weak-doublets), where the decay width has suppression of $1 / N_D = 1 / 4$ and hence of order $O(100) \text{ GeV}$.
independently of the details of the skyrmion profile, solely due to the dilatonic nature of the SM Higgs (Here, $v^2$ in Eq.10 has been rescaled by $N$ in the context of large $N$ limit arguments). In the limit $a \to \infty$ the entire physics of the $\rho$ is traded for the Skyrme term with the coefficient $e^2 = g_{\text{HLS}}^2$, see Eq. (135). In this limit the complex scalar skyrmion mass $M_{X_s}$ (and hence the coupling $g_{X_s X_s}$) and its square of the mean radius $\langle r_{X_s}^2 \rangle_{X_s}$ have been calculated as [1, 12]:

$$M_{X_s} \simeq 35 \frac{F_\pi}{g_{\text{HLS}}} \simeq 11 \text{ GeV} \times \left(\frac{780}{g_{\text{HLS}}}\right), \quad \lambda_{X_s X_s} = \left(\frac{35}{g_{\text{HLS}}}\right)^2 = 0.002 \times \left(\frac{780}{g_{\text{HLS}}}\right)^2,$$

which would imply

$$g_{\text{HLS}} \simeq 780,$$

and

$$\langle r_{X_s}^2 \rangle_{X_s} \simeq \left(\frac{2.2}{g_{\text{HLS}} F_\pi}\right)^2 \simeq 1.3 \times 10^{-10} \text{ GeV}^{-2} \times \left(\frac{780}{g_{\text{HLS}}}\right)^2.$$

This leads to the annihilation cross section of the skyrmion dark matter and the relic abundance $\Omega_{X_s} h^2$ [142]:

$$\langle \sigma_{\text{ann}} \bar{v}_{\text{rel}} \rangle_{\text{radius}} \simeq 4 \pi \cdot \langle r_{X_s}^2 \rangle_{X_s} \simeq 1.7 \times 10^{-9} \text{ GeV}^{-2},$$

$$\Omega_{X_s} h^2 \simeq \mathcal{O}(0.1),$$

which is roughly consistent with the observed cold dark matter relic abundance $\Omega_{X_s} h^2 \simeq 0.12$. Then the popular belief that “the dark matter is the physics beyond the SM” will be no more than a folklore. For $a < \infty$, by $1/a$ expansion we can explicitly show that the results are rather stable against changing $a$ all the way down to $a \sim 2$ [12]. Note the cutoff is $\Lambda = e^{-4/3} \Lambda \simeq e^{-4/3} \cdot M_\rho = \mathcal{O}(10^2 \text{ TeV})$, where $M_\rho = g_{\text{HLS}} \cdot F_\pi$ is a typical mass scale (no longer the “on-shell” mass, since the SM rho is deeply off-shell).

In either scenario, the phenomenologically interesting nonperturbative SM physics has typical strong SM rho gauge coupling $g_{\text{HLS}} \simeq 1/3 - 10^3$, which corresponds to the cutoff $\Lambda = \mathcal{O}(10^0 - 10^2) \text{ TeV}$, or the quadratic divergence corrections to the weak scale (see Eq.155):

$$\delta F_\pi^2 \sim 4 \Lambda^2 \left(\frac{4\pi}{\Lambda^2}\right) \sim (0.1 \text{ TeV})^2 - (10^2 \text{ TeV})^2,$$

thus resolving the naturalness problem without recourse to the BSM, in sharp contrast to the pSM. (As already mentioned in the Introduction, the HLS as a dynamically generated gauge symmetry in the SM is trivially anomaly-free, since the SM fermions have no HLS charges.) In this sense the SM with nonperturbative dynamics may be “dual” to some underlying BSM theory on that scale, similarly to the hadron-quark duality (nonlinear sigma model/chiral Lagrangian vs QCD). Immediate candidate for such a BSM would be the walking technicolor also being nothing but a (pseudo-) dilaton (technidilaton) [17, 18] (See Summary and Discussions).

VII. SUMMARY AND DISCUSSIONS

In this paper, we found that the “SM rho” as the gauge bosons $\rho(x)$ of the $O(3) \simeq SU(2)_V$ Hidden Local Symmetry (HLS) [6, 11] within the Standard Model (SM) Higgs Lagrangian, though auxiliary field at classical level, acquire the kinetic term at quantum level by the nonperturbative dynamics in the large $N$ limit, becoming fully propagating dynamical gauge bosons.

We first recapitulated the previous observation [2] that the SM Higgs Lagrangian is rewritten straightforwardly into a nonlinear realization based on the manifold $G/H = O(4)/O(3) \simeq [SU(2)_L \times SU(2)_R]/SU(2)_V$ and also a nonlinear realization of the (approximate) scale symmetry, with the SM Higgs $\varphi$ being nothing but a (pseudo-) dilaton near the BPS limit. The $G/H$ part is further gauge-equivalent to another Lagrangian having a larger symmetry $G_{\text{global}} \times H_{\text{local}} = O(4)_{\text{global}} \times O(3)_{\text{local}} \simeq [SU(2)_L \times SU(2)_R]_{\text{global}} \times [SU(2)_V]_{\text{local}}$, with $H_{\text{local}}$ being the HLS [6, 10], a spontaneously broken gauge symmetry existing in any nonlinear sigma model [8, 9].
We then studied nonperturbative dynamics of the SM Higgs Lagrangian in this HLS form in the large $N$ limit, by extending it to a scale-invariant version of the Grassmannian model $G/H = O(N) / [O(N-3) \times O(3)]$, which is gauge equivalent to $O(N)_{\text{global}} \times [O(N-3) \times O(3)]_{\text{local}}$. Our starting point is the most general HLS Lagrangian of such a scale-invariant version of the Grassmannian model presented as an extension of the SM, see Eq. (29) with Eq. (30), Eq. (31) and Eq. (32). The model is simply reduced to the SM Higgs Lagrangian when we take $N = 4 , p = 3$, Eq. (33), where $L_V^{(N-p)}$ term is missing. Also the dynamical generation of the $O(N-3)_{\text{local}}$ HLS gauge bosons is not possible in the large $N$ arguments where all the planar diagrams come into play and hence are uncontrollable. If the kinetic term is not generated, then the term Eq. (32) is simply solved away by the equation of motion of the $O(N-3)_{\text{local}}$ HLS gauge bosons as the bare auxiliary field. So we focused on the dynamical generation of the HLS gauge boson of the $O(3)_{\text{local}}$, the SM rho, with the simplified Lagrangian omitting Eq. (32), consisting of the standard form of the HLS Lagrangian, $L_A + aL_V$, besides the dilatonic factor of the dilaton (SM Higgs) $\varphi$. The $L_A$ is the original nonlinear sigma model without HLS gauge bosons, $aL_V$ an additional term associated with the HLS.

For the convenience in taking the large $N$ limit, we followed the standard way as in the $CP^{N-1}$ model, to rewrite the classical Lagrangian $L_A + aL_V$ into the form of the covariant derivative, Eq. (34), and used Lagrange multiplier $\eta(x)$ for the constraint for the nonlinear realization (in the scale-invariant form). This we showed directly corresponds to the $a = 2$ choice in the HLS model, although it is equivalent to any value of $a$ through the use of the equation of motion of the auxiliary field $\rho_\mu(x)$ (or adding $L_V(=0)$ with arbitrary weight to change $a$), as far as the classical level without kinetic term of $\rho_\mu$ is concerned. The theory at classical level does not depend on $a$ as a matter of course.

Based on Eq. (35), we obtained the effective action in large $N$ limit for the $D$ dimensions with $2 \leq D \leq 4$, which yields the gap equation, Eq. (36), in a form similar to that of the NJL model and to other Grassmannian models in $D$ dimensions. Namely, the inverse coupling $1/G = g^2/\pi^2N$ receives quantum correction (power divergence) denoted by $1/G_{\text{crit}} = \Lambda^{D-2}/g_{\text{crit}}$ as $v^2 = 1/G - 1/G_{\text{crit}} (G < G_{\text{crit}})$, and $1/G - 1/G_{\text{crit}} = -v^2/\eta (G > G_{\text{crit}})$.

As such, it changes the phase continuously from the broken phase ($\nu \neq 0, v^2_\eta = 0 = \eta = (\eta(x))$, the same as the bare theory) in the weak coupling region, Eq. (37), into the unbroken phase ($\nu = 0, \eta \neq 0$, genuine quantum theory) in the strong coupling one, Eq. (38). The phase transition is the second order as is the case for other Grassmannian models including the $CP^{N-1}$ model and the NJL model. Hence the critical point $G_{\text{crit}}$ is a nontrivial ultraviolet fixed point for the dimensionless coupling $g = G\Lambda^{D-2}$, see Eq. (39):

$$\beta(g) = -(D - 2) \frac{g}{g_{\text{crit}}} (g - g_{\text{crit}}), \quad g_{\text{crit}} = (4\pi)^{D/2} \left( \frac{D}{2} - 1 \right) \Gamma \left( \frac{D}{2} \right), \quad (143)$$

and the same form for the renormalized coupling $g^{(R)}$ defined in Eq. (40), for $D \neq 4$.

However, for $D = 4$, the “renormalized coupling” $g^{(R)}$, after tuning $v^2$ through the quadratic divergence, actually still has a log divergence, Eq. (41), which is regularized here by the cutoff $\Lambda$ (Such a cutoff is needed to define the dynamically generated HLS gauge boson kinetic term any way, which is absent in the tree-level SM Higgs Lagrangian as a counter term.):

$$D = 4: \quad g^{(R)}(\mu) = \frac{(4\pi)^2}{\ln(\Lambda^2/\mu^2)} \rightarrow 0 \quad \left( \frac{\Lambda}{\mu} \rightarrow \infty \right); \quad \rightarrow \infty \quad (\mu \rightarrow \Lambda), \quad \beta(g^{(R)}(\mu)) = \frac{\partial g^{(R)}(\mu)}{\partial \mu} = \frac{2}{(4\pi)^2} \left( g^{(R)}(\mu) \right)^2. \quad (144)$$

This implies a trivial infrared fixed point, although the bare coupling $g(\Lambda)$ has a nontrivial UV fixed point $g(\Lambda) \rightarrow g_{\text{crit}}$ (now understood as a Gaussian fixed point, trivial theory). The cutoff $\Lambda$ is nothing but a Landau pole, corresponding to the extra free parameter to define the nonperturbative quantum theory for the dynamically induced HLS gauge boson kinetic term absent in the tree SM Higgs Lagrangian.

We then found that similarly to the $CP^{N-1}$ model, the HLS gauge boson is dynamically generated in the large $N$ limit for the $D$ dimensions with $2 \leq D \leq 4$ as in Eq. (42), which takes the form of Eq. (43) in the broken phase, and that of Eq. (44) in the unbroken phase, respectively. For $2 \leq D < 4$ the theory is renormalizable and no extra free parameters are induced.

On the other hand, in $D = 4$ the log divergence in the nonperturbatively generated kinetic term as regularized by the cutoff $\Lambda$ cannot be renormalized in a usual sense due to absence of the counter term of the kinetic term in the SM Lagrangian, thus giving rise to an extra free parameter, the induced HLS gauge coupling $g_{\text{HLS}}$ related to $\Lambda$, in sharp contrast to the perturbative SM which never generates such a kinetic term of the HLS gauge boson.

In the broken phase, the induced gauge coupling $g_{\text{HLS}}$ for the kinetic term Eq. (45) is given in Eq. (46) and $\rho_\mu$ has a mass $M_\rho^2 = 2g_{\text{HLS}}^2 F_\pi^2$, Eq. (47), in a typical form of the Higgs mechanism.
In the unbroken phase, on the other hand, kinetic term with \(g_{\text{HLS}}\) is given in Eq. (79) and the unbroken gauge symmetry is realized by the massless \(\rho\), namely, the unbroken gauge symmetry is no longer “hidden” but explicit, as is the case in the well-known phenomenon of \(CP^{N-1}\) model. The NG boson \(\pi\) in the classical theory is no longer the NG boson but has a mass \(M_\pi^2 = \eta\), Eq. (88).

Then our main results for the SM Higgs case were obtained as the \(D = 4\) and \(N \to 4\) case of the above generic results. The dynamically generated kinetic term and the mass of the SM rho \(\rho\) read as Eq. (80) and Eq. (87):

\[
\begin{align*}
\text{SM} : & \quad \frac{1}{\lambda_{\text{HLS}}(\mu^2)} = \frac{1}{N g_{\text{HLS}}^2(\mu^2)} = \frac{1}{3} \frac{1}{(4\pi)^2} \ln \left( \frac{\tilde{\Lambda}^2}{\mu^2} \right), \\
M_\rho^2(\mu^2) = & \ g_{\text{HLS}}^2(\mu^2) \cdot F_\rho^2, \\
F_\rho^2 = & \ 2 \cdot N \mu^2 = 2 \cdot F_\pi^2 \approx 2 \cdot (246 \text{ GeV})^2 \approx (350 \text{ GeV})^2, 
\end{align*}
\]

(145)

where \(\mu^2 = M_\rho^2 = \eta \neq 0\) is understood in the unbroken phase with \(M_\rho^2(\mu^2) \equiv 0\), and the “on-shell” \(M_\rho^2\) in the broken phase with \(\eta \equiv 0\) is defined by the solution of \(M_\rho^2 = M_\rho^2(\mu^2) = g_{\text{HLS}}^2(M_\rho^2) \cdot F_\rho^2\). Then the induced HLS gauge coupling \(\alpha_{\text{HLS}}(\mu^2) = g_{\text{HLS}}^2(\mu^2)/(4\pi)\) has a Landau pole \(\alpha_{\text{HLS}}(\mu^2) \to \infty (\mu^2 \to \tilde{\Lambda}^2)\) as in Eq. (85), and is asymptotically non-free, i.e., has an infrared zero:

\[
\begin{align*}
\text{SM} : & \quad \beta(\alpha_{\text{HLS}}(\mu^2)) = \mu^2 \frac{\partial \alpha_{\text{HLS}}(\mu^2)}{\partial \mu^2} = \frac{N}{12\pi} \alpha_{\text{HLS}}^2(\mu^2) > 0, 
\end{align*}
\]

(146)

similarly to the \(g_R(\mu)\) in Eq. (144), which is in accord with the fact that the UV fixed point for the original bare coupling \(g = g_{\text{crit}} = (4\pi)^2\) is a Gaussian fixed point (free theory). In fact both \(\pi\) and \(\rho\) become massless free particles just on the fixed point with vanishing coupling \(\alpha_{\text{HLS}}(\mu^2) \to 0\) (\(g \to g_{\text{crit}}\)) as \(\tilde{\Lambda}^2/\mu^2 \to \infty\).

We further studied possible \(a\)–dependence of our results. The classical theory without the kinetic term is obviously independent of \(a\), since the auxiliary field \(\rho_\mu(x)\) can simply be solved away via equation of motion. On the other hand, the theory at quantum level sometimes crucially depend on \(a\). The different parameterization of the same classical theory may lead to different quantum theory, particularly in the nonperturbative dynamics.

In fact, as we repeatedly emphasized (see discussions related to Eqs. (79), (110) and (121)), in the case \(a = 0\), i.e., the CCWZ nonlinear realization without gauge symmetry, the HLS, the quantum theory becomes ill-defined in the unbroken phase, where the transversality of the two-point function clearly indicates the existence of the massless vector meson pole, while were it not for the gauge symmetry, the HLS, it cannot be inverted to the well-defined propagator. This is indeed consistent with the Weinberg-Witten theorem which forbids the massless particle with spin \(J \geq 1\) within the positive definite Hilbert space (i.e., without gauge symmetry).

Even in the broken phase, the quantum theory which acquired the kinetic term could in principle depend on \(a\). The classical equation of motion is generally violated as dictated by the Ward-Takahashi identities as in Eq. (113). More explicitly we showed the \(\rho_\mu\) propagator acquires an unusual \(a\)–dependent contact term in Eq. (107), in conformity with Eq. (113).

An outstanding off-shell physics of such in the broken phase is the skyrmion physics, which does depend on \(a\). In the limit \(a \to \infty\) we recover the classical equation of motion \(\rho_\mu = \alpha_{\mu,||} = i \frac{\partial}{\partial \rho_\mu} \phi_\mu \phi^\dagger\) even at the quantum level, see Eq. (113), or explicit calculations of each corresponding propagator, Eq. (118), and Eq. (119) (or Eq. (107)). Then the dynamically generated kinetic term of the SM rho \(\rho_\mu\) is entirely replaced by the Skyrme term, Eq. (135).

For all those \(a\)–dependences, however, we also showed that as far as the on-shell quantities are concerned, all the \(a\)–dependences are (apparently) miraculously cancelled out in the large \(N\) limit, as in Eq. (107), to leave them completely independent of \(a\), Eq. (110), in sharp contrast to the simple one-loop result in Eq. (98).

We further showed notable \(a\)–independent relations, Eq. (125), Eq. (126) and Eq. (127) as generalized form of the KSRF I relation, the universality of the \(\rho_\mu\) coupling, and the KSRF II relation, respectively. To our surprise, the outstanding off-shell physics, vector meson dominance (VMD), is also realized independently of \(a\), see Eq. (130).

Now to the possible phenomenological implications of the SM rho which were discussed in the subsection V.1D. First of all, as we mentioned in the Introduction, we emphasize that the success of the conventional perturbative SM (pSM) results is intact in our unconventional parameterization based on the nonlinear realization Eq. (12), which is equivalent to the standard one Eq. (3) as far as the perturbation is concerned. See e.g., Ref. [31] for more generic parameterization for the perturbative calculations. The same is true also for its gauge equivalent HLS Lagrangian.
since the perturbation does not generate the kinetic term of the HLS gauge boson with the $a\mathcal{L}_V$ term identically zero without any physical effects. Also note that the pSM in the infrared region are intact even by the nonperturbative dynamics in the ultraviolet region, somewhat analogously to the QCD where success of the perturbative QCD in the ultraviolet does not contradict the nonperturbative QCD in the infrared (reversed infrared-ultraviolet is due to the asymptotically free QCD versus asymptotically non-free (infrared free) SM Higgs, or infrared Landau pole vs. ultraviolet Landau pole).

Then what about the physical effects of the nonperturbative dynamics, namely the dynamical gauge boson SM rho $\rho_\mu$? As given in the subsection VTD, we may expect $\rho_\mu$ signature at either collider physics or dark matter physics, depending on the possible value of a single free parameter $M_\rho$, or equivalently $g_{HLS}$ or the cutoff $\Lambda = e^{-1/3} \Lambda$; either $\Lambda < M_\rho (g_{HLS} > 6.7, M_\rho > 2.3 \text{TeV})$ or $\Lambda > M_\rho (g_{HLS} < 6.7, M_\rho < 2.3 \text{TeV})$ (see Eq. [139]).

1) “Low $M_\rho$ scenario” ($M_\rho < 2.3 \text{TeV}, \Lambda > M_\rho$, collider detection):

A typical example is $M_\rho = 2 \text{TeV} (g_{\rho\pi\pi} \simeq 5.7)$, which is a simple scale-up of the QCD $\rho$ meson, thus is perfectly natural with $\Lambda \simeq 3.3 \text{TeV} \simeq 4\pi F_\pi$. This yields the “broad width” $\Gamma_\rho \simeq \Gamma_{\rho \to WW} \simeq g_{\rho\pi\pi}^2 M_\rho/(4\pi) \simeq 433 \text{GeV}$, which, although a scale-up of the $\rho$ meson width, may be barely detectable at LHC. For larger (smaller) $M_\rho$ the width gets larger (smaller) as $\sim M_\rho^2$, and the production cross section gets smaller (larger) as $\sim 1/M_\rho^2$, thus more difficult for $M_\rho > 2 \text{TeV}$ to be seen at LHC. The SM rho with narrow resonance $\Gamma_\rho \lesssim 100 \text{GeV}$ if any could be detected at LHC for $M_\rho \lesssim 1.2 \text{TeV}$, which corresponds to $g_{HLS} \lesssim 3.5$ and $\Lambda \gtrsim 50 \text{TeV}$.

2) “High $M_\rho$ scenario” ($M_\rho \gg 2.3 \text{TeV}, \Lambda < M_\rho$, as a stabilizer of the skyrmion dark matter $X_s$):[1]

Even if no direct evidence were seen at the collider experiments, physical effects of the dynamical $\rho_\mu$ are still observable through the skyrmion dark matter $X_s$ in the SM. In fact the SM skyrmion is stabilized by the off-shell $\rho_\mu$ in the short distance physics as shown in Ref. [1], the result of which corresponds to $a \to \infty$ calculation, while the results are numerically similar even for $a \sim 2$.[12]. The HLS coupling is extremely large $g_{HLS} = \mathcal{O}(10^3)$, which yields $M_{X_s} \lesssim \mathcal{O}(10) \text{GeV}$ consistent with the direct detection of the dark matter, and in rough agreement with the relic abundance of the dark matter: $\Omega_{X_s} h^2 \simeq 0.1$[1][12]. Note the cutoff is $\Lambda = e^{-1/3} \Lambda \simeq e^{-1/3} \cdot M_\rho = \mathcal{O}(10^3 \text{TeV})$, where $M_\rho = g_{HLS} \cdot F_\rho$ is a typical mass scale (no longer the “on-shell” mass, since the SM rho is deeply off-shell).

In either scenario, the phenomenologically interesting nonperturbative SM physics has typical strong SM rho gauge coupling $g_{HLS} \simeq 1/3 - 10^{-3}$, which will have the cutoff $\Lambda = \mathcal{O}(10^6 - 10^2)$ TeV close to the weak scale, thus resolving the naturalness problem. Note that $g_{HLS}$ diverges at the near low scale Landau pole $\Lambda = e^{4/3} \Lambda \simeq 3.8 \text{GeV}$, even if the Higgs self coupling is still perturbative and pSM perfectly makes sense.

This indicates that the quadratic divergence corrections to the weak scale $\delta F_\pi^2 \sim 4 \cdot \Lambda^2/(4\pi)^2 \sim (0.1 \text{TeV} - 10 \text{TeV})^2$ (see the gap equation Eq. [58]). This also suggests a possibility that the SM in the full nonperturbative formulation eventually reveals itself as a “dual” to a possible BSM underlying theory with such a scale, similarly to the hadron-quark duality (nonlinear sigma model/chiral Lagrangian vs QCD), or as an analogue of the Seiberg duality to be discussed below.

Another possible phenomenological impact would be the phase transition from the broken phase to the unbroken phase in the early Universe; the unbroken phase consists of the 3 massless HLS gauge bosons $\rho_\mu$’s and the 3 massive $\pi$’s (no longer the NG bosons), plus 3 massive $\tilde{\rho}$’s (no longer the would-be NG bosons to be absorbed into $\rho_\mu$ in the broken phase), plus 6 other spinless massive modes (corresponding to the 6 constraints in the broken phase), one of which is a pseudo-dilaton in the unbroken phase, see Eq. [24]. This is quite different from the conventional linear sigma model picture of the unbroken phase having degenerate massive 3 $\pi$’s and 3 $\tilde{\rho}$’s in Eq. [3]. Although the zero temperature phase transition is the second order, the finite temperature phase transition could be different, in which case electroweak phase transition in the early Universe would be quite different from the conventional one.

As is easily seen from the calculations in the present paper, it is straightforward to do the same analyses as done here for other Grassmannian manifolds $U(N)/[U(N-p) \times U(p)]$ including $CP^{N-1}$ based on $G/H = U(N)/[U(N-1) \times U(1)]$, the result being precisely the same for the above relations, i.e., $\rho$-universality, KSRF I,II, and VMD.

One notable example is the SUSY QCD with $N_c, N_f$ for $N_c < N_f < 3 N_c/2$, near the conformal window, whose effective theory is described by the Grassmannian manifold $G/H = SU(N_f)/[SU(N_c) \times SU(N_f-N_c)]$ and further by the “magnetic gauge theory” in the Seiberg duality.[13] It was already pointed out [10, 33] that the HLS is a concrete realization of the magnetic gauge theory of the Seiberg duality in such a way that $G/H = SU(N_f)/[SU(N_c) \times SU(N_f-N_c)] \simeq G_{\text{global}} \times H_{\text{local}} = [SU(N_f)]_{\text{global}} \times [SU(N_c) \times SU(N_f-N_c)]_{\text{local}}$ for $N_c < N_f < 3 N_c/2$. It in fact corresponds to $N = N_f \to \infty$ with $p = N_f - N_c$ fixed in the present case. We have mentioned that such a limit realizes the dynamical generation of the HLS only for $SU(N_f-N_c)_{\text{local}} = SU(p)_{\text{local}}$ but not $SU(N_c)_{\text{local}} = SU(N-p)_{\text{local}}$.

Incidentally, this limit is nothing but the “anti-Veneziano limit” [20] of the “large $N_f$ QCD”, $N_c, N_f \to \infty$ with $N_f/N_c = \text{fixed} (> 1)$, near conformal window, a concrete realization of the walking technicolor which predicted a technidilaton[14], a composite Higgs behaving in the same way as the SM Higgs as a pseudo-dilaton described in the present paper. So this limit is relevant to the SQCD near the conformal window as well. In fact the magnetic
gauge theory $SU(N_f - N_c)$ in the Seiberg duality is an IR free theory, similarly to the present case Eq.(143). It was further argued \cite{44} that Seiberg duality implies the VMD in the SQCD. Here we showed that the same result is indeed realized dynamically in the large $N$ limit, independently of the parameter $a$, even in the non-SUSY QCD.

So, whatever underlying theory beyond the SM Higgs might be, the SM Higgs equivalent to the Grassmannian model as it stands should be regarded in its own right as a consistent quantum theory on the basis of the nonperturbative formulation. Hence all the nonperturbative results given in the present paper are the dynamical consequences of the SM itself and must also be realized in a possible underlying theory such as the walking technicolor, as far as such a theory has the same symmetry realization $G/H$.

Furthermore, on the basis of the duality between macroscopic theory and its microscopic one, if there exists an underlying theory of the Standard Model, it must also have a spontaneously broken approximate scale symmetry to realize the 125 GeV Higgs as a pseudo-dilaton, besides its internal symmetry $G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V \simeq O(4)/O(3)$, as given in the form of Eq.(12). An immediate candidate for such a UV completion is the walking technicolor \cite{17, 18}, \#28, which indeed has a spontaneously broken approximate scale symmetry and its pseudo-dilaton, “technidilaton”, as a composite Higgs, and at the same time has a large anomalous dimension $\gamma_m \simeq 1$ to suppress the problematic Flavor-Changing Neutral Currents (FCNC) when extended to include the mass of the SM quarks and leptons.

In fact, possible candidate gauge theories for the walking technicolor have been searched for on the lattice, particularly in the “large $N_f$ QCD” with $N_f \gg N_c = 3$, which is expected to be close to the conformal window in accord with the anti-Veneziano limit mentioned above, $N_c,N_f \to \infty$ with $N_f/N_c$ = fixed ($>1$). In particular, it was discovered \cite{64, 65} that $N_f = 8$ QCD has a light flavor-singlet scalar on the lattice as a candidate for the technidilaton (Such a light scalar was also found in $N_f = 12$ QCD \cite{66, 67}, possibly as a remnant of the conformal window). Further studies on this line will be decisive for revealing a possible underlying theory beyond the SM.

On the same token, it would be even more important to check whether or not the dynamical generation of the HLS gauge boson $\rho_0$ of the SM presented here is the case on the lattice, not just in the large $N$ limit dynamics. So far only triviality studies were made based on the conventional linear sigma model parameterization Eq.(Higgs1).

However, different parameterization of the same classical theory could lead to different quantum theory, as we have seen in the present mode which would become ill-defined in the unbroken phase, unless the gauge symmetry, the HLS, is explicitly introduced. It is well known that latticizing gauge theories is equivalent to nonlinear realization or vice versa \cite{68}. In other words, the nonlinear realization is nothing but the gauge theories on the lattice, the same as the HLS. Thus the fully nonperturbative lattice simulations of the SM Higgs Lagrangian in terms of the parameterization based on the nonlinear realization Eq.(12) (and its HLS version Eq.(A22)) could be different from the conventional parameterization Eq.(49).

Incidentally, one might think that the SM, when combined with the Yukawa coupling and the electroweak gauge coupling, has no Landau pole below the Planck scale in the perturbative calculations, so that there would be no urgent motivation for studying the nonperturbative dynamics. This arguments would make sense, if these logically independent different parts theories in the SM were inter-correlated at deeper level such as in “the final theory”, which is unfortunately no more than a dream theory at this moment. Otherwise, such a result is just a phenomenon of the accidental cancellation in the running coupling coefficients among separate theories having a Landau pole of their own, namely, the result itself would need logical explanation.

We now conclude with a generic remark: although useful, the concept of HLS itself in the broken phase is not literally of a rigorous physical sense. This is actually the case for any spontaneously broken gauge symmetry including the SM electroweak gauge symmetry, namely, the concept of the spontaneously broken “gauge symmetry” does not make real sense, unless the coupling is very small !! It is really the convenience of the description, while the dynamical formation of the massive vector bound is a real fact, independently of the “gauge symmetry” as we showed in Eq.(120). As in the case of the SM electroweak theory, however, there is a notable exception where the HLS is very useful even in the broken phase, though not mandatory, that is, near the phase transition point, Eq.(83), where the induced HLS coupling is (conceptually) small: $N_{\rho} g^2_{\text{HLS}} \simeq 3(4\pi)^2/\ln(\Lambda^2/M^2_\rho) \ll 1 (\Lambda^2/M^2_\rho \gg 1)$, or $g \to g_{\text{crit}} (\Lambda^2/\mu^2 \to \infty)$. Indeed,

\#28 Similar studies for suppressing the FCNC are also made without notion of the anomalous dimension and the scale symmetry/technidilaton \cite{68}. For a recent review of the walking technicolor and technidilaton in view of the LHC experiments on the Higgs boson see Ref.\cite{69}.
this is the case for the $\rho$ meson in the QCD, where the HLS is rather useful even though the $\rho$ coupling is not very small numerically (similarly to the QCD itself where the large $N_c$ limit works well for $1/N_c = 1/3$, not very small numerically). Also some off-shell dynamics like skyrmion physics, the HLS is a useful concept even for the massive (unstable) $\rho$, which is far away from the phase transition point.

On the other hand, in the unbroken phase, the gauge symmetry is mandatory, not just useful, namely the HLS comes into a rigorous reality. Were it not for the HLS, the quantum theory in the unbroken phase is ill-defined as in the case of $CP^{N-1}$, consistently with the Weinberg-Witten Theorem [22] which forbids massless spin $J \geq 1$ particles in quantum theory on the positive definite Hilbert space (i.e., without gauge symmetry), see Appendix [21].

So, if a theory ought to be a well-defined quantum theory independently of all possible different phases in the nonperturbative sense, we necessarily have to introduce the HLS, notwithstanding the fact that its presence in the broken phase is no more than the convenience of description (redundancy of the description), except for the near phase transition point and some off-shell physics like the skyrmion physics.

Finally, our results are not restricted to the SM Higgs Lagrangian but to the generic nonlinear sigma model of the same $G/H = O(4)/O(3) \simeq [SU(2)_L \times SU(2)_R]/SU(2)_V$, with/without nonlinearly realized (approximate) scale symmetry, since we showed that the dynamical results obtained in the large $N$ limit are not sensitive to the presence of the pseudo-dilaton $\phi$. Then it is readily applied to the two-flavored QCD in the chiral limit. #29

In particular, the so-called successful $a = 2$ results of the $\rho$ meson, i.e., $\rho$-universality, KSRF I and II, and vector meson dominance (VMD), are now proved to be realized for any $a$ for the dynamical gauge boson of the HLS, and thus are simply nonperturbative dynamical consequences in the large $N$ limit but not a mysterious parameter choice $a = 2$.

The dynamically generated kinetic term has a new free parameter, the $\rho$ coupling (related to the cutoff or Landau pole, Eq. [1]), which is adjusted to the reality as $g_{\rho \pi} = 9_{\text{HLS}} \simeq 5.9$ corresponding to $m_\rho = g_{\rho \pi} f_\rho = \sqrt{2} g_{\text{HLS}} f_\pi \simeq 770$ MeV ($f_\pi \simeq 92$ MeV), Eq. [2]. This implies the cutoff (related to the Landau pole) $\Lambda = \Lambda e^{-4/3} = m_\rho e^{4(4\pi^2)/(8g_{\text{HLS}}^2)} e^{-4/3} \simeq 1.1$ GeV which coincides with the breakdown scale of the chiral perturbation theory $\Lambda \chi \simeq 4\Lambda f_\pi$.

The fact is a most remarkable triumph of the nonlinear sigma model as an effective field theory including full nonperturbative dynamics. It in fact becomes a direct evidence of the dynamically generation of the HLS gauge boson in QCD!! Phrased differently, QCD knows the Grassmannian manifold! Or, Nature chooses Grassmannian manifold as the effective theory of QCD-like theories.

If the phase transition at zero temperature shown in this paper is also applied to the finite temperature or finite density phase transition, the unbroken phase is quite different from the conventional view of the relevant QCD phase transition. The unbroken phase in the chiral limit would be accompanied by massless $\rho$ mesons of the unbroken HLS gauge symmetry (“magnetic gauge symmetry” of the Seiberg dual, as pointed out in [16, 33]), while $\pi$’s are no longer the NG bosons and hence are all massive, degenerate with the $\rho$’s which are no longer the would-be NG bosons absorbed into the $\rho_\mu$. The $\rho$’s, if existed, have exotic quantum number and hence are not simple $q \bar{q}$ bound states, maybe $q q$ (color-flavor locking?), or multi-quarks or even glueballs, or mixtures of them? The quark-gluon plasma discovered at RHIC is seemingly still a strongly coupled system (non-Abelian Coulomb phase?), and may retain some bound states such as those in the unbroken phase found in the present work. We shall come back to this point in the future.

The result obtained here in the large $N$ limit is based on the equivalence $G/H = SU(2)_L \times SU(2)_R/SU(2)_V \simeq O(4)/O(3)$ and its extension to $G/H = O(N)/(O(N - 3) \times O(3))$ in the large $N$ limit. As it stands, such a large $N$ limit is not available for the $N_f \geq 3$ QCD, just the same situation as the skyrmion model whose $N_f \geq 3$ extension is highly involved. Extension of the present result to the $N_f \geq 3$ QCD is certainly a challenging project for the future.

Acknowledgments

We would like to thank Taichiro Kugo for very helpful numerous discussions including crucial suggestions and some concrete calculations, without which the paper would not have been materialized in its present form. We also thank Shinya Matsuzaki and Hiroshi Ohki for collaborations on Ref. [1, 42], which motivated the present work, and also for fruitful discussions. Stimulating discussions with Masayasu Harada and Masaharu Tanabashi are also appreciated.

#29 Incidentally, our results also imply that some “composite Higgs model” based on $G/H = SO(N)/(SO(N - p) \times SO(p))$ (e.g., $SO(5)/SO(4)$ [60]) should have the dynamical gauge boson of $SO(p)_{\text{local}} \text{HLS}$ by the nonperturbative dynamics of the large $N$ in a way independent of $a$, without recourse to the UV completion.
Here we review the generic CCWZ nonlinear realization \cite{12,13} and its generic HLS version \cite{8,9} and its scale-invariant version equivalent to the SM Higgs Lagrangian.

1. CCWZ Nonlinear realization for the SM

The system having the symmetry \(G\) spontaneously broken into \(H(\subset G)\) may be described in terms of the Nambu-Goldstone variables \(\pi\) in the CCWZ nonlinear realization:

\[
\xi(\pi) = e^{i\pi X_a/F_a}(\in G/H), \quad T_A = \{ S_a \in G, X_a \in G - H \}, \quad \text{tr}(T_AT_B) = \frac{1}{2}\delta_{AB}, \quad \pi = \pi_a X_a, \quad (A1)
\]

which transforms under \(G\) as

\[
\xi(\pi) \rightarrow h(\pi,g) \xi(\pi) g^\dagger, \quad (h \in H \text{ and } g \in G). \quad (A2)
\]

We define Maurer-Cartan one-form taking Lie-algebra value \(\alpha_\mu(\pi) = \alpha_\mu^a(\pi)\):

\[
\alpha_\mu(\pi) = \partial_\mu \xi(\pi) \cdot \xi^\dagger(\pi)/i = \frac{1}{i} \left[ \frac{i}{F_\mu} \partial_\mu \pi + \frac{1}{2!} \left( \frac{i}{F_\mu} \right)^2 [\pi, \partial_\mu \pi] + \frac{1}{3!} \left( \frac{i}{F_\mu} \right)^3 [\pi, [\pi, \partial_\mu \pi]] + \cdots \right]
\]

\[
= \alpha_{\mu,\|}(\pi) + \alpha_{\mu,\perp}(\pi),
\]

\[
\alpha_{\mu,\perp}(\pi) \equiv 2\text{tr}(\alpha_\mu(\pi)X_a) \cdot X_a = \frac{1}{F_\mu} \partial_\mu \pi + \cdots,
\]

\[
\alpha_{\mu,\|}(\pi) \equiv 2\text{tr}(\alpha_\mu(\pi)S_a) \cdot S_a = \frac{i}{2F_\mu} [\pi, \partial_\mu \pi] + \cdots, \quad (A3)
\]

where we confined ourselves to the symmetric coset space \([G-H, G-H] \subset H\) such that \(\alpha_{\mu,\perp}(\pi)\) contains only odd number of \(\pi\) and \(\alpha_{\mu,\|}(\pi)\) does even number of \(\pi\), respectively. They transform as

\[
\alpha_\mu(\pi) \rightarrow h(\pi,g) \alpha_\mu(\pi) h^\dagger(\pi,g) - i\partial_\mu h(\pi,g) \cdot h^\dagger(\pi,g),
\]

\[
\alpha_{\mu,\|}(\pi) \rightarrow h(\pi,g) \alpha_{\mu,\|}(\pi) h^\dagger(\pi,g) - i\partial_\mu h(\pi,g) \cdot h^\dagger(\pi,g),
\]

\[
\alpha_{\mu,\perp}(\pi) \rightarrow h(\pi,g) \alpha_{\mu,\perp}(\pi) h^\dagger(\pi,g),
\]

and hence the invariant as the Lagrangian takes the form

\[
\mathcal{L}_{\text{CCWZ}} = F_\pi^2 \cdot \text{tr}(\alpha_{\mu,\perp}(\pi))^2 = \text{tr}((\partial_\mu \pi)^2 + \cdots) = \frac{1}{2} (\partial_\mu \pi_a)^2 + \cdots. \quad (A7)
\]

For \(G/H = SU(2)_L \times SU(2)_R/SU(2)_V\), with \(S^a/X^a = (T^a_2 \pm T^a_L)/2\), we may write

\[
U(x) = e^{\frac{2\pi}{F_\pi} \xi(\pi)} = \xi(\pi) \cdot \xi^\dagger(\pi) = [\xi^\dagger(\pi)]^\dagger \cdot \xi(\pi), \quad \left( \xi(\pi) = e^{i\frac{2\pi}{F_\pi}} \right),
\]

\[
\rightarrow g_L U(x) g_R^\dagger,
\]

\[
(\xi(\pi), \xi^\dagger(\pi)) \rightarrow h(\pi,g) (\xi(\pi), \xi^\dagger(\pi)) g_R^\dagger. \quad (A9)
\]

The Maurer-Cartan one-form reads

\[
\alpha_\mu(\pi)_{(R,L)} = (\partial_\mu \xi(\pi) \cdot \xi^\dagger(\pi), \partial_\mu \xi^\dagger(\pi) \cdot \xi(\pi))/i,
\]

\[
\alpha_{\mu,\|}(\pi) = (\partial_\mu \xi(\pi) \cdot \xi^\dagger(\pi) + \partial_\mu \xi^\dagger(\pi) \cdot \xi(\pi))/(2i),
\]

\[
\alpha_{\mu,\perp}(\pi) = (\partial_\mu \xi(\pi) \cdot \xi^\dagger(\pi) - \partial_\mu \xi^\dagger(\pi) \cdot \xi(\pi))/(2i). \quad (A10)
\]

and then an invariant yields the CCWZ Lagrangian:

\[
\mathcal{L}_{\text{CCWZ}} = F_\pi^2 \text{tr} \left( \alpha_{\mu,\perp}(\pi) \right) = \frac{F_\pi^2}{4} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right), \quad (A11)
\]
where use has been made of $\alpha_{\mu,\perp}(\tau) = (\partial_\mu \xi(\tau) \cdot \xi^\dagger(\tau) - \partial_\mu \xi^\dagger(\tau) \cdot \xi(\tau))/2i = \xi^\dagger(\tau)\partial_\mu U(x)\xi(\tau)/(2i) = \xi(\tau)\partial_\mu U^\dagger(x)\xi^\dagger(\tau)/(2i)$. Then the SM Lagrangian with $F_\tau = v$, Eq. (12), is further rewritten as

$$L_{SM} = \chi^2(\varphi) \cdot \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + v^2 \text{tr} (\alpha_{\mu,\perp}(\tau))^2 \right] - V(\varphi). \quad \text{(A12)}$$

2. SM Higgs Lagrangian as a Scale-invariant HLS model [2]

The SM Higgs Lagrangian was further shown [2] to be gauge equivalent to the scale-invariant version [14] of the Hidden Local Symmetry (HLS) Lagrangian [6, 9, 10], which contains possible new vector boson, “SM rho”, hidden behind the SM Higgs Lagrangian, as an analogue of the QCD scale-invariant HLS Lagrangian.

In the generic case, nonlinear sigma model based on the manifold $G/H$, the HLS model having a symmetry $G_{global} \times H_{local}$ can be written in terms of the base $\xi(x)$ transforming as $39$

$$\xi(x) \rightarrow h(x) \cdot \xi(x) \cdot g^\dagger, \quad h(x) \in H_{local}, \quad g \in G_{global}, \quad \text{A13}$$

which may be parameterized as

$$\xi(x) = \xi(\hat{\rho}) \cdot \xi(\pi), \quad \xi(\hat{\rho}) = \exp \left( \frac{i}{F_\rho} \hat{\rho} \right), \quad \xi(\pi) = \exp \left( \frac{i}{F_\pi} \pi \right), \quad \text{A14}$$

with $\hat{\rho}$ and $F_\rho$ being the would-be NG boson to be absorbed into the HLS gauge boson $\rho_{\mu}$ by the Higgs mechanism and its decay constant, respectively (See the discussions below, where $\xi(\hat{\rho}) = \text{e}^{i\hat{\rho}/F_\rho}$ transforms as $\xi(\hat{\rho}) \rightarrow h(x) \cdot \xi(\hat{\rho}) \cdot h^\dagger(\pi, g)$ and the CCWZ base $\xi(\pi)$ does as $\xi(\pi) \rightarrow h(\pi, g) \cdot \xi(\pi) \cdot g^\dagger$. The Maurer-Cartan one-form reads:

$$\alpha_{\mu}(x) = \frac{1}{i} \partial_\mu \xi(x) \cdot \xi^\dagger(x) = \frac{1}{i} \partial_\mu \xi(\hat{\rho}) \cdot \xi^\dagger(\hat{\rho}) + \xi(\hat{\rho}) \cdot \alpha_{\mu}(\pi) \cdot \xi^\dagger(\hat{\rho}),$$

$$\alpha_{\mu,\perp}(x) = \frac{2}{i} \text{tr} \left( \partial_\mu \xi(x) \cdot \xi^\dagger(x) X_a \right) X_a = \xi(\hat{\rho}) \alpha_{\mu,\perp}(\pi) \xi^\dagger(\hat{\rho}),$$

$$\alpha_{\mu,||}(x) = \frac{2}{i} \text{tr} \left( \partial_\mu \xi(x) \cdot \xi^\dagger(x) S_a \right) S_a = \frac{1}{i} \partial_\mu \xi(\hat{\rho}) \cdot \xi^\dagger(\hat{\rho}) + \xi(\hat{\rho}) \alpha_{\mu,||}(\pi) \xi^\dagger(\hat{\rho})$$

$$= \frac{1}{i} \left[ \frac{1}{F_\rho} \partial_\mu \hat{\rho} + \frac{1}{2i} \left( \frac{i}{F_\rho} \right)^2 \left[ \hat{\rho}, \partial_\mu \hat{\rho} \right] + \frac{1}{2i} \left( \frac{i}{F_\rho} \right)^3 \left[ \hat{\rho}, \left[ \hat{\rho}, \partial_\mu \hat{\rho} \right] \right] + \cdots \right] + \xi(\hat{\rho}) \alpha_{\mu,||}(\pi) \xi^\dagger(\hat{\rho}), \quad \text{A15}$$

where use has been made of $\text{tr} \left( \partial_\mu \xi(\hat{\rho}) \cdot \xi^\dagger(\hat{\rho}) X_a \right) = 0$.

When we fix the gauge of HLS as $\xi(x) = \xi(\pi)$ (unitary gauge $\xi(\hat{\rho}) = 1$, $\hat{\rho} = 0$). $H_{local}$ and $G_{global} \subset G_{global}$ get simultaneously broken spontaneously (Higgs mechanism), leaving the diagonal subgroup $H = H_{local} \perp H_{global}$, which is nothing but the subgroup $H$ of the original $G$ of $G/H$; $H \subset G$. Then the extended symmetry $G_{global} \times H_{local}$ is simply reduced back to the original nonlinear realization of $G$ on the manifold $G/H$, both are gauge equivalent to each other.

The HLS gauge boson, $\rho_{\mu}(x)$, is introduced as usual by a covariant derivative as

$$D_\mu \xi(x) = \partial_\mu \xi(x) - i \rho_{\mu}(x) \xi(x), \quad \rho_{\mu} = \rho^a_{\mu} S_a, \quad \text{A16}$$

which transforms in the same way as $\xi(x)$. The covariantized Maurer-Cartan one-form reads:

$$\hat{\alpha}(x) = \frac{1}{i} D_\mu \xi(x) \cdot \xi^\dagger(x) = \alpha_{\mu}(x) - \rho_{\mu}(x) = \hat{\alpha}_{\mu,\perp}(x) + \hat{\alpha}_{\mu,||}(x),$$

$$\hat{\alpha}_{\mu,\perp}(x) = \frac{2}{i} \text{tr} \left( D_\mu \xi(x) \cdot \xi^\dagger(x) X_a \right) X_a = \alpha_{\mu,\perp}(x) = \xi(\hat{\rho}) \alpha_{\mu,\perp}(\pi) \xi^\dagger(\hat{\rho}),$$

$$\hat{\alpha}_{\mu,||}(x) = \frac{2}{i} \text{tr} \left( D_\mu \xi(x) \cdot \xi^\dagger(x) S_a \right) S_a = \alpha_{\mu,||}(x) - \rho_{\mu}(x). \quad \text{A17}$$

They both transform as

$$\hat{\alpha}_{\mu,\perp,||} \rightarrow h(x) \cdot \hat{\alpha}_{\mu,\perp,||} \cdot h^\dagger(x). \quad \text{A18}$$
Thus the HLS Lagrangian consists of two invariants:

\[ \mathcal{L}_{\text{HLS}} = \mathcal{L}_A + a \mathcal{L}_V, \]

\[ \mathcal{L}_A = F^2_\pi \text{tr} \left( \alpha^2_{\mu,\perp}(x) \right) = F^2_\pi \text{tr} \left( \alpha^2_{\mu,\perp}(\pi) \right) - \text{tr} \left( \left( \partial_\mu \pi \right)^2 + \cdots \right) = \frac{1}{2} (\partial_\mu \pi_a)^2 + \cdots, \]

\[ \mathcal{L}_V = F^2_\pi \text{tr} \left( \alpha^2_{\mu,||}(x) \right) = F^2_\pi \text{tr} \left( \alpha_{\mu,||}(x) - \rho_\mu(x) \right)^2 \]

\[ = F^2_\pi \text{tr} \left[ \frac{1}{2} \partial_\mu \hat{\rho} - \frac{i}{2 F^2_\pi} [\partial_\mu \hat{\rho}, \hat{\rho}] - \frac{i}{2 F^2_\pi} [\partial_\mu \pi, \pi] \right] - \rho_\mu(x) + \cdots \right]. \tag{A19} \]

Note that \( \mathcal{L}_A \) is identical to the original nonlinear sigma model based on \( G/H \), since \( \xi(\hat{\rho}) \) carrying the gauge transformation in the representation Eq.\([A14]\) has been traced out in \( \mathcal{L}_A \), which is equivalent to taking the unitary gauge \( \xi(\hat{\rho}) = 1(\hat{\rho} = 0) \). On the other hand, a new term \( a \mathcal{L}_V \) contains the kinetic term of the \( \hat{\rho} \), which is normalized as the canonical one by the requirement:

\[ a = \frac{F^2_\pi}{F^2_\pi}. \tag{A20} \]

The \( \rho_\mu \) field at classical level is merely an auxiliary field without kinetic term and hence can be solved away by the equation of motion

\[ \rho_\mu = \alpha_{\mu,||}(x), \text{ i.e., } a \mathcal{L}_V \equiv 0. \tag{A21} \]

Thus the HLS Lagrangian Eq.\([A19]\) at classical level is gauge equivalent to the original nonlinear sigma model based on \( G/H \): \( \mathcal{L}_{\text{HLS}} = \mathcal{L}_A \).

In the case at hand, the SM in the form of Eq.\([12]\), the gauge-equivalent HLS Lagrangian having the SM rho, \( \rho_\mu \), reads:

\[ \mathcal{L}_{\text{SM-HLS}} = \chi^2(\varphi) \cdot \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + \mathcal{L}_A + a \mathcal{L}_V \right] - V(\varphi), \]

\[ \mathcal{L}_A = F^2_\pi \text{tr} \left( \alpha_{\mu,\perp}(x) \right)^2 = F^2_\pi \text{tr} \left( \alpha_{\mu,\perp}(\pi) \right)^2 = \text{tr} \left( (\partial_\mu \pi)^2 + \cdots \right), \]

\[ a \mathcal{L}_V = F^2_\pi \text{tr} \left( \alpha_{\mu,||}(x) \right)^2 = F^2_\pi \text{tr} \left[ \left( \rho_\mu - \frac{1}{F^2_\pi} \partial_\mu \hat{\rho} \right) - \frac{1}{2 F^2_\pi} [\partial_\mu \hat{\rho}, \hat{\rho}] - \frac{i}{2 F^2_\pi} [\partial_\mu \pi, \pi] + \cdots \right]^2, \]

\[ F^2_\rho = a F^2_\pi = a v^2, \tag{A22} \]

which is obviously reduced back to Eq.\([12]\) (and hence the original SM Lagrangian Eq.\([3]\)) by solving away the auxiliary field \( \rho_\mu \) as Eq.\([A21]\) with fixing the gauge \( \xi(\hat{\rho}) = 1(\hat{\rho} = 0) \) and/or using the parameterization Eq.\([A14]\).

i) Parameterization for \( SU(2)_L \times SU(2)_R/SU(2)_V \equiv [SU(2)_L \times SU(2)_R]_{\text{global}} \times [SU(2)_V]_{\text{local}} \):

A more familiar notation in this case is by dividing \( U(x) \) into two parts \([10,10]\):

\[ U(x) = e^{2 \chi(x)/\sqrt{\rho}} = \xi^1_L(x) \cdot \xi_R(x), \tag{A23} \]

where \( \xi_{R,L}(x) \) transform under \( G_{\text{global}} \times H_{\text{local}} \) as

\[ \xi_{R,L}(x) \rightarrow h(x) \cdot \xi_{R,L}(x) \cdot g_{R,L}^\dagger, \quad U(x) \rightarrow g_L U(x) g_R^\dagger, \quad (h(x) \in H_{\text{local}}, g_{R,L} \in G_{\text{global}}). \tag{A24} \]

The \( H_{\text{local}} \) is regarded as a gauge symmetry of group \( H \) arising from the redundancy (gauge symmetry) how to divide \( U \) into two parts. \( \xi_{R,L} \), which can be parameterized as

\[ (\xi_R(x), \xi_L(x)) = (\xi(\hat{\rho}) \cdot (\xi(\pi), \xi(\pi)^\dagger) \tag{A25} \]

The covariant derivative reads

\[ D_\mu \zeta_{R,L}(x) = \partial_\mu \zeta_{R,L}(x) - i \rho_\mu(x) \zeta_{R,L}(x), \quad \rho_\mu = \rho_\mu \frac{\tau^a}{2}, \tag{A26} \]
and the Maurer-Cartan one-forms are
\[
\{\hat{\alpha}_{\mu,R,L}(x), \hat{\alpha}_{\mu,\perp}(x)\} \rightarrow h(x) \cdot \{\hat{\alpha}_{\mu,R,L}(x), \hat{\alpha}_{\mu,\perp}(x)\} \cdot h^\dagger(x),
\]
\[
\hat{\alpha}_{\mu,R,L}(x) = \frac{1}{i} D_\mu \xi_{R,L}(x) \cdot \xi^\dagger_{R,L}(x) = \frac{1}{i} \partial_\mu \xi_{R,L}(x) \cdot \xi^\dagger_{R,L}(x) - \rho_\mu(x),
\]
\[
\hat{\alpha}_{\mu,\perp}(x) = \frac{1}{2} (\hat{\alpha}_{\mu,R}(x) \pm \hat{\alpha}_{\mu,L}(x)) = \left\{ \frac{\alpha_{\mu}^{|}(x) - \rho_\mu(x)}{\alpha_{\mu}(x)} \right\},
\]
(A27)

where
\[
\hat{\alpha}_{\mu}^{|}(x) = \frac{1}{2i} \left( D_\mu \xi_{R}(x) \cdot \xi^\dagger_{R}(x) + D_\mu \xi_{L}(x) \cdot \xi^\dagger_{L}(x) \right) = \frac{1}{F_\rho} \partial_\mu \hat{\rho} - \frac{i}{2F_\rho^2} [\partial_\mu \hat{\rho}, \hat{\rho}] - \frac{i}{2F_\rho^2} [\partial_\mu \pi, \pi] + \cdots,
\]
\[
\hat{\alpha}_{\mu}(x) = \frac{1}{2i} \left( D_\mu \xi_{R}(x) \cdot \xi^\dagger_{R}(x) - D_\mu \xi_{L}(x) \cdot \xi^\dagger_{L}(x) \right) = \alpha_{\mu}(x) = \xi(\hat{\rho}) \cdot \alpha_{\mu}(\pi) \cdot \xi^\dagger(\hat{\rho})
\]
\[
= \frac{1}{2i} \xi_{L} \left( \partial_\mu U \cdot U^\dagger \right) \xi^\dagger_{L} = \frac{1}{2i} \xi_{R} \left( \partial_\mu U^\dagger \cdot U \right) \xi^\dagger_{R}.
\]
(A28)

Then \( L_A \) takes a familiar form:
\[
L_A = F_\rho^2 \text{tr} \left( \alpha_{\mu,\perp}^2(x) \right) = F_\rho^2 \text{tr} \left( \alpha_{\mu,\perp}^2(\pi) \right) = \frac{F_\rho^2}{4} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right),
\]
(A29)

where again the gauge-variant field \( \xi(\hat{\rho}) \) in the parameterization Eq. (A25) is traced out in \( L_A \), which is equivalent to taking the unitary gauge. While \( aL_V \) takes the same form as Eq. (A22).

ii) Parameterization for \( O(4)/O(3) \simeq O(4)_{\text{global}} \times O(3)_{\text{local}} \):

The CCWZ base \( \xi(x) \) for the \( O(4)/O(3) \) given in Eq. (18) is extended to the HLS base \( \xi^{(4)}(x) = \xi^{(4)}(\hat{\rho}) \cdot \xi^{(4)}(\pi) \) as in the generic case Eq. (A14), with a different normalization \( \text{tr}(T_A T_B) = 2\delta_{AB}, T_A^\dagger = -T_A \). Everything is the same as the generic case, Eqs. (A15) - (A19), except for the normalization and we have the HLS version of the SM in terms of this parameterization corresponding to Eq. (A22):

\[
L_{\text{SM-HLS}} = \chi^2(\varphi) \cdot \left[ \frac{1}{2} \left( \partial_\mu \varphi \right)^2 + L_A + aL_V \right] - V(\varphi),
\]
\[
L_A = \frac{F_\rho^2}{4} \text{tr} \left( \alpha_{\mu,\perp}^{(4)}(x) \right)^2 = \frac{F_\rho^2}{4} \text{tr} \left( \alpha_{\mu,\perp}^{(4)}(\pi) \right)^2 = \frac{1}{4} \text{tr} \left( (\partial_\mu \pi)^2 + \cdots \right),
\]
\[
aL_V = \frac{F_\rho^2}{4} \text{tr} \left( \hat{\alpha}_{\mu,\perp}(x) \right)^2 = \frac{F_\rho^2}{4} \text{tr} \left[ \left( \rho_\mu - \frac{1}{F_\rho} \partial_\mu \hat{\rho} \right) - \frac{i}{2F_\rho^2} [\partial_\mu \hat{\rho}, \hat{\rho}] - \frac{i}{2F_\rho^2} [\partial_\mu \pi, \pi] + \cdots \right]^2,
\]
\[
a_\rho^2 = aF_\rho^2 = av^2.
\]
(A30)

This form is the basis for the Grassmannian N-extension \( O(N)/[O(N-p) \times O(p)] \simeq O(N)_{\text{global}} \times [O(N-p) \times O(p)]_{\text{local}} \) as the main target of the present paper.

3. Physical Implications of the Dynamical Generation of the HLS Gauge Boson

If the SM rho as an auxiliary field \( \rho_\mu \) acquires the kinetic term
\[
L^{(\rho)}_{\text{kinetic}} = -\frac{1}{2g_{\text{HLS}}^2} \text{tr} \rho_{\mu\nu}^2
\]
by the quantum corrections, with \( g_{\text{HLS}} \) being the induced gauge coupling of the HLS, then the quantum theory for the SM Higgs would take the form:
\[
L^{\text{quantum}}_{\text{SM-HLS}} = \chi^2 \cdot \left( \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{v^2}{4} \cdot \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + F_\rho^2 \cdot \text{tr} \left( \rho_\mu^2 - \alpha_{\mu}^2 \right) \right) - V(\varphi)
\]
\[
- \frac{1}{2g_{\text{HLS}}^2} \text{tr} \rho_{\mu\nu}^2 + \cdots.
\]
(A32)
This form of the Lagrangian is the same as that of the effective theory of the one-family ($\rho \pi \pi$ gauge fields to a case), we would have for when ($R$ with $L$) are treated as external source fields (thus NG bosons $\pi$ are not absorbed into $W/Z$ as in the QCD case), we would have for $a = 2$ the celebrated “vector meson dominance (VMD)” where direct coupling of electroweak gauge fields to $\pi \pi$ are cancelled between those from $L_A$ and $L_V$ terms, so that the couplings go only through mixing with the SM rho $\rho_\mu$: 

$$g(\rho_\mu/L_\mu)\pi\pi = 1 - \frac{a}{2} = 0, \quad (a = 2), \quad \text{(Vector Meson Dominance)}$$

Note that the HLS gauge boson acquires the scale-invariant mass term thanks to the dilaton factor $\chi^2$, the nonlinear realization of the scale symmetry, in sharp contrast to the Higgs (pseudo-dilaton) which acquires mass only from the explicit breaking of the scale symmetry.

The electroweak gauge bosons ($\in R_\mu(L_\mu)$) are introduced by extending the covariant derivative of Eq. (A25) this time by gauging $G_{\text{global}}$, which is independent of $H_{\text{local}}$ in the HLS extension:

$$D_\mu \xi_{R,L}(x) \Rightarrow \hat{D}_\mu \xi_{R,L}(x) \equiv \partial_\mu \xi_{R,L}(x) - i \rho_\mu(x) \xi_{R,L}(x) + i \xi_{R,L}(x) R_\mu(L_\mu).$$

We then finally have a gauged s-HLS version of the Higgs Lagrangian (gauged-s-HLS): 

$$L_{\text{Higgs-HLS}}^{\text{gauged}} = \chi^2(x) \cdot \left[ \frac{1}{2} (\partial_\mu \varphi)^2 + \hat{L}_A + a \hat{L}_V \right] - V(\varphi) + L^{(\rho,L,R)}_{\text{kinetic}} + \cdots,$$

with

$$\hat{L}_{A,V} = L_{A,V} \left( D_\mu \xi_{R,L}(x) \Rightarrow \hat{D}_\mu \xi_{R,L}(x) \right).$$

When ($R_\mu,L_\mu$) are treated as external source fields (thus NG bosons $\pi$ are not absorbed into $W/Z$ as in the QCD case), we would have for $a = 2$ the celebrated “vector meson dominance (VMD)” where direct coupling of electroweak gauge fields to $\pi \pi$ are cancelled between those from $L_A$ and $L_V$ terms, so that the couplings go only through missing with the SM rho $\rho_\mu$: 

$$g(\rho_\mu/L_\mu)\pi\pi = 1 - \frac{a}{2} = 0, \quad (a = 2), \quad \text{(Vector Meson Dominance)}$$

---

#30 This form of the Lagrangian is the same as that of the effective theory of the one-family ($N_F = 8$) walking technicolor [14], except for the shape of the scale-violating potential $V(\varphi)$ which has a scale dimension 4 (trace anomaly) in the case of the walking technicolor instead of 2 of the SM Higgs case (Lagrangian mass term).
In the present SM case, the standard Higgs mechanism in $\mathcal{L}_A$ yields the conventional $W/Z/\gamma$ mass mixings in the SM, while $\mathcal{L}_V$ term yields additional mass mixing $\rho - W/Z/\gamma$ with the $\gamma$ mass staying exactly zero after diagonalization as usual. The VMD in Eq. (A35) implies that $q\bar{q} \rightarrow W/Z/\gamma \rightarrow W_L W_L/W_L Z_L(\pi\pi)$, can only go through the Drell-Yang process as a production of $\rho_\mu$ due to the $W/Z/\gamma - \rho_\mu$ mixing: $W/Z/\gamma \rightarrow \rho_\mu \rightarrow W_L W_L/W_L Z_L$, with the coupling $\sim \alpha_{em} g_\rho / M_\rho = \alpha_{em} F_\rho / M_\rho = \alpha_{em} / g_{HLS}$. This is similar to the walking technirho described also by the s-HLS effective theory \[ g_{HLS} \equiv \frac{\alpha_{em} g_\rho}{M_\rho} \].

Eq. (A35) also yields a notable $a$-independent relation (KSRF I) between the $\rho - \gamma$ mixing strength $g_\rho$ and $g_{\rho\pi\pi}$ from the mass term $\chi^2 a \mathcal{L}_V$ \[ [2, 4] \] (low energy theorem of HLS \[ 8, 9 \]: Proof in Ref. \[ 55 \]):

\[
g_\rho = M_\rho F_\rho = g_{HLS} F_\rho^2 = \left( \frac{2}{a} g_{\rho\pi\pi} \right) \cdot a F_\pi^2 = 2 g_{\rho\pi\pi} F_\pi^2 \quad (a \text{- independent}) , \quad \text{(KSRF I)} ,
\]

in accord with the $a$-independence to be shown later in the case of the dynamically generated HLS gauge boson.

Also note that the extension to $W/Z/\gamma - \rho$ mixing strength should be intact in the scale-invariant mass term which simply carries the extra dilaton factor $\chi^2$. In fact the mass terms including the couplings of all the SM particles, except for the Higgs mass term $V(\varphi)$, are dimension 4 operators due to $\chi^2$ and thus are scale-invariant, yet giving the same mass as in the case without the $\chi^2$ factor.

A salient feature of the scale-invariance of the $\rho_\mu$ mass term in $\chi^2 \cdot a \mathcal{L}_V$ is the absence of the coupling of $\rho_\mu - W/Z - \varphi$ (“conformal barrier”), also similarly to the walking technirho \[ 2, 4 \]. The SM Higgs $\varphi$ resides in the overall factor $\chi^2$ and hence can be coupled to the each gauge boson $\rho_\mu, W/Z$ only in the form of the simultaneous mass diagonalization, hence has no off-diagonal couplings $\rho_\mu - W/Z$. This is in sharp contrast to many other vector meson models having the scale-violating mass term of $\rho_\mu^2$ (dimension 2), dominance of which would lead to the so-called “equivalence theorem result” $\Gamma(\rho_\mu \rightarrow WW/WZ) \sim \Gamma(\rho_\mu \rightarrow WZ + \varphi)$. Thus the $\rho_\mu$ in the present case with only scale-invariant mass term predominantly decays to diboson channels $\rho_\mu \rightarrow W_L W_L/W_L Z_L$ with the coupling $g_{\rho\pi\pi} = g_{HLS}$ for universality.

Appendix B: Dynamical Generation of the HLS Gauge Boson in $CP^{N-1}$ Model

In this appendix we elaborate on Refs. \[ 9, 10 \] for the dynamical generation of the HLS gauge boson in $CP^{N-1}$ model. The $CP^{N-1}$ model is renormalizable in $D$ dimensions for $2 \leq D < 4$, and thus is a well-defined laboratory to test the nonperturbative quantum effects. It is in fact well established \[ 3, 2, 21, 27, 28 \] that the $U(1)$ hidden local symmetry (HLS) gauge boson introduced as an auxiliary field at classical level of the $CP^{N-1}$ model does in fact generate the kinetic term at quantum level by the nonperturbative dynamics in $1/N$ expansion. See e.g., Ref. \[ 22 \] for an excellent description of this phenomenon in $D = 2$. For $D = 4$ the theory is a cutoff theory, nevertheless the dynamical generation of the kinetic term of the HLS gauge boson is operative in exactly the same manner as for $2 < D < 4$, see \[ 2, 10 \]. The cutoff formulation we adopt here in $D = 4$ is also established in a slightly different formulation (“effective theory” made finite by all possible counter terms, i.e., with extra free parameters, as in the Chiral Perturbation Theory \[ 23 \]).

The $CP^{N-1}$ model is a nonlinear sigma model based on the complex Grassmannian coset space $G/H = U(N)/[U(N - p) \times U(p)] |_{p=1} \simeq SU(N)/[SU(N-1) \times U(1)]$ written in terms of the massless NG bosons living in the manifold $G/H$ at classical level. As we emphasized in the text, any nonlinear sigma model has HLS \[ 9 \], and as such the classical $CP^{N-1}$ Lagrangian is most commonly given in the form invariant under $G_{\text{global}} \times H_{\text{local}} = SU(N)_{\text{global}} \times U(1)_{\text{local}}$, with $H_{\text{local}} = U(1)_{\text{local}}$ being the Hidden Local Symmetry (HLS) ($H \subset G$):

\[
\mathcal{L}_{\text{HLS}} = D_\mu \phi^\dagger D^\mu \phi - \eta(x) \left( \phi^\dagger \phi - N/G \right) , \quad \text{(B1)}
\]

where $\phi$ is an $N$-component complex scalar field $\phi$ with a constraint:

\[
\hat{\phi} \equiv (\phi_1, \phi_2, \ldots, \phi_N) , \quad \phi_a \in \mathbb{C} , \quad \phi^\dagger \phi = N/G \quad (G : \text{coupling constant}) , \quad \text{(B2)}
\]

where the field $\eta(x)$ is a Lagrange multiplier, and $D_\mu \phi$ is the $U(1)_{\text{hidden}}$ covariant derivative given by $D_\mu \phi = (\partial_\mu - i A_\mu) \phi$. The reason why $SU(N)_{\text{local}}$ is not usually discussed is that its gauge boson carries the index running
The dimensions of the quantities $O$ are as follows: $d_\phi = D/2 - 1 = d_v, d_\eta = 2, d_{A_\mu} = 1, d_G = 2 - D, d_{F_{\mu\nu}} = 4$. 

1, \ldots, N - 1, so that the all planar graphs not just one-loop are involved in the large $N$ limit, which makes the analyses impossible in contrast to the $U(1)_{\text{local}}$. This is the same situation as for the $O(N - 3)_{\text{local}}$ in the Grassmannian model $G/H = O(N)/[O(N - 3) \times O(3)]$ discussed in the main text of the present paper. Note that the theory also has a scale symmetry at classical level.  

Since the HLS gauge field $A_\mu$ is an auxiliary field having no kinetic term, it can be eliminated by using the equation of motion:

$$A_\mu = -i \frac{G}{2N} \phi^\dagger \partial_\mu \phi \left( f \partial_\mu g - f \partial_\mu g \right).$$

Then the classical Lagrangian \[\text{Eq.}(B1)\] is equivalent to

$$L^{\text{HLS}} = \partial_\mu \phi^\dagger \partial^\mu \phi + \frac{G}{4N} \left( \phi^\dagger \partial^\mu \phi \right)^2 - \eta \left( \phi^\dagger \phi - N/G \right),$$

which still retains the $U(1)_{\text{local}}$ invariance: $\phi'(x) = e^{i\theta(x)} \phi(x)$. Since $\phi$ has $2N$ real components and is constrained by one real condition in Eq. \[\text{Eq.}(B2)\], then reduced to $2N - 1$ degrees of freedom, and by the $U(1)_{\text{local}}$ gauge invariance we can gauge away one further component of $\phi$, leaving $2N - 2$ degrees of freedom which are exactly the dimension of the manifold $CP^{N-1} = SU(N)/[SU(N - 1) \times U(1)]$. Thus by gauge fixing of the HLS $U(1)_{\text{local}}$ symmetry, Eq. \[\text{Eq.}(B1)\] is further reduced to the genuine $CP^{N-1}$ nonlinear sigma model based on the manifold $G/H = SU(N)/[SU(N - 1) \times U(1)]$:

$$L^{\text{NLS}} = \partial_\mu v^\dagger \partial^\mu v \left[ 1 + \frac{G}{4N} v^\dagger v \right]^{-2} + G \left( v^\dagger \partial^\mu v \right)^2 \left[ 1 + \frac{G}{4N} v^\dagger v \right]^{-4},$$

where $v \equiv (v^1, v^2, \ldots, v^{N-1})$ is an unconstrained $N - 1$ component complex variables standing for $2N - 2$ independent degrees of freedom.

Eq. \[\text{Eq.}(B5)\] is certainly gauge equivalent to the HLS model Eq. \[\text{Eq.}(B2)\] at classical level, in exactly the same sense as the gauge equivalence between Eq. \[\text{Eq.}(3)\] (and hence Eq. \[\text{Eq.}(12)\]) and the HLS Lagrangian Eq. \[\text{Eq.}(A22)\] in the case of the SM. Note that the classical theory takes the form of the spontaneously broken phase, broken down to $H$, written in terms of the NG boson fields living on the coset $G/H$. At quantum level, however, the theory gets in broken phase in the strong coupling region (for all coupling region in $D = 2$ dimensions) where both become different in such a way that Eq. \[\text{Eq.}(D3)\] becomes ill-defined, while Eq. \[\text{Eq.}(B1)\] is well-defined, due to the very existence of the dynamical generation of the massless HLS gauge boson acquiring the kinetic term through the nonperturbative dynamics, as we see in the below.

Let us consider the effective action for the Lagrangian Eq. \[\text{Eq.}(B1)\] with a symmetry $G_{\text{global}} \times H_{\text{local}} = SU(N)_{\text{global}} \times U(1)_{\text{local}}$: In the leading order of the $1/N$ expansion it is evaluated as

$$\Gamma [\phi, \lambda] = \int d^Dx \left[ D_\mu \phi^\dagger D^\mu \phi - \eta \left( \phi^\dagger \phi - N/G \right) \right] + iN \text{Tr} \ln (-D_\mu D^\mu - \eta).$$

Because of the $SU(N)$ symmetry, the VEV of $\varphi$ can be written in the form

$$\langle \phi(x) \rangle = \left( 0, 0, \ldots, \sqrt{N} \right).$$

Then the effective action \[\text{Eq.}(B5)\] gives the effective potential for $v$ and $\eta \equiv \langle \eta(x) \rangle \langle A_\mu \rangle = 0$ as

$$\frac{1}{N} V(v, \eta) = \eta (v^2 - 1/G) + \int \frac{d^Dk}{i(2\pi)^D} \ln (k^2 - \eta).$$

The stationary conditions of this effective potential are given by

$$\frac{1}{N} \frac{\partial V}{\partial v} = 2\lambda v = 0,$$

$$\frac{1}{N} \frac{\partial V}{\partial \eta} = v^2 - 1/G + \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{\eta - k^2} = 0.$$
Similarly to the USO so that the equation reads a finite relation for $2 \leq D < 4$.

The first condition (B11) is realized in either of the cases

$$\begin{align*}
\eta &= 0 \ (v \neq 0) \ , \ \text{case (i)} , \\
v &= 0 \ (\eta \neq 0) \ , \ \text{case (ii)} .
\end{align*}$$

(B11)

The case (i) corresponds to the broken phase of the $SU(N)_{\text{global}} \times U(1)_{\text{local}}$ symmetry and scale symmetry, and case (ii) does to the unbroken phase of $SU(N)_{\text{global}} \times U(1)_{\text{local}}$ symmetry but the broken phase of the scale symmetry.

The second stationary condition (B10) gives relation between $\eta$ and $v$. By putting $\eta = v = 0$ in Eq. (B10), the critical point $G(\equiv G(\Lambda)) = G_{\text{crit}}(\equiv G_{\text{crit}}(\Lambda))$ separating the two phases in Eq. (B11) is determined as

$$\frac{1}{G_{\text{crit}}} = \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{-k^2} = \frac{1}{G} - \frac{1}{G_{\text{crit}}} = \frac{1}{G(R)} - \frac{1}{G_{\text{crit}}(\mu)} ,$$

(B12)

which implies that $G_{\text{crit}} \to 0$ for $D \to 2$.

Substituting Eq. (B12) into the second stationary condition (B10), we obtain

$$v^2 - \int \frac{d^Dk}{i(2\pi)^D} \left( \frac{1}{-k^2} - \frac{1}{\eta - k^2} \right) = \frac{1}{G} - \frac{1}{G_{\text{crit}}} = \frac{1}{G(R)} - \frac{1}{G_{\text{crit}}(\mu)} ,$$

(B13)

where we have defined the renormalized coupling at renormalization point $\mu^2$ as

$$\begin{align*}
\frac{1}{G(R)} &= \frac{1}{G(R)(\mu)} = \frac{1}{G} - \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{\mu^2 - k^2} , \\
\frac{1}{G_{\text{crit}}(\mu)} &= \int \frac{d^Dk}{i(2\pi)^D} \left( \frac{1}{-k^2} - \frac{1}{\mu^2 - k^2} \right) \\
&= \Gamma\left(2 - D/2\right) \frac{\mu^{D/2}}{(4\pi)^{D/2}} ,
\end{align*}$$

(B14)

so that the equation reads a finite relation for $2 \leq D < 4$ as it should since it is a renormalizable theory.

The stationary condition in Eq. (B13), combined with Eq. (B9), leads to the cases (i) (broken phase of $SU(N)_{\text{global}} \times U(1)_{\text{local}}$) and (ii) (unbroken phase of $SU(N)_{\text{global}} \times U(1)_{\text{local}}$) in Eq. (B11), respectively;

$$\begin{align*}
\text{(i)} \ G < G_{\text{cr}} \Rightarrow \langle \varphi_N \rangle &= \sqrt{N} v \neq 0 \ , \ \langle \eta(x) \rangle = \eta = 0 \\
&= \frac{1}{G(\Lambda)} - \frac{1}{G_{\text{crit}}(\Lambda)} = \frac{1}{G(R)(\mu)} - \frac{1}{G_{\text{crit}}(\mu)} = v^2 > 0 ,
\end{align*}$$

(iii) $G > G_{\text{cr}} \Rightarrow \langle \varphi_N \rangle = \sqrt{N} v = 0 \ , \ \langle \eta(x) \rangle = \eta \neq 0$

$$\begin{align*}
\frac{1}{G(\Lambda)} - \frac{1}{G_{\text{crit}}(\Lambda)} &= \frac{1}{G(R)(\mu)} - \frac{1}{G_{\text{crit}}(\mu)} \\
&= -\frac{\Gamma\left(2 - D/2\right) v^{D/2 - 1}}{(4\pi)^{D/2}} \equiv -v_0^2 < 0 .
\end{align*}$$

(B15)

(B16)

The gap equations Eq. (B15) and Eq. (B16) take the same form as that of the $D$-dimensional NJL model which is also renormalizable for $2 \leq D < 4$, with opposite sign and the same sign, respectively. (See also Eq. (C3) for $D = 4$ NJL model). Both (i) and (ii) are in broken phase of the scale symmetry by $v \neq 0$ and $\eta \neq 0$, respectively. #32

The case (i) is the perturbative phase where the classical theory structure remains. Eq. (B15) is the gap equation for the spontaneous breaking of the symmetry $SU(N)_{\text{global}} \times U(1)_{\text{local}}$, or equivalently (after gauge fixing) the coset space $SU(N)/[SU(N-1) \times U(1)]$, with the Higgs mechanism of $U(1)_{\text{local}}$ yielding the “mass” of $A_\mu$: $(M_0^2)^2 = 2N v^2$ (with mass dimension $D - 2$), as read from Eq. (B1), with Eq. (B2). The scale symmetry is also spontaneously broken.

#32 Similarly to the $D$-dimensional NJL model [48, 49], we may define dimensionless coupling $g$ as $G \equiv g/\Lambda^{D-2}$ and $g_{\text{crit}} = (D/2 - 1)\Gamma(D/2)/(4\pi)^{D/2}$, where the beta function in both phases takes the form similar to that in the $D$-dimensional NJL model [48, 49]: $\beta(g) = \frac{\Lambda g}{\Lambda g_{\text{fixed}}} = -(D-2) g (g - g_{\text{crit}})/g_{\text{crit}}$ (the same form for $g^{(R)}(\mu) = G(R)(\mu)^{D-2}$ at $D < 4$), where $g_{\text{crit}}$ is a UV fixed point.
by the same $v \neq 0$, with the pseudo-dilaton identified with the real part of the $\phi_N$ (its mass from the trace anomaly due to the regularization with $\Lambda$, or the renormalization with $\mu$).

The case (ii) is a genuine nonperturbative phase in strong coupling $G > G_{\text{crit}}$. It implies that the quantum theory is actually in the unbroken phase of $SU(N)_{\text{global}} \times U(1)_{\text{local}}$, although the theory at classical level is written in terms of the NG boson variables $\phi$ in the coset $SU(N)/[SU(N-1) \times U(1)]$ as if it were in the broken phase. The HLS gauge symmetry $U(1)_{\text{local}}$ thus is never spontaneously broken and the gauge boson if exists as a particle should be massless. In fact, the would-be NG bosons $\phi$ at classical level are no longer the NG bosons at quantum level by the nonperturbative dynamics (at large $N$) and acquire dynamically the mass

$$M^2_\phi = \eta = \langle \eta(x) \rangle \neq 0, \quad (G > G_{\text{crit}}, \quad v = 0),$$

as readily seen from Eq. (B11). Note that $\langle \eta(x) \rangle$ in Eq. (B11) breaks no internal symmetry but the scale symmetry. Writing $\eta(x) = e^\varphi(x)/\eta$, we may regard $\varphi(x)$ as a pseudo-dilaton in this phase (its mass from the trace anomaly due to the regularization with $\Lambda$, or the renormalization with $\mu$).

Special attention should be paid to $D = 2$ dimensions, where $G_{\text{crit}} = 0$ and hence the case (i) (the classical/perturbative phase, broken phase with $v \neq 0$) does not exist at all, in accord with the Mermin-Wagner-Cooper theorem on absence of the spontaneous symmetry breaking in $D = 2$ dimensions. On the other hand, the gap equation Eq. (B10) with $D = 2$ takes the form $\frac{1}{D} = \frac{1}{2} \ln \frac{\Lambda}{\mu}$, or:

$$\langle \eta(x) \rangle = \Lambda^2 \cdot \exp\left(-\frac{4\pi}{G(\Lambda)}\right) = \mu^2 \cdot \exp\left(-\frac{4\pi}{G^{(R)}(\mu)}\right),$$

where the scale symmetry appears to be spontaneously broken by $\langle \eta(x) \rangle \neq 0$ in the same sense as $D > 2$ (up to explicit breaking due to the trace anomaly), but actually undergoes the BKT phase transition similarly to the $D = 2$ NJL model (Gross-Neveu model), see footnote #18.

We now discuss the dynamical generation of the kinetic term of the $U(1)$ HLS gauge boson.

First we take a look at the case (ii): $(G > G_{\text{crit}}, v = 0, \langle \eta(x) \rangle \neq 0)$, where the massless gauge boson of unbroken $U(1)_{\text{local}}$ HLS gauge symmetry does appear dynamically. Particularly for $D = 2$ this is a whole story, since case (i) does not exist there.

The (amputated) two-point vertex function of the HLS gauge field $A_\mu$ is an auxiliary field at classical level: $\Gamma_{\mu\nu}(x) = \langle A_\mu(x)A_\nu(0)\rangle_{\text{amp}}$. At quantum level at the $1/N$ leading order it has one-loop contributions of the fundamental particles $\phi$. Since the Lagrangian Eq. (B11) has the $U(1)_{\text{local}}$ symmetry, $\Gamma_{\mu\nu}(x)$ must have the form invariant under the gauge symmetry such that:

$$\Gamma_{\mu\nu}(p) = \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \cdot f(p^2).$$

(B19)

Since $\phi$ are now massive, $M^2_\phi = \langle \eta(x) \rangle \neq 0$, in the unbroken phase, the only singularity of $f(p^2)$ arises from the two-$\phi$ threshold $p^2 = (2M_\phi)^2 > 0$ and beyond, and hence has no singularity at $p^2 = 0$, namely $f(0) \neq 0$. Then we see that the two-point Green function develops a genuine massless pole:

$$F(x)(T(A_\mu(x)A_\nu(0))) = -\Gamma_{\mu\nu}(p)^{-1} = \frac{g_{\mu\nu} - f^{-1}(0)}{p^2} + \text{gauge terms},$$

where the “gauge terms” depend on the gauge fixing, and the residue $-f^{-1}(0) > 0$ is characterized by $M^2_\phi = \langle \eta(x) \rangle = \eta \neq 0$. Thus the HLS gauge boson kinetic term reads

$$L_{\text{HLS}}^{(\text{kin})} = -\frac{1}{4g_{\text{HLS}}^2} F^2_{\mu\nu}, \quad \frac{1}{2} g_{\text{HLS}} = -f(0) = \frac{N}{3} \frac{\Gamma(2-D/2)}{(4\pi)^{D/2} \Gamma(2)} M_\phi^{D-4}.$$  

(B21)

Hence the kinetic term of the HLS gauge boson indeed has been dynamically generated by the nonperturbative dynamics at $1/N$ leading order!! Note that the scale symmetry existing at classical level has been broken by the kinetic term, with the scale dimension 4 (not $D$) operator, which is traced back to the spontaneous scale-symmetry breaking due to $\langle \eta(x) \rangle \neq 0$. #33

#33 We may write the kinetic term in a scale-invariant form through the dilaton field $\phi_\eta(x)$:

$$-\frac{1}{4g_{\text{HLS}}^2} \cdot \chi_\eta^{D-4} \cdot F^2_{\mu\nu} = -\frac{1}{4g_{\text{HLS}}^2} \cdot F^2_{\mu\nu} + \cdots,$$

where $\sqrt{\eta(x)} = \sqrt{\Sigma} \cdot \chi_\eta(x)$ with $\chi_\eta(x) \equiv e^{\phi_\eta(x)} / \sqrt{\eta}$. The dimensionless field $\chi_\eta(x)$ has a scale-dimension 1, similarly to $\chi(x)$ in Eq. (11) in the SM case.
Alternatively, it may be better to parameterize freedom due to the absence of the gauge symmetry. Thus the quantum theory of the Lagrangian Eq.(B4) is simply

\[ g \]

which is divergent because of the zero eigenvalue of \( - \). It does not yield the same quantum theory in the nonperturbative dynamics, depending on the parameterization, even if arriving at the same perturbative result.

Corresponding to Eq.(B20), let us look at the two-point Green function for the composite vector operator \( u^i \partial_\mu u \) of the Lagrangian Eq.(B4):

\[ T_{\mu\nu} = \mathcal{F} \mathcal{T} \langle T(u^i(x)\partial_\mu u(x)u^j(0)\partial_\nu u(0)) \rangle \quad (B22) \]

in \( D = 2 \) case where only the unbroken phase \( g > g_{\text{crit}} = 0 \) exists. At leading order of \( 1/N \), \( T_{\mu\nu} \) is given by the infinite geometric series of the one-loop bubble diagram contribution \( B_{\mu\nu} \) as a solution of the equation:

\[
T_{\mu\nu} = B_{\mu\nu} + B^\mu_\rho T_{\rho\nu}, \quad B_{\mu\nu} = g_{\mu\nu} + (g_{\mu\nu}p^2 - p_\mu p_\nu)f(p^2), \quad (B23)
\]

where \( B_{\mu\nu} \) has a gauge non-invariant term \( g_{\mu\nu} \) due to the lack of gauge symmetry in Eq.(B4). Then the solution reads

\[
T_{\mu\rho} = (g_{\mu\nu} - B_{\mu\nu})^{-1} \cdot B^\nu_\rho = - (g_{\mu\nu}p^2 - p_\mu p_\nu)^{-1} f^{-1}(p^2) \cdot B^\nu_\rho, \quad (B24)
\]

which is divergent because of the zero eigenvalue of \( g_{\mu\nu}p^2 - p_\mu p_\nu \), namely non-invertible, since there is no gauge-fixing freedom due to the absence of the gauge symmetry. Thus the quantum theory of the Lagrangian Eq.(B4) is simply ill-defined due to the lack of the gauge symmetry.

The result is in perfect conformity with the Weinberg-Witten theorem [52] which forbids the dynamical generation of the massless particles with spin \( J \geq 1 \). The theorem is proved in the Hilbert space with positive definite metric and hence without gauge symmetry. This is in sharp contrast to the HLS Lagrangian Eq.(B1) which does have a gauge symmetry thus is quantized with indefinite metric Hilbert space, and hence generates a massless gauge boson without conflict with the Weinberg-Witten theorem.

Thus the lesson is: when the theory is parameterized differently and still equivalent to each other at classical level, it may not yield the same quantum theory in the nonperturbative dynamics, depending on the parameterization, even if arriving at the same perturbative result.

Now to the case (i): \( G < G_{\text{crit}}(\neq 0), v \neq 0, (\eta(x)) = 0 \), which exists only for \( D > 2 \), and is more similar to the HLS Lagrangian Eq.(A19) or the SM HLS Lagrangian as its scale-invariant version Eq.(A22) which is gauge equivalent to the original SM Higgs Lagrangian, Eqs.(12) and (3). In this case there remains the symmetry structure at classical level, namely both SU(\( N \))_\text{global} and U(1)_\text{hidden} are spontaneously broken, in such a way that the theory is gauge equivalent to the model based on the manifold \( G/H = SU(N)/[SU(N-1) \times U(1)] \) without HLS. There exist \( 2N - 2 \) massless NG bosons (\( \phi_1, \cdots, \phi_{N-1} \)), and at classical level the HLS gauge boson has a (bare) mass \( (M^2_\text{HLS})^2 = 2Nv^2 \) by the Higgs mechanism absorbing the would-be NG boson \( \hat{\pi} \) in the parametrization \( \phi_N = \hat{\sigma} + i\hat{\pi} \) corresponding to in Eq.(B7), where the \( \hat{\sigma} \) corresponds to \( \hat{\sigma} \) in the SM case, Eq.(B3) #34.

Yet the quantum theory also develops the kinetic term of the HLS gauge boson precisely in the same manner as in the unbroken phase in case (ii):

\[
\Gamma_{\mu\nu}(p) = (p^2g_{\mu\nu} - p_\mu p_\nu) \cdot f(p^2) + g_{\mu\nu} \cdot (2Na^2), \quad (B25)
\]

up to the additional mass term. Similarly to the unbroken phase, it reads the kinetic term \(-\frac{1}{g_{\text{HLS}}^2}F_{\mu\nu}^2 \cdot (g^2_{\text{HLS}} = -f^{-1}(\mu^2)) \) at the scale \( \mu \). Again the dynamically generated kinetic term is a dimension 4 operator and thus breaks the scale symmetry, similarly to \( (\eta(x)) \neq 0 \) in the case (ii), this time by \( \langle \phi_N \rangle = \sqrt{N}v \neq 0 \) (see footnote #33). In this case the HLS gauge boson would have a mass at quantum level,

\[
M^2_\text{HLS} = f^{-1}(M^2_\text{HLS}) (2Na^2) = g^2_{\text{HLS}} \cdot (2Na^2) \quad (B26)
\]

#34 Alternatively, it may be better to parameterize \( \varphi_i(x) = \sigma(x) \cdot e^{i\theta(x)}z_i(x) \), with \( \Sigma_{i=1}^N z_i = N/G \) and \( \langle z_i \rangle = 0 (i = 1, \cdots, N - 1) \), \( z_N = 1/G \), where \( \theta \) is the would-be NG boson absorbed into HLS gauge boson, and \( \sigma = \sqrt{N} \Sigma_{i=1}^N \phi_i \), with the dilaton \( \varphi \) similar to \( \varphi \) as the SM Higgs boson in Eq.(12) with Eq.(11).
(now dimension 2), in precisely the standard form of the Higgs mechanism, with the factor 2 characteristic to all the Grassmannian models, a remnant of the KSRF II relation (see e.g., Eq. (A3)). The result is the same as that read from the Lagrangian at quantum level including the induced kinetic term, after it is rescaled to the canonical form $-\frac{1}{4g^2_{\text{HLS}}^2} F_{\mu\nu}^2 \rightarrow -\frac{1}{4} F_{\mu\nu}^2$.

Actually, the pole of the HLS gauge boson is moved into the Euclidean region having imaginary part of the mass, in such a way that it decays into the massless NG bosons with threshold at $p^2 = (2M_\phi)^2 = 0$, although the generation of the kinetic term is operative to the off-shell dynamics, such as the soliton like the skyrmion as in Ref. [1, 12].

Finally, the $D = 4$ case, which is a cutoff theory, not a renormalizable theory in the usual sense (see Ref. [29] for the effective theory approach). In the same sense as in the D=4 NJL model discussed in Appendix C, we identify the cutoff à la Ref. [30] as a Landau pole, where the dynamically generated kinetic term of the HLS gauge boson disappears, or the induced gauge coupling diverges.

In the case (ii) $G > G_{\text{crit}}$, the gap equation Eq. (B13) or Eq. (B16) with $D = 4$ reads

$$\frac{1}{G} - \frac{1}{G_{\text{crit}}} = -\frac{1}{(4\pi)^2} \frac{\Lambda^2}{\eta} $$

and the kinetic term of the massless HLS gauge boson reads

$$\mathcal{L}_{\text{HLS}}^{(\text{kin})} = -\frac{1}{4g_{\text{HLS}}^2} F_{\mu\nu}^2 \cdot \frac{1}{g_{\text{HLS}}^2} = -f(0) = \frac{N}{48\pi^2} \ln \frac{\Lambda^2}{\eta}.$$ 

which is compared with the renormalizable theory Eq. (B21) with $2 \leq D < 4$. When the momentum integration is done from $\Lambda$ down to $\mu$ in the sense of Wilsonian renormalization group, the gauge coupling reads $1/g_{\text{HLS}}^2 = [N/(48\pi^2)] \ln(\Lambda^2/\mu^2)$. Now the gauge coupling $g_{\text{HLS}}^2$ has a Landau pole at $\mu = \Lambda$ where the dynamically generated kinetic term does vanish: $1/g_{\text{HLS}}^2 \rightarrow 0$ as $\mu \rightarrow \Lambda$. Similarly, in the case (ii) $G < G_{\text{crit}}$ (perturbative/broken phase), the kinetic term of the massive HLS gauge boson is generated as $\mathcal{L}_{\text{HLS}}^{(\text{kin})} = -\frac{1}{4g_{\text{HLS}}^2} F_{\mu\nu}^2$, with $1/g_{\text{HLS}}^2 = -f^{-1}(\mu^2)$, and the mass reads $M_A^2 = f(0)(2N\mu^2) = g_{\text{HLS}}^2 \cdot (2N\mu^2)$.

**Appendix C: Dynamical Generation of the Kinetic Term of the Auxiliary Fields in the NJL Model**

Here we summarize the dynamical generation of the auxiliary fields in the nonperturbative quantum theory of the $SU(2)_L \times SU(2)_R$ NJL model, in a way [34–36] particularly developed by Ref. [36] to reformulate the top quark condensate model of Ref. [53] which has the $SU(2)_R$ violating four-fermion coupling. The quantum dynamical phenomenon discussed below is essentially the same in both cases. Further details including the top quark condensate case are given in, e.g., [4, 5] and references therein.

The NJL Lagrangian reads [33]:

$$\mathcal{L}_{\text{NJL}}^{\text{classical}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + \frac{G}{4} \left[ (\bar{\psi} \gamma^5 \tau^a \psi)^2 + (\bar{\psi} i\tau^a \gamma^5 \tau^a \psi)^2 - \frac{1}{2G} \right] \left( \hat{\sigma} + \frac{G}{\sqrt{2}} \bar{\psi} i\tau^a \gamma^5 \tau^a \psi \right)^2 - \frac{1}{2G} \left( \bar{\psi} i\gamma^5 \tau^a \psi \right)^2 \frac{m_0^2}{G} \left( \hat{\sigma} + \frac{G}{\sqrt{2}} \bar{\psi} \right)$$

where the equation of motion for the auxiliary fields $\hat{\sigma} = -G \bar{\psi} \psi / \sqrt{2}$ and $\hat{\pi}_a = -G \bar{\psi} i\gamma^5 \tau_a \psi / \sqrt{2}$ may be plugged in the second line to get back to the original Lagrangian on the first line.

By integrating out the high frequency modes from the cutoff scale $\Lambda$ down to $\mu$ in the Wilsonian sense at leading order of $1/N_c$ s.t. $N_c \rightarrow \infty$ with $N_c G$ fixed, we have a quantum theory which does generate kinetic term of $\hat{\sigma} / \hat{\pi}_a$ and the quartic coupling:

$$\mathcal{L}_{\text{NJL}}^{1/N_c} = \bar{\psi} \left( i\gamma^\mu \partial_\mu - \frac{1}{\sqrt{2}} (\hat{\sigma} + i\gamma^5 \tau^a \hat{\pi}_a) \right) \psi - \frac{m_0^2(\mu)}{2} \left( \hat{\sigma}^2 + \hat{\pi}_a^2 \right) - \frac{1}{2} Z_\phi(\mu) [ (\partial_\mu \hat{\sigma})^2 + (\partial_\mu \hat{\pi}_a)^2 ] - \frac{\lambda_0(\mu)}{4} \left[ (\hat{\sigma})^2 + (\hat{\pi}_a^2) \right]^2,$$

$$\lambda_0(\mu) = Z_\phi(\mu) = \frac{N_c}{8\pi^2} \ln \frac{\Lambda^2}{\mu^2}, \quad m_0^2(\mu) = \frac{1}{G} - \frac{N_c}{4\pi^2} (\Lambda^2 - \mu^2),$$

(C2)
with \( \tilde{\sigma}, \tilde{\pi}_a \) now being the dynamical tachyon \( m_0^2(0) < 0 \) (in contrast to \( m_0^2 > 0 \)) for \( G > G_{\text{crit}} = \frac{\Lambda^2}{\sqrt{N_c}} \), the phase change from the unbroken phase into the broken one by the quantum effects (quadratic divergence), in accord with the gap equation for the dynamically generated fermion mass \( M_F \neq 0 \) (valid in the formal limit \( \mu \sim M_F \ll v = O(\sqrt{N_c}M_F) \ll \Lambda \)) where \( m_0^2(M_F) \simeq m_0^2(0) \): \[\text{(C3)}\]

\[
\frac{1}{G} - \frac{1}{G_{\text{crit}}} = -2M_F^2 \left( \frac{N_c}{8\pi^2} \ln \frac{\Lambda^2}{M_F^2} \right) = -v^2 < 0, \quad (G > G_{\text{crit}}).
\]

Taking \( \mu \to \Lambda \) we get back to the original classical theory, Eq.\[\text{(C1)}\], with \( m_0^2(\mu) \to \frac{1}{\Lambda} \), \( \lambda_0(\mu) = Z_\phi(\mu) \to 0. \)

After rescaling the kinetic term \( Z_\phi^{1/2}(\mu) (\tilde{\sigma}, \tilde{\pi}_a) \to (\tilde{\sigma}, \tilde{\pi}_a) \) to the canonical one, we have

\[
\mathcal{L}_{\text{NJL}}^{1/N_c} = \bar{\psi} \left( i\gamma^\mu \partial_\mu - \frac{g_Y}{\sqrt{2}} (\tilde{\sigma} + i\gamma^5 \tau^a \tilde{\pi}_a) \right) \psi - \frac{m_0^2}{2} (\hat{\sigma}^2 + \hat{\pi}_a^2) \frac{1}{\Lambda^2} \ln \frac{\Lambda^2}{\mu^2} = g_Y^2 \sqrt{2}, \quad m^2 = m_0^2(\mu) \cdot Z_\phi^{-1}(\mu), \quad (C4)
\]

which is precisely the same form as the SM Higgs Lagrangian Eq.\[\text{(3)}\], plus the Yukawa term, both having the Landau pole \( \lambda = g_Y^2 \to \infty \) at \( \mu \to \Lambda \) ("compositeness condition") \[\text{(4)}\], where \( v^2 = (\sigma^2)_M = (\sigma^2 + \pi^2_a) = \frac{\sqrt{2}}{\sqrt{N_c}}m_F = -m_0^2(M_F) \).

Note that there appear extra free parameter, \( \lambda \) and/or \( g_Y \), which is absent at classical level but does exist in the quantum theory originating from the cutoff \( \Lambda \) in \( D = 4 \), the phenomenon also realized in the SM case with \( g_Y \) (rescaling of the kinetic term) and \( \lambda \) (quartic coupling) corresponding to an extra free parameter \( g_{\text{HLS}} \) of the kinetic term of the SM rho.

The result is of course the same as the popular (original NLJ) dynamical calculation, the gap equation for the fermion mass generation \( M_F \neq 0 \) and also the Bethe-Salpeter equation summing up the infinite geometric series of the one-loop bubble diagram producing the bound states, massless NG boson \( \sigma \) and massive \( \sigma \) (Higgs as a dilaton \( \phi \)) (physical modes, not \( \tilde{\sigma} \) and \( \hat{\pi}_a \), see discussions below Eq.\[\text{(10)}\]) in the large \( N_c \) limit. (The relation \( \lambda = g_Y^2 \) is a dynamical consequence of the NJL model specific to the large \( N_c \) limit (subject to modification at higher orders), which leads to the famous NJL mass relation \( M_\phi^2 = 2\lambda v^2 = 2g_Y^2 v^2 = 4(g_Y v/\sqrt{2})^2 = (2m_F)^2 \).

**Appendix D: Direct calculation for the \( \rho \) universality**

We here confirm that the \( \rho \)-universality is realized independently of \( a \) in the large \( N \) limit, Eq.\[\text{(12)}\], by direct calculations \#35

Let us study the \( \rho \pi \pi \) vertex

\[
\Gamma_{\rho \pi \pi}(q, k, q + k)|_{\phi-\text{amputated}} = \left[ \frac{a}{2} g_{\mu \nu} + \frac{a}{2} \left( B_{\mu \nu}(q) + B_{\mu \lambda}(q) \cdot \left( \frac{a}{2} - 1 \right) \frac{G}{N} B_{\nu}^\lambda(q) + \cdots \right) \right] \left( \frac{a}{2} - 1 \right) \frac{G}{N} \cdot (q + 2k)\nu = \frac{a}{2} g_{\mu \nu} + \frac{a}{2} \left( B_{\mu \nu}(q) - \frac{a N}{2 G} \cdot g_{\mu \nu} \right) \left( \frac{a}{2} - 1 \right) \frac{G}{N} \cdot (q + 2k)\nu = [1 \cdot g_{\mu \nu} + \Gamma_{\mu \nu}(q) \left( 1 - \frac{2}{a} \right) \frac{G}{N} \cdot (q + 2k)\nu, \quad (D1)
\]

where \( \Gamma_{\mu \nu}(q) = \frac{1}{2} \Gamma_{\mu \nu}(q) = -\langle \rho_\mu \rho_\nu \rangle^{-1}(q) \) is given in Eq.\[\text{(104)}\]. We then multiply the \( \rho_\mu \) propagator \( \langle \rho_\mu \rho_\nu \rangle(q) \) in

\#35 The calculations here are largely owe to Taichiro Kugo, private communication.
Eq. (107) to get

\[
\langle \rho_\mu \rho_\nu \rangle (q) \cdot \Gamma^{\rho \pi \pi, \nu} (q, k, q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} = \left[ (\rho_\mu \rho_\nu) (q) - g_{\mu \nu} \cdot \left( 1 - \frac{2}{a} \right) \frac{G}{N} \right] \cdot (q + 2k)^\nu,
\]

which is \( a \)-independent, where \( g_{\text{HLS}}^{-2} (q^2, \eta) = -2N f (q^2, \eta) \) and \( F_\rho^2 = NF_\pi^2 = 2N a^2 \). Although both \( (\rho_\mu \rho_\nu) (q) \) and \( \Gamma^{\rho \pi \pi, \nu} (q, k, q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} \) have \( a \)-dependence, it is cancelled out in the combination in the Green function. This we already have seen for the \( \pi \pi \) scattering amplitude \( T_{\mu \nu} (q) \) for \( a = 0 \) in Eq. (120), which has contact terms cancellation between the tree and the infinite bubble sum, thus realizing the VMD. The present example clearly shows that it is also the case for the \( \rho_\mu \) vertex for arbitrary \( a \).

Actually, the result is nothing but the manifestation of the Ward-Takahashi identity:

\[
0 = \int D\phi \frac{\delta}{\delta \rho_\mu (x)} \left( \phi (y) \phi (z) \cdot e^{iS [\phi]} \right) = \int D\phi \left( \frac{aN}{2G} \right) \left( \rho_\mu (x) - \alpha_{\mu, ||} (x) \right) \cdot \phi (y) \phi (z) \cdot e^{iS [\phi]} \tag{3.3}\]

(cf. Eq. (3.3) for the two-point functions), which yields

\[
\langle \rho_\mu \rho_\nu \rangle (q) \cdot \Gamma^{\rho \pi \pi, \nu} (q, k, q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} = \langle \rho_\mu (q) \phi (k) \phi (q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} = \langle \alpha_{\mu, ||} (q) \phi (k) \phi (q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} \tag{4.4}
\]

where the last term is independent of the auxiliary field \( \rho_\mu \) and hence is obviously independent of \( a \). So the \( \rho \pi \pi \) Green function is \( a \)-independent as it should be.

Now we define those quantities for the "renormalized" \( \rho_\mu^{(R)} \) by rescaling the "kinetic term" to the canonical one by \( g_{\text{HLS}}^{-2} (q^2, \eta) = -2N f (q^2, \eta) \to 1:\)

\[
\rho_\mu^{(R)} = g_{\text{HLS}}^{-1} (q^2, \eta) \cdot \rho_\mu = \left[ -2N f (q^2, \eta) \right]^{\frac{1}{2}} \cdot \rho_\mu,
\]

\[
\langle \rho_\mu^{(R)} (q) \rho_\nu^{(R)} (q) \rangle = g_{\text{HLS}}^{-2} (q^2, \eta) \cdot \langle \rho_\mu \rho_\nu \rangle (q), \quad \Gamma^{\rho \pi \pi, \nu}^{(R)} (q, k, q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} = g_{\text{HLS}}^{-1} (q^2, \eta) \cdot \langle \rho_\mu^{(R)} (q) \cdot \Gamma^{\rho \pi \pi, \nu} (q, k, q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} \tag{5.5}
\]

by which the renormalized \( \rho \pi \pi \) Green function (intrinsically \( a \)-independent as mentioned above) is a product of the canonical \( \rho_\mu \) propagator times the renormalized \( \rho \pi \pi \) coupling \( g_{\rho \pi \pi} (q^2) \) defined for the renormalized \( \rho_\mu^{(R)} \):

\[
\langle \rho_\mu^{(R)} (q) \phi (k) \phi (q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} = 2 \cdot \frac{g_{\mu \nu} - g_{\text{HLS}}^{-1} (q^2, \eta) F_\rho^2}{q^2 - g_{\text{HLS}}^{-2} (q^2, \eta) F_\rho^2} \left[ g_{\rho \pi \pi} (q^2) \cdot (q + 2k)^\nu \right], \tag{6.6}
\]

On the other hand, Eq. (122) yields the same renormalized Green function as

\[
\langle \rho_\mu^{(R)} (q) \phi (k) \phi (q + k) \big|^{k^2 = (k+q)^2 = 0}_{\phi - \text{amputated}} = \frac{g_{\mu \nu} - g_{\text{HLS}}^{-1} (q^2, \eta) F_\rho^2}{q^2 - g_{\text{HLS}}^{-2} (q^2, \eta) F_\rho^2} \left[ g_{\rho \pi \pi} (q^2, \eta) \cdot (q + 2k)^\nu \right], \tag{7.7}
\]

which implies:

\[
g_{\rho \pi \pi} (q^2) = g_{\text{HLS}} (q^2, \eta), \quad (\text{universality, } a \text{ - independent}). \tag{8.8}
\]

In particular, the on-shell \( \rho_\mu \) coupling reads:

\[
g_{\rho \pi \pi} = g_{\text{HLS}} = \left[ -2N f (M_\rho^2, 0) \right]^{-1/2}, \quad (\text{broken phase}) \]

\[
g_{\rho \pi \pi} = \left[ -2N f (0, \eta) \right]^{-1/2}, \quad (\text{symmetric phase}). \tag{9.9}
\]
[59] M. Benayoun, P. David, L. DelBuono and F. Jegerlehner, Eur. Phys. J. C \textbf{72}, 1848 (2012).
[60] Y. Igarashi, M. Johnmura, A. Kobayashi, H. Otsu, T. Sato and S. Sawada, Nucl. Phys. B \textbf{259}, 721 (1985).
[61] E. Aprile \textit{et al.} [XENON Collaboration], Phys. Rev. Lett. \textbf{119}, no. 18, 181301 (2017).
[62] X. Cui \textit{et al.} [PandaX-II Collaboration], Phys. Rev. Lett. \textbf{119}, no. 18, 181302 (2017)
[63] B. Holdom, Phys. Lett. B \textbf{150}, 301 (1985); T. Akiba and T. Yanagida, Phys. Lett. B \textbf{169}, 432 (1986); T. W. Appelquist, D. Karabali and L. C. R. Wijewardhana, Phys. Rev. Lett. \textbf{57}, 957 (1986).
[64] Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K. Miura, K.i. Nagai, H. Ohki, E. Rinaldi, A. Shibata, K. Yamawaki and T. Yamazaki [LatKMI Collaboration], Phys. Rev. D \textbf{89}, 111502 (2014);
Y. Aoki, T. Aoyama, E. Bennett, M. Kurachi, T. Maskawa, K. Miura, K.i. Nagai, H. Ohki, E. Rinaldi, A. Shibata, K. Yamawaki and T. Yamazaki [LatKMI Collaboration], Phys. Rev. D \textbf{96}, no. 1, 014508 (2017).
[65] T. Appelquist \textit{et al.} [LSD Collaboration], Phys. Rev. D \textbf{93}, no. 11, 114514 (2016).
[66] Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K. Miura, K.i. Nagai, H. Ohki, E. Rinaldi, A. Shibata, K. Yamawaki and T. Yamazaki [LatKMI Collaboration], Phys. Rev. Lett. \textbf{111}, no. 16, 162001 (2013).
[67] J. Kuti, PoS LATTICE \textbf{2013}, 004 (2014).
[68] W. A. Bardeen, R. B. Pearson and E. Rabinovici, Phys. Rev. D \textbf{21}, 1037 (1980).
[69] K. Agashe, R. Contino and A. Pomarol, Nucl. Phys. B \textbf{719}, 165 (2005).