APPROXIMATIONS OF STOCHASTIC 3D TAMED NAVIER-STOKES EQUATIONS

XUHUI PENG
MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University
Hunan, 410081, China

RANGRANG ZHANG*
School of Mathematics and Statistics, Beijing Institute of Technology
Beijing, 100081, China

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Abstract. In this paper, we are concerned with 3D tamed Navier-Stokes equations with periodic boundary conditions, which can be viewed as an approximation of the classical 3D Navier-Stokes equations. We show that the strong solution of 3D tamed Navier-Stokes equations driven by Poisson random measure converges weakly to the strong solution of 3D tamed Navier-Stokes equations driven by Gaussian noise on the state space $D([0,T];\mathbb{H}^1)$.

1. Introduction. In this paper, we are interested in stochastic 3D tamed Navier-Stokes equations with periodic boundary conditions, which were proposed by Röckner and Zhang in [22]. Fix any $T > 0$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ be a stochastic basis. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ is assumed to be complete. We use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. We consider the following Cauchy problem of the 3D tamed Navier-Stokes equations driven by stochastic forcing

\begin{equation}
\begin{cases}
du - \nu \Delta u dt + (u \cdot \nabla)u dt + g_N(|u(t)|^2)u(t) dt + \nabla p dt = \sum_{i=1}^{m} \sigma_i(u) dW^i(t) & \text{in } T^3 \times (0,T) \\
\text{div } u(t,x) = 0 & \text{in } T^3 \times (0,T) \\
u(0) = h,
\end{cases}
\end{equation}

where $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ is a vector function, $p$ denotes the pressure, $g_N(\cdot)$ is a smooth function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $\sigma^i, i = 1, \ldots, m$ are measurable functions and $W = (W^1(t), \ldots, W^m(t))$ is an $m$--dimensional standard Brownian

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* Corresponding author.
motion defined on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)\). \(\nu\) is the viscosity coefficient. \(T^3 \subset \mathbb{R}^3\) denotes the three dimensional torus (suppose the periodic length is 1).

As is well known, the stochastic Navier-Stokes equations in 2D case has been studied extensively in the literature (see [5, 7, 9, 26] and references therein), however, there exist serious obstacles in dealing with the stochastic 3D Navier-Stokes equations. For example, Flandoli and Gatarek [12] proved the existence of martingale solutions and stationary solutions of the stochastic 3D Navier-Stokes equations in a bounded domain. Later, Mikulevicius and Rozovskii [17] established the existence of a global weak (martingale) solution of the stochastic 3D Navier-Stokes equations in \(\mathbb{R}^d (d \geq 2)\). However, the uniqueness of the stochastic 3D Navier-Stokes equations still remains open. The motivation to study (1.1) originates from the deterministic case, i.e., when the noise is zero. In that case, a bounded strong solution of the classical 3D Navier-Stokes equations coincides with the solution of (1.1) for large enough \(N\) (see [21]). Mathematically, the taming term can effectively control the estimation of the nonlinear terms, as a result, plenty of results on this model are obtained. For instance, Röckner and Zhang established the existence of a unique strong solution (strong in the probabilistic sense and weak in the PDE sense) to equation (1.1) in the whole space as well as in the periodic boundary case indirectly by employing the Yamada-Watanabe Theorem in [22].

In recent years, Stochastic Partial Differential Equations (SPDEs) driven by jump-type noises such as Lévy-type or Poisson-type perturbations become extremely popular for modeling financial, physical and biological phenomena. In some circumstances, purely Brownian motion perturbation has many imperfections while capturing some large moves and unpredictable events. Therefore, jump-type perturbations come to the stage to reproduce the performance of those natural phenomena in some real world models.

In the present paper, we also consider the following 3D tamed Navier-Stokes equations driven by Poisson random measures, which can be written as

\[
\begin{align*}
&\begin{cases}
    du - \nu \Delta u dt + (u \cdot \nabla) u dt + g_N(|u|^2(t))u(t)dt + \nabla pdt \\
    = \sum_{i=1}^{m} \int_{E_i} \sigma_i(u(t-), z) \tilde{N}_i(dt, dz) & \text{in } T^3 \times (0,T) \\
    \text{div } u(t, x) = 0 & \text{in } T^3 \times (0,T) \\
    u(0) = h, & \text{in } T^3 \times (0,T)
\end{cases}
\end{align*}
\] (1.2)

\(T^3 \subset \mathbb{R}^3\) denotes the three dimensional torus (suppose the periodic length is 1). \(\nu\) is the viscosity coefficient. \(T^3 \subset \mathbb{R}^3\) denotes the three dimensional torus (suppose the periodic length is 1).
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where $\tilde{N}^i, i = 1, \ldots, m$ are mutually independent compensated time homogeneous Poisson random measures on a certain locally compact Polish space (see subsection 2.4). Recently, Dong and Zhang [10] obtained the existence and uniqueness of strong solutions of (1.2) and proved its large deviation principles on the state space $\mathcal{D}([0, T]; \mathbb{H}^1)$.

The purpose of this paper is to show that the solution of (1.2) converges weakly to the solution of (1.1), which can be applied to the numerical simulations of the 3D tamed Navier-Stokes equations driven by Poisson random measures. This topic has attracted a lot of people’s interests. For instance, Nunno and Zhang [8] obtained such an approximation for a general class of SPDEs. However, their results can not cover some important models in fluid mechanics such as stochastic 2D Navier-Stokes equations, 3D tamed Navier-Stokes equations and so on. For the stochastic 2D Navier-Stokes equations, Shang and Zhang [25] established such an approximation on the state space $\mathcal{D}([0, T]; \mathbb{H}^1)$. In this paper, we aim to prove the same result for the stochastic 3D tamed Navier-Stokes equations on the state space $\mathcal{D}([0, T]; \mathbb{H}^1)$. To achieve it, we proceeds along the same lines as [25]. However, the treatment of nonlinear terms is different from that in [25], as the cancellation property does not hold in $\mathbb{H}^1$. Moreover, as an important part of the proof, we need to show the tightness of the approximating equations on the space $\mathcal{D}([0, T]; \mathbb{H}^1)$, which requires some estimates of high order Sobolev norms, such as $\| \cdot \|_{\mathbb{H}^1}, \| \cdot \|_{\mathbb{H}^2}$. They are highly nontrivial.

Our paper is organized as follows. The mathematical formulation of stochastic 3D tamed Navier-Stokes equations and the statement of our main result are presented in Section 2. In Section 3, we establish some important a priori estimates and show the weak convergence result of stochastic 3D tamed Navier-Stokes equations under a stronger condition. At last, in Section 4, we show the proof process of the main result without this stronger condition. Throughout the paper, we will denote various generic positive constants by the same letter $C$, although the constants may differ from line to line.

2. Formulations. Keeping in mind that we are working on a time interval $[0, T]$. Let $u(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x))$ be a vector function on $\mathbb{T}^3$. The following notations will be used: $|u|^2 = \sum_{i=1}^{3} |u^i|^2$, $\text{div } u := \sum_{i=1}^{3} \partial_i u^i$ and $(u \cdot \nabla) u = \sum_{i=1}^{3} u^i \partial_i u$. Throughout the paper, $g_N(\cdot)$ will denote a fixed smooth function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that for some $N > 0$,

$$
\begin{align*}
&g_N(r) = 0, \quad \text{if } r \leq N, \\
g_N(r) = \frac{r - N}{\nu}, \quad \text{if } r \geq N + 1, \\
&0 \leq g_N(r) \leq C, \quad \text{if } r \geq 0.
\end{align*}
$$

Without loss of generality, we assume the viscosity coefficient $\nu = 1$.

2.1. Functional spaces. Let $\mathcal{L}(K_1; K_2)$ (resp. $\mathcal{L}_2(K_1; K_2)$) be the space of bounded (resp. Hilbert-Schmidt) linear operators from the Hilbert space $K_1$ to $K_2$, whose norm is denoted by $\| \cdot \|_{\mathcal{L}(K_1; K_2)}(\| \cdot \|_{\mathcal{L}_2(K_1; K_2)})$. For a topological space $\mathcal{E}$, denote the corresponding Borel $\sigma$–field by $\mathcal{B}(\mathcal{E})$. For a metric space $X$, $C([0, T]; X)$ stands for the space of continuous functions from $[0, T]$ into $X$ and $\mathcal{D}([0, T]; X)$ represents the space of right continuous functions with left limits from $[0, T]$ into $X$. For a metric space $Y$, denote by $M_b(Y), C_b(Y)$ the space of real valued bounded $Y$–measurable maps and real valued bounded continuous functions, respectively.
Let \( C^\infty(\mathbb{T}^3) = C^\infty(\mathbb{T}^3; \mathbb{R}^3) \) denote the set of all smooth periodic functions from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \). For \( p \geq 1 \), \( L^p(\mathbb{T}^3) = L^p(\mathbb{T}^3; \mathbb{R}^3) \) stands for the vector valued \( L^p \) space in which the norm is denoted by \( \| \cdot \|_{L^p} \). For a non-negative integer \( m \geq 0 \), let \( H^m \) be the usual Sobolev space on \( \mathbb{T}^3 \) with values in \( \mathbb{R}^3 \), i.e., the closure of \( C^\infty(\mathbb{T}^3) \) with respect to the norm:

\[
\| u \|_{H^m}^2 = \int_{\mathbb{T}^3} |(I - \Delta)^{\frac{m}{2}} u|^2 \, dx,
\]

where

\[
(I - \Delta)^{\frac{m}{2}} u := \left( (I - \Delta) u, (I - \Delta)^{\frac{1}{2}} u, (I - \Delta)^{\frac{3}{2}} u \right),
\]

is defined by Fourier transformation.

For \( m \in \mathbb{N}_0 \), set

\[
\mathbb{H}^m := \{ u \in H^m : \text{div} \, u = 0 \}.
\]

Then the norm of \( H^m \) restricted to \( \mathbb{H}^m \) will be denoted by \( \| \cdot \|_{\mathbb{H}^m} \). In particular, \( \mathbb{H}^0 \) is a closed linear subspace of the Hilbert space \( L^2(\mathbb{T}^3) = H^0 \). Let \( P \) be the orthogonal projection from \( L^2(\mathbb{T}^3) \) to \( \mathbb{H}^0 \). It is well-known that \( P \) commutes with the derivative operators and that \( P \) can be restricted to a bounded linear operator from \( H^m \) to \( \mathbb{H}^m \) (see [16]). For any \( u \in \mathbb{H}^0 \) and \( v \in L^2(\mathbb{T}^3) \), we have

\[
\langle u, v \rangle_{\mathbb{H}^0} := \langle u, P v \rangle_{\mathbb{H}^0} = \langle u, v \rangle_{L^2}.
\]

Moreover, for \( u \in \mathbb{H}^0 \) and \( v \in \mathbb{H}^2 \), the inner product \( \langle u, v \rangle_{\mathbb{H}^1} \) is taken in the generalized sense, i.e.,

\[
\langle u, v \rangle_{\mathbb{H}^1} := \langle u, (I - \Delta) v \rangle_{\mathbb{H}^0}.
\]

For any \( u \in H^2 \cap \mathbb{H}^1 \), define

\[
Au := -P \Delta u. \tag{2.2}
\]

It is well-known that the Stokes operator \( A \) is a positive self-adjoint operator in \( \mathbb{H}^0 \) with a compact resolvent. Let \( \{e_i\}_{i=1}^\infty \subset \mathbb{H}^0 \) be an orthonormal basis of \( \mathbb{H}^0 \) composed of eigenfunctions of \( A \) with corresponding eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \) satisfies \( Ae_i = \lambda_i e_i \), which is also an orthogonal base in \( \mathbb{H}^1 \). Let \( \lambda_k = \frac{1}{\sqrt{\lambda_k}} \), we know that \( \mathcal{G} := \{l_k : k \in \mathbb{N} \} \) is an orthonormal basis of \( \mathbb{H}^1 \).

For any \( u, v \in \mathbb{H}^1 \), set

\[
B(u, v) := \langle u \cdot \nabla, v \rangle.
\]

If \( u = v \), we write \( B(u) = B(u, u) \). By the incompressibility condition, it gives

\[
\langle B(u, v), v \rangle_{\mathbb{H}^0} = 0, \quad \langle B(u, v), w \rangle_{\mathbb{H}^0} = -\langle B(u, w), v \rangle_{\mathbb{H}^0},
\]

for \( u, v, w \in \mathbb{H}^1 \).

Letting the operator \( P \) act on both sides of (1.1), we get the following abstract evolution equations:

\[
\begin{cases}
    du + Audt + B(u)dt + \mathcal{P}(g_N(|u|^2(t))u(t))dt = \sum_{i=1}^{m} \sigma^i(u(t))dW^i(t), \\
    \text{div} \, u(t, x) = 0, \quad t > 0, \\
    u(0) = h \in \mathbb{H}^1,
\end{cases} \tag{2.3}
\]

where we have denoted by the same symbols the projection of \( \sigma^i, i = 1, \ldots, m \) with a slight abuse of notations.
2.2. Some estimates. In this part, we present some estimates of nonlinear terms of stochastic 3D tamed Navier-Stokes equations. Let
\[
F(u) := -Au - B(u) - \mathcal{P}(g_N(|u|^2)u).
\]
Referring to Lemma 2.3 in [22] and (5.21)-(5.23) in [19], we have the following estimates.

**Lemma 2.1.** (1) For \( u \in \mathbb{H}^2 \), it gives
\[
\langle F(u), u \rangle_{\mathbb{H}^1} \leq -\frac{1}{2} \|u\|^2_{\mathbb{H}^2} - \frac{1}{2} \|u\| \|\nabla u\|^2_{L^2} + C_N \|\nabla u\|^2_{H^0} + \|u\|^2_{L^2}.
\]
(2) For \( u_1, u_2 \in \mathbb{H}^2 \), it gives
\[
2 \langle F(u_1) - F(u_2), u_1 - u_2 \rangle_{\mathbb{H}^1} \leq -\frac{1}{2} \|u_1 - u_2\|^2_{\mathbb{H}^2} + C_0 \left(1 + \|u_1\|^2_{\mathbb{H}^1} + \|u_2\|^2_{\mathbb{H}^1} + \|u_2\|^2_{\mathbb{H}^2}\right) \|u_1 - u_2\|^2_{H^1}.
\]
We emphasize the constant \( C_0 > 1 \) as it will appear in the proof of Theorem 2.8.

**Remark 1.** We stress that Lemma 2.1 can not be obtained for 3D tamed Navier-Stokes equations in the case of bounded domain with zero Dirichlet boundary conditions (see [16] or [18]).

2.3. Poisson random measure. Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a filtered probability space with expectation \( \mathbb{E} \). Let \( \vartheta \) be a \( \sigma \)-finite positive measure on the measurable space \((\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))\), where \( \mathbb{R}_0 = \mathbb{R} \setminus \{0\} \). Set \( C_c(\mathbb{R}_0) \) be the space of continuous functions on \( \mathbb{R}_0 \) with compact supports. Denote
\[
\mathcal{M}_{FC}(\mathbb{R}_0) := \left\{ \text{measures } \vartheta \text{ on } (\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0)) \text{ such that } \vartheta(K) < \infty \right. \\
\left. \quad \text{for every compact } K \text{ in } \mathbb{R}_0 \right\}.
\]
Endow \( \mathcal{M}_{FC}(\mathbb{R}_0) \) with the weakest topology such that for every \( f \in C_c(\mathbb{R}_0) \), the function \( \vartheta \to \langle f, \vartheta \rangle = \int_{\mathbb{R}_0} f(u) d\vartheta(u), \vartheta \in \mathcal{M}_{FC}(\mathbb{R}_0) \) is continuous. This topology can be metrized such that \( \mathcal{M}_{FC}(\mathbb{R}_0) \) is a Polish space (see [6]). Let \( T > 0 \). Fix a measure \( \vartheta \in \mathcal{M}_{FC}(\mathbb{R}_0) \) and let \( \vartheta_T = \lambda_T \otimes \vartheta \), where \( \lambda_T \) is Lebesgue measure on \([0, T]\). We recall the following definition of Poisson random measure from [14].

**Definition 2.2.** We call measure \( N \) a Poisson random measure on \( \mathbb{R}_0 \times [0, T] \) with intensity measure \( \vartheta_T \), if it is a \( \mathcal{M}_{FC}(\mathbb{R}_0) \)-valued random variable and satisfies
1. for each \( B \in \mathcal{B}(\mathbb{R}_0 \times [0, T]) \) with \( \vartheta_T(B) < \infty \), \( N(B) \) is a Poisson distribution with mean \( \vartheta_T(B) \),
2. for disjoint \( B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}_0 \times [0, T]) \), \( N(B_1), \ldots, N(B_k) \) are mutually independent random variables.

We will denote by \( \tilde{N} = N - \vartheta_T \) the compensated time homogeneous Poisson random measure associated to \( N \). Assume \((H, |\cdot|_H)\) is a Hilbert space. Let \( L^2(\Omega \times [0, T]; L^2(\mathbb{R}_0, \vartheta; H)) \) be the space of progressively measurable process \( X : \mathbb{R}^+ \times \mathbb{R}_0 \times \Omega \to H \) satisfying
\[
\mathbb{E} \int_0^T \int_{\mathbb{R}_0} |X(r, z)|^2_H \vartheta(dz) dr < \infty, \quad T > 0.
\]
Then, it follows from [4] that for every \( X \in L^2(\Omega \times [0, T]; L^2(\mathbb{R}_0, \vartheta; H)) \),
\[
\mathbb{E} \left| \int_0^t \int_{\mathbb{R}_0} X(r, z) \tilde{N}(dr, dz) \right|^2_H = \mathbb{E} \int_0^t \int_{\mathbb{R}_0} |X(r, z)|^2_H \vartheta(dz) dr, \quad t \geq 0.
\]
The main tool in the present paper is the following Itô formula, whose proof can be found in [4].

**Lemma 2.3.** Assume that $E$ is a Hilbert space with norm $\| \cdot \|_E$. Let $X$ be a process given by

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t \int_{\mathbb{R}_0} f(s,z)\tilde{N}(ds,dz), \quad t \geq 0,$$

where $a$ is an $E$-valued progressively measurable process on the space $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \times \mathcal{F})$ such that for all $t \geq 0$, $\int_0^t |a(s,\omega)|_E ds < \infty$, $\mathbb{P}$-a.s. and $f$ is a predictable process on $E$ with $\mathbb{E} \int_0^t \int_{\mathbb{R}_0} |f(s,z)|_E^2 \vartheta(dz)ds < \infty$, for each $t > 0$. Denote by $G$ a separable Hilbert space. Let $\phi: E \to G$ be a function of class $C^1$ such that the first derivative $\phi': E \to \mathcal{L}(E;G)$ is $\alpha$-Hölder continuous with $\alpha > 0$.

Then for every $t > 0$, we have $\mathbb{P}$-a.s.

$$\phi(X_t) = \phi(X_0) + \int_0^t \phi'(X_s)(a(s))ds + \int_0^t \int_{\mathbb{R}_0} [\phi'(X_s-)f(s,z)]\tilde{N}(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}_0} [\phi(X_{s-} + f(s,z)) - \phi(X_{s-})] - \phi'(X_{s-})f(s,z)]\tilde{N}(ds,dz).$$

### 2.4. Hypotheses and statement of the main result

In order to obtain the existence and uniqueness of equation (2.3), referring to [22], it requires the following conditions:

Hypothesis H1. There exists a constant $C > 0$ such that for any $\xi = (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3$ and $j = 1, 2, 3$

$$\sum_{i=1}^m |\sigma^i(\xi)|^2 \leq C|\xi|^2,$$

where $\sigma^i, i = 1, \ldots, m$ are measurable mappings from $\mathbb{H}^1$ (resp. $\mathbb{H}^0$) to $\mathbb{H}^1$ (resp. $\mathbb{H}^0$) satisfying

$$\sum_{i=1}^m \|\sigma^i(u)\|_{\mathbb{H}^0}^2 \leq C\|u\|_{\mathbb{H}^0}^2, \quad \sum_{i=1}^m \|\sigma^i(u)\|_{\mathbb{H}^1}^2 \leq C\|u\|_{\mathbb{H}^1}^2.$$

**Definition 2.4.** A continuous $\mathbb{H}^1$-valued ($\mathcal{F}_t$)-adapted process $u = \{u(t), t \geq 0\}$ is said to be a solution of (2.3) if for any $T > 0$, $u \in L^2([0,T] \times \Omega, dt \times d\mathbb{P}, \mathbb{H}^2)$ and for any $t \geq 0$, the following equation holds in $\mathbb{H}^0$, $\mathbb{P}$-a.s.,

$$u(t) = h - \int_0^t A u(s)ds - \int_0^t B(u(s))ds - \int_0^t \mathcal{P}(g_N(|u(s)|^2)u(s))ds$$

$$+ \sum_{i=1}^m \int_0^t \sigma^i(u(s))dW^i(s).$$

Recall the global well-posed result of (2.3) from [22] as follows:

**Theorem 2.5.** Assume Hypothesis H1 holds and the initial value $h \in \mathbb{H}^1$. Then there exists a unique strong solution to (2.3). Moreover, it satisfies the following energy inequality:

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} \|u(t)\|_{\mathbb{H}^1}^2 \right) + \mathbb{E} \int_0^T \|u(t)\|_{\mathbb{H}^2}^2 dt < \infty.$$

In this paper, we consider the approximations of (2.3) by 3D tamed Navier-
Stokes equations driven by Poisson random measures. Concretely, for \( \varepsilon > 0 \), let \( \sigma^{i,\varepsilon} : \mathbb{H}^1 \times \mathbb{R}_0 \rightarrow \mathbb{H}^1 \) be given measurable mappings, we consider the following 3D tamed Navier-Stokes equations driven by Poisson random measures:

\[
\begin{align*}
\frac{du^\varepsilon}{dt} + Au^\varepsilon dt + B(u^\varepsilon)dt + \mathcal{P}g_N(|u^\varepsilon|^2(t))u^\varepsilon(t)dt \\
= \sum_{i=1}^m \sigma^{i,\varepsilon}(u^\varepsilon(t-), z)\tilde{N}^i(dt, dz) \quad \text{in } \mathbb{T}^3 \times (0, T) \\
\text{div } u(t, x) = 0 \quad \text{in } \mathbb{T}^3 \times (0, T) \\
u^\varepsilon(0) = h \in \mathbb{H}^1,
\end{align*}
\]

(2.10)

where \( \mathcal{P} \) is defined in subsection 2.1. Let \( \vartheta^i(dx), i = 1, \ldots, m \) be \( \sigma \)-finite measures on the measurable space \((\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))\) and \( N^i, i = 1, \ldots, m \) be mutually independent \( \mathcal{F}_t \)-Poisson random measures on \([0, T] \times \mathbb{R}_0\) with intensity measure \( dt \times \vartheta^i(dx) \), respectively. For \( U \in \mathcal{B}(\mathbb{R}_0) \) with \( \vartheta^i(U) < \infty \), \( \tilde{N}^i((0, t] \times U) = N^i((0, t] \times U) - t\vartheta^i(U) \) is the compensated time homogeneous Poisson random measure associated to \( N^i \).

To obtain the global well posedness of (2.10), we need to impose the following conditions on \( \sigma^{i,\varepsilon} \).

Hypothesis H2: For any \( \varepsilon > 0 \), let \( \sigma^{i,\varepsilon}, i = 1, \ldots, m \) be measurable mappings from \( \mathbb{H}^1 \times \mathbb{R}_0 \rightarrow \mathbb{H}^1 \) (resp. \( \mathbb{H}^0 \times \mathbb{R}_0 \rightarrow \mathbb{H}^0 \)). There exists a positive constant \( C > 0 \) independent of \( \varepsilon \) such that for \( u, u_1, u_2 \in \mathbb{H}^0 \),

\[
\begin{align*}
\sum_{i=1}^m \int_{\mathbb{R}_0} \|\sigma^{i,\varepsilon}(u, z)\|_{\mathbb{H}^0}^2 \vartheta^i(dz) &\leq C(1 + \|u\|_{\mathbb{H}^0}^2), \\
\sum_{i=1}^m \int_{\mathbb{R}_0} \|\sigma^{i,\varepsilon}(u_1, z) - \sigma^{i,\varepsilon}(u_2, z)\|_{\mathbb{H}^0}^2 \vartheta^i(dz) &\leq C\|u_1 - u_2\|_{\mathbb{H}^0}^2,
\end{align*}
\]

(2.11)

and for \( u, u_1, u_2 \in \mathbb{H}_1 \),

\[
\begin{align*}
\sum_{i=1}^m \int_{\mathbb{R}_0} \|\sigma^{i,\varepsilon}(u, z)\|_{\mathbb{H}_1}^2 \vartheta^i(dz) &\leq C(1 + \|u\|_{\mathbb{H}_1}^2), \\
\sum_{i=1}^m \int_{\mathbb{R}_0} \|\sigma^{i,\varepsilon}(u_1, z) - \sigma^{i,\varepsilon}(u_2, z)\|_{\mathbb{H}_1}^2 \vartheta^i(dz) &\leq C\|u_1 - u_2\|_{\mathbb{H}_1}^2,
\end{align*}
\]

(2.13)

Now, we introduce the definition of a strong solution to (2.10).

**Definition 2.6.** The system (2.10) has a strong solution if for every stochastic basis \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) and time homogeneous Poisson random measures \( \tilde{N}^i \) on \((\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))\) over the stochastic basis with intensity measure \( \vartheta^i, i = 1, 2, \ldots, m \), there exists a progressively measurable process \( u^\varepsilon : [0, T] \times \Omega \rightarrow \mathbb{H}^1 \) satisfying

\[
u^\varepsilon(\cdot, \omega) \in \mathcal{D}([0, T]; \mathbb{H}^1) \cap L^2([0, T]; \mathbb{H}^2), \quad \mathbb{P} - \text{a.s.}
\]

(2.16)

such that for all \( t \in [0, T] \), the following identity

\[
u^\varepsilon(t) = h - \int_0^t Au^\varepsilon(s)ds - \int_0^t B(u^\varepsilon(s))ds - \int_0^t \mathcal{P}(g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s))ds
\]
\begin{align*}
&+ \sum_{i=1}^{m} \int_{0}^{t} \int_{\mathbb{R}_{0}} \sigma^{i,\varepsilon}(u^{\varepsilon}(s-),z)\tilde{N}^{i}(ds,dz), \quad \mathbb{P} \text{ a.s.}
\end{align*}

holds in \( \mathbb{H}^{0} \).

Referring to [10], it states that

**Theorem 2.7.** Assume Hypothesis H2 is in force and the initial value \( h \in \mathbb{H}^{1} \), then there exists a unique strong solution of (2.10).

To prove the solution of (2.10) converges weakly to the solution of (2.3), we need some additional conditions on \( \sigma^{i,\varepsilon} \) and \( \sigma^{i} \).

Hypothesis H3: (i): For each \( i \in \{1, \ldots, m\} \) and any \( M > 0 \),
\[
\sup_{\|u\|_{\mathbb{H}}^{1} \leq M} \sup_{z \in \mathbb{R}_{0}} \|\sigma^{i,\varepsilon}(u, z)\|_{\mathbb{H}^{1}} \to 0, \quad \varepsilon \to 0. \tag{2.17}
\]

(ii): For each \( i \in \{1, \ldots, m\} \), \( l_{j}, l_{k} \in \mathcal{G} \) and \( u \in \mathbb{H}^{1} \),
\[
\int_{\mathbb{R}_{0}} (\sigma^{i,\varepsilon}(u, z), l_{k})_{\mathbb{H}^{1}} (\sigma^{i,\varepsilon}(u, z), l_{j})_{\mathbb{H}^{1}} \vartheta^{i}(dz) \to \langle \sigma^{i}(u), l_{k} \rangle_{\mathbb{H}^{1}} \langle \sigma^{i}(u), l_{j} \rangle_{\mathbb{H}^{1}}, \quad \varepsilon \to 0. \tag{2.18}
\]

Hypothesis H4: For each \( i \in \{1, \ldots, m\} \) and every \( u \in \mathbb{H}^{1} \),
\[
\int_{\mathbb{R}_{0}} \|\sigma^{i,\varepsilon}(u, z)\|_{\mathbb{H}^{1}}^{2} \vartheta^{i}(dz) \to \|\sigma^{i}(u)\|_{\mathbb{H}^{1}}^{2}, \quad \varepsilon \to 0. \tag{2.19}
\]

**Remark 2.** The examples satisfying Hypotheses (H2)-(H4) can be constructed similarly to Section 4 in [25].

Denote by \( \mu^{\varepsilon}, \mu \) the laws of \( u^{\varepsilon} \) and \( u \) on the space \( \mathcal{D}([0, T]; \mathbb{H}^{1}) \) and \( C([0, T]; \mathbb{H}^{1}) \), respectively. Our main result of this paper reads as

**Theorem 2.8.** Assume Hypotheses H1–H4 hold and the initial value \( h \in \mathbb{H}^{1} \). Then, for any \( T > 0 \), \( \mu^{\varepsilon} \) converges weakly to \( \mu \), as \( \varepsilon \to 0 \), on the space \( \mathcal{D}([0, T]; \mathbb{H}^{1}) \) equipped with the Skorohod topology.

To prove the above main result, we adopt the method from [25]. We firstly show the result holds under the following stronger condition Hypothesis H5, then we relax it to Hypothesis H4.

Hypothesis H5: For any \( \varepsilon > 0 \), \( \sigma^{i,\varepsilon} \) map \( \mathbb{H}^{2} \times \mathbb{R}_{0} \) to \( \mathbb{H}^{2} \) and there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that
\[
\sum_{i=1}^{m} \int_{\mathbb{R}_{0}} \|\sigma^{i,\varepsilon}(u, z)\|_{\mathbb{H}^{2}}^{2} \vartheta^{i}(dz) \leq C(1 + \|u\|_{\mathbb{H}^{1}}^{2}). \tag{2.20}
\]

In the rest of this paper, we let \( m = 1 \) and omit the subscript \( i \) of \( \sigma^{i}, \tilde{N}^{i}, \vartheta^{i} \). The case of \( m > 1 \) does not cause extra difficulties.

3. **A priori estimates.**

**Lemma 3.1.** Assume Hypothesis H2 holds and \( h \in \mathbb{H}^{1} \), then for the solution \( u^{\varepsilon} \) of (2.10), there exists a constant \( C = C(N, T) > 0 \) such that

\begin{align*}
\sup_{\varepsilon > 0} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^{\varepsilon}(t)\|_{\mathbb{H}^{1}}^{2} + \mathbb{E} \int_{0}^{T} \|u^{\varepsilon}(t)\|_{\mathbb{H}^{1}}^{2} dt \right\} &\leq C(1 + \|h\|_{\mathbb{H}^{1}}^{2}), \quad \text{(3.1)}
\end{align*}

\begin{align*}
\sup_{\varepsilon > 0} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^{\varepsilon}(t)\|_{\mathbb{H}^{1}}^{6} + \mathbb{E} \int_{0}^{T} \|u^{\varepsilon}(t)\|_{\mathbb{H}^{1}}^{6} \|u^{\varepsilon}(t)\|_{\mathbb{H}^{1}}^{2} dt \right\} &\leq C(1 + \|h\|_{\mathbb{H}^{1}}^{5}). \quad \text{(3.2)}
\end{align*}
Applying Lemma 2.3 to (2.10) with the function \( \varphi(x) = |x|^2 \) and by \( \varphi(x + y) - \varphi(x) = \langle y, \nabla \varphi(x) \rangle = |y|^2 \), we deduce that for \( 0 \leq t \leq T \),
\[
||u^\varepsilon(t)||^2_{H^1} = ||h||^2_{H^1} + 2 \int_0^t \left< u^\varepsilon(s), F(u^\varepsilon(s)) \right>_{H^1} ds
+ 2 \int_0^t \int_{\mathbb{R}^d} \langle u^\varepsilon(s, \cdot), \sigma^\varepsilon(u^\varepsilon(s, \cdot), z) \rangle_{H^1} \tilde{N}(ds, dz)
+ \int_0^t \int_{\mathbb{R}^d} \|\sigma^\varepsilon(u^\varepsilon(s, \cdot), z)\|^2_{H^1} N(ds, dz).
\]

With the aid of (2.4), we deduce that
\[
\sup_{0 \leq s \leq t} ||u^\varepsilon(s)||^2_{H^1} + \int_0^t ||u^\varepsilon(s)||^2_{H^2} ds + \int_0^t ||u^\varepsilon|| \cdot |\nabla u^\varepsilon||^2_{L^2} ds
\leq ||h||^2_{H^1} + 2CN \int_0^t ||u^\varepsilon(s)||^2_{H^1} ds
+ 2 \sup_{0 \leq s \leq t} \int_{\mathbb{R}^d} \left( u^\varepsilon(r - \cdot), \sigma^\varepsilon(u^\varepsilon(r - \cdot), z) \right)_{H^1} \tilde{N}(dr, dz)
+ \sup_{0 \leq s \leq t} \int_{\mathbb{R}^d} ||\sigma^\varepsilon(u^\varepsilon(r - \cdot), z)||^2_{H^1} N(dr, dz).
\]

Applying the Burkholder-Davis-Gundy inequality (see [13]), (2.13), the Hölder inequality and the Young inequality, we get
\[
\mathbb{E}\left[ 2 \sup_{0 \leq s \leq t} \int_{\mathbb{R}^d} \left( u^\varepsilon(r - \cdot), \sigma^\varepsilon(u^\varepsilon(r - \cdot), z) \right)_{H^1} \tilde{N}(dr, dz) \right]
\leq C\mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} ||u^\varepsilon(s)||^2_{H^1} ||\sigma^\varepsilon(u^\varepsilon(s), z)||^2_{H^1} \vartheta(dz) ds \right]^{\frac{1}{2}}
\leq C \left[ \mathbb{E} \sup_{0 \leq s \leq t} ||u^\varepsilon(s)||^2_{H^1} \right]^{\frac{1}{2}} \left[ \mathbb{E} \int_0^t (1 + ||u^\varepsilon(s)||^2_{H^1}) ds \right]^{\frac{1}{2}}
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} ||u^\varepsilon(s)||^2_{H^1} + C\mathbb{E} \int_0^t (1 + ||u^\varepsilon(s)||^2_{H^1}) ds
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} ||u^\varepsilon(s)||^2_{H^1} + CT + C\mathbb{E} \int_0^t \sup_{0 \leq r \leq s} ||u^\varepsilon(r)||^2_{H^1} ds.
\]

Taking into account that the process
\[
t \mapsto \int_0^t \int_{\mathbb{R}^d} ||\sigma^\varepsilon(u^\varepsilon(s, \cdot), z)||^2_{H^1} N(ds, dz)
\]
has only positive jumps and by (2.13), we deduce that
\[
\mathbb{E} \sup_{0 \leq s \leq t} \int_{\mathbb{R}^d} ||\sigma^\varepsilon(u^\varepsilon(r - \cdot), z)||^2_{H^1} N(dr, dz)
\leq \mathbb{E} \int_0^t \int_{\mathbb{R}^d} ||\sigma^\varepsilon(u^\varepsilon(r, z)||^2_{H^1} N_i(dr, dz) = \mathbb{E} \int_0^t \int_{\mathbb{R}^d} ||\sigma^\varepsilon(u^\varepsilon(s, z)||^2_{H^1} \vartheta(dz) ds
\leq C\mathbb{E} \int_0^t (1 + ||u^\varepsilon(s)||^2_{H^1}) ds \leq CT + C \int_0^t \mathbb{E} \sup_{0 \leq r \leq s} ||u^\varepsilon(r)||^2_{H^1} ds.
\]
Collecting the above estimates, we arrive at
\[
\mathbb{E} \sup_{0 \leq s \leq t} \|u^x(s)\|^2_{\mathcal{H}_1} + \mathbb{E} \int_0^t \|u^x(s)\|^2_{\mathcal{H}_2} \, ds
\]
\[
\leq \|h\|^2_{\mathcal{H}_1} + CT + (2C_N + C) \mathbb{E} \int_0^t \sup_{0 \leq r \leq s} \|u^x(r)\|^2_{\mathcal{H}_1} \, ds.
\]
By Gronwall inequality, it gives that
\[
\mathbb{E} \sup_{0 \leq s \leq T} \|u^x(s)\|^2_{\mathcal{H}_1} + \mathbb{E} \int_0^T \|u^x(s)\|^2_{\mathcal{H}_2} \, ds
\]
\[
\leq \left( \|h\|^2_{\mathcal{H}_1} + CT \right) \exp \left\{ (2C_N + C)T \right\} \leq C(N, T)(1 + \|h\|^2_{\mathcal{H}_1}),
\]
which implies (3.1).

Applying the Itô formula to the function \(\|u^x(t)\|^6_{\mathcal{H}_1}\), it yields
\[
\|u^x(t)\|^6_{\mathcal{H}_1} = \|h\|^6_{\mathcal{H}_1} + 6 \int_0^t \|u^x(s)\|^4_{\mathcal{H}_1} \langle u^x(s), F(u^x(s)) \rangle_{\mathcal{H}_1} ds
\]
\[
+ 6 \int_0^t \int_{\mathbb{R}_0} \|u^x(s)\|^2_{\mathcal{H}_1} \langle u^x(s), \sigma^x(u^x(s), \sigma^x(u^x(s))), 1 \rangle_{\mathcal{H}_1} \tilde{N}(ds, dz)
\]
\[
+ \int_0^t \int_{\mathbb{R}_0} \left( \|u^x(s)\|^2_{\mathcal{H}_1} + \sigma^x(u^x(s), z) \right) ds
\]
\[
- 6 \|u^x(s)\|^4_{\mathcal{H}_1} \langle u^x(s), \sigma^x(u^x(s), z) \rangle_{\mathcal{H}_1} \tilde{N}(ds, dz)
\]
\[
=: \|h\|^6_{\mathcal{H}_1} + I_n^1(t) + I_n^2(t) + I_n^3(t). 
\]
(3.3)

By (2.4), we deduce that
\[
\mathbb{E} \sup_{0 \leq s \leq t} I_n^1(s) \leq -3 \int_0^t \|u^x(s)\|^4_{\mathcal{H}_1} \|u^x(s)\|^2_{\mathcal{H}_2} \, ds
\]
\[
- 3 \int_0^t \|u^x(s)\|^4_{\mathcal{H}_1} \|u^x(s)\| \cdot \|\nabla u^x(s)\|^2_{L^2} \, ds
\]
\[
+ C_N \int_0^t \|u^x(s)\|^6_{\mathcal{H}_1} \, ds.
\]

Applying the Burkholder-Davis-Gundy inequality, (2.13), the Hölder inequality and the Young inequality, it follows that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} I_n^2(s) \right] \leq 6C \mathbb{E} \left[ \int_0^t \|u^x(s)\|^8_{\mathcal{H}_1} \|u^x(s)\|^2_{\mathcal{H}_2} \|\sigma^x(u^x(s), z)\|^2_{\mathcal{H}_1} \vartheta(dz) \, ds \right]^{\frac{1}{2}}
\]
\[
\leq 6C \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|u^x(s)\|^3_{\mathcal{H}_1} \left( C \int_0^t (1 + \|u^x(s)\|^6_{\mathcal{H}_1}) \, ds \right)^{\frac{1}{2}} \right]
\]
\[
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|u^x(s)\|^6_{\mathcal{H}_1} + CT + C \mathbb{E} \int_0^t \|u^x(s)\|^6_{\mathcal{H}_1} \, ds.
\]

By the Taylor formula, we have
\[
\left| x + h^{2p} - |x|^{2p} - 2p|x|^{2(p-1)}(x, h) \right| \leq C_p(|x|^{2(p-1)}|h|^2 + |h|^{2p}), 
\]
(3.4)
where \(C_p\) is a finite positive constant.
With the help of (3.4), (2.13) and (2.15), we deduce that
\[
E \left[ \sup_{0 \leq s \leq t} I_2^\ve(s) \right] 
\leq E \int_0^t \int_{\mathbb{R}^d} \left| u^\ve(s-) + \sigma^\ve(u^\ve(s-), z) \right| \mathcal{H}^4_{21} - \left| u^\ve(s-) \right| \mathcal{H}^4_{21} 
- 6 \left| u^\ve(s-) \right| \mathcal{H}^4_{21}(u^\ve(s-), \sigma^\ve(u^\ve(s-), z)) \mathcal{N}(ds, dz) 
\leq E \int_0^t \int_{\mathbb{R}^d} \left( \left| u^\ve(s-) \right| \mathcal{H}^4_{21}, \left| \sigma^\ve(u^\ve(s-), z) \right| \mathcal{H}^2_{21} + \left| \sigma^\ve(u^\ve(s-), z) \right| \mathcal{H}^4_{21} \right) \vartheta(dz)ds 
\leq C E \int_0^t (1 + \left| u^\ve(s) \right| \mathcal{H}^4_{21})ds \leq CT + C E \int_0^t \sup_{0 \leq r \leq s} \left| u^\ve(r) \right| \mathcal{H}^6_{21} ds.
\]

By (3.3), it follows that
\[
E \sup_{0 \leq s \leq t} \left| u^\ve(s) \right| \mathcal{H}^6_{21} + 6E \int_0^t \left| u^\ve(s) \right| \mathcal{H}^4_{21} \left| u^\ve(s) \right| \mathcal{H}^2_{21} ds 
+ 6E \int_0^t \left| u^\ve(s) \right| \mathcal{H}^4_{21} \left| u(s) \cdot |\nabla u^\ve(s)| \right| \mathcal{H}^2_{21} ds 
\leq \left| h \right| \mathcal{H}^6_{21} + CT + (C_N + C) \int_0^t E \sup_{0 \leq r \leq s} \left| u^\ve(r) \right| \mathcal{H}^6_{21} ds.
\]

Using the Gronwall inequality, we get
\[
E \sup_{0 \leq s \leq T} \left| u^\ve(s) \right| \mathcal{H}^6_{21} + E \int_0^T \left| u^\ve(s) \right| \mathcal{H}^4_{21} \left| u^\ve(s) \right| \mathcal{H}^2_{21} ds \leq C(N, T)(1 + \left| h \right| \mathcal{H}^6_{21}),
\]
which implies (3.2).

\[\Box\]

**Lemma 3.2.** Assume Hypothesis H2 and Hypothesis H5 hold. Let the initial value \( h \in \mathbb{H}^2 \) and for any constant \( M > 0 \), define
\[
\tau^\ve_M := T \land \inf \left\{ t \geq 0 : \int_0^t \left| u^\ve(s) \right| \mathcal{H}^2_{21} ds > M \right\} \land \inf \left\{ t \geq 0 : \left| u^\ve(t) \right| \mathcal{H}^2_{21} > M \right\},
\]
where we set \( \inf \emptyset = \infty \), we have
\[
\sup_{\ve > 0} \left\{ E \sup_{0 \leq s \leq \tau^\ve_M} \left| u^\ve(s) \right| \mathcal{H}^2_{21} + E \int_0^{\tau^\ve_M} \left| u^\ve(s) \right| \mathcal{H}^2_{21} ds \right\} \leq C(T, M, N)(1 + \left| h \right| \mathcal{H}^2_{21}). \tag{3.5}
\]

**Proof.** Firstly, we assume that for any \( \ve > 0 \) and \( h \in \mathbb{H}^2 \), the equation (2.10) admits a unique solution \( u^\ve \in L^\infty([0, T]; \mathbb{H}^2) \cap L^2([0, T]; \mathbb{H}^3) \) with probability one. In the following, we devote to proving (3.5).

Applying the Itô formula to \( \left| u^\ve(t) \right| \mathcal{H}^2_{21} \) (cf. [4, 7, 23]), we get
\[
\left| u^\ve(t) \right| \mathcal{H}^2_{21} = \left| h \right| \mathcal{H}^2_{21} + 2 \int_0^t \langle F(u^\ve(s)), u^\ve(s) \rangle \mathcal{H}^2_{21} ds 
+ 2 \int_0^t \int_{\mathbb{R}^d} \langle u^\ve(s-), \sigma^\ve(u^\ve(s-), z) \rangle \mathcal{H}^2_{21} \mathcal{N}(ds, dz) 
+ \int_0^t \int_{\mathbb{R}^d} \left| \sigma^\ve(u^\ve(s-), z) \right| \mathcal{H}^2_{21} \mathcal{N}(ds, dz)
\]
where
\[ \langle F(u^\varepsilon(s)), u^\varepsilon(s) \rangle_{H^2} = -\langle Au^\varepsilon(s), u^\varepsilon(s) \rangle_{H^2} - \langle B(u^\varepsilon(s)), u^\varepsilon(s) \rangle_{H^2} - \langle \mathcal{P}(g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s)), u^\varepsilon(s) \rangle_{H^2}. \]

Clearly, it holds that
\[ -\langle Au^\varepsilon(s), u^\varepsilon(s) \rangle_{H^2} = -\langle -\Delta u^\varepsilon(s), (I - \Delta)^2 u^\varepsilon(s) \rangle_{H^2} + \langle u^\varepsilon, (I - \Delta)^2 u^\varepsilon \rangle_{H^2} = -\|u^\varepsilon\|_{H^2}^2 + \|u^\varepsilon\|_{L^2}^2 + 2\|\nabla u^\varepsilon\|_{L^2}^2 + \|\Delta u^\varepsilon\|_{L^2}^2. \]

Utilizing Cauchy-Schwarz inequality, we obtain
\[ -\langle B(u^\varepsilon(s)), u^\varepsilon(s) \rangle_{H^2} = -\langle (I - \Delta)^2 (u^\varepsilon \cdot \nabla) u^\varepsilon, (I - \Delta)^2 u^\varepsilon \rangle_{H^2} \leq \frac{1}{4}\|u^\varepsilon\|_{H^2}^2 + \frac{1}{2}\langle (I - \Delta)(u^\varepsilon \cdot \nabla) u^\varepsilon, (u^\varepsilon \cdot \nabla) u^\varepsilon \rangle_{H^2}. \]

By the same method as in the proof of (2.10) in [22], it yields
\[ \langle (I - \Delta)(u^\varepsilon \cdot \nabla) u^\varepsilon, (u^\varepsilon \cdot \nabla) u^\varepsilon \rangle_{H^2} = \langle \nabla u^\varepsilon \rangle_{H^2} \leq \|\nabla u^\varepsilon\|_{L^2}(\|\Delta u^\varepsilon\|_{L^2}^2 + \|\nabla u^\varepsilon\|_{L^2}^2 + \|\nabla u^\varepsilon\|_{H^2}^2) + 2\int_{T^3} |\nabla u^\varepsilon| |u^\varepsilon| \Delta u^\varepsilon |dx. \]

Hence, it follows that
\[ -\langle B(u^\varepsilon(s)), u^\varepsilon(s) \rangle_{H^2} \leq \frac{1}{4}\|u^\varepsilon\|_{H^2}^2 + \frac{1}{2}\|u^\varepsilon\|_{H^2}^2 + \frac{1}{2}\|u^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|u^\varepsilon\|_{H^2}^2 + \int_{T^3} |\nabla u^\varepsilon| |u^\varepsilon| \Delta u^\varepsilon |dx. \]

Similarly to (2.10) in [22], we have
\[ -\langle \mathcal{P}(g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s)), u^\varepsilon(s) \rangle_{H^2} = -\langle (I - \Delta)[g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s)], (I - \Delta)u^\varepsilon(s) \rangle_{H^2} \]
\[ = -\langle g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), (I - \Delta)u^\varepsilon(s) \rangle_{H^2} + \langle \Delta g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), (I - \Delta)u^\varepsilon(s) \rangle_{H^2} \leq -\|u^\varepsilon\|_{H^2}^2 + CN\|\nabla u^\varepsilon\|_{H^2}^2 + \|\Delta g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), (I - \Delta)u^\varepsilon(s) \rangle_{H^2} \]
\[ = -\|u^\varepsilon\|_{H^2}^2 + CN\|\nabla u^\varepsilon\|_{H^2}^2 - \|\nabla g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), \nabla u^\varepsilon(s) \rangle_{H^2} - \langle \Delta g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), \Delta u^\varepsilon(s) \rangle_{H^2} \]
\[ \leq -\|u^\varepsilon\|_{H^2}^2 + CN\|\nabla u^\varepsilon\|_{H^2}^2 - \|u^\varepsilon\|_{H^2}^2 + CN\|\nabla u^\varepsilon\|_{H^2}^2 - \langle \Delta g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), \Delta u^\varepsilon(s) \rangle_{H^2}. \]

Now, we devote to making estimates of \(-\langle \Delta g_N(|u^\varepsilon(s)|^2)u^\varepsilon(s), \Delta u^\varepsilon(s) \rangle_{H^2}\). For simplicity, we denote by \(u^\varepsilon := u\). By the definition of \(g_N(r)\) in (2.1), it follows that
\[ -\langle \Delta g_N(|u|^2)u, \Delta u \rangle_{H^2}. \]
The last inequality in (3.7) is derived by using the Young inequality, where we have used the Young inequality and sup $\epsilon$

Combining (3.6) and (3.7), we deduce that

Thus, we conclude that

Thus, we conclude that

Thus, we conclude that

where we have used the Young inequality and $\sup_x |u(x)|^2 \leq C \|u\|_{H^1}^2 \|u\|_{H^2}^2$.

Based on the above estimates, we reach

where

The last inequality in (3.7) is derived by using the Young inequality,

and

Combining (3.6) and (3.7), we deduce that

and

Combining (3.6) and (3.7), we deduce that

$$
\|u^\epsilon(t)\|_{H^2}^2 + \int_0^t \|u^\epsilon(s)\|_{H^3}^2 ds \\
\leq \|h\|_{H^2}^2 + C \int_0^t \|u^\epsilon\|_{H^2}^2 ds + C N \int_0^t (1 + \|u^\epsilon\|_{H^1}^2) ds + C \int_0^t \|u^\epsilon\|_{H^2}^2 ds
$$
\[ + C \int_0^t \|u^\varepsilon\|_{H^2}^2 \|\sigma^\varepsilon\|_{H^2}^2 ds + C \int_0^t \|u^\varepsilon\|_{H^2}^2 \|\sigma^\varepsilon\|_{H^2}^2 ds + I(t) + J(t). \]

Taking the supremum on time from 0 to \(\tau_M\), it yields that
\[
\sup_{0 \leq t \leq \tau_M} \|u^\varepsilon(t)\|_{H^2}^2 + \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 ds \leq \|h\|_{L^2}^2 + CMT + CN(T + TM) + C \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 \|\sigma^\varepsilon(s)\|_{H^2}^2 ds + C(M^2 + M^4) \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 ds + \sup_{0 \leq t \leq \tau_M} I(t) + \sup_{0 \leq t \leq \tau_M} J(t).
\]

By the Gronwall inequality, we get
\[
\sup_{0 \leq t \leq \tau_M} \|u^\varepsilon(t)\|_{H^2}^2 + \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 ds \leq \left[\|h\|_{L^2}^2 + C_1(M, N, T) + \sup_{0 \leq t \leq \tau_M} I(t) + \sup_{0 \leq t \leq \tau_M} J(t)\right] \times \exp\{C(M^2 + M^4)T + C \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 ds\} \leq \left[\|h\|_{L^2}^2 + C_1(M, N, T) + \sup_{0 \leq t \leq \tau_M} I(t) + \sup_{0 \leq t \leq \tau_M} J(t)\right] \exp\{C_2(M, T)\},
\]

where \(C_1(M, N, T) = CMT + CN(T + TM)\) and \(C_2(M, T) = C(M^2 + M^4)T + CM\).

Taking the expectation to the above equation, we obtain
\[
\mathbb{E} \sup_{0 \leq t \leq \tau_M} \|u^\varepsilon(t)\|_{H^2}^2 + \mathbb{E} \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 ds \leq \left[\|h\|_{L^2}^2 + C_1(M, N, T) + \mathbb{E} \sup_{0 \leq t \leq \tau_M} I(t) + \mathbb{E} \sup_{0 \leq t \leq \tau_M} J(t)\right] \exp\{C_2(M, T)\}. \tag{3.8}
\]

Applying Burkholder-Davis-Gundy inequality, by Hypothesis H5 and using the Young inequality, it follows that
\[
\mathbb{E} \sup_{0 \leq s \leq \tau_M} |I(s)| \leq 2 \mathbb{E} \sup_{0 \leq s \leq \tau_M} \|u^\varepsilon(s)\|_{H^2} \left[ \int_0^{\tau_M} \|\sigma^\varepsilon(s, u^\varepsilon(s-), z)\|_{H^2}^2 \vartheta(dz) ds \right]^{\frac{1}{2}} \leq 2 \left( \mathbb{E} \sup_{0 \leq s \leq \tau_M} \|u^\varepsilon(s)\|_{H^2}^2 \right)^{\frac{1}{2}} \left[ \mathbb{E} \int_0^{\tau_M} \|\sigma^\varepsilon(s, u^\varepsilon(s-), z)\|_{H^2}^2 \vartheta(dz) ds \right]^{\frac{1}{2}} \leq 2C \left( \mathbb{E} \sup_{0 \leq s \leq \tau_M} \|u^\varepsilon(s)\|_{H^2}^2 \right)^{\frac{1}{2}} \left[ \mathbb{E} \int_0^{\tau_M} (1 + \|u^\varepsilon(s)\|_{H^2}^2) ds \right]^{\frac{1}{2}} \leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq \tau_M} \|u^\varepsilon(s)\|_{H^2}^2 + C(T + M).
\]

With the help of Hypothesis H5 and (2.7), it yields
\[
\mathbb{E} \sup_{0 \leq t \leq \tau_M} |J(t)| \leq 2 \mathbb{E} \int_0^{\tau_M} \|\sigma^\varepsilon(u^\varepsilon, z)\|_{H^2}^2 \mathbb{N}(ds, dz)
\]
\[
\leq C \mathbb{E} \int_0^{\tau_M} \int_{\mathbb{R}^3} \|\sigma(\varepsilon^\tau, z)\|_{H^2}^2 \vartheta(dz) ds \\
\leq C \mathbb{E} \int_0^{\tau_M} (1 + \|u^\varepsilon(s)\|_{H^2}^2) ds \leq C(T, M).
\]

Combining the above two estimates and (3.8), we conclude that

\[
\mathbb{E} \sup_{0 \leq s \leq \tau_M} \|u^\varepsilon(s)\|_{H^2}^2 + \mathbb{E} \int_0^{\tau_M} \|u^\varepsilon(s)\|_{H^2}^2 ds \leq C(M, N, T)(1 + \|h\|_{H^2}^2),
\]

which is the desired result.

To complete the proof, it remains to prove the existence and uniqueness of solutions \(u^\varepsilon \in L^\infty([0, T]; H^2) \cap L^2([0, T]; \mathbb{H}^3)\) to (2.10), which can be achieved by applying the Galerkin approximation method similarly to the proof of Theorem 3.1 in [19] and the above \(H^2\)-norm estimates. We omit the proof.

**Lemma 3.3.** Assume Hypothesis H2 and Hypothesis H5 hold. Let the initial value \(h \in \mathbb{H}^2\), then the family \(\{u^\varepsilon; \varepsilon > 0\}\) is tight on the space \(\mathcal{D}([0, T]; \mathbb{H}^1)\).

**Proof.** Clearly, we know that \(\mathbb{H}^2\) is compactly embedded into \(\mathbb{H}^1\). Due to Aldous’ tightness criterion (see Theorem 1 in [1]), it suffices to show

1. for any 0 < \(\eta < 1\), there exists \(L_\eta > 0\) such that

\[
\sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^2} > L_\eta \right) < \eta, \quad (3.9)
\]

2. for any stopping time 0 ≤ \(\xi^\varepsilon \leq T\) with respect to the natural filtration generated by \(\{u^\varepsilon(s), s \leq t\}\), and any \(\eta > 0\),

\[
\lim_{\delta \to 0} \sup_{\varepsilon > 0} \mathbb{P}\left( \|u^\varepsilon(\xi^\varepsilon + \delta) - u^\varepsilon(\xi^\varepsilon)\|_{\mathbb{H}^1} > \eta \right) = 0, \quad (3.10)
\]

where \(\xi^\varepsilon + \delta := T \land (\xi^\varepsilon + \delta)\). Firstly, we proceed with (1). Note that for any \(M > 0\),

\[
\sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^2} > L \right) \leq \sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^2} > L, \tau_M = T \right) + \sup_{\varepsilon > 0} \mathbb{P}\left( \tau_M < T \right) \leq \sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq \tau_M} \|u^\varepsilon(t)\|_{H^2} > L \right) + \sup_{\varepsilon > 0} \mathbb{P}\left( \tau_M < T \right). \quad (3.11)
\]

By the definition of \(\tau_M\), we deduce that

\[
\sup_{\varepsilon > 0} \mathbb{P}\left( \tau_M < T \right) \leq \sup_{\varepsilon > 0} \mathbb{P}\left( \int_0^T \|u^\varepsilon(t)\|_{H^2}^2 dt > M \right) + \sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^1} > M \right) \leq \frac{1}{M} \sup_{\varepsilon > 0} \mathbb{E} \int_0^T \|u^\varepsilon(t)\|_{H^2}^2 dt + \frac{1}{M} \sup_{\varepsilon > 0} \mathbb{E} \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^1}^2 \leq \frac{C}{M}. \quad (3.12)
\]

By the Chebyshev inequality, we get

\[
\sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq \tau_M} \|u^\varepsilon(t)\|_{H^2} > L \right) \leq \frac{1}{L^2} \sup_{\varepsilon > 0} \mathbb{E} \sup_{0 \leq t \leq \tau_M} \|u^\varepsilon(t)\|_{H^2}^2 \leq \frac{C_M}{L^2}.
\]

Combining the above estimates, we deduce from (3.11) that

\[
\sup_{\varepsilon > 0} \mathbb{P}\left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_{H^2} > L \right) \leq \frac{C}{M} + \frac{C_M}{L^2}.
\]
For any \( \eta > 0 \), we can firstly choose \( M \) to be sufficiently large, then choosing sufficiently large \( L \) such that \( \frac{CM}{M} + \frac{C}{M} < \eta \). Hence, (i) is verified.

Now, we focus on the verification of (2). For any \( \eta > 0 \),

\[
\begin{aligned}
&\sup_{\epsilon > 0} \mathbb{P}\left( \left\| u^{\epsilon}(\zeta^\epsilon + \delta) - u^{\epsilon}(\zeta^\epsilon) \right\|_{H^1} > \eta \right) \\
&\leq \sup_{\epsilon > 0} \mathbb{P}\left( \left\| \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} A u^\epsilon(s) ds \right\|_{H^1} > \frac{\eta}{4} \right) + \sup_{\epsilon > 0} \mathbb{P}\left( \left\| \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} B(u^\epsilon(s)) ds \right\|_{H^1} > \frac{\eta}{4} \right) \\
&\quad + \sup_{\epsilon > 0} \mathbb{P}\left( \left\| \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \mathcal{P}(g_N(|u^\epsilon(s)|^2) u^\epsilon(s)) ds \right\|_{H^1} > \frac{\eta}{4} \right) \\
&\quad + \sup_{\epsilon > 0} \mathbb{P}\left( \left\| \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \int_{\mathbb{R}^3} \sigma^\epsilon(u^\epsilon(s-), z) \tilde{N}(ds, dz) \right\|_{H^1} > \frac{\eta}{4} \right) \\
&=: K_1 + K_2 + K_3 + K_4.
\end{aligned}
\]

With the help of (3.5) and (3.12), it follows that for any \( M > 0 \),

\[
K_1 \leq \sup_{\epsilon > 0} \mathbb{P}\left( \delta \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| A u^\epsilon(s) \right\|_{H^1}^2 ds > \frac{\eta^2}{16} \right)
\]

\[
\leq \sup_{\epsilon > 0} \mathbb{P}\left( \delta \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| A u^\epsilon(s) \right\|_{H^1}^2 ds > \frac{\eta^2}{16}, \tau_M = T \right)
\]

\[
+ \sup_{\epsilon > 0} \mathbb{P}\left( \delta \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| A u^\epsilon(s) \right\|_{H^1}^2 ds > \frac{\eta^2}{16}, \tau_M < T \right)
\]

\[
\leq \sup_{\epsilon > 0} \mathbb{P}\left( \epsilon \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| u^\epsilon(s) \right\|_{H^1}^2 ds > \frac{\eta^2}{16}, \tau_M = T \right)
+ \sup_{\epsilon > 0} \mathbb{P}\left( \tau_M < T \right)
\]

\[
\leq \sup_{\epsilon > 0} \mathbb{P}\left( \epsilon \int_{0}^{\tau_M} \left\| u^\epsilon(s) \right\|_{H^1}^2 ds > \frac{\eta^2}{16} \right) + \frac{C}{M}
\]

\[
\leq \frac{16}{\eta^2} \sup_{\epsilon > 0} \mathbb{E} \left[ \int_{0}^{\tau_M} \left\| u^\epsilon(s) \right\|_{H^1}^2 ds \right] + \frac{C}{M} \leq \frac{C M \delta}{\eta^2} + \frac{C}{M}.
\]

By \( \sup_{x} |u^\epsilon(x)|^2 \leq C \| u^\epsilon \|_{H^1} \| u^\epsilon \|_{H^2} \), we deduce that

\[
\| B(u^\epsilon) \|_{H^1}^2 = \langle (I - \Delta) B(u^\epsilon), B(u^\epsilon) \rangle_{L^2}
\]

\[
= \| B(u^\epsilon) \|_{L^2}^2 + \| \nabla B(u^\epsilon) \|_{L^2}^2
\]

\[
\leq \int_{T^3} |u^\epsilon|^2 |\nabla u^\epsilon|^2 dx + 2 \int_{T^3} |u^\epsilon|^2 |\Delta u^\epsilon|^2 dx + 2 \int_{T^3} |\nabla u^\epsilon|^4 dx
\]

\[
\leq \| u^\epsilon \|_{L^4}^2 \| \nabla u^\epsilon \|_{L^2}^4 + 2 \sup_{x} |u^\epsilon(x)|^2 \| u^\epsilon \|_{H^1}^2 + 2 \| \nabla u^\epsilon \|_{L^4}^4
\]

\[
\leq C \| u^\epsilon \|_{H^1}^2 \| u^\epsilon \|_{H^2}^2 + C \| u^\epsilon \|_{H^1} \| u^\epsilon \|_{H^3}^3 + C \| u^\epsilon \|_{H^1} \| u^\epsilon \|_{H^2}^3
\]

\[
\leq C \| u^\epsilon \|_{H^1}^2 \| u^\epsilon \|_{H^2}^2 + C \| u^\epsilon \|_{H^1} \| u^\epsilon \|_{H^2}^3.
\]

Notice that

\[
K_2 \leq \sup_{\epsilon > 0} \mathbb{P}\left( \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| B(u^\epsilon(s)) \right\|_{H^1} ds > \frac{\eta}{4} \right)
\]

\[
\leq \sup_{\epsilon > 0} \mathbb{P}\left( \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| u^\epsilon \right\|_{H^1}^\frac{1}{2}, \left\| u^\epsilon \right\|_{H^2}^\frac{3}{2} ds > \frac{\eta}{8C} \right) + \sup_{\epsilon > 0} \mathbb{P}\left( \int_{\zeta^\epsilon}^{\zeta^\epsilon + \delta} \left\| u^\epsilon \right\|_{H^1}^\frac{5}{4}, \left\| u^\epsilon \right\|_{H^2}^\frac{3}{4} ds > \frac{\eta}{8C} \right)
\]
Utilizing (3.1) and (3.5), it yields

\[ K_{2,1} \leq \sup_{\epsilon > 0} \mathbb{P} \left( \int_{\xi^*}^{\xi + \delta} \left( \| u^{\epsilon} \|^2 \right)_{H^2} ds > \frac{\eta}{8C}, \tau^\epsilon_M = T \right) + \sup_{\epsilon > 0} \mathbb{P}(\tau^\epsilon_M < T) \]

\[ \leq \sup_{\epsilon > 0} \mathbb{P} \left( \int_{\xi^*}^{(\xi^* + \delta) \land \tau^\epsilon_M} \left( \| u^{\epsilon} \|^2 \right)_{H^2} ds > \frac{\eta}{8C} \right) + \frac{C}{M} \]

\[ \leq \frac{8C}{\eta} \sup_{\epsilon > 0} \left( \mathbb{E} \int_{\xi^*}^{(\xi^* + \delta) \land \tau^\epsilon_M} \left( \| u^{\epsilon} \|^2 \right)_{H^2} ds \right)^{\frac{1}{2}} \left( \mathbb{E} \int_{\xi^*}^{(\xi^* + \delta) \land \tau^\epsilon_M} \left( \| u^{\epsilon} \|^2 \right)_{H^2} ds \right)^{\frac{1}{2}} + \frac{C}{M} \]

\[ \leq \frac{C\delta}{\eta} + \frac{C}{M}. \]

By the same method as above, we get \( K_{2,1} \leq \frac{C\delta}{\eta} + \frac{C}{M}. \) Hence, it yields \( K_2 \leq \frac{C\delta}{\eta} + \frac{C}{M}. \)

Since \( 0 \leq g_N'(r) \leq C, \) applying the interpolation formula, it follows that

\[ \| \mathcal{P}(g_N(|u^{\epsilon}|^2)u^{\epsilon}) \|_{H^1} \]

\[ \leq \left( \int_{\Omega} \| g_N(|u^{\epsilon}|^2)|u^{\epsilon}|^2 \right) dx \]

\[ + \int_{\Omega} \left[ \frac{2g_N'(|u^{\epsilon}|^2)|u^{\epsilon}| \cdot \nabla u^{\epsilon}|^2 + g_N(|u^{\epsilon}|^2)|\nabla u^{\epsilon}|^2 \right] dx \]

\[ \leq C \left( \| u^{\epsilon} \|_{L^6}^3 + \int_{\Omega} \left( |u^{\epsilon}|^2 |\nabla u^{\epsilon}|^2 \right) dx \right)^{\frac{1}{2}} \]

\[ \leq C \| u^{\epsilon} \|_{H^1}^3 + C \sup_{0 \leq s \leq T} \| u^{\epsilon} \|_{H^1}. \]

Due to (3.13), by using (3.2) and (3.5), we obtain

\[ K_3 \leq \sup_{\epsilon > 0} \mathbb{P} \left( \int_{\xi^*}^{\xi^* + \delta} \left( \mathbb{E} \int_{\xi^*}^{(\xi^* + \delta) \land \tau^\epsilon_M} \left( \| u^{\epsilon} \|^2 \right)_{H^1} ds \right)^{\frac{1}{2}} \left( \mathbb{E} \int_{\xi^*}^{(\xi^* + \delta) \land \tau^\epsilon_M} \left( \| u^{\epsilon} \|^2 \right)_{H^2} ds \right)^{\frac{1}{2}} \right) \]

\[ + \mathbb{E} \int_{\xi^*}^{(\xi^* + \delta) \land \tau^\epsilon_M} \left( \| u^{\epsilon} \|^3 \right)_{H^1} ds + \frac{C}{M} \]

\[ \leq \frac{C\delta}{\eta} \sup_{\epsilon > 0} \left( \mathbb{E} \sup_{t \in [0,T]} \| u^{\epsilon} \|^4_{H^1} \right)^{\frac{1}{2}} \left( \mathbb{E} \sup_{t \in [0,T]} \| u^{\epsilon} \|^2_{H^2} \right)^{\frac{1}{2}} + \mathbb{E} \sup_{t \in [0,T]} \| u^{\epsilon} \|^3_{H^1} + \frac{C}{M}. \]
Combining all the above estimates, we conclude that

\[ \leq \frac{C \delta}{\eta} + \frac{C}{M}. \]

By the Burkholder-Davis-Gundy inequality, we deduce that

\[ K_4 \leq \frac{16}{\eta^2} \sup_{\varepsilon > 0} \mathbb{E} \left( \left\| \int_{\zeta_t}^{\zeta_{t+\delta}} \int_{\mathbb{R}_0} \sigma^\varepsilon(u^\varepsilon(s,-), z) \hat{N}(ds, dz) \right\|_{\mathbb{H}_1}^2 \right) \]

\[ \leq \frac{C}{\eta} \sup_{\varepsilon > 0} \mathbb{E} \left( \int_{\zeta_t}^{\zeta_{t+\delta}} \left\| \sigma^\varepsilon(u^\varepsilon(s,-), z) \right\|_{\mathbb{H}_1}^2 \right) \leq \frac{C \delta}{\eta^2}. \]

Combining all the above estimates, we conclude that

\[ \sup_{\varepsilon > 0} \mathbb{P} \left( \left\| u^\varepsilon(\zeta_t + \delta) - u^\varepsilon(\zeta_t) \right\|_{\mathbb{H}_1} > \eta \right) \leq \frac{C \delta}{\eta^2} + \frac{C \delta}{\eta} + \frac{C}{M}. \]

Let \( \delta \to 0 \), then taking \( M \to \infty \), we obtain (3.10). We complete the proof. \( \square \)

Denote by \( \mu^\varepsilon, \mu \) the laws of \( u^\varepsilon \) and \( u \) on the space \( \mathcal{D}([0, T]; \mathbb{H}_1) \) and \( \mathcal{C}([0, T]; \mathbb{H}_1) \), respectively. Based on Lemmas 3.1–3.3, we are able to establish the following result.

**Proposition 1.** Assume Hypotheses H1, H2, H3, H5 hold. Let \( h \in \mathbb{H}_2 \), then for any \( T > 0 \), \( \mu^\varepsilon \) converges weakly to \( \mu \), as \( \varepsilon \to 0 \), on the space \( \mathcal{D}([0, T]; \mathbb{H}_1) \) equipped with the Skorohod topology.

**Proof.** According to Lemma 3.3, the family \( \{\mu^\varepsilon; \varepsilon > 0\} \) is tight in \( \mathcal{D}([0, T]; \mathbb{H}_1) \). Let \( \mu_0 \) be the weak limit of any convergence subsequence \( \{\mu^{\varepsilon_n}\} \). We will show that \( \mu_0 = \mu \). The proof is divided into three steps.

**Step 1:** \( \mu_0 \) is supported on the space \( C([0, T]; \mathbb{H}_1) \).

For any \( \eta > 0, M > 0 \), we have

\[ \mathbb{P} \left( \sup_{0 < t \leq T} \left\| u^\varepsilon(t) - u^\varepsilon(t-0) \right\|_{\mathbb{H}_1} > \eta \right) \leq \mathbb{P} \left( \sup_{0 < t \leq T} \sup_{z \in \mathbb{R}_0} \left\| \sigma^\varepsilon(u^\varepsilon(t-0), z) \right\|_{\mathbb{H}_1} > \eta \right) \]

\[ \leq \mathbb{P} \left( \sup_{0 < t \leq T} \sup_{z \in \mathbb{R}_0} \left\| \sigma^\varepsilon(u^\varepsilon(t-0), z) \right\|_{\mathbb{H}_1} > \eta, \sup_{0 \leq t \leq T} \left\| u^\varepsilon(t) \right\|_{\mathbb{H}_1} \leq M \right) \]

\[ + \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| u^\varepsilon(t) \right\|_{\mathbb{H}_1} > M \right) \]

\[ \leq \mathbb{P} \left( \sup_{\|z\|_{\mathbb{H}_1} \leq M} \sup_{z \in \mathbb{R}_0} \left\| \sigma^\varepsilon(x, z) \right\|_{\mathbb{H}_1} > \eta \right) + \frac{1}{M^2} \sup_{\varepsilon > 0} \mathbb{E} \sup_{0 \leq t \leq T} \left\| u^\varepsilon(t) \right\|_{\mathbb{H}_1}^2. \]

With the help of (2.17) and (3.1), we firstly let \( \varepsilon \to 0 \), then taking \( M \to \infty \) to get

\[ \sup_{0 < t \leq T} \left\| u^\varepsilon(t) - u^\varepsilon(t-0) \right\|_{\mathbb{H}_1} \to 0 \quad \text{in probability as} \quad \varepsilon \to 0. \]

Hence, we deduce from Theorem 13.4 in [2] that \( \mu_0 \) is supported on the space \( C([0, T]; \mathbb{H}_1) \). As a consequence, the finite dimensional distributions of \( \mu^{\varepsilon_n} \) converges to that of \( \mu_0 \).

**Step 2:** \( \mu_0 \) is a solution of a martingale problem.

Recall that \( \{l_j\}_{j \geq 1} \) is an orthonormal basis of \( \mathbb{H}_1 \). For \( k, j \in \mathbb{N} \), let

\[ f(x) := \langle x, l_j \rangle_{\mathbb{H}_1} \langle x, l_k \rangle_{\mathbb{H}_1}, \quad x \in \mathbb{H}_1. \]
The gradient of \( f \) (denoted by \( \nabla f \)) and the second derivative of \( f \) (denoted by \( f'' \)) are given by

\[
\nabla f(x) = \langle x, l_j \rangle_{H^1} l_k + \langle x, l_k \rangle_{H^1} l_j, \quad (3.14)
\]

\[
f''(x) = l_j \otimes l_k + l_k \otimes l_j. \quad (3.15)
\]

Set

\[
L^\varepsilon f(x) := -(A\nabla f(x), x)_{H^1} - \langle \nabla f(x), B(x) \rangle_{H^1} - \langle \nabla f(x), \mathcal{P}g_N(|x|^2)x \rangle_{H^1} \notag
\]

\[
+ \int_{\mathbb{R}_0} \left[ f(x + \sigma\varepsilon(x,z)) - f(x) - \langle \nabla f(x), \sigma\varepsilon(x,z) \rangle_{H^1} \right] \vartheta(dz), \quad (3.16)
\]

\[
L f(x) := -(A\nabla f(x), x)_{H^1} - \langle \nabla f(x), B(x) \rangle_{H^1} - \langle \nabla f(x), \mathcal{P}g_N(|x|^2)x \rangle_{H^1} \notag
\]

\[
+ \frac{1}{2}\langle f''(x)\sigma(x), \sigma(x) \rangle_{H^1}. \quad (3.17)
\]

By the Itô formula in Lemma 2.3, we deduce that

\[
f(u^\varepsilon(t)) - f(h) - \int_0^t L^\varepsilon f(u^\varepsilon(s))ds
\]

\[
= \int_0^t \int_{\mathbb{R}_0} \left[ f(u^\varepsilon(s-)) + \sigma\varepsilon(u^\varepsilon(s-), z)) - f(u^\varepsilon(s-)) \right] \tilde{N}(ds, dz) \quad (3.18)
\]

is a martingale. Denote by \( X_t(\omega) := \omega(t), \omega \in \mathcal{D}([0, T]; \mathbb{H}^1) \) the coordinate process on \( \mathcal{D}([0, T]; \mathbb{H}^1) \). By the martingale property of (3.18), we have for any \( 0 \leq s_0 < s_1 < \cdots < s_n \leq s < t \) and \( f_0, f_1, \ldots, f_n \in \mathcal{C}_b(\mathbb{H}^1), \)

\[
\mathbb{E}^{\mu_\varepsilon} \left[ \left( f(X_t) - f(X_s) - \int_s^t L^\varepsilon f(X_r)dr \right)f_0(X_{s_0}) \cdots f_n(X_{s_n}) \right] = 0. \quad (3.19)
\]

For \( x \in \mathbb{H}^1, \) define

\[
G_\varepsilon(x) := \left| \int_{\mathbb{R}_0} \langle \sigma\varepsilon(x, z), l_k \rangle_{H^1} \langle \sigma\varepsilon(x, z), l_j \rangle_{H^1} \vartheta(dz) - \langle \sigma(x), l_k \rangle_{H^1} \langle \sigma(x), l_j \rangle_{H^1} \right|. \quad (3.20)
\]

By (3.16) and (3.17), we have

\[
|L^\varepsilon f(X_r) - L f(X_r)| = G_\varepsilon(X_r). \quad (3.21)
\]

We claim that

\[
\lim_{n \to \infty} \mathbb{E}^{u^\varepsilon_n} \left[ \int_s^t \left| L^\varepsilon f(X_r) - L f(X_r) \right|dr \right] = \lim_{n \to \infty} \int_s^t \mathbb{E}G_\varepsilon_n(X^\varepsilon_n)dr = 0. \quad (3.22)
\]

Since

\[
\sup_{\varepsilon > 0} G_\varepsilon(x) \leq C(1 + \|x\|_{H^1}^2), \quad (3.23)
\]

by the dominated convergence theorem and (3.1), to prove (3.22), it suffices to prove that for every \( r \in [0, T], \)

\[
\lim_{n \to \infty} \mathbb{E}G_\varepsilon_n(u^\varepsilon_n(r)) = 0. \quad (3.24)
\]

Since the finite dimensional distributions of \( \mu^\varepsilon_n \) converge weakly to that of \( \mu_0, \) by the Skorohod’s representation theorem, we can assume that \( u^\varepsilon_n(r) \) converges almost surely to a \( \mathbb{H}^1 \)-valued random variable \( u^0. \) In view of (3.1), we know that \( \{\|u^\varepsilon_n(r)\|_{H^1}\}_{n \geq 1} \) is uniformly integrable, and therefore, we can deduce that \( u^0 \in L^2(\Omega; \mathbb{H}^1) \) and

\[
\lim_{n \to \infty} \mathbb{E}\|u^\varepsilon_n(r) - u^0(r)\|_{H^1}^2 = 0. \quad (3.25)
\]
By the dominated convergence theorem, it follows from (2.18) and (3.23) that
\[
\lim_{n \to \infty} \mathbb{E} G_{\varepsilon_n}(u^0) = 0. \tag{3.26}
\]
Hence, in order to prove (3.24), it suffices to prove
\[
\lim_{n \to \infty} \mathbb{E} |G_{\varepsilon_n}(u^\varepsilon_n(r)) - G_{\varepsilon_n}(u^0)| = 0. \tag{3.27}
\]
Note that
\[
\mathbb{E} |G_{\varepsilon_n}(u^\varepsilon_n(r)) - G_{\varepsilon_n}(u^0)| \\
\leq \mathbb{E} \left| \int_{R_0} \langle \sigma^{\varepsilon_n}(u^\varepsilon_n(r), z), l_k \rangle_{H^1} \langle \sigma^{\varepsilon_n}(u^\varepsilon_n(r), z), l_j \rangle_{H^1} \vartheta(dz) - \int_{R_0} \langle \sigma^{\varepsilon_n}(u^0, z), l_k \rangle_{H^1} \langle \sigma^{\varepsilon_n}(u^0, z), l_j \rangle_{H^1} \vartheta(dz) \right| \\
+ \mathbb{E} \left| \langle \sigma(u^\varepsilon_n(r)), l_k \rangle_{H^1} \langle \sigma(u^\varepsilon_n(r)), l_j \rangle_{H^1} - \langle \sigma(u^0), l_k \rangle_{H^1} \langle \sigma(u^0), l_j \rangle_{H^1} \right|
\]
\[=: I_1 + I_2. \]
In view of (2.13) and (2.14), we have
\[
I_1 \leq \mathbb{E} \int_{R_0} \left| \langle \sigma^{\varepsilon_n}(u^\varepsilon_n(r), z), l_k \rangle_{H^1} \langle \sigma^{\varepsilon_n}(u^\varepsilon_n(r), z), l_j \rangle_{H^1} \vartheta(dz) - \langle \sigma^{\varepsilon_n}(u^0, z), l_k \rangle_{H^1} \langle \sigma^{\varepsilon_n}(u^0, z), l_j \rangle_{H^1} \vartheta(dz) \right|
\]
\[+ \mathbb{E} \int_{R_0} \left| \langle \sigma^{\varepsilon_n}(u^\varepsilon_n(r), z) - \sigma^{\varepsilon_n}(u^0, z), l_k \rangle_{H^1} \langle \sigma^{\varepsilon_n}(u^0, z), l_j \rangle_{H^1} \vartheta(dz) \right|
\]
\[\leq C \left[ \mathbb{E} \int_{R_0} \|\sigma^{\varepsilon_n}(u^\varepsilon_n(r), z)\|^2_{H^1} \vartheta(dz) \right]^{1/2}
\]
\[\times \left[ \mathbb{E} \int_{R_0} \|\sigma^{\varepsilon_n}(u^\varepsilon_n(r), z) - \sigma^{\varepsilon_n}(u^0, z)\|^2_{H^1} \vartheta(dz) \right]^{1/2}
\]
\[+ \left[ \mathbb{E} \int_{R_0} \|\sigma^{\varepsilon_n}(u^\varepsilon_n(r), z) - \sigma^{\varepsilon_n}(u^0, z)\|^2_{H^1} \vartheta(dz) \right]^{1/2} \left[ \mathbb{E} \int_{R_0} \|\sigma^{\varepsilon_n}(u^0, z)\|^2_{H^1} \vartheta(dz) \right]^{1/2}
\]
\[\leq C \left[ 1 + \mathbb{E} \|u^0\|^2_{H^1} \right]^{1/2} + \sup_{\varepsilon_n} \left( 1 + \mathbb{E} \|u^\varepsilon_n\|^2_{H^1} \right)^{1/2} \left( \mathbb{E} \|u^\varepsilon_n(r) - u^0\|^2_{H^1} \right)^{1/2}.
\]
Taking into account (3.1) and (3.25), we deduce that $I_1 \to 0$, as $n \to \infty$. Using a similar method as above, we get $I_2 \to 0$. Therefore, (3.27) holds. Hence, the claim (3.22) is proved.

Next, we prove that
\[
M_{k,\langle j \rangle}(t) := f(X_t) - f(h) - \int_0^t Lf(X_r)dr 
\tag{3.28}
\]
is a martingale under $\mu_0$. This is equivalent to proving that
\[
\mathbb{E}^{\mu_0} \left[ \left( f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr \right) f_0(X_{s_0}) \cdots f_n(X_{s_n}) \right] = 0. \tag{3.29}
\]
Since the finite dimensional distribution of $u^\varepsilon_n$ converges to that of $u_0$ and $|f(x)| \leq C|x|^{1/2}$, using (3.1), we deduce from Theorem 1.6.8 in [11] that
\[
\mathbb{E}^{\mu_0} \left[ f(X_t) f_0(X_{s_0}) \cdots f_n(X_{s_n}) \right] = \lim_{n \to \infty} \mathbb{E}^{\mu_0} \left[ f(X_t) f_0(X_{s_0}) \cdots f_n(X_{s_n}) \right]. \tag{3.30}
\]
In the following, we devote to proving that $Lf(x)$ is a bounded continuous function on $H^1$. 
Note that
\[ Lf(x) := -\left(\nabla f(x), Ax\right)_{\mathbb{H}} - \left(\nabla f(x), B(x)\right)_{\mathbb{H}} - \left(\nabla f(x), \mathcal{P}(g_N(|x|^2)x)\right)_{\mathbb{H}} + \frac{1}{2}\left(f''(x)\sigma(x), \sigma(x)\right)_{\mathbb{H}}. \]

Using Hölder inequality and the interpolation inequality, we have
\[ \left|\left(\nabla f(x), Ax\right)_{\mathbb{H}}\right| \leq C\|x\|_{\mathbb{H}}^2 \|l_k\|_{\mathbb{H}} \|l_j\|_{\mathbb{H}} + C\|x\|_{\mathbb{H}}^2 \|l_j\|_{\mathbb{H}} \|l_k\|_{\mathbb{H}} \leq C(\lambda_k, \lambda_j)\|x\|_{\mathbb{H}}^3. \]

Since \(l_j \in \mathbb{H}^3\), we have
\[ \left|\left(\nabla f(x), Ax\right)_{\mathbb{H}} + \left(\nabla f(x), l_j\right)_{\mathbb{H}}\right| \leq C\|x\|_{\mathbb{H}}^2 \|l_k\|_{\mathbb{H}} \|l_j\|_{\mathbb{H}} + C\|x\|_{\mathbb{H}}^2 \|l_j\|_{\mathbb{H}} \|l_k\|_{\mathbb{H}} \leq C(\lambda_k, \lambda_j)\|x\|_{\mathbb{H}}^3. \]

Using Hölder inequality and the interpolation inequality, we have
\[ \left|\left(\nabla f(x), (I - \Delta)k\right)_{\mathbb{H}}\right| = \left|\left(\nabla f(x), (I - \Delta)k\right)_{\mathbb{H}}\right|_{\mathbb{H}^0} \leq C\|\nabla (I - \Delta)k\|_{L^2} \|x\|_{L^2} \|x\|_{L^6} \leq C\|l_k\|_{\mathbb{H}} \|x\|_{\mathbb{H}}^\frac{1}{3} \|x\|_{\mathbb{H}}^\frac{1}{6} \|x\|_{\mathbb{H}}^\frac{1}{6} \leq C(\lambda_k)\|x\|_{\mathbb{H}}^3, \]

which implies that
\[ \left|\left(B(x), l_k\right)_{\mathbb{H}}\right| \leq C(\lambda_k)\|x\|_{\mathbb{H}}^3. \]

Because \(0 \leq g_N'(r) \leq C\), by the interpolation inequality, it follows that
\[ \left|\mathcal{P}(g_N(|x|^2)x)\right|_{\mathbb{H}^0} = \left(\int_{\mathbb{T}^3} |\mathcal{P}(g_N(|x|^2)x)|^2 dx\right)^\frac{1}{2} \leq C\left(\int_{\mathbb{T}^3} |x|^4 |x|^2 dx\right)^\frac{1}{2} \leq C\|x\|_{L^6}^3 \leq C\|x\|_{\mathbb{H}}^3, \]

which implies
\[ \left|\mathcal{P}(g_N(|x|^2)x), l_k\right|_{\mathbb{H}} \leq \left|\mathcal{P}(g_N(|x|^2)x), l_k\right|_{\mathbb{H}} \leq C(\lambda_k)\|x\|_{\mathbb{H}}^3. \]

Hence,
\[ \left|\mathcal{P}(g_N(|x|^2)x), l_k\right|_{\mathbb{H}} + \left|\mathcal{P}(g_N(|x|^2)x), l_j\right|_{\mathbb{H}} \leq C(\lambda_k, \lambda_j)\|x\|_{\mathbb{H}}^3. \]

By (2.9), we obtain
\[ \left|\sigma(x), l_k\right|_{\mathbb{H}} \leq C\|x\|_{\mathbb{H}}^2 \|l_j\|_{\mathbb{H}} \|l_k\|_{\mathbb{H}} = C(\lambda_k, \lambda_j)\|x\|_{\mathbb{H}}^2. \]

Combining all the above estimates, we deduce that \(L(x)\) is a bounded function on \(\mathbb{H}^1\). Moreover, in a similar way, one can show the continuity of \(L(x)\) with respect to \(x \in \mathbb{H}^1\). Thus, we conclude that \(L(x) \in C_0(\mathbb{H}^1)\) and
\[ |L(x)| \leq C(\lambda_k, \lambda_j)(1 + \|x\|_{\mathbb{H}}^3). \]

Therefore, we get for every \(r \in [s, t]\),
\[ \mathbb{E}^{\mu_0} \left[Lf(X_r)f_0(X_{s_0})\ldots f_n(X_{s_n})\right] = \lim_{n \to \infty} \mathbb{E}^{\mu_0^n} \left[Lf(X_r)f_0(X_{s_0})\ldots f_n(X_{s_n})\right]. \]
With the help of (3.31) and (3.32), by the Fubini theorem and the dominated convergence theorem, it follows that
\[
\mathbb{E}^{\mu_0} \left[ \left( \int_s^t Lf(X_r)dr \right) f_0(X_{s_0}) \ldots f_n(X_{s_n}) \right] = \lim_{n \to \infty} \mathbb{E}^{\mu_n} \left[ \left( \int_s^t Lf(X_r)dr \right) f_0(X_{s_0}) \ldots f_n(X_{s_n}) \right].
\] (3.33)

Using (3.30), (3.33), (3.22) and (3.19), we deduce that
\[
\mathbb{E}^{\mu_0} \left[ \left( f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr \right) f_0(X_{s_0}) \ldots f_n(X_{s_n}) \right] = \lim_{n \to \infty} \mathbb{E}^{\mu_n} \left[ \left( f(X_t) - f(X_s) - \int_s^t Lf(X_r)dr \right) f_0(X_{s_0}) \ldots f_n(X_{s_n}) \right] = 0.
\]

Hence, \( M_{k,j}(t) \) defined by (3.28) is a martingale under \( \mu_0 \).

Now, for \( k \in \mathbb{N} \), we define
\[
g(x) = \langle x, l_k \rangle_{\mathbb{H}^1}, \quad x \in \mathbb{H}^1.
\]

Using the similar method as above, we can show that
\[
M_k(t) := g(X_t) - g(h) - \int_0^t Lg(X_r)dr = \langle X_t, l_k \rangle_{\mathbb{H}^1} - \langle h, l_k \rangle_{\mathbb{H}^1} + \int_0^t \langle X_s, Al_k \rangle_{\mathbb{H}^1} ds + \int_0^t \langle B(X_s), l_k \rangle_{\mathbb{H}^1} ds + \int_0^t \langle \mathcal{P}(g_N(|X_s|^2)X_s), l_k \rangle_{\mathbb{H}^1} ds
\] (3.34)

is a martingale under \( \mu_0 \).

Step 3: \( \mu_0 \) is the law of a weak solution of (2.3).

Applying the Itô formula to \( g(X_t) \) and by (3.34), we get
\[
< M_k, M_j > (t) = \int_0^t \langle \sigma(X_s), l_k \rangle_{\mathbb{H}^1}, \langle \sigma(X_s), l_j \rangle_{\mathbb{H}^1} ds,
\] (3.35)

where \(< \cdot, \cdot >\) stands for the sharp bracket of two martingales. According to Theorem 18.12 in [15], there exists a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) with a filtration \( \mathcal{F}'_t \) such that on the extension \((\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathcal{F}_t \times \mathcal{F}'_t, \mu_0 \times \mathbb{P}') \) of \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), there exists a one-dimensional Brownian motion \( W(t), t \geq 0 \) such that
\[
M_k(t) = \int_0^t \langle \sigma(X_s), l_k \rangle_{\mathbb{H}^1} dW(s),
\] (3.36)

which means that
\[
\langle X_t, l_k \rangle_{\mathbb{H}^1} - \langle h, l_k \rangle_{\mathbb{H}^1} = - \int_0^t \langle Al_k, X_s \rangle_{\mathbb{H}^1} ds - \int_0^t \langle B(X_s), l_k \rangle_{\mathbb{H}^1} ds - \int_0^t \langle \mathcal{P}(g_N(|X_s|^2)X_s), l_k \rangle_{\mathbb{H}^1} ds + \int_0^t \langle \sigma(X_s), l_k \rangle_{\mathbb{H}^1} dW(s),
\]
for every $k \geq 1$. Thus, under $\mu_0$, $X_t$ is a solution to (2.3). Due to the uniqueness of (2.3), we conclude $\mu_0 = \mu$. We complete the proof. □

4. Proof of the main result. In the sequel, we relax the Hypothesis $H5$ to Hypothesis $H4$.

Proof of Theorem 2.8. For each $n \in \mathbb{N}$, let $h_n, \sigma_n, \sigma_n^\varepsilon$ be the orthogonal projections of $h, \sigma, \sigma^\varepsilon$ into the $n$-dimensional space $\text{span}\{e_1, \ldots, e_n\}$, respectively. Then, it is easy to know that for each $n \in \mathbb{N}$, $\sigma_n$ satisfies Hypothesis H1 and $\sigma_n^\varepsilon$ satisfies Hypotheses H2–H5. Moreover, there exists a constant $C$ independent of $n$ such that for every $u, u_1, u_2 \in \mathbb{H}^1$,

$$\sup_{n \in \mathbb{N}} \|\sigma_n(u)\|_{\mathbb{H}^1}^2 + \sup_{n \in \mathbb{N}, t > 0} \int_{\mathbb{R}_0} \|\sigma_n^\varepsilon(u, z)\|_{\mathbb{H}^1}^2 \vartheta(dz) \leq C(1 + \|u\|_{\mathbb{H}^1}^2),$$

(4.1)

$$\sup_{n \in \mathbb{N}} \|\sigma_n(u_1) - \sigma_n(u_2)\|_{\mathbb{H}^1}^2 + \sup_{n \in \mathbb{N}, t > 0} \int_{\mathbb{R}_0} \|\sigma_n^\varepsilon(u_1, z) - \sigma_n^\varepsilon(u_2, z)\|_{\mathbb{H}^1}^2 \vartheta(dz) \leq C\|u_1 - u_2\|_{\mathbb{H}^1}^2.$$  

(4.2)

Let $u^{n, \varepsilon}$ be the solution of the following equation

$$
    u^{n, \varepsilon}(t) = h_n - \int_0^t A u^{n, \varepsilon}(s) ds - \int_0^t B(u^{n, \varepsilon}(s)) ds \\
    - \int_0^t P(g_N(|u^{n, \varepsilon}(s)|^2) u^{n, \varepsilon}(s)) ds \\
    + \int_0^t \int_{\mathbb{R}_0} \sigma_n^\varepsilon(u^{n, \varepsilon}(s-), z) \tilde{N}(ds, dz),
$$

(4.3)

and $u^n$ satisfies that

$$
    u^n(t) = h_n - \int_0^t A u^n(s) ds - \int_0^t B(u^n(s)) ds \\
    - \int_0^t P(g_N(|u^n(s)|^2) u^n(s)) ds + \int_0^t \sigma_n(u^n(s)) dW(s).
$$

(4.4)

According to Proposition 1, we have for each $n \in \mathbb{N}$,

$$
    u^{n, \varepsilon} \to u^n \text{ in distribution on the space } D([0,T];\mathbb{H}^1).
$$

(4.5)

Moreover, using the same method as the proof of (3.1) and (3.2), we have

$$
    \sup_{\varepsilon > 0} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^{n, \varepsilon}(t)\|_{\mathbb{H}^1}^2 + \mathbb{E} \int_0^T \|u^{n, \varepsilon}(t)\|_{\mathbb{H}^2}^2 dt \right\} \leq C(1 + \|h\|_{\mathbb{H}^1}^2),
$$

(4.6)

$$
    \sup_{\varepsilon > 0} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^{n, \varepsilon}(t)\|_{\mathbb{H}^1}^4 + \mathbb{E} \int_0^T \|u^{n, \varepsilon}(t)\|_{\mathbb{H}^2}^4 \|u^{n, \varepsilon}(t)\|_{\mathbb{H}^2}^2 dt \right\} \leq C(1 + \|h\|_{\mathbb{H}^1}^2),
$$

(4.7)

and

$$
    \sup_{\varepsilon > 0} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^n(t)\|_{\mathbb{H}^1}^2 + \mathbb{E} \int_0^T \|u^n(t)\|_{\mathbb{H}^2}^2 dt \right\} \leq C(1 + \|h\|_{\mathbb{H}^1}^2),
$$

(4.8)

$$
    \sup_{\varepsilon > 0} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} \|u^n(t)\|_{\mathbb{H}^1}^4 + \mathbb{E} \int_0^T \|u^n(t)\|_{\mathbb{H}^2}^4 \|u^n(t)\|_{\mathbb{H}^2}^2 dt \right\} \leq C(1 + \|h\|_{\mathbb{H}^1}^6).
$$

(4.9)
We claim that for any $\delta > 0$,
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} P \left( \sup_{0 \leq t \leq T} \|u_{n, \varepsilon}(t) - u(t)\|_{H^1} > \delta \right) = 0,
\] (4.10)
\[
\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq T} \|u_n(t) - u(t)\|_{H^1} > \delta \right) = 0.
\] (4.11)
Because of similarity, we only prove (4.10).
Recall from (2.5) that
\[
2 \langle F(u_1) - F(u_2), u_1 - u_2 \rangle_{H^1}
\leq -\frac{1}{2} \|u_1 - u_2\|^2_{H^2} + C_0 (1 + \|u_1\|^4_{H^1} + \|u_2\|^4_{H^1} + \|u_2\|^2_{H^2}) \|u_1 - u_2\|^2_{H^1}.
\] (4.12)
Define
\[
\rho(t) := \int_0^t (1 + \|u_{n, \varepsilon}(s)\|^4_{H^1} + \|u^\varepsilon(s)\|^4_{H^1} + \|u^\varepsilon(s)\|^2_{H^2}) ds.
\]
Applying the Itô formula ([24], proof of Theorem 4.1), we have
\[
e^{-C_0 \rho(t)} \|u_{n, \varepsilon}(t) - u(t)\|^2_{H^1}
= \|h_n - h\|^2_{H^1} - C_0 \int_0^t e^{-C_0 \rho(s)} \|u_{n, \varepsilon}(s) - u^\varepsilon(s)\|^2_{H^1} ds
\times (1 + \|u_{n, \varepsilon}(s)\|^4_{H^1} + \|u^\varepsilon(s)\|^4_{H^1} + \|u^\varepsilon(s)\|^2_{H^2}) ds
+ 2 \int_0^t e^{-C_0 \rho(s)} \langle F(u_{n, \varepsilon}) - F(u^\varepsilon), u_{n, \varepsilon} - u^\varepsilon \rangle_{H^1} ds
+ 2 \int_0^t \int_{\mathbb{R}^d} e^{-C_0 \rho(s)} \langle \sigma_n(u_{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^\varepsilon(s), z), u_{n, \varepsilon} - u^\varepsilon \rangle_{H^1} N(ds, dz)
+ \int_0^t \int_{\mathbb{R}^d} e^{-C_0 \rho(s)} \|\sigma_n(u_{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^\varepsilon(s), z)\|^2_{H^1} N(ds, dz)
= \|h_n - h\|^2_{H^1} + I_1^{n, \varepsilon}(t) + I_2^{n, \varepsilon}(t) + I_3^{n, \varepsilon}(t) + I_4^{n, \varepsilon}(t).
\]
With the aid of (4.12), it follows that
\[
I_1^{n, \varepsilon}(t) + I_2^{n, \varepsilon}(t) \leq -\frac{1}{2} \int_0^t e^{-C_0 \rho(s)} \|u_{n, \varepsilon}(s) - u^\varepsilon(s)\|^2_{H^1} ds.
\] (4.13)
Using the Burkholder-Davis-Gundy inequality, the triangle inequality and (4.2), we deduce that
\[
E \sup_{0 \leq s \leq t} |I_3^{n, \varepsilon}(s)|
\leq CE \left[ \int_0^t \int_{\mathbb{R}^d} e^{-2C_0 \rho(s)} \|\sigma_n(u_{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^\varepsilon(s), z)\|^2_{H^1} \|u_{n, \varepsilon} - u^\varepsilon\|^2_{H^1} \theta(dz) ds \right]^\frac{1}{2}
\leq CE \left[ \sup_{0 \leq s \leq t} e^{-C_0 \rho(s)} \|u_{n, \varepsilon}(s) - u^\varepsilon(s)\|_{H^1}
\times \left( \int_0^t \int_{\mathbb{R}^d} e^{-C_0 \rho(s)} \|\sigma_n(u_{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^\varepsilon(s), z)\|^2_{H^1} \theta(dz) ds \right)^\frac{1}{2} \right]
\leq \frac{1}{2} E \sup_{0 \leq s \leq t} e^{-C_0 \rho(s)} \|u_{n, \varepsilon}(s) - u^\varepsilon(s)\|^2_{H^1}
+ CE \int_0^t \int_{\mathbb{R}^d} e^{-C_0 \rho(s)} \|\sigma_n(u_{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^\varepsilon(s), z)\|^2_{H^1} \theta(dz) ds
\]
\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 + C \mathbb{E} \int_0^t e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 \, ds
\]

\[+
C \mathbb{E} \int_0^t \int_{\mathbb{R}^n} e^{-C_{0p}(s)} \|\sigma_n^\epsilon(u^{n, \epsilon}(s), z) - \sigma^\epsilon(u^{n, \epsilon}(s), z)\|_{H^1}^2 \, \vartheta(dz) \, ds.
\]

Utilizing (4.2) and by the triangle inequality, we obtain

\[
\mathbb{E} \sup_{0 \leq s \leq t} \|I_n^{n, \epsilon}(s)\| \leq \mathbb{E} \int_0^t \int_{\mathbb{R}^n} e^{-C_{0p}(s)} \|\sigma_n^\epsilon(u^{n, \epsilon}(s), z) - \sigma^\epsilon(u^{n, \epsilon}(s), z)\|_{H^1}^2 \, \vartheta(dz) \, ds
\]

Combining all the above estimates, we get for \(t \leq T\),

\[
\mathbb{E} \sup_{0 \leq s \leq T} e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 + \mathbb{E} \int_0^T e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 \, ds
\]

\[\leq \frac{1}{2} \|h_n - h\|_{H^1}^2 + C \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \|\sigma_n^\epsilon(u^{n, \epsilon}(s), z) - \sigma^\epsilon(u^{n, \epsilon}(s), z)\|_{H^1}^2 \, \vartheta(dz) \, ds.
\]

Applying Gronwall inequality, we obtain

\[
\mathbb{E} \sup_{0 \leq s \leq T} e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 + \mathbb{E} \int_0^T e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 \, ds
\]

\[\leq C(T) \left[ \|h_n - h\|_{H^1}^2 + \mathbb{E} \int_0^T \int_{\mathbb{R}^n} \|\sigma_n^\epsilon(u^{n, \epsilon}(s), z) - \sigma^\epsilon(u^{n, \epsilon}(s), z)\|_{H^1}^2 \, \vartheta(dz) \, ds \right].
\]

If

\[\lim_{n \to \infty} \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^n} \|\sigma_n^\epsilon(u^{n, \epsilon}(s), z) - \sigma^\epsilon(u^{n, \epsilon}(s), z)\|_{H^1}^2 \, \vartheta(dz) \, ds = 0.
\]

then we deduce from (4.14) that

\[\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \sup_{0 \leq s \leq T} e^{-C_{0p}(s)} \|u^{n, \epsilon}(s) - u^{\varepsilon}(s)\|_{H^1}^2 = 0.
\]

In the following, we aim to prove (4.15). Let

\[\Gamma_n^\epsilon(x) := \int_{\mathbb{R}^n} \|\sigma_n^\epsilon(x, z) - \sigma^\epsilon(x, z)\|_{H^1}^2 \, \vartheta(dz), \quad x \in \mathbb{H}^1.
\]

Clearly,

\[\sup_{n \in \mathbb{N}, \varepsilon > 0} \Gamma_n^\epsilon(x) \leq C(1 + \|x\|_{\mathbb{H}^1}^2).
\]

By (4.6) and the dominated convergence theorem, to prove (4.15), it suffices to show that for every \(s \in [0, T]\),

\[\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \Gamma_n^\epsilon(u^{n, \epsilon}(s)) = 0.
\]

Note that (4.19) will follow if we can prove the following three equalities:

\[\lim_{\varepsilon \to 0} \mathbb{E} \Gamma_n^\epsilon(u^{n, \epsilon}(s)) = \lim_{\varepsilon \to 0} \mathbb{E} \Gamma_n^\epsilon(u^n(s)), \quad \forall n \in \mathbb{N},
\]
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \Gamma_n^\varepsilon(u^n(s)) = \lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \Gamma_n^\varepsilon(u(s)), \tag{4.21}
\]
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \Gamma_n^\varepsilon(u(s)) = 0. \tag{4.22}
\]

We firstly prove (4.20). Since \(u_n\) is continuous, by (4.5), we have for each \(n \in \mathbb{N}, \ s \in [0, T]\),
\[
\|u^{n, \varepsilon}(s) - u^n\|_{\mathbb{H}^1} \to 0 \text{ almost surely as } \varepsilon \to 0.
\]
Hence, we can use Skorohod’s representative theorem to assume that \(\|u^{n, \varepsilon}(s) - u^n\|_{\mathbb{H}^1} \to 0\) almost surely as \(\varepsilon \to 0\). In view of (4.6), \(\{\|u^{n, \varepsilon}(s)\|_{\mathbb{H}^1}^2\}_{\varepsilon > 0}\) is uniformly integrable, and therefore, we deduce that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \|u^{n, \varepsilon}(s) - u^n(s)\|_{\mathbb{H}^1}^2 = 0. \tag{4.24}
\]

Note that
\[
\mathbb{E} \Gamma_n^\varepsilon(u^{n, \varepsilon}(s)) - \Gamma_n^\varepsilon(u^n(s))
\]
\[
\leq \mathbb{E} \int_{\mathbb{R}_0} \left| \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right| \phi(dz)
\]
\[
\leq \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz)
\]
\[
\times \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz)
\]
\[
\leq \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz)
\]
\[
\times \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz)
\]
\[
\leq \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz)
\]
\[
\times \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz)
\]
\[
= \left[ \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz) \right]^2
\]
where we have used
\[
|a-b|^2 - |c-d|^2 = (a-b-c+d)(a-b+c-d) \leq (|a-c| + |b-d|)(|a-b| + |c-d|).
\]

For \(I_2\), by (2.13), (4.1), (4.6) and (4.7), we deduce that
\[
\sup_{\varepsilon > 0} \mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz) < \infty.
\]

For \(I_1\), with the help of (2.14), (4.2) and (4.24), we deduce that
\[
\mathbb{E} \int_{\mathbb{R}_0} \left( \sigma_n^\varepsilon(u^{n, \varepsilon}(s), z) - \sigma^\varepsilon(u^{n, \varepsilon}(s), z) \right) \phi(dz) < \infty.
\]

Therefore, (4.20) holds.

In view of (4.11), using a similar method, we obtain
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \left| \Gamma_n^\varepsilon(u^n(s)) - \Gamma_n^\varepsilon(u(s)) \right| = 0.
\]
Hence, (4.21) holds. Utilizing Hypothesis H3 and Hypothesis H4, we deduce that
\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \int_{\mathbb{R}_0} \left| \sigma_n^\varepsilon(x, z) - \sigma^\varepsilon(x, z) \right| \phi(dz)
\]
\[
= \lim_{n \to \infty} \lim_{\varepsilon \to 0} \left[ \int_{\mathbb{R}_0} \left| \sigma^\varepsilon(x, z) \right| \phi(dz) - \int_{\mathbb{R}_0} \left| \sigma_n^\varepsilon(x, z) \right| \phi(dz) \right]
\]
\[
= \left| \sigma(x) \right| \phi(\mathbb{R}_0) - \lim_{n \to \infty} \left| \sigma_n(x) \right| \phi(\mathbb{R}_0) = 0, \quad \forall x \in \mathbb{H}^1. \tag{4.25}
\]
Thus, by the dominated convergence theorem, we deduce from (4.25) and (4.18) that (4.22) holds. Then (4.19) holds. Therefore, (4.15) is proved, which implies that (4.16) holds.

Now, we are ready to prove (4.10). In view of (4.7) and (4.9), for any given \( \delta_1 > 0 \), we can choose a positive constant \( M_1 \) such that

\[
\sup_{n \in \mathbb{N}, \varepsilon > 0} \mathbb{P} \left( \sup_{0 \leq t \leq T} \| u^{n, \varepsilon}(t) - u^\varepsilon(t) \|_{H^1} > \delta \right)
\]

\[
\int_0^T (1 + \| u^{n, \varepsilon}(s) \|^4_{H^1} + \| u^n(s) \|^4_{H^1} + \| u^n(s) \|_{H^2}^2) ds > M_1
\]

\[
\leq \sup_{n \in \mathbb{N}, \varepsilon > 0} \mathbb{P} \left( \int_0^T (1 + \| u^{n, \varepsilon}(s) \|^4_{H^1} + \| u^n(s) \|^4_{H^1} + \| u^n(s) \|_{H^2}^2) ds > M_1 \right)
\]

\[
\leq \frac{1}{M_1} \left[ T + T \sup_{n \in \mathbb{N}, \varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \| u^{n, \varepsilon}(t) \|^4_{H^1} + T \sup_n \mathbb{E} \sup_{t \in [0, T]} \| u^n(t) \|^4_{H^1} \right.
\]

\[
+ \sup_n \mathbb{E} \int_0^T \| u^n(s) \|_{H^2}^2 ds \leq \frac{C}{M_1} \leq \delta_1.
\]

Moreover, using (4.16), it follows that

\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{0 \leq t \leq T} \| u^{n, \varepsilon}(t) - u^\varepsilon(t) \|_{H^1} > \delta \right)
\]

\[
\int_0^T (1 + \| u^{n, \varepsilon}(s) \|^4_{H^1} + \| u^n(s) \|^4_{H^1} + \| u^n(s) \|_{H^2}^2) ds \leq M_1
\]

\[
\leq \lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{-C_0 \rho(t)} \| u^{n, \varepsilon}(s) - u^\varepsilon(s) \|_{H^1}^2 \right) \geq e^{-C_0 M_1 \delta^2} \]

\[
\leq \frac{e^{-C_0 M_1 \delta^2}}{\delta^2} \lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{-C_0 \rho(t)} \| u^{n, \varepsilon}(t) - u^\varepsilon(t) \|_{H^1}^2 \right) = 0.
\]

Combining (4.26) and (4.27), we deduce that

\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{0 \leq t \leq T} \| u^{n, \varepsilon}(t) - u^\varepsilon(t) \|_{H^1} > \delta \right) \leq \delta_1.
\]

Since \( \delta_1 \) is arbitrary, we conclude that (4.10) holds.

Finally, we are ready to prove that \( \mu^\varepsilon \) converges weakly to \( \mu \). Let \( \mu^\varepsilon_n, \mu_n \) denote the laws of \( u^{n, \varepsilon} \) and \( u^n \) on \( S := D([0, T]; \mathbb{H}^1) \), respectively. Let \( G \) be any given bounded, uniformly continuous function on \( S \). For any \( n \geq 1 \), we have

\[
\int_S G(\omega) \mu^\varepsilon(\omega, d\omega) - \int_S G(\omega) \mu(\omega, d\omega)
\]

\[
= \int_S G(\omega) \mu^\varepsilon(\omega, d\omega) - \int_S G(\omega) \mu^\varepsilon_n(\omega, d\omega) + \int_S G(\omega) \mu^\varepsilon_n(\omega, d\omega) - \int_S G(\omega) \mu_n(\omega, d\omega)
\]

\[
+ \int_S G(\omega) \mu_n(\omega, d\omega) - \int_S G(\omega) \mu(\omega, d\omega)
\]

\[
= \mathbb{E} \left[ G(u^\varepsilon) - G(u^{n, \varepsilon}) \right] + \int_S G(\omega) \mu^\varepsilon_n(\omega, d\omega) - \int_S G(\omega) \mu_n(\omega, d\omega)
\]

\[
+ \mathbb{E} \left[ G(u^n) - G(u) \right].
\]

(4.29)
Fix \( \delta > 0 \). Since \( G \) is uniformly continuous, there exists \( \delta_1 > 0 \) such that for any \( n \geq 1 \) and \( \varepsilon > 0 \),
\[
\mathbb{E} \left[ (G(u^\varepsilon) - G(u^{n_1, \varepsilon})) I_{\{ \sup_{0 \leq s \leq T} \| u^{n_1, \varepsilon}(s) - u^\varepsilon(s) \|_{H^1} \leq \delta_1 \}} \right] \leq \frac{\delta}{4}. \tag{4.30}
\]
Due to (4.10), there exist \( n_1 > 0 \) and \( \varepsilon_{n_1} \) such that
\[
\sup_{\varepsilon \leq \varepsilon_{n_1}} \mathbb{E} \left[ (G(u^\varepsilon) - G(u^{n_1, \varepsilon})) I_{\{ \sup_{0 \leq s \leq T} \| u^{n_1, \varepsilon}(s) - u^\varepsilon(s) \|_{H^1} > \delta_1 \}} \right] \leq C \sup_{\varepsilon \leq \varepsilon_{n_1}} \mathbb{P} \left( \sup_{0 \leq s \leq T} \| u^{n_1, \varepsilon}(s) - u^\varepsilon(s) \|_{H^1} > \delta_1 \right) \leq \frac{\delta}{4}. \tag{4.31}
\]
Combining (4.30) and (4.31), we have
\[
\mathbb{E} \left[ G(u^\varepsilon) - G(u^{n_1, \varepsilon}) \right] \leq \frac{\delta}{2}. \tag{4.32}
\]
Similarly, by using (4.11), there exists \( n_2 > n_1 \) such that
\[
\mathbb{E} \left[ G(u^{n_2}) - G(u) \right] \leq \frac{\delta}{2}. \tag{4.33}
\]
Moreover, with the help of (4.5), there exists \( \varepsilon_1 > 0 \) such that for \( \varepsilon \leq \varepsilon_1 \),
\[
\left| \int_S G(\omega) \mu^\varepsilon_{n_2}(d\omega) - \int_S G(\omega) \mu_{n_2}(d\omega) \right| \leq \frac{\delta}{4}. \tag{4.34}
\]
Combining (4.32)-(4.34), and by (4.29), we deduce that for \( \varepsilon \leq \min\{\varepsilon_{n_1}, \varepsilon_1\} \),
\[
\left| \int_S G(\omega) \mu^\varepsilon(d\omega) - \int_S G(\omega) \mu(d\omega) \right| \leq \delta.
\]
Since \( \delta > 0 \) is arbitrary small, we deduce that
\[
\lim_{\varepsilon \to 0} \int_S G(\omega) \mu^\varepsilon(d\omega) = \int_S G(\omega) \mu(d\omega).
\]
We complete the proof. \( \square \)

REFERENCES

[1] D. Aldous, Stopping times and tightness, Ann. Probab., 6 (1978), 335–340.
[2] P. Billingsley, Convergence of Probability Measure, 2nd edition, John Wiley & Sons, Inc., New York, 1999.
[3] Z. Brzeźniak and G. Dhariwal, Stochastic Tamed Navier-Stokes Equations on \( \mathbb{R}^3 \): The Existence and the Uniqueness of Solutions and the Existence of an Invariant Measure, J. Math. Fluid Mech., 22 (2020), 54 pp.
[4] Z. Brzeźniak, E. Hausenblas and J. Zhu, 2D Navier-Stokes equations driven by jump noise, Nonlinear Anal., 79 (2013), 122–139.
[5] A. Bensoussan and R. Temam, Équations stochastiques du type Navier-Stokes, J. Funct. Anal., 13 (1973), 195–222.
[6] A. Budhiraja, P. Dupuis and V. Maroulas, Variational representations for continuous time processes, Ann. Inst. Henri Poincaré Probab. Stat., 47 (2011), 725–747.
[7] Z. Brzeźniak, W. Liu and J. Zhu, Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise, Nonlinear Anal. Real World Appl., 17 (2014), 283–310.
[8] G. Di Nunno and T. Zhang, Approximations of stochastic partial differential equations, Ann. Appl. Probab., 26 (2016), 1443–1466.
[9] Z. Dong, J. Xiong, J. Zhai and T. Zhang, A moderate deviation principle for 2-D stochastic Navier-Stokes equations driven by multiplicative Lévy noises, J. Funct. Anal., 272 (2017), 227–254.
[10] Z. Dong and R. Zhang, 3D tamed Navier-Stokes equations driven by multiplicative Lévy noise: Existence, uniqueness and large deviations, J. Math. Anal. Appl., 492 (2020), 124404.
[11] R. Durrett, Probability: Theory and Examples, 4th edition, Cambridge University Press, Cambridge, 2010.
[12] F. Flandoli and D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations, Probab. Theory Related Fields, 102 (1995), 367–391.
[13] A. Ichikawa, Some inequalities for martingales and stochastic convolutions, Stoch. Anal. Appl., 4 (1986), 329-339.
[14] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, 2nd edition, North-Holland Mathematical Library, 1989.
[15] O. Kallenberg, Foundations of Modern Probability, 2nd edition, Springer-Verlag, New York, 2002.
[16] P. L. Lions, Mathematical Topics in Fluid Mechanics, Clarendon Press, Oxford, 1996.
[17] R. Mikulevicius and B. L. Rozovskii, Global $L^2$ solution of stochastic Navier-Stokes equations, Ann. Probab., 33 (2005), 137–176.
[18] M. T. Mohan, K. Sakthivel and S. S. Sritharan, Ergodicity for the 3D stochastic Navier-Stokes equations perturbed by Lévy noise, Math. Nachr., 292 (2019), 1056–1088.
[19] M. Röckner and T. Zhang, Stochastic 3D tamed Navier-Stokes equations: existence, uniqueness and small time large deviation principles, J. Differ. Equ., 252 (2012), 716–744.
[20] M. Röckner, T. Zhang and X. Zhang, Large deviations for stochastic tamed 3D Navier-Stokes equations, Appl. Math. Optim., 61 (2010), 267–285.
[21] M. Röckner and X. Zhang, Tamed 3D Navier-Stokes equation: existence, uniqueness and regularity, Infin. Dimens. Anal. Quantum Probab. Relat. Top., 12 (2009), 525–549.
[22] M. Röckner and X. Zhang, Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and ergodicity, Probab. Theory Related Fields, 145 (2009), 211–267.
[23] B. L. Rozovskii, Stochastic Evolution Systems, Kluwer Academic, Dordrecht, 1990.
[24] B. Schmalfuss, Qualitative properties for the stochastic Navier-Stokes equation, Nonlinear Anal., 28 (1997), 1545–1563.
[25] S. Shang and T. Zhang, Approximations of stochastic Navier-Stokes equations, Stochastic Process. Appl., 130 (2020), 2407–2432.
[26] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.
[27] X. Zhang, A tamed 3D Navier-Stokes equation in uniform $C^2$–domains, Nonlinear Anal., 71 (2009), 3093–3112.

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E-mail address: xhpeng@hunnu.edu.cn
E-mail address: rrzhang@amss.ac.cn