Twisted D-branes of the $\widehat{\mathfrak{su}}(N)_K$ WZW model and level-rank duality

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Abstract

We analyze the level-rank duality of $\omega_c$-twisted D-branes of $\widehat{\mathfrak{su}}(N)_K$ (when $N$ and $K > 2$). When $N$ or $K$ is even, the duality map involves $\mathbb{Z}_2$-cominimal equivalence classes of twisted D-branes. We prove the duality of the spectrum of an open string stretched between $\omega_c$-twisted D-branes, and ascertain the relation between the charges of level-rank-dual $\omega_c$-twisted D-branes.

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1 Introduction

Level-rank duality is a relationship between various quantities in bulk Wess-Zumino-Witten models with classical Lie groups \([1, 2, 3]\). It has recently been shown \([4, 5]\) that level-rank duality also applies to untwisted and to certain twisted D-branes in the corresponding boundary WZW models \([6]-[31]\). (For a review of D-branes on group manifolds, see ref. \([32]\).)

In this paper, we extend this work to include all charge-conjugation-twisted D-branes of the \(\hat{\mathfrak{su}}(N)_K\) WZW model.

Untwisted (i.e., symmetry-preserving) D-branes of WZW models are labelled by the integrable highest-weight representations \(V_\lambda\) of the affine Lie algebra. For \(\hat{\mathfrak{su}}(N)_K\), these representations belong to cominimal equivalence classes generated by the \(\mathbb{Z}_N\) simple current of the WZW model, and therefore so do the untwisted D-branes of the model. Level-rank duality is a one-to-one correspondence between cominimal equivalence classes (or simple-current orbits) of integrable representations of \(\hat{\mathfrak{su}}(N)_K\) and \(\hat{\mathfrak{su}}(K)_N\), and therefore induces a map between cominimal equivalence classes of untwisted D-branes.

The spectrum of an open string stretched between D-branes labelled by \(\alpha\) and \(\beta\) is specified by the coefficients of the partition function

\[
Z_{\alpha\beta}^{\text{open}}(\tau) = \sum_{\lambda \in P^K_+} n_{\beta \lambda}^\alpha \chi_\lambda(\tau) \tag{1.1}
\]

where \(\chi_\lambda(\tau)\) is the affine character of the integrable highest-weight representation \(V_\lambda\). For untwisted D-branes, the coefficients \(n_{\beta \lambda}^\alpha\) are equal to the fusion coefficients of the bulk WZW theory \([33]\), so the well-known level-rank duality of the fusion rules \([1, 2, 3]\) implies the duality of the open-string spectrum between untwisted branes.

Untwisted D-branes of \(\hat{\mathfrak{su}}(N)_K\) possess a conserved D0-brane charge belonging to \(\mathbb{Z}_{x_{N,K}}\):

\[
Q_\lambda = (\dim \lambda)_{\mathfrak{su}(N)} \mod x_{N,K} \tag{1.2}
\]

where \([15, 17]\)

\[
x_{N,K} \equiv \frac{N + K}{\gcd\{N + K, \text{lcm}\{1, \ldots, N - 1\}\}}. \tag{1.3}
\]

The charges of cominimally-equivalent untwisted D-branes are equal up to sign \([17]\)

\[
Q_{\sigma(\lambda)} = (-1)^{N-1} Q_\lambda \mod x_{N,K} \tag{1.4}
\]

where \(\sigma\) is the \(\mathbb{Z}_N\) simple current of \(\hat{\mathfrak{su}}(N)_K\). It was shown in refs. \([4, 5]\) that the charges of level-rank-dual untwisted D-branes of \(\hat{\mathfrak{su}}(N)_K\) and \(\hat{\mathfrak{su}}(K)_N\) are related by

\[
\tilde{Q}_\lambda = \begin{cases} (-1)^{r(\lambda)} Q_\lambda \mod x & \text{for } N + K \text{ odd} \\ Q_\lambda \mod x & \text{for } N + K \text{ even (except for } N = K = 2^m) \end{cases} \tag{1.5}
\]

where \(r(\lambda)\) is the number of boxes in the Young tableau associated with the representation \(\lambda\), where \(\tilde{\lambda}\) is the level-rank-dual representation of \(\hat{\mathfrak{su}}(K)_N\) associated with the transposed tableau, and \(x = \min\{x_{N,K}, x_{K,N}\}\). For the remaining case, it was conjectured that

\[
\tilde{Q}_\lambda = \begin{cases} (-1)^{r(\lambda)/N} Q_\lambda \mod x & \text{when } N \mid r(\lambda) \\ Q_\lambda \mod x & \text{when } N \not\mid r(\lambda) \end{cases} \tag{1.6}
\]

for \(N = K = 2^m\).
on the basis of numerical evidence.

In addition to untwisted D-branes, most WZW models contain twisted D-branes, whose charges also belong to \( \mathbb{Z}_{N,K} \). The coefficients \( n_{\beta \lambda}^\alpha \) of the partition function (1.1) of an open string stretched between twisted D-branes \( \alpha \) and \( \beta \) are given by

\[
n_{\beta \lambda}^\alpha = \sum_{\mu \in \mathcal{E}} \frac{\psi_{\alpha \mu}^* S_{\mu \lambda} \psi_{\beta \mu}}{S_{0 \mu}} \tag{1.7}
\]

where \( \psi_{\alpha \mu} \) is the modular-transformation matrix of the associated twisted affine Lie algebra.

One such class of D-branes for \( \hat{\mathfrak{su}}(N) \) are those twisted by the charge-conjugation symmetry \( \omega_c \), which exist for all \( N > 2 \). This paper will analyze the level-rank duality of \( \omega_c \)-twisted D-branes of \( \hat{\mathfrak{su}}(N) \) and in particular, the relationship between the open-string partition function coefficients (1.7), and between the D-brane charges. (In ref. [5], level-rank duality of \( \omega_c \)-twisted D-branes was examined in the special case that \( N \) and \( K \) were both odd.)

As shown in ref. [21], and reviewed in sections 4 and 5, the \( \omega_c \)-twisted D-branes of \( \hat{\mathfrak{su}}(2n) \) (resp. \( \hat{\mathfrak{su}}(2n+1) \)) are labelled by a subset of integrable highest-weight representations of \( \hat{\mathfrak{so}}(2n+1) \) (resp. \( \hat{\mathfrak{so}}(2n+1) \)), or alternatively, by a subset of integrable highest-weight representations of \( \hat{\mathfrak{sp}}(n) \) (resp. \( \hat{\mathfrak{sp}}(n) \)). In section 4, we show that, like untwisted D-branes, \( \omega_c \)-twisted D-branes of \( \hat{\mathfrak{su}}(2n) \) belong to cominimal equivalence classes, but now generated by the \( \mathbb{Z}_2 \) simple-current of \( \hat{\mathfrak{so}}(2n+1) \). As shown in section 7, cominimally-equivalent \( \omega_c \)-twisted D-branes of \( \hat{\mathfrak{su}}(2n) \) have equal and opposite charges (mod \( x_{2n,K} \)).

In section 6, we describe a one-to-one map \( \alpha \rightarrow \hat{\alpha} \) between the \( \omega_c \)-twisted D-branes (or cominimal equivalence classes of branes) of \( \hat{\mathfrak{su}}(N) \) and the \( \omega_c \)-twisted D-branes (or cominimal equivalence classes of branes) of \( \hat{\mathfrak{su}}(K) \). The exact form of the level-rank map depends on whether \( N \) and \( K \) are even or odd. We then show the equality of the open string partition function coefficients (1.7) for level-rank-dual \( \omega_c \)-twisted D-branes. Because the level-rank map involves cominimal equivalence classes in the case of \( \hat{\mathfrak{su}}(2n) \), the natural quantity to consider in that case is

\[
s_{\beta \lambda}^\alpha = \left( \frac{1}{2} \right)^{\frac{1}{2}(t(\alpha)+t(\beta))+1} \left[ n_{\beta \lambda}^\alpha + n_{\beta \lambda} \sigma(\alpha) + n_{\sigma(\beta)\lambda}^\alpha + n_{\sigma(\beta)\lambda} \sigma(\alpha) \right] \tag{1.8}
\]

where \( \sigma \) is the \( \mathbb{Z}_2 \) simple-current symmetry of \( \hat{\mathfrak{so}}(2n+1) \), and \( t(\alpha) \) is defined in eq. (4.3).

In section 7, we ascertain the relationship between the charges of level-rank-dual \( \omega_c \)-twisted D-branes.

Sections 2 and 3 contain some necessary background material on twisted states in WZW models and on integrable representations of \( \hat{\mathfrak{so}}(2n+1) \), and concluding remarks comprise section 8.

## 2 Twisted D-branes of WZW models

In this section, we review some aspects of twisted D-branes of WZW models and their relation to the twisted Cardy and twisted Ishibashi states of the closed-string sector, drawing on refs. [7] [8] [9] [19] [21].
The WZW model, which describes strings propagating on a group manifold, is a rational
conformal field theory whose chiral algebra (for both left- and right-movers) is the (untwisted)
affine Lie algebra \( \hat{g}_K \) at level \( K \). We only consider WZW theories with a diagonal closed string spectrum:

\[
\mathcal{H}^{\text{closed}} = \bigoplus_{\lambda \in P^+_K} V_\lambda \otimes \overline{V}_{\lambda^*}
\]

where \( V \) and \( \overline{V} \) represent left- and right-moving states respectively, and \( \lambda^* \) denotes the representation conjugate to \( \lambda \). \( V_\lambda \in P^+_K \) are integrable highest-weight representations of \( \hat{g}_K \), whose highest weight \( \lambda \) has non-negative Dynkin indices \( (a_0, a_1, \cdots, a_n) \) satisfying \( \sum_{i=0}^{n} m_i a_i = K \) (where \( n = \text{rank} \ g \) and \( (m_0, m_1, \cdots, m_n) \) are the dual Coxeter labels of \( \hat{g}_K \)).

D-branes of the WZW model may be studied algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We label the allowed boundary conditions (and therefore the D-branes) by \( \alpha, \beta, \cdots \).

We consider boundary conditions on the currents of the affine Lie algebra of the form

\[
[J^a(z) - \omega \overline{J}^a(\bar{z})] \bigg|_{z=\bar{z}=0} = 0 \quad (2.2)
\]

where \( \omega \) is an automorphism of the Lie algebra \( g \). These boundary conditions leave unbroken the \( \hat{g}_K \) symmetry, as well as the conformal symmetry, of the theory. Untwisted D-branes correspond to \( \omega = 1 \). Open-closed string duality allows one to correlate the boundary conditions (2.2) of the boundary WZW model with coherent states \( \langle B \rangle_{\omega} \in \mathcal{H}^{\text{closed}} \) of the bulk WZW model satisfying

\[
[J^a_m + \omega \overline{J}^a_{-m}] \langle B \rangle_{\omega} = 0, \quad m \in \mathbb{Z} \quad (2.3)
\]

where \( J^a_m \) are the modes of the affine Lie algebra generators.

Solutions of eq. (2.3) that belong to a single sector \( V_{\mu} \otimes \overline{V}_{\omega(\mu)} \) of the bulk WZW theory are known as \( \omega \)-twisted Ishibashi states \( \langle \mu \rangle_{I}^{\omega} \). (Solutions corresponding to \( \omega = 1 \) are the ordinary untwisted Ishibashi states \([37]\).) Since we are considering the diagonal closed-string theory (2.1), these states only exist when \( \mu = \omega(\mu) \), so the \( \omega \)-twisted Ishibashi states are labelled by \( \mu \in \mathcal{E}^{\omega} \), where \( \mathcal{E}^{\omega} \subset P^+_K \) are the integrable highest-weight representations of \( \hat{g}_K \) that satisfy \( \omega(\mu) = \mu \). Equivalently, \( \mu \) corresponds to an integrable highest-weight representation of \( \hat{g} \), the orbit Lie algebra \([38]\) associated with \( \hat{g}_K \).

A coherent state \( \langle B \rangle_{\omega} \) that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions \([33]\). Solutions of eq. (2.3) that also satisfy the Cardy conditions are denoted \( \omega \)-twisted Cardy states \( \langle \alpha \rangle_{C}^{\omega} \), where the labels \( \alpha \) take values in some set \( \mathcal{B}^{\omega} \). The \( \omega \)-twisted D-branes of \( \hat{g}_K \) correspond to \( \langle \alpha \rangle_{C}^{\omega} \) and are therefore also labelled by \( \alpha \in \mathcal{B}^{\omega} \). These states correspond \([9]\) to integrable highest-weight representations of the \( \omega \)-twisted affine Lie algebra \( \hat{g}_K^{\omega} \) (but see ref. [24]).

The \( \omega \)-twisted Cardy states may be expressed as linear combinations of \( \omega \)-twisted Ishibashi states

\[
\langle \alpha \rangle_{C}^{\omega} = \sum_{\mu \in \mathcal{E}^{\omega}} \frac{\psi_{\alpha \mu}}{\sqrt{S_{0\mu}}} \langle \mu \rangle_{I}^{\omega} \quad (2.4)
\]

where \( S_{\lambda \mu} \) is the modular transformation matrix of \( \hat{g}_K \), \( 0 \) denotes the identity representation, and the coefficients \( \psi_{\alpha \mu} \) may be identified \([9]\) with the modular transformation matrices of
characters of the twisted affine Lie algebra \( \hat{\mathfrak{g}}_K \), as may be seen, for example, by examining the partition function of an open string stretched between an \( \omega \)-twisted and an untwisted D-brane [19, 21]. Using arguments presented, e.g., in ref. [21], the coefficients of the open string partition function (1.1) may be expressed as

\[
n_{\beta \lambda}^\alpha = \sum_{\mu \in \mathcal{E}} \psi_{\alpha \mu}^\ast S_{\lambda \mu} \psi_{\beta \mu} S_{0 \mu}.
\]  

(2.5)

3 Integrable representations of \( \hat{\mathfrak{so}}(2n+1)_{K'} \)

This section presents details about integrable highest-weight representations of \( \hat{\mathfrak{so}}(2n+1)_{K'} \) that will be needed for the discussion of \( \omega_c \)-twisted states of the \( \hat{\mathfrak{su}}(N)_K \) WZW model.

Integrable representations of \( \hat{\mathfrak{so}}(2n+1)_{K'} \) have Dynkin indices \((a_0, a_1, \ldots, a_n)\) that satisfy

\[
\sum_{i=0}^{n} a_i = K',
\]

where \( m_i \) are the dual Coxeter labels of the extended Dynkin diagram for \( \mathfrak{so}(2n+1) \) (with the dual Coxeter labels shown adjacent to each node), that is,

\[
a_0 + a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n = K'.
\]  

(3.1)

An even or odd value of \( a_n \) corresponds to a tensor or spinor representation respectively. With each tensor representation of \( \mathfrak{so}(2n+1) \) may be associated a Young tableau whose row lengths \( \ell_i \) are given by

\[
\ell_i = \begin{cases} 
\frac{1}{2} a_n + \sum_{j=i}^{n-1} a_j & \text{for } 1 \leq i \leq n-1 \\
\frac{1}{2} a_n & \text{for } i = n
\end{cases}
\]

(3.2)

with total number of boxes \( r = \sum_{i=1}^{n} \ell_i \). We also formally use eq. (3.2) to define row lengths for a spinor representation. These row lengths are all half-integers, and correspond to a “Young tableau” with a column of “half-boxes.” The integrability condition (3.1) corresponds to the constraint \( \ell_1 + \ell_2 \leq K' \) on the row lengths of the tableau.

The extended Dynkin diagram of \( \mathfrak{so}(2n+1) \) has a \( \mathbb{Z}_2 \) symmetry that interchanges the 0th and 1st nodes. This symmetry induces a simple-current symmetry (denoted by \( \sigma \)) of the \( \hat{\mathfrak{so}}(2n+1)_{K'} \) WZW model that pairs integrable representations related by \( a_0 \leftrightarrow a_1 \), with the other Dynkin indices unchanged. Their respective Young tableaux are related by \( \ell_1 \rightarrow K' - \ell_1 \). Under \( \sigma \), tensor representations are mapped to tensors, and spinor representations to spinors, and the modular transformation matrix \( S' \) of \( \hat{\mathfrak{so}}(2n+1)_{K'} \) obeys

\[
S'_{\sigma(\alpha), \mu'} = \pm S_{\alpha', \mu} \quad \text{for } \mu' \text{ a } \begin{cases} \text{tensor} \\
\text{spiner}
\end{cases}
\]

representation.

(3.3)

\[\footnote{Note: throughout this paper, by \( \hat{\mathfrak{so}}(3)_{K'} \) we mean the affine Lie algebra \( \hat{\mathfrak{su}}(2)_{2K'} \). Its integrable representations have \( \mathfrak{so}(3) \) Young tableaux that obey \( \ell_1 \leq K' \). Since \( \ell_1 = \frac{1}{2} a_1 \), this means that eq. (3.1) is replaced with \( a_0 + a_1 = 2K' \) when \( n = 1 \).}
Representations related by $\sigma \in \mathbb{Z}_2$ belong to a simple-current orbit, or cominimally self-equivalence class.

In this paper, we will refer to representations of $\hat{\text{so}}(2n+1)_{K'}$ with $\ell_1 < \frac{1}{2}K'$, $\ell_1 = \frac{1}{2}K'$, and $\ell_1 > \frac{1}{2}K'$ as being of types I, II, and III respectively. Type II representations are cominimally self-equivalent, and are tensors (resp. spinors) when $K'$ is even (resp. odd). Each simple-current orbit of $\hat{\text{so}}(2n+1)_{K'}$ contains either a type I and type III representation, or a single type II representation.

4 Twisted states of the $\hat{\text{su}}(2n)_K$ model

The invariance under reflection of the Dynkin diagram of the finite Lie algebra $\text{su}(N)$ gives rise (when $N > 2$) to an order-two automorphism $\omega_c$ of the Lie algebra, under which the Dynkin indices $a_i$ ($i = 1, \cdots, N - 1$) of an irreducible representation are mapped to $a_{N-i}$, corresponding to charge conjugation. This automorphism lifts to an automorphism of the affine Lie algebra $\hat{\text{su}}(N)_K$ that leaves the zero$^\text{th}$ node of the extended Dynkin diagram invariant. It gives rise (for $N > 2$) to a set of $\omega_c$-twisted Ishibashi states and $\omega_c$-twisted Cardy states of the bulk $\hat{\text{su}}(N)_K$ WZW model, and a corresponding class of $\omega_c$-twisted D-branes of the boundary model. In this section and the next, we review these twisted states for $\hat{\text{su}}(2n)_K$ and $\hat{\text{su}}(2n+1)_K$ respectively. Much of this material is a summary of ref. [21].

Twisted Ishibashi states of $\hat{\text{su}}(2n)_K$

Recall from section [2] that the $\omega_c$-twisted Ishibashi states $|\mu\rangle_{\omega_c}^\ell$ of the $\hat{\text{su}}(2n)_K$ WZW model ($n > 1$ is understood throughout this section) are labelled by self-conjugate integrable highest-weight representations $\mu \in \mathcal{E}_{\omega_c}$ of $\hat{\text{su}}(2n)_K = (A_{2n-1}^{(1)})_K$. These representations have Dynkin indices $(\mu_0, \mu_1, \cdots, \mu_n, \mu_n, \mu_{n-1}, \cdots, \mu_1)$ that satisfy

$$\mu_0 + 2(\mu_1 + \cdots + \mu_{n-1}) + \mu_n = K. \quad (4.1)$$

Equivalently, the $\omega_c$-twisted Ishibashi states of $\hat{\text{su}}(2n)_K$ may be characterized [38] by the integrable highest weight representations of the associated orbit Lie algebra $\hat{g} = (D_{n+1}^{(2)})_K$, whose Dynkin diagram is

![Dynkin diagram](image)

with the integers adjacent to each node indicating the dual Coxeter label $m_i$. The representation $\mu \in \mathcal{E}_{\omega_c}$ corresponds to the $(D_{n+1}^{(2)})_K$ representation with Dynkin indices $(\mu_0, \mu_1, \cdots, \mu_n)$, whose integrability condition is precisely (4.1).

Each $\omega_c$-twisted Ishibashi state $\mu$ of $\hat{\text{su}}(2n)_K$ may be mapped [21] to an integrable highest-weight representation $\mu'$ of the untwisted affine Lie algebra $\hat{\text{so}}(2n+1)_{K+1}$ with Dynkin indices $(\mu_0 + \mu_1 + 1, \mu_1, \cdots, \mu_n)$. The constraint (4.1) translates into the constraint $\ell_1(\mu') \leq \frac{1}{2}K$ on the $\hat{\text{so}}(2n+1)_{K+1}$ Young tableaux. This means that $\omega_c$-twisted Ishibashi states of $\hat{\text{su}}(2n)_K$ are in one-to-one correspondence with the set of type I tensor and type I spinor representations of $\hat{\text{so}}(2n+1)_{K+1}$. 

6
Twisted Cardy states of $\tilde{su}(2n)_K$

Recall that the $\omega_i$-twisted Cardy states $|\alpha\rangle_{C}^{\omega_i}$ (and therefore the $\omega_i$-twisted D-branes) of the $\tilde{su}(2n)_K$ WZW model are labelled $\mathfrak{g}$ by the integrable highest-weight representations $\alpha \in \mathcal{B}^{\omega_i}$ of the twisted affine Lie algebra $\hat{\mathfrak{g}}^K = (A_{2n-1}^{(2)})_K$, whose Dynkin diagram is

![Dynkin diagram](image)

The Dynkin indices $(a_0, a_1, \cdots, a_n)$ of the highest weights $\alpha$ thus satisfy

$$a_0 + a_1 + 2(a_2 + \cdots + a_n) = K.$$  \hfill (4.2)

(For $n = 2$, the twisted affine Lie algebra is instead $D_3^{(2)}$ with nodes 1 and 2 interchanged $[21]$, but the condition (4.2) remains valid.)

The $\omega_i$-twisted Cardy state $\alpha \in \mathcal{B}^{\omega_i}$ of $\tilde{su}(2n)_K$ may be associated $[21]$ with an integrable highest-weight spinor representation $\alpha'$ of the untwisted affine Lie algebra $\tilde{so}(2n + 1)_{K+1}$ with Dynkin indices $(a_0, a_1, \cdots, a_{n-1}, 2a_n + 1)$. The constraint (4.2) is precisely the condition on integrable representations of $\tilde{so}(2n + 1)_{K+1}$. (In terms of $so(2n + 1)$ Young tableau row lengths, this constraint reads $\ell_1(\alpha') + \ell_2(\alpha') \leq K + 1$.) Therefore, there is a one-to-one correspondence between the $\omega_i$-twisted D-branes of $\tilde{su}(2n)_K$ and integrable spinor representations of $\tilde{so}(2n + 1)_{K+1}$ of type I, type II (when $K$ is even), and type III. For later convenience, we define

$$t(\alpha) = \begin{cases} 0, & \text{if } \ell_1(\alpha') \neq \frac{1}{2}(K + 1) \quad \text{(types I and III)} \\ 1, & \text{if } \ell_1(\alpha') = \frac{1}{2}(K + 1) \quad \text{(type II)} \end{cases} \quad (4.3)$$

Even though the $\omega_i$-twisted Cardy states and the $\omega_i$-twisted Ishibashi states of $\tilde{su}(2n)_K$ are characterized differently in terms of integrable representations of $\tilde{so}(2n + 1)_{K+1}$, they are equal in number. The $\omega_i$-twisted Cardy states $\alpha$ may be written as linear combinations of $\omega_i$-twisted Ishibashi states $\mu$, with the transformation coefficients $\psi_{\alpha \mu}$ given by the modular transformation matrix of $(A_{2n-1}^{(2)})_K$. In ref. $[21]$, it was shown that, for $\tilde{su}(2n)_K$, these coefficients are proportional to matrix elements of the (real) modular transformation matrix $S'$ of the untwisted affine Lie algebra $\tilde{so}(2n + 1)_{K+1}$:

$$\psi_{\alpha \mu} = \sqrt{2} S'_{\alpha' \mu'} = \sqrt{2} S'^*_{\alpha' \mu'} \quad (4.4)$$

where $\alpha'$ and $\mu'$ are the $\tilde{so}(2n + 1)_{K+1}$ representations related to $\alpha$ and $\mu$ as described above.

Since the finite Lie algebra associated with the twisted affine Lie algebra $(A_{2n-1}^{(2)})_K$ is $C_n$, the representations of $(A_{2n-1}^{(2)})_K$ form $C_n$-multiplets at each level. More specifically $[21]$, each $\omega_i$-twisted Cardy state $\alpha \in \mathcal{B}^{\omega_i}$ of $\tilde{su}(2n)_K$ may be associated with an integrable highest-weight representation $\alpha''$ of the untwisted affine Lie algebra $\tilde{sp}(n)_{K+n-1}$ with (finite) Dynkin indices $(a_1, \cdots, a_n)$. The row lengths of the $sp(n)$ Young tableau associated with $\alpha''$ are equal to those of the $so(2n + 1)$ Young tableau associated with $\alpha'$ reduced by one-half:

$^4$Throughout this paper, our convention is $sp(n) = C_n$. 

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\( \ell_i(\alpha'') = \ell_i(\alpha') - \frac{1}{2} \). Therefore, an alternative characterization of the \( \omega_c \)-twisted D-branes of \( \widehat{\mathfrak{su}}(2n)K \) is as the subset of integrable representations of \( \widehat{\mathfrak{sp}}(n)_{K+n-1} \) characterized by Young tableaux with row lengths satisfying \( \ell_1(\alpha'') + \ell_2(\alpha'') \leq K \).

**Equivalence classes of \( \omega_c \)-twisted D-branes of \( \widehat{\mathfrak{su}}(2n)K \)**

The \( \mathbb{Z}_2 \) simple current symmetry \( \sigma \) of \( \widehat{\mathfrak{so}}(2n+1)_{K+1} \) relates type I and type III representations in pairs. Using the 1-1 correspondence between integrable \( \widehat{\mathfrak{so}}(2n+1)_{K+1} \) spinor representations and \( \omega_c \)-twisted Cardy states, we lift the map \( \sigma \) to the twisted D-branes of \( \widehat{\mathfrak{su}}(2n)K \), and refer to \( \sigma(\alpha) \) as cominimally equivalent to \( \alpha \). (In section 7 we will show that \( \alpha \) and \( \sigma(\alpha) \) have equal and opposite D0-brane charges, modulo \( x_{2n,K} \).) Therefore, the cominimal equivalence classes of \( \omega_c \)-twisted D-branes of \( \widehat{\mathfrak{su}}(2n)K \) are in one-to-one correspondence with the set of type I spinor representations of \( \widehat{\mathfrak{so}}(2n+1)_{K+1} \) when \( K \) is odd, and with type I and type II spinor representations of \( \widehat{\mathfrak{so}}(2n+1)_{K+1} \) when \( K \) is even.

**Twisted open string partition function of \( \widehat{\mathfrak{su}}(2n)K \)**

The coefficients of the partition function of an open string stretched between \( \omega_c \)-twisted D-branes \( \alpha \) and \( \beta \) of \( \widehat{\mathfrak{su}}(2n)K \) are given by

\[
n_{\beta\lambda}^{\alpha} = \sum_{\mu'=\{\text{tensors}\}}^{\{\text{spinors}\}} \frac{2S'_{\alpha'\mu'}S_{\lambda\mu}S'_{\beta'\mu'}}{S_{0\mu}} \tag{4.5}
\]

using eqs. (2.5) and (4.4). Since the \( \omega_c \)-twisted D-branes of \( \widehat{\mathfrak{su}}(2n)K \) belong to \( \mathbb{Z}_2 \)-cominimal equivalence classes, we also define the linear combination

\[
s_{\beta\lambda}^{\alpha} = \left( \frac{1}{2} \right)^{\frac{1}{2}[(\alpha)+\ell(\beta)]+1} \left[ n_{\beta\lambda}^{\alpha} + n_{\beta\lambda}^{\sigma(\alpha)} + n_{\sigma(\beta)\lambda}^{\alpha} + n_{\sigma(\beta)\lambda}^{\sigma(\alpha)} \right] \]

\[
= \left( \frac{1}{2} \right)^{\frac{1}{2}[(\alpha)+\ell(\beta)]-2} \sum_{\mu'=\{\text{tensors}\}}^{\{\text{spinors}\}} \frac{S'_{\alpha'\mu'}S_{\lambda\mu}S'_{\beta'\mu'}}{S_{0\mu}} \tag{4.6}
\]

where, as a result of eq. (3.3), the sum over spinor representations drops out. (The normalization is chosen so that \( s_{\beta\lambda}^{\alpha} = n_{\beta\lambda}^{\alpha} \) when \( \alpha \) and \( \beta \) are both type II, and therefore belong to single-element cominimal equivalence classes.) The quantity \( s_{\beta\lambda}^{\alpha} \) is the more natural one to consider in the context of level-rank duality.

5 **Twisted states of the \( \widehat{\mathfrak{su}}(2n+1)_K \) model**

**Twisted Ishibashi states of \( \widehat{\mathfrak{su}}(2n+1)_K \)**

Recall from section 2 that the \( \omega_c \)-twisted Ishibashi states \( |\mu\rangle_{I}^{\omega_c} \) of the \( \widehat{\mathfrak{su}}(2n+1)_K \) WZW model are labelled by self-conjugate integrable highest-weight representations \( \mu \in \mathcal{E}_{\omega_c}^{\omega_c} \) of \( \widehat{\mathfrak{su}}(2n+1)_K = (A_{2n}^{(1)})_K \). The Dynkin indices \( (\mu_0, \mu_1, \cdots, \mu_{n-1}, \mu_n, \mu_n, \mu_{n-1}, \cdots, \mu_1) \) of these representations satisfy

\[
\mu_0 + 2(\mu_1 + \cdots + \mu_n) = K. \tag{5.1}
\]
Equivalently, the $\omega_c$-twisted Ishibashi states of $\hat{\mathfrak{su}}(2n+1)_K$ may be characterized by the integrable highest weight representations of the associated orbit Lie algebra $\hat{\mathfrak{g}} = (A_{2n}^{(2)})_K$, whose Dynkin diagram is (the right-hand diagram is for $n=1$) 

![Dynkin Diagram]

The representation $\mu \in \mathcal{E}^{\omega_c}$ corresponds to the $(A_{2n}^{(2)})_K$ representation with Dynkin indices $\{\mu_0, \mu_1, \ldots, \mu_n\}$. Consistency with eq. (5.1) requires that the dual Coxeter labels be $\{m_0, m_1, \ldots, m_n\} = (1, 2, 2, \ldots, 2)$, and hence we must choose as the zeroth node the right-most node of the Dynkin diagrams above (consistent with ref. [21], but differing from refs. [20, 31]).

Each $\omega_c$-twisted Ishibashi state $\mu$ of $\hat{\mathfrak{su}}(2n+1)_K$ may be mapped [21] to an integrable highest-weight spinor representation $\mu'$ of the untwisted affine Lie algebra $\hat{\mathfrak{so}}(2n+1)_{K+2}$ with Dynkin indices $\{\mu_0 + \mu_1 + 1, \mu_1, \ldots, \mu_{n-1}, 2\mu_n + 1\}$. The constraint (5.1) translates into the constraint $\ell_1(\mu') \leq \frac{1}{2}(K+1)$ on the $\hat{\mathfrak{so}}(2n+1)_{K+2}$ Young tableau. This means that $\omega_c$-twisted Ishibashi states of $\hat{\mathfrak{su}}(2n+1)_K$ are in one-to-one correspondence with the set of type I spinor representations of $\hat{\mathfrak{so}}(2n+1)_{K+2}$.

**Twisted Cardy states of $\hat{\mathfrak{su}}(2n+1)_K$**

Recall that the $\omega_c$-twisted Cardy states $|\alpha\rangle_{\mathcal{C}^c}$ (and therefore the $\omega_c$-twisted D-branes) of the $\hat{\mathfrak{su}}(2n+1)_K$ WZW model are labelled [29] by the integrable highest-weight representations $\alpha \in \mathcal{B}^{\omega_c}$ of the twisted affine Lie algebra $\hat{\mathfrak{g}}_K = (A_{2n}^{(2)})_K$ (but see ref. [24]). We adopt the same convention as above for the labelling of the nodes of the Dynkin diagram of $(A_{2n}^{(2)})_K$. Thus the Dynkin indices $(a_0, a_1, \ldots, a_n)$ of the highest weights $\alpha$ must satisfy

$$a_0 + 2(a_1 + \cdots + a_n) = K.$$  

(5.2)

The $\omega_c$-twisted Cardy state $\alpha \in \mathcal{B}^{\omega_c}$ of $\hat{\mathfrak{su}}(2n+1)_K$ may be associated [21] with an integrable highest-weight spinor representation $\alpha'$ of the untwisted affine Lie algebra $\hat{\mathfrak{so}}(2n+1)_{K+2}$ with Dynkin indices $\{a_0 + a_1 + 1, a_1, \ldots, a_{n-1}, 2a_n + 1\}$. The constraint (5.2) translates into the constraint $\ell_1(\alpha') \leq \frac{1}{2}(K+1)$ on the $\hat{\mathfrak{so}}(2n+1)_{K+2}$ Young tableaux. This means that $\omega_c$-twisted D-branes of $\hat{\mathfrak{su}}(2n+1)_K$ are in one-to-one correspondence with the set of type I spinor representations of $\hat{\mathfrak{so}}(2n+1)_{K+2}$.

Since $\omega_c$-twisted Cardy states of $\hat{\mathfrak{su}}(2n+1)_K$ correspond only to type I spinor representations of $\hat{\mathfrak{so}}(2n+1)_{K+2}$, there is no notion of cominimal equivalence of $\omega_c$-twisted Cardy states in this case.

In the case of $\hat{\mathfrak{su}}(2n+1)_K$, the total number of $\omega_c$-twisted Cardy states is manifestly equal to the total number of $\omega_c$-twisted Ishibashi states. The coefficients $\psi_{\alpha\mu}$ relating $\omega_c$-twisted Cardy states $\alpha$ to $\omega_c$-twisted Ishibashi states $\mu$ are given by the modular transformation matrix of $(A_{2n}^{(2)})_K$. In ref. [21], it was shown that, for $\hat{\mathfrak{su}}(2n+1)_K$, these coefficients are proportional to matrix elements of the modular transformation matrix $S'$ of the untwisted

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5See the note regarding $\hat{\mathfrak{so}}(3)_K$ in footnote 3.

6For $n = 1$, $\mu'$ has Dynkin indices $(2\mu_0 + 2\mu_1 + 3, 2\mu_1 + 1)$.

7For $n = 1$, $\alpha'$ has Dynkin indices $(2a_0 + 2a_1 + 3, 2a_1 + 1)$. 

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affine Lie algebra $\hat{\text{so}}(2n+1)_{K+2}$:

$$\psi_{\alpha\mu} = 2S'_{\alpha'\mu'}$$

(5.3)

where $\alpha'$ and $\mu'$ are the $\hat{\text{so}}(2n+1)_{K+2}$ representations related to $\alpha$ and $\mu$ as described above.

Since the finite Lie algebra associated with the twisted affine Lie algebra $(A^{(2)}_{2n})_K$ is $C_n$, the representations of $(A^{(2)}_{2n})_K$ form $C_n$-multiplets at each level. More specifically [21], each $\omega_c$-twisted Cardy state $\alpha \in B^{\omega_c}$ of $\hat{\text{su}}(2n+1)_K$ may be associated with an integrable highest-weight representation $\alpha''$ of the untwisted affine Lie algebra $\hat{\text{sp}}(n)_K$ with (finite) Dynkin indices $(a_1, \cdots, a_n)$. The row lengths of the $\text{sp}(n)$ Young tableau associated with $\alpha''$ are equal to those of the $\text{so}(2n+1)$ Young tableau associated with $\alpha'$ reduced by one-half: $\ell_i(\alpha'') = \ell_i(\alpha') - \frac{1}{2}$. Therefore, an alternative characterization of the $\omega_c$-twisted D-branes of $\hat{\text{su}}(2n+1)_K$ is as the subset of integrable representations of $\hat{\text{sp}}(n)_K$ characterized by Young tableaux with row lengths satisfying $\ell_1(\alpha') \leq \frac{1}{2}K$.

Twisted open string partition function of $\hat{\text{su}}(2n+1)_K$

The coefficients of the partition function of an open string stretched between $\omega_c$-twisted D-branes $\alpha$ and $\beta$ of $\hat{\text{su}}(2n+1)_K$ are given by

$$n_{\beta\lambda} = \sum_{\mu'=\text{spinors}} 4S'_{\alpha'\mu'}S_{\lambda\mu}S'_{\beta'\mu'}S_{0\mu'}$$

(5.4)

using eqs. (2.5) and (5.3).

Special case of $\hat{\text{su}}(2n+1)_{2k+1}$

Note that in the special case of odd level, the $\omega_c$-twisted Cardy states $\alpha$ and $\omega_c$-twisted Ishibashi states $\mu$ of $\hat{\text{su}}(2n+1)_{2k+1}$ are in one-to-one correspondence with the integrable representations $\alpha''$ and $\mu''$ of $\hat{\text{sp}}(n)_k$ with finite Dynkin indices $(a_1, \cdots, a_n)$ and $(\mu_1, \cdots, \mu_n)$ respectively. Moreover, it was observed [38, 20, 21] in this case that the Cardy/Ishibashi coefficients may be expressed as

$$\psi_{\alpha\mu} = S''_{\alpha''\mu''}$$

(5.5)

where $S''_{\alpha''\mu''}$ are elements of the modular transformation matrix of $\hat{\text{sp}}(n)_k$.

6 Level-rank duality of the twisted D-branes of $\hat{\text{su}}(N)_K$

This section is the heart of the paper, in which we present the level-rank map between the $\omega_c$-twisted D-branes of $\hat{\text{su}}(N)_K$ and $\hat{\text{su}}(K)_N$. We use this to show the level-rank duality of the spectrum of an open string stretched between $\omega_c$-twisted D-branes.

As in the case of untwisted D-branes, the level-rank correspondence involves cominimal equivalence classes (unless $N$ and $K$ are both odd). The details of the correspondence differ markedly depending on whether $N$ and $K$ are even or odd, so we must treat three cases separately. In refs. [4, 5], the tilde ($\tilde{\cdot}$) notation was used to denote the level-rank dual of an untwisted state, because the duality map was given by transposition of the associated Young tableaux. Here, in all cases, we will use the hat ($\hat{\cdot}$) notation to denote the level-rank dual of an $\omega_c$-twisted state, but the specific form of the duality map depends on whether $N$
and $K$ are even or odd, and on whether we are considering $\omega_c$-twisted Cardy or $\omega_c$-twisted Ishibashi states.

**Duality of twisted states of $\tilde{\mathfrak{su}}(2n)_{2k} \leftrightarrow \tilde{\mathfrak{su}}(2k)_{2n}$**

As we saw in section 4, the cominimal equivalence classes of $\omega_c$-twisted Cardy states (and therefore of $\omega_c$-twisted D-branes $\alpha$) of $\tilde{\mathfrak{su}}(2n)_{2k}$ correspond to type I and type II spinor representations $\alpha'$ of $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$. The number of equivalence classes of $\omega_c$-twisted D-branes of $\tilde{\mathfrak{su}}(2n)_{2k}$ is equal to the number of equivalence classes of $\omega_c$-twisted D-branes of $\tilde{\mathfrak{su}}(2k)_{2n}$, and there is a natural map $\alpha \to \tilde{\alpha}$ between them (when $n$, $k > 1$). This map is defined in terms of the map $\alpha' \to \tilde{\alpha}'$ between the corresponding spinor representations of $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$ and $\tilde{\mathfrak{o}}(2k+1)_{2n+1}$, as follows:

- reduce each of the row lengths of $\alpha'$ by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- take the complement with respect to an $k \times n$ rectangle,
- add $\frac{1}{2}$ to each of the row lengths.

(The map $\alpha'' \to \tilde{\alpha}''$ between the corresponding representations of $\tilde{\mathfrak{sp}}(n)_{2k+n-1}$ and $\tilde{\mathfrak{sp}}(k)_{2n+k-1}$ is given by the middle two steps above.) The map $\alpha' \to \tilde{\alpha}'$ was first described in the appendix of ref. 3 in the context of level-rank duality of $\mathfrak{o}(N)_K$ WZW models. It takes type I (resp. type II) spinor representations of $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$ to type II (resp. type I) spinor representations of $\tilde{\mathfrak{o}}(2k+1)_{2n+1}$. Hence,

$$t(\alpha) + \tilde{t}(\tilde{\alpha}) = 1$$  

(6.1)

for all $\omega_c$-twisted Cardy states $\alpha$ of $\tilde{\mathfrak{su}}(2n)_{2k}$, where $t(\alpha)$ is defined in eq. (1), and $\tilde{t}(\tilde{\alpha})$ is the corresponding quantity in $\tilde{\mathfrak{su}}(2k)_{2n}$. The map $\alpha' \to \tilde{\alpha}'$ lifts to a one-to-one map $\alpha \to \tilde{\alpha}$ between cominimal equivalence classes of $\omega_c$-twisted D-branes of $\tilde{\mathfrak{su}}(2n)_{2k}$ and cominimal equivalence classes of $\omega_c$-twisted D-branes of $\tilde{\mathfrak{su}}(2k)_{2n}$.

Next, we turn to the level-rank map for $\omega_c$-twisted Ishibashi states of $\tilde{\mathfrak{su}}(2k)_{2n}$. As we saw in section 4, $\omega_c$-twisted Ishibashi states $\mu$ of $\tilde{\mathfrak{su}}(2n)_{2k}$ correspond to type I tensor and type I spinor representations $\mu'$ of $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$. The level-rank map $\mu \to \tilde{\mu}$ between $\omega_c$-twisted Ishibashi states of $\tilde{\mathfrak{su}}(2n)_{2k}$ and those of $\tilde{\mathfrak{su}}(2k)_{2n}$ is defined only for states that correspond to type I tensor representations. The map between $\mu'$ and $\tilde{\mu}'$, the corresponding $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$ and $\tilde{\mathfrak{o}}(2k+1)_{2n+1}$ representations, is simply given by transposition of the tensor tableaux; that is, $\tilde{\mu}' = (\mu')'$. There is no level-rank map between $\omega_c$-twisted Ishibashi states that correspond to type I spinor representations, for the simple reason that these sets of representations are not equal in number. (Moreover, the map described above for $\omega_c$-twisted Cardy states maps type I spinor representations of $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$ to type II spinor representations of $\tilde{\mathfrak{o}}(2k+1)_{2n+1}$, which do not correspond to $\omega_c$-twisted Ishibashi states of $\tilde{\mathfrak{su}}(2k)_{2n}$.)

Having defined the level-rank map between $\mu$ and $\tilde{\mu}$ in terms of the corresponding tensor representations of $\tilde{\mathfrak{o}}(2n+1)_{2k+1}$, one may show that

$$\hat{\mu} = \sigma^{-r(\mu)/(2n)}(\tilde{\mu})$$  

(6.2)
that is, \( \tilde{\mu} \) is in the same \( \tilde{\mathfrak{su}}(2n)_{2k} \) cominimal equivalence class (simple-current orbit) as \( \tilde{\mu} \),
where \( \tilde{\mu} \) is the transpose\(^8\) of the Young tableau of the self-conjugate representation \( \mu \) of \( \tilde{\mathfrak{su}}(2n)_{2k} \), and \( r(\mu) \) is the number of boxes of this \( \tilde{\mathfrak{su}}(2n)_{2k} \) tableau. (Note that \( \tilde{\mu} \) is, in general, not self-conjugate, while \( \tilde{\mu} \) necessarily is.) The proof of eq. (6.2) is very similar to one given in section 6 of ref. [3]. A consequence of eq. (6.2) is that the \( \tilde{\mathfrak{su}}(2n)_{2k} \) modular transformation matrix \( S \) is related to the \( \mathfrak{su}(2k)_{2n} \) modular transformation matrix \( \tilde{S} \) by

\[
S_{\lambda\mu}^* = \sqrt{\frac{k}{n}} \tilde{S}_{\tilde{\lambda}\tilde{\mu}}
\]

which follows from [2, 3]

\[
S_{\lambda\mu}^* = \sqrt{\frac{k}{n}} e^{2\pi i r(\lambda)r(\mu)/(4nk)} \tilde{S}_{\tilde{\lambda}\tilde{\mu}}, \quad
\tilde{S}_{\tilde{\lambda}\tilde{\mu}} = e^{-2\pi i r(\lambda)r(\mu)/(4nk)} S_{\lambda\mu}.
\]

Having defined level-rank maps for the \( \omega_c \)-twisted Cardy and Ishibashi states of \( \tilde{\mathfrak{su}}(2n)_{2k} \), we now turn to the duality of the open-string spectrum between \( \omega_c \)-twisted D-branes. The coefficients of the partition function of an open string stretched between \( \omega_c \)-twisted D-branes \( \alpha \) and \( \beta \) are real numbers so we may write (4.6) as

\[
s_{\beta\lambda}^\alpha = \left( \frac{1}{2} \right)^{\frac{1}{2}l(\alpha)+l(\beta)} \sum_{\mu' = \text{tensors}} \frac{S'_{\alpha'\mu'} S_{\lambda\mu}^* S'_{\beta'\mu'}}{S_{0\mu}^*}.
\]

In ref. [3], the spinor-tensor components \( S'_{\alpha'\mu'} \) of the modular transformation matrix of \( \tilde{\mathfrak{so}}(2n+1)_{2k+1} \) were shown to be related to the spinor-tensor components \( \tilde{S}'_{\alpha'\mu'} \) of \( \tilde{\mathfrak{so}}(2k+1)_{2n+1} \) by

\[
S'_{\alpha'\mu'} = 2^{l(\alpha)-\frac{1}{2}} (-1)^r(\mu') \tilde{S}'_{\alpha'\tilde{\mu}'} = 2^{\frac{1}{2}l(\alpha)-\frac{1}{2}l(\beta)} (-1)^r(\mu') \tilde{S}'_{\alpha'\tilde{\mu}'}
\]

where we have used eq. (6.1). Using eqs. (6.3) and (6.6), we find

\[
s_{\beta\lambda}^\alpha = \left( \frac{1}{2} \right)^{\frac{1}{2}l(\alpha)+l(\beta)} \sum_{\tilde{\mu}' = \text{tensors}} \frac{\tilde{S}'_{\alpha'\tilde{\mu}'} \tilde{S}_{\lambda\tilde{\mu}} \tilde{S}'_{\beta'\tilde{\mu}'}}{\tilde{S}_{0\tilde{\mu}}^*} = \tilde{s}_{\beta\lambda}^\alpha.
\]

Thus the (linear combination of) coefficients (4.6) of the open-string partition function of \( \omega_c \)-twisted D-branes of \( \tilde{\mathfrak{su}}(2n)_{2k} \) are equal to those of \( \tilde{\mathfrak{su}}(2k)_{2n} \) under the level-rank duality map acting on \( \omega_c \)-twisted D-branes.

**Duality of twisted states of** \( \tilde{\mathfrak{su}}(2n+1)_{2k+1} \leftrightarrow \tilde{\mathfrak{su}}(2k+1)_{2n+1} \)

As we saw in section 5 the \( \omega_c \)-twisted Cardy states (and therefore the \( \omega_c \)-twisted D-branes \( \alpha \)) of \( \tilde{\mathfrak{su}}(2n+1)_{2k+1} \) map one-to-one to type I spinor integrable representations \( \alpha' \) of \( \tilde{\mathfrak{so}}(2n+1)_{2k+3} \), and also to integrable representations \( \alpha'' \) of \( \tilde{\mathfrak{sp}}(n)_{k} \). We define the level-rank duality map \( \alpha \rightarrow \tilde{\alpha} \) for \( \omega_c \)-twisted Cardy states by transposition of the associated \( \tilde{\mathfrak{sp}}(n)_{k} \) tableaux: that is, \( \tilde{\alpha}'' \equiv (\alpha'') \). (In ref. [5], we therefore denoted this map simply by \( \alpha \rightarrow \tilde{\alpha} \).) Exactly similar statements hold for the \( \omega_c \)-twisted Ishibashi states \( \mu \) of \( \tilde{\mathfrak{su}}(2n+1)_{2k+1} \).

\(^8\)If \( \mu \) has \( \ell_1 = 2k \), \( \tilde{\mu} \) is obtained by stripping off leading columns of length \( 2k \) from the transpose of \( \mu \).
The equality of the Cardy/Ishibashi coefficients of $\hat{su}(2n+1)_{2k+1}$ and $\hat{su}(2k+1)_{2n+1}$

$$\psi_{a\mu} = \tilde{\psi}_{\hat{a}\hat{\mu}}$$

(6.8)

follows immediately from eq. (5.5) together with level-rank duality of the $\hat{sp}(n)_k$ WZW model $^3$

$$S''_{a''\mu''} = \tilde{S''}_{\hat{a}''\hat{\mu}''}$$

(6.9)

where $S''$ and $\tilde{S}''$ are the modular transformation matrices of $\hat{sp}(n)_k$ and $\hat{sp}(k)_n$ respectively. Moreover, by eq. (5.3), we have

$$S'_{a'\mu'} = \tilde{S}'_{\hat{a}'\hat{\mu}'}$$

(6.10)

where $S'$ and $\tilde{S}'$ are the modular transformation matrices of $\hat{so}(2n+1)_{2k+3}$ and $\hat{so}(2k+1)_{2n+3}$ respectively, and the map $\alpha' \rightarrow \hat{\alpha}'$ from $\hat{so}(2n+1)_{2k+3}$ to $\hat{so}(2k+1)_{2n+3}$ (induced from the transposition map $\alpha'' \rightarrow \hat{\alpha}''$) is:

- reduce each of the row lengths of $\alpha'$ by $\frac{1}{2}$, so that they all become integers,
- transpose the resulting tableau,
- add $\frac{1}{2}$ to each of the row lengths.

and equivalently for $\mu' \rightarrow \hat{\mu}'$. (Note that this map differs from spinor map defined in the last subsection by the omission of the complement map.) Note that eq. (6.10) differs from the standard level-rank duality of WZW models $^3$, which relates $\hat{so}(N)_K$ to $\hat{so}(K)_N$.

Finally, we turn to the duality of the open-string spectrum between $\omega_c$-twisted D-branes of $\hat{su}(2n+1)_{2k+1}$ and $\hat{su}(2k+1)_{2n+1}$. In ref. $^3$, $\hat{sp}(n)_k$ level-rank duality (6.9) was used to show the level-rank duality of the coefficients of the open string partition function. We can equivalently use eqs. (5.4) and (6.10) to show the same result

$$n_{\beta\lambda}^{\alpha} = \sum_{\mu'='spins} \frac{4}{1} \frac{S'_{\alpha'\mu'} S^{*}_{\lambda\mu} S'_{\beta'\mu'}}{S^{*}_{\mu}} = \sum_{\hat{\mu}'='spins} \frac{4}{1} \frac{\tilde{S}'_{\hat{\alpha}'\hat{\mu}'} \tilde{S}^{*}_{\lambda\hat{\mu}} \tilde{S}'_{\beta'\hat{\mu}'}}{\tilde{S}_{0\hat{\mu}}} = \tilde{n}_{\hat{\beta}\hat{\lambda}}^{\hat{\alpha}}$$

(6.11)

since $\mu' \rightarrow \hat{\mu}'$ maps type I spinor representations of $\hat{so}(2n+1)_{2k+3}$ to type I spinors of $\hat{so}(2k+1)_{2n+3}$, and we have also used

$$S^{*}_{\lambda\mu} = \sqrt{\frac{2k+1}{2n+1}} \tilde{S}_{\lambda\hat{\mu}}$$

(6.12)

which was proved in ref. $^5$.

**Duality of twisted states of $\hat{su}(2n+1)_{2k} \leftrightarrow \hat{su}(2k)_{2n+1}$**

Recall that the $\omega_c$-twisted D-branes $\alpha$ of $\hat{su}(2n+1)_{2k}$ correspond to type I spinor representations $\alpha'$ of $\hat{so}(2n+1)_{2k+2}$, and the equivalence classes of $\omega_c$-twisted D-branes $\hat{\alpha}$ of $\hat{su}(2k)_{2n+1}$ correspond to type I spinor representations $\hat{\alpha}'$ of $\hat{so}(2k+1)_{2n+2}$. The number of such spinor representations is equal, and we define the one-to-one level-rank map $\alpha' \rightarrow \hat{\alpha}'$ from $\hat{so}(2n+1)_{2k+2}$ to $\hat{so}(2k+1)_{2n+2}$ (for $k > 1$) as follows:
• reduce each of the row lengths of $\alpha'$ by $\frac{1}{2}$, so that they all become integers,
• transpose the resulting tableau,
• take the complement with respect to an $k \times n$ rectangle,
• add $\frac{1}{2}$ to each of the row lengths.

(By comparison, the definition of $\alpha' \to \hat{\alpha}'$ from $\tilde{s}\tilde{o}(2n + 1)_{2k+1}$ to $\tilde{s}\tilde{o}(2k + 1)_{2n+1}$ is the same, but in that case type I spinors are mapped to type II spinors and vice versa.) The map $\alpha' \to \hat{\alpha}'$ lifts to a one-to-one map $\alpha \to \hat{\alpha}$ between $\omega_c$-twisted D-branes of $\tilde{s}\tilde{u}(2n + 1)_{2k}$ and equivalence classes of $\omega_c$-twisted D-branes of $\tilde{s}\tilde{u}(2k)_{2n+1}$. (The map $\alpha'' \to \hat{\alpha}''$ between the corresponding representations of $\tilde{s}\tilde{p}(n)_{2k+n}$ and $\tilde{s}\tilde{p}(k)_{2n+k}$ is given by the middle two steps above.)

Next, we turn to the level-rank map between $\omega_c$-twisted Ishibashi states. The $\omega_c$-twisted Ishibashi states $\mu$ of $\tilde{s}\tilde{u}(2n + 1)_{2k}$ correspond to type I spinor representations $\mu'$ of $\tilde{s}\tilde{o}(2n + 1)_{2k+2}$. The $\omega_c$-twisted Ishibashi states $\hat{\mu}$ of $\tilde{s}\tilde{u}(2k)_{2n+1}$ correspond to type I tensor and type I spinor representations $\hat{\mu}'$ of $\tilde{s}\tilde{o}(2k + 1)_{2n+2}$. The number of such representations on each side is not equal, and the level-rank map $\mu' \to \hat{\mu}'$ takes type I spinor representations of $\tilde{s}\tilde{o}(2n + 1)_{2k+2}$ to only the type I tensor representations of $\tilde{s}\tilde{o}(2k + 1)_{2n+2}$. (Just as for $\tilde{s}\tilde{u}(2k)_{2n}$, there is no level-rank correspondence for the spinor Ishibashi states of $\tilde{s}\tilde{u}(2k)_{2n+1}$.) The map $\mu' \to \hat{\mu}'$ from $\tilde{s}\tilde{o}(2n + 1)_{2k+2}$ to $\tilde{s}\tilde{o}(2k + 1)_{2n+2}$ is defined as follows:

• reduce each of the row lengths of $\mu'$ by $\frac{1}{2}$, so that they all become integers, and
• transpose the resulting tableau.

The map $\mu' \to \hat{\mu}'$ then lifts to a map $\mu \to \hat{\mu}$ between $\omega_c$-twisted Ishibashi states of $\tilde{s}\tilde{u}(2n + 1)_{2k}$ and a subset of $\omega_c$-twisted Ishibashi states of $\tilde{s}\tilde{u}(2k)_{2n+1}$. One may show that

$$\hat{\mu} = \sigma^{-r(\mu)/(2n+1)}(\tilde{\mu})$$  \hspace{1cm} (6.13)

where $\tilde{\mu}$ is the transpose\(^9\) of the Young tableau of the self-conjugate representation $\mu$ of $\tilde{s}\tilde{u}(2n + 1)_{2k}$, and $r(\mu)$ is the number of boxes of this $\tilde{s}\tilde{u}(2n + 1)_{2k}$ tableau. The proof of eq. (6.13) is very similar to one given in section 6 of ref. [5]. Consequently, the modular transformation matrices $S$ of $\tilde{s}\tilde{u}(2n + 1)_{2k}$ and $\tilde{S}$ of $\tilde{s}\tilde{u}(2k)_{2n+1}$ are related by

$$S^*_{\lambda\mu} = \sqrt{\frac{2k}{2n + 1}} \tilde{S}^*_{\hat{\lambda}\hat{\mu}}$$  \hspace{1cm} (6.14)

which follows from [2, 3]

$$S^*_{\lambda\mu} = \sqrt{\frac{2k}{2n + 1}} e^{2\pi i r(\lambda)r(\mu)/(2n+1)(2k)} \tilde{S}^*_{\hat{\lambda}\hat{\mu}},$$

$$\tilde{S}^*_{\hat{\lambda}\hat{\mu}} = e^{-2\pi i r(\lambda)r(\mu)/(2n+1)(2k)} \tilde{S}^*_{\hat{\lambda}\hat{\mu}}.$$  \hspace{1cm} (6.15)

\(^9\)If $\mu$ has $\ell_1 = 2k$, $\tilde{\mu}$ is obtained by stripping off leading columns of length $2k$ from the transpose of $\mu$. 

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Finally, in the appendix of this paper, we show that

$$S'_{α' μ'} = (-1)^{r(μ') + k} S'_{α' μ'}$$

(6.16)

where \( S'_{α' μ'} \) and \( \tilde{S}'_{α' μ'} \) are modular transformation matrices of \( \hat{\mathfrak{s}\mathfrak{o}}(2n + 1)_{2k + 2} \) and \( \hat{\mathfrak{s}\mathfrak{o}}(2k + 1)_{2n + 2} \) respectively. As before, we observe that eq. (6.16) is not the standard \( \hat{\mathfrak{s}\mathfrak{o}}(N)_K \leftrightarrow \hat{\mathfrak{s}\mathfrak{o}}(K)_N \) duality of WZW models.

Equations (6.14) and (6.16) may be used to establish the level-rank duality of the coefficients of the partition functions (6.1) and (6.10) of an open string stretched between \( ω_c \)-twisted D-branes of \( \hat{\mathfrak{s}\mathfrak{u}}(2n + 1)_{2k} \) and \( \hat{\mathfrak{s}\mathfrak{u}}(2k)_{2n + 1} \)

$$n_{βλ}^{α} = \sum_{μ' = \text{spinors}} \frac{4 S'_{α' μ'} S_{α' μ'} S'_{β' μ'}}{S_{βμ}} = \sum_{β' = \text{tensors}} \frac{4 \tilde{S}'_{α' μ'} \tilde{S}_{α' μ'} \tilde{S}'_{β' μ'}}{\tilde{S}_{βμ}} = \tilde{S}_{β' μ'}$$

(6.17)

where the last equality follows because \( \hat{α} \) and \( \hat{β} \) are both type I spinor representations of \( \hat{\mathfrak{s}\mathfrak{o}}(2k + 1)_{2n + 2} \), so that \( \tilde{t}(\hat{α}) = \tilde{t}(\hat{β}) = 0 \).

7 Level-rank duality of twisted D-brane charges

In this section, we ascertain the relationship between the charges of level-rank-dual \( ω_c \)-twisted D-branes of \( \hat{\mathfrak{s}\mathfrak{u}}(N)_K \) and \( \hat{\mathfrak{s}\mathfrak{u}}(K)_N \). Recall from ref. [22] that the D0-brane charge of the \( ω_c \)-twisted D-brane of \( \hat{\mathfrak{s}\mathfrak{u}}(N)_K \) labelled by \( \alpha \) is given by

$$Q_{α''}^{α'} = \text{(dim } α'')_{\text{sp}(n)} \mod x_{N,K} \quad \text{for } \hat{\mathfrak{s}\mathfrak{u}}(N)_K$$

(7.1)

where \( α'' \) is the \( \text{sp}(n) \) representation corresponding to the \( ω_c \)-twisted Cardy state \( α \) of \( \hat{\mathfrak{s}\mathfrak{u}}(2n)_K \) or \( \hat{\mathfrak{s}\mathfrak{u}}(2n + 1)_K \), as described in sections 4 and 5.

Since the charges of \( \hat{\mathfrak{s}\mathfrak{u}}(N)_K \) D-branes (both untwisted and twisted) are defined only modulo \( x_{N,K} \), and those of \( \hat{\mathfrak{s}\mathfrak{u}}(K)_N \) D-branes modulo \( x_{K,N} \), comparison of charges of level-rank-dual D-branes is only possible modulo \( x \equiv \text{gcd}\{x_{N,K}, x_{K,N}\} = \min\{x_{N,K}, x_{K,N}\} \). In refs. [4, 5], the charges of untwisted D-branes of the \( \hat{\mathfrak{s}\mathfrak{u}}(N)_K \) model and those of the level-rank-dual \( \hat{\mathfrak{s}\mathfrak{u}}(K)_N \) model were shown to be equal modulo \( x \), up to a (known) sign (1.5), (1.6). In ref. [5], the charges of \( ω_c \)-twisted D-branes of the \( \hat{\mathfrak{s}\mathfrak{u}}(2n + 1)_{2k + 1} \) model and those of the level-rank-dual \( \hat{\mathfrak{s}\mathfrak{u}}(2k + 1)_{2n + 1} \) model were also shown to be equal, modulo \( x \). As we will see below, the relationship between charges of level-rank-dual \( ω_c \)-twisted D-branes of \( \hat{\mathfrak{s}\mathfrak{u}}(N)_K \) and \( \hat{\mathfrak{s}\mathfrak{u}}(K)_N \) is more complicated when \( N \) and \( K \) are not both odd.

Charges of cominimally-equivalent twisted D-branes of \( \hat{\mathfrak{s}\mathfrak{u}}(2n)_K \)

Since level-rank duality is a correspondence between \( \mathbb{Z}_2 \)-cominimal equivalence classes of \( ω_c \)-twisted D-branes when either \( N \) or \( K \) is even, we must first demonstrate that cominimally-equivalent \( ω_c \)-twisted D-branes of \( \hat{\mathfrak{s}\mathfrak{u}}(2n)_K \) have the same charge (modulo sign and modulo \( x_{2n,K} \)). The \( \text{sp}(n) \) representation \( α'' \) is related to the \( \text{so}(2n + 1) \) representation \( α' \) by reducing each row length of the tableau for the latter by one-half. As demonstrated in appendix A of ref. [22] (see also ref. [42]), the respective dimensions of these representation are related by the “miraculous dimension formula”

$$\text{(dim } α')_{\text{so}(2n + 1)} = 2^n \text{ (dim } α'')_{\text{sp}(n)}.$$
Next, in appendix B of ref. [22], it is shown that

\[(\dim \sigma(\lambda))_{so(2n+1)} = - (\dim \lambda)_{so(2n+1)} \mod x_{2n,K}\]  \hspace{1cm} (7.3)

where \(\sigma(\lambda)\) is the \(\hat{so}(2n+1)_{K+1}\) representation cominimally-equivalent to \(\lambda\). Using conjecture B\(^{\text{spin}}\) of ref. [22], and the facts that the dimensions of all spinor representations of \(so(2n+1)\) are multiples of \(2^n\) and that \((\dim \sigma(0))_{so(2n+1)} = -1 \mod x_{2n,K}\) [22], eq. (7.3) may be strengthened to

\[(\dim \sigma(\alpha'))_{so(2n+1)} = - (\dim \alpha')_{so(2n+1)} \mod 2^n x_{2n,K}\]  \hspace{1cm} (7.4)

for \(\alpha'\) a spinor representation of \(\hat{so}(2n+1)_{K+1}\). Together with eq. (7.2), this implies that the charges of cominimally-equivalent \(\omega_c\)-twisted D-branes of \(\hat{su}(2n)_K\) are related by

\[Q_{\sigma(\alpha)}^{\omega_c} = - Q_{\alpha}^{\omega_c} \mod x_{2n,K}\]  \hspace{1cm} (7.5)

analogous to eq. (1.4) for untwisted D-branes.

Finally, we turn to the relationship between the charges of level-rank-dual \(\omega_c\)-twisted D-branes.

**Duality of twisted D-brane charges under \(\hat{su}(2n+1)_{2k+1} \longleftrightarrow \hat{su}(2k+1)_{2n+1}\)**

Let \(x = \gcd\{x_{2n+1,2k+1}, x_{2k,2n+1}\}\). In ref. [5], it was shown that

\[(\dim \alpha'')_{sp(n)} = (\dim \hat{\alpha}'')_{sp(k)} \mod x\]  \hspace{1cm} (7.6)

where \(\hat{\alpha}''\) is obtained from \(\alpha''\) by tableau transposition. Since \(\hat{\alpha}''\) is the \(sp(k)\) representation corresponding to the level-rank-dual \(\omega_c\)-twisted D-brane \(\hat{\alpha}\) of \(\hat{su}(2k+1)_{2n+1}\), it immediately follows from eq. (7.1) that the charges of level-rank-dual \(\omega_c\)-twisted D-branes are equal

\[Q_{\omega_c}^{\alpha} = \tilde{Q}_{\hat{\alpha}}^{\omega_c} \mod x.\]  \hspace{1cm} (7.7)

This was previously presented in ref. [5] and is included here for completeness.

**Duality of twisted D-brane charges under \(\hat{su}(2n+1)_{2k} \longleftrightarrow \hat{su}(2k)_{2n+1}\)**

Let \(x = \gcd\{x_{2n+1,2k}, x_{2k,2n+1}\}\). We begin with the relationship

\[(\dim \Lambda_s)_{sp(n)} = (\dim \Lambda_s)_{su(2n+1)} - (\dim \Lambda_{s-1})_{su(2n+1)}\]  \hspace{1cm} (7.8)

where \(\Lambda_s\) is the completely antisymmetric representation with Young tableau \(s\)\(^{\text{\_antisym}}\). Next, as shown in ref. [4],

\[(\dim \Lambda_s)_{su(2n+1)} = (-1)^s (\dim \Lambda_s)_{su(2k)} \mod x\]  \hspace{1cm} (7.9)

where \(\Lambda_s\) is the completely symmetric representation with Young tableau \(\text{sym} \ s\). Finally,

\[(\dim \Lambda_s)_{so(2k+1)} = (\dim \Lambda_s)_{su(2k)} + (\dim \Lambda_{s-1})_{su(2k)}\]  \hspace{1cm} (7.10)

Combining these three equations, we obtain

\[(\dim \Lambda_s)_{sp(n)} = (-1)^s (\dim \Lambda_s)_{so(2k+1)} \mod x.\]  \hspace{1cm} (7.11)
This result can be used in the determinantal formulas (A.44) and (A.60) of ref. [43], following the approach of ref. [5], to establish a relationship between arbitrary representations of \( \text{sp}(n) \) and \( \text{so}(2k + 1) \),
\[
\text{dim } \tilde{\alpha}''_{k} \equiv (-1)^{r(\tilde{\alpha}'')} (\text{dim } \tilde{\alpha}''_{k})_{\text{so}(2k+1)} \text{ mod } x \tag{7.12}
\]
where \( \tilde{\alpha}'' \) is the transpose of the tableau of \( \alpha'' \).

Now, from the level-rank map of section 6, the representation \( \tilde{\alpha}'' \) is related to the representation \( \tilde{\alpha}' \) that corresponds to the level-rank dual \( \omega_{c} \)-twisted D-brane by taking the complement of the tableau with respect to a \( k \times (n + \frac{1}{2}) \) rectangle. This maps a type I tensor representation of \( \tilde{\text{so}}(2k+1)_{2n+2} \) to a type I spinor representation. We conjecture a relationship
\[
\text{dim } \tilde{\alpha}''_{k} \equiv (-1)^{k(k+1)/2} (\text{dim } \tilde{\alpha}'_{k})_{\text{so}(2k+1)} \text{ mod } x_{2k,2n+1} \text{ (conjecture) \ (7.13)}
\]
between the dimensions of \( \tilde{\alpha}'' \) and \( \tilde{\alpha}' \). To justify this, consider the expression for the dimension of the \( \text{so}(2k + 1) \) representation \( \tilde{\alpha}'' \):
\[
(\text{dim } \tilde{\alpha}''_{k})_{\text{so}(2k+1)} \equiv \frac{\prod_{i=1}^{k} (2\phi_{i}) \prod_{i<j} (\phi_{i} - \phi_{j})(\phi_{i} + \phi_{j})}{\prod_{i=1}^{k} (2k + 1 - i) \prod_{i<j} (j - i)(2k + 1 - i - j)} \tag{7.14}
\]
where \( \phi_{i} = \ell_{i}(\tilde{\alpha}'') - \frac{1}{2} + k - i \). All the factors in parentheses are integers. The row lengths of \( \tilde{\alpha}' \) are related to those of \( \tilde{\alpha}'' \) by \( \ell_{i}(\tilde{\alpha}') = n + \frac{1}{2} - \ell_{k+1-i}(\tilde{\alpha}'') \). Hence
\[
(\text{dim } \tilde{\alpha}'_{k})_{\text{so}(2k+1)} \equiv \frac{\prod_{i=1}^{k} (X - 2\phi_{k+1-i})(\phi_{k+1-j} - \phi_{k+1-i})(X - \phi_{k+1-i} - \phi_{k+1-j})}{\prod_{i=1}^{k} (2k + 1 - i) \prod_{i<j} (j - i)(2k + 1 - i - j)} \tag{7.15}
\]
where \( X \equiv 2n + 2k + 1 \). Then
\[
(\text{dim } \tilde{\alpha}''_{k})_{\text{so}(2k+1)} - (-1)^{k(k+1)/2} (\text{dim } \tilde{\alpha}'_{k})_{\text{so}(2k+1)} \equiv XR \tag{7.16}
\]
where \( R \) is a rational number with denominator \( \prod_{i=1}^{k} (2k + 1 - i) \prod_{i<j} (j - i)(2k + 1 - i - j) \). If \( X \) is prime, then none of the factors in the denominator of \( R \) (which are all less than \( 2k+1 \)) divide \( X \), and since the left-hand-side is an integer, \( R \) must also be an integer, in which case the left-hand-side is a multiple of \( X \). This establishes eq. (7.13) when \( X \) is prime, since \( x_{2k,2n+1} = X \) in that case. When \( X \) is not prime, some of the factors in the denominator of \( R \) may divide \( X \), but we believe (proved for \( k = 2 \), and based on strong numerical evidence for \( k = 3, 4, \text{ and } 5 \), with arbitrary \( n \) ) that the right-hand-side of eq. (7.16) is always a multiple of \( x_{2k,2n+1} \), and therefore that the conjecture (7.13) holds.

Finally, from eq. (7.2), we have
\[
(\text{dim } \tilde{\alpha}'_{k})_{\text{so}(2k+1)} = 2^{k} (\text{dim } \tilde{\alpha}''_{k})_{\text{sp}(k)}. \tag{7.17}
\]
Putting together eqs. (7.12), (7.13), and (7.17), we obtain the relationship between the charge of the \( \omega_{c} \)-twisted D-brane \( \alpha \) of \( \text{sl}(2n+1)_{2k} \) and the level-rank-dual \( \omega_{c} \)-twisted D-brane \( \alpha \) of \( \text{sl}(2k)_{2n+1} \)
\[
Q_{\tilde{\alpha}}^{\omega_{c}} = 2^{k} (-1)^{r(\tilde{\alpha}'' + k(k+1)/2) Q_{\tilde{\alpha}}^{\omega_{c}}} \text{ mod } x \tag{7.18}
\]

\[\text{After v1 of this paper appeared, we learned that an equivalent version of this relationship has been independently conjectured by Stefan Fredenhagen and collaborators [44].}\]
whose validity is subject only to the conjectured relation (7.13).  

**Duality of twisted D-brane charges under** $\hat{s\bar{u}}(2n)_{2k} \leftrightarrow \hat{s\bar{u}}(2k)_{2n}$  

Let $x = \gcd\{x_{2n,2k}, x_{2k,2n}\}$. As shown in ref. [5], if $n = k$, then $x = 4$ if $n = 2^m$, otherwise $x = 1$. If $n \neq k$, then $x = 2$ if $n + k = 2^m$, otherwise $x = 1$.

We saw above that the charges of level-rank dual $\omega_c$-twisted D-branes of $\hat{s\bar{u}}(N)_K$ and $\hat{s\bar{u}}(K)_N$ are equal (modulo $x$) when both $N$ and $K$ are odd. This equality (modulo $x$) no longer holds if either $N$ or $K$ is even. When both $N$ and $K$ are even, the charges are again not equal (even modulo $x$ and modulo sign), as may be checked in a specific case (e.g., $\hat{s\bar{u}}(4)_4$, with $\alpha'' = \hat{\omega}$ and $\hat{\omega}'' = \omega$, since $5 \neq \pm 10$ mod 4). On the basis of eq. (7.18), one might expect a relationship such as

$$2^n Q^{\omega_c}_\alpha = \pm 2^k \hat{Q}^{\omega_c}_\alpha \mod x.$$  

However, any such relationship is trivially satisfied, since $\omega_c$-twisted branes exist only when $n, k > 1$, and $x$ is either 1, 2, or 4.

**8 Conclusions**  

In this paper, we have considered D-branes of the $\hat{s\bar{u}}(N)_K$ WZW model twisted by the charge-conjugation symmetry $\omega_c$. Such D-branes exist for all $N > 2$, and possess integer D0-brane charge, defined modulo $x_{N,K}$.

For $\hat{s\bar{u}}(2n)_K$ and $\hat{s\bar{u}}(2n + 1)_K$, the $\omega_c$-twisted D-branes are labelled by a subset of the integrable representations of $\hat{s\bar{o}}(2n + 1)_{K+1}$ and $\hat{s\bar{o}}(2n + 1)_{K+2}$ respectively. In the former case, the D-branes belong to cominimal equivalence classes generated by the $\mathbb{Z}_2$ simple current symmetry of $\hat{s\bar{o}}(2n + 1)_{K+1}$. We showed that the D0-brane charges of cominimally equivalent D-branes are equal and opposite modulo $x_{2n,K}$.

We then showed that level-rank-duality of $\hat{s\bar{u}}(N)_K$ WZW models extends to the $\omega_c$-twisted D-branes of the theory when both $N$ and $K$ are greater than two. In particular, we demonstrated a one-to-one mapping $\alpha \rightarrow \hat{\alpha}$ between the $\omega_c$-twisted D-branes for $N$ odd (or cominimal equivalence classes of D-branes for $N$ even) of $\hat{s\bar{u}}(N)_K$ and the $\omega_c$-twisted D-branes for $K$ odd (or cominimal equivalence classes of D-branes for $K$ even) of $\hat{s\bar{u}}(K)_N$.

We then showed that the spectrum of an open string stretched between $\omega_c$-twisted D-branes is invariant under level-rank duality. More precisely, we showed that the coefficients $n_{\beta\lambda}^\alpha$ of the open-string partition function (or $s_{\beta\lambda}^\alpha$, the appropriate linear combination of those coefficients corresponding to cominimal equivalence classes of $\omega_c$-twisted D-branes of $\hat{s\bar{u}}(2n)_K$) are invariant under $\alpha \rightarrow \hat{\alpha}$, $\beta \rightarrow \hat{\beta}$, and $\lambda \rightarrow \hat{\lambda}$. The proof of this required the existence of a *partial* level-rank mapping between the $\omega_c$-twisted Ishibashi states of each theory. (That is, the map only involved a subset of the $\omega_c$-twisted Ishibashi states of $\hat{s\bar{u}}(2n)_K$.)

Finally, we analyzed the relation between the D0-brane charges of level-rank-dual $\omega_c$-twisted D-branes (or cominimal equivalence classes thereof), modulo $x = \gcd\{x_{N,K}, x_{K,N}\}$. When $N$ and $K$ are both odd, the charges are equal mod $x$ (as previously demonstrated in ref. [5]), but in other cases this simple relationship does not hold. For $N = 2n + 1$ and
\( K = 2k \); the relation between the charges of level-rank-dual \( \omega_c \)-twisted D-branes is
\[
Q^{\omega_c}_\alpha = 2^k (\alpha') + k(k+1)2^{k/2} \tilde{Q}^{\omega_c}_{\hat{\alpha}} \mod x
\]
subject to the validity of a certain conjecture (7.13) stated in section 7.

It would interesting to know whether level-rank duality extends to any of the other twisted D-branes of the \( \hat{\mathfrak{su}}(N)_K \) WZW model [23].

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**Appendix**

In this appendix, we establish the relationship between certain matrix elements of the modular transformation matrices of \( \hat{\mathfrak{so}}(2n+1)_{2k+2} \) and \( \hat{\mathfrak{so}}(2k+1)_{2n+2} \) through the use of Jacobi’s theorem, following the approach of ref. [3]. Note that this is not the usual level-rank duality between \( \hat{\mathfrak{so}}(N)_K \) and \( \hat{\mathfrak{so}}(K)_N \).

Let \( \alpha' (\mu') \) be an integrable type I spinor representation of \( \hat{\mathfrak{so}}(2n+1)_{2k+2} \) corresponding to a \( \omega_c \)-twisted Cardy (Ishibashi) state of \( \hat{\mathfrak{su}}(2n+1)_{2k} \). The \( \hat{\mathfrak{so}}(2n+1)_{2k+2} \) modular transformation matrix has the matrix element [45]
\[
S'_{\alpha'\mu'} = (-1)^{n(n-1)/2}2^{n-1}(2k+2n+1)^{-n/2} \det M
\]

where \( M \) is an \( n \times n \) matrix with matrix elements
\[
M_{ij} = \sin \left( \frac{\pi \phi_i(\alpha') \phi_j(\mu')}{k+n+i} \right), \quad \phi_i(\alpha') = \ell_i(\alpha') + n + \frac{1}{2} - i, \quad i = 1, \cdots, n. \quad (A.2)
\]

Let \( \hat{\alpha}' (\hat{\mu}') \) be the integrable type I spinor (tensor) representation of \( \hat{\mathfrak{so}}(2k+1)_{2n+2} \) related to \( \alpha' (\mu') \) by the level-rank duality map described in section 6. The \( \hat{\mathfrak{so}}(2k+1)_{2n+2} \) modular transformation matrix has matrix element
\[
\tilde{S}'_{\hat{\alpha}'\hat{\mu}'} = (-1)^{k(k-1)/2}2^{k-1}(2k+2n+1)^{-k/2} \det \tilde{M}
\]

where \( \tilde{M} \) is a \( k \times k \) matrix with matrix elements
\[
\tilde{M}_{ij} = \sin \left( \frac{\pi \tilde{\phi}_i(\hat{\alpha}') \tilde{\phi}_j(\hat{\mu}')}}{k+n+i} \right), \quad \tilde{\phi}_i(\hat{\alpha}') = \ell_i(\hat{\alpha}') + k + \frac{1}{2} - i, \quad i = 1, \cdots, k. \quad (A.4)
\]

Next, define the index sets for the \( \omega_c \)-twisted Cardy states
\[
\mathcal{I} = \{ \phi_i(\alpha'), \quad i = 1, \cdots, n \}, \quad \mathcal{I} = \{ \tilde{\phi}_i(\hat{\alpha}'), \quad i = 1, \cdots, k \}. \quad (A.5)
\]
Using the level-rank duality map $\alpha' \rightarrow \hat{\alpha}'$ given in section 6, one may establish that $I$ and $\bar{T}$ are complementary sets of integers:

\[ I \cup \bar{T} = \{1, 2, \ldots, n + k\}, \quad I \cap \bar{T} = 0. \quad (A.6) \]

Also, define the index sets for the $\omega$-twisted Ishibashi states

\[ J = \{ \phi_j(\mu'), \quad j = 1, \ldots, n\}, \]

\[ \bar{J} = \{ n + k + \frac{1}{2} - \tilde{\phi}_j(\hat{\mu}'), \quad j = 1, \ldots, k\}. \quad (A.7) \]

Using the level-rank duality map $\mu' \rightarrow \hat{\mu}'$ given in section 6, one may also establish that $J$ and $\bar{J}$ are complementary sets of integers:

\[ J \cup \bar{J} = \{1, 2, \ldots, n + k\}, \quad J \cap \bar{J} = 0. \quad (A.8) \]

Now, define the $L \times L$ matrix $\Omega$ with matrix elements

\[ \Omega_{ij} = \sin \left( \frac{\pi ij}{L + \frac{1}{2}} \right), \quad i, j = 1, \ldots, L \quad (A.9) \]

where $L = n + k$. This matrix has determinant

\[ \det \Omega = (-1)^{L(L-1)/2} \left( \frac{2L + 1}{4} \right)^{L/2} \quad (A.10) \]

and obeys

\[ \Omega^{-1} = \left( \frac{4}{2L + 1} \right) \Omega. \quad (A.11) \]

Define $(\Omega)_{IJ}$ to be the $n \times n$ submatrix obtained from the larger $\Omega$ by considering only those rows indexed by the elements of $I$ and those columns indexed by the elements of $J$. Jacobi’s theorem [46] states that

\[ \det[(\Omega^{-1})^T]_{IJ} = (-)^{\Sigma_I + \Sigma_J} (\det \Omega)^{-1} \det(\Omega)_{\bar{I}\bar{J}}, \quad (A.12) \]

where

\[ \Sigma_I = \sum_{i \in I} i \quad \text{and} \quad \Sigma_J = \sum_{j \in J}. \quad (A.13) \]

One may observe that

\[ \det \mathbf{M} = \det(\Omega)_{IJ}, \quad \det \tilde{\mathbf{M}} = (-1)^{k+\Sigma_J + k(k-1)/2} \det(\Omega)_{\bar{I}\bar{J}}, \quad (A.14) \]

where the last contribution to the sign results from reversing the ordering of the rows of $\tilde{\mathbf{M}}$. Assembling eqs. (A.1), (A.3), (A.10), (A.11), (A.12), and (A.14), and using

\[ (-1)^{\Sigma_I + \Sigma_J + \Sigma_T} = (-1)^{\Sigma_T} = (-1)^{nk+k(k-1)/2+r(\hat{\mu}')} \quad (A.15) \]

one concludes that

\[ S'_{\alpha' \mu'} = (-1)^{r(\hat{\mu}') + k} \tilde{S}'_{\hat{\alpha}' \hat{\mu}'} \quad (A.16) \]

which is used in proving the level-rank duality of the open string spectrum in the last subsection of section 6.
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