Random matrix ensembles from nonextensive entropy

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The classical Gaussian ensembles of random matrices can be constructed by maximizing Boltzmann-Gibbs-Shannon’s entropy, $S_{BGSS} = -\int dH P(H) \ln [P(H)]$, with suitable constraints. Here we construct and analyze random-matrix ensembles arising from the generalized entropy $S_q = (1 - \int dH [P(H)]^q) / (q - 1)$ (thus $S_1 = S_{BGSS}$). The resulting ensembles are characterized by a parameter $q$ measuring the degree of nonextensivity of the entropic form. Making $q \to 1$ recovers the Gaussian ensembles. If $q \neq 1$, the joint probability distributions $P(H)$ cannot be factorized, i.e., the matrix elements of $H$ are correlated. In the limit of large matrices two different regimes are observed. When $q < 1$, $P(H)$ has compact support, and the fluctuations tend asymptotically to those of the Gaussian ensembles. Anomalies appear for $q > 1$: Both $P(H)$ and the marginal distributions $P(H_{ij})$ show power-law tails. Numerical analyses reveal that the nearest-neighbor spacing distribution is also long-tailed (not Wigner-Dyson) and, after proper scaling, very close to the result for the $2 \times 2$ case—a generalization of Wigner's surmise. We discuss connections of these “nonextensive” ensembles with other non-Gaussian ones, like the so-called Lévy ensembles and those arising from soft-confinement.

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I. INTRODUCTION

The Gaussian ensembles of Random Matrix Theory provide the standard statistical description of spectral fluctuations in a multiplicity of quantum systems ranging from nuclei to disordered mesoscopic conductors and classically chaotic systems [1, 2, 3, 4, 5].

Gaussian ensembles can be obtained from two postulates: the invariance of the joint distribution probability $P(H)$ with respect to changes of bases and the statistical independence of matrix elements. An alternative and more appealing way of constructing random matrix ensembles uses a maximum entropy principle [1, 6]. One constraint is normalization,

$$\int dH P(H) = 1. \hspace{1cm} (1)$$

The other one has the purpose of confining the spectrum, but is otherwise arbitrary (as long as the integral converges),

$$\int dH P(H) \text{tr} V(H) = 1 \hspace{1cm} (2)$$

(the trace ensures rotational invariance). For instance, the Gaussian ensemble of real symmetric matrices is obtained by the simplest choice

$$V(H) = H^2. \hspace{1cm} (3)$$

It has been proven that, in the limit of large matrices, and for a strong enough confining potential $V$, local fluctuation properties tend to those of the Gaussian case, whatever the shape of $V$ [7, 8].

To escape from Gaussian universality one must consider soft-confinement potentials [9, 10], or breaking rotational invariance. The latter case typically arises when matrix elements $H_{ij}$ are independent (non Gaussian) random variables. For instance, Cizeau and Bouchaud constructed anomalous “Lévy ensembles” by drawing $H_{ij}$ from a long-tailed distribution [11].

The purpose of this paper is to present a new way of constructing non-Gaussian ensembles while preserving rotational invariance. The idea is to use a maximum entropy approach with the usual constraints but with the nonextensive entropy [12]:

$$S_q [P(H)] = \frac{1 - \int dH [P(H)]^q}{q - 1}, \hspace{1cm} (4)$$
where \( q \) is a free real parameter \((q = 1 \text{ recovers Shannon’s standard entropy})\). This scheme produces a variety of ensembles, with \( q \) controlling the degree of confinement. Some ensembles belong to the Gaussian universality class but others exhibit anomalous behavior, characterized by distributions having power-law tails.

The explicit construction of these \( q \)-ensembles is presented in Sect. III where we also derive expressions for marginal distributions and the joint density of eigenvalues. Remarkably, for large matrices, the \( q \)-ensembles can be represented as a superposition of Gaussian ensembles. This allows us to obtain closed analytical formulas for the eigenvalue density, level-spacing probability distributions, etc (Sect. III). The comparison of analytical results with numerical simulations is the subject Sect. IV. We present in Sect. V the concluding remarks.

II. THE GENERALIZED ENSEMBLES

For simplicity we restrict our analysis to ensembles of real and symmetric matrices \( \mathbf{H} \) — extensions are straightforward. The volume element in this space is

\[
d\mathbf{H} = \prod_{i=1}^{N} dH_{ii} \prod_{i<j}^{N} dH_{ij} ,
\]

where it is understood that matrices are of size \( N \times N \). Generalized ensembles are obtained by maximizing the entropy of Eq. \( \text{(4)} \) subjected to normalization, Eq. \( \text{(4)} \), and

\[
\frac{\int d\mathbf{H} \, \text{tr} \mathbf{H}^2 \, |P(\mathbf{H})|^q}{\int d\mathbf{H} \, |P(\mathbf{H})|^q} = \sigma^2 ,
\]

with \( \sigma \) a constant having units of energy (we are assuming that \( \mathbf{H} \) is a Hamiltonian). Equation \( \text{(6)} \) is the generalization of the usual constraint that leads to the Gaussian ensembles in the standard maximum entropy approach. Arguments justifying the use of the escort probabilities \( P^q \), and applications of this generalized maximum entropy scheme to various problems can be found in Ref. \( \text{(12)} \).

Using the Lagrange multiplier technique it is straightforward to find the distribution of maximum entropy:

\[
P(\mathbf{H}) \propto \exp_q ( - \lambda \text{tr} \mathbf{H}^2 ) ,
\]

where we have defined the \( q \)-exponential function \( \exp_q \) [12]

\[
\exp_q(x) \equiv [(1 + (1 - q)x)_{+}]^{1/(1-q)} ,
\]

with

\[
\kappa_{+} = \max\{\kappa, 0\}
\]

[\text{note that } \exp_q(x) = \exp(x) \text{ if } \kappa = 0 \]. The omitted normalization constant in \( \text{(7)} \) and the parameter \( \lambda \) can be determined from the constraints \( \text{(11)} \) and \( \text{(15)} \). (Some preliminary results along these lines have been obtained by Evans and Michael \( \text{(13)} \).)

The ensemble defined by Eq. \( \text{(7)} \) will be called the “\( q \)-Orthogonal Ensemble” \( (q\text{OE}) \), as it can be seen in \( \text{(7)} \) that the probability distribution depends only on \( \text{tr} \mathbf{H}^2 \), an orthogonal invariant. When \( q \to 1 \) the \( q \)-exponential function tends to the usual exponential, and one recovers the Gaussian Orthogonal Ensemble \( \text{(GOE)} \). Except for the \( q = 1 \) case, the \( q \)-exponential in \( \text{(7)} \) cannot be factorized into a product of (marginal) distributions for individual matrix elements \( H_{ij} \), which are then correlated. We can already verify that the cases \( q < 1 \) and \( q > 1 \) are qualitatively different. Equations \( \text{(8)} \) and \( \text{(9)} \) show that for \( q < 1 \) the distributions have compact support; if \( q > 1 \), there are always power-law tails (we are assuming \( \lambda > 0 \), see below).

To proceed with the analysis of \( q\text{OE} \) it will be convenient to think of matrices \( \mathbf{H} \) as points in a \( d \)-dimensional euclidean space \( \mathbb{R}^d \). The first \( N \) components of a point \( \mathbf{r} \) correspond to diagonal elements \( H_{ii} \), the last ones to the upper triangle \( H_{ij} \), \( i < j \):

\[
\mathbf{r} = (H_{11}, \cdots, H_{NN}, \sqrt{2}H_{12}, \cdots, \sqrt{2}H_{N-1,N}) .
\]

The dimension of this space equals the number of independent matrix elements of \( \mathbf{H} \), i.e.,

\[
d = \frac{N(N+1)}{2} .
\]

The scaling of \( H_{ij} \) by \( \sqrt{2} \) makes the probability distribution \( \text{(10)} \) spherically symmetric in \( \mathbb{R}^d \), i.e., \( P_{\text{qOE}}(\mathbf{r}) \) is the product of a uniform distribution in the angles, and a radial distribution \( \text{(11)} \):

\[
\mathcal{P}(r; q, \sigma, N) \propto r^{d-1} \exp_q ( - \lambda r^2 ) ,
\]

where

\[
r^2 \equiv \mathbf{r} \cdot \mathbf{r} = \text{tr} \mathbf{H}^2 .
\]

The observations above imply that \( q\text{OE} \) belong to the wider category of “spherical ensembles” recently studied by Le Caër and Delannay \( \text{(14, 15)} \).

For \( q > 1 \), the distribution \( \text{(12)} \) has a power-law tail that goes like \( 1/r^{(1+\mu)} \) with

\[
\mu = \frac{2}{q - 1} - d .
\]

Then the normalization condition cannot be satisfied for all values of \( q \), but only by those making \( \mu > 0 \), i.e.,

\[
-\infty < q < 1 + \frac{2}{d} .
\]

(Note the formal similarity between this problem and the generalized random walker in \( d \) dimensions \( \text{(17)} \).

The Lagrange multiplier \( \lambda \) is given by

\[
\lambda = \frac{1}{\sigma^2} \frac{d}{2 - d(q - 1)} .
\]
Inside the region \((i.e.,\text{ normalizability})\) \(\lambda\) is always positive.

Integrating Eq. \(\text{[7]}\) over all variables but one, we obtain

\[
P(H_{ii}) \propto \exp_q\left(-\lambda' H_{ii}^2\right), \quad (17)
\]

\[
P(H_{ij}) \propto \exp_q\left(-2\lambda' H_{ij}^2\right), \quad (18)
\]

where

\[
q' = \frac{2 - (d - 3)(q - 1)}{2 - (d - 1)(q - 1)}
\]

and

\[
\lambda' = \frac{d}{2\sigma^2} \frac{2 - (d - 1)(q - 1)}{2 - d(q - 1)}.
\]

The following properties can be easily verified. The parameter \(q'\) is an increasing function of \(q\), and around the critical value \(q = 1\) one has

\[
q' = q + O\left[(q - 1)^2\right]. \quad (21)
\]

In addition, \(\lambda'\) is always positive. Then, in parallel with the global \(P(H)\), the marginal distributions also decay as power laws or have compact support, depending on \(q\) being larger or smaller than one, respectively. We remark that the matrix elements are not independent, so \(P(H)\) cannot be reconstructed from the marginal probabilities \(17,18\).

The joint density of eigenvalues can be obtained in a straightforward way: \(\text{[1, 13]}\)

\[
P(\varepsilon_1, \cdots, \varepsilon_N) \propto \prod_{i<j=1}^N |\varepsilon_j - \varepsilon_i| \exp_q\left(-\lambda \sum_{i=1}^N \varepsilon_i^2\right).
\]

(22)

The part that is responsible for level repulsion is identical to that in GOE because it arises only from orthogonal symmetry. The difference is in the confinement term, which in the present case is a non-separable \(g\)-exponential. Thus the “potential” that confines the spectrum is not a single-particle quadratic well, as in GOE. It is rather a mean field, proportional to the moment of inertia \(\sum \varepsilon_i^2\).

We can get a clear view of the generalized ensembles by noting that these are connected to the so-called Fixed Trace Ensembles (FTE) and, for large \(N\), to the Gaussian ensembles. In fact, recall that FTE are defined by \(\text{[1, 13]}\)

\[
P_{\text{FTE}}(H; r, N) \propto \delta(\text{tr}H^2 - r^2).
\]

(23)

Let \(f(H)\) be an arbitrary function and consider the averages in both ensembles qOE and FTE, namely,

\[
\langle f(H) \rangle_{\text{qOE}}(q, \sigma, N) = \int dHP_{\text{qOE}}(H; q, \sigma, N)f(H), \quad (24)
\]

and

\[
\langle f(H) \rangle_{\text{FTE}}(r, N) = \int dHP_{\text{FTE}}(H; r, N)f(H). \quad (25)
\]

Then we have the relation

\[
\langle f(H) \rangle_{\text{qOE}}(q, \sigma, N) = \int_0^\infty dr\mathcal{P}(r; q, \sigma, N)\langle f(H) \rangle_{\text{FTE}}(r, N).
\]

(26)

The average over qOE can be calculated in two stages. First do the average over the angles, for a fixed radius \(r\). This correspond to a FTE average. Then average over radii, with the weighting function \(\mathcal{P}(r)\). Of course, the same is true for the GOE, which corresponds to the particular case \(q = 1\). The relationship between qOE (or GOE) and FTE is analogous to that between the canonical and microcanonical ensembles of Statistical Mechanics.

Equation \(\text{[26]}\) involves no approximations. Although exact, it is not very useful because it requires the knowledge of fixed-trace averages. However, if one is interested in the limit of large matrices, important simplifications can be made.

### III. THE LARGE \(N\) LIMIT

The key point is that, for \(N\) large enough, the FTE average in the r.h.s. of \(\text{[26]}\) can be approximated by an average in a GOE having the property \(\text{tr}H^2 = r^2\). Then, if we know the GOE average of a given function, its corresponding qOE average can in principle be calculated by doing just one integration. We will analyze in detail two spectral statistics: the eigenvalue density,

\[
\rho(\varepsilon; q, \sigma, N) = \left\langle \sum_{i=1}^N \delta(\varepsilon - \varepsilon_i) \right\rangle,
\]

(27)

and the distribution of level spacings,

\[
p(s; q, \sigma, N) = \langle \delta(\varepsilon_{i+1} - \varepsilon_i - s) \rangle.
\]

(28)

In the last equation \(\varepsilon_i\) and \(\varepsilon_{i+1}\) are two consecutive eigenvalues lying at the center of the band, i.e., \(\varepsilon_i \approx 0\). It is (or will become) clear that other statistics, e.g., two-point correlation functions, can also be considered along the same lines.

In order to obtain the qOE averages of \(\text{[27]}\) and \(\text{[28]}\) we need the corresponding FTE expressions, to be further averaged with \(\mathcal{P}(r; q, \sigma, N)\), as indicated by Eq. \(\text{[26]}\). However, we will approximate FTE averages by the corresponding GOE ones. Then, the basic ingredients become the “semicircle law” (for the eigenvalue density),

\[
\rho(\varepsilon; N) = \frac{N^2}{2\pi^2 r^2} \sqrt{4r^2 - \varepsilon^2}, \quad (29)
\]
and Wigner’s surmise,
\[
p(s; N, r) = \frac{N^3 s}{2 \pi r^2} \exp \left( - \frac{N^3 s^2}{4 r^2} \right) ,
\]
(30)
giving the level-spacing distribution. Equations (29) and (30) are good approximations for both GOE and FTE distributions when \( N \) is large [15].

We recall that if \( q > 1 \), the normalization condition limits the value of \( q \) and the case \( q < 1 \) does not present such a problem. So, we analyze both cases separately.

### A. Ensembles with \( q < 1 \)

Except for providing an energy scale, \( \sigma \) plays no special role. From now on, without loss of generality, we set \( \sigma = 1 \). If desired, \( \sigma \) can be restored at any time by dimensional analysis.

When \( N \to \infty \) (\( q \) fixed) the radial distribution of qOE tends to
\[
\mathcal{P}(r; q, N) \propto r^{d-1} \left[ 1 - r^2 \right]^{1/(1-q)} ,
\]
(31)
limited to the domain \( 0 \leq r \leq 1 \). As \( d \) grows, the distribution is squeezed against \( r = 1 \), being concentrated in a small region below \( r = 1 \), of width \( \mathcal{O}(1/d) = \mathcal{O}(1/N^2) \). It can be verified that both the level density [20] and the spacing distribution [20], when considered as functions of \( r \), have widths which are \( \mathcal{O}(1) \). Thus the radial distribution is much narrower and we can safely approximate
\[
\mathcal{P}(r; q < 1, N \to \infty) \simeq \delta(r - 1) .
\]
(32)
We conclude that, when \( q < 1 \) and \( N \to \infty \) the ensembles qOE tend to the GOE [as far as it concerns the distributions being studied, namely Eqs. (27) and (28)].

### B. Ensembles with \( q > 1 \)

When \( q > 1 \) the possible ensembles are restricted to a region in the plane \( q - N \) that gets thinner as \( N \to \infty \) (see Fig. 1). The natural coordinates in this region are \( N \) (or \( d \)) and \( \mu \) [see Eq. (14)], the latter controlling the tails of \( \mathcal{P}(r) \) and other distributions. For instance, substituting (14) and (15) into (17) or (18), one immediately verifies that the marginal distributions behave asymptotically like
\[
P(H_{ij}) \sim \frac{1}{H_{ij}^{1+\mu}} .
\]
(33)
The radial distribution [12], as a function of \( r, \mu, N \) becomes:
\[
\mathcal{P}(r, \mu, N) \propto r^{d-1} \left[ 1 + \frac{d}{\mu} r^2 \right]^{-(d+\mu)/2} .
\]
(34)

![FIG. 1: When \( q > 1 \) the ensembles qOE lie in the region limited by the axes and the curve \( \mu = 0 \) (normalization frontier). As \( N \) becomes large, the maximum q allowed tends to one. Lines correspond to families of ensembles having the same power-law tails (labeled by \( \mu \)).](image)

This expression allows the identification of some well known ensembles as special members of the qOE class: the Cauchy-Lorentz ensemble corresponds to \( \mu = 1 \). An integer \( \mu > 1 \) produces Student’s ensembles (see [17, 18] and references therein; see also [20]).

Now we analyze the limit \( N \to \infty \) while keeping \( \mu > 0 \) fixed, i.e., we move upwards along the curves of Fig. 1. As in the case \( q < 1 \), examined before, there is a limiting distribution. Some simple algebra leads to
\[
\mathcal{P}(r; \mu, N \to \infty) \propto r^{-(1+\mu)} \exp \left( -\frac{\mu}{2r^2} \right) .
\]
(35)
Only when \( \mu \to \infty \), \( \mathcal{P} \) tends to the delta function (32), and GOE is recovered. For finite \( \mu \) the width of \( \mathcal{P}(r) \) is at least \( \mathcal{O}(1) \). In any case, the average of a given GOE distribution with \( \mathcal{P}(r) \) gives the corresponding qOE distribution (via Eq. (26) with \( \langle f(H) \rangle_{\text{FTE}} \sim \langle f(H) \rangle_{\text{GOE}} \), when \( N \to \infty \)). Let us first consider the density of states. Inserting Eqs. (29) and (35) into (26) we obtain
\[
\rho(\varepsilon; \mu) \propto \int_{\varepsilon}^{\infty} dr \frac{\sqrt{4r^2 - N\varepsilon^2}}{r^{\mu+3}} \exp \left( -\frac{\mu}{2r^2} \right) .
\]
(36)
This integral can not be expressed in terms of elementary functions. However, some information can be extracted analytically. Setting \( \varepsilon = 0 \) one obtains the qOE density of states at the center of the band,
\[
\rho(0; \mu) = \frac{N^{3/2}}{\pi} \frac{\Gamma[(\mu + 1)/2]}{\Gamma[\mu/2]} \sqrt{\frac{2}{\mu}} .
\]
(37)
The behavior for large \( \varepsilon \) can be easily recognized by making the change of variables \( 2r = \sqrt{N} \varepsilon z \) in (36), which leads to
\[
\rho(\varepsilon; \mu) \propto \varepsilon^{-(1+\mu)} \int_{1}^{\infty} dz \frac{\sqrt{z^2 - 1}}{z^{\mu+3}} \exp \left( -\frac{2\mu}{Nz^2\varepsilon^2} \right) .
\]
(38)
Evidently the tails vanish like $\varepsilon^{-(1+\mu)}$. This is also
the behavior observed by Cizeau and Bouchaud in their
"Lévy ensembles" of matrices having independent entries
distributed according to the same law of Eq. (33) [11].
We note, however, that the analogies can not be pushed
further because our ensembles are rotationally invariant
and Lévy ensembles are not (the ensembles of Ref. [11]
belong to the so-called $\alpha$-symmetric class [13]).

The calculation of the spacing distribution proceeds as
before. We have to insert Eqs. (30) and (35) into (26).
The result is

$$p(s; \mu) \propto s \int_0^\infty dr r^{-(\mu+3)} \exp \left[ -\frac{\mu}{2r^2} \left( 1 + \frac{N^3 s^2}{2\pi\mu} \right) \right].$$

(39)

The dependence on $s$ can be easily isolated by a change
of variables, so we can write:

$$p(s; \mu) \propto s \left( 1 + \frac{N^3 s^2}{2\pi\mu} \right)^{-(1+\mu/2)},$$

(40)
or alternatively

$$p(s; \mu) \propto s \exp q_s (-as^2),$$

(41)

where

$$q_s \equiv \frac{\mu+4}{\mu+2} \quad \text{and} \quad \alpha \equiv \frac{N^3}{4\pi} \frac{\mu+2}{\mu}.$$  

(42)

The function of Eq. (41) (or Eq. (41)) is identical in shape
with the exact level-spacing distribution of the $2\times 2$ qOE
having the same $\mu$ (see appendix). Then, both distributions
can be collapsed by a simple scaling of the arguments.
This curious result constitutes a generalization of
Wigner’s surmise to qOE.

Remark. When analyzing spectral statistics it is usual
to normalize energies so that the (local) average spacing
is one (the spectrum is “unfolded”). This amounts to
measuring energies in units of

$$\Delta \equiv \int_0^\infty ds s p(s).$$

(43)

Note, however, that in qOE the first moment of $p(s)$ does
not exist for $\mu \leq 1$. In these cases, instead of $\Delta$, one may
alternatively use the energy scale

$$\tilde{\Delta} \equiv \left[ \int_0^\infty ds s^{-1} p(s) \right]^{-1}.$$  

(44)

Due to level repulsion there is no singularity at $s = 0$, and
$\tilde{\Delta}$ always exists, thus representing a characteristic energy
of qOE. It is close to the inverse of the level density at
$\varepsilon = 0$:

$$\tilde{\Delta} = \frac{2}{\pi \rho(0; \mu)},$$

(45)

with $\rho(0; \mu)$ given in [15].

IV. NUMERICAL RESULTS

When thought of as clouds in $R^d$, via the map of
Eq. (10), both ensembles qOE and GOE are spherically
symmetric. This means that qOE can be constructed
just by rescaling the radii of all points in the GOE cloud
[13, 16]. Thus, the construction of a qOE matrix $H_1$
(with parameters $q, \sigma, N$) can be done in three steps. (i)
Construct a GOE matrix $H_0$ of size $N \times N$. In this case
matrix elements are independent and can be calculated
using Eqs. (17) and (18) with $q = 1$. The radius of $H_0$ is

$$r_0 = \sqrt{\text{tr} H_0^2}.$$  

(46)

(ii) Choose a radius $r_1$ randomly according to the radial
probability distribution $P(r_1, q, \sigma, N)$ of Eq. (12). (iii)
Define $H_1$ as

$$H_1 = H_0 \frac{r_1}{r_0}.$$  

(47)

This is the recipe we followed for constructing qOE matrices.
(If, instead of being a random variable, $r_1$ is fixed, we
obtain a matrix belonging to FTE.) The only difficulty is
to devise the random number generator, especially when $P(r)$
has very long tails. For this purpose we used a com-
bination of the rejection method and the transformation
method, as explained in Ref. [21].

In Fig. 2 we show histograms representing densities of
states obtained from diagonalization of qOE matrices. It
is clear that they are very well described by the formula
[38], which was evaluated by direct numerical integration.
The statistics of level spacings is exhibited in

FIG. 2: Density of states in the ensembles qOE (normalized to
one). We compare histograms generated numerically (dots)
with the theoretical result of Eq. (45) (curves). Each his-
togram was generated from a set of $10^5$ matrices. We used
the following values of $\mu$: 0.5, 1.5, 2.5, 6.0, $\infty$. Densities with
larger $\mu$’s have larger values at $\varepsilon = 0$ and decay faster. In all
cases $N = 20$ and $\sigma = 1$. The dashed line corresponds to the
GOE semicircle ($N \to \infty$).
In accordance to the following criterion. When \( \mu \) integer moments diverge. For \( \mu = 0.5 \) all integer moments diverge. For \( \mu = 1.5 \) (\( \mu = 2.5 \)) the first (second) moment exists but higher ones diverge. The case \( \mu = 6.0 \) is intended to represent an ensemble qOE approaching GOE.

In both figures we observe some small deviations, which may be attributed to the relatively small size of the matrices considered (\( N=20 \)). The values of \( \mu \) were chosen in accordance to the following criterion. When \( \mu = 0.5 \) all integer moments diverge. For \( \mu = 1.5 \) (\( \mu = 2.5 \)) the first (second) moment exists but higher ones diverge. The case \( \mu = 6.0 \) is intended to represent an ensemble qOE approaching GOE.

![Histograms were obtained by binning data from 10⁵ matrices. Each matrix contributed with the “central” spacing between levels \( \varepsilon_{N/2} \) and \( \varepsilon_{N/2+1} \).](image)

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Equation (A3) plays the role of Wigner’s surmise for qOE. It is more useful to rewrite (A3) in terms of the parameter $\mu$ of Eq. (14). The result is

$$p(s; q, \sigma) \propto s \left( 1 + \frac{3s^2}{2\mu} \right)^{-(1+\mu/2)}. \quad (A6)$$

In Sect. III this expression is compared with the spacing distribution for the large $N$ case (Eq. (40)).

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