A STABILITY RESULT FOR RIESZ POTENTIALS IN HIGHER DIMENSIONS

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ABSTRACT. We prove a stability estimate, with the optimal quadratic error term, for the Coulomb energy of a set in \( \mathbb{R}^n \) with \( n \geq 3 \). This estimate extends to a range of Riesz potentials.

1. INTRODUCTION

Let \( \phi \) be a strictly radially decreasing, nonnegative measurable function on \( \mathbb{R}^n \) that vanishes at infinity, and consider the convolution functional

\[
\mathcal{E}(f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) f(y) \phi(x - y) \, dx \, dy,
\]

defined for measurable functions \( f \) on \( \mathbb{R}^n \) that satisfy a suitable integrability condition. It is well-known that the value of \( \mathcal{E} \) can only increase under symmetric decreasing rearrangement of \( f \) \([22, 23]\): If \( f^* \) is the function equimeasurable with \( |f| \) that decreases radially about the origin, then the Riesz–Sobolev inequality implies that

\[
\mathcal{E}(f^*) \geq \mathcal{E}(f).
\]

Equality occurs (for a finite, non-zero value of \( \mathcal{E} \) and nonnegative \( f \)) if only if \( f \) itself is symmetric decreasing about some point \( x_0 \in \mathbb{R}^n \), that is, \( f(x) = f^*(x - x_0) \) \([18]\). A natural question is whether near-equality implies that \( f \) must be close to a translate of \( f^* \)? If so, how close must it be?

In this paper, we investigate near-equality cases in the Riesz–Sobolev inequality when \( \phi = \phi_\lambda \) is a Riesz potential

\[
\phi_\lambda(x) = \frac{1}{c_\lambda} |x|^{-(n-\lambda)},
\]

and \( f = \chi_A \) is the characteristic function of a set \( A \subset \mathbb{R}^n \) of finite positive volume \( |A| \). Here, \( 0 < \lambda < n \), and \( c_\lambda \) is a particular normalizing constant, see Eq. (2.1) below. The symmetric decreasing rearrangement of \( \chi_A \) is the characteristic function of \( A^\ast \), the centered ball of the same volume as \( A \).

With a slight abuse of notation, we write

\[
\mathcal{E}_\lambda(A) = \int_A \int_A \phi_\lambda(x - y) \, dx \, dy.
\]

Our main result provides a lower bound on the deficit

\[
\delta(A) := \left( \frac{|B^n|}{|A|} \right)^{2 - \frac{\lambda}{n}} (\mathcal{E}(A^\ast) - \mathcal{E}(A))
\]

in terms of the Fraenkel asymmetry

\[
\alpha(A) := \left( \frac{|B^n|}{|A|} \right) \inf_{x \in \mathbb{R}^n} \{|A \Delta (x + A^\ast)|\}.
\]

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Here, \( x + A^* \) is the ball of the same volume as \( A \), centered at \( x \), and \( \Delta \) denotes the symmetric difference of two sets. Both the deficit and the asymmetry are invariant under translation, rotation, and dilation.

**Theorem 1.1 (Stability).** Fix the dimension \( n \geq 2 \). For \( \lambda \in (1, n) \), let \( \phi_\lambda \) be the Riesz potential defined by Eqs. (1.1) and (2.1), and let \( \mathcal{E}_\lambda \) be as in Eq. (1.2). There exists a constant \( C_\lambda > 0 \) such that

\[
\delta(A) \geq C_\lambda \alpha^2(A)
\]

for every subset \( A \subset \mathbb{R}^n \) of finite positive volume.

The exponent 2 is best-possible, see Example 3.3 but we do not determine the value of the sharp constant \( C_\lambda \). Theorem 1.1 extends our previous results on the Newton potential from [3]. In that work, we used techniques that had been developed for the quantitative isoperimetric inequality [15] along with reflection positivity of the Coulomb kernel to prove Theorem 1.1 for \( \lambda = 2 \) in dimension \( n = 3 \). In higher dimensions, we obtained a weaker inequality (with a non-sharp exponent) from Talenti’s comparison principle for solutions of Poisson’s equation [25]. The proof of Theorem 1.1 that we present here instead uses an approach of Fuglede to obtain explicit estimates for near-spherical sets, in combination with lengthy but straightforward global rearrangements.

Shape optimization problems involving convolution functionals appear in physical and biological models for pair interactions between large numbers of particles or individuals. Geometric stability results for such non-local functionals have many potential applications, from dynamical stability for isotropic steady states in stellar dynamics [4], to the construction of continuum limits in statistical mechanics [5], and flocking in biological aggregation models [12]. Regardless of their importance, such problems are not as well-understood as the classical isoperimetric inequality and other inequalities for gradient functionals. Fewer explicit estimates are available for the integral equations which characterize optimal shapes than for elliptic PDE arising from gradient functionals. Very recently, a number of results have begun to address these questions.

The fundamental stability question for the Riesz–Sobolev inequality, in the case where all three functions in the convolution integral are symmetrized simultaneously, has been settled by M. Christ in a series of papers since 2013. In [6], he proves a sharp result, where the geometric asymmetry of a triple of sets is controlled by the square root of their deficit in the Riesz–Sobolev inequality. This estimate is rather delicate because it can hold only when the three sets are comparable in size. Frank and Lieb in [11] extend Christ’s results (in the case of a radially decreasing integral kernel) from sets to densities taking values in the interval \([0, 1]\). In a different direction, Figalli and Jerison obtain geometric stability results for the Brunn–Minkowski inequality [9], an affine invariant inequality for the volume of sum sets, which can be seen as a limiting case of the Riesz–Sobolev inequality. For the important case when one summand is a ball, the Brunn–Minkowski inequality becomes a non-local isoperimetric inequality. Here, sharp stability estimates are due to Figalli, Maggi, and Mooney [10]. While the results described above are motivated by insight from additive combinatorics, convex geometry, and geometric measure theory, their proofs tend to rely on direct estimates on how different parts of a set contribute to the integral functional under consideration. The approach of Fuglede that we employ here has also been used to give a new proof of the quantitative isoperimetric inequality; see the article [14] by N. Fusco for a survey of all of these techniques.

We close out this section with some open questions.

**Question 1.2.** Does Theorem 1.1 extend to \( \lambda \leq 1 \)?

The spherical integral that we use as a toy model for \( \mathcal{E}_\lambda \) makes sense only for \( \lambda > 1 \) (see Section 2, particularly Eq. 2.5). We suspect that the case of \( \lambda = 1 \) can be resolved by taking appropriate limits in our proof. However, for \( \lambda \in (0, 1) \), our method breaks down, and a new idea is needed.
**Question 1.3.** What is the analogous result for functions?

As discussed in the opening lines, the Riesz–Sobolev inequality implies that the functional $E(\lambda)(f)$ can only increase (for $f \geq 0$) if $f$ is replaced by its symmetric decreasing rearrangement, $f^\ast$. It increases strictly, unless $f$ is a translate of $f^\ast$. Measuring the deficit of $f$ as above by $\delta(f) = E(\lambda)(f^\ast) - E(\lambda)(f)$, what is the correct measure of asymmetry, and what is the correct sharp stability inequality?

It has been conjectured that $E(\lambda)(f^\ast) - E(\lambda)(f) \geq C(\lambda) \inf_{a \in \mathbb{R}^n} \int (f^\ast(x) - f(x-a))(f^\ast(y) - f(y-a)) \phi(\lambda)(x-y) \, dx \, dy$ for some positive constant $C(\lambda)$ (which depends on $n$) [4]. The right hand side in this inequality equals the squared distance of $f$ from the translates of $f^\ast$ in the negative Sobolev space $H^{-\frac{n}{2}}$, a space of distributions that contains $L^\frac{2n}{n+\lambda}$. It is known that this distance is small whenever the left hand side is small, but no explicit bounds are known.

The energy functional $E(\lambda)$ is closely connected with the notion of Riesz capacity of a compact subset $A \subset \mathbb{R}^n$, given by

$$\text{Cap}_{n-\lambda}(A) = \sup_\mu \left( \int \int \phi(\lambda)(x-y) \, d\mu(x) \, d\mu(y) \right)^{-1},$$

where the supremum is taken over all probability measures $\mu$ supported on $A$. This quantity has been studied extensively in the literature; it is used in Harmonic Analysis and Metric Geometry to obtain lower bounds on the Hausdorff dimension of sets. For further discussion, we direct the reader to Chapter 6 of [8].

**Question 1.4.** Among sets of given volume, are near-minimizers of the Riesz capacity close to balls?

For the special case of the Newton potential ($\lambda = 2$) in dimension $n \geq 3$, the Riesz capacity agrees with the more common notion of capacity, defined by minimizing the Dirichlet integral of the potential generated by the mass distribution under suitable constraints. For that notion of capacity, balls are indeed minimal, and stability follows via the co-area formula from from results on the classical isoperimetric inequality [20, 16].

Balls are known to be the unique minimizers of the Riesz capacity also in the range $\lambda \in (0, 2)$ [26, 2, 21]; it is not clear whether this continues to hold for $\lambda > 2$. Theorem [1, 1] does not apply, because the equilibrium measure is generally far from uniform, concentrating on the boundary for $\lambda \geq 2$ and near the boundary for $\lambda \in (0, 2)$. Still, the estimates in Sections 4 and 5 may prove useful.

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While preparing this manuscript, the authors became aware that Theorem 4 in preprint just posted by Frank and Lieb [12] implies the main stability result of this article. Since their proof proceeds along different lines, and this article explores other aspects of the problem, we feel that the results presented here are still relevant and of interest.

2. Outline of the Proof

We work in Euclidean space $\mathbb{R}^n$ of a fixed dimension $n \geq 2$ (which we routinely suppress in the notation). The volume and surface area of the unit ball $B^n$ are given by

$$|B^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}, \quad |S^{n-1}| = n|B^n| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$
We use lowercase Roman symbols to denote points in Euclidean space (e.g., \( x \in \mathbb{R}^n \)), and Greek symbols for points on the unit sphere (e.g., \( \xi \in S^{n-1} \)). On the unit sphere, we denote by \(|\xi - \eta|\) the chordal distance, by \(d\xi\) integration against the uniform measure inherited from the Lebesgue measure on \(\mathbb{R}^n\), and by \(\| \cdot \|\) the \(L^2\)-norm.

For \(0 < \lambda < n\), the Riesz potentials are defined by

\[
\phi_{\lambda}(x) = \frac{1}{c_{\lambda}} |x|^{-(n-\lambda)}
\]
on \(\mathbb{R}^n\) (see [24] and [17]). The constant is conventionally chosen as

\[
(2.1) \quad c_{\lambda} = 2^\lambda \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{n-\lambda}{2}\right)} .
\]

With this normalization, their Fourier transform is given by \(\hat{\phi}_{\lambda}(p) = (2\pi |p|)^{-\lambda}\) (see, for example, [19] Theorem 5.9), and \(\phi_{\lambda} \ast \phi_{\mu} = \phi_{\lambda+\mu}\). Since \(\phi_{\lambda}\) has a positive Fourier transform, the convolution functional is positive definite, that is,

\[
\iint f(x)f(y)\phi_{\lambda}(x-y) \, dx \, dy > 0
\]
for any non-zero function \(f\) such that the integral exists. The value of this double integral equals \(\|f\|_{H^{-\lambda}}^2\), the squared Sobolev norm of fractional order \(-\lambda\). The case \(\lambda = 2\)

\[
\phi_2(x) = \frac{1}{(n-2)|S^{n-1}|} |x|^{-(n-2)} , \quad n \geq 3
\]
is the Newton potential, which plays a special role in Mathematical Physics, with many applications to gravitation and electrostatics. It is also the fundamental solution of the (negative) Laplacian.

Consider minimizing the deficit among all sets \(A \subset \mathbb{R}^n\) of given asymmetry \(\alpha > 0\). Since both \(\delta\) and \(\alpha\) are bounded functionals, it suffices to analyze the situation when \(\alpha\) is small. One may also scale \(A\) to a given volume, and translate it so that the infimum in the definition of the Fraenkel asymmetry occurs at \(x = 0\).

Suppose that \(A\) is squeezed between two balls,

\[
(2.2) \quad e^{-\varepsilon} B^n \subset A \subset e^\varepsilon B^n ,
\]
where \(\varepsilon\) is a small positive number that will later be chosen as a function of the asymmetry (with \(\varepsilon \to 0\) as \(\alpha \to 0\), see Proposition 6.1). Denote by \(\Phi_{\lambda} := \mathcal{X}_{B^n} \ast \phi_{\lambda}\) the potential of the unit ball. We expand the difference as \(\mathcal{E}(B^n) - \mathcal{E}(A) = \mathcal{V}(A) - \mathcal{W}(A)\), with

\[
(2.3) \quad \mathcal{V}(A) = 2 \int (\mathcal{X}_{B^n}(x) - \mathcal{X}_A(x)) \Phi_{\lambda}(x) \, dx ,
\]
and

\[
(2.4) \quad \mathcal{W}(A) = \iint (\mathcal{X}_{B^n}(x) - \mathcal{X}_A(x)) (\mathcal{X}_{B^n}(y) - \mathcal{X}_A(y)) \phi_{\lambda}(x-y) \, dy \, dx .
\]

Since \(\Phi_{\lambda}\) is strictly radially decreasing, \(\mathcal{V}(A) > 0\) by the bathtub principle. On the other hand, \(\mathcal{W}(A) > 0\) since the Riesz potential is positive definite. We seek a positive lower bound on the difference \(\mathcal{V} - \mathcal{W}\) in terms of the asymmetry.

Viewing \(\mathcal{V}\) as the first variation of \(\mathcal{E}\) about \(B^n\), and \(\mathcal{W}\) as the second variation, one would expect \(\mathcal{V}(A)\) to vanish linearly and \(\mathcal{W}(A)\) to vanish quadratically as \(\alpha \to 0\), and thus \(\mathcal{E}(B^n) - \mathcal{E}(A)\) to be comparable to \(\mathcal{V}(A)\) for \(A\) sufficiently close to \(B^n\). However, if \(A\) has the same volume as \(B^n\), then in fact \(\mathcal{V}(A)\) also vanishes quadratically, since the value of \(\Phi_{\lambda}\) is almost constant in a neighborhood of
the unit sphere. In order to obtain a positive lower bound on \( V(A) - W(A) \), the competing terms have to be estimated with care.

Our strategy, based on work of Fuglede [13], is to approximate \( V \) and \( W \) by integral functionals on the unit sphere, and then solve the resulting minimization problem by expanding in spherical harmonics. For each direction \( \xi \in S^{n-1} \), consider the contribution of the ray through \( \xi \) to the volume of \( A \setminus B^n \) and \( B^n \setminus A \), respectively, given by

\[
M_+(\xi) = \int_1^\infty \mathcal{X}_A(r\xi) r^{n-1} \, dr, \quad M_-(\xi) = \int_0^1 (1 - \mathcal{X}_A(r\xi)) r^{n-1} \, dr.
\]

By construction, \(|A| - |B^n| = \int_{S^{n-1}} (M_+ - M_-) \, d\xi\). We scale \( A \) to the same volume as \( B^n \) and center it near the origin to obtain

\[
\int_{S^{n-1}} M_+(\xi) \, d\xi = \int_{S^{n-1}} M_-(\xi) \, d\xi \geq \frac{\alpha}{2},
\]

We next express \( \delta(A) = V(A) - W(A) \) in terms of \( M_+ \) and \( M_- \). The first variation decreases if mass from \( A \) is moved inwards along the rays towards the origin. As \( \varepsilon \to 0 \), we obtain

\[
V(A) \gtrsim -\Phi_\lambda \bigg|_{S^{n-1}} \int_{S^{n-1}} (M_+ - M_-) \, d\xi + |\nabla \Phi_\lambda| \bigg|_{S^{n-1}} (\|M_+\|^2 + \|M_-\|^2)
\]

\[
=: V(M_+, M_-),
\]

where \( \| \cdot \| \) denotes the \( L^2 \)-norm on \( S^{n-1} \). For the second variation, we assume that \( \lambda > 1 \) and approximate

\[
W(A) \approx \int_{S^{n-1}} \int_{S^{n-1}} M(\xi) M(\eta) \phi_\lambda(\xi - \eta) \, d\xi d\eta =: W(M),
\]

where \( M = M_+ - M_- \). Thus \( \delta(A) \gtrsim V(M_+, M_-) - W(M_+ - M_-) \).

**The toy model.** Consider minimizing \( V - W \) over pairs of nonnegative functions \( M_+, M_- \) on the sphere, subject to the constraints in Eq. (2.6). We diagonalize the quadratic form \( W \) by expanding \( M_+ \) and \( M_- \) in spherical harmonics. The eigenvalues of \( W \) are the Funk-Hecke multipliers associated with \( \phi_\lambda \), which form a strictly decreasing sequence \( \beta_k_{k \geq 0} \) satisfying \( \lim \beta_k = 0 \), see Eq. (5.7). Since the constraints in Eq. (2.6) require \( M = M_+ - M_- \) to be orthogonal to the spherical harmonics of degree zero and one,

\[
W(M_+ - M_-) \leq \beta_2 \|M_+ - M_-\|^2 \leq \beta_2 (\|M_+\|^2 + \|M_-\|^2),
\]

see Lemma 5.3. We have used that \( M_+ \) and \( M_- \) are nonnegative to drop the mixed term. Returning to Eq. (2.7), note that the first integral vanishes since \(|A| = |B^n|\), and the coefficient of the second term is given by \( |\nabla \Phi_\lambda| \bigg|_{S^{n-1}} = \beta_1 > \beta_2 \), see Eq. (5.1). Therefore

\[
V(M_+, M_-) \geq \beta_1 (\|M_+\|^2 + \|M_-\|^2).
\]

We conclude that

\[
V(M_+, M_-) - W(M_+ - M_-) \geq (\beta_1 - \beta_2) (\|M_+\|^2 + \|M_-\|^2).
\]

Since \( \|M_+\|^2 + \|M_-\|^2 \geq \frac{\alpha^2}{2|S^{n-1}|} \) by Schwarz’ inequality and Eq. (2.6), this establishes the conclusion of the theorem in the toy model. Note that the toy model makes sense only for \( \lambda > 1 \), where the Riesz potential is locally integrable on \( S^{n-1} \).
The main part of the paper is dedicated to justifying the approximations of \( \mathcal{V} \) and \( \mathcal{W} \) by the spherical integrals \( V \) and \( W \) under the assumption of (2.2). This is done in Sections 4 and 5. For \( \mathcal{V} \), we rearrange \( A \setminus B^n \) and \( B^n \setminus A \) into thin ring-shaped sets adjacent to the unit sphere, and then Taylor expand \( \Phi_\lambda(r\xi) \) about \( r = 1 \). For \( \mathcal{W} \), we cannot simply expand \( \phi_\lambda(r\xi - s\eta) \) about \( r = s = 1 \) because of the singularity at \( \xi = \eta \). Instead, we represent \( \mathcal{X}_\lambda(r\xi) \) in spherical harmonics and control \( \mathcal{W}(A) - \bar{W}(M) \) as \( \varepsilon \to 0 \) by bounds on the multiplication operator associated with \( \phi_\lambda \), see Lemma 5.5. In Section 6 we establish the geometric conditions in Eqs. (2.2) and (2.6). Finally, Theorem 1.1 is proved in Section 7.

3. Some deformations of the ball

Setting aside for the moment the validation of the toy model, we illustrate its use by computing some examples. The first two examples determine the coefficients in Eq. (2.7) in terms of the multipliers \( \beta_0 \) and \( \beta_1 \).

**Example 3.1 (Translation).** Let \( A = -\varepsilon e_n + B^n \). Clearly, \( \delta(A) = 0 \), and thus \( \mathcal{V}(A) = \mathcal{W}(A) \). The functions \( M_+ \) and \( M_- \) from Eq. (2.5) are the positive and negative parts of \( M(\xi) = \varepsilon e_n \cdot \xi + O(\varepsilon^2) \). For the first variation, Eq. (2.7) yields

\[
\mathcal{V}(A) \approx V(M_+, M_-) \approx \frac{1}{n} |\nabla \Phi_\lambda| \bigg|_{S^{n-1}} |S^{n-1}| \varepsilon^2,
\]

and for the second variation, Eq. (2.8) yields

\[
\mathcal{W}(A) \approx W(M) \approx \frac{1}{n} \beta_1 |S^{n-1}| \varepsilon^2,
\]

since \( M \) is to leading order a spherical harmonic of degree one. The errors are of order \( O(\varepsilon^3) \) and \( o(\varepsilon^2) \), respectively, as \( \varepsilon \to 0 \), see Propositions 4.1 and 5.1. Comparing coefficients, we conclude that

\[
(3.1) \quad |\nabla \Phi_\lambda| \bigg|_{S^{n-1}} = \beta_1.
\]

**Example 3.2 (Dilation).** Let \( A = e^\varepsilon B^n \), where \( \varepsilon > 0 \) is small. By scaling, \( |A| - |B^n| = (e^{\varepsilon n} - 1)|B^n| \), and

\[
\mathcal{V}(A) - \mathcal{W}(A) = \mathcal{E}_\lambda(B^n) - \mathcal{E}_\lambda(A) = -(e^{(n+\lambda)\varepsilon} - 1)\mathcal{E}_\lambda(B^n).
\]

Obviously \( M_+ \equiv \varepsilon + O(\varepsilon^2) \) while \( M_- \) vanishes (the roles would be reversed if \( \varepsilon < 0 \)). For the first variation, Eq. (2.7) yields

\[
\mathcal{V}(A) \approx 2 \Phi_\lambda \bigg|_{S^{n-1}} \bigg( |B^n| - |A| \bigg) + |\nabla \Phi_\lambda| \bigg|_{S^{n-1}} \bigg( \|M_+\|^2 + \|M_-\|^2 \bigg)
\]

\[
\approx -\Phi_\lambda \bigg|_{S^{n-1}} |S^{n-1}| (2\varepsilon + \varepsilon^2 n) + \beta_1 |S^{n-1}| \varepsilon^2.
\]

In the first term, we have expanded to second order in \( \varepsilon \) and replaced \( n|B^n| \) with \( |S^{n-1}| \). In the second term, we have inserted the value of the gradient from Eq. (5.1). For the second variation, Eq. (2.8) yields

\[
\mathcal{W}(A) \approx W(M) \approx \beta_0 |S^{n-1}| \varepsilon^2,
\]

since \( M = M_+ \) is constant on \( S^{n-1} \). As in the previous example, all errors are of order \( o(\varepsilon^2) \), see Propositions 4.1 and 5.1. Comparing the coefficients of \( \varepsilon \) and \( \varepsilon^2 \), we conclude that

\[
(3.2) \quad \Phi_\lambda \bigg|_{S^{n-1}} = \frac{\beta_0 - \beta_1}{\lambda}, \quad \mathcal{E}_\lambda(B^n) = 2 \frac{\beta_0 - \beta_1}{\lambda(n + \lambda)} |S^{n-1}|.
\]

The next example demonstrates that the quadratic scaling in Theorem 1.1 is sharp.
Therefore, the deficit is given by the asymmetry $\alpha$ of $A$. 

**Example 3.3 (Ring).** Take $M_+ = M_- \equiv c\varepsilon$, where $c = (2|S^{n-1}|)^{-\frac{1}{2}}$ so that $\|M_+\|^2 + \|M_-\|^2 = \varepsilon^2$. Let $A \setminus B^n$ and $B^n \setminus A$ be narrow annuli of the appropriate width that meet at the unit sphere. For the asymmetry $\alpha = \int_{S^{n-1}}(M_+ + M_-)d\xi$, we find $\alpha^2(A) \approx 2|S^{n-1}|\varepsilon^2$, since the functions $M_\pm$ are constant. Further, $\mathcal{W}(A) \approx 0$ since $M \equiv 0$, and by Eqs. (2.7) and (3.1)

$$\delta(A) \approx \mathcal{V}(A) \approx \beta_1 \varepsilon^2 \approx \frac{\beta_1}{2|S^{n-1}|} \alpha^2(A)$$

up to errors of order $o(\varepsilon^2)$.

In the final example, the ball is squeezed into an approximate ellipsoid.

**Example 3.4 (Squeeze).** Let $M(\xi) = c\varepsilon(n(\xi \cdot e_n)^2 - 1)$, where $c = \left(\frac{4n}{n+2}|S^{n-1}|\right)^{-\frac{1}{2}}$ is chosen so that $\|M\| = \varepsilon$, and let $M_\pm$ its positive and negative parts. Define the corresponding set $A$ by its radial function, $R(\xi)$. Since $M$ is a spherical harmonic of degree two, $\int_{S^{n-1}}Md\xi = 0$, and the deformation is volume-preserving. By Schwarz’ inequality, the asymmetry satisfies $\alpha^2(A) = \left(\int_{S^{n-1}}|M|d\xi\right)^2 \leq |S^{n-1}|\varepsilon^2$. Eqs. (2.7) and (2.8) yield $\mathcal{V}(A) \approx \beta_1 \varepsilon^2$ and $\mathcal{W}(A) \approx \beta_2 \varepsilon^2$, with errors of order $o(\varepsilon^2)$. Therefore, the deficit is given by

$$\delta(A) \approx (\beta_1 - \beta_2)\varepsilon^2 \geq \frac{\beta_1}{2|S^{n-1}|} \alpha^2(A) .$$

4. First Variation

In this section, we estimate $\mathcal{V}(A)$ for sets that satisfy Eq. (2.2), and justify the approximation in Eq. (2.7). Fix $\lambda \in (0, n)$, and recall that $\Phi_\lambda = \phi_\lambda \star X_{B^n}$ denotes the potential of the unit ball.

**Proposition 4.1 (Projection to the sphere).** For $A \subset \mathbb{R}^n$, define $M_+$ and $M_-$ by Eq. (2.3). Let $\mathcal{V}(A)$ be the first variation in Eq. (2.3), and let $V(M_+, M_-)$ be the spherical integral on the right hand side of Eq (2.7). Then

(4.1) $\mathcal{V}(A) \geq V(M_+, M_-) + O(\varepsilon)\left(\|M_+\|^2 + \|M_-\|^2\right)$

as $\varepsilon \to 0$ for every $A$ satisfying Eq. (2.2).

**Proof.** By definition,

(4.2) $\mathcal{V}(A) = 2 \left(\int_{B^n \setminus A} \Phi_\lambda(x)dx - \int_{A \setminus B^n} \Phi_\lambda(x)dx \right)$.

In polar coordinates, the second integral on the right hand side takes the form

$$\int_{A \setminus B^n} \Phi_\lambda(x)dx = \int_{S^{n-1}} \int_1^{\infty} X_A(r\xi)\Phi_\lambda(r\xi) r^{n-1} drd\xi .$$

Consider a rearrangement of $A \setminus B^n$ where mass is moved inwards along rays in such a way that for each $\xi \in S^{n-1}$ the value of $M_+(\xi)$ is preserved and the intersection with the ray through $\xi$ becomes an interval $[1, 1 + R_+(\xi)]$. By construction, $0 \leq R_+ \leq e\varepsilon - 1$, and $M_+ = \frac{1}{n}\left((1 + R_+)^n - 1\right)$. Since $\Phi_\lambda$ is radially decreasing, this rearrangement increases the value of the integral,

$$\int_{A \setminus B^n} \Phi_\lambda(x)dx \leq \int_{S^{n-1}} \int_1^{1+R_+(\xi)} \Phi_\lambda(r\xi) r^{n-1} drd\xi .$$
From the Taylor expansion of $\Phi_\lambda(r\xi)$ about $r = 1$, we obtain for the inner integral
\[
\int_1^{1+R_+(\xi)} \Phi_\lambda(r\xi) r^{n-1} dr = \Phi_\lambda(\xi) M_+(\xi) + (\xi \cdot \nabla \Phi_\lambda(\xi) + O(\varepsilon)) \int_1^{1+R_+(\xi)} (r-1) r^{n-1} dr
\]
\[
= \Phi_\lambda(\xi) M_+(\xi) - \frac{1}{2} (|\nabla \Phi_\lambda(\xi)| + O(\varepsilon) |M_+(\xi)|^2).
\]
In the last step, we have integrated explicitly and used that $R_+ = (1 + O(\varepsilon)) M_+$. The quadratic term appears with a negative sign, because $\Phi_\lambda$ is radially decreasing. All error estimates hold uniformly for $0 \leq R_+ \leq e^\varepsilon - 1$ and $\xi \in S^{n-1}$. Integration over $S^{n-1}$ yields
\[
(4.3) \quad \int_{A \cap B^n} \Phi_\lambda(x) \, dx \leq \left( \Phi_\lambda \big|_{S^{n-1}} \right) |A \setminus B^n| - \frac{1}{2} \left( |\nabla \Phi_\lambda| \big|_{S^{n-1}} + O(\varepsilon) \right) ||M_+||^2.
\]

Similarly, the first integral on the right hand side of Eq. (4.2) takes the form
\[
\int_{B^n \setminus A} \Phi_\lambda(x) \, dx = \int_{S^{n-1}} \int_0^1 (1-\mathcal{X}_A(r\xi)) \Phi_\lambda(r\xi) r^{n-1} dr d\xi.
\]
This decreases under the rearrangement of $A \cap B^n$ that moves mass inwards such that the intersection with each ray defined by a direction $\xi \in S^{n-1}$ is replaced with the interval $[0, 1 - R_-(\xi)]$, where $0 \leq R_- \leq 1 - e^{-\varepsilon}$, and $M_- = \frac{1}{n} (1 - (1 - R_-)^n)$,
\[
\int_{B^n \setminus A} \Phi_\lambda(x) \, dx \geq \int_{S^{n-1}} \int_0^1 \Phi_\lambda(r\xi) r^{n-1} dr d\xi.
\]
As above, we use the Taylor expansion of $\Phi_\lambda(r\xi)$ about $r = 1$ to obtain
\[
(4.4) \quad \int_{B^n \setminus A} \Phi_\lambda(x) \, dx \geq \left( \Phi_\lambda \big|_{S^{n-1}} \right) |B^n \setminus A| + \frac{1}{2} \left( |\nabla \Phi_\lambda| \big|_{S^{n-1}} + O(\varepsilon) \right) ||M_-||^2.
\]
This time, the quadratic term appears with the positive sign. The proof is completed by subtracting Eq. (4.3) from Eq. (4.4). \hfill \Box

5. SECOND VARIATION

In this section, we estimate $\mathcal{W}(A)$ for sets satisfying Eq. (2.2), and justify the approximation in Eq. (2.8). Fix $\lambda \in (1, n)$.

**Proposition 5.1** (Projection to the sphere). For $A \subseteq \mathbb{R}^n$, define $M_+$ and $M_-$ by Eq. (2.5). Let $\mathcal{W}(A)$ be the first variation in Eq. (2.4), and let $W(M_+ - M_-)$ be the spherical integral on the right hand side of Eq (2.8). Then
\[
(5.1) \quad \mathcal{W}(A) = W(M_+ - M_-) + o(1) \left( ||M_+||^2 + ||M_-||^2 \right)
\]
as $\varepsilon \to 0$ for every $A$ satisfying Eq. (2.2).

As in the previous section, we work in polar coordinates.

**Lemma 5.2** (Error term). For $\lambda \in (1, n)$ and $\varepsilon > 0$, set
\[
\psi_{\lambda, \varepsilon}(\xi, \eta) := e^{(n-\lambda)\varepsilon} \phi_\lambda(\xi - \eta) - e^{-2(n-\lambda)\varepsilon} \phi_\lambda(e^{-2\varepsilon} \xi - \eta),
\]
and let $L_{\lambda, \varepsilon}$ be the linear operator defined by
\[
(5.2) \quad (L_{\lambda, \varepsilon} M)(\xi) := \int_{S^{n-1}} \psi_{\lambda, \varepsilon}(\xi, \eta) M(\eta) \, d\eta
\]
for $M \in L^2$ and $\xi \in S^{n-1}$. If $A \subset \mathbb{R}^n$ satisfies Eq. (2.2), then
\[
|W(A) - W(M)| \leq 2\|L_{\lambda,\varepsilon}\|_{L^2 \to L^2} \left(\|M_+\| + \|M_-\|\right)^2.
\]

**Proof.** By decomposing $\mathcal{X}_{B^n} - \mathcal{X}_A$ into its positive and negative parts and using Schwarz’ inequality, it suffices to prove that
\[
D := \left| \int_{A_+} \phi_{\lambda}(x - y) \, dxdy - \int_{S^{n-1}} \int_{S^{n-1}} M_i(\xi)M_j(\eta)\phi_{\lambda}(\xi - \eta) \, d\xi d\eta \right| \leq \|L_{\lambda,\varepsilon}\|_{L^2 \to L^2} \|M_i\| \|M_j\|
\]
for $i, j \in \{+, -\}$, where $A_+ = A \setminus B^n$, and $A_- = B^n \setminus A$. In polar coordinates,
\[
D = \int_{S^{n-1}} \int_{S^{n-1}} \left( \int_0^\infty \int_0^{2\pi} \mathcal{X}_{A_i}(r\xi)\mathcal{X}_{A_j}(s\eta)\left(\phi_{\lambda}(r\xi - s\eta) - \phi_{\lambda}(\xi - \eta)\right) \, (rs)^{n-1} \, dr \, d\theta \right) \, d\xi d\eta.
\]
We claim that for $r, s \in [e^{-\varepsilon}, e^\varepsilon]$,
\[
e^{-\left(n-\lambda\right)e}(\xi, \eta) \leq \phi_{\lambda}(r\xi - s\eta) \leq e^{2\left(n-\lambda\right)e}(\xi, \eta).
\]
The upper bound holds since
\[
|r\xi - s\eta|^2 = (r - s)^2 + rs|\xi - \eta|^2 \geq e^{-2\varepsilon}|\xi - \eta|^2,
\]
and the lower bound follows from
\[
|r\xi - s\eta|^2 = rs\left(\frac{r^2}{s^2} + \frac{s^2}{r^2} - 2\xi \cdot \eta\right) \leq e^{2\varepsilon}(e^{2\varepsilon} + e^{-2\varepsilon} - 2\xi \cdot \eta) = e^{4\varepsilon}|e^{-2\varepsilon}\xi - \eta|^2.
\]
In the middle step, we have used the convexity of $t \mapsto t + t^{-1}$ to replace $t^2$ with $e^{-2\varepsilon}$.

Since $\phi_{\lambda}(\xi - \eta)$ lies between the upper and lower bound, Eq. (5.4) implies that
\[
\sup_{r, s \in [e^{-\varepsilon}, e^\varepsilon]} |\phi_{\lambda}(r\xi - s\eta) - \phi_{\lambda}(\xi - \eta)| \leq \psi_{\lambda,\varepsilon}(\xi, \eta).
\]
Therefore, for each $\xi, \eta \in S^{n-1}$, the inner integral satisfies
\[
\left| \int_0^\infty \int_0^{2\pi} \mathcal{X}_{A_i}(r\xi)\mathcal{X}_{A_j}(s\eta)\left(\phi_{\lambda}(r\xi - s\eta) - \phi_{\lambda}(\xi - \eta)\right) \, (rs)^{n-1} \, dr \, d\theta \right| \leq M_i(\xi)M_j(\eta)\psi_{\lambda,\varepsilon}(\xi, \eta).
\]
Integration over the spherical variables yields Eq. (5.3). \qed

The remainder of the section is dedicated to computing $W(M)$ and bounding the norm of $L_{\lambda,\varepsilon}$. Since $W$ is rotation-invariant, we will expand $M$ in spherical harmonics. We briefly summarize the pertinent facts about spherical harmonics, following the conventions in [1]. By definition, a spherical harmonic is the restriction of a harmonic homogeneous polynomial to the unit sphere. Explicitly, $Y_k$ is a spherical harmonic of degree $k \geq 0$ on $S^{n-1}$, if and only if the function $P_k(x) = |x|^k Y_k(\frac{x}{|x|})$ is a harmonic homogeneous polynomial of degree $k$ on $\mathbb{R}^n$.

The spherical harmonics simultaneously diagonalize all rotation-invariant linear operators on $L^2(S^{n-1})$. We first consider the functional $W$. By the Funk-Hecke formula,
\[
\int_{S^{n-1}} Y_k(\eta)\phi_{\lambda}(\xi - \eta) \, d\eta = \beta_k Y_k(\xi), \quad \xi \in S^{n-1}
\]
for every spherical harmonic $Y_k$ of degree $k$ on $S^{n-1}$, with multipliers $\beta_k$ that depend only on $k$, $n$, and $\lambda$. In terms of these multipliers,
\[
W(M) = \sum_{k \geq 0} \beta_k \|Y_k\|^2, \quad M \in L^2(S^{n-1}),
\]
Proof. Expand in spherical harmonics
\[ \beta(e_n \cdot \eta) \]

(5.7) The functional equation for Gamma yields the recursion relation
\[ \beta_k = \frac{1}{c_\lambda} \int_{S^{n-1}} |e_n - \eta|^{-n+\lambda} d\eta > 0 . \]

For \( k \geq 1 \) in dimension \( n \geq 3 \), we apply the Funk-Hecke formula to the zonal harmonic \( Z_k(\eta) = C_k(e_n \cdot \eta) \), where \( C_k \) is a Gegenbauer polynomial of order \( \nu = \frac{n-2}{2} \), and evaluate the integral at \( \xi = e_n \). By definition, \( C_k(t) \) is a polynomial of degree \( k \), given by the \( k \)-th Taylor coefficient of \( (1 + r^2 - 2rt)^{\frac{n-2}{2}} \) at \( r = 0 \). In particular, \( Z_k(e_n) = C_k(1) = \frac{\Gamma(n-2+k)}{k!\Gamma(n-2)} \). Note that \( C_k \) is even when \( k \) is even, and odd otherwise. We compute
\[
\beta_k = \frac{1}{c_\lambda Z_k(e_n)} \int_{S^{n-1}} Z_k(\eta)|e_n - \eta|^{-n+\lambda} d\eta
\]
\[
= \frac{|S^{n-2}|}{C_k(1)c_\lambda} \int_0^\pi C_k(\cos \theta)(2 - 2 \cos \theta)^{-\frac{(n-\lambda)}{2}} (\sin \theta)^{n-2} d\theta
\]
\[
= \frac{|S^{n-2}|}{c_\lambda 2^{\frac{n-\lambda}{2}} C_k(1)} \int_{-1}^1 C_k(-t)(1+t)^{\frac{\lambda-3}{2}} (1-t)^{\frac{n-2}{2}} dt
\]
\[
= (-1)^k \frac{c_{n,\lambda}}{\Gamma\left(\frac{n-2}{2} - k\right)} \Gamma\left(\frac{\lambda+n-2}{2} + k\right)
\]

for some constant \( c_{n,\lambda} > 0 \) that does not depend on \( k \). Here, \( \theta \) is the angle from the north pole, \( t = -\cos \theta \), and we have expanded \( \sin^2 \theta = (1-t)(1+t) \). In the last line, we have turned to the tables of Erdelyi et al [7] and applied Eq. (3) on p. 280 (with parameter values \( n = k, \nu = \frac{n-2}{2}, \beta = \frac{\lambda-3}{2} \)). The functional equation for Gamma yields the recursion relation
\[
(5.7) \quad \beta_{k+1} = \frac{n - \lambda + 2k}{n + \lambda + 2k - 2} \beta_k, \quad k \geq 0, n \geq 3 .
\]

Clearly, the sequence \( (\beta_k) \) is positive, decreasing, and converges to zero. The order of decay is \( O(k^{-(\lambda-1)}) \).

In dimension \( n = 2 \), the zonal harmonics are given by \( Z_k = \cos(k\theta) = T_k(\cos \theta) \), where \( T_k \) is a Chebychev polynomial. Here, we use Eq. (1) on p. 271 (with parameter values \( n = k, a = \frac{\lambda-3}{2} \)) to derive the slightly different recursion relation
\[
\beta_{k+1} = \frac{(2k+2)(2k+2 - \lambda)}{(2k+1)(2k+\lambda)} \beta_k, \quad k \geq 0, n = 2 .
\]

The sequence is decreasing for \( k \geq 1 \); if \( \lambda > \frac{\beta}{3} \) then also \( \beta_1 < \beta_0 \).

**Proposition 5.3** (Bound on \( W \)). Let \( \lambda \in (1, n) \). If \( F \) is a square-integrable function on \( S^{n-1} \) that satisfies
\[ \int_{S^{n-1}} F(\xi) d\xi = 0 , \quad \int_{S^{n-1}} \xi F(\xi) d\xi = 0 , \]
then \( W(F) \leq \beta_2 \|F\|^2 \).

**Proof.** Expand in spherical harmonics \( F = \sum_{k \geq 0} Y_k \), and apply the Funk-Hecke formula,
\[ W(F) = \sum_{k \geq 2} \beta_k \|Y_k\|^2 \leq \beta_2 \sum_{k \geq 2} \|Y_k\|^2 = \beta_2 \|F\|^2 . \]
Here, the first and last identities hold since $Y_0 = Y_1 = 0$ by assumption, and the middle inequality follows since $\beta_k < \beta_2$ for all $k > 2$. \hfill \Box

To bound the norm of $L_{\lambda, \varepsilon}$, we also need estimates on the multipliers $b_k(r)$ associated with the kernels $\phi_\lambda(r\xi - \eta)$. The multipliers are defined by the property that
\[
(5.8) \quad \int_{S^{n-1}} Y_k(\eta) \phi_\lambda(r\xi - \eta) \, d\eta = b_k(r)Y_k(\xi) \quad \xi \in S^{n-1}
\]
for every spherical harmonic $Y_k$. We note in passing that, for any pair of spheres of different radii $0 < r \leq s$, the multipliers are given by $s^{-(n-2)}b_k(s)$. When $\lambda = 2$, the multipliers are easily computed:

**Lemma 5.4** (Funk-Hecke multipliers for the Newton potential). For $n \geq 3$, $\lambda = 2$,
\[
b_k(r) = \frac{r^k}{n + k - 2}, \quad k \geq 0, r \in [0, 1].
\]

*Proof.* At $r = 1$, Eq. (5.6) and the recursion in Eq. (5.7) yield $\beta_k = \frac{1}{n+k-2}$.

Let $Y_k$ be a spherical harmonic of order $k \geq 0$. Since $\phi_2$ is the fundamental solution of the Laplacian, the function
\[
u(x) := b_k(|x|)Y_k\left(\frac{x}{|x|}\right) = \int_{S^{n-1}} Y_k(\eta)\phi_2(x - \eta) \, d\eta, \quad (x \in B^n)
\]
is harmonic on the unit ball, with boundary values on $S^{n-1}$ given by $\beta_kY_k$. But the unique harmonic extension of the spherical harmonic $\beta_kY_k$ to the unit ball is the homogeneous harmonic polynomial of degree $k$ that defines it. Therefore $\nu(r\xi) = \beta_k r^k Y_k(\xi)$, proving the claim. \hfill \Box

As a consequence, for $\lambda = 2$ Proposition [5.7] holds with an explicit error estimate of order $O(\varepsilon)$:

**Proof of Proposition [5.7] for the Newton potential.** Let $n \geq 3$ and $\lambda = 2$. By Lemma [5.2] and Schwarz’ inequality,
\[
|W(A) - W(M)| \leq 2\|L_{2, \varepsilon}\|_{L^2 \to L^2}(\|M_+\|^2 + \|M_-\|^2).
\]
By the definition of $L_{2, \varepsilon}$ and Lemma [5.4] the operator is represented by the multipliers
\[
e^{(n-2)\varepsilon} \beta_k - e^{-2(n-2)\varepsilon} b_k(e^{-2\varepsilon}) = \frac{e^{(n-2)\varepsilon} - e^{-2(n+k-2)\varepsilon}}{n + k - 2}, \quad k \geq 0.
\]
Its operator norm is the norm of the sequence of multipliers in $\ell^\infty$,
\[
\|L_\varepsilon\|_{L^2 \to L^2} = \sup_{k \geq 0} \frac{e^{(n-2)\varepsilon} - e^{-2(n+k-2)\varepsilon}}{n + k - 2} = \frac{e^{(n-2)\varepsilon} - e^{-2(n-2)\varepsilon}}{n - 2}.
\]
In particular, $\|L_\varepsilon\|_{L^2 \to L^2} = O(\varepsilon)$ as $\varepsilon \to 0$. \hfill \Box

For $\lambda \neq 2$, we have the following estimate.

**Lemma 5.5** (Funk-Hecke multipliers for radius $r < 1$). Let $\lambda \in (1, n)$. For $k \geq 0$, define $b_k(r)$ by Eq. (5.8). Then $b_k$ is continuous on $[0, 1]$, and
\[
b_k(r) \leq \left(\frac{2}{1 + r^2}\right)^{\frac{2}{n-2}} \beta_k, \quad k \geq 0, r \in [0, 1],
\]
where $\beta_k$ is as in Eqs. (5.6) and (5.7). Equality holds for $r = 1$.\n
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Proof. Fix $\lambda \in (1, n)$ and $k \geq 0$. We evaluate Eq. 5.8 with $Y_k = Z_k$ (the zonal harmonic) at $\xi = e_n$,

$$b_k(r) = \frac{1}{c_k Z_k(e_n)} \int_{S^{n-1}} Z_k(\eta)|r e_n - \eta|^{-(n-\lambda)} d\eta. \quad (5.9)$$

For $\eta \neq e_n$, the Riesz potential is bounded by

$$\phi_\lambda(r e_n - \eta) = \frac{1}{r^\frac{n+\lambda}{2}} \left( \frac{1 + r^2}{r} - 2e_n \cdot \eta \right)^{-\frac{n+\lambda}{2}} \leq r^{-\frac{n+\lambda}{2}} \phi_\lambda(e_n - \eta).$$

Since each $Z_k$ is bounded on $S^{n-1}$, the Dominated Convergence Theorem implies that $b_k(r)$ is continuous on $[0, 1]$, and that $b_k(1) = \beta_k$.

To obtain the upper bound on $b_k(r)$, we use the binomial expansion,

$$b_k(r) = \int_{S^{n-1}} Z_k(\eta)(e_n \cdot \eta)^{k} d\eta = 0, \quad \text{if } \ell < k \text{ or } \ell - k \text{ is odd},$$

because $Z_k$ is orthogonal to all polynomials of order less than $k$ and contains only monomials of the same parity as $k$. For $\ell = k + 2j$, the integrals can be evaluated exactly in terms of Gamma functions.

In dimension $n \geq 3$, we apply Eq. (2) on p. 280 of [7]; in dimension $n = 2$ we interpret $\eta \in S^1$ as a complex variable, write $Z_k(\eta) = \text{Re}|\eta|^k$ and $\eta \cdot e_2 = \frac{1}{2}(\eta + \eta^{-1})$, and apply Cauchy’s formula. In either case,

$$d_{k,j} := \int_{S^{n-1}} Z_k(\eta)(e_n \cdot \eta)^{k+2j} d\eta > 0.$$

Since all coefficients in the series

$$(1 + r^2)^{\frac{n+\lambda}{2}} b_k(r) = \frac{1}{c_k Z_k(e_n)} \sum_{j=0}^{\infty} a_{k+2j} d_{k,j} \left( \frac{2r}{1 + r^2} \right)^{k+2j}$$

are positive, it defines an increasing function of $r$ on $[0, 1]$. The proof is completed by comparing with the value at $r = 1$.  

The proof of Lemma 5.5 shows that $b_k(r) = O(r^k)$ as $r \to 0$. We suspect that $b_k$ itself may be increasing on $[0, 1]$ but could not find a reference.

Proof of Proposition 5.7 Let $n \geq 2$ and fix $\lambda \in (1, n)$. By Lemma 5.2 and Schwarz’ inequality,

$$|\mathcal{W}(A) - W(M)| \leq 2\|L_{\lambda, \varepsilon}\|_{L^2 \rightarrow L^2} \left(\|M_{+}\|^2 + \|M_{-}\|^2\right).$$

By definition, the operator $L_{\lambda, \varepsilon}$ is represented by the multipliers $e^{(n-\lambda)\varepsilon} \beta_k - e^{-2(n-\lambda)\varepsilon} b_k(e^{-2\varepsilon}) > 0$. Its norm is bounded by

$$\|L_{\lambda, \varepsilon}\|_{L^2 \rightarrow L^2} = \sup_{k \geq 0} \left\{ e^{(n-\lambda)\varepsilon} \beta_k - e^{-2(n-\lambda)\varepsilon} b_k(e^{-2\varepsilon}) \right\}. $$

By Lemma 5.5

$$e^{-2(n-\lambda)\varepsilon} b_k(e^{-2\varepsilon}) \leq \left( \frac{2e^{-2(n-\lambda)\varepsilon}}{1 + e^{-4\varepsilon}} \right)^{\frac{n+\lambda}{2}} \beta_k \leq \beta_k.$$
In particular, the multipliers are positive. Since $\beta_k$ is decreasing, we have for any $K > 0$

$$\|L_{\lambda, \varepsilon}\| \leq \max \left\{ \max_{k \leq K} \left( b_k(r) - \beta_k \right), e^{(n-\lambda)\varepsilon} \beta_k \right\}.$$ 

Since $\lim_{\varepsilon \to 0} b_k = \beta_k$ for each $k$ by Lemma 5.5, it follows that

$$\limsup_{\varepsilon \to 0} \|L_{\lambda, \varepsilon}\| \leq \beta_k.$$ 

We finally take $K \to \infty$ and recall that $\lim \beta_k = 0$. \qed

6. Geometric reduction

In this section, we reduce the proof of Theorem 1.1 to sets that are squeezed between two balls, as in Eq. (2.2), and satisfy the constraints in Eq. (2.6). In the estimates, we use the notation $a \lesssim b$ (and equivalently $b \gtrsim a$) to signify that $a \leq Cb$ for some constant $C$ that depends only on $n$ and $\lambda$. We say that a subset $A \subset \mathbb{R}^n$ is scaled if $|A| = |B^n|$, or equivalently, $A^s = B^n$. It is centered if $\alpha(A) = |A \Delta A^s|$.

Proposition 6.1 (Auxiliary properties). For every subset $A \subset \mathbb{R}^n$ of finite positive volume there exists a scaled subset $\tilde{A} \subset \mathbb{R}^n$ and $\varepsilon \lesssim (\alpha(A))^{\frac{1}{n}}$ such that

(P1) $\delta(\tilde{A}) \leq \delta(A)$,
(P2) $\alpha(\tilde{A}) = \alpha(A)$,
(P3) $e^{-\varepsilon} B^n \subset \tilde{A} \subset e^\varepsilon B^n$,
(P4) $\int \frac{x}{\tilde{A} |x|} \, dx = 0$.

In the proof of Proposition 6.1, we assume that $A$ is scaled and centered, and move parts of its mass towards the origin to achieve (P1) and (P3). To ensure (P2), we leave a narrow neighborhood of the unit circle unchanged. In the last step, a small translation yields (P4).

The first two lemmas will be used to establish Property (P1). As in Section 2 we expand

$$\delta(A) - \delta(\tilde{A}) = \mathcal{V}(A) - \mathcal{V}(\tilde{A}) + \mathcal{W}(A) - \mathcal{W}(\tilde{A}),$$

and separately estimate the contributions of $\mathcal{V}$ and $\mathcal{W}$.

Lemma 6.2 (Moving mass inwards, first variation). Let $\frac{1}{2} \leq R_1 < R_2 \leq \frac{3}{2}$, and let $A, \tilde{A} \subset \mathbb{R}^n$ be scaled subsets with $A \setminus \tilde{A} \subset (\mathbb{R}^n \setminus R_2 B^n)$ and $\tilde{A} \setminus A \subset R_1 B^n$. Then

$$\mathcal{V}(A) - \mathcal{V}(\tilde{A}) \gtrsim (R_2 - R_1) |A \setminus \tilde{A}|.$$ 

Proof. By Eq. (2.3),

$$\mathcal{V}(A) - \mathcal{V}(\tilde{A}) = 2 \int \left( \mathcal{X}_{\tilde{A} \setminus A}(x) - \mathcal{X}_{A \setminus \tilde{A}}(x) \right) \Phi_\lambda \, dx.$$ 

Since $\Phi_\lambda$ is smooth and radially decreasing, it follows that

$$\mathcal{V}(A) - \mathcal{V}(\tilde{A}) \geq 2 \left( \Phi_\lambda \right)_{|x| = R_2} |A \setminus \tilde{A}| - 2 \left( \Phi_\lambda \right)_{|x| = R_1} |A \setminus \tilde{A}|$$

$$\geq \left( \inf_{\frac{1}{2} \leq |x| \leq \frac{3}{2}} |\nabla \Phi_\lambda| \right) (R_2 - R_1) |A \setminus \tilde{A}|.$$ 

In the last line, we have used that $\frac{1}{2} \leq R_1 < R_2 \leq \frac{3}{2}$ and $|A \setminus \tilde{A}| = |A \setminus \tilde{A}| = \frac{1}{2} |A \setminus \tilde{A}|$. \qed
**Lemma 6.3** (Moving mass inwards, second variation). If $A, \tilde{A} \subset \mathbb{R}^n$ satisfy $|A \Delta B^n| \leq \alpha$, $|\tilde{A} \Delta B^n| \leq \alpha$, then

$$|\mathcal{W}(A) - \mathcal{W}(\tilde{A})| \lesssim \alpha^{\frac{1}{2}} |\tilde{A} \Delta A|.$$ 

**Proof.** By Eq. (2.4),

$$\mathcal{W}(A) - \mathcal{W}(\tilde{A}) = \iint (\mathcal{X}_A(x) - \mathcal{X}_A(x))(\mathcal{X}_\tilde{A}(y) + \mathcal{X}_A(y) - 2\mathcal{X}_B(y))\phi(x, y) \, dx \, dy.$$ 

The first factor in the integral is supported on $\tilde{A} \Delta A$, where it takes the values $\pm 1$. The second factor is supported on a set of measure at most $2\alpha$, where it takes values in $\{0, \pm 1, \pm 2\}$. By the Riesz–Sobolev inequality,

$$\mathcal{W}(A) - \mathcal{W}(\tilde{A}) \leq \iint |\mathcal{X}_A(x) - \mathcal{X}_A(x)| |\mathcal{X}_\tilde{A}(y) + \mathcal{X}_A(y) - 2\mathcal{X}_B(y)|\phi(x, y) \, dx \, dy$$

$$\leq \int_{|B^n| < |\tilde{A} \Delta A|} \int_{|B^n| < 2\alpha} 2\phi(x - y) \, dx \, dy$$

$$\leq 2(2\alpha)^{\frac{1}{2}} \Phi(0) |\tilde{A} \Delta A|.$$ 

In the last line, we have rescaled the inner integral to range over the unit ball, and then used that $\Phi(\lambda)$ is radially decreasing. \hfill \Box

The next two lemmas will be used to establish Property (P2). They show that mass can be moved around without changing the asymmetry, so long as a suitable neighborhood of the unit circle is left untouched.

**Lemma 6.4** (Symmetric difference of balls). For any $y \in \mathbb{R}^n$, we have

$$|B^n \Delta (y + B^n)| \geq \min\{|y|, 2\} |B^n|.$$ 

**Proof.** We may take $y = (t, 0, \ldots, 0)$ with $t \geq 0$. For $t \geq 2$, the balls are disjoint and their symmetric difference equals $2|B^n|$. For $t \in [0, 2]$, let

$$f(t) := |B^n \cap (te_1 + B^n)| = 2 \left| \{x \in B^n \mid x_1 \geq \frac{t}{2}\} \right|.$$ 

Clearly, $f(0) = |B^n|$ and $f(2) = 0$. The derivative $f'(t)$ is given by a negative multiple of the cross-sectional area of $B^n$ at $x_1 = \frac{t}{2}$. Since $f'(t)$ is increasing on $[0, 2]$, $f$ is convex. Therefore

$$f(t) \leq \left(1 - \frac{t}{2}\right) f(0) + \frac{t}{2} f(2) = \left(1 - \frac{t}{2}\right) |B^n|.$$ 

Since $|B^n \Delta (x + B^n)| = 2(|B^n| - f(|x|))$, this proves the claim. \hfill \Box

**Lemma 6.5** (Preserving asymmetry while moving mass). Let $\rho \in [0, 1)$, and let $A \subset \mathbb{R}^n$ be a scaled and centered subset with asymmetry $\alpha(A) \leq \frac{\rho}{2} |B^n|$. If

$$\tilde{A} \cap (1 + \rho) B^n = A \cap (1 + \rho) B^n,$$

then $\tilde{A}$ is centered and $\alpha(\tilde{A}) = \alpha(A)$. The same conclusion holds if, instead,

$$\tilde{A} \setminus (1 - \rho) B^n = A \setminus (1 - \rho) B^n.$$
Proof. We need to show that $|\tilde{A}\Delta(y+B^n)| \geq |\tilde{A}\Delta B^n| = \alpha(A)$ for all $y \in \mathbb{R}^n$. If $\tilde{A}$ agrees with $A$ on $(1+\rho)B^n$, then
\[
|\tilde{A}\Delta(y+B^n)| = 2|(y+B^n)\setminus \tilde{A}| = 2|(y+B^n)\setminus A| \geq \alpha(A), \quad (|y| \leq \rho),
\]
since $y+B^n \subset (1+\rho)B^n$. Similarly, if $\tilde{A}$ agrees with $A$ on the complement of $(1-\rho)B^n$, then
\[
|\tilde{A}\Delta(y+B^n)| = 2|\tilde{A}\setminus(y+B^n)| = 2|A\setminus(y+B^n)| \geq \alpha(A), \quad (|y| \leq \rho),
\]
since $y+B^n \supset (1-\rho)B^n$. In either case, for $y = 0$ we have $|\tilde{A}\Delta B^n| = \alpha(A)$. Moreover, by the reverse triangle inequality and Lemma 6.4
\[
|\tilde{A}\Delta(y+B^n)| \geq |B^n\Delta(y+B^n)| - |\tilde{A}\Delta B^n| \geq \rho|B^n| - \alpha(A) > \alpha(A), \quad (|y| > \rho),
\]
completing the proof. □

The next lemma will be used to establish Property (P4).

Lemma 6.6 (Median). For $n \geq 2$, let $A \subset \mathbb{R}^n$ be a bounded set of positive measure. There is a unique point $x_0 \in \mathbb{R}^n$ such that
\[
\int_{x_0+A} \frac{y}{|y|} dy = 0.
\]
If $A$ is scaled and centered, with asymmetry $\alpha(A) = \alpha$, then $|x_0| \lesssim \alpha(A)$.

Proof. In dimension $n \geq 2$, the function
\[
f(x) = \int_{x+A} |y| dy = \int_A |y-x| dy, \quad (x \in \mathbb{R}^n).
\]
is continuously differentiable and strictly convex. Since $f$ grows at infinity, it has a unique minimizer, $x_0$, which is characterized by the variational equation
\[
0 = \nabla f(x_0) = \int_{x_0+A} \frac{y}{|y|} dy.
\]
Suppose that $A$ is scaled and centered, with asymmetry $\alpha(A) = \alpha$, where $\alpha > 0$ is small. Let $f$ be the function defined above, and let $g$ be the corresponding function for the unit ball. By the triangle inequality,
\[
f(x) - f(0) = \int_A (|y-x| - |y|) dy \\
\geq \int_{B^n} (|y-x| - |y|) dy - \int_{A\Delta B^n} ||y-x|-|y|| dy \\
\geq g(x) - g(0) - \alpha|x|.
\]
In dimension $n \geq 2$, the function $g$ is twice continuously differentiable, strictly radially increasing, and strictly convex. We find its Hessian by differentiating under the integral,
\[
D^2 g(x) = \int_{B^n} \frac{1}{|y-x|^n} P_{y-x}^\perp dy = \int_{x+B^n} \frac{1}{|y|^n} P_{y}^\perp dy.
\]
Here $P_{y}^\perp$ denotes the matrix of the orthogonal projection onto to hyperplane normal to $y$ for $y \neq 0$. The integral converges and defines a positive definite matrix that depends continuously on $x$. In particular,
\[
\sum_{j=1}^n \partial^2_i g(0) = (n-1) \int_{B^n} \frac{1}{|y|} dy = |B^n|.
\]
By radial symmetry, $D^2 g(0) = \frac{1}{n} |B^n| I$. For $|x| \leq \frac{1}{n}$, we use that $x + B^n \supset \frac{n-1}{n} B^n$ Eq. (6.3) to conclude

$$D^2 g(x) \geq \int_{\frac{n-1}{n} B^n} \frac{1}{|y|} P(y) \, dy \geq \left( \frac{n-1}{n} \right)^{n-1} D^2 g(0) \geq \frac{|B^n|}{n e} I, \quad (|x| \leq \frac{1}{n})$$

as quadratic forms, and thus $g(x) - g(0) \geq \frac{|B^n|}{2ne} |x|^2$. By Eq. (6.2), this implies

$$f(x) - f(0) \geq \left( \frac{|B^n|}{2ne} |x| - \alpha \right) |x|, \quad (|x| \leq \frac{1}{n}).$$

If $\alpha \leq \frac{|B^n|}{2n e}$, then $f(x) \geq f(0)$ for $|x| = \frac{1}{n}$, and by convexity for all $|x| \geq \frac{1}{n}$. In that case, the minimal value of $f$ lies below $f(0)$, and hence $|x_0| < \frac{2ne}{|B^n|} \alpha$. □

**Proof of Proposition 6.1** Given a set $A$ of asymmetry $\alpha(A) = \alpha > 0$ and deficit $\delta(A) = \delta$. We may assume that $A$ is scaled and centered. and that $\alpha$ is small. The set $\tilde{A}$ will be constructed in three steps. First, the portion of $A$ that lies in the complement of a ball $(1 + R) B^n$ is moved into a narrow annulus $(1 + r) B^n \setminus (1 + \rho) B^n$, to create a set $A'$. Then the portion of $A'$ in the annulus $(1 - \rho) B^n \setminus (1 - r) B^n$ is moved into the ball $(1 - R) B^n$ to create $A''$. Here, $R = C \alpha^\frac{1}{n}$, where $C$ will be determined below. Once $C$ has been chosen, we take $\alpha$ small enough that $R \leq \frac{1}{2}$, and set $\rho = 2\alpha / |B^n|$. By construction, $(1 - R) B^n \subset A \subset (1 + R) B^n$, as required by (P3). Finally, we perform a translation that re-centers $A''$ to $\tilde{A}$ to obtain Property (P4).

**Step 1.** Define

$$A' = (A \cap (1 + R) B^n) \cup ((1 + r) B^n \setminus (1 + \rho) B^n),$$

where $r \geq \rho$ is uniquely determined by the condition that $|A'| = |A|$. By construction,

$$A' \cap (1 + \rho B^n) = A \cap (1 + \rho B^n), \quad A' \subset (1 + R) B^n.$$ 

In particular, $|A' \Delta A| \leq \frac{1}{2} \alpha$. Since $\rho \lesssim \alpha$ and

$$|B^n| = |A'| \geq |B^n| - \frac{1}{2} \alpha + ((1 + r)^n - (1 + \rho)^n)|B^n|,$$

it follows that $r \lesssim \alpha$. Consider the expansion for $\delta(A) - \delta(A')$ from Eq. (6.1). By Lemmas 6.2 and 6.3

$$\mathcal{V}(A) - \mathcal{V}(A') \gtrsim (R - r) |A' \Delta A| \gtrsim C \alpha^\frac{1}{n} |A' \Delta A|,$$

$$|\mathcal{W}(A) - \mathcal{W}(A')| \lesssim \alpha^\frac{1}{n} |A' \Delta A|.$$ 

By choosing $C$ is sufficiently large, we ensure that $\mathcal{V}(A) - \mathcal{V}(A') \geq |\mathcal{W}(A) - \mathcal{W}(A')|$, and consequently $\delta(A') \leq \delta$. Since the implied constants in Eq. (6.4) depend only on $n$ and $\lambda$, the same is true for $C$. Moreover, by Lemma 6.5, $A'$ is centered and $\alpha(A') = \alpha$.

**Step 2.** Define

$$A'' := (A' \cup (1 - R) B^n) \setminus ((1 - \rho) B^n \setminus (1 - r) B^n),$$

where $r \geq \rho$ is uniquely determined by requiring that $|A''| = |A'|$. By construction,

$$A'' \setminus (1 - \rho) B^n = A' \setminus (1 - \rho) B^n, \quad (1 - R) B^n \subset A'' \subset (1 + R) B^n.$$ 

In particular, $|A'' \Delta A'| \leq \frac{1}{2} \alpha$. Since $\rho \lesssim \alpha$ and

$$|B^n| = |A''| \leq (1 - (1 - \rho)^n + (1 - r)^n)|B^n| + \frac{1}{2} \alpha,$$

it follows that $r \lesssim \alpha$. As in Step 1, Lemmas 6.2 and 6.3 imply that $\delta(A'') \leq \delta(A') \leq \delta$ for $C$ sufficiently large. Moreover, by Lemma 6.5, $A''$ is centered and $\alpha(\tilde{A}) = \alpha$. Setting

$$\varepsilon = - \log (1 - R) \lesssim \alpha^\frac{1}{n},$$


we see that $A''$ satisfies (P1)-(P3).

Step 3. By Lemma 6.6 there exists $x_0 \in \mathbb{R}^n$ such that
\[
\int_{x_0 + A''} \frac{x}{|x|} \, dx = 0.
\]
Setting $\tilde{A} = x_0 + A''$ yields Property (P3). Since $|x_0| \lesssim \alpha$, Property (P3) remains in force after replacing $R$ with $R + |x_0|$ and adjusting $\varepsilon$ according to Eq. (6.5). \qed

7. PROOF OF THEOREM 1.1

Given a subset $A \subset \mathbb{R}^n$ of asymmetry $\alpha(A) = \alpha$ and deficit $\delta(A) = \delta$, we need to show that $\delta \gtrsim \alpha^2$. We may assume that $\alpha$ is small, and that $A$ is scaled to have volume $|A| = |B^n|$. By Proposition 6.1, we may further assume that $A$ is squeezed between two balls as in Eq. (2.2), and that $\int_A \frac{x}{|x|} \, dx = 0$.

Define the functions $M_+$ and $M_-$ by Eq. (2.5). Since $|A| = |B^n|$, $\int_{S^{n-1}} M_+(\xi) \, d\xi = |A \setminus B^n| = \frac{1}{2} |A \Delta B^n| \geq \frac{\alpha}{2}$, and correspondingly for $M_-$. This verifies the first line of the constraints in Eq. (2.6). For the second line, we compute in polar coordinates
\[
\int_{S^{n-1}} \xi (M_+(\xi) - M_-(\xi)) \, d\xi = \int_{S^{n-1}} \int_0^\infty \xi (\chi_A(r\xi) - \chi_{B^n}(r\xi)) r^{n-1} \, dr \, d\xi
\]

As described in Section 2, we split the deficit into the first and second variation, $\delta = V - W$, and then compare these with the corresponding spherical integrals $V$ and $W$. By Proposition 4.1
\[
V \geq V(M_+, M_-) + O(\varepsilon)(\|M_+\|^2 + \|M_\|-^2),
\]
and by Proposition 5.1
\[
W \leq W(M_+ - M_-) + o(1)(\|M_+\|^2 + \|M_-\|^2)
\]
as $\varepsilon \to 0$. Since $\varepsilon \lesssim \alpha^\lambda$, the $o(1)$ error term converges to zero uniformly as $\alpha \to 0$. By the analysis of the toy model in Eq. (2.9),
\[
V(M_+, M_-) - W(M_+ - M_-) \geq (\beta_1 - \beta_2)(\|M_+\|^2 + \|M_-\|^2),
\]
where $\beta_1 > \beta_2 > 0$ are determined by $n$ and $\lambda$ through Eq. (5.7). It follows that
\[
\delta \geq (\beta_1 - \beta_2 - o(1)) (\|M_+\|^2 + \|M_-\|^2)
\]
as $\alpha \to 0$. Finally, by Schwarz’ inequality and Eq. (2.6), $\|M_+\|^2 + \|M_-\|^2 \geq \frac{\alpha^2}{2|S^{n-1}|}$. \qed
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