Thermodynamics of the quantum $su(1, 1)$ Landau–Lifshitz model

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Abstract. We present the thermodynamics of the quantum $su(1, 1)$ Landau–Lifshitz model, following our earlier exposition of the quantum integrability of the theory, which is based on the construction of self-adjoint extensions, leading to a regularized quantum Hamiltonian for an arbitrary $n$-particle sector. Starting from general discontinuity properties of the functions used to construct the self-adjoint extensions, we derive the thermodynamic Bethe ansatz equations. We show that due to the non-symmetric and singular kernel, the self-consistency implies that only negative chemical potential values are allowed, which leads to the conclusion that, unlike its $su(2)$ counterpart, the $su(1, 1)$ LL theory at $T = 0$ has no instabilities.

Keywords: integrable quantum field theory, quantum integrability (Bethe ansatz), thermodynamic Bethe ansatz

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1. Introduction

There has been a renewed interest in the Landau–Lifshitz (LL) model in recent years, mainly due to its integrability, and, thus, its role as a toy model for checking the gauge/string duality as both having underlying integrable structures (see for recent reviews [1, 2]). A large amount of literature has been devoted to this subject, mostly concerning its classical integrability aspects. At this point, there is a relatively complete understanding of the classical integrable properties of the LL model [3]–[5]. The quantum integrability, on the other hand, has been shown to be surprisingly complex and non-trivial. Firstly, as demonstrated by Sklyanin [7], one has to clearly distinguish the $su(1,1)$ and the $su(2)$ models, as they have completely different physical properties. This is due to construction of the positive-definite metric in the corresponding Hilbert spaces. So far, only the $su(1,1)$ case has been addressed in the literature from this point of view, as in the $su(2)$ case the construction of the positive-definite metric seems to be a quite complicated mathematical task [8]. The next interesting issue arises in the anisotropic LL model, where the algebra of observables has to be modified for consistency to be a quadratic one. While this is natural from the lattice regularization point of view [9, 10], it is not clear how to derive it directly within the framework of the continuous integrable model. And finally, the LL model is distinctly different from other known integrable models due to its highly singular nature in the quantum mechanical description [11]. Indeed, it gives rise to $\delta''(x)$ kinds of interactions which are not mathematically easy to deal with [12]. As a result, it was not possible to obtain the quantum Hamiltonian until recently.

In our previous paper [6], we have solved one of the problems outlined above, and by constructing the self-adjoint extensions with the help of the unusual vector-like states in Hilbert space, specified by the non-trivial scalar product, we explicitly presented the quantum Hamiltonian, and derived the spectrum of the $su(1,1)$ isotropic model, in complete agreement with all known results. In the current paper we work out the thermodynamics of the model, relying on our previous construction. We re-derive the Bethe ansatz equations, find the integral equation describing the excitation and finally construct the thermodynamics, following the standard methods. We show that unlike for other models, in this case one has to start from a finite temperature case, and only

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1 See an extensive list of literature in [6].

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then consider a careful $T \to 0$ limit to avoid contradictory results. One interesting consequence of the thermodynamic Bethe ansatz equations for the $su(1,1)$ LL model is that there is no Fermi point, which means that only negative values for the chemical potential are allowed. We use this feature to argue that the $su(1,1)$ field theory at $T = 0$ has no instabilities, unlike the perturbative results [13] for the $su(2)$ model.

We emphasize that we use the LL model as a testing ground for such singular potentials, as the above mentioned aspects and difficulties of the LL model can be equally transferred to other interesting models, e.g. the fermionic AAF model [14], the quantum mechanical description of which has the same highly singular $\delta''(x)$ behavior. Our goal here is to learn to deal with quantum integrability directly in the continuous case, without resorting to the lattice regularization schemes, which can be a daunting task in itself. We also consider only the $su(1,1)$ isotropic LL model, leaving the essential complications of the $su(2)$ and anisotropic cases to future work.

Our paper is organized as follows. In section 2, we give a short description of the Landau–Lifshitz model, and give only essential details of our method of constructing the self-adjoint extensions, and give the quantum Hamiltonian, as well as the exact spectrum. On this basis, in section 3, we use the discontinuity properties of the functions used to construct the Hilbert space, re-derive in this way the Bethe equations, and following this initial setup work out the thermodynamics. Finally, in the conclusion, we remark that the results of the current paper open several interesting questions, and we outline possible directions.

2. The LL model, quantum Hamiltonian and self-adjoint extensions

The classical Hamiltonian of the full anisotropic Landau–Lifshitz model\(^2\) has the form

$$H = \frac{\epsilon}{2} \int \left[ -(\partial_x S, \partial_x S) + 4\gamma^2 ((S^3)^2 - 1) \right]. \quad (1)$$

Here the scalar product $(S, S) \equiv (S^3)^2 - \epsilon(S^1)^2 - \epsilon(S^2)^2 = 1$ for the vector $S = (S^1, S^2, S^3)$ depends on the parameter $\epsilon = \pm 1$, which gives rise to two different models corresponding to the $su(1,1)$ and $su(2)$ cases. In what follows we will consider only the $su(1,1)$ ($\epsilon = 1$) and isotropic ($\gamma = 0$) case, as there are, as we noted above, significant difficulties with $su(2)$ or anisotropic cases in the quantum theory. The classical dynamics is governed by the Poisson structure:

\[
\{ S^+(x), S^\pm(y) \} = \pm iS^\pm(x)\delta(x-y)
\]

which is replaced, in the quantum theory, by the commutation relations for the $S$-operators\(^3\):

\[
[S^+(x), S^\pm(y)] = \pm S^\pm(x)\delta(x-y)
\]

\[
[S^-(x), S^+(y)] = 2S^3(x)\delta(x-y).
\]

\(^2\) We use the notation of [7].

\(^3\) This is true only for $\gamma = 0$. In general, the algebra (3) should be modified to a quadratic Sklyanin algebra [9].

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The representations of (3), in the ferromagnetic vacuum \(S^3(x)|0\rangle = |0\rangle; S^-(x)|0\rangle = 0\), can be constructed using the vectors of the form
\[
|f_n\rangle = \int \mathrm{d}x_1 \cdots \mathrm{d}x_n f_n(x_1 \ldots x_n) S^+(x_1) \cdots S^+(x_n)|0\rangle
\]  
(4)

where \(f_n(x_1 \ldots x_n)\) are continuous functions, decreasing sufficiently fast for the integral (4) to be well defined. It can be shown that only the \(su(1,1)\) case guarantees a positive-definite scalar product \(\langle g_n|f_n\rangle\).

The problem that one generally faces in continuous integrable models is that the quantum trace identities, defined via the standard procedure of first finding the transfer matrix \(T(u)\) and then taking the trace, lead, unlike for the classical case, to polynomials which contain operator products at the same point, and therefore are not well defined. This issue, however, was neglected for a long time in the majority of the classical integrable models, for example the Thirring model endowed with the \(\delta(x)\) type of potential, by using various (mathematically not well defined) procedures of taking regularizations such as \(\epsilon(x)\delta(x) = 0\). In the LL model, as well as in the fermionic AAF model \(14\), this can no longer be done, and formal manipulations with singular functions lead to meaningless expressions. As a result of these significant difficulties, the quantum Hamiltonian for the LL model, as well as other charges, could not be extracted from the trace identities until recently.

In our previous paper we partially solved this problem, by finding the regularized continuous quantum Hamiltonian\(^4\) and giving the complete description of the resulting Hilbert space. We refer the reader to the publication \(6\) for complete details, and only present here the necessary formulae and essential results. There are essentially two steps involved in our construction. First, we regularize the continuous Hamiltonian by the split-point method, so the quantum Hamiltonian has the form
\[
H_Q = \lim_{\varepsilon \to 0} H_\varepsilon
\]  
(5)

where
\[
H_\varepsilon = \frac{1}{2} \int \mathrm{d}u \, \mathrm{d}v \, F_\varepsilon(u, v) \left[ -\partial_u S^3 \partial_v S^3 + \partial_u S^+ \partial_v S^- + \partial_u \partial_v (S^3(u)\delta(u-v)) - \partial_u \delta(u-v) \right].
\]  
(6)

Here the function \(F(u, v)\) is any sufficiently rapidly decreasing smooth function which makes the integral well defined, and the dependence on an arbitrary parameter (or set of parameters) \(\varepsilon\) is not essential as long as we require that
\[
\lim_{\varepsilon \to 0} F_\varepsilon(u, v) = \delta(u-v).
\]  
(7)

This condition simply means that the quantum Hamiltonian \(H_Q\) is local. It can be shown that the action of the Hamiltonian \(H_Q\) on an arbitrary \(n\)-particle state
\[
|f_n\rangle = \int \mathrm{d}^n x f(x) S^+(x_1) \cdots S^+(x_n)|0\rangle
\]  
(8)

\(^4\) We will present a method for constructing the regularized charges in all orders in the upcoming paper \(15\).
has the form
\[ H_Q|f_n\rangle = -\int d\vec{x}\ (\Delta f(\vec{x})) \prod_{i=1}^{n} S^+(x_i)|0\rangle \]
\[ + \sum_{i>j} \int \prod_{k\neq j} dx_k \left( (\partial_j f(\vec{x}) - \partial_i f(\vec{x})) |_{x_j = x_i + \epsilon} - \partial_i \partial_j f(\vec{x}) |_{x_j = x_i} \right) \prod_{i=1}^{n} S^+(x_i)|x_i = x_j \rangle 0\rangle \]

(here the Laplacian \( \Delta = \sum_{i=1}^{N} \partial_i^2 \)) leading to the following matching conditions:
\[ - (\partial_j f(\vec{x}) - \partial_i f(\vec{x})) |_{x_j = x_i + \epsilon} - \partial_i \partial_j f(\vec{x}) |_{x_j = x_i} = 0, \quad \forall i > j. \]

This was shown to be equivalent to self-adjointness of some Hamiltonian \( \hat{H} \) (for the explicit expression for \( \hat{H} \) see [6]; here we will not need it) acting on the Hilbert space generated by vectors of the type
\[ \Psi = \left( f_1(x_1) \right. \left. f_2(x_1, x_2) \right) \]
with the scalar product defined e.g. for \( n = 2 \) case as
\[ \langle \Phi | \Psi \rangle = \frac{1}{2} \int_{-\infty}^{\infty} g_1^*(x) f_1(x) \, dx + \int_{-\infty}^{\infty} g_2^*(x, y) f_2(x, y) \, dx \, dy \]

which was shown to reproduce beautifully Sklyanin’s earlier attempts to deal with the ill-defined object in the LL model. The matching conditions (10) were also shown to correspond to the \( n \)-particle \( S \)-matrix factorization property
\[ S_n(p_1, \ldots, p_n) = \prod_{i \neq j} S_2(p_i, p_j) \]
where the two-particle scattering \( S \)-matrix for the \( su(1, 1) \) LL model can be derived from (8) for \( n = 2 \) and has the form
\[ S_2 = \frac{2(p_1 - p_2) + ip_1 p_2}{2(p_1 - p_2) - ip_1 p_2}. \]

Let us also note here that in terms of the cluster fields \( \Psi_1^+(x_1) \) and \( \Psi_2^+(x_1) \) (see for details [7]) the state \( \Psi \) (11) for \( n = 2 \) can be written in the form
\[ \Psi_{n=2} = \left( \int dx \ e^{ip_1 p_2 x} \Psi_2^+(x) \right. \]
\[ + \left. \int_{x_1 > x_2} dx_1 dx_2 \left( c(p_1, p_2) e^{ip_1 x_1 + p_2 x_2} + \bar{c}(p_1, p_2) e^{ip_1 x_2 + p_2 x_1} \right) \Psi_1^+(x_1) \Psi_1^+(x_2) \right) |0\rangle \]

where \( c(p_1, p_2) = (2(p_1 - p_2) + ip_1 p_2)/(2(p_1 - p_2)) \). Thus, the \( S \)-matrix has the meaning of the scattering of the quanta corresponding to the \( \Psi_1^+(x) \) cluster field. Note also that the first term is usually absent in other models.

To sum up, our method, being mathematically strict, gives a clear systematic description and understanding of the properties of integrable systems endowed with a singular potential.

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3. Thermodynamics

Having obtained the spectrum and the $S$-matrix, we proceed to the thermodynamics of the model, following the standard methods (see, for example, [11, 16, 17]). One can easily repeat the analysis of [11], and we present only the final results and emphasize the differences from the non-linear Schrödinger case. Note, that while the analysis of the basic thermodynamic properties is quite similar to that for other models, there are several interesting features that require a careful analysis. In particular, we show below that one cannot start directly from the $T = 0$ case, as it leads to contradictory results. Instead, due to the singular behavior of the kernel, one has to consider the general $T \neq 0$ case, and then take the $T \to 0$ limit. Note also that essential differences will arise when considering the correlation functions at finite temperature, as this would involve the properties of the correct Hilbert space (11) with the scalar product (12).

Using the results of the previous section, we obtain the Bethe equations for the $su(1,1)$ LL model in the system of length $L$:

$$e^{ip_iL} = \prod_{i \neq j}^N \frac{2(p_i - p_j) + i p_i p_j}{2(p_i - p_j) - i p_i p_j}.$$  

(16)

Note that the $S$-matrix for the $su(1,1)$ that we consider here is different from the one obtained in $su(2)$ by a factor $i - 2$ in front of $(p_i - p_j)$ [13]. (This is not surprising, considering that these are two different models.) While the absolute value of this factor is not of great importance, the sign plays a big role, essentially leading to different physics: it immediately follows from (16) that unlike for the $su(2)$ case, in the $su(1,1)$ theory the momenta $p$ are always real and as a consequence the theory does not have bound states. It is convenient to introduce variables $u \equiv 1/p$. This is because in the general anisotropic case, the Bethe equations have a simpler form in terms of the $u$ variable, which is associated with the spectral parameter in the inverse scattering method. Thus, (16) can also be written in the equivalent form

$$\frac{L}{u_j} + \sum_{k \neq j}^N \Theta(u_j - u_k) = 2\pi n_j$$  

(17)

where $\Theta(u) = \ln[(i/2 - u)/(i/2 + u)]$, with the numbers $n_j = n'_j + (N - 1)/2, n'_j \in \mathbb{Z}, j = 1 \cdots N$. Since the kernel is

$$K(u) = \partial_u \Theta(u) = -\frac{1}{1/4 + u^2}$$  

(18)

we see that the function $\Theta(u)$ is a monotonically decreasing function of $u$. This is contrary to the behavior of the closely related (in fact, equivalent on the classical level) non-linear Schrödinger model, for which the corresponding $\Theta$ function is a monotonically increasing

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5 The perturbative calculations for the $su(2)$ model lead to the $S$-matrix $S = \prod_{i \neq j}^N ((p_i - p_j) - i p_i p_j)/(p_i - p_j + i p_i p_j)$.

6 It can be shown that this result is true in general for the anisotropic $su(1,1)$ LL model as well.

7 It is obvious already from (16) that $p_i = 0$ is a special point in momentum space. For the moment, we will consider this space as $\mathbb{R} \setminus \{0\}$, so that the $u$-variables are well defined, and extend to the full $\mathbb{R}$ later.
function of momenta. Nevertheless, the solutions can be shown to exist and be unique due to the existence of the convex Yang–Yang type of action

\[ S_{\text{YY}} = -L \sum_j \ln u_j + 2\pi \sum_j n_j u_j - \frac{2}{2} \sum_{k \neq j} \int_0^{u_j - u_k} \Theta(v) \, dv. \]  

(19)

Using (17) and (18) one can obtain the estimate

\[ \frac{2\pi(n_j - n_k)}{L} \geq \frac{1}{u_j} - \frac{1}{u_k} = p_j - p_k \geq \frac{2\pi(n_j - n_k)}{L(1 + D)} \geq \frac{2\pi}{L(1 + D)}. \]  

(20)

This is exactly the same estimate as in the case of the NLS model, which allows us to consider the thermodynamic limit of the model. To proceed, one introduces the density of states in complete analogy as

\[ \rho(p_k) = \frac{1}{L(p_k + 1 - p_k)} \]  

(21)

and then takes the thermodynamic limit \( L \to \infty; N \to \infty; D \equiv N/L = (\text{const}) \). Because the zero-temperature case is rigorously obtained as a limit of the \( T \neq 0 \) system, we will first consider this general situation, i.e., the derivation of the thermodynamic Bethe ansatz (TBA) equation. The initial steps of the construction are parallel to those in [11] for the NLS system, and we only mention one interesting difference, that in the process of the minimization of the partition function one has to be more careful, since the effective kernel \( \widetilde{K}(p_j, p_k) \equiv -(1/(p_j)^2)K(p_j, p_k) = (1/p_j^2)(1/(1/4 + (1/p_j - 1/p_k)^2)) \) that appears in the equations in our case is not symmetric under the exchange of the momenta \( p_j \) and \( p_k \). The resulting TBA equation has the form

\[ \varepsilon(p) = p^2 - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} d\mu \, \widetilde{K}(\mu, p) \ln(1 + e^{-\varepsilon(\mu)/T}). \]  

(22)

At this point we should recall that the point \( p = 0 \) has been excluded so far from our momentum space. Thus, we have to verify that the solution to (22) is continuous at this point. This will lead to a severe constraint on the chemical potential \( h \). To demonstrate this, let us note that the naive conclusion that \( \varepsilon(p) \to -h \) as \( p \to 0 \) (which seems to follow from the fact that \( \widetilde{K}(\mu, 0) = 0 \)) is not correct. In fact, we have the following behavior of the kernel when \( p \to \pm 0 \):

\[ \lim_{p \to \pm 0} \widetilde{K}(\mu, p) = 2\pi \delta(\mu \mp 0). \]  

(23)

Using (23) in (22), we obtain

\[ \varepsilon(\pm 0) = T \ln(e^{-h/T} - 1). \]  

(24)

The first consequence of (24) is that if a solution exists, then it is continuous at the point \( p = 0 \). Now we have two distinct cases:

(i) \( h \geq 0 \). In this case a solution does not exist because \( (e^{-h/T} - 1) \) is always negative or zero.

(ii) \( h < 0 \). This range of the chemical potential is allowed. So we can proceed with the same iterative proof as in the case of NLS to establish the existence and uniqueness of the solution of (22).
Here we have one of the most striking differences between LL and NLS models: while in the latter, all the values of $h$ are allowed (although the thermodynamics, and especially $T \to 0$ limit, drastically depend on its sign), in the former, only negative values of the chemical potential are allowed.

Before we move to discussing the case of zero temperature, $T = 0$, it is instructive to compare our situation with NLS when parameter $c$ (the strength of the delta-functional potential) is going to zero. This is relevant, because all the specifics of our model are due to the effective turning-off of the interaction when one of the momenta is approaching zero. In the case of the NLS model, the TBA equation takes exactly the same form (22), where the kernel now is given by

$$K_c(\mu, p) = \frac{2c}{c^2 + (p - \mu)^2}. \quad (25)$$

Once again, the naive setting of $c$ to zero would lead to a theory of free bosons. But it is easy to see that this is not the case for $c \to +0$, because of the property of $K_c(\mu, p)$ analogous to (23):

$$\lim_{c \to +0} K_c(\mu, p) = 2\pi \delta(\mu - p) \quad (26)$$

which allows to solve TBA exactly with the solution being

$$\varepsilon(p) = T \ln(e^{(p^2 - h)/T} - 1). \quad (27)$$

Notice the striking similarity between (24) and (27)! The analysis of this equation also demonstrates the clear difference of the NLS system for $h \geq 0$ and $h < 0$, as well as shedding light on the physical interpretation of the negative chemical potential for the LL model. Indeed, as we have mentioned earlier, and as is clear from the comparison of the $S$-matrices for two models, the LL model can be thought of as the NLS model with the momentum-dependent coupling constant $c$. Now it is obvious that the physics of the NLS model in the $c \to 0$ limit and the LL model for $p \to 0$ should have very similar properties. Since the NLS model in the limit $c \to 0$ clearly corresponds to free bosons, we expect the chemical potential to be negative. And in fact, this immediately follows from the analysis of (27): only negative values of the chemical potential are allowed as physical solutions of (27).\(^8\) We, thus, expect the same behavior for the LL model.

Now we give a brief consideration of the zero-temperature limit for the LL model in the case of the allowed values of the chemical potential $h \in (-\infty, 0)$. From (24) one can immediately see that in the zero-temperature limit

$$\varepsilon(0) = -h > 0. \quad \text{(28)}$$

Using the standard iterative proof of Yang and Yang (see the comment after (24)), we easily establish that this is the minimum of the function $\varepsilon(p)$.

Then, from (22) we find that in this limit we have

$$\varepsilon(p) = p^2 - h > 0. \quad \text{(28)}$$

In fact, equation (28) is exactly what one has in the case of NLS model for $h < 0$ and $T = 0$, so all the results of the NLS model are applicable here. The main difference is that

\(^8\) Though formally (27) admits non-negative chemical potential solutions, they are not physical, as can be seen from the iterative proof of the Yang–Yang for the NLS model, thus leaving $h = 0$ as the only possible solution in this case.
the other sector of \( h \), namely, the \( h \in [0, \infty) \) sector, is missing in the \( su(1,1) \) LL model. As a consequence, \( \varepsilon(p) \) is always positive for \( T = 0 \), which guarantees the stability of the model. Let us emphasize that for the \( su(2) \) continuous Heisenberg model the quantum inverse scattering has not yet been developed. The main difficulty is the construction of the positive-definite metric in this case, which is mathematically quite non-trivial problem [7]. The claims of instability in the \( su(2) \) model are based on the perturbative calculations of [13] and the assumption of quantum integrability of the model (which was verified in the first non-trivial order in [18]), where it was shown that there are negative energy bound states present, signaling instabilities in the model. In the \( su(1,1) \) model, as we have shown above, there are no bound states. However, from our experience with the \( su(1,1) \) counterpart, we already know that some subtleties, such as extended Hilbert space and self-adjointness of the quantum Hamiltonian, can be fully understood only within the inverse scattering method framework. Thus, the true comparison between the \( su(2) \) and \( su(1,1) \) models is only possible after developing the inverse scattering method for the former model, and we plan to do this in the future.

To construct the excitations, we should again start from the general \( T \neq 0 \) case. Using the continuous form of (17) one can write an analog of the Lieb integral equation:

\[
1 + \int_{-\infty}^{\infty} \tilde{K}(p, \lambda) \rho_p(\lambda) \, d\lambda = 2\pi \rho_t(p)
\]

(29)

where, as usual, \( \rho_p(p) \) and \( \rho_t(p) \) are the distribution densities respectively of particles and vacancies.

The excitations are obtained from (17) in the standard manner; we refer the reader to the monograph [11] for complete details of the similar derivation for the NLS model. For example, a one-particle excitation with the momentum \( \lambda_p \) is described by the equation

\[
-\Theta(1/p - 1/\lambda_p) + 2\pi F(p|\lambda_p) - \int_{-\infty}^{\infty} \tilde{K}(\mu, p) \frac{F(\mu|\lambda_p)}{1 + e^{\varepsilon(\mu)/T}} \, d\mu = 0
\]

(30)

where the shift function \( F(p|\lambda_p) \equiv (p - \tilde{p})/\partial p \) describes the transition from the set of initial momenta \( \{p\} \) to the set of excited momenta \( \{\tilde{p}\} \). Similarly, one can easily write the equations describing the hole excitation as well. Using (30), the excitation energy takes the form

\[
\Delta E(\lambda_p) = \varepsilon_0(\lambda_p) - \int_{-\infty}^{\infty} \varepsilon'_0(\mu) \frac{F(\mu|\lambda_p)}{1 + e^{\varepsilon(\mu)/T}} \, d\mu
\]

(31)

where the energy of the free particle \( \varepsilon_0(p) = p^2 - h \). It can be shown that \( \Delta E(p) = \varepsilon(p) \), where the latter function is determined from the TBA integral equation (22). As usual, this demonstrates that \( \varepsilon(p) \) is indeed the particle excitation energy. The proof consists of manipulating with TBA equations (22) and (30). In particular, using the equation (28), it is easily seen that \( \Delta E(p) \) and \( \varepsilon(p) \) have the same \( p^2 - h \) asymptotic behavior in the \( T \to 0 \) limit.

Finally, we just mention here that the computation of the correlation functions at the finite temperature \( \langle \Omega_T | S^+(x) S^-(y) | \Omega_T \rangle \) requires a separate investigation, as the unusual Hilbert space structure (11) and the scalar product (12) will essentially modify the results. We leave this to a future investigation.

\[9\] It can be seen from (30) that the function \( F(p|\lambda_p) \) has a discontinuity at \( p = 0 \). However, it is easy to see that the integral (31) is well defined.
4. Conclusion

We have presented the essential steps for constructing the thermodynamics for the isotropic $su(1, 1)$ LL model, which is an interesting toy model due to its singular potential structure. We only used our method of constructing the self-adjoint extensions and regularized Hamiltonian to derive the Bethe ansatz equations, as well as the $S$-matrix factorization property. This enables us to derive the spectrum and proceed to the derivation of the thermodynamics. Several key features make the thermodynamics of the LL model an interesting system. Firstly, we showed, using the non-symmetric and singular behavior of the kernel, that the self-consistency of the TBA equations restricts the chemical potential to being a negative function. This leads to a rather restrictive character of the spectrum. We also showed that the $su(1, 1)$ theory is stable, unlike its $su(2)$ counterpart. Another essential conclusion was that one has to start from $T \neq 0$ and only then consider a careful zero-temperature limit.

One interesting question, which we mention just briefly, is to compare the finite size corrections for $T = 0$ for the energy with the corresponding corrections at infinite length and finite temperature. Let us recall, that, e.g., in the NLS model it can be shown that in the first case the finite size corrections have the form

$$E = R \int d\mu \rho_\mu(\mu) (\mu^2 - h) + \frac{\text{const}}{R} v_F + \cdots \quad (32)$$

where $R = 1/T$, and $v_F$ is the Fermi velocity. This is to be compared with $E' \equiv TF(T)/L$, at the limit $L \rightarrow \infty$. In this case the expansion has the form

$$E' = R \int d\mu \rho_\mu(\mu) (\mu^2 - h) + \frac{\text{const}}{R} C_V + \cdots \quad (33)$$

where in the second term $C_V$ is the heat capacity at constant volume. The reason that these two expressions do not coincide is, of course, the lack of Lorentz invariance in the NLS model. However, these computations can be easily repeated for the relativistic AAF model, and we expect these two expressions to coincide [19]. On the other hand, in the case of the LL model we encounter a problem, since the Fermi point and, therefore, the Fermi velocity are not defined. In this case, one has to consider the finite size correction at finite temperature and then consider the $L \rightarrow \infty$ limit. We leave this to a future investigation.

We again emphasize that the LL model should be considered as a toy model for understanding the nuances of the singular potentials. It will be interesting, as a next step, to apply the results of this paper to the AAF model [14], which shares many general features with the LL model, but which is more interesting due to its relativistic invariance, as well as its non-trivial bound state structure. Other interesting and related developments of our approach include the generalization of our method to the anisotropic LL model and, what should be more challenging due to the problems mentioned above, to the $su(2)$ LL model. All these problems are currently under investigation, and we hope to report on the results in the future.

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