Spanning $k$-ended trees of 3-regular connected graphs

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Abstract

A vertex of degree one is called an end-vertex and the set of end-vertices of $G$ is denoted by $\text{End}(G)$. For a positive integer $k$, a tree $T$ be called $k$-ended tree if $|\text{End}(T)| \leq k$. In this paper, we obtain sufficient conditions for spanning $k$-trees of 3-regular connected graphs. We give a construction sequence of graphs satisfying the condition. At the end, we present a conjecture about spanning $k$-ended trees of 3-regular connected graphs.

Keywords: Spanning tree, $k$-ended tree, leaf, 3-regular graph, connected graph

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1. Introduction

Throughout this article we consider only finite undirected labeled graphs without loops or multiple edges. The vertex set and edge set of graph $G$ is denoted by $V = V(G)$ and $E = E(G)$, respectively. For $u, v \in V$, an edge joining two vertices $u$ and $v$ is denoted by $uv$ or $vu$. The neighbourhood $N_G(v)$ or $N(v)$ of vertex $v$ is the set of all $u \in V$ which are adjacent to $v$. The degree of a vertex $v$, denoted by $\deg_G(u) = |N_G(v)|$.

The minimum degree of a graph $G$ is denoted $\delta(G)$ and the maximum degree is denoted $\Delta(G)$. If all vertices of $G$ have same degree $k$, then the graph $G$ is called $k$-regular. The distance between vertices $u$ and $v$, denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest path between $u$ and $v$. A Hamiltonian path of a graph is a path passing through all vertices of the graph. A graph is
Hamiltonian-connected if every two vertices are connected with a Hamiltonian path. In graph $G$, an independent set is a subset $S$ of $V(G)$ such that no two vertices in $S$ are adjacent. A maximum independent set is an independent set of largest possible size for a given graph $G$. This size is called the independence number of $G$, that denoted by $\alpha(G)$.

A vertex of degree one is called an end-vertex, and the set of end-vertices of $G$ is denoted by $\text{End}(G)$. If $T$ is a tree, an end-vertex of $T$ usually called a leaf of $T$ and the set of leaves of $T$ is denoted by $\text{leaf}(T)$. A spanning tree is called independence if $\text{End}(G)$ is independent in $G$.

For a positive integer $k$, a tree $T$ is said to be a $k$-ended tree if $|\text{End}(T)| \leq k$. We define $\sigma_k(G) = \min \{d(v_1) + \ldots + d(v_k) \mid \{v_1, \ldots, v_k\} \text{ is an independent set in } G\}$. Clearly, $\sigma_1(G) = \delta(G)$.

By using $\sigma_2(G)$, Ore [4] obtain the following famous theorem on Hamiltonian path. Notice that a Hamiltonian path is spanning 2-ended tree. A Hamilton cycle can be interpreted as a spanning 1-ended tree. In particular, $K_2$ is hamiltonian and is a 1-ended tree.

**Theorem 1.1.** [4] Let $G$ be a connected graph, if $\sigma_2(G) \geq |G| - 1$, then $G$ has Hamiltonian path.

The following theorem of Las Vergnas Broersma and Tuinstra [1] gives a similar sufficient condition for a graph $G$ to have a spanning $k$-ended tree.

**Theorem 1.2.** [2] Let $k \geq 2$ be an integer, and let $G$ be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then $G$ has a spanning $k$-ended tree.

Win [10] obtained a sufficient condition related to independent number for $k$-connected graph that confirms a conjecture of Las Vergnas Broersma and Tuinstra [1] gave a degree sum condition for a spanning $k$-ended tree.

**Theorem 1.3.** [10] Let $k \geq 2$ and let $G$ be a $m$-connected graph. If $\alpha(G) \leq m + k - 1$, then $G$ has a spanning $k$-ended tree.

A closure operation is useful in the study of existence of Hamiltonian cycles, Hamiltonian path and other spanning subgraphs in graph. It was first introduced by Bondy and Chavatal.

**Theorem 1.4.** [1] Let $G$ be a graph and let $u$ and $v$ be two nonadjacent vertices of $G$ then,

1. Suppose $\deg_G(u) + \deg_G(v) \geq |G|$. Then $G$ has a Hamiltonian cycle if and only if $G + uv$ has a Hamiltonian cycle.
2. Suppose $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then $G$ has a Hamiltonian path if and only if $G + uv$ has a Hamiltonian path.

After [1], many researchers have defined other closure concepts for various graph properties.

More on $k$-ended tree and spanning tree can be found in [6, 7, 8, 9]. In this paper, we obtain sufficient conditions for spanning $k$-ended trees of 3-regular connected graphs and with construction sequence of graphs like $G_m$, we will show this condition is sharp. At the end, we present a conjecture about spanning $k$-ended trees of 3-regular connected graphs.
2. Our results

**Lemma 2.1.** Let $T$ be a tree with $n$ vertices such that $\Delta(T) \leq 3$. If $|\text{leaf}(T)| = k$ and $p$ be the number of vertices of degree 3 in $T$, then $k = p + 2$.

*Proof.* It is easy by the induction on $p$. □

**Lemma 2.2.** Let $G$ be a labelled graph and $k \geq 3$ be the smallest integer such that $G$ has a spanning tree $T$ with $k$ leaves. Then, no two leaves of $T$ are adjacent in $G$.

*Proof.* Put $S = \{v_1, v_2, \ldots, v_k\}$ be the set of all leaves of $T$. By contradiction, suppose that $v_1$ and $v_2$ are adjacent vertices in $G$. If $T_1 = T + v_1v_2$, then $T_1$ contains a unique cycle as $C : v_1v_2c_1c_2 \ldots c_\ell v_1$ where $c_i \in G$ for $1 \leq i \leq \ell$. Since $k \geq 3$ then there exist vertex $v_s \in G$ such that it is not a vertex of $C$. Let $P$ be the shortest path of vertex $v_s$ to the cycle $C$ such that its intersection with cycle $C$ is $c_j$ for $1 \leq j \leq \ell$. Now, we omit the edge $c_{j-1}c_j$ of $T_1$, (if $j = 1$ put $c_{j-1} = v_2$). Let $T_2 = T_1 - c_{j-1}c_j$. Then $T_2$ is a spanning subtree of $G$ such that $\deg_{T_2}(c_j) \geq 2$. The vertices of degree one in spanning subtree $T_2$ is equal to the set $\{v_3, v_4, \ldots, v_k\}$ either $\{v_3, v_4, \ldots, v_k, c_{j-1}\}$. That is a contradiction by minimality of $k$. □

**Theorem 2.1.** Let $G$ be a labeled 3-regular connected graph such that $|V(G)| = n \geq 6$. Then $G$ has a spanning $\lfloor \frac{n+2}{4}\rfloor$-ended tree.

*Proof.* For the graph $T$, we denote the vertices of degree 1 with the set $A_1$, the vertices of degree 2 with the set $A_2$ and the vertices of degree 3 with the set $A_3$.

If $v \in A_3$ then the two adjacent edges to $v$ (those were in $G$ but are not in $T$), each one connects $v$ to a vertex of $A_2$ in $G$, because by Lemma 2.2 it cannot connect $v$ to a member of $A_1$. So, for each vertex in $A_1$ there exist two vertices in $A_2$ such that they are connected to $v$ in $G$ but not in $T$. Now, we have $2 \times |A_1| \leq |A_2|$. Let $|A_1| = k$, $|A_2| = s$ and $|A_3| = p$. By Lemma 2.1 we have $k = p + 2$ and since $2|A_1| \leq |A_2|$ then $2k \leq s$.

We have 

$$n = p + s + k = k - 2 + s + k \geq k - 2 + 2k + k = 4k - 2,$$

Then $k \leq \lfloor \frac{n+2}{4}\rfloor$. □

3. Some concluding remarks

Now we construct the sequence $G_m$ of 3-regular graphs, For $m = 1$, Consider the graph $G_1$ as Figure 1.

Clearly $G_1$ has spanning subtree like $T$ that has 3 leaves and $G$ has no spanning subtree with less than 3 leaves. Every part of $G_1$ like subgraph induced by vertices $\{1, 2, 3, 4, 5\}$ is called a branch, so $G_1$ has 3 branch. Let $H$ be a branch of $G_1$ with vertices $\{1, 2, 3, 4, 5\}$ and set of edges $\{12, 15, 23, 24, 34, 35, 45\}$. Since the edge $\{01\}$ is a cut edge in $G_1$, So $T$ must has a vertex with degree one in $H$. Also in every other branches of $G_1$, $T$ must has a vertex with degree one. so $G_1$ is 3-ended tree and has no spanning tree with less than 3 leaves. Now, we counteract 3-regular graph
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Figure 1. The 3-regular graph $G_1$ with 3 branch.

Figure 2. One part of $G_2$ constructed from $G_1$.

$G_2$, consider $G_1$ and for each branch of that like $H$ defined as above, we removed two vertices \{3, 4\} and add 8 new vertices $\{v_1, \ldots, v_8\}$ then we construct new 3-regular graph as Figure 2.

Clearly $|G_2| = 16 + 3 \times 6$ and minimum number leaves in every spanning subtree of $G_2$ is at least $2 \times 3$ and obviously $G_2$ has spanning subtree with $2 \times 3$ leaves.

Let the number of vertices of $G_m$ is equal $n$ and the number of branches of $G_m$ is equal $k$, then we have the table 1.

| $m$ | $n$ | $k$ |
|-----|-----|-----|
| $G_1$ | 16  | 3   |
| $G_2$ | $16 + 3 \times 6$ | $2 \times 3$ |
| $G_3$ | $16 + 3 \times 6 + 2 \times 3 \times 6$ | $2 \times 2 \times 3$ |
| \ldots | \ldots | \ldots |
| $G_m$ | $16 + 3 \times 6 + \ldots + 2^{m-2} \times 3 \times 6$ | $2^{m-1} \times 3$ |

Table 1. The number of vertices and branches of $G_m$ for $m \in \mathbb{N}$.

It obvious for each $m \in \mathbb{N}$ if the number of vertices of $G_m$ is equal $n$ and the number of branches of $G_m$ is equal $k$, then $\frac{n+2}{6} = k$, and so $G_m$ is $\frac{n+2}{6}$-ended tree (such that $\frac{n+2}{6}$ is the minimum number for that $G_m$ is $\frac{n+2}{6}$-ended tree).
Conjecture 1. There exists \( n \in \mathbb{N} \) such that each 3-regular graph with at least \( n \) vertices has a spanning \( \lfloor \frac{n+2}{6} \rfloor \)-ended tree.

References

[1] J.A. Bondy, and V. Chvátal, A method in graph theory, *Discrete Math.* **15** (2) (1976), 111–135.

[2] H. Broersma and H. Tuinstra, Independence trees and Hamilton cycles, *J. Graph Theory* **29** (1998), 227–237.

[3] M. Kano, A. Kyaw, H. Matsuda, K. Ozeki, A. Saito and T. Yamashita, Spanning trees with a bounded number of leaves in a claw-free graph, submitted.

[4] O. Ore, Note on Hamilton circuits, *Amer. Math. Monthly* **67** (1960), 55.

[5] M. Las Vergnas, Sur une proprié des arbres maximaux dans un graphe, *C. R. Acad. Sci. Paris Sr. A* **272** (1971), 1297–1300.

[6] J. Akiyama and M. Kano, *Factors and factorizations of graphs*, Lecture Note in Mathematics (LNM 2031), Springer, 2011 (Chapter 8).

[7] A. Czygrinow, G. Fan, G. Hurlbert, H.A. Kierstead and W.T. Trotter, Spanning trees of bounded degree, *Electron. J. Combin.* **8** (1) (2001) 12. R33.

[8] K. Ozeki and T. Yamashita, Spanning trees: a survey, *Graphs Combin.* **27** (2011), 1–26.

[9] G. Salamon and G. Wiener, On finding spanning trees with few leaves, *Inform. Process. Lett.* **105** (2008), 164–169.

[10] S. Win, On a conjecture of Las Vergnas concerning certain spanning trees in graphs, *Result. Math.* **2** (1979), 215–224.