Analytical inversion of one-dimensional operators of diffraction of electromagnetic waves in Sobolev spaces

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Abstract. The paper develops a method for analytical inversion in Sobolev spaces on a segment of one-dimensional integral diffraction operators that depend on a parameter. The method is based on positive definiteness of operators and the presence of an orthonormal basis. For a special case, the operator of diffraction of an H–polarized wave on a band or on the surface of a vibratory antenna, the inverse operator is explicitly constructed. It is possible to study the physical properties of surface currents using the inverse operator, as well as to construct an effective numerical method for solving integral diffraction equations.

1. Introduction
The study of the diffraction of electromagnetic waves on cylindrical surfaces is based on the solution of integral equations for surface currents [1]. Knowledge of the surface currents makes it possible to find the main characteristics of diffraction: the field in the far zone, the field in the near zone, the active and reactive parts of the radiation power. The kernels of integral equations have singularities; therefore, the numerical solution of integral equations presents certain difficulties. The original operator of problem B is represented as a sum of two operators A and K to overcome them. The requirement for operator A is that its matrix can be found analytically. The kernel of the operator K is either continuous or has a weak singularity, and its matrix elements are efficiently calculated on a computer using standard programs. For the successful solution of diffraction problems, it seems important to study the properties of the operator A. This work is devoted to the analytical inversion of this operator and the study of properties based on the inverse operator.

2. Problem statement
Let us introduce the definitions of the spaces [2–6]. $H_s(R)$ is the space of generalized functions $u(t)$ whose Fourier transform $\hat{u}(\xi)$ is locally integrable in the sense of Lebesgue and satisfies the inequality

$$
\|u\|_s^2 = \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 (1 + |\xi|)^{2s} d\xi < +\infty.
$$

(1)

$H_s(−1,1)$ is the subspace of $H_s(R)$ consisting of the functions $u(t)$ supported in the closed interval [-1,1]. Finite and infinitely differentiable functions $C_0^\infty(−1,1)$ are dense in the space $H_s(−1,1)$ in the norm (1). $\overline{H}_s(−1,1)$ is the space $H_s(R)$ of generalized functions $f$ admitting an extension of $f$ to $R$ belonging to the space $H_s(R)$. The norm in the space $\overline{H}_s(−1,1)$ is defined by the formula

$$
\|f\|_{\overline{s}} = \inf \|f\|_s.
$$

(2)

We introduce the integral operator into consideration...
\[(A_s u)(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\xi|^2 \hat{u}(\xi) \exp(-i\tau \xi) d\xi, -1 \leq \tau \leq 1 \] (3)

at \( s \geq 0 \). As shown in [5], this integral operator is continuous, as an operator from the space \( \tilde{H}_s(-1,1) \) into space \( \tilde{H}_{-s}(-1,1) \). Note that this statement is true for any real \( s \). Further, the quadratic form of the operator \( A_s \) is positive definite [7], which means that the following inequality holds

\[(A_s u, u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\xi|^2 |\hat{u}(\xi)|^2 d\xi \geq \gamma ||u||_{s}^2, \gamma > 0. \] (4)

The dual space of \( H_s(-1,1) \) is isomorphic to the space \( \tilde{H}_{-s}(-1,1) \). Therefore, the conditions of the Lax-Milgram theorem [8] are satisfied. This theorem implies that the operator \( A_s \) maps the spaces \( H_s(-1,1) \) to the entire space \( \tilde{H}_s(-1,1) \). Moreover, it is bijective, like a positive operator. These two properties make it possible to apply Banach's inverse operator theorem and derive the following theorem

**Theorem 1** The continuous operator \( A_s \) for \( s \geq 0 \) maps one-to-one the space \( H_s(-1,1) \) onto the entire space \( \tilde{H}_{-s}(-1,1) \) and the inverse operator \( A_s^{-1} \) is also bounded.

The purpose of this work is to construct and study the inverse operator \( A_s^{-1} \).

### 3. Method of analytical inversion of integral operators

We use the basis of the space \( H_{s}(-1,1) \) to invert operators. In [7], a system of functions \( \{\varphi_n\}_{n=1}^{+\infty} \) was constructed, which is dense in this space and orthonormal in the following sense

\[(A_s \varphi_m, \varphi_n) = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases} \] (5)

It is necessary to solve the equation to invert the operator \( A_s \)

\[A_s u = f. \] (6)

We expand the unknown function in terms of the complete system of functions

\[u = \sum_{n=1}^{+\infty} c_n \varphi_n. \]

We substitute this expansion in (6), multiply by \( \varphi_n \) scalarsly in the space \( L_2[-1,1] \) and use the normalization property (5). As a result, we get

\[u(\tau) = (A_s^{-1} f)(\tau) = \sum_{n=1}^{+\infty} (f, \varphi_n) \varphi_n(\tau). \] (7)

Formula (7) represents the inverse operator in general form, the sum of a series. For special cases, the sum of the series can be found and the inverse operator can be written in the form of an integral operator. Elements of the space \( \tilde{H}_{-s}(-1,1) \) are generalized functions that manifest themselves in the action on test functions. The theorem holds

**Theorem 2.** If \( f \in \tilde{H}_{-s}(-1,1), s \geq 0 \), then the inequality is true

\[\sum_{n=1}^{+\infty} |(f, \varphi_n)|^2 < +\infty. \] (8)

The converse is true; if (8) holds, then the element

\[u(\tau) = \sum_{n=1}^{+\infty} (f, \varphi_n) \varphi_n(\tau) \]

Belongs to the space \( H_s(-1,1) \). Proof.

Since the operator \( A_s \) maps the space \( H_s(-1,1) \) to the entire space \( \tilde{H}_s(-1,1) \), then for each element \( f \in \tilde{H}_{-s}(-1,1) \) there is such an element \( u \in H_s(-1,1) \) such that \( A_s u = f \). Expanding \( u \) into a series (7) and taking into account (5), we obtain

\[(A_s u, u) = \sum_{n=1}^{+\infty} (f, \varphi_n) A_s \varphi_n \sum_{m=1}^{+\infty} (f, \varphi_m) \varphi_m(\tau) = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} (f, \varphi_n) (\overline{f}, \varphi_m) (A_s \varphi_n, \varphi_m) = \sum_{n=1}^{+\infty} |(f, \varphi_n)|^2. \] (9)

Since the left side is finite, the right side (9) is also limited. Conversely, if the right-hand side is bounded, then, due to the positive definiteness, the norm of the element \( u \) is finite. The theorem is proved.

### 4. Integral equation for the diffraction of an H-polarized wave on a strip. Isolation and inversion of the main part

Let us consider the problem of diffraction of electromagnetic waves on an ideally conducting strip. The primary field is perpendicular to the edge of the strip and does not depend on the longitudinal
component. Accordingly, the surface currents are perpendicular to the edge of the strip and satisfy the equation [9]

\[(Bu)(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sqrt{\xi^2 - 1} \tilde{u}(\xi) \exp(-i\tau\xi) d\xi = f(\tau), -1 \leq \tau \leq 1.\]  

(10)

We draw cuts from the branch points parallel to the imaginary axis \([-1, -1 + i\infty)\) and \([1, 1 - i\infty)\) to isolate the single-valued branch of the function \(\xi^2 - 1\) on the complex plane \(\xi\). The function \(\sqrt{\xi^2 - 1}\) takes real values for |\(\xi| \geq 1\), and when |\(\xi| \leq 1\), the branch is chosen for which the equality is \(\sqrt{\xi^2 - 1} = i\sqrt{1 - \xi^2}\). With this choice of the branch of the multivalued function on the whole line, the real part of the function \(\sqrt{\xi^2 - 1}\) is non-negative. Using this, we represent the left side of (10) as the sum of two operators

\[
(Bu)(\tau) = (Au)(\tau) + (Ku)(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\xi| \tilde{u}(\xi) \exp(-i\tau\xi) d\xi + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\sqrt{\xi^2 - 1} - |\xi|\right) \tilde{u}(\xi) \exp(-i\tau\xi) d\xi = f(\tau), -1 \leq \tau \leq 1.

The operator \(A\) is called the main operator, the functional space for the solution of the equation is determined and the properties of the solution of the integral equation are investigated with its help.

Operator \(A\) belongs to the family of operators \(As\) for \(s = \frac{1}{2}\). It follows from Theorem 1 that the operator \(A\) maps the space \(H\left(-1, 1\right)\) to the entire space \(\tilde{H}_{\frac{1}{2}}(-1, 1)\) and the inverse operator \(A^{-1}\) is also bounded. The latter maps the space to the entire space \(\tilde{H}_{\frac{1}{2}}(-1, 1)\). As for the operator \(K\), it is compact. We use the system of functions to construct \(A^{-1}\) by formula (7)

\[
q_n(\tau) = \sqrt{\frac{\tau}{\pi n}} \sin(n\arccos(\tau)), n = 1, 2, 3, ...
\]

(11)

The functions \(q_n(\tau)\) are the product of polynomials by the weight function \(\sqrt{1 - \tau^2}\) and, at \(\tau \to \mp 1\), turn to zero according to the root law. Surface currents have the same behavior, as is known from the solution of the problem of diffraction of electromagnetic waves on a half-plane. Using formula (7), we obtain

\[
u(\tau) = (A^{-1}f)(\tau) = \sum_{n=1}^{\infty} (f, q_n) q_n(\tau) = \sum_{n=1}^{\infty} (f, q_n)q_n(\tau)
\]

(12)

Formula (12) presents the sum of trigonometric functions, which can be found using tabular sums [10]. Calculating this sum after elementary transformations, we obtain

\[
u(\tau) = \frac{2}{\pi} \int_{-1}^{1} f(t) \left(\arccos t - \frac{1}{2} \ln|\tau - t|\right) dt = -\frac{1}{\pi} \int_{-1}^{1} f(t) \ln \left|\frac{1-\tau t + \sqrt{1 - t^2}\sqrt{1 - \tau^2}}{\tau - t}\right| dt.
\]

(13)

Based on Theorem 1 and formula (13), we obtain the theorem

**Theorem 3.** Integral operator

\[(A^{-1}f)(\tau) = \frac{1}{\pi} \int_{-1}^{1} f(t) \ln \left|\frac{1-\tau t + \sqrt{1 - t^2}\sqrt{1 - \tau^2}}{\tau - t}\right| dt
\]

(14)

maps the space \(\tilde{H}_{\frac{1}{2}}(-1, 1)\) in a one-to-one and continuous manner onto the entire space \(H\left(-1, 1\right)\).

Analysis of the integral operator (14) makes it possible to study the properties of surface currents.

5. **Integral equation of an E-polarized wave on a strip**

Let us now consider the problem of diffraction of E-polarized electromagnetic waves by an ideally conducting band. The primary field is parallel to the edge of the strip and is also independent of the longitudinal component. Accordingly, the surface currents are parallel to the edge of the strip and satisfy the equation [4]

\[(Bu)(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\xi^2 - 1}} \tilde{u}(\xi) \exp(-i\tau\xi) d\xi = f(\tau), -1 \leq \tau \leq 1.
\]

(15)
Equation (15) will also be investigated in Sobolev spaces for the parameter $s = \frac{1}{2}$. Using the asymptotics of the integrand and the well-known formula for the logarithm [10]

$$
\ln \frac{1}{|r-t|} = C + \int_0^{+1} \cos \left(\frac{\pi}{4} (r-t) \xi \right) d\xi + \int_1^{+\infty} \cos \left(\frac{\pi}{4} (r-t) \xi \right) d\xi, \quad C = 0.5772,
$$

we select the main part and write it in coordinate form

$$
(Lu)(\tau) = \frac{1}{\pi} \int_{-1}^{+1} u(t) \ln \frac{1}{|r-t|} dt, -1 \leq \tau \leq 1. \quad (16)
$$

The operators $B$ and $L$, as follows from the results of [11], differ by a completely continuous operator. In addition, operator $L$ is positive. It can be proved that the operator $L$ maps the space $H_{-\frac{1}{2}}(1,1)$ to the entire space $\tilde{H}_{-\frac{1}{2}}(1,1)$. We use the complete system of functions to construct the inverse operator

$$
\varphi_n(\tau) = \begin{cases} 
\frac{1}{\sqrt{\pi n^2 \sqrt{1-\tau^2}}} \tau, & n \geq 1, \\
\frac{2n \cos(n \pi \arccos(\tau))}{\sqrt{\pi}}, & n \geq 1.
\end{cases} \quad (17)
$$

This system of functions has a remarkable property [12]

$$
(L\varphi_n)(\tau) = \begin{cases} 
\ln(2\sqrt{1-\tau^2}) \varphi_0, & n = 0, \\
\frac{1}{n} \sqrt{1-\tau^2} \varphi_n, & n \geq 1.
\end{cases} \quad (18)
$$

And the property (18) implies the orthonormality

$$
(A_n \varphi_m, \varphi_n) = \begin{cases} 
1, & m = n, \\
0, & m \neq n.
\end{cases} \quad (19)
$$

According to (18), the operator $L$ maps the functions defined by formula (17) and turning to infinity at the ends of the segment into polynomials. This implies the completeness of systems of functions (17) in the space $H_{-\frac{1}{2}}(1,1)$. Similarly to Theorem 3, we prove the theorem

**Theorem 4** The operator $L$ maps continuously the space $H_{-\frac{1}{2}}(1,1)$ onto the entire space $\tilde{H}_{\frac{1}{2}}(1,1)$, the inverse operator $L^{-1}$ is bounded and is defined by the formula

$$
(A^{-1} f)(\tau) = \frac{1}{\pi \sqrt{1-\tau}} \int_{-1}^{1} \frac{1-\tau^2 f(t)}{\sqrt{1-\tau^2}} dt + \frac{1}{\pi n^2 \sqrt{1-\tau^2}} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-\tau^2}} dt. \quad (20)
$$

And maps the entire space $\tilde{H}_{\frac{1}{2}}(1,1)$ onto the entire space $H_{-\frac{1}{2}}(1,1)$.

The operator $L^{-1}$ is integro-differential. Formula (20) was obtained by many authors, in particular [12, p. 591], [13, p. 224]. The statement that the integro-differential operator $L^{-1}$ maps the bounded space $\tilde{H}_{\frac{1}{2}}(1,1)$ to the entire space $H_{-\frac{1}{2}}(1,1)$ is new.

Formula (20) contains the derivative $f''(t)$. In this regard, we note an important fact from the theory of Sobolev spaces: the space $\tilde{H}_{\frac{1}{2}}(1,1)$ coincides with the space $H_{1/2}(1,1)$ [14, p. 71]. The reason is that the operator $L^{-1}$ is well defined on the dense in space set $C_0^{+\infty}(1,1)$ and then continues by continuity.

The operator is not only positive, but also positive definite. This property is a consequence of the positivity and limited reversibility [15].

6. Conclusion

Thus, in this paper, a general inversion method for one class of integral operators is constructed and it is implemented for two special cases. The method is based on the fundamental properties of spaces and operators: the space $\tilde{H}_\delta(-1,1)$ is isomorphic to the space dual to $H_\delta(-1,1)$; the operator $\delta_s$ for $s \geq 0$ maps the space $H_\delta(-1,1)$ one-to-one to the entire space $\tilde{H}_\delta(-1,1)$; an orthonormal basis is constructed for the spaces $H_\delta(-1,1)$. The inverse operator is presented as the sum of a series in the basis.
The inverse operator is also represented in the form of an integral or integro-differential operator for problems of diffraction of electromagnetic waves on an ideally conducting strip. The kernels of these operators make it possible to study the properties of surface currents both at the boundary and in the regions of application of the primary field. Knowledge of inverse operators also allows constructing effective numerical-analytical methods for solving integral equations. In these methods, the solution to the integral equation is sought in the form of two terms, one of which is found numerically on a computer, and the second analytically. The last element is defined using the inverse operator.

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