OPENNESS OF UNIFORM K-STABILITY IN FAMILIES OF $\mathbb{Q}$-FANO VARIETIES

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Abstract. We show that uniform K-stability is a Zariski open condition in $\mathbb{Q}$-Gorenstein families of $\mathbb{Q}$-Fano varieties. To prove this result, we consider the behavior of the stability threshold in families. The stability threshold (also known as the delta-invariant) is a recently introduced invariant that is known to detect the K-semistability and uniform K-stability of a $\mathbb{Q}$-Fano variety. We show that the stability threshold is lower semicontinuous in families and provide an interpretation of the invariant in terms of the K-stability of log pairs.

Throughout, we work over a characteristic zero algebraically closed field $k$.

1. INTRODUCTION

In this article, we consider the behavior of K-stability in families of $\mathbb{Q}$-Fano varieties. Recall that K-stability is an algebraic notion introduced by Tian [Tia97] and later reformulated by Donaldson [Don02] to detect certain canonical metrics on complex projective varieties. In the special case of complex $\mathbb{Q}$-Fano varieties, the Yau-Tian-Donaldson conjecture states that a complex $\mathbb{Q}$-Fano variety is K-polystable iff it admits a Kähler-Einstein metric. (By a $\mathbb{Q}$-Fano variety, we mean a projective variety that has at worst klt singularities and anti-ample canonical divisor.) For smooth complex Fano varieties, this conjecture was recently settled in the work of Chen-Donaldson-Sun and Tian [CDS15, Tia15] (see also [DS16, BBJ15, CSW15]).

One motivation for understanding the K-stability of $\mathbb{Q}$-Fano varieties is to construct compact moduli spaces for such varieties. It is expected that there is a proper good moduli space parametrizing K-polystable $\mathbb{Q}$-Fano varieties of fixed dimension and volume. For smoothable $\mathbb{Q}$-Fano varieties, such a moduli space is known to exist [LWX16] (see also [SSY16, Oda15]). A key step in constructing the moduli space of K-polystable Fano varieties is verifying the Zariski openness of K-semistability.

Theorem A. If $\pi : X \to T$ is a projective family of varieties such that $T$ is normal, $\pi$ has normal connected fibers, and $-K_{X/T}$ is $\mathbb{Q}$-Cartier and $\pi$-ample, then

1. $\{ t \in T \mid X_t$ is uniformly K-stable $\}$ is a Zariski open subset of $T$, and
2. $\{ t \in T \mid X_t$ is K-semistable $\}$ is a countable intersection of Zariski open subsets of $T$.

The notion of uniform K-stability is a strengthening of K-stability introduced in [BHJ17, Der16]. In [BBJ15], it was shown that a smooth Fano variety $X$ with discrete automorphism group is uniformly K-stable iff there exists a Kähler-Einstein metric on $X$. K-semistability is strictly weaker than K-(poly)stability and corresponds to being almost Kähler-Einstein [Li17a, BBJ15].

In [BX18], the first author and Xu show that the moduli functor of uniformly K-stable $\mathbb{Q}$-Fano varieties of fixed volume and dimension is represented by a separated Deligne-Mumford stack, which has a coarse moduli space that is separated algebraic space. The proof of the result combines Theorem A.1 with a boundedness statement in [Jia17] (that uses ideas from [Bir16]) and a separatedness statement in [BX18].

For smooth families of Fano varieties, Theorem A is not new. Indeed, for a smooth family of complex Fano varieties with discrete automorphism group, the K-stable locus is Zariski open by [Oda13b, Don15]. In [LWX16], it was shown that the K-semistable locus is Zariski open in
families of smoothable $\mathbb{Q}$-Fano varieties. These results all rely on deep analytic tools developed in [CDS15, Tia15].

Unlike the previous results, our proof of Theorem A is purely algebraic. (A different algebraic proof of Theorem A.2 was also given in [BL18] using a characterization of K-semistability in terms of the normalized volume of the affine cone over a $\mathbb{Q}$-Fano variety [Li17b, LL16, LX16].) Furthermore, the result holds for all $\mathbb{Q}$-Fano varieties, including those that are not smooth(able), and also log Fano pairs. The argument relies on new tools for characterizing the uniform K-stability and K-semistability of Fano varieties [BJJ17, Li17b, Fuji16b, FO16, BJ17].

Our approach to proving Theorem A is through understanding the behavior of the stability threshold (also known as $\delta$-invariant or basis log canonical threshold) in families. We recall the definition of this new invariant.

Let $X$ be projective klt variety and $L$ an ample Cartier divisor on $X$. Set

$$|L|_Q := \{D \in \text{Div}(X)_Q \mid D \geq 0 \text{ and } mD \sim mL \text{ for some } m \in \mathbb{Z}_{>0}\}.$$ 

Following [FO16], we say that $D \in |L|_Q$ is an $m$-basis type divisor of $L$ if there exists a basis \{s_1, \ldots, s_{N_m}\} of $H^0(X, \mathcal{O}_X(mL))$ such that

$$D = \frac{1}{mN_m}(\{s_1 = 0\} + \cdots + \{s_{N_m} = 0\}).$$

For $m \in M(L) := \{m \mid h^0(X, \mathcal{O}_X(mL)) \neq 0\}$, set

$$\delta_m(X; L) := \inf_{D \text{ m-basis type}} \text{lct}(X; D),$$

where lct($X; D$) denotes the log canonical threshold of $D$. The stability threshold of $L$ is

$$\delta(X; L) := \limsup_{M(L) \ni m \to \infty} \delta_m(X; L).$$

In fact, the above limsup is a limit by [BJJ17]. If $X$ is a $\mathbb{Q}$-Fano variety, we set $\delta(X) := r\delta(X; -rK_X)$, where $r \in \mathbb{Z}_{>0}$ is such that $-rK_X$ is Cartier. (The definition is independent of the choice of $r$.) The stability threshold is closely related to global log canonical threshold of $L$, which is an algebraic version of Tian’s $\alpha$-invariant. Recall that the global log canonical threshold of $L$ is

$$\alpha(X; L) := \inf_{D \in |L|_Q} \text{lct}(X; D).$$

The two thresholds satisfy

$$\frac{n+1}{n} \alpha(X; L) \leq \delta(X; L) \leq (n+1)\alpha(X; L).$$

where $n = \dim(X)$.

The stability threshold was introduced in the $\mathbb{Q}$-Fano case by K. Fujita and Y. Odaka to characterize the K-stability of $\mathbb{Q}$-Fano varieties [FO16]. More generally, the invariant coincides with an invariant suggested by R. Berman and defined in [BoJ18]. As the name suggests, the stability threshold characterizes the stability of $\mathbb{Q}$-Fano varieties.

**Theorem 1.1.** [FO16, BJJ17] Let $X$ be a $\mathbb{Q}$-Fano variety.

1. $X$ is uniformly K-stable iff $\delta(X) > 1$.
2. $X$ is K-semistable iff $\delta(X) \geq 1$.

In light of the previous statement, Theorem A is a consequence of the following result.

**Theorem B.** Let $\pi : X \to T$ be a projective family of varieties and $L$ a $\pi$-ample Cartier divisor on $X$. Assume $T$ is normal, $X_t$ is a klt variety for all $t \in T$, and $K_{X/T}$ is $\mathbb{Q}$-Cartier. Then, the functions

$$T \ni t \mapsto \delta(X_t, L_T) \quad \text{and} \quad T \ni t \mapsto \alpha(X_t, L_T)$$
are lower semicontinuous.

Let us note the main limitation of Theorem B. While the statement implies \( \{ t \in T \mid \delta(X_T; L_T) > a \} \) is open for each \( a \in \mathbb{R}_{\geq 0} \), it does not imply \( t \mapsto \delta(X_T; L_T) \) takes finitely many values. Hence, we are unable to prove \( \{ t \in T \mid \delta(X_T; L_T) \geq a \} \) is open and cannot verify the openness of K-semistability in families of \( \mathbb{Q} \)-Fano varieties. The openness of K-semistability is an immediate consequence of Theorem B and the following conjecture (see [BL18, Conjecture 2] for a local analogue).

**Conjecture 1.2.** If \( \pi : X \to T \) is a projective family of varieties such that \( T \) is normal, \( X_t \) is klt for all \( t \in T \), and \(-K_{X/T}\) is \( \mathbb{Q} \)-Cartier and ample, then \( T \ni t \mapsto \delta(X_T) \) takes finitely many values.

We also provide a new interpretation of the stability threshold in terms of (log) K-stability. The result provides further motivation for studying this invariant. Note that a similar result is obtained independently by Cheltsov, Rubinstein and Zhang in [CRZ18, Lemma 5.8].

**Theorem C.** Let \( X \) be a \( \mathbb{Q} \)-Fano variety. We have:

\[
\min \{1, \delta(X)\} = \sup \{ \beta \in (0, 1] \mid (X, (1 - \beta)D) \text{ is K-semistable for some } D \in | -K_X|_\mathbb{Q} \} = \sup \{ \beta \in (0, 1] \mid (X, (1 - \beta)D) \text{ is uniformly K-stable for some } D \in | -K_X|_\mathbb{Q} \}
\]

To conclude the introduction, we briefly explain the proof of Theorem B for the stability threshold. The strategy is similar in spirit to the proof of [BL18, Theorem 1].

1. We define a modification of \( \delta_m(X_T, L_T) \), denoted by \( \hat{\delta}_m(X_T, L_T) \), defined in terms of \( N \)-filtrations of \( H^0(X_T, \mathcal{O}_X(mL_T)) \) rather than bases of this vector space (see §4.3 for the precise definition). The advantage of working with \( N \)-filtrations of \( H^0(X_T, \mathcal{O}_X(mL_T)) \) is that \( N \)-filtrations of bounded length are simply flags. Hence, they are parametrized by a proper variety.

2. We show \( \hat{\delta}_m \) is lower semicontinuous for \( m \gg 0 \) (Proposition 6.4) and \( (\hat{\delta}_m)_m \) converges to \( \delta \) as \( m \to \infty \) (Theorem 4.17).

3. To show that \( \delta \) is lower semicontinuous, it is sufficient to show that \( (\hat{\delta}_m)_m \) converges to \( \delta \) uniformly. We prove a convergence statement (Theorem 5.2) that implies the lower semicontinuity of \( \delta \). The statement is an extension of a convergence result in [BLJ17] whose proof relies on Nadel vanishing and properties of multiplier ideals.

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2. Preliminaries

2.1. Conventions. We work over an algebraically closed characteristic zero field \( k \). A variety will mean an integral separated scheme of finite type over \( k \). For a variety \( X \), a point \( x \in X \) will mean a scheme theoretic point. A geometric point \( \overline{x} \in X \) will mean a map from the spectrum of an algebraically closed field to \( X \).

A pair \((X, \Delta)\) is a composed of a normal variety \( X \) and an effective \( \mathbb{Q} \)-divisor \( \Delta \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. If \((X, \Delta)\) is a pair and \( f : Y \to X \) a proper birational morphism with \( Y \) normal, we write \( \Delta_Y \) for the \( \mathbb{Q} \)-divisor on \( X \) such that

\[ K_Y + \Delta_Y = f^*(K_X + \Delta). \]

Let \((X, \Delta)\) be a pair and \( f : Y \to X \) a log resolution of \((X, \Delta)\). The pair \((X, \Delta)\) is lc (resp., \(\varepsilon\)-lc) if \( \Delta_Y \) has coefficients \( \leq 1 \) (resp., \( \leq 1 - \varepsilon \)). The pair \((X, \Delta)\) is klt if \( \Delta_Y \) has coefficients \( < 1 \). A pair \((X, \Delta)\) is log Fano if \( X \) is projective, \(- (K_X + \Delta) \) is ample, and \((X, \Delta)\) is klt. A variety \( X \) is \( \mathbb{Q} \)-Fano if \((X, 0)\) is a log Fano pair.
2.2. **K-stability.** Let \((X, \Delta)\) be a pair such that \(-K_X - \Delta\) is ample. We refer the reader to [BHJ17] for the definition of K-semistability and uniform K-stability of \((X, \Delta)\) in terms of test configurations.\(^1\) In this article, we will use a characterization of K-semistability and uniform K-stability in terms of the stability threshold (see Theorem 4.8).

2.3. **Families of klt pairs.** A \(\mathbb{Q}\)-Gorenstein family of klt pairs \(\pi : (X, \Delta) \to T\) over a normal base will mean a flat surjective morphism of varieties \(\pi : X \to T\) and a \(\mathbb{Q}\)-divisor \(\Delta\) on \(X\) not containing any fibers satisfying:

1. \(T\) is normal and \(f\) has normal, connected fibers (hence, \(X\) is normal as well),
2. \(K_X/T + \Delta\) is \(\mathbb{Q}\)-Cartier, and
3. \((X_t, \Delta_t)\) is a klt pair for all \(t \in T\).

We briefly explain the definition of \(\Delta_t\) mentioned above (since \(\Delta\) is not necessarily \(\mathbb{Q}\)-Cartier, the definition of the pullback of \(\Delta\) to \(X_t\) may not be obvious). Let \(U \subseteq X\) denote the smooth locus of \(f\). The assumption that \(K_X/T + \Delta\) is \(\mathbb{Q}\)-Cartier implies \(\Delta\big|_U\) is \(\mathbb{Q}\)-Cartier on \(U\), while the assumption that \(X_t\) is normal implies \(\text{codim}(X_t, X_t \setminus (X_t \cap U)) \geq 2\). Hence, we may define \(\Delta_t\) as the unique \(\mathbb{Q}\)-divisor on \(X_t\) such that its restriction to \(X_t \cap U\) is the pullback of \(\Delta_t\) to \(X_t \cap U\).

2.4. **Valuations.** Let \(X\) be a variety. A valuation on \(X\) will mean a valuation \(v : K(X)^\times \to \mathbb{R}\) that is trivial on \(k\) and has center on \(X\). Recall, \(v\) has center on \(X\) if there exists a point \(\xi \in X\) such that \(v \geq 0\) on \(O_{X, \xi}\) and \(v > 0\) on the maximal ideal of \(O_{X, \xi}\). Since \(X\) is assumed to be separated, such a point \(\xi\) is unique, and we say \(v\) has center \(c_X(v) := \xi\). We use the convention that \(v(0) = +\infty\).

We write \(\text{Val}_X\) for the set valuations on \(X\), and \(\text{Val}_X^*\) for the set of non-trivial valuations. (The trivial valuation is the 0 map \(K(X)^\times \to \mathbb{R}\).) The set \(\text{Val}_X\) may be equipped with the topology of pointwise convergence as in [JM12, BldFFU15], but we will not use this additional structure.

To any valuation \(v \in \text{Val}_X\) and \(\lambda \in \mathbb{R}\) there is an associated valuation ideal \(a_\lambda(v)\) defined as follows. For an affine open subset \(U \subseteq X\), \(a_\lambda(v)(U) = \{f \in O_X(U) \mid v(f) \geq \lambda\}\) if \(c_X(v) \in U\) and \(a_\lambda(v)(U) = O_X(U)\) otherwise.

For an ideal \(a \subseteq O_X\) and \(v \in \text{Val}_X\), we set

\[
v(a) := \min\{v(f) \mid f \in a \cdot O_{X, c_X(v)}\} \in [0, +\infty].\]

We can also make sense of \(v(s)\) when \(L\) is a line bundle and \(s \in H^0(X, L)\). After trivializing \(L\) at \(c_X(v)\), we write \(v(s)\) for the value of the local function corresponding to \(s\) under this trivialization; this is independent of the choice of trivialization.

Similarly, we can define \(v(D)\) when \(D\) is an effective \(\mathbb{Q}\)-Cartier divisor on \(X\). Pick \(m \geq 1\) such that \(mD\) is Cartier and set \(v(D) = m^{-1}v(f)\), where \(f\) is a local equation of \(mD\) at the center of \(v\) on \(X\). Note that \(v(D) = m^{-1}v(O_X(-mD))\).

2.5. **Divisorial valuations.** If \(\pi : Y \to X\) is a proper birational morphism, with \(Y\) normal, and \(E \subseteq Y\) is a prime divisor (called a prime divisor over \(X\), then \(E\) defines a valuation \(\text{ord}_E : K(X)^\times \to \mathbb{Z}\) in \(\text{Val}_X\) given by the order of vanishing at the generic point of \(E\). Note that \(c_X(\text{ord}_E)\) is the generic point of \(\pi(E)\). Any valuation of the form \(v = c \cdot \text{ord}_E\) with \(c \in \mathbb{R}_{>0}\) will be called divisorial. We write \(\text{DivVal}_X \subset \text{Val}_X\) for the set of divisorial valuations.

2.6. **Graded sequences of ideals.** A graded sequence of ideals is a sequence \(a_\bullet = (a_p)_{p \in \mathbb{Z}_{\geq 0}}\) of ideals on \(X\) satisfying \(a_p \cdot a_q \subseteq a_{p+q}\) for all \(p, q \in \mathbb{Z}_{>0}\). We will always assume \(a_p \neq (0)\) for some \(p \in \mathbb{Z}_{>0}\). We write \(M(a_\bullet) := \{p \in \mathbb{Z}_{>0} \mid a_p \neq (0)\}\). By convention, \(a_0 := O_X\).

\(^1\)While these notions are defined for polarized pairs, we will always mean K-stability with respect to the anti (log) canonical polarization \(L = -K_X - \Delta\).
Let $a_\bullet$ be a graded graded sequence of ideals on $X$ and $v \in \text{Val}_X$. It follows from Fekete’s Lemma that the limit

$$v(a_\bullet) := \lim_{M(a_\bullet) \ni m \to \infty} \frac{v(a_m)}{m}$$

exists, and equals $\inf_m v(a_m)/m$; see [JM12].

The following statement concerns a type of graded sequence of ideals which will arise in §3.9.

**Proposition 2.1.** Let $a_1, \ldots, a_m$ be ideals on a variety $X$. For each $p \in \mathbb{N}$, set

$$b_p := \sum_b a_1^{b_1} \cdots a_m^{b_m},$$

where the sum runs through all $b = (b_1, \ldots, b_m) \in \mathbb{N}^m$ such that $\sum_{i=1}^m ib_i \geq p$. The following hold:

1. $b_\bullet$ is a graded sequence of ideals on $X$.
2. There exists $N$ such that $b_{Np} = b_{Np}^p$ is for all $p \in \mathbb{N}$.

Before proving the proposition, we state the following lemma.

**Lemma 2.2.** Let $R = k[X_1, \ldots, X_m]$ and $A$ be the graded sub-algebra of $R[T]$ where

$$A = \bigoplus_{p \in \mathbb{N}} A_p \subseteq \bigoplus_{p \in \mathbb{N}} RT^p = R[T],$$

and $A_p$ is generated over $A_0 = R$ by monomials $\{T^p X_1^{b_1} \cdots X_m^{b_m} | \sum_{i=1}^m ib_i \geq p\}$. There exists $N$ so that $A_{Np} = A_{Np}^p$ for all $p > 0$.

**Proof.** Note that $A$ is the coordinate ring of an affine toric variety. Indeed, fix a lattice $N \cong \mathbb{Z}^{m+1}$ with basis $e_0, e_1, \ldots, e_m$. Consider the cone $\sigma \subset M_R = N^\vee_R$ cut out by the equations

$$\sum_{i=1}^m ie_i \geq p, \quad e_0 \geq 0, \quad e_1 \geq 0, \quad \ldots, \quad e_m \geq 0.$$

Now, $A \cong k[\sigma \cap M]$ and is finitely generated by Gordon’s lemma. Since $A_0$ is a Noetherian ring and $A$ is finitely generated over $A_0$, there exists $N$ so that $A_{Np}^p = A_{Np}$ for all $n > 0$.

**Proof of Proposition 2.1.** Statement (1) is clear. To approach (2), we apply the previous lemma to find $N$ so that $A_{Np} = A_{Np}^p$ for all $p > 0$. We claim that $b_{Np} = b_{Np}^p$ for all $p > 0$. Indeed, $b_{Np}^p \subseteq b_{Np}$ by (1). To show the reverse inclusion, fix $p > 0$ and choose $b = (b_1, \ldots, b_m) \in \mathbb{N}^m$ such that $\sum ib_i \geq Np$. We will proceed to show $a^b := a_1^{b_1} \cdots a_m^{b_m} \subseteq b_{Np}^p$.

Since $A_{Np} = A_{Np}^p$, we may find $c^{(1)}, \ldots, c^{(p)} \in \mathbb{N}^m$ such that $\sum_{i=1}^m ic_i^{(j)} \geq N$ for $1 \leq j \leq p$ and

$$T^{Np} X^b \in \left( T^N X^{c^{(1)}} \right) \cdots \left( T^N X^{c^{(p)}} \right) R_0.$$

Hence, $b_i \geq \sum_{j=1}^p c_i^{(j)}$ for each $1 \leq i \leq m$.

Now, consider the ideal

$$a^{c^{(j)}} := a_1^{c_1^{(j)}} \cdots a_m^{c_m^{(j)}}.$$

For each $j$, the inequality $\sum_{i=1}^m ic_i^{(j)} \geq p$ implies $a^{c^{(j)}} \subseteq b_N$. Therefore,

$$a^{c^{(1)}} \cdots a^{c^{(p)}} \subseteq b_{Np}^p.$$

The inequality $b_i \geq \sum_{j=1}^p c_i^{(j)}$ implies

$$a^{b} \subseteq a^{c^{(1)}} \cdots a^{c^{(p)}}.$$

Since the latter is contained in $b_{Np}^p$, the proof is complete.
2.7. Log discrepancies. Let \((X, \Delta)\) be a pair. If \(\pi : Y \to X\) is a projective birational morphism with \(Y\) normal and \(E \subset Y\) a prime divisor, then the log discrepancy of \(\text{ord}_E\) with respect to \((X, \Delta)\) is defined by

\[
A_{X, \Delta}(\text{ord}_E) := 1 - (\text{coefficient of } E \text{ in } \Delta_Y).
\]

Following [JM12, BdFFU15], the function \(A_{X, \Delta} : \text{DivVal}_X \to \mathbb{R}\) may be extended to a lower semicontinuous function \(A_{X, \Delta} : \text{Val}_X \to \mathbb{R} \cup \{+\infty\}\). (See [Blu18, §3.2] for the setting of log pairs.)

We will frequently use the following facts: A pair \((X, \Delta)\) is klt iff \(A_{X, \Delta}(v) > 0\) for all \(v \in \text{Val}_X^+\). If \((X, \Delta)\) is a pair and \(D\) an effective \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(X\), then \(A_{X, \Delta + D}(v) = A_{X, \Delta}(v) - v(D)\) [Blu18, Proposition 3.2.4].

2.8. Log canonical thresholds. Let \((X, \Delta)\) be a klt variety. Given a nonzero ideal \(a \subseteq O_X\), the log canonical threshold of \(a\) is given by

\[
\text{lct}(X, \Delta; a) := \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{v(a)} = \inf_{v \in \text{Val}_X} \frac{A_{X, \Delta}(v)}{v(a)}.
\]

If \(f : Y \to X\) is a log resolution of \((X, \Delta, a)\), then the above infimum is achieved by a divisorial valuation \(\text{ord}_F\), where \(F\) is a divisor on \(Y\). If \(D\) is \(\mathbb{Q}\)-divisor on \(X\), then

\[
\text{lct}(X, \Delta; D) := \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{v(D)} = \inf_{v \in \text{Val}_X} \frac{A_{X, \Delta}(v)}{v(D)}
\]

and is equal to \(\sup\{c \in \mathbb{R}_{>0} \mid (X, \Delta + cD)\}\) is lc.

Let \(a_*\) be graded sequence of ideals on \(X\). Following [Blu18, §3.4] (which extends result of [JM12] to the setting of klt pairs), the log canonical threshold of \(a_*\) is given by

\[
\text{lct}(X, \Delta; a_*) := \lim_{M(a_*) \ni m \to \infty} m \cdot \text{lct}(X, \Delta; a_m) = \sup_{m \geq 1} m \cdot \text{lct}(X, \Delta; a_m).
\]

We have

\[
\text{lct}(X, \Delta; a_*) = \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{v(a_*)} = \inf_{v \in \text{Val}_X} \frac{A_{X, \Delta}(v)}{v(a_*)}
\]

by [Blu18, Propositions 3.4.3-3.4.4]. We say \(v^* \in \text{Val}_X\) computes \(\text{lct}(X, \Delta; a_*)\) if \(\text{lct}(a_*) = A(v^*)/v^*(a_*)\). Given a graded sequence \(a_*\), such a valuation always exists [JM12, Theorem A] [Blu18, Theorem 3.4.10].

**Lemma 2.3.** [Blu18, Lemma 3.4.9] If \(v \in \text{Val}_X^+\), then \(\text{lct}(X, \Delta; a_*(v)) \leq A_{X, \Delta}(v)\).

3. Filtrations

In this section, we recall information on filtrations of section rings. Much of the content appears in [BlJ17, §2] and relies on results in [BC11].

Throughout, let \(X\) be a normal projective variety of dimension \(n\) and \(L\) a big Cartier divisor on \(X\). Write

\[
R = R(X, L) = \bigoplus_{m \in \mathbb{N}} R_m = \bigoplus_{m \in \mathbb{N}} H^0(X, O_X(mL))
\]

for the section ring of \(L\). Set

\[
N_m := \dim H^0(X, O_X(mL)) \quad \text{and} \quad M(L) := \{m \in \mathbb{N} \mid H^0(X, O_X(mL)) \neq 0\}.
\]
3.1. Graded linear series. A graded linear series \( W_* = \{W_m\}_{m \in \mathbb{Z}_{>0}} \) of \( L \) is a collection of \( k \)-vector subspaces \( W_m \subseteq H^0(X, \mathcal{O}_X(mL)) \) such that

\[
R(W_*) := \bigoplus_{m \in \mathbb{N}} W_m \subseteq \bigoplus_{m \in \mathbb{N}} R_m
\]

is a graded sub-algebra of \( R(X, L) \). By convention, \( W_0 := H^0(X, \mathcal{O}_X) \).

A graded linear series \( W_* \) of \( L \) is said to be birational if for all \( m \gg 0 \), \( W_m \neq 0 \) and the rational map \( X \dashrightarrow \mathbb{P}(W^*_m) \) is birational onto its image. A graded linear series \( W_* \) of \( L \) is said to contain an ample series if \( W_m \neq 0 \) for all \( m \gg 0 \), and there exists a decomposition \( L = A + E \) where \( A, E \) are \( \mathbb{Q} \)-divisors with \( A \) ample and \( E \) effective such that

\[
H^0(X, \mathcal{O}_X(mA)) \subseteq W_m \subseteq H^0(X, \mathcal{O}_X(mL))
\]

for all \( m \) sufficiently large and divisible. See [LM09, §2.3] for further details.

**Example 3.1.** Fix a vector subspace \( V \subseteq H^0(X, \mathcal{O}_X(L)) \).

1. For each \( m > 0 \), set \( V_m := \text{im}(S^mV \to H^0(X, mL)) \). Then \( V_* \) is a graded linear series of \( L \) and \( R(V_*) \) is a finitely generated \( k \)-algebra. If the rational map \( X \dashrightarrow \mathbb{P}(V^*_m) \) is birational, then the graded linear series \( V_* \) is birational.

2. For each \( m > 0 \), set \( \bar{V}_m = H^0(X, mL \otimes \mathfrak{b}_m) \), where \( \mathfrak{b}_m \) denotes the integral closure of the \( m \)-th power of the base ideal of \( |V| \). Now, \( \bar{V}_* \) is a graded linear series of \( L \). If the rational map \( X \dashrightarrow \mathbb{P}(V^*) \) is birational, then the graded linear series \( \bar{V}_* \) is birational.

3.2. Volume of graded linear series. Let \( W_* \) be a graded linear series of \( L \). The Hilbert function of \( W_* \) is the function \( HF_{W_*} : \mathbb{N} \to \mathbb{N} \) defined by

\[
HF_{W_*}(m) = \dim(W_m).
\]

When \( V \subseteq H^0(X, \mathcal{O}_X(L)) \) is a linear series, we set \( HF_V := HF_{V_*} \), where \( V_* \) is the grade linear series defined in Example 3.1.1.

The volume of \( W_* \) is given by

\[
\text{vol}(W_*) := \limsup_{M(W_*) \ni m \to \infty} \frac{\dim W_m}{m^n/n!},
\]

where \( M(W_*) := \{m \in \mathbb{N} \mid \dim(W_m) \neq 0\} \). Equivalently, \( \text{vol}(W_*) = \limsup_{M(W_*) \ni m \to \infty} (HF_{W_*}(m))/(m^n/n!) \).

The previous limsup are in fact limits [LM09, KK12].

**Proposition 3.2.** Let \( V \subseteq H^0(X, \mathcal{O}_X(L)) \) be a nonzero vector subspace and \( \pi : Y \to X \) a proper birational morphism with \( Y \) normal such that \( b(|V|) : \mathcal{O}_Y = \mathcal{O}_Y(-E) \) with \( E \) a Cartier divisor on \( Y \). If the map \( X \dashrightarrow \mathbb{P}(V^*) \) is birational, then \( \text{vol}(V_*) = \text{vol}(\bar{V}_*) = (\pi^* L - E)^n \).

**Proof.** We first show \( \text{vol}(V_*) = (\pi^* L - E)^n \). Consider the rational map \( \varphi : X \dashrightarrow \mathbb{P}(V^*) \) and write \( Z \) for the closure of the image. The rational map extends to a morphism \( \tilde{\varphi} : Y \to \mathbb{P}(V^*) \) with the property \( \tilde{\varphi}^* \mathcal{O}_{\mathbb{P}(V^*)}(1) \simeq \pi^* L - E \). Since \( Z = \text{Proj}(R(V_*) \mathcal{O}_X) \) and \( \tilde{\varphi} \) is birational,

\[
\text{vol}(V_*) = \mathcal{O}_Z(1)^n = (\varphi^* \mathcal{O}_Z(1))^n = (\pi^* L - E)^n.
\]

We next show \( \text{vol}(\bar{V}_*) = (\pi^* L - E)^n \). Since \( \pi_* \mathcal{O}_Y(-mE) \subseteq \mathcal{O}_X \) is the integral closure of the \( m \)-power of \( b(|V|) \), \( \bar{V}_m \simeq H^0(Y, \mathcal{O}_Y(m(\pi^* L - E))) \). Hence, \( \text{vol}(\bar{V}_*) = \text{vol}(\pi^* L - E) \). Since \( \pi^* L - E \) is base point free and, hence, nef, \( \text{vol}(\pi^* L - E) = (\pi^* L - E)^n \). \( \square \)
3.3. Filtrations.

**Definition 3.3.** For \( m \in \mathbb{N} \), a filtration \( F \) of \( R_m \) we will mean a family of \( k \)-vector subspaces \( F^\bullet R_m = (F^\lambda R_m)_{\lambda \in \mathbb{R}} \) of \( R_m \) such that

1. \( F^\lambda R_m \subseteq F^\lambda' R_m \) when \( \lambda \geq \lambda' \);
2. \( F^\lambda R_m = \cap_{\lambda' < \lambda} F^\lambda' R_m \) for \( \lambda > 0 \);
3. \( F^0 R_m = R_m \) and \( F^\lambda R_m = 0 \) for \( \lambda \gg 0 \).

A filtration \( F \) of \( R \) is the data of a filtration \( F \) of \( R_m \) for each \( m \in \mathbb{N} \) such that

4. \( F^\lambda R_m \cdot F^\lambda' R_{m'} \subseteq F^{\lambda + \lambda'} R_{m+m'} \) for all \( m, m' \in \mathbb{N} \) and \( \lambda, \lambda' \in \mathbb{R}_{\geq 0} \).

A filtration \( F \) of \( R_m \) is trivial if \( F^\lambda R_m = 0 \) for all \( \lambda > 0 \). A filtration \( F \) of \( R \) is trivial if \( F^\bullet R_m \) is trivial for all \( m \in \mathbb{N} \).

3.4. Jumping numbers. Let \( F \) be a filtration of \( R_m \) where \( m \in M(L) \). The *jumping numbers* of \( F \) are given by

\[
0 \leq a_{m,1} \leq \cdots \leq a_{m,N_m} = mT_m(F)
\]

where

\[
a_{m,j} = a_{m,j}(F) = \inf \{ \lambda \in \mathbb{R}_{\geq 0} \mid \dim F^\lambda R_m \geq j \}
\]

for \( 1 \leq j \leq N_m \). The scaled average of the jumping numbers and the maximal jumping number are given by

\[
S_m(F) := \frac{1}{mN_m} \sum_{j=0}^{N_m} a_{m,j}(F) \quad \text{and} \quad T_m(F) := \frac{a_{m,N_m}}{m}.
\]

3.5. Induced graded linear series. Given a filtration \( F \) of \( R \), there is an induced family of graded linear series \( V^F_s \) indexed by \( s \in \mathbb{R}_{\geq 0} \) and defined by

\[
V^F_s := F^m s H^0(X, \mathcal{O}_X(mL)).
\]

To reduce notation, we will often write \( V^s \) for \( V^F_s \) when the choice of filtration is clear.

By unravelling our definitions, we see

\[
T_m(F) = \sup \{ s \in \mathbb{R}_{\geq 0} \mid V^F_s \neq 0 \}, \quad \text{and} \quad S_m(F) = \frac{1}{N_m} \int_0^{T_m(F)} \dim V^F_s ds
\]

for \( m \in M(L) \). Since property (F4) implies \( T_{m+1}(F) \geq \frac{m_1}{m_1+m_2} T_{m_1}(F) + \frac{m_2}{m_1+m_2} T_{m_2}(F) \), the limit

\[
T(F) := \lim_{M(L) \ni m \to \infty} T_m(F) \in [0, +\infty]
\]

exists by Fekete’s Lemma [JM12, Lemma 2.3] and equals \( \sup_{m \in M(L)} T_m(F) \). We say \( F \) is linearly bounded if \( T(F) < +\infty \).

The following two propositions are a consequence of [BC11, §1.3]. For the second proposition, see [BLJ17, Lemma 2.9] for the result stated in our terminology.

**Proposition 3.4.** Let \( F \) be a linearly bounded filtration of \( R \).

1. \( V^F_s \) contains an ample series for \( s \in [0, T(F)] \).
2. The function \( s \mapsto \text{vol}(V^F_s)^{1/n} \) is a decreasing concave function on \( [0, T(F)] \) and vanishes on \( (T(F), +\infty) \).

**Proposition 3.5.** For any linearly bounded filtration \( F \) of \( R \), we have

\[
\lim_{M(L) \ni m \to \infty} S_m(F) = \frac{1}{\text{vol}(L)} \int_0^{T(F)} \text{vol}(V^F_s) ds.
\]
Given the above proposition, we set $S(F) := \lim_{M(L) \ni m \to \infty} S_m(F)$. The following lemma follows easily from our definitions.

**Lemma 3.6.** Let $F$ be a linearly bounded filtration of $R$. We have:

1. $0 \leq S_m(F) \leq T_m(F)$ for all $m \in M(L)$.
2. $0 \leq S(F) \leq T(F)$.

We next consider a variant of $S_m(F)$ that is more asymptotic in nature. For $s \in [0, T(F))$ and $m \in M(L)$, consider the graded linear series $V^{F,s}_{m,k}$, where

$$\tilde{V}^{F,s}_{m,k} = H^0(X, \mathcal{O}_X(kmL) \otimes b(|V^{F,s}_m|)^k)$$

as in Example 3.1.2. We set

$$\tilde{S}_m(F) := \frac{1}{\text{vol}(L)} \int_0^{T(F)} \frac{\text{vol}(\tilde{V}^{F,s}_{m,k})}{m^n} ds.$$

**Proposition 3.7.** For any linearly bounded filtration $F$ of $R(X, L)$, we have

$$S(F) = \lim_{M(L) \ni m \to \infty} \tilde{S}_m(F).$$

**Proof.** We claim that for $s \in [0, T(F))$,

$$\text{vol}(V^{F,s}_m) = \lim_{m \to \infty} \frac{\text{vol}(\tilde{V}^{F,s}_{m,k})}{m^n}. \quad (3.1)$$

If we assume the claim and note that $\text{vol}(\tilde{V}^{F,s}_{m,k})/m^n \leq \text{vol}(L)$, we see that the proposition now follows from the dominated convergence theorem.

To prove the above claim, note that $V^{F,s}_m$ contains an ample series for $s \in [0, T(F))$ by Proposition 3.4.1. Now, we may apply [LM09, Theorem D] to see

$$\text{vol}(V^{F,s}_m) = \lim_{m \to \infty} \frac{\text{vol}(\tilde{V}^{F,s}_{m,k})}{m^n}, \quad (3.2)$$

where $V^{F,s}_{m,p} := \text{im}(S^nV^{F,s}_m \to R_{mp})$ as in Example 3.1.1. Combining (3.2) with Proposition 3.2 completes the claim. \qed

### 3.6. Filtrations induced by valuations.

Given $v \in \text{Val}_X$, we set

$$F^\lambda_v R_m = \{ s \in H^0(X, \mathcal{O}_X(mL)) \mid v(s) \geq \lambda \}$$

for each $\lambda \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{N}$. Equivalently, $F^\lambda_v R_m = H^0(X, \mathcal{O}_X(mL) \otimes a_\lambda(v))$. Note that $F_v$ is a filtration of $R$.

**Proposition 3.8.** [BlJ17, Lemma 3.1] Let $(X, \Delta)$ be a projective klt pair and $L$ a big Cartier divisor on $X$. If $v \in \text{Val}_X$ and $A_{X,\Delta}(v) < +\infty$, then the filtration $F_v$ of $R(X, L)$ is linearly bounded.

**Definition 3.9.** Let $v$ be a valuation on $X$ such that $F_v$ is a linearly bounded filtration of $R$.

1. The **maximal vanishing** (or pseudo-effective threshold) of $L$ along $v$ is $T(L; v) := T(F_v)$.
2. The **expected vanishing** of $L$ along $v$ is $S(L; v) := S(F_v)$.

When the choice of $L$ is clear, we simply write $T(v)$ and $S(v)$ for the $T(L; v)$ and $S(L; v)$. Similarly, we also write $T_m(v)$, $S_m(v)$, and $\tilde{S}_m(v)$ for $T_m(F_v)$, $S_m(F_v)$, and $\tilde{S}_m(F_v)$.

**Remark 3.10.** Let $\pi : Y \to X$ be a proper biration morphism with $Y$ normal. If $E$ is prime divisor on $Y$, then

$$S(L; \text{ord}_E) := \frac{1}{\text{vol}(L)} \int_0^\infty \text{vol}(\pi^*L - xE) dx$$
and 

\[ T(L; \text{ord}_E) := \sup \{ x \in \mathbb{R}_{>0} \mid \pi^*L - xE \text{ is pseudo-effective} \} . \]

**Proposition 3.11.** [BlJ17, Lemma 3.7] Let \( v \) be a valuation on \( X \) of linear growth.

1. For \( c \in \mathbb{R}_{>0} \), \( S(L; cv) = cS(L; v) \) and \( T(L; cv) = cT(L; v) \).
2. For \( m \in \mathbb{Z}_{>0} \), \( S(mL; v) = mS(L; v) \) and \( T(mL; v) = T(L; v) \).
3. If \( \pi : Y \to X \) is a projective birational morphism with \( Y \) normal, then \( S(\pi^*L; v) = S(L; v) \) and \( T(\pi^*L; v) = T(L; v) \).
4. If \( P \) is a Cartier divisor on \( X \) numerically equivalent to \( L \), then \( S(P; v) = S(L; v) \) and \( T(P; v) = T(L; v) \).

**Remark 3.12.** If \( L \) is a big \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \) and \( v \in \text{Val}_X \) is a valuation of linear growth, then we set \( S(L; v) := (1/m)S(mL; v) \), where \( m \in \mathbb{Z}_{>0} \) is chosen so that \( mL \) is a Cartier divisor.

By Proposition 3.11.2, \( S(L; v) \) is independent of the choice of \( m \).

### 3.7. \( \mathbb{N} \)-filtrations.

**Definition 3.13.** A filtration \( \mathcal{F} \) of \( R_m \) is an \( \mathbb{N} \)-filtration if all its jumping numbers are integers. Equivalently,

\[ \mathcal{F}^\lambda R_m = \mathcal{F}[^{\lambda}] R_m \]

for all \( \lambda \in \mathbb{R}_{\geq 0} \). An \( \mathbb{N} \)-filtration of \( R \) is a filtration of \( \mathcal{F} \) of \( R \) if \( \mathcal{F}^* R_m \) is an \( \mathbb{N} \)-filtration for each \( m \in \mathbb{N} \).

Note that an \( \mathbb{N} \)-filtration of \( R \) is equivalent to the data of subspaces \( (\mathcal{F}^\lambda R_m)_{m, \lambda \in \mathbb{N}} \) such that (F1), (F3), and (F4) of Definition 3.3 are satisfied. We say that an \( \mathbb{N} \)-filtration \( \mathcal{F} \) of \( R \) is **finitely generated** if the multigraded ring

\[ \bigoplus_{(\lambda, m) \in \mathbb{Z} \times \mathbb{N}} \mathcal{F}^\lambda R_m \]

is finitely generated over \( k \).

Any filtration \( \mathcal{F} \) of \( R \) induces an \( \mathbb{N} \)-filtration \( \mathcal{F}_\mathbb{N} \) defined by setting

\[ \mathcal{F}^\lambda _\mathbb{N} R_m := \mathcal{F}[^{\lambda}] R_m . \]

Indeed, conditions (F1)-(F3) are trivially satisfied for \( \mathcal{F}_\mathbb{N} \) and (F4) follows from the inequality \( \lceil \lambda_1 \rceil + \lceil \lambda_2 \rceil \geq \lceil \lambda_1 + \lambda_2 \rceil \).

**Proposition 3.14.** [BlJ17, Proposition 2.11] If \( \mathcal{F} \) is a filtration of \( R \) with linear growth, then

\[ T_m(\mathcal{F}_\mathbb{N}) = \lfloor m T_m(\mathcal{F}) \rfloor / m \quad \text{and} \quad S_m(\mathcal{F}) - 1/m \leq S_m(\mathcal{F}_\mathbb{N}) \leq S_m(\mathcal{F}) . \]

Hence, \( S(\mathcal{F}) = S(\mathcal{F}_\mathbb{N}) \) and \( T(\mathcal{F}) = T(\mathcal{F}_\mathbb{N}) \).

### 3.8. Base ideals of filtrations.

In this subsection, we assume \( L \) is ample. Recall that if \( V \subseteq H^0(X, \mathcal{O}_X(mL)) \) is a \( k \)-vector subspace, then the base ideal of \( |V| \) is given by

\[ b(|V|) := \text{im}(V \otimes_k \mathcal{O}_X(-mL) \to \mathcal{O}_X) , \]

where the previous map is given by multiplication of sections.

To a filtration \( \mathcal{F} \) of \( R \), we associate a graded sequence of base ideals. For \( \lambda \in \mathbb{R}_{\geq 0} \) and \( m \in M(L) \), set

\[ b_{\lambda, m}(\mathcal{F}) := b(|\mathcal{F}^\lambda H^0(X, \mathcal{O}_X(mL))|) . \]

**Lemma 3.15.** [BlJ17, Lemma 3.17 and Corollary 3.18] The sequence of ideals \( (b_{\lambda, m}(\mathcal{F}))_{m \in M(L)} \) has a unique maximal element, which we denote by \( b_\lambda(\mathcal{F}) \). Furthermore,

1. \( b_\lambda(\mathcal{F}) = b_{\lambda, m}(\mathcal{F}) \) for \( m \gg 0 \), and
2. \( b_\bullet(\mathcal{F}) = (b_p(\mathcal{F}))_{p \in \mathbb{N}} \) is a graded sequence of ideals.

We state some basic properties of these ideal sequences.
Lemma 3.16. [BlJ17, Lemma 3.19] If $v \in \text{Val}_X$, then $b_\lambda(F_v) = a_\lambda(v)$ for all $\lambda \in \mathbb{R}_{\geq 0}$.

Proposition 3.17. Let $v$ be a valuation on $X$ and $F$ a filtration of $R$. If $F_v$ and $F$ are both of linear growth, we have

$$S(v) \geq v(b_*(F))S(F) \quad \text{and} \quad T(v) \geq v(b_*(F))T(F).$$

In the case when $F$ is an $\mathbb{N}$-filtration,

$$S_m(v) \geq v(b_*(F))S_m(F) \quad \text{and} \quad T_m(v) \geq v(b_*(F))T_m(F)$$

for all $m \in M(L)$.

Proof. It is sufficient to prove the inequalities after replacing $v$ with a scalar multiple. Hence, we may consider the case when $v(b_*(F)) = 1$. Now, [BlJ17, Lemma 3.20] gives

$$F^p R_m \subseteq F^p R_m$$

for all $m \in M(L)$ and $p \in \mathbb{N}$. Therefore,

$$a_{p,m}(F_{\mathbb{N}}) \leq a_{p,m}(F_{v,\mathbb{N}})$$

for all $m \in M(L)$ and $0 \leq p \leq N_m$. The previous inequality combined with Proposition 5.2 gives

$$S_m(F_{\mathbb{N}}) \leq S_m(F_{v,\mathbb{N}}) \leq S_m(F_v) := S_m(v).$$

If $F = F_{\mathbb{N}}$ (which is the case when $F$ is an $\mathbb{N}$-filtration), we see $S_m(F) \leq S_m(F_v)$. More generally, Proposition 3.14 implies $S(F) \leq S(v)$. The inequalities for $T_m(F)$ and $T(F)$ follow from the same argument. \qed

3.9. Extending filtrations. In this subsection, we again assume $L$ is ample. Fix $m' \in M(L)$ and consider a $\mathbb{N}$-filtration $F$ of $R_{m'}$ such that $T_{m'}(F) < +\infty$. Set $r' := m'T_{m'}(F)$.

Definition 3.18. We write $\hat{F}$ for the $\mathbb{N}$-filtration of $R$ defined as follows:

(i) For $m < m'$,

$$\hat{F}^p R_m := \begin{cases} R_m & \text{for } p = 0 \\ 0 & \text{for } p > 0. \end{cases}$$

(ii) For $m = m'$,

$$\hat{F}^p R_m := F^p R_m \text{ for } p \geq 0.$$

(iii) For $m > m'$,

$$\hat{F}^p R_m := \sum_{b} \left( (F^1 R_{m'})^{b_1} \cdots (F^{r'} R_{m'})^{b_{r'}} \right) \cdot R_{m-m'} \sum_{b_i}$$

where the previous sum runs through all $b = (b_1, \ldots, b_{r'}) \in \mathbb{N}^{r'}$ such that $\sum_{i=1}^{r'} ib_i \geq p$ and $m \geq m' \sum_{i=1}^{r'} b_i$. \hspace{1cm} (3.4)

It is clear that $\hat{F}$ is a filtration of $R$. Furthermore, $\hat{F}$ is the minimal filtration of $R$ such that $\hat{F}$ and $F$ give the same filtration of $R_{m'}$.

Remark 3.19. The previous definition is related to the definition of $\chi^{(k)}$ in [Szé15, §3.2].

Lemma 3.20. The following hold:

1. $S_m(F) = S_{m'}(\hat{F})$.

2. Since $F$ is decreasing, taking the sum in (3.4) over all $b = (b_1, \ldots, b_{r'}) \in \mathbb{N}^{r'}$ such that $\sum_{i=1}^{r'} ib_i = p$ and $m \geq m' \sum_{i=1}^{r'} b_i$ yields the same filtration.
(2) For each $p \in \mathbb{N}$,
\[ b_p(\hat{\mathcal{F}}) = \sum_{b} a_i^{b_1} \cdots a_r^{b_r'} \]
where the sum runs through all $b = (b_1, \ldots, b_r') \in \mathbb{N}^{r'}$ such that $b_1 + 2b_2 + \cdots + r' b_{r'} \geq p$ and $a_i = b((\mathcal{F}^\mathcal{R} R_m))$ for each $1 \leq i \leq r'$.

Proof. Statement (1) follows immediately from the fact that $\mathcal{F}$ and $\hat{\mathcal{F}}$ give the same filtration on $R_m$. We now show (2). Taking base ideals of the left and right sides of (3.4) gives the inclusion “$\subseteq$”. For the reverse inclusion, fix $b = (b_1, \ldots, b_r') \in \mathbb{N}^{r'}$ such that $\sum_{i=1}^{r'} i \cdot b_i \geq p$. Choose $M \in \mathbb{N}$ so that $\mathcal{O}_X(mL)$ is globally generated for all $m \geq M$. Now, if $m \geq m'|b| + M$, then (3.4) gives
\[ a_1^{b_1} \cdots a_r^{b_r'} \subseteq b_{p,m}(\hat{\mathcal{F}}). \]
Since $b_p(\hat{\mathcal{F}}) = b_{p,m}(\hat{\mathcal{F}})$ for $m \gg 0$, we conclude $a_1^{b_1} \cdots a_r^{b_r'} \subseteq b_p(\hat{\mathcal{F}})$. \qed

4. Thresholds

Let $(X, \Delta)$ be a klt pair and $L$ a big Cartier divisor on $X$. Associated to $L$ are two thresholds that measure the singularities of members of $|mL|$ as $m \to \infty$.

4.1. The global log canonical threshold. For $m \in M(L)$, we set
\[ \alpha_m(X, \Delta; L) = \inf_{D \in |mL|} \text{let}(X, \Delta, D). \]
The global log canonical threshold of $L$ is
\[ \alpha(X, \Delta; L) = \inf_{m \in M(L)} \alpha_m(X, \Delta, L). \]
When the choice of pair $(X, \Delta)$ is clear, we will often write $\alpha(L)$ for the above threshold. As explained by Demailly in [CS08, Theorem A.3], the global log canonical threshold may be written as a generalization of the $\alpha$-invariant introduced by Tian.

As shown in [Amb16, BJ17], the global log canonical threshold may be expressed in terms of valuations. (See [Blu18] for the level of generality stated below.)

Proposition 4.1. For $m \in M(L)$,
\[ \alpha_m(X, \Delta, L) = \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{T_m(v)} = \inf_{v} \frac{A_{X, \Delta}(v)}{T(v)}, \]
where the second infimum runs through all valuations $v \in \text{Val}_X^\ast$ with $A_{X, \Delta}(v) < +\infty$.

Proposition 4.2. We have
\[ \alpha(X, \Delta, L) = \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{T(v)} = \inf_{v} \frac{A_{X, \Delta}(v)}{T(v)}, \]
where the second infimum runs through all valuations $v \in \text{Val}_X^\ast$ with $A_{X, \Delta}(v) < +\infty$.

4.2. The stability threshold. Given $m \in M(L)$, we say that $D \in |L|_Q$ is a $m$-basis type divisor of $L$ if there exists a basis $\{s_1, \ldots, s_{Nm}\}$ of $H^0(X, \mathcal{O}_X(mL))$ such that
\[ D = \frac{1}{mNm} (\{s_1 = 0\} + \cdots + \{s_{Nm} = 0\}). \]
Set
\[ \delta_m(X, \Delta; L) := \inf \{\text{let}(X, \Delta; D) | D \text{ is a } m\text{-basis type divisor of } L\}. \]
The stability threshold of $L$ is
\[ \delta(X, \Delta; L) := \limsup_{M(L) \ni m \to \infty} \delta_m(X, \Delta; L). \]
When the pair \((X, \Delta)\) is clear, we will simply write \(\delta(L)\) for \(\delta(X, \Delta; L)\).

The previous definition of stability threshold was introduced in [FO16] by K. Fujita and Y. Odaka in the log Fano case. The invariant was designed to characterize the K-stability of log Fano varieties in terms of singularities of anti-canonical divisors.

**Proposition 4.3.** [BlJ17, Proposition 4.3] For \(m \in M(L)\),

\[
\delta_m(X, \Delta : L) = \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{S_m(v)} = \inf_{v \in \text{Val}_X^*} \frac{A_{X, \Delta}(v)}{S(v)},
\]

where the second infimum runs through all valuations \(v \in \text{Val}_X^*\) with \(A_{X, \Delta}(v) < +\infty\).

**Theorem 4.4.** [BlJ17, Theorem C] We have

\[
\delta(X, \Delta : L) = \inf_{v \in \text{DivVal}_X} \frac{A_{X, \Delta}(v)}{S(v)} = \inf_{v \in \text{Val}_X^*} \frac{A_{X, \Delta}(v)}{S(v)},
\]

where the second infimum runs through all valuations \(v \in \text{Val}_X^*\) with \(A_{X, \Delta}(v) < +\infty\). Furthermore, the limit \(\lim_{M(L) \ni m \to \infty} \delta_m(X, \Delta; L)\) exists.

**Remark 4.5.** If we further assume that the base field \(k = \mathbb{C}\) and \(L\) is ample, there exists \(v^* \in \text{Val}_X^*\) with \(A_{X, \Delta}(v^*) < +\infty\) such that \(\delta(X, \Delta; L) = A_{X, \Delta}(v^*)/S(v^*)\) [BlJ17, Theorem E]. We will not use this result.

**Remark 4.6.** We can also make sense of \(\delta(X, \Delta; L)\) when \(L\) is a big \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor. In this case, we set

\[
\delta(X, \Delta; L) := r\delta(X, \Delta; rL),
\]

where \(r \in \mathbb{Z}_{>0}\) is chosen so that \(rL\) is a Cartier divisor. As a consequence of Theorem 4.4 and Proposition 3.11.2, \(\delta(X, \Delta; L)\) is independent of the choice of \(r\).

**Proposition 4.7.** [BlJ17, Theorem A] We have

\[
\alpha(X, \Delta; L) \leq \delta(X, \Delta; L) \leq (n + 1)\alpha(X, \Delta; L),
\]

where \(n = \text{dim}(X)\). Furthermore, when \(L\) is ample \(((n + 1)/n)\alpha(X, \Delta; L) \leq \delta(X, \Delta; L)\).

When \((X, \Delta)\) is a log Fano pair, we set

\[
\delta(X, \Delta) := \delta(X, \Delta; -K_X - \Delta).
\]

Using K. Fujita and C. Li’s valuative criterion for (log) K-stability [Fuj16b, Li17b], Theorem 4.4 implies

**Theorem 4.8.** [FO16, Theorem 0.3] [BlJ17, Theorem B] Let \((X, \Delta)\) be a log Fano pair.

1. \((X, \Delta)\) is K-semistable if \(\delta(X, \Delta) \geq 1\).
2. \((X, \Delta)\) is uniformly K-stable if \(\delta(X, \Delta) > 1\).

**Remark 4.9.** In [BlJ17], the previous statements were proved in the case when \(\Delta = 0\). The more general case follows from the same approach (see [Blu18, CP18]). Furthermore, in [BlJ17], all varieties are defined over \(\mathbb{C}\). While the uncountability of the base field is needed to prove [BlJ17, Theorem E] (Remark 4.5), the other main theorems hold over any algebraically closed characteristic zero field.
4.3. The stability threshold in terms of filtrations. We now proceed to interpret the stability threshold in terms of filtrations. We restrict ourselves to the case when $L$ is ample.

**Proposition 4.10.** If $(X, \Delta)$ is a projective klt pair and $L$ an ample Cartier divisor on $X$, then

$$\delta(X, \Delta; L) = \inf_F \frac{\text{lct}(X, \Delta; b_\cdot(F))}{S(F)},$$

where the infimum runs through all non-trivial linearly bounded filtrations of $R$

**Remark 4.11.** In [BoJ18], $\delta(X, \Delta; L)$ is expressed in terms of Radon probability measures on the Berkovich analytification of $X$. Such probability measures are closely related to filtrations of $R$.

**Proof.** The statement is an immediate consequence of the following lemma and Theorem 4.4. \qed

**Lemma 4.12.** Let $(X, \Delta)$ be a projective klt pair and $L$ an ample Cartier divisor on $X$.

1. If $v \in \text{Val}_X$ with $A_{X, \Delta}(v) < +\infty$, then

$$\frac{\text{lct}(X, \Delta; b_\cdot(F_v))}{S(F_v)} \leq \frac{A_{X, \Delta}(v)}{S(v)}.$$

2. If $F$ is a non-trivial filtration of $R(X, L)$ with $T(F) < +\infty$ and $w \in \text{Val}_X$ computes $\text{lct}(b_\cdot(F))$, then

$$\frac{A_{X, \Delta}(w)}{S(w)} \leq \frac{\text{lct}(X, \Delta; b_\cdot(F))}{S(F)}.$$

**Proof.** In order to prove (1), note that Lemmas 2.3 and 3.16 combine to show

$$\text{lct}(b_\cdot(F_v)) = \text{lct}(a_\cdot(v)) \leq A(v).$$

Since $S(v) := S(F_v)$, the desired inequality follows.

For (2), recall that $w$ computes $\text{lct}(b_\cdot(F))$ means $\text{lct}(b_\cdot(F)) = A(w)/v(b_\cdot(F))$. Combining the previous inequality with the inequality $S(v) \geq v(b_\cdot(F))S(F)$ (Proposition 3.17) completes the proof. \qed

Next, we introduce a variant on $\delta_m(X, \Delta; L)$, which is defined using filtrations of $R_m$ rather than bases for the vector space.

**Definition 4.13.** For $m \in M(L)$, set

$$\hat{\delta}_m(X, \Delta; L) = \inf_F \frac{\text{lct}(X, \Delta; b_\cdot(\hat{F}))}{S_m(\hat{F})},$$

where the infimum runs through all non-trivial $\mathbb{N}$-filtrations $F$ of $R_m$ with $T_m(\hat{F}) \leq 1$. (Recall, $\hat{F}$ is the extension of $F$ to a filtration of $R(X, L)$ as defined in §3.9.)

**Theorem 4.14.** If $(X, \Delta)$ is a projective klt pair and $L$ an ample Cartier divisor on $X$, then

$$\delta(X, \Delta; L) = \lim_{M(L) \ni m \to \infty} \hat{\delta}_m(X, \Delta; L).$$

To prove the above theorem, we will use the following statements.

**Lemma 4.15.** Keep the assumptions of Theorem 4.14. Fix $v \in \text{Val}_X$ with $A_{X, \Delta}(v) < +\infty$. For $m \in M(L)$, let $F_{v, m}$ denote the $\mathbb{N}$-filtration of $R_m$ given by $F_{v, m}^\lambda R_m := F_v^\lambda R_m$. The following hold:

1. $\text{lct}(X, \Delta; b_\cdot(\hat{F}_{v, m})) \leq A_{X, \Delta}(v)$ and
2. $S_m(v) - 1/m \leq S_m(F_{v, m}).$
Proof. To show (1), we first note that $b_p(\hat{F}_{v, m}) \subseteq a_p(v)$ for all $p \in \mathbb{N}$, since $b(|F^p R_m|) \subseteq a_p(v)$. Therefore, $\text{lct}(b_*(\hat{F}_{v, m})) \leq \text{lct}(a_*(v))$. Since $\text{lct}(a_*(v)) \leq A_{X, \Delta}(v)$ by Lemma 3.20.2, (1) is complete. Statement (2) follows from Proposition 3.14 and the fact that $S_m(F_{v, m}) = S_m(F_{v, N})$. □

Lemma 4.16. Keep the assumptions of Theorem 4.14. Fix $m \in M(L)$. If $F$ is a non-trivial $\mathbb{N}$-filtration of $R_m$ and $w$ computes $\text{lct}(X, \Delta; b_*(\hat{F}))$, then

$$\frac{A_{X, \Delta}(w)}{S_m(w)} \leq \frac{\text{lct}(X, \Delta; b_*(\hat{F}))}{S_m(F)}.$$\n
Proof. Since $w$ computes $\text{lct}(b_*(F))$, $\text{lct}(b_*(F)) = A_{X, \Delta}(w)/v(b_*(F))$. Combining the previous inequality with the inequality $S_m(v) \geq v(b_*(F)) S_m(F)$ (Proposition 3.17) completes the proof. □

Proposition 4.17. Keep the assumptions of Theorem 4.14. For $m \in M(L)$,

$$\frac{1}{\delta_m(X, \Delta; L)} - \frac{1}{m \cdot A_{X, \Delta}(v)} \leq \frac{1}{\delta_m(X, \Delta; L)} \leq \frac{1}{\delta_m(X, \Delta; L)}.$$

Proof. We begin by showing the first inequality. Fix $\varepsilon > 0$ and choose $v \in \text{Val}^*_X$ such that $A_{X, \Delta}(v) < +\infty$ and $A_{X, \Delta}(v)/S(v) < \delta(X, \Delta; L) + \varepsilon$. After replacing $v$ with a scalar multiple, we may assume $T(v) = 1$. Let $F_{v, m}$ denote the $\mathbb{N}$-filtration of $R_m$ as defined in Lemma 4.15. Note that the assumption $T(v) = 1$ implies $T(F_{v, m}) \leq 1$. Therefore,

$$\frac{1}{\delta_m(X, \Delta; L)} \geq \frac{S_m(F_{v, m})}{\text{lct}(X, \Delta; b_*(\hat{F}_{v, m}))} \geq \frac{S_m(v) - 1/m}{A_{X, \Delta}(v)}$$

by the Lemma 4.15. Now, our choice of $v$ implies

$$\geq \frac{1}{\delta_m(X, \Delta; L) + \varepsilon} - \frac{1}{m \cdot A_{X, \Delta}(v)}$$

and the inequality $\alpha(X, \Delta; L) \leq A_{X, \Delta}(v)/T(v)$ combined with $T(v) = 1$ gives

$$\geq \frac{1}{\delta_m(X, \Delta; L) + \varepsilon} - \frac{1}{m \cdot \alpha(X, \Delta; L)}.$$\n
Sending $\varepsilon \to 0$ completes the first inequality.

We move on to the second inequality. Let $F$ be a nontrivial $\mathbb{N}$-filtration of $R_m$ satisfying $T_m(F) \leq 1$. After choosing $v \in \text{Val}^*_X$ computing $\text{lct}(b_*(\hat{F}))$, we apply Lemma 4.16 to see

$$\frac{\text{lct}(b_*(\hat{F}))}{S_m(F)} \geq \frac{A_{X, \Delta}(v)}{S_m(v)} \geq \delta_m(X, \Delta; L),$$

where the last inequality follows from Proposition 4.17. Hence, $\hat{\delta}_m(X, \Delta; L) \geq \delta_m(X, \Delta, L)$ and the proof is complete. □

Proof of Theorem 4.14. The statement follows immediately from combining Theorem 4.4 with Proposition 4.17. □

5. Convergence results

The goal of this section is to prove the following results.

Theorem 5.1. Let $\pi : (X, \Delta) \to T$ be a projective $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base and $L$ a $\pi$-ample Cartier divisor on $X$. For each $\varepsilon > 0$, there exists a positive integer $M = M(\varepsilon)$ such that

$$0 \leq \alpha_m(X_\tau, \Delta_\tau; L_\tau) - \alpha(X_\tau, \Delta_\tau; L_\tau) \leq \varepsilon$$
for all positive integers \( m \geq M \) and \( t \in T \).

**Theorem 5.2.** Let \( \pi: (X, \Delta) \to T \) be a projective \( \mathbb{Q} \)-Gorenstein family of klt pairs over a normal base and \( L \) a \( \pi \)-ample Cartier divisor on \( X \). For each \( \varepsilon > 0 \), there exists a positive integer \( M = M(\varepsilon) \) such that

\[
\delta_m(X_T, \Delta_T; L_T) - \delta(X_T, \Delta_T; L_T) \leq \varepsilon
\]

for all positive integers \( m \) divisible by \( M \) and \( t \in T \).

While both results are deduced from statements in [BLJ17], the proof of the latter result is significantly more involved.

5.1. Bounding the global log canonical threshold in families. In this section we prove a boundedness statement for the global log canonical threshold in bounded families. The result is well known (for example, see [Oda13b, Proposition 2.4] for a related statement).

**Proposition 5.3.** Let \( \pi: (X, \Delta) \to T \) be a projective \( \mathbb{Q} \)-Gorenstein family of klt pairs over a normal base and \( L \) is a \( \pi \)-ample Cartier divisor on \( X \). There exist constants \( c_1, c_2 > 0 \) so that

\[
c_1 < \alpha(X_T, \Delta_T; L_T) < c_2
\]

for all \( t \in T \).

To prove the result, we will need the following statements.

**Lemma 5.4.** Let \( (X, \Delta) \) be a projective klt pair that is \( \varepsilon \)-log canonical and \( L \) an ample Cartier divisor on \( X \). If \( f: Y \to X \) a log resolution of \( (X, \Delta) \), then:

1. \( \varepsilon \cdot A_Y(v) \leq A_X(v) \) for all \( v \in \text{Val}_X \), and
2. \( \varepsilon \cdot \alpha(Y, 0; f^*L) \leq \alpha(X, \Delta; L) \).

**Proof.** For statement (1), we recall an argument in [BHJ17, Proof of Theorem 9.14]. Since \( (X, \Delta) \) is \( \varepsilon \)-lc, \( \Delta_Y \leq (1 - \varepsilon)\Delta_{Y, \text{red}} \). Now, \( \operatorname{lct}(Y, 0; \Delta_{Y, \text{red}}) = 1 \), since \( \operatorname{Supp}(\Delta_Y) \) is snc. Thus, if \( v \in \text{Val}_X \), then \( v(\Delta_{Y, \text{red}}) \leq A_{Y, 0}(v) \). Therefore,

\[
v(\Delta_Y) \leq (1 - \varepsilon)v(\Delta_{Y, \text{red}}) \leq (1 - \varepsilon)A_{Y, 0}(v),
\]

and we see

\[
A_{X, \Delta}(v) = A_{Y, 0}(v) - v(\Delta_Y) \geq \varepsilon A_{Y, 0}(v).
\]

Statement (2) follows from statement (1) combined with Propositions 3.11.3 and 4.2. \( \square \)

**Lemma 5.5.** Let \( X \) be a smooth variety of dimension \( n \) and \( L \) an ample line Cartier divisor on \( X \). If \( A \) is very ample Cartier divisor on \( X \), then \( \alpha(X, 0; L) \geq 1/(L \cdot A^n) \).

**Proof.** The statement follows from [Vie95, Corollary 5.11]. \( \square \)

**Proof of Proposition 5.3.** We will show that there exists a dense open set \( U \subseteq T \) and constants \( c_1, c_2 > 0 \) so that \( c_1 < \alpha(X_t, \Delta_t; L_t) < c_2 \) for all \( t \in U \). By induction on the dimension of \( T \), the proposition will follow.

Let \( f: Y \to X \) be a projective log resolution of \( (X, \Delta) \) and write

\[
K_{Y/T} + \Delta_Y = g^*(K_{X/T} + \Delta).
\]

Choose a dense open set \( U \subseteq T \) such that \( U \) is smooth and affine, \( Y \to X \) is smooth over \( U \), and \( \operatorname{Exc}(\pi) + \Delta \) has relative simple normal crossing over \( U \). Thus, \( Y_t \to X_t \) is a log resolution of \( (X_t, \Delta_t) \) for all \( t \in U \). Since the fibers of \( (X, \Delta) \) along \( \pi \) are klt, we may find \( 0 < \varepsilon \ll 1 \) so that \( \Delta_Y|_U \) has coefficients \( \leq 1 - \varepsilon \). Hence, \( (X_t, \Delta_t) \) is \( \varepsilon \)-lc for all \( t \in U \).

Now, since \( Y_U \to U \) is projective and \( U \) is affine, there exists a Cartier divisor \( A \) on \( Y_U \) that is very ample over \( U \). Replacing \( A \) with a high enough power, we may assume \( \pi^*L + A \) is very ample over \( U \) as well. Now,

\[
\alpha(X_t, \Delta_t; L_t) \geq \varepsilon \cdot \alpha(Y_t, 0, f^*L_t) \geq \alpha(Y, 0, f^*L_t + A_t) \geq 1/(f^*L_t + A_t) \cdot A_t^{n-1}
\]
Since $Y_U \to U$ is smooth, and, hence, flat, $U \ni t \mapsto 1/(f^*L_t + A_t) \cdot A_t^{p-1}$ is constant. Hence, we may find $c_1 > 0$ so that $\alpha(X, \Delta, L_t) > c_1$ for all $t \in U$.

We move onto finding an upper bound. Since $L$ is $\pi$-ample and $U$ is affine, there exists a divisor $\Gamma \in \mathbb{Z}_{>0}$ such that $\Gamma$ does not contain a fiber. Now, $t \mapsto \text{lct}(X, \Delta, \Gamma) \leq m/L$ is finite. Hence, we may find $c_2$ so that $\alpha(X, \Delta, L_t) < c_2$ for all $t \in U$. □

### 5.2 A finiteness result for Hilbert functions.

**Theorem 5.6.** Let $\pi : X \to T$ be a flat projective family of varieties and $L$ a $\pi$-ample Cartier divisor on $X$ such that $R^i \pi_* (O_X(mL)) = 0$ for all $i, m \geq 1$. Then, the set of functions

$$\bigcup_{t \in T} \{ HF_W : \mathbb{N} \to \mathbb{N} \mid W \leq h^0(X, \mathcal{O}_{X,T}(L_t)) \},$$

is finite.

The theorem is a consequence of the following proposition and the use of the Grassmanian to parametrize the set of linear series in question.

**Proposition 5.7.** Keep the setup of Theorem 5.6 and fix a sub $\mathcal{O}_T$-module $W \subseteq \pi_* \mathcal{O}_X(L)$. For each geometric point $\overline{t} \in T$, set

$$W_{\overline{t}} := \text{im}(W \otimes k(\overline{t}) \to \pi_* \mathcal{O}_{X,T}(L_t) \otimes k(\overline{t}) \simeq h^0(X, \mathcal{O}_{X,T}(mL_\overline{t})).$$

Then, the set of functions

$$\{ HF_{W_{\overline{t}}} : \mathbb{N} \to \mathbb{N} \mid \overline{t} \in T \}$$

is finite.

**Proof.** We prove the statement by induction on the dimensions of $T$. If $\dim(T) = 0$, the statement is trivial. Next, assume $\dim(T) > 0$. We show that there is a dense open set $U \subseteq T$ such that $HF_{W_\overline{t}}$ is independent of geometric point $\overline{t} \in U$.

Let $R = \bigoplus_{m \geq 0} R_m$ denote the graded $\mathcal{O}_T$-algebra where $R_0 = \mathcal{O}_T$ and $R_m = \pi_* \mathcal{O}_X(mL)$ for $m > 0$. Note that our assumption on the vanishing of higher cohomology implies $\pi_* \mathcal{O}_X(mL)$ is a vector bundle and commutes with base change for all $m \geq 1$.

Viewing $W$ as a subset of $R$, let $J \subseteq R$ denote the homogeneous ideal generated by $W$. Note that

$$J^m \cap R_m = \text{im}(S^m(W) \to \pi_* \mathcal{O}_X(mL)).$$

Hence, for each $\overline{t} \in T$ and $m > 0$, we have

$$(W_{\overline{t}})_m := \text{im}((J^m \cap R_m) \otimes \mathcal{O}_{\overline{t}} k(\overline{t}) \to R_m \otimes k(\overline{t}) \simeq h^0(X, \mathcal{O}_{X,T}(mL_\overline{t}))).$$

(5.1)

Now, consider the graded $\mathcal{O}_T$-algebra given by $\text{gr}_J R := J^m/J^{m+1}$. Since $\text{gr}_J R$ is a finitely generated $\mathcal{O}_T$-algebra, we may apply generic smoothness to find a dense open set $U \subseteq T$ so that $\text{gr}_J R_{\mid U} = \text{flat over } U$. Applying the following lemma gives that $(J^m \cap R_m)_{\mid U}$ is flat over $U$ and the natural map

$$(J^m \cap R_m) \otimes k(\overline{t}) \to R_m \otimes k(\overline{t}) \simeq h^0(X, \mathcal{O}_{X,T}(mL_\overline{t}))$$

(5.2)

is injective for all $m \geq 0$ and $t \in U$. Since $(J^m \cap R_m)_{\mid U}$ is flat over $U$, $U \ni \overline{t} \mapsto \text{im}((J^m \cap R_m) \otimes k(\overline{t}))$ is constant. Therefore, (5.1) and (5.2) combine to show $\dim((J^m \cap R_m) \otimes k(\overline{t})) = \dim(W_{\overline{t}}^m)$, and we conclude $HF_{W_{\overline{t}}}$ is independent of $\overline{t} \in U$.

To finish the proof, note that $\{ HF_{W_{\overline{t}}} \mid \overline{t} \in T \}\setminus U$ is finite by our inductive hypothesis. Combining this with the statement that $HF_{W_{\overline{t}}}$ is independent of $\overline{t} \in U$ completes the proof. □
Lemma 5.8. Let $A \to B$ be a flat morphism of Noetherian rings, $I \subseteq B$ an ideal and $M$ an $A$-module. If the graded ring $\text{gr}_B B = \bigoplus_{m \geq 0} I^m/I^{m+1}$ is flat over $A$, then for each $m \geq 0$

(1) $I^m$, viewed as a $B$-module, is flat over $A$ and
(2) the natural map $I^m \otimes_A M \to B \otimes_A M$ is injective.

Proof. The statement is trivial for $m = 0$. Now, consider the short exact sequence

$$0 \to I^{m+1} \to I^m \to I^m/I^{m+1} \to 0,$$

and assume the statement holds for $I^m$. Since the latter two terms of (5.3) are flat over $B$, so is $I^{m+1}$. By the flatness of $I^m/I^{m+1}$, (5.3) remains exact after applying $\otimes_A M$ and we have

$$0 \to I^{m+1} \otimes_A M \to I^m \otimes_A M \to I^m/I^{m+1} \otimes_A M \to 0$$

Thus, the injectivity of $I^m \otimes M \to B \otimes M$ implies the injectivity of $I^{m+1} \otimes M \to B \otimes M$. □

We are now ready to prove Theorem 5.6.

Proof of Theorem 5.6. It suffices to show that the set

$$\bigcup_{t \in T} \{ HF_W : N \to N | W \subseteq H^0(X_T, O_{X_T}(L_T)) \text{ with } \dim(W) = r \}$$

is finite for each $r \leq \text{rank } \pi_* O_X(L)$. Hence, we consider the Grassmanian $\rho : \text{Gr}(r, \pi_* O_X(L)) \to T$ parameterizing rank $r$ sub vector bundles of $\pi_* O_X(L)$. Note that there is a correspondence between $k(\mathfrak{t})$-valued points of $\text{Gr}(r, \pi_* O_X(L))$ and rank $r$ subspaces $W \subseteq H^0(X_T, L_T).

Set $X' := \text{Gr}(r, \pi_* O_X(L)) \times_T X$ and let $\pi'$ and $\rho'$ denote the projection maps to $\text{Gr}(r, \pi_* O_X(L))$ and $X$. Write $W_u \subseteq \rho^* (\pi_* O_X(L))$ for the universal sub-bundle of the Grassmanian. For a geometric point $\sigma \in \text{Gr}(r, \pi_* O_X(L))$, set

$$W_\sigma := \text{im} (W_u \otimes k(\sigma) \to \rho^* (\pi_* O_X(L)) \otimes k(\sigma) \simeq H^0(X_{\sigma}, O_{X_{\sigma}}(L_{\sigma}))).$$

To complete the proof, it is sufficient to show that

$$\{ HF_{W_\sigma} : N \to N | \sigma \in \text{Gr}(r, \pi_* O_X(L)) \}$$

is finite.

Set $L' = \rho^* L$. By flat base change, $R^i \pi'_* O_{X'}(mL') \simeq \rho^* R^i \pi_* O_X(mL)$ for all $i, m \geq 0$. Hence, our assumption that $R^i \pi'_* O_{X'}(mL) = 0$ for all $i, m \geq 1$ implies $R^i \pi_* O_X(mL) = 0$ for all $i, m \geq 1$. Additionally, since $\pi'_* O_{X'}(L') \simeq \rho^* \pi_* O_X(L)$, we may view $W$ as a sub vector bundle of $\pi'_* O_{X'}(L')$. Therefore, we may apply Proposition 5.7 to see that (5.4) is a finite set. □

The following corollary will be used in the proof of Theorem 5.13.

Corollary 5.9. Let $\pi : X \to T$ be a flat projective family of varieties and $L$ a $\pi$-ample Cartier divisor on $X$ such that $R^i \pi_* (O_X(mL)) = 0$ for all $i, m \geq 1$. Given any $\varepsilon > 0$, there exists $M = M(\varepsilon)$ so that the following holds: If $t \in T$ and $V \subseteq H^0(X_T, O_{X_T}(L_T))$, then

$$\left| \frac{\text{vol}(V_\sigma)}{\text{vol}(L_T)} - \frac{\text{dim}(V_m)}{h^0(O_{X_T}(mL_T))} \right| < \varepsilon$$

for all $m \geq M$. (The integer $M$ is independent of the choice of $t$ and $V$.)

Proof. Given any $t \in T$ and $V \subseteq H^0(X_T, O_{X_T}(L_T))$,

$$\lim_{m \to \infty} \left( \frac{\text{dim} HF_V(m)}{h^0(X_T, O_{X_T}(mL_T))} \right) = \frac{\text{vol}(V_\sigma)}{\text{vol}(L_T)},$$

since

$$\lim_{m \to \infty} \frac{\text{dim} HF_V(m)}{m^n/n!} = \text{vol}(V_\sigma) \text{ and } \lim_{m \to \infty} \frac{h^0(X_T, O_{X_T}(mL_T))}{m^n/n!} = \text{vol}(L_T).$$
Now, the result follows from the fact that the set of functions
\[ \bigcup_{t \in T} \{ HF_V(m) : \mathbb{N} \to \mathbb{N} | V \subseteq H^0(X, \mathcal{O}_X(mL_t)) \} \]
is finite by Theorem 5.6 and \( h^0(X, \mathcal{O}_X(mL_t)) \) is independent of \( t \) by our assumption that \( R^i \pi_* (\mathcal{O}_X((mL)) = 0 \) for all \( i, m \geq 1 \).

5.3. Approximations of \( S \) and \( T \).

**Theorem 5.10** ([BIL17]). Let \((X, \Delta)\) be a projective klt pair and \( L \) an ample Cartier divisor on \( X \). There exists a positive constant \( C \) such that
\[
0 \leq T(L; v) - T_m(L; v) \leq \frac{C A_{X, \Delta}(v)}{m} \quad \text{and} \quad 0 \leq S(L; v) - \tilde{S}_m(L; v) \leq \frac{C \cdot A_{X, \Delta}(v)}{m}
\]
for all \( m \in \mathbb{Z}_{>0} \) and \( v \in \text{Val}_X^\bigstar \) with \( A_{X, \Delta}(v) < \infty \).

Furthermore, fix \( r \in \mathbb{N} \) such that \( r(K_X + B) \) is a Cartier divisor. If \( b, c \in \mathbb{Z}_{>0} \) are chosen so that
1. \( \mathcal{O}_X(cL) \) is globally generated,
2. \( bL - K_X - \Delta \) is big and nef, and
3. \( H^0(X, \mathcal{O}_X((c + nb)L) \otimes \text{Jac}_X \mathcal{O}_X(-r\Delta)) \neq 0 \),
then the result holds with \( C = 1 + (c + \dim(X)b)/\alpha(X, \Delta; L) \).

See [BIL18, §5.4.3] for the precise result listed above.

**Lemma 5.11.** Let \( \pi : (X, \Delta) \to T \) be a projective \( \mathbb{Q} \)-Gorenstein family of klt pairs over a normal base and \( L \) a \( \pi \)-ample Cartier divisor on \( X \). Given \( r \in \mathbb{N} \), there exists a positive integer \( m_0 = m_0(r) \) so that
\[
H^0(X, \mathcal{O}_X(mL) \otimes \text{Jac}_X \mathcal{O}_X(-r\Delta)) \neq 0
\]
for all \( m \geq m_0 \) and \( t \in T \).

**Proof.** Since \( L \) is \( \pi \)-ample, there exists \( m_0 \) so that
\[
\pi^* \pi_* (\mathcal{O}_X(mL) \otimes \text{Jac}_X \mathcal{O}_X(-r\Delta)) \to \mathcal{O}_X(mL) \otimes \text{Jac}_X \mathcal{O}_X(-r\Delta)
\]
is surjective for all \( m \geq m_0 \). Hence,
\[
H^0(X, \mathcal{O}_X(mL) \otimes \text{Jac}_X \mathcal{O}_X(-r\Delta)|_{X_T}) \neq 0
\]
for all \( m \geq m_0 \) and \( t \in T \). Since \( \mathcal{O}_X(-r\Delta) \cdot \mathcal{O}_X \subseteq \mathcal{O}_X(-r\Delta_T) \) and \( \text{Jac}_{X/T} \mathcal{O}_X = \text{Jac}_X \), the result follows.

**Proposition 5.12.** Let \( \pi : (X, B) \to T \) be a projective \( \mathbb{Q} \)-Gorenstein family of klt pairs over a normal base and \( L \) a \( \pi \)-ample Cartier divisor on \( X \). There exists a positive constant \( C \) such that the following holds:

For each \( t \in T \),

(i) \( 0 \leq T(L_T; v) - T_m(L_T; v) \leq \frac{C A_{X_T, \Delta_T}(v)}{m} \), and

(ii) \( 0 \leq S(L_T; v) - \tilde{S}_m(L_T; v) \leq \frac{C A_{X_T, \Delta_T}(v)}{m} \)

for all \( m \in \mathbb{Z}_{>0} \) and \( v \in \text{Val}_{X_T}^\bigstar \) with \( A_{X_T, \Delta_T}(v) < \infty \). (The value \( C \) is independent of the choice of \( t, v, \) and \( m \).)

**Proof.** We seek to find positive integers \( b, c \) so that (i)-(iii) of Theorem 5.10 are satisfied for each \( t \in T \). First, fix \( r \in \mathbb{N} \) so that \( r(K_X/T + \Delta) \) is a Cartier divisor and apply Lemma 5.11 to find \( m_0 \) so that
\[
H^0(X, \mathcal{O}_X(mL) \otimes (\text{Jac}_X \mathcal{O}_X(-r\Delta_T)) \neq 0
\]
for all \( m \geq m_0 \) and \( t \in T \). Since \( L \) is \( \pi \)-ample, we may find \( b, c \in \mathbb{Z}_{>0} \) so that \( cL_T \) and \( bL_T - K_{X_T} - \Delta_T \) are very ample for all \( t \in T \) and \( c + nb \geq m_0 \), where \( n := \dim(X) - \dim(T) \).

With the above choices, we have:

(i) \( \mathcal{O}_{X_T}(cL_T) \) is globally generated,

(ii) \( bL_T - K_{X_T} - \Delta_T \) is big and nef,

(iii) \( H^0(X_T, \mathcal{O}_{X_T}((c + nb)L_T) \otimes (\text{Jac}_{X_T} \cdot \mathcal{O}_{X_T}(-r\Delta_T))) \neq 0 \),

for all \( t \in T \). Next, set \( \gamma := \inf_{t \in T} \alpha(X_T, \Delta_T; L_T) \), which is \( > 0 \) by Proposition 5.3. Theorem 5.10 now implies that the desired inequalities will hold with \( C := (c + nb)/\gamma \).

We seek to apply Proposition 5.12 to prove

**Theorem 5.13.** Let \( \pi : (X, \Delta) \to T \) be a \( \mathbb{Q} \)-Gorenstein of klt pairs and \( L \) a \( \pi \)-ample Cartier divisor on \( X \). Given \( \epsilon > 0 \), there exists a positive integer \( N := N(\epsilon) \) such that the following holds: For each \( t \in T \),

\[
S(L_T; v) - S_{mN}(L_T; v) \leq \epsilon \Delta_{X_T}(v)
\]

for all \( m \geq 1 \) and \( v \in \text{Val}_{X_T} \) with \( \Delta_{X_T}(v) < \infty \). (The integer \( N \) is independent of \( t \), \( m \), and \( v \).)

Before proving the theorem, we need the following statements.

**Proposition 5.14.** Keep the hypotheses of Theorem 5.13. There exists a positive integer \( D \) so that the following holds: If \( t \in T \) and \( v \in \text{Val}_{X_T} \) with \( \Delta_{X_T}(v) < \infty \), then

\[
|V^s_m| \text{ is birational when } m \geq 1 \text{ and } 0 \leq s \leq T(v) - \frac{DA_{X_T}(v)}{m},
\]

where \( V^s_m \) is abbreviated notation for the linear series \( F_v^{ms}H^0(X_T, \mathcal{O}_{X_T}(mL_T)) \).

The proof relies on Proposition 5.12 and an argument from the proof of [BC11, Lemma 1.6].

**Proof.** Fix a positive constant \( C \) satisfying the conclusion of Proposition 5.12, and set \( \gamma \) equal to the minimum of \( \inf_{t \in T} \alpha(X_T, \Delta_T; L_T) \) and \( 1 \), which is \( > 0 \) by Proposition 5.3. Since \( L \) is \( \pi \)-ample, we may find \( a \in \mathbb{Z}_{>0} \) so that \( \mathcal{O}_{X_T}(aL_T) \) is very ample for all \( t \in T \).

With the previous choices, set \( D := C + a/\gamma \). Now, fix \( t \in T \) and \( v \in \text{Val}_{X_T} \) with \( A(v) < +\infty \). To shorten notation, set

\[
V^s_m := F_v^{ms}H^0(X_T, \mathcal{O}_{X_T}(mL_T)).
\]

We will proceed to show \( |V^s_m| \) is birational when \( 0 \leq s \leq T(v) - \frac{DA(v)}{m} \).

For \( m \leq a \), the statement is vacuous. Indeed, since \( \gamma \leq \alpha(X_T, \Delta_T; L_T) \leq A(v)/T(v) \), we see \( T(v) - DA(v)/m < 0 \) when \( m \leq a \). For \( m > a \), consider the inclusion:

\[
V^0_a \cdot V^{sm/(m-a)}_{m-a} \subseteq V^s_m.
\]

Note that \( |V^0_a| \) is birational, since \( V^0_a = H^0(X_T, \mathcal{O}_{X_T}(aL_T)) \) and \( aL_T \) is very ample. Therefore, inclusion (5.5) implies \( V^s_m \) is birational as long as \( V^{sm/(m-a)}_{m-a} \) is nonzero, which is equivalent to the condition \( sm/(m-a) \leq T_{m-a}(v) \). Therefore, it is sufficient to show

\[
s \left( \frac{m}{m-a} \right) \leq T_{m-a}(v) \quad \text{whenever } m > a \text{ and } 0 \leq s \leq T(v) - \frac{DA(v)}{m}.
\]

Now, if \( s \leq T(v) - DA(v)/m \), then

\[
s \left( \frac{m}{m-a} \right) \leq \left( T(v) - \frac{DA(v)}{m} \right) \left( \frac{m}{m-a} \right) = T(v) + \left( \frac{a}{m-a} \right) T(v) - \frac{(a/\gamma + C)A(v)}{m-a}
\]
Now, the inequality $\gamma \leq \alpha(X_\mathcal{T}, \Delta_\mathcal{T}; L_\mathcal{T}) \leq A(v)/T(v)$ implies

$$\leq T(v) + \left(\frac{a}{m-a}\right) \frac{A(v)}{\gamma} - \frac{(a/\gamma + C)A(v)}{m-a} \leq T(v) - \frac{CA(v)}{m-a} \leq T_{m-a}(v)$$

and the proof is complete. □

**Proposition 5.15.** Keep the hypotheses of Theorem 5.13. There exists a positive integer $E$ so that the following holds: If $t \in T$ and $v \in \text{Val}_{X_\mathcal{T}}$ with $A_{X_\mathcal{T},\Delta_\mathcal{T}}(v) < \infty$, then

$$S(v) \leq \frac{1}{\text{vol}(L_{\mathcal{T}})} \int_0^{T(v)} \frac{\text{vol}(V_{m,s}^s)}{m^n} \, ds + \frac{EAX_{X_\mathcal{T},\Delta_\mathcal{T}}(v)}{m}$$

for all $m \geq 0$.

where $V_{m,s}^s$ is shortened notation for the $F_v^m H^0(X_\mathcal{T}, \mathcal{O}_{X_\mathcal{T}}(mL_{\mathcal{T}}))$.

**Proof.** Fix constants $C$ and $D$ satisfying the conclusions of Proposition 5.12 and 5.14. We will show that the theorem holds with $E = C + D$.

Fix $t \in T$ and $v \in \text{Val}_{X_\mathcal{T}}$ with $A(v) < +\infty$. The desired inequality is trivial when $m \leq EA(v)/T(V)$, since $T(v) \geq S(v)$. In the case when $m \geq EA(v)/T(v)$, we have

$$S(v) \leq \tilde{S}_m(v) + \frac{CA(v)}{m}$$

$$= \frac{1}{\text{vol}(L_{\mathcal{T}})} \int_0^{T(v)} \left(\frac{\text{vol}(\tilde{V}_{m,s}^s)}{m^n}\right) \, ds + \frac{CA(v)}{m}$$

$$\leq \frac{1}{\text{vol}(L_{\mathcal{T}})} \int_0^{T(v) - DA(v)/m} \left(\frac{\text{vol}(\tilde{V}_{m,s}^s)}{m^n}\right) \, ds + \frac{DA(v)}{m} + \frac{CA(v)}{m},$$

since $\text{vol}(\tilde{V}_{m,s}^s) \leq m^n \text{vol}(L_{\mathcal{T}})$. Since $\text{vol}(V_{m,s}^s) = \text{vol}(\tilde{V}_{m,s}^s)$ for all $0 \leq s \leq T(v) - DA(v)/m$ as a consequence of our choice of $D$ and Proposition 3.2,

$$= \frac{1}{\text{vol}(L_{\mathcal{T}})} \int_0^{T(v) - DA(v)/m} \left(\frac{\text{vol}(V_{m,s}^s)}{m^n}\right) \, ds + \frac{DA(v)}{m} + \frac{CA(v)}{m}$$

$$\leq \frac{1}{\text{vol}(L_{\mathcal{T}})} \int_0^{T(v)} \left(\frac{\text{vol}(V_{m,s}^s)}{m^n}\right) \, ds + \frac{DA(v)}{m} + \frac{CA(v)}{m},$$

and the proof is complete. □

We are now ready to prove Theorem 5.13.

**Proof of Theorem 5.13.** Fix $E$ satisfying the conclusion of Proposition 5.15 and choose a positive integer $p_0 \geq 2E/\varepsilon$. After replacing $p_0$ with a high enough multiple, we may assume $R^i\pi_* \mathcal{O}_X(mp_0L) = 0$ for all $i, m \geq 0$. Next, set $\gamma := \inf_{t \in T} \alpha(X_\mathcal{T}, \Delta_\mathcal{T}; L_\mathcal{T})$, which is positive by Proposition 5.3.

By Corollary 5.9, we may find a positive integer $M$ so that if $t \in T$ and $V \subseteq H^0(X_\mathcal{T}, \mathcal{O}_{X_\mathcal{T}}(p_0L_{\mathcal{T}}))$, then

$$\left| \frac{\text{dim}(V_m)}{\text{vol}(p_0L_{\mathcal{T}})} - h^0(\mathcal{O}_{X_\mathcal{T}}(mp_0L_{\mathcal{T}})) \right| < \frac{\varepsilon \gamma}{2} \quad (5.6)$$

for all $m \geq M$.

With the previous choices, consider $t \in T$ and a valuation $v \in \text{Val}_{X_\mathcal{T}}$ such that $A_{X_\mathcal{T},\Delta_\mathcal{T}}(v) < \infty$. To simplify notation, write $V_{m}^s$ for the graded linear series of $L_{\mathcal{T}}$ defined by $V_{m}^s = F_v^m H^0(X_\mathcal{T}, \mathcal{O}_{X_\mathcal{T}}(mL_{\mathcal{T}}))$. 
For $k \geq M$, we have

$$S(v) \leq \left( \frac{1}{\text{vol}(L_T)} \right) \int_0^{T(v)} \left( \frac{\text{vol}(V_{p_0,s}^s)}{p_0^s} \right) ds + \frac{EA(v)}{p_0}$$

$$\leq \left( \frac{1}{\text{vol}(L_T)} \right) \int_0^{T(v)} \left( \frac{\text{vol}(V_{p_0,k}^s)}{p_0^s} \right) ds + \frac{\varepsilon A(v)}{2}$$

by our choice of $p_0$. Next, (5.6) implies

$$\leq \int_0^{T(v)} \left( \frac{\dim(V_{p_0,k}^s)}{h^0(\mathcal{O}_{X_T}(p_0kL_T))} \right) ds + \frac{T(v)\varepsilon \gamma}{2} + \frac{\varepsilon A(v)}{2}$$

and the inequality $T(v) \leq A(v)/\alpha(X_T, \Delta_T; L_T) \leq A(v)/\gamma$ implies

$$\leq \int_0^{T(v)} \left( \frac{\dim(V_{p_0,k}^s)}{h^0(\mathcal{O}_{X_T}(p_0kL_T))} \right) ds + \varepsilon A(v).$$

Since $V_{p_0,k}^s \subseteq V_{p_0,k}^s$, we then have

$$\leq \int_0^{T(v)} \left( \frac{\dim(V_{p_0,k}^s)}{h^0(\mathcal{O}_{X_T}(p_0kL_T))} \right) ds + \varepsilon A(v)$$

$$= S_{p_0,k}(v) + \varepsilon A(v).$$

Therefore, the desired inequality holds with $N = p_0 M$. \hfill \Box

5.4. Proofs of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. We claim that for any $\varepsilon > 0$, there exists $M = M(\varepsilon)$ so that

$$0 \leq \alpha(X_T, \Delta_T; L_T)^{-1} - \alpha_m(X_T, \Delta_T; L_T)^{-1} \leq \varepsilon$$

for all $t \in T$ and $m \geq M(\varepsilon)$. Since $T \ni t \mapsto \alpha(X_T, \Delta_T; L_T)$ is bounded from above thanks to Proposition 5.3, the above claim implies the theorem.

To prove the claim, fix a positive constant $C$ satisfying the conclusion of Proposition 5.12. Now, consider $t \in T$. For $v \in \text{Val}_{X_T}^*$ with $A(v) < +\infty$, our choice of $C$ implies

$$0 \leq \frac{T(v)}{A(v)} - \frac{T_m(v)}{A(v)} \leq \frac{C}{m}.$$ 

Combining the previous inequality with Propositions 4.1 and 4.2 now gives

$$0 \leq \alpha(X_T, \Delta_T; L_T)^{-1} - \alpha_m(X_T, \Delta_T; L_T)^{-1} \leq C/m.$$ 

Therefore, the claim holds when $M(\varepsilon) = \lceil C/\varepsilon \rceil$. \hfill \Box

Proposition 5.16. Let $\pi : (X, \Delta) \rightarrow T$ be a projective $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base and $L$ a $\pi$-ample Cartier divisor on $X$. For $\varepsilon > 0$, there exists an integer $M = M(\varepsilon)$ such that

$$\delta_m(X_T, \Delta_T; L_T) - \delta(X_T, \Delta_T; L_T) \leq \varepsilon$$

for all positive integer $m$ divisible by $M$ and $t \in T$.

Proof. We claim that for any $\varepsilon > 0$, there exists $M = M(\varepsilon)$ so that

$$\delta(X_T, \Delta_T; L_T)^{-1} - \delta_m(X_T, \Delta_T; L_T)^{-1} \leq \varepsilon$$

for all $t \in T$ and $m$ divisible by $M(\varepsilon)$. Since $T \ni t \mapsto \delta(X_T, \Delta_T; L_T)$ is bounded from above (see Propositions 4.7 and 5.3), the above claim implies the proposition.
To prove the claim, choose an integer $N(\varepsilon)$ satisfying the conclusion of Theorem 5.13. Now, consider $t \in T$. For $v \in \text{Val}_{X_T}^*$ with $A(v) < +\infty$, our choice of $C$ implies
\[
\frac{S(v)}{A(v)} - \frac{S_m(v)}{A(v)} \leq \varepsilon
\]
for $m$ divisible by $N(\varepsilon)$. Combining the previous inequality with Proposition 4.3 and Theorem 4.3 gives
\[
\delta(X_T, \Delta_T; L_T)^{-1} - \delta_m(X_T, \Delta_T; L_T)^{-1} \leq \varepsilon.
\]
for $m$ divisible by $N(\varepsilon)$ and the proof is complete.

\textbf{Proof of Theorem 5.2.} We claim that for any $\varepsilon > 0$, there exists $M = M(\varepsilon)$ so that
\[
\widehat{\delta}(X_T, \Delta_T; L_T)^{-1} - \delta_m(X_T, \Delta_T; L_T)^{-1} \leq \varepsilon/2
\]
for all $t \in T$ and $m$ divisible by $M$. As in the previous proof, the above claim implies the theorem.

To prove the claim, apply Proposition 6.1 to choose an integer $M_1$ so that
\[
\delta(X_T, \Delta_T; L_T)^{-1} - \delta_m(X_T, \Delta_T; L_T)^{-1} \leq \varepsilon/2
\]
(5.7) for all $t \in T$ and $m$ divisible by $M_1$. Combining (5.7) with Proposition 4.17, we see
\[
\widehat{\delta}(X_T, \Delta_T; L_T)^{-1} - \delta_m(X_T, \Delta_T; L_T)^{-1} \leq \varepsilon/2 + (1/m)\alpha(X_T, \Delta_T; L_T)^{-1}
\]
for all $t \in T$ and $m$ divisible by $M_1$. Thanks to Proposition 5.3, there exists a positive integer $M_2$ so that $(1/m)\alpha(X_T, \Delta_T; L_T)^{-1} < \varepsilon/2$ for all $t \in T$ and $m \geq M_2$. Hence, the desired statement holds with $M = M_1 \cdot M_2$. \qed

6. Lower Semicontinuity Results

6.1. Lower semicontinuous functions. Recall that a function $f : X \to \mathbb{R}$, where $X$ is a topological space, is lower semicontinuous iff \{ $x \in X \mid f(x) > a$ \} is open for every $a \in \mathbb{R}$. The following elementary real analysis result will be used to show that our thresholds are lower semicontinuous in families.

\textbf{Proposition 6.1.} Let $X$ be a topological space and $(f_m : X \to \mathbb{R})_m$ a sequence of functions converging pointwise to a function $f : X \to \mathbb{R}$ such that:

(1) For $m \gg 0$, $f_m$ is lower semicontinuous;

(2) For each $\varepsilon > 0$, there exists a positive integer $M(\varepsilon)$ so that for each $x \in X$
\[f_m(x) \leq f(x) + \varepsilon \quad \text{for all } m \geq M(\varepsilon).\]

Then $f$ is lower semicontinuous.

6.2. Semicontinuity of the global log canonical threshold.

\textbf{Proposition 6.2.} Let $\pi : (X, \Delta) \to T$ be a $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base and $L$ a $\pi$-ample Cartier divisor on $X$. For $m \gg 0$, the function $T \ni t \mapsto \alpha_m(X_T, \Delta_T; L_T)$ is lower semicontinuous and takes finitely many values.

\textbf{Proof.} Fix $m \gg 0$, so that $R^i\pi_* \mathcal{O}_X(mL) = 0$ for all $i > 0$. Hence, for $m \geq M$, $\pi_* \mathcal{O}_X(mL)$ is a vector bundle and $\pi_* \mathcal{O}_X(mL)$ commutes with base-change.

Consider the projective bundle $\rho : W = \mathbb{P}(\pi_* \mathcal{O}_X(mL)^*) \to T$. For $t \in T$, we have a bijection between $k(\mathcal{T})$-valued points of $W_T$ and $D \in |mL_T|$. Let $\Gamma$ be the universal divisor on $Y \times_T X$ with respect to this correspondence.

By [KP17, Lemma 8.10], the function $\overline{\gamma} \in W \mapsto \text{lc}(X_T, \Delta_T; \Gamma_{\overline{T}})$ is lower semicontinuous and takes finitely many values. Hence, there exists finitely many rational numbers $a_1 > a_2 > \cdots > a_s$ and a sequence of closed sets
\[Y = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_s \supseteq Z_{s+1} = \emptyset\]
such that if $y \in Z_i \setminus Z_{i+1}$, then $\lct(X_{\overline{y}}, \Delta_{\overline{y}}; \Gamma_{\overline{y}}) = a_i$. Therefore, \(\{\alpha_m(X_{\overline{y}}, \Delta_{\overline{y}}; L_{\overline{y}}) \mid t \in T\} \subseteq \{a_1, \ldots, a_s\}\).

To see the lower semicontinuity of the function, we will show
\[ \{t \in T \mid \alpha_m(X_{\overline{y}}, \Delta_{\overline{y}}; L_{\overline{y}}) \leq a_i\} \]
is closed for each $i \in \{1, \ldots, s\}$. Set
\[ Y_i = \rho(Z_i \cup Z_{i+1} \cup \cdots \cup Z_s). \]
Since, for $t \in T$, $t \in Y_i$ if and only if $(Z_i \cup Z_{i+1} \cup \cdots \cup Z_s)_{\overline{y}}$ contains a $k(\overline{t})$-valued point, we see
\[ Y_i = \{t \in T \mid \alpha_m(X_{\overline{y}}, \Delta_{\overline{y}}; L_{\overline{y}}) \leq a_i\}. \]
Since $\rho$ is proper and $Z_i \cup \cdots \cup Z_s$ is closed, $Y_i$ is closed.

**Theorem 6.3.** If $\pi : (X, \Delta) \to T$ is a $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base and $L$ a $\pi$-ample Cartier divisor on $X$, then the function $T \ni t \mapsto \alpha(X_{\overline{t}}, \Delta_{\overline{t}}; L_{\overline{t}})$ is lower semicontinuous.

**Proof.** The result follows from combining Theorem 5.1 and Proposition 6.2 with Proposition 6.1. \(\square\)

### 6.3. Semicontinuity of the stability threshold.

**Proposition 6.4.** Let $\pi : (X, \Delta) \to T$ be a $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base and $L$ a $\pi$-ample Cartier divisor on $X$. For $m \gg 0$, the function $T \ni t \mapsto \hat{\delta}_m(X_{\overline{t}}, \Delta_{\overline{t}}; L_{\overline{t}})$ is lower semicontinuous and takes finitely many values.

To approach the above proposition, we seek to parametrize $\mathbb{N}$-filtrations of $H^0(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}(mL_{\overline{y}}))$ satisfying $T_m(\mathcal{F}) \leq 1$. Recall that such a filtration is equivalent to the data of a length $m$ decreasing sequence of subspaces of $H^0(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}(mL_{\overline{y}}))$.

Fix $m \gg 0$ so that $R^i\pi_*\mathcal{O}_X(mL) = 0$ for all $i > 0$. Set $N_m = \text{rank}(\pi_*\mathcal{O}_X(mL))$. Hence, $\pi_*\mathcal{O}_X(mL)$ is a vector bundle of rank $N_m$ and commutes with base change. For each sequence of integers $\ell = (\ell_1, \ldots, \ell_m) \in \mathbb{N}^m$ satisfying
\[ N_m \geq \ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 0, \tag{6.1} \]
let $\rho_{\ell} : \text{Fl}^{m, \ell} \to T$ denote the relative flag variety for $\pi_*\mathcal{O}_X(mL)$ that parametrizes flags of signature $\ell$. Hence, for a geometric point $\overline{t} \in T$, there is a bijection between $k(\overline{t})$-valued points of $\text{Fl}^{m, \ell} \mathcal{F}$ and $\mathbb{N}$-filtrations $\mathcal{F}$ of $H^0(X_{\overline{t}}, \mathcal{O}_{X_{\overline{t}}}(mL_{\overline{t}}))$ satisfying
\[ \dim_{k(\overline{t})}(\mathcal{F}^iH^0(X_{\overline{t}}, \mathcal{O}_{X_{\overline{t}}}(mL_{\overline{t}}))) = \begin{cases} \ell_i & \text{for } 1 \leq i \leq m \\ 0 & \text{for } i > m \end{cases}. \]

For a geometric point $\overline{y} \in \text{Fl}^m$, we write $\mathcal{F}_{\overline{y}}$ for the corresponding filtration of $H^0(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}(mL_{\overline{y}}))$.

Let $\text{Fl}^m$ denote the disjoint union $\sqcup_{\ell} \text{Fl}^{m, \ell}$, where the union runs through all $0 \neq \ell \in \mathbb{N}^m$ satisfying (6.1). Hence, for $t \in T$, there is a bijection between $k(\overline{t})$-valued points of $\text{Fl}^m$ and non-trivial $\mathbb{N}$-filtrations $\mathcal{F}$ of $H^0(X_{\overline{t}}, \mathcal{O}_{X_{\overline{t}}}(mL_{\overline{t}}))$ satisfying $T_m(\mathcal{F}) \leq 1$. Let $\rho : \text{Fl}^m \to T$ denote the map induced by the $\rho_{\ell}$.

**Lemma 6.5.** The function $\text{Fl}^m \ni \overline{y} \mapsto \lct(b_*(\mathcal{F}_{\overline{y}}))/S_m(\mathcal{F}_{\overline{y}})$ is lower semicontinuous and takes finitely many values.

**Proof.** Note that $\overline{y} \mapsto S_m(\mathcal{F}_{\overline{y}})$ is constant on each irreducible component of $\text{Fl}^m$. Indeed, for any $\overline{y} \in \text{Fl}^{m, \ell}$, $S_m(\mathcal{F}_{\overline{y}}) = \frac{1}{mN_m} \sum_{i=1}^m \ell_i$. Hence, we are reduced to showing that $\overline{y} \mapsto \lct(b_*(\mathcal{F}_{\overline{y}}))$ is lower semicontinuous and takes finitely many values.

Set $X' := X \times_T \text{Fl}^m$, and write $\pi'$ and $\rho'$ for the projection maps.
Set $L' := \rho^*(L)$, and note that $\pi'_*\mathcal{O}_{X'}(mL') \simeq \rho^*\pi_*\mathcal{O}_X(mL)$ by the projection formula and flat base change.

On $\text{Fl}^m$ there is a universal flag

$$\pi'_*\mathcal{O}_{X'}(mL') \supseteq W_{u,1} \supseteq \cdots \supseteq W_{u,m}.$$  

such that $\mathcal{F}_\mathcal{Y} H^0(X_{\mathcal{Y}}, \mathcal{O}_{X_{\mathcal{Y}}}(mL_{\mathcal{Y}}))$ is the image of the map

$$W_{u,i} \otimes k(\mathcal{Y}) \rightarrow \pi'_*\mathcal{O}_{X'}(-mL) \rightarrow \mathcal{O}_{X'}$$

for each $\mathcal{Y} \in \text{Fl}^m$ and $i \in \{1, \ldots, m\}$. The universal flag gives rise a universal sequence of base ideals. Indeed, for each $i \in \{1, \ldots, m\}$, set

$$a_{u,i} := \operatorname{im}(\pi'^* (W_{u,i}) \otimes \mathcal{O}_{X'}(-mL) \rightarrow \mathcal{O}_{X'})$$

where the previous map is induced by the map $\pi'^* \pi_* \mathcal{O}_{X'}(mL') \otimes k(\mathcal{Y}) \simeq H^0(X_{\mathcal{Y}}, \mathcal{O}_{X_{\mathcal{Y}}}(mL_{\mathcal{Y}}))$. Note that the base ideal of $\mathcal{F}_\mathcal{Y} H^0(X_{\mathcal{Y}}, \mathcal{O}_{X_{\mathcal{Y}}}(mL_{\mathcal{Y}}))$ is the image of the map $a_{u,i} \otimes k(\mathcal{Y}) \rightarrow \mathcal{O}_{X_{\mathcal{Y}}}$.

Now, set

$$b_{u,p} = \sum c_{u,1}^1 \cdots c_{u,m}^m$$

where the sum runs through all $c = (c_1, \ldots, c_m) \in \mathbb{N}^m$ such that $\sum ic_i \geq p$. By Lemma 3.20,

$$b_p(\mathcal{F}_{\mathcal{Y}}) = b_{u,p} \cdot \mathcal{O}_{X_{\mathcal{Y}}}$$

for all $\mathcal{Y} \in \text{Fl}^m$ and $p \in \mathbb{N}$.

Next, apply Lemma 2.1 to find $N \in \mathbb{N}>0$ so that $b_{u,N} = b_{u,N}$ for all $p > 0$. By (6.2), this implies $b_{Np}(\mathcal{F}_\mathcal{Y}) = b_N(\mathcal{F}_\mathcal{Y})^p$ for all $\mathcal{Y} \in \text{Fl}^m$ and $p > 0$. Therefore,

$$\operatorname{lct}(b_{u,p}(\mathcal{F}_{\mathcal{Y}})) = \operatorname{lct}(b_{N}(\mathcal{F}_\mathcal{Y})) = \operatorname{lct}(b_{u,N} \cdot \mathcal{O}_{X_{\mathcal{Y}}})$$

for all $\mathcal{Y} \in \text{Fl}^m$. Hence, it suffices to show $\mathcal{Y} \mapsto \operatorname{lct}(b_{N}(\mathcal{F}_{\mathcal{Y}}))$ is lower semicontinuous and takes finitely many values. Since the latter holds by the lower semicontinuity of the log canonical threshold in flat proper families [KP17, Lemma 8.10], the proof is complete.

\textbf{Proof of Proposition 6.4.} Since $L$ is $\pi$-ample, there exists $M > 0$ such that $R^i \pi_* \mathcal{O}_X(mL) = 0$ for all $i > 0$ and $m \geq M$. Hence, $\pi_* \mathcal{O}_X(mL)$ is a vector bundle and commutes with base change for all $m \geq M$.

Fix $m \geq M$, and consider $\text{Fl}^m$ as defined above. By Lemma 6.5, there exist finitely many rational numbers $a_1 > a_2 > \cdots > a_s$ and a sequence of closed sets

$$\text{Fl}^m = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_s \supseteq Z_{s+1} = \emptyset$$

such that if $\mathcal{Y} \in Z_i \setminus Z_{i+1}$, then $a_i := \operatorname{lct}(b_{u,i}(\mathcal{F}_{\mathcal{Y}}))/S_m(\mathcal{F}_{\mathcal{Y}})$.

Recall that for $t \in T$, there is a bijection between non-trivial $\mathbb{N}$-filtrations of $H^0(X_T, \mathcal{O}_{X_T}(mL_T))$ and $k(\mathcal{Y})$-valued points of $\text{Fl}^m$. Therefore, $\{\delta_m(X_T, \Delta_T, L_T) \mid t \in T\} \subseteq \{a_1, \ldots, a_s\}$.

We now seek to show

$$\{t \in T \mid \delta_m(X_T, \Delta_T; L_T) \leq a_i\}$$

is closed in $T$ for each $i \in \{1, \ldots, s\}$. Note that $\delta_m(X_T, \Delta_T; L_T) \leq a_i$ if and only if $(Z_i \cup Z_{i+1} \cup \cdots \cup Z_s)_{\mathcal{Y}} \neq \emptyset$. Set

$$Y_i := \rho(Z_i \cup Z_{i+1} \cup \cdots \cup Z_s).$$
Since $t \in (Y_i)$ if and only if $(Z_i \cup Z_{i+1} \cup \cdots \cup Z_s)_T$ contains a $k(\overline{T})$-valued point,

$$Y_i = \{ t \in T \mid \hat{\delta}_m(X_T, \Delta_T; L_T) \leq a_i \}.$$  

Since $\rho$ is proper and each $Z_i$ is closed, each $Y_i$ is closed. \hfill \Box

**Theorem 6.6.** If $\pi : (X, \Delta) \to T$ be a projective $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base and $L$ a $\pi$-ample Cartier divisor on $X$, then the function $T \ni t \mapsto \delta(X_T, \Delta_T; L_T)$ is lower semicontinuous.

**Proof.** Consider the sequence of functions $f_m : T \to \mathbb{R}$ defined by $f_m(t) := \hat{\delta}(m!) (X_T, \Delta_T; L_T)$ if $m \in M(L_T)$ and 0 otherwise. By Proposition 4.14, $\hat{\delta}(X_T, \Delta_T; L_T) = \lim_{m \to \infty} f_m(t)$ for each $t \in T$. By Theorem 5.2 and Proposition 6.4, we see that the sequence $(f_m)$ satisfies the hypotheses of Proposition 6.1. Therefore, $T \ni t \mapsto \delta(X_T, \Delta_T; L_T)$ is lower semicontinuous. \hfill \Box

**Remark 6.7.** In [CP18, Proposition 4.14], it was shown that the stability threshold is constant on very general points. The result also follows from Theorem 6.6.

**Proof of B.** The statement is a special case of Theorems 6.3 and 6.6. \hfill \Box

### 6.4. Openness of uniform K-stability

The following result follows from Theorems 4.8 and 6.6.

**Theorem 6.8.** If $(X, \Delta) \to T$ be a projective $\mathbb{Q}$-Gorenstein family of klt pairs over a normal base such that $-K_X/T - \Delta$ is $\pi$-ample, then

1. $\{ t \in T \mid X_T \text{ is uniformly K-stable} \}$ is an open subset of $T$, and
2. $\{ t \in T \mid X_T \text{ is K-semistable} \}$ is a countable intersection of open subsets of $T$.

**Proof.** By Theorem 6.6 with $L := -K_X - \Delta$, we see

$$\{ t \in T \mid \delta(X_T, \Delta_T; -K_X/T - \Delta_T) > 1 \}$$

is open in $T$ and

$$\bigcap_{m \geq 1} \{ t \in T \mid \delta(X_T, \Delta_T; -K_X/T - \Delta_T) > 1 - 1/m \}$$

is a countable intersection of open subsets of $T$. Applying Theorem 4.8 completes the proof. \hfill \Box

**Proof of Theorem A.** Let $V \subseteq T$ denote the locus of point $t \in T$ such that $X_T$ is klt. The set $V$ is open in $T$ [Kol13, Corollary 4.10.2] and contains all K-semistable geometric fibers [Oda13a, Theorem 1.3]. Applying Theorem 6.8 to the family $X_V \to V$ with $\Delta = 0$ completes the proof. \hfill \Box

### 7. The stability threshold and K-stability for log pairs

We first give a motivation from complex geometry. For a Fano manifold $X$, the greatest Ricci lower bound (or $\beta$-invariant$^3$) of $X$ is defined as

$$\beta(X) := \sup \{ t \in [0, 1] \mid \text{there exists a Kähler metric } \omega \in c_1(X) \text{ such that } \text{Ric}(\omega) > t\omega \}.$$  

This invariant was studied by Tian in [Tia92], although it was not explicitly defined there. It was first explicitly defined by Rubinstein in [Rub08, Rub09] and was later further studied by Székelyhidi [Szé11], Li [Li11], Song and Wang [SW16], and Cable [Cab18]. (Note that $\beta(X)$ is denoted by $R(X)$ in some papers.) In the following result, Song and Wang study the relationship between $\beta(X)$ and the existence of conical Kähler-Einstein metrics.

**Theorem 7.1.** [SW16, Theorem 1.1] Let $X$ be a Fano manifold.

---

$^3$The $\beta$-invariant of a Fano manifold defined here is different from the $\beta$-invariant of a divisorial valuation introduced by Fujita in [Fuj16b].
(1) For any \( \beta \in [\beta(X),1] \) and smooth divisor \( D \in |-mK_X| \) with \( m \in \mathbb{N} \), there does not exist a smooth conical Kähler-Einstein metric \( \omega \) with
\[
\text{Ric}(\omega) = \beta \omega + \frac{1-\beta}{m}[D]
\] (7.1)
if \( \beta(X) < 1 \).

(2) For any \( \beta \in (0,\beta(X)) \), there exists a smooth divisor \( D \in |-mK_X| \) for some \( m \in \mathbb{N} \) and a smooth conical Kähler-Einstein metric \( \omega \) satisfying (7.1).

It is shown by Berman, Boucksom and Jonsson [BBJ18] and independently by Cheltsov, Rubinstein and Zhang [CRZ18] that \( \beta(X) = \min\{1,\delta(X)\} \) for any Fano manifold \( X \). In this section, we prove the following result which can be viewed as a K-stability analogue of Theorem 7.1. Note that a similar result is proved independently in [CRZ18].

**Theorem 7.2.** Let \((X, \Delta)\) be a log Fano pair.

1. For any rational number \( \beta \in (\delta(X, \Delta), 1] \) and any \( D \in |-K_X - \Delta|_{\mathbb{Q}} \), the pair \((X, \Delta + (1 - \beta)D)\) is not K-semistable when \( \delta(X, \Delta) < 1 \). Moreover, the pair \((X, \Delta + (1 - \beta)D)\) is not uniformly K-stable when \( \beta = \delta(X, \Delta) \leq 1 \).

2. For any rational number \( \beta \in (0, \min\{1, \delta(X, \Delta)\}) \), there exists an effective \( \mathbb{Q} \)-divisor \( D \sim_{\mathbb{Q}} -(K_X + \Delta) \) such that the pair \((X, \Delta + (1 - \beta)D)\) is uniformly K-stable.

**Proof.** (1) Assume \( \delta(X, \Delta) \leq 1 \) and fix \( \beta \in [\delta(X, \Delta), 1] \) and \( D \in |-K_X - \Delta|_{\mathbb{Q}} \). If \((X, \Delta + (1 - \beta)D)\) is not klt, then pair is not K-semistable by [BHIJ17, Corollary 9.6]. We move onto the case when \((X, \Delta + (1 - \beta)D)\) is klt.

Fix \( v \in \text{Val}_X \) with \( A_{X,\Delta}(v) < +\infty \). Since \(- (K_X + \Delta + (1 - \beta)D) \sim_{\mathbb{Q}} -\beta(K_X + \Delta)\), we have
\[
\text{S}(-(K_X + \Delta + (1 - \beta)D); v) = \beta \text{S}(-K_X - \Delta; v)
\]
by Proposition 3.11.2. We also have
\[
A_{X,\Delta+(1-\beta)D}(v) = A_{X,\Delta}(v) - (1 - \beta)v(D) \leq A_{X,\Delta}(v).
\]
Hence,
\[
\delta(X, \Delta + (1 - \beta)D) = \inf_v \frac{A_{X,\Delta+(1-\beta)D}(v)}{\text{S}(-(K_X + \Delta + (1 - \beta)D); v)} \leq \inf_v \frac{A_{X,\Delta}(v)}{\beta \text{S}(-K_X - \Delta; v)} = \frac{\delta(X, \Delta)}{\beta}.
\]
If \( \beta > \delta(X, \Delta) \), we have \( \delta(X, \Delta + (1 - \beta)D) < 1 \). If \( \beta = \delta(X, \Delta) \), then \( \delta(X, \Delta + (1 - \beta)D) \leq 1 \). Applying Theorem 4.8 completes the proof of (1).

(2) Fix \( \beta \in (0, \min\{1, \delta(X, \Delta)\}) \). Let \( m \in \mathbb{N} \) be chosen so that \(-m(K_X + \Delta)\) is a Cartier divisor and the linear system \(|-m(K_X + \Delta)|\) is base point free. Then, for a general \( \mathbb{Q} \)-divisor \( D \in \frac{1}{m}[-(K_X + \Delta)] \) the pair \((X, \Delta + mD)\) is lc by [KM98, Lemma 5.17]. In particular, \( A_{X,\Delta}(v) \geq mv(D) \) for any \( v \in \text{Val}_X \).

Consider \( v \in \text{Val}_X \) with \( A_{X,\Delta}(v) < +\infty \). we have
\[
A_{X,\Delta+(1-\beta)D}(v) = A_{X,\Delta}(v) - (1 - \beta)v(D) \geq (1 - (1 - \beta)/m)A_{X,\Delta}(v).
\]
As in the proof of (1), we also have \( \text{S}(-(K_X + \Delta + (1 - \beta)D); v) = \beta \text{S}(-K_X - \Delta; v) \). Therefore,
\[
\delta(X, \Delta + (1 - \beta)D) \geq \inf_v \frac{(1 - (1 - \beta)/m)A_{X,\Delta}}{\beta \text{S}(-K_X - \Delta; v)} = \frac{1 - (1 - \beta)/m}{\beta} \delta(X, \Delta).
\]
Thus, if \( m \) was chosen sufficiently large and divisible, then \( \delta(X, \Delta + (1 - \beta)D) > 1 \). Hence, \((X, \Delta + (1 - \beta)D)\) is uniformly K-stable by Theorem 4.8. \( \square \)

**Proof of Theorem C.** The statement follows immediately from Theorem 7.2.2. \( \square \)
The proof of Theorem 7.2.2 implies the following result which can be viewed as a K-stability analogue of [SW16, Proposition 1.1].

**Proposition 7.3.** Let \((X, \Delta)\) be a log Fano pair with and \(m\) an integer \(\geq 2\). If \(D = m^{-1}H\), where \(H \in \mathbb{J}^m(K_X + \Delta)\), satisfies that \((X, \Delta + H)\) is log canonical, then \((X, \Delta + (1 - \beta)D)\) is uniformly K-stable for any \(\beta \in (0, \frac{(m - 1) \min \{1, \beta(X, \Delta)\}}{m - \min \{1, \beta(X, \Delta)\}})\).

The next theorem is an application of Theorem 7.2. The result may also be deduced from Theorem 6.6.

**Theorem 7.4.** Assume the Zariski openness of uniform K-stability in \(\mathbb{Q}\)-Gorenstein flat families of log Fano pairs. Then for any \(\mathbb{Q}\)-Gorenstein flat family \(\pi : (X, \Delta) \to T\) of log Fano pairs, the function \(T \ni t \mapsto \min \{1, \delta(X_T, \Delta_T)\}\) is lower semicontinuous in the Zariski topology.

**Proof.** It suffices to show that for any rational number \(\beta \in (0, 1)\), the locus \(\{t \in T \mid \delta(X_{T\!\!\!/t}, \Delta_{T\!\!\!/t}) > \beta\}\) is Zariski open. Assume that \(\delta(X_{T\!\!\!/t}, \Delta_{T\!\!\!/t}) > \beta\) for some point \(o \in T\). Then by Theorem 7.2.2, there exists an effective \(\mathbb{Q}\)-divisor \(D_T \sim \mathbb{Q}_{\mathbb{Q}} (K_{X/T} + \Delta_T)\) such that \((X_{T\!\!\!/t}, \Delta_{T\!\!\!/t} + (1 - \beta)D_T)\) is uniformly K-stable. Let us choose \(m \in \mathbb{N}\) sufficiently divisible such that \(mD_T\) is Cartier, \(-m(K_{X/T} + \Delta)\) is Cartier, and \(\pi_*\mathcal{O}_X(-m(K_{X/T} + \Delta))\) is locally free on \(T\). The projective bundle \(W := \mathbb{P}_T(\pi_*\mathcal{O}_X(-m(K_{X/T} + \Delta)))\) over \(T\) parametrizes effective \(\mathbb{Q}\)-divisors \(D_T \in \frac{1}{m!} - m(K_{X/T} + \Delta)\) on \(X_T\). Since \((X_{T\!\!\!/t}, \Delta_{T\!\!\!/t} + (1 - \beta)D_T)\) is uniformly K-stable, the openness of uniform K-stability we can find an open set \(U\) of \(W\) containing \(D_T\), such that for any \(D_T \in U\) the pair \((X_T, \Delta_T + (1 - \beta)D_T)\) is uniformly K-stable. Denote by \(\psi : W \to T\) the projection morphism, then \(\psi(U)\) is an open neighborhood of \(o\) in \(T\) since \(\psi\) is flat. Hence part (1) of Theorem 7.2 implies that \(\delta(X_{T\!\!\!/t}, \Delta_{T\!\!\!/t}) > \beta\) for any \(t \in \psi(U)\). \(\square\)

**Remark 7.5.** Using the weak openness of K-semistability from [BL18] and Theorem 7.2, the above proof implies the weak lower semicontinuity of \(T \ni t \mapsto \min \{1, \delta(X_{T\!\!\!/t}, \Delta_{T\!\!\!/t})\}\).

### 7.1. The toric case

In this section, we will explain that a stronger version of Theorem 7.2 holds in the toric setting. Throughout, we will freely use results and notation of [Ful93] for toric varieties.

#### 7.1.1. Setup

Fix a projective toric variety \(X = X(\Sigma)\) given by a rational fan \(\Sigma \subset N_\mathbb{R}\), where \(N \simeq \mathbb{Z}^n\) is a lattice and \(N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}\). We write \(M = \text{Hom}(N, \mathbb{Z})\), \(M_Q = M \otimes \mathbb{Q}\), and \(M_\mathbb{Q} = M \otimes \mathbb{Q}\) for the corresponding dual lattice and vector spaces.

Let \(v_1, \ldots, v_d\) denote the primitive generators of the one-dimensional cones in \(\Sigma\) and \(D_1, \ldots, D_d\) be the corresponding torus invariant divisors on \(X\). When the context is clear, we will a bit abusively write \(v_i\) for the valuation \(\text{ord}_{D_i}\).

Fix toric invariant \(\mathbb{Q}\)-divisors
\[
\Delta = \sum_{i=1}^d b_i D_i \quad \text{and} \quad L = \sum_{i=1}^d c_i D_i,
\]
so that (i) \(\Delta\) has coefficients in \([0, 1)\), \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier, and (ii) \(L\) is \(\mathbb{Q}\)-Cartier and ample. Assumption (i) implies \((X, \Delta)\) is klt.

Associated to \(L\) is the convex polytope
\[
P_L = \{u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geq -c_i \text{ for all } 1 \leq i \leq d\}.
\]

Let \(\overline{u} \in M_\mathbb{Q}\) denote the barycenter of \(P_L\). Recall that there is a correspondence between points in \(P_L \cap M_\mathbb{Q}\) and effective torus invariant \(\mathbb{Q}\)-divisors \(\mathbb{Q}\)-linearly equivalent to \(L\), under which \(u \in P_L \cap M_\mathbb{Q}\) corresponds to
\[
D_u := L + \sum_{i=1}^d \langle u, v_i \rangle D_i := \sum_{i=1}^d (\langle u, v_i \rangle + c_i) D_i.
\]
7.1.2. The stability threshold. We recall the following result from [BLJ17, §7] (and [Blu18] for the setting of log pairs) on the value of the stability threshold in the toric case.

**Proposition 7.6.** With the above setup,
\[ A_{X,\Delta}(v_i) = 1 - b_i \quad \text{and} \quad S(L; v_i) = \langle \overline{\pi}, v_i \rangle + c_i \]
for each \( i \in \{1, \ldots, d\} \).

**Theorem 7.7.** With the above setup,
\[ \delta(X, \Delta; L) = \min_{i=1, \ldots, d} \frac{A_{X,\Delta}(v_i)}{S(L; v_i)} \]

7.1.3. Log Fano toric pairs. We keep the previous setup, but will additionally assume \((X, \Delta)\) is a toric log Fano pair. Hence \(-K_X - \Delta = \sum_{i=1}^d (1 - b_i)D_i\) is ample. The vector \(\overline{\pi}\) will denote the barycenter of
\[ P_{-K_X - \Delta} := \{ u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geq -1 + b_i \text{ for all } 1 \leq i \leq d \}. \]

The following statement appeared [BLJ17, §7.6] in the \(\mathbb{Q}\)-Fano case. As we will explain, the more general result follows from the same argument.

**Proposition 7.8.** Let \((X, \Delta)\) be a toric log Fano pair and \(\overline{\pi}\) denote the barycenter of \(P_{-K_X - \Delta}\).

1. If \(\overline{\pi}\) is the origin, then \(\delta(X, \Delta) = 1\).
2. If \(\overline{\pi}\) is not the origin, then
\[ \delta(X, \Delta) = \frac{c}{1 + c}, \]
where \(c\) is the largest real number such that \(-c\pi \in P_{-K_X - \Delta}\).

**Proof.** Theorem 7.7 in the case when \(L = -K_X - \Delta\) gives
\[ \delta(X, \Delta) = \min_{i=1, \ldots, d} \frac{1 - b_i}{1 - b_i + \langle \overline{\pi}, v_i \rangle}. \quad (7.2) \]
Statement (1) follows immediately from (7.2). For (2), we claim that if \(\overline{\pi}\) is not the origin, then
\[ 0 < \langle \overline{\pi}, v_i \rangle + (1 - b_i) \leq (1 - b_i)/c + (1 - b_i) \]
for each \(i = 1, \ldots, d\) and equality holds in the last inequality for some \(i\). Statement (2) now follows from the claim and (7.2).

We now prove the claim. Since \(\overline{\pi}\) lies in the interior of \(P_{-K_X - \Delta}\), \(\langle \overline{\pi}, v_i \rangle > -1 + b_i\) for all \(i\). Since \(-c\pi\) lies on the boundary of \(P_{-K_X}\),
\[ -c\langle \overline{\pi}, v_i \rangle = -c\langle \overline{\pi}, v_i \rangle \geq -1 + b_i \]
and the last inequality holds for some \(i\). This completes the proof.

The following statements are inspired by results in complex geometry (specifically, [SW16, Theorem 3.3.2] and [LS14, Theorem 1.14]). We thank Song Sun for bringing our attention to the previous results and suggesting the existence of algebraic analogs.

**Proposition 7.9.** Let \((X, \Delta)\) be a toric log Fano pair that is not \(K\)-semistable. There exists a toric invariant \(\mathbb{Q}\)-divisor \(D^* \in -K_X - \Delta|_\mathbb{Q}\) such that
1. \((X, \Delta + (1 - \delta(X, \Delta))D^*)\) is a log Fano pair and
2. \(\delta(X, \Delta + (1 - \delta(X, \Delta))D^*) = 1\).

**Proof.** Let \(\overline{\pi}\) denote the barycenter of \(P_{-K_X - \Delta}\) and \(c\) the largest real number such that \(-cu \in P_{-K_X - \Delta}\). Recall that \(\delta(X, \Delta) = c/(1 + c)\) by Proposition 7.8.2. Set
\[ D := D_{-cu} = \sum_{i=1}^d ((1 - b_i) + \langle -c\overline{\pi}, v_i \rangle) D_i \in -K_X - \Delta|_\mathbb{Q}. \]
We first show statement (1). For \( i = 1, \ldots, d \), we compute
\[
A_{X, \Delta + (1 - \delta(X, \Delta))D^*}(v_i) = 1 - b_i - (1 - c/(c + 1))(1 - b_i + \langle -c\overline{u}, v_i \rangle)
= (c/(c + 1))(1 - b_i + (\overline{u}, v_i)).
\]

Since \( \overline{u} \) is in the interior of \( P_{-K_X - \Delta} \), \( \langle \overline{u}, v_i \rangle > -1 + b_i \). Hence, the above log discrepancies are > 0 and the pair is klt. Since \(- (K_X + \Delta + (1 - \delta(X, \Delta))D^*) \sim_{\mathbb{Q}} -\delta(X, \Delta)(K_X + \Delta) \) is ample, \((X, \Delta + (1 - \delta(X, \Delta))D^*) \) is log Fano.

To prove (2), we compute
\[
S(-(K_X + \Delta + (1 - \delta(X, \Delta))D^*); v_i) = \delta(X, \Delta)S(-(K_X - \Delta); v_i)
= (c/(c + 1))(1 - b_i + (\overline{u}, v_i)).
\]

Proposition 7.6 and our previous computations imply \( \delta(X, \Delta + (1 - \delta(X, \Delta))D^*) = 1 \).

**Theorem 7.10.** Let \((X, \Delta)\) be a toric log Fano pair. If \( m \in \mathbb{Z}_{>0} \) is sufficiently divisible and \( D = m^{-1}H \), where \( H \in | - m(K_X + \Delta)| \) is very general, then
\[
(X, \Delta + (1 - \beta)D) \text{ is uniformly K-stable for } \beta \in (0, \delta(X, \Delta)).
\]
Moreover, \( (X, \Delta + (1 - \delta(X, \Delta))D) \) is K-semistable.

**Proof.** If \( \delta(X, \Delta) = 1 \), the statement follows from Proposition 7.3. From now on, assume \( \delta(X, \Delta) < 1 \).

Let \( D^* \in | - (K_X + \Delta)|_{\mathbb{Q}} \) denote a toric invariant \( \mathbb{Q} \)-divisor satisfying Proposition 7.9. Fix an integer \( m \geq 2 \) so that \( mD^* \) is a Cartier divisor and \( | - m(K_X + \Delta)| \) is base point free.

We claim that for a very general \( H \in | - m(K_X + \Delta)| \) (i) \( (X, \Delta + H) \) is lc and (ii) \( \delta(X, \Delta + (1 - \delta(X, \Delta))m^{-1}H) \geq 1 \). Indeed, since \( | - m(K_X + \Delta)| \) is base point free, (i) follows from [KM98, Lemma 5.17]. Since
\[
\bigcap_{q \in \mathbb{Z}_{>0}} \left\{ H \in | - m(K_X + \Delta)| \left| \delta(X, \Delta + (1 - \delta(X, \Delta))m^{-1}H) > 1 - 1/q \right. \right. \}
\]
is a countable intersection of open sets (by Theorem 6.6) and each contains \( mD^* \), (ii) holds.

Now, consider a very general element \( H \in | - m(K_X + \Delta)| \) satisfying (i) and (ii). Set \( D := m^{-1}H \). We claim that \( \delta(X, \Delta + (1 - \beta)D) \) is decreasing in \( \beta \) on \((0, 1] \). Assuming the claim, (ii) implies \( \delta(X, \Delta + (1 - \beta)D) > 1 \) and, hence, uniformly K-stable for \( \beta \in (0, \delta(X, \Delta)) \).

To prove the above claim, it suffices to show that for each \( v \in \text{Val}_X^* \) with \( A_{X, \Delta}(v) < +\infty \),
\[
\frac{A_{X, \Delta + (1 - \beta)D}(v)}{S(-(K_X + \Delta + (1 - \beta)D); v)} = \frac{d}{d\beta} \frac{A_{X, \Delta + (1 - \beta)D}(v)}{S(-(K_X + \Delta + (1 - \beta)D); v)} \tag{7.3}
\]
is a differentiable function in \( \beta \) with derivative bounded away from 0 by \(-((m - 1)/m)\delta(X, \Delta)/\beta^2 \).

The see this, we compute
\[
\frac{d}{d\beta} \left( \frac{A_{X, \Delta + (1 - \beta)D}(v)}{S(-(K_X + \Delta + (1 - \beta)D); v)} \right) = \frac{d}{d\beta} \left( \frac{A_{X, \Delta + D}(v) + \beta v(D)}{\beta S(-K_X - \Delta; v)} \right)
= A_{X, \Delta + D}(v) \frac{d}{d\beta} \left( \frac{A_{X, \Delta + D}(v)}{\beta S(-K_X - \Delta; v)} \right)
\]
Since \((X, \Delta + mD)\) is lc by (i), \( A_{X, \Delta + mD}(v) = A_{X, \Delta}(v) - mv(D) \geq 0 \). Thus, \( A_{X, \Delta + D}(v) = A_{X, \Delta}(v) - v(D) \geq (m - 1)m^{-1}A_{X, \Delta}(v) \), and
\[
\leq - \left( \frac{m - 1}{m} \right) \frac{A_{X, \Delta}(v)}{\beta^2 S(-K_X - \Delta; v)} \leq - \left( \frac{m - 1}{m} \right) \frac{\delta(X, \Delta)}{\beta^2}.
\]
Question 7.11. Does the conclusion of Theorem 7.10 hold for all log Fano pairs?

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