DEFORMATIONS OF CIRCLE-VALUED MORSE FUNCTIONS
ON 2-TORUS

BOHDAN FESHCHENKO

Abstract. In this paper we give an algebraic description of fundamental groups of orbits of circle-valued Morse functions on $T^2$ with respect to the action of the group of diffeomorphisms of $T^2$.

1. Introduction

Circle-valued Morse functions are natural generalizations of (ordinary) Morse functions. They can be viewed as multi-valued Morse functions, their values is locally defined up to an additive integer. Therefore locally circle-valued Morse function can be viewed as Morse functions but global properties of such functions are different from real-valued case.

A modern theory of circle-valued Morse functions originates in a series of papers [23, 24] by S. Novikov in 80’s. It was motivated by the study of multi-valued Lagrangians in some problems of theoretical physics and leaded him to develop a generalization of a Morse theory for circle-valued Morse functions and more generally a theory (now called Morse-Novikov theory) for differential 1-forms. This theory has many applications, e.g., in questions of fibrations of manifolds over $S^1$ [26], Lagrangian intersections [4], knot theory [28], Seiberg-Witten [7, 8] theory, etc. The reader can find more on this theory and its applications in the book by A. Pajitnov [25].

Main subjects of study in this paper are orbits and stabilizers of (circle-valued) Morse functions on surfaces. We start with necessary precise definitions.

Let $M$ be a smooth compact surface, $X$ be a closed (possible empty) subset of $M$. By $P$ we also denote $\mathbb{R}$ or $S^1$. The group $\mathcal{D}(M, X)$ of diffeomorphisms of $M$ fixed on $X$ acts from the right on the space of smooth maps $C^\infty(M, P)$ by the rule

\[ \gamma : C^\infty(M, P) \times \mathcal{D}(M, X) \to C^\infty(M, P), \quad (f, h) \mapsto f \circ h. \]

With respect to $\gamma$ we denote by

\[ \mathcal{S}(f, X) = \{ h \in \mathcal{D}(M, X) \mid f \circ h = f \}, \]
\[ \mathcal{O}(f, X) = \{ f \circ h \mid h \in \mathcal{D}(M, X) \} \]

the stabilizer and the orbit of $f \in C^\infty(M, P)$. Endow strong Whitney $C^\infty$-topologies on $C^\infty(M, P)$ and $\mathcal{D}(M, X)$; then for a map $f \in C^\infty(M, P)$ these topologies induce some topologies on $\mathcal{S}(f, X)$ and $\mathcal{O}(f, X)$. We denote by $\mathcal{D}_{id}(M, X)$, $\mathcal{S}_{id}(f, X)$ connected components of the identity map $\mathcal{D}(M, X)$ and $\mathcal{S}(f, X)$ respectively, and by $\mathcal{O}_{f}(f, X)$ a connected component of $\mathcal{O}(f, X)$ containing $f$. If $X = \emptyset$ we omit...
the symbol “∅” from our notation, i.e., we will write \( D(M) \) and \( S(f) \) instead of \( D(M, ∅) \) and \( S(f, ∅) \) and so on.

A smooth map \( f : M \to P \) is called a \( P \)-valued Morse function, if it satisfies the following conditions:

- all critical points of \( f \) are non-degenerate and belong \( \text{Int}(M) \),
- the map \( f \) takes values on each boundary component of \( M \).

The class of all \( P \)-valued functions on \( M \) will be denoted by \( \mathcal{F}(M, P) \).

An \( \mathbb{R} \)-valued Morse function is ordinary called a Morse function and an \( S^1 \)-valued Morse function is also called a circle-valued Morse function. In the further text the adjective “\( P \)-valued’ will always be omitted since the target \( P \) is unique determined from the notation, but we will indicate it where necessary.

E. Kudryavtseva [10, 11] and S. Maksymenko [14, 15, 16, 18, 19] studied homotopy groups of stabilizers and orbits of \( P \)-valued Morse functions on surfaces. The following theorem describes the general homotopy properties of orbits.

**Theorem 1.1** ([27, 14, 18, 17]). Let \( f : M \to P \) be a Morse function on a smooth compact surface \( M, X \) be a closed (possible empty) subset of \( M \) consisting of finitely many connected components of some level-set of \( f \) and some critical points of \( f \). Then the following statements hold.

1. The map
   \[
   \zeta_f : D(M, X) \to O(f, X), \quad \zeta_f(h) = f \circ h
   \]
   is a locally trivial principal fibration with the fiber \( S(f, X) \). The restriction \( \zeta_f \big|_{Dal(M, X)} : D_{al}(M, X) \to O(f, X) \) is also a locally trivial principal fibration with the fiber
   \[
   S'(f, X) = S(f) \cap D_{al}(M, X).
   \]
   The orbit \( O_f(f, X) \) is a Fréchet manifold, so it has a homotopy type of a CW complex.

2. \( O_f(f, X) = O_f(f, X \cup \partial M) \), and so
   \[
   \pi_kO_f(f, X) \cong \pi_kO_f(f, X \cup \partial M)
   \]
   for \( k \geq 1 \).

3. Suppose that either \( f \) has a saddle point or \( M \) is non-orientable surface. Then \( S_{al}(f) \) is contractible, \( \pi_kO_f(f) \cong \pi_kM, k \geq 3 \), \( \pi_2O_f(f) = 0 \), and for \( \pi_1O_f(f) \) the following short sequence\(^1\) of groups
   \[
   \pi_1D_{al}(M) \xrightarrow{\xi_1} \pi_1O_f(f) \xrightarrow{\partial_1} \pi_0S'(f)
   \]
   is exact. Moreover, \( p(\pi_1D_{al}(M)) \) contains in the center of \( \pi_1O_f(f) \).

4. If \( \chi(M) < \text{card}(X) \), then \( \pi_1D_{al}(M, X) \) is contractible, \( \pi_kO_f(f, X) = 0 \) for \( k \geq 2 \) and the boundary map
   \[
   \pi_1O_f(f, X) \xrightarrow{\partial_1} \pi_0S'(f, X)
   \]
   is an isomorphism.

5. If \( f \) is generic, then \( O_f(f) \) homotopy equivalent to \( (S^1)^m \) if \( M \neq S^2, \mathbb{R}P^2 \);
   to \( S^2 \) is \( M = S^2 \) and \( f \) has only two critical points, and to \( SO(3) \times (S^1)^m \) otherwise for some \( m \geq 0 \) depending on \( f \).

\(^1\)Throughout the text → and ← mean mono- and epimorphism respectively.
(6) If $f$ has exactly $n$ critical points, then $O_f(f)$ if homotopy equivalent to some covering space of $n$th configuration space of $M$ and $\pi_1 O_f(f)$ is a subgroup of $n$th braid group $B_n(M)$.

Notice that a sequence (2) is nonzero part of a long exact sequence of homotopy groups of the fibration $\zeta_f$. Moreover there is an isomorphism
\begin{equation}
\zeta : \pi_1(\mathcal{D}_{id}(M), S'(f)) \longrightarrow \pi_1 O_f(f), \quad \zeta : [[h_t]] \longmapsto [[f \circ h_t]],
\end{equation}
where $h_t : M \to M$, $t \in [0, 1]$ is an isotopy of $M$ with $h_0 = id_M$ and $h_1 \in S'(f)$.

If $M$ is a closed compact and oriented surface of genus $\geq 2$, then $\mathcal{D}_{id}(M)$ is contractible [19], and so an epimorphism $\partial_1$ from (2) is an isomorphism. Note that an algebraic structure of groups $\pi_1 O_f(f)$ for Morse functions on a 2-disks $D^2$ and a cylinder $S^1 \times [0, 1]$ are described in [19].

An algebraic structure of $\pi_1 O_f(f)$ for the Morse functions on 2-torus was studied in the series of papers by S. Maksymenko and the author [21, 13, 20, 3]. It is well known that $\mathcal{D}_{id}(T^2)$ is not contractible, so the image of $\pi_1 \mathcal{D}_{id}(T^2)$ is nontrivial in $\pi_1 O_f(f)$. In this case, an algebraic structure of $\pi_1 O_f(f)$ is reduced to studying the restrictions of a given function to subsurfaces of $T^2$ being 2-disks and cylinders, for which this structure is known. These cylinders and 2-disks are in some sense building blocks which carry the “combinatorial symmetries” of the function. In particular, for the case of functions on $T^2$ the sufficient conditions when the sequence (2) splits was described in [13]. A good overview the reader can find in [19] where these results are presented in the form of so-called crystallographic and Bieberbach sequences.

Our main goal is to generalize the results on algebraic structure of $\pi_1 O_f(f)$ for Morse functions on $T^2$ to the case of circle-valued Morse function on $T^2$. Theorem 2.1 is our main result. Note that if $f$ is a function from $\mathcal{F}(T^2, P)$ without critical points, then $f : T^2 \to S^1$ is a locally trivial fibration and the homotopy types of orbits and stabilizers are known, see [14, Theorem 1.9].

In subsequent parts of the present section we will study homotopy properties orbits of null-homotopic Morse functions on smooth compact oriented surfaces and Kronrod-Reeb graphs of circle-valued functions on $T^2$.

1.2. Null-homotopic $S^1$-valued Morse functions and their orbits. A function $f : M \to S^1$ homotopic to a constant map, a map to some point of $S^1$, will be called null-homotopic.

It is easy to see that homotopy properties of null-homotopic Morse functions to $S^1$ are the same as for its “universal” lift, an ordinary Morse function arising from a universal cover of $S^1$. To be more precise, consider a universal covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. Then by a lifting property for maps [6, Proposition 1.33], there exist a unique smooth function $\tilde{f} : M \to \mathbb{R}$ (up to a choice of appropriate triple of points) such that $p \circ \tilde{f} = f$. Note that the function $\tilde{f}$ is also an ordinary Morse function in the sense of our definition. This function $\tilde{f}$ will be called a universal lift of $f$. The fact that homotopy properties of $O_f(f)$ are the same as for $O_f(\tilde{f})$ follows from the proposition.

**Proposition 1.3.** Let $f : M \to S^1$ be a null-homotopic Morse function and $\tilde{f} : M \to \mathbb{R}$ be its universal lift. Then $O(f)$ and $O(\tilde{f})$ are homeomorphic, and hence $O_f(f) \cong O_f(\tilde{f})$. 
Proof. Recall that $\zeta_f : \mathcal{D}(M) \to \mathcal{O}(f)$ and $\zeta_{\tilde{f}} : \mathcal{D}(M) \to \mathcal{O}(\tilde{f})$ are locally trivial principal fibrations with fibers $\mathcal{S}(f)$ and $\mathcal{S}(\tilde{f})$ respectively. Then $\zeta_f$ decomposes as the composition

$$\mathcal{D}(M) \xrightarrow{\zeta_f} \mathcal{O}(f) \xrightarrow{\alpha_f} \mathcal{D}(M)/\mathcal{S}(f),$$

where $\alpha_f(h) = \overline{h}$ is an open quotient-map, and $\beta_f(\overline{h}) = \zeta_f(h)$ is an induced by $\zeta_f$ homeomorphism, where $\overline{h}$ is a coset of $h \in \mathcal{D}(M)$ modulo $\mathcal{S}(f)$. The same decomposition holds for $\zeta_f$. So $\mathcal{O}(f)$ and $\mathcal{O}(\tilde{f})$ are homeomorphic to quotient groups $\mathcal{D}(M)/\mathcal{S}(f)$ and $\mathcal{D}(M)/\mathcal{S}(\tilde{f})$ respectively. S. Maksymenko [19, Lemma 5.3] showed that there is a homeomorphism $\mathcal{S}(f) \cong \mathcal{S}(\tilde{f})$. Then quotient-spaces $\mathcal{D}(M)/\mathcal{S}(f)$ and $\mathcal{D}(M)/\mathcal{S}(\tilde{f})$ are homeomorphic, which implies that $\mathcal{O}(f)$ is homeomorphic to $\mathcal{O}(\tilde{f})$, and so $\mathcal{O}_f(f) \cong \mathcal{O}_f(\tilde{f})$. \hfill \Box

By Proposition 1.3 the homotopy properties of $\mathcal{O}_f(f)$ for null-homotopic Morse functions $f : M \to S^1$ are “the same” as for $\mathcal{O}_f(\tilde{f})$ where $\tilde{f} : M \to \mathbb{R}$ is a universal lift of $f$. This case have been studying in a series of papers [21, 13, 20, 3] mentioned before, therefore we will focus on the case when $f$ is not null-homotopic.

1.4. Graphs of $P$-valued Morse functions. Kronrod-Reed graphs are important tools to study Morse functions since they carry a lot of informations about its combinatorial structure. We recall this definition. Let $f : M \to P$ be a Morse function on a smooth compact oriented surface $M$ and $c \in P$. A connected component $C$ of a level-set $f^{-1}(c)$ is called critical, if $C$ contains at most one critical point of $f$, and $C$ is called regular otherwise. Let $\Xi$ be a partition of $M$ into connected components of level-sets of $f$. It is well-known that the quotient-space $M/\Xi$ denoted by $\Gamma_f$ has a structure of an 1-dimensional CW complex called a Kronrod-Reeb graph of $f$, or simply a graph of $f$.

Let $q_f : M \to \Gamma_f$ be a quotient-map. Then the map $f : M \to P$ can be presented as the composition of the projection $q_f$ onto $\Gamma_f$ and the map $f_\Gamma$ induced by $f$:

$$f = f_\Gamma \circ q_f : M \xrightarrow{q_f} \Gamma_f \xrightarrow{f_\Gamma} P. \quad (4)$$

Graphs of $S^1$-valued Morse function on $T^2$. Graphs of ordinary Morse functions on $T^2$ were studied in [9].

Lemma 1.5 (cf. Lemma 2.3 [2]). Let $f : T^2 \to S^1$ be a Morse function. If $f$ is null-homotopic, then $\Gamma_f$ is either a tree or contains a cycle. Otherwise, $\Gamma_f$ only contains a cycle.

Proof. Assume that $f$ is a null-homotopic Morse function. Let $\tilde{f} : M \to \mathbb{R}$ be its universal lift. For an ordinary Morse functions this result is proved in [2].

Therefore assume that $f$ is not null-homotopic. It is easy to show that $\Gamma_f$ is either a tree or contains a cycle. We claim that if $f$ is not null-homotopic, then $\Gamma_f$ is not a tree. Assume the converse holds and consider the sequence of fundamental
groups induced by (4):

\[
\begin{array}{c}
\pi_1T^2 \xrightarrow{q^*_T} \pi_1\Gamma_f \xrightarrow{f^*_T} \pi_1S^1 \\
\mathbb{Z}^2 \xrightarrow{q^*_f} 0 \xrightarrow{f^*_f} \mathbb{Z}
\end{array}
\]

The homomorphism \( f^*_T \) is a zero-map, so the homomorphism \( f^* = f^*_T \circ q^*_f \) induced by \( f \) is also a zero map. This leads to the contradiction that \( f \) is not null-homotopic, so \( \Gamma_f \) contains a cycle. \( \square \)

The case of Morse functions \( f : T^2 \to S^1 \) whose graphs are trees is a special case – it is only possible when \( f \) is null-homotopic. So in further text we will study only the case of Morse functions whose graphs contain a cycle. Examples of such functions are given in Figure 1.

![Figure 1. Null-homotopic (left) and not null-homotopic (right) functions \( T^2 \to S^1 \)](image)

Note that functions in Figure 1 allow some “combinatorial symmetries” preserving the given functions which play an essential role in the description of \( \pi_1 \mathcal{O}_f(f) \).

2. MAIN RESULT

To state our main result we need a notion of wreath product of groups of a special kind. Let \( G \) be a group, \( n \geq 1 \) be an integer. A semi-direct product \( G^n \rtimes \mathbb{Z} \) with respect to a non-effective \( \mathbb{Z} \)-action \( \alpha \) on \( G^n \) by cyclic shifts

\[
\alpha(b_0, b_1, \ldots, b_{n-1}; k) = (b_k, b_{1+k}, \ldots, b_{n+k-1}),
\]

where all indexes are taken modulo \( n \), will be denoted by \( G \wr_n \mathbb{Z} \) and called a wreath product of \( G \) with \( \mathbb{Z} \) under \( n \). Notice that this definition differs from the standard one, [22].

The following theorem describes an algebraic structure of \( \pi_1 \mathcal{O}_f(f) \) for Morse function \( f : T^2 \to S^1 \) whose graphs contain a cycle. It will be proved in Section 4.

**Theorem 2.1.** Let \( f \) be a function from \( \mathcal{F}(T^2, P) \) with at least one critical point and whose graph \( \Gamma_f \) contains a cycle. Then there exist a cylinder \( Q \subset T^2 \) and \( n \in \mathbb{N} \) such that \( f|_Q : Q \to S^1 \) is a Morse function and there is an isomorphism

\[
\pi_1 \mathcal{O}_f(f) \cong \pi_0 \mathcal{S}'(f|_Q, N(\partial Q)) \wr_n \mathbb{Z},
\]
where $N(\partial Q)$ is some regular neighborhood of $\partial Q$ containing no critical points of $f$.

**Remark 2.2.** Theorem 2.1 generalizes the main result of the paper [20]. To prove [20, Theorem 1.6] we mainly use properties of diffeomorphisms of subsurfaces of $T^2$ and then "glue" them together to obtain global ones. So Theorem 2.1 can be proved step-by-step by the same arguments and strategy. In the present paper we give more straightforward proof of this result separating algebraic methods from topological ones. Proofs of some known facts will be given only for the sake of completeness.

2.3. **The structure of the paper.** In Section 3 we will give some preliminary constructions and definitions needed to prove Theorem 2.1 in Section 4. Section 5 contains the result about the image of the generator $\mathbf{M}$ of $\pi_1\mathbb{D}_{id}(T^2)$ in $\pi_1\mathcal{O}_f(f)$. In Section 6 we discuss results about other spaces containing the homotopy information about $\pi_1\mathcal{O}_f(f)$ which can be obtained from Theorem 2.1.

3. **Axillary constructions and definitions**

3.1. **Curves on $T^2$.** Let $f$ be a function from $\mathcal{F}(T^2, P)$ whose graph $\Gamma_f$ contains a unique cycle denoted by $\Lambda$, let also $q_f : T^2 \to \Gamma_f$ be a projection induced by $f$. Let $z$ be a point in $\Lambda$, $c = f(q_f^{-1}(z))$ be a point in $S^1$, and $C$ be a regular connected component of $f^{-1}(c)$. Note that $f^{-1}(c)$ consists of finitely many connected components and it is invariant under the action of $S'(f)$. We set $\mathcal{C} = \{h(C) | h \in S'(f)\}$. Since $\mathcal{C}$ has finite cardinality we can cyclically enumerate elements of the set $\mathcal{C}$ $\mathcal{C} = \{C_0 = C, C_1, C_2, \ldots, C_{n-1}\}$ for some $n \in \mathbb{N}$. Curves from $\mathcal{C}$ are mutually disjoint and do not separate $T^2$, and each pair $C_i$ and $C_{i+1}$ bounds a cylinder $Q_i \subset T^2$, where all indexes are taken modulo $n$.

3.2. **$f$-adapted neighborhoods.** We regard $S^1$ and $T^2$ as a quotient-spaces $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2/\mathbb{Z}^2$ respectively. By a proper choose of coordinates on $T^2$ one can assume that the following conditions hold:

- $C_i = \{\frac{i}{n}\} \times S^1 \subset \mathbb{R}^2/\mathbb{Z}^2 = T^2$, so we can regard each curve $C_i$ as a meridian of $T^2$, and the curve $C' = \{0\} \times S^1$ as a longitude of $T^2$.
- there exists $\varepsilon > 0$ such that for all $t \in \left(\frac{1}{n}, \frac{1}{n} + \varepsilon\right)$ the curve $\{t\} \times S^1$ is regular connected component of some level-set of $f$.

This assumption makes possible to define special neighborhoods of curves from $\mathcal{C}$. A neighborhood $V$ of $C \in \mathcal{C}$ will be called $f$-adapted if

- $V$ is diffeomorphic to $S^1 \times [0, 1]$ via a diffeomorphism, say $\phi$,
- $\phi^{-1}(S^1 \times \{t\})$ is a connected component of some level set of $f$, $t \in [0, 1]$,
- $V$ does not contain critical points of $f$.

In this place we fix two families of $f$-adapted neighborhoods $V_i$ and $W_i$ of $C_i$, $i = 0, \ldots, n - 1$ needed in the further text, so that $V_i \cap V_j = \emptyset$, $V_i \subset \text{Int}(W_j)$ for $i \neq j$ and for each $i, j$ there exists $h \in S'(f)$ such that $h(V_i) = V_j$. In particular, unions $V = \bigcup_{i=0}^{n-1} V_i$ and $W = \bigcup_{i=0}^{n-1} W_i$ are $S'(f)$-invariant.

3.3. **Generators of $\pi_1\mathbb{D}_{id}(T^2)$.** Let $\mathbf{L}, \mathbf{M} : T^2 \times [0, 1] \to T^2$ be two isotopies defined by

$$L(x, y, t) = (x + t \mod 1, y), \quad M(x, y, t) = (x, y + t \mod 1) \quad (5)$$
for \( x \in C', y \in C_k, k = 0, 1, \ldots, n - 1 \). Geometrically \( L \) is a rotation of \( T^2 \) along its longitude, and \( M \) is a rotation along meridians. Isotopies \( L \) and \( M \) can be regarded as loops in \( \mathcal{D} \text{id}(T^2) \). It is well known that \( L \) and \( M \) commute and \( \pi_1 \mathcal{D} \text{id}(T^2) = \langle L \rangle \times \langle M \rangle \), see [1, 5].

3.4. Dehn twists and slides along curves from \( C \). Let \( Q = S^1 \times [0, 1] \) be a cylinder and \( C \) be the curve \( S^1 \times \{0\} \), and \( \alpha, \beta : [-1, 1] \to [0, 1] \) be two smooth functions such that

\[
\alpha(x) = \begin{cases} 
0, & x \in [-1, -1/2], \\
1, & x \in [1/2, 1],
\end{cases} \\
\beta(x) = \begin{cases} 
0, & x \in [-1, -2/3] \cup [2/3, 1], \\
1, & x \in [-1/3, 1/3].
\end{cases}
\]

Define two diffeomorphisms of \( Q \) by formulas:

\[
\tau(z, t) = (ze^{\alpha(t)}, t) \quad \theta(z, t) = (ze^{\beta(t)}, t), \quad (z, t) \in Q;
\]

the diffeomorphisms \( \tau \) and \( \theta \) are called a Dehn twist and a slide along \( C = S^1 \times \{0\} \). Note that \( \tau \) is fixed on some neighborhood of \( \partial Q \), and \( \theta \) is fixed on some neighborhood of \( C \cup \partial Q \). A diffeomorphism of a smooth surface \( M \) supported in some cylindrical neighborhood of a simple closed and two-sided curve \( C \subset M \) isotopic to a Dehn twist with respect to the boundary of this neighborhood will be called a Dehn twist along \( C \) on \( M \). Similarly the notion of slide along \( C \) can be extended to the case of surfaces.

Recall that a vector field \( F \) on a smooth oriented surface \( M \) is called Hamiltonian-like for a Morse function \( f \) if the following conditions satisfy:

- singular points of \( F \) correspond to critical points of \( f \),
- \( f \) is constant along \( F \),
- Let \( z \) be a critical point of \( f \). Then there exists a local coordinate system \((x, y)\) such that \( f(z) = 0, f(x, y) = \pm x^2 \pm y^2 \) near \( z \), and in this coordinates \( F \) has the form \( F(x, y) = -f'_x \partial /\partial x + f'_y \partial /\partial y \).

Fix a Hamiltonian-like vector field \( F \) of the given function \( f : T^2 \to S^1 \), and let \( F_t : T^2 \to T^2, t \in \mathbb{R} \) be its flow. The set \( W \) does not consist singular points of \( F \) and it is \( F \)-invariant and consists of periodic orbits. So one can assume that periods of all trajectories of \( F \) are equal to 1 on \( W \).

Let \( \theta_i : T^2 \to T^2 \) be a slide along \( C_i \) supported on \( W_i - V_i, i = 0, 1, \ldots, n - 1 \), and set \( \theta = \theta_0 \circ \theta_1 \circ \ldots \circ \theta_{n-1} \). We proved that there exists a smooth function \( \sigma : T^2 \to \mathbb{R} \) which satisfies

- \( \sigma \) is constant along trajectories of \( F \),
- \( \sigma = 1 \) on \( V \), \( \sigma = 0 \) on \( T^2 - W \), and
- \( \theta = F_\sigma \),

and therefore \( \theta^k = F_{k\sigma} \), see [20, Lemma 5.2]. A free abelian group generated by \( \theta \) will be denoted by \( \langle \theta \rangle \).

3.5. Algebraic preliminaries. In this paragraph we recall conditions when the group \( G \) splits into a direct product of its subgroups and discuss when \( G \) splits into a wreath product as in Section 2. Let \( G \) be a group and \( G_1, \ldots, G_n \) be their subgroups. It is well-known that the group \( G \) splits into a direct product \( G_1 \times G_2 \times \ldots \times G_n \) if the following three conditions satisfied:

(D1) \( G_i \cap G_j = \{ e \} \), for \( i \neq j = 1, 2, \ldots, n \), where \( e \) is the unit of \( G \),

(D2) \( G_i G_j = G_j G_i \) for all \( i, j = 1, 2, \ldots, n \),

(D3) groups \( G_1, G_2 \) and \( G_n \) generate \( G \).
The following lemma gives conditions when the group $G$ splits into a wreath product $L_0 \wr_n Z$ for some subgroup $L_0 \subset G$.

**Lemma 3.6** (Lemma 2.3 [19]). Let $\phi : G \to \mathbb{Z}$ be an epimorphism and $L_0$ be a subgroup of $\ker \phi$. Let also $g \in G$ be with $\phi(g) = 1$. Assume that for some $n \in \mathbb{Z}$ the following conditions hold:

1. $g^n$ commutes with $\ker \phi$,
2. $\ker \phi$ splits into a direct product of $L_0$, $L_1 = g^{-1}L_0g$, ..., $L_{m-1} = g^{-(n-1)}L_0g^{n-1}$.

Then the map $\xi : L_0 \wr_n \mathbb{Z} \to G$ given by the formula

$$
\xi(b_0, b_1, \ldots, b_{n-1}, k) = b_0(g^{-1}b_1g)(g^{-2}b_2g^2)\ldots(g^{-n+1}b_{n-1}g^{n-1})g^k
$$

is an isomorphism.

4. **Proof of Theorem 2.1**

4.1. **Structure of the proof.** Our main proof-tool is Lemma 4.7. So we need to define a “natural” epimorphism $\phi : \pi_1 O_f(f) \to \mathbb{Z}$, an element $g$ from $\ker \phi$, and groups $L_i$, $i = 0, 1, \ldots, n-1$ such in Lemma 3.6.

4.2. **Epimorphism $\phi$ and its kernel.** Let $V_i$ and $W_i$ be fixed $f$-adapted neighborhoods of $C_i \subset \mathbb{C}$, $i = 0, 1, \ldots, n-1$ such in Subsection 3.2.

**Proposition 4.3** (Theorem 6.1 [20]). There exists an epimorphism $\phi : \pi_1 O_f(f) \to \mathbb{Z}$ with the kernel of $\phi$ isomorphic to $\pi_0 S'(f, W)$, i.e., the following sequence of groups

$$
\pi_0 S'(f, W) \longrightarrow \pi_1 O_f(f) \longrightarrow \mathbb{Z}
$$

is exact.

**Proof.** This result is proved in [20], but for completeness of our exposition we will recall the construction of an epimorphism $\phi$. Let $q : \mathbb{R} \times S^1 \to S^1 \times S^1 = T^2$ be a covering map by $q(x, y) = (\frac{x}{2} \mod 1, y)$. Then $q(\{i\} \times S^1) = C_i \times \mathbb{Z}$, $\mathbb{Z} = S^1 \times \mathbb{Z}$.

Let $\omega : [0, 1] \to O_f(f)$ be a loop and $h : T^2 \times [0, 1] \to T^2$ be an isotopy such that $\omega_t = f \circ h_t$ and $h_0 = \text{id}_{T^2}$, $h_1 \in S'(f)$. There exist an isotopy $\tilde{h} : (\mathbb{R} \times S^1) \times [0, 1] = \mathbb{R} \times S^1$ such that $\tilde{h}_0 = \text{id}_{\mathbb{R} \times S^1}$ and $q \circ \tilde{h}_t = h_t \circ q$ for all $t \in [0, 1]$. Since $h_1(C) = C$ then from the definition of $q$ we have $\tilde{h}_1(\mathbb{Z} \times S^1) = \mathbb{Z} \times S^1$. Then there exists an integer $\phi_t$ such that

$$
\tilde{h}_1(\{i\} \times S^1) = \{i + \phi_t\} \times S^1.
$$

The number $\phi_t$ depends only on the homotopy class of $h_t$, and so on the isotopy class of the loop $\omega$. Is is easy to see that the correspondence

$$
\begin{align*}
\{[w]\} & \longrightarrow \phi_t \in \mathbb{Z}
\end{align*}
$$

defined by (7) is an epimorphism $\phi : \pi_1 O_f(f) \to \mathbb{Z}$.

The kernel of $\phi$ consists of homotopy classes of isotopies $h : T^2 \times [0, 1] \to T^2$ such that $h_1$ leaves invariant each curve $C_i$ from $C$, i.e., $h_1(C_i) = C_i$. It was shown [14, Lemma 4.14] that each such $h_1$ can be isotoped in $S'(f)$ to the diffeomorphism

$$
\begin{align*}
\begin{cases}
\text{rotate } 2\pi \text{ around } C_i, \\
\text{fix otherwise.}
\end{cases}
\end{align*}
$$

and then $\phi$ is an epimorphism.


Let \( g \) denote by \( g \) of Proposition 4.4. To prove (1) and (2) we need to replace \( \phi \) is an isomorphism. So the kernel of \( \phi \) is isomorphic to \( \pi_0S'(f, W) \).

**4.4. Special isotopy and subgroups of** \( \ker \phi \). The following proposition holds true.

**Proposition 4.5** (Theorem 6.1 (c), [20]). There exist an isotopy \( g : T^2 \times [0, 1] \to T^2 \) such that

1. \( g_1 \in S'(f, W) \),
2. \( g_1^n = \text{id}_{T^2} \),
3. \( g_1(Q_i) = Q_{i+1} \), \( i = 0, 1, \ldots, n-1 \),
4. \( \phi([f \circ g_1]) = 1 \).

**Proof.** By definition of the set \( \mathcal{C} \), there exists \( g_1 \in S'(f) \) such that \( g(Q_i) = Q_{i+1} \)

So (3) is obvious. To prove (1) and (2) we need to replace \( g_1 \) to \( \sigma = L_{1/n} \) on \( W \) in \( S'(f) \); this was done in (c) of [20, Theorem 6.1]. The resulting diffeomorphism we also denote by \( g_1 \).

(4) Let \( g : T^2 \times [0, 1] \to T^2 \) be an isotopy between \( g_1 \) and \( \text{id}_{T^2} \). So \( \phi([g]) = an + 1 \) for some \( n \in \mathbb{Z} \). If \( a \neq 0 \) then we replace \([g_1]\) by \([g_1 \circ L_{-a}^n]\) in order to have \( \phi([f \circ g_1]) = 1 \).

Denote by \( X_i^- \) and \( X_i^+ \) the following intersections \( Q_i \cap W_i \), \( Q_i \cap W_{i+1} \), and set \( X_i = X_i^- \cup X_i^+ \), \( U_i = Q_i - X_i \) for \( i = 0, 1, \ldots, n-1 \). The set \( X_i \) is an \( f \)-adapted neighborhood of the boundary \( \partial Q_i \) of the cylinder \( Q_i \subset T^2 \); see Figure 2.

![Figure 2](image)

We denote by \( L_i \) the following subgroup of the kernel of \( \phi \):

\[
L_i = \pi_0S'(f, T^2 - U_i), \quad i = 0, 1, \ldots, n-1.
\]

Let \( g : T^2 \times [0, 1] \to T^2 \) be an isotopy from Proposition 4.5. Since \( g_1 \) satisfies (1)-(4) of Proposition 4.5, it follows that

\[
L_i = [g_i^{-1}]L_0[g_i], \quad i = 0, 1, \ldots, n-1.
\]

Since each diffeomorphism \( h \in S'(f, T^2 - U_i) \) is fixed on \( T^2 - U_i \), it follows that the restriction \( h \to h \mid_{Q_i} \) induces an isomorphism

\[
\beta_i : \pi_0S'(f, T^2 - U_i) \to \pi_0S'(f \mid_{Q_i}, X_i)
\]

given by the restriction map \( \beta_i(h) = [h \mid_{Q_i}] \). So we will not distinguish between groups \( \pi_0S'(f, T^2 - U_i) \) and \( \pi_0S'(f \mid_{Q_i}, X_i) \).
4.6. The end of the proof. The following lemma completes the proof.

**Lemma 4.7.** The data of an epimorphism \( \phi \) and the element \( g \) from Proposition 4.5, and groups \( L_i = \pi_0 S'(f|Q_i, X_i) \) from (8) satisfy conditions of Lemma 3.6. So \( \pi_1 O_f(f) \) is isomorphic to \( \pi_0 S'(f|Q_0, X_0) \wr \mathbb{Z} \) and this isomorphism is given by the formula (6).

**Proof.** Let \( g : T^2 \times [0, 1] \to T^2 \) is an isotopy defined from Proposition 4.5. By (2) of Proposition 4.5 the diffeomorphism \( g_1 = \text{id}_{T^2} \), and so it commutes with \( \ker \phi \). Thus (1) of Lemma 3.6 holds true. To verify (2) of Lemma 3.6 we need to check that the three conditions (D1)–(D3) from subsection 3.5 satisfied:

1. \( L_i \cap L_j = [\text{id}_{T^2}], i \neq j \)
2. \( L_i L_j = L_j L_i \)
3. groups \( L_0, L_1, \ldots, L_{n-1} \) generates \( \ker \phi \),

for all \( i, j = 0, 1, \ldots, n-1 \). Conditions (1) and (2) follows from the fact that \( \text{supp}(h_i) \cap \text{supp}(h_j) = \emptyset \) for \( [h_i] \in L_i, [h_j] \in L_j, i \neq j = 0, 1, \ldots, n-1 \). To see that (3) holds suppose \( h \) is a diffeomorphism from \( S'(f, W) \), and \( h_i = h|Q_i \) is a restriction of \( h \) onto \( Q_i, i = 0, 1, \ldots, n-1 \). Note that \( h_i \) is fixed on \( X_i \) and so \( h_i \) can be extended to a diffeomorphism of \( T^2 \) by the identity map. It is easy to see that \([h]\) can be represented as the composition:

\[
[h] = \beta_0^{-1}([h_0]) \circ \beta_1^{-1}([h_1]) \circ \ldots \circ \beta_{n-1}^{-1}([h_{n-1}]).
\]

Therefore groups \( L_0, L_1, \ldots, L_{n-1} \) generates \( \ker \phi \).

\[ \square \]

5. INTERMEZZO. THE KERNEL OF \( \phi \) AND AN ISOTOPY \( M \)

Note that the group \( \langle M \rangle \) contains in the kernel of \( \phi \), but to prove Theorem 2.1 we do not need the explicit form of the element of \( \ker \phi \) representing \( M \). Let \( V \) and \( W \) be fixed in subsection 3.2 \( f \)-adapted neighborhoods of \( C \). Next proposition describes the relationship between \( \ker \phi \) and the group \( M \).

**Proposition 5.1.** There is an isomorphism \( \pi_0 S'(f, W) \cong \langle M \rangle \times \pi_0 S'(f, V) \).

**Proof.** In [20] we showed that the isotopy class of \( M \) can be represented as the isotopy class of \( \theta = \theta_0 \circ \theta_1 \circ \ldots \circ \theta_{n-1} \), where \( \theta_i \) is a slide along \( C_i \) supported in \( W_i - V_i, i = 0, 1, 2, \ldots, n-1 \).

To prove this proposition we need to check the conditions (D1)–(D3) from subsection 3.5 hold for groups \( \langle \theta \rangle \) and \( \pi_0 S'(f, V) \). Conditions (D1) and (D2) obviously hold since \( \text{supp}(\theta) \cap \text{supp}(h) = \emptyset \), where \( h \in S'(f, V) \). So, it remains to check that for each \( h \) in \( S'(f, W) \) there is a unique decomposition

\[
[h] = [g^{k(h)}] \circ [h'],
\]

where \( h' \in S'(f, V) \) and \( k(h) \in \mathbb{Z} \). To define this decomposition we use the same arguments such in [2, Theorem 5.5].

Let \( h \) be a diffeomorphism from \( S'(f, W) \). Since \( f \) is fixed on \( W \), it follows from Lemma [19, Lemma 6.1], there exists a smooth function \( \alpha : W \to \mathbb{R} \) such that \( h = F \alpha \). The restriction of \( h \) and \( \alpha \) onto \( W_i \) are denoted by \( h_i \) and \( \alpha_i \), respectively. Since periods of all trajectories of \( F|W \) is equal to 1, it follows that \( \alpha_i \) takes an integer number \( k_i(h) \in \mathbb{Z} \). \( x \in W_i \). The diffeomorphism \( h_i|Q_i \) is isotopic relative \( W \cap Q_i \) to a Dehn twist \( \tau^{a_i} \) supported on \( W \cap Q_i \), where \( a_i = \alpha(C_{i+1}) - \alpha(C_i), i = 0, 1, \ldots, n-1 \). Since \( h \in S'(f, W) \), it follows that \( h|Q_i \) is isotopic to the identity
map of $Q$. Hence $a_i = \alpha(C_{i+1}) - \alpha(C_i) = k_i(h) = k_i(h) = 0$. Then numbers $k_i(h)$ pairwise equal for $i = 0, 1, \ldots, n - 1$, so they all coincide, i.e., $k_i(h) = k(h)$.

Define an isotopy $H^t : T^2 \to T^2$ between $h$ and $\theta^{-1}(h) \circ h$ by the formula

$$H^t(h) = F_{t(k)}^{-1} \circ h.$$

A diffeomorphism $H^t(h)$ is fixed on $V$ for all $t \in [0, 1]$. Then we have the following decomposition

$$[h] = [\theta^k(h)] \circ [\theta^{-1}(h) \circ h] = [\theta^k(h)] \circ [H^1(h)]$$

which coincides with (10) for $h' = H^t(h)$. □

6. Conclusion remarks

In this section we discuss some consequences of Theorem 2.1 – an algebraic structure of groups which carry some informations about $\pi_1 O_f(f)$ and classes of groups $\pi_1 O_f(f)$.

An algebraic structure of $\pi_1 O_f(f)$ for a Morse function $f : T^2 \to S^1$ is partially determined by “symmetries” of the function $f$, i.e., by the group $\pi_0 S'(f)$ and its subgroups. Note that each $h \in S(f)$ preserves the partition of $T^2$ into connected component of level-sets of $f$. So it induces an automorphism $\rho(h)$ of the graph $\Gamma_f$ of $f$, i.e., the following map $\rho : \pi : S'(f) \to \text{Aut}(\Gamma_f)$ is a homomorphism. We denote by $G(f)$ the image of $S'(f)$ with respect to $\rho$. This group $G(f)$ is a finite subgroup of $\text{Aut}(\Gamma_f)$. A homomorphism $\rho$ induces an epimorphism $\pi_0 S'(f) \to G(f)$ which we also denote by $\rho$. Let $\Delta(f)$ be a subgroup of $S(f)$ consisting of diffeomorphisms which leave invariant every connected component of each level-set of $f$ and $\Delta(f)$ be the intersection $\Delta(f) \cap \text{Diff}(T^2)$. Note that the kernel of $\rho$ is a free abelian group and $\ker \rho = \pi_0 \Delta'(f)$. Groups $\pi_0 \Delta'(f)$ and $G(f)$ encodes “combinatorially” trivial and non-trivial information about $\pi_0 S'(f)$, and so about $\pi_1 O_f(f)$. An algebraic structure of these groups can be obtained by using only algebraic technique and the structure of $\pi_1 O_f(f)$, [2, Theorem 3]. It can be easily generalized to the circle-valued case: for non null-homotopic Morse functions $T^2 \to S^1$ the statement (2) of [2, Theorem 3] holds. We also mention that classes of isomorphisms of groups $G(f)$ and $\pi_1 O_f(f)$ for Morse functions on $T^2$ were studied in [9] and [12]. These results can be also generalized to circle-valued case and we left them to the reader.

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