Metric characterizations of superreflexivity in terms of word hyperbolic groups and finite graphs

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Abstract. We show that superreflexivity can be characterized in terms of bilipschitz embeddability of word hyperbolic groups. We compare characterizations of superreflexivity in terms of diamond graphs and binary trees. We show that there exist sequences of series-parallel graphs of increasing topological complexity which admit uniformly bilipschitz embeddings into a Hilbert space, and thus do not characterize superreflexivity.

Keywords: bi-Lipschitz embedding; diamond graphs; series-parallel graph; superreflexivity; word hyperbolic group

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1 Introduction

The theory of superreflexive Banach spaces was created by James (see [20, 21]), an important building block for its foundation was added by Enflo [15]. One of the equivalent definitions is: A Banach space X is superreflexive if and only if it has an equivalent uniformly convex norm. The theory of superreflexive spaces is a rich theory, one of the reasons for this richness is that superreflexive spaces can be characterized in many different ways, see accounts in [3, 4, 11, 13, 27, 29].

In addition to having many “linear” characterizations, the class of superreflexive Banach spaces admits many purely metric characterizations (that is, characterizations which do not refer to the linear structure of the space). The purpose of this

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paper is to obtain some new results on metric characterizations of superreflexivity. We start by reminding the known metric characterizations of superreflexivity.

The first metric characterization of superreflexivity was discovered by Bourgain \[5\]. Recall that a binary tree of depth \(n\) is a finite graph \(T_n\) in which each vertex is represented by a finite (possibly empty) sequence of 0s and 1s, of length at most \(n\). Two vertices are adjacent if the sequence corresponding to one of them is obtained from the sequence corresponding to the other by adding one term on the right. We consider a binary tree of depth \(n\) as a metric space whose elements are vertices of the tree, and the distance is the shortest path distance.

**Theorem 1.1 \([5]\).** A Banach space \(X\) is nonsuperreflexive if and only if it admits bilipschitz embeddings with uniformly bounded distortions of finite binary trees of all depths.

In Bourgain’s proof the difficult direction is the “if” direction, the “only if” is an easy consequence of the theory of superreflexive spaces. Recently Kloeckner \[23\] found a simple proof of the “if” direction.

We are going to use the following terminology:

**Definition 1.2.** Let \(\mathcal{P}\) be a class of Banach spaces and let \(T = \{T_\alpha\}_{\alpha \in A}\) be a set of metric spaces. We say that \(T\) is a set of test-spaces for \(\mathcal{P}\) if the following two conditions are equivalent:

1. \(X \notin \mathcal{P}\);
2. The spaces \(\{T_\alpha\}_{\alpha \in A}\) admit bilipschitz embeddings into \(X\) with uniformly bounded distortions.

### 1.1 Versions of Bourgain’s characterization with one test-space

Baudier \[2\] strengthened the “only if” part of the Bourgain theorem and got the following result. Denote by \(T_\infty\) an infinite graph in which each vertex is represented by a finite (possibly empty) sequence of 0s and 1s. Two vertices are adjacent if the sequence corresponding to one of them is obtained from the sequence corresponding to the other by adding one term on the right. We consider \(T_\infty\) as a set of vertices endowed with the shortest path distance.

**Theorem 1.3 \([2]\).** A Banach space \(X\) is nonsuperreflexive if and only if it admits bilipschitz embedding of \(T_\infty\).

**Remark 1.4.** There exists a much simpler result stating that there exists a one-test-space characterization superreflexivity, and even that the test-space can be chosen to be a tree, see Section \[5\]. An important feature of Theorem 1.3 is that the test-space can be chosen to be \(T_\infty\).

A more general than Theorem 1.3 result was proved in \[26\] (recall that a metric space is called locally finite if all balls of finite radius in it have finite cardinalities):
Theorem 1.5 ([26], also [27, Sections 2.1 and 2.3]). Let $A$ be a locally finite metric space whose finite subsets admit uniformly bilipschitz embeddings into a Banach space $X$. Then $A$ admits a bilipschitz embedding into $X$.

Remark 1.6. In [27, Definition 2.1] locally finite spaces are required to be uniformly discrete, but this property is not used in the proof of Theorem 1.5.

Theorem 1.5 is more general than Theorem 1.3 in the sense that combining Theorem 1.5 with Theorem 1.1 we immediately get Theorem 1.3. The first goal of this paper is to show that any infinite finitely generated word hyperbolic group (word hyperbolic in the sense of Gromov [17]) which does not contain $\mathbb{Z}$ as a finite index subgroup, is also a test-space for superreflexivity:

Theorem 1.7. (a) Let $G$ be a finitely generated word hyperbolic group. Then the Cayley graph of $G$ admits a bilipschitz embedding into an arbitrary nonsuperreflexive Banach space.

(b) Let $G$ be an infinite finitely generated word hyperbolic group which does not have a finite index subgroup isomorphic to $\mathbb{Z}$. If $G$ admits a bilipschitz embedding into a Banach space $X$, then $X$ is nonsuperreflexive.

In (b) we need to exclude groups which contain $\mathbb{Z}$ as a finite index subgroup because Cayley graphs of such groups admit bilipschitz embeddings into $\mathbb{R}$, and hence into any Banach space of dimension at least 1 (this fact is well known, because we do not know a suitable reference we prove it in Proposition 2.5 for completeness).

We prove Theorem 1.7 in Section 2.

1.2 Characterizations of superreflexivity using different sequences of finite graphs

Johnson and Schechtman [22] proved that binary trees in Theorem 1.1 may be replaced by some other sequences of finite graphs, for example, by so-called diamond graphs introduced in (the conference version of) [18]. Diamond graphs can be defined as follows: The diamond graph of level 0 is denoted $D_0$. It has two vertices joined by an edge. $D_i$ is obtained from $D_{i-1}$ as follows. Given an edge $uv \in E(D_{i-1})$, it is replaced by a quadrilateral $u, a, v, b$. We endow vertex sets of $D_n$ with their shortest path metrics (each edge is assumed to have length 1). In this context we consider diamonds as finite metric spaces.

Theorem 1.8 ([22]). A Banach space $X$ is nonsuperreflexive if and only if it admits bilipschitz embeddings with uniformly bounded distortions of diamonds $\{D_n\}_{n=1}^{\infty}$ of all sizes.

Our next purpose it to show that this characterization of Johnson and Schechtman is independent of the Theorem 1.1 in the sense that Theorem 1.9 and the statement below it hold.

Let $\{M_n\}_{n=1}^{\infty}$ and $\{R_n\}_{n=1}^{\infty}$ be two sequences of metric spaces. We say that $\{M_n\}_{n=1}^{\infty}$ admits uniformly bilipschitz embeddings into $\{R_n\}_{n=1}^{\infty}$ if for each $n \in \mathbb{N}$
there is \( m(n) \in \mathbb{N} \) and a bilipschitz map \( f_n : M_n \to R_{m(n)} \) such that the distortions of \( \{ f_n \} \) are uniformly bounded.

**Theorem 1.9.** Binary trees \( \{ T_n \}_{n=1}^{\infty} \) do not admit uniformly bilipschitz embeddings into diamonds \( \{ D_n \}_{n=1}^{\infty} \).

The fact that diamonds \( \{ D_n \} \) do not admit uniformly bilipschitz embeddings into binary trees \( \{ T_n \} \) is well known; it follows immediately from the result of Rabinovich and Raz [30, Corollary 5.3] stating that the distortion of any embedding of an \( n \)-cycle into any tree is \( \geq \frac{n}{3} - 1 \), and the fact that \( D_n \ (n \geq 1) \) contains a cycle of length \( 4^n \) isometrically.

We prove Theorem 1.9 in Section 3.

**Remark 1.10.** In [22] it was proved that a characterization similar to Theorem 1.8 can be proved for the sequence of Laakso graphs. An analogue of Theorem 1.9 for Laakso graphs is easy because Laakso graphs are doubling and binary trees are not, see [25, Theorem 1.5] for an interesting related result.

Our next result is related to the following remark of Johnson and Schechtman [22, Remark 6, p. 188]: “In light of [7], it might very well be that Theorem 1.8 extends to any series-parallel graph.” We show that at least in one direction this is not true, namely, we prove Theorem 1.12. We use the following equivalent definition of series-parallel graphs (see [18, p. 243]).

**Definition 1.11.** A weighted graph is called **series-parallel** if it can be obtained in the following way:

- We start with an edge.
- In each step we add a new vertex and attach it to end vertices of an already existing edge.
- At the end of the construction we remove an arbitrary set of edges.

Series-parallel graphs can be also defined as \( K_4 \)-excluded graphs, see [12, Theorem 31.3.7]. Another equivalent definition of series-parallel graphs is in terms of so-called series and parallel compositions, see [16]; this definition explains the name.

**Theorem 1.12.** There exists a sequence of series-parallel graphs \( \{ W_n \}_{n=1}^{\infty} \) which satisfies the following conditions.

1. The underlying unweighted graphs contain \( \{ D_n \}_{n=1}^{\infty} \) as subgraphs.
2. \( \{ W_n \}_{n=1}^{\infty} \) admits uniformly bilipschitz embeddings into \( \ell_2 \).

The point of the first condition is to show that graphs \( \{ W_n \} \) are topologically complicated (it is clear that such series-parallel graphs as paths admit uniformly bilipschitz embeddings into any nonzero Banach space). We prove Theorem 1.12 in Section 4.
We do not know whether the second part of the Johnson-Schechtman remark holds. Namely we do not know whether series-parallel graphs admit uniformly bilipschitz embeddings into any nonsuperreflexive space. It is worth mentioning that any counterexample should involve a non-reflexive Banach space with nontrivial type because Gupta, Newman, Rabinovich, and Sinclair \[18\] Section 4.1 proved that series-parallel graphs admit uniformly bilipschitz embeddings into $\ell_1$.

2 Characterization of superreflexivity in terms of hyperbolic groups

The purpose of this section is to prove Theorem 1.7.

Proof. To prove part (a) we are going to use the Buyalo–Dranishnikov–Schroeder \[8\] result on the metric structure of hyperbolic groups. We need the following definitions. A map $f : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is called a quasi-isometric embedding if there are $a_1, a_2 > 0$ and $b \geq 0$, such that

$$a_1d_X(u, v) - b \leq d_Y(f(u), f(v)) \leq a_2d_X(u, v) + b$$

for all $u, v \in X$. By a binary tree we mean an infinite tree in which each vertex has degree 3. By a product of trees, denoted $(\oplus_{i=1}^n T(i))_1$, we mean (see \[9\] Remark 12.1.2) their Cartesian product with the $\ell_1$-metric, that is,

$$d(\{u_i\}, \{v_i\}) = \sum_{i=1}^n d_{T(i)}(u_i, v_i).$$

Theorem 2.1 (\[8\]). Every Gromov hyperbolic group admits a quasi-isometric embedding into the product of finitely many copies of the binary tree.

This result shows that in order to prove part (a) of Theorem 1.7 it suffices to find

1. A bilipschitz embedding of a product of binary trees in the sense (2) into any nonsuperreflexive space.

2. A way to modify a quasi-isometric embedding of a hyperbolic group into a Banach space in order to get a bilipschitz embedding.

To achieve the first goal we use Theorem 1.5 stating that embeddability of a locally finite metric space into a Banach space is finitely determined. It is clear that the product (in the sense (2)) of finitely many binary trees is a locally finite metric space. Therefore to achieve the goal of item 1 it suffices to find bilipschitz embeddings of products of $n$ binary trees of depth $k$ into any nonsuperreflexive space $X$ with distortions bounded independently of $k$ (for a fixed $n$).

Bourgain \[5\] Section 3] showed that there is an absolute constant $B$ such that for any $k$ and any nonsuperreflexive Banach space $X$ there is a $B$-bilipschitz embedding.
of a binary tree of depth \( k \) into \( X \) (see \[29\] Section 9.1 for a more detailed description of this embedding). To embed a finite sum of finite binary trees \( (\oplus_{i=1}^{n} T(i))_1 \) (where each \( T(i) \) is a binary tree of depth \( k \)) into an arbitrary nonsuperreflexive Banach space \( X \) we do the following. First we find a finite-dimensional subspace \( X_1 \subset X \) containing a \( B \)-bilipschitz image of \( T(1) \) (where \( B \) is the absolute constant mentioned above). We assume that this embedding, and the embeddings of other \( T(i) \) introduced below have Lipschitz constants \( \leq B \), and their inverses have Lipschitz constants \( \leq 1 \).

**Remark 2.2.** Using the argument of James \[19\] (see \[3\] p. 261, \[20\], and \[31\]) one can show that \( B \) can be chosen to be any constant in \((1, \infty)\).

Let \( \lambda \in (0,1] \). Recall that a subspace \( M \subset X^* \) is called \( \lambda \)-norming over a subspace \( Y \subset X \) if

\[
\forall y \in Y \sup\{|f(y)| : f \in M, \|f\| \leq 1\} \geq \lambda\|y\|.
\]

Pick any \( \lambda \in (0,1) \). Let \( M_1 \subset X^* \) be a finite-dimensional subspace which is \( \lambda \)-norming over \( X_1 \). The existence of such subspace can be shown as follows. Let \( \{x_i\}_{i=1}^{n} \) be an \((1-\lambda)\)-net in the unit sphere of \( X_1 \) and let \( M_1 \) be the linear span of functionals \( x_i^* \) satisfying the conditions \( \|x_i^*\| = 1 \) and \( x_i^*(x_i) = 1 \). The verification that \( M_1 \) is \( \lambda \)-norming is immediate.

The subspace \( (M_1)^\perp := \{x \in X : \forall x^* \in M_1 x^*(x) = 0\} \) is of finite codimension. It is clear that for \( x_1 \in X_1 \) and \( x_2 \in (M_1)^\perp \) we have \( \|x_1 + x_2\| \geq \lambda\|x_1\| \).

It is also clear that the subspace \( (M_1)^\perp \) is nonsuperreflexive. Hence we can find in it a finite-dimensional subspace \( X_2 \) containing a \( B \)-bilipschitz copy of \( T(2) \). Now let \( M_2 \subset X^* \) be a finite-dimensional subspace which is \( \lambda \)-norming over the linear span of \( X_1 \cup X_2 \). We continue in an obvious way (e.g. we let \( X_3 \) to be a finite-dimensional subspace of \((M_2)^\perp \) containing a \( B \)-bilipschitz image of \( T(3) \)).

The finite-dimensional subspaces \( \{X_i\}_{i=1}^{n} \) constructed in this manner are such that for any choice of \( x_i \in X_i, i = 1, \ldots, n \), the inequalities

\[
\lambda \max\{\|x_1\|, \|x_1 + x_2\|, \ldots, \|x_1 + \cdots + x_{n-1}\|\} \leq \|x_1 + \cdots + x_n\|
\]

\[
\leq \|x_1\| + \|x_2\| + \cdots + \|x_n\| 
\]

(3)

hold. The leftmost inequality implies that for each \( 1 \leq t < n \) we have

\[
\|x_{t+1} + \cdots + x_n\| \leq \|x_1 + \cdots + x_n\| + \|x_1 + \cdots + x_t\|
\]

\[
\leq \frac{\lambda + 1}{\lambda} \|x_1 + \cdots + x_n\|.
\]

Applying (3) again we get

\[
\|x_{t+1}\| \leq \frac{\lambda + 1}{\lambda^2} \|x_1 + \cdots + x_n\|.
\]
We conclude that
\[ \|x_1 + \cdots + x_n\| \geq \lambda^2 \frac{\lambda^2}{\lambda + 1} \max_{1 \leq j \leq n} \|x_j\| \geq \lambda^2 \frac{\lambda^2}{\lambda + 1} \sum_{i=1}^{n} \|x_i\|. \]

So the \( \ell_1 \)-sum \( (\oplus_{i=1}^{k} X_i)_1 \) admits a natural bilipschitz embedding into \( X \) with distortion \( \leq \frac{(\lambda+1)n}{\lambda^2} \). The image of this embedding is the linear span of the spaces \( X_i \) in \( X \). Combining this embedding with the \( B \)-bilipschitz embeddings of \( T(i) \) into \( X_i \) we get a \( B \lambda^2 \)-bilipschitz embedding of \( (\oplus_{i=1}^{n} T(i))_1 \) into \( X \). Applying Theorem 1.5 we get that a finite product of (infinite) binary trees admits a bilipschitz embedding into \( X \).

It remains to modify the quasi-isometric embedding in order to get a bilipschitz embedding. Easy examples show that this cannot be done for general metric spaces; our purpose is to show that this can be done in the case where we embed a locally finite uniformly discrete metric space into an infinite-dimensional Banach space (observe that a finite-dimensional Banach space cannot be nonsuperreflexive).

**Lemma 2.3.** Let \( M \) be a locally finite uniformly discrete metric space admitting a quasi-isometric embedding into an infinite dimensional Banach space \( X \). Then \( M \) admits a bilipschitz embedding into \( X \).

**Proof.** We may assume that the distance between any two distinct elements in \( M \) is at least 1 (multiplying all distances in \( M \) by some positive number, if necessary). Let \( T : M \to X \) be a quasi-isometric embedding with constants \( a_1, a_2, \) and \( b \) (see (1)). Since the distance between any distinct elements of the \( M \) is at least 1, it is clear that \( T \) is \( (a_2 + b) \)-Lipschitz.

To get the desired estimate from below we perturb the map in order to make it injective and to make its image uniformly discrete. This is done in the following way. We consider a 1-net \( N \) in \( X \), that is, a set of points \( \{x_i\}_{i=1}^{\infty} \) such that \( \|x_i - x_j\| \geq 1 \) for \( i \neq j \) and \( \forall x \in X \exists i \in N \|x - x_i\| \leq 1 \) (without loss of generality we may assume that \( X \) is separable since the linear span of \( T(M) \) is separable). Since \( X \) is infinite-dimensional, it is easy to see that the ball of radius 4 centered at any \( x \in X \) contains infinitely many points of \( N \). In fact, one can show that a ball of radius 3 centered at \( x \) contains an infinite sequence \( \{s_i\}_{i=1}^{\infty} \) satisfying \( \|s_i - s_j\| \geq \frac{5}{3} \). This implies that points of \( N \) which are close to \( \{s_i\} \) are distinct and are inside the ball of radius 4 centered at \( x \). (Using results of Kottman [24] and Elton-Odell [14] one can improve the constant 4 in this statement, but we do not need this improvement here). On the other hand, \( T(M) \), as an image of a locally finite metric space under a quasi-isometric embedding, is locally finite (recall that we do not require a locally finite metric space to be uniformly discrete), and each point in it has at most finitely many pre-images. Therefore for each \( u \in M \) we can find a point \( R(u) \in N \) such that \( \|T(u) - R(u)\| \leq 4 \) and \( \|R(u) - R(v)\| \geq 1 \) if \( u \neq v, u, v \in M \). It is easy to
verify that $R$ is also a quasi-isometric embedding ($\mathbf{(1)}$ is satisfied with the same $a_1$ and $a_2$ and with $b' = b + 8$), hence it is also a Lipschitz map.

To show that it is bilipschitz it remains to get an estimate for $\|R(u) - R(v)\|$ from below. We observe that

$$a_1 d_M(u, v) - b' \leq \|R(u) - R(v)\|$$

(4)

implies the estimate $\|R(u) - R(v)\| \geq \frac{a_1}{2} d_M(u, v)$ if $d_M(u, v) \geq \frac{2b'}{a_1}$. If $b' = 0$, there is nothing to prove. If $b' > 0$ and $d_M(u, v) \leq \frac{2b'}{a_1}$, we have $\|R(u) - R(v)\| \geq \frac{a_1}{2b'} d_M(u, v)$ just because $\|R(u) - R(v)\| \geq 1$. Thus $R$ is a bilipschitz embedding.

Now we turn to the proof of part (b). To prove this result we need another result on the structure of hyperbolic groups. We use the metric geometry terminology of [6]. Let $K \geq 0$. A subset $Y$ of a geodesic metric space $X$ is called $K$-quasiconvex if any geodesic path in $X$ with endpoints in $Y$ lies in the $K$-neighborhood of $Y$. A subset $Y$ is called quasiconvex if it is $K$-quasiconvex for some $K < \infty$.

**Theorem 2.4.** Let $G$ be a finitely generated word hyperbolic group which does not contain an infinite cyclic subgroup of finite index. Then $G$ contains a free subgroup $H$ with two generators as a quasiconvex subgroup.

In the case of torsion free groups this result was obtained by Arzhantseva [1]. The general case follows by combining results of Dahmani, Guirardel, Osin [10, Theorem 6.14] and Sisto [32, Theorem 2].

A proof of part (b) of Theorem 1.7 can be derived from Theorem 2.4 as follows. Suppose that a Banach space $X$ admits a bilipschitz embedding of an infinite word hyperbolic group $G$ satisfying the conditions of Theorem 2.4. By Theorem 2.4 $G$ contains a free subgroup $H$ with two generators, we denote them $a$ and $b$, as a quasiconvex subgroup. The quasiconvexity implies that the identical map of $H$ endowed with the metric of the graph $\text{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$ onto $H$ endowed with the metric induced from $G$ is a quasi-isometric embedding, see [6, Lemma 3.5].

Thus we get a quasi-isometric embedding of the tree $\text{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$ into the Banach space $X$. It is easy to see that $X$ cannot be finite-dimensional. Using Lemma 2.3 we modify the quasi-isometric embedding and get a bilipschitz embedding of the tree $\text{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$ into $X$. By the result of Bourgain [5] (see also a short proof in [23]), we get that the Banach space $X$ is nonsuperreflexive.

The proof of the next proposition is well known and elementary. It is included because I have not found a suitable reference. We would like to emphasize that in Proposition 2.5 we consider the Cayley graph as a set of elements of $G$ with the shortest path distance (and not as a 1-dimensional simplicial complex).

**Proposition 2.5.** Let $G$ be a group containing $\mathbb{Z}$ as a finite index subgroup. Then the Cayley graph $\text{Cay}(G, S)$ with respect to any finite set of generators admits a bilipschitz embedding into $\mathbb{R}$. 8
Proof. Let $x$ be the generator of $\mathbb{Z}$. We use multiplicative notation, so

$$Z = \{ \ldots, x^{-n}, \ldots, x^{-1}, e, x, \ldots, x^{n}, \ldots \}.$$  

Let $g_1 = e, g_2, \ldots, g_m$ be representatives of left cosets of $Z$ in $G$. We consider the most straightforward embedding. We map

$$\ldots, x^{-n}, \ldots, x^{-1}, e, x, \ldots, x^{n}, \ldots$$

to

$$\ldots, -n, \ldots, -1, 0, 1, \ldots, n, \ldots$$

respectively; and map

$$x^n, g_2 x^n, \ldots, g_m x^n$$

to points

$$n, n + \frac{1}{m}, \ldots, n + \frac{m - 1}{m},$$

respectively.

We consider the following generating set $S = \{ x, x^{-1}, g_2, g_2^{-1}, \ldots, g_m, g_m^{-1} \}$ and let $Cay(G, S)$ be the right-invariant Cayley graph. Since word metrics for different sets of generators are bilipschitz-equivalent, it suffices to show that the described above embedding, let us denote it $\varphi$, is bilipschitz as an embedding of $Cay(G, S)$ into $\mathbb{R}$.

To estimate the Lipschitz constant of $\varphi : Cay(G, S) \to \mathbb{R}$ we need to estimate from above the distance between images of adjacent vertices, that is, $|\varphi(u) - \varphi(x^{\pm 1}u)|$ and $|\varphi(u) - \varphi(g_i^{\pm 1}u)|$, $i = 2, \ldots, m$.

We have $u = g_i x^k$ for some $i$ and $k$. Observe that $g_j g_i = g_{k(i,j)} x^{p(i,j)}$, $g_j^{-1} g_i = g_{i(j)} x^{r(i,j)}$, $x g_i = g_{k(i)} x^{p(i)}$, and $x^{-1} g_i = g_{l(i)} x^{r(i)}$. This implies that $x^{\pm 1} u$ and $g_i^{\pm 1} u$ are of the form $g_s x^{k+t}$, where the absolute value of $t$ is bounded above by

$$T := \max_{i,j} \{ |p(i, j)|, |r(i, j)|, |p(i)|, |r(i)| \}.$$  

The number $T$ depends on the group $G$ and the choice of the coset representatives, but not on $i$ and $k$. The desired estimates of $|\varphi(u) - \varphi(x^{\pm 1}u)|$ and $|\varphi(u) - \varphi(g_i^{\pm 1}u)|$, $i = 2, \ldots, m$ from above follow.

Now we estimate the Lipschitz constant of $\varphi^{-1}$. If $k = l$, we have $|\varphi(g_i x^k) - \varphi(g_j x^l)| \geq \frac{1}{m}$, and the word distance between $g_i x^k$ and $g_j x^k$ is 2, and we are done in this case. Observe that for $k > l$ we have $|\varphi(g_i x^k) - \varphi(g_j x^l)| \geq |k - l - \frac{m - 1}{m}|$.

On the other hand, we can reach $g_i x^k$ from $g_j x^l$ traversing $k - l + 2$ edges (first we traverse the edge corresponding to $g_j^{-1}$, then edges corresponding to $x$ ($(k - l)$ times since $k > l$), then $g_i$). The estimate for the Lipschitz constant of $\varphi^{-1}$ follows. \hfill $\square$
3 Binary trees do not admit uniformly bilipschitz embeddings into diamonds

Proof of Theorem 1.9. We are going to show that for any \( k \in \mathbb{N} \), no matter how we choose the numbers \( m(n) \in \mathbb{N} \) and \( p(n) \in \mathbb{N} \cup \{0\} \), it is impossible to find maps \( F_n : T_n \to D_{m(n)} \) such that

\[
\forall n \forall u, v \in V(T_n) \quad 2^{p(n)} d_{T_n}(u, v) \leq d_{D_{m(n)}}(F_n u, F_n v) \leq 2^k \cdot 2^{p(n)} d_{T_n}(u, v). \tag{5}
\]

Let us remind that we normalize distances in diamonds in such a way that each edge has length 1.

Note: We do not have to consider negative \( p(n) \) because in such cases we may replace \( D_{m(n)} \) by \( D_{m(n)} - p(n) \) and use the natural map of \( D_{m(n)} \) into \( D_{m(n)} - p(n) \), which multiplies all distances by \( 2^{-p(n)} \).

We are going to use the notion of a subdiamond. It is defined as a part of a diamond which evolved from an edge. A subdiamond has naturally defined top and bottom. (These notions are defined up to the choice of the bottom and the top of \( D_0 \).) The \textit{height} of a subdiamond is the distance from its top to its bottom.

**Lemma 3.1.** The cardinality of a \( 2^{p(n)} \)-separated set (i.e. a set satisfying \( d(u, v) \geq 2^{p(n)} \) for any \( u \neq v \)) in a subdiamond of height \( 2^h \) does not exceed \( 2 \cdot 4^{h-p(n)} \).

**Proof.** It is easy to see that each subdiamond of height \( 2^{p(n)} \) contains at most two vertices out of each \( 2^{p(n)} \)-separated set. The number of subdiamonds of height \( 2^{p(n)} \) in a diamond of height \( 2^h \) is equal to the number of edges in the diamond of height \( 2^{h-p(n)} \). This number of edges is \( 4^{h-p(n)} \), because in each step of the construction of diamonds the number of edges quadruples. \( \square \)

By a subtree in \( T_n \) we mean the subgraph consisting of some vertex and all of its descendants. By Lemma 3.1 if there is a subtree \( S \) of \( T_n \) whose order (number of vertices) is \( > 2 \cdot 4^{h-p(n)} \) and whose root is mapped by \( F_n \) into a subdiamond \( M \) of height \( 2^h \), some of the vertices of \( S \) have to be mapped outside the subdiamond, and so some of the root-leaf paths in \( S \) have to leave \( M \).

Our next observation is: there are two exits in a subdiamond. We mean that there are two vertices in any subdiamond \( M \) whose deletion would separate the subdiamond from the rest of the diamond. This happens just because the subdiamond evolved from an edge. In the proof of Lemma 3.2 we combine this fact with the bilipschitz condition on \( F_n \) and get the conclusion that two root-leaf paths in \( S \) with small intersection cannot leave \( M \) through the same exit if they stay in \( M \) for long time before leaving. Here we say that a path \( P \) in \( T_n \) leaves \( M \) through the exit \( v \) if there are two consecutive vertices \( u \) and \( w \) in \( P \), such that \( F_n u \in M \), \( F_n w \notin M \) and

\[
d_{D_{m(n)}}(F_n u, v) + d_{D_{m(n)}}(v, F_n w) \leq 2^k \cdot 2^{p(n)}. \tag{6}
\]
Observe that the bilipschitz condition (5) implies that if some vertices of $P$ are in $M$, and some other vertices are not, then $P$ should leave $M$ through one of the two exits.

**Lemma 3.2.** Let $S$ be a subtree of $T_n$ whose root $s$ is mapped into a subdiamond $M$. Suppose that $F_n$ maps at least $2^k + 2$ generations of descendants of $s$ into $M$. Let $s_1, s_2, s_3$, and $s_4$ be 4 grandchildren of $s$, denote by $S_1, S_2, S_3$, and $S_4$ the sets of those descendants of $s_1, s_2, s_3$, and $s_4$, respectively, which belong to generations $2^k + 3$ and later ones, where we mean that $s$ is generation 0. Then only two out of the four sets $S_1, S_2, S_3$, and $S_4$ can have vertices whose $F_n$-images are not in $M$.

**Proof.** It is easy to check that the pairwise distances between $F_n$-images of $S_1, S_2, S_3$, and $S_4$ in $D_{m(n)}$ are $\geq 2^p(n)(2^{k+1} + 4)$. Suppose that at least three of the sets $S_1, S_2, S_3, S_4$ contain vertices whose $F_n$-images are not in $M$. We may assume that the vertices are $i_1 \in S_1, i_2 \in S_2$, and $i_3 \in S_3$. Consider paths $\{P_i\}_{i=1}^3$ in $S$ joining these vertices and $s$. Each of these paths leaves $M$. Since there are two exits and three paths, two of the paths should leave $M$ through the same exit $v$. Since the paths cannot leave $M$ within the first $2^k + 2$ generations from $s$, each of these two $P_i$ contains consecutive vertices $u_i$ and $v_i$ satisfying (6) such that $F_n(u_i) \in M, F_n(v_i) \notin M$, and $u_i$ are in the generation at least $2^k + 2$ from $s$ and $w_i$ are in the generation at least $2^k + 3$. It is clear that this implies the existence of two vertices, $t$ and $s$, among these $u_i$ and $v_i$ such that $d_{T_n}(t, s) \geq 2^{k+1} + 2$ and $d_{D_{m(n)}}(F_nt, F_n s) \leq d_{D_{m(n)}}(F_nt, v) + d_{D_{m(n)}}(v, F_n s) \leq 2^{k+2p(n)}$. This contradicts the bilipschitz condition (5).

Lemma 3.2 suggests the following plan for completion of the proof. We find a subdiamond $M$ with height $2^h$ in $D_{m(n)}$ which contains all of the $F_n$-images of the first $2^k + 2$ generations of a rooted in $s$ subtree $S$ (of $T_n$), and the total number of generations in $T_n$ which contain descendants of $s$ is more than $2(h - p(n) + 1)$.

In fact, if we find such an $M$, by Lemma 3.2 more that half of the descendants of $s$ should have their $F_n$-images in $M$. The number of such descendants is $> 2 \cdot 4^{h-p(n)}$, which is more than the possible number of $2^{p(n)}$-separated points in $M$ estimated by Lemma 3.1 and so this would complete the proof.

So it remains to find a suitable subdiamond $M$. The reasons for which it is possible is that, on one hand, vertices which appear later (in diamond’s construction) are “dense” in the diamond and, on the other hand, they have neighborhoods which are contained in subdiamonds of controlled size. We present details below.

Now we introduce generations of vertices in a diamond. We label them from the end. Generation number 1 is the set of vertices appeared in the last step of the construction of $D_{m(n)}$. Generation number 2 is the set of vertices appeared in the previous step of the construction, so on, there are $m(n)$ generations (two original vertices do not belong to any of the generations.) The following is clear from the construction:
Observation 3.3. (1) Let \( v \) be a vertex of generation number \( r, r \in \{1, \ldots, m(n)\} \). Then the \( 2^{r-1} \)-neighborhood of \( v \) is contained in a subdiamond of height \( 2^r \).

(2) Let \( Z_r \) be the set of all vertices of generation number \( r \). Then the connected components of \( D_{m(n)} \setminus Z_r \) have diameters \( < 2^r \).

Now we consider the binary tree \( T_n \) of depth \( n = L(k) \), where \( L(k) \) is a “large” number depending only on \( k \), we shall specify our choice of \( L(k) \) later.

Let \( q = q(k) \in \mathbb{N} \) (\( q(k) \) also will be chosen later). No matter how we choose the image of the root of \( T_n \) in \( D_{m(n)} \), by Observation 3.3 (2), within \( 2^q \) steps, following \( F_n \)-images of any of the descending paths in \( T_n \), we shall “pass over” (recall that we make “steps” of lengths between \( 2^{p(n)} \) and \( 2^{k+p(n)} \) for each edge) a vertex \( z \) belonging to the generation \( Z_{q+p(n)} \), we can “miss” \( z \) by \( 2^{k+p(n)-1} \), let \( s \) be the vertex in a generation with number \( \leq 2^q \) of \( T_n \) such that \( d_{D_{m(n)}}(F_n(s), z) \leq 2^{k+p(n)-1} \). After that we consider the subtree of descendants of \( s \) for \( 2^k + 2 \) generations (as we did in Lemma 3.2). All of them are in the \( 2^{p(n)} \cdot (2^{k-1} + (2^k + 2)2^k) \)-neighborhood of \( z \). Now we pick \( q = q(k) \) in such a way that

\[
2^{q(k)-1} \geq 2^{k-1} + (2^k + 2)2^k.
\]

Then, by Observation 3.3 (1), all of the first \( 2^k + 2 \) generations of descendants of \( s \) (including \( s \)) are mapped into a subdiamond \( M \) of height \( 2^{q(k)+p(n)} \). Now we pick \( L(k) > 2^{q(k)}+2(q(k)+1) \). With this choice, the subdiamond \( M \) and the vertex \( s \) have the desired properties. We mean that the total number of generations of \( T_n = T_{L(k)} \) which contain descendants of \( s \) is more than \( 2(q(k)+1) = 2((q(k)+p(n))-p(n)+1) \), see the discussion following Lemma 3.2.

4 Example of a series-parallel family admitting uniformly bilipschitz embeddings into \( \ell_2 \)

Proof of Theorem 1.12. The graphs which we use for this construction are close to diamond graphs (see the beginning of Section 1.2), one of the differences is that we do not remove edges of graphs which appear earlier in the sequence. Also the metric is obtained by introducing weights of edges and the corresponding shortest weighted path distance (subdividing edges one can avoid using weights).

We pick a number \( \varepsilon \in (0, \frac{1}{2}) \). The sequence \( \{W_n\}_{n=0}^{\infty} \) of weighted diamonds is defined in terms of diamonds \( \{D_n\}_{n=0}^{\infty} \) as follows:

- \( W_0 \) is the same as \( D_0 \)
- \( W_1 = D_1 \cup W_0 \) with edges of \( D_1 \) given weights \( (\frac{1}{2} + \varepsilon) \); weight of the edge of \( W_0 \) stays as 1 (as it was in the first step of the construction).
- \( W_2 = D_2 \cup W_1 \) with edges of \( D_2 \) given weights \( (\frac{1}{2} + \varepsilon)^2 \); weights of the edges of \( W_1 \) stay as they were in the previous step of the construction.
• \[ W_n = D_n \cup W_{n-1} \]\ with edges of \( D_n \) given weights \((\frac{1}{2} + \varepsilon)^n\); weights of the edges of \( W_{n-1} \) stay as they were in the previous step of the construction.

We consider \( \{W_n\} \) as finite metric spaces whose elements are vertices and the distance between two vertices is the weighted length of the shortest path. We define embeddings \( F_n : W_n \rightarrow \ell_2 \) as follows:

• The map \( F_0 \) maps the vertices of \( D_0 \) to 0 and \( e_0 \), respectively (where \( \{e_i\}_{i=0}^{\infty} \) is the unit vector basis of \( \ell_2 \)). It is clear that \( F_0 \) is an isometric embedding.

• The map \( F_n, n \geq 1, \) is an extension of the map \( F_{n-1} \). The description is generic for all \( n \geq 1 \). Each vertex \( w \) of \( W_n \setminus W_{n-1} \) corresponds to two vertices of \( W_{n-1} \): \( w \) is the vertex of the 2-edge path joining \( u \) and \( v \). We map the vertex \( w \) to \( \frac{1}{2}(F_{n-1}(u) + F_{n-1}(v)) \pm \omega_n e_{uv}, \) where \( e_{uv} \) is an element of \( \{e_i\}_{i=1}^{\infty} \) picked for the edge \( uv \) (we pick different \( e_i \) for different edges) and \( \pm \) are picked differently for the pair of vertices \( w, \tilde{w} \) which corresponds to the same edge \( uv \) (our construction is such that there are two such vertices); and \( \omega_n \) is picked in such a way that \( ||F_n(u) - F_n(w)|| = d_{W_n}(u, w) \), so \( \omega_n = \sqrt{\varepsilon + \varepsilon^2} \left(\frac{1}{2} + \varepsilon\right)^{n-1} \).

Now we estimate the distortion. Observe that \( F_n \) is distance-preserving on edges. Therefore \( \text{Lip}(F_n) \leq 1 \), and we need to estimate the Lipschitz constant of \( F_{n-1} \) only.

Let us start with \( \text{Lip}(F_1^{-1}) \), it is attained on the only pair of vertices of \( W_1 \) which is not an edge, and it is therefore

\[
\text{Lip}(F_1^{-1}) = \frac{1 + 2\varepsilon}{2\sqrt{\varepsilon + \varepsilon^2}}.
\]

The estimate of \( \text{Lip}(F_n^{-1}) \), \( n \geq 2, \) can be done in general in the following way. We consider any shortest path between two vertices in \( W_n \).

**Note 4.1.** Some caution is needed for arguments about shortest paths in \( W_m \), because they can be quite different from shortest paths in diamonds. For example, if \( m \) is such that

\[
\left(\frac{1}{2} + \varepsilon\right) + \left(\frac{1}{2} + \varepsilon\right)^2 + \cdots + \left(\frac{1}{2} + \varepsilon\right)^m \geq 1 + \left(\frac{1}{2} + \varepsilon\right)^m
\]

(such number \( m \) obviously exists if \( \varepsilon > 0 \)), then a shortest path between two vertices in \( W_m \) can consist of the edge of \( D_0 \) and one more edge of length \( \left(\frac{1}{2} + \varepsilon\right)^m \).

**Claim 4.2.** A shortest path between two vertices in \( W_n \) can contain edges of each possible length:

\[
1, \left(\frac{1}{2} + \varepsilon\right), \left(\frac{1}{2} + \varepsilon\right)^2, \left(\frac{1}{2} + \varepsilon\right)^3, \ldots
\]

at most twice, actually for 1 this can happen only once because there is only one such edge.
Proof. Let $e$ be one of the longest edges in the path and $(\frac{1}{2} + \varepsilon)^k$ be its length. As for diamonds, we define weighted subdiamonds as subsets evolving from edges (as sets of vertices they coincide with the subdiamonds defined before). The edge from which a subdiamond evolved is called its diagonal. Consider the subdiamond $S$ containing $e$ with diagonal of length $(\frac{1}{2} + \varepsilon)^{k-1}$ (here we assume that $k \neq 0$, for $k = 0$ the statement is trivial). Let $e = uv$, without loss of generality we may assume that $u$ is the bottom of $S$ (turning the graph upside down, if needed).

The rest of the path consists of two pieces: (1) The one which starts at $v$; (2) The one which starts at $u$. We claim that the part which starts at $v$ can never leave $S$. It obviously cannot leave through $u$, it cannot leave through the top of $S$, let us denote it $t$, because otherwise the piece of the path between $u$ and $t$ could be replaced by the diagonal of $S$, which is strictly shorter.

This implies that the part of the path in $S$ can contain edges only shorter than $(\frac{1}{2} + \varepsilon)^k$ (except $e$). For the next edge in this part of the path we can repeat the argument and get (by induction) that lengths of edges in the remainder of the path in $S$ are strictly decreasing.

The part of the path which starts at $u$ can be considered similarly.

Now we need to analyze the “directions” in which the path can go. Let $e = uv$, the subdiamond $S$, $t$, and the two parts of the shortest path be as above.

If the part of the path which starts at $u$ does not enter $S$, the estimate of the Lipschitz constant can be based just on the part contained in $S$. Here is the estimate:

- By Claim 4.2 we get that the length of the path is $\leq 2 (\frac{1}{2} + \varepsilon)^k \cdot \frac{1}{\frac{1}{2} - \varepsilon}$

- On the other hand the $e_{ut}$-coordinate of the path

  (a) Does not change along the path which starts at $u$ and leaves $S$.

  (b) Changes by $(\frac{1}{2} + \varepsilon)^{k-1} \sqrt{\varepsilon + \varepsilon^2}$ when we traverse $e$.

  (c) Cannot change by more than

  $$\left(1 - \left(\frac{1}{2}\right)^m\right) \left(\frac{1}{2} + \varepsilon\right)^{k-1} \sqrt{\varepsilon + \varepsilon^2}$$

  on the path which starts at $v$, where $m \in \mathbb{N}$ is the least number satisfying (7).

Only the statement (c) requires a proof. We observe that traversing the image of an edge of length $(\frac{1}{2} + \varepsilon)^{k+d} (d = 0, 1, 2, \ldots)$ of this path we change the $e_{ut}$ coordinate by $(\frac{1}{2})^d (\frac{1}{2} + \varepsilon)^{k-1} \sqrt{\varepsilon + \varepsilon^2}$ (this follows by induction from our
definition of $F_n$). Therefore, if the change exceeds (8), it implies that the path starting at $v$ contains edges of all lengths 
\[
\left(\frac{1}{2} + \varepsilon\right)^{k+1}, \left(\frac{1}{2} + \varepsilon\right)^{k+2}, \ldots, \left(\frac{1}{2} + \varepsilon\right)^{k+m}.
\]

But then, because of (7), it is not a shortest path between its ends (using the diagonal we get a shorter path).

The same argument works in the case where the path starting at $u$ stays inside $S$, because in this case (as is easy to see) the sign of the $e_{ul}$-coordinate on this part of the path is different from its sign on the first part of the path.

Therefore in both cases we get that the quotient of (the length of the path)/(the distance between the images) does not exceed 
\[
\frac{2^{m+1}}{\left(\frac{1}{2} - \varepsilon\right)\sqrt{\varepsilon + \varepsilon^2}\left(\frac{1}{2} + \varepsilon\right)}.
\]
Since $m$ depends only on the choice of $\varepsilon$ (and not on $n$), we get the desired estimate for $\sup_n \text{Lip}(F_n^{-1})$.

\section{Characterizations using one test-space}

Our purpose is to show that a class of Banach spaces for which there is a sequence of finite test-spaces has one test-space, similar results hold for test-spaces satisfying some additional conditions. The proof is very simple (it simplifies some of the results of [28]).

\textbf{Proposition 5.1.} \textbf{(a)} Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of finite test-spaces for some class $\mathcal{P}$ of Banach spaces containing all finite-dimensional Banach spaces. Then there is a metric space $S$ which is a test-space for $\mathcal{P}$.

\textbf{(b)} If $\{S_n\}_{n=1}^{\infty}$ are

\begin{itemize}
  \item unweighted graphs,
  \item trees,
  \item graphs with uniformly bounded degrees,
\end{itemize}

then $S$ also can be required to have the same property.

\textit{Proof.} In all cases the space $S$ will contain subspaces isometric to each of $\{S_n\}_{n=1}^{\infty}$. Therefore the only implication which is nontrivial is that the embeddability of $\{S_n\}_{n=1}^{\infty}$ implies the embeddability of $S$.

Each finite metric space can be considered as a weighted graph with it shortest path distance. In all cases we construct the space $S$ as an infinite graph by joining $S_n$
with $S_{n+1}$ with a path $P_n$ whose length is $\geq \max\{\text{diam} S_n, \text{diam} S_{n+1}\}$. To be more specific, we pick in each $S_n$ a vertex $O_n$ and let each of the paths mentioned above be a path joining $O_n$ with $O_{n+1}$. We endow the infinite graph $S$ with the shortest path distance. It is clear that all of the conditions are satisfied. It remains only to show that each infinite-dimensional Banach space which admits bilipschitz embeddings of $\{S_n\}_{n=1}^{\infty}$ with uniformly bounded distortions admits a bilipschitz embedding of $S$.

So let $X$ be an infinite dimensional Banach space admitting bilipschitz embeddings of spaces $\{S_n\}_{n=1}^{\infty}$ with uniformly bounded distortions. Let $F_n : S_n \rightarrow X$ be the corresponding embeddings. We may assume that $\text{Lip}(F_n) = 1$ for each $n$ and $\text{Lip}((F_n)^{-1})$ are uniformly bounded. Since the considered metric spaces are finite and all of the hyperplanes in $X$ are isomorphic (with uniform bound on all such isomorphisms) we may assume that images of all of $F_n$ are contained in a fixed hyperplane $H$ in $X$. We may assume that $H$ is a kernel of a functional $x^* \in X^*$, $\|x^*\| = 1$ which attains its norm on some vector $x \in X$, $\|x\| = 1$. We may assume also that $F_n(O_n) = 0$ for each $n$. Now we modify $F_n$ by shifting each of the $F_n(S_n)$, $n \geq 2$, by $\lambda_n x$, where $\lambda_n$ is the sum of the lengths of the paths $P_1, \ldots, P_{n-1}$, and we map vertices of paths $P_1, \ldots, P_{n-1}$ to the one-dimensional space spanned by $x$ in such a way that the distance between any two adjacent vertices is 1. The Lipschitz constant of the obtained map $F : S \rightarrow X$ is equal to 1 since it does not increase distance between endpoints of edges.

To estimate $\text{Lip}(F^{-1})$ consider any two vertices $u$ and $v$ in $S$. It is clear that if $u, v$ are in the same $S_n$, then $d_S(u, v) \leq \text{Lip}((F_n)^{-1})\|F(u) - F(v)\|$. It is also clear that the embedding $F$ is an isometry on the union of the paths $\{P_n\}$. Therefore it remains to consider the cases (i) $u$ and $v$ are in different $S_n$; (ii) $u$ is in one of the $\{S_n\}$ and $v$ is in one of the $\{P_n\}$.

**Case (i).** Let $u \in S_n$, $v \in S_m$, $n < m$. In this case $d_S(u, v) = d_{S_n}(u, O_n) + (\text{the sum of lengths of paths } P_n, \ldots, P_{m-1}) + d_{S_m}(O_m, v)$, and $\|F(u) - F(v)\| \geq |x^*(F(u) - F(v))| = (\text{the sum of lengths of paths } P_n, \ldots, P_{m-1})$. Recalling the way in which we choose the lengths of $\{P_n\}$, we get the desired conclusion.

**Case (ii).** Let $u \in S_n$, $v \in P_m$. In this case we have to consider two subcases: $d_S(v, O_n) \geq \frac{1}{2\text{Lip}((F_n)^{-1})}d_S(u, O_n)$ and $d_S(v, O_n) < \frac{1}{2\text{Lip}((F_n)^{-1})}d_S(u, O_n)$. In the first subcase we have

$$\|F(u) - F(v)\| \geq |x^*(F(u) - F(v))| = d_S(v, O_n) \geq \frac{1}{4\text{Lip}((F_n)^{-1})}d(u, v).$$
In the second subcase we get

\[ \|F(u) - F(v)\| \geq \|F(u) - F(O_n)\| - \|F(v) - F(O_n)\| \]
\[ \geq \frac{1}{\text{Lip}((F_n)^{-1})} d_S(u, O_n) - d_S(v, O_n) \]
\[ > \frac{1}{2\text{Lip}((F_n)^{-1})} d_S(u, O_n) \]
\[ \geq \frac{1}{4\text{Lip}((F_n)^{-1})} d_S(u, v) \]

\[ \square \]

6 References

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