INVERSE BOUNDARY PROBLEM FOR THE TWO PHOTON ABSORPTION
TRANSPORT EQUATION

PLAMEN STEFANOV AND YIMIN ZHONG

Abstract. We study the inverse boundary problem for the nonlinear two photon absorption radiative transport equation. We show that the absorption coefficients and the scattering coefficient can be uniquely determined from the albedo operator. If the scattering is absent, we do not require smallness of the incoming source and the reconstruction of the absorption coefficients is explicit.

1. Introduction

In this work we study the inverse boundary problem for the two photon absorption radiative transport equation. Two photon absorption happens when it takes two photons to excite a molecule from one state to another [24,30]. The probability of a two photon absorption at a given point is proportional to the light intensity there regardless of the incoming direction, which makes the corresponding term quadratic. One of the applications of two photon absorption is in medical imaging: the human body is not transparent to optical rays but it is more transparent to infrared ones. Then fluorescent dyes with good two photon absorption rates can be used successfully with such a large wavelength excitation, see, e.g., [15,22]. Other applications are pointed out in [22]; for example: microscopy, microfabrication, three-dimensional data storage, etc. For applications to photoacoustic imaging, we refer to [7] and the references there.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded convex set with a $C^1$ boundary $\partial \Omega$, and let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Then our model is the following equation, see also [23],

$$\theta \cdot \nabla x u(x, \theta) + (\sigma_a(x, \theta) + \sigma_b(x, \theta)|\langle u \rangle|)u(x, \theta) - Ku(x, \theta) = 0 \quad \text{in } \Omega \times S^{n-1},$$

$$u(x, \theta) = f_-(x, \theta) \quad \text{on } \Gamma_-,$$

where $\langle u \rangle$ is the average of $u(x, \theta)$ over the angular variable $\theta$; that is,

$$\langle u \rangle := \int_{S^{n-1}} u(x, \theta)d\theta,$$

with $d\theta$ being the normalized surface measure on $S^{n-1}$. When $u \geq 0$, the absolute value in $|\langle u \rangle|$ does not matter, of course but for general solutions, we include it to have a well-posed problem. The linear operator $K$ is defined by

$$Ku(x, \theta) := \int_{S^{n-1}} k(x, \theta', \theta)u(x, \theta')d\theta'.$$

The coefficients $\sigma_a(x, \theta), k(x, \theta', \theta)$ are the usual total absorption and scattering coefficients, respectively. The coefficient $\sigma_b$ stands for strength of the nonlinear effect of two photon absorption, the term $\sigma_a + \sigma_b \langle u \rangle$ can be understood as the effective total absorption coefficient dependent on the

P.S. partially supported by the National Science Foundation under grant DMS-1900475.
solution. They are all assumed to be non-negative, and we impose smallness assumptions on $k$, $\sigma_b$ and $f_-$, see Definition \ref{def:smallness}.

If the direct problem (\ref{eq:transport}) is uniquely solvable, one can define the usual albedo operator

$$ A : f_- \mapsto f_+, $$

where $f_+(x, \theta) := u(x, \theta)|_{\Gamma_+}$ denotes the exiting photon density. This albedo operator is nonlinear, and we are interested in finding out whether the albedo operator $A$ determines uniquely the coefficients $\sigma_a(x, \theta), \sigma_b(x, \theta), k(x, \theta', \theta)$.

When $\sigma_b = 0$, the equation (\ref{eq:transport}) is linear. Uniqueness and recovery formulas for $\sigma_a$ and $k$, when $\sigma_a$ depends on $x$ only, were established in \cite{guo1994} for $n \geq 3$ and in \cite{Stefanov2022} for $n = 2$ under a smallness assumption on $k$. The general case of $\sigma = \sigma(x, \theta)$ for $n \geq 3$ was resolved in \cite{Stefanov2022}. Stability estimates were proved in \cite{Stefanov2022}. Inverse radiative transport in the Riemannian setting was studied in \cite{Stefanov2022, Stefanov2022, Stefanov2022}, and for a different dynamical system, see \cite{Stefanov2022}, there are also many other works regarding different types of boundary measurement, see \cite{Stefanov2022, Stefanov2022, Stefanov2022, Stefanov2022, Stefanov2022} and the references therein. References to earlier works can be found in the survey \cite{Stefanov2022}.

Inverse problems for non-linear versions of the transport equation (different from the one we study here) are studied in \cite{Stefanov2022, Stefanov2022}. In \cite{Stefanov2022}, the authors considered the inverse medium problem under the same nonlinear model as (\ref{eq:transport}) and showed the uniqueness and stability of the reconstruction of absorption coefficients from internal data.

The main result is the following. We show that we can recover $\sigma_a$, $k$, and $\sigma_b$ given the nonlinear operator $A$. The idea of the proof is the following. If we take $f_-$ small, then we are in the linear regime and can use the result in \cite{guo1994} to recover $\sigma_a$ if it depends on $x$ only, and $k$. The latter requires $n \geq 3$, see also \cite{Stefanov2022} for the 2D case. Next, we can take $f_- = f_0 + \delta f_1$, see \cite{Stefanov2022}, with $0 < \delta \ll 1$ and $f_0 > 0$ smooth but $f_1$ singular in the $\theta$ variable only. Then $f_0$ would not create singularities in solution at order $O(\delta)$ but the effective absorption coefficient would involve $\sigma_b(u_0)$, where $u_0$ is the leading $O(1)$ term of the solution which is determined by $f_0$, see \cite{Stefanov2022} and \cite{Stefanov2022}. This is the reason we require $f_0 > 0$, so that $\langle u_0 \rangle > 0$ and we can divide by it eventually to recover $\sigma_b$. Then choosing $f_1$ concentrated near a single $\theta'$ (and independent of $x$), allows us to reconstruct the X-ray transform of $\sigma_b$, and therefore $\sigma_b$ itself; see Theorem \ref{thm:recover-sigma-b}.

Particularly, when $k = 0$, one can solve the equation (\ref{eq:transport}) directly with $f_-$ in the form of $f_- = v_-(x) \delta_{\theta_0}(\theta)$ (a collimated source), see \cite{Stefanov2022}, where $v_- > 0$ smooth. Then we are solving a Riccati ODE along each line $s \mapsto (x_0 + s\theta_0, \theta_0)$. This allows us to recover $\sigma_a$ if it depends on $x$ only, and $\sigma_b$ through their attenuated X-ray transforms without the smallness assumption on $f_-$ (or of the perturbation of $f_-$ as in \cite{Stefanov2022}), see Theorem \ref{thm:recover-sigma-b}. This way, we may work with signals which are not necessarily small and will be less sensitive to additive background noise.

The rest of the paper is organized as follows. In Section \ref{sec:prelim}, we state the preliminary results about the well posedness of the two photon absorption radiative transport model (\ref{eq:transport}). Section \ref{sec:main-results} consists of the main theorems about the reconstructions of the absorption and scattering coefficients, respectively. The scattering free case $k = 0$ is discussed in Section \ref{sec:scattering-free}.

2. Preliminaries

We first study the well posedness of (\ref{eq:transport}) and of the albedo operator $A$. Define $\tau_{\pm}(x, \theta) := \min\{t \geq 0 \mid x + t\theta \in \partial \Omega\}$, which stands for the distance between $x$ and the boundary $\partial \Omega$ along $\pm \theta$. Set $\tau(x, \theta) = \tau_-(x, \theta) + \tau_+(\theta)$, and define the boundary measure $d\xi = |n(x) \cdot \theta| d\mu(x) d\theta$, where $d\mu(x)$ is the Lebesgue measure on $\partial \Omega$. Define the function space

$$ H^1(\Omega \times S^{n-1}) := \{ f \mid f \in L^1(\Omega \times S^{n-1}) \text{ and } \theta \cdot \nabla_x f \in L^1(\Omega \times S^{n-1}) \}. $$
Lemma 2.3. If the coefficients are admissible, then for any $H(x)$ and let $u^m(x, \theta) = f(x - \tau_m(x, \theta)\theta, \theta)$ we define the following operators:

$$
\Sigma(m) := \left\{ f \mid f \in L^1(\Gamma, d\xi) \text{ and } \|f\|_\ast < \infty \right\},
$$

where the $\| \cdot \|_\ast$ norm is defined by

$$
\|f\|_\ast := \left\| \int_{\Gamma} |f(x - \tau_m(x, \theta)\theta, \theta)| d\theta \right\|_{L^\infty(\Omega)}.
$$

Definition 2.1. We call the tuple of functions $(\sigma_a, \sigma_b, k, f_-)$ admissible if

1. $\sigma_a, \sigma_b \in L^\infty(\Omega \times S^{n-1})$, $\sigma_a \geq 0$ and $\sigma_b \geq 0$,
2. $0 \leq k(x, \theta', \theta) \in L^\infty(\Omega \times S^{n-1} \times S^{n-1})$ and there exists a constant $\mu \in [0, 1)$ such that
   $$
   \|\tau\|_{L^\infty(\Omega \times S^{n-1})} \|k\|_{L^\infty(\Omega \times S^{n-1} \times S^{n-1})} \leq \mu,
   $$
3. $f_- \in L^1(\Gamma, d\xi)$ and there exists $\nu \in [0, 1)$ such that
   $$
   \|\tau\|_{L^\infty(\Omega \times S^{n-1})} \|\sigma_b\|_{L^\infty(\Omega \times S^{n-1})} \|f_-\|_\ast \leq \nu(1 - \mu)^2.
   $$

Definition 2.2. We define the following operators:

$$
Tu := -\theta \cdot \nabla x u, \quad Su := \langle u \rangle, \quad \Sigma(m)u := -(\sigma_a + \sigma_b m)u.
$$

Then $\Sigma(m) = -(\sigma_a + \sigma_b m)$. Let $J(m) : L^1(\Gamma, d\xi) \to L^1(\Omega \times S^{n-1})$ be defined by

$$
J(m)f_- (x, \theta) = f_- (x - \tau_m(x, \theta)\theta, \theta) \exp \left( \int_{0}^{\tau_m(x, \theta)} \Sigma(m)(x - l\theta, \theta) dl \right),
$$

and let $H(m) : L^1(\Omega \times S^{n-1}) \to L^1(\Omega \times S^{n-1})$ be defined by

$$
H(m)u(x, \theta) = \int_{0}^{\tau_m(x, \theta)} \exp \left( \int_{0}^{l} \Sigma(m)(x - s\theta, \theta) ds \right) K u(x - l\theta, \theta) dl.
$$

Lemma 2.3. If the coefficients are admissible, then for any $m \in L^\infty(\Omega),

$$
|H(|m|)u(x, \theta)| \leq \mu \|\langle u \rangle\|_{L^\infty(\Omega)}.
$$

Proof. Since $K|u|(x, \theta) \leq \|k\|_{L^\infty(\Omega \times S^{n-1} \times S^{n-1})} \|\langle u \rangle\|_{L^\infty(\Omega)},$ we can derive

$$
|H(|m|)u(x, \theta)| = \left| \int_{0}^{\tau_m(x, \theta)} \exp \left( \int_{0}^{l} \Sigma(|m|)(x - s\theta, \theta) ds \right) K u(x - l\theta, \theta) dl \right|
$$

$$
\leq \int_{0}^{\tau_m(x, \theta)} \exp \left( \int_{0}^{l} \Sigma(|m|)(x - s\theta, \theta) ds \right) K|u|(x - l\theta, \theta) dl
$$

$$
\leq \int_{0}^{\tau_m(x, \theta)} \|k\|_{L^\infty(\Omega \times S^{n-1} \times S^{n-1})} \|\langle u \rangle\|_{L^\infty(\Omega)} \leq \mu \|\langle u \rangle\|_{L^\infty(\Omega)}.
$$

In particular, this shows that the operator $H(|m|)$ is a contraction in $L^\infty(\Omega, L^1(S^{n-1}))$. □
Lemma 2.4. If \((\sigma_a, \sigma_b, k, f_-)\) is admissible and if \(m \in L^\infty(\Omega)\), then the linear initial value problem
\[
(T + \Sigma(|m|) + K) u = 0 \quad \text{in} \ \Omega \times S^{n-1},
\]
\[
u(x, \theta) = f_-(x, \theta) \quad \text{on} \ \Gamma_-
\]
has a unique solution \(u \in L^\infty(\Omega, L^1(S^{n-1})) \cap H^1(\Omega \times S^{n-1})\), and this solution satisfies
\[
\|\langle u \rangle\|_{L^\infty(\Omega)} \leq \frac{1}{1 - \mu}\|f_-\|_*.
\]

Proof. The solution \(u\) to (9) satisfies
\[
u(x, \theta) = H(|m|)u(x, \theta) + J(|m|)f_-(x, \theta),
\]
and vice-versa, every solution to (10) solves (9) (in a weak sense). Take the absolute value on both sides of (10) and apply the operator \(S\) to get, for every \(x \in \Omega\),
\[
\langle |u| \rangle(x) \leq \int_{S_{(0)}} |H(|m|)u(x, \theta)|d\theta + \int_{S_{(0)}} |J(|m|)f_-(x, \theta)|d\theta
\]
\[
\leq \mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + \int_{S_{(0)}} |J(|0|)f_-(x, \theta)|d\theta
\]
\[
\leq \mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + \|f_-\|_*.
\]
where we used the Lemma 2.3. The supremum on the left-hand-side satisfies
\[
\|\langle |u| \rangle\|_{L^\infty(\Omega)} \leq \mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + \|f_-\|_*.
\]
In particular, this shows that the operator \(H(|m|)\) is a contraction in \(L^\infty(\Omega, L^1(S^{n-1}))\), and that \(J(|m|)f_-(x, \theta)\) belongs to that space, thus (10) is solvable in \(L^\infty(\Omega, L^1(S^{n-1}))\). Moreover, it satisfies the estimate in the lemma by (12). Then we can apply \(T\) to (10) to conclude that \(u \in H^1\) and solves (11) in strong sense. \(\Box\)

Corollary 2.5. Under the assumptions of Lemma 2.4, the solution \(u\) to (9) also satisfies
\[
\left\| \int_{S_{0}} \int_{0}^{\tau_-(x, \theta)} |u(x - s\theta, \theta)|dsd\theta \right\|_{L^\infty(\Omega)} \leq \frac{\|\tau\|_{L^\infty(\Omega \times S^{n-1})}}{1 - \mu}\|f_-\|_*.
\]

Proof. Using the estimate from Lemma 2.3 and the equation (10), \(\forall s \in [0, \tau_-(x, \theta)]\),
\[
|u(x - s\theta, \theta)| \leq \mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + |f_-(x - s\theta - \tau_-(x, \theta)\theta, \theta)|
\]
\[
= \mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + |f_-(x - \tau_-(x, \theta)\theta, \theta)|.
\]
Apply the integrals with respect to \(s\) and \(\theta\), we obtain
\[
\int_{S_{(0)}} \int_{0}^{\tau_-(x, \theta)} |u(x - s\theta, \theta)|dsd\theta \leq \tau_-(x, \theta) \left(\mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + \int_{S_{(0)}} |f_-(x - \tau_-(x, \theta)\theta, \theta)|d\theta\right).
\]
Then take the supremum on both sides and use the conclusion of Lemma 2.4 to get
\[
\left\| \int_{S_{0}} \int_{0}^{\tau_-(x, \theta)} |u(x - s\theta, \theta)|dsd\theta \right\|_{L^\infty(\Omega)} \leq \|\tau\|_{L^\infty(\Omega \times S^{n-1})} \left(\mu\|\langle |u| \rangle\|_{L^\infty(\Omega)} + \|f_-\|_*\right)
\]
\[
\leq \|\tau\|_{L^\infty(\Omega \times S^{n-1})} \frac{1}{1 - \mu}\|f_-\|_*.
\] \(\Box\)
Lemma 2.6. If \((\sigma_a, \sigma_b, k, f_-)\) is admissible, then the radiative transport equation (11) permits a unique solution \(u(x, \theta) \in H^1(\Omega \times S^{n-1})\), and
\[
\|\langle |u| \rangle\|_{L^\infty(\Omega)} \leq \frac{1}{1 - \mu} \|f_-\|_*,
\]
(16)

In addition, if there exists a constant \(c_0 \geq 0\) such that \(f_- \geq c_0\), then there is a constant \(C = C(\Omega, \sigma_a, \sigma_b, k, f_-) > 0\) such that \(u(x, \theta) \geq Cc_0\).

Proof. The proof is based on the Banach fixed point theorem. Define the mapping \(C : L^\infty(\Omega) \mapsto L^\infty(\Omega)\) by \(C(m) := \langle u \rangle\), where \(u(x, \theta)\) solves (9). Define the sets of functions \(\mathcal{M}\) and \(\mathcal{M}_+\) by
\[
\mathcal{M} := \left\{ m \in L^\infty(\Omega) : |m(x)| \leq \frac{1}{1 - \mu} \|f_-\|_* \right\},
\]
and
\[
\mathcal{M}_+ := \left\{ m \in L^\infty(\Omega) : 0 \leq m(x) \leq \frac{1}{1 - \mu} \|f_-\|_* \right\}.
\]

We prove that \(C\) is a contraction mapping on \(\mathcal{M}\) (resp. \(\mathcal{M}_+\)) with the \(L^\infty(\Omega)\) metric. First we show that \(C : \mathcal{M} \to \mathcal{M}\). If \(m \in \mathcal{M}\), the solution to (9) will satisfy (10). Take the absolute value on both sides of (10) and apply the operator \(S\). By (16), \(\|u\| \in \mathcal{M}\), hence \(\langle u \rangle \in \mathcal{M}\). When \(f_- \geq 0\), from the theory of linear transport (10), the solution \(u(x, \theta)\) to (10) is non-negative as well through a fixed point iteration, thus we have the mapping \(C : \mathcal{M}_+ \to \mathcal{M}_+\). In the next, we show \(C\) is indeed a contraction mapping on both sets. Let \(m_1, m_2 \in \mathcal{M}\) (resp. \(\mathcal{M}_+\)) and \(u_1, u_2\) be the solutions to (9), respectively. Denote \(w = u_1 - u_2\), then
\[
(T + \Sigma(|m_1|) + K)w = \sigma_b u_2(|m_1| - |m_2|), \quad \text{in } \Omega \times S^{n-1},
\]
\[
w(x, \theta) = 0 \quad \text{on } \Gamma_-.\]

Let \(q(x, \theta) := \sigma_b u_2(|m_1| - |m_2|)\), then the solution \(w(x, \theta)\) solves
\[
w(x, \theta) = H(|m_1|)w(x, \theta) + \int_0^{\tau_-(x, \theta)} \exp\left(\int_0^l \Sigma(|m_1|)(x - s \theta)ds\right) q(x - l \theta, \theta)dl.
\]
(18)

Apply the integral operator \(S\) on the second term on the right-hand-side to get
\[
\left| \int_{S^{n-1}} \int_0^{\tau_-(x, \theta)} \exp\left(\int_0^l \Sigma(|m_1|)(x - s \theta)ds\right) q(x - l \theta, \theta)dl d\theta \right|
\leq \|\sigma_b(|m_1| - |m_2|)\|_{L^\infty(\Omega \times S^{n-1})} \left| \int_{S^{n-1}} \int_0^{\tau_-(x, \theta)} |u_2(x - l \theta, \theta)| dl d\theta \right|
\leq \|\tau\|_{L^\infty(\Omega \times S^{n-1})} \|\sigma_b(|m_1| - |m_2|)\|_{L^\infty(\Omega \times S^{n-1})} \frac{1}{1 - \mu} \|f_-\|_*.
\]
The last inequality comes directly from Corollary 2.5. As in the proof of Lemma 2.4,
\[
\langle |w| \rangle(x) \leq \mu \|\langle |w| \rangle\|_{L^\infty(\Omega)} + \frac{\|\tau\|_{L^\infty(\Omega \times S^{n-1})} \|\sigma_b|m_1 - m_2\|_{L^\infty(\Omega \times S^{n-1})}}{1 - \mu} \|f_-\|_*,
\]
(19)

where we have used the triangle inequality \(|m_1| - |m_2| \leq |m_1 - m_2|\). Then use \(|\langle w \rangle(x)\| \leq \langle |w| \rangle(x)\) and \(\langle u_2 \rangle \in \mathcal{M}\) (resp. \(\mathcal{M}_+\) when \(f_-(x, \theta) \geq 0\)) to get
\[
\langle |w| \rangle(x) \leq \frac{\|\tau\|_{L^\infty(\Omega \times S^{n-1})} \|\sigma_b|m_1 - m_2\|_{L^\infty(\Omega \times S^{n-1})}}{(1 - \mu)^2} \|f_-\|_*.
\]
(20)
By condition (3) in Definition 2.1, \( \mathcal{C} \) is a contraction mapping on both \( \mathcal{M} \) and \( \mathcal{M}_+ \) with the \( L^\infty(\Omega) \) metric. Then by the Banach fixed point theorem, \( \mathcal{C} \) has a unique fixed point in \( \mathcal{M} \) (resp. \( \mathcal{M}_+ \) when \( f_-(x, \theta) \geq 0 \)). Then \( \ref{lem:Banach} \) follows from Lemma 2.4. In particular, when \( f_-(x, \theta) \geq c_0 > 0 \), then \( u(x, \theta) \geq 0 \) and \( 0 \leq \langle u \rangle \leq \frac{1}{1 - \mu} \| f_- \|_* \), therefore

\[
\begin{align*}
\langle u(x, \theta) \rangle &= H(\langle u \rangle)u(x, \theta) + J(\langle u \rangle)f_-(x, \theta) \\
&\geq J\left(\frac{1}{1 - \mu} \| f_- \|_* \right) f_-(x, \theta) \\
&\geq c_0 \exp \left(-\text{diam}(\Omega) \left(\| \sigma_a \|_{L^\infty(\Omega \times S^{n-1})} + \frac{1}{1 - \mu} \| \sigma_b \|_{L^\infty(\Omega \times S^{n-1})} \| f_- \|_* \right)\right).
\end{align*}
\]

\[ \square \]

**Remark 2.7.** The mapping \( \mathcal{C} \) may not be compact when \( f_- \in L^1_S(\Gamma_-, d\xi) \), therefore the Schauder fixed point theorem does not apply.

### 3. Main Theorems

In this section, we show that the nonlinear albedo operator determines the three coefficients \( \sigma_a, \sigma_b, \) and, under the condition \( \sigma_a(x, \theta) = \sigma_a(x) \) and \( \sigma_b(x, \theta) = \sigma_b(x) \). In the following, we consider a source function \( f_-(x, \theta) \) in the form of

\[
f_-(x, \theta) = f_0(x, \theta) + \delta f_1(x, \theta)
\]

with \( \delta \to 0 \) a scaling parameter, with \( f_i \in L^1_S(\Gamma_-, d\xi) \) non-negative, \( i = 1, 2 \). Formally, the non-negative solution \( u \) expands as

\[
u(x, \theta) = u_0(x, \theta) + \delta u_1(x, \theta) + \delta^2 u_2(x, \theta) + \cdots.
\]

Then \( u_0 \) and \( u_1 \) will satisfy the equations

\[
\begin{align*}
(T + \Sigma(\langle u_0 \rangle) + K)u_0 &= 0 & \text{in } \Omega \times S^{n-1}, \\
u_0(x, \theta) &= f_0(x, \theta) & \text{on } \Gamma_-, \\
\end{align*}
\]

and

\[
\begin{align*}
(T + \Sigma(\langle u_0 \rangle) + K)u_1 &= -\sigma_b(u_1)u_0 & \text{in } \Omega \times S^{n-1}, \\
u_1(x, \theta) &= f_1(x, \theta) & \text{on } \Gamma_-. \\
\end{align*}
\]

When the coefficients are admissible and \( f_0 = 0 \), then the equation \( \ref{eq:alpha} \) has unique solution \( u_0 = 0 \); and the equation \( \ref{eq:beta} \) becomes the linear transport equation. Then one can follow the method in \ref{estimates} to decompose the singularities, which leads to the reconstruction of \( \sigma_a \) and \( k \), the latter requires dimension \( n \geq 3 \). After the coefficients \( \sigma_a \) and \( k \) are recovered, we can select arbitrary nonzero \( f_0 \in L^1_S(\Gamma_-, d\xi) \) such that \( u_0 \) is non-singular. Then in the equation \( \ref{eq:beta} \), the most singular part in the solution will come from the source \( f_1 \) if we select it to be singular in angular variable \( \theta \). Therefore \( \Sigma(\langle u_0 \rangle) \) can be recovered, and then \( u_0 \) can be solved from \( \ref{eq:alpha} \), which finally reconstructs \( \sigma_b \). In the following, we rigorously prove these claims.

#### 3.1. Reconstruction of \( \sigma_a \)

In next theorem, we show that we can recover the X-ray transform of \( \sigma_a(x, \theta) \). As a corollary, if \( \sigma_a \) is \( \theta \)-independent, one recovers it through the inverse X-ray transform \( \ref{estimates} \).

Here and below, we take sources approximating singular ones in the spirit of \ref{estimates}. Let \( B_1 \) be the unit ball centered at origin in \( \mathbb{R}^n \), \( h \in C_0^\infty(B_1) \) with \( 0 \leq h \leq 1 \) and \( h \equiv 1 \) near origin be a cut-off.
function. Given \( \theta' \in S^{n-1} \), define the source function
\[
(26) \quad f_{\epsilon, \delta}(x, \theta; \theta') = \frac{\delta}{\omega_{n-1} \epsilon^{n-1}} h \left( \frac{\theta - \theta'}{\epsilon} \right),
\]
where \( \delta, \epsilon > 0 \) are small parameters such that \( f_{\epsilon, \delta} \in L^\infty_\gamma(\Gamma_-, d\xi) \) and \( \omega_{n-1} \) is the constant defined by
\[
(27) \quad \omega_{n-1} := \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{1}{\epsilon^{n-1}} h \left( \frac{\theta - \theta'}{\epsilon} \right) d\theta.
\]

We view \( f_{\epsilon, \delta} \) as \( \delta \) times an approximation (a Friedrichs’ mollifier) of the delta function \( \delta_{\theta'}(\theta) \) on the sphere. Then \( f_{\epsilon, \delta} \) plays the role of \( \delta f_1 \) in (22) with \( f_0 = 0 \) there.

**Theorem 3.1.** Let \( f_- = f_{\epsilon, \delta} \) and assume the tuple \((\sigma_a, \sigma_b, k, f_-)\) is admissible, then
\[
\lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{u_{\epsilon, \delta}(x, \theta)}{\delta} h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta = \exp \left( - \int_{0}^{\tau_-(x, \theta')} \sigma_a(x - s\theta', \theta') ds \right),
\]
where \( u_{\epsilon, \delta} \) is the unique solution to (11) with boundary condition \( f_{\epsilon, \delta} \).

**Proof.** Let \( w_\epsilon \) be the unique solution to the following radiative transport equation:
\[
(28) \quad (T + \Sigma(0) + K)w_\epsilon = 0 \quad \text{in } \Omega \times S^{n-1},
\]
\[
\quad w_\epsilon(x, \theta) = \frac{1}{\omega_{n-1} \epsilon^{n-1}} h \left( \frac{\theta - \theta'}{\epsilon} \right) \quad \text{on } \Gamma_-.
\]
The solution \( w_\epsilon \) then satisfies
\[
(29) \quad w_\epsilon(x, \theta) = \frac{1}{\omega_{n-1} \epsilon^{n-1}} h \left( \frac{\theta - \theta'}{\epsilon} \right) \exp \left( - \int_{0}^{\tau_-(x, \theta)} \sigma_a(x - s\theta, \theta) ds \right) + H(0)w_\epsilon,
\]
where \(|H(0)w_\epsilon| \leq \mu \|w_\epsilon\|_{L^\infty(\Omega)}\) which is uniformly bounded from Lemma 2.3. Therefore, the following iterated limit holds
\[
\lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} w_\epsilon(x, \theta) h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta
\]
\[
- \lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{1}{\omega_{n-1} \epsilon^{n-1}} h \left( \frac{\theta - \theta'}{\epsilon} \right) \exp \left( - \int_{0}^{\tau_-(x, \theta)} \sigma_a(x - s\theta, \theta) ds \right) h \left( \frac{\theta - \theta'}{\epsilon} \right) d\theta
\]
\[
+ \lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} H(0)w_\epsilon(x, \theta) h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta
\]
\[
= \exp \left( - \int_{0}^{\tau_-(x, \theta')} \sigma_a(x - s\theta', \theta') ds \right).
\]
The term containing \( H(0) \) vanishes because when \( \gamma \to 0 \),
\[
(30) \quad \left| \int_{S^{n-1}} H(0)w_\epsilon(x, \theta) h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta \right| \leq \mu \|w_\epsilon\|_{L^\infty(\Omega)} \int_{S^{n-1}} h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta \to 0.
\]

Denote \( \phi = \frac{1}{\delta} u_{\epsilon, \delta} - w_\epsilon \), then
\[
(31) \quad (T + \Sigma(\|u_{\epsilon, \delta}\|) + K)\phi = \sigma_b \langle u_{\epsilon, \delta} \rangle |w_\epsilon| \quad \text{in } \Omega \times S^{n-1},
\]
\[
\phi(x, \theta) = 0 \quad \text{on } \Gamma_-.
\]
Then one can show that $\phi(x, \theta) = L_1(x, \theta) + L_2(x, \theta)$, where

$$L_1 = \int_0^{\tau_-(x, \theta)} \exp \left( \int_0^l \Sigma(||u^{\varepsilon, \delta}||)(x - s\theta, \theta)ds \right) K\phi(x - l\theta, \theta)dl,$$

$$L_2 = -\int_0^{\tau_-(x, \theta)} \exp \left( \int_0^l \Sigma(||u^{\varepsilon, \delta}||)(x - s\theta, \theta)ds \right) \sigma_b||u^{\varepsilon, \delta}||w^\varepsilon(x - l\theta, \theta)dl.$$ (33)

The first term $L_1$ is uniformly bounded in $L^\infty$ norm, this could be derived from the Lemma 2.6 and Lemma 2.6 by observing that

$$\frac{1}{c^{n-1}} \int_{S^{n-1}} h \left( \frac{\theta - \theta'}{\varepsilon} \right) d\theta = \int_{S^{n-1}} h(\theta - \theta')d\theta \leq c|\partial B_1|$$

for some absolute constant $c > 0$. Therefore,

$$\int_{S^{n-1}} L_1(x, \theta)h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta = \mathcal{O}(\gamma^{n-1}) \to 0, \text{ as } \gamma \to 0.$$ (35)

For the second term $L_2$ we have

$$\left| \int_{S^{n-1}} L_2(x, \theta)h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta \right| \leq \int_{S^{n-1}} \int_0^{\tau_-(x, \theta)} \sigma_b||u^{\varepsilon, \delta}||w^\varepsilon(x - l\theta, \theta)h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta dl$$

$$\leq \|\sigma_b(u^{\varepsilon, \delta})\|_{L^\infty(\Omega)} \int_{S^{n-1}} \int_0^{\tau_-(x, \theta)} \omega^\varepsilon(x - l\theta, \theta)h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta dl.$$ (36)

Note that $\|\sigma_b(u^{\varepsilon, \delta})\|_{L^\infty(\Omega)} = \mathcal{O}(\delta)$ by Lemma 2.6 and the integral part is uniformly bounded by the decomposition for $w^\varepsilon$ in (29), therefore

$$\lim_{\gamma \to 0} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{S^{n-1}} L_2(x, \theta)h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta = 0.$$ (37)

Combining (30), (35) and (36), we arrive at our conclusion.

### 3.2. Reconstruction of $k$. We show next that once $\sigma_a$ is known, one can recover $k$ pointwise.

When $n \geq 3$, we let $\theta, \theta' \in S^{n-1}$ such that $\theta \not\parallel \theta'$ and denote $\pi_{\theta, \theta'}(x)$ the projection of $x$ onto the subspace $\Theta$ spanned by $\theta, \theta'$. Let $\theta'_1 \in \Theta = \text{span}(\theta, \theta')$ be the unit vector such that $\theta'_1 \cdot \theta' = 0$. Take any $\varphi \in C^\infty_0(-1, 1)$ that $0 \leq \varphi \leq 1$ and $\int_{\mathbb{R}} \varphi(t)dt = 1$. We then define the test function

$$\phi_{\gamma_1, \gamma_2}(x, \theta, \theta') = \frac{1}{\gamma_1} \varphi \left( \frac{x \cdot \theta'_1}{\gamma_1 \theta \cdot \theta'_1} \right) h \left( \frac{x - \pi_{\theta, \theta'}(x)}{\gamma_2} \right),$$

we also define the source function $f^{-\varepsilon, \varepsilon'}_{-\delta}$ in the form of

$$f^{-\varepsilon, \varepsilon'}_{-\delta}(x, \theta; x', \theta') = \frac{\delta}{\omega_n^{-1} c^{n-1}} h \left( \frac{x - x'}{\varepsilon'} \right) h \left( \frac{\theta - \theta'}{\varepsilon} \right),$$

such that $f^{-\varepsilon, \varepsilon'}_{-\delta} \in L^1(\Gamma_-, d\xi)$, the constant $\omega_n^{-1}$ is defined by (27).

**Theorem 3.2.** Let $n \geq 3$, set $f_\varepsilon = f^{-\varepsilon, \varepsilon'}_{-\delta}$, and assume the tuple $(\sigma_a, \sigma_b, k, f_-)$ is admissible. Then

$$\lim_{\gamma_1 \to 0} \lim_{\gamma_2 \to 0} \lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{\partial \Omega} u^{\varepsilon, \varepsilon'}_{-\delta}(x + \tau_+(x, \theta)\theta; x', \theta') h \left( \frac{x - x'}{\varepsilon'} \right) f^{-\varepsilon, \varepsilon'}_{-\delta}(x', \theta') d\mu(x')$$

$$= \exp \left( -\int_0^{\tau_+(x, \theta)} \sigma_a(x + s\theta)ds \right) \exp \left( -\int_0^{\tau_-(x, \theta')} \sigma_a(x - s\theta')ds \right) k(x, \theta', \theta),$$
where \( u^{\varepsilon, \varepsilon, \delta}(x, \theta; x', \theta') \) is the unique solution to (1) with boundary condition \( f_-^{\varepsilon, \varepsilon, \delta} \). The limit holds in \( L^1_{\text{loc}}(\Omega \times (S^{n-1} \times S^{n-1} \setminus D)) \) where \( D = \{(\theta, \theta') \in S^{n-1} \times S^{n-1} \mid \theta \parallel \theta'\}\).

**Proof.** Similar to the section 3 of [9], we can write the solution decomposed as

\[
\begin{align*}
\varepsilon & \frac{L(x, \theta, \theta)}{\varepsilon^{m-1}} \phi_{\gamma_1, \gamma_2}(x' - x + \tau_-(x, \theta', \theta, \theta') d\mu(x') \\
\varepsilon & = \int \frac{1}{\omega_{n-1}^2 \varepsilon^{n-1}} h \left( \frac{x - \tau_-(x, \theta, \theta')}{\varepsilon} \right) h \left( \frac{\theta - \theta'}{\varepsilon} \right) \\
\varepsilon & \times \exp \left( - \int_0^{\tau(x, \theta)} \Sigma(|u^{\varepsilon, \delta}|)(x - s\theta, \theta) ds \right) \phi_{\gamma_1, \gamma_2}(x' - x + \tau_-(x, \theta', \theta, \theta') d\mu(x') \\
\varepsilon & = 0.
\end{align*}
\]

Next, we compute the contribution of the single-scattering term. Let \( E(x, y, m) \) denote

\[
E(x, y, m) = \exp \left( |x - y| \int_0^1 \Sigma(m)(x + s(y - x)) ds \right).
\]

In order to make the derivation concise, we also introduce the following notation

\[
\begin{align*}
x_{\pm, \theta} & = x \pm \tau_{\pm}(x, \theta) \theta, \\
y_{\perp, \theta} & = x_{\perp, \theta} - \theta, \\
z_{\perp, \theta, \theta'} & = y_{\perp, \theta} - \tau_-(y_{\perp, \theta'}, \theta') \theta'.
\end{align*}
\]

Then we could write

\[
\begin{align*}
\lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{\partial \Omega} \frac{L_2(x + \tau_+(x, \theta, \theta), \theta)}{\varepsilon^{m-1}} \phi_{\gamma_1, \gamma_2}(x' - x + \tau_-(x, \theta', \theta, \theta') d\mu(x') \\
= \lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{\partial \Omega} \int_{\varepsilon'^{n-1}} E(x_{\perp, \theta}, y_{\perp, \theta}, |u^{\varepsilon', \delta}|) E(y_{\perp, \theta}, z_{\perp, \theta, \theta'}, |u^{\varepsilon', \delta}|) \times \\
\frac{1}{\omega_{n-1}^{n-1}} h \left( \frac{x' - y'}{\varepsilon'} \right) \frac{1}{\omega_{n-1}^{n-1}} h \left( \frac{\theta' - \theta}{\varepsilon} \right) k(y_{\perp, \theta}, \theta', \theta') \\
\phi_{\gamma_1, \gamma_2}(x' - x + \tau_-(x, \theta', \theta, \theta') d\mu(x') \\
= \lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{\partial \Omega} \int_{\varepsilon'^{n-1}} E(x_{\perp, \theta}, y_{\perp, \theta}, 0) E(y_{\perp, \theta}, z_{\perp, \theta, \theta'}, 0) \times \\
\frac{1}{\omega_{n-1}^{n-1}} h \left( \frac{x' - y'}{\varepsilon'} \right) \frac{1}{\omega_{n-1}^{n-1}} h \left( \frac{\theta' - \theta}{\varepsilon} \right) k(y_{\perp, \theta}, \theta', \theta') \\
\phi_{\gamma_1, \gamma_2}(x' - x + \tau_-(x, \theta', \theta, \theta') d\mu(x').
\end{align*}
\]
The right-hand-side has the limit

\[
\lim_{\gamma_1 \to 0, \gamma_2 \to 0} \int_{\Omega} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0) \int_{\Omega} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0)
\]

To show that the multi-scattering contribution is zero, we only need to show that

\[
\frac{1}{\omega_{n-1} \varepsilon^{m-1}} \int_{0}^{(x,\theta)} \int_{0}^{(x,\theta)} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0) \int_{\Omega} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0)
\]

The right-hand-side has the limit

\[
\int_{\Omega} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0) \int_{\Omega} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0)
\]

To show that the multi-scattering contribution is zero, we only need to show that \(\frac{1}{\varepsilon^{m-1}} L_3(x,\theta) \in L^1(\Omega \times S^{n-1})\) uniformly, hence uniform bounded in \(L^1(\Gamma_+, d\xi)\). Given any \(\chi \in C^\infty_0(\Omega \times (S^{d-1} \times S^{d-1} \setminus D))\), we have

\[
\int_{S^{n-1} \times S^{n-1}} L_3(x + \gamma_2 (x,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0) \int_{\Omega} E(x + \gamma_2 (y,0,0) E(y,0,0)) k(y,0,0) \times \phi_{\gamma_1,\gamma_2}(y,0)
\]

where \(T_{\gamma_2} = \{ (x,0,0) \in \Omega \times S^{n-1} \times S^{n-1} \cap \text{supp} \chi \} \). When \(\frac{1}{\varepsilon^{m-1}} L_3(x,\theta) \) is uniformly bounded in \(L^1(\Gamma_+, d\xi)\), the integrand of (43) is an \(\varepsilon\) function. On the other hand, \(meas(T_{\gamma_2}) = 0\) as \(\gamma_2 \to 0\), therefore the integral vanishes as \(\gamma_2 \to 0\). In the following, we prove \(\frac{1}{\varepsilon^{m-1}} L_3(x,\theta) \in L^1(\Omega \times S^{n-1})\) with a uniform bound there with respect to \(\varepsilon \ll 1\) and \(\delta \ll 1\).

Since \((I - H(||u_{\varepsilon,\varepsilon',\delta}'||))^{-1}\) is a uniformly bounded operator in \(L^1(\Omega \times S^{n-1})\), we merely have to show that \(\frac{1}{\varepsilon^{m-1}} H^2(||u_{\varepsilon,\varepsilon',\delta}'||) J(||u_{\varepsilon,\varepsilon',\delta}'||) f_-\) is also uniformly bounded, see (39). Let \(y_{i,\theta} = x - \theta, z_{s,\theta''} = y_{i,\theta} - s\theta', w_{y^m} = z_{s,\theta''} - \gamma_2 (z_{s,\theta''}, \theta'' \theta'')\). Then

\[
\int_{S^{n-1} \times S^{n-1}} H^2(||u_{\varepsilon,\varepsilon',\delta}'||) J(||u_{\varepsilon,\varepsilon',\delta}'||) f_- (x,\theta) \]

\[
\leq \int_{-\gamma_2}^{\gamma_2} \int_{S^{n-1} \times S^{n-1}} E(x,y_{i,\theta}, ||u_{\varepsilon,\varepsilon',\delta}'||) E(y_{i,\theta}, z_{s,\theta''}, ||u_{\varepsilon,\varepsilon',\delta}'||) \times \int_{S^{n-1} \times S^{n-1}} k(y_{i,\theta}, z_{s,\theta''}, \theta'', \theta'' \theta'') f_- (w_{y^m}, \theta'' \theta'' \theta'') d\theta'' d\theta' d\theta' d\theta'' d\theta' d\theta'' dl
\]

\[
\leq \int_{-\gamma_2}^{\gamma_2} \int_{S^{n-1} \times S^{n-1}} k(y_{i,\theta}, \theta'', \theta'') k(z_{s,\theta''}, \theta'', \theta'') f_- (w_{y^m}, \theta'' \theta'') d\theta'' d\theta' d\theta'' d\theta' d\theta'' dl.
\]
Since $z_{s,\theta''} = y_{l,\theta} - s\theta''$, we change the variable that $dz_{s,\theta''} = s^{n-1}dsd\theta''$, and recall the formula

$$
\int_{\Omega \times S^{n-1}} g(x, \theta) dx d\theta = \int_{\Gamma_-} \int_0^\tau(x', \theta) g(x' + t\theta, \theta) dt d\xi(x', \theta),
$$

see \cite{9}, with $x' = x - \tau_-(x, \theta).$ We obtain

$$
\int_0^{\tau_-(x, \theta)} \int_{S^{n-1}} k(y_{l,\theta}, \theta''') f_-(w_{\theta''', \theta''})(y_{l,\theta}, \theta'')k(z_{s,\theta''}, \theta''')|df_-(w_{\theta''}, \theta'')| d\theta''' dsd\theta d\theta' d\theta'' d\theta''' d\theta'' d\theta'''
$$

\begin{align}
&= \int_0^{\tau_-(x, \theta)} \int_{S^{n-1}} k(y_{l,\theta}, \theta'')k(w_{\theta'''} + t\theta''), \theta'') \\
&\quad \cdot \frac{1}{s^{n-1}} f_-(w_{\theta''}, \theta'')|s^{1-n} dtd\xi(w_{\theta''}, \theta'')| d\theta'' d\theta'' d\theta'' d\theta'' d\theta'' d\theta'' d\theta'' d\theta'' d\theta'' d\theta''
\end{align}

$$
\leq C \left\| \frac{1}{s^{n-1}} f_- \right\|_{L^1(\Gamma_-, d\xi)} \int_0^{\tau_-(x, \theta)} \int_{S^{n-1}} \frac{s^{1-n} dtd\xi}{s^{n-1}} \in L^1(\Omega \times S^{n-1}),
$$

which is uniformly bounded in $L^1(\Omega \times S^{n-1})$ with respect to $\varepsilon'$, where $s = |y_{l,\theta} - (x' + t\theta')|$ and $C = \|k\|^2_{L^\infty(\Omega \times S^{n-1} \times S^{n-1})}. \square$

3.3. Reconstruction of $\sigma_b$. Let the source function $f_-$ be chosen in the following form

$$
f_-^{\varepsilon, \delta}(x, \theta; \theta') = c_0 \frac{\delta}{\omega_{n-1} \varepsilon^{n-1}} h\left(\frac{\theta - \theta'}{\varepsilon}\right),
$$

where $c_0$ is a positive constant and $\delta, \varepsilon$ are positive small parameters. Compared with \cite{22}, here we have added $f_0 = c_0$ in \cite{22}.

**Theorem 3.3.** Let $f_- = f_-^{\varepsilon, \delta}$ and assume the tuple $(\sigma_a, \sigma_b, k, f_-)$ is admissible, then

$$
\lim_{\delta \to 0} \lim_{\gamma \to 0} \lim_{\varepsilon \to 0} \int_{S^{n-1}} u^{\varepsilon, \delta}(x, \theta) h\left(\frac{\theta - \theta'}{\gamma}\right) ds = \exp\left(\int_0^{\tau_-(x, \theta')} \Sigma(|\langle w \rangle|)(x - \theta') ds\right),
$$

where $u^{\varepsilon, \delta}$ is the unique solution to \cite{11} with boundary condition $f_-$ and $w$ is the unique solution to \cite{11} with the boundary condition $f_- = c_0$.

**Proof.** Let $w(x, \theta)$ be the solution to the following equation

$$
(T + \Sigma(|\langle w \rangle|) + K)w = 0 \quad \text{in} \quad \Omega \times S^{n-1},
$$

$$
w(x, \theta) = c_0 \quad \text{on} \quad \Gamma_-.
$$

Then $w \in L^\infty(\Omega \times S^{n-1})$, which implies

$$
\lim_{\delta \to 0} \lim_{\gamma \to 0} \lim_{\varepsilon \to 0} \int_{S^{n-1}} w(x, \theta) h\left(\frac{\theta - \theta'}{\gamma}\right) ds = 0.
$$

We denote $\phi = \frac{1}{\delta}(u^{\varepsilon, \delta} - w)$. It satisfies

$$
(T + \Sigma(|\langle u^{\varepsilon, \delta} \rangle|) + K)\phi = \sigma_b\left(\frac{|\langle u^{\varepsilon, \delta} \rangle| - |\langle w \rangle|}{\delta}\right) w \quad \text{in} \quad \Omega \times S^{n-1},
$$

$$
\phi(x, \theta) = \frac{1}{\omega_{n-1} \varepsilon^{n-1}} h\left(\frac{\theta - \theta'}{\varepsilon}\right) \quad \text{on} \quad \Gamma_-.
$$
Therefore, the solution $\phi$ can be written in the following form,

$$
\phi(x, \theta) = \exp \left( \int_0^{\tau_-(x, \theta)} \Sigma((\|u^{\epsilon, \delta}\|))(x - s\theta)ds \right) \frac{1}{\omega_{n-1} \epsilon^{n-1} h} \left( \frac{\theta - \theta'}{\epsilon} \right)
$$

$$
+ \int_0^{\tau_-(x, \theta)} \exp \left( \int_0^{\tau_-(x, \theta)} \Sigma((\|u^{\epsilon, \delta}\|))(x - s\theta)ds \right) K\phi(x - l\theta, \theta)dl
$$

$$
- \int_0^{\tau_-(x, \theta)} \exp \left( \int_0^{l'} \Sigma((\|u^{\epsilon, \delta}\|))(x - s\theta)ds \right) \left[ \sigma_b \left( \frac{|\langle u^{\epsilon, \delta} \rangle| - |\langle w \rangle|}{\delta} \right) w(x - l\theta, \theta) \right] dl
$$

$$
= L_1(x, \theta) + L_2(x, \theta) + L_3(x, \theta).
$$

Integrate $\phi(x, \theta)$ over $S^{n-1}$ and note $||\langle u^{\epsilon, \delta} \rangle| - |\langle w \rangle|| \leq \delta ||\phi||$, to obtain

$$
\|\langle \phi \rangle\|_{L^\infty(\Omega)} \leq \frac{1}{(1 - \mu)(1 - \nu)} \int_{S^{n-1}} \frac{1}{\omega_{n-1} \epsilon^{n-1} h} \left( \frac{\theta - \theta'}{\epsilon} \right) d\theta.
$$

This implies that $L_2, L_3$ are both uniformly bounded in $L^\infty(\Omega \times S^{n-1})$, hence

$$
\lim_{\delta \to 0} \lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} \phi(x, \theta) h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta = \lim_{\delta \to 0} \lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} L_1(x, \theta) \frac{1}{\omega_{n-1} \epsilon^{n-1} h} \left( \frac{\theta - \theta'}{\epsilon} \right) h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta
$$

$$
= \lim_{\delta \to 0} \exp \left( \int_0^{\tau_-(x, \theta')} \Sigma((\langle u^{\epsilon, \delta} \rangle))(x - s\theta')ds \right)
$$

$$
= \exp \left( \int_0^{\tau_-(x, \theta')} \Sigma((\langle w \rangle))(x - s\theta')ds \right).
$$

Combine this with (50) to obtain

$$
\lim_{\delta \to 0} \lim_{\gamma \to 0} \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{u^{\epsilon, \delta}(x, \theta)}{\delta} h \left( \frac{\theta - \theta'}{\gamma} \right) d\theta = \exp \left( \int_0^{\tau_-(x, \theta')} \Sigma((\langle w \rangle))(x - s\theta')ds \right).
$$

Theorem 3.3 implies that $\Sigma(|\langle w \rangle|)$ can be reconstructed from the albedo operator. Therefore the solution $w$ of (19) can be uniquely determined and there exists a constant $C > 0$ such that $w(x, \theta) \geq C\epsilon_0$ by Lemma 2.6 and when $\sigma_a$ is known, one can find $\sigma_b = (\Sigma(|\langle w \rangle|) - \sigma_a)/|\langle w \rangle|$.

4. Scattering-free media

For media with $k = 0$, there exists a more direct explicit reconstruction method. Moreover, no smallness assumptions on the boundary source are needed. Equation (1) reduces to

$$
\theta \cdot \nabla u + \sigma_a u + \sigma_b \langle w \rangle u = 0.
$$

Choose the boundary condition

$$
f_- = v_-(x)\delta\theta_0(\theta).
$$
in (11) with some \( v_- (x) \geq 0 \) in \( C^1 \). We are going to look for a non-negative weak solution, i.e., for a solution of the integrated equation

\[
(57) \quad u(x, \theta) = v_-(x - \tau_-(x, \theta) \theta) \delta_{\theta_0}(\theta) \exp \left( - \int_0^{\tau_-(x, \theta)} (\sigma_a + \sigma_b(u))(x - \theta s) ds \right)
\]

in the following class: \( u(x, \theta) \) is a measure-valued function in \( \theta \), \( C^1(\Omega) \cap C(\bar{\Omega}) \) in the \( x \) variable. Then \( \langle u \rangle(x) \) is in the latter space. By (57), \( u = \delta_{\theta_0}(\theta)v \) with \( v \in C(\bar{\Omega} \times \mathbb{S}^{n-1}) \); and also, \( v \) is \( C^1 \) except for \((x, \theta)\) such that \( x \in \partial \Omega \) and \( \theta \) is tangent to \( \partial \Omega \) (which is \( \partial \Gamma_0 \)). Clearly, only the value of \( v \) at \( \theta = \theta_0 \) matters for \( u \). With some abuse of notation, we denote \( v(x, \theta_0) \) by \( v(x) \). Then by (56), \( v \) must satisfy the boundary condition \( v = v_- \) on \( \partial \Omega \).

In view of the \( C^1 \) regularity of \( v \) as stated above, we can differentiate (57) to get back to the differential form (55), which in this case reduces to

\[
(58) \quad (\theta_0 \cdot \nabla v + \sigma_a v + \sigma_b v^2) = 0,
\]

since \( \langle u \rangle = v \). Here, \( \sigma_a \) and \( \sigma_b \) can depend on \( \theta \) as well; then \( \theta = \theta_0 \) above. Therefore, on each line \( s \mapsto (x_0 + s \theta_0, \theta_0) \), the equation reduces to

\[
(59) \quad v' + \sigma_a v + \sigma_b v^2 = 0.
\]

This is a homogeneous Riccati equation. For each initial condition \( v(0) = v_- (x_0) \), we measure \( v(\tau_+(x, \theta_0)) \).

Let \( \mu(t) = \exp \left( - \int_0^t \sigma_a(s) ds \right) \); then \( 1/\mu \) is the integrating factor. Multiply (59) by \( 1/\mu \) to get

\[
(60) \quad (v/\mu)' + \sigma_b v^2/\mu = 0.
\]

This is a separable ODE for \( v/\mu \) and the solution satisfies

\[
(61) \quad \frac{\mu}{v} = \frac{1}{v_- (x_0)} + \int_0^s \mu(t) \sigma_b(t) dt,
\]

therefore,

\[
(62) \quad v(s) = \mu(s) \left( \frac{1}{v_- (x_0)} + \int_0^s \mu(t) \sigma_b(t) dt \right)^{-1}.
\]

Hence, at \( s = \tau_+(x_0, \theta_0) \) we recover the attenuated X-ray transform of \( \sigma_b \) with attenuation \( \sigma_a \), assuming \( \sigma_a \) known. One way to recover \( \sigma_a \) is to replace \( v_- (x_0) \) by \( \delta v_- (x_0) \) as in the previous section with \( \delta \to 0 \), then we get the X-ray transform \( - \log \mu(\tau_+(x, \theta_0)) \) of \( \sigma_a \); and by varying \( \theta \), we can recover \( \sigma_a \). Then we recover \( \sigma_b \) by inverting the attenuated X-ray transform of \( \sigma_b \), see [3][21].

If we do not want to deal with small signals which may be corrupted by background noise too much, we can proceed as following. To reconstruct \( \sigma_a \), we choose two distinct boundary sources \( f_{-j} = v_{-j} (x) \delta_{\theta_0}(\theta), \ j = 1, 2 \) such that \( \forall x \in \partial \Omega, \ v_{-1}(x) > v_{-2}(x) \). Let \( v_1, v_2 \) be the solutions to (59) with \( v_j(0) = v_{-j} (x_0) \), then from (62) we observe

\[
(63) \quad \frac{1}{v_j(s)} = \frac{1}{\mu(s)} \left( \frac{1}{v_{-j} (x_0)} + \int_0^s \mu(t) \sigma_b(t) dt \right), \ \ j = 1, 2.
\]

Subtracting the above formulas with \( j = 1, 2 \), we obtain

\[
(64) \quad \frac{1}{v_1(s)} - \frac{1}{v_2(s)} = \frac{1}{\mu(s)} \left( \frac{1}{v_{-1} (x_0)} - \frac{1}{v_{-2} (x_0)} \right),
\]
which implies

\[
\mu(s) = \left( \frac{1}{v_1(s)} - \frac{1}{v_2(s)} \right)^{-1} \left( \frac{1}{v_{-1}(x_0)} - \frac{1}{v_{-2}(x_0)} \right).
\]

Take \( s = \tau_+(x_0, \theta_0) \) to get \( \mu(\tau_+(x_0, \theta_0)) = \exp(-X\sigma_a(x_0, \theta_0)) \), where \( X \) is the X-ray transform, can be determined by (65). Therefore, we can recover \( \sigma_a \) first by varying \( \theta_0 \) and inverting the X-ray transform of \( \sigma_a \) as above. After that, we recover \( \sigma_b \) as above.

Also, one can take \( v_-(x_0) \) approximating \( \delta_{x_0}(x) \), this corresponds to a single beam.

Therefore, we proved the following.

**Theorem 4.1.** Assume \( k = 0 \). Let \( \sigma_a \) and \( \sigma_b \) depend on \( x \) only and be in \( C^0(\Omega) \). Then \( A \) acting on \( f_- \) as in (56), determines \( \sigma_a, \sigma_b \) uniquely by inverting their attenuated, respectively the non-attenuated X-ray transforms, which can be determined by [62] and (65).

**REFERENCES**

[1] G. Bal. Inverse transport theory and applications. *Inverse Problems*, 25(5):053001, 48, 2009.
[2] G. Bal and A. Jollivet. Stability estimates in stationary inverse transport. *Inverse Probl. Imaging*, 2(4):427–454, 2008.
[3] G. Bal and A. Jollivet. Generalized stability estimates in inverse transport theory. *Inverse Probl. Imaging*, 12(1):59–90, 2018.
[4] G. Bal, A. Jollivet, I. Langmore, and F. Monard. Angular average of time-harmonic transport solutions. *Comm. Partial Differential Equations*, 36(6):1044–1070, 2011.
[5] G. Bal, I. Langmore, and F. Monard. Inverse transport with isotropic sources and angularly averaged measurements. *Inverse Probl. Imaging*, 2(1):23–42, 2008.
[6] G. Bal and F. Monard. Inverse transport with isotropic time-harmonic sources. *SIAM J. Math. Anal.*, 44(1):134–161, 2012.
[7] P. Bardsley, K. Ren, and R. Zhang. Quantitative photoacoustic imaging of two-photon absorption. *Journal of biomedical optics*, 23(1):016002, 2018.
[8] A. A. Bukhgeim and S. G. Kazantsev. Inversion formula for the Fan-Beam attenuated Radon transform in a unit disk. *Sobolev Institute of Mathematics, Siberian Branch of Russian Acad. Sci.*, Novosibirsk, preprint No. 99, 2002.
[9] M. Choulli and P. Stefanov. An inverse boundary value problem for the stationary transport equation. *Osaka J. Math.*, 36(1):87–104, 1999.
[10] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology: Volume 6 Evolution Problems II*. Springer Science & Business Media, 2012.
[11] C. Klingenberg, R.-Y. Lai, and Q. Li. Reconstruction of the emission coefficient in the nonlinear radiative transfer equation. *SIAM J. Appl. Math.*, 81(1):91–106, 2021.
[12] R.-Y. Lai and Q. Li. Parameter reconstruction for general transport equation. *arXiv:1904.10049*, 2019.
[13] R.-Y. Lai, Q. Li, and G. Uhlmann. Inverse problems for the stationary transport equation in the diffusion scaling. *SIAM Journal on Applied Mathematics*, 79(6):2340–2358, 2019.
[14] R.-Y. Lai, G. Uhlmann, and Y. Yang. Reconstruction of the collision kernel in the nonlinear Boltzmann equation. *SIAM J. Appl. Math.*, 53(1):1049–1069, 2021.
[15] N. S. Makarov, M. Drobizheva, and A. Rebane. Two-photon absorption standards in the 550–1600 nm excitation wavelength range. *Opt. Express*, 16(6):4029–4047, Mar 2008.
[16] S. McDowall, P. Stefanov, and A. Tamasan. Gauge equivalence in stationary radiative transport through media with varying index of refraction. *Inverse Probl. Imaging*, 4(1):151–167, 2010.
[17] S. McDowall, P. Stefanov, and A. Tamasan. Stability of the gauge equivalent classes in inverse stationary transport. *Inverse Problems*, 26(2):025006, 19, 2010.
[18] S. McDowall, P. Stefanov, and A. Tamasan. Stability of the gauge equivalent classes in inverse stationary transport in refractive media. *Contemporary Math.*, 559:85–100, 2011.
[19] S. R. McDowall. An inverse problem for the transport equation in the presence of a Riemannian metric. *Pacific J. Math.*, 216(2):303–326, 2004.
[20] S. R. McDowall. Optical tomography on simple Riemannian surfaces. *Comm. Partial Differential Equations*, 30(7-9):1379–1400, 2005.
[21] R. G. Novikov. An inversion formula for the attenuated X-ray transformation. *Ark. Mat.*, 40(1):145–167, 2002.

[22] M. Pawlicki, H. A. Collins, R. G. Denning, and H. L. Anderson. Two-photon absorption and the design of two-photon dyes. *Angewandte Chemie International Edition*, 48(18):3244–3266, 2009.

[23] K. Ren and Y. Zhong. Unique determination of absorption coefficients in a semilinear transport equation. *arXiv:2007.09516*, 2020.

[24] M. Rumi and J. W. Perry. Two-photon absorption: an overview of measurements and principles. *Advances in Optics and Photonics*, 2(4):451–518, 2010.

[25] V. A. Sharafutdinov. Inverse problem of determining a source in the stationary transport equation on a Riemannian manifold. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 239(Mat. Vopr. Teor. Rasprostr. Voln. 26):236–242, 270, 1997.

[26] P. Stefanov. Inverse problems in transport theory. In *Inside out: inverse problems and applications*, volume 47 of *Math. Sci. Res. Inst. Publ.*, pages 111–131. Cambridge Univ. Press, Cambridge, 2003.

[27] P. Stefanov and A. Tamasan. Uniqueness and non-uniqueness in inverse radiative transfer. *Proc. Amer. Math. Soc.*, 137(7):2335–2344, 2009.

[28] P. Stefanov and G. Uhlmann. Optical tomography in two dimensions. *Methods Appl. Anal.*, 10(1):1–9, 2003.

[29] P. Stefanov and G. Uhlmann. An inverse source problem in optical molecular imaging. *Anal. PDE*, 1(1):115–126, 2008.

[30] E. W. Van Stryland, H. Vanherzeele, M. A. Woodall, M. Soileau, A. L. Smirl, S. Guha, and T. F. Boggess. Two photon absorption, nonlinear refraction, and optical limiting in semiconductors. *Optical Engineering*, 24(4):244613, 1985.

[31] H. Zhao and Y. Zhong. Instability of an inverse problem for the stationary radiative transport near the diffusion limit. *SIAM Journal on Mathematical Analysis*, 51(5):3750–3768, 2019.

Email address: stefanop@purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907

Email address: yimin.zhong@duke.edu

Department of Mathematics, Duke University, Durham, NC 27710