The Hamilton-Jacobi analysis for higher-order modified gravity

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The Hamilton-Jacobi [HJ] study for the Chern-Simons [CS] modification of general relativity [GR] is performed. The complete structure of the Hamiltonians and the generalized brackets are reported, from these results the HJ fundamental differential is constructed and the symmetries of the theory are found. By using the Hamiltonians we remove an apparent Ostrogradsky’s instability and the new structure of the hamiltonian is reported. In addition, the counting of physical degrees of freedom is developed and some remarks are discussed.

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I. INTRODUCTION

It is well-known that GR is a successful framework for describing the classical behavior of the gravitational field and its relation with the geometry of space-time [12]. From the canonical point of view, GR is a background independent gauge theory with diffeomorphisms invariance; the extended Hamiltonian is a linear combination of first class constraints and propagates two physical degrees of freedom [7]. From the quantum point of view, the quantization program of gravity is a difficult task to perform. In fact, from the nonperturbative scheme, the non-linearity of the gravitational field, manifested in the constraints, obscures the quantization making the complete description of a nonperturbative quantum theory of gravity still an open problem [8, 9]. On the other hand, the perturbative point of view of the path-integral method leads to the non-renormalizability problem [10, 11] with all the tools that have been developed in quantum field theory have not worked successfully. In this respect, it is common to study modified theories of gravity in order to obtain insights in the classical or quantum regime; with the expectation that these theories will provide new ideas or allow the development of new tools to carry out the quantization program, with an example of this being the so-called higher order theories [12, 15]. In fact, higher-order theories are good candidates for fixing the infinities that appear in the renormalization problem of quantum gravity. It is claimed that adding higher order terms quadratic in the curvature to gravity could help avoid this problem; since these terms have a dimensionless coupling constant, which ensures...
that the final theory is divergence-free \cite{16,17}. The study of higher-order theories is a modern topic in physics, these theories are relevant in dark energy physics \cite{18,19}, generalized electrodynamics \cite{20,22} and string theories \cite{23,24}. Furthermore, an interesting model in four dimensions can be found in the literature, in which the Einstein-Hilbert \([EH]\) action is extended by the addition of a Chern-Simons four-current coupled with an auxiliary field, thus, under a particular choice of the auxiliary field the resulting action will be a close model to \(GR\) \cite{25}. In fact, at Lagrangian level the theory describes the propagation of two degrees of freedom corresponding to gravitational waves traveling with velocity \(c\), but these propagate with different polarization intensities violating spatial reflection symmetry. Moreover, the Schwarzchild metric is a solution of the equations of motion, thus, the modified theory and the \(EH\) action share the same classical tests. On the other hand, at hamiltonian level the theory is a higher-order gauge theory \cite{26} whose Hamiltonian analysis is known not to be easy to perform. In this respect, the analysis of constrained higher-order systems is usually developed by using the Ostrogradsky-Dirac \([OD]\) \cite{27,30} or the Gitman-Lyakhovich-Tyutin \([GLT]\) \cite{31,32} methods. \(OD\) scheme is based on the extension of the phase space by considering to the fields and their velocities as canonical coordinates and then introducing an extension to their canonical momenta. However, the identification of the constraints is not easy to develop; in some cases, the constraints are fixed by hand in order to obtain a consistent algebra \cite{33} and this yields the opportunity to work with alternative methods. On the other hand, the \(GLT\) framework is based on the introduction of extra variables which transforms a problem with higher time derivatives to one with only first-order ones then, by using the Dirac brackets the second class constraints and the extra variables can be removed \cite{34}.

Nevertheless, there is an alternative scheme for analyzing higher-order theories: the so-called Hamilton-Jacobi method. The \(HJ\) scheme for regular field theories was developed by Güler \cite{35,36} and later extended for singular systems in \cite{37,38}. It is based on the identification of the constraints, called Hamiltonians. These Hamiltonians can be either involutive or non-involutive and they are used for constructing a generalized differential, where the characteristic equations, the gauge symmetries, and the generalized \(HJ\) brackets of the theory can be identified. It is important to remark that the identification of the Hamiltonians is performed by means of the null vectors, thus, the Hamiltonians will have the correct structure without fix them by hand as is done in other approaches, then the identification of the symmetries will be, in general, more economical than other schemes \cite{39,43}.

With all of above the aims of this paper is to develop a detailed \(HJ\) analysis of the theory reported in \cite{25}. In fact, we shall analyze this model beyond the Lagrangian approach reported in \cite{25}; we shall see that the Jackiw-Yi \([JY]\) model is a higher-order theory and it is mandatory to study this theory due to its closeness with \(GR\). However, it is well-known that in higher-order theories could be present ghost degrees of freedom associated to Ostrogradsky’s instabilities \cite{44}, namely, the hamiltonian function is unbounded and this is reflected with the presence of linear terms in the canonical momenta in the hamiltonian. In this respect, it is important to comment that if there are constraints, then it is possible to heal those instabilities \cite{45,46}; in our case the \(JY\) model will show an apparent Ostrogradsky’s instability since linear terms in the momenta will appear, however, we
will heal the theory by using the complete set of Hamiltonians, thereby exorcising the associated ghosts.

The paper is organized as follows. In Sect. II, we start with the CS modification of GR, we will work in the perturbative context, say, we will expand the metric around the Minkowski background. We shall observe that the modified theory is of higher-order in the temporal derivatives, then we shall introduce a change of variables in order to express the action in terms of only first-order temporal derivatives. The change of variable will allows us to develop the HJ analysis in an easy way; the identification of the Hamiltonians, the construction of the generalized differential and the symmetries will be identified directly. In Sect. III we present the conclusions and some remarks.

II. THE HAMILTON-JACOBI ANALYSIS

The modified EH action is given by

$$S[g_{\mu\nu}] = \int_M \left( R\sqrt{-g} + \frac{1}{4} \theta^\tau R^\tau_{\mu\nu} R^\sigma_{\sigma\mu\nu} \right) d^4x,$$

where $M$ is the space-time manifold, $g_{\mu\nu}$ the metric tensor, $R$ the scalar curvature, $g$ the determinant of the metric, $R^\tau_{\beta\mu\nu}$ the Riemman tensor and $\theta$ is a coupling field. In general, $\theta$ can be viewed as an external quantity or as a local dynamical variable, however, in order to obtain an action close to GR we are going to choose $\theta = \frac{4}{M}$. Along the paper we will use greek letters for labeling space-time indices $\mu = 0, 1, 2, 3$ and latin letters for space indices $i = 1, 2, 3$. In addition, we will work within the perturbative context expanding the metric around the Minkowski background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is the perturbation. By substituting the expression for $\theta$ and by taking into account eq. (2) in (1) we obtain the following linearized action

$$S[h_{\mu\nu}] = -\frac{1}{2} \int_M h^{\mu\nu} \left( G^{lin}_{\mu\nu} + C^{lin}_{\mu\nu} \right) d^4x,$$

where $G^{lin}_{\mu\nu}$ is the linearized version of the Einstein tensor and $C^{lin}_{\mu\nu}$ is a linearized Cotton-type tensor

$$C^{lin}_{\mu\nu} = -\frac{1}{16}[\epsilon_{\mu\nu\lambda\gamma} \partial^\lambda(\Box h_{\nu\lambda} - \partial_\nu \partial_\lambda h^{\alpha\gamma}) + \epsilon_{\mu\nu\lambda\gamma} \partial^\lambda(\Box h_{\lambda\nu} - \partial_\lambda \partial_\nu h^{\alpha\gamma})$$

defined in four-dimensions. Now we shall suppose that the space-time has a topology $M \cong \mathbb{R} \times \Sigma$, where $\mathbb{R}$ is an evolution parameter and $\Sigma$ is a Cauchy hypersurface. Hence, by performing the $3 + 1$ decomposition of the action (3) we write down the corresponding Lagrangian density

$$\mathcal{L} = \int \left[ \frac{1}{2} h_{ij} \dot{h}^{ij} - \partial_j h_{0i} \partial^j h^{0i} - \frac{1}{2} \partial_k h_{ij} \partial^k h^{ij} - \frac{1}{2} \dot{h}_i \dot{h}^{ij} + \partial^i \dot{h}_0 \partial_j h^{ij} + \frac{1}{2} \partial_k h_i \partial^k h^{ij} - 2 \dot{h}_i \dot{h}_j + 

- \partial_i h_0 \partial_j \dot{h}^{ij} - \dot{h}_i \dot{h}^{ij} \partial_j h_k + 2 \partial_j h_0 \dot{h}^{ij} + \partial_i h_0 \partial_j h^{0j} + \partial_k h_i \partial_j h^{ij} + \frac{1}{\mu} \epsilon^{ijk} (\dot{h}^{ij} \partial_j h_k) + 2 \dot{h}_i \partial_j h^{ij} + \partial_k h m \partial_m \partial_j h^{k} + \nabla^2 h_0 \partial_j h_{0k} + \nabla^2 h^m \partial_j h_{mk} \right] d^3x,$$
by introducing the following change of variable

\[ K_{ij} = \frac{1}{2}(\dot{h}_{ij} - \partial_i h_{0j} - \partial_j h_{0i}), \]

(5)

here \( K_{ij} \) is related with the so-called extrinsic curvature \([47, 48]\). Thus, by substituting (5) into (4) we rewrite the Lagrangian in the following new fashion

\[ L = \int \left[ 2K_{ij}K_{ij} - 2K_i^iK_j^j - h_{00}R_{ij}^{ij} - h_{ij}R^{ij} + \frac{1}{2}h^{ij}R_{ij}^{ij} + \frac{1}{\mu}\epsilon^{ijk}(4K_i^i\partial_j K_{kl} + \partial^m h_{im}\partial_j \partial^k h_{kl} + \nabla^2 h_i^m \partial_j h_{km} + \psi^{ij}(\dot{h}_{ij} - \partial_i h_{0j} - \partial_j h_{0i} - 2K_{ij})) \right] d^3x, \]

(6)

where we have added the Lagrange multipliers \( \psi^{ij} \) enforcing the relation (5), and the expressions \( R_{ij}^{ij} \) and \( R_{ij} \) are defined in the following way

\[ R_{ij}^{ij} = \partial^i \partial^j h_{ij} - \nabla^2 h_i^i, \]

(7)

\[ R_{ij} = \frac{1}{2}(\partial_i \partial^k h_{jk} + \partial_j \partial^k h_{ik} - \partial^i \partial^j h_{kk} - \nabla^2 h_{ij}). \]

(8)

Now, we calculate the canonical momenta associated with the dynamical variables

\[ \pi^{00} = \frac{\partial L}{\partial \dot{h}_{00}} = 0, \]

(9)

\[ \pi^{0i} = \frac{\partial L}{\partial \dot{h}_{0i}} = 0, \]

(10)

\[ \pi^{ij} = \frac{\partial L}{\partial \dot{h}_{ij}} = \psi^{ij}, \]

(11)

\[ P^{ij} = \frac{\partial L}{\partial \dot{K}_{ij}} = 0, \]

(12)

\[ \Lambda^{ij} = \frac{\partial L}{\partial \dot{\psi}_{ij}} = 0. \]

(13)

Thus, from the equations (9) - (13) we identify the following HJ Hamiltonians of the theory

\[ \mathcal{H}' = \mathcal{H}_0 + \Pi = 0, \]

(14)

\[ H_1^{00} = \pi^{00} = 0, \]

(15)

\[ H_2^{0i} = \pi^{0i} = 0, \]

(16)

\[ H_3^{ij} = \pi^{ij} - \psi^{ij} = 0, \]

(17)

\[ H_4^{ij} = P^{ij} = 0, \]

(18)

\[ H_5^{ij} = \Lambda^{ij} = 0, \]

(19)

where \( \mathcal{H}_0 \) is the canonical hamiltonian defined as usual \( \mathcal{H}_0 = \dot{h}_{\mu\nu} \pi^{\mu\nu} + \dot{K}_{ij}P^{ij} + \dot{\psi}_{ij} \Lambda^{ij} - \mathcal{L} \) and \( \Pi = \partial_0 S \). Moreover, the fundamental Poisson brackets \([PB]\) between the canonical variables
are given by
\[
\{ h_{\mu\nu}, \pi^{\alpha\beta} \} = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu)\delta^3(x-y), \tag{20}
\]
\[
\{ K_{ij}, \pi^{kl} \} = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^k_j \delta^l_i)\delta^3(x-y), \tag{21}
\]
\[
\{ \psi^{ij}, \Lambda_{kl} \} = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^k_j \delta^l_i)\delta^3(x-y). \tag{22}
\]
Furthermore, in the \( HJ \) scheme, the dynamics of the system is governed by the fundamental differential defined as
\[
dF = \{ F, H_I \} d\omega^I, \tag{23}
\]
where \( F \) is any function defined on the phase space, \( H_I \) is the set of all Hamiltonians \( (14)-(19) \) and \( \omega^I \) are the parameters related to them. It is important to remark, that in the \( HJ \) method the Hamiltonians are classified as involutive and non-involutive. Involutive ones are those whose integrability conditions, the non-involutive Hamiltonians are removed from the fundamental differential \( (23) \) by introducing the so-called generalized brackets, these new brackets are given by
\[
\{ f, g \}^* = \{ f, g \} - \{ f, H' \} C_{a'b'}^{-1} \{ H', g \}, \tag{24}
\]
where \( C^{a'b'} \) is the matrix formed with the \( PB \) between all non-involutive Hamiltonians. From \( (14)-(19) \) the non-involutive Hamiltonians are \( H^3_{ij} \) and \( H^5_{ij} \), whose \( PB \) is
\[
\{ H^3_{ij}, H^5_{ij} \} = -\frac{1}{2}(\eta^{ik} \eta^{jl} + \eta^{il} \eta^{kj})\delta^3(x-y), \tag{25}
\]
therefore, the matrix \( C^{a'b'} \) given by
\[
C^{a'b'} = \begin{pmatrix}
0 & -\frac{1}{2}(\eta^{ik} \eta^{jl} + \eta^{il} \eta^{kj}) \\
\frac{1}{2}(\eta^{ik} \eta^{jl} + \eta^{il} \eta^{kj}) & 0
\end{pmatrix} \delta^3(x-y), \tag{26}
\]
and its inverse \( C_{a'b'}^{-1} \) takes the form
\[
C_{a'b'}^{-1} = \begin{pmatrix}
0 & \frac{1}{2}(\eta^{ik} \eta^{jl} + \eta^{il} \eta^{kj}) \\
-\frac{1}{2}(\eta^{ik} \eta^{jl} + \eta^{il} \eta^{kj}) & 0
\end{pmatrix} \delta^3(x-y). \tag{27}
\]
In this manner, the following non-vanishing generalized brackets between the fields arise
\[
\{ h_{\mu\nu}, \pi^{\alpha\beta} \}^* = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu)\delta^3(x-y), \tag{28}
\]
\[
\{ K_{ij}, P^{kl} \}^* = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^k_j \delta^l_i)\delta^3(x-y), \tag{29}
\]
\[
\{ h_{\mu\nu}, \psi^{\alpha\beta} \}^* = \frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu)\delta^3(x-y), \tag{30}
\]
\[
\{ \psi_{ij}, \Lambda^{kl} \}^* = 0, \tag{31}
\]
we observe from \( (31) \) that the canonical variables \( (\psi_{ij}, \Lambda^{kl}) \) can be removed which implies that we can perform the substitution of \( \pi^{ij} = \psi^{ij} \) and \( \Lambda^{ij} = 0 \), hence, the canonical hamiltonian takes the
form

\[ \mathcal{H}_0 = \int \left[ 2K^i_iK^j_j - 2K_{ij}K^{ij} + h_{00}R_{ij}^{ij} + h_{ij}R^{ij} - \frac{1}{2}h^i_iR_{ij}^{ij} - \frac{1}{\mu} \epsilon_{ijk} (4K^i_i \partial_j K_{kl} + \partial^m h_{im} \partial_j \partial^o h_{kl} + \nabla^2 h_i^m \partial_j h_{km}) - 2h_{0j} \partial_i \pi^{ij} + 2K_{ij} \pi^{ij} \right] d^3x. \]  

(32)

It is worth to comment, that the canonical Hamiltonian has linear terms in the momenta \( \pi^{ij} \) and this fact could be related to Ostrogradsky’s instabilities. Nevertheless, it is well-known that those instabilities could be healed by means the correct identification of the constraints \[43, 46\]. In this respect, an advantage of the \( HJ \) scheme is that the constraints are identified directly and it is not necessary to fix them by hand, then with the generalized brackets and the identification of the Hamiltonians we can remove the linear canonical momenta terms. In fact, by using the Hamiltonians \([14]-[19]\) the canonical Hamiltonian takes the following form

\[ \mathcal{H}_0' = \int \left[ \frac{1}{2} \pi^{ij} \pi_{ij} - \frac{1}{4} \pi^i_i \pi_j^j + h_{ij}R^{ij} - \frac{1}{\mu} \epsilon_{ijk} (4K^i_i \partial_j K_{kl} + \partial^m h_{im} \partial_j \partial^o h_{kl} + \nabla^2 h_i^m \partial_j h_{km}) - \frac{1}{4 \mu^2} (2\partial^i K_{ij} \partial^j K_{ik}^k + 2\partial^i K_{ij} \partial_i K_{jk}^k - 2\partial^i K_{ij} \partial_i K_{k}^k - \partial^i K_{ij} \partial^k K_{ij} - \partial_k K_{ij} \partial^k K_{ij}^j) \right] d^3x. \]

hence, the Ostrogradsky instability has been healed and the associated ghost was exorcised.

On the other hand, with all these results we rewrite the fundamental differential in terms of either involutive Hamiltonians or generalized brackets, this is

\[ dF = \int \left[ \{F, H'\}^* dt + \{F, H'_{10}\}^* d\omega^1_{00} + \{F, H'_{20}\}^* d\omega^2_{01} + \{F, H'_{1i}\}^* d\omega^3_{ij} + \{F, H'_{ij}\}^* d\omega^4_{ij} \right] d^3y. \]  

(33)

thus, we will search if there are more Hamiltonians in the theory. For this aim, we shall take into account either the generalized differential \([32]\) or the Frobenius integrability conditions which, ensure that system is integrable, this is

\[ dH_a = 0, \]  

(34)

where \( H_a \equiv (H'_{10}, H'_{20}, H'_{ij}) \) are all involutive Hamiltonians. From integrability conditions \([34]\) the following 10 new Hamiltonians arise

\[ H'_{60} \equiv \nabla^2 h_i^i - \partial^i \partial^j h_{ij} = 0, \]  

(35)

\[ H'_{70} \equiv \partial_i \pi^{ij} = 0, \]  

(36)

\[ H'_{80} \equiv \pi^{ij} - 2K^{ij} + 2\eta^{ij} K^k_k - \frac{2}{\mu} (\epsilon_{ikl} \eta^{jm} + \epsilon_{ikl} \eta^{jm}) \partial_k K_{jm} = 0, \]  

(37)

Now, we observe that the Hamiltonians \( H'_{4i}, H'_{60} \) and \( H_8 \) are non-involutive, therefore they will be removed by introducing a new set of generalized brackets. In this respect, if we calculate the matrix whose entries will be all generalized brackets, say \([28]-[31]\), between the non-involutive Hamiltonians, we will find null vectors, say \( \psi^i = (\frac{1}{2} \partial_i \partial_j \zeta, \partial^i_k \zeta, 0) \), where \( \zeta \) is an arbitrary function. Hence, from the contraction of the null vectors with the Hamiltonians \([42]-[43]\), we will find the following involutive Hamiltonian

\[ H_0 = \nabla^2 h_i^i - \partial^i \partial^j h_{ij} + \frac{1}{2} \partial_i \partial_j \psi^{ij}, \]  

(38)
thus, there are only 12 non-involutive Hamiltonians \((H^j_4, H^j_8)\) whose generalized brackets are given by

\[
\{H^j_4, H^j_8\}^* = \frac{1}{2\mu}(\epsilon_{km}n^{jl} + \epsilon_{km}n^{jk} + \epsilon_{lm}n^{jk} + \epsilon_{lm}n^{jk})\partial_m + \frac{1}{2}(\eta^{ik}n^{jl} + 
+ n^{jk}n^{il})\delta^3(x - y).
\]

In this manner, we proceed to construct the new set of \(HJ\) generalized brackets, namely \{ , \}**, in the same way as we did before with the brackets (28)-(31). The non-trivial new generalized brackets are given by

\[
\{h_{ij}, \pi^{kl}\}^* = \frac{1}{2}(\delta^k_i\delta_j^l + \delta^l_j\delta^k_i)\delta^3(x - y),
\]

\[
\{K_{ij}, P^{kl}\}^* = 0,
\]

\[
\{h_{ij}, K_{kl}\}^* = \frac{1}{4}(\eta_{ik}\eta_{jl} + n_{il}n_{jk} - \eta_{ij}\eta_{kl})\delta^3(x - y) + \frac{\mu^2}{4\Xi}[[\eta_{ik}\eta_{jl} + n_{il}n_{jk} - \eta_{ij}\eta_{kl}]\nabla^2 + (\eta_{ij}\partial_k\partial_l
+ \eta_{kl}\partial_i\partial_j)](\nabla^2 + \mu^2) - 3\partial_i\partial_j\partial_k\partial_l - \frac{3\mu^2}{4}(\eta_{ik}\partial_j\partial_l + n_{il}n_{jk} + \eta_{ij}\partial_k\partial_l + \eta_{kl}\partial_i\partial_j)
+ \frac{\mu}{4}[(\epsilon_{ik}n^{jl} + \epsilon_{jk}n^{il} + \epsilon_{il}n^{jk} + \epsilon_{jl}n^{ik})(\nabla^2 + \mu^2) + 3(\epsilon_{ik}n^{jl} + \epsilon_{jk}n^{il} + \epsilon_{il}n^{jk} + \epsilon_{jl}n^{ik})]\nabla^2 + \mu^2)]\partial_m\delta^3(x - y),
\]

where \(\Xi \equiv -\mu^2(\nabla^2 + \mu^2)/(\nabla^2 + \frac{\mu^2}{4})\). It is worth commenting, that some brackets were reported in [26], however, there are some differences. In fact, in this paper we have used an alternative analysis and new variables were introduced; the introduction of the variables allowed us to identify the brackets directly and they have a more compact form than those reported in [26]. Moreover, the tedious classification of the constrains into first class and second class as usually is done, in the \(HJ\) scheme is it not necessary. Thus, we can observe that the \(HJ\) is more economical.

With the new set of either involutives Hamiltonians or generalized brackets, the fundamental differential takes the following new form

\[
dF = \int \left[\{F, H'(y)\}**dt + \{F, H'^{00}_1(y)\}**d\omega^{00}_0 + \{F, H'^{0i}_1(y)\}**d\omega^{0i}_0 + \{F, H'^{0i}_2(y)\}**d\omega^{0i}_0 + \{F, H'^{0i}_7(y)\}**d\omega^{0i}_0 + \right.
+ \left.\{F, H_9(y)\}**d\omega^9_0\right]d^3y,
\]

where

\[
H'^{00}_1 = \pi^{00},
\]

\[
H'^{0i}_2 = \pi^{0i},
\]

\[
H'^{0i}_7 = \partial_i\pi^{ij},
\]

\[
H_9 = \nabla^2h^i_i - \partial^i\partial^i h_{ij}.
\]

From integrability conditions of \(H'^{0i}_7\) and \(H_9\) we find

\[
dH'^{0i}_2 = 0,
\]

\[
dH_9 = -\partial_i\partial_j\pi^{ij} = -\partial_iH'^{0i}_7 = 0,
\]
we identify the equations of motion for \( h \) associated with the perturbation. In this manner, we calculate the number of physical degrees of freedom as follows: there are 12 canonical variables \( (h, \pi) \) obtain a second order time equation for \( h \), and its momentum \( \pi \). In fact, by taking \( d\theta^7 = 0 \) and \( d\theta^9 = 0 \), we obtain

\[
\begin{align*}
\dot{h}_{ij} &= 2K_{ij} + \partial_i h_{0j} + \partial_j h_{0i}, \\
\dot{\pi}^{ij} &= \eta^{ij}\nabla^2 h_{00} - \partial^i \partial^j h_{00} - \eta^{ij} R_{kl} k^{kl} - 2R^{ij} - \frac{1}{2}[\delta^{ij} \partial_j + \delta^{ij} \partial_i] \partial_k \partial^m h_{tm} \\
&\quad - (\epsilon^{ikl} \partial^{im} + \epsilon^{jkl} \partial^{jm}) \partial_k \nabla^2 h_{tm}], \\
\dot{K}_{ij} &= -\frac{1}{2} \partial_i \partial_j h_{00} - R_{ij} + \frac{1}{4} \eta_{ij} R_{kl} k^{kl}.
\end{align*}
\]

We observe that \( K_{ij} \) corresponds to the definition of \( K_{ij} \), thus, if we use \( K_{ij} \) and \( \dot{K}_{ij} \) we will obtain a second order time equation for \( h_{ij} \) as expected, then there are six degrees of freedom associated with the perturbations. In this manner, we calculate the number of physical degrees of freedom as follows: there are 12 canonical variables \( (h_{ij}, \pi^{ij}) \) and eight involutive Hamiltonians \( (H_{ij}^{90}, H_{ij}^{9i}, H_{ij}^{9j}, H_9) \), thus

\[
DOF = \frac{1}{2}[12 - 8] = 2,
\]

and thus, the theory has two physical degrees of freedom just like GR.

On the other hand, if in the characteristics equations we take \( dt = 0 \), then we identify the following
canonical transformations

\[ \delta h_{00} = \delta \omega^1_{00}, \]
\[ \delta h_{0i} = \frac{1}{2} \delta \omega^2_{0i}, \]
\[ \delta h_{ij} = -\frac{1}{2} (\delta^k_j \partial_j + \delta^k_i \partial_i) \delta \omega^7_{0k}, \]

moreover, we can then identify the corresponding gauge transformations of the theory by considering that the Lagrangian (6) will be invariant under (61)-(63) if the variation \( \delta S = 0 \). This is

\[ \delta S = \left[ \frac{\partial S}{\partial h_{\mu \nu}} \delta h_{\mu \nu} + \frac{\partial S}{\partial (\partial_\alpha h_{\mu \nu})} \delta (\partial_\alpha h_{\mu \nu}) + \frac{\partial S}{\partial (\partial_\alpha \partial_\beta h_{\mu \nu})} \delta (\partial_\alpha \partial_\beta h_{\mu \nu}) \right] \]
\[ = \int \left[ \left( -\Box h^{\mu \nu} + \Box h^\lambda \eta^{\mu \nu} - \partial_\alpha \partial_\lambda h^{\alpha \lambda \eta^{\mu \nu} - \partial^\mu \partial^\nu h^\lambda \chi + 2 \partial^\mu \partial^\nu h^{\lambda \chi} + \frac{1}{\mu} \epsilon^{\alpha \mu \lambda \gamma} (\partial^\nu \partial_\alpha \partial_\lambda h^{\alpha \gamma} \right) \right) \delta h_{\mu \nu} \right] d^4 x = 0, \]

thus, by taking account (61)-(63) into the variation, we obtain the following

\[ \delta S = \int [R_{ij}^{\quad ij} \delta \omega^1_{00} + \frac{1}{2} 2 \Box^2 h^0_{\nu} + 2 \partial^i \partial^j h_{0i} - 2 \partial^i \partial^j h_{0j} - 2 \partial^i \partial^j h_{0ij} - \frac{1}{\mu} \epsilon^{ijk} (\partial_j \Box^2 h_{0k} - \partial_j \partial^i h_{0k})] \delta \omega^2_{0i} \]
\[ - \frac{1}{2} [\delta h_{ij} \eta^{ij} + \frac{1}{2} \delta h_{00} \eta^{ij} - 2 \partial^i \partial^j h_{0ij} - \partial^i \delta h_{0j} - \partial^j \delta h_{0i} + \frac{1}{2} \mu] \epsilon^{ijk} (\partial_k \Box^2 h^0_{ij}) \delta (\partial_i \omega^\gamma_{0j} + \partial_j \omega^\gamma_{0i}) d^4 x = 0. \]

Now, we define \( \partial_0 \xi \equiv \delta \omega^1_{00} \), so after long algebraic work we find that the variation takes the form

\[ \delta S = \int [-\partial_j \delta h_{ij} + \partial^i \partial^j h_{0j} + \Box^2 h^0_{ij} - \partial^i \partial^j h_{0ij} + \frac{1}{2} \mu \epsilon^{ijk} (\partial_j \Box^2 h_{0k} - \partial_j \partial^i h_{0k})] (\partial_i \omega^\gamma_{0j} + \partial_j \omega^\gamma_{0i}) d^4 x = 0, \]

hence, the action will be invariant under (61)-(63) if the the parameters \( \omega^\gamma \)'s obey

\[ \delta \omega^2_{0i} = -\partial_0 \delta \omega^7_{0i} + \partial_i \xi. \]

Now, we will write (68) in a new fashion. In fact, we introduce the following 4-vector \( \xi_\mu \equiv (\frac{1}{2} \xi, \frac{1}{2} \delta \omega^7_{0i}) \equiv (\xi_0, \xi_i) \); then \( \xi = 2 \xi_0 \) and \( \delta \omega^7_{0i} = -2 \xi_i \). Hence, the relation (68) takes the form

\[ \frac{1}{2} \delta \omega^2_{0i} = \partial_0 \delta \xi_i + \partial_i \xi_0, \]

finally, from the equations (61)-(63) and (68) the following gauge transformations are identified

\[ \delta h_{\mu \nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \]

all these results are in agreement with those reported in [26], thus, our study complete and extends those reported in the literature.

III. CONCLUSIONS AND REMARKS

In this paper a detailed HJ analysis for the higher-order modified gravity has been performed.

We introduced a new set of variables in a different way than other approaches and reported in
the literature, then the full set of involutive and non-involutive Hamiltonians were identified. The correct identification of the Hamiltonians allow us to avoid the Ostrogradsky instability by removing the terms with linear momenta, healing the canonical Hamiltonian. Furthermore, the $HJ$ generalized brackets and the fundamental differential were obtained from which the characteristic equations and the gauge symmetries were identified. The complete identification of the Hamiltonians allowed us to carry out the counting of the physical degrees of freedom, concluding that the modified theory and $GR$ shares the same number of physical degrees of freedom. In this manner, we have all elements to analyze the theory in the quantum context. In fact, with our perturbative $HJ$ study either constraints or the generalized brackets are under control, thus, we could use the tools developed in the canonical quantization of field theories in order to make progress in this program [51]. Furthermore, our analysis will be relevant for the study of the theory in the non-perturbative scenario. In fact, now the modified theory will be full background independent then we will compare the differences between the canonical structure of $GR$ reported in the literature [8, 9] and that for the modified theory. However, all those ideas are still in progress and will be reported soon [52].

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