A SELF-SIMILAR MEASURE WITH DENSE ROTATIONS, SINGULAR PROJECTIONS AND DISCRETE SLICES

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Abstract. We construct a planar homogeneous self-similar measure, with strong separation, dense rotations and dimension greater than 1, such that there exist lines for which dimension conservation does not hold and the projection of the measure is singular. In fact, the set of such directions is residual and the typical slices of the measure, perpendicular to these directions, are discrete.

1. Introduction and statement of results

Let $R$ be a $2 \times 2$ rotation matrix, with $R^n \neq Id$ for all $n \geq 1$, and let $r \in (0, 1)$. Consider a homogeneous IFS on $\mathbb{R}^2$

$$\{ \varphi_i(x) = rRx + a_i \}_{i \in I},$$

with the strong separation condition (SSC), and a self-similar measure

$$\mu = \sum_{i \in I} p_i \cdot \varphi_i \mu.$$

It is among the most basic planar self-similar measures. Hence it is a natural question in fractal geometry to study the dimension and continuity of the projections $\{ P_u \mu \}_{u \in S}$ and slices

$$\{ \mu_{u,x} : x \in \mathbb{R}^2, u \in S \}.$$

Here $S$ is the unit circle of $\mathbb{R}^2$, $P_u$ is the orthogonal projection onto the line spanned by $u$, and $\{ \mu_{u,x} \}_{x \in \mathbb{R}^2}$ is the disintegration of $\mu$ with respect to $P_u^{-1}(B)$, where $B$ is the Borel $\sigma$-algebra of $\mathbb{R}^2$. A more elaborate description of these disintegrations is given in Section 2.

Dimensionwise, the behaviour of the projections is as regular as possible. Indeed, Hochman and Shmerkin [HS] have proven that $P_u \mu$ is exact dimensional, with

$$\dim P_u \mu = \min \{ 1, \dim \mu \},$$

2000 Mathematics Subject Classification. Primary: 28A80, Secondary: 28A78.

Key words and phrases. Self-similar measure, singular measure, dimension conservation.

Supported by ERC grant 306494.
for each $u \in S$. A version of this, for self-similar sets with dense rotations, was first proven by Peres and Shmerkin \[PS\]. Considering the continuity of the projections, Shmerkin and Solomyak \[SS\] have shown, assuming $\dim \mu > 1$, that the set
\[ E = \{ u \in S : P_u \mu \text{ is singular} \} \]
has zero Hausdorff dimension.

Let us turn to discuss the concept of dimension conservation and the dimension of slices. A Borel probability measure $\nu$ on $\mathbb{R}^2$ is said to be dimension conserving (DC), with respect to the projection $P_u$, if
\[ \dim_H \nu = \dim_H P_u \nu + \dim_H \nu_{u,x} \text{ for } \nu\text{-a.e. } x \in \mathbb{R}^2, \]
where $\dim_H$ stands for Hausdorff dimension. It always holds that $\nu$ is DC with respect to $P_u$ for almost every $u \in S$. This follows from results, valid for general measures, regarding the typical dimension of projections (see \[HK\]) and slices (see \[JM\]). Falconer and Jin \[FJ1\] have shown that $\nu$ is DC, with respect to $P_u$ for all $u \in S$, whenever $\nu$ is self-similar with a finite rotation group. An analogous statement, for self-similar sets with the SSC, was first proven by Furstenberg \[F\].

Another related result for sets is due to Falconer and Jin \[FJ2\]. They showed that if $K \subset \mathbb{R}^2$ is self-similar, with $\dim K > 1$ and a dense rotation group, then for every $\epsilon > 0$ there exists $N_\epsilon \subset S$, with $\dim_H N_\epsilon = 0$, such that for $u \in S \setminus N_\epsilon$ the set
\[ \{ x \in \text{span}\{u\} : \dim_H (K \cap P_u^{-1}\{x\}) > \dim K - 1 - \epsilon \} \]
has positive length.

Taking these results into account, it is natural to ask whether the sets $E$, defined above, and
\[ F = \{ u \in S : \mu \text{ is not DC with respect to } P_u \}, \]
must be empty whenever the dimension of $\mu$ exceeds 1. A version, for self-similar sets, of this folklore question regarding $E$ is asked in Section 4 of \[BFVZ\]. The purpose of this paper is to show that $E$ and $F$ are not necessarily empty, and in fact can both be topologically large. The following theorem is our main result. Recall that a measure $\nu$ is said to be discrete if it is supported on a countable set.

**Theorem 1.1.** There exist $r \in (0, 1)$, a $2 \times 2$ rotation matrix $R$ with $R^n \neq \text{Id}$ for all $n \geq 1$, and a homogeneous planar self-similar IFS
\[ \{ \varphi_i(x) = rRx + a_i \}_{i \in I} \]
with the SSC, such that the self-similar measure $\mu = \sum_{i \in I} |I|^{-1} \cdot \varphi_i \mu$ satisfies $\dim \mu > 1$, and each of the sets
\[ \{ u \in S : P_u \mu \text{ is singular} \} \]
and

\[(1.2) \quad \{ u \in S : \mu_{u,x} \text{ is discrete for } \mu\text{-a.e. } x \in \mathbb{R}^2 \}\]

contains a dense $G_\delta$ subset of $S$.

If $\nu$ is a discrete measure on $\mathbb{R}^2$ then clearly $\dim \nu = 0$, hence we get the following corollary.

**Corollary 1.2.** Let $\mu$ be the self-similar measure from Theorem 1.1 then the set

\[ \{ u \in S : \mu \text{ is not DC with respect to } P_u \} \]

contains a dense $G_\delta$ subset of $S$.

Theorem 1.1 is related to an example obtained by Nazarov, Peres and Shmerkin [NPS]. They have presented a planar self-affine measure $\nu$, with the SSC and having dimension greater than 1, such that the set of $u \in S$ for which $P_u \nu$ is singular contains a $G_\delta$ subset. We do not pursue this, but our argument can probably be used for showing that, for a residual set of directions $u \in S$, the slices $\{ \nu_{u,x} \}_{x \in \mathbb{R}^2}$ are $\nu$-typically discrete. Also related to Theorem 1.1 is a recent paper, by Simon and Vágó [SV], in which certain one-parameter families of self-similar measures $\nu_\alpha$ on the line are constructed. For these families it holds that the similarity dimension of $\nu_\alpha$ is greater than 1 for every $\alpha$, but the set of parameters for which $\nu_\alpha$ is singular is topologically large.

In our construction of $\mu$ the rotational part $rR$, of the maps in the IFS, comes from a reciprocal of a complex Pisot number (see definition 2.1 below). While dealing with parametric families of measures, Pisot numbers have been used before, in several situations, in order to demonstrate the existence of exceptional parameters for which the corresponding measures are singular. This was first done by Erdős [E], who proved that the Bernoulli convolution corresponding to $\lambda$, i.e. the distribution of the random sum $\sum_n \pm \lambda^n$, is singular whenever $\lambda^{-1} \in (1,2)$ is Pisot. The example from [NPS], mentioned above, also utilizes real Pisot numbers. In [SX], complex Pisot numbers are used in order to obtain examples of singular complex Bernoulli convolutions.

As a by-product of our construction, we obtain information on the Hausdorff measure of typical slices of self-similar sets at the critical dimension. Let $K$ be a planar self-similar set with the SSC, and denote by $m$ the Haar measure of $S$. Write $s$ for the Hausdorff dimension of $K$, and assume $s > 1$. For $t \geq 0$ denote by $\mathcal{H}^t$ the $t$-dimensional Hausdorff measure. Given $u \in S$ and $x \in K$ write $K_{u,x}$ for the slice $K \cap (x + \text{span}\{u\})$. Since $0 < \mathcal{H}^s(K) < \infty$, the Hausdorff dimension of $K_{u,x}$ is equal to $s - 1$, with finite $\mathcal{H}^{s-1}$-measure, for $\mathcal{H}^s \times m$-a.e. $(x, u) \in K \times S$.
(see Theorem 10.11 in [M]). However, it was not known whether the set
\[ Q = \{ (x, u) \in K \times S : \mathcal{H}^{s-1}(K_{u,x}) > 0 \} \]
must have positive \( \mathcal{H}^s \times m \)-measure. In Corollary 2.3 from [R] the author has shown that if this holds, and \( K \) has dense rotations, then \( P_u(\mathcal{H}^s|_K) \) is absolutely continuous for all \( u \in S \). In our example from Theorem 1.1 we shall have \( \mu = C \cdot \mathcal{H}^s|_K \), where \( C > 0 \) is a normalizing constant. Hence we obtain the following corollary.

**Corollary 1.3.** Let \( \{ \varphi_i \}_{i \in I} \) be the IFS constructed in Theorem 1.1. Denote by \( K \) its attractor, and write \( s \) for the Hausdorff dimension of \( K \). Then \( s > 1 \) and,
\[ \mathcal{H}^s \times m \{ (x, u) \in K \times S : \mathcal{H}^{s-1}(K_{u,x}) > 0 \} = 0. \]

It is interesting to note that, in contrast with Corollary 1.3, if \( K \subset \mathbb{R}^n \) is self-similar, with the SSC, finite rotation group and dimension \( s \) greater than \( 2m \), then \( \mathcal{H}^{s-m}(K \cap V) > 0 \) for typical affine \((n - m)\)-planes \( V \subset \mathbb{R}^n \) (see Corollary 2.2 in [R]).

The rest of the paper is organized as follows. In Section 2 the measure \( \mu \) from Theorem 1.1 is constructed. In Section 3 we show that the set defined in (1.1) is residual. In Section 4 we complete the proof by establishing this for the set appearing in (1.2).

**Acknowledgment.** This paper is a part of the author’s PhD thesis conducted at the Hebrew University of Jerusalem. I would like to thank my advisor, Michael Hochman, for his support and useful suggestions. I would also like to thank Or Landesberg for helpful discussions.

## 2. Construction of \( \mu \)

In this section we carry out the construction of the measure \( \mu \) from Theorem 1.1. It will be convenient to identify \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \). We shall use some simple facts from the theory of field extensions, for which we refer to chapters 17 and 18 in [I]. Our example involves complex Pisot numbers, which we now define.

**Definition 2.1.** An algebraic integer \( \theta \in \mathbb{C} \) is called a complex Pisot number if \( \theta \notin \mathbb{R} \), \( |\theta| > 1 \), and all of the Galois conjugates of \( \theta \) (i.e. the other roots of the minimal polynomial of \( \theta \)), except \( \overline{\theta} \), are less than one in modulus.

Given algebraic numbers \( \alpha_1, ..., \alpha_n \in \mathbb{C} \), we denote by \( \mathbb{Q}[\alpha_1, ..., \alpha_n] \) the smallest subfield of \( \mathbb{C} \) containing \( \alpha_1, ..., \alpha_n \). If \( F \subset E \) are subfields of \( \mathbb{C} \), we write \([E:F]\) for the degree of the field extension \( E/F \). The next lemma is probably known, but
Lemma 2.2. Let $\theta$ be a complex Pisot number with $[Q[\theta] : Q] = 3$, then $\arg \theta \notin \pi Q$.

Now let $\theta$ be a complex Pisot number such that,
- $\arg \theta \notin \pi Q$;
- $|\theta|$ lies in $(3, 4)$;
- the minimal polynomial of $\theta$ has constant term $1$ or $-1$.

For example, the polynomial $f(X) = X^3 + X^2 + 10X + 1$ has three roots,
$$\theta_0 \approx -0.45 + 3.11i, \quad \theta_0' \approx -0.45 - 3.11i$$
and $\alpha \approx -0.1$.

Since $f$ doesn’t have a root in $\mathbb{Z}$, it follows from Gauss’s lemma that $f$ is irreducible over $Q$. Hence $f$ is the minimal polynomial of $\theta_0$ over $Q$, and
$$[Q[\theta_0] : Q] = \deg f = 3.$$ This shows that $\theta_0$ is a complex Pisot number, and from Lemma 2.2 we get that $\arg \theta_0 \notin \pi Q$. Since $3 < |\theta_0| < 4$ and the constant term of $f$ is $1$, the number $\theta_0$ satisfies all of the required properties.

Write $\lambda = \theta^{-1}$ and note that $\lambda$ may be thought of as a $2 \times 2$ matrix $rR$, where $r = |\lambda|$ and $R$ is a planar rotation by angle $\arg \lambda$. From $\arg \lambda \notin \pi Q$ it follows $R^n \neq Id$ for all $n \geq 1$. Let $V$ be the set of all $(a_1, a_2) \in \mathbb{C}^2$ for which the IFS
$$\{z \to \lambda \cdot z + (-1)^k a_j : k, j \in \{1, 2\}\}$$
satisfies the strong separation condition (SSC). Since $|\lambda| < \frac{1}{3}$, it is not hard to see that $(\frac{2}{3}, \frac{2}{3}) \in V$ and in particular that $V \neq \emptyset$. The next lemma is proven at the end of this section.

Lemma 2.3. The set $V := \{k \cdot \lambda^l : k, l \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

Clearly $V$ is open in $\mathbb{C}^2$, hence from $V \neq \emptyset$ and Lemma 2.3 it follows that there exists
$$(a_1, a_2) \in V \cap (V \times V).$$
For $k, j \in \{1, 2\}$ and $z \in \mathbb{C}$ set
$$\varphi_{k,j}(z) = \lambda \cdot z + (-1)^k a_j,$$
then the IFS
$$\Phi := \{\varphi_{k,j} : k, j \in \{1, 2\}\}$$
satisfies the SSC. Denote by $\mathcal{M}(\mathbb{C})$ the collection of all compactly supported Borel probability measures on $\mathbb{C}$. Let $\mu$ be the unique member of $\mathcal{M}(\mathbb{C})$ with
$$\mu = \frac{1}{4}(\varphi_{1,1} \mu + \varphi_{1,2} \mu + \varphi_{2,1} \mu + \varphi_{2,2} \mu),$$
then
\[ \dim_H \mu = \frac{\log 4}{\log |\theta|} > 1. \]

Denote by \( \langle \cdot, \cdot \rangle \) the Euclidean inner product on \( \mathbb{C} \), i.e. \( \langle z, w \rangle = \text{Re}(z \cdot \overline{w}) \) for \( z, w \in \mathbb{C} \). Write
\[ S = \{ z \in \mathbb{C} : |z| = 1 \}, \]
and \( P_z w = \langle w, z \rangle \) for \( z \in S \) and \( w \in \mathbb{C} \). Note that \( P_z \mu \) is, up to affine equivalence, the pushforward of \( \mu \) by the orthogonal projection onto the line \( z \cdot \mathbb{R} \). The following proposition is proven in Section 3.

**Proposition 2.4.** There exists a dense \( G_\delta \) subset \( B \) of \( S \), such that \( P_z \mu \) is singular for all \( z \in B \).

Let \( B \) be the Borel \( \sigma \)-algebra of \( \mathbb{C} \). For \( z \in S \) let \( \{ \mu_{z,w} \}_{w \in \mathbb{C}} \subset \mathcal{M}(\mathbb{C}) \) be the disintegration of \( \mu \) with respect to \( P_z^{-1}(B) \), as defined in Theorem 5.14 in [EW]. This means that \( \mu_{z,w} \) is supported on \( P_z^{-1}(P_z w) \) for \( w \in \mathbb{C} \), and for each bounded \( B \)-measurable \( f : \mathbb{C} \to \mathbb{R} \)
\[ \int f \, d\mu_{z,w} = E_\mu(f \mid P_z^{-1}(B))(w) \]  for \( \mu \)-a.e. \( w \in \mathbb{C} \).

Here the right hand side is the conditional expectation of \( f \) given \( P_z^{-1}(B) \) with respect to \( \mu \). In order to prove Theorem 1.1 it remains to establish the following proposition, which is done in Section 4.

**Proposition 2.5.** There exists a dense \( G_\delta \) subset \( B \) of \( S \), such that for each \( z \in B \) it holds that \( \mu_{z,w} \) is discrete for \( \mu \)-a.e. \( w \in \mathbb{C} \).

**Proofs of Lemmas 2.2 and 2.3.**

**Proof of Lemma 2.2.** By the assumptions on \( \theta \) there exists \( \alpha \in \mathbb{C} \), with \( |\alpha| < 1 \), such that \( \overline{\theta} \) and \( \alpha \) are the Galois conjugates of \( \theta \). Set \( E = \mathbb{Q}[\theta, \overline{\theta}, \alpha] \), let \( f \in \mathbb{Q}[X] \) be the minimal polynomial of \( \theta \) over \( \mathbb{Q} \), and let \( G \) be the Galois group of the field extension \( E/\mathbb{Q} \). Note that \( E \) is a splitting field for \( f \) over \( \mathbb{Q} \), and that the roots of \( f \) are \( \theta, \overline{\theta}, \alpha \). It follows, by Lemma 18.3 in [I], that the action of \( G \) on \( \{ \theta, \overline{\theta}, \alpha \} \) induces an isomorphism from \( G \) into a subgroup of \( S_3 \), where \( S_3 \) is the symmetric group on 3 letters. It also follows, by Theorem 18.13 in [I], that the extension \( E/\mathbb{Q} \) is Galois. Hence, from Corollary 18.19 and Lemma 17.6 in [I], we get
\[ |G| = [E : \mathbb{Q}] = [E : \mathbb{Q}(\theta)] \cdot [\mathbb{Q}(\theta) : \mathbb{Q}] = [E : \mathbb{Q}(\theta)] \cdot 3, \]
which shows that 3 divides \( |G| \). Let \( \sigma \in G \) be with \( \sigma(\beta) = \overline{\beta} \) for \( \beta \in E \), then \( \sigma \) has order 2. This implies that 2 divides \( |G| \), and so it must hold that \( G \) is isomorphic to \( S_3 \).
Now assume by contradiction that \( \arg \theta \notin \pi \mathbb{Q} \), then \( \theta^n \in \mathbb{R} \) for some \( n \geq 1 \). Let \( \tau \in G \) be such that \( \tau(\theta) = \theta, \tau(\overline{\theta}) = \alpha \) and \( \tau(\alpha) = \overline{\theta} \). Since \( \tau \) and \( \sigma \) are distinct, both have order 2, and \( G \) is isomorphic to \( S_3 \), it follows that the group generated by \( \tau \) and \( \sigma \) is \( G \). Clearly \( \tau(\theta^n) = \theta^n \) and from \( \theta^n \in \mathbb{R} \) we get \( \sigma(\theta^n) = \theta^n \), hence \( \eta(\theta^n) = \theta^n \) for all \( \eta \in G \). Let \( \eta \in G \) be with \( \eta(\theta) = \alpha \), then
\[
\theta^n = \eta(\theta^n) = \eta(\theta)^n = \alpha^n.
\]
But we also have \( |\theta^n| > 1 > |\alpha^n| \), which yields a contradiction, and so it must holds that \( \arg \theta \notin \pi \mathbb{Q} \).

**Proof of Lemma 2.3.** Let \( z \in \mathbb{C} \) and \( \epsilon > 0 \) be given, and let \( N \geq 1 \) be with \( |\lambda_N| < \epsilon \).

Since \( \arg \lambda = -\arg \theta \notin \pi \mathbb{Q} \), we have that
\[
\{l \cdot \arg \lambda \mod 2\pi\}_{l=1}^{\infty}
\]
is dense in \([0, 2\pi)\). It follows there exists \( l \geq N \) with
\[
|\exp(i \cdot \arg(\lambda^l)) - \exp(i \cdot \arg z)| < \epsilon.
\]
Let \( k \geq 0 \) be the integer with \( k \cdot |\lambda^l| \leq |z| < (k + 1) \cdot |\lambda^l| \), then
\[
||z| - k \cdot |\lambda^l|| \leq |\lambda^N| < \epsilon.
\]
From this, from \( \arg(k \cdot \lambda^l) = \arg(\lambda^l) \), and from (2.1), the lemma follows.

3. **Proof of Proposition 2.4**

Let \( \theta, \lambda, (a_1, a_2), \Phi := \{\varphi_{k,j} : k, j \in \{1, 2\}\} \) and \( \mu \), as obtained in Section 2. We shall show that there exists a dense \( G_\delta \) subset \( B \) of \( S \), such that for every \( z \in B \) the Fourier transform of \( P_z \mu \) does not decay to 0 at infinity.

**Lemma 3.1.** There exist constants \( \rho \in (0, 1) \) and \( C > 0 \), with
\[
(3.1) \quad \text{dist}(2\text{Re}(\theta^n), \mathbb{Z}) \leq C \cdot \rho^{|n|} \text{ for all } n \in \mathbb{Z}.
\]

**Proof.** Let \( \theta_3, ..., \theta_m \) be the Galois conjugates of \( \theta \) other than \( \overline{\theta} \). Since \( \theta \) is an algebraic integer,
\[
\theta^n + \overline{\theta^n} + \sum_{j=3}^{m} \theta_j^n \in \mathbb{Z} \text{ for all } n \in \mathbb{N}.
\]
From \( |\theta_j| < 1 \) for \( 3 \leq j \leq m \), it follows there exists \( \rho \in (0, 1) \) and \( C > 0 \) such that (3.1) holds for \( n \in \mathbb{N} \). Since \( |\theta| > 1 \) and for each integer \( n < 0 \)
\[
\text{dist}(2\text{Re}(\theta^n), \mathbb{Z}) \leq 2|\theta|^n,
\]
the lemma follows.

\( \square \)
Given $\nu \in \mathcal{M}(\mathbb{C})$ let $F(\nu)$ be the Fourier transform of $\nu$ as a measure on $\mathbb{R}^2$, i.e.
for $\xi \in \mathbb{C}$
\[
F(\nu)(\xi) = \int_{\mathbb{C}} e^{i(z,\xi)} \, d\nu(z) = \int_{\mathbb{C}} \exp(i\text{Re}(z \cdot \xi)) \, d\nu(z).
\]
The proof of the following proposition resembles the argument given by Erdős in [E].

**Proposition 3.2.** There exists a constant $c > 0$ with $|F(\mu)(4\pi \theta^N)| > c$ for all $N \in \mathbb{N}$.

**Proof.** Let $X_1, X_2, \ldots$ be i.i.d. random variables with,
\[
P(X_1 = (-1)^k a_j) = \frac{1}{4} \text{ for } k, j \in \{1, 2\}.
\]
Since $\mu$ is the unique Borel probability measure on $\mathbb{C}$ with
\[
\mu = \frac{1}{4}(\varphi_{1,1} \mu + \varphi_{1,2} \mu + \varphi_{2,1} \mu + \varphi_{2,2} \mu),
\]
it is equal to the distribution of the random sum $\sum_{n=0}^{\infty} \lambda^n \cdot X_n$. Hence for every $\xi \in \mathbb{C},$
\[
F(\mu)(\xi) = \prod_{n=0}^{\infty} F\left(\frac{1}{4} \sum_{j=1}^{2} (\delta_{\lambda^n a_j} + \delta_{-\lambda^n a_j})\right)(\xi)
\]
\[
= \prod_{n=0}^{\infty} \frac{1}{4} \cdot \sum_{j=1}^{2} (\exp(i\text{Re}(\lambda^n a_j \cdot \xi)) + \exp(i\text{Re}(-\lambda^n a_j \cdot \xi)))
\]
\[
= \prod_{n=0}^{\infty} \frac{1}{2} \cdot (\cos(4\pi \text{Re}(\lambda^n a_1 \cdot \xi)) + \cos(4\pi \text{Re}(\lambda^n a_2 \cdot \xi))).
\]
Since $a_1, a_2 \in \mathcal{Y}$, where $\mathcal{Y}$ is defined in Lemma 2.3, for $j = 1, 2$ there exist $k_j, l_j \in \mathbb{N}$ with $a_j = k_j \cdot \theta^{-l_j}$. Hence for $N \in \mathbb{N},$
\[
(3.2) \quad F(\mu)(4\pi \theta^N) = \prod_{n=-\infty}^{N} \frac{1}{2} \cdot (\cos(4\pi \text{Re}(\theta^n a_1)) + \cos(4\pi \text{Re}(\theta^n a_2)))
\]
\[
= \prod_{n=-\infty}^{N} \frac{1}{2} \cdot (\cos(4\pi k_1 \cdot \text{Re}(\theta^{n-l_1})) + \cos(4\pi k_2 \cdot \text{Re}(\theta^{n-l_2}))).
\]
Let us show that $b_n \neq 0$ for every $n \in \mathbb{Z}$, where
\[
b_n := \frac{1}{2} \cdot (\cos(4\pi k_1 \cdot \text{Re}(\theta^{n-l_1})) + \cos(4\pi k_2 \cdot \text{Re}(\theta^{n-l_2}))).
\]
Recall that the set of algebraic integers is closed under addition, subtraction and multiplication. The product of $\theta$ with its Galois conjugates is equal to the constant term of the minimal polynomial of $\theta$, which is $\pm 1$ by assumption. These conjugates are all algebraic integers, hence $\theta^{-1}$ is an algebraic integer, and so $\theta^n$ is an algebraic
integer for all \( n \in \mathbb{Z} \). Let \( n \in \mathbb{Z} \), then from the identity
\[
\cos \beta + \cos \gamma = 2 \cos\left(\frac{\beta + \gamma}{2}\right) \cos\left(\frac{\beta - \gamma}{2}\right) \text{ for all } \beta, \gamma \in \mathbb{R},
\]
we obtain
\[
(3.3) \quad b_n = \cos\left(2\pi \cdot \text{Re}(k_1 \theta^{n-l_1} + k_2 \theta^{n-l_2})\right) \cdot \cos\left(2\pi \cdot \text{Re}(k_1 \theta^{n-l_1} - k_2 \theta^{n-l_2})\right).
\]
Since \( 2\text{Re}(k_1 \theta^{n-l_1} + k_2 \theta^{n-l_2}) \) is equal to
\[
k_1 \theta^{n-l_1} + k_2 \theta^{n-l_2} + k_1 \overline{\theta^{n-l_1}} + k_2 \overline{\theta^{n-l_2}},
\]
it is an algebraic integer, and so it can’t be of the form \( k + \frac{1}{2} \) with \( k \in \mathbb{Z} \). It follows the first term in the product \( (3.3) \) is nonzero. In a similar manner the second term in \( (3.3) \) is nonzero, which shows \( b_n \neq 0 \).

Fix \( n \in \mathbb{Z} \) and \( j \in \{1, 2\} \), and let \( d \in \mathbb{Z} \) be with
\[
|2\text{Re}(\theta^{n-l_j}) - d| = \text{dist}(2\text{Re}(\theta^{n-l_j}), \mathbb{Z}).
\]
Let \( C \) and \( \rho \) be the constants from Lemma 3.1, and write
\[
C_0 := 2\pi C \cdot \max\{k_1, k_2\} \cdot \rho^{-\max\{1, l_2\}}.
\]
From Lemma 3.1
\[
|\cos(4\pi k_j \cdot \text{Re}(\theta^{n-l_j})) - 1| = |\cos(4\pi k_j \cdot \text{Re}(\theta^{n-l_j})) - \cos(2\pi k_j d)|
\leq 2\pi k_j \cdot |2\text{Re}(\theta^{n-l_j}) - d| = 2\pi k_j \cdot \text{dist}(2\text{Re}(\theta^{n-l_j}), \mathbb{Z})
\leq 2\pi k_j C \cdot \rho^{\max\{1, l_2\}} \leq C_0 \cdot \rho^{\max\{1, l_2\}}.
\]
This shows,
\[
|b_n| \geq 1 - \frac{1}{2} \sum_{j=1}^{2} \cos(4\pi k_j \cdot \text{Re}(\theta^{n-l_j})) - 1 \geq 1 - C_0 \cdot \rho^{\max\{1, l_2\}}.
\]
Now let \( M \geq 1 \) be such that \( C_0 \cdot \rho^{\max\{1, l_2\}} < 1 \) for all \( n \in \mathbb{Z} \) with \( |n| \geq M \). Then from \( (3.2) \) it follows that for each \( N \geq 0 \),
\[
|\mathcal{F}(\mu)(4\pi \theta^N)| \geq \prod_{n=-\infty}^{-M} |b_n| \prod_{n=1-M}^{M-1} |b_n| \prod_{n=M}^{\infty} |b_n|
\geq \prod_{n=M}^{\infty} (1 - C_0 \cdot \rho^{\max\{1, l_2\}})^2 \cdot \prod_{n=1-M}^{M-1} |b_n| > 0,
\]
which completes the proof. \( \square \)
Let \( \mathcal{M}(\mathbb{R}) \) be the collection of all compactly supported Borel probability measures on \( \mathbb{R} \). Given \( \nu \in \mathcal{M}(\mathbb{R}) \) let \( \mathcal{F}(\nu) \) be the Fourier transform of \( \nu \), i.e.
\[
\mathcal{F}(\nu)(r) = \int_{\mathbb{R}} e^{i x r} \, d\nu(x) \quad \text{for} \quad r \in \mathbb{R}.
\]
Recall that
\[
S = \{ z \in \mathbb{C} : |z| = 1 \},
\]
and \( P_z w = \langle w, z \rangle \) for \( z \in S \) and \( w \in \mathbb{C} \). For \( n \geq 1 \) write
\[
U_n = \{ z \in S : \sup_{r \geq n} |\mathcal{F}(P_z \mu)(r)| > c \},
\]
where \( c \) is the constant from Proposition 3.2.

**Lemma 3.3.** Let \( n \geq 1 \), then \( U_n \) is an open and dense subset of \( S \).

**Proof.** Note that for \( z \in S \) and \( r \in \mathbb{R} \)
\[
\mathcal{F}(P_z \mu)(r) = \int_{\mathbb{R}} \exp(i x r) \, dP_z \mu(x) = \int_{\mathbb{R}} \exp(i \langle w, rz \rangle) \, d\mu(w) = \mathcal{F}(\mu)(rz),
\]
hence
\[
U_n = \{ z \in S : \sup_{r \geq n} |\mathcal{F}(\mu)(rz)| > c \}.
\]
Now since \( \mathcal{F}(\mu) \) is continuous it follows \( U_n \) is open in \( S \). Set \( \eta = \exp(-i \arg \theta) \), then from Proposition 3.2
\[
|\mathcal{F}(P^k \mu)(4\pi|\theta|^k)| = |\mathcal{F}(\mu)(4\pi\bar{\theta}^k)| > c
\]
for every integer \( k \geq 0 \). Let \( N \geq 1 \) be with \( |4\pi\theta^N| \geq n \), then \( \{ \eta^k \}_{k=N}^\infty \subset U_n \) by (3.4). By assumption \( \arg \theta \notin \pi \mathbb{Q} \), hence \( \{ \eta^k \}_{k=N}^\infty \) is dense in \( S \), which proves the lemma. \( \square \)

We can now complete the proof of Proposition 2.4.

**Proof of Proposition 2.4.** Set \( B = \cap_{n=1}^\infty U_n \), then \( B \) is a dense \( G_\delta \) subset of \( S \) by Lemma 3.3 and Baire’s theorem. Let \( z \in B \), then \( \mathcal{F}(P_z \mu)(r) \) does not tend to 0 as \( r \to \infty \). Hence, by the Riemann-Lebesgue lemma, \( P_z \mu \) is not absolutely continuous. From the law of pure types (see Theorem 3.26 in [B]) it now follows \( P_z \mu \) is singular, which completes the proof of the Proposition. \( \square \)

**4. Proof of Proposition 2.5**

In order to prove Proposition 2.5 we shall use the following theorem due to Wiener (see Section VI.2.12 of [K]).

**Theorem 4.1.** For every \( \nu \in \mathcal{M}(\mathbb{R}) \),
\[
\sum_{x \in \mathbb{R}} (\nu(x))^2 = \lim_{M \to \infty} \frac{1}{2M} \int_{-M}^M |\mathcal{F}(\nu)(\xi)|^2 \, d\xi.
\]
Let $\theta$, $\Phi$ and $\mu$ be as in Section 2. For $z \in S$ write $z^\perp = e^{-i\pi/2}z$. Given $n, k \in \mathbb{N}$ set

$$J_{n,k} = \{z \in S : \langle 4\pi\overline{\theta^k}, z^\perp \rangle \in (n, n + 1)\},$$

and let $V_n = \cup_{k \in \mathbb{N}} J_{n,k}$.

**Lemma 4.2.** Let $n \in \mathbb{N}$, then $V_n$ is a dense open subset of $S$.

**Proof.** Since $J_{n,k}$ is open in $S$ for every $k \in \mathbb{N}$ the same holds for $V_n$. Let $a \in \mathbb{R}$ and $0 < \epsilon < 1$ be given. For $E \subset \mathbb{R}$ write $q(E) = E + 2\pi\mathbb{Z}$. Since $|\theta| > 1$ and $\arg \theta \notin \pi\mathbb{Q}$, there exists $k \in \mathbb{N}$ with

$$|4\pi\theta^k| > \frac{n + 1}{\cos(\frac{\pi}{2} - \epsilon)} \quad \text{and} \quad \arg(4\pi\theta^k) \in q(a - \epsilon, a + \epsilon).$$

Set $w = 4\pi\theta^k$ and for $t \in \mathbb{R}$ write $f(t) = \langle w, e^{it} \rangle$. It holds that

$$f(\arg w^\perp) = \langle w, \frac{w^\perp}{|w^\perp|} \rangle = 0,$$

$$f(\arg w^\perp + \epsilon) = f(\arg w - \frac{\pi}{2} + \epsilon) = \langle w, \frac{w}{|w|} \cdot e^{i(\epsilon - \pi/2)} \rangle = \Re(w \cdot \frac{\overline{w}}{|w|} \cdot e^{i(\pi/2 - \epsilon)}) = |w| \cdot \cos(\frac{\pi}{2} - \epsilon) > n + 1,$$

and

$$[\arg w^\perp, \arg w^\perp + \epsilon] \subset q[a - \epsilon - \frac{\pi}{2}, a + 2\epsilon - \frac{\pi}{2}].$$

Hence, since $f$ is continuous and $2\pi$-periodic, there exists $t \in [a - \epsilon - \frac{\pi}{2}, a + 2\epsilon - \frac{\pi}{2}]$ with $f(t) \in (n, n + 1)$. Set $z = \exp(i(t + \frac{\pi}{2}))$, then

$$\langle w, z^\perp \rangle = \langle w, e^{it} \rangle = f(t) \in (n, n + 1),$$

and so $z \in J_{n,k} \subset V_n$. Now since $a$ and $\epsilon$ are arbitrary and

$$\arg z \in q\{t + \frac{\pi}{2}\} \subset q[a - \epsilon, a + 2\epsilon],$$

it follows that $V_n$ is dense in $S$, which proves the lemma. \qed

Set $B = \bigcap_{n \in \mathbb{N}} V_n$, then $B$ is a dense $G_\delta$ subset of $S$ by Lemma 13. Fix $z \in B$ and recall that $\{\mu_{z,w}\}_{w \in \mathbb{C}}$ is the disintegration of $\mu$ with respect to $P_z^{-1}(B)$, where $B$ is the Borel $\sigma$-algebra of $\mathbb{C}$. In order to prove the proposition, it suffices to show that $\mu_{z,w}$ is discrete for $\mu$-a.e. $w \in \mathbb{C}$. Write

$$(4.1) \quad \tau_w(\xi) = \xi - w \quad \text{and} \quad R \xi = \overline{z^\perp} \cdot \xi \quad \text{for} \ w, \xi \in \mathbb{C},$$

for each $w \in \mathbb{C}$ let $\nu_w = R\tau_w \mu_{z,w}$, and note that $\nu_w \in \mathcal{M}(\mathbb{R})$. 

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Lemma 4.3. Let $c > 0$ be the constant from Proposition 3.2, then for each $n \in \mathbb{N}$ there exists $t_n \in (n, n + 1)$ with

\begin{equation}
(4.2) \quad \int |F(v_n)(t_n)|^2 \, d\mu(w) > c^2.
\end{equation}

Proof. Let $n \in \mathbb{N}$. Since $z \in V_n$ there exists $k_n \in \mathbb{N}$ and $t_n \in (n, n + 1)$ with $\langle 4\pi\theta^k_n, z \rangle = t_n$. Write $\eta = 4\pi\theta^k_n$, then by Proposition 3.2

\begin{equation}
(4.3) \quad c < |F(\mu)(\eta)| \leq \int \left| \int e^{i \langle \xi, \eta \rangle} \, d\mu_{z,w}(\xi) \right| \, d\mu(w).
\end{equation}

Let $Q_{z \perp}$ be the orthogonal projection onto $z \perp \mathbb{R}$, i.e.

\[ Q_{z \perp} \xi = \langle \xi, z \rangle z \perp \quad \text{for} \quad \xi \in \mathbb{C}. \]

From (4.3) and since $\tau_w \mu_{z,w}$ is supported on $z \perp \mathbb{R}$ for $w \in \mathbb{C}$,

\[ c < \int \left| \int e^{i \langle \xi + w \cdot \eta \rangle} \, d\tau_w \mu_{z,w}(\xi) \right| \, d\mu(w) = \int \left| \int e^{i \langle w, \eta \rangle} \, dQ_{z \perp} \tau_w \mu_{z,w}(\xi) \right| \, d\mu(w) \]

\[ = \int \left| \int \exp(i \langle Q_{z \perp} \xi, \eta \rangle) \, d\tau_w \mu_{z,w}(\xi) \right| \, d\mu(w). \]

Now since $Q_{z \perp}$ is self-adjoint, $\langle \eta, z \perp \rangle$ is equal to $t_n$, and $R$ from (4.1) is a rotation,

\[ c < \int \left| \int \exp(i \langle \xi, t_n \cdot z \perp \rangle) \, d\tau_w \mu_{z,w}(\xi) \right| \, d\mu(w) = \int \left| \int \exp(i \langle R \xi, t_n \cdot R z \perp \rangle) \, d\tau_w \mu_{z,w}(\xi) \right| \, d\mu(w) \]

\[ = \int \left| \int e^{i t_n} \, d\nu_w(\xi) \right| \, d\mu(w) = \int |F(v_n)(t_n)| \, d\mu(w). \]

From this and Jensen’s inequality the lemma follows. \qed

Let us define the set,

\begin{equation}
(4.4) \quad E_z = \{ w \in \mathbb{C} : \mu_{z,w}\{w\} > 0 \}.
\end{equation}

Lemma 4.4. It holds that $\mu(E_z) > 0$.

Proof. Let $\{t_n\}_{n \in \mathbb{N}}$ be the numbers obtained in Lemma 4.3. Since $\text{supp}(\mu)$ is compact and

\[ \text{supp}(\mu_{z,w}) \subset \text{supp}(\mu) \quad \text{for} \quad \mu\text{-a.e.} \quad w \in \mathbb{C}, \]

there exists $M > 0$ such that $\nu_w$ is supported on $[-M, M]$ for $\mu$-a.e. $w \in \mathbb{C}$. It follows that $F(\nu_w)$ is $M$-Lipschitz for $\mu$-a.e. $w \in \mathbb{C}$. Hence there exist $\delta > 0$ and
intervals \( \{A_n\}_{n \in \mathbb{N}} \), such that for every \( n \in \mathbb{N} \) it holds
\[
t_n \in A_n \subset (n, n + 1),
\]
\( A_n \) has length \( \delta \), and for \( \mu \)-a.e. \( w \in \mathbb{C} \),
\[
\left| |F(\nu_w)(t_n)|^2 - |F(\nu_w)(x)|^2 \right| < \frac{c^2}{2} \text{ for } x \in A_n.
\]
We now get from (4.2) that for each \( N \geq 1 \),
\[
c^2 \leq \int \frac{1}{N} \sum_{n=0}^{N-1} |F(\nu_w)(t_n)|^2 d\mu(w)
\]
\[
= \int \frac{1}{\delta N} \sum_{n=0}^{N-1} \int_{A_n} |F(\nu_w)(t_n)|^2 dx d\mu(w)
\]
\[
\leq \int \frac{1}{\delta N} \sum_{n=0}^{N-1} \int_{A_n} |F(\nu_w)(x)|^2 + \frac{c^2}{2} dx d\mu(w)
\]
\[
\leq \int \frac{1}{\delta N} \int_{-N}^{N} |F(\nu_w)(x)|^2 dx d\mu(w) + \frac{c^2}{2},
\]
which gives,
\[
\frac{\delta c^2}{4} \leq \int \frac{1}{2N} \int_{-N}^{N} |F(\nu_w)(x)|^2 dx d\mu(w).
\]
Now by Theorem 4.1 and the bounded convergence theorem,
\[
\int \sum_{\xi \in \mathbb{C}} \mu_{z,w}\{\xi\} d\mu(w) = \int \sum_{x \in \mathbb{R}} \nu_w\{x\} d\mu(w)
\]
\[
= \int \lim_{N \to \infty} \frac{1}{2N} \int_{-N}^{N} |F(\nu_w)(\xi)|^2 d\xi d\mu(w)
\]
\[
= \lim_{N \to \infty} \int \frac{1}{2N} \int_{-N}^{N} |F(\nu_w)(\xi)|^2 d\xi d\mu(w) \geq \frac{\delta c^2}{4} > 0.
\]
This gives \( \mu(F_z) > 0 \), where
\[
F_z = \{ w \in \mathbb{C} : \mu_{z,w}\{\xi\} > 0 \text{ for some } \xi \in \mathbb{C} \}.
\]
Let \( w \in F_z \), then there exists
\[
\xi \in \text{supp}(\mu_{z,w}) \subset w + z^{-1}\mathbb{R}
\]
with \( \mu_{z,w}\{\xi\} > 0 \). Since \( \mu_{z,\xi} = \mu_{z,w} \) it follows \( \xi \in E_z \), where \( E_z \) is defined in (4.4), and so
\[
\mu_{z,w}(E_z) \geq \mu_{z,w}\{\xi\} > 0.
\]
Now from \( \mu(F_z) > 0 \) we get
\[
\mu(E_z) \geq \int_{E_z} \mu_{z,w}(E_z) d\mu(w) > 0,
\]
where $B \in [FH]$ we get that for each $m$ $F$ or $F$(4.5) hence, from this and (4.5) it follows that for $\mu$ which proves the lemma. □

for $w$ $\in$ $\lambda$ $\K$ $\Phi$ $\K$ $\alpha$ $(\{0,1\}) = \lim_{m \to \infty} \mu_{z_m} \in S \text{ and } F_{m,n}(w) = \mu_{z_m,w}(K_{\alpha_n(w)})$.

For $m \in N$, $w \in C$ and $\delta > 0$ let $V^m_w(\delta) = w + z_m^{-1} \cdot \mathbb{R} + B(0, \delta)$, where $B(0, \delta)$ is the open disk in $\mathbb{C}$ with centre 0 and radius $\delta$. From Lemma 3.3 in [FH] we get that for each $m \in N$ and $A \in \mathcal{B}$, $\mu_{z_m,w}(A) = \lim_{\delta \downarrow 0} \frac{\mu(V^m_w(\delta) \cap A)}{\mu(V^m_w(\delta))}$ for $\mu$-a.e. $w \in C$.

Fix $m,n \in N$, then for $\mu$-a.e. $w \in K$

(4.5) $F_{m,n}(T^m w) = \frac{\mu_{z_m,T^m w}(K_{\alpha_n(T^m w)})}{\mu_{z_m,T^m w}(K_{\alpha_n(T^m w)})} = \lim_{\delta \downarrow 0} \frac{\mu(V^m_{T^m w}(\delta) \cap K_{\alpha_n(T^m w)})}{\mu(V^m_{T^m w}(\delta) \cap K_{\alpha_n(T^m w)})}$.

Since $\mu$ satisfies the SSC,

$$\varphi_\alpha(K) \cap \varphi_{\alpha_m(w)}(K) = \emptyset \text{ for } \alpha \in I \setminus \{\alpha_m(w)\},$$

hence,

$$\mu(\varphi^{-1}_\alpha(\varphi_{\alpha_m(w)}(K))) = 0 \text{ for } \alpha \in I \setminus \{\alpha_m(w)\}.$$

From this and (4.5) it follows that for $\mu$-a.e. $w \in K$,

$$F_{m,n}(T^m w) = \lim_{\delta \downarrow 0} \frac{\sum_{\alpha \in I} |I|^{-m} \cdot \mu(\varphi^{-1}_\alpha(\varphi_{\alpha_m(w)}(V^m_{T^m w}(\delta) \cap K_{\alpha_n(T^m w)})))}{\sum_{\alpha \in I} |I|^{-m} \cdot \mu(\varphi^{-1}_\alpha(\varphi_{\alpha_m(w)}(V^m_{T^m w}(\delta) \cap K_{\alpha_n(T^m w)})))}.$$
Now since $\mu = \sum_{a \in T} |I|^{-m} \cdot \varphi_a \mu$.

\[(4.6) \quad F_{m,n}(T^m w) = \lim_{\delta \downarrow 0} \frac{\mu(\varphi_{\alpha_m}(w)(V_{T^m w}(\delta) \cap K_{\alpha_n}(T^m w)))}{\mu(\varphi_{\alpha_m}(w)(V_{T^m w}(\delta) \cap K_{\alpha_n}(T^m w)))} \text{ for } \mu \text{-a.e. } w \in K.\]

For every $\delta > 0$,

$$\varphi_{\alpha_m}(w)(V_{T^m w}(\delta)) = \varphi_{\alpha_m}(w)(T^m w) + \lambda^m \cdot z^\perp + \lambda^m \cdot B(0, \delta) = w + z^\perp + B(0, |\delta \lambda^m|) = V_w^m(|\delta \lambda^m|).$$

Hence from (4.6) it follows that for $\mu$-a.e. $w \in K$,

\[(4.7) \quad F_{m,n}(T^m w) = \lim_{\delta \downarrow 0} \frac{\mu(V_w^m(|\delta \lambda^m|) \cap K_{\alpha_{m+n}}(w))}{\mu(V_w^m(|\delta \lambda^m|) \cap K_{\alpha_n}(w))} = \frac{\mu_{z,w}(K_{\alpha_{m+n}}(w))}{\mu_{z,w}(K_{\alpha_n}(w))},\]

where we have used the fact that $\mu_{z,w}(K_{\alpha_n}(w)) > 0$ for $\mu$-a.e. $w \in K$.

Let $m \in \mathbb{N}$, then from (4.7) we get that mod $\mu$ it holds

$$E_z = \{w \in K : \lim_{n} \frac{\mu_{z,w}(K_{\alpha_{m+n}}(w))}{\mu_{z,w}(K_{\alpha_n}(w))} > 0\}$$

$$= \{w \in K : \lim_{n} F_{m,n}(T^m w) > 0\} \subseteq T^{-m}(B_K),$$

which shows,

$$E_z \subseteq \cap_{m \in \mathbb{N}} T^{-m}(B_K) \mod \mu.$$

Now since $(K, B_K, T, \mu)$ is isomorphic to a Bernoulli shift, it follows that $\mu(E_z) = 0$ or 1. But by Lemma 4.3 we have $\mu(E_z) > 0$, which completes the proof of the lemma. \qed

We can now complete the proof of Proposition 2.5.

Proof of Proposition 2.5. As mentioned above, it suffices to show $\mu_{z,w}$ is discrete for $\mu$-a.e. $w \in C$. By lemma 4.3 we have $\mu(E_z) = 1$, and so $\mu_{z,w}(E_z) = 1$ for $\mu$-a.e. $w \in C$. Fix such a $w \in C$ and let $A = w + z^\perp \mathbb{R}$. Since $\mu_{z,w}(A) = 1$ and $\mu_{z,w} = \mu_{z,\xi}$ for $\xi \in A$,

$$1 = \mu_{z,w}(E_z \cap A) = \mu_{z,w}(\{\xi \in A : \mu_{z,w}(\{\xi\}) > 0\}).$$

This shows that $\mu_{z,w}$ is discrete, which completes the proof. \qed

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