What is $NP$?
- Interpretation of a Chinese paradox white horse is not horse

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Abstract

The notion of nondeterminism has disappeared from the current definition of $NP$, which has led to ambiguities in understanding $NP$, and caused fundamental difficulties in studying the relation $P$ versus $NP$. In this paper, we question the equivalence of the two definitions of $NP$, the one defining $NP$ as the class of problems solvable by a nondeterministic Turing machine in polynomial time, and the other defining $NP$ as the class of problems verifiable by a deterministic Turing machine in polynomial time, and reveal cognitive biases in this equivalence. Inspired from a famous Chinese paradox white horse is not horse, we further analyze these cognitive biases. The work shows that these cognitive biases arise from the confusion between different levels of nondeterminism and determinism, due to the lack of understanding about the essence of nondeterminism. Therefore, we argue that fundamental difficulties in understanding $P$ versus $NP$ lie firstly at cognition level, then logic level.

Keywords: recognition of problem; $P$ versus $NP$; nondeterminism; determinism; polynomial time verifiability; nondeterministic Turing machine; deterministic Turing machine; white horse is not horse

1 Introduction

One of the main issues in the theory of $NP$-completeness is to study whether a $NP$ problem can be efficiently solved by an algorithm, i.e., the existence of a polynomial time algorithm to solve a $NP$ problem.

However, not only does the general public find it difficult to understand the theory of $NP$-completeness, but also a large number of computer scientists and students feel it half-comprehended. In our opinion, a major cause for this phenomenon is that there exist ambiguities in understanding $NP$. Initially, $NP$ refers to Nondeterministic Polynomial time, but the notion of nondeterminism is lost in the current definition of $NP$. Consequently, although one can give out the definition of
NP as in the literature, but one cannot give a satisfactory explanation about what NP is. Furthermore, $P$ versus $NP$ has always remained one of the most perplexing problems in the theory of computational complexity since it was raised in 1971 [1].

In this paper, we investigate the recognition of $NP$ by questioning the equivalence of the two definitions of $NP$.

The paper is organized as follows. In Section 2, we carry out a brief survey of the definition of $NP$. In Section 3, we make a preliminary discussion about the recognition of problem with the help of the etymology in English and in Chinese. In Section 4, we interpret two well-known arguments for supporting the equivalence of the two definitions of $NP$. In Section 5, we make use of a famous Chinese paradox white horse is not horse to further analyze the cognitive biases in this equivalence. In Section 6, we conclude the paper.

2 Overview of the definition of $NP$

Concerning the definition of $NP$, it can be traced to the 60’s [2][3], where a great number of applicable and significant problems were discovered for which no polynomial algorithms could be found to solve them. As time passed and much effort was expended on attempts at efficiently solving these problems, it began to be suspected that there was no such solution.

Then, the theory of computational complexity was born in order to study these problems. The class of these problems was defined as problems solvable by nondeterministic Turing machine in polynomial time, denoted as $NP$ (Nondeterministic Polynomial time), to distinguish from those known problems solvable by deterministic Turing machine in polynomial time, denoted as $P$ (Polynomial time).

Cook’s theorem, presented in a paper entitled The complexity of theorem proving procedures [1] in 1971, laid the foundation for the theory of NP-completeness that becomes the core of the current theory of computational complexity. Since then, there exist two definitions for $NP$:

- **Solvability-based definition**: $NP$ is the class of problems solvable by a nondeterministic Turing machine in polynomial time.

- **Verifiability-based definition**: $NP$ is the class of problems verifiable by a deterministic Turing machine in polynomial time.

In order to facilitate the following discussion, we denote the class defined by the verifiability-based definition as $P_{\text{verifiable}}$, and keep $NP$ to denote the class defined by the solvability-based definition.

The current academic community generally considers the two above definitions to be equivalent [4], that is, $NP = P_{\text{verifiable}}$:

*The two definitions of $NP$ as the class of problems solvable by a nondeterministic Turing machine in polynomial time and the class of problems verifiable by a deterministic Turing machine in polynomial time are equivalent. The proof is described by many textbooks, for example Sipser’s Introduction to the Theory of Computation, section 7.3 [5].*
Therefore, the verifiability-based definition is used to replace the solvability-based one, and has become the standard definition of $NP$:

$P$ is the class of problems solvable by a deterministic Turing machine in polynomial time, while $NP$ is the class of problems verifiable by a deterministic Turing machine in polynomial time.

Clearly, $P \subseteq NP$, since for each problem in $P$ its solutions can be verified in polynomial time by a deterministic Turing machine.

Furthermore, the famous problem of $P$ versus $NP$ consists in asking whether a problem verifiable by a polynomial algorithm is solvable by a polynomial algorithm, denoted as $NP = P$?

Such a problem has become a major unsolved problem in the theory of computational complexity [3][6][7][8], and was selected as one of the seven Millennium Prize Problems by the Clay Mathematics Institute [9]. Although a lot of effort has been done (see a comprehensive list maintained by Woeginger (2010) [10]), no substantial progress has been achieved until now.

William Gasarch conducted two polls about the future of $P$ versus $NP$ [11], with 100 participants in 2002 and 152 participants in 2012. The result (see Table 1) shows that:

| Year | $P \neq NP$ | $P = NP$ | Ind | DC | DK | DK and DC | Other |
|------|-------------|---------|-----|----|----|-----------|------|
| 2002 | 61(61%)     | 9(9%)   | 4(4%) | 1(1%) | 22(22%) | 0(0%) | 3(3%) |
| 2012 | 126(83%)    | 12(9%)  | 5(3%) | 5(3%) | 1(0.6%) | 1(0.6%) | 1(0.6%) |

Table 1: Result of two polls about the opinions about $P$ versus $NP$, where Ind stands for Independent of ZFC, DC stands for Do not Care, DK stands for Do not Know.

There is a definite trend - more people think $P \neq NP$ now than they did in 2002. How strongly held are these opinions? Although I did not ask people what the strength of their opinion was in either poll (1) in 2002, 7 out of the 61 $P \neq NP$ votes (11%) said they had some doubts that $P \neq NP$, (2) in 2011, 16 out of the 125 (again 11%) said they had some doubts that $P \neq NP$. Hence, of the people that think $P \neq NP$, the level of confidence is about the same.

In our opinion, the difficulty in understanding $P$ versus $NP$ lies firstly at cognition level, then logic level. In fact, due to the fact that the verifiability-based definition has been accepted as the standard definition of $NP$, the notion of nondeterminism has disappeared from $NP$, which has caused ambiguities in understanding $NP$. So if we hope to get an insight into $P$ versus $NP$, we need at first to question the equivalence of the two definitions of $NP$.

In the following, we start from a preliminary discussion about the recognition of problem, then we interpret two well-known arguments for supporting $NP = P_{\text{verifiable}}$.

3 Preliminary discussion about the recognition of problem

Problem is the center of thought. Around a problem, all ideas flourish. So without a problem, there would be no thought. However, if one wants to ask some general questions about problem
such as, *where does a problem come from? what is the essence of a problem? how to define a problem?* one would find it difficult to answer these questions, because such an investigation is absent from the traditional philosophy.

Starting from the etymology of *problem* in English and in Chinese, we make a preliminary discussion about the recognition of problem.

### 3.1 Etymology of *problem*

#### 3.1.1 In English

In Greek, proballo = pro (in face of) + ballo (to throw).

*Problem* means an obstacle to prevent from reaching a goal.

#### 3.1.2 In Chinese

In Chinese, *problem* consists of two characters, and each character represents an image that is composed by several basic characters [12][13]:

- 问题 = 问 + 题: problem
- 问 = 门 (door) + 口 (mouth): ask for
- 题 = 是 (affirmation) + 头 (head): important subject

Then, the Chinese characters of *problem* describe a scene where a person outside a door knocks and asks for a question, and a person inside the door opens and answers the question.

Therefore, *problem* refers to an interface between the questioner and the responder, which means questions about the essence of things.

#### 3.1.3 Interpretation of *problem* in the context of *problem-solving*

In the context of *problem-solving*, *problem* can be interpreted in two ways inspired by the above etymology.

When a problem has been recognized out, it can be clearly defined by specifying the input data and the output result, and solved by designing an algorithm. In this case, such a problem is not really problematique.

When a problem has not been recognized out, it is obscure, so it needs to question about the essence of things in order to recognize it. In this case, such a problem becomes problematique.

The two interpretations are complementary and both contribute to *problem-solving*.

### 3.2 Recognition of *NP*

From the view of cognition, *recognition* is *cognition* again, that is, *recognition* means to distinguish something new from some previously known things, rather than identify it as those known things.
Such a principe of recognition can be stated as *Yin-Yang principe* in the traditional Chinese thought. We refer to a text from *Dao De Jing*, a Chinese classic, to clarify this idea [14]:

*When the people of the world all know beauty as beauty, there arises the recognition of ugliness. When they all know the good as good, there arises the recognition of bad.*

*Therefore being and non-being produce each other; difficult and easy complete each other; long and short contrast each other; High and low distinguish each other; sound and voice harmonize with each other; beginning and end follow each other.*

Concerning the recognition of *NP*, the key is to determine the property that can distinguish *NP* from *P*.

The concept of *NP* (*Nondeterministic Polynomial time*) is defined in terms of nondeterministic Turing machine, that is, *NP* is defined at *algorithm* level rather than *problem* level. *NP* implies the two properties, the *nondeterminism* referring to the multiple choices of transition, and the *polynomial time verifiability* referring to the verifiability of solution in polynomial time.

About the concept of *nondeterminism*, it is under investigation, so the definition at algorithm level is different from that at problem level in the view of cognition, as discussed in the above two interpretations of *problem*. Therefore, the *nondeterminism* in terms of the multiple choices of transition just expresses the formal sense of nondeterminism at algorithm level, but not the essential sense of nondeterminism at problem level. In order to continue our discussion, we temporarily use such nondeterminism as denotation of concept to recognize *NP* from *P*. For the concept of *polynomial time verifiability*, it concerns the *determinism* that is already clear in the view of cognition.

The equivalence of the two definitions of *NP* suggests that the property of *Polynomial time verifiability* is used to recognize *NP*. Let us interpret two well-known arguments for supporting this equivalence.

### 4 Interpretation of two well-known arguments for *NP = P*$_{\text{verifiable}}$

#### 4.1 A popular argument for *NP = P*$_{\text{verifiable}}$

A popular argument accepted by the general public can be stated as:

*Because the solution of a *NP* problem is verifiable in polynomial time, so *NP* is the class of problems verifiable in polynomial time (*NP = P*$_{\text{verifiable}}$).*

The argument consists of two basic propositions. The first, *the solution of a *NP* problem is verifiable in polynomial time*, means that a *NP* problem is a member of the set *P*$_{\text{verifiable}}$ and possesses the property of polynomial time verifiability. The second, *NP is the class of problems verifiable in polynomial time*, means that the set *NP* is identified as the set *P*$_{\text{verifiable}}$, i.e., *NP = P*$_{\text{verifiable}}$. The argument uses the polynomial time verifiability that defines *P*$_{\text{verifiable}}$ to recognize *NP* as *P*$_{\text{verifiable}}$. 

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Although a NP problem is verifiable in polynomial time at logic level, but this property is not able to recognize NP from P at cognition level, because the essence of this property is the determinism and it is shared by P, since a P problem is verifiable in polynomial time. Therefore, this argument is not valid. In other words, if the polynomial time verifiability is used to define NP, the nondeterminism would disappear at cognition level, as the determinism and the nondeterminism are relative at the same level of concept.

4.2 A professional argument for \( NP = P_{\text{verifiable}} \)

A professional argument accepted in the theory of NP-completeness can be stated as:

Because a deterministic Turing machine can be considered as a special case of a nondeterministic Turing machine, so \( P \subseteq NP \) thus \( NP = P_{\text{verifiable}} \).

In fact, the first proposition, a deterministic Turing machine can be considered as a special case of a nondeterministic Turing machine, is a pure mathematical deduction, that is, if \( n = 1 \) and \( n > 1 \) then \( n \geq 1 \), where \( n = 1 \) refers to only one transition, \( n > 1 \) several possible transitions, and \( n \geq 1 \) one or several possible transitions. In other words, this proposition is just valid at mathematical logic level, because the sense of the transition number \( n \) is not taken into account. For the second proposition, it concerns the recognition of NP and is situated at cognition level.

Therefore, although \( n = 1 \) and \( n > 1 \) can be mixed to be \( n \geq 1 \) at mathematical logic level, but \( P \) and \( NP \) cannot be confused in terms of \( P \subseteq NP \) at cognition level, because the relation \( P \subseteq NP \) itself needs to be proved. This question begging leads to \( NP = P_{\text{verifiable}} \), then the nondeterminism disappears from \( NP \).

5 Interpretation of a Chinese paradox white horse is not horse

The cognitive biases in the above recognition of NP are quite typical and widespread. For example, a famous Chinese paradox called white horse is not horse [15], illustrates such cognitive biases.

The paradox of white horse is not horse was proposed by a great Chinese logician Gongsun Long (325 - 250 BC), and it tells a history that, one day, Gongsun Long went to a city on riding a white horse. At the gate of the city, the guard said to him that, your white horse is horse, in according to regulations, horses were not allowed to enter the city. Thus, Gongsun Long began his argument - white horse is not horse, finally he persuaded the guard, and entered the city riding on his white horse.

The argument of Gongsun Long was presented in his writing known as White Horse Dialogue between two speakers. Here is a partial summary:
The issue: Can it be that white horse is not horse?

Advocate: It can.

Objector: How so?

Advocate: Horse is named by means of the shape. White is named by means of the color. What names the color is not what names the shape. Hence, I say that White horse is not Horse.

Objector: Having a white horse cannot be said as having no horses. As it cannot be said as having no horses, it means having a horse. Having a white horse is having a horse, how can a white one not be horse?

Advocate: If one wants a horse, that extends to yellow or black horses. But if one wants a white horse, that does not extend to yellow or black horses. Suppose that a white horse were horse, then what one wants would be horse. If what one wants were horse, then a white horse would not differ from a horse. If what one wants does not differ, then how is it that a yellow or black horse is sometimes acceptable and sometimes unacceptable? It is clear that acceptable and unacceptable are mutually contrary. Hence, given a yellow or black horse, one can respond that there are horses, but one cannot respond that there are white horses. Thus, it is evident that white horse is not horse.

We analyze this history. For the guard, his responsibility was to recognize horses and to prevent any horse from entering the city, whether it was a white, yellow or black horse. So when he said that your white horse is horse, in fact he meant that the white horse of Gongsun Long is a member of the set of horses. However, for Gongsun Long, his objective was to recognize the set of white horses, that is, he wanted to distinguish the set of white horses with the set of yellow or black horses, so the term of white horse in his argument white horse is not horse refers to the set of white horses instead of his white horse.

The proposition of the guard and that of Gongsun Long are right respectively. But if they are confused, there would arise cognitive biases, such as saying because a white horse is horse, so white horse is horse, because the two propositions situate at two different levels. In the writing of White Horse Dialogue, Gongsun Long described such cognitive bias as a bird flying in water.

Gongsun Long just took advantage of such cognitive bias that was not perceived by the guard, and confuse the guard, so that the guard forgot his position and slided to the position of Gongsun Long, finally let the white horse of Gongsun Long enter the city.

If we make an analogy between the paradox white horse is not horse and the issue of the recognition of NP, we can clearly see their similarity.

6 Conclusion

We question the equivalence of the two definitions of NP and argue that there exist cognitive biases in this equivalence that arise from the confusion of levels of nondeterminism and determinism, due to the lack of understanding about the essence of nondeterminism. Rightly in this sense, we argue that the difficulty in understanding P versus NP lies firstly at cognition level, then logic level.
This work is the first step of our work of the interpretation of the theory of \(NP\)-completeness [16], and undertaken by another work of the interpretation of Cook’s theorem [17]. Moreover, it provides an interpretation of a deeper work discussed in [18][19].

We hope that this work can evoke reflections from different angles about some fundamental problems in cognitive science, and contribute to understand \(P \text{ versus } NP\). Furthermore, we hope that this work can help to understand the complementarity of Chinese thought and Western philosophy.

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