A general computer program for the Bell detection loophole

by

Roberto M. Basoalto and Ian C. Percival

Department of Physics
Queen Mary and Westfield College, University of London
Mile End Road, London E1 4NS, England

i.c.percival@qmw.ac.uk

Abstract

The difference between ideal experiments to test Bell’s weak nonlocality and the real experiments leads to loopholes. Ideal experiments involve either inequalities (Bell) or equalities (Greenberger, Horne and Zeilinger). Every real experiment has its own critical inequalities, which are almost all more complicated than the corresponding ideal inequalities or equalities. If one of these critical inequalities is violated, then the detection loophole is closed, with no further assumptions. If all the critical inequalities are satisfied, then it remains open, unless further assumptions are made. The computer program described here and published on the website

http://www.strings.ph.qmw.ac.uk/QI/main.htm

obtains the critical inequalities for any real experiment, given the number of allowed settings of the angles and the corresponding possible output signals for a single run. Given all the necessary conditional probabilities or rates, it tests whether all these inequalities are satisfied.
Weak nonlocality, or nonlocality in the sense of Bell, applies to systems whose properties cannot be explained using local hidden variables [1]. It is defined in [2] and in the next section in terms of transfer functions for input-output systems. The original Bell inequalities and their generalizations lead to tests for weak nonlocality in the laboratory.

There have been many experiments designed to do this. It is thirty-six years since Bell obtained the first inequality and tentatively suggested an experiment, and about three decades since the first experiments. Yet there is still no published unambiguous experimental demonstration of weak nonlocality, because of the detection loophole [3,4,5,6,7,8], which we now describe.

Real experiments have outputs that are excluded in ideal experiments. For example, in the original ideal Bell experiment, two entangled particles are detected for every run. But in a real experiment, because of imperfect detectors, or losses due to absorption or imperfect collimation, only one of these particles may be detected. Such outputs, that are present in the real experiment, but not in the ideal experiment, affect the inequalities, and the tests of nonlocality. The experiments can be analysed by making assumptions about detection efficiency, and using the original inequalities, but these assumptions may not be justified, leaving a loophole. This is the detection loophole.

Since nonlocality is unique to quantum measurement, and appears nowhere else in physics, we cannot afford to leave such a loophole. We need to close it by improving the experiments and analysing them unambiguously. In some special cases, as suggested by Clauser, Horne, Shimony and Holt, the real and ideal experiments are the same, and so are the corresponding inequalities [9,10]. In other cases a real experiment has critical inequalities that apply to all the outputs, including those that are absent in the ideal experiment which it simulates. Weak nonlocality follows unambiguously from the violation of any one of these critical inequalities. They are almost always different from the original inequalities for the ideal experiment, and sometimes much more complicated. These critical inequalities have been obtained for particular experiments, for example [11,12], but in general are relatively difficult to find.

This letter briefly describes the method used to obtain the critical inequalities, and a computer program which provides them, for any Bell experiment with two spatially separated subsystems. A more general and detailed description and program will appear in a later paper. Zukowski et al. [13,14] and more recently Pitowski and Svozil [15] have proposed similar programs and applied them to special cases. Our program also provides an unambiguous test of weak nonlocality for the results of all such experiments with adequate statistics. This part of the program does not use all the critical inequalities, and so can be much faster than the one which does.

Some of the methods used here were developed for special cases by Wigner [16],
Garg and Mermin [11,12]. The general mathematical theory was first developed by Froissart [17] and then Pitowsky [18]. A general account of the relevant theory of input-output systems is given in [19].

Nonlocality

There is a profound distinction between experiments to test weak nonlocality by the violation of Bell inequalities and most other experiments on quantum systems.

Consider for example an experiment to determine the spectrum of an atom, or a differential cross section for the scattering of an electron by a molecule, or an experiment to determine the band gaps of a solid, or to find a new particle. The aim of all these experiments is to determine the properties of quantum systems. The classical apparatus used to prepare the system and to make the necessary measurements is essential, but secondary to obtaining these quantum properties.

In Bell experiments the converse is true. The aim is to test for violation of the inequalities, which are derived from the (statistical) properties of classical events, such as the setting of the apparatus, which is a classical input, or the detection of a particle by an electron avalanche, which is a classical output. The probabilities of the outputs, given the inputs, are what appear in the Bell inequalities, and it is the location of these events in spacetime that determine the locality or nonlocality. The classical events are connected by an ancillary quantum system, whose function is to produce the unusual statistical properties of these classical events. The quantum properties, like the entanglement of particles, or the polarization of photons, or the spins of atoms, are essential, but secondary. The primary result is the violation of an inequality, which depends on the properties of the classical inputs and outputs.

We will only be concerned with those systems which have a finite number of possible inputs and a finite number of possible outputs, so we may use integers to label them, and there is also only a finite number of the transfer functions $F$ described below.

Suppose a deterministic system consists of two subsystems $A,B$, with corresponding inputs $\alpha, \beta$ and outputs $a,b$. They are related by a transfer function $F$ such that

$$ (a,b) = F(\alpha, \beta). $$

(1)

If the subsystems do not interact, then we expect $a$ to be a function of $\alpha$ alone, independent of $\beta$, and $b$ to be independent of $\alpha$, giving the separate relations
\[
\begin{align*}
a &= F_A(\alpha), \\
b &= F_B(\beta),
\end{align*}
\]  
which is a very special case. If the output event \(b\) is outside the forward lightcone of the input event \(\alpha\) and the output event \(a\) is outside the forward lightcone of the input event \(\beta\), this independence is an inevitable consequence of special relativity, for otherwise there would be signals faster than the velocity of light. For this special case, \(F\) is called a \textit{local} transfer function, and the remainder are \textit{nonlocal}. Deterministic nonlocal transfer functions are never seen.

For stochastic systems, the relation between the inputs and outputs can be expressed in terms of the measurable transition or conditional probabilities

\[
\Pr(\alpha, \beta \rightarrow a, b) = \Pr(a, b | \alpha, \beta).
\]

The input-output properties of the system can also be expressed in terms of the possible transfer functions \(F\). We can suppose that the stochastic system sometimes behaves like a deterministic system with one transfer function, and sometimes like another. The probability that the stochastic system behaves like a deterministic system with transfer function \(F\) is the \textit{transfer probability} \(\Pr(F)\). The transition probabilities are then

\[
\Pr(\alpha, \beta \rightarrow a, b) = \sum_F \Pr(F) \delta((a, b), F(\alpha, \beta)),
\]

where the delta function is equal to 1 if equations (1) are satisfied for \((a, b)\) and \(F(\alpha, \beta)\) and equal to zero otherwise. The stochasticity may be produced entirely by classical fluctuations, in which case the sum on the right needs no nonlocal transfer functions. Their probabilities can always be put to zero, as a consequence of classical locality.

\section*{General Bell inequalities}

According to quantum dynamics and the quantum theory of measurement, there could be systems for which it is impossible to express the transition probabilities in terms of local transfer functions alone. At least one nonlocal transfer function may have to have a nonzero probability \(\Pr(F)\). This is weak nonlocality. This definition of weak nonlocality in terms of transfer functions is equivalent to Bell’s original definition in terms of hidden variables [19,1].

A general Bell experiment is an experiment to test this weak nonlocality for experimental transition probabilities. The general Bell inequalities are inequalities relating transition probabilities, which follow from equation (4) when only
local transfer functions are included in the sum on the right hand side. The transfer probabilities themselves have to sum to unity and lie in the range

$$0 \leq \text{Pr}(F) \leq 1.$$  \hspace{1cm} (5)

When the sum over transfer probabilities is unconstrained, these inequalities put no constraints on the transition probabilities other than the obvious ones that they must also sum to unity for each input, and lie in this range. But if the sum is constrained to include only local transfer functions, there are additional constraints on the transition probabilities. These are the Bell inequalities. All Bell inequalities are of this type.

For simple cases, like the original ideal Bell experiment, there is little difficulty in deriving the inequalities, but as the numbers of inputs and outputs increases, the number of inequalities, and the difficulty in deriving them, increase very rapidly. Since real experiments have more outputs than the ideal ones, this is a practical theoretical problem which has to be solved before the data can be analysed for an unambiguous test of weak nonlocality.

Hence the need for a general computer program that derives the critical inequalities effectively, and provides a test of weak nonlocality for a given experimental set of transition probabilities.

**Inputs and outputs**

To apply the theory to experiment, we have to know the inputs and outputs. For each run of an experiment, they are all classical events. The classical inputs are the settings of the devices that determine what quantum variable is to be measured. For example, when the Paris group of Aspect and his collaborators [20] measured the polarization of a photon, it was the setting of the effective angle of orientation of the polarizers. When the Geneva group of Tittel measured the delay of a photon, it was the time difference in passing through the long or short arm of an interferometer [21]. When Fry et al. in Texas come to measure the spin of their atoms, it will be the polarization angle of the laser that excites each atom [8]. In each case, only a finite number of possible angles are used, and these can be labelled by an integer.

In all experiments to date, the classical outputs occur in particle detectors. Suppose there are $\mathcal{N}(A,d)$ detectors in subsystem $A$ and $\mathcal{N}(B,d)$ in $B$. For each run of the experiment, a given detector may or may not fire, so altogether the number of possible outputs for the whole system is

$$\mathcal{N}(a)\mathcal{N}(b) = 2^{\mathcal{N}(A,d)+\mathcal{N}(B,d)}.$$  \hspace{1cm} (6)
In practice some of the outputs for the whole system may have negligible probability, in which case they could be ignored, and the effective number of outputs reduced. But this simplification complicates the use of the computer program and is not recommended.

To use the program, first establish the notation. For each subsystem, label the detectors by an integer $d$, where

$$d = 0, \ldots, \mathcal{N}(A, d) - 1$$

for subsystem $A$, and

$$d = 0, \ldots, \mathcal{N}(B, d) - 1$$

for subsystem $B$. 

(7)

(8)

For one detector the output is a binary digit $f$, which is equal to one if the detector fires in a given run, and zero otherwise. For a single run we denote this output for the detector labelled $d$ of subsystem $A$ by $f(A, d)$, and similarly for $B$. The output for the subsystem $A$ or $B$ is represented by the integer

$$a = \sum_{d=0}^{\mathcal{N}(A, d)-1} f(A, d)2^d$$

or

(9)

$$b = \sum_{d=0}^{\mathcal{N}(B, d)-1} f(B, d)2^d$$

(10)

and the total number of possible outputs for each subsystem is

$$\mathcal{N}(a) = 2^{\mathcal{N}(A, d)} \quad \text{or} \quad \mathcal{N}(b) = 2^{\mathcal{N}(B, d)}.$$ 

(11)

If required, the output $(a, b)$ for the whole system can be represented by the single integer $\mathcal{N}(b)a + b$.

**The experiment and the computer program**

Bell experiments test Bell inequalities or one of its generalizations. Every Bell experiment has classical inputs and outputs. The inputs are the settings of the apparatus that determine the measurements that are to be made. Typically these settings are orientations of polarizers. The outputs are the detections of particles. Typically these are electron avalanches produced by incident photons, or possibly detection of ions or electrons.

The settings must be sufficiently late in time, and the detections sufficiently early, that there are limits on the possible propagation of signals between them. The inputs and outputs belong to a finite number $\mathcal{N}(s)$ of subsystems, such that for any one of these subsystems, no signal can go from any input or output.
of one subsystem to any input or output of another. For example in the original Bell experiment [1], or the CHSH experiment [9], $N(s) = 2$, for entanglement-swapping $N(s) = 3$, and for Greenberger-Horne-Shimony-Zeilinger (GHZ) experiments either $N(s) = 3$ or $N(s) = 4$ [22]. For the ideal GHZ experiments, weak nonlocality follows from equalities, but for the real experiments, these become inequalities.

In general, experiments that test weak nonlocality by means of Bell inequalities can be classified by a string of positive integers:

$$(N(s), N(0, d), N(1, d), \ldots, N(N(s) - 1, d), N(i))$$

where $N(s)$ is the total number of subsystems, $N(k, d)$ is the maximum number of detectors in subsystem $k$, and $N(i)$ is the total number of settings, which we assume to be the same for all subsystems. In the first edition the program can be used to analyse only those experiments for which $N(s) = 2$ and therefore the integer triplet

$$(N(A, d), N(B, d), N(i))$$

characterizes any such setup. The generalization to an arbitrary number of subsystems, which will be required for later editions, is straightforward.

BellTest is a computer program. We will describe the first edition which performs two independent functions: (i) it provides a test for the violation of locality using raw experimental data, and (ii) it produces all the Bell inequalities for a given experimental setup.

The program operates in the following manner: For a given integer triplet of the form (12), the set of all possible inputs and outputs for this class of experiments is generated. Although the program is restricted to cases involving equal numbers of inputs, in general $N(A, d) \neq N(B, d)$ and can therefore handle unequal numbers of outputs. A particular experiment is defined by specifying the set of settings of the classical measurement apparatus. The program then proceeds to carry out functions (i) and (ii).

For function (i) one must specify a set of conditional probabilities, obtained from the raw experimental data, that may or may not satisfy the corresponding Bell inequalities. Nothing in the raw data must be neglected, failing to do so results in biased statistics. This test amounts to the linear programming problem of checking for the possibility to satisfy a given set of constraints. In our case the quantities to be varied are the transfer probabilities (5), subject to the equalities that the linear combination must be the same as the given conditional probability, their sum must be unity, and that every transfer probability must be greater than zero. The linear programming analysis is done using a freely available software called LP Solve [23].

Mathematically, such a space is constrained to exist inside a unit-hypercube; that being the generalization of the ordinary cube to dimensions greater than
3, and with vertices whose coordinates are composed of 0/1 entries only. Geometrically, the conditional probability space is a bounded convex polyhedron, otherwise known as a polytope, which is given as the convex hull of its vertices. Each vertex has coordinates that are strings of 0’s and 1’s and represent transfer functions of deterministic systems. For a general account of polytope theory the reader is advised to read Ziegler’s excellent text on the topic [24].

Polytopes can be described in terms of their vertices ($V$-representation), or in terms of equations and facet-defining inequalities ($H$-representation), where a facet is a surface of dimension one less than that of the polytope itself. The generalized Bell inequalities define such surfaces. The problem of transforming from the $V$-representation to the $H$-representation is called facet enumeration and the reverse of this is known as vertex enumeration. We are interested in the former. Computational geometers have developed many tools for the analysis of polytopes, and we make use of a software called Polyhedron Representation Transformation Algorithm (PORTA) [25] to generate all the Bell inequalities for a given experimental setup. However, polytopes of this type are infested with large numbers of facets, and as a result it may take a long time to enumerate them all [26].

An Example

Let us consider an example so as to put BellTest into perspective. Suppose that our experiment is of the class (2,2,2). The integer triplet has $N(i) = 2$, which defines two possible apparatus settings for each subsystem. These are labeled 0 and 1. In general, for $N(i)$ settings we define the following integer set of possible apparatus settings $0, 1, \ldots, N(i) - 1$. Each subsystem is assigned the same set of integers. However the set of integers for subsystem $A$ do not necessarily define the same set of apparatus settings as defined by the integer set assigned to subsystem $B$. For example, Clauser, Horne, Shimony and Holt [9] proposed an experiment where each adjustable apparatus has two possible settings, that is subsystem $A$ has settings $a$ and $a'$, and subsystem $B$ has settings $b$ and $b'$. With BellTest, subsystem $A$ has settings 0, representing $a$, and 1, representing $a'$. Subsystem $B$ also has settings 0 and 1 but here they represent $b$ and $b'$ respectively. For the system as a whole, that is $A + B$, there are four inputs $(\alpha, \beta)$, which define the set of possible experiments that we may carry out.

\[
(0, 0) \quad (0, 1) \\
(1, 0) \quad (1, 1)
\]

Suppose that in our experiment the adjustable apparatus of each subsystem is prepared such that we only investigate setups (0,0), (0,1), and (1,1). For completion, let us suppose that $\alpha$ and $\beta$ are the orientations of a polarizer, perhaps a calcite crystal. Each subsystem has two detectors, each of which
has two states, triggered (labelled 1) and not triggered (labelled 0), that record the emergence or non-emergence of a photon in the ordinary or extraordinary ray. All possible combinations of detector events for each subsystem define an output. For the case at hand each subsystem has four possible outputs, which we label 0, 1, 2, and 3, and therefore, the whole system has 16 possible outputs \((a, b)\).

For this example, the space of all conditional probabilities is given by the convex hull of 256 transfer functions. Using PORTA, BellTest lists 48 inequalities, some of which are of the Bell type.

Given raw data in mode (i), BellTest checks if a given set of conditional probabilities satisfy the corresponding Bell inequalities.

**Closing Remarks**

In this letter we have described a tool to test for violations of the critical Bell inequalities and therefore the closure of the detection loophole for any Bell experiment with two subsystems. The analysis involves making use of the entire raw data set. In addition, the user can use BellTest to list all the Bell inequalities if desired, but it must be stressed that the number of such inequalities grows very rapidly with respect to the number of transfer functions representing deterministic systems for a particular class of experiments (12). We invite anyone who is interested in using BellTest to download the first edition from http://www.strings.ph.qmw.ac.uk/QI/main.htm.

**Acknowledgements**

We thank Ed Fry, Nicolas Gisin, Serge Haroche, Monique Laurent, Boris Tsirelson, Thomas Walther and Ting Yu for helpful and stimulating communications and the Leverhulme Trust, the European Science Foundation and PPARC for support.

**References**

[1] J.S. Bell, Physics 1 (1964) 195.
[2] I.C. Percival, Phys. Lett. A 244 (1998) 495.
[3] P. Pearle, Phys. Rev. D 2, (1970) 1418.
[4] E. Santos, Phys. Rev. A 46 (1992) 3646.
[5] P.G. Kwiat, P.H. Eberhard, A.M. Steinberg, and R.Y. Chiao, Phys. Rev. A 49 (1994) 3209.
[6] E.S. Fry, T. Walther and S. Li, Phys. Rev. A 52 (1995) 4381.

[7] N. Gisin, and B. Gisin, Phys. Lett. A (1999) 323.

[8] E.S. Fry and T. Walther, in Adv. Atom. Molec. and Opt. Phys. 42 (Academic Press, New York, 2000) 1.

[9] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Phys. Rev. Lett. 23 (1969) 880.

[10] J.F. Clause, M.A. Horne, Phys. Rev. D 10 (1974) 526.

[11] A. Garg and N.D. Mermin, Found. Phys. 14 (1984) 1.

[12] A. Garg and N.D. Mermin, Phys. Rev. D 35 (1987) 3831.

[13] M. Zukowski, D. Kaszlikowski, A. Baturo, J. A. Larsson, quant-ph/9910058.

[14] D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski and A. Zeilinger, quant-ph/0005028.

[15] I. Pitowski and K. Svozil, quant-ph/0011060.

[16] E. Wigner, Am. J. Phys. 38 (1970) 1005.

[17] M. Froissart, Nuovo Cimento B 64, 241 (1981).

[18] I. Pitowsky, Quantum Probability - Quantum Logic. Lecture Notes in Physics 321 (Springer-Verlag, Berlin, 1989).

[19] I.C. Percival, quant-ph/9906005.

[20] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49 (1982) 1804.

[21] W. Tittel, J. Brendel, H. Gisin, and H. Zbinden, Phys. Rev. A 59 (1999) 4150.

[22] D.M. Greenberger, M.A. Horne and A. Zeilinger, in Bell’s Theorem, Quantum Theory and Conceptions of the Universe, ed. M. Kafatos (Kluwer Academic Publishers, 1989) 69.

[23] M. Berkelaar, LP_Solve Version 3, ftp://ftp.es.ele.tue.nl/pub/lp_solve/.

[24] G.M. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics 152 (Springer-Verlag, New York, Berlin Heidelberg, 1995).

[25] T. Christof and A. Lobel, PORTA,http://elib.zib.de/pub/Packages/mathprog/polyth/porta/.

[26] G.M. Ziegler, Lectures on 0/1-Polytopes, http://www.math.tu-berlin.de/~ziegler/.