The Probability of a Run

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Abstract
We deduce the explicit formula for the probability of a run of \( r \) successes in \( n \) trials.

1 Introduction

A famous problem in classical probability was first stated in De Moivre’s treatise, *The Doctrine of Chances* [2] as Problem LXXIV:

“To find the Probability of throwing a Chance assigned a given number of times without intermission, in any given number of Trials.”

We formulate this more explicitly as follows:

In a series of independent trials, an event \( E \) has the constant probability \( p \). If, in this series, \( E \) occurs at least \( r \) times in succession, we say that there is a run of \( r \) successes. What is the probability of having a run of \( r \) successes in \( n \) trials, where naturally \( n > r \)?

Let us denote by \( y_n \) the unknown probability of a run of \( r \) in \( n \) trials. The classical solution to the problem, (Feller [3], Uspensky [11]), consists of deducing the following difference equation for the complementary probability \( z_n := 1 - y_n \):

\[
z_{n+1} - z_n + q p^r z_{n-r} = 0
\]

(1)

where \( q := 1 - p \), and then concluding that the generating function:

\[
G(x) := z_0 + z_1 x + z_2 x^2 + \cdots + z_n x^n + [\cdots]
\]

(2)

is, in fact, the rational function

\[
G(x) = \frac{1 - p x^r}{1 - x + q p^r x^{r+1}}
\]

(3)

The coefficient of \( x^n \) gives the general formula for \( z_n \). We will prove:
Theorem 1. Let

\[ \beta_{n,r} := \sum_{l=0}^{\lfloor \frac{n}{r+1} \rfloor} (-1)^l \binom{n-lr}{l} (q^r)^l \]  

(4)

where \([\alpha] := \text{the greatest integer contained in } \alpha\). Then

\[ z_n = \beta_{n,r} - p^r \beta_{n-r,r} \]  

(5)

This explicit formula for \(z_n\), which one would expect to be at least as famous as the problem, itself, is amazingly hard to find in the literature. Although Todhunter [9] details the solutions of De Moivre, Condorcet, and Laplace, none of them gives the formula, although they all give versions of the generating function.

Nor is it to be found in the classical text of Markoff [10].

When we looked at the modern texts of Chung [1], Feller [3], Gnedenko [4], Hoel [5], Parzen [7], Ross [8], Tucker [10], and Uspensky [11], we were able to find only a statement, and that \textit{without proof}, of Theorem 1, only in Uspensky [11]. The most detailed presentation of the theory of runs on the internet is to be found in Weisstein [12], but the explicit formula is not even mentioned there!

Uspensky, (11, page 79), states that the formula can be found “. . . according to the known rules.” Following Feller (3, pp. 275-276), if we write

\[ G(x) = \frac{U(x)}{V(x)} \]  

(6)

then the formula for \(z_n\) is:

\[ z_n = \frac{\rho_1}{x_1^{n+1}} + \frac{\rho_2}{x_2^{n+1}} + \cdots + \frac{\rho_{r+1}}{x_{r+1}^{n+1}} \]  

(7)

where \(x_1, x_2, \cdots, x_{r+1}\) are the \(r+1\) distinct roots of \(V(x) = 0\) and

\[ \rho_k = -\frac{U(x_k)}{V'(x_k)}. \]  

(8)

Unfortunately, the equation \(V(x) = 0\) cannot, in general, be solved explicitly for its \(r+1\) roots, and so “. . . the known rules” are useless in this case.

We therefore offer the following simple derivation of the formula for \(z_n\), based on the binomial theorem.
2 Proof of Theorem 1

It suffices to prove the formula for $\beta_{n,r}$ since the formula for $z_n$ is an immediate consequence of it. Now,

$$\sum_{n=0}^{\infty} \beta_{n,r} x^n = \frac{1}{1 - x + qp^r x^{r+1}}$$

$$= \frac{1}{1 - x(1 - qp^r x^r)}$$

$$= \sum_{k=0}^{\infty} \{x(1 - qp^r x^r)\}^k$$

$$= \sum_{k=0}^{\infty} x^k \sum_{l=0}^{k} \binom{k}{l} (-qp^r x^r)^l$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (qp^r)^l x^{rl+k}.$$  

We must now determine the coefficient, $\beta_{n,r}$, of $x^n$ in this series. Thus, for fixed $r$ and $n$, we must find all pairs of integers $(l, k)$, with $0 \leq l \leq k$ which satisfy the equation:

$$rl + k = n, \quad (9)$$

since each such pair contributes the summand

$$(-1)^l \binom{k}{l} (qp^r)^l = (-1)^l \binom{n-lr}{l} (qp^r)^l \quad (10)$$

to the final value of $\beta_{n,r}$. By inspection we note that

$$n = r \cdot 0 + n$$
$$= r \cdot 1 + (n - r)$$
$$= r \cdot 2 + (n - 2r)$$
$$= r \cdot 3 + (n - 3r)$$
$$= \cdots$$
$$= (n - r) + r \cdot 1.$$  

and these equations correspond to the pairs

$$(l, k) = (0, n), (1, n-r), (2, n-2r), \cdots, (n-r, r),$$
respectively. We do not include \((l,k) = (n,0)\) since the corresponding summand has the value 0.

By (9) and the final equation in the list above, the largest value of \(l\) occurs when \(l = k\) and thus satisfies the equation:

\[
lr + r = n
\]

and we conclude that

\[
l = \left\lfloor \frac{n}{r+1} \right\rfloor.
\]

Therefore, \(l\) takes on the values 0, 1, 2, \(\cdots\), \(\left\lfloor \frac{n}{r+1} \right\rfloor\), and the proof is complete.

References

[1] **Chung, K.L.** *Elementary Probability Theory With Stochastic Processes*, Springer-Verlag, New York, 1979.

[2] **De Moivre, A.** *Doctrine of Chances*, Chelsea, New York, 1965.

[3] **Feller, W.** *An Introduction to Probability Theory and Its Applications, Third Edition, Vol. I*, John Wiley and Sons, New York, 1968.

[4] **Gnedenko, B.V.** *The Theory of Probability*, Chelsea, New York, 1962.

[5] **Hoel, P.** *An Introduction to Probability Theory*, Houghton Mifflin Company, New York, 1971.

[6] **Markoff, A.A.** *Wahrscheinlichheitsrechnung*, translated from the second Russian edition, Teubner, Leipzig and Berlin, 1912.

[7] **Parzen, E.** *Modern Probability and Its Applications*, John Wiley and Sons, New York, 1992.

[8] **Ross, F.M.** *Introduction to Probability and Statistics for Engineers*, John Wiley and Sons, New York, 1987.

[9] **Todhunter, I.** *A History of the Mathematical Theory of Probability: From the Time of Pascal to that of Laplace*, Macmillan, New York, 1865.

[10] **Tucker, H.G.** *A Graduate Course in Probability*, Academic Press, New York, 1967.

[11] **Uspensky, J.V.** *Introduction to Mathematical Probability*, McGraw-Hill, New York, 1937.

[12] **Weisstein, E.W.** *Run*, MathWorld, A Wolfram Web Resource, [http://mathworld.wolfram.com/Run.html](http://mathworld.wolfram.com/Run.html)