Three-block exceptional collections over Del Pezzo surfaces

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Abstract

We study complete exceptional collections of coherent sheaves over Del Pezzo surfaces, which consist of three blocks such that inside each block all Ext groups between the sheaves are zero. We show that the ranks of all sheaves in such a block are the same and the three ranks corresponding to a complete 3-block exceptional collection satisfy a Markov-type Diophantine equation that is quadratic in each variable. For each Del Pezzo surface, there is a finite number of these equations; the complete list is given. The 3-string braid group acts by mutations on the set of complete 3-block exceptional collections. We describe this action. In particular, any orbit contains a 3-block collection with the sum of ranks that is minimal for the solutions of the corresponding Markov-type equation, and the orbits can be obtained from each other via tensoring by an invertible sheaf and with the action of the Weyl group. This allows us to compute the number of orbits up to twisting.

Introduction

Recall that a sheaf $E$ is called \textit{exceptional} if $\text{Hom}(E, E) \cong \mathbb{C}$ and $\text{Ext}^i(E, E) = 0$ for $i > 0$. An ordered collection of sheaves $(E_1, \ldots, E_\alpha)$ is called \textit{exceptional} if all $E_j$’s are exceptional and $\text{Ext}^i(E_k, E_j) = 0 \ \forall i, \ \forall k > j$.

The theory of exceptional sheaves is being developed for ten years. For the first time, exceptional vector bundles over $\mathbb{P}^2$ appeared in the paper by J.-M. Drezet and J. Le Potier in 1985 [6]. In this paper, with the help of discrete parameters $(r, c_1, c_2)$ of exceptional vector bundles, the boundary of the set of these parameters for semistable sheaves over $\mathbb{P}^2$ was constructed (here, $r$ is the rank, $c_1, c_2$ are the Chern classes). In the subsequent papers by Drezet [7–9], exceptional bundles were used in studying the moduli space of semistable sheaves over $\mathbb{P}^2$.

In another direction, the theory of exceptional sheaves was being developed in Moscow by the participants of the seminar of Profs. A. N. Rudakov and A. N. Tyurin. Originally, a general setting of the problem was to describe in a reasonable way the set of exceptional sheaves over a given variety. Towards this end, mutations of exceptional sheaves were used yielding new exceptional sheaves. Also, the notion of a helix was introduced that ensured the existence of mutations [3]. The first and most brilliant results were obtained for $\mathbb{P}^2$ [4]; namely, any exceptional sheaf is contained in some helix and any helix can be obtained from a helix consisting of invertible sheaves by a finite number of mutations. The possibility to obtain a given exceptional collection from a certain canonical-form elementary one (e.g., for the case of $\mathbb{P}^2$), from that consisting of invertible

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sheaves) by mutations is called *constructivity*. The proof of the constructivity of helices over $\mathbb{P}^2$ is based on the following fact: The ranks $x, y, z$ of three successive bundles in a helix obey the Markov equation

$$x^2 + y^2 + z^2 = 3xyz; \quad (1)$$

moreover, helix mutations exactly correspond to numerical mutations of the equation solutions. As for the solutions in positive integers, A. A. Markov [15] has shown that they can be reduced by mutations to the solution $(1, 1, 1)$ with the minimum sum $x + y + z$.

This correspondence seems to be unique in some sense since for all known cases except $\mathbb{P}^2$, there is no universal Diophantine equation on the ranks of elements of a foundation of a helix. Therefore, to examine the constructivity problem for other varieties, different methods were used. Thus, the proof of the constructivity of helices and exceptional bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ due to Rudakov [19] was based on geometric constructions on the plane Pic$(\mathbb{P}^1 \times \mathbb{P}^1) \otimes \mathbb{Q}$, these constructions being so subtle that they could hardly be employed for other varieties. In the same paper, symmetric helices were studied, i.e., those invariant under the involution which transposes the systems of generators of $\mathbb{P}^1 \times \mathbb{P}^1$. Of four bundles forming a foundation of a helix, two are invariant under this involution and two others are mapped onto another. Let $x$ and $y$ be the ranks of the first two bundles and $z$ be the rank of two others. Then, as is shown in [19], the triple $(x, y, z)$ is a solution of the Diophantine equation

$$x^2 + y^2 + 2z^2 = 4xyz. \quad (2)$$

All positive integer solutions of this equation can also be reduced by mutations to $(1, 1, 1)$.

Using same methods, S. Yu. Zyuzina [10] proved the constructivity of exceptional pairs over $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the fact that any exceptional pair is contained in some helix and, therefore, can be obtained by mutations of invertible sheaves.

One of the authors devised the technique allowing one to prove constructivity of not exceptional collections themselves, but their images in the Grothendieck group $K_0$. This technique proved to be especially fruitful for varieties with $\text{rk} K_0 = 4$, namely, rational ruled surfaces, $\mathbb{P}^3$, 3-dimensional quadric, and Fano 3-folds $V_5$ and $V_{22}$ (see [16, 17]).

Presently, the approach due to S. A. Kuleshov using rigid ($\text{Ext}^1(F, F) = 0$) and superrigid ($\text{Ext}^i(F, F) = 0$, $i > 0$) sheaves seems to be most effective. Thus, over a Del Pezzo surface, any rigid sheaf is isomorphic to a direct sum of exceptional sheaves. Basing on this, the constructivity of exceptional collections of any length over a Del Pezzo surface is proved in [11]. The extension of this result to the case of a surface with anticanonical class free of basis components is presented in [12]. With the help of this approach, Kuleshov obtained a new solution of the constructivity problem for $\mathbb{P}^2$ not employing the Markov equation (see [13]). This can open prospects in investigating the case of $\mathbb{P}^n$.

The aim of the present paper is to describe such classes of exceptional collections over Del Pezzo surfaces, for which Markov-type equations exist that play the same role as equations (1) and (2) play for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. These are 3-block exceptional collections.

An exceptional collection $\mathcal{E} = (E_1, \ldots, E_\alpha)$ such that $\text{Ext}^i(E_j, E_k) = 0$ $\forall i$, $\forall j \neq k$, is called a block. In Sec. 1, we show that the ranks of all sheaves contained in a block are equal; this number will be denoted by $r(\mathcal{E})$. A 3-block collection is an exceptional collection

$$(\mathcal{E}, \mathcal{F}, \mathcal{G}) = (E_1, \ldots, E_\alpha, F_1, \ldots, F_\beta, G_1, \ldots, G_\gamma)$$
consisting of three blocks. The mutations of such collections preserving the 3-block structure are described in Sec. 2. These mutations define the action of the 3-string braid group on the set of 3-block collections (see Sec. 2.6).

The subject of our further study is not all 3-block collections but only complete ones, i.e., those generating the derived category of coherent sheaves over a surface under consideration. The main result of Sec. 3 is that the ranks \( x = r(E) \), \( y = r(F) \), \( z = r(G) \) of a complete 3-block collection \((E, F, G)\) obey the Markov-type equation

\[
\alpha x^2 + \beta y^2 + \gamma z^2 = \sqrt{K^2 \alpha \beta \gamma xyz},
\]

where \( K^2 \) is the self-intersection index of the canonical class of a surface. As is readily seen, the coefficients in the left-hand side are the numbers of sheaves in the blocks \( E \), \( F \), \( G \). The coefficient \( \sqrt{K^2 \alpha \beta \gamma} \) in a right-hand side is an integer. This imposes certain restrictions on \( \alpha \), \( \beta \), and \( \gamma \), thereby making the complete list of possible equations (3) for Del Pezzo surfaces quite limited. The list is given in Sec. 3.5. There, we also show that any positive integer solution of each equation (3) can be obtained by mutations from the minimum solution, i.e., the solution with the minimum sum \( x + y + z \). This means that any 3-block collection with given \( \alpha \), \( \beta \), and \( \gamma \) can be obtained by mutations preserving the 3-block structure from the collection with the same \( \alpha \), \( \beta \), and \( \gamma \) and minimum sum of ranks. Note that the equations (1) and (2) are particular cases of (3).

In Sec. 4, we show that for each equation (3), a complete 3-block collection with given \( \alpha \), \( \beta \), and \( \gamma \) exists which corresponds to the minimum solution of this equation. Moreover, any two such collections can be obtained one from another by action of the Weyl group and tensoring by an invertible sheaf (Sec. 5). This implies the following: Consider the orbits of the action of the braid group on the set of complete 3-block collections. If one joins into one class the orbits that differ by twisting only, the number of classes corresponding to a fixed equation (3) will be finite. These numbers are computed in Sec. 5.7.

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1. Preliminaries

1.1. Notations and agreements. In what follows, \( S \) is a Del Pezzo surface over \( \mathbb{C} \). By definition (see [14, Chapter IV]), \( S \) is a birationally trivial smooth surface with ample anticanonical class \((-K_S)\).

For a coherent sheaf \( F \), we use the following discrete invariants: the rank and Chern classes \( r(F) \), \( c_1(F) \), \( c_2(F) \), the slope \( \mu(F) = \frac{c_1(F) \cdot (-K_S)}{r(F)} \), \( \nu(F) = \frac{c_1(F)}{r(F)} \) (a point in \( \text{Pic}(S) \otimes \mathbb{Q} \)), the degree \( d(F) = c_1(F) \cdot (-K_S) \).

For any coherent sheaves \( E \) and \( F \), denote by \( \chi(E, F) \) the alternated sum

\[
\chi(E, F) = \sum (-1)^i \dim \text{Ext}^i(E, F)
\]

which defines a bilinear form over \( K_0(S) \). For torsion-free sheaves, the relation

\[
\chi(E, F) = r(E)r(F) \left( \chi(O_S) + \frac{\mu(F) - \mu(E)}{2} + q(F) + q(E) - \frac{c_1(E) \cdot c_1(F)}{r(E)r(F)} \right)
\]

(4)
holds, which follows from the Riemann–Roch theorem. Here, \( q(F) = \frac{c_1^2(F) - 2c_2(F)}{2r(F)} = \frac{\text{ch}_2(F)}{r(F)} \). Note that for Del Pezzo surfaces, \( \chi(\mathcal{O}_S) = 1 \).

The existence of a locally free resolvent for any coherent sheaf and additivity of both sides of (4) imply the validity of this relation for any two coherent sheaves. Removing the parentheses, we deduce

\[
\chi(E, F) = r(E) r(F) \chi(\mathcal{O}_S) + \frac{r(E) d(F)}{2} - \frac{r(F) d(E)}{2} + r(E) \text{ch}_2(F) + r(F) \text{ch}_2(E) - c_1(E) \cdot c_1(F). 
\]

The relation (4) immediately implies the expression for the skew-symmetric part of the form \( \chi \),

\[
\chi_-(E, F) \triangleq \chi(E, F) - \chi(F, E) = r(E) r(F) (\mu(F) - \mu(E)).
\]

If at least one of the ranks \( r(E) \) and \( r(F) \) is zero, one can use another form of (6),

\[
\chi_-(E, F) = \left| \begin{array}{cc} r(E) & r(F) \\ d(E) & d(F) \end{array} \right|.
\]

We denote the bounded derived category of coherent sheaves over \( S \) by \( D^b(S) \). Additive functions (in particular, \( r \) and \( d \)) have natural extensions over \( D^b(S) \). Namely, if an object \( A \) of the derived category is represented by a complex \( K \) with cohomology sheaves \( H^i(K) \) and \( s \) is an additive function, then

\[
s(A) = \sum_i (-1)^i s(K^i) = \sum_i (-1)^i s(H^i(K)).
\]

1.2. Lemma. Let \( 0 \to E \to F \to G \to 0 \) be an exact sequence of sheaves over \( S \). Then \( \chi_-(E, F) = \chi_-(F, G) = \chi_-(E, G) \).

The statement of the lemma immediately follows from (7).

1.3. Exceptional sheaves. Recall that a sheaf \( E \) is called exceptional if \( \text{Hom}(E, E) \cong \mathbb{C} \), \( \text{Ext}^i(E, E) = 0 \), \( i > 0 \).

In [11], it is proved that an exceptional sheaf over a Del Pezzo surface either is locally free, or is a torsion sheaf of the form \( \mathcal{O}_\ell(m) \), where \( m \in \mathbb{Z} \) and \( \ell \) is an exceptional curve (or \((-1\)-curve)), i.e., an irreducible rational curve with \( \ell^2 = \ell \cdot K_S = -1 \).

An exceptional vector bundle is Mumford–Takemoto stable with respect to \((-K_S)\), and therefore, is uniquely determined up to an isomorphism by its point \( \nu \) (see [5]). On the other hand, computation shows that \( \text{ch}_2(\mathcal{O}_\ell(m)) = \frac{1}{2} - m \). Thus, the statement below is valid.

Proposition. An exceptional sheaf over a Del Pezzo surface \( S \) is uniquely determined up to an isomorphism by its image in \( K_0(S) \).

The notion of exceptionality is naturally extended to the derived category. An object \( A \in D^b(S) \) is called exceptional if \( \text{Hom}^0_{D^b(S)}(A, A) \cong \mathbb{C} \), \( \text{Hom}^i_{D^b(S)}(A, A) = 0 \), \( i \neq 0 \). It is known (see [11]) that an object of a derived category over a Del Pezzo surface is exceptional if and only if it is isomorphic to \( \delta E[i] \), where \( E \) is an exceptional sheaf, \( \delta \) denotes the canonical embedding of the ground category in the derived one, \( [i] \) denotes the translation in \( D^b(S) \).

Define the slope of an exceptional torsion sheaf as

\[
\mu(\mathcal{O}_\ell(m)) = +\infty.
\]
1.4. Lemma. Let $E$ and $F$ be exceptional sheaves over $S$. Then
\[ \operatorname{sgn} \chi_-(E, F) = \operatorname{sgn} (\mu(F) - \mu(E)). \]

**Proof.** For $r(E)r(F) \neq 0$, the statement immediately follows from (6). If $r(E) = 0$, i.e., $E = \mathcal{O}_\ell(m)$, we have $d(E) = -K \cdot \ell = 1$, and (7) implies $\chi_-(E, F) = -r(F)$. Similarly, for $r(F) = 0$, we have $\chi_-(E, F) = r(E)$. Clearly, this implies the desired equality.

1.5. Exceptional collections. Recall that an ordered collection of sheaves $(E_1, \ldots, E_n)$ is called exceptional if all $E_j$'s are exceptional sheaves, and for $1 \leq j < k \leq n$, $\operatorname{Ext}^i(E_k, E_j) = 0$ for any $i$. Similarly, an exceptional collection of objects of $D^b(S)$ is defined.

An exceptional collection of sheaves such that for $j \neq k$, $\operatorname{Ext}^i(E_j, E_k) = 0$ for any $i$ is called a block. Evidently, a collection obtained from a block by a permutation of its elements is also a block.

**Definition.** An $m$-block collection is an exceptional collection
\[(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m) = (E_{11}, \ldots, E_{1\alpha_1}, E_{21}, \ldots, E_{2\alpha_2}, \ldots, E_{m1}, \ldots, E_{ma_m})\]
such that all its subcollections $\mathcal{E}_i = (E_{i1}, \ldots, E_{i\alpha_i})$ are blocks. The type and structure of such a collection are, respectively, the ordered and unordered collections of numbers $(\alpha_1, \alpha_2, \ldots, \alpha_m)$. We will sometimes call $\alpha_i$ the length of $\mathcal{E}_i$.

1.6. Proposition. For a block $(E_1, \ldots, E_{\alpha})$,
(a) $r(E_1) = \ldots = r(E_\alpha)$;
(b) $d(E_1) = \ldots = d(E_\alpha)$;
(\text{these invariants are denoted by } r(\mathcal{E}) \text{ (the rank of the block)} \text{ and } d(\mathcal{E}) \text{ respectively.})
(c) if $r(\mathcal{E}) = 0$, then $\mathcal{E} = (\mathcal{O}_{\ell_1}(D), \ldots, \mathcal{O}_{\ell_\alpha}(D))$, where $\ell_i$'s are pairwise nonintersecting exceptional curves;
(d) the divisors $c_{ij} = c_1(E_i) - c_1(E_j)$, $i \neq j$, satisfy the relations $c_{ij}^2 = -2$ and $c_{ij} \cdot K_S = 0$.

**Proof.** The definition immediately implies that $\chi_-(E_j, E_k) = 0$ $\forall j \neq k$.

If $r(E_j) = 0$, then $E_j \cong \mathcal{O}_\ell(m)$, and $c_1(E_j) = \ell$ is a $(-1)$-curve, whence $d(E_j) = 1$. Therefore, by (7), $E_k$ cannot be locally free and thus has the same invariants $r = 0$ and $d = 1$.

Next, assume that $E_j$ is locally free. Then $E_k$ also is, and by (6),
\[ \frac{d(E_i)}{r(E_i)} = \frac{d(E_k)}{r(E_k)} . \]

The restriction of the exceptional bundle $E$ to a smooth elliptic curve from the linear system $| - K_S |$ is a simple bundle of degree $d(E)$ (see, e.g., [11]), and the rank and degree of a simple bundle over an elliptic curve are coprime [1]. Hence, the validity of (a) and (b) follows.

Let us prove (c). We have $\mathcal{E} = (\mathcal{O}_{\ell_1}(m_1), \ldots, \mathcal{O}_{\ell_\alpha}(m_\alpha))$. The formula (5) under $r(E) = r(F) = 0$ yields $\chi(E, F) = -c_1(E) \cdot c_1(F)$. Hence, $\ell_i \cdot \ell_j = 0$, $i \neq j$. Finally, $\mathcal{O}_{\ell_i}(m_i) \cong \mathcal{O}_{\ell_i}(D)$, where $D = -\sum_{i=1}^{\alpha} m_i \ell_i$.

To prove (d), consider two cases.
(i) $r(\mathcal{E}) = 0$. By (c), we have $c_{ij} = \ell_i - \ell_j$, where $\ell_i^2 = \ell_j^2 = \ell_i \cdot K_S = \ell_j \cdot K_S = -1$ and $\ell_i \cdot \ell_j = 0$. Now the desired statement is verified by direct computations.
(ii) $r(\mathcal{E}) = r \neq 0$. Then the equality $\chi(E_i, E_i) = 1$ together with (4) implies

$$q(E_i) = \frac{1}{2} \left( \frac{1}{r^2} + \frac{c_1(E_i)^2}{r^2} - 1 \right), \quad i = 1, \ldots, \alpha.$$ 

It follows from (a) and (b) that $\mu(E_i) = \mu(E_j)$. Then

$$0 = \chi(E_i, E_j) = r^2 \left( 1 + q(E_i) + q(E_j) - \frac{c_1(E_i) \cdot c_1(E_j)}{r^2} \right) = 1 + \frac{1}{2} \left( c_1(E_i) - c_1(E_j) \right)^2.$$ 

Therefore, $c_{ij}^2 = -2$. The equality $c_{ij} \cdot K_S = 0$ is a direct consequence of (b). The proposition is proved.

**Corollary.** Let $(\mathcal{E}, \mathcal{F}) = (E_1, \ldots, E_\alpha, F_1, \ldots, F_\beta)$ be a two-block exceptional collection. Then $\chi(E_j, F_k) = \chi_-(E_j, F_k)$ does not depend on $j \in \{1, \ldots, \alpha\}$ and $k \in \{1, \ldots, \beta\}$.

We denote this quantity by $\chi(\mathcal{E}, \mathcal{F})$.

**1.7. Mutations** (see [5]). Let $(A, B)$ be an exceptional pair of objects of $D^b(S)$. Consider an object $L^0_A B$ which completes the canonical morphism $R\mathrm{Hom}(A, B) \otimes A \xrightarrow{\text{can}} B$ to a distinguished triangle

$$L^0_A B[-1] \rightarrow R\mathrm{Hom}(A, B) \otimes A \xrightarrow{\text{can}} B \rightarrow L^0_A B.$$  

(8)

It is known [5] that $(L^0_A B, A)$ is an exceptional pair in $D^b(S)$.

**Definition.** The left mutation is the mapping $(A, B) \mapsto (L^0_A B, A)$ of the set of exceptional pairs of objects of $D^b(S)$ onto itself.

The left mutation of a pair $(A, B)$ is the pair $(L^0_A B, A)$. The object $L^0_A B$ is referred to as the result of a mutation, result of a shift, or just a (left) shift of $B$ over $A$.

Next, let $(E, F)$ be an exceptional pair of sheaves over $S$. It is known [11, 2.11] that among the spaces $\mathrm{Ext}^i(E, F)$, either only $\mathrm{Hom}$, or only $\mathrm{Ext}^1$ can be nonzero. By the definition of an exceptional pair, $\mathrm{Ext}^i(F, E) = 0 \forall i$, whence

$$\chi(E, F) = \chi_-(E, F).$$

Using Lemma 1.4, we obtain the known [5] classification of exceptional pairs in terms of the slopes. Namely, any pair $(E, F)$ has one of the following types:

| hom-pair: | $\mathrm{Hom}(E, F) \neq 0$, $\mathrm{Ext}^i(E, F) = 0$, $i = 1, 2$, | $\iff \mu(E) < \mu(F)$; |
| ext-pair: | $\mathrm{Ext}^1(E, F) \neq 0$, $\mathrm{Ext}^i(E, F) = 0$, $i = 0, 2$, | $\iff \mu(E) > \mu(F)$; |
| zero-pair: | $\mathrm{Ext}^i(E, F) = 0 \forall i$ | $\iff \mu(E) = \mu(F)$. |

Thus, a distinguished triangle (8) for a pair $(A, B) = (\delta E, \delta F)$ is always reduced to one of the following exact triples:

$$0 \rightarrow L_E F \rightarrow \mathrm{Hom}(E, F) \otimes E \rightarrow F \rightarrow 0 \quad (\text{division});$$

$$0 \rightarrow \mathrm{Hom}(E, F) \otimes E \rightarrow F \rightarrow L_E F \rightarrow 0 \quad (\text{recoil});$$

$$0 \rightarrow F \rightarrow L_E F \rightarrow \mathrm{Ext}^1(E, F) \otimes E \rightarrow 0 \quad (\text{extension}).$$

Since the pair $(L^0_{\delta E} \delta F, \delta E)$ of objects of $D^b(S)$ is exceptional, the same is true for the pair of sheaves $(L_E F, E)$. 

Definition. The left sheaf mutation is the mapping \((E, F) \mapsto (L_E F, E)\) of the set of exceptional pairs of sheaves onto itself. Three types of sheaf mutations are distinguished, namely, \textit{division}, \textit{recoil}, and \textit{extension}, depending on which of the exact triples given above takes place.

Dually, right mutations are defined; if \((A, B)\) is an exceptional pair in \(D^b(S)\), then the object \(R^D_B A\) completes the canonical morphism \(A \to R\text{Hom}^*(A, B) \otimes B\) to the distinguished triangle

\[
R^D_B A \to A \to R\text{Hom}^*(A, B) \otimes B \to R^D_B A[1].
\]

The mapping \((A, B) \mapsto (B, R^D_B A)\) is called the right mutation. The pair \((B, R^D_B A)\) is also exceptional.

For an exceptional pair of sheaves \((E, F)\), the distinguished triangle above is reduced to one of the following triples:

- \((\text{division})\):
  \[
  0 \to E \to \text{Hom}(E, F)^* \otimes F \to R_F E \to 0
  \]

- \((\text{recoil})\):
  \[
  0 \to R_F E \to E \to \text{Hom}(E, F)^* \otimes F \to 0
  \]

- \((\text{extension})\):
  \[
  0 \to \text{Ext}^1(E, F)^* \otimes E \to R_F E \to E \to 0
  \]

The right sheaf mutation is the mapping \((E, F) \mapsto (F, R_F E)\) of the set of exceptional pairs onto itself.

The following obvious statement establishes the relation between mutations in the derived category and sheaf mutations.

1.8. Proposition. If a sheaf mutation of a pair \((E, F)\) is either recoil or extension, then

\[
L^D_E \delta F = \delta L_E F \quad \text{and} \quad R^D_F \delta E = \delta R_F E.
\]

For the case of division, \(L^D_E \delta F = \delta L_E F[1]\) and \(R^D_F \delta E = \delta R_F E[-1]\).

Further on, we identify the category of coherent sheaves over \(S\) with its image in \(D^b(S)\) under the canonical embedding. The symbol \(\delta\) is usually omitted. Under this agreement, the latter proposition reads that a sheaf mutation other than a division coincides with the mutation in \(D^b(S)\), and for a division-type mutation, the shift by \(\pm 1\) in \(D^b(S)\) occurs.

1.9. Proposition. The type of a left sheaf mutation is described in terms of slopes as follows:

- \((\text{division})\) \iff \(\mu(L_E F) < \mu(E) < \mu(F)\);
- \((\text{recoil})\) \iff \(\mu(E) \leq \mu(F) \leq \mu(L_E F)\);
- \((\text{extension})\) \iff \(\mu(F) \leq \mu(L_E F) \leq \mu(E)\).

Proof. For \(\mu(E) = \mu(F)\), i.e., \(\text{Ext}^i(E, F) = 0\ \forall i\), the left mutation of the pair \((E, F)\) can be regarded as either trivial recoil or trivial extension and is just the transposition of \(E\) and \(F\). Lemmas 1.2 and 1.4 yield \(\mu(L_E F) = \mu(E) = \mu(F)\).

Now, let \(\mu(E) \neq \mu(F)\). If \(\mu(E) < \mu(F)\), i.e., \((E, F)\) is a hom-pair, then the left mutation of this pair is either division or recoil. For the case of division, the correspondent exact triple shows that the pair \((L_E F, E)\) is also of hom type, whence the desired inequality follows. For a recoil,

\[
\chi_-(F, L_E F) \geq \chi_-(E \otimes \text{Hom}(E, F), F) = \text{dim Hom}(E, F) \cdot \chi_-(E, F) > 0,
\]

and it remains to apply Lemma 1.4.
Finally, if $\mu(E) > \mu(F)$, i.e., $(E, F)$ is an ext-pair, then the left mutation of this pair is an extension. The correspondent exact triple shows that $(L_E F, E)$ is a hom-pair, and

$$
\chi_-(F, L_E F) = \chi_-(F, \text{Ext}^1(E, F) \otimes E) = \dim \text{Ext}^1(E, F) \cdot \chi_-(F, E) > 0,
$$

which completes the proof.

1.10. Let $\tau = (E_1, \ldots, E_n)$ be an exceptional collection of sheaves (or objects of $D^b(S)$). It is known [2, 5], that the mappings

$$
\tau \mapsto (E_1, \ldots, E_{i-1}, L_{E_i}^{(D)} E_{i+1}, E_i, E_{i+2}, \ldots, E_n)
$$

and

$$
\tau \mapsto (E_1, \ldots, E_{i-1}, R_{E_i}^{(D)} E_i, E_{i+1}, E_{i+2}, \ldots, E_n),
$$

$i = 1, \ldots, n - 1$, result in exceptional collections. These mappings are called mutations of exceptional collections.

1.11. Helices. Consider a bi-infinite extension of an exceptional collection of sheaves $(E_1, \ldots, E_n)$ defined recursively by

$$
E_{i+n} = R_{E_{i+n-1}} \ldots R_{E_{i+1}} E_i, \quad E_{-i} = L_{E_{i-1}} \ldots L_{E_{n-1-i}} E_{n-i}, \quad i \geq 1.
$$

The sequence $\{E_m\}_{m \in \mathbb{Z}}$ thus constructed is called a (sheaf) helix of period $n$ if

$$
E_i = E_{n+i} \otimes K_S \quad \forall i \in \mathbb{Z}.
$$

A foundation of a helix is any its subcollection of the form $(E_{m+1}, \ldots, E_{m+n})$, where $m \in \mathbb{Z}$.

**Definition.** An exceptional collection $(E_1, \ldots, E_n)$ is called complete if it generates $D^b(S)$.

For collections over a variety with ample anticanonical class, it is known [2] that an exceptional collection is complete if and only if it is a foundation of a helix.

Below, we study complete 3-block exceptional collections over Del Pezzo surfaces.

2. Mutations of block collections

2.1. Let $(\mathcal{E}, \mathcal{F}) = (E_1, \ldots, E_\alpha, F_1, \ldots, F_\beta)$ be a two-block collection. Define the shift of $F_j$ over $\mathcal{E}$ as

$$
L_{\mathcal{E}} F_j = L_{E_1} \ldots L_{E_\alpha} F_j, \quad j = 1, \ldots, \beta.
$$

According to [5, Sec. 4], $\text{Ext}^i(L_{\mathcal{E}} F_j, L_{\mathcal{E}} F_k) = \text{Ext}^i(F_j, F_k) = 0 \forall i, \forall j \neq k$. Hence, the collection of sheaves

$$
L_{\mathcal{E}} \mathcal{F} \triangleq (L_{\mathcal{E}} F_1, \ldots, L_{\mathcal{E}} F_\beta)
$$

is a block; we will sometimes call it the shift of $\mathcal{F}$ over $\mathcal{E}$. Consider the sheaves

$$
R_{\mathcal{F}} E_i = R_{F_\beta} \ldots R_{F_1} E_i
$$

and the collection $R_{\mathcal{F}} \mathcal{E} = (R_{\mathcal{F}} E_1, \ldots, R_{\mathcal{F}} E_\alpha)$ which is a block too.
One can easily see that the two-block collections \((L_{\mathcal{E}} \mathcal{F}, \mathcal{E})\) and \((\mathcal{F}, R_{\mathcal{F}} \mathcal{E})\) are obtained from the initial collection \((\mathcal{E}, \mathcal{F})\) by a finite number of mutations in the sense of 1.10 and, therefore, are exceptional.

We will also use the block \(L^D_{\mathcal{E}} \mathcal{F}\) obtained as a result of replacing sheaf mutations in the definition of \((L_{\mathcal{E}} \mathcal{F}, \mathcal{E})\) with mutations in the derived category,

\[
L^D_{\mathcal{E}} \mathcal{F} \triangleq (L^D_{\mathcal{E}} F_1, \ldots, L^D_{\mathcal{E}} F_{\beta}), \quad \text{where} \quad L^D_{\mathcal{E}} F_j = L^D_{E_{\mathcal{E}} F_j}.
\]

**Definition.** The mappings \((\mathcal{E}, \mathcal{F}) \mapsto (L_{\mathcal{E}} \mathcal{F}, \mathcal{E})\) and \((\mathcal{E}, \mathcal{F}) \mapsto (\mathcal{F}, R_{\mathcal{F}} \mathcal{E})\) of the set of two-block collections onto itself are called, respectively, the *left and right mutations of two-block collections.*

**2.2. Proposition.** 1. The sheaf \(L_{\mathcal{E}} F_j\) is contained in one of the following exact triples:

\[
0 \to L_{\mathcal{E}} F_j \to \bigoplus_{i=1}^{\alpha} \left( \text{Hom} \left( E_i, F_j \right) \otimes E_i \right) \xrightarrow{\text{can}} F_j \to 0 \quad \text{(division)},
\]

\[
0 \to \bigoplus_{i=1}^{\alpha} \left( \text{Hom} \left( E_i, F_j \right) \otimes E_i \right) \xrightarrow{\text{can}} F_j \to L_{\mathcal{E}} F_j \to 0 \quad \text{(recoil)},
\]

\[
0 \to F_j \to L_{\mathcal{E}} F_j \to \bigoplus_{i=1}^{\alpha} \left( \text{Ext}^1 \left( E_i, F_j \right) \otimes E_i \right) \to 0 \quad \text{(extension)}.
\]

2. These three cases are described in terms of the discrete invariants as follows:

\[
\text{(division)} \iff \alpha \chi(\mathcal{E}, \mathcal{F}) r(\mathcal{E}) > r(\mathcal{F}),
\]

\[
\text{(recoil)} \iff \chi(\mathcal{E}, \mathcal{F}) \geq 0 \text{ and } \alpha \chi(\mathcal{E}, \mathcal{F}) r(\mathcal{E}) \leq r(\mathcal{F}),
\]

\[
\text{(extension)} \iff \chi(\mathcal{E}, \mathcal{F}) \leq 0,
\]

In particular, the type of the exact triple of item 1 does not depend on \(j \in \{1, \ldots, \beta\}\).

**Proof.** Identify the category of coherent sheaves over \(S\) with its image under the canonical embedding in \(D^b(S)\). Consider the direct sum of canonical morphisms

\[
\bigoplus_{i=1}^{\alpha} \left( \text{RHom} \left( E_i, F_j \right) \otimes E_i \right) \to F_j
\]

which can be completed to a distinguished triangle

\[
X[-1] \to \bigoplus_{i=1}^{\alpha} \left( \text{RHom} \left( E_i, F_j \right) \otimes E_i \right) \to F_j \to X. \quad (9)
\]

Applying the functor \(\text{RHom}(E_{i'}, \cdot)\) to it and taking into account that \(\text{RHom}(E_{i'}, E_i) = 0\) for \(i \neq i'\), we obtain the long exact sequence

\[
\ldots \to \text{Hom}^{k-1}_{D^b(S)}(E_{i'}, X) \to \text{Hom}^k_{D^b(S)}(E_{i'}, \text{RHom}(E_{i'}, F_j) \otimes E_{i'}) \xrightarrow{f_k} \text{Hom}^k_{D^b(S)}(E_{i'}, F_j) \to \text{Hom}^k_{D^b(S)}(E_{i'}, X) \to \ldots,
\]

where all \(f_k\)'s are canonical identities due to the exceptionality of \(E_{i'}\). Hence, all spaces \(\text{Hom}^k_{D^b(S)}(E_{i'}, X)\) are zero, i.e.,

\[
\text{RHom} \left( E_{i'}, X \right) = 0 \quad \forall i' \in \{1, \ldots, \alpha\}.
\]

(10)
Next, applying the functor $\text{RHom}(F_j, \cdot)$ to (9) and taking into account that $\text{RHom}(F_j, E_i) = 0 \ \forall i \in \{1, \ldots, \alpha\}$, we obtain the long exact sequence consisting of spaces $\text{Hom}^k_{D^b(S)}(F_j, \cdot)$, whence we conclude that

$$\text{Hom}^0_{D^b(S)}(F_j, F_j) \cong \text{Hom}^0(F_j, X) \cong \mathbb{C} \ \text{and} \ \text{Hom}^k_{D^b(S)}(F_j, X) = 0, k \neq 0. \quad (11)$$

Finally, applying $\text{RHom}(\cdot, X)$ to (9), we obtain $\text{Hom}^0_{D^b(S)}(X, X) \cong \text{Hom}^0(F_j, X) \cong \mathbb{C}$ and $\text{Hom}^k_{D^b(S)}(X, X) = 0, k \neq 0$, i.e., $X$ is an exceptional object in $D^b(S)$.

Let $T = \text{Tr}(E_1, \ldots, E_\alpha, F_j)$ be a complete triangulated subcategory in $D^b(S)$ generated by the corresponding collection of objects; it contains $T_\mathcal{E} = \text{Tr}(E_1, \ldots, E_\alpha)$. The distinguished triangle (9) and the equality (10) mean that $X$ belongs to the intersection of the right orthogonal subcategory $T^\perp_\mathcal{E}$ (see [2, Sec. 3]) and $T$, this intersection being generated by the exceptional object $L_\mathcal{E}F_j$. Since $X$ is exceptional, it is quasi-isomorphic to $L_\mathcal{E}F_j[p]$, the latter being a complex with $L_\mathcal{E}F_j$ in the $p$th position and zeroes in the others.

Therefore, any object in (9) has only one nonzero cohomology sheaf; hence, this distinguished triangle is reduced to one of the exact triples from the proposition statement. This proves item 1.

The assertions of item 2 are obvious due to Lemma 1.4, except only for the fact that under $\alpha \chi(\mathcal{E}, \mathcal{F}) \leq r(\mathcal{F})$, the division cannot take place. Let us show this. Indeed, if the contrary holds, $r(L_\mathcal{E}\mathcal{F}) = 0$, and hence, $r(\mathcal{E}) = 0$ since a torsion sheaf is never a subsheaf of a locally free one. Hence, $r(\mathcal{F}) = 0$ too. Then, by (7), $\chi(\mathcal{E}, \mathcal{F}) = 0$, i.e., the mutation is a trivial recoil and $L_\mathcal{E}F_j = F_j$. Note that in this case the collection $(\mathcal{E}, \mathcal{F})$ is actually a single block. The proposition is proved.

Dual reasoning easily proves the analogous facts for right mutations.

2.3. Proposition. 1. The sheaf $R_\mathcal{F}E_i$ is contained in one of the exact triples

$$0 \rightarrow E_i \xrightarrow{\text{can}} \bigoplus_{i=1}^\beta (\text{Hom}^*(E_i, F_j) \otimes F_j) \rightarrow R_\mathcal{F}E_i \rightarrow 0 \quad \text{(division)},$$

$$0 \rightarrow R_\mathcal{F}E_i \rightarrow E_i \xrightarrow{\text{can}} \bigoplus_{i=1}^\beta (\text{Hom}^*(E_i, F_j) \otimes F_j) \rightarrow 0 \quad \text{(recoil)},$$

$$0 \rightarrow \bigoplus_{i=1}^\beta (\text{Ext}^1(E_i, F_j)^* \otimes F_j) \rightarrow R_\mathcal{F}E_i \rightarrow E_i \rightarrow 0 \quad \text{(extension)}.$$

2. These three cases can be described in terms of the discrete invariants as follows:

$$\begin{align*}
(\text{division}) & \iff r(\mathcal{E}) \leq \beta \chi(\mathcal{E}, \mathcal{F}) r(\mathcal{F}), \\
(\text{recoil}) & \iff r(\mathcal{E}) > \beta \chi(\mathcal{E}, \mathcal{F}) r(\mathcal{F}) \geq 0, \\
(\text{extension}) & \iff \chi(\mathcal{E}, \mathcal{F}) \leq 0,
\end{align*}$$

In particular, the type of the exact triple of item 1 does not depend on $i \in \{1, \ldots, \alpha\}$.

Note that, for the simplest case where each of the blocks $\mathcal{E}$ and $\mathcal{F}$ consists of one sheaf, exact triples of Propositions 2.2 and 2.3 coincide with those of Sec. 1.7.

2.4. A mutation of a two-block collection $(\mathcal{E}, \mathcal{F})$ is called a division, recoil, or extension depending on the type of the corresponding exact triples given in the latter two propositions.
Our next aim is to show that a non-division-type mutation of a two-block collection coincides with the mutation in $D^b(S)$, and under a division, a shift of grading by ±1 occurs.

Below, precise statements for left mutations are presented; we leave the case of right mutation to the reader.

For convenience, renumber a block $E$ in the reverse order, $E = (E_\alpha, \ldots, E_1)$; then $L_\alpha F_j = L_{E_\alpha} \ldots L_{E_1} F_j$. Consider the sequence of sheaves

$$L^0 F_j = F_j, \quad L^i F_j = L_{E_i} \ldots L_{E_1} F_j, \quad i = 1, \ldots, \alpha.$$

Note that $L^\alpha F_j = L_\alpha F_j$ is the shift of $F_j$ over the block $E$, and $L^i F_j$ is the shift of $F_j$ over $E_i = (E_i, \ldots, E_1)$. Denote by $L^i$ the left (sheaf) mutation of the pair $(E_i, L^{i-1} F_j)$. Then the sheaf $L_\alpha F_j$ is a result of applying the sequence of mutations $L^1, \ldots, L^\alpha$.

2.5. Proposition. 1. If the left mutation of a two-block collection $(E, F)$ is not a division, then all mutations $L^i$, $i = 1, \ldots, \alpha$, are not divisions for any $F_j \in F$.

2. If the left mutation of $(E, F)$ is a division, then exactly one mutation in the sequence $L^i$ is a division.

Proof. Let $\chi(E, F) > 0$. Then, by 2.2, the left mutation of the two-block collection $(E, F)$ is either a division or a recoil. For the case of a division, we obtain by Lemma 1.2 that

$$\chi_-(L_{E} F_j, \alpha \sum_{i=1}^{\alpha} \text{Hom}(E_i, F_j) \otimes E_i) = \chi_-(\alpha \sum_{i=1}^{\alpha} \text{Hom}(E_i, F_j) \otimes E_i, F_j),$$

whence $\chi(L_E F, E) > 0$ and $\mu(L_E F) < \mu(E)$.

Consider the sequence $\mu_i = \mu(L^i F_j)$, where $\mu_0 = \mu(F) > \mu(E)$ and $\mu_\alpha = \mu(L_\alpha F) < \mu(E)$. Let $p$ be the least number where the change of the sign of $\mu_i - \mu(E)$ occurs, i.e.,

$$\mu_{p-1} - \mu(E) > 0 \quad \text{and} \quad \mu_p - \mu(E) < 0.$$

By Proposition 1.9, each mutation $L^i$, $1 \leq i \leq p - 1$, can be none other than a recoil, and $L^p$ is a division. Then, again by Proposition 1.9, all mutations $L^i$, $p + 1 \leq i \leq \alpha$, are extensions.

Similarly, for the case where the left block mutation of $(E, F)$ is a recoil, we have $\mu(L_E F) > \mu(F) > \mu(E)$. Proposition 1.9 implies that all $L^i$'s are recoils in this case.

Let now $\chi(E, F) < 0$. Computation shows that $\mu(F) < \mu(L_E F) < \mu(E)$, and by Proposition 1.9, all mutations $L^i$ are extensions.

Finally, for $\chi(E, F) = 0$, evidently, all $L^i$'s are trivial recoils (or trivial extensions). This completes the proof of the proposition.

Corollary. An object $X$ in the distinguished triangle (9) coincides with $L_E^p F_j$.

Proof. According to the proof of Proposition 2.2, $X$ is quasi-isomorphic to $L_E^p F_j[p]$ for some $p$. Applying the preceding proposition and 1.8, we obtain that $L_E^p F_j = L_\alpha F_j$ if the left mutation of the two-block collection $(E, F)$ is not a division, and $L_E^p F_j = L_\alpha F_j[1]$ otherwise. Examining the correspondence between the distinguished triangle (9) and exact triples of Proposition 2.2, we obtain the desired statement.

2.6. Action of the braid group. Let $\tau = (E_1, \ldots, E_m)$ be an $m$-block collection. Define left and right mutations of $m$-block collections as mappings of the set of $m$-block collections.
onto itself,
\[ L_i : \tau \mapsto (\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, L_{\mathcal{E}_i} \mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_m) \]
and
\[ R_i : \tau \mapsto (\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, R_{\mathcal{E}_i} \mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_m), \]
where \( i = 1, \ldots, m - 1 \). Note that the structure of an \( m \)-block collection defined in 1.5 is preserved under mutations \( L_i \) and \( R_i \). Hence, any orbit under the action of the braid group is contained in the set of \( m \)-block collections of a correspondent structure. We say that the mutations of \( \tau \) in the sense of 1.10 do not preserve the 3-block structure. The statement below is an analog of [2, 2.3] for block collections.

**Proposition.** 1. Mutations \( R_i \) and \( L_i \) are inverse, i.e., \( R_i \circ L_i = \text{id} \).

2. Right (and left) mutations define the action of the \( m \)-string braid group, i.e., the generating relations of the braid group hold,
\[ R_i \circ R_{i+1} \circ R_i = R_{i+1} \circ R_i \circ R_{i+1}, \quad L_i \circ L_{i+1} \circ L_i = L_{i+1} \circ L_i \circ L_{i+1}. \]

**Proof.** To prove item 1, it suffices to check that for a two-block collection \( (\mathcal{E}, \mathcal{F}) \), the equality \( R_\mathcal{E}(L_\mathcal{E} \mathcal{F}) = \mathcal{F} \) holds. This easily follows from the fact that left sheaf mutations are inverse to right ones.

To prove item 2, it suffices to check the following statement: If \( (\mathcal{E}, \mathcal{F}, \mathcal{G}) \) is a 3-block collection, then
\[ R_\mathcal{G}(R_\mathcal{F} \mathcal{E}) = R_{\mathcal{G} \mathcal{F} \mathcal{E}}. \]
Indeed, according to [2], the block in the left-hand side of the latter equality is the right shift of \( \mathcal{E} \) over the category \( \text{Tr}(\mathcal{F}, \mathcal{G}) = \text{Tr}(\mathcal{G}, R_\mathcal{G} \mathcal{F}) \) and does not depend on the choice of a basis in it.

**2.7.** In conclusion, note that one can use a “matrix notation” for mutations of two-block collections. For example, a division-type left mutation of \( (\mathcal{E}, \mathcal{F}) \) corresponds to the sequence
\[
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} \mapsto
\begin{pmatrix}
L_{\mathcal{E} F_1} \\
\vdots \\
L_{\mathcal{E} F_\beta}
\end{pmatrix} \mapsto
\begin{pmatrix}
\text{Hom}(E_1, F_1) & \ldots & \text{Hom}(E_\alpha, F_1) \\
\vdots & \ddots & \vdots \\
\text{Hom}(E_1, F_\beta) & \ldots & \text{Hom}(E_\alpha, F_\beta)
\end{pmatrix} \odot
\begin{pmatrix}
E_1 \\
\vdots \\
E_\alpha
\end{pmatrix} \mapsto
\begin{pmatrix}
F_1 \\
\vdots \\
F_\beta
\end{pmatrix} \mapsto
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
(12)
where the result of the “multiplication” \( \odot \) of the matrix \( \left( \text{Hom}(E_i, F_j) \right) \) by the column of sheaves is the column of sheaves with
\[
\left( \text{Hom}(E_1, F_j) \ldots \text{Hom}(E_\alpha, F_j) \right) \odot
\begin{pmatrix}
E_1 \\
\vdots \\
E_\alpha
\end{pmatrix} = \bigoplus_{i=1}^\alpha \left( \text{Hom}(E_i, F_j) \otimes E_i \right)
\]
in the \( j \)th position.

It is quite natural to denote a matrix consisting of vector spaces \( \text{Hom}(E_i, F_j) \) by \( \text{Hom}(\mathcal{E}, \mathcal{F}) \). Then the middle term in (12) takes the form \( \text{Hom}(\mathcal{E}, \mathcal{F}) \odot \mathcal{E} \), and exact sequences like (12) that correspond to various types of mutations of \( (\mathcal{E}, \mathcal{F}) \) can be obtained from the sequences of Sec. 1.7 (which define mutations of an exceptional pair of sheaves \( (E, F) \)) by replacing \( E \), \( F \), and the symbol of tensor product with \( \mathcal{E} \), \( \mathcal{F} \), and \( \odot \).
3. Markov-type equations for complete 3-block collections

In this section, we always assume

\[(\mathcal{E}, \mathcal{F}, \mathcal{G}) = (E_1, \ldots, E_\alpha, F_1, \ldots, F_\beta, G_1, \ldots, G_\gamma)\]

to be a complete 3-block collection of sheaves over a Del Pezzo surface \(S\) (in the sense of 1.5). The completeness is equivalent to the fact that this ordered collection of sheaves is a foundation of a helix (see 1.11).

3.1. Consider a \(\mathbb{Z}\)-module \(K_0(S)\) with the bilinear form \(\langle x, y \rangle = \chi(x, y)\). Let \(\lambda : K_0(S) \rightarrow K_0(S)^*\) be defined as \(\lambda x = \langle \cdot, x \rangle\). For any additive functions \(s\) and \(t\) on \(K_0(S)\), define

\[\langle s, t \rangle \triangleq (\lambda^{-1}s, \lambda^{-1}t).\]  

Consider the additive functions \(r\) and \(d\), where \(r\) is the rank and \(d(U) = c_1(U) \cdot (-K_S)\). Then \(\lambda^{-1}r = \mathcal{O}_p\), the latter being a structure sheaf of a point. Direct computations show that \(\lambda^{-1}d\) lies in the linear span of \(\mathcal{O}_{-K_S}\) and \(\mathcal{O}_p\). Therefore,

\[\langle r, r \rangle = \chi(\mathcal{O}_p, \mathcal{O}_p) = r(\mathcal{O}_p) = 0,\]
\[\langle r, d \rangle = \langle \mathcal{O}_p, \lambda^{-1}d \rangle = d(\mathcal{O}_p) = 0,\]
\[\langle d, r \rangle = r(\lambda^{-1}d) = 0.\]

Let \((e_1, \ldots, e_n)\) be a semiorthogonal basis of \(K_0(S)\) (i.e., \(\langle e_i, e_i \rangle = 1\), \(\langle e_j, e_i \rangle = 0\), \(j > i\)), \((e_1^\vee, \ldots, e_n^\vee)\) be a dual semiorthogonal basis, i.e., such that \(\langle e_i, e_j^\vee \rangle = \delta_{ij}\). Then, for additive functions \(s\) and \(t\), one has

\[\langle s, t \rangle = \sum_{i=1}^{n} s(e_i)t(e_i^\vee).\]  

As a basis \((e_1, \ldots, e_n)\), consider now the image of \((\mathcal{E}, \mathcal{F}, \mathcal{G})\) in \(K_0(S)\). It is semiorthogonal since the corresponding collection of sheaves is exceptional.

3.2. Proposition. The image in \(K_0(S)\) of the collection of objects of \(D^b(S)\),

\[\sigma = (G_\gamma \otimes K[2], \ldots, G_1 \otimes K[2], L_2^D F_\beta, \ldots, L_2^D F_1, E_\alpha, \ldots, E_1),\]

is a basis dual to \((e_1, \ldots, e_n)\).

Proof. The collection \(\sigma\) is exceptional since it is obtained from the initial collection \((\mathcal{E}, \mathcal{F}, \mathcal{G})\) by mutations in \(D^b(S)\). Namely, if one introduces uniform indexing for the sheaves of the initial collection, i.e.,

\[(E_1, \ldots, E_\alpha, F_1, \ldots, F_\beta, G_1, \ldots, G_\gamma) = (A_1, \ldots, A_n) \subset D^b(S),\]

then the completeness, helix properties (Sec. 1.11), and triviality of intrablock mutations imply that

\[\sigma = (L_{A_1} \ldots L_{A_{n-1}} A_n, \ldots, L_{A_1} L_{A_2} A_3, L_{A_1} A_2, A_1) \triangleq (A_n^\vee, \ldots, A_1^\vee).\]

Since \(\sigma\) is exceptional, its image in \(K_0(S)\) is semiorthogonal.

It remains to show that \(\chi(A_i, A_j^\vee) = \delta_{ij}\).
Consider the collections of objects of $D^b(S)$,

$$(G_\gamma \otimes K[2], \ldots, G_1 \otimes K[2], E_1, \ldots, E_\alpha, F_1, \ldots, F_\beta)$$

and

$$(L_\varepsilon^D F_\beta, \ldots, L_\varepsilon^D F_1, E_1, \ldots, E_\alpha, G_1, \ldots, G_\gamma).$$

One easily sees that they are obtained by mutations from the initial collection $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ and, therefore, are exceptional. Hence,

$$
\begin{align*}
\chi(E_i, L_\varepsilon^D F_j) &= 0, \\
\chi(E_i, G_j \otimes K[2]) &= 0, \\
\chi(F_i, E_j) &= 0, \\
\chi(F_i, G_j \otimes K[2]) &= 0, \\
\chi(G_i, L_\varepsilon^D F_j) &= 0, \\
\chi(G_i, E_j) &= 0, \\
\chi(E_i, E_j) &= \delta_{ij}.
\end{align*}
$$

The latter equality easily follows from the fact that $\mathcal{E}$ is a block. By Corollary 2.5 and equalities (11), $\chi(F_i, L_\varepsilon^D F_j) = \delta_{ij}$. By the Serre duality, $\chi(G_i, G_j \otimes K[2]) = \chi(G_j, G_i) = \delta_{ij}$, which completes the proof.

### 3.3. Derivation of Markov-type equations.

Introduce the notations

$$
\begin{align*}
x &= r(\mathcal{E}), \\
y &= r(\mathcal{F}), \\
z &= r(\mathcal{G}), \\
y' &= r(L_\varepsilon^D \mathcal{F}); \\
a &= \chi(\mathcal{F}, \mathcal{G}), \\
b &= \chi(\mathcal{G} \otimes K[2], \mathcal{E}) = \chi(\mathcal{G} \otimes K, \mathcal{E}), \\
c &= \chi(\mathcal{E}, \mathcal{F}).
\end{align*}
$$

From the exact sequences of Proposition 2.2 and from Corollary 2.5,

$$
y' = y - ca x, \quad d(L_\varepsilon^D \mathcal{F}) = d(\mathcal{F}) - cad(\mathcal{E}).
$$

Let us compute values of the bilinear form (13) with the help of the representation (14). As a basis in $K_0(S)$, let us use the image of the initial collection $(\mathcal{E}, \mathcal{F}, \mathcal{G})$; and as a dual basis, the image of $\sigma$ from the preceding proposition.

The equality $\langle r, r \rangle = 0$ means that $\alpha x^2 + \beta y y' + \gamma z^2 = 0$, or

$$
\alpha x^2 + \beta y^2 + \gamma z^2 = c \alpha \beta x y.
$$

By the assumption, $x, y, z \geq 0$ and $\alpha, \beta, \gamma > 0$. The ranks $x$, $y$, and $z$ cannot be zero simultaneously since the image of $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ is a basis in $K_0(S)$. Hence, $c > 0$ and $x, y, z > 0$.

Next, rewrite the equality $\langle r, d \rangle - \langle d, r \rangle = 0$ as follows:

$$
axd(\mathcal{E}) + \beta yd(L_\varepsilon^D \mathcal{F}) + \gamma zd(\mathcal{G} \otimes K[2]) - cad(\mathcal{E})x - \beta d(\mathcal{F})y' - \gamma zd(\mathcal{G}) =
$$

$$
= \beta \left( y(d(\mathcal{F}) - cad(\mathcal{E})) - d(\mathcal{F})(y - ca x) \right) + \gamma z \left( d(\mathcal{G} \otimes K) - d(\mathcal{G}) \right) =
$$

$$
= c \alpha \beta (d(\mathcal{F})x - d(\mathcal{E})y) - \gamma z^2 K^2 = c^2 \alpha \beta - \gamma z^2 K^2 = 0.
$$

Hence,

$$
c = z \sqrt{\frac{K^2 \gamma}{\alpha \beta}}. \quad (16)
$$

Substituting this into (15), we arrive at the statement below.

**Theorem.** The ranks $x, y, z$ and the numbers $\alpha, \beta, \gamma$ of sheaves in blocks of a complete collection $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ over a Del Pezzo surface satisfy the relation

$$
\alpha x^2 + \beta y^2 + \gamma z^2 = \sqrt{K^2 \alpha \beta \gamma} xyz,
$$

(3)
where \( K^2 \) is a square of the canonical class of the surface, and the coefficient in the right-hand side is an integer.

Let us explain the latter assertion. As we have seen above, \( x, y, z > 0 \), whence \( \sqrt{K^2} \alpha \beta \gamma \in \mathbb{Q} \). All rooted factors are integer; hence, \( \sqrt{K^2} \alpha \beta \gamma \in \mathbb{Z} \).

Furthermore, applying (16) to 3-block sheaf foundations \( (\mathcal{F}, \mathcal{G}, \mathcal{E} \otimes (-K)) \) and \( (\mathcal{G} \otimes K, \mathcal{E}, \mathcal{F}) \), we obtain

\[
a = x \sqrt{\frac{K^2\alpha}{\beta \gamma}} \quad \text{and} \quad b = y \sqrt{\frac{K^2\beta}{\alpha \gamma}}.
\]

Then, expressing \( x \), \( y \), and \( z \) in terms of \( a \), \( b \), and \( c \), and substituting these expressions into (3), we get the equation on the dimensions of the interblock Hom spaces as follows:

\[
\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = abc.
\]

### 3.4. Corollary

1. For any pair of blocks \( (\mathcal{E}, \mathcal{F}) \) contained in a 3-block collection, \( \text{Hom}(E_i, F_j) \neq 0 \). In other words, there is no complete two-block collection over a Del Pezzo surface.

2. Any mutation of a complete 3-block collection is a division.

3. Any sheaf contained in a complete 3-block collection is locally free.

**Proof.** Item 1 follows from the fact that \( c = \chi(\mathcal{E}, \mathcal{F}) > 0 \). Then, in a pair \( (\mathcal{E}, \mathcal{F}) \), a mutation of the extension type can never occur. Hence, an inverse mutation of the recoil type is also impossible. Item 3 follows from the classification of exceptional sheaves (Sec. 1.3) and the inequalities \( x, y, z > 0 \) obtained above.

### 3.5. The list of equations

It is well known [14], that a Del Pezzo surface \( S \) is isomorphic to either \( \mathbb{P}^1 \times \mathbb{P}^1 \) or \( X_m \), the latter being a plane with \( m \) generic points blown-up, where \( 0 \leq m \leq 8 \).

Assume there is a complete 3-block collection \( (\mathcal{E}, \mathcal{F}, \mathcal{G}) \) over \( S \). Then, as above, the ranks \( x \), \( y \), and \( z \) are positive and obey the equation (3). Moreover, the image of a given collection in \( K_0(S) \) forms a basis, whence \( \alpha + \beta + \gamma = \text{rk} K_0(S) = 12 - K^2 \). Therefore, \( \alpha \), \( \beta \), \( \gamma \), and the squared anticanonical class of \( S \) satisfy the following system of conditions:

\[
\begin{aligned}
\alpha + \beta + \gamma + K^2 &= 12, \\
K^2 \alpha \beta \gamma \text{ is a square of an integer}, \\
\alpha, \beta, \gamma &\geq 1, \\
1 &\leq K^2 \leq 9.
\end{aligned}
\]

The system is solved by the finite exhausting. As an answer, we present the complete list of equations (3) with indication of the correspondent surfaces. For some equations, a common multiplier can be cancelled, but we do not do this in order to leave as coefficients the numbers \( \alpha \), \( \beta \), and \( \gamma \) of sheaves in the correspondent blocks. One can easily check that the blocks with given numbers of sheaves can be arbitrarily reordered by mutations. (For instance, the first block of \( (L_{\mathcal{E}} \mathcal{F}, \mathcal{E}, \mathcal{G}) \) consists of \( \beta \) sheaves, the second one consists of \( \alpha \) sheaves, and the third one, of \( \gamma \) sheaves.) Therefore, without loss of generality, we assume that \( \alpha \leq \beta \leq \gamma \).

---

1. We prove below that for each equation, the corresponding complete 3-block collection of sheaves exists.
**Definition.** A minimum solution of the equation (3) is a positive integer solution with the minimum possible sum \( x + y + z \).

**Proposition.** All equations of type (3) with \( \alpha \leq \beta \leq \gamma \), together with their minimum solutions, are presented in the table below.

| Number of the equation | Surface | Equation | Minimum solution |
|------------------------|---------|----------|-----------------|
| (1) \( P^2 \)         |         | \( x^2 + y^2 + z^2 = 3xyz \) | (1,1,1)         |
| (2) \( P^1 \times P^1 \) |         | \( x^2 + y^2 + 2z^2 = 4xyz \) | (1,1,1)         |
| (3) \( X_3 \)         |         | \( x^2 + 2y^2 + 3z^2 = 6xyz \) | (1,1,1)         |
| (4) \( X_4 \)         |         | \( x^2 + y^2 + 5z^2 = 5xyz \) | (1,2,1) and (2,1,1) |
| (5) \( X_5 \)         |         | \( 2x^2 + 2y^2 + 4z^2 = 8xyz \) | (1,1,1)         |
| (6.1) \( X_6 \)       |         | \( 3x^2 + 3y^2 + 3z^2 = 9xyz \) | (1,1,1)         |
| (6.2) \( X_6 \)       |         | \( x^2 + 2y^2 + 6z^2 = 6xyz \) | (2,1,1)         |
| (7.1) \( X_7 \)       |         | \( x^2 + y^2 + 8z^2 = 4xyz \) | (2,2,1)         |
| (7.2) \( X_7 \)       |         | \( 2x^2 + 4y^2 + 4z^2 = 8xyz \) | (2,1,1)         |
| (7.3) \( X_7 \)       |         | \( x^2 + 3y^2 + 6z^2 = 6xyz \) | (3,1,1)         |
| (8.1) \( X_8 \)       |         | \( x^2 + y^2 + 9z^2 = 3xyz \) | (3,3,1)         |
| (8.2) \( X_8 \)       |         | \( x^2 + 2y^2 + 8z^2 = 4xyz \) | (4,2,1)         |
| (8.3) \( X_8 \)       |         | \( 2x^2 + 3y^2 + 6z^2 = 6xyz \) | (3,2,1)         |
| (8.4) \( X_8 \)       |         | \( x^2 + 5y^2 + 5z^2 = 5xyz \) | (5,2,1) and (5,1,2) |

**Proof.** We omit verification of the fact that all possible equations of type (3) are presented here. Let us show that the right-hand column actually contains minimum solutions.

For the equations that have a solution (1,1,1), this is obvious. For the equations (4), (6.2), and (7.2), this is true since (1,1,1) is not a solution to these equations. Consider the remaining cases.

(7.1): It is easily seen that \( x \) and \( y \) are of the same parity, but they cannot be odd since \( x^2 + y^2 \) is divisible by 4.

(7.3): \( x \) is divisible by 3.

(8.1): \( x^2 + y^2 \) is divisible by 3, but since a square's residual modulo 3 equals either 0 or 1, both \( x \) and \( y \) are divisible by 3.

(8.2): \( x \) is even; hence, \( 2y^2 \) is divided by 4. Then \( y \) is even, and hence, \( x^2 \) is divisible by 8, i.e., \( x \) is also divisible by 4.

(8.3): \( x \) is divisible by 3, and \( y \) is even.

(8.4): \( x \) is divisible by 5. Let \( x = 5\tilde{x} \), then the equation takes the form \( 5\tilde{x}^2 + y^2 + z^2 = 5\tilde{x}yz \), which coincides with (4) up to a designation.

**Remark.** The surface \( X_1 \) has \( K^2 = 8 \), but there are no complete exceptional collections over \( X_1 \). Indeed, the total number of sheaves in a complete collection should be equal to
\( \text{rk} K_0(X_1) = 4 \), i.e., one block should consist of two sheaves. By Proposition 1.6, the difference \( c \) of the first Chern classes of these sheaves should satisfy the relations \( c^2 = -2 \) and \( c \cdot K = 0 \). But there is no such a divisor over \( X_1 \).

3.6. Solution mutations. Let \((x, y, z)\) be a solution of \((3)\). The solution mutation in the variable \( y \) is the mapping \( M_y : (x, y, z) \mapsto (x, y', z) \), where

\[
y' = \sqrt{\frac{K^2 \alpha \gamma}{\beta - xz}} - y = c\alpha x - y = a\gamma z - y.
\]

Similarly, the solution mutations \( M_x \) and \( M_z \) (in \( x \) and \( z \) respectively) are defined.

For any complete 3-block collection with structure \( \{\alpha, \beta, \gamma\} \), define the correspondent solution of \((3)\) as a triple of numbers \( (r_\alpha, r_\beta, r_\gamma) \), where the first number is the rank of sheaves in the block of length \( \alpha \), etc. If \( \alpha < \beta < \gamma \), a single solution corresponds to the 3-block collection, and for \( \alpha = \beta \) or \( \beta = \gamma \), more than one solution may correspond to one collection. Moreover, the same solution may correspond to collections of different types obtained by various permutations of \( \{\alpha, \beta, \gamma\} \).

Let \((x, y, z)\) be a solution corresponding to a collection \((E, F, G)\) of type \( \{\alpha, \beta, \gamma\} \). Then the solution \((x, y', z)\) described above corresponds to the collections \((L_E F, E, G)\) and \((E, G, R_G F)\). This follows from Corollary 3.4 and Proposition 2.2. Note that \( L_E F = R_G F \otimes K \). In the general case, one can easily check that mutations of the collection which change the block of length \( \alpha \) \( (\beta \) or \( \gamma \) \) induce the correspondent solution mutations in \( x \) \( (y \) or \( z \) \).

Thus, mutations of complete 3-block collections are concordant with mutations of correspondent solutions.

3.7. Proposition. (a) Any solution of any of equation from Proposition 3.5 can be reduced by mutations to a minimum solution.

(b) Moreover, for a nonminimum solution \((x, y, z)\), one mutation reduces and two others increase the sum \( x + y + z \).

Proof. For the equations \((1)\) and \((2)\), (a) is well known; see \([15, 18, 19]\). Let us verify (b) for these equations. Since \((1)\) is symmetric in \( x, y, \) and \( z \), we may assume that \( x = \max\{x, y, z\} \). Then \( y + y' = 3xz \geq 3yz \geq 3y > 2y \), whence \( y' > y \), i.e., the mutation in \( y \) increases \( x + y + z \). For the mutation in \( z \), the reasoning is similar.

The equation \((2)\) is symmetric in \( x, y \). Put, for definiteness, \( x \geq y \). Consider the case \( x \geq z \). Then \( y + y' = 4xz \geq 4yz \geq 4y > 2y \Longrightarrow y' > y \), and also \( z + z' = 2xy \geq 2yz \geq 2z \Longrightarrow z' \geq z \). The equality \( z' = z \) is possible only if \( x = z \) and \( y = 1 \). Putting this into \((2)\), we get \( x = 1 \). Hence, if \((x, y, z) \neq (1, 1, 1) \), then \( z' > z \). For the case \( x < z \), the reasoning is similar.

Next, consider the equations \((3)\) and \((4)\). Introduce the notations

\[
\Phi(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2 - \sqrt{K^2 \alpha \beta \gamma x y z}, \quad \varphi_y(t) = \Phi(x, t, z),
\]

\((x, y, z)\) being a fixed solution of \((3)\). Here, \( t_1 = y \) and \( t_2 = y' \) are the roots of the quadratic equation \( \varphi_y(t) = 0 \). Consider also the functions \( \varphi_x(t) = \Phi(t, y, z) \) and \( \varphi_z(t) = \Phi(x, y, t) \). The main tool for proving (a) is the following obvious statement:

\[
y' \geq y \iff \varphi_y(t) \geq 0, \forall t \leq y,
\]

and the analogous statements for \( x \) and \( z \) as well.
The condition $y' \geq y$ means that the mutation of the solution $(x, y, z)$ in $y$ does not decrease $y$.

Let $(x, y, z)$ be a solution of (3) such that none of its mutation reduces $x + y + z$. Let us show that $(x, y, z) = (1, 1, 1)$ then. Consider the cases below.

(i) $x \geq y \geq z$. Then $0 \leq \varphi_x(y) = 3y^2 + 3z^2 - 6y^2z \leq 6y^2 - 6y^2z$, whence $z = 1$ and $\varphi_x(y) = 3 - 3y^2 \geq 0$. Therefore, $y = 1$, and (3) directly implies $x = 1$.

(ii) $x \geq z > y$. Then $0 \leq \varphi_x(z) = 4z^2 + 2y^2 - 6z^2y < 6z^2 - 6y^2z$, which is impossible. The other cases are also impossible:

(iii) $y > x \geq z \implies 0 < \varphi_y(x) = 3x^2 + 3z^2 - 6x^2z \leq 6x^2 - 6x^2z$;

(iv) $z > x \geq y \implies 0 < \varphi_z(x) = 4x^2 + 2y^2 - 6x^2y \leq 6x^2 - 6x^2y$;

(v) $y \geq z > x \implies 0 < \varphi_y(z) = x^2 + 5z^2 - 6z^2x < 6z^2 - 6z^2x$;

(vi) $z > y > x \implies 0 < \varphi_z(y) = x^2 + 5y^2 - 6y^2x < 6y^2 - 6y^2x$.

Thus, for any nonminimum solution of (3), a mutation exists which reduces $x + y + z$. Hence, for (3), (a) is proved. Let us prove (b), assuming $x = \max\{x, y, z\}$ (in other cases, the reasoning is similar). We have $y + y' = 3xz \geq 3yz \geq 3y > 2y \implies y' > y$, and also $z + z' = 2xy > 2yz \geq 2z \implies z' > z$. The equality $z' = z$ is possible only if $x = z$ and $y = 1$, whence the minimality of $(x, y, z)$ follows.

Now, let $(x, y, z)$ be a solution of (4) such that none of its mutations reduces $x + y + z$. Let us show that $(x, y, z)$ coincides with one of the minimum solutions, $(2, 1, 1)$ or $(1, 2, 1)$. The variables $x$ and $y$ are equivalent, so we assume $x \geq y$.

(i) $x \geq y \geq z$. Then $0 \leq \varphi_x(y) = 2y^2 + 5z^2 - 5y^2z \leq 7y^2 - 5y^2z$, whence $z = 1$ and $\varphi_x(y) = 5 - 3y^2 \geq 0$. Hence, $y = 1$, and then $x = 2$, which directly follows from (4).

(ii) $x \geq z > y$. Then $0 \leq \varphi_x(z) = 6z^2 + y^2 - 5z^2y < 7z^2 - 5z^2y$, whence $y = 1$. Consider the solution mutation in $z$, $z' = xy - z = x - z$. By the assumption, $z' \geq z$; hence, $x \geq 2z$. Then $0 \leq \varphi_x(z) = 9z^2 + 1 - 10z^2 = 1 - z^2$, i.e., $z = 1$, which provides a contradiction.

(iii) $z > x \geq y$. Then $0 \leq \varphi_z(x) = 6x^2 + y^2 - 5x^2y \leq 7x^2 - 5x^2y$, whence $y = 1$, and the solution mutation in $z$ yields $z' = x - z < 0$, a contradiction.

Thus, for any nonminimum solution of (4), a mutation exists which reduces $x + y + z$. This proves (a). Let us prove (b), assuming $x \geq y$ as before.

(i) $x \geq y$. Then $y + y' = 5xz \geq 5yz \geq 5y > 2y \implies y' > y$, i.e., the mutation in $y$ increases $x + y + z$. Let $y \geq 2$, then $z + z' = xy \geq 2x \geq 2z$, whence $z' \geq z$. Here, the equality is possible only if $y = 2$ and $x = z$, which implies $x = z = 1$. Hence, the mutation in $z$ reduces $z$ if $(x, y, z) \neq (1, 2, 1)$. Now, let $y = 1$. Let us show that the mutations in $x$ and $z$ cannot reduce $x + y + z$ simultaneously. Indeed, if $x' = 5z - x < x$, then $5z < 2x$, whence $2z < x$, and $z' = x - z < z$.

(ii) $z > x$. In this case, $y + y' = 5xz \geq 5xy \geq 5y > 2y$, whence $y' > y$, i.e., the mutation in $y$ increases $x + y + z$. This completes the proof of (b) for the equation (4) due to its symmetry in $x, y$.

Thus, the proposition is valid for the first four equations. Each of the others can be reduced to one of the first four by a change of variables, which is possible in each case (according to the proof of Proposition 3.5). Namely,

(6.2) is reduced to (3) under $x = 2x$;
(7.1) is reduced to (2) under $x = 2x$, $y = 2y$;
(7.2) is reduced to (2) under $x = 2x$;
(7.3) is reduced to (3) under $x = 3x$;
(8.1) is reduced to (1) under $x = 3\bar{x}$, $y = 3\bar{y}$;
(8.2) is reduced to (2) under $x = 4\bar{x}$, $y = 2\bar{y}$;
(8.3) is reduced to (3) under $x = 3\bar{x}$, $y = 2\bar{y}$;
(8.4) is reduced to (4) under $x = 5\bar{x}$.
This completes the proof of the proposition.

3.8. Groups of equations. As one can see from the proof of the preceding proposition, any equation starting from (5) either can be obtained from one of the first four equations by a change of variables, or is proportional to one of them. Let us join the equations in groups as follows:

Group I: (1), (6.1), (8.1);
Group II: (2), (5), (7.1), (7.2), (8.2);
Group III: (3), (6.2), (7.3), (8.3);
Group IV: (4), (8.4).

For each equation, consider a pseudograph whose vertices are the solutions, and two vertices are joined by an edge if and only if the solutions can be obtained one from another by one of the mutations $M_x$, $M_y$, or $M_z$. Obviously, the pseudographs for equations of the same group are isomorphic. One can easily deduce from the preceding proposition that the pseudographs of the solutions have the form as follows: for the group I, $\Gamma_1$ (see Fig. 1); for the groups II and III, $\Gamma_2$; and for the group IV, the pseudograph consists of two connected components isomorphic to $\Gamma_2$. The point $P_0$ denotes the minimum solution. Moreover, $\Gamma_1$ and $\Gamma_2$ have no cycles, i.e., $\Gamma_1$ is actually a graph. By abuse of language, we will say “solution graph” instead of “solution pseudograph” although $\Gamma_2$ is not a graph (since it has a single loop $M$ starting and ending at the minimum solution).

3.9. Definition. A complete 3-block collection $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ is called minimum if the sum $r(\mathcal{E}) + r(\mathcal{F}) + r(\mathcal{G})$ is minimum of all sums for 3-block collections of the same structure.

3.10. Theorem. Any complete 3-block collection can be obtained by mutations from a minimum collection of type $(\alpha, \beta, \gamma)$, where $\alpha \leq \beta \leq \gamma$.

Proof. According to 3.6 and 3.7, any complete 3-block collection can be reduced by mutations to a minimum one. Invertibility of mutations imply that a given collection can be obtained by mutations from a minimum one. Hence, it suffices to show that a minimum collection whose
type is an arbitrary permutation of \((\alpha, \beta, \gamma)\) can be obtained from a minimum collection \((E, F, G)\) of type \((\alpha, \beta, \gamma)\), where \(\alpha \leq \beta \leq \gamma\). The sequences of mutations \(L_1L_2\) and \(R_2R_1\) (cf. the notations of Sec. 2.6) take \((E, F, G)\) to \((G(K), E, F)\) and \((F, G, E(-K))\) respectively. The types of these collections are cyclic transpositions of \((\alpha, \beta, \gamma)\). Thus, the theorem is valid for \(\alpha = \beta\) or \(\beta = \gamma\). This condition holds for all equations except those of group III. For the equations of this group, the mutation that preserves the minimum solution is induced by the mutation of the minimum collection which performs a transposition of \((\alpha, \beta, \gamma)\). The theorem is proved.

Thus, the problem on the action of the braid group and, in particular, on the set of orbits, reduces to the problem on the set of minimum collections.

Below, we study minimum 3-block collections, namely, prove their existence and describe the action of the Weyl group on collections of a given structure.

### 4. Existence of minimum collections.

#### 4.1. Agreements.

Fix generic points \(x_1, \ldots, x_8\) on a projective plane and denote by \(\sigma_r: X_r \to \mathbb{P}^2\) the monoidal transform with center \(\{x_1, \ldots, x_r\}\). Let \(\ell_0\) be a divisor class on \(X_r\) equal to the lifting of the class of a line on \(\mathbb{P}^2\). Let \(\ell_i, i = 1, \ldots, r\), be the classes of exceptional curves \(\sigma_{r-1}^{-1}(x_i)\). For \(1 \leq i \leq r\), each of \(\ell_i\)’s contains a single divisor, namely, the curve \(\sigma_{r-1}^{-1}(x_i)\) itself. We denote this curve by the same symbol \(\ell_i\).

The surface \(X_{r+1}\) is obtained as \(X_r\) with a blown-up point \(\sigma_{r-1}^{-1}(x_{r+1})\), so we have the diagram

\[
X_8 \longrightarrow X_7 \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \mathbb{P}^2.
\]  

(18)

In correspondence with it, we will consider \(\mathbb{Z}\)-modules \(\text{Pic} X_r\) to be embedded in one another, i.e.,

\[
\mathbb{Z} \cong \text{Pic} \mathbb{P}^2 \subset \text{Pic} X_1 \subset \ldots \subset \text{Pic} X_8,
\]

where \(\text{Pic} X_r = \mathbb{Z}\ell \oplus \mathbb{Z}\ell_1 \oplus \ldots \oplus \mathbb{Z}\ell_r\). The intersection form is defined by

\[
\ell_i^2 = 1, \quad \ell_i^2 = -1, \quad i \geq 1, \quad \ell_i \cdot \ell_j = 0, \quad i \neq j.
\]

The canonical class of \(X_r\) is

\[
\omega_r = -3\ell + \sum_{i=1}^{r} \ell_i.
\]

Denote by \(\sigma: X_r \to X_p, \ r > p\), the composition of morphisms in (18), i.e., blowing up the points \(\sigma_{p-1}^{-1}(x_{p+1}), \ldots, \sigma_{p-1}^{-1}(x_r)\). This will not cause ambiguity since it will always be clear from a context which \(p\) and \(r\) are meant. Under our agreements, \(\text{Pic} X_p \subset \text{Pic} X_r\) is the orthogonal complement to the linear span of \(\ell_{p+1}, \ldots, \ell_r\).

To any divisor class \(a\ell_0 + \sum_{i=1}^{r} b_i\ell_i\) modulo linear equivalence, a unique class of invertible sheaves modulo isomorphism corresponds which we denote by \(\mathcal{O}_{X_r}(a\ell_0 + \sum_{i=1}^{r} b_i\ell_i)\). Here,

\[
\sigma^* \mathcal{O}_{X_p}(a\ell_0 + \sum_{i=1}^{p} b_i\ell_i) = \mathcal{O}_{X_r}(a\ell_0 + \sum_{i=1}^{p} b_i\ell_i).
\]
In this section, we show that over the surfaces \( X_r, \ 3 \leq r \leq 8 \), complete 3-block exceptional collections corresponding to minimum solutions of Markov-type equations (see Sec. 3.5) exist, and all such collections can be obtained by a procedure which may be called “lifting.” The general scheme of this procedure is as follows. It is known that if \( \tau \) is a complete exceptional collection over \( X_p \), then the collection \( \sigma^* \tau \) complemented from the left by the sheaves \( \mathcal{O}_{\ell_{p+1}}(-1), \ldots, \mathcal{O}_{\ell_r}(-1) \) (or from the right by \( \mathcal{O}_{\ell_{p+1}}, \ldots, \mathcal{O}_{\ell_r} \)), form a complete exceptional collection over \( X_r \). Here, if \( \tau \) is 3-block, the latter collection over \( X_r \) is 4-block. In some cases\(^2\), it turns out to be possible to perform such mutations of the 4-block collection \( (\sigma^* \tau, \mathcal{O}_{\ell_{p+1}}, \ldots, \mathcal{O}_{\ell_r}) \) (or \( (\mathcal{O}_{\ell_{p+1}}, \ldots, \mathcal{O}_{\ell_r}, \sigma^* \tau) \)) that in an obtained 4-block collection, two neighboring blocks can be joined into one, which actually gives a 3-block collection over \( X_r \).

Recall that a minimum collection over a Del Pezzo surface is a complete 3-block collection whose block ranks form a minimum solution of the correspondent Markov-type equation.

**4.2. Proposition.** Over the surfaces \( X_r, \ 3 \leq r \leq 8 \), there exist minimum collections of types \((\alpha, \beta, \gamma)\), where \( \alpha \leq \beta \leq \gamma \).

**Proof.** We could just present the collections required, but it will be essential for us that all of them can be obtained by the “lifting” described above. Therefore, we present the corresponding sequences of mutations using the notations of Sec. 2.6. For known discrete invariants \( r(E_i) \) and \( \chi(E_i, E_j) J < i \), Propositions 2.2 and 2.3 make it possible to determine the type of any mutation of a 4-block collection \((E_1, E_2, E_3, E_4)\) and compute these invariants for the collection obtained by a mutation. Note that if a collection \((E_1, E_2, E_3, E_4)\) is complete, the sequences of mutations

\[
R_1^{(3)} = R_3 \circ R_2 \circ R_1 \quad \text{and} \quad L_3^{(3)} = L_1 \circ L_2 \circ L_3
\]

take it to \((E_2, E_3, E_4, E_1(-K))\) and \((E_4(K), E_1, E_2, E_3)\) respectively.

We enumerate the items of the proof in the same way as the Markov-type equations in the table of Sec. 3.5. For brevity, we denote a block \( \mathcal{O}_{\ell_{p+1}}, \ldots, \mathcal{O}_{\ell_r}, \ p < r \), consisting of torsion sheaves by \( \mathcal{E}_{p:r} \), or by \( \mathcal{E}_{p,p+1} \) for \( r = p + 1 \). We arrange the final 3-block collection into a table \( \begin{array}{c|c|c}
E & F & G
\end{array} \), where the block obtained as a result of the last operation (joining two blocks into one) is divided by a dotted line into the parts of which it is composed. We call this block distinguished. Introduce the notation

\[
\tau_0 = (\mathcal{O}_{p^2}(-1), \mathcal{O}_{p^2}, \mathcal{O}_{p^2}(1))
\]

for a well-known foundation of a helix over \( \mathbb{P}^2 \).

(3). Consider the sequence of mutations

\[
(\sigma_3^* \tau_0, \mathcal{L}_{1:3}) = (\mathcal{O}_{X_3}(-\ell), \mathcal{O}_{X_3}, \mathcal{O}_{X_3}(\ell), \mathcal{L}_{1:3}) \xrightarrow{R_1^{(3)}}
\rightarrow (\mathcal{O}_{X_3}, \mathcal{O}_{X_3}(\ell), \mathcal{L}_{1:3}, \mathcal{O}_{X_3}(2\ell - \ell_1 - \ell_2 - \ell_3)) \xrightarrow{R_3}
\rightarrow \begin{bmatrix}
\mathcal{O}_{X_3} & \mathcal{O}_{X_3}(\ell) & \mathcal{O}_{X_3}(2\ell - \ell_2 - \ell_3) \\
\mathcal{O}_{X_3}(2\ell - \ell_1 - \ell_2 - \ell_3) & \mathcal{O}_{X_3}(2\ell - \ell_2 - \ell_3) & \mathcal{O}_{X_3}(2\ell - \ell_1 - \ell_2)
\end{bmatrix} = \tau_{(3)}.
\]

\(^2\)Namely, if the difference between the slopes of two blocks in \( \tau \) equals the slope of the anticanonical class of \( X_r \).
This is the desired collection over a plane with three blown-up points.

(4). Over $X_4$, we have

$$(\sigma^*\tau_0, L_{1:4}) \xrightarrow{R^{(3)}_1} (O_{X_4}, O_{X_4}(\ell), L_{1:4}, O_{X_4}(-\omega_4 - \ell)) \xrightarrow{R_3}$$

$$\xrightarrow{L_2} (O_{X_4}, O_{X_4}(\ell), O_{X_4}(-\omega_4 - \ell), \{O_{X_4}(\ell_i - \omega_4 - \ell)\}_{i=1,2,3,4}) = \tau(4).$$

This is the collection corresponding to the solution $(1,2,1)$ of (4). To the solution $(1,1,2)$, the collection $R_1 \circ R_2 \circ R_2(\tau(4))$ corresponds.

The bundle $F$ is obtained as a universal extension

$$0 \longrightarrow O_{X_4}(-\omega_4 - \ell) \longrightarrow F \longrightarrow O_{X_4}(\ell) \longrightarrow 0.$$

Here, $c_1(F) = -\omega_4$, $r(F) = 2$.

In other cases, we present the starting and final collections only and the sequence of mutations. Verifying details is left to an interested reader.

(5). Over $X_5$, the mutations $R^{(3)}_1 \circ R_1$ of the 4-block collection $(L_{4:5}(-1), \sigma^*\tau(3))$ result in the desired collection

$$\tau(5) = \begin{bmatrix}
O_{X_5}(\ell_4) & O_{X_5}(\ell) & O_{X_5}(-\omega_5) \\
O_{X_5}(\ell_5) & O_{X_5}(2\ell - \ell_1 - \ell_2 - \ell_3) & \end{bmatrix}.$$

(6.1). The sequence of mutations $R^{(3)}_1 \circ L_3 \circ R_1$ takes the 4-block collection $(L_{4:6}(-1), \sigma^*\tau(3))$ over $X_6$ to the desired one,

$$\tau(6.1) = \begin{bmatrix}
O_{X_6}(\ell_4) & O_{X_6}(\ell - \ell_1) & O_{X_6}(-\omega_6) \\
O_{X_6}(\ell_5) & O_{X_6}(\ell - \ell_2) & O_{X_6}(\ell) \\
O_{X_6}(\ell_6) & O_{X_6}(\ell - \ell_3) & O_{X_6}(2\ell - \ell_1 - \ell_2 - \ell_3) \\
\end{bmatrix}.$$

(6.2). The 4-block collection $(L_{1:6}(-1), \sigma^*\tau_0)$ over $X_6$ is taken by $R^{(3)}_1 \circ R^{(3)}_1 \circ R_1 \circ R_2$ to the desired one,

$$\tau(6.2) = \begin{bmatrix}
T_6 & O_{X_6}(\ell) & O_{X_6}(\ell - \omega_6) & O_{X_6}(\ell_1 - \omega_6) & O_{X_6}(\ell_2 - \omega_6) \\
O_{X_6}(\ell_3 - \omega_6) & O_{X_6}(\ell_4 - \omega_6) & O_{X_6}(\ell_5 - \omega_6) & O_{X_6}(\ell_6 - \omega_6) \\
\end{bmatrix}.$$
Here, \( T_6 = \sigma^*_6 \mathbb{P}^2(-1) \), \( c_1(T_6) = \ell \), \( r(T_6) = 2 \), and \( \mu(T_6) = \frac{3}{2} \).

(7.1). In this case, the way found by the authors is rather long, and we divide it into two parts. The 4-block collection \( (\sigma^* \tau_0, \mathcal{L}_{1+7}) \) is taken by \( L_3^{(3)} \circ L_3 \circ R_1 \) to

\[
\left( \mathcal{O}_{X_7}(\ell + \omega_7), \mathcal{O}_{X_7}, T_7, \{ \mathcal{O}_{X_7}(\ell - \ell_i) \}_{i=1,\ldots,7} \right),
\]

where \( T_7 = \sigma^*_7 \mathbb{P}^2(-1) \) with the same \( r \), \( c_1 \), and \( \mu \) as those of \( T_6 \). The latter collection is taken by \( R_1^{(3)} \circ R_1 \) to the desired collection

\[
\tau(7.1) = \begin{array}{c|c|c}
E_7 & T_7 & O_{X_7}^{(-\omega_7)} \\
\hline
& & \ldots \ldots \ldots \\
& & O_{X_7}(\ell - \ell_1), \ldots, O_{X_7}(\ell - \ell_7)
\end{array}
\]

The bundle \( E_7 \) is obtained as the extension

\[
0 \longrightarrow \mathcal{O}_{X_7} \longrightarrow E_7 \longrightarrow \mathcal{O}_{X_7}(\ell + \omega_7) \longrightarrow 0.
\]

Then \( c_1(E_7) = \ell + \omega_7 \), \( r(E_7) = 2 \), and \( \mu(E_7) = \frac{1}{2} \). In the corresponding sequence of mutations, \( R_1 \circ L_1 = \text{id} \), i.e.,

\[
R_1^{(3)} \circ R_1 \circ (L_1 \circ L_2 \circ L_3) \circ L_3 = R_1^{(3)} \circ L_2 \circ L_3 \circ L_3.
\]

(7.2). The sequence \( L_3^{(3)} \circ R_1 \circ R_3 \) takes \( (\mathcal{L}_{1+7}, \sigma^* \tau(3)) \) to the 4-block collection

\[
\left( \begin{array}{c}
\mathcal{O}_{X_7}(\ell + \omega_7) \\
\mathcal{O}_{X_7}(\omega_7 - \ell - \omega_3) \\
\mathcal{O}_{X_7}(\omega_7 - \ell - \omega_3)
\end{array} \right), \quad \left( \begin{array}{ccc}
\mathcal{O}_{X_7}(\ell_4) & \mathcal{O}_{X_7}(\ell_5) & \mathcal{O}_{X_7}(\ell - \ell_1) \\
\mathcal{O}_{X_7}(\ell_5) & \mathcal{O}_{X_7}(\ell_6) & \mathcal{O}_{X_7}(\ell - \ell_2) \\
\mathcal{O}_{X_7}(\ell_6) & \mathcal{O}_{X_7}(\ell_7) & \mathcal{O}_{X_7}(\ell - \ell_3)
\end{array} \right),
\]

which is taken by \( R_1^{(3)} \circ R_1 \) to the desired collection

\[
\tau(7.2) = \begin{array}{c|c|c}
E_7 & E_7' & O_{X_7}(\ell_4) \\
\hline
& & O_{X_7}(\ell_5) \\
& & O_{X_7}(\ell_6) \\
& & O_{X_7}(\ell_7) \\
E_7 & E_7' & O_{X_7}(\ell - \ell_1) \\
& & O_{X_7}(\ell - \ell_2) \\
& & O_{X_7}(\ell - \ell_3)
\end{array}
\]

Here \( E_7 \) is the rank-2 bundle described above and \( E_7' \) is the result of the right shift of the pair \( (\mathcal{O}_{X_7}(\omega_7 - \ell - \omega_3), \mathcal{O}_{X_7}) \),

\[
0 \longrightarrow \mathcal{O}_{X_7} \longrightarrow E_7' \longrightarrow \mathcal{O}_{X_7}(\omega_7 - \ell - \omega_3) \longrightarrow 0.
\]

We have \( c_1(E_7') = \omega_7 - \ell - \omega_3 = -\ell + \ell_4 + \ell_5 + \ell_6 + \ell_7 \), \( r(E_7') = 2 \), and \( \mu(E_7') = \frac{1}{2} \).
(7.3). The 4-block collection \( (\mathcal{O}_{\ell_7}(-1), \sigma^*\tau_{(6.1)}) \) over \( X_7 \) is taken by \( R_1^{(3)} \circ R_1 \) to the desired one,
\[
\tau_{(7.3)} = \begin{array}{ccc}
\mathcal{O}_{X_7}(\ell) \\
\mathcal{O}_{X_7}(2\ell - \ell_1 - \ell_2 - \ell_3) \\
\mathcal{O}_{X_7}(-\omega_6) = \mathcal{O}_{X_7}(\ell_7 - \omega_7) \\
\mathcal{O}_{X_7}(\ell_6 - \omega_7) \\
\mathcal{O}_{X_7}(\ell_5 - \omega_7) \\
\mathcal{O}_{X_7}(\ell_4 - \omega_7)
\end{array}
\]
Here, the bundle \( E_{7}'' \) is the result of the right shift of the torsion sheaf \( \mathcal{O}_{\ell_7}(-1) \) over the block \( (\mathcal{O}_{X_7}(\ell_4), \mathcal{O}_{X_7}(\ell_5), \mathcal{O}_{X_7}(\ell_6)) \),
\[
0 \rightarrow (\mathcal{O}_{X_7}(\ell_4) \oplus \mathcal{O}_{X_7}(\ell_5) \oplus \mathcal{O}_{X_7}(\ell_6)) \rightarrow E_{7}'' \rightarrow \mathcal{O}_{\ell_7}(-1) \rightarrow 0.
\]
We have \( c_1(E_{7}''') = \ell_4 + \ell_5 + \ell_6 + \ell_7, \ r(E_{7}''') = 3 \), and \( \mu(E_{7}''') = \frac{4}{3} \).

(8.1). Over \( X_8 \), the collection \( (\sigma^*\tau_{(7.1)}), \mathcal{L}_{1:8}) \) is taken by \( L_3^{(3)} \circ L_3 \circ R_1 \) to the 4-block collection
\[
\left( \mathcal{O}_{X_8}(-\omega_8), \mathcal{O}_{X_8}(-\ell), \sigma^*(\mathbb{TP}^2(-2)), \{ \mathcal{O}_{X_i}(-\ell_i) \} \right)_{i=1,\ldots,8}.
\]
Applying \( L_1 \circ L_1 \) to the latter collection, we obtain the desired one,
\[
\tau_{(8.1)} = \begin{array}{ccc}
E_8 \\
F_8 \\
\mathcal{O}_{X_8}(-\omega_8) \\
\mathcal{O}_{X_8}(-\ell_1), \ldots, \mathcal{O}_{X_8}(-\ell_8)
\end{array}
\]
Here \( E_8 = L_{\mathcal{O}_{X_8}(-\omega_8)} \mathcal{O}_{X_8}(-\ell) \) and \( F_8 = L_{\mathcal{O}_{X_8}(-\omega_8)} \sigma^*(\mathbb{TP}^2(-2)) \). Both mutations are extensions, \( c_1(E_8) = -2\omega_8 - \ell, \ c_1(F_8) = -\omega_8 - \ell, \ r(E_8) = r(F_8) = 3, \ \mu(E_8) = -\frac{5}{3}, \) and \( \mu(F_8) = -\frac{4}{3} \).

(8.2). Applying \( L_3 \circ L_3 \) to \( (\mathcal{L}_{1:8}(-1), \sigma^*\tau_{(3)}) \), we obtain the 4-block collection
\[
\left( \mathcal{L}_{1:8}(-1), \mathcal{O}_{X_8}, \mathcal{T}_8, \mathcal{T}'_8 \right) = \left( \mathcal{O}_{X_8}(\ell - \ell_1), \mathcal{O}_{X_8}(\ell - \ell_2), \mathcal{O}_{X_8}(\ell - \ell_3) \right).
\]
Here, the bundle \( T_8 = \sigma^*\mathbb{TP}^2(-1) \) is the left shift of \( \mathcal{O}_{X_8}(2\ell - \ell_1 - \ell_2 - \ell_3) \) over the block \( \{ \mathcal{O}_{X_i}(\ell - \ell_i) \} \). This is the unique (according to 1.3) exceptional bundle of rank 2 over \( X_3 \) with \( c_1 = \ell \). The bundle \( T_8' \) is the left shift of \( \mathcal{O}_{X_8}(\ell) \) over the same block. Then \( c_1(T_8') = -\omega_3 - \ell, \ r(T_8') = 2, \) and \( \mu(T_8') = \mu(T_8) = \frac{3}{2} \).

Applying \( R_1^{(3)} \circ R_1 \circ R_1 \) to the latter 4-block collection results in the desired collection
\[
\tau_{(8.2)} = \begin{array}{ccc}
E_8' \\
T_8 \\
\mathcal{O}_{X_8}(\ell_4 - \omega_8) \mathcal{O}_{X_8}(\ell_5 - \omega_8) \mathcal{O}_{X_8}(\ell_6 - \omega_8) \\
\mathcal{O}_{X_8}(\ell_7 - \omega_8) \mathcal{O}_{X_8}(\ell_8 - \omega_8) \\
\mathcal{O}_{X_8}(\ell_1) \mathcal{O}_{X_8}(\ell_2) \mathcal{O}_{X_8}(\ell_3)
\end{array}
\]
For applied. Moreover, the same procedure can be applied to an arbitrary minimum 3-block collection into two blocks, and to the obtained 4-block collection, the inverse sequence of mutations is revertible. The reverse procedure is the following: a distinguished block is divided clear which block is distinguished, and into what parts it should be divided. One easily sees that the lifting procedure used to obtain all minimum 3-block collections in the latter Over \( X_8 \), the 4-block collection \( (\mathcal{L}_{7,8}(-1), \sigma^*\tau(6.1)) \) is taken by \( R_1^{(3)} \circ L_3 \circ R_1 \) to the desired collection

\[
\tau(8.3) = \begin{array}{|c|c|c|}
\hline
\sigma^* E_7'' & T_8 & O_{X_8}(\ell - \omega_6) O_{X_8}(\ell - \omega_6)
\hline
E_8'' & T_8' & O_{X_8}(\ell - \omega_5) O_{X_8}(\ell - \omega_5)
\hline
\end{array}
\]

The bundle \( \sigma^* E_7'' \) is the right shift of the torsion sheaf \( O_{\ell_i}(-1) \) over \( \{O_{X_8}(\ell_i)\}_{i=4,5,6} \). This mutation is the lifting under \( \sigma: X_8 \to X_7 \) of the mutation described in item \( (7.3) \). The bundle \( E_8'' \) is the right shift of \( O_{\ell_i}(-1) \) over the same block \( \{O_{X_8}(\ell_i)\}_{i=4,5,6} \). The correspondent mutation is an extension, and we have \( c_1(E_8'') = \ell_4 + \ell_5 + \ell_6 + \ell_8 \), \( r(E_8'') = 4 \), and \( \mu(E_8'') = \mu(\sigma^* E_7'') = \frac{4}{3} \).

The bundles \( T_8 \) and \( T_8' \) are described in the previous item, and \( T_8'' \) is the left shift of \( O_{X_8}(-\omega_6) \) over \( \{O_{X_8}(\ell - \ell_i)\}_{i=1,2,3} \) (the middle block in \( \tau(6.1) \)). We have \( c_1(T_8'') = \omega_6 - \omega_3 \), \( r(T_8'') = 2 \), and \( \mu(T_8'') = \mu(T_8') = \mu(T_8) = \frac{3}{2} \).

\( (8.4) \). Applying \( R_1^{(3)} \circ L_3 \circ R_1 \circ L_2 \), to \( (\sigma^*\tau(3), \mathcal{L}_{4+8}) \), we obtain the collection corresponding to the solution \( (5,2,1) \).

\[
\tau(8.4) = \begin{array}{|c|c|c|}
\hline
E_8'' & F_{48} & O_{X_8}(\ell - \omega_3)
\hline
F_{48} & F_{58} & O_{X_8}(2\ell - \ell_1 - \ell_2 - \ell_3)
\hline
F_{58} & F_{68} & O_{X_8}(\ell - \ell_1 - \omega_6)
\hline
F_{68} & F_{78} & O_{X_8}(\ell - \ell_2 - \omega_8)
\hline
F_{78} & F_{88} & O_{X_8}(\ell - \ell_3 - \omega_8)
\hline
\end{array}
\]

To the solution \( (5,1,2) \), the collection \( L_4 \circ L_1 \circ L_1(\tau(8.4)) \) corresponds.

Here, \( E_8'' \) is the right shift of \( O_{X_8} \) over \( \{O_{X_8}(\ell - \ell_i)\}_{i=1,2,3} \). We have \( c_1(E_8'') = -2\omega_3 \), \( r(E_8'') = 5 \), and \( \mu(E_8'') = \frac{25}{7} \). The bundles \( F_{48} \) are obtained as the shifts of the torsion sheaves \( O_{\ell_i} \) over \( \{O_{X_8}(\ell), O_{X_8}(2\ell - \ell_1 - \ell_2 - \ell_3)\} \). We have \( c_1(F_{48}) = -\omega_3 - \ell_i \), \( i = 4,5,6,7,8 \), \( r(F_{48}) = 2 \), and \( \mu(F_{48}) = \frac{25}{7} \). This completes the proof of the proposition.

Note that the lifting procedure used to obtain all minimum 3-block collections in the latter proof, is revertible. The reverse procedure is the following: a distinguished block is divided into two blocks, and to the obtained 4-block collection, the inverse sequence of mutations is applied. Moreover, the same procedure can be applied to an arbitrary minimum 3-block collection \( (\mathcal{E}, \mathcal{F}, \mathcal{G}) \) over a Del Pezzo surface as well since the invariants \( (r(\mathcal{E}), r(\mathcal{F}), r(\mathcal{G})) \), \( (\alpha, \beta, \gamma) \), and \( (\chi(\mathcal{E}, \mathcal{F}), \chi(\mathcal{F}, \mathcal{G})) \) are determined by the equation itself and, therefore, take the same values as for one of the minimum collections obtained in the latter proof (see also \( (16), (17) \)). Here, it is clear which block is distinguished, and into what parts it should be divided. One easily sees that the ranks of sheaves contained in an exceptional collection and the dimensions of \( \text{Ext} \) spaces defined by \( (6) \) can be uniquely determined for the collection obtained by a mutation. Thus, applying the above-mentioned procedure to \( (\mathcal{E}, \mathcal{F}, \mathcal{G}) \), we get a 4-block collection, one of whose blocks consists of zero-rank sheaves. Taking into account 1.6, we arrive at the statement below.
4.3. Proposition. Let $S$ be a Del Pezzo surface other than $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, $\tau$ be a minimum 3-block collection of type $(\alpha, \beta, \gamma)$, $\alpha \leq \beta \leq \gamma$, over $S$. Then a divisor $D$ over $S$ and a sequence of mutations (which does not preserve the 3-block structure) exist such that this sequence takes $\tau(D)$ to $(\sigma^* \tau', (\mathcal{O}_{e_1}, \ldots, \mathcal{O}_{e_m}))$, where $e_1, \ldots, e_m$ are pairwise nonintersecting exceptional curves, $\sigma : S \to S'$ is the monoidal transform with center $\{e_1, \ldots, e_m\}$, and $\tau'$ is a minimum 3-block collection over $S'$.

5. The action of the Weyl group on complete 3-block collections

5.1. In this section, we denote by $X_r$ a Del Pezzo surface of degree $9 - r$, i.e., we do not fix an identification of $X_r$ with a plane with $r$ blown-up points. Denote by $I_r \subset \text{Pic} X_r$ the set of classes of exceptional curves, the latter being characterized by the equalities $e^2 = e \cdot \omega_r = -1$. Let $\mathcal{R}_r \subset \text{Pic} X_r$ be the set of vectors $s$ such that $s^2 = -2$ and $s \cdot \omega_r = 0$. The form obtained by changing sign of the intersection form on $\text{Pic} X_r \otimes \mathbb{R}$ induces the structure of a Euclidean space on the orthogonal complement to $\omega_r$. The set $\mathcal{R}_r$ is a root system in it. The Weyl group $W(\mathcal{R}_r)$ of this system, which is generated by symmetries with respect to the roots, coincides with the group of automorphisms of the lattice $\text{Pic} X_r$ which preserve $\omega_r$ and the intersection form and also with the group of permutations of elements of $I_r$ which preserve pairwise intersection indices of the elements (see [14]). We denote this group by $W_r$ and call it the Weyl group. Over $\mathbb{P}^1 \times \mathbb{P}^1$, there is only one (up to the sign) divisor with square $-2$, i.e., the Weyl group for the quadric is $\mathbb{Z}_2$.

Agreement. All statements of this section concerning $X_r$ and $W_r$, except Proposition 5.4, concern $\mathbb{P}^1 \times \mathbb{P}^1$ as well.

Fix the isomorphism

$$v : K_0(X_r) \longrightarrow \mathbb{Z} \oplus \text{Pic} X_r \oplus \mathbb{Z}, \quad E \mapsto (r(E), c_1(E), 2c_2(E)),$$

where $c_2(E) = c_2^2(E)/2 - c_2^2(E)$ is the second component of the Chern character. The action of the Weyl group on $\text{Pic} X_r$ is in a natural way extended to the action on $K_0(X_r)$, and $r$ and $2c_2$ do not change therewith. We want to define the action of $W_r$ on the set of exceptional collections in concordance with the action on $K_0(X_r)$.

Put

$$g\mathcal{O}_{X_r}(D) \triangleq \mathcal{O}_{X_r}(gD) \quad \text{and} \quad g\mathcal{O}_\ell(m) \triangleq \mathcal{O}_{g\ell}(m).$$

5.2. Lemma. Let $(E, F)$ and $(E', F')$ be exceptional pairs of sheaves, where $v(E') = gv(E)$ and $v(F') = gv(F)$. Then the statements below hold,

(a) $v(R_r E') = g\mathcal{O}_{R_r E}^v E$ and $v(L_{E'} F') = g\mathcal{O}_{L_{F'} E'}(F)$;

(b) $\dim \text{Ext}^i(E', F') = \dim \text{Ext}^i(E, F)$, $\forall i$.

Proof. A pair $(E, F)$ is the elementary two-block collection. Propositions 2.2 and 2.3 imply that the mutation types of this pair depend on $r(E), r(F), c_1(E) \cdot K$, and $c_1(F) \cdot K$ only (the value of $\chi(E, F) = \chi_-(E, F)$ is determined by (6)). The action of the Weyl group preserves the intersection form and the canonical class. Hence, the above-mentioned invariants for $(E', F')$ take the same values as for $(E, F)$. Thus, both pairs have the same type of left and right mutations. Now, (a) is verified by direct computation using the exact sequences of Sec. 1.7, additivity of $v$, and linearity of $g$. The validity of (b) follows from the equality $\chi(E, F) = \chi(E', F')$ and the classification of exceptional pairs (see Sec. 1.7). The lemma is proved.
5.3. Proposition–definition. Let $\tau = (E_1, \ldots, E_n)$ be an exceptional collection and $g \in W_r$. Then a unique exceptional collection $\tau' = (E'_1, \ldots, E'_n)$ exists such that $v(E'_i) = g v(E_i)$ for $i = 1, \ldots, n$.

Put $g \tau \triangleq \tau'$ and $g E_i \triangleq E'_i$.

Proof. According to [11, 6.11], $\tau$ is included in a complete exceptional collection. Therefore, without loss of generality, we assume $\tau$ to be complete. By the constructivity theorem [11, 7.7], $\tau$ is obtained by a sequence of mutations (in the sense of Sec. 1.10) from the complete collection

$$\tau_1 = (O_{e_1}(-1), \ldots, O_{e_r}(-1), O_{X_r}, O_{X_0}, O_{X_0}(2)),$$

for which the proposition obviously holds. Applying this sequence of mutations to the complete exceptional collection $g \tau_1$, we obtain the desired exceptional collection according to the preceding lemma. The uniqueness follows from Proposition 1.3.

Thus, the Weyl group acts on the set of exceptional collections and preserves the ranks of sheaves and the dimensions of Ext spaces. According to 5.2a, this action commutes with mutations. In what follows, we are interested in the action of $W_r$ on complete 3-block collections of sheaves.

5.4. Proposition. Let $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ and $(\mathcal{E'}, \mathcal{F'}, \mathcal{G'})$ be two minimum collections of type $(\alpha, \beta, \gamma)$, $\alpha \leq \beta \leq \gamma$, corresponding to the same minimum solution (the latter is essential for the equations of group IV). Then an element $g \in W_r$ and a divisor $D \in \text{Pic} X_r$ exist such that

$$(\mathcal{E}, \mathcal{F}, \mathcal{G}) = g(\mathcal{E}'(D), \mathcal{F}'(D), \mathcal{G}'(D)).$$

The proof is carried out by induction on $r$. For $r = 0$, the statement is trivial due to triviality of $W_0$. Let the proposition hold for all $p < r$. By 4.3, there exist a divisor $D_1 \in \text{Pic} X_r$ and a sequence of mutations $\Phi$ (which does not preserve the 3-block structure) that takes $(\mathcal{E}(D_1), \mathcal{F}(D_1), \mathcal{G}(D_1))$ to a 4-block collection of the form $(\sigma^* \tau, L_e)$. Here $e = \{e_1, \ldots, e_{r-p}\}$ is a set of $(−1)$-curves with $e_i \cdot e_j = 0$ for $i \neq j$, the block $L_e$ consists of the sheaves $O_{e_i}, i = 1, \ldots, r − p$, the morphism $\sigma : X_r \to X_p$ is the blowing down for $e$, and $\tau$ is a 3-block collection over $X_p$. The same sequence of mutations $\Phi$ takes $(\mathcal{E}'(D_2), \mathcal{F}'(D_2), \mathcal{G}'(D_2))$ for some $D_2 \in \text{Pic} X_r$ to an analogous collection $(\sigma'^* \tau', L_{e'})$, where $e' = \{e'_1, \ldots, e'_{r-p}\}$ is another set of pairwise nonintersecting $(−1)$-curves, $\sigma' : X_r \to X_p$ is the blowing down for $e'$, and $\tau'$ is a minimum collection over $X_p$ of the same type as $\tau$.

Consider an element $g_0 \in W_r$ that takes $e'_i$ to $e_i, i = 1, \ldots, r − p$. Then

$$g_0(\sigma'^* \tau', L_{e'}) = (\sigma^* \tau'', L_e).$$

(19)

Here, $\tau''$ is the minimum collection over $X_p$ of the same type as $\tau$. By the induction assumption, an element $h \in W_p$ and a divisor $D_0 \in \text{Pic} X_p$ exist such that $\tau = h(\tau''(D_0))$. Identifying $W_p$ with a subgroup in $W_r$ that preserves all $(−1)$-curves $e_i, i = 1, \ldots, r − p$, and identifying $\text{Pic} X_p$ with the orthogonal complement to $e$ in $\text{Pic} X_r$ (with respect to the intersection form), we obtain

$$(\sigma^* \tau, L_e) = h(\sigma^* \tau''(D_0), L_e) = h(\sigma^* \tau''(D_0), L_e'(D_0))$$

(20)

(note that $O_{e_i}(D_0) = O_{e_i}$ since $e_i \cdot D_0 = 0$).

Combining (19) and (20) together, we get

$$(\sigma^* \tau, L_e) = h((g_0 \sigma^* \tau)(D_0), L_e(D_0)) = h(g_0(\sigma'^* \tau'(g_0^{-1} D_0)), L_{e'}(g_0^{-1} D_0)).$$
Applying to both sides of this equality the sequence of mutations $\Phi^{-1}$, and taking into account that mutations commute with twisting and the action of $W_r$, we obtain
\[(\mathcal{E}(D_1), \mathcal{F}(D_1), \mathcal{G}(D_1)) = hg_0(\mathcal{E}'(D_2 + g_0^{-1}D_0), \mathcal{F}'(D_2 + g_0^{-1}D_0), \mathcal{G}'(D_2 + g_0^{-1}D_0)).\]

Hence, the desired statement easily follows; it suffices to put $g = hg_0$ and $D = D_2 + g_0^{-1}D_0 + g^{-1}D_1$.

5.5. On the set of complete 3-block collection,
1. the group $\text{Pic} X_r$ acts by tensoring by invertible sheaves;
2. the 3-string braid group $B(3)$ acts by mutations;
3. the Weyl group $W_r$ acts.

The action of the braid group commutes with the action of $\text{Pic} X_r$ and $W_r$.

We call two 3-block collections equivalent, if one is obtained from another by tensoring. The preceding proposition means that $W_r$ acts transitively on the equivalence classes of minimum collections of type $(\alpha, \beta, \gamma)$, where $\alpha \leq \beta \leq \gamma$. By Theorem 3.10, any orbit with respect to $B(3)$ contains minimum collections of this type. Hence, $W_r$ acts transitively on $(B(3) \times \text{Pic} X_r)$-orbits of complete 3-block collections of a given structure.

For brevity, we call the set of all collections obtained from a complete 3-block collection by mutations and tensoring (i.e., its orbit under the action of $B(3) \times \text{Pic} X_r$) the orbit of this collection. The number of orbits of complete 3-block collections of a given structure is finite since $W_r$ is finite. We are going to compute these numbers. Note that a structure $\{\alpha, \beta, \gamma\}$ of a complete 3-block collection corresponds to a Markov-type equation in the table of Sec. 3.5.

5.6. Proposition. The number $C$ of equivalence classes of minimum collections of type $(\alpha, \beta, \gamma)$, $\alpha \leq \beta \leq \gamma$, which lie in one orbit is finite. These numbers are given in the bottom row of the table below (in the top row, equation numbers are given).

| (1) | (2) | (3) | (4) | (5) | (6.1) | (6.2) | (7.1) | (7.2) | (7.3) | (8.1) | (8.2) | (8.3) | (8.4) |
|-----|-----|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1   | 1   | 1   | 2   | 2   | 3     | 1     | 2     | 2     | 1     | 1     | 1     | 1     | 2     |

Proof. For the equation (1), this follows from the well-known fact that any exceptional collection over $\mathbb{P}^2$ which consists of invertible sheaves has the form $(\mathcal{O}_{\mathbb{P}^2}(m-1), \mathcal{O}_{\mathbb{P}^2}(m), \mathcal{O}_{\mathbb{P}^2}(m+1))$. For (2), the statement is also well-known and follows from the description of exceptional collections over $\mathbb{P}^1 \times \mathbb{P}^1$ that consist of invertible sheaves (see [5, 5.6]). Let us consider the remaining equations.

The main tool in the proof is mutations of 3-block helices; see 1.11. A helix $[\mathcal{E}, \mathcal{F}, \mathcal{G}]$ generated by a complete 3-block collection $[\mathcal{E}, \mathcal{F}, \mathcal{G}]$ is an infinite sequence of blocks
\[
(\ldots, \mathcal{E}(K), \mathcal{F}(K), \mathcal{G}(K), \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{E}(-K), \mathcal{F}(-K), \mathcal{G}(-K), \ldots).
\]

Any three successive blocks of this sequence form a complete 3-block exceptional collection called a foundation of the helix. Evidently, a helix is uniquely determined by any foundation. Mutations of foundations in the sense of Sec. 2 define mutations of helices. For example, the mappings $[\mathcal{E}, \mathcal{F}, \mathcal{G}] \mapsto [L_\mathcal{E} \mathcal{F}, \mathcal{E}, \mathcal{G}]$ and $[\mathcal{E}, \mathcal{F}, \mathcal{G}] \mapsto [\mathcal{E}, \mathcal{G}, R_\mathcal{G} \mathcal{F}]$ are helix mutations which coincide since $L_\mathcal{E} \mathcal{F} = L_\mathcal{E} L_\mathcal{G} R_\mathcal{G} \mathcal{F} = R_\mathcal{G} L_\mathcal{F}$. The ranks and lengths in any foundation of a helix satisfy one of the equations (3), and mutations of a helix induce mutations of the solutions of this equation, i.e., a route along the corresponding graph of solutions.
Lemma. Assume that a sequence of mutations $\Phi$ of a 3-block helix $[E, F, G]$ induces a cyclic route along the solution graph, the route containing no loop at the minimum solution. Then $\Phi$ preserves the 3-block helix.

Proof. Let $\Psi$ be the sequence of solution mutations induced by $\Phi$, and a solution $w$ be the starting and final point of the route. The solution graph contains no cycles; hence, $\Psi = M\Psi_1M$. Here, $M$ is one of the mutations $M_x, M_y, M_z$, and $\Psi_1$ is a sequence of solution mutations inducing a cyclic route that starts and finishes at $M(w)$. Using induction on the number of mutations in $\Psi$, we may consider that the sequence of helix mutations which induces $\Psi_1$ preserves the helix $\tilde{M}[E, F, G]$, where $\tilde{M}$ induces $M$. Hence, $\Phi[E, F, G] = \tilde{M}^2[E, F, G]$. The square of a solution mutation over one of the variables can only be induced by one of the equations of group IV), and $\Phi$ be a sequence of mutations that takes the first collection to the second one. Consider the sequence $\Psi$ of mutations of solution $s$ of the corresponding Markov-type equation which is induced by $\Phi$. It defines a cyclic route along the solution graph with starting and final point at the minimum solution. Two cases are possible.

(a) If $\Psi$ contains an even number of loops, then, as is proved above, the collections $(E, F, G)$ and $(E', F', G')$ are foundations of the same helix. For all equations except (6.1), $\alpha < \beta$ or $\beta < \gamma$, and coincidence of types of these collections implies their equivalence, i.e., $(E, F, G) = (E'(mK), F'(mK), G'(mK))$.

(b) If $\Psi$ contains an odd number of loops, then $\Psi = \Psi_1M\Psi_2$, where $M$ is the loop and $\Psi_2$ are the routes that start and finish at the minimum solution and contain even numbers of loops. Then $\Phi = \Phi_1\tilde{M}\Phi_2$, where $\Phi_1$ and $\Phi_2$ are sequences of mutations of 3-block collections which induce $\Psi_1$ and $\Psi_2$, and $\tilde{M}$ induces $M$. As is proved above, $\Phi_1$ and $\Phi_2$ preserve the 3-block helix. Hence,

$$[E', F', G'] = \Phi[E, F, G] = \tilde{M}[E, F, G].$$

Let us consider each group of equations separately.

I. The solution graph is loopless; hence, the case (a) is possible only.

For the equation (6.1), $\alpha = \beta = \gamma = 3$, and one can consider (up to tensoring by $mK_{X_6}$) that the collection $(E', F', G')$ coincides with one of the collections $(E, F, G), (G(K), E, F), (F(K), G(K), E)$. It is easy to check that these collections are not equivalent for $(E, F, G) = \tau_{(6.1)}$. Hence, by 5.4, they are not equivalent in a general case as well.

For (8.1), the statement is proved in (a).

II. Consider the equation (5). In the case (a), collections $(E, F, G)$ and $(E', F', G')$ are equivalent. In the case (b), applying the action of the Weyl group and tensoring the second collection by $mK_{X_5}$, we can obtain $(E, F, G) = \tau_{(5)}$ and $(E', F', G') = R_1R_2R_2\tau_{(5)}$. Direct computations show that the collections $\tau_{(5)}$ and $R_1R_2R_2\tau_{(5)}$ are not equivalent. Hence, in the case (b), the collections $(E, F, G)$ and $(E', F', G')$ are not equivalent too.
For (7.1) and (7.2), the reasoning is similar.

For (8.2), we have $\alpha < \beta < \gamma$. The types of foundations obtained by a single mutation from $[\mathcal{E}, \mathcal{F}, \mathcal{G}]$ are odd permutations of $(\alpha, \beta, \gamma)$. Hence, the case (b) is impossible.

III. For all equations of this group, we have $\alpha < \beta < \gamma$, and the reasoning is similar to that for (8.2) given above.

IV. For equations of this group, different minimum solutions correspond to the helices $[\mathcal{E}, \mathcal{F}, \mathcal{G}]$ and $\tilde{M}[\mathcal{E}, \mathcal{F}, \mathcal{G}]$, so the case (b) is also impossible here. But the number of equivalence classes of the minimum collections under consideration equals two, as well as the number of the minimum solutions. This completes the proof of the proposition.

5.7. The number of orbits. In this final subsection, we describe the computation of the number of orbits under the action of $B(3) \times \text{Pic} X_r$ on complete 3-block collections of a given structure. For $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, there is one orbit; this is shown in [4] and [19]. In the other cases, we find at first the number $N$ of equivalence classes of minimum collections of type $(\alpha, \beta, \gamma)$, where $\alpha \leq \beta \leq \gamma$. By 4.3, such a collection $\tau$ over a Del Pezzo surface $S$ with the help of tensoring and a sequence of mutations (determined by the type of the collection from the proof of Proposition 4.2) can be reduced to the form $(\sigma^* \tau', (O_{e_1}, \ldots, O_{e_m}))$, where $e_1, \ldots, e_m$ are pairwise nonintersecting exceptional curves, $\sigma: S \to S'$ is the monoidal transform with center $\{e_1, \ldots, e_m\}$, and $\tau'$ is a minimum 3-block collection over $S'$. Note that this sequence of mutations does not preserve the 3-block structure; and before applying it, one should find the distinguished block and divide it into two blocks, the length of one of them being $m$. Thus, we have

$$N \cdot \binom{n}{m} = N' \cdot \text{(the number of sets } \{e_1, \ldots, e_m\}, e_i \cdot e_j = 0),$$

where $n$ is the length of the distinguished block and $N'$ is the number of equivalence classes of minimum collections over $S'$ with the same type as that of $\tau'$. Computing the values of $N$ for all Markov-type equations with the help of this formula and dividing them by the correspondent values of $C$ from Proposition 5.6, we obtain the number of orbits. The results are presented in the table below.
| Equation number | Surface | $(\alpha, \beta, \gamma)$ | $N$ | $C$ | Number of orbits |
|-----------------|---------|-----------------|-----|-----|-----------------|
| (1)             | $\mathbb{P}^2$ | (1, 1, 1)       | 1   | 1   | 1               |
| (2)             | $\mathbb{P}^1 \times \mathbb{P}^1$ | (1, 1, 2)       | 1   | 1   | 1               |
| (3)             | $X_3$   | (1, 2, 3)       | 1   | 1   | 1               |
| (4)             | $X_4$   | (1, 1, 5)       | 2   | 2   | 1               |
| (5)             | $X_5$   | (2, 2, 4)       | 20  | 2   | 10              |
| (6.1)           | $X_6$   | (3, 3, 3)       | 240 | 3   | 80              |
| (6.2)           | $X_6$   | (1, 2, 6)       | 36  | 1   | 36              |
| (7.1)           | $X_7$   | (1, 1, 8)       | 72  | 2   | 36              |
| (7.2)           | $X_7$   | (2, 4, 4)       | 2520| 2   | 1260            |
| (7.3)           | $X_7$   | (1, 3, 6)       | 672 | 1   | 672             |
| (8.1)           | $X_8$   | (1, 1, 9)       | 1920| 1   | 1920            |
| (8.2)           | $X_8$   | (1, 2, 8)       | 8640| 1   | 8640            |
| (8.3)           | $X_8$   | (2, 3, 6)       | 80640| 1  | 80640           |
| (8.4)           | $X_8$   | (1, 5, 5)       | 96768| 2  | 48384           |

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