Quantum measurement and uncertainty relations in photon polarization

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Abstract

Recent theoretical and experimental studies have given rise to new aspects in quantum measurements and error-disturbance uncertainty relations. After a brief review of these issues, we present an experimental test of the error-disturbance uncertainty relations in photon polarization measurements. Using a generalized, strength-variable measurement of a single photon polarization state, we experimentally evaluate the error and disturbance in the measurement process and demonstrate the validity of recently proposed uncertainty relations.

Keywords: quantum optics, quantum measurement, uncertainty relation, photon polarization

1. Introduction

Measurement (or observation) is the most fundamental issue in science and technology. We understand the laws of nature only through measurements. In classical science, we tacitly assume the existence of perfect measurement, in which one can measure a physical observable, e.g., a particle’s position, very accurately without disturbing the object’s state. This property ensures the objectivity of the measurement outcome and thus the reality of physical observable. However, in quantum mechanics, such perfect measurement does not exist; any measurement with finite accuracy inevitably disturbs the object’s state. Heisenberg considered such a situation in his famous thought experiment on the gamma-ray microscope [1]. Since then, problems on quantum measurement and the uncertainty relation have long been discussed in fundamental quantum physics. However, surprisingly, no commonly agreed definitions have been established on the measurement error and disturbance, and thus on uncertainty relations in quantum measurements. Recent theoretical progress in quantum measurements and uncertainty relations has revealed the new aspects of these issues [2–6]. In addition, the concept of weak measurement and weak value [7] has attracted great attention.

Photons are the quanta of electromagnetic waves. A photon propagating in a vacuum has two degrees of freedom for its polarization, corresponding to two orthogonal directions of its oscillatory electric fields perpendicular to the propagation direction. Thus the polarization state of a photon can be treated as a two-level qubit system, one of the simplest and most fundamental systems in quantum physics. This polarization qubit held in a single photon is extremely useful in quantum information and communication technologies. However, as mentioned above, quantum measurement of even the simplest system has not yet been fully understood. In this article, I describe our recent experiments in which we realize the generalized measurement of the photon polarization qubit and evaluate its uncertainty relations.

This article is organized as follows. In section 2, a quantum-mechanical view of photon polarization is presented, mainly for non-specialists. In section 3, the general theory of quantum measurement is briefly reviewed. In section 4, definitions of error and disturbance in quantum measurements are introduced and discussed. In section 5, uncertainty relations in quantum measurement are introduced, by contrast with the uncertainty relations in quantum state preparation. In section 6, our experimental results on the generalized measurement of photon polarization are presented.
and compared with the uncertainty relations. Section 7 is the summary.

2. Quantum optics in photon polarization

Here we consider a plane electromagnetic wave propagating along the z-axis. Then, we take the x- and y-axes as the two orthogonal directions in the oscillatory field plane. The field along each axis is expressed by a harmonic oscillator and thus the fields in the x–y plane are expressed by the two-dimensional harmonic oscillator. Let \( \hat{a}_x \) and \( \hat{a}_y \) (i.e., \( x \) or \( y \)) be the creation and annihilation operators of the field parallel to \( x \) or \( y \), respectively and \( \hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \) the corresponding number operator. A simultaneous eigenstate of \( \hat{n}_x \) and \( \hat{n}_y \) can be expressed as \([n, m]\), where \( n \) and \( m \) are the eigenvalues of \( \hat{n}_x \) and \( \hat{n}_y \), i.e., the photon numbers in the \( x \) and \( y \) polarization modes. In general, a pure state \( |\psi\rangle \) of the two-mode field can be expressed as

\[
|\psi\rangle = \sum_{n,m} c_{n,m} |n, m\rangle. \tag{1}
\]

In classical optics, the polarization state is often characterized by the Stokes parameters \([8]\). In quantum optics, the Stokes parameters turn out to be a set of operators, i.e., the Stokes operators \( \hat{S}_0, \hat{S}_1, \hat{S}_2, \hat{S}_3 \) can be expressed as

\[
\hat{S}_0 = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y = \hat{n}_x + \hat{n}_y, \tag{2}
\]

\[
\hat{S}_1 = \hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_x = \hat{n}_x - \hat{n}_y, \tag{3}
\]

\[
\hat{S}_2 = \hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger \hat{a}_x, \tag{4}
\]

\[
\hat{S}_3 = -i(\hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_x), \tag{5}
\]

where \( \hat{S}_0 \) corresponds to the total photon number, \( \hat{S}_1 \) and \( \hat{S}_2 \) present the degree of polarization in the \( x \)–\( y \) plane, \( \pm45^\circ \), and left and right circular polarizations, respectively. The Stokes operators are often used to characterize the polarization state of the quantized optical field.

Hereafter, we consider single photon polarization states, in which a single photon stays in one of the polarization modes, i.e., \( n + m = 1 \). A pure state of the single photon polarization is expressed by the linear combination of \( |1, 0\rangle \) and \( |0, 1\rangle \):

\[
|\psi\rangle = \alpha|1, 0\rangle + \beta|0, 1\rangle, \tag{6}
\]

where \( |\alpha|^2 + |\beta|^2 = 1 \). For this state, we find

\[
\langle \hat{a}_x \rangle = \langle \hat{a}_x^\dagger \rangle = \langle \hat{a}_y \rangle = \langle \hat{a}_y^\dagger \rangle = 0 \tag{7}
\]

and thus the mean values of all the field amplitudes are zero. Nonetheless, the mean value of the Stokes operators are obtained as:

\[
\langle \hat{S}_0 \rangle = |\alpha|^2 + |\beta|^2 = 1, \tag{8}
\]

\[
\langle \hat{S}_1 \rangle = |\alpha|^2 - |\beta|^2, \tag{9}
\]

These are equivalent to those of the classical pure polarization state associated with the field amplitude vector (Jones vector) \((\alpha, \beta)\). Thus the single photon polarization state (6) is a pure polarization state satisfying

\[
\langle \hat{S}_0 \rangle^2 + \langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 = \langle \hat{S}_3 \rangle^2, \tag{12}
\]

where the Stokes vector \((\langle \hat{S}_1 \rangle, \langle \hat{S}_2 \rangle, \langle \hat{S}_3 \rangle)\) reaches the surface of the Poincaré sphere with the radius \(\langle \hat{S}_0 \rangle\).

The two bases, \(|1, 0\rangle\) and \(|0, 1\rangle\), in (6) are often written as

\[
|1, 0\rangle \equiv |H\rangle, \quad |0, 1\rangle \equiv |V\rangle, \tag{13}
\]

where \(H\) and \(V\) mean horizontal and vertical polarizations, respectively. Then, the single photon polarization state can be expressed as

\[
|\psi\rangle = \alpha|H\rangle + \beta|V\rangle. \tag{14}
\]

It is also convenient to define the linear polarization states along \(\pm45^\circ\) directions, \(|D\rangle\) and \(|A\rangle\), as

\[
|D\rangle \equiv \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \quad |A\rangle \equiv \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle), \tag{15}
\]

and the left and right circular polarization states, \(|L\rangle\) and \(|R\rangle\), as

\[
|L\rangle \equiv \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle), \quad |R\rangle \equiv \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle). \tag{16}
\]

These are the eigenstates of \(\hat{S}_1\), \(\hat{S}_2\), and \(\hat{S}_3\):

\[
\hat{S}_1|H\rangle = |H\rangle, \quad \hat{S}_1|V\rangle = -|V\rangle, \tag{17}
\]

\[
\hat{S}_2|D\rangle = |D\rangle, \quad \hat{S}_2|A\rangle = -|A\rangle, \tag{18}
\]

\[
\hat{S}_3|L\rangle = |L\rangle, \quad \hat{S}_3|R\rangle = -|R\rangle. \tag{19}
\]

The single photon polarization state is expressed by these bases in the two-dimensional Hilbert space; it is thus expressed by SU(2) algebra as in the spin 1/2 system. In practice, using \(|H\rangle\) and \(|V\rangle\) as the bases, the matrix representations of the Stokes operators result in the Pauli matrices:

\[
|\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |\sigma_1 = \sigma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{19}
\]

\[
|\sigma_2 = \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad |\sigma_3 = \sigma_5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{20}
\]

Thus, when the single photon polarization states are concerned, the Stokes operators are equivalent to the Pauli matrices. Accordingly, the Stokes vector and the Poincaré sphere for the single photon polarization states are equivalent to the Bloch vector and the Bloch sphere for a two-level qubit system, respectively.

\[\text{Reference:}\] Defined in equations (2)–(5), the Stokes operators correspond to the classical Stokes parameters given, e.g., in \([8]\). Note that we may use an alternative definition of Stokes operators, where \(\hat{\delta}_1, \hat{\delta}_2\) and \(\hat{\delta}_3\) are defined as the right-hand sides of equations (4), (5) and (3) respectively.

\[\text{Reference:}\] The labeling of Pauli matrices in equation (20) is chosen so that \(\sigma_0, \sigma_2\) and \(\sigma_3\) represent the corresponding Stokes operators \(\hat{\delta}_1, \hat{\delta}_2\) and \(\hat{\delta}_3\) defined in equations (3)–(5), respectively. If the alternative definition was used, we would obtain \(\sigma_1 = \sigma_3, \sigma_2 = \sigma_5, \text{ and } \sigma_3 = \sigma_1\).
with a discrete spectrum and a



\[ \hat{\mathcal{M}}_m |\psi\rangle \]

on the probe. Because of \( \hat{U} \), the observable \( \hat{A} \) on the signal can be correlated with the observable \( \hat{M} \) on the probe. Then \( \hat{M} \) is measured on the probe. This procedure indirectly measures \( \hat{A} \) on the signal.

### 3. Quantum measurement

#### 3.1. Projective measurement

Here we consider a model of the quantum measurement of an observable \( \hat{A} \) with a discrete spectrum and a finite dimension. The spectral decomposition of \( \hat{A} \) is expressed as

\[ \hat{A} = \sum_j \lambda_j \hat{\Pi}_j, \]

where \( \lambda_j \) is the eigenvalue of \( \hat{A} \) and \( \hat{\Pi}_j \equiv |j\rangle \langle j| \) is the projector to the corresponding eigenstate \( |j\rangle \). The projective measurement of \( \hat{A} \) observes the state in one of the eigenstates \( |j\rangle \), and assigns the outcome \( \lambda_j \). When the projective measurement acts on the state \( |\psi\rangle \), the probability \( P(j) \) to find the state in \( |j\rangle \) is given by

\[ P(j) = \langle \psi | \hat{\Pi}_j | \psi \rangle = \langle \hat{\Pi}_j \rangle, \]

where the average \( \langle ... \rangle \) is taken over the signal state \( |\psi\rangle \). The mean value of the measurement of \( \hat{A} \) for the state \( |\psi\rangle \) is obtained as

\[ \langle \hat{A} \rangle = \sum_j \lambda_j \langle \hat{\Pi}_j \rangle. \]

#### 3.2. Generalized measurement

We introduce a model of a measurement instrument in which the signal \( |\psi\rangle \) interacts with a probe (meter) state \( |\xi\rangle \) through the interaction unitary operator \( \hat{U} \), as shown in figure 1. After the interaction, the initial state \( |\Psi\rangle = |\psi\rangle \otimes |\xi\rangle \) is converted to

\[ |\Psi\rangle \xrightarrow{\hat{U}} |\Psi'\rangle = \hat{U} (|\psi\rangle \otimes |\xi\rangle). \]

Then, we make the projective measurement of observable \( \hat{M} \) on the probe state; the spectral decomposition of \( \hat{M} \) is given by

\[ \hat{M} = \sum_m \mu_m |m\rangle \langle m|, \]

where \( \mu_m \) is the eigenvalue of \( \hat{M} \) and \( |m\rangle \) the corresponding eigenvector. The state \( |\Psi'\rangle \) after the interaction can be decomposed in terms of \( |m\rangle \):

\[ |\Psi'\rangle = \sum_m (\hat{M}_m |\psi\rangle) \otimes |m\rangle, \]

where \( \hat{M}_m \) is the measurement operator acting on the signal state \( |\psi\rangle \):

\[ \hat{M}_m = \langle m| \hat{U} |\xi\rangle. \]

The probability \( P(m) \) to find the probe state in \( |m\rangle \) is

\[ P(m) = \langle \hat{M}_m^\dagger \hat{M}_m \rangle = \langle \hat{E}_m \rangle \]

where \( \hat{E}_m = \hat{M}_m^\dagger \hat{M}_m \) is the positive operator valued measure (POVM) element satisfying

\[ \sum_m \hat{E}_m = \hat{1}, \]

where \( \hat{1} \) is the identity operator. Thus the POVM element \( \hat{E}_m \) determines the probability of finding the measurement outcome in \( m \). The signal state is changed from its initial state \( |\psi\rangle \) by \( \hat{M}_m \); the measurement operator characterizes the backaction of the measurement. The measurement system characterized by equation (26) is referred to as generalized measurement. When \( \hat{A} \) and \( \hat{M} \) have the same spectrum and if \( \hat{M}_j = \hat{E}_j \equiv \hat{\Pi}_j \) by substituting \( m = j \), equation (28) is equivalent to (22). Thus, in this special case, the measurement turns out to be the projective measurement of \( \hat{A} \). However, in the context of generalized measurement, we can design not only the projective measurement but also weak and approximate measurement in which we control the measurement strength and the backaction caused by the measurement.

#### 3.3. Measurement of photon polarization

As described in section 2, the Pauli matrices are the observables of the single photon polarization quibit. Their measurement outcomes are \pm 1, each of which corresponds to one of the two orthogonal polarization states where the single photon is found. For instance, \( \sigma_z \) measures the polarization state in \( |H\rangle \) or \( |V\rangle \), and \( \sigma_z \) measures the polarization state in \( |D\rangle \) or \( |A\rangle \).

In experimental optics, polarization beamsplitters (PBS) are commonly used for polarization measurements. A PBS transmits one polarization component (p-component) parallel to the plane of incidence and reflects the other component (s-component) perpendicular to the plane of incidence. Hereafter, we take the laboratory coordinates so that p-component is horizontal (\( H \)) and the s-component is vertical (\( V \)). Thus, a PBS treats the polarization degrees of freedom (\( H \) or \( V \)) and the path degrees of freedom (1 or 2). The input–output relation between the field operators is

\[ \begin{pmatrix} \hat{a}_{H1}^\dagger \\ \hat{a}_{H2}^\dagger \\ \hat{a}_{V1}^\dagger \\ \hat{a}_{V2}^\dagger \end{pmatrix} = \hat{U}_{\text{PBS}} \begin{pmatrix} \hat{a}_{H1} \\ \hat{a}_{H2} \\ \hat{a}_{V1} \\ \hat{a}_{V2} \end{pmatrix}, \]

where

- \( \hat{a}_{HI} \) is the horizontal input mode.
- \( \hat{a}_{HI} \) is the horizontal output mode.
- \( \hat{a}_{VI} \) is the vertical input mode.
- \( \hat{a}_{VI} \) is the vertical output mode.
where the measurement operators of this model apparatus are
\[ \hat{M}_+ = \cos \theta |H \rangle \langle H| + \sin \theta |V \rangle \langle V| = \frac{1}{\sqrt{2}} (\alpha \hat{I} + \beta \hat{\sigma}_z), \]
\[ \hat{M}_- = \sin \theta |H \rangle \langle H| + \cos \theta |V \rangle \langle V| = \frac{1}{\sqrt{2}} (\alpha \hat{I} - \beta \hat{\sigma}_z), \]
where \( \alpha = \cos(\pi/4 - \theta) \) and \( \beta = \sin(\pi/4 - \theta) \). The corresponding POVM elements are
\[ \hat{E}_+ = \cos^2 \theta |H \rangle \langle H| + \sin^2 \theta |V \rangle \langle V| = \frac{1}{2} (\hat{I} + \cos 2\theta \hat{\sigma}_z), \]
\[ \hat{E}_- = \sin^2 \theta |H \rangle \langle H| + \cos^2 \theta |V \rangle \langle V| = \frac{1}{2} (\hat{I} - \cos 2\theta \hat{\sigma}_z). \]

The measurement is a projective measurement for \( \theta = 0 \), since in this case \( \hat{M}_+ \) and \( \hat{M}_- \) are the projectors to \(|H\rangle \) and \(|V\rangle \), respectively. On the other hand, the measurement is the null measurement for \( \theta = \pi/4 \), in this case the measurement for any state returns either outcome with even probability. The measurement strength is characterized by \( s = \cos 2\theta \) \((0 \leq s \leq 1)\). Thus this protocol realizes the generalized measurement with variable measurement strength.

The optical implementation of the generalized measurement described above is shown in figure 3, which we call a variable polarization beamsplitter (VPBS) [10, 11]. Here, the signal to be measured is the single photon polarization qubit, and the probe is the path qubit, i.e., the use of path degrees of freedom \(|+1\rangle \) and \(|-1\rangle \) of the photon, the roles of which are the same as those of a PBS. The quantum circuit model of this apparatus is shown in figure 3 (b). Although the circuit is a bit different from that in figure 2, these are statistically equivalent, i.e., both give the same measurement operators given in (37) and (38) for the same probe input \(|\xi\rangle = |+1\rangle\).

\[ \Delta x_{\text{rms}} = \sqrt{\langle (x - x_0)^2 \rangle}. \]

Note that \( \Delta x_{\text{rms}} \) includes both accuracy and precision, as the distance between the true value and the mean value of the outcomes on one hand, and as the distribution of the measurement outcomes on the other hand.

However, in quantum measurement, we cannot assume the true value in general. Instead, the measurement results are
Figure 3. (a) Generalized polarization measurement using a variable polarizing beamsplitter (VPBS) [10, 11]. PBS and HWP stand for polarization beamsplitters and half-wave plates, respectively. The signal is a photon polarization qubit $|\psi\rangle$, and the probe is a path qubit $|\xi\rangle$. A photon having polarization $|\psi\rangle$ is injected from the path $|\xi\rangle = |+1\rangle$, and exits from either of the output path $|m\rangle = |+1\rangle$ or $|−1\rangle$ depending on the measurement outcome of the polarization qubit. (b) Quantum circuit model of the VPBS.

generally probabilistic and their distribution depends on the signal’s state. Also, in general, the object’s state is disturbed because of the back-action of the measurement, resulting in the disturbance on the sequential or joint measurement. Because of this property of quantum measurement, no commonly agreed definitions of the measurement error and disturbance have been established to date. Nevertheless, there are a number of proposals on the definitions of error in quantum measurement. Some of them are defined as state-dependent, i.e., the amounts of error and disturbance are dependent on the state of the object to be measured, and others are defined as state-independent. Also, some of them are defined based on the RMS of measurement outcomes, and others are defined based on information-theoretic quantities. There are active discussions and debates on this matter [3, 5, 6]. Here, we introduce a state-dependent definition of the measurement error given in the general theory of quantum instruments proposed by Ozawa [2].

We consider the generalized measurement model introduced in section 3.2 (figure 1). The signal state $|\psi\rangle$ is subjected to the measurement of an observable $\hat{A}$ by an instrument in which $|\psi\rangle$ interacts with the probe state $|\xi\rangle$ through $\hat{U}$. The measurement outcome is obtained by observing the probe’s observable $\hat{M}$ after the interaction. After the measurement of $\hat{A}$, the signal is then subjected to the projective measurement of another observable $\hat{B}$. Using the Heisenberg picture, the observables $\hat{M}$ and $\hat{B}$ acting on the input state $|\psi\rangle = |\psi\rangle \otimes |\xi\rangle$ are:

$$\hat{M}_A = \hat{U}^{-1} (\hat{1} \otimes \hat{M}) \hat{U},$$

$$\hat{M}_B = \hat{U}^{-1} (\hat{B} \otimes \hat{1}) \hat{U}.$$  

These are the observables corresponding to what is actually measured by this instrument. The noise operator $\hat{N}(A)$ and the disturbance operator $\hat{D}(B)$ are defined as the difference between the observables that we actually measure and that we want to measure:

$$\hat{N}(A) = \hat{M}_A − \hat{A},$$

$$\hat{D}(B) = \hat{M}_B − \hat{B}.$$  

Here and hereafter, we use the abbreviation: $\hat{A} \otimes \hat{I}$ as simply $\hat{A}$, and $\hat{B} \otimes \hat{I}$ as $\hat{B}$. Then the error $\epsilon(\hat{A})$ and the disturbance $\eta(\hat{B})$ in the measurement of $\hat{A}$ and $\hat{B}$ are defined as the RMS of $\hat{N}$ and $\hat{D}$ [2]:

$$\epsilon(\hat{A}) = \sqrt{\langle \hat{N}(\hat{A})^2 \rangle}, \quad \eta(\hat{B}) = \sqrt{\langle \hat{D}(\hat{B})^2 \rangle}.$$  

These definitions of error and disturbance were given by Ozawa [2]. The same or similar definitions were proposed and widely used by Arthurs and Kelly [12], Arthurs and Goodman [13], Appleby [14, 15], Hall [16], and Branciard [4], etc. It is important to note that, if $\hat{M}_A$ and $\hat{A}$ commute, equation (46) corresponds to the classical RMS error [3]. In this sense, equation (46) is considered to be the generalization of the classical RMS error to the quantum measurement.

4.2. Evaluation of error and disturbance

From the definition of the measurement error (46), we get

$$\epsilon(\hat{A})^2 = \langle (\hat{M}_A − \hat{A})^2 \rangle = \langle \hat{M}_A^2 \rangle − \langle \hat{A}^2 \rangle = \langle \hat{M}_A\hat{A} + \hat{A}\hat{M}_A \rangle.$$  

and a similar relation for $\eta(\hat{B})^2$. The first two terms of (47) can be evaluated experimentally or theoretically. Using (28) and (21)

$$\langle \hat{M}_A^2 \rangle = \sum_m \mu_m^2 \langle \hat{E}_m \rangle, \quad \langle \hat{A}^2 \rangle = \sum_j \lambda_j^2 \langle \hat{I}_j \rangle.$$  

For instance, in the case of qubit measurement, where $\mu_m = \lambda_j = ±1$, these terms turn out to be unity. However, the experimental evaluation of the third term, which presents the correlation between $\hat{M}_A$ and $\hat{A}$, is not so straightforward.
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One method is to transform the third term of (47) as
\[
\langle \hat{M}_z \hat{A} + \hat{A} \hat{M}_z \rangle = \langle (\hat{I} + \hat{A}) \hat{M}_z (\hat{I} + \hat{A}) \rangle - \langle \hat{A} \hat{M}_z \hat{A} \rangle - \langle \hat{M}_z \hat{A} \rangle. \tag{49}
\]
In this form, the first term of the right-hand side is the expected value of \( \hat{M}_z \hat{A} \) for the signal state \((\hat{A} + \hat{I})|\psi\rangle\). Also, the second and third terms are those of the states \(\hat{A}|\psi\rangle\) and \(|\psi\rangle\), respectively. Thus, if these three states are prepared, one can evaluate the experimental error by (47) and (49), as illustrated in figure 4(a). This procedure, called as the three-state method, is given by Ozawa [17]. Similarly, we show that one may use the relation
\[
2 \langle \hat{M}_z \hat{A} + \hat{A} \hat{M}_z \rangle = \langle (\hat{I} + \hat{A}) \hat{M}_z (\hat{I} + \hat{A}) \rangle - \langle (\hat{I} - \hat{A}) \hat{M}_z (\hat{I} - \hat{A}) \rangle, \tag{50}
\]
preparing \((\hat{I} + \hat{A})|\psi\rangle\) and \((\hat{I} - \hat{A})|\psi\rangle\) as the signal states (two-state method). In the case of qubit measurement, \(\hat{A} = \sigma_z\); for instance, \(\sigma_z\) presents the rotation on the Bloch sphere, \((\hat{I} \pm \sigma_z)/2\) are the projectors to the two eigenstates of \(\sigma_z\). Thus, it is not difficult to prepare these states in experiments. In practice, the three-state method was used to evaluate the error and disturbance in the measurement of neutron spin [18] and photon polarization [11]. However, in general cases, it is difficult to implement the operation \(\hat{I} \pm \hat{A}\) or even \(\hat{A}\) in practical experiments, and thus the applicability of (49) or (50) is not so obvious.

The other method is to use weak values. Using the POVM elements \(\hat{E}_w\) of the measurement \(\hat{M}_w\), we obtain
\[
\langle \hat{M}_z \hat{A} + \hat{A} \hat{M}_z \rangle = 2 \text{Re} \langle \hat{E}_w \hat{A} \rangle = 2 \sum_m \mu_m \text{Re} \langle \hat{E}_w \hat{A} \rangle = 2 \sum_{j,m} \lambda_j \mu_m \text{Re} \langle \hat{E}_w \hat{A} \rangle. \tag{51}
\]
In the last expression, \(\text{Re} \langle \hat{E}_w \hat{A} \rangle \equiv P_w(j, m)\) is called the weak-valued joint probability [19], which is related to the weak value [7, 20]:
\[
\text{Re} \langle \hat{A} \rangle_w = \frac{\text{Re} \langle \hat{E}_w \hat{A} \rangle}{\langle \hat{E}_w \rangle} = \sum_j \lambda_j P_w(j, m). \tag{52}
\]
Using the relations (47), (48), (51), and
\[
\langle \hat{E}_w \rangle = \sum_j P_w(j, m), \quad \langle \hat{E}_w \rangle = \sum_m P_w(j, m), \tag{53}
\]
one finds [9, 21]
\[
\epsilon(A)^2 = \sum_{j,m} (\mu_m - \lambda_j)^2 P_w(j, m). \tag{54}
\]
Thus, we can evaluate the measurement error if we know every weak-valued joint probability \(P_w(j, m)\). Lund and Wiseman [9] derived (54) and presented a practical example using the generalized qubit probability \(P_w(j, m)\) as described in section 3.3. However, for general \(\hat{A}\), the experimental evaluation of every weak-valued joint probability is usually impractical.

In what follows, we show a more practical and general procedure to evaluate the measurement error and disturbance using weak measurement or weak probe (weak probe method, illustrated in figure 4(b)). In order to evaluate \(\text{Re} \langle \hat{E}_w \hat{A} \rangle\), one may use a qubit as a probe that interacts with the signal via
\[
\hat{U} = \exp(i \hat{g} \hat{A} \otimes \sigma_z), \tag{55}
\]
where \(g (\geq 0)\) is the coupling strength. The measurement is done by detecting the probe state in either of the eigenstates of \(\sigma_z\), i.e., \(|+\rangle\) or \(|-\rangle\). Assuming the initial probe state is \((|+\rangle + |-\rangle)/\sqrt{2}\), the corresponding measurement operators are
\[
\hat{W}_w = \frac{1}{\sqrt{2}} [\cos(g \hat{A}) \pm \sin(g \hat{A})]. \tag{56}
\]
When the coupling is sufficiently weak, i.e., \(g \langle \hat{A} \rangle \ll 1\), equation (56) can be approximated as
\[
\hat{W}_w \approx \frac{1}{\sqrt{2}} (\hat{I} \pm g \hat{A}). \tag{57}
\]
After this weak measurement, the signal is subject to the main measurement presented by the POVM elements \(\hat{E}_w\). The joint probability \(P(w, m)\) obtaining the outcomes of the weak probe in \(w = \pm 1\) and the main measurement in \(m\) is given by
\[
P(w, m) = \langle \hat{W}_w \hat{E}_w \hat{W}_w \rangle \approx \frac{1}{2} \langle (\hat{I} \pm g \hat{A}) \hat{E}_w (\hat{I} \pm g \hat{A}) \rangle = \frac{1}{2} (\hat{E}_w \pm \frac{g}{2} \hat{A}) \hat{E}_w \hat{A} + \hat{A} \hat{E}_w \hat{A} + \frac{g^2}{2} (\hat{A} \hat{E}_w \hat{A}). \tag{58}
\]
Thus
\[
\sum_w P(w, m) \approx g (\hat{E}_w \hat{A} + \hat{A} \hat{E}_w \hat{A}) = 2g \text{Re} \langle \hat{E}_w \hat{A} \rangle. \tag{59}
\]
Using (47), (51) and (59), \(\text{Re} \langle \hat{E}_w \hat{A} \rangle\) and thus the measurement error can be evaluated by measuring the probability

![Figure 4](https://example.com/figure4.png)

Figure 4. Methods to evaluate measurement error and disturbance: (a) three-state method and (b) weak probe method.
\[ P(w, p) \text{ within the weak coupling limit:} \]
\[ \epsilon(A)^2 \simeq \langle \hat{M}_A^2 \rangle + \langle \hat{A}^2 \rangle - \frac{1}{g} \sum_{w, m} w \mu_m P(w, m). \]  
(60)

When the signal observable is also a qubit, e.g., \( \hat{A} = \sigma_z \) (more generally when \( \hat{A} = \hat{I} \)), (56) reduces to
\[ \hat{W}_\pm = \frac{1}{\sqrt{2}} (\cos g \hat{I} \pm \sin g \sigma_z) = \frac{1}{\sqrt{2}} (\alpha \hat{I} \pm \beta \sigma_z), \]
(61)
where we put \( \alpha = \cos g \) and \( \beta = \sin g \). Thus, in this case, \( \hat{W}_\pm \) is equivalent to the measurement operators (37) and (38) for the generalized qubit measurement described in section 3.3. We get
\[ P(w, m) = \langle \hat{W}_w \hat{E}_m \hat{W}_h \rangle \]
\[ = \frac{1}{2} \langle (\alpha \hat{I} \pm \beta \sigma_z) \hat{E}_m (\alpha \hat{I} \pm \beta \sigma_z) \rangle \]
\[ = \alpha^2 \langle \hat{E}_m \rangle + \alpha \beta \langle \sigma_z \hat{E}_m \rangle + \beta^2 \langle \hat{E}_m \sigma_z \rangle, \]
\[ = \alpha^2 \langle \hat{E}_m \rangle - \alpha^2 \beta \langle \hat{E}_m \sigma_z \rangle, \]
\[ = \alpha^2 \langle \hat{E}_m \rangle - \alpha^2 \beta \langle \hat{E}_m \sigma_z \rangle, \]
\[ \sum_w w \mu_w P(w, m) = 2 \alpha \beta \text{Re} \langle \hat{E}_m \sigma_z \rangle. \]  
(62)
Thus we obtain
\[ \epsilon(A)^2 = 2 - \frac{1}{2} \sum_w w \mu_w P(w, m) \]
\[ = 2 - \frac{2}{2} \sum_w w \mu_w P(w, m). \]  
(63)
Here, \( 2 \alpha \beta = \sin 2g \) is the measurement strength of the weak probe. Note that in the qubit case the coupling strength \( g \) is not necessarily weak; equations (63) and (64) are valid for any \( g \). Thus, we can even use the projective measurement where \( \alpha = \beta = 1/\sqrt{2} \). In this case, it is interesting to observe that (63) is equivalent to the procedure obtained in the two-state method (50).

5. Uncertainty relations in quantum measurement

5.1. Heisenberg’s relation

In 1927, Heisenberg considered the relationship between the measurement error and disturbance in his famous thought experiment of gamma-ray microscope [1]. His relation is written as
\[ \epsilon(x) \eta(p) \geq \frac{\hbar}{2}, \]  
(65)
where \( \epsilon(x) \) is the measurement error of the position \( x \), and \( \eta(p) \) the disturbance in the momentum \( p \). He obtained equation (65), under some assumptions [22], from the relation proven by Kennard [23]
\[ \sigma(x) \sigma(p) \geq \frac{\hbar}{2}, \]  
(66)
between the standard deviations (e. g., \( \sigma(x) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \) of \( x \) and \( p \). Later, Arthurs and Kelly [12] quantified and confirmed the relation (65) in the case where the measurement of \( x \) and \( p \) are both unbiased, i.e., the mean value of the measurement results are the same as that of the corresponding observable for every state. The generalization of equation (66) is
\[ \sigma(A) \sigma(B) \geq C, \]
(67)
where
\[ C = \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|. \]  
(68)
Equation (67) was proven by Robertson [24] and is referred to as Robertson’s relation. The relation between the standard deviations as in (66) and (67) is sometimes referred to as the uncertainty relation in state preparation, or preparation uncertainty relation. On the other hand, the relation between the measurement error and disturbance as in (65) is referred to as the uncertainty relation in joint measurement, error-disturbance relation (EDR), or measurement-disturbance relation. The generalization of Heisenberg’s relation (65) corresponding to (67) is [13, 25, 26]
\[ \epsilon(A) \eta(B) \geq C. \]  
(69)
Equation (69) is sometimes referred to also as Heisenberg’s relation. It is known that the relation (69) is valid under the assumption that the measurements are both unbiased for \( \hat{A} \) and \( \hat{B} \) [13, 25, 26]. However, it is noteworthy that the relation (65) or (69) may be violated if this assumption does not hold.

5.2. Ozawa’s and Branciard’s relations

In 2003, using the definitions of error and disturbance in (46) and Robertson’s relation (67), Ozawa derived the universally valid relation [2]
\[ \epsilon(A) \eta(B) + \frac{1}{2} |\langle [\hat{N}(A), \hat{B}] \rangle + \langle [\hat{A}, \hat{D}(B)] \rangle| \geq C. \]  
(70)
When \( \hat{N}(A) \) and \( \hat{D}(B) \) both give constant values regardless of the state, i.e.,
\[ \langle \hat{N}(A) \rangle = \langle \hat{M}_A \rangle - \langle \hat{A} \rangle = a, \]  
(71)
\[ \langle \hat{D}(B) \rangle = \langle \hat{M}_B \rangle - \langle \hat{B} \rangle = b, \]  
(72)
the second and the third terms of (70) vanish. Thus, in this case, equation (70) is reduced to Heisenberg’s relation (69). If \( M_A - a \) and \( M_B - b \) are redefined as \( M_A \) and \( M_B \), respectively, we get
\[ \langle \hat{M}_A \rangle - \langle \hat{A} \rangle = 0, \]  
(73)
\[ \langle \hat{M}_B \rangle - \langle \hat{B} \rangle = 0. \]  
(74)
These are nothing else than the unbiased conditions of the measurements of \( \hat{M}_A \) and \( \hat{M}_B \). Thus, equation (70) is regarded as the generalization of Heisenberg’s relation (69), specifying the forming condition of Heisenberg’s relation. From (70), Ozawa derived another universally valid relation [2]
\[ \epsilon(A) \eta(B) + \epsilon(A) \sigma(B) + \sigma(A) \eta(B) \geq C \]  
(75)
between the measurement error, disturbance and standard deviations. In the left-hand sides, (70) and (75) contain additional terms that are absent from Heisenberg’s relation (69), while the right-hand sides are the same. Thus, the term \( \epsilon(A)\eta(B) \) itself may be smaller than the right-hand side, suggesting that it is possible to violate Heisenberg’s relation (69). Given the definitions of measurement error and disturbance in (46), equations (70) and (75) are more general relations including Heisenberg’s relation (69) as a special case.

Although the relation (75) is universally valid, it is not tight in general; the left-hand side is always greater than the right-hand side. It may be tight only for cases, where \( \epsilon(A) = 0 \) or \( \eta(B) = 0 \) [2]. In 2013, based on the definitions of measurement error and disturbance (46), Branciard derived the stronger relation [4]

\[
\epsilon(A)^2\sigma(B)^2 + \sigma(A)^2\eta(B)^2 + 2\epsilon(A)\eta(B)\sqrt{\sigma(A)^2\sigma(B)^2 - C^2} \geq C^2.
\]

Branciard’s relation (76) is proven to be universally valid and tight, i.e., there exist cases where the left and right-hand sides are equal, for general joint measurements of \( \hat{A} \) and \( \hat{B} \). A simpler expression can be derived from (76):

\[
\epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq C,
\]

which is just the second and third terms of Ozawa’s relation (75). Hence, Branciard’s relation (76) is the stronger relation that includes Ozawa’s relation (75). In addition, Branciard derived the even stronger relation

\[
\tilde{\epsilon}(A)^2 + \tilde{\eta}(B)^2 + 2\tilde{\epsilon}(A)\tilde{\eta}(B)\sqrt{1 - C^2} \geq C^2,
\]

where \( \tilde{\epsilon} = \epsilon\sqrt{1 - \epsilon^2}/4 \) and \( \tilde{\eta} = \eta\sqrt{1 - \eta^2}/4 \). The relation (78) is valid when, as in the case of our photon polarization measurement, the spectra of \( \hat{A} \), \( \hat{B} \) and \( \hat{M} \) are all \( \pm 1 \), and \( \langle A \rangle = \langle B \rangle = 0 \) (hence \( \sigma(A) = \sigma(B) = 1 \)).

Branciard’s relations (76) and (78) are known to be tight for pure signal states. These relations can be modified to more general relations that are tight even for mixed signal states [27]. To do so, in (76) and (78), we just replace \( C \) defined in (68) with \( D \):

\[
C \rightarrow D = \frac{1}{2} \text{Tr} \sqrt{\rho} [\hat{A}, \hat{B}] \sqrt{\rho},
\]

where \( \rho \) is the density operator of the input state and \( \{\hat{X}\} \) for an operator \( \hat{X} \) is a non-negative Hermitian operator given by the polar decomposition: \( \hat{X} = U |\hat{X}| \).

5.3. Other relations

As mentioned above, there are active discussions and debates on the definitions of error and disturbance in quantum measurement. Correspondingly, there are a number of proposals for the measurement uncertainty relations based on different definitions of error and disturbance. Busch et al [5, 28, 29] proposed a definition of measurement error based on the RMS distance of the distributions between the original (\( \hat{A} \)) and the measurement (\( \hat{M}_n \)) observables. The state-independent error is defined by taking the supremum of the RMS distance with respect to all the input states. They derived the EDR for the qubit case [29]

\[
\Delta(A)^2 + \Delta(B)^2 \geq \sqrt{2} (\|a - b\| + \|a + b\| - 2).
\]

where \( \Delta(A) \) and \( \Delta(B) \) are the error and disturbance, under their definition, in the measurements of \( \hat{A} \) and \( \hat{B} \), respectively. In the right-hand side, \( a \) and \( b \) are unit vectors on the Bloch sphere; the projection operators \( \hat{M}_n^A \) and \( \hat{M}_n^B \) for the observables \( \hat{A} \) and \( \hat{B} \) are expressed in terms of \( a \) and \( b \) as

\[
\hat{M}_n^A = \frac{1}{2}(I \pm a \cdot \sigma), \quad \hat{M}_n^B = \frac{1}{2}(I \pm b \cdot \sigma),
\]

where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) in (20). They also showed that in the qubit case \( \Delta(A) \) and \( \Delta(B) \) coincide with \( \epsilon(A) \) and \( \eta(B) \), respectively, and thus

\[
\epsilon(A)^2 + \eta(B)^2 \geq \sqrt{2} (\|a - b\| + \|a + b\| - 2).
\]

When \( a \perp b \), the right-hand side of (82) is maximized as

\[
\epsilon(A)^2 + \eta(B)^2 \geq 2(2 - \sqrt{2}).
\]

Information-theoretic definitions of error, disturbance and EDR in quantum measurements were recently considered by Hofmann [30], Watanabe et al [31], Buscemi et al [6], Coles and Furrer [32], and Sulyok et al [33]. These are based on the information-theoretic quantities, e.g., conditional entropy, that quantify the uncertainty in the estimation (or retrodiction) of the value of original observable from the measurement outcome. Some of them assume that the input state is completely unknown, i.e., it is in a fully mixed state [6, 32]; thus the corresponding definitions of error, disturbance and EDR are state-independent. Since the definitions are different, the translation of these relations to the RMS-based relations is not straightforward. For instance, Buscemi et al [6] translated their relation for the qubit case to the RMS-based relation as

\[
\left[ \epsilon(A)^2 + \frac{1}{3} [\eta(B)^2 + 1] \right] \geq \left( \frac{4}{\pi} \right)^2 \approx 0.219.
\]

However, this relation is weaker than the tight relation predicted for the maximally mixed states defined in (78) and (79) [27]. Recently, a tight relation within the framework of the information-theoretic definition was proposed [33], but its translation to the RMS-based relation is not apparent.

6. Experiments on the measurement error, disturbance, and uncertainty relations

To date, experimental evaluation of the measurement error and disturbance in qubit systems has been reported using neutron spin [18, 33, 34] and photon polarization [11, 35–38]. Here, we review our experiments [11, 37] in which the measurement error, disturbance and uncertainty relations were examined in generalized, strength-variable measurement of a single photon polarization.

The main measurement apparatus is based on the generalized qubit measurement [9] described in section 3.3. The
optical implementation of the measurement apparatus is shown in figure 3, i.e., a VPBS [10, 11]. Using the VPBS, we want to measure the photon’s polarization in the $|H\rangle$ or $|V\rangle$ basis, i.e., $\hat{A} = \sigma_x$ for the polarization qubit. The measurement is done by observing the photons either of the output path $|+\rangle$ or $|-\rangle$, i.e., $\hat{M} = \sigma_y$ for the path qubit. Then, we make the successive measurement on the photons polarization in the $|D\rangle$ or $|A\rangle$ basis as the observable $\hat{B}$, i.e., $\hat{B} = \sigma_z$ for the polarization qubit. In this case, $C$ in (68) that appears in uncertainty relations is $C = \left\{\left[|\sigma_x, \sigma_y\rangle\right]\right\}/2 = \left\{|\sigma_y\rangle\right\}$. Thus, the initial state of the signal, polarization qubit, is chosen to be $|L\rangle$ (or $|R\rangle$), so that it maximizes $C = 1$ in the uncertainty relations to be examined.

Under the condition described above, the expected values of the noise and disturbance operators defined in (44) and (45) are

\[
\langle \hat{N}(A) \rangle = \langle \hat{M}(A) \rangle - \langle \hat{A} \rangle = (\cos 2\theta - 1) \langle \sigma_y \rangle, \tag{85}
\]

\[
\langle \hat{D}(B) \rangle = \langle \hat{M}(B) \rangle - \langle \hat{B} \rangle = (\sin 2\theta - 1) \langle \sigma_y \rangle, \tag{86}
\]

where $\theta$ is the parameter given in (36), defining the measurement strength as $s = \cos 2\theta$. We see that the expected values (85) and (86) are dependent on the input signal state and thus the forming conditions (71) and (72) for Heisenberg’s relation (69) are not fulfilled. The expected measurement error and disturbance defined in (46) are calculated to be [9, 11, 37]

\[
\epsilon(A) = 2 \sin \theta, \quad \eta(B) = 2 \sin \left(\frac{\pi}{4} - \theta\right). \tag{87}
\]

Thus, for this particular measuring apparatus, both the error and the disturbance are independent of the input signal state. The error $\epsilon(A)$ and the disturbance $\eta(B)$ remain finite even when the other goes to zero when $\theta = 0$ or $\pi/4$, since the error and disturbance are given by RMS difference between $\pm 1$-valued observables. At this point, the violation of Heisenberg’s relation (69) is already apparent.

In the experiment [37], we use the weak probe method to evaluate the measurement error and disturbance. The experimental setup is illustrated in figure 5. We use the VPBS for the weak probe ($W_A$ or $W_B$) and the main apparatus ($M_A$) and the PBS for the post measurement of $\hat{B}$. In this experiment, we employed the displaced Sagnac configuration that provides much higher phase stability than the Mach–Zehnder configuration (figure 3) used in our previous experiment [11]. Using this apparatus, we evaluated the measurement error $\epsilon(A)$ and disturbance $\eta(B)$ by varying the measurement strength $s = \cos 2\theta$ of $M_A$ from the null measurement $s = 0$ to the projective measurement $s = 1$. In the experiment, the measurement strength of the weak probe ($W_A$ or $W_B$) was set to $\cos 2\theta_{wa} = 0.104$ that produced very small disturbance in the initial signal state. In practice, for the signal state after the weak probe, we expected $C = 0.995$, which was close to the ideal value $C = 1$.

The quantities of $\epsilon(A)$ and $\eta(B)$ thus obtained are shown in figure 6. The dashed curves represent the theoretical calculations of $\epsilon(A)$ and $\eta(B)$ assuming the ideal instrument, and the solid curves are those in which the imperfect extinction ratio of the PBS is taken into account (a detailed discussion is given in [9, 11]). The experimentally measured error and disturbance present good agreement with the theoretical calculations. We clearly see the trade-off relation between the error and disturbance; as the measurement strength increases, $\epsilon(A)$ decreases while $\eta(B)$ increases.

![Figure 5](https://via.placeholder.com/150)

**Figure 5.** Schematic diagram of the experimental setup. The photon source is a strongly attenuated laser diode (LD) and the polarization is set to $|\psi\rangle = (|H\rangle + i|V\rangle)/\sqrt{2}$. The weak probe ($W_A$ or $W_B$) and the main apparatus ($M_A$) are based on the VPBS depicted in figure 3 but in this experiment they are modified to the Sagnac configuration. The projective measurement of $\hat{B}$ is implemented by the PBS. To evaluate the measurement error in $\hat{A}$ (disturbance in $\hat{B}$), the weak probe $W_A$ ($W_B$) is chosen to probe $\hat{A}$ ($\hat{B}$), with a weak measurement strength ($s_{wa} = 0.104$). Then the main apparatus ($M_A$) measures $\hat{A} = \sigma_y$ (in $H$ or $V$ polarization) with the variable measurement strength ($s = 0$–1). Finally, the post measurement apparatus $M_B$ measures $\hat{B} = \sigma_z$ (in $\pm 45^\circ$ polarization). Since each apparatus has two output paths as the measurement outcomes, the photon is finally detected in either of the $2^2 = 8$ output path modes, depending on the measurement outcomes of $W_A$ ($W_B$), $M_A$, and $M_B$. Details are shown in [37].

![Figure 6](https://via.placeholder.com/150)

**Figure 6.** Experimental results. The error $\epsilon(A)$ (blue circles) and disturbance $\eta(B)$ (red squares) are plotted as functions of the measurement strength $\cos 2\theta$. Dashed curves are the theoretically calculated error and disturbance for perfect implementation of the quantum circuit presented in figures 2 or 3(b). Solid curves are the theoretical values after the non-ideal extinction ratio of a PBS is taken into account.
In figure 7, we plot the predicted lower bounds of the EDRs in equations (69), (75), (76), (78), (83) and (84), together with the experimental data. Under Heisenberg’ EDR the error or disturbance must be infinite when the other goes to zero, while other EDRs allow finite error or disturbance even when the other is zero. We see that the experimental data clearly violate Heisenberg’(s) EDR, yet satisfy other recently proposed EDRs. In particular, our experimental data were close to Branciard’s bound (dot chain curve) given in equation (78), which could be saturated by ideal experiments.

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7. Conclusions

We experimentally implemented the generalized, strength-variable measurement of photon polarization, and evaluated the measurement error and disturbance making use of weak measurement with minimum disturbance that keeps the initial signal state practically unchanged. Our measurement results were compared with various EDRs predicted thus far, demonstrating the violation of Heisenberg’s EDR and the validity of Ozawa’s and other recently proposed EDRs.

Measurement error, disturbance, and the uncertainty relations are fundamentals to our observation of the quantum world. Recent progress in the quantum theory of measurement has revealed the new aspects of these issues, providing more precise and fundamental understanding of what we can take from nature through measurements. Although the experiments thus far carried out on these issues are still limited to qubit systems such as a neutron spin or a photon polarization, experimental investigation extending to other systems will be essential not only for understanding fundamentals of physical measurement but also for developing novel quantum information and communication protocols.

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