THE BRILL-NOETHER CURVE AND PRYM-TYURIN VARIETIES

ANGELA ORTEGA

ABSTRACT. We prove that the Jacobian of a general curve $C$ of genus $g = 2a + 1$, with $a \geq 2$, can be realized as a Prym-Tyurin variety for the Brill-Noether curve $W_{a+2}^r(C)$. As a consequence of this result we are able to compute the class of the sum of secant divisors of the curve $C$, embedded with a complete linear series $g_{2a-2}^{a-1}$.

1. Introduction

Consider a smooth general curve $C$ of genus $g \geq 5$. The locus $W_d^r(C)$ parametrizing line bundles $L$ of degree $d$ over $C$ with $h^0(L) \geq r + 1$, is an irreducible variety of dimension equal to the Brill-Noether number $\rho = \rho(g, r, d)$. In particular, $W_d^r(C)$ is a smooth curve when $\rho = 1$. In the case of $g = 5$, $r = 1$, the involution $i : L \mapsto \omega_C \otimes L^{-1}$, induces an automorphism of the curve $W_4^1(C)$, which is of genus 11. Since $C$ is general, the quotient map $W_4^1(C) \to W_4^1(C)/\iota$ is an étale double covering over a curve of genus 6. If $P$ denotes the Prym variety associated to this covering, it is known that $P$ is isomorphic to the Jacobian $JC$ as a principally polarized abelian variety ([1]). The main result of this paper shows that this situation generalizes to curves of higher odd genus, obtaining in this way Prym-Tyurin varieties.

Recall that a principally polarized abelian variety (ppav) $(P, \Xi)$ is a Prym-Tyurin variety if there exists a smooth projective curve $X$, such that $P$ is an abelian subvariety of the Jacobian $JX$ and the restriction of the principal polarization of $JX$ to $P$ is algebraically equivalent to $e\Xi$, where $e \in \mathbb{Z}_{>0}$ is the exponent of $P$ in $JX$. In that case, we say that $P$ is a Prym-Tyurin variety for the curve $X$ with exponent $e$.

Let $g = 2a + 1$, for $a \geq 2$. The locus $W_{a+2}^1(C)$ is a smooth curve, which from now on will be called the Brill-Noether curve. We define a correspondence $\gamma$ on $W := W_{a+2}^1(C)$, hence an endomorphism of the Jacobian $JW$, by means of the multiplication of sections. More precisely,

$L \mapsto \gamma(L) := \{L' \in W \mid H^0(L) \otimes H^0(\omega_C \otimes (L')^{-1}) \to H^0(\omega_C \otimes L \otimes (L')^{-1}) \text{ is not injective}\}$.

Let $P := \text{Im}(1 - \gamma) \subset JW_{a+2}^1(C)$. We prove the following theorem.

Theorem 1.1. Let $C$ be a general curve of genus $g = 2a + 1$. The subvariety $P := \text{Im}(1 - \gamma)$ is a Prym-Tyurin variety for the Brill-Noether curve $W_{a+2}^1(C)$ of exponent the Catalan number

$$\frac{(2a)!}{a!(a+1)!}.$$ 

Moreover, $P \simeq JC$ as principally polarized abelian varieties.

This result can also be interpreted from the point of view of enumerative geometry. It is reasonable to expect that, under suitable generality assumptions, a linear series $L \in W_d^r(C)$ has finitely many $(2r-2)$-secant $(r-2)$-planes that is, divisors $D \in C^{(2r-2)}$ such that $h^0(L(-D)) \geq 2$. In that case the number of secants is computed by the Castelnuovo formula ([11] Chapter VIII). Then one can associate to every linear series $g_d^{a-1}$ an element of $\text{Pic}(C)$, namely the class of the
sum of the secant divisors. By the results of Ciliberto (2) it is natural to expect that this class should depend only on the canonical divisor and the $g'_d$. For instance, when $\rho(g, r, d) = 0$ one can assign to the curve the class of the sum of the elements in $W^1_{a+2}(C)$. In this situation, Franchetta’s conjecture implies that the sum is a multiple of the canonical bundle (see [3]).

For $a \geq 4$, the residual linear system of $L \in W^1_{a+2}(C)$ defines an embedding $C \hookrightarrow \mathbb{P}^{a-1}$, whose image admits finitely many $(2a-4)$-secant $(a-3)$-planes. These secants are in bijection with the elements of $\gamma(L)$ by setting $L' = \omega_C \otimes L^{-1}(-D) \in \gamma(L)$, where $D$ is the divisor defined by a secant plane of the embedded curve. As an application of the Theorem 1.1, we are able to determine the class in $\text{Pic}(C)$ of the sum of the secant divisors.

**Theorem 1.2.** Let $C$ be a general smooth curve of genus $2a + 1$. For any line bundle $L \in W^1_{a+2}(C)$ we have that

$$\bigotimes_{L \in \gamma(L)} L' = \omega_C^{\alpha} \otimes L^{1-e},$$

where $e = \frac{(2a)!}{a!(a+1)!}$ and $\alpha = \frac{a+2}{4a}(g(W) - 1 - e(g-1))$.

A principally polarized abelian variety can always be realized as a Prym-Tyurin variety for some curve, but with a very large exponent (see [3, Corollary 12.2.4]). In fact, the curve for which a ppav is a Prym-Tyurin variety is not uniquely determined. It is an open problem to find, for a fixed $g$, the smallest integer $m$ such that any ppav of dimension $g$ is a Prym-Tyurin variety of exponent $e \leq m$. For instance, Mumford’s results ([12]) show that the general ppav of dimension $g = 4, 5$ is a Prym-Tyurin variety of exponent $2$ for a curve of genus $2g - 1$. Another example of Prym-Tyurin varieties of small exponent can be found in [10], where the authors exhibit a family of Prym-Tyurin varieties of dimension $6$ and exponent $6$.

2. **The Brill-Noether curve**

Let $C$ be a general curve of genus $g = 2a + 1$ satisfying Petri’s theorem. Let $\omega_C$ denote the canonical line bundle on $C$. Consider the Brill-Noether locus $W := W^1_{a+2}(C)$ consisting of line bundles $L$ on $C$ of degree $a+2$, with $h^0(C, L) \geq 2$. Since $C$ is general and $\rho(g, 1, a + 2) = 1$, the locus $W$ is a smooth irreducible curve naturally embedded in $\text{Pic}^{a+2}(C)$. The genus of the Brill-Noether curve is computed by the formula ([9, Theorem 4]):

$$g(W) = \frac{a}{a + 2} \cdot \frac{2g!}{a!(a+1)!} + 1. \tag{2.1}$$

We fix a point $L_0 \in \text{Pic}^{a+2}(C)$ and consider the embedding

$$\varphi : W \to JC, \quad L \mapsto L \otimes L_0^{-1}.$$

**Lemma 2.1.** The curve $W$ generates $JC$ as an abelian group.

**Proof.** The embedding $\varphi : W \to JC$ induces a morphism $\tilde{\varphi} : JW \to JC$. It suffices to show that $\tilde{\varphi}$ is surjective. It has been shown in [9], that the induced map

$$\varphi_* : H_1(W, \mathbb{Z}) \longrightarrow H_1(JC, \mathbb{Z}) \simeq H_1(C, \mathbb{Z})$$

is surjective. This map corresponds to the rational representation of $\tilde{\varphi}$ and it determines it completely. Hence, $\tilde{\varphi}$ is surjective. \hfill $\square$

Thus we have a short exact sequence

$$0 \longrightarrow K^1_{a+2}(C) \longrightarrow JW^1_{a+2}(C) \longrightarrow \tilde{\varphi} \longrightarrow JC \longrightarrow 0, \tag{2.2}$$
where \( \tilde{\varphi} \) is the map which takes a class of equivalence of divisors of degree zero in \( W_{a+2}^1 \) to its linear equivalence class as a divisor on the curve \( C \). The following result is proved in [3, Theorem 1.1].

**Theorem 2.2.** For a general curve \( C \) of genus \( g \geq 3 \), the abelian variety \( K_{a+2}^1(C) \) is connected and has no non-trivial endomorphisms which are rationally determined.

By rationally determined we mean defined over the field of rational functions of \( M_{g,1} \), the moduli space of smooth pointed curves of genus \( g \).

Let us denote \( \theta_C : JC \to \hat{JC} \) (respectively \( \theta_W \) ) the principal polarization of \( JC \) (respectively that of \( JW \)). By dualizing the exact sequence (2.2), we find that

\[
\varphi^* = \theta_W^{-1} \circ \hat{\varphi} \circ \theta_C : JC \to JW,
\]

is an embedding since \( \hat{\varphi} \) is also one (see [5, Prop. 2.4.2]). We shall show that the image of \( \varphi^* \) defines an abelian subvariety of \( JW \), which is a Prym-Tyurin variety for \( W \). A polarized abelian variety \( (P, \Xi) \) is a Prym-Tyurin variety for a curve \( C \) if there is an embedding \( i : P \to JC \) such that \( i^* \Theta \equiv e \Xi \); the integer \( e \) is called the exponent of \( P \). We will use Welters’ criterion for Prym-Tyurin varieties ([5, Theorem 12.2.2]).

**Theorem 2.3.** (Welters’ Criterion). Let \( (P, \Xi) \) be a ppav of dimension \( g \) and \( C \) smooth curve. Then \( (P, \Xi) \) is a Prym-Tyurin variety of exponent \( e \) for \( C \) if and only if it exists a morphism \( \phi : C \to P \) such that

a) \( \phi^* : P \to JC \) is an embedding,

b) \( \phi_*[C] = \frac{e}{(g-1)!} \wedge^{g-1}[\Theta_C] \) in \( H^{2g-2}(P, \mathbb{Z}) \).

**Theorem 2.4.** The Jacobian \( JC \) is a Prym-Tyurin variety for \( W \) of exponent the Catalan number

\[
e = \frac{(2a)!}{a!(a+1)!}.
\]

**Proof.** We apply the Criterion 2.3 to the embedding \( \varphi^* : JC \to JW \). It suffices to show that \( \varphi_*[W] \) has the required cohomology class. The class of the curve \( W \) in \( H^{2g-2}(\text{Pic}^{a+2}(C), \mathbb{Z}) \) is given by (1[p. 320])

\[
[W_{a+2}^1(C)] = \frac{1}{a!(a+1)!} \wedge^{g-1}[\Theta_C].
\]

Hence

\[
\varphi_*[W] = \frac{1}{a!(a+1)!} \wedge^{g-1}[\Theta_C] = \frac{e}{(2a)!} \wedge^{g-1}[\Theta_C]
\]

in \( H^{2g-2}(JC, \mathbb{Z}) \). \( \square \)

3. A correspondence on the Brill-Noether curve

We define the following correspondence on the Brill-Noether curve \( W \):

\[
\gamma : L \mapsto \{ L' \in W \mid \mu : H^0(C, L) \otimes H^0(C, \omega_C \otimes (L')^{-1}) \to H^0(C, \omega_C \otimes L \otimes (L')^{-1}) \text{ is not injective} \},
\]

where \( \mu \) denotes the multiplication of sections. It has been shown in [3] that this correspondence is non-empty for any \( a \geq 2 \). The correspondence \( \gamma \) defines an endomorphism (denoted by the same symbol) \( \gamma \in \text{End}(JW) \) by

\[
\left[ \sum n_i L_i \right] \mapsto \left[ \sum n_i \gamma(L_i) \right],
\]
where \( L_i \) are points on the curve \( W \) (corresponding to line bundles of degree \( a + 2 \)). Using the base-point-free-pencil trick, one checks that \( L' \in \gamma(L) \) if and only if \( H^0(C, \omega_C \otimes L^{-1} \otimes (L')^{-1}) \neq 0 \). So, we can rewrite the correspondence \( \gamma \) as

\[
\gamma(L) = \{ L' \in W \mid H^0(C, \omega_C \otimes L^{-1} \otimes (L')^{-1}) \neq 0 \}.
\]

From this description follows that \( \gamma \) is symmetric. Moreover, since \( C \) is general the Gieseker-Petri Theorem ([II, p. 215]) ensures that the multiplication map

\[
H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \to H^0(C, \omega_C)
\]

is injective for any line bundle \( L \in W_{a+2}(C) \). Thus the correspondence \( \gamma \) has no fixed points, i.e. \( \gamma \) does not intersect the diagonal \( \Delta \subset W \times W \). This also shows that the induced endomorphism of \( JW \), is not a multiple of the identity, since these endomorphisms are induced by divisors of the form \( n\Delta \), for \( n \in \mathbb{Z} \).

For instance, for \( a = 2 \) the correspondence \( \gamma \) induces an involution on the curve \( W \) of genus 11, namely \( \iota : L \mapsto \omega_C \otimes L^{-1} \). It is known that the corresponding Prym variety associated to the étale double covering \( W \to W/\iota \) is an abelian subvariety of \( JW \) of dimension 5 isomorphic to the Jacobian of \( C \) ([III]).

**Lemma 3.1.** Let \( a \geq 3 \) and \( g = 2a + 1 \). The degree of the correspondence \( \gamma \) is given by the Castelnuovo number

\[
C(a - 1, 3a - 2, 2a + 1) = \sum_{i=0}^{a-2} \binom{-1}{i} \binom{a}{a-2-i} \binom{2a-i}{a-1-i}.
\]

**Proof.** Let \( L \in W_{a+2}(C) \) and set \( M = \omega_C \otimes L^{-1} \in W_{3a-2}(C) \). An element \( L' \in W_{a+2}(C) \) is in \( \gamma(L) \) if and only if \( H^0(M \otimes (L')^{-1}) \neq 0 \), that is, if \( M \otimes (L')^{-1} = O_C(D) \) for an effective divisor \( D \) of degree \( 2a - 4 \). So \( L' \) is of the form \( M(-D) \) with \( h^0(M(-D)) \geq 2 \). Hence the degree is given by the degree of the degeneracy locus in \( C_d \) (the \( d \)-symmetric power of \( C \)) of the divisors of degree \( d = 2a - 4 \) that impose at most \( d - r = a - 2 \) conditions on \( |M| \). Thus one can interpret the degree of \( \gamma \) as the number of the \((2a - 4)\)-secant \((a - 3)\)-planes in the linear system \( |M| \). For the general curve there are finitely many of such \((a - 3)\)-planes ([V]) and their number is given by the Castelnuovo formula ([II, Chapter VIII]).

The endomorphism \( \gamma \) also defines a map \( W_{a+2}(C) \to \text{Pic}^m(C) \), where \( m := (a + 2)(\text{deg} \gamma) \) by considering \( \gamma(L) \) as a tensor product of line bundles on \( C \). More precisely, if \( D_i \in \text{Div}^{2a-4}(C) \), for \( i = 1, \ldots, \text{deg} \gamma \), are the secant divisors of the image of \( C \) in \( |\omega_C \otimes L^{-1}|^* \cong \mathbb{P}^{a-1} \), we set \( L_i := M(-D_i) \in \text{Pic}^{a+2}(C) \) and \( \gamma(L) \) can be viewed as the line bundle

\[
\bigotimes_{i=1}^{\text{deg} \gamma} L_i \in \text{Pic}^m(C).
\]

We denote \( \mathcal{P} \) the Zariski open subset consisting of all equivalence classes \( (C, x) \in \mathcal{M}_{g,1} \), with \( C \) a curve having no non-trivial automorphisms and satisfying the Petri condition. There exists a smooth scheme \( G_{a+2}^d \) and a morphism \( G_{a+2}^d \to \mathcal{P} \) such that the fiber over any closed point \( t \in \mathcal{P} \) is isomorphic to \( G_d(C) \), the variety parametrizing all the \( g_d \)'s on \( C \). For \( a \geq 4 \), set \( r = 1, d = a + 2 \) and \( \mathcal{G} := G_{a+2}^d \). Now, let \( \mathcal{H} \) be the Hilbert scheme of curves of degree \( 3a - 2 \) and of genus \( 2a + 1 \) in \( \mathbb{P}^{a-1} \) and \( \mathcal{H}_1 \) the open set of a component of \( \mathcal{H} \) with general moduli, parametrizing curves without nontrivial automorphisms. For every point \( z = ((C, x), D) \in \mathcal{G} \) denote by \( \Gamma \subset \mathbb{P}^{a-1} \) the image of \( C \) by the residual series \( |\omega_C \otimes L^{-1}| \), with \( L = O_C(D) \) and by \( [\Gamma] \) the corresponding point in \( \mathcal{H}_1 \). Let \( \mathcal{H}_z \) be a closed subset of \( \mathcal{H}_1 \) given by the orbit of \( [\Gamma] \) under the action of \( PGL(a, \mathbb{C}) \) (see [2, §3]). The map \( z \mapsto \gamma(L) \) induces a regular section \( \mathcal{H}_z \to \text{Pic}^m(C) \). By varying the
curve $C$ we obtain a rationally determined line bundle $L$ on the universal family $\mathcal{F}$ over $\mathcal{H}_1$, such that the restriction of $L$ to the fiber over $z$ is isomorphic to $\gamma(L) \in \text{Pic}^m(C)$, where $\gamma(L)$ is the tensor product $\mathcal{L}$. As a consequence of the Theorem 2.2 one has the following result ([3, Theorem 1.2]).

**Theorem 3.2.** Let $\mathcal{H}$ be any component of the Hurwitz scheme of coverings of $\mathbb{P}^1$ of degree $d$ and genus $g \geq 3$ containing curves with general moduli and with $\rho = 1$. Then the group of rationally determined line bundles of the universal family over $\mathcal{H}$ is generated by the relative canonical bundle and the hyperplane bundle.

It follows that there exist integers $\alpha, \beta$ such that

$$(3.3) \quad \gamma(L) = \omega_C^{\otimes \alpha} \otimes L^{\otimes \beta} \in \text{Pic}^m(C).$$

We are able to deduce the coefficients $\alpha$ and $\beta$ as an application of Theorem 3.3. Set $P := \text{Im}(1 - \gamma) \subset J_W$. On the light of the Theorem 2.2, one does not expect other subvarieties of $J_W$, other that the obvious ones. More precisely, we prove:

**Theorem 3.3.** The subvariety $P$ is isomorphic to $JC$. In particular, $P$ is a Prym-Tyurin variety for $W$ of exponent $e$.

**Proof.** Consider the map $\tilde{\varphi}|_P : P \to JC$ and suppose it is non-zero. Then by Theorem 2.2 $\tilde{\varphi}|_P$ is an isogeny. The embedding $\varphi^* : JC \to J_W$ gives then an isomorphism $JC \simeq P$. In particular, $(P, \Theta_{C|P})$ is a Prym-Tyurin variety of exponent $e$ for $W$. If the restriction of $\tilde{\varphi}$ to $P$ is zero, the complementary subvariety $Z$ of $P$ with respect to $\Theta_{W}$ is isogenous to $JC$, via the restriction $\tilde{\varphi}|Z : Z \to JC$. In this case $\varphi^*(JC) = Z$ and $Z$ is a Prym-Tyurin of exponent $e$ for $W$. Moreover, $Z = \text{Im}(e - 1 + \gamma)$. Using the formula in [5, Corollary 5.3.10], one computes that

$$\dim Z = \frac{(e - 1)g(W) + \deg \gamma}{e}.$$ 

Since $\dim Z = \dim JC = g$, we have $(e - 1)g(W) + \deg \gamma = eg$. By Lemma 3.4 we obtain that

$$(e - 2)g(W) = -2\deg \gamma,$$

which is a contradiction since $e \geq 2$ and $\deg \gamma > 0$. Therefore $JC \simeq \text{Im}(1 - \gamma)$. $\square$

**Lemma 3.4.** The equation $g(W) - \deg(\gamma) = eg$ holds.

**Proof.** A direct computation. $\square$

4. **The equivalence class of the sum of secants to a curve**

For any line bundle $L \in W_{a+2}(C)$, consider the product

$$\gamma(L) = \bigotimes_{i=1}^{\deg \gamma} L_i$$

as defined in §3.

**Theorem 4.1.** Let $C$ be a general smooth curve of genus $2a + 1$. For any line bundle $L \in W_{a+2}(C)$ we have that

$$\gamma(L) = \bigotimes_{i=1}^{\deg \gamma} L_i = \omega_C^{\alpha} \otimes L^{1-e},$$

where $e = \frac{(2a)!}{a!(a+1)!}$ and $\alpha = \frac{a+2}{4a} (g(W) - 1 - e(g - 1))$. 
Proof. The norm-endomorphism corresponding to the subvariety $P \subset JW$ is $1 - \gamma$. It satisfies $(1 - \gamma)^2 = e(1 - \gamma)$, or equivalently, the quadratic equation

$$(4.1) \quad (1 - e - \gamma)(1 - \gamma) = \gamma^2 + (e - 2)\gamma - (e - 1) = 0$$

on the Jacobian $JW$. Consider the projection $\tilde{\varphi}$ from $P = \operatorname{Im}(1 - \gamma)$ to $JC$. Fix $M \in W_{a+2}^1(C)$. Then, by (3.3), there exists an integer $\beta$ such that

$$\tilde{\varphi}(1 - \gamma)(L - M) = L \otimes M^{-1} \otimes L^{-\beta} \otimes M^\beta.$$ 

Since the relation (4.1) holds on $JC$ as well, we obtain

$$(1 - e - \gamma)(L \otimes M^{-1}) \otimes 1^{1-\beta} = (L \otimes M^{-1}) \otimes (1 - e - \beta) \otimes (L \otimes M^{-1}) \otimes 1^{1-\beta} = (L \otimes M^{-1}) \otimes 1^{1-\beta} = \mathcal{O}_C$$

for all $L \in W$. Therefore $(1 - \beta)(1 - e + \beta) = 0$. If $\beta = 1$, $\tilde{\varphi}(1 - \gamma) = 0$, which is a contradiction to the fact that $\tilde{\varphi}_| P$ is surjective. Hence $\beta = 1 - e$. In order to compute the value of $\alpha$ one compares the degrees in the equation (3.3) and uses Lemma 6.4.

For example, for a general line bundle $L \in W_2^0(C)$ on a curve $C$ of genus 7, the image of the map $\phi : C \to |\omega_C \otimes L^{-1}|$ is a plane curve with 8 nodes. Let $p_i, q_i, \text{ for } i = 1, \ldots, 8,$ denote the pre-images of the nodes. Set $M := \omega_C \otimes L^{-1} \in W_6^0(C)$. Hence

$$\gamma(L) = \bigotimes_{i=1}^{8} M(-p_i - q_i) \in \operatorname{Pic}^{40}(C).$$

By the adjunction formula we have that

$$\omega_C = M^4(- \sum_{i=1}^{8} p_i + q_i),$$

that is,

$$\bigotimes_{i=1}^{8} M(-p_i - q_i) = \omega_C^5 \otimes L^{-4},$$

which is predicted by Theorem 4.1 since the Catalan number is equal to 5. A less trivial example is the case of a general curve $C$ of genus 9 embedded in $\mathbb{P}^3$ by the linear system $|M|$, with $M \in W_{10}^3(C)$. The space curve admits 43 4-secant lines, the genus of the curve $W_4^3(C)$ is 169 and the exponent of the Prym-Tyurin variety is 14. Let us denote $D_i \in \operatorname{Div}^4(C)$ the corresponding divisors. By Theorem 4.2 we obtain $\alpha = 21, \beta = -13$ and

$$\bigotimes_{i=1}^{43} \mathcal{O}_C(D_i) = M^{30} \otimes \omega_C^{-8}.$$ 

Remark 4.2. For a general curve $C$, the subring of $H^*\left(C_2, \mathbb{Q}\right)$ generated by the fundamental classes of algebraic cycles on $C_2$ is generated by the class of a fiber of the projection $\pi_1 : C_2 \to C$ and the diagonal (H, p. 359). In the situation of the Brill-Noether curve, such subring of $H^*(W_2, \mathbb{Q})$ has an extra generator induced by the correspondence $\gamma \subset \mathbb{W} \times \mathbb{W}$.

Remark 4.3. It would be interesting to study the properties of the curve $W$, for instance determine its gonality or if it has a special Brill-Noether behavior.
Acknowledgements. I would like to thank A. Beauville, C. Ciliberto, G. Farkas, E. Izadi, G.-P. Pirola, O. Serman and A. Verra for stimulating conversations. This research is partially supported by the Sonderforschungsbereich 647 “Raum - Zeit - Materie”.

REFERENCES

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, Geometry of algebraic curves, Volume I. Grundlehren der Math. Wiss. 267, Springer - Verlag (1985).
[2] C. Ciliberto, On rationally determined line bundles on a family of projective curves with general moduli. Duke Math. 55 (1987), 909-917.
[3] C. Ciliberto, J. Harris, M. Teixidor i Bigas, On the endomorphisms of Jac(W^1_d(C)) when \(\rho = 1\) and \(C\) has general moduli. Classification of irregular varieties (Trento, 1990), 41-67. Lecture Notes in Math. 1515, Springer - Verlag (1992).
[4] A. Beauville, Diviseurs spéciaux et intersections de cycles dans la jacobienne d’une courbe algébrique. Enumerative geometry and classical algebraic geometry, PM 24, 133–142, Birhäuser (1982).
[5] Ch. Birkenhake, H. Lange, Complex Abelian Varieties. Second edition, Grundlehren der Math. Wiss. 302, Springer - Verlag (2004).
[6] D. Eisenbud, J. Harris, The Kodaira dimension of the moduli space of curves of genus \(\geq 23\). Inventiones Math. 90 (1987), 359–387.
[7] G. Farkas, Higher ramification and varieties of secant divisors on the generic curve. Journal of the London Mathematical Society 78 (2008), 418-440.
[8] G. Farkas, A. Ortega, The maximal rank conjecture and rank two Brill-Noether theory. Pure and Applied Math. Quarterly 7 (2011), 1265–1296.
[9] W. Fulton, R. Lazarsfeld, On connectedness of degeneracy loci and special divisors. Acta Math. 146, (1981), 116–147.
[10] H. Lange, A. Rojas, A Galois-theoretic approach to Kanev’s correspondence. Manuscripta Math. 125 (2008), no. 2, 225-240.
[11] Masiewicki, Leon, Universal properties of Prym varieties with an application to algebraic curves of genus five. Trans. Amer. Math. Soc. 222 (1976), 221–240.
[12] D. Mumford, Prym varieties I. In L.V. Ahlfors, I. Kra, B. Maskit, and L. Niremberg, editors, Contributions to Analysis. Academic Press (1974), 325–350.

ANGELA ORTEGA, INSTITUT FÜR MATHEMATIK, HUMBOLDT UNIVERSITÄT ZU BERLIN, GERMANY
E-mail address: ortega@math.hu-berlin.de