Finding Octonionic Eigenvectors
Using Mathematica

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Abstract

The eigenvalue problem for $3 \times 3$ octonionic Hermitian matrices contains some surprises, which we have reported elsewhere [1]. In particular, the eigenvalues need not be real, there are 6 rather than 3 real eigenvalues, and the corresponding eigenvectors are not orthogonal in the usual sense. The nonassociativity of the octonions makes computations tricky, and all of these results were first obtained via brute force (but exact) Mathematica computations. Some of them, such as the computation of real eigenvalues, have subsequently been implemented more elegantly; others have not. We describe here the use of Mathematica in analyzing this problem, and in particular its use in proving a generalized orthogonality property for which no other proof is known.

Key words: octonions, eigenvectors, Mathematica

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1 INTRODUCTION

Finding the eigenvalues and eigenvectors of a given matrix is one of the basic techniques in linear algebra, with countless applications. The familiar case of Hermitian (complex) matrices is very important, for instance in quantum mechanics, where the fact that such matrices have real eigenvalues allows them to represent physically observable quantities.
The eigenvalue problem is usually formulated over the complex numbers \( \mathbb{C} \), including the reals \( \mathbb{R} \) as a special case. In recent work [1], we considered the generalization to the other normed division algebras, namely the quaternions \( \mathbb{H} \) and the octonions \( \mathbb{O} \). Most of the basic properties are retained, provided they are reinterpreted to take into account the lack of commutativity of \( \mathbb{H} \) and \( \mathbb{O} \), and the lack of associativity of \( \mathbb{O} \). However, there are a number of surprises, including the fact that such matrices admit non-real eigenvalues.

Our most important results concern the eigenvalue problem for \( 3 \times 3 \) octonionic Hermitian matrices, the \emph{Jordan matrices}. It turns out [1] that such matrices admit more than the expected 3 real eigenvalues, and that eigenvectors corresponding to different eigenvalues fail in general to be orthogonal in the usual sense, although they do seem to be orthogonal in a generalized sense.

Because of the lack of both commutativity and associativity, working with octonionic matrices is rather tricky. All of the above results were initially discovered using a \emph{Mathematica} package we have developed over the years for just this purpose. While we have subsequently been able to derive more elegant derivations by hand for some of these results, we have not succeeded in doing this for all. In particular, the only current proof of the generalized orthogonality property consists of a lengthy, brute force \emph{Mathematica} computation, which used 6 hours of CPU time on a Sparc20 with 224 Mb of memory. This article describes both the mathematics behind this result, and the \emph{Mathematica} computation used to obtain it.

We set the stage in Section 2 by reviewing some basic properties about both the standard eigenvalue problem and the octonions, as well as introducing our \emph{Mathematica} package. We then summarize the theory of \( 3 \times 3 \) octonionic Hermitian matrices in Section 3, pointing out that the only known proof of some of the results in this section involves direct computation using \emph{Mathematica}. In Section 4 we give an explicit example, and we discuss our results in Section 5.

2 BACKGROUND

2.1 The Standard Eigenvalue Problem

We begin by collecting some standard results about the standard eigenvalue problem. We give the details of some of the proofs in order to emphasize the use of both the commutativity and associativity of \( \mathbb{C} \).

The eigenvalue problem as usually stated is to find solutions \( \lambda, v \) to the equa-
Av = \lambda v \quad (1)

for a given square matrix A. The basic properties of the eigenvalue problem for \( n \times n \) complex Hermitian matrices are well-understood.

**Lemma 1** An \( n \times n \) complex Hermitian matrix \( A \) has \( n \) eigenvalues (counting multiplicity), all of which are real.

**PROOF.** We give here only the proof that the eigenvalues are real. Let \( A, v, \lambda \) satisfy (1), with \( A^\dagger = A \). Then

\[
\overline{\lambda} v^\dagger v = (Av)^\dagger v = v^\dagger Av = \lambda v^\dagger v \quad (2)
\]

so that if \( v \neq 0 \) we have \( v^\dagger v \neq 0 \), which forces \( \overline{\lambda} = \lambda \). □

**Lemma 2** Eigenvectors of an \( n \times n \) complex Hermitian matrix \( A \) corresponding to different eigenvalues are orthogonal.

**PROOF.** For \( m = 1, 2 \), let \( v_m \) be an eigenvector of \( A = A^\dagger \) with eigenvalue \( \lambda_m \). By the previous lemma, \( \lambda_m \in \mathbb{R} \). Then

\[
\lambda_1 v_1^\dagger v_2 = (Av_1)^\dagger v_2 = v_1^\dagger Av_2 = \lambda_2 v_1^\dagger v_2 \quad (3)
\]

Then either \( \lambda_1 = \lambda_2 \) or \( v_1^\dagger v_2 = 0 \). □

**Lemma 3** For any \( n \times n \) complex Hermitian matrix \( A \), there exists an orthonormal basis of \( \mathbb{C}^n \) consisting of eigenvectors of \( A \).

**PROOF.** If all eigenvalues have multiplicity one, the result follows from the previous lemma. But the Gram-Schmidt orthogonalization process can be used on any eigenspace corresponding to an eigenvalue with multiplicity greater than one. □

These lemmas are equivalent to the standard result that a complex Hermitian matrix can always be diagonalized by a unitary transformation. It is important for what follows to realize that the form of the proofs given above relies on both the commutativity and the associativity of \( \mathbb{C} \).
Combining the above results, it is easy to see that any (complex) Hermitian matrix $A$ admits a decomposition in terms of an orthonormal basis of eigenvectors.

**Theorem 4** Let $A$ be an $n \times n$ complex Hermitian matrix. Then $A$ can be expanded as

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger$$

(4)

where $\{v_m; m = 1, \ldots, n\}$ is an orthonormal basis of eigenvectors corresponding to eigenvalues $\lambda_m$.

**PROOF.** By the previous lemma, there exists an orthonormal basis $\{v_m\}$ of eigenvectors. It then suffices to check that

$$\sum_{m=1}^{n} \lambda_m v_m v_m^\dagger v_k = \lambda_k v_k$$

(5)

But this follows by direct computation using orthonormality. □

Furthermore, the set of eigenvalues $\{\lambda_m\}$ is unique, and the (unit) eigenvectors are unique up to unitary transformations in the separate eigenspaces (which reduce to multiplication by a complex phase for eigenvalues of multiplicity one).

2.2 Octonions

The quaternions $\mathbb{H}$ double the dimension of the complex numbers by adding two additional square roots of $-1$, usually denoted $j$ and $k$. The multiplication table follows from

$$i^2 = j^2 = k^2 = -1 \quad ij = k = -ji$$

(6)

and associativity; note that $\mathbb{H}$ is not commutative. Equivalently, $\mathbb{H}$ can be viewed as the sum of 2 copies of the complex numbers

$$\mathbb{H} = \mathbb{C} + k\mathbb{C}$$

(7)

with $j$ being defined by $j = ki$. 
Fig. 1. The representation of the octonionic multiplication table using the 7-point projective plane. Each of the 7 oriented lines gives a quaternionic triple.

The octonions \( \mathbb{O} \) in turn can be viewed as the direct sum of two copies of the quaternions \( \mathbb{H} \)

\[
\mathbb{O} = \mathbb{H} + \mathbb{H} \ell = (\mathbb{C} + k\mathbb{C}) + (\mathbb{C} + k\mathbb{C})\ell
\]  

(8)

where \( \ell \) is yet another square root of \(-1\). The octonions are thus spanned by the identity element \(1\) and the 7 imaginary units \(\{i, j, k, k\ell, j\ell, i\ell, \ell\}\). These units can be grouped into (the imaginary parts of) quaternionic subspaces in 7 different ways; these will be referred to as “triples”. Any three of these imaginary units which do not lie in a such a triple anti-associate. The multiplication table can be neatly summarized by appropriately labeling the 7-point projective plane, as shown in Figure 1.

Even though the octonions are not associative, since any 2 octonions lie in a quaternionic subspace, products involving only 2 different octonions (and their octonionic conjugates) do associate. For example,

\[
p(pq) = p^2q
\]  

(9)

which is a weak form of associativity known as *alternativity*.

---

1 This construction of a new division algebra from 2 copies of another is a special case of the Cayley-Dickson process; for modern treatments, see [2–6].
We use the notation $\overline{a}$ to denote the (octonionic) conjugate of the octonion $a$, 

$$|a|^2 := a\overline{a} \quad (10)$$

to denote the squared norm of $a$, $A^\dagger$ to denote the (octonionic) Hermitian conjugate of the matrix $A$,

$$[a, b] := ab - ba \quad (11)$$

to denote the commutator of $a$ and $b$, and

$$[a, b, c] := (ab)c - a(bc) \quad (12)$$

to denote the associator of $a$, $b$, $c$. Both the commutator and the associator are purely imaginary, totally antisymmetric, and change sign if any one of their arguments is replaced by its conjugate. Another octonionic product with these properties is given by the *associative 3-form* [2,7]

$$\Phi(a, b, c) = 1/2 \text{Re}((a(\overline{bc}) - c(\overline{ba})) \quad (13)$$

which reduces to the vector triple product when $a$, $b$, $c$ are imaginary quaternions.

$$\Phi(a, b, c) = 1/2 \text{Re}([a, \overline{b}]c) \quad (14)$$

### 2.3 Mathematica Package

We needed a way to easily manipulate octonions and octonionic matrices — it is quite difficult to unlearn associativity! There are 2 complementary approaches, depending on whether it is desired to manipulate abstract octonions or whether an explicit basis can be used. For our purposes, it was initially quite sufficient to work with an abstract basis: We define an octonion to be a list with 8 elements

$$a = a_1 + a_2 i + a_3 j + a_4 k - a_5 k\ell - a_6 j\ell - a_7 i\ell + a_8 \ell = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\} \quad (15)$$

where the signs are conventional, and where we have emphasized the role of the imaginary units as “basis vectors” by writing them in boldface. Octonionic multiplication can then be expressed as a series of rules of the form

$$i \ast j = k \quad (16)$$
In[8]:= Omult[O3, bar[O3]]
Out[8]= \(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_8^2\)

In[9]:= Expand[assoc[O1, O2, O3]]
Out[9]= 2 \(a_2 b_3 c_8\) \(k\)

In[10]:= Expand[\[Phi][O1, O2, O3]]
Out[10]= \(a_2 b_3 c_4\)

In[11]:= \[Phi][i, j, k]
Out[11]= \(1\)

Fig. 2. A sample Mathematica session illustrating basic manipulation of octonions using an explicit basis representation, where \(O_1, O_2, O_3\) are defined as in (18).

where we have introduced the symbol \(*\) to represent octonionic multiplication. Octonionic conjugation can be defined as a series of rules of the form

\[
\bar{i} = -i
\]

(17)

and everything else can be defined in terms of these 2 basic operations.

Furthermore, as already stated, computations involving small numbers of octonions can be dramatically simplified. For instance, 3 arbitrary octonions can be assumed to take the form

\[
\begin{align*}
O_1 &= a_1 + a_2 i \\
O_2 &= b_1 + b_2 i + b_3 j \\
O_3 &= c_1 + c_2 i + c_3 j + c_4 k + c_8 \ell
\end{align*}
\]

(18)

When implementing these ideas using Mathematica, it turned out to be more efficient to define the 2 fundamental operations directly on lists, rather than building them up in terms of rules. For instance, conjugation is more easily defined by

\[
\text{bar}[\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \ldots\}] := \{x_1, -x_2, -x_3, -x_4, -x_5, -x_6, -x_7, -x_8, \ldots\}
\]

(19)

and there is an analogous definition of octonionic multiplication, called Omult. A illustration of computation in an explicit basis appears in Figure 2.
Matrices can now be constructed as nested lists, and operations such as matrix multiplication can easily be defined. However, matrix expressions involving several octonions quickly become unwieldy. We therefore supplemented the above basic definitions in terms of an explicit basis with an alternative set of definitions using abstract octonions. In the process, we took full advantage of the formatting capabilities in Mathematica 3.0. An illustration of computation using the abstract approach appears in Figure 3.

Even though we chose not to implement the concrete version of the basic octonionic operations in terms of rules, Mathematica’s ability to manipulate such rules was crucial in constructing the abstract version. Especially when teaching Mathematica to both manipulate and print matrices of lists (i.e. octonions), the ability to easily modify the code to recognize special cases proved extremely helpful. (Octonions are closely related to Clifford algebras; see for instance [8], in which the representation theory of Clifford algebras is extended so as to employ octonions. There are a number of packages for manipulating Clifford algebras, some of which are described in [9]. Several of these can handle octonions, such as the program CLICAL by Pertti Lounesto [10] and the Maple package CLIFFORD by Rafal Ablamowicz [11], both of which introduce the octonions as paravectors over an appropriate Clifford algebra [12]. There is also an older Maple package Octonion (and the companion package Clifford) by Jörg Schray [13].)
3 3 \times 3 OCTONIONIC HERMITIAN MATRICES

It is not immediately obvious that 3 \times 3 octonionic Hermitian matrices have a well-defined determinant, let alone a characteristic equation. We therefore first discuss some of the properties of these matrices before turning to the eigenvalue problem.

3.1 Jordan Matrices

The 3 \times 3 octonionic Hermitian matrices, henceforth referred to as the Jordan matrices, form the exceptional Jordan algebra under the Jordan product

A \circ B := \frac{1}{2}(AB + BA) \quad (20)

which is commutative, but not associative. A special case of this is

A^2 \equiv A \circ A \quad (21)

and we define

A^3 := A^2 \circ A = A \circ A^2 \quad (22)

Remarkably, with these definitions, Jordan matrices satisfy the usual characteristic equation (see e.g. [2])

A^3 - (\text{tr } A) A^2 + \sigma(A) A - (\det A) I = 0 \quad (23)

where \sigma(A) is defined by

\sigma(A) := \frac{1}{2} \left( (\text{tr } A)^2 - \text{tr } (A^2) \right) \quad (24)

and where the determinant \det A of A is defined abstractly in terms of the Freudenthal product [14,15]

A \ast B = A \circ B - \frac{1}{2} (A \text{ tr } (B) + B \text{ tr } (A)) + \frac{1}{2} (\text{tr } (A) \text{ tr } (B) - \text{tr } (A \circ B)) \quad (25)
\begin{align*}
\text{In}[10] &:= \ A = \text{MakeHMatrix}[[p, m, n, a, b, c]] \\
\text{Out}[10] &:= \begin{pmatrix} p & a & \bar{b} \\ a & m & c \\ b & c & n \end{pmatrix} \\
\text{In}[11] &:= \text{Tr}[A] \\
\text{Out}[11] &= m + n + p \\
\text{In}[12] &:= \text{JDet}[A] \\
\text{Out}[12] &= mn p + b*(a*c) + (\bar{c} \ast \bar{a}) \ast \bar{b} - n |a|^2 - m |b|^2 - p |c|^2 \\
\text{In}[13] &:= A = A /. \{a \to O1, b \to O2, c \to O3\} \\
\text{Out}[13] &= \begin{pmatrix} p & a_1 + a_2 i & b_1 - b_2 i - b_3 j \\ a_3 - a_2 i & m & c_1 + c_2 i + c_3 + c_4 k + c_8 l \\ b_1 + b_2 i + b_3 j & c_1 - c_2 i - c_3 j - c_4 k - c_8 l & n \end{pmatrix} \\
\text{In}[14] &:= \text{JDet}[A] \\
\text{Out}[14] &= 2a_1 b_1 c_1 - 2a_2 b_2 c_1 - 2a_3 b_3 c_1 - 2a_1 b_2 c_2 - 2a_1 b_3 c_2 + 2a_2 b_3 c_4 - b_5 m - b_2 m - b_5 m - a_3^2 n - a_2^2 n - c_2^2 p - c_2^2 p - c_2^2 p - c_6^2 p + m n p \\
\end{align*}

Fig. 4. A Mathematica session showing the calculation of the determinant for a generic Jordan matrix, both abstractly and in terms of an explicit basis.

which leads to

\begin{equation}
\text{det}(A) = \frac{1}{3} \text{tr} \left( (A \ast A) \circ A \right)
\end{equation}

Concretely, if

\begin{equation}
A = \begin{pmatrix} p & a & \bar{b} \\ \bar{a} & m & c \\ b & \bar{c} & n \end{pmatrix}
\end{equation}

with \(p, m, n \in \mathbb{R}\) and \(a, b, c \in \mathbb{O}\), then

\begin{align*}
\text{tr} A &= p + m + n \\
\sigma(A) &= pm + pm + mn - |a|^2 - |b|^2 - |c|^2 \\
\text{det} A &= pmn + b(ac) + \overline{b(ac)} - n|a|^2 - m|b|^2 - p|c|^2 \\
\end{align*}

The determinant calculation is illustrated in Figure 4.
3.2 The Real Eigenvalue Problem

Each division algebra can be rewritten as a real matrix algebra of appropriate dimension (see e.g. [1]). Under this identification, a Hermitian matrix over any of the division algebras becomes a real symmetric matrix. It is therefore clear that a $3 \times 3$ octonionic Hermitian matrix must have $8 \times 3 = 24$ real eigenvalues [16]. However, as we now discuss, instead of having (a maximum of) 3 distinct real eigenvalues, each with multiplicity 8, there appear to be (a maximum of) 6 distinct real eigenvalues, each with multiplicity 4.

The reason for this is that, somewhat surprisingly, a (real) eigenvalue $\lambda$ of a Jordan matrix $A$ does not in general satisfy the characteristic equation. To see this, consider the eigenvalue equation (1), with $A$ as in (27), $\lambda \in \mathbb{R}$, and where $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Assuming without loss of generality that $z \neq 0$, explicit computation yields [1]

$$
\left[ \det(\lambda I - A) \right] z \equiv \left[ \lambda^3 - (\text{tr} A) \lambda^2 + \sigma(A) \lambda - \det A \right] z \\
= b\left(a(cz)\right) + \overline{\tau}\left(\overline{\pi(bz)}\right) - \left[ b(ac) + \left(\overline{\tau \pi}\right)\overline{b} \right] z 
$$

(31)

If $a$, $b$, $c$, and $z$ associate, the RHS of (31) vanishes, and $\lambda$ does indeed satisfy the characteristic equation (23); this will not happen in general. However, since the LHS of (31) is a real multiple of $z$, this must also be true of the RHS, so that

$$
b\left(a(cz)\right) + \overline{\tau}\left(\overline{\pi(bz)}\right) - \left[ b(ac) + \left(\overline{\tau \pi}\right)\overline{b} \right] z = rz \\
r \in \mathbb{R} 
$$

(32)

which can be solved to yield a quadratic equation for $r$ as well as constraints on $z$.

**Theorem 5 (Dray & Manogue [1])** The real eigenvalues of the $3 \times 3$ octonionic Hermitian matrix $A$ satisfy the modified characteristic equation

$$
\det(\lambda I - A) = \lambda^3 - (\text{tr} A) \lambda^2 + \sigma(A) \lambda - \det A = r 
$$

(33)

where $r$ is either of the two roots of

$$
r^2 + 4\Phi(a, b, c) r - \left|[a, b, c]\right|^2 = 0 
$$

(34)

with $a, b, c$ as defined by (27) and where $\Phi$ was defined in (14).
Furthermore, provided that \([a, b, c] \neq 0\), each of \(x, y, z\) can be shown to admit an expansion of the form (given for \(z\) only)

**Corollary 6** With \(A\) and \(r\) as above, and assuming \([a, b, c] \neq 0\),

\[
z = (\alpha a + \beta b + \gamma c + \delta) \left(1 + \frac{[a, b, c]r}{|[a, b, c]|^2}\right)
\]

(35)

with \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\).

**PROOF.** Both of these results were obtained using *Mathematica* to solve (32) by brute force for real \(r\) and octonionic \(z\) given generic octonions \(a, b, c\). An outline of the computation appears in Figure 5. \(\square\)

The real parameters \(\alpha, \beta, \gamma, \delta\) may be freely specified for one (nonzero) component, say \(z\); the remaining components \(x, y\) have a similar form.

The solutions of (32) are real, since the corresponding \(24 \times 24\) real symmetric matrix has 24 real eigenvalues. We will refer to the 3 real solutions of (32) corresponding to a single value of \(r\) as a *family* of eigenvalues of \(A\). There are thus 2 families of real eigenvalues, each corresponding to a 4 independent (over \(\mathbb{R}\)) eigenvectors.

We note several intriguing properties of these results. If \(A\) is in fact complex, then the only solution of (34) is \(r = 0\), and we recover the usual characteristic equation with a unique set of 3 (real) eigenvalues. If \(A\) is quaternionic, then one solution of (34) is \(r = 0\), leading to the standard set of 3 real eigenvalues and their corresponding quaternionic eigenvectors. However, unless \(a, b, c\) involve only two independent imaginary quaternionic directions (in which case \(\Phi(a, b, c) = 0 = [a, b, c]\)), there will also be a nonzero solution for \(r\), leading to a second set of 3 real eigenvalues. Finally, if \(A\) is octonionic (so that in particular \([a, b, c] \neq 0\)), then there are two distinct solutions for \(r\), and hence two different sets of real eigenvalues, with corresponding eigenvectors. Note that if \(\det A = 0 \neq [a, b, c]\) then all of the eigenvalues of \(A\) will be nonzero!

### 3.3 Orthogonality

The final surprise lies with the orthogonality condition for eigenvectors \(v, w\) corresponding to different eigenvalues. It is *not* true that \(v^\dagger w = 0\), although the real part of this expression does vanish \([1]\). But, at least in the \(2 \times 2\) case \([1]\), it is straightforward to show that what is needed to ensure a decomposition
Definition 7 (Dray & Manogue [1]) Let \( v \) and \( w \) be two octonionic vectors. We will say that \( w \) is orthogonal to \( v \) if
\[
(vv^\dagger)w = 0
\] (36)

In the \( 3 \times 3 \) case, a lengthy, direct computation verifies that eigenvectors with different real eigenvalues satisfy (36) provided that the same value of \( r \) is used for both eigenvectors.
Theorem 8 (Dray & Manogue [1]) If \( v \) and \( w \) are eigenvectors of the \( 3 \times 3 \) octonionic Hermitian matrix \( A \) corresponding to different real eigenvalues in the same family (same \( r \) value), then \( v \) and \( w \) are mutually orthogonal in the sense of (36).

**PROOF.** The modified characteristic equation (33) can be used to eliminate cubic and higher powers of \( \lambda \) from any expression. Furthermore, given two distinct eigenvalues \( \lambda_1 \neq \lambda_2 \), subtracting the two versions of (33) and factoring the result leads to the equation

\[
(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) - (\text{tr} A)(\lambda_1 + \lambda_2) + \sigma(A) = 0
\]

which can be used to eliminate quadratic terms in one of the eigenvalues. We used Mathematica to implement these simplifications in a brute force verification of (36) in this context, which used 6 hours of CPU time on a SUN Sparc20 with 224 Mb of RAM. A summary of the computation appears in Figures 6 and 7. □

For Jordan matrices, we thus obtain two decompositions of the form (4), corresponding to the two sets of real eigenvalues. For each, the eigenvectors are fixed up to orthogonal transformations which preserve the form (35) of \( z \).

Theorem 9 (Dray & Manogue [1]) Let \( A \) be a \( 3 \times 3 \) octonionic Hermitian matrix. Then \( A \) can be expanded as in (4), where \( \{v_1, v_2, v_3\} \) is an orthonormal basis, as per (36), of eigenvectors of \( A \) corresponding to the real eigenvalues \( \lambda_m \), which belong to the same family (same \( r \) value).

**PROOF.** Fix a family of real eigenvalues of \( A \) by fixing \( r \). If the eigenvalues are distinct, then the previous theorem guarantees the existence of an orthonormal basis of eigenvectors, and the result follows.

If the eigenvalues are the same, the family consists of a single real eigenvalue \( \lambda \) with multiplicity 3. Then \( \text{tr}(A) = 3\lambda \) and \( \sigma(A) = 3\lambda^2 \). Writing out these two equations in terms of the components (27) of \( A \), and inserting the first into the second, results in a quadratic equation for \( \lambda \); the discriminant \( D \) of this equation satisfies \( D \leq 0 \). But \( \lambda \) is assumed to be real, which forces \( D = 0 \), which in turn forces \( A \) to be a multiple of the identity matrix, for which the result holds.

The remaining case is when one eigenvalue, say \( \mu \), has multiplicity 2 and one has multiplicity 1. Letting \( v \) be a (normalized) eigenvector with eigenvalue \( \mu \), consider the matrix

\[
X = A - \alpha vv^\dagger
\]

\( \alpha \) is a parameter which is chosen so that \( \langle v, w \rangle = 0 \) for \( \{v, w\} \) any set of eigenvectors corresponding to \( \mu \) and \( \lambda \), respectively.
In[17]:= 
\(Z_0 := \text{Expand}[\text{Denominator}[Oz[[8]]]]\)

In[18]:= 
\(X[i_] := \text{Expand}[\text{Mult}[a, \text{Mult}[c, Z_0]] + (q[i]-m) \text{Mult}[b, Z_0]]\)
\(Y[i_] := \text{Expand}[\text{Mult}[c, \text{Mult}[b, Z_0]] + (q[i]-p) \text{Mult}[c, Z_0]]\)
\(Z[i_] := \text{Expand}[((q[i]-p) (q[i]-m) - \text{Onormsq}[a]) Z_0]\)

In[21]:= 
\(V[i_] := \text{MakeVector}[[X[i], Y[i], Z[i]]]\)

In[22]:= 
\{r1, r2\} = r /. \text{RRule};
\(r2Rule = r^2 \rightarrow \text{Expand}[(r1+r2)*r-r1*r2];\)
\(r3Rule = r^3 \rightarrow \text{Expand}[(r (r^2 /. r2Rule))/.r2Rule];\)

In[25]:= 
\(qRule = q[2]^2 \rightarrow \text{Expand}[(\text{Tr}[A][[1]] - q[1]) (q[1]+q[2]) - \text{sigma}[A][[1]])];\)
\(q3Rule = q[1]^3 \rightarrow \text{Expand}[\text{Tr}[A][[1]] q[1]^2 - \text{sigma}[A][[1]] q[1]+J\text{det}[A]+r];\)
\(q4Rule = q[1]^4 \rightarrow \text{Expand}[(q[1]*q[1]^3 /. q3Rule)) / . q3Rule];\)

In[28]:= 
ZeroVector == \text{Expand}[(\text{MMult}[A, V[1]] - q[1] V[1]] / . q3Rule / . r2Rule]

Out[28]=
True

In[29]:= 
\(VV = \text{Expand}[\text{Expand}[\text{MSq}[V[1]]]] / . \{q4Rule, q3Rule\}
\/. \{r3Rule, r2Rule\}];\)

Fig. 6. The construction of eigenvectors \(V[i]\) of \(A\) with different real eigenvalues \(q[i]\).
The characteristic equation (33) is implemented via \(q3Rule\); the condition (14) on \(r\)
is given by \(r2Rule\), and \(qRule\) is the extra condition (37). Finally, \(VV = V[1] V[1]^{\dagger}\).

with \(\alpha \in \mathbb{R}\). For most values of \(\alpha\), \(X\) will have 3 distinct real eigenvalues, whose
eigenvectors will be orthogonal by the previous theorem. But this means that
eigenvectors of \(X\) are also eigenvectors of \(A\); the required decomposition of \(A\)
is obtained from that of \(X\) simply by solving (38) for \(A\). □

Note in particular that for some quaternionic matrices with determinant equal
to zero, one and only one of these two decompositions will contain the eigen-
value zero.

15
In[7]:=
    VVW0 = Expand[MMult[VV, W]]; 

In[9]:=
    Clear[W, VV] 

In[10]:=
    VVW011 = VVW0[[1, 1]]; 
    VVW021 = VVW0[[2, 1]]; 
    VVW031 = VVW0[[3, 1]]; 
    Clear[VVW0]; 

In[15]:=
    VVW11 = Expand[Expand[ 
        VVW011/. {q4Rule, q3Rule}]/. {r3Rule, r2Rule}]; 
    Clear[VVW011]; 

In[19]:=
    VVW21 = Expand[Expand[ 
        VVW021/. {q4Rule, q3Rule}]/. {r3Rule, r2Rule}]; 
    Clear[VVW021]; 

In[23]:=
    VVW31 = Expand[Expand[ 
        VVW031/. {q4Rule, q3Rule}]/. {r3Rule, r2Rule}]; 
    Clear[VVW031]; 

In[27]:=
    VVW = MakeVector[{VVW11, VVW21, VVW31}]; 

In[30]:=
    VVW === ZeroVector 

Out[30]=
    True 

In[33]:=
    TimeUsed[]/3600. 

Out[33]=
    5.90069 

Fig. 7. A summary of the *Mathematica* 2.2 computation used to show that eigenvectors $V, W$ of Jordan matrices with different real eigenvalues are orthogonal in the generalized sense. Note that each term was simplified separately.
4 Example

Let $s$ be given by

$$s = \cos \theta + k \ell \sin \theta$$

and consider the matrix

$$B = \begin{pmatrix} p & iq & kqs \\ -iq & p & jq \\ -kqs & -jq & p \end{pmatrix}$$

noting that $B$ is quaternionic if $\theta = 0$. Turning first to the equation for $r$, (32) becomes

$$r^2 + 4q^3 r \cos \theta - 4q^6 \sin^2 \theta = 0$$

with solutions

$$r_\pm = -2q^3 (\cos \theta \pm 1)$$

Since

$$\text{tr} \ B = 3p$$
$$\sigma(B) = 3(p^2 - q^2)$$
$$\det B = p^3 - 3pq^2 + 2q^3 \cos \theta$$

the eigenvalue equation (33) becomes

$$0 = \lambda^3 - 3p \lambda^2 + 3(p^2 - q^2) \lambda - (p^3 - 3pq^2 \mp 2q^3)$$
$$= (\lambda - p \mp q)^2 (\lambda - p \pm 2q)$$

An orthonormal basis of eigenvectors associated with these eigenvalues is

$$\lambda_u = p \pm q : \quad u_\pm = \begin{pmatrix} i \\ 0 \\ j \end{pmatrix} (f_u R_\pm)$$

$$\lambda_v = p \pm q : \quad v_\pm = \begin{pmatrix} j \\ 2ks \\ i \end{pmatrix} (f_v R_\pm)$$
Fig. 8. A *Mathematica* computation illustrating that the vectors $u_+, v_+, w_+$ given in (48)–(50) are indeed eigenvectors of the Jordan matrix $B$ given in (40) with the given eigenvalues (with $f_u = 1$, $f_v = i$, and $f_w = j$), that these eigenvectors are only orthogonal in the generalized sense of Theorem 8, and that they lead to a decomposition of $B$ as implied by Theorem 9. (Normalization factors have been added in the final computation, since the given vectors are not unit vectors.)
$$\lambda_w = p \mp 2q : \quad w_\pm = \begin{pmatrix} j \\ -ks \\ i \end{pmatrix} (f_w R_\pm)$$  \hspace{1cm} (50)$$

where \( f_u, f_v, f_w \) are arbitrary linear combinations of \( a = iq, b = -kqs, c = jq \), i.e.

$$f_u, f_v, f_w \in \langle 1, a, b, c \rangle$$  \hspace{1cm} (51)$$

and where \( R_\pm \) is given by

$$R_\pm = \begin{cases} 
\sin \frac{\theta}{2} + k\ell \cos \frac{\theta}{2} \\
\cos \frac{\theta}{2} - k\ell \sin \frac{\theta}{2}
\end{cases}$$  \hspace{1cm} (52)$$

Note that in the limiting case \( \theta \to 0 \), each \( f \) is quaternionic, and \( R_- \) reduces to 1 while \( R_+ \) becomes \( k\ell \).

As expected, provided one fixes a family of eigenvectors and eigenvalues arising from a given choice of \( r \), these eigenvectors satisfy the orthogonality property (36) and thus lead to a decomposition of the form (4). A partial verification of this using Mathematica is given in Figure 8.

### 5 Discussion

There are 2 quite different surprising aspects of our work: the mathematical changes needed to extend the eigenvalue problem to the octonions, and the fact that we have only been able to prove one of our key results using computer algebra. We discuss each of these in turn.

It is of course intriguing that the eigenvalue problem over the octonions changes so much, for instance in that there are unexpectedly many real eigenvalues. But we find it remarkable that so much of the standard structure remains, provided it is reinterpreted appropriately. The most striking example of this is the need to generalize what is meant by orthogonality.

We can relate our notion of orthonormality to the usual one by noting that a basis of \( \mathbb{O}^a \) which is orthonormal in the sense (36) satisfies

$$vv^\dagger + \ldots + ww^\dagger = I$$  \hspace{1cm} (53)$$

which follows directly from the definition. If we define a matrix \( Q \) whose
columns are just $v, ..., w$, then this statement is equivalent to

$$QQ^\dagger = I$$  \hspace{1cm} (54)

Over the quaternions, left matrix inverses are the same as right matrix inverses, and we would also have

$$Q^\dagger Q = I$$  \hspace{1cm} (55)

or equivalently

$$v^\dagger v = 1 = ... = w^\dagger w; \hspace{0.5cm} v^\dagger w = 0 = ...$$  \hspace{1cm} (56)

which is just the standard notion of orthogonality. These two notions of orthogonality fail to be equivalent over the octonions; we have been led to view the former as more fundamental.

Turning to our proof-by-computer, we reiterate that the only proof we currently have of our main orthogonality result, namely Theorem 8, uses Mathematica to explicitly perform a horrendous, but exact, algebraic computation. While one could hope for a more elegant mathematical proof of this result, the Mathematica computation nevertheless establishes a result which would otherwise remain for the moment merely a conjecture. This is a good example of being able to use the computer to verify one’s intuition when it may not be possible to do so otherwise.

But even more is true: Throughout our work with the octonions, the ability to manipulate octonionic expressions quickly and accurately has been crucial in developing our intuition in the first place. We do not feel that we would have been able to reach anything like our current understanding of the applications of the octonions to physics, on which we continue to be working actively, without the availability of a package such as the one described here.

Finally, this computation was initially done in 1996 using Mathematica 2.2. While preparing this paper, we attempted to reproduce the computation using Mathematica 3.0 — and couldn’t! Even for Mathematica 2.2, it was necessary to massage the computation by hand in order to succeed. One way this was done (between Figure 6 and Figure 7) was by saving some intermediate steps to files and then restarting the kernel. Another technique was not to simplify all the components of an expression at the same time. For instance, in Figure 7, the vector $VVW = (VV^\dagger)W$ contains 3 octonions, each of which requires roughly 8 Mb.

Comparing the computations both versions of Mathematica could handle, Mathematica 3.0 appears to require nearly 4 times as much CPU time for
the same computation; this was for identical inputs, with a minimum of special formatting. It is unfortunate that the many nice features of Mathematica 3.0 appear to require such a high price.

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