Mixed frameworks and structure preserving integration for coupled 
electro-elastodynamics

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In the present contribution new approaches for the design of structure preserving time integrators for nonlinear coupled problems are proposed. Polyconvexity inspired energy functionals are obtained by using the rediscovered tensor cross product which greatly simplifies the algebra, see [1]. In this connection an extended kinematic set, consisting of the right Cauchy-Green tensor, its co-factor and its Jacobian, is introduced. On this basis coupled problems like e.g. non-linear thermo-elastodynamics, see [2] or electro-elastodynamics, see [3], can be considered. Furthermore the formulations are readily extendible for mixed Hu-Washizu type formulations where the extended kinematic set is introduced as unknown field. In particular in [1] an elegant cascade system of kinematic constraints was introduced for elastodynamics, crucial for the satisfaction of the required conservation properties of a new family of energy momentum (EM) consistent time integrators. The objective of the present contribution is the introduction of new mixed variational principles for EM consistent time integrators in electro-elastodynamics, hence bridging the gap between the previous works [3] and [1], opening the possibility to a variety of new finite element implementations, see [4]. The following characteristics of the proposed EM consistent time integrator make it very appealing: (i) the new family of time integrators can be shown to be thermodynamically consistent and second order accurate; (ii) piecewise discontinuous interpolation of the mixed fields is carried out in order to obtain a computational cost comparable to that of standard displacement, electric potential formulations. Eventually, the superior numerical performance of the proposed formulation is demonstrated.

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1 Polyconvexity inspired framework

A novel polyconvexity inspired formulation based on an extended kinematic set of strain-type variables is introduced (see e.g. [1]). The kinematic set consists of the fibre map (deformation gradient F), the area map (cofactor H) and the volume map (Jacobian determinant J). In particular the transformations of fiber elements \( d\mathbf{x} = \mathbf{F} \, d\mathbf{X} \), area elements \( d\mathbf{a} = \mathbf{H} \, d\mathbf{A} \) and volume elements \( d\mathbf{v} = J \, d\mathbf{V} \) are employed using the following definitions

\[
\begin{align*}
\mathbf{F} & = \partial_{\mathbf{X}} \varphi, \\
\mathbf{H} & = \text{cof}(\mathbf{F}) = \frac{1}{2} \mathbf{F} \times \mathbf{F}, \\
J & = \det(\mathbf{F}) = \frac{1}{2} (\mathbf{F} : \mathbf{F}) : \mathbf{F}.
\end{align*}
\]

In the above \( \mathbf{X} \) denotes the placement of a particle in the reference configuration and \( \varphi \) is the deformation mapping. Furthermore the tensor cross product is defined by \( (A \times B)_{ij} = \varepsilon_{\alpha\beta\gamma} A_{\alpha a} B_{\beta b} \) (for more informations see e.g. [1]) where \( \varepsilon_{ijk} \) denotes the permutation symbol and the summation convention applies. For an objective formulation symmetric strain measures are introduced as

\[
\begin{align*}
\mathbf{C} & = \mathbf{F}^T \mathbf{F}, \\
\mathbf{G} & = \frac{1}{2} \mathbf{C} \times \mathbf{C}, \\
\mathbf{C} & = \frac{1}{2} \mathbf{G} : \mathbf{C},
\end{align*}
\]

where \( \mathbf{C} \) is the right Cauchy-Green tensor, \( \mathbf{G} \) its cofactor and \( \mathbf{C} \) its determinant. The introduction of the extended kinematic set facilitates the use of polyconvexity inspired formulations of strain energy density functions. The unsophisticated structure of such energy density functions makes possible a simplified variation and linearisation.

2 Electro-elastodynamics

The balance of linear momentum is given by

\[
\text{Div}(\mathbf{FS}) + \mathbf{B} = \rho_0 \ddot{\varphi},
\]

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where \( S \) denotes the 2nd PK stress tensor comprised of pure mechanical and electromechanical contributions, \( B \) denotes the volume load density and \( \rho_0 \) denotes the mass density. In addition to the above equation Gauss’s law and Faraday’s law

\[
\text{Div}(D) = \rho_0, \quad E = -\partial_X \phi,
\]

characterize the electrostatic case which is assumed herein to describe the motion of electroactive polymers. In equation (4) \( D \) denotes the electric displacement, \( \rho_0 \) denotes the volume charge density, \( E \) denotes the electric field and \( \phi \) denotes the electric potential. An objective reexpression of the internal energy density is employed \( u(F,D) = u(C,G,C,D) \). On this basis we provide a static three-field variational principle

\[
\Pi(\phi, \phi, D) = \int_{B_0} u(\phi, D) + \partial_X \phi : D \ dV - \Pi^{\text{ext}}(\phi, \phi)
\]

Furthermore a more sophisticated mixed Hu-Washizu type approach is postulated (for more informations see [4]) by providing the following nine-field augmented potential

\[
\Pi(\phi, \phi, D, \mathcal{D}, \Lambda) = \int_{B_0} u(C, G, C, D) + D : \partial_X \phi + \Lambda_C : \Phi_C + \Lambda_G : \Phi_G + \Lambda_C \Phi_C \ dV - \Pi^{\text{ext}}(\phi, \phi)
\]

using the extended kinematic set \( \mathcal{D} = \{ C, G, C \} \). A beneficial cascade form of the constraints (see [1])

\[
\Phi_C = \partial_X \phi^T \partial_X \phi - C, \quad \Phi_G = \text{cof}(C) - G, \quad \Phi_C = \frac{1}{3} C : G - C,
\]

with corresponding Lagrange multipliers \( \Lambda = \{ \Lambda_C, \Lambda_G, \Lambda_C \} \) is employed (see [4]). By using Hamilton’s principle, the above three- and nine-field formulations are extended to dynamic formulations (for more informations see [3, 4]).

### 3 Discretisation

For the weak forms of the coupled problems a structure preserving integration is conceived. In particular the concept of partitioned discrete derivatives is applied which leads to energy and momentum consistency (see [4]). For the spatial discretisation isoparametric finite elements are employed. A static condensation of fields \( D, \mathcal{D}, \Lambda \) is possible and leads to similar costs when compared to standard three-field \((v, \varphi, \phi)\)-formulations.

### 4 Numerical Examples

An H-shaped actuator is considered with applying electrical Neumann boundary conditions and an initial velocity. Snapshots of the configuration and contour plots of third component of \( D \) at \( t = \{ 0, 0.8, 1.6, 2.4, 3.2, 4.0 \} \) are shown in Fig. 1. Total angular momentum and energy evolution are shown for the energy-momentum scheme and the midpoint rule (see [4]).

![Fig. 1: Snapshots of \( D \) at \( t = \{ 0, 0.8, 1.6, 2.4, 3.2, 4.0 \} \) (left), time evolution of total angular momentum \( ||J|| \) and energy \( H \) (right).](image)

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