A UNIFORM FIELD-OF-DEFINITION/FIELD-OF-MODULI BOUND FOR DYNAMICAL SYSTEMS ON $\mathbb{P}^N$

JOHN R. DOYLE AND JOSEPH H. SILVERMAN

Abstract. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be an endomorphism of degree $d \geq 2$ defined over $\mathbb{Q}$ or $\mathbb{Q}_p$, and let $K$ be the field of moduli of $f$. We prove that there is a field of definition $L$ for $f$ whose degree $[L : K]$ is bounded solely in terms of $N$ and $d$.

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1. Introduction

We start with an informal description of a fundamental problem. Let $K$ be an algebraically closed field, for convenience of characteristic 0, and let $X$ be an algebraic “object” defined over $K$. The field of moduli (FOM) of $X$ is the smallest subfield $K \subset \bar{K}$ with the property that for every $\sigma \in \text{Gal}(\bar{K}/K)$, there is a $\bar{K}$-isomorphism from $X^\sigma$ to $X$. A field of definition (FOD) for $X$ is a subfield $K \subset \bar{K}$ with the property that there is an “object” $Y$ defined over $K$ such that $Y$
is $K$-isomorphic to $X$. It is easy to see that every FOD contains the FOM. The field-of-moduli versus field-of-definition problem is to determine whether the FOM is itself already a FOD, and if not, to describe the extent to which one must extend the FOM in order to obtain a FOD.

The FOM versus FOD problem arises in many areas of arithmetic geometry, including for example the theories of abelian varieties [13, 20], curves and their covering maps [11, 3], sets of $n$ points [12], automorphic functions on $\mathbb{P}^1$ [19], and dynamical systems [21]. (This list of references is meant to be illustrative, and is far from exhaustive.) Our primary goal in this paper is to prove a uniform bound for the minimal degree of a FOD over the FOM for dynamical systems on $\mathbb{P}^N$.

We start with some notation and formal definitions, then we state our main theorem and briefly survey earlier results on the FOM-versus-FOD problem in dynamics.

Let $K$ be a field of characteristic 0, $\bar{K}$ an algebraic closure of $K$, $G_K$ the Galois group $\text{Gal}(\bar{K}/K)$, $V/K$ an algebraic variety that is defined over $K$, $\text{End}(V)$ the monoid of $\bar{K}$-endomorphisms $f : V \to V$, $\text{Aut}(V)$ the group of $\bar{K}$-automorphisms $\varphi : V \to V$.

We let $\text{Aut}(V)$ act on $\text{End}(V)$ by conjugation, i.e., for $f \in \text{End}(V)$ and $\varphi \in \text{Aut}(V)$, we define

$$f^\varphi := \varphi^{-1} \circ f \circ \varphi.$$ 

This is the correct action for dynamics, since it commutes with iteration,

$$(f \circ f \circ \cdots \circ f)^\varphi = f^\varphi \circ f^\varphi \circ \cdots \circ f^\varphi.$$ 

**Definition.** Let $f \in \text{End}(V)$. The field of moduli (FOM) of $f$ is the fixed field of the following subgroup of $G_K$:

$$\{ \sigma \in G_K : \text{there exists a } \varphi \in \text{Aut}(V) \text{ so that } f^\sigma = f^\varphi \}.$$ 

**Definition.** Let $f \in \text{End}(V)$. A subfield $L$ of $\bar{K}$ is a field of definition (FOD) for $f$ if there is an automorphism $\varphi \in \text{Aut}(V)$ so that the conjugate $f^\varphi$ is defined over $L$.

For a given $f \in \text{End}(V)$, the following group of automorphisms of $f$ plays a key role in studying the FOM and FODs for $f$. More precisely, the analysis is generally much easier to prove if one assumes that $\text{Aut}(f)$ is trivial.
Definition. Let \( f \in \text{End}(V) \). The automorphism group of \( f \) is the subgroup of \( \text{Aut}(V) \) that commutes with \( f \), i.e.,
\[
\text{Aut}(f) := \{ \alpha \in \text{Aut}(V) : f^\alpha = f \}.
\]

It is clear that the FOM of \( f \) is contained in every FOD, but the FOM need not be a FOD. The \textit{FOM-versus-FOD problem} is to describe situations in which FOM = FOD, or to characterize the amount by which they may differ. The main result of the present note is a uniform bound for the minimal degree of a FOD over the FOM for endomorphisms of \( \mathbb{P}^N \). Our bound applies to all maps, including those having non-trivial automorphism group. For ease of exposition, we state a special case of our theorem here and refer the reader to Theorem \ref{thm:main} for the general statement.

**Theorem 1.** Fix integers \( N \geq 1 \) and \( d \geq 2 \). There is a constant \( C(N, d) \) such that the following holds: Let \( K \) be a number field or the completion of a number field, and let \( f : \mathbb{P}^N \rightarrow \mathbb{P}^N \) be an endomorphism of degree \( d \) defined over \( \overline{K} \) whose field of moduli is contained in \( K \). Then there is a field of definition \( L \) for \( f \) satisfying
\[
[L : K] \leq C(N, d).
\]

For endomorphisms of \( \mathbb{P}^1 \), i.e., for \( N = 1 \), much stronger results are known. If we let \( C(N, d) \) denote the smallest value making Theorem \ref{thm:main} true, then
\[
C(1, d) = \begin{cases} 
1 & \text{if } d \text{ is even} \ [21], \\
2 & \text{if } d \text{ is odd} \ [8].
\end{cases}
\]

In other words, even degree self-maps of \( \mathbb{P}^1 \) have FOM = FOD, while odd degree maps require at most a quadratic extension, and in all odd degrees there do exist maps with FOM \( \neq \) FOD. In order to handle maps having non-trivial automorphisms, both \[21\] and \[8\] require a detailed case-by-case analysis using the classical classification of finite subgroups of \( \text{PGL}_2(K) \).

For maps \( f : \mathbb{P}^N \rightarrow \mathbb{P}^N \) satisfying \( \text{Aut}(f) = 1 \), Hutz and Manes \[9\] generalized the earlier \( C(1, 2d) = 1 \) result to higher dimensions. It is also not hard in the setting of Theorem \[4\] to show that if \( \text{Aut}(f) = 1 \), then \( f \) has a FOD of degree at most \( N + 1 \) over its FOM; see Theorem \ref{thm:main}(b). But the situation becomes significantly more complicated for maps \( f \) possessing non-trivial automorphisms, and indeed Hutz and Manes give examples showing that their main theorem is false for maps with \( \text{Aut}(f) \neq 1 \).

**Question 2.** As noted earlier, Hidalgo \[8\] proved the \( N = 1 \) case of Theorem \[4\] with the explicit constant \( C(1, d) = 2 \). Thus our Theorem \ref{thm:main}
may be viewed as a higher dimensional version of Hidalgo’s theorem, although our result is neither as explicit nor as uniform as his $\mathbb{P}^1$ result, and our general result (Theorem 13) further requires a technical condition on the Brauer group of the base field $K$. It is striking that Hidalgo’s bound $C(1,d) = 2$ does not depend on $d$. This raises the natural question of whether Theorem 1 is true for all $N$ with a constant $C(N,d)$ that depends only on $N$.

**Remark 3.** A propos Question 2, we remark that Theorem 13(a) shows that the FOD/FOM bound in Theorem 1 can be replaced with a bound of the form

$$[L : K] \leq C''(N, \# \text{Aut}(f)).$$

(1)

It is then a theorem of Levy [11] that $\# \text{Aut}(f)$ may be bounded solely in terms of $N$ and $d$, but [11] yields a stronger result if, for example, one varies over a collection of maps of increasing degree whose automorphism groups have bounded size.

**Remark 4.** A primary application of the main result of this paper is to the Uniform Boundeness Conjecture [14] for preperiodic points. In a subsequent paper [4] we construct moduli spaces for dynamical systems with portraits, and we use the FOD/FOM results from the present paper to relate the Uniform Boundeness Conjecture to the existence of algebraic points of bounded degree on these dynamical portrait moduli spaces. We briefly describe this connection in Section 2 and refer the reader to [4] for complete details.

We close this introduction with a summary of the contents of this paper and a brief sketch of the steps that go into the proof of Theorem 1. As already noted, Section 2 briefly discusses dynamical moduli spaces the connection with the uniform boundedness conjecture. In Section 3 we review some facts about Brauer groups and the period–index problem, and we prove a cohomology splitting result (Proposition 9) involving a finite subgroup of an algebraic group and its normalizer and centralizer. Section 4 deals with the FOD/FOM problem for maps $f : V \rightarrow V$ of general varieties, and proves a key criterion (Proposition 10) for the $1$-cocycle $\varphi : G_K \rightarrow \text{Aut}(V)$ associated to $f$ to take values in the normalizer of $\text{Aut}(f)$ in $\text{Aut}(V)$. In Section 5 we state two Lemmas, which are actually theorems of Brauer and Levy, that will be needed to prove our main result. This leads to the proof in Section 6 of our main result, Theorem 13, which gives a uniform FOD/FOM bound for all $f : \mathbb{P}^N \rightarrow \mathbb{P}^N$, and also a more precise, and much more easily proven, FOD/FOM bound for maps satisfying $\text{Aut}(f) = 1$. The proof of Theorem 13 involves successively moving the $1$-cocycle from $\text{PGL}_{N+1}$...
to the normalizer of $\text{Aut}(f)$ in $\text{PGL}_{N+1}$ to the centralizer of $\text{Aut}(f)$ in $\text{PGL}_{N+1}$. We also lift $\text{Aut}(f)$ from $\text{PGL}_{N+1}$ to $\text{GL}_{N+1}$, decompose the resulting representation into a sum of irreducible representations, and apply a general version of Schur’s lemma and Hilbert’s theorem 90 to map the 1-cocycle associated to $f$ into a product of Brauer groups. Finally, in Section 7 we prove a result on endomorphisms, quotients, and twists (Proposition 14) and a result on uniform existence of periodic points off of specified subvarieties (Proposition 15) that we feel may be useful in further study of dynamical FOD/FOM problems.

2. DYNAMICAL MODULI SPACES, FOM-VERSUS-FOD, AND THE DYNAMICAL UNIFORM BOUNDEDNESS CONJECTURE

This section indicates how the FOD/FOM bound in Theorem 1 may be interpreted in terms of the existence of algebraic points of bounded degree on fibers of dynamical moduli spaces, and briefly describes an application to the Uniform Boundedness Conjecture. We refer the reader to [4] for details of this application. The material in this section is not used elsewhere in this paper.

Let $\text{End}^N_d$ denote the space of degree $d$ endomorphisms $f : \mathbb{P}^N \to \mathbb{P}^N$, and let $\varphi \in \text{PGL}_{N+1}(\bar{K})$ act on $\text{End}^N_d(\bar{K})$ by conjugation. The space $\text{End}^N_d$ has a natural structure as an affine variety, and one can show that the quotient $\mathcal{M}^N_d := \text{End}^N_d \sslash \text{PGL}_{N+1}$ also has the structure of an affine variety in the sense of geometric invariant theory. See [11, 15, 22] for details. We write $\langle \cdot \rangle : \text{End}^N_d \to \mathcal{M}^N_d$ for the quotient map. Then the FOM of $f \in \text{End}^N_d(\bar{K})$ may equally well be defined as the field generated by the coordinates of the point $\langle f \rangle \in \mathcal{M}^N_d(\bar{K})$, and similarly a field $L$ is a FOD for $f$ if $\langle f \rangle$ is in the image of $\text{End}^N_d(L)$. The FOM \text{FOD} phenomenon arises due to the fact that the map

$$\langle \cdot \rangle : \text{End}^N_d(\bar{K}) \to \mathcal{M}^N_d(\bar{K})$$

need not be surjective.

More generally, the authors have constructed spaces $\text{End}^N_d[\mathcal{P}]$ and $\mathcal{M}^N_d[\mathcal{P}]$ that classify maps together with a list of points modeling a given portrait $\mathcal{P}$; see [4]. These dynamical moduli spaces can be used to formulate the following uniform boundedness conjecture.

**Conjecture 5** (Strong Moduli Boundedness Conjecture). Fix integers $D \geq 1$, $N \geq 1$, and $d \geq 2$. Then there is a constant $C_1(D,N,d)$ such that for all number fields $K/\mathbb{Q}$ satisfying $[K : \mathbb{Q}] \leq D$ and all preperiodic portraits $\mathcal{P}$ containing at least $C_1(D,N,d)$ points, we have

$$\mathcal{M}^N_d[\mathcal{P}](K) = \emptyset.$$
This may be compared with the usual uniform boundedness conjecture for dynamical systems on $\mathbb{P}^N$.

**Conjecture 6 (Strong Uniform Boundedness Conjecture).** (Silverman–Morton [14]) Fix integers $D \geq 1$, $N \geq 1$, and $d \geq 2$. Then there is a constant $C_2(D, N, d)$ such that for all number fields $K/\mathbb{Q}$ satisfying $[K : \mathbb{Q}] \leq D$ and all endomorphisms $f \in \text{End}_d^N(K)$, we have

$$\# \left( \text{PrePer}(f) \cap \mathbb{P}^N(K) \right) \leq C_2(D, N, d).$$

Here $\text{PrePer}(f)$ denotes the set of points in $\mathbb{P}^N(\overline{K})$ having finite forward $f$-orbit, i.e., the set of preperiodic points for $f$.

It is easy to see that Conjecture 5 implies Conjecture 6 but in order to prove the converse, one needs a uniform FOD/FOM bound. And indeed, one of the motivations for the present paper was to provide this key step in proving the equivalence of Conjectures 5 and 6 in [4].

3. Preliminary Results on Group Cohomology and Brauer Groups

We start with a standard result for finite Galois modules, whose elementary proof we recall for the convenience of the reader.

**Lemma 7.** Let $A$ be a finite group with a continuous $G_K$-action, and let $c : G_K \to A$ be a continuous 1-cocycle. Then there exists an extension $L/K$ satisfying

$$[L : K] \leq \# A \cdot \# \text{Aut}(A) \quad \text{and} \quad c_\sigma = 1 \text{ for all } \sigma \in G_L.$$

In particular, $[L : K]$ is bounded by a constant that depends only on the order of the group $A$.

*Proof.* The action of $G_K$ on $A$ is given by a group homomorphism $G_K \to \text{Aut}(A)$. The fixed field of the kernel of this homomorphism has degree over $K$ bounded by $\# \text{Aut}(A)$. Replacing $K$ by this fixed field, we may assume that $G_K$ acts trivially on $A$. Then the 1-cocycle condition on $c$ says that $c : G_K \to A$ is a homomorphism. Taking $L$ to be the fixed field of the kernel of this homomorphism, we have $[L : K] \leq \# A$, and the homomorphism $c$ becomes trivial on $G_L$. \qed

We recall two definitions.

**Definition.** Let $\xi \in \text{Br}(K) = H^2(G_K, K^*)$. The period, respectively index, of $\xi$ are the quantities

$$\text{Period}(\xi) := \text{the order of } \xi \text{ as an element of } \text{Br}(K),$$

$$\text{Index}(\xi) := \min \left\{ [L : K] : \text{Res}_{L/K}(\xi) = 0 \text{ in } \text{Br}(L) \right\}.$$
Definition. Let $K$ be a field. We define the *Brauer period-index exponent* of $K$ to be the smallest integer $\beta(K) \geq 1$ with the property that every element $\xi \in \text{Br}(K)$ has the property that

$$\text{Index}(\xi) \text{ divides } \text{Period}(\xi)^{\beta(K)}.$$ 

(If no such integer exists, we set $\beta(K) = \infty$.) We note that the period always divides the index, so $\beta(K) \geq 1$, and thus

$$\text{Period}(\xi) = \text{Index}(\xi) \text{ for all } \xi \in \text{Br}(K) \iff \beta(K) = 1.$$ 

See for example [16, Proposition 1.5.17].

Remark 8. We summarize some standard properties relating the period and the index of elements of $\text{Br}(K)$. For additional information, see for example [6].

(a) If $K$ is a global field or a local field then $\beta(K) = 1$; see [16, Theorems 1.5.34 and 1.5.36].

(b) Let $K$ be an extension of an algebraically closed field $k$ of characteristic $0$. If tr. deg. $(K/k) = 1$, then Tsen’s theorem says that $\text{Br}(K) = 0$, and if tr. deg. $(K/k) = 2$, then $\beta(K) = 1$; see [2]. More generally, it is known [2] that $\beta(K) \geq \text{tr. deg.}(K/k) - 1$, and it is conjectured that this is always an equality.

Proposition 9. Let $K$ be a field, and suppose that we are given the following quantities:

- $\mathcal{G}/K$ an algebraic group defined over $K$.
- $\mathcal{A}/K$ a finite subgroup of $\mathcal{G}(\bar{K})$ that is defined over $K$.
- $\mathcal{N}/K$ the normalizer of $\mathcal{A}$ in $\mathcal{G}(\bar{K})$.
- $\mathcal{C}/K$ the centralizer of $\mathcal{A}$ in $\mathcal{G}(\bar{K})$.
- $\xi$ a cohomology class in the pointed set $H^1(\mathcal{G}_K, \mathcal{A}\setminus\mathcal{N})$.

Then there is a finite extension $L/K$ and a constant $c = c(#\mathcal{A})$ depending only on the order of the group $\mathcal{A}$ such that the following three statements are true:

$$\mathcal{A} \subset \mathcal{G}(L).$$ (2)

$$\text{Res}_{L/K}(\xi) \in \text{Image}(H^1(\mathcal{G}_L, \mathcal{C}) \to H^1(\mathcal{G}_L, \mathcal{A}\setminus\mathcal{N})).$$ (3)

$$[L : K] \leq c \cdot (\mathcal{C} \cap \mathcal{A})^{\beta(K)}.$$ (4)

Proof. To ease notation during the proof, when we replace $K$ by an extension field whose degree is bounded by a function of $\#A$, we again

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1Following [16], we define a *local field* to be a finite extension of one of $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_p((t))$, and a *global field* to be a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p((t))$. 
 denote the extension field by $K$. We also let $m = \text{the exponent of the finite group } \mathcal{A}$.

We first adjoin a primitive $m$’th root of unity to $K$, which gives an extension of degree at most $\varphi(m)$, which is less than $\# \mathcal{A}$. Next, the fact that $\mathcal{A}$ is finite and defined over $K$ means that the action of $G_K$ on $\mathcal{A}$ gives a homomorphism $G_K \to \text{Aut}(\mathcal{A})$. Hence replacing $K$ with a finite extension whose degree is bounded by $\# \text{Aut}(\mathcal{A})$, we may assume that $G_K$ acts trivially on $\mathcal{A}$. So we are reduced to the case that $\mathcal{A} \subset \mathcal{G}(K)$ and $\mu_m \subset K$.

For an abstract group $G$ and subgroup $A \subseteq G$ with normalizer $N$ and centralizer $C$, the elements of $N$ induce (inner) automorphisms of $A$, so more-or-less by definition we have an exact sequence

$$1 \longrightarrow C \longrightarrow N \xrightarrow{\gamma \mapsto (\alpha \mapsto \gamma^{-1} \alpha \gamma)} \text{Aut}(A).$$

(5)

We always have $A \subset N$, but the inclusion $A \subset C$ is equivalent to the statement that $A$ is abelian. So the exact sequence (5), taken modulo $A$, yields

$$1 \longrightarrow A \setminus AC \longrightarrow A \setminus N \xrightarrow{\gamma \mapsto (\alpha \mapsto \gamma^{-1} \alpha \gamma)} A \setminus \text{Aut}(A).$$

(6)

Applying (6) with $G = \mathcal{G}(\bar{K})$ and $A = \mathcal{A}$, we find that

$$\mathcal{A}C \setminus N \hookrightarrow A \setminus \text{Aut}(A).$$

(7)

We consider the exact sequence of groups

$$1 \longrightarrow \mathcal{A} \setminus AC \longrightarrow \mathcal{A} \setminus \mathcal{N} \longrightarrow \mathcal{A}C \setminus \mathcal{N} \longrightarrow 1.$$ 

Taking Galois cohomology gives the exact sequence of cohomology sets

$$H^1(G_K, \mathcal{A} \setminus AC) \longrightarrow H^1(G_K, \mathcal{A} \setminus \mathcal{N}) \longrightarrow H^1(G_K, \mathcal{A}C \setminus \mathcal{N}).$$

(8)

We know from (7) that the group $\mathcal{A}C \setminus \mathcal{N}$ is finite and has order bounded by $\# \mathcal{A} \text{Aut}(\mathcal{A})$, so the order of $\mathcal{A}C \setminus \mathcal{N}$ is bounded by a function of $\# \mathcal{A}$. Applying Lemma 7, we can replace $K$ by a finite extension such that the degree of the extension is bounded by a function of $\# \mathcal{A}$ and such that the image of $\xi$ in $H^1(G_K, \mathcal{A}C \setminus \mathcal{N})$ is trivial. Then the exact sequence (8) tells us that $\xi \in H^1(G_K, \mathcal{A} \setminus AC)$.

We use the basic isomorphism

$$\mathcal{A} \setminus AC \cong (C \cap \mathcal{A}) \setminus C.$$ 

The fact that $C \cap \mathcal{A}$ is in the center of $C$ means that when we take cohomology of the exact sequence

$$1 \longrightarrow C \cap \mathcal{A} \longrightarrow C \longrightarrow (C \cap \mathcal{A}) \setminus C \longrightarrow 1,$$
then as explained in [15, Chapter VII, Appendix, Proposition 2], we get an exact sequence with a connecting homomorphism to an $H^2$ term,

$$H^1(G_K, \mathcal{C}) \longrightarrow H^1(G_K, (\mathcal{C} \cap \mathcal{A}) \setminus \mathcal{C}) \longrightarrow H^2(G_K, \mathcal{C} \cap \mathcal{A}). \quad (9)$$

We write the finite abelian group $\mathcal{C} \cap \mathcal{A}$ as a product of cyclic groups,

$$\mathcal{C} \cap \mathcal{A} \cong \mu_{n_1} \times \cdots \times \mu_{n_t}.$$ 

We note that this is an isomorphism of $G_K$-modules, with all $G_K$-actions trivial, since we have already arranged matters so that $G_K$ acts trivially on $\mathcal{A}$ and on $\mu_m$, and since every $n_i$ divides the exponent $m$ of $\mathcal{A}$. Hence the right-hand cohomology group in the exact sequence (9) is

$$H^2(G_K, \mathcal{C} \cap \mathcal{A}) \cong \prod_{i=1}^{t} H^2(G_K, \mu_{n_i}) \cong \prod_{i=1}^{t} \text{Br}(K)[n_i].$$

The image of $\xi$ in $H^2(G_K, \mathcal{C} \cap \mathcal{A})$ gives a $t$-tuple

$$(\zeta_1, \ldots, \zeta_t) \in \prod_{i=1}^{t} \text{Br}(K)[n_i].$$

The element $\zeta_i$ has period $n'_i$ for some integer dividing $n_i$, so by definition of the Brauer period-index exponent $\beta(K)$, we see that $\zeta_i$ becomes trivial over an extension of $K$ of degree dividing $(n'_i)^{\beta(K)}$. Applying this reasoning to each of $\zeta_1, \ldots, \zeta_t$ and taking the compositum of the fields, we see that there is an extension $L/K$ of degree at most

$$(n'_1 n'_2 \cdots n'_t)^{\beta(K)} \leq (n_1 n_2 \cdots n_t)^{\beta(K)} = \#(\mathcal{C} \cap \mathcal{A})^{\beta(K)}.$$

such that the image of $\text{Res}_{L/K}(\xi)$ in $H^2(G_L, \mathcal{C} \cap \mathcal{A})$ is trivial.

To recapitulate, we have constructed an extension $L/K$ whose degree satisfies (1) and such that

$$\text{Res}_{L/K}(\xi) \longrightarrow 0 \quad \text{in the cohomology group } H^2(G_L, \mathcal{C} \cap \mathcal{A}).$$

It follows from the exact sequence (9) the we can lift $\text{Res}_{L/K}(\xi)$ to an element of the cohomology set $H^1(G_L, \mathcal{C})$, which is the desired conclusion. \hfill \Box

4. FOD/FOM for General Varieties

We recall that we have fixed a field $K$ of characteristic 0 and an algebraic variety $V/K$, and we are looking at morphisms $f : V \to V$ defined over an algebraic closure $\overline{K}$ of $K$. To ease notation, we let

$$\mathcal{A}_V := \text{Aut}(V) \quad \text{and} \quad \mathcal{A}_f := \text{Aut}(f),$$
and we also define

\[ N_f := \text{the normalizer of } A_f \text{ in } A_V, \]
\[ C_f := \text{the centralizer of } A_f \text{ in } A_V. \]

Let \( f : V \to V \) be an endomorphism whose field of moduli contains \( K \). By definition of FOM, for each \( \sigma \in G_K \) there exists an automorphism \( \varphi_\sigma \in A_V \) satisfying \( f^\sigma = f^{\varphi_\sigma} \), and the automorphism \( \varphi_\sigma \) is determined up to left composition by an element of \( A_f \). In this way \( f \) determines a well-defined map \( \varphi : G_K \to A_f \backslash A_V \), \( f^\sigma = f^{\varphi_\sigma} \) for all \( \sigma \in G_K \).

From the definition, it is easy to verify that \( \varphi \) is a “1-cocycle relative to the subgroup \( A_f \),” i.e., it satisfies

\[ \varphi^{-1}\varphi_\sigma \varphi_\tau \in A_f \quad \text{for all } \sigma, \tau \in G_K. \]

In particular, if \( A_f = 1 \), then \( \varphi \) is a \( G_K \)-to-\( A_V \) 1-cocycle, and thus represents an element of the cohomology set \( H^1(G_K, A_V) \). But in general \( \varphi \) is a sort of 1-cocycle taking values in the quotient \( A_f \backslash A_V \), which need not be a group. However, if \( A_f \) is defined over \( K \), then the situation is better, which is the first part of the following proposition.

**Proposition 10.** With notation as above, we make the following two assumptions:

1. The automorphism group \( A_f \) is finite.  \hspace{1cm} (10)
2. The group \( A_f \) is defined over \( K \).  \hspace{1cm} (11)

Then the following are true:

(a) The image \( \varphi(G_K) \) of \( \varphi \) is contained in \( N_f \), the normalizer of \( A_f \) in \( A_V \), and hence

\[ \varphi : G_K \to A_f \backslash N_f \]

is a 1-cocycle taking values in a group. This in turn gives an element of the cohomology set \( H^1(G_K, A_f \backslash N_f) \).

(b) The following are equivalent:

1. There is a \( \gamma \in A_V \) such that \( f^\gamma \) is defined over \( K \), i.e., \( K \) is a FOD for \( f \).
2. There is a \( \delta \in A_V \) such that \( \varphi_\sigma = A_f \delta^{-1}\delta^\sigma \) for all \( \sigma \in G_K \), i.e., \( \varphi \) is a \( G_K \)-to-\( A_f \backslash N_f \) coboundary.

\[ ^2 \text{If we ever need to indicate the fact that } \varphi \text{ depends on } f, \text{ we will write } \varphi_{f,\sigma}. \]
Proof. (a) For \( \alpha \in A_f \subset A_V \) and \( \sigma \in G_K \), the assumption \( (\ref{eq:assumption}) \) says that \( \alpha \sigma \in A_f \), which allows us to compute

\[
 f^{\varphi^{-1}} \alpha \varphi = (f^{\sigma^{-1}})^{\alpha \varphi} = ((f^{\alpha \sigma})^{\sigma^{-1}})^{\varphi} = (f^{\sigma^{-1}})^{\varphi} = (f^{\varphi^{-1}})^{\varphi} = f.
\]

Hence \( \varphi^{-1} \alpha \varphi \in A_f \), which proves that \( \varphi \sigma \in N_f \). Next, for \( \sigma, \tau \in G_K \) we compute

\[
 f^{\varphi_{\sigma \tau}} = (f^{\varphi_{\sigma \tau}})^{\tau} = (f^{\varphi_{\sigma}})^{\tau_{\sigma}} = f^{\varphi_{\sigma} \tau_{\sigma}}.
\]

Hence

\[
 \varphi_{\sigma \tau} \equiv \varphi_{\tau} \varphi_{\sigma} \pmod{A_f},
\]

so \( \varphi \) is a \( G_K \)-to-\( A_f \) \( 1 \)-cocycle.

(b) Suppose first that \( (1) \) holds, so we have some \( \gamma \in A_V \) such that \( \varphi^{\gamma} \) is defined over \( K \). It follows that for every \( \sigma \in K \) we have

\[
 f^{\gamma} = (f^{\gamma})^{\sigma} = (f^{\sigma})^{\gamma} = (f^{\varphi_{\sigma}})^{\gamma} = f^{\varphi_{\sigma} \gamma}.
\]

Hence \( \varphi_{\sigma} \gamma \gamma^{-1} \in A_f \), and we may take \( \delta = \gamma^{-1} \).

We next prove that \( (2) \) implies \( (1) \), so we assume that \( \delta \in A_V \) has the property that \( \varphi_{\sigma} = A_f \delta^{-1} \delta^{\sigma} \) for all \( \sigma \in G_K \). We set \( \gamma = \delta^{-1} \), so \( \varphi_{\sigma} \gamma \gamma^{-1} \in A_f \), and we use this to compute

\[
 (f^{\gamma})^{\sigma} = (f^{\sigma})^{\gamma} = (f^{\varphi_{\sigma}})^{\gamma} = f^{\varphi_{\sigma} \gamma} = f^{\gamma}.
\]

Hence \( f^{\gamma} \) is defined over \( K \). \( \square \)

5. Two Other Preliminary Results

In this section we state two results that are needed for the proof of Theorem \( (\ref{theorem:main}) \). We denominate them as lemmas, although they are in fact non-trivial theorems in their own right.

**Lemma 11.** (Brauer’s Theorem) Let \( \bar{K} \) be an algebraically closed field of characteristic 0, let \( \Gamma \subset GL_{N+1}(\bar{K}) \) be a finite group, let \( m \) be the exponent of \( \Gamma \), and let \( \zeta_m \in \bar{K} \) be a primitive \( m \)’th root of unity. Then there exists an element \( A \in GL_{N+1}(\bar{K}) \) such that \( A^{-1} \Gamma A \subset GL_{N+1}(\mathbb{Q}(\zeta_m)) \).

**Proof.** See, for example, [17, Theorem 24, §12.3]. \( \square \)

**Lemma 12.** (Levy [11]) Let \( \bar{K} \) be an algebraically closed field of characteristic 0. There is a constant \( C_3(N,d) \) such that every \( f \in End(\mathbb{P}^N) \) of degree \( d \) satisfies

\[
 \# \text{Aut}(f) := \# \{ \varphi \in PGL_{N+1}(\bar{K}) : f^\varphi = f \} \leq C_3(N,d).
\]

**Proof.** This is due to Levy [11], or see [24, Theorem 2.53]. \( \square \)
6. A FOD/FOM Bound for \( \mathbb{P}^N \) Endomorphisms

We recall that Theorem 1 in the introduction was stated only for number fields \( K \) and their completions, and that the bound for the FOD/FOM degree of \( f \) then depended only on \( \dim(\mathbb{P}^N) \) and \( \deg(f) \). For general fields of characteristic 0, we give a bound for the FOD/FOM degree that depends also on the period-index exponent of the Brauer group of \( K \).

**Theorem 13.** Let \( N \geq 1 \) and \( d \geq 2 \) be integers, let \( K \) be a field of characteristic 0, and let \( f : \mathbb{P}^N \to \mathbb{P}^N \) be an endomorphism of degree \( d \) defined over \( \bar{K} \) whose field of moduli is contained in \( K \).

(a) There is a field of definition \( L \) for \( f \) satisfying

\[
[L : K] \leq C_4(N, \#A_f) \cdot (\#A_f \cdot e^{(N+1)/e})^{\beta(K)},
\]

where as the notation indicates, the constant \( C_4(N, \#A_f) \) depends only on \( N \) and the order of the automorphism group \( A_f \).

(b) There is a field of definition \( L \) for \( f \) satisfying

\[
[L : K] \leq C_5(N, d)^{\beta(K)}.
\]

(c) Suppose further that \( \text{Aut}(f) = 1 \). Then there is a field of definition \( L \) for \( f \) satisfying

\[
[L : K] \leq (N + 1)^{\min(\beta(K), N)}.
\]

**Proof.** (a) We assume without loss of generality that \( K \) contains the group \( \mu_{N+1} \) of \((N+1)\)'st roots of unity. We start with the exact sequence

\[
1 \longrightarrow \mu_{N+1} \longrightarrow \text{SL}_{N+1}(\bar{K}) \longrightarrow \text{PGL}_{N+1}(\bar{K}) \longrightarrow 1.
\]

We define \( \hat{A}_f \subset \text{SL}_{N+1}(\bar{K}) \) to be the pull-back of \( A_f \), and similarly we let \( \hat{C}_f \subset \text{SL}_{N+1}(\bar{K}) \) be the pull-back of \( C_f \). We note that \( \hat{A}_f \) is an extension of \( A_f \) by \( \mu_{N+1} \), so

\[
\#\hat{A}_f = (N + 1)\#A_f.
\]

For the remainder of the proof we let

\[
m = m(\hat{A}_f) := \text{the exponent of the finite group } \hat{A}_f,
\]

so \( m \) is also bounded in terms of \( N \) and \( \#A_f \). In particular, we may assume that \( \mu_m \subset K \).

\[
3\text{We recall that the Brauer period-index exponent } \beta(K) \text{ is the smallest positive integer such that every } \xi \in \text{Br}(K) \text{ satisfies } \text{Index}(\xi) | Period(\xi)^{\beta(K)}. \text{ In particular, as noted in Remark 8, we have } \beta(K) = 1 \text{ for number fields and their completions, so Theorem 1 as stated in the introduction is a special case of Theorem 13(b).}
\]
Viewing \( \hat{A}_f \) as a subgroup of \( \text{GL}_{N+1}(\bar{K}) \), and using the fact that the exponent of a group divides its order, we apply Brauer’s theorem (Lemma 11) to find a matrix \( A \in \text{GL}_{N+1}(\bar{K}) \) with the property that

\[
A^{-1} \hat{A}_f A \subset \text{SL}_{N+1}(\mathbb{Q}(\mu_m)) \subset \text{SL}_{N+1}(K).
\]

Using the fact that \( \text{Aut}(\hat{A}_f) = A^{-1} \hat{A}_f A \), we see that if we replace \( f \) with \( f^A := A^{-1} \circ f \circ A \), then \( \hat{A}_f \subset \text{PGL}(K) \). So we may assume henceforth that

\[
\hat{A}_f \subset \text{PGL}_{N+1}(K) \quad \text{and} \quad \mu_{N+1} \subset K. \tag{12}
\]

We next apply Proposition 10 to the variety \( V = \mathbb{P}^N \), which we can do since \( \hat{A}_f \) is finite and since (12) tells us in particular that \( \hat{A}_f \) is defined over \( K \). We thus get a 1-cocycle \( \varphi : G_K \rightarrow \hat{A}_f \backslash N_f \) characterized by

\[
f^\sigma = f^{\varphi_\sigma}.
\]

Thus \( \varphi \) defines an element of the cohomology set \( H^1(G_K, \hat{A}_f \backslash N_f) \). It follows from Proposition 9 that we can replace \( K \) with an extension whose degree is bounded by \( C_6(N, \#A_f) \cdot (\#A_f)^{\delta(K)} \) so that \( \varphi \in H^1(G_K, \hat{A}_f \backslash N_f) \) comes from an element of \( H^1(G_K, \mathcal{C}_f) \). In other words, there is a 1-cocycle

\[
\varphi' : G_K \rightarrow \mathcal{C}_f
\]

whose image in \( H^1(G_K, \hat{A}_f \backslash N_f) \) is cohomologous to \( \varphi \). This means that there is an element \( \gamma \in \mathcal{N}_f \) with the property that

\[
\varphi'_\sigma = \hat{A}_f \gamma^{-1} \varphi_\sigma \gamma^\sigma \quad \text{for all} \quad \sigma \in G_K.
\]

(We note that since \( \gamma \in \mathcal{N}_f \), we can multiply both sides by \( \gamma \) to get \( \gamma \varphi'_\sigma \in \hat{A}_f \varphi_\sigma \gamma^\sigma \).)

We replace \( f \) with \( f^\gamma \). This has the effect of replacing \( \hat{A}_f \) by \( \hat{A}_f^\gamma \), but this is just \( \hat{A}_f \), since \( \gamma \in \mathcal{N}_f \). To determine the 1-cocycle associated to \( f^\gamma \), we compute

\[
(f^\gamma)^\sigma = (f^\sigma)^\gamma^\sigma = (f^{\varphi_\sigma})^{\gamma^\sigma} = f^{\varphi_\sigma} \gamma^\sigma = f^{\gamma \varphi'_\sigma}.
\]

Hence the 1-cocycle associated to \( f^\gamma \) is the composition

\[
G_K \xrightarrow{\varphi'} \mathcal{C}_f \rightarrow \hat{A}_f \backslash N_f.
\]

By abuse of notation, we write \( f \) instead of \( f^\gamma \), and we write \( \varphi : G_K \rightarrow \mathcal{C}_f \) for \( \varphi' \), which is a lift of the 1-cocycle for \( f \) to a 1-cocycle taking values in \( \mathcal{C}_f \). It remains to find an appropriate extension of \( K \) over which \( \varphi \) becomes a coboundary.

Our next task is to pin down more precisely the structure of \( \mathcal{C}_f \). We construct a pairing

\[
\langle \cdot, \cdot \rangle : \mathcal{C}_f \times \hat{A}_f \rightarrow \bar{K}^* \tag{13}
\]
as follows. Let $\gamma \in C_f$ and $\alpha \in A_f$. Lift $\gamma$ and $\alpha$ to elements $\hat{\gamma} \in \hat{C}_f$ and $\hat{\alpha} \in \hat{A}_f$. Then the fact that $\alpha \gamma = \gamma \alpha$ in $\text{PGL}_{N+1}(\bar{K})$ implies that
\[
\hat{\alpha} \hat{\gamma} = c(\hat{\alpha}, \hat{\gamma}) \hat{\gamma} \hat{\alpha}
\]
for some scalar $c(\hat{\alpha}, \hat{\gamma}) \in \bar{K}^*$. Choosing different lifts of $\alpha$ and $\gamma$ clearly has no effect on $c(\hat{\alpha}, \hat{\gamma})$, so we define
\[
\langle \alpha, \gamma \rangle := c(\hat{\alpha}, \hat{\gamma}) \text{ using any choice of lifts.}
\]
It is easy to see from the definition that the pairing (13) is a group homomorphism in each coordinate, and that it is $G_K$-equivariant. We define $C^\circ_f$ to be the left-kernel, i.e.,
\[
C^\circ_f := \{ \gamma \in C_f : \langle \gamma, \alpha \rangle = 1 \text{ for all } \alpha \in A_f \},
\]
and we let $\hat{C}^\circ_f$ be the pull-back of $C^\circ_f$ to $\text{SL}_{N+1}(\bar{K})$. By definition we then have
\[
\hat{\gamma} \hat{\alpha} = \hat{\alpha} \hat{\gamma} \text{ for all } \hat{\gamma} \in \hat{C}^\circ_f \text{ and all } \hat{\alpha} \in \hat{A}_f.
\]
The pairing induces a homomorphism from $C_f$ to the dual of $A_f$ with kernel $C^\circ_f$, so we obtain a natural $G_K$-invariant injective homomorphism
\[
C_f/C^\circ_f \longrightarrow A_f^\gamma := \text{Hom}_K(A_f, K^*), \quad \gamma \longrightarrow \langle \cdot, \gamma \rangle.
\] (14)

We recall that we have a cocycle $\varphi : G_K \to C_f$. We consider the exact sequence of groups
\[
1 \longrightarrow C^\circ_f \longrightarrow C_f \longrightarrow C^\circ_f/C_f \longrightarrow 1,
\]
leading to an exact sequence of cohomology sets
\[
H^1(G_K, C^\circ_f) \longrightarrow H^1(G_K, C_f) \longrightarrow H^1(G_K, C^\circ_f/C_f).
\]
From (14) we obtain the bound
\[
\#(C^\circ_f/C_f) \leq \#A_f^\gamma \leq \#A_f,
\]
so applying Lemma 7 we can replace $K$ by a finite extension whose degree is bounded in terms of $\#A_f$ so that the 1-cocycle
\[
G_K \xrightarrow{\varphi} C_f \longrightarrow C^\circ_f/C_f
\]
is trivial, i.e., so that $\varphi_\sigma \in C^\circ_f$ for all $\sigma \in G_K$. This reduces us to the case that $\varphi$ is a 1-cocycle of the form
\[
\varphi : G_K \longrightarrow C^\circ_f.
\] (15)

We next want to use some basic representation theory to describe $C^\circ_f$, but we need to be a bit careful, since the projective linear group $\text{PGL}_{N+1}(\bar{K})$ does not act on $\bar{K}^{N+1}$. So instead we use the lifts $\hat{A}_f$ and $\hat{C}_f$, which live in $\text{SL}_{N+1}(\bar{K})$ and thus do act on $\bar{K}^{N+1}$. We let $W_1, \ldots, W_r$ be the distinct irreducible representations of $\hat{A}_f$ over the
field \( \bar{K} \). Further, since \( \hat{\mathcal{A}} \subset \text{SL}_{N+1}(K) \), and since we have already arranged that \( K \) contains an \( m' \)th root of unity, where \( m \) is the exponent of the group \( \hat{\mathcal{A}}_f \), Brauer’s theorem (Lemma 11) says that we may assume that the \( W_i \) are defined over \( K \). (More precisely, there are \( K \)-vector spaces \( W'_i \) on which \( \hat{\mathcal{A}}_f \) act such that \( W_i \cong W'_i \otimes_K \bar{K} \) as \( \bar{K}[\hat{\mathcal{A}}_f, G_K] \)-bimodules.)

We decompose the representation
\[
\hat{\mathcal{A}}_f \hookrightarrow \text{SL}_{N+1}(K)
\]
into a direct sum of irreducible representations, i.e., we choose a \( \bar{K}[\hat{\mathcal{A}}_f] \)-isomorphism
\[
\psi : \bigoplus_{i=1}^r W_{e_i}^i \rightarrow \bar{K}^{N+1}.
\] (16)

In this isomorphism, we know that the \( W_i \) are defined over \( K \) and that the maps in \( \hat{\mathcal{A}}_f \) are defined over \( K \), so Hilbert’s Theorem 90 says that we can find a \( \psi \) that is defined over \( K \), i.e., so that the map \( \psi \) in (16) is an isomorphism of \( \bar{K}[\hat{\mathcal{A}}_f, G_K] \)-bimodules.\(^4\)

By definition, the group \( \hat{\mathcal{C}}_f \) is the subgroup of \( \text{SL}_{N+1}(\bar{K}) \) that commutes with \( \hat{\mathcal{A}}_f \). It is convenient at this point to extend \( \hat{\mathcal{C}}_f \) to include the center of \( \text{GL}_{N+1}(\bar{K}) \), i.e., to include all diagonal matrices, so we look at \( \bar{K}^*\hat{\mathcal{C}}_f \). This is the commutator subgroup of \( \hat{\mathcal{A}}_f \) in \( \text{GL}_{N+1}(\bar{K}) \), i.e.,
\[
\bar{K}^*\hat{\mathcal{C}}_f = \text{Aut}_{\bar{K}[\hat{\mathcal{A}}_f]}(\bar{K}^{N+1}) \subset \text{GL}_{N+1}(\bar{K}).
\]
Using the \( \bar{K}[\hat{\mathcal{A}}_f] \)-isomorphism (16) yields an isomorphism
\[
\text{Aut}_{\bar{K}[\hat{\mathcal{A}}_f]} \left( \bigoplus_{i=1}^r W_{e_i}^i \right) \cong \bar{K}^*\hat{\mathcal{C}}_f.
\] (17)

Applying a general version of Schur’s lemma \([10, \text{Section XVII.1}]\) to the left-hand side, we find that
\[
\text{Aut}_{\bar{K}[\hat{\mathcal{A}}_f]} \left( \bigoplus_{i=1}^r W_{e_i}^i \right) \cong \prod_{i=1}^r \text{Aut}_{\bar{K}[\hat{\mathcal{A}}_f]}(W_{e_i}^i) \cong \prod_{i=1}^r \text{GL}_{e_i}(\bar{K}).
\] (18)

\(^4\)This is standard, so we just sketch the proof. Schur’s lemma says that it suffices to work with the power \( W^e \) of a single irreducible representation. Let \( \tau_j : W \rightarrow W^e \) be injection on the \( j \)’th factor and \( \pi_k : W^e \rightarrow W \) projection on the \( k \)’th factor. Then for every \( \sigma \in G_K \), the map \( \pi_k \psi^{-1} \psi^\sigma \tau_j \in \text{GL}(W) \) commutes with the action of \( \hat{\mathcal{A}}_f \), hence Schur’s lemma tells us that it is scalar multiplication, say by \( \lambda_{jk}(\sigma) \). Then \( \sigma \mapsto (\lambda_{jk}(\sigma))_{j,k} \) is a \( G_K \)-to-\( \text{GL}_e(\bar{K}) \) 1-cocycle, hence by Hilbert’s Theorem 90 it is the coboundary of some \( M \in \text{GL}_e(\bar{K}) \). Using \( M \) to define a map \( M : W^e \rightarrow W^e \) in the obvious way, we find that \( \psi \circ M \) is defined over \( K \).
Alternatively, using the classical version of Schur’s lemma \[(17)\] Section 2.2, the first isomorphism in \[(18)\] is a consequence of the fact that for distinct $i$ and $j$, the only $\hat{A}_f$-equivariant map from $W_i$ to $W_j$ is the 0 map, and the second isomorphism follows from the fact that for a given $i$, the only $\hat{A}_f$-equivariant maps from $W_i$ to $W_i$ are scalar multiplications. Combining \[(17)\] and \[(18)\], we have identifications

\[
\tilde{K}^{N+1} \overset{\sim}{\rightarrow} \bigoplus_{i=1}^r W_i^{e_i} \quad \text{and} \quad \tilde{K}^* \hat{C}_f^\circ \overset{\sim}{\rightarrow} \prod_{i=1}^r \text{GL}_{e_i}(\tilde{K}). \quad (19)
\]

We recall that we have a cocycle $\varphi : G_K \rightarrow \hat{C}_f^\circ$.

Using the identifications \[(19)\] and the fact that the group $\tilde{K}^* \hat{C}_f^\circ$ is the $\text{GL}_{N+1}(\tilde{K}) \rightarrow \text{PGL}_{N+1}(\tilde{K})$ pull-back of $\hat{C}_f^\circ$, we find that our cocycle has the form $\varphi : G_K \rightarrow \tilde{K}^* \left/ \prod_{i=1}^r \text{GL}_{e_i}(\tilde{K}) \right.$.

We next consider the exact sequence

\[
1 \rightarrow \tilde{K}^* \left/ \prod_{i=1}^r \text{GL}_{e_i}(\tilde{K}) \right. \rightarrow \tilde{K}^* \rightarrow \tilde{K} \rightarrow 1.
\]

We observe that the quotient group on the left is isomorphic to an $(r - 1)$-fold product of copies of $\tilde{K}^*$, and that Hilbert’s theorem 90 tells us that $H^1(G_K, (\tilde{K}^*)^{r-1}) = 0$. Hence taking Galois cohomology yields an injection of pointed sets,

\[
H^1(G_K, \tilde{K}^* \left/ \prod_{i=1}^r \text{GL}_{e_i}(\tilde{K}) \right. ) \hookrightarrow \prod_{i=1}^r H^1(G_K, \text{PGL}_{e_i}(\tilde{K})).
\]

Each of the pointed cohomology sets in the right-hand product admits an injection into a Brauer group,

\[
H^1(G_K, \text{PGL}_{e_i}(\tilde{K})) \hookrightarrow \text{Br}(K)[e_i],
\]

so we obtain an injection

\[
H^1(G_K, \tilde{K}^* \left/ \prod_{i=1}^r \text{GL}_{e_i}(\tilde{K}) \right. ) \hookrightarrow \prod_{i=1}^r \text{Br}(K)[e_i].
\]

We write the image of our 1-cocycle $\varphi$ in the product of Brauer groups as

\[
\varphi \mapsto (\varphi_1, \ldots, \varphi_r) \in \prod_{i=1}^r \text{Br}(K)[e_i].
\]
Let \( e'_i \) be the period of \( \varphi_i \), where \( e'_i \) divides \( e_i \). By definition of the Brauer period-index exponent, for each \( i \) we can find an extension of \( K \) of degree at most \( (e'_i)^{\beta(K)} \) that trivializes \( \varphi_i \), and hence we can find an extension of \( K \) of degree at most \( (e'_1 \cdots e'_r)^{\beta(K)} \) so that the image of \( \varphi \) is trivial in \( \prod \Br(K)[e_i] \). We can estimate this degree using the fact that

\[
N + 1 = \dim \left( \bigoplus_{i=1}^{r} W_i^{e_i} \right) = \sum_{i=1}^{r} e_i \dim(W_i) \geq \sum_{i=1}^{r} e_i,
\]

so the arithmetic-geometric inequality and elementary calculus yield

\[
e'_1 \cdots e'_r \leq e_1 \cdots e_r \leq \left( \frac{1}{r} \sum_{i=1}^{r} e_i \right)^r \leq \left( \frac{N + 1}{r} \right)^r \leq e^{(N+1)/e}.
\]

(The \( e \) in the right-hand expression is the usual \( 2.71828 \ldots \)) This completes the proof of Theorem 13(a).

(b) This follows directly from (a) and Levy’s theorem (Lemma 12) which says that \( \mathcal{A}_f \) is a finite group whose order is bounded by a function of \( N \) and \( d \).

(c) The assumption that \( \text{Aut}(f) = 1 \) means that we have a cocycle \( \varphi : G_K \to \text{PGL}_{N+1}(\bar{K}) \) determined by \( f^\sigma = f^{\varphi(x)} \) whose triviality in \( H^1(G_K, \text{PGL}_{N+1}) \) is equivalent to \( K \) being a FOD for \( f \). The connecting homomorphism \( \delta : H^1(G_K, \text{PGL}_{N+1}) \to \Br(K)[N+1] \) sends \( \varphi \) to an element \( \delta(\varphi) \) of period dividing \( N + 1 \). The definition of \( \beta \) says that the index of \( \delta(\varphi) \) divides \( (N + 1)^{\beta(K)} \), and the definition of index says that there is an extension \( L/K \) of degree dividing \( (N + 1)^{\beta(K)} \) such that \( \text{Res}_{L/K} \delta(\varphi) = 1 \) in \( \Br(L) \). It follows that \( \text{Res}_{L/K}(\varphi) = 1 \) in \( H^1(G_L, \text{PGL}_{N+1}) \), and hence that \( L \) is a FOD for \( f \). This proves half of (c).

For the other half, we use the theory of Severi–Brauer varieties, i.e., varieties \( X \) that are defined over \( K \) and admit a \( \bar{K} \)-isomorphism to \( \mathbb{P}^N \). We refer the reader to [9] or [12, X §6] for the basic facts that we use. The cocycle \( \varphi : G_K \to \text{PGL}_{N+1}(\bar{K}) \) is associated to a Severi–Brauer variety \( X_\varphi \). We claim that there is a field \( L/K \) satisfying

\[
X_\varphi(L) \neq \emptyset \quad \text{and} \quad [L : K] \leq (N + 1)^N.
\]

From this it will follow that \( X_\varphi \times_K L \) is a trivial Severi-Brauer variety [12, X §6], i.e., \( X_\varphi \) is \( L \)-isomorphic to \( \mathbb{P}^N \), and hence that the cocycle \( \varphi \) trivializes over \( L \). To prove our claim, we note that since \( X_\varphi \) is defined over \( K \) and is \( \bar{K} \)-isomorphic to \( \mathbb{P}^N \), the anti-canonical bundle \( \mathcal{K}_{X_\varphi}^{-1} \) on \( X_\varphi \) is defined over \( \bar{K} \) and is very ample. The associated linear system has dimension equal to \( \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(N + 1)) = \binom{2N+1}{N}, \)
so we obtain an embedding

\[ \iota : X_\varphi \longrightarrow |K^{-1}| \cong \mathbb{P}_K^{(2N+1) - 1} \tag{20} \]

that is defined over \( K \). The degree of the embedding (20), i.e., the number of geometric points in the intersection of \( \iota(X_\varphi) \) with a generic linear subspace of complementary dimension, is \( (N+1)^N \); cf. [7, Exercise I.7.1(a)]. Intersecting \( \iota(X_\varphi) \) with a linear subspace defined over \( K \) gives points on \( \iota(X_\varphi) \) defined over a field of degree \( L \) with \( [L : K] \leq (N+1)^N \). □

7. An Alternative Approach using Quotient Varieties

The material in this section may be useful in an alternative approach to FOD/FOM problems for endomorphisms \( f : V \rightarrow V \) in which one tries to rigidify the map \( f \) by specifying the position of marked points, e.g., (pre)periodic points. One way to do this is to look at the map that \( f \) induces on the quotient variety \( V/\mathcal{A}_f \), and twist \( V/\mathcal{A}_f \) to obtain a map defined over the FOM of \( f \), as in the following result.

**Proposition 14.** We continue with the notation from Section 4 and the assumptions in Proposition 10.

(a) The quotient variety

\[ \bar{V}_f := V/\mathcal{A}_f \]

is defined over \( K \), and \( f \) descends to give a \( \bar{K} \)-morphism\(^5\)

\[ \bar{f} : \bar{V}_f \rightarrow \bar{V}_f. \]

(b) Composing the 1-cocycle \( \varphi \) with the map \( \mathcal{A}_f \setminus \mathcal{N}_f \rightarrow \text{Aut}(\bar{V}_f) \) gives a 1-cocycle

\[ \hat{\varphi} : G_K \xrightarrow{\varphi} \mathcal{A}_f \setminus \mathcal{N}_f \hookrightarrow \text{Aut}(\bar{V}_f). \]

Let \( \bar{V}_f^\varphi \) be the \( \bar{K}/K \)-twist of \( \bar{V}_f \) determined by \( \hat{\varphi} \), and let \( F \) be a \( \bar{K} \)-isomorphism

\[ F : \bar{V}_f^\varphi \xrightarrow{\sim/\bar{K}} \bar{V}_f \text{ satisfying } \hat{\varphi}_\sigma \circ F^\sigma = F. \]

Then the map

\[ \bar{f}^F : \bar{V}_f^\varphi \rightarrow \bar{V}_f^\varphi \]

is defined over \( K \), where as usual \( \bar{f}^F \) is our notation for \( F^{-1} \circ \bar{f} \circ F \).

\(^5\)To be notationally consistant, we should use the horrible notation \( \bar{f}_f \) for this map, but instead will simply use \( \bar{f} \).
(c) Let $P \in V(K)$ be a point such that $F^{-1}(P) \in V^f(K)$. Then for all $\sigma \in G_K$ we have
\[ \phi_\sigma^{-1}(P) = A_f P^\sigma, \]
where this notation indicates that since $\phi_\sigma \in A_f \backslash N_f$, the function $\phi_\sigma^{-1}$ sends a point in $V(K)$ to the $A_f$-orbit of a point.

**Proof.** (a) We are given (10) that $A_f$ is finite, and in the category of algebraic varieties, quotients by finite groups of automorphisms always exist. Then the assumption (11) that $A_f$ is defined over $K$ implies that the quotient variety is defined over $K$.

(b) We compute
\[ (\hat{\phi} F)^\sigma = (F^\sigma)^{-1} \hat{\phi} F^{\sigma} = (\hat{\phi}_\sigma^{-1} F)^{-1}(\hat{\phi}_\sigma^{-1} \hat{\phi}) (\hat{\phi}_\sigma^{-1} F) = \hat{f} F. \]
Hence $\hat{f} F$ is defined over $K$.

(c) We compute
\[ F^{-1}(\bar{P}) = F^{-1}(P)^\sigma \quad \text{since } F^{-1}(P) \text{ is defined over } K, \]
\[ = (F^{-1})^\sigma(\bar{P}^\sigma) \]
\[ = F^{-1} \circ \hat{\phi}_\sigma(\bar{P}^\sigma) \quad \text{since } \hat{\phi}_\sigma \circ F^\sigma = F. \]
Applying $\hat{\phi}_\sigma^{-1} \circ F$ to both sides, we find that
\[ \hat{\phi}_\sigma^{-1}(\bar{P}) = \bar{P}^\sigma. \]
Lifting this to $V$, it says precisely that $\phi_\sigma^{-1}(P)$ is the $A_f$-orbit of $P^\sigma$. [Closed]

The next result says that we can find large numbers of periodic points that avoid any specified proper closed subvariety, where in general for a morphism $f : V \to V$, we use the standard notation,
\[ \Per_n(f) := \{ P \in V(K) : f^n(P) = P \}. \]

**Proposition 15.** Let $d \geq 2$, and let $Z \subseteq \mathbb{P}^N$ be a proper closed subvariety of $\mathbb{P}^N$. Then for every $r \geq 1$ there exists an $n = n(N, d, r, Z)$ such that
\[ \#(\Per_n(f) \setminus Z) \geq r \quad \text{for all } f \in \End_d^N. \]

**Proof.** We set the notation
\[ \Per^*_n(t)(f) := \Per_n(f) \setminus \bigcup_{i=1}^t \Per_i(f), \]
i.e., $\Per^*_n(t)(f)$ is the set of periodic points of $f$ whose exact period divides $n$ and is at least equal to $t + 1$.  

For $n \geq 1$ and $t \geq 1$, define a (possibly reducible) subvariety $Y_{n,t} \subseteq \text{End}_d^N$ by the condition

$$Y_{n,t} := \{ f \in \text{End}_d^N : \text{Per}_{n,t}^*(f) \subseteq Z \}.$$

We note that $Y_{n,t}$ is a subvariety, since the map $f \to \text{Per}_n(f)$ is a morphism from $\text{End}_d^N$ to an appropriate Chow variety, and the condition that $\text{Per}_{n,t}^*(f) \subseteq Z$ leads, via elimination theory, to an algebraic condition on the coefficients of $f$. We let

$$X_{n,t} := \bigcap_{k=1}^n Y_{k,t}.$$

Equivalently, the set $X_{n,t}$ is characterized by

$$X_{n,t} = \left\{ f \in \text{End}_d^N : \text{every periodic point of } f \text{ of exact period between } t+1 \text{ and } n \text{ lies on the subvariety } Z \right\}.$$

We observe that

$$X_{1,t} \supseteq X_{2,t} \supseteq X_{3,t} \supseteq \cdots.$$

A decreasing sequence of varieties must stabilize, and hence there is an $m = m(N, d, t, Z)$ having the property that

$$X_{m+i,t} = X_{m,t} \text{ for all } i \geq 0.$$

We claim that $X_{m,t} = \emptyset$. Suppose not. Then we can find a map

$$f \in X_{m,t} = \bigcap_{k=1}^{\infty} X_{k,t}.$$

It would follow that all but finitely many periodic points of $f$ lie on $Z$, i.e., every $f$-periodic point of period strictly larger than $t$ would lie on $Z$. However, by assumption, $Z$ is a proper closed subvariety of $\mathbb{P}^N$, so this contradicts a theorem of Fakhruddin [5, Corollary 5.3] stating that the periodic points of $f$ are Zariski dense in $\mathbb{P}^N$.

We now know that for every $t \geq 1$ there is an $m = m(N, d, t, Z)$ so that $X_{m,t} = \emptyset$. Hence every $f \in \text{End}_d^N$ has a periodic point $P_f$ whose exact period satisfies

$$t < \text{Period}(P_f) \leq m(t),$$

where to ease notation, we write $m(t)$ for $m(N, d, t, Z)$, since $N$, $d$, and $Z$ are fixed.

We apply this last statement recursively. Thus we start with $t = 1$, so for every $f$ can find a point $P_{f,1} \notin Z$ whose exact period is less than $m(1)$. We then apply the statement with $t = m(1)$, which gives us a point $P_{f,2} \notin Z$ satisfying

$$m(1) < \text{Period}(P_{f,2}) \leq m^{o2}(1),$$
where as usual, \( m^2(1) \) means \( m(m(1)) \). We observe that \( P_{f,2} \neq P_{f,1} \), since \( \text{Period}(P_{f,1}) \leq m(1) \) and \( \text{Period}(P_{f,2}) > m(1) \). Repeating the process with \( t = m^2(1) \) yields a third periodic point \( P_{f,3} \notin \mathbb{Z} \) with exact period between \( m^2(1) + 1 \) and \( m^3(1) \), hence distinct from \( P_{f,1} \) and \( P_{f,2} \). Proceeding in this fashion, we see that for every \( f \in \text{End}_{d}^{N} \) we can find distinct periodic points \( P_{f,1}, \ldots, P_{f,r} \) for \( f \) that do not lie on \( \mathbb{Z} \) and with periods at most \( m^{or}(1) \). We observe that \( m^{or}(1) \) depends only on \( N, d, r \) and \( \mathbb{Z} \). Hence taking

\[
n := \text{LCM}(1, 2, \ldots, m^{or}(1))
\]

completes the proof of Proposition 15. \( \square \)

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E-mail address: jdoyle@latech.edu

MATHEMATICS & STATISTICS DEPARTMENT, LOUISIANA TECH UNIVERSITY, RUSTON, LA 71272 USA

E-mail address: jhs@math.brown.edu

MATHEMATICS DEPARTMENT, BOX 1917 BROWN UNIVERSITY, PROVIDENCE, RI 02912 USA. ORCID: https://orcid.org/0000-0003-3887-3248