FUNCTIONAL CALCULUS OF OPERATORS WITH GENERALISED GAUSSIAN BOUNDS ON NON-DOUBLING MANIFOLD WITH ENDS

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Abstract. Let $\Delta$ be the Laplace–Beltrami operator associated to a non-doubling manifold with two ends $\mathbb{R}^m \sharp \mathbb{R}^n$ with $m > n \geq 3$. We say that a non-negative self-adjoint operator $L$ on $L^2(\mathbb{R}^m \sharp \mathbb{R}^n)$ has a generalised Gaussian bounds if the semigroup $e^{-tL}$ has a similar upper bound as $e^{-t\Delta}$. This class of operators includes the Schrödinger operator $L = \Delta + V$ where $V$ is an arbitrary non-negative potential. We then obtain upper bounds of the Poisson semigroup kernel of $L$ and its time derivatives and use them to show the weak type $(1, 1)$ estimate for the holomorphic functional calculus $m(\sqrt{L})$ where $m$ is a function of Laplace transform type. Our result covers the purely imaginary powers $L^{is}, s \in \mathbb{R}$, as a special case and serves as a model case for weak type $(1, 1)$ estimates of singular integrals with non-smooth kernels on non-doubling spaces.

1. Introduction

In the last fifty years, the theory of Calderón-Zygmund singular integrals has been a central part and a success story of modern harmonic analysis. This theory has had extensive influence on other fields of mathematics such as complex analysis and partial differential equations. A main part of this theory aims to study boundedness of certain singular integrals on Lebesgue and other function spaces. Here, we will focus on the operators which are known to be bounded on some $L^p$ space and study whether those operators are bounded on other $L^p$ spaces with $p \neq p_0$.

Assume that $T$ is a bounded operator on a space $L^2(X)$ (for convenience we take $p_0 = 2$) where $X$ is equipped with a distance $d$ and a measure $\mu$. Also assume that $T$ has an associated kernel $k(x, y)$ in the sense that

$$Tf(x) = \int_X k(x, y)f(y)d\mu(y) \quad (1.1)$$

for any continuous function $f$ with compact support and for $x$ not in the support of $f$.

We note that the operator $T$ is not defined by the associated kernel $k(x, y)$ since different operators could have the same associated kernels. For example, two operators $T_1$ and $T_2$ would have the same associated kernels when $T_1f(x) - T_2f(x) = m(x)f(x)$ for some bounded function $m$.

The theory of Calderón-Zygmund singular integrals established sufficient conditions on the space $X$ and the associated kernel $k(x, y)$ for such an operator $T$ to be bounded on $L^p(X)$ for $p \neq 2$. There are 2 key conditions:

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Doubling condition: a measure $\mu$ on the metric (or quasi-metric) space $X$ is said to be doubling if there exists some positive constant $C$ such that
\begin{equation}
0 < \mu(B(x,2r)) \leq C\mu(B(x,r)) < +\infty
\end{equation}
for all $x \in X$ and $r > 0$, where $B(x,r)$ denotes the ball centered at $x$ and with radius $r > 0$.

Hörmander condition: the associated kernel $k(x,y)$ is said to satisfy the (almost $L^1$) Hörmander condition if there exist positive constants $c$ and $C$ such that
\begin{equation}
\int_{d(x,y_1) \geq cd(y_1,y_2)} |k(x,y_1) - k(x,y_2)|d\mu(x) \leq C
\end{equation}
uniformly of $y_1, y_2$.

Under the doubling condition (1.2) and the Hörmander condition (1.3), it is known that $T$ is of weak type $(1,1)$. By Marcinkiewicz interpolation, $T$ is bounded on $L^p(X)$ for $1 < p \leq 2$. If the Hörmander condition (1.3) is satisfied with $x$ and $y$ swapped, then $T$ is bounded on $L^p(X)$ for $2 \leq p < \infty$.

While the theory of Calderón-Zygmund singular integrals has been a great success, there are still many important singular integral operators which do not belong to this class. Within the last twenty years, there were two main directions of development which study operators beyond this Calderón-Zygmund class.

1. Singular integrals on non-homogeneous spaces: substantial progress has been made by F. Nazarov, S. Treil, A. Volberg, X. Tolsa, T. Hytönen and others in showing that many features of the classical Calderón-Zygmund theory still hold without assuming the doubling property. More specifically, the doubling condition on $\mathbb{R}^d$ can be replaced by the polynomial growth condition: for some fixed positive constants $C$ and $n \in (0,d]$, one has
\begin{equation}
\mu(B(x,r)) \leq Cr^n
\end{equation}
for all $x \in \mathbb{R}^d, r > 0$.

If the measure $\mu$ satisfies the condition (1.4), then the space $(\mathbb{R}^d,\mu)$ is called a non-homogeneous space. Calderón-Zygmund operator theory has been developed on such non-homogeneous spaces; see for example [20, 21, 22, 27]. For the BMO and $H^1$ function space, the Littlewood–Paley theory, and weighted norm inequalities on such non-homogeneous spaces, see [2, 23, 26, 28]; for Morrey spaces, Besov spaces and Triebel-Lizorkin spaces in this setting, see [8, 14, 24]. See also [2, 16, 17, 18] for recent work in this direction which studies a more general setting for non-homogeneous analysis on metric spaces $(X,d,\mu)$, where $(X,d)$ is said to be geometrically doubling.

However, to obtain boundedness of singular integrals in this setting, one needs certain strong regularity on the associated kernels in terms of the upper doubling measure, i.e., $r^n$ as in (1.4) rather than $\mu(B(x,r))$. For example Hölder continuity on the space variables of the kernels is needed for weak type $(1,1)$ estimate.

2. Singular integrals with non-smooth kernels: A lot of work has been carried out to study singular integrals whose associated kernels are not smooth enough to satisfy the Hörmander condition. Substantial progress has been made by X. Duong, A. McIntosh, S. Hofmann, L. Yan, J. Martell, P. Auscher, T. Coulhon and others. The Hörmander condition was replaced by a weaker one to obtain the weak type $(1,1)$ estimates. The study in this direction also gives rise to the study of function spaces associated operators, notably the theory of Hardy...
and BMO spaces associated to operators which contain the classical Hardy and BMO spaces as special cases. See for example [1, 3, 9, 11, 12, 15].

The achievements in this direction are mostly obtained for operators acting on doubling spaces. In some specific cases, boundedness of singular integrals can be obtained for operators acting on non-doubling domains of doubling spaces. See [9].

A natural question arises: How about singular integrals with non-smooth kernels on non-doubling spaces? It is obvious that this is a very difficult question when both the key conditions of Calderón-Zygmund theory are missing but it is certainly an interesting and important topic in harmonic analysis. This paper is an effort in this direction in which we study certain singular integrals with non-smooth kernels acting on non-doubling spaces. Our model here is the holomorphic functional calculus of Laplace transform type for the Schrödinger operator which is the sum of a non-negative potential and the Laplace-Beltrami operator on a non-doubling manifold with two ends.

Let us recall manifolds with two ends as in [13]. For $m > n \geq 3$, consider $\mathbb{R}^m$ and $\mathbb{R}^n := \mathbb{R}^n \times S^{m-n}$ where $S^{m-n}$ is the unit sphere in $\mathbb{R}^{m-n}$. Then one takes out the unit balls at the origin of $\mathbb{R}^m$ and $\mathbb{R}^n \times S^{m-n}$ and glue the two ends smoothly by a cylinder $K$ of length 1. This creates the non-doubling manifold with two ends $\mathbb{R}^m \sharp \mathbb{R}^n$. In [13], Grigor’yan and L. Saloff-Coste studied the heat kernel estimates for the Laplace-Beltrami operator $\Delta$ on $\mathbb{R}^m \sharp \mathbb{R}^n$ and obtained by probabilistic methods upper and lower bounds for the heat kernels $p_t(x, y)$. However, no further information on $p_t(x, y)$ are known, for example we do not know if some (good) pointwise estimates on the time derivatives and space derivatives of $p_t(x, y)$ exist. Indeed, the standard method of extending the Gaussian upper bound on the heat kernel with $t > 0$ to complex $z = t + is$ in the case of doubling space like $\mathbb{R}^n$ or spaces of homogeneous type is not applicable to the upper bound of the heat kernel of the Laplace-Beltrami operator on the non-doubling $\mathbb{R}^m \sharp \mathbb{R}^n$.

We view the upper bound of the heat kernel of the Laplace-Beltrami operator on $\mathbb{R}^m \sharp \mathbb{R}^n$ as the standard upper bound in this setting which plays the role of the Gaussian bound on doubling spaces. We introduce the concept of generalised Gaussian bound as follows.

**Definition:** Let $L$ be a non-negative self-adjoint operator on $L^2(\mathbb{R}^m \sharp \mathbb{R}^n)$. We say that $L$ has a generalised Gaussian upper bound if the kernel of $e^{-tL}$ satisfies similar upper bound to the kernel of $e^{-t\Delta}$ as in Theorem A.

**Remark 1.1.** (a) In our definition, the generalised Gaussian upper bound is of similar form to the upper bound of the kernel of $e^{-t\Delta}$, possibly with different values of the constants $C$ and $c_0$ as those in Theorem A.

(b) The operators with generalised Gaussian upper bound includes the Schrödinger operator $L = \Delta + V$ where $V$ is a non-negative potential. Indeed the semigroup $e^{-tL} = e^{-t(\Delta+V)}$ is dominated by the semigroup $e^{-t\Delta}$, hence the upper bound on $p_t(x, y)$ still holds for the kernel $h_t(x, y)$ of $e^{-t(\Delta+V)}$. However, the lower bound on the heat kernel is no longer valid.

(c) There is no assumption on the smoothness of the heat kernel in the definition of generalised Gaussian upper bound. In the specific case of the Schrödinger operator, due to the effect of the non-negative potential $V$, it is possible that the kernel $h_t(x, y)$ of $e^{-t(\Delta+V)}$ is discontinuous hence regularity estimate such as Hölder continuity are false for $h_t(x, y)$ in general.
We note that in [10], the authors obtained the weak type \( (1, 1) \) estimates for the maximal operator \( T(f) = \sup_{t>0} |e^{-t\Delta}f| \) by using the upper bounds of the heat kernels. The proof was a direct consequence of the sharp upper bounds on heat kernels in [13]. In [4], the authors obtained some estimates which showed that spectral multipliers for a function \( m \) with compact support are bounded on \( L^p \) spaces on a space \( X \) which includes the case of non-doubling manifolds with ends. While the result in [4] is applicable to large class of underlying spaces \( X \), the condition that the function \( m \) having compact support is quite restrictive. Indeed, the model case of the function \( m(z) = z^s \), \( s \) real, which gives rise to the purely imaginary power \( \Delta^s \) is not covered by the result of [4]. This indicates that the method in [4] is not strong enough to deal with singular integrals.

The following theorem is our main result.

**Theorem 1.2.** Let \( L \) be an operator with generalised Gaussian upper bound. Let \( m(\sqrt{L}) \) be the holomorphic functional calculus of Laplace transform type of \( \sqrt{L} \) defined by

\[
m(\sqrt{L})f = \int_0^\infty \sqrt{L} \exp(-t\sqrt{L})f m(t) dt
\]

in which \( m(t) \) is a bounded function on \([0, \infty)\), i.e. \( |m(t)| \leq C_0 \), where \( C_0 \) is a constant. Then \( m(\sqrt{L}) \) is of weak type \((1,1)\). Hence by interpolation and duality, the operator \( m(\sqrt{L}) \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

**Remark 1.3.**

(a) In Theorem 1.2 we prove the weak type \((1,1)\) estimate for \( m(\sqrt{L}) \) for a function \( m \) of Laplace transform type. While the \( L^p \) boundedness of \( m(\sqrt{\Delta}) \) for \( 1 < p < \infty \) can be obtained by the Littlewood–Paley theory [25] or transference method [6], the end-point weak \((1,1)\) estimate of \( m(L) \) is new even for the case when \( L = \Delta \). Our main result includes the operators \( L^{is} \), \( s \) real, as a special case and it is a good example for singular integrals whose kernels do not satisfy the Hörmander condition (1.3) and acting on non-doubling spaces.

(b) By using the same approach and similar techniques in the proof of our main result, Theorem 1.2, we can also obtain the weak type \((1,1)\) estimate for the Littlewood–Paley square function defined via the Poisson semigroup generated by \( L \) as follows:

\[
g(f)(x) = \left( \int_0^\infty |(t\sqrt{L})^\kappa e^{-t\sqrt{L}}(f)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \quad \kappa \in \mathbb{N}, \kappa \geq 1.
\]

In addition to standard techniques of harmonic analysis of real variables, there are two key elements in our method of proofs in this paper.

(i) Since the pointwise estimates on space and time derivatives of the semigroup \( e^{-tL} \) are not known, we overcome this problem by using the subordination formula to obtain the upper bound on the time derivatives of the kernel of Poisson semigroup \( e^{-t\sqrt{L}} \) via the known upper bound of the kernel of the heat semigroup \( e^{-tL} \). Then we approach the holomorphic functional calculus \( m(\sqrt{L}) \) through the Poisson semigroup \( e^{-t\sqrt{L}} \).

(ii) The standard Calderón–Zygmund decomposition on non-homogeneous spaces (such as [21, 27]) or the Calderón–Zygmund decomposition associated to an operator on non-doubling domains of space of homogeneous type (such as [9]) do not work for our singular integral \( m(\sqrt{L}) \) in this setting. To overcome this problem, we carry out a number of subtle decomposition and meticulous estimates that match the Poisson kernel upper bounds with the blowing up of volumes of balls due to the non-doubling volume.
The method in this paper relies only on good upper bounds on the Poisson semigroup kernel and its time derivatives which can be derived from the heat semigroup kernels. It does not require, for example the contraction property of the semigroup on $L^p$ spaces, hence can be applied to other differential operators. We believe that our method can be developed further to study boundedness of singular integrals with non-smooth kernels acting on non-doubling spaces in other settings.

2. Poisson semigroup and its time-derivatives

Let $\Delta$ be the Laplace Beltrami operator acting on the manifold $\mathbb{R}^m \sharp \mathbb{R}^n$ and $\exp(-t\Delta)$ the heat propagator corresponding to $\Delta$. Here and throughout the whole paper, we use $\mathbb{R}^m \setminus K$ to denote the large end of the manifold $\mathbb{R}^m \sharp \mathbb{R}^n$, $\mathbb{R}^n \setminus K$ to denote the small end, and $K$ to denote the centre part of the manifold.

**Theorem A ([13]).** The heat kernel $h_t(x, y)$ satisfies the following estimates:

1. For $t \leq 1$ and all $x, y \in \mathbb{R}^m \sharp \mathbb{R}^n$,
   $$h_t(x, y) \approx \frac{C}{V(x, \sqrt{t})} \exp \left( -c_0 \frac{d(x, y)^2}{t} \right);$$

2. For $t > 1$ and all $x, y \in K$,
   $$h_t(x, y) \approx \frac{C}{t^{n/2}} \exp \left( -c_0 \frac{d(x, y)^2}{t} \right);$$

3. For $t > 1$ and $x \in \mathbb{R}^m \setminus K$, $y \in K$,
   $$h_t(x, y) \approx C \left( \frac{1}{t^{n/2} |x|^{n-2}} + \frac{1}{t^{m/2}} \right) \exp \left( -c_0 \frac{d(x, y)^2}{t} \right);$$

4. For $t > 1$ and $x \in \mathbb{R}^n \setminus K$, $y \in K$,
   $$h_t(x, y) \approx C \left( \frac{1}{t^{n/2} |x|^{n-2}} + \frac{1}{t^{m/2} |y|^{n-2}} \right) \exp \left( -c_0 \frac{d(x, y)^2}{t} \right);$$

5. For $t > 1$ and $x \in \mathbb{R}^m \setminus K$, $y \in \mathbb{R}^n \setminus K$,
   $$h_t(x, y) \approx C \left( \frac{1}{t^{n/2} |x|^{m-2}} + \frac{1}{t^{m/2} |y|^{n-2}} \right) \exp \left( -c_0 \frac{d(x, y)^2}{t} \right);$$

6. For $t > 1$ and $x, y \in \mathbb{R}^m \setminus K$,
   $$h_t(x, y) \approx \frac{C}{t^{n/2} |x|^{m-2} |y|^{m-2}} \exp \left( -c_0 \frac{|x|^2 + |y|^2}{t} \right) + \frac{C}{t^{m/2}} \exp \left( -c_0 \frac{d(x, y)^2}{t} \right);$$

7. For $t > 1$ and $x, y \in \mathbb{R}^n \setminus K$,
   $$h_t(x, y) \approx \frac{C}{t^{n/2} |x|^{n-2} |y|^{n-2}} \exp \left( -c_0 \frac{|x|^2 + |y|^2}{t} \right) + \frac{C}{t^{m/2}} \exp \left( -c_0 \frac{d(x, y)^2}{t} \right).$$

We now recall the following result.
Theorem B ([10]). Let $T$ be the maximal operator defined by $T(f)(x) := \sup_{t>0} |\exp(-t\Delta) f(x)|$. Then $T$ is weak type $(1, 1)$ and for any function $f \in L^p(\mathbb{R}^m \times \mathbb{R}^n)$, $1 < p \leq \infty$, the following estimates hold

$$\|Tf\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)}.$$

Remark 2.1. Let $L$ be a non-negative self-adjoint operator which has generalised Gaussian bound. Then Theorem B also holds for the maximal operator via the heat semigroup generated by $L$, i.e., $T_L(f)(x) := \sup_{t>0} |\exp(-tL) f(x)|$ is of weak type $(1, 1)$ and bounded on $L^\infty(\mathbb{R}^m \times \mathbb{R}^n)$, and hence it is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $1 < p \leq \infty$.

Next, we study the properties of the Poisson semigroup generated by $L$. Let $k \in \mathbb{N}$, we denote by $P_{t,k}(x,y)$ the kernel of $(t\sqrt{L})^k e^{-t\sqrt{L}}$. For $k = 0$, we write $P_t(x,y)$ instead of $P_{t,0}(x,y)$.

Theorem 2.2. For $k \in \mathbb{N}$, set $k \vee 1 = \max\{k, 1\}$. Then the kernel $P_{t,k}(x,y)$ satisfies the following estimates:

1. For $x, y \in K$,

$$|P_{t,k}(x,y)| \leq \frac{C}{t^m} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1} + \frac{C}{t^n} \left( \frac{t}{t+d(x,y)} \right)^{n+k\vee 1};$$

2. For $x \in \mathbb{R}^m \setminus K$, $y \in K$,

$$|P_{t,k}(x,y)| \leq \frac{C}{t^m} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1} + \frac{C}{t^n} \left( \frac{t}{t+d(x,y)} \right)^{n+k\vee 1};$$

3. For $x \in \mathbb{R}^n \setminus K$, $y \in K$,

$$|P_{t,k}(x,y)| \leq \frac{C}{t^m} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1} + \frac{C}{t^n} \left( \frac{t}{t+d(x,y)} \right)^{n+k\vee 1};$$

4. For $x \in \mathbb{R}^m \setminus K$, $y \in \mathbb{R}^n \setminus K$,

$$|P_{t,k}(x,y)| \leq \frac{C}{t^m} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1} + \frac{C}{t^n} \left( \frac{t}{t+d(x,y)} \right)^{n+k\vee 1} + \frac{C}{t^n|y|^{m-2}} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1};$$

5. For $x, y \in \mathbb{R}^m \setminus K$,

$$|P_{t,k}(x,y)| \leq \frac{C}{t^m} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1} + \frac{C}{t^n} \left( \frac{t}{t+d(x,y)} \right)^{n+k\vee 1};$$

6. For $x, y \in \mathbb{R}^n \setminus K$,

$$|P_{t,k}(x,y)| \leq \frac{C}{t^m} \left( \frac{t}{t+d(x,y)} \right)^{m+k\vee 1} + \frac{C}{t^n} \left( \frac{t}{t+d(x,y)} \right)^{n+k\vee 1}.$$

Proof. By the subordination formula we have

$$e^{-t\sqrt{L}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty te^{-\frac{t^2}{2v}} e^{-vL}dv$$

(2.1)
and hence
\[(t\sqrt{L})^k e^{-t\sqrt{T}} = (-1)^k \frac{t^k}{2\sqrt{\pi}} \int_0^\infty \partial_t^k (te^{-\frac{t^2}{4v}}) e^{-vL} \frac{du}{v^{3/2}}\]
\[= (-1)^k \frac{t^k}{\sqrt{\pi}} \int_0^\infty \partial_t^{k+1} (e^{-\frac{t^2}{4v}}) e^{-vL} \frac{dv}{v}.\]

This yields that
\[P_{t,k}(x,y) = (-1)^k \frac{t^k}{\sqrt{\pi}} \int_0^\infty \partial_t^{k+1} (e^{-\frac{t^2}{4v}}) H_v(x,y) \frac{dv}{v}\]
where \(H_v(x,y)\) is the kernel of \(e^{-vL}\).

Let \(s > 0\) and \(k \in \mathbb{N}\). By Faà di Bruno’s formula, we can write
\[\partial_t^{k+1} e^{-\frac{t^2}{4v}} = \sum \frac{(-1)^{m_1+m_2}}{2 \cdot m_1! m_2!} e^{-t^2/s} \left( \frac{1}{s} \right)^{m_1} \left( \frac{1}{s} \right)^{m_2},\]
where the sum is taken over all pairs \((m_1, m_2)\) of nonnegative integers satisfying \(m_1 + 2m_2 = k + 1\). For such a pair \((m_1, m_2)\), there exists \(C > 0\) so that
\[e^{-\frac{t^2}{4} \left( \frac{1}{u} \right)^{m_1} \left( \frac{1}{u} \right)^{m_2}} = e^{-\frac{t^2}{4s}} \left( \frac{1}{s} \right)^{m_1} \left( \frac{1}{s} \right)^{m_2} \]
\[\leq Ce^{-\frac{t^2}{s} \frac{1}{s}} \frac{1}{s^{(k+1)/2}} \max \left\{ 1, \left( \frac{t}{s} \right)^{k+1} \right\} \].

This implies that
\[|\partial_t^{k+1} e^{-\frac{t^2}{4v}}| \leq Ce^{-\frac{t^2}{s} \frac{1}{s}} \frac{1}{s^{(k+1)/2}}.\]

From (2.2) and (2.4) we deduce that
\[|P_{t,k}(x,y)| \leq \frac{C}{4\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} \left( \frac{t}{\sqrt{v}} \right)^k |H_v(x,y)| \frac{dv}{v}.\]

We now give the estimates for \(P_{t,k}(x,y)\) with \(k \geq 1\) only, since the remaining case \(k = 0\) can be done similarly.

We have
\[|P_{t,k}(x,y)| \leq \frac{C}{4\sqrt{\pi}} \int_0^1 e^{-\frac{t^2}{4s}} \left( \frac{t}{\sqrt{v}} \right)^k |H_v(x,y)| \frac{dv}{v} + \frac{C}{4\sqrt{\pi}} \int_1^\infty e^{-\frac{t^2}{4s}} \left( \frac{t}{\sqrt{v}} \right)^k |H_v(x,y)| \frac{dv}{v}\]
\[=: \mathbb{E}_1(x,y) + \mathbb{E}_2(x,y).\]

Applying the upper bound in Point 1 in Theorem A for \(|H_v(x,y)|\), we have
\[\mathbb{E}_1(x,y) \leq C \int_0^1 e^{-\frac{t^2}{4s}} \left( \frac{t}{\sqrt{v}} \right)^k \frac{1}{v^{m/2}} \exp \left( -\frac{d(x,y)^2}{c^2 v} \right) dv\]
\[\leq C \int_0^1 \left( \frac{t}{\sqrt{v}} \right)^k \frac{1}{v^{m/2}} \exp \left( -\frac{d(x,y)^2}{c^2} + t^2 \right) dv\]
\[\leq C \left( \frac{1}{\sqrt{v}} + \frac{1}{d(x,y)^2 + t^2} \right)^k \left( \frac{t}{\sqrt{v}} \right)^k \frac{1}{v^{m/2}} \exp \left( -\frac{d(x,y)^2}{c^2} + t^2 \right) dv\]
\[\leq \frac{C}{t^m} \left( \frac{t}{t + d(x,y)} \right)^{m+k}.\]
For the term $E_2(x, y)$, we consider the following 6 cases:

Case 1: $x, y \in K$.
Applying the upper bound in Point 2 in Theorem A for $|H_v(x, y)|$, we have

$$|p_v(x, y)| \leq \frac{C}{v^{m/2}} \exp \left(-c_0 \frac{d(x, y)^2}{v}\right).$$

Arguing similarly to the estimate of $E_1(x, y)$ we obtain

$$E_2(x, y) \leq \frac{C}{t^n} \left(\frac{t}{t + d(x, y)}\right)^{n+k}.$$

Case 2: $x \in \mathbb{R}^m \setminus K$, $y \in K$.
Applying the upper bound in Point 3 in Theorem A for $|H_v(x, y)|$, we get that

$$|H_v(x, y)| \leq C \left(\frac{1}{v^{m/2}|x|^{m-2}} + \frac{1}{v^{m/2}}\right) \exp \left(-c_0 \frac{d(x, y)^2}{v}\right).$$

Hence, we get

$$E_2(x, y) \leq C \int_1^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k \left(\frac{1}{v^{m/2}|x|^{m-2}} + \frac{1}{v^{m/2}}\right) \exp \left(-c_0 \frac{d(x, y)^2}{v}\right) dv$$

$$\leq \frac{C}{t^n} \left(\frac{t}{t + d(x, y)}\right)^{n+k} + C \left(\frac{t}{t + d(x, y)}\right)^{m+k}.$$

Case 3: $x \in \mathbb{R}^n \setminus K$, $y \in K$.
Applying the upper bound in Point 4 in Theorem A for $|H_v(x, y)|$, we have

$$E_2(x, y) \leq C \int_1^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k \left(\frac{1}{v^{n/2}|x|^{n-2}} + \frac{1}{v^{n/2}}\right) \exp \left(-c_0 \frac{d(x, y)^2}{v}\right) dv$$

$$\leq \frac{C}{t^n} \left(\frac{t}{t + d(x, y)}\right)^{n+k} \leq \frac{C}{t^n} \left(\frac{t}{t + d(x, y)}\right)^{n+k}.$$

Case 4: $x \in \mathbb{R}^m \setminus K$, $y \in \mathbb{R}^n \setminus K$.
Applying the upper bound in Point 5 in Theorem A for $|H_v(x, y)|$, we get that

$$E_2(x, y) \leq C \int_1^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k \left(\frac{1}{v^{m/2}|x|^{m-2}} + \frac{1}{v^{m/2}|y|^{n-2}}\right) \exp \left(-c_0 \frac{d(x, y)^2}{v}\right) dv$$

$$\leq \frac{C}{t^n|x|^{m-2}} \left(\frac{t}{t + d(x, y)}\right)^{n+k} + C \left(\frac{t}{t + d(x, y)}\right)^{m+k}.$$

Case 5: $x, y \in \mathbb{R}^m \setminus K$.
Applying the upper bound in Point 6 in Theorem A for $|H_v(x, y)|$, we find that

$$E_2(x, y) \leq C \int_1^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k \frac{1}{v^{m/2}|x|^{m-2}|y|^{m-2}} \exp \left(-c_0 \frac{|x|^2}{v}\right) dv$$

$$+ C \int_1^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k \frac{1}{v^{m/2}} \exp \left(-c_0 \frac{d(x, y)^2}{v}\right) dv.$$
defines the complex Poisson semigroup (which is unique by analyticity). Hence is real, the formula (2.6) coincides with the Poisson semigroup (2.1), hence the formula (2.6) complex Poisson semigroup and its time derivatives (\(z \sqrt{L}\))^k \exp(-z \sqrt{L})

Case 6: \(x, y \in \mathbb{R}^n \setminus K\).

Applying the upper bound in Point 7 in Theorem A for \(|H_v(x, y)|\), we obtain that

\[
\mathbb{E}_2(x, y) \leq C \int_1^\infty e^{-\frac{z^2}{4}} \left(\frac{t}{\sqrt{v}}\right)^{\frac{k}{2}} \frac{1}{v^{n/2}|x|^{n-2}|y|^{n-2}} \exp\left(-c_0 \frac{|x|^2 + |y|^2}{v}\right) \frac{dv}{v} + \int_1^\infty e^{-\frac{z^2}{4}} \left(\frac{t}{\sqrt{v}}\right)^{\frac{k}{2}} \frac{1}{v^{n/2}} \exp\left(-c_0 \frac{d(x, y)^2}{v}\right) \frac{dv}{v} \\
\leq \frac{C}{t^n} |x|^{n-2}|y|^{n-2} \left(\frac{t}{t + d(x, y)}\right)^{n+k} + \frac{C}{t^n} \left(\frac{t}{t + d(x, y)}\right)^{n+k}.
\]

\[
\leq \frac{C}{t^n} \left(\frac{t}{t + d(x, y)}\right)^{n+k}.
\]

We observe that the proof of Theorem 2.2 can be extended to obtain the estimates for the complex Poisson semigroup and its time derivatives (\(z \sqrt{L}\))^k \exp(-z \sqrt{L}). Indeed, we have the following result.

**Theorem 2.3.** Fix \(0 < \mu < \frac{\pi}{4}\), let \(S^0_\mu = \{z \in \mathbb{C} : |\arg z| < \mu\}\) and choose \(z \in S^0_\mu\). The complex Poisson semigroup and its time derivatives (\(z \sqrt{L}\))^k \exp(-z \sqrt{L}) exist and satisfy the upper bounds as in Theorem 2.2 with \(t\) to be replaced by \(|z|\).

**Proof.** Fix \(0 < \mu < \frac{\pi}{4}\). For \(z \in S^0_\mu\), define

\[
e^{-z\sqrt{T}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty z e^{-\frac{z^2}{4v^2}} v^{-1/2} e^{-vL} dv.
\]

For \(0 < |\arg z| < \mu < \frac{\pi}{4}\) we have \(\Re z^2 > 0\), hence the integral in (2.6) converges. When \(z\) is real, the formula (2.6) coincides with the Poisson semigroup (2.1), hence the formula (2.6) defines the complex Poisson semigroup (which is unique by analyticity). Hence

\[
(z \sqrt{L})^k e^{-z\sqrt{L}} = (-1)^k \frac{z^k}{2\sqrt{\pi}} \int_0^\infty \partial_z^k (z e^{-\frac{z^2}{4v^2}}) e^{-vL} dv \\
= (-1)^k \frac{z^k}{\sqrt{\pi}} \int_0^\infty \partial_z^{k+1} (e^{-\frac{z^2}{4v^2}}) e^{-vL} dv \\
\]

This yields that

\[
P_{z,k}(x, y) = (-1)^k \frac{z^k}{\sqrt{\pi}} \int_0^\infty \partial_z^{k+1} (e^{-\frac{z^2}{4v^2}}) H_v(x, y) \frac{dv}{\sqrt{v}}
\]

where \(H_v(x, y)\) is the kernel of \(e^{-vL}\). The rest of the proof is similar to Theorem 2.2.

We now obtain a weak type \((1, 1)\) estimate for a maximal operator.

**Proposition 2.4.** Fix \(0 < \mu < \frac{\pi}{4}\). Let \(T_k\) be the operator defined by

\[
T_k(f)(x) := \sup_{z \in S^0_\mu} |(z \sqrt{L})^k \exp(-z \sqrt{L}) f(x)|
\]
for an integer \( k \geq 0 \) and \( f \in L^p(\mathbb{R}_m^\# \mathbb{R}^n) \). Then \( T_k \) is of weak type \((1, 1)\) and bounded on \( L^p(\mathbb{R}_m^\# \mathbb{R}^n) \) for \( 1 < p \leq \infty \).

**Proof.** We point out that with the upper bound of the kernel \( P_{z,k}(x, y) \) of \((z\sqrt{L})^k \exp(-z\sqrt{L})\), the weak type \((1, 1)\) estimate of the maximal operator \( T_k \) follows from the same idea and approach in proof of Theorem B. For more details, we refer to [10]. \( \square \)

The concept of approximation to the identity plays an important role in harmonic analysis. For a family of approximation to the identity \( \phi_t * f \) in a doubling space like \( \mathbb{R}^n \), the upper bound on \( \phi(x) \) can be taken as the Gaussian bound with exponential decay or Poisson bound with polynomial decay. However, in a non-doubling space like a manifold with ends \( \mathbb{R}_m^\# \mathbb{R}^n \), it is not obvious which type of bound is deemed as natural. Here we suggest to use the Poisson kernels in the definition of an approximation to the identity in this setting. We note that in the case of \( \mathbb{R}^n \), the term \( \frac{1}{t^n} \) in the Poisson kernel \( \frac{1}{t^n} \times \frac{c_n}{(1 + |x - y|^2)^{n+\frac{1}{2}}} \) is independent of \( x \) and \( y \), the corresponding term in the case of \( \mathbb{R}_m^\# \mathbb{R}^n \) might depend on \( x \) and \( y \).

**Definition 2.5.** A family of kernels \( \phi_t(x, y), t > 0 \), is said to be a generalised approximation to the identity if it satisfies the same upper bound as \( CP_{t,k}(x, y) \) in Theorem 2.2 for some positive constants \( C, k \) and \( \alpha \).

We note that in the proof of our main result, Theorem 1.2, we use \( e^{-t\sqrt{L}} \) as an approximation to the identity. The following result is similar to the basic result in \( \mathbb{R}^n \) that the operator \( \sup_{t>0} \| \phi_t * f \| \) is bounded on \( L^p \), \( 1 < p < \infty \) for a family of approximation to identity \( \phi_t \).

**Proposition 2.6.** Assume that \( \phi_t(x, y), t > 0 \), is a generalised approximation to the identity on \( \mathbb{R}_m^\# \mathbb{R}^n \). Define the family of operators \( D_t \) by

\[
D_t f = \int_{\mathbb{R}_m^\# \mathbb{R}^n} \phi_t(x, y) f(y) \, d\mu(y)
\]

for \( f \in L^p(\mathbb{R}_m^\# \mathbb{R}^n), 1 < p \leq \infty \). Then the operator \( T(f)(x) := \sup_{t>0} |D_t f(x)| \) is bounded on \( L^p(\mathbb{R}_m^\# \mathbb{R}^n), 1 < p < \infty \) and is of weak type \((1, 1)\).

**Proof.** Proposition 2.6 follows from Proposition 2.4 and domination argument. \( \square \)

**Remark 2.7.** Theorem 2.3, Propositions 2.4 and 2.6 are of independent interest as they are useful for the study of harmonic analysis on the setting of non-doubling manifolds with ends.

### 3. Proof of main result: Theorem 1.2

To begin with, we first recall the standard definition of the maximal function and its properties. For any \( p \in [1, \infty] \) and any function \( f \in L^p \) we set

\[
\mathcal{M} f(x) = \sup \left\{ \frac{1}{|B(y, r)|} \int_{B(y, r)} |f(z)| \, dz : x \in B(y, r) \right\}.
\]

**Theorem C ([10]).** The maximal function operator is of weak type \((1, 1)\) and bounded on all \( L^p \) spaces for \( 1 < p \leq \infty \).
Similarly, we have that the measure in the left-hand side of (3.3) satisfies

\[
|x \in \mathbb{R}^m \setminus \mathbb{R}^n : |m(\sqrt{L})f(x)| > \lambda| \leq C \frac{\|f\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}}{\lambda}.
\]

Then, to prove (3.1), it suffices to verify the following three inequalities:

\[
|x \in \mathbb{R}^m \setminus \mathbb{R}^n : |m(\sqrt{L})f(x)| > \lambda| \leq C \frac{\|f\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}}{\lambda},
\]

(3.2)

\[
|x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f(x)| > \lambda| \leq C \frac{\|f\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}}{\lambda},
\]

(3.3)

\[
|x \in K : |m(\sqrt{L})f(x)| > \lambda| \leq C \frac{\|f\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}}{\lambda}.
\]

(3.4)

We now set

\[
 f_1(x) := f(x)\chi_{\mathbb{R}^m \setminus K}, \quad f_2(x) := f(x)\chi_{\mathbb{R}^n \setminus K}, \quad \text{and} \quad f_3(x) := f(x)\chi_{K}.
\]

Thus, \( f \) can be written as

\[
f = f_1 + f_2 + f_3.
\]

Then, since \( m(\sqrt{L}) \) is a linear operator, the measure in the left-hand side of (3.2) satisfies

\[
|x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f(x)| > \lambda| \leq \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_1(x)| > \frac{\lambda}{3} \right\} \right| + \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_2(x)| > \frac{\lambda}{3} \right\} \right| + \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_3(x)| > \frac{\lambda}{3} \right\} \right| =: I_1 + I_2 + I_3.
\]

Similarly, we have that the measure in the left-hand side of (3.3) satisfies

\[
|x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f(x)| > \lambda| \leq \left| \left\{ x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f_1(x)| > \frac{\lambda}{3} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f_2(x)| > \frac{\lambda}{3} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f_3(x)| > \frac{\lambda}{3} \right\} \right| =: II_1 + II_2 + II_3
\]

and that the measure in the left-hand side of (3.4) satisfies

\[
|x \in K : |m(\sqrt{L})f(x)| > \lambda| \leq \left| \left\{ x \in K : |m(\sqrt{L})f_1(x)| > \frac{\lambda}{3} \right\} \right| + \left| \left\{ x \in K : |m(\sqrt{L})f_2(x)| > \frac{\lambda}{3} \right\} \right| + \left| \left\{ x \in K : |m(\sqrt{L})f_3(x)| > \frac{\lambda}{3} \right\} \right|.
\]
\[ + \left| \left\{ x \in K : |m(\sqrt{L})f_3(x)| > \frac{\lambda}{3} \right\} \right| \]
\[ =: III_1 + III_2 + III_3. \]

For each of the terms above, it then suffices to prove an estimate on it of the form \( C \|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)} \).

3.1. **Estimate of \( I_1 \).** In this case, since \( x \) is in \( \mathbb{R}^m \setminus K \) and the function \( f_1 \) is also supported in \( \mathbb{R}^m \setminus K \), we can restrict to the setting \( \mathbb{R}^m \setminus K \), where the measure now becomes the standard Lebesgue measure on \( \mathbb{R}^m \setminus K \) which is doubling. However, the non-homogeneous property shows up in the kernel estimate in this case. The Poisson kernel here is not bounded by the classical upper bound and the main difficulty comes from the term \( C \|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)} \).

To begin the proof, we now restrict the setting on \( \mathbb{R}^m \setminus K \), and \( f_1 \) is in \( L^1(\mathbb{R}^m \setminus K) \). We now extend \( f_1 \) to the whole of \( \mathbb{R}^m \) by zero extension, i.e., define \( f_1(x) := 0 \) when \( x \in K \).

We now consider the standard Calderón–Zygmund decomposition as follows. Recall that the standard dyadic cubes in \( \mathbb{R}^m \) are of the form \( [2^k a_1, 2^k(a_1 + 1)) \times \cdots \times [2^k a_m, 2^k(a_m + 1)) \), where \( k, a_1, \ldots, a_m \) are integers. Decompose \( \mathbb{R}^m \) into a mesh of equal size disjoint dyadic cubes so that

\[ |Q| \geq \frac{1}{\lambda} \|f_1\|_{L^1(\mathbb{R}^m)} \]

for every cube in the mesh. Subdivide each cube in the mesh into \( 2^m \) congruent cubes by bisecting each of its sides. We now have a new mesh of dyadic cubes. Select a cube in the new mesh if

\[ \frac{1}{|Q|} \int_Q |f_1(x)|dx > \lambda. \]  \hspace{1cm} (3.5)

Let \( S \) be the set of all these selected cubes. Now subdividing each non-selected cube into \( 2^m \) congruent subcubes by bisecting each side as before. Then select one of these new cubes if (3.5) holds. Put all these selected cubes of this generation into the set \( S \). Repeat this procedure indefinitely.

Then we have \( S = \bigcup_j Q_j \), where all these \( Q_j \)'s are disjoint, and we further have

\[ |S| = \sum_j |Q_j| \leq \frac{1}{\lambda} \sum_j \int_{Q_j} |f_1(x)|dx \leq \frac{1}{\lambda} \|f_1\|_{L^1(\mathbb{R}^m)}. \]
We now define
\[ b_j(x) := \left( f_1(x) - \frac{1}{|Q_j|} \int_{Q_j} f_1(y) dy \right) \chi_{Q_j}(x) \]
and
\[ b(x) := \sum_j b_j(x), \quad g(x) := f_1(x) - b(x). \]

For a selected \( Q_j \), there exists a unique non-selected dyadic cube \( Q' \) with twice its side length that contains \( Q_j \). Since \( Q' \) is not selected, we get that
\[ \frac{1}{|Q'|} \int_{Q'} |f_1(y)| dy \leq \lambda, \]
which implies that
\[ \frac{1}{|Q_j|} \int_{Q_j} |f_1(y)| dy \leq \frac{2^m}{|Q'|} \int_{Q'} |f(y)| dy \leq 2^m \lambda. \]

For the good part \( g(x) \), since \( b = 0 \) on \( F := \mathbb{R}^m \setminus S \), we have
\[ g(x) = f_1(x) \text{ on } F, \quad \text{and} \quad g(x) = \frac{1}{|Q_j|} \int_{Q_j} f_1(x) dx \text{ on } Q_j. \]

Then it is easy to verify that
\[ \|g\|_{L^1(\mathbb{R}^m)} \leq \|f\|_{L^1(\mathbb{R}^m)} \quad \text{and} \quad \|g\|_{L^\infty(\mathbb{R}^m)} \leq C\lambda. \]

We now have
\[ I_1 \leq \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})g(x)| > \frac{\lambda}{6} \right\} \right| \]
\[ + \left| \left\{ x \in (\mathbb{R}^m \setminus K) \cup \bigcup_i 8Q_i : |m(\sqrt{L})(\sum_j b_j)(x)| > \frac{\lambda}{6} \right\} \right| \]
\[ + |\cup_i 8Q_i| \]
\[ =: I_{11} + I_{12} + I_{13}. \]

As for \( I_{11} \), by using the \( L^2 \) boundedness of \( m(\sqrt{L}) \), we obtain that
\[ I_{11} = \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})g(x)| > \frac{\lambda}{6} \right\} \right| \]
\[ \leq \frac{C}{\lambda^2} \|g\|_{L^2(\mathbb{R}^m \setminus K)}^2 \]
\[ \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}, \]
where we use the fact that \( |g(x)| \leq C\lambda \).

As for \( I_{13} \), note that we have the doubling condition in this case. So we get that
\[ I_{13} \leq C \sum_i |Q_i| \leq C \frac{\|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}}{\lambda}. \]

As for \( I_{12} \), we now split all the \( Q_i \)'s in the set \( S \) into two groups:
\[ \mathcal{I}_1 := \{ i : \text{none of the corners of } Q_i \text{ is the origin} \}, \]
and
\[ I_2 = \{ i : \text{one of the corners of } Q_i \text{ is the origin} \}. \]

Then we write
\[ m(\sqrt{L})\left( \sum_{i} b_i \right)(x) = \sum_{i \in I_1} m(\sqrt{L})b_i(x) + \sum_{i \in I_2} m(\sqrt{L})b_i(x). \]

For each \( i \in I_1 \), we further decompose
\[ m(\sqrt{L})b_i(x) = m(\sqrt{L})e^{-t_i \sqrt{T} b_i}(x) + m(\sqrt{L}) \left( I - e^{-t_i \sqrt{T}} \right) b_i(x), \]

where \( \{e^{-t \sqrt{T}}\}_{t>0} \) is the Poisson semigroup of \( L \) as studied in Section 2, and for each \( i \), \( t_i \) is the side length of the cube \( Q_i \).

Then we have
\[
I_{12} \leq \left\| \sum_{i \in I_1} m(\sqrt{L}) \left( \sum_{j \in I_2} e^{-t_j \sqrt{T} b_i} \right)(x) > \frac{\lambda}{18} \right\|
+
\left\| \sum_{i \in I_1} m(\sqrt{L}) \left( \sum_{j \in I_2} (I - e^{-t_j \sqrt{T}}) b_i \right)(x) > \frac{\lambda}{18} \right\|
+
\left\| \sum_{i \in I_1} m(\sqrt{L}) (\sum_{j \in I_2} b_i(x)) > \frac{\lambda}{18} \right\|
=: I_{121} + I_{122} + I_{123}.
\]

We now first estimate \( I_{121} \). To see this, we claim that
\[
\left\| \sum_{i \in I_1} e^{-t_i \sqrt{T} b_i} \right\|_{L^2(R^n)} \leq C \lambda^{\frac{1}{2}} \| f_1 \|_{L^1(R^n)},
\]

To see this claim, it suffices to show the following 3 cases:
\[
\left\| \sum_{i \in I_1} e^{-t_i \sqrt{T} b_i} \right\|_{L^2(K)} \leq C \lambda^{\frac{1}{2}} \| f_1 \|_{L^1(K)},
\]
\[
\left\| \sum_{i \in I_1} e^{-t_i \sqrt{T} b_i} \right\|_{L^2(R^n \setminus K)} \leq C \lambda^{\frac{1}{2}} \| f_1 \|_{L^1(R^n \setminus K)},
\]
and
\[
\left\| \sum_{i \in I_1} e^{-t_i \sqrt{T} b_i} \right\|_{L^2(K \setminus K)} \leq C \lambda^{\frac{1}{2}} \| f_1 \|_{L^1(K \setminus K)}.
\]

Hence, combining the estimates of (3.7), (3.8) and (3.9), for \( I_{121} \), we get that
\[
I_{121} \leq \left\| \sum_{i \in I_1} e^{-t \sqrt{T} b_i} \right\|_{L^2(R^n)}^2 \leq C \lambda \| f_1 \|_{L^1(R^n)}.
\]
We first estimate (3.7). We first consider the function \( e^{-t_i \sqrt{T} b_i(x)} \) for \( x \in \mathbb{R}^m \setminus K \). Since
\[
e^{-t_i \sqrt{T} b_i(x)} = \int_{\mathbb{R}^m \setminus K} P_{t_i}(x, y) b_i(y) dy,
\]
applying the upper bound in Point 5 in Theorem 2.2 for \( P_{t_i}(x, y) \) we obtain that
\[
|e^{-t_i \sqrt{T} b_i(x)}| \leq \int_{\mathbb{R}^m \setminus K} |P_{t_i}(x, y)| |b_i(y)| dy
\]
\[
\leq C \int_{\mathbb{R}^m \setminus K} \left( \frac{t_i}{|x|m-2|y|m-2(t_i + |x| + |y|)^n+1} + \frac{t_i}{(t_i + d(x, y))^{n+1}} \right) |b_i(y)| dy
\]
\[
= C \int_{Q_i} \frac{t_i}{|x|m-2|y|m-2(t_i + |x| + |y|)^n+1} |b_i(y)| dy
\]
\[
+ C \int_{\mathbb{R}^m \setminus R^n} \frac{t_i}{(t_i + d(x, y))^{n+1}} |b_i(y)| dy
\]
\[
=: F_{1,i} + F_{2,i}.
\]

We first estimate the term \( F_{2,i} \). By noting that in this case, we have that \( x \in \mathbb{R}^m \setminus K \) and that \( Q_i \subset \mathbb{R}^m \setminus K \), dyadic, with none of the corners of \( Q_i \) is the origin. This implies that
\[
F_{2,i} \leq C \sup_{z \in Q_i} \frac{t_i}{(t_i + d(x, z))^{m+1}} \int_{\mathbb{R}^m \setminus R^n} |b_i(y)| dy
\]
\[
\leq C \inf_{z \in Q_i} \frac{t_i}{(t_i + d(x, z))^{m+1}} \lambda|Q_i|
\]
\[
\leq C \lambda \int_{\mathbb{R}^m \setminus R^n} \frac{t_i}{(t_i + d(x, z))^{m+1}} \chi_{Q_i}(z) dz,
\]
where \( \chi_{Q_i} \) is the characteristic function of \( Q_i \).

For any \( h \in L^2(\mathbb{R}^m \setminus K) \) with \( \|h\|_{L^2(\mathbb{R}^m \setminus K)} = 1 \), we get that
\[
\langle F_{2,i}, h \rangle = C \lambda \int_{\mathbb{R}^m \setminus R^n} \int_{\mathbb{R}^m \setminus K} \frac{t_i}{(t_i + d(x, z))^{m+1}} h(x) dx \chi_{Q_i}(z) dz
\]
\[
\leq C \lambda \langle M(h), \chi_{Q_i} \rangle.
\]
As a consequence we obtain that
\[
\langle \sum_{i \in I_1} F_{2,i}, h \rangle \leq C \lambda \langle M(h), \sum_{i \in I_1} \chi_{Q_i} \rangle,
\]
which yields
\[
\left\| \sum_{i \in I_1} F_{2,i} \right\|_{L^2(\mathbb{R}^m \setminus K)} \leq C \lambda \left\| \sum_{i \in I_1} \chi_{Q_i} \right\|_{L^2(\mathbb{R}^m \setminus K)} \leq C \lambda \left( \sum_{i \in I_1} |Q_i| \right)^{1/2}
\]
\[
\leq C \lambda \|f_1\|_{L^1(\mathbb{R}^m \setminus R^n)}^{1/2}
\]
\[
\leq C \lambda \|f_1\|_{L^1(\mathbb{R}^m \setminus R^n)}^{1/2}.
\]
To handle $F_{1,i}$, we note that the distance of $Q_i$ to the center $K$ is comparable to the side length of $Q_i$ for $i \in I_2$ since none of the corner of $Q_i$ is the origin. Hence, we obtain that

$$\sup_{z \in Q_i} |z| \approx \inf_{z \in Q_i} |z|.$$  \hspace{1cm} (3.10)

Thus, we further obtain that

$$F_{1,i} \leq C \sup_{z \in Q_i} \frac{t_i}{|x|^{m-2}|z|^{m-2}(t_i + |x| + |z|)^{n+1}} \int_{\mathbb{R}^m \setminus \mathbb{R}^n} |b_i(y)| dy$$

$$\leq C \inf_{z \in Q_i} \frac{t_i}{|x|^{m-2}|z|^{m-2}(t_i + |x| + |z|)^{n+1}} \lambda |Q_i|$$

$$\leq C \lambda \int_{\mathbb{R}^m \setminus \mathbb{R}^n} \frac{1}{|x|^{m-2}|z|^{m-2}(t_i + |x| + |z|)^{n+1}} \lambda |Q_i| dz.$$  

Then we obtain that for any $h \in L^2(\mathbb{R}^m \setminus K)$ with $\|h\|_{L^2(\mathbb{R}^m \setminus K)} = 1$,

$$|\langle F_{1,i}, h \rangle| \leq C \lambda \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{t_i}{|x|^{m-2}|z|^{m-2}(t_i + |x| + |z|)^{n+1}} |h(x)| dx \chi_{Q_i}(z) dz$$

$$\leq C \lambda \int_{\mathbb{R}^m \setminus K} G(h)(z) \chi_{Q_i}(z) dz,$$

where $G$ is an operator defined as

$$G(h)(z) := \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{m-2}|z|^{m-2}(t_i + |x| + |z|)^{n}} |h(x)| dx.$$

Next, it is direct to see that $G$ is a bounded operator on $L^2(\mathbb{R}^m \setminus K)$:

$$\|G(h)\|^2_{L^2(\mathbb{R}^m \setminus K)} \leq \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{2m-4}|z|^{2m-4}(|x| + |z|)^{2n}} dx \|h\|_{L^2(\mathbb{R}^m \setminus K)}^2 dz$$

$$\leq \|h\|^2_{L^2(\mathbb{R}^m \setminus K)} \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{2m-4}|z|^{2m-4}(|x| + |z|)^{2n}} dxdz$$

$$\leq \|h\|^2_{L^2(\mathbb{R}^m \setminus K)} \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{2m-4}} dx \int_{\mathbb{R}^m \setminus K} \frac{1}{|z|^{2m-4}} dz$$

$$\leq C \|h\|^2_{L^2(\mathbb{R}^m \setminus K)},$$

where in the last inequality we use the condition that $m > n > 2$.

As a consequence, similar to the estimates for $\sum_{i \in I_2} F_{2,i}$, we obtain that

$$\left\| \sum_{i \in I_2} F_{1,i} \right\|_{L^2(\mathbb{R}^m \setminus K)} \leq C \lambda \left\| \sum_{i \in I_2} \chi_{Q_i} \right\|_{L^2(\mathbb{R}^m \setminus K)}$$

$$\leq C \lambda \left( \sum_{i} |Q_i| \right)^{1/2}$$

$$\leq C \lambda^{\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}.$$  

Combining the estimates with respect to $F_{1,i}$ and $F_{2,i}$ above, we deduce that

$$\left\| \sum_{i \in I_1} e^{-t_i \Delta} b_i \right\|_{L^2(\mathbb{R}^m \setminus K)} \leq \left\| \sum_{i \in I_2} F_{1,i} \right\|_{L^2(\mathbb{R}^m \setminus K)} + \left\| \sum_{i \in I_2} F_{2,i} \right\|_{L^2(\mathbb{R}^m \setminus K)}.$$
which shows that the claim (3.7) holds.

Next, it is direct to see that

\[ |b_i| \leq C =: C e^{-a_i}, \]

where the operator \( H \) is defined as

\[ |b_i| = \sup_{x \in Q_i} |b_i(x)|. \]

We now estimate (3.8). Consider the function \( e^{-t_i \sqrt{L} b_i(x)} \) for \( x \in \mathbb{R}^n \setminus K \). Since

\[ e^{-t_i \sqrt{L} b_i(x)} = \int_{\mathbb{R}^m \setminus K} P_{t_i}(x, y) b_i(y) dy, \]

applying the upper bound in Point 4 in Theorem 2.2 for \( |P_{t_i}(x, y)| \) we obtain that

\[ |e^{-t_i \sqrt{L} b_i(x)}| \leq \int_{\mathbb{R}^m \setminus K} |P_{t_i}(x, y)| |b_i(y)| dy \]

\[ \leq C \int_{\mathbb{R}^m \setminus K} \left( \frac{t_i}{t_i + d(x, y)} \right)^{m+1} + \frac{1}{|x|^{m-2}} \left( \frac{t_i}{t_i + d(x, y)} \right)^{m+1} + \frac{1}{y^{m-2}} \left( \frac{t_i}{t_i + d(x, y)} \right)^{n+1} |b_i(y)| dy \]

\[ \leq C \int_{\mathbb{R}^m \setminus K} \frac{t_i}{t_i + d(x, y)} |b_i(y)| dy + C \int_{\mathbb{R}^m \setminus K} \frac{1}{y^{m-2}} \left( \frac{t_i}{t_i + d(x, y)} \right)^{n+1} |b_i(y)| dy \]

\[ =: G_{1,i} + G_{2,i}. \]

By the equivalence in (3.10), we have

\[ G_{2,i} \leq C \int_{\mathbb{R}^m \setminus K} \frac{t_i}{|z|^{m-2}} \left( \frac{t_i}{t_i + d(x, z)} \right)^{n+1} |b_i(y)| dy \]

\[ \leq C \sup_{z \in Q_i} \frac{1}{|z|^{m-2}} \left( \frac{t_i}{t_i + d(x, z)} \right)^{n+1} \int_X |b_i(y)| dy \]

\[ \leq C \inf_{z \in Q_i} \frac{1}{|z|^{m-2}} \left( \frac{t_i}{t_i + d(x, z)} \right)^{n+1} \int_X |b_i(y)| dy \]

\[ \leq C \lambda \int_{\mathbb{R}^m \setminus K} \frac{1}{|z|^{m-2}} \left( \frac{t_i}{t_i + d(x, z)} \right)^{n+1} \chi_{Q_i}(z) dz. \]

Then we obtain that for any \( h \in L^2(\mathbb{R}^m \setminus K) \) with \( \|h\|_{L^2(\mathbb{R}^m \setminus K)} = 1 \),

\[ |\langle G_{2.i}, h \rangle| \leq C \lambda \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{1}{|z|^{m-2}} \left( \frac{t_i}{t_i + d(x, z)} \right)^{n+1} |h(x)| dx \chi_{Q_i}(z) dz \]

\[ \leq C \lambda \int_{\mathbb{R}^m \setminus K} |h(x)| \chi_{Q_i}(z) dz, \]

where the operator \( \mathcal{H} \) is defined as

\[ \mathcal{H}(h)(z) := \int_{\mathbb{R}^m \setminus K} \frac{1}{|z|^{m-2}} \left( \frac{t_i}{t_i + d(x, z)} \right)^{n+1} |h(x)| dx. \]

Next, it is direct to see that \( \mathcal{H} \) is a bounded operator on \( L^2(\mathbb{R}^m \setminus K) \):

\[ \| \mathcal{H}(h) \|_{L^2(\mathbb{R}^m \setminus K)} \leq \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{1}{|z|^{2m-4}} \left( \frac{t_i}{t_i + |x| + |z|} \right)^{2(n+1)} dx \|h\|_{L^2(\mathbb{R}^m \setminus K)}^2 dz \]

\[ \leq \|h\|_{L^2(\mathbb{R}^m \setminus K)}^2 \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{1}{t_i} \left( \frac{t_i}{t_i + |x|} \right)^{n+2} dx \frac{1}{|z|^{2m+n-4}} dz. \]
where in the last inequality we use the condition that $m > n > 2$.

As a consequence we obtain that

$$
\left\| \sum_{i \in I_1} G_{2,i} \right\|_{L^2(\mathbb{R}^n)} \leq C \lambda \left\| \sum_{i \in I_1} \chi_{Q_i} \right\|_{L^2(\mathbb{R}^m)} \\
\leq C \lambda \left( \sum_{i} |Q_i| \right)^{1/2} \\
\leq C \lambda^{3/2} \| f_1 \|_{L^1(\mathbb{R}^m)}^{1/2}.
$$

Similar to the estimates for the terms $F_{2,i}$, we also obtain

$$
\left\| \sum_{i \in I_1} G_{1,i} \right\|_{L^2(\mathbb{R}^n)} \leq C \lambda^{3/2} \| f_1 \|_{L^1(\mathbb{R}^m)}^{1/2}.
$$

Combining the estimates for $G_{1,i}$ and $G_{2,i}$ above, we obtain that (3.8) holds.

We now verify (3.9). Consider the function $e^{-t_i \sqrt{\mathcal{L}}} b_i(x)$ for $x \in K$. Since

$$
e^{-t_i \sqrt{\mathcal{L}}} b_i(x) = \int_{\mathbb{R}^m} P_{t_i}(x,y)b_i(y)dy,
$$

applying the upper bound in Point 5 in Theorem 2.2 for $|P_{t_i}(x,y)|$ we obtain that

$$
|e^{-t_i \sqrt{\mathcal{L}}} b_i(x)| \leq \int_X |P_{t_i}(x,y)||b_i(y)|dy
\leq C \int_X (t_i + d(x,y))^{m+1} |b_i(y)|dy + C \int_X \frac{1}{|y|^{m-2}} \frac{t}{(t_i + d(x,y))^{m+1}} |b_i(y)|dy
=: H_{1,i} + H_{2,i}.
$$

Arguing similarly to the estimates for the terms $G_{1,i}$ and $G_{2,i}$, we get the same estimates for the terms $H_{1,i}$ and $H_{2,i}$, respectively. This implies that (3.9) holds.

We now consider the term $I_{122}$. Note that

$$
I_{122} \leq \left\{ x \in (\mathbb{R}^m) \setminus \bigcup_i 8Q_i : \left| m(\sqrt{\mathcal{L}}) \left( \sum_j (I - e^{-t_j \sqrt{\mathcal{L}}})b_j \right)(x) \right| > \frac{\lambda}{12} \right\}
\leq \frac{C}{\lambda} \sum_j \int_{(8Q_j)^c} \left| m(\sqrt{\mathcal{L}})(I - e^{-t_j \sqrt{\mathcal{L}}})b_j(x) \right| dx.
$$

Note that for each $j$, we get that

$$
(3.11) \int_{(8Q_j)^c} \left| m(\sqrt{\mathcal{L}})(I - e^{-t_j \sqrt{\mathcal{L}}})b_j(x) \right| dx \\
\leq \int_{(8Q_j)^c} \int_{Q_j} |k_{t_j}(x,y)||b_j(y)|dydx \\
= \int_{Q_j} \int_{(8Q_j)^c} |k_{t_j}(x,y)|dx |b_j(y)|dy
$$

where we use $k_{t_j}(x,y)$ to denote the kernel of of the operator $m(\sqrt{\mathcal{L}})(I - e^{-t_j \sqrt{\mathcal{L}}})$. 

By definition, we have
\[
m(\sqrt{L})(I - e^{-t\sqrt{L}}) = \int_0^\infty \sqrt{L} e^{-s\sqrt{L}} m(s) ds \int_0^{t_j} - \frac{d}{dt} e^{-t\sqrt{L}} dt
\]
\[
= \int_0^\infty \sqrt{L} e^{-s\sqrt{L}} m(s) ds \int_0^{t_j} \sqrt{L} e^{-t\sqrt{L}} dt
\]
\[
= \int_0^{t_j} \int_0^\infty (\sqrt{L})^2 e^{-(s+t)\sqrt{L}} m(s) ds dt
\]
\[
= \int_0^{t_j} \int_0^\infty (s + t)^2 e^{-(s+t)\sqrt{L}} \frac{m(s)}{(s + t)^2} ds dt.
\]
Hence, we obtain that
\[
k_{t_j}(x, y) = \int_0^{t_j} \int_0^\infty P_{s+t,2}(x, y) \frac{m(s)}{(s + t)^2} ds dt.
\]
We now claim that there exists an absolute positive constant \(C\) such that
\[
\int_{(8Q_j)^c} |k_{t_j}(x, y)| dx \leq C.
\]
To see this, applying the kernel expression above and Case 5 in Theorem \([2,2]\) for \(P_{t,2}(x, y)\), we get that
\[
\int_{(8Q_j)^c} |k_{t_j}(x, y)| dx
\]
\[
\leq \int_{(8Q_j)^c} \int_0^{t_j} \int_0^\infty |P_{s+t,2}(x, y)| \frac{|m(s)|}{(s + t)^2} ds dt dx
\]
\[
\leq \int_{(8Q_j)^c} \int_0^{t_j} \int_0^\infty C \left( \frac{s + t}{s + t + d(x, y)} \right)^{m+2} \frac{1}{(s + t)^2} ds dt dx
\]
\[
+ \int_{(8Q_j)^c} \int_0^{t_j} \int_0^\infty C \left( \frac{s + t}{s + t + |x| + |y|} \right)^{n+2} \frac{1}{(s + t)^2} ds dt dx =: E_1 + E_2.
\]
We first consider the term \(E_1\). Note that
\[
E_1 \leq C \int_0^{t_j} \int_0^\infty \int_{d(x,y) \geq 2t_j} \frac{1}{(s + t)^m} \left( \frac{s + t}{s + t + d(x, y)} \right)^{m+2} \frac{1}{(s + t)^2} dx ds dt
\]
\[
\leq C \int_0^{t_j} \int_0^\infty \int_{d(x,y) \geq 2t_j} \left( \frac{1}{s + t + d(x, y)} \right)^{m+2} dx ds dt
\]
\[
+ C \int_0^{t_j} \int_0^\infty \int_{d(x,y) \geq 2t_j} \frac{1}{r^{m+2}} \left( \frac{s + t}{s + t + d(x, y)} \right)^{m+2} dx \frac{1}{(s + t)^2} ds dt
\]
\[
\leq C \int_0^{t_j} \int_0^\infty \int_{t_j}^{\infty} \frac{1}{r^{m+2}} r^{m-1} dr ds dt + C \int_0^{t_j} \int_0^\infty \frac{1}{(s + t)^2} ds dt
\]
\[
\leq C,
\]
where in the last inequality, we use polar coordinates to estimate the first term and we use the following fact for the second term
\[
\int_{d(x,y) \geq 2t_j} \frac{1}{(s + t)^m} \left( \frac{s + t}{s + t + d(x, y)} \right)^{m+2} dx \leq C.
\]
Next we consider the term $E_2$. Note that

$$E_2 \leq C \int_0^{t_j} \int_0^{t_j} \int_{d(x,y) \geq 2t_j} \frac{1}{x^{m-2}|y|^{m-2}} \left( \frac{1}{|x| + |y|} \right)^{n+2} \, dx \, dsdt$$

$$+ C \int_0^{t_j} \int_0^{t_j} \int_{d(x,y) \geq 2t_j} \frac{1}{(s + t)^n|x^{m-2}|y^{m-2}} \left( \frac{s + t}{s + t + |x| + |y|} \right)^{n+2} \, dx \, \frac{1}{(s + t)^2} \, dsdt$$

$$=: E_{21} + E_{22}.$$

As for $E_{21}$, we first suppose $t_j \geq 1$. Then by noting that $d(x,y) \leq |x| + |y|$ and that $|x| \geq 1$ and $|y| \geq 1$, we have

$$\frac{1}{|x|^{m-2}} \leq \frac{1}{d(x,y)^{m-2}} \text{ or } \frac{1}{|y|^{m-2}} \leq \frac{1}{d(x,y)^{m-2}},$$

which implies that

$$\frac{1}{|x|^{m-2}|y|^{m-2}} \leq \frac{1}{d(x,y)^{m-2}}.$$

As a consequence,

$$E_{21} \leq C \int_0^{t_j} \int_0^{t_j} \int_{d(x,y) \geq 2t_j} \frac{1}{d(x,y)^{m-2}} \left( \frac{1}{d(x,y)} \right)^{n+2} \, dx \, dsdt$$

$$\leq C \int_0^{t_j} \int_0^{t_j} \int_{t_j}^{\infty} \frac{1}{r^{m+n}r^{m-1}} \, drdsdt \leq C \frac{t_j^2}{t_j} \leq C.$$

We now suppose $t_j < 1$. Then it is direct that

$$E_{21} \leq C \int_0^{1} \int_0^{1} \int_{d(x,y) \geq 2t_j} \frac{1}{|x|^{m-2}|y|^{m-2}} \left( \frac{1}{|x| + |y|} \right)^{n+2} \, dx \, dsdt$$

$$\leq C \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{m-2}} \left( \frac{1}{|x|} \right)^{n+2} \, dx \leq C.$$

As for $E_{22}$, again noting that $d(x,y) \leq |x| + |y|$, we have

$$E_{22} \leq C \int_0^{t_j} \int_0^{t_j} \int_{d(x,y) \geq 2t_j} \frac{1}{(s + t)^n|x^{m-2}|y^{m-2}} \left( \frac{s + t}{s + t + |x| + |y|} \right)^{n+2} \, dx \, \frac{1}{(s + t)^2} \, dsdt$$

$$\leq C \int_0^{t_j} \int_0^{t_j} \int_{d(x,y) \geq 2t_j} \frac{1}{(s + t)^n|x^{m-2}|y^{m-2}} \left( \frac{s + t}{s + t + |x| + |y|} \right)^n \, dx \, \frac{1}{(s + t)^2} \, dsdt$$

$$\leq C \int_0^{t_j} \int_0^{t_j} \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{m-2}} \left( \frac{1}{|x|} \right)^n \, dx \, \frac{1}{(s + t)^2} \, dsdt$$

$$\leq C \frac{t_j}{(s + t)^2} \, dsdt$$

$$\leq C.$$

Combining the estimates of $E_{22}$, $E_{12}$ and $E_1$, we obtain that the claim (3.12) holds. As a consequence, from (3.11), we obtain that for each $j$,

$$\int_{(8Q_j)^c} |m(\sqrt{L})(I - e^{-t_j \sqrt{L}})b_j(x)| \, dx \leq \int_{Q_j} |b_j(y)| \, dy \leq C\lambda|Q_j|,$$
which implies that
\[
I_{122} \leq \frac{C}{\lambda} \sum_j \int_{(8Q)_i} |m(\sqrt{L})(I - e^{-t_j \sqrt{L}}) b_j(x)| \, dx
\]
\[
\leq C \sum_j |Q_j| \leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}.
\]

We now consider the term \( I_{123} \). Note that for each \( i \in \mathcal{I}_2 \) we have \( t_i \geq 1/2 \). Fix \( i \in \mathcal{I}_2 \). Denote by \( k_{m(\sqrt{L})}(x, y) \) the associated kernel of \( m(\sqrt{L}) \). For \( x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i \) and \( y \in Q_i \), by Point 5 in Theorem 2.2 we have
\[
|k_{m(\sqrt{L})}(x, y)| \leq \int_0^\infty |P_t(x, y)| \frac{dt}{t}
\]
\[
\leq \int_0^\infty \frac{C}{tm+1} \left( \frac{t}{t + d(x, y)} \right)^{m+1} dt + \int_0^\infty \frac{C}{tm+1} \left( \frac{t}{t + |x|} \right)^{m+1} dt
\]
\[
=: K_1(x, y) + K_2(x, y).
\]
Since \( d(x, y) \sim d(x, x_{Q_i}) \), we have
\[
K_1(x, y) \leq \int_0^\infty \frac{C}{tm+1} \left( \frac{t}{t + d(x, x_{Q_i})} \right)^{m+1} dt
\]
\[
\leq \int_0^{d(x, x_{Q_i})} \frac{C}{d(x, x_{Q_i})^{m+1}} dt + \int_0^\infty \frac{C}{tm+1} dt
\]
\[
\leq \frac{C}{d(x, x_{Q_i})^m}.
\]
Using the fact that \( |x|/|y| \geq |x| + |y| \geq d(x, y) \geq d(x, x_{Q_i}) \) we have
\[
K_2(x, y) \leq \int_0^\infty \frac{C}{tn+1} \left( \frac{t}{t + d(x, x_{Q_i})} \right)^{n+1} dt
\]
\[
\leq \int_0^{d(x, x_{Q_i})} \frac{C}{d(x, x_{Q_i})^{m+n-1}} dt + \int_0^\infty \frac{C}{tn+1} \left( \frac{t}{t + d(x, x_{Q_i})} \right)^{m-2} dt
\]
\[
\leq \frac{C}{d(x, x_{Q_i})^{m+n-2}}
\]
where in the last inequality we used \( d(x, x_{Q_i}) \geq 2t_i \geq 1 \).

From the estimates of \( K_1(x, y) \) and \( K_2(x, y) \), for each \( i \in \mathcal{I}_2 \) and \( x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i \) we have
\[
\sup_{y \in Q_i} |k_{m(\sqrt{L})}(x, y)| \leq \frac{C}{d(x, x_{Q_i})^m}.
\]
Moreover, observe that since \( i \in \mathcal{I}_2 \) and \( x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i \) we have
\[
\frac{1}{d(x, x_{Q_i})^m} \sim \frac{1}{|x|^m}.
\]
As a consequence, for each \( i \in \mathcal{I}_2 \) and \( x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i \) we have
\[
\sup_{y \in Q_i} |k_{m(\sqrt{L})}(x, y)| \leq \frac{C}{|x|^m}.
\]
This implies that for each \( x \in (\mathbb{R}^m \setminus K) \setminus \cup_i 8Q_i \), we have
\[
\left| \sum_{i \in I_2} m(\sqrt{L})b_i(x) \right| \leq C \frac{\sum_{i \in I_2} \| b_i \|_{L^1(\mathbb{R}^m \oplus \mathbb{R}^n)}}{|x|^m}.
\]
Therefore,
\[
I_{123} \leq \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \cup_i 8Q_i : \left| \sum_{i \in I_2} m(\sqrt{L})b_i(x) \right| > \frac{\lambda}{18} \right\}
\leq \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \cup_i 8Q_i : C|x|^{-m} \left( \sum_{i \in I_2} \| b_i \|_{L^1(\mathbb{R}^m \oplus \mathbb{R}^n)} \right) > \frac{\lambda}{18} \right\}
\leq C \frac{\| f \|_{L^1(\mathbb{R}^m \oplus \mathbb{R}^n)}}{\lambda}.
\]
Combining all cases of \( I_{11}, I_{13}, I_{121}, I_{122} \) and \( I_{123} \), we obtain that
\[
I_1 \leq C \frac{\| f \|_{L^1(\mathbb{R}^n)}}{\lambda}.
\]

3.2. Estimate of \( I_2 \). We now consider the term \( I_2 \). Note that in the case, \( x \) is in the large end \( \mathbb{R}^m \setminus K \) and the function \( f_2 \) is supported in the small end \( \mathbb{R}^n \setminus K \), and hence the measure will become non-doubling since if we enlarge a ball contained in \( \mathbb{R}^n \setminus K \), then the enlargement can be partially contained in \( \mathbb{R}^m \setminus K \). The standard Calderón–Zygmund decomposition on non-homogeneous space such as in [21, 27] does not apply since in that decomposition, we only know the existence of a sequence of Calderón–Zygmund cubes but we do not know where they are exactly. And the Poisson kernel upper bound depends heavily on the position of the variables \( x \) and \( y \) in different ends.

Thus, to deal with this case, we use a Whitney type decomposition of the level set \( \Omega \) below and then we make clever use of the Poisson kernel upper bound in this case to handle the weak type estimate, without enlarging those cubes, which avoids the case of non-doubling measure. The genesis of this approach is an adaptation of an idea from [21].

Note that \( f_2 \) is supported in \( \mathbb{R}^n \setminus K \). We now split \( \mathbb{R}^n \setminus K \) into two parts according to \( f_2 \). Define
\[
F := \{ x \in \mathbb{R}^n \setminus K : M_2(f_2) \leq \lambda \}
\]
and
\[
\Omega := \{ x \in \mathbb{R}^n \setminus K : M_2(f_2) > \lambda \},
\]
where \( M_2 \) is the Hardy–Littlewood maximal function defined on \( \mathbb{R}^n \setminus K \).

Then we define
\[
f_{2,\lambda}(x) := f_2(x)\chi_F(x) \quad \text{and} \quad f_2^\lambda(x) := f_2(x)\chi_\Omega(x).
\]
Then we have
\[
I_2 \leq \left\{ x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f_{2,\lambda}(x)| > \frac{\lambda}{6} \right\}
+ \left\{ x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})f_2^\lambda(x)| > \frac{\lambda}{6} \right\}
=: I_{21} + I_{22}.
\]
As for $I_{21}$, by using the $L^2$ boundedness of $m(\sqrt{L})$, we obtain that

$$I_{21} = \left| \{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_{2,\lambda}(x)| > \frac{\lambda}{6} \} \right|$$

$$\leq \frac{C}{\lambda^2} \| f_{2,\lambda} \|_{L^2(\mathbb{R}^n \setminus K)}^2$$

$$\leq \frac{C}{\lambda} \| f \|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)},$$

where we use the fact that $|f_{2,\lambda}(x)| = |f_2(x)| \chi_F(x) \leq |M_2(f_2)(x)| \chi_F(x) \leq \lambda$.

As for $I_{22}$, we consider the function $f^\lambda_2$. We now apply a covering lemma in [5] (see also [7, Lemma 5.5]) for the set $\Omega$ in the homogeneous space $\mathbb{R}^n$ to obtain a collection of balls $\{ Q_i := B(x_i, r_i) : x_i \in \Omega, r_i = d(x_i, \Omega^c)/2, i = 1, \ldots \}$ so that

(i) $\Omega = \bigcup_i Q_i$;
(ii) $\{ B(x_i, r_i/5) \}_{i=1}^\infty$ are disjoint;
(iii) there exists a universal constant $K$ so that $\sum_k \chi_{Q_k}(x) \leq K$ for all $x \in \Omega$.

Hence, we can further decompose

$$f^\lambda_2(x) = \sum_i f^\lambda_{2,i}(x),$$

where $f^\lambda_{2,i}(x) = \frac{\chi_{Q_i}(x)}{\sum_k \chi_{Q_k}(x)} f^\lambda_2(x)$.

Next, note that for $x \in \mathbb{R}^m \setminus K$,

$$|m(\sqrt{L})(f^\lambda_{2,i})(x)| = \left| \int_0^\infty t\sqrt{L} \exp(-t\sqrt{L})(f^\lambda_{2,i})(x)t \frac{dt}{t} \right|$$

$$\leq \int_0^\infty \int_{Q_i} |P_{1,1}(x, y)| |m(t)| |f^\lambda_{2,i}(y)| dy dt.$$

Applying the upper bound in point 4 in Theorem 2.2 for $|P_{1,1}(x, y)|$ we obtain that

$$|m(\sqrt{L})(f^\lambda_{2,i})(x)| \leq C \int_0^\infty \int_{Q_i} \left( 	frac{t}{(t + d(x, y))^{m+1}} + \frac{1}{|x|^m (t + d(x, y))^{n+1}} + \frac{1}{|y|^{m-2} (t + d(x, y))^{n+1}} \right) |f^\lambda_{2,i}(y)| dy dt$$

$$=: E_1 + E_2 + E_3.$$

As for the term $E_1$, note that $d(x, y) \approx |x| + |y|$ since $x \in \mathbb{R}^m \setminus K$ and $y \in \mathbb{R}^n \setminus K$. Hence, we obtain that

$$\frac{t}{(t + d(x, y))^{m+1}} \leq C \frac{1}{|x|^m},$$

which implies that

$$E_1 \leq C \int_{Q_i} \frac{1}{|x|^m} |f^\lambda_{2,i}(y)| dy.$$

As for the term $E_2$, again, from the fact that $d(x, y) \approx |x| + |y|$ since $x \in \mathbb{R}^m \setminus K$, we have

$$\frac{1}{|x|^{m-2} (t + d(x, y))^{n+1}} \leq C \frac{1}{|x|^{m-2} |x|^{n}} \leq C \frac{1}{|x|^m}$$

since $m > n > 2$. As a consequence, we have

$$E_2 \leq C \int_{Q_i} \frac{1}{|x|^m} |f^\lambda_{2,i}(y)| dy.$$
As for the term $E_3$, by noting that $|y| > 1$, we obtain that $E_3$ is bounded by $E_1$, which, together with the estimate of $E_1$, yields

$$E_3 \leq C \frac{1}{|x|^m} \int_{Q_i} |f^\lambda_{2,i}(y)| dy.$$ 

Combining the estimates of $E_1$, $E_2$ and $E_3$, we obtain that

$$|m(\sqrt{L})f^\lambda_{2,i}(x)| \leq C \frac{1}{|x|^m} \sum_i \int_{Q_i} |f^\lambda_{2,i}(y)| dy,$$

which implies that

$$|m(\sqrt{L})f^\lambda_2(x)| \leq \sum_i |m(\sqrt{L})f^\lambda_{2,i}(x)| \leq C \frac{1}{|x|^m} \sum_i \int_{Q_i} |f^\lambda_{2,i}(y)| dy \leq C \frac{1}{|x|^m} \|f_2\|_{L^1(\mathbb{R}^m)}.$$

Hence, we obtain that

$$I_{22} = \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f^\lambda_2(x)| > \frac{\lambda}{6} \right\} \right|$$

$$\leq \left| \left\{ x \in \mathbb{R}^m \setminus K : C \frac{1}{|x|^m} \|f_2\|_{L^1(\mathbb{R}^m)} > \frac{\lambda}{6} \right\} \right|$$

$$= \left| \left\{ x \in \mathbb{R}^m \setminus K : |x|^m < \frac{6C}{\lambda} \|f_2\|_{L^1(\mathbb{R}^m)} \right\} \right|$$

$$\leq \frac{C \|f_2\|_{L^1(\mathbb{R}^m)}}{\lambda} \leq \frac{C \|f\|_{L^1(\mathbb{R}^m)}}{\lambda}.$$

### 3.3. Estimate of $I_3$.

For the term $I_3$, we point out that we can handle this case by using the same approach as in the estimates for the term $I_2$. We sketch the proof as follows.

Define $F := \{ x \in K : M_3(f_3) \leq \lambda \}$ and $\Omega := \{ x \in K : M_3(f_3) > \lambda \}$, where $M_3$ is the Hardy–Littlewood maximal function defined on $K$. Then let $f_{3,\lambda}(x) := f_3(x)\chi_F(x)$ and $f^\lambda_3(x) := f_3(x)\chi_\Omega(x)$.

Then we have

$$I_3 \leq \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_{3,\lambda}(x)| > \frac{\lambda}{6} \right\} \right|$$

$$+ \left| \left\{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f^\lambda_3(x)| > \frac{\lambda}{6} \right\} \right|$$

$$=: I_{31} + I_{32}.$$

Using the $L^2$ boundedness of $m(\sqrt{L})$ and the fact that $|f_{3,\lambda}(x)| \leq \lambda$, we obtain that $I_{31} \leq \frac{\lambda}{6} \|f\|_{L^1(\mathbb{R}^m)}$.

As for $I_{32}$, by using the Whitney decomposition, we obtain that $\Omega = \bigcup_i Q_i$ such that $\sum_i |Q_i| = |\Omega|$, which gives

$$f^\lambda_3(x) = \sum_i f^\lambda_{3,i}(x),$$

where $f^\lambda_{3,i}(x) = f^\lambda_3(x)\chi_{Q_i}(x)$.

Next, for $x \in \mathbb{R}^m \setminus K$,

$$|m(\sqrt{L})(f^\lambda_{3,i})(x)| \leq \int_0^\infty \int_{Q_i} |P_{1,t}(x,y)| |m(t)| |f^\lambda_{3,i}(y)| dy dt.$$
Applying the upper bound in Point 2 in Theorem 2.2 for $|P_{t,1}(x,y)|$ we obtain that
\[
|m(\sqrt{L})(f^\lambda_{1,i})| \leq C \int_0^\infty \int_{Q_i} \left( \frac{t}{(t+d(x,y))^{m+1}} + \frac{1}{|x|^{m-2}(t+d(x,y))^{n+1}} \right) |f^\lambda_{3,i}(y)| \, dy \, dt
\]
\[
\leq \frac{C}{|x|^m} \int_{Q_i} |f^\lambda_{3,i}(y)| \, dy,
\]
where the last inequality follows from the same estimates for $E_1, E_2$ in Subsection 3.2.

3.4. Estimate of $II_1$. For the term $II_1$, we point out that we can handle this case by using similar way as in the estimates for the term $I_2$. We sketch the proof as follows.

Define $F := \{ x \in \mathbb{R}^m \setminus K : M_1(f_1) \leq \lambda \}$ and $\Omega := \{ x \in \mathbb{R}^m \setminus K : M_1(f_1) > \lambda \}$, where $M_1$ is the Hardy–Littlewood maximal function defined on $\mathbb{R}^m \setminus K$. Then let $f_{1,\lambda}(x) := f_1(x)\chi_F(x)$ and $f^\lambda_1(x) := f_1(x)\chi_{\Omega}(x)$.

Then we have
\[
II_1 \leq \left| \{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_1(x)| > \frac{\lambda}{6} \} \right| + \left| \{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f_1^\lambda(x)| > \frac{\lambda}{6} \} \right|
\]
\[
= II_{11} + II_{12}.
\]
Using the $L^2$ boundedness of $m(\sqrt{L})$ and the fact that $|f_{1,\lambda}(x)| \leq \lambda$, we obtain $II_{11} \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}$.

As for $II_{12}$, by using the Whitney decomposition, we obtain $\Omega = \bigcup_i Q_i$ such that $\sum_i |Q_i| = |\Omega|$, which gives
\[
f^\lambda_1(x) = \sum_i f^\lambda_{1,i}(x),
\]
where $f^\lambda_{1,i}(x) = f^\lambda_1(x)\chi_{Q_i}(x)$.

Next, for $x \in \mathbb{R}^m \setminus K$,
\[
|m(\sqrt{L})(f^\lambda_{1,i})| \leq \int_0^\infty \int_{Q_i} |P_{t,1}(x,y)| \, m(t) \, |f^\lambda_{1,i}(y)| \, dy \, dt.
\]
Applying the upper bound in Point 4 in Theorem 2.2 for $|P_{t,1}(x,y)|$ we obtain that
\[
|m(\sqrt{L})(f^\lambda_{1,i})| \leq C \int_0^\infty \int_{Q_i} \left( \frac{t}{(t+d(x,y))^{m+1}} + \frac{1}{|x|^{m-2}(t+d(x,y))^{n+1}} \right) \, dy \, dt
\]
\[
\leq \frac{C}{|x|^n} \int_{Q_i} |f^\lambda_{1,i}(y)| \, dy,
\]
where the last inequality follows from similar estimates for $E_1, E_2$ and $E_3$ in Subsection 3.2. This implies that
\[
|m(\sqrt{L})(f^\lambda_{1})| \leq \sum_i |m(\sqrt{L})(f^\lambda_{1,i})| \leq C \frac{1}{|x|^n} \sum_i \int_{Q_i} |f^\lambda_{1,i}(y)| \, dy \leq C \frac{1}{|x|^n} \|f_1\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)}.
\]

Hence, we obtain that
\[
II_{12} = \left| \{ x \in \mathbb{R}^m \setminus K : |m(\sqrt{L})f^\lambda_1(x)| > \frac{\lambda}{6} \} \right| \leq \left| \{ x \in \mathbb{R}^m \setminus K : C \frac{1}{|x|^n} \|f_1\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)} > \frac{\lambda}{6} \} \right|
\]
\[
= \left| \{ x \in \mathbb{R}^m \setminus K : |x|^n < 6C \|f_1\|_{L^1(\mathbb{R}^m \setminus \mathbb{R}^n)} \frac{\lambda}{6} \} \right|.
\]
Then we get
\[ \|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)} \leq \frac{C\|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}}{\lambda}. \]

### 3.5. Estimate of $I_2$. We will apply a similar approach as that in [11] and using similar estimates for the term $I_1$ in our Section 3.1 to estimate $I_2$.

We restrict the setting on $\mathbb{R}^m \setminus K$, and $f_2$ is in $L^1(\mathbb{R}^n \setminus K)$. We now extend $f_2$ to the whole $\mathbb{R}^n$ by zero extension, i.e., define $f_2(x) := 0$ when $x \in K$.

Similar to the Calderón–Zygmund decomposition in $I_1$, we get
\[ f_2(x) = g(x) + \sum_j b_j(x) \]
with $\|g\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}$ and $\|g\|_{L^\infty(\mathbb{R}^m \times \mathbb{R}^n)} \leq C\lambda$ and
\[ b_j(x) := \left( f_2(x) - \frac{1}{|Q_j|} \int_{Q_j} f_2(y) dy \right) \chi_{Q_j}(x). \]

Then we get
\[ I_2 \leq \left| \left\{ x \in \mathbb{R}^n \setminus K : |m(\sqrt{L})g(x)| > \frac{\lambda}{6} \right\} \right| + \left| \left\{ x \in (\mathbb{R}^n \setminus K) \cup i 8Q_i : |m(\sqrt{L})(\sum_i b_i)(x)| > \frac{\lambda}{6} \right\} \right| + \cup_i 8Q_i \]
\[ =: I_{21} + I_{22} + I_{23}. \]

By using the $L^2$ boundedness of $m(\sqrt{L})$ and the fact that $\|g\|_{L^\infty(X)} \leq C\lambda$, we obtain that $I_{21} \leq \frac{\lambda}{6}\|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}$. Next, from the doubling condition in this case, we get that $I_{23} \leq C\sum_i |Q_i| \leq C\frac{\|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}}{\lambda}$. For the term $I_{22}$, we have
\[ I_{22} \leq \left| \left\{ x \in (\mathbb{R}^n \setminus K) \cup i 8Q_i : \left| m(\sqrt{L})\left( \sum_i e^{-t_i \sqrt{L}}b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right| \]
\[ + \left| \left\{ x \in (\mathbb{R}^n \setminus K) \cup i 8Q_i : \left| m(\sqrt{L})\left( \sum_i (I - e^{-t_i \sqrt{L}})b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right| \]
\[ =: I_{221} + I_{222}, \]
where for each $i$, $t_i$ is the side length of the cube $Q_i$. Note that the term $I_{222}$ can be handled similarly by using the same approach as that for $I_{122}$ and using upper bound in Point 6 in Theorem 2.2 for $|P_{t,2}(x,y)|$, which yields that $I_{222}$ is bounded by $C\frac{\|f\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}}{\lambda}$.

As for $I_{221}$, we first claim that
\[ \left\| \sum_i e^{-t_i \sqrt{L}}b_i \right\|_{L^2(\mathbb{R}^m \times \mathbb{R}^n)} \leq C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}^{\frac{1}{2}}. \]

To see this claim, it suffices to show the following 3 cases:
\[ \left\| \sum_i e^{-t_i \sqrt{L}}b_i \right\|_{L^2(\mathbb{R}^m \setminus K)} \leq C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}^{\frac{1}{2}}, \]
\[ \left\| \sum_i e^{-t_i \sqrt{L}}b_i \right\|_{L^2(\mathbb{R}^n \setminus K)} \leq C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m \times \mathbb{R}^n)}^{\frac{1}{2}}, \]
and

\[
\left\| \sum_{i} e^{-t_{i}^{\frac{1}{2}}T_{b_{i}}} \right\|_{L^{2}(K)} \leq C \lambda^{\frac{1}{2}} \| f_{2} \|_{L^{1}(\mathbb{R}^{n} \times \mathbb{R}^{n})}^{\frac{1}{2}}.
\]

We now point out that (3.14) can be obtained by using similar estimates as those for (3.8) and that (3.16) can be obtained by using similar estimates as those for (3.9). We omit the details.

As for (3.15), applying the upper bound in Point 6 in Theorem 2.2 for \( |P_{t_{i}}(x, y)| \) we obtain that

\[
|e^{-t_{i}^{\frac{1}{2}}T_{b_{i}}}(x)| \leq \int_{\mathbb{R}^{n} \setminus K} |P_{t_{i}}(x, y)| |b_{i}(y)| \, dy
\]

\[
\leq C \int_{\mathbb{R}^{n} \setminus K} \left( \frac{t_{i}}{(t_{i} + d(x, y))^{m+1}} + \frac{t_{i}}{(t_{i} + d(x, y))^{n+1}} \right) |b_{i}(y)| \, dy
\]

\[
=: F_{1,i} + F_{2,i}.
\]

For the term \( F_{2,i} \), by using similar technique of the sup–inf estimate as in the estimate for \( F_{2,i} \) in Subsection 3.1, we obtain that \( \left\langle \sum_{i} F_{2,i}, h \right\rangle \leq C \lambda \langle M_{2}(h), \sum_{i} \chi_{Q_{i}} \rangle \) for any \( h \) with \( \| h \|_{L^{2}(\mathbb{R}^{n} \setminus K)} = 1 \), which yields that

\[
\left\| \sum_{i} F_{2,i} \right\|_{L^{2}(\mathbb{R}^{m} \setminus K)} \leq C \lambda^{\frac{1}{2}} \| f_{2} \|_{L^{1}(\mathbb{R}^{m} \times \mathbb{R}^{n})}^{\frac{1}{2}}.
\]

For the term \( F_{2,i} \), we consider the position of \( Q_{i} \), the support of \( b_{i} \), as follows: if one of the corners of \( Q_{i} \) is origin, then \( t_{i} \geq \frac{1}{2} \), since otherwise the function \( f_{2} \) on \( Q_{i} \) is zero which yields that this \( Q_{i} \) can not be chosen from the Calderón–Zygmund decomposition; if none of the corners of \( Q_{i} \) is origin, then if \( t_{i} < 1, d(Q_{i}, 0) \geq \frac{1}{2} \). Combining all these cases, we get that

\[
\frac{t_{i}}{(t_{i} + d(x, y))^{m+1}} \leq C \frac{t_{i}}{(t_{i} + d(x, y))^{n+1}},
\]

which shows that

\[
\left\| \sum_{i} F_{1,i} \right\|_{L^{2}(\mathbb{R}^{m} \setminus K)} \leq C \left\| \sum_{i} F_{2,i} \right\|_{L^{2}(\mathbb{R}^{m} \setminus K)} \leq C \lambda^{\frac{1}{2}} \| f_{2} \|_{L^{1}(\mathbb{R}^{m} \times \mathbb{R}^{n})}^{\frac{1}{2}}.
\]

3.6. Estimate of \( III_{3}, \ III_{1}, \ III_{2}, \ III_{3} \). We point out that the estimates of \( III_{3} \) follows from the upper bound in Point 3 in Theorem 2.2 for \( |P_{t_{i}}(x, y)| \) and from similar estimates as for \( I_{3} \) in Subsection 3.3. The estimates of \( III_{1} \) and \( III_{2} \) can be obtained by using similar techniques as in \( I_{3} \) and \( III_{3} \), respectively. \( III_{3} \) can also be obtained using similar approaches as in \( III_{1} \). We omit the details here.

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