Graded Primary Submodules over Multiplication Modules

Malik Bataineh
Mathematics Department
Jordan University of Science and Technology
Irbid 22110, Jordan e-mail: msbataineh@just.edu.jo

Ala’ Lutfi Khazaaleh
Mathematics Department
Jordan University of Science and Technology
Irbid 22110, Jordan

Abstract: Let $G$ be an abelian group with identity $e$, $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module where all modules are unital. Various generalizations of graded prime ideals and graded submodules have been studied. For example, a proper graded ideal $I$ is a graded weakly (resp; almost) prime ideal if $0 \neq ab \in I$ (resp; $ab \in I - I^2$) then $a \in I$ or $b \in I$. Also a proper graded submodule $N$ of $M$ is graded primary submodule if $rm \in N$, then either $m \in N$ or $r \in \sqrt{(N : M)}$.

Throughout this work, we define that a proper graded submodule is a graded weakly (resp; almost) primary submodule if $0 \neq rm \in N$ (resp; $rm \in N - (N : M)N$), then either $m \in N$ or $r \in \sqrt{(N : M)}$. We give some properties and characterizations of graded weakly (resp; almost) primary submodules. We show that graded weakly primary submodules enjoy analogs of many of the properties of prime submodules and primary submodules.

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1. INTRODUCTION

Several authors have extended the notion of prime ideals to modules [2], [3]. Almost prime ideals were introduced by S. M. Bhatwadekar and P. K. Sharma [6]. Graded almost prime ideals in a graded commutative ring with non-zero identity have been introduced and studied by A. Jabeer, M. Bataineh and H. Khasan [1, 4]. Moreover, graded primary submodules in a graded commutative ring with non-zero identity have been studied by many authors such as S. Ebrahimi Atani [7] and K. H. Oral, V. Tekir and A. G. Agargun [5]. In this study, we introduce some properties and characterizations of graded weakly (almost) primary submodules.
Moreover, we study graded weakly (almost) primary submodules over multiplication modules and give some of their properties and characterizations.

2. Graded Weakly Primary Submodules

Graded weakly primary ideals in a graded commutative ring with non-zero identity have been introduced and studied by S. E. Atani [8]. Graded primary submodules in a graded module over a graded commutative ring had been studied by many authors like K. H. Oral, U. Tekir and A. G. Agargun [5] and S. E. Atani [7]. In this section we study graded weakly (resp., almost) primary submodules under multiplication and torsion free modules.

Now we give the definition of a graded weakly primary and graded almost primary submodules over a graded commutative ring.

**Definition 2.1** Let \( N \) be a graded submodule of \( M \) and \( g \in G \).

1. \( Ng \) is a **weakly** \( g \)-primary submodule of the \( R \)-module \( Mg \), if \( Ng \neq Mg \); and whenever \( a \in R \) and \( m \in Mg \) with \( 0 \neq am \in Ng \), then either \( m \in Ng \) or \( a^k \in (Ng : Mg) \) for some positive integer \( k \).

2. \( N \) is a graded weakly primary submodule of \( R \)-module \( M \), if \( N \neq M \); and whenever \( a \in h(R) \) and \( m \in h(M) \) with \( 0 \neq am \in N \), then either \( m \in N \) or \( a^k \in (N : M) \) for some positive integer \( k \).

3. \( Ng \) is an almost **g**-primary submodule of the \( R \)-module \( Mg \), if \( Ng \neq Mg \); and whenever \( a \in R \) and \( m \in Mg \) with \( am \in Ng \), then either \( m \in Ng \) or \( a^k \in (Ng : Mg) \) for some positive integer \( k \).

4. \( N \) is a graded almost primary submodule of \( R \)-module \( M \), if \( N \neq M \); and whenever \( a \in h(R) \) and \( m \in h(M) \) with \( am \in N \), then either \( m \in N \) or \( a^k \in (N : M) \) for some positive integer \( k \).

Clearly a graded weakly (resp., almost) prime submodule is a graded weakly (resp., almost) primary submodule, but the converse need not to be true.

Next we give an example of a graded weakly primary submodule which is not graded weakly prime.

**Example 2.2** Let \( R = \mathbb{Z}[i] \) be a graded ring and \( M = \mathbb{Z}_{12}[i] \) a graded \( R \)-module. Take \( N = \langle 4 + 4i \rangle \). Then \( N : M = 4R \). We note that \( N \) is a graded weakly primary submodule, but not graded weakly prime since \( 0 \neq 2(2i) = 4i \in N \) with \( 2 \notin (N : M) \) and \( 2i \notin N \).

The following theorem characterizes the homogeneous components of a graded weakly primary submodules.

**Theorem 2.3** Let \( N \) be a graded submodule of \( M \) and \( g \in G \). Then the following are equivalent.

1. \( Ng \) is a weakly \( g \)-primary submodule of \( Mg \).
2. For \( m \in Mg - Ng \), then \( \sqrt{(Ng : m)} = \sqrt{(Ng : Mg)} \cup \sqrt{(0 : m)} \).
(3) For $m \in Mg - Ng$, then $\sqrt{(Ng : m)} = \sqrt{(Ng : Mg)}$ or $\sqrt{(Ng : m)} = \sqrt{(0 : m)}$.

(4) If whenever $0 \neq PK \subseteq Ng$ with $P$ an ideal of $R$, and $K$ a submodule of $Mg$, then $P \subseteq \sqrt{(Ng : Mg)}$ or $K \subseteq Ng$.

**Proof** (1 $\Rightarrow$ 2) Let $m \in Mg - Ng$ and $a \in \sqrt{(Ng : m)}$, then $a^k \in (Ng : m)$ for some positive integer $k$ and so $a^k m \in Ng$. If $a^k m \neq 0$, then $a^k \in (Ng : Mg)$, since $Ng$ is a weakly $g$-primary submodule and $m \notin Ng$. If $a^k m = 0$, implies $a \notin \sqrt{(0 : m)}$. The reverse inclusion, if $a \in \sqrt{(Ng : Mg)}$, then $a^k Mg \subseteq Ng$ for some positive integer $k$, so $a^k m \in Ng$, hence $a \in \sqrt{(Ng : m)}$. If $a \in \sqrt{(0 : m)}$, then $a^k m = 0 \subseteq Ng$ then $a \in (Ng : m)$. Therefore, $\sqrt{(Ng : m)} \equiv \sqrt{(Ng : Mg)} \cup \sqrt{(0 : m)}$.

(2 $\Rightarrow$ 3) Clearly.

(3 $\Rightarrow$ 4) Let $P$ be an ideal of $R$, and $K$ be a submodule of $Mg$ such that $PK \subseteq Ng$, $P \bigcup \sqrt{(Ng : Mg)}$ and $K \bigcup Ng$. Want $PK = 0$. Let $a \in P$ and $m \in K$. **Case 1** assume that $m \notin N_g$. If $a \notin \sqrt{(Ng : Mg)}$, since $am \notin Ng$, we have $\sqrt{(Ng : m)} \neq \sqrt{(Ng : Mg)}$. By assuming $\sqrt{(Ng : m)} = \sqrt{(0 : m)}$. So $a^k m = 0$ where $k$ is the smallest positive integer, if $k = 1$, then $am = 0$. If $k > 1$, then $a^{k-1} m \notin N_g - \{0\}$ implies $a^{k-1} \notin \sqrt{(Ng : Mg)}$, since $m \notin N_g$ and so $a \in \sqrt{(Ng : Mg)}$ which is a contradiction. Hence $am = 0$. If $a \in P \cap \sqrt{(Ng : Mg)}$. Let $b \in P - \sqrt{(Ng : Mg)}$. Then $a + b \in P - \sqrt{(Ng : Mg)}$. By previous case, we have $bm = 0$ and $(a + b)m = 0$. So $am = 0$. **Case 2** assume that $m \in N_g$. Let $m' \in K - N_g$. Then $m + m' \in K - N_g$. By Case 1, $am' = 0$ and $a(m + m') = 0$. Hence $am = 0$ for arbitrary $a \in P$ and $m \in K$. Therefore $PK = 0$.

(4 $\Rightarrow$ 1) Let $0 \neq am \in Ng$ with $a \in R$, and $m \in Mg$. Take $P \equiv \langle a \rangle$ and $K \equiv R_m$, then $0 \neq PK \subseteq Ng$. So by assuming $P \subseteq \sqrt{(Ng : Mg)}$ or $K \subseteq Ng$. Hence $a^k \in (Ng : Mg)$ for some positive integer $n$ or $m \in Ng$.

**Theorem 2.4** If $N$ is a graded weakly primary submodule of $M$, then for $\forall g \in G$ (with $Mg \neq 0$), $Ng$ is a weakly $g$-primary submodule of $Mg$ for all $g \in G$.

**Proof** Let $0 \neq am \in Ng$ for some $g \in G$, where $a \in R$, and $m \in Mg$. Then $0 \neq am \in N$. Since $N$ is a graded weakly primary submodule we get $m \in N$ or $a^k \in (N : M)$ for some positive integer $k$. If $m \in N$ and $m \in Mg$ then $m \in N \cap Mg \equiv Ng$ and so $m \in Ng$. If $a \in (N : M)$, then $a \in (Ng : Mg)$. Therefore, $Ng$ is a weakly $g$-primary submodule of $Mg$ for all $g \in G$.

**Theorem 2.5** Assume that $N$ and $K$ are graded submodules of $M$ such that $K \subseteq N \not\subseteq M$. Then

1. If $N$ is a graded weakly primary submodule of $M$, then $N/K$ is also a graded weakly primary submodule of $M/K$. 


(2) If $K$ and $N/K$ are graded weakly primary submodules, then $N$ is also a graded weakly primary submodule of $M$.

**proof** (1) Let $\bar{0} \neq a(m + K) = am + K \in N/K$ where $a \in h(R)$, and $m \in h(M)$. Then $am + K \in N/K$ and so $am \in N$. If $am = 0$, then $am + K = \bar{0}$ which is a contradiction. If $am \neq 0$, $N$ is a graded weakly primary submodule gives either $m \in N$ or $a^k \in (N : M)$ for some positive integer $k$, hence either $m + K \in N/K$ or $a^k \in (N / K : M / K)$.

(2) Let $\bar{0} \neq am \in N$, where $a \in h(R)$, and $m \in h(M)$. So $a(m + K) \in N/K$. If $am \in K$. Since $am \neq 0$ and $K$ is a graded weakly primary submodule then either $a^k \in (K : M) \subseteq (N : M)$ for some positive integer $k$ or $m \in K \subseteq N$. If $am \notin K$, we get $0 \neq a(m + K) \in N/K$. Since $N/M$ is a graded weakly primary submodule we get either $m \in N$ or $a^k \in (N / K : M / K) \subseteq (N : M)$ for some positive integer $k$.

**Theorem 2.6** Let $N$ be a graded submodule of $M$. Then $N$ is a graded almost primary submodule of $M$ if and only if $N/(N : M)N$ is a graded weakly primary submodule of $M/(N : M)N$.

**proof** Let $r \in h(R)$ and $m \in h(M)$ such that $0 \neq r(m + (N : M)N) \in N/(N : M)N$, then $rm \in N - (N : M)N$. So $m \in N$ or $r^k \in (N : M)$ for some positive integer $k$, since $N$ is a graded almost primary submodule in $M$. Then $m + (N : M)N \in N/(N : M)N$ or $r^k \in (N : M) \equiv (N/(N : M)N : M/(N : M)N)$.

Conversely, let $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$. Then $0 \neq r(m + (N : M)N)$. So $m \in N$ or $r^k (N/(N : M)N : M/(N : M)N) \equiv (N : M)$ for some positive integer $k$.

In the next theorems, we give a characterizations of graded weakly primary prime submodules in one kind of cancellation graded modules. We need the following definitions.

**Definition 2.7** An $R$-module $M$ is a **multiplication graded module** provided that for every graded submodule $N$ of $M$, there is an ideal $I$ of $R$ such that $N = IM$ (or $N = (N : M)M$).

**Definition 2.8** An $R$-module $M$ is a **cancellation graded module** of $R$ if for all graded ideals $I$ and $J$ of $R$, $IM = JM$, implies that $I = J$.

**Definition 2.9** An $R$-module $M$ is a **faithful graded module** if it is annihilator ann $(M)$ is 0.

**Definition 2.10** A graded $R$-module $M$ is said to be **finitely generated** if there exists a finite set $A = \{m_1, m_2, ..., m_n\}$ in $M$ such that any element $m$ in $M$ can be written as $m = a_1m_1 + a_2m_2 + ... + a_nm_n$, where $a_i's \in R$.

**Theorem 2.11** Let $M$ be a finitely generated faithful multiplication graded $R$-module and $N$ be a proper graded submodule of $M$. Then $N$ is a graded almost primary submodule of $M$ if and only if $(N : M)$ is a graded almost primary ideal of $R$.

**proof** Let $ab \in (N : M) - (N : M)^2$ where $a, b \in h(R)$. Then $ab\overline{M}(N : M)N$ because $ab \notin (N : M)^2 = ((N : M)N : M)$, since $M$ is a cancellation module. Then $abM \subseteq N$ and $ab\overline{M}(N : M)N$. Since $N$ is a graded almost primary submodule we get
\( a^k \in (N : M) \) for some positive integer \( k \) or \( bM \subseteq N \). Hence \( a^k \in (N : M) \) or \( b \in (N : M) \). Therefore, \( (N : M) \) is a graded almost primary ideal.

Conversely, let \( rm \in N - (N : M)N \) where \( r \in h(R) \) and \( m \in h(M) \). We not that 
\[
\text{If } r(\langle m \rangle : M) \subseteq \langle (N : M) \rangle M \text{ and } \text{then } r(\langle m \rangle : M) \subseteq \langle (N : M) \rangle M . \\
\text{But } r(\langle m \rangle : M)M = r(m), \text{ since } M \text{ is a}
\]
cancellation module which is a contradiction. Since \( (N : M) \) is a graded almost primary ideal, we obtain \( r^k \in (N : M) \) or \( (\langle m \rangle : M) \subseteq \langle (N : M) \rangle \), then \( (\langle m \rangle : M)M \subseteq \langle (N : M) \rangle M = N \). Therefore, \( N \) is a graded almost primary submodule.

**Theorem 2.12** Let \( M \) be a finitely generated faithful multiplication graded \( R \)-module and \( N \) be a proper graded submodule of \( M \). Then \( N \) is a graded almost primary submodule of \( M \) if and only if whenever \( A \) and \( B \) are graded submodules of \( M \) such that \( AB \subseteq N \) and \( AB\hat{\cup}(N : M)N \), then either \( A \subseteq N \) or \( B \subseteq \sqrt{N} \).

**Proof** Let \( N \) and \( K \) be submodules of \( M \). Since \( M \) is a cancellation module then 
\[
N = (N : M)M \text{ and } K = (K : M)M \text{ and so } NK = (N : M)(K : M)M . \\
\text{Suppose } AB \subseteq N \text{ and } \text{then } (A : M)\hat{\cup}(N : M)N \text{ but } A\hat{\cup}N \text{ and } B\hat{\cup}\sqrt{N} . \\
\text{Then } (A : M)\hat{\cup}(N : M) \\
\text{and } (B : M)^n = (B^r : M)\hat{\cup}(N : M) \text{ for some positive integer } n . \\
\text{Since } (N : M) \text{ is a graded almost primary ideal then either } (A : M)(B : M)\hat{\cup}(N : M) \text{ or } (A : M)(B : M) \subseteq \langle (N : M) \rangle . \\
\text{If } (A : M)(B : M)\hat{\cup}(N : M) , \text{ we get } AB = (A : M)(B : M)\hat{\cup} (N : M)M = N \text{ which is a contradiction. If } (A : M)(B : M) \subseteq \langle (N : M) \rangle \text{ then } AB = (A : M)(B : M)M \subseteq \langle (N : M) \rangle N = \langle (N : M) \rangle \text{ which is a contradiction.}
\]

Conversely, by Theorem 2.11 it is enough to prove that \( (N : M) \) is a graded almost primary ideal of \( R \). Let \( rs \in (N : M) - \langle (N : M) \rangle \) where \( r, s \in h(R) \) want \( r^k \in (N : M) \) for some positive integer \( k \) or \( s \in (N : M) \). Let \( A = \langle r \rangle M \) and \( B = \langle s \rangle M \). Then \( AB = \langle r \rangle \langle s \rangle M \subseteq N \) and \( AB\hat{\cup}(N : M)N . \) If \( AB \subseteq N )M \), then \( \langle r \rangle \langle s \rangle \subseteq \langle (N : M) \rangle N \).
\[
\text{which is a contradiction. By assuming either } \langle r^k \rangle M = \langle (r)M \rangle ^k \subseteq N \text{ or } \langle s \rangle M \subseteq N \text{ implies } r^k \in (N : M) \text{ or } s \in (N : M) . \\ \text{Therefore } (N : M) \text{ is a graded almost primary ideal and so } N \text{ is a graded almost primary submodule of } M .
\]

**Corollary 2.13** Let \( M \) be a finitely generated faithful multiplication graded \( R \)-module and \( N \) be a proper graded submodule of \( M \). Then \( N \) is a graded almost primary submodule of \( M \) if and only if \( mm' \in N - (N : M)N \) implies \( m^k \in N \) for some positive integer \( k \) or \( m' \in N \) for \( m, m' \in h(M) \).

**Definition 2.14** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module and \( N \) a graded submodule of \( M \). We say that \( M \) is a graded torsion free \( R \)-module whenever \( a \in h(R) \) and \( m \in h(M) \) with \( am = 0 \) implies that either \( a = 0 \) or \( m = 0 \).

**Theorem 2.15** Let \( M \) be a graded torsion free \( R \)-module and \( N \) a graded submodule of \( M \). If \( N \) is a graded weakly prime submodule, then \( M / N \) is torsion free \( R(\langle N : M \rangle) \)-module.

**Proof** Let \( (r + (N : M))(m + N) = N \) where \( r \in h(R/(N : M)) \) and \( m \in h(M/N) \). If \( rm = 0 \), then \( r \in (N : M) \) or \( m \in N \), since \( M \) is a graded torsion free module. If \( rm \neq 0 \), then \( r \in (N : M) \) or \( m \in N \).
since \( N \) is a graded weakly prime submodule we obtain \( m \in N \) or \( r \in (N:M) \). Therefore, \( M/N \) is a graded torsion free module.

**Theorem 2.16** Let \( M \) be a graded torsion free \( R \)-module and \( N \) a graded submodule of \( M \). If \( M/N \) is a graded torsion free \( R/(N:M) \)-module. Then \( N \) is a graded weakly prime submodule of \( M \) if and only if \( (N:M) \) is a graded weakly prime ideal of \( R \).

**proof** Let \( 0 \neq ab \in (N:M) \) where \( a,b \in h(R) \). Then \( abM \subseteq N \). Since \( M \) is a graded torsion free module \( abM \neq 0 \). Since \( N \) is a graded weakly prime submodule \( a \in (N:M) \) or \( b \in (N:M) \).

Conversely, let \( 0 \neq rm \in N \) where \( r \in h(R) \) and \( m \in h(M) \). Then \( rm + N = (r + (N:M))(m + N) = N \). Since \( M/N \) is a graded torsion free module \( r \in (N:M) \) or \( m \in N \).

**Lemma 2.17** Let \( R \) be a graded domain and \( I \) a graded ideal of \( R \). If \( I \) is a graded weakly primary ideal of \( R \) then \( \sqrt{I} \) is a graded weakly primary ideal of \( R \).

**proof** Let \( 0 \neq ab \in \sqrt{I} \) and \( a \notin \sqrt{I} \) where \( a,b \in h(R) \). Then there is some positive integer \( n \) such that \( (ab)^n = a^n b^n \in I \) since \( R \) is a graded domain. As \( a^n \notin I \), \( I \) graded weakly primary gives \( b^n \in I \). Hence \( b \in \sqrt{I} \). Therefore \( \sqrt{I} \) is a graded weakly prime ideal of \( R \).

**Theorem 2.18** Let \( M \) be a graded torsion free \( R \)-module and \( N \) a graded submodule of \( M \). If \( N \) is a graded weakly primary submodule of \( M \) then \( (N:M) \) is a graded weakly primary ideal of \( R \).

**proof** Let \( 0 \neq ab \in (N:M) \) with \( b \notin (N:M) \) where \( a,b \in h(R) \), so there is \( m \in h(M) – h(N) \) such that \( bm \notin N \). As \( abm \in N \) and \( abm \neq 0 \) since \( M \) is a torsion free module. Then \( a^n \in (N:M) \) since \( N \) is a graded weakly primary submodule.

**Theorem 2.19** Let \( M \) be a graded multiplication torsion free \( R \)-module and \( N \), \( A \) and \( B \) be graded submodules of \( M \). Then \( N \) is a graded weakly primary submodule of \( M \) if and only if \( 0 \neq AB \subseteq N \) implies \( A \subseteq N \) or \( B \subseteq \sqrt{N} \).

**proof** Let \( N \) be a graded weakly primary submodule of \( M \) and \( 0 \neq AB \subseteq N \) for some graded submodules \( A \), \( B \) of \( M \). Suppose \( A = IM \) and \( B = JM \) for some graded ideals \( I \) and \( J \) of \( R \). Then we obtain \( 0 \neq AB = (IJ)M \subseteq N \). So \( IJ \subseteq (N:M) \). As \( 0 \neq AB = (IJ)M \), \( M \) is graded torsion free gives \( IJ \neq 0 \). Since \( (N:M) \) is a graded weakly primary ideal we get \( I \subseteq (N:M) \) or \( J \subseteq \sqrt{(N:M)} \). Then \( A \subseteq N \) or \( B \subseteq \sqrt{N} \).

Conversely, suppose if \( 0 \neq AB \subseteq N \) where \( A \) and \( B \) are graded submodules of \( M \) then \( A \subseteq N \) or \( B^n \subseteq N \) for some positive integer \( n \). Let \( IK \subseteq N \) for some graded ideal \( I \) of \( R \) and graded submodule \( K \) of \( M \). Since \( M \) is a multiplication module there is a graded ideal \( J \) of \( R \) such that \( K = JM \). Then we obtain \( 0 \neq (IM)(JM) = IJM = IK \subseteq N \) and hence \( JM \subseteq N \) or \( IM \subseteq \sqrt{N} = \sqrt{(N:M)}M = \sqrt{(N:M)}M \). Therefore, \( K \subseteq N \) or \( I \subseteq \sqrt{(N:M)} \).
Corollary 2.20 Let $M$ be a graded multiplication torsion free $R$–module and $N$ be a graded submodule of $M$. Then $N$ is a graded weakly primary submodule of $M$ if and only if for any $m,m' \in h(M)$ such that $0 \neq mm' \subseteq N$ implies $m' \subseteq N$ or $m^n \subseteq N$ for some positive integer $n$.

Theorem 2.21 Let $M$ be a graded multiplication torsion free $R$–module and $N$ be a graded submodule of $M$. Then $N$ is a graded weakly primary submodule of $M$ if and only if $M \neq N$ and every graded zero divisor in $M$ is nilpotent.

Proof Let $N$ be a proper graded weakly primary submodule. Then $M/N \neq 0$. Assume that $\overline{a} = a+N \in M/N$ is a graded zero divisor. Then there is a homogeneous element $\overline{b} = b+N \in M/N$ such that $\overline{ab} = 0$. Then $ab \subseteq N$, $b \notin N$. If $ab = 0$, then $a = 0$ since $M$ is a graded torsion free module. So $\overline{a} = 0$. If $ab \neq 0$ and since $N$ is a graded weakly primary submodule we get $a^4 \subseteq N$. Hence $\overline{a^k} = 0$.

Conversely, suppose $M/N \neq 0$ and every graded zero divisor in $M/N$ is nilpotent. Now let $0 \neq ab \subseteq N$ and $b \notin N$ where $a,b \in h(M)$. Then $(a+N)(b+N) = ab + N = \overline{0}$ and $b+N \neq \overline{0}$. By assuming, we get $a^k + N = (a+N)^k = 0$ for some positive integer $k$. Thus $a^k \subseteq N$.

Theorem 2.22 Let $M$ be a graded multiplication torsion free $R$–module and $N$ a graded submodule of $M$. If $N$ is a graded weakly primary submodule of $M$ then $\sqrt[N]{N}$ is a graded weakly prime submodule of $M$.

Proof Assume that $N$ is a graded weakly primary submodule. Then by Theorem 2.18, $(N:M)$ is a graded weakly primary ideal of $R$. And by Lemma 2.17, $\sqrt{(N:M)}$ is a graded weakly prime ideal. But $\sqrt[N]{(N:M)} = (M-\text{rad}(N:M))$. By Theorem 2.16, we obtain $M-\text{rad}(N)$ as a graded weakly prime submodule.

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