Research Article

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A numerical study of anomalous electro-diffusion cells in cable sense with a non-singular kernel

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Abstract: The time-fractional cable model is solved using an extended cubic B-spline (ECBS) collocation strategy. The B-spline function was used for space partitioning, while the Caputo-Fabrizio (CF) was used for temporal discretization. The finite difference technique was used to discretize the CF operator. For the first time in cable modeling, the CF operator has been used. In terms of time, the convergence of order $\tau$.

An ECBS collocation approach is investigated by numerical example at different values, and comparisons with published work are made. The numerical results show that the scheme performed well, and the graphical representations show that the results are very close to exact values. The Von Neumann technique is applied to investigate the stability of the proposed scheme.

Keywords: B-spline function, Caputo-Fabrizio, time-fractional cable model

MSC 2020: 35R11, 26A33, 65N12, 65N06

1 Introduction

Anomalous diffusion in cells has been observed in several research experiments, and this anomalous diffusion is primarily exhibited by fractional derivatives. Brownian motion can be used to explain the process of signal transmission to the insides of cells. As a consequence of the irregularity, the diffusion in the cells can become aberrant. The voltage transmission in axons is affected in this situation [1]. The fractional diffusion model has caught the interest of many academicians in recent years because of its numerous applications in various areas. The special case of time-fractional cable model [2] is one of these models that we will look at in this article.

\[
\frac{\partial U(y, t)}{\partial t} + C D_{t, t}^{\gamma} U(y, t) - \frac{\partial^2 U(y, t)}{\partial y^2} + U(y, t) = F(y, t), \quad y \in \Omega, \quad t > 0,
\]

with initial and boundary value conditions, respectively,

\[
\begin{cases}
U(y, 0) = U_0(y), & y \in \Omega, \\
U(y, t)|_{y=0} = 0, & t \geq 0,
\end{cases}
\]

where $\gamma > 0$ is a constant, $\Omega = [a, b]$, $U_0F$, is the smooth function. Also, $C D_{t, t}^{\gamma}$ is the $\gamma$th order Caputo-Fabrizio (CF) operator and can be defined as [3]:
where $N(y)$ is a normalizing function, such that $N(0) = N(1) = 1$.

If $G(t)$ is a constant function, then the CF derivative of $F(t)$ is zero, which is the same as the Caputo derivative. As a result, no singularity exists in the kernel.

The CF derivative has added a new dimension to the study of fractional differential equations (FDEs). The Caputo, Riemann-Liouville, and other singular kernel operators have kernels for power law that have limitations in representing physical situations [4]. It is produced using the exponential function and regular derivative convolution and it retains the same intrinsic inspirational qualities of heterogeneity and configuration for many scales as the Caputo and Riemann-Liouville fractional derivatives [5–8]. This derivative has already been mentioned in a number of articles, such as fractional Maxwell fluid [9], fluid flows [10], diffusive transport system [6], non-linear Fisher’s diffusion equation [11], and electric circuits [5]. To explain the electrotonic characteristics of spiny neural dendrites, Henry et al. [12] presented cable equations with fractional order temporal operators. The anomalous electrodiffusion can be alternatively modeled by a fractal-fractional model in [13,14]. He’s fractional derivative for the evolution equation has been adopted by Wang and Yao [15].

The time-fractional cable model is equivalent to the classic cable equation other than that the order of derivative is fractional with regard to time. Many fractional models have no analytical solutions due to the non-locality of fractional derivatives. Hence, numerical solutions to fractional differential model are important in terms of both science and practical application. As a result, several investigators have recently taken fractional differential models as a challenge and discovered numerical solutions to such problems. Two implicit compact difference techniques for solving fractional cable model were developed by Hu and Zhang [16]. Yu and Jiang [17] studied time-fractional cable model using fourth-order compact difference method. The local discontinuous Galerkin technique has been developed by Zheng and Zhao [18]. Yang et al. [19] proposed a temporal-spatial spectral-tau technique for solving fractional cable and its inverse model. Liu et al. [20,21] solved the fractional cable equations via finite element approximations and Grünwald difference approximations. Zhu et al. [22] presented the Galerkin finite element and convolution quadrature methods for the solution of fractional cable equation. Various researchers have worked on the solution of the FDEs by using numerical technique based on spline function [23–26]. Li [27] has studied the Wavelet collocation approach based on B-spline functions for solving the FDEs. B splines helped to resolve the unknowns in the equation. It was shown that the B-spline finite element is useful for the numerical solutions of FDE, particularly when continuity of the solutions was essential. Saad et al. [28] used a Atangana-Baleanu fractional derivative with spectral collocation methods to approximate the solution of the fractional Fisher’s type equations. The results indicate that the proposed methodology is a simple and effective tool for studying nonlinear equations with local and non-local singular kernels. Akram et al. [29,30] investigated time fractional cable equation and sub-diffusion equation using B-spline methods, Riemann-Liouville, and Caputo derivatives. The stability and convergence of the proposed methods have also been examined. Moshtaghi and Saadatmandi [31] developed the numerical technique for time fractional cable problem using Sinc-Bernoulli and Riemann-Liouville methods. A Galerkin method with spline function for solving time fractional diffusion equation has been proposed by Pezza and Pitolli [32]. Tabriz et al. [33] developed the operational matrix method based on Caputo derivative to solve fractional optimal control problem.

Despite the fact that B-spline has been used to solve fractional partial differential equations in several articles, we are not aware of any research that has used B-spline to solve time-fractional cable model equations. The B-spline function has very suitable properties for designing such convex hull and continuity properties [34]. Thus, the main objective of this research is to use the extended cubic B-spline (ECBS) to develop the collocation approach for the time-fractional cable model. The strategy’s main feature is that it converts such problems into algebraic system of equations that can be used in programming. The following is the organization of this research article: Section 2 presents the extended cubic B-spline (ECBS) functions and time approximation. The ECBS collocation method is presented in Section 3. The stability analysis is proved in Section 4. Sections 5 and 6 discuss the numerical experiment as well as the conclusion.
2 Preliminaries

2.1 ECBS functions

Assume that \( \{y_j\} \) is a division of equal length depending on the given interval with \( j \in \mathbb{Z} \). The assumed interval at the endpoints is split into \( N \) equivalent subintervals using \( y_j = y_0 + j\sigma \), where \( \sigma \) is the grid length. The basis function at the nodes \( y_j \) over the given interval is described as follows:

\[
E_k(y, \delta) = \frac{1}{24\sigma^5} \begin{cases}
4\sigma(1-\delta)(y-y_{j-2})^3 + 3\delta(y-y_{j-2})^4, & y \in [y_{j-2}, y_{j-1}), \\
(4-\delta)\sigma^4 + 12\sigma^2(y-y_{j-1}) + 6\sigma^2(2+\delta)(y-y_{j-1})^2 & y \in [y_{j-1}, y_j), \\
-12\sigma(y-y_{j-1})^3 - 3\delta(y-y_{j-1})^4 & y \in [y_j, y_{j+1}), \\
(4-\delta)\sigma^4 + 12\sigma^2(y_{j+1} - y) + 6\sigma^2(2+\delta)(y_{j+1} - y)^2 & y \in [y_{j+1}, y_{j+2}), \\
-12\sigma(y_{j+1} - y)^3 - 3\delta(y_{j+1} - y)^4 & y \in \{y_{j+1}, y_{j+2}\}, \\
4\sigma(1-\delta)(y_{j+2} - y)^3 + 3\delta(y_{j+2} - y)^4 & y \in [y_{j+2}, y_{j+3}), \\
0, & \text{otherwise},
\end{cases}
\]  

where \( j = -1(1)N + 1, \delta \in \mathbb{R} \) represent the free parameter in the \([-8, 1]\). \( y \in \mathbb{R} \) represents a variable. The extended and cubic B-spline functions have the same features for \( \delta \in [-8, 1] \). The ECBS changes to a cubic B-spline when \( \delta = 0 \). For a function \( U(y, t) \) there is a unique \( \hat{U}(y, t) \), which assures the given requirements, as

\[
\hat{U}(y, t) = \sum_{j=k-1}^{k+1} d_j(t)E_j(y, \delta).
\]  

The ECBS functions (4) and equation (5) yield the equations as follows:

\[
\hat{U}(y, t) = \sum_{j=k-1}^{k+1} d_j(t)E_j(y, \delta) = \left( \frac{4 - \delta}{2\sigma} \right) d_{k-1} + \left( \frac{8 + \delta}{12\sigma} \right) d_k + \left( \frac{4 - \delta}{2\sigma} \right) d_{k+1},
\]

\[
\hat{U}'(y, t) = \sum_{j=k-1}^{k+1} d_j(t)E_j'(y, \delta) = \left( -\frac{1}{2\sigma} \right) d_{k-1} + \left( \frac{1}{2\sigma} \right) d_{k+1},
\]

\[
\hat{U}''(y, t) = \sum_{j=k-1}^{k+1} d_j(t)E_j''(y, \delta) = \left( \frac{2 + \delta}{2\sigma^2} \right) d_{k-1} + \left( -\frac{4 + 2\delta}{2\sigma^2} \right) d_k + \left( \frac{2 + \delta}{2\sigma^2} \right) d_{k+1}.
\]

2.2 Difference method for the CF

In this section, we look at using CF to discretize fractional time derivatives. Assume that \( t_l = t_0 + l\tau \), \( l = 0, 1, \ldots, L \), where \( \tau = \frac{T}{L} \) is the grid size in time direction. The difference method is applied to discretize CF in the bounded domain. The CF can be expressed as follows using equation (3):

\[
\frac{\partial U(y, t_{l+1})}{\partial \tau} = \frac{1}{1 - \gamma} \int_0^{b_1} \frac{\partial U(y, \eta)}{\partial \eta} \exp \left[ -\frac{\gamma}{1 - \gamma} (t_{l+1} - \eta) \right] d\eta.
\]

The discretized formulation [23] is as follows:

\[
\frac{\partial U(y, t_{l+1})}{\partial \tau} = \frac{1}{\tau} \sum_{m=0}^l \epsilon_m [U(y, t_{l+m+1}) - U(y, t_{l+m})] \left( 1 - \exp \left[ -\frac{\gamma}{1 - \gamma} \right] \right) + G^{l+1}_{l+1}.
\]
The truncation error is [23] as follows:

\[ |G^{[r]}_t| \leq R\tau^2, \]  
(10)

where \( e_m = \exp\left[ -\frac{\tau}{1-\gamma} \right] \); it is simple to demonstrate the characteristics of \( e_m \) [23]:

- \( e_0 = 1 \),
- \( e_0 > e_1 > e_2 > \ldots > e_m, e_m \to 0 \) as \( m \to \infty \),
- \( e_m > 0 \) for \( m = 0, 1, \ldots, l \),
- \( \sum_{m=0}^{l}(e_m - e_{m+1}) + e_{l+1} = (1 - e_1) + \sum_{m=0}^{l-1}(e_m - e_{m+1}) + e_l = 1 \).

3 Derivation of the ECBS

To develop the numerical strategy for solving the cable model, we use the finite difference method, the CF, and the EBSM in this section. By plugging equations (9) and (4) in equation (1), we obtain

\[
\frac{\hat{U}(y, t_{l+1}) - \hat{U}(y, t_l)}{\tau} + \frac{1}{\tau y} \sum_{m=0}^{l} e_m \left( \hat{U}(y, t_{m+l}) - \hat{U}(y, t_m) \right) \left( 1 - \exp\left[ -\frac{y}{1-\gamma} \right] \right) - \hat{U}''(y, t_{l+1}) \\
+ \hat{U}''(y, t_{l+1}) = F(y, t_{l+1}).
\]  
(11)

The nonlinear term is linearized by Taylor’s series expansion as follows:

\[
(U^2)^{l+1} = (U^2)^l + \tau(U^2)^l + O(\tau)^2 \\
= (U^2)^l + \tau(2UU'') + O(\tau)^2 \\
= (U^2)^l + \tau \left( 2U'' \left( \frac{U^{l+1} - U^l}{\tau} + O(\tau) \right) \right) + O(\tau)^2 \\
= (U^2)^l + 2U''U^{l+1} - 2U''U^l + O(\tau)^2 \\
= 2U''U^{l+1} - U''U^l + O(\tau)^2,
\]

which implies that

\[
(U^2)^{l+1} = 2U''U^{l+1} - (U^2)^l.
\]  
(12)

Using equation (12) in (11), we obtain

\[
\frac{\hat{U}(y, t_{l+1}) - \hat{U}(y, t_l)}{\tau} + \frac{1}{\tau y} \sum_{m=0}^{l} e_m \left( \hat{U}(y, t_{m+l}) - \hat{U}(y, t_m) \right) \left( 1 - \exp\left[ -\frac{y}{1-\gamma} \right] \right) - \hat{U}''(y, t_{l+1}) \\
+ 2\hat{U}''(y, t_{l+1})\hat{U}(y, t_l) - \hat{U}''(y, t_l) = F(y, t_{l+1}).
\]

The aforementioned equation can be written in another form as:

\[
s\hat{U}^{l+1} + s_1 \tau \hat{U}^{j+1} - s_1 \tau \hat{U}''^{j+1} + 2rs \hat{U}''^{j+1} = s\hat{U}^l + s_1 \tau \hat{U}^l + s_1 \tau \sum_{m=1}^{l} e_m [\hat{U}^{l-m+1} - \hat{U}^{l-m}] + \tau sF^{l+1},
\]

where \( s_1 = \left( 1 - \exp\left[ -\frac{y}{1-\gamma} \right] \right) \) and \( s = \gamma \tau \).

\[
s \sum_{j=k+1}^{k+1} d_j^{l+1}E_j + s_1 \tau \sum_{j=k+1}^{k+1} d_j^{l+1}E_j - s_1 \tau \sum_{j=k-1}^{k+1} d_j^{l+1}E_j + 2rs \sum_{j=k-1}^{k+1} d_j^{l+1}E_j + 2rs \sum_{j=k-1}^{k+1} d_j^{l+1}E_j \\
= s \sum_{j=k-1}^{k+1} d_j^{l+1}E_j + s_1 \tau \sum_{j=k-1}^{k+1} d_j^{l+1}E_j + s_1 \tau \sum_{j=k-1}^{k+1} d_j^{l+1}E_j \\
+ \tau sF^{l+1} - s_1 \tau \sum_{m=1}^{l} e_m \left( \sum_{j=k-1}^{k+1} d_j^{l-m+1}E_j - \sum_{j=k-1}^{k+1} d_j^{l-m}E_j \right) \\
= s \sum_{j=k-1}^{k+1} d_j^{l+1}E_j + s_1 \tau \sum_{j=k-1}^{k+1} d_j^{l+1}E_j + s_1 \tau \sum_{j=k-1}^{k+1} d_j^{l+1}E_j \\
+ \tau sF^{l+1} - s_1 \tau \sum_{m=1}^{l} e_m \left( \sum_{j=k-1}^{k+1} d_j^{l-m+1}E_j - \sum_{j=k-1}^{k+1} d_j^{l-m}E_j \right). \\
\]  
(13)
The aforementioned system of equations have the order \((N + 1) \times (N + 3)\). In order to obtain a unique solution, two equations are required which can be derived from the boundary conditions. The following two boundary conditions are needed to solve the aforementioned system uniquely.

\[
\begin{bmatrix}
\left(\frac{4 - \delta}{24}\right) d_{-1} + \left(\frac{8 + \delta}{12}\right) d_{0} + \left(\frac{4 - \delta}{24}\right) d_{1} = U(y_{0}, t), \\
\left(\frac{4 - \delta}{24}\right) d_{N-1} + \left(\frac{8 + \delta}{12}\right) d_{N} + \left(\frac{4 - \delta}{24}\right) d_{N+1} = U(y_{N}, t).
\end{bmatrix}
\]

Hence, we have the \((N + 3) \times (N + 3)\) order system. Now, we will employ the initial vector from the initial conditions to start the iteration on system (13).

### 3.1 Algorithm

- Describe the model.
- Discretize the modeled problem using the CF operator in time direction.
- Linearize the nonlinear terms.
- Discretize the problem using ECBS formulation in space direction.
- Apply the two boundary conditions.
- Resulting system of order \((N + 3) \times (N + 3)\) is obtained.
- Solve the aforementioned system using Mathematica.

### 4 Stability and convergence analysis

The stability concept is linked to computation approach errors that do not increase as the procedure continues. The Von Neumann stability technique also known as Fourier stability is a method utilized to analyze the stability of difference techniques. Here, we will use the Von Neumann technique to investigate the stability of the proposed method. Assume that \(\eta^{l}\) is the growth factor in the Fourier mode form, and that \(\hat{\eta}^{l}\) is the obtained value. As a result, we defined the error concept at the \(l\)th time step as

\[
\zeta^{l} = \eta^{l} - \hat{\eta}^{l}.
\]

To linearize the non-linear term in (11), we use \(U^{0} = aU\), while \(a\) is a positive constant, we obtain

\[
\dot{U}(y, t_{i+1}) - \dot{U}(y, t_{i}) + \frac{\tau}{s} \sum_{m=0}^{l} \varepsilon_{ml} \dot{U}(y, t_{i-m}) - \dot{\eta}^{l} (y, t_{i+1}) + \tau a \dot{\eta}^{l} (y, t_{i+1}) = \tau \dot{F}(y, t_{i+1}).
\]

The aforementioned equation becomes

\[
s \dot{\zeta}^{l+1} + s_{i} \tau \dot{\zeta}^{l+1} - st \dot{\zeta}^{l+1} + ta \dot{\zeta}^{l+1} = s \dot{\zeta}^{l} + s_{i} \tau \dot{\zeta}^{l} - s_{i} \tau \sum_{m=1}^{l} \varepsilon_{ml} [\dot{\zeta}^{l-m+1} - \dot{\zeta}^{l-m}] + \tau \dot{F}^{l+1}.
\]

By inserting equation (14) in equation (15), we obtain the following error equation:

\[
s \dot{\zeta}^{l+1} + s_{i} \tau \dot{\zeta}^{l+1} - st \dot{\zeta}^{l+1} + ta \dot{\zeta}^{l+1} = s \dot{\zeta}^{l} + s_{i} \tau \dot{\zeta}^{l} - s_{i} \tau \sum_{m=1}^{l} \varepsilon_{ml} [\dot{\zeta}^{l-m+1} - \dot{\zeta}^{l-m}] + \tau \dot{F}^{l+1}.
\]

Consider the error solution for the B-spline can be written in one Fourier mode as

\[
\zeta^{l} = \Phi^{l} e^{i \omega y},
\]

\(\omega\) is the wave number.
where $\Phi, \sigma, \rho$ and $i = \sqrt{-1}$ are the Fourier coefficient, element size, and mode number, respectively. By combining equation (17) and B-spline functions in (16), we obtain the following result:

$$s\Phi^{l+1}(E_e^{l+1}(j-1)\alpha p + E_\rho^{l+1}(j-1)\alpha p) + s_i\tau \Phi^{l+1}(E_e^{l}(j-1)\alpha p + E_\rho^{l}(j-1)\alpha p)$$

$$- s\tau(E_e^{l+1}(j-1)\alpha p + E_\rho^{l+1}(j-1)\alpha p)\Phi^{l+1} + s\tau(E_e^{l}(j-1)\alpha p + E_\rho^{l}(j-1)\alpha p)\Phi^{l+1}$$

$$= s\Phi^{l}(E_e^{l}(j-1)\alpha p + E_\rho^{l}(j-1)\alpha p) + s_i\tau \Phi^{l}(E_e^{l}(j-1)\alpha p + E_\rho^{l}(j-1)\alpha p)$$

$$- s\tau \sum_{m=1}^{l} E_e^{m}(E_e^{l}(j-1)\alpha p + E_\rho^{l}(j-1)\alpha p)\Phi^{l-m} - \Phi^{l-m},$$

where $E_1 = \frac{4-\delta}{2\sigma}, E_2 = \frac{8+\delta}{12}, E_3 = \frac{1}{2\sigma} + \frac{2+\delta}{2\sigma^2}, E_5 = -\frac{2+\delta}{\sigma^2}$. By calculating and rearranging, we obtain

$$(s + s_i\tau + \tau\alpha)\Phi^{l+1}(E_2 + 2E_1\cos(\varepsilon\sigma)) - s\tau(E_2 + 2E_1\cos(\varepsilon\sigma))\Phi^{l+1}$$

$$= s\Phi^{l}(E_2 + 2E_1\cos(\varepsilon\sigma)) + s_i\tau E_2(E_2 + 2E_1\cos(\varepsilon\sigma))\Phi^{l} + s\tau(E_2 + 2E_1\cos(\varepsilon\sigma))\sum_{m=0}^{l-1} [\varepsilon m - \varepsilon m] \Phi^{l-m}. $$

Taking the part that is identical on both sides and dividing it by $(s + s_i\tau + \tau\alpha)\Phi^{l+1}$, we obtain

$$[(s + s_i\tau + \tau\alpha) + \tau\beta] \Phi^{l+1} = s\Phi^{l} + s_i\tau E_2\Phi^{l} + \sum_{m=0}^{l-1} [\varepsilon m - \varepsilon m] \Phi^{l-m},$$

(18)

where $\beta = \frac{\delta(2+\delta)}{\sigma(6+2\delta)\sin(\sigma\delta/2)} > 0, \delta \not= -2$.

**Proposition 4.1.** Consider $\Phi^l, l = 0, 1, ..., L$ be the solution of model problem (1), we write

$$|\Phi^l| \leq |\Phi^0|, \quad l = 0, 1, ..., L.$$

(19)

**Proof.** Here, we use mathematical induction to validate this result. The following equation can be obtained when we choose $l = 0$ in equation (18):

$$[(s + s_i\tau + \tau\alpha) + \tau\beta] \Phi^{l+1} = s\Phi^{l} + s_i\tau E_2\Phi^{l} + \sum_{m=0}^{l-1} [\varepsilon m - \varepsilon m] \Phi^{l-m},$$

$$\Phi^{l+1} = \frac{s + s_i\tau E_2}{s + s_i\tau + \tau\alpha + \tau\beta} \Phi^{l},$$

$$|\Phi^l| \leq |\Phi^0|. \quad \therefore \varepsilon_0 = 1.$$

Suppose that $|\Phi^0| \leq |\Phi^0|$ is true for $l = 0, 1, ..., L - 1$. For $l + 1$, we have

$$[(s + s_i\tau + \tau\alpha) + \tau\beta] \Phi^{l+1} = s\Phi^{l} + s_i\tau E_2\Phi^{l} + \sum_{m=0}^{l-1} [\varepsilon m - \varepsilon m] \Phi^{l-m},$$

$$\leq s\Phi^{l} + s\tau \left( \sum_{m=0}^{l-1} [\varepsilon m - \varepsilon m] \right)|\Phi^0|,$$

$$|\Phi^{l+1}| \leq |\Phi^0|.$$  

Thus, $|\Phi^{l+1}| = |\varepsilon_1^{l+1}| \leq |\Phi^0| = |\varepsilon_1^0|$, so that $|\Phi^{l+1}|_2 = |\Phi^0|_2$. As a result, the presented approach is unconditionally stable.

**Theorem 1.** Let $U(y_j, \tau^l)$ be the exact solution and $\tilde{U}^l$ be the time discrete result of the given equation, then

$$\|Y^{l+1}\| \leq Br,$$

(20)

where $Y^{l+1} = U(y_j, \tau^l) - \tilde{U}^{l+1}$. 
Proof. The non-linear term in (11) can be linearized as \( U^2 = \alpha U \), where \( \alpha \) is a positive constant, we obtain the difference equation of time discrete solution and exact solution as follows:

\[
SY^{l+1} + s_1rY^{l+1} - srY^l + \tau \alpha Y^{l+1} = SY^l + s_1rY^0 + s_1r \sum_{m=1}^{l-1} [e_m - e_{m+1}]Y^{l-m} + \tau R^{l+1}.
\]  

(21)

For \( l = 0 \), we obtain

\[
SY^1 + s_1rY^1 - srY^0 + \tau \alpha Y^1 = SY^0 + s_1rY^0 + \tau R^1, \quad \therefore Y = 0.
\]

Taking inner product of the aforementioned equation with \( Y \), we obtain

\[
s\langle Y^1, Y^1 \rangle + s_1r\langle Y^1, Y^1 \rangle - sr\langle Y^0, Y^1 \rangle + \tau \alpha \langle Y^1, Y^1 \rangle = \tau \langle R^1, Y^1 \rangle.
\]

For \( \langle u_{yy}, u \rangle = -\langle u_y, u_y \rangle, \langle u, u \rangle = \|u\|^2 \), and \( \langle u, u \rangle \leq \|u\| \cdot \|u\| \), the following result is obtained:

\[
s\|Y^1\|^2 + s_1r\|Y^0\|^2 + \tau \alpha \|Y^1\|^2 \leq \tau \|R^1\|\|Y^1\|,
\]

\[
\|Y^1\|^2 \leq \frac{\tau}{s + s_1r + \tau \alpha} \|R^1\| \|Y^1\|,
\]

which implies that

\[
\|Y^1\| \leq B_1 \tau, \tag{22}
\]

where \( B_1 \) is a constant free of \( \tau \). Assume that (22) is true when \( l = 0, 1, \ldots, L - 1 \). Taking the inner product of (21) with \( Y^{l+1} \), we obtain

\[
s\langle Y^{l+1}, Y^{l+1} \rangle + s_1r\langle Y^{l+1}, Y^{l+1} \rangle - sr\langle Y^l, Y^{l+1} \rangle + \tau \alpha \langle Y^{l+1}, Y^{l+1} \rangle
\]

\[
= s\langle Y^l, Y^{l+1} \rangle + s_1r\langle Y^0, Y^{l+1} \rangle + s_1r \sum_{m=1}^{l-1} [e_m - e_{m+1}]\langle Y^{l-m}, Y^{l+1} \rangle + \tau \langle R^{l+1}, Y^{l+1} \rangle.
\]

Using \( Y^0 = 0, \langle u_{yy}, u \rangle = -\langle u_y, u_y \rangle, \langle u, u \rangle = \|u\|^2 \), and \( \langle u, u \rangle \leq \|u\| \cdot \|u\| \), we obtain

\[
s\langle Y^{l+1}, Y^{l+1} \rangle + s_1r\langle Y^{l+1}, Y^{l+1} \rangle + \tau \alpha \langle Y^{l+1}, Y^{l+1} \rangle
\]

\[
= -sr\langle Y^l, Y^{l+1} \rangle + s\langle Y^l, Y^{l+1} \rangle + s_1r \sum_{m=1}^{l-1} [e_m - e_{m+1}]\langle Y^{l-m}, Y^{l+1} \rangle + \tau \langle R^{l+1}, Y^{l+1} \rangle,
\]

which implies that

\[
s\|Y^{l+1}\|^2 + s_1r\|Y^{l+1}\|^2 + \tau \alpha \|Y^{l+1}\|^2 \leq s\|Y^l\|^2 + s_1r \sum_{m=1}^{l-1} [e_m - e_{m+1}]\|Y^{l-m}\|^2 \|Y^{l+1}\|^2 + \tau \|R^{l+1}\|\|Y^{l+1}\|.
\]

By dividing throughout with \|\|Y^{l+1}\|\|, we obtain

\[
s\|Y^{l+1}\| + s_1r\|Y^{l+1}\| + \tau \alpha \|Y^{l+1}\| \leq s\|Y^l\|^2 + s_1r \sum_{m=0}^{l-1} [e_m - e_{m+1}]\|Y^{l-m}\|^2 + \tau \|R^{l+1}\|, \tag{22}
\]

\[
\frac{s + s_1r + \tau \alpha}{s_1r} \|Y^{l+1}\| \leq \frac{s}{s_1r} \|Y^l\|^2 + \sum_{m=0}^{l-1} [e_m - e_{m+1}]\|Y^{l-m}\|^2 + \frac{\|R^{l+1}\|}{s_1}.
\]
By Gronwall’s inequality, we obtain
\[
\frac{s + s \tau + \alpha \tau}{s \tau} \| Y^{t+1} \| \leq \left( \frac{s}{s \tau} \| Y \| + \| R^{t+1} \| \right) \exp \left( \sum_{m=0}^{t-1} (\varepsilon_m - \varepsilon_{m+1}) \right)
\]
\[
\| Y^{t+1} \| \leq \left( \frac{s \| Y \| + \tau \| R^{t+1} \|}{s + s \tau + \alpha \tau} \right) \exp(1 - \varepsilon_t),
\]
\[
\| Y^{t+1} \| \leq \left( \frac{s \| Y \| + \tau R^{t+1} \tau^2}{s + s \tau + \alpha \tau} \right) \exp(1 - \varepsilon_t),
\]
\[
\| Y^{t+1} \| \leq B \tau.
\]

5 Numerical results

In this section, we incorporate the simulation findings of given problem by using the provided scheme. The numerical findings are used to test the claimed method validity and efficiency. The main goal of these examples is to examine the degree of convergence of the estimated results for \( y \).

Example 1.

\[
\frac{\partial U(y, t)}{\partial t} + \frac{\partial}{\partial y} \left( D_y U(y, t) \right) - \frac{\partial^2 U(y, t)}{\partial y^2} + U^2(y, t) = F(y, t), \quad y \in \Omega = [0, 2\pi], \ t \in [0, 1],
\]

with

\[ U(y, 0) = U_0 = 0, \]

and

\[ F(y, t) = 2 \left( t + \frac{t}{y} - \frac{(1 - y)}{y} \left( 1 - e^{i \pi y} \right) \right) \sin y + t^2 \sin y(1 + t^2 \sin y). \]

| \( r \) | \( L_2 \) [2] | Proposed method \( L_2 \) | Proposed method \( L_\infty \) | Order |
|---|---|---|---|---|
| 1/5 | 1.4927 \times 10^{-1} | 3.94746 \times 10^{-3} | 3.64701 \times 10^{-2} | ... |
| 1/10 | 4.0466 \times 10^{-2} | 2.16374 \times 10^{-3} | 1.79683 \times 10^{-2} | 1.02126 |
| 1/20 | 1.0758 \times 10^{-2} | 1.30405 \times 10^{-3} | 8.44512 \times 10^{-3} | 1.08926 |
| 1/40 | 2.7868 \times 10^{-3} | 4.66346 \times 10^{-4} | 4.16857 \times 10^{-3} | 1.01856 |

| \( r \) | \( L_2 \) [2] | Proposed method \( L_2 \) | Proposed method \( L_\infty \) | Order |
|---|---|---|---|---|
| 1/5 | 1.5542 \times 10^{-1} | 8.99805 \times 10^{-3} | 5.85797 \times 10^{-2} | ... |
| 1/10 | 4.3922 \times 10^{-2} | 3.37174 \times 10^{-3} | 2.84031 \times 10^{-2} | 1.04435 |
| 1/20 | 1.1966 \times 10^{-2} | 1.72413 \times 10^{-3} | 1.36493 \times 10^{-3} | 1.05722 |
| 1/40 | 3.1359 \times 10^{-3} | 9.17447 \times 10^{-4} | 6.59511 \times 10^{-3} | 1.04936 |
The exact solution of the problem is:

\[ U(y, t) = t^2 \sin y. \]

Figure 1: Exact and approximated solution of Example 1 for value of \( \gamma = 0.5 \), at \( T = 1 \).

Figure 2: Space-time absolute error plot for value of \( \gamma = 0.2, 0.4 \), respectively, at \( T = 1 \).
Figure 3: Space-time absolute error plot for value of $\gamma = 0.5, 0.6$, respectively, at $T = 1$.

By taking $y = 0.5, \sigma = 0.05$, in Table 1, we show the $L_2, L_\infty$ and convergence order at different time-steps. The comparison of $L_2$ error norm has also presented at $T = 1$. In Table 2, we demonstrate the $L_2, L_\infty$ and convergence order and comparison of $L_2$ error for fixed $\sigma = \frac{L}{50}, y = 0.01$ when $T = 1$, Figure 1 depicts the comparison of approximated and exact results with $y = 0.5, \frac{L}{50}$ when $T = 1$. Figures 2 and 3 show that the absolute error graph of Example 1 for $\sigma = \frac{2\pi}{50}, y = 0.2, 0.4, 0.5, 0.6$ at $T = 1$. Figure 4 displays the contour plot for $y = 0.5, \sigma = \frac{2\pi}{25}, L = 25$ at $T = 1$ (Table 3).

6 Conclusion

This study describes an ECBS collocation strategy for solving the time-fractional cable model. For space partitioning, the ECBS was used, whereas for temporal discretization, the CF was applied. The CF operator was discretized using the finite difference technique. The CF operator has been utilized for the first time in the cable model. The technique has order $\tau$ in time direction. Therefore, the ECBS with CF has reliable computational outcomes; the theoretical and numerical results indicate its compatibility with each other. The graphical representations also show that the results are very near to exact values.
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