Unifying Lower Bounds on Prediction Dimension of Consistent Convex Surrogates

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Abstract
Given a prediction task, understanding when one can and cannot design a consistent convex surrogate loss, particularly a low-dimensional one, is an important and active area of machine learning research. The prediction task may be given as a target loss, as in classification and structured prediction, or simply as a (conditional) statistic of the data, as in risk measure estimation. These two scenarios typically involve different techniques for designing and analyzing surrogate losses. We unify these settings using tools from property elicitation, and give a general lower bound on prediction dimension. Our lower bound tightens existing results in the case of discrete predictions, showing that previous calibration-based bounds can largely be recovered via property elicitation. For continuous estimation, our lower bound resolves an open problem on estimating measures of risk and uncertainty.

1. Introduction
A surrogate loss function is an error measure that is related but not identical to one’s target problem of interest. Selecting a hypothesis by minimizing surrogate risk is one of the most widespread techniques in supervised machine learning. There are two main reasons why a surrogate loss is necessary: (1) the target loss does not satisfy some desiderata, such as convexity, or (2) the goal is to estimate some target statistic and there is no target loss, as in many continuous estimation problems. In both settings, a key criteria for choosing a surrogate loss is consistency, a precursor to excess risk bounds and convergence rates. Roughly speaking, consistency means that minimizing surrogate risk corresponds to solving the target problem of interest, i.e. in (1) the target risk is also minimized, or in (2) the continuous prediction approaches the true conditional statistic.

Despite the ubiquity of surrogate losses, we lack general frameworks to design and analyze consistent surrogates. This state of affairs is especially dire when one seeks low prediction dimension, the dimension of the surrogate prediction domain. For example, in multiclass classification with \( n \) labels, the prediction domain might be \( \mathbb{R}^n \). In many type (1) settings, such as structured prediction and extreme classification, the prediction dimension can easily become intractably large, forcing one to sacrifice consistency for computational efficiency. To understand whether this sacrifice is necessary, recent work developed tools like the feasible subspace dimension to lower bound the prediction dimension of any consistent convex surrogate (Ramaswamy and Agarwal, 2016). Challenges of type (2) include risk measures such as conditional value at risk (CVaR), with applications in financial regulation, robust
engineering design, and algorithmic fairness. Risk measures provably cannot be specified via a target loss, and thus we seek a surrogate loss of low (or at least finite) prediction dimension. Recent work (Fissler et al., 2016; Frongillo and Kash, 2020) gives prediction dimension bounds for some of these risk measures, but without the requirement that the surrogate be convex: bounds for convex surrogates are left as a major open question.

We present a unification of existing techniques to bound the prediction dimension of consistent convex surrogates in both settings above. Applied to settings of type (1), we recover the feasible subspace dimension result of Ramaswamy and Agarwal (2016), and give an example where our bound is even tighter. For type (2), we give the first prediction dimension bounds for risk measures with respect to convex surrogates, addressing the open question above. Our framework rests on property elicitation, a weaker condition than calibration, as a tool to understand consistency across a wide variety of domains.

The “four quadrants” of problem types Above, we discuss a significant divergence in previous frameworks: constructing a surrogate given a target loss versus a target statistic. In addition to the two possible targets, we may have one of two domains: a discrete (i.e. finite) target prediction space, like a classification problem, or a continuous one, like a regression or estimation problem. We informally refer to the four resulting cases—target loss vs. target statistic, and discrete vs. continuous predictions—as the “four quadrants” of supervised learning problems, shown in Table 1. For further examples, see Appendix E.

Literature on consistency and calibration We focus on the construction of consistent surrogate losses $L : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R}$, roughly meaning that minimizing $L$-loss corresponds to solving the target problem of interest. When given a target loss $\ell$, we roughly define $L$ to be consistent if minimizing $L$, and applying a link function, minimizes $\ell$ (Definition 5) (Zhang, 2004; Bartlett et al., 2006; Tewari and Bartlett, 2007; Steinwart, 2007; Ramaswamy and Agarwal, 2016). When given a target statistic such as the conditional quantile or variance, but no target loss, we introduce a notion of consistency in line with classical statistics (Definition 6) (Györfi et al., 2006; Fan and Yao, 1998; Ruppert et al., 1997). Here we will define $L$ to be consistent if minimizing $L$ and applying a link function yields estimates converging to the correct value.

A priori, it is not clear that compatible definitions of consistency could be given for both target statistics and target losses. In fact, we observe that consistency for target losses is a special case of consistency for target statistics (§ 3). This observation suggests property elicitation (see § 2.1) as a useful tool to study general lower bounds.

As definitions of consistency are relatively intractable to apply directly, the literature often focuses on a weaker condition called calibration, which only applies when given a discrete target loss, e.g. Quadrants 1 and 3. Particularly, Zhang (2004); Lin (2004); Bartlett et al. (2006); Tewari and Bartlett (2007); Ramaswamy and Agarwal (2016) show the equivalence of consistency and calibration in Quadrant 1, where one is given a target statistic and discrete prediction set. We discuss the additional relationship of elicitation and calibration in Appendix A, and derive Theorem 8 via calibration.

Contributions First, we formalize a notion of consistency with respect to a target statistic (Definition 6) and show its relationship to consistency with respect to a target loss (Lemma 7). We then show indirect elicitation is a necessary condition for consistency (Theorem 8). With
Table 1: The four quadrants of problem types, with an example of interest for each.

| Discrete prediction | Target loss | Target statistic |
|---------------------|-------------|------------------|
| Q1: Classification  | Q2: Risk-averse classification (Appendix E) |
| Continuous estimation | Q3: Least-squares regression |
| Q4: Variance estimation |

Figure 1: Flow and implications of our results. Compared to calibration, we suggest indirect elicitation as a simpler but almost-as-powerful necessary condition for consistency. In particular, we obtain a testable necessary condition, based on \( d \)-flats, for whether there exists a \( d \)-dimensional consistent convex surrogate. This condition recovers and strengthens existing calibration-based results.

these tools in hand, we present a new framework for deriving lower bounds on the prediction dimension of consistent convex surrogates (Corollaries 12 and 13) via indirect elicitation. These bounds are the first to our knowledge that can be applied in all four quadrants. Moreover, our framework can also give tighter bounds than previously existed in the literature. We illustrate this sharpness with new bounds for well-studied problems such as abstain loss (§ 5) and variance, CVaR, and other measures of risk and uncertainty (§ 6). See Figure 1 for a roadmap of our main results.

2. Setting

We consider supervised learning problems in the space \( \mathcal{X} \times \mathcal{Y} \), for some feature space \( \mathcal{X} \) and a label space \( \mathcal{Y} \), with data drawn from a distribution \( D \) over \( \mathcal{X} \times \mathcal{Y} \). The task is to produce a hypothesis \( f : \mathcal{X} \to \mathcal{R} \), for some prediction space \( \mathcal{R} \), which may be different from \( \mathcal{Y} \). For example, in ranking problems, \( \mathcal{R} \) may be all \(|\mathcal{Y}| \) ! permutations over the \(|\mathcal{Y}| \) labels forming \( \mathcal{Y} \). As we focus on conditional distributions \( p := D_x = \Pr[Y \mid X = x] \) over \( \mathcal{Y} \) given some \( x \in \mathcal{X} \), we often abstract away \( x \), working directly with a convex set of distributions over outcomes \( \mathcal{P} \subseteq \Delta_Y \). We then write e.g. \( \mathbb{E}_p L(\cdot, Y) \) to mean the expectation when \( Y \sim p \).

If given, we use \( \ell : \mathcal{R} \times \mathcal{Y} \to \mathbb{R} \) to denote a target loss, with predictions \( r \in \mathcal{R} \). Similarly, \( L : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R} \) will typically denote a surrogate loss, with surrogate predictions \( u \in \mathbb{R}^d \). We write \( \mathcal{L}_d \) for the set of \( \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Y} \)-measurable and lower semi-continuous surrogates \( L : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R} \) such that \( \mathbb{E}_{Y \sim p} L(u, Y) < \infty \) for all \( u \in \mathbb{R}^d, p \in \mathcal{P} \), that are minimizable in that \( \arg\min_u \mathbb{E}_p L(u, Y) \) is nonempty for all \( p \in \mathcal{P} \). Moreover, \( \mathcal{L}_{cvx}^d \subseteq \mathcal{L}_d \) is the set of convex (in \( \mathbb{R}^d \) for every \( y \in \mathcal{Y} \)) losses in \( \mathcal{L}_d \). Set \( \mathcal{L} = \bigcup_{d \in \mathbb{N}} \mathcal{L}_d \), and \( \mathcal{L}_{cvx} = \bigcup_{d \in \mathbb{N}} \mathcal{L}_{cvx}^d \). A loss \( \ell : \mathcal{R} \times \mathcal{Y} \to \mathbb{R} \) is discrete if \( \mathcal{R} \) is a finite set. For a given \( p \in \mathcal{P} \), the (conditional) regret, or
excess risk, of a loss $L$ is given by $R_L(u,p) := \mathbb{E}_p L(u,Y) - \inf_{u^*} \mathbb{E}_p L(u^*,Y)$. Typically, we notate finite report sets $\mathcal{R}$.

### 2.1. Property elicitation

Arising from the statistics and economics literature, property elicitation is similar to calibration, but only characterizes exact minimizers of a surrogate (Savage, 1971; Osband and Reichelstein, 1985; Lambert et al., 2008; Lambert and Shoham, 2009; Lambert, 2018; Frongillo and Kash, 2015, 2014). Specifically, given a statistic or property $\Gamma$ of interest, which maps a distribution $p \in \mathcal{P} \subseteq \Delta \mathcal{Y}$ to the set of desired or correct predictions, the minimizers of $L$ should precisely coincide with $\Gamma$. For example, squared loss $L(r,y) = (r - y)^2$ elicits the mean $\Gamma(p) = \mathbb{E}_p Y$. For intuition, to relate to consistency, one can think of $p = \Pr[Y \mid X = x]$ as a conditional distribution, though the definition is also applied to point prediction settings.

**Definition 1 (Property, elicits)**: A property is a set-valued function $\Gamma : \mathcal{P} \to 2^{\mathcal{R}} \setminus \{\emptyset\}$, which we denote $\Gamma : \mathcal{P} \rightrightarrows \mathcal{R}$. A loss $L : \mathcal{R} \times \mathcal{Y} \to \mathbb{R}$ elicits the property $\Gamma$ if

$$\forall p \in \mathcal{P}, \Gamma(p) = \arg \min_{u \in \mathcal{R}} \mathbb{E}_p L(u,Y). \quad (1)$$

An example is the mean, $\Gamma(p) = \{\mathbb{E}_p Y\}$. The level set of $\Gamma$ at value $r \in \mathcal{R}$ is $\Gamma_r := \{p \in \mathcal{P} : r \in \Gamma(p)\}$. We call a property $\Gamma : \mathcal{P} \rightrightarrows \mathcal{R}$ discrete if $\mathcal{R}$ is a finite set, as in Quadrants 1 and 2. A property is single-valued if $|\Gamma(p)| = 1$ for all $p \in \mathcal{P}$, in which case we may write $\Gamma : \mathcal{P} \to \mathcal{R}$ and $\Gamma(p) \in \mathcal{R}$. The mean is single-valued. We define the range of a property by $\text{range} \Gamma = \bigcup_{p \in \mathcal{P}} \Gamma(p) \subseteq \mathcal{R}$. When $L \in \mathcal{L}$, we use $\Gamma := \text{prop}_P[L]$ to denote the unique property elicited by $L$ (for distributions in $\mathcal{P}$) from eq. (1). Typically, we denote the target property by $\gamma$, and the surrogate by $\Gamma$.

To relate property elicitation to consistency, we need to allow for a link function, which gives rise to the notion of indirect elicitation. For single-valued properties, this definition reduces to the natural requirement $\gamma = \psi \circ \Gamma$.

**Definition 2 (Indirect Elicitation)**: A surrogate loss and link $(L, \psi)$ indirectly elicit a property $\gamma : \mathcal{P} \rightrightarrows \mathcal{R}$ if $L$ elicits a property $\Gamma : \mathcal{P} \rightrightarrows \mathcal{R}^d$ such that for all $u \in \mathcal{R}^d$, we have $\Gamma_u \subseteq \gamma_{\psi(u)}$. We say $L$ indirectly elicits $\gamma$ if such a link $\psi$ exists.

An important caveat to the above definitions is that, since $\Gamma = \text{prop}_P[L]$ is nonempty everywhere, we must have $L \in \mathcal{L}$, meaning that $\mathbb{E}_p L(\cdot,Y)$ always achieves a minimum. This restriction is also implicit in e.g. (Agarwal and Agarwal, 2015). While some popular surrogates such as logistic and exponential loss are not minimizable, these losses are still covered in Corollary 13 and Theorem 17 as $\Gamma(p) \neq \emptyset$ when $p \in \mathcal{P} := \text{relint}(\Delta \mathcal{Y})$; moreover, by thresholding $L'(u,y) = \max(L(u,y),\epsilon)$ for sufficiently small $\epsilon > 0$ we can achieve $L' \in \mathcal{L}$ for both. We expect that a generalization of property elicitation which allows for “infinite” predictions (e.g., along a prescribed ray), thereby ensuring a minimum is always achieved for convex losses, would allow us to lift the minimizable restriction entirely.

### 2.2. Convex consistency dimension and elicitation complexity

Various works have studied the minimum prediction dimension $d$ needed in order to construct a consistent surrogate loss $L : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R}$, typically through proxies such as
calibration (Steinwart and Christmann, 2008; Agarwal and Agarwal, 2015; Ramaswamy and Agarwal, 2016) and property elicitation (Frongillo and Kash, 2015; Fissler et al., 2016; Frongillo and Kash, 2020). In Quadrant 1, Ramaswamy and Agarwal (2016) introduce a special case of convex consistency dimension (Definition 3), which led to consistent convex surrogates for discrete prediction problems such as hierarchical classification (Ramaswamy et al., 2015) and classification with an abstain option (Ramaswamy et al., 2018).

**Definition 3 (Convex Consistency Dimension)**  Given target loss $\ell : \mathcal{R} \times \mathcal{Y} \to \mathcal{R}$ or property $\gamma : \mathcal{P} \Rightarrow \mathcal{R}$, its convex consistency dimension $\text{cons}_{\text{cvx}}(\cdot)$ is the minimum dimension $d$ such that $\exists L \in \mathcal{L}_d^{\text{cvx}}$ and link $\psi$ such that $(L, \psi)$ is consistent with respect to $\ell$ or $\gamma$.

Consistency is defined for a target loss in Definition 5 and for a target property in Definition 6.

In the case of a target property $\gamma$, i.e. a statistic, Lambert et al. (2008) similarly introduce the notion of elicitation complexity, later generalized by Frongillo and Kash (2020), which captures the lowest prediction dimension of a surrogate which indirectly elicits $\gamma$. This notion is quite general as it includes continuous estimation settings and does not inherently depend on a target loss being given.

**Definition 4 (Convex Elicitation Complexity)**  Given a target property $\gamma$, the convex elicitation complexity $\text{elic}_{\text{cvx}}(\gamma)$ is the minimum dimension $d$ such that there is a $L \in \mathcal{L}_d^{\text{cvx}}$ indirectly eliciting $\gamma$.

Agarwal and Agarwal (2015, Corollary 10) provide a necessary condition for the direct convex elicitation of single-valued properties, yielding bounds on the dimensionality of level sets. Moreover, Finocchiaro et al. (2019) study surrogate losses which embed a discrete loss, which is a special case of indirect elicitation. Finocchiaro et al. (2020) further introduce the notion of embedding dimension, which is a lower bound on both convex elicitation complexity of discrete properties and convex consistency dimension of discrete losses and finite statistics.

### 3. Consistency implies indirect elicitation

In this section, we connect consistency of any surrogate to an indirect elicitation requirement. This will allow us to show indirect elicitation gives state-of-the-art lower bounds on the prediction dimension of consistent convex surrogates.

We start by formalizing consistency in two ways that generalize across our four quadrants. First, given a target loss $\ell$, we say $L$ is consistent if optimizing $L$ and applying a link $\psi$ optimizes $\ell$ (Definition 5). Second, given a target property $\gamma$, such as the $\alpha$-quantile, we say $L$ is consistent if optimizing $L$ implies approaching, in some sense, the correct statistic $\gamma(D_x)$ of the conditional distributions $D_x = \Pr[Y \mid X = x]$ (Definition 6). We then observe that Definition 5 is subsumed by Definition 6, and use this to show consistency implies $L$ indirectly elicits $\text{prop}_P[\ell]$ or $\gamma$ respectively.

**Definition 5 (Consistent: loss)**  A loss $L \in \mathcal{L}$ and link $(L, \psi)$ are $\mathcal{D}$-consistent for a set $\mathcal{D}$ of distributions over $\mathcal{X} \times \mathcal{Y}$ with respect to a target loss $\ell$ if, for all $D \in \mathcal{D}$ and all sequences of measurable hypothesis functions $\{f_m : \mathcal{X} \to \mathcal{R}\}$,

$$
\mathbb{E}_D L(f_m(X), Y) \to \inf_f \mathbb{E}_D L(f(X), Y) \implies \mathbb{E}_D \ell((\psi \circ f_m)(X), Y) \to \inf_f \mathbb{E}_D \ell((\psi \circ f)(X), Y).
$$
For a given convex set \( P \subseteq \Delta_Y \), we simply say \((L, \psi)\) is consistent if it is \( \mathcal{D}\)-consistent for some \( \mathcal{D} \) satisfying the following: for all \( p \in P \), there exists \( D \in \mathcal{D} \) and \( x \in X \) such that \( D \) has a point mass on \( x \) and \( p = D_x \).

Instead of a target loss \( \ell \), one may want to learn a target property, i.e. a conditional statistic such as the expected value, variance, or entropy. In this case, following the tradition in the statistics literature on conditional estimation (Győrfi et al., 2006; Fan and Yao, 1998; Ruppert et al., 1997), we formalize consistency as converging to the correct conditional estimates of the property. Convergence is measured by functions \( \mu(r, p) \) that formalize how close \( r \) is to “correct” for conditional distribution \( p \). In particular we should have

\[
\mu(r, p) = 0 \iff r \in \gamma(p).
\]

**Definition 6 (Consistent: property)** Suppose we are given a loss \( L \in \mathcal{L} \), link function \( \psi : \mathbb{R}^d \to \mathcal{R} \), and property \( \gamma : \mathcal{P} \Rightarrow \mathbb{R} \). Moreover, let \( \mu : \mathcal{R} \times \mathcal{P} \to \mathbb{R}_+ \) be any function satisfying \( \mu(r, p) = 0 \iff r \in \gamma(p) \). We say \((L, \psi)\) is \((\mu, \mathcal{D})\)-consistent with respect to \( \gamma \) if, for all \( D \in \mathcal{D} \) and sequences of measurable functions \( \{f_m : X \to \mathcal{R}\} \),

\[
\mathbb{E}_D L(f_m(X), Y) \to \inf_f \mathbb{E}_D L(f(X), Y) \implies \mathbb{E}_X \mu(\psi \circ f_m(X), D_X) \to 0.
\] (2)

We simply say \((L, \psi)\) is \( \mu \)-consistent if it is \((\mu, \mathcal{D})\)-consistent for some \( \mathcal{D} \) satisfying the following: for all \( p \in P \), there exists \( D \in \mathcal{D} \) and \( x \in X \) such that \( D \) has a point mass on \( x \) and \( p = D_x \). Additionally, we say \((L, \psi)\) is consistent if there is a \( \mu \) such that \((L, \psi)\) is \( \mu \)-consistent.

Typical definitions of consistency require \( \mathcal{D} \) to be the set of all distributions over \( X \times Y \), while our conditions are much weaker. As the main focus of this paper is lower bounds on the prediction dimension, i.e., showing that surrogates of a certain prediction dimension cannot exist, these weaker conditions translate to stronger impossibility statements.

Given a target loss \( \ell \), we can define a statistic \( \gamma \), the property it elicits. Intuitively, consistency of a surrogate \( L \) with respect to \( \ell \) and \( \gamma \) are equivalent, i.e. in both cases estimates should converge to values that minimize \( \ell \)-loss. We formalize this by letting \( \mu \) be the \( \ell \)-regret, yielding Lemma 7, proven in Appendix D.

**Lemma 7** Let a convex \( P \subseteq \Delta_Y \) be given. Given a surrogate loss \( L \in \mathcal{L} \), link \( \psi \), and target loss \( \ell \), set \( \mu(r, p) := R_\ell(r, p) \). Then there is a \( \mathcal{D} \) such that \((L, \psi)\) is \( \mathcal{D}\)-consistent with respect to \( \ell \) if and only if \((L, \psi)\) is \((\mu, \mathcal{D})\)-consistent with respect to \( \gamma := \text{prop}_P[\ell] \).

Because each target loss in \( \mathcal{L} \) elicits some property, but not all target properties can be elicited by a loss (e.g. the variance), consistency with respect to a property is the strictly broader notion. This points to indirect elicitation as a natural necessary condition for consistency, as formalized in Theorem 8.

**Theorem 8** For a surrogate \( L \in \mathcal{L} \), if the pair \((L, \psi)\) is consistent with respect to a property \( \gamma : \mathcal{P} \Rightarrow \mathbb{R} \) or a loss \( \ell \) eliciting \( \gamma \), then \((L, \psi)\) indirectly elicits \( \gamma \).
Proof By Lemma 7, it suffices to show the result for consistency with respect to a property \( \gamma \), setting \( \gamma := \text{prop}_P[\ell] \) if \( \ell \) is given instead. We show the contrapositive; suppose \((L, \psi)\) does not indirectly elicit \( \gamma \), meaning we have some \( p \in \mathcal{P} \) so that \( u \in \Gamma(p) \) but \( \psi(u) \notin \gamma(p) \), where \( \Gamma := \text{prop}_P[L] \). Observe that we use the fact \( \Gamma(p) \neq \emptyset \). By definition, if we had consistency, there must be some distribution \( D \) on \( \mathcal{X} \times \mathcal{Y} \) with a point mass on some \( x \in \mathcal{X} \) and \( D_x = p \). Consider a constant sequence \( \{f_m\} \) with \( f_m = f' \) such that \( f'(x) = u \), so that \( \mathbb{E}_D L(f_m(X), Y) = \mathbb{E}_{D_x} L(f_m(x), Y) = \mathbb{E}_p L(u, Y) \). Since \( u \in \Gamma(p) \), we have \( \mathbb{E}_p L(u, Y) = \inf_f \mathbb{E}_{D_x} L(f(x), Y) = \inf_f \mathbb{E}_D L(f(X), Y) \). In particular, we have \( \mathbb{E}_D L(f_m(X), Y) \rightarrow \inf_f \mathbb{E}_D L(f(X), Y) \). However, we have \( \mathbb{E}_X \mu(\psi \circ f_m(X), DX) = \mu(f_m(x), p) = \mu(\psi(u), p) 
eq 0 \), since \( \psi(u) \notin \gamma(p) \). Therefore \((L, \psi)\) is not consistent with respect to \( \gamma \) (Definition 6).

This result allows us to state elicitation complexity as a lower bound for convex consistency dimension.

Corollary 9 Given a property \( \gamma : \mathcal{P} \Rightarrow \mathcal{R} \) or loss \( \ell : \mathcal{R} \times \mathcal{Y} \rightarrow \mathbb{R} \) eliciting \( \gamma \), we have \( \text{elic}_{\text{cvx}}(\gamma) \leq \text{cons}_{\text{cvx}}(\gamma) = \text{cons}_{\text{cvx}}(\ell) \).

4. Prediction Dimension of Consistent Convex Surrogates

We now turn to the question of bounding the prediction dimension of a consistent convex surrogate. From Theorem 8, given a target property \( \gamma \) or loss \( \ell \) with \( \gamma = \text{prop}_P[\ell] \), this task reduces to lower bounding the prediction dimension of a convex surrogate indirectly eliciting \( \gamma \). We now explore two tools, Corollaries 12 and 13, for proving such convex elicitation lower bounds. The key idea, crystallized from the proofs of Ramaswamy and Agarwal (2016, Theorem 16) and Agarwal and Agarwal (2015, Theorem 9), is to consider a particular distribution \( p \) and surrogate prediction \( u \in \mathbb{R}^d \) with is optimal for \( p \). Theorem 11 will show that if \( d \) is small, then the level set \( \{p \in \mathcal{P} : u \in \arg \min_{u'} \mathbb{E}_p L(u', Y)\} \) must be large; in fact, it must roughly contain a high-dimensional flat. By definition of indirect elicitation, there is some level set \( \gamma_r \) (where \( u \) is linked to \( r \)) containing this flat as well. The use of this result is to leverage the contrapositive: if \( \gamma \) has a level set intricate enough to not contain any high-dimensional flats, then \( \gamma \) cannot have a low-dimensional consistent surrogate.

Definition 10 (Flat) For \( d \in \mathbb{N} \), a \( d \)-flat, or simply flat, is a nonempty set \( F = \ker W \) such that \( \{q \in \mathcal{P} : \mathbb{E}_q W = 0\} \) for some measurable \( W : \mathcal{Y} \rightarrow \mathbb{R}^d \).

We state our elicitation lower bounds in Corollaries 12 and 13, which when combined with Theorem 8, yield consistency bounds. A similar result is Agarwal and Agarwal (2015, Theorem 9), which bounds the dimension of level sets of a single-valued \( \text{prop}_P[L] \). Corollaries 12 and 13 instead bound the dimension of flats contained in the level sets, an additional power which we leverage in our examples.

Lemma 11 Let \( \Gamma : \mathcal{P} \Rightarrow \mathbb{R}^d \) be (directly) elicited by \( L \in \mathcal{L}_d^{\text{cvx}} \) for some \( d \in \mathbb{N} \). Let \( \mathcal{Y} \) be either a finite set, or \( \mathcal{Y} = \mathbb{R} \), in which case we assume each \( p \in \mathcal{P} \) admits a Lebesgue density supported on the same set for all \( p \in \mathcal{P} \). For all \( u \in \text{range} \Gamma \) and \( p \in \Gamma_u \), there is some \( V_{u,p} : \mathcal{Y} \rightarrow \mathbb{R}^d \) such that \( p \in \ker V_{u,p} \subseteq \Gamma_u \).

1. This assumption is largely for technical convenience, to ensure that \( V_{u,p} \) does not depend on \( p \). Any such assumption would suffice, and we suspect even that condition can be relaxed.
Proof As $L$ is convex and elicits $\Gamma$, we have $u \in \Gamma(p) \iff 0 \in \partial \mathbb{E}_p L(u, Y)$. We proceed in two cases, depending on $|\mathcal{Y}|$.

Finite $\mathcal{Y}$: If $\mathcal{Y}$ is finite, this is additionally equivalent to $0 \in \oplus_y p_y \partial L(u, y)$, where $\oplus$ denotes the Minkowski sum (Hiriart-Urruty and Lemaréchal, 2012, Theorem 4.1.1).\footnote{\partial denotes the subdifferential $\partial f(x) = \{z : f(x') - f(x) \geq \langle z, x' - x \rangle \ \forall x\}$.}

Expanding, we have $\oplus_y p_y \partial L(u, y) = \{\sum_y y p_y x_y | x_y \in \partial L(u, y) \ \forall y \in \mathcal{Y}\}$, and thus $W p = \sum_y y p_y x_y = 0$ where $W = [x_1, \ldots, x_n] \in \mathbb{R}^{d \times n}$; cf. (Ramaswamy and Agarwal, 2016, A\textsuperscript{m} in Theorem 16). Let $V_{u,p} : \mathcal{Y} \to \mathbb{R}^d, y \mapsto W_y$ be the function encoding the columns of $W$. Observe that $\mathbb{E}_p V_{u,p} = 0$.

$\mathcal{Y} = \mathbb{R}$: Any $L \in \mathcal{L}\textsuperscript{cvx}_d$ satisfies the assumptions of Ioffe and Tikhomirov (1969), so we may interchange subdifferentiation and expectation. Specifically, letting $V_{u,p} = \{V : \mathcal{Y} \to \mathbb{R}^d | V \text{ measurable, } V(y) \in \partial L(u, y) \ p\text{-a.s.}\}$, we have $\partial \mathbb{E}_p L(u, Y) = \{f V(y)dp(y) | V \in V_{u,p}\}$. As $0 \in \partial \mathbb{E}_p L(u, Y)$, in particular, there is some $V_{u,p} \in V_{u,p}$ such that $\mathbb{E}_p V_{u,p} = 0$. For any $q \in \mathcal{P}$, as by assumption $q$ is supported on the same set as $p$, we have $V_{u,p}(y) \in \partial L(u, y) q\text{-a.s.}$, so that $V_{u,p} \in V_{u,q}$. Thus, $\mathbb{E}_q V_{u,p} = 0$ implies $0 \in \partial \mathbb{E}_q L(u, Y)$ by the above.

In both cases, we take the flat $F := \ker_p V_{u,p}$ and have $p \in F$ by construction. To see $F \subseteq \Gamma_u$, from the chain of equivalences above, we have for any $q \in \mathcal{P}$ that $q \in \ker_p V_{u,p} \implies 0 \in \partial \mathbb{E}_q L(u, Y) \implies u \in \Gamma(q) \implies q \in \Gamma_u$.

Knowing indirect elicitation implies the existence of such a flat, we now apply Theorem 8 and Lemma 11 to construct lower bounds on convex consistency dimension.

Corollary 12 Let target property $\gamma : \mathcal{P} \Rightarrow \mathcal{R}$ and $d \in \mathbb{N}$ be given. Let $\mathcal{Y}$ be either a finite set, or $\mathcal{Y} = \mathbb{R}$, in which case we assume each $p \in \mathcal{P}$ admits a Lebesgue density supported on the same set for all $p \in \mathcal{P}$. Let $p \in \mathcal{P}$ with $|\gamma(p)| = 1$, and take $\gamma(p) = \{r\}$. If there is no $d$-flat $F$ with $p \in F \subseteq \gamma_r$, then $\text{cons}_{\text{cvx}}(\gamma) \geq \text{elic}_{\text{cvx}}(\gamma) \geq d + 1$.

Proof Let $(L, \psi)$ indirectly elicit $\gamma$, where $L \in \mathcal{L}\textsuperscript{cvx}_d$, and let $\Gamma = \text{prop}_p[L]$. As $\Gamma$ is non-empty, there is some $u \in \Gamma(p)$. Since $\gamma$ is single-valued at $p$, we have $r = \psi(u)$; by Lemma 11, we know there is a $d$-flat $F = \ker_p V_{u,p}$ so that $p \in F \subseteq \Gamma_u$. By definition of indirect elicitation, we additionally have $\Gamma_u \subseteq \gamma_r$. Thus, we have $p \in F \subseteq \gamma_r$. If no flat $F$ satisfies the above conditions, then no $L \in \mathcal{L}\textsuperscript{cvx}_d$ indirectly elicits $\gamma$, so $\text{elic}_{\text{cvx}}(\gamma) \geq d + 1$, and recall $\text{cons}_{\text{cvx}}(\gamma) \geq \text{elic}_{\text{cvx}}(\gamma)$ by Corollary 9.

Corollary 13 Let an elicitable target property $\gamma : \mathcal{P} \Rightarrow \mathcal{R}$ be given, where $\mathcal{P} \subseteq \Delta \mathcal{Y}$ is defined over a finite set of outcomes $\mathcal{Y}$, and let $d \in \mathbb{N}$. Let $p \in \text{relint}(\mathcal{P})$. If there is no $d$-flat $F$ with $p \in F \subseteq \gamma_r$, then $\text{cons}_{\text{cvx}}(\gamma) \geq \text{elic}_{\text{cvx}}(\gamma) \geq d + 1$.

Proof Let $(L, \psi)$ indirectly elicit $\gamma$ and the convex function $L$ and eliciting $\Gamma$. As $\Gamma$ is non-empty, there is some $u \in \Gamma(p)$, and suppose $r' = \psi(u)$. Take $F \subseteq \Gamma_u$ to be the flat that exists by Lemma 11. If $r = r'$, then $p \in F \subseteq \Gamma_u \subseteq \gamma_r$ by indirect elicitation. Otherwise, by Lemma 39, for elicitable properties with $p \in \gamma_r \cap \gamma_{r'}$, we observe $p \in F \subseteq \gamma_r \iff p \in F \subseteq \gamma_{r'}$.

As above, if no flat $F$ satisfies the above conditions, then no $L \in \mathcal{L}\textsuperscript{cvx}_d$ indirectly elicits $\gamma$, so $\text{cons}_{\text{cvx}}(\gamma) \geq \text{elic}_{\text{cvx}}(\gamma) \geq d + 1$, recalling Corollary 9 for the first inequality.
5. Discrete-valued predictions

The main known technique for lower bounds on surrogate dimensions is given by Ramaswamy and Agarwal (2016) for the Quadrant 1 (target loss and discrete predictions). The proof heavily builds around the “limits of sequences” in the definition of calibration. By restricting slightly to the broad class of minimizable losses $\mathcal{L}^{\text{cvx}}$, we show their bound follows relatively directly from Corollary 13. (We conjecture that the minimizability restriction to $\mathcal{L}^{\text{cvx}}$ can be lifted; see § 7.) Ramaswamy and Agarwal (2016) construct what they call the subspace of feasible directions and give bounds in terms of its dimension.

**Definition 14 (Subspace of feasible directions)** The subspace of feasible directions $S_c(p)$ of a convex set $C \subseteq \mathbb{R}^n$ at $p \in C$ is $S_c(p) = \{v \in \mathbb{R}^n : \exists \epsilon > 0 \text{ such that } p + \epsilon v \in C \forall \epsilon \in (-\epsilon_0, \epsilon_0)\}$.

Ramaswamy and Agarwal (2016) gives a lower bound on the dimensionality of all consistent convex surrogates, i.e. $\text{cons}_{\text{cvx}}(\ell) \geq \|p\|_0 - \dim(S_{\gamma_r}(p)) - 1$ for all $p$ and $r \in \gamma(p)$, particularly in the setting where one is given a discrete prediction problem and target loss over finite outcomes. It turns out that the subspace of feasible directions is essentially a special case of a flat described by Lemma 11. So, by making a slight restriction to the class of minimizable convex surrogates $\mathcal{L}^{\text{cvx}}$, we can derive this lower bound from our general technique in a way that we find shorter and simpler.

**Corollary 15 (Ramaswamy and Agarwal (2016) Theorem 18)** Let $\ell : \mathcal{R} \times \mathcal{Y} \rightarrow \mathbb{R}$ be a discrete loss eliciting $\gamma : \Delta_{\mathcal{Y}} \Rightarrow \mathcal{R}$ with $\mathcal{Y}$ finite. Then for all $p \in \Delta_{\mathcal{Y}}$ and $r \in \gamma(p)$,

$$\text{cons}_{\text{cvx}}(\gamma) \geq \|p\|_0 - \dim(S_{\gamma_r}(p)) - 1.$$  \hfill (3)

**Proof** [Sketch] If $\text{cons}_{\text{cvx}}(\gamma) \leq d$, then there is a $L \in \mathcal{L}^{\text{cvx}}_d$ so that $L$ is consistent with respect to $\gamma$, and in turn, indirectly elicits $\gamma$. Lemma 11 says that there is some $d$-flat $F = \ker_p V$ such that $p \in F \subseteq \gamma_r$. In particular, if $p \in \text{relint}(\Delta_{\mathcal{Y}})$, we can see $\dim(F) = \dim(S_{\gamma_r}(p))$. Since $\text{affhull}(\Delta_{\mathcal{Y}})$ has dimension $|\mathcal{Y}| - 1 = \|p\|_0 - 1$, by rank-nullity and $\dim(V) \leq d$ (more precisely, the corresponding linear map $q \mapsto \mathbb{E}_q V$) we have $d \geq \|p\|_0 - 1 - \dim(S_{\gamma_r}(p))$.

When $p \notin \text{relint}(\Delta_{\mathcal{Y}})$, we can project down to the subsimplex on the support of $p$, again of dimension $\|p\|_0 - 1$, and modify $L$ and $\ell$ accordingly. Now $p$ is in the relative interior of this subsimplex, so the above gives $\text{cons}_{\text{cvx}}(\gamma) \geq \|p\|_0 - 1 - \dim(S_{\gamma_r}(p))$, where now $S$ is relative to $\mathbb{R}^{\text{supp}(p)}$. Finally, the feasible subspace dimension in the projected space is the same as in the original space because of $p$’s location on a face of $\Delta_{\mathcal{Y}}$.

There are some cases where the bound provided by Corollaries 12 and 13 is strictly tighter than the bound provided by feasible subspace dimension in Corollary 15. For an example of how Corollary 12 applies to a discrete property for which there is no target loss – a non-elicitable property, i.e. Quadrant 2, which is not considered by Ramaswamy et al. (2018) – we refer the reader to Appendix E.

**Example: High-confidence classification.** Given the target loss $\ell_{\text{abs}}(r, y) := I[r \notin \{y, \perp\}] + (1/2)I[r = \perp]$, we can consider the abstain property it elicits, where one predicts the most likely outcome $y$ if $Pr[Y = y|x] \geq 1/2$ and “abstain” by predicting $\perp$ otherwise. Ramaswamy and Agarwal (2016) present a convex surrogate for the abstain loss that takes as input a
prediction whose dimension is logarithmic in the number of outcomes, yielding new upper bounds on $\text{cons}_{\text{cvx}}(\ell_{\text{abs}})$ which are an exponential improvement over previous results, e.g., Crammer and Singer (2001).

To lower bound the dimension of convex surrogates, we can consider two different distributions; in the first, our bound yields a strict gap over the feasible subspace dimension bound, and in the second, the bounds are equal. First, we choose $p = \bullet$ to be the uniform distribution (see Figure 2). In this case, the bound by feasible subspace dimension yields $\text{cons}_{\text{cvx}}(\ell_{\text{abs}}) \geq 3 - 2 - 1 = 0$, as the feasible subspace dimension is 2 since we are on the relative interior of the level set and simplex, as shown in Figure 2 (L).

However, consider any 1-flat containing $\bullet$. When intersected with the simplex, one can see that any line (a 1-flat, since $\bullet \in \text{relint}(\Delta_Y)$) in the simplex through $\bullet$ also leaves the cell $\gamma_\perp$, which contains $p$. See Figure 2 (R) for intuition; a 1-flat through $p \in \text{relint}(\Delta_Y)$ would be a line in such a figure. Therefore, we have no 1-flat containing $p$ staying in $\gamma_\perp$, so we obtain a better lower bound, $\text{cons}_{\text{cvx}}(\ell_{\text{abs}}) \geq 2$. Combining this with the upper bounds given by Ramaswamy et al. (2018), we observe the bound $\text{cons}_{\text{cvx}}(\ell_{\text{abs}}) = 2$ is tight in this case with $|Y| = 3$.

Our bounds sometimes match those of (Ramaswamy and Agarwal, 2016); consider the distribution $\star = (1/4,1/4,1/2)$, shown in Figure 2. The feasible subspace dimension of both $\gamma_\perp$ and $\gamma_3$ at $\star$ is 1, since one only moves toward the distributions $(0,1/2,1/2)$ and $(1/2,0,1/2)$ without leaving the level sets, and the three points are collinear in affhull($\Delta_Y$), suggesting $S_{\gamma_\perp}(q) = 1$. This yields $\text{cons}_{\text{cvx}}(\ell_{\text{abs}}) \geq 3 - 1 - 1 = 1$. The same line segment defines a flat contained in both $\gamma_\perp$ and $\gamma_3$, so we have $\text{cons}_{\text{cvx}}(\ell_{\text{abs}}) \geq 1$ by Corollary 13, matching the feasible subspace dimension bound.

Bounds using $d$-flats appear to work well at distributions where previous bounds via feasible subspace dimension would have been vacuous. In essence, flats allow us a “global” view of the property we are eliciting, while the feasible subspace method only permits a “local” look at the property, so we find our method works better for distributions in $\text{relint}(\Delta_Y)$.

6. Continuous-valued predictions

In continuous estimation problems, often one is not given a target loss, but instead a target (conditional) statistic of the data one wishes to estimate. Examples include estimating the
mean or variance of $y$ conditioned on a given $x$. In this setting, Lemma 11 gives lower bounds on the prediction dimension of convex losses with a link to the desired conditional statistic, i.e., the convex elicitation complexity. In particular, Theorem 17 below yields new bounds on the convex elicitation complexity of statistics which quantify risk or uncertainty such as variance, entropy, or financial risk measures.

These bounds address an open question of Frongillo and Kash (2020), that of developing a theory of elicitation complexity with respect to convex-elicitable properties. The lower bounds of previous work are essentially all with respect to identifiable properties; a property is $d$-identifiable if its level sets are all $d$-flats. Frongillo and Kash (2020) rely on finding a dimension $d$ such that the level sets of certain risk measures $\gamma$ have too much curvature to contain any $d$-flat. Thus, the elicitation complexity with respect to identifiable properties is greater than $d$.

In contrast, properties elicited by non-smooth convex losses are generally not identifiable. For example, the properties elicited by hinge loss and the abstain surrogate are not identifiable, as their level sets are not flats (see Figure 2). It therefore might appear that entirely new ideas are needed. Our framework is closely related to identifiability, however; Lemma 11 states that the level sets of $d$-dimensional convex-elicitable properties, if not $d$-flats themselves, are unions of $d$-flats. Thus, the general logic of Frongillo and Kash (2020) can still apply. In particular, we recover their main lower bound for the large class of Bayes risks.

**Definition 16** Given loss function $L : \mathcal{R} \times Y \to \mathbb{R}$ for some report set $\mathcal{R}$, the Bayes risk of $L$ is defined as $L(p) := \inf_{r \in \mathcal{R}} \mathbb{E}_p L(r, Y)$.

**Condition 1** For some $r \in \text{range } \Gamma$, the level set $\Gamma_r = \text{ker } \mathcal{P} V$ is a $d$-flat presented by some $V : Y \to \mathbb{R}^d$ such that $0 \in \text{int } \{ \mathbb{E}_p V : p \in \mathcal{P} \}$.

**Theorem 17** Let $\mathcal{P}$ be a set of Lebesgue densities supported on the same set for all $p \in \mathcal{P}$. Let $\Gamma : \mathcal{P} \to \mathbb{R}^d$ satisfy Condition 1 for some $r \in \mathbb{R}^d$. Let $L \in \mathcal{L}_{\text{cvx}}$ elicit $\Gamma$ such that $L$ is non-constant on $\Gamma_r$. Then $\text{cons}_{\text{cvx}}(L) \geq \text{elic}_{\text{cvx}}(L) \geq d + 1$.

We now illustrate the theorem with two important examples: variance and conditional value at risk. Several other applications from Frongillo and Kash (2020), such as spectral risk measures, entropy, and norms, follow similarly.

**Example: Variance.** As a warm-up, let us see how to show $\text{elic}_{\text{cvx}}(\text{Var}) = 2$, meaning the lowest dimension of a convex loss to estimate conditional variance is 2. This lower bound will follow from Theorem 17 using that variance is the Bayes risk of squared loss $L(r, y) = (r - y)^2$, which elicits the mean $\Gamma(p) = \mathbb{E}_p Y$. Interestingly, while perhaps intuitively obvious, even this simple result is novel. In particular, the well-known fact that the variance is not elicitable does not yield a lower bound of 2, as it does not rule out the variance being a link of a real-valued convex-elicitable property; cf. Frongillo and Kash (2020, Remark 1).

**Corollary 18** Let $\mathcal{P}$ be a set of continuous Lebesgue densities on $Y = \mathbb{R}$ with all $p \in \mathcal{P}$ having the same support. If there exist $p, q, q' \in \mathcal{P}$ with $\mathbb{E}_p Y = \mathbb{E}_q Y \neq \mathbb{E}_q' Y$ and $\text{Var}(p) \neq \text{Var}(q)$, then $\text{cons}_{\text{cvx}}(\text{Var}) = \text{elic}_{\text{cvx}}(\text{Var}) = 2$. 

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Proof For the upper bound, we may elicit the first two moments via the convex loss $L(r, y) = (r_1 - y)^2 + (r_2 - y'^2)$, and recover the variance via $\psi(r) = r_2 - r'^2$, giving $\text{elic}_{\text{cvx}}(\text{Var}) \leq 2$. Now for the lower bound. Without loss of generality, $E_y Y < E'_q Y$. Let $r = \frac{1}{2}E_q Y + \frac{1}{2}E'_q Y$, and define $V : Y \to \mathbb{R}, y \mapsto y - r$. Then $\text{ker}_p V = \{ p' \in \mathcal{P} \mid E_p Y = r \} = \Gamma_r$ where $\Gamma : p' \mapsto E_p' Y$ is the mean. As $E_q Y < r < E'_q Y$, we conclude $E_q V < 0 < E'_q V$. We have now satisfied Condition 1 for $d = 1$. To apply Theorem 17, it remains to show that $\Gamma$ is non-constant on $\Gamma_r$. By our assumptions and the definition of $\text{Var}$, we have $E_p Y^2 \neq E'_q Y^2$. Letting $p_1 = \frac{1}{2}q + \frac{1}{2}q'$, $p_2 = \frac{1}{2}p + \frac{1}{2}q'$, we have $E_{p_i} Y = r$ for $i \in \{1, 2\}$, but $E_{p_1} Y^2 = \frac{1}{2}E_q Y^2 + \frac{1}{2}E'_q Y^2 \neq \frac{1}{2}E_p Y^2 + \frac{1}{2}E'_q Y^2 = E_{p_2} Y^2$. As $p_1, p_2$ have the same mean but different second moments, we conclude $\text{Var}(p_1) \neq \text{Var}(p_2)$.

Example: Conditional Value at Risk. Frongillo and Kash (2020) observe that one of the most prominent financial risk measures, the conditional value at risk (CVaR), can be expressed as a Bayes risk. In particular, for $0 < \alpha < 1$, we may define

$$\text{CVaR}_\alpha(p) = \inf_{r \in \mathbb{R}} E_p \left\{ \frac{1}{\alpha} (r - Y) 1_{r \geq Y} - r \right\},$$

which is the Bayes risk of the transformed pinball loss $L_\alpha(r, y) = \frac{1}{\alpha} (r - y) 1_{r \geq y} - r$. In turn, $L_\alpha$ elicits the $\alpha$-quantile, the quantity $q_\alpha(p)$ such that $P_{r_p}[Y \geq q_\alpha(p)] = \alpha$. Following Frongillo and Kash (2020), we will restrict to the set $\mathcal{P}_q$ of probability measures over $\mathbb{R}$ with connected support and whose CDFs are strictly increasing on their support, so that $q_\alpha$ is single-valued. Under mild assumptions, we find that there is no consistent real-valued convex surrogate for $\text{CVaR}_\alpha$.

Corollary 19 Let $\mathcal{P}$ be a set of continuous Lebesgue densities on $Y = \mathbb{R}$ with all $p \in \mathcal{P}$ having support on the same interval. If we have $p_1, p_2, p_3, p'_2 \in \mathcal{P}$ with $q_\alpha(p_1) < q_\alpha(p_2) < q_\alpha(p_3)$ and $\text{CVaR}_\alpha(p_2) \neq \text{CVaR}_\alpha(p'_2)$, then $\text{cons}_{\text{cvx}}(\text{CVaR}_\alpha) \geq \text{elic}_{\text{cvx}}(\text{CVaR}_\alpha) \geq 2$.

As first shown by Fissler et al. (2016), the pair $(\text{CVaR}_\alpha, q_\alpha)$ is jointly identifiable and elicitable, but not by any convex loss (Fissler, 2017, Prop. 4.2.31). We conjecture the stronger statement $\text{elic}_{\text{cvx}}(\text{CVaR}_\alpha) \geq 3$, which if true would constitute an interesting gap between elicitation complexity for identifiable and convex-elicitable properties.

7. Conclusions and future work

In this work, we show that indirect property elicitation can be a powerful necessary condition for the existence of a consistent surrogate loss (Theorem 8). Furthermore, we introduce a new lower bound (Corollaries 12 and 13) on convex consistency dimension that is generally applicable and extends previous results from both the discrete (Corollary 15) and continuous (Corollaries 18 and 19) estimation settings.

Several important questions remain open. Particularly for the discrete settings, we would like to know whether one can lift the restriction that surrogates always achieve a minimum; we conjecture positively. Of course, we would like to characterize $\text{cons}_{\text{cvx}}$ and $\text{elic}_{\text{cvx}}$ and develop a general framework for constructing surrogates achieving the best possible prediction dimension. Moreover, the practical reason why consistency is desired is to ensure the guarantee of empirical risk minimization (ERM) rates; however, the relationship between ERM rates and property elicitation has not been studied.
References

Arpit Agarwal and Shivani Agarwal. On consistent surrogate risk minimization and property elicitation. In JMLR Workshop and Conference Proceedings, volume 40, pages 1–19, 2015. URL http://www.jmlr.org/proceedings/papers/v40/Agarwal15.pdf.

Peter L. Bartlett, Michael I. Jordan, and Jon D. McAuliffe. Convexity, classification, and risk bounds. Journal of the American Statistical Association, 101(473):138–156, 2006. URL http://amstat.tandfonline.com/doi/abs/10.1198/016214505000000907.

Koby Crammer and Yoram Singer. On the algorithmic implementation of multiclass kernel-based vector machines. Journal of machine learning research, 2(Dec):265–292, 2001.

Jianqing Fan and Qiwei Yao. Efficient estimation of conditional variance functions in stochastic regression. Biometrika, 85(3):645–660, 09 1998. ISSN 0006-3444. doi: 10.1093/biomet/85.3.645. URL https://doi.org/10.1093/biomet/85.3.645.

Jessie Finocchiaro, Rafael Frongillo, and Bo Waggoner. An embedding framework for consistent polyhedral surrogates. In Advances in neural information processing systems, 2019.

Jessie Finocchiaro, Rafael Frongillo, and Bo Waggoner. Embedding dimension of polyhedral losses. The Conference on Learning Theory, 2020.

Tobias Fissler. On higher order elicitability and some limit theorems on the Poisson and Wiener space. PhD thesis, 2017.

Tobias Fissler, Johanna F Ziegel, and others. Higher order elicitability and Osband’s principle. The Annals of Statistics, 44(4):1680–1707, 2016.

Gerald B Folland. Real analysis: modern techniques and their applications, volume 40. John Wiley & Sons, 1999.

Rafael Frongillo and Ian Kash. General truthfulness characterizations via convex analysis. In Web and Internet Economics, pages 354–370. Springer, 2014.

Rafael Frongillo and Ian Kash. Vector-Valued Property Elicitation. In Proceedings of the 28th Conference on Learning Theory, pages 1–18, 2015.

Rafael Frongillo and Ian A Kash. Elicitation Complexity of Statistical Properties. Biometrika, 11 2020. ISSN 0006-3444. doi: 10.1093/biomet/asaa093. URL https://doi.org/10.1093/biomet/asaa093.

László Györfi, Michael Kohler, Adam Krzyzak, and Harro Walk. A distribution-free theory of nonparametric regression. Springer Science & Business Media, 2006.

Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. Fundamentals of convex analysis. Springer Science & Business Media, 2012.

Aleksandr Davidovich Ioffe and Vladimir Mikhailovich Tikhomirov. On minimization of integral functionals. Functional Analysis and Its Applications, 3(3):218–227, 1969.
Nicolas S. Lambert. Elicitation and evaluation of statistical forecasts. 2018. URL https://web.stanford.edu/˜nlambert/papers/elicitability.pdf.

Nicolas S. Lambert and Yoav Shoham. Eliciting truthful answers to multiple-choice questions. In Proceedings of the 10th ACM conference on Electronic commerce, pages 109–118, 2009.

Nicolas S. Lambert, David M. Pennock, and Yoav Shoham. Eliciting properties of probability distributions. In Proceedings of the 9th ACM Conference on Electronic Commerce, pages 129–138, 2008.

Yi Lin. A note on margin-based loss functions in classification. Statistics & probability letters, 68(1):73–82, 2004.

Kent Osband and Stefan Reichelstein. Information-eliciting compensation schemes. Journal of Public Economics, 27(1):107–115, June 1985. ISSN 0047-2727. doi: 10.1016/0047-2727(85)90031-3. URL http://www.sciencedirect.com/science/article/pii/0047272785900313.

Kent Harold Osband. Providing Incentives for Better Cost Forecasting. University of California, Berkeley, 1985.

Harish Ramaswamy, Ambuj Tewari, and Shivani Agarwal. Convex calibrated surrogates for hierarchical classification. In International Conference on Machine Learning, pages 1852–1860, 2015.

Harish G Ramaswamy and Shivani Agarwal. Convex calibration dimension for multiclass loss matrices. The Journal of Machine Learning Research, 17(1):397–441, 2016.

Harish G Ramaswamy, Ambuj Tewari, Shivani Agarwal, et al. Consistent algorithms for multiclass classification with an abstain option. Electronic Journal of Statistics, 12(1):530–554, 2018.

David Ruppert, M. P. Wand, Ulla Holst, and Ola H˚osjer. Local polynomial variance-function estimation. Technometrics, 39(3):262–273, 1997. doi: 10.1080/00401706.1997.10485117. URL https://www.tandfonline.com/doi/abs/10.1080/00401706.1997.10485117.

L.J. Savage. Elicitation of personal probabilities and expectations. Journal of the American Statistical Association, pages 783–801, 1971.

Ingo Steinwart. How to compare different loss functions and their risks. Constructive Approximation, 26(2):225–287, 2007.

Ingo Steinwart and Andreas Christmann. Support Vector Machines. Springer Science & Business Media, September 2008. ISBN 978-0-387-77242-4. Google-Books-ID: HUnqr-pYt4IC.

Ambuj Tewari and Peter L. Bartlett. On the consistency of multiclass classification methods. The Journal of Machine Learning Research, 8:1007–1025, 2007. URL http://dl.acm.org/citation.cfm?id=1390325.

Tong Zhang. Statistical behavior and consistency of classification methods based on convex risk minimization. Annals of Statistics, pages 56–85, 2004.
Appendix A. Notes on calibration

When given a discrete target loss, such as for classification-like problems, direct empirical risk minimization is typically NP-hard, forcing one to find a more tractable surrogate. To ensure consistency, the literature has embraced the notion of calibration from Steinwart and Christmann (2008, Chapter 3), which aligns with the definition in Tewari and Bartlett (2007) for multiclass classification, and its generalizations to arbitrary discrete target losses (Agarwal and Agarwal, 2015; Ramaswamy and Agarwal, 2016). Calibration is more tractable and weaker than consistency, yet the two are equivalent under suitable assumptions (Tewari and Bartlett, 2007; Ramaswamy and Agarwal, 2016), notably in Quadrant 1. Intuitively, calibration says one cannot achieve the optimal surrogate loss while linking to a suboptimal target prediction.

**Definition 20 (Calibrated: Quadrant 1)** Let $\ell : \mathcal{R} \times \mathcal{Y} \rightarrow \mathbb{R}$ be a discrete target loss. A surrogate loss $L : \mathbb{R}^d \times \mathcal{Y} \rightarrow \mathbb{R}$ and link $\psi : \mathbb{R}^d \rightarrow \mathcal{R}$ pair $(L, \psi)$ is $\mathcal{P}$-calibrated with respect to $\ell$ if

\[
\forall p \in \mathcal{P} : \inf_{u \in \mathbb{R}^d, \psi(u) \notin \arg\min_{r \in \mathcal{R}} \mathbb{E}_p \ell(r, Y)} \mathbb{E}_p L(u, Y) > \inf_{u \in \mathbb{R}^d} \mathbb{E}_p L(u, Y). \tag{5}
\]

We simply say $L$ is calibrated if $\mathcal{P} = \Delta_{\mathcal{Y}}$.

Many works characterize calibrated surrogates for specific discrete target losses (Zhang, 2004; Lin, 2004; Bartlett et al., 2006; Tewari and Bartlett, 2007), including the canonical 0-1 loss for binary and multiclass classification. We give another definition of calibration which is a special case of calibration via Steinwart and Christmann (2008), and show it is equivalent to Definition 20 in discrete prediction settings, but can be applied in continuous estimation settings as well. We use this more general definition of calibration when proving statements about the relationship between consistency, calibration, and indirect elicitation.

The close connection between indirect elicitation and consistency was first explored by Agarwal and Agarwal (2015). In particular, calibration of $L \in \mathcal{L}$ with respect to $\ell$ implies indirect elicitation quite directly: take $u \in \mathbb{R}^d$ and $p \in \Gamma_u$, implying $u \in \Gamma(p)$. From eq. (1), $\mathbb{E}_p L(u, Y) = \inf_{u' \in \mathbb{R}^d} \mathbb{E}_p L(u', Y)$, so we must have $\psi(u) \in \gamma(p)$ from eq. (5), as desired.

**Definition 21 (Calibrated: Quadrants 1 and 3)** A loss $L : \mathbb{R}^d \times \mathcal{Y} \rightarrow \mathbb{R}$ is $\mathcal{P}$-calibrated with respect to a loss $\ell : \mathcal{R} \times \mathcal{Y} \rightarrow \mathbb{R}$ if there is a link $\psi : \mathbb{R}^d \rightarrow \mathcal{R}$ such that, for all distributions $p \in \mathcal{P}$, there exists a function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\zeta$ continuous at $0^+$ and $\zeta(0) = 0$ such that for all $u \in \mathbb{R}^d$, we have

\[
\ell(\psi(u); p) - \ell(p) \leq \zeta(\mathbb{E}_p L(u, Y) - L(p)). \tag{6}
\]

If $\mathcal{P} = \Delta_{\mathcal{Y}}$, we simply say $(L, \psi)$ is calibrated.

Consider the following four conditions: Suppose we are given $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

A $\zeta$ satisfies $\zeta : 0 \mapsto 0$ and is continuous at $0$.

B $\epsilon_m \rightarrow 0 \implies \zeta(\epsilon_m) \rightarrow 0$. 

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C. Given $\zeta : \mathbb{R} \rightarrow \mathbb{R}_+$, for all $u \in \mathbb{R}^d$, $R_\ell(\psi(u); p) \leq \zeta(R_L(u; p))$.

D. For all $p \in \mathcal{P}$ and sequences $\{u_m\}$ so that $R_L(u_m; p) \rightarrow 0$, we have $R_\ell(\psi(u_m); p) \rightarrow 0$.

The existence of a function $\zeta$ so that $(A \land C)$ defines calibration as in Definition 21, and we show $A \iff B$ in Lemma 23. Lemma 24 shows calibration if and only if $D$, which yields a condition equivalent to calibration without dependence on the function $\zeta$.

Proposition 22. When $\mathcal{R}$ and $\mathcal{Y}$ are finite, a continuous loss and link $(L, \psi)$ are $\mathcal{P}$-calibrated with respect to a target loss $\ell$ via Definition 21 if and only if they are $\mathcal{P}$-calibrated via Definition 20.

Proof $\implies$ We prove the contrapositive; if $(L, \psi)$ is not calibrated with respect to $\ell$ by Definition 20, then it is not calibrated via Definition 21 either. If $(L, \psi)$ are not calibrated with respect to $\ell$ by Definition 20, then there is a $p \in \mathcal{P}$ so that $\inf_{\psi \in \mathcal{P}} E_p L(u,Y) = \inf_u E_p L(u,Y)$. Thus there is a sequence $\{u_m\}$ so that $\lim_{m \rightarrow \infty} \psi(u_m) \neq \gamma(p)$ and $E_p L(u_m, Y) \rightarrow L(p)$. Now we have $R_L(u_m; p) \rightarrow 0$ but $R_\ell(\psi(u_m); p) \not\rightarrow 0$, so by Lemma 24, we contradict calibration by Definition 21.

$\iff$ Suppose there was a function $\zeta$ satisfying the bound in eq. (6) for a fixed distribution $p \in \mathcal{P}$. Observe the bound in eq. (5) can be written as $R_L(u, p) > 0$ for all $p \in \Delta_\mathcal{Y} \land u$ such that $\psi(u) \neq \gamma(p)$. By eq. (6), for any sequence $\{u_m\}$ so that $\psi(u_m) \not\rightarrow \gamma(p)$, we have $\zeta(R_L(\psi(u_m); p)) \not\rightarrow 0$ as we would otherwise contradict the bound in eq. (6) since $R_\ell(\psi(u), p) \not\rightarrow 0$. Therefore $R_L(u_m, p) \not\rightarrow 0$; thus, the strict inequality holds.

The following Lemma shows that conditions $A$ and $B$ are equivalent, so that we can use condition $B$ in lieu of condition $A$ in the proof of Lemma 24.

Lemma 23. A function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at 0 and $\zeta(0) = 0$ if and only if the sequence $\{u_m\} \rightarrow 0$.

Proof $\implies$ Suppose we have a sequence $\{u_m\} \rightarrow 0$. By continuity, we have $\lim_{u_m \rightarrow 0} \zeta(u_m) = \zeta(0) = 0$, so $\zeta(u_m) \rightarrow 0$.

$\iff$ Suppose $\zeta(0) \neq 0$ but $\zeta$ was continuous at 0. The constant sequence $\{u_m\} = 0$ then converges to 0, but as $\zeta$ is continuous at 0, we must have $\lim_{m \rightarrow \infty} \zeta(u_m) = \zeta(0) \neq 0$, so $\zeta(u_m) \not\rightarrow 0$.

Now suppose $\zeta(0) = 0$ but $\zeta$ was not continuous at 0. There must be a sequence $\{u_m\} \rightarrow 0$ so that $\lim_{m \rightarrow \infty} \zeta(u_m) \neq \zeta(0) = 0$, so $\zeta(u_m) \not\rightarrow 0$.  

Lemma 24 now gives a condition equivalent to calibration without requiring one to already have a function $\zeta$ in mind.

Lemma 24. A continuous surrogate and link $(L, \psi)$ are $\mathcal{P}$-calibrated (via definition 21) with respect to $\ell$ if and only if, for all $p \in \mathcal{P}$ and sequences $\{u_m\}$ so that $R_L(u_m; p) \rightarrow 0$, we have $R_\ell(\psi(u_m); p) \rightarrow 0$.

Proof $\implies$ Take a sequence $\{u_m\}$ so that $R_L(u_m; p) \rightarrow 0$. Since $\zeta(0) = 0$ and $\zeta$ is continuous at 0, we have $\zeta(R_L(u_m; p)) \rightarrow 0$. As the bound from Equation (6) is satisfied for
all \( u \in \mathbb{R}^d \) by assumption, we observe
\[
\forall m, 0 \leq R_\ell(\psi(u_m); p) \leq \zeta(R_L(u_m; p)) \\
\implies 0 \leq \lim_{m \to \infty} R_\ell(\psi(u_m); p) \leq \lim_{m \to \infty} \zeta(R_L(u_m; p)) = 0 \\
\implies 0 = \lim_{m \to \infty} R_\ell(\psi(u_m); p).
\]

\( \iff \) Fix \( p \in \mathcal{P} \), and consider \( \zeta(c) := \sup_{u:R_L(u;p) \leq c} R_\ell(\psi(u); p) \). We will show \( R_L(u_m; p) \to 0 \implies R_\ell(\psi(u_m); p) \to 0 \) gives calibration via the function \( \zeta \) constructed above. With \( \zeta \) as constructed, we observe that the bound in equation (6) is satisfied for all \( u \in \mathbb{R}^d \) and apply Lemma 23 to observe that if there is a sequence \( \{\epsilon_m\} \to 0 \) so that \( \zeta(\epsilon_m) \not\to 0 \), it is because \( R_L(u_m, p) \not\to 0 \iff R_\ell(\psi(u_m), p) \to 0 \).

Now, we observe that the bound in Equation (6) is satisfied for all \( u \in \mathbb{R}^d \) by construction of \( \zeta \). Let \( S(v) := \{u' \in \mathbb{R}^d : R_L(u';p) \leq R_L(v,p)\} \). Showing \( R_\ell(\psi(u); p) \leq \sup_{u' \in S(u)} R_\ell(\psi(u'); p) \) for all \( u \in \mathbb{R}^d \) gives the condition \( C \). As \( u \) is in the space over which the supremum is being taken (as \( R_L(u;p) \leq R_L(u; p) \)), we then have calibration by definition of the supremum.

Now suppose there exists a sequence \( \{\epsilon_m\} \to 0 \) so that \( \zeta(\epsilon_m) \not\to 0 \). Consider \( S(\epsilon) = \{u \in \mathbb{R}^d : R_L(u, p) \leq \epsilon\} \).

\[
\epsilon_1 \leq \epsilon_2 \implies S(\epsilon_1) \subseteq S(\epsilon_2) \\
\implies \zeta(\epsilon_1) \leq \zeta(\epsilon_2).
\]

Now suppose there exists a sequence \( \{u_m\} \) so that \( R_L(u_m, p) \to 0 \). Then for all \( \epsilon > 0 \), there exists a \( m' \in \mathbb{N} \) so that \( R_L(u_m, p) < \epsilon \) for all \( m \geq m' \). Since this is true for all \( \epsilon \), we have \( S(\epsilon) \) nonempty for all \( \epsilon > 0 \), and therefore \( \zeta(\epsilon) \) is discrete for all \( \epsilon > 0 \). Now if \( \zeta(\epsilon_m) \not\to 0 \), it must be because \( R_\ell(\psi(u_m), p) \not\to 0 \) for some sequence converging to zero surrogate regret, and therefore we contradict the statement \( R_L(u_m, p) \to 0 \implies R_\ell(\psi(u_m), p) \to 0 \).

Moreover, we argue that such a sequence of \( \{u_m\} \) with converging surrogate regret always exists by continuity and boundedness from below of the surrogate loss, since we can take the constant sequence at the (attained) infimum.  

**A.1. Relating calibration, consistency, and indirect elicitation.**

Even with the more general notion of calibration that extends beyond discrete predictions, we still have consistency implying calibration.

**Proposition 25** If a loss and link \((L, \psi)\) are consistent with respect to a loss \( \ell \), then they are calibrated with respect to \( \ell \).

**Proof** We show the contrapositive. If \((L, \psi)\) are not calibrated with respect to \( \ell \), then there is a sequence \( \{u_m\} \) such that \( R_L(u_m; p) \to 0 \) but \( R_\ell(\psi(u_m); p) \not\to 0 \) via Lemma 24. Suppose \( D \sim \mathcal{X} \times \mathcal{Y} \) has only one \( x \in \mathcal{X} \) with \( P_D(X = x) > 0 \) so that \( p := D_x \) and \( \mathbb{E}_D f(X, Y) = \mathbb{E}_p f(x, Y) \). Consider any sequence of functions \( \{f_m\} \to f \) with \( f_m(x) = u_m \) for all \( f_m \). Now we have \( \mathbb{E}_D L(f_m(X), Y) \to \inf_{f} \mathbb{E}_D L(f(X), Y) \), but \( \mathbb{E}_D \ell(\psi \circ f(X), Y) \not\to \inf_{f} \mathbb{E}_D \ell(\psi \circ f(X), Y) \), and therefore \((L, \psi)\) is not consistent with respect to \( \ell \).
Moreover, we have calibration implying indirect elicitation.

**Lemma 26** If a surrogate and link \((L, \psi)\) with \(L \in \mathcal{L}\) are calibrated with respect to a loss \(\ell : \mathcal{R} \times \mathcal{Y} \to \mathbb{R}\), then \(L\) indirectly elicits the property \(\gamma := \text{prop}_\mathcal{L}[\ell]\).

**Proof** Let \(\Gamma\) be the unique property directly elicited by \(L\), and fix \(p \in \Delta_\mathcal{Y}\) with \(u\) such that \(p \in \Gamma_u\). We know such a \(u\) exists since \(\Gamma(p) \neq \emptyset\). As \(p \in \Gamma_u\), then \(\zeta(\mathbb{E}_p L(u, Y) - L(p)) = \zeta(0) = 0\), we observe the bound \(\ell(\psi(u); p) \leq \ell(p)\). We also have \(\ell(\psi(u); p) \geq \ell(p)\) by definition of \(\ell\), so we must have \(\ell(\psi(u); p) = \ell(p) = \ell(\gamma(p); p)\), and therefore, \(p \in \gamma_\psi(u)\). Thus, we have \(\Gamma_u \subseteq \gamma_\psi(u)\), so \(L\) indirectly elicits \(\gamma\). \qed

Combining the two results, we can observe the result of Theorem 8 another way: through calibration.

**Appendix B. Reconstructing Ramaswamy and Agarwal (2016, Thm. 16)**

**Lemma 27** Let the \(d\)-flat \(F \subseteq \mathcal{P}\) (defined over finite \(\mathcal{Y}\)) contain some \(p \in \text{relint}(\mathcal{P})\). Then

(i) \(p \in \text{relint}(F)\);

(ii) \(\dim(\mathcal{S}_F(p)) \geq \dim(\text{affhull}(\mathcal{P})) - d\).

**Proof**

As \(F\) is a \(d\)-flat, we have some \(W : \mathcal{Y} \to \mathbb{R}^d\) such that \(F = \ker_\mathcal{P} W\). Throughout, given a point (typically a distribution) \(p\) and convex set \(\mathcal{P}\), we define \(\mathcal{P}_p := \mathcal{P} - \{p\}\). Define \(T_W : \text{span}(\mathcal{P}_p) \to \mathbb{R}^d, v \mapsto E_{\mathcal{P}} v W\).

(i) Since \(p \in \text{relint}(\mathcal{P})\), for all \(q \in \mathcal{P}\), there is some small enough \(\epsilon > 0\) so that for all \(\alpha \in (-\epsilon, \epsilon)\), the point \(q_\alpha := p - \alpha(q - p)\) is still in \(\mathcal{P}\). In particular, for \(q \in F\), we claim \(q_\alpha \in F\). As \(p, q \in F\), we have \(E_q W = E_{q_\alpha} W = 0\). By linearity of expectation, we then have \(E_{q_\alpha} W = 0\). This implies \(q_\alpha \in F\), and therefore \(p \in \text{relint}(F)\).

(ii) We first show \(\text{span}(F_p) = \mathcal{S}_F(p)\). First, take \(v \in \mathcal{S}_F(p)\), and take \(\epsilon_0\) as in the definition. For \(\epsilon = \epsilon_0/2\), we then have \(p + \epsilon v \in F \implies \epsilon v \in F_p\), and therefore, \(v \in \text{span}(F_p)\).

Now take \(v \in \text{span}(F_p)\). Since \(p \in \text{relint}(F)\) (i), we have \(0 \in \text{relint}(F_p)\). Therefore there is an \(\epsilon_0 > 0\) so that \(\epsilon v \in F_p\) for all \(\epsilon \in (-\epsilon_0, \epsilon_0)\) by convexity of \(F\). Therefore, \(v \in \mathcal{S}_F(p)\), and we observe \(\mathcal{S}_F(p) = \text{span}(F_p)\).

We now show \(\mathcal{S}_F(p) = \ker(T_W)\). Observe that \(\mathcal{S}_F(p) \subseteq \ker(T_W)\) follows trivially from the definitions of the two functions. Now let \(v \in \ker(T_W)\), and \(v' \in F_p\). This means \(E_{\mathcal{P}} v W = 0\), so it suffices to show \(v = \alpha v' \in F_p\), thus showing \(v \in \mathcal{S}_F(p)\). Since \(p \in \text{relint}(\mathcal{P})\), we must have \(0 \in \text{relint}(F_p)\), so we know there is some small enough \(\epsilon > 0\) so that \(-\alpha v' \in F_p\) for \(\alpha \in (-\epsilon, \epsilon)\). Take \(c = -\alpha\), and we conclude \(v \in \mathcal{S}_F(p)\). Therefore, \(\ker(T_W) = \mathcal{S}_F(p)\).

We finally want to show \(\dim(\text{affhull}(\mathcal{P})) = \dim(\text{span}(\mathcal{P}_p))\). Consider that any \(q \in \text{span}(\mathcal{P}_p)\) can be written as a scalar multiple of an element of \(\mathcal{P}_p\), which can be written as a convex combination of elements of the minimal basis \(\mathcal{P}_p\). In particular, since \(0 \in \mathcal{P}_p\), it can be written as an affine combination of elements of the basis, so \(\dim(\text{affhull}(\mathcal{P})) \geq \dim(\text{span}(\mathcal{P}_p))\). We also have \(\text{affhull}(\mathcal{P}) - \{p\} \subseteq \text{span}(\mathcal{P}_p)\), so \(\dim(\text{affhull}(\mathcal{P})) = \dim(\text{affhull}(\mathcal{P}) - \{p\}) \leq \text{span}(\mathcal{P}_p)\). Therefore, \(\dim(\text{affhull}(\mathcal{P})) = \dim(\text{span}(\mathcal{P}_p))\).
As \( \mathcal{Y} \) is a finite set, \( \text{span}(\mathcal{P}_p) \) is a finite-dimensional vector space. The rank-nullity theorem states \( \dim(\text{im}(T_W)) + \dim(\ker(T_W)) = \dim(\text{span}(\mathcal{P}_p)) = \dim(\text{affhull}(\mathcal{P})) \). As \( \dim(\text{im}(T_W)) \leq d \), and we have shown above that \( \mathcal{S}_F(p) = \text{span}(\mathcal{F}_p) = \ker(T_W) \), the conclusion follows.

**Corollary 28 (Ramaswamy and Agarwal (2016) Theorem 18)** Let \( \ell : \mathcal{R} \times \mathcal{Y} \to \mathcal{R} \) be a discrete loss eliciting \( \gamma : \Delta_\mathcal{Y} \Rightarrow \mathcal{R} \) with \( \mathcal{Y} \) finite. Then for all \( p \in \Delta_\mathcal{Y} \) and \( r \in \gamma(p) \),

\[
\text{cons}_{\text{cvx}}(\gamma) \geq \|p\|_0 - \dim(\mathcal{S}_{\gamma_r}(p)) - 1.
\]

**Proof** Let \( L \in \mathcal{L}_d^\text{cvx} \) be a calibrated surrogate for \( \ell \), and let \( \Gamma := \text{prop}_{\Delta_\mathcal{Y}}[L] \). Consider \( \mathcal{Y}' := \{y \in \mathcal{Y} : p_y > 0\} \) and \( p' = (p_y)_{y \in \mathcal{Y}'} \in \Delta_\mathcal{Y}' \). Take \( \mathcal{L}' := L|_{\mathcal{Y}'} \) and \( \ell' := \ell|_{\mathcal{Y}'} \). Define \( h : \mathcal{R}^{\mathcal{Y}'} \to \mathcal{R}^{\mathcal{Y}'} \) such that \( h(q') = q \) such that \( q_y = q_{y}' \) for \( y \in \mathcal{Y}' \) and \( q_y = 0 \) otherwise. Take \( \Gamma' = \Gamma \circ h \), \( \gamma' = \gamma \circ h \).

We wish to first show \( \mathcal{L}' \) indirectly elicits \( \gamma' \). Since \( L \) indirectly elicits \( \gamma \), we have a link \( \psi \) such that for all \( u \in \mathcal{R}^d, \Gamma_u \subseteq \gamma_{\psi(u)} \). As \( \Gamma'(q) = \Gamma(h(q)) \) and \( \gamma'(q) = \gamma(h(q)) \), we have \( q \in \Gamma'_u \iff h(q) \in \Gamma_u \iff h(q) \in \gamma_{\psi(u)} \iff (q_y)_{y \in \mathcal{Y}'} \in \gamma'_{\psi(u)} \), and therefore, \( \mathcal{L}' \) indirectly elicits \( \gamma' \) via the link \( \psi \circ \text{proj}(\mathcal{Y}') \), where \( \text{proj}(\mathcal{Y}') : q \mapsto (q_y)_{y \in \mathcal{Y}'} \).

We aim to show \( \dim(\mathcal{S}_{\gamma_r}(p)) \geq \dim(\mathcal{S}_{\gamma'_r}(p')) \). We do this by showing that \( h(\mathcal{S}_{\gamma_r}(p)) \subseteq \mathcal{S}_{\gamma'_r}(p') \) and the result holds as \( h \) is linear and injective. Suppose \( v \in h(\mathcal{S}_{\gamma_r}(p)) \), then there exists a \( v' \) so that \( v = h(v') \) and an \( \epsilon > 0 \) such that \( ev' + p' \in \gamma'_r \) for all \( \epsilon \in (-\epsilon_0, \epsilon_0) \). Since \( h \) is linear and recall \( h(\gamma'_r) \subseteq \gamma_r \), this implies \( ev + p \in \gamma_r \) for all \( \epsilon \in (-\epsilon_0, \epsilon_0) \). Therefore \( v \in \mathcal{S}_{\gamma_r}(p) \), and the result follows.

As \( \mathcal{L}' \) indirectly elicits \( \gamma' \), by Corollary 13, we know there exists a \( d \)-flat \( F \) with \( p' \in F \subseteq \gamma'_r \). Taking \( \mathcal{P} = \Delta_{\mathcal{Y}'} \), we know \( p' \in \text{relint}(\Delta_{\mathcal{Y}'}) \) by construction, so we can apply Lemma 27(ii), which gives \( \dim(\mathcal{S}_{\gamma_r}(p')) \geq \dim(\text{affhull}(\Delta_{\mathcal{Y}'}) - d = \|p\|_0 - 1 - d \). Additionally, \( \mathcal{S}_{\gamma_r}(p) \subseteq \mathcal{S}_{\gamma'_r}(p') \) by subset inclusion of the sets themselves. Chaining these results, we obtain

\[
\dim(\mathcal{S}_{\gamma_r}(p)) \geq \dim(\mathcal{S}_{\gamma'_r}(p')) \geq \dim(\mathcal{S}_F(p')) \geq \|p\|_0 - 1 - d.
\]

**Appendix C. Proof of Theorem 17**

**C.1. General setting of elicitation complexity**

We briefly introduce the general notion of elicitation complexity, of which Definition 4 is a special case, as some statements are more naturally made in this general setting.

**Definition 29** \( \Gamma' \) refines \( \Gamma \) if for all \( r' \in \text{range} \Gamma' \) there exists \( r \in \text{range} \Gamma \) with \( \Gamma_{r'} \subseteq \Gamma_r \).

Equivalently, \( \Gamma' \) refines \( \Gamma \) if there is a link function \( \psi : \text{range} \Gamma' \to \text{range} \Gamma \) such that \( \Gamma_{r'} \subseteq \Gamma_{\psi(r')} \) for all \( r' \in \text{range} \Gamma' \).

\[3\] To reason about \( \dim(\text{affhull}(\Delta_{\mathcal{Y}'}) \) = \|p\|_0 - 1, observe that the uniform distribution on \( \Delta_{\mathcal{Y}'} \) has full support and therefore requires \( \|p\|_0 - 1 \) elements in its basis.

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Definition 30  For \( k \in \mathbb{N} \cup \{\infty\} \), let \( \mathcal{E}_k(\mathcal{P}) \) denote the class of all elicitable properties \( \Gamma : \mathcal{P} \to \mathbb{R}^k \), and \( \mathcal{E}(\mathcal{P}) := \bigcup_{k \in \mathbb{N} \cup \{\infty\}} \mathcal{E}_k(\mathcal{P}) \). When \( \mathcal{P} \) is implicit we simply write \( \mathcal{E} \).

Definition 31  Let \( \mathcal{C} \) be a class of properties. The elicitation complexity of a property \( \Gamma \) with respect to \( \mathcal{C} \), denoted \( \text{elic}_\mathcal{C}(\Gamma) \), is the minimum value of \( k \in \mathbb{N} \cup \{\infty\} \) such that there exists \( \Gamma' \in \mathcal{C} \cap \mathcal{E}_k(\mathcal{P}) \) that refines \( \Gamma \).

C.2. Supporting statements

Proposition 32 (Osband (1985))  Let \( \Gamma \) be elicitable. Then \( \Gamma_r \) is convex for all \( r \in \text{range} \Gamma \).

Lemma 33 (Set-valued extension of Frongillo and Kash (2020, Lemma 4))  If \( \Gamma' \) refines \( \Gamma \) then \( \text{elic}_\mathcal{C}(\Gamma') \geq \text{elic}_\mathcal{C}(\Gamma) \).

Proof  As \( \Gamma' \) refines \( \Gamma \), we have some \( \psi : \text{range} \Gamma' \to \text{range} \Gamma \) such that for all \( r' \in \text{range} \Gamma' \) we have \( \Gamma'_{r'} \subseteq \psi(\Gamma_{r'}) \). Suppose we have \( \Gamma' \in \mathcal{C} \) and \( \varphi : \text{range} \Gamma' \to \text{range} \Gamma' \) such that for all \( u \in \text{range} \Gamma \) we have \( \Gamma'_u \subseteq \Gamma'_{\varphi(u)} \). Then for all \( u \in \text{range} \Gamma \) we have \( \Gamma'_u \subseteq \Gamma'_{\varphi(u)} \subseteq \Gamma_{\psi(\varphi(u))} \).

In particular, if \( \text{elic}_\mathcal{C}(\Gamma') = m \), then we have such a \( \Gamma : \mathcal{P} \to \mathbb{R}^m \), and hence \( \text{elic}_\mathcal{C}(\Gamma) \leq m \).

Lemma 34 (Frongillo and Kash (2020, Lemma 8))  Suppose \( L \in \mathcal{L} \) elicits \( \Gamma : \mathcal{P} \to \mathcal{R} \) and has Bayes risk \( L \). Then for any \( p, p' \in \mathcal{P} \) with \( \Gamma(p) \neq \Gamma(p') \), we have \( L(\lambda p + (1-\lambda)p') > \lambda L(p) + (1-\lambda)L(p') \) for all \( \lambda \in (0, 1) \).

Lemma 35 (Adapted from Frongillo and Kash (2020, Theorem 4))  If \( L \) elicits a single-valued \( \Gamma \), and \( \hat{\Gamma} \) refines \( \mathcal{L} \), then \( \hat{\Gamma} \) refines \( \Gamma \).

Proof  Suppose for a contradiction that \( \hat{\Gamma} \) does not refine \( \Gamma \). Then we have some \( u \in \text{range} \hat{\Gamma} \) such that for all \( r \in \text{range} \Gamma \) we have \( \hat{\Gamma}_r \nsubseteq \Gamma_r \). In particular, recalling that \( \Gamma \) is single-valued, we must have \( p, p' \in \Gamma_u \) such that \( \Gamma(p) \neq \Gamma(p') \). Moreover, as \( \hat{\Gamma} \) refines \( \mathcal{L} \), we also have \( L(p) = L(p') \). From Lemma 34 and \( \lambda = 1/2 \) we have \( L(q) > \frac{1}{2}L(p) + \frac{1}{2}L(p') = L(p) \), where \( q = \frac{1}{2}p + \frac{1}{2}p' \). As the level set \( \hat{\Gamma}_u \) is convex by Proposition 32, we also have \( q \in \hat{\Gamma}_u \), and hence \( L(q) = L(p) \), a contradiction.

Lemma 36 (Minor modifications from Frongillo and Kash (2020))  Let \( \mathcal{V} \) be a real vector space. Let \( f : \mathcal{V} \to \mathbb{R}^k \) be linear and \( \mathcal{C} \subseteq \mathcal{V} \) convex with \( \text{span} \mathcal{C} = \mathcal{V} \), and let \( m \in \mathbb{N} \). Suppose that \( 0 \in \text{int} f(C) \), and for all \( v \in S := C \cap \text{ker} f \), there exists a linear \( \hat{f}_v : \mathcal{V} \to \mathbb{R}^m \) with \( v \in C \cap \text{ker} \hat{f}_v \subseteq S \). Then \( \mathcal{V} \geq k \). If \( m = k \), we additionally have \( 0 \in \text{int} \hat{f}_v(C) \) for some \( v \in S \).

Proof  The condition \( 0 \in \text{int} f(C) \) is equivalent to the existence of some \( v_1, \ldots, v_{k+1} \in C \) such that \( 0 \in \text{int conv}\{f(v_i) : i \in \{1, \ldots, k+1\}\} \). Let \( \alpha_1, \ldots, \alpha_{k+1} > 0 \), \( \sum_{i=1}^{k+1} \alpha_i = 1 \), such that \( \sum_{i=1}^{k+1} \alpha_i f(v_i) = 0 \). As these are barycentric coordinates, this choice of \( \alpha_i \) is unique, a fact which will be important later. We will take \( v = \sum_{i=1}^{k+1} \alpha_i v_i \), an element of \( C \) by convexity, and thus an element of \( S \) as \( f(v) = 0 \).
Let \( \hat{f}_v : \mathcal{V} \to \mathbb{R}^m \) be linear with \( v \in \hat{S} := C \cap \ker \hat{f}_v \subseteq S \). Let \( \beta_1, \ldots, \beta_{k+1} \in \mathbb{R} \), \( \sum_{i=1}^{k+1} \beta_i = 0 \), such that \( \sum_{i=1}^{k+1} \beta_i \hat{f}_v(v_i) = 0 \). We will show that the \( \beta_i \) must be identically zero, i.e. that \( \{ \hat{f}_v(v_i) : i \in \{1, \ldots, k+1\} \} \) are affinely independent. By construction, \( v' := \sum_{i=1}^{k+1} \beta_i v_i \in \ker \hat{f}_v \), and as \( v \in \ker \hat{f}_v \) for all \( \lambda > 0 \) we have \( v_\lambda := v + \lambda v' = \sum_{i=1}^{k+1} (\alpha_i + \lambda \beta_i) v_i \in \ker \hat{f}_v \). Taking \( \lambda \) sufficiently small, we have \( \gamma_i := \alpha_i + \lambda \beta_i > 0 \) for all \( i \), and \( \sum_{i=1}^{k+1} \gamma_i = \sum_{i=1}^{k+1} \alpha_i + \lambda \sum_{i=1}^{k+1} \beta_i = 1 \). By convexity of \( C \), we have \( v_\lambda \in C \). Now \( v_\lambda \in C \cap \ker \hat{f}_v \subseteq S = C \cap \ker f \), and in particular \( v_\lambda \in \ker f \). Thus, \( f(v_\lambda) = \sum_{i=1}^{k+1} \gamma_i f(v_i) = 0 \).

By the uniqueness of barycentric coordinates, for all \( i \in \{1, \ldots, k+1\} \), we must have \( \gamma_i = \alpha_i \) and thus \( \beta_i = 0 \), as desired.

As \( \hat{f}_v(C) \) contains \( k+1 \) affinely independent points, we have \( m \geq \dim \text{im} \hat{f}_v \geq k \). When \( m = k \), by affine independence, the set \( \text{conv} \{ \hat{f}_v(v_i) : i \in \{1, \ldots, k+1\} \} \) has dimension \( k \) in \( \mathbb{R}^k \). As \( 0 = \hat{f}_v(v) = \sum_{i=1}^{k+1} \alpha_i \hat{f}_v(v_i) \), and \( \alpha_i > 0 \) for all \( i \), we conclude \( 0 \in \text{int conv} \{ \hat{f}_v(v_i) : i \in \{1, \ldots, k+1\} \} \subseteq \text{int} \hat{f}_v(C) \). 

Lemma 37 (Frongillo and Kash (2020, Lemma 14)) Let \( \mathcal{V} \) be a real vector space. Let \( f : \mathcal{V} \to \mathbb{R}^k \) be linear, \( C \subseteq \mathcal{V} \) convex with \( \text{span} C = \mathcal{V} \), and let \( S = C \cap \ker f \). If \( 0 \in \text{int} f(C) \) then \( \text{span} S = \ker f \).

C.3. Proving the lower bound for spectral risks

Let \( C^*_d \) be the class of properties \( \Gamma \) which are elicited by a convex loss \( L \in \mathcal{L}^\text{cvx}_d \) for some \( d \in \mathbb{N} \), and let \( C^* := \bigcup_{d \in \mathbb{N}} C^*_d \). Then for all properties \( \gamma \), if \( \text{elicit}_{C^*} (\gamma) < \infty \), we have \( \text{elicit}_{C^*} (\gamma) = \text{elicit}_{\text{cvx}} (\gamma) \), a fact we use tacitly in the proof.

Theorem 17 Let \( \mathcal{P} \) be a set of Lebesgue densities supported on the same set for all \( p \in \mathcal{P} \). Let \( \Gamma : \mathcal{P} \to \mathbb{R}^d \) satisfy Condition 1 for some \( r \in \mathbb{R}^d \). Let \( L \in \mathcal{L}^\text{cvx} \) elicit \( \Gamma \) such that \( L \) is non-constant on \( \Gamma_r \). Then \( \text{cons}_{\text{cvx}} (L) \geq \text{elicit}_{\text{cvx}} (L) \geq d + 1 \).

Proof Let \( V : \mathcal{V} \to \mathbb{R}^d \) and \( r \) be given by the statement of the theorem and from Condition 1. Let \( m = \text{elicit}_{C^*_m} (L) \), so that we have \( \hat{\Gamma} \in C^*_m \) which refines \( L \). By Lemma 35 we have \( \hat{\Gamma} \) refines \( \Gamma \).

We now establish the conditions of Lemma 36 for \( C = \mathcal{P} \). Let \( f : \text{span} \mathcal{P} \to \mathbb{R}^d, p \mapsto \mathbb{E}_p V \). From Condition 1, we have \( 0 \in \text{int} f(\mathcal{P}) \) and \( \ker f \cap \mathcal{P} = \ker \mathbb{E}_p V = \Gamma_r \). Now let \( p \in \Gamma_r \) be arbitrary, and take any \( u \in \hat{\Gamma}(p) \). As \( \Gamma \) is single-valued, \( r \in \text{range} \Gamma \) is the unique value with \( p \in \Gamma_r \). As \( \hat{\Gamma} \) refines \( \Gamma \), there exists \( r' \in \text{range} \Gamma \) with \( \hat{\Gamma}_u \subseteq \Gamma_{r'} \), and since \( p \in \hat{\Gamma}_u \), we conclude \( r' = r \) from the above. From Lemma 11, we have some \( \hat{V}_{u,p} \) with \( p \in \ker \mathbb{E}_p \hat{V}_{u,p} \subseteq \hat{\Gamma}_u \subseteq \Gamma_r = \ker \mathbb{E}_p V \). Letting \( \hat{f}_p : \text{span} \mathcal{P} \to \mathbb{R}^d, p \mapsto \mathbb{E}_p \hat{V}_{u,p} \), we have now satisfied the conditions of Lemma 36. We conclude \( m \geq d \), and moreover, if \( m = d \), then there exists some \( q \in \Gamma_r \) such that \( 0 \in \text{int} \hat{f}_q(\mathcal{P}) \).

Now suppose \( m = d \) for a contradiction. Let \( \hat{S} := \ker \hat{f}_q \cap \mathcal{P} \). Applying Lemma 37 to the functions \( f \) and \( \hat{f}_q \) we have \( \text{span ker } f = \text{span} \Gamma_r \) and \( \text{span ker } \hat{f}_q = \text{span} \hat{S} \). As \( \hat{S} \subseteq \Gamma_r \), we have \( \ker \hat{f}_q \) is linear, \( \hat{S} \subseteq \text{span} \Gamma_r = \ker f \). By the first isomorphism theorem, we also have \( \text{codim} \ker \hat{f}_q = \text{codim} \ker f = d \), as the images of these linear maps span all of \( \mathbb{R}^d \). By the third isomorphism theorem we conclude \( \Gamma_r = \hat{S} \). Moreover, as \( \hat{S} \subseteq \Gamma_u \subseteq \Gamma_r \), we have \( \hat{S} = \Gamma_u = \Gamma_r \).
We now see that \( L \) is constant on \( \Gamma_r \) since there is some link function \( \psi : \mathbb{R}^m \to \mathbb{R} \) such that \( \Gamma_r = \Gamma_u \subseteq \mathcal{L}_{\psi(u)} \), meaning \( L(p) = \psi(u) \) for all \( p \in \Gamma_r \). This statement contradicts the assumption that \( L \) is non-constant on \( \Gamma_r \).

\[ \square \]

Appendix D. Miscellaneous omitted proofs

**Lemma 7** Let a convex \( \mathcal{P} \subseteq \Delta_Y \) be given. Given a surrogate loss \( L \in \mathcal{L} \), link \( \psi \), and target loss \( \ell \), set \( \mu(r,p) := R(\ell(r,p)) \). Then there is a \( \mathcal{D} \) such that \( (L, \psi) \) is \( \mathcal{D} \)-consistent with respect to \( \ell \) if and only if \( \mu(r,p) \) is \( (\mu, \mathcal{D}) \)-consistent with respect to \( \gamma := \text{prop}_\mathcal{P}[\ell] \).

**Proof** First, observe that \( \mu(r,p) = 0 \iff \mathbb{E}_p \ell(r,Y) = \inf_{r'} \mathbb{E}_p \ell(r',Y) \iff r \in \gamma(p) \).

Now suppose \( (L, \psi) \) are consistent with respect to \( \ell \), and take any sequence \( \{f_m\} \) of measurable hypotheses. Rewriting the right-hand side of Definition 5,

\[ \mathbb{E}_{f_m(X)} \rightarrow \mathbb{E}_D \ell(\psi \circ f_m(X), Y) \rightarrow \mathbb{E}_D \mu(r,p) \implies \mathbb{E}_D \ell(r,Y) \to 0 \]

Therefore, \( \mathbb{E}_D(L(f_m(X), Y) \to \inf_{r,p} \mathbb{E}_D L(f(X), Y) \) implies (7) if and only if it implies (8). Observe that the assumptions on \( \mathcal{L} \) allow us to apply the Fubini-Tonelli Theorem (Folland, 1999, Theorem 2.37), which yields the equivalence of eq. 7 to the next line.

A hyperplane weakly separates two sets if its two closed halfspaces respectively contain the two sets.

**Lemma 38** If \( \gamma : \mathcal{P} \Rightarrow \mathcal{R} \) is an elicitable property, then for any pair of predictions \( r, r' \in \mathcal{R} \) where \( \gamma_r \neq \gamma_{r'} \), there is a hyperplane \( H = \{ x \in \mathbb{R}^q : v \cdot x = 0 \} \), for some \( v \in \mathbb{R}^q \), that weakly separates \( \gamma_r \) and \( \gamma_{r'} \), and has \( \gamma_r \cap H = \gamma_{r'} \cap H = \gamma_r \cap \gamma_{r'} \).

**Proof** Let \( \ell \) elicit \( \gamma \). Let \( v = \ell(r, \cdot) - \ell(r', \cdot) \), interpreted as a nonzero vector in \( \mathbb{R}^q \). Let \( H = \{ q : v \cdot q = 0 \} \). If \( v \cdot q < 0 \), then \( r' \) cannot be optimal, so \( q \notin \gamma_{r'} \). So \( \gamma_{r'} \subseteq \{ q : v \cdot q \geq 0 \} \).

Symmetrically, \( \gamma_r \subseteq \{ q : v \cdot q \leq 0 \} \). This is weak separation, and it immediately implies that \( \gamma_r \cap \gamma_{r'} \subseteq H \). Finally, if and only if \( v \cdot q = 0 \), i.e. \( q \in H \), by definition the expected losses of both reports are the same. So \( \gamma _r \cap H \iff q \in \gamma_{r'} \cap H \). This gives \( \gamma_r \cap H = \gamma_{r'} \cap H = \gamma_r \cap \gamma_{r'} \).

\[ \square \]

**Lemma 39** Suppose we are given an elicitable property \( \gamma : \mathcal{P} \Rightarrow \mathcal{R} \), where \( \mathcal{Y} \) is finite, and distribution \( p \in \text{relint}(\mathcal{P}) \) such that \( p \in \gamma_r \cap \gamma_{r'} \) for \( r, r' \in \mathcal{R} \). Then for any flat \( F \) containing \( p, F \subseteq \gamma_r \iff F \subseteq \gamma_{r'} \).

**Proof** If \( \gamma_r = \gamma_{r'} \), we are done. Otherwise, Lemma 38 gives a hyperplane \( H = \{ x \in \mathbb{R}^q : v \cdot x = 0 \} \) and a guarantee that \( \gamma_r \subseteq \{ q \in \mathbb{R}^q : v \cdot q \leq 0 \} \), while \( \gamma_{r'} \subseteq \{ q \in \mathbb{R}^q : v \cdot q \geq 0 \} \), and finally \( \gamma_r \cap \gamma_{r'} \subseteq H \).

Suppose \( F \subseteq \gamma_r \); we wish to show \( F \subseteq \gamma_{r'} \). Let \( q \in F \). By Lemma 27(i), we have \( p \in \text{relint}(F') \), so there exists \( \epsilon > 0 \) so that \( q' = p - \epsilon(q - p) \in F' \).
Now, suppose for contradiction that $q \notin \gamma_r$. Then $v \cdot q < 0$: containment in $\gamma_r$ gives $v \cdot q \leq 0$, and if $v \cdot q = 0$ then $q \in \gamma_r \cap H \implies q \in \gamma_r'$, a contradiction. But, noting that $p \in H$, we have $v \cdot q' = -\epsilon(v \cdot q) > 0$, so $q'$ is not in $\gamma_r$. This contradicts the assumption $F \subseteq \gamma_r$. Therefore, we must have $q \in \gamma_r'$, so we have shown $F \subseteq \gamma_r'$. Because $r$ and $r'$ were completely symmetric, this completes the proof.

Appendix E. Omitted Examples

Discrete problem with no target loss (Quadrant 2). Consider the following scenario where someone is deciding how to dress for the weather based on a meteorologist’s forecast. Consider the three outcomes $\mathcal{Y} = \{\text{rainy, sunny, snowy}\}$, and we suppose we want to have some bias towards health and safety, so the meteorologist should only predict sunny weather if $Pr[\text{sunny} | \text{weather data}] \geq \frac{3}{4}$. Otherwise, they should predict whatever is more likely given the weather data: rain or snow.

We can now model this problem by a property with the reports $\mathcal{R} = \mathcal{Y}$, and have

$$
\gamma(p) = \begin{cases} 
\text{sunny} & p_{\text{sunny}} \geq \frac{3}{4} \\
\text{rainy} & p_{\text{sunny}} \leq \frac{3}{4} \land p_{\text{rainy}} \geq p_{\text{snowy}} \\
\text{snowy} & p_{\text{sunny}} \leq \frac{3}{4} \land p_{\text{snowy}} \geq p_{\text{rainy}}
\end{cases}
$$

shown in Figure 3. Since the cells of elicitable properties in the simplex form a power diagram (Lambert and Shoham, 2009), we know that there is actually no target loss that directly elicits this problem. Constructing a consistent surrogate for this task is ill-defined without Definition 6. The function $\mu(r, p) = I\{r \notin \gamma(p)\}$ now allows us to use Definition 6 to think about consistent surrogates for this task.

Intuitively, since the feasible subspace dimension bound would be lowest at the distribution $p = (1/8, 3/4, 1/8)$, we might want to test Corollary 12 or Corollary 13 at $p$. However, we cannot apply either at $p$ since $\gamma(p) = \{\text{rainy, snowy, sunny}\}$ but the property is not elicitable. Ramaswamy and Agarwal (2016, Theorem 16) cannot draw any conclusions about this property for two reasons: first, we are given a target property instead of a target loss. Second, since the property is not elicitable (hence why there can be no target loss), we observe $\dim(S_{\text{rainy}}(p)) \neq \dim(S_{\text{sunny}}(p))$, contradicting the requirements of Ramaswamy and Agarwal (2016, Lemma 23).

However, our bounds from Corollary 12 on the distribution $q = (1/8, 3/4 - \epsilon, 1/8 + \epsilon)$ for a small enough $\epsilon > 0$, which we can apply since $\gamma(q) = \{\text{snowy}\}$, suggest that the convex elicitation complexity $\text{elic}_{cvx}(\gamma) \geq 2$, since there is no way to draw a 1-flat (a line, since $q \in \text{relint}(\Delta_{\mathcal{Y}})$) through $q$ while staying in just one level set on the simplex.

This example also extends to other decision-tree-like properties that do not have an explicit or easily constructed target loss.
Figure 3: A meteorology example with a bias towards citizen safety.