Potential games are games in which preferences of all players are aligned with a global objective.
- easy to analyze
- pure Nash equilibrium exists
- simple dynamics converge to an equilibrium

How “close” is a game to a potential game?

What is the topology of the space of preferences?

Are there “natural” decompositions of games?
Potential Games

- We consider finite games in strategic form:

  \[ G = \langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle \]

- \( G \) is an exact potential game if there exists \( \Phi : E \to \mathbb{R} \) such that

  \[
  u^m(x^m, x^{-m}) - u^m(y^m, x^{-m}) = \Phi(x^m, x^{-m}) - \Phi(y^m, x^{-m})
  \]

- Weaker notion: ordinal potential game, if the utility differences above agree only in sign.

- Potential \( \Phi \) aggregates and explains incentives of all players.

- Examples: congestion games, etc.
A global maximum of an ordinal potential is a pure Nash equilibrium.
Every finite potential game has a pure equilibrium.
Many learning dynamics (e.g., better-reply dynamics, fictitious play, spatial adaptive play) “converge” to a pure Nash equilibrium [Monderer and Shapley 96], [Young 98], [Hofbauer, Sandholm 00], [Marden, Arslan, Shamma 06, 07].
Potential Games

- When is a given game a potential game?
- More important, what are the obstructions, and what is the underlying structure?
Existence of Exact Potential

A path is a collection of strategy profiles $\gamma = (x_0, \ldots, x_N)$ such that $x_i$ and $x_{i+1}$ differ in the strategy of exactly one player where $x_i \in E$ for $i \in \{0, 1, \ldots, N\}$. For any path $\gamma$, let

$$I(\gamma) = \sum_{i=1}^{N} u^{m_i}(x_i) - u^{m_i}(x_{i-1}),$$

where $m_i$ denotes the player changing its strategy in the $i$th step.

**Theorem ([Monderer and Shapley 96])**

A game $G$ is an exact potential game iff for all simple closed paths $\gamma$, $I(\gamma) = 0$. Moreover, it is sufficient to check closed paths of length 4.

A linear condition, thus the set of exact potential games is a subspace.
Game Flows

A key reformulation: instead of utilities, a flow on a graph

- Nodes are strategy profiles
- Edges between comparable strategy profiles
- Labeled by utility differences
- Isomorphic to a direct product of $M$ cliques (one per player)
- E.g., for (modified) battle-of-the-sexes:

|     | O  | F  |
|-----|----|----|
| O   | 4, 2 | 0, 0 |
| F   | 1, 0 | 2, 3 |

$(O, O) \xleftarrow{2} (O, F)$

$(O, O) \xrightarrow{3} (F, O)$

$(O, F) \xrightarrow{2} (O, F)$

$(F, O) \xrightarrow{3} (F, F)$
Game Flows: 3-Player Example

- $E^m = \{a, b\}$ for all $m \in \mathcal{M}$, and payoff of player $i$ be $-1$ if its strategy is the same with its successor, $0$ otherwise.
- This game is neither an exact nor an ordinal potential game.

![Graph of Game Flows: 3-Player Example](image-url)
Global Structure of Preferences

- What is the global structure of these cycles?
- Equivalently, topological structure of aggregated preferences.
- Conceptually similar to structure of (continuous) vector fields.
- A well-developed theory from algebraic topology, we need the combinatorial analogue (e.g., [Jiang-Lim-Yao-Ye 08]).
Helmholtz (Hodge) Decomposition

The Helmholtz Decomposition allows orthogonal decomposition of a vector field into three vector fields:

- Gradient flow (globally acyclic component)
- Harmonic flow (locally acyclic but globally cyclic component)
- Curl flow (locally cyclic component).

Figure: Helmholtz Decomposition
Helmholtz decomposition (a cartoon)
Decomposition example

(a) Original game.

(b) Potential Component.

(c) Harmonic Component.
Decomposition

\[ \mathcal{G}_{\mathcal{M},E} \cong C_0^M \]

\[ \begin{array}{c}
  C_0 \\
  \delta_0 \\
  D \\
  C_1 \\
  \delta_1 \\
  C_2
\end{array} \]

Pull-back through \( D \) the Helmholtz decomposition of the flows \((C_1)\):

\[ \mathcal{P} \triangleq \{ u \in C_0^M \mid u = \Pi u \text{ and } Du \in \text{im } \delta_0 \} \]

\[ \mathcal{H} \triangleq \{ u \in C_0^M \mid u = \Pi u \text{ and } Du \in \ker \delta_0^* \} \]

\[ \mathcal{N} \triangleq \{ u \in C_0^M \mid u \in \ker D \}. \]

where \( \Pi = D^\dagger D \).
Decomposition: Potential, Harmonic, and Nonstrategic

Decomposition of game flows induces a similar partition of the space of games:

- When going from utilities to flows, the nonstrategic component is removed.
- If we start from utilities (not preferences), always locally consistent.
- Therefore, only two flow components: potential and harmonic

Thus, the space of games has a canonical direct sum decomposition:

\[ \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H} \]

where the components are orthogonal subspaces.
Bimatrix games

For two-player games, simple explicit formulas. Assume the game is given by matrices \((A, B)\), and (for simplicity), the non-strategic component is zero (i.e., \(1^T A = 0, B1 = 0\)). Define

\[
S := \frac{1}{2}(A + B), \quad D := \frac{1}{2}(A - B), \quad \Gamma := \frac{1}{2n}(A11^T - 11^T B).
\]

- Potential component:
  \[(S + \Gamma, \quad S - \Gamma)\]

- Harmonic component:
  \[(D - \Gamma, \quad -D + \Gamma)\]

Notice that the harmonic component is zero sum.
Harmonic games

Very different properties than potential games. Agreement between players is never a possibility!

- Simple examples: rock-paper-scissors, cyclic games, etc.
- Essentially, sums of cycles.
- Generically, *never* have pure Nash equilibria.
- Uniformly mixed profile (for all players) is mixed Nash.

Other interesting static and dynamic properties (e.g., correlated equilibria, best-response dynamics, etc.)
## Potential vs. harmonic

| Subspaces       | Potential Games | Harmonic Games |
|-----------------|-----------------|----------------|
| $\mathcal{P} \oplus \mathcal{N}$ | $\mathcal{H} \oplus \mathcal{N}$ |

| Flows            | Globally consistent | Locally consistent but globally inconsistent |
|------------------|----------------------|---------------------------------------------|

| Pure NE          | Always exists        | Generically does not exist                 |
| Mixed NE         | Always exists        | - Uniformly mixed strategy is always a mixed NE  
|                  |                      | - Players do not strictly prefer their equilibrium strategies. |

| Special cases    | -(two players) Set of mixed Nash equilibria coincides with the set of correlated equilibria  
|                  | -(two players & equal number of strategies) Uniformly mixed strategy is the unique mixed NE |

Consequences

Nice and beautiful. But (if that’s not enough!) why should we care?

- Provides classes of games with simpler structures, for which stronger results can be proved.
- Yields natural mechanisms for approximation, for both static and dynamical properties.

Let’s see this...
Projection onto the Set of Exact Potential Games

- Since the set of exact potential games is a subspace, can easily find “closest” potential game \( \hat{G} \) to a given game \( G \):

  \[
  \hat{G} := \text{arg min}_{h \in \mathcal{H}} \| G - h \|
  \]

- For \( L_2 \)-type distances, closed-form expressions, in terms of a Laplacian-like operator.
Equilibria of a Game and its Projection

Theorem

Let $G$ be a game and $\hat{G}$ be its projection. Any equilibrium of $\hat{G}$ is an $\epsilon$-equilibrium of $G$ for some $\epsilon \leq \sqrt{2} \cdot d(G)$ (and viceversa).

- If projection distance is small, equilibria of the projected game are “close” to the equilibria of the initial game.
- Thus, near-potential games have pure $\epsilon$-equilibria
- Similar results for dynamics: for “near-potential” games, natural game dynamics will converge to “near-equilibria”.
Summary

- Analysis of the global structure of preferences
- Decomposition: nonstrategic, potential and harmonic components
- Projection to “closest” potential game
- Preserves $\epsilon$-approximate equilibria and dynamics
- Enables extension of many tools to non-potential games

Want to know more?

- Candogan, Menache, Ozdaglar, P., “Flow representations of games: harmonic and potential games,” Math. of OR, to appear. arXiv:1005.2405.
- Candogan, Menache, Ozdaglar, P., “Near-optimal power control in wireless networks: a potential game approach,” INFOCOM 2010.
- Candogan, Ozdaglar, P., “Dynamics in near-potential games,” in preparation.