Black Hole Thermodynamics in Carathéodory’s Approach

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Abstract

We show that, in the framework of Carathéodory’s approach to thermodynamics, one can implement black hole thermodynamics by realizing that there exists a quasi-homogeneity symmetry of the Pfaffian form \( \delta Q_{\text{rev}} \) representing the infinitesimal heat exchanged reversibly by a Kerr-Newman black hole; this allows us to calculate readily an integrating factor, and, as a consequence, a foliation of the thermodynamic manifold can be recovered.

Key words: Black Hole Thermodynamics, Thermodynamics
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1 Introduction

We consider black hole thermodynamics in the framework of Carathéodory’s approach to thermodynamics, which postulates the integrability of the Pfaffian form \( \delta Q_{\text{rev}} \) representing the infinitesimal heat exchanged reversibly [1,2]. The integrability of \( \delta Q_{\text{rev}} \) means that there exist an integrating factor \( \mu \) and a function \( \sigma \) (called “empirical entropy” [1]) such that \( \delta Q_{\text{rev}} = \mu d\sigma \). In standard thermodynamics, the integrability of \( \delta Q_{\text{rev}} \) is a consequence of Clausius inequality, which ensures that the absolute temperature \( T \) is an integrating factor for \( \delta Q_{\text{rev}} \), in such a way that \( \delta Q_{\text{rev}} = T dS \), where \( S \) is the entropy (“metrical entropy” [1]). The existence of the entropy function is part of the second law (the principle of entropy increase for thermodynamic processes of closed and thermally insulated systems is the other part [2]; it is related to

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the strict inequality, which holds in irreversible processes, in Clausius inequality). If the integrable Pfaffian form $\delta Q_{\text{rev}}$ displays a symmetry, in a sense to be described in the following, it is also possible to calculate explicitly and readily an integrating factor by means of elementary tools of differential geometry [3,4,5]. As a consequence, an explicit construction of the foliation of the thermodynamic manifold into disconnected adiabatic hypersurfaces satisfying $\delta Q_{\text{rev}} = 0$ is readily calculated.

By postulating a natural form for $\delta Q_{\text{rev}}$ in the case of black holes of the Kerr-Newman family, we can introduce a notion of temperature and of entropy for black holes without referring a priori to the laws of black hole mechanics, thanks to the integrability of $\delta Q_{\text{rev}}$ and to the presence of a quasi-homogeneity symmetry of the Pfaffian form $\delta Q_{\text{rev}}$. Particularly, we can generate a potential which is then related to the entropy of the black hole. Both the entropy and the temperature appear as derived quantities. We point out also that the following construction for black holes, being made an explicit use both of the integrability condition and of the symmetry of $\delta Q_{\text{rev}}$, is of interest also for thermodynamicists.

2 Pfaffian form and symmetry

We recall that, for a Kerr-Newman black hole, due to the no-hair theorem, the only parameters available are $M, Q, J$ (mass, charge and angular momentum); for an insulated black hole they play the role of conserved charges, and are chosen as independent variables in the thermodynamic domain, which is assumed to be the non-extremal manifold $M^4 - M^2 Q^2 - J^2 > 0$ (the extremal sub-manifold $M^4 - M^2 Q^2 - J^2 = 0$ is a boundary of the former, and is temporarily not taken into account. Some more discussion on this topic is found in sect. 4). We look for a natural infinitesimal reversible form for the first law. Then, we have to find out a Pfaffian form such that the first law in its infinitesimal form is implemented, and moreover, such that it is integrable (thus, the first part of the second law is ensured). In the case of a charged rotating system, it would be natural to define $\delta Q_{\text{rev}} = dU + pdV - \phi dq - \omega dl - \mu dN$, where $q$ is the electric charge, $l$ is the angular momentum, $\phi$ is the electrostatic potential and $\omega$ is the angular velocity. We introduce then the angular velocity

$$\Omega = \frac{J}{M^2 - Q^2 + 2M \sqrt{M^2 - Q^2 - J^2 / M^2}}$$

and the electric potential

$$\Phi = \frac{Q(M + \sqrt{M^2 - Q^2 - J^2 / M^2})}{2M^2 - Q^2 + 2M \sqrt{M^2 - Q^2 - J^2 / M^2}}$$

(1)
of the black hole. Both $\Omega$ and $\Phi$ can be assigned on a purely geometrical footing, without any a priori knowledge of black hole thermodynamics. Analogy with standard thermodynamics leads us to associate infinitesimal variations of $Q, J$ with work terms $-\Phi dQ$ and $-\Omega dJ$; infinitesimal variations of $M$ can naturally play the role of $dU$ in standard thermodynamics, where $U$ is the internal energy. We then define

$$\delta Q_{\text{rev}} \equiv dM - \Phi dQ - \Omega dJ. \quad (3)$$

Definition (3) is natural, in fact the (rest) mass can be identified with (a term of) the internal energy (the rest mass of a fluid can be considered as a term of the internal energy in standard thermodynamics; see e.g. [6]); moreover, as seen, the work terms appear as standard work terms. The most evident difference with the case of a standard system consists in the absence, for the black hole case, of the $pdV$ term. This lack is associated with the lack of a notion of volume in the black hole case, as well known. Even a pressure cannot be defined, and the same is true for the particle number $N$. It is stressed that the infinitesimal variation $dM - \Phi dQ - \Omega dJ$ is taken along stationary black hole solutions of the Kerr-Newman family, because of (1), (2); this means that the Einstein equations are satisfied for each state involved in the aforementioned variation. Moreover, these solutions of the Einstein equations are considered as black hole equilibrium states, to be compared with equilibrium states of standard thermodynamics.

The Pfaffian form $\delta Q_{\text{rev}}$ is everywhere non-singular, i.e., there is no point of the thermodynamic domain where all the coefficients of the differential form vanish (this property holds also for points of the extremal boundary). It is easy to show that $\delta Q_{\text{rev}}$ is smooth on the non-extremal manifold and is completely integrable, that is, it satisfies the condition $\delta Q_{\text{rev}} \wedge d(\delta Q_{\text{rev}}) = 0$, i.e.

$$-\partial_J \Phi + \partial_Q \Omega + \Phi \partial_M \Omega - \Omega \partial_M \Phi = 0. \quad (4)$$

Being $\delta Q_{\text{rev}}$ a one-form in three variables, this integrability condition is surely non-trivial (it would be trivial in the case of two variables). Notice that a different choice for the sign of the work terms in (3) would lead to a non-integrable Pfaffian form, as it is easy to verify. In the appendix, we extend the present considerations to the case where a magnetic monopole charge $P$ is allowed. It is remarkable that integrability is not postulated, but it simply follows from considering infinitesimal “on shell” variations, i.e. variations along the aforementioned solutions.

We can also find an integrating factor by using the quasi-homogeneity symmetry of the Pfaffian form (3).\footnote{A Pfaffian form $\omega = \sum_{i=1}^{n} \omega_i(x^1, \ldots, x^n) dx^i$ is quasi-homogeneous of degree $r \in \mathbb{R}$ and weights $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ if, under the scaling $x^1, \ldots, x^n \mapsto \lambda^{\alpha_1} x^1, \ldots, \lambda^{\alpha_n} x^n$} In fact, under the quasi-homogeneous transfo-
mation [7] (also called “similarity transformation” and “stretching transformation” ) \( M \mapsto \lambda^\alpha M; \ Q \mapsto \lambda^\alpha Q; \ J \mapsto \lambda^{2\alpha} J, \) one obtains \( \delta Q_{\text{rev}} \mapsto \lambda^\alpha \delta Q_{\text{rev}}, \) i.e., \( \delta Q_{\text{rev}} \) is quasi-homogeneous of degree \( \alpha. \) \( (\alpha, \alpha, 2\alpha) \) are defined to be the weights of \( M, Q, J \) respectively and they have to be determined. Let us define the so-called Euler vector field [7], which is infinitesimal generator of the transformation

\[
D_\alpha \equiv \alpha M \frac{\partial}{\partial M} + \alpha Q \frac{\partial}{\partial Q} + 2\alpha J \frac{\partial}{\partial J};
\]  

(5)

we have introduced above a label \( \alpha \) which underlines that \( \alpha \) is not yet fixed \( \{D_\alpha\}_\alpha \) is a one-parameter family of Euler vector fields; let the corresponding Lie derivative be \( L_{D_\alpha}; \) then, the quasi-homogeneous transformation is a symmetry for \( \delta Q_{\text{rev}} \) (see e.g. [8,9]), in the sense that

\[
(L_{D_\alpha} \delta Q_{\text{rev}}) \wedge \delta Q_{\text{rev}} = 0.
\]  

(6)

In fact, \( L_{D_\alpha} \delta Q_{\text{rev}} = \alpha \delta Q_{\text{rev}}. \) An integrating factor \( f_\alpha \) such that the form \( \delta Q_{\text{rev}}/f_\alpha \) is exact is \( f_\alpha \equiv i_{D_\alpha} \delta Q_{\text{rev}} = \delta Q_{\text{rev}}(D_\alpha). \) For a proof that \( f_\alpha \) is an integrating factor see Ref. [3] and, for the homogeneous case, see e.g. Ref. [9] and also Ref. [4], where an application to ordinary thermodynamics can be found. In our case, one obtains \( f_\alpha = \alpha (M - \Phi Q - 2\Omega J), \) explicitly \( f_\alpha = \alpha \sqrt{M^2 - Q^2 - J^2/M^2}, \) which is not identically vanishing, thus \( D_\alpha \) is associated with a transversal (or non-trivial) symmetry [3] (i.e., \( D_\alpha \) does not belong to the distribution of codimension one associated with the kernel of \( \delta Q_{\text{rev}}). \) We remark that the integrating factor \( f_\alpha \) is proportional to the horizon coordinate \( c \) introduced by B.Carter in [10]. Then, \( f_\alpha \) is constant on the horizon.

3 Foliation of the thermodynamic manifold

Frobenius theorem for the Pfaffian form \( \delta Q_{\text{rev}} \) on the non-extremal manifold can be invoked and a foliation of the non-extremal manifold can be generated thanks to the integrability property (4). The non-extremal thermodynamic space is foliated by the submanifolds (of codimension one) which are solutions of the Pfaffian equation \( \delta Q_{\text{rev}} = 0. \) The leaves of the foliation of codimension one are surfaces where the potential associated with \( \delta Q_{\text{rev}}/f_\alpha \) is constant. Let \( (M_0, Q_0, J_0) \) be a reference state and \( \Gamma \) be any path connecting the reference

one finds \( \omega \mapsto \lambda^r \omega. \) This happens if and only if \( \omega_i(x^1, \ldots, x^n) \) is quasi-homogeneous of degree \( \beta_i = r - \alpha_i \) for all \( i = 1, \ldots, n, \) which means that \( \omega_i(\lambda^{\alpha_1} x^1, \ldots, \lambda^{\alpha_n} x^n) = \lambda^{r-\alpha_i} \omega_i(x^1, \ldots, x^n). \)
state to the state $(M, Q, J)$ of interest. By choosing e.g. a rectangular path
$(M_0, Q_0, J_0) \rightarrow (M, Q_0, J_0) \rightarrow (M, Q, J_0) \rightarrow (M, Q, J)$ contained in the non-extremal manifold, one finds

$$\hat{S}_\alpha(M, Q, J) - \hat{S}_\alpha(M_0, Q_0, J_0) \equiv \int_{\Gamma} \frac{\delta Q_{\text{rev}}}{f_\alpha}$$

$$= \frac{1}{2\alpha} \log \left( \frac{M^2b^2(M, Q, J) + J^2/M^2}{M_0^2b(M_0, Q_0, J_0) + J_0^2/M_0^2} \right) \quad (7)$$

where $b(M, Q, J) \equiv (1 + \sqrt{1 - Q^2/M^2 - J^2/M^4})$. The argument of the logarithm is proportional to the black hole area $A = 4\pi(M^2b^2(M, Q, J) + J^2/M^2)$. We have generated a foliation of the parameter space of Kerr-Newman black holes. The leaves are the surfaces $A = \text{const.}$, as expected, but we cannot yet determine the so-called metrical entropy [1] in the case of black holes.

We now introduce an assumption which requires some discussion. The above procedure is a generalization, discussed in Ref. [3], of the procedure one can develop for standard thermodynamics [4]. In the case of standard thermodynamics of homogeneous systems, the Pfaffian form $\delta Q_{\text{rev}} = dU + pdV - \mu dN$ in Gibbsian variables $(U, V, N)$ is homogeneous (for the definition of homogeneous differential form see e.g. Ref. [9]). The generator of the symmetry is the “Liouville” operator $Y = U\partial_U + V\partial_V + N\partial_N$ and the integrating factor is $\delta Q_{\text{rev}}(Y) = U + pV - \mu N$. For a standard thermodynamic system one finds that $d\hat{S} \equiv \delta Q_{\text{rev}}/f = dS/S$, where $S$ is an extensive function which coincides with the metrical entropy of the system and corresponds to the fundamental relation in the entropy representation [4]. This deduction is corroborated by appealing to the homogeneity of $S$ in Gibbs’ approach, which allows to find $TS = U + pV - \mu N = \delta Q_{\text{rev}}(Y)$, i.e., the integrating factor coincides with $TS$.

We proceed by analogy with the formalism of thermodynamics just sketched, which means that we assume that the metrical entropy is the unique $^2$ quasi-homogeneous function $S$ of degree one which satisfies $d\hat{S} \equiv \delta Q_{\text{rev}}/f = dS/S$.

We refer to [3] for a proof that such an $S$ exists and is unique. This assumption about the role of $S$ such that $\delta Q_{\text{rev}}/f = dS/S$ is correct both in the case of standard thermodynamics in the Gibbs space [4] and in the case of black hole thermodynamics, and the latter case is analyzed in the following.

The potential $S_\alpha$ such that $d\hat{S}_\alpha = dS_\alpha/S_\alpha$ for the black hole case is

$$S_\alpha = c_\alpha A^{1/2\alpha} \quad (8)$$

where $c_\alpha$ is an undetermined constant. We have a one-parameter family of possible metrical entropies (and also fundamental relations in the entropy representation) which satisfy $D_\alpha S_\alpha = S_\alpha$, in analogy with $YS = S$ of standard thermodynamics.

\[ ^2 \] Uniqueness holds within a multiplicative constant. See [3].
thermodynamics. Our result (8) agrees with the result contained in Ref. [11] but we work in a more general framework where no reference to the laws of black hole mechanics is made [notice also that in our expression for $S$ no additive constant appears, due to quasi-homogeneity symmetry]. There is still an ambiguity due to the undetermined value of $\alpha$, which means that we know the ratio between the weights of $M, Q, J$ but not yet the weights themselves. Notice that this ambiguity does not occur in the case of standard thermodynamics, where the weights of $(U, V, N)$ are known and they are all equal to one. We recall that the metrical entropy is assumed to belong to the one-parameter family $\{S_\alpha\}_\alpha$. For each $\alpha$ the temperature is $T_\alpha = (\partial S_\alpha/\partial M)^{-1}$ and it is a quasi-homogeneous function of degree $\alpha - 1$ and weights $(\alpha, \alpha, 2\alpha)$.

It is useful to realize that

$$S_\alpha = \frac{c_\alpha}{(c_{1/2})^{1/2\alpha}(c_{1/2}A)^{1/2\alpha}} = \frac{c_\alpha}{(c_{1/2})^{1/2\alpha}(S_{1/2})^{1/2\alpha}}$$

(9)

$$T_\alpha = 2\alpha\frac{(c_{1/2})^{1/2\alpha}}{c_\alpha}(S_{1/2})^{1-1/2\alpha}T_{1/2}$$

(10)

For any $\alpha$ one gets $T_\alpha dS_\alpha = dM - \Phi dQ - \Omega dJ$. The black hole area is known to be a superadditive function of $M, Q, J$. Superadditivity of the entropy, which plays a fundamental role when one considers the merging of two black holes, does not fix $\alpha$. \(^3\) Cf. also [11] for a discussion concerning an analogous ambiguity (our $1/2\alpha$ is $\gamma$ therein). The Hawking effect is necessary in order to give us an actual thermodynamic meaning to our calculation; it also fixes $\alpha$, in fact, in order to identify the temperature of the black hole with the Hawking one it is mandatory to choose $\alpha = 1/2$. There is also a multiplicative constant (namely, $c_{1/2}$) which has to be determined. By comparison with the Hawking effect, one finds that $c_{1/2} = 1/4$. The above ambiguity can also be resolved phenomenologically, in perfect agreement with the phenomenological nature of a thermodynamic approach [3]; one should determine $M, Q, J$ and then plot $T(M, Q, J)$ from measurements of the temperature. $\alpha = 1/2$ (and also $c_{1/2} = 1/4$) should come out again.

It is remarkable that, as a consequence of the quasi-homogeneity of black hole entropy, one gets the following generalized Gibbs-Duhem equation, which is analogous to the Gibbs-Duhem equation of standard thermodynamics:

$$M^2d\left(\frac{1}{2MT}\right) - Q^2d\left(\frac{\Phi}{2QT}\right) - Jd\left(\frac{\Omega}{T}\right) = 0.$$ 

(11)

\(^3\) $0 < \alpha \leq 1/2$ is a sufficient condition for preserving superadditivity.
This follows from $S = i_D(\delta Q_{\text{rev}}/T)$ and from $dS = \delta Q_{\text{rev}}/T$, where $D \equiv D_{1/2}$. It is easy to show that this generalized Gibbs-Duhem equation corresponds to

$$-i_D d\left(\frac{\delta Q_{\text{rev}}}{T}\right) = 0.$$ (12)

In fact, $dS = -i_D d(\delta Q_{\text{rev}}/T) + L_D(\delta Q_{\text{rev}}/T) = -i_D d(\delta Q_{\text{rev}}/T) + \delta Q_{\text{rev}}/T$; from the latter equality and from $dS = \delta Q_{\text{rev}}/T$ equation (12) follows. A simple rearrangement of the terms one obtains by making explicit (12) gives then (11). See [3] for a general setting and [12] for the case of standard thermodynamics.

Contrarily to the naive expectation, the laws of black hole mechanics give no unique hints about the value of $\alpha$, they don’t fix uniquely the metrical entropy and the absolute temperature of the black hole. For any $\alpha$ one gets $T_\alpha dS_\alpha = dM - \Phi dQ - \Omega dJ$, to be compared with the differential form of the first law. Moreover, one finds that $f_\alpha = T_\alpha S_\alpha$, which implies $\alpha(M - \Phi Q - 2\Omega J) = T_\alpha S_\alpha = 2\alpha T_{1/2} S_{1/2}$. By comparison with the first law in the finite form one realizes that $T_{1/2} = k/(8\pi c_{1/2})$. The choice of a generic $\alpha$ is equivalent to the the substitutions $A \mapsto \bar{A}_\alpha$ and $k \mapsto \bar{k}_\alpha$, where $\bar{A}_\alpha = A^{1/2\alpha}$ and $\bar{k}_\alpha = 2\alpha k/A^{1/2\alpha-1}$, which implement both the differential form and the finite form of the first law (the latter appears as $\bar{k}_\alpha \bar{A}_\alpha = 8\pi \alpha (M - \Phi Q - 2\Omega J)$ which is equivalent to the well-known one). Notice that $\bar{k}_\alpha$ is constant on the horizon, thus the zeroth law of black hole mechanics is not sufficient in order to select $\alpha = 1/2$.

4 The extremal boundary

The extremal submanifold is very problematic. It is easy to show that $\delta Q_{\text{rev}} = 0$ on the extremal submanifold, i.e. the extremal submanifold is still an integral submanifold of the Pfaffian form [13]. Nevertheless, there is an important property which fails in the case of states belonging to the extremal submanifold. In fact, given a point of the extremal submanifold, there exist two kinds of adiabatic paths having the given state as initial point. One is a path lying on the extremal submanifold, the other is an “isoreal” path, i.e. a path starting from the extremal submanifold and reaching non-extremal states each of which has the same area as the initial extremal state [13]. In absence of the latter class of solutions, the extremal states would represent a leaf of a foliation, thus they would be adiabatically disconnected from the non-extremal states. Instead, their presence can jeopardize the second law of thermodynamics. A detailed discussion of this topic and of the third law in black hole thermodynamics is the subject of Ref. [13]. See also [14] for the case of standard thermodynamics.
5 Conclusions

The approach to black hole thermodynamics by means of Pfaffian forms we have discussed (Carathéodory’s formalism) represents a further corroboration of the fact that black hole thermodynamics is a form of thermodynamics, even if to large extent exceptional. Quasi-homogeneity symmetry of $\delta Q_{\text{rev}}$ plays an important role in allowing to calculate an integrating factor and, then, to generate a thermodynamic potential depending on an undetermined parameter $\alpha$ which can nevertheless be fixed phenomenologically, as seen. Notice that our approach can be extended in a straightforward way to KN-AdS black holes [15]. Also in this case, there is a quasi-homogeneity structure in the Pfaffian form, as it can be easily realized. It is remarkable that quasi-homogeneity is a symmetry occurring also in the thermodynamics of other self-gravitating systems, like non-relativistic fermionic matter and self-gravitating radiation. Independent thermodynamic variables and their weights change \(^4\), but a quasi-homogeneous symmetry appears again. See [3] on the latter topic. The corresponding lack of homogeneity can be related to the purely attractive nature of gravity.

This kind of thermodynamic approach can be insightful also from the point of view of a more general discussion concerning horizon thermodynamics. The availability of a meaningful $\delta Q_{\text{rev}}$ allows to discriminate between cases where there is a complete thermodynamic structure at hand, which can be associated with the Einstein equations (e.g. because $\delta Q_{\text{rev}}$ is integrable “on shell”) and cases where Hawking temperature seems to be related simply to kinematics. Cf. [16]. The case of black holes belongs to the former class, a full thermodynamic structure exists and the role of the equations of General Relativity in ensuring the laws of thermodynamics, enhanced e.g. in Ref. [16], is corroborated in this framework. Notice that, from this point of view, the behavior of General Relativity is, to some extent, intermediate with respect to a macroscopic phenomenological approach, like classical thermodynamics, and a microscopic approach, like statistical mechanics. A macroscopic (“thermodynamic”) point of view is adopted in treating variables like $M, Q, J$; on the other hand, field equations furnish $\Phi, \Omega$ and ensure an integrability condition which, for standard systems, should be an outcome of statistical mechanics [statistical mechanics should calculate the analytic form of the functions $\Phi, \Omega$ (they are phenomenological interpolations for thermodynamics); moreover, it should justify an integrability condition which is only a postulate in standard thermodynamics; statistical mechanics should allow to determine both the metrical entropy and the weights of the variables $M, Q, J$ [3]]. As far as a “cosmological horizon” like De Sitter (one parameter) is concerned, it corre-

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\(^4\) One has \((U, V, N)\) as independent variables in the fermionic matter case, and \((U, V)\) in the self-gravitating radiation case.
sponds to a solution of General Relativity and \( \delta Q_{\text{rev}} \) can be given (but a too tight thermodynamic space does not allow a non-trivial integrability condition). On the other hand, in the “acceleration horizon” case (like e.g. Rindler case) there is a too poor thermodynamic structure, in the sense that there is no first law. The same comment holds true in the case of “acoustic horizons” (no first law is known).

A Magnetically charged black holes

An extension is represented by a rotating charged black hole:

\[ \delta Q_{\text{rev}} = dM - \Phi dQ - \Psi dP - \Omega dJ, \tag{A.1} \]

where \( P \) is the magnetic monopole charge and

\[
\Omega = \frac{J}{M} \sqrt{2M^2 - Q^2 - P^2 + 2M \sqrt{M^2 - Q^2 - P^2 - J^2/M^2}}, \tag{A.2}
\]

\[
\Phi = \frac{Q(M + \sqrt{M^2 - Q^2 - P^2 - J^2/M^2})}{2M^2 - Q^2 - P^2 + 2M \sqrt{M^2 - Q^2 - P^2 - J^2/M^2}}, \tag{A.3}
\]

\[
\Psi = \frac{P(M + \sqrt{M^2 - Q^2 - P^2 - J^2/M^2})}{2M^2 - Q^2 - P^2 + 2M \sqrt{M^2 - Q^2 - P^2 - J^2/M^2}}. \tag{A.4}
\]

Notice that \( \delta Q_{\text{rev}} \) is “on shell”, i.e., it is taken along (stationary) solutions of General Relativity. \( \delta Q_{\text{rev}} \wedge d(\delta Q_{\text{rev}}) = 0 \) corresponds to four integrability conditions:

\[
l_{MQP} \equiv -\partial_P \Phi + \partial_Q \Psi + \Phi \partial_M \Psi - \Psi \partial_M \Phi = 0, \tag{A.5}
\]

\[
l_{MQJ} \equiv -\partial_J \Phi + \partial_Q \Omega + \Phi \partial_M \Omega - \Omega \partial_M \Phi = 0, \tag{A.6}
\]

\[
l_{MPJ} \equiv -\partial_J \Psi + \partial_P \Omega + \Psi \partial_M \Omega - \Omega \partial_M \Psi = 0, \tag{A.7}
\]

\[
l_{QPJ} \equiv \Phi(\partial_J \Psi - \partial_P \Omega) - \Psi(\partial_J \Phi - \partial_Q \Omega) + \Omega(\partial_P \Phi - \partial_Q \Psi) = 0. \tag{A.8}
\]

Even in this case, the integrability is verified and \( \delta Q_{\text{rev}} \) is a quasi-homogeneous Pfaffian form. The Euler vector field is given by

\[
F = \frac{1}{2} M \frac{\partial}{\partial M} + \frac{1}{2} Q \frac{\partial}{\partial Q} + \frac{1}{2} P \frac{\partial}{\partial P} + J \frac{\partial}{\partial J}. \tag{A.9}
\]

The integrating factor is easily found, and, again the area law can be obtained.
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