Nash Equilibrium Seeking in Subnetwork Zero-Sum Games with Switching Communication Graphs

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Abstract

In this paper, we investigate a distributed Nash equilibrium seeking problem for a time-varying multi-agent network consisting of two subnetworks. We propose a subgradient-based distributed algorithm to seek a Nash equilibrium of a zero-sum game, where the two subnetworks share the same sum objective function. We show that the proposed distributed algorithm with homogenous stepsize can achieve a Nash equilibrium under uniformly jointly strongly connected (UJSC) weight-balanced digraphs. Then for weight-unbalanced graphs, we show, by considering a special case, that generally a Nash equilibrium cannot be achieved with homogenous stepsize unless certain conditions on the objective functions hold. Next we show that for any UJSC digraphs, there always exist stepsizes under which a Nash equilibrium can be achieved. Finally, for two general weight-unbalanced cases, we show that by adaptively updating the stepsize along with the arc weights in the proposed algorithm, a convergence to the Nash equilibrium can still be guaranteed.

Keywords: multi-agent systems, Nash equilibrium, weight-unbalanced graphs, adaptive stepsize, joint connection

1 Introduction

In recent years, distributed control and optimization of multi-agent systems have drawn much research attention. Research on multi-agent cooperation problems such as consensus, rendezvous,
and formation via local information exchange and suitable distributed algorithms have been extensively studied [8, 9, 10, 21, 17, 20]. It has been well-known that connectivity of a multi-agent network plays a key role in distributed multi-agent cooperation. Due to the communication failure or energy saving, the network topology is usually changing in practice. A time-varying communication structure in the network brings fundamental challenges in the design and analysis to distributed solutions, and various efforts have been made over the last decade to achieve the coordination of multi-agent systems with switching interconnections [8, 21, 9, 10, 19, 37].

Distributed optimization has been widely investigated in sensor networks and distributed computation [37, 36, 12, 16]. One of the important problems is for multi-agent systems cooperatively to minimize a sum of convex objective functions [34, 29, 32, 33, 31, 35, 36, 13, 14, 11], using subgradient-based algorithms with each node computing a subgradient of its own objective function. A distributed iterative algorithm that avoids choosing a diminishing stepsize was proposed in [42]. Moreover, both deterministic and randomized versions of distributed projection-based protocols were studied in [38, 39], while a pairwise equalizing protocol was proposed in [35]. In existing works, most of the results were obtained for a switching weight-balanced graph case [32, 30, 36] because there usually exists a common Lyapunov function for the analysis of algorithm convergence in this case. However, for time-varying weight-unbalanced graphs, coordination or optimization problems become extremely difficult since there may not exist a common Lyapunov function or it may be very hard to construct one even if exists. In the consensus study, it is known that there exists no quadratic common Lyapunov function for weight-unbalanced graphs in [22]. To solve distributed optimization problems, a push-sum algorithm was proposed for weight-unbalanced graphs in [15]. However, to the best of our knowledge, results on distributed optimization or computation for weight-unbalanced cases are still very few.

Furthermore, distributed optimization algorithms in the presence of adversaries have gained rapidly growing interest due to many practical applications in engineering [4] and social studies [3]. A non-model based approach was proposed for seeking a Nash equilibrium of a noncooperative game in [13], while distributed methods for seeking Nash equilibria with the almost sure convergence under stochastic measurement noises were developed in [14]. On the other hand, saddle point searching problems arise in a wide variety of applications, including zero-sum games, Lagrangian dual problems, and mathematical programming [25, 26, 28, 24, 27]. In fact, the Nash equilibrium of a zero-sum game is equal to the saddle-point set of some payoff function [2]. Related to Nash equilibrium seeking, a diminishing stepsize method was developed for saddle point computation in [20, 28]. Moreover, advanced algorithms for solving the saddle-point problem
emerged in recent years. For example, an approximate saddle point problem with a constant stepsize was considered, and the corresponding per-iteration convergence rate was estimated in [24]. An algorithm based on two subnetworks, with one to minimize the objective function and the other to maximize it, was proposed in [40], where a continuous-time set-valued dynamical system solution to seek a Nash equilibrium was first designed for undirected graphs and then for weight-balanced directed graphs by introducing a new system parameter. However, there are still few results obtained on distributed computation to reach a saddle-point Nash equilibrium under switching weight-unbalanced graphs.

In this paper, we consider the distributed Nash equilibrium seeking problem proposed in [40], where a multi-agent network consisting of two subnetworks plays a zero-sum game. Each agent in the network has its own objective function, and the two subnetworks share the same sum objective function. When the two subnetworks are viewed as two (groups of) players, a zero-sum game is defined and the objective of the agents is to achieve a Nash equilibrium via distributed computation based on local information. In fact, the agents in the two different subnetworks play antagonistic roles against each other, while the agents in the same subnetwork behave cooperatively. The contribution of this paper is summarized as follows.

- We propose a subgradient-based distributed algorithm for a multi-agent network to seek a saddle-point Nash equilibrium under time-varying topologies. We show that our proposed algorithm with homogeneous stepsizes can achieve a saddle-point Nash equilibrium under uniformly jointly strongly connected (UJSC) weight-balanced digraphs.

- Most results on distributed optimization were obtained for weight-balanced graphs, but the weight-balanced condition is quite restrictive and may not be easy to verify in a distributed setup. In weight-unbalanced cases, existing distributed homogeneous-stepsize algorithms may fail. In fact, we show that even for the completely identical subnetworks case, the proposed algorithm with homogeneous stepsizes converges, but may not converge to the desired Nash equilibrium unless the saddle point sets of all the objective functions are the same.

- To solve the distributed Nash equilibrium seeking problems in general weight-unbalanced cases, we propose a heterogeneous stepsize rule and study its possibility to cooperatively seek the desired Nash equilibrium. We first show that for any UJSC time-varying digraphs, there always exist (heterogeneous) stepsizes to make the network achieve a Nash equilibrium. Then we construct an adaptive algorithm to update the stepsizes in two standard cases: the cases with a common eigenvector and with periodically switching topologies, to
ensure the convergence to a saddle-point Nash equilibrium under time-varying digraphs.

The paper is organized as follows. Section 2 gives some basic concepts and preliminary results. Then Section 3 formulates the distributed Nash equilibrium seeking problem and proposes a distributed algorithm for it. Section 4 gives conditions to achieve a saddle-point Nash equilibrium in the weight-balanced digraph case. Section 5 discusses the limitations of homogeneous stepsize algorithms by considering a special case of the proposed algorithm, while Section 6 presents results for weight-unbalanced digraphs. Following that, Section 7 provides numerical simulations for illustration. Finally, some concluding remarks are given in Section 8.

Notations: $| \cdot |$ denotes the Euclidean norm; $\langle \cdot , \cdot \rangle$ denotes the Euclidean inner product; $B(z, \varepsilon) \triangleq \{ y \mid \| y - z \| \leq \varepsilon \}$ denotes a ball with $z$ as the center and $\varepsilon > 0$ as the radius; $z^T$ denotes the transpose of vector $z$; $1 = (1, \ldots, 1)^T$ of appropriate dimension; $A_{ij}$ denotes the $i$-th row and $j$-th column entry of matrix $A$; $\mathcal{S}_n^+ = \{ \omega \mid \omega_i > 0, \sum_{i=1}^n \omega_i = 1 \}$ denotes the set of all $n$-dimensional positive unit stochastic vectors with $\ell_1$-norm; $\text{diag}\{c_1, \ldots, c_n\}$ denotes the diagonal matrix with diagonal elements $c_1, \ldots, c_n$.

2 Preliminaries

In this section, we give preliminaries on graph theory [5], convex analysis [6], Nash equilibrium [2], and sequence convergence.

2.1 Graph Theory

A multi-agent network consisting of $n$ agents can be described by a digraph (directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ formed by node set $\mathcal{V} = \{1, \ldots, n\}$ and arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Associated with graph $\mathcal{G}$, there is a (weighted) adjacency matrix $\bar{A} = (\bar{a}_{ij}) \in \mathbb{R}^{n \times n}$ with nonnegative adjacency elements $\bar{a}_{ij}$, which is positive if and only if $(j, i) \in \mathcal{E}$. Node $j$ is a neighbor of node $i$ if $(j, i) \in \mathcal{E}$. Assume $(i, i) \in \mathcal{E}$ for $i = 1, \ldots, n$. A path in $\mathcal{G}$ from $i_1$ to $i_p$ is an alternating sequence $i_1i_2i_3\cdots i_{p-1}i_p$ of nodes $i_r, 1 \leq r \leq p$ and arcs $e_r = (i_r, i_{r+1}) \in \mathcal{E}, 1 \leq r \leq p-1$. Graph $\mathcal{G}$ is said to be bipartite if $\mathcal{V}$ can be partitioned into two disjoint parts $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{E} \subseteq \bigcup_{i=1}^3 (\mathcal{V}_i \times \mathcal{V}_{3-i})$.

Consider a multi-agent network $\Xi$ consisting of two subnetworks $\Xi_1$ and $\Xi_2$ with respective $n_1$ and $n_2$ agents. $\Xi$ is described by a digraph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which contains self-loops, i.e., $(i, i) \in \mathcal{E}$ for each $i$. Here $\mathcal{G}$ can be partitioned into three digraphs: $\mathcal{G}_\ell = (\mathcal{V}_\ell, \mathcal{E}_\ell)$ with $\mathcal{V}_\ell = \{ \omega_1^\ell, \ldots, \omega_{n_\ell}^\ell \}$, $\ell = 1, 2$, and a bipartite graph $\mathcal{G}_\infty = (\mathcal{V}, \mathcal{E}_\infty)$, where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_\infty$. That is to say, $\Xi_1$ and $\Xi_2$ are described by the two digraphs, $\mathcal{G}_1$ and $\mathcal{G}_2$.
respectively, and the interconnection between $\Xi_1$ and $\Xi_2$ is described by $G_{\infty}$. Here graph $G_{\infty}$ is called bipartite without isolated nodes if, for any $i \in V_\ell$, there is at least one node $j \in V_{3-\ell}$ such that $(j, i) \in E$ for $\ell = 1, 2$. Let $A_\ell$ denote the adjacency matrix of $G_\ell$, $\ell = 1, 2$. Digraph $G_\ell$ is said to be strongly connected if there is a path in $G_\ell$ from $i$ to $j$ for any pair node $i, j \in V_\ell$. A node is said to be a root node if there is at least a path from this node to any other node. In the sequel, we will write $i \in V_\ell$ instead of $\omega^\ell_i \in V_\ell$, $\ell = 1, 2$ for simplicity if there is no confusion.

Digraph $G_\ell$ is said to be weight-balanced if $\sum_{j \in V_\ell} a_{ij} = \sum_{j \in V_\ell} a_{ji}$ for $i \in V_\ell$; and it is weight-unbalanced otherwise.

A vector is said to be stochastic if all its components are non-negative and the sum of its components is one. A matrix is a stochastic matrix if its each row vector is stochastic. A stochastic vector is positive if all its components are positive.

Let $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be a stochastic matrix. Define $G_B = \{(1,...,n), E_B\}$ as the graph associated with $B$, where $(j, i) \in E_B$ if and only if $b_{ij} > 0$ (its adjacency matrix is $B$). According to Perron-Frobenius theorem [1], there is a unique positive stochastic left eigenvector of $B$ associated with eigenvalue one if $G_B$ is strongly connected. We call this eigenvector the Perron vector of $B$. The following lemma shows that the converse in some sense is also true, whose proof is in Appendix.

**Lemma 2.1** For any $\mu = (\mu_1 \cdots \mu_n)^T \in S_n^+$, there is a stochastic matrix $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ such that $G_B$ is strongly connected and $\mu^T B = \mu^T$.

**2.2 Convex Analysis**

A set $K \subseteq \mathbb{R}^m$ is said to be convex if $\lambda z_1 + (1 - \lambda)z_2 \in K$ for any $z_1, z_2 \in K$ and $0 < \lambda < 1$.

A point $z$ is said to be an interior point of $K$ if $B(z, \varepsilon) \subseteq K$ for some $\varepsilon > 0$. For a closed convex set $K$ in $\mathbb{R}^m$, we can associate with any $z \in \mathbb{R}^m$ a unique element $P_K(z) \in K$ satisfying $|z - P_K(z)| = \inf_{y \in K} |z - y|$ (denoted as $|z|_K$), where $P_K$ is the projection operator onto $K$.

We have the following lemma for the projection operator $P_K$.

**Lemma 2.2** Let $K_0 \subseteq K$ be two closed convex sets in $\mathbb{R}^m$. Then

(i) $|P_K(y) - P_K(z)| \leq |y - z|$ for any $y$ and $z$;

(ii) $|P_K(y) - z| \leq |y - z|$ for any $y \in \mathbb{R}^m$ and any $z \in K$.

Clearly, (i) is the non-expansiveness property; (ii) comes from Lemma 1 (b) in [30].
A function $\varphi(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is said to be (strictly) convex if $\varphi(\lambda z_1 + (1 - \lambda)z_2)(<) \leq \lambda \varphi(z_1) + (1 - \lambda)\varphi(z_2)$ for any $z_1 \neq z_2 \in \mathbb{R}^m$ and $0 < \lambda < 1$. A function $\varphi$ is said to be (strictly) concave if $-\varphi$ is (strictly) convex. A convex function $\varphi : \mathbb{R}^m \to \mathbb{R}$ is continuous.

For a convex function $\varphi$, $v(\hat{z}) \in \mathbb{R}^m$ is a subgradient of $\varphi$ at point $\hat{z}$ if

$$\varphi(z) \geq \varphi(\hat{z}) + \langle z - \hat{z}, v(\hat{z}) \rangle, \ \forall z \in \mathbb{R}^m,$$

For a concave function $\varphi$, $v(\hat{z}) \in \mathbb{R}^m$ is a subgradient of $\varphi$ at $\hat{z}$ if

$$\varphi(z) \leq \varphi(\hat{z}) + \langle z - \hat{z}, v(\hat{z}) \rangle, \ \forall z \in \mathbb{R}^m.$$ The set of all subgradients of (convex or concave) function $\varphi$ at $\hat{z}$ is denoted by $\partial \varphi(\hat{z})$, which is called the subdifferential of $\varphi$ at $\hat{z}$.

### 2.3 Saddle Point and Nash Equilibrium

Function $\phi(\cdot, \cdot) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is said to be (strictly) convex-concave if it is (strictly) convex for first argument and (strictly) concave for second argument. Given a point $(\hat{x}, \hat{y})$, we denote by $\partial_x \phi(\hat{x}, \hat{y})$ the subdifferential of convex function $\phi(\cdot, \hat{y})$ at $\hat{x}$ and $\partial_y \phi(\hat{x}, \hat{y})$ the subdifferential of concave function $\phi(\hat{x}, \cdot)$ at $\hat{y}$.

A pair $(x^*, y^*) \in X \times Y$ is a saddle point of $\phi$ on $X \times Y$ if

$$\phi(x^*, y) \leq \phi(x^*, y^*) \leq \phi(x, y^*), \ \forall x \in X, y \in Y.$$

The next lemma presents a necessary and sufficient condition of characterizing the saddle point, which is Proposition 2.6.1 on page 132 in [45].

**Lemma 2.3** Let $X \subseteq \mathbb{R}^{m_1}, Y \subseteq \mathbb{R}^{m_2}$ be two closed convex sets. Then a pair $(x^*, y^*)$ is a saddle point of $\phi$ on $X \times Y$ if and only if $\sup_{y \in Y} \inf_{x \in X} \phi(x, y) = \inf_{x \in X} \sup_{y \in Y} \phi(x, y) = \phi(x^*, y^*)$, and $x^*$ is an optimal solution of optimization problem

$$\text{minimize } \sup_{y \in Y} \phi(x, y) \quad \text{subject to } x \in X, \quad (1)$$

while $y^*$ is an optimal solution of optimization problem

$$\text{maximize } \inf_{x \in X} \phi(x, y) \quad \text{subject to } y \in Y. \quad (2)$$

From Lemma 2.3, we find that all saddle points of $\phi$ on $X \times Y$ yield the same value. Moreover, the next lemma can be obtained from Lemma 2.3.

**Lemma 2.4** If $(x^*_1, y^*_1)$ and $(x^*_2, y^*_2)$ are two saddle points of $\phi$ on $X \times Y$, then $(x^*_1, y^*_2)$ and $(x^*_2, y^*_1)$ are also saddle points of $\phi$ on $X \times Y$. 

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Remark 2.1 Denote by $\bar{Z}$ the set of all saddle points of function $\phi$ on $X \times Y$, $\bar{X}$ and $\bar{Y}$ the optimal solution sets of optimization problems (1) and (2), respectively. Then from Lemma 2.3 it is not hard to find that if $\bar{Z}$ is nonempty, then $\bar{X}$, $\bar{Y}$ are nonempty, convex, and $\bar{Z} = \bar{X} \times \bar{Y}$. Moreover, if $X$ and $Y$ are convex and compact, $\phi$ is convex-concave, then $\bar{Z}$ is nonempty (see Proposition 2.6.9 on page 150 in [45]).

The saddle point computation can be related to a zero-sum game. In fact, a (strategic) game is described as a triple $(I, W, U)$, where $I$ is the set of all players; $W = W_1 \times \cdots \times W_n$, $n$ is the number of players, $W_i$ is the set of actions available to player $i$; $U = (u_1 \cdots u_n)$, $u_i : W \to \mathbb{R}$ is the payoff function of player $i$. The game is said to be zero-sum if $\sum_{i=1}^{n} u_i = 0$. A profile action $w^* = (w^*_1 \cdots w^*_n)$ is said to be a Nash equilibrium if $u_i(w^*_i, w^*_{-i}) \geq u_i(w_i, w^*_{-i})$ for each $i \in V$ and $w_i \in W_i$, where $w^*_{-i}$ denotes the actions of all players other than $i$. The set of all Nash equilibria of a two-person zero-sum game ($n = 2, u_1 + u_2 = 0$) is exactly the set of all saddle points of payoff function $u_2$ [2].

2.4 Sequence Lemmas

We introduce two lemmas for the following convergence analysis. The first lemma is the deterministic version of Lemma 11 on page 50 in [1]. The second one is Lemma 7 in [30].

Lemma 2.5 Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be non-negative sequences with $\sum_{k=0}^{\infty} b_k < \infty$. If $a_{k+1} \leq a_k + b_k - c_k$ holds for any $k$, then $\lim_{k \to \infty} a_k$ is a finite number.

Lemma 2.6 Let $0 < \lambda < 1$ and $\{a_k\}$ be a positive sequence. If $\lim_{k \to \infty} a_k = 0$, then $\lim_{k \to \infty} \sum_{r=0}^{k} \lambda^{k-r} a_r = 0$. Moreover, if $\sum_{k=0}^{\infty} a_k < \infty$, then $\sum_{k=0}^{\infty} \sum_{r=0}^{k} \lambda^{k-r} a_r < \infty$.

3 Nash Equilibrium Seeking and Distributed Algorithm

In this section, we introduce a distributed Nash equilibrium seeking problem and then propose a subgradient-based distributed algorithm.

Consider a network $\Xi$ consisting of two subnetworks $\Xi_1$ and $\Xi_2$. Each agent $i$ in $\Xi_1$ is associated with a convex-concave objective function $f_i(x, y) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$, and each agent $i$ in $\Xi_2$ is associated with a convex-concave objective function $g_i(x, y) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$. Each agent only knows its own objective function. The two subnetworks have a common sum objective function with closed convex constraint sets $X \subseteq \mathbb{R}^{m_1}$, $Y \subseteq \mathbb{R}^{m_2}$:

$$U(x, y) = \sum_{i=1}^{n_1} f_i(x, y) = \sum_{i=1}^{n_2} g_i(x, y), \quad x \in X, \ y \in Y. \quad (3)$$
Then the network is engaged in a (generalized) zero-sum game \(\{\Xi_1, \Xi_2\}, X \times Y, u\), where \(\Xi_1\) and \(\Xi_2\) are viewed as two players, their respective payoff functions are \(u_{\Xi_1} = -\sum_{i=1}^{n_1} f_i\) and \(u_{\Xi_2} = \sum_{i=1}^{n_2} g_i\). The objective of \(\Xi_1\) and \(\Xi_2\) is to achieve a Nash equilibrium of the zero-sum game.

**Remark 3.1** The original problem definition for the two-subnetwork zero-sum game was introduced in [40]. Notice that the Nash equilibria set of this zero-sum game is the saddle point set of \(U\) on \(X \times Y\). The Nash equilibrium seeking problem is related to many interesting problems such as the power allocation [4, 40] and saddle point computation of Lagrangian functions of constrained optimization problems [27, 24, 36].

In the paper, we assume that the saddle point set of each objective function \(f_i\) or \(g_i\) on \(X \times Y\) is nonempty, which holds if \(X\) and \(Y\) are bounded. However, in this paper we do not require \(X\) and \(Y\) are bounded. Let \(X^* = X^* \times Y^* \subseteq X \times Y\) denote the set of all saddle points of \(U\) on \(X \times Y\). Notice that \(X^* \times Y^*\) is also the set of Nash equilibria of the modified zero-sum game.

Time is slotted for \(k = 0, 1, \ldots\). Each node \(i \in V_1\) holds a state \(x_i(k) \in \mathbb{R}^{m_1}\) at time \(k\), and each node \(i \in V_2\) holds a state \(y_i(k) \in \mathbb{R}^{m_2}\) at time \(k\). In this paper, we are interested in how to achieve a Nash equilibrium.

**Definition 3.1** The network \(\Xi\) is said to achieve a Nash equilibrium if, for any initial condition \(x_i(0) \in \mathbb{R}^{m_1}, i \in V_1\) and \(y_i(0) \in \mathbb{R}^{m_2}, i \in V_2\), there are \(x^* \in X^*\) and \(y^* \in Y^*\) such that

\[
\lim_{k \to \infty} x_i(k) = x^*, \quad i \in V_1, \quad \lim_{k \to \infty} y_i(k) = y^*, \quad i \in V_2.
\]

The interconnection in the network \(\Xi\) is time-varying and modeled as three digraph sequences: \(G_1 = \{G_1(k)\}\), \(G_2 = \{G_2(k)\}\), and \(G_\infty = \{G_\infty(k)\}\), where \(G_1(k) = (V_1, E_1(k))\) and \(G_2(k) = (V_2, E_2(k))\) are the graphs to describe \(\Xi_1\) and \(\Xi_2\), respectively, and \(G_\infty(k) = (V, E_\infty(k))\) is the bipartite graph to describe the interconnection between \(\Xi_1\) and \(\Xi_2\) at time \(k \geq 0\). For \(k_2 > k_1 \geq 0\), denote \(\hat{G}_\infty([k_1, k_2])\) as the union graph with node set \(V\) and arc set \(\bigcup_{k_1 \leq s < k_2} E_\infty(s)\), and \(\hat{G}_\ell([k_1, k_2])\) as the union graph with node set \(V_\ell\) and arc set \(\bigcup_{k_1 \leq s < k_2} E_\ell(s)\) for \(\ell = 1, 2\). For convenience, we still use \(G_j([k_1, k_2])\) for \(\hat{G}_j([k_1, k_2])\) for \(j = 1, 2, \infty\) when there is no confusion. The following assumption on connectivity is made.

**A1 (Connectivity)** (i) The graph sequence \(G_\infty\) is uniformly jointly bipartite (UJB), that is, there exists an integer \(T_\infty > 0\) such that \(G_\infty([k, k + T_\infty])\) is bipartite without isolated nodes for \(k \geq 0\).

(ii) For \(\ell = 1, 2\), the graph sequence \(G_\ell\) is uniformly jointly strongly connected (UJSC), that is, there is an integer \(T_\ell > 0\) such that \(G_\ell([k, k + T_\ell])\) is strongly connected for \(k \geq 0\).
To handle the distributed Nash equilibrium seeking problem, we propose the following subgradient-based algorithm.

**Distributed Nash Equilibrium Seeking Algorithm:**

\[
\begin{align*}
  x_i(k+1) &= P_x(\tilde{x}_i(k) - \alpha_{i,k} q_{1i}(k)), \\
  q_{1i}(k) &\in \partial_x f_i(\tilde{x}_i(k), \hat{x}_i(k)), \quad i \in \mathcal{V}_1, \\
  y_i(k+1) &= P_y(\tilde{y}_i(k) + \beta_{i,k} q_{2i}(k)), \\
  q_{2i}(k) &\in \partial_y g_i(\tilde{y}_i(k), \hat{y}_i(k)), \quad i \in \mathcal{V}_2,
\end{align*}
\]

with

\[
\begin{align*}
  \hat{x}_i(k) &= \sum_{j \in \mathcal{N}_i^\ell(k)} a_{ij}(k)x_j(k), \quad \hat{x}_i(k) &= \sum_{j \in \mathcal{N}_i^\ell(\tilde{k}_i)} a_{ij}(\tilde{k}_i)y_j(\tilde{k}_i), \quad i \in \mathcal{V}_1, \\
  \hat{y}_i(k) &= \sum_{j \in \mathcal{N}_i^\ell(k)} a_{ij}(k)y_j(k), \quad \hat{y}_i(k) &= \sum_{j \in \mathcal{N}_i^\ell(\tilde{k}_i)} a_{ij}(\tilde{k}_i)x_j(\tilde{k}_i), \quad i \in \mathcal{V}_2,
\end{align*}
\]

where \( \alpha_{i,k} > 0, \beta_{i,k} > 0 \) are the stepsizes at time \( k \), \( a_{ij}(k) \) is the time-varying weight of arc \( (j, i) \), \( \mathcal{N}_i^\ell(k) \) is the set of neighbors in \( \mathcal{V}_\ell \) of node \( i \) at time \( k \) and \( \tilde{k}_i \) is the last time before \( k \) when node \( i \in \mathcal{V}_\ell \) has at least one neighbor in \( \mathcal{V}_{3-\ell} \), namely

\[
\tilde{k}_i = \max \{ s | s \leq k, \mathcal{N}_i^{3-\ell}(s) \neq \emptyset \} \leq k. \quad (5)
\]

We introduce an assumption on the weights, which is also used in [22, 33, 51].

**A2 (Weights Rule)** (i) There is \( 0 < \eta < 1 \) such that \( a_{ij}(k) \geq \eta \) for all \( i, k \) and \( j \in \mathcal{N}_i^\ell(k) \cup \mathcal{N}_j^\ell(k) \).

(ii) \( \sum_{j \in \mathcal{N}_i^\ell(k)} a_{ij}(k) = 1 \) for all \( i \) and \( k \in \mathcal{V}_\ell, \ell = 1, 2 \).

(iii) \( \sum_{j \in \mathcal{N}_i^{3-\ell}(\tilde{k}_i)} a_{ij}(\tilde{k}_i) = 1 \) for \( i \in \mathcal{V}_\ell, \ell = 1, 2 \).

The conditions (ii) and (iii) in A2 state that the information from agents’ neighbors is used in a weighted average way. The next assumption is about subgradients of objective functions.

**A3 (Boundedness of Subgradients)** There is \( L > 0 \) such that, for each \( i, j \)

\[
|q| \leq L, \quad \forall q \in \partial_x f_i(x, y) \cup \partial_y g_j(x, y), \quad \forall x \in X, y \in Y.
\]

In fact, a bounded assumption similar to A3, which is naturally satisfied if \( X, Y \) are bounded, has been widely used (see [21, 32, 80, 29]).

**Remark 3.2** Different from the algorithm given in [40], where the nodes in \( \Xi_\ell \) connect directly with those in \( \Xi_{3-\ell} \) for all the time, we only require that the nodes in two subnetworks be connected at least once in each interval of length \( T_\infty \) according to A1 (i). In fact, it may be practically
hard for the agents of different subnetworks to maintain communications all the time. Moreover, even if each agent in $\Xi_\ell$ can receive the information from $\Xi_{3-\ell}$, agents may just send or receive once during a period of length $T$, in the aim of standing as much time as possible under a communication energy budget.

In the proposed algorithm (4), one problem is how to select the stepsizes $\{\alpha_{i,k}\}$ and $\{\beta_{i,k}\}$. One simple case, called homogenous stepsize case, is to take a certain positive sequence $\{\gamma_k\}$ such that $\alpha_{i,k} = \beta_{j,k} = \gamma_k$ for $i \in V_1, j \in V_2$ and all $k$. We give an assumption for this sequence $\{\gamma_k\}$.

**A4** $\{\gamma_k\}$ is non-increasing, $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$.

The condition, that is, $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$, in **A4** is also well-known in the selection of the stepsize for distributed subgradient algorithms (e.g., [30, 31, 36]). In many results for distributed optimization [30, 31, 34, 36], their algorithms are of homogeneous-stepsize and mainly for weight-balanced graphs, but the stepsize in our setup is heterogenous, that is, the stepsizes may be different for different nodes, mainly to deal with general unbalanced cases.

In what follows, we start with the homogeneous stepsize case to achieve a Nash equilibrium for weight-balanced graphs (in Section 4). Then we focus on a special case to show how a homogeneous-stepsize algorithm fails to achieve our aim for weight-unbalanced graphs (in Section 5). Following that, we demonstrate that the heterogeneity of stepsize can help us achieve a Nash equilibrium in some cases for weight-unbalanced graphs (in Section 6).

**Remark 3.3** Distributed problems under weight-unbalanced directed graphs have been studied in recent years, for example, a push-sum protocol is employed to solve the average consensus problems using non-doubly stochastic matrices in [46], while a similar push-sum (homogeneous-stepsize) algorithm is developed to eliminate the imbalance caused by the weight-unbalanced graphs for distributed optimization problems in [13], where each node is required to know its out-degree all the time. With a different viewpoint, in our algorithm to handle unbalanced cases, we take (adaptive) heterogeneous stepsizes to achieve a Nash equilibrium.

### 4 Weight-balanced Graphs

In this section, we first present some preliminary analysis for the proposed algorithm (4) and then give the main results for the weight-balanced graphs with related convergence analysis.
4.1 A Simplified Model and Preliminary Results

Basically, the two dynamics of algorithm 4 are in the same form. Let us check the first one, that is,

\[ x_i(k+1) = P_X(\hat{x}_i(k) - \alpha_{i,k} q_{i1}(k)), \quad q_{i1}(k) \in \partial x f_i(\hat{x}_i(k), \bar{x}_i(k)), \quad i \in V_1. \]  

(6)

By regarding the term containing \( y_j \) (\( j \in V_2 \)) as “disturbance”, we can transform (6) to a simplified model in the following form with disturbance \( w_i \):

\[ x_i(k+1) = P_X \left( \sum_{j \in N_i^1(k)} a_{ij}(k) x_j(k) + w_i(k) \right), \quad i = 1, ..., n_1, \]  

(7)

where \( w_i(k) = -\alpha_{i,k} q_{i1}(k) \).

In this section, we provide some lemmas for the simplified model (7) with UJSC graphs, which are very useful to deal with the joint-connection topology for (4) in the following sections.

We first introduce a lemma about stochastic matrices [23].

**Lemma 4.1** Let \( B = (b_{ij}) \in \mathbb{R}^{n \times n} \) be a stochastic matrix and \( h(\mu) = \max_{1 \leq i,j \leq n} |\mu_i - \mu_j|, \mu = (\mu_1 \cdots \mu_n)^T \in \mathbb{R}^n \). Then \( h(B \mu) \leq \mu(B) h(\mu) \), where \( \mu(B) = 1 - \min_{j_1,j_2} \sum_{i=1}^n \min \{b_{j_1i}, b_{j_2i}\} \), is called “the ergodicity coefficient” of \( B \).

Clearly, system (7) can be rewritten as

\[ x_i(k+1) = \sum_{j \in N_i^1(k)} a_{ij}(k) x_j(k) + \epsilon_i(k), \quad i = 1, ..., n_1, \]  

(8)

where \( \epsilon_i(k) = P_X \left( \sum_{j \in N_i^1(k)} a_{ij}(k) x_j(k) + w_i(k) \right) - \sum_{j \in N_i^1(k)} a_{ij}(k) x_j(k) \). Since \( x_j(k) \in X \) and \( X \) is convex, \( \sum_{j \in N_i^1(k)} a_{ij}(k) x_j(k) \in X \). From Lemma 2.2 (i), \( |\epsilon_i(k)| \leq |w_i(k)| \).

Without loss of generality, we assume \( n_1 = 1 \) in this section for notational simplicity. Denote \( x(k) = (x_1(k) \cdots x_{n_1}(k))^T, \epsilon(k) = (\epsilon_1(k) \cdots \epsilon_{n_1}(k))^T \). Then system (8) can be written in a compact form:

\[ x(k+1) = A_1(k)x(k) + \epsilon(k), k \geq 0. \]  

(9)

Define transition matrix \( \Phi^\ell(k,s) = A_\ell(k)A_\ell(k-1) \cdots A_\ell(s), k \geq s, \ell = 1, 2 \). Therefore,

\[ x(k+1) = \Phi^1(k,s)x(s) + \sum_{r=s}^{k-1} \Phi^1(k,r+1)\epsilon(r) + \epsilon(k), k \geq 0. \]  

(10)

We next introduce four lemmas, whose proofs are given in the Appendix. The first one gives an estimation for the decay of

\[ h_1(k) = \max_{1 \leq i,j \leq n_1} |x_i(k) - x_j(k)| \quad \text{and} \quad h_2(k) = \max_{1 \leq i,j \leq n_2} |y_i(k) - y_j(k)| \]

over a bounded interval.
Lemma 4.2 Suppose A1 (ii), A2 and A3 hold. Then for \( \ell = 1, 2 \) and any \( p \geq 1, 0 \leq q \leq T^\ell - 1 \),

\[
h_\ell(pT^\ell + q) \leq (1 - \eta T^\ell)h_\ell((p - 1)T^\ell + q) + 2L \sum_{r=(p-1)T^\ell+q}^{pT^\ell+q-1} \lambda_\ell, \tag{11}
\]

where \( \lambda_\ell = \bar{\alpha}_\ell := \max_{1 \leq i \leq n_1} \alpha_{i,r}, \lambda_\ell^2 = \bar{\beta}_\ell := \max_{1 \leq i \leq n_2} \beta_{i,r}, T^\ell = (n_\ell(n_\ell - 2) + 1)T_\ell \), \( T_\ell \) is the constant in A1.

The second lemma is an extension of Lemma 8 (a) in [30] dealing with weight-balanced graph sequence to general graph sequence (may be weight-unbalanced).

Lemma 4.3 Suppose A1 (ii), A2 and A3 hold. If \( \sum_{k=0}^{\infty} \bar{\alpha}_k^2 < \infty \) and \( \sum_{k=0}^{\infty} \bar{\beta}_k^2 < \infty \), then \( \sum_{k=0}^{\infty} \bar{\alpha}_k h_1(k) < \infty, \sum_{k=0}^{\infty} \bar{\beta}_k h_2(k) < \infty. \)

The third lemma is given about the consensus within the subnetworks, which will be frequently used in the sequel.

Lemma 4.4 Suppose A1 (ii), A2 and A3 hold. If for each \( i \), \( \lim_{k \to \infty} \alpha_{i,k} = 0 \) and \( \lim_{k \to \infty} \beta_{i,k} = 0 \), then the subnetworks \( \Xi_1 \) and \( \Xi_2 \) achieve a consensus, respectively, that is, \( \lim_{k \to \infty} h_1(k) = 0 \) and \( \lim_{k \to \infty} h_2(k) = 0. \)

Remark 4.1 In algorithm [4], the \( x_i \) dynamics and \( y_i \) dynamics are coupled by the subgradient optimization terms. However, when the stepsize is vanishing, the coupling term (as disturbance) is also vanishing. In fact, Lemma 4.4 shows that the consensus can be achieved when the coupling/disturbance term is vanishing under some suitable assumptions. Therefore, the two dynamics can be viewed as two independent systems in some asymptotical sense. In fact, similar ideas have been used in many multi-agent problems including robust consensus [15] and convex intersection computation [37].

The fourth lemma provides estimation of errors between the states of agents and their average, which plays an important role in the following convergence analysis.

Lemma 4.5 Suppose A1 (ii), A2 and A3 hold, \( \sum_{k=0}^{\infty} \bar{\alpha}_k^2 < \infty \) and \( \sum_{k=0}^{\infty} \bar{\beta}_k^2 < \infty \). Then for each \( i \in V_1, \)

(i) \( \sum_{k=0}^{\infty} \bar{\alpha}_k |\bar{x}_i(k) - \bar{x}(k)| < \infty \) and \( \sum_{k=0}^{\infty} \bar{\beta}_k |\bar{y}_i(k) - \bar{y}(k)| < \infty, \) where \( \bar{x}(k) = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i(k), \)

\( \bar{y}(k) = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i(k); \)

(ii) If the sequences \( \{\bar{\alpha}(k)\}, \{\bar{\beta}(k)\} \) are non-increasing, then \( \sum_{k=0}^{\infty} \bar{\alpha}_k |\bar{x}_i(k) - \bar{y}(k)| < \infty \) and \( \sum_{k=0}^{\infty} \bar{\beta}_k |\bar{y}_i(k) - \bar{x}(k)| < \infty. \)
Remark 4.2 From the proof we find that Lemma 4.5 (ii) still holds when the non-increasing condition of \( \{\bar{\alpha}_k\} \) and \( \{\bar{\beta}_k\} \) is replaced with that there exist positive integer \( T^* \) and \( c^* > 0 \) such that \( \bar{\alpha}_{k+T^*} \leq c^* \bar{\alpha}_k \) and \( \bar{\beta}_{k+T^*} \leq c^* \bar{\beta}_k \) for each \( k \).

Although the above lemmas are obtained for a simplified model \( \mathcal{G} \), they are, in fact, very useful in the analysis of the Nash equilibrium seeking in what follows.

4.2 Weight-balanced Graphs

In this subsection, we consider the algorithm (4) with homogeneous stepsizes (that is, \( \alpha_{i,k} = \beta_{i,k} = \gamma_k \) for each \( k \)) for weight-balanced digraphs.

The following result, in fact, provides two sufficient conditions to achieve a Nash equilibrium under switching weight-balanced digraphs.

Theorem 4.1 Suppose A1–A4 hold and digraph \( \mathcal{G}_\ell(k) \) is weight-balanced for \( k \geq 0 \) and \( \ell = 1, 2 \). Then the multi-agent network \( \Xi \) achieves a Nash equilibrium by the algorithm (4) with the homogeneous stepsizes \( \{\gamma_k\} \) if either of the following two conditions holds:

(i) \( U \) is strictly convex-concave;

(ii) \( X^* \times Y^* \) contains an interior point.

Remark 4.3 The authors in [40] developed a continuous-time dynamical system solution to solve the Nash equilibrium seeking problem for fixed weight-balanced digraphs, and showed that the network converges to a Nash equilibrium for a differentiable strictly convex-concave sum objective function \( U \). Theorem 4.1 is a generalization of the result in [40] in the sense that we allow time-varying communication structures and non-smooth objective function \( U \). The same result might continue to hold for the continuous-time solution in [40] under these generalizations, but the analysis would be much more involved (cf. the treatment to a simpler problem in [41]).

Theorem 1.1 are established following the three main steps:

S1. The consensus is achieved in each subnetwork (see Lemma 4.4)

S2. The system states are bounded (see Lemma 4.6)

S3. The Nash equilibrium is achieved (see the proof of Theorem 4.1)

Before giving the detailed proof of Theorem 4.1, we first present a useful lemma, which demonstrates the boundedness of dynamics (4).
Lemma 4.6 For algorithm (4) with A1–A4, if digraph $\mathcal{G}_\ell(k)$ is weight-balanced for $k \geq 0$ and $\ell = 1, 2$, then the system states $(x_i(k), y_j(k))$ for $k \geq 0$ are bounded.

Proof: Take $(x, y) \in X \times Y$. By (11) and Lemma 2.2 (ii), we have

$$|x_i(k+1) - x|^2 \leq |\dot{x}_i(k) - \gamma_k q_{1i}(k) - x|^2$$

$$= |\dot{x}_i(k) - x|^2 + 2\gamma_k \langle \dot{x}_i(k) - x, -q_{1i}(k) \rangle + \gamma_k^2 |q_{1i}(k)|^2, \quad q_{1i}(k) \in \partial_x f_i(\hat{x}_i(k), \bar{x}_i(k)), i = 1, ..., n_1;$$

$$|y_i(k+1) - y|^2 \leq |\dot{y}_i(k) + \gamma_k q_{2i}(k) - y|^2$$

$$= |\dot{y}_i(k) - y|^2 + 2\gamma_k \langle \dot{y}_i(k) - y, q_{2i}(k) \rangle + \gamma_k^2 |q_{2i}(k)|^2, \quad q_{2i}(k) \in \partial_y g_i(\hat{y}_i(k), \bar{y}_i(k)), i = 1, ..., n_2.$$  \hspace{1cm} (12)

Because $|\cdot|^2$ is a convex function, $|\dot{x}_i(k) - x|^2 \leq \sum_{j \in N_i^1(k)} a_{ij}(k) |x_j(k) - x|^2$. Moreover, since $q_{1i}(k)$ is a subgradient of $f_i(\cdot, \hat{x}_i(k))$ at $\hat{x}_i(k)$, we have $\langle x - \hat{x}_i(k), q_{1i}(k) \rangle \leq f_i(x, \hat{x}_i(k)) - f_i(\hat{x}_i(k), \bar{x}_i(k))$. Thus, based on (12) and A3,

$$|x_i(k+1) - x|^2 \leq \sum_{j \in N_i^1(k)} a_{ij}(k) |x_j(k) - x|^2 + L^2 \gamma_k^2$$

$$+ 2\gamma_k \left( f_i(x, \hat{x}_i(k)) - f_i(\bar{x}_i(k), \bar{x}_i(k)) \right).$$ \hspace{1cm} (14)

From A3 again, we have

$$|f_i(x, y_1) - f_i(x, y_2)| \leq L |y_1 - y_2|, \quad |f_i(x_1, y) - f_i(x_2, y)| \leq L |x_1 - x_2|, \quad \forall x, x_1, x_2 \in X, y, y_1, y_2 \in Y,$$

which implies

$$|f_i(x, \hat{x}_i(k)) - f_i(x, \bar{y}(k))| \leq L |\hat{x}_i(k) - \bar{y}(k)|,$$

$$|f_i(\hat{x}_i(k), \hat{x}_i(k)) - f_i(\hat{x}_i(k), \bar{y}(k))| \leq |f_i(\hat{x}_i(k), \hat{x}_i(k)) - f_i(\bar{x}_i(k), \hat{x}_i(k))|$$

$$+ |f_i(\bar{x}_i(k), \hat{x}_i(k)) - f_i(\bar{x}_i(k), \bar{y}(k))|$$

$$\leq L (|\hat{x}_i(k) - \bar{x}_i(k)| + |\hat{x}_i(k) - \bar{y}(k)|).$$ \hspace{1cm} (16)

where $\bar{x}(k) = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i(k) \in X$ and $\bar{y}(k) = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i(k) \in Y$. Therefore, we have

$$|x_i(k+1) - x|^2 \leq \sum_{j \in N_i^1(k)} a_{ij}(k) |x_j(k) - x|^2 + 2\gamma_k \left( f_i(x, \bar{y}(k)) - f_i(\bar{x}_i(k), \bar{y}(k)) \right)$$

$$+ L^2 \gamma_k^2 + 2L \gamma_k e_{i1}(k).$$ \hspace{1cm} (17)

with $e_{i1}(k) = 2|\hat{x}_i(k) - \bar{y}(k)| + |\hat{x}_i(k) - \bar{x}(k)|$. 

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Because $G_1(k)$ is weight balanced, taking the sum for the two sides of (17) over $i = 1, ..., n_1$ yields to
\begin{equation}
\sum_{i=1}^{n_1} |x_i(k+1) - x|^2 \leq \sum_{i=1}^{n_1} |x_i(k) - x|^2 + 2\gamma_k \left( U(x, \bar{y}(k)) - U(\bar{x}(k), \bar{y}(k)) \right) + n_1L^2\gamma_k^2 + 2L\gamma_k \sum_{i=1}^{n_1} e_{i1}(k).
\end{equation}

Similarly,
\begin{equation}
\sum_{i=1}^{n_2} |y_i(k+1) - y|^2 \leq \sum_{i=1}^{n_2} |y_i(k) - y|^2 + 2\gamma_k \left( U(\bar{x}(k), \bar{y}(k)) - U(\bar{x}(k), y) \right) + n_2L^2\gamma_k^2 + 2L\gamma_k \sum_{i=1}^{n_2} e_{i2}(k),
\end{equation}
where $e_{i2}(k) = 2|\bar{y}_i(k) - \bar{x}(k)| + |\bar{y}_i(k) - \bar{y}(k)|$. Let $(x, y) = (x^*, y^*) \in X^* \times Y^*$. Denote $\xi(k, x^*, y^*) = \sum_{i=1}^{n_1} |x_i(k) - x|^2 + \sum_{i=1}^{n_2} |y_i(k) - y|^2$. Then adding (18) and (19) together leads to
\begin{align}
\xi(k+1, x^*, y^*) &\leq \xi(k, x^*, y^*) + 2\gamma_k \left( U(x^*, \bar{y}(k)) - U(\bar{x}(k), y^*) \right) \\
&\quad + (n_1 + n_2)L^2\gamma_k^2 + 2L\gamma_k \left( \sum_{i=1}^{n_1} e_{i1}(k) + \sum_{i=1}^{n_2} e_{i2}(k) \right) \\
&= \xi(k, x^*, y^*) - 2\gamma_k \left( U(x^*, y^*) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(x^*, y^*) \right) \\
&\quad + (n_1 + n_2)L^2\gamma_k^2 + 2L\gamma_k \sum_{\ell=1}^{2} \sum_{i=1}^{n_{\ell}} e_{i\ell}(k).
\end{align}

Since $(x^*, y^*)$ is a saddle point of $U$ on $X \times Y$, $U(x^*, y^*) - U(x^*, \bar{y}(k)) \geq 0$, $U(\bar{x}(k), y^*) - U(x^*, y^*) \geq 0$ for $k \geq 0$. Therefore, $U(x^*, y^*) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(x^*, y^*) \geq 0$. Moreover, by Lemma 4.5
\begin{equation}
\sum_{k=0}^{\infty} \gamma_k \sum_{\ell=1}^{2} \sum_{i=1}^{n_{\ell}} e_{i\ell}(k) < \infty.
\end{equation}
Thus, by virtue of $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$, (21), (20) and Lemma 2.5 $\lim_{k \to \infty} \xi(k, x^*, y^*)$ is a finite number, denoted as $\xi(x^*, y^*)$. Thus, the conclusion follows. $\square$

We are in a position to prove Theorem 4.1.

**Proof of Theorem 4.1.** To get the conclusion, we first show that the limit points of all agents satisfy certain sum objective function equations, and then prove the Nash equilibrium convergence under either of the two conditions, (i) and (ii).
It is known that \((x_i(k), y_j(k)), k \geq 0\) are bounded according to Lemma 4.6. Moreover, it also follows from (20) that
\[
2 \sum_{r=0}^{k} \gamma_r \left( U(x^*, y^*) - U(x^*, \bar{y}(r)) + U(\bar{x}(r), y) - U(x^*, y^*) \right)
\]
\[
\leq \xi(0, x^*, y^*) - \xi(k + 1, x^*, y^*) + (n_1 + n_2)L^2 \sum_{r=0}^{k} \gamma_r^2 + 2L \sum_{r=0}^{k} \gamma_r \sum_{i=1}^{n_2} \epsilon_i(r)
\]
\[
\leq \xi(0, x^*, y^*) + (n_1 + n_2)L^2 \sum_{r=0}^{k} \gamma_r^2 + 2L \sum_{r=0}^{k} \gamma_r \sum_{i=1}^{n_2} \epsilon_i(r),
\]
and then by \(\sum_{k=0}^{\infty} \gamma_k^2 < \infty\) and (21) we have
\[
0 \leq \sum_{k=0}^{\infty} \gamma_k \left( U(x^*, y^*) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(x^*, y^*) \right) < \infty. \quad (22)
\]

The stepsize condition \(\sum_{k=0}^{\infty} \gamma_k = \infty\) and (22) lead to
\[
\liminf_{k \to \infty} \left( U(x^*, y^*) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(x^*, y^*) \right) = 0.
\]
As a result, there is a subsequence \(\{k_r\}\) such that \(U(x^*, \bar{y}(k_r)) \to U(x^*, y^*)\) and \(U(\bar{x}(k_r), y^*) \to U(x^*, y^*)\) as \(r \to \infty\). Let \((\bar{x}, \bar{y})\) be any limit pair of \(\{(\bar{x}(k_r), \bar{y}(k_r))\}\) (noting that the finite limit pairs exist by Lemma 4.6). Because \(U(\cdot, \cdot), U(\cdot, y^*)\) are continuous and the Nash equilibrium point \((x^*, y^*)\) is taken from \(X^* \times Y^*\) freely, the limit pair \((\bar{x}, \bar{y})\) must satisfy
\[
U(x^*, \bar{y}) = U(\bar{x}, y^*) = U(x^*, y^*), \quad \forall (x^*, y^*) \in X^* \times Y^*.
\]

We complete the proof by discussing the proposed two sufficient conditions, (i) and (ii):

(i). For the strictly convex-concave function \(U\), we claim that \(X^* \times Y^*\) is a single point set. If it contains two different points \((x_1^*, y_1^*)\) and \((x_2^*, y_2^*)\) (without loss of generality, assume \(x_1^* \neq x_2^*\)), then it also contains the point \((x_2^*, y_1^*)\) (by Lemma 2.4). Thus, for all \(x \in X, U(x_1^*, y_1^*) \leq U(x, y_1^*)\) and \(U(x_2^*, y_1^*) \leq U(x, y_1^*)\), which yields a contradiction since \(U(\cdot, y_1^*)\) is strictly convex and the minimizer of \(U(\cdot, y_1^*)\) is unique. Thus, \(X^* \times Y^*\) contains only a single-point, denoted as \((x^*, y^*)\).

Then from (23), we have \(\bar{x} = x^*, \bar{y} = y^*\). Consequently, each limit pair of \(\{(\bar{x}(k_r), \bar{y}(k_r))\}\) is \((x^*, y^*)\), that is, \(\lim_{r \to \infty} \bar{x}(k_r) = x^*\) and \(\lim_{r \to \infty} \bar{y}(k_r) = y^*\). According to Lemma 4.6, \(\lim_{r \to \infty} x_i(k_r) = x^*\), \(i \in V_1\) and \(\lim_{r \to \infty} y_j(k_r) = y^*, j \in V_2\). Moreover, \(\lim_{k \to \infty} \xi(k, x^*, y^*) = \xi(x^*, y^*)\) as given in Lemma 4.6, so \(\xi(x^*, y^*) = \lim_{r \to \infty} \xi(k_r, x^*, y^*) = 0\), which in return implies \(\lim_{k \to \infty} x_i(k) = x^*\) for \(i \in V_1\) and \(\lim_{k \to \infty} y_i(k) = y^*\) for \(i \in V_2\).

(ii). In Lemma 4.6, we have proved that \(\lim_{k \to \infty} \xi(k, x^*, y^*) = \xi(x^*, y^*)\) for any \((x^*, y^*) \in X^* \times Y^*\). We check the existence of the two limits \(\lim_{k \to \infty} \bar{x}(k)\) and \(\lim_{k \to \infty} \bar{y}(k)\). Let \((x^+, y^+)\)
be an interior point of $X^* \times Y^*$ for which $B(x^+, \varepsilon) \subseteq X^*$ and $B(y^+, \varepsilon) \subseteq Y^*$ for some $\varepsilon > 0$. Clearly, any two limit pairs $(\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)$ of $\{(\bar{x}(k), \bar{y}(k))\}$ must satisfy

$$n_1|\hat{x}_1 - x|^2 + n_2|\hat{y}_1 - y|^2 = n_1|\hat{x}_2 - x|^2 + n_2|\hat{y}_2 - y|^2, \ \forall x \in B(x^+, \varepsilon), \ y \in B(y^+, \varepsilon).$$

Take $y = y^+$. Then

$$n_1|\hat{x}_1 - x|^2 = n_1|\hat{x}_2 - x|^2 + n_2(|\hat{y}_2 - y^+|^2 - |\hat{y}_1 - y^+|^2), \ \forall x \in B(x^+, \varepsilon). \ (24)$$

Taking the gradient with respect to $x$ on the two sides of (24) yields $2n_1(x - \hat{x}_1) = 2n_1(x - \hat{x}_2)$, namely, $\hat{x}_1 = \hat{x}_2$. Similarly, we can show $\hat{y}_1 = \hat{y}_2$. Thus, the limits, $\lim_{k \to \infty} \bar{x}(k) = \bar{x} \in X$ and $\lim_{k \to \infty} \bar{y}(k) = \bar{y} \in Y$, exist. Based on Lemma 4.4, $\lim_{k \to \infty} x_i(k) = \bar{x}, i \in V_1$ and $\lim_{k \to \infty} y_i(k) = \bar{y}, i \in V_2$.

We claim that $(\bar{x}, \bar{y}) \in X^* \times Y^*$. It follows from (18) that, for any $x \in X$,

$$\sum_{k=0}^{\infty} \gamma_k \sum_{i=1}^{n_1} (U(\bar{x}(k), \bar{y}(k)) - U(x, \bar{y}(k))) < \infty.$$ 

Moreover, recalling $\sum_{k=0}^{\infty} \gamma_k = \infty$, we obtain

$$\liminf_{k \to \infty} (U(\bar{x}(k), \bar{y}(k)) - U(x, \bar{y}(k))) \leq 0. \ (25)$$

Then from $\lim_{k \to \infty} \bar{x}(k) = \bar{x}, \lim_{k \to \infty} \bar{y}(k) = \bar{y}$, the continuity of $U$ and (25) we have $U(\bar{x}, \bar{y}) - U(x, \bar{y}) \leq 0$ for all $x \in X$. Similarly, we can show $U(\bar{x}, y) - U(\bar{x}, \bar{y}) \leq 0$ for all $y \in Y$. Thus, $(\bar{x}, \bar{y})$ is a saddle point of $U$ on $X \times Y$, which implies $(\bar{x}, \bar{y}) \in X^* \times Y^*$.

Thus, the proof is completed. \qed

5 Homogenous Stepsize vs. Unbalanced Graph

In the last section, we showed that a Nash equilibrium can be achieved with homogeneous stepsize conditions when the graphs of two subnetworks are weight-balanced. Here we demonstrate, even in a special case, that the homogenous stepsize algorithm may not be enough to guarantee the Nash equilibrium convergence for general weight-unbalanced digraphs, unless certain conditions about the objective functions hold.

Here we consider a special case called the identical subnetwork case, which is equivalent to a distributed saddle-point computation problem, to discuss homogeneous stepsizes and the weight-balanced condition, as follows:

$(o)$ $\Xi_1$ and $\Xi_2$ are completely identical: $n_1 = n_2, f_i = g_i, i = 1, \ldots, n_1, A_1(k) = A_2(k), k \geq 0$. 17
Obviously, the considered problem under the special case (□) is equivalent to the following case. Consider network $\Xi_1$ consisting of $n_1$ agents with node set $\mathcal{V}_1 = \{1, \ldots, n_1\}$, the objective of this network is to seek a saddle point of sum objective function $\sum_{i=1}^{n_1} f_i(x, y)$ by a distributed way, where $f_i$ can only be known by node $i$. The algorithm (4) with homogeneous stepsize $\{\gamma_k\}$ will be simplified as the following distributed saddle point computation algorithm:

$$
\begin{align*}
  x_i(k+1) &= \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)x_j(k) - \gamma_k q_1_i(k), \\
  y_i(k+1) &= \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)y_j(k) + \gamma_k q_2_i(k),
\end{align*}
$$

(26)

where $(x_i, y_i)$ is the state of node $i$, $q_1_i(k) \in \partial_x f_i(\hat{x}_i(k), \hat{y}_i(k))$, and $q_2_i(k) \in \partial_y f_i(\hat{x}_i(k), \hat{y}_i(k))$.

**Remark 5.1** Similar distributed saddle point computation algorithms have been proposed in the literature (for example, distributed saddle point computation for the Lagrange function of constraint optimization problems in [36]). Moreover, the algorithm (26) can be viewed as a distributed version of the following centralized algorithm:

$$
\begin{align*}
  x(k+1) &= P_X(x(k) - \gamma q_1(k)), \quad q_1(k) \in \partial_x U(x(k), y(k)), \\
  y(k+1) &= P_Y(y(k) + \gamma q_2(k)), \quad q_2(k) \in \partial_y U(x(k), y(k)),
\end{align*}
$$

which is proposed to solve the approximate saddle point problem with a constant stepsize in [24].

We first show that the algorithm (4) with homogeneous stepsize (or equivalently (26)) cannot seek the desired Nash equilibrium though it is convergent, even for fixed weight-unbalanced graphs.

**Proposition 5.1** Under A2–A4, suppose that $f_i$, $i = 1, \ldots, n_1$ are strictly convex-concave and the graph is fixed with $\mathcal{G}_1(0)$ strongly connected. Then, with (26), all the agents converge to the unique saddle point, denoted as $(\bar{x}, \bar{y})$, of an objective function $\sum_{i=1}^{n_1} \mu_i f_i$ on $X \times Y$, where $\mu = (\mu_1 \cdots \mu_{n_1})^T$ is the Perron vector of the adjacency matrix $A_1(0)$ of graph $\mathcal{G}_1(0)$.

**Proof:** The result follows from almost the same procedure in the proof of Theorem 4.1 by replacing $\sum_{i=1}^{n_1} \mu_i |x_i(k) - \bar{x}|^2$, $\sum_{i=1}^{n_1} \mu_i |y_i(k) - \bar{y}|^2$ and $\sum_{i=1}^{n_1} \mu_i f_i(x, y)$ for $\sum_{i=1}^{n_1} |x_i(k) - x^*|^2$, $\sum_{i=1}^{n_2} |y_i(k) - y^*|^2$ and $U(x, y)$, respectively.

Although it is hard to achieve the desired Nash equilibrium with the homogeneous-stepsize algorithm in general, we can still achieve it in some cases. Here we can give a necessary and sufficient condition to achieve a Nash equilibrium for any UJSC switching digraph sequence.

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Proposition 5.2 Suppose A2–A4 hold and $f_i, i = 1, \ldots, n_1$ are strictly convex-concave. Then the multi-agent network $\Xi$ achieves a Nash equilibrium by algorithm (26) for any UJSC switching digraph sequence $G_1$ if and only if the (unique) saddle point of $f_i, i = 1, \ldots, n_1$ on $X \times Y$ are the same.

Proof: (Necessity) Let $(x^*, y^*)$ be the unique saddle point of strictly convex-concave function $U$ on $X \times Y$. Take $\mu = (\mu_1 \cdots \mu_{n_1})^T \in S_{n_1}^{+}$. By Lemma 2.4, there is a stochastic matrix $A_1$ such that $\mu^T A_1 = \mu^T$ and $G_{A_1}$ is strongly connected. Let $G_1 = \{ G_1(k) \}$ be the graph sequence of algorithm (1) with $G_1(k) = G_{A_1}$ for $k \geq 0$, and $A_1$ being the adjacency matrix of $G_1(k)$. Clearly, $\{ G_1 \}$ is UJSC. On the one hand, by Proposition 5.1 all agents converge to the unique saddle point of $\sum_{i=1}^{n_1} \mu_i f_i$ on $X \times Y$. On the other hand, the necessity condition states that $\lim_{k \to \infty} x_i(k) = x^*$ and $\lim_{k \to \infty} y_i(k) = y^*$ for $i = 1, \ldots, n_1$. Therefore, $(x^*, y^*)$ is the saddle point of $\sum_{i=1}^{n_1} \mu_i f_i$ on $X \times Y$.

Because $\mu$ is taken from $S_{n_1}^{+}$ freely, we have that, for any $\mu \in S_{n_1}^{+}$,

$$\sum_{i=1}^{n_1} \mu_i f_i(x^*, y) \leq \sum_{i=1}^{n_1} \mu_i f_i(x^*, y^*) \leq \sum_{i=1}^{n_1} \mu_i f_i(x, y^*), \quad \forall x \in X, y \in Y. \quad (27)$$

We next show by contradiction that, given any $i = 1, \ldots, n_1$, $f_i(x^*, y^*) \leq f_i(x, y^*)$ for all $x \in X$. Suppose there are $i_0$ and $\hat{x} \in X$ with $f_{i_0}(x^*, y^*) > f_{i_0}(\hat{x}, y^*)$. Let $\mu_i, i \neq i_0$ be sufficiently small such that $|\sum_{i \neq i_0} \mu_i f_i(x^*, y^*) - f_{i_0}(x^*, y^*)| < \frac{\mu_{i_0}}{2}(f_{i_0}(x^*, y^*) - f_{i_0}(\hat{x}, y^*))$ and $|\sum_{i \neq i_0} \mu_i f_i(\hat{x}, y^*)| < \frac{\mu_{i_0}}{2}(f_{i_0}(x^*, y^*) - f_{i_0}(\hat{x}, y^*))$. Then $\sum_{i=1}^{n_1} \mu_i f_i(x^*, y^*) > \frac{\mu_{i_0}}{2}(f_{i_0}(x^*, y^*) + f_{i_0}(\hat{x}, y^*)) > \sum_{i=1}^{n_1} \mu_i f_i(\hat{x}, y^*)$, which contradicts the second inequality of (27). Thus, $f_i(x^*, y^*) \leq f_i(x, y^*)$ for all $x \in X$. Similarly, we can show from the first inequality of (27) that for each $i = 1, \ldots, n_1$, $f_i(x^*, y) \leq f_i(x^*, y^*)$ for all $y \in Y$. Thus, we obtain that

$$f_i(x^*, y) \leq f_i(x^*, y^*) \leq f_i(x, y^*), \quad \forall x \in X, y \in Y,$$

or equivalently, $(x^*, y^*)$ is the saddle point of $f_i, i = 1, \ldots, n_1$ on $X \times Y$.

(Sufficiency) Let $(x^*, y^*)$ be the unique saddle point of $f_i, i = 1, \ldots, n_1$ on $X \times Y$. Similar to (17), we can show

$$|y_i(k+1) - y^*|^2 \leq \sum_{j \in \mathcal{N}_i^1(k)} a_{ij}(k)|y_j(k) - y^*|^2 + 2\gamma_k \left( f_i(\bar{x}(k), \bar{y}(k)) - f_i(\bar{x}(k), y^*) \right) + L^2 \gamma^2_k + 2L \gamma_k u_{i2}(k), \quad (28)$$

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where \( u_{i2}(k) = 2|\dot{x}_i(k) - \bar{x}(k)| + |\dot{y}_i(k) - \bar{y}(k)| \). Merging (17) and (28) gives

\[
\zeta(k + 1) \leq \zeta(k) + 2\gamma_k \max_{1 \leq i \leq n_1} \left( f_i(x^*, y^*) - f_i(\bar{x}(k), y^*) \right) + 2L^2\gamma_k^2 + 2L\gamma_k \max_{1 \leq i \leq n_1} (u_{i1}(k) + u_{i2}(k))
\]

\[
= \zeta(k) + 2\gamma_k \max_{1 \leq i \leq n_1} \left( f_i(x^*, y^*) - f_i(x^*, y^*) + f_i(x^*, y^*) - f_i(\bar{x}(k), y^*) \right) + 2L^2\gamma_k^2 + 2L\gamma_k \max_{1 \leq i \leq n_1} (u_{i1}(k) + u_{i2}(k)),
\]

where \( \zeta(k) = \max_{1 \leq i \leq n_1} \left( |x_i(k) - x^*|^2 + |y_i(k) - y^*|^2 \right) \), \( u_{i1}(k) = |\dot{x}_i(k) - \bar{x}(k)| + 2|\dot{y}_i(k) - \bar{y}(k)| \).

Since \( f_i(x^*, y^*) - f_i(x^*, y^*) \leq 0 \) and \( f_i(x^*, y^*) - f_i(\bar{x}(k), y^*) \leq 0 \) for all \( i, k \), the second term in (29) is non-positive. By Lemma 2.3,

\[
\lim_{k \to \infty} \zeta(k) = \zeta^* \geq 0
\]

for a finite number \( \zeta^* \), which implies that \( (x_i(k), y_i(k)) \) (for \( k \geq 0 \)) are bounded.

From (29), we also have

\[
0 \leq 2 \sum_{l=0}^{k} \gamma_l \min_{1 \leq i \leq n_1} \left( f_i(x^*, y^*) - f_i(x^*, \bar{y}(l)) + f_i(\bar{x}(l), y^*) - f_i(x^*, y^*) \right)
\]

\[
\leq \zeta(0) - \zeta(k + 1) + 2L^2 \sum_{l=0}^{k} \gamma_l + 2L \sum_{l=0}^{k} \gamma_l \max_{1 \leq i \leq n_1} \left( u_{i1}(l) + u_{i2}(l) \right), \quad k \geq 0
\]

and then

\[
0 \leq \sum_{k=0}^{\infty} \gamma_k \min_{1 \leq i \leq n_1} \left( f_i(x^*, y^*) - f_i(x^*, \bar{y}(k)) + f_i(\bar{x}(k), y^*) - f_i(x^*, y^*) \right) < \infty.
\]

The stepsize condition \( \sum_{k=0}^{\infty} \gamma_k = \infty \) implies that there is a subsequence \( \{k_r\} \) such that

\[
\lim_{r \to \infty} \min_{1 \leq i \leq n_1} \left( f_i(x^*, y^*) - f_i(x^*, \bar{y}(k_r)) + f_i(\bar{x}(k_r), y^*) - f_i(x^*, y^*) \right) = 0.
\]

Without loss of generality, we suppose \( \lim_{r \to \infty} \bar{x}(k_r) = \dot{x}, \lim_{r \to \infty} \bar{y}(k_r) = \dot{y} \) for some \( \dot{x}, \dot{y} \) (otherwise we can find a subsequence of \( \{k_r\} \) recalling the boundedness of system states). Due to the finiteness of the number of agents and the continuity of \( f_i, i = 1, \ldots, n_1 \), there exists \( i_0 \) such that \( f_{i_0}(x^*, y^*) = f_{i_0}(x^*, \dot{y}) \) and \( f_{i_0}(\dot{x}, y^*) = f_{i_0}(x^*, y^*) \). It follows from the strict convexity-concavity of \( f_{i_0} \) that \( \dot{x} = x^*, \dot{y} = y^* \).

Since the consensus is achieved within two subnetworks, \( x_i(k_r) \to x^* \) and \( y_i(k_r) \to y^* \) as \( r \to \infty \), which leads to \( \zeta^* = 0 \) based on (30). Thus, the conclusion follows.

\( \square \)

**Remark 5.2** The proof reveals that the necessity of Proposition 5.2 does not require that each objective function \( f_i \) to be strictly convex-concave, and the strict convexity-concavity of the sum objective function \( \sum_{i=1}^{n_1} f_i \) suffices to guarantee the necessity of Proposition 5.2.
6 Weight-unbalanced Graphs

The results in the preceding sections showed that the homogenous-stepsize algorithm may not make a weight-unbalanced network achieve its Nash equilibrium unless certain conditions (for example, the saddle points of $f_i$, $i = 1, \ldots, n_1$ on $X \times Y$ are the same in a special case) hold. Here we first show that the proposed algorithm with heterogeneous stepsize works for weight-unbalanced graphs under the UJSC connectivity condition, and then design adaptive algorithms to update the (heterogeneous) stepsize to achieve the Nash equilibrium in two standard cases.

To show the main results, we introduce some lemmas. We first give a lemma about the limit of transition matrix sequence $\Phi^\ell(k, s) = A_k(k)A_k(k-1) \cdots A_k(s)$, $k \geq s$, $\ell = 1, 2$, where (i), (ii) and (iv) are taken from Lemma 4 in [32], while (iii) can be obtained from Lemma 2 in [32].

**Lemma 6.1** Suppose A1 (ii) and A2 (i), (ii) hold. Then for $\ell = 1, 2$, we have

(i) The limit $\bar{\Phi}^\ell(s) = \lim k \to \infty \Phi^\ell(k, s)$ exists for each $s$.

(ii) There is a stochastic vector $\phi^\ell(s)$ such that $\Phi^\ell(s) = 1(\phi^\ell(s))^T$.

(iii) For every $i = 1, \ldots, n_\ell$ and $s$, $\phi^\ell_i(s) \geq \eta^{(n_\ell-1)}T_\ell$.

(iv) For every $i$, the entries $\Phi^\ell(k, s)_{ij}$, $j = 1, \ldots, n_\ell$ converge to the same limit $\phi^\ell_j(s)$ with a geometric rate, i.e., for every $i = 1, \ldots, n_\ell$ and all $s \geq 0$,

$$\big| \Phi^\ell(k, s)_{ij} - \phi^\ell_j(s) \big| \leq C_\ell \rho^k s$$

for all $k \geq s$ and $j = 1, \ldots, n_\ell$, where $C_\ell = 2^{1+n_\ell-M_\ell} \rho_\ell = (1 - \eta^{M_\ell})^{\frac{1}{\rho_\ell}}$ and $M_\ell = (n_\ell - 1)T_\ell$.

The next lemma is about a limit for the two subnetworks.

**Lemma 6.2** Consider algorithm (4) with A2 (ii) and A3. If $\lim k \to \infty \bar{\alpha}_k \sum_{s=0}^{k-1} \bar{\alpha}_s = \lim k \to \infty \bar{\beta}_k \sum_{s=0}^{k-1} \bar{\beta}_s = 0$, then for any $x, y$, $\lim k \to \infty \bar{\alpha}_k \max_{1 \leq i \leq n_1} |x_i(k) - x| = \lim k \to \infty \bar{\beta}_k \max_{1 \leq i \leq n_2} |y_i(k) - y| = 0$, where $\bar{\alpha}_k = \max_{1 \leq i \leq n_1} \alpha_{i,k}$, $\bar{\beta}_k = \max_{1 \leq i \leq n_2} \beta_{i,k}$.

**Proof:** We will only show $\lim k \to \infty \bar{\alpha}_k \max_{1 \leq i \leq n_1} |x_i(k) - x| = 0$ since the one about $\bar{\beta}_k$ can be proved similarly. At first, from $\lim k \to \infty \bar{\alpha}_k \sum_{s=0}^{k-1} \bar{\alpha}_s = 0$, we have $\lim k \to \infty \bar{\alpha}_k = 0$. Recall (3):

$$x_i(k+1) = \sum_{j \in N_i^n(k)} a_{ij}(k)x_j(k) + \epsilon_i(k), \ i = 1, \ldots, n_1,$$

where $|\epsilon_i(k)| \leq \bar{\alpha}_k L$ from A3. From the above equation and A2 (ii) we have $\max_{1 \leq i \leq n_1} |x_i(k+1) - x| \leq \max_{1 \leq i \leq n_1} |x_i(k) - x| + \bar{\alpha}_k L, \forall k$. Therefore, $\max_{1 \leq i \leq n_1} |x_i(k) - x| \leq \max_{1 \leq i \leq n_1} |x_i(0) -
where \( \alpha \) and \( s \) algorithm (4).

**Lemma 6.3** Let \( \{B_k\} \) be a sequence of stochastic matrices. Suppose \( B_k, k \geq 0 \) have a common left eigenvector \( \mu \) with eigenvalue one and the associated graph sequence \( \{G_k\} \) is UJSC. Then, for each \( s \), \( \lim_{k \to \infty} B_k \cdots B_s = \mathbf{1} \mu^T / (\mu^T \mathbf{1}) \).

**Proof:** Since \( \mu \) is the common left eigenvector of \( B_r, r \geq s \) associated with eigenvalue one, \( \mu^T \lim_{k \to \infty} B_k \cdots B_s = \lim_{k \to \infty} \mu^T B_k \cdots B_s = \mu^T \). In addition, by Lemma 6.1 for each \( s \), the limit of \( \lim_{k \to \infty} B_k \cdots B_s := \phi^T(s) \) exists. Therefore, \( \mu^T (1 \phi^T(s)) = (\mu^T \mathbf{1}) \phi^T(s) = \mu^T \), which implies (\( \mu^T \mathbf{1} \)) exists. Thus, the conclusion follows. □

Here is the main result for general graphs (may be weight-unbalanced) for the existence of a heterogeneous-stepsizes design to make the network achieve a Nash equilibrium.

**Theorem 6.1** Suppose \( \mathbf{A2} - \mathbf{A3} \) hold and \( U \) is strictly convex-concave. Then for any time-varying communication graphs \( \mathcal{G}_\ell, \ell = 1, 2 \) and \( \mathcal{G}_{01} \) that satisfy \( \mathbf{A1} \), there always exist stepsizes sequences \( \{\alpha_{i,k}\} \) and \( \{\beta_{i,k}\} \) such that the multi-agent network \( \Xi \) achieves a Nash equilibrium by algorithm (3).

**Proof:** First by Lemma 6.1 (i), (ii), the limit \( \lim_{r \to \infty} \Phi^\ell(r, k) := 1 (\phi^\ell(k))^T \) exists for each \( k \). We design the stepsizes \( \alpha_{i,k} \) and \( \beta_{i,k} \) as follows:

\[
\alpha_{i,k} = \frac{1}{\alpha_k^1} \gamma_k, \quad \beta_{i,k} = \frac{1}{\beta_k^1} \gamma_k, \tag{32}
\]

where \((\alpha_k^1 \cdots \alpha_k^{n_1})^T = \phi^1(k+1), (\beta_k^1 \cdots \beta_k^{n_2})^T = \phi^2(k+1), \{\gamma_k\}\) satisfies the following conditions:

\[
\lim_{k \to \infty} \gamma_k \sum_{s=0}^{k-1} \gamma_s = 0, \quad \{\gamma_k\} \text{ is non-increasing}, \quad \sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty. \tag{33}
\]

Let \((x^*, y^*)\) be the unique Nash equilibrium. From (17) we have

\[
|x_i(k + 1) - x^*|^2 \leq \sum_{j \in N_i^\ell(k)} \alpha_{ij}(k) |x_j(k) - x^*|^2 + 2\alpha_{i,k} \left( f_i(x^*, y(k)) - f_i(x(k), y(k)) \right) + L^2 \alpha_{i,k}^2 + 2L\alpha_{i,k} e_{i1}(k). \tag{34}
\]
Similarly, \[
|y_i(k + 1) - y^*|^2 \leq \sum_{j \in \mathcal{N}^+_i(k)} a_{ij}(k) |y_j(k) - y^*|^2 + 2\beta_{i,k}\left(g_i(x_i(k), y_i(k)) - g_i(\bar{x}(k), y^*)\right) + L^2\beta^2_{i,k} + 2L\beta_{i,k}e_i^2(\ell).
\] (35)

Denote \(\Lambda^1_k = \text{diag}\{\frac{1}{\gamma_i} \cdots \frac{1}{\gamma_n}\}, \Lambda^2_k = \text{diag}\{\frac{2}{\gamma_i} \cdots \frac{2}{\gamma_n}\}\); \(\psi(\ell)(k) = (\psi_1(\ell)(k) \cdots \psi_n(\ell)(k))^T, \ell = 1, 2,\)
\(\psi_1(k) = |x_i(k) - x^*|^2, \psi_2(k) = |y_i(k) - y^*|^2; \vartheta(\ell)(k) = (\vartheta_1^\ell(k) \cdots \vartheta_n^\ell(k))^T, \vartheta_1^\ell(k) = f_i(x_i(k), y_i(k)) - f_i(x^*, y_i(k)), \vartheta_2^\ell(k) = g_i(\bar{x}(k), y^*) - g_i(\bar{x}(k), y_i(k)); e_\ell(k) = (e_{1\ell}(k) \cdots e_{n\ell}(k))^T\). Then it follows from (34) and (35) that
\[
\psi(\ell)(k + 1) \leq A_\ell(k)\psi(\ell)(k) - 2\gamma_k\Lambda^1_k\vartheta(\ell)(k) + \delta^2_kL^2\gamma_k^21 + 2\delta_kL\gamma_ek_\ell(k), k \geq 0,
\] (36)

where \(\delta_k = \sup_{i,k}\{1/\alpha_k^i, 1/\beta_k^i\}\). By Lemma 6.1 (iii), \(\alpha_k^i \geq \eta^{(n_1 - 1)T_1}, \beta_k^i \geq \eta^{(n_2 - 1)T_2}, \forall i, k\) and then \(\delta_k\) is a finite number. Therefore, by (36) we have
\[
\psi(\ell)(k + 1) \leq \Phi(\ell)(k, r)\psi(\ell)(r) - 2\sum_{s=r}^{k-1} \gamma_s\Phi(\ell)(k, s + 1)\Lambda^1_s\vartheta(\ell)(s) + \delta^2_kL^2\sum_{s=r}^{k} \gamma^2_s1
+ 2\delta_kL\sum_{s=r}^{k-1} \gamma_s\Phi(\ell)(k, s + 1)e_\ell(s) - 2\gamma_k\Lambda^1_k\vartheta(\ell)(k) + 2\delta_kL\gamma_ek_\ell(k).
\] (37)

Then (37) can be written as
\[
\psi(\ell)(k + 1) \leq \Phi(\ell)(k, r)\psi(\ell)(r) - 2\sum_{s=r}^{k-1} \gamma_s1(\vartheta(\ell)(s + 1))\Lambda^1_s\vartheta(\ell)(s) + \delta^2_kL^2\sum_{s=r}^{k} \gamma^2_s1
+ 2\delta_kL\sum_{s=r}^{k-1} \gamma_s1(\vartheta(\ell)(s + 1))e_\ell(s) + 2\sum_{s=r}^{k-1} \gamma_s\left[1(\vartheta(\ell)(s + 1))^T - \vartheta(\ell)(k, s + 1)\right]\Lambda^1_s\vartheta(\ell)(s)
- 2\gamma_k\Lambda^1_k\vartheta(\ell)(k) + 2\delta_kL\gamma_ek_\ell(k) + 2\delta_kL\sum_{s=r}^{k-1} \gamma_s\left[\Phi(\ell)(k, s + 1) - 1(\vartheta(\ell)(s + 1))^T\right]e_\ell(s).
\] (38)

The subsequent proof is as follows. First, we will show that the designed stepsizes (32) can eliminate the imbalance caused by the unbalanced graphs (see the second term in (35)), then we shall prove that all the terms from the third one to the last one in (38) is summable based on the geometric rate convergence of transition matrices. Finally, we will show the desired convergence based on inequality (38), as (20) for the weight-balance case in Theorem 4.1.

Clearly, \(1(\vartheta(\ell)(s + 1))^T\Lambda^1_s = 11^T, \ell = 1, 2\). From Lemma 6.1 (iv) we also have that for \(\ell = 1, 2\), every \(i = 1, \ldots, n_\ell\) and \(s \geq 0\), \(|\Phi(\ell)(k, s)_{ij} - \vartheta(\ell)_j(s)| \leq C\rho^{-s}\) for all \(k \geq s\) and \(j = 1, \ldots, n_\ell\), where \(C = \max\{C_1, C_2\}, 0 < \rho = \max\{\rho_1, \rho_2\} < 1\). Moreover, by A3, for \(i \in \mathcal{V}_1\), \(\gamma_k|\vartheta_1^i(k)| = \gamma_k|f_i(x_i(k), y_i(k)) - f_i(x^*, y_i(k))| \leq L\gamma_k|x_i(k) - x^*|\), and for \(i \in \mathcal{V}_2\), \(\gamma_k|\vartheta_2^i(k)| = L\gamma_k|x_i(k) - x^*|\),
\[ \gamma_k |f_i(\bar{x}(k), y^*) - f_i(\bar{x}(k), \bar{y}(k))| \leq L \gamma_k |y_i(k) - y^*|. \]

Then based on these conclusions, multiplying \(1/n_\ell 1^{T}\) on the both sides of (39) and taking the sum over \(\ell = 1, 2\) yield

\[
\sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \psi^f(k + 1)
\leq \sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \Phi^f(k, r) \psi^f(r) - 2 \sum_{s=r}^{k-1} \sum_{i=1}^{n_\ell} \vartheta_i^f(s) + 2\delta^2 L^2 \sum_{s=r}^{k} \gamma_s^2 + 2\delta_s L \sum_{s=r}^{k-1} \gamma_s \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} \varepsilon_i^f(s)
\]

\[
+ 2CL\delta_s(n_1 + n_2) \rho^{k-s-1} \gamma_s \varsigma(s) + 2LC\delta_s \sum_{s=r}^{k-1} \gamma_s \rho^{k-s-1} \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} \varepsilon_i^f(s)
\]

\[
+ 2\delta_s \gamma_k \varsigma(k) + 2\delta_s L \gamma_k \sum_{\ell=1}^{2} \frac{1}{n_\ell} \sum_{i=1}^{n_\ell} \varepsilon_i^f(k) \tag{39}
\]

\[
:= \sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \Phi^f(k, r) \psi^f(r) - 2 \sum_{s=r}^{k-1} \sum_{i=1}^{n_\ell} \vartheta_i^f(s) + \sum_{s=r}^{k} \theta_s \tag{40}
\]

where \(\varsigma(s) = \max\{|x_i(s) - x^*|, i \in V_1, |y_j(s) - y^*|, j \in V_2\}\), \(\sum_{s=r}^{k} \theta_s \) is the sum of all terms from the third one to the last one in (39).

We next show \(\sum_{s=r}^{k} \theta_s < \infty\). Lemma 6.2 and Remark 4.2 imply \(\sum_{s=r}^{k} \gamma_s \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} e_i^f(s) < \infty\) and then \(\lim_{k \to \infty} \gamma_k \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} e_i^f(k) = 0\). Noticing \(0 < \rho < 1\), we obtain for each \(k\),

\[
\sum_{s=r}^{k-1} \gamma_s \rho^{k-s-1} \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} e_i^f(s) \leq \sum_{s=r}^{k} \gamma_s \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} e_i^f(s) < \infty.
\]

Moreover, by Lemma 6.2 we have \(\lim_{s \to \infty} \gamma_s \varsigma(s) = 0\), which combined with Lemma 6.2 leads to \(\lim_{k \to \infty} \sum_{s=r}^{k} \rho^{k-s-1} \gamma_s \varsigma(s) = 0\). From the previous conclusions we have \(\sum_{s=r}^{k} \theta_s < \infty\). Then from (40) \(\sum_{s=r}^{k} \gamma_s \sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} \vartheta_i^f(s) < \infty\).

Clearly, \(\sum_{\ell=1}^{2} \sum_{i=1}^{n_\ell} \vartheta_i^f(s) = U(\bar{x}(k), \bar{y}(k)) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(\bar{x}(k), \bar{y}(k)) = U(x^*, y^*) - U(x^*, \bar{y}(k)) + U(\bar{x}(k), y^*) - U(x^*, y^*) \geq 0\). Then by similar procedures in the proof of Theorem 4.1 (i), we can show that there is a subsequence \(\{k_1\}\) such that \(\lim_{i \to \infty} \bar{x}(k_i) = x^*\), \(\lim_{i \to \infty} \bar{y}(k_i) = y^*\).

Now we show \(\lim_{k \to \infty} \sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \psi^f(k) = 0\). It follows from \(\sum_{s=r}^{k} \theta_s < \infty\) that for any \(\varepsilon > 0\), there exists sufficiently large \(l_0\) such that when \(l \geq l_0\), \(\sum_{s=k}^{\infty} \theta_s \leq \varepsilon\). Moreover, by noticing that the consensus within the two subnetworks are achieved, the number \(l_0\) can be selected sufficiently large such that for each \(i\), \(|x_i(k_{l_0}) - x^*| \leq \varepsilon\) and \(|y_i(k_{l_0}) - y^*| \leq \varepsilon\). Then based on (40) we have for each \(k \geq k_1\),

\[
\sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \psi^f(k + 1) \leq \sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \Phi^f(k, k_1) \psi^f(k_1) + \sum_{s=k_1}^{\infty} \theta_s \leq 2\varepsilon + \varepsilon,
\]

from which we have \(\lim_{k \to \infty} \sum_{\ell=1}^{2} \frac{1}{n_\ell} 1^{T} \psi^f(k) = 0\). Therefore, \(\lim_{k \to \infty} x_i(k) = x^*\), \(i \in V_1\) and \(\lim_{k \to \infty} y_i(k) = y^*\), \(i \in V_2\).

Thus, the proof is completed. \(\square\)
Remark 6.1 The stepsize design strategy in Theorem 6.1 is motivated by the following two ideas. On one hand, agents need to eliminate the imbalance caused by the unbalanced graphs, which is done by \( \{1/\alpha_k^i\}, \{1/\beta_k^i\} \), while on the other hand, agents also need to achieve a consensus within each subnetwork and a cooperative optimization behavior, which is done by \( \{\gamma_k\} \) as that in the balanced graph case.

Remark 6.2 The condition (33) can be satisfied by letting \( \gamma_k = \frac{1}{k+1} \) for \( k \geq 0 \). Moreover, from the proof of Theorem 6.1 we can find that, if the system states are bounded, which are naturally true if the sets \( X \) and \( Y \) are bounded, the condition (33) can be relaxed as the following condition: \( \{\gamma_k\} \) is non-increasing, \( \sum_{k=0}^{\infty} \gamma_k = \infty \), \( \sum_{k=0}^{\infty} \gamma_k^2 < \infty \).

Clearly, the above choice of stepsizes at time \( k \) is based on the limits of \( \lim_{r \to \infty} \Phi_1^1(r, k+1) \) and \( \lim_{r \to \infty} \Phi_2^2(r, k+1) \), respectively, which depend on the adjacency matrix sequences \( \{A_1(s)\}_{s \geq k+1} \) and \( \{A_2(k)\}_{s \geq k+1} \). Therefore, it is not easy to use this approach in practical applications. In what follows, we consider how to design adaptive stepsize sequences \( \{\alpha_{i,k}\} \) and \( \{\beta_{i,k}\} \) such that the Nash equilibrium can be achieved, where the stepsizes at time \( k \) just depend on the local information nodes can obtain before time \( k \) for some special cases.

Take
\[
\alpha_{i,k} = \frac{1}{\hat{\alpha}_i^k(k)} \gamma_k, \quad \beta_{i,k} = \frac{1}{\hat{\beta}_i^k(k)} \gamma_k,
\]
(41)
where \( \{\gamma_k\} \) satisfies (33). The only difference between stepsize selection rule (41) and (32) is that \( \alpha_k^i \) and \( \beta_k^i \) are replaced with \( \hat{\alpha}_i^k \) and \( \hat{\beta}_i^k \), respectively. Then we consider how to design distributed adaptive algorithms for \( \hat{\alpha}_i^k \) and \( \hat{\beta}_i^k \) such that
\[
\hat{\alpha}_i^k(k) = \hat{\alpha}_i^k(a_{ij}(s), j \in N_i^1(s), s \leq k), \quad \hat{\beta}_i^k(k) = \hat{\beta}_i^k(a_{ij}(s), j \in N_i^2(s), s \leq k),
\]
(42)
and
\[
\lim_{k \to \infty} \left( \hat{\alpha}_i^k(k) - \alpha_k^i \right) = 0, \quad \lim_{k \to \infty} \left( \hat{\beta}_i^k(k) - \beta_k^i \right) = 0.
\]
(43)
Notice that \( (\alpha_k^1 \cdots \alpha_k^{n_1})^T \) and \( (\beta_k^1 \cdots \beta_k^{n_2})^T \) are the Perron vectors of the limits of \( \lim_{r \to \infty} \Phi_1^1(r, k+1) \) and \( \lim_{r \to \infty} \Phi_2^2(r, k+1) \), respectively.

The next theorem shows that, in two particular cases, we can design distributed adaptive algorithms satisfying (42) and (43) in order to ensure that network \( \Xi \) can achieve a Nash equilibrium. How to design them is given in the proof of the next theorem.

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**Theorem 6.2** Consider algorithm (4) with stepsize selection rule (41). Suppose A1–A4 hold, U is strictly convex-concave. For the following two cases, with the adaptive distributed algorithms satisfying (42) and (43), network Ξ can achieve a Nash equilibrium.

(i) For ℓ = 1, 2, the adjacency matrices \( A_ℓ(k), k \geq 0 \) have a common left eigenvector with eigenvalue one;

(ii) For ℓ = 1, 2, the adjacency matrices \( A_ℓ(k), k \geq 0 \) are switching periodically, that is, there exist positive integers \( S^ℓ \) and two finite sets of stochastic matrices \( A^0_ℓ, ..., A^{S^ℓ-1}_ℓ \) such that \( A_ℓ(rS^ℓ + s) = A^s_ℓ \) for \( r \geq 0 \) and \( s = 0, ..., S^ℓ - 1 \).

**Proof:** (i). In this case we design another auxiliary state \( \alpha^i = (\alpha^1_i \cdots \alpha^n_i)^T \) for node \( i \in V_1 \) to estimate the desired stepsize; similarly, design \( \beta^i = (\beta^1_i \cdots \beta^n_i)^T \) for node \( i \in V_2 \), where for each \( p \) (\( p = 1, ..., n_1 \)), the dynamics of \( \alpha^i_p, i = 1, ..., n_1 \) and \( \beta^i_p, i = 1, ..., n_2 \) are given by

\[
\begin{align*}
\alpha^i_p(k + 1) &= \sum_{j \in \mathcal{N}_i^p} a_{ij}(k)\alpha^j_p(k), \\
\beta^i_p(k + 1) &= \sum_{j \in \mathcal{N}_i^p} a_{ij}(k)\beta^j_p(k), \quad k \geq 0
\end{align*}
\]

with the initial values \( \alpha^0_p(0) = 1, \alpha^p_i(0) = 0, \forall i \neq p; \beta^0_p(0) = 1, \beta^p_i(0) = 0, \forall i \neq p \). Clearly, algorithm (44) satisfies (42).

Then for each \( i \) and \( k \), let \( \dot{\alpha}^i(k) = \alpha^i_1(k), \dot{\beta}^i(k) = \beta^i_1(k) \).

First by A2 (i) and algorithm (44) we have that for each \( r, \alpha^i_1(k) \geq \eta^k > 0 \) and \( \beta^i_1(k) \geq \eta^k > 0 \), which guarantees that the stepsize selection rule (41) is well-defined. Let \( \phi^r = (\phi^1 \cdots \phi^n)^T \) be the common left eigenvector of \( A_ℓ(r), r \geq 0 \) with eigenvalue one, where \( \sum_{i=1}^n \phi^i = 1 \). According to Lemma 6.3 for each \( k \geq 0 \), \( \lim_{r \to \infty} \Phi^r(r, k) = \lim_{r \to \infty} A_ℓ(r) \cdots A_ℓ(k) = 1(\phi^r)^T \). Then \( \alpha^k_1 = \phi^1, i = 1, ..., n_1; \beta^k_1 = \phi^2, i = 1, ..., n_2 \) for all \( k \).

For \( p = 1, ..., n_1 \), let \( \theta_p(k) = (\alpha^1_p(k) \cdots \alpha^n_p(k))^T \). From algorithm (44) we have \( \theta_p(k + 1) = A_1(k)\theta_p(k) \) and then \( \lim_{k \to \infty} \theta_p(k) = \lim_{k \to \infty} \Phi^1(0)\theta_p(0) = 1(\phi^1)^T\theta_p(0) = \phi^1_11 \). Therefore, \( \lim_{k \to \infty} \alpha^i_1(k) = \phi^1_1 \) for \( i \in V_1 \). Similarly, we can show \( \lim_{k \to \infty} \beta^i_1(k) = \phi^2_1 \) for \( i \in V_2 \). Noticing \( \alpha^k_1 = \phi^1_1 \) and \( \beta^k_1 = \phi^2_1 \) for all \( k \), (43) holds. Moreover, by Lemma 6.1 (iv) the above convergence is achieved with geometric rate. Without loss of generality, suppose \( |\alpha^i_1(s) - \phi^1_1| \leq \bar{C}\rho^s \) and \( |\beta^i_1(s) - \phi^2_1| \leq \bar{C}\rho^s \) for \( C > 0, 0 < \rho < 1 \), and all \( i, s \).

The only difference between the models in Theorem 6.1 and the current one is that the terms \( \alpha^k_1 \) and \( \beta^k_1 \) (equal to \( \phi^1_1 \) and \( \phi^2_1 \) in case (i), respectively) in stepsize selection rule (32) are replaced with \( \dot{\alpha}^i(s) \) and \( \dot{\beta}^i(s) \) (equal to \( \alpha^i_1(s) \) and \( \beta^i_1(s) \), respectively) in stepsize selection rule (41). We can find that all lemmas involved in the proof of Theorem 6.1 still hold under the new stepsize selection rule (41). Moreover, all the analysis is almost the same as that in Theorem 6.1.
except that the new stepsize selection rule will yield an error term, denoted as \( \bar{\omega}^f(k,r) \), on the right-hand side of (37). In fact,

\[
\bar{\omega}^f(k,r) = 2 \sum_{s=r}^{k-1} \gamma_s \Phi^f(k,s+1) \bar{\omega}^f(s) + 2 \gamma_k \bar{\omega}^f(k),
\]

where \( \bar{\omega}^1_s = \text{diag}\{ \frac{1}{\phi^1_1} - \alpha^1(s) \cdots \frac{1}{\phi^1_n} - \alpha^1_n(s) \} \), \( \bar{\omega}^2_s = \text{diag}\{ \frac{1}{\phi^2_1} - \beta^1(s) \cdots \frac{1}{\phi^2_n} - \beta^1_n(s) \} \). Moreover, since \( \lim_{s \to \infty} \alpha^1_i(s) = \bar{\phi}^1_i \), \( \alpha^1(s) \geq \phi^1_i/2 \geq \eta/(n-1)T_1/2 \) and then

\[
\left| \frac{1}{\alpha^1_i(s)} - \frac{1}{\phi^1_i} \right| \leq \left| \frac{1}{\alpha^1_i(s)} - \frac{1}{\phi^1_i} \right| \leq \frac{2}{(\eta/(n-1)T_1)^2} \beta^s
\]

for sufficiently large \( s \). Similarly, we have

\[
\left| \frac{1}{\beta^1(s)} - \frac{1}{\phi^1_i} \right| \leq \frac{2}{(\eta/(n-1)T_2)^2} \beta^s. \tag{45}
\]

where \( \varepsilon_1 = \max\{ \gamma_s, \frac{1}{\eta/(n-1)T_1}, \frac{1}{\eta/(n-1)T_2} \} \), \( \varepsilon_2 = \sup_s \{ \gamma_s, \max_{i,j} \{ |x_i(s) - x^*|, |y_j(s) - y^*| \} \} \) is finite since \( \lim_{s \to \infty} \gamma_s \max_{i,j} \{ |x_i(s) - x^*|, |y_j(s) - y^*| \} = 0 \) by Lemma 6.2. From the proof of Theorem 6.1, we can find that the relation (45) makes all the arguments hold and then a Nash equilibrium is achieved for case (i).

(ii). Here we design \( S^1 \) auxiliary states \( \alpha^{(v)} = (\alpha_1^{(v)} \cdots \alpha_n^{(v)})^T \), \( \nu = 0, ..., S^1 - 1 \) for node \( i \in V_1 \) to estimate the desired stepsize; similarly, design \( S^2 \) auxiliary states \( \beta^{(v)} = (\beta_1^{(v)} \cdots \beta_n^{(v)})^T \), \( \nu = 0, ..., S^2 - 1 \) for node \( i \in V_2 \), where for each \( p \) and \( \nu \), the dynamics of \( \alpha^{(p)} \), \( i = 1, ..., n_1 \) and \( \beta^{(p)} \), \( i = 1, ..., n_2 \) are given by

\[
\begin{align*}
\alpha_p^{(v)}(s+1) &= \sum_{j \in N_i^v} a_{ij}(s) \alpha^{(v)}_j(s), \quad s \geq \nu + 1, \\
\beta^{(p)}_i(s+1) &= \sum_{j \in N_i^p} a_{ij}(s) \beta^{(p)}_j(s), \quad s \geq \nu + 1
\end{align*}
\]

with the initial values \( \alpha_p^{(v)}(\nu + 1) = 1, \alpha^{(v)}_i(\nu + 1) = 0, \forall i \neq p; \beta^{(p)}_i(\nu + 1) = 1, \beta^{(p)}_i(\nu + 1) = 0, \forall i \neq p \) and \( \alpha^{(v)}_i(s) = 1, i = 1, ..., n_1; \beta^{(p)}_i(s) = 1, i = 1, ..., n_2 \) for all \( 0 \leq s \leq \nu \).

Then, for each \( r \geq 0 \), let \( \hat{\alpha}^i(rS^1 + \nu) = \alpha^{(v)}_i(rS^1 + \nu) \) for \( i \in V_1 \), \( \nu = 0, ..., S^1 - 1 \); let \( \hat{\beta}^i(rS^2 + \nu) = \beta^{(p)}_i(rS^2 + \nu) \) for \( i \in V_2 \), \( \nu = 0, ..., S^2 - 1 \).

Note that A1 implies that the union graphs \( \bigcup_{s=0}^{S^1-1} G_{A_i^1}, \ell = 1, 2 \) are strongly connected. Let \( \phi^{(0)}_i \) be the Perron vector of the limit of transition matrix \( \Phi^f(rS^\ell - 1,0) \), i.e., \( \lim_{r \to \infty} \Phi^f(rS^\ell - 1,0) \).
1, 0) = \lim_{r \to \infty} (A^{S-1}_x \cdots A^0_y)^r = 1(\phi^{(0)})^T. \text{ Then for } \nu = 1, \ldots, S^\ell - 1,

\lim_{r \to \infty} \Phi^\ell(rS^\ell + \nu - 1, \nu) = \lim_{r \to \infty} (A^{\nu-1}_x \cdots A^0_x A^{S-1}_x \cdots A^{\nu+1}_x A^r_y)^r

= \lim_{r \to \infty} (A^{\nu-1}_x \cdots A^0_y A^{S-1}_x \cdots A^{\nu+1}_x A^r_y

= 1(\phi^{(0)})^T A^{S-1}_x \cdots A^{\nu+1}_x A^r_y

:= 1(\phi^{(\nu)})^T. \tag{47}

Then for each \( r \geq 0, \alpha_i^{r,1}_{S^1+\nu} = \phi_i^{1(\nu+1)}, \nu = 0, 1, \ldots, S^1 - 2, \alpha_i^{r,1}_{S^1+S^1-1} = \phi_i^{1(0)}; \beta_i^{r,2}_{r+S^2+\nu} = \phi_i^{2(\nu+1)}, \nu = 0, 1, \ldots, S^2 - 2 \) and \( \beta_i^{r,2}_{S^2+S^2-1} = \phi_i^{2(0)} \). Moreover, from \( 46 \) and \( 47 \) we obtain that for \( \nu = 0, 1, \ldots, S^1 - 1 \),

\[ \lim_{r \to \infty} \theta^{\nu}(r) = \lim_{r \to \infty} \Phi^1(r, \nu + 1) \theta^{\nu}(\nu + 1) \]

\[ = \lim_{r \to \infty} \Phi^1(r, S^1) A^{S-1}_x \cdots A^{\nu+1}_x \theta^{\nu}(\nu + 1) \]

\[ = 1(\phi^{1(\nu+1)})^T \theta^{\nu}(\nu + 1), \]

where \( \phi^{1(S^1)} = \phi^{1(0)}, \theta^{\nu}(r) = (\alpha^{(\nu)}_p(1) \cdots \alpha^{(\nu)}_p(m_1)(r))^T, \theta^{\nu}(\nu + 1) \) is the \( p \)-th unit vector \((p\)-th component is one and all other components are zero\). Then \( \lim_{r \to \infty} \alpha_i^{(\nu)}(r) = \phi_i^{1(\nu+1)} \) for \( i \in V_1 \). Similarly, we can show that \( \lim_{r \to \infty} \beta_i^{(\nu+1)}(r) = \phi_i^{2(\nu+1)} \) for \( i \in V_2 \). Thus, we show that

\[ \lim_{r \to \infty} (\hat{\alpha}_i^{1(rS^1+\nu)} - \alpha_i^{r,1}_{S^1+\nu}) = 0, \nu = 0, \ldots, S^1 - 1; \lim_{r \to \infty} (\hat{\beta}_i^{1(rS^2+\nu)} - \beta_i^{r,2}_{r+S^2+\nu}) = 0, \nu = 0, \ldots, S^2 - 1. \]

Moreover, the convergence is achieved with geometric rate. Then applying the similar idea used in showing (i) can show the conclusion (ii).

This concludes the proof and the desired conclusion follows. \( \square \)

7 Numerical Examples

In this section, we provide examples to illustrate the obtained results in both balanced and unbalanced graph cases.

Consider a network of five agents, where \( n_1 = 3, n_2 = 2, m_1 = m_2 = 1, X = Y = [-5, 5] \), \( f_1(x, y) = x^2 - (20 - x^2)(y - 1)^2, f_2(x, y) = |x - 1| - |y|, f_3(x, y) = (x - 1)^3 - 2y^2 \) and \( g_1(x, y) = (x - 1)^4 - |y| - \frac{5}{2}y^2 - \frac{1}{2}(20 - x^2)(y - 1)^2, g_2(x, y) = x^2 + |x - 1| - \frac{3}{4}y^2 - \frac{1}{2}(20 - x^2)(y - 1)^2 \). Notice that \( \sum_{i=1}^3 f_i = \sum_{i=1}^2 g_i \) and all objective functions are strictly convex-concave on \([ -5, 5 ] \times [-5, 5] \). The unique saddle point of sum objective function \( g_1 + g_2 \) on \([ -5, 5 ] \times [-5, 5] \) is \((0.6102, 0.8844)\).

Take initial conditions \( x_1(0) = 2, x_2(0) = -0.5, x_3(0) = -1.5 \) and \( y_1(0) = 1, y_2(0) = 0.5 \). When \( \dot{x}_2(k) = 1 \), we take \( q_{12}(k) = 1 \in \partial_x f_2(1, \dot{x}_2(k)) = [-1, 1] \); when \( \dot{y}_1(k) = 0 \), we take
\( q_{21}(k) = -1 + (20 - \hat{y}_i^2(k)) \in \partial_y g_1(\hat{y}_1(k), 0) = \{ r + (20 - \hat{y}_i^2(k)) | -1 \leq r \leq 1 \} \). Let \( \gamma_k = 1/(k+50) \), \( k \geq 0 \), which satisfies \( A4 \).

**Example 7.1** The communication graph is switching periodically over the two graphs \( G^e, G^o \) given in Fig. 1, where \( G(2k) = G^e, G(2k+1) = G^o, k \geq 0 \). Denote \( G^e_1 \) and \( G^o_1 \) by the two subgraphs of \( G^e \) describing the communications within the two subnetworks. Similarly, the subgraphs of \( G^o \) are denoted as \( G^o_1 \) and \( G^o_2 \). Here the adjacency matrices of \( G^e_1, G^e_2 \) and \( G^o_1 \) are as follows:

\[
A_1(2k) = \begin{pmatrix}
0.6 & 0.4 & 0 \\
0.4 & 0.6 & 0 \\
0 & 0 & 1
\end{pmatrix},
A_1(2k + 1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.7 & 0.3 \\
0 & 0.3 & 0.7
\end{pmatrix},
A_2(2k) = \begin{pmatrix}
0.9 & 0.1 \\
0.1 & 0.9
\end{pmatrix}.
\]

Clearly, with the above adjacency matrices, the three digraphs \( G^e_1, G^e_2 \) and \( G^o_1 \) are weight-balanced. Let the stepsize be \( \alpha_{i,k} = \beta_{j,k} = \gamma_k \) for all \( i, j \) and \( k \geq 0 \). Fig. 2 shows that the agents converge to the unique Nash equilibrium \( (x^*, y^*) = (0.6102, 0.8844) \).

**Example 7.2** We consider the same switching graphs given in Example 7.1 except that \( G^e_1 \) is added a new arc \((2, 3)\), and the new graph is still denoted as \( G^e_1 \) for simplicity. Here the adjacency matrices of the three digraphs \( G^e_1, G^e_2 \) and \( G^o_1 \) are given by

\[
A_1(2k) = \begin{pmatrix}
0.8 & 0.2 & 0 \\
0.7 & 0.3 & 0 \\
0 & 0.6 & 0.4
\end{pmatrix},
A_1(2k + 1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.3 & 0.7 \\
0 & 0.4 & 0.6
\end{pmatrix},
A_2(2k) = \begin{pmatrix}
0.9 & 0.1 \\
0.1 & 0.9
\end{pmatrix}.
\]

In this case, \( G^e_1, G^e_2 \) and \( G^o_1 \) are weight-unbalanced with \( (\alpha^1_{2k}, \alpha^2_{2k}, \alpha^3_{2k}) = (0.5336, 0.1525, 0.3139), (\alpha^1_{2k+1}, \alpha^2_{2k+1}, \alpha^3_{2k+1}) = (0.5336, 0.3408, 0.1256) \) and \( (\beta^1_k, \beta^2_k) = (0.8889, 0.1111) \), \( \forall k \geq 0 \). We design the heterogeneous stepsize as follows:

\[
\alpha_{i,2k} = \frac{1}{\alpha^1}_1 \gamma^2_{2k}, \quad \alpha_{i,2k+1} = \frac{1}{\alpha^0}_0 \gamma^2_{2k+1}, i = 1, 2, 3; \quad \beta_{i,k} = \frac{1}{\beta^0}_0 \gamma_k, i = 1, 2;
\]
Figure 2: The Nash equilibrium is achieved (i.e., $x_i \rightarrow x^*$ and $y_i \rightarrow y^*$) for the time-varying weight-balanced digraphs with homogeneous stepsizes. 

Fig. 3 shows that the agents converge to the unique Nash equilibrium under the above designed heterogeneous stepsize.

Figure 3: The Nash equilibrium is achieved for weight-unbalanced digraphs with heterogeneous stepsizes.

**Example 7.3** In this example, we verify the result obtained in Theorem 6.2 (ii). Consider Example 7.2 where $S^1 = S^2 = 2$. Here we design adaptive stepsize algorithms: for $p = 1, 2, 3$
and \( \nu = 0, 1, \)
\[
\theta_p^\nu(r) = A_1(r) \cdots A_1(\nu + 1)\theta_p^{\nu}(\nu + 1), \quad r \geq \nu + 1,
\]
where \( \theta_p^\nu(r) = (\alpha_p^{(\nu)1}(r), \alpha_p^{(\nu)2}(r), \alpha_p^{(\nu)3}(r))^T, \theta_1^\nu(\nu + 1) = (1, 0, 0)^T, \theta_2^\nu(\nu + 1) = (0, 1, 0)^T \) and 
\[
\theta_3^\nu(\nu + 1) = (0, 0, 1)^T; \quad \theta_p^0(0) = \theta_p^1(0) = \theta_p^2(1) = (1, 1, 1)^T, \quad p = 1, 2, 3; \text{ for } p = 1, 2 \text{ and } \nu = 0, 1,
\]
\[
\vartheta_p^\nu(r) = A_2(r) \cdots A_2(\nu + 1)\vartheta_p^{\nu}(\nu + 1), \quad r \geq \nu + 1,
\]
where \( \vartheta_p^\nu(r) = (\beta_p^{(\nu)1}(r), \beta_p^{(\nu)2}(r))^T, \theta_1^\nu(\nu + 1) = (1, 0)^T \) and \( \theta_2^\nu(\nu + 1) = (0, 1)^T; \quad \vartheta_p^0(0) = \vartheta_p^1(0) = \vartheta_p^2(1) = (1, 1)^T, \quad p = 1, 2.
\]

Let \( \hat{\alpha}_i = \alpha_i^{(0)i}(2k), \hat{\alpha}_i = \alpha_i^{(1)i}(2k + 1), \hat{\beta}_i = \beta_i^{(0)i}(2k), \hat{\beta}_i = \beta_i^{(1)i}(2k + 1) \) and
\[
\alpha_i, 2k = \frac{1}{\hat{\alpha}_i}, \quad \alpha_i, 2k + 1 = \frac{1}{\hat{\alpha}_i}, \quad i = 1, 2, 3,
\]
\[
\beta_i, 2k = \frac{1}{\hat{\beta}_i}, \quad \beta_i, 2k + 1 = \frac{1}{\hat{\beta}_i}, \quad i = 1, 2,
\]

Fig. 4 shows that the agents converge to the unique Nash equilibrium under the above designed adaptive stepsize.

Figure 4: The Nash equilibrium is achieved for weight-unbalanced digraphs by adaptive stepsize.

8 Conclusions

A subgradient-based distributed algorithm was proposed to solve a distributed Nash equilibrium seeking problem, which can be formulated as a zero-sum game. Sufficient conditions were
provided to achieve a Nash equilibrium for switching weight-balanced digraphs under homogeneous stepsizes. In the case of weight-unbalanced graphs, it was shown that the algorithm with homogeneous stepsizes may fail the algorithm to reach the Nash equilibrium. Furthermore, it was shown that there always exist heterogeneous stepsizes such that a Nash equilibrium can be achieved by our algorithm. Also, we designed respective adaptive algorithms to update the stepsizes for the Nash equilibrium seeking for two special cases. Other interesting problems, including the extension of the current idea of designing heterogeneous stepsizes to eliminate the unbalance caused by the weight-unbalanced graphs to other distributed optimization algorithms, for example, the distributed dual averaging algorithm [47], the Newton-Raphson consensus algorithm [48], are still under investigation.

Appendix

In this appendix, we present the proofs of Lemmas 2.1, 4.2, 4.3, 4.4 and 4.5

A. Proof of Lemma 2.1

Take \( \mu = (\mu_1 \cdots \mu_n)^T \in \mathbb{S}^+_n \). Without loss of generality, we assume \( \mu_1 = \min_{1 \leq i \leq n} \mu_i \) (otherwise we can rearrange the index of agents). Let \( B \) be a stochastic matrix such that the graph \( G_B \) associated with \( B \) is a directed cycle: \( 1e_n n \cdots 2e_1 1 \) with \( e_r = (r+1, r), 1 \leq r \leq n-1 \) and \( e_n = (1, n) \). Clearly, \( G_B \) is strongly connected. Then all nonzero entries of \( B \) are \( \{b_{ii}, b_{i(i+1)}, 1 \leq i \leq n-1, b_{nn}, b_{n1} \} \) and \( \mu^T B = \mu^T \) can be rewritten as

\[
\begin{align*}
(1 - b_{22}) \mu_2 &= (1 - b_{11}) \mu_1 \\
(1 - b_{33}) \mu_3 &= (1 - b_{11}) \mu_1 \\
& \vdots \\
(1 - b_{nn}) \mu_n &= (1 - b_{11}) \mu_1
\end{align*}
\]

\( (48) \)

Let \( b_{11} = b_{11}^* \) with \( 0 < b_{11}^* < 1 \). Clearly, there is a solution to \( (48) \): \( b_{11} = b_{11}^*, b_{rr} = 1 - (1 - b_{11}^*) \mu_i / \mu_r, 2 \leq r \leq n \). We complete the proof.

B. Proof of Lemma 4.2

Consider \( n_1(n_1 - 2) + 1 \) time intervals \([0, T_1 - 1], [T_1, 2T_1 - 1], ..., [n_1(n_1 - 2)T_1, (n_1(n_1 - 2) + 1)T_1 - 1]\). By the definition of UJSC graph, \( G_1([pT_1, (p + 1)T_1 - 1]) \) contains a root node for \( 0 \leq p \leq n_1(n_1 - 2) \). Clearly, the set of the \( n_1(n_1 - 2) + 1 \) root nodes contains at least one node, say \( i_0 \), at least \( n_1 - 1 \) times. Without loss of generality, assume \( i_0 \) is a root node of \( G_1([pT_1, (p + 1)T_1 - 1]), p = p_0, ..., p_{n_1 - 2} \).
Take \( j_0 \neq i_0 \) from \( \{1, ..., n_1\} \). It is not hard to show that there is a node set \( \{j_1, ..., j_q\} \) and time set \( \{k_0, ..., k_{q-1}, k_q\}, q \leq n_1 - 2 \) such that \( (j_{r+1}, j_r) \in E_1(k_{q-r}), 0 \leq r \leq q - 1 \) and \( (i_0, j_q) \in E_1(k_0), \) where \( k_0 < \cdots < k_{q-1} < k_q \) and all \( k_r \)'s belong to different intervals \([p_rT_1, (p_r + 1)T_1 - 1], 0 \leq r \leq n_1 - 2\).

Noting that all diagonal elements of all adjacency matrices are positive and moreover, for matrices \( D_1, D_2 \in \mathbb{R}^{n_1 \times n_1} \) with nonnegative entries, if \( (D_1)_{r_1r_1} > 0 \) and \( (D_2)_{r_2r_2} > 0 \), then \( (D_1D_2)_{r_1r_2} > 0 \), and we have \( \Phi_1(T^1 - 1, 0)_{j_0i_0} > 0 \). Because \( j_0 \) is taken from \( \{1, ..., n_1\} \) freely, \( \Phi_1(T^1 - 1, 0)_{j_i0} > 0 \) for \( j = 1, ..., n_1 \). As a result, \( \Phi_1(T^1 - 1, 0)_{j_0i_0} \geq \eta_1^{r} \) for \( j = 1, ..., n_1 \) with \( A_2 \).

Clearly, \( \mu(\Phi_1(T^1 - 1, 0)) \leq 1 - \eta_1^{r} \) by the definition of ergodicity coefficient given in Lemma 4.1. According to (10), the relation \( h(\mu + \nu) \leq h(\mu) + 2\max_\nu, \) Lemma 4.1 and A3,

\[
\begin{align*}
 h_1(T^1) & \leq h_1(\Phi_1(T^1 - 1, 0)z(0)) + 2L \sum_{r=0}^{T^1-1} \tilde{\alpha}_r \\
 & \leq \mu(\Phi_1(T^1 - 1, 0))h_1(0) + 2L \sum_{r=0}^{T^1-1} \tilde{\alpha}_r \\
 & \leq (1 - \eta_1^{r})h_1(0) + 2L \sum_{r=0}^{T^1-1} \tilde{\alpha}_r,
\end{align*}
\]

which shows (11) for \( \ell = 1, p = 1, q = 0 \). Similarly, we can show (11) for \( \ell = 2, \) and any \( p \geq 1, 0 \leq q \leq T^1 - 1 \). \( \square \)

### C. Proof of Lemma 4.3

Here we only show \( \sum_{k=0}^{\infty} \tilde{\alpha}_k h_1(k) < \infty \) since \( \sum_{k=0}^{\infty} \tilde{\beta}_k h_2(k) < \infty \) can be shown similarly. Clearly, \( \sum_{k=0}^{\infty} \tilde{\alpha}(k) h_1(k) = \sum_{q=0}^{T^1-1} \sum_{p=0}^{\infty} \tilde{\alpha}(pT^1 + q)h_1(pT^1 + q). \)

Let us first estimate \( \sum_{p=0}^{\infty} \tilde{\alpha}(pT^1)h_1(pT^1). \) Denote \( \eta_0 = 1 - \eta_1 \) \((0 < \eta_0 < 1)\), \( c_{p-1} = \sum_{r=(p-1)T^1}^{pT^1-1} \tilde{\alpha}(r). \) By (11), \( h_1(pT^1) \leq \eta_0 h_1((p-1)T^1) + 2c_{p-1}L \) for all \( p \) and hence, \( h_1(pT^1) \leq \eta_0^p h_1(0) + 2L \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r. \) Take \( \alpha^* = \sup_{k \geq 0} \tilde{\alpha}(k), \) which is a finite number due to \( \sum_{k=0}^{\infty} \tilde{\alpha}^2(k) < \infty. \) Then \( \sum_{p=0}^{\infty} \tilde{\alpha}(pT^1) \eta_0^p h_1(0) \leq \alpha^* h_1(0)/1 - \eta_0. \) Thus,

\[
\begin{align*}
\sum_{p=0}^{\infty} \tilde{\alpha}(pT^1)h_1(pT^1) & \leq \sum_{p=1}^{\infty} \tilde{\alpha}(pT^1) \eta_0^p h_1(0) + 2L \sum_{p=1}^{\infty} \tilde{\alpha}(pT^1) \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r \\
& \leq \frac{\alpha^* h_1(0)}{1 - \eta_0} + 2L \sum_{p=1}^{\infty} \tilde{\alpha}(pT^1) \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r.
\end{align*}
\]

Using the well-known inequality \( 2ab \leq a^2 + b^2, \) we have

\[
\tilde{\alpha}(pT^1) \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r \leq \frac{1}{2} \left( \tilde{\alpha}^2(pT^1) \sum_{r=0}^{p-1} \eta_0^{p-1-r} + \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r^2 \right),
\]

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and then
\[
\sum_{p=1}^{\infty} \tilde{a}(pT^1) \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r \leq \frac{1}{2(1-\eta_0)} \sum_{p=1}^{\infty} \tilde{a}^2(pT^1) + \frac{1}{2} \sum_{p=1}^{\infty} \sum_{r=0}^{p-1} \eta_0^{p-1-r} c_r^2
\]
\[
= \frac{1}{2(1-\eta_0)} \sum_{p=1}^{\infty} \tilde{a}^2(pT^1) + \frac{1}{2\eta_0} \left( \sum_{p=1}^{\infty} \sum_{r=0}^{P} \eta_0^{p-r} c_r^2 - \sum_{p=1}^{\infty} c_r^2 \right)
\]
\[
< \infty \tag{50}
\]
based on \(\sum_{r=0}^{\infty} c_r^2 \leq T^1 \sum_{r=0}^{\infty} \tilde{a}^2(r) < \infty\) and Lemma 2.6. By (49) and (50), \(\sum_{p=0}^{\infty} \tilde{a}(pT^1)h_1(pT^1) < \infty\). Similarly, we can show \(\sum_{p=0}^{\infty} \tilde{a}(pT^1 + q)h_1(pT^1 + q) < \infty\) for \(1 \leq q \leq T^1 - 1\). Thus, \(\sum_{k=0}^{\infty} \tilde{a}(k)h_1(k) < \infty\). \(\square\)

**D. Proof of Lemma 4.4**

We only show \(\lim_{k \to \infty} h_1(k) = 0\) since \(\lim_{k \to \infty} h_2(k) = 0\) can be proved similarly. By Lemma 4.2, \(\limsup_{p \to \infty} h_1(pT^1) \leq \limsup_{p \to \infty} 2 \sum_{r=(p-1)T^1}^{pT^1-1} \alpha_r L/\eta^T\). With (10), we have that, for any \(pT^1 \leq k < (p+1)T^1\), \(h_1(k) \leq h_1(pT^1) + 2 \sum_{r=pT^1}^{(p+1)T^1-1} \alpha_r\). Thus, the conclusion follows from \(\lim_{k \to \infty} \tilde{a}_k = 0\). \(\square\)

**E. Proof of Lemma 4.3**

(i) It is obvious based on \(|\hat{x}_i - \bar{x}| \leq h_1\), \(|\tilde{y}_i - \bar{y}| \leq h_2\) and Lemma 4.3.

(ii) We only need to show the first conclusion since the second one can be obtained in the same way. At first, we have
\[
\sum_{k=0}^{\infty} \tilde{b}_k |\hat{x}_i(k) - \bar{y}(\tilde{k}_i)| = \sum_{k=0}^{\infty} \tilde{b}_k \left| \sum_{j \in N^2(\tilde{k}_i)} a_{ij}(\tilde{k}_i) y_j(\tilde{k}_i) - \bar{y}(\tilde{k}_i) \right|
\]
\[
\leq \sum_{k=0}^{\infty} \tilde{b}_k \sum_{j \in N^2(\tilde{k}_i)} \left| a_{ij}(\tilde{k}_i) y_j(\tilde{k}_i) - \bar{y}(\tilde{k}_i) \right|
\]
\[
\leq \sum_{k=0}^{\infty} \tilde{b}_k h_2(\tilde{k}_i), \tag{51}
\]
where the second inequality follows from A2 (iii). Let \(\{s_{ir}, r \geq 0\}\) be the set of all moments when \(N^2(s_{ir}) \neq \emptyset\). Recalling the definition of \(\tilde{k}_i\) in (5), \(\tilde{k}_i = s_{ir}\) when \(s_{ir} \leq k < s_{i(r+1)}\). Since \(\{\tilde{b}_k\}\) is non-increasing and \(\sum_{k=0}^{\infty} \tilde{b}_k h_2(\tilde{k}_i) < \infty\) (by Lemma 4.3), we have \(\sum_{k=0}^{\infty} \tilde{b}_k h_2(\tilde{k}_i) \leq \sum_{k=0}^{\infty} \tilde{b}_k h_2(\tilde{k}_i) = \sum_{r=0}^{\infty} \tilde{b}_r |s_{i(r+1)} - s_{ir}| h_2(s_{ir}) \leq T_{\infty} \sum_{r=0}^{\infty} \tilde{b}_r h_2(s_{ir}) \leq T_{\infty} \sum_{k=0}^{\infty} \tilde{b}_k h_2(k) < \infty\), where \(T_{\infty}\) is the constant given in A1. Thus, (51) implies \(\sum_{k=0}^{\infty} \tilde{b}_k |\hat{x}_i(k) - \bar{y}(\tilde{k}_i)| < \infty\).
Since \( y_i(k) \in Y \) for all \( i \) and \( Y \) is convex, \( \bar{y}(k) \in Y \). Then, from Lemma 2.2 (i),

\[
|\bar{y}(k + 1) - \bar{y}(k)| = \left| \sum_{i=1}^{n_2} \left[ P_Y \left( \sum_{j \in N_i^2(k)} a_{ij}(k)y_j(k) + \beta_{i,k}q_{2i}(k) \right) - P_Y(\bar{y}(k)) \right] \right|
\]

\[
\leq \frac{1}{n_2} \sum_{i=1}^{n_2} \left| \sum_{j \in N_i^2(k)} a_{ij}(k)y_j(k) + \beta_{i,k}q_{2i}(k) - \bar{y}(k) \right|
\]

\[
\leq h_2(k) + \bar{\beta}_kL. \tag{52}
\]

Based on (52) and \( \tilde{k}_i \geq k - T_{\infty} + 1 \), we also have

\[
\sum_{k=0}^{\infty} \tilde{\beta}_k |\tilde{y}(\tilde{k}_i) - \bar{y}(k)| \leq \sum_{k=0}^{\infty} \tilde{\beta}_k \sum_{r=\tilde{k}_i}^{k-1} |\tilde{y}(r) - \bar{y}(r + 1)|
\]

\[
\leq \sum_{k=0}^{\infty} \tilde{\beta}_k \sum_{r=\tilde{k}_i}^{k-1} (h_2(r) + \bar{\beta}_r L)
\]

\[
\leq \sum_{k=0}^{\infty} \tilde{\beta}_k \sum_{r=\tilde{k}_i - T_{\infty} + 1}^{k-1} (h_2(r) + \bar{\beta}_r L)
\]

\[
\leq \sum_{k=0}^{\infty} \tilde{\beta}_k \sum_{r=\tilde{k}_i - T_{\infty} + 1}^{\infty} h_2(r) + \frac{1}{2}(T_{\infty} - 1)L \sum_{k=0}^{\infty} \tilde{\beta}_k^2 + \frac{1}{2}L \sum_{k=0}^{\infty} \sum_{r=\tilde{k}_i - T_{\infty} + 1}^{\infty} \bar{\beta}_r^2
\]

\[
\leq (T_{\infty} - 1) \sum_{k=0}^{\infty} \tilde{\beta}_k h_2(k) + \frac{1}{2}(T_{\infty} - 1)L \sum_{k=0}^{\infty} \tilde{\beta}_k^2 + \frac{1}{2}(T_{\infty} - 1)L \sum_{k=0}^{\infty} \bar{\beta}_k^2
\]

\[
< \infty,
\]

where \( \bar{\beta}_r = 0 \) for \( r < 0 \). Since \( |\tilde{x}_i(k) - \bar{y}(k)| \leq |\tilde{x}_i(k) - \bar{y}(\tilde{k}_i)| + |\bar{y}(\tilde{k}_i) - \bar{y}(k)| \), the conclusion follows. □

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