Any strongly controllable group system or group shift or any linear block code is isomorphic to a generator group

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ABSTRACT

Consider any sequence of finite groups $A^t$, where $t$ takes values in an integer index set $\mathbb{Z}$. A group system $A$ is a set of sequences with components in $A^t$ that forms a group under componentwise addition in $A^t$, for each $t \in \mathbb{Z}$. As shown previously, any strongly controllable complete group system $A$ can be decomposed into generators. We study permutations of the generators when sequences in the group system are multiplied. We show that any strongly controllable complete group system $A$ is isomorphic to a generator group $(U, \circ)$. The set $U$ is a set of tensors, a double Cartesian product space of sets $G^t_k$, with indices $k$, for $0 \leq k \leq \ell$, and time $t$, for $t \in \mathbb{Z}$. $G^t_k$ is a set of unique generator labels for the generators in $A$ with nontrivial span for the time interval $[t, t+k]$. We show the generator group contains a unique elementary system, an infinite collection of elementary groups, one for each $k$ and $t$, defined on small subsets of $U$, in the shape of triangles, which form a tile like structure over $U$. There is a homomorphism from each elementary group to any elementary group defined on smaller tiles of the former group. The group system $A$ may be constructed from either the generator group or elementary system. These results have application to linear block codes, any algebraic system that contains a linear block code, group shifts, and harmonic theory in mathematics, and systems theory, coding theory, control theory, and related fields in engineering.
1. INTRODUCTION

Linear systems theory is a well-established field in engineering. Willems reenvisioned linear system theory by starting with a set of sequences which forms a group, called here a group system, and derived many properties of the system from this set [2]. Following the work of Willems, Forney and Trott [3] described the state group and state code of a group system, which they called a group code. They showed that any group code that can be characterized by its local behavior (complete [3]) can be wholly specified by a sequence of connected labeled group trellis sections, a trellis, which may vary in time. They explained the important idea of “shortest length code sequences” or generators. In a controllable group code, a generator is a code sequence which is nontrivial over a finite time interval \( [t, t+k] \), for some time \( t \in \mathbb{Z} \) and some finite integer \( k \), and which is not a combination of shorter code sequences. In a strongly controllable group code, all the generators have a finite length which is uniformly bounded by a finite integer \( \ell \). Any sequence in the code can be found by a composition of generators. At any time \( t \), any letter in a code sequence is obtained by a combination of generators with nontrivial letters at time \( t \).

Loeliger and Mittelholzer [4] obtain a complementary approach to the work of Forney and Trott by starting with a trellis having group properties instead of a set of sequences.

A time invariant group system is essentially a group shift [18]. Therefore the results here have application to group shifts. The study of group shifts originated with the work of Kitchens [1] in symbolic dynamics theory, and preceded the work of Willems. Kitchens [1] showed that any group shift has finite memory, i.e., it is a shift of finite type [18].

We consider the general time varying group system. A general time varying group system can have nontrivial values over a finite time interval, in which case it is often called a block code over a group [3] or linear block code [15]. Therefore the results here also apply to block codes.

In the setting of group systems, a natural definition of a linear system is a homomorphism from a group of inputs to the group system, an output. Both Willems and Forney and Trott have taken the approach to systems theory that all properties of the system can be derived starting from the sequences in the system themselves. Willems has shown for linear systems that this approach gives a notion of an input sequence. Similarly Forney and Trott have shown that there is a notion of input for a group system, the sequence of first components of the generators. However as discussed in [3], they do not show there is a homomorphic map from the input sequence space to the output sequence space. Therefore, they do not show a group system is a linear system. Likewise, as also discussed in [3], the work of Brockett and Willsky [14] on “finite group homomorphic sequential systems” does not suffice to realize general linear systems over groups. One result of this paper is to show that for a strongly controllable complete group system, the generators themselves form a group, the generator group, and there is an isomorphism from the generator group, an input group, to the group system, the output group. Therefore a group system
is a linear system. This fills a gap in the theory of Willems and Forney and Trott. Moreover, since there is an isomorphism from the generator group to the group system, the generator group can be used to study the group system. The generator group is very revealing of the structure of group systems and seems to give new insights into related studies such as linear systems, group codes, group shifts, linear block codes, and any algebraic structure which contains a linear block code.

Consider any sequence of finite groups \( A^t \), where \( t \) takes values in an integer index set \( Z \). A group system \( A \) is a set of sequences with components in \( A^t \), for each \( t \in Z \), that forms a group under componentwise addition in \( A^t \) [3]. We only consider strongly controllable group systems \( A \), in which there is a fixed least integer \( \ell \) such that for any time \( t \), for any pair of sequences \( a, \bar{a} \in A \), there is a sequence \( \bar{a} \in A \) that agrees with \( a \) on \( (-\infty, t] \) and agrees with \( \bar{a} \) on \( [t + \ell, \infty) \) [3]. We only study group systems \( A \) that are complete [3]. These are group systems which can be completely characterized by local behavior, i.e., no global constraints are needed.

As shown in [3], any strongly controllable complete group system \( A \) can be decomposed into generators. A generator \( g^{[t, t] + k} \) is a sequence in \( A \) which is the identity except for a nontrivial span on the time interval \( [t, t + k] \) [3]. A generator is a coset representative in a coset decomposition chain of \( A \). We can form any sequence \( a \in A \) by selecting one coset representative from each coset in the coset decomposition chain, for each time \( t, t \in Z \), for \( 0 \leq k \leq \ell \).

The list of generators for sequence \( a \) forms a tensor \( r \). The set of tensors \( r \) formed by \( a \in A \) is \( R \). There is a 1-1 correspondence between sequences \( a \in A \) and tensors \( r \in R \). We study permutations of the generators in \( r \) when sequences \( a \) in the group system \( A \) are multiplied. If \( \bar{a}, \bar{a} \) are two sequences in \( A \), then their product \( \bar{a} \bar{a} \) is another sequence in \( A \). Let \( \bar{a} \to \bar{a} \) and \( \bar{r} \to \bar{a} \) under the bijection \( R \to A \). We define an operation \( * \) on \( R \) by \( \bar{r} \to \bar{a} \) if \( \bar{r} \to \bar{a} \) under the bijection \( R \to A \). The set of tensors \( R \) with operation \( * \) forms a group \(( R, *) \) called a decomposition group of \( A \), and \( A \simeq (R, *) \).

An element \( r \in R \) is equivalent to a sequence of generators \( g^{[t, t] + k} \) for \( 0 \leq k \leq \ell \), for each \( t \in Z \). We replace each generator \( g^{[t, t] + k} \) in \( r \in R \) with a generator label \( g^{[t, t] + k} \). Under the assignment \( g^{[t, t] + k} \to g^{[t, t] + k} \), for \( 0 \leq k \leq \ell \), for each \( t \in Z \), a tensor \( r \in R \) becomes a tensor \( u, r \to u \). Let \( U \) be the set of tensors \( u \) obtained from \( R \) this way. The set \( U \) is a double Cartesian product space of sets \( G^k \), with indices \( k \), for \( 0 \leq k \leq \ell \), and time \( t \), for \( t \in Z \), where \( G^k \) is a set of unique generator labels for the generators in \( A \) with nontrivial span for the time interval \( [t, t + k] \). The operation \( \circ \) in \(( R, *) \) determines an operation \( \circ \) on \( U \), and this gives a group \(( U, \circ) \) isomorphic to \(( R, *) \), called a group generator; we have \( A \simeq (R, *) \simeq (U, \circ) \). Therefore any strongly controllable complete group system \( A \) is isomorphic to a generator group \(( U, \circ) \).

We regard the coset decomposition chain of \( A \) used in [3] as a spectral domain decomposition of \( A \). We use a different coset decomposition chain of \( A \) which we think of as a time domain decomposition. We show the time and spectral domain decompositions of \( A \) can be more easily obtained as the coset representative chains of two different coset decomposition chains of the generator group of \( A \). We show the generator group has many other coset decomposition chains which give other decompositions. This gives a harmonic theory of group systems.

We show the multiplication of two sequences in \( A \) can be broken into local groups in \(( U, \circ) \). We show there is an infinite collection of local elementary groups \(( V^k \cup_t (U), @_{0, k}^t ) \) of \(( U, \circ) \), one for each \( k \) such that \( 0 \leq k \leq \ell \), and each \( t \in Z \), defined on small subsets of \( U \), in the shape of triangles, which form a nested tile like structure over \( U \). There is a homomorphism from each elementary group to any elementary group nested in the former group. We define the infinite collection of elementary groups, together with a homomorphism condition, to be an \(( \ell + 1) \)-depth elementary system \( E_A \). The homomorphism condition is
that for each \( k \) such that \( 0 \leq k < \ell \), for each \( t \in \mathbb{Z} \), there is a homomorphism from elementary group \((\bigtriangledown_{0,k}^t(U), \circ_{0,k}^t)\) to the next two largest elementary groups nested in \((\bigtriangledown_{0,k}^t(U), \circ_{0,k}^t)\). Then we have shown that the generator group of any \( \ell \)-controllable complete group system \( A \) contains an elementary system \( \mathcal{E}_A \).

Given an \((\ell + 1)\)-depth elementary system \( \mathcal{E}_A \) of a group system \( A \), we can always recover the \((\ell + 1)\)-depth generator group \((U, \circ)\) of \( A \), up to an isomorphic and essentially identical group, using a generalization of the first homomorphism theorem to group systems. Therefore any \( \ell \)-controllable complete group system \( A \) can be constructed from either the generator group \((U, \circ)\) of \( A \) or the \((\ell + 1)\)-depth elementary system \( \mathcal{E}_A \) of \( A \). Moreover, starting from any constructed \((\ell + 1)\)-depth elementary system \( \mathcal{E} \), the first homomorphism theorem for group systems can always construct an \( \ell \)-controllable complete group system. Therefore the study and construction of \( \ell \)-controllable complete group systems is also essentially the study and construction of elementary systems. Since the \((\ell + 1)\)-depth elementary system has finite depth and is nested by depth, the construction of any \((\ell + 1)\)-depth elementary system is very simple.

In this paper, we study group systems as a permutation of the generators, without any explicit use of the concept of state as in conventional systems theory and group codes [3] or of the future cover as in symbolic dynamics and group shifts [18]. Nevertheless, we can replicate many of the results in these fields. For example, the local elementary group \((\bigtriangledown_{0,0}^t(U), \circ_{0,0}^t)\) is an analog of the branch group in [3] and future cover in [18].

In Section 2, we review some of the definitions in group systems and previous work. In Section 3, we find a normal chain of \( A \), the generators of \( A \) for this normal chain, the set \( R \), and a formula to calculate a component \( a^t \) of \( A \) using a matrix in \( R \). In Section 4, we find the decomposition group \((R, *)\), the generator group \((U, \circ)\), the largest elementary group of \((U, \circ)\), and the component group of \((R, *)\). We also show how to recover \( A \) from the generator group using a first homomorphism theorem for group systems. In Section 5, we find the nested elementary groups of \((U, \circ)\) and their homomorphism relation. We show that finite or infinite sets of elementary groups of \((U, \circ)\) form a group. Then we find normal chains of the generator group. In Section 6, we discuss the elementary system, the global group of the elementary system, how to find all \( \ell \)-controllable complete group systems \( A \) from the elementary system, and how to construct elementary systems.
2. GROUP SYSTEMS

This section gives a very brief review of some fundamental concepts in [3], and introduces some definitions used here. We follow the notation of Forney and Trott [3] as closely as possible. One significant difference is that subscript \( k \) in [3] denotes time; we use \( t \) (an integer) in place of \( k \). Further, time is always indicated with a superscript. We always associate notation \( 1 \) with the identity of any group or group of sequences; for example \( g_1 \) will be the identity of group \( G \).

Forney and Trott study a collection of sequences with time axis defined on the set of integers \( \mathbb{Z} \), whose components \( a^t \) are taken from an alphabet group or alphabet \( A^t \) at each time \( t, t \in \mathbb{Z} \). The set of sequences is a group under componentwise addition in \( A^t \) [3]. We call this a \textit{group system} \( A \). A sequence \( a \) in \( A \) is given by

\[
a = \ldots, a^{t-1}, a^t, a^{t+1}, \ldots,
\]

where \( a^t \in A^t \) is the component at time \( t \). We assume that all elements of \( A^t \) are represented in \( A \) at time \( t \). The identity of \( A^t \) is \( a^1_1 \), and the identity of \( A \) is \( a_1 \).

We apply the standard definition of group isomorphism for finite groups to group systems. Let \( G \) be a group or group system. Then \( A \simeq G \) if and only if there is a bijection \( A \to G \) such that if \( \hat{a}, \hat{a} \in A \) and \( \hat{a} \to \hat{g}, \hat{a} \to \hat{g} \), then \( \hat{a} \hat{a} \to \hat{g} \hat{g} \). There may be other definitions of isomorphism for group systems that are more suitable [22 v11-v12].

In this paper, we study \textit{complete} group systems [2 3]. A complete group system can be characterized by its local behavior; in particular complete systems can be generated by their trellis diagrams [3]. Forney and Trott construct their canonical encoder for a complete strongly controllable group system. Completeness is called closure in symbolic dynamics [3]. Therefore a time invariant complete group system \( A \) is the same thing as a group shift in symbolic dynamics [3]. For an incomplete group system, a global constraint is required to fully specify the group system. Some examples of group systems that require a global constraint are given in [3].

As in [3], we use conventional notation for time intervals. If \( m \leq n \), the time interval \([m, n]\) starts at time \( m \), ends at time \( n \), and has length \( n - m + 1 \). We also write time interval \([m, n]\) as \([m, n + 1]\). The time interval \([m, m]\) or \([m, m + 1]\) has length 1 and is also written just \( m \).

Let \( A \) be a group system, and let \( a \) be a sequence in \( A \). Using (1), define the projection map at time \( t \), \( \chi^t : A \to A^t \), by the assignment \( a \to a^t \). Define the projection map \( \chi^{[t_1, t_2]} : A \to A^{t_1} \times \cdots \times A^{t_2} \) by the assignment \( a \to (a^{t_1}, \ldots, a^{t_2}) \). In general, we say that sequence \( a \) has \textit{span} \( t_2 - t_1 + 1 \) if \( a \) is the same as the identity sequence except for a finite segment \( (a^{t_1}, \ldots, a^{t_2}) \) of length \( t_2 - t_1 + 1 \), where \( a^{t_1} \neq a_1^{t_1} \) and \( a^{t_2} \neq a_1^{t_2} \). We define \( A^{[t_1, t_2]} \) to be the sequences in \( A \) which are the identity outside time interval \([t_1, t_2]\).

A group system \( A \) is \((m, n)\)-controllable if for any \( \hat{a}, \hat{a} \in A \), there exists a sequence \( a \in A \) with \( \chi^{(-\infty, m)}(a) = \chi^{(-\infty, m)}(\hat{a}) \) and \( \chi^{[n, +\infty)}(a) = \chi^{[n, +\infty)}(\hat{a}) \). Then the finite segment \( \chi^{[m, n)}(a) \) of length \( n - m \) in \( a \) connects the past \( \chi^{(-\infty, m)}(\hat{a}) \) of \( \hat{a} \) to the future \( \chi^{[n, +\infty)}(\hat{a}) \) of \( \hat{a} \) [3]. A group system \( A \) is \textit{l-controllable} if there is an integer \( l > 0 \) such that \( A \) is \([t, t + l)\)-controllable for all \( t \in \mathbb{Z} \). A group system \( A \) is strongly \textit{controllable} if it is \( l \)-controllable for some integer \( l \). The least integer \( l \) for which a group system \( A \) is strongly controllable is denoted as \( \ell \). In this paper we study strongly controllable group systems.

For each \( t \in \mathbb{Z} \), define \( X^t \) to be the set of all sequences \( a \) in \( A \) for which \( a^n = a^n_1 \) for \( n < t \), where \( a^n_1 \) is the identity of \( A^n \) at time \( n \) (see Figure 4). For each \( t \in \mathbb{Z} \), define \( Y^t \) to be the set of all sequences \( a \) in \( A \) for which \( a^n = a^n_1 \) for \( n > t \). The \textit{canonic state space} \( \Sigma^t \) of \( A \) at time \( t \) is defined to be

\[
\Sigma^t \overset{\text{def}}{=} \frac{A}{Y_{t-1}X^t}.
\]
(Note that \( A \) is the same as \( C \) in [3], \( Y^{t-1} \) is the same as \( C^{-} \), and \( X^{t} \) is the same as \( C^{+} \).) From Figure 1, it is evident the definition of the state at time \( t \) involves a split between time \( t - 1 \) and time \( t \) [3]. The canonic state space is isomorphic to the direct product of quotient groups \( \Gamma \). The canonic state space is isomorphic to the identity sequence. The group system satisfies the **axiom of state**: whenever two sequences pass through the same state at a given time, the concatenation of the past of either with the future of the other is a valid sequence [3]. In this paper, we just use the identity state or zero state of \( \Sigma^{t} \) at each time \( t \in \mathbb{Z} \).

![Figure 1: Definition of \( Y^{t-1} \) and \( X^{t} \).](image)

We now review some results from [3] on the construction of \( A \) from its fundamental components, the generators. Assume a group system \( A \) is \( \ell \)-controllable and complete. Forney and Trott [3] define the \( k \)-controllable subcode \( A_{k} \) of an \( \ell \)-controllable group code \( A \), for \( 0 \leq k \leq \ell \). The \( k \)-controllable subcode \( A_{k} \) of a group code \( A \) is defined as the set of combinations of sequences of span \( k + 1 \) or less:

\[
A_{k} = \prod_{t} A^{[t,t+k]}.
\]

They show

\[
A_{0} \subset A_{1} \subset \ldots A_{k-1} \subset A_{k} \subset \ldots A_{\ell} = A
\] (2)

is a normal series. A chain coset decomposition yields a one-to-one correspondence

\[
A \leftrightarrow A_{0} \times (A_{1}/A_{0}) \times \cdots \times (A_{k}/A_{k-1}) \times \cdots \times (A_{\ell}/A_{\ell-1}).
\] (3)

For \( 1 \leq k \leq \ell \), the quotient groups \( (A_{k}/A_{k-1}) \) may be further evaluated as follows. In their Code Granule Theorem [3], they show \( A_{k}/A_{k-1} \) is isomorphic to a direct product of quotient groups \( \Gamma^{[t,t+k]} \),

\[
A_{k}/A_{k-1} \simeq \prod_{t} \Gamma^{[t,t+k]},
\] (4)

where \( \Gamma^{[t,t+k]} \) is defined by

\[
\Gamma^{[t,t+k]} \overset{\text{def}}{=} \frac{A^{[t,t+k]}}{A^{[t,t+k]} A^{(t,t+k)}}.
\]

\( \Gamma^{[t,t+k]} \) is called a **granule**. A coset representative of \( \Gamma^{[t,t+k]} \) is called a Forney-Trott **generator** \( g^{[t,t+k]} \) of \( A \). The coset representative of \( A^{[t,t+k]} A^{[t,t+k]} \) is always taken to be the identity sequence. In case \( \Gamma^{[t,t+k]} \) is isomorphic to the identity group, the identity sequence is the only coset representative. A non-identity generator is an element of \( A^{[t,t+k]} \) but not of \( A^{[t,t+k]} \) or of \( A^{[t,t+k]} \), so its span is exactly \( k + 1 \). Thus every nonidentity generator is a codeword that cannot be expressed as a combination of shorter codewords [3].

If \( Q \) is any quotient group, let \([Q]\) denote a set of coset representatives of \( Q \), or a transversal of \( Q \). Let \( \{ \Gamma^{[t,t+k]} \} \) be a transversal of \( \Gamma^{[t,t+k]} \). It follows from (4) that the set \( \prod_{t} \{ \Gamma^{[t,t+k]} \} \) is a set of coset representatives for the cosets of \( A_{k-1} \) in \( A_{k} \). We know that the set of coset representatives of the granule \( \Gamma^{[t,t+k]} \), or of transversal \( \{ \Gamma^{[t,t+k]} \} \), is a set of generators \( \{ g^{[t,t+k]} \} \). Then from (3), any sequence \( a \) can be uniquely evaluated as a product

\[
a = A_{0} \prod_{k=1}^{\ell} \prod_{t=-\infty}^{+\infty} g^{[t,t+k]}.
\]
Any element of $A^{[t,t]}$ is a coset representative $g_{FT}^{[t,t]}$. Then $A_0 = \prod_t \{ g_{FT}^{[t,t]} \}$. It follows that (Generator Theorem 3) every sequence $a$ can be uniquely expressed as a product

$$a = \prod_{k=0}^\ell \prod_{t=-\infty}^{+\infty} g_{FT}^{[t,t+k]}$$

(5)

de of generators $g_{FT}^{[t,t+k]}$. Thus every sequence $a$ is a product of some sequence of generators, and conversely, every sequence of generators corresponds to some sequence $a$. Then a component $a^t$ of $a$ is given by

$$a^t = \prod_{k=0}^\ell \left( \prod_{j=0}^k \chi_{FT}^{[t-j,t-j+k]} \right).$$

(6)

A basis of $A$ is a smallest set of shortest length generators that is sufficient to generate the group system $A \{9\}$. It follows from the encoder in \{3\} that a basis of $A$ is a set of coset representatives of $[\Gamma^{[t,t+k]}]$, for $0 \leq k \leq \ell$, for each $t \in \mathbb{Z}$. The encoder in \{5\} forms output $a$ from a sequence of generators selected from the basis. For each time $t \in \mathbb{Z}$, for each $k$ such that $0 \leq k \leq \ell$, a single generator $g_{FT}^{[t,t+k]}$ is selected from set $[\Gamma^{[t,t+k]}]$. In general, the set of sequences that can be selected from the basis to form $A$ is a proper subset of the double Cartesian product \{3\}

$$\bigotimes_{t=+\infty}^{t=-\infty} \bigotimes_{0 \leq k \leq \ell} [\Gamma^{[t,t+k]}].$$

(7)

But if $A$ is complete, then the set of sequences that forms $A$ is exactly the double Cartesian product \{7\}, \{9\}.
3. THE NORMAL CHAIN OF $A$

Recall that for each $t \in \mathbb{Z}$, we have defined $X^t$ to be the set of all sequences $a \in A$ for which $a^n = a_1^n$ for $n < t$, where $a_1^n$ is the identity of $A^n$ at time $n$. And for each $t \in \mathbb{Z}$, we have defined $Y^t$ to be the set of all sequences $a \in A$ for which $a^n = a_1^n$ for $n > t$.

It is clear that $X^t < A$ and $Y^t < A$ for each $t \in \mathbb{Z}$. Then the group $A$ has two normal series (and chief series)

$$a_1 \cdots \subset X^{t+1} \subset X^t \subset \cdots \subset X^{t-j+1} \subset X^{t-j} \subset \cdots \subset X^{t-\ell+1} \subset X^{t-\ell} \subset \cdots A,$$

(8)

and

$$a_1 \cdots \subset Y^{t-1} \subset Y^{t-1} \subset \cdots \subset Y^{t+k-1} \subset Y^{t+k} \subset \cdots \subset Y^{t+\ell-1} \subset Y^{t+\ell} \subset \cdots A.$$

(9)

In (8) we have chosen index $j$ such that $0 \leq j \leq \ell$ and in (9) we have chosen index $k$ such that $0 \leq k \leq \ell$. The Schreier refinement theorem used to prove the Jordan-Hölder theorem [16] shows how to obtain a refinement of two normal series by inserting one into the other. We can obtain a refinement of (8) by inserting (9) between each two successive terms of (8). This gives the normal series by inserting one into the other. We can obtain a refinement of (8) by inserting (9) between each two successive terms of (8). This gives the normal series shown in (10). The normal series is given by the $\ell+3$ rows in the middle of (10) and the rows denoted by the two vertical ellipses near the top and bottom. The bottom row and top row in (10) are limiting groups, explained further below. The normal series is an infinite series of groups, with each column an infinite series of groups. We have only shown $\ell+1$ columns of the infinite series in (10). Since (8) and (9) are chief series, the normal chain (10) is a chief series.

We now show that (10) is indeed a refinement of (8). First we show that in each column the group in the bottom row is contained in all groups in the infinite column of groups, and the group in the top row contains all groups in the infinite column of groups. Any term in (10) is of the form

$$X^{i+1}(X^i \cap Y^{i+m})$$

for some integer pair $i, m \in \mathbb{Z}$. For example, term $X^{t-j+1}(X^{t-j} \cap Y^{t+k-j})$ is of the form $X^{i+1}(X^i \cap Y^{i+m})$ for $i = t - j$ and $m = k$. Fix $i \in \mathbb{Z}$. For any $m \in \mathbb{Z}$, we have $a_{i+1} \subset X^{i+1}(X^i \cap Y^{i+m})$. Therefore the group in the bottom row is contained in all groups in the infinite column of groups. For any $m \in \mathbb{Z}$, it is clear that $X^{i+1}(X^i \cap Y^{i+m}) \subset X^{i+1}(X^i)$. Therefore the group in the top row contains all groups in the infinite column of groups. Now note that $X^{i+1}(X^i) = X^i$. This means that a group in the top row is the same as the group in the bottom row in the next column. But now note the groups in the bottom row form the same sequence as (8). Therefore (10) is indeed a refinement of (8).

The factor $X^{t-j+1}$ in term $X^{t-j+1}(X^{t-j} \cap Y^{t+k-j})$ in (10) is called an integration factor, and the factor $(X^{t-j} \cap Y^{t+k-j})$ in term $X^{t-j+1}(X^{t-j} \cap Y^{t+k-j})$ is called a derivative factor. The derivative factors in each row of a finite band of rows in (10), absent the normal chain given here, are essentially what is used in [3] to find a coset decomposition chain for the spectral domain (see also [4]). The product of the derivative factors in each row of the finite band corresponds to a $k$-controllable subcode of [3].

We now show the normal chain (10) contains all the sequences $a \in A$. Since $A$ is complete, any $a \in A$ can be specified by its projection over finite time intervals [3]. Then we just have to show that the normal chain contains the projection of $a$ over any finite time interval $[t, t+n]$. $\chi^{[t, t+n]}(a)$. But since $A$ is $\ell$-controllable, the group $(X^i \cap Y^{i+m})$ contains $\chi^{[t, t+n]}(a)$ for $i \leq t - \ell$ and $i + m \geq t + n + \ell$. Therefore the group $X^{i+1}(X^i \cap Y^{i+m})$ in (10) contains $\chi^{[t, t+n]}(a)$ for $i \leq t - \ell$ and $i + m \geq t + n + \ell$. This gives the following.

**Theorem 1** Any sequence $a \in A$ is completely specified by the normal chain (10).
\[
\begin{array}{cccccc}
X^{t+1}(X^t) & X^t(X^{t-1}) & \ldots & X^{t-j+1}(X^{t-j}) & \ldots & X^{t-\ell+1}(X^{t-\ell}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+\ell+1}) & X^t(X^{t-1} \cap Y^{t+\ell}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t+\ell-j+1}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t+\ell+1}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+\ell}) & X^t(X^{t-1} \cap Y^{t+\ell-1}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t+\ell-j}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+\ell-1}) & X^t(X^{t-1} \cap Y^{t+\ell-2}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t+\ell-j-1}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t-1}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+k}) & X^t(X^{t-1} \cap Y^{t+k-1}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t+k-j}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t+k-\ell}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+k-1}) & X^t(X^{t-1} \cap Y^{t+k-2}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t+k-j-1}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t+k-1}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+j}) & X^t(X^{t-1} \cap Y^{t+j-1}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t-j}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t+j-\ell}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+j-1}) & X^t(X^{t-1} \cap Y^{t+j-2}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t-1}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t+j-1}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t+1}) & X^t(X^{t-1} \cap Y^{t}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t-j+1}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t+1}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1}(X^t \cap Y^{t}) & X^t(X^{t-1} \cap Y^{t-1}) & \ldots & X^{t-j+1}(X^{t-j} \cap Y^{t-1}) & \ldots & X^{t-\ell+1}(X^{t-\ell} \cap Y^{t}) \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
\cup & \cup & \ldots & \cup & \ldots & \cup \\
X^{t+1} & X^t & \ldots & X^{t-j+1} & \ldots & \cup \\
\hline
\end{array}
\]

(10)
Since any sequence \( a \in A \) is specified by the normal chain \((10)\), we can use the coset decomposition chain of \((10)\) to find any sequence \( a \) in \( A \). In any normal chain, we may form the quotient group of two successive groups in the chain. Then a normal chain of groups, as in \((10)\), gives a series of quotient groups. A general quotient group obtained from \((10)\) is of the form

\[
\Lambda^{[i,i+m]} \overset{\text{def}}{=} \frac{X^{i+1}(X^i \cap Y^{i+m})}{X^{i+1}(X^i \cap Y^{i+m-1})}
\]  

(11)

for any integer pair \( i, m \in \mathbb{Z} \). We call \( \Lambda^{[i,i+m]} \) the time domain granule. Note that the time domain granule has half infinite extent while the granule \( \Gamma^{[i,i+k]} \) of \( 3 \) has finite extent. We wish to find a transversal of the time domain granule. The coset representatives of the time domain granule are called generators. Since \((X^{i+1} \cap Y^{i+m}) \subset X^{i+1} \), we have

\[
X^{i+1}(X^i \cap Y^{i+m-1}) = X^{i+1}(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})
\]

and we may rewrite \((11)\) as

\[
\Lambda^{[i,i+m]} = \frac{X^{i+1}(X^i \cap Y^{i+m})}{X^{i+1}(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})}.
\]

(12)

We now find a transversal of \((12)\). An element of the numerator group is of the form \( xy \), where \( x \in X^{i+1} \) and \( y \in (X^i \cap Y^{i+m}) \). Then a coset of normal subgroup \( X^{i+1}(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) is

\[
xyX^{i+1}(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) = X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})
\]

\[
= X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}).
\]

Theorem 2 A coset of normal subgroup \( X^{i+1}(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) is \( X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) where \( y \in (X^i \cap Y^{i+m}) \). A coset representative of coset \( X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) is \( xy \) where \( x \) is in \( X^{i+1} \) and \( y \) is in \( y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \). A transversal of the quotient group \((12)\) is a selection of one coset representative \( xy \) from each coset \( X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \).

We may always select a coset representative \( xy \) of coset \( X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) to be \( a^1y \) where \( a^1 \) is the identity of \( X^{i+1} \). But note that \( y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) is a coset of normal subgroup \( (X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}) \) in quotient group

\[
\frac{(X^i \cap Y^{i+m})}{(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})}.
\]

(13)

Then \( y \) is a coset representative of \((13)\). This gives the following.

Corollary 3 A transversal of the quotient group \((13)\) is a transversal of the quotient group \((12)\), but the reverse is only true if \( xy = a^1y \) or \( x = a^1 \) for each coset representative \( xy \) of \((12)\).

We have just shown that a transversal of \((13)\) is a transversal of \((12)\). It is not surprising then that there is a homomorphism from \((12)\) to \((13)\). In fact this result is an application of the Zassenhaus lemma used in the proof of the Schreier refinement theorem \((16)\). We first restate the following lemma, excised from the proof of the Zassenhaus lemma (see p. 100 of \((16)\)).

Lemma 4 (from proof of Zassenhaus lemma) Let \( U \triangleleft U^* \) and \( V \triangleleft V^* \) be four subgroups of a group \( G \). Then \( D = (U^* \cap V)(U \cap V^*) \) is a normal subgroup of \( U^* \cap V^* \). If \( g \in U(U^* \cap V^*) \), then \( g = uu^* \) for \( u \in U \) and \( u^* \in U^* \cap V^* \). Define function \( f : U(U^* \cap V^*) \to (U^* \cap V^*)/D \) by \( f(g) = f(uu^*) = Du^* \). Then \( f \) is a well defined homomorphism with kernel \( U(U^* \cap V) \) and

\[
\frac{U(U^* \cap V^*)}{U(U^* \cap V)} \simeq \frac{U^* \cap V^*}{D}.
\]
Note that (12) is equivalent to (11). We now use Lemma 4 to show there is a homomorphism from (11) to (13). Let $U = X^{i+1}$ and $U^* = X^i$. Let $V = Y^{i+m-1}$ and $V^* = Y^{i+m}$. Note that $U \triangleleft U^*$ and $V \triangleleft V^*$. Then

$$
\frac{U(U^* \cap V^*)}{U(U^* \cap V)} = \frac{X^{i+1}(X^i \cap Y^{i+m})}{X^{i+1}(X^i \cap Y^{i+m-1})},
$$

and

$$
\frac{U^* \cap V^*}{D} = \frac{U^* \cap V^*}{(U \cap V^*)(U \cap V)} = \frac{(X^i \cap Y^{i+m})}{(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})}.
$$

Now the function $f$ of Lemma 4 is a homomorphism from (11) to (13).

**Theorem 5** There is a homomorphism from the quotient group (11), or equivalently (12), to the quotient group (13), given by the function $f$ of the Zassenhaus lemma.

We now choose a coset representative or generator of the time domain granule (11) formed by the normal chain (10) for any integer pair $i, m \in \mathbb{Z}$. Fix any $i \in \mathbb{Z}$. There are three cases to consider for (11), depending on the value of $m$. For $m < 0$, we have $(X^i \cap Y^{i+m}) = a_1$ and so (11) reduces to $(X^{i+1}(a_1))/(X^{i+1}(a_1)) = X^{i+1}/X^{i+1}$. Then coset representative $\hat{x}\hat{y}$ is $xa_1$. We may always select $\hat{x}$ to be $a_1$. Then for $m < 0$, we can choose the coset representative or generator of (11) to be $a_1$.

We next consider the case $0 \leq m \leq \ell$. If we select a coset representative $\hat{x}\hat{y}$ of coset $X^{i+1}y(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})$ to be $a_1\hat{y}$ where $a_1$ is the identity of $X^{i+1}$, then a transversal of the quotient group (13) is a transversal of the quotient group (12). But for $0 \leq m \leq \ell$ note that (13) is the same as the Forney-Trott granule $\Gamma^{[i,i+m]}$, or spectral domain granule, for $A$.

$$
\Gamma^{[i,i+m]} = \frac{A^{[i,i+m]}}{A^{[i,i+m-1]}A^{[i+1,i+m]}}.
$$

(14)

This gives the following.

**Corollary 6** For $0 \leq m \leq \ell$, a transversal of the spectral domain granule (14) is a transversal of the time domain granule (11), but the reverse is only true if $\hat{x}\hat{y} = a_1\hat{y}$ or $\hat{x} = a_1$ for each coset representative $\hat{x}\hat{y}$ of the time domain granule.

Then for $0 \leq m \leq \ell$, we can choose the coset representative or generator of (11) to be $a_1\hat{y}$. In this case, the coset representative or generator of (11) is the same as the coset representative $g^{[i,i+m]}$ of the spectral domain granule (14) of $A$. Therefore the coset representative of the time domain granule can be chosen to be the same as the Forney-Trott generator of $A$. Note that for $0 \leq m \leq \ell$, Theorem 5 shows there is a homomorphism from the time domain granule to the spectral domain granule given by the function $f$ of the Zassenhaus lemma.

Lastly we consider the case $m > \ell$. Since $A$ is $\ell$-controllable, in (13) there can be no elements of $(X^i \cap Y^{i+m})$ that are not elements of $(X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m})$. Then

$$
X^i \cap Y^{i+m} = (X^i \cap Y^{i+m-1})(X^{i+1} \cap Y^{i+m}),
$$

and quotient group (13) is trivial. Then quotient group (12) and (11) is trivial. Then we can choose the coset representative $\hat{x}\hat{y}$ or generator of (11) to be $\hat{x}\hat{y} = a_1a_1$ or $a_1$.

We have found generators of the time domain granule for each $i \in \mathbb{Z}$ and ranges $m < 0$, $0 \leq m \leq \ell$, and $m > \ell$. We can arrange the selected set of generators in the same order as (10), as shown in (15). For $m < 0$, there are an
Theorem 7. The sequence \((17)\) is a well defined product in \(A\). Therefore \((10)\) is a well defined sequence in \(A\), where \(a\) is given by

\[
a = \prod_{i=-\infty}^{+\infty} \left( \prod_{m=0}^{\ell} g[i,i+m] \right).
\]

**Proof.** Each term in \((10)\) is a sequence in \(A\). The sequences in product \((10)\) are multiplied according to the definition of product in \(A\), which means component by component multiplication for \(-\infty < i < +\infty\). The infinite product \((10)\) is well defined if for each \(i \in \mathbb{Z}\), there are only finitely many terms \(g[i,i+m]\) having a nontrivial component \([3]\). Therefore the final result is a well defined sequence in \(A\).

Choosing coset representatives or generators \(g[i,i+m]\) of the time domain granules \(\Lambda[i,i+m]\) formed by the normal chain \((10)\) forms a basis \(\mathcal{B}\) of \(A\). If a basis \(\mathcal{B}\) of \(A\) is chosen, then \(\mathcal{R}\) is chosen.

**Theorem 8.** Find a basis \(\mathcal{B}\) of \(A\). Then \(\mathcal{R}\) is chosen, and there is a bijection \(\alpha : \mathcal{R} \rightarrow A\) given by assignment \(\alpha : r \mapsto a\), where \(a\) is an encoding of \(r\) using product \((17)\) on the generators in \(r\).
which corrects its sequence with a new estimate at each time \( t \) of input granules and state granules in the same way as the spectral domain encoder. The time domain encoder has a granule based construction of the time domain granule of the Trott generator of the spectral domain granule of (19) use the same generators and the alphabet group change of the double product in the spectral domain encoder. If both (6) and comparing (6) and (19), we see that the time domain encoder is just an inter-

of a time convolution, reminiscent of a linear system. The encoder (6) in [3] does not have a convolution property. For this reason, we say the encoder (17) in parentheses is some function of time, say \( \chi \) for \( 0 \leq t \leq a \) Then a component Proof. Since (10) forms a coset decomposition chain of \( A \), and since (17) uses a selection of one representative from each nontrivial quotient group in the chain, then any \( a \in A \) can be obtained by the composition (17). The assignment \( \alpha : r \mapsto a \) is a bijection since any unique selection of coset representatives gives a unique \( a \in A \).

A component \( a^t \) of \( a \) at any time \( t \) can be obtained by the composition of the time \( t \) component \( \chi^t \) of generators in (14). There are only a finite number of generators in (15) with nontrivial components at time \( t \). These are generators \( g_{t-j,t-j+k} \), for \( 0 \leq j \leq \ell \), for \( j \leq k \leq \ell \), shown in the triangular matrix (20). The components at time \( t \) of all other generators in (15) are the identity element \( a^t_1 \) of \( A^t \). This gives the following.

**Lemma 9** The only quotient groups, or time domain granules, formed from (10) that have transversals with nontrivial components at time \( t \) are of the form

\[
\Lambda^{t-j,t-j+k} = \frac{X^{t-j+1}(X^{t-j} \cap Y^{t+k-j})}{X^{t-j+1}(X^{t-j} \cap Y^{t+k-j-1})}
\]

for \( 0 \leq j \leq \ell \), for \( j \leq k \leq \ell \).

Then a component \( a^t \) of \( a \) can be obtained by the composition of components \( \chi^t(g_{t-j,t-j+k}) \) for \( 0 \leq j \leq \ell \), for \( j \leq k \leq \ell \), as

\[
a^t = \prod_{j=0}^{\ell} \left( \prod_{k=j}^{\ell} \chi^t(g_{t-j,t-j+k}) \right).
\]  

(19)

The encoder in (19) can be written as \( a^t = \prod_{j=0}^{\ell} h_{j}^{t-j} \), where the inner term in parentheses is some function of time, say \( h_{j}^{t-j} \). Thus the encoder has the form of a time convolution, reminiscent of a linear system. The encoder (6) in [3] does not have a convolution property. For this reason, we say the encoder (17) or (19) is a *time domain encoder*, and the encoder (15) or (6) in [3] a *spectral domain encoder*. The time domain encoder has a granule based construction of input granules and state granules in the same way as the spectral domain encoder in [3]. Developed in another way [22], the encoder (19) is an estimator which corrects its sequence with a new estimate at each time \( t \).

We have seen from Corollary (6) (see also [21]) that the coset representative of the time domain granule of \( A \) can be chosen to be the same as the Forney-Trott generator of the spectral domain granule of \( A \). In this case, the generator \( g_{t-j,t-j+k} \) in (19) can be the same as the generator \( g_{FT}^{t-j,t-j+k} \) in (6). Then comparing (6) and (19), we see that the time domain encoder is just an interchange of the double product in the spectral domain encoder. If both (6) and (19) use the same generators and the alphabet group \( A^t \) is abelian, then the
Then the two generators differ by a sequence \( (a_1, a_2) \) and

Using (11) and (12), (18) can be rewritten as

\[ \text{Proof.} \]

But \( \chi \) is in a generator \( g[t-j,t-j+k] \) which is some letter \( a^t \in A^t \).

A transversal \( [\Lambda[t-j,t-j+k]] \) of \( \Lambda[t-j,t-j+k] \) is a selection of one representative from each coset of \( [\Lambda[t-j,t-j+k]] \). We always take the coset representative \( g[t-j,t-j+k] \) of the denominator in (13) to be the identity \( a_1 \) of \( A \). Since \( \chi(g[t-j,t-j+k]) = r_{j,k}^t \), we let \( \{r_{j,k}^t\} \) be the set of representatives at time \( t \) in generators \( g[t-j,t-j+k] \) in \( [\Lambda[t-j,t-j+k]] \).

**Lemma 10** Fix \( t \in \mathbb{Z} \). Fix \( j \) such that \( 0 \leq j \leq \ell \) and fix \( k \) such that \( j \leq k \leq \ell \). There is a bijection \( [\Lambda[t-j,t-j+k]] \rightarrow \{r_{j,k}^t\} \).

**Proof.** Using (11) and (12), (13) can be rewritten as

\[ \Lambda[t-j,t-j+k] = \frac{X^{t-j+1}(X^{t-j} \cap Y^{t+k-j})}{X^{t-j+1}(X^{t-j} \cap Y^{t+k-j})} \] (22)

We use proof by contradiction. Suppose two generators \( g_1[t-j,t-j+k] \) and \( g_2[t-j,t-j+k] \) in set \( [\Lambda[t-j,t-j+k]] \) share the same representative in set \( \{r_{j,k}^t\} \). Then there is a sequence \( a[t-j,t-1] \) in \( A \) which is trivial outside the interval \( [t-j, t-1] \), and a sequence \( a[t+1,t-j+k] \) in \( A \) which is trivial outside the interval \( [t+1, t-j+k] \), such that \( g_1[t-j,t-j+k] = (a[t-j,t-1])g_2[t-j,t-j+k] \).

But \( a[t-j,t-1] \) is an element of denominator term \( (X^{t-j} \cap Y^{t+k-j-1}) \) in (22), and \( a[t+1,t-j+k] \) is an element of denominator term \( (X^{t-j+1} \cap Y^{t+k-j}) \) in (22). Then the two generators differ by a sequence \( (a[t-j,t-1])a[t+1,t-j+k] \) which is an element in the denominator of (22). Then both generators must be in the same coset of (22), a contradiction.

Since \( \chi(g[t-j,t-j+k]) = r_{j,k}^t \), we can write the time \( t \) components of the
generators in (20) as the triangular matrix (23).

\[
\begin{array}{cccccc}
r_{0,\ell} & r_{1,\ell}^{-1} & \ldots & \ldots & r_{t-1,\ell}^{-1} & r_{t,\ell}^{-\ell} \\
r_{0,\ell-1} & r_{1,\ell-1}^{-1} & \ldots & \ldots & r_{t-1,\ell-1}^{-1} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_{0,k} & r_{1,k}^{-1} & \ldots & \ldots & r_{t-1,k}^{-1} & r_{t,k}^{-k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_{0,2} & r_{1,2}^{-1} & r_{2,2}^{-2} \\
r_{0,1} & r_{1,1}^{-1} \\
r_{0,0} \\
\end{array}
\]  

(23)

Since all the entries in $\nabla_{0,0}(r)$ are alphabet letters in alphabet $A^t$, we call (23) the alphabet matrix. Then we see that each sequence $r \in \mathcal{R}$ gives a sequence of alphabet matrices

\[
\ldots, \nabla_{0,0}^t(r), \nabla_{0,0}^{t-1}(r), \ldots
\]  

(24)

Let $\nabla_{0,0}^t(\mathcal{R})$ be the set of all alphabet matrices $\nabla_{0,0}^t(r)$. In other words, $\nabla_{0,0}^t(\mathcal{R}) \equiv \{ \nabla_{0,0}^t(r) : r \in \mathcal{R} \}$.

We have written a sequence $a \in A$ as $\ldots, a^{t-1}, a^t, \ldots$. We have chosen to write the sequence of alphabet matrices (24) in reverse time order since it reflects the order of indices in (15), which reflects the order of indices in (10). It is a natural ordering for this problem. But writing the sequence in reverse time order on a piece of paper does not change the physics of time: the component at time epoch $t - 1$ still occurs before the component at time epoch $t$, as it does for $a$.

Using the alphabet matrix $\nabla_{0,0}^t(r)$, we can rewrite (19) in the equivalent form as

\[
a^t = \prod_{j=0}^{t} \left( \prod_{k=j}^{t} a^{(t-j,t+j+k)} \right)
\]  

(25)

\[
= \prod_{j=0}^{t} \left( \prod_{k=j}^{t} r_{j,k}^{t-j} \right),
\]  

(26)

where the inner product in parentheses in (26) is just the product of terms in the $j$-th column of $\nabla_{0,0}^t(r)$. By the convention used here, equation (26) is evaluated as

\[
a^t = r_{0,0}^{t} r_{0,1}^{t} r_{0,2}^{t-1} \ldots r_{0,\ell,1,1}^{t-2} \ldots r_{1,\ell,2,2}^{t-3} \ldots r_{\ell,1,\ell,\ell}^{t-2} \ldots r_{j,1,\ell}^{j} \ldots r_{j,\ell}^{j} \ldots r_{\ell,\ell-1,\ell}^{t-1} r_{\ell-1,\ell}^{t-1} r_{\ell,\ell},
\]  

(27)

Using (26) we can give the following enhancement of Theorem 8.

**Theorem 11** Find a basis $\mathcal{B}$ of $A$. Then $\mathcal{R}$ is chosen, and there is a bijection $\alpha : \mathcal{R} \rightarrow A$ given by assignment $\alpha : r \mapsto a$, where component $a^t$ of $a$ is an encoding of the representatives in $\nabla_{0,0}^t(r)$ of $r$ using (20), for each $t \in \mathbb{Z}$.

For each $t \in \mathbb{Z}$, define a map $\alpha^t : \nabla_{0,0}^t(\mathcal{R}) \rightarrow A^t$ by the assignment $\alpha^t : \nabla_{0,0}^t(r) \mapsto a^t$ if $a^t$ is given by the map in encoders (25)–(26). In general we have $|\nabla_{0,0}^t(\mathcal{R})| > |A^t|$ so this is a many to one map. This gives the following corollary to Theorem 11.

**Corollary 12** Find a basis $\mathcal{B}$ of $A$. Then $\mathcal{R}$ is chosen, and there is a bijection $\alpha : \mathcal{R} \rightarrow A$ given by assignment $\alpha : r \mapsto a$, where component $a^t$ of $a$ is given by the assignment $\alpha^t : \nabla_{0,0}^t(r) \mapsto a^t$, for each $t \in \mathbb{Z}$.
The encoder in (25)-(26) forms output $a$ from a sequence of generators selected from basis $B$, which forms a tensor $r \in R$. For each time $t \in \mathbb{Z}$, for each $k$ such that $0 \leq k \leq \ell$, a single generator $g^{[t,t+k]}$ is selected from set $[A^{[t,t+k]}]$. Since $A$ is complete [3], the set of tensors $r \in R$ that can be selected in this way to form $A$ is the double Cartesian product

$$
\bigotimes_{t=-\infty}^{t=+\infty} \bigotimes_{0 \leq k \leq \ell} [A^{[t,t+k]}].
$$

We have let $r \in R$ be the infinite series of finite columns given by generators in (15). For each $t \in \mathbb{Z}$, for $0 \leq k \leq \ell$, let $r^{[t,t+k]}_g$ be the tensor $r \in R$ when all the generators in (15) other than $g^{[t,t+k]}$ are trivial. Then $\alpha : r^{[t,t+k]}_g \mapsto g^{[t,t+k]}$. 

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4. THE DECOMPOSITION GROUP AND GENERATOR GROUP

4.1 The decomposition group

Let \( B \) be a basis of \( A \). Let \( R \) be the set of tensors determined by \( B \). From Theorem 8, there is a bijection \( \alpha : R \rightarrow A \) with assignment \( \alpha : r \mapsto a \) if \( a \) is an encoding of \( r \) using product (17) on the generators in \( r \). We define an operation \( * \) on \( R \) by using the operation in \( A \). Let \( \tilde{r}, \tilde{t} \in R \). Let \( \alpha : \tilde{r} \mapsto \tilde{a} \) and \( \alpha : \tilde{t} \mapsto \tilde{a} \). Define an operation \( * \) on \( R \) by

\[
\tilde{r} * \tilde{t} = \tilde{r}
\]

if \( \alpha : \tilde{r} \mapsto \tilde{a} \tilde{a} \).

Lemma 13 The operation \( * \) is well defined.

Proof. Let \( \tilde{r}, \tilde{t} \in R \) such that \( \tilde{r} = \tilde{r} \) and \( \tilde{t} = \tilde{r} \). We have to show that \( \tilde{r} * \tilde{t} = \tilde{r} * \tilde{r} \). But if \( \tilde{r} = \tilde{r} \), then \( \alpha : \tilde{r} \mapsto \tilde{a} \), and similarly \( \alpha : \tilde{t} \mapsto \tilde{a} \). Then both \( \tilde{r} \) and \( \tilde{t} \) are determined by \( \tilde{a} \tilde{a} \).

Theorem 14 The set \( R \) with operation \( * \) forms a group \( (R, *) \), and \( A \simeq (R, *) \) under the bijection \( \alpha : R \rightarrow A \).

Proof. We first show the operation \( * \) is associative. Let \( r, \tilde{r}, \tilde{t} \in R \). We need to show

\[
(r * \tilde{r}) * \tilde{t} = r * (\tilde{r} * \tilde{t}).
\]

Let \( \alpha : r \mapsto a, \alpha : \tilde{r} \mapsto \tilde{a}, \) and \( \alpha : \tilde{t} \mapsto \tilde{a} \). Then (30) is the same as showing

\[
(a \tilde{a}) \tilde{a} = a(\tilde{a} \tilde{a}).
\]

But this follows since operation in \( A \) is associative.

Let \( a_1 \) be the identity of \( A \). Let \( \alpha : r_1 \mapsto a_1 \). We show \( r_1 \) is the identity of \( (R, *) \). Let \( r \in R \). We need to show \( r_1 * r = r \) and \( r * r_1 = r \). But this is the same as showing \( a_1 a = a \) and \( aa_1 = a \), where \( \alpha : r \mapsto a \). But this follows since \( a_1 \) is the identity of \( A \).

Let \( r \in R \). We show \( r \) has an inverse in \( (R, *) \). Let \( \alpha : r \mapsto a \). The group element \( a \) has an inverse \( \tilde{a} \) in \( A \) such that \( \tilde{a} a = a_1 \) and \( a a_1 = a \). Let \( \alpha : \tilde{r} \mapsto \tilde{a} \). It follows that \( \tilde{r} * r = r_1 \) and \( r * \tilde{r} = r_1 \).

Together these results show that \( (R, *) \) is a group. By the definition of operation \( * \) given in (29), \( (R, *) \) is just an isomorphic copy of \( A \) under bijection \( \alpha \).

If \( \alpha : r \mapsto a \), then \( r \) is the decomposition of \( a \) into its generators. For this reason, we call \( (R, *) \) the decomposition group of \( A \).

In a sense, the decomposition group \( (R, *) \) shows the permutation of the generators when sequences in \( A \) are multiplied. However the group \( (R, *) \) is a global group defined on sequences \( r \) in \( R \). It is not clear that the permutation of the generators in the infinite sequences \( r \in R \) under multiplication determines a component group on set \( \oplus_{0,0}(R) \) for each \( t \in Z \). We show that \( (R, *) \) has a component group in Subsection 4.4.

4.2 The generator group

First we discuss how to label each generator in basis \( B \) uniquely. The time index \( t \) and length \( k \) specify each generator \( g^{(t,k)} \) in basis \( B \) uniquely up to the set of generators of the same length \( k \) and time \( t \). The generators are coset representatives in the normal chain (10) of \( A \), and therefore they must be unique. The set of generators of the same length \( k \) and time \( t \) are the representatives...
in \([A^{[t,t+k]}]\). For generators of the same length \(k\) and time \(t\), we define a set of unique identifiers \(G'_k \overset{\text{def}}{=} \{g^t_k\}\). We call each \(g^t_k\) in set \(G'_k\) a generator label. Then there is a bijection \([A^{[t,t+k]}] \to G'_k\) with assignment \(g^{t,t+k}_r \mapsto g^t_k\).

Previously we constructed a tensor set \(\mathcal{R}\). A tensor \(r \in \mathcal{R}\) is equivalent to a collection of generators \(g^{t,t+k}_r\) for \(0 \leq k \leq \ell\), for each \(t \in \mathbb{Z}\), as shown by the nontrivial generators in the middle rows of (31). For \(0 \leq k \leq \ell\), for each \(t \in \mathbb{Z}\), we now replace each generator \(g^{t,t+k}_r\) in \(r \in \mathcal{R}\) with a single generator label \(g^t_k\), where \(g^{t,t+k}_r \mapsto g^t_k\) in the bijection \([A^{[t,t+k]}] \to G'_k\). Under the assignment \(g^{t,t+k}_r \mapsto g^t_k\) for \(0 \leq k \leq \ell\), for each \(t \in \mathbb{Z}\), a tensor \(r \in \mathcal{R}\), as shown by the middle rows of (32), becomes a tensor \(u\) as shown in (32). The tensor \(u\) has the same time reverse ordering as \(r\). Let \(\mathcal{U}\) be the set of tensors \(u\) obtained from \(\mathcal{R}\) this way. This gives the following.

**Lemma 15** There is a bijection \(\beta: \mathcal{R} \to \mathcal{U}\) given by the assignment \(\beta: r \mapsto u\) where each generator \(g^{t,t+k}_r\) in \(r\) is replaced by a single generator label \(g^t_k\) in \(u\), for \(0 \leq k \leq \ell\), for each \(t \in \mathbb{Z}\), where \(g^{t,t+k}_r \mapsto g^t_k\) in the bijection \([A^{[t,t+k]}] \to G'_k\).

Using (31) with bijections \(\beta: \mathcal{R} \to \mathcal{U}\) and \([A^{[t,t+k]}] \to G'_k\), set \(\mathcal{U}\) is just the double Cartesian product

\[
\bigotimes_{t=-\infty}^{t=+\infty} \bigotimes_{0 \leq k \leq \ell} G'_k.
\]

Using (32) with bijections \(\beta: \mathcal{R} \to \mathcal{U}\) and \([A^{[t,t+k]}] \to G'_k\), set \(\mathcal{U}\) is just the double Cartesian product

\[
\bigotimes_{t=-\infty}^{t=+\infty} \bigotimes_{0 \leq k \leq \ell} G'_k.
\]

We say an element \(u \in \mathcal{U}\) is a nontrivial generator \(u^{t,k}_{g,k}\) of \((\mathcal{U}, \circ)\) if \(u^{t,k}_{g,k}\) contains one and only one nontrivial generator label \(g^t_k\) for some \(k\) such that \(0 \leq k \leq \ell\) and some time \(t \in \mathbb{Z}\). For each \(k\) such that \(0 \leq k \leq \ell\) and each \(t \in \mathbb{Z}\), we always assume there is a trivial generator \(u^{t,k}_1\) of \((\mathcal{U}, \circ)\) which is the identity \(u^{t,k}_1\) of \((\mathcal{U}, \circ)\). Under the bijection \(\beta: \mathcal{R} \to \mathcal{U}\), a generator \(r^{t,t+k}_g\) of \((\mathcal{R}, \ast)\) is mapped to a generator \(u^{t,k}_{g,k}\) of \((\mathcal{U}, \circ)\), or \(\beta: r^{t,t+k}_g \mapsto u^{t,k}_{g,k}\).

We now develop a compressed version of \((\mathcal{R}, \ast)\) called \((\mathcal{U}, \circ)\). The operation \(\ast\) in \((\mathcal{R}, \ast)\) determines an operation \(\circ\) on \(\mathcal{U}\). Let \(\bar{u}, \bar{u} \in \mathcal{U}\). Let \(\beta: \bar{r} \mapsto \bar{u}\) and \(\beta: \bar{r} \mapsto \bar{u}\). Define an operation \(\circ\) on \(\mathcal{U}\) by

\[
\bar{u} \circ \bar{u} \overset{\text{def}}{=} \bar{u}
\]

if \(\beta: \bar{r} \ast \bar{r} \mapsto \bar{u}\).

**Lemma 16** The operation \(\circ\) is well defined.

**Proof.** The proof is similar to the proof of Lemma 15.

**Theorem 17** The set \(\mathcal{U}\) with operation \(\circ\) forms a group \((\mathcal{U}, \circ)\), and \((\mathcal{R}, \ast) \simeq (\mathcal{U}, \circ)\) under the bijection \(\beta: \mathcal{R} \to \mathcal{U}\).

**Proof.** The proof is similar to the proof of Theorem 14.
We know that each \( r \in \mathcal{R} \) in group \((\mathcal{R}, \ast)\) corresponds to a sequence of generators. We see that \( u \in \mathcal{U} \) in group \((\mathcal{U}, \circ)\) demonstrates this sequence using a single generator label in place of each generator. For this reason we call \((\mathcal{U}, \circ)\) the generator group of \( A \).

### 4.3 The elementary groups of \((\mathcal{U}, \circ)\)

In this subsection we define a component group on \((\mathcal{U}, \circ)\). We first define a matrix \( \nabla_{0,0}^t(u) \) in \( u \) which is congruent in shape to alphabet matrix \( \nabla_{0,0}^t(r) \) in \( r \). Given \( u \) in \((\mathcal{U}, \circ)\), we define \( \nabla_{0,0}^t(u) \) to be the triangle in \( u \) with lower vertex \( r^0_t \) and upper vertices \( g^j_t \) and \( g^t_{t,j} \), as shown in \((34)\). Note that all the entries in alphabet matrix \( \nabla_{0,0}^t(r) \) are at the same time \( t \), but in matrix \( \nabla_{0,0}^t(u) \), all entries in the same column are at the same time, but entries in different columns are at different times.

\[
\begin{align*}
g^0_t & \quad g^0_{t-1} & \quad \ldots & \quad \ldots & \quad g^0_{t-j} & \quad \ldots & \quad \ldots & \quad g^0_{t-k} & \quad \ldots & \quad \ldots & \quad g^0_{t-\ell} \\
g^1_t & \quad g^1_{t-1} & \quad \ldots & \quad \ldots & \quad g^1_{t-j} & \quad \ldots & \quad \ldots & \quad g^1_{t-k} \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots \\
g^k_t & \quad g^k_{t-1} & \quad \ldots & \quad g^k_{t-j} & \quad g^k_{t-k} & \quad \ldots & \quad \ldots & \quad \ldots \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots \\
\ell & \quad \ell & \quad \ldots & \quad \ldots & \quad \ell & \quad \ldots & \quad \ldots & \quad \ell & \quad \ldots & \quad \ldots & \quad \ell & \quad \ldots & \quad \ldots & \quad \ell \\
\end{align*}
\]

Note that we can think of tensor \( u \) in \((\mathcal{U}, \circ)\) as a sequence of matrices \( \ldots, \nabla_{0,0}^t(u), \nabla_{0,0}^{t-1}(u), \ldots \) which overlap to form \( u \). Define \( \nabla_{0,0}^t(\mathcal{U}) \) to be the set \( \{ \nabla_{0,0}^t(u) : u \in \mathcal{U} \} \).

Define a mapping \( \beta^t : \nabla_{0,0}^t(\mathcal{R}) \rightarrow \nabla_{0,0}^t(\mathcal{U}) \) given by assignment \( \beta^t : \nabla_{0,0}^t(r) \rightarrow \nabla_{0,0}^t(u) \) if \( \beta : r \rightarrow u \) in the bijection \( \beta : \mathcal{R} \rightarrow \mathcal{U} \).

**Theorem 18** For each \( t \in \mathbb{Z} \), the map \( \beta^t : \nabla_{0,0}^t(\mathcal{R}) \rightarrow \nabla_{0,0}^t(\mathcal{U}) \) is a bijection.

**Proof.** We first show the map \( \beta^t \) is well defined. Let \( r, \tilde{r} \in \mathcal{R} \) such that \( \nabla_{0,0}^t(r) = \nabla_{0,0}^t(\tilde{r}) \). Let \( \beta : r \rightarrow u \) and \( \beta : \tilde{r} \rightarrow \tilde{u} \). Then \( \beta^t : \nabla_{0,0}^t(r) \rightarrow \nabla_{0,0}^t(u) \) and \( \beta^t : \nabla_{0,0}^t(\tilde{r}) \rightarrow \nabla_{0,0}^t(\tilde{u}) \). We need to show \( \nabla_{0,0}^t(u) = \nabla_{0,0}^t(\tilde{u}) \).

If \( \nabla_{0,0}^t(r) = \nabla_{0,0}^t(\tilde{r}) \), then for each \( j, 0 \leq j \leq \ell \), and each \( k, j \leq k \leq \ell \), the representative \( \check{r}^t_{j,k} \) in \( \nabla_{0,0}^t(r) \) in \((23)\) is the same as the representative \( \check{r}^t_{j,k} \) in \( \nabla_{0,0}^t(\tilde{r}) \). Since from Lemma \(11\) for each \( j, 0 \leq j \leq \ell \), and each \( k, j \leq k \leq \ell \), there is a bijection \( [\mathcal{A}^{\ell-j,j-t-k}] \rightarrow \{r^t_{j,k}\} \), then for these same indices, generators \( g^t_{t-j,t-j+k} \) in \( r \) and generators \( \check{g}^t_{t-j,t-j+k} \) in \( \tilde{r} \) must be the same. Therefore the mapping \( \beta : r \rightarrow u \) and \( \beta : \tilde{r} \rightarrow \tilde{u} \) must give \( \nabla_{0,0}^t(u) = \nabla_{0,0}^t(\tilde{u}) \).

We now show the map \( \beta^t \) is 1-1. Suppose \( \nabla_{0,0}^t(u) = \nabla_{0,0}^t(\tilde{u}) \). Let \( \beta : r \rightarrow u \) and \( \beta : \tilde{r} \rightarrow \tilde{u} \). We need to show \( \nabla_{0,0}^t(r) = \nabla_{0,0}^t(\tilde{r}) \). If \( \nabla_{0,0}^t(u) = \nabla_{0,0}^t(\tilde{u}) \), then for each \( j, 0 \leq j \leq \ell \), and each \( k, j \leq k \leq \ell \), the generator label \( g^t_{t-j} \) in \( \nabla_{0,0}^t(u) \) in \((34)\) is the same as the generator label \( \check{g}^t_{t-j} \) in \( \nabla_{0,0}^t(\tilde{u}) \). Since there is a bijection \( [\mathcal{A}^{\ell-k}] \rightarrow G^t_{k} \) for each \( t \in \mathbb{Z} \) and each \( k, 0 \leq k \leq \ell \), then for each \( j, 0 \leq j \leq \ell \), and each \( k, j \leq k \leq \ell \), generators \( g^t_{t-j,j-t+k} \) in \( r \) and generators \( \check{g}^t_{t-j,j-t+k} \) in \( \tilde{r} \) must be the same. But this means \( \nabla_{0,0}^t(r) = \nabla_{0,0}^t(\tilde{r}) \).

We now show the map \( \beta^t \) is onto \( \nabla_{0,0}^t(\mathcal{U}) \). Let \( \nabla_{0,0}^t(u) \) be any element of \( \nabla_{0,0}^t(\mathcal{U}) \). It is clear there is an \( r \in \mathcal{R} \) such that \( \beta : r \rightarrow u \). But if \( \beta : r \rightarrow u \), then we have \( \beta^t : \nabla_{0,0}^t(r) \rightarrow \nabla_{0,0}^t(u) \) by definition. Therefore \( \beta^t \) is onto. Therefore \( \beta^t \) is a bijection \( \nabla_{0,0}^t(\mathcal{R}) \rightarrow \nabla_{0,0}^t(\mathcal{U}) \).

We use this result in Subsection 4.4 to define a component group of \((\mathcal{R}, \ast)\).
We now define a local group of \((\mathcal{U}, \circ)\). Recall that for each \(t \in \mathbb{Z}\), we have defined \(X^t\) to be the set of all sequences \(a\) in \(A\) for which \(a^n = a^t\) for \(n < t\), where \(a^n\) is the identity of \(A^n\) at time \(n\). And for each \(t \in \mathbb{Z}\), we have defined \(Y^t\) to be the set of all sequences \(a\) in \(A\) for which \(a^n = a^t\) for \(n > t\). It is clear that \(X^t \triangleleft \mathcal{U}\) and \(Y^t \triangleleft \mathcal{U}\) for each \(t \in \mathbb{Z}\).

Fix \(t \in \mathbb{Z}\). In \(\mathcal{R}\), \(X^t\) is the set of all \(r \in \mathcal{R}\) with trivial generators \(g_{[t', t'+k]}^t\), \(0 < k < \ell\), for all \(t' \in \mathbb{Z}\) except for \(t' \geq t\). Call this subset of \(\mathcal{R}\) as \(\mathcal{R}^{t^+}\). Since \(X^t\) is a normal subgroup of \(A\), then \((\mathcal{R}^{t^+}, *)\) is a normal subgroup of \((\mathcal{R}, *)\). In \(\mathcal{U}\), this is the set of all \(u\) with trivial generator labels \(g_{[t', t'+k]}^t\), \(0 < k < \ell\), except for \(t' \geq t\). Call this subset of \(\mathcal{U}\) as \(\mathcal{U}^{t^+}\). Since \((\mathcal{R}^{t^+}, *)\) is a normal subgroup of \((\mathcal{R}, *)\), then \((\mathcal{U}^{t^+}, \circ)\) is a normal subgroup of \((\mathcal{U}, \circ)\), under the bijection \(\beta\) of the isomorphism from \((\mathcal{R}, *)\) to \((\mathcal{U}, \circ)\). This shows that \((\mathcal{U}^{t^+}, \circ)\) is the analog in \((\mathcal{U}, \circ)\) of \(X^t\) in \(A\).

In \(\mathcal{R}\), \(Y^t\) is the set of all \(r \in \mathcal{R}\) with trivial generators \(g_{[t', t'+k]}^t\), \(0 < k < \ell\), for all \(t' \in \mathbb{Z}\) except for \(t' < t - k\). Call this subset of \(\mathcal{R}\) as \(\mathcal{R}^{t^-}\). Since \(Y^t\) is a normal subgroup of \(A\), then \((\mathcal{R}^{t^-}, *)\) is a normal subgroup of \((\mathcal{R}, *)\). In \(\mathcal{U}\), this is the set of all \(u\) with trivial generator labels \(g_{[t', t'+k]}^t\), \(0 < k < \ell\), except for \(t' < t - k\). Call this subset of \(\mathcal{U}\) as \(\mathcal{U}^{t^-}\). Since \((\mathcal{R}^{t^-}, *)\) is a normal subgroup of \((\mathcal{R}, *)\), then \((\mathcal{U}^{t^-}, \circ)\) is a normal subgroup of \((\mathcal{U}, \circ)\), under the bijection \(\beta\) of the isomorphism from \((\mathcal{R}, *)\) to \((\mathcal{U}, \circ)\). This shows that \((\mathcal{U}^{t^-}, \circ)\) is the analog in \((\mathcal{U}, \circ)\) of \(Y^t\) in \(A\).

Note that \((\mathcal{U}^{t^+})^{t^+}\) and \((\mathcal{U}^{t^-})^{t^-}\) cover \(\mathcal{U}\) except for \(\mathcal{V}_{0,0}^{t^+}\). In other words the union of the set \(\mathcal{U}^{t^+}\) and set \(\mathcal{U}^{t^-}\) is the set of all \(u \in \mathcal{U}\) which are trivial in \(\mathcal{V}_{0,0}^{t^+}(u)\); call this subset of \(\mathcal{U}\) as \(\mathcal{V}_{0,0}^{t^-}\). Then the product of \((\mathcal{U}^{t^+})^{t^+}\) and \((\mathcal{U}^{t^-})^{t^-}\) is a normal subgroup \((\mathcal{V}_{0,0}^{t^+}, \circ)\) of \((\mathcal{U}, \circ)\). Then we have a quotient group \((\mathcal{U}, \circ)/(\mathcal{V}_{0,0}^{t^+}, \circ)\). The projection of the cosets in \((\mathcal{U}, \circ)/(\mathcal{V}_{0,0}^{t^+}, \circ)\) on set \(\mathcal{V}_{0,0}^{t^+}(u)\) is \(\mathcal{V}_{0,0}^{t^-}(u)\). We can use \(\mathcal{V}_{0,0}^{t^+}(u)\) to define a local group on \((\mathcal{U}, \circ)\).

Fix time \(t\). Let \(\mathcal{V}_{0,0}^{t^+}(u), \mathcal{V}_{0,0}^{t^-}(u) \in \mathcal{V}_{0,0}^{t^+}(u)\). Define an operation \(\circ_{0,0}\) on set \(\mathcal{V}_{0,0}^{t^+}(u)\) by

\[
\mathcal{V}_{0,0}^{t^+}(u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u) \overset{\text{def}}{=} \mathcal{V}_{0,0}^{t^+}(u \circ u).
\]

**Lemma 19** Fix time \(t\). The operation \(\circ_{0,0}\) on set \(\mathcal{V}_{0,0}^{t^+}(u)\) is well defined.

**Proof.** Choose any \(u, u' \in \mathcal{U}\) such that \(\mathcal{V}_{0,0}^{t^+}(u) = \mathcal{V}_{0,0}^{t^+}(u')\) and \(\mathcal{V}_{0,0}^{t^-}(u) = \mathcal{V}_{0,0}^{t^-}(u')\). To show the operation is well defined, we need to show

\[
\mathcal{V}_{0,0}^{t^+}(u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u') = \mathcal{V}_{0,0}^{t^+}(u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u),
\]

or what is the same, \(\mathcal{V}_{0,0}^{t^+}(u \circ u') = \mathcal{V}_{0,0}^{t^+}(u \circ u')\). But if \(\mathcal{V}_{0,0}^{t^+}(u) = \mathcal{V}_{0,0}^{t^+}(u')\), then \(u\) and \(u'\) are in the same coset of \((\mathcal{U}, \circ)/(\mathcal{V}_{0,0}^{t^+}, \circ)\). And if \(\mathcal{V}_{0,0}^{t^+}(u) = \mathcal{V}_{0,0}^{t^+}(u')\), then \(u\) and \(u'\) are in the same coset of \((\mathcal{U}, \circ)/(\mathcal{V}_{0,0}^{t^-}, \circ)\). But then \(u \circ u\) and \(u' \circ u'\) are in the same coset of \((\mathcal{U}, \circ)/(\mathcal{V}_{0,0}^{t^+}, \circ)\). Then \(\mathcal{V}_{0,0}^{t^+}(u \circ u') = \mathcal{V}_{0,0}^{t^+}(u \circ u')\).

**Theorem 20** Fix time \(t\). The set \(\mathcal{V}_{0,0}^{t^+}(u)\) with operation \(\circ_{0,0}\) forms a group \((\mathcal{V}_{0,0}(\mathcal{U}), \circ_{0,0})\).

**Proof.** First we show the operation \(\circ_{0,0}\) is associative. Let \(\mathcal{V}_{0,0}^{t^+}(u), \mathcal{V}_{0,0}^{t^-}(u), \mathcal{V}_{0,0}^{t^-}(u) \in \mathcal{V}_{0,0}^{t^+}(u)\). We need to show

\[
(\mathcal{V}_{0,0}^{t^+}(u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u)) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u) = \mathcal{V}_{0,0}^{t^+}(u) \circ_{0,0} (\mathcal{V}_{0,0}^{t^-}(u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u)).
\]

But using (35) we have

\[
(\mathcal{V}_{0,0}^{t^+}(u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u)) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u) = \mathcal{V}_{0,0}^{t^+}(u \circ u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u) = \mathcal{V}_{0,0}^{t^+}(u \circ u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u) = \mathcal{V}_{0,0}^{t^+}(u \circ u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u) = \mathcal{V}_{0,0}^{t^+}(u \circ u) \circ_{0,0} \mathcal{V}_{0,0}^{t^-}(u).
\]
and
\[ \nabla_{0,0}^t(u) \circ_{0,0} (\nabla_{0,0}^t(u) \circ_{0,0} \nabla_{0,0}^t(\bar{u})) = \nabla_{0,0}^t(u) \circ_{0,0} \nabla_{0,0}^t(u \circ \bar{u}) \]
\[ = \nabla_{0,0}^t(u \circ (u \circ \bar{u})). \]

Therefore the operation \( \circ_{0,0}^t \) is associative since the operation \( \circ \) in group \((\mathcal{U}, \circ)\) is associative.

Let \( u_1 \) be the identity of \((\mathcal{U}, \circ)\). We show \( \nabla_{0,0}^t(u_1) \) is the identity of \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\). Let \( u \in \mathcal{U} \) and \( \nabla_{0,0}^t(u) \in \nabla_{0,0}^t(\mathcal{U}) \). But using (35) we have
\[ \nabla_{0,0}^t(u_1) \circ_{0,0}^t \nabla_{0,0}^t(u) = \nabla_{0,0}^t(u_1 \circ u) \]
\[ = \nabla_{0,0}^t(u) \]
and
\[ \nabla_{0,0}^t(u) \circ_{0,0}^t \nabla_{0,0}^t(u_1) = \nabla_{0,0}^t(u \circ u_1) \]
\[ = \nabla_{0,0}^t(u). \]

Fix \( u \in \mathcal{U} \). Let \( \bar{u} \) be the inverse of \( u \) in \((\mathcal{U}, \circ)\). We show \( \nabla_{0,0}^t(\bar{u}) \) is the inverse of \( \nabla_{0,0}^t(u) \) in \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\). But using (35) we have
\[ \nabla_{0,0}^t(\bar{u}) \circ_{0,0}^t \nabla_{0,0}^t(u) = \nabla_{0,0}^t(\bar{u} \circ u) \]
\[ = \nabla_{0,0}^t(u_1) \]
and
\[ \nabla_{0,0}^t(u) \circ_{0,0}^t \nabla_{0,0}^t(\bar{u}) = \nabla_{0,0}^t(u \circ \bar{u}) \]
\[ = \nabla_{0,0}^t(u_1). \]

Together these results show \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\) is a group. ●

Corollary 21 For each \( t \in \mathbb{Z} \), the group \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\) determined by \((\mathcal{U}, \circ)\) is unique.

Proof. From the proof of Lemma 19 for any \( \nabla_{0,0}^t(u), \nabla_{0,0}^t(\bar{u}) \in \nabla_{0,0}^t(\mathcal{U}) \), the operation \( \nabla_{0,0}^t(u) \circ_{0,0}^t \nabla_{0,0}^t(\bar{u}) \) is unique. ●

We say the group \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\), for each \( t \in \mathbb{Z} \), is an upper elementary group of \((\mathcal{U}, \circ)\).

For each \( t \in \mathbb{Z} \), define a map \( \theta^t : \mathcal{U} \to \nabla_{0,0}^t(\mathcal{U}) \) by the assignment \( \theta^t : u \mapsto \nabla_{0,0}^t(u) \). Note that \( \theta^t \) is not a time projection since the generator labels in \( \nabla_{0,0}^t(\mathcal{U}) \) occur at different times. Therefore the map \( \theta^t \) includes memory of the generator labels in \( \mathcal{U} \). In this sense, it is different from the projection \( \chi^t \) of \( A \).

Theorem 22 For each \( t \in \mathbb{Z} \), there is a homomorphism from \((\mathcal{U}, \circ)\) to \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\) given by the map \( \theta^t : \mathcal{U} \to \nabla_{0,0}^t(\mathcal{U}) \) with assignment \( \theta^t : u \mapsto \nabla_{0,0}^t(u) \). Also there is an isomorphism \((\mathcal{U}, \circ)/\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t) \simeq (\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\).

Proof. The homomorphism follows immediately from the definition of operation \( \circ_{0,0}^t \) in (35). We have \((\mathcal{U}, \circ)/\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t) \simeq (\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\) since \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\) is the kernel of the homomorphism. ●

We now show that the group \((\mathcal{U}, \circ)\) and the collection of upper elementary groups \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\) contain redundant information in that the global operation in \((\mathcal{U}, \circ)\) can be recovered from the collection of upper elementary groups.

Lemma 23 The product operation \( \bar{u} \circ \bar{u} \) in \((\mathcal{U}, \circ)\) is uniquely determined by the evaluation of
\[ \nabla_{0,0}^t(\bar{u}) \circ_{0,0}^t \nabla_{0,0}^t(\bar{u}) \]
in group \((\nabla_{0,0}^t(\mathcal{U}), \circ_{0,0}^t)\), for \( t = +\infty \) to \( -\infty \).
Proof. The product $\hat{u} \circ \hat{u}$ is uniquely determined by the evaluation of $\nabla^t_{0,0}(\hat{u} \circ \hat{u})$ for $t = +\infty$ to $-\infty$. But by definition we have

$$\nabla^t_{0,0}(\hat{u} \circ \hat{u}) = \nabla^t_{0,0}(\hat{u}) \otimes \nabla^t_{0,0}(\hat{u})$$

for each $t \in \mathbb{Z}$.

Evidently the local information in the upper elementary group $(\nabla^t_{0,0}(\mathcal{U}), \otimes^t_{0,0})$, for each $t \in \mathbb{Z}$, is enough to determine the multiplication of the infinite sequences $u \in \mathcal{U}$. This means $(\mathcal{U}, \circ)$ is completely specified by local operations on local sets.

### 4.4 The component group of $(\mathcal{R}, *)$

Fix time $t$. Let $\nabla^t_{0,0}(\hat{r}), \nabla^t_{0,0}(\hat{r}) \in \nabla^t_{0,0}(\mathcal{R})$. Let $\beta : \hat{r} \mapsto \hat{u}$ and $\beta : \hat{r} \mapsto \hat{u}$. Define an operation $\otimes^t_{0,0}$ on set $\nabla^t_{0,0}(\mathcal{R})$ by

$$\nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}) = (\beta^t)^{-1}(\nabla^t_{0,0}(\hat{u} \circ \hat{u})).$$

(37)

Lemma 24 Fix time $t$. The operation $\otimes^t_{0,0}$ on set $\nabla^t_{0,0}(\mathcal{R})$ is well defined.

Proof. Choose any $\hat{r}, \hat{r} \in \mathcal{R}$ such that $\nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r})$ and $\nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r})$. Let $\beta : \hat{r} \mapsto \hat{u}$ and $\beta : \hat{r} \mapsto \hat{u}$. To show the operation is well defined, we need to show

$$\nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}),$$

or what is the same,

$$(\beta^t)^{-1}(\nabla^t_{0,0}(\hat{u} \circ \hat{u})) = (\beta^t)^{-1}(\nabla^t_{0,0}(\hat{u} \circ \hat{u})).$$

(38)

But if $\nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r})$ and $\nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r})$, then $\nabla^t_{0,0}(\hat{u}) = \nabla^t_{0,0}(\hat{u})$ and $\nabla^t_{0,0}(\hat{u}) = \nabla^t_{0,0}(\hat{u})$. But the operation $\otimes$ in group $(\nabla^t_{0,0}(\mathcal{U}), \otimes^t_{0,0})$ is well defined, so

$$\nabla^t_{0,0}(\hat{u}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{u}) = \nabla^t_{0,0}(\hat{u}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{u}).$$

This gives $\nabla^t_{0,0}(\hat{u} \circ \hat{u}) = \nabla^t_{0,0}(\hat{u} \circ \hat{u})$, which proves (38).

Fix time $t$. Let $\nabla^t_{0,0}(\hat{r}), \nabla^t_{0,0}(\hat{r}) \in \nabla^t_{0,0}(\mathcal{R})$. Define an operation $\otimes^t_{0,0}$ on set $\nabla^t_{0,0}(\mathcal{R})$ by

$$\nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}).$$

(39)

Lemma 25 Fix time $t$. The operation $\otimes^t_{0,0}$ on set $\nabla^t_{0,0}(\mathcal{R})$ is well defined.

Proof. Since $\beta^t : \nabla^t_{0,0}(\mathcal{R}) \mapsto \nabla^t_{0,0}(\mathcal{U})$ is a bijection, definition (37) is equivalent to definition (39).

Theorem 26 Fix time $t$. The set $\nabla^t_{0,0}(\mathcal{R})$ with operation $\otimes^t_{0,0}$ forms a group $(\nabla^t_{0,0}(\mathcal{R}), \otimes^t_{0,0})$.

Proof. The proof is exactly the same as for Theorem 20 with a change of notation.

Corollary 27 For each $t \in \mathbb{Z}$, the group $(\nabla^t_{0,0}(\mathcal{R}), \otimes^t_{0,0})$ determined by $(\mathcal{R}, *)$ is unique.

Proof. Let $\hat{r}, \hat{r} \in \mathcal{R}$. Fix time $t$. Let $\nabla^t_{0,0}(\hat{r}), \nabla^t_{0,0}(\hat{r}) \in \nabla^t_{0,0}(\mathcal{R})$. Choose any $\hat{r}, \hat{r} \in \mathcal{R}$ such that $\nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r})$ and $\nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r})$. From Lemma 25 since the operation $\otimes^t_{0,0}$ is well defined, we have

$$\nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}) = \nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r}),$$

This means for any $\nabla^t_{0,0}(\hat{r}), \nabla^t_{0,0}(\hat{r}) \in \nabla^t_{0,0}(\mathcal{R})$, the operation $\nabla^t_{0,0}(\hat{r}) \otimes^t_{0,0} \nabla^t_{0,0}(\hat{r})$ is unique. Therefore the group $(\nabla^t_{0,0}(\mathcal{R}), \otimes^t_{0,0})$ is unique.
We say the group \((\nabla_{0,0}^t, \circlearrowleft_{0,0}^t)\), for each \(t \in \mathbb{Z}\), is a \textit{component group} of \((R, \ast)\).

**Corollary 28** For each \(t \in \mathbb{Z}\), there is an isomorphism \((\nabla_{0,0}^t(R), \circlearrowleft_{0,0}^t) \simeq (\nabla_{0,0}^t(U), \circlearrowleft_{0,0}^t)\) under the bijection \(\beta^t: \nabla_{0,0}^t(R) \to \nabla_{0,0}^t(U)\) with assignment \(\beta^t: \nabla_{0,0}^t(r) \to \nabla_{0,0}^t(u)\) if \(\beta: R \to U\) under the bijection \(R \to U\).

**Proof.** Let \(\hat{r}, \hat{r} \in R\). Let \(\beta: \hat{r} \mapsto \hat{u}\) and \(\beta: \hat{r} \mapsto \hat{u}\) under the bijection \(R \to U\). Then \(\nabla_{0,0}^t(\hat{r}) \mapsto \nabla_{0,0}^t(\hat{u})\) and \(\nabla_{0,0}^t(\hat{r}) \mapsto \nabla_{0,0}^t(\hat{u})\). We have to show that
\[
\nabla_{0,0}^t(\hat{r}) \circlearrowleft_{0,0}^t \nabla_{0,0}^t(\hat{r}) \mapsto \nabla_{0,0}^t(\hat{u}) \circlearrowleft_{0,0}^t \nabla_{0,0}^t(\hat{u}).
\]
But this is the same as showing
\[
\nabla_{0,0}^t(\hat{r} \ast \hat{r}) \mapsto \nabla_{0,0}^t(\hat{u} \circ \hat{u}).
\]
But if \(\beta: \hat{r} \mapsto \hat{u}\) and \(\beta: \hat{r} \mapsto \hat{u}\), then \(\beta: \hat{r} \ast \hat{r} \mapsto \hat{u} \circ \hat{u}\), and so (40) holds.

For each \(t \in \mathbb{Z}\), define a map \(\phi^t: R \to \nabla_{0,0}^t(R)\) by the assignment \(\phi^t: r \mapsto \nabla_{0,0}^t(r)\). The map \(\phi^t\) is the projection of \((R, \ast)\) at time \(t\). Note that \(\phi^t(R) = \nabla_{0,0}^t(R)\).

**Theorem 29** There is a homomorphism from \((R, \ast)\) to \((\nabla_{0,0}^t(R), \circlearrowleft_{0,0}^t)\) given by the map \(\phi^t: R \to \nabla_{0,0}^t(R)\), for each \(t \in \mathbb{Z}\).

**Proof.** This follows immediately from the definition of operation \(\circlearrowleft_{0,0}^t\) in (20).

**Theorem 30** There is a homomorphism from \((\nabla_{0,0}^t(R), \circlearrowleft_{0,0}^t)\) to \(A^t\) given by the map \(\alpha^t\), for each \(t \in \mathbb{Z}\).

**Proof.** Let \(\alpha: \hat{r} \mapsto \hat{a}\) and \(\alpha: \hat{r} \mapsto \hat{a}\). Then \(\alpha: \hat{r} \ast \hat{r} \mapsto \hat{a} \ast \hat{a}\). From Corollary 12 we have \(\alpha^t: \nabla_{0,0}^t(\hat{r}) \mapsto \hat{a}^t\) and \(\alpha^t: \nabla_{0,0}^t(\hat{r}) \mapsto \hat{a}^t\). Let \(\hat{a}^t = \hat{a}^t\). Then from Corollary 12 we have \(\alpha^t: \nabla_{0,0}^t(\hat{r} \ast \hat{r}) \mapsto \hat{a}^t \ast \hat{a}^t\). But by definition \(\nabla_{0,0}^t(\hat{r} \ast \hat{r}) = \nabla_{0,0}^t(\hat{r}) \circlearrowleft_{0,0}^t \nabla_{0,0}^t(\hat{r})\). Then \(\alpha^t: \nabla_{0,0}^t(\hat{r}) \circlearrowleft_{0,0}^t \nabla_{0,0}^t(\hat{r}) \mapsto \hat{a}^t \ast \hat{a}^t\), where \(\alpha^t: \nabla_{0,0}^t(\hat{r}) \mapsto \hat{a}^t\) and \(\alpha^t: \nabla_{0,0}^t(\hat{r}) \mapsto \hat{a}^t\).

The following lemma is proven in the same way as Lemma 23.

**Lemma 31** The product operation \(\hat{r} \ast \hat{r}\) in \((R, \ast)\) is uniquely determined by the evaluation of
\[
\nabla_{0,0}^t(\hat{r}) \circlearrowleft_{0,0}^t \nabla_{0,0}^t(\hat{r})
\]
for \(t = +\infty\) to \(-\infty\).

We have found a component group \((\nabla_{0,0}^t(R), \circlearrowleft_{0,0}^t)\) of \((R, \ast)\) from an elementary group of \((U, \circlearrowleft)\). Evidently the local information in the component group, for each \(t \in \mathbb{Z}\), is enough to determine the multiplication of the infinite sequences \(r \in R\). The component group gives the local permutation of the generators in \(R\) when sequences in \(A\) with these generators are multiplied. The component group \((\nabla_{0,0}^t(R), \circlearrowleft_{0,0}^t)\) resembles the branch group of a group trellis discussed in [3], but is not the same. The branch group is obtained by a brute force determination of the state group of the system [3], but the group \((\nabla_{0,0}^t(R), \circlearrowleft_{0,0}^t)\) is obtained from permutations of the local generators active at time \(t\) when sequences in \(A\) are multiplied.

Collecting Theorems 11 and 17 we have the following result.

**Theorem 32** There is an isomorphism from \(A\) to \((U, \circlearrowleft)\) given by the composition map \(\xi \defeq \beta \ast^{-1}: A \to U\), where isomorphism \(A \simeq (R, \ast)\) is given by the bijection \(\alpha: R \to A\) in Theorem 14 and isomorphism \((R, \ast) \simeq (U, \circlearrowleft)\) is given by the bijection \(\beta: R \to U\) in Theorem 17.
Then we can summarize the results of this paper so far by the chain

\[ A \xrightarrow{\alpha} (\mathcal{R}, \ast) \xrightarrow{\beta} (\mathcal{U}, \circ) \]  

(42)

where \( A \simeq (\mathcal{R}, \ast) \text{ and } (\mathcal{R}, \ast) \simeq (\mathcal{U}, \circ) \) under bijections \( \alpha : \mathcal{R} \to A \) and \( \beta : \mathcal{R} \to \mathcal{U} \), respectively.

Note that \((\mathcal{R}, \ast)\) is not a group system but it is easily made into one. Since \((\mathcal{R}, \ast)\) has a component group, we can make group \((\mathcal{R}, \ast)\) into a group system in a natural way. All the entries in \( \nabla^t_{0,0}(\mathcal{R}) \) occur at time \( t \). Therefore it is natural to define a group system by the assignment

\[ r \mapsto \ldots, \nabla^t_{0,0}(r), \nabla^{t-1}_{0,0}(r), \ldots \]  

(43)

It is easy to see the resulting set of sequences with a componentwise group addition defined by \((\nabla^t_{0,0}(\mathcal{R}), \circ^t_{0,0})\) is a group system.

In a similar way, note that \((\mathcal{U}, \circ)\) is a group but it is not a group system. However it has elementary groups \((\nabla^t_{0,0}(\mathcal{U}), \circ^t_{0,0})\) which overlap in time. If we separate the elementary groups in time using the map

\[ u \mapsto \ldots, \nabla^t_{0,0}(u), \nabla^{t-1}_{0,0}(u), \ldots \]  

(44)

then the resulting set of sequences is a group system with component groups \( (\nabla^t_{0,0}(\mathcal{U}), \circ^t_{0,0}) \).

The assignment (43) is a 1-1 correspondence between the set of \( r \in \mathcal{R} \) and the set of sequences formed by \( r \). Similarly, the assignment (44) is a 1-1 correspondence between the set of \( u \in \mathcal{U} \) and the set of sequences formed by \( u \). Consider the bijection \( \beta : \mathcal{R} \to \mathcal{U} \). If \( \beta : r \mapsto u \), under the bijection \( \beta^t : \nabla^t_{0,0}(\mathcal{R}) \to \nabla^t_{0,0}(\mathcal{U}) \) for each \( t \in \mathbb{Z} \), a sequence of triangles \( \ldots, \nabla^t_{0,0}(r), \nabla^{t-1}_{0,0}(r), \ldots \) of \( r \) in (43) becomes a sequence of triangles \( \ldots, \nabla^t_{0,0}(u), \nabla^{t-1}_{0,0}(u), \ldots \), which then overlap to form \( u \) in (44). We can also go in reverse. This gives the following.

**Corollary 33** Let \( \beta : \mathcal{R} \to \mathcal{U} \) with assignment \( \beta : r \mapsto u \). Given \( u \in \mathcal{U} \), we can find \( r \in \mathcal{R} \) such that \( \beta : r \mapsto u \) by first finding the sequence of triangles in (43), and then finding \( (\beta^t)^{-1} : \nabla^t_{0,0}(u) \mapsto \nabla^t_{0,0}(r) \) for each \( t \in \mathbb{Z} \) to find the sequence of triangles

\( \ldots, \nabla^t_{0,0}(r), \nabla^{t-1}_{0,0}(r), \ldots \)  

(45)

which then finds \( r \) in (43).

**4.5 Recovery of group system \( A \) from generator group \((\mathcal{U}, \circ)\)**

We show in this subsection we can recover \( A \) from its generator group \((\mathcal{U}, \circ)\).

We prove the following simple extension of the first homomorphism theorem to show that it is possible to use a homomorphism to construct a group system from any input group. We can think of the following construction theorem as a *first homomorphism theorem for group systems*. In Subsection 6.3, we use the first homomorphism theorem for group systems to find all \( \ell \)-controllable complete group systems \( A \) up to isomorphism.

**Theorem 34** Consider any group \( G \). Suppose there is a homomorphism \( p^t : G \to G^t \) from \( G \) to a group \( G^t \) for each \( t \in \mathbb{Z} \). In general group \( G^t \) may be different for each \( t \in \mathbb{Z} \). Define the direct product group \((G_{1t}, +)\) by \( G_{1t}, +) \) \( \equiv \ldots \times G^t \times G^{t+1} \times \cdots \). There is a homomorphism \( p : G \to G_{1t} \), from \( G \) to the direct product group \((G_{1t}, +)\), defined by

\[ p(g) \equiv \ldots, p^t(g), p^{t+1}(g), \ldots \]  

(46)
Define $g_{\underline{u}} \overset{\text{def}}{=} \ldots, p^t(g), p^{t+1}(g), \ldots$. Then $p : G \to G_{\underline{u}}$ with assignment $p : g \mapsto g_{\underline{u}}$. Then

\[ G/G_K \simeq \text{im} \, p, \]

where group im $p$ is the image of the homomorphism $p$, and where $G_K$ is the kernel of the homomorphism $p$. We have im $p$ is a group system defined by a componentwise operation in $G^t$ for each $t \in \mathbb{Z}$. Lastly we have $G \simeq \text{im} \, p$ if and only if the kernel $G_K$ of the homomorphism $p$ is the identity.

**Proof.** Since there is a homomorphism $p^t : G \to G'$ from $G$ to a group $G'$ for each $t \in \mathbb{Z}$, we must have $p^t(g\circ g) = p^t(g)p^t(g)$ for each $t \in \mathbb{Z}$. Then

\[ p(g\circ g) = \ldots, p^t(g\circ g), p^{t+1}(g\circ g), \ldots = \ldots, p^t(g), p^{t+1}(g), p^{t+1}(g), \ldots = p(g) + p(g). \]

Then there is a homomorphism $p$ from $G$ to the direct product group $(G_{\underline{u}}, \circ)$. We have $G/G_K \simeq \text{im} \, p$ from the first homomorphism theorem. We have im $p$ is a group system since the global operation $\circ$ in im $p$ is defined by a componentwise operation in $G^t$ for each $t \in \mathbb{Z}$. \hfill ●

We refer to (16) by the notation $p = \ldots, p^t, p^{t+1}, \ldots$. We now ask whether we can reverse the chain in (12), i.e., starting with a generator group $(\mathcal{U}, \circ)$, can we recover $A$. We now show that we can recover $A$ directly from group $(\mathcal{U}, \circ)$ using the first homomorphism theorem for group systems.

**Theorem 35** The first homomorphism theorem for group systems constructs a group system im $f_\mathcal{U}$ with component group $A^t$ from an $(\ell + 1)$-depth generator group $(\mathcal{U}, \circ)$ using a homomorphism $f_\mathcal{U}$.

**Proof.** The composition $\alpha^t \circ (\beta^t)^{-1} \circ \theta^t$ is a map $\mathcal{U} \to \nabla^t_{0,0}(\mathcal{U}) \to \nabla^t_{0,0}(\mathcal{R}) \to A^t$ given by the assignment $\alpha^t \circ (\beta^t)^{-1} \circ \theta^t : u \mapsto \nabla^t_{0,0}(u) \mapsto \nabla^t_{0,0}(r) \mapsto a^t$. From Theorem 22 we know there is a homomorphism from $(\mathcal{U}, \circ)$ to $(\nabla^t_{0,0}(\mathcal{U}), \circ_{0,0})$ given by the map $\theta^t$; from Corollary 25 we know there is an isomorphism from $(\nabla^t_{0,0}(\mathcal{U}), \circ_{0,0})$ to $(\nabla^t_{0,0}(\mathcal{R}), \circ_{0,0})$ given by the map $(\beta^t)^{-1}$; and from Theorem 40 we know there is a homomorphism from $(\nabla^t_{0,0}(\mathcal{R}), \circ_{0,0})$ to $A^t$ given by the map $\alpha^t$. Let $f_\mathcal{U}^t \overset{\text{def}}{=} \alpha^t \circ (\beta^t)^{-1} \circ \theta^t$. Then $f_\mathcal{U}^t : \mathcal{U} \to A^t$ is a homomorphism from $(\mathcal{U}, \circ)$ to $A^t$ for each $t \in \mathbb{Z}$. Consider the Cartesian product $A_{\underline{u}} \overset{\text{def}}{=} \ldots \times A^t \times A^{t+1} \times \ldots$ (note here $A^t$ is interpreted as a set). Define the direct product group $(A_{\underline{u}}, \circ)$ by $(A_{\underline{u}}, \circ) \overset{\text{def}}{=} \ldots \times A^t \times A^{t+1} \times \ldots$. Then from Theorem 33 using $(\mathcal{U}, \circ)$ for information group $G$ and alphabet group $A^t$ for $G^t$, $t \in \mathbb{Z}$, there is a homomorphism $f_\mathcal{U} : \mathcal{U} \to A_{\underline{u}}$, from $(\mathcal{U}, \circ)$ to the direct product group $(A_{\underline{u}}, \circ)$, defined by

\[ f_\mathcal{U}(u) \overset{\text{def}}{=} \ldots, f_\mathcal{U}^t(u), f_\mathcal{U}^{t+1}(u), \ldots \]

Define

\[ a_{\underline{u}} \overset{\text{def}}{=} \ldots, f_\mathcal{U}^t(u), f_\mathcal{U}^{t+1}(u), \ldots = \ldots, a^t, a^{t+1}, \ldots \]

Then $f_\mathcal{U} : \mathcal{U} \to A_{\underline{u}}$ with assignment $f_\mathcal{U} : u \mapsto a_{\underline{u}}$. We can think of $a_{\underline{u}}$ as a “sliding block” mapping of $u$. Applying the first homomorphism theorem for groups, we have

\[ (\mathcal{U}, \circ)/(\mathcal{U}, \circ)_K \simeq \text{im} \, f_\mathcal{U}, \]

where group im $f_\mathcal{U}$ is the image of the homomorphism $f_\mathcal{U}$, and where $(\mathcal{U}, \circ)_K$ is the kernel of the homomorphism $f_\mathcal{U}$. Since group im $f_\mathcal{U}$ is a subgroup of the direct product group $(A_{\underline{u}}, \circ)$, then im $f_\mathcal{U}$ is a group system where global operation $\circ$ defines the componentwise operation in group $A^t$ for each $t \in \mathbb{Z}$. \hfill ●
Theorem 36 The homomorphism $f_u$ is a bijection $f_u : U \to A$. Then $\text{im} f_u = A$. Therefore the group system $\text{im} f_u$ is $\ell$-controllable and complete.

Proof. Fix $u \in U$. Let

$$r_{\text{tot}} = \ldots, \nabla_{0,0}(r), \nabla_{0,0}^{-1}(r), \ldots,$$

where $\nabla_{0,0}(r)$ is given by the assignment $(\beta)^{-1} \cdot \theta^t : u \mapsto \nabla_{0,0}(u) \mapsto \nabla_{0,0}(r)$ for each $t \in \mathbb{Z}$ of Theorem 35. From Corollary 33 the assignment $(\beta)^{-1} \cdot \theta^t : u \mapsto \nabla_{0,0}(u) \mapsto \nabla_{0,0}(r)$ for each $t \in \mathbb{Z}$ is the same as the assignment $\beta^{-1} : u \mapsto r_{\text{tot}}$ of bijection $\beta^{-1} : U \to \mathbb{R}$. Then $r_{\text{tot}} \in \mathbb{R}$. Let

$$a_{\text{tot}}' = \ldots, a', a_{\text{tot}}^t, \ldots,$$

where $a'$ is given by the assignment $\alpha^t : \nabla_{0,0}(r) \mapsto a'$ for each $t \in \mathbb{Z}$ of Theorem 35. From Corollary 12 the assignment $\alpha^t : \nabla_{0,0}(r) \mapsto a'$ for each $t \in \mathbb{Z}$ is the same as the assignment $\alpha : r_{\text{tot}} \mapsto a_{\text{tot}}'$ of bijection $\alpha : \mathbb{R} \to A$. Then $a_{\text{tot}}' \in A$. Then the assignment $\alpha' \cdot (\beta)^{-1} \cdot \theta^t : u \mapsto \nabla_{0,0}(u) \mapsto \nabla_{0,0}(r) \mapsto a'$ for each $t \in \mathbb{Z}$ of Theorem 35 is the same as the assignment $\alpha \cdot \beta^{-1} : U \to \mathbb{R} \to A$. But the map $\alpha' \cdot (\beta)^{-1} \cdot \theta^t$ is the same as the map $f_u$. Then the bijection $\alpha \cdot \beta^{-1} : U \to A$ is the same as map $f_u$, since $f_u = \ldots, f^t, f_{u}^{t+1}, \ldots$. Then $f_u(u) = a_{\text{tot}}'$, and $f_u$ is a bijection $f_u : U \to A$. Then $\text{im} f_u = A$.

Since $\text{im} f_u = A$, the group system $\text{im} f_u$ is $\ell$-controllable and complete. •

With (42), these results give the chain shown in (47). As shown we can recover $A$ directly from group $(U, \circ)$ using the first homomorphism theorem for group systems.

\[
\begin{array}{cccc}
\downarrow & \leftarrow & \leftarrow & \text{im} f_u \\
\downarrow & \leftarrow & \leftarrow & \uparrow \circ \ u = \ldots, f^t, f_{u}^{t+1}, \ldots \\
A & \cong & (R, \ast) & \cong (U, \circ)
\end{array}
\]  

(47)

Corollary 37 We have $(U, \circ) \cong \text{im} f_u = A$.

Proof. The kernel of the homomorphism $f_u$ is the identity so $(U, \circ) \cong \text{im} f_u = A$. •
5. PROPERTIES OF THE GENERATOR GROUP

5.1 The nested elementary groups of \((\mathcal{U}, \circ)\)

For \(0 \leq k \leq \ell\) and \(t \in \mathbb{Z}\), and for each \(u \in \mathcal{U}\), we let \(\triangledown_{0,k}^t(u)\) be the triangle of generator labels in \(u\) in \([32]\) with lower vertex \(r_{0,k}^t\) and upper vertices \(r_{\ell-k}^t\) and \(r_{\ell}^t\). Note that for \(0 \leq k \leq \ell\), \(\triangledown_{0,k}^t(u)\) is a subtriangle in \(\triangledown_{0,0}^t(u)\) in \([34]\), and for \(k = 0\), \(\triangledown_{0,k}^t(u)\) is the same as \(\triangledown_{0,0}^t(u)\) in \([34]\). For \(0 \leq k \leq \ell\) and \(t \in \mathbb{Z}\), let \(\triangledown_{0,k}^t(\mathcal{U})\) be the set of all possible triangles \(\triangledown_{0,k}^t(u), \triangledown_{0,k}^t(\mathcal{U}) \triangleq \{ \triangledown_{0,k}^t(u) : u \in \mathcal{U}\}\). Note that for \(k = 0\), \(\triangledown_{0,k}^t(\mathcal{U})\) is just the set \(\triangledown_{0,0}^t(\mathcal{U})\) previously defined.

We now show there are smaller elementary groups on sets \(\triangledown_{0,k}^t(\mathcal{U})\) nested in \((\triangledown_{0,0}^t(\mathcal{U}), \triangledown_{0,0}^t)\), for \(0 \leq k \leq \ell\), for each \(t \in \mathbb{Z}\). We can use the same approach as for defining \((\triangledown_{0,t}^t(\mathcal{U}), \triangledown_{k,0}^t)\). We restate this approach now. Recall that we have defined \((\mathcal{U}^{t+}, \circ)\) and \((\mathcal{U}^{t-}, \circ)\) for each \(t \in \mathbb{Z}\), and these are normal subgroups of \((\mathcal{U}, \circ)\). Fix \(t \in \mathbb{Z}\). Fix \(k\) such that \(0 \leq k \leq \ell\). Note that \(\mathcal{U}(t+1)\) and \(\mathcal{U}(t+k-1)\) cover \(\mathcal{U}\) except for \(\triangledown_{0,k}^t(\mathcal{U})\). In other words the union of set \(\mathcal{U}(t+1)\) and set \(\mathcal{U}(t+k-1)\) is the set of all \(u \in \mathcal{U}\) which are trivial in \(\triangledown_{0,k}^t(u)\); call this subset of \(\mathcal{U}\) as \(\triangledown_{0,k}^t \mathcal{K}\). Then the product of \((\mathcal{U}^{t+}, \circ)\) and \((\mathcal{U}^{t+k-1}, \circ)\) is a normal subgroup \((\triangledown_{0,k}^t \mathcal{K}, \circ)\) of \((\mathcal{U}, \circ)\). Then there is a quotient group \((\mathcal{U}, \circ) / (\triangledown_{0,k}^t \mathcal{K}, \circ)\). The projection of the cosets in \((\mathcal{U}, \circ) / (\triangledown_{0,k}^t \mathcal{K}, \circ)\) on set \(\triangledown_{0,k}^t(\mathcal{U})\) is \(\triangledown_{0,k}^t(\mathcal{U})\). We use this to define a group on \(\triangledown_{0,k}^t(\mathcal{U})\).

Fix time \(t\). Fix \(k\) such that \(0 \leq k \leq \ell\). Let \(\triangledown_{0,k}^t(\mathcal{U}), \triangledown_{0,k}^t(\mathcal{U}) \in \triangledown_{0,k}^t(\mathcal{U})\).

Define an operation \(\circ_{0,k}^t\) on \(\triangledown_{0,k}^t(\mathcal{U})\) by

\[
\triangledown_{0,k}^t(u) \circ_{0,k}^t \triangledown_{0,k}^t(\mathcal{U}) \triangleq \triangledown_{0,k}^t(\mathcal{U} \circ \mathcal{U}).
\]

**Lemma 38** Fix time \(t\). Fix \(k\) such that \(0 \leq k \leq \ell\). The operation \(\circ_{0,k}^t\) on \(\triangledown_{0,k}^t(\mathcal{U})\) is well defined.

**Proof.** The proof is the same as Lemma 19.

**Theorem 39** Fix time \(t\). Fix \(k\) such that \(0 \leq k \leq \ell\). The set \(\triangledown_{0,k}^t(\mathcal{U})\) with operation \(\circ_{0,k}^t\) forms a group \((\triangledown_{0,k}^t(\mathcal{U}), \circ_{0,k}^t)\).

**Proof.** The proof is the same as Theorem 20.

**Corollary 40** For each \(t \in \mathbb{Z}\), for \(0 \leq k \leq \ell\), the group \((\triangledown_{0,k}^t(\mathcal{U}), \circ_{0,k}^t)\) determined by \((\mathcal{U}, \circ)\) is unique.

**Proof.** The proof is the same as Corollary 21.

We call the groups \((\triangledown_{0,k}^t(\mathcal{U}), \circ_{0,k}^t)\), for each \(t \in \mathbb{Z}\), for \(0 \leq k \leq \ell\), the upper elementary groups of \((\mathcal{U}, \circ)\). The upper elementary group \((\triangledown_{0,t}^t(\mathcal{U}), \circ_{0,t}^t)\), for each \(t \in \mathbb{Z}\), is the component group of \((\mathcal{U}, \circ)\).

The operation in elementary group \((\triangledown_{0,k}^t(\mathcal{U}), \circ_{0,k}^t)\) just depends on the elements in set \(\triangledown_{0,k}^t(\mathcal{U})\), and is otherwise independent of the remaining portion of \(\mathcal{U}\). Then we can think of the collection of elementary groups \((\triangledown_{0,k}^t(\mathcal{U}), \circ_{0,k}^t)\), for \(0 \leq k \leq \ell\), for \(t \in \mathbb{Z}\), as a decomposition of \((\mathcal{U}, \circ)\) into local groups since each group \((\triangledown_{0,k}^t(\mathcal{U}), \circ_{0,k}^t)\) is wholly defined by a local set \(\triangledown_{0,k}^t(\mathcal{U})\).

Essentially \((\mathcal{U}, \circ)\) is defined to be the permutation of the generator labels in \(\mathcal{U}\) when two sequences in \(A\) with these generators are multiplied. The group \((\triangledown_{0,t}^t(\mathcal{U}), \circ_{0,t}^t)\) shows that when two sequences in \(A\) are multiplied, the longest generator \(g_{\ell}^t\) of the result only depends on the longest generators at time \(t\) of the two multiplicands. Next, the group \((\triangledown_{0,\ell-1}^t(\mathcal{U}), \circ_{0,\ell-1}^t)\) contains generator labels of two longest generators and one next longest generator. This group shows that when two sequences in \(A\) are multiplied, the next longest generator...
\( g_{t-1} \) of the result only depends on the longest generators at times \( t \) and \( t-1 \) and next longest generators at time \( t \) of the two multiplicands. And so on. This approach can be used to calculate \( \hat{u} \circ \hat{u} \) from all the upper elementary groups.

We now give some homomorphism properties of upper elementary groups. For each \( t \in \mathbb{Z}, \) for \( 0 \leq k \leq \ell, \) define a map \( \theta_t^{u,k} : U \to \nabla_{0,k}^t(U) \) by the assignment \( \theta_t^{u,k} : u \mapsto \nabla_{0,k}^t(u). \) From the definition of operation \( \odot_{0,k}^t \) in upper elementary group \( (\nabla_{0,k}^t(U), \odot_{0,k}^t) \), the following homomorphism is clear.

**Theorem 41** Fix any \( t \in \mathbb{Z}. \) Fix \( k \) such that \( 0 \leq k \leq \ell. \) There is a homomorphism from \( (U, \circ) \) to any upper elementary group \( (\nabla_{0,k}^t(U), \odot_{0,k}^t) \) given by the map \( \theta_t^{u,k} : U \to \nabla_{0,k}^t(U) \) with assignment \( \theta_t^{u,k} : u \mapsto \nabla_{0,k}^t(u). \) Also there is an isomorphism \( \theta_t^{u,k}(U)/(\nabla_{0,k}^t K, \circ) \simeq (\nabla_{0,k}^t(U), \odot_{0,k}^t). \)

**Proof.** The proof is the same as Theorem 22.

Next we show there are homomorphisms among the upper elementary groups. Fix any \( t \in \mathbb{Z}. \) Fix \( k \) such that \( 0 \leq k \leq \ell. \) Consider set \( \nabla_{0,k}^t(U). \) For any \( u \in U, \) \( \nabla_{0,k}^{t-j}(u) \) is a subtriangle of \( \nabla_{0,k}^t(u) \) for \( 0 \leq j \leq k. \) Then \( \nabla_{0,k}^{t-j}(U) \) is nested inside set \( \nabla_{0,k}^t(U) \) for \( 0 \leq j \leq k. \) Define the projection map \( \theta_{k,j}^{u} : \nabla_{0,k}^t(U) \to \nabla_{0,j}^{t-j}(U) \) by the assignment \( \theta_{k,j}^{u} : \nabla_{0,k}^t(u) \mapsto \nabla_{0,j}^{t-j}(u). \)

**Lemma 42** The projection map \( \theta_{k,j}^{u} : \nabla_{0,k}^t(U) \to \nabla_{0,j}^{t-j}(U) \) is well defined.

**Proof.** Consider any other \( \hat{u} \in U \) such that \( \nabla_{0,k}^t(u) = \nabla_{0,k}^t(\hat{u}). \) Since \( u \) and \( \hat{u} \) agree on \( \nabla_{0,k}^t(U) \), then they must agree on \( \nabla_{0,j}^{t-j}(U). \) Then we must have \( \nabla_{0,j}^{t-j}(u) = \nabla_{0,j}^{t-j}(\hat{u}), \) and so \( \theta_{k,j}^{u} \) is well defined.

**Theorem 43** Fix any \( t \in \mathbb{Z}. \) Fix \( k \) such that \( 0 \leq k \leq \ell. \) Consider group \( (\nabla_{0,k}^t(U), \odot_{0,k}^t). \) For any \( 0 \leq j \leq k, \) there is a homomorphism from group \( (\nabla_{0,k}^t(U), \odot_{0,k}^t) \) to group \( (\nabla_{0,j}^{t-j}(U), \odot_{0,j}^{t-j}) \) given by the projection map \( \theta_{k,j}^{u} : \nabla_{0,k}^t(U) \to \nabla_{0,j}^{t-j}(U) \) with assignment \( \theta_{k,j}^{u} : \nabla_{0,k}^t(u) \mapsto \nabla_{0,j}^{t-j}(u). \)

**Proof.** Let \( \nabla_{0,k}^t(\hat{u}), \nabla_{0,k}^t(\hat{u}) \in \nabla_{0,k}^t(U). \) Consider the projections \( \theta_{k,j}^{u} : \nabla_{0,k}^t(\hat{u}) \mapsto \nabla_{0,j}^{t-j}(\hat{u}) \) and \( \theta_{k,j}^{u} : \nabla_{0,k}^t(\hat{u}) \mapsto \nabla_{0,j}^{t-j}(\hat{u}) \). We need to show

\[
\theta_{k,j}^{u}(\nabla_{0,k}^t(\hat{u}) \odot_{0,k}^t \nabla_{0,k}^t(\hat{u})) = \theta_{k,j}^{u}(\nabla_{0,k}^t(\hat{u} \circ \hat{u}))
\]

But

\[
\begin{align*}
\theta_{k,j}^{u}(\nabla_{0,k}^t(\hat{u}) \odot_{0,k}^t \nabla_{0,k}^t(\hat{u})) &= \theta_{k,j}^{u}(\nabla_{0,k}^t(\hat{u} \circ \hat{u})) \\
&= \nabla_{0,j}^{t-j}(\hat{u} \circ \hat{u}) \\
&= \nabla_{0,j}^{t-j}(\hat{u}) \odot_{0,j}^{t-j} \nabla_{0,j}^{t-j}(\hat{u}) \\
&= \theta_{k,j}^{u}(\nabla_{0,k}^t(\hat{u})) \odot_{0,j}^{t-j} \theta_{k,j}^{u}(\nabla_{0,k}^t(\hat{u})).
\end{align*}
\]

We say the homomorphism in Theorem 43 from a larger upper elementary group to a smaller one nested inside is a nested homomorphism property. This gives the following.

**Theorem 44** The generator group \( (U, \circ) \) of any \( \ell \)-controllable complete group system \( A \) is a collection of upper elementary groups \( (\nabla_{0,k}^t(U), \odot_{0,k}^t) \) of \( (U, \circ) \) for each \( t \in \mathbb{Z}, \) for \( 0 \leq k \leq \ell, \) which have a nested homomorphism property.

Because the upper elementary groups of \( (U, \circ) \) are indexed by \( 0 \leq k \leq \ell, \) we say generator group \( (U, \circ) \) is an \( (\ell + 1) \)-depth generator group.
Theorem 45 Any \( (\ell + 1) \)-depth generator group \((\mathcal{U}, \circ)\) determines a unique collection of upper elementary groups.

Proof. This follows from Corollary 411.

There are upper elementary groups \((\triangledown^t_{0,k}(\mathcal{R}), \oplus^t_{0,k})\) with a nested homomorphism property in \((\mathcal{R}, \ast)\) as well. These can be found by the same approach as used in Subsection 4.4 to find \((\triangledown^t_{0,0}(\mathcal{R}), \oplus^t_{0,0})\) from \((\triangledown^t_{0,0}(\mathcal{U}), \circ^t_{0,0})\). The nested upper elementary groups \((\triangledown^t_{0,k}(\mathcal{R}), \oplus^t_{0,k})\) in \((\mathcal{R}, \ast)\) are isomorphic to the nested upper elementary groups \((\triangledown^t_{0,0}(\mathcal{U}), \circ^t_{0,0})\) in \((\mathcal{U}, \circ)\); more details can be found in [22].

The matrix \(\triangledown^t_{0,0}(\mathcal{U})\) has an upper triangular form. We now define a matrix with a lower triangular form. Given \(u \in \mathcal{R}\), for each \(t \in \mathbb{Z}\), we define \(\Delta^t_{0,k}(u)\) to be the triangle in \(u\) with lower vertices \(g_0^{t+k}\) and \(g_0^t\) and upper vertex \(g_k^t\), as shown in (49).

\[
\begin{array}{ccccccc}
g_t^t & g_{t+1}^t & \cdots & \cdots & g_k^t \\
g_t^{t+1} & g_{t+1}^{t+1} & \cdots & \cdots & g_k^{t+1} \\
& \vdots & \ddots & \ddots & \ddots \\
g_k^{t+k} & \cdots & g_k^t & g_k^{t+1} \\
\end{array}
\]

(49)

In general, for \(0 \leq k \leq \ell\) and \(t \in \mathbb{Z}\), and for each \(u \in \mathcal{U}\), we let \(\Delta_{0,k}(u)\) be the triangle of generator labels in \(u\) in (52) with lower vertices \(g_0^{t+k}\) and \(g_0^t\) and upper vertex \(g_k^t\). Note that for \(0 \leq k \leq \ell\), \(\Delta^t_{0,k}(u)\) is a subtriangle in \(\Delta_{0,t}(u)\) in (49), and for \(k = 0\), \(\Delta^t_{0,k}(u)\) is the same as \(\Delta_{0,t}(u)\) in (49). For \(0 \leq k \leq \ell\) and \(t \in \mathbb{Z}\), let \(\Delta^t_{0,k}(U)\) be the set of all possible triangles \(\Delta^t_{0,k}(u)\), \(\Delta_{0,k}(U) \defeq \{ \Delta^t_{0,k}(u) : u \in U \}\).

Fix \(t \in \mathbb{Z}\) and consider \(A^{[t,t+k]}\); this is a normal subgroup of \(A^3\). In \(\mathcal{R}\), this is the set of all \(r \in \mathcal{R}\) with trivial generators for all \(t' \in \mathbb{Z}\) except for generators \(g_t^{[m,n]}\) where \(t \leq m \leq n \leq t + k\). Call this subset of \(\mathcal{R}\) as \(R^t_{0,k}\). Since \(A^{[t,t+k]}\) is a normal subgroup of \(A\), then \(R^t_{0,k}\) is a normal subgroup of \((\mathcal{R}, \ast)\), under the bijection \(\alpha^{-1}\) of the isomorphism from \(A\) to \((\mathcal{R}, \ast)\). In \(U\), this is the set of all \(u\) with trivial entries except for those in \(\Delta^t_{0,k}(u)\); define this subset of \(U\) to be \(\Delta^t_{0,k}(U)\). Since \(R^t_{0,k}\) is a normal subgroup of \((\mathcal{R}, \ast)\), then \(\Delta^t_{0,k}(U, \circ)\) is a normal subgroup of \((U, \circ)\), under the bijection \(\beta\) of the isomorphism from \((\mathcal{R}, \ast)\) to \((U, \circ)\). This shows that \(\Delta^t_{0,k}(U, \circ)\) is the analog in \((U, \circ)\) of \(A^{[t,t+k]}\) in \(A\). We say the group \(\Delta^t_{0,k}(U, \circ)\), for each \(t \in \mathbb{Z}\), is a lower elementary group of \((U, \circ)\).

Note that a lower elementary group \(\Delta^t_{0,k}(U, \circ)\) is defined on a subset of \(U\), but an upper elementary group \((\triangledown^t_{0,0}(U), \circ^t_{0,0})\) is defined on a local set \(\triangledown^t_{0,0}(U)\) of \(U\).

The groups \((\Delta^t_{0,k+1}(U, \circ))\) and \((\Delta^t_{0,k}(U, \circ))\) are normal subgroups of \((U, \circ)\). Then the product \(\Delta^t_{0,k+1}(U, \circ)\) is a normal subgroup of \((U, \circ)\). Then the product is a normal subgroup of \((\Delta^t_{0,k+1}(U, \circ))\) of \((U, \circ)\). The quotient group \(\Delta^t_{0,k+1}(U, \circ) / \Delta^t_{0,k}(U, \circ)\) is the analog of \(A^{[t,t+k]}\) in \(A\). Unlike the upper elementary groups, the lower elementary groups are defined on \(U\), and there is no nested homomorphism property from \(\Delta^t_{0,k+1}(U, \circ)\) to \(\Delta^t_{0,k+1}(U, \circ)\) and \(\Delta^t_{0,k}(U, \circ)\).

5.2 Set of elementary groups.

The tensor \(u\) in (52) is indexed by ordered pairs of the form \((k, t)\), where subscript \(k\) satisfies \(0 \leq k \leq \ell\) and superscript \(t \in \mathbb{Z}\). Then tensor \(u\) gives an
index tensor $n$ of ordered pairs $(k, t)$, as shown in (50).

\[
\begin{array}{cccccc}
(\ell, t) & (\ell, t-1) & \cdots & (\ell, t-j) & \cdots & (\ell, t-\ell) \\
(\ell-1, t) & (\ell-1, t-1) & \cdots & (\ell-1, t-j) & \cdots & (\ell-1, t-\ell) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(k, t) & (k, t-1) & \cdots & (k, t-j) & \cdots & (k, t-\ell) \\
(k-1, t) & (k-1, t-1) & \cdots & (k-1, t-j) & \cdots & (k-1, t-\ell) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(j, t) & (j, t-1) & \cdots & (j, t-j) & \cdots & (j, t-\ell) \\
(j-1, t) & (j-1, t-1) & \cdots & (j-1, t-j) & \cdots & (j-1, t-\ell) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(1, t) & (1, t-1) & \cdots & (1, t-j) & \cdots & (1, t-\ell) \\
(0, t) & (0, t-1) & \cdots & (0, t-j) & \cdots & (0, t-\ell) \\
\end{array}
\]

We use a similar notation to describe triangular subsets in index tensor $n$ that was previously used for tensor $r \in R$ and $u \in U$. For $0 \leq k \leq \ell$ and $t \in \mathbb{Z}$, let upper index triangle $\nabla_{\ell, k}^r(n)$ be the ordered pairs $(k, t)$ in (50) specified by the triangle with lower vertex $(k, t)$ and upper vertices $(\ell, t)$ and $(\ell, (\ell-1) - k)$. As an example, $\nabla_{\ell, 0}^r(n)$ in (51) has the index triangle $\nabla_{\ell, 0}^r(n)$ shown in (51).

\[
\begin{array}{cccccc}
(0, \ell) & (1, \ell) & \cdots & (j, \ell) & \cdots & (\ell-1, \ell) \\
(0, \ell-1) & (1, \ell-1) & \cdots & (j, \ell-1) & \cdots & (\ell-1, \ell-1) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(0, k) & (1, k) & \cdots & (j, k) & \cdots & (k, k) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(0, 2) & (1, 2) & (2, 2) \\
(0, 1) & (1, 1) \\
(0, 0) & \\
\end{array}
\]

Similarly, for $0 \leq k \leq \ell$ and $t \in \mathbb{Z}$, we let lower index triangle $\Delta_{\ell, k}^l(n)$ be the ordered pairs $(k, t)$ in (50) specified by the triangle with lower vertices $(0, t + k)$ and $(0, t)$ and upper vertex $(k, t)$.

We define a sequence of ordered pairs $(k, t)$ in index tensor $n$. Let $t$ be a sequence of times $t = . . . , t', . . .$, where times $t, t' \in \mathbb{Z}$. The sequence $t = . . . , t', . . .$ is written in reverse time order, so that $t' < t$ as in (50). The sequence $t$ may be the integers $\mathbb{Z}$ or a finite or infinite subset of $\mathbb{Z}$, in reverse time order. Let $k$ be a sequence of integers $\ldots, k, k', . . .$, with each integer $k$ satisfying $0 \leq k \leq \ell$. The paired sequence $(k, t)$ is a sequence of ordered pairs $(k, t)$ in $n$, defined by

\[
(k, t) \overset{\text{def}}{=} \ldots, (k, t), (k, t'), . . . ,
\]

such that each $k$ in sequence $k$ is paired with a $t$ in sequence $t$ and vice versa.

Fix index tensor $n$. We next consider a sequence of lower elementary triangles in $n$ indexed by the paired sequence $(k, t)$. Define $\Delta^{t}(n)$ to be the sequence of lower elementary triangles

\[
\Delta^{t}(n) \overset{\text{def}}{=} \ldots, \Delta_{0, k}^l(n), \Delta_{0, k'}^l(n), . . . ,
\]

indexed by paired sequence

\[
(k, t) = \ldots, (k, t), (k', t'), . . . ,
\]

Note that triangles in $\Delta^{t}(n)$ may overlap. We assume the sequence (52) is purged so that any two triangles in sequence $\Delta^{t}(n)$ are disjoint, i.e., there are
entries in one triangle that are not in the other. In other words, there is no triangle in the sequence that is wholly contained in another triangle.

Next consider a sequence of upper elementary triangles in \( n \) indexed by the paired sequence \((k_u, t_u)\). Define \( \upsilon_{k_u}^t(n) \) to be the sequence of upper elementary triangles

\[
\upsilon_{k_u}^t(n) \overset{\text{def}}{=} \ldots, \upsilon_{0,k_u}^{t_u}(u), \upsilon_{0,k_u'}^{t_u'}(n), \ldots
\]  

indexed by paired sequence

\[(k_u, t_u) = \ldots, (k_u, t_u), (k'_u, t'_u), \ldots\]

Note that triangles in \( \upsilon_{k_u}^t(n) \) may overlap. We assume the sequence \( \upsilon_{k_u}^t(n) \) is purged so that any two triangles in sequence \( \upsilon_{k_u}^t(n) \) are disjoint.

We now show that given a sequence of lower elementary triangles \( u \) with paired sequence \((k, t)\), there is a sequence of upper elementary triangles \( \upsilon_{k_u}^t(n) \) with paired sequence \((k_u, t_u)\), and the two sequences of triangles partition \( n \). In this case, we say \((k_u, t_u)\) is the \textit{complementary paired sequence of} \((k, t)\). The reverse is also true, and we say \((k, t)\) is the complementary paired sequence of \((k_u, t_u)\).

\textbf{Lemma 46} For any paired sequence \((k, t)\), there is a complementary paired sequence \((k_u, t_u)\) of \((k, t)\), and \( \upsilon_{k_u}^t(n) \) and \( \upsilon_{k_u}^t(n) \) partition \( n \) into two sawtooth patterns. For any paired sequence \((k', t')\), there is a complementary paired sequence \((k', t')\) of \((k', t')\), and \( \upsilon_{k_u}^t(n) \) and \( \upsilon_{k_u}^t(n) \) partition \( n \) into two sawtooth patterns.

\textbf{Proof.} We prove the first assertion. Assume we are given a sequence of triangles \( \Delta_{k_u}(n) \) indexed by paired sequence \((k, t)\). We need to show there is a complementary paired sequence \((k_u, t_u)\) of \((k, t)\), and \( \upsilon_{k_u}^t(n) \) and \( \upsilon_{k_u}^t(n) \) partition \( n \) into two sawtooth patterns. Let \((k^*, t^*)\) be any ordered pair in \( n \) not contained in \( \Delta_{k_u}(n) \). If \((k^*, t^*)\) is not in \( \Delta_{k_u}(n) \), then it is readily seen that triangle \( \upsilon_{k_u}^{t_u}(n) \) cannot intersect with any triangle \( \Delta_{0,k^*}(n) \) in \( \Delta_{k_u}(n) \), otherwise \((k^*, t^*)\) would be in \( \Delta_{k_u}(n) \). Consider the union of all triangles \( \upsilon_{k_u}^{t_u}(n) \) over all \((k^*, t^*)\) not in \( \Delta_{k_u}(n) \). From this union, purge triangles that are wholly contained in larger triangles. Then we obtain a sequence of disjoint triangles \( \upsilon_{k_u}^t(n) \) which is indexed by a paired sequence \((k_u, t_u)\), none of which intersect with any triangle \( \Delta_{0,k^*}(n) \) in \( \Delta_{k_u}(n) \). Clearly all \((k^*, t^*)\) not in \( \Delta_{k_u}(n) \) are in \( \upsilon_{k_u}^t(n) \). Then \( \Delta_{k_u}(n) \) and \( \upsilon_{k_u}^t(n) \) partition \( n \), and \((k_u, t_u)\) is a complementary paired sequence of \((k, t)\).

The set \( \upsilon_{k_u}^t(n) \) forms a sawtooth pattern in \( n \) with lower triangular teeth indexed by paired sequence \((k, t)\). The set \( \upsilon_{k_u}^t(n) \) forms a sawtooth pattern in \( n \) with upper triangular teeth indexed by complementary paired sequence \((k_u, t_u)\) of \((k, t)\). Since an ordered pair must be in either set, the two sets partition \( n \) into two sawtooth patterns.

The second assertion is proven in an analogous way. \( \blacksquare \)

Fix tensor \( u \in \mathcal{U} \). We next consider a sequence of triangles in \( u \) indexed by \( \Delta_{k_u}(n) \). Define \( \Delta_{k_u}(u) \) to be the sequence of lower elementary triangles

\[
\Delta_{k_u}(u) \overset{\text{def}}{=} \ldots, \Delta_{0,k_u}(u), \Delta_{0,k_u'}(u), \ldots
\]  

indexed by paired sequence \((k, t) = \ldots, (k, t), (k', t')\).

Next consider a sequence of triangles in \( u \) indexed by \( \upsilon_{k_u}^t(n) \). Define \( \upsilon_{k_u}^t(u) \) to be the sequence of upper elementary triangles

\[
\upsilon_{k_u}^t(u) \overset{\text{def}}{=} \ldots, \upsilon_{0,k_u}(u), \upsilon_{0,k_u'}(u), \ldots
\]  

indexed by paired sequence \((k_u, t_u) = \ldots, (k_u, t_u), (k'_u, t'_u)\). Define \( \upsilon_{k_u}^t(\mathcal{U}) \) to be the set of sequences \( \{\upsilon_{k_u}^t(u) : u \in \mathcal{U}\} \).

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Define any paired sequence \((k, t)\). Consider the product

\[
\prod_{t \in t} A^{[t, t+k]}
\]

for ordered pairs \((k, t)\) in \((k, t)\). Each group \(A^{[t, t+k]}\) is a normal subgroup of \(A\).

Then the product \((\Delta^{t}_{0, k}L, \circ)\) is a normal subgroup of \(A\). We have seen the image of \(A^{[t, t+k]}\) in \(A\) is the lower elementary group \((\Delta^{t}_{0, k}L, \circ)\) in \((U, \circ)\), which is normal in \((U, \circ)\). Then the image of the product \((\Delta^{t}_{0, k}L, \circ)\) in \(A\) is the product

\[
(\Delta^{t}_{k}L, \circ) \overset{\text{def}}{=} \prod_{t \in t} (\Delta^{t}_{0, k}L, \circ)
\]

in \((U, \circ)\), for ordered pairs \((k, t)\) in \((k, t)\). Each group \((\Delta^{t}_{0, k}L, \circ)\) is a normal subgroup of \((U, \circ)\). Then the product \((\Delta^{t}_{k}L, \circ)\) is a normal subgroup of \((U, \circ)\). Note that \(\Delta^{t}_{k}L\) is a subset of \(U\), whereas \(\nabla^{t}_{k_0}(u)\) is a sequence of triangles of \(u \in U\) as in (53).

**Lemma 47** For any paired sequence \((k, t)\), \((\Delta^{t}_{k}L, \circ)\) is a normal subgroup of \((U, \circ)\).

Consider the single normal subgroup \((\Delta^{t}_{0, k}L, \circ)\) of \((U, \circ)\). Let \((k^*, t^*)\) be the paired sequence given by the single ordered pair \((k, t)\). The complementary paired sequence \((k^*_u, t^*_u)\) of \((k^*, t^*)\) is the paired sequence

\[
(k^*_u, t^*_u) = \ldots, (0, t+k+3), (0, t+k+2), (0, t+k+1), \ldots, (0, t-1), (0, t-2), (0, t-3), \ldots.
\]

We know \(\Delta^{t}_{0, k}L\) is the subset of all \(u \in U\) with trivial entries except for those in \(\Delta^{t}_{0, k}(u)\). Then \(\Delta^{t}_{0, k}L\) is the subset of all \(u \in U\) which are the identity on \(\nabla^{t}_{k_0}(U)\), or \(\{u \in U : \nabla^{t}_{k_0}(u) = \nabla^{t}_{k_0}(u_1)\}\), where \(u_1\) is the identity of \((U, \circ)\).

Now let \((k, t)\) be the paired sequence used to define \((\Delta^{t}_{k}L, \circ)\). Then we see that \(\Delta^{t}_{k}L\) is the subset of all \(u \in U\) which are the identity on \(\nabla^{t}_{k_0}(U)\), or \(\{u \in U : \nabla^{t}_{k_0}(u) = \nabla^{t}_{k_0}(u_1)\}\), where \((k_u, t_u)\) is the complementary paired sequence of \((k, t)\). This gives the following.

**Theorem 48** The normal subgroup \((\Delta^{t}_{k}L, \circ)\) of \((U, \circ)\) defined by paired sequence \((k, t)\) is the normal subgroup formed by the set of all \(u \in U\) which are the identity on \(\nabla^{t}_{k_0}(U)\), where \((k_u, t_u)\) is the complementary paired sequence of \((k, t)\).

We have just seen that the generator group \((U, \circ)\) of any \(\ell\)-controllable complete group system \(A\) is a collection of elementary groups \((\nabla^{t}_{0, k}(U), \circ^{t}_{0, k})\) of \((U, \circ)\) for each \(i \in Z\), for \(0 \leq k \leq \ell\). In the remainder of this subsection, we show that a finite or infinite set of elementary groups in the generator group also forms a group, with properties similar to an elementary group.

Since \((\Delta^{t}_{k}L, \circ)\) is normal in \((U, \circ)\), there is a quotient group \((U, \circ)/(\Delta^{t}_{k}L, \circ)\). The projection of the cosets in \((U, \circ)/(\Delta^{t}_{k}L, \circ)\) on set \(\nabla^{t}_{k_0}(U)\) is \(\nabla^{t}_{k_0}(U)\). We now define a group on \(\nabla^{t}_{k_0}(U)\).

Let \((k_u, t_u)\) be any paired sequence. Let \(\nabla^{t}_{k_0}(\tilde{u}), \nabla^{t}_{k_0}(\tilde{u}) \in \nabla^{t}_{k_0}(U)\). Define an operation \(\oplus^{t}_{k_0}\) on \(\nabla^{t}_{k_0}(U)\) by

\[
\nabla^{t}_{k_0}(\tilde{u}) \oplus^{t}_{k_0} \nabla^{t}_{k_0}(\tilde{u}) \overset{\text{def}}{=} \nabla^{t}_{k_0}(\tilde{u} \circ \tilde{u}).
\]

Note that the operation \(\oplus^{t}_{k_0}\) in (58) can be evaluated as

\[
\nabla^{t}_{k_0}(\tilde{u}) \oplus^{t}_{k_0} \nabla^{t}_{k_0}(\tilde{u}) = \nabla^{t}_{k_0}(\tilde{u} \circ \tilde{u})
\]

\[
= \ldots, \nabla^{t}_{0, k}(\tilde{u} \circ \tilde{u}), \nabla^{t}_{0, k}(\tilde{u} \circ \tilde{u}), \ldots
\]

\[
= \ldots, \nabla^{t}_{0, k}(\tilde{u}) \circ \nabla^{t}_{0, k}(\tilde{u}), \nabla^{t}_{0, k}(\tilde{u}) \circ \nabla^{t}_{0, k}(\tilde{u}), \ldots
\]

for each \(\nabla^{t}_{k_0}(\tilde{u}), \nabla^{t}_{k_0}(\tilde{u}) \in \nabla^{t}_{k_0}(U)\), and the last line is easy to evaluate using groups \((\nabla^{t}_{0, k}(U), \circ^{t}_{0, k})\) isomorphic to elementary groups.
Lemma 49  The operation $\oplus_{k_n}^{t_n}$ is well defined.

Proof. Let $\nabla_{k_n}^{t_n}(u), \nabla_{k_n}^{t_n}(\tilde{u}) \in \nabla_{k_n}^{t_n}(U)$. Let $\nabla_{k_n}^{t_n}(u), \nabla_{k_n}^{t_n}(\tilde{u}) \in \nabla_{k_n}^{t_n}(U)$ such that $\nabla_{k_n}^{t_n}(u) = \nabla_{k_n}^{t_n}(\tilde{u})$ and $\nabla_{k_n}^{t_n}(\tilde{u}) = \nabla_{k_n}^{t_n}(\tilde{u})$. To show the operation is well defined, we need to show

$$\nabla_{k_n}^{t_n}(u) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u}) = \nabla_{k_n}^{t_n}(u) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u}),$$

or $\nabla_{k_n}^{t_n}(u \circ \tilde{u}) = \nabla_{k_n}^{t_n}(\tilde{u} \circ \tilde{u})$. From (39), this is the same as showing

$$\cdots, \nabla_{0,k}^{t_n}(u) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}), \nabla_{0,k}^{t_n}(\tilde{u}) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}), \ldots = \cdots, \nabla_{0,k}^{t_n}(u) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}), \nabla_{0,k}^{t_n}(\tilde{u}) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}), \ldots,$$

or

$$\cdots, \nabla_{0,k}^{t_n}(u) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}) = \nabla_{0,k}^{t_n}(u) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}),$$

$$\nabla_{0,k}^{t_n}(\tilde{u}) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}) = \nabla_{0,k}^{t_n}(u) \oplus_{0,k}^{t_n} \nabla_{0,k}^{t_n}(\tilde{u}), \ldots$$

(60)

But if $\nabla_{k_n}^{t_n}(\tilde{u}) = \nabla_{k_n}^{t_n}(\tilde{u})$, then

$$\cdots, \nabla_{0,k}^{t_n}(\tilde{u}), \nabla_{0,k}^{t_n}(\tilde{u}), \ldots = \cdots, \nabla_{0,k}^{t_n}(\tilde{u}), \nabla_{0,k}^{t_n}(\tilde{u}), \ldots,$$

or

$$\cdots, \nabla_{0,k}^{t_n}(\tilde{u}) = \nabla_{0,k}^{t_n}(\tilde{u}), \nabla_{0,k}^{t_n}(\tilde{u}) = \nabla_{0,k}^{t_n}(\tilde{u}), \ldots$$

(61)

Similarly, if $\nabla_{k_n}^{t_n}(\tilde{u}) = \nabla_{k_n}^{t_n}(\tilde{u})$, then

$$\cdots, \nabla_{0,k}^{t_n}(\tilde{u}) = \nabla_{0,k}^{t_n}(\tilde{u}), \nabla_{0,k}^{t_n}(\tilde{u}) = \nabla_{0,k}^{t_n}(\tilde{u}), \ldots$$

(62)

From Lemma 38, the operation $\oplus_{0,k}^{t_n}$ on $\nabla_{0,k}^{t_n}(U)$ is well defined. Then using (61) and (62), we have verified (60).

Theorem 50  The set $\nabla_{k_n}^{t_n}(U)$ with operation $\oplus_{k_n}^{t_n}$ forms a group $(\nabla_{k_n}^{t_n}(U), \oplus_{k_n}^{t_n})$.

Proof. The proof is essentially the same as Theorem 26. We recapitulate this proof now. First we show the operation $\oplus_{k_n}^{t_n}$ is associative. Let $\nabla_{k_n}^{t_n}(u), \nabla_{k_n}^{t_n}(\tilde{u}), \nabla_{k_n}^{t_n}(\tilde{u}) \in \nabla_{k_n}^{t_n}(U)$. We need to show

$$(\nabla_{k_n}^{t_n}(u) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u})) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u}) = \nabla_{k_n}^{t_n}(u) \oplus_{k_n}^{t_n} (\nabla_{k_n}^{t_n}(\tilde{u}) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u})).$$

But using (39) we have

$$(\nabla_{k_n}^{t_n}(u) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u})) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u}) = \nabla_{k_n}^{t_n}(u \circ \tilde{u}) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u}) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(\tilde{u})$$

$$= \nabla_{k_n}^{t_n}(u \circ (\tilde{u} \circ \tilde{u})).$$

Therefore the operation $\oplus_{k_n}^{t_n}$ is associative since the operation $\circ$ in group $(U, \circ)$ is associative.

Let $u_1$ be the identity of $(U, \circ)$. We show $\nabla_{k_n}^{t_n}(u_1)$ is the identity of $(\nabla_{k_n}^{t_n}(U), \oplus_{k_n}^{t_n})$. Let $u \in U$ and $\nabla_{k_n}^{t_n}(u) \in \nabla_{k_n}^{t_n}(U)$. But using (39) we have

$$\nabla_{k_n}^{t_n}(u_1) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(u) = \nabla_{k_n}^{t_n}(u) \oplus_{k_n}^{t_n} \nabla_{k_n}^{t_n}(u)$$

$$= \nabla_{k_n}^{t_n}(u \circ u)$$

$$= \nabla_{k_n}^{t_n}(u)$$

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\[ \mathbf{\nabla}_{k_u}^t (\mathbf{u}) \oplus \mathbf{\nabla}_{k_u}^t (\mathbf{u}_1) = \mathbf{\nabla}_{k_u}^t (\mathbf{u} \circ \mathbf{u}_1) = \mathbf{\nabla}_{k_u}^t (\mathbf{u}). \]

Fix \( \mathbf{u} \in \mathcal{U} \). Let \( \bar{\mathbf{u}} \) be the inverse of \( \mathbf{u} \) in \( (\mathcal{U}, \circ) \). We show \( \mathbf{\nabla}_{k_u}^t (\bar{\mathbf{u}}) \) is the inverse of \( \mathbf{\nabla}_{k_u}^t (\mathbf{u}) \) in \( (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \). But using \( 39 \) we have

\[ \mathbf{\nabla}_{k_u}^t (\bar{\mathbf{u}}) \oplus_{k_u} \mathbf{\nabla}_{k_u}^t (\mathbf{u}) = \mathbf{\nabla}_{k_u}^t (\bar{\mathbf{u}} \circ \mathbf{u}) = \mathbf{\nabla}_{k_u}^t (\mathbf{u}_1) \]

and

\[ \mathbf{\nabla}_{k_u}^t (\mathbf{u}) \oplus_{k_u} \mathbf{\nabla}_{k_u}^t (\bar{\mathbf{u}}) = \mathbf{\nabla}_{k_u}^t (\mathbf{u} \circ \bar{\mathbf{u}}) = \mathbf{\nabla}_{k_u}^t (\mathbf{u}_1) \]

Together these results show \( (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \) is a group.

We have the following analogue of Theorem \( 41 \)

**Theorem 51** There is a homomorphism from \( (\mathcal{U}, \circ) \) to \( (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \) given by the map \( \omega_{k_u}^t : \mathcal{U} \rightarrow \mathbf{\nabla}_{k_u}^t (\mathcal{U}) \) with assignment \( \omega_{k_u}^t : \mathbf{u} \mapsto \mathbf{\nabla}_{k_u}^t (\mathbf{u}) \).

**Proof.** This follows immediately from the definition of operation \( \oplus_{k_u} \).

**Corollary 52** For any paired sequence \( (k, t) \) and its complementary paired sequence \( (k_u, t_u) \) of \( (k, t) \), we have \( (\mathcal{U}, \circ)/(\mathbf{\nabla}_k^t \mathcal{L}, \circ) \simeq (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \).

**Proof.** Note that \( (\mathbf{\nabla}_k^t \mathcal{L}, \circ) \) is the kernel of the homomorphism \( \omega_{k_u}^t \). Now the result follows from the first homomorphism theorem.

We now show there is a homomorphism from \( A \) to \( (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \). Combining Theorem \( 52 \) and Theorem \( 51 \) gives the following.

**Theorem 53** There is a homomorphism from \( A \) to \( (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \) given by the composition map \( \omega_{k_u}^t \circ \xi : A \rightarrow \mathbf{\nabla}_{k_u}^t (\mathcal{U}) \), where isomorphism \( A \simeq (\mathcal{U}, \circ) \) is given by the bijection \( \xi : A \rightarrow \mathcal{U} \) in Theorem \( 52 \), and the homomorphism from \( (\mathcal{U}, \circ) \) to \( (\mathbf{\nabla}_{k_u}^t (\mathcal{U}), \oplus_{k_u}) \) is given by the map \( \omega_{k_u}^t : \mathcal{U} \rightarrow \mathbf{\nabla}_{k_u}^t (\mathcal{U}) \) in Theorem \( 51 \).

5.3 Normal chains of the generator group (harmonic theory)

A filling sequence \( \mathbf{f} \) of \( \mathbf{n} \) is a walk of ordered pairs in \( \mathbf{n} \) that includes all ordered pairs in \( \mathbf{n} \) once and only once,

\[ \mathbf{f} \overset{\text{def}}{=} (k', t'), (k'', t''), \ldots, (k''', t'''), \ldots \]

The ordered pairs \( (k', t') \) in \( \mathbf{f} \) are called filled ordered pairs in \( \mathbf{n} \). Clearly, the filling sequence \( \mathbf{f} \) is specified by a sequence of filled ordered pairs. The filled ordered pairs \( (k', t') \) in \( \mathbf{f} \) can be in any time order.

For each time epoch \( \tau, \tau > 0 \), we let \( \mathbf{f}^\tau \) be the filling subsequence of \( \mathbf{f} \) of the \( \tau \) ordered pairs at the start of the filling sequence. Then \( \mathbf{f}^1 = (k', t'), \mathbf{f}^2 = (k', t'), (k'', t''), \mathbf{f}^3 = (k', t'), (k'', t''), (k''', t'''), \mathbf{f}^4 = (k', t'), (k'', t''), (k''', t'''), \ldots \), and so on. Fix time epoch \( \tau, \tau > 0 \), and consider any filling subsequence \( \mathbf{f}^\tau \) of \( \mathbf{f} \) at time epoch \( \tau \),

\[ \mathbf{f}^\tau = (k', t'), (k'', t''), \ldots \]

The ordered pairs in \( \mathbf{f}^\tau \) are filled ordered pairs in \( \mathbf{n} \) at time epoch \( \tau \), and the ordered pairs not in \( \mathbf{f}^\tau \) are called unfilled ordered pairs in \( \mathbf{n} \) at time epoch \( \tau \).
The ordered pairs in \( f^{r+1} \) are the filled ordered pairs in \( f^r \) and one unfilled ordered pair in \( n \) at time epoch \( \tau \).

We now subject the filling subsequence \( f^r \) of \( f \) at time epoch \( \tau \) to another constraint. We require that the filled ordered pairs in \( f^r \) form a sequence \( \triangle_{k^r}(n) \) of lower elementary triangles

\[
\triangle_{k^r}(n) = \ldots, \Delta_{0,0^r}(n), \Delta_{0,1^r}(n), \ldots, \tag{64}
\]

indexed by a paired sequence

\[(k^r, t^r) = \ldots, (k''^r, t''^r), (k'^r, t'^r), \ldots.\]

We call a filling subsequence \( f^r \) with this property a \textit{normal filling subsequence} of \( f \). If \( f^r \) is a normal filling subsequence of \( f \) for all \( \tau > 0 \), we say \( f \) is a \textit{normal filling sequence} of \( n \). If \( f \) is a normal filling sequence of \( n \), the filled ordered pairs in \( f^r \) must form a sequence \( \triangle_{k^r}(n) \) of lower elementary triangles, and the filled ordered pairs in \( f^{r+1} \) must form a sequence \( \triangle_{k^{r+1}}(n) \) of lower elementary triangles. This means the sequence of lower elementary triangles \( \triangle_{k^{r+1}}(n) \) is formed from the sequence of lower elementary triangles \( \triangle_{k^r}(n) \) and the unfilled ordered pair in \( n \) at time epoch \( \tau \) that is in \( f^{r+1} \) but not \( f^r \).

**Theorem 54** Let \( f \) be a normal filling sequence of \( n \). For each \( \tau > 0 \), let \( f^{r} \) be a normal filling subsequence of \( f \) with a paired sequence \((k^r, t^r)\). For each \( \tau > 0 \), \( (\triangle_{k^r}, \mathcal{L}, \circ) \) is a normal subgroup of \((\mathcal{U},\circ)\).

**Proof.** This follows from Lemma 47.

Suppose \((k^r, t^r)\) is the filled ordered pair of \( f^{r+1} \) that is not filled in \( f^r \). Then we must have a filled triangle \( \Delta_{0,k^r}(n) \) in \( \triangle_{k^{r+1}}(n) \). If we are to have a filled triangle \( \Delta_{0,k^r+1}(n) \) in \( \triangle_{k^{r+1}}(n) \), then triangles \( \Delta_{0,k^r} \) and \( \Delta_{0,k^r+1} \) must be filled in \( \triangle_{k^r}(n) \). Triangle \( \Delta_{0,k^r+1} \) cannot be a subtriangle of a larger triangle in \( \triangle_{k^r}(n) \) because \((k^r, t^r)\) is unfilled in \( \triangle_{k^r}(n) \). However triangle \( \Delta_{0,k^r+1} \) can be a subtriangle of a larger triangle \( \Delta_{0,k^r+1+m^+} \) in \( \triangle_{k^r}(n) \), where \( m^+ > 0 \) and \( k^r + 1 + m^+ \leq \ell \). Then \( \triangle_{k^r}(n) \) must be the sequence of triangles

\[
\triangle_{k^r}(n) = \ldots, \Delta_{0,0^r}(n), \ldots, \Delta_{0,k^r+1}(n), \Delta_{0,k^r+1+m^+}(n), \ldots, \Delta_{0,k^r+1}(n), \ldots, \tag{65}
\]

indexed by paired sequence

\[(k^r, t^r) = \ldots, (k'', t''), (k^r+1, t^r+1), (k^r+1+m^+, t^r+1), \ldots, (k^r, t^r), \ldots,\]

where \( m^+ \geq 0 \) and \( k^r + 1 + m^+ \leq \ell \), and \( t^r < t^r + m^+ \). If \( m^+ = 0 \), then \( \triangle_{k^{r+1}}(n) \) is given by the sequence of triangles

\[
\triangle_{k^{r+1}}(n) = \ldots, \Delta_{0,0^r}(n), \ldots, \Delta_{0,k^r+1}(n), \ldots, \Delta_{0,k^r+1}(n), \ldots, \tag{66}
\]

indexed by paired sequence

\[(k^{r+1}, t^{r+1}) = \ldots, (k'', t''), (k^r+1, t^r+1), (k^r+1+m^+, t^r+1), \ldots, (k^r, t^r), \ldots,\]

If \( m^+ > 0 \), then \( \triangle_{k^{r+1}}(n) \) is given by the sequence of triangles

\[
\triangle_{k^{r+1}}(n) = \ldots, \Delta_{0,0^r}(n), \ldots, \Delta_{0,k^r+1}(n), \Delta_{0,k^r+1+m^+}(n), \ldots, \Delta_{0,k^r+1}(n), \ldots, \tag{67}
\]

indexed by paired sequence

\[(k^{r+1}, t^{r+1}) = \ldots, (k'', t''), (k^r+1, t^r+1), (k^r+1+m^+, t^r+1), \ldots, (k^r, t^r), \ldots,\]

In either case, we see that the sequence of triangles \( \triangle_{k^r}(n) \) in \( \text{(65)} \) is contained in the sequence of triangles \( \triangle_{k^{r+1}}(n) \) in \( \text{(66)} \) or \( \text{(67)} \), which we write as \( \triangle_{k^r}(n) \subset \triangle_{k^{r+1}}(n) \).
Lemma 55 Let $f$ be a normal filling sequence of $n$. For each $\tau > 0$, let $f^\tau$ be a normal filling subsequence of $f$ with a paired sequence $(k^\tau, t^\tau)$. At each time epoch $\tau > 0$, we have $(A_{k^\tau}^t, L, o) \subset (A_{k^\tau+1}^{t+1}, L, o)$. In addition, we have $u_1 \subset (A_{k^t}^1, L, o)$.

Proof. The filled ordered pairs in $n$ at time epoch $\tau$ are the sequence of triangles $A_{k^\tau}^t (n)$, and the filled ordered pairs in $n$ at time epoch $\tau + 1$ are the sequence of triangles $A_{k^\tau+1}^{t+1} (n)$.

The sequence of triangles $A_{k^\tau}^t (n)$ is contained in the sequence of triangles $A_{k^\tau+1}^{t+1} (n)$, or $A_{k^\tau}^t (n) \subset A_{k^\tau+1}^{t+1} (n)$. This means $(A_{k^\tau}^t, L, o) \subset (A_{k^\tau+1}^{t+1}, L, o)$. In a similar way, we have $n \subset A_{k^t}^1 (n)$ and then $u_1 \subset (A_{k}^1, L, o)$.

Since $(A_{k}^1, L, o)$ is a normal subgroup of $(U, o)$ for any paired sequence $(k^\tau, t^\tau)$ at any time epoch $\tau$, then from Lemma 55 for any normal filling sequence of $n$ we may construct a normal chain

$$u_1 \subset (A_{k}^t, L, o) \subset \cdots \subset (A_{k}^t, L, o) \subset (A_{k}^{t+1}, L, o) \subset \cdots \subset (U, o).$$

We say (68) is the normal chain of a normal filling sequence $f$ of $n$. This gives the following.

Theorem 56 Any normal filling sequence $f$ of $n$ gives a normal chain (68) of $(U, o)$.

Each normal chain (68) of $(U, o)$ gives a coset decomposition chain of $(U, o)$. We now find coset representatives of the coset decomposition chain. First we find coset representatives of the quotient group $(A_{k^\tau+1}^{t+1}, L, o) / (A_{k^\tau}^t, L, o)$. A coset representative of $(A_{k^\tau+1}^{t+1}, L, o) / (A_{k^\tau}^t, L, o)$ is an element of $(A_{k^\tau+1}^{t+1}, L, o)$ that is not an element of $(A_{k^\tau}^t, L, o)$. Suppose $(k^\tau, t^\tau)$ is the filled ordered pair of $f^\tau$ that is not filled in $f^\tau$. Clearly generator $u_{g, k^\tau}^t$ of $(U, o)$ is an element of $(A_{k^\tau+1}^{t+1}, L, o)$ that is not an element of $(A_{k^\tau}^t, L, o)$. Then the set of generators $\{u_{g, k^\tau}^t : g_k^\tau \in G_{k^\tau}^t\}$ are elements of $(A_{k^\tau+1}^{t+1}, L, o)$ that are not elements of $(A_{k^\tau}^t, L, o)$. For any $g_k^\tau \in G_{k^\tau}^t$, we show that $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ is a coset of $(A_{k^\tau+1}^{t+1}, L, o)$ in $(A_{k^\tau+1}^{t+1}, L, o)$. First, it is clear that $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ is a coset of $(A_{k^\tau}^t, L, o)$. Next we show the coset is contained in $(A_{k^\tau+1}^{t+1}, L, o)$.

Lemma 57 We have that $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ is a coset of $(A_{k^\tau}^t, L, o)$. The coset $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ is contained in $(A_{k^\tau+1}^{t+1}, L, o)$, and the cosets $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ for each $g_k^\tau \in G_{k^\tau}^t$ are disjoint.

Proof. Let $u \in (A_{k^\tau}^t, L, o)$. We have seen above that if $(k^\tau, t^\tau)$ is the filled ordered pair of $f^\tau$ that is not filled in $f^\tau$, then $A_{k^\tau}^t (u) \subset A_{k^\tau+1}^{t+1} (u)$. This means that the sequence of triangles $A_{k^\tau}^t (u)$ is contained in the sequence of triangles $A_{k^\tau+1}^{t+1} (u)$. In other words, we have $u \in (A_{k^\tau+1}^{t+1}, L, o)$. Clearly $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ is contained in $(A_{k^\tau+1}^{t+1}, L, o)$. Then $u_{g, k^\tau}^t \circ u$ is in $(A_{k^\tau+1}^{t+1}, L, o)$. Since $u_{g, k^\tau}^t \circ u \in (A_{k^\tau+1}^{t+1}, L, o)$, the cosets $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ are disjoint for each $g_k^\tau \in G_{k^\tau}^t$.

Lemma 58 The cosets $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ for each $g_k^\tau \in G_{k^\tau}^t$ are all the cosets of $(A_{k^\tau}^t, L, o)$ in $(A_{k^\tau+1}^{t+1}, L, o)$.

Proof. The cardinality of $(A_{k^\tau+1}^{t+1}, L, o)$ is $|G_{k^\tau}^t| \times |(A_{k^\tau}^t, L, o)|$. But this is the same as the cardinality of all the cosets $u_{g, k^\tau}^t \circ (A_{k^\tau}^t, L, o)$ for each $g_k^\tau \in G_{k^\tau}^t$. 

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Similarly the coset representatives of \((\mathbb{A}^n_1, \mathbb{L}, \circ)/u_1\) are the generators \(\{u^l_{g,k} : g^l_{k,1} \in G^l_{k,1}\}\) where \((k^1, t^1)\) is the filled ordered pair in \(f^1\). Combining Lemmas 57 and 58 gives the following.

**Theorem 59** The generators of \((\mathcal{U}, \circ)\) form a coset representative chain of any coset decomposition chain of \((\mathcal{U}, \circ)\) formed by the normal chain (68) of \((\mathcal{U}, \circ)\) of any normal filling sequence \(f\) of \(n\).

Under the bijection \(\beta^{-1} : \mathcal{U} \to \mathcal{R}\) in Theorem 17, a generator \(u^l_{g,k}\) in \((\mathcal{U}, \circ)\) is taken to a generator \(r^l_{y,k} \in \mathcal{R}\), with assignment \(\beta^{-1} : u^l_{g,k} \mapsto r^l_{y,k}\).

Under the bijection \(\alpha : \mathcal{R} \to A\) in Theorem 14 a generator \(r^l_{y,k}\) in \((\mathcal{R}, \ast)\) is taken to a generator \(g^l\) in \(A\), with assignment \(\alpha : r^l_{y,k} \mapsto g^l\).

Thus the bijection \(\alpha \circ \beta^{-1} : \mathcal{U} \to A\) takes the set of generators \(u^l_{g,k}\) in \((\mathcal{U}, \circ)\) to the set of generators \(g^l\) in \(A\). Furthermore, the bijection \(\alpha \circ \beta^{-1}\) is an isomorphism between \((\mathcal{U}, \circ)\) and \(A\). Therefore a normal chain of \((\mathcal{U}, \circ)\) is taken to a normal chain of \(A\), and a coset representative in \((\mathcal{U}, \circ)\) is taken to a coset representative in \(A\).

Then we can reconstruct the set of elements \(A\) using the coset representatives of the coset decomposition chain of \((\mathcal{U}, \circ)\) obtained from any normal filling sequence of \(n\). This gives the following.

**Theorem 60** We can reconstruct \(A\) using the generators of \((\mathcal{U}, \circ)\) as the coset representatives of the coset decomposition chain of \((\mathcal{U}, \circ)\) obtained from any normal filling sequence of \(n\).

We now give some examples of this result.

In general there are many normal filling sequences \(f\) of \(n\) and therefore many normal chains of \((\mathcal{U}, \circ)\). We now give 4 important examples of normal filling sequences of \(n\). We specify the filling sequence by the sequence of filled ordered pairs. If we go left to right in (69), the time index decreases. We call this **reverse time**. If we go right to left, the time index increases. We call this **forward time**.

For the first example of a sequence of filled ordered pairs, we go up the columns in (60), from left to right. This gives the sequence of filled ordered pairs \((k, t),\)

\[
\ldots, (0, t), (1, t), (2, t), \ldots, (j, t), \ldots, (\ell, t), (0, t-1), (1, t-1), (2, t-1), \ldots, (j, t-1), \ldots, (\ell, t-1), \ldots
\]

(69)

It can be easily verified that this gives a normal filling sequence. We call this the **time domain normal filling sequence of \(n\) in reverse time**. For the second example of a sequence of filled ordered pairs, we go up the diagonals in (60), from right to left. This gives the sequence of filled ordered pairs \((k, t),\)

\[
\ldots, (0, t), (1, t-1), (2, t-2), \ldots, (j, t-j), \ldots, (\ell, t-\ell), (0, t+1), (1, t), (2, t-1), \ldots, (j, t-j+1), \ldots, (\ell, t-\ell+1), \ldots
\]

We call this the **time domain normal filling sequence of \(n\) in forward time**. Note that we go up the columns in reverse time and up the diagonals in forward time because of the triangular shape of \(\mathbb{V}_{0,0}^l(u)\). Next we go row by row in (60), from bottom row to top row and left to right. This gives the sequence of filled ordered pairs \((k, t),\)

\[
\ldots, (0, t), (0, t-1), \ldots, \ldots, (1, t), (1, t-1), \ldots, (2, t), (2, t-1), \ldots, (j, t), (j, t-1), \ldots, (\ell, t), (\ell, t-1), \ldots
\]

(70)

We call this the **spectral domain normal filling sequence of \(n\) in reverse time**. Finally we may go row by row in (60), from bottom row to top row and right to left. This gives the sequence of filled ordered pairs \((k, t),\)

\[
\ldots, (0, t-1), (0, t), \ldots, (1, t-1), (1, t), \ldots, (2, t-1), (2, t), \ldots, (j, t-1), (j, t), \ldots, (\ell, t-1), (\ell, t), \ldots
\]

We call this the **spectral domain normal filling sequence of \(n\) in forward time**. The examples given here can be viewed as limiting cases; in general a normal filling sequence need not be time ordered in forward or reverse time.
If we use generators of $\langle U, \circ \rangle$ obtained from the normal chain of the time domain normal filling sequence of $n$ in reverse time (69), we obtain the time domain encoder of $A$ in (10). If we use generators of $\langle U, \circ \rangle$ obtained from the normal chain of the spectral domain normal filling sequence of $n$ in reverse time (70), we obtain the Forney-Trott spectral domain encoder of $A$ in (6). Thus the time domain encoder of $A$ discussed in Section 3 and the spectral domain encoder of $A$ in (3) can be more easily obtained just from the normal chain of their respective normal filling sequence of $n$. In addition, there are many other encoders and coset representative chains of $A$ that can be obtained from the normal chain of many other normal filling sequences of $n$.

Let $(k_u, t_u)$ be any paired sequence. Let group $\langle \mathbf{\nabla}_k^{t_u} U, \circ_{k_u} \rangle$ be the set of elementary groups formed from paired sequence $(k_u, t_u)$ as in Subsection 5.2. We now find a normal chain of the group $\langle \mathbf{\nabla}_k^{t_u} U, \circ_{k_u} \rangle$ in much the same way as we just found normal chains of $\langle U, \circ \rangle$ using the normal filling sequence.

From Lemma (16) for any paired sequence $(k_u, t_u)$, there is a complementary paired sequence $(k', t')$ of $(k_u, t_u)$, and $\mathbf{\Delta}_k^{t_u}(n)$ and $\mathbf{\nabla}_k^{t_u}(n)$ partition $n$ into two sawtooth patterns. A filling sequence $f$ of $n$ is a walk of ordered pairs in $n$ that includes all ordered pairs in $n$ once and only once. To study $\langle \mathbf{\nabla}_k^{t_u} U, \circ_{k_u} \rangle$, we consider a filling sequence $f_{\chi}$ of $n$,

$$f_{\chi} \overset{\text{def}}{=} [\mathbf{\Delta}_k^{t_u}(n)], (k', t'), (k'', t''), \ldots, (k'''', t''', \ldots),$$

where $[\mathbf{\Delta}_k^{t_u}(n)]$ is any sequence of all the ordered pairs in $\mathbf{\Delta}_k^{t_u}(n)$, and the remaining sequence $(k', t'), (k'', t''), \ldots, (k'''', t''', \ldots)$ are all the ordered pairs in $\mathbf{\nabla}_k^{t_u}(n)$.

For each time epoch $\tau = 0, \ldots$, we let $f_{\chi}^\tau$ be the filling subsequence of $f_{\chi}$ of the ordered pairs in $[\mathbf{\Delta}_k^{t_u}(n)]$ and the first $\tau$ ordered pairs after the ordered pairs in $[\mathbf{\Delta}_k^{t_u}(n)]$. Then $f_{\chi}^0 = [\mathbf{\Delta}_k^{t_u}(n)], (k', t'), f_{\chi}^1 = [\mathbf{\Delta}_k^{t_u}(n)], (k', t'), (k'', t''),$ and so on. Fix time epoch $\tau = 0, \ldots$, and consider any filling subsequence $f_{\chi}^\tau$ of $f_{\chi}$ at time epoch $\tau$,

$$f_{\chi}^\tau = [\mathbf{\Delta}_k^{t_u}(n)], (k', t'), (k'', t''), \ldots.$$  \hspace{1cm} (71)

The ordered pairs in $f_{\chi}^\tau$ are filled ordered pairs in $\mathbf{\Delta}_k^{t_u}(n)$ and $\mathbf{\nabla}_k^{t_u}(n)$ at time epoch $\tau$, and the ordered pairs not in $f_{\chi}^\tau$ are unfilled ordered pairs in $\mathbf{\nabla}_k^{t_u}(n)$ at time epoch $\tau$.

We now subject the filling subsequence $f_{\chi}^\tau$ of $f_{\chi}$ at time epoch $\tau$ to another constraint. We require that the filled ordered pairs in $f_{\chi}^\tau$ form a sequence $\mathbf{\Delta}_k^{t_u}(n)$ of lower elementary triangles

$$\mathbf{\Delta}_k^{t_u}(n) \overset{\text{def}}{=} \ldots, \Delta_{0,k'''}(n), \ldots, \Delta_{0,k'}(n), \ldots, \Delta_{0,k}(n), \ldots,$$  \hspace{1cm} (72)

indexed by a paired sequence

$$(k_u^\chi, t_u^\chi) = \ldots, (k'', t''), \ldots, (k', t'), \ldots, (k', t'), \ldots$$  \hspace{1cm} (73)

In (73), the ordered pair $(k', t')$ is in $(k', t')$, and the ordered pairs $(k'', t''), (k', t')$ are not in $(k', t')$. Note that the ordered pairs in $[\mathbf{\Delta}_k^{t_u}(n)]$ already form a sequence of lower elementary triangles $\mathbf{\Delta}_k^{t_u}(n)$ indexed by $(k', t')$, and so we just need to ensure that we can use $(k', t')$ and the additional ordered pairs $(k', t'), (k'', t''), \ldots$ in $f_{\chi}^\tau$ to form a paired sequence $(k_u^\chi, t_u^\chi)$ and a sequence $\mathbf{\Delta}_k^{t_u}(n)$ of lower elementary triangles. We call a filling subsequence $f_{\chi}^\tau$ with this property a normal filling subsequence of $f_{\chi}$. If $f_{\chi}^\tau$ is a normal filling subsequence of $f_{\chi}$ for all $\tau = 0, \ldots$, we say $f_{\chi}$ is a normal filling sequence of $n$.

**Theorem 61** Let $f_{\chi}$ be a normal filling sequence of $n$. For each $\tau = 0, \ldots$, let $f_{\chi}^\tau$ be a normal filling subsequence of $f_{\chi}$ with a paired sequence $(k_u^\chi, t_u^\chi)$. For each $\tau = 0, \ldots$, $(\mathbf{\Delta}_k^{t_u}(n), \circ_{k_u^\chi})$ is a normal subgroup of $\langle U, \circ \rangle$.

**Proof.** This follows from Lemma (17).
Previously we have seen that if \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\), then \(\Delta_k^r(n) \subset \Delta_{k^+}^{r+1}(n)\), where \((k^+, t^+)\) is the paired sequence of \(f^r\) and \((k^{r+1}, t^{r+1})\) is the paired sequence of \(f^{r+1}\). Now suppose \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). Then in a similar way, we have \(\Delta_k^r(n) \subset \Delta_{k^+}^{r+1}(n)\), where \((k^+, t^+)\) is the paired sequence of \(f^r\) and \((k^{r+1}, t^{r+1})\) is the paired sequence of \(f^{r+1}\). This gives the following.

**Lemma 62** Let \(f^r\) be a normal filling sequence of \(n\). For each \(\tau > 0\), let \(f^r_{\tau}\) be a normal filling subsequence of \(f^r\) with a paired sequence \((k^r_{\tau}, t^r_{\tau})\). At each time epoch \(\tau > 0\), we have \((\Delta^r_{k^r_{\tau}} \cup \tau, o) \subset (\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\). In addition, we have \((\Delta^r_{k^r_{\tau}} \cup \tau, o) \subset (\Delta_{k^r_{\tau+1}} \cup \tau, o)\). Further if \((\Delta^r_{k^r_{\tau}} \cup \tau, o) = (\Delta_{k^r_{\tau+1}} \cup \tau, o)\) then \((\Delta^r_{k^r_{\tau}} \cup \tau, o) \subset (\Delta_{k^r_{\tau+1}} \cup \tau, o)\). This gives the following.

**Theorem 63** Any normal filling sequence \(f^r\) of \(n\) gives a normal chain \((\cup, o)\) of \(U\).

From Lemmas 57 and 58 we have seen the cosets \(u_{g,k^+}^r \circ (\Delta^r_{k^r_{\tau}} \cup \tau, o)\) for each \(g^r_{k^+} \in g^r_{k^+}\) are all the cosets of \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) in \((\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\), where \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). \(u^r_{g,k^+} \circ (\Delta^r_{k^r_{\tau}} \cup \tau, o)\) for each \(g^r_{k^+} \in g^r_{k^+}\) are all the cosets of \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) in \((\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\), where \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). \(u^r_{g,k^+} \circ (\Delta^r_{k^r_{\tau}} \cup \tau, o)\) for each \(g^r_{k^+} \in g^r_{k^+}\) are all the cosets of \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) in \((\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\), where \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). \(u_{g,k^+}^r \circ (\Delta^r_{k^r_{\tau}} \cup \tau, o)\) for each \(g^r_{k^+} \in g^r_{k^+}\) are all the cosets of \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) in \((\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\), where \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). \(u_{g,k^+}^r \circ (\Delta^r_{k^r_{\tau}} \cup \tau, o)\) for each \(g^r_{k^+} \in g^r_{k^+}\) are all the cosets of \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) in \((\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\), where \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). \(u_{g,k^+}^r \circ (\Delta^r_{k^r_{\tau}} \cup \tau, o)\) for each \(g^r_{k^+} \in g^r_{k^+}\) are all the cosets of \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) in \((\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\), where \((k^+, t^+)\) is the filled ordered pair of \(f^{r+1}\) that is not filled in \(f^r\). This gives the following.

**Theorem 64** The generators \(u_{g,k^+}^r\) of \((\cup, o)\) where \((k^+, t^+)\) is in \(\Delta^r_{k^r_{\tau}}(n)\), form a coset representative chain of any coset decomposition chain of \((\cup, o)\) formed by the normal chain \(\Delta^r_{k^r_{\tau}}(n)\) of \((\cup, o)\) of any normal filling sequence \(f^r\) of \(n\).

We know that \((\Delta^r_{k^r_{\tau}} \cup \tau, o)\) is a normal subgroup of \((\cup, o)\). There is a homomorphism from \((\cup, o)\) to \((\Delta^r_{k^r_{\tau}}(n), \oplus^r_{k^r_{\tau}}(n))\) given by the map \(\omega^r_{k^r_{\tau}} : \cup \rightarrow \Delta^r_{k^r_{\tau}}(n)\) with assignment \(\omega^r_{k^r_{\tau}} : u \mapsto \Delta^r_{k^r_{\tau}}(n)\). Then \(\omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau}} \cup \tau, o)\) is a normal subgroup of \(\Delta^r_{k^r_{\tau}}(n)\). Further if \((\Delta^r_{k^r_{\tau}} \cup \tau, o) \subset (\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\) at each time epoch \(\tau > 0\), then \(\omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau}} \cup \tau, o) \subset \omega^r_{k^r_{\tau+1}}(\Delta^r_{k^r_{\tau+1}} \cup \tau, o)\) at each time epoch \(\tau > 0\). Finally we know that \(\omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau}} \cup \tau, o) = (\Delta^r_{k^r_{\tau}}(n_{1}), \oplus^r_{k^r_{\tau}}(n_{1}))\). Then using normal chain \(\Delta^r_{k^r_{\tau}}(n_{1})\), we have the normal chain

\[
(\Delta^r_{k^r_{\tau}}(n_{1}), \oplus^r_{k^r_{\tau}}(n_{1})) = \omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau}} \cup \tau, o) \subset \omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau+1}} \cup \tau, o) \subset \cdots \subset \omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau+1}} \cup \tau, o) \subset \omega^r_{k^r_{\tau}}(\Delta^r_{k^r_{\tau+1}} \cup \tau, o) \subset \cdots \subset \omega^r_{k^r_{\tau}}(\cup, o) = (\Delta^r_{k^r_{\tau}}(n), \oplus^r_{k^r_{\tau}}(n)) \quad (75)
\]
of \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\).

We have just seen that the cosets \(u_{g,k}^{t^+} \circ (\Delta_{g,k}^{t^+} L, \circ)\) for each \(g_{k}^{t} \in G_{k}^{t^+}\) are all the cosets of \((\Delta_{g,k}^{t^+} L, \circ)\) in \((\Delta_{g,k}^{t^+} L, \circ)\), where \((k^+, t^+)\) is the filled ordered pair of \(f_{k}^{t^+}\) that is not filled in \(f_{k}^{t}\), and \(u_{g,k}^{t^+}\) is a generator of \((U, \circ)\) where \((k^+, t^+)\) is in \(\nabla^t_{\kappa^u}(n)\). It follows that the cosets of \(\omega_{k_n}^{t^+}(\Delta_{g,k}^{t^+} L, \circ)\) in \(\omega_{k_n}^{t^+}(\Delta_{g,k}^{t^+} L, \circ)\) are \(\omega_{k_n}^{t^+}(u_{g,k}^{t^+} \circ (\Delta_{g,k}^{t^+} L, \circ))\), where \((k^+, t^+)\) is the filled ordered pair of \(f_{k}^{t^+}\) that is not filled in \(f_{k}^{t}\), and \(u_{g,k}^{t^+}\) is a generator of \((U, \circ)\) where \((k^+, t^+)\) is in \(\nabla^t_{\kappa^u}(n)\). We have

\[
\omega_{k_n}^{t^+}(u_{g,k}^{t^+} \circ (\Delta_{g,k}^{t^+} L, \circ)) = \omega_{k_n}^{t^+}(u_{g,k}^{t^+}) \circ_{k_n} \omega_{k_n}^{t^+}((\Delta_{g,k}^{t^+} L, \circ)).
\]

We say the projection \(\omega_{k_n}^{t^+}(u_{g,k}^{t^+})\) of the generator \(u_{g,k}^{t^+}\) of \((U, \circ)\) is a generator of \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\). This gives the following.

**Theorem 65** The generators of \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\) form a coset representative chain of any coset decomposition chain of \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\) formed by the normal chain \(\{\nabla^t_{\kappa^u}(U), \circ_{\kappa^u}\}\) of any normal filling sequence \(f_{k}\) of \(n\).

Note that a nontrivial generator \(\omega_{k_n}^{t^+}(u_{g,k}^{t^+})\) of \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\) has all trivial entries except for \(g_{k}^{t^+}\). A special case of group \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\) is group \((\nabla^t_{0,0}(U), \circ_{0,0})\). In this case, a nontrivial generator of \((\nabla^t_{0,0}(U), \circ_{0,0})\) has all trivial entries except for one entry. We call this an eigentriangle of \((\nabla^t_{0,0}(U), \circ_{0,0})\); we say the identity generator is also an eigentriangle. Then group \((\nabla^t_{0,0}(U), \circ_{0,0})\) has an expansion in terms of a coset representative chain of eigentriangles. From Corollary 28 there is an isomorphism from \((\nabla^t_{0,0}(R), \circ_{0,0})\) to \((\nabla^t_{0,0}(U), \circ_{0,0})\) with assignment \(\beta^t : \nabla^t_{0,0}(R) \rightarrow \nabla^t_{0,0}(U)\), for each \(t \in \mathbb{Z}\), if \(\beta : r \rightarrow u\) is the assignment of the isomorphism \((R, \ast) \simeq (U, \circ)\). Then the assignment \(\beta^t : \nabla^t_{0,0}(R) \rightarrow \nabla^t_{0,0}(U)\) shows that an eigentriangle of \((\nabla^t_{0,0}(U), \circ_{0,0})\) is an eigentriangle of \((\nabla^t_{0,0}(R), \circ_{0,0})\). Therefore the group \((\nabla^t_{0,0}(R), \circ_{0,0})\) also has an expansion in terms of eigentriangles, which are generators of \((\nabla^t_{0,0}(R), \circ_{0,0})\).

Any linear block code \(B\) of length \(L\) is isomorphic to a strongly controllable complete group system \(A_B\) which is nontrivial on the time interval \([0, L - 1]\) and trivial elsewhere. The generator group \((U, \circ)\) of \(A_B\) is isomorphic to a direct product group on the sets of generators in \(A_B\), whose nontrivial portions on the interval \([0, L - 1]\) are the sets of generators in \(B\). In the typical case, \(B\) has a generator of length \(L\). Then \(A_B\) is \(\ell\)-controllable where \(\ell = L - 1\). In this case, the generator group of \(A_B\) is simply the lower elementary group \((\Delta_{0,\ell}^t L, \circ)\), where \(t = 0\). Then we can study the block code \(B\) using the lower elementary group \((\Delta_{0,\ell}^t L, \circ)\) for \(t = 0\).

Note that we can construct a normal filling sequence \(f\) of \(n\) which first fills the single index triangle \(\Delta_{0,\ell}^t(n)\) with a subsequence \(f'\) before filling the rest of \(n\). Let \(f'\) have paired sequence \((k^\tau, t^\tau)\). Then \((\Delta_{k^\tau}^t L, \circ)\) is a normal subgroup of \((U, \circ)\), and in fact, \((\Delta_{k^\tau}^t L, \circ) = (\Delta_{0,\ell}^t L, \circ)\). Then the normal subchain

\[
u_1 \subset (\Delta_{k^\tau}^t L, \circ) \subset \cdots \subset (\Delta_{k^\tau}^t L, \circ)
\]

of normal chain \(\{\nabla^t_{\kappa^u}(U), \circ_{\kappa^u}\}\) is a normal chain of \((\Delta_{0,\ell}^t L, \circ)\), and therefore of block code \(B\). It is clear that \(\tau = (\ell + 1)!\). In general, if \(\ell\) is large, there are many ways to fill the index triangle \(\Delta_{0,\ell}^t(n)\) that give a subsequence \(f'\) of a normal filling sequence \(f\) of \(n\), and therefore there are many normal chains of \(B\).

In addition, we can study the structure of the block code \(B\) using sets of elementary groups \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\) discussed in Subsection 5.2, the normal chains of \((\nabla^t_{\kappa^u}(U), \circ_{\kappa^u})\) just discussed above, and the homomorphism in Theorem \(55\).

More details and examples can be found in \(22\) v9-v12.
6. THE ELEMENTARY SYSTEM

6.1 The elementary system

Fix $k$ such that $0 \leq k \leq \ell$. For each $t \in \mathbb{Z}$, let $V^t_k$ be a collection of elements $v^t_k$, which may be integers or any arbitrary objects; besides this, there is no requirement on set $V^t_k$. Consider the set $\mathcal{V}$, which is the double Cartesian product

$$
\bigotimes_{t=-\infty}^{t=+\infty} \bigotimes_{0 \leq k \leq \ell} V^t_k.
$$

We call $\mathcal{V}$ the elementary set. A row in $\mathcal{V}$ is assumed to be written in time reverse order as $\ldots, v^t_k, v^{t-1}_k, \ldots$.

Let $v$ be an element in $\mathcal{V}$. Because $v$ has the same form as $u$ in (32), we can use the same triangle notation $\nabla^t_{0,k}(v)$ and $\nabla^t_{0,k}(\mathcal{V})$ for $v$ and $\mathcal{V}$, for $0 \leq k \leq \ell$, for $t \in \mathbb{Z}$, as $\nabla^t_{0,k}(u)$ and $\nabla^t_{0,k}(u)$ for $u$ and $\mathcal{U}$. Note that the entry in index position $(0,k)$ in $\nabla^t_{0,k}(v)$ is $v^t_k$.

The elementary list $\mathcal{L}$ is an infinite collection of groups defined on triangular subsets $\nabla^t_{0,k}(v)$ of the elementary set $\mathcal{V}$. The groups in the elementary list $\mathcal{L}$ are

$$
\{(\nabla^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : 0 \leq k \leq \ell, t \in \mathbb{Z}\}.
$$

The $(\ell + 1)$-depth elementary system $\mathcal{E}$ is an elementary set $\mathcal{V}$, the elementary list $\mathcal{L}$, and a homomorphism condition. The homomorphism condition is that for each $k$ such that $0 \leq k \leq \ell$, for each $t \in \mathbb{Z}$, there is a homomorphism from $(\nabla^t_{0,k}(\mathcal{V}), \circ^t_{0,k})$ to $(\nabla^t_{0,k+1}(\mathcal{V}), \circ^t_{0,k+1})$ and $(\nabla^t_{0,k+1}(\mathcal{V}), \circ^t_{0,k+1})$ under the projection map from set $\nabla^t_{0,k}(\mathcal{V})$ to set $\nabla^t_{0,k+1}(\mathcal{V})$ and set $\nabla^t_{0,k+1}(\mathcal{V})$ given by the assignment $\nabla^t_{0,k}(v) \mapsto \nabla^t_{0,k+1}(v)$ and $\nabla^t_{0,k}(v) \mapsto \nabla^t_{0,k+1}(v)$.

The $(\ell + 1)$-depth elementary system $\mathcal{E}$ is nested. For example, the list $\{(\nabla^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : k = \ell, t \in \mathbb{Z}\}$ forms a 1-depth elementary system $\mathcal{E}_\ell$. For this trivial case, the homomorphism condition is vacuous. The list $\{(\nabla^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : \ell - 1 \leq k \leq \ell, t \in \mathbb{Z}\}$ forms a 2-depth elementary system $\mathcal{E}_{\ell-1}$. In general the following holds.

**Theorem 66** The list $\{(\nabla^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : \ell - m + 1 \leq k \leq \ell, t \in \mathbb{Z}\}$ forms an $m$-depth elementary system $\mathcal{E}_{\ell - m + 1}$.

In Subsection 6.2 we show that we can form an $(\ell + 1)$-depth global group $(\mathcal{V}, \cdot)$ from any $(\ell + 1)$-depth elementary system $\mathcal{E}$ with elementary set $\mathcal{V}$. Then in Subsection 6.3, we show that we can start with the $(\ell + 1)$-depth elementary system $\mathcal{E}_A$ of any $\ell$-controllable complete group system $A$ and use the global group formed from $\mathcal{E}_A$ to obtain an isomorphic copy of the generator group $(\mathcal{U}, \circ)$ of $A$ and hence recover $A$. In fact, we show that we can recover an $\ell$-controllable complete group system $A$ from any $(\ell + 1)$-depth elementary system $\mathcal{E}$. Therefore the study of $\ell$-controllable complete group systems $A$ is essentially the study of $(\ell + 1)$-depth elementary systems $\mathcal{E}$. Finally in Subsection 6.4, we give a brief discussion of how to construct all $(\ell + 1)$-depth elementary systems $\mathcal{E}$.

6.2 The global group $(\mathcal{V}, \cdot)$

We now define a global operation $\cdot$ on the elementary set $\mathcal{V}$ using the infinite collection of groups in the elementary system $\mathcal{E}$. We show this forms a group $(\mathcal{V}, \cdot)$. We say $(\mathcal{V}, \cdot)$ is the global group of elementary system $\mathcal{E}$.

We define the product operation $\tilde{v} \cdot \tilde{v}$ to be the element $v \in \mathcal{V}$ given by

$$
\nabla^t_{0,0}(\tilde{v}) \overset{\text{def}}{=} \nabla^t_{0,0}(\mathcal{V}) \circ^t_{0,0} \nabla^t_{0,0}(\tilde{v}).
$$

for $t = +\infty$ to $-\infty$. 

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Lemma 67  The product (77) for \( t = +\infty \) to \(-\infty \) gives an element in \( V \).

**Proof.** To show that the product (77) for \( t = +\infty \) to \(-\infty \) gives an element \( \tilde{v} \) in \( V \), we first show that if \( \forall_{0,0}(\tilde{v}) \) and \( \forall^t_{0,0}(\tilde{v}) \) intersect, they are the same on their intersection. Assume without loss in generality that \( t' < t \). Then the intersection is a triangle \( \forall_{0,k}(\tilde{v}) \) for some \( k < \ell \). But \( \forall_{0,0}(\tilde{v}) = \forall_{0,0}(\tilde{v}) \odot_0 \forall^t_{0,0}(\tilde{v}) \) and \( \forall^t_{0,0}(\tilde{v}) = \forall_{0,0}(\tilde{v}) \odot_0 \forall_{0,0}(\tilde{v}) \). But by finite induction the homomorphism condition ensures there is a homomorphism from \( \forall_{0,0}(V) \), \( \odot_0 \) to \( \forall_{0,k}(V), \odot_0 \), and from \( \forall^t_{0,0}(V), \odot_0 \) to \( \forall^t_{0,k}(V), \odot_0 \). Therefore \( \forall_{0,0}(\tilde{v}) \odot_0 \forall^t_{0,0}(\tilde{v}) \) and \( \forall^t_{0,0}(\tilde{v}) \odot_0 \forall_{0,0}(\tilde{v}) \) give the same result on their intersection \( \forall_{0,0}(V) \odot_0 \forall^t_{0,0}(\tilde{v}) \). Therefore \( \forall_{0,0}(\tilde{v}) \) and \( \forall^t_{0,0}(\tilde{v}) \) are the same on their intersection \( \forall_{0,0}(V) \). But if the products in (77) for \( t = +\infty \) to \(-\infty \) are consistent on their intersections, then it is clear that \( \tilde{v} \) is just some element of the double Cartesian product \( V \).

\[ \Box \]

**Theorem 69** The set \( V \) with operation \( \cdot \) forms a group \((V,\cdot)\).

**Proof.** First we show the operation \( \cdot \) is associative. Let \( v, \tilde{v}, \overline{v} \in V \). We need to show
\[ (v \cdot \tilde{v}) \cdot \overline{v} = v \cdot (\tilde{v} \cdot \overline{v}). \]
(78)
To find the left hand side of (78), we evaluate
\[ (\forall^t_{0,0}(v) \odot_{0,0} \forall^t_{0,0}(\tilde{v})) \odot_{0,0} \forall^t_{0,0}(\overline{v}) \]
(79)
for \( t = +\infty \) to \(-\infty \). And to find the right hand side of (78), we evaluate
\[ \forall^t_{0,0}(v) \odot_{0,0} (\forall^t_{0,0}(\tilde{v}) \odot_{0,0} \forall^t_{0,0}(\overline{v})) \]
(80)
for \( t = +\infty \) to \(-\infty \). But we know group \((\forall^t_{0,0}(V), \odot_{0,0})\) is associative so (79) is the same as (80). This means the left hand side and right hand side of (78) evaluate to the same element in \( V \).

We show \( v_1 \) is the identity of \((V,\cdot)\). Let \( v \in V \). We need to show \( v_1 \cdot v = v \) and \( v \cdot v_1 = v \). First, to find \( v_1 \cdot v \) we evaluate
\[ \forall^t_{0,0}(v_1) \odot_{0,0} \forall^t_{0,0}(v) \]
(81)
for \( t = +\infty \) to \(-\infty \). But we know that \( \forall^t_{0,0}(v_1) \) is the identity of group \((\forall^t_{0,0}(V), \odot_{0,0})\). Then (81) reduces to
\[ \forall^t_{0,0}(v_1) \odot_{0,0} \forall^t_{0,0}(v) = \forall^t_{0,0}(v) \]
(82)
for \( t = +\infty \) to \(-\infty \). Then \( v_1 \cdot v = v \). A similar argument shows that \( v \cdot v_1 = v \).

Let \( v \in V \). We show \( v^{-1} \) is the inverse of \( v \) in \((V,\cdot)\). We need to show \( v^{-1} \cdot v = v_1 \) and \( v \cdot v^{-1} = v_1 \). First, to find \( v^{-1} \cdot v \) we evaluate
\[ \forall^t_{0,0}(v^{-1}) \odot_{0,0} \forall^t_{0,0}(v) \]
(83)
for \( t = +\infty \) to \(-\infty \). But we know that \( \forall^t_{0,0}(v^{-1}) \) is the inverse of \( \forall^t_{0,0}(v) \) in \((\forall^t_{0,0}(V), \odot_{0,0})\). Then (83) reduces to
\[ \forall^t_{0,0}(v^{-1}) \odot_{0,0} \forall^t_{0,0}(v) = \forall^t_{0,0}(v_1) \]
(84)
for \( t = +\infty \) to \(-\infty \). Then \( v^{-1} \cdot v = v_1 \). A similar argument shows that \( v \cdot v^{-1} = v_1 \).

Together these results show \((V,\cdot)\) is a group.

\[ \Box \]
We call the group \((V, \cdot)\) formed from an \((\ell + 1)\)-depth elementary system \(E\) an \((\ell + 1)\)-depth global group. The groups \((\nabla_{0,k}^t(V), \odot_{0,k}^t)\) in the elementary system are the elementary groups of \((V, \cdot)\). We have just shown the following.

**Theorem 70** Any \((\ell + 1)\)-depth elementary system \(E\) forms an \((\ell + 1)\)-depth global group \((V, \cdot)\) by the procedure just described.

**Theorem 71** The \((\ell + 1)\)-depth global group \((V, \cdot)\) formed from the \((\ell + 1)\)-depth elementary system \(E\) is uniquely determined by \(E\).

**Proof.** By definition, the global operation \(-\) in \((V, \cdot)\) is uniquely determined by the elementary groups \(\{(\nabla_{0,k}^t(U), \odot_{0,k}^t) : t \in \mathbb{Z}\}\) in the elementary list of \(E\). 

The \((\ell + 1)\)-depth global group is nested. For example, the top row of an \((\ell + 1)\)-depth global group \((V, \cdot)\) is a 1-depth global group \((V, \cdot)_{\ell}\). The top two rows of an \((\ell + 1)\)-depth global group \((V, \cdot)\) is a 2-depth global group \((V, \cdot)_{\ell-1}\).

In general the following holds.

**Theorem 72** The top \(m\) rows of an \((\ell + 1)\)-depth global group \((V, \cdot)\) form an \(m\)-depth global group \((V, \cdot)_{\ell-m+1}\).

To summarize, in Subsection 6.2 we have constructed the chain

\[
E \rightarrow (V, \cdot),
\]

where \(E\) is an \((\ell + 1)\)-depth elementary system, and \((V, \cdot)\) is an \((\ell + 1)\)-depth global group formed from \(E\).

### 6.3 Construction of all \(\ell\)-controllable complete group systems \(A\) from the elementary system

We now show that any \(\ell\)-controllable complete group system \(A\) can be reduced to an elementary system \(E_A\).

**Theorem 73** The \((\ell + 1)\)-depth generator group \((U, \circ)\) of any \(\ell\)-controllable complete group system \(A\) contains a unique \((\ell + 1)\)-depth elementary system \(E_A\) with elementary set \(U\) and elementary list \(\{(\nabla_{0,k}^t(U), \odot_{0,k}^t) : 0 \leq k \leq \ell, t \in \mathbb{Z}\}\).

**Proof.** First we show that \(U\) can be considered to be an elementary set. We have seen that \(U\) is just the double Cartesian product \(\prod\). Comparing set \(U\) in \(\prod\) to set \(V\) in \(\sigma\), we see there is a bijection between the two sets provided there is a bijection between the sets \(G_k^n\) and \(V_k^n\), for \(0 \leq k \leq \ell\), for each \(t \in \mathbb{Z}\). Then \(U\) is an elementary set. Lasty, from Theorem 44 the upper elementary groups \((\nabla_{0,k}^t(U), \odot_{0,k}^t)\) of \((U, \circ)\), for \(0 \leq k \leq \ell\) and \(t \in \mathbb{Z}\), satisfy the homomorphism condition of groups in an \((\ell + 1)\)-depth elementary system. We define the elementary system \(E_A\) to be the elementary set \(U\) and elementary list \(\{(\nabla_{0,k}^t(U), \odot_{0,k}^t) : 0 \leq k \leq \ell, t \in \mathbb{Z}\}\). The elementary system \(E_A\) is unique from Theorem 45.

We call \(E_A\) the elementary system of \(A\). Essentially the elementary system \(E_A\) of \(A\) is just the generator group \((U, \circ)\) of \(A\) stripped of its global operation \(\circ\). Then we can summarize the results of this paper so far by the chain

\[
\begin{array}{ccc}
\downarrow & \leftarrow & \longleftarrow \\
\downarrow & & \\
A & \xrightarrow{\cong} & (\mathcal{R}, \ast) & \xrightarrow{\cong} & (U, \circ) & \rightarrow & E_A
\end{array}
\]

(86)

where \((\mathcal{R}, \ast)\) is a decomposition group; \((U, \circ)\) is a generator group; and \(E_A\) is an elementary system. In the remainder of this subsection, we ask whether we can reverse the chain in (86), i.e., can we recover \(A\) from \(E_A\). Then we define a new notion of isomorphism for group systems and show how to obtain all \(\ell\)-controllable complete group systems \(C\) up to this new isomorphism from the set of all \((\ell + 1)\)-depth elementary systems \(E\).
Theorem 74. We know the generator group \((\mathcal{U}, \circ)\) of any \(\ell\)-controllable complete group system \(A\) forms an \((\ell + 1)\)-depth elementary system \(\mathcal{E}_A\) with elementary set \(\mathcal{U}\). Form the \((\ell + 1)\)-depth global group \((\mathcal{U}, \ast)\) of \(\mathcal{E}_A\). The global operation \(\mathbf{u} \ast \mathbf{u}\) in \((\mathcal{U}, \ast)\) is the same as the global operation \(\mathbf{u} \circ \mathbf{u}\) in \((\mathcal{U}, \circ)\). Therefore there is an isomorphism \((\mathcal{U}, \ast) \cong (\mathcal{U}, \circ)\) under the 1-1 correspondence \(\mathcal{U} = \mathcal{U}\) given by the assignment \(\mathbf{u} = \mathbf{u}\).

Proof. From Theorem 73, we have seen that \(\mathcal{E}_A\) is an elementary system with elementary set \(\mathcal{U}\) and elementary groups \((\bigvee_{0,0}^t(\mathcal{U}), \otimes_{0,k}^t)\). We use \(\mathcal{E}_A\) to define a global group \((\mathcal{U}, \ast)\). Let \(\mathbf{u}, \mathbf{u} \in \mathcal{U}\). By definition, the product operation \(\mathbf{u} \ast \mathbf{u}\) in global group \((\mathcal{U}, \ast)\) is uniquely determined by the evaluation of

\[
\nabla^t_{0,0}((\mathbf{u}) \otimes_{0,0}^t, \nabla^t_{0,0}(\mathbf{u})), \quad (87)
\]

for \(t = +\infty\) to \(-\infty\). From Lemma 73, the product operation \(\mathbf{u} \circ \mathbf{u}\) in generator group \((\mathcal{U}, \circ)\) is uniquely determined by the evaluation of

\[
\nabla^t_{0,0}(\mathbf{u}) \otimes_{0,0}^t, \nabla^t_{0,0}(\mathbf{u}) \quad (88)
\]

for \(t = +\infty\) to \(-\infty\). We see that \(\nabla(\bullet)\) and \(\nabla(\circ)\) are the same. Therefore \(\mathbf{u} \ast \mathbf{u}\) and \(\mathbf{u} \circ \mathbf{u}\) are the same. Therefore there is an isomorphism \((\mathcal{U}, \ast) \cong (\mathcal{U}, \circ)\) under the 1-1 correspondence \(\mathcal{U} = \mathcal{U}\) given by the assignment \(\mathbf{u} = \mathbf{u}\). 

For \(\mathbf{u}, \mathbf{u} \in \mathcal{U}\), the global operation \(\mathbf{u} \ast \mathbf{u}\) in \((\mathcal{U}, \ast)\) is the same as the global operation \(\mathbf{u} \circ \mathbf{u}\) in \((\mathcal{U}, \circ)\). This is a little stronger condition than isomorphism, and we say that \((\mathcal{U}, \ast)\) is essentially identical to \((\mathcal{U}, \circ)\), written \((\mathcal{U}, \ast) \equiv (\mathcal{U}, \circ)\).

Lemma 75. The global group \((\mathcal{U}, \ast)\) of \(\mathcal{E}_A\) is essentially identical to the generator group \((\mathcal{U}, \circ)\) of \(A\), \((\mathcal{U}, \ast) \equiv (\mathcal{U}, \circ)\).

We have just constructed the chain

\[\mathcal{E}_A \rightarrow (\mathcal{U}, \ast), \quad (89)\]

where \(\mathcal{E}_A\) is an \((\ell + 1)\)-depth elementary system, and \((\mathcal{U}, \ast)\) is an \((\ell + 1)\)-depth global group. We can incorporate the chain \(\mathcal{E}_A \rightarrow (\mathcal{U}, \ast)\) into chain \((\mathcal{U}, \ast) \equiv (\mathcal{U}, \circ)\) as follows.

\[
\downarrow \quad \downarrow \quad \nabla \quad \nabla \quad \nabla \quad \nabla
\]

\[
A \quad (\mathcal{R}, \ast) \quad (\mathcal{U}, \circ) \quad \mathcal{E}_A
\]

\[
\uparrow \quad \uparrow \quad \langle \mathbf{u} \rangle \quad \langle \mathbf{u} \rangle \quad \langle \mathbf{u} \rangle
\]

This gives the following.

Theorem 76. We may recover any \(\ell\)-controllable complete group system \(A\) from the \((\ell + 1)\)-depth elementary system \(\mathcal{E}_A\) of \(A\) using the chain \(\mathcal{E}_A \rightarrow (\mathcal{U}, \ast)\).

\[
\downarrow \quad \nabla \quad \nabla \quad \nabla \quad \nabla \quad \nabla
\]

\[
A \quad (\mathcal{U}, \ast) \quad \mathcal{E}_A
\]

Proof. Since \((\mathcal{U}, \circ) \equiv (\mathcal{U}, \ast)\), we can easily recover the generator group \((\mathcal{U}, \circ)\) of \(A\) from the global group \((\mathcal{U}, \ast)\) of \(\mathcal{E}_A\). Then we can recover \(A\) from \((\mathcal{U}, \circ)\) using the homomorphism \(f_\mathbf{u}\) in the first homomorphism theorem for group systems as done previously in Subsection 6.5.

In fact, we can recover \(A\) directly from \((\mathcal{U}, \ast)\) using the first homomorphism theorem for group systems. The homomorphism \(f_\mathbf{u}\) in the top half of chain \((\mathcal{U}, \ast)\) just uses the primary elementary groups \((\bigvee_{0,0}^t(\mathcal{U}), \otimes_{0,0}^t)\) of \((\mathcal{U}, \circ)\) for each \(t \in \mathbb{Z}\).
But the latter groups are already available in \((\mathcal{U}, *)\). In fact these groups are available in \(\mathcal{E}_A\), so \(A\) can be recovered directly from the elementary system \(\mathcal{E}_A\) as well.

So far we have shown that the set of all elementary systems \(\mathcal{E}_A\) of all \(\ell\)-controllable complete group systems \(A\), or \(\{\mathcal{E}_A\}\), is contained in the set of all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\), \(\{\mathcal{E}\}\), or \(\{\mathcal{E}_A\} \subset \{\mathcal{E}\}\). We now show that we can construct at least one \(\ell\)-controllable complete group system \(A\) from any \((\ell + 1)\)-depth elementary system \(\mathcal{E}\).

Assume we are given any \((\ell + 1)\)-depth elementary system \(\mathcal{E}\). First find the \((\ell + 1)\)-depth global group \((\mathcal{V}, \cdot)\) of \(\mathcal{E}\). We now show that we can always construct a special \(\ell\)-controllable complete group system \((\mathcal{V}_s, \tau)\) from any \((\ell + 1)\)-depth global group \((\mathcal{V}, \cdot)\) of \(\mathcal{E}\). Previously, we recovered \(A\) from \((\mathcal{U}, \circ)\) using the first homomorphism theorem for group systems with \((\mathcal{U}, \circ)\) as an input group. The construction of \((\mathcal{V}_s, \tau)\) uses the first homomorphism theorem for group systems with \((\mathcal{V}, \cdot)\) as an input group, as summarized in chain \((32)\). We give this construction now.

For each \(t \in \mathbb{Z}\), define a map \(\theta^t_\mathcal{V} : \mathcal{V} \to \mathcal{V}^t_{0,0}(\mathcal{V})\) with assignment \(v \mapsto \mathcal{V}^t_{0,0}(v)\). Using \((77)\), the map \(\theta^t_\mathcal{V}\) is a homomorphism from \((\mathcal{V}, \cdot)\) to \((\mathcal{V}^t_{0,0}(\mathcal{V}), \circ_{0,0}^t)\). Consider the Cartesian product

\[
\mathcal{V}_\mathcal{H} \overset{\text{def}}{=} \cdots \times \mathcal{V}^t_{0,0}(\mathcal{V}) \times \mathcal{V}^t_{0,0}(\mathcal{V}) \times \cdots.
\]

Define the direct product group \((\mathcal{V}_\mathcal{H}, \tau)\) by

\[
(\mathcal{V}_\mathcal{H}, \tau) \overset{\text{def}}{=} \cdots \times (\mathcal{V}^t_{0,0}(\mathcal{V}), \circ_{0,0}^t) \times (\mathcal{V}^{t+1}_{0,0}(\mathcal{V}), \circ_{0,0}^{t+1}) \times \cdots.
\]

Then from Theorem \((34)\) using \((\mathcal{V}, \cdot)\) for information group \(\mathcal{G}\) and the primary elementary group \((\mathcal{V}^t_{0,0}(\mathcal{V}), \circ_{0,0}^t)\) for \(\mathcal{G}^t, t \in \mathbb{Z}\), there is a homomorphism \(\theta_v : \mathcal{V} \to \mathcal{V}_\mathcal{H}\), from \((\mathcal{V}, \cdot)\) to the direct product group \((\mathcal{V}_\mathcal{H}, \tau)\), defined by

\[
\theta_v(v) \overset{\text{def}}{=} \ldots, \theta^t_\mathcal{V}(v), \theta^{t+1}_v(v), \ldots.
\]

Define

\[
v_s \overset{\text{def}}{=} \ldots, \theta^t_\mathcal{V}(v), \theta^{t+1}_v(v), \ldots
\]

\[= \ldots, \mathcal{V}^t_{0,0}(v), \mathcal{V}^{t+1}_{0,0}(v), \ldots.
\]

Then \(\theta_v : \mathcal{V} \to \mathcal{V}_\mathcal{H}\) with assignment \(\theta_v : v \mapsto v_s\). We can think of \(v_s\) as the sequence of triangles \(\mathcal{V}^t_{0,0}(v)\) of \(v\), now written in conventional time order and not overlapped. Then

\[(\mathcal{V}, \cdot)/(\mathcal{V}, \cdot)_K \simeq \text{im} \theta_v,
\]

where \(\text{im} \theta_v\) is the image of the homomorphism \(\theta_v\), and where \((\mathcal{V}, \cdot)_K\) is the kernel of the homomorphism \(\theta_v\). Since \(\text{im} \theta_v\) is a subgroup of the direct product group \((\mathcal{V}_\mathcal{H}, \tau)\), then \(\text{im} \theta_v\) is a group system where global operation \(\tau\) is defined by the componentwise operation \(\circ_{0,0}^t\) in group \((\mathcal{V}^t_{0,0}(\mathcal{V}), \circ_{0,0}^t)\) for each \(t \in \mathbb{Z}\). We denote group \(\text{im} \theta_v\) by \((\mathcal{V}_s, \tau)\), where \(\mathcal{V}_s\) is the subset of the Cartesian product \(\mathcal{V}_\mathcal{H}\) determined by \(\mathcal{V} \in \mathcal{V}\), or equivalently the subset of \(\mathcal{V}_\mathcal{H}\) defined by \(\text{im} \theta_v\). We call \((\mathcal{V}_s, \tau)\) the \emph{global group system} of \((\mathcal{V}, \cdot)\). We can summarize the preceding discussion as follows.

**Theorem 77** The first homomorphism theorem for group systems constructs a group system \((\mathcal{V}_s, \tau)\) with component group \((\mathcal{V}^t_{0,0}(\mathcal{V}), \circ_{0,0}^t)\) from an \((\ell + 1)\)-depth generator group \((\mathcal{V}, \cdot)\) using a homomorphism \(\theta_v\), where \(\theta_v = \ldots, \theta^t_\mathcal{V}, \theta^{t+1}_v, \ldots\), and \(\theta^t_\mathcal{V}\) is a homomorphism \(\theta^t_\mathcal{V} : (\mathcal{V}, \cdot) \to (\mathcal{V}^t_{0,0}(\mathcal{V}), \circ_{0,0}^t)\), for each \(t \in \mathbb{Z}\). The homomorphism \(\theta_v\) is a bijection. We have \((\mathcal{V}, \cdot) \simeq \text{im} \theta_v = (\mathcal{V}_s, \tau)\) under the assignment \(\theta_v : v \mapsto v_s\) given by the bijection \(\theta_v : \mathcal{V} \to \mathcal{V}_s\).
Proof. For each \( v_s \in V_s \), there can be only one \( v \in V \) such that \( \theta_v : v \to v_s \) because the sequence \( v_s = \ldots, v_{t,0}^t(v), v_{t,0}^{t+1}(v), \ldots \) defines a unique \( v \). Then \( \theta_v : V \to V_s \) given by the assignment \( \theta_v : v \to v_s \) is a bijection. Then the kernel \( (V_s, \cdot)_K \) is the identity and \( (V_s, \cdot) \cong \text{im} \theta_v \equiv (V_s, \cdot) \).

We can summarize the construction in Theorem 77 as shown in chain (92).

\[
\begin{align*}
\downarrow & \quad \cong \quad \text{im} \theta_v \\
\downarrow & \quad \uparrow \theta_v = \ldots, \theta_v^t, \theta_v^{t+1}, \ldots \\
(V_s, \cdot) & \quad \leftarrow \mathcal{E}
\end{align*}
\]

The chain (92) forms a linear system with input group \((V, \cdot)\), homorphism \(\theta_v\), and output group \((V_s, \cdot)\). We say a linear system is invertible if the homomorphism is a bijection. Since the homomorphism \(\theta_v\) in (92) is a bijection, then the linear system in (92) is invertible. If the linear system in (92) is invertible, then the input \( v \in V \) can be discovered from the output \( v_s \in V_s \).

We say an element \( v \in V \) is a nontrivial generator \( v_{g,k}^t \) of \((V, \cdot)\) if \( v_{g,k}^t \) contains one and only one nontrivial element \( v_k^t \) for some \( k \) such that \( 0 \leq k \leq \ell \) and some time \( t \in \mathbb{Z} \). For each \( k \) such that \( 0 \leq k \leq \ell \) and each \( t \in \mathbb{Z} \), we always assume there is a trivial generator \( v_{g,k}^t \) of \((V, \cdot)\) which is the identity \( v_1 \) of \((V, \cdot)\).

Lemma 78 Let \( \Xi \equiv (v_{g,k}^t : 0 \leq k \leq \ell, t \in \mathbb{Z}) \) be the set of generators in \((V, \cdot)\). The sequences in set \( \theta_v(\Xi) \) are generators of \((V_s, \cdot)\) and form a basis \( B \) of \((V_s, \cdot)\).

Proof. Let \( v \) be any element in \((V, \cdot)\). Then \( \theta_v : v \to v_s \), where \( v_s \) is an element in \((V_s, \cdot)\). For \( v_{g,k} \in \Xi \), let \( \theta_v : v_{g,k}^t \to v_{g,k}^{t+t+k} \), where \( v_{g,k}^{t+t+k} \in (V_s, \cdot) \).

We now show any element in \( \theta_v(\Xi) \) is a generator of \((V_s, \cdot)\). Fix \( k \) such that \( 0 \leq k \leq \ell \). Fix nontrivial generator \( v_{g,k}^t \) in \( \Xi \). Under the bijection \( \theta_v \) of the first homomorphism theorem for group systems shown in (92), nontrivial element \( v_k^t \) in \( v_{g,k}^t \) lies in \((k + 1)\) for primary elementary groups \((v_{0,0}^t(V), \circ_{0,0}^t), (v_{0,0}^{t+1}(V), \circ_{0,0}^{t+1}), \ldots, (v_{0,0}^{t+k}(V), \circ_{0,0}^{t+k}) \), and therefore generator \( v_{g,k}^t \) becomes a sequence \( v_{s,g,k}^{t+t+k} \) of span \( k + 1 \) in \( V_s \). To show \( v_{g,k}^{t+t+k} \) is a generator, we have to show \( v_{s,g,k}^{t+t+k} \) is a coset representative of the time domain granule (11). It is sufficient to show \( v_{s,g,k}^{t+t+k} \) is a coset representative of (13). But \( v_{s,g,k}^{t+t+k} \) is a sequence of span \( k + 1 \) in \( V_s \). Therefore it is a member of the denominator of (13) but not of either term in the denominator. Therefore \( v_{s,g,k}^{t+t+k} \) is a coset representative of (13). Clearly, \( \theta_v : v_1 \to v_{s,1} \) where \( v_{s,1} \) is the identity of \((V_s, \cdot)\).

We have shown \( \theta_v(\Xi) \) is a generator of \((V_s, \cdot)\) for \( 0 \leq k \leq \ell \) and \( t \in \mathbb{Z} \). Therefore \( \theta_v(\Xi) \) is a basis \( B \) of \((V_s, \cdot)\).

We see that the first homomorphism theorem for group systems constructs generators in \((V_s, \cdot)\) from generators in \((V, \cdot)\). There is a bijection between the generators of group system \((V_s, \cdot)\) and the generators in \((V, \cdot)\) given by the restriction of \( \theta_v \) to the generators of \((V, \cdot)\).

Theorem 79 The group system \((V_s, \cdot) = \text{im} \theta \) constructed by Theorem 77 is \( \ell \)-controllable and complete.

Proof. Since the generator group \((V, \cdot)\) is \((\ell + 1)\)-depth, there is at least one generator \( v_{g,1}^t \) in \((V, \cdot)\) which has a nontrivial label \( v_{g,1}^t \). Under the bijection \( \theta_v \) of the first homomorphism theorem for group systems shown in (92), \( v_{g,1}^t \) lies in \((\ell + 1)\) for primary elementary groups \((v_{0,0}^t(V), \circ_{0,0}^t), (v_{0,0}^{t+1}(V), \circ_{0,0}^{t+1}), \ldots, (v_{0,0}^{t+k}(V), \circ_{0,0}^{t+k}) \), and therefore generator \( v_{g,1}^t \) becomes a sequence \( v_{s,g,1}^{t+t+1} \) of span \( k + 1 \) in \( V_s \). From the same argument used to show \( v_{s,g,1}^{t+t+1} \) is a generator of span \( k + 1 \) in the proof of Lemma 78 we know \( v_{s,g,1}^{t+t+1} \) is a generator of span \( \ell + 1 \). Therefore \((V_s, \cdot)\) is \( \ell \)-controllable.
The group system \((V_s, \cdot)\) is complete since it is determined by component groups \((\nabla^l_{0,0}(V), \odot^l_{0,0})\) in \((V, \cdot)\), which have no global constraints.

**Lemma 80** The generator group of \((V_s, \cdot)\) is \((V, \cdot)\).

**Proof.** Under the bijection \(\theta^{-1}_v : V_s \to V\), the group system \((V_s, \cdot)\) collapses to the generator group, which is \((V, \cdot)\).

Theorem 81 shows that we can always construct a special \(\ell\)-controllable complete group system from any \((\ell + 1)\)-depth generator group \((V, \cdot)\), namely \((V_s, \cdot)\). We can summarize these results as follows.

**Theorem 81** Given any \((\ell + 1)\)-depth elementary system \(\mathcal{E}\), we may always use the chain \([22]\) to construct an \(\ell\)-controllable complete group system \((V_s, \cdot)\) from the \((\ell + 1)\)-depth global group \((V, \cdot)\) of \(\mathcal{E}\). The homomorphism \(\theta_v\) is a bijection, and we have \((V, \cdot) \simeq \text{im} \theta_v = (V_s, \cdot)\). The generator group of \((V_s, \cdot)\) is \((V, \cdot)\), and the elementary system \(\mathcal{E}(V_s, \cdot)\) of \((V_s, \cdot)\) is \(\mathcal{E}\).

Taken together, Theorems 76 and 81 give the following.

**Theorem 82** The set of all elementary systems \(\mathcal{E}_A\) of all \(\ell\)-controllable complete group systems \(A\), or \(\{\mathcal{E}_A\}\), is the same as the set of all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\), \(\{\mathcal{E}\}\), or \(\{\mathcal{E}_A\} = \{\mathcal{E}\}\).

**Proof.** Theorem 76 shows that \(\{\mathcal{E}_A\} \subset \{\mathcal{E}\}\). The group system \((V_s, \cdot)\) of Theorem 81 is an \(\ell\)-controllable complete group system \(A\). Then Theorem 81 shows that given any \((\ell + 1)\)-depth elementary system \(\mathcal{E}\), we can find an \(\ell\)-controllable complete group system \(A\) whose elementary system \(\mathcal{E}_A\) is \(\mathcal{E}\); then \(\{\mathcal{E}\} \subset \{\mathcal{E}_A\}\).

We can use Theorem 81 and chain \([22]\) to find all \(\ell\)-controllable complete group systems \(A\) up to isomorphism from the set of all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\). First find the set of all \((\ell + 1)\)-depth global groups \((V, \cdot)\) from the set of all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\). To construct all \(\ell\)-controllable complete group systems \(A\) up to isomorphism, we divide the set of all \((\ell + 1)\)-depth global groups into equivalence classes \([V, \cdot]\). Pick one representative \((V, \cdot)\) from each equivalence class. Construct the \(\ell\)-controllable complete global group system \((V_s, \cdot)\) of \((V, \cdot)\) using chain \([22]\) of Theorem 81. The set of all global group systems \((V_s, \cdot)\) obtained this way, one for each equivalence class \([V, \cdot]\), is the set of all \(\ell\)-controllable complete group systems \(A\) up to isomorphism \([22]\) v9-v12.

In general, if we use the first homomorphism theorem for group systems with input group \((V, \cdot)\) and a homomorphism from \((V, \cdot)\) to an alphabet group \(A^t\) other than \((\nabla^l_{0,0}(V), \odot^l_{0,0})\), for each \(t \in \mathbb{Z}\), then we obtain an \(\ell\)-controllable group system, where \(l\) may be less than \(\ell\). And the \(\ell\)-controllable group system with \(l < \ell\) can be isomorphic to \((V_s, \cdot)\) \([22]\) v9-v12. This example shows the definition of isomorphism of finite groups is somewhat defective for group systems; for one reason it does not include the notion of time used in group systems. There is a more restrictive notion of isomorphism for the elementary system and global group, called list isomorphism \([22]\) v11-v12. Two group systems constructed from list isomorphic elementary systems and global groups must be \(\ell\)-controllable for the same \(l\). We can find all \(\ell\)-controllable complete group systems \(A\) up to list isomorphism from the set of all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\) up to list isomorphism in the same way as discussed just above for isomorphism \([22]\) v11-v12.

### 6.4 Construction of any elementary system \(\mathcal{E}\)

In the previous subsection, we discussed how to construct all \(\ell\)-controllable complete group systems \(C\) from all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\). We
now give a brief discussion of how to construct all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\). We first discuss how to construct a single elementary system \(\mathcal{E}\). Since an elementary system \(\mathcal{E}\) is nested, to construct an \((\ell + 1)\)-depth elementary system \(\mathcal{E}\), we first construct a 1-depth elementary system \(\mathcal{E}_1 = \{ (\nu^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : k = \ell, t \in \mathbb{Z} \} \); then a 2-depth elementary system \(\mathcal{E}_{\ell-1} = \{ (\nu^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : \ell - 1 \leq k \leq \ell, t \in \mathbb{Z} \} \), where there is a homomorphism from groups \((\nu^t_{0,\ell-1}(\mathcal{V}), \circ^t_{0,\ell-1})\) in \(\mathcal{E}_{\ell-1}\) to groups \((\nu^t_{0,\ell}(\mathcal{V}), \circ^t_{0,\ell})\) and \((\nu^{t-1}_{0,\ell}(\mathcal{V}), \circ^{t-1}_{0,\ell})\) in \(\mathcal{E}_\ell\) for each \(t \in \mathbb{Z}\); and continue on. In this way, we obtain a sequence of elementary systems \(\mathcal{E}_1, \mathcal{E}_{\ell-1}, \ldots, \mathcal{E}_m, \ldots, \mathcal{E}_1, \mathcal{E}_0 = \mathcal{E}\) which ends in \(\mathcal{E}_0 = \mathcal{E}\), where \(\mathcal{E}_m = \{ (\nu^t_{0,k}(\mathcal{V}), \circ^t_{0,k}) : m \leq k \leq \ell, t \in \mathbb{Z} \}\).

We may construct the sequence \(\mathcal{E}_1, \mathcal{E}_{\ell-1}, \ldots, \mathcal{E}_m, \ldots, \mathcal{E}_1, \mathcal{E}_0\) in the following way. Assume we have found the partial sequence \(\mathcal{E}_1, \mathcal{E}_{\ell-1}, \ldots, \mathcal{E}_{m+1}\) for some \(m\), \(0 < m \leq \ell\). We show how to find \(\mathcal{E}_m\). To find \(\mathcal{E}_m\) we have to find groups \((\nu^t_{0,m}(\mathcal{V}), \circ^t_{0,m})\) for each \(t \in \mathbb{Z}\), such that there is a homomorphism from \((\nu^t_{0,m}(\mathcal{V}), \circ^t_{0,m})\) to \((\nu^t_{0,m+1}(\mathcal{V}), \circ^t_{0,m+1})\) and \((\nu^{t-1}_{0,m+1}(\mathcal{V}), \circ^{t-1}_{0,m+1})\) in the elementary list, under the projection map from set \(\nu^t_{0,m}(\mathcal{V})\) to sets \(\nu^t_{0,m+1}(\mathcal{V})\) and \(\nu^{t-1}_{0,m+1}(\mathcal{V})\). The elementary groups \((\nu^t_{0,m+1}(\mathcal{V}), \circ^t_{0,m+1})\) and \((\nu^{t-1}_{0,m+1}(\mathcal{V}), \circ^{t-1}_{0,m+1})\) intersect and form the subdirect product group [22] \(^{17}\) \[
(\nu^t_{0,m+1}(\mathcal{V}), \circ^t_{0,m+1}) \ltimes (\nu^{t-1}_{0,m+1}(\mathcal{V}), \circ^{t-1}_{0,m+1}).
\]

Note that set \(\nu^t_{0,m}(\mathcal{V})\) is the same as set \(\nu^t_{0,m+1}(\mathcal{V}) \ltimes \nu^{t-1}_{0,m+1}(\mathcal{V})\) except for the addition of entries \(\nu^t_m\) at index position \((0, m)\) in \(\nu^t_{0,m}(\mathcal{V})\). Since there must be a homomorphism from \((\nu^t_{0,m}(\mathcal{V}), \circ^t_{0,m})\) to each individual group \((\nu^t_{0,m+1}(\mathcal{V}), \circ^t_{0,m+1})\) and \((\nu^{t-1}_{0,m+1}(\mathcal{V}), \circ^{t-1}_{0,m+1})\), there must be a homomorphism to subdirect product group [23]. If \(K\) is the kernel of this homomorphism, then \((\nu^t_{0,m}(\mathcal{V}), \circ^t_{0,m})\) is an extension of [23] by \(K\). This approach gives the construction of group \((\nu^t_{0,m}(\mathcal{V}), \circ^t_{0,m})\). Continuing in this way, we finally obtain \(\mathcal{E}_0 = \mathcal{E}\). More construction details can be found in v2-v6 of [22].

To construct all \((\ell + 1)\)-depth elementary systems \(\mathcal{E}\), we just iterate the above approach, first constructing all 1-depth elementary systems, then all 2-depth elementary systems for each of the 1-depth elementary systems, and so on. The approach in this subsection can also construct all linear block codes [22] v9-v12.
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