The Smoothed Possibility of Social Choice

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Abstract

We develop a framework to leverage the elegant “worst average-case” idea in smoothed complexity analysis to social choice, motivated by modern applications of social choice powered by AI and ML. Using our framework, we characterize the smoothed likelihood of some fundamental paradoxes and impossibility theorems as the number of agents increases. For Condorcet’s paradox, we prove that the smoothed likelihood of the paradox either vanishes at an exponential rate, or does not vanish at all. For the folklore impossibility on the non-existence of voting rules that satisfy anonymity and neutrality, we characterize the rate for the impossibility to vanish, to be either polynomially fast or exponentially fast. We also propose a novel easy-to-compute tie-breaking mechanism that optimally preserves anonymity and neutrality for even number of alternatives in natural settings. Our results illustrate the smoothed possibility of social choice—even though the paradox and the impossibility theorem hold in the worst case, they may not be a big concern in practice in certain natural settings.

1 Introduction

Dealing with paradoxes and impossibility theorems is a major challenge in social choice theory, because “the force and widespread presence of impossibility results generated a consolidated sense of pessimism, and this became a dominant theme in welfare economics and social choice theory in general”, as the eminent economist Amartya Sen commented in his Nobel prize lecture [45].

Many paradoxes and impossibility theorems in social choice are based on worst-case analysis. Take perhaps the earliest impossibility theorem, Condorcet’s (voting) paradox [11], for example. Condorcet’s paradox states that, when there are at least three alternatives, it is impossible for pairwise majority aggregation to be transitive. The proof is done by explicitly constructing a worst-case scenario—a profile $P$ that contains a Condorcet cycle. For example, in $P = \{a \succ b \succ c, b \succ c \succ a, c \succ a \succ b\}$, there is a cycle $a \succ b, b \succ c, c \succ a$ of pairwise majority. Condorcet’s paradox is closely related to the celebrated Arrow’s impossibility theorem: if Condorcet’s paradox can be avoided, then the pairwise majority rule can avoid Arrow’s impossibility theorem.

As another example, a folklore impossibility theorem states that no single-winner voting rule $r$ can satisfy both anonymity ($r$ is insensitive to the identities of agents) and neutrality ($r$ is insensitive to the identities of alternatives). The proof is done by analyzing a worst-case scenario $P = \{a \succ b, b \succ a\}$. Suppose for the sake of contradiction that $r$ satisfies both anonymity and neutrality. Without loss of generality, suppose $r(P) = a$. After exchanging $a$ and $b$, the winner should be $b$ according to neutrality. But since the permuted profile still contains one vote for $a \succ b$ and one vote for $b \succ a$, the winner should be $a$ according to anonymity, which is a contradiction.

There is an enormous literature in social choice on circumventing the impossibilities, most of which belongs to the following two approaches. (1) Domain restrictions, namely, agents’ reported preferences are assumed to come from a subset of all linear orders such as single-peaked preferences [5][2][45][38][12][6][17]; and (2) likelihood analysis, where impossibility theorems are evaluated by the likelihood of their occurrence in profiles randomly generated from a distribution such as the i.i.d. uniform distribution, a.k.a. Impartial Culture (IC) [21][24]. Both approaches have been...
criticized for making strong and unrealistic assumptions on the domain and on the probability distributions, respectively. In particular, IC has received much criticism, see e.g. \cite{14}, yet no widely-accepted probabilistic model exists to the best of our knowledge.

The worst-case nature of the impossibility theorems might be desirable for high-stakes, less-frequent applications such as political elections, but its practical relevance is questionable, especially in the AI era, where AI systems can assist in group decision-making, for example by learning agents’ preferences \cite{50}. AI-powered social choice could be a promising solution to the turnout problem, which is one of the major problems social choice theory faces, as pointed out by Sen \cite{45}. Such AI systems can promote democracy on a larger scale and with a higher frequency. Additionally, despite the mathematical impossibilities, a social choice has to be made in many cases, so a more practical and discriminative—ideally quantitative—measure beyond worst-case analysis is desirable. This motivates us to ask the following key question:

**How can we measure the practical relevance of paradoxes and impossibility theorems in large-scale frequently-used applications of social choice?**

The large-scale feature naturally leads to the analysis of asymptotic occurrence of the impossibility theorems, and the frequently-used feature naturally leads to the analysis of average likelihood of the impossibility theorems. But in light of the criticism of the classical likelihood analysis approach discussed above, what is a realistic model to answer the key question?

Interestingly, computer science has encountered a similar challenge and has gone through a similar way in the analysis of practical performance of algorithms. Initially, the analysis mostly focused on the worst case, as in the spirit of big $O$ notation and NP-hardness. Domain restrictions have been a popular approach beyond the worst-case analysis. For example, while SAT is NP-hard, its restriction 2-SAT is in P. Likelihood analysis, in particular average-case analysis, has also been a popular approach, yet it suffers from the same criticism as its counterpart in social choice—the distribution used in the analysis may well be unrealistic \cite{48}.

The challenge was addressed by the **smoothed (complexity) analysis** introduced by Spielman and Teng \cite{47}, which focuses on the “worst average-case” scenario that combines the worst-case analysis and the average-case analysis. The idea is based on the fact that the input $\vec{x}$ of an algorithm is often a noisy perception of the ground truth $\vec{x}^*$. Therefore, the worst-case is analyzed by assuming that an adversary chooses a ground truth $\vec{x}^*$ and then Nature adds a noise $\vec{\epsilon}$ (e.g. a Gaussian noise) to it, such that the algorithm’s input becomes $\vec{x} = \vec{x}^* + \vec{\epsilon}$. The smoothed runtime of an algorithm is therefore $\max_{\vec{x}^*} \mathbb{E}_{\vec{\epsilon}} \text{Runtime}(\vec{x}^* + \vec{\epsilon})$, in contrast to the worst-case runtime $\max_{\vec{x}^*} \text{Runtime}(\vec{x}^*)$ and the average-case runtime $\mathbb{E}_{\vec{x}^*, \vec{\epsilon}} \text{Runtime}(\vec{x}^* + \vec{\epsilon})$, where $\pi$ is a given distribution over data.

**Our Contributions.** We are not aware of a previous work on smoothed analysis in classical social choice, and our conceptual contribution is a framework that leverages the elegant idea of smoothed complexity analysis to answer the key question above. In social choice, the data is a profile, which consists of agents’ reported preferences that are often represented by linear orders over a set $A$ of $m$ alternatives. Like in the smoothed complexity analysis, in our framework there is an adversary who controls agents’ “ground truth” preferences, which may be ordinal (as rankings over alternatives) or cardinal (as utilities over alternatives). Then, Nature adds a “noise” to the ground truth preferences and outputs a preferences profile, which consists of linear orders over alternatives.

We use a statistical model to model Nature’s noising procedure for each agent. As in many smoothed-analysis approaches, we assume that noises are independently generated, yet agents’ ground truth preferences can be arbitrarily correlated, which constitutes the basis for the worst-case analysis. Using our smoothed analysis framework, we obtain dichotomy theorems on the asymptotic smoothed likelihood of Condorcet’s paradox and the folklore impossibility theorem under mild assumptions, when the number of alternatives $m$ is fixed and the number of agents $n$ goes to infinity.

**Theorem 1** (Smoothed Condorcet’s paradox, informally put). The smoothed likelihood of Condorcet’s Paradox either vanishes at an exponential rate, or does not vanish at all.

**Theorem 2** (Smoothed folklore (im)possibility theorem, informally put). The theorem has two parts. The smoothed possibility part states that there exist voting rules under which the impossibility theorem either vanishes at an exponential rate or at a polynomial rate. The smoothed impossibility part states that there does not exist a voting rule under which the impossibility theorem vanishes faster than under the rules in the smoothed possibility part.
Both theorems are quite general and their formal statements also characterize conditions for each case. Such conditions in Theorem 1 will tell us when Condorcet’s Paradox vanishes (at an exponential rate), which is positive. The smoothed possibility part of Theorem 2 is positive too, because it states that the folklore impossibility theorem vanishes as the number of agents \( n \) increases. The smoothed impossibility part of Theorem 2 is mildly negative, because it states that no voting rule can do better, though the impossibility may still vanish as \( n \) increases. Together, our results illustrate the smoothed possibility of social choice—even though the paradox and the impossibility theorem hold in the worst case, they may not be a big concern in practice in some natural settings.

Our framework also allows us to develop a novel easy-to-compute tie-breaking mechanism called most popular singleton ranking (MPSR) tie-breaking, which tries to break ties using a linear order that uniquely occurs most often in the profile (Definition 8). We prove that MPSR can significantly reduce the smoothed likelihood of the folklore impossibility theorem with the commonly-used lexicographic tie-breaking and fixed-agent tie-breaking, from \( n^{-0.5} \) to \( n^{-5} \), for many commonly-studied voting rules under natural assumptions (Proposition 1 and Theorem 3), and is optimal for all even number of alternatives \( m \) (Theorem 2 and Lemma 2).

**Proof Techniques.** Standard approximation techniques such as Berry- Esseen theorem and its high-dimensional counterparts Lyapunov-type of results, due to their \( O(n^{-0.5}) \) error terms, are too coarse for the (tight) bound in Theorem 2. To prove our theorems, we first model various events of interest as systems of linear constraints. Then, we develop a technical tool (Lemma 1) to provide a dichotomy characterization for the histogram of a randomly generated profile to satisfy the constraints, which might be of independent interest. We further show in Appendix 14 that Lemma 1 is a powerful tool for analyzing smoothed likelihood of many other commonly-studied events in social choice (Table 4), which are otherwise hard to analyze. The smoothed (avoidance of) paradoxes and (im)possibility theorem, and the technical lemma are our technical contributions.

**Related Work and Discussions.** Smoothed analysis has been applied to a wide range of problems in mathematical programming, machine learning, numerical analysis, discrete math, combinatorial optimization, and equilibrium analysis and price of anarchy [10], see [48] for a survey. In a recent blue-sky paper, Baumeister et al. [4] proposed to conduct smoothed complexity analysis in computational aspects of social choice.

In our model, agents’ ground truth preferences can be arbitrarily correlated, and the randomness comes from independent noises. This is a standard assumption in smoothed complexity analysis as well as in the literature in psychology, economics, and behavioral science, such as in random utility models, logistic regression, etc. As another justification, the adversary can be seen as a manipulator who wants to affect agents’ reported preferences, but is only able to do it in a probabilistic way.

Our framework leverages the worst average-case idea in smoothed complexity analysis but is technically different in two aspects. First, the noising procedure is different. Our framework uses a statistical model to capture the noising procedure for each agent, while smoothed complexity analysis often assumes that a noise is directly added to the ground truth. Second, our results characterize asymptotic smoothed likelihood of events, while smoothed complexity analysis focuses on studying the relationship between the noise level and the smoothed runtime of an algorithm. In our framework the asymptotic anonymized profile can be viewed as the ground truth plus an Gaussian-like noise as in smoothed complexity analysis, but this viewpoint does not allow us to accurately analyze the smoothed likelihood of events due to the coarse \( O(n^{-0.5}) \) error in Gaussian approximations.

Our technical results are quite general and can be easily applied to classical likelihood analysis in social choice under i.i.d. distribution, to answer open questions, obtain new results, and provide new insights. For example, a straightforward application of Lemma 1 gives an asymptotic answer to an open question by Tsetlin et al. [49] (see the discussion after Corollary 2 in Appendix 14). As another example, we are not aware of a previous work that characterizes asymptotic likelihood of the folklore impossibility theorem even under IC, which is a special case of Theorem 2. We also show in Appendix 10.1 that our proof techniques for Theorem 2 can be used to provide an alternative proof for an elegant characterization for the folklore impossibility theorem to hold [39] Problem 1.

As discussed above, there is a large literature on domain restrictions and the likelihood analysis toward circumventing impossibility theorems, see, for example, the book by Gehrlein and Lepelley [25] for a recent survey. In particular, there is a large literature on the likelihood of Condorcet voting paradox and the likelihood of (non)-existence of Condorcet winner under i.i.d. distributions.
especially IC [15, 43, 22, 49, 27, 24, 8, 9]. The IC assumption has also been used to prove quantita-
tive versions of other impossibility theorems in social choice such as Arrow’s impossibility the-
orem [28, 29, 36] and Gibbard-Satterthwaite theorem [19, 37], as well as in judgement aggrega-
tion [40, 18]. Other works have studied social choice problems when each agent’s preferences are
represented by a probability distribution [3, 25, 23, 51, 41, 31]. These works have focused on de-
ing efficient algorithms for computing the outcome, which is quite different from our goal. Our
smoothed analysis framework is also related to the line of research on distortion in social choice,
which aims at characterizing the loss of social welfare due to the inefficiency in communicating
agents’ preferences to the center [42, 1, 32, 30]. Again, the goal there is quite different from ours.

2 Preliminaries

Basic Setting. Let \(\mathcal{A} = [m] = \{1, \ldots, m\}\) denote the set of \(m \geq 3\) alternatives. Let \(\mathcal{L}(\mathcal{A})\) denote
the set of all linear orders (a.k.a. rankings) over \(\mathcal{A}\). Let \(n \in \mathbb{N}\) denote the number of agents. Each
agent uses a linear order to represent his or her preferences. The vector of preferences is called a
(profile). For any profile \(P\), let Hist\((P) \in \mathbb{Z}_{\geq 0}^n\) denote the anonymized profile of \(P\), sometimes also called the histogram of \(P\), which counts the
number of each linear order in \(P\). A voting rule \(r\) is a mapping from each profile to a single winner in \(\mathcal{A}\). A voting correspondence \(c\) is a mapping from each profile to a non-empty set of co-winners.

Tie-breaking Mechanisms. Many commonly-studied voting rules are defined as correspondences
combined with a tie-breaking mechanism. For example, a positional scoring correspondence is
characterized by a scoring vector \(s = (s_1, \ldots, s_m)\) with \(s_1 \geq s_2 \geq \cdots \geq s_m\) and \(s_1 > s_m\). For
any alternative \(a\) and any linear order \(R \in \mathcal{L}(\mathcal{A})\), we let \(s(R, a) = s_i\), where \(i\) is the rank of \(a\)
in \(R\). Given a profile \(P\), the positional scoring correspondence \(c_s\) chooses all alternatives \(a\) with
maximum \(\sum_{R \in P} s(R, a)\). For example, Plurality uses the scoring vector \((1, 0, \ldots, 0)\) and Borda
uses the scoring vector \((m-1, m-2, \ldots, 0)\). The positional scoring rule \(c_s\) chooses a single
alternative by further applying a tie-breaking mechanism. The lexicographic tie-breaking, denoted by
LEX\(R\) where \(R \in \mathcal{L}(\mathcal{A})\), breaks ties in favor of alternatives ranked higher in \(R\). The fixed-agent
tie-breaking, denoted by FA\(-j\) where \(1 \leq j \leq n\), uses agent \(j\)’s preferences to break ties.

(Un)weighted Majority Graphs. For any profile \(P\) and any pair of alternatives \(a, b\), let \(P[a \succ b] \in \mathcal{L}(\mathcal{A})\)
denote the number of rankings in \(P\) where \(a\) is preferred to \(b\). Let WMG\((P)\) denote the weighted
majority graph of \(P\), where the vertices are \(\mathcal{A}\) and the edge weights are \(w_p(a, b) = P[a \succ b] - P[b \succ a]\). For any distribution \(\pi\) over \(\mathcal{L}(\mathcal{A})\), let WMG\((\pi)\) denote the weighted majority graph where \(\pi\) is treated as a fractional profile, where for each \(R \in \mathcal{L}(\mathcal{A})\) there are \(\pi(R)\) copies of \(R\).

The unweighted majority graph (UMG) of a profile \(P\), denoted by UMG\((P)\), is the unweighted
directed graph where the vertices are the alternatives and there is an edge \(a \rightarrow b\) if and only if
\(P[a \succ b] > P[b \succ a]\). If \(a\) and \(b\) are tied, then there is no edge between \(a\) and \(b\). UMG\((\pi)\) is defined
similarly. A Condorcet cycle of a profile \(P\) is a cycle in UMG\((P)\). A weak Condorcet cycle of a profile
\(P\) is a cycle in a supergraph of UMG\((P)\).

Axiomatic Properties. A voting rule \(r\) satisfies anonymity, if the winner is insensitive to the identity
of the voters. That is, for any pair of profiles \(P\) and \(P'\) with Hist\((P) = Hist(P')\), we have \(r(P) = r(P')\). \(r\) satisfies neutrality if the winner is insensitive to the identity of the alternatives. That is, for any permutation \(\sigma\) over \(\mathcal{A}\), we have \(r(\sigma(P)) = \sigma(r(P))\).

Single-Agent Preference Models. A statistical model \(\mathcal{M} = (\Theta, \mathcal{S}, \Pi)\) has three components: the
parameter space \(\Theta\), which contains the “ground truth”; the sample space \(\mathcal{S}\), which contains all
possible data; and the set of probability distributions \(\Pi\), which contains a distribution \(\pi_\theta\) over \(\mathcal{S}\)
for each \(\theta \in \Theta\). In this paper we use single-agent preference models, where \(\mathcal{S} = \mathcal{L}(\mathcal{A})\).

Definition 1. A single-agent preference model is denoted by \(\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)\). \(\mathcal{M}\) is strictly
positive if there exists \(d > 0\) such that the probability of any ranking under any distribution in \(\Pi\) is
greater than \(d\). \(\mathcal{M}\) is closed if \(\Pi\) is a closed set. \(\mathcal{M}\) is neutral if for any \(\theta \in \Theta\) and any permutation
\(\sigma\) over \(\mathcal{A}\), there exists \(\eta \in \Theta\) such that for all \(V \in \mathcal{L}(\mathcal{A})\), we have \(\pi_\theta(V) = \pi_\eta(\sigma(V))\).

For example, in a single-agent Mallows’ model \(\mathcal{M}_{\text{M}}\), \(\Theta = \mathcal{L}(\mathcal{A}) \times [0, 1]\), where in each \((R, \varphi) \in \Theta\), \(R\) is the central ranking and \(\varphi\) is the dispersion parameter. For any \(W \in \mathcal{L}(\mathcal{A})\), we have \(\pi(R, \varphi) = \varphi^{\text{KT}(R,W)} / Z_\varphi\) where \(\text{KT}(R,W)\) is the Kendall Tau distance between \(R\) and \(W\), namely the number of pairwise disagreements between \(R\) and \(W\), and \(Z_\varphi = \sum_{W \in \mathcal{L}(\mathcal{A})} \varphi^{\text{KT}(R,W)}\) is the
normalization constant. For any $0 < \epsilon \leq 1$, we let $M^{[e,1]}_{\text{Ma}}$ denote the Mallows’ model where the parameter space is $L(A) \times [e, 1]$. As another example, in the single-agent Plackett-Luce model $M_{\text{Pl}}$, let

$$\Theta = \{ \tilde{\theta} \in [0, 1]^m : \tilde{\theta} \cdot 1 = 1 \}.$$ 

For any $\tilde{\theta} \in \Theta$ and any $R = \sigma(1) \succ \sigma(2) \succ \cdots \succ \sigma(m)$, we have $\pi_{\tilde{\theta}}(R) = \prod_{i=1}^{m-1} \frac{\theta_{\sigma(i)}}{\sum_{j=1}^{m} \theta_{\sigma(i)}}$. For any $0 < \epsilon \leq 1$, we let $M^{[e,1]}_{\text{Pl}}$ denote the Plackett-Luce model where $\Theta = \{ \tilde{\theta} \in [e, 1]^m : \tilde{\theta} \cdot 1 = 1 \}$. It follows that for any $0 < \epsilon \leq 1$, $M^{[e,1]}_{\text{Ma}}$ and $M^{[e,1]}_{\text{Pl}}$ are strictly positive, closed, and neutral.

### 3 Smoothed Analysis Framework and The Main Technical Lemma

Many commonly-studied axioms and events in social choice, denoted by $X$, are defined in the following way. Let $r$ denote a voting rule or correspondence and let $P$ denote a profile. There is a function $S_X(r, P) \in \{0, 1\}$ that indicates whether $X$ holds for $r$ at $P$. Then, $r$ satisfies $X$ if $\forall P, S_X(r, P) = 1$, or equivalently, $\min_P S_X(r, P) = 1$. For example, for anonymity, let $S_{\text{ano}}(r, P) = 1$ iff for all profiles $P'$ with Hist$(P') = \text{Hist}(P)$, we have $r(P') = r(P)$. For neutrality, let $S_{\text{neu}}(r, P) = 1$ iff for all permutation $\sigma$ over $A$, we have $\sigma(r(P)) = r(\sigma(P))$. For non-existence of Condorcet cycle, let $S_{\text{NCC}}(P) = 1$ iff there is no Condorcet cycle in $P$.

Our smoothed analysis framework assumes that each of the $n$ agents’ preferences are chosen from a single-agent preference model $M = (\Theta, L(A), \Pi)$ by the adversary.

**Definition 2 (Smoothed likelihood of events).** Given a single-agent preference model $M = (\Theta, L(A), \Pi), n \in \mathbb{N}$ agents, a function $S_X$ that characterizes an axiom or event $X$, and a voting rule (or correspondence) $r$, the smoothed likelihood of $X$ is defined as $\min_{\pi \in \Pi^n} \mathbb{E}_{r \sim \pi} S_X(r, P)$.

For example, $\min_{\pi \in \Pi^n} \mathbb{E}_{r \sim \pi} S_{\text{NCC}}(P)$ is the smoothed likelihood of non-existence of Condorcet cycle, which corresponds to the avoidance of Condorcet’s paradox. $\min_{\pi \in \Pi^n} \mathbb{E}_{r \sim \pi} S_{\text{amo}}(P) + S_{\text{neu}}(P) = 2$ is the smoothed likelihood of satisfaction of anonymity + neutrality, which corresponds to the avoidance of the folklore impossibility theorem.

We use the following simple example to show how to model events of interest as a system of linear constraints. The first event is closely related to the smoothed Condorcet’s paradox (Theorem 1) and the second event is closely related to the smoothed folklore theorem (Theorem 2).

**Example 1.** Let $m = 3$ and $A = \{1, 2, 3\}$. For any profile $P$, let $x_{123}$ denote the number of $1 \succ 2 \succ 3$ in $P$. The event “there is a Cordorcet cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$” can be represented by:

$$
\begin{align*}
& x_{123} + x_{132} + x_{312} - (x_{213} + x_{231} + x_{321}) > 0 & (1) \\
& x_{213} + x_{231} + x_{132} - (x_{312} + x_{321} + x_{123}) > 0 & (2) \\
& x_{312} + x_{321} + x_{123} - (x_{132} + x_{132} + x_{231}) > 0 & (3)
\end{align*}
$$

Equation (1) (respectively, (2) and (3)) states that UMG$(P)$ has edge $1 \rightarrow 2$ (respectively, $2 \rightarrow 3$ and $3 \rightarrow 1$). As another example, the event “Hist$(P)$ is invariant to the permutation $\sigma$ over $A$ that exchanges $1$ and $2$” can be represented by $\{x_{123} - x_{213} = 0, x_{132} - x_{312} = 0, x_{312} - x_{321} = 0\}$.

Notice that in this example each constraint has the form $\tilde{a} \cdot \tilde{x} = 0$ or $\tilde{b} \cdot \tilde{x} > 0$, where $\tilde{a} \cdot 1 = 0$ and $\tilde{b} \cdot 1 = 0$. More generally, our main technical lemma upper-bounds the smoothed likelihood for Hist$(P)$ to satisfy a similar system of linear constraints.

**Definition 3.** Let $q, n \in \mathbb{N}$. For any vector of $n$ distributions $\vec{\pi} = (\pi_1, \ldots, \pi_n)$, each of which is over $[q]$, let $\vec{Y} = (Y_1, \ldots, Y_n)$ denote the vector of $n$ random variables distributed as $\pi_1, \ldots, \pi_n$, respectively, and let $\vec{X}_\vec{\pi} = \text{Hist}(\vec{Y})$, which is a $q$-dimensional random vector counting the numbers of occurrence of each $i \leq q$ in $\vec{Y}$.

**Definition 4.** Let $C_{\vec{A}\vec{B}}(\vec{x}) = \{ A \cdot (\vec{x})^T = (\vec{0})^T \text{ and } B \cdot (\vec{x})^T > (\vec{0})^T \}$, where $A$ is a $K \times q$ integer matrix and $B$ is an $L \times q$ integer matrix with $K + L \geq 1$. Let $C_{\vec{A}\vec{B}}(\vec{x}) = \{ A \cdot (\vec{x})^T = (\vec{0})^T \text{ and } B \cdot (\vec{x})^T \geq (\vec{0})^T \}$ denote the relaxation of $C_{\vec{A}\vec{B}}(\vec{x})$. Let Sol$_{\vec{A}\vec{B}}$ and $\overline{\text{Sol}}_{\vec{A}\vec{B}}$ denote the solutions to $C_{\vec{A}\vec{B}}(\vec{x})$ and $\overline{C}_{\vec{A}\vec{B}}(\vec{x})$, respectively.

**Lemma 1 (Main technical lemma).** Let $q \in \mathbb{N}$ and let $\Pi$ be a closed set of strictly positive distributions over $[q]$. Let CH$(\Pi)$ denote the convex hull of $\Pi$.

**Upper bound.** For any $n \in \mathbb{N}$ and any $\vec{\pi} \in \Pi^n$,
The surprising part of the theorem is the degree of polynomial \( O(\|\pi\|) \) because the center multinomial-Gaussian. Then, the zero part of Lemma 1 is trivial; the exponential part makes sense theorem as follows. Roughly, \( \vec{X}_\pi \) is distributed around \( \vec{\pi} \cdot \vec{1} \) (which is a \( q \)-dimensional vector) like a multinomial-Gaussian. Then, the zero part of Lemma 1 is trivial; the exponential part makes sense because the center \( \vec{\pi} \cdot \vec{1} \) is \( \Theta(n) \) away from any vector in \( \text{Sol}_{AB} \); and the last part is expected to be \( O(n^{-0.5}) \) because the center \( \vec{\pi} \cdot \vec{1} \) satisfies the \( A \) part of \( C_{AB} \).

At a high level the theorem is quite natural and follows the intuition of multivariate central limit theorem as follows. Suppose for all \( \pi \in \Pi \), \( UMG(\pi) \) does not contain a weak Condorcet cycle. Then, for any \( n \in \mathbb{N} \), we have:

\[
\Pr \left( \vec{X}_\pi \in \text{Sol}_{AB} \right) = \begin{cases} 
0 & \text{if } \text{Sol}_{AB} = \emptyset \\
\exp(-\Omega(n)) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset \\
\Omega\left(n^{-\frac{\text{rank}(A)}{2}}\right) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) \neq \emptyset
\end{cases}
\]

**Tightness of the upper bound.** There exists a constant \( C \) such that for any \( n' \in \mathbb{N} \), there exists \( n' \leq n \leq Cn' \) and \( \vec{\pi} \in \Pi^n \) such that

\[
\Pr \left( \vec{X}_\pi \in \text{Sol}_{AB} \right) = \begin{cases} 
\exp(-O(n)) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset \\
\Omega\left(n^{-\frac{\text{rank}(A)}{2}}\right) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) \neq \emptyset
\end{cases}
\]

Lemma 1 is quite general, as the assumptions on \( \Pi \) are mild, and \( A \) and \( B \) are general enough to model a wide range of events in social choice as we will see later in the paper. The theorem provides asymptotically tight upper bounds on the probability for the histogram of a randomly generated profile from \( \vec{\pi} \in \Pi^n \) to satisfy all constraints in \( C_{AB} \). The bounds provide a trichotomy: if no vector satisfies all constraints in \( C_{AB} \), i.e. \( \text{Sol}_{AB} = \emptyset \), then the upper bound is \( 0 \); otherwise if \( \text{Sol}_{AB} \neq \emptyset \) and its relaxation \( \overline{\text{Sol}}_{AB} \) does not contain a vector in the convex hull of \( \Pi \), then the upper bound is exponentially small; otherwise the upper bound is polynomially small in \( n \), and the degree of polynomial is determined by the rank of \( A \), namely \( \frac{\text{rank}(A)}{2} \). The tightness part of the theorem states that the upper bounds cannot be improved for all \( n \).

4 Smoothed Likelihood of Condorcet’s Paradox and Smoothed Folklore (Im)possibility Theorem

We first apply the main technical lemma (Lemma 1) to characterize the smoothed likelihood of Condorcet’s paradox in the following dichotomy theorem.

**Theorem 1 (Smoothed likelihood of Condorcet’s paradox).** Let \( M = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model.

**Smoothed avoidance of Condorcet’s paradox.** Suppose for all \( \pi \in \text{CH}(\Pi) \), \( UMG(\pi) \) does not contain a weak Condorcet cycle. Then, for any \( n \in \mathbb{N} \), we have:

\[
\min_{\pi \in \Pi^n} \mathbb{E}_{P \sim \pi} S_{NCC}(P) = 1 - \exp(-\Omega(n))
\]

**Smoothed Condorcet’s paradox.** Suppose there exists \( \pi \in \text{CH}(\Pi) \) such that \( UMG(\pi) \) contains a weak Condorcet cycle. Then, there exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\min_{\pi \in \Pi^n} \mathbb{E}_{P \sim \pi} S_{NCC}(P) = 1 - \Omega(1)
\]

The first part of the theorem (smoothed avoidance) is positive news: if there is no weak Condorcet cycle in the UMG of any distribution in the convex hull of \( \Pi \), then no matter how the adversary sets agents’ ground truth preferences, the probability for Condorcet’s paradox to hold, which is \( 1 - \min_{\pi \in \Pi^n} \mathbb{E}_{P \sim \pi} S_{NCC}(P) = \max_{\pi \in \Pi^n} \Pr_{P \sim \pi}(S_{NCC}(P) = 0) \), vanishes at an exponential rate as \( n \to \infty \). The second part (smoothed paradox) states that otherwise the adversary can make Condorcet’s paradox occur with constant probability. The proof is done by modeling \( S_{NCC}(P) \approx 0 \) as systems of linear constraints as in Definition 4, each of which represents a target UMG with a weak Condorcet cycle as in Example 1, then applying Lemma 1. The full proof is in Appendix B.
We now turn to the smoothed folklore impossibility theorem. We will reveal a relationship between all \(n\)-profiles and all permutation groups over \(A\) after recalling some basic notions in group theory. The symmetric group over \(A = [m]\), denoted by \(S_A\), is the set of all permutations over \(A\). A permutation group \(G\) is a subgroup of \(S_A\) where the identity element is denoted by \(\text{Id}\), and for any \(\sigma, \eta \in S_A\), \(\sigma \circ \eta\) is the permutation such that for any \(R \in \mathcal{L}(A)\), \((\sigma \circ \eta)(R) = \sigma(\eta(R))\).

**Definition 5.** For any profile \(P\), let \(\text{Perm}(P)\) denote the set of all permutations \(\sigma\) over \(A\) that maps \(\text{Hist}(P)\) to itself. Formally, \(\text{Perm}(P) = \{\sigma \in S_A : \text{Hist}(P) = \sigma(\text{Hist}(P))\}\).

See Appendix [9] for additional notation and examples about group theory. It is not hard to see that \(\text{Perm}(P)\) is a permutation group. We now define a special type of permutation groups that “cover” all alternatives in \(A\), which are closely related to the impossibility theorem.

**Definition 6.** For any permutation group \(U \subseteq S_A\) and any alternative \(a \in A\), we say that \(U\) covers \(a\) if there exists \(\sigma \in U\) such that \(a \neq \sigma(a)\). We say that \(U\) covers \(A\) if it covers all alternatives in \(A\). For any \(m\), let \(U_m\) denote the set of all permutation groups that cover \(A\).

For example, when \(m = 3\), \(U_3\) contains only one permutation group \(\{\text{Id}, (1, 2, 3), (1, 3, 2)\}\), where \((1, 2, 3)\) is the circular permutation \(1 \rightarrow 2 \rightarrow 3 \rightarrow 1\). See Example [4] in Appendix [9] for the list of all permutation groups for \(m = 3\).

**Theorem 2 (Smoothed folklore (im)possibility theorem).** Let \(A = (\Theta, \mathcal{L}(A), \Pi)\) be a strictly positive and closed single-agent preference model. Let \(U_m^\Pi = \{U : U \in \mathcal{U}_m : \exists \pi \in \text{CH}(\Pi), \forall a \in A, \sigma(\pi) = \pi, \text{ and when } U_m^\Pi \neq \emptyset, \text{ let } l_{\text{min}} = \min_{U \in \mathcal{U}_m}(|U|) \text{ and } l_1 = \frac{l_{\text{min}} - 1}{m}!\).

**Smoothed possibility.** There exist an anonymous voting rule \(r_{\text{ano}}\) and a neutral voting rule \(r_{\text{neu}}\) such that for any \(r \in \{r_{\text{ano}}, r_{\text{neu}}\}\), any \(n\), and any \(\hat{\sigma} \in \Pi^n\), we have:

\[
\Pr_{P \sim \hat{\sigma}}(S_{\text{ano}}(r, P) + S_{\text{neu}}(r, P) < 2) = \begin{cases} O(n^{-\frac{m}{2}}) & \text{if } U_m^\Pi \neq \emptyset \\ \exp(-\Omega(n)) & \text{otherwise} \end{cases}
\]

**Smoothed impossibility.** For any voting rule \(r\), there exist infinitely many \(n \in \mathbb{N}\) such that:

\[
\max_{\hat{\sigma} \in \Pi^n} \Pr_{P \sim \hat{\sigma}}(S_{\text{ano}}(r, P) + S_{\text{neu}}(r, P) < 2) = \begin{cases} \Omega(n^{-\frac{m}{2}}) & \text{if } U_m^\Pi \neq \emptyset \\ \exp(-\Omega(n)) & \text{otherwise} \end{cases}
\]

We note that for any profile \(P\), \(S_{\text{ano}}(r, P) + S_{\text{neu}}(r, P) < 2\) if and only if at least one of anonymity or neutrality is violated at \(P\). In other words, if \(S_{\text{ano}}(r, P) + S_{\text{neu}}(r, P) = 2\) then both anonymity and neutrality are satisfied at \(P\). Therefore, the first part of Theorem 2 is called “smoothed possibility” because it states that no matter how the adversary sets agents’ ground truth preferences, the probability for \(r_{\text{ano}}\) (respectively, \(r_{\text{neu}}\)) to satisfy both anonymity and neutrality converges to 1. The second part (smoothed impossibility) shows that the rate of convergence in the first part is asymptotically tight for all \(n\). This is a mild impossibility theorem because violations of anonymity or neutrality may still vanish (at a slower rate) as \(n \rightarrow \infty\).

The proof proceeds in the following three steps. Step 1. For any \(m\) and \(n\), we define a set of profiles, denoted by \(T_{m,n}\), that represent the source of impossibility. In fact, \(T_{m,n}\) is the set of all \(n\)-profiles \(P\) such that \(\text{Perm}(P)\) covers \(A\), i.e. \(\text{Perm}(P) \in \mathcal{U}_m\). Step 2. To prove the smoothed possibility part, we define \(r_{\text{ano}}\) and \(r_{\text{neu}}\) that satisfy both anonymity and neutrality for all profiles that are not in \(T_{m,n}\). Then, we apply Lemma 1 to upper-bound the probability of \(T_{m,n}\). Step 3. The smoothed impossibility part is proved by applying the tightness part of Lemma 1 to the probability of \(T_{m,n}\). The full proof can be found in Appendix 10.

In general \(l_{\text{min}}\) in Theorem 2 can be hard to characterize. The following lemma provides a lower bound on \(l_{\text{min}}\) by characterizing \(l_{\text{min}} = \min_{U \in \mathcal{U}_m}(|U|)\), whose group-theoretic proof is in Appendix 11.

**Lemma 2.** For any \(m \geq 2\), let \(l^* = \min_{U \in \mathcal{U}_m}(|U|)\). We have \(l^* = 2\) if \(m\) is even; \(l^* = 3\) if \(m\) is odd and \(3 \mid m\); \(l^* = 5\) if \(m\) is odd, \(3 \mid m\), and \(5 \mid m\); and \(l^* = 6\) for other \(m\).

We note that both Theorem 1 and Theorem 2 are quite general as they require mild assumptions on \(\Pi\). A notable special case of Theorem 2 is \(\pi_{\text{uni}} \in \text{CH}(\Pi)\), where \(\pi_{\text{uni}}\) is the uniform distribution over \(\mathcal{L}(A)\). We note that for any permutation \(\sigma, \pi_{\text{uni}} = \sigma(\pi_{\text{uni}})\), which means that \(U_{m,\text{uni}} = \mathcal{U}_m\). Therefore, only the polynomial bound in Theorem 2 remains, with \(l_1 = \frac{l^* - 1}{m}!\). In particular, \(\pi_{\text{uni}} \in \text{CH}(\Pi)\) for all neutral single-agent preference models, such as \(\mathcal{M}_{\text{M}, \epsilon}^{[1]}\), \(\mathcal{M}_{\text{P}, \epsilon}^{[1]}\) where \(0 < \epsilon \leq 1\), and IC, which corresponds to \(\Pi = \{\pi_{\text{uni}}\}\). See Corollary [1] in Appendix 11.1 for the formal statement.
5 Optimal Tie-Breaking for Anonymity + Neutrality

While \( r_{\text{ano}} \) and \( r_{\text{neu}} \) in Theorem 3 are asymptotically optimal w.r.t. anonymity + neutrality, they may be hard to compute. The following proposition shows that the commonly-used LEX and FA mechanisms are far from being optimal for positional scoring rules.

**Proposition 1.** Let \( r \) be a voting rule obtained from a positional scoring correspondence by applying LEX or FA. Let \( \mathcal{M} = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model where \( \pi_{anu} \in \mathcal{CH}(\Pi) \). There exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\max_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}} (S_{\text{ano}}(r, P) + S_{\text{neu}}(r, P) < 2) = \Omega(n^{-0.5})
\]

The proof is done by modeling ties under positional scoring correspondences as systems of linear constraints, then applying the tightness of the polynomial bound in Lemma 1. The full proof can be found in Appendix 12. We now introduce a new class of easy-to-compute tie-breaking mechanisms based on MPSR.

**Definition 7 (Most popular singleton ranking).** Given a profile \( P \), we define its most popular singleton ranking (MPSR) as \( \text{MPSR}(P) = \arg \max_R P[R] : \exists W \neq R \text{ s.t. } P[W] = P[R] \).

Put differently, a ranking \( R \) is called a **singleton** in a profile \( P \), if there does not exist another linear order that occurs for the same number of times in \( P \). MPSR \( P \) is the singleton ranking that occurs most frequently in \( P \). If no singleton exists, then we let MPSR \( P = \emptyset \). We now define tie-breaking mechanisms based on MPSR.

**Definition 8 (MPSR tie-breaking mechanism).** For any voting correspondence \( c \), any profile \( P \), and any backup tie-breaking mechanism TB, the MPSR-then-TB mechanism uses MPSR \( P \) to break ties whenever MPSR \( P \neq \emptyset \); otherwise it uses TB to break ties.

**Theorem 3.** Let \( \mathcal{M} = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model, where \( \pi_{anu} \in \mathcal{CH}(\Pi) \). For any voting correspondence \( c \) that satisfies anonymity and neutrality, let \( r_{\text{MPSR}} \) denote the voting rule obtained from \( c \) by MPSR-then-TB. For any \( n \) and any \( \vec{\pi} \in \Pi^n \),

\[
\Pr_{P \sim \vec{\pi}} (S_{\text{ano}}(r_{\text{MPSR}}, P) + S_{\text{neu}}(r_{\text{MPSR}}, P) < 2) = O(n^{-1/4})
\]

Moreover, if TB satisfies anonymity (respectively, neutrality) then so does \( r_{\text{MPSR}} \).

The proof is done by showing that (1) anonymity and neutrality are preserved when MPSR \( P \neq \emptyset \), and (2) any profile \( P \) with MPSR \( P = \emptyset \) can be represented by a system of linear constraints, whose smoothed likelihood is upper-bounded by the polynomial upper bound in Lemma 1. The full proof can be found in Appendix 12.

Note that when \( m \) is even, the \( O(n^{-1/4}) \) upper bound in Theorem 3 matches the optimal upper bound in light of Theorem 2 and Lemma 2. This is good news because it implies that any anonymous and neutral correspondence can be made an asymptotically optimal voting rule w.r.t. anonymity+neutrality by MPSR tie-breaking. When \( m \) is odd, the \( O(n^{-1/4}) \) upper bound in Theorem 3 is suboptimal but still significantly better than that of the popular lexicographic or fixed-agent tie-breaking mechanisms, which is \( \Omega(n^{-0.5}) \) (Proposition 1).

**Future work.** We have only touched the tip of the iceberg of smoothed analysis in social choice. There are at least three major dimensions for future work: (1) other social choice axioms and impossibility theorems, for example Arrow’s impossibility theorem [28, 29, 36] and the Gibbard-Satterthwaite theorem [19, 37], (2) computational aspects in social choice [7, 4] such as the smoothed complexity of winner determination and complexity of manipulation, and (3) other social choice settings such as judgement aggregation [40, 18], matching, resource allocation, etc.

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7 Appendix: Proof of Lemma 1

Lemma 1 (Main technical lemma). Let $q \in \mathbb{N}$ and $\Pi$ be a closed set of strictly positive distributions over $[q]$. Let $\text{CH} (\Pi)$ denote the convex hull of $\Pi$.

Upper bound. For any $n \in \mathbb{N}$ and any $\vec{\pi} \in \Pi^n$,

$$
\Pr \left( \vec{X}_{\vec{\pi}} \in \text{Sol}_{AB} \right) = \begin{cases} 
0 & \text{if } \text{Sol}_{AB} = \emptyset \\
\exp(-\Omega(\frac{1}{n})) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset \\
\Omega(n^{-\frac{\text{Rank}(\Pi)}{2}}) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) \neq \emptyset 
\end{cases}
$$

Tightness of the upper bound. There exists a constant $C$ such that for any $n' \in \mathbb{N}$, there exists $n' \leq n \leq Cn'$ and $\vec{\pi} \in \Pi^n$ such that

$$
\Pr \left( \vec{X}_{\vec{\pi}} \in \text{Sol}_{AB} \right) = \begin{cases} 
\exp(-\Omega(n)) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset \\
\Omega(n^{-\frac{\text{Rank}(\Pi)}{2}}) & \text{if } \text{Sol}_{AB} \neq \emptyset \text{ and } \text{Sol}_{AB} \cap \text{CH}(\Pi) \neq \emptyset 
\end{cases}
$$

Proof. The $\text{Sol}_{AB} = \emptyset$ case trivially holds. Let $o = \text{Rank}(A)$. Let $\vec{X} = \text{Hist}(\overrightarrow{Y}) = (X_{\vec{\pi},1}, \ldots, X_{\vec{\pi},q})$. That is, for any $i \leq q$, $X_{\vec{\pi},i}$ represents the number of occurrences of outcome $i$ in $\overrightarrow{Y}$. For any $n \in \mathbb{N}$ and any $\vec{\pi} \in \Pi^n$, let $\vec{\mu}_{\vec{\pi}} = (\mu_{\vec{\pi},1}, \ldots, \mu_{\vec{\pi},q}) = \mathbb{E}(\sum_{j=1}^{n} \vec{X}_{\vec{\pi}}/n)$ denote the mean of $\vec{X}_{\vec{\pi},i}$ and let $\vec{d}_{\vec{\pi}} = (\sigma_{\vec{\pi},1}, \ldots, \sigma_{\vec{\pi},q})$, where for each $i \leq q$, $\sigma_{\vec{\pi},i} = \sqrt{\text{Var}(X_{\vec{\pi},i})}/n$. Because $\Pi$ is strictly positive, there exists $d_1 > 0, d_2 > 0$ such that for all $n$, all $\vec{\pi} \in \Pi^n$, and all $i \leq q$, we have $d_1 < \mu_{\vec{\pi},i} < d_2$ and $d_1^2 < \sigma_{\vec{\pi},i} < d_2^2$.

Upper bound when $\text{Sol}_{AB} \neq \emptyset$ and $\text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset$. It is not hard to see that $\text{Sol}_{AB}$ is convex and closed. Because $\Pi$ is closed and bounded, $\text{CH}(\Pi)$ is convex, closed, and compact. Because $\text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset$, by the strict hyperplane separation theorem, there exists a hyperplane that strictly separates $\text{Sol}_{AB}$ and $\text{CH}(\Pi)$. Therefore, there exists $\epsilon' > 0$ such that for any $\vec{x}_1 \in \text{Sol}_{AB}$ and any $\vec{x}_2 \in \text{CH}(\Pi)$, we have $|\vec{x}_1 - \vec{x}_2|_\infty > \epsilon'$, where $|x|_\infty$ is the $L_\infty$ norm. This means that any solution to $\text{CH}(\vec{x})$ is at least $\epsilon' n$ away from $n \cdot \vec{\mu}_{\vec{\pi}}$ in $L_\infty$. Therefore, we have:

$$
\Pr \left( \vec{X}_{\vec{\pi}} \in \text{Sol}_{AB} \right) \leq \Pr \left( |\vec{X}_{\vec{\pi}} - n \cdot \vec{\mu}_{\vec{\pi}}|_\infty > \epsilon' n \right) \leq \sum_{i=1}^{q} \Pr(|X_{\vec{\pi},i} - n \mu_{\vec{\pi},i}| > \epsilon' n)
$$

$$
\leq 2q \exp \left( - \frac{(\epsilon')^2 n}{(1 - 2d_2)^2} \right)
$$

The last inequality follows after Hoeffding’s inequality (Theorem 2 in [26]), where $\delta$ is a constant such that any distribution in $\Pi$ is bounded above 0 by $\delta$.

Upper bound when $\text{Sol}_{AB} \neq \emptyset$ and $\text{Sol}_{AB} \cap \text{CH}(\Pi) \neq \emptyset$. Let $C_A (\vec{x}) = \{ A \cdot \vec{x} = (\vec{0})^\top \}$ denote the relaxation of $C_A (\vec{x})$ by removing the $B$ part. Let $C_A (\vec{X})$ denote the event that $\vec{X}_{\vec{\pi}}$ satisfies all constraints in $C_A$. It follows that each vector in $\text{Sol}_{AB}$ is a solution to $C_A (\vec{x})$, which means that $\Pr(\vec{X}_{\vec{\pi}} \in \text{Sol}_{AB}) \leq \Pr(\text{C}_{A}(\vec{X}))$. Therefore, it suffices to prove that for any $n \in \mathbb{N}$ and any $\vec{\pi} \in \Pi^n$, $\Pr(\text{C}_{A}(\vec{X}_{\vec{\pi}})) = O(n^{-\frac{\text{Rank}(\Pi)}{2}})$. Because $A \cdot (\vec{I})^\top = (\vec{0})^\top$, it follows that $\vec{I} = \{ 1 \}^q$ is linearly independent with the row vectors of $A$. Therefore, the rank of $A' = \begin{bmatrix} A \\ I \end{bmatrix}$ is $o + 1$. Let $C_A (\vec{x}) = \{ A \cdot \vec{x} = 0 \text{ and } \vec{1} \cdot \vec{x} = \epsilon \}$.

From basic linear algebra, in particular the reduced row echelon form (a.k.a. row canonical form) [33] of $A'$ computed by Gauss-Jordan elimination where $\epsilon$ is treated as a constant, we know that there exist $I_0 \subseteq [q]$ and $I_1 \subseteq [q]$ such that $I_0 \cap I_1 = \emptyset$, $|I_0| = \text{Rank}(A') = \text{Rank}(A) + 1$, and an $|I_0| \times (|I_1| + 1)$ matrix $D$ in $\mathbb{Q}$ such that $C_A (\vec{x})$ is equivalent to $(\vec{x}_{I_0})^\top = D \cdot [\vec{x}_{I_1}, n]^\top$, where $\vec{x}_{I_0}$ is the subvector of $\vec{x}$ that contains variables whose subscripts are in $I_0$. In other words, a vector $\vec{x}$ satisfies $C_A (\vec{x})$ if and only if $(\vec{x}_{I_0})^\top = D \cdot [\vec{x}_{I_1}, n]^\top$. See Example 5 in Appendix 7.1 for an example of deriving $D$ from $A'$.

We note that $I_1 \cup I_0 = [q]$. For the sake of contradiction suppose this is not true, which means that the reduced row echelon form of $A'$ has a column of zeros for some variable $x_j$. However, this means that $\vec{1}$ is not a linear combination of the rows of the reduced row echelon form of $A'$, which is
a contradiction because $A'$, which includes $\bar{t}$, can be obtained from a series of linear transformations on its reduced row echelon form. W.l.o.g. in the remainder of this proof we let $I_0 = \{1, \ldots, o + 1\}$ and $I_1 = \{o + 2, \ldots, q\}$.

The hardness in bounding $\Pr(C_A(\mathbf{x}_\#))$ is that elements of $\mathbf{x}_\#$ are not independent, and typical asymptotical tools such as Lyapunov-type bound are too coarse. To solve this issue, we use the following alternative representation of $Y_1, \ldots, Y_n$. For each $j \leq n$, we use a binary random variable $Z_j \in \{0, 1\}$ to represent whether the outcome of $Y_j$ is in $I_0$ (corresponding to $Z_j = 0$) or is in $I_1$ (corresponding to $Z_j = 1$). Then, we use another random variable $W_j \in [q]$ to represent the outcome of $Y_j$ conditioned on $Z_j$. See Figure 1 for an illustration.

At a high level, this addresses the independent issue because components of $\mathbf{W}$ are conditionally independent given $\mathbf{Z}$, and as we will see below, concentration happens in $\mathbf{W}$ when $\mathbf{Z}$ contains $\Theta(n)$ many 0’s.

**Definition 9 (Alternative representation of $Y_1, \ldots, Y_n$).** For each $j \leq n$, we define a Bayesian network with two random variables $Z_j \in \{0, 1\}$ and $W_j \in [q]$, where $Z_j$ is the parent of $W_j$, and

- for each $l \in \{0, 1\}$, $\Pr(Z_j = l) = \Pr(Y_j \in I_l)$;
- for each $l \in \{0, 1\}$ and each $t \leq q$, $\Pr(W_j = t|Z_j = l) = \Pr(Y_j = t|Y_j \in I_l)$.

In particular, if $t \notin I_l$ then $\Pr(W_j = t|Z_j = l) = 0$.

It follows that $W_j$ has the same distribution as $Y_j$. For any $\mathbf{z} \in \{0, 1\}^n$, we let $\text{Ind}_0(\mathbf{z}) \subseteq [n]$ denote the indices of $\mathbf{z}$ that equals to 0. Given $\mathbf{z}$, we let $\mathbf{W}_\text{Ind}_0(\mathbf{z})$ denote the set of all $W_j$’s with $z_j = 0$, and let $\text{Hist}(\mathbf{W}_\text{Ind}_0(\mathbf{z}))$ denote the vector of $o + 1$ random variable that correspond to the histogram of $\mathbf{W}_\text{Ind}_0(\mathbf{z})$. Similarly, we let $\text{Hist}(\mathbf{W}_\text{Ind}_1(\mathbf{z}))$ denote the vector of $q - o - 1$ random variables that correspond to the histogram of $\mathbf{W}_\text{Ind}_1(\mathbf{z})$. Recall that each $\pi_j$ is bounded above 0 by $d > 0$. We have the following calculation on $\Pr(C_A(\mathbf{x}_\#))$, i.e. the probability that $\text{Hist}(\mathbf{Y})$ satisfies all constraints in $C_A$.

$$
\Pr(C_A(\mathbf{x}_\#)) = \sum_{\mathbf{z} \in \{0, 1\}^n} \Pr(\mathbf{z} = \mathbf{z}) \Pr\left(C_A(\text{Hist}(\mathbf{W})) \mid \mathbf{z} = \mathbf{z}\right) \tag{total probability}
$$

$$
= \sum_{\mathbf{z} \in \{0, 1\}^n} \Pr(\mathbf{z} = \mathbf{z}) \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_0(\mathbf{z}))^\top = \mathbf{D} \cdot \text{Hist}(\mathbf{W}_\text{Ind}_0(\mathbf{z})), n)^\top \mid \mathbf{z} = \mathbf{z}\right)
$$

$$
= \sum_{\mathbf{z} \in \{0, 1\}^n} \Pr(\mathbf{z} = \mathbf{z}) \sum_{\mathbf{x} \in \mathbb{Z}_{\geq 0}^{n-o-1}} \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_1(\mathbf{z})) = \mathbf{x} \mid \mathbf{z} = \mathbf{z}\right)
\times \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_0(\mathbf{z}))^\top = \mathbf{D} \cdot [\mathbf{x}, n]^\top \mid \mathbf{z} = \mathbf{z}\right)
$$

$$
= \sum_{\mathbf{z} \in \{0, 1\}^n} \Pr(\mathbf{z} = \mathbf{z}) \sum_{\mathbf{x} \in \mathbb{Z}_{\geq 0}^{n-o-1}} \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_1(\mathbf{z})) = \mathbf{x} \mid \mathbf{z}_\text{Ind}_1(\mathbf{z}) = \mathbf{1}\right)
\times \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_0(\mathbf{z}))^\top = \mathbf{D} \cdot [\mathbf{x}, n]^\top \mid \mathbf{z}_\text{Ind}_0(\mathbf{z}) = \mathbf{0}\right) \tag{4}
$$

$$
= \sum_{\mathbf{z} \in \{0, 1\}^n} \Pr(\mathbf{z} = \mathbf{z}) \sum_{\mathbf{x} \in \mathbb{Z}_{\geq 0}^{n-o-1}} \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_1(\mathbf{z})) = \mathbf{x} \mid \mathbf{z}_\text{Ind}_1(\mathbf{z}) = \mathbf{1}\right)
\times \Pr\left(\text{Hist}(\mathbf{W}_\text{Ind}_0(\mathbf{z}))^\top = \mathbf{D} \cdot [\mathbf{x}, n]^\top \mid \mathbf{z}_\text{Ind}_0(\mathbf{z}) = \mathbf{0}\right) \tag{5}
$$
To simplify notation, we write (6) as follows.

\[
\Pr\left(Z_l = \bar{z} \mid \Hist(W) = \bar{x} \right) \sum_{\bar{z} \in Z_{\ind_0}^{q-1} \cap \Hist(\bar{x})} \Pr\left(Z_{\ind_1}^{q-1} = \bar{z} \right)
\]

\[
\times \Pr\left(\Hist(W) = \bar{x} \right) = D \cdot [\bar{x}, n] + \Pr\left((\Hist(Z))_0 < 0.9dn \right)
\]

where \(|\Hist(\bar{z})|_0\) is the number of 0's in \(\bar{z}\). [4] holds because \(W_j\)'s are independent of each other given \(Z_j\)'s, which means that for any \(\bar{z} \in \{0, 1\}^n\), \(\Hist(W_{\ind_0})\) and \(\Hist(W_{\ind_1})\) are independent given \(\bar{z}\). [4] holds because \(W_{\ind_0}\) (respectively, \(W_{\ind_1}\)) is independent of \(\Hist(\bar{z})\) (respectively, \(\Hist(\bar{z})\)) in light of independence in the Bayesian network.

To simplify notation, we write [4] as follows.

\[
\sum_{\bar{z} \in \{0, 1\}^n : \Hist(\bar{z}) \geq 0.9dn} \Pr\left(Z_l = \bar{z} \mid \Hist(W) = \bar{x} \right) \times F_1(\bar{x}, \bar{z}) + F_3,
\]

where

\[
F_1(\bar{x}, \bar{z}) = \Pr\left(\Hist(W_{\ind_1}) = \bar{x} \mid \Hist(W_{\ind_0}) = \bar{z} \right)
\]

\[
F_2(\bar{x}, \bar{z}) = \Pr\left(\Hist(W_{\ind_0}) = \bar{x} \mid \Hist(W_{\ind_1}) = \bar{z} \right)
\]

\[
F_3 = \Pr\left((\Hist(Z))_0 < 0.9dn \right)
\]

We now show that given \(\Hist(\bar{z})_0 \geq 0.9dn\), for any \(\bar{x} \in \mathbb{Z}_{\geq 0}^{q-1}\),

\[
F_2(\bar{x}, \bar{z}) = O((0.9dn)^{-2}) = O(n^{-2}),
\]

which follows after the following claim, where \(n = |\ind(\bar{z})|\) and \(q^* = o + 1\). The claim can be seen as a multi-value extension of the Littlewood-Offord-Erdős anti-concentration bound, because it says that the probability for \(\Hist(W_{\ind_0}) = \bar{x} \) to take a specific value \(D \cdot [\bar{x}, n] + \bar{z}\) (which is a constant vector given \(\bar{z}\)) is bounded above by \(O(n^{-2})\).

**Claim 1.** Given \(q^* \in \mathbb{N}\) and \(d > 0\). There exists a constant \(C^* > 0\) such that for any \(n \in \mathbb{N}\) and any vector \(Y_j = (Y_{j1}, \ldots, Y_{jn})\) of \(n\) independent random variables over \([q^*]\), each of which is bounded above by \(n\), and any vector \(\bar{p} \in \mathbb{Z}_{\geq 0}^{q^*}\), we have \(\Pr\left(\Hist(Y_j) = \bar{p} \right) < C^* n^{1-2q^*}\).

**Proof.** When \(\bar{p} \cdot \bar{1} \neq n\) the inequality holds for any \(C^* > 0\). Suppose \(\bar{p} \cdot \bar{1} = n\), we prove the claim by induction on \(q^*\). When \(q^* = 1\) the claim holds for any \(C^* > 1\). Suppose the claim holds for \(q^* = q^* - 1\). W.l.o.g. suppose \(p_1 \leq p_2 \leq \cdots \leq p_{q^*}\). When \(q^* = q'\), we use the following representation of \(Y_j = (Y_{j1}, \ldots, Y_{jn})\) that is similar to Definition 9. For each \(j \leq n\), let \(Y_j'\) be represented by the Bayesian network with two random variables \(Z_j' \in (0, 1)\) and its child \(W_j' \in \{\ldots, q'\}\). Let \(Pr(Z_j' = 0) = Pr(Y_j' \in (1, \ldots, q' - 1))\) and \(Pr(Z_j' = 1) = Pr(Y_j' = q')\). Let \(Pr(W_j' = q') = 1\) and for all \(l \leq q' - 1\), \(Pr(W_j = l | Z_j' = 0) = Pr(Y_j' = l | Y_j' \in (1, \ldots, q' - 1))\).

It follows that \(Pr(Z_j' = 0, W_j' = q') = 0\), \(Pr(Y_j' = q') = Pr(Z_j' = 1, W_j' = q')\), and for all \(l \leq q' - 1\), \(Pr(Y_j' = l) = Pr(Z_j' = 0, W_j' = l)\) and \(Pr(Z_j' = 1, W_j' = l) = 0\). In other words, \(Z_j'\) determines whether \(Y_j' \in (1, \ldots, q' - 1)\) (corresponding to \(Z_j' = 0\)) or \(Y_j' = q'\) (corresponding to \(Z_j' = 1\)), and \(W_j'\) determines the value of \(Y_j'\) conditioned on \(Z_j'\). By the law of total probability, we have:

\[
\Pr\left(\Hist(Y_j) = \bar{p} \right) = \sum_{\bar{z} \in \{0, 1\}^n : \bar{z} \cdot \bar{1} = p_{q^*}} \Pr\left(\Hist(W) = \bar{p} \mid Z_j' = \bar{z} \right) \cdot \Pr\left(Z_j' = \bar{z} \right)
\]

We note that \(W_j\)'s are independent of each other given \(Z_j',\) and \(Pr(W_j = q' | Z_j = 1) = 1\). Therefore, for any \(\bar{z} \in \{0, 1\}^n\) with \(\bar{z} \cdot \bar{1} = p_{q^*}\), we have

\[
\Pr\left(\Hist(W_j) = \bar{p} \mid Z_j = \bar{z} \right) = \Pr\left(\Hist(W_j) = \bar{p}_{q^*} \mid Z_{\ind_0}(\bar{z}) = 0 \right)
\]

\[
\leq C_{q^*-1} |n - p_{q^*}|^{(2-q')/2} \leq C_{q^*-1} (q'-1)^{2-q'/2} n^{(2-q')/2}/2
\]
where $\text{Hist}(\vec{Z}^\prime)_{-q'}$ is the subvector of $\text{Hist}(\vec{Z}^\prime)$ by taking out the $q'$-th component and $\vec{p}_{-q'} = (p_1, \ldots, p_{q'-1})$. The first inequality follows after the induction hypothesis, because each $W_j^\prime$ with $j \in \text{Ind}_0(\vec{z})$ is a random variable over $\{1, \ldots, q - 1\}$ that is strictly above 0 by $d$. The second step uses the assumption that $p_1 \geq p_2 \geq \cdots \geq p_{q'}$, which means that $p_{q'} \leq \frac{1}{q} n$. The next claim, which can be seen as an extension of the Littlewood-Offord-Erdős anti-concentration bound to Poisson binomial distributions, proves that $\Pr(\vec{Z}^\prime \cdot \vec{l} = p_{q'}) = \mathcal{O}(n^{-1/2})$.

**Claim 2.** There exists a constant $C'$ that does not depend on $n$ or $q'$ such that $\Pr(\vec{Z}^\prime \cdot \vec{l} = p_{q'}) \leq C' n^{-1/2}$.

**Proof.** For any $j \leq n$, recall that $\Pr(Z_j = 0) \geq d$ and $\Pr(Z_j = 1) \geq d$. Therefore, for all $j \leq n$, $\text{Var}(Z_j) \geq d(1 - d)$ and there exists a constant $\rho$ such that for all $j \leq n$, $\mathbb{E}((Z_j - \mathbb{E}(Z_j))^2) < \rho$. Let $\mu = \sum_{j=1}^n Z_j$ and $\sigma = \sqrt{\sum_{j=1}^n \text{Var}(Z_j)}$. We have $\sigma \geq \sqrt{n d(1 - d)}$. By Berry-Esseen theorem (Theorem 4 in Appendix [15]), there exists a constant $C_0$ such that:

$$
\Pr(\vec{Z}^\prime \cdot \vec{l} = p_{q'}) \leq \Pr(p_{q'} - 1 < \vec{Z}^\prime \cdot \vec{l} \leq p_{q'} + 1) = \Pr\left(\frac{p_{q'} - 1}{\sigma} \leq \frac{\vec{Z}^\prime \cdot \vec{l}}{\sigma} \leq \frac{p_{q'} + 1}{\sigma}\right)
$$

$$
\leq (\text{Hist}\left(\frac{p_{q'} + 1}{\sigma}\right) - \text{Hist}\left(\frac{p_{q'} - 1}{\sigma}\right)) + 2C_0(n d(1 - d))^{-3/2} n \rho
$$

$$
\leq \frac{2}{\sigma} + 2C_0(n d(1 - d))^{-3/2} n \rho \leq \left(\frac{2}{\sqrt{d(1 - d)}} + \frac{2C_3 \rho}{(d(1 - d))^{3/2}}\right) n^{-1/2}
$$

The claim follows by letting $C' = \frac{2}{\sqrt{d(1 - d)}} + \frac{2C_3 \rho}{(d(1 - d))^{3/2}}$. \hfill \Box

Finally, we have

$$
\Pr(\text{Hist}(\vec{Z}^\prime) = \vec{p}^{\prime}) \leq C_{q' - 1}^{\ast} (\frac{q' - 1}{q'})^{(2 - q')/2} n^{(2 - q')/2} \sum_{\vec{z} \in \{0, 1\}^n : \vec{l} \cdot \vec{z} = p_{q'}} \Pr(\vec{Z}^\prime = \vec{z})
$$

$$
\leq C' n^{-1/2} C_{q' - 1}^{\ast} (\frac{q' - 1}{q'})^{(2 - q')/2} n^{(2 - q')/2}
$$

This proves the $q^* = q'$ case by letting $C_{q'}^{\ast} = C' C_{q' - 1}^{\ast} (\frac{q' - 1}{q'})^{(2 - q')/2}$. This completes the proof of Claim[1]. \hfill \Box

Because random variables in $\vec{Y}$ are bounded, for all $j \leq n$, $Z_j$ takes 0 with probability at least $d$. Therefore, $\mathbb{E}(\vec{Z} \cdot \vec{l}) > d n$. By Hoeffding’s inequality, $F_3 = \Pr(\left|\text{Hist}(\vec{Z})\right|_0 < 0.9 d n)$ is exponentially small in $n$, which means that it is $O(n^{-\frac{2}{3}})$. We also note that for any $\vec{z}$ and $\vec{x}$ we have $F_1(\vec{z}, \vec{x}) \leq 1$. Therefore, continuing the proof, we have:

$$
\Pr(C_{\vec{X}^{\prime}}(\vec{X}^{\prime}_\vec{y})) \leq \sum_{\vec{z} \in \{0, 1\}^n : \left|\text{Hist}(\vec{z})\right|_0 \geq 0.9 d n} \Pr(\vec{Z} = \vec{z}) \sum_{\vec{x} \in \mathbb{Z}^{\mathbb{R}}_{\geq 0}^{n-1}} F_1(\vec{z}, \vec{x}) \times F_2(\vec{z}, \vec{x}) + F_3
$$

$$
\leq \sum_{\vec{z} \in \{0, 1\}^n : \left|\text{Hist}(\vec{z})\right|_0 \geq 0.9 d n} \Pr(\vec{Z} = \vec{z}) O(n^{-\frac{2}{3}}) + O(n^{-\frac{2}{3}}) = O(n^{-\frac{2}{3}})
$$

This proves the upper bound when $\text{Sol}_{AB} \neq \emptyset$ and $\overline{\text{Sol}}_{AB} \cap \text{CH}(\Pi) \neq \emptyset$.

**Tightness of the upper bound when $\text{Sol}_{AB} \neq \emptyset$ and $\overline{\text{Sol}}_{AB} \cap \text{CH}(\Pi) = \emptyset$.** We first prove the following claim that will be frequently used in this proof. The claim follows after a straightforward application of Theorem 1(i) in [13]. It states that for any $\vec{x} \in \text{Sol}_{AB}$ and any $l > 0$, $\text{Sol}_{AB}$ contains an integer vector that is close to $l \cdot \vec{x}$.

**Claim 3.** Suppose $\text{Sol}_{AB} \neq \emptyset$. There exists a constant $C'$ such that for any $\vec{x} \in \text{Sol}_{AB}$, there exists an integer vector $\vec{x}^{\prime} \in \text{Sol}_{AB}$ with $|\vec{x}^{\prime} - l \cdot \vec{x}|_\infty < C'$. 

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Proof. We first prove that $\Sol_{AB}$ contains an integer vector. Because $\Sol_{AB} \neq \emptyset$ and $A$ is an integer matrix, $\Sol_{AB}$ contains a rational solution $\vec{x}'$, which can be chosen from the neighborhood of any vector in $\Sol_{AB}$. It follows that for any $L > 0$, $L \cdot \vec{x}' \in \Sol_{AB}$ because $A \cdot (L \cdot \vec{x}')^\top = (0)^\top$ and $B \cdot (L \cdot \vec{x}')^\top > (0)^\top$. It is not hard to see that there exists $L \in \mathbb{N}$ such that $L \cdot \vec{x}' \in \mathbb{Z}_{\geq 0}^n$.

The claim then follows after Theorem 1(i) in [13] by replacing $B \cdot (\vec{x}')^\top > (0)^\top$ with equivalent constraints $B \cdot (L \cdot \vec{x}')^\top \geq (\vec{1})^\top$.

Let $C'$ denote the constant in Claim 3. W.l.o.g. let $\vec{x} \in \Sol_{AB}$ denote an arbitrary solution such that $\vec{x}' > C' \cdot \vec{1}$; otherwise we consider $\vec{x} + (\vec{x}'_\infty + C') \cdot \vec{1} \in \Sol_{AB}$. For any $l \in \mathbb{N}$, let $\vec{x}'_l$ denote the integer vector in $\Sol_{AB}$ that is guaranteed by Claim 3; let $n_l = \vec{x}'_l \cdot \vec{1}$; and let $\vec{y}'_l \in \Pi^{n_l}$ denote an arbitrary vector of $n_l$ distributions, each of which is chosen from $\Pi$. Let $\vec{y}' \in [q]^{n_l}$ denote an arbitrary vector with $\Hist(\vec{y}') = \vec{x}'_l$. We note that for any $l \in \mathbb{N}$, each distribution in $\vec{y}'_l$ is bounded above by $d$ in $\vec{y}'$. Therefore, $\Pr(\vec{X}_{\vec{y}'_l} = \vec{y}') \geq d^{n_l} = \exp(n_l \log d)$, which is $\exp(-O(n_l))$.

It is not hard to verify that for any $l \in \mathbb{N}$, $\vec{x}'_l$ is strictly positive and $\frac{n_l-1}{2}$ is bounded above by a constant, denoted by $\tilde{C}$. Let $C = \max(n_1, \tilde{C})$. This proves the tightness of the upper bound when $\Sol_{AB} \neq \emptyset$ and $\Sol_{AB} \cap \CH(\Pi) = \emptyset$.

Tightness of the upper bound when $\Sol_{AB} \neq \emptyset$ and $\Sol_{AB} \cap \CH(\Pi) \neq \emptyset$. Let $\vec{x}'_* \in \Sol_{AB} \cap \CH(\Pi)$ and write $\vec{x}'_* = \sum_{i=1}^k a_i \pi_i$ as a linear combination of vectors in $\Pi$. The tightness of the upper bound will be proved in the following four steps. Step 1. For every $l \in \mathbb{N}$ we prove that there exists an integer vector $\vec{y}' \in \Sol_{AB}$ that is $\Theta(\sqrt{l})$ away from $l \cdot \vec{x}'_*$. Then, we define $\vec{y}'_l = (\vec{y}'_1, \ldots, \vec{y}'_k) \in \Pi^{n_l}$, where for all $l \leq k$, $\vec{y}'_l$ is approximately $\alpha_l$ copies of $\pi_i$, which means that $n_l = \Theta(l)$. Step 2. We identify $\Omega\left(\frac{\sqrt{l}}{\sqrt{n_l}}\right)^k$ combinations of values of $\Hist(\vec{X}_{\vec{y}'_1}), \ldots, \Hist(\vec{X}_{\vec{y}'_k})$, such that the sum of each such combination is no more than $\Omega(l)$ away from $\vec{y}'_l$ and is a solution to $C_{AB}(\vec{x}')$. Step 3. We prove that the probability of each such combination is $\Omega\left(\frac{n_l}{\sqrt{l}}\right)$. Finally, we will have $\Pr(\vec{X}_{\vec{x}} \in \Sol_{AB}) \geq \Omega\left(\frac{(k-1)(k-2)\ldots(2)}{2} \frac{\sqrt{l}}{\sqrt{n_l}}\right)^k = \Omega\left(\frac{n_l}{\sqrt{l}}\right)^k$.

Step 1. Let $\vec{x}'_* \in \Sol_{AB}$ and $\vec{x}'_* \in \Sol_{AB} \cap \CH(\Pi)$. W.l.o.g. suppose $\vec{x}'_*$ is strictly positive; otherwise let $l > 0$ denote an arbitrary number such that $\vec{x}'_* + l \cdot \vec{1}$ is strictly positive, and because $A \cdot (\vec{1})^\top = (0)^\top$ and $B \cdot (\vec{1})^\top = (0)^\top$, we have $\vec{x}'_* + l \cdot \vec{1} \in \Sol_{AB}$. Because $\vec{x}'_* \in \Sol_{AB} \cap \CH(\Pi)$, we can write $\vec{x}'_* = \sum_{i=1}^k a_i \pi_i$, where for all $l \leq k$, $\alpha_i > 0$ and $\pi_i \in \Pi$, and $\sum_{i=1}^k a_i = 1$. We note that $\vec{x}'_* > d \cdot \vec{1}$, because $\Pi$ is bounded above by $0$ by $d$.

For any sufficiently large $l \in \mathbb{N}$, we let $\vec{y}'_l \in \Sol_{AB}$ denote the integer approximation to $\sqrt{l} \cdot \vec{x}'_*$ that is guaranteed by Claim 3; let $\vec{y}'_* \in \Sol_{AB}$ denote the integer approximation to $l \cdot \vec{x}'_*$ that is guaranteed by Claim 3, where we merge all rows $\vec{B}$ in $\vec{A}$ with $\vec{B} \cdot \vec{x}'_* = 0$ to $\vec{A}$ before applying the claim. It follows that $\vec{y}'_* \in \Sol_{AB}$. Let $C$ denote an arbitrary constant such that $|\vec{y}'_* - \sqrt{l} \cdot \vec{x}'_*|_\infty < C$ and $|\vec{y}'_* - l \cdot \vec{x}'_*|_\infty < C$.

Let $\vec{y}' = \vec{y}'_* + \vec{y}'_l$ and let $n_l = \vec{y}'_l \cdot \vec{1}$. It follows that $\vec{y}'$ is a strictly positive integer solution to $C_{AB}(\vec{x}')$ and each of its elements is $\Theta(n_l)$. We now define $\vec{y}'_l$ that is approximately $l$ copies of $\vec{x}'_*$. Formally, for each $l \leq k$, let $\vec{y}'_l$ denote the vector of $\beta_i = \lfloor l a_i \rfloor$ copies of $\pi_i$. Let $\vec{y}'_l$ denote the vector of $\beta_k = n_l - \sum_{i=1}^{k-1} \beta_i$ copies of $\pi_k$. It follows that for any $l \leq k - 1$, $\beta_i - \lfloor l a_i \rfloor \leq 1$, and $\beta_k - \lfloor l a_k \rfloor \leq k + \beta_k - \vec{1} = O(\sqrt{l}) = O(\sqrt{n_l})$. Let $\vec{y}'_l = (\vec{y}'_1, \ldots, \vec{y}'_k)$ denote the vector of $n_l$ distributions. It follows that $|\vec{E}(\vec{X}_{\vec{y}'_l}) - n_l|_\infty = O(\sqrt{n_l})$.

Step 2. Let $\vec{X}_{\vec{y}'_l} \subseteq [N]^{k}$ be a set of $k$ vectors $(\vec{x}_1, \ldots, \vec{x}_k)$ such that $\vec{x}_i = (x_{i1}, \ldots, x_{i\alpha}) \in [N]^\alpha$ will be used as a target value for $\vec{X}_{\vec{y}'_l}$ soon in the proof. Let $\mathcal{X}_l$ denote the set of all integer vectors $(\vec{x}_1, \ldots, \vec{x}_k)$ that satisfy the following three conditions.
(i) Let \( \zeta > 0 \) be a constant whose value will be specified later. For any \( i \leq k - 1 \) and any \( j \leq q - 1 \), we require \(|x_{ij} - \pi_{ij}\beta_i| < \zeta \sqrt{n_i}\), where \( \pi_i = (\pi_{i1}, \ldots, \pi_{iq}) \). Therefore, \(|x_{ij} - \pi_{ij}\beta_i| < (q - 1)\zeta \sqrt{n_i}\), because \( \sum_{j \leq q} x_{ij} = \beta_i \).

(ii) Let \( \rho \) be the least common multiple of the denominators of all entries in \( D \). For example, in Example 3 we have \( \rho = 2 \). For each \( o + 2 \leq j \leq q \), we require \(|x_{kj} - (y_j - \sum_{i=1}^{k-1} x_{ij})| < \zeta \sqrt{n_j}\) and \( \rho \) divides \( x_{kj} - (y_j - \sum_{i=1}^{k-1} x_{ij}) \).

(iii) \( \vec{x} = \sum_{i=1}^{k} \vec{x}_i \) is a solution to \((\vec{x}_0)^\top = D \cdot [\vec{x}_1, n_1]^\top \).

Each element \((\vec{x}_1, \ldots, \vec{x}_k)\) of \( X_k \) can be generated in the following three steps. First, \( \vec{x}_1, \ldots, \vec{x}_k-1 \) are arbitrarily chosen according to condition (i) above. Second, \( x_{k(o+2)}, \ldots, x_{kq} \) are chosen according to condition (ii) above given \( \vec{x}_1, \ldots, \vec{x}_{k-1} \). This guarantees that for each \( o + 2 \leq j \leq q \), \( \sum_{i=1}^{k} x_{ij} \) is no more than \( O(\sqrt{n_j}) \) away from \( y_j \) and \( \sum_{i=1}^{k} x_{ij} - y_j \) is divisible by \( \rho \). Finally, \( x_{k1}, \ldots, x_{k(o+1)} \) are determined by condition (iii) together with other components of \( \vec{x} \) that are specified in the first and second step. More precisely,

\[
\begin{bmatrix}
  x_{k1} \\
  \vdots \\
  x_{k(o+1)}
\end{bmatrix} = D \cdot \begin{bmatrix}
  \sum_{i=1}^{k} x_{i(o+2)} \\
  \vdots \\
  \sum_{i=1}^{k} x_{iq}/n_i
\end{bmatrix} - \sum_{i=1}^{k-1} \begin{bmatrix}
  x_{i1} \\
  \vdots \\
  x_{i(o+1)}
\end{bmatrix}.
\]

\( x_{k1}, \ldots, x_{k(o+1)} \) are integers because for all \( o + 2 \leq j \leq q \), \( \rho \) divides \( y_j - \sum_{i=1}^{k} x_{ij} \) and \( D \cdot [\vec{y}_1, n_1]^\top = [\vec{y}_1, n_1]^\top \). We let \( \zeta > 0 \) be a sufficiently small constant that does not depend on \( n_i \), such that the following two conditions hold.

(1) Each \((\vec{x}_1, \ldots, \vec{x}_k) \in X_k \) is strictly positive. This can be achieved by assigning \( \zeta \) a small positive value, because \( |\sum_{i=1}^{k} \vec{x}_i - \vec{y}|_\infty = O(\zeta \sqrt{n_i}) \) and each element in \( \vec{y} \) is strictly positive and is \( \Theta(n_i) \).

(2) For any \((\vec{x}_1, \ldots, \vec{x}_k) \in X_k \), we have \( \sum_{i=1}^{k} \vec{x}_i \in \text{Sol}_{AB} \). Let \( \vec{x} = \sum_{i=1}^{k} \vec{x}_i \). By definition we have \( A \cdot (\vec{x})^\top = (0)^\top \). We also have \( B \cdot (\vec{x})^\top = B \cdot (\vec{x} - \vec{y})^\top + B \cdot (\vec{y}^\#)^\top \geq B \cdot (\vec{x} - \vec{y})^\top + B \cdot (\vec{y}^\#)^\top \).

Notice that \( |\sum_{i=1}^{k} \vec{x}_i - \vec{y}|_\infty = O(\zeta \sqrt{n_i}) \), and each element in \( B \cdot (\vec{y}^\#)^\top \) is \( \Theta(n_i) \). Therefore, when \( \zeta \) is sufficiently small we have \( B \cdot (\vec{x})^\top > 0 \). This means that when \( \zeta > 0 \) is sufficiently small we have \( \vec{x} = \sum_{i=1}^{k} \vec{x}_i \in \text{Sol}_{AB} \).

For any \( l \) with \( \frac{\zeta \sqrt{n_i}}{\rho} > 1 \), we have:

\[ |\mathcal{X}_l| \geq \frac{1}{(\rho)^q - 1} \left( q^a - 1 \right)^{(q-1)(k-1)+q-a-1} n_i^{(q-1)(k-1)+q-a-1}/2 = \Omega(n_i^{(q-1)(k-1)+q-a-1}/2) \]

This is because according to condition (i) there are at least \((\zeta \sqrt{n_i})^{(q-1)(k-1)}\) combinations of values for \( \vec{x}_1, \ldots, \vec{x}_{k-1} \), and according to condition (ii) there are at least \((\zeta \sqrt{n_i})^{q-a-1}\) combinations of values for \( x_{k(o+2)}, \ldots, x_{kq} \).

**Step 3.** By the definition of \( X_k \), there exists a constant \( \alpha > 0 \) that does not depend on \( n_1 \) such that for each \((\vec{x}_1, \ldots, \vec{x}_k) \in X_k \) and each \( i \leq k \), we have \( |\vec{x}_i - \beta_i n_i| < \alpha \sqrt{n_i} \). Also because all agents’ preferences are independently generated, we have: \( \Pr(\forall i \leq k, X_{\vec{x}_i} = \vec{x}_i) = \prod_{i=1}^{k} \Pr(X_{\vec{x}_i} = \vec{x}_i) \).

We note that for each \( i \leq k \), \( X_{\vec{x}_i} \) is the histogram of \( \beta_i \) i.i.d. random variables, each of which is distributed as \( \pi_i \). The next claim implies that for each \( i \leq k \), \( \Pr(X_{\vec{x}_i} = \vec{x}_i) = \Omega(n_i^{(1-q)/2}) \).

**Claim 4.** Given \( q \in \mathbb{N}, d > 0, \alpha > 0 \). There exists a constant \( \beta > 0 \) such that for any distribution \( \pi \) over \([q]\) that is bounded above by \( d \), any \( n \in \mathbb{N} \), and any vector \( \vec{p} \in \mathbb{Z}_{\geq 0}^q \) with \( \vec{p} \cdot \vec{1} = n \) and \( |\vec{p} - n\pi|_\infty < \alpha \sqrt{n} \), we have \( \Pr(X_{\vec{x}_i} = \vec{p}) > \beta n^{-\frac{q-a-1}{2}} \), where \( X_{\vec{x}} \) is the histogram of \( n \) i.i.d. random variables, each of which is distributed as \( \pi \).
Proof. Let \( \tilde{\rho} = (p_1, \ldots, p_q) \), and let \( \tilde{d} = \tilde{\rho} - n\pi \). We have:

\[
\Pr(\tilde{X}_\pi = \tilde{\rho}) = \frac{n!}{p_1 \ldots (p_q-1)} \cdot \frac{\prod_{i=1}^{q} n_i^{p_i}}{\prod_{i=1}^{q} n_i^{p_i}} \geq \frac{1}{\sqrt{2\pi \lambda n}} \left( \frac{n}{e} \right)^n \prod_{i=1}^{q} \frac{n_i^{p_i}}{p_i} = Cn^{\frac{1}{2}} \prod_{i=1}^{q} \frac{n_i^{p_i}}{p_i}
\]

Inequality (7) is due to Robbins’ Stirling approximation \([44]\), where \( \lambda \) represents ratio of circumference to diameter to distinguish from the distribution \( \pi \). \( C \) is a constant that does not depend on \( n \).

Inequality (8) is because \( \prod_{i=1}^{q} \left( \frac{n_i^{p_i}}{p_i} \right) \) is non-negative, meaning that \( \prod_{i=1}^{q} \left( \frac{n_i^{p_i}}{p_i} \right) \geq 1 \). Let \( \pi_{\min} = \min_i \pi_i \).

\[
\prod_{i=1}^{q} \left( \frac{n_i^{p_i}}{p_i} \right)^{d_i} = \prod_{i=1}^{q} \left( 1 + \frac{d_i}{n\pi_i} \right)^{d_i} \geq \prod_{i=1}^{q} \left( 1 + \frac{d_i}{n\pi_i} \right)^{d_i} \geq \left( 1 - \frac{\alpha}{n\pi_{\min}} \right)^{q\alpha^2/\pi_{\min}}
\]

As \( \lim_{x \to \infty} (1 - \frac{1}{e^x}) = 1 \), when \( n \) is large enough we have \( \prod_{i=1}^{q} \left( \frac{n_i^{p_i}}{p_i} \right)^{d_i} > \left( \frac{1}{e^\alpha} \right)^{q\alpha^2/\pi_{\min}} \) which is a constant that does not depend on \( n \). This proves that \( \Pr(\tilde{X}_\pi = \tilde{\rho}) = \Omega(n^{1/2}) \). \( \square \)

By Claim 4 we have \( \Pr(\tilde{X}_\pi = \tilde{X}_i) = \Omega(p_i^{(1-\eta)/2}) = \Omega(n_i^{(1-\eta)/2}) \).

Finally, we have \( \Pr(\tilde{X}_\pi \in \text{Sol}_{AB}) \geq \Omega(n_i \frac{\eta}{2(1-\eta)} \frac{\sqrt{n_i}}{p_i}) = \Omega(n_i^{1/2}) \). Note that we require \( l \) to be sufficiently large such that \( \frac{\sqrt{n_i}}{p_i} > 1 \) in order to guarantee that \( |X_i| \) is large enough. Because \( n_i = \Theta(l) \), there exists a constant \( \hat{C} \) such that for any \( l \in \mathbb{N}, \frac{n_i}{p_i} < \hat{C} \). Let \( C = \max(n_i+1, \hat{C}) \). This proves the tightness of the upper bound when \( \text{Sol}_{AB} \neq \emptyset \) and \( \overline{\text{Sol}}_{AB} \cap \text{CH}(\Pi) \neq \emptyset \). \( \square \)

7.1 Examples

Example 2. Let \( m = 3 \) and \( A = \{1, 2, 3\} \). For any profile \( P \), let \( x_{123} \) denote the number of \( 1 \gg 2 \gg 3 \) in \( P \). The event “alternatives 1 and 2 are co-winners under Borda as well as the only two weak Condorcet winner” can be represented by the following constraints.

\[
2(x_{123} + x_{132}) + x_{213} + x_{312} = 2(x_{213} + x_{231}) + x_{123} + x_{321} \quad (9)
\]

\[
x_{123} + x_{132} + x_{312} = x_{213} + x_{231} + x_{321} \quad (10)
\]

\[
2(x_{123} + x_{132}) + x_{213} + x_{312} > 2(x_{312} + x_{321}) + x_{132} + x_{231} \quad (11)
\]

\[
x_{123} + x_{132} + x_{312} > x_{312} + x_{321} + x_{321} \quad (12)
\]

\[
x_{213} + x_{231} + x_{123} > x_{312} + x_{321} + x_{132} \quad (13)
\]

Equation (9) says that the Borda scores of 1 and 2 are the same; equation (10) says that 1 and 2 are tied in their head-to-head competition; inequality (11) says that the Borda score of 1 is strictly higher than the Borda score of 3; and inequalities (12) and (13) requires that 1 and 2 beat 3 in their head-to-head competitions, respectively.

Example 3. We show how to obtain \( D \) using the setting in Example 2. Let \( \bar{x} = [x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}] \). The reduced echelon form of \( A' \) and its corresponding equa-
tions can be calculated as follows.

\[
\begin{bmatrix}
1 & 2 & -1 & -2 & 1 & -1 & 0 \\
1 & 1 & -1 & -1 & 1 & -1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & -1 & -2 & 1 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 2 & 3 & 2 & n \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & 0 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & n \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

The text above each arrow represents the matrix operations. \( R_1, R_2, R_3 \) represents the first, second, and third row vector of the matrix on the left. For example, “\( R_1; R_2; R_3; R_1 \)” represents that in the right matrix, the first row is \( R_1 \) of the left matrix; the second row is obtained by subtracting \( R_1 \) from \( R_2 \); and the third row is obtained from subtracting \( R_1 \) from \( R_3 \). Let \( I_0 = [x_{123}, x_{132}, x_{213}] \) and \( I_1 = [x_{231}, x_{312}, x_{321}] \). We have \( D = \begin{bmatrix} -1 & -1 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & \frac{1}{2} \end{bmatrix} \) and

\[
\begin{bmatrix}
x_{123} \\
x_{132} \\
x_{213} \\
\end{bmatrix} = D \times 
\begin{bmatrix}
x_{231} \\
x_{312} \\
x_{321} \\
\end{bmatrix}.
\]

8 Appendix: Proof of Theorem 1

Theorem 1 (Smoothed likelihood of Codorcet’s paradox). Let \( \mathcal{M} = (\Theta, L(\mathcal{A}), \Pi) \) be a strictly positive and closed single-agent preference model.

**Smoothed avoidance of Codorcet’s paradox.** Suppose for all \( \pi \in CH(\Pi) \), \( UMG(\pi) \) does not contain a weak Condorcet cycle. Then, for any \( n \in \mathbb{N} \), we have:

\[
\min_{\pi \in \Pi} \mathbb{E}_{P \sim \pi} S_{NCC}(P) = 1 - \exp(-\Omega(n))
\]

**Smoothed Condorcet’s paradox.** Suppose there exists \( \pi \in CH(\Pi) \) such that \( UMG(\pi) \) contains a weak Condorcet cycle. Then, there exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\min_{\pi \in \Pi} \mathbb{E}_{P \sim \pi} S_{NCC}(P) = 1 - \Omega(1)
\]

**Proof.** The theorem is proved by applying Lemma 1 multiple times, where \( C_{AB} \) represents the profiles whose UMG contains a specific weak Condorcet cycle. Formally, we have the following definitions.

**Definition 10** (Variables and pairwise constraints). For any linear order \( R \in L(\mathcal{A}) \), let \( x_R \) be a variable that represents the number of times \( R \) occurs in a profile. Let \( \mathcal{X}_A = \{ x_R : R \in L(\mathcal{A}) \} \) and let \( \bar{x}_A \) denote the vector of elements of \( \mathcal{X}_A \) w.r.t. a fixed order. For any pair of different alternatives \( a, b \), let \( \text{Pair}_{a,b}(\bar{x}_A) \) denote the linear combination of variables in \( \bar{x}_A \), where for any \( R \in L(\mathcal{A}) \), the coefficient of \( x_R \) is 1 if \( a \succ_R b \); otherwise the coefficient is \(-1\).

For any profile \( P \), \( \text{Pair}_{a,b}(\text{Hist}(P)) \) is the weight on the \( a \succ b \) edge in WMG(\( P \)). It is not hard to check that \( \text{Pair}_{a,b}(\bar{1}) = 0 \).

**Definition 11.** For any unweighted directed graph \( G \) over \( \mathcal{A} \), we define \( C_G \) to be the constraints as in Definition 4 that is based on matrices \( A_G = \{ \text{Pair}_{a,b}(\bar{x}_A) = 0 : (a, b) \notin G \text{ and } (b, a) \notin G \} \) and \( B_G = \{ \text{Pair}_{a,b}(\bar{x}_A) > 0 : (a, b) \in G \} \). Let \( \text{Sol}_G \) and \( \overline{\text{Sol}}_G \) denote the solutions to \( C_G \) and its relaxation \( \overline{C}_G \) as in Definition 4, respectively.

We immediately have the following claim.

**Claim 5.** For any profile \( P \) and any unweighted directed graph \( G \) over \( \mathcal{A} \), \( \text{Hist}(P) \in \text{Sol}_G \) if and only if \( G = UMG(P) \); \( \text{Hist}(P) \in \overline{\text{Sol}}_G \) if and only if \( UMG(P) \) is a subgraph of \( G \). Moreover, \( \text{Rank}(A_G) \) equals to the number of ties in \( G \).
Proof. The necessary and sufficient conditions for Hist($P$) ∈ Sol$_C$ and Hist($P$) ∈ $\overline{\text{Sol}}_C$ follow from the definitions. Because the number of equations in $A_G$ equals to the number of ties in $G$, it suffices to prove that the equations in $A_G$ are independent. This is true because for any pair of alternatives $(a, b)$, the UMG of the following profile of two rankings contains one edge $a \rightarrow b$: $a \succ b \succ$ others and rev $\succ a \succ b$, where rev is the reverse order of others.

Proof of the smoothed avoidance part. Let $\mathcal{C}$ denote the set of all $C_G$, where $G$ is an unweighted directed graph without weak Condorcet cycles. We have the following observations. (1) For any profile $P$, $S_{\text{NCC}}(P) = 0$ if and only if there exists $G \in \mathcal{C}$ such that Hist($P$) ∈ Sol$_C$. To see this, when $S_{\text{NCC}}(P) = 0$, we have Hist($P$) ∈ Sol$_{\text{UMG}(P)}$ and UMG($P$) ∈ $\mathcal{C}$; and vice versa, if Hist($P$) ∈ Sol$_C$ for some $G \in \mathcal{C}$, then $S_{\text{NCC}}(P) = 0$. (2) For any graph $G$, Sol$_C \neq \emptyset$ by McGarvey’s theorem. (3) The total number of UMGs over $\mathcal{A}$ only depends on $m$ not on $n$, which means that $|\mathcal{C}|$ can be seen as a constant that does not depend on $n$.

The three observations imply that for any distribution of $P$, we have

$$\Pr(S_{\text{NCC}}(P) = 0) \leq \sum_{G \in \mathcal{C}} \Pr(\text{Hist}(P) \in \text{Sol}_G)$$

Therefore, based on observation (3) above, to prove the smoothed avoidance part of the theorem, it suffices to prove that for any $n$, any $\bar{\pi} \in \Pi_n$, and any $G \in \mathcal{C}$, we have $\Pr_{\pi \sim \bar{\pi}}(\text{Hist}(P) \in \text{Sol}_G) = \exp(-\Omega(n))$. This follows after the exponential case in Lemma 1 applied to $C_G$. To see this, we first note that Sol$_G \neq \emptyset$ according to observation (2) above. Also, for each $\pi \in \Pi$, because UMG($\pi$) does not contain a weak Condorcet cycle, UMG($\pi$) is not a subgraph of $G$, which contains a Condorcet cycle. Therefore, Hist($\pi$) $\notin$ Sol$_G$ due to Claim 5, which means that Sol$_G \cap \text{CH}(\pi) = \emptyset$.

Proof of the smoothed paradox part. Let $\pi \in \text{CH}(\Pi)$ denote any such distribution which contains UMG($\pi$) contains a weak Condorcet cycle. Let $G$ denote an arbitrary supergraph of UMG($\pi$) that contains a Condorcet cycle, e.g. by completing the weak Condorcet cycle in UMG($\pi$). The weak smoothed paradox part is proved by applying the tightness of the polynomial case in Lemma 1 applied to Sol$_G$ following a similar argument with the proof for the smoothed avoidance part.

9 Additional Preliminaries and Examples of Group Theory

The symmetric group over $\mathcal{A} = [m]$, denoted by $S_\mathcal{A}$, is the set of all permutations over $\mathcal{A}$. A permutation $\sigma$ that maps each $a \in \mathcal{A}$ to $\sigma(a)$ can be represented in two ways.

- **Two-line form:** $\sigma$ is represented by a $2 \times m$ matrix, where the first row is $(1, 2, \ldots, m)$ and the second row is $(\sigma(1), \sigma(2), \ldots, \sigma(m))$.
- **Cycle form:** $\sigma$ is represented by non-overlapping cycles over $\mathcal{A}$, where each cycle $(a_1, \ldots, a_k)$ represents $a_{i+1} = \sigma(a_i)$ for all $i \leq k - 1$, and with $a_1 = \sigma(a_k)$.

It follows that any cycle in the cycle form $(a_1, \ldots, a_k)$ is equivalent to $(a_2, \ldots, a_k, a_1)$. Following the convention, in the cycle form $a_1$ is the smallest elements in the cycle. For example, all permutations in $S_3$ are represented in two-line form and cycle form respective in the Table 1.

| Two-line | Cycle |
|----------|-------|
| $(1, 2, 3)$ | $(1, 2)$ |
| $(1, 2, 3)$ | $(2, 3)$ |
| $(1, 3)$ | $(1, 3)$ |
| $(1, 2, 3)$ | $(1, 2, 3)$ |
| $(1, 2, 3)$ | $(1, 3, 2)$ |

A permutation group $G$ is a subgroup of $S_\mathcal{A}$ where the identity element is the identity permutation Id, and for any $\sigma, \eta \in S_\mathcal{A}$, $\sigma \circ \eta$ is the permutation where for any linear order $V \in \mathcal{L}(\mathcal{A})$, $(\sigma \circ \eta)(V) = \sigma(\eta(V))$. For example, $(1, 2) \circ (2, 3) = (1, 2, 3)$, because $1 \succ 2 \succ 3$ is first mapped to $1 \succ 3 \succ 2$ by permutation $(2, 3)$, then to $2 \succ 3 \succ 1$ by permutation $(1, 2)$. There are six subgroups of $S_{[3]}$: \{Id\}, \{Id, (1, 2)\}, \{Id, (2, 3)\}, \{Id, (1, 3)\}, \{Id, (1, 2, 3), (1, 3, 2)\}, and $S_{[3]}$.

Example 4. Table 2 shows all permutation groups over $S_{[3]}$ as the result of Perm($P$) for some profiles.
Table 2: Examples of Perm($P$), where 123 represents 1 ∼ 2 ∼ 3.

| $P$ | $P[123]$ | $P[132]$ | $P[213]$ | $P[231]$ | $P[312]$ | $P[321]$ | Perm($P$) |
|-----|----------|----------|----------|----------|----------|----------|------------|
| $P_1$ | 1 2 2 | 2 2 2 | 2 2 2 | 2 2 2 | 2 2 2 | 2 2 2 | $\{\text{Id}\}$ |
| $P_2$ | 3 5 5 | 3 5 5 | 3 5 5 | 3 5 5 | 3 5 5 | 3 5 5 | $\{\text{Id}, (1, 2)\}$ |
| $P_3$ | 3 5 4 | 4 4 5 | 5 5 3 | 3 5 5 | 3 5 5 | 3 5 5 | $\{\text{Id}, (1, 3)\}$ |
| $P_4$ | 3 5 4 | 4 5 4 | 4 5 4 | 4 5 4 | 4 5 4 | 4 5 4 | $\{\text{Id}, (2, 3)\}$ |
| $P_5$ | 3 5 3 | 3 5 3 | 3 5 3 | 3 5 3 | 3 5 3 | 3 5 3 | $\{\text{Id}, (1, 2, 3), (1, 3, 2)\}$ |
| $P_6$ | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | $S_{[3]}$ |

$P_1$ in the table consists of one ranking for 1 ∼ 2 ∼ 3 and two rankings for each of the remaining five linear orders. Perm($P_1$) only contains Id because if it contains any other permutation $\sigma$, then we must have $P[1 ∼ 2 ∼ 3] = P[\sigma(1 ∼ 2 ∼ 3)]$, which is impossible. (12) $\in$ Perm($P_2$) because $P_2[1 ∼ 2 ∼ 3] = P_2[2 ∼ 1 ∼ 3] = 3$, $P_2[1 ∼ 3 ∼ 2] = P_2[2 ∼ 3 ∼ 1] = 5$, and $P_2[3 ∼ 1 ∼ 2] = P_2[3 ∼ 2 ∼ 1] = 4$. It is not hard to verify that no other permutations except Id belong to Perm($P_2$).

10 Proof of Theorem 2

Theorem 2 (Smoothed folklore (im)possibility theorem). Let $\mathcal{M} = (\Theta, \mathcal{L}(A), \Pi)$ be a strictly positive and closed single-agent preference model. Let $\mathcal{U}_{m}^{\Pi} = \{U \in \mathcal{U}_{m} : \exists \pi \in \chi(\Pi), \forall \sigma \in U, \sigma(\pi) = \pi\}$, and when $\mathcal{U}_{m}^{\Pi} \neq \emptyset$, let $l_{\min} = \min_{U \in \mathcal{U}_{m}^{\Pi}} |U|$ and $l_{\Pi} = \frac{l_{\min} - 1}{l_{\max}}$.

Smoothed possibility. There exist an anonymous voting rule $r_{\text{anno}}$ and a neutral voting rule $r_{\text{neu}}$ such that for any $r \in \{r_{\text{anno}}, r_{\text{neu}\}$, any $n$, and any $\# \in \Pi^n$, we have:

$$\Pr_{r \sim \#}(S_{\text{anno}}(r, P) + S_{\text{neu}}(r, P) < 2) = \begin{cases} O(n^{-\frac{1}{4}}) & \text{if } \mathcal{U}_{m}^{\Pi} \neq \emptyset \\ \exp(-\Omega(n)) & \text{otherwise} \end{cases}$$

Smoothed impossibility. For any voting rule $r$, there exist infinitely many $n \in \mathbb{N}$ such that:

$$\max_{\# \in \Pi^n} \Pr_{r \sim \#}(S_{\text{anno}}(r, P) + S_{\text{neu}}(r, P) < 2) = \begin{cases} \Omega(n^{-\frac{1}{4}}) & \text{if } \mathcal{U}_{m}^{\Pi} \neq \emptyset \\ \exp(-O(n)) & \text{otherwise} \end{cases}$$

Proof. Step 1. Identifying the source of the folklore impossibility theorem.

Definition 12. For any $n \in \mathbb{N}$ and any $m \geq 2$, let $\mathcal{T}_{m,n}$ denote the set of $n$-profiles $P$ such that $\text{Perm}(P) \in \mathcal{U}_{m}$. That is,

$$\mathcal{T}_{m,n} = \{P \in \mathcal{L}(A)^n : \text{Perm}(P) \text{ covers } A\}$$

The following lemma states that profiles in $\mathcal{T}_{m,n}$ are the intrinsic source of the folklore impossibility theorem.

Lemma 3. For any voting rule $r$, any $n \in \mathbb{N}$, any $m \geq 2$, and any $P \in \mathcal{T}_{m,n}$, we have:

$$S_{\text{anno}}(r, P) + S_{\text{neu}}(r, P) \leq 1$$

Proof. We first partition $\mathcal{T}_{m,n}$ to $\text{Hist}^{-1}(H_1) \cup \cdots \cup \text{Hist}^{-1}(H_J)$, where $H_1, \ldots, H_J$ are some histograms of $n$ votes. The partition exists because for any $P \in \mathcal{T}_{m,n}$ and any profile $P'$ with Hist($P'$) = Hist($P$), we have Perm($P'$) = Perm($P$) $\in \mathcal{U}_{m}$, which means that $P' \in \mathcal{T}_{m,n}$. For any $j \leq J$, if there exist $P_1, P_2 \in \text{Hist}^{-1}(H_j)$ such that $r(P_1) \neq r(P_2)$, then for all $P \in \text{Hist}^{-1}(H_j)$ we have $S_{\text{anno}}(P) = 0$, which means that inequality (14) holds for all $P \in \text{Hist}^{-1}(H_j)$. Otherwise if $r(P) = a$ for all $P \in H_j$, then by the definition of $\mathcal{T}_{m,n}$ there exists a permutation $\sigma$ over $A$ such that Hist($\sigma(P)$) = Hist($P$) and $\sigma(a) \neq a$. This means that $\sigma(P) \in H_j$, and therefore, $r(\sigma(P)) = a \neq r(P)$, which means that $S_{\text{neu}}(P) = 0$. Again, inequality (14) holds for $P \in \text{Hist}^{-1}(H_j)$. This proves the lemma. \qed
Step 2. Smoothed possibility. First, we define $r_{\text{ano}}$ and $r_{\text{neu}}$ by extending a voting rule $r^*$ that is defined on $\mathcal{L}(A)^n \setminus T_{m,n}$ and is guaranteed to satisfy anonymity and neutrality simultaneously for all profiles that are not in $T_{m,n}$. More precisely, for any pair of profiles $P$ and $P'$, we write $P \equiv P'$ if and only if there exists a permutation $\sigma \in S_A$ such that $\text{Hist}(P') = \sigma(\text{Hist}(P))$. Let $(\mathcal{L}(A) \setminus T_{m,n}) = D_1 \cup \cdots \cup D_{L'}$ denote the partition w.r.t. $\equiv$. For each $l \leq L'$, let $P_l \in D_l$ denote an arbitrary profile. Because $P_l \not\in T_{m,n}$, there exists an alternative $a_l$ that is not covered by $\text{Perm}(P_l)$. Then, for any $P' \in D_l$ and any permutation $\sigma$ such that $\text{Hist}(P') = \sigma(\text{Hist}(P))$, we let $r^*(P') = \sigma(a_l)$. We note that the choice of $\sigma$ does not matter, because all permutations $\sigma$ that map $\text{Hist}(P)$ to $\text{Hist}(P')$ have the same image for $a_l$. To see this, for the sake of contradiction, suppose $\text{Hist}(P') = \sigma_1(\text{Hist}(P)) = \sigma_2(\text{Hist}(P))$ where $\sigma_1(a_l) = \sigma_2(a_l)$. Then, we have $\sigma_1^{-1} \circ \sigma_2 \in \text{Perm}(P)$, and $(\sigma_1^{-1} \circ \sigma_2)(a_l) = a_l$, which is a contradiction to the assumption that $\text{Perm}(P)$ does not cover $a_l$.

Claim 6. For any profile $P \not\in T_{m,n}$ we have $S_{\text{ano}}(r^*, P) = 1$ and $S_{\text{neu}}(r^*, P) = 1$.

Proof. By definition, for any profiles $P \not\in T_{m,n}$, $r^*(P)$ only depends on $\text{Hist}(P)$, which means that $S_{\text{ano}}(r^*, P) = 1$. To prove $S_{\text{neu}}(r^*, P) = 1$, let $P \equiv P_l$ for some $l \leq L$ and let $\sigma \in S_A$ denote a permutation over alternatives. We first prove that $\sigma(P) \in D_l$. Let $P' = \sigma(P)$. For the sake of contradiction suppose $P' \not\in D_l$. Then we have $P' \in T_{m,n}$. Therefore, for any alternative $a \in A$, there exists $\eta \in \text{Perm}(P')$ such that $\eta(\sigma(a)) = \sigma(a)$. It follows that $\eta^{-1} \circ \eta \circ \sigma \in \text{Perm}(P)$ and $(\sigma^{-1} \circ \eta \circ \sigma)(a) = a$. Therefore, $\text{Perm}(P)$ covers $A$, which contradicts the assumption that $P \not\in T_{m,n}$. Let $P = \zeta(P_l)$, where $P_l$ is the profile chosen in the definition of $r^*$. It follows that $P' = \zeta(P_l)$. Therefore, we have $r^*(P') = \zeta(a_l) = \sigma(\zeta(a_l)) = r^*(P)$. This proves the claim. 

For any profile $P \not\in T_{m,n}$, we let $r_{\text{ano}}(P) = r_{\text{neu}}(P) = r^*(P)$. For any profile $P$ in $T_{m,n}$, we let $r_{\text{ano}}(P) = a$ for an arbitrary fixed alternative $a$, and let $r_{\text{neu}}(P)$ be the top-ranked alternative of agent $a$. By Claim 6, for any $r \in \{r_{\text{ano}}, r_{\text{neu}}\}$ any $n$, and any $\bar{\theta} \in \Pi^n$, $\Pr_{P \sim \bar{\theta}}(S_{r}(r, P) \neq 0) \geq 2 \cdot \Pr_{P \sim \bar{\theta}}(S_{r}(r, P) \neq 2) \geq 2 \cdot \Pr_{P \sim \bar{\theta}}(S_{r}(r, P) < 2)$. Therefore, it suffices to prove the following lemma.

Lemma 4. For any $r \in \{r_{\text{ano}}, r_{\text{neu}}\}$ any $n$, and any $\bar{\theta} \in \Theta^n$, we have:

$$\Pr_{P \sim \bar{\theta}}(P \in T_{m,n}) = \frac{O(n^{-\frac{1}{2}})}{\exp(-\Omega(n))} \quad \text{if } \sum_{i=1}^{m} \Pi_{\text{ano}} \neq \emptyset.\quad \text{ otherwise}
$$

Proof. The lemma is proved by applying the upper bound in Lemma 1 as in the proof of Theorem 1. For any $U \in \mathcal{U}_m$, we define $C_U$ as follows.

Definition 13. For any $U \in \mathcal{U}_m$, we define $C_U$ as in Definition 7 that is based on $A_U$ and $B_U$, where $A_U$ represents $\{x_R - x_{\sigma(R)} = 0 : \forall R \in \mathcal{L}(A), \forall \sigma \in U\}$ and $B_U = \emptyset$. Let $S_{O_U}$ and $\overline{S_{O_U}}$ denote the solutions to $C_U$ and its relaxation $\overline{C_U}$ as in Definition 7 respectively.

By definition, $A_U(x_R) = 0$ if and only if for all $\sigma \in U$, $x_R = x_{\sigma(R)}$. Because $B_U = \emptyset$, we have $C_U = \overline{C_U}$, which means that $S_{O_U} = \overline{S_{O_U}}$. We have the following claim about $C_U$.

Claim 7. For any profile $P$ and any $U \in \mathcal{U}_m$, $\text{Hist}(P) \in S_{O_U}$ if and only if for all $\sigma \in U$, $\text{Hist}(P) = \sigma(\text{Hist}(P))$. Moreover, $\text{Rank}(A_U) = (1 - \frac{1}{|U|})m!$. 

Proof. The “if and only if” part follows after the definition. We now prove that $\text{Rank}(A_U) = (1 - \frac{1}{|U|})m!$. Let $\equiv_U$ denote the relationship over $\mathcal{L}(A)$ such that for any pair of linear orders $R_1, R_2, R_1 \equiv_U R_2$ if and only if there exists $\sigma \in U$ such that $R_1 = \sigma(R_2)$. Because $U$ is a permutation group, $\equiv_U$ is an equivalence relationship that partitions $\mathcal{L}(A)$ into $\frac{m!}{|U|}$ groups, each of which has $|U|$ linear orders. It is not hard to see that each equivalent class is characterized by $|U| - 1$ linearly independent equations represented by rows in $A_U^T$, which means that $\text{Rank}(A_U) \leq (|U| - 1)(\frac{m!}{|U|}) = (1 - \frac{1}{|U|})m!$. For any $s < (1 - \frac{1}{|U|})m!$ and any combination of $s$ rows of $A_U^T$, denoted by $A$, it is not hard to construct $\bar{x}$ such that $A \times (\bar{x})^T = (\bar{0})^T$ but $A_U \times (\bar{x})^T \neq (\bar{0})^T$, which proves that $\text{Rank}(A_U) \geq (1 - \frac{1}{|U|})m!$. This proves the claim. 

\[ \Box \]
Let $C$ denote the set of all $C_U$ where $U \in \mathcal{U}_m$ as in Definition 13. We have the following observations. (1) $C$ characterizes $\mathcal{T}_{m,n}$, because by Claim 7 for any $P \in \mathcal{T}_{m,n}$ we have $\text{Hist}(P) \in \text{Sol}_{\text{perm}(P)}$ and $\text{Perm}(P) \in C$; and vice versa, for any $C_U \in C$ and any $P$ such that $\text{Hist}(P) \in \text{Sol}_U$, by Claim 7 we have $\text{Perm}(P) \supseteq U$, which means that $\text{Perm}(P)$ covers $A$, and it follows that $P \in \mathcal{T}_{m,n}$. (2) For any $U \in \mathcal{U}_m$, $\text{Sol}_U \neq \emptyset$ because $\bar{I} \in \text{Sol}_U$. (3) $|C|$ can be seen as a constant that does not depend on $n$, because it is no more than the total number of permutation groups over $A$.

We now prove the polynomial upper bound when $\mathcal{U}_m^{\Pi} \neq \emptyset$. For any $U \in \mathcal{U}_m^{\Pi}$, by Claim 7 we have $\text{Rank}(A^U) \geq (1 - \frac{1}{l_{\text{min}}})m! \geq (1 - \frac{1}{l_{\text{min}}})m!$. By applying the polynomial upper bound in Lemma 1 to all $C_U \in C$, for any $\bar{P} \in \Pi^n$, we have:

$$\Pr_{P \sim \bar{P}}(P \in \mathcal{T}_{m,n}) \leq \sum_{C_U \in C} \Pr_{P \sim \bar{P}}(\text{Hist}(P) \in \text{Sol}_U) = \sum_{C_U \in C} O(n^{-\frac{(l_{\text{min}} - 1)}{2l_{\text{min}}}}m!) \leq |C|O(n^{-\frac{(l_{\text{min}} - 1)}{2l_{\text{min}}}}m!)
$$

The exponential upper bound is proved similarly, by applying the exponential upper bound in Lemma 1 to all $C_U \in C$. This proves the lemma.

**Step 3. Smoothed impossibility.** Lemma 3 implies that for any $\bar{P} \in \Pi^n$, $\Pr_{P \sim \bar{P}}(\text{Sol}_{\text{ano}}(r,P) + S_{\text{neu}}(r,P) < 2) \geq \Pr_{P \sim \bar{P}}(P \in \mathcal{T}_{m,n})$. Therefore, it suffices to prove the following lemma.

**Lemma 5.** For any voting rule $r$, there exist infinitely many $n \in \mathbb{N}$ and corresponding $\bar{P} \in \Pi^n$, such that:

$$\Pr_{P \sim \bar{P}}(P \in \mathcal{T}_{m,n}) = \begin{cases} \Omega(n^{-\frac{m}{l_{\text{min}}}}) & \text{if } \mathcal{U}_m^{\Pi} \neq \emptyset \\ \exp(-\mathcal{O}(n)) & \text{otherwise} \end{cases}$$

**Proof.** Let $C$ be the set as defined in the proof of Lemma 4.

**Applying the lower bound in Lemma 1.** We first prove the polynomial lower bound. Suppose $\mathcal{U}_m^{\Pi} \neq \emptyset$ and let $U \in \mathcal{U}_m^{\Pi}$ denote the permutation group with the minimum size, i.e. $|U| = l_{\text{min}}$. By Claim 7 $\text{Rank}(A^U) = (1 - \frac{1}{l_{\text{min}}})m!$. By applying the tightness of the polynomial bound part in Lemma 1 to $C_U$, we have that there exists constant $C > 0$ such that for any $n' \in \mathbb{N}$, there exists $n' \leq n \leq Cn'$ and $\bar{P} \in \Pi^n$ such that

$$\Pr_{P \sim \bar{P}}(\text{Hist}(P) \in \text{Sol}_U) = \Omega(n^{-\frac{(l_{\text{min}} - 1)}{2l_{\text{min}}}}m!)
$$

Note that for any profile $P$ such that $\text{Hist}(P) \in \text{Sol}_U$, we have $\text{Perm}(P) \supseteq U$, which means that $\text{Perm}(P)$ covers $A$ and therefore $P \in \mathcal{T}_{m,n}$. Therefore, we have:

$$\Pr_{P \sim \bar{P}}(P \in \mathcal{T}_{m,n}) \geq \Pr_{P \sim \bar{P}}(\text{Hist}(P) \in \text{Sol}_U) \geq \Omega((Cn)^{-\frac{1}{l_{\text{min}}}}m!) = \Omega(n^{-\frac{1}{l_{\text{min}}}}m!)
$$

The exponential lower bound is proved similarly, by applying the tightness of the exponential bound part in Lemma 1 to an arbitrary $U \in \mathcal{U}_m$. This proves the lemma.

This finishes the proof of Theorem 2.

**10.1 Connection to the (Non-)Existence of Anonymous and Neutral Voting Rules**

As a side note, Lemma 3 and Claim 6 can be used to prove Moulin’s characterization of existence of anonymous and neutral voting rules, which was stated as Problem 1 [39], where only very high-level ideas about the proof was presented, from which it is hard to tell how similar our proof below is with the proof there.

**Claim 8 (Problem 1 [39]).** Fix any $m \geq 2$ and $n \geq 2$, there exists a voting rule that satisfies anonymity and neutrality if and only if $m$ cannot be written as the sum of $n$’s nontrivial divisors.
Proof. By Lemma and Claim, there exists a voting rule that satisfies anonymity and neutrality if and only if \( T_{m,n} = \emptyset \). Therefore, it suffices to prove that \( T_{m,n} = \emptyset \) if and only if \( m \) cannot be written as the sum of \( n \)'s nontrivial divisors.

The “if” direction. Suppose for the sake of contradiction that \( T_{m,n} \neq \emptyset \). Let \( P \in T_{m,n} \neq \emptyset \) denote an arbitrary profile. We now partition \( A \) according to the following equivalence relationship \( \equiv \), \( a \equiv b \) if and only if there exists \( \sigma \in \text{Perm}(P) \) such that \( \sigma(a) = b \). The partition is well defined because \( \text{Perm}(P) \) is a permutation group. In other words, if \( a \equiv b \) then \( b \equiv a \), because the \( \text{Perm}(P) \) is closed under inversion. If \( a \equiv b \) (via \( \sigma_1 \)) and \( b \equiv c \) (via \( \sigma_2 \)) then \( a \equiv c \) because \( c = (\sigma_2 \circ \sigma_1)(a) \), and \( \sigma_2 \circ \sigma_1 \in \text{Perm}(P) \).

Suppose \( A = A_1 \cup \cdots \cup A_L \) is divided into \( L \) parts according to \( \equiv \). Because \( P \in T_{m,n} \), which means that \( \text{Perm}(P) \) covers \( A \), for all \( l \leq L \), \( |A_l| \geq 2 \). It is not hard to check that for any \( l \leq L \), we have \( |A_l| \) divides \( |\text{Perm}(P)| \). Also it is not hard to see that \( \text{Perm}(P) \) divides \( n \). This means that \( m = \sum_{l=1}^{L} |A_l| \) which contradicts the assumption. This proves the “if” direction.

The “only if” direction. Suppose for the sake of contradiction that \( m = m_1 + \cdots + m_L \) where each \( m_l \geq 2 \) and divides \( n \). Consider the permutation \( \sigma \) whose cycle form consists of \( L \) cycles whose sizes are \( m_1, \ldots, m_L \), respectively. Let \( U \) denote the permutation group generated by \( \sigma \). It follows that \( |U| = \prod_{l=1}^{L} m_l \), which means that \( |U| \) divides \( n \). Therefore, it is not hard to construct an \( n \)-profile \( P \) such that for any ranking \( R \) and any \( \sigma' \in U \), we must have \( P[R] = P[\sigma'(R)] \). It follows that \( U \subseteq \text{Perm}(P) \), which means that \( \text{Perm}(P) \) covers \( A \), and therefore \( \in T_{m,n} \neq \emptyset \). This contradicts the assumption that \( \in T_{m,n} = \emptyset \), which proves the “only if” direction.

\[\square\]

### 11 Proof of Lemma

**Lemma.** For any \( m \geq 2 \), let \( l^* = \min_{U \subseteq S_m} |U| \). We have \( l^* = 2 \) if \( m \) is even; \( l^* = 3 \) if \( m \) is odd and \( 3 \leq m \); \( l^* = 5 \) if \( m \) is odd, \( 3 \leq m \) and \( 5 \mid m \); and \( l^* = 6 \) for other \( m \).

**Proof.** We prove the lemma by discussing the following cases.

**Case 1:** \( m \mid 2 \). Let \( \sigma = (1, 2)(3, 4) \cdots (m-1, m) \) be a permutation that consists of 2-cycles. It follows that \( \text{Id} = \sigma^2 \) and \( \{\text{Id}, \sigma\} \) is a permutation group that covers \( m \). This means that \( l^* = 2 \).

**Case 2:** \( 2 \nmid m \) and \( 3 \mid m \). Let \( \sigma = (1, 2, 3)(4, 5, 6) \cdots (m-2, m-1, m) \) be a permutation that consists of 3-cycles. It follows that \( \text{Id} = \sigma^2 \) and \( \{\text{Id}, \sigma, \sigma^2\} \) is a permutation group that covers \( m \). This means that \( l^* \leq 3 \). We now prove that \( l^* \) cannot be 2 by contradiction. Suppose for the sake of contradiction that \( l^* = 2 \) and let \( G \) denote a permutation group with \( |G| = 2 \). Table\textsuperscript[3] (part of Table 26.1 in \textsuperscript{[20]}) lists all groups of orders 1 through 5 up to isomorphism. The order of a group is the number of its elements. For any \( l \in \mathbb{N} \), a permutation group \( G \) of order \( l \) is isomorphic to \( Z_l \) if and only if \( G = \{\text{Id}, \sigma, \ldots, \sigma^{l-1}\} \), where \( \sigma \) is an order-\( l \) permutation, that is, \( \sigma^l = \text{Id} \) and for all \( 1 \leq i < l \), \( \sigma^i \neq \text{Id} \). A permutation group \( G \) of order 4 is isomorphic to \( Z_4 \), and only if \( G = \{\text{Id}, \sigma, \eta, \sigma \circ \eta\} \), where \( \sigma, \eta \), and \( \sigma \circ \eta \) are three order-2 permutations.

| \begin{tabular}{c|c|c|c} \\[-1em] Group & 1 & 2 & 3 \\ \hline \{\text{Id}\} & Z_2 & Z_3 & Z_4 or \( Z_2 \oplus Z_2 \) \\ \hline \end{tabular} |
|---|---|---|---|
| 4 | \( Z_5 \) |

Therefore, \( G = \{\text{Id}, \sigma\} \). Because \( G \) covers \( A \), the cycle form of \( \sigma \) must consist of 2-cycles that cover all alternatives in \( A \), which means that \( 2 \mid m \), a contradiction.

**Case 3:** \( 2 \nmid m \), \( 3 \nmid m \) and \( 5 \mid m \). Let \( \sigma = (1, 2, 3, 4, 5) \cdots (m-4, m-3, m-2, m-1, m) \) be a permutation that consists of 5-cycles. It follows that \( \text{Id} = \sigma^2 \) and \( \{\text{Id}, \sigma, \sigma^3, \sigma^4, \sigma^5\} \) is a permutation group that covers \( m \). This means that \( l^* \leq 5 \). We now prove by contradiction that \( l^* \) cannot be 2, 3, or 4. Suppose for the sake of contradiction that \( l^* = 2 \) and let \( G \) denote a permutation group whose order is no more than 4. Because \( 2 \nmid m \) and \( 3 \nmid m \), following a similar argument with Case 2 we know that \( G \) cannot be isomorphic to \( Z_2, Z_3 \), or \( Z_4 \). Therefore, by Table\textsuperscript[5] \( G \) must be isomorphic to \( Z_2 \oplus Z_2 \). However, the next claim shows that this is impossible. Therefore, \( l^* = 5 \).
Claim 9. For any $m$ with $2 \nmid m$, no permutation group in $G \in G_m$ is isomorphic to $Z_2 \oplus Z_2$, where $G_m$ is the set of all permutation groups over $A = [m]$.

Proof. We prove the claim by contradiction. Suppose for the sake of contradiction $2 \nmid m$ and there exists $G \in G_m$ that is isomorphic to $Z_2 \oplus Z_2$. This means that $G = \{\text{Id}, \sigma, \eta, \sigma \circ \eta\}$, where $\sigma$, $\eta$, and $\sigma \circ \eta$ only contain 2-cycles in their cycle forms, respectively. Because $2 \nmid m$, at least one alternative is not involved in any cycle in $\eta$. W.l.o.g. we let $\{1, \ldots, k\}$ denote the alternatives that are not involved in any cycle in $\eta$, which means that they are mapped to themselves by $\eta$. It follows that the remaining $m - k$ alternatives are covered by 2-cycles in $\eta$, which means that $k$ is an odd number. For any $i \leq k$, because $G$ covers $A$, $i$ must be involved in a 2-cycle in $\sigma$, otherwise none of $\eta, \sigma$, or $\sigma \circ \eta$ will map $i$ to a different alternative. Suppose $(i, j)$ is a 2-cycle in $\sigma$. We note that $(\sigma \circ \eta)(i) = j$, which means that $(ij)$ is a 2-cycle in $\sigma \circ \eta$. This means that $\eta(j) = j$, that is, $j \leq k$. Therefore, $\{1, \ldots, k\}$ consists of 2-cycles in $\sigma$. This means that $2 \nmid k$, which is a contradiction.

Case 4: $2 \nmid m$, $3 \nmid m$, and $5 \nmid m$. Let $\sigma = (1, 2, 3)(4, 5) \cdots (m - 1, m)$. It follows that the order of $\sigma$ is 6, which means that $\{\text{Id}, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$ is a permutation group that covers $[m]$. This means that $t^* \leq 6$. The proof for $t^* = 6$ is similar with the proof in Case 3. Suppose for the sake of contradiction there exists $G \in G_m$ with $|G| \leq 5$. Then, by Table 3 $G$ must be isomorphic to $Z_2, Z_3, Z_4, Z_5$, or $Z_2 \oplus Z_2$. However, this is impossible because none of 2, 3, or 5 can divide $m$, and by claim 2 $G$ is not isomorphic to $Z_2 \oplus Z_2$. This proves that $t^* = 6$.

11.1 A Corollary of Theorem 2 and Lemma 2

Corollary 1. Let $M = (\Theta, L(A), \Pi)$ be a strictly positive and closed single-agent preference model where $\pi_{\text{uni}} \in \text{CH}(\Pi)$. Let $t^*$ be the number in Lemma 2.

Smoothed possibility. Let $r \in \{r_{\text{ano}}, r_{\text{new}}\}$. For any $n$ and any $\vec{r} \in \Pi^n$, we have:

$$\Pr_{P \sim \vec{r}}(S_{\text{ano}}(r, P) + S_{\text{new}}(r, P) < 2) = O(n^{-\frac{(m - 1)}{m!}})$$

Smoothed impossibility. For any voting rule $r$, there exist infinitely many $n \in \mathbb{N}$ such that:

$$\max_{\vec{r} \in \Pi^n} \Pr_{P \sim \vec{r}}(S_{\text{ano}}(r, P) + S_{\text{new}}(r, P) < 2) = \Omega(n^{-\frac{(m - 1)}{m!}})$$

In particular, Corollary 1 applies to all neutral, strictly positive, and closed models, which contains $M^{[r, 1]}_{\text{Ma}}, M^{[r, 1]}_{\Pi}$, and IC as a special case. This is because for any neutral model, let $\pi \in \Pi$ denote an arbitrary distribution. Then, for any permutation $\sigma$ over $A$, $\sigma(\pi) \in \Pi$. Therefore, $\pi_{\text{uni}} = \frac{1}{m!} \sum_{\sigma \in S_A} \sigma(\pi) \in \text{CH}(\Pi)$.

12 Proof of Proposition 1

Proposition 1. Let $r$ be a voting rule obtained from a positional scoring correspondence by applying LEX or FA. Let $M = (\Theta, L(A), \Pi)$ be a strictly positive and closed single-agent preference model where $\pi_{\text{uni}} \in \text{CH}(\Pi)$. There exist infinitely many $n \in \mathbb{N}$ such that:

$$\max_{\vec{r} \in \Pi^n} \Pr_{P \sim \vec{r}}(S_{\text{ano}}(r, P) + S_{\text{new}}(r, P) < 2) = \Omega(n^{-0.5})$$

Proof. Suppose $r$ is obtained from a correspondence $c$ by applying LEX or FA. We first prove that any profiles $P$ with $|c(P)| = 2$ violates anonymity or neutrality under $r$. In other words, $S_{\text{ano}}(r, P) + S_{\text{new}}(r, P) < 2$. W.l.o.g. suppose $c(P) = \{1, 2\}$. Suppose $r$ is obtained from $c$ by applying LEX-$R$. W.l.o.g. suppose $1 \succ_R 2$. Let $\sigma$ denote the permutation that exchanges 1 and 2. Because $c$ is neutral, $c(\sigma(P)) = c(\sigma(P')) = \{1, 2\}$. By LEX-$R$, $c(r(P)) = c(\sigma(r(P))) = 1 \neq 2 = c(r(P))$, which violates neutrality. Suppose $r$ is obtained from $c$ by applying FA-$j$. W.l.o.g. suppose $1 \succ 2$ in the $j$-th vote in $P$. Because $c(P) = \{1, 2\}$, there exists a vote in $P$ where $2 \succ 1$. Suppose $2 \succ 1$ in the $j'$-th vote. Let $P'$ denote the profile obtained from $P$ by switching $j$-th and $j'$-th vote. Because $c$ is anonymous, we have $c(P') = \{1, 2\}$. By FA-$j$, $r(P') = 2 \neq 1 = r(P)$, which violates anonymity.
Therefore, for any \( n \) and any \( \vec{\pi} \in \Pi^n \), we have \( \Pr_{P \sim \vec{\pi}}(S_{\text{ano}}(r, P) + S_{\text{neu}}(r, P) < 2) \geq \Pr_{P \sim \vec{\pi}}(|c(P)| = 2) \). By applying the tightness of polynomial bound part in Lemma 1, it is not hard to prove that for any positional scoring correspondence \( c \), there exist infinitely many \( n \in \mathbb{N} \) and corresponding \( \vec{\pi} \in \Pi^n \) such that \( \Pr_{P \sim \vec{\pi}}(|c(P)| = 2) = \Omega(n^{-0.5}) \). This proves the proposition. \( \square \)

13 Proof of Theorem 3

**Theorem** Let \( \mathcal{M} = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model, where \( \pi_{\text{uni}} \in CH(\Pi) \). For any voting correspondence \( c \) that satisfies anonymity and neutrality, let \( r_{\text{MPSR}} \) denote the voting rule obtained from \( c \) by MPSR-then-TB. For any \( n \) and any \( \vec{\pi} \in \Pi^n \), we have:

\[
\Pr_{P \sim \vec{\pi}}(S_{\text{ano}}(r_{\text{MPSR}}, P) + S_{\text{neu}}(r_{\text{MPSR}}, P) < 2) = O(n^{-\frac{1}{2}})
\]

Moreover, if TB satisfies anonymity (respectively, neutrality) then \( r_{\text{MPSR}} \) also satisfies anonymity (respectively, neutrality).

**Proof.** The upper bound is proved in the following two steps. First, we show that for any profile \( P \) such that MPSR(\( P \)) \( \neq \emptyset \),

\[
S_{\text{ano}}(r_{\text{MPSR}}, P) + S_{\text{neu}}(r_{\text{MPSR}}, P) = 2
\]

(15)

If \( |c(P)| = 1 \), then we have \( r_{\text{MPSR}}(P) = c(P) \). (15) holds because \( c \) satisfies neutrality and anonymity. If MPSR(\( P \)) \( \neq \emptyset \), then for any permutation \( \sigma \) over \( A \), we have \( c(\sigma(P)) = \sigma(c(P)) \) and it is not hard to see that MPSR(\( \sigma(P) \)) = MPSR(\( P \)). Let MPSR(\( P \)) = \( V \). Therefore, \( r_{\text{MPSR}}(\sigma(P)) \) is the alternative in \( \sigma(c(P)) \) that is ranked highest in \( \sigma(V) \), which means that \( r_{\text{MPSR}}(\sigma(P)) = r_{\text{MPSR}}(P) \). This means that \( S_{\text{neu}}(r_{\text{MPSR}}, P) = 1 \). For any profile \( P' \) with Hist(\( P' \)) = Hist(\( P \)), it is not hard to see that MPSR(\( P' \)) = MPSR(\( P \)). Because \( c \) satisfies anonymity, we have \( c(\sigma(P)) = c(P) \). This means that \( S_{\text{ano}}(r_{\text{MPSR}}, P) = 1 \).

Second, we show that \( \Pr_{P \sim \vec{\pi}}(\text{MPSR}(P) = \emptyset) = O(n^{-\frac{1}{2}}) \) by applying the polynomial upper bound in Lemma 1 in a way similar to the proof of Theorem 2. For any partition \( Q = \{Q_1, \ldots, Q_L\} \) of \( \mathcal{L}(A) \), we define \( C_Q \) as in Definition 4 based on \( A_Q \), which represents \( \{x_R - x_R' = 0 : \forall l \leq L, \forall R, R' \in Q_l\} \), and \( B_Q = \emptyset \). We have \( \text{Rank}(A_Q) = n! - |Q| \). Let \( \mathcal{C} \) denote the set of all \( C_Q \), where each set in \( Q \) contains at least two linear orders. It is not hard to verify that \( \mathcal{C} \) characterizes all \( P \) with MPSR(\( P \)) \( \neq \emptyset \), for any \( C_Q \in \mathcal{C} \), \( S_{\text{ano}}(Q) \neq \emptyset \) and \( \text{Rank}(A_Q) \geq m! \), and \( |\mathcal{C}| \) does not depend on \( n \). Because for any \( C_Q \in \mathcal{C} \), \( \pi_{\text{uni}} \in \text{Sol}_Q \cap CH(\Pi) \), we can apply the polynomial upper bound in Lemma 1 to all \( C_Q \in \mathcal{C} \), which gives us \( \Pr_{P \sim \vec{\pi}}(\text{MPSR}(P) = \emptyset) = O(n^{-\frac{1}{2}}) \).

The “moreover” part follows after noticing that (1) for any pair of profiles \( P \) and \( P' \) such that MPSR(\( P \)) = \( \emptyset \) and Hist(\( P' \)) = Hist(\( P \)), we have MPSR(\( P' \)) = \( \emptyset \), and (2) for any profile \( P \) with MPSR(\( P \)) = \( \emptyset \) and any permutation \( \sigma \) over \( A \), we have MPSR(\( \sigma(P) \)) = \( \emptyset \). \( \square \)

14 Smoothed Likelihood of Other Commonly-Studied Events in Social Choice

In this section we show how to apply Lemma 1 to obtain dichotomy results on smoothed likelihood of various social choice events. Let us start with a dichotomy result on the non-existence of Condorcet cycles.

**Proposition 2** Let \( \mathcal{M} = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model.

**Upper bound.** For any \( n \in \mathbb{N} \) and any \( \vec{\pi} \in \Pi^n \), we have:

\[
\Pr_{P \sim \vec{\pi}}(S_{\text{NCC}}(P) = 1) = \begin{cases} O(1) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } \text{UMG}(\pi) \text{ is acyclic} \\ \exp(-\Omega(n)) & \text{otherwise} \end{cases}
\]

**Tightness of the upper bound.** There exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\max_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(S_{\text{NCC}}(P) = 1) = \begin{cases} \Omega(1) & \text{if } \exists \pi \in CH(\Pi) \text{ s.t. } \text{UMG}(\pi) \text{ is acyclic} \\ \exp(-O(n)) & \text{otherwise} \end{cases}
\]
Proof. The proof proceeds in the following three steps.

First step: defining $C$. Let $C$ denote the set of all $C_g$ where $G$ is an acyclic unweighted directed graph, as in Definition 11. We have the following observations. (1) For any profile $P$, $S_{\text{NCC}}(P) = 1$ if and only if $\text{Hist}(P) \subseteq \bigcup_{g \in C} \text{Sol}_g$. To see this, if $S_{\text{NCC}}(P) = 1$ then $\text{UMG}(P)$ is acyclic, which means that $\text{Hist}(P) \subseteq \text{Sol}_{\text{UMG}(P)}$, where $C_{\text{UMG}(P)} \in C$; and conversely, if $\text{Hist}(P) \subseteq \text{Sol}_g$ for an acyclic graph $G$, then $S_{\text{NCC}}(P) = 1$. (2) For any graph $G$, $\text{Sol}_G \neq \emptyset$ by McGarvey’s theorem. (3) $|C|$ only depends on $m$, which means that $|C|$ can be seen as a constant that does not depend on $n$.

Second step, the $O(1)$ case. The $O(1)$ upper bound is straightforward. To prove its tightness, suppose there exists $\pi \in \text{CH}(\Pi)$ such that $\text{UMG}(\pi)$ is acyclic. This means that there exists a topological ordering of $\text{UMG}(\pi)$. Let $G$ denote an arbitrary complete acyclic supergraph of $\text{UMG}(\pi)$. It follows that $C_g \in C$, and $\pi \in \text{Sol}_G$ due to Claim 3. Therefore, $C_g \cap \text{CH}(\Pi) \neq \emptyset$. Also note that $A_G = \emptyset$, which means that $\text{Rank}(A_G) = 0$. The tightness follows after applying the tightness of the polynomial part in Lemma 11 to $C_g$.

Third step, the exponential case. For any $C_g \in C$ and any $\pi \in \text{CH}(\Pi)$, we first show that $\text{Sol}_G \cap \text{CH}(\Pi) = \emptyset$. Suppose for the sake of contradiction there exists an acyclic graph $G$ such that $\pi \in \text{Sol}_G \cap \text{CH}(\Pi)$. Then, by Claim 3, UMG$(\pi)$ is a subgraph of $G$, which means that UMG$(\pi)$ is acyclic. This contradicts the assumption on CH$(\Pi)$ in the exponential case. The upper bound (respectively, its tightness) follows after applying the exponential upper bound (respectively, its tightness) in Lemma 11 to all $C_g \in C$ (respectively, an arbitrary $C_g \in C$).

The exponential case of Proposition 3 is exactly the very strong smoothed paradox part of Theorem 11 because $\mathbb{E}_{P \sim \pi} S_{\text{NCC}}(P) = Pr_{P \sim \pi}(S_{\text{NCC}}(P) = 1)$.

The dichotomy results in this section will be presented by the following template exemplified by Proposition 3. In the template, we will specify three components: (1) EVENT, which is an event of interest that depends on the profile $P$, (2) CONDITION, which is often about the existence of $\pi \in \text{CH}(\Pi)$ that satisfies a weaker version of EVENT, and (3) a number $l_{H_{11}}$ that depends on the statistical model $\mathcal{M}$.

Template for dichotomy results (Smoothed likelihood of EVENT). Let $\mathcal{M} = (\Theta, \mathcal{L}(A), \Pi)$ be a strictly positive and closed single-agent preference model.

Upper bound. For any $n \in \mathbb{N}$ and any $\bar{\pi} \in \Pi^n$, we have:

$$Pr_{P \sim \bar{\pi}}(\text{EVENT}) = \begin{cases} O(n^{-\frac{1}{4}}) & \text{if CONDITION holds} \\ \exp(-\Omega(n)) & \text{otherwise} \end{cases}$$

Tightness of the upper bound. There exist infinitely many $n \in \mathbb{N}$ such that:

$$\max_{\bar{\pi} \in \Pi^n} Pr_{P \sim \bar{\pi}}(\text{EVENT}) = \begin{cases} \Omega(n^{\frac{1}{4}}) & \text{if CONDITION holds} \\ \exp(-O(n)) & \text{otherwise} \end{cases}$$

For example, in Proposition 3 EVENT is “there is no Condorcet cycle”, CONDITION is “there exists $\pi \in \text{CH}(\Pi)$ such that UMG$(\pi)$ is acyclic”, and $l_{H_{11}} = 0$. Table 3 summarizes the dichotomy results using the template. A Condorcet winner is the alternative who beats every other alternative in their head-to-head competition. If a Condorcet winner exists, then it must be unique. A weak Condorcet winner is an alternative who never loses in head-to-head competitions. Weak Condorcet winners may not be unique.

Any dichotomy result using the template is quite general, because it applies to all strictly positive and closed single-agent preference models, any $n$, and any combination of distributions. In particular, it is not hard to verify that if $\pi_{\text{uni}} \in \text{CH}(\Pi)$, where $\pi_{\text{uni}}$ is the uniform distribution over $\mathcal{L}(A)$, then CONDITION is satisfied for all propositions in Table 3, which means that the polynomial bounds apply. For example, because $\pi_{\text{uni}} \in \text{CH}(\Pi)$ for any neutral model (for any $\pi \in \Pi$, the average of $m!$ distributions obtained from $\pi$ by applying all permutations is $\pi_{\text{uni}}$), we have the following corollary.

Corollary 2. The polynomial bounds in Table 3 apply to all neutral, strictly positive, and closed models including $\mathcal{M}_A^\epsilon$ and $\mathcal{M}_p^{\epsilon,1}$ for all $0 < \epsilon \leq 1$, and IC, which corresponds to $\Pi = \{\pi_{\text{uni}}\}$.
Table 4: Summary of dichotomy results on smoothed likelihood of events. CH(Π) is the convex hull of Π. UMG is the unweighted majority graph.

| Prop. | EVENT | CONDITION | \( l_\Pi \) |
|-------|-------|-----------|-----------|
| 2     | No Condorcet cycles | \( \exists \pi \in \text{CH}(\Pi) \) s.t. UMG(\( \pi \)) is acyclic | 0 |
| 3     | \( \exists \) Condorcet cycle of length \( k \) | \( \exists \pi \in \text{CH}(\Pi) \) s.t. UMG(\( \pi \)) contains a weak Condorcet cycle of length \( k \) | 0 |
| 4     | \( \exists \) Condorcet winner | \( \exists \pi \in \text{CH}(\Pi) \) that has at least one weak Condorcet winner | 0 |
| 5     | No Condorcet winner | \( \exists \pi \in \text{CH}(\Pi) \) and a supergraph \( G \) of UMG(\( \pi \)) that has no weak Condorcet winner | 0 or 1 |
| 6     | \( \exists \) exactly \( k \) weak Condorcet winners | \( \exists \pi \in \text{CH}(\Pi) \) that contains at least \( k \) weak Condorcet winners | \( k(k-1)/2 \) |
| 7     | No weak Condorcet winners | \( \exists \pi \in \text{CH}(\Pi) \) and a supergraph \( G \) of UMG(\( \pi \)) that has no weak Condorcet winner | 0 |

In light of Corollary 2 and as a result of Proposition 3, the likelihood of Condorcet voting paradox is asymptotically maximized under IC, among all i.i.d. distributions over \( \mathcal{L}(\mathcal{A}) \). This gives an asymptotic answer to an open questions by Tsetlin et al. [49].

If EVENT is desirable, such as “Condorcet winner” or “No Condorcet cycles”, then a polynomial (sometimes \( \Theta(1) \)) smoothed likelihood is desirable; if EVENT is undesirable, such as “No Condorcet winner” or “there exists a Condorcet cycle of length \( k \)”, then an exponential likelihood is desirable.

**Overview of proof techniques.** All propositions are proved by applying Lemma 1 in the following three steps exemplified by the proof of Proposition 2. First, we define a set \( C \) of constraints \( C_{AB} \)'s such that (1) EVENT is characterized by \( \bigcup_{C_{AB} \in C} \text{Sol}_{AB} \) in the sense that EVENT holds for a profile \( P \) if and only if \( \text{Hist}(P) \in \text{Sol}_{AB} \) for some \( C_{AB} \in C \), (2) for each \( C_{AB} \in C \), \( \text{Sol}_{AB} \neq \emptyset \), and (3) \( |C| \) is a constant that does not depend on \( n \) (but may depend on \( m \)). Second, for the polynomial bound, the upper bound is proved by applying the polynomial upper bound in Lemma 1 to all (constant number of) \( C_{AB} \in C \). The tightness is proved by explicitly choosing \( C_{AB} \in C \), often as a function of some \( \pi \in \text{CH}(\Pi) \), so that \( \pi \in \text{Sol}_{AB} \), which implies \( \text{Sol}_{AB} \cap \text{CH}(\Pi) \neq \emptyset \), and then applying the tightness of the polynomial bound in Lemma 1. Third, for the exponential bound, we first prove that for any \( C_{AB} \in C \), \( \text{Sol}_{AB} \cap \text{CH}(\Pi) = \emptyset \). Then, the upper bound (respectively, its tightness) is proved by applying the exponential bound (respectively, its tightness) in Lemma 1 to all \( C_{AB} \in C \) (respectively, an arbitrary \( C_{AB} \in C \)).

For any \( 3 \leq k \leq m \), we let \( S_{\text{CC}=k}(P) = 1 \) (respectively, \( S_{\text{WCC}=k}(P) = 1 \)) if there exists a Condorcet cycle (respectively, weak Condorcet cycle) of length \( k \) in \( P \); otherwise \( S_{\text{CC}=k}(P) = 0 \) (respectively, \( S_{\text{WCC}=k}(P) = 0 \)).

**Proposition 3 (Smoothed likelihood of existence of Condorcet cycles with length \( k \)).** Let \( \mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi) \) be a strictly positive and closed single-agent preference model.

**Upper bound.** For any \( n \in \mathbb{N} \), any \( \bar{\pi} \in \Pi^n \), and any \( 3 \leq k \leq m \), we have:

\[
\Pr_{P \sim \bar{\pi}}(S_{\text{CC}=k}(P)) = \begin{cases} \frac{O(1)}{\exp(-\Omega(n))} & \text{if } \exists \pi \in \text{CH}(\Pi) \text{ s.t. } S_{\text{WCC}=k}(\pi) = 1 \\ \text{otherwise} & \end{cases}.
\]

**Tightness of the upper bound.** For any \( 3 \leq k \leq m \), there exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\max_{\bar{\pi} \in \Pi^n} \Pr_{P \sim \bar{\pi}}(S_{\text{CC}=k}(P)) = \begin{cases} \frac{\Omega(1)}{\exp(-\Omega(n))} & \text{if } \exists \pi \in \text{CH}(\Pi) \text{ s.t. } S_{\text{WCC}=k}(\pi) = 1 \\ \text{otherwise} & \end{cases}.
\]

**Proof.** First step, defining \( C \). For any length-\( k \) cycle \( p = a_1 \to a_2 \to \cdots \to a_k \to a_1 \) in \( \mathcal{A} \), we define \( C_p \), where \( A_p = \emptyset \) and \( B_p \) represents the \( k \) constraints \( \text{Pair}_{a_1,a_2}(\bar{x},A) > 0, \text{Pair}_{a_2,a_3}(\bar{x},A) > \)
0, . . . , Pair_{a_k,a_1}(\vec{x}_A) > 0\}. Let C denote all such C_p. We have the following observations. (1) For any profile \( P \), \( S_{CC=k}(P) = 1 \) if and only if Hist(\( P \)) \( \in \bigcup_{C_p \in \text{Sol}_p} \). To see this, if \( S_{CC=k}(P) = 1 \) then there exists a length-k cycle \( p \) in UMG(\( P \)), which means that Hist(\( P \)) \( \in \text{Sol}_p \), where C_p \( \in \mathcal{C} \); and conversely, if Hist(\( P \)) \( \in \text{Sol}_p \) for some length-k cycle \( p \), then \( p \) is a length-k Condorcet cycle in \( P \), which means that \( S_{CC=k}(P) = 1 \). (2) by McGarvey’s theorem [34], for each C_p \( \in \mathcal{C} \), there exists a profile \( P \) where \( p \) is a cycle in UMG(\( P \)), which means that Sol_p \( \cap \mathcal{C} \neq \emptyset \). (3) The total number of length-k cycles in \( A \) only depends on \( m \) and \( k \), which means that |\( \mathcal{C} \)| can be seen as a constant that does not depend on \( n \).

**Second step, the polynomial case.** The O(1) upper bound is straightforward. To prove the tightness, suppose there exists \( \pi \in CH(\Pi) \) with \( S_{WCC=k}(\pi) = 1 \). Let \( p \) denote an arbitrary length-k weak Condorcet cycle in UMG(\( \pi \)). It follows that \( B_p \cdot (\pi) \geq 0 \), which means that \( \text{Sol}_p \cap CH(\Pi) \neq \emptyset \), because A_p \( = \emptyset \). The tightness follows after applying the tightness of the polynomial lower bound in Lemma[1] to \( C_p \), where \text{Rank}(A_p) = 0 \).

**Third step, the exponential case.** For any C_p \( \in \mathcal{C} \) and any \( \pi \in CH(\Pi) \), we first prove that \( \text{Sol}_p \cap CH(\Pi) = \emptyset \). Suppose for the sake of contradiction that there exists a length-k cycle \( p \) such that \( \pi \in \text{Sol}_p \cap CH(\Pi) \). Then, because \( B_p \cdot (\pi) \geq 0 \), for any edge \( a_i \rightarrow a_{i+1} \) in \( p \) we must have \( \pi[a_i, a_{i+1}] - \pi[a_{i+1}, a_i] \geq 0 \), which means that \( p \) is a length-k weak Condorcet cycle in UMG(\( \pi \)), meaning that \( S_{WCC=k}(\pi) = 1 \). This contradicts the assumption on CH(\( \Pi \)) in the exponential case. The upper bound (respectively, its tightness) follows after applying the exponential bound (respectively, its tightness) in Lemma[1] to all (respectively, an arbitrary) C_p \( \in \mathcal{C} \).

Let \( S_{CW}(P) = 1 \) if \( P \) has a Condorcet winner; otherwise let \( S_{CW}(P) = 0 \).

**Proposition 4 (Smoothed likelihood of existence of Condorcet winner).** Let \( \mathcal{M} = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model.

**Upper bound.** For any \( n \in \mathbb{N} \) and any \( \vec{\pi} \in \Pi^n \), we have:

\[
\Pr_{P \sim \vec{\pi}}(S_{CW}(P) = 1) = \begin{cases} 
O(1) \\
\text{exp}(-\Omega(n))
\end{cases}
\]

if \( \exists \pi \in CH(\Pi) \) s.t. WCW(\( \pi \)) \geq 1.

**Tightness of the upper bound.** There exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\max_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}}(S_{CW}(P) = 1) = \begin{cases} 
\Omega(1) \\
\text{exp}(-\Omega(n))
\end{cases}
\]

if \( \exists \pi \in CH(\Pi) \) s.t. WCW(\( \pi \)) \geq 1.

**Proof.** **First step: defining \( C \).** Let \( \mathcal{C} = \{C_G : S_{CW}(G) = 1\} \), that is, \( \mathcal{C} \) contains all C_G (Definition[1]) where G contains a Condorcet winner. We have the following observations. (1) For any profile \( P \), \( S_{CW}(P) = 1 \) if and only if Hist(\( P \)) \( \in \bigcup_{G \in \text{Sol}_G} G \). To see this, if \( S_{CW}(P) = 1 \) then Hist(\( P \)) \( \in \text{Sol}_{UMG(G)} \subseteq \mathcal{C} \), where C_G \( \in \mathcal{C} \); and conversely, if Hist(\( P \)) \( \in \text{Sol}_{G} \) for some C_G \( \in \mathcal{C} \), then \( P \) has a Condorcet winner. (2) For any G with \( S_{CW}(G) = 1 \), \( \text{Sol}_G \neq \emptyset \) due to McGarvey’s theorem[34]. (3) |\( \mathcal{C} \)| can be seen as a constant that does not depend on \( n \).

**Second step: the polynomial case.** The O(1) upper bound trivially holds. To prove the tightness, suppose there exits \( \pi \in CH(\Pi) \) such that WCW(\( \pi \)) \( \geq 1 \). Let a denote an arbitrary weak Condorcet winner in UMG(\( \pi \)). We obtain a complete graph \( G^* \) from UMG(\( \pi \)) by adding \( a \rightarrow b \) for all tied pairs (\( a, b \)) in UMG(\( \pi \)), and then adding arbitrary edges between other tied pairs in UMG(\( \pi \)). It follows that \( S_{CW}(G^*) = 1 \) and A_{G^*} \( = \emptyset \) because \( G^* \) is complete, which means that \text{Rank}(A_{G^*}) = 0 \). The tightness follows after applying the tightness of the polynomial bound in Lemma[1] to C_{G^*}.

**Third step: the exponential case.** For any C_G \( \in \mathcal{C} \) and any \( \pi \in CH(\Pi) \), we now prove that \( \text{Sol}_G \cap CH(\Pi) = \emptyset \). Suppose for the sake of contradiction there exist C_G \( \in \mathcal{C} \) and \( \pi \in \text{Sol}_G \cap CH(\Pi) \). By Claim[5] UMG(\( \pi \)) is a subgraph of G, which means that the Condorcet winner in G is a weak Condorcet winner in UMG(\( \pi \)), which contradicts the assumption that WCW(\( \pi \)) = 0 in the exponential case. The upper bound (respectively, its tightness) follows after applying the exponential bound (respectively, its tightness) in Lemma[1] to all (respectively, an arbitrary) C_G \( \in \mathcal{C} \).

**Proposition 5 (Smoothed likelihood of non-existence of Condorcet winner).** Let \( \mathcal{M} = (\Theta, \mathcal{L}(A), \Pi) \) be a strictly positive and closed single-agent preference model. Let \( \mathcal{G} \) denote the set of all unweighted directed graphs G over A such that (1) there exists \( \pi \in CH(\Pi) \) such that
UMG(\pi) \subseteq G$, and (2) $SC\omega (G) = 0$. When $\mathcal{G}_\Pi \neq \emptyset$, we let $l_{\Pi} = \min_{G \in \mathcal{G}} Ties(G)$, where $Ties(G)$ denote the number of unordered pairs that are tied in $G$.

**Upper bound.** For any $n \in \mathbb{N}$ and any $\pi \in \Pi^n$, we have:

$$\Pr_{P \sim \pi} (SC\omega (P) = 0) = \begin{cases} O(n^{-\frac{\omega}{2}}) \exp(-\Omega(n)) & \text{if } \mathcal{G}_\Pi \neq \emptyset \\ \exp(-\Omega(n)) & \text{otherwise} \end{cases}.$$

**Tightness of the upper bound.** There exist infinitely many $n \in \mathbb{N}$ such that:

$$\max_{\pi \in \Pi^n} \Pr_{P \sim \pi} (SC\omega (P) = 0) = \begin{cases} \Omega(n^{-\frac{\omega}{2}}) \exp(-O(n)) & \text{if } \mathcal{G}_\Pi \neq \emptyset \\ \exp(-O(n)) & \text{otherwise} \end{cases}.$$

**Proof.** **First step: defining $C$.** Let $C = \{C_G : SC\omega (G) = 0\}$, where $C_G$ is defined in Definition [11]. That is, $C$ contains all $C_G$ where $G$ does not contain a Condorcet winner. We have the following observations. (1) For any profile $P$, $SC\omega (P) = 0$ if and only if $Hist(P) \in \bigcup_{C_G \in C} Sol_G$. To see this, if $SC\omega (P) = 0$ then $Hist(P) \in Sol_{UMG(P)}$, where $UMG(P) \in C$; and conversely, if $Hist(P) \in Sol_G$ for some $C_G \in C$, then $P$ does not have a Condorcet winner. (2) For any $G$ with $SC\omega (G) = 0$, we have $Sol_G \neq \emptyset$ due to McGarvey’s theorem [34]. (3) $|C|$ can be seen as a constant that does not depend on $n$.

**Second step: the polynomial case.** To prove the upper bound, we note that for any $G$ such that (1) there is no Condorcet winner and (2) $G$ is a supergraph of the UMG of some $\pi \in CH(\Pi)$, the number of ties in $G$ is at least $l_{\Pi}$, which means that $Rank(A_C) \geq l_{\Pi}$. Therefore, according to observation (1) above, we have:

$$\Pr_{P \sim \pi} (WC\omega (P) = k) \leq \sum_{G \in C_G \in C} \Pr_{P \sim \pi} (Hist(P) \in Sol_G) = O(n^{-\frac{\omega}{2}}).$$

The last part follows after applying the polynomial upper bound in Lemma [1] to all $C_G \in C$ and the observation (3) above. In particular, for any graph $G$ that is not a supergraph of the UMG of any $\pi \in CH(\Pi)$, $Pr_{P \sim \pi} (Hist(P) \in Sol_G)$ is exponentially small due to Claim [5] and the exponential upper bound in Lemma [1] applied to $C_G$.

To prove the tightness, let $\pi \in CH(\Pi)$ denote a distribution such that there exists a supergraph $G^* \in \mathcal{G}_\Pi$ of UMG(\pi) where $G^*$ contains $l_{\Pi}$ ties. By Claim [5], $\pi \in Sol_{G^*}$, which means that $Sol_G \cap CH(\Pi) \neq \emptyset$. Also by Claim [5], $Rank(A_{G^*}) = l_{\Pi}$. The tightness follows after applying the polynomial tightness in Lemma [1] to $C_G$.

**Third step: the exponential case.** For any $C_G \in C$ and any $\pi \in CH(\Pi)$, we first prove that $Sol_G \cap CH(\Pi) = \emptyset$. Suppose for the sake of contradiction that such $C_G \in C$ and $\pi \in Sol_G \cap CH(\Pi)$ exist. It follows from Claim [5] that UMG(\pi) is a subgraph of $G$, which means that $G \in \mathcal{G}_\Pi$. This contradicts the assumption that $G = \emptyset$. The upper bound (respectively, its tightness) follows after applying the exponential bound (respectively, its tightness) in Lemma [1] to all (respectively, an arbitrary) $C_G \in C$.

The following claim implies that $l$ in Proposition [5] can only be 0 or 1.

**Claim 10.** For any unweighted directed graph $G$ over $A$ that does not contain a Condorcet winner, there exists a supergraph of $G$, denoted by $G^*$, such that $G^*$ does not contain a Condorcet winner, and the number of ties in $G^*$ is no more than one. The upper bound of one is tight.

**Proof.** Let $G^*$ denote a supergraph of $G$ without a Condorcet winner and with the minimum number of ties. If there is no tie in $G^*$ then the claim is proved. For any pair of tied alternatives $a$ and $b$ in $G^*$, adding $a \rightarrow b$ to $G^*$ must lead to a Condorcet winner due to the minimality of $G^*$, and $a$ must be the Condorcet winner. This means that $a$ beats all alternatives other than $b$ in $G^*$. Similarly, $b$ beats all alternatives other than $a$ in $G^*$. This means that $\{a, b\}$ is the only tie in $G^*$ because if there exists another tie $\{c, d\}$, then adding $c \rightarrow d$ to $G^*$ will not introduce a Condorcet winner, which contradicts the minimality of $G^*$. Therefore, the number of ties in $G^*$ is upper bounded by 1. The tightness of the upper bound is proved by letting $G$ be a graph where $a$ and $b$ are the only weak Condorcet winners. \qed
For any profile \( P \), let WCW\((P)\) denote the number of weak Condorcet winners in \( P \).

**Proposition 6 (Smoothed likelihood of exactly \( k \) weak Condorcet winners).** Let \( \mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi) \) be a strictly positive and closed single-agent preference model.

**Upper bound.** For any \( n \in \mathbb{N} \), any \( 1 \leq k \leq m \), and any \( \pi \in \Pi^n \), we have:

\[
\Pr_{P \sim \pi}(\text{WCW}(P) = k) = \begin{cases} 
O(n^{-\frac{k(k-1)}{2}}) \exp(-\Omega(n)) & \text{if } \exists \pi \in \text{CH}(\Pi) \text{ s.t. WCW}(\pi) \geq k \\
\text{otherwise}
\end{cases}
\]

**Upper bound.** For any \( 1 \leq k \leq m \), there exist infinitely many \( n \in \mathbb{N} \) such that:

\[
\max_{\pi \in \Pi^n} \Pr_{P \sim \pi}(\text{WCW}(P) = k) = \begin{cases} 
\Omega(n^{-\frac{k(k-1)}{2}}) \exp(-O(n)) & \text{if } \exists \pi \in \text{CH}(\Pi) \text{ s.t. WCW}(\pi) \geq k \\
\text{otherwise}
\end{cases}
\]

**Proof.** First step: defining \( C \). Let \( C = \{ C_G : \text{WCW}(G) = k \} \), where \( C_G \) is defined in Definition 11. That is, \( C \) contains all \( C_G \) where \( G \) has exactly \( k \) weak Condorcet winners. We have the following observations. (1) For any profile \( P \), WCW\((P)\) = \( k \) if and only if Hist\((P)\) \( \in \bigcup_{C_G \in C} \text{Sol}_G \). To see this, if WCW\((P)\) = \( k \) then Hist\((P)\) \( \in \text{Sol}_{\text{UMG}(G)} \), where \( \text{UMG}(G) \in C \), and conversely, if Hist\((P)\) \( \in \text{Sol}_G \), then \( G = \text{UMG}(P) \) contains exactly \( k \) weak Condorcet winners, which means that WCW\((P)\) = \( k \). (2) By McGarvey’s theorem [24], for each \( C_G \in C \), there exists a profile \( P \) with UMG\((P)\) = \( G \), which means that \( \text{Sol}_G \neq \emptyset \). (3) The total number of unweighted directed graphs over \( A \) only depends on \( m \), which means that \( |C| \) can be seen as a constant that does not depend on \( n \).

**Second step: the polynomial case.** To prove the upper bound, we note that for any \( G \) that has exactly \( k \) weak Condorcet winners, the number of ties is at least \( \frac{k(k-1)}{2} \), which means that

\[
\text{Rank}(A_G) \geq \frac{k(k-1)}{2}.
\]

Therefore, according to observation (1) above, we have:

\[
\Pr_{P \sim \pi}(\text{WCW}(P) = k) \leq \sum_{G: C_G \in C} \Pr_{P \sim \pi}(\text{Hist}(P) \in \text{Sol}_G) = O(n^{-\frac{k(k-1)}{2}})
\]

The last part follows after applying the polynomial upper bound in Lemma 1 to all \( C_G \in C \) and the observation (3) above.

To prove the tightness, suppose there exists \( \pi \in \text{CH}(\Pi) \) with WCW\((\pi)\) \( \geq k \). This means that there exists a supergraph of \( \text{UMG}(\pi) \) over \( A \), denoted by \( G^* \), that has exactly \( k \) weak Condorcet winners, and there is an edge from any weak Condorcet winner to any other alternative. By Claim 5, \( \pi \in \text{Sol}_{G^*} \) and \( \text{Rank}(A_{G^*}) = \frac{k(k-1)}{2} \). The tightness follows after applying the tightness of the polynomial bound in Lemma 1 to \( C_{G^*} \).

**Third step: the exponential case.** For any \( C_G \in C \) and any \( \pi \in \text{CH}(\Pi) \), we first prove that \( \text{Sol}_G \cap \text{CH}(\Pi) = \emptyset \). Suppose for the sake of contradiction there exist \( C_G \in C \) and \( \pi \in \text{Sol}_G \cap \text{CH}(\Pi) \). It follows that \( \text{UMG}(\pi) \) is a subgraph of \( G \), which means that all weak Condorcet winners in \( G \) must also be weak Condorcet winners in \( \pi \). Because there are \( k \) weak Condorcet winners in \( G \), we have WCW\((\pi)\) \( \geq k \), which is a contradiction. The lower (respectively, upper) bound follows after applying Lemma 1 to an arbitrary (respectively, all) \( C_G \in C \).

Consider the special case where \( \pi_{uni} \in \text{CH}(\Pi) \). Notice that all edge weights in \( \pi_{uni} \) are 0, which means that \( \pi_{uni} \) satisfies all pairwise constraints \( \text{Pair}_{a,b} \) defined in Definition 10. Consequently, for any \( \mathcal{A} \) and \( \mathcal{B} \) that only contain pairwise constraints, we have \( \text{Sol}_{\mathcal{A}\mathcal{B}} \cap \text{CH}(\Pi) \neq \emptyset \). This observation leads to the following corollary of Proposition 6.

**Corollary 3.** Let \( \mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi) \) be a strictly positive and closed single-agent preference model, where \( \pi_{uni} \in \text{CH}(\Pi) \). For any \( 1 \leq k \leq m \), any \( n \in \mathbb{N} \), and any \( \theta \in \Pi^n \), we have

\[
\Pr_{P \sim \theta}(\text{WCW}(P) = k) = O(n^{-\frac{k(k-1)}{2}}).
\]

The bound is tight for infinitely many \( n \in \mathbb{N} \) and corresponding \( \theta \in \Pi^n \).

When \( \mathcal{M} \) is neutral, we have \( \pi_{uni} \in \text{CH}(\Pi) \). This is because for any \( \pi \in \Pi \), the average of \( m! \) distributions obtained from \( \pi \) by applying all permutations is \( \pi_{uni} \). Therefore, Corollary 1 applies to all neutral, strictly positive, and closed models including \( \mathcal{M}^{\epsilon}_{\mathcal{A}} \) and \( \mathcal{M}^{[\epsilon,1]}_{\mathcal{P}i} \) for all \( 0 < \epsilon \leq 1 \).
Proposition 7 (Smoothed likelihood of non-existence of weak Condorcet winners). Let $\mathcal{M} = (\Theta, \mathcal{L}(A), \Pi)$ be a strictly positive and closed single-agent preference model.

Upper bound. For any $n \in \mathbb{N}$ and any $\bar{\pi} \in \Pi^n$, we have:

$$
\Pr_{P \sim \bar{\pi}}(WCW(P) = 0) = \begin{cases} 
O(1) & \text{if } \exists \pi \in CH(\Pi) \text{ and } G \supseteq UMG(\pi) \text{ s.t. } WCW(G) = 0 \\
\exp(-\Omega(n)) & \text{otherwise}
\end{cases}.
$$

Tightness of the upper bound. There exist infinitely many $n \in \mathbb{N}$ such that:

$$
\max_{\bar{\pi} \in \Pi^n} \Pr_{P \sim \bar{\pi}}(WCW(P) = 0) = \begin{cases} 
\Omega(1) & \text{if } \exists \pi \in CH(\Pi) \text{ and } G \supseteq UMG(\pi) \text{ s.t. } WCW(G) = 0 \\
\exp(-O(n)) & \text{otherwise}
\end{cases}.
$$

Proof. First step: defining $C$. Let $C = \{C_G : WCW(G) = 0\}$, where $C_G$ is defined in Definition [11]. That is, $C$ contains all $C_G$ where $G$ has no weak Condorcet winners. We have the following observations. (1) For any profile $P$, $WCW(P) = 0$ if and only if $Hist(P) \in \bigcup_{G \in C} Sol_G$. To see this, if $WCW(P) = 0$ then $Hist(P) \in Sol_{UMG(P)}$, where $UMG(P) \in C$; and conversely, if $Hist(P) \in Sol_G$, then $G = UMG(P)$ does not contain a weak Condorcet winner, which means that $WCW(P) = 0$. (2) $Sol_G \neq \emptyset$ due to McGarvey’s theorem [34]. (3) $|C|$ can be seen as a constant that does not depend on $n$.

Second step: the polynomial case. The upper bound trivially holds. To prove the tightness, suppose there exist $\pi \in CH(\Pi)$ and a supergraph $G$ of $UMG(\pi)$ that does not contain a Condorcet winner. Let $G^*$ denote an arbitrary complete supergraph of $G$. It follows that $G^*$ is a supergraph of $UMG(\pi)$ and $G^*$ does not contain a Condorcet winner, which means that $C_{G^*} \in C$. By Claim [5] $\pi \in Sol_{G^*}$, which means that $Sol_{G^*} \cap CH(\Pi) \neq \emptyset$. Note that $A_{G^*} = 0$, which means that $Rank(A_{G^*}) = 0$. The tightness follows after applying the tightness of the polynomial bound in Lemma [1] to all (respectively, an arbitrary) $C_G \in C$.

Third step: the exponential case. For any $C_G \in C$ and any $\pi \in CH(\Pi)$, we first prove that $Sol_G \cap CH(\Pi) = \emptyset$. Suppose for the sake of contradiction that there exist $C_G \in C$ and $\pi \in Sol_G \cap CH(\Pi)$, which means that $WCW(G) = 0$. By Claim [5] $UMG(\pi)$ is a subgraph of $G$, which contradicts the assumption of the exponential case, that all supergraphs of $UMG(\pi)$ contains at least one weak Condorcet winner. The upper bound (respectively, its tightness) follows after applying the exponential bound (respectively, its tightness) in Lemma [1] to all (respectively, an arbitrary) $C_G \in C$.

15 Berry-Esseen Theorem

Theorem 4 (Berry-Esseen Theorem [16]). Let $Y_1, \ldots, Y_n$ be independent random variables where for all $j \leq n$, $\mathbb{E}((Y_j - \mathbb{E}(Y_j))^2) = \sigma_j^2 > 0$, and $\mathbb{E}(|Y_j - \mathbb{E}(Y_j)|^3) = \rho_j < \infty$. Let $F_n(x)$ denote the cumulative distribution function (CDF) of $\sum_{j=1}^{n} Y_j / \sqrt{\sum_{j=1}^{n} \sigma_j^2}$ and let $Hist(x)$ denote the CDF of the standard normal distribution. Let $\psi_0 = (\sum_{j=1}^{n} \sigma_j^2)^{-3/2} \cdot \sum_{j=1}^{n} \rho_j$. Then, there exists a constant $C_0$ such that for all $x \in \mathbb{R}$,

$$
|F_n(x) - Hist(x)| \leq C_0 \psi_0
$$