Spherically symmetric perturbation of ultrarelativistic fluid in homogeneous and isotropic universe

Yu. G. Ignat’ev & A. A. Popov*
Department of Geometry, Kazan State Pedagogical University,
Mezhlauk 1, Kazan 420021, Russia

Abstract
A solution of the linearized Einstein’s equations for a spherically symmetric perturbation of the ultrarelativistic fluid in the homogeneous and isotropic universe is obtained. Conditions on the boundary of the perturbation are discussed. The examples of particle-like and wave-like solutions are given.

1 Introduction
The description of metric perturbations in relativistic cosmology is significance in the theory of creation of the universe large-scale structure. Spherically symmetric perturbations are an important and interesting class of that perturbations.

It is easy to understand that the spherically symmetric perturbation may be represent as outgoing and ingoing waves (traveling from and into the center of the configuration). The ingoing wave amplitude increases as soon as the wave approaches to the center of the configuration. Therefore the nonlinear stage of the development of the spherically symmetric perturbation must appear more quickly in the contrast to the case considered by Lifschitz [1] for flatwave perturbations. As the wave velocity in the ultrarelativistic fluid is equal to the velocity of sound $c/\sqrt{3}$ so the time $T$ of appearing of the nonlinear stage is only determined by the typical size $L$ of an initial perturbation: $T = \sqrt{3}Lc^{-1}$. Below we describe this process explicitly.

2 Ultrarelativistic fluid in spherically symmetric spacetime. Rigorous equations
The spherically symmetric metric may be taken in the form [2]

$$ds^2 = e^\nu d\eta^2 - e^\lambda \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \tag{1}$$

*E-mail: popov@kspu.ksu.ras.ru
where the metric parameters $\nu$ and $\lambda$ are the functions of the time coordinate $\eta$ and a radial space coordinate $r$. With this notations the nontrivial Einstein’s equations take the form ($c = G = 1$, $\varepsilon = 3p$)

\[
\frac{e^{-\lambda}}{2} \left[ \frac{1}{2} \lambda''^2 + \lambda' \nu' + \frac{2}{r} (\lambda' + \nu') \right] - \frac{e^{-\nu}}{3} \left( \frac{\dot{\lambda}}{2}^2 - \frac{3}{4} \lambda''^2 \right) = 8\pi p \left( 1 + 4v^2 \right),
\]

\[
\frac{e^{-\lambda}}{4} \left[ 2 (\lambda'' + \nu'') + \nu'^2 + \frac{2}{r} (\lambda' + \nu') \right] - \frac{e^{-\nu}}{3} \left( \frac{\dot{\lambda}}{2}^2 - \frac{3}{4} \lambda''^2 \right) = 8\pi p,
\]

\[
- e^{-\lambda} \left( \lambda'' + \frac{1}{4} \lambda''^2 + \frac{2}{r} \lambda' \right) + \frac{e^{-\nu}}{4} \lambda'^2 = 8\pi p \left( 3 + 4v^2 \right),
\]

\[
\frac{e^{-\lambda}}{2\sqrt{3}} \left( 2 \lambda' - \nu' \lambda \right) = 8\pi e^{(\nu - \lambda)/2} 4p \sqrt{1 + v^2},
\]

where $\varepsilon$ is the energy density and $p$ is the pressure; $v = u^r e^{\lambda/2}$ is the frame projection of radial velosity and $u^r$ is the radial coordinate of 4-velocity of fluid; a dot and a prime denote partial derivatives with respect to

\[
\tau = \frac{\eta}{\sqrt{3}}
\]

and $r$, respectively.

There are two consequences of these equations

\[
\frac{1}{2} e^{-\lambda} \left[ \lambda'' - \nu'' + \frac{1}{2} \lambda' \nu' - \frac{1}{2} \nu'^2 + \frac{1}{r} (\lambda' + \nu') \right] = 8\pi v^2 4p,
\]

\[
- e^{-\lambda} \left[ 2 (\lambda'' + \nu'') + \nu'^2 + \frac{2}{r} (\lambda' + \nu') \right] = 8\pi p \left( 3 + 4v^2 \right),
\]

\[
+ e^{-\nu} \left( \dot{\lambda} + \lambda^2 - \frac{1}{2} \dot{\lambda}' \right) = 0.
\]

### 3 Background spacetime

In isotropic spherical coordinates the metric of the background space-time is

\[
ds^2 = a^2 \left\{ d\eta^2 - \frac{1}{y^4} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right] \right\},
\]

where

\[
y = \sqrt{1 + Kr^2b^{-2}},
\]

\[
a = \begin{cases} 
  a_0 \tau, & \text{for } K = 0; \\
  a_0 \sinh(\sqrt{12} \tau b^{-1}), & \text{for } K = -1; \\
  a_0 \sin(\sqrt{12} \tau b^{-1}), & \text{for } K = +1,
\end{cases}
\]

$K = 0, \pm 1, b, a_0$ are the constant parameters of the background space-time.

The energy density $\varepsilon_0$, the pressure $p_0$ and the frame projection of radial velocity of the backgound fluid $v_0$ are

\[
\varepsilon_0 = 3p_0 = \frac{1}{8\pi} \left[ 4 \frac{y^3}{a^2} \left( y'' - \frac{2}{y} y' \, y'^2 + \frac{2}{r} y' \right) + \frac{9}{a^4} \right],
\]

\[
v_0 = 0.
\]
4 Perturbation equations and general solution

Let us define small perturbation quantities as follows

\[ \delta \varepsilon = 3 \delta p = \varepsilon - \varepsilon_0, \quad (14) \]
\[ \delta \nu = \nu - \ln \left( a^2 \right), \quad (15) \]
\[ \delta \lambda = \lambda - \ln \left( a^2 / y^4 \right). \quad (16) \]

We will assume that the frame projection of radial velocity of fluid \( \nu \) is the small quantity too. Substituting relations (14-16) into Eq.(7) and keeping only the first-order quantities in perturbation we obtain

\[ - (\delta \lambda'' + \delta \nu'') + \frac{1}{r} (\delta \lambda' + \delta \nu') - \frac{4y'}{y} (\delta \lambda' + \delta \nu') = 0. \quad (17) \]

The solution of this equation is

\[ \delta \lambda + \delta \nu = \begin{cases} 
C_1 + C_2 r^2, & \text{for } K = 0; \\
C_3 + C_4 b^2 K^{-1} y^{-2}, & \text{for } K = \pm 1, 
\end{cases} \quad (18) \]

where \( C_1, C_2, C_3 \) and \( C_4 \) are the arbitrary functions of the time coordinate \( \tau \).

If we require that

\[ \delta \lambda = \delta \nu = 0 \quad \text{for } r > r_0(\tau), \quad (19) \]

where \( r_0 \) is the some function of \( \tau \), i.e. the space-time is asymptotically homogeneous and isotropic, then

\[ C_1 = C_2 = C_3 = C_4 = 0 \quad (20) \]

and

\[ \delta \lambda = - \delta \nu. \quad (21) \]

Substituting the Eqs.(13), (16) and (21) into Eq.(8), we get in the linear approximation

\[ - y^4 \left[ \delta \lambda'' + 2 \left( \frac{1}{r} - \frac{y'}{y} \right) \delta \lambda' \right] + \delta \ddot{\lambda} + 4 \frac{\ddot{a}}{a} \delta \dot{\lambda} + 4 \frac{\dddot{a}}{a} \delta \lambda = 0. \quad (22) \]

If we introduce new radial coordinate

\[ L(r) = \int_0^r \frac{dr}{1 + K (r/b)^2} = \begin{cases} 
K = 0; \\
K = -1; \\
K = +1 
\end{cases} \quad (23) \]

and make the substitution

\[ \delta \lambda = \frac{y^2}{a r} \frac{\partial}{\partial \tau} \left( \frac{\varphi}{a} \right), \quad (24) \]

where \( \varphi = \varphi(\tau, L) \), then the equation (22) can be rewritten in the form

\[ \frac{\partial}{\partial \tau} \left[ \frac{1}{a} \left( \ddot{\varphi} - \frac{\partial^2 \varphi}{\partial L^2} - \frac{4K}{b^2} \varphi \right) \right] = 0. \quad (25) \]
One can easily integrate this equation
\[ \dot{\varphi} - \frac{\partial^2 \varphi}{\partial L^2} - \frac{4K}{b^2} \varphi = aF(L), \] (26)
where \( F(L) \) is an arbitrary function of \( L \).

A general solution of Eq. (24) is the sum of the general solution of the equation
\[ \dot{\varphi} - \frac{\partial^2 \varphi}{\partial L^2} - \frac{4K}{b^2} \varphi = 0 \] (27)
and the particular solution of the equation (23). But this particular solution does not change the form \( \delta \lambda \), because it can be rewritten as \( aW \), where \( W = W(L) \) and
\[ - \frac{d^2 W}{dL^2} - \frac{16K}{b^2} W = F. \] (28)

Therefore we can substitute the solution of equation (27) but not equation (26) into the expression (24). In the case \( K = 0 \) this solution is
\[ \varphi(\tau, L) = \Phi_+(\tau + L) + \Phi_-(\tau - L) \] (29)
and in the case \( K = \pm 1 \)
\[ \varphi(\tau, L) = \int_0^{\tau+L} \int_0^{\tau-L} \left[ \frac{d^3 f(t)}{dt^3} + \frac{d^3 g(t)}{dt^3} + b \left( \frac{d^3 (f + g)}{dt^3} \right) \right] \left[ \frac{2i}{b} \sqrt{(\tau - L)(\tau + L - t)} \right] dt \]
\[ + b \left( \frac{d^3 (f + g)}{dt^3} \right) \left. \right|_{t=0} \left[ \frac{2i}{b} \sqrt{(\tau - L)(\tau + L - t)} \right] \]
\[ = \int_0^{\tau+L} \int_0^{\tau-L} \left( \frac{2i}{b} \sqrt{(\tau - L)(\tau + L - t)} \right) dt \]
\[ = J_0 \left( \frac{2i}{b} \sqrt{(\tau - L)(\tau + L - t)} \right), \] (30)
where \( \Phi_+, \Phi_-, f \) and \( g \) are arbitrary functions and \( J_0 \) is the Bessel function.

Thus the relations (24) and (23) describe the spherically symmetric perturbation of the metric satisfying the condition (19) on the background of spacetime (1), if we take into account the relations (29) and (30).

The perturbations of the density \( \delta \varepsilon \) and the frame projection of the radial velocity \( v \) are
\[ 8\pi \delta \varepsilon = 8\pi 3\delta p = \frac{\dot{a}}{a^2} \left( \dot{\lambda} + \frac{\dot{a}}{a} \delta \lambda \right) - \frac{y^4}{a^2} \left[ 2 \left( \frac{1}{r} - \frac{y}{y'} \right) \delta \lambda' \right] + \frac{4}{y} \left( y'' - 2 \frac{y'^2}{y} + \frac{2}{r} y' \right) \delta \lambda, \] (31)
\[ 8\pi v = \frac{\sqrt{3} y^2}{4\varepsilon_0} \left( \delta \lambda' + \frac{\dot{a}}{a} \delta \lambda \right). \] (32)

In the case
\[ K = 0, \ \Phi_+(x) = D_+ + D_1 x + D_2 x^2 + D_3 x^3, \ \Phi_-(x) = D_- + D_1 x - D_2 x^2 + D_3 x^3, \] (33)
where $D_+, D_-, D_1, D_2, D_3$ are constants, we obtain a Newtonian potential

\[ \delta \lambda = - \delta \nu = \frac{2m}{ar} \]  

(34)

caused by a particle of variable mass

\[ m = - \frac{\dot{a}}{2a^2} (D_+ + D_-) + \frac{\tau^2}{2a} 2D_3. \]  

(35)

In the conclusion of this section we would like to note that the perturbations in the homogeneous and isotropic universe are gauge-dependent [4]. Therefore, we should know whether the perturbative quantities $\delta \varepsilon, \nu, \delta \lambda, \delta \nu$ obtained in this section are the actual physical perturbations or merely an artifact of gauge.

In consequence of the symmetry of problem the perturbations described in this paper are scalar ones. The gauge-invariant amplitude of density perturbation with the notations of the paper [4] is

\[ \epsilon_g = \delta - 3(1 + w) \frac{1}{kS} \left( B^{(0)} - \frac{1}{k} \dot{H}_T^{(0)} \right) \]  

(36)

The choice of the isotropic coordinates (1) in our paper corresponds to a longitudinal gauge. With the notations of the paper [4] this give

\[ B^{(0)} = H_T^{(0)} = 0. \]  

(37)

By comparing the corresponding expressions in two papers we find

\[ \epsilon_g Q^{(0)} = \delta Q^{(0)} = \delta \varepsilon / \varepsilon_0. \]  

(38)

The relations between the other perturbative quantities can be determined analogously

\[ v_s^{(0)} Q^{(0)} = \left[ v^{(0)} - \frac{1}{k} \dot{H}_T^{(0)} \right] Q^{(0)} = v^{(0)} Q^{(0)} = v, \]  

(39)

\[ \Phi_A Q^{(0)} = \left[ A + \frac{1}{k} \dot{B}^{(0)} + \frac{1}{kS} B^{(0)} - \frac{1}{k^2} \left( \dot{H}_T^{(0)} + \frac{\dot{S}}{S} \dot{H}_T^{(0)} \right) \right] Q^{(0)} = AQ^{(0)} = \frac{\delta \nu}{2}, \]  

(40)

\[ \Phi_H Q^{(0)} = \left[ H_L + \frac{1}{3} \dot{H}^{(0)} + \frac{1}{kS} B^{(0)} - \frac{1}{k^2} \frac{\dot{S}}{S} \dot{H}_T^{(0)} \right] Q^{(0)} = H_L Q^{(0)} = \frac{\delta \lambda}{2}. \]  

(41)

From these relations we find that the perturbation quantities introduced in this section are actualy physical as well as the gauge-invariant quantities $\epsilon_g Q^{(0)}, v_s^{(0)} Q^{(0)}, \Phi_A Q^{(0)}, \Phi_H Q^{(0)}$.

### 5 Boundary conditions

The solution [34] is physically unacceptable for the description of the gravitational field of perturbation which arose as a result of fluctuation at the moment of "time" $\tau = \tau_0$, since it is inconsistent with the principle of causality. Therefore, the boundary conditions should be formulated in the way that at least beyond the light horizon, the potential $\delta \lambda$ should vanish together with its derivatives. Such boundary conditions are in accordance with the "birth" of a perturbation as a result of the redistribution of Robertson-Walker matter. In fact, however,
the horizon of the perturbation is not the light cone but the sound cone, since the density perturbations extend with velocity of sound. Thus the boundary conditions at the sound horizon $L = \tau - \tau_0$ are taken in the form

$$
\delta \lambda |_{L=\tau-\tau_0} = \frac{\partial}{\partial L} \delta \lambda \bigg|_{L=\tau-\tau_0} = \frac{\partial}{\partial \tau} \delta \lambda \bigg|_{L=\tau-\tau_0} = 0. \quad (42)
$$

Substituting Eqs. (29) and (30) into this relation, we obtain in the case $K = 0$

$$
\Phi_+'(2\tau - \tau_0) + \Phi_-'(0) - \frac{1}{\tau} [\Phi_+'(2\tau - \tau_0) + \Phi_-(0)] = 0, \quad (43)
$$

$$
\Phi_+''(2\tau - \tau_0) - \Phi_-'(0) - \frac{2}{\tau} \Phi_+'(2\tau - \tau_0) + \frac{1}{\tau^2} [\Phi_+'(2\tau - \tau_0) + \Phi_-(0)] = 0, \quad (44)
$$

$$
\Phi_+''(2\tau - \tau_0) + \Phi_-'(0) - \frac{3}{\tau} [\Phi_+'(2\tau - \tau_0) + \Phi_-(0)] + \frac{3}{\tau^2} [\Phi_+'(2\tau - \tau_0) + \Phi_-(0)] = 0, \quad (45)
$$

and in the case $K = \pm 1$

$$
\frac{f'''}{2} (2\tau - \tau_0) + g'''(0) - \frac{\dot{a}}{a} \left\{ f'''(2\tau - \tau_0) - f''(0) + b [f''(0) + g''(0)] \right\}
- \frac{2\tau}{b^2} f''(0) + \frac{f'(2\tau - \tau_0) - f'(0)}{b^2} = 0, \quad (46)
$$

$$
\frac{f'''}{2} (2\tau - \tau_0) - g'''(0) - \frac{\dot{a}}{a} [f'''(2\tau - \tau_0) - g''(0)] - \frac{8\tau^2}{b^2} [f''(0) + g''(0)]
- \frac{2\tau}{b^2} g''(0) + \left( \frac{2\tau^2}{b^4} - \frac{\dot{a}^2 \tau}{a b^2} \right) f''(0) + \frac{\dot{a}}{a b^2} f'(2\tau - \tau_0) + \left( \frac{2\tau}{b^4} - \frac{\dot{a}}{a b^2} \right) f'(0)
- \frac{1}{b^4} [f(2\tau - \tau_0) - f(0)] = 0, \quad (47)
$$

$$
\frac{f'''}{2} (2\tau - \tau_0) + g'''(0) + \left( \frac{8\tau^2}{b^2} + \frac{12k}{b} \right) [f''(0) + g''(0)] + \frac{2\tau}{b^2} g''(0)
+ \frac{2}{b^2} (1 + 6k) [f''(2\tau - \tau_0) - f''(0)] - \frac{2\tau^2}{b^4} f''(0) - \frac{2\tau}{b^4} f'(0)
+ \frac{1}{b^4} [f(2\tau - \tau_0) - f(0)] = 0, \quad (48)
$$

where a prime denotes a derivative of the function with respect to its argument.

## 6 Case of spatially flat universe ($K = 0$)

Integrating Eqs. (43)-(45), we obtain

$$
\Phi_+ (\tau) = -\frac{x^2}{2} \Phi''(\tau_0) - x \left[ \Phi_-'(\tau_0) - \tau_0 \Phi_-'(\tau_0) \right] - \frac{\tau_0^2}{2} \Phi''(\tau_0) + \tau_0 \Phi_-'(\tau_0) - \Phi_-'(\tau_0). \quad (49)
$$
We denote a new function
\[
\Psi (\tau - r - \tau_0) = \Phi_- (\tau - r) - \Phi_- (\tau_0) - \frac{1}{1!} \Phi_- (\tau_0) \cdot (\tau - r - \tau_0) - \frac{1}{2!} \Phi_- (\tau_0) \cdot (\tau - r - \tau_0)^2.
\]
(50)

Then it is possible to reduce the relation (24) to
\[
\delta \lambda = \frac{1}{ar} \frac{\partial}{\partial \tau} \left[ \frac{\Psi (\tau - r - \tau_0)}{a} \right]
\]
(51)
and the conditions (43)-(45) to
\[
\Psi (0) = \Psi' (0) = \Psi'' (0) = 0.
\]
(52)

If the function \( \Psi (x) \) satisfies to the condition
\[
\Psi (x) = 0 \quad \text{for} \quad x < 0,
\]
(53)
then the relations (51)-(52) describe the spherically symmetric fluctuation of gravitational field in region \( \tau - r - \tau_0 > 0 \) that satisfies to the conditions
\[
\delta \lambda \bigg|_{r=r-\tau_0} = \frac{\partial}{\partial r} \delta \lambda \bigg|_{r=r-\tau_0} = \frac{\partial}{\partial \tau} \delta \lambda \bigg|_{r=r-\tau_0} = 0.
\]
(54)

For more detailed study of the solution near the boundary we represent the function \( \Psi (x) \) as a power series
\[
\Psi (x) = \sum_{n=0}^{\infty} \Psi_n x^n H(x),
\]
(55)
where \( \Psi_k \) are constants and
\[
H(x) = \begin{cases} 
0, & x < 0, \\
1, & x \geq 0.
\end{cases}
\]
(56)

Taking the relation (52) into account, we obtain
\[
Psi_0 = Psi_1 = Psi_2 = 0.
\]
(57)

Then relation (51) reduces to
\[
\delta \lambda = \frac{1}{ar} \sum_{n=3}^{\infty} \frac{\Psi_n}{a} (\tau - r - \tau_0)^{n-1} \left( n - 1 + \frac{r + \tau_0}{\tau} \right) H(\tau - r - \tau_0).
\]
(58)

The solution obtained in [3]
\[
\delta \lambda = \Psi_3 \frac{2\tau^2}{a} \left( \frac{1}{ar} - \frac{3}{2a\tau} + \frac{r^2}{2a\tau^3} \right) H(\tau - r)
\]
(59)
corresponds to the first term of the series (58) for \( \tau_0 = 0 \). The solution (51) is a particle-like one if the function
\[
m = \frac{\partial}{\partial \tau} \left[ \frac{\Psi (\tau - \tau_0)}{2a} \right]
\]
(60)
is not equal to zero. This function describes the mass of particle in the center of configuration and in the case considered in [5] it is

$$m = \frac{\Psi_3 \tau^2}{a}.$$  \hspace{1cm} (61)

The wave traveling from the center of configuration

$$\delta \lambda = \begin{cases} \frac{\Psi_0}{a \tau \ell} \left[ \frac{4\pi}{\ell} \sin \frac{2\pi}{\ell} x + \frac{1}{\tau} \cos \frac{2\pi}{\ell} x - \frac{1}{\tau} \right] \left[ 1 - \cos \frac{2\pi}{\ell} x \right], & \text{for } 0 \leq x \leq l; \\ 0, & \text{for } x < 0, x > l, \end{cases}$$  \hspace{1cm} (62)

where $x = \tau - r - \tau_0, \Psi_0$ and $l$ are constants, may be an example of nonparticle-like solution in the case $\tau > \tau_0 + l$. The function $\Psi$ that corresponds to that solution is

$$\Psi(x) = \begin{cases} \Psi_0 \left[ 1 - \cos \frac{2\pi}{\ell} x \right]^2, & \text{for } 0 \leq x \leq l; \\ 0, & \text{for } x < 0, x > l. \end{cases}$$  \hspace{1cm} (63)

7 Conclusion

We have considered the spherically symmetric perturbation of the cosmological perfect fluid with the equation of state $p = \varepsilon / 3$. The general solution of the linearized Einstein’s equations which describe this perturbation has been obtained. The solution is represented as outgoing and ingoing waves moving with velocity of sound. In order to consider the perturbation in causally connected region we have imposed the boundary conditions on the solution. In this case we have investigated the behavior of solution near the boundary which is the sound horizon. We have given the examples of particle-like and wave-like solutions.

We would like to finish our article by the following remark. The point of view that the large-scale universe structure is pancake-like is well-known [4]. This point of view is based on the fact that flat perturbations (with the wave length more than the Jeans one) begin to be increasing only with the decoupling moment on the nonrelativistic stage of the universe extension. However, the obtained solution shows that spherical perturbations increase already on the ultrarelativistic universe extension stage. Thus it may be turned out that at the decoupling moment the spherical perturbations will have a bigger amplitude than the flat ones, and in this case we will have to revise the scenario of creation of large-scale universe structure.

References

[1] E.M.Lifschitz and I.M.Khalatnikov, Adv. Phys. 12 (1963) 185.
[2] C.W.Mizner, K.S.Thorne and J.A.Wheeler, Gravitation (W.H.Freeman, San Francisco, 1973)
[3] G.C.McVittie, Monthly Notices of the Royal Astronom. Society 93 (1933) 325.
[4] J.M. Bardeen, Phys. Rev. D22 (1980) 1882.
[5] Yu.G.Ignat’ev and A.A.Popov, Astrophysics and Space Science 163 (1990) 153.
[6] Ya.B.Zeldovich, Structura Vselennoj, Itogi nauki i techniki, Astronomiya 22 (VINITI, Moscow, 1983)