Buckling analysis of square Functionally Graded Material (FGM) plate using Discrete Kirchhoff Mindlin Triangular (DKMT) element

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Abstract. This paper presents the convergence behavior of Discrete Kirchhoff Mindlin Triangular (DKMT) element in buckling analysis under uniaxial compression of square plate problems. The DKMT element has a good result for a thin plate and a thick plate. For the Functionally Graded Material (FGM) problem, the DKMT element is reformulated. FGM is a graded composite material that has high-temperature and structural flexibility resistance. The numerical results of mechanical buckling of square FGM plate under uniaxial compression using the DKMT element are reported. The critical buckling of the square FGM plate is compared to the reference existing solutions. The effects of parametric variation, such as the type of meshing, boundary conditions, power-law index, and ratio $L/h$ are presented. The results show that the DKMT element gives good results on the buckling analysis of square FGM plate problems.

1. Introduction
Functionally Graded Material (FGM) was first found in 1984 by scientists from Sendai Japan. This material has been widely applied in engineering such as nuclear, civil, combustion chambers, racing vehicle frame, rocket nozzle, wings, engine casing, helicopter components or aerospace structure, etc. [1]–[8]. The materials that are usually used in FGM are ceramic and metal. Ceramic is resistant to high-temperature and corrosion, while metal is resistant to structural flexibility and fracture toughness [9]–[11]. FGM can increase the bond strength, reduces the crack force, and eliminates stress differences of its forming materials (delamination). This paper uses the power-law rule to adjust the gradient of FGM [1]–[5].

The Discrete Kirchhoff Mindlin Triangular (DKMT) element was introduced by Katili in 1993. At the same time, the quadrilateral element which is called DKMQ (Discrete Kirchhoff Mindlin Quadrilateral) is also introduced by Kaliti. Both of them give the excellent results in isotropic and orthotropic material [12]–[21]. The DKMT element can be used for thin and thick plate structures. The shear-locking phenomenon is eliminated by using the Assumed Natural Strain method [22]–[25]. In this paper, we reformulate the DKMT element for buckling analysis on the square FGM plate problem. Wong et al. [26] have developed the DKMQ element for buckling analysis of square isotropic plate bending. The buckling analysis on isotropic and FGM plate structures has also been conducted in [8], [27]–[33].
In this paper, the convergence behavior of the DKMT element on buckling analysis under uniaxial compression of square plate problems is presented. The effects of parametric variation, such as the types of mesh, boundary conditions, the power-law index, and the aspect ratio $L/h$ are applied. The results of these problems are compared then with the reference existing solutions.

2. Theoretical formulation

2.1. Functionally Graded Material (FGM)

Functionally graded material consists of two materials in this paper (Figure 1), i.e. ceramic (top) and metal (bottom). The volume fraction changes continuously through the structure’s thickness [2].

![Figure 1. Geometry of FGM](image1)

The volume fractions with the power-law rule (P-FGM) give the equation:

$$ P(z) = (P_C - P_M)V_C(z) + P_M $$

$$ V_C = \left(\frac{z}{h} + \frac{1}{2}\right)^n $$

with $P_C$ and $P_M$ are the material properties of ceramic and metal, respectively. $h$ is the plate’s thickness, $z$ is the thickness coordinate of the plate, $n$ is the power-law index, and $V$ is a variation of the volume fractions (Figure 2). The modulus of elasticity using the power-law rule is given by:

$$ E_m = E_s \int_{-h/2}^{h/2} E(z)dz = \left(\frac{E_C - E_M}{n+1} + E_M\right)h $$

$$ E_p = \int_{-h/2}^{h/2} E(z)z^2dz = \left(\frac{n^2 + n + 2}{4(n+1)(n+2)(n+3)}(E_C - E_M) + \frac{E_M}{12}\right)h^3 $$

$$ E_{mb} = \int_{-h/2}^{h/2} E(z)zdz = \left(\frac{n}{2(n+1)(n+2)}(E_C - E_M)\right)h^2 $$

2.2. The formulation of the DKMT element

In this paper, we reformulated the DKMT element from its original version in 1993. There are two additional degrees of freedom (d.o.f) that are $u$ and $v$. So, the DKMT element has five d.o.f on each node. The formulations of the DKMT element on the composite plates can be seen in [22], [23]. Figure 3 shows the illustration of the DKMT element with the kinematic variables.
Figure 3. The kinematic variables of the DKMT element

The interpolations of displacements in the DKMT element are given by:

\[
\begin{align*}
\mathbf{u} &= \sum_{i=1}^{3} N_i \mathbf{u}_i; \\
\mathbf{v} &= \sum_{i=1}^{3} N_i \mathbf{v}_i; \\
\mathbf{w} &= \sum_{i=1}^{3} N_i \mathbf{w}_i; \\
\end{align*}
\]

(6)

\[
\begin{align*}
\beta_x &= \sum_{i=1}^{3} N_i \beta_{xi} + \sum_{k=1}^{6} P_k C_k \Delta \beta_{xk}; \\
\beta_y &= \sum_{i=1}^{3} N_i \beta_{yi} + \sum_{k=1}^{6} P_k S_k \Delta \beta_{yk}; \\
\end{align*}
\]

(7)

The linear shape functions and the quadratic shape functions are given by:

\[
\begin{align*}
N_i &= \lambda = 1 - \xi - \eta; \\
N_2 &= \xi; \\
N_3 &= \eta; \\
N_4 &= 4\lambda \xi; \\
P_1 &= 4\lambda \xi; \\
P_5 &= 4\lambda \eta; \\
\end{align*}
\]

(8)

(9)

The stiffness matrix is formulated by the Hu-Washizu principle as follows:

\[
\begin{align*}
\Pi_{st} &= \Pi_{st}^m + \Pi_{st}^b + \Pi_{st}^{mb} + \Pi_{st}^t \\
\Pi_{st}^m &= \frac{1}{2} \langle \mathbf{u}_n \rangle \mathbf{k}_m \langle \mathbf{u}_n \rangle \\
\Pi_{st}^b &= \frac{1}{2} \langle \mathbf{u}_n \rangle \mathbf{k}_b \langle \mathbf{u}_n \rangle \\
\Pi_{st}^{mb} &= \frac{1}{2} \langle \mathbf{u}_n \rangle \mathbf{k}_{mb} \mathbf{y} \langle \mathbf{u}_n \rangle \\
\Pi_{st}^t &= \frac{1}{2} \langle \mathbf{u}_n \rangle \mathbf{k}_t \langle \mathbf{u}_n \rangle \\
\end{align*}
\]

(10)

(11)

(12)

(13)

(14)

where the equation (11) until the equation (14) are respectively the membrane energy, the bending energy, the membrane-bending energy and the shear energy.

For buckling analysis, we need the geometric stiffness matrix below [2], [26]:

\[
\begin{align*}
\Pi_\sigma &= \frac{1}{2} h \langle \nabla \mathbf{u} \rangle \mathbf{[s]} \{ \nabla \mathbf{u} \} dA + \frac{1}{2} h \langle \nabla \mathbf{v} \rangle \mathbf{[s]} \{ \nabla \mathbf{v} \} dA + \frac{1}{2} h \langle \nabla \mathbf{w} \rangle \mathbf{[s]} \{ \nabla \mathbf{w} \} dA \\
&+ \frac{1}{2} h^3 \int_{\Delta} \langle \nabla \beta_x \rangle \mathbf{[s]} \{ \nabla \beta_x \} dA + \frac{1}{2} h^3 \int_{\Delta} \langle \nabla \beta_y \rangle \mathbf{[s]} \{ \nabla \beta_y \} dA \\
\Pi_\sigma &= \frac{1}{2} \langle \mathbf{u}_n \rangle \mathbf{k}_c \langle \mathbf{u}_n \rangle
\end{align*}
\]

(15)

(16)
The membrane stresses matrix is a mechanical load matrix, as follows:

\[
\left[ \sigma_n \right] = \begin{bmatrix}
\sigma_x^0 & \tau_{xy}^0 \\
\tau_{xy}^0 & \sigma_y^0
\end{bmatrix}
\quad \sigma^0 = \frac{N^0}{h}
\]  

(17)

The eigenvalue equation for critical buckling load \( N_{cr} \) is given by:

\[
(\left[ k \right] - N_{cr} \left[ k_G \right])\{u_n\} = 0
\]  

(18)

3. Numerical test

The convergence behavior of the DKMT element for buckling analysis under uniaxial compression on the square plate (Figure 4) is presented to predict the critical buckling load. We present the non-dimensional critical buckling load results by normalizing them that use the equations below:

\[
N_{cr} = N_{cr}L^2 \left( \pi^2 D \right)^{\frac{1}{2}}
\]  

(19)

\[
D = Eh^3 \left( 12 \left( 1 - \nu^2 \right) \right)^{-1}
\]  

(20)

**Figure 4.** The illustration of uniaxial compression on the square plate

The material properties used in this problem are Al/ZrO_2 with which \( E_m = 200000 \) and \( E_c = 70000 \). Both materials have a constant Poisson’s ratio, \( \nu = 0.3 \). We analyze the square plate with \( L=1000 \) and the aspect ratio \( (L/h=10) \) and \( (L/h=100) \). The index power-law that we used are \( n=0; 0.5; 1; 2; 5; 10; \) and infinite \( \infty \). The types of meshes are Mesh A, Mesh B, and Mesh C (Figure 5) with the mesh index \( N\times N \times 2 \) \( (N=4, 8, 16, 32, 64, \) and 128). The boundary conditions are hard simply supported (SSSS) and clamped supported (CCCC).

**Figure 5.** Square plates with \( N=4 \) with (a) CCCC Mesh A, (b) SSSS Mesh B, and (c) SSSS Mesh C
The reference solutions used in this paper are given in [26]–[30]. Table 1 shows the results of the DKMT element on the square isotropic plate ($n=0$) under uniaxial compression. The convergence behaviors of the SSSS square plates are presented in Figure 6 and Figure 7. The results show that the DKMT element has a good correlation compared to the reference existing solutions with the different types of meshes, the mesh index, and the aspect ratio $L/h$. We also observe about the best convergence speed. For the aspect ratio $L/h=10$ (thick plate), the best convergence speed is by using Mesh B while for ratio $L/h=100$ (thin plate) is by using Mesh A. The difference between the results of the DKMT element and the FSDT-Meshfree using $L/h=10$ is about 0.05%. When compared with the analytical solution [28] using $L/h=100$, the difference is about 0.08%.

Table 1. Non-dimensional critical buckling load of the SSSS square isotropic plate under uniaxial compression

| Method          | Mesh Index, $N$ | $L/h = 10$ | $L/h = 100$ |
|-----------------|----------------|------------|-------------|
|                 | Mesh A | Mesh B | Mesh C | Mesh A | Mesh B | Mesh C |
| Present         | 4      | 3.74743 | 3.72888 | 3.76629 | 4.00130 | 3.98911 | 4.03463 |
|                 | 8      | 3.73509 | 3.72967 | 3.73894 | 3.99787 | 3.99483 | 4.00644 |
|                 | 16     | 3.73068 | 3.72925 | 3.73155 | 3.99730 | 3.99659 | 3.99944 |
|                 | 32     | 3.72944 | 3.72907 | 3.72965 | 3.99732 | 3.99701 | 3.99770 |
|                 | 64     | 3.72912 | 3.72903 | 3.72917 | 3.99716 | 3.99706 | 3.99723 |
|                 | 128    | 3.72904 | 3.72901 | 3.72905 | 3.99709 | 3.99706 | 3.99711 |
| DKMQ[26]        |        | 3.750   |         |         | 4.013   |         |         |
| Analytical Solution |      | 3.741[27] |        |         | 4.000[28] |        |         |
| Meshfree[29]    |        | 3.727   |         |         | -       |         |         |
| pb-2 Ritz[30]   |        | 3.787   |         |         | -       |         |         |

Figure 6. Non-dimensional critical buckling load on the simply supported isotropic square plate $L/h=10$

Figure 7. Non-dimensional critical buckling load on the simply supported isotropic square plate $L/h=100$
Table 2. Non-dimensional critical buckling load of the square FGM plates under uniaxial compression ($L/h=100$)

| Boundary Condition | Type of Mesh | Index Power-Law, $n$ | 0   | 0.5 | 1   | 2   | 5   | 10  | $\infty$ |
|--------------------|--------------|---------------------|-----|-----|-----|-----|-----|-----|---------|
| SSSS               | Mesh A       |                     | 3.99709 | 2.91720 | 2.48963 | 2.18989 | 1.98452 | 1.83649 | 1.39898 |
|                    | Mesh B       |                     | 3.99706 | 2.91718 | 2.48961 | 2.18987 | 1.98451 | 1.83647 | 1.39897 |
|                    | Mesh C       |                     | 3.99711 | 2.91721 | 2.48964 | 2.18989 | 1.98453 | 1.83649 | 1.39899 |
| CCCC               | Mesh A       |                     | 10.04765 | 7.33384 | 6.25897 | 5.50498 | 4.98781 | 4.61548 | 3.51668 |
|                    | Mesh B       |                     | 10.04758 | 7.33379 | 6.25893 | 5.50494 | 4.98778 | 4.61545 | 3.51665 |
|                    | Mesh C       |                     | 10.04743 | 7.33366 | 6.25881 | 5.50482 | 4.98768 | 4.61537 | 3.51660 |

Table 2 and Figure 8 show that the critical buckling load decreases when the power-law index increases. When we increase the numbers of the element, the critical buckling load will independent and almost constant. We also evaluate that the result used boundary condition clamped supported (CCCC) is greater than simply supported (SSSS). The DKMT element gives a good result in buckling analysis by using square plates.

Figure 8. Non-dimensional critical buckling load on clamped supported FGM square plate $L/h=10$

4. Conclusion
DKMT element gives a good result and converges well with various parameters in buckling analysis on the square FGM plates. Increasing the ratio $L/h$ and the power-law index will decrease the non-dimensional critical buckling load. Non-dimensional critical buckling load using the boundary condition ‘clamped supported’ (CCCC) is greater than hard simply supported (SSSS). And the DKMT element also shows that it can be used on the thick and thin square FGM plates in buckling analysis.

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