ODE TO $L^p$ NORMS

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Abstract. In this paper we relate the geometry of Banach spaces to the theory of differential equations, apparently in a new way. We will construct Banach function space norms arising as weak solutions to ordinary differential equations of first order. This provides as a special case a new way of defining varying exponent $L^p$ spaces, different from the Orlicz type approach. We explain heuristically how the definition of the norm by means of the particular ODE is justified. The resulting class of spaces includes the classical $L^p$ spaces as a special case. It turns out that the duality of these spaces behaves in an anticipated way, same as the uniform convexity and uniform smoothness. We study the arising duality of the ODEs and also the duality of their solutions. We present an ODE-free means of defining the norms investigated. Extensions of the definitions to several directions are discussed at the end.

1. Introduction

In this paper we will introduce a novel way of defining function space norms by means of weak solutions to ordinary differential equations (ODE). This provides a new perspective for looking at varying exponent $L^p$ spaces. This leads to looking at the geometry and duality of Banach spaces in terms of the properties of corresponding differential equations.

The classical Orlicz norms were defined in 1930’s and since then there have been various generalizations of the these norms to several directions. For example, this approach leads to several varying exponent $L^{p(·)}$ type constructions, e.g. for sequence spaces, Lebesgue spaces, Hardy spaces and Sobolev spaces. There is a vast literature on these topics, see [6], [7], [8] and [9] for samples.

Let us recall that the (generalized) Luxemburg type norm are defined as follows:

$$
\|f\| = \inf \left\{ \lambda > 0 : \int_{\Omega} \phi \left( \frac{|f(x)|}{\lambda}, x \right) \, dm(x) \leq 1 \right\}.
$$

Here $s \mapsto \phi(s, x)$ is a positive convex function and $\phi$ must satisfy suitable structural conditions. For instance, $\phi(s, x) = s^{p(x)}$, $1 \leq p(\cdot) < \infty$, produces a norm that can be seen as a varying exponent $L^p$ norm. These norms enjoy the attractive property of being rearrangement invariant in the sense that applying a measure-preserving transformation $T : \Omega \to \Omega$ onto such that $\phi(|f(x)|, x) = \phi(|f \circ T(x)|, T(x))$ for a.e. $x \in \Omega$ does not change the value of the norm. However, one might argue that the rearrangement invariance and apparent simplicity of the definition of the norm come with a cost. Namely, the definition of the norm is opaque in the sense that it involves an infimum with an integral formula inequality. For instance, by looking at

\textit{Date:} April 18, 2014.

\textit{2010 Mathematics Subject Classification.} 46E30; 46B10; 34A12; 31B10.

\textit{Key words and phrases.}
the definition of the norm it is difficult to decide how adding \(1_\Delta, \Delta \subseteq \Omega, m(\Delta) > 0\), to \(f\) contributes to the norm, even if \(\Delta\) is in some sense conveniently displaced.

The ‘virtues and vices’ of the norms that we are about to introduce are mirror images of the ones mentioned above. The ODE driven norms here, in comparison, will typically not be rearrangement invariant in the above sense, and in particular they do not reduce to the above Luxemburg type (cf. an example in [10]). On the other hand, our norms will be ‘localized’ in the sense that one can analyze the (infinitesimal) contribution of a single coordinate to the norm, a built-in feature of the construction. To make a point, it is possible to compute these norms by solving the defining ODE numerically for continuous functions \(f\) and \(p\). (It is, of course, also straight-forward to compute the above infimum numerically but we stress the fact that the methods needed to solve our first order ODE are linear in nature and elementary.) Thus, our approach to the definition of varying exponent \(L^p\) space norms is rather inductive than global.

Next we will discuss the motivating ideas behind the ODE driven norms. The author studied in [10] varying exponent \(L^p(\cdot)\) spaces formed in the following naïve fashion. As usual, we denote by \(X \oplus_p Y\) the direct sum of Banach spaces \(X\) and \(Y\) with the norm given by

\[
\|(x, y)\|_{X \oplus_p Y}^p = \|x\|_X^p + \|y\|_Y^p, \quad x \in X, \ y \in Y, \ 1 \leq p < \infty.
\]

Let \(p: \mathbb{N} \rightarrow [1, \infty)\) be a ‘varying exponent’. Define first a 2-dimensional Banach space by \(\mathbb{R} \oplus_{p(1)} \mathbb{R}\), then a 3-dimensional one \((\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}\) and proceed recursively to obtain an \(n\)-dimensional spaces

\[
\ldots((\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}) \oplus_{p(3)} \ldots) \oplus_{p(n-1)} \mathbb{R},
\]

and, finally, by taking an inverse limit, this yields a space which can be written formally as

\[
\ldots((\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}) \oplus_{p(3)} \ldots) \oplus_{p(n)} \mathbb{R}) \oplus_{p(n+1)} \ldots.
\]

Thus, this is a space normed by taking a limit of semi-norms corresponding to the \(n\)-dimensional spaces above. The recursive construction of the spaces can be regarded trivial at each step but the end result may exhibit some peculiar properties (depending on the selection of the sequence \((p(n))_n\), see [10]). In any case, this is a rather natural way of constructing Banach sequence spaces (cf. [11,4]).

The main aim of this paper is to study ‘a continuous version’ of the above class of sequence spaces \(L^p(\cdot)\), thus a space of suitable functions \(f: [0, 1] \rightarrow \mathbb{R}\), instead of sequences. The idea is somewhat similar here, knowing the norm of \(f\) up to a coordinate \(0 < t < 1\), i.e. \(\|1_{[0,t]}f\|\), and knowing the value \(|f(t^+)|\) is sufficient information in predicting the accumulation of the norm right after \(t\), i.e. knowing \(\|1_{[0,t+\delta t]}f\|\). For example, if \(f(r) = 0\) for \(t < r < s\), then we should have \(\|1_{[0,t]}f\| = \|1_{[0,s]}f\|\), and if \(|f(t^+)| > 0\), then \(\|1_{[0,t]}f\| < \|1_{[0,s]}f\|\), and so on. This intuitive description of the accumulation of the norm can be encaptured by means of a suitable ODE in such a way that its weak solution, \(\varphi_f: [0, 1] \rightarrow [0, \infty)\), shall represent the norm as follows: \(\varphi_f(t) = \|1_{[0,t]}f\|\), so that in particular \(\varphi_f(0) = 0\) and \(\varphi_f(1) = \|f\|\). The basic idea how to accomplish this and the motivations appear shortly, see Section 1.2.

The required mathematical machinery in this paper is classical and there is no apparent reason, why this alternative approach could have not been experimented
with much earlier. Also, our approach does not lead to excessively technical considerations, so hopefully it is accessible to a wide range of analysts.

1.1. Preliminaries. We will usually consider the unit interval $[0, 1]$ endowed with the Lebesgue measure. Here for almost every (a.e.) refers to $m$-a.e., unless otherwise specified. Denote by $L^0$ the space of Lebesgue-to-Borel measurable functions on the unit interval. We denote by $\ell^0(\mathbb{N})$ the vector space of sequences of real numbers with point-wise operations. We refer to [2], [3] and [7] for suitable background information.

We will study Carathéodory’s weak formulation to ODEs, that is, in the sense of Picard type integral formulation where solutions are required to be only absolutely continuous. This means that, given an ODE

$$\varphi(0) = x_0, \quad \varphi'(t) = \Theta(\varphi(t), t), \quad \text{for a.e. } t \in [0, 1],$$

we call $\varphi$ a weak solution in the sense of Carathéodory if $\varphi$ is absolutely continuous, $t \mapsto \Theta(\varphi(t), t)$ is measurable and

$$\varphi(T) = x_0 + \int_0^T \Theta(\varphi(t), t) \, dt$$

holds for all $T \in [0, 1]$.

**Lemma 1.1.** Let $\mathcal{F}$ be a collection of absolutely continuous functions $\varphi : [0, 1] \to \mathbb{R}$. Suppose that

$$\varphi_0(t) = \sup_{\varphi \in \mathcal{F}} \varphi(t)$$

is defined as a real valued function and is non-decreasing. Then there is a sequence of non-decreasing absolutely continuous functions $\varphi_n : [0, 1] \to \mathbb{R}$ such that $\varphi_n \not\uparrow \varphi_0$ uniformly as $n \to \infty$. Moreover, $\varphi_0$ is continuous.

**Proof.** For each rational number $r \in [0, 1]$ we pick a countable subset $\mathcal{F}_r \subset \mathcal{F}$ such that

$$\sup_{\varphi \in \mathcal{F}_r} \varphi(r) = \varphi_0(r).$$

Enumerate $\{\psi_i\}_{i \in \mathbb{N}} = \bigcup_r \mathcal{F}_r$. Then

$$\varphi_n = \min \{ f : [0, 1] \to \mathbb{R} : f \geq \max_{1 \leq i \leq n} \psi_i, \ f \text{ non-decreasing} \}$$

gives the required sequence.

Indeed, assume to the contrary that for some $\varepsilon > 0$ it holds that $\|\varphi_0 - \varphi_n\|_{\infty} > \varepsilon$ for all $n \in \mathbb{N}$. Then, assuming that $\varphi_0$ is continuous, a compactness argument yields a point $s \in [0, 1]$ such that $\varphi_0(s) - \lim_{n \to \infty} \varphi_n(s) \geq \varepsilon$. Pick $\varphi \in \mathcal{F}$ with $\varphi(s) > \varphi_0(s) - \varepsilon/4$ and $r < s$ such that $\varphi(r) > \varphi_0(s) - \varepsilon/2$. It follows that $\varphi_n(s) \geq \varphi_n(r) > \varphi_0(s) - \varepsilon/2$ for sufficiently large $n$, a contradiction.

In a similar way we see that $\varphi_0$ is in fact continuous. $\square$

The following result refines Carathéodory’s existence theorem in a sense.

**Lemma 1.2.** Suppose that there is a non-decreasing function $\psi : [0, 1] \to [0, \infty)$ satisfying:

$$\psi(0) = x_0 \geq 0, \quad \psi'(t) \geq \Psi(\psi(t), t) \geq 0, \quad \text{for a.e. } t \in [0, 1].$$

Consider a differential equation

$$\varphi(0) = y_0 \geq 0, \quad \varphi'(t) = \Phi(\varphi(t), t) \geq 0 \quad \text{for a.e. } t \in [0, 1]$$

for a.e. $t \in [0, 1]$. 


where \( \Phi \) is continuous and decreasing on the first coordinate, \( y_0 \leq x_0 \) and \( \Phi(x, t) \leq \Psi(x, t) \). Then the differential equation has an absolutely continuous maximal Carathéodory’s weak solution given by

\[
\varphi_0(t) = \sup_{\varphi} \varphi(t)
\]

where the supremum is taken over absolutely continuous functions \( \varphi \) such that \( \varphi(0) = y_0 \) and \( \varphi'(t) \leq \Phi(\varphi(t), t) \) for a.e. \( t \).

**Proof.** Let \( \varphi_0 \) be as above. Note that \( \varphi_0 \) is dominated by \( \psi \) because \( \varphi(t) \geq \psi(t) \) implies

\[
\varphi'(t) \leq \Phi(\varphi(t), t) \leq \Psi(\varphi(t), t) \leq \Psi(\psi(t), t) \leq \psi'(t).
\]

Indeed, here \( \psi \) may be non absolutely continuous and in fact it may even have jump discontinuities, but since it is non-decreasing we obtain the claimed dominance.

By Lemma 1.1 we find a sequence \( (\varphi_n) \) of admissible functions such that \( \varphi_n \nearrow \varphi_0 \) uniformly as \( n \to \infty \). Then by the selection \( \varphi_n \)’s and monotone convergence theorem we get

\[
\varphi_0(t) = y_0 + \lim_{n \to \infty} \int_0^t \varphi_n'(s) \, ds \leq y_0 + \lim_{n \to \infty} \int_0^t \Phi(\varphi_n(s), s) \, ds = y_0 + \int_0^t \Phi(\varphi_0(s), s) \, ds.
\]

Here \( t \to y_0 + \int_0^t \Phi(\varphi_0(s), s) \, ds \) defines an absolutely continuous increasing function.

Moreover, \( \Phi(\varphi_0(t), t) \leq \Phi(\varphi(t), t) \) where \( \varphi \) are as above, since \( \Phi(\varphi, t) \) are non-increasing. This way we see that \( \varphi_0 \) is absolutely continuous. Namely,

\[
\varphi_0(r) - \varphi_0(t) = \lim_{n \to \infty} \int_t^r \varphi_n'(s) \, ds \leq \lim_{n \to \infty} \int_t^r \Phi(\varphi_n(s), s) \, ds = \int_t^r \Phi(\varphi_0(s), s) \, ds.
\]

This also shows that \( \varphi_0'(t) \leq \Phi(\varphi_0(t), t) \) a.e.

Finally, we claim that \( \varphi_0'(t) = \Phi(\varphi_0(t), t) \) a.e. Assume to the contrary, then we can find by Lebesgue’s differentiation theorem a point of density \( t_0 \) of a set of the type

\[
\{ t \in [0, 1] : (1 + \varepsilon)\varphi_0'(t) < \Phi((1 + \varepsilon)\varphi_0(t), t) \}
\]

for some \( \varepsilon > 0 \). Then we redefine \( \varphi_0'(t) \) around this point to be \( (1 + \varepsilon)\varphi_0'(t) \) in the above set and 0 on the complement. Then the condition \( \varphi_0'(t) \leq \Phi(\varphi_0(t), t) \) still holds. By virtue of the density property this modification yields an increase in the values of \( \varphi_0 \) in small intervals around \( t_0 \), contradicting the maximality of \( \varphi_0 \).

\[ \square \]

Whenever we make a statement about a derivative we implicitly state that it exists. We will write \( F \leq G \), involving elements of \( L^0 \), if \( F(t) \leq G(t) \) for a.e. \( t \in [0, 1] \). We denote the characteristic function or indicator function by \( 1_A \) defined by \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) otherwise.

**Lemma 1.3.** Suppose that \( \varphi, \psi \in L^0 \) are absolutely continuous with \( \varphi'(t) \leq \psi'(t) \) for a.e. \( t \in [0, 1] \) such that \( \varphi(t) \geq \psi(t) \). Then \( \varphi \leq \psi \).

**Proof.** Observe that

\[
\varphi'(t) \leq (\min(\varphi, \psi))', \quad \text{for a.e. } t \in [0, 1].
\]

\[ \square \]
We will frequently calculate terms of the form \((a^p + b^p)^\frac{1}{p}\) where \(a, b \geq 0\) and \(1 \leq p < \infty\). We will adopt from [10] the following short hand notation for this:

\[ a \mathbin{\oplus}_p b = (a^p + b^p)^\frac{1}{p}. \]

This defines a commutative semi group on \(\mathbb{R}_+\), in particular associativity,

\[ a \mathbin{\oplus}_p (b \mathbin{\oplus}_p c) = (a \mathbin{\oplus}_p b) \mathbin{\oplus}_p c, \]

is useful. In taking a sequence of \(\mathbin{\oplus}_p\) or \(\mathbin{\otimes}_p\) operations we always perform the operations from left to right, unless there are parentheses indicating another order. We will also use the following operation:

\[ \bigoplus_{1 \leq i \leq n} x_i x_1 \mathbin{\oplus}_p x_2 \mathbin{\oplus}_p \ldots \mathbin{\oplus}_p x_n = \left( \sum_{i=1}^n x_i^p \right)^\frac{1}{p}, \quad x_1, \ldots, x_n \in \mathbb{R}_+. \]

The space \(L^p(\cdot) \subset L^0, p : \mathbb{N} \to [1, \infty)\), consists of those elements \((x_n)\) such that the following limit of a non-decreasing sequence exists and is finite:

\[ \lim_{n \to \infty} \left( \ldots \left( \left( |x_1| \mathbin{\oplus}_p |x_2| \mathbin{\oplus}_p \ldots \mathbin{\oplus}_p |x_n| \right) \mathbin{\oplus}_p \ldots \mathbin{\oplus}_p |x_{n+1}| \right) \right) \]

and the above limit becomes the norm of the space, see [10]. Throughout we will use the convention \(0^p = 0\) for \(p \in \mathbb{R}\).

1.2. Arriving at the varying exponent \(L^p\) norm ODE. Let us ‘derive’ heuristically our basic differential equation for varying exponent \(L^p\) norm. As mentioned in the introduction, we wish to extend the varying exponent \(L^p(\cdot)\) norm in the sense of [10] to continuous setting. Although the motivation for the task here involves the above sequence spaces, we are only required to look at simple structures \(X \mathbin{\oplus}_p Y\) one at a time due to the infinitesimal nature of the enterprise.

We will assume a Platonist approach on developing the definition of the varying exponent norms here. Thus we wish to find a function space norm following the gist of \(L^p(\cdot)\) space norms. This leads to thought experiments on the right behavior of the function \(t \mapsto \|1_{[0,t]}f\|\). The resulting ODE will be in a sense a very robust one and this allows us to write arguments in this paper in a concise fashion, not paying very much attention on the general theory of the ODEs involved.

Suppose that we have a varying exponent, i.e. a measurable function \(p : [0,1] \to [1, \infty)\) and \(f : [0,1] \to \mathbb{R}\) is another measurable function, a possible candidate to lie in the function space. We wish to arrange matters in such a way that we have an absolutely continuous non-decreasing function \(\varphi_f : [0,1] \to [0, \infty)\) such that

\[ \varphi_f(t) = \|1_{[0,t]}f\|, \quad 0 \leq t \leq 1, \]

so \(\varphi_f(0) = 0\) and \(\varphi_f(1) = \|f\| < \infty\). We will study Carathéodory’s weak formulation to ODEs. It is convenient to work with absolutely continuous solutions since this way we may apply usual tools such as Fatou’s lemma and Lebesgue’s convergence theorems on the solutions (sometimes implicitly). We are only interested here in Banach lattice norms, therefore \(\varphi_f\) is non-decreasing. In fact, we will require a modified version of Carathéodory’s weak formulation, tailor-made specifically to our setting.

We are aiming at a recursive like formula for \(\varphi_f\), similarly as in [10], so suppose that we have defined the function \(\varphi_f\) up to the interval \([0,t_0]\). Then we are not interested in the values of \(f\) and \(p\) on \([0,t_0]\), a Markovian property. Suppose, as a
thought experiment, that $f$ and $p$ are constant on an interval $[t_0, t_0 + \Delta]$, $\Delta > 0$. Then we should have
\begin{equation}
\varphi(t_0 + \Delta) = (\varphi(t_0))^{p(t_0)} + \Delta |f(t_0)|^{p(t_0)} \frac{1}{p(t_0)}
\end{equation}
(1.1)
which is analogous to the $\ell^p(\cdot)$ construction, and actually to the usual $L^p$ norm formula, since
\[ \left( \int_{t_0}^{t_0 + \Delta} |f(s)|^p \, dm(s) \right)^{\frac{1}{p}} = \left( \int_{t_0}^{t_0 + \Delta} |f(s)|^p \, dm(s) \right)^{\frac{1}{p}} \Delta^{1/p(t_0)} |f(t_0)|^{p(t_0)} \]
where the right-most term is $\Delta^{1/p(t_0)} |f(t_0)|$. Thus, by differentiating (1.1) we find a natural candidate for the norm-determining differential equation:
\begin{equation}
\partial^* \varphi(t_0) := \frac{d}{d\Delta} \varphi(t_0 + \Delta) \bigg|_{\Delta=0} = \frac{|f(t_0)|^{p(t_0)}}{p(t_0)} \varphi(t_0)^{1-p(t_0)}.
\end{equation}
(1.2)
Here we set $\Delta = 0$ because we are interested in (infinitesimal) increments around $t_0$. So, the above equation is right if $f$ and $\varphi$ are constant on the interval $[t_0, t_0 + \Delta]$ but the equation does not concern the values of $f$ and $p$ beyond $t_0$.

In formulating the differential equation we do not require $f$ or $p$ to continuous anywhere but motivated by Lusin’s theorem and related considerations we will use the above formula in any case and aim to define $\varphi$ by
\begin{equation}
\varphi(0) = 0, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0, 1].
\end{equation}
(1.3)
This formulation has the drawback that $0^{1-p(t)}$ is not defined. Also, it has a trivial solution $\varphi \equiv 0$, regardless of the values of $f$ if we use the convention $0^0 = 0$ and $p \equiv 1$. The behavior of the solutions is difficult to anticipate in the case where $\varphi(t)$ is small and $p(t)$ is large.

To fix these issues, we will use initial values $\varphi(0) = x_0 > 0$ and to correct the error incurred we let $x_0 \searrow 0$. It turns out that the corresponding unique solutions $\varphi(x_0)$ decreasingly converge point-wise to $\varphi$ which again satisfies the same ODE (where applicable). So, this procedure yields a unique solution $\varphi$ which we will formulate, by slight abuse of notation, as
\begin{equation}
\varphi(0) = 0^+, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0, 1].
\end{equation}
(1.4)
There is more to the above procedure than merely picking a maximal solution; it turns out that in many situations it is convenient to look at positive-initial-value-solutions first.

We define the varying exponent space $L^{p(\cdot)} \subset L^0$ as the space of those functions $f \in L^0$ such that $\varphi_f(1) < \infty$ where $\varphi_f$ is an absolutely continuous solution to (1.4) and the norm of $f$ will be $\varphi_f(1)$. We will study the properties of these spaces in an abstract setting shortly more carefully.

The above ODE is a separable one for a constant $p(\cdot) \equiv p$, $1 \le p < \infty$, and solving it yields $\varphi_f(1)^p = \int_0^1 |f(t)|^p \, dt$, compatible with the classical definition of the $L^p$ norm. If $p(\cdot)$ is locally bounded and $|f(t)|^{p(t)}$ is locally integrable, then Picard iteration performed locally yields a unique solution for each initial value $\varphi(0) = a > 0$. (More precisely, here local means that every point has an open
neighborhood such that the function in question has the required property when restricted to the neighborhood and the conclusion holds on some segment \([0, r), 0 < r \leq 1\), since a priori it is possible that \(\varphi(s) \to \infty\) as \(s \to r\).)

The above procedure, treating \(f(t)\) and \(p(t)\) constant after some point \(t_0\) and then differentiating by \(\frac{d}{dt} \phi(t_0 + \Delta)|_{\Delta=0}\) gives rise to a differential operator \((\partial^\ast \phi)(t_0)\). For example, \((1.1)\) immediately gives
\[
(\partial^\ast \varphi)(t_0) = |f(t_0)|^{p(t_0)}.
\]
If \(\varphi\) is differentiable in the sense of \(\partial^\ast\) at \(t_0\), then direct calculation yields
\[
(\partial^\ast \varphi)(t_0) = p(t_0)(\partial^\ast \varphi)(t_0)^{P(t_0)-1}.
\]
These two lines reproduce \((1.2)\). This operator can be seen as the composition of a usual differential operator and an operator which maps variable functions to their time stopped versions. Thus we know that the operation is consistent, simply because all the calculations take place in the space of stopped functions.

2. General ODE driven norms of function spaces

Guided by the prototypical norm-determining ODE minted above we will formulate a more general class of Banach lattice function spaces ODEs. The main point in this section is to isolate the key properties of the norm-determining ODE.

Suppose that \(f \in L^0([0, 1])\) and we are dealing with Carathéodory’s weak solutions \(\varphi_f\) of the following differential equation:
\[
(2.1) \quad \varphi_f(0) = x_0 > 0, \quad \varphi_f'(t) = \Upsilon(\varphi(t), |f(t)|, t), \quad \text{for a.e. } t \in [0, 1]
\]
where we make the following structural assumptions on \(\Upsilon\):

(i) \((0, \infty) \times [0, \infty) \to (0, \infty), (s, x) \mapsto \Upsilon(s, x, t)\) is jointly continuous, non-increasing on \(s\) and non-decreasing on \(x\) for \(t \in [0, 1]\).

(ii) \((s, x) \mapsto \Upsilon(s, x, t)\) is positively homogeneous and \(\Upsilon(s, 0, t) = 0\) for \(t \in [0, 1]\).

(iii) \(\varphi_f\) is absolutely continuous.

(iv) If \(\varphi_f, \varphi_g\) exist then also \(\varphi_{f+g}\) exists in the same sense and \(\varphi_{f+g} \leq \varphi_f + \varphi_g\).

(v) \(\varphi_f(1) > \varphi_f(0)\) for small \(x_0\), unless \(f(t) = 0\) a.e.

(vi) Given \(f_n := 1_{A_n}, \varphi_{f_n}(1) \to 0\) implies \(m(A_n) \to 0\) as \(n \to \infty\).

Conditions (i)-(iv) imply that \(|f| = \varphi_f(1)\) is a semi-norm, (v) additionally gives a norm. Under conditions (i)-(iii) the assumption that \(\Upsilon\) is concave on \(x\), (iv*), we have (iv). Condition (vi) is used in proving the completeness of the norm.

The first part of condition (iv) may fail even for some \(L^p(\cdot)\) spaces, so that the corresponding functions form a class, rather than a space. We will return to this issue.

The differential equation \((1.3)\) discussed in the previous section with
\[
\Upsilon(\varphi(t), |f(t)|, t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)}
\]
satisfies the above conditions. Unfortunately, looking at this ODE it is hard to decide if the triangle inequality holds. However, this can be accomplished by Lemma 3.6 and subsequent observations. Some other examples are
\[
\Upsilon(\varphi(t), |f(t)|, t) = tf(t)^2 \varphi(t)^{-1},
\]
\[
\Upsilon(\varphi(t), |f(t)|, t) = \max_{p \in [p_1, p_2]} \frac{|f(t)|^p}{p} \varphi(t)^{1-p},
\]
1 \leq p_1 < p_2 < \infty, and 
\begin{align*}
\Upsilon(\varphi(t), |f(t)|, t) = \max(|f(t)| - \varphi_f(t), 0).
\end{align*}
In the last case it is easy to check that
\begin{equation}
(2.2) \quad \varphi_{f+g}(t) \geq \varphi_f(t) + \varphi_g(t) \quad \implies \quad \varphi'_{f+g}(t) \leq \varphi'_f(t) + \varphi'_g(t),
\end{equation}
so that (iv) holds.
Some of the above assumptions are self-explanatory. The assumption that \( \Upsilon \) is decreasing on \( s \) requires the most attention. From the geometry of the norm point of view, it contributes to the validity of the triangle inequality, as in \((2.2)\), and hence to the convexity of the unit ball. It also tends to ‘flatten’ the unit ball, similarly as in the case of \( L^p \) spaces with \( p < \infty \) large. Therefore it can be viewed to have a smoothing effect on the norm as well, as opposed to the extremal non-smooth case \( p(\cdot) \equiv 1 \) where \( s \mapsto \Upsilon(s, x, t) \) is a constant, since \( \varphi(t)^{1-p(t)} \equiv 1 \).

From the point of view of differential equations, the assumption that \( \Upsilon \) is non-increasing on \( s \) also has a remarkably stabilizing effect on the solutions \( \varphi \). The corresponding differential equation exhibits a mean reverting phenomenon as follows. Suppose that \( \varphi \) and \( \psi \) are different solutions, possibly with different initial values, say \( x_0, y_0 > 0 \), respectively. Then \( \varphi'(t) \leq \psi'(t) \) if \( \varphi(t) \geq \psi(t) \) and vice versa. Recall Lemma \( \text{(1.3)} \) Then we easily obtain the following facts:
\begin{equation}
(2.3)
\begin{align*}
(1) & \quad \text{If } \varphi(t_0) = \psi(t_0) \text{ for some } t_0 \text{ then it follows that } \varphi(t) = \psi(t) \text{ for } t > t_0. \\
(2) & \quad \text{If } x_0 \geq y_0, \text{ then } \varphi \geq \psi. \\
(3) & \quad |\varphi(t) - \psi(t)| \leq |x_0 - y_0| \text{ for all } t \in [0, 1]. \\
(4) & \quad \text{If a solution exists for one initial value } x_0 > 0, \text{ then it exists for all initial values } y_0 > x_0. \\
(5) & \quad \text{The solution corresponding to the initial condition } \varphi(0) = 0^+ \text{ exists and is unique if the solutions exist separately for all initial values } x_0 > 0.
\end{align*}
\end{equation}

The existence of the particular initial condition solution is seen by using the facts that \( \Upsilon \) is continuous with respect to \( (s, x) \), non-increasing with respect to \( s \), and then using the Lebesgue’s monotone convergence theorem on the weak formulation as follows:
\begin{align*}
\varphi_{x_0}(T) = x_0 + \int_0^T \Upsilon(\varphi_{x_0}(t), |f(t)|, t) \, dt \quad \leq \quad \int_0^T \Upsilon(\varphi(t), |f(t)|, t) \, dt = \varphi(T)
\end{align*}
where \( \varphi_{x_0}(t) \searrow \varphi(t) \) and \( \Upsilon(\varphi_{x_0}(t), |f(t)|, t) \nearrow \Upsilon(\varphi(t), |f(t)|, t) \) as \( x_0 \searrow 0 \).

We denote by \( L^T \) the set of all functions \( f \in L^0([0, 1]) \) such that \((2.1)\) and the listed assumption hold for \( \varphi_f \) with initial value \( 0^+ \). As customary, we will identify functions \( f \in L^T \) which vanish a.e. with the zero constant function.

**Theorem 2.1.** Assume that \( L^T \) is a set of functions such that conditions \((i)-(vi)\) hold. Then \( L^T \) is a Banach lattice when endowed with point-wise linear operations, point-wise order and the norm \( \|f\| = \varphi_f(1) \).

**Proof of Theorem 2.1.** Note that if \( f \in L^T \) is not zero a.e., then \( \varphi' \) is strictly positive in subset of positive measure for a given initial value, thus \( \varphi(1) - \varphi(0) > 0 \). The same holds for initial value \( 0^+ \), since \( \Upsilon \) is decreasing on \( \varphi \). It is clear by using
is a solution \( \psi \) by the proof of Lemma 1.2 we observe that any for any initial value \( x \), as
\( n,m \) double subsequence can be additionally chosen in such a way that it contains infinitely many members of both subsequences.)

Thus, given solutions \( \varphi_f \) and \( \varphi_g \) with \( \varphi_f(0) = \varphi_g(0) = x_o > 0 \) we have a majorizing function \( \varphi_f + \varphi_g \) for the differential equation
\[
\varphi_f+g(0) = x_0, \quad \varphi_{f+g}(t) = \Upsilon(\varphi_f(t), |f+g(t)|, t) \quad \text{for a.e. } t \in [0,1].
\]
By Lemma 1.2 we obtain a maximal solution \( \varphi_{f+g} \) which is necessarily majorized by \( \varphi_f + \varphi_g \). By letting \( x_0 \to 0^+ \) we observe the triangle inequality and that the solutions for \( f + g \) satisfy (iii).

By the construction of the norm it is clear that \( \| f \| = \| f \| \). Moreover, \( |f| \leq |g| \) implies \( \| f \| \leq \| g \| \), provided that both the functions are in the space. The above monotonicity of the norm is seen as follows: suppose that \( \varphi_f \) and \( \varphi_g \) are the solutions and \( \varphi_f(t) \geq \varphi_g(t) \) on \( t \in A \subset [0,1] \), then \( \Upsilon(\varphi_f(t), |f(t)|, t) \leq \Upsilon(\varphi_g(t), |g(t)|, t) \) for a.e. \( t \in A \), since \( \Upsilon \) is decreasing on the first coordinate and increasing on the second. Thus Lemma 1.3 applies and \( L^T \) has a lattice norm.

In fact, for every measurable \( A \subset [0,1] \) the mapping \( R: f \mapsto f - 1.4.2f \) defines an isometric reflection operator which induces a bicontractive projection.

Next we will verify the completeness of the norm. Let \( (f_n) \subset L^T \) be a Cauchy sequence. Let \( f(t) = \lim_{n \to \infty} f_n(t) \) if the limit exists and \( f(t) = 0 \) otherwise. In order to prove that \( \| f_n - f \| \to 0 \) as \( n \to \infty \) it suffices to prove that \( f \in L^T \) and for any subsequence \( n_k \) there is a further subsequence \( n_{k_j} \) such that \( \| f_{n_{k_j}} - f \| \to 0 \) as \( j \to \infty \).

Towards this, fix a subsequence \( n_k \). Let \( A_{n,m,\varepsilon} = \{ t \in [0,1] : |f_n(t) - f_m(t)| > \varepsilon \} \) for \( n, m \in \mathbb{N}, \varepsilon > 0 \). According to (v) we obtain that \( a_{i,j} := \sup_{j \geq i} m(A_{n, m, \varepsilon}) \to 0 \) as \( i, j \to \infty \), since
\[
\varepsilon \| 1_{A_{n,m,\varepsilon}} \| \leq \| f_n(t) - f_m(t) \| \to 0
\]
as \( n, m \to \infty \). Thus we may construct a subsequence \( n_{k_j} \) and \( \varepsilon_j \to 0 \) such that \( \sum_j a_{n_{k_j}, \varepsilon_j} < \infty \) and \( \sum_j \| f_{n_{k_j+1}} - f_{n_{k_j}} \| < \infty \).

(If \( n_k \) and \( m_k \) are subsequences involving a Cauchy sequence, then the above double subsequence can be additionally chosen in such a way that it contains infinitely many members of both subsequences.)

Put \( A_{n_{k_j}} = \bigcup_{j \geq j} A_{n_{k_j}, \varepsilon_j} \), and note that this is a decreasing sequence of sets whose measures tend to 0. Thus \( f(t) = \lim_{j \to \infty} f_{n_{k_j}}(t) \) exists a.e.

Note that \( \varphi_{f_{n_{k_j}}} \) is Cauchy on \( C([0,1]) \). Indeed, assumption \( \varphi_{f_{n_{k_j+1}}} - \varphi_{f_{n_{k_j}}} = \delta \) and the triangle inequality for any given \( t \) yield that \( \varphi_{f_{n_{k_j+1}}} - f_{n_{k_j}}(t) \geq \delta \), so that \( \varphi_{f_{n_{k_j+1}}} - f_{n_{k_j}}(1) \geq \delta \). Thus \( \varphi = \lim_{j \to \infty} \varphi_{f_{n_{k_j}}} \) exists.

Therefore the continuity of \( \Upsilon \) gives that \( \Upsilon(\varphi_{f_{n_{k_j}}}, |f_{n_{k_j}}(t)|, t) \to \Upsilon(\varphi(t), |f(t)|, t) \) for a.e. \( t \) as \( j \to \infty \). Fatou’s lemma gives that \( \int_0^1 \Upsilon(\varphi(t), |f(t)|, t) \, dt < \infty \) exists. Thus, by the proof of Lemma 1.2 we observe that any for any initial value \( x_0 > 0 \) there is a solution \( \psi_{x_0, f} \) with \( \psi'_{x_0, f}(t) = \Upsilon(\psi(t), |f(t)|, t) \leq \Upsilon(\varphi(t), |f(t)|, t) \). Indeed, the
above inequality holds for $\psi(t) \geq \varphi(t)$ and the opposite situation is not possible due to Lemma 1.3. Hence $f \in L^p$. 

Finally, we wish to verify that

$$
\|f - f_{n_k}\| = \left\| \text{a.e.} - \lim_{m \to \infty} \sum_{i=j}^{m} (f_{n_{k_i+1}} - f_{n_{k_i}}) \right\| \leq \sum_{i \geq j} \|f_{n_{k_i+1}} - f_{n_{k_i}}\| \to 0
$$
as $j \to \infty$.

Let $g_0 = |f_{n_{k_1}}|$ and $g_j = |f_{n_{k_{j+1}}} - f_{n_{k_j}}|$ for $j \in \mathbb{N}$. Put $h_{i} = \sum_{j=0}^{i} g_j$, $i \in \{1, \ldots, \infty\}$. By using (iv), the construction of the double subsequence and the same triangle inequality argument as above we obtain that $\varphi_{h_i} \not\to \varphi$ uniformly as $i \to \infty$.

By Lebesgue’s monotone convergence theorem we obtain that

$$
\Upsilon(\varphi(t), h_i, t) \to \Upsilon(\varphi(t), h_\infty, t)
in L^1$as $i \to \infty$.

Similarly as above, for any $x_0 > 0$ there is a solution $\varphi_{x_0, f-f_{n_{k_j}}}$ with

$$
\varphi'_{x_0, f-f_{n_{k_j}}} = \Upsilon(\varphi_{x_0, f-f_{n_{k_j}}}, |f - f_{n_{k_j}}|, t) \leq \Upsilon(\varphi(t), h_\infty, t),
$$
since $|f - f_{n_{k_j}}| \leq h_\infty$. We may select the initial value as small as we wish, same as $\varphi(1)$ by repeating the same arguments starting with the index $k_j$ with $j$ large, in place of $k_1$. This means that for each $\epsilon > 0$ there is $j_0$ such that the corresponding 0+-initial value solution $\varphi_{f-f_{n_{k_j}}}$, $j \geq j_0$, is bounded from above by $\epsilon$. Thus $\|f - f_{n_{k_j}}\| \to 0$ as $j \to \infty$.

Since $(f_n)$ was Cauchy, we see that $f$ does not depend on the selection of $(n_k)$. \qed

3. Basic properties of $L^{p(\cdot)}$ spaces

In this section we will study only spaces of the type $L^{p(\cdot)}$ with $p: [0, 1] \to [1, \infty)$ measurable. Some of the unbounded functions $p$ actually produce a class of functions, rather than a linear space. We will first restrict our considerations to those $L^{p(\cdot)}$ classes which satisfy the axioms of the previous section. The norms of these spaces were described in the introductory part and we will rely on the previous section for the fact that $L^{p(\cdot)}$ is a Banach lattice.

Let us (re)verify briefly that taking solutions is non-expansive operation with respect to initial values.

**Lemma 3.1.** The solutions $x_0 \to \varphi$ are 1-Lipschitz with respect to the initial value condition $\varphi(0) = x_0$.

**Proof.** We may apply the following mean-value principle: given $0 \leq t < t + \Delta \leq 1$ there is $b \geq 0$ such that

$$
\varphi(t + \Delta) = (\varphi(t))^{p(t)} + \Delta b^{p(t)} /^{1/p(t)}.
$$

The statement follows from the fact that

$$
\frac{\partial}{\partial t} \left( a^p + \Delta b^p \right)^{1/p} \not\to 1
$$
as $\Delta b^p \to 0^+$, $1 \leq p < \infty$. \qed
Proposition 3.2. Suppose that $f \in L^{p_1(\cdot)}$ and $g \in L^{p_2(\cdot)}$. Then they satisfy Hölder’s inequality:

$$\int_0^1 f(t)g(t) \, dt \leq \|f\|_{p_1(\cdot)}\|g\|_{p_2(\cdot)}, \quad p_1, p_2 : [0, 1] \to (1, \infty), \ p_1^*(\cdot) = p_2(\cdot).$$

Proof. We obtain from the usual Hölder’s inequality (for a suitable atomic measure $\mu$ with 2 atoms) that

$$\int_0^{t+\Delta} f(s)g(s) \, ds \leq (\varphi_f(t)^{p(t)} + \Delta|f(t)|^{p(t)})^{1/p(t)} (\varphi_g^*(t)^{p(t)} + \Delta|g(t)|^{p(t)})^{1/p(t)}$$

provided that

$$\int_0^t f(s)g(s) \, ds \leq \varphi_f(t) \varphi_g^*(t)$$

holds and $f$, $g$ and $p$ are constant on $[t, t + \Delta]$. Differentiating with respect to $\Delta$ on both sides and setting $\Delta = 0$ then yields $|f(t)g(t)| \leq (\varphi_f^* \varphi_g^*)^*$ for a.e. $t$.

The inequality (3.2) can be arranged by first choosing positive initial values.

3.1. Building blocks and estimates. We may define simple semi-norms as follows. First we define a very simple semi-norm by the formula

$$|f|^p_{\mu, \mu} = \int \|f\|^p \, d\mu$$

where $\mu$ is a Lebesgue measure with support restricted to a measurable subset of $[0, 1]$. If $p_i \in [1, \infty)$ and $\text{supp}(\mu_i) \subseteq \text{supp}(\mu_{i+1})$, $1 \leq i \leq n - 1$, then we may define a composite semi-norm as follows

$$\|f\|_{(\cdots \cap \mu_{p_1(\cdot)} \cap \mu_{p_2(\cdot)} \cap \cdots \cap \mu_{p_{n-1}(\cdot)} \cap \mu_{p_n(\cdot)})} = \left( \cdots \left( \left( \|f\|_{p_1(\cdot)} \cap \mu_{p_2(\cdot)} \cap \cdots \right. \cap \left. \mu_{p_{n-1}(\cdot)} \cap \mu_{p_n(\cdot)} \right) \right).$$

For example, if $p(\cdot) \equiv p_1$ on $[0, t_0]$ and $p(\cdot) \equiv p_2 = r_2$ on $[t_0, 1]$ and $f \in L^{p(\cdot)}$ then

$$\|f\|_{L^{p(\cdot)}} = \|f\|_{L^{p_1(\cdot)} \cap \mu_{p_2(\cdot)} \cap \mu_{L^p(\cdot)}}$$

where $\text{supp}(\mu_1) = [0, t_0]$ and $\text{supp}(\mu_2) = [t_0, 1]$ (see subsequent Lemma 3.3).

It is easy to see that then $\|1\|_{p(\cdot)} \to 2$ as $p_1 \to \infty$, $t_0 \to 0^+$ and $p_2 = 1$. This is perhaps a bit surprising, since always $\|1\|_p = 1$ in the constant $p$ case. We may also reverse the above example as follows, letting above $f \equiv 1/t_0$ on $[0, t_0]$ and $f \equiv 1$ on $[t_0, 1]$ with $p_1 = 1$ and $r_2 = p_2 \to \infty$ and $t_0 \to 0^+$ yields that $\|f\|_{p(\cdot)} \to 1$ whereas $\|f\|_{1} \to 2$.

We suspect that the above examples are typical in the sense that

$$\frac{1}{2} \|f\|_1 \leq \|f\|_{p(\cdot)} \leq 2 \|f\|_{1}$$

should always hold (so that constant 2 would be the best possible according to the above examples). In any case, the above inequalities hold with other constants in place of 2. Namely, suppose that $\varphi_f(t_0) = \|f\|_{1}$. Then

$$\varphi(t_0) = \|f\|_{1}, \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)}$$

for a.e. $t_0 \leq t \leq 1$.
yields
\[ \varphi'(t) \leq \varphi(t), \quad \text{for a.e. } t_0 \leq t \leq 1. \]

Observe that \( \varphi(1) \leq y(1) \) where \( y \) is the solution to \( y' = y \) with \( y(0) = \|f\|_\infty \), that is, \( y(x) = \|f\|_\infty e^x \).

Let \( a \in (1,2) \) be the solution to \( a^a = e \). This satisfies that \( \frac{ae}{x} \) is increasing on \( x \geq 1 \) for all \( b > a \). Suppose that \( 1_{p(x) \geq p} f = f \in L^p, 1 \leq p < \infty \), with \( \|f\|_p = 1 + a \).

We are mainly interested in the case where \( \varphi_{p(\cdot), f}(1) \leq 1 \). Let \( \Delta = [r,1] \) be a maximal interval such that \( \varphi_{p(\cdot), f}(t) \leq \varphi_{p, f}(t) \) on \( \Delta \). Let \( A_0 \subset [0,1] \) be the set where \( |f(t)| > a \). On \( \Delta \cap A_0 \cap \{ t : p(t) \geq p \} \) we have
\[
\frac{|f(t)|^{p(t)}}{p(t)} \varphi_{p(\cdot), f}(t)^{1-p(t)} \geq \frac{|f(t)|^p}{p} \varphi_{p, f}(t)^{1-p}.
\]
Thus
\[
\varphi_{p, f}(1) - \varphi_{p(\cdot), f}(1) \leq \|f\|_p - \|1_{[0,1] \setminus (\Delta \cap A_0 \cap \{ t : p(t) \geq p \})} f\|_p
\leq \|1_{\Delta \cap A_0 \cap \{ t : p(t) \geq p \}} f\|_p \leq \|a1_{[0,1]}\|_p = a.
\]
Thus \( \varphi_{p(\cdot), f}(1) \geq (1 + a) - a = 1 \). An estimate downwards and in varying exponent case on both sides are obtained in a similar fashion. We conclude the following.

**Proposition 3.3.** The following inequalities hold whenever defined:

1. \( \frac{1}{1 + a} \|1_{p(x) \geq p} f\|_p \leq \|1_{p(\cdot) \geq p} f\|_{p(\cdot)} \),
2. \( \frac{1}{1 + a} \|1_{p_1(\cdot) \geq p_2(x)} f\|_{p_2(\cdot)} \leq \|1_{p_1(\cdot) \geq p_2(\cdot)} f\|_{p_1(\cdot)} \),
3. \( \|f\|_{p(\cdot)} \leq e \|f\|_\infty \).

The following fact connects our varying exponent norm to the Luxemburg type norms.

**Proposition 3.4.** Let \( f \in L^{p(\cdot)} \). Then
\[
\|f\|_{p(\cdot)} \geq \inf \left\{ \lambda > 0 : \int \frac{1}{p(t)} \left( \frac{|f(t)|}{\lambda} \right)^{p(t)} \, dt \leq 1 \right\}.
\]

**Proof.** Suppose that \( 0 < \varphi_f(1) = \lambda \). Then
\[
\varphi'(t) \geq \frac{|f(t)|^{p(t)}}{p(t)} \lambda^{1-p(t)},
\]
(with strict inequality in a set of positive measure if \( f \neq 0 \)), so that
\[
\lambda = \varphi_f(1) \geq \int \frac{|f(t)|^{p(t)}}{p(t)} \lambda^{1-p(t)} \, dt = \int \lambda \frac{1}{p(t)} \left( \frac{|f(t)|}{\lambda} \right)^{p(t)} \, dt.
\]
This is equivalent to
\[
\int \frac{1}{p(t)} \left( \frac{|f(t)|}{\lambda} \right)^{p(t)} \, dt \leq 1.
\]
(The inequality is typically strict, both here and in the formulation of the result.)
We can use the above ideas to construct counter-examples as well. Let \( p : [0,1] \to [1,\infty) \) be a function defined by
\[
\frac{1}{p(t)} \left( \frac{2}{3} \right)^{1-p(t)} = \frac{1}{t - \frac{1}{2}}
\]
if \( t \in \left( \frac{1}{2},1 \right] \) and 1 otherwise. Then the constant function \( f \equiv 1 \) is not in \( L^p(\cdot) \). Assuming the contrary, clearly \( \varphi_f(\frac{1}{2}) \) would be \( \frac{1}{2} \). Suppose that \( t_0 > \frac{1}{2} \) is a point such that \( \varphi_f(t) \leq \frac{2}{3} \) for \( \frac{1}{2} \leq t \leq t_0 \). Then we should have
\[
\frac{2}{3} - \frac{1}{2} \geq \varphi_f(t_0) - \varphi_f\left( \frac{1}{2} \right) = \int_{\frac{1}{2}}^{t_0} \varphi'_f \, dt \geq \int_{\frac{1}{2}}^{t_0} \frac{1}{t - \frac{1}{2}} \, dt = \infty,
\]
contradicting the assumption that \( f \) was in the class, that is, having an absolutely continuous solution \( \varphi_f \). However, if we are allowed initial values \( x_0 \geq \frac{1}{2} \), then we have nice corresponding solutions.

Also note that \( 1_{[0,\frac{1}{4}]} + 1_{[0,1]} \in L^p(\cdot) \). This means that in general the \( L^p(\cdot) \) class need not be an ideal as a function class (cf. Banach lattice theory), i.e. \( g \in L^p(\cdot) \), \( f \in L^0 \), \( f \leq g \) does not imply \( f \in L^p(\cdot) \).

In the above example, we have \( 1_{[0,\frac{1}{4}]} + 1_{[0,1]} \), \( 1_{[0,\frac{1}{4}]} \in L^p(\cdot) \) and \( 1_{[0,\frac{1}{4}]} + 1_{[0,1]} - 1_{[0,\frac{1}{4}]} = 1_{[0,1]} \notin L^p(\cdot) \). This shows that for some \( p(\cdot) \) the class \( L^p(\cdot) \) is not a linear space.

3.2. Transcending from discrete to continuous state. Consider simple semi-norms of the type
\[
\| f \|_{N} = \| f \|_{\left( \cdots (L^{p_1}(\mu_1))_{\| \cdot \|_{r_1}} L^{p_2}(\mu_2))_{\| \cdot \|_{r_2}} \cdots \right)_{\| \cdot \|_{r_n}} L^{p_n}(\mu_n)}
\]
where \( \text{supp} (\mu_i) \leq \inf \text{supp}(\mu_{i+1}) \) and denote this collection by \( N \). We say that a semi-norm is of standard form if \( r_{i+1} = p_{i+1} \) for all \( i \in \{1,\ldots,n-1\} \). In this case we may, in a sense, extend the elements \( N \) of \( N \) to a \( L^{\tilde{p}(\cdot)} \) norm by putting \( \tilde{p}(t) = p_i \) for \( t \in \text{supp}(\mu_i) \) and \( \tilde{p}(t) = 1 \) otherwise. Note that this extension is unique. If \( \bigcup_i \text{supp} \mu_i = [0,1] \) then a corresponding standard form norm \( N \in N \) satisfies \( \| f \|_{N} = \| f \|_{\tilde{p}(\cdot)} \) by Lemma 3.3.

Observe that the semi-norms are decreasing on \( r \)'s and increasing on \( p \)'s. Consider point-wise intervals \([p_t, r_t] \) as follows: \( p_t = p_{t+1} \) and \( r_t = r_{t+1} \) on the support of \( \mu_{t+1} \). We denote by \( N \leq p(\cdot) \) whenever \( p(t) \in [p_t, r_t] \) for all \( t \) such that the interval is defined.

We may define a partial order on \( N \) by setting \( N \leq M \) if the partition each of the supports of the measures corresponding to \( N \) is a union of supports of the measures corresponding to \( M \) and \([p_{M,t}, r_{M,t}] \subset [p_{N,t}, r_{N,t}] \) for each \( t \) such that the left hand interval is defined. We note that \( N_{\leq p(\cdot)} = \{ N \in N : N \leq p(\cdot) \} \) is a directed poset, there are natural upper bound and lower bound operations yielding for each \( N, M \in N \) the bounds \( N \vee M \) and \( N \wedge M \). These are obtained by taking a coarsest common refinement (resp. finest common coarsening) of the supports of measures and then taking \( p_t = \max(p_{N,t}, p_{M,t}) \) and \( r_t = \min(r_{N,t}, r_{M,t}) \) (resp. \( p_t = \min(p_{N,t}, p_{M,t}) \) and \( r_t = \max(r_{N,t}, r_{M,t}) \)) applied on the respective domains.
We may define a functional as follows:
\[
\rho(f) := \limsup_{N \to \infty} \frac{\|f\|_N}{\|\cdot\|_{N,\leq p(t)}}.
\]
It turns out that in some cases the functions \( f \in L^0 \) with \( \rho(f) < \infty \) form a normed space.

**Lemma 3.5.** Let \( \rho(f) < \infty \). Suppose that there are compact subsets \( C_i \subset [0,1], \)
\( 1 \leq i \leq n, \max C_i \leq \min C_{i+1} \) such that \( p|_{C_i} \equiv p_i \in [1,\infty) \). Assume additionally that \( f = 1_{[0,1]}f \) and \( p|_{[0,1]\setminus \bigcup C_i} \equiv 1 \). Then the mapping \( \varphi: [0,1] \to \mathbb{R} \) given by \( \varphi(t) = \rho(1_{[0,t]}f) \) is absolutely continuous and satisfies
\[
\varphi(0) = 0, \quad \varphi'(t) = \frac{|f(t)|^p}{p(t)}(t)^{1-p(t)} \quad \text{for a.e. } t \in [0,1].
\]

**Proof.** First we observe the analogous claim on an interval with constant \( p \) by studying the following differential equation:
\[
\varphi(a) = c, \quad \varphi'(t) = \frac{|f(t)|^p}{p(t)}(t)^{1-p(t)} \quad \text{for a.e. } t \in [a,b] \subset [0,1].
\]

We use the separability of the differential equation to obtain
\[
\int_a^b p\varphi'(t)\varphi^{p-1}(t) \, dt = \int_a^b |f(t)|^p \, dt = \int_a^b p\varphi^{p-1} \, d\varphi = \left[ \varphi^p \right]_a^b = \int_a^b |f(t)|^p \, dt.
\]
Indeed, we see immediately that \( \varphi \) defined in the formulation of the Lemma is absolutely continuous in this special case. The above calculation considered in backward order shows also that in the constant \( p \) case \( \varphi \) arises as a solution to the above differential equation on that interval.

From this we obtain the analogous compact subset \( C \subset [0,1] \) case by passing to function of the type \( 1_{C}f \). It is clear that the resulting \( \varphi \) is absolutely continuous and the derivative is
\[
\varphi'(t) = 1_{C}(t)\frac{|f(t)|^p}{p(t)}(t)^{1-p(t)} = \frac{|1_{C}(t)f(t)|^p}{p(t)}(t)^{1-p(t)}.
\]
This way we easily see that the semi-norm accumulation functions
\[
t \mapsto \|1_{[0,t]}f\|_{L^{p^1}(\mu_1)\oplus \ldots \oplus L^{p^n}(\mu_n)}
\]
can be seen as solutions to
\[
\varphi(0) = 0, \quad \varphi'(t) = \frac{|f(t)|^p(t)}{p(t)}(t)^{1-p(t)} \quad \text{for a.e. } t \in [0,1]
\]
where \( p(t) = p_i \) for \( t \in C_i \) and \( \varphi'(t) = 0 \) for \( t \in [0,1]\setminus \bigcup C_i \).

Note that the sup in the lim sup is actually attained in this simple case with \( p(\cdot) \) essentially piecewise constant. \( \square \)

Given a measurable \( p: [0,1] \to [1,\infty) \), by Lusin’s theorem there is for each \( \varepsilon > 0 \) a compact set \( C \subset [0,1] \) with \( m([0,1]\setminus C) < \varepsilon \) such that \( p|_{C} \) is continuous, thus uniformly continuous and bounded. Thus we can find a sequence of compact subsets \( C_m \subset [0,1] \) such as above with \( m(C_m) \to 1 \). In the following lemma we will study a technical tool, namely a functional given by
\[
\|f\|_{L^p(C)} = \sup_C \rho(1_{C}f)
\]
where the supremum is taken over compact subsets $C$ such that $p|C$ is continuous.

It is easy to see that the above functional gives a complete norm in the space of functions $f \in L^p$ with $\|f\|_{L^p(C)} < \infty$. This space is denoted by $L^p(C)$.

**Lemma 3.6.** Let $f \in L^p(C)$. Then $\|f\|_{p(C)} = \|f\|_{\frac{1}{p(C)}}$.

**Proof.** Fix $p : [0, 1] \to [1, \infty)$ and $f \in L^p(C)$. First, suppose that $f \in L^\infty$. We will later select the supports of $\mu_i$'s inside these sets $C_m$. We can find an increasing sequence $N_n$ such that

$$\phi_{n,m}(t) := \|1_{[0,t]} \cap C_m f\|_{N_n} \leq \phi_m := \|1_{[0,t]} \cap C_m f\|_{L^p(C)} \quad m \in \mathbb{N},$$

$$\limsup_{n \to \infty} \phi_{n,m} = \phi_m, \quad m \in \mathbb{N}.$$

Indeed, here we apply the fact that $\phi_m$ have finite variation and the basic properties (e.g Markovian property) of the semi-norms $\|\cdot\|_N$. Moreover, the above sequence can be chosen in such a way that the diameters of the supports of the measures $\mu_i$ tend to 0 as $n \to \infty$. By a diagonal argument we may assume that the above lim sup is in fact lim.

By the uniform continuity of $p|C_m$ we observe that the $r_i$ and $p_i$ appearing in the definition of $N_n$ semi-norms satisfy $r_i - p_i \to 0$ uniformly on $i$ as $n \to \infty$.

Recall Lemma 3.5. Since $f|_{C_m}$ and $p|_{C_m}$ are bounded we observe that

$$\phi'_{n,m}(t) \geq \frac{|1_{C_m} f|_{p(t)}}{p(t)} \phi_m^{1-p(t)}(t),$$

uniformly on $C_m \cap \{t : \inf_{n,m} \phi_{n,m}(t) > \varepsilon\}$ as $n \to \infty$. The same effect can be accomplished by using a joint initial value $x_0 = \varepsilon > 0$. This means that $\phi'_m$ exists a.e. and satisfies

$$\phi'_m(t) = \frac{|1_{C_m} f|_{p(t)}}{p(t)} \phi_m^{1-p(t)}(t).$$

In letting $x_0 \to 0$ the monotone convergence theorem applies. Therefore $\phi_m$ is an admissible solution and $\|1_{[0,t]} \cap C_m f\|_{p(C)} = \phi_m$.

Let $\phi = \limsup_{m \to \infty} \phi_m$. By the preceding observation and collecting all the subsets $C_m$ we have that

$$\phi(T) \geq \int_0^T \frac{|f(t)|_{p(t)}}{p(t)} \phi_m^{1-p(t)}(t) \, dt.$$

Now, by the selection of the functions $\phi_{n,m}$ we have that

$$\phi_{n,m}(T) \leq \sup_m \int_0^T \frac{|1_{C_m} f(t)|_{p(t)}}{p(t)} \phi_m^{1-p(t)}(t) \, dt = \sup_m \|1_{[0,T]} \cap C_m f\|_{p(C)} = \|1_{[0,T]} f\|_{p(C)},$$

where the last equality follows from the absolute continuity of the solution $\varphi_f$. We conclude that $\phi = \varphi_f$.

Next we will treat the case where $f \in L^p(C)$ may be unbounded. By Lusin’s theorem, let $D_i \subset [0,1]$ be a sequence of compact subsets such that both $f$ and $p$ are uniformly continuous on $D_i$, $D_i \subset C_i$ and $m(D_i) \to 1$ as $i \to \infty$. Let $\psi_i$ be defined by

$$\psi_i(t) = \rho(1_{[0,t]} \cap D_i, f) = \|1_{[0,t]} \cap D_i f\|_{p(C)}$$

where we applied the conclusion obtained in the first part of the proof.
Let $\psi = \limsup_{i \to \infty} \psi_i$. We immediately observe that $\psi \leq \varphi_f$, since $\| \cdot \|_{p(t)}$ is a lattice norm. On the other hand,

$$\psi(T) = \limsup_{i \to \infty} \int_0^T \left| \frac{1}{D_i(t)} f(t) \right|^{p(t)} \psi_i^{1-p(t)} \, dt \geq \lim_{i \to \infty} \int_0^T \left| \frac{1}{D_i(t)} f(t) \right|^{p(t)} \varphi_f^{1-p(t)} \, dt$$

$$= \int_0^T \left| f(t) \right|^{p(t)} \varphi_f^{1-p(t)} \, dt = \varphi_f(T).$$

Thus $\psi = \varphi_f$.

Finally, note that

$$\|1_{[0,t]} \cap C_m \cap D_i f\|_{L_r^{p(t)}} \to \|1_{[0,t]} \cap C_m f\|_{L_r^{p(t)}}, \quad i \to \infty$$

since the analogous fact holds for all simple semi-norms $N \leq p(t)$ separately. Indeed, this is due to the fact that the accumulation functions corresponding to the simple semi-norms are absolutely continuous.

We conclude that $\| f \|_{p(t)} = \| f \|_{L_r^{p(t)}}$ for all $f \in L_r^{p(t)}$.

\[\square\]

Remark 3.7. According to the previous result the $\| \cdot \|_{L_r^{p(t)}}$ functional satisfies the triangle inequality.

Remark 3.8. By adapting the end of the proof of the previous result we can conclude the following: For each $f \in L_r^{p(t)}$ there is a sequence of simple functions $(f_n)$ such that $f - f_n \to 0$ a.e. and $|f_n| \nrightarrow |f|$ a.e. and $\|f_n\|_{L_r^{p(t)}} \nrightarrow \|f\|_{L_r^{p(t)}}$ as $n \to \infty$.

3.3. The essentially bounded exponent case. Let us take a look at the nice case where $\text{ess sup}_t p(t) < \infty$ as it turns out that the corresponding spaces have less pathological properties. In this subsection we impose throughout $\overline{p} = \text{ess sup}_t p(t) < \infty$.

We observed previously that $L_r^{p(t)}$ spaces need not have the ideal property in general. However, in this case assumptions $g \in L_r^{p(t)}$, $f \leq g$ imply that also $f \in L_r^{p(t)}$. This follows immediately from the following observation.

Remark 3.9. Suppose that $g \in L_r^{p(t)}$, $f \leq g$ and $x_0 > 0$ is a given initial value. Then a simple evaluation yields that

$$|\varphi_{f,g,x_0}'| \leq |\varphi_{g,g,x_0}'| \left( \frac{x_0}{\varphi_{g,g,x_0}(1)} \right)^{1-\overline{p}}.$$

In particular we remain within the class if we restrict supports to a measurable subset. Therefore it is possible in principle to formulate negative initial value solutions simply by setting the function $f \in L_r^{p(t)}$ to zero in a suitable initial segment.

Lemma 3.10. Let $f \in L_r^{p(t)}$ and $A_n \subset [0,1]$ a sequence of measurable subsets such that $m(A_n) \to 0$ as $n \to \infty$. Then $\|1_{A_n} f\|_{p(t)} \to 0$ as $n \to \infty$.

Proof. Fix $\varepsilon > 0$. We claim that given initial value $x_0 = \varepsilon$, there is $n_0 \in \mathbb{N}$ such that

$$\varphi_{1_{A_n},f,x_0}(1) < 2\varepsilon, \quad n \geq n_0. \quad (3.3)$$

This clearly suffices for the statement of the lemma.

The absolute continuity of the solution $\varphi_{f,x_0}$ implies that

$$\int_{A_n} \varphi_{f,x_0}'(t) \, dt \to 0, \quad n \to \infty.$$
Then this observation together with the previous Remark yields (3.3).

Then \(L^\infty \subset L^{p(\cdot)}\) is dense by the triangle inequality.

The following fact follows rather easily from the above observations.

**Proposition 3.11.** In our case \(\widehat{L^{p(\cdot)}} = L^{p(\cdot)}\).

Then for a general exponent \(p(\cdot)\) we have a linear space

\[ X = \bigcup_{n} \{1_{p(t) \leq n} f : f \in L^{p(\cdot)}\} \subset \widehat{L^{p(\cdot)}} \]

and by going back to the arguments in Section 2 it can be verified that \(X \subset L^{p(\cdot)}\) and is a complete space.

4. Duality

Given a function \(g: [0, 1] \rightarrow \mathbb{R}\) with finite variation, let us denote a special ‘variation norm’ as follows:

\[ \bigvee_{p(\cdot)\ast} m_g = \bigvee_{p(\cdot)\ast} g = \sup \left\{ \int_0^1 f \, dm_g : f \in C[0, 1], \|f\|_{p(\cdot)} \leq 1 \right\}. \]

Here \(m_g\) is the Lebesgue-Stieltjes measure induced by \(g\). For a continuously differentiable \(g\) the notable special cases are

\[ \bigvee_{(p=1)\ast} g = \text{Lip}(g), \]

the best Lipschitz constant of \(g\), and the usual total variation

\[ \bigvee_{(p=x)\ast} g = \bigvee g. \]

The above notion is applied somewhat tautologically in the following result.

**Theorem 4.1.** Let

\[ X = \overline{C[0, 1]} \subset L^{p(\cdot)}. \]

Then the dual space \(X^\ast\) elements are Lebesgue-Stieltjes measures \(m_g\) with finite \(\bigvee_{p(\cdot)\ast} m_g\) variation. The dual space is endowed with the norm

\[ \|m_g\|_{X^\ast} = \bigvee_{p(\cdot)\ast} m_g \]

and the duality is given by

\[ \langle F, x \rangle = \int_0^1 x(t) \, dm_g(t), \quad x \in X, \]

the Lebesgue integral with Lebesgue-Stieltjes measure \(m_g\) induced by \(g(t) = F(1_{[0,t]}\bigcap \mathbb{R})\) for \(F \in X^\ast\).

**Proof.** Let us begin by studying continuous linear functionals \(F\) on the normed space \(C[0, 1] \subset L^{p(\cdot)}\). Since \(\|p(\cdot)\|_{X^\ast} \leq \|f\|_{X}\) we obtain that each \(F \in (C[0, 1], \|\cdot\|_{p(\cdot)})^\ast\) is also bounded with respect to the norm \(\|\cdot\|_{X}\). Thus \(F \in (C[0, 1], \|\cdot\|_{p(\cdot)})^\ast \subset (C[0, 1], \|\cdot\|_{X})^\ast\) with the usual duality

\[ \langle F, f \rangle = \int f(t) \, dg(t), \quad g(t) = F(1_{[0,t]}\bigcap \mathbb{R}) \]
and
\[ \sqrt{g} \leq e^{\|F\|_{(C[0,1],\|\cdot\|_{L_p})^*}}. \]

To allow integrating non-continuous functions without any difficulty we will integrate in the more general Lebesgue-Stieltjes sense in taking duality. Thus, let \( m_g \) be the Lebesgue-Stieltjes measure induced by \( g \). We note that \( F \) is a continuous linear functional on \( (C[0,1],\|\cdot\|_{L_p}) \), the above duality holds, if and only if
\[ \langle F, f \rangle = \int f \, dm_g, \quad f \in C[0,1], \]
\( g(t) = F(1_{[0,t]}). \) Here \( \|F\|_{X^*} = \sqrt[p]{m_g} \) by the definition of the special variation.

Let us verify that the above integral representations extends continuously to the closure \( C[0,1] \subset L_p^r \) for each \( F \in X^* \). Fix \( x \in \overline{C[0,1]} \). Pick \( (x_n) \subset C[0,1] \) such that \( \|x_n - x\|_{L_p} \to 0 \) as \( n \to \infty \). Since \( (x_n) \) is Cauchy we can extract a subsequence \( (n_j) \) such that \( x_{n_j} + \sum_j x_{n_{j+1}} - x_{n_j} = x \) unconditionally in the \( L_p^r \)-norm and \( \sum_j \|x_{n_{j+1}} - x_{n_j}\|_p < \infty \). It follows from the definition of \( \sqrt[p]{m_g} \) that then
\[ \sum_j \left| \int (x_{n_{j+1}} - x_{n_j})(t) \, dm_g(t) \right| < \infty. \]

By passing to a further subsequence and modifying all the functions \( x_{n_j} \) and \( x \) in a \( m_g \)-null set we may assume that \( x_{n_j}(t) = \sum_j (x_{n_{j+1}} - x_{n_j})(t) = x(t) \) for every \( t \).

Consider the Banach space \( L^1(m_g) \). We obtain from \( (4.1) \) that \( x_{n_j} \to y \) in the norm \( \|\cdot\|_{L^1(m_g)} \). Also, we observe that \( y(t) = x(t) \) for \( m_g \)-a.e. \( t \) by convergence in \( m_g \)-measure considerations. We conclude that
\[ \int x_{n_j}(t) \, dm_g(t) \to \int x(t) \, dm_g(t), \quad j \to \infty. \]
It is easy to see that the above convergence does not depend on the particular selection of the approximating Cauchy sequence of continuous functions. \( \Box \)

Next we will return to the considerations involving the H"older’s inequality. Let us denote by \( J \) the ‘duality map’ \( J : L^p(r) \to L^0 \),
\[ J(x)[t] = \text{sign}(x(t))|x(t)|^{\frac{p(t)}{p^*(t)}}, \quad p, p^* \in L^0, \quad \frac{1}{p(t)} + \frac{1}{p^*(t)} = 1. \]

Next we will perform some calculations. We will use rules consistent with the definition of the solutions \( \varphi \), that is, in terms of the differential operator \( \partial^* \); when calculating a weak derivatives with respect to \( t \) we will treat \( f \) and \( p \) being constant on some interval \([t, t + \Delta]\), \( \Delta > 0 \).

Going back to \( (3.2) \), we are interested here in the equality which corresponds to the duality case. According to Young’s inequality and its proof it is known that the equality can only hold if both \( \varphi_x^{p(t)} = \varphi_x^{p^*(t)} \) and \( |x(t)|^{p(t)} = |x^*(t)|^{p^*(t)} \) hold for \( \text{a.e. } t \).

Let us calculate
\[ \partial^* \varphi_x^{p(t)} = p(t) \varphi_x^{p(t)} \partial_x \varphi_x^{p(t)-1} = p(t) \frac{|x(t)|^{p(t)}}{p(t)} \varphi_x^{1-p(t)} \varphi_x^{p(t)-1} = |x(t)|^{p(t)} \]
and similarly
\[ \partial^* \varphi_x^{p^*(t)} = |x^*(t)|^{p^*(t)} = (|x(t)|^{\frac{p(t)}{p^*(t)}})^{p^*(t)} = |x(t)|^{p(t)}. \]
Therefore we conclude that $\varphi_x^{p(t)} = \varphi_{x,\ast}^{p \ast(t)}$ a.e. if the solutions $\varphi_x$ and $\varphi_{x,\ast}$ specified in (1.4) exist. By the above calculation $|x(t)|^{p(t)} = |x^{\ast(t)}|^{p \ast(t)}$ a.e. in this case.

Using the above observations on $\varphi'_x$ and $\varphi'_{x,\ast}$, written as in the basic ODE, provides us with the following identity

\begin{equation}
\varphi'_{x,\ast}(t) = \varphi'_x(t) \frac{p(t) \varphi^{\ast}_{x}(t)}{p^{\ast}(t) \varphi_x(t)}.
\end{equation}

As a side remark, looking at the above identity shows how $p \equiv p^* \equiv 2$ (first with a joint positive initial value) results in a self-dual situation $\varphi_x = \varphi_{x,\ast}$.

Note that in the case where $1 < \text{ess inf}_t p(t) \leq \text{ess sup}_t p(t) < \infty$ and $\varphi$'s have a positive initial value we have that

$$\partial^* \ln \varphi_x(t) = \frac{\varphi'_{x,\ast}(t)}{\varphi_{x,\ast}(t)} \leq C \varphi'_x(t)$$

for a suitable constant $C > 0$. This inequality together with a suitable approximation of $x$ and $x^{\ast}$ from below by increasing supports, as in Lemma 3.6, gives that $J$ maps $L^p(\cdot) \to L^{p^*(\cdot)}$ for $p(\cdot)$ such as above.

Next we wish to justify that

$$\int_0^T x(t) x^*(t) \, dt = \varphi_x(T) x_{x,\ast}(T), \quad 0 \leq T \leq 1.$$ 

Recall that $\frac{p(t)}{p^*(t)} = p(t) - 1$. By using that $\phi_x^{p(t)} = \phi_{x,\ast}^{p(t)}$, we get $\phi_{x,\ast}(t) = \phi_{x,\ast}^{\frac{p(t)}{p^*(t)}}$ and thus

\begin{equation}
\varphi_x(t) x_{x,\ast}(t) = \phi_x^{1 + \frac{p(t)}{p^*(t)}}.
\end{equation}

We will differentiate the right hand side to obtain the claim as follows:

$$\partial^* \phi_x^{1 + \frac{p(t)}{p^*(t)}} = \left(1 + \frac{p(t)}{p^*(t)}\right) \varphi'_x(t) \phi_x^{\frac{p(t)}{p^*(t)}} = \left(\frac{p(t)}{p^*(t)}\right) |x(t)|^{p(t)} |x|^{p^*(t)-1} = |x(t)|^{p(t)} |x(t)||x^*(t)|$$

which justifies the claim. These observation yield the following fact.

**Lemma 4.2.** Suppose that $1 < \text{ess inf}_t p(t) \leq \text{ess sup}_t p(t) < \infty$. Then the duality mapping

$$J(x) = \text{sign}(x(t)) |x(t)|^{\frac{p(t)}{p^*(t)}}$$

satisfies

$$\langle J(x), x \rangle = \|x\|_{p(\cdot)} \|J(x)\|_{p^*(\cdot)}.$$ 

Let $p: [0, 1] \to (1, \infty)$ be a measurable function. Let $X_n \subset L^{p^n}$, $n \in \mathbb{N}$, be the images of the contractive projections $P_n: L^{p(\cdot)} \to X_n$, $(P_n f)(t) = 1_{1+1/n \leq p(t) \leq n} f(t)$. Recall that we denote by

$$L^{p(\cdot)}_0 = \bigcup_n X_n \subset L^{p(\cdot)}.$$ 

Actually,

$$L^{p(\cdot)}_0 = \bigcup_n \{1_{p(t) \leq n} f : f \in L^{p(\cdot)}\} := X.$$ 

This is seen as follows: we claim that for each $f \in X$ we have

$$\|f - P_n f\| + \|P_n f\| \to \|f\|, \quad n \to \infty.$$
For fixed initial value \( a > 0 \) and \( \varphi_{f-P_n f}(0) = \varphi_{P_n f}(0) = \varphi_f(0) = a \) the analogous statement follows easily since \( \varphi^{1-p(t)} \to 1 \) as \( p(t) \searrow 1 \). By the absolute continuity of \( \varphi \) we obtain \( \varphi_{P_n f}(1) \to \varphi_f(1) \) as \( n \to \infty \) for any given initial value \( a > 0 \). Thus \( \varphi_{f-P_n f}(1) \to a \) as \( n \to \infty \) for any initial value \( a > 0 \). By a diagonal argument we find a sequence \( a_n \searrow 0 \) of initial values such that \( \varphi_{f-P_n f,a_n}(1) \to 0 \) as \( n \to \infty \). Since the solutions are non-decreasing with respect to their initial values we obtain \( \|f-P_n f\| \to 0 \) as \( n \to \infty \).

\[ \square \]

**Theorem 4.3.** If \( 1 < \text{ess} \inf_t p(t) \leq \text{ess} \sup_t p(t) < \infty \) then for each \( F \in (L^{p(\cdot)})^* \) there is \( f \in L^{p(\cdot)} \) such that

\[ \langle F, x \rangle = \int x(t) f(t) \, dm(t), \quad \text{for all } x \in L^{p(\cdot)} \]

and the above duality induces an isometric isomorphism \( (L^{p(\cdot)})^* \to L^{p(\cdot)} \). Moreover, \( (L_0^{p(\cdot)})^* \) is isometric to \( L^{p(\cdot)} \) with the above duality for a general \( p : [0, 1] \to (1, \infty) \).

**Proof.** It follows from an easy adaptation of Hölder’s inequality that \( L^{p(\cdot)} \subset (L^{p(\cdot)})^* \) in the sense that

\[ |F(x)| \leq \int \|x\|_{p(\cdot)} \|f\|_{p^*(\cdot)}, \quad x \in L^{p(\cdot)} \]

whenever \( f \in L^{p^*(\cdot)} \) is regarded as a function and \( F \) is in the subspace with the usual identification (4.4).

Let us begin by verifying the statement in the reflexive case, i.e. \( \text{ess} \inf_t p(t) > 1 \) and \( \text{ess} \sup_t p(t) < \infty \) (see Theorem 5.2), so that we are actually studying a space \( X_n \) for a given \( n \). Let \( F \in (L^{p(\cdot)})^* \). By modifying the standard proof (see e.g. [3, Prop. 2.17]) of the statement in the usual constant exponent case we obtain that there is \( f \) such that

\[ (4.4) \quad \langle F, x \rangle = \int x(t) f(t) \, dm(t) \]

holds for every measurable bounded \( x \).

Note that \( \|F\|_{X^*} \leq \|f\|_{p^*(\cdot)} \). By applying Lebesgue’s monotone convergence theorem we observe that we may approximate

\[ \int x_n f \, dm \to \int x f \, dm \]

by simple functions \( x_n \) and that (4.4) holds for all \( x \in L^{p(\cdot)} \).

We claim that \( \|F\|_{(L^{p(\cdot)})^*} = \|f\|_{p^*(\cdot)} \). Indeed, by Theorem 5.2 we know that \( L^{p(\cdot)} \) is reflexive, strictly convex and smooth, thus we may apply Lemma 1.2 to get the claim. Namely, if \( \|F\|_{(L^{p(\cdot)})^*} = 1 \) then there is by reflexivity a point \( x \in L^{p(\cdot)} \), \( \|x\|_{p(\cdot)} = 1 \), with \( F(x) = 1 \) and by strict convexity this point is unique. Then we apply \( J \) to \( x \) to get a functional which is norm-attaining at \( x \). By smoothness and the Smulyan lemma \( J(x) \) is the unique functional attaining its norm at \( x \), so that

\[ J(x) = F. \]

Consequently, \( \|x\|_{p(\cdot)} = \|J(x)\|_{p^*(\cdot)} = \|F\|_{(L^{p(\cdot)})^*} \). This way we also see that \( (L^{p(\cdot)})^* = L^{p^*(\cdot)} \).

Next we treat the non-reflexive case. As pointed above, it follows from Hölder’s inequality that \( L^{p^*(\cdot)} \subset (L_0^{p(\cdot)})^* \) and

\[ \|F\|_{(L^{p(\cdot)})^*} \leq \|f\|_{L^{p^*(\cdot)}} \]

Pick \( f \in L^{p^*(\cdot)} \).
Restrict the corresponding $F$ to the subspace $\bigcup_n X_n$. This does not change the operator norm, since the subspace is dense. It is easy to see that $\|P_n f\|_{L^p(\cdot)} \to \|f\|_{\widetilde{L}^p(\cdot)}$ as $n \to \infty$. Hence we may pick for each $\varepsilon > 0$ such $n$ and $x \in X_n$, $\|x\|_{L^{p(\cdot)}} = 1$, that $|(P_n^* f)(x)| > 1 - \varepsilon$. Thus we observe that $\widetilde{L}^p(\cdot) \subset (L^p_0(\cdot))^*$ is an isometric subspace. Pick $F \in (L^p_0(\cdot))^*$. Restrict $F$ to $\bigcup_n X_n$. Since the projections $P_n$ commute, this produces a natural candidate for the representation, namely $f = \lim_n P_n^* F$, the limit taken point-wise a.e. Since each $P_n F \in L^p(\cdot)$ and $\|P_n F\|_{L^p(\cdot)} \leq \|F\|_{(L^p_0(\cdot))^*}$ we obtain that $f \in \widetilde{L}^p(\cdot)$, although the above limit does not, a priori, exist in the $\widetilde{L}^p(\cdot)$ norm. Let us verify that $f$ presents $F$. Pick $x \in L^p_0(\cdot)$. Then

$$F(x) = \int x(t) f(t) \, dm(t) = F(x-P_n x) + F(P_n x) - \left( \int (x-P_n x) \, f \, dm + \int P_n x \, f \, dm \right).$$

Here $F(x-P_n x) \to 0$ by the continuity of the functional and $\int (x-P_n x) \, f \, dm \to 0$ by Hölder’s inequality. On the other hand

$$F(P_n x) = (P_n^* f)(x) = \int 1_{1+1/n \leq p(t) \leq n} \, f(t) \, x(t) \, dm(t) = \int (P_n x)(t) \, f(t) \, dm(t).$$

The previous result gives an anticipated duality for $L^p(\cdot)$ spaces and in its preparations it turned out that the solutions satisfy $\varphi_x^p(t) = \varphi_x^p(t)$. This can now be given an interpretation in terms of duality. Namely, we have the following commuting diagram:

$$x \xrightarrow{J} x^* \quad \varphi \quad \varphi \quad \varphi_x \xrightarrow{J} \varphi_x^*$$

Interestingly, here not only the functions $x$ and $x^* = J(x)$ are dual in the sense of the usual pairing but also the solutions are dual to each other:

$$\langle \varphi_x^*, \varphi_x \rangle = \|\varphi_x\|_{L^p(\cdot)} \|\varphi_x^*\|_{L^p(\cdot)}.$$

5. Extensions of the definition of the spaces

5.1. Change of variable. Let us consider an equivalent measure $\mu \sim m$ on the unit interval and $\frac{d\mu}{dm}$ with

$$\mu(A) = \int_A \frac{d\mu}{dm} \, dm(t)$$

for all Borel sets $A$. The above Radon-Nikodym derivative need not be integrable. Going back to the heuristical derivation of the norm-determining ODE and repeating the considerations with $L^p(\mu)$ in place of $L^p$ under the assumption that $\frac{d\mu}{dm}(t)$ is a continuous function, we arrive at the following ODE:

$$\varphi(0) = 0^+, \quad \varphi'(t) = \frac{d\mu}{dm}(t) \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)}$$

for $m$-a.e. $t \in [0,1]$.

Similarly as above we can define a space of functions together with a norm and we denote this space by $L^p(\cdot)(\mu)$. This can be regarded as a ‘weighted’ $L^p(\cdot)$ space...
and provide examples in the wider class of spaces introduced in Section [2]. Recall that $L^p([0,1])$ and $L^p(\mathbb{R})$ are isometric; the same reasoning extends to our setting.

**Proposition 5.1.** Let $p: [0,1] \to [1, \infty)$ be measurable. Let $\mu$ be as above. Then the mapping

$$T: f(t) \mapsto \left(\frac{d\mu}{dm}\right)^{-\frac{1}{p(t)}} f(t)$$

is a surjective linear isometry $L^{p(\cdot)} \to L^{p(\cdot)}(\mu)$.

**Proof.** Clearly the mapping is linear. Isometry follows by calculation:

$$\varphi_{\mu,T(f)}'(t) = \frac{d\mu}{dm}(t) \left| \left(\frac{d\mu}{dm}\right)^{-\frac{1}{p(t)}} f(t) \right|^{p(t)} \varphi_{\mu,T(f)}(t)^{1-p(t)} = \frac{|f(t)|^{p(t)}}{p(t)} \varphi_{\mu,T(f)}(t)^{1-p(t)} = \varphi_{m,f}'(t).$$

Indeed, a moments reflection involving a joint positive initial value justifies the fact $\varphi_{\mu,T(f)} = \varphi_{m,f}$. Surjectivity follows by observing that

$$f \mapsto \left(\frac{d\mu}{dm}\right)^{\frac{1}{p(t)}} f(t)$$

defines the inverse of the operator. \hfill \Box

**Theorem 5.2.** Let $p: [0,1] \to (1, \infty)$ be measurable. The following conditions are equivalent:

1. $L^{p(\cdot)}$ is uniformly convex and uniformly smooth.
2. $L^{p(\cdot)}$ is reflexive.
3. $L^{p(\cdot)}_0$ contains neither $\ell^1$, nor $c_0$ almost isometrically.
4. $\text{ess inf}_t p(t) > 1$ and $\text{ess sup}_t p(t) < \infty$.

**Proof.** The implications (1) $\implies$ (2) $\implies$ (3) are clear. The direction (3) $\implies$ (4). Suppose that $\text{ess inf}_t p(t) = 1$. We will show that then $L^{p(\cdot)}_0$ contains an isomorphic copy of $\ell^1$ for any isomorphism constant $C > 1$.

By the compactness of the unit interval we can find a point $t_0$ such that

$$\text{ess inf}_t 1_{(t_0-\varepsilon,t_0+\varepsilon)}(t)p(t) = 1 \quad \text{for each } \varepsilon > 0.$$

Indeed, assume that this is not the case and consider a suitable open cover of open intervals $(t_0-\varepsilon, t_0+\varepsilon)$, so that there is a finite subcover contradicting $\text{ess inf}_t p(t) = 1$. Therefore we may extract a sequence $(A_n)$ of measurable subsets of the unit interval with positive measure such that the following conditions hold:

1. $\sup p|_{A_n} \not\to 1$ as $n \to \infty$.
2. Either $\max A_n < \min A_{n+1}$ for all $n$ or $\max A_n > \min A_{n+1}$ for all $n$.

Fix a rapidly decreasing sequence of exponents $p_i \not\to 1$ such that

$$\prod_i |I: \ell^p(2) \to \ell^1(2)| < 1 + \varepsilon.$$

We can find a strictly increasing sequence $(n_i)$ such that $p_i \geq p|_{A_{n_i}}$ for each $i \in \mathbb{N}$. 

Let $\mu$ be an equivalent measure on the unit interval such that $\mu(A_n) = 1$ for $i \in \mathbb{N}$. In proving the claim it suffices study $L^{p(\cdot)}(\mu)$ in place of $L^{p(\cdot)}$, since these spaces are isometric. Put $\tilde{p}(t) = \max(1, \sum_i p_i 1_{A_{n,i}}(t))$.

Define a mapping $T : \ell^1 \to L^{p(\cdot)}(\mu)$ by putting

$$T((x_i)) = \sum_i x_i 1_{A_{n,i}}$$

where the sum is defined point-wise a.e.

We follow the arguments in [10] involving sequence space semi-norms arising as follows. For $(x_n) \in \ell^0$ we put

$$(\ldots (|x_1| \|p_1\| |x_2|) \|p_2\| |x_3|) \|p_3\| \ldots \|p_{n-1}\| |x_n|) \|p_n\| |x_{n+1}|,$$

in case $(A_n)$ is increasing, or the analogous left-handed version if $(A_n)$ is decreasing:

$$|x_1| \|p_1\| (|x_2| \|p_2\| (|x_3| \|p_3\| \ldots \|p_{n-2}\| (|x_{n-1}| \|p_{n-1}\| (|x_n| \|p_n\| |x_{n+1}|) \ldots),$$

we observe that one may control inductively the difference of norms when one changes the values of $p_i$'s by using (5.1) That is,

$$\left\| \sum_i x_i 1_{A_{n,i}} \right\|_{L^{p(\cdot)}(\mu)} \geq \frac{1}{1 + \varepsilon} \sum_i \|x_i 1_{A_{n,i}}\|_{L^{p(\cdot)}(\mu)}.$$ 

Thus, $\|T^{-1} : T(\ell^1) \to \ell^1\| \leq 1 + \varepsilon$.

Similarly, by passing to subsequences of $(A_n)$ multiple times we obtain that

$$\sum_i \|x_i 1_{A_{n,i}}\|_{L^1(\mu)} = \left\| \sum_i x_i 1_{A_{n,i}} \right\|_{L^1(\mu)} \leq (1 + \varepsilon) \left\| \sum_i x_i 1_{A_{n,i}} \right\|_{L^{p(\cdot)}(\mu)} \leq (1 + 2\varepsilon) \left\| x_n \right\|_{L^p} \leq (1 + 3\varepsilon) \left\| x_n \right\|_{L^p} \leq (1 + 4\varepsilon) \left\| x_n \right\|_{L^1} = (1 + 4\varepsilon) \sum_i \|x_i 1_{A_{n,i}}\|_{L^1(\mu)}.$$ 

Indeed, analyzing the $L^{p(\cdot)}$-differential equation shows that for a constant function the values of the derivative uniformly approximate $|f(t)|$ as $p(t) \searrow 1$. Thus $\|T\| \leq 1 + \varepsilon$. This shows that the space contains $\ell^1$ almost isometrically.

Next, assume that $\text{ess sup}_t p(t) = \infty$. We will show that $L^{p(\cdot)}_0$ contains $c_0$ almost isometrically. We may again without loss of generality make some assumptions about the equivalent measure, namely, that $\mu([0,1]) = 1$ and

$$\mu(\{t \in [0,1]: p(t) > r\})^1_r \rightarrow 1, \quad r \rightarrow \infty.$$ 

We will partition each set $\{t \in [0,1]: n < p(t) \leq n + 1\}$ to measurable subsets of equal $\mu$-measure, call them $A_{n,0}^{(1)}$ and $A_{n,1}^{(1)}$. (Possibly both the subsets have measure 0.) Divide $A_{n,1}^{(1)}$ again to two subsets of equal measure, $A_{n,0}^{(2)}$ and $A_{n,1}^{(2)}$. We proceed recursively in this manner to construct sets $A_{n,\theta}^{(k)}$, $k,n \in \mathbb{N}$, $\theta \in \{0,1\}$.

Let $A_{j}^{(k)} = \bigcup_{n \geq j} A_{n,\theta}^{(k)}$. Observe that

$$\mu(A_{j}^{(k)}) = 2^{-k} \mu(\{t \in [0,1]: p(t) > j\}), \quad k,j \in \mathbb{N}.$$
Note that
\[
(5.2) \quad \lim_{j \to \infty} \mu(A_j^{(k)})^\frac{1}{p} = \lim_{j \to \infty} (2^{-k})^\frac{1}{p} \mu(\{t \in [0, 1]: p(t) > j\})^\frac{1}{p} = 1, \quad k \in \mathbb{N}.
\]
Assume first that \(1_{A_j^{(n)}} \in L^{p(\cdot)}(\mu)\), although this is not necessarily the case (see remarks in Section 3). Define an operator \(T: c_0 \to L^{p(\cdot)}(\mu)\) by
\[
T((x_n)) = \sum_n x_n 1_{A_j^{(n)}}
\]
defined point-wise a.e. Clearly \(\|T\| \leq \|1\|_{L^{p(\cdot)}(\mu)}\). In fact, by choosing \(j\) large enough we get that \(\|T\| \leq 1 + \varepsilon\). Indeed, observe that if \(\varphi(t) \geq 1\) then \(\frac{1}{j} \varphi^{1-j}(t)\) becomes small for \(j\) large. Thus
\[
(1 + \varepsilon) \max_n |x_n| \geq \|T((x_n))\|_{L^{p(\cdot)}(\mu)} \geq \max_n \|T(x_ne_n)\|_{L^{p(\cdot)}(\mu)}.
\]
Here \((T(e_n))_n \subset L^{p(\cdot)}(\mu)\) is a 1-unconditional sequence. To show the claim it is required to check that
\[
\|T(e_n)\|_{L^{p(\cdot)}(\mu)} \geq 1 - \varepsilon, \quad n \in \mathbb{N}.
\]
This is seen as follows, first observe that
\[
\|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)} \geq \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)}.
\]
Then observe that for each \(\varepsilon > 0\) there is \(j \in \mathbb{N}\) such that
\[
\frac{1}{p(\cdot)} \varphi(t) \geq \frac{1}{j} \varphi(t) + \varepsilon, \quad p(\cdot) \geq j, \quad \varphi(1) + \varepsilon \leq 1.
\]
This reads
\[
(5.3) \quad \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)} \geq \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)} - \varepsilon
\]
and further
\[
(5.4) \quad \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)} \geq \limsup_{j \to \infty} \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)}.
\]
Recall that
\[
(5.5) \quad \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)} = (2^{-n})^\frac{1}{p} \mu(\{t \in [0, 1]: p(t) > j\})^\frac{1}{p} \to 1, \quad j \to \infty.
\]
We made an additional assumption during the course of the proof that \(1_{A_j^{(n)}}\) is included in the space. This assumption can be removed by observing that we may restrict the support of these functions to suitable sets \(\{t: p(t) \leq p(n)\}\), so that the positive initial value solutions become Lipschitz with a large constant and such that simultaneously (5.3) and (5.5) hold up to an extra \(\varepsilon\). Thus \(L^{p(\cdot)}_0\) contains \(c_0\) almost isometrically.

The direction (4) \(\implies\) (1). Here we will follow the analogous argument in the setting of \(L^{p(\cdot)}\) spaces. We will require the notions of upper \(p\)-estimate and lower \(q\)-estimate of Banach lattices. If \(X\) is a Banach lattice and \(1 \leq p \leq q < \infty\) then the upper \(p\)-estimate and the lower \(q\)-estimate, respectively, are defined as follows:
\[
\|\sum_{1 \leq i \leq n} x_i\| \leq \oplus_{1 \leq i \leq n}^p \|x_i\|,
\]
\[
\|\sum_{1 \leq i \leq n} x_i\| \geq \oplus_{1 \leq i \leq n}^q \|x_i\|,
\]
respectively, for any vectors $x_1, \ldots, x_n \in X$ with pairwise disjoint supports. These estimates involve multiplicative coefficients which are taken to be $1$ in this treatment. We will apply the fact that a Banach lattice, which satisfies an upper $p$-estimate and a lower $q$-estimate for some $1 < p < q < \infty$ with constants $1$ is both uniformly convex and uniformly smooth (with the respective power types), see [1] 1.f.1, 1.f.7.

Let $1 < p = \text{ess inf}_t p(t)$ and $\text{ess sup}_t p(t) = q < \infty$. We claim that $L^p(\cdot)$ satisfies the respective estimates for these $p$ and $q$. To check the upper $p$-estimate, let $f_k$, $1 \leq k \leq n$, be disjointly supported functions in $L^p(\cdot)$.

By a simple argument using the definition of outer measure we see that each simple semi-norm can be approximated point-wise from below with another semi-norms of the type $\| \cdot \|_{(\ldots (\| L^{p_1} (\mu_1) \|_{L^{p_2} (\mu_2)} \| \ldots \|_{L^{p_m} (\mu_m)} ) )}$, ess inf$_t p(t) \leq r_i \leq$ ess sup$_t p(t)$, such that only one of $f_k$s is supported on the support of a given $\mu_i$. We may interpret the values of the semi-norms as norms of finite $L^p(\cdot)$ sequences

$$f \mapsto (\| f \|_{L^{p_1} (\mu_1)}, \| f \|_{L^{p_2} (\mu_2)}, \ldots, \| f \|_{L^{p_m} (\mu_m)})$$

and then the supports of the sequences are disjoint for disjointly supported $f_k$s. We apply the fact proved in [10] which states that for disjointly supported $L^p(\cdot)$ sequences we have the upper $p$-estimate for $p = \text{inf}_t p$. From these considerations it follows that also disjointly supported $L^p(\cdot)$ functions satisfy the upper $p$-estimate for $p = \text{ess inf}_t p(t)$.

The argument for lower $q$-estimates is analogous. This concludes the proof. □

Next our aim is to build a kind of universal $L^p(\cdot)$ space. We will study a modification of Topologist’s Sine Curve as follows:

$$p_0(t) = \frac{1}{1-t} \sin\left(\frac{1}{1-t}\right) + \frac{1}{1-t} + 1, \quad 0 < t < 1.$$  

**Theorem 5.3.** Let $p_0$ be as above. We consider a Borel measure $\mu$ on $[0,1]$ given by $\frac{d\nu}{dm}(t) = |p'_0(t)|$ for $m$-a.e. $t$. Let $p: [0,1] \rightarrow [1, \infty)$ be a $C^1$ function, not constant on any proper interval and such that $p'$ changes its sign finitely many times on each interval $[0, a] \subset [0,1)$. Then there is an isometric linear embedding $L^p(\cdot) \rightarrow L^{p_0(\cdot)}(\mu)$ onto a projection band.

We note that $p_0$ satisfies the assumptions made about $p$ and that the construction can be easily modified to accommodate the case where $p$ is prescribed to be constant on some intervals.

**Proof.** By the assumptions we find a sequence of open subintervals $\Delta_n \subset [0,1]$, $n \in \mathbb{N}$, with sup $\Delta_n = \inf \Delta_{n+1}$ such that the sign of $p'$ does not properly change on the intervals $\Delta_n$. Moreover, we may assume that $|p'(x)| > 0$ for $x \in \bigcup_n \Delta_n$. We may choose this collection to be almost a cover in the sense that $m ([0,1] \setminus \bigcup_n \Delta_n) = 0$.

Now, $p$ is monotone on each $\Delta_n$. By the construction of $p_0$ we can find a sequence of open intervals $\Delta'_n \subset [0,1]$, $n \in \mathbb{N}$, with sup $\Delta'_n \leq \inf \Delta'_{n+1}$ such that there is a $C^1$ diffeomorphism $T_n : \Delta_n \rightarrow \Delta'_n$ with $p|\Delta_n = p_0 \circ T_n$.

By taking the union of the graphs of $T_n$, i.e. by ‘gluing together’ these mappings, we define a mapping $T$ defined a.e. on $[0,1]$, which has the property that $p(x) = p_0(T(x))$ for a.e. $x \in [0,1]$.

Let us define a measure $\nu$ on $[0,1]$ by $\frac{d\nu}{dm}(x) = |p'(x)|$. Both $\nu$ and $\mu$ can be thought as variation measures corresponding to $p$ and $p_0$, respectively. Thus it is
easy to see that $T$ is a $\nu$-$\mu$-measure-preserving mapping. Thus we get
$$\|1_{T([0,1])}f\|_{L^p(\mu)} = \|f \circ T\|_{L^p(\mu)} = \|g\|_{L^p(\nu)}$$
for $f \in L^p(\mu)$ such that $f \circ T = g \in L^p(\nu)$. Indeed, by using the absolute continuity of the $\varphi$’s we observe that values of $f$ outside $T([0,1])$ do not influence the norm.

By making suitable identifications we may consider $L^p(\nu)$ as a subspace of $L^p(\mu)$ and disregarding the measure outside $T([0,1])$, we may completely identify these spaces. This way we may apply Proposition 5.1 to observe that $G: L^p(\nu) \to L^p(\mu)$ given by
$$G(f)(t) = \left(\frac{d\nu}{dt}\right)^{-1} \hat{\nu}\left((f \circ T^{-1})(t) = \left(|p'(t)|\right)^{\frac{1}{p(n)}} (f \circ T^{-1})(t) \quad \text{if } t \in T([0,1]),
$$
and $G(f)(t) = 0$ otherwise, defines the required isometry.

5.2. Extension to several dimensions. The definition of the $L^p(\nu)$ spaces appears to be fundamentally 1-dimensional, being essentially on ODE, and a priori it is not clear if it extends naturally to $n$-dimensional setting, i.e. spaces of the type $L^p(\mathbb{R}^n)$.

By a similar heuristical motivation as we have seen in the ‘derivation’ of the differential equation with $\varphi'$, one finds the following way of defining norms in domains $\Omega \subset \mathbb{R}^n$, instead of the unit interval. Suppose that $p: \Omega \to [0, \infty)$ and $f: \Omega \to \mathbb{R}$ are measurable. Our aim is to define
$$\varphi(x) = \|1_{y \leq x}(y)f(y)\|_{L^p(\Omega)}, \quad x = (x_1, \ldots, x_n) \in \Omega$$
where $y \leq x$ is the partial order of coordinate-wise dominance. Here $\varphi$ is going to be absolutely continuous on every line (ACL) parallel to coordinate axes.

Let us study the equality
$$\hat{\varphi}(x) = \int_{y \leq x, y_i = x_i}^\infty \frac{|f(y)|^{p(y)}}{p(y)} \varphi^{1-p(y)} dm_{n-1}, \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (5.6)$$
The convenient feature about (5.6) is that if $\Omega \subset \{y \in \mathbb{R}^n: z_0 \leq y\}$ for some $z_0 \in \mathbb{R}^n$ then (5.6) defines a function $\varphi$ similarly as in the 1-dimensional case for analogous limit process on the initial condition $\varphi(z_0) = 0^+$ and suitable $f$. (The key idea in analyzing the accumulation of the values of $\varphi$ is that in adding small new blocks
$$\{y: y_j = x_j, j \neq i, y_i \in [x_i, x_i + \Delta]\}$$
they ‘interact’ with the big block $\{y \in \mathbb{R}^n: y \leq x\}$ in an essential way but the mutual interaction of the small blocks vanishes rapidly as $\Delta$ tends to 0.)

Let us study the properties of (5.6). The integral can be seen as a $n - 1$-fold integral and by applying the fundamental theorem of calculus $n - 1$-times (at each step disregarding an $m_1$-null set of coordinates), we obtain that
$$\hat{\varphi}(x) = \int_{y \leq x, y_i = x_i}^\infty \frac{|f(y)|^{p(y)}}{p(y)} \varphi^{1-p(y)} dm_{n-1}, \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (5.7)$$
The $n - 1$-fold integral can be written in any order by Fubini’s theorem, and, consequently, so can the derivatives $\hat{\varphi}_j, j \neq i$. Thus, by looking at the right hand side we observe immediately that the order of taking the derivatives on the left hand side does not matter a.e.
Therefore we have

\[
\hat{c}_1 \ldots \hat{c}_n \varphi(x) = \frac{|f(x)|^{p(x)}}{p(x)} \varphi^{1-p(x)}, \quad \text{for a.e. } x \in \mathbb{R}^n.
\]

By going backwards by integration we observe that the above PDE in fact characterizes an ACL \( \varphi \) described by (5.6).

This suggests defining a norm corresponding to any given open set \( \Omega \subset \mathbb{R}^n \) and any Borel measure \( \mu \ll \text{dim} \) on it as follows

\[
\|f\|_{L^\Psi(\Omega, \mu)} = \inf \sup_{\psi \in \mathcal{L}} \psi(x)
\]

where the infimum is taken over all ACL functions \( \psi: \Omega \to (0, \infty) \) satisfying

\[
\hat{c}_1 \ldots \hat{c}_n \psi(x) = \frac{d\mu}{d\text{dim}}(x) \Psi(\psi(x), |f(x)|, x), \quad \text{for a.e. } x \in \Omega
\]

and \( \Psi \) satisfies similar structural conditions as above.

We leave studying the properties of these function spaces to future research.

6. Discussion

As mentioned in the introduction, the approach taken to function space norms here is inductive or local, rather than global. One can also conceive a ‘left-handed’ version of the \( L^\Psi \) spaces where the process travels backwards, i.e.

\[
\varphi'(t) = \Psi(\varphi(t), |f(1-t)|, t), \quad \text{for a.e. } t \in [0, 1].
\]

In applied sciences the Hilbertian norm has of course numerous uses but also other \( L^p \) norms are widely used. For example, larger values of \( p \) can be used in penalizing large deviations in fitting. This has an interesting side effect; although the norm distinguishes between the maximal absolute values, it does not distinguish so much between the measures of the supports of the (near)maximum value. That is, if there are already some recorded deviations of certain magnitude, then other such observations do not change the norm that much. This phenomenon can be analyzed more formally by using the framework of this paper, the basic ODE tells exactly how the accrued norm accumulation (e.g. after observations) slows down the future norm accumulation. Perhaps the ODE (or just the \( p \)-norms) could be also used in modeling the economics of innovation where the first occurrences of so-far highest achievements have a disproportionate impact on the wealth (of an individual or society). For \( p \) close to infinity this metaphor becomes most pronounced, with a ‘winner takes it all’ logic, the norm accumulation depends at each moment of time on the highest value until then (actually only on the first attainment of that value). Turning away from potential applications, we will discuss the infinite \( p \) case next.

In this paper we have only studied absolutely continuous solutions \( \varphi \) to ODEs. It appears that one could extend the class of the norms studied here by admitting \( \varphi' \) to be a distribution. This could allow the exponent \( p \) to attain the value \( \infty \). We leave this for future work. On the other hand, the ‘approximate monotonicity’ of the semi-norms with respect to the exponent suggests the following natural extension:

\[
\|f\|_{L^p(\cdot)} := \lim_{\tilde{p} \to \infty} \sup_{p \in L^\infty, \tilde{p} \leq p} \|f\|_{L^{\tilde{p}}}, \quad p: [0, 1] \to [1, \infty].
\]

Let us still take a brief look at the asymptotics of the equation

\[
\frac{d\varphi}{dt} = \frac{|f(t)|^p}{p} \varphi(t)^{1-p}
\]
as \( p \) tends to \( \infty \). It is clear that if \( |f(t)| \leq \varphi(t) \), then the limit is 0. This, of course,
is the anticipated outcome when thinking how the hypothetical norm accumulation function \( \varphi \) should behave in the \( L^\infty \) case. If \( |f(t_0^+)| = \varphi(t_0) + a \), \( a > 0 \), then we would expect in the \( L^\infty \) case that
\[
d\varphi = a\delta_{t_0}
\]
holds at \( t_0 \). The following calculation
\[
d\varphi = (\varphi^p(t_0) + \Delta(\varphi(t_0) + a)^p)^{\frac{1}{p}} - \varphi(t_0) \xrightarrow{p \to \infty} a, \quad \forall \Delta > 0
\]
 appears to point to the direction of (6.1).

So far we have applied the symbol \( \boxplus_p \) as a short hand notation only. Next we will briefly look into the relationship between the algebraic structure of this operation and the basic ODE studied above.

We observe that

\[
h_{z_1, p}: \mathbb{C} \times \mathbb{C} \to \mathbb{C}
\]
given by
\[
h_{z_1, p}(z, \Delta) = (z^p + \Delta z_1^p)^{\frac{1}{p}}, \quad p \in \mathbb{C}\setminus\{0\}
\]
defines a group action where we consider \( \mathbb{C} \) in the \( \Delta \)-coordinate with its additive structure and the other copies of the complex plane as sets.

This way we may define the ‘left roots’ corresponding to the \( \cdot \boxplus_p \) \( t \) operation as follows:

\[
(r \boxplus_p \left[ \begin{array}{c} p \cr 1 \leq i \leq n \end{array} \right] t = s \boxplus_p t
\]

where

\[
r = (h_{t, p}(\cdot, -n^{-1}) \circ \ldots \circ h_{t, p}(\cdot, -1))(s) \quad (n \text { -times}).
\]

The formula of \( \varphi' \) essentially provides us with the formal derivative of the semigroup \((S_t)_{t \geq 0}\) given by \( S_t(x) = (x^p + ty^p)^{\frac{1}{p}}\), \( y \) being a constant. Namely,

\[
\frac{dS_t}{dt} = \frac{d}{dt} (x^p + ty^p)^{\frac{1}{p}}, \quad \frac{dS_t}{dt} \bigg|_{t=0} = \frac{yp}{p}x^{1-p}.
\]

However, the infinitesimal generator does not exist for \( x = 0 \) since

\[
\frac{d}{dt} t^{\frac{1}{p}} = t^{\frac{1}{p} - 1} \frac{1}{p} \to \infty, \quad t \to 0^+
\]

unless \( p = 1 \), the case where the accumulation of \( \varphi(t) \) depends on \( |f(t)| \) solely. This appears to be related to the fact that the ODEs studied here are not very stable around initial value 0.

There are obviously many open questions related to these \( L^p(\cdot) \) spaces. It is known that the Luxemburg norms do not coincide isometrically with the norms introduced here but we do not know if something can be said about the equivalence of the norms. In fact, we do not know whether \( L^p(\cdot) \) and \( L^{p_2}(\cdot) \) are isomorphic whenever there is a measure-preserving mapping \( T: [0, 1] \to [0, 1] \) such that \( p_1 = p_2 \circ T \).

We would like to see a classification for the surjective linear isometries acting on \( L^p(\cdot) \) spaces, or at least find some examples of \( p(\cdot) \) such that these isometries have the trivial form, i.e. \( f \to \sigma f \) with \( \sigma: [0, 1] \to \{-1, 1\} \) measurable. We suspect that this is the case when \( p(\cdot) \) is not a constant a.e. on any subinterval \([a, b]\) \( \subset [0, 1] \).

We do not know if \( L^p(\cdot) \) is strictly convex. We suspect that for \( L^p(\cdot) \) spaces superreflexivity is equivalent to being Asplund.
We do not know any criteria for the uniform convexity or uniform smoothness of the norms (or reflexivity, RNP, Asplund, LUR) for a general Υ.

Finally, we would like to call attention to the duality pairing perceived as an ODE in this context. This may have an interesting form. For instance, what is the relationship in the reflexive case between Υ solutions $\varphi = \varphi_f$ and $\varphi^*_J(f)$ (here $J$ is the duality mapping, cf. (1.2))?

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