ON THE BEAUVILLE–BOGOMOLOV DECOMPOSITION IN CHARACTERISTIC $p \geq 0$

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Abstract. We prove a variant of the Beauville–Bogomolov decomposition for weakly ordinary, or generally globally $F$-split, varieties $X$ with $K_X \sim 0$, in characteristic $p > 0$. If the assumption $K_X \sim 0$ is replaced by $-K_X$ being semi-ample, we show the weaker statement that all closed fibers of the Albanese morphism are isomorphic. In the appendix written jointly with Giulio Codogni, we also deduce the singular version of the latter statement, in characteristic zero. Finally, we apply our main theorem to draw consequences to the behavior of rational points and fundamental groups of weakly ordinary $K$-trivial varieties in positive characteristic.

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1. INTRODUCTION

1.1. Statements for smooth varieties

A weak version of the classical Beauville–Bogomolov decomposition (see [Bog74, Bea83]) was already observed by Calabi for Kähler manifolds in [Cal57]. It states that for every smooth projective variety $X$ defined over $\mathbb{C}$ with $K_X \sim 0$ there exists a finite étale cover

$$V \times B \rightarrow X,$$

where $B$ is an abelian variety and $V$ is a smooth projective simply connected variety with $K_V \sim 0$. Already this statement has substantial corollaries concerning geometry of varieties with trivial canonical class. For example, it directly implies that the fundamental group of such variety is virtually abelian, that is, admits a finite index abelian subgroup.

Instead of requiring $V$ to be simply connected, the decomposition (1.0.a) can be alternatively also characterized up to an étale cover of $V$ by using the notion of augmented irregularity. By definition, the augmented irregularity of $X$ is

$$\hat{q}(X) := \max \{ \dim \text{Alb}_{X'} | X' \rightarrow X \text{ is a finite étale morphism} \},$$

for which one needs to prove that the above maximum exists. This follows from Calabi’s original statement, or from [Kaw81, Thm 1] in a more general singular setting. Then, the above mentioned alternative characterization of (1.0.a) is by requiring that $\hat{q}(X) = \dim B$ and $\hat{q}(V) = 0$.

In the present article, we show a positive characteristic version of the decomposition (1.0.a) using the augmented irregularity. Our base field $k$ is perfect and of characteristic $p > 0$, and we also need the following positive characteristic notion: a projective variety $X$ over $k$ is strong weakly ordinary if the action of the absolute Frobenius morphism $F_X$ on $H^{\dim X}(X, \mathcal{O}_X)$ is bijective. This is a genericity notion. That is, being weakly ordinary is an open condition in positive equicharacteristic and it is typically dense in moduli. Additionally, it is conjectured to be dense over mixed characteristic bases that are finite type over $\mathbb{Z}$ [MS11]. For a smooth projective weakly ordinary variety $X$ with $K_X \sim 0$ we define the augmented irregularity just as in (1.0.b), where the existence of the maximum is guaranteed by [Eji17, Thm 1.1].

Also, already when $X$ is smooth, our decomposition statement uses the notion of strongly $F$-regular singularities, which is a characteristic $p$ class of mild singularities. In particular, it is contained both in the class of klt and rational singularities. We refer to the many surveys in the topic for a detailed introduction on the notions of $F$-singularities [ST12, PST17, Pat16]

**Theorem 1.1** (Smooth & weakly ordinary case of the Beauville–Bogomolov decomposition, special case of Theorem 11.6). Let $X$ be smooth projective variety over $k$, such that $K_X \sim 0$ and $X$ is weakly ordinary. Then there is a composition

$$B \times V \rightarrow Z \rightarrow X$$

of two finite covers, such that

1. $Z \rightarrow X$ is étale, $B \times V \rightarrow Z$ is a torsor under $\prod_{i=1}^{\hat{q}(X)} \mu_{p^i}$, for some integers $j_i \geq 0$,
2. $B$ is an abelian variety with $\dim B = \hat{q}(X)$, and
3. $V$ is a weakly ordinary projective Gorenstein variety over $k$ with strongly $F$-regular singularities, $K_V \sim 0$ and $\hat{q}(V) = 0$. 
Remark 1.2. There are two major differences between Theorem 1.1 and the original characteristic zero statement mentioned above:

1. $B \times V \rightarrow X$ in Theorem 1.1 is not étale as in characteristic zero, but has an infinitesimal part as well, and
2. $V$ is not necessarily smooth, but has only strongly $F$-regular singularities.

These two phenomena are in fact interconnected, as both are caused by the possible presence of non-reduced polarized automorphism groups. However, we do not know an example of a smooth projective weakly ordinary variety $X$ with $K_X \sim 0$ for which any of (1) or (2) occurs. On the other hand, we do have mildly singular examples for which phenomenon (1) occurs: see Example 13.3, which satisfies the assumptions of our singular statements in Section 1.2. In Question 13.6 we state precise properties of a hypothetical singular Calabi–Yau with a $\mu_p$ action under which the quotient is a smooth Calabi–Yau, the existence of such variety would yield a smooth example to both of the above two phenomena.

Remark 1.3. In fact, one can make the statement of Theorem 1.1 slightly more precise. That is, according to Theorem 11.6, by adding the following additional requirements, the decomposition of Theorem 1.1 still exists: the action of $G := \prod_{i=1}^{n} \mu_{p^{f_i}}$ on $Y$ is the diagonal action induced by an action on $V$ and an action on $B$, respectively, such that

1. $G$ acts freely and faithfully on $B$, and
2. $G$ acts faithfully on $V$.

Remark 1.4. If $X$ is a normal, projective variety over $k$ such that $K_X \sim 0$, then $X$ is weakly ordinary if and only if it is globally $F$-split. That is if and only if the structure morphism $O_X \rightarrow F^* O_X$ splits as a homomorphism of $O_X$-modules. This is the framework in which the statement of Theorem 1.1 generalizes to the case when $K_X \equiv 0$. Additionally, in this framework, one can allow also singularities. Hence, in Section 1.2, and in general in most parts of the article, the weakly ordinary condition will be replaced with global $F$-splitting.

In Theorem 1.6, we are also able to prove, the positive characteristic version of the main result of [Cao17], with the caveat that if we are not over a finite field or its algebraic closure, then we have to replace nef by semi-ample. This results, i.e. Theorem 1.6, uses Theorem 1.5, which might be of interest independently as well.

These statements are also impossible to state completely without the use of the local and global $F$-singularity notions. The notion of global $F$-splitting generalizes weak ordinarity, as explained in Remark 1.4, and strong $F$-regularity generalizes smoothness, as mentioned before Theorem 1.1.

**Theorem 1.5** (Isotriviality over curves, special case of Corollary 8.5 and Corollary 8.7.). Let $f : X \rightarrow T$ be a surjective morphism from a smooth projective variety to a smooth projective curve with strongly $F$-regular general fiber such that either

1. $-K_{X/T}$ is semi-ample, or
2. $k \subseteq \mathfrak{F}_p$ and $-K_{X/T}$ is nef.

Then $f$ has isomorphic closed fibers over $\overline{k}$.

Luckily, in the situation of Theorem 1.6, we can prove that the general fibers of the Albanese morphism are strongly $F$-regular (see Theorem 6.3). This, together with Theorem 1.5, leads then to the following statement:

**Theorem 1.6** (Isotriviality of the Albanese morphism, special case of Theorem 9.2). Let $X$ be a smooth projective globally $F$-split variety such that either

1. $-K_X$ is semi-ample, or
2. $k \subseteq \mathfrak{F}_p$ and $-K_X$ is nef.

Then the Albanese morphism $f : X \rightarrow A$ has isomorphic closed fibers over $\overline{\mathbb{F}}$.
Remark 1.7. The general version of Theorem 1.6 provided in Theorem 9.2, makes a more precise statement: $f$ gets trivialized over a specific torsor under the polarized automorphism group of (any) fiber.

We conclude the present section with a few applications of our results. We start with a conjecture that mirrors conjectures over number fields for Calabi-Yau varieties:

**Conjecture 1.8.** If $X$ is a smooth projective Calabi-Yau variety over $\mathbb{F}_q$ (that is, $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$), then $X(\mathbb{F}_q) \neq \emptyset$.

We are able to deduce from our main results a statement in the direction of Conjecture 1.8:

**Corollary 1.9** (special case of Corollary 12.9). If $X$ is a smooth projective weakly ordinary 3-fold over $\mathbb{F}_q$ with $K_X \sim 0$, $\hat{q}(X) \neq 0$ and $q \geq 83$, then $X(\mathbb{F}_q) \neq \emptyset$.

Our other application of our main results is towards the conjecture, the characteristic zero counterpart of which is well known, that $K$-trivial smooth projective varieties have virtually abelian étale fundamental groups:

**Corollary 1.10** (special case of Corollary 12.5). If $X$ is a smooth projective weakly ordinary variety with $K_X \sim 0$ and $\hat{q}(X) = \dim X - 2$, then $\pi_1^\text{et}(X)$ is virtually abelian.

Remark 1.11. We note that in the full statements, that is, in Corollary 12.9 and in Corollary 12.5 we are able to also include the $K_X$ numerically trivial case by assuming $F$-purity instead of weak ordinarity.

1.2. **Statements for singular varieties**

As explained before, the main recipe of turning the statements concerning smooth varieties provided in Section 1.1 into statements about singular varieties is to replace every occurrence of smooth by strongly $F$-regular, and every occurrence of weakly ordinary by globally $F$-split. Hence for Theorem 1.1 we obtain:

**Theorem 1.12** (Singular Beauville–Bogomolov decomposition, Theorem 11.6). Let $(X, \Delta)$ be globally $F$-split projective pair over $k$ with strongly $F$-regular singularities, such that $K_X + \Delta \equiv 0$. Then there is a composition $Y \to Z \to X$ of two finite covers such that $Z \to X$ is quasi-étale, $Y \to Z$ is a torsor under $\prod_{i=1}^\text{dim}(X) \mu_{p^j_i}$ for some integers $j_i \geq 0$, and such that

$$(Y, \Delta_Y) \cong (V, \Delta_V) \times B,$$

where

1. $B$ is an abelian variety with $\dim B = \hat{q}(X)$.
2. $(V, \Delta_V)$ is a globally $F$-split projective pair over $k$ with strongly $F$-regular singularities, $K_V + \Delta_V \equiv 0$ and $\hat{q}(V) = 0$.

The proof of the theorem actually works in a more general setting. We managed to show the following result under the assumption that $-K_X - \Delta$ is nef/semi-ample.

**Theorem 1.13** (Isotriviality of the Albanese morphism, Theorem 9.2). Let $(X, \Delta)$ be a projective globally $F$-split pair with strongly $F$-regular singularities such that $K_X + \Delta$ is Q-Cartier with index prime-to-$p$, and either

(a) $-(K_X + \Delta)$ is semi-ample, or
(b) $k \subseteq \overline{\mathbb{F}}_p$ and $-(K_X + \Delta)$ is nef.

Then, $(X_t, \Delta_t) \cong (X_s, \Delta_s)$ for every $s, t \in \mathbb{A}(\overline{k})$.

While proving Theorem 1.13, we also obtain the following theorem concerning fibrations over curves.
Theorem 1.14 (Isotriviality over curves, Corollary 8.5 and Corollary 8.7.). Let \( f : (X, \Delta) \to T \) be a surjective morphism from a projective normal pair to a smooth projective curve such that \( \Delta \) is an effective \( \mathbb{Q} \)-divisor, \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, the general fiber \( (X_t, \Delta_t) \) is strongly \( F \)-regular and either

(a) \(- (K_{X/T} + \Delta)\) is semi-ample, or
(b) \( k \subseteq \mathbb{F}_p \) and \(- (K_{X/T} + \Delta)\) is nef.

Then, \( (X_t, \Delta_t) \cong (X_s, \Delta_s) \) for every \( s, t \in T \left( \overline{k} \right) \).

Remark 1.15. Historical remarks (in characteristic zero): As mentioned above, the original smooth version of the Beauville–Bogomolov decomposition was shown in \([\text{Bog74}, \text{Bea83}]\). The singular Beauville–Bogomolov decomposition has recently attracted a serious amount of attention which culminated in the series of papers \([\text{GKP16}, \text{GGK17}, \text{Dru16}, \text{HP17}]\) leading to the full decomposition theorem for klt varieties with numerically trivial canonical class. The weak Beauville–Bogomolov condition was shown even in the singular logarithmic setting in the papers \([\text{Kaw81}, \text{Amb05}]\).

Remark 1.16. For the additional statement about trivialization over a flat torsor, mentioned in Remark 1.7, we refer to Corollary 8.5, Corollary 8.7 and Theorem 9.2.

1.3. Statements in characteristic zero

Using the techniques developed during the positive characteristic considerations we also managed to prove the following theorem.

Theorem 1.17 (Corollary A.14.). Let \((X, \Delta)\) be a klt \( \mathbb{Q} \)-factorial pair over an algebraically closed field of characteristic zero such that \(-K_X - \Delta\) is nef. Then the Albanese morphism \( \pi : (X, \Delta) \to \text{Alb}_X \) is an isotrivial fibration.

This generalizes to the singular and log setting the recent result of Cao \([\text{Cao17}]\) resolving in the projective case the question posed in \([\text{DPS96}]\) (see Remark 1.15). The problem stated by Demailly, Peternell and Schneider originates from the works \([\text{CP91, DPS94}]\) concerning the structure of smooth varieties with nef tangent bundle.

1.4. Outline of the proof

In the following section, we give an outline of the proof of Theorem 1.1. This includes all the techniques necessary to get the most general results, i.e., the statements of Section 1.2. However, for the purpose of clarity we avoid some technical difficulties. Let \( X \) be a smooth weakly ordinary projective variety defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). Suppose that the canonical divisor satisfies the condition \( K_X \sim 0 \).

1.4.1. Albanese morphism and augmented irregularity in characteristic \( p \). We begin our proof with the application of the results of Ejiri provided in \([\text{Eji17}]\). For this purpose, we first apply the standard result of Mehta and Ramanathan \([\text{MR85, Proposition 9}]\) to see that a weakly ordinary variety satisfying the condition \( K_X \sim 0 \) is globally \( F \)-split. Then, using \([\text{Eji17, Theorem 1.1 and Theorem 1.2}]\) we see the Albanese \( X \to \text{Alb}_X \) is in fact a relatively normal and \( F \)-split surjective algebraic fibre space. Consequently, we know that the general fibres of \( X \to \text{Alb}_X \) are normal, and all the fibres are \( F \)-pure and hence reduced.

Moreover, since the condition of \( F \)-splitting is preserved under étale covers, this also implies that the augmented irregularity as defined in (1.0.b) is finite. We may therefore take a Galois étale cover \( Z \to X \) such that \( \dim \text{Alb}_Z = \hat{q}(X) \). We note that as \( K_Z \sim 0 \) and \( Z \) is globally \( F \)-split, for \( Z \to \text{Alb}_Z \) also hold the above features of \( X \to \text{Alb}_X \). We shall prove that the Albanese morphism \( Z \to \text{Alb}_Z \) becomes a product after taking an étale cover and a further diagonalizable torsor over the base.
1.4.2. General fibers are strongly $F$-regular. The standard tools to control the behaviour of fibrations such as $Z \to \text{Alb}_Z$ are the semi-positivity results for the relative canonical sheaves provided in characteristic $p$ in the paper of the first author [Pat14]. The ubiquitous requirement for the application of the aforementioned results is the strong $F$-regularity of the general fibres of the investigated morphism. Using the arguments above, until now we only managed to prove that the fibres are $F$-pure. In order to improve the situation and get that the general fibres of $Z \to \text{Alb}_Z$ are strongly $F$-regular it suffices to show that the Frobenius pullbacks $Z^e = Z \times_{F^e \text{Alb}_Z} \text{Alb}_Z$ are strongly $F$-regular along the generic fibres of natural projections $Z^e \to \text{Alb}_Z$, for every $e > 0$ [PSZ18]. The natural tool now is the theory of test ideals $\tau(Y) \subseteq O_Y$, associated to normal varieties $Y$, developed by Hochster and Huneke (see [HH89] for the original work and [ST12] for a comprehensible survey). The ideals control strong $F$-regularity of $Y$ in the sense that the condition $\tau(Y) = O_Y$ is satisfied exactly along the locus where $Y$ is strongly $F$-regular. In our situation, to prove that $\tau(Z^e) = O_{Z^e}$ along the generic fibre of the projection we apply the transformation rule for test ideals under finite maps provided in [ST1] to the relative Frobenius $F^*_Z : Z \to Z^e$. We emphasize that, as required by the results of loc.cit, $Z^e$ is normal along the generic fibre of the projection (see the middle of Section 1.4.1). For the precise argument, we refer to Section 6.

1.4.3. Flatness. As a next step towards the proof, in Section 4 we show that the Albanese morphism of $Z$ is in fact flat. For this purpose, we mimic the characteristic zero arguments given in [LTZZ10, Theorem]. Our contribution here is mainly the realization that in characteristic $p$ there are appropriate semi-positivity results (see Section 3) to execute the strategy. To sum up, our current arguments state that the morphism $Z \to \text{Alb}_Z$, which we intend to prove becomes a product, is a flat algebraic fibre space with strongly $F$-regular general fibre.

1.4.4. Restriction to curves. We consequently proceed to the proof of the next approximation of the desired result, that is, we show that the family $Z \to \text{Alb}_Z$ is isotrivial over a general complete intersection curve $T \subset \text{Alb}_Z$ that goes through an arbitrarily fixed closed point $t_{\text{special}} \in \text{Alb}_Z$ and a general fixed closed point $0 \in \text{Alb}_Z$. We set $V = T \times_{\text{Alb}_Z} Z$. Since the all fibres of $Z \to \text{Alb}_Z$ are reduced, and the general fibre is normal and strongly $F$-regular, we see that $V$ is normal and the morphism $V \to T$ is a flat fibration with strongly $F$-regular general fibre. Moreover, using the base change formulas for the relative canonical divisor we see that $K_{V/T} \sim (K_Z/\text{Alb}_Z)|_V \sim 0$. We note that in general this part of the argument requires a little care. We provide the relevant base change results for the relative canonical in Section 2.8. The details of the argument are provided as the part of the main proof given in Section 9.

1.4.5. Numerical flatness. In order to prove that the morphism $f : V \to T$ is isotrivial we first show that there exists an appropriate $f$-ample divisor on $L$ on $V$ such that the relative section sheaves $f_* O_V(mL)$, for $m > 0$, satisfy certain notion of triviality called numerical flatness. A vector bundle $E$ on a projective smooth scheme $X$ is numerically flat if both $E$ and $E^\vee$ are nef. We refer to Section 2.5 for a more detailed description of the notion.

In our context, we prove that the sheaves $f_* O_V(mL)$ are numerically flat if $L$ is an $f$-ample divisor such that $L^{d+1} = 0$, where $d+1 = \dim V$. For the detailed proof we refer to Section 5. In this paragraph we give a brief description of the arguments.

First of all, we show that an $f$-ample divisor $L$ satisfying $L^{d+1} = 0$ is in fact nef. In fact, it is enough to show that $L + f^* \varepsilon H$ is nef for fixed ample divisor $H$ on $T$ and for every $\varepsilon > 0$. The point is that a Riemann-Roch computation shows that in this case there is an effective $\Gamma \sim_0 L + f^* \varepsilon H$, see Lemma 5.3. Then, semi-positivity theory applied to $\varepsilon T = (K_V + \Delta_V + \varepsilon T) + (\varepsilon K_V - \Delta_V)$ yields the above nefness using that $-K_V - \Delta_V \sim_0 0$, see Theorem 5.4 and Theorem 3.1. Here $\Delta_V$ is a natural boundary on $V$ that makes the linear equivalence $K_X/T|_V \sim_0 K_V + \Delta_V$, see Section 2.8. Having shown the nefness of $L$, the nefness of $f_* O_V(mL)$ follows from standard semi-positivity theory again, see Theorem 5.10.
Second, we show that \( f_! \mathcal{O}_V(mL) \) is anti-nef. So, assume the contrary. According to Lemma 5.6, this is equivalent to the lowest piece \( E \) of the Harder-Narasimhan filtration of \( F^l_! f_! \mathcal{O}_V(mL) \) having positive degree for every \( l \gg 0 \). Now, by choosing \( l \gg 0 \) we may assume that \( E \) is strongly semi-stable, and then \( E \otimes r \) is semi-stable for every \( r > 0 \) [Lan04, Thm 6.1]. The main idea is that by going to \( r \gg 0 \), via the multiplication map \( E \otimes r(-t) \to (f_! \mathcal{O}_X(rmL))(-t) \) this yields a section in \( H^0(T, f_! \mathcal{O}_X(rmL)(-t)) \cong H^0(X, rmL - X_1) \) that should not exist as \( L^{d+1} = 0 \), see Theorem 5.8.

### 1.4.6. Isotriviality for finite fields

In characteristic \( p > 0 \), the numerical flatness turns out to be a particularly strong notion if the underlying variety \( X \) is defined over the algebraic closure of the finite field. More precisely, using the results of Langer and the classical theorem of Lange and Stühler one can prove that a numerically flat bundle defined on a variety \( X/F_q \) is trivializable on the cover \( Y \to X \) which is a composition of a finite étale morphism and a power of the Frobenius (see Lemma 2.5). Assuming that the variety \( V \) is defined over the algebraic closure of a finite field and taking such a cover \( \tau: S \to T \) for the bundle \( f_! \mathcal{O}_V(mL) \), where \( m \) is chosen so that the natural multiplication maps \( \text{Sym}^d f_! \mathcal{O}_V(mL) \to f_! \mathcal{O}_V(dmL) \) are surjective, we see that the relative canonical ring

\[
R_{V_S/S}(mL_S) = \bigoplus_{d \in \mathbb{N}} f_! \mathcal{O}_{V_S}(dmL_S) \cong \bigoplus_{d \in \mathbb{N}} \tau^* f_! \mathcal{O}_V(dmL)
\]

consists of numerically flat bundles which are quotients of trivial bundles. Such bundles are in fact trivial themselves and hence, since \( S \) is a projective curve, \( R_{V_S/S}(mL_S) \) comes as pullback of a ring defined over the base field. This implies that \( V_S \to S \) is a product family, and hence gives isotriviality over curves \( T \) if the initial variety \( X \) is defined over the algebraic closure of a finite field. The precise statements are presented in Section 8. Similar arguments actually lead to the proof of Theorem 1.14.

### 1.4.7. Reduction to finite fields

In order to reduce to the case where the base field is the algebraic closure of a finite field, we use the spreading out technique and the base change properties for the suitable relative polarized isomorphism scheme. We first observe that above isotriviality result implies that the natural map

\[
\text{Isom}_T ((V, mL), (V_t \times_k T, mL_t \times_k T)) \to T,
\]

where \( t \in T \) is a \( k \)-rational base point, is surjective if the base field \( k \) is the algebraic closure of a finite field. In order to get a similar statement in general, for arbitrary perfect base field \( k \), we take a spreading out \((V \to T, \mathcal{L}, \sigma: \text{Spec}(R) \to T)\) over a finitely generated \( \mathbb{F}_q \)-algebra \( R \) of the morphism \( V \to T \) along with the divisor \( \mathcal{L} \) and a choice of a base point \( t \in T \). We consider the relative isomorphism scheme

\[
\mathcal{I} = \text{Isom}_T \left((V, m\mathcal{L}), (V_\sigma \times_{\text{Spec}(R)} T, m\mathcal{L}_\sigma \times_{\text{Spec}(R)} T)\right) \to T.
\]

By base change property of isomorphism schemes, we see that the result of the previous section implies that the morphism \( \mathcal{I} \to T \) defined over \( R \) is surjective when restricted to every closed point of \( \text{Spec}(R) \). By a standard scheme theoretic argument, this yields surjectivity at the geometric generic point, and hence the required isotriviality even in the polarized setting. We note that the number \( m \) showing up in Section 1.4.6 can be chosen in advance before the spreading out so that it yields surjectivity of the multiplication map for every finite field reduction.

### 1.4.8. From isotriviality over curves to the isotriviality over \( \text{Alb}_Z \) and to the splitting over a flat cover

First, we observe that the choice of a line bundle \( L \) in Section 1.4.5 and the number \( m \) in Section 1.4.6 can in fact be performed uniformly, see Theorem 9.1 and Theorem 9.2. Then, we consider the isomorphism scheme

\[
\mathcal{I} = \text{Isom}_{\text{Alb}_Z} \left((Z, mL_Z), (\text{Alb}_Z \times_k Z_0, \text{Alb}_Z \times_k mL_0)\right)
\]
where $0 \in \text{Alb}_Z$ is a fixed $k$-rational base point and $L_0 = L_{Z|Z_0}$. By the base-change properties of the isomorphism scheme, and the above explained isotriviality over the curves $T \subseteq \text{Alb}_Z$, the image of $I \to \text{Alb}_Z$ contains every such $T$. However, as $t_{\text{special}} \in T$ was arbitrarily fixed, this means that $\pi : I \to \text{Alb}_Z$ is surjective. Additionally, by using again the base-change properties of the isomorphism scheme, we see that the morphism $Z \to \text{Alb}_Z$ becomes a product (even in the polarized way) after the base change along $\pi$. The formal version of the argument is provided in the proof of Theorem 9.1.

We remark that in the logarithmic setting the actual argument is quite delicate. We provide the details of the necessary base change results for the logarithmic version of the isomorphism scheme in Section 7.

1.4.9. Finiteness of automorphism groups. The above argument does not say anything about the structure of the trivializing cover $\pi : I \to \text{Alb}_Z$. In order to rectify this situation, we observe that $\pi$ is in fact a torsor under the polarized automorphism scheme $G = \text{Aut}(Z_0, L_0)$ of the fibre $Z_0$. By the previous considerations we see that $Z_0$ is a strongly $F$-regular, Gorenstein variety satisfying $K_{Z_0} \sim 0$. In Section 10 we show that polarized automorphisms schemes of such varieties are in fact finite group schemes. The proof is by contradiction. Assuming otherwise, we infer that $G$ admits a subgroup isomorphic to the additive or multiplicative group acting on $Z_0$ with generically trivial stabilizers. Using the classical observation of Rosenlicht [Ros56, Theorem 2 and 10] this easily imply that $Z_0$ is ruled which gives a contradiction using a simple argument based on Kodaira dimension. We emphasize that the last part of argument requires our strong bounds on singularities of $Z_0$.

1.4.10. Nori fundamental group scheme. We are now ready to finish the proof. In the previous section, we showed that the trivializing morphism $I \to \text{Alb}_Z$ is in fact a torsor under a finite flat groups schemes over $k$. Such objects were extensively studied by Nori in his works concerning generalizations of étale fundamental groups. In particular, in [Nor83, Proposition] it is proven that the reduced part of every torsor under a finite flat group scheme over an abelian variety $A$ is dominated by the $A[n]$-torsor given by the multiplication map $[n] : A \to A$, for some $n \in \mathbb{N}$. Applying this to our situation, this means the morphism $I_{\text{red}} \to \text{Alb}_Z$ is covered by the multiplication map $[n] : \text{Alb}_Z \to \text{Alb}_Z$, which in turn implies that $Z \to \text{Alb}_Z$ becomes a product after taking a base change under $[n]$. We conclude the proof by observing that $[n]$ is in fact a composition of an étale morphism and a diagonalizable torsor because $\text{Alb}_Z$ is $F$-split. The details of the above argument are provided in Section 11.

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2. General preliminaries

In this section we gather some preliminary results required in the following considerations.

2.1. Base-field

The field $k$ is perfect, and apart from Section 7 and Appendix A, of characteristic $p > 0$. 

2.2. Basic notation and definitions

A variety is a separated, integral scheme of finite type over our fixed base field $k$. A fibration means a surjective proper morphism $f: X \to T$ between varieties, such that the natural map $\mathcal{O}_T \to f_*\mathcal{O}_X$ is an isomorphism. An open set $U$ of a Noetherian scheme $X$ is big if $\text{codim}_X(X \setminus U) \geq 2$.

For a variety $X$ defined over a field $k$, by $H^i(X_{\text{ét}}, \mathbb{Q}_l)$ (resp. $H^i_f(X_{\text{ét}}, \mathbb{Q}_l)$) we denote the $\ell$-adic cohomology (resp. $\ell$-adic cohomology with compact support) given by the formula $H^i(X_{\text{ét}}, \mathbb{Q}_l) = \lim_{n \to \infty} H^i_{\text{ét}}(X_{\text{ét}}, \mathbb{Q}_l) / \ell^n \mathbb{Z} \otimes \mathbb{Q}_l$.

2.2.1. $F$-singularities. The following definition is a summary of the classical definitions concerning $F$-singularities given for example in [PST17, Definition 4.1 and Definition 4.2].

**Definition 2.1.** Let $X$ be a normal variety defined over a perfect field of characteristic $p > 0$, and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor.

1. The pair $(X, \Delta)$ is globally $F$-split if there exists an integer $e > 0$ for which the natural map $\mathcal{O}_X \to F^e\mathcal{O}_X((p^e-1)\Delta)$ splits.
2. The pair $(X, \Delta)$ is sharply $F$-pure if it can be covered by globally $F$-split open subvarieties.
3. The pair $(X, \Delta)$ is globally $F$-regular if for every divisor $D$ and for every integer $e \gg 0$ the natural map $\mathcal{O}_X \to F^e\mathcal{O}_X((p^e-1)\Delta + D)$ splits.
4. The pair $(X, \Delta)$ is strongly $F$-regular if it can be covered by globally $F$-regular open subvarieties.

**Remark 2.2.** It turns out that strong $F$-regularity can be verified using the concept of test ideals (see [HII89, ST12]). More precisely, a pair $(X, \Delta)$ is strongly $F$-regular if and only if the corresponding test ideal $\tau(X,\Delta)$ is isomorphic to $\mathcal{O}_X$ (see [ST12, Theorem 3.19] for a divisor free statement). We will use the transformation rules for test ideals under finite maps, proven in [ST14], in order to analyze the singularities of the Albanese morphism (see Section 6).

2.3. Reflexive sheaves and reflexive operations

If $\mathcal{F}$ is a coherent sheaf on a normal variety $X$, then $\text{Sym}^{[a]}(\mathcal{F}) := (\text{Sym}^a(\mathcal{F}))^{**}$ is the reflexive symmetric power. The most important thing to remember is that all types of reflexive operations are determined in codimension 1, that is, in this particular case if $\mathcal{T} \subseteq \mathcal{F}$ is the maximal torsion subsheaf, and $\iota : U \hookrightarrow X$ is the locus where $\mathcal{F} / \mathcal{T}$ is locally free, then $\text{Sym}^{[a]}(\mathcal{F}) = \iota_* \left( \text{Sym}^a \left( \mathcal{F} / \mathcal{T} \right)_U \right)$ [Har80]. We define similarly reflexive pullbacks $f^*[\mathcal{F}] := (f^*\mathcal{F})^{**}$, where $f : Y \to X$ is a morphism of finite type, or reflexive tensor products $\mathcal{F} \otimes \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{**}$, where $\mathcal{G}$ is also a coherent sheaf on $X$. For the latter reflexive operations similar extension property holds in terms of $\iota_*$ as for the reflexive symmetric product.

In general we use the basic statements about reflexive coherent sheaves on normal varieties [Har80, page 124-129] without giving a precise reference each time.

2.4. General, generic and geometric generic fibers

If $f: X \to T$ is a fibration, then the generic fiber is the fiber $X_{\eta}$ over the generic point $\eta \in T$. Similarly, the geometric generic fiber is $X_{\overline{\eta}}$, where $\overline{\eta} = \text{Spec} k(\eta)$. On the other hand, we say that a property holds for a general fiber, if there is a dense open set $U \subseteq T$ such that the given property holds for all $X_t$ where $t \in U$ is a closed point. The general pattern is that, as $k$ is assumed to be perfect, singularity properties holds for general fibers if and only if they hold for the geometric generic fiber. See [PW17, Prop 2.1] for the incarnation of this pattern for normality, reducedness and regularity.
2.5. Numerically flat bundles

In the main part of the paper, we will use the theory of numerically flat bundles in characteristic $p$ provided by Langer in [Lan11, Lan12].

**Definition 2.3.** Let $T$ be a smooth projective variety over a perfect field $k$. We say that a vector bundle $E$ on $T$ is numerically flat, if both $E$ and $E^*$ are nef.

**Proposition 2.4** ([Lan11, Proposition 5.1]). Let $T$ be a smooth variety over $k$, let $H$ be an ample divisor and let $E$ be a vector bundle. The following conditions are equivalent:

1. $E$ is numerically flat,
2. $E$ is nef and of degree zero.
3. $E$ is strongly $H$-semistable with $c_1(E).H^{n-1} = c_2(E).H^{n-2} = 0$.

We recall that by definition strong $H$-semistability is equivalent to $H$-semistability of all Frobenius pullbacks. We also note that by the above result numerically flat bundles are in fact strongly semistable with respect to any polarization.

**Lemma 2.5.** Suppose $E$ is a numerically flat vector bundle on a normal projective variety $T$ over $\mathbb{F}_q$. Then there is a finite cover $\tau: S \to T$ by a normal projective variety such that $\tau^*E \cong O^{\deg E}_S$. The cover $\tau$ might be chosen to be a composition of a finite étale covering and an iteration of the Frobenius morphism.

**Proof.** According to [Lan12, Thm 1.1], there is a scheme $M$ of finite type over $\mathbb{F}_q$ parameterizing numerically flat vector bundles of rank $r$ over $T$. So, for each numerically vector bundle $F$ over $T$ correspond a few $\mathbb{F}_q$-rational points of $M$. As $|M(\mathbb{F}_q)| < \infty$, we obtain that there are integers $e \geq 0$ and $e' > 0$ such that $(F^{e+e'})^* \cong (F^e)^*$. Hence by the main result of [LS77], there is a finite étale cover $\rho: S \to T$ such that $\rho^*(F^e)^* \cong E$. Set then $\tau := F^e \circ \rho$. 

2.6. Albanese morphism

For the arithmetic applications, we shall work over perfect non-necessarily algebraically closed field. In this context the *Albanese morphism* of a normal complete variety $X$ is an initial object in the category of morphism $X \to T$, where $T$ are torsors over abelian varieties. It exists by the detailed discussion provided in the Mathoverflow answer [Nfd17].

**Proposition 2.6.** Let $X$ be a normal projective variety defined over a perfect field $k$. Then the dimension of the Albanese variety $\dim \text{Alb}_X$ is equal to $\frac{1}{2} \dim H^1(X_{et},\mathbb{Q}_l)$.

**Proof.** After base extension, we may assume that $X$ has a rational point. Then the Albanese variety is isomorphic to the dual $\text{Pic}^0(X)^{\vee}_{\text{red}}$ (see [Bâd01, Theorem 5.3]), and consequently the statement follows from Proposition 2.7. 

**Proposition 2.7.** Let $X$ be a normal projective variety defined over a perfect field $k$. Then the identity component $\text{Pic}^0(X)$ of the Picard scheme is projective. Its reduced part $\text{Pic}^0(X)_{\text{red}}$ is an abelian variety of dimension $\frac{1}{2} \dim H^1(X_{et},\mathbb{Q}_l)$.

**Proof.** We need to prove $\text{Pic}^0(X)$ is of dimension $\frac{1}{2} \dim H^1(X_{et},\mathbb{Q}_l)$. For this purpose, we consider an exact sequence of étale sheaves on $X_{et}$:

$$0 \longrightarrow \mathbb{Z}/\ell^n \mathbb{Z} \longrightarrow G_m \longrightarrow G_m \longrightarrow 1.$$ 

The long exact sequence of cohomology yields an isomorphism $H^1_{et}(X_{et},\mathbb{Z}/\ell^n \mathbb{Z}) \cong \text{Pic}(X_{et})[\ell^n]$. As $|\text{Pic}(X_{et})[\ell^n]| / |\text{Pic}^0_{\text{red}}(X_{et})[\ell^n]|$ is bounded in $N$, and $|\text{Pic}^0_{\text{red}}(X_{et})[\ell^n]| \cong (\mathbb{Z}/\ell^n \mathbb{Z})^{2 \dim \text{Pic}^0(X)}$, taking a limit with $N \to \infty$ we get the desired claim. 

**Corollary 2.8.** Let $X \to Y$ be a universal homeomorphism between normal schemes defined over a perfect field $k$. Then the Albanese dimension of $X$ and $Y$ are equal.
2.7. $F$-splittings of varieties and morphisms

We shall need the following relative version of Definition 2.1 (1). Let $f: X \to S$ be a morphism of schemes over a perfect field $k$. We recall that the $e$-th relative Frobenius morphism $F^e_{X/S}$ is defined by the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{F^e_{X/S}} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
X_S & \xrightarrow{F^e_S} & X_S
\end{array}
\]

**Definition 2.9 (Eji17, Definition 5.1).** We say that a pair $(f, \Delta)$ is $F$-split if the map:

\[
\mathcal{O}_{X^e_S} \to F^e_{X/S} \mathcal{O}_X \to F^e_{X/S,e} \mathcal{O}_X([p^e - 1] \Delta).
\]

admits a splitting. In particular, a scheme $X$ defined in characteristic $p > 0$ is $F$-split if the natural map $X \to \text{Spec}(F_p)$ is $F$-split (equivalently, the natural map $\mathcal{O}_X \to F^e_\mathcal{O}_X$ is split, cf. Definition 2.1).

Suppose $f: X \to S$ is an $F$-split morphism, and let $T \to S$ be a morphism. Taking a base change of a splitting we see that $f_T: X_T \to T$ is also $F$-split. In particular, fibres of an $F$-split morphism are $F$-split [Eji17, Proposition 5.6].

**Proposition 2.10.** Let $X$ be a normal scheme defined over a perfect field $k$. Then for every Weil divisor $D$ there exists an isomorphism

\[
\text{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(D), \mathcal{O}_X) \simeq F^e_* \mathcal{O}_X \left((1 - p^e) \left(K_X + \frac{D}{p^e - 1}\right)\right).
\]

In particular, there is a bijection between the morphisms $F^e_* \mathcal{O}_X(D) \to \mathcal{O}_X$ and global sections of the reflexive sheaf $F^e_* \mathcal{O}_X \left((1 - p^e) \left(K_X + \frac{D}{p^e - 1}\right)\right)$.

**Proof.** We first remark that the divisor $(1 - p^e) \left(K_X + \frac{D}{p^e - 1}\right)$ in the statement is a Weil divisor with integer coefficients, and so the statement does make sense. We may restrict our considerations to the smooth locus $X_{\text{sm}}$ because $X$ is normal and hence all the sheaves in question are reflexive. Using the Grothendieck duality we thus obtain a bijection:

\[
\text{Hom}(F^e_* \mathcal{O}_{X_{\text{sm}}}(D), \mathcal{O}_{X_{\text{sm}}}) \simeq \text{Hom}(F^e_* \mathcal{O}_{X_{\text{sm}}}(D), \omega_{X_{\text{sm}}}) \otimes \omega_{X_{\text{sm}}}^{-1} \simeq F^e_* \mathcal{O}_{X_{\text{sm}}} \left((1 - p^e) K_{X_{\text{sm}}} - D\right)
\]

**Grothendieck duality and projection formula.**

The final statement follows by taking global sections. \qed

We are now ready to recall the following results which we will use to understand the singularities of the fibres of the Albanese morphism.

**Proposition 2.11 (Eji17, Theorem 1.2).** Let $(X, \Delta)$ be a pair of a normal variety $X$ and an effective $\mathbb{Q}$-Weil divisor $\Delta$. Then $(X, \Delta)$ is $F$-split if and only if the Albanese morphism $X \to A$ is $F$-split with respect to $\Delta$ (see Definition 2.9), and $A$ is ordinary.

**Proposition 2.12.** Let $(X, \Delta)$ be a pair of a normal variety $X$ and an effective $\mathbb{Q}$-Weil divisor $\Delta$ of Weil index coprime to $p$ (the divisor $(p^e - 1) \Delta$ is Weil for some $e > 0$). Moreover, let $S$ be a normal variety and $f: (X, \Delta) \to S$ an $F$-split morphism. Then the generic fibre of $f$ is regular in codimension one.
Proof. The proof is the same as that of [Eji17, Proposition 5.6] after restricting to the big open subset where $\Delta$ is $\mathbb{Q}$-Cartier of index prime to $p$ (e.g. regular locus of $X$).

**Proposition 2.13** ([MR85, Proposition 9]). Let $X$ be a normal Cohen–Macaulay variety with trivial canonical class defined over a perfect field of characteristic $p > 0$. Then $X$ is weakly ordinary if and only if $X$ is $F$-split.

Proof. For the convenience of the reader we recall the proof. First, we observe that the dualizing sheaf $\omega_X$ is $S_2$ and agrees with $\mathcal{O}_X(K_X)$ on the regular locus, and hence $X$ is Gorenstein and $\omega_X \cong \mathcal{O}_X$. Consequently, the natural map $\mathcal{O}_X \to F_0\mathcal{O}_X$ is split if and only if the trace map given by the Grothendieck dual

$$\text{Tr}_X: F_0\omega_X \cong \mathcal{H}om(F_0\mathcal{O}_X, \omega_X) \to \omega_X$$

is split. Since $\omega_X$ is trivial, this happens if and only if the corresponding map on global sections is surjective. Using Serre duality this is equivalent to $H^d(X, \mathcal{O}_X) \to H^d(X, \mathcal{O}_X)$ being injective, and hence bijective since $k$ is perfect.

### 2.8. The relative canonical bundle

For a surjective morphism $f: X \to T$ of varieties we define the relative canonical sheaf $\omega_{X/T}$ as $\omega_{X/T} := h^{-r}(f^* O_T)$, where $r$ is the relative dimension $\dim X - \dim T$, $f^*$ is the upper shriek functor of Grothendieck duality as defined in [Har66], and $h^{-r}(\_)$ means taking the $(-r)$-th cohomology sheaf.

In general the philosophy is that whenever we may associate a divisor to $\omega_{X/T}$, we call it $K_{X/T}$. This philosophy is made precise below in two cases: when $f$ is equidimensional and $X$ and $T$ are normal, and when $T$ is smooth.

#### 2.8.1. The case of an equidimensional morphism

To be precise, here equidimensional means that there is an integer $r > 0$, such that all the fibers of $f$ have pure dimension $r$. In this case, note that if $U \subseteq T$ is any big open set, and $\iota: f^{-1}U \to T$ is the natural embedding (note that $f^{-1}U \subseteq T$ is also big by the equidimensional assumption), then $\omega_{X/T} \cong \iota_* \omega_{f^{-1}U/U}$.

Indeed, this statement is local, so we may shrink arbitrarily the base, and then we may take a Noether normalization of $f$: $T \leftarrow \frac{\mathbb{P}^n_T}{\mathbb{P}^n_T}$

Then

$$\tau_* \omega_{X/T} \cong \mathcal{H}om_{\mathbb{P}^n_T} \left( \tau_* \mathcal{O}_X, \omega_{\mathbb{P}^n_T/T} \right) \cong \mathcal{j}_* \mathcal{H}om_{f^{-1}U} \left( \tau_* \mathcal{O}_X, \omega_{\mathbb{P}^n_T/T} \right) \cong \mathcal{j}_* \left( \iota|_{f^{-1}U} \right)_* \omega_{f^{-1}U/U} \cong \iota_* \omega_{f^{-1}U/U}.$$

\[\text{Grothendieck duality}\]

\[\text{Grothendieck duality}\]

#### 2.8.2. The case of equidimensional morphism, normal base and normal total space

Assume now additionally that $X$ and $T$ are normal. Let $U \subseteq T$ be the big open set which is the intersection of the regular locus of $T$ and the flat locus of $f$. Then, the above isomorphism $\omega_{X/T} \cong \iota_* \omega_{f^{-1}U/U}$ yields that

$$\omega_{X/T} \cong \iota_* \omega_{f^{-1}U/U} \cong \iota_* \left( \omega_{f^{-1}U} \otimes f^* \omega_U^{-1} \right) \cong \iota_* \mathcal{O}_X(K_{f^{-1}U} - f^{-1}K_U) \cong \mathcal{O}_X(K_X - f^*K_T),$$

where the pullback means the usual pullback for equidimensional surjective morphism between normal varieties (i.e., doing it over the flat locus which is big in the target, and then extending uniquely).
2.8.3. The case of smooth base and normal total space. If $T$ is smooth, $X$ is normal, but $f$ possibly non-equidimensional, $\omega_{X/T}$ is still a divisorial sheaf. Indeed, in this case $\omega_{X/T} \cong \omega_X \otimes f^*(\omega_T^{-1})$ by [Pat15, Lem 2.4]. Hence, as $\omega_X$ is reflexive [KM98, Cor. 5.69] we obtain that so is $\omega_{X/T}$ and hence it is divisorial. By the above isomorphism we also have $K_{X/T} = K_X - f^*K_T$.

2.8.4. Adjunction when the base is smooth and the total space is normal. In the case of Section 2.8.3, let $S \subseteq T$ be a smooth one codimensional subvariety, and assume that the scheme theoretical preimage $f^{-1}S$ is integral at the codimension 1 points of $X$. That is, we assume that the divisorial pullback $Y := f^*S$ is integral. Let $\tau: Z \rightarrow Y$ be the normalization, and set $g := f|_Y$. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $Y \subseteq \text{Supp} \Delta$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Finally, let $\Delta_Z$ be the different of $\Delta + Y$, that is, in particular $K_Z + \Delta_Z \sim_{\mathbb{Q}} K_X + Y + \Delta|_Z$. Then, we have

\begin{equation}
(2.13.a) \quad K_{Z/S} + \Delta_Z = K_Z + \Delta_Z - g^*K_S = K_X + \Delta + Y|_Y - g^*(K_T + S|_S))
\end{equation}

\[ \uparrow \]

\[ \begin{align*}
&= K_X + \Delta + Y - f^*(K_T + S)|_Y = K_X + \Delta - f^*K_T|_Y = K_{X/T} + \Delta|_Y \\
&\uparrow \\
&Y = f^*S \text{ as divisors}
\end{align*} \]

3. Semi-positivity engine

The base-field in this section and in Section 4 and Section 5 is perfect and of characteristic $p > 0$. In some of the proofs we claim that we may assume that the base field is algebraically closed. This is justified since the extension from a perfect field to one of its algebraic closures preserves all positivity and singularity properties.

We will use the following technical statement multiple times throughout the article.

Recall that pseudo-effectivity of non-$\mathbb{Q}$-Cartier divisors is defined as follows: a divisor $D$ on a normal projective variety $X$ is pseudo-effective if for any (resp. all) ample divisor $H$ and any $\varepsilon > 0$, $D + \varepsilon H$ is $\mathbb{Q}$-effective. In particular for a divisor $D$ on $X$:

- if $C$ is a general element of a (generically irreducible) covering family of curves on $X$, such that $C$ is contained in the $\mathbb{Q}$-Cartier locus of $D$, then $C \cdot D \geq 0$, and
- if there is a $\mathbb{Q}$-effective Weil divisor $E$ such that $D + \varepsilon E$ is $\mathbb{Q}$-effective for every rational number $\varepsilon > 0$, then $D$ is pseudo-effective.

**Theorem 3.1.** Assume we are in the following situation:

1. $f: X \rightarrow T$ is an equidimensional fibration between normal projective varieties with normal general fibers,
2. $U \subseteq T$ is a non-empty open set,
3. $\Gamma \geq 0$ be a $\mathbb{Q}$-divisor such that $K_{X/T} + \Gamma|_{f^{-1}U}$ is $\mathbb{Q}$-Cartier,
4. $L$ is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $K_{X/T} + \Gamma + L$ is $f$-nef over $U$, and
5. either
   - (i) $(X_t, \Gamma_t)$ is strongly $F$-regular for every closed point $t \in U$, or
   - (ii) $(X_t, \Gamma_t)$ is sharply $F$-pure for every closed point $t \in U$ with the Cartier index of $K_{X/T} + \Gamma|_U$ prime-to-$p$.

Then $K_{X/T} + \Gamma + L$ is pseudo-effective.

**Case (‘*’):** If $T$ is a curve, and $K_{X/T} + \Gamma + L$ is $f$-nef $\mathbb{Q}$-Cartier (so globally, not only over $U$), then $K_{X/T} + \Gamma + L$ is not only pseudo-effective but also nef.

**Proof.** By base extension we may assume that $k$ is algebraically closed. We do this because we reduce the statement to [Eji15, Thm 5.1], which is stated over an algebraically closed base field. There are two main issues during this reduction. These issues and our strategy to deal with them are the following:
(a) One issue is that there is no $L$ in [Eji15, Thm 5.1]. We solve this by the following trick: we perturb $L$ with a little ample $\varepsilon H$ so that it becomes ample, and then by choosing a general effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to $L + \varepsilon H$ we merge the perturbed $L + \varepsilon H$ into the boundary. The key here is that this process does not make the singularities on the general fibers worse by the results of [SZ13].

(b) The other main issue is that one has to guarantee two conditions of [Eji15, Thm 5.1]. First, [Eji15, Thm 5.1] requires the Weil-index to be prime-to-$p$, and second (ii) of [Eji15, Thm 5.1] requires that $S^0 \left( X, \Gamma ; m \left( K_X + \Gamma \right) \right) = H^0 \left( m \left( K_X + \Gamma \right) \right)$ for every divisible enough integer $m > 0$, where $\eta$ is the generic point of $T$. These two conditions are closely related because according to [Pat14, Prop 2.23], the second condition is satisfied if the index is not divisible by $p$, assuming the perturbation described in point (a).

We deal with these two requirements differently in the two cases of singularity assumptions. In the case of assumption (5)(ii)), the above requirements are almost satisfied except there can be non prime-to-$p$ Weil index away from the general fibers. This is easy to solve by removing the vertical components of $\Gamma$ from the boundary.

In the case of assumption (5)(i)) however we have to work more. We use [Pat14, Lem 3.15] in this case to perturb to the (even Cartier-)index prime-to-$p$ situation.

Below we work out the details of the argument.

First, we need to separate the vertical and the horizontal parts of $\Gamma$ in the case of assumption (5)(ii)), because there is no way we can guarantee in that case that the Weil index of $\Gamma$ in the vertical part is prime-to-$p$.

In the case of assumption (5)(i)) however we have to work more. We use [Pat14, Lem 3.15] in this case to perturb to the (even Cartier-)index prime-to-$p$ situation.

Let $H$ be an ample divisor on $X$. Then $L + \varepsilon H$ is ample for every $\varepsilon > 0$. So, for every fixed $0 < \varepsilon \in \mathbb{Q}$ the denominator of which is prime-to-$p$, there is a prime-to-$p$ integer $m > 0$ such that $|m(L + \varepsilon H)|$ is very ample. Choose a general $D \in |m(L + \varepsilon H)|$. Then for a general $t \in T$ we have that $D_t \in |mL_t|$ is also general. In particular, in the case of the respective singularity assumptions we have that:

\[(1) \left( X_t, (\Gamma_h)_t + \frac{1}{m} D_t \right) \text{ is also strongly } F\text{-regular according to [SZ13, Corollary 6.10 (iii)]}, \text{ and} \]
\[(3.1.a) \]
\[(2) \left( X_t, (\Gamma_h)_t + \frac{1}{m} D_t \right) \text{ is also sharply } F\text{-pure with Gorenstein index prime-to-} p \text{ according to [SZ13, Corollary 6.10 (i)].} \]

The above perturbation is adequate for solving issue (a). So, set $\Gamma_\varepsilon = \frac{1}{m} D$. Besides the singularity properties stated in (3.1.a), the only feature of $\Gamma_\varepsilon$ important to remember is that

\[(3.1.b) \]
\[\Gamma_\varepsilon \sim L + \varepsilon H. \]

To also deal with issue (b) in the case of assumption (5)(i)), we add a further divisor $\Gamma_{e,\varepsilon}$ to the boundary. This is defined as follows in the respective cases of singularity assumptions:

\[(1) \text{ Choose } E \in |K_{X/T} + aH| \text{ for some fixed } a > 0 \text{ independent of } \varepsilon > 0. \text{ Set then } \Gamma_{e,\varepsilon} := \frac{1}{p^m}(E + \Gamma_h + \Gamma_\varepsilon) \text{ for some } e \gg 0 \text{ (warning: } K_{X/T} + aH \text{ is not Cartier, so no general choice of } E \text{ is possible, it must go through all the points where } K_{X/T} + aH \text{ is not Cartier).} \]
\[(3.1.c) \]
\[(2) \text{ Set } \Gamma_{e,\varepsilon} := 0. \]
Finally, in either cases set \( \Lambda := \Gamma_h + \Gamma_e + \Gamma_{e,e} \). Summing up in the case of assumption (5)(iii) we obtain:

\[
(3.1.d) \quad K_{X/T} + \Lambda + \Gamma_v \sim_{\mathbb{Q}} K_{X/T} + \Gamma_h + \Gamma_e + \Gamma_{e,e} + \Gamma_v \sim_{\mathbb{Q}} K_{X/T} + \Gamma + L + \varepsilon H,
\]

and in the case of assumption (5)(i) we obtain

\[
(3.1.e) \quad K_{X/T} + \Lambda + \Gamma_v \sim_{\mathbb{Q}} K_{X/T} + \Gamma_h + \Gamma_e + \Gamma_{e,e} + \Gamma_v \sim_{\mathbb{Q}} K_{X/T} + \Gamma + L + \varepsilon H,
\]

and by \( Eji15 \) (3.1.c) in \( Pat14 \) Eji15 (3.1.e) we obtain

\[
(3.1.a) \quad \Gamma = \Gamma_h + \Gamma_v, \quad (3.1.b) \text{ and } (3.1.e)
\]

\[
\Gamma_v + \frac{p^e}{p^e - 1} (K_{X/T} + \Gamma_h + L) + \left( \varepsilon \frac{p^e}{p^e - 1} + a \frac{1}{p^e - 1} \right) H \rightarrow K_{X/T} + \Gamma + L
\]

\[
\varepsilon \rightarrow 0, e \rightarrow \infty
\]

In particular, in either case, it is enough to show that \( K_{X/T} + \Lambda + \Gamma_v \) is pseudo-effective (resp. nef in case (\( *) \)) for fixed \( \varepsilon = \frac{1}{2} \), where \( r > 0 \) is divisible enough, and \( e \gg 0 \), where \( e \) is chosen to be very big after choosing a very divisible \( r \). So, from now we change our goal to prove the pseudo-effectivity (resp. nefness) of \( K_{X/T} + \Lambda + \Gamma_v \) with the above fixed choices of \( \varepsilon \) and \( e \).

The next step is to apply [Eji15, Thm 5.1] to \((X, \Lambda)\). So, we have to verify the requirements of [Eji15, Thm 5.1] one by one. Let us start with the Weil index being prime-to-\( p \). In the case of assumption (5)(ii), \( \Lambda = \Gamma_h + \frac{1}{m} D \), where \( m \) is prime-to-\( p \) and \( \Gamma_h \) has prime-to-\( p \) Weil index. So, the Weil index of \( \Lambda \) being prime-to-\( p \) is satisfied in this case.

In the case of assumption (5)(i), we achieved this requirement by adding \( \Gamma_{e,e} \) into \( \Lambda \). In fact, in this case even the Cartier index of \( K_{X/T} + \Lambda \) is prime-to-\( p \) and also \((X_t, \Lambda_t)\) has strongly \( F \)-regular singularities for \( t \) in \( T \) general closed point, by (3.1.a) and [Pat14, Lem 3.15]. To be precise, to go from a single \( t \) in \( T \) here one has to invoke [PSZ18, Thm B] as well.

Additionally, in both cases we have \((X_{\pi}, \Lambda_{\pi})\) is sharply \( F \)-pure with Cartier index not divisible by \( p \) (by the previous paragraph in the case of assumption (5)(i)) and by (3.1.a) in the other case). In particular, by [Pat14, Prop 2.23], in both cases, if \( \eta \) is the generic point of \( T \), for every integer \( m > 0 \) divisible enough we have

\[
S^0 \left( X_{\pi}, \Lambda_{\pi}, m \left( K_{X_{\pi}} + \Lambda_{\pi} \right) \right) = H^0 \left( X_{\pi}, m \left( K_{X_{\pi}} + \Lambda_{\pi} \right) \right).
\]

Furthermore, \( f \) is separable, because \( g_* \mathcal{O}_X \cong \mathcal{O}_T \) and the geometric generic fiber of \( h \) is normal. So, all condition of [Eji15, Thm 5.1] are satisfied (the finite generation comes out of being ample on the general fiber, by (3.1.d) and (3.1.e)). In particular, we obtain that \( f_* \mathcal{O}_X (m(K_{X/T} + \Lambda)) \) is weakly positive for every \( m \) divisible enough. To be precise, [Eji15, Thm 5.1] states that \( f_* \mathcal{O}_X (m(K + \Lambda)) \otimes \omega^{-q} \) is weakly positive. However, this is the same statement as the weak positivity of \( f_* \mathcal{O}_X (m(K_{X/T} + \Lambda)) \), as weak positivity is decided on any big open sets and over the regular locus of \( T \), the two sheaves are the same (see Section 2.8.2).

By definition, the above weak positivity means that for a fixed ample divisor \( A \) on \( T \) there is an integer \( b > 0 \) such that \( \text{Sym}^b (f_* \mathcal{O}_X (m(K_{X/T} + \Lambda)) \otimes \mathcal{O}_S (bA)) \) is generically globally generated [Eji15, Def 4.2]. However, then the following natural composition homomorphism (which is non-zero for \( m > 0 \) divisible enough by the relative ampleness along the general...
fibers) shows that $K_{X/T} + \Lambda + \frac{1}{m} f^*A$ is $\mathbb{Q}$-effective:

\[(3.1.f) \quad f^* \left( \text{Sym}^b(f_*\mathcal{O}_Y(m(K_{X/T} + \Lambda))) \otimes \mathcal{O}_T(bA) \right) \rightarrow f^* \left( f_*\mathcal{O}_Y(bm(K_{X/T} + \Lambda)) \otimes \mathcal{O}_T(bA) \right) \]

Obtained by applying $f^* (\_ \otimes \mathcal{O}_T(bA))$ to $\text{Sym}^b(f_*\mathcal{O}_Y(m(K_{X/T} + \Lambda))) \rightarrow f_*\mathcal{O}_Y(bm(K_{X/T} + \Lambda))$. The latter is constructed by noting that $f_*\mathcal{O}_Y(bm(K_{X/T} + \Lambda))$ is reflexive, because $f$ is equidimensional. Hence the multiplication map $\text{Sym}^b(f_*\mathcal{O}_Y(m(K_{X/T} + \Lambda))) \rightarrow f_*\mathcal{O}_Y(bm(K_{X/T} + \Lambda))$ factors through $\text{Sym}^b(f_*\mathcal{O}_Y(m(K_{X/T} + \Lambda)))$.

\[ \cong f^* f_* \mathcal{O}_X(bm(K_{X/T} + \Lambda) + f^*bA) \rightarrow \mathcal{O}_X(bm(K_{X/T} + \Lambda) + f^*bA). \]

(projection formula, as $A$ is Cartier)

As $\Gamma_v$ is effective, $K_{X/T} + \Lambda + \Gamma_v + \frac{1}{m} f^*A$ is also $\mathbb{Q}$-effective. As this is true for every $m > 0$ divisible enough, $K_{X/T} + \Lambda + \Gamma_v$ is pseudo-effective. This concludes the main case of the lemma.

For case $(\ast)$ note that we only have to prove that on every horizontal curve $C$ on $X$ we have $(K_{X/T} + \Lambda + \Gamma_v) \cdot C \geq 0$. However, by the $f$-ample assumption and by the generic global generation of $\text{Sym}^b(f_*\mathcal{O}_Y(m(K_{X/T} + \Lambda))) \otimes \mathcal{O}_T(bA)$, the homomorphism of (3.1.f) is generically surjective over $C$ for every $q > 0$ divisible enough. Hence,

\[ (K_{X/T} + \Lambda + \Gamma_v + \frac{1}{m} f^*A) \cdot C \geq (K_{X/T} + \Lambda + \frac{1}{m} f^*A) \cdot C \geq 0 \]

for every $m > 0$ divisible enough. In particular, $(K_{X/T} + \Lambda + \Gamma_v) \cdot C \geq 0$, which concludes also Case $(\ast)$. \hfill \Box

\section{Flatness}

The proof of the next theorem closely follows the argument of [LTZZ10, Theorem], with little modifications. Our main contribution is the realization that there exists a semi-positivity statement in positive characteristic (Theorem 3.1) that is needed to implement the argument of [LTZZ10, Theorem] in positive characteristic.

\textbf{Theorem 4.1.} Let $f : (X, \Delta) \rightarrow T$ be a surjective fibration from a normal projective pair to a smooth normal variety such that the general fibers are normal. $-(K_{X/T} + \Delta)$ is a nef $\mathbb{Q}$-Cartier divisor and either

1. $(X_t, \Delta_t)$ is strongly $F$-regular for $t \in T$ general, or
2. $(X_t, \Delta_t)$ is sharply $F$-pure for $t \in T$ general and the the Cartier index of $K_{X/T} + \Delta$ is prime to $p$ over some non-empty open set of $T$.

Then $f$ is equidimensional.

\textbf{Proof.} By base extension we may assume that $k$ is algebraically closed. We do this because at some point we will take general hyperplane sections.

If $f$ is equidimensional, then there is nothing to prove. Hence, we may assume that $f$ is not equidimensional.

We claim that we may also assume that there is a closed point $t_0 \in T$ such that $\text{codim}_X f^{-1}(t_0) = 1$. Suppose that there is no $t_0$ as in the claim. Choose then $t_0$ with $\text{codim}_X f^{-1}(t_0)$ minimal. Let $T'$ be a general hypersurface of $T$ through $t_0$, and let $X'$ be the normalization of the scheme theoretic preimage $f^{-1}(T')$. The above claim then follows from our second claim that $f' : X' \rightarrow T'$ satisfies the assumptions of the theorem, with adequately chosen boundary $\Delta_{X'}$ (made explicit below) and with $\text{codim}_{X'} (f')^{-1}(t_0) = \text{codim}_X f^{-1}(t_0) - 1$. First we note that $Z := f^{-1}(T')$ is irreducible with normal general fibers: let $\text{Exc}(\_)$ denote the locus where
the fibers of the given morphism do not have the expected dimension, which is a closed set [Har77, Exc. II.3.22.d], and let $g : Z \rightarrow T'$ be the induced morphism. Then, by Krull's Hauptidealsatz and the genericity of $T'$, $Z$ is equidimensional with $\dim Z = \dim X - 1$. Additionally,

$$(4.1.a) \quad \dim \left( \text{Exc}(g) \setminus g^{-1}(t_0) \right) \leq \dim \text{Exc}(f) - 1 < \dim X - 1 = \dim Z,$$

and

$$(4.1.b) \quad \dim g^{-1}(t_0) = \dim f^{-1}(t_0) < \dim X - 1 = \dim Z.$$
\((D\) cannot have divisorial image in \(T\) because then \(E\) would intersect \(X \setminus \text{Exc}(f)\)). By the smoothness of \(T\), \(\text{coeff}_D \Sigma < 0\), and as \(E\) maps to a divisor of \(X\), \(\text{coeff}_E \Theta \geq 0\). Hence,

\[
(4.1.f) \quad \text{coeff}_E \Delta_Y = \text{coeff}_E \Theta - \text{coeff}_E h^* \Sigma > 0.
\]

It follows that, we may write

\[
(4.1.g) \quad \Delta_Y = \Gamma + H - G,
\]

where \(\Gamma, G\) and \(H\) are effective, and \(\Gamma\) agrees with \(\Delta_Y\) over general fibers, \(G\) is \(\rho\)-exceptional and \(H\) is the part of \(\Delta_Y\) that is supported on the strict transforms of the divisorial components of \(\text{Exc}(f)\). Furthermore, by \((4.1.c)\) and \((4.1.f)\), \(H \neq 0\). Theorem 3.1 then yields that the following divisor is pseudo-effective:

\[
(4.1.h) \quad K_{Y/S} + \Gamma + (-\rho^*(K_{X/T} + \Delta)) \sim_{\mathbb{Q}} -H + G
\]

By intersecting with the preimage of a general complete intersection curve on \(X\), we see that \(-H + G\) is not pseudo-effective. This is a contradiction. \(\square\)

We also provide a general proposition concerning reducedness of fibres avoiding the statements on relative \(F\)-splitting of the Albanese morphism.

**Proposition 4.2.** Let \(f: (X, \Delta) \to T\) be a flat fibration from a projective normal pair to a smooth variety such that the divisor \(-K_{X/T} - \Delta\) is \(\mathbb{Q}\)-Cartier and nef, and the general fibre \((X_t, \Delta_t)\) is strongly \(F\)-regular. Then the following statements hold true:

1. every geometric fibre of \(f\) is reduced,
2. the components of \(\Delta\) do not contain any irreducible components of the fibers of \(f\).

**Proof.** We first prove (1). For the sake of contradiction, we assume that \(t \in T\) is such that \(f^{-1}(t)\) is non-reduced. We choose a general curve \(C\) passing through \(t\) and intersecting the strongly \(F\)-regular locus of \(f\). There is a finite map \(\tau: S \to C\) from a smooth projective curve such that \(Z\) is the normalization of \(X_S\), then \(g: Z \to S\) has reduced fibers \([dJS03\text{, Prop 2.1}]\). Let \(\Delta_Z\) be the induced crepant boundary on \(Z\) given by \([CP18\text{, Prop 2.1}]\). The construction of \([CP18\text{, Prop 2.1}]\), is based on \([KP17, \text{Lem 9.13}]\), and in particular \(\Delta_Z\) contains the divisor of the map \(\omega_{Z/S} \hookrightarrow \beta^* \omega_{X_S/S}\) induced by the Grothendieck trace \(\beta_* \omega_{Z/S} \to \omega_{X_S/S}\), where \(\beta: Z \to X_S\) is the natural morphism. Using that \(S\) is Gorenstein this is the same as the map \(\omega_Z \to \beta^* \omega_{X_S}\) induced by the Grothendieck trace \(\beta_* \omega_Z \to \omega_{X_S}\). However, the divisor of \(\omega_{Z} \hookrightarrow \beta^* \omega_{X_S}\) is exactly the divisorial part of the conductor for \(G_1\) schemes \([PW17, \text{Lem 2.14}]\) and \(X_S\) is \(G_1\), because \(X\) is \(G_1\), hence by the smoothness of \(T\), \(X\) is also relatively Gorenstein over \(T\) in codimension 1, and hence so is \(X_S\) over \(S\), which finally yields that \(X_S\) is \(G_1\) by the smoothness of \(T\).

So, \(\text{Supp} \Delta_Z\) includes the conductor of \(Z \to X_S\), and as \(X_S\) has at least one non-reduced fiber, this conductor has to contain at least one vertical divisor. Hence, \(\Delta_Z \geq E\), where \(E\) is a non-zero effective vertical \(\mathbb{Q}\)-divisor. Then, the following equation shows that Theorem 3.1
implies that the divisor $-E$ is pseudo-effective.

\[(4.2.i) \quad K_{Z/S} + (\Delta_Z - E) + (-K_{Z/S} + \Delta_Z) \equiv -E \]

This is a contradiction. Hence, the fibers of $f$ are indeed reduced.

In order to prove (2) we reason as above to reduce the situation to the case where the base schemes $S$ is a curve, and then use the same argument using an equivalent of (4.2.i). \[\square\]

5. Numerical flatness

In the following chapter we prove the main technical result of this paper concerning numerical flatness of relative section rings.

**Notation 5.1.** Let $f : (X, \Delta) \to T$ be a fibration from a normal pair of dimension $d + 1$ to a smooth projective curve with normal general fiber and with $-K_X/T - \Delta$ a nef $\mathbb{Q}$-Cartier divisor. Let us assume additionally that the general fibers $(X_t, \Delta_t)$ are strongly $F$-regular. (Note that this requirement for general fibers is equivalent to the analogous requirement for the geometric generic fiber by [PSZ18, Theorem B])

Let $\mathcal{L}$ be a very ample divisor on $X$ such that $R^i f_* \mathcal{O}_X (\mathcal{L}) = 0$ for every $i > 0$, and let $G$ be a general fiber of $f$. By Lemma 5.2, we have $L^{d+1} = 0$ for $L := n\mathcal{L} - mG$, where $n = (d + 1)L^d_t$ and $m = N^{d+1}$. This ends the description of our notation.

**Lemma 5.2.** Let $X$ be a projective variety of dimension $d + 1$, let $f : X \to T$ be a fibration onto a curve $T$, let $G$ be the general fiber of the fibration, and let $N$ be an ample Cartier divisor on $X$. Then $(nN - mG)^{d+1} = 0$ for $n = (d + 1)N^d_t$ and $m = N^{d+1}$.

**Proof.** We have

\[(5.2.a) \quad (nN - mG)^{d+1} = n^{d+1} N^{d+1} - (d + 1)n^d mN^d \cdot G + \sum_{2 \leq i \leq d+1} \binom{d+1}{i} n^{d+1-i} m^i N^{d+1-i} \cdot G^i \]

\[
\uparrow \quad \uparrow
\]

\[
G^i = 0, \text{ for } i \geq 2 \quad N^d \cdot G = N^d_t
\]

The next lemma is shown in [CP18, Lem A.2 & Rem A.3] in characteristic zero. Unfortunately, the argument there is based on a Grothendieck–Riemann–Roch argument, and hence cannot be performed without passing to a resolution of singularities of the total space. Hence, here we have to make an alternative argument using the weaker statement of asymptotic Riemann–Roch [Kol96, Thm IV.2.15], which works for singular schemes of any characteristic. An unfortunate side-product is that the argument is a bit more cumbersome than that of [CP18, Lemma A.2].

**Lemma 5.3.** If $f : X \to T$ is a projective morphism from a variety of dimension $d + 1$ to a smooth projective curve and $L$ is an $f$-ample $\mathbb{Q}$-Cartier divisor on $X$, then for integers $m > 0$ divisible enough,

\[h^0(X, mL) \geq \frac{m^{d+1}}{(d + 1)!} L^{d+1} + O \left( m^d \right) .\]
**Proof.** The main issue solved during the argument is that [Kol96, Thm IV.2.15] does not apply directly to \( L \) as it is not necessarily nef. Nevertheless, it is \( f \)-ample, and hence there is \( \mathbb{Q} \ni a > 0 \) such that \( L + af^*H \) is nef, where \( H \) is an ample divisor on \( T \). So, we have to apply [Kol96, Thm IV.2.15] to \( L + af^*H \), and then we have to connect on the both ends of the following computation the invariants of \( L \) to those of \( L + af^*H \). First, we do the part of the computation pertaining to \( L + af^*H \). That is, for every integer \( m > 0 \) divisible enough we have:

\[
\chi(m(L + af^*H)) = \chi(f_*\mathcal{O}_X(m(L + af^*H))) = \chi(\mathcal{O}_T(maH) \otimes f_*\mathcal{O}_X(mL))
\]

Then, yields the above version with \( \chi(f_*\mathcal{O}_X(mL)) + ma(\deg H)(\text{rk } f_*\mathcal{O}_X(mL)) \)

This then yields the statement by the following computation:

\[
\begin{align*}
\frac{m^{d+1}}{(d+1)!}L^{d+1} & = \frac{m^{d+1}}{(d+1)!}(L + af^*H)^{d+1} - \frac{m^{d+1}}{d!}L^d_a \deg H + O(m^d) \\
& = \chi(m(L + af^*H)) - \frac{m^{d+1}}{d!}L^d_a \deg H + O(m^d) \\
& = \chi(f_*\mathcal{O}_X(mL)) + ma(\deg H)(\text{rk } f_*\mathcal{O}_X(mL)) - \frac{m^{d+1}}{d!}L^d_a \deg H + O(m^d) \\
& = \chi(f_*\mathcal{O}_X(mL)) + O(m^d) \leq h^0(T, f_*\mathcal{O}_X(mL)) + O(m^d) = h^0(X, mL) + O(m^d)
\end{align*}
\]

This then yields the statement by the following computation:

\[
\begin{align*}
\frac{m^{d+1}}{(d+1)!}L^{d+1} & = \frac{m^{d+1}}{(d+1)!}(L + af^*H)^{d+1} - \frac{m^{d+1}}{d!}L^d_a \deg H + O(m^d) \\
& = \chi(m(L + af^*H)) - \frac{m^{d+1}}{d!}L^d_a \deg H + O(m^d) \\
& = \chi(f_*\mathcal{O}_X(mL)) + ma(\deg H)(\text{rk } f_*\mathcal{O}_X(mL)) - \frac{m^{d+1}}{d!}L^d_a \deg H + O(m^d) \\
& = \chi(f_*\mathcal{O}_X(mL)) + O(m^d) \leq h^0(T, f_*\mathcal{O}_X(mL)) + O(m^d) = h^0(X, mL) + O(m^d)
\end{align*}
\]

**Theorem 5.4.** *In the situation of Notation 5.1, \( L \) is nef.*

**Proof.** First, note that for any \( 0 < \varepsilon \ll 1 \), the following is true: there is a \( \varepsilon' > 0 \), depending on \( \varepsilon \), such that for all \( t \in T \) general (where generality also depends on \( \varepsilon \)) and for all \( \Gamma \in |L_t| \), \( (X_t, \Delta_t + \varepsilon'\Gamma) \) is strongly F-regular. Indeed, the above statement is clear for \( t \) and \( \Gamma' \) fixed. Then, the openness of strong F-regularity [PSZ18, Theorem B] and Noetherian induction yields the above version with \( t \) and \( \Gamma' \) not fixed.

So, fix \( \varepsilon > 0 \). Consider \( L + \varepsilon f^*H \). It is enough to prove, by limiting \( \varepsilon \to 0 \), that \( L + \varepsilon f^*H \) is nef. Set \( N := q(L + \varepsilon f^*H) \), for \( q \gg 0 \). By Lemma 5.3,

\[
h^0(X, N) \geq q^{d+1} \frac{(L + \varepsilon f^*H)^{d+1}}{(d+1)!} + O(q).
\]

As

\[
(L + \varepsilon f^*H)^{d+1} = \varepsilon(d+1)(\deg H)(L^n_t) > 0,
\]

we obtain that \( h^0(X, N) \neq 0 \) (using \( q \gg 0 \)). So, choose \( \Gamma \in |N| \). Then, \( (X_t, \Delta_t + \varepsilon'\Gamma_t) \) strongly F-regular for \( t \in T \) general. Hence, according to Theorem 3.1, the following divisor
is nef
\[
\frac{K_{X/T} + \Delta + \epsilon T}{(X_t, \Delta_t + \epsilon T_t)} \text{ is } \text{f-ample} \quad \text{for} \quad t \in T \text{ general} \\
\frac{-(K_{X/T} + \Delta)}{\text{is nef}} = \epsilon' q(L + \epsilon f^* H)
\]
Hence \( L + \epsilon f^* H \) is nef, which concludes our statement. \( \square \)

Recall that a vector bundle \( E \) on a smooth, projective curve \( T \) is strongly semi-stable if \((F^t)^* E\) is semi-stable for all integers \( n \geq 0 \).

**Notation 5.5.** Let \( E \) be a vector bundle on a projective curve \( T \). Let
\[
0 = E_{l,0} \subseteq E_{l,1} \subseteq \ldots E_{l,s_l} = F^{l,s_l} E.
\]
be the Harder–Narasimhan filtration of \( F^{l,s} E \). Then, it is shown [Lan04, Claim 2.7.1] that the sequences \( \mu_{\max}(F^{l,s} E) \) and \( \mu_{\min}(F^{l,s} E) \) are stabilizing and for \( l \gg 0 \), the quotients \( E_{l,i}/E_{l,i-1} \) are strongly semi-stable for all \( i \). So one defines \( L_{\max}(E) \) and \( L_{\min}(E) \) as the stable values of the above sequences.

**Lemma 5.6.** Let \( E \) be a vector bundle on a smooth, projective curve \( X \). Using Notation 5.5, the following are equivalent

1. \( E \) is nef
2. \( L_{\min}(E) \geq 0 \)

and similarly

1. \( E^* \) is nef
2. \( L_{\max}(E) \leq 0 \)

**Proof.** We use Notation 5.5 throughout the proof. Furthermore, we are using the following definition of nefness: \( E \) is nef, if for every finite morphism \( \tau : Y \to X \) from a smooth projective curve, and every quotient line bundle \( \tau^* E \to L \), \( \deg L \geq 0 \). Also, we show only the first equivalence, as the second equivalence follows from it by applying \((F^{l,s} E)^* \equiv F^{l,s} E^*\).

Now, assume that \( E \) is nef. Then, so is \( F^{l,s} E \) for every \( l \gg 0 \), and so are all the quotient bundles of \( F^{l,s} E \) (for example using the above definition). However, \( F^{l,s} E/E_{l,s_l-1} \) is a quotient of \( F^{l,s} E \) with slope equal to \( p/l \min(E) \) for \( l \gg 0 \). This shows that \( \min(E) \geq 0 \).

In the other direction, assume that \( L_{\min} \geq 0 \), and let \( \tau : Y \to X \) be a finite morphism from a smooth projective curve, and \( \tau^* E \to L \) a quotient line bundle. For proving that \( \deg L \geq 0 \) we may freely pull-back further this quotient. Hence, we may assume that \( \tau \) factors through \( F^l \) for \( l \gg 0 \) as:

\[
\begin{array}{c}
Y \\
\xrightarrow{\rho} X \\
\xrightarrow{\tau} F^l X
\end{array}
\]
Hence, \( L \) will have a non-zero map from \( \rho^* E_{l,i}/\rho^* E_{l,i-1} \cong \rho^* (E_{l,i}/E_{l,i-1}) \) for some \( i \). As \( E_{l,i}/E_{l,i-1} \) is strongly semi-stable with slope at least \( p/l \min(E) \), this implies that \( \deg L \geq 0 \). \( \square \)

\footnote{Zoold: I am sure there is an earlier reference for the curve case which we should mention, Gieseker? I looked through Langer’s references. He says that Gieseker constructed stable non-strongly semi-stable bundles, then Shepherd-Barron was considering destabilizing subsheaves of Frobenius pullbacks (but mainly got the non-triviality of Frobenius descent obstructions). It is Langer’s Prop. 2.4 attributed to Shpeherd-Barron.}
Notation 5.7. We recall our notation for the Frobenius pullback

\[
\begin{array}{c}
\xymatrix{
X^t \ar[r]^-{F^t_X} & X \\
Z := X_{T^t} \ar[d]_-{f_{T^t}} \ar[r]^-{\square} & f \\
S := T^t \ar[r]_-{F^t_{\Delta}} & T.
}
\end{array}
\]

Theorem 5.8. In the situation of Notation 5.1, for any integer \( a > 0 \), \( (f_* O_X(aL))^* \) is nef.

Proof. Assume \( (f_* O_X(aL))^* \) is not nef. According to Lemma 5.6, and using Notation 5.7, this means that for every \( l \gg 0 \),

\[
0 < \mu_{\text{max}} \left( F^l_* f_* O_X(aL) \right) = \mu_{\text{max}} \left( g_* O_X(\alpha L_{T^t}) \right)
\]

Furthermore, as \( l \gg 0 \), we also have that

\[
\mu_{\text{max}} \left( g_* O_X(\alpha L_{T^t}) \right) = \mu(E),
\]

where \( E \) is a strongly semi-stable subbundle of \( g_* O_X(\alpha L_{T^t}) \). According to [Lan04, Thm 6.1], \( E^{\otimes r} \) is semi-stable with slope \( r \mu(E) \). In particular, we may fix \( r > 0 \) such that \( \mu(E^{\otimes r}) > 2g \left( T^t \right) + 1 = 2g(T) + 1 \). Consider then the composition

\[
E^{\otimes r} \twoheadrightarrow (g_* O_X(\alpha L_{T^t}))^{\otimes r} \to g_* O_X(ra L_{T^t}).
\]

This is non-zero (diagonal tensors do not go to zero). So, we obtain that

\[
\mu_{\text{max}} \left( g_* O_X(ra L_{T^t}) \right) > 2g \left( T^t \right) + 1 = 2g(T) + 1.
\]

Hence for any closed point \( t \in T^t \):

\[
0 \neq H^0 \left( T^t, (g_* O_X(ra L_{T^t}))(-t) \right) = H^0(ra L_{T^t} - X_t).
\]

Choose \( 0 \neq \Gamma \in ra L_{T^t} - X_t \). Let \( Z \) be the normalization of \( X_{T^t} \), and let \( \Delta_Z \) be the induced crepant boundary given by [CP18, Prop 2.1] (where the general characteristic zero assumption of [CP18] is not used). That is, if \( \rho : Z \to X \) is the induced morphism, then we have

\[
K_{Z/T^t} + \Delta_Z \sim_{Q} \rho^* (K_{X/T} + \Delta).
\]

Additionally, by [CP18, Prop 2.1.(iii)], over the locus where \( f \) has normal fibers, \( X_{T^t} \) is already normal, and \( \Delta_Z \) just the usual pullback of \( \Delta \). Hence, over this locus the fibers \( (Z_t, (\Delta_Z)_t) \) are strongly \( F \)-regular. However, then for any \( 0 < \varepsilon \ll 1 \), \( (Z_t, (\Delta_Z)_t + \varepsilon \Gamma_t) \) is strongly \( F \)-regular pair for \( t \in T \) general, where the notion of generality depends on \( \varepsilon \) [PSZ18, Theorem B]. In particular, according to Theorem 3.1, the following divisor is nef:

\[
\varepsilon \Gamma \sim_{Q} \underbrace{K_{Z/T^t} + \Delta_Z + \varepsilon \Gamma}_{\text{is strongly } F\text{-regular for } t \in T \text{ general}} \underbrace{+ (- (K_{Z/T^t} + \Delta_Z))}_{\text{f-ample}}
\]

\[\]
However, then
\[
\Gamma^{d+1} = (raL_T^t - Z_t)^{d+1} = -(n + 1)raL_t^d < 0,
\]
which contradicts nefness of \(\Gamma\). Hence, our initial assumption was false, which concludes our proof.

We leave the proof of the following lemma to the reader.

**Lemma 5.9.** Let \(E\) be a vector bundle on a smooth projective curve \(T\). Then, \(E\) is nef if and only if for every integer \(e > 0\), \(((F^e)^* E)(t)\) is nef for some closed point \(t \in T\).

Recall that a vector bundle \(E\) on a projective scheme \(X\) is called numerically flat, if both \(E\) and \(E^*\) are nef (see Definition 2.3).

**Theorem 5.10.** In the situation of Notation 5.1, \(F := f_*\mathcal{O}_X(mL)\) is numerically flat for every integer \(m > 0\).

**Proof.** By base extension we may assume that \(k\) is algebraically closed. We do this, because at some point we need to use [Pat14, Prop 3.6], which is stated over algebraically closed fields.

By Theorem 5.8, \(F^*\) is nef. Furthermore, using Theorem 5.4, \(L\) is nef. Hence, if the Cartier index of \(K_{X/T} + \Delta\) is prime to \(p\), we may apply [Pat14, Prop 3.6] to obtain that \(\mathcal{F}\) is nef, because
\[
K_{X/T} + \Delta + (- \langle (K_{X/T} + \Delta) \rangle + mL) \equiv mL.
\]
However, if the above index is divisible by \(p\), we need a subtle perturbation argument. This is allowed by the fact that \((- \langle (K_{X/T} + \Delta) \rangle + mL)\) is \(f\)-ample and nef, and hence a perturbation coming from the base makes it ample. The precise argument is as follows.

First, we see that by Proposition 4.2 all the fibres of \(f\) are reduced. We now approach the nefness of \(\mathcal{F}\). According to Lemma 5.9 it is enough to show that for every \(e > 0\), \(((F^e)^* F)(t)\) is nef for some closed point \(t \in T\). So, fix an integer \(e > 0\). By our claim the fibers of \(g : V := X_{T^e} \to T^e\) are reduced. As we also assume that the general fibers are normal, it follows that \(V\) is normal. Then, for \(\rho : V \to X\) we have
\[
(F^e)^* F \cong g_* \mathcal{O}_V(m\rho^* L).
\]
Choose \(t \in T^e\). Then \(V_t + m\rho^* L\) is ample. According to [Pat14, Lem 3.15], we may choose an ample divisor \(A\) on \(Z\), a sequence \(0 < \varepsilon_n \to 0\), and \(\Gamma \in |K_Z + A|\), such that \((V, \Delta_V + \varepsilon_n(\Gamma + \Delta_V))\) has index prime-to-\(p\) and \((V_t, (\Delta_V + \varepsilon_n(\Gamma + \Delta_V))_t)\) is strongly \(F\)-regular for general closed point \(t \in T^e\). Then, for \(n > 0\), [Pat14, Prop 3.6] implies that the following sheaf is nef
\[
g_* \mathcal{O}_V\left(\left(V_t, (\Delta_V + \varepsilon_n(\Gamma + \Delta_V))_t\right)\right)\text{ is strongly }\ F\text{-regular for general closed point } t \in T^e, \text{ and Cartier index is prime to } p
\]
\[
\cong g_* \mathcal{O}_V(m\rho^* L + V_t - \varepsilon_n(\Gamma + \Delta_V)) \cong g_* \mathcal{O}_V(m\rho^* L + V_t) \cong ((F^e)^* F)(t)
\]
This concludes our proof.

6. **Singularities of the General Fibers of Sharply \(F\)-Pure Fibrations**

In this section, we prove a general statement roughly saying that if a variety \(X\) that has strongly \(F\)-regular singularities is fibered over \(T\) such that the general fiber \(X_{t_{\text{gen}}}\) is normal and sharply \(F\)-pure, then \(X_{t_{\text{gen}}}\) is in fact strongly \(F\)-regular. This statement is local on \(X\).
Nevertheless, we would like to use it primarily in the following global situation: $f: X \to T$ is the Albanese morphism and $X$ is globally $F$-split. To apply in this situation the results of Section 4 and Section 5, we do need to prove that the general fibers are strongly $F$-regular (see the assumptions of the statements in the above sections). This is the main motivation for the results of the present section. Note that even if we assume that $X$ is smooth we do have to deal with the above issues because the results of [Eji17] only tell us even in this case that the general fibers are normal and $F$-pure.

We begin by recalling the necessary statements from [ST14] where the authors analyze the behaviour of test ideals under finite maps. First, mimicking the approach from loc.cit (see Setting 6.19 therein), we describe the setting in which we work.

**Setting 6.1.** Let $\pi: Y \to X$ be a surjective finite map between normal varieties over a perfect field $k$ of characteristic $p > 0$. Let $L = \mathcal{O}_Y(L)$ be a line bundle on $Y$ and let $\mathfrak{T}: \pi_* \mathcal{L} \to \mathcal{O}_X$

be a non-zero homomorphism of $\mathcal{O}_X$-modules (we call it the trace). By the Grothendieck duality (on the normal locus of $X$) for finite maps we obtain:

$$
(6.1.a) \quad \mathcal{H}om(\pi_* \mathcal{L}, \mathcal{O}_X) \simeq \mathcal{H}om(\pi_* \mathcal{L}, \omega_X/k)[\otimes] \omega_X^* \simeq \pi_* \mathcal{H}om(\mathcal{L}, \omega_Y/k)[\otimes] \omega_X^*
$$

$\mathcal{H}om(\pi_* \mathcal{L}, \mathcal{O}_X)$ is reflexive, hence we may use projection formula on the regular locus of $X$ and then we can extend the isomorphism using reflexivity, Grothendieck duality for $\pi$

$$
\simeq \pi_* \left( \mathcal{L}^* \otimes \omega_Y/k[\otimes] \pi^*[\omega_X^*/k] \right) = \pi_* \mathcal{O}_Y(-L + K_Y - \pi^* K_X).
$$

Consequently, the homomorphism $\mathfrak{T} \in \text{Hom}(\pi_* \mathcal{L}, \mathcal{O}_X)$ corresponds to an effective divisor

$$
R_{\mathfrak{T}} \equiv -L + K_Y - \pi^* K_X.
$$

In fact, if we quotient out by multiplication by a unit on the homomorphism side, this correspondence becomes 1-to-1 by (6.1.a).

Choose now an affine open set $U \subseteq X$ and a trivialization $\xi: \mathcal{L}|_{f^{-1}U} \simeq \mathcal{O}_{f^{-1}U}$. This yields a homomorphism

$$
\mathfrak{T}_\xi: \pi_* \mathcal{O}_{f^{-1}U} \xrightarrow{\mathfrak{T}_{\mid_U \circ \pi_* (\xi^{-1})}} \mathcal{O}_U
$$

It follows from the above definition that $R_{\mathfrak{T}_\xi} = R_{\mathfrak{T}}|_U$.

Next, we cite an immediate consequence of [ST14, Theorem 6.25]. This consequence can be obtained by applying [ST14, Theorem 6.25] to $\Sigma_U$, where $\{U\}$ is an open cover of $X$ with trivializations $\zeta_U$ of $\mathcal{L}$ over $f^{-1}U$. More precisely, $\Sigma_{\zeta}$ is a homomorphism $\pi_* \mathcal{O}_{f^{-1}U} \to \mathcal{O}_U$, which extends uniquely to a $K(X)$-module homomorphism $\pi_* K(Y) \to K(X)$. We apply [ST14, Theorem 6.25] to the latter over $U$. Doing this over each element $U$ of our chosen open cover yields the following global statement:

**Theorem 6.2 ([ST14, Theorem 6.25]).** Let $\pi: Y \to X$ and $\mathfrak{T}: \pi_* \mathcal{L} \to \mathcal{O}_X$ be as in Setting 6.1, and set $\Delta_{Y/\mathfrak{T}} = \pi^* \Delta_X - R_{\mathfrak{T}}$. Then we have the following relation between the respective test ideals:

$$
\mathfrak{T} \left( \pi_* (\tau(Y, \Delta_{Y/\mathfrak{T}}, \mathcal{L})) \right) = \tau(X, \Delta_X).
$$

In particular, if the pair $(Y, \Delta_{Y/\mathfrak{T}})$ is strongly $F$-regular and $\mathfrak{T}$ is surjective then $(X, \Delta_X)$ is strongly $F$-regular too.
**Theorem 6.3.** Let \( f : (X, \Delta) \to T \) be a fibration from a normal to a regular variety such that \( K_X + \Delta \) has Cartier index prime-to-\( p \), \((X, \Delta)\) is strongly \( F \)-regular and the geometric generic fiber \((X_\eta, \Delta_\eta)\) of \( f \) is normal and \( F \)-pure, then \((X_\eta, \Delta_\eta)\) is strongly \( F \)-regular.

**Proof.** For the entire proof we use the notation associated to iterated Frobenius pullbacks introduced in Notation 5.7, with the only difference being that we use \( e \) instead of \( l \) for the index of the iteration.

First, we observe that by [PSZ18, last 3 lines of Thm A], it suffices to prove that for every divisible enough integer \( e > 0 \) the Frobenius pullback \((X_{T^e}, \Delta_{T^e})\) is strongly \( F \)-regular in a neighborhood of the generic fiber. So, from now, \( e > 0 \) is always any divisible enough integer. In particular, \( L := \mathcal{O}_X \left( (1 - p^e) \left( K_{X/T} + \Delta \right) \right) \) is a line bundle. We consider now the relative logarithmic Grothendieck trace, one of the definitions ([PSZ18, Def 2.8-Rem 2.11]) of which is that it is the \( \mathcal{O}_X \)-linear homomorphism

\[
\varphi^e_{X/T, \Delta} : F^e_{X/T} \ast L \to \mathcal{O}_{X_T^e},
\]
such that, using the language of Setting 6.1, we have

\[
(6.3.b) \quad R^{\varphi^e_{X/T, \Delta}} = (p^e - 1)\Delta.
\]

According to [PSZ18, Lemma 2.16 & 1st-3rd paragraph of Sec 2.3], using that \( \overline{k(\eta)}^{1/p^e} = k(\eta) \), we may identify the geometric generic fiber of \( \varphi^e_{X/T, \Delta} \) with the trace for the geometric generic fiber, that is, we have:

\[
(6.3.c) \quad \left( \varphi^e_{X/T, \Delta} \right)_\pi = \varphi^e_{X_\pi, \Delta_\pi}.
\]

Since \((X_\eta, \Delta_\eta)\) is sharply \( F \)-pure, we see that \( \varphi^e_{X_\eta, \Delta_\eta} \) is surjective and hence, after potentially shrinking \( T \), we may assume that \( \varphi^e_{X/T, \Delta} \) is surjective too. Also, by possibly shrinking \( T \), as \( X_\eta \) is normal, we may assume that \( X_{T^e} \) is normal. Consequently, we may apply Theorem 6.2 for \( X, Y, \pi \) and \( \Delta_X \) of that theorem set to be \( X_{T^e}, X, F^e_{X/T}, \Delta_{T^e} \) of the present proof, and by setting additionally \( L := \mathcal{O}_X \left( (1 - p^e) \left( K_{X/T} + \Delta \right) \right) \) and \( \mathfrak{T} := \varphi^e_{X/T, \Delta} \). By a direct computation in the context of Setting 6.1 we verify that

\[
\Delta_{X/\varphi^e_{X/T}} = \left( F^e_{X/T} \right)^* (\Delta_{T^e}) - R^{\varphi^e_{X/T, \Delta}} = (F^e)^* \Delta - (p^e - 1)\Delta = p^e \Delta - (p^e - 1)\Delta = \Delta.
\]

(6.3.b)

This means that the test ideal \( \tau(X_{T^e}, \Delta_{T^e}) = \mathcal{O}_{X_{T^e}} \) and hence \((X_{T^e}, \Delta_{T^e})\) is strongly \( F \)-regular. This concludes our proof. \( \square \)

**7. The Log-Isom Scheme**

In this section we define precisely the variants of Isom schemes that we use in this article. The main issue is defining the corresponding logarithmic Isom scheme, i.e., how one interprets in a modular way the restriction that in the logarithmic setting the boundary divisor should be invariant under automorphisms. Modular here means that the construction should be compatible with base-change. This compatibility for closed points and flat covers of the base is essential for our applications. Nevertheless, the present section is admittedly technical, and it is reasonable to skip it for the first read.

One main issue is to deal with the changing of the coefficients after restrictions. For example one could have coefficients 1/2 and 1 in the boundary which become all 1 after either restricting to a fiber or an inseparable base-change. In particular, it can happen that some components of the boundary were not allowed to be swapped by an automorphism originally, but they are allowed after the base-change. The only way we are able to remedy this issue is to assume that the log-canonical divisors are \( \mathbb{Q} \)-Cartier. That is, given a pair \((X, \Delta)\) such that \( m(K_X + \Delta) \) is Cartier, we encode \((X, \Delta)\) by the induced homomorphism...
we consider the following moduli functors. This is similar to the approach presented in [KP17, Sec. 6], the only significant difference being that the polarization was log canonical, but here it is given by a line bundle $\mathcal{L}$ independent of the log canonical divisor. We note that there is also another approach based on the work of Kollár [Kol19].

**Setting 7.1.** Consider the following situation:

1. $f^r : X^r \to T$ (for $r = 1, 2$) are two flat families of geometrically normal projective varieties of dimension $n$ over a normal, Noetherian base,

2. $\mathcal{L}_r$ are $f^r$-ample line bundles on $X^r$,

3. $\Delta_r$ are effective divisors on $X^r$, such that
   1. no irreducible component of any fiber $X^r_t$ is contained in $\text{Supp} \Delta_r$, and
   2. $K_{X^r_r/T} + \Delta_r$ is Cartier

4. we fix an integer $m > 0$, such that $m(K_{X^r_r/T} + \Delta_r)$ is Cartier for both $r = 1$ and $2$, we set $\mathcal{M}_r := \mathcal{O}_{X^r}(m(K_{X^r_r/T} + \Delta_r))$, and we fix induced homomorphisms $\iota_{\Delta_r, m} : \omega_{X^r_r/T}^{\otimes m} \to \mathcal{M}_r$

5. let $U^r \subseteq X^r$ be the open set where $f^r$ is smooth. By assumption (1) $U^r$ is a relatively big open set over $T$.

**Definition 7.2.** In the situation of Setting 7.1, we consider the following moduli functors $\mathcal{S} \mathcal{C}_{\mathcal{T}} \to \mathcal{S} \mathcal{E} \mathcal{T}$ from the category of schemes over $T$ to the category of sets. Here, $S$ is any scheme over $T$, and $p_r : X^r \times_T S \to X^r$ is the projection onto the first factor. Also, we define the functors only on the objects of $\mathcal{S} \mathcal{C}_{\mathcal{T}}$. On arrows the functors go to the usual pullback maps.

$$\text{Isom}_T \left( \left( X^1, \mathcal{L}_1 \right), \left( X^2, \mathcal{L}_2 \right) \right)(S) := \left\{ \alpha : X^1 \times_T S \to X^2 \times_T S \mid \alpha \text{ is an isomorphism over } S, \text{ and } \quad \begin{array}{c} \alpha^* \mathcal{L}_1 \cong_S \alpha^* \mathcal{L}_2 \
\text{or}
\end{array} \right\},$$

(7.2.a) $\text{Isom}_T \left( \left( X^1, \Delta_1; \mathcal{L}_1 \right), \left( X^2, \Delta_2; \mathcal{L}_2 \right) \right)(S) := \left\{ \alpha : X^1 \times_T S \to X^2 \times_T S, \quad \xi : \alpha^* (\mathcal{M}_2 \times_T S) \to \mathcal{M}_1 \times_T S \mid \begin{array}{c} \alpha \text{ and } \xi \text{ are an isomorphisms such that} \\
\text{(1) } p_1^* \mathcal{L}_1 \cong_S \alpha^* p_2^* \mathcal{L}_2,
\text{(2) the following diagram commutes:} \\
\begin{array}{ccc}
\alpha^* \omega_{U^2 \times_T \mathcal{S}/S}^{\otimes m} & \xrightarrow{\iota_{\Delta_2, m, U^2 \times_T \mathcal{S}}} & \alpha^* (\mathcal{M}_2 |_{U^2 \times_T S}) \\
\downarrow d (\alpha^* \omega_{U^1 \times_T \mathcal{S}/S}) & & \downarrow \xi |_{U^1 \times_T S} \\
\omega_{U^1 \times_T \mathcal{S}/S}^{\otimes m} & \xrightarrow{\iota_{\Delta_1, m, U^1 \times_T S}} & \mathcal{M}_1 |_{U^1 \times_T S}
\end{array}
\end{array} \right\}.$$

We also set

$$\text{Isom}_T \left( \left( X^1, \Delta_1; \mathcal{L}_1 \right) \right) := \text{Isom}_T \left( \left( X^1, \Delta_1; \mathcal{L}_1 \right), \left( X^1, \Delta_1; \mathcal{L}_1 \right) \right).$$

**Remark 7.3.** We note the following:

1. $\mathcal{L} \cong_S \mathcal{L}'$ for two line bundles $\mathcal{L}$ and $\mathcal{L}'$ on $X^r \times_T S$ means that $\mathcal{L}' \otimes \mathcal{L}^{-1} \cong p_S^* \mathcal{M}$ for some line bundle $\mathcal{M}$ on $S$.

2. For ease of notation we omitted the restrictions to $U^1 \times_T S$ on $\alpha$ in the commutative diagram of (7.2.a).

3. As $U^r$ is the smooth locus of $f^r$, $U^r \times_T S$ is the smooth locus of $f^r \times S T$. In particular, $\alpha(U^1 \times_T S) = U^2 \times_T S$. Hence, we may regard $\alpha$ also as a morphism $U^1 \times_T S \to U^2 \times_T S$ over $S$.

4. As we only deal with the relative canonical bundle of smooth morphisms in the commutative diagram of (7.2.a), we take them to be the particular model $\wedge^{\dim T/S} \Omega_{U^r \times_T S/S}$. 
In particular, the homomorphism $d\left(\alpha_{\mathcal{U}^1\times T}S\right)$ is uniquely defined, not only up to an isomorphism. In particular, if $S' \to S$ is a morphism, then we obtain that $d\left(\left(\alpha_{\mathcal{U}^1\times T}S\right) \times S S'\right) = d\left(\alpha_{\mathcal{U}^1\times T}S\right) \times S S'$.

(5) as $\mathcal{U}^n$ is relatively big, the commutativity of the diagram of (7.2.a) is equivalent to the commutativity of

\[
\begin{array}{ccc}
\omega_{\mathcal{U}^1\times T/S} & \longrightarrow & \alpha^* \left(\mathcal{M}_2|_{\mathcal{U}^2\times T}S\right) \\
\uparrow & & \uparrow \xi \\
\omega_{\mathcal{X}^1\times T/S} & \longrightarrow & \mathcal{M}_1|_{\mathcal{U}^1\times T}S \\
\end{array}
\]

where the unlabeled arrows are the unique extensions of the maps of (7.2.a). We note that similarly to the absolute setting, in the relative setting reflexive sheaves are also equivalent to relatively $S_2$ flat sheaves ([HK04, Prop 3.1 - Cor 3.8] and [PSZ18, Appendix]).

(6) $\text{Isom}_T\left(\left(X^1, \Delta_1; \mathcal{L}_1\right), \left(X^2, \Delta_2; \mathcal{L}_2\right)\right)$ is a slight abuse of notation as it does not only depend on $\Delta$, but on the actual choice of $\iota_{\Delta, m}$. Nevertheless, the different choices of $\iota_{\Delta, m}$ differ by a multiplication by a unit, and hence they induce isomorphic $\text{Isom}$ functors via unique isomorphisms (one composes $\xi$ with the above mentioned units).

**Remark 7.4.** In the situation of Definition 7.2, $\text{Isom}_T\left(\left(X^1, \Delta_1; \mathcal{L}_1\right)\right)$ is a group scheme over $T$, and both $\text{Isom}_T\left(\left(X^1, \Delta_1; \mathcal{L}_1\right)\right)$ and $\text{Isom}_T\left(\left(X^2, \Delta_2; \mathcal{L}_2\right)\right)$ acts on $\text{Isom}_T\left(\left(X^1, \Delta_1; \mathcal{L}_1\right), \left(X^2, \Delta_2; \mathcal{L}_2\right)\right)$.

**Construction 7.5.** Here we describe the construction of the fine moduli space for the functor $\text{Isom}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right)$ of Definition 7.2. This moduli space exists as a quasi-projective scheme over $T$. We start the construction with the functor $\text{Isom}^{\text{pre}}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right)(S)$

\[
:= \left\{ \alpha : X^1 \times T S \to X^2 \times T S | \alpha \text{ is an isomorphism, and for every } m \in \mathbb{Z} \text{ and every } s \in S : \chi\left((p_1^s \mathcal{L}_1 \otimes \alpha^* p_2^s \mathcal{L}_2)^{\otimes m}\right) = \chi\left((p_1^s \mathcal{L}_1^{\otimes 2m})\right) \right\}
\]

According to [Gro62, page 221-20], $\text{Isom}^{\text{pre}}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right)$ exists as a quasi-projective scheme over $T$. Then, one can define a morphism $\text{pol} : \text{Isom}^{\text{pre}}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right) \to \text{Pic}(X_1/T)$

by the following assignment for every $S \in \text{Sch}_T$:

\[
\text{pol}(S) : \text{Isom}^{\text{pre}}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right)(S) \ni \alpha \\
\mapsto p_1^s \mathcal{L}_1 \otimes \alpha^* p_2^s \mathcal{L}_2^{-1} \in \text{Pic}(X_1/T)(S) \left(= \text{Pic}(X_1 \times T S)/\text{Pic}(S)\right).
\]

by definition, [Gro62, Prop V.2.1, p 232-04]

As both $\text{Isom}^{\text{pre}}_T(X, \mathcal{L})$ and $\text{Pic}(X/T)$ are representable [Gro62, Thm V.3.1] [AK80, Thm 6.3] by locally quasi-projective schemes, $\text{pol}$ is in fact a morphism of schemes. Additionally as functors

\[
\text{Isom}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right) \cong \text{Isom}^{\text{pre}}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right) \times_{\text{Pic}(X_1/T)} T.
\]

using the morphism $\text{pol}$ on the left side, and on the right side the morphism $T \to \text{Pic}(X_1/T)$ induced by $\mathcal{O}_{X_1}^T$.

Hence, $\text{Isom}^{\text{pre}}_T\left(\left(X^1, \mathcal{L}_1\right), \left(X^2, \mathcal{L}_2\right)\right)$ is a closed subfunctor of $\text{Isom}^{\text{pre}}_T(X, \mathcal{L})$ and its fine moduli space exists as a scheme and it is given by the same formula as (7.5.b).
Definition 7.2. The fine moduli space for the functor $\text{Isom}_T \left( (X^1, \Delta_1; \mathcal{L}_1), (X^2, \Delta_2; \mathcal{L}_2) \right)$ of Definition 7.2 also exists as a quasi-projective scheme over $T$. To see this, we have to go through a longer construction, a big part of the notation of which is shown on the following diagram.

First, set $I := \text{Isom}_T \left( (X^1, \mathcal{L}_1), (X^2, \mathcal{L}_2) \right)$, and let $\beta : X^1 \times_T I \to X^2 \times_T I$ be the universal isomorphism. In particular, we have $\beta^* (\mathcal{L}_2 \times_T I) \cong_{I} (\mathcal{L}_1 \times_T I)$. Set $J := \text{Isom}_T (\beta^* (\mathcal{L}_2 \times_T I), (\mathcal{M}_1 \times_T I))$, where $\text{Isom}_T (\_, \_)$ is the open set of $\text{Hom}_T (\_, \_)$ [Kol08, Def-Lem 33] parametrizing isomorphisms only. Let $\zeta : (\beta \times_I J)^* (\mathcal{M}_2 \times_T J) \to \mathcal{M}_1 \times_T J$ be the universal family. Let

$$\eta : (\beta \times_I J)^* \omega_{X^2 \times_T J/J} \cong (\beta \times_I J)^*[\omega_{X_2^2 \times_T J/J}] \to \omega_{X^1 \times_T J/J}$$

the induced homomorphism. Set $H := \text{Hom}_J \left( (\beta \times_I J)^*[\omega_{X_2^2 \times_T J/J}], \mathcal{M}_1 \times_T J \right)$, and the following sections of $H$ over $J$:

$$\gamma := (t_{\Delta_1, m} \times_T J) \circ \eta \in H(J), \text{ and } \delta := \zeta \circ (\beta \times_I J)^*[t_{\Delta_2, m} \times_T J] \in H(J).$$

Set then the scheme $\text{Isom}_T \left( (X^1, \Delta_1; \mathcal{L}_1), (X^2, \Delta_2; \mathcal{L}_2) \right) \subseteq J$ be the closed subscheme where the sections $\gamma$ and $\delta$ of $H \to J$ agree (or with other words $\gamma^{-1}(\delta)$ or $\delta^{-1}(\gamma)$, where the $\gamma$ and $\delta$ in the parentheses are regarded as a closed subscheme).

To see that the above construction in fact yields a fine moduli space for the functor $\text{Isom}_T \left( (X^1, \Delta_1; \mathcal{L}_1), (X^2, \Delta_2; \mathcal{L}_2) \right)$, fix

$$\left( \alpha : X^1 \times_T S \to X^2 \times_T S, \xi : \alpha^* (\mathcal{M}_2 \times_T S) \to \mathcal{M}_1 \times_T S \right) \in \text{Isom}_T \left( (X^1, \Delta_1; \mathcal{L}_1), (X^2, \Delta_2; \mathcal{L}_2) \right) (S).$$
Then, there is an induced morphism $S \to I$ over $T$ given by $\alpha$ such that $\beta \times_I S = \alpha$. But then
\[ \alpha^*(M_2 \times_T S) = (\beta \times_I S)^*(M_2 \times_T S) = (\beta^*(M_2 \times_T I)) \times_I S. \]
So, $\xi$ is an isomorphism $(\beta^*(M_2 \times_T I)) \times_I S \to (M_1 \times_T I) \times_I S$. Hence, by the universal property of $J$, $S \to I$ lifts to $S \to J$ such that $\xi = \zeta \times_J S$. Note that, as $I$ and $J$ are fine moduli spaces themselves as well, To conclude showing that $\text{Isom}_T \left( (X^1, \Delta_1; L_1), (X_2, \Delta_2; L_2) \right)$ is a fine moduli space for its functor, we have to show that $S \to J$ factors through $\gamma^{-1}(\delta)$ if and only if it satisfies the commutative diagram of (7.2.a). By Lemma 7.7, the condition of $S \to J$ factoring through $\gamma^{-1}(\delta)$ is equivalent to $\gamma_S = \delta_S$. However, as the target of $\gamma_S$ and $\delta_S$ are line bundles and $X_1 \times_T S \to S$ is a relatively $S$ morphism, the $\gamma_S = \delta_S$ holds if and only if $\gamma_S|_{U_1 \times_T S} = \delta_S|_{U_1 \times_T S}$ (see also point (5) of Remark 7.3). However, the latter is exactly the commutativity of the diagram of (7.2.a).

**Lemma 7.7.** Let $f : X \to T$ be a morphism with two sections $\sigma_i : T \to X$ ($i = 1, 2$). Let $V := \sigma_i^{-1}(\alpha(T)) = \sigma_i^{-1}(\beta(T))$. Then any morphism $S \to T$ factors through $V$ if and only if $(\sigma_1)_S = (\sigma_2)_S$.

**Proof.** By considering affine charts, we may assume that all spaces and morphisms are affine. Let $B$, $A$ and $C$ be the rings corresponding to $X$, $T$ and $C$, respectively, and let $\lambda_i : B \to A$ be the homomorphisms corresponding to $\sigma_i$. Set $I_i := \ker(\lambda_i)$. In the affine language, $S \to T$ factoring through $V$ means that $\lambda_1(I_2) \subseteq \ker(A \to C)$. On the other hand, $(\sigma_1)_S = (\sigma_2)_S$ corresponds to $\lambda_1 \otimes_A C(I_2 \otimes_A C) = 0$ or equivalently to $\lambda_1 \otimes_A C(I_2 \otimes_A 1) = 0$. However, $\lambda_1(I_2) \subseteq \ker(A \to C)$ and $\lambda_1 \otimes_A C(I_2 \otimes_A 1) = 0$ are equivalent because they phrase the conditions that the image of $I_2$ via the two routes from the top left corner to the bottom right corner of the following commutative diagram is $0$:

\[
\begin{array}{ccc}
B & \to & B \otimes_A C \\
\downarrow & & \downarrow \lambda_1 \otimes_A C \\
A & \to & C
\end{array}
\]

**Proposition 7.8.** In the situation of Setting 7.1:

1. $\text{Isom}_T \left( (X^1, \Delta_1; L_1), (X_2, \Delta_2; L_2) \right) \times_T S$ is compatible with regular base change. That is if $S \to T$ is a morphism from a regular variety, then

\[
\text{Isom}_T \left( (X^1, \Delta_1; L_1), (X_2, \Delta_2; L_2) \right) \times_T S \cong \text{Isom}_S \left( (X_1^1, (\Delta_1)_S; (L_1)_S), (X_1^2, (\Delta_2)_S; (L_2)_S) \right).
\]

2. Over regular bases, the sections of $\text{Isom}_T \left( (X^1, \Delta_1; L_1), (X_2, \Delta_2; L_2) \right)$ correspond to the intuitive definition. That is, for every morphism $S \to T$ from a regular variety we have

\[
\text{Isom}_T \left( (X^1, \Delta_1; L_1), (X_2, \Delta_2; L_2) \right)(S) = \left\{ \alpha \in \text{Isom}_S \left( (X_1)_S, (X_2)_S \right) | \alpha^* (L_2)_S \cong (L_1)_S, \alpha^* (\Delta_2)_S = (\Delta_1)_S \right\}
\]

**Proof.** All these points follow immediately from the definition. For the last one, use Remark 7.3.

**Remark 7.9.** We would like to emphasize that the naive description stated in point (2) of Proposition 7.8 does not even make much sense over non-reduced bases. There the functor can be regarded directly via the definition in Definition 7.2.

8. **Isotriviality over curves in special cases**

In this section we show that in the situation of Notation 5.1, the morphism $f$ has isomorphic fibers, in the following two special cases: if the base field $k$ is finite, or if $-(K_{X/T} + \Delta)$
is semi-ample. Unfortunately, because numerical flatness of vector bundles is not an open condition, we are not able to show similar statement without some additional assumptions as the above two. In particular, this works if the family is obtained by pulling back over a general curve in the Albanese variety of a pair \((X, \Delta)\) with \(-(K_X + \Delta)\) semi-ample, leading to one of our main theorems, Theorem 9.2.

We precede the actual arguments with a few lemma concerning vector bundles of degree zero.

**Lemma 8.1.** If \(E \subseteq O^\oplus_T =: F\) is a vector bundle of degree zero on a smooth projective curve \(T\) over any field \(k\), then \(E = O_T \otimes_k V\) for some \(V \subseteq H^0(T, F)\).

**Proof.** We prove it by induction on \(m - \text{rk} E\). If \(m - \text{rk} E = 0\), then \(E\) has full rank, and so the only way \(\deg E = 0\) can happen is that \(E = F\). Hence we may assume that \(\text{rk} E < m\), and that we know the statement for the higher co-rank case. Then we may find \(O_T \hookrightarrow F\) such that \(O_T \cap E = 0\). In particular, the quotient \(F/O_T =: F' \cong O^\oplus_T/m\) induces an embedding \(E \hookrightarrow F'\). So, we obtain by the induction hypothesis that \(E \cong O^\oplus_T/mE\). Then the image of \(H^0(T, E) \to H^0(T, F')\) gives the required \(V\). \(\Box\)

**Lemma 8.2.** Let \(f: X \to T\) be a morphism of a projective scheme onto a smooth projective curve over \(k\), and let \(L\) be an \(f\)-ample line bundle. Assume that the following conditions are satisfied

1) the multiplication map \(\text{Sym}^m f_*O_X(L) \to O_X(mL)\) is surjective, for every \(m > 0\),
2) the sheaves \(f_*O_X(mL)\) are degree zero vector bundles, for every \(m \geq 1\),
3) the bundle \(f_*O_X(L)\) is numerically flat.

Then if \(k = \mathbb{F}_q\) and \(t \in T(\mathbb{F}_q)\), then there is a finite cover \(\tau: S \to T\) such that \(X_S \cong S \times_k X_t\) over \(S\). In fact, the natural map \(\text{Isom}_\tau((X, L), (X_t \times T, L_t \times T)) \to T\) is surjective. In particular, the conclusion holds if \(f_*O_X(mL)\) is numerically flat for every \(m > 0\).

**Proof.** As numerically flat vector bundles have degree 0, we need to prove only the main statement. By assumptions the bundle \(f_*O_X(L)\) is numerically flat. Consequently Lemma 2.5 yields a cover \(\tau: S \to T\) such that \(\tau^*f_*O_X(L) \cong O^\oplus_S\). Let \(\rho: X_S \to X\) and \(g: X_S \to S\) be the induced morphisms. Introduce the notation \(\xi_m\) for the surjection

\[\xi_m: \text{Sym}^m \tau^*f_*O_X(L) \cong \text{Sym}^m g_*O_V(\rho^*L) \to g_*O_V(m\rho^*L).\]

Then \(\ker \xi_m\) is a degree zero vector bundle in the trivial vector bundle \(\text{Sym}^m g_*O_V(\rho^*L)\). Hence, Lemma 8.1 yields that it is induced by some \(V \subseteq H^0(S, \text{Sym}^m g_*O_V(\rho^*L))\). Then, it follows that the map \(\text{Sym}^m g_*O_V(\rho^*L) \to g_*O_V(m\rho^*L)\) is also induced by global sections. However, then we obtain that the multiplication maps \(\text{Sym}^m g_*O_X(\rho^*L) \to g_*O_X(m\rho^*L)\) themselves are induced by the global section maps

\[\text{Sym}^m H^0(S, g_*O_V(\rho^*L)) \to H^0(S, g_*O_V(m\rho^*L)).\]

This implies that for the relative canonical ring \(R_S(V, \rho^*L)\) we have \(R_S(V, \rho^*L) \cong R_t(X_t, L_t) \times \mathbb{F}_q\) on \(S\). This yields the desired claim. \(\Box\)

**Lemma 8.3.** Consider a commutative diagram as follows:

\[
\begin{tikzcd}
X & T \arrow{r}{g} \arrow{d}{\sigma} & S, \arrow{dl}{h}
\end{tikzcd}
\]

where

1) \(S\) is the spectrum of a finitely generated algebra over \(\mathbb{Z}\),
2) \(\sigma\) is a section of \(g\),
3) \(f\) and \(g\) are flat and projective,
Hence, our assumption tells us that $V$ is surjective and flat on $T_s$.

Then whenever $\eta \to S$ is a spectrum of a field mapping to the generic point of $S$, the homomorphism $\text{Isom}_{T_s} \left( (X_s, \Delta; \mathcal{L}_s), \left( X_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s, \mathcal{L}_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s \right) \right)$ is surjective and flat on $T_s$.

If we assume additionally that

(6) all geometric fibers of $f$ are normal,

(7) there is an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ not containing any fiber in its support such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier,

and instead of we assume that

(5)' for every $s \in S$ with $|k(s)| < \infty$,

$$\text{Isom}_{T_s} \left( (X_s, \Delta; \mathcal{L}_s), \left( X_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s, \psi^*_s \Delta; \mathcal{L}_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s \right) \right)$$

is surjective and flat on $T_s$, where $\psi : X \times_{T, \sigma_{\text{og}}} T \to X$ is the projection morphism onto the first factor.

Then, for $\eta \to S$ as above, we obtain that

$$\text{Isom}_{T_\eta} \left( (X_\eta, \Delta_\eta; \mathcal{L}_\eta), \left( X_{\sigma(\eta)} \times \text{Spec} k(\eta) \mathcal{T}_\eta, \psi^*_\eta \Delta; \mathcal{L}_{\sigma(\eta)} \times \text{Spec} k(\eta) \mathcal{T}_\eta \right) \right) \to T_\eta$$

is surjective and flat.

**Proof.** Set in the respective cases

$$V := \text{Isom}_T \left( (X, \mathcal{L}), (X \times_{T, \sigma_{\text{og}}} T, \mathcal{L} \times_{T, \sigma_{\text{og}}} T) \right).$$

or

$$V := \text{Isom}_T \left( (X, \Delta; \mathcal{L}), (X \times_{T, \sigma_{\text{og}}} T, \psi^* \Delta; \mathcal{L} \times_{T, \sigma_{\text{og}}} T) \right).$$

Then, for all $s \in S$, we have the respective canonical isomorphisms

$$V_s \cong \text{Isom}_{T_s} \left( (X_s, \mathcal{L}_s), \left( X_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s, \mathcal{L}_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s \right) \right),$$

or

$$V_s \cong \text{Isom}_{T_s} \left( (X_s, \Delta; \mathcal{L}_s), \left( X_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s, \psi^*_s \Delta, \mathcal{L}_{\sigma(s)} \times \text{Spec} k(s) \mathcal{T}_s \right) \right).$$

Hence, our assumption tells us that $V_s \to T_s$ is surjective, whenever $k(s)$ is finite, and we have to deduce that $V_\eta \to T_\eta$ is surjective. As $V \to T$ is finite type, the image $W$ of $V \to T$ is constructible. So, we know that $W_s = T_s$ whenever $k(s)$ is a finite field, that is, for all closed points $s \in S$, and for our surjectivity statements we have to show that then $W_\eta = T_\eta$ as well. Assume the contrary. Then $W^c := T \setminus W$ is a constructible set, which avoids all fibers of $g$ over closed points, but it intersects the generic fiber non-trivially. Hence, the constructible set $g(W^c)$ avoids all closed points of $S$, but contains the generic point. This is a contradiction, concluding our surjectivity statements.

For the flatness statements we use that the locus $\tilde{W} := \{ t \in T | V \to T$ is flat at $t \in T \}$ is constructible (combination of openness of the flat locus in $V$ [Gro66, Théorème 11.3.1] and Chevalley’s theorem about the image of constructible set is constructible). Just as above for $W$ we know that $\tilde{W}_s = T_s$ whenever $|k(s)| < \infty$, and then just as above for $W$ we obtain that $\tilde{W}_\eta = T_\eta$.

**Lemma 8.4.** If $Y$ and $S$ are normal projective varieties, and $\Gamma$ is an effective $\mathbb{Q}$-divisor on $Y \times S$, such that $K_{Y \times S} + \Gamma$ is an anti-nef $\mathbb{Q}$-Cartier divisor, then $(Y \times S, \Gamma) \cong (X, \Sigma) \times_k S$ for some effective $\mathbb{Q}$-divisor $\Sigma$ on $X$. 

\[ \square \]
Proof. As divisors are determined in dimension 1, we may replace $Y$ by its regular locus, and hence we may assume that $Y$ is regular. This way we do lose the projectivity of $Y$, but this actually does not matter as we only consider $h$-nefness in the present proof, where $h: Y \times S \to Y$ is the natural projection. On the other hand, since $Y$ is regular, we gain that $K_Y$ is Cartier. This is used essentially in the following computation, which implies that every $h$-vertical curve that intersects $\operatorname{Supp} \Gamma$ is necessarily contained in $\operatorname{Supp} \Gamma$: 

$$
\Gamma = (K_{Y \times S} + \Gamma) - K_{Y \times S/S} = (K_{Y \times S} + \Gamma) - h^* K_Y \begin{smallmatrix}
\text{\text{h-anti-nef}} \\
\text{\text{h-numerically trivial}} \\
\end{smallmatrix} \begin{smallmatrix}
\text{\text{Q-Cartier and h-anti-nef}} \\
\end{smallatrix}.
$$

In particular, every component of $\Gamma$ is vertical over $Y$, and hence horizontal over $S$. This concludes our proof. 

Corollary 8.5. In the situation of Notation 5.1, if $k = \mathbb{F}_q$ and $t \in T(\mathbb{F}_q)$, then there is a finite cover $\tau: S \to T$ by a smooth projective curve such that $(X_S, ρ^*Δ; ρ^*L) \cong_S S \times_k (X_t, Δ_t, L_t)$, where $ρ: X_S \to X$ is the projection morphism induced by (8.5.a). (Here isomorphism over $S$ for the line bundle means that we allow twist by pull-backs of line bundles on $S$, in accordance with Definition 7.2.)

Addendum: in particular, $\operatorname{Isom}_T((X, Δ, L), (X_t \times T, Δ_T \times T, L_t \times T)) \to T$ is surjective and flat.

Proof. It follows directly from Lemma 8.2 using Theorem 5.10 that there is $τ: S \to T$ such that

$$
(8.5.a) \quad (X_S, ρ^*L) \cong_S S \times_k (X_t, L_t).
$$

We claim that in fact for the same $τ$, we have $(X_S, ρ^*Δ; ρ^*L) \cong S \times_k (X_t, Δ_t; L_t)$. To show this claim, note first that by (8.5.a), we obtain that the fibers of $X \to T$ are all normal. Hence, $K_{X_S} + ρ^*Δ = ρ^*(K_X/T + Δ)$ by [CP18, Prop 2.1]. In particular, $K_{X_S} + ρ^*Δ$ is anti-nef. Then, Lemma 8.4 concludes the main statement.

For the addendum, use Proposition 7.8 and in the case of flatness combine it with a corollary of the existence of a flattening stratification stating that if a morphism is flat after finite base-change, it is flat already without applying the base-change [Sta, Tag 0533].

Lemma 8.6. In the situation of Notation 5.1, if $−(K_X/T + Δ)$ is semi-ample (e.g., $K_X/T + Δ \sim_{Q} 0$) and $t \in T(k)$ is a rational point, then natural morphism $\operatorname{Isom}_T((X, Δ, L), (X_t \times T, Δ_T \times T, L_t \times T)) \to T$ is surjective and flat.

Proof. Choose a model $(X_S, Δ_S) \xrightarrow{f_S} TS \xrightarrow{g_S} S$ of $(X, Δ)$ $\xrightarrow{f} T \xrightarrow{g} \operatorname{Spec} k$ over the spectrum $S$ of a regular finitely generated $\mathbb{F}_p$-algebra, such that for every closed point $s \in S$, $f_s: (X_s, Δ_s) \to T_s$ satisfies the assumptions of Notation 5.1. This is doable by the semi-ample assumption, and by the openness of strongly $F$-regular singularities [PSZ18, Thm B]. Moreover, take a spreading out $L$ of the line bundle $L$ such that the bundles $L_s$ satisfy $L^{d+1}_s = 0$. Then Corollary 8.5 and Proposition 7.8 tell us that for every closed point $s \in S$, the structure morphism $\operatorname{Isom}_{T_s}((X_s, Δ_s; L_s), (X_{σ(s)} \times \operatorname{Spec} k(s), T_s, ψ^*_s Δ, ψ^*_s L_{σ(s)})) \to T_s$ is surjective, where $φ$ is the projection $X \times_{T, σ(s) g_S} T \to X$. We emphasize that the assumption on normality of the fibres is satisfied because we may first prove that the family is trivial in the boundary free setting. Additionally, Corollary 8.5 and Proposition 7.8, also tells us that the same structure morphism is flat after a finite base, change. Hence flattening stratification, or the more elementary algebra lemma [Sta, Tag 0533] tells us that it is also flat without applying the base-change. Then Lemma 8.3 concludes our proof.

Corollary 8.7. Let $f: (X, Δ) \to T$ be a fibration from a normal pair of dimension $d + 1$ to a smooth projective curve with normal general fiber and with $−K_X/T − Δ$ a semi-ample
Q-Cartier divisor. Assume additionally that the general fibers of \( f \) are strongly \( F \)-regular. Then \((X_\mathcal{T}, \Delta_\mathcal{T}) \cong (X_\mathcal{T}', \Delta_\mathcal{T}')\) for any geometric points \( \mathcal{T}, \mathcal{T}' \in \mathcal{T} \) with the same residue fields.

**Proof.** According to Lemma 5.2, we may choose a line bundle \( L \) on \( X \) such that \( L^{n+1} = 0 \), and hence the morphism \( f : (X, \Delta) \to T \) together with a line bundle \( L \) satisfies the conditions of Notation 5.1. Consequently, we may apply Lemma 8.6 to conclude. \( \square \)

9. **ISOTRIVIALITY OF THE ALBANESE MORPHISM**

In this section we combine the result of the previous sections into the main theorem of the present paper.

**Theorem 9.1** (Main decomposition theorem – general version). Let \((X, \Delta)\) be a Cohen–Macaulay pair \((X, \Delta)\) such that either

(a) \(-K_X - \Delta\) is a semi-ample \( \mathbb{Q} \)-Cartier divisor, or

(b) \(k \subseteq \mathbb{P}_p\) and \(-K_X - \Delta\) is a nef \( \mathbb{Q} \)-Cartier divisor.

Assume that \( A \) admits a \( k \)-point \( 0 \in A(k) \) and the Albanese morphism \( f : X \to A \) is surjective and that the general fibre \((X_t, \Delta_t)\) is strongly \( F \)-regular. Then,

\[ (X, \Delta) \times_A I \cong_I (X_0, \Delta_0) \times_k I, \]

where

1. \( I := \text{Isom}_A((X, \Delta; L_X), (X_0 \times_k A, \Delta_0 \times_k A; L_0 \times_k A)), \)

for some relatively ample line bundle \( L_X \) on \( X, X_0 := f^{-1}(\{0\}), \Delta_0 := \Delta|_{X_0} \) and \( L_0 := L_X|_{X_0}. \)

2. \( I \to A \) is surjective and flat.

In particular, all fibers \((X_t, \Delta_t)\) of \( f \) over \( t \in A(k) \) are isomorphic.

**Proof.** Note first that \( \text{pr}_2 : I \times_A I \to I \) has the natural diagonal section. Hence, points (1) and (2) of Proposition 7.8 yield automatically point (1) of the present theorem. In particular, it is enough to show point (2), that is, that \( I \to A \) is surjective and flat.

Second, note that as the base-field \( k \) is perfect, by base-changing to the algebraic closure, we may assume that in fact \( k \) is algebraically closed. Here we are using Proposition 7.8.

By applying Theorem 4.1 we obtain that \( f \) is equidimensional and hence flat because \( X \) is Cohen–Macaulay. Let us fix then the following notation and the basic setting. We assume that the dimension of \( X \) is equal to \( n \), and the relative dimension of \( f \) is equal to \( d \). We take a very ample divisor \( \tilde{L} \) on \( X \) satisfying the conditions:

\[ R^i f_* \mathcal{O}_X \left( j \tilde{L} \right) = 0, \text{ for } i > 0 \text{ and } j > 0, \]

\[ \text{Sym}^m \mathcal{O}_X \left( \tilde{L} \right) \to f_* \mathcal{O}_X \left( m \tilde{L} \right) \text{ is surjective for every } m > 0. \]

This ensures first that the sheaves \( f_* \mathcal{O}_X \left( j \tilde{L} \right) \) are vector bundles the formation of which commutes with every base change. Second, it ensures that the relative section ring

\[ \bigoplus_{m \geq 0} f_* \mathcal{O}_X \left( m \tilde{L} \right), \]

and hence the whole morphism \( f \) as well are determined by \( f_* \mathcal{O}_X \left( \tilde{L} \right) \) and the natural relations given by the kernel of the multiplication map \( \text{Sym}^d \mathcal{O}_X \left( \tilde{L} \right) \to f_* \mathcal{O}_X \left( d \tilde{L} \right). \)

Furthermore we choose a very ample divisor \( G \) on \( A \), such that for any two closed points \( x, y \in A \), and for general \( G_1, \ldots, G_{n-d-1} \in |G| \) through \( x \) and \( y \), the curve \( C_{x,y} := \bigcap_{i=1}^{n-d-1} G_i \) is smooth and irreducible.
Set $H = f^*G$. Along the lines of Notation 5.1 and Lemma 5.2, we observe that $H^{n-d+1} = (f^*G)^{n-d+1} = 0$, and consequently we may find positive integers $m$ and $n$ such that the divisor $L_X = mL - nH$ is $f$-ample and satisfies the condition
\[(9.1.a)\]
$$L_X^{d+1} \cdot H^{n-d-1} = 0$$
We remark that $L_X$ is clearly $f$-ample since $H$ is a pullback of a divisor from the base $A$. By the projection formula it also satisfies the conditions explained above:
\[(9.1.b)\]
$$R^if_*\mathcal{O}_X(jL_X) = 0, \text{ for } i > 0 \text{ and } j > 0,$$
\[(9.1.c)\]
$$\text{Sym}^d \mathcal{O}_X(L_X) \rightarrow f_*\mathcal{O}_X(mL_X) \text{ is surjective for every } d > 0.$$
We prove the surjectivity and the flatness of $I \rightarrow A$ using the results of Section 5 by restricting to the curves $C_{x,0} \sim G^{n-d-1}$ introduced above, where $x$ is an arbitrary closed point. For the ease of notation, fix this $x \in X$, set $C := C_{x,0} \subset A$, and consider the pullback pair $(Y, \Delta_Y) = (X \times_A C, \Delta_Y)$ and the Cartier divisor $L = L_X|_Y$. To explain what $\Delta_Y$ is, note first that as $X$ is Cohen-Macaulay and $C$ is a smooth curve in $A$, $Y$ is also Cohen-Macaulay. Furthermore, as $C$ goes through a general point of $A$, the general fibers of $Y \rightarrow C$ are normal. Taking into account that the special fibers are reduced, we obtain that $Y$ is normal (e.g., [CP18, Lemma 6.2]), as well as by [CP18, Proposition 2.1] there is a $\Delta_Y$ on $Y$ such that
\[(9.1.d)\]
$$K_{X/A} + \Delta_Y \sim_Q K_{Y/C} + \Delta_Y.$$ By the base change properties of the log-isomorphism scheme presented in Section 7, and by flattening decomposition [Mum66, Lecture 8] it suffices prove that the morphism
$$\text{Isom}_C((Y, \Delta_Y; L), (X_0 \times C, \Delta_0 \times C; L_0 \times_A C)) \simeq I \times_A C \rightarrow C$$
is surjective.
To this aim, we want to apply the results of Section 8. For this purpose, we verify that the morphism $g: (Y, \Delta_Y) \rightarrow C$ and the Cartier divisor $L$ satisfy all the requirements described in Notation 5.1. First, the normality of $Y$ was shown above. Second, if $t \in C$ is general, by [CP18, Lemma 6.2], $(Y_t, (\Delta_Y)_t) = (X_t, \Delta_t)$. In particular, as we have seen above that the latter is strongly $F$-regular, so is the former. Third, the condition $L^{d+1} = 0$ is shown by the following computation:
\[(9.1.a)\]
$$L^{d+1} = L_X^{d+1} \cdot Y = L_X^{d+1} \cdot H^{n-d-1} = 0.$$ Finally, we have the following identification
\[(9.1.d)\]
$$-K_{Y/C} - \Delta_Y \sim_Q (-K_{X/A} - \Delta)|_Y \sim (-K_X - \Delta)|_Y,$$
which implies that the divisor $-K_{Y/C} - \Delta_Y$ is nef in the case of assumption (b) and semi-ample in the case of assumption (a). In either, case it is nef as required.
In order to finish the proof, we may now apply directly apply Lemma 8.6 in the case of assumption (a) and Corollary 8.5 in the case of assumption (b). 

**Theorem 9.2 (Main decomposition theorem).** Let $(X, \Delta)$ be a strongly $F$-regular and globally $F$-split projective pair over $k$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier of index coprime to $p$ and either
\[(a)\] $-K_X - \Delta$ is semi-ample, or
\[(b)\] $k \subseteq \mathbb{F}_p$ and $-K_X - \Delta$ is nef.
Let \( X \to A \) be the Albanese morphism of \( X \), and assume that \( A \) admits a \( k \)-point \( 0 \in A(k) \). Then,
\[
(X, \Delta) \times_A I \cong f(X_0, \Delta_0) \times_k I,
\]
where
\[
I := \text{Isom}_A((X, \Delta; L_X), (X_0 \times_k A, \Delta_0 \times_k A; L_0 \times_k A)),
\]
for some relatively ample line bundle \( L_X \) on \( X \), \( X_0 := f^{-1}(\{0\}) \), \( \Delta_0 := \Delta|_{X_0} \) and \( L_0 := L_X|_{X_0} \).

(1) \( I \to A \) is surjective and flat.

(2) \( I \to A \) is surjective and flat.

(3) \( (X_0, \Delta_0) \) is strongly \( F \)-regular and globally \( F \)-split such that \( -(K_X + \Delta) \) has Cartier index coprime to \( p \).

In particular, all fibers \( (X_t, \Delta_t) \) of \( f \) over \( t \in A(k) \) are isomorphic.

**Proof.** First, just as at the beginning of the proof of Theorem 9.1 we may assume that \( k \) is algebraically closed.

Second, we claim that the general fibre \((X_t, \Delta_t)\) of \( f \) satisfies the same assumptions as \((X, \Delta)\). Since \((X, \Delta)\) is \( F \)-split by Proposition 2.11 we know that \( f \) is \( F \)-split relative to \( \Delta \) (see Section 2.7 or [Eji17, Def 5.1] for the definition of \( f \) being \( F \)-split). Therefore all the fibres \((X_t, \Delta_t)\) are \( F \)-split, and hence reduced. Consequently, using the assumption on the Cartier index of \( -K_X - \Delta \), we may use Proposition 2.12 to deduce that the general fibre is regular in codimension one. Since \( X \) is strongly \( F \)-regular, it is also Cohen–Macaulay and thus the general fibre is Cohen–Macaulay too. This means that the general fibre satisfies the condition in Serre’s criterion for normality and is therefore normal. This allows us to apply Theorem 6.3 to see that the general fibre \((X_t, \Delta_t)\) is strongly \( F \)-regular. Finally, we observe that \( -K_X - \Delta_t \) is semi-ample using the adjunction formula described in Section 2.8.4. This concludes our claim.

We use [Eji17, Theorem 1.1] now to see that \( f \) is surjective. By the above claim, Theorem 9.1 can be applied to \((X, \Delta)\), yielding our result. \(\square\)

10. **Finite Automorphisms**

The first author has learned the idea of Proposition 10.1 and in particular the reference [Ros56] from János Kollár. Note that the proof works in any characteristic.

**Proposition 10.1.** If \((X, \Delta)\) is a projective klt pair over an algebraically closed field \( k \) such that \( K_X + \Delta \) is pseudo-effective and \( L \) is a line bundle, then
\[
\text{Aut}(X, \Delta; L) := \{ \sigma \in \text{Aut}(X, \Delta) \mid \sigma^*L \cong L \}
\]
is finite.

**Proof.** Assume now that \( \text{Aut}(X, \Delta; L) \) is not finite. Write \( \Delta = \sum_{i=1}^r c_i \Delta_s \), where \( c_i \) are distinct and \( \Delta_s \) are reduced. Since \( L^\otimes q \) is very ample, for some divisible enough integer \( q \) \( 0, \text{Aut}(X, \Delta; L) \) can be identified with the \( k \)-points of the linear algebraic group
\[
G := \left\{ \alpha \in \text{PGL} \left(h^0(L^\otimes q), k\right) \mid \alpha(X) = X, \forall s : \alpha(\Delta_s) = \Delta_s \right\},
\]
In particular, \( G \) has infinitely many \( k \)-points, and hence it is positive dimensional. Hence, it contains an algebraic sub-group \( H \) isomorphic either to \( \mathbb{G}_m \) or to \( \mathbb{G}_a \). Hence we have an action \( \sigma : H \times X \to X \). The stabilizer sub-group scheme \( S \to X \) of \( \sigma \) is a closed, proper (so not the whole \( H \times X \)) sub-group scheme of the group scheme \( H \times X \to X \) over \( X \).

We may assume that \( S \) contains no sub-group scheme of the form \( G \times X \to X \) for some finite \( k \)-subgroup scheme of \( G \) of \( H \), as then we could replace \( H \) by \( H/G \) (which is still a 1-dimensional linear algebraic group and hence isomorphic to \( \mathbb{G}_a \) or \( \mathbb{G}_m \)).

We claim that there is an open set of \( X \), where the stabilizer of \( \sigma \) is trivial, that is, the group scheme \( S \) introduced above is trivial over this open set. As \( H \times X \) is
irreducible, the only way $S$ can be not equal to it is if $\dim S < \dim H \times X = \dim X + 1$. Hence, there is an open set $U_{\text{fin}} \subseteq X$ over which $S$ is finite. However, then over $U_{\text{fin}}$ there is an integer $m$, such that $|S| U_{\text{fin}}$ is $m$-torsion. So, $|S| U_{\text{fin}}$ is a sub-group scheme of $H[m] \times U_{\text{fin}} \rightarrow U_{\text{fin}}$, where $H[m]$ denotes the $m$-torsion sub-group scheme of $H$. In particular, we obtain a morphism $\varphi : U_{\text{fin}} \rightarrow \text{Hilb}_k(H[m])$. Assume now that we are able to prove that there are only finitely many $k$-subgroup schemes $G$ of $H[m]$. Then, $\text{Hilb}_k(H[m])$ would be of dimension 0, and hence, as $U_{\text{fin}}$ is irreducible, $\varphi$ would map it to a single point of $\text{Hilb}_k(H[m])$. That is, $|S| U_{\text{fin}} = G \times U_{\text{fin}}$ would hold for some $k$-subgroup scheme $G$ of $H[m]$. Then, using that $S$ is closed in $H \times X$, we would obtain that $G \times X$ is a sub-group scheme of $S$ over $X$ too. This would be a contradiction.

Hence, to prove the above claim, we only have to prove that $H[m]$ has only finitely many $k$-subgroup schemes. So, write $m = ip^l$, where $\gcd(i, p) = 1$ if $\text{char } k = p > 0$ and $i := m$ otherwise. Let $\tilde{H}$ be an arbitrary sub-group scheme of $H[m]$. As we are over a prefect field (in fact, even algebraically closed), $\tilde{H}$ splits as $\tilde{H}_{\text{et}} \times \tilde{H}_{\text{inf}}$, where $\tilde{H}_{\text{et}}$ is étale and hence a subgroup(scheme) of $H[i]$ and $\tilde{H}_{\text{inf}}$ is connected and hence a sub-group scheme of $H[p^l]$. So, it is enough to prove that $H[p^l]$ has only finitely many $k$-subgroup schemes separately. This is immediate for the former, however one has to be slightly careful with the latter as there are finite connected group schemes over $k$ that have infinitely many subgroup schemes, e.g., $\alpha_p \times \alpha_p$. Luckily, $H[p^l] \cong \alpha_{p^l}$ or $\mu_{p^l}$, and then it has a unique chain of subgroup schemes given by the $p^l$-torsion subgroup schemes for $1 \leq l \leq j$. To make a precise argument, we proceed by induction on $j$ proving that $H[p^l]$ has only finitely many subgroup schemes. For $j = 1$ this is immediate by dimension reasons. Then, it is enough to prove that every non-zero sub-group scheme $\tilde{H} \subseteq H[p^l]$ contains $H[p]$, as then $\tilde{H}/H[p]$ becomes a subgroup scheme of $H[p^l]/H[p] \cong H[p^{l-1}]$ (the isomorphism given by multiplication by $p$), for which we already know the statement. However, the latter statement is quite straightforward: let $l > 0$ be the smallest integer such that $p^l \cdot \tilde{H} = \{0\}$. Then, $p^{l-1} \cdot \tilde{H}$ is a non-zero subgroup scheme of both $\tilde{H}$ and of $H[p]$, and by dimension reasons then it has to be equal to the latter. This finishes the proof of our claim.

Note now that $U := X \setminus \text{Supp } \Delta$ is necessarily $H$ invariant. By intersecting it with the open set found in the above claim, we may also assume that $H$ acts freely on an open set $U$ contained in $X \setminus \text{Supp } \Delta$. Further, by [Ros56, Theorem 2 and 10] or [Ros67], we may assume that $V := U/H$ exists as a scheme and that there is a section $s : V \rightarrow U$. However, then $U \cong V \times H$ necessarily. This yields a birational morphism $X \dashrightarrow \mathbb{P}^1 \times Y$, where $Y$ is a normal compactification of $V$, $U$ is in the domain of $f$, and $U$ is mapped isomorphically to $H \times V$ for some fixed embedding $H \subseteq \mathbb{P}^1$. Let $Z$ be the normalization of the graph of this map. That is, we have the following commutative diagram, where $f$ and $g$ are birational, proper and the other morphisms are open embeddings, over the images of which $f$ and $g$ are isomorphisms:

![Diagram](image)

One can write

\[(10.1.a) \quad f^*(K_X + \Delta) = K_Z + \Gamma\]
for compatible choices of $K_X$ and $K_Z$. Let $C$ be a general $\mathbb{P}^1$ in $\mathbb{P}^1 \times Y$. Then:

\[(10.1.b) \quad K_X + \Delta \text{ is pseudo-effective } \Rightarrow K_Z + \Gamma \text{ is pseudo-effective}
\]

\[(10.1.a) \quad \Rightarrow K_{\mathbb{P}^1 \times Y} + g_* \Gamma \text{ is pseudo-effective } \Rightarrow (K_{\mathbb{P}^1 \times Y} + g_* \Gamma) \cdot C \geq 0
\]

$K_{\mathbb{P}^1 \times Y} + g_* \Gamma = g_*(K_Z + \Gamma)$ as $C$ is the general element of a moving family of curves.

As $f$ and $g$ are isomorphisms over $U$ and $U \cap \text{Supp} \Delta = \emptyset$, we obtain that

\[(10.1.c) \quad g(\text{supp} \Gamma) \subseteq \mathbb{P}^1 \times Y \setminus j(H \times V).
\]

As $(X, \Delta)$ is klt, all coefficients of $\Gamma$ are smaller than 1. By (10.1.c), the only possible components of $g_* \Gamma$ that $C$ meets are the ones that are the components of $(\mathbb{P}^1 \setminus H) \times Y$. As $C$ meets there are at most two such components, and $C$ meets them in multiplicity 1, we obtain that

\[(K_{\mathbb{P}^1 \times Y} + g_* \Gamma) \cdot C = \deg(K_{\mathbb{P}^1} + g_* \Gamma) \cdot C < -2 + 2 \cdot 1 = 0.
\]

This contradicts (10.1.b). Hence, our assumption was false, and $\text{Aut}(X, \Delta; L)$ is finite.

\section{11. Weak Bogomolov–Beauville decomposition}

In this chapter we apply the results from two previous chapters to get the weak Beauville–Bogomolov decomposition for weakly ordinary varieties with trivial canonical class. First, we generalize to the setting of Theorem 9.2 the necessary definition which appeared before in [GKP16]. We say that a morphism is quasi-étale if it is étale in codimension one.

\subsection{11.1. Quasi-étale maps and augmented irregularity}

Before defining the augmented irregularity in our situation, we need a lemma:

\textbf{Lemma 11.1.} Let $(X, \Delta)$ be pair a normal pair, and let $f: X' \to X$ be a quasi-étale morphism of normal varieties. Assume that $(X, \Delta)$ satisfies one of the properties:

1) sharply $F$-pure,
2) globally $F$-split,
3) strongly $F$-regular,
4) globally $F$-regular.

Then the pair $(X', \Delta')$, where $\Delta' := f^* \Delta$, satisfies the same property.

\textbf{Proof.} We first prove the results concerning sharp $F$-purity and global $F$-splitting. Since sharp $F$-purity is just a local version of global $F$-splitting, we may focus on latter notion. We take a Frobenius splitting $s: F^* \mathcal{O}_X([p^e - 1] \Delta) \to \mathcal{O}_X$ of the pair $(X, \Delta)$. Let $f_U: U' \to U$ be a restriction of $f$ to the intersection of the regular locus of $X$ and the étale locus of $f$. By restricting the splitting $s$ to $U$ and taking the pullback along $f^*_U$ we obtain a Frobenius splitting

\[f^*_U(s|_U): f^*_U F^* \mathcal{O}_{U'} \to f^*_U F^* \mathcal{O}_U \simeq \mathcal{O}_{U'}
\]

defined on the open subset $U'$. Since the relevant sheaves are reflexive, we may extend $s|_{U'}$ to a splitting on the whole $X'$, which concludes this part of the proof.

We now proceed to the statements concerning $F$-regularity. As above it suffices to approach the global case. In this situation, we take a Cartier divisor $C \subset X$ containing the
non-étale locus of \( f \) and such that \( X \setminus C \) is smooth and affine (and hence globally strongly \( F \)-regular). We denote by \( C' \) the preimage \( f^* C \). By [SS10, Theorem 3.9] it suffices to prove that \( X' \setminus C' \) is globally strongly \( F \)-regular and the map

\[(11.1.a) \quad \mathcal{O}_{X'} \to F_*^e \mathcal{O}_{X'} \left( \left[ (p^e - 1) \Delta' \right] + C' \right) \]

splits. The former statement is clear since \( X' \setminus C' \) is affine and smooth. In order to see the latter, by using [SS10, Theorem 3.9] again we obtain that \( F_*^e \mathcal{O}_X \left( \left[ (p^e - 1) \Delta \right] + C \right) \to \mathcal{O}_X \) splits. Then as above by taking pullbacks of this splitting along the étale locus we obtain the splitting of (11.1.a).

We recall that using [Eji17, Theorem 1.1] the Albanese morphism of a pair \((X, \Delta)\) satisfying the conditions of Theorem 9.2 is surjective. Furthermore, by Lemma 11.1, these conditions hold also for quasi-étale covers. This justifies the existence of the maximum in the following definition, which is in fact bounded from above by \( \dim X \).

**Definition 11.2** (Augmented irregularity). Let \((X, \Delta)\) be a strongly \( F \)-regular and globally \( F \)-split projective pair over \( k \) such that the Cartier index of \( K_X + \Delta \) is \( p \)-coprime. We define the augmented irregularity \( \hat{q}(X) \) by the formula:

\[
\hat{q}(X) = \max \{ \dim \text{Alb}_{X'} | X' \to X \text{ is a finite quasi-étale morphism of normal varieties} \}.
\]

Note that if \( X \) happens to be smooth then it is enough to consider only étale morphisms in the above definition by the purity of branch locus.

**Proposition 11.3.** Let \( X \) and \( Y \) be normal projective varieties. Assume that \( X \to Y \) is either a finite quasi-étale map or a universal homeomorphism. Then \( \hat{q}(X) = \hat{q}(Y) \).

**Proof.** The statement concerning quasi-étale maps is immediate from Definition 11.2. We therefore proceed to the case when \( X \to Y \) is a universal homeomorphism. First we claim that the normalized pullback map induces an equivalence between the categories of finite quasi-étale normal coverings of \( X \) and \( Y \). Indeed, let \( Y' \to Y \) be a normal variety and let \( Y'' \to Y' \) be a finite quasi-étale map. The scheme \( X' \) defined as a normalization of the pullback \( X \times_Y Y'' \) is normal and admits a finite quasi-étale map to \( X \). This yields a functor in one direction. In order to describe an inverse, we take a finite quasi-étale covering \( \pi: X' \to X \). By [Sta, Tag 04DY] we see that \( \pi \) can be uniquely descended to a finite étale covering \( \pi_U: U' \to U \) of a big open subset \( U \subset Y \). The morphism \( \pi_U \) can be uniquely extended to a covering of \( Y \) by taking the normalization of \( Y \) inside the fraction field of \( U \). This yields the required inverse functor. We leave the details to the reader. We conclude the proof of the proposition by observing that for a pair of finite quasi-étale maps \( X' \to X \) and \( Y' \to Y \) associated to each other via the above equivalence the natural map \( X' \to Y' \) is a universal homeomorphism, and hence \( \dim \text{Alb}_{X'} = \dim \text{Alb}_{Y'} \), by Corollary 2.8.

11.2. **The proof of the weak Beauville–Bogomolov decomposition**

In the proof of the weak Beauville–Bogomolov decomposition for \(-K_X - \Delta\) numerically trivial we use the following result which implies that the torsors arising from Theorem 9.2 are particularly simple, that is, they are dominated by torsors induced by \( n \)-torsion points, for \( n \in \mathbb{N} \).

**Proposition 11.4** ([Nor83, Proposition]). Let \( A \) be an abelian variety over a field \( k \), let \( G \) be a finite group scheme over \( k \), and let \( P \to A \) be a \( G \)-torsor with a \( k \)-rational point over \( 0 \in A \). Then there exists an integer \( n \in \mathbb{N} \), a homomorphism of group schemes \( \varphi: A[n] \to G \), and a morphism \( A \to P \), which is \( A[n] \)-equivariant with the \( A[n] \) action on \( P \) induced by \( \varphi \).

The statement of Proposition 11.4 is summarized in the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{A[n] \text{ equivariant}} & P \\
\downarrow & \downarrow & \downarrow \\
n_A & \to & A
\end{array}
\]
Before stating Theorem 11.6, we note the following lemma stating that the pairs in Theorem 11.6 automatically have log-canonical divisors with Cartier index prime-to-\( p \).

**Lemma 11.5.** Let \( (X, \Delta) \) be a globally \( F \)-pure pair such that \( K_X + \Delta \equiv 0 \), then the Cartier index of \( K_X + \Delta \) is prime-to-\( p \).

**Proof.** By Proposition 2.10, a global Frobenius splitting \( F^*_X \mathcal{O}_X ([ (p^e - 1) \Delta ]) \rightarrow \mathcal{O}_X (X, \Delta) \) corresponds to \( \Gamma \) such that

\[
(11.5.a) \quad 0 \leq \Gamma \sim [(1 - p^e)(K_X + \Delta)] = (1 - p^e)(K_X + \Delta) + \underbrace{\left( [ (1 - p^e)\Delta ] - (1 - p^e)\Delta \right)}_{=0, \text{ where } D \leq 0}
\]

Hence, in the sum \( \Gamma + (-D) \) both summands are effective divisors, and the sum itself is numerically trivial. This implies that \( \Gamma = D = 0 \), and then by (11.5.a), \( (1 - p^e)(K_X + \Delta) \sim 0 \).

**Theorem 11.6** (Weak Beauville–Bogomolov decomposition). Let \( (X, \Delta) \) be a globally \( F \)-split pair with strongly \( F \)-regular singularities such that \( -K_X - \Delta \equiv 0 \). Then there exists a composition

\[
Y \rightarrow W \rightarrow X
\]

such that

1. \( W \rightarrow X \) is a finite quasi-étale morphism,
2. \( Y \rightarrow W \) is a finite infinitesimal torsor under \( G^0 = \prod_{i=1}^{\hat{q}(X)} \mu_{p^{j_i}} \) for some integers \( j_i \geq 0 \), and
3. \( (Y, \Delta_Y) \simeq (Z_0, \Delta_0) \times B \) where
   - \( B \) is an abelian variety of dimension \( \hat{q}(X) \), and
   - \( (Z_0, \Delta_0) \) is a globally \( F \)-split pair with strongly \( F \)-regular singularities such that
     \( K_{Z_0} + \Delta_{Z_0} \equiv 0 \) and \( \hat{q}(Z_0) = 0 \).

Additionally, we can assume that \( G^0 \) acts diagonally on the factors such that the action is faithful on \( Z_0 \) and it is free on \( B \).

**Proof.** We show the statement using Theorem 9.2. First, take a quasi-étale morphism \( \pi: Z \rightarrow X \) such that \( Z \) is normal and \( \dim \text{Alb}_Z = \hat{q}(X) \). As \( k \) is perfect, we can perform any additional finite base-extensions, and we can count them into the quasi-étale part of the cover we construct. Hence, we may assume that \( Z \) has a rational point (and hence the Albanese variety has one) and we may also assume that for the identity component of the \( p^r \) torsion subgroup of \( \text{Alb}_Z \) we have \( (\text{Alb}_Z[p^r])^0 \cong \mu_{p^r \dim \text{Alb}_Z} \), for some integer specified later [Mum70, “The p-rank”, p. 146].

We consider the Albanese morphism \( f: Z \rightarrow \text{Alb}_Z \), where \( \text{Alb}_Z \) is an abelian variety. By Lemma 11.1 we see that the assumptions concerning \( (X, \Delta) \) are also satisfied for \( (Z, \Delta_Z) \), where \( \Delta_Z = \pi^* \Delta \), and therefore by Theorem 9.2, there exists a flat, surjective morphism \( P \rightarrow \text{Alb}_Z \) such that

\[
(Z, \Delta_Z) \times_{\text{Alb}_Z} P \simeq P \times (Z_0, \Delta_0),
\]

where \( (Z_0, \Delta_0) \) possesses all the required properties. The flatness of \( P \rightarrow \text{Alb}_Z \) means that \( P \) is in fact a torsor under \( \text{Isom}(Z_0, \Delta_{Z_0}, L_0) \) for some ample polarization \( L \) on \( Z \). By Proposition 10.1, \( \text{Isom}(Z_0, \Delta_{Z_0}, L_0) \) is a finite group scheme. Additionally, from the precise description of \( P \) as an \( \text{Isom} \) scheme in Theorem 9.2, using the base-change properties of \( \text{Isom} \) stated in Proposition 7.8, we see that \( P \) has a \( k \)-rational point over \( 0 \in A(k) \) given by the identity automorphism. Therefore, by Proposition 11.4 there exists an torsor \( B \rightarrow \text{Alb}_Z \) under a finite abelian group-scheme \( G \) and a morphism \( B \rightarrow P \) that is \( G \)-equivariant via a group-scheme homomorphism \( G \rightarrow \text{Isom}(Z_0, \Delta_{Z_0}, L_0) \). Additionally, the total space of \( B \) is also an abelian variety and \( G = B[p^r] \) for some \( r > 0 \), which is the integer \( r \) that was promised at the beginning of the proof to get fixed later. In particular, the identity component of \( G \) is isomorphic to \( \prod_{i=1}^{\hat{q}(X)} \mu_{p^{j_i}} \) for some integers \( j_i \geq 0 \), which is the assumption
we keep instead of insisting on \( G = B[\gamma] \). Replacing now both \( B \) and \( G \) by the quotient with \( \ker (G \to \text{Isom}(Z_0, \Delta_{Z_0}, L_0)) \) we may also assume that \( G \to \text{Isom}(Z_0, \Delta_{Z_0}, L_0) \) is injective. Set now \( Y := Z \times_{\text{Alb}_Z} B \). As \( B \to X \) factors through \( P \to X \), \( P \times_{\text{Alb}_Z} B \to B \) has a section. Then, using the Proposition 7.8 and the precise description of \( P \) as an Isom scheme from Theorem 9.2, we obtain that \((Y, \Delta_Y) \cong_B (Z_0, \Delta_{Z_0}) \times B\).

Our next task is to define \( W \). The group scheme \( G \) is abelian and can be decomposed as a direct sum \( G = G^0 \oplus G^\text{et} \), where \( G^0 \) is infinitesimal and \( G^\text{et} \) is étale. We set \( W := Y/G^0 \). As \( W \to Z \) is a quotient by \( G^\text{et} \), \( W \to Z \) is étale, and hence \( W \to X \) is quasi-étale. We also have \( \dim B = \dim \text{Alb}_Z = \bar{q}(X) \).

To show \( \bar{q}(Z_0) = 0 \), consider first the following line of equalities

\[
\dim B = \bar{q}(X) = \bar{q}(W) = \bar{q}(Z_0 \times B)
\]

(11.6.b) Proposition 11.3, as \( W \to X \) is quasi-étale

Assume now that \( \bar{q}(Z_0) > 0 \) holds. Then, there is a quasi-étale cover \( V \to Z_0 \) with \( q(V) > 0 \). Hence, \( V \times B \) is a quasi-étale cover of \( Z_0 \times B \) with \( q(V \times B) > \dim B \). This contradicts (11.6.b).

Concerning the addendum: \( G^0 \) acts freely on \( B \) as \( G^0 \subseteq G \) and \( G \) acts freely on \( B \). Furthermore, \( G^0 \) acts faithfully on \( Z_0 \) as it is a subgroup of \( \text{Isom}(Z_0, \Delta_{Z_0}, L_0) \).

**Proof of Theorem 1.1.** By Proposition 2.13 weakly ordinary varieties with trivial canonical class are globally \( F \)-split, and hence Theorem 11.6 can be directly applied for \( \Delta = 0 \). □

**Remark 11.7.** We claim that every two decompositions as in Theorem 11.6 can be dominated via a composition of a quasi-étale map and an abelian infinitesimal torsor by a third one, if we relax the condition stated in the addendum of Theorem 11.6 about the free/faithful action on the respective factors. So, take two such decomposition \( Z_0^1 \times B^1 \) and \( Z_0^2 \times B^2 \) as in the following diagram. The other parts of the diagram are constructed after the diagram itself, and the numbers on the arrows denote the number of the step of this construction in which the given arrow is constructed.

\[
\begin{array}{c}
B^1 \\
\downarrow^{(1)} \\
\text{Alb}_{W^1} \\
\downarrow^{(1)} \\
z_0^1 \times B^1 \\
\downarrow^{\text{inf.ab}} \\
W^1 \\
\downarrow^{\beta^1} \\
X \\
\downarrow^{q-\text{ét}} \\
(11.7.c)
\end{array}
\text{Proposition 11.3, as } W \to X \text{ is quasi-étale}
\begin{array}{c}
B \\
\downarrow^{(1)} \\
\text{Alb}_W \\
\downarrow^{(1)} \\
z_0 \times B \\
\downarrow^{\text{inf.ab}} \\
W \\
\downarrow^{\beta^2} \\
X \\
\downarrow^{q-\text{ét}} \\
\text{Proposition 11.3, as } Z_0 \times B \to W \text{ is a universal homeomorphism}
\end{array}
\begin{array}{c}
B^2 \\
\downarrow^{(1)} \\
\text{Alb}_{W^2} \\
\downarrow^{(1)} \\
z_0^2 \times B^2 \\
\downarrow^{\text{inf.ab}} \\
W^2 \\
\downarrow^{\beta^2} \\
X \\
\downarrow^{q-\text{ét}} \\
\text{Proof of Theorem 11.6 starts by taking a high enough quasi-étale cover, we may find a composition } Z_0 \times B \to W \to X \text{ as in Theorem 11.6, such that } W \text{ dominates both } W^1 \text{ and } W^2.
\end{array}
\]

(1) As \( \alpha_i \) are inseparable, \( \dim \text{Alb}_{W^i} = \dim \text{Alb}_{Z_0^i \times B^i} = \dim B_i \). Hence, using the functoriality of the Albanese morphisms we obtain the isogenies \( B^i \to \text{Alb}_{W^i} \) as in (11.7.c).

(2) As the proof of Theorem 11.6 starts by taking a high enough quasi-étale cover, we may find a composition \( Z_0 \times B \to W \to X \) as in Theorem 11.6, such that \( W \) dominates both \( W^1 \) and \( W^2 \).

(3) As in point (1) we obtain the isogeny \( B \to \text{Alb}_W \).
(4) By the universal property of Albanese morphisms we obtain the isogenies \( \text{Alb}_W \to \text{Alb}_{W^1} \) and \( \text{Alb}_W \to \text{Alb}_{W^2} \).

(5) By possibly increasing \( B \to \text{Alb}_W \) by a higher inseparable isogeny, we obtain also the isogenies \( B \to B^1 \) and \( B \to B^2 \).

(6) As \( Z_0 \times B^i \cong W^i \times \text{Alb}_{W^i} \), we obtain the morphisms \( \gamma^i \).

(7) It is immediate from the construction, that \( \gamma^i \) map the fibers of the projection \( Z_0 \times B \to B \) to the fibers of the projection \( Z_0 \times B \to B^i \), as these projections, \( \gamma_i \) and the isogenies \( B \to B^i \) commute. However, it is not clear from the construction if there are similar morphisms \( Z_0 \to Z_0^i \) with which \( \gamma^i \) and the corresponding projections commute.

12. Corollaries

We start with a short summary of the classification of smooth surfaces with trivial canonical class defined over an algebraically closed field of characteristic \( p > 0 \) provided by Bombieri and Mumford, with a special emphasis on the relation between augmented irregularity and \( F \)-splitting (see [Lie13, BM77, BM76]).

Proposition 12.1. Let \( S \) be a smooth surface defined over an algebraically closed field of characteristic \( p > 0 \) with trivial canonical class. Then \( S \) belongs to one of the following classes of surfaces having the described properties:

(1) abelian surfaces,

(2) \( K3 \) surfaces are simply connected and hence satisfying \( \tilde{q}(S) = 0 \), with the invariants:

\[
h^1(S, O_S) = 0, \quad b_1(S) = 0, \quad b_2(S) = 22,
\]

(3) non-classical Enriques surfaces in characteristic \( p = 2 \), with invariants:

\[
h^1(S, O_S) = h^2(S, O_S) = 1, \quad b_1(S) = 0, \quad b_2(S) = 10
\]

the surfaces are called singular (resp. supersingular) if equivalently the Frobenius action on \( H^2(S, O_S) \) is bijective (resp. zero) or the surface is globally \( F \)-split (resp. not globally \( F \)-split). The fundamental group of an Enriques surface is non-trivial if and only the surface is singular, it is then isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

(4) hyperelliptic surfaces, given as a quotient of a product of two elliptic curves by an action of a potentially non-reduced abelian group scheme, with invariants:

\[
h^1(S, O_S) = 2, \quad p_g(S) = 1, \quad b_1(S) = b_2(S) = 2,
\]

and hence satisfying with \( \tilde{q}(S) \geq 1 \).

(5) a quasi-hyperelliptic surfaces in characteristic \( p = 2, 3 \), given as a quotient of a product of \( \mathbb{P}^1 \) and an elliptic curve by a finite group scheme. We have \( b_1(S) = 2, b_2(S) = 2, \tilde{q}(S) = 1, \) and \( S \) admits a fibration \( S \to E \) to an elliptic curve such that the fibers are all rational curves with a single cusp singularity. As general fibers of fibrations of globally \( F \)-split varieties are globally \( F \)-split [GLP+15, Cor 2.5], and since cusps are not \( F \)-pure, we see that \( S \) is not globally \( F \)-split.

Lemma 12.2. Let \( X \) be a smooth projective \( F \)-split variety with numerically trivial canonical class and \( \tilde{q}(X) = \dim X - 2 \) (e.g. a threefold with \( \tilde{q}(X) = 1 \)). Then there exists a generically finite map \( A \times Z \to X \) where \( A \) is an abelian variety of dimension \( \tilde{q}(X) \) and \( Z \) is a globally \( F \)-split \( K3 \) surface.

Proof. We use Theorem 1.1 to see that there exists a composition of an étale map and a finite universal homeomorphism \( X' \to X \) such that \( X' \simeq A \times Y \) where \( A \) is an abelian variety of dimension \( \tilde{q}(X) \) and \( Y \) is an \( F \)-split Gorenstein strongly \( F \)-regular surface with trivial canonical class and \( \tilde{q}(Y) = 0 \). In particular, \( Y \) has DuVal singularities. Let \( Y' \to Y \) be the minimal resolution. As the exceptional divisors of \( Y' \to Y \) are all smooth rational curves, \( \tilde{q}(Y') = \tilde{q}(Y) = 0 \). By Proposition 12.1, \( Y' \) is either a K3 surface or a singular Enriques. In the first case, set \( Z := Y' \) and the second one set \( Z \to Y' \) to be the \( \mathbb{Z}/2\mathbb{Z} \) Galois étale cover.
stated to exist in Proposition 12.1. Note that in the latter case, $Z$ is again a K3 surface, and it is globally $F$-split by Lemma 11.1.(3). This concludes our proof. □

12.1. Fundamental groups

In characteristic zero, as direct corollaries of the Beauville–Bogomolov decomposition, one obtains the following:

**Corollary 12.3** (of the Beauville–Bogomolov decomposition in char. zero). Let $X$ be a smooth projective variety with numerically trivial canonical class over $\mathbb{C}$. Then

1. $\pi_1(X)$ and $\pi_1^{et}(X)$ are virtually abelian – the abelian part comes from the abelian variety in the decomposition,
2. and consequently the condition $\tilde{q}(X) \neq 0$ is equivalent to $|\pi_1(X)| = |\pi_1^{et}(X)| = \infty$.

It is natural to expect that similar behaviour can be exhibited in characteristic $p > 0$ under some favorable arithmetic assumptions, for example some version of ordinarity. Unfortunately, we have not been able to get any unconditional results in this direction. However, using the corollaries of Theorem 11.6, we can show that the above conditions hold true for weakly ordinary varieties with numerically trivial class and the augmented irregularity satisfying $\tilde{q}(X) \geq \dim X - 2$. In the proof we need the following:

**Lemma 12.4** ([Del71, Lemme 4.4.17]). Let $f : Y \to X$ be a generically finite morphism of normal varieties. Then the image of $f_* : \pi_1^{et}(Y) \to \pi_1^{et}(X)$ is of finite index.

**Corollary 12.5.** Let $X$ be a smooth projective $F$-split variety with numerically trivial canonical class and $\tilde{q}(X) \geq \dim X - 2$ (e.g. a threefold with $\tilde{q}(X) \neq 0$). Then the fundamental group $\pi_1^{et}(X)$ is virtually abelian.

**Proof.** By Lemma 12.2 there exists a generically finite map $A \times Z \to X$, where $A$ is an abelian variety and $Z$ is a K3 surface. This gives the statement by Lemma 12.4. □

**Example 12.6.** In order to prove the statements of Corollary 12.3 in characteristic $p > 0$, one needs to exclude the following group as a fundamental group of an ordinary variety with trivial canonical class. Let $F_n$ be a group with $n \geq 2$ free generators. Then we claim that the pro-$p$ completion $F_n^{\wedge p}$ is not virtually abelian but every open subgroup admits a surjection on $\mathbb{Z}/p\mathbb{Z}$. For the first part, we just observe that there exists a non-abelian group generated by two elements (extension of $\mathbb{Z}/p^n\mathbb{Z}$ by itself) and hence a quotient of $F_n^{\wedge p}$ with no abelian subgroups of small index. For the second, we use the fact that every finite $p$-group has a non-trivial center, and therefore admits a surjection on $\mathbb{Z}/p\mathbb{Z}$. We note that under the ordinarity assumption, using the Artin–Schreier sequence, the $\mathbb{Z}/p\mathbb{Z}$-quotients of the fundamental group is directly determined by the size of $H^1(X, \mathcal{O}_X)$. Unfortunately, in presence of a non-reduced Picard scheme, this cannot control the dimension of the Albanese variety.

12.2. Betti numbers

In this section we prove that the Betti numbers of $F$-split threefolds with numerically trivial canonical class are bounded from above. Before proceeding with the actual argument we recall the following classical result.

**Lemma 12.7** ([Kle68, Proposition 1.2.4]). Let $f : X \to Y$ be a surjective morphism of smooth varieties defined over an algebraically closed field $k$. Then for every $i \geq 0$ the natural maps $f^* : H^i_*(Y_{\text{ét}}, \mathbb{Q}_\ell) \to H^i_*(X_{\text{ét}}, \mathbb{Q}_\ell)$ and $f^! : H^i(Y_{\text{ét}}, \mathbb{Q}_\ell) \to H^i(X_{\text{ét}}, \mathbb{Q}_\ell)$ are injective.

**Theorem 12.8.** Let $X$ be a smooth projective $F$-split threefold with numerically trivial canonical class and $\tilde{q}(X) \geq 1$. Then the Betti numbers satisfy the following inequalities:

$$b_1(X) = b_5(X) \leq 6, \quad b_2(X) = b_6(X) \leq 23, \quad b_3(X) \leq 44.$$
Proof. In the case of $\hat{q}(X) = 1$, by Lemma 12.2 there exists a generically finite morphism $E \times Z \to X$, where $E$ is an elliptic curve and $Z$ is a K3 surface. In the case of $\hat{q}(X) \geq 2$, by Theorem 11.6, there exists a finite morphism $A \to X$ where $A$ is an abelian 3-fold. By Lemma 12.7 this implies that $b_i(X) \leq \max\{b_i(E \times Z), b_i(A)\}$. The claim now follows from the Künneth formula:

$$
\begin{align*}
&b_0(E) = b_2(E) = 1, \quad b_1(E) = 2, \\
&b_0(Z) = b_4(Z) = 1, \quad b_1(Z) = b_3(Z) = 0, \quad b_2(Z) = 22,
\end{align*}
$$

and the general formula for the Betti number of abelian threefolds:

$$
\begin{align*}
&b_1(A) = 6, \quad b_2(A) = 15, \quad b_3 = 20.
\end{align*}
$$

□

Corollary 12.9. If $X$ is a smooth projective $F$-split threefold with numerically trivial canonical class and $\hat{q}(X) \geq 1$ over a finite field $\mathbb{F}_q$, then we have

$$
(12.9.a) \quad X(q) \leq 1 + 6q^{1/2} + 23q + 44q^{3/2} + 23q^2 + 6q^{5/2} + q^3.
$$

and

$$
(12.9.b) \quad |X(q) - 1 - q^3| \leq 6q^{1/2} + 23q + 44q^{3/2} + 23q^2 + 6q^{5/2}.
$$

In particular,

$$
(12.9.c) \quad X(q) \neq \emptyset \text{ for } q \geq 83.
$$

Proof. (12.9.a) and (12.9.b) direct consequence of Theorem 12.8, and of the Weil conjectures. Then (12.9.c) follows from (12.9.b) by directly computing that for every integer $q \geq 83$,

$$
1 + q^3 > 6q^{1/2} + 23q + 44q^{3/2} + 23q^2 + 6q^{5/2}.
$$

□

13. Examples

For simplicity in this section the base-field $k$ is algebraically closed, and as usual of characteristic $p > 0$.

In Remark 1.2, we explained that there are two main differences between our theorems and the characteristic zero statements:

1. In our case, the cover $Z \to X$, for which $Z$ splits as a product, can have inseparability.
2. The stably Albanese trivial factor of $Z$ might be singular even if we start with a smooth variety.

We do have examples that the former phenomenon does occur at least for singular $X$ (Example 13.3). On the other hand, we do not have an example showing that the latter phenomenon occurs, and we also do not have smooth examples of the former phenomenon. Nevertheless we are able to isolate a question about the existence of $K$-trivial varieties with prescribed infinitesimal automorphisms (Question 13.6), a positive answer to which would yield an example of both phenomena.

To be more precise, in the case of phenomenon (1), we give examples of $K$-trivial globally $F$-split and strongly $F$-regular varieties $Y$ such that for the Albanese morphism $f : Y \to A$, if $B \to A$ is an isogeny such that $Y \times_A B \to B$ is a product, then $B \to A$ is not separable.

Construction 13.1. Let $S$ be a projective variety such that

1. $K_S \sim 0$,
2. $S$ is globally $F$-split,
3. $S$ has strongly $F$-regular singularities,
4. $\hat{q}(S) = 0$, and
5. $\mu_p \subseteq \text{Aut}(S)$.
Let $E$ be an ordinary elliptic curve, and let $\nu_p \subseteq E$ be the kernel of the Frobenius endomorphism of $E$. We set then $Y := S \times E/\mu_p$, where $\mu_p$ acts diagonally. We note that the fibers of $f : Y \to E' := E/\mu_p$ are all isomorphic to $S$. In particular, the natural morphism $S \times E \to Y \times E'$ is a morphism of fibrations over $E$ mapping the fibers (which are isomorphic to $S$) isomorphically. In particular, this morphism is an isomorphism.

From $f : Y \to E'$ having fibers isomorphic to $S$ it follows that $Y$ has strongly $F$-regular singularities [Sch09]. As $\mu_p$-quotients preserve global $F$-splitting [AIS17, Thm 4.1 & Rem 4.9], $Y$ is globally $F$-split. Also, as $\mu_p$ acts freely on $E$, so it does on $S \times E$, and hence, $k_Y \sim 0$ [CR17, Thm A]. Additionally, $f : Y \to E'$ is the Albanese morphism by using that the fibers of $f$ are isomorphic to $S$ in conjunction with assumption (4) above.

Let $L$ be an ample Cartier divisor on $S$ and let $p_S : S \times E \to S$ be the first projection. Then $p_S^*(pL)$ descends to a Cartier divisor $M$ on $Y$. We set $I := \text{Isom}_{E'} \left( (S \times E', p_S^*(pL)), (Y, M) \right)$. We note that

\[(13.1.a) \quad I = \text{Isom}(S, pL) \times E/\mu_p,\]

where the action is again diagonal, and via post-composition on the left factor.

**Proposition 13.2.** In the situation of Construction 13.1, if $\text{Isom}(S, pL)$ is a finite group scheme such that $\mu_p \subseteq \text{Isom}(S, pL)$ is a closed subgroup scheme, then as schemes over $E'$ we have an isomorphism $I_{\text{red}} \cong E' \bigcup_{\text{finite}} E$. In particular, a base-change $T \to E'$ trivializes $Y \to E'$ if and only if it factors through $E$.

**Proof.** For ease of notation, set $H := \text{Isom}(S, pL)$, and let $G$ be the identity component of $H$. Note that as schemes over $E$ we have a $\mu_p$-equivariant isomorphism $H \times E \cong E \bigcup G \times E$, such that the $\mu_p$-action is diagonal on each of the $G \times E$ factors on the right. Hence, by (13.1.a), it is enough to show that $(G \times E/\mu_p)_{\text{red}} \cong E'$. As $\mu_p \to G$ is a closed embedding, so is $\mu_p \times E/\mu_p \to G \times E/\mu_p$. Hence, it is enough to show that $\mu_p \times E/\mu_p \cong E$. However, this is immediate by identifying this quotient with the surjective group homomorphism $\mu_p \times E \to E$ given by $(x, y) \mapsto x - y$. \qed

**Example 13.3.** Consider the example constructed by Matsumoro [Mat18, Example 9.5], that is a surface:

\[S := \left\{ u^3 + x^5y + y^5z + z^5x + ax^2y^2z^2 = 0 \right\} \subseteq P := \mathbb{P}_{k}(3, 1, 1, 1)_{w, x, y, z}, \]

where $k$ is an algebraically closed field of characteristic $p = 7$ and $a^3 \neq 1$. This surface $S$ is a K3 surface with only Du-Val singularities and a $\mu_p$-action. In particular, all the assumptions of Construction 13.1 are automatically satisfied [Har98], except possibly that $S$ is globally $F$-split.

We claim that in fact $S$ is also globally $F$-split. Let $R$ be the graded ring $k[w, x, y, z]$, where $\deg w = 3$, and $\deg x = \deg y = \deg z = 1$, and let $k[w, z, y, z]_d$ denote the homogeneous part of degree $d$. Then, $H^0 \left( P, \omega_P^{1-p} \right) \to H^0(P, \mathcal{O}_P) = k$ can be identified with the $k$-linear homomorphism $k[w, x, y, z]_{(p-1)} \to k$ sending $wxyz^{p-1}$ to 1 and all the other monomials to 0. Consider now the following diagram:

\[
\begin{array}{ccc}
\kappa \cong H^0 \left( P, \omega_P^{1-p} \right) & \longrightarrow & H^0 \left( P, \omega_P^{1-p} \right) \\
\downarrow \cong & & \downarrow \cong \\
\kappa \cong H^0 \left( S, \omega_S^{1-p} \right) & \longrightarrow & H^0 \left( S, \mathcal{O}_S \right) \cong \kappa
\end{array}
\]

The diagram shows that the one can show that $X$ is globally $F$-split by computing that the composition of the top horizontal maps is bijective. Hence, one needs to compute whether the coefficient of $wxyz^{p-1} = wxyz^6$ in

\[
\left( u^3 + x^5y + y^5z + z^5x + ax^2y^2z^2 \right)^{p-1} = \left( u^3 + x^5y + y^5z + z^5x + ax^2y^2z^2 \right)^6
\]
is non-zero. However, this coefficient is \( (w^2)^3 \cdot (x^5y) \cdot (y^5z) \cdot (z^5x) \) and for \((w^2)^3 \cdot (x^2y^2z^2)^3\):

\[
\frac{6!}{3!} + \frac{a^3 \cdot 6!}{3 \cdot 3!} = 120 + \frac{a^3 \cdot 20}{3} = 1 - a^3 \neq 0
\]

\[\text{char } k = 7 \quad a^3 \neq 1 \text{ by assumption}\]

This concludes our claim above.

Hence, all conditions of Construction 13.1 are satisfied, and according to Proposition 13.2 Y \(\to E'\) constructed using \(S\) of the present example yields an example where in Theorem 11.6 we must have an inseparable base-change.

Example 13.3 shows that phenomenon (1) from the beginning of the section does appear in the singular case. However, we do not know if this phenomenon appears in the smooth case. However, following basically the same construction, we can construct a non globally \(F\)-split smooth example for both phenomena (1) and (2).

**Example 13.4.** Let \(S\) be a classical Enriques surface defined over a perfect field \(k\) of characteristic two [Lie15, Def 1.2]. By [Lie15, Section 1], we know that \(S\) admits a \(\mu_2\)-torsor \(\pi: S' \to S\) such that \(S'\) is a K3-like surface (see loc.cit.), i.e., \(S'\) is Gorenstein (not necessarily normal) and satisfies \(\omega_{S'} \simeq \mathcal{O}_{S'}\) and \(H^1(S, \mathcal{O}_{S'}) = 0\). Moreover, we take \(E\) to be an ordinary elliptic curve over \(k\). Since \(E\) is ordinary, it admits a subgroup scheme isomorphic to \(\mu_2\). We now consider a threefold defined as quotient of the product \(S' \times E\) by the diagonal \(\mu_2\)-action:

\[X = S' \times E/\mu_2.\]

**Proposition 13.5.** The threefold \(X\) of Example 13.4 satisfies the following properties:

(1) \(X\) is smooth,

(2) \(K_X\) is 2-torsion Cartier divisor,

(3) \(X \to E/\mu_2\) is the Albanese morphism of \(X\), which then has singular general fiber.

**Proof.** First, classical Enriques surfaces are defined as having \(K_X\) 2-torsion [Lie15, Def 2.1]. This shows point (2).

Second, we observe that \(X\) admits two morphisms arising from quotients of natural projections of \(S' \times E\). The first one is a \(\mu_2\)-quotient of the morphism \(S' \times E \to S'\), and hence a fibration \(X \to S'/\mu_2 = S\) with fibre \(E\). Since both the base and the fibre are smooth, we see that \(X\) is in fact smooth. This yields point (1). The second morphism is a fibration \(\pi: X \to E'\) onto an elliptic curve \(E' = E/\mu_2\) with fibre isomorphic to \(S'\). Since \(S'\) does not admit non-trivial morphisms to abelian varieties, the map \(\pi\) is in fact the Albanese morphism of \(X\). This gives point (3), and finishes the proof of the proposition.

So, let \(f: X \to E' := E/\mu_2\) be the Albanese morphism of Example 13.3. Then, the base-change of \(X \to E'\) along \(E \to E'\) yields a Beauville-Bogomolov decomposition, that is, we have \(X \times_{E'} E \cong S' \times E\). In particular, if \(L\) is an \(f\)-ample line bundle on \(X\), then \(\text{Isom}_{E'}((X, L), (X_1 \times E', L_2 \times E')) \to E'\) is inseparable. This shows that in fact, one must have an infinitesimal part in the above base-change to obtain a Beauville-Bogomolov type decomposition, by Proposition 13.2.

However, unfortunately the classical Enriques surfaces in characteristic two are not globally \(F\)-split (as \(K_X\) has Cartier index prime-to-\(p\) for a globally \(F\)-split variety \(X\), see Proposition 2.10). So, in particular, Example 13.4 does not show that the infinitesimal part of the finite cover in Theorem 11.6 is necessary. In order to apply the same construction to get an example that does show that infinitesimal part of the finite cover in Theorem 11.6 is necessary we would need to find a variety as described by the following question.
Question 13.6. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Does there exist a finite \( \mu_p \)-quotient \( X \to Y \) such that \( X \) is a singular, projective Gorenstein \( K \)-trivial variety and \( Y \) is a smooth, weakly ordinary, projective \( K \)-trivial variety over \( k \) with \( \hat{q}(Y) = 0 \)?

We note that for any \( X \) as in Question 13.6 we have \( \hat{q}(X) = 0 \) by Proposition 11.3.

The examples asked for in Question 13.6 do not exist in dimension two. We learned the following argument from Yuya Matsumoto whose paper [Mat18] gives a detailed description of \( \mu_p \) and \( \alpha_p \) quotients of K3 surfaces with canonical singularities. We start with a lemma describing the behaviour of Betti numbers under resolutions.

Lemma 13.7. Let \( f: Y \to X \) be a minimal log resolution of a proper canonical surface defined over an algebraically closed field. Then \( b_1(Y) = b_1(X) \) and \( b_2(Y) = b_2(X) + \sum_{x \in X_{\text{sing}}} d_x \), where \( d_x \) is the number of irreducible components in the minimal resolution of the germ of \( X \) at \( x \in X_{\text{sing}} \).

Proposition 13.8 (Matsumoto, private communication). Let \( X \to Y \) be a generically finite map of normal Gorenstein surfaces with trivial canonical class and at most canonical singularities defined over an algebraically closed field of characteristic \( p > 0 \). Assume that \( Y \) is smooth and weakly ordinary (equiv. globally \( F \)-split). Then \( X \) is smooth as well.

Proof. Assume by contradiction that \( X \) is not smooth. Let \( X' \to X \) be the minimal resolution of singularities. Since \( X \) was canonical the resolution is crepant and therefore \( X' \) is a smooth Calabi–Yau variety. The map \( X' \to Y \) is generically finite, and therefore by Lemma 12.7 and Lemma 13.7 the Betti numbers satisfy the inequalities \( b_1(X') \geq b_1(Y) \), \( b_2(X') > b_2(Y) \). Both surfaces \( X' \) and \( Y \) are Calabi–Yau and therefore by the classification of surfaces (see Proposition 12.1) we obtain a table where the entries contain the contradictory inequalities of Betti numbers for the associated possibilities for types of \( X' \) and \( Y \) (respectively in rows and columns).

|       | K3       | abelian   | Enriques  | hyperelliptic |
|-------|----------|----------|-----------|--------------|
| K3    | 22 = b_2(X') \not\equiv b_2(Y) = 22 | 0 = b_1(X') \not\equiv b_1(Y) = 4 | No contradiction | 0 = b_1(X') \not\equiv b_1(Y) = 2 |
| abelian | 2 = b_2(X') \not\equiv b_2(Y) = 22 | 4 = b_1(X') \not\equiv b_1(Y) = 4 | 2 = b_2(X') \not\equiv b_2(Y) = 10 | No contradiction |
| Enriques | 10 = b_2(X') \not\equiv b_2(Y) = 22 | 0 = b_1(X') \not\equiv b_1(Y) = 4 | 10 = b_2(X') \not\equiv b_2(Y) = 10 | 0 = b_1(X') \not\equiv b_1(Y) = 2 |
| hyperelliptic | 2 = b_2(X') \not\equiv b_2(Y) = 22 | 2 = b_1(X') \not\equiv b_1(Y) = 4 | 2 = b_2(X') \not\equiv b_2(Y) = 10 | 2 = b_2(X') \not\equiv b_2(Y) = 2 |
| quasi-elliptic | 2 = b_2(X') \not\equiv b_2(Y) = 22 | 2 = b_1(X') \not\equiv b_1(Y) = 4 | 2 = b_2(X') \not\equiv b_2(Y) = 10 | 2 = b_1(X') \not\equiv b_1(Y) = 2 |

This leaves us with the following cases:

(1) \( X' \) is abelian and \( Y \) is hyperelliptic, which gives a contradiction because \( X \) was singular canonical, and hence \( X' \) contains a rational curve, or

(2) \( X' \) is a K3 surface and \( Y \) is an Enriques surface.

In the latter case, we first observe that if \( p > 2 \) we may consider the cartesian diagram

\[
\begin{array}{ccc}
X' & \times_Y & Y' \\
\text{ét} & \downarrow & \text{ét} \\
X' & \rightarrow & Y.
\end{array}
\]

where \( Y' \to Y \) is the canonical étale K3 cover of \( Y \). Since \( X' \) is a K3 surface, it is simply connected, and hence the map \( X' \times_Y Y' \to X' \) admits a section yielding a generically finite morphism \( X' \to Y' \). The map \( X' \to Y' \) contracts a rational curve and therefore \( b_2(X') > b_2(Y') \) which is a contradiction, because both \( X' \) and \( Y' \) are K3 surfaces. If \( p = 2 \), we use the weakly ordinary assumption to see that \( Y \) is in fact a singular Enriques surface (see Proposition 12.1 for the definition of singular Enriques surfaces), and hence we may repeat the above argument because a required \( Y' \) also exists. \( \square \)
Remark 13.9. Without the weak ordinarity assumption the last part of the above argument breaks down. There are examples Enriques surfaces in characteristic two (non $F$-split and called supersingular) which are covered by singular varieties with trivial canonical class.

**Appendix A. Characteristic zero results — jointly with Giulio Codogni**

The aim of this appendix is to generalize the result of [Cao17, Theorem 1.2] [Cao17, Proposition 2.8] concerning the structure of the Albanese morphism of varieties with nef anticanonical class to the boundary and singular setting. The boundary free problems were already posed in [DPS96, Section 4]. In this section we work over the field of complex numbers denoted by $\mathbb{C}$.

**Definition A.1.** Let $f: X \to T$ be a flat projective morphism of varieties defined over $\mathbb{C}$. We say that $f$ is *isotrivial* if it is locally trivial (isomorphic to the product family) in the étale topology. We say that a morphism $(X, \Delta) \to T$ of a pair is *isotrivial* if it is locally trivial in the étale topology as a morphism of pairs.

We first recall the necessary isotriviality statement in the analytic topology. We say that a flat fibration $f: X \to T$ of projective varieties is *analytically isotrivial relative to a divisor $L$ on $X$* if there exists a trivializing cover $\pi: U \to X$ in the analytic topology such that the pair $(X_U, \pi^*L)$ is isomorphic to $(X_t \times U, L_t \times U)$ for a closed point $t \in T$. The definition clearly extends to the setting with boundary.

We shall need some results concerning numerically flat bundles in characteristic zero. We recall that by definition, a vector bundle $\mathcal{E}$ on a smooth projective variety $X$ is *numerically flat* if both $\mathcal{E}$ and $\mathcal{E}^\vee$ are nef vector bundles. In characteristic zero this condition turns out to be equivalent to $H$-semistability and the vanishing

$$c_1(\mathcal{E}).H^{n-1} = c_2(\mathcal{E}).H^{n-2} = 0,$$

for every (equiv. any) ample line bundle $H$ on $X$. Additionally, in characteristic zero, it is also equivalent to the condition that $\mathcal{E}$ comes from an extension of irreducible unitary flat bundles (equiv. irreducible unitary representations of $\pi_1(X)$). In the proof of the following proposition, we summarize the known results and justify some of the above claims. We refer to [Lan11, §1.3 and §5] for a detailed discussion.

**Proposition A.2** (Numerically flat bundles and local systems). Let $T$ be a smooth projective variety. Then for every numerically flat bundle $\mathcal{E}$ there exists a canonical choice of a holomorphic integrable connection on $\mathcal{E}$. Moreover, every $\mathcal{O}_X$-linear homomorphism of numerically flat bundles is parallel with respect to the canonical connections.

**Proof.** First of all, we observe using [DPS94, Thm 1.18 & Cor 1.19] that every numerically flat bundle is an extension of unitary local systems and hence also a semistable bundle with vanishing Chern classes. The association $\mathcal{E} \to (\mathcal{E}, 0)$, where 0 denotes the zero morphism $\mathcal{E} \to \mathcal{E} \otimes \Omega^1_T$, yields a fully faithful functor from the category of numerically flat bundles to the category of semistable Higgs sheaves $\mathcal{E}$ satisfying $c_1(\mathcal{E}) = c_2(\mathcal{E}) = 0$. By [Sim92, Corollary 3.10], the latter category is in fact equivalent to the category of local systems. By the remark at the end of subsection “Examples” of [Sim92, Section 3], the equivalence does not change the holomorphic structure of the bundles in our case which consequently gives the claim of the proposition. \qed

**Proposition A.3** ([Cao17, Proposition 2.8]). Let $f: X \to T$ be a flat projective fibration of a normal variety $X$ onto a smooth projective variety $T$, and let $L$ be an $f$-ample line bundle. Assume that $f_1\mathcal{O}_X(nL)$ is a numerically flat vector bundle for every integer $n \geq 1$. Then, the morphism $f$ is analytically isotrivial relative to some multiple of the divisor $L$.

Assume further that $X$ is $\mathbb{Q}$-factorial and equipped with a boundary divisor $\Delta$ whose components do not contain any fibres of $f$ and such that $(X, \Delta)$ is klt. Then the morphism $f: (X, \Delta) \to T$ is analytically isotrivial relative to some multiple of the divisor $L$ if the
sheaves $f_*\mathcal{O}_X(nL - D)$ are numerically flat (in particular locally free), for every irreducible component $D$ of the boundary $\Delta$.

**Proof.** Since $L$ is $f$-ample, we observe that for $m > 0$ sufficiently large the formation of $f_*\mathcal{O}_X(nmL)$ commutes with base change and that the family $f: X \to T$ is determined by the sheaf of rings $\mathcal{R}_{X/T} = \bigoplus_{n \in \mathbb{N}} f_*\mathcal{O}_X(nmL)$. Under our assumptions the bundles $f_*\mathcal{O}_X(nmL)$ forming $\mathcal{R}_{X/T}$ are numerically flat. By Proposition A.2 this implies that they admit a structure of local systems which in turn means they are trivialized on the universal cover. Taking a sufficiently large integer $m > 0$ we may assume that the natural multiplication maps

$$Sym^n f_*\mathcal{O}_X(mL) \to f_*\mathcal{O}_X(nmL)$$

are surjective. Using [Laz04, Theorem 6.2.12 (iii)], we see that the symmetric power bundles are also numerically flat, and consequently the multiplication is a surjective morphism of numerically flat bundles. Consequently, by Proposition A.2 the homomorphism is parallel with respect to canonical connections. In particular, the kernels of multiplications maps, describing the family entirely, are also local systems and hence are trivialized on the universal cover. Hence, $\mathcal{R}_{X/T}$ itself is a sheaf of rings trivialized on the universal cover. This concludes the proof of the first part of the statement.

For the second part of the statement, our assumptions yield that the sheaf of ideals $\bigoplus_{n \in \mathbb{N}} f_*\mathcal{O}_X(nmL - D) =: \mathcal{I}_{D/X} \subseteq \mathcal{R}_{X/T}$ is also numerically flat. Hence as above, $\mathcal{I}_{D/X}$ trivializes on the universal cover with $\mathcal{I}_{D/X} \hookrightarrow \mathcal{R}_{X/T}$ being horizontal as well. So, it is enough to show that for every $t \in T$ we have $\mathcal{I}_{D/X}|_{\Delta} \cong I_{D_t/\Delta}$, as then not only the fibers of $X \to T$ are identified on the universal cover, but also the ideals of the fibers of the components of the boundary. We defer the proof of this statement to the following Lemma A.4 since the result will actually be used in a few other places.

We also remark that we crucially take advantage of numerical flatness of the bundles in question. A morphism of arbitrary local systems is not necessarily flat. For examples on varieties admitting non-trivial global differential forms there are many non-isomorphic flat structures on the trivial bundle, the identity map is the not parallel. \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill \hfill

**Lemma A.4.** Let $f: X \to T$ be a flat projective fibration of a klt pair $(X, \Delta)$ onto a smooth projective variety $T$, and let $D$ be a $\mathbb{Q}$-Cartier divisor which does not contain any irreducible components of the fibres of $f$. Suppose that $L$ is an $f$-ample line bundle. Then for every regular base change $S \to T$ the natural morphism

$$\mathcal{O}_X(nL - D)|_{X_S} \to \mathcal{O}_X(nL_S - D_S)$$

is an isomorphism for every $n \in \mathbb{Z}$. Moreover, for $n \gg 0$ the higher direct images satisfy the vanishing

$$R^i f_*\mathcal{O}_X(nL - D) = 0$$

for $i > 0$ and universally, that is, after any base change. For all such $n$ the sheaves $f_*\mathcal{O}_X(nL - D)$ are locally free with formation commuting with any regular base change, and

**Proof.** First, we claim that for $n$ sufficiently large we have the appropriate base change property for $\mathcal{O}_X(nL - D)$. We recall that $X_S$ denotes the associated base change $X \times_T S$. Since $(X, \Delta)$ is klt and $D$ is $\mathbb{Q}$-Cartier, for every $n > 0$ the sheaves $\mathcal{O}_X(nL - D)$ are Cohen–Macaulay (see [KM98, Corollary 5.25]). This property is preserved under regular base change, and therefore the natural map

$(1.4.a)$

$$\mathcal{O}_X(nL - D)|_{X_S} \simeq \mathcal{O}_{X_S}(nL_S - D_S)$$

The sheaf $\mathcal{O}_X(nL - D)$ is Cohen–Macaulay, and hence reflexive of rank one after every base change. The associated divisor is equal to $nL_S - D_S$ because $D$ does not contain any fibres.
is an isomorphism. By [BHPS13, Corollary 2.14] the sheaves $\mathcal{O}_X(nL - D)$ are flat. Consequently using (1.4.a) in conjunction with the $f$-ampleness of $L$ and the cohomology and base change theorem, for $n \gg 0$, we have the necessary vanishing of higher direct images and the identification $f_*\mathcal{O}_X(nL - D)|_{X_s} \cong f_{S*}\mathcal{O}_{X_S}(nLS - DS)$. □

**Proposition A.5** (Algebraization of Proposition A.3). Let $f : X \to T$ be a flat projective fibration of a normal variety $X$ onto a smooth projective variety, and let $L$ be an $f$-ample line bundle. Assume that the bundles $f_*\mathcal{O}_X(nL)$ are numerically flat, for every $n > 0$. Then the morphism $f$ is isotrivial over $T$ relative to some multiple of $L$.

The same statement holds true for a morphism of pairs $(X, \Delta) \to T$ provided that $X$ is $\mathbb{Q}$-factorial, no irreducible components of the boundary $\Delta$ contain any fibers of $f$, the pair $(X, \Delta)$ is klt and the sheaves $f_*\mathcal{O}_X(nL - D)$ are numerically flat vector bundles for every integer $n > 0$.

**Proof.** We begin by using Proposition A.3 to see that the family is analytically isotrivial relative to some multiple of $L$. Using GAGA this means that there exists an integer $m$ such that for every two points $t, t' \in T$ the pairs $(X_t, mL_t)$ and $(X_{t'}, mL_{t'})$ are isomorphic. Consequently, the relative isomorphism scheme $\text{Isom}_{T}((X, mL), (X_t \times T, mL_t \times T))$ is a torsor over $T$ under a finite type smooth (we work in characteristic zero) group scheme $\text{Aut}(X_t, mL_t)$. The map $\text{Isom}_{T}((X, mL), (X_t \times T, mL_t \times T)) \to T$ is therefore smooth and hence admits sections étale locally. This finishes the proof of the first part of the proposition. The same argument works in the boundary setting by substituting the relevant isomorphism scheme with its boundary version. See Construction 7.6 for the construction, and Proposition 7.8 for the appropriate base change properties. □

**Remark A.6.** From the proof (see the final remarks in the proof of Proposition A.3) we directly see that the multiplying factor $m$ necessary to get analytic isotriviality with respect to line bundle $mL$ is equal to one if the natural multiplication map $\text{Sym}^n \mathcal{O}_X(L) \to \mathcal{O}_X(nL)$ is surjective for every $n \geq 1$.

Since we develop tools to get numerical flatness of relative section bundles only over curves, we need the following lemma for bases of higher dimension.

**Lemma A.7** (Algebraization of Proposition A.3). Let $f : X \to T$ be a flat projective fibration defined over $\mathbb{C}$, and let $L$ be an $f$-ample line bundle. Suppose that direct images of $\mathcal{O}_X(nL)$ satisfy the following conditions:

- $f_*\mathcal{O}_X(nL)$ are locally free with formation commuting with any base change,
- the multiplication map $\text{Sym}^m \mathcal{O}_X(L) \to \mathcal{O}_X(mL)$ is surjective, for every $m \geq 1$.

Moreover, assume that for a covering family of smooth projective curves $\{C_s \subset X\}_{s \in S}$ the vector bundle

$$f_*\mathcal{O}_X(nL)|_{C_s} \cong f_{C_s*}\mathcal{O}_{X_{C_s}}(nL_{|C_s})$$

**[formation of pushforward commutes with base change]**

is numerically flat, for every $n > 0$ and for every $s \in S$. Then the morphism $f$ is isotrivial over $T$ relative to $L$.

**Proof.** As in the proof of Proposition A.5 we consider the relative isomorphism scheme $\text{Isom}_{T}((X, L), (X_t \times T, L_t \times T))$. In order to get our claim, it suffices to prove that the natural map from the isomorphism scheme is surjective onto the base $T$. We denote by $X_{C_s}$ the base change $X \times_T C_s$, and by $f_{C_s}$ the natural projection. Since the family $\{C_s\}_{s \in S}$ covers $T$, the bundles $f_{C_s*}\mathcal{O}_{X_{C_s}}(nL_{|C_s})$ are isomorphic to $f_*\mathcal{O}_X(nL)|_{C_s}$, and the surjectivity of the multiplication map which is inherited by the bundles $f_{C_s*}\mathcal{O}_{X_{C_s}}(nL_{|C_s})$ (by the base change and the assumptions), using base change theorem for the isomorphism scheme (see Proposition 7.8) we may reduce to the case of $T = C_s$. This finishes the proof using Proposition A.5 along with Remark A.6. □
Remark A.8. We note that the cohomological criteria necessary in Lemma A.7 can be guaranteed by taking a sufficient power of the line bundle $L$. Moreover, the statement of Lemma A.7 holds true for a morphism of pairs $(X, \Delta) \to T$ provided that every irreducible component $D$ of the boundary $\Delta$ intersect every fibre of $f$ properly and the formation of $f_*\mathcal{O}_X(nL - D)$ commutes with the base change $C_s \to T$ yielding numerically flat bundles

$$f_*\mathcal{O}_X(nL - D)|_{C_s} \simeq f_{C_s*}\mathcal{O}_{X_{C_s}}(nL_{C_s} - D_{C_s})$$

for every sufficiently large integer $n > 0$, where $f_{C_s} : X_{C_s} \to C_s$ is the base change of $f$ along the inclusion $C_s \to T$. This happens for example if the divisor $D$ is $\mathbb{Q}$-Cartier and the pair $(X, \Delta)$ is klt. The proof is provided in Lemma A.4.

A.1. Semi-positivity in characteristic zero

Here we restate and provide the reference for the characteristic zero version of the main result of Section 3.

**Theorem A.9.** Assume we are in the following situation:

1. $f : X \to T$ is an equidimensional fibration between normal projective varieties with normal general fibers,
2. $U \subseteq T$ is a non-empty open set,
3. $\Gamma \geq 0$ be a $\mathbb{Q}$-divisor such that $K_{X/T} + \Gamma | f^{-1}U$ is $\mathbb{Q}$-Cartier,
4. $L$ is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $K_{X/T} + \Gamma + L$ is $\mathbb{Q}$-nef over $U$, and
5. $(X_t, \Delta_t)$ is klt for every closed point $t \in U$.

Then $K_{X/T} + \Gamma + L$ is pseudo-effective.

**Case ($\ast$):** If $T$ is a curve, and $K_{X/T} + \Gamma + L$ is $\mathbb{Q}$-nef $\mathbb{Q}$-Cartier (so globally, not only over $U$), then $K_{X/T} + \Gamma + L$ is not only pseudo-effective but also nef.

**Proof.** Fix an ample divisor $H$ on $X$. As in the proof of Theorem 3.1, it is enough to show that for every rational number $\varepsilon > 0$ and every divisible enough integer $m > 0$ (where divisible enough depends on $\varepsilon$), $f_*\mathcal{O}_X(m(K_{X/T} + \Gamma + L + \varepsilon H))$ is weakly positive. So, we prove this statement in the rest of the proof, for which we fix a rational number $\varepsilon > 0$. Note that by fixing a general effective $\mathbb{Q}$-divisor $\Gamma' \sim_{\mathbb{Q}} L + \varepsilon H$ and setting $\Delta := \Gamma + \Gamma'$, it is enough to show that $f_*\mathcal{O}_X(m(K_{X/T} + \Delta))$ is weakly positive. Additionally, $(X_t, \Delta_t)$ is klt for every $t \in U$, after possibly shrinking $U$.

Let $\pi : Y \to X$ be a log resolution of $(Y, \Delta)$, and set $g := f \circ \pi$. Let $\Delta_Y$ be the crepant boundary on $Y$. Then, it is enough to show that $g_*\mathcal{O}_Y(m(K_{Y/T} + \{\Delta_Y\}))$ is weakly positive for every integer $m > 0$ divisible enough:

$$K_Y + \{\Delta_Y\} = \pi^*(K_Y + \Delta) - [\Delta_Y]$$

However, the semi-positivity of $g_*\mathcal{O}_Y(m(K_{Y/T} + \{\Delta_Y\}))$ is exactly the statement of [Fuj14, Thm 1.1].

A.2. Flatness and reducedness of fibres

In this subsection we state the generalization of the result of [LTZZ10] to the case of an arbitrary map $(X, \Delta) \to T$ where $(X, \Delta)$ is klt and $-K_{X/T} - \Delta$ is nef. As described in Section 4 our argument is very similar to the one given in loc.cit.

**Proposition A.10.** Let $f : (X, \Delta) \to T$ be a morphism from a normal, projective klt pair to a smooth variety such that $-K_{X/T} - \Delta$ is nef. Then $\Delta$ does not contain any irreducible components of the fibres of $f$, the morphism $f$ is flat and all its fibres are reduced.
Proposition A.11. Let \( f: (X, \Delta) \to T \) be a fibration from a normal, projective pair to a smooth curve. Assume that \( -K_{X/T} - \Delta \) is \((\mathbb{Q}\text{-Cartier})\) nef, and that the general fibre \((X_t, \Delta_t)\) is klt. Let \( L \) be an \( f \)-ample line bundle on \( X \) satisfying \( L^{\dim X} = 0 \). Then the following statements hold true.

1. For every \( m > 0 \) the vector bundle \( f_*\mathcal{O}_X(mL) \) is numerically flat.
2. Let \( D \) be an irreducible component of the divisor \( \text{Supp} \Delta \). Assume that \( D \) is \( \mathbb{Q}\text{-Cartier} \). Then for every \( m > 0 \) such that \( mL - D \) is \( f \)-ample, the sheaf \( f_*\mathcal{O}_X(mL - D) \) is a numerically flat vector bundle.

Proof. The following argument is a characteristic zero combination of Theorem 5.4 and Theorem 5.10. We recall the argument for the sake of completeness and because some characteristic \( p > 0 \) subtleties are not relevant in the present proof.

We first prove statement (1). We observe that by Proposition A.10 the map \( f \) is flat with reduced fibres, and hence we can use the semi-positivity result from Theorem A.9. We now proceed to the proof of the fact that \( f_*\mathcal{O}_X(mL) \) is nef. We begin by showing that \( L \) is nef. We fix \( \varepsilon > 0 \). Consider \( L + \varepsilon f^*H \). It is enough to prove, by limiting \( \varepsilon \to 0 \), that \( L + \varepsilon f^*H \) is nef. Set \( N := q(L + \varepsilon f^*H) \), for \( q \gg 0 \). By Lemma 5.3, we know that

\[
h^0(X, N) \geq q^{d+1} \frac{(L + \varepsilon f^*H)^{d+1}}{(d+1)!} + O(q).
\]

Since

\[
(L + \varepsilon f^*H)^{d+1} = \varepsilon(d+1)(\deg H)(L^n) > 0,
\]

we obtain that \( h^0(X, N) \neq 0 \) (using \( q \gg 0 \)). So, choose \( \Gamma \in |N| \). For \( \varepsilon' \) small enough, \((X_t, \Delta_t + \varepsilon'T)\) is klt for \( t \in T \) general. Hence, according to Theorem A.9, the following divisor is nef

\[
K_{X/T} + \Delta + \varepsilon'T \quad \text{(}(X_t, \Delta_t + \varepsilon'T)\ \text{is klt for} \ t \in T \ \text{general)}
\]

\[
= \varepsilon'q(L + \varepsilon f^*H)
\]

\[
f\text{-ample}
\]

Hence \( L + \varepsilon f^*H \) is nef, which concludes our statement.

Now we deduce that \( f_*\mathcal{O}_X(mL) \) is nef. For this purpose, we observe that

\[
mL = K_{X/T} + \Delta + mL + (-K_{X/T} - \Delta)
\]

and hence by [CP18, Proposition 6.3] the pushforward \( f_*\mathcal{O}_X(mL) \) is nef.

We now show that \( f_*\mathcal{O}_X(mL)^* \) is nef. Assume for the sake of contradiction that this is not the case. According to the characteristic zero version of Lemma 5.6 (FIND char. zero reference for that), this means that

\[
\mu_{\max}(f_*\mathcal{O}_X(aL)) > 0
\]

We take the maximal destabilizing sheaf \( \mathcal{E} \) of \( f_*\mathcal{O}_X(aL) \). According to [Laz04, Corollary 6.4.14], \( \mathcal{E}^\otimes r \) is semi-stable with slope \( r\mu(\mathcal{E}) \). In particular, we may fix \( r > 0 \) such that
\[ \mu(E^{\otimes r}) > 2g(T) + 1. \] Consider then the composition
\[ E^{\otimes r} \hookrightarrow (f_*O_X(aL)^{\otimes r}) \rightarrow f_*O_X(raL). \]
This is non-zero (diagonal tensors do not go to zero). So, we obtain that
\[ \mu_{\text{max}}(f_*O_X(raL)) > 2g(T) + 1. \]

Hence for any closed point \( t \in T \), by \([CP18, \text{Prop} 5.7]\), we have \( h^0(T, (f_*O_X(raL))(-t)) \neq 0 \).

Choose \( 0 \neq \Gamma \in [raL - X_t] \). Then for any \( 0 < \varepsilon \ll 1 \), \((X_t, \Delta_t + \varepsilon \Gamma_t)\) is a klt pair for \( t \in T \) general, where the notion of generality depends on \( \varepsilon \). In particular, according to \([CP18, \text{Corollary 6.4}]\), the following divisor is nef:
\[
\varepsilon \Gamma \sim_{Q} \begin{cases} K_{X/T} + \Delta + \varepsilon \Gamma & (X_t, \Delta_t + \varepsilon \Gamma_t) \text{ is klt for } t \in T \text{ general} \\ \text{f-ample} & \text{nef} \end{cases}
\]

However, then
\[ \Gamma^{d+1} = (raL - X_t)^{d+1} = -(n + 1)raL^d_t < 0, \]
which contradicts nefness of \( \Gamma \). Hence, our initial assumption was false, which concludes the proof of (1).

We now proceed to the proof of (2).

First, we prove that for \( m \gg 0 \) sufficiently large the divisor \( mL - D \) is nef. Let \( \varepsilon = 1/m \) and choose \( m \gg 0 \) such that \( L - \varepsilon D \) is f-ample. The claim follows from the semipositivity result and the relation
\[ L - \varepsilon D = \begin{cases} K_{Y/T} + \Delta - \varepsilon D & (X_t, \Delta_t - \varepsilon D_t) \text{ is klt for } t \in T \text{ general, and } m \gg 0 \\ \text{f-ample} & \text{nef} \end{cases} \]

In order to see that \( f_*O_X(mL - D) \) is nef, we consider the relation
\[ mL - D = K_{X/T} + \Delta + (-K_{X/T} - \Delta) + mL - D \]
and apply \([CP18, \text{Proposition 6.3}]\) as above. However, this already implies that \( f_*O_X(mL - D) \) is numerically flat. Indeed, the vector bundle \( f_*O_X(mL - D) \) is nef, and hence of non-negative degree. It is also a torsion-free subsheaf of \( f_*O_X(mL) \) which is numerically flat, and hence semistable of degree zero. This implies that \( f_*O_X(mL - D) \) is of degree zero and hence numerically flat by the result of Langer (see \([Lan11, \text{Proposition 5.1}]\)) recalled in Proposition 2.4.

\( \square \)

A.4. The proof of the main result

**Theorem A.12.** Let \( f : (X, \Delta) \rightarrow T \) be a fibration of a normal projective \( \mathbb{Q} \)-factorial pair onto a smooth projective variety. Assume that the \(-K_{X/T} - \Delta \) is nef and that \((X, \Delta)\) is klt. Then \( f \) is isotrivial, and in particular all the fibres \( f \) are isomorphic.

**Proof.** By Proposition A.10, \( f \) is flat with reduced fibres, and \( \Delta \) does not contain any fiber. Let \( d \) be the relative dimension of \( f \). As in the proof of Theorem 9.2 we find an f-ample divisor \( \tilde{L} \) such that \( \tilde{L}^{d+1} \cdot H^{n-d-1} = 0 \), for a given very ample divisor \( H \) on \( T \), and the following conditions are satisfied
(a) the bundles \( f_*O_X(n\tilde{L}) \) are locally free with formation commuting with any base change, 
(b) the multiplication map \( \text{Sym}^n O_X(n\tilde{L}) \rightarrow O_X(n\tilde{L}) \) is surjective, for every \( n > 0 \)
(c) the line bundle $L$ is sufficiently $f$-ample (so that $nL - D$ is $f$-ample for every $n > 0$) and satisfies the vanishing $R^i f_* \mathcal{O}_X(nL - D) = 0$ for every irreducible component $D$ of $\text{Supp} \Delta$ and $i > 0$.

The intersection condition implies that for a general smooth curve $C$ in the intersection $H^{n-d-1}$ the line bundle $L = \mathcal{L}|_{X_C}$, where $X_C = X \times_T C$, satisfies the equality $L^{d+1} = 0$. Here general means that it is not contained in the proper closed subvariety of the base over which the fibers are not normal.

We claim that the morphism $f_C : X_C \to C$ satisfies the assumptions of Proposition A.11. Indeed, we see that the fibers of $f$ are reduced by Proposition A.10 and hence $X_C$ is normal. Moreover the general fibre of $(X_C, \Delta_C) \to C$ is klt, and by [CP18, Proposition 2.1] we have the base change formula $-K_{X_C/C} - \Delta_C = (-K_{X/T} - \Delta)|_{X_C}$ yielding nefness of $-K_{X/C} - \Delta_C$.

As a result, we see using the base change and Proposition A.11 that the line bundle $\mathcal{L}$ and the family of smooth curves in $H^{n-d-1}$ satisfies the assumption of Lemma A.7 which implies that the morphism $f : X \to T$ is isotrivial over $T$. In order to extend this to the boundary setting, we observe that the conclusion of Remark A.8 and Proposition A.11 could be used in the boundary setting, that is, the required sufficient $f$-ampleness property holds uniformly for a family of curves covering a big open subset of $T$. More precisely, the condition (c) above and the $(X, \Delta)$ klt assumption ensure that both the base change property for the relative section rings (see Lemma A.4) and the $f$-ampleness of $nL - D$ necessary in Proposition A.11 are satisfied. We may extend isotriviality from the open subset covered by a family of smooth complete intersection to the whole $T$ by taking a closure, because the divisors are uniquely determined in codimension one.

\[ \square \]

**Theorem A.13.** Let $f : (X, \Delta) \to T$ be a fibration of a normal $\mathbb{Q}$-factorial projective pair onto a normal projective variety. Assume that the $-K_{X/T} - \Delta$ is nef and that $(X, \Delta)$ is klt. Then $f$ is generically locally isotrivial.

**Proof.** The proof is very similar to the argument given above. We claim that the fibrations is isotrivial over the smooth locus $U$ of $T$. Indeed, as in the proof of Theorem A.12, we construct a divisor $\mathcal{L}$ and a family of complete intersection curves covering $T$. Since $T$ is normal, its singular locus is of codimension at least two, and therefore there exists a subfamily of smooth curves $\{C_s\}_{s \in S}$ covering $U$. We conclude as above. \[ \square \]

**Corollary A.14.** Let $(X, \Delta)$ be a projective $\mathbb{Q}$-factorial klt pair such that $-K_X - \Delta$ is nef. Then the Albanese morphism $\pi : (X, \Delta) \to \text{Alb} X$ is an isotrivial fibration.

**Proof.** First, using the result of Section 2.8.2, we see that $-K_{X/T} - \Delta$ is nef. Since $(X, \Delta)$ is klt, the general fibre $(X, \Delta_t)$ of $\pi$ is klt as well, and we may therefore apply Theorem A.12 to conclude. \[ \square \]

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