Isoparametric submanifolds in Hilbert spaces and holonomy maps

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Abstract

Let \( \pi : P \to B \) be a smooth \( G \)-bundle over a compact Riemannian manifold \( B \) and \( c \) a smooth loop in \( B \) of constant seed \( a(>0) \), where \( G \) is compact semi-simple Lie group. In this paper, we prove that the holonomy map \( \text{hol}_c : \mathcal{A}^H_0 P \to G \) is a homothetic submersion of coefficient \( a \), where \( s \) is a non-negative integer, \( \mathcal{A}^H_s P \) is the Hilbert space of all \( H^s \)-connections of the bundle \( P \). In particular, we prove that, if \( s = 0 \), then \( \text{hol}_c \) has minimal regularizable fibres. From this fact, we can derive that each component of the inverse image of any equifocal submanifold in \( G \) by the holonomy map \( \text{hol}_c : \mathcal{A}^H_0 P \to G \) is an isoparametric submanifold in \( \mathcal{A}^H_0 P \). As the result, we obtain a new systematic construction of isoparametric submanifolds in a Hilbert space.

1 Introduction

R. S. Palais and C. L. Terng \cite{PaTe} introduced the notion of a proper Fredholm submanifold in a (separable) Hilbert space as an immersed (Hilbert) submanifold of finite codimension satisfying the following conditions:

(PF-i) The normal exponential map \( \exp^\perp \) of the submanifold is a Fredholm map;

(PF-ii) The restriction of \( \exp^\perp \) to the ball normal bundle of any radius is proper.

Since the shape operators of a proper Fredholm submanifold are self-adjoint and compact operators, the set of all eigenvalues of each shape operator is bounded and has no accumulating point other than 0. Also, the multiplicities of the nonzero eigenvalues are finite. Hence, for each unit normal vector \( \xi \) of the submanifold, the set of of all focal radii (which are equal to the inverses of the nonzero eigenvalues of the shape operator \( A_\xi \) of the submanifold along the normal geodesic \( \gamma_\xi \) of \( \xi \)-direction has no accumulating points and the multiplicities of each focal radii are finite. Thus a proper Fredholm submanifold has a good focal structure similar to finite dimensional submanifolds. Furthermore,
they (PaTe) defined the notion of an isoparametric submanifold in a Hilbert space as a proper Fredholm submanifold satisfying the following conditions:

(I-i) The normal holonomy group of the submanifold is trivial;

(I-ii) For any parallel normal vector field \( \tilde{\xi} \) of the submanifold, the set of all eigenvalues of the shape operator \( A_{\tilde{\xi}} \) is independent of the choice of the base point \( p \) with considering the multiplicities.

See GH, H, HL, PaTe, PiTh, T1, T2 and TT about the study of isoparametric submanifolds in a Hilbert space.

Next we recall the definitions of the regularized traces of a self-adjoint and compact operator of a (separable) Hilbert space in the sense of KT and HLO. Let \( A \) be a self-adjoint operator of a (separable) Hilbert space \((V, \langle \cdot, \cdot \rangle)\) and

\[
-\lambda_1^- \leq -\lambda_2^- \leq \cdots \leq 0 \leq \cdots \leq \lambda_2^+ \leq \lambda_1^+
\]

be the spectrum of A. C. King and C. L. Terng (KT) defined the \( L^s \)-norm \( \|A\|_s \) (\( s > 1 \)) of of a self-adjoint and compact operator \( A \) by

\[
\|A\|_s := \left( \sum_{i=1}^{\infty} (\lambda_i^+)^s + (\lambda_i^-)^s \right)^{\frac{1}{s}}.
\]

They called that \( A \) is regularizable if \( \|A\|_s < \infty \) for all \( s > 1 \) and if

\[
(1.1) \lim_{s \downarrow 1} \sum_{i=1}^{\infty} ((\lambda_i^+)^s - (\lambda_i^-)^s)
\]

exists. Also, they called the limit in (1.1) the regularized trace of \( A \). In HLO, this regularized trace was called the \( \zeta \)-regularized trace. Later, E. Heintze, X. Liu and C. Olmos (HLO) called that \( A \) is regularizable if \( \|A\|_2 < \infty \) and

\[
(1.2) \sum_{i=1}^{\infty} (\lambda_i^+ - \lambda_i^-)
\]

exists. Also, they called the limit in (1.2) the regularized trace of \( A \). The regularized trace in the sense of HLO is easier to handle than one in the sense of KT. We denote the regularized trace (in the sense of HLO) (resp. the \( \zeta \)-regularized trace) of \( A \) by \( \text{Tr}_r A \) (resp. \( \text{Tr}_\zeta A \)), where the subscript “r” in \( \text{Tr}_r \) implies the initial letter of “regularized”. In almost all relevant cases, these regularized traces coincide. In this paper, we shall use the terminologies ”regularized trace” and “regularizable” in the sense of HLO. Here we give an example of a self-adjoint and compact operator \( A \) such that \( \|A\|_s \) (\( s > 1 \)) and \( \text{Tr}_r A \) exist but \( \text{Tr}_r A \) does not exist.

Example We consider the case where the spectrum of \( A \) is given by

\[
-1 < -\frac{1}{2} < \cdots < -\frac{1}{k} < \cdots \leq 0 \leq \cdots < \frac{2}{k} < \cdots < 1 < 2,
\]
where we note that the multiplicities of all eigenvalues of $A$ are equal to 1. Then we have

$$
\|A\|_s = \sum_{k=1}^{\infty} \left( \left( \frac{2}{k} \right)^s + \left( \frac{1}{k} \right)^s \right) = \sum_{k=1}^{\infty} \frac{2^s + 1}{k^s} $$

$$
= (2^s + 1) \left( 1 + \sum_{k=2}^{\infty} \frac{1}{k^s} \right) < (2^s + 1) \left( 1 + \int_1^{\infty} \frac{1}{x^s} \, dx \right) $$

$$
= \frac{s(2^s + 1)}{s - 1} \quad (s > 1)
$$

and

$$
\text{Tr}_A = \lim_{s \downarrow 1} \sum_{k=1}^{\infty} \left( \left( \frac{2}{k} \right)^s - \left( \frac{1}{k} \right)^s \right) = \lim_{s \downarrow 1} \sum_{k=1}^{\infty} \frac{2^s - 1}{k^s} $$

$$
= \lim_{s \downarrow 1} (2^s - 1) \left( 1 + \sum_{k=2}^{\infty} \frac{1}{k^s} \right) < \lim_{s \downarrow 1} (2^s - 1) \left( 1 + \int_1^{\infty} \frac{1}{x^s} \, dx \right) $$

$$
= \lim_{s \downarrow 1} \frac{s(2^s - 1)}{s - 1} = 1 + 2 \ln 2.
$$

Thus both $\|A\|_s$ $(s > 1)$ and $\text{Tr}_A$ exists. Hence $A$ is regularizable in the sense of [KT]. On the other hand, we have

$$
\text{Tr}_A = \sum_{k=1}^{\infty} \left( \frac{2}{k} - \frac{1}{k} \right) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.
$$

Thus $\text{Tr}_A$ does not exist. Hence $A$ is not regularizable in the sense of [HLO].

Let $M$ be a proper Fredholm submanifold in $(V, \langle , \rangle)$ immersed by $f$. If, for any $p \in M$ and any normal vector $\xi$ of $M$ at $p$, there exist the regularized trace $\text{Tr}_A (A_p)_{\xi}$ of the shape operator $(A_p)_{\xi}$ of $M$ for $\xi$ and the trace $\text{Tr} (A_p)_{\xi}^2$ of $(A_p)_{\xi}^2$, then $M$ is called a regularizable submanifold. In particular, if, for any $p \in M$ and any normal vector $\xi$ of $M$ at $p$, $\text{Tr}_A (A_p)_{\xi}$ vanishes, then $M$ is called a minimal regularizable submanifold.

Y. Maeda, S. Rosenberg and P. Tondeur ([MRT]) studied the submanifold structure of the gauge orbit in the Hilbert space of all $H^0$-connections of a $G$-bundle $P$ over a compact Riemannian manifold $(B, g_B)$, where $G$ is a compact semi-simple Lie group. Denote by $A^H_0$ the Hilbert space of all $H^0$-connections of $P$ and $\mathcal{G}^H_0$ the $H^1$-gauge transformation group of $P$. Note that, if dim $B \geq 2$, then $\mathcal{G}^H_0$ is not a smooth Hilbert Lie group and the gauge action $\mathcal{G}^H_0 \rtimes A^H_0$ is not smooth. Take a $C^\infty$-curve $c : [0, 1] \to B$ and a $C^\infty$-curve $\sigma : [0, 1] \to P$ with $\pi \circ \sigma = c$. Let $c^*P$ be the pull-back bundle of $P$ by $c$, which is identified with the trivial $G$-bundle $[0, 1] \times G$ over $[0, 1]$ by $\sigma$. Hence the Hilbert space of all $H^0$-connections of $c^*P$ is identified with the Hilbert space $H^0([0, 1], \mathfrak{g})$ of all
\(H^0\)-curves in the Lie algebra \(\mathfrak{g}\) of \(G\) by \(\sigma\). Also, the \(H^1\)-gauge transformation group of \(e^*P\) is identified with the Hilbert Lie group \(H^3([0, 1], G)\) of all \(H^1\)-curves in \(G\). The parallel transport map for \(G\) is a map \(\phi : H^0([0, 1], \mathfrak{g}) \to G\) defined by

\[
\phi(u) := g_u(1) \quad (u \in H^0([0, 1], \mathfrak{g}))
\]

where \(g_u\) is the element of \(H^1([0, 1], G)\) with \(g_u(0) = e\) (\(e\) : the identity element of \(G\)) and \((R_{g_u(t)})_{s=1}^{-1}(g_u'(t)) = u(t) \ (t \in [0, 1])\). The group \(H^1([0, 1], G)\) acts on \(H^0([0, 1], \mathfrak{g})\) as the (usual) action of the gauge transformation group on the space of the connections and the closed subgroup \(\Lambda_{H^1}(G)\) (of \(H^1([0, 1], G)\)) of all \(H^1\)-loops at \(e\) in \(G\) acts on \(H^0([0, 1], \mathfrak{g})\) as its subaction. This subaction \(\Lambda_{H^1}(G) \cap H^0([0, 1], \mathfrak{g})\) is transitive and the orbit space \(H^0([0, 1], \mathfrak{g})/\Lambda_{H^1}(G)\) is identified with \(G\). Furthermore the orbit map of this subaction \(\Lambda_{H^1}(G) \cap H^0([0, 1], \mathfrak{g})\) is identified with the parallel transport map \(\phi\). Fix an \(\text{Ad}(G)\)-invariant inner product \(\langle\ ,\ \rangle_{\mathfrak{g}}\) of \(\mathfrak{g}\). The parallel transport map \(\phi\) is a Riemannian submersion of the Hilbert space \(H^0([0, 1], \mathfrak{g})\) equipped with the \(H^0\)-inner product defined by \(\langle\ ,\ \rangle_{\mathfrak{g}}\) onto \(G\) equipped with a bi-invariant metric defined by \(\langle\ ,\ \rangle_{\mathfrak{g}}\). See [M1], [M2], [PaTe], [PiTh], [TT], [T2], [TT] and [K] about the study related to the parallel transport map.

In a finite dimensional complete Riemannian manifold \((\tilde{M}, \tilde{\mathfrak{g}})\), the notion of an equifocal submanifold is defined as an immersed compact submanifold satisfying the following conditions:

(i) The submanifold has abelian normal bundle (i.e., flat section);
(ii) The normal holonomy group of the submanifold is trivial;
(iii) The focal structure of the submanifold is invariant under parallel translations with respect to the normal connection.

According to Theorem 1.10 of [TT], if \(M\) is an equifocal submanifold in \(G\) equipped with the bi-invariant metric induced from a fixed \(\text{Ad}(G)\)-invariant inner product \(\langle\ ,\ \rangle_{\mathfrak{g}}\) of \(\mathfrak{g}\), then each component of \(\phi^{-1}(M)\) is an isoparametric submanifold in \(H^0([0, 1], \mathfrak{g})\) equipped with the \(L^2\)-inner product induced from \(\langle\ ,\ \rangle_{\mathfrak{g}}\).

Assume that \(s > \frac{1}{2} \dim B - 1\). Let \(A_P^{H^s}\) be the Hilbert space of all \(H^s\)-connections of \(P\) and \(G_P^{H^{s+1}}\) the group of all \(H^{s+1}\)-gauge transformations of \(P\). Then, according to Section 9 of [P], \(G_P^{H^{s+1}}\) is a smooth Hilbert Lie group. Also, according to Lemma 1.2 of [U], the gauge action \(G_P^{H^{s+1}} \curvearrowright A_P^{H^s}\) is smooth. However, \(G_P^{H^{s+1}}\) does not act isometrically on the Hilbert space \((A_P^{H^s}, \langle\ ,\ \rangle_{\omega_0})\), where \(\langle\ ,\ \rangle_{\omega_0}\) denotes the \(L^2\)-inner product defined by using an arbitrarily fixed \(C^\infty\)-connection \(\omega_0\) of \(P\) (see Section 2 about this definition). Define a Riemannian metric \(g_\omega\) on \(A_P^{H^s}\) by \((g_\omega)_{\omega} := \langle\ ,\ \rangle_{\omega}\ (\omega \in A_P^{H^s})\), where \(\langle\ ,\ \rangle_{\omega}\) denotes the \(L^2\)-inner product defined by using \(\omega\) instead of \(\omega_0\). This Riemannian metric \(g_\omega\) is non-flat. The gauge transformation group \(G_P^{H^{s+1}}\) acts isometrically on the Riemannian Hilbert manifold \((A_P^{H^s}, g_\omega)\), where we note that the Hilbert space \((A_P^{H^s}, \langle\ ,\ \rangle_{\omega_0})\) is regarded as the tangent space of \((A_P^{H^s}, g_\omega)\) at \(\omega_0\). Hence
we can give the moduli space \( M_{orbisubmersion} \) such that the orbit map \( \pi_{M_p} : (A^H_p, g_s) \rightarrow (M_{orbisubmersion}, \overline{g}_s) \) is a Riemannian orbisubmersion.

Fix an \( \text{Ad}(G) \)-invariant inner product \( \langle , \rangle_\mathfrak{g} \) of \( \mathfrak{g} \). Let \( \langle , \rangle_0 \) be the \( L^2 \)-inner product of \( A^H_p \) defined by using the Riemannian metric \( g_B \) of \( B \) and \( \langle , \rangle_\mathfrak{g} \) and \( \mathfrak{g}_G \) the bi-invariant metric of \( G \) defined by using \( \langle , \rangle_\mathfrak{g} \).

In this paper, we prove the following fact.

**Theorem A.** Let \( \pi : P \rightarrow B \) be a smooth \( G \)-bundle over a compact Riemannian manifold \((B, g_B)\) and \((A^H_p, g_s)\) the Riemannian Hilbert manifold consisting of all \( H^s \)-connections of the bundle \( P \), where \( s \) is an arbitrary non-negative number. Then, for any \( C^\infty \)-loop \( c : S^1 \rightarrow B \) of constant speed \( a(>0) \), the following statements (i) and (ii) hold:

(i) The holonomy map \( \text{hol}_c : (A^H_p, g_s) \rightarrow (G, g_G) \) along \( c \) is a homothetic submersion of coefficient \( a \). In particular, \( \text{hol}_c : (A^H_0, g_0) \rightarrow (G, g_G) \) has minimal regularizable fibres;

(ii) For an equifocal submanifold \( M \) in \((G, g_G)\), each component of \( \text{hol}_c^{-1}(M) \) is a \((G^H_p)_{c(0)}\)-invariant isoparametric submanifold in \((A^H_p, g_0)\), where \((G^H_p)_{c(0)}\) is the based gauge transformation group of class \( H^1 \) at \( c(0) \) of the bundle \( P \). Furthermore, if \( M \) is \( \text{Ad}(G) \)-invariant, then each component of \( \text{hol}_c^{-1}(M) \) is \( G^H_p \)-invariant, where \( \text{Ad} \) is the adjoint action of \( G \) on oneself and \( G^H_p \) is the gauge transformation group of class \( H^1 \) of the bundle \( P \).

## 2 The holonomy map and the pull-back connection map along a loop

In this section, we define the holonomy map along loops and the pull-back connection maps by loops for smooth principal bundle. Let \( \pi : P \rightarrow B \) be a \( G \)-bundle of class \( C^\infty \) over a compact Riemannian manifold \((B, g_B)\), where \( G \) is a compact semi-simple Lie group. Fix an \( \text{Ad}(G) \)-invariant inner product \( \langle , \rangle_\mathfrak{g} \) (for example, the \((-1)\)-multiple of the Killing form of \( \mathfrak{g} \)) of the Lie algebra \( \mathfrak{g} \) of \( G \), where \( \text{Ad} \) denotes the adjoint representation of \( G \). Denote by \( g_G \) the bi-invariant metric of \( G \) induced from \( \langle , \rangle_\mathfrak{g} \). Let \( \mathcal{A}_\mathfrak{g}^\infty \) be the affine Hilbert space of all \( C^\infty \)-connections of \( P \) and \( \Omega_{\Gamma_1}(P, g) \) be the Hilbert space of all \( \mathfrak{g} \)-valued tensorial \( i \)-form of class \( C^\infty \) on \( P \) (see [KN] about the definition of a \( \mathfrak{g} \)-valued tensorial \( i \)-form). Also, let

\[
\Omega_{\Gamma_1}^\infty(B, \text{Ad}(P)) \equiv \Gamma^\infty((\wedge^i T^* B) \otimes \text{Ad}(P))
\]

be the space of all \( \text{Ad}(P) \)-valued \( i \)-forms of class \( C^\infty \) over \( B \), where \( \text{Ad}(P) \) denotes the adjoint bundle \( P \times_{\text{Ad}} \mathfrak{g} \). The space \( \mathcal{A}_\mathfrak{g}^\infty \) is the affine space having \( \Omega_{\Gamma_1}^\infty(P, g) \) as the associated vector space. Furthermore, \( \Omega_{\Gamma_1}^\infty(P, g) \) is identified with \( \Omega_{\Gamma_1}^\infty(B, \text{Ad}(P)) \) under the correspondence \( A \leftrightarrow A \) defined by \( u \cdot A_u(v) = 5 \)
\[ \widehat{A}_{x(u)}(\pi_x(v)) \quad (u \in P, \ v \in T_u P) \], where we note that \( \widehat{A} \) is the section of \( \text{Ad}(P) \)
defined from the pullbacks of \( A \) by local sections of \( P \).

According to [GP2], we shall define the \( H^s \)-completion of \( \Omega^\infty_{T^1}(P, g) \), where \( s \geq 0 \). Denote by \( A^{w,s}_P \) the space of all \( s \)-times weak differentiable connections of \( P \) and \( \Omega^{w,s}_{T^1}(P, g) \) the space of all \( s \)-times weak differentiable tensorial \( i \)-forms on \( P \). Fix a \( C^\infty \)-connection \( \omega_0 \) of \( P \) as the base point of \( A^{w,s}_P \). Define an operator \( \Box_{\omega_0} : \Omega^{w,s}_{T^1}(P, g) \rightarrow \Omega^{w,s-2}_{T^1}(P, g) \) by

\[
\Box_{\omega_0} := \begin{cases} d_{\omega_0} \circ d_{\omega_0}^* + d_{\omega_0}^* \circ d_{\omega_0} + \text{id} & (i \geq 1) \\
\quad d_{\omega_0}^* \circ d_{\omega_0} + \text{id} & (i = 0),
\end{cases}
\]

where \( d_{\omega_0} \) denotes the covariant exterior derivative with respect to \( \omega_0 \) and \( d_{\omega_0}^* \)
denotes the adjoint operator of \( d_{\omega_0} \) with respect to the \( L^2 \)-inner products of \( \Omega^i_{s-j}(B, \text{Ad}(P)) \) \((i \geq 0, \ j \geq 1) \). The \( L^2 \)-inner product \( \langle , \rangle_{s}^{\omega_0} \) of

\[ T_u A^{w,s} = \Omega^w_{T^1}(P, g) \approx \Omega^w(B, \text{Ad}(P)) \approx \Gamma^w(T^* B \otimes \text{Ad}(P)) \]
is defined by

\[
\langle A_1, A_2 \rangle_{s}^{\omega_0} := \int_{x \in B} \langle (\widehat{A}_1)_x, (\Box_{\omega_0}^s(A_2))_x \rangle_{B, g} \ dv_B
\]

\[ (A_1, A_2 \in \Omega^{w,s}_{T^1}(P, g)), \]

where \( \Box_{\omega_0}^s(A_2) \) denotes the \( \text{Ad}(P) \)-valued 1-form over \( B \) corresponding to \( \Box_{\omega_0}^s(A_2) \),

\[ \langle , \rangle_{B, g} \]
denotes the fibre metric of \( T^* B \otimes \text{Ad}(P) \) defined naturally from \( g_B \) and \( \langle , \rangle_{g} \) and \( dv_B \)
denotes the volume element of \( g_B \). Let \( \Omega^H_{T^1}(P, g) \) be the completion of \( \Omega^\infty_{T^1}(P, g) \) with respect to \( \langle , \rangle_{s}^{\omega_0} \). Also, set

\[ A^H_P := \{ \omega_0 + A | A \in \Omega^H_{T^1}(P, g) \}. \]

Let \( \Omega^{H^s}_{T^1}(B, \text{Ad}(P)) \) be the completion of \( \Omega^\infty_{T^1}(B, \text{Ad}(P)) \) corresponding to \( \Omega^H_{T^1}(P, g) \).

\[
\begin{array}{c}
\omega \\
\downarrow \Psi
\end{array}
\quad
\rightarrow
\quad
\begin{array}{c}
A(\omega = \omega - \omega_0) \\
\downarrow \Psi
\end{array}
\quad
\rightarrow
\quad
\begin{array}{c}
\hat{A} \\
(\omega_0 : \text{the base point of } \Omega^H_P)
\end{array}
\]

Let \( \hat{G}^\infty_P \) be the group of all \( C^\infty \)-gauge transformations \( g \)'s of \( P \) with \( \pi g = \pi \).

For each \( g \in \hat{G}^\infty_P \), \( \hat{g} \in C^\infty(P, G) \) is defined by \( \hat{g}(u) = u \hat{g}(u) \quad (u \in P) \). This element \( \hat{g} \)
satisfies

\[ \hat{g}(ug) = \text{Ad}(g^{-1})(\hat{g}(u)) \quad (\forall u \in P, \ \forall g \in G), \]

where \( \text{Ad} \) denotes the homomorphism of \( G \) to \( \text{Aut}(G) \) defined by \( \text{Ad}(g_1)(g_2) := g_1 \cdot g_2 \cdot g_1^{-1} \quad (g_1, g_2 \in G) \). Under the correspondence \( g \leftrightarrow \hat{g} \), \( \hat{G}^\infty_P \) is identified with

\[ \hat{G}^\infty_P := \{ \hat{g} \in C^\infty(P, G) | \hat{g}(ug) = \text{Ad}(g^{-1})(\hat{g}(u)) \quad (\forall u \in P, \ \forall g \in G) \}. \]
For \( \hat{g} \in \mathcal{G}_P^\infty \), the \( C^\infty \)-section \( g \) of the associated \( G \)-bundle \( P \times_{Ad} G \) is defined by \( g(x) := u \cdot \hat{g}(u) \ (x \in B) \), where \( u \) is any element of \( \pi^{-1}(x) \). Under the correspondence \( \hat{g} \leftrightarrow g \), \( \mathcal{G}_P^\infty (= \mathcal{P}_P^\infty) \) is identified with the space \( \Gamma^\infty(P \times_{Ad} G) \) of all \( C^\infty \)-sections of \( P \times_{Ad} G \). The gauge action \( \mathcal{G}_P^\infty \curvearrowright \mathcal{A}_P^\infty \) is given by

\[
(\mathbf{g} \cdot \omega)_u = \text{Ad}(\hat{g}(u)) \circ \omega_u - (R_{\hat{g}(u)})_s^{-1} \circ \hat{g}_su
\]

(2.2)

The \( H^{s+1} \)-completion of \( \Gamma^\infty(P \times_{Ad} G) \) was defined in [GP1] (see Section 1 (P668)). Denote by \( \Gamma^{H^{s+1}}(P \times_{Ad} G) \) this \( H^{s+1} \)-completion. Also, denote by \( \mathcal{G}_P^{H^{s+1}} \) (resp. \( \mathcal{G}_P^{H^{s+1}} \)) the \( H^{s+1} \)-completion of \( \mathcal{G}_P^\infty \) (resp. \( \mathcal{G}_P^\infty \)) corresponding to \( \Gamma^{H^{s+1}}(P \times_{Ad} G) \). If \( s > \frac{1}{2} \dim B - 1 \), then the \( H^{s+1} \)-gauge transformation group \( \mathcal{G}_P^{H^{s+1}} \) of \( P \) is a smooth Hilbert Lie group and the gauge action \( \mathcal{G}_P^{H^{s+1}} \curvearrowright \mathcal{A}_P^{H^{s+1}} \) is smooth as stated in Introduction. However, by this action, \( \mathcal{G}_P^{H^{s+1}} \) does not act isometrically on the Hilbert space \( (\mathcal{A}_P^{H^{s+1}}, \langle \cdot, \cdot \rangle_\omega) \). Define a Riemannian metric \( g_s \) on \( \mathcal{A}_P^{H^{s+1}} \) by \( (g_s)_\omega := \langle \cdot, \cdot \rangle_\omega \ (\omega \in \mathcal{A}_P^{H^{s+1}}) \), where \( \langle \cdot, \cdot \rangle_\omega \) is the \( L_2^s \)-inner product defined as in (2.1) by using \( \omega \) instead of \( \omega_0 \). This Riemannian metric \( g_s \) is non-flat. The gauge transformation group \( \mathcal{G}_P^{H^{s+1}} \) acts isometrically on the Riemannian Hilbert manifold \( (\mathcal{A}_P^{H^{s+1}}, g_s) \), where we note that the Hilbert space \( (\mathcal{A}_P^{H^{s+1}}, \langle \cdot, \cdot \rangle_\omega) \) is regarded as the tangent space of \( (\mathcal{A}_P^{H^{s+1}}, g_s) \) at \( \omega_0 \). Hence we can give the moduli space \( \mathcal{M}_{\mathcal{A}_P^{H^{s+1}}} := \mathcal{A}_P^{H^{s+1}}/\mathcal{G}_P^{H^{s+1}} \) the Riemannian oribisubmersion.

\[
\begin{array}{ccc}
\mathcal{G}_P^{H^{s+1}} & \approx & \mathcal{G}_P^{H^{s+1}} \\
\cup & & \cup
\end{array} \quad 
\begin{array}{ccc}
\hat{g} & \approx & \Gamma^{H^{s+1}}(P \times_{Ad} G) \\
\setminus & & \setminus
\end{array}
\]

Let \( S^1 := \{e^{2\pi \sqrt{-1}t} \mid t \in [0, 1]\} \). Define \( z : [0, 1] \rightarrow S^1 \) by \( z(t) := e^{2\pi \sqrt{-1}t} \ (t \in [0, 1]) \). Fix \( x_0 \in B \) and \( u_0 \in \pi^{-1}(x_0) \). Define the based \( C^\infty \)-loop spaces \( \Lambda_\infty^\infty(B) \) and \( \Lambda^\infty_u(P) \) by

\[
\Lambda_\infty^\infty(B) := \{ c \in C^\infty(S^1, B) \mid c(1) = x_0 \}
\]

and

\[
\Lambda^\infty_u(P) := \{ \sigma \in C^\infty(S^1, P) \mid \sigma(1) = u_0 \},
\]

respectively. Similarly define the based \( C^\infty \)-loop group \( \Lambda_\infty^\infty(G) \) at the identity element \( e \) of \( G \) and the based \( C^\infty \)-loop algebra \( \Lambda_0^\infty(g) \) at the zero vector \( 0 \) of \( g \) by

\[
\Lambda_\infty^\infty(G) := \{ g \in C^\infty(S^1, G) \mid g(1) = e \},
\]

and

\[
\Lambda_0^\infty(g) := \{ u \in C^\infty(S^1, g) \mid u(1) = 0 \},
\]

respectively. Let \( \sigma \) be the horizontal lift of \( c \circ z \) starting from \( u_0 \) with respect to \( \omega_0 \). Denote by \( \pi^c : c^*P \rightarrow S^1 \) the induced bundle of \( P \) by \( c \), which is identified with the trivial \( G \)-bundle \( P_o := S^1 \times G \) over \( S^1 \) by \( \sigma \). Define an immersion \( \iota_c \) of the induced bundle \( c^*P \) into \( P \) by \( \iota_c(z(t), u) = u \ (z(t), u) \in c^*P) \).
Definition 2.1. Define a map $\text{hol}_c : A_H^s \to G$ by

$$(P_{\text{coz}}^\omega \circ (P_{\text{coz}}^\omega)^{-1})(u_0) = u_0 \cdot \text{hol}_c(\omega),$$

where $s$ is any non-negative integer and $P_{\text{coz}}^\omega$ (resp. $P_{\text{coz}}^\omega$) denotes the parallel translation along $c \circ z$ with respect to $\omega$ (resp. $\omega_0$). We call this map $\text{hol}_c$ the holonomy map along $c$.

In particular, in the case of the trivial $G$-bundle $P_o := S^1 \times G$, the adjoint bundle $A_{o}^s$ is identified with the Hilbert space $H^s(\Omega^1(S^1, Ad(P_o)))$ is identified with the space $C^\infty(S^1, g)$ of all $C^\infty$-maps of $S^1$ into $g$ and $G$ is identified with the group $C^\infty(S^1, G)$ of all $C^\infty$-maps of $S^1$ into $G$. Let $s$ be a non-negative integer. The Hilbert space $A_{o}^s(\Omega^1(S^1, Ad(P_o)))$ is identified with the Hilbert space $H^s([0, 1], g)$ of all $H^s$-maps of $[0, 1]$ into $g$ and $G$ is identified with the Hilbert Lie group $H^{s+1}([0, 1], G)$ of all $H^{s+1}$-maps of $[0, 1]$ into $G$. Here we note that, for any non-negative integer $s$, the action $G \rtimes A_{o}^s$ is a smooth action because $S^1$ is of one-dimension.

Remark 2.1. We shall explain why $A_{o}^s(\Omega^1(S^1, Ad(P_o)))$ is identified with the Hilbert space $H^s([0, 1], g)$. The space $C^\infty(S^1, g)$ is embedded into $C^\infty([0, 1], g)$ by assigning $c \circ z$ to each $c \in C^\infty(S^1, g)$. The orthonormal basis of the $H^s$-completion $H^s([0, 1], g)$ of $C^\infty([0, 1], g)$ is given as a family of $C^\infty$-loops in $g$ (having $[0, 1]$ as the domain). Hence the Fourier expansion of each element of the $H^s$-completion $H^s([0, 1], g)$ of $C^\infty([0, 1], g)$ is given as a series consisting of $C^\infty$-loops in $g$. Therefore, we see that $H^s([0, 1], g)$ is the completion of $C^\infty(S^1, g)$.

As stated in Introduction, the parallel transport map for $G$ has been defined as a map of $H^0([0, 1], g)$ onto $G$. In more general, we can define the parallel transport map for $G$ as the map of $H^s([0, 1], g)$ onto $G$ similarly, where $s$ is any non-negative integer.

Let $c$ and $\sigma$ be as above. Then the pull-back bundle $c^*P$ of $P$ by $c$ is identified with the trivial $G$-bundle $S^1 \times G$ over $S^1$ by $\sigma$ as follows. Define a map $\eta : S^1 \times G \to c^*P$ by

$$\eta(z(t), g) := (z(t), \sigma(t)g) \quad ((t, g) \in [0, 1] \times G).$$

It is clear that $\eta$ is a bundle isomorphism. Throughout this bundle isomorphism $\eta$, $c^*P$ is identified with $S^1 \times G$. Similarly, a bundle isomorphism $\bar{\eta} : [0, 1] \times G \to (c \circ z)^*P$ is defined by

$$\bar{\eta}(t, g) := (t, \sigma(t)g) \quad ((t, g) \in [0, 1] \times G).$$

Throughout this bundle isomorphism $\bar{\eta}$, $(c \circ z)^*P$ is identified with the trivial bundle $[0, 1] \times G$. The natural embedding $\iota_\sigma : [0, 1] \times G \to P$ by

$$\iota_\sigma(t, g) := \sigma(t)g \quad ((t, g) \in [0, 1] \times G).$$
As above, define a map $A((0, u) \in H)$ be the element of $H$ orbit space $G$.

Proof. Take any $\omega \in \mathcal{A}^H_p$. By the one-to-one correspondence $\omega$ we call this map $\Lambda^{(-1)}(0) = \{ (\sigma, \omega) | \sigma = \omega(0) \}$ is identified with the Hilbert space $H^s([0, 1], g)$ of all $H^s$-paths in $g$.

Definition 2.2. Define a map $\mu_c : \mathcal{A}^H_p \to H^s([0, 1], g)$ by

$$\mu_c(\omega)(t) := \omega(\sigma(t)) \quad (t \in [0, 1], \omega \in \mathcal{A}^H_p).$$

As above, $\omega_{\xi}(\sigma(t))$ is equal to $(\sigma^*\omega)(t, g)$ of the connection $\mu_c$. From this fact, we call this map $\mu_c$ the pull-back connection map by $c$.

Lemma 2.1. Among holc, $\phi$ and $\mu_c$, the relation $\text{hol}_c = \phi \circ \mu_c$ holds.

Proof. Take any $\omega \in \mathcal{A}^H_p$. For the simplicity, set $u := \mu_c(\omega)$. Let $g_u : [0, 1] \to G$ be the element of $H^{s+1}([0, 1], G)$ satisfying $g_u(0) = e$ and $(R_{g_u(t)})^{-1}(\mu_c'(t)) = u(t) \quad (t \in [0, 1])$.

In $\sigma = \mu_c(\omega)$, $g_u$ is the projection of the horizontal curve starting from $(0, e)$ with respect to the connection $\mu_c^*\omega$ of $(c \circ z)^*P = [0, 1] \times G$ onto the $G$-component. From this fact, we see that $g_u(1)(=\phi(u))$ is equal to $\text{hol}_c(\omega)$. Hence we obtain $\text{hol}_c(\omega) = (\phi \circ \mu_c)(\omega)$.

By the one-to-one correspondence

$$g \longleftrightarrow t \mapsto g(t, \sigma(t)) \quad (t \in [0, 1]),$$

$G_{(c, z)}^*P$ is identified with the Hilbert space $H^{s+1}([0, 1], G)$ of all $H^s$-paths in $G$, where $g_0$ is the element of $H^{s+1}([0, 1], G)$ corresponding to $g \in G_{(c, z)}^*P$. Set

$$\Lambda^{s+1}(G) := \{ g \in H^{s+1}([0, 1], G) | g(0) = g(1) = e \},$$

and

$$\Lambda^{s+1}(G) := \{ g \in H^{s+1}([0, 1], G) | g(0) = g(1) \}.$$

Note that the orbit space $H^s([0, 1], g)/\Lambda^{s+1}(G)$ of the subaction $\Lambda^{s+1}(G)$ of $H^s([0, 1], g)$ of the gauge action $H^{s+1}([0, 1], G)$ of $H^s([0, 1], g)$ is isomorphic to $G$ and that the orbit map of this subaction coincides with the parallel transport map $\phi$ for $G$. Also, note that the orbit space $H^s([0, 1], g)/\Lambda^{s+1}(G)$ of this subaction $\Lambda^{s+1}(G)$ of $H^s([0, 1], g)$ is isomorphic to $G/\text{Ad}(G)$ and that the orbit space $H^s([0, 1], g)/\Lambda^{s+1}(G)$ of this subaction $\Lambda^{s+1}(G)$ of $H^s([0, 1], g)$
is isomorphic to \( G \), where \( \text{Ad} \) is the adjoint action of \( G \) on oneself. In fact, according to the proof of Theorem 4.1 of [TT], we have

\[
\phi \circ g = L_{g(0)} \circ R_{g(1)}^{-1} \circ \phi
\]

for any \( g \in H^{s+1}([0,1], G) \), where \( g \) in the left-hand side means the diffeomorphism of \( H^s([0,1], g) \) onto oneself defined by the action of \( H^{s+1}([0,1], G) \) on \( H^s([0,1], g) \) and \( L_{g(0)} \) and \( R_{g(1)} \) are the left translation by \( g(0) \) and the right translation by \( g(1) \), respectively.

**Definition 2.3.** Define a map \( \lambda_c : \mathcal{G}_P^{H^{s+1}} \to H^{s+1}([0,1], G) \) by

\[
\lambda_c(g)(t) := \hat{g}(\sigma(t)) \quad (t \in [0,1], \ g \in \mathcal{G}_P^{H^{s+1}}).
\]

The based gauge transformation group \((\mathcal{G}_P^{H^{s+1}})_x\) at \( x \in B \) is defined by

\[
(\mathcal{G}_P^{H^{s+1}})_x := \{ g \in \mathcal{G}_P^{H^{s+1}} \mid \hat{g}(\sigma^{-1}(x)) = \{ e \} \}.
\]

Denote by \( \mathcal{M}_P^{H^s} \) the moduli space \( \mathcal{A}_P^{H^s}/\mathcal{G}_P^{H^{s+1}} \) and \( \pi_{\mathcal{M}_P} \) the orbit map of the action \( \mathcal{G}_P^{H^{s+1}} \triangleleft \mathcal{A}_P^{H^s} \). Also, denote by \((\mathcal{M}_P^{H^s})_x\) the moduli space \( \mathcal{A}_P^{H^s}/(\mathcal{G}_P^{H^{s+1}})_x \) and \( \pi_{(\mathcal{M}_P)_x} \) the orbit map of the action \( (\mathcal{G}_P^{H^{s+1}})_x \triangleleft \mathcal{A}_P^{H^s} \).

Throughout \( \lambda_c \), \( \mathcal{G}_P^{H^{s+1}} \) acts on \( H^s([0,1], g) \). Also, \( \mathcal{G}_P^{H^{s+1}} \) acts on \( G \) by

\[
g \cdot g := \text{Ad}(g(\sigma(1)))(g) \quad (g \in \mathcal{G}_P^{H^{s+1}}, \ g \in G).
\]

From Lemma 2.1, (2.2), (2.4) and the definitions of \( \mu_c \) and \( \lambda_c \), we can derive the following facts.

**Lemma 2.2.** (i) The pull-back connection map \( \mu_c \) is \( \mathcal{G}_P^{H^{s+1}} \)-equivariant, that is, the following relation holds:

\[
\mu_c(g \cdot \omega) = \lambda_c(g) \cdot \mu_c(\omega) \quad (g \in \mathcal{G}_P^{H^{s+1}}, \ \omega \in \mathcal{A}_P^{H^s}).
\]

(ii) The holonomy map \( \text{hol}_c \) is \( \mathcal{G}_P^{H^{s+1}} \)-equivariant, that is, the following relation holds:

\[
\text{hol}_c(g \cdot \omega) = \text{Ad}(g(\sigma(1)))(\text{hol}_c(\omega)) \quad (g \in \mathcal{G}_P^{H^{s+1}}, \ \omega \in \mathcal{A}_P^{H^s}).
\]

(iii) Set \( x_0 := c(0) \). The pull-back connection map \( \mu_c \) maps \( (\mathcal{G}_P^{H^{s+1}})_{x_0} \)-orbits in \( \mathcal{A}_P^{H^s} \) to \( \Lambda_c^{H^{s+1}}(G) \)-orbits in \( H^s([0,1], g) \) and hence there uniquely exists the map \( \overline{\pi}_c \) between the orbit spaces \( (\mathcal{M}_P^{H^s})_{x_0} \) and \( H^s([0,1], g)/\Lambda_c^{H^{s+1}}(G) \) (\( \approx G \)) satisfying

\[
\overline{\pi}_c \circ \pi_{(\mathcal{M}_P)_{x_0}} = \phi \circ \mu_c(= \text{hol}_c)
\]

(see Diagram 2.1).
(iv) The pull-back connection map \( \mu_c \) maps \( \mathcal{G}_P^{H^{+1}} \)-orbits in \( \mathcal{A}_P^{H^*} \) to \( \Lambda^{H^{+1}}(G) \)-orbits in \( H^*([0,1], g) \) and hence there uniquely exists the map \( \overline{\mu}_c \) between the orbit spaces \( \mathcal{M}_P^{H^*} \) and \( H^*([0,1], g)/\Lambda^{H^{+1}}(G) (\cong G/\Ad(G)) \) satisfying

\[
\overline{\mu}_c \circ \pi_{\mathcal{M}_P} = \pi_{\Ad} \circ \phi \circ \mu_c = \pi_{\Ad} \circ \hol_c =: \overline{\hol}_c
\]

(see Diagram 2.2), where \( \pi_{\Ad} \) denotes the natural projection of \( G \) onto \( G/\Ad(G) \).

**Proof.** First we shall show (2.6). From (2.2) and the definition of \( \mu_c \), we have

\[
\mu_c(g \cdot \omega)(t) = (g \cdot \omega)_{\sigma(t)}(\sigma'(t))
\]

\[
= (\Ad(g(\sigma(t)))(\omega_{\sigma(t)}(\sigma'(t))) - (R_{g(\sigma(t))})^{-1}((\hat{g} \circ \sigma)'(t)).
\]

On the other hand, from (2.2) and the definitions of \( \lambda_c \) and \( \mu_c \), we have

\[
(\lambda_c(g) \cdot \mu_c(\omega))(t) = (\hat{g} \circ \sigma \cdot (\omega \circ \sigma'))(t)
\]

\[
= \Ad(g(\sigma(t)))(\omega_{\sigma(t)}(\sigma'(t))) - (R_{g(\sigma(t))})^{-1}((\hat{g} \circ \sigma)'(t)).
\]

Therefore we obtain \( \mu_c(g \cdot \omega)(t) = (\lambda_c(g) \cdot \mu_c(\omega))(t) \).

Next we shall show (2.7). From \( \hol_c = \phi \circ \mu_c \) (by Lemma 2.1) and (2.4), we have

\[
\hol_c(g \cdot \omega) = \phi((g \cdot \omega) \circ \sigma')
\]

\[
= \phi((\Ad(g \circ \sigma)(\omega \circ \sigma') - (R_{g \circ \sigma})^{-1} \circ (\hat{g} \circ \sigma)'))
\]

\[
= \phi((g \circ \sigma) \cdot (\omega \circ \sigma'))
\]

\[
= (L_{g(\sigma(0))} \circ R_{g(\sigma(1))}^{-1})(\phi(\omega \circ \sigma'))
\]

\[
= \Ad(g(\sigma(1)))(\hol_c(\omega)).
\]

Next we shall show the statements (iii) and (iv). It is clear that \( \lambda_c((\mathcal{G}_P^{H^{+1}})_{x_0}) = \Lambda_c^{H^{+1}}(G) \) and \( \lambda_c((\mathcal{G}_P^{H^{+1}})) = \Lambda^{H^{+1}}(G) \) hold. From these facts and (2.6), we can derive the statements (iii) and (iv) directly. \( \square \)

\[\text{Diagram 2.1: The map } \overline{\mu}_c \text{ induced from } \mu_c\]
Diagram 2.2: The map \( \pi_c \) induced from \( \mu_c \)

3 Proof of Theorem A

In this section, we shall prove Theorem A stated in Introduction. Fix \( \omega_0 \in A^\infty_P \).

Let \( \mathcal{V}_m \) be the vertical distribution on \( A^H_P \) with respect to \( \pi_M_P \), that is,

\[
\mathcal{V}_m := \ker (\pi_M_P) (\omega) \quad (\omega \in A^H_P)
\]

and \( \mathcal{H}_m \) the horizontal distribution on \( A^H_P \) with respect to \( \langle , \rangle_{\omega_0} \), that is,

\[
\mathcal{H}_m := \{ v \in T_{\omega}A^H_P \mid \langle v, w \rangle_{\omega_0} = 0 \quad (\forall w \in \mathcal{V}_m) \}
\]

Also, let \( \mathcal{V}_P \) be the vertical distribution on \( P \), that is,

\[
\mathcal{V}_P := \ker \pi_u \quad (u \in P)
\]

and \( \mathcal{H}_P \) the horizontal distribution on \( P \) with respect to \( \omega_0 \), that is, \( \mathcal{H}_P := \ker (\omega_0)_u \) \( (u \in P) \). For \( u \in P, v \in T_uP \setminus \mathcal{V}_u \) and \( \xi \in \mathfrak{g} \), define a linear map \( \eta_{u,\xi} : T_uP \to \mathfrak{g} \) by

\[
\eta_{u,\xi}(w) := \begin{cases} 
\xi & (w = v) \\
0 & (w \in \mathcal{V}_P \oplus (\mathcal{H}_P \cap \text{Span}\{v\}^\perp))
\end{cases}
\]

Define \( \delta_{\eta_{u,\xi}} \in \Omega^H_P(P, \mathfrak{g}) \) by

\[
\langle \delta_{\eta_{u,\xi}}, A \rangle_0 = \langle \tilde{\eta}_{u,\xi}, \tilde{A}_{\pi(u)} \rangle_{B, \mathfrak{g}} \quad (\forall A \in \Omega^H_{T,1}(P, \mathfrak{g}))
\]

where \( \tilde{\eta}_{u,\xi} \) is an element of \( T^*_{\pi(u)}B \otimes \text{Ad}(P)_{\pi(u)} \) corresponding to \( \eta_{u,\xi} \).

Lemma 3.1. Let \( c \) and \( \sigma \) be as in the statement of Theorem A. Then we have

\[
(Ker(\mu_c)_{\omega_0})^\perp = \left\{ \int_0^1 \delta_{\eta_{u,\xi}(\gamma(t))} dt \mid \xi \in H^s([0,1], \mathfrak{g}) \right\}.
\]
Proof. First we note that \( \mu_c \) is linear. Since
\[
\text{Ker}(\mu_c)_\omega = \left\{ A \in \Omega_{T^*P}^s(P, g) \mid A_{\sigma(t)}(\sigma'(t)) = 0 \quad (\forall t \in [0, 1]) \right\},
\]
\( A \in \Omega_{T^*P}^s(P, g) \) belongs to Ker \((\mu_c)_\omega^\perp\) if and only if \( A \) vanishes over
\[
\left( TP \setminus (TP|_\sigma(S^1)) \cup \left( \bigcup_{t \in [0, 1]} (H^P_{\sigma(t)} \cap \text{Span}\{\sigma'(t)\})^\perp \right) \right),
\]
that is, \( A \) is expressed as \( A = \int_0^1 \delta_{\eta'(t), \xi(t)} \, dt \) for some \( \xi \in H^*([0, 1], g) \). Therefore, we obtain the desired relation. \( \square \)

Set
\[
\Lambda^H_0(g) := \{ u \in H^*([0, 1], g) \mid u(0) = u(1) \}
\]
and
\[
\Lambda^H_0(g) := \{ u \in H^*([0, 1], g) \mid u(0) = u(1) = 0 \}.
\]
Let \( g_\omega^s \) be the Riemannian metric of \( H^*([0, 1], g)(\approx \Lambda^H_0) \) defined in similar to \( g_\omega \).

**Proposition 3.2.** For any non-negative integer \( s \), the pull-back connection map
\[
\mu_c : (\Lambda^H_0, g_\omega) \to (H^*([0, 1], g), g_\omega^s)
\]
along \( c \) is a homothetic submersion of coefficient \( a \) onto the dense linear subspace \( \Lambda^H_0(g) \) of \( H^*([0, 1], g) \). Also, if \( s = 0 \), then the fibres of \( \mu_c \) are totally geodesic.

Proof. In this proof, we abbreviate \( \hat{A} \in \Gamma^H(T^*B \otimes \text{Ad}(P)) \) corresponding to \( A \in \Omega_{T^*P}^s(P, g) \) as \( \hat{A} \) and \( \hat{A}_{\pi(*)} \in T^*B \otimes \text{Ad}(P)_{\pi(*)} \) corresponding to \( A_{\pi(*)} \in T_{\pi(*)} \otimes g \). Then A is expressed as \( A := \int_0^1 \delta_{\eta'(t), \xi(t)} \, dt \) for some \( \xi \in H^*([0, 1], g) \). Then we have
\[
\begin{align*}
\langle g_\omega^s(A, A) \rangle_0 &= \int_0^1 \langle \eta'(t), \xi(t) \rangle \, dt, \\
&= \frac{1}{a^2} \int_0^1 \left( \int_0^1 \delta_{\eta'(t), \xi(t)} \, dt \right) \, dt \\
&= \frac{1}{a^2} \int_0^1 \int_0^1 \langle \eta'(t), \xi(t) \rangle \, dt \\
&= \frac{1}{a^2} \int_0^1 \langle \eta'(t), \xi(t) \rangle \left( \delta_{\eta'(t), \xi(t)} \right)_{\text{Ad}(P)_{\eta'(t)}} \, dt
\end{align*}
\]
(3.1)
Also, it is clear that $\mu$. From (3), we have
\[ \Lambda^H(\Pi, \xi(t)) = (\xi, \Gamma^t_{\mu_t}(\xi))_0, \]
where $\langle \cdot, \cdot \rangle_{\text{Ad}(P)(c(\xi(t)))}$ denotes the fibre metric of the bundle $\text{Ad}(P)$ at $c(z(t))$.

On the other hand, we have
\[ \langle (\mu_t)_*\omega(A)(t) \rangle = \int_0^1 \langle \delta_{\eta(t)}(\xi(t)) \sigma(t) \rangle d\xi \]
and hence
\[ \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle = \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle_{t}^* \]
\[ = \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle_{t}^* \]
\[ = \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle_{t}^* \]
(3.2)
\[ = \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle_{t}^* \]
\[ = \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle_{t}^* \]

From (3.1) and (3.2), we obtain
\[ \langle (\mu_t)_*\omega(A), (\mu_t)_*\omega(A) \rangle = a^2 (g_\omega)(A, A). \]

Also, it is clear that $\mu (A^H) = \Lambda^H(g)$ holds. These facts imply that $\mu$ is a homothetic submersion of coefficient $a$ of $\Lambda^H_{s+1}$ onto a dense linear subspace $\Lambda^H(\Pi, \xi(t)) = (H^s([0, 1], g), g_\omega)$. In particular, in the case of $s = 0$, $\Lambda^H_{s+1}$ and $(\Pi, \xi(t)) = (H^s([0, 1], g), g_\omega)$ are Hilbert spaces and $\mu$ is a linear map between these Hilbert spaces. Therefore the fibres of $\mu$ are affine subspaces of $\Lambda^H_{s+1}$ and hence they are totally geodesic in $(\Pi, \xi(t))$. This completes the proof. \(\square\)

For the parallel transport map, we have the following fact.

**Proposition 3.3** For any non-negative integer $s$, the parallel transport map $\phi : (H^s([0, 1], g), g_\omega) \rightarrow (G, \pi_G)$ is a Riemannian submersion.

**Proof.** The gauge action $H^s([0, 1], G) \rightharpoonup (H^s([0, 1], g), g_\omega)$ is isometric (and free) and $\phi$ is the orbit map of this subaction $\Lambda^H_{s+1}(G) \rightharpoonup H^s([0, 1], g)$, there exists the metric $\tilde{G}$ of $G$ such that $\phi : (H^s([0, 1], g), g_\omega) \rightarrow (G, \tilde{G})$ is a Riemannian submersion. It is easy to show that $\tilde{G}$ is equal to the bi-invariant metric $g_G$. Therefore we obtain the statement of this proposition. \(\square\)

Let $K$ be a symmetric subgroup of $G$, that is, a closed subgroup of $G$ such that $(\text{Fix}\theta)_0 \subset K \subset \text{Fix}\theta$ holds for some involution of $G$, where $\text{Fix}\theta$ denotes the fixed point set of $\theta$ and $(\text{Fix}\theta)_0$ denotes the identity component of $\text{Fix}\theta$. Since the natural $K$-action on $(G, g_G)$ is isometric, there exists a unique Riemannian metric $g_{G/K}$ on $G/K$ such that the natural projection $\pi_{G/K} : (G, g_G) \rightarrow (G/K, g_{G/K})$ is a Riemannian submersion. The Riemannian manifold $(G/K, g_{G/K})$ is a symmetric space of compact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
be the canonical decomposition associated to the symmetric pair \((G, K)\). The space \(\mathfrak{p}\) is identified with the tangent space \(T_{eK}(G/K)\) through \((\pi_{G/K})_e|_p^*\). Take a maximal abelian subspace \(\mathfrak{a}\) of \(\mathfrak{p}\) and let

\[
\mathfrak{p} = \mathfrak{a} \oplus \left( \bigoplus_{\alpha \in \Delta_+} \mathfrak{p}_\alpha \right) \quad \text{and} \quad \mathfrak{k} = \mathfrak{z}(\mathfrak{a}) \oplus \left( \bigoplus_{\alpha \in \Delta_+} \mathfrak{t}_\alpha \right)
\]

be the root space decomposition of \(\mathfrak{p}\) (resp. \(\mathfrak{k}\)) with respect to \(\mathfrak{a}\) (\(\Delta_+\) : the positive root system), where \(\mathfrak{z}(\mathfrak{a})\) is the centralizer of \(\mathfrak{a}\) in \(\mathfrak{k}\) and \(\mathfrak{p}_\alpha\) (resp. \(\mathfrak{t}_\alpha\)) is the root space for \(\alpha\), that is,

\[
\mathfrak{p}_\alpha = \{ w \in \mathfrak{p} | \text{ad}(v)^2(w) = -\alpha(v)^2w \quad (\forall v \in \mathfrak{a}) \},
\]

\[
\mathfrak{t}_\alpha = \{ w \in \mathfrak{t} | \text{ad}(v)^2(w) = -\alpha(v)^2w \quad (\forall v \in \mathfrak{a}) \}.
\]

Set \(\tilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{z}(\mathfrak{a})\). For \(v \in \mathfrak{g}\), denote by \(\tilde{v}\) the element of \(H^0([0,1], \mathfrak{g})\) defined as the constant map at \(v\). Let \(e^\alpha\) be a unit vector belonging to \(\mathfrak{p}_\alpha\) and \(e^0\) a unit vector belonging to \(\tilde{\mathfrak{a}}\). Define \(l_{e^\alpha, j}^i \in H^0([0,1], \mathfrak{g})\) \((i = 1, 2, j \in \mathbb{Z} \setminus \{0\})\) by

\[
l_{e^\alpha, j}^1(z(t)) := e^\alpha \cos(2j\pi t) - (e^\alpha_t) \sin(2j\pi t),
\]

\[
l_{e^\alpha, j}^2(z(t)) := e^\alpha \sin(2j\pi t) + (e^\alpha_t) \cos(2j\pi t)
\]

\((t \in [0,1])\), where \(e^\alpha_t\) is the element of \(\mathfrak{t}_\alpha\) such that

\[
\text{ad}(v)(e^\alpha) = \alpha(v) e^\alpha \quad \text{and} \quad \text{ad}(v)(e^0) = -\alpha(v) e^0
\]

for any \(v \in \mathfrak{a}\). Also, define \(l_{e^0, j}^i \in H^0([0,1], \mathfrak{g})\) \((i = 1, 2, j \in \mathbb{N} \setminus \{0\})\) by

\[
l_{e^0, j}^1(z(t)) := \cos(2j\pi t) \cdot e^0 \quad (t \in [0,1]),
\]

\[
l_{e^0, j}^2(z(t)) := \sin(2j\pi t) \cdot e^0 \quad (t \in [0,1]).
\]

Then, from the definition of the \(L^2\)-inner product \(\langle \cdot \rangle_{\mathfrak{g}^0}\) of \(H^0([0,1], \mathfrak{g})\), we can show the following facts directly.

**Lemma 3.4.** Let \(\{e_j^\alpha\}_{j=1}^{m_\alpha}\) be an orthonormal basis of \(\mathfrak{p}_\alpha\) and \(\{e_j^0\}_{j=1}^{m_0}\) that of \(\tilde{\mathfrak{a}}\). Then the statements (i), (ii) and (iii) hold:

(i) The system

\[
\{\tilde{e}_j^0 | 1 \leq j \leq m_0\} \bigcup \left( \bigcup_{\alpha \in \Delta_+} \{ e_j^\alpha | 1 \leq j \leq m_\alpha \} \right)
\]

\[
\bigcup \left( \bigcup_{\alpha \in \Delta_+} \{ l_{e_j^\alpha, k}^i | i = 1, 2, 1 \leq j \leq m_\alpha, k \in \mathbb{N} \setminus \{0\} \} \right)
\]

\[
\bigcup \left( \bigcup_{\alpha \in \Delta_+} \{ l_{e_j^0, k}^i | i = 1, 2, 1 \leq j \leq m_\alpha, k \in \mathbb{Z} \setminus \{0\} \} \right)
\]

is an orthonormal basis of \((H^0([0,1], \mathfrak{g}), \mathfrak{g}^0)\).

(ii) The system

\[
\{\tilde{e}_j^0 | 1 \leq j \leq m_0\} \bigcup \left( \bigcup_{\alpha \in \Delta_+} \{ e_j^\alpha | 1 \leq j \leq m_\alpha \} \right)
\]

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is an orthonormal basis of the horizontal space (which is denoted by $H_{0}^{\phi}$) of $\phi$ at $\hat{0}$.

(iii) The system

$$\left\{ l_{i}^{j,k} \mid i = 1, 2, \ 1 \leq j \leq m_{0}, \ k \in \mathbb{N} \setminus \{0\} \right\}$$

is an orthonormal basis of the vertical space (which is denoted by $V_{0}^{\phi}$) of $\phi$ at $\hat{0}$.

By using (2.4), we can show that the action $H^{1}([0,1],G) \rhd (H^{0}([0,1],g), g_{0}^{0})$ is isometric and transitive and each element of $H^{1}([0,1],G)$ maps the fibres of $\phi$ to them. Hence the fibres of $\phi$ are congruent to one another. We suffice to focus attention on the fibre $\phi^{-1}(e)$ (which passes through $\hat{0}$). Denote by $A^{F}$ the shape tensor of $M := \phi^{-1}(e)$.

**Lemma 3.5.** Let $v \in T_{e}G(=g)$ and $\hat{v}(\in H^{0}([0,1],g))$ the constant path at $v$. Then the following relations hold:

$$A_{0}^{F}(l_{i}^{j,k}) = 0 \quad (i = 1, 2, \ j = 1, \cdots, m_{0}, \ k \in \mathbb{N} \setminus \{0\}),$$

$$A_{0}^{F}(l_{\alpha}^{j,k}) = -\frac{\alpha(v)}{2k\pi} l_{\alpha}^{j,k} \quad (i = 1, 2, \ j = 1, \cdots, m_{\alpha}, \ k \in \mathbb{Z} \setminus \{0\}).$$

**Proof.** Denote by $\tilde{\nabla}$ the Riemannian connection of the Hilbert space $(H^{0}([0,1],g), g_{0}^{0})$. From the relations in the proof of Propositions 3.1 and 3.2 of [K], we obtain

$$(3.5) \quad \tilde{\nabla}_{l_{e_{i}^{j,k}}} v^{L} = 0 \quad \text{and} \quad \tilde{\nabla}_{\alpha e_{i}^{j,k}} v^{L} = -\frac{\alpha(v)}{2k\pi} l_{\alpha}^{j,k}.$$ 

where $v^{L}$ is the horizontal lift of $v$. Furthermore, from these relations, we obtain the desired relations. \qed

From this lemma, we can derive the following fact.

**Proposition 3.6.** The parallel transport map $\phi : (H^{0}([0,1],g), g_{0}^{0}) \rightarrow (G, g_{G})$ is a Riemannian submersion with minimal regularizable fibres.

**Proof.** From the relations in Lemma 3.5, it follows that $M = \phi^{-1}(e)$ is a minimal regularizable submanifold in $(H^{0}([0,1],g), g_{0}^{0})$. Hence so are all the fibres. \qed

According to Theorem 6.5 of [HLQ], the following fact holds.

**Theorem 3.7** ([HLQ]). Let $\psi$ be a Riemannian submersion with minimal regularizable fibres of a Hilbert space $V$ onto a finite dimensional Riemannian manifold $N$. Then, for a compact submanifold $M$ in $N$, the following statements are equivalent:
(i) $M$ is equifocal;
(ii) $\psi^{-1}(M)$ is isoparametric.

By using Propositions 3.2, 3.3 and 3.6 and Theorem 3.7, we prove Theorem A.

**Proof of Theorem A.** Since $\text{hol}_c = \phi \circ \mu_c$ holds, it follows from Propositions 3.2 and 3.3 that $\text{hol}_c : (A^H_0, g_0) \to (G, g_G)$ is a homothetic submersion of coefficient $a$. We consider the case of $s = 0$. Then it is clear that $\mu_c$ is an affine map and the operator norm of the linear part is equal to $a$ (and hence it is bounded). Therefore, the fibres of $\mu_c$ are affine subspaces of $A^H_0$ and hence they are totally geodesic in $(A^H_0, g_0)$. On the other hand, according to Proposition 3.6, $\phi : (H^0([0, 1], g), g_0^0) \to (G, g_G)$ has a Riemannian submersion with minimal regularizable fibres. Therefore, $\text{hol}_c : (A^H_0, g_0) \to (G, g_G)$ is a homothetic submersion of coefficient $a$ with minimal regularizable fibres. Thus the statement (i) of Theorem A is shown.

Let $M$ be an equifocal submanifold in $(G, g_G)$. Since $\text{hol}_c : (A^H_0, g_0) \to (G, \frac{1}{a} g_G)$ is a Riemannian submersion with minimal regularizable fibres and $M$ is equifocal in $(G, \frac{1}{a} g_G)$, it follows from Theorem 3.7 that $\text{hol}_c^{-1}(M)$ is isoparametric in $(A^H_0, g_0)$. The $G^H_0$-invariance of $\text{hol}_c^{-1}(M)$ follows from (2.8). Also, in the case where $M$ is $\text{Ad}(G)$-invariant, the $G^H_0$-invariance of $\text{hol}_c^{-1}(M)$ follows from (2.9).

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