Exchange relations of Level-two vertex operators of $U_q(\hat{sl}_2)$

Wen-Li Yang$^{a,b}$

$^a$ Institute of Modern Physics, Northwest University, Xian 710069, P.R. China
$^b$ Physikalisches Institut der Universität Bonn, Nussallee 12, 53115 Bonn, Germany

Abstract

We calculate the exchange relations of vertex operators of $U_q(\hat{sl}_2)$ at level-two from its bosonic realization. The corresponding invertibility relation of type I vertex operators is also studied.

Mathematics Subject Classifications (1991): 81R10, 17B37, 16W30

$^1$E-mail: wlyang@th.physik.uni-bonn.de
1 Introduction

Infinite-dimensional highest weight representations and the corresponding vertex operator of quantum affine (super) algebras are two ingredients of great importance in algebraic analysis of lattice integrable models [2, 3]. The exchange relation of the vertex operators and its invertibility relation play a key role to construct bosonization of the corresponding lattice models both in the bulk case [4, 5, 6] and the boundary case [7, 8, 9].

A powerful approach for studying the highest weight representations and vertex operators is the bosonization technique [10, 11] which allows one to explicitly construct these objects in terms of the q-deformed free bosonic and fermionic fields. In this paper, we study the exchange relations of vertex operators of $U_q(\hat{sl}_2)$ at level-two $2$ and the invertibility relations from its bosonization which can be realized by a q-deformed bosonic free field and a fermionic free field [13, 14].

2 The quantum affine algebra $U_q(\hat{sl}_2)$

The symmetric Cartan matrix of the affine Lie algebra $\hat{sl}_2$ is

\[(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\]

where $i, j = 0, 1$. Quantum affine algebra $U_q(\hat{sl}_2)$ is a $q$-analogue of the universal enveloping algebra of $\hat{sl}_2$ generated by the Chevalley generators $\{e_i, f_i, t_i^{\pm 1}, d| i = 0, 1\}$, where $d$ is the usual derivation operator. The defining relations are [15]

\[
t_i t_j = t_j t_i, \quad t_i d = d t_i, \quad [d, e_i] = \delta_{i,0} e_i, \quad [d, f_i] = -\delta_{i,0} f_i,
\]

\[
t_i e_j t_i^{-1} = q^{a_{ij}} e_j, \quad t_i f_j t_i^{-1} = q^{-a_{ij}} f_j,
\]

\[
[e_i, f_j] = \delta_{ij} t_i - t_i^{-1},
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right] (e_i)^r e_j (e_i)^{1-a_{ij}-r} = 0, \quad \text{if } i \neq j,
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right] (f_i)^r f_j (f_i)^{1-a_{ij}-r} = 0, \quad \text{if } i \neq j,
\]

where

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1] \cdots [1],
\]

The exchange relation of the type I vertex operators of the 19-vertex model at critical regime has been recently studied by T. Kojima [12].
where the corresponding Drinfeld currents 

\[ \Delta(t_i) = t_i \otimes t_i, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \]

\[ \epsilon(t_i) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0, \]

\[ S(e_i) = -t_i^{-1}e_i, \quad S(f_i) = -f_it_i, \quad S(t_i^{\pm 1}) = t_i^{\mp 1}, \quad S(d) = -d. \]

\( U_q(\hat{sl}_2) \) is a quasi-triangular Hopf algebra endowed with Hopf algebra structure:

\[ \Delta(t_i) = t_i \otimes t_i, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \]

\[ \epsilon(t_i) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0, \]

\[ S(e_i) = -t_i^{-1}e_i, \quad S(f_i) = -f_it_i, \quad S(t_i^{\pm 1}) = t_i^{\mp 1}, \quad S(d) = -d. \]

\( U_q(\hat{sl}_2) \) can also be realized by the Drinfeld generators \[ \{d, X_\pm^m, a_n, K^{\pm 1}, \gamma^{\pm 1/2} | m \in \mathbb{Z}, n \in \mathbb{Z} \neq 0 \} \]. The relations read

\[ \gamma \text{ is central}, \quad [K, a_n] = 0, \quad [d, K] = 0, \quad [d, a_n] = na_n, \]

\[ [a_m, a_n] = \delta_{m+n,0} \frac{\lfloor 2m \rfloor (\gamma^m - \gamma^{-m})}{m(q - q^{-1})}, \]

\[ KX_\pm^m = q^{\pm 2}X_\pm^mK, \quad [d, X_\pm^m] = mX_\pm^m, \]

\[ [a_m, X_\pm^m] = \pm \frac{\lfloor 2m \rfloor}{m} \gamma^{\mp m/2}X_\pm^{m}, \]

\[ [X_\pm^m, X_-^n] = \frac{1}{q - q^{-1}}(\gamma^{(m-n)/2}\psi_\pm^m - \gamma^{-(m-n)/2}\psi_-^m), \]

\[ (z - wq^{\pm 2})X_\pm^m(z)X_-^n(w) = (zq^{\pm 2} - w)X_\pm^m(w)X_-^n(z), \]

where the corresponding Drinfeld currents \( \psi_\pm(z) \) and \( X_\pm(z) \) are defined by

\[ \psi_\pm(z) = \sum_{m=0}^{\infty} \psi_\pm^m z^{-m} = Kexp\{(q - q^{-1}) \sum_{k=1}^{\infty} a_k z^{-k}\}, \]

\[ \psi^-_m(z) = \sum_{m=0}^{\infty} \psi^-_m z^{-m} = K^{-1}exp\{-(q - q^{-1}) \sum_{k=1}^{\infty} a_{-k} z^{k}\}, \]

\[ X_\pm(z) = \sum_{m \in \mathbb{Z}} \psi_\pm^m z^{-m-1}. \]

The Chevalley generators are related to the Drinfeld generators by the formulae:

\[ t_1 = K, \quad e_1 = X_0^+, \quad t_0 = \gamma K^{-1}, \quad f_1 = X_0^-, \]

\[ e_0 = X_1^-t_1^{-1}, \quad f_0 = t_1 X_1^+. \]

2.1 Bosonization of \( U_q(\hat{sl}_2) \) at level-two

Let us introduce the \( q \)-bosonic-oscillators \( \{a_n, Q, P | n \in \mathbb{Z} - \{0\} \} \) and Neveu Schwartz sector of \( q \)-fermionic-oscillators \( \{b_r | r \in \mathbb{Z} + \frac{1}{2}\} \) which satisfy the commutation relations

\[ [a_m, a_n] = \delta_{m+n,0} \frac{\lfloor 2m \rfloor^2}{m}, \quad m, n \neq 0, \quad [P, a_m] = [Q, a_m] = 0, \quad [P, Q] = 1, \]

\[ \{b_r, b_s\} = \frac{[4r]}{2[2r]}\delta_{r+s,0}. \]
Then we have [13, 14]

**Theorem 1** The Drinfeld currents of \( U_q(\hat{sl}_2) \) at level-two are realized as

\[
\gamma = q^2, \quad K = q^{2P},
\]

\[
\psi_+^*(z) = q^{2P}e^{(q - q^{-1})\sum_{n=1}^{\infty}a_n z^{-n}}
\]

\[
\psi_-^*(z) = q^{-2P}e^{-(q - q^{-1})\sum_{n=1}^{\infty}a_n z^n}
\]

\[
X^\pm(z) = \sqrt{2}B(z)E^\pm(z),
\]

where

\[
E^\pm(z) = \exp\left\{ \pm \sum_{n=1}^{\infty} \frac{a_n}{[2n]} q^{zn} z^{-n} \right\} \exp\left\{ \pm \sum_{n=1}^{\infty} \frac{a_n}{[2n]} q^{zn} z^{n} \right\} \epsilon^\pm z^P,
\]

\[
B(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r z^{-r - \frac{1}{2}}.
\]

### 2.2 Level-two vertex operators

Let \( V \) be the 3-dimensional (or spin-1) evaluation representation of \( U_q(\hat{sl}_2) \), \( \{v_1, v_0, v_{-1}\} \) be the basis vectors of \( V \). Then the 3-dimensional level-0 representation \( V_z \) of \( U_q(\hat{sl}_2) \) is given by [17]

\[
e_1 v_m = [1 + m]v_{m-1}, \quad f_1 v_m = [1 - m]v_{m+1}, \quad e_0 = z f_1, \quad f_0 = z^{-1} e_1. \quad (2.6)
\]

We define the dual modules \( V_z^* \) of \( V_z \) by \( \pi_{V^*}(a) = \pi_V(S(a))^t, \forall a \in U_q(\hat{sl}_2) \), where \( t \) is the transposition operation.

Throughout, we denote by \( V(\lambda) \) a level-two irreducible highest weight \( U_q(\hat{sl}_2) \)-module with the highest weight \( \lambda \). Consider the following intertwines of \( U_q(\hat{sl}_2) \)-modules:

\[
\Phi_\lambda^V(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z, \quad \Phi_\lambda^{V^*}(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z^*,
\]

\[
\Psi_\lambda^V(z) : V(\lambda) \rightarrow V_z \otimes V(\mu), \quad \Psi_\lambda^{V^*}(z) : V(\lambda) \rightarrow V_z^* \otimes V(\mu).
\]

They are intertwines in the sense that for any \( x \in U_q(\hat{sl}_2) \),

\[
\Theta(z) \cdot x = \Delta(x) \cdot \Theta(z), \quad \Theta(z) = \Phi(z), \Phi^*(z), \Psi(z), \Psi^*(z).
\]

\( \Phi(z) (\Phi^*(z)) \) is called type I (dual) vertex operator and \( \Psi(z) (\Psi^*(z)) \) type II (dual) vertex operator.

We expand the vertex operators as

\[
\Phi(z) = \sum_{j=1,0,-1} \Phi_j(z) \otimes v_j, \quad \Phi^*(z) = \sum_{j=1,0,-1} \Phi_j^*(z) \otimes v_j^*,
\]

\[
\Psi(z) = \sum_{j=1,0,-1} v_j \otimes \Psi_j(z), \quad \Psi^*(z) = \sum_{j=1,0,-1} v_j^* \otimes \Psi_j^*(z).
\]
Define the operators \( \phi_j(z), \phi_j^*(z), \psi_j(z) \) and \( \psi_j^*(z) \) \((j = 1, 0, -1) \) bosonized by

\[
\phi_1(z) = \exp\left\{ \sum_{n=1}^{\infty} \frac{q^{5n}z^n}{[2n]} a_{-n} \right\} \exp\left\{ -\sum_{n=1}^{\infty} \frac{q^{-3n}z^{-n}}{[2n]} a_n \right\} e^Q(-q^4)P, \tag{2.7}
\]

\[
\phi_0(z) = [\phi_1(z), f_1]q^2, \quad \phi_{-1}(z) = \frac{1}{2} [\phi_0(z), f_1], \quad \phi_j^*(z) = \phi_{-i}(zq^{-2}), \tag{2.8}
\]

\[
\psi_{-1}(z) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{q^n z^n}{[2n]} a_{-n} \right\} \exp\left\{ \sum_{n=1}^{\infty} \frac{q^{-3n}z^{-n}}{[2n]} a_n \right\} e^{-Q}(-q^2)^{-P}, \tag{2.9}
\]

\[
\psi_0(z) = [\psi_{-1}(z), e_1]q^2, \quad \psi_1(z) = \frac{1}{2} [\psi_0(z), e_1], \quad \psi_j^*(z) = \psi_{-i}(zq^{-2}), \tag{2.10}
\]

where \([a, b]_x = ab - xba\). Introduce \( \phi(z), \phi^*(z), \psi(z), \psi^*(z) \) by

\[
\phi(z) = \sum_{j=1,0,-1} \phi_j(z) \otimes v_j, \quad \phi^*(z) = \sum_{j=1,0,-1} \phi_j^*(z) \otimes v_j^*,
\]

\[
\psi(z) = \sum_{j=1,0,-1} v_j \otimes \psi_j(z), \quad \psi^*(z) = \sum_{j=1,0,-1} v_j^* \otimes \psi_j^*(z).
\]

Using the method of Idzumi [4], we have the following result.

**Proposition 1** The operators \( \phi(z), \phi^*(z), \psi(z), \psi^*(z) \) satisfy the same commutation relations as \( \Phi^V_\lambda(z), \Phi^V^*_\lambda(z), \Psi^V_\lambda(z), \Phi^{V*}_\lambda(z) \) respectively have, respectively.

To prove the proposition, the relations in appendix B are useful.

**Remark.** The vertex operators (both type I and type II) can almost be determined by the method used for the level-one bosonization of the vertex operators of \( U_q(\widehat{sl}_N} [3\). But the contribution to the vertex operators of the fermionic part can not be determined by studying the commutation relations with the q-bosonic-oscillators. However the commutation relations: \([\Phi_1(z), X^+(w)] = [\Psi_{-1}(z), X^-(w)] = 0\) enables us to determine them uniquely up to some scalar factor.

### 3 Level-two highest weight \( U_q(\widehat{sl}_2) \)-modules and the corresponding intertwines

Set \( F_1 = \oplus_{n \in \mathbb{Z}} C[a_{-1}, a_{-2}, \ldots; b_{-\frac{1}{2}}, b_{-\frac{3}{2}}, \ldots] e^{nQ}|0\rangle \), where the Fock vacuum vector \(|0\rangle \) is defined by

\[
a_n|0\rangle = 0, \quad \text{for } n > 0, \quad P|0\rangle = 0,
\]

\[
b_{l+\frac{1}{2}}|0\rangle = 0 \quad \text{for } l \geq 0.
\]

It can be shown that the bosonized action of \( U_q(\widehat{sl}_2) \) on \( F_1 \) is closed. Hence the Fock space constitutes a \( U_q(\widehat{sl}_2) \)-module at level-two. However, it is not irreducible. In order to
obtain the irreducible subspace in $\mathcal{F}_1$, we should introduce the GSO-like projectors [13]:

$$P_\pm = \frac{1 \pm \exp\{-2\pi i \, d\}}{2},$$  \hspace{1cm} (3.1)

where

$$d = -\sum_{n=1}^{\infty} \frac{n^2}{(2n)^2} a_n a_n - \sum_{l=0}^{\infty} \frac{(2l+1)[2l+1]}{[4l+2]} b_{-l-\frac{1}{2}} b_{l+\frac{1}{2}} - \frac{P^2}{2}. $$

Note that $[d, a_n] = na_n$, $[d, b_r] = rb_r$, $[d, Q] = P$, we have the following relations

$$x^{-d} \phi_1(z) x^d = x^{\frac{1}{2}} \phi_1(xz), \quad x^{-d} \psi_{-1}(z) x^d = x^{\frac{1}{2}} \psi_{-1}(xz), \hspace{1cm} (3.2)$$

$$x^{-d} X^\pm(z) x^d = x X^\mp(xz). \hspace{1cm} (3.3)$$

Then we have $P_\pm X = XP_\pm$ for any $X \in U_q(\widehat{sl}_2)$. Define $\mathcal{F}^{(0)} = P_+ \mathcal{F}_1$ and $\mathcal{F}^{(1)} = P_- \mathcal{F}_1$. Then we have [13]

**Theorem 2** \( \mathcal{F}^{(0)} \) and \( \mathcal{F}^{(1)} \) are the irreducible highest weight \( U_q(\widehat{sl}_2) \)-modules with the highest weights \( 2\Lambda_0 \) and \( 2\Lambda_1 \) respectively, namely, \( \mathcal{F}^{(0)} = V(2\Lambda_0) \) and \( \mathcal{F}^{(1)} = V(2\Lambda_1) \).

One can further find that \( P_\pm \Theta(z) = \Theta(z) P_\pm \) for \( \Theta(z) = \phi_i(z), \phi^*_i(z), \psi_i(z), \psi^*_i(z) \). Hence, we have the following equality:

$$\Phi_{2\Lambda_0}^{2\Lambda_1} V(z) = P_- \phi(z) P_+, \quad \Phi_{2\Lambda_1}^{2\Lambda_0} V(z) = P_+ \phi(z) P_-$$  \hspace{1cm} (3.4)

$$\Psi_{2\Lambda_1} V^{2\Lambda_0}(z) = P_- \psi(z) P_+, \quad \Psi_{2\Lambda_0} V^{2\Lambda_1}(z) = P_+ \psi(z) P_- $$  \hspace{1cm} (3.5)

$$P_\pm \Theta(z) P_\pm = 0, \quad \text{for} \ \Theta(z) = \phi_i(z), \phi^*_i(z), \psi_i(z), \psi^*_i(z). \hspace{1cm} (3.6)$$

### 4 Exchange relations of vertex operators

In this section, we derive the exchange relations of the type I and type II operators of \( U_q(\widehat{sl}_2) \) at level-two from their bosonization.

#### 4.1 The R-matrix

Let \( R(z) \in \text{End}(V \otimes V) \) be the R-matrix of \( U_q(\widehat{sl}_2) \) defined by

$$R(z)(v_i \otimes v_j) = \sum_{k,l} R_{kl}^i(z) v_k \otimes v_l.$$
It can be given explicitly by

\[
R(z) = r(z) \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & e & 0 & 0 & 0 & 0 \\
0 & 0 & d & 0 & g & 0 & f & 0 \\
0 & \tilde{e} & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{g} & 0 & a & 0 & h & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & e \\
0 & 0 & \tilde{f} & 0 & \tilde{h} & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{e} & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\] (4.1)

Here the normalized partition is

\[
r(z) = \frac{1 - zq^2}{z - q^2},
\] (4.2)

which is just that of Ref. [17] with the level \(k = 2\). The other nonzero elements are given by

\[
a = \frac{qz^2 - (2q + 2q^{-1} - q^3 - q^{-3})z + q^{-1}}{(zq^2 - q^{-2})(zq - q^{-1})},
\] (4.3)

\[
b = \frac{z - 1}{zq^2 - q^{-2}}, \quad d = \frac{(z - 1)(zq^{-1} - q)}{(zq^2 - q^{-2})(zq - q^{-1})},
\] (4.4)

\[
e = \frac{q^2 - q^{-2}}{zq^2 - q^{-2}}, \quad \tilde{e} = \frac{(q^2 - q^{-2})z}{zq^2 - q^{-2}},
\] (4.5)

\[
f = \frac{(q^2 - q^{-2})(qz^2 - (q + q^{-1})z + q)}{(zq^2 - q^{-2})(zq - q^{-1})},
\] (4.6)

\[
\tilde{f} = \frac{(q^2 - q^{-2})(q^{-3}z^3 - q^{-3} + q - q^{-1})z}{(zq^2 - q^{-2})(zq - q^{-1})},
\] (4.7)

\[
g = \frac{(q^2 - q^{-2})(q^2 + 1)(z - 1)}{(zq^2 - q^{-2})(zq - q^{-1})},
\] (4.8)

\[
\tilde{h} = \frac{(q^2 - q^{-2})(q^2 + 1)(z - 1)z}{(zq^2 - q^{-2})(zq - q^{-1})},
\] (4.9)

\[
\tilde{g} = \frac{q^2 - q^{-2}}{[2]_q^2}, \quad h = \frac{q^{-2}}{[2]_q^2} g.
\] (4.10)

The R-matrix satisfies the Yang-Baxter equation on \(V \otimes V \otimes V\)

\[
R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),
\]

and the initial condition \(R(1) = P\) with \(P\) being the permutation operator.
4.2 The exchange relations

Define

$$\oint dz f(z) = \text{Res}(f) = f_{-1}, \quad \text{for formal series function } f(z) = \sum_{n \in \mathbb{Z}} f_n z^n. \quad (4.11)$$

Then the chevalley generators of $U_q(\hat{sl}_2)$ can be expressed by the integrals

$$e_1 = \oint dz X^+(z), \quad f_1 = \oint dz X^-(z). \quad (4.12)$$

From the normal order relations in appendix A, one can also obtain the integral expression of the vertex operators defined in (2.7-2.9)

$$\phi_0(z) = \sqrt{2} \oint dw \frac{(q^2 - q^{-2})}{(z q^2)(1 - \frac{w}{z q^2})(1 - \frac{z q^2}{w})} : \phi_1(z) E^-(w) : B(w), \quad (4.13)$$

$$\phi_{-1}(z) = \frac{2}{[2]} \oint dw \oint d\eta \frac{(q^2 - q^{-2})}{(z q^2)(1 - \frac{w}{z q^2})(1 - \frac{z q^2}{w})} : \phi_1(z) E^-(w) E^-(\eta) : \times \left\{ \frac{w - \eta q^2}{(-z q^4)(1 - \frac{n}{z q^4})} B(w) B(\eta) - \frac{\eta - w q^2}{\eta(1 - \frac{z q^2}{\eta})} B(\eta) B(w) \right\}. \quad (4.14)$$

$$\psi_0(z) = \sqrt{2} \oint dw \frac{(q^2 - q^{-2})}{(1 - \frac{w}{z q^2}) w(1 - \frac{z}{w})} : \psi_{-1}(z) E^+(w) : B(w), \quad (4.15)$$

$$\psi_1(z) = \frac{2}{[2]} \oint dw \oint d\eta \frac{(q^2 - q^{-2})}{(1 - \frac{w}{z q^2}) w(1 - \frac{z}{w})} : \phi_1(z) E^+(w) E^+(\eta) : \times \left\{ \frac{-w(1 - \frac{w q^2}{z q^4})}{(z q^2)(1 - \frac{z q^4}{w})} B(w) B(\eta) - \frac{\eta(1 - \frac{w q^2}{\eta})}{\eta(1 - \frac{z q^4}{\eta})} B(\eta) B(w) \right\}. \quad (4.16)$$

By the “weak equality” technique proposed in [18], using the normal order relations given in appendix A and the relations in appendix B, after tedious calculations, we can show that the bosonic vertex operators defined in (2.7-2.9) satisfy the Faddeev-Zamolodchikov (ZF) algebra

$$\phi_j(z_2) \phi_i(z_1) = \sum_{kl} R(\frac{z_1}{z_2})^{kl}_{ij} \phi_k(z_1) \phi_l(z_2),$$

$$\psi_i(z_1) \psi_j(z_2) = \sum_{kl} R(\frac{z_1}{z_2})^{kl}_{ij} \psi_l(z_2) \psi_k(z_1),$$

$$\psi_i(z_1) \phi_j(z_2) = \tau(\frac{z_1}{z_2}) \phi_j(z_2) \psi_i(z_1),$$

where $\tau(z) = -1$. Furthermore, we can show that the type I bosonic vertex operators have the following invertibility relation

$$\phi_i(z) \phi_i^*(z) = f_i \text{ id},$$
where the scalar factors \( \{ f_i \} \) are 
\[
f_1 = \frac{1}{z(q^2 - 1)q^2}, \quad f_0 = -\frac{2}{z(q^2 - 1)q^2}, \quad f_{-1} = \frac{q^2}{z(q^2 - 1)q^2}.
\]
In the derivation of the above relations we have used the identity: 
\[
\phi_1(z) \phi_1(z q^{-2}) E^{-}(z q^4) E^{-}(z q^2) := id.
\]

From the theorem 2 and the properties of bosonized vertex operators (3.6), we have

**Proposition 2** The vertex between the irreducible level-two highest weight \( U_q(\hat{sl}_2) \)-module \( V(2\Lambda_0) \) and \( V(2\Lambda_1) \) satisfy the ZF algebra and invertibility relation

\[
\Phi_j(z_2) \Phi_i(z_1) = \sum_{kl} R\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Phi_k(z_1) \Phi_l(z_2), \quad (4.17)
\]

\[
\Psi_i(z_1) \Psi_j(z_2) = \sum_{kl} R\left(\frac{z_1}{z_2}\right)_{ij}^{kl} \Psi_k(z_2) \Psi_l(z_1), \quad (4.18)
\]

\[
\Psi_i(z_1) \Phi_j(z_2) = \tau\left(\frac{z_1}{z_2}\right) \Phi_j(z_2) \Psi_i(z_1), \quad (4.19)
\]

\[
\Phi_i(z) \Phi_i^*(z) = f_i \ id. \quad (4.20)
\]

This agrees with the results of [17] which were obtained from the solution of Q-KZ equation.

**Remark.** We can similarly derive that the level-two vertex operators among \( V(\Lambda_0 + \Lambda_1) \)s with the 3-dimensional evaluation representation in Ref.[14] satisfies the ZF algebraic relation with the same R-matrix.

**Acknowledgments.** The author would like to thank Prof. G. von Gehlen for his encouragements and useful comments. This work has been supported by the Alexander von Humboldt Foundation.

**Appendix A**

In this appendix, we give the normal order relations of the fundamental bosonic fields including the fermionic field:

\[
\phi_1(z) \phi_1(w) = (-z q^4)(1 - \frac{w q^2}{z}) : \phi_1(z) \phi_1(w) :,
\]

\[
\phi_1(z) E^+ (w) = (-z q^4)(1 - \frac{w}{z q^4}) : \phi_1(z) E^+ (w) := E^+ (w) \phi_1(z),
\]

\[
\phi_1(z) E^- (w) = \frac{1}{(-z q^4)(1 - \frac{w}{z q^2})} : \phi_1(z) E^- (w) :,
\]

\[
E^- (w) \phi_1(z) = \frac{1}{w(1 - \frac{z q^2}{w})} : \phi_1(z) E^- (w) :,
\]

\[
\psi_{-1}(z) \psi_{-1}(w) = (-z q^2)(1 - \frac{w}{z q^2}) : \psi_{-1}(z) \psi_{-1}(w) :,
\]

\[
\psi_{-1}(z) E^- (w) = (w - z q^2) : \psi_{-1}(z) E^- (w) := E^- (w) \psi_{-1}(z),
\]
\[\psi_{-1}(z)E^+(w) = -\frac{1}{(zq^2)(1 - \frac{w}{zq})} : \psi_{-1}(z)E^+(w) :;\]
\[E^+(w)\psi_{-1}(z) = \frac{1}{w(1 - \frac{z}{w})} : \psi_{-1}(z)E^+(w) ;;\]
\[E^+(z)E^+(w) = (z - wq^{-2}) : E^+(z)E^+(w) ;;\]
\[E^-(z)E^-(w) = (z - wq^2) : E^-(z)E^-(w) ;;\]
\[E^+(z)E^-(w) = \frac{1}{z(1 - \frac{w}{z})} : E^+(z)E^-(w) ;;\]
\[E^-(w)E^+(z) = \frac{1}{w(1 - \frac{z}{w})} : E^-(w)E^+(z) ;;\]
\[B(z)B(w) = \frac{[2]}{2} \frac{1 - \frac{w}{z}}{z(1 - \frac{wq^2}{z})(1 - \frac{w}{zq^2})} + : B(z)B(w) ;.\]

**Appendix B**

By means of the bosonic realization of $U_q(\widehat{sl}_2)$, the integral expressions of the vertex operators and the “weak equality” technique given in Ref. [18], one can check the following relations.

- For the type I vertex operators
  \[[\phi_1(z), e_1] = 0, \ [\phi_0(z), e_1] = [2]t_1\phi_1(z), \]
  \[[\phi_{-1}(z), e_1] = t_1\phi_0(z), \ [\phi_{-1}(z), f_1]_{q^{-2}} = 0, \]
  \[t_1\phi_1(z) = q^2\phi_1(z)t_1, \ t_1\phi_0(z) = \phi_0(z)t_1, \ t_1\phi_{-1}(z) = q^{-2}\phi_{-1}(z)t_1.\]

- For the type II vertex operators
  \[[\psi_{-1}(z), f_1] = 0, \ [\psi_0(z), f_1] = [2]t_1^{-1}\psi_{-1}(z), \]
  \[[\psi_1(z), f_1] = t_1^{-1}\psi_0(z), \ [\psi_1(z), e_1]_{q^{-2}} = 0, \]
  \[t_1\psi_1(z) = q^2\psi_1(z)t_1, \ t_1\psi_0(z) = \psi_0(z)t_1, \ t_1\psi_{-1}(z) = q^{-2}\psi_{-1}(z)t_1.\]

**References**

[1] I.B. Frenkel, N.Yu. Reshetikhin, *Comm. Math. Phys.* **146** (1992), 1.

[2] M. Jimbo, T. Miwa, *Algebraic analysis of solvable lattice model*, CBMS Regional Conference Series in Mathematics, **Vol. 85** (AMS, Providence, 1994).

[3] W.-L. Yang, Y.-Z. Zhang, *Nucl. Phys.* **B547** (1999), 599.
[4] B. Davies, O. Foda, M. Jimbo, T. Miwa, A. Nakayashiki, *Commun. Math. Phys.* **151** (1993), 89.

[5] Y. Koyama, *Comm. Math. Phys.* **164** (1994), 277.

[6] B. Y. Hou, W.-L. Yang, Y.-Z. Zhang, *Nucl. Phys. B**556** (1999), 485.

[7] M. Jimbo, R. Kedem, T. Kojima, H. Konno, T. Miwa, *Nucl. Phys. B**441** (1995), 437.

[8] H. Furutsu, T. Kojima, *J. Math. Phys.* **41** (2000), 4413.

[9] W.-L. Yang, Y.-Z. Zhang, *Nucl. Phys. B**596** (2001), 495.

[10] I.B. Frenkel, N. Jing, *Proc. Nat'l. Acd. Sci. USA** 85** (1988), 9373.

[11] D. Bernard, *Lett. Math. Phys.* **17** (1989), 239.

[12] T. Kojima, *Int. J. Mod. Phys.* **A16** (2001), 1559.

[13] A.H. Bougourzi, R. Weston, *Nucl. Phys. B**417** (1994), 439.

[14] M. Idzumi, *Int. J. Mod. Phys.* **A9** (1994), 4449.

[15] V. Chari, A. Pressley, *Comm. Math. Phys.* **196** (1998), 461.

[16] V.G. Drinfeld, *Sov. Math. Dokl.* **36** (1988), 212.

[17] M. Idzumi, T. Tokihiro, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima, *Int. J. Mod. Phys. A**8** (1993), 1479.

[18] Y. Asai, M. Jimbo, T. Miwa, Y. Pugai, *J. Phys. A**29** (1996), 6595.