ANALYTIC COHOMOLOGY IN A BANACH SPACE

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ABSTRACT. Let $X$ be a Banach space with a countable unconditional basis (e.g., $X = \ell_2$ Hilbert space), $\Omega \subset X$ pseudoconvex open. We show that the sheaf cohomology groups $H^q(\Omega, S)$ vanish for $q \geq 1$ if $S$ is a member of a fairly inclusive class of sheaves of $\mathcal{O}$-modules over $\Omega$. In particular, we have the above vanishing if $S = I$ or $S = (\mathcal{O}_E)^0$, where $I$ is the ideal sheaf of a split complex Banach submanifold $M$ of $\Omega$, $E \to M$ is a locally trivial holomorphic Banach vector bundle, and $(\mathcal{O}_E)^0$ is the zero extension to $\Omega$ of the sheaf $\mathcal{O}_E$ of germs of holomorphic sections of $E \to M$. Some applications are also given.

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1. INTRODUCTION.

It was around 55 years ago (ca. 1950) when Karl Stein defined the notion of Stein manifolds, and Cartan, Oka, and Serre proved two fundamental, and long classical, theorems about Stein manifolds (and Stein spaces) called ever since Theorems A and B. One way to express (a substantial part of) Theorem B is to say that over $X = \mathbb{C}^n$, $n \geq 1$, the sheaf cohomology groups $H^q(X, S)$ vanish for all $q \geq 1$ if $S \to X$ is a coherent analytic sheaf. Sheaf cohomology vanishing theorems hold the key to many global results about complex manifolds, especially to those that can be solved first locally and then globally by patching the local solutions to global solutions.

In this paper we look at a class of sheaves, called sheaves of type $(S)$ or $(S)$-sheaves, over suitable complex Banach spaces that in a way mimics

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2 To my dear Younger Brother on his birthday.
the class of coherent analytic sheaves in finite dimensions, and for which we can prove vanishing in infinite dimensions. While the class of \((S)\)-sheaves is far from being as perfect as the class of coherent analytic sheaves in finite dimensions, it is, arguably, about the best class for which vanishing in infinite dimensions can be proved with current technology, and it also contains the most immediately geometrically relevant analytic sheaves, e.g., the sheaves of germs of holomorphic sections of holomorphic Banach vector bundles, and ideal sheaves of split complex Banach submanifolds. The definition of the class of \((S)\)-sheaves is fairly long, so it is given in its own section §4.

Following [L2] by Lempert we say that **plurisubharmonic domination** holds in a complex Banach manifold \(\Omega\) if for every \(u : \Omega \to \mathbb{R}\) locally upper bounded there is a \(\psi : \Omega \to \mathbb{R}\) continuous and plurisubharmonic such that \(u(x) < \psi(x)\) for all \(x \in \Omega\).

**Theorem 1.1.** (Lempert, [L2]) If \(X\) is a Banach space with a countable unconditional basis, and \(\Omega \subset X\) is pseudoconvex open, then plurisubharmonic domination holds in \(\Omega\).

Here we prove Theorems 1.2, 1.3, and 1.4 below.

**Theorem 1.2.** Let \(X\) be a Banach space with a Schauder basis, \(\Omega \subset X\) pseudoconvex open, \(M \subset \Omega\) a split complex Banach submanifold of \(\Omega\), \(Z\) a Banach space, \(I^Z \to \Omega\) the sheaf of germs of holomorphic functions \(\Omega \to Z\) that vanish on \(M\), \(E \to M\) a locally trivial holomorphic Banach vector bundle, \(O^E \to M\) the sheaf of germs of holomorphic sections of \(E \to M\), and \((O^E)^0 \to \Omega\) the zero extension to \(\Omega\) of the sheaf \(O^E\). If plurisubharmonic domination holds in every pseudoconvex open subset of \(\Omega \times X\), then the following hold.

(a) The sheaf \(I^Z \to \Omega\) is an \((S)\)-sheaf.

(b) The sheaf \((O^E)^0 \to \Omega\) is an \((S)\)-sheaf.

The main theorem of this paper is Theorem 1.3 below.

**Theorem 1.3.** Let \(X, \Omega\) be as in Theorem 1.2, and \(S \to \Omega\) an \((S)\)-sheaf. Then we have the following.

(a) The sheaf cohomology groups \(H^q(\Omega, S)\) vanish for all \(q \geq 1\).

(b) There is a short exact sequence

\[
0 \to K \to O^{Z_1} \to S \to 0
\]

of locally convex analytic sheaves over \(\Omega\), where \(Z_1\) is a Banach space, and \(K\) is an \((S)\)-sheaf, such that over any pseudoconvex open subset \(U\) of \(\Omega\) and for any Banach space \(Z\) the image of (1.1) under the functor \(\text{Hom}(O^Z, -)\) satisfies that

\[
0 \to \text{Hom}(O^Z, K) \to \text{Hom}(O^Z, O^{Z_1}) \to \text{Hom}(O^Z, S) \to 0
\]
is exact over $U$ both on the level of germs and on the level of global sections.

(c) There is a long exact sequence

$$
\ldots \to O^{Z^n} \to O^{Z^n-1} \to \ldots \to O^{Z_1} \to S \to 0
$$

of locally convex analytic sheaves over $\Omega$, where $Z_n$, $n \geq 1$, is a Banach space, such that over any pseudoconvex open subset $U$ of $\Omega$ and for any Banach space $Z$ the image of (1.3) under the functor $\Hom(O^Z, -)$ satisfies that

$$
\ldots \to \Hom(O^Z, O^{Z^n}) \to \Hom(O^Z, O^{Z^n-1}) \to \ldots \to \Hom(O^Z, O^{Z_1}) \to \Hom(O^Z, S) \to 0
$$

is exact over $U$ both on the level of germs and on the level of global sections.

(d) If ‘$(S')$-sheaves’ make up any class of locally convex analytic sheaves over pseudoconvex open subsets of $\Omega$, and for any $(S')$-sheaf $S$ parts (a) and (b) above holds with ‘$(S)$-sheaf’ replaced by ‘$(S')$-sheaf,’ then any $(S')$-sheaf $S \to \Omega$ is in fact an $(S)$-sheaf $S \to \Omega$.

Theorem 1.4 below is a geometric corollary of Theorem 1.3 above.

**Theorem 1.4.** With the notation and hypotheses of Theorem 1.2 the following hold.

(a) The sheaf $I^Z$ is acyclic over $\Omega$.

(b) Any holomorphic function $f : M \to Z$ can be extended to a holomorphic function $\tilde{f} : \Omega \to Z$ with $\tilde{f}(x) = f(x)$ for $x \in M$.

(c) The sheaf $(O^E)^0$ is acyclic over $\Omega$, and thus the sheaf $O^E$ is acyclic over $M$.

(d) There is a Banach space $Z_1$ and a holomorphic function $T : \Omega \to Z_1^*$ into the dual Banach space $Z_1^*$ of $Z_1$ such that as point sets $M = \{x \in \Omega : T(x) = 0\}$.

(e) For any open $U \subset \Omega$ with $M \subset U$, there is a pseudoconvex open $\omega \subset \Omega$ with $M \subset \omega \subset U$.

(f) There is a holomorphic neighborhood retraction $r : \omega \to M$, where $\omega$ is pseudoconvex open with $M \subset \omega \subset \Omega$ and $r$ is holomorphic with $r(x) = x$ for $x \in M$.

The proof of Theorem 1.3 follows broadly the classical proof of Theorem B by Cartan, Oka, and Serre, with exhaustions coming from [L1, L2] and [P4], patching and dimension shifting from [P3], and amalgamation of syzygies from [G], [Lt1], and [P4]. For background see [L1–L4, P1].

2. EXHAUSTION.
This section describes a way to exhaust a pseudoconvex open subset $\Omega$ of a Banach space $X$ that is convenient for proving vanishing results for sheaf cohomology over $\Omega$. We follow here [L4, §2].

We say that a function $\alpha$, call their set $\mathcal{A}'$, is an admissible radius function on $\Omega$ if $\alpha : \Omega \to (0, 1)$ is continuous and $\alpha(x) < \text{dist}(x, X \setminus \Omega)$ for $x \in \Omega$. We say that a function $\alpha$, call their set $\mathcal{A}$, is an admissible Hartogs radius function on $\Omega$ if $\alpha \in \mathcal{A}'$ and $-\log \alpha$ is plurisubharmonic on $\Omega$. Call $\mathcal{A}$ cofinal in $\mathcal{A}'$ if for each $\alpha \in \mathcal{A}'$ there is a $\beta \in \mathcal{A}$ with $\beta(x) < \alpha(x)$ for $x \in \Omega$.

**Proposition 2.1.** Plurisubharmonic domination holds in $\Omega$ if and only if $\mathcal{A}$ is cofinal in $\mathcal{A}'$.

**Proof.** Write $\alpha = e^{-u} \in \mathcal{A}'$ and $\beta = e^{-\psi} \in \mathcal{A}$. As plurisubharmonic domination holds on $\Omega$ for $u$ continuous if and only if for $u$ locally upper bounded, the proof of Proposition 2.1 is complete.

Put $B_X(x_0, r) = \{x \in X : |x - x_0| < r\}$ for a ball in a Banach space $X$, where $x_0 \in X$, and $0 < r \leq \infty$.

It will be often useful to look at coverings by balls $B_X(x, \alpha(x))$, $x \in \Omega$, $\alpha \in \mathcal{A}'$, and shrink their radii to obtain a finer covering by balls $B_X(x, \beta(x))$, $x \in \Omega$, $\beta \in \mathcal{A}$.

Let $e_n$, $n \geq 1$, be a Schauder basis in the Banach space $(X, \| \cdot \|)$. One can change the norm $\| \cdot \|$ to an equivalent norm so that $\| \sum_{i=1}^N x_i e_i \| \leq \| \sum_{i=M}^N x_i e_i \|$ for $0 \leq M \leq m \leq n \leq N \leq \infty$, $x_i \in \mathbb{C}$. Introduce the projections $\pi_N : X \to X$, $\pi_N \sum_{i=1}^\infty x_i e_i = \sum_{i=1}^N x_i e_i$, $x_i \in \mathbb{C}$, $\pi_0 = 0$, $\pi_\infty = 1$, $\varrho_N = 1 - \pi_N$, and define for $\alpha \in \mathcal{A}$ and $N \geq 0$ integer the sets

$$
\begin{align*}
D_N(\alpha) &= \{\xi \in \Omega \cap \pi_N X : (N + 1)\alpha(\xi) > 1\}, \\
\Omega_N(\alpha) &= \{x \in \pi_N^{-1} D_N(\alpha) : \varrho_N x \| < \alpha(\pi_N x)\}, \\
D_N^\times(\alpha) &= \pi_{N+1} X \cap \Omega_N(\alpha), \\
\Omega_N^\times(\alpha) &= \{x \in \pi_{N+1}^{-1} D_N^\times(\alpha) : \varrho_{N+1} x \| < \alpha(\pi_N x)\}, \\
\mathfrak{B}(\alpha) &= \{B_X(x, \alpha(x)) : x \in \Omega\}, \\
\mathfrak{B}_N(\alpha) &= \{B_X(x, \alpha(x)) : x \in \Omega_N(\alpha)\}.
\end{align*}
$$

(2.1)

These $\Omega_N(\alpha)$ are pseudoconvex open in $\Omega$, and they will serve to exhaust $\Omega$ as $N = 0, 1, 2, \ldots$ varies.

**Proposition 2.2.** (Lempert) Let $\alpha \in \mathcal{A}$, and suppose that plurisubharmonic domination holds in $\Omega$.

(a) There is an $\alpha' \in \mathcal{A}$, $\alpha' < \alpha$, with $\Omega_n(\alpha') \subset \Omega_N(\alpha)$ for all $N \geq n$. So any $x_0 \in \Omega$ has a neighborhood contained in all but finitely many $\Omega_N(\alpha)$. 
(b) There are $\beta, \gamma \in \mathcal{A}$, $\gamma < \beta < \alpha$, so that for all $N$ and $x \in \Omega_N(\gamma)$

\begin{equation}
B_X(x, \gamma(x)) \subset \Omega_N(\beta) \cap \pi_N^{-1}B_X(\pi_Nx, \beta(x)) \subset B_X(x, \alpha(x)).
\end{equation}

(c) If $8\alpha \in \mathcal{A}$, $Y \subset X$ is a finite dimensional complex affine subspace, then $Y \cap \overline{\Omega_N(\alpha)}$ is plurisubharmonically convex in $Y \cap \Omega$.

(d) We have that $\Omega_N(\alpha) \subset \Omega^N(\alpha)$. If $4\alpha \in \mathcal{A}$, then $\Omega^N(\alpha) \subset \Omega_N(2\alpha)$.

(e) There is a $\beta \in \mathcal{A}$, $\beta < \alpha$, with $\Omega_N(\gamma) \subset \Omega_N(\alpha) \cap \Omega_{N+1}(\alpha)$ for $N \geq 0$.

(f) There is an $\alpha' \in \mathcal{A}$, $\alpha' < \alpha$, such that the covering $B_N(\alpha)|_{\Omega_N(\alpha')}$ has a finite basic refinement for all $N \geq 0$.

**Proof.** For (a) and (b) see [L4, Prop. 2.1], and [L3, Prop. 4.3], for (c) [L3, Prop. 4.3], for (d) [L3, Prop. 4.4], for (e) [L4, Prop. 2.3], and for (f) see [P3, Prop. 3.2(c)]. The proof of Proposition 2.2 is complete. (Remark for the record that (f) was not explicitly formulated by Lempert.)

The meaning of Proposition 2.2(b) is that certain refinement maps exist between certain open coverings, while (cd) are useful for Runge type approximation, and (ef) for exhaustion.

### 3. MODEL SHEAVES AND THEIR HOMOMORPHISMS.

In this section we look at the simplest kinds of sheaves of $\mathcal{O}$-modules, their topology on their spaces of sections, and their continuous homomorphisms.

A **complex Banach manifold** $\Omega$ modelled on a Banach space $X$ is a paracompact Hausdorff space with an atlas of biholomorphically related charts onto open subsets of $X$. Many of the complex analytic properties of $\Omega$ can be studied by looking at the sheaves $\mathcal{O}^Z \rightarrow \Omega$ of germs of holomorphic functions $\Omega \rightarrow Z$, where $Z$ is any Banach space. We call any such sheaf $\mathcal{O}^Z$ a model sheaf over $\Omega$.

The vector space $\mathcal{O}(U, Z) = \mathcal{O}^Z(U)$ of global sections of $\mathcal{O}^Z$ over any open $U \subset \Omega$ carries a natural complete locally convex vector topology induced by the family of seminorms $p_K$, where $K$ runs through all compact subsets of $U$, defined by $p_K(f) = \sup_{x \in K} \|f(x)\|_Z$ for $f \in \mathcal{O}(U, Z)$. As the point evaluations $\mathcal{O}(U, Z) \ni f \mapsto f(x) \in Z$, $x \in U$, are continuous linear functionals in this topology, the space $\mathcal{O}(U, Z)$ is indeed Hausdorff. If $\Omega$ is finite dimensional, then the resulting locally convex spaces $\mathcal{O}(U, Z)$ are in fact Fréchet spaces. If $\Omega$ is infinite dimensional, then $\mathcal{O}(U, Z)$ may not be a Fréchet spaces.

We denote by $H = \text{Hom}(\mathcal{O}^{Z_1}, \mathcal{O}^{Z_2})$ the sheaf of $\mathcal{O}$-linear continuous sheaf homomorphisms over $\Omega$ from $\mathcal{O}^{Z_1}$ to $\mathcal{O}^{Z_2}$, i.e., the sections $\tau \in H(U)$ over an open $U \subset \Omega$ are $\mathcal{O}$-linear maps $\tau : \mathcal{O}^{Z_1}|_U \rightarrow \mathcal{O}^{Z_2}|_U$ that induce continuous linear maps $\tau : \mathcal{O}(V, Z_1) \rightarrow \mathcal{O}(V, Z_2)$ in the natural topology discussed above.
for all open $V \subset U$. (For our purposes it is enough to consider topology on $\mathcal{O}(V, Z)$ only for arbitrarily small coordinate pseudoconvex open neighborhoods $V$ of every point of $\Omega$.)

Any holomorphic operator function $T \in \mathcal{O}(\Omega, \text{Hom}(Z_1, Z_2))$ induces a sheaf homomorphism $\tau = \hat{T} \in \text{Hom}(\mathcal{O}^Z_1, \mathcal{O}^Z_2) \to \Omega$ defined by $\hat{T}f = g$, where $f \in \mathcal{O}(U, Z_1)$, $g \in \mathcal{O}(U, Z_2)$, and $g(x) = T(x)f(x)$ for $x \in U$, $U \subset \Omega$ open.

We go on to show that any $\tau$ arises as $\tau = \hat{T}$ for a unique $T$ at least if $X$ is nice enough.

**Proposition 3.1.** Let $\Omega$ be a complex Banach manifold modelled on a Banach space $X$ with a Schauder basis, $Z_1, Z_2$ Banach spaces, and $\tau : \mathcal{O}^Z_1 \to \mathcal{O}^Z_2$ a sheaf homomorphism of $\mathcal{O}$-module sheaves over $\Omega$. Suppose that $\tau$ is sequentially continuous, i.e., if for each point $x_0 \in \Omega$ there is a coordinate pseudoconvex open set $V$ with $x_0 \in V \subset \Omega$ such that for every coordinate pseudoconvex open $U$ with $x_0 \in U \subset V$, $f_n, f \in \mathcal{O}(U, Z_1)$, $n = 1, 2, 3, \ldots$, and $f_n \to f$ uniformly on compact subsets of $U$ as $n \to \infty$, then $\tau f_n \to \tau f$ uniformly on compact subsets of $U$ as $n \to \infty$. Then $\tau$ is of the form $\tau = \hat{T}$ for a unique $T \in \mathcal{O}(\Omega, \text{Hom}(Z_1, Z_2))$.

Note that for any such $T$ the induced sheaf homomorphism $\hat{T}$ is (sequentially) continuous, since $\|T(x)\|$ is bounded for $x$ in any compact subset $K$ of $\Omega$. The proof of Proposition 3.1 will occupy us for a while.

**Proposition 3.2.** (a) Let $X$ be a Banach space with a Schauder basis, and Schauder projections $\pi_n, \varrho_n : X \to X$ as in §2. Then for any $x \in X$ the sequence $\|\varrho_n(x)\|$ decreases down to zero as $n \to \infty$.

(b) If $f_n : K \to [0, \infty)$ are continuous functions on a compact space $K$, and for each $x \in K$ the sequence $f_n(x)$ decreases down to zero as $n \to \infty$, then $f_n \to 0$ uniformly on $K$.

(c) For any compact $K \subset X$ we have that $\sup_{x \in K} \|\varrho_n(x)\| \to 0$ as $n \to \infty$.

(d) Let $B_X(x_0, R)$ be an open ball in $X$, $Z$ a Banach space, and $f : B_X(x_0, R) \to Z$ holomorphic, or, even just holomorphic on all complex affine one dimensional slices of $B_X(x_0, R)$. If $f$ is bounded on $B_X(x_0, R)$, then $f$ is Lipschitz continuous on $B_X(x_0, r)$ for $r < R/4$.

(e) If $f \in \mathcal{O}(B_X(0, r), Z)$ is Lipschitz continuous, then $f_n \in \mathcal{O}(B_X(0, r), Z)$ defined by $f_n(x) = f(\pi_n(x))$ tends to $f$ uniformly on compact subsets of $B_X(0, r)$.

**Proof.** (a) See the paragraph of (2.1).

(b) This is a classical theorem of Dini. Given any $\varepsilon > 0$, for any $x_0 \in K$ let $N_\varepsilon(x_0)$ be the smallest index $N \geq 1$ with $f_N(x_0) < \varepsilon$. As the inequality
\( f_N(x) < \varepsilon \) persists for \( x \) in an open neighborhood of \( x_0 \), we see that the function \( x_0 \mapsto N_\varepsilon(x_0) \) is locally upper bounded on \( K \). A locally upper bounded function on a compact space is in fact globally upper bounded. Hence there is an integer \( M_\varepsilon \geq 1 \) such that \( N_\varepsilon(x) \leq M_\varepsilon \) for all \( x \in K \), i.e., for all \( n \geq M_\varepsilon \) we have that \( 0 \leq f_n(x) \leq f_{M_\varepsilon}(x) < \varepsilon \) for all \( x \in K \), or, \( f_n \to 0 \) uniformly on \( K \).

(c) By (a) part (b) is applicable to \( f_n(x) = \| g_n(x) \| \).

(d) By assumption there is an \( M > 0 \) with \( \| f(x) \| < M \) for \( \| x \| < R \). If \( \| x \| < r \), \( \| y \| < r \), \( x \neq 0 \), \( |\lambda| < 3r \), then \( \| x + \lambda \frac{y - x}{\| y - x \|} \| \leq \| x \| + |\lambda| < r + 3r = 4r < R \). Let \( z^* \in Z^* \) be a continuous linear functional on the Banach space \( Z \) with \( \| z^* \| \leq 1 \). Thus the function \( \varphi(\lambda) = z^* (f(x + \lambda \frac{y - x}{\| y - x \|}) - f(x)) \), \( |\lambda| < 3r \), is a numerical holomorphic function, and satisfies that \( \varphi(0) = 0 \), and \( |\varphi(\lambda)| \leq 2M \) for \( |\lambda| < 3r \). The classical Schwarz lemma implies that \( |\varphi(\lambda)| \leq \frac{2M}{3r} |\lambda| \) for \( |\lambda| < 3r \). In particular, for \( \lambda = \| y - x \| < 2r \) we have that \( |\varphi(\| y - x \|)| = |z^* (f(y) - f(x))| \leq \frac{2M}{3r} \| y - x \| \). Taking supremum for \( z^* \in Z^* \) with \( \| z^* \| \leq 1 \) we find that \( \| f(y) - f(x) \| \leq \frac{2M}{3r} \| y - x \| \) as claimed.

(e) Let \( L \) be a Lipschitz constant for \( f \), i.e., \( \| f(y) - f(x) \| \leq L \| y - x \| \) for \( \| x \|, \| y \| < r \). As \( \| f_n(x) - f(x) \| \leq \| f(\pi_n(x)) - f(x) \| \leq L \| \pi_n(x) - x \| \leq L \| g_n(x) \| \) an application of (c) completes the proof of Proposition 3.2.

**Proposition 3.3.** If \( T : \Omega \to \text{Hom}(Z_1, Z_2) \) satisfies that \( T(x)z_1 \) is holomorphic in \( x \in \Omega \) for each fixed \( z_1 \in Z_1 \), then \( T \) is holomorphic, i.e., \( T \in \mathcal{O}(\Omega, \text{Hom}(Z_1, Z_2)) \).

**Proof.** This is a well-known classical statement. As the desired holomorphy of \( T \) is a local property, we may assume that \( \Omega = B_X(0, 1) \). If \( x_n \to x_0 \) in \( \Omega \), then \( T(x_n)z_1 \to T(x_0)z_1 \). So by, say, the Banach–Steinhaus theorem \( \| T(x_n) \| \) is bounded as \( n \to \infty \), i.e., \( \| T(x) \| \) is a locally bounded function of \( x \in \Omega \).

We need to show that \( T \) is holomorphic on one dimensional complex affine slices of \( \Omega \). Fix \( x_0, x_1 \in X, \| x_1 \| = 1, \) and look at the function \( T_\lambda = T(x_0 + \lambda x_1) \) of \( \lambda \) on the open set \( \Lambda \) of the \( \lambda \)-plane where it is defined. Before we can show the desired holomorphy of \( T_\lambda \) for \( \lambda \in \Lambda \) we check that \( T_\lambda \) is locally Lipschitz continuous on \( \Lambda \). To that end let \( M \) be a bound of \( \| T(x) \| \) in a neighborhood in \( \Omega \) of the point \( x_0 + \lambda_0 x_1 \) for any fixed \( \lambda_0 \in \Lambda \), and \( z_1 \in Z_1 \) any vector with \( \| z_1 \| \leq 1 \). On applying the Schwarz lemma as in the proof of Proposition 3.2(d) to the holomorphic function \( T_\lambda z_1 \) in a small disc about \( \lambda_0 \) we see that an estimate \( \| T_\lambda z_1 - T_{\lambda_1} z_1 \| \leq L |\lambda_2 - \lambda_1| \) holds, where \( L \) is independent of \( z_1 \) and \( \lambda_1, \lambda_2 \). Taking supremum for \( z_1 \in Z_1 \), \( \| z_1 \| \leq 1 \), we obtain that \( \| T_{\lambda_2} z_1 - T_{\lambda_1} z_1 \| \leq L |\lambda_2 - \lambda_1| \), i.e., \( T_\lambda \) is indeed locally Lipschitz continuous for \( \lambda \in \Lambda \). Thus the vector valued Riemann integral
Proposition 3.5 says that if \( \tau \) remains to show that \( O \) uniformly on (compact subsets of) \( Z \) we can write \( h(\zeta) = f(\sum_{i=1}^{n} \zeta_{i}e_{i}) \) defined and holomorphic for \( \zeta = (\zeta_{1}, \ldots , \zeta_{n}) \) in an open neighborhood of the origin in \( \mathbb{C}^{n} \). As \( h(0) = 0 \) we can write \( h(\zeta) = \sum_{i=1}^{n} \zeta_{i}h_{i}(\zeta) \) in a neighborhood of the origin, where \( h_{i}, \ i = 1, \ldots , n \), is holomorphic in a neighborhood of the origin, either by power series expansion, or by looking at \( h(\zeta) = h(\zeta) - h(0) = \int_{t=0}^{1} \frac{d}{dt} h(t\zeta) dt \). Hence 
\[
g(x) = \sum_{i=1}^{n} \xi_{i}(x)g_{i}(x) \text{ for } x \text{ in a neighborhood of the origin of } X, \text{ where the } g_{i} \text{ are holomorphic}. \]
Since \( (\tau g)(x) = \sum_{i=1}^{n} \xi_{i}(x)(\tau g_{i})(x) \), on setting \( x = 0 \) we get that \( (\tau g)(0) = 0 \). The proof of Proposition 3.4 is complete.

Proposition 3.5. With the notation and hypotheses of Proposition 3.1 if \( f \in \mathcal{O}(U, Z_{1}) \), \( f(x_{0}) = 0 \), \( x_{0} \in U \subset \Omega \) open, then \( (\tau f)(x_{0}) = 0 \).

Proof. Without loss of generality we may assume that \( x_{0} = 0 \in X, \ U = B_{X}(0, 1) \) and \( f \) is Lipschitz continuous on \( U \). Define \( f_{n} \in \mathcal{O}(U, Z_{1}) \) by 
\[
f_{n}(x) = f(\pi_{n}(x)). \]
Since \( f_{n}(0) = 0 \) Proposition 3.4 implies that \( (\tau f_{n})(0) = 0 \) for \( n \geq 1 \). Proposition 3.2(e) shows that \( (\tau f_{n}) \to (\tau f) \) uniformly on compact subsets \( K \) of \( U \). In particular, letting \( K = \{0\} \) yields that \( 0 = (\tau f_{n})(0) \to (\tau f)(0) \) as \( n \to \infty \). Thus \( (\tau f)(0) = 0 \), and the proof of Proposition 3.5 is complete.

Proof of Proposition 3.1. Letting \( f_{z_{1}} \) be various constant functions \( f_{z_{1}}(x) = z_{1} \in Z_{1} \) we see that \( \tau(f_{z_{1}})(x) = T(x)z_{1} \) for a unique linear map \( T(x) : Z_{1} \to Z_{2} \). If a sequence \( z_{1}^{n} \to z_{1} \) converges in \( Z_{1} \) in norm, then \( f_{z_{1}^{n}} \to f_{z_{1}} \) uniformly on (compact subsets of) \( X \), so \( T(x) \in \text{Hom}(Z_{1}, Z_{2}) \) is indeed a bounded linear operator, and \( T \in \mathcal{O}(\Omega, \text{Hom}(Z_{1}, Z_{2})) \) by Proposition 3.3. It remains to show that \( \tau = \tilde{T}, \text{ i.e., } \tau f = \tilde{T} f \) for \( f \in \mathcal{O}(U, Z_{1}), U \subset \Omega \) open. Proposition 3.5 says that if \( f(x_{0}) = 0 \) for an \( x_{0} \in U \), then \( (\tau f)(x_{0}) = 0 \) as well. For a general \( f \) write \( f(x) = (f(x) - f(x_{0})) + f(x_{0}) \). If we regard \( f(x_{0}) \)
as a constant member of $O(U, Z_1)$, then $\tau f = \tau(f - f(x_0)) + \dot{T}f(x_0)$, whose value at $x_0$ is $(\tau f)(x_0) = 0 + T(x_0)f(x_0)$. Hence $\tau = \dot{T}$ and the proof of Proposition 3.1 is complete.

Note that in the above proof of Proposition 3.1 the sequential continuity of $\tau$, and topology on section spaces, were used only on arbitrarily small open neighborhoods $U$ of any point $x_0$ of $\Omega$, where $U$ is biholomorphic to a ball in $X$.

4. (S)-SHEAVES.

In this section we define a class of analytic sheaves called (S)-sheaves that form the major object of study in this paper, and we also look at some of their first properties.

Let $\Omega$ be a complex Banach manifold modelled on a Banach space $X$. We call an open $U \subset \Omega$ coordinate pseudoconvex if $U$ is biholomorphic to a pseudoconvex open subset of $X$. An open subset $U$ of $X$ is coordinate pseudoconvex if and only if it is pseudoconvex. We call an open covering $\mathcal{U}$ of $\Omega$ an (S)-covering if all intersections $\bigcap U_i$ of finitely many members $U_i$ of $\mathcal{U}$ are coordinate pseudoconvex. It is easy to see that any open covering $\mathcal{V}$ of $\Omega$ has a refinement $\mathcal{U}$ that is an (S)-covering.

Let $S \to \Omega$ be a sheaf of $O$-modules. We call $S$ a locally convex analytic sheaf over $\Omega$ if $\Omega$ has an (S)-covering $\mathcal{U}$ such that for any $U \in \mathcal{U}$, $V \subset U$ coordinate pseudoconvex open, the set of sections $S(V)$ carries a complete Hausdorff locally convex topological vector space structure so that the $O$-module multiplication $O(V) \times S(V) \to S(V)$ is continuous, and the restriction maps $V_2 \subset V_1$ induce continuous linear maps $S(V_1) \to S(V_2)$.

Let $S_1$ and $S_2$ be two locally convex analytic sheaves over $\Omega$. Let $\mathcal{U}_i$ be an (S)-covering of $\Omega$ that can serve in the above definition for $S_i$, $i = 1, 2$. Let $\mathcal{U}$ be an (S)-covering of $\Omega$ that is a common refinement of $\mathcal{U}_1$ and $\mathcal{U}_2$. Let $\tau : S_1 \to S_2$ be a sheaf homomorphism of $O$-module sheaves over $\Omega$. We call $\tau$ continuous over $\Omega$ and write $\tau \in \text{Hom}(S_1, S_2)(\Omega)$ if for each $U \in \mathcal{U}$, $V \subset U$ coordinate pseudoconvex open, the map induced by $\tau$ on sections $S_1(V) \to S_2(V)$ is continuous in the given topologies of $S_1$ and $S_2$. The set of all continuous sheaf homomorphisms $\tau$ as above form a sheaf $\text{Hom}(S_1, S_2)$ over $\Omega$, whose sections over any open $U \subset \Omega$ are all the continuous sheaf homomorphisms $S_1|U \to S_2|U$ of locally convex analytic sheaves. This sheaf $\text{Hom}(S_1, S_2)$ may or may not be a locally convex analytic sheaf over $\Omega$.

The model sheaf $O^Z \to \Omega$, where $Z$ is any Banach space, is, with its natural topology, a locally convex analytic sheaf over $\Omega$, and the sheaf of continuous homomorphisms $\text{Hom}(O^{Z_1}, O^{Z_2})$, where $Z_1, Z_2$ are Banach spaces, is naturally isomorphic to the sheaf of holomorphic operator functions $O^{\text{Hom}(Z_1, Z_2)}$. 

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by Proposition 3.1 if $X$ has a Schauder basis. The main reason to look at topology on spaces of sections, and continuity of sheaf homomorphisms is precisely the above identification of $\text{Hom}(\mathcal{O}Z_1, \mathcal{O}Z_2)$ with $\mathcal{O}^{\text{Hom}(Z_1, Z_2)}$; a triviality if both $Z_1$ and $Z_2$ are finite dimensional, as in the case of classical Stein theory.

The only locally convex analytic sheaves $S$ that will interest us in this paper are locally of the form $\mathcal{O}Z/K$, where $K$ is a subsheaf of $\mathcal{O}Z$ with $K(U)$ being a closed subspace of $\mathcal{O}(U, Z)$, where $U$ is a small enough coordinate pseudoconvex open neighborhood of any point of $\Omega$. Such sheaves $S$ are indeed locally convex analytic sheaves over $\Omega$.

Let $X$ be a Banach space, $\Omega \subset X$ pseudoconvex open, and $M$ a complex Banach manifold modelled on $X$. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of $\mathcal{O}$-module sheaves over $\Omega$.

We say that (4.1) is 1-exact or a short 1-exact sequence over a pseudoconvex open subset $U$ of $\Omega$ if (4.1) is exact on the germ level at any point $x \in U$, and on the level of global sections over any pseudoconvex open $V \subset U$.

Let (4.1) be a short exact sequence of $\mathcal{O}$-module sheaves over $M$. We say that (4.1) is locally 1-exact or a short locally 1-exact sequence over $M$ if there is an (S)-covering $\mathcal{U}$ of $M$ such that for all $U \in \mathcal{U}$ our (4.1) is a short 1-exact sequence over $U$ in the above sense.

Let (4.1) be a short exact sequence of locally convex analytic sheaves (and their continuous homomorphisms) over $\Omega$. We say that (4.1) is 2-exact or a short 2-exact sequence over a pseudoconvex open $U \subset \Omega$ if for any Banach space $Z$ the image of (4.1) under the functor $\text{Hom}(\mathcal{O}Z, -)$ satisfies that

$$0 \to \text{Hom}(\mathcal{O}Z, A) \to \text{Hom}(\mathcal{O}Z, B) \to \text{Hom}(\mathcal{O}Z, C) \to 0$$

is a short 1-exact sequence over $U$.

Let (4.1) be a short exact sequence of locally convex analytic sheaves over $M$. We say that (4.1) is locally 2-exact or a short locally 2-exact sequence over $M$ if there is an (S)-covering $\mathcal{U}$ of $M$ such that (4.1) is a short 2-exact sequence over each $U \in \mathcal{U}$ in the above sense.

Let $i = 1, 2,$ and

$$\cdots \to A_n \xrightarrow{\tau_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{\tau_1} A_0 \to 0$$

a long exact sequence of sheaves of $\mathcal{O}$-modules over $\Omega$ for $i = 1$, and a long exact sequence of locally convex analytic sheaves over $\Omega$ for $i = 2$,
\( K_p = \text{Ker} \, \tau_n, \, p \geq 1, \, K_0 = A_0, \) the associated kernel sheaves, and

\[
(4.4) \quad 0 \to K_{p+1} \to A_{p+1} \to K_p \to 0,
\]

\( p \geq 0, \) the sequence of short exact sequences over \( \Omega \) associated to (4.3).

If (4.4) is a short \( i \)-exact sequence over a pseudoconvex open \( U \subset \Omega \) for all \( p \geq 0, \) then we say that (4.3) is \( i \)-exact or a long \( i \)-exact sequence over \( U. \)

Let \( i = 1, 2, \) and (4.3) a long exact sequence of sheaves of \( \mathcal{O} \)-modules over \( M \) for \( i = 1, \) and a long exact sequence of locally convex analytic sheaves over \( M \) for \( i = 2, \) and let \( K_p, \) and (4.4) as above. If there is an \( (S) \)-covering \( \mathfrak{U} \) of \( M \) so that for all \( U \in \mathfrak{U} \) our (4.4) is a short \( i \)-exact sequence over \( U \) for all \( p \geq 0, \) then we say that (4.3) is locally \( i \)-exact or a long locally \( i \)-exact sequence over \( M. \)

Let \( Z_n, \, n \geq 1, \) be a Banach space, \( S \to M \) a sheaf of \( \mathcal{O} \)-modules. A long exact sequence

\[
(4.5) \quad \cdots \to \mathcal{O}^{Z_n} \to \mathcal{O}^{Z_{n-1}} \to \cdots \to \mathcal{O}^{Z_1} \to S \to 0
\]

of sheaves of \( \mathcal{O} \)-modules over \( M \) is called a 1-resolution (by model sheaves) over \( M \) if (4.5) is a long locally 1-exact sequence over \( M. \)

Let \( Z_n, \, n \geq 1, \) be a Banach space, \( S \to M \) a locally convex analytic sheaf. A long exact sequence (4.5) of locally convex analytic sheaves over \( M \) is called a 2-resolution or an \((S)\)-resolution (by model sheaves) over \( M \) if (4.5) is a long locally 2-exact sequence over \( M. \)

We call a locally convex analytic sheaf \( S \to M \) a sheaf of type \((S)\) or an \((S)\)-sheaf if there is an \( (S) \)-covering \( \mathfrak{U} \) of \( M \) such that over each \( U \in \mathfrak{U} \) our sheaf \( S|U \) admits an \( (S) \)-resolution as above.

To deal with \((S)\)-resolutions and \((S)\)-sheaves Theorem 5.4 and the following two theorems come in handy.

**Theorem 4.1.** ([P4, Thm. 1.3]) Let \( X \) be a Banach space with a Schauder basis, \( \Omega \subset X \) pseudoconvex open, \( E \to \Omega \) a holomorphic Banach vector bundle with a Banach space \( Z \) for fiber type. If plurisubharmonic domination holds in \( \Omega, \) then we have the following.

(a) \( H^q(\Omega, \mathcal{O}^Z) = 0 \) for \( q \geq 1. \)

(b) Let \( Z_1 = \ell_p(Z), \, 1 \leq p < \infty. \) Then \( E \oplus (\Omega \times Z_1) \) and \( \Omega \times Z_1 \) are holomorphically isomorphic over \( \Omega. \)

(c) \( H^q(\Omega, \mathcal{O}^E) = 0 \) for \( q \geq 1. \)

(d) If \( E \) is continuously trivial over \( \Omega, \) then \( E \) is holomorphically trivial over \( \Omega. \)
**Theorem 4.2.** ([P3, Thm. 4.3]) Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, and suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$. Let $(4.5)$ be a 1-resolution of a sheaf $S$ of $\mathcal{O}$-modules over $\Omega$, and $K_p, p \geq 0$, the associated sequence of kernel sheaves over $\Omega$. Then the following hold.

(a) $H^q(\Omega, K_p) = 0$ for all $q \geq 1$ and $p \geq 0$.

(b) The sequence $(4.5)$ is exact over $\Omega$ on the level of global sections.

Next we show that locally exact global resolutions are in fact globally exact.

**Theorem 4.3.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $i = 1, 2$, $S \to \Omega$ a sheaf of $\mathcal{O}$-modules for $i = 1$, and a locally convex analytic sheaf for $i = 2$. Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$. If $(4.5)$ is an $i$-resolution of $S$ over $\Omega$, $K_p, p \geq 0$, are the associated kernel sheaves, then $K_p, p \geq 0$, are acyclic over any pseudoconvex open $U \subset \Omega$. In particular, $(4.5)$ is a long $i$-exact sequence over $\Omega$, and for $i = 2$ the sheaves $K_p, p \geq 0$, are $(S)$-sheaves over $\Omega$.

**Proof.** For $i = 1$ this follows from Theorem 4.2. For $i = 2$ this follows from the case $i = 1$ above applied to $(4.5)$, and then to its image under the functor $\text{Hom}(\mathcal{O}^Z, -)$, where $Z$ is any Banach space. Finally, the sheaf $K_p$ is an $(S)$-sheaf, since

$$\ldots \to \mathcal{O}^Z_n \to \mathcal{O}^Z_{n-1} \to \ldots \to \mathcal{O}^Z_{p+1} \to K_p \to 0$$

is an $(S)$-resolution of $K_p, p \geq 0$, over $\Omega$, where all maps are as in $(4.5)$. The proof of Theorem 4.3 is complete.

Our goal is to show eventually that any $(S)$-sheaf $S \to \Omega$ as in Theorem 4.3 admits in fact an $(S)$-resolution over all of $\Omega$.

**Proposition 4.4.** (a) The direct sum of two $(S)$-resolutions

$$\ldots \to \mathcal{O}^{Z'_n} \xrightarrow{\tau'_n} \mathcal{O}^{Z'_n-1} \to \ldots \to \mathcal{O}^{Z'_1} \xrightarrow{\tau'_1} S' \to 0,$$

$$\ldots \to \mathcal{O}^{Z''_n} \xrightarrow{\tau''_n} \mathcal{O}^{Z''_n-1} \to \ldots \to \mathcal{O}^{Z''_1} \xrightarrow{\tau''_1} S'' \to 0$$

over the same complex Banach manifold $M$ is an $(S)$-resolution $(4.5)$ over $M$, where $Z_n = Z'_n \oplus Z''_n$, $\tau_n = (\tau'_n, 0)$, $n \geq 1$, and $S = S' \oplus S''$.

(b) If in an $(S)$-resolution $(4.5)$ over a complex Banach manifold $M$ the map $\tau_1 : \mathcal{O}^{Z_1} \to S$ is replaced by $\tau_1 A : \mathcal{O}^{Z_1} \to S$, where $A \in \mathcal{O}(M, \text{GL}(Z_1))$, and the other maps $\tau_n, n \geq 2$, are unchanged, then we get another $(S)$-resolution of $S$ over $M$. 
Proof. As both parts are clear from the definitions, the proof of Proposition 4.4 is complete.

5. EXAMPLES OF (S)-SHEAVES.

In this section we show that some of the simplest and most immediately geometrically relevant analytic sheaves are (S)-sheaves.

In infinite dimensions no analog of the classical Oka coherence theorem seems to be currently available. It is therefore difficult to verify whether a sheaf is an (S)-sheaf, requiring a case by case study, and we can do it here in only a few, but useful, cases.

The definition of an (S)-sheaf is purely local. We only need to exhibit an (S)-resolution on small enough coordinate pseudoconvex open neighborhoods of each point of the ground Banach manifold.

As an aside, note that over a finite dimensional complex manifold any coherent analytic sheaf is an (S)-sheaf, as it is easy to see using the basic theorems of Stein theory, such as the Oka coherence theorem and Theorem B. Also there are many (S)-sheaves that are not of a finite rank, let alone coherent analytic.

Proposition 5.1. Let $\mathcal{M}$ be a complex Banach manifold, and $\mathcal{Z}$ a Banach space. Then $\mathcal{O}_Z \to \mathcal{M}$ is an (S)-sheaf.

Proof. As it is easy to check that the trivial sequence $\ldots \to 0 \to 0 \to \ldots \to 0 \to \mathcal{O}_Z \to 0$ is a long locally 2-exact sequence over $\mathcal{M}$, the proof of Proposition 5.1 is complete.

Note that to make any use of (S)-sheaves one has to have a thorough understanding of the model sheaves $\mathcal{O}_Z$, in particular, one has to know that over any pseudoconvex open subset of the ground Banach space the sheaves $\mathcal{O}_Z$ are acyclic.

Proposition 5.2. Let $\mathcal{M}$ be a complex Banach manifold, and $E \to \mathcal{M}$ a holomorphic Banach vector bundle with a Banach space $\mathcal{Z}$ for fiber type. Then the sheaf $\mathcal{O}_E$ of germs of holomorphic sections $E \to \mathcal{M}$ is an (S)-sheaf over $\mathcal{M}$.

Proof. Restricting $E$ to members of an (S)-covering of $\mathcal{M}$ by coordinate balls $U$ over which $E$ is holomorphically isomorphic to $U \times \mathcal{Z}$, we see that it is enough to apply Proposition 5.1 to $\mathcal{O}_E|U$ to conclude the proof of Proposition 5.2.

Theorem 5.3. Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open. Suppose that plurisubharmonic domination holds in every
pseudoconvex open subset of $\Omega \times X$. Let $Z$ be a Banach space, $X = X' \times X''$ a direct decomposition of Banach spaces, and $I^Z \to X$ the sheaf of germs of holomorphic functions $X \to Z$ that vanish on $X'$. Then $I^Z$ is an (S)-sheaf over $\Omega$.

The canonical Koszul resolution of $I^Z$ is an (S)-resolution, as we will see shortly. In fact, this Koszul resolution served as model for the notion of (S)-resolution.

Let $\Lambda^Z_p$ the Banach space of all continuous complex $p$-linear alternating maps $X'' \to Z$ for $p \geq 0$; $\Lambda^Z_0 = \Lambda^Z_{-1} = Z$; and $\mathcal{O}\Lambda^Z_p \to X$ the sheaf of germs of holomorphic functions $X \to \Lambda^Z_p$. Let $E$ be the Euler vector field on $X''$ defined by $E(x'') = x''$, and $i_E$ the inner derivation determined by the vector field $E$, i.e., $i_E$ is the contraction of $p$-forms with $E$: if $f$ is a local section of $\mathcal{O}\Lambda^Z_p$, then let $i_E f$ be the local section of $\mathcal{O}\Lambda^Z_{p-1}$ given for $p \geq 1$ by $(i_E f)(x', x'')(\xi''_1, \xi''_2, \ldots, \xi''_{p-1}) = f(x', x'')(x'', \xi''_1, \ldots, \xi''_{p-1})$, and for $p = 0$ by $(i_E f)(x', x'') = f(x', 0)$. We consider the Koszul complex

$$
\cdots \to \mathcal{O}\Lambda^Z_p \to \mathcal{O}\Lambda^Z_{p-1} \to \cdots \to \mathcal{O}\Lambda^Z_1 \to I^Z \to 0 \tag{5.1}
$$

of locally convex analytic sheaves over $X$, where each map is $i_E$. Let $K_p$, $p \geq 0$, be the corresponding sequence of kernel sheaves: $K_p(U) = \{f \in \mathcal{O}(U, \Lambda^Z_p) : i_E f = 0 \text{ on } U\}$, $U \subset X$ open; $K_0 = I^Z$.

**Theorem 5.4.** [P3, Thm. 5.1] Let $X', X'', Z$ be Banach spaces, $\Omega \subset X = X' \times X''$ pseudoconvex open, $I^Z$ the sheaf of germs of holomorphic functions $\Omega \to Z$ that vanish on $X'$. Suppose that $X$ has a Schauder basis, and that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega \times X$. Then

(a) the Koszul complex (5.1) is exact on the germ level and on the level of global sections over $\Omega$, and

(b) the $K_p$ are acyclic over $\Omega$: $H^q(\Omega, K_p) = 0$ for all $q \geq 1$ and $p \geq 0$.

**Proof of Theorem 5.3.** Look at the image

$$
\cdots \to \mathcal{O}\text{Hom}(Z', \Lambda^Z_p) \to \mathcal{O}\text{Hom}(Z', \Lambda^Z_{p-1}) \to \cdots \to \mathcal{O}\text{Hom}(Z', \Lambda^Z_1) \to \text{Hom}(\mathcal{O}Z', I^Z) \to 0 \tag{5.2}
$$

of (5.1) under the functor $\text{Hom}(\mathcal{O}Z', -)$, where $Z'$ is any Banach space. Let $Z'' = \text{Hom}(Z', Z)$, and note that the Banach spaces $\text{Hom}(Z', \Lambda^Z_p)$ and $\Lambda^Z_p$ are canonically isomorphic for $p \geq 0$, and that the sheaves $\text{Hom}(\mathcal{O}Z'', I^Z)$ and $I^Z$ are canonically isomorphic over $X$. Moreover, the sequence (5.2) is canonically isomorphic to the sequence

$$
\cdots \to \mathcal{O}\Lambda^Z_p \to \mathcal{O}\Lambda^Z_{p-1} \to \cdots \to \mathcal{O}\Lambda^Z_1 \to I^Z \to 0 \tag{5.3}
$$
over $X$, which, being just another Koszul complex, is a long 1-exact sequence over $X$ by Theorem 5.4, i.e., (5.1) is a long 2-exact sequence over $X$. The proof of Theorem 5.3 is complete.

**Theorem 5.5.** Let $M$ be a complex Banach manifold modelled on a Banach space $X$ with a Schauder basis, and suppose that plurisubharmonic domination holds in any pseudoconvex open subset of $B_X(0,1) \times X$. Let $N$ be a closed split complex Banach submanifold of $M$, $Z$ a Banach space, and $I^Z$ the sheaf of germs of holomorphic functions $M \rightarrow Z$ that vanish on $N$. Then $I^Z$ is an $(S)$-sheaf over $M$.

**Proof.** Let $\mathfrak{U}$ be an $(S)$-covering of $M$ so that if $U \in \mathfrak{U}$ meets $N$, then the pair $(U, U \cap N)$ is biholomorphic to the pair $(B_{X'}(0,1) \times B_{X''}(0,1), B_{X'}(0,1) \times \{0\})$, where $X = X' \times X''$ is a direct decomposition of Banach spaces. As $I^Z|U$ has an $(S)$-resolution by Theorem 5.3, the proof of Theorem 5.5 is complete.

We now turn to zero extensions of analytic sheaves from split complex Banach submanifolds, and show in many cases that the zero extension of an $(S)$-sheaf over the submanifold is an $(S)$-sheaf over the ambient manifold.

Let $M$ be a complex Banach manifold, $N$ a split complex Banach submanifold of $M$, $S \rightarrow N$ a sheaf of $\mathcal{O}$-modules over the submanifold $N$. The **zero extension** or **trivial extension** $S^0 \rightarrow M$ of $S \rightarrow N$ is the sheaf of the canonical presheaf $S^0$ defined over $M$ by letting $S^0(U) = S(U \cap N)$, $U \subset M$ open. Note that $S(\emptyset) = 0$ as usual.

**Theorem 5.6.** Let $X$ be a Banach space with a Schauder basis, $X = X' \times X''$ a direct decomposition of Banach spaces, $\Omega = B_{X'}(0,1) \times B_{X''}(0,1)$, $\Omega' = B_{X'}(0,1) \times \{0\}$, regard $\Omega'$ as a split complex Banach submanifold of $\Omega$, and let $S \rightarrow \Omega'$ be an $(S)$-sheaf with a long 2-exact sequence (4.5) of locally convex analytic sheaves over $\Omega'$. If plurisubharmonic domination holds in any pseudoconvex open subset of $\Omega \times X$, then the zero extension $S^0 \rightarrow \Omega$ of $S \rightarrow \Omega'$ has a long 2-exact sequence of model sheaves over $\Omega$.

**Proof.** Denote the maps in (4.5) by $\tau_n^* : \mathcal{O}^{\mathcal{Z}_n} \rightarrow \mathcal{O}^{\mathcal{Z}_{n-1}}$ for $n \geq 2$, and $\tau_1^* : \mathcal{O}^{\mathcal{Z}_1} \rightarrow \mathcal{O}^{\Omega}$, $\text{Hom}(\mathcal{Z}_n, \mathcal{Z}_{n-1}))$ with $\tau_n' = T_n'$ for $n \geq 2$. Let $T_n(x',x'') = T_n'(x')$ be the trivial extension $T_n \in \mathcal{O}(\Omega, \text{Hom}(\mathcal{Z}_{n}, \mathcal{Z}_{n-1}))$ of $T_n'$ for $n \geq 2$. Let $i_E$ be the inner derivation $i_E \in \mathcal{O}(X, \text{Hom}(\Lambda_p^{Z_x}, \Lambda_p^{Z_x}))$ of the Koszul complex (5.1) for $n \geq 1, p \geq 1$. Note that

\begin{equation}
T_{n-1}T_n = 0, \quad T_mi_E = i_E T_m, \quad i_E i_E = 0,
\end{equation}

where $n \geq 3$ in the first part, $T_mi_E f = i_E T_m f$ for $m \geq 2$, $f \in \mathcal{O}(U, \Lambda_p^{Z_x})$, $n \geq 1, p \geq 1$, $U \subset \Omega$ open, in the second part, since both sides have the value $T_m'(x')(x', x'')(x'', x_1'', \ldots, x_{p-1}'')$ at a point $(x',x'') \in U$.
Let \( \tilde{Z}_n = \bigoplus_{p=1}^n \Lambda_{n-p} \tau \) for \( n \geq 1 \), and define \( \tilde{\tau}_n : \mathcal{O} \tilde{Z}_n \rightarrow \mathcal{O} \tilde{Z}_{n-1} \) for \( n \geq 2 \), and \( \tilde{\tau}_1 : \mathcal{O} \tilde{Z}_1 \rightarrow S^0 \) over \( \Omega \) by letting \( \tilde{\tau}_1(f_1) = \tau_1^f(f_1|\Omega' \cap U) \in S(\Omega' \cap U) = S^0(U) \) for \( f_1 \in \mathcal{O}(U, \tilde{Z}_1) \), \( \tilde{Z}_1 = Z_1, U \subset \Omega \) pseudoconvex open, and \( \tilde{\tau}_n(f_1, \ldots, f_n) = (g_1, \ldots, g_{n-1}), n \geq 2 \), by

\[
g_k = T_{k+1}f_{k+1} + (-1)^k i_E f_k,
\]

where \( f_k \in \mathcal{O}(U, \Lambda_{n-k} \tau) \) for \( k = 1, \ldots, n \), and \( g_k \in \mathcal{O}(U, \Lambda_{n-k} \tau) \) for \( k = 1, \ldots, n-1 \). Then \( \tilde{\tau}_1 \tilde{\tau}_2(f_1, f_2) = \tilde{\tau}_1(T_2f_2 - i_E f_1) = \tau_1^f(f_2|\Omega' \cap U) + 0 = 0 \) since (4.5) is a complex, \( \tilde{\tau}_n-1 \tilde{\tau}_n(f_1, \ldots, f_n) = (h_1, \ldots, h_{n-2}) \), where \( n \geq 3, h_k = T_{k+1}g_{k+1} + (-1)^k i_E g_k = T_{k+1}(T_{k+2}f_{k+2} + (-1)^k i_E f_{k+1}) + (-1)^k i_E (T_{k+1}f_k) = 0 + (-1)^k (T_{k+1}i_E f_{k+1} - i_E T_{k+1}f_k) + 0 = 0 \), taking (5.4) into account.

Let now \( g_1 \in \text{Ker} \tilde{\tau}_1 \), i.e., \( \tau_1^f(g_1|\Omega' \cap U) = 0 \), where \( g_1 \in \mathcal{O}(U, \tilde{Z}_1) \), \( \tilde{Z}_1 = Z_1, U \subset \Omega \) pseudoconvex open. As (4.5) is exact on the level of global sections over \( \Omega' \cap U \), there is a representation \( g_1|\Omega' \cap U = T_2f'_2 \), where \( f'_2 \in \mathcal{O}(\Omega' \cap U, Z_2) \). Theorem 5.4(b) gives an extension \( f_2 \in \mathcal{O}(U, Z_2) \) with \( f_2 = f'_2 \) on \( \Omega' \cap U \). Then \( g_1 - T_2f_2 \) vanishes on \( \Omega' \), so by Theorem 5.4(a) it can be written as \( g_1 - T_2f_2 = -i_E f_1 \), where \( f_1 \in \mathcal{O}(U, \Lambda_{1} \tau) \), i.e., \( g_1 = \tilde{\tau}_2(f_1, f_2) \).

Let now \( f \in \text{Ker} \tilde{\tau}_n, n \geq 2 \). Consider the system of equations

\[
\begin{align*}
g_1 &= T_2f_2 - i_E f_1 = 0 \\
g_2 &= T_3f_3 + i_E f_2 = 0 \\
&\vdots \\
g_{n-1} &= T_n f_n + (-1)^{n-1} i_E f_{n-1} = 0
\end{align*}
\]

for the unknowns \( f_n, f_{n-1}, \ldots, f_1 \). We must show that all solutions are of the form

\[
\begin{align*}
f_1 &= T_2h_2 - i_E h_1 \\
f_2 &= T_3h_3 + i_E h_2 \\
&\vdots \\
f_n &= T_{n+1}h_{n+1} + (-1)^n i_E h_n
\end{align*}
\]

We will look at the equations \( g_{n-1} = 0, \ldots, g_1 = 0 \) in this order and produce the representatives \( h_{n+1}, \ldots, h_1 \) of \( f_n, \ldots, f_1 \) in this order. Restricting the equation \( g_{n-1} = 0 \) to \( \Omega' \cap U \) we find that \( T_n(f_n|\Omega' \cap U) = 0 \), i.e., \( f_n|\Omega' \cap U = T_{n+1}h_{n+1} \), where \( h_{n+1} \in \mathcal{O}(\Omega' \cap U, \Lambda_0 Z_{n+1}) \) since (4.5) is exact on the level of global sections over \( \Omega' \cap U \) by assumption and Theorem 4.3. Theorem 5.4(b) provides an extension \( h_{n+1} \in \mathcal{O}(U, \Lambda_0 Z_{n+1}) \) such that \( h_{n+1} = h'_{n+1} \) on \( \Omega' \cap U \). Thus \( f_n - T_{n+1}h_{n+1} \) vanishes on \( \Omega' \cap U \). Theorem 5.4(a) then provides an \( h_n \in \mathcal{O}(U, \Lambda_1 Z_{n+1}) \) with \( f_n = T_{n+1}h_{n+1} + (-1)^n i_E h_n \). Looking at
the equation $g_{n-1} = 0$ again, we get that $0 = g_{n-1} = (-1)^n i_E (T_n h_n - f_{n-1})$
where we plugged in the above form of $f_n$, and used the first two identities in (5.4). As $T_n h_n - f_{n-1}$
is in the kernel of $i_E$ over the pseudoconvex open set $U$, another application of Theorem 5.4(a)
gives an $h_{n-1}$ with $f_{n-1} = T_n h_n + (-1)^{n-1} i_E h_{n-1}$. Plugging this form of $f_{n-1}$ into the
equation $g_{n-2} = 0$ we obtain the compatibility condition that $0 = g_{n-2} = (-1)^{n-1} i_E (T_{n-1} h_{n-1} - f_{n-2})$
so as above we find an $h_{n-2}$ with $f_{n-2} = T_{n-1} h_{n-1} + (-1)^{n-2} i_E h_{n-2}$. Continuing in this way we find $h_{n+1}, h_n, \ldots, h_1$
one after another by Theorem 5.4(a). Thus any solution of $\tilde{\tau}_n(f) = 0$ is of the
form $f = \tilde{\tau}_{n+1}(h)$. Hence our sequence

$$(5.5) \quad \ldots \to \mathcal{O} \tilde{Z}_n \overset{\tilde{\tau}_n}{\to} \mathcal{O} \tilde{Z}_{n-1} \to \ldots \to \mathcal{O} \tilde{Z}_1 \overset{\tilde{\tau}_1}{\to} \mathcal{O} S^0 \to 0$$

is a long 1-exact sequence of locally convex analytic sheaves over $\Omega$. Taking the image of (5.5) under the functor $\text{Hom}(\mathcal{O}^Z, -)$, where $Z$ is any Banach space, we obtain a similar sequence

$$(5.6) \quad \ldots \to \mathcal{O} \text{Hom}(Z, \tilde{Z}_n) \overset{\tilde{\tau}_n}{\to} \mathcal{O} \text{Hom}(Z, \tilde{Z}_{n-1}) \to \ldots \to \mathcal{O} \text{Hom}(Z, \tilde{Z}_1) \overset{\tilde{\tau}_1}{\to} \text{Hom}(\mathcal{O}^Z, S^0) \to 0,$$

which is a long 1-exact sequence over $\Omega$ by essentially the same reasoning as above for (5.5) noting that in all our arguments we could carry a linear parameter $z \in Z$. The proof of Theorem 5.6 is complete.

**Theorem 5.7.** Let $M$ be a complex Banach manifold modelled on a Banach space $X$ with a Schauder basis, and suppose that plurisubharmonic domination holds in any pseudoconvex open subset of $B_X(0,1) \times X$. Let $N$ be a closed split complex Banach submanifold of $M$, $S \to N$ and $(S)$-sheaf, and $S^0 \to M$ the zero extension of $S \to N$. Then $S^0$ is an $(S)$-sheaf over $M$.

**Proof.** This follows from Theorem 5.6 in a way similar to the proof of Theorem 5.5. The proof of Theorem 5.7 is complete.

**6. AMALGAMATION OF SYZYGIES.**

In this section we paste together resolutions over neighboring pseudoconvex open sets.

Let $X$ be a Banach space, we call a pair of pseudoconvex open subsets $U', U''$ of $X$ a **(C)-pair** if $U' \cup U''$ is also pseudoconvex open in $X$. A fairly typical example of a (C)-pair can be obtained as follows. Let $\Omega \subset X$ be pseudoconvex open, $f \in \mathcal{O}(\Omega)$, and $-\infty < a' < a'' < b' < b'' < \infty$ constants, and define $U' = \{x \in \Omega : a' < \text{Re} f(x) < b'\}$ and $U'' = \{x \in \Omega : a'' < \text{Re} f(x) < b''\}$. Then $U', U''$ is a (C)-pair in $X$.

**Theorem 6.1.** Let $X$ be a Banach space with a Schauder basis, $U', U'' \subset X$ a **(C)-pair**, $U = U' \cup U''$, $V = U' \cap U''$, $S \to U$ a locally convex analytic sheaf.
Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $U$.

(a) If $S|U'$ has an $(S)$-resolution over $U'$, and $S|U''$ has an $(S)$-resolution over $U''$, then there is a short exact sequence

$$0 \to K \to \mathcal{O}^Z_1 \to S \to 0 \tag{6.1}$$

of locally convex analytic sheaves over $U$, where $Z_1$ is a Banach space, such that the restriction of (6.1) to $U'$ is a short 2-exact sequence over $U'$, and the restriction of (6.1) to $U''$ is a short 2-exact sequence over $U''$, and $K|U'$ has an $(S)$-resolution over $U'$, and $K|U''$ has an $(S)$-resolution over $U''$.

(b) There is an $(S)$-resolution (4.5) over $U$.

Proof. Let

$$\cdots \to \mathcal{O}^{Z'_n} \xrightarrow{\tau'_n} \mathcal{O}^{Z'_{n-1}} \to \cdots \to \mathcal{O}^{Z'_1} \xrightarrow{\tau'_1} S|U' \to 0, \tag{6.2}$$

$$\cdots \to \mathcal{O}^{Z''_n} \xrightarrow{\tau''_n} \mathcal{O}^{Z''_{n-1}} \to \cdots \to \mathcal{O}^{Z''_1} \xrightarrow{\tau''_1} S|U'' \to 0 \tag{6.3}$$

be $(S)$-resolutions over $U'$ and $U''$, and

$$\cdots \to \mathcal{O}^{\text{Hom}(Z''_1, Z'_1)} \xrightarrow{\tau''_1} \mathcal{O}^{\text{Hom}(Z''_2, Z'_2)} \to \cdots \to \mathcal{O}^{\text{Hom}(Z''_1, Z'_1)} \xrightarrow{\tau''_1} \text{Hom}(\mathcal{O}^{Z'_1}, S)|U' \to 0, \tag{6.4}$$

$$\cdots \to \mathcal{O}^{\text{Hom}(Z'_1, Z''_1)} \xrightarrow{\tau''_1} \mathcal{O}^{\text{Hom}(Z'_2, Z''_2)} \to \cdots \to \mathcal{O}^{\text{Hom}(Z'_1, Z''_1)} \xrightarrow{\tau''_1} \text{Hom}(\mathcal{O}^{Z'_1}, S)|U'' \to 0 \tag{6.5}$$

the image of (6.2) under the functor $\text{Hom}(\mathcal{O}^{Z''_1}, -)$, and the image of (6.3) under the functor $\text{Hom}(\mathcal{O}^{Z'_1}, -)$. As (6.4) is exact, by Theorem 4.3, on the level of global sections over $V$ there is for $\tau''_1 \in \text{Hom}(\mathcal{O}^{Z''_1}, S)(V)$ an $A \in \mathcal{O}(V, \text{Hom}(Z''_1, Z'_1))$ such that $\tau'_1 A = \tau''_1$ over $V$. As (6.5) is exact, by Theorem 4.3, on the level of global sections over $V$ there is for $\tau'_1 \in \text{Hom}(\mathcal{O}^{Z'_1}, S)(V)$ a $B \in \mathcal{O}(V, \text{Hom}(Z'_1, Z''_1))$ such that $\tau''_1 B = \tau'_1$ over $V$. (The above intertwining property is the main reason to look at the notion of 2-exactness.)

Let the direct sum of (6.2) with the trivial $(S)$-resolution

$$\cdots \to 0 \to 0 \to \mathcal{O}^{Z''_1} \xrightarrow{1} \mathcal{O}^{Z''_1} \to 0 \to 0$$

over $U'$ be

$$\cdots \to \mathcal{O}^{Z'_n} \xrightarrow{\tau'_n} \mathcal{O}^{Z'_{n-1}} \to \cdots \to \mathcal{O}^{Z'_1} \xrightarrow{\tau'_1} \mathcal{O}^{Z''_1} \xrightarrow{(\tau'_1, 0)} \mathcal{O}^{Z'_1} \oplus Z''_1 \xrightarrow{(\tau'_1, 0)} S|U' \to 0, \tag{6.6}$$
which is another (S)-resolution by Proposition 4.4(a).

Let the direct sum of the trivial (S)-resolution

\[ \ldots \to 0 \to 0 \to O^{Z_1} \xrightarrow{1} O^{Z_1'} \to 0 \to 0 \]

with (6.3) over \( U'' \) be

\[ \ldots \to O^{Z_n''} \xrightarrow{\tau''_n} O^{Z''_{n-1}} \to \ldots \]

\[ \to O^{Z_3''} \xrightarrow{\begin{pmatrix} 0 \\ tA(x) \end{pmatrix}} \to O^{Z_2''} \xrightarrow{\begin{pmatrix} 1 & 0 \\ tB(x) & 1 \end{pmatrix}} O^{Z_1''} \xrightarrow{\tau''_1} S|U'' \to 0, \]

which is another (S)-resolution by Proposition 4.4(a).

Let \( Z_1 = Z_1' \oplus Z_1'' \), and consider the holomorphic operator function \( C_t \in O(V, GL(Z_1)) \) defined for \( 0 \leq t \leq 1 \) by

\[ C_t(x) = \begin{bmatrix} 1 & tA(x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -tB(x) & 1 \end{bmatrix}, \]

whose inverse is \( C_t(x)^{-1} = \begin{bmatrix} 1 & 0 \\ tB(x) & 1 \end{bmatrix} \begin{bmatrix} 1 & tA(x) \\ 0 & 1 \end{bmatrix} \). Note that \( (\tau'_1, 0)C_1 = (0, \tau''_1) \) over \( V \). Look at the holomorphic Banach vector bundle \( E \to U \) whose fiber type is \( Z_1 \), and whose transition function relative to the open covering \( \{U', U''\} \) of \( U \) is \( C_1 \). This \( E \) is continuously trivial over \( U \) due to the homotopy \( C_t \) from \( C_1 \) to \( C_0 = 1 \). By Theorem 4.1(d) our holomorphic Banach vector bundle \( E \) is holomorphically trivial over \( U \). So there are holomorphic operator functions \( C' \in O(U', GL(Z_1)) \), and \( C'' \in O(U'', GL(Z_1)) \) with \( C_1(x) = C'(x)C''(x)^{-1} \) for \( x \in V \). Thus the homomorphisms \( (\tau'_1, 0)C' \in Hom(O^{Z_1}, S)(U') \), and \( (0, \tau''_1)C'' \in Hom(O^{Z_1}, S)(U'') \) fit together to a homomorphism \( \tau_1 \in Hom(O^{Z_1}, S)(U) \).

Replace \( (\tau'_1, 0) \) in (6.6) by \( \tau_1 = (\tau'_1, 0)C' \) to obtain

\[ \ldots \to O^{Z_n''} \xrightarrow{\tau''_n} O^{Z''_{n-1}} \to \ldots \]

\[ \to O^{Z_3''} \xrightarrow{\begin{pmatrix} 0 \\ \tau'_3 \end{pmatrix}} O^{Z_2''} \xrightarrow{\begin{pmatrix} 1 & 0 \\ \tau'_2 & 1 \end{pmatrix}} O^{Z_1''} \xrightarrow{\tau''_1} S|U' \to 0. \]

Replace \( (0, \tau''_1) \) in (6.7) by \( \tau_1 = (0, \tau''_1)C'' \) to obtain

\[ \ldots \to O^{Z_n''} \xrightarrow{\tau''_n} O^{Z''_{n-1}} \to \ldots \]

\[ \to O^{Z_3''} \xrightarrow{\begin{pmatrix} 0 \\ \tau'_3 \end{pmatrix}} O^{Z_2''} \xrightarrow{\begin{pmatrix} 1 & 0 \\ \tau'_2 & 1 \end{pmatrix}} O^{Z_1''} \xrightarrow{\tau''_1} S|U'' \to 0. \]

Both (6.8) and (6.9) are (S)-resolutions by Proposition 4.4(b). Let \( K \subset O^{Z_1} \) be the kernel of \( \tau_1 \in Hom(O^{Z_1}, S)(U) \), \( K' \subset O^{Z_1} \) the kernel of \( \tau'_1 \in Hom(O^{Z_1}, S)(U') \), and \( K'' \subset O^{Z''_1} \) the kernel of \( \tau''_1 \in Hom(O^{Z''_1}, S)(U'') \).

Then \( K|U' \cong K' \oplus O^{Z''_1} \) has an (S)-resolution by (6.8), and \( K|U'' \cong O^{Z''_1} \oplus K'' \) has an (S)-resolution by (6.9). Thus (6.1) is as claimed in part (a).
(b) As the kernel $K$ in (a) satisfies the same conditions as $S$ does, we can find by repeated application of (a) short exact sequences $0 \to K_n \to \mathcal{O}^{Z_n} \to K_{n-1} \to 0$ over $U$ for $n \geq 1$, where $K_0 = S$, and $Z_n$ is a Banach space. Splicing together these short exact sequences we get an (S)-resolution (4.5) as claimed. The proof of Theorem 6.1 is complete.

Let $Y$ be a Banach space, $\pi : \mathbb{C}^N \times \to \mathbb{C}^N \times Y$ the projection $\pi(\zeta, y) = (\zeta, 0)$, $D \subset \subset \mathbb{C}^N$ pseudoconvex open, $R : D \to (0, \infty)$ continuous and bounded away from zero with $-\log R$ plurisubharmonic on $D$, and

\begin{align}
(6.10) \quad \Omega(D, R) = \{ (\zeta, y) \in \mathbb{C}^N \times Y : \| y \| < R(\zeta) \}.
\end{align}

Let $\mathcal{U}$ be an (S)-covering of $D$, $\Omega = \Omega(D, R)$, and

\begin{align}
(6.11) \quad \mathcal{U}(\Omega) = \{ U(\Omega) = \pi^{-1}(U) \cap \Omega : U \in \mathcal{U} \}
\end{align}
a basic covering of $\Omega$.

Let $Q = \{ \zeta \in \mathbb{C}^N : A_j' \leq \Re \zeta_j \leq A_j'', B_j' \leq \Im \zeta_j \leq B_j'' \}$ be a compact ‘cube’ (rectangular box) in $\mathbb{C}^N$ with $A_j' < A_j'', B_j' < B_j''$ for $j = 1, \ldots, N$. A simple subdivision of $Q$ into subcubes $\{Q_{k_1,\ldots,k_N}^{l_1,\ldots,l_N}\}$ is a choice of subdivisions $A_j' = a_j0 < a_j1 < \ldots < a_jm = A_j''$, $B_j' = b_j0 < b_j1 < \ldots < b_jm = B_j''$ of the edges $[A_j', A_j'']$, $[B_j', B_j'']$ of $Q$, $j = 1, \ldots, N$, where $Q_{k_1,\ldots,k_N}^{l_1,\ldots,l_N} = \{ \zeta \in \mathbb{C}^N : a_{j,k_j} \leq \Re \zeta_j \leq a_{j,k_j} + b_{j,l_j} - b_{j,l_j-1} \leq \Im \zeta_j \leq b_{j,l_j} \}$ for $k_1, \ldots, k_N, l_1, \ldots, l_N = 1, \ldots, m$. A simple covering $\mathcal{V} = \{ V_{k_1,\ldots,k_N}^{l_1,\ldots,l_N} \}$ of $Q$ is an open covering obtained by fattening up the cubes $Q_{k_1,\ldots,k_N}^{l_1,\ldots,l_N}$ of a simple subdivision of $Q$ by a small amount $0 < \varepsilon < \frac{1}{4} \min \{a_{j,k} - a_{j,k-1}, b_{j,k} - b_{j,k-1} : j = 1, \ldots, N, k = 1, \ldots, m\}$, where $V_{k_1,\ldots,k_N}^{l_1,\ldots,l_N} = \{ \zeta \in \mathbb{C}^N : a_{j,k} - \varepsilon \leq \Re \zeta_j \leq a_{j,k} + \varepsilon, b_{j,l} - \varepsilon \leq \Im \zeta_j \leq b_{j,l} + \varepsilon \}$. A (C)-covering $\mathcal{W}$ of a pseudoconvex open subset $\Omega$ of a Banach space $X$ is an (S)-covering $\mathcal{W}$ of the form $\mathcal{W} = f^{-1}(\mathcal{V}) = \{ f^{-1}(V_{k_1,\ldots,k_N}^{l_1,\ldots,l_N}) \}$, where $f \in \mathcal{O}(\Omega, \mathbb{C}^N)$, $Q$ is a cube in $\mathbb{C}^N$ that contains $f(\Omega)$, and $\mathcal{W}$ is a simple covering of $Q$. (If convenient, we may throw away those $f^{-1}(V_{k_1,\ldots,k_N}^{l_1,\ldots,l_N})$ that are empty.) A (C)-covering plays well with (C)-pairs.

**Proposition 6.2.** With the above notation and hypotheses suppose that $Y$ has a Schauder basis, and plurisubharmonic domination holds in every pseudoconvex subset of $\Omega = \Omega(D, R)$, and $\mathcal{U}$ has a (finite) refinement $\mathcal{W}$ that is a (C)-covering of $D$. Let $S \to \Omega$ be a sheaf that has an (S)-resolution over each member of $\mathcal{U}(\Omega)$. Then $S$ has an (S)-resolution over $\Omega$.

**Proof.** As this can be proved by the usual induction process of Cousin and Cartan relying on Theorem 6.1 to amalgamate syzygies over larger and larger (C)-pairs, the proof of Proposition 6.2 is complete.
Theorem 6.3 Let $X$ be a Banach space with a Schauder basis, $\Omega \subseteq X$ pseudoconvex open, $\omega_N \subset \omega_{N+1} \subset \Omega$ pseudoconvex open, $N \geq 1$, and $\bigcup_{N=1}^{\infty} \omega_N = \Omega$. Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$. Let $S \to \Omega$ be a sheaf such that there are (S)-resolutions

\[ \ldots \to \mathcal{O}^{Z_N}_n \xrightarrow{\tau^N_n} \mathcal{O}^{Z_{N-1}}_n \to \ldots \to \mathcal{O}^{Z_k}_1 \xrightarrow{\tau^N_1} S|\omega_N \to 0 \]

for all $N \geq 1$. Then there is a Banach space $Z'$ so that the following hold.

(a) There is a short exact sequence $0 \to K \to \mathcal{O}^{Z'} \to S \to 0$ over $\Omega$ that is 2-exact over $\omega_N$ for all $N \geq 1$, and $K|\omega_N$ has an (S)-resolution

\[ \ldots \to \mathcal{O}^{Z'} \to \mathcal{O}^{Z'} \to K|\omega_N \to 0 \]

with all Banach spaces equal to $Z'$ for all $N \geq 1$.

(b) There is an (S)-resolution

\[ \ldots \to \mathcal{O}^{Z'} \to \mathcal{O}^{Z'} \to S \to 0 \]

over $\Omega$ with all Banach spaces equal to $Z'$.

Proof. Let $1 \leq P < \infty$, and $Z'$ the $\ell_P$-sum of countably infinitely many copies of $Z^n_N$ for $n, N \geq 1$. Then $Z' \oplus Z' \cong Z'$, $Z^n_N \oplus Z' \cong Z'$ for $n, N \geq 1$, and $\ell_P(Z') \cong Z'$, where the isomorphisms are effected by isometries that permute the coordinates. We may easily achieve that in (6.12) all the Banach spaces are equal to $Z'$ by taking the direct sum of (6.12) and the following two trivial (S)-resolutions

\[ \ldots \to 0 \to \mathcal{O}^{Z'} \xrightarrow{1} \mathcal{O}^{Z'} \to 0 \to \mathcal{O}^{Z'} \xrightarrow{1} \mathcal{O}^{Z'} \to 0 \to 0 \]

\[ \ldots \to \mathcal{O}^{Z'} \xrightarrow{1} \mathcal{O}^{Z'} \to 0 \to \mathcal{O}^{Z'} \xrightarrow{1} \mathcal{O}^{Z'} \to 0 \to 0 \]

over $\Omega$. Suppose now that we have (S)-resolutions

\[ \ldots \to \mathcal{O}^{Z'} \xrightarrow{\tau^N_1} \mathcal{O}^{Z'} \xrightarrow{\tau^N_1} S|\omega_N \to 0 \]

with all Banach spaces equal to $Z'$ for all $N \geq 1$. Looking at the image of (6.13) under the functor $\text{Hom}(\mathcal{O}^{Z'}, -)$ we find, by the 2-exactness of (6.13) and Theorem 4.3, for $\tau^{N+1}_1 \in \text{Hom}(\mathcal{O}^{Z'}, S)(\omega_N)$ an $A_N \in \mathcal{O}(\omega_N, \text{End}(Z'))$ with $\tau^{N+1}_1 = \tau^N_1 A_N$ over $\omega_N$ for $N \geq 1$. Similarly, we find a $B_N \in \mathcal{O}(\omega_N, \text{End}(Z'))$ with $\tau^N_1 = \tau^{N+1}_1 B_N$ over $\omega_N$ for $N \geq 1$. We regard $Z'$ as $\ell_P(Z')$, and similarly as in (6.6) and (6.7) by adding one or two trivial (S)-resolutions to (6.13) we can arrange that the (S)-resolution (6.13) takes the form

\[ \ldots \to \mathcal{O}^{Z'} \xrightarrow{\sigma^N_1} \mathcal{O}^{Z'} \xrightarrow{\sigma^N_1} S|\omega_N \to 0 \]

over $\omega_N$, where the $N$th entry of $\sigma^N_1 = (0, \ldots, \tau^N_1, 0, \ldots)$ is $\tau^N_1$ for $N \geq 1$.  

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Consider the holomorphic operator matrices $A_N^t \in \mathcal{O}(\omega_N, \text{GL}(\ell_P(Z')))$, $B_N^t \in \mathcal{O}(\omega_N, \text{GL}(\ell_P(Z'))) \text{ defined for } 0 \leq t \leq 1 \text{ and } x \in \omega_N$ by

$$A_N^t(x) = [\delta_i^j + tA_N(x)\delta_i^j \delta_{N+1}]_{i,j \geq 1}, \quad B_N^t(x) = [\delta_i^j - tB_N(x)\delta_i^j \delta_{N+1}]_{i,j \geq 1},$$

where $\delta_{ij}$ is the Kronecker delta. Note that the above operators are in $\delta_{ij}^N$ over $x$.

Look at the holomorphic Banach vector bundle $\mathcal{E}$ whose defining cocycle with respect to the covering $p$ is holomorphically trivial over $\Omega$, i.e., $\mathcal{E}$ is the Kronecker delta. Note that the above operators are in $\delta_{ij}^N$ over $x$.

Indeed,

$$[\sigma_1^N \tilde{C}_N^1]_{k \geq 1} = \left[ \sum_{i,j} \delta_i^j \tau_1^N(\delta_i^j + A_N \delta_i^j \delta_{N+1})(\delta_j^k - B_N \delta_j^N \delta_{N+1}^k) \right]$$

$$= \left[ \sum_{j} \tau_1^N(\delta_j^N + A_N \delta_j^N \delta_{N+1})(\delta_j^k - B_N \delta_j^N \delta_{N+1}^k) \right]$$

$$= \left[ \tau_1^N \sum_{j} (\delta_j^N \delta_j^k + A_N \delta_j^N \delta_{N+1} \delta_j^k - B_N \delta_j^N \delta_{N+1} \delta_j^k) \right]$$

$$= \left[ \tau_1^N \delta_j^N + \tau_1^N \delta_{N+1}^k - 0 - \tau_1^N \delta_j^N \right]$$

$$= \left[ \tau_1^N \delta_{N+1}^k \right]_{k \geq 1} = \sigma_1^N$$

over $\omega_N$, where we used that $\tau_1^N A_N B_N = \tau_1^N$.

Let $p \land q = \min\{p, q\}$, and define $\tilde{C}_{pq}^t \in \mathcal{O}(\omega_{p \land q}, \text{GL}(\ell_P(Z'))) \text{ for } 0 \leq t \leq 1$, $x \in \omega_{p \land q}$, and $p, q \geq 1$ by

$$\tilde{C}_{pq}^t(x) = \begin{cases} 
\tilde{C}_{p}^t(x)\tilde{C}_{p+1}^t(x)\ldots \tilde{C}_{q}^t(x) & \text{if } p < q \\
1 & \text{if } p = q \\
(\tilde{C}_{q}^t(x)\tilde{C}_{q+1}^t(x)\ldots \tilde{C}_{p-1}^t(x))^{-1} & \text{if } p > q
\end{cases}$$

Note that $\sigma_1^p \tilde{C}_{pq}^t = \sigma_1^q$ over $\omega_{p \land q}$ for $p, q \geq 1$. Then $\tilde{C}_{pq}^t$ is a holomorphic cocycle, i.e., $\tilde{C}_{pq}^t \tilde{C}_{qr}^t \tilde{C}_{rp}^t = 1$ over $\omega_{p \land q \land r}$ for $p, q, r \geq 1$ as it is easy to verify.

Look at the holomorphic Banach vector bundle $E \to \Omega$ with fiber type $\ell_P(Z')$ whose defining cocycle with respect to the covering $\{\omega_N : N \geq 1\}$ of $\Omega$ is $\tilde{C}_{pq}^1$. This Banach vector bundle $E$ is topologically trivial over $\Omega$, due to the homotopy $\tilde{C}_{pq}^t$ of cocycles from $\tilde{C}_{pq}^1$ to $\tilde{C}_{pq}^0 = 1$. By Theorem 4.1(d) our holomorphic Banach vector bundle $E$ is homologically trivial over $\Omega$, i.e., there are $D_p \in \mathcal{O}(\omega_p, \text{GL}(\ell_P(Z'))) \text{ such that } \tilde{C}_{pq}^1(x) = D_p(x)D_q(x) \text{ for } x \in \omega_{p \land q}$, and $p, q \geq 1$. Then $\sigma_1^p D_p = \sigma_1^q D_q$ patch up to a homomorphism $\sigma_1 \in \text{Hom}(\mathcal{O}(\ell_P(Z'), S)(\Omega)$. Replacing $\sigma_1^N$ by $\sigma_1$ in (6.14) we get another (S)-resolution

$$(6.15) \quad \ldots \to \mathcal{O}Z' \xrightarrow{\sigma_2^N} \mathcal{O}Z' \xrightarrow{\sigma_1} S|\omega_N \to 0$$
of $S|\omega_N$ for $N \geq 1$ by Proposition 4.4(b), where we regard $\ell_P(Z')$ as $Z'$. Let $K = \text{Ker} \sigma_1 \subset \mathcal{O}^{Z'}$ over $\Omega$, $K_N = \text{Ker} \sigma_1^N \subset \mathcal{O}^{Z'}$ over $\omega_N$, $N \geq 1$. Then $K|\omega_N \cong K_N \oplus \mathcal{O}^{Z'}$ by (6.15).

(b) As the kernel $K$ in (a) satisfies the same conditions as $S$ does, we can find by repeated application of (a) short exact sequences $0 \to K_n \to \mathcal{O}^{Z'} \to K_{n-1} \to 0$ over $\Omega$ for $n \geq 1$, where $K_0 = S$. Splicing together these short exact sequences we get an $(S)$-resolution of the type claimed. The proof of Theorem 6.3 is complete.

7. GLOBAL $(S)$-RESOLUTIONS.

In this section we show in many cases that an $(S)$-sheaf over a pseudoconvex open subset of a Banach space has a global $(S)$-resolution.

**Proposition 7.1.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $S \to \Omega$ an $(S)$-sheaf, and suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$. Then there is an admissible Hartogs function $\beta \in \mathcal{A}$ as in §2 such that there are $(S)$-resolutions

$$(7.1) \quad \ldots \to \mathcal{O}^{Z_n^N} \to \mathcal{O}^{Z_{n-1}^N} \to \ldots \to \mathcal{O}^{Z_1^N} \to S|\Omega_N\langle \beta \rangle$$

for all $N \geq 1$.

**Proof.** Let $\mathcal{U}$ be a covering of $\Omega$ by balls $U = B_X(x, r(x))$ for $x \in \Omega$ with continuous radius function $r \in C(\Omega, (0, 1))$ so small that over each $U$ our $S|U$ admits an $(S)$-resolution. By plurisubharmonic domination in $\Omega$ there is an $\alpha$ with $10\alpha \in \mathcal{A}$, $10\alpha < r$, such that $\mathcal{B}(\alpha)$ as in (2.1) is a refinement of $\mathcal{U}$. Proposition 2.2(f) gives a $\beta \in \mathcal{A}$, $\beta < \alpha$, such that the covering $\mathcal{U}_N = \mathcal{B}_N(\alpha)|\Omega_N\langle \beta \rangle$ has a finite basic refinement $\mathcal{U}_N$ for all $N \geq 1$. Any finite basic covering $\mathcal{U}_N$ of $\Omega_N\langle \beta \rangle$ has a finite refinement $\mathcal{W}_N$ which is a $(C)$-covering of $\Omega_N\langle \beta \rangle$ for all $N \geq 1$. Proposition 6.2 gives us an $(S)$-resolution (7.1) for all $N \geq 1$. The proof of Proposition 7.1 is complete.

We now prepare to apply Theorem 6.3.

**Proposition 7.2.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $\beta \in \mathcal{A}$, and suppose that plurisubharmonic domination holds in $\Omega$. Then there are $\omega_p \subset \omega_{p+1} \subset \Omega$ pseudoconvex open for $p \geq 1$ with $\bigcup_{p=1}^{\infty} \omega_p = \Omega$, and $\omega_p \subset \Omega_p\langle \beta \rangle$ for $p \geq 1$.

**Proof.** Define a function $m : \Omega \to \{1, 2, \ldots\}$ as follows. For $x \in \Omega$ let $m(x)$ be the least integer $N \geq 1$ such that $x$ has an open neighborhood $U$ in $\Omega$ with $U \subset \bigcap_{i=0}^{\infty} \Omega_{N+i}\langle \beta \rangle$. Such a number exists by Proposition 2.2(a), and $m$ is a locally upper bounded function of $x \in \Omega$ since $m(y) \leq m(x)$ if $y \in U$, where $U$ is as above. By plurisubharmonic domination in $\Omega$ we find
a continuous plurisubharmonic function $\psi : \Omega \to \mathbb{R}$ with $m(x) < \psi(x)$ for $x \in \Omega$. Let $\omega_p = \{x \in \Omega : \psi(x) < p\}$ for $p = 1, 2, \ldots$. If $x \in \omega_p$, then $m(x) < \psi(x) < p$, so $x \in \Omega_p(\beta)$, i.e., $\omega_p \subset \Omega_p(\beta)$ for $p \geq 1$. As $\omega_p \subset \omega_{p+1}$, and $\bigcup_{p=1}^{\infty} \omega_p = \Omega$ clearly hold, the proof of Proposition 7.2 is complete.

**Theorem 7.3.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, and $S \to \Omega$ an (S)-sheaf. If plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$, then there is an (S)-resolution (4.5) over $\Omega$.

**Proof.** Proposition 7.1 gives a $\beta \in \mathcal{A}$ and (S)-resolutions (7.1) over $\Omega_N(\beta)$ for $N \geq 1$. Proposition 7.2 yields $\omega_N \subset \Omega_N(\beta)$ for $N \geq 1$. Since the restriction of (7.1) to $\omega_N$ is an (S)-resolution for $N \geq 1$, an application of Theorem 6.3 completes the proof of Theorem 7.3.

**8. THE PROOF OF THEOREM 1.2.**

In this section we complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Part (a) follows from Theorem 5.3. (b) Proposition 5.2 shows that $\mathcal{O}^E \to M$ is an (S)-sheaf over $M$, the zero extension $(\mathcal{O}^E)^0 \to \Omega$ of which then by Theorem 5.6 is an (S)-sheaf over $\Omega$. The proof of Theorem 1.2 is complete.

The same reasoning proves the following Theorem 8.1.

**Theorem 8.1.** With the notation and hypotheses of Theorem 1.2 let $S \to M$ be an (S)-sheaf, and $S^0 \to \Omega$ its zero extension to $\Omega$ Then $S^0$ is an (S)-sheaf over $\Omega$, and Theorem 1.3 holds with $\Omega$ replaced by $M$, and pseudoconvex open sets $U$ replaced by open sets $U$ of the form $U = \tilde{U} \cap M$, where $\tilde{U}$ is any pseudoconvex open subset of $\Omega$.

**9. THE PROOF OF THEOREM 1.3.**

In this section we complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Part (a) follows from Theorem 7.3. Part (b) follows from (c) on letting $K = \text{Ker}(\mathcal{O}^Z_1 \to S)$. Part (a) follows on applying Theorem 4.2 to a global (S)-resolution in (c). To prove (d) we see by repeated application of (b) for (S')-sheaves that $S$ admits a global 2-resolution (1.3) over $\Omega$, hence $S$ is an (S)-sheaf over $\Omega$. The meaning of (d) is that the class of (S)-sheaves is the largest subclass of locally convex analytic sheaves for which Theorem 1.3(b), i.e., a natural condition, holds. The proof of Theorem 1.3 is complete.

**10. THE PROOF OF THEOREM 1.4.**

In this section we complete the proof of Theorem 1.4.
Proof of Theorem 1.4. (a) As $I$ is an (S)-sheaf over $\Omega$ by Theorem 1.2(a) an application of Theorem 1.3(a) shows the acyclicity of $I$ over $\Omega$.

(b) As there are local extensions $\tilde{f}_U$ of $f$, and the cocycle $(\tilde{f}_V - \tilde{f}_U)$ of $I$ over $\Omega$ can be resolved by (a), part (b) is proved.

(c) Since $(O^E)^0$ is an (S)-sheaf over $\Omega$ by Theorem 1.2(b), we see by Theorem 1.3(a) that $H^q(\Omega, (O^E)^0)$ for $q \geq 1$. As $H^q(\Omega, (O^E)^0)$ and $H^q(M, O^E)$ are canonically isomorphic for $q \geq 1$, the latter are zero, too.

(d) As the ideal sheaf $I = I_C$ of $M$ in $\Omega$ is an (S)-sheaf over $\Omega$ by Theorem 1.2(a), there is by Theorem 1.3(b) a short 2-exact sequence

\[ 0 \to K \to O^{Z_1} \to I \to 0 \]

over $\Omega$. Proposition 3.1 gives a $T \in O(\Omega, Hom(Z_1, \mathbb{C}))$ that induces the homomorphism $\tau : O^{Z_1} \to O$ as $\tau = T$. Let $M' = \{ x \in \Omega : T(x) = 0 \}$.

We claim that $M = M'$ as point sets.

Let $x_0 \in M$ and suppose for a contradiction that $x_0 \notin M'$. As there is a $z_1 \in Z_1$ with $T(x_0)z_1 \neq 0$, there is a small open ball $U$ with $x_0 \in U \subset \Omega$ such that the function $g \in O(U, Z_1)$ defined by $g(x) = T(x)z_1$ is nonzero for $x \in U$. As $g \in I(U)$ and $x_0 \in M$ we find the contradiction that $g(x_0) = 0$.

Let $x_0 \in M'$ and suppose for a contradiction that $x_0 \notin M$. There is a small open ball $U$ with $x_0 \in U \subset \Omega$ that is disjoint from the closed set $M$. As the constant $1 \in I(U)$, there is a $g \in O(U, Z_1)$ with $1 = T(x)g(x)$ for $x \in U$. Letting $x = x_0$ we find the contradiction that $1 = 0$.

Hence $M = M'$, and (d) is proved.

The meaning of part (d) is that $M$ can be defined by a global holomorphic equation $T(x) = 0$ in $\Omega$.

(e) This follows from (d) upon applying [P2, Thm. 1.2].

(f) This follows by the Grauert–Docquier type argument in the proof of [P2, Thm. 6.2] together with (c) and (e).

The proof of Theorem 1.4 is complete.

11. APPLICATIONS.

In this section we discuss some applications of the theorems in §1. See [P4, §14] for additional applications.

Theorem 11.1. Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a split complex Banach submanifold of $\Omega$, and $E \to M$ a holomorphic Banach vector bundle with a Banach space $Z$ for
fiber type. If plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$, then we have the following.

(a) Let $Z_1 = \ell_p(Z), 1 \leq p < \infty$. Then $E \oplus (M \times Z_1)$ and $M \times Z_1$ are holomorphic isomorphic over $M$.

(b) $H^q(M, \mathcal{O}^E) = 0$ for $q \geq 1$.

(c) If $E$ is continuously trivial over $M$, then $E$ is holomorphically trivial over $M$.

Proof. By Theorem 1.4(f) there is a holomorphic rectraction $r : \omega \to M$, where $\omega$ is pseudoconvex open with $M \subset \omega \subset \Omega$. Apply Theorem 4.1 to the pull back bundle $r^*E \to \omega$, and then restrict back to $M$. That proves (b) and (c). In part (a) we also need to know in advance that $E \oplus (M \times Z_1)$ is continuously trivial. This follows since $r^*(E \oplus (M \times Z_1)) = (r^*E) \oplus (\omega \times Z_1)$ is continuously trivial by [P4, Prop. 7.1]. Then (a) follows from this and (c). The proof of Theorem 11.1 is complete.

Proposition 11.2 will be useful in the proof of Theorem 11.4 below.

Proposition 11.2. Let $X$ be a Banach space with a Schauder basis, and $\Omega \subset X$ open.

(a) If $\Omega' \subset X$ is pseudoconvex open, $\Omega \subset \Omega'$ is open, and for each boundary point $x_0$ of $\Omega$ that is not a boundary point of $\Omega'$ there is an open set $U \subset X$ with $x_0 \in U$, and $\Omega \cap U$ pseudoconvex open in $X$, then $\Omega$ is pseudoconvex open in $X$.

(b) If $\Omega$ is biholomorphic to a pseudoconvex open subset of $X$, then $\Omega$ is pseudoconvex open in $X$.

(c) If $X = X' \times X''$ is a direct decomposition of Banach spaces, $\pi : X \to X' \times \{0\}$ is the projection $\pi(x',x'') = (x',0)$, $\Omega \subset X$ pseudoconvex open, $\Omega' \subset X' \times \{0\}$ is pseudoconvex open (relative to $X'$), then $\pi^{-1}(\Omega') \cap \Omega$ is pseudoconvex open in $X$.

(d) If $M \subset \Omega$ is a split complex Banach submanifold, and $r : \Omega \to M$ is a holomorphic retraction, then for each $x_0 \in M$ there is a ball $U = B_X(x_0, \varepsilon)$ in $X$ and a direct decomposition $X = X' \times X''$ of Banach spaces such that in $U$ there is a biholomorphic coordinate system in which the retraction $r$ can be written as a linear projection $\pi$ as in (c).

(e) Let $\Omega \subset X$ be pseudoconvex open, $M \subset \Omega$ a split complex Banach submanifold of $\Omega$. Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$. Let $D \subset M$ be a relatively open subset of $M$. If in $M$ at every relative boundary point $x_0 \in \partial D$ there is a coordinate ball $U$ in $M$ with $x_0 \in U$, and $D \cap U$ coordinate pseudoconvex, then there is a pseudoconvex open subset $\tilde{D}$ of $\Omega$ with $D = \tilde{D} \cap M$.

(f) Let $\Omega \subset X$ be pseudoconvex open, $M \subset \Omega$ a split complex Banach
submanifold of Ω. Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of Ω. Let D ⊂ M be a relatively open subset of M, E → D be a continuous Banach vector bundle with a Banach space Z for fiber type, and Z₁ = ℓ_p(Z) for 1 ≤ p < ∞. Then E ⊕ (D × Z₁) is continuously isomorphic to D × Z₁.

(g) Let M be a complex Banach manifold modelled on a Banach space X, Z₁, Z₂ Banach spaces, and g ∈ O(M, Hom(Z₁, Z₂)). If g(x) ∈ Hom(Z₁, Z₂) is an epimorphism with split kernel for each x ∈ M, then the set K = Ker g = \{(x, ζ₁) ∈ M × Z₁ : g(x)ζ₁ = 0\} is a holomorphic Banach vector bundle over M.

(h) Let X be a Banach space, X* its dual space, ξₙ ∈ X*, n ≥ 1, with ∥ξₙ∥ → ∞ as n → ∞. Then there is an x₀ ∈ X with |ξₙ(x₀)| unbounded as n → ∞.

(i) If U₁, U₂ are open subsets of a complex Banach manifold M, and plurisubharmonic domination holds in U₁, and in U₂, then plurisubharmonic domination holds in V = U₁ ∩ U₂.

(j) With the notation and hypotheses of (e) plurisubharmonic domination holds in M.

Proof. (a) A well-known criterion for the pseudoconvexity of Ω in X runs as follows. An open set Ω is pseudoconvex in X if and only if X = Ω or else for each boundary point x₀ ∈ ∂Ω in X there are an open set V with x₀ ∈ V, a sequence of points xₙ ∈ V ∩ Ω with xₙ → x₀ as n → ∞, and a holomorphic function f ∈ O(V ∩ Ω) with |f(xₙ)| unbounded as n → ∞. Such a function f is called a local holomorphic function on Ω singular at x₀.

We will use the above criterion to show that our Ω is pseudoconvex. Let x₀ be a boundary point of Ω in X. If x₀ is a boundary point of Ω', then there is a holomorphic function f ∈ O(Ω') that is singular at x₀. If x₀ is not a boundary point of Ω', then there is an open set U with x₀ ∈ U, and Ω ∩ U pseudoconvex open in X. As x₀ is a boundary point of the pseudoconvex open set Ω ∩ U, there is a holomorphic function f ∈ O(Ω ∩ U) that is singular at x₀. Hence Ω is pseudoconvex by the above criterion.

(b) This is a well-known statement.

(c) As π⁻¹(Ω') ∩ Ω is the intersection of two pseudoconvex open subsets π⁻¹(Ω') = Ω' × X'' and Ω of X, the statement follows.

(d) It is easy to see that at any point x₀ ∈ M the Fréchet differential dr(x₀) ∈ End(X) is a linear projection with split kernel. Hence is the statement.

(e) Theorem 1.4(f) gives us a holomorphic retraction r : Ω' → M, where Ω'
is pseudoconvex open in $X$ with $M \subset \Omega' \subset \Omega$. There is a continuous radius function $\varepsilon' \in C(M,(0,1))$ so small that over the ball $B_X(x_0,\varepsilon'(x_0)) \subset \Omega'$ the retraction $r$ can be linearized to a linear projection by a biholomorphism for all $x_0 \in M$. By a standard argument with a partition of unity there is a continuous radius function $\varepsilon'' \in C(M,(0,1))$ so small that $\varepsilon'' < \varepsilon'$, and over the $\varepsilon''$-balls a doubling inequality holds for $\varepsilon'$, i.e., for all $x_0 \in M$ we have that $\varepsilon''(x_0) \times \varepsilon'(x_0)$, and $\frac{1}{2} \varepsilon'(z) \times \varepsilon'(y) \times 2 \varepsilon'(z)$ for all $y,z \in B_X(x_0,\varepsilon''(x_0)) \cap M$. Theorem 1.4(e) gives a pseudoconvex open subset $\omega$ of $X$ with $M \subset \omega \subset \{x \in \Omega' : \|x - r(x)\| < \frac{1}{4} \varepsilon''(r(x))\}$. Let $\tilde{D} = r^{-1}(D) \cap \omega$. Then $\tilde{D}$ is an open subset of $\Omega$, and $\tilde{D} \cap M = D$ since if $x \in \tilde{D} \cap M$, then $r(x) = x \in D$, and if $x \in D$, then $x \in r^{-1}(D) \cap M \subset \tilde{D} \cap M$.

We proceed to show via (a) that $\tilde{D}$ is pseudoconvex open in $X$. If $\tilde{D} = X$, then $\tilde{D}$ is pseudoconvex open in $X$. If $\tilde{D} \neq X$, then let $x_0$ be any boundary point of $\tilde{D}$. As $\tilde{D}$ is an open subset of the pseudoconvex open subset $\Omega'$ of $X$ in order to apply (a) it is enough to check that at any boundary point $x_0$ of $\tilde{D}$ in $X$ that is not a boundary point of $\Omega'$ there is an open set $U \subset X$ such that $\tilde{D} \cap U$ is pseudoconvex open in $X$, and $x_0$ is a limit point of $\tilde{D} \cap U$, but not a point of $\tilde{D} \cap U$. As $x_0$ is not a boundary point of $\Omega'$, i.e., $x_0 \in \Omega'$, our retraction $r$ is defined at $x_0$. Since there are points $x_n \in \tilde{D}$ with $x_n \to x_0$ as $n \to \infty$, we see that $r(x_n) \to r(x_0)$ as $n \to \infty$, i.e., $r(x_0)$ is in the closure of $D$ relative to $M$.

We claim that if $V \subset B_X(r(x_0),\frac{1}{4} \varepsilon''(r(x_0))) \cap M$, then $r^{-1}(V) \cap \omega \subset B_X(r(x_0),\varepsilon'(r(x_0)))$.

Indeed, we must show for $x \in r^{-1}(V) \cap \omega$ that $\|x - r(x_0)\| < \varepsilon'(r(x_0))$. As $r(x) \in V$, and $x \in \omega$, we have the inequalities

$$\|r(x) - r(x_0)\| < \frac{1}{4} \varepsilon''(r(x_0)),$$

$$\|x - r(x)\| < \frac{1}{4} \varepsilon''(r(x)),$$

adding up which implies that

$$\|x - r(x_0)\| < \frac{1}{4} \varepsilon''(r(x_0)) + \frac{1}{4} \varepsilon''(r(x))$$

$$< \frac{1}{4} \varepsilon'(r(x_0)) + \frac{1}{4} \varepsilon'(r(x))$$

$$< \frac{1}{4} \varepsilon'(r(x_0)) + \frac{2}{4} \varepsilon'(r(x_0))$$

$$< \varepsilon'(r(x_0)),$$

where we applied in the penultimate inequality the doubling property of $\varepsilon'$ on the ball $B_X(r(x_0),\frac{1}{4} \varepsilon''(r(x_0))) \cap M$.

If $r(x_0) \in D$, then the point $r(x_0)$ is contained in a coordinate ball $V \subset B_X(r(x_0),\frac{1}{4} \varepsilon''(r(x_0))) \cap D \subset D$ relative to $M$. By the claim above the set $U = r^{-1}(V) \cap \omega$ is contained in a ball $B_X(r(x_0),\varepsilon'(r(x_0)))$ in which $r$ can be linearized to a linear projection. Then $U = \tilde{D} \cap U$ is pseudoconvex open in
$X$ by (b) and (c). We now show that $x_0$ is limit point of $\tilde{D} \cap U$. Indeed, let $x_n \in \tilde{D}$ be any sequence with $x_n \rightarrow x_0$ as $n \rightarrow \infty$. As $r(x_n) \rightarrow r(x_0) \in V$, and $V$ is open relative to $M$, there is an $N$ with $r(x_n) \in V$ for all $n \geq N$. So $x_n \in r^{-1}(V) \cap \omega = U$ for $n \geq N$, i.e., $x_0$ is a limit point of $U = \tilde{D} \cap U$.

If $r(x_0) \in \partial D$ is in the boundary of $D$ relative to $M$, then there is a coordinate ball $V \subset B_X(r(x_0), \varepsilon''(r(x_0))) \cap M$ relative to $M$ with $x_0 \in V$ and $V \cap D$ coordinate pseudoconvex open relative to $M$. By the claim above the set $U = r^{-1}(V) \cap \omega$ is contained in a ball $B_X(r(x_0), \varepsilon'(r(x_0)))$ in which $r$ can be linearized to a linear projection. Then $\tilde{U}$ and $\tilde{D} \cap U = r^{-1}(V \cap D) \cap \omega$ are pseudoconvex open in $X$ by (b) and (c). We now show that $x_0$ is a limit point of $\tilde{D} \cap U$. Let $x_n \in \tilde{D} = r^{-1} \cap \omega$ be any sequence with $x_n \rightarrow x_0$ as $n \rightarrow \infty$. As $r(x_n) \rightarrow r(x_0) \in D \cap V$, and $D \cap V$ is open relative to $M$, there is an $N$ with $r(x_n) \in D \cap V$ for all $n \geq N$. Then $x_n \in r^{-1}(D \cap V) \cap \omega = \tilde{D} \cap U$ for $n \geq N$, i.e., $x_0$ is a limit point of $\tilde{D} \cap U$. Thus as our $\tilde{D}$ is pseudoconvex open in $X$ by (a), the proof of (e) is complete.

(f) By Theorem 1.4(f) there is a holomorphic retraction $r : \omega \rightarrow M$, where $\omega$ is pseudoconvex open $X$ with $M \subset \omega \subset \Omega$. Look at $r^*(E \oplus (D \times Z_1)) = (r^*E) \oplus (r^{-1}(D) \times Z_1)$ and apply [P4, Prop.7.1], then restrict back to $D$.

(g) This follows from the inverse function theorem for holomorphic maps of Banach spaces. Note that any closed finite dimensional or finite codimensional subspace of $Z_1$ is split, and so is any closed subspace of $Z_1$ if $Z_1$ is a Hilbert space.

(h) This follows from the principle of uniform boundedness or the principle of condensation of singularities in linear functional analysis.

(i) Let $u : V \rightarrow \mathbb{R}$ be a locally upper bounded function, $U = U_1 \cup U_2$, and $\chi_1, \chi_2 : U \rightarrow \mathbb{R}$ a continuous partition of unity subordinate to the open covering $\{U_1, U_2\}$ of $U$. As $u\chi_i$ is a locally upper bounded function on $U_i$, plurisubharmonic domination in $U_i$ gives a continuous plurisubharmonic function $\psi_i : U_i \rightarrow \mathbb{R}$ with $u(x)\chi_i(x) < \psi_i(x)$ for $x \in U_i, i = 1, 2$. As $u(x) = u(x)\chi_1(x) + u(x)\chi_2(x) < \psi_1(x) + \psi_2(x)$ for $x \in V$, the continuous plurisubharmonic function $\psi_1 + \psi_2 : V \rightarrow \mathbb{R}$ dominates $u$ on $V$.

(j) Let $u : M \rightarrow \mathbb{R}$ be a locally upper bounded function, and define $u' : \Omega \rightarrow \mathbb{R}$ by $u'(x) = u(x)$ if $x \in M$, and $u'(x) = 0$ if $x \in \Omega \setminus M$. As $u'$ is easily seen locally upper bounded (since $M$ is relatively closed in $\Omega$), plurisubharmonic domination in $\Omega$ gives a continuous plurisubharmonic function $\psi' : \Omega \rightarrow \mathbb{R}$ with $u'(x) < \psi'(x)$ for $x \in \Omega$. Then $\psi = \psi'|M$ is a continuous plurisubharmonic function on $M$ that dominates $u$.

The proof of Proposition 11.2 is complete.
Among finite dimensional complex manifolds the class of Stein manifolds can be characterized by cohomological criteria. There are also cohomological criteria for open subsets of a Stein manifold $M$ to be themselves Stein. Here is one such criterion by Leiterer.

**Theorem 11.3.** (Leiterer, [Lt2]) Let $M$ be a Stein manifold of complex dimension $n$, and $D \subset M$ open. Then the following are equivalent.

(a) $D$ is a Stein manifold.

(b) $H^1(D, \mathcal{O}) = 0$, and any topologically trivial holomorphic vector bundle over $D$ is holomorphically trivial over $D$.

(c) $H^1(D, \mathcal{O}) = 0$, and for every corank 1 holomorphic vector subbundle $E$ of $D \times \mathbb{C}^{2n+1}$ such that for some $m$ the bundle $E \oplus (D \times \mathbb{C}^m)$ is topologically trivial over $D$, there is a topologically trivial holomorphic vector bundle $F \to D$ with $E \oplus F$ holomorphically trivial over $D$.

(d) $H^1(D, \mathcal{O}^E) = 0$ for every corank 1 holomorphic vector subbundle $E$ of $D \times \mathbb{C}^{2n+1}$.

(e) For every choice of holomorphic functions $g_1, \ldots, g_{2n+1} \in \mathcal{O}(D)$ without common zeros in $D$ there are holomorphic functions $f_1, \ldots, f_{2n+1} \in \mathcal{O}(D)$ with $\sum_{i=1}^{2n+1} f_i(x)g_i(x) = 1$ for $x \in D$.

We give in Theorem 11.4 below an analog of Theorem 11.3 above.

**Theorem 11.4.** Let $X$ be a Banach space with a Schauder basis, $\Omega \subset X$ pseudoconvex open, $M$ a split complex Banach submanifold of $\Omega$, and $D \subset M$ relatively open. Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of $\Omega$. Then the following are equivalent.

(a) There is a pseudoconvex open subset $\tilde{D}$ of $X$ with $D = M \cap \tilde{D}$.

(b) $H^1(D, \mathcal{O}^Z) = 0$ for any Banach space $Z$, and any continuously trivial holomorphic Banach vector bundle over $D$ is holomorphically trivial over $D$.

(c) $H^1(D, \mathcal{O}^Z) = 0$ for any Banach space $Z$, and for any corank 1 holomorphic Banach vector subbundle $E$ of $D \times X$ over $D$ there is a holomorphic Banach vector bundle $F \to D$ with $E \oplus F$ holomorphically trivial over $D$.

(d) $H^1(D, \mathcal{O}^E) = 0$ for every corank 1 Banach vector subbundle $E$ of $D \times X$ over $D$.

(e) For every $g \in \mathcal{O}(D, X)$ with $g(x) \neq 0$ for $x \in D$ there is an $f \in \mathcal{O}(D, X^*)$ with $f(x) \cdot g(x) = 1$ for $x \in D$, where $X^*$ is the Banach space dual to $X$, and the dot denotes the natural pairing $X^* \times X \to \mathbb{C}$.

(f) Plurisubharmonic domination holds in $D$.

**Proof.** (a $\Rightarrow$ b) As $D = M \cap \tilde{D}$ we see that $D$ is a split complex Banach submanifold of the pseudoconvex open set $\tilde{D}$ in $X$. Thus (b) follows from Theorem 11.1(b).
(a ⇒ c) Let \( F = D \times Z_1 \). Then (c) holds by Theorem 11.1(a).

(a ⇒ d) Part (d) follows from Theorem 11.1(b).

(a ⇒ e) As \( E = \text{Ker} \, g = \{(x, \xi) \in D \times X^* : \xi g(x) = 0\} \) is the kernel of the epimorphism (with split kernel) in \( \mathcal{O}(D, \text{Hom}(X^*, \mathbb{C})) \) defined by \((x, \xi) \mapsto \xi g(x)\), Proposition 11.2(g) shows that \( E \) is a holomorphic Banach vector bundle over \( D \). As \( g(x_0) \neq 0 \) for \( x_0 \in D \), there is a continuous linear functional \( \xi_0 \in X^* \) with \( \xi_0 g(x_0) = 1 \). Then there is an open neighborhood \( U_{x_0} \) of \( x_0 \) in \( D \) with \( \xi_0 g(x) \neq 0 \) for \( x \in U_{x_0} \). Define \( f_{x_0} \in \mathcal{O}(U, X^*) \) by \( f_{x_0}(x) = \frac{1}{\xi_0 g(x)} \xi_0 \). Then \( f_{x_0} \cdot g = 1 \) on \( U_{x_0} \). Let \( \mathcal{U} = \{ U_{x_0} : x_0 \in D \} \), and look at the cocycle \((f_{y_0} - f_{x_0}) \in Z^1(\mathcal{U}, \mathcal{O}^E)\). As \( H^1(D, \mathcal{O}^E) = 0 \) by Theorem 11.1(b) we have a cochain \( h_{x_0} \in C^0(\mathcal{U}, \mathcal{O}^E) \) with \( f_{y_0} - f_{x_0} = h_{y_0} - h_{x_0} \) on \( U_{x_0} \cap U_{y_0} \). Then \( f = f_{x_0} - h_{x_0} = f_{y_0} - h_{y_0} \) patch up to a well defined function \( f \in \mathcal{O}(D, X^*) \) with \( f(x)g(x) = f_{x_0}(x)g(x) - h_{x_0}(x)g(x) = 1 - 0 \) for \( x \in U_{x_0}, x_0 \in D \). Thus (e) follows.

(a ⇒ f) As \( D = \bar{D} \cap M \) Proposition 11.2(j) shows that plurisubharmonic domination holds in \( D \).

(b ⇒ c) Let \( F = D \times Z_1 \), and apply Proposition 11.2(f). Then \( E \oplus F \to D \) is continuously trivial, hence it is holomorphically trivial by (b), and so (c) follows.

(d ⇒ e) See the proof of (a ⇒ e) above.

(e ⇒ a) At any boundary point \( x_0 \) of \( D \) relative to \( M \) there is a holomorphic function \( \varphi \in \mathcal{O}(D) \) that is singular at \( x_0 \). Indeed, look at \( g \in \mathcal{O}(D, X) \) defined by \( g(x) = x - x_0 \). As \( g(x) \neq 0 \) for \( x \in D \), there is an \( f \in \mathcal{O}(D, X^*) \) with \( f(x)g(x) = 1 \) for \( x \in D \). Let \( x_n \in D, n \geq 1 \), be any sequence of points with \( x_n \to x_0 \) as \( n \to \infty \). As \( g(x_n) \to 0 \) as \( n \to \infty \), we find that \( \|f(x_n)\| \) may not be bounded as \( n \to \infty \) (since otherwise \( 1 = f(x_n)g(x_n) \to 0 \) as \( n \to \infty \) would hold). Proposition 11.2(h) gives a point \( \bar{x} \in X \) with \( |f(x_n)\bar{x}| \) unbounded as \( n \to \infty \). Define \( \varphi \in \mathcal{O}(D) \) by \( \varphi(x) = f(x)\bar{x} \). Then \( \varphi \) is singular at \( x_0 \).

Let \( U \) be a coordinate ball relative to \( M \) with \( x_0 \in U \). Then \( U \cap D \) is coordinate pseudoconvex open relative to \( M \). Indeed, any boundary point \( y_0 \) of \( U \cap D \) relative to \( M \) is a boundary point of \( U \) relative to \( M \) or a boundary point of \( D \) relative to \( M \). In either case there is a holomorphic function \( \varphi \in \mathcal{O}(U) \) or \( \varphi \in \mathcal{O}(D) \) that is singular at \( y_0 \). Thus \( U \cap D \) is coordinate pseudoconvex open relative to \( M \), as claimed, by the criterion in the proof of Proposition 11.2(a).

Proposition 11.2(e) thus applies and gives us a pseudoconvex open \( \bar{D} \) in \( X \) with \( D = M \cap \bar{D} \), completing the proof of (e ⇒ a).
(c ⇒ a) The bundles \( E = \text{Ker} g \) introduced in the proof of \((a \Rightarrow e)\) are corank 1 Banach vector subbundles of \( D \times X \), and (c) provides a holomorphic Banach vector bundle \( F \to D \) with \( E \oplus F \cong D \times Z \) holomorphically trivial over \( D \), where \( Z \) is a Banach space. Thus by (c) we see that 
\[
0 = H^1(D, \mathcal{O}^2) = H^1(D, \mathcal{O}^E) \oplus H^1(D, \mathcal{O}^F), \text{i.e., } H^1(D, \mathcal{O}^E) = 0.
\]
Thus (c ⇒ e), and as (e ⇒ a), we find that (c ⇒ a).

(f ⇒ a) Let \( x_0 \in \partial D \) be any boundary point of \( D \) relative to \( M \). There is a small enough ball \( U = B_X(x_0, r) \) in \( \Omega \) with \( U \cap M \) coordinate pseudoconvex open relative to \( M \). Then plurisubharmonic domination holds in \( U \cap M \) by Proposition 11.2(j), and in \( (U \cap M) \cap D = U \cap D \) by Proposition 11.2(i), hence \( U \cap D \) is coordinate pseudoconvex open relative to \( M \). Thus Proposition 11.2(e) applies and gives a pseudoconvex open subset \( \tilde{D} \) of \( \Omega \) with \( D = \tilde{D} \cap M \).

The proof of Theorem 11.4 is complete.

**Theorem 11.5.** Let \( M \) be a complex Banach manifold modelled on a Banach space \( X \) with a Schauder basis, \( S \to M \) an \((S)\)-sheaf, and \( \mathcal{U} \) an \((S)\)-covering of \( M \). Suppose that plurisubharmonic domination holds in every pseudoconvex open subset of \( X \). Then the following hold.

(a) The covering \( \mathcal{U} \) is a Leray covering of \( M \) for the sheaf \( S \), and \( H^q(M, S) \) is naturally isomorphic to the cohomology group \( H^q(\mathcal{U}, S) \) of alternating cochains of \( \mathcal{U} \) for all \( q \geq 1 \).

(b) If \( \mathcal{U} \) is finite, say, it has \( n = 1, 2, \ldots \) elements, then \( H^q(M, S) = 0 \) for \( q \geq n \).

(c) Let \( M \subset X \) be pseudoconvex open, \( f_i \in \mathcal{O}(M) \), \( U_i = \{ x \in M : f_i(x) \neq 0 \}, \) \( i = 1, \ldots, n, n \geq 1 \), \( \mathcal{U} = \{ U_i : i = 1, \ldots, n \} \), \( M_0 = \bigcap_{i=1}^n (M \setminus U_i) \), and \( S_0 \to M \setminus M_0 \) an \((S)\)-sheaf. Then \( H^q(M \setminus M_0, S_0) \) is naturally isomorphic to \( H^q(\mathcal{U}, S_0) \) for \( q \geq 1 \), and is, in particular, zero for \( q \geq n \).

**Proof.** As (a) follows from Theorem 1.3(a), while (b) from (a), and (c) from (b) on looking at the \((S)\)-covering \( \mathcal{U} \), the proof of Theorem 11.5 is complete.

12. DISCUSSION.

In this section we make some informal remarks on the methods adopted in this paper.

To prove cohomology vanishing for analytic sheaves via the method of amalgamation of syzygies it is necessary to be able to obtain local resolutions, to paste local resolutions to make resolutions over bigger sets, and to pass to the limit. The Oka coherence theorem is the main tool in finite dimensions to obtain local resolutions. No analog of that fundamental theorem of Oka seems to be known currently in infinite dimensions. To obtain our local
resolutions we simply define away the problem. To paste local resolutions together we need to lift local epimorphisms to holomorphic operator valued functions through another local epimorphism. Once again we define away the difficulty in the notion of 2-exactness. Once the liftings are in hand we need the fact that a holomorphic Banach vector bundle that is topologically trivial is also holomorphically trivial in certain cases. This was one of the main raison d’être of [P4], where this was shown.

To pass to the limit there are two basic ways. One requires us to show that our sheaf is acyclic over the members of the exhaustion and to pass to the limit in the first cohomology of the sheaf at hand. This could be done in either of two ways. The one is to do cohomology with bounds for the sheaf — this seems very difficult in infinite dimensions. The other is to use the multiplicative Runge type approximation Hypothesis($X, \text{GL}(Z)$) in [P1], where $X, Z$ are Banach spaces, which hypothesis is still not proved. The second way of passage to the limit is to obtain a global resolution. This is the path that we took here. The reason for that is that we can reduce the problem to proving that a topologically trivial holomorphic Banach vector bundle is holomorphically trivial in certain cases, i.e., to something already available. Once a global resolution is available the proof of the acyclicity of the sheaf can be effected through the use of the ensuing dimension shifting formula, which in finite dimensions is sufficient by itself, while in infinite dimensions it can be combined with exhaustions whose members have finite Leray coverings. This was done in [P3] explicitly for this purpose.

The notion of 2-exactness could be fine tuned by requiring exactness under the functor $\text{Hom}(\mathcal{O}^Z, -)$ for not all Banach spaces $Z$, but just for a class of them, which suffices for local resolutions, and local amalgamations, and in the resolutions we could allow only special type of epimorphisms. E.g., if $S \to \Omega$ is an analytic sheaf over a pseudoconvex open set $\Omega$ that admits local 1-resolutions (4.5) with all Banach spaces $Z_n$ finite dimensional, then we could look at $\text{Hom}(\mathcal{O}^Z, -)$ with $Z$ finite dimensional only, in which case the lifting property is trivial. In the limit we could allow ourselves to wind up with a slightly different type of global resolution, e.g., in the above example we can produce a global 1-resolution

$$\ldots \to \mathcal{O}^{\ell_2} \xrightarrow{\tau_3} \mathcal{O}^{\ell_2} \xrightarrow{\tau_1} S \to 0$$

over $\Omega$, where, locally or over $U = \omega_p$ as in Theorem 7.3, each map $\tau_n$ is of the form $\tau_n = \text{diag}(0, \sigma_n)A_n : \mathcal{O}^{\ell_2} \oplus \mathbb{C}^{N_n} \to \mathcal{O}^{\ell_2} \oplus \mathbb{C}^{N_{n-1}}$, where $A_n \in \mathcal{O}(U, \text{GL}(\ell_2 \oplus \mathbb{C}^{N_n})), \sigma_n \in \text{Hom}(\mathcal{O}^{N_n}, \mathcal{O}^{N_{n-1}})(U)$ for $n \geq 2$, and $\tau_1 = (0, \sigma_1)A_1 : \mathcal{O}^{\ell_2} \oplus \mathbb{C}^{N_1} \to S, \sigma_1 \in \text{Hom}(\mathcal{O}^{N_1}, S)(U)$, and $N_n$ are nonnegative integers for $n \geq 1$.

Some words on analytic subsets of Banach spaces are in order. The no-
tion of a Banach analytic set in the most general sense seems pathological. Douady \([D]\) showed that any compact metric space can be embedded in a suitable Banach space as a Banach analytic set. Later Pestov \([Ps1, Ps2]\) proved the same for any complete metric space. Even closed linear subspaces may raise problems. Let \(A\) be a closed linear subspace of a Banach space \(X\). Is \(A\) a Banach analytic subset of \(X\)? Formally, of course, yes, since one can write \(A = \{\pi = 0\}\), where \(\pi : X \to X/A\) is the natural projection. If the associated short exact sequence \(0 \to A \to X \to X/A \to 0\) splits, then well and good, \(A\) has a direct complement. If the above sequence does not split, can one still understand some of the complex analytic properties of \(A\) from those of \(X\)? E.g., can we understand much about any separable Banach space \(A\) by embedding it into a universal space like \(X = C[0, 1]\)? Well, not most directly, but perhaps plurisubharmonic domination in \(C[0, 1]\), if available, would have a positive impact on the study of \(A\). A nice class of Banach analytic sets seems to be that of the split complex Banach submanifolds, (and possibly their finite branched coverings), which can be well studied, as demonstrated in this paper, with current technology.

The results of this paper naturally raise some questions about \((S)\)-sheaves and complex Banach manifolds. Here is one for instance. Let \(D\) be a complex Hilbert manifold, and suppose that \(H^q(D, S) = 0\) for any \(q \geq 1\) and any \((S)\)-sheaf \(S\) over \(D\). Can \(D\) be holomorphically embedded as a closed complex Hilbert submanifold in the Hilbert space \(\ell_2\)? Recently, Aaron B. Zerhusen has shown, based on the notion of holomorphic domination in \([L2]\), that if \(D\) is a pseudoconvex open subset of \(\ell_2\), then \(D\) can be holomorphically embedded in \(\ell_2\) as a Hilbert submanifold. So in particular, if \(D\) is as in Theorem 11.4 with \(X = \ell_2\), then the answer to the above question for \(D\) is yes.

In conclusion we would like to remark that parts of the present paper, especially Theorems 6.1 and 6.3, were inspired by \([Lt1]\) by Leiterer. While we could not manage to cite any part of it, we are glad that we could at least quote \([Lt2]\) by him instead.

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