Abstract. Let $k$ be a field, $S$ be a bigraded $k$-algebra, and $S_\Delta$ denote the diagonal subalgebra of $S$ corresponding to $\Delta = \{(cs, es) \mid s \in \mathbb{Z}\}$. It is known that the $S_\Delta$ is Koszul for $c, e \gg 0$. In this article, we find bounds for $c, e$ for $S_\Delta$ to be Koszul, when $S$ is a geometric residual intersection. Furthermore, we also study the Cohen-Macaulay property of these algebras. Finally, as an application, we look at classes of perfect ideals of height two in a polynomial ring, show that all their powers have a linear resolution, and study the Koszul, and Cohen-Macaulay property of the diagonal subalgebras of their Rees algebras.

Introduction

Let $k$ be a field, $c, e \in \mathbb{N}$, and $\Delta = \{(cs, es) \mid s \in \mathbb{Z}\}$ be the $(c, e)$-diagonal of $\mathbb{Z}^2$. Given a bigraded $k$-algebra $S = \bigoplus_{(u,v) \in \mathbb{Z}^2_{\geq 0}} S_{(u,v)}$, one can associate a graded $k$-algebra $S_\Delta = \bigoplus_{s \in \mathbb{Z}} S_{(cs, es)}$, the $(c, e)$-diagonal subalgebra of $S$. In a special case, when $S = k[x_1, \ldots, x_n; y_1, \ldots, y_p]$ with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$, then $S_\Delta$ is the homogeneous coordinate ring of the image of the correspondence under the Segre embedding $\mathbb{P}^{n-1} \times \mathbb{P}^{p-1} \hookrightarrow \mathbb{P}^{np-1}$ (cf. [15]).

The motivation for the study of diagonal subalgebras comes from algebraic geometry. Many authors have studied the algebraic properties (for instance, normality, Cohen-Macaulayness, defining equations etc.) of rational surfaces obtained by blowing up projective space $\mathbb{P}^2$ at a finite set of points, for instance see [8, 9, 10]. Simis, Trung and Valla introduced diagonal subalgebra to study the normality and the Cohen-Macaulayness of rational surfaces obtained by blowing up projective space $\mathbb{P}^2$ along a subvariety (See [15]). Conca, Herzog, Trung and Valla used diagonal subalgebras as an effective tool for the study of normality, Cohen-Macaulayness, Gorensteinness and Koszulness of embedded $(n - 1)$-fold obtained by blowing up $\mathbb{P}^{n-1}$ along a subvariety (cf. [6]).

The notion of residual intersections was introduced and initially studied by Artin-Nagata ([2]), and Huneke-Ulrich ([11]). Our motivation for studying geometric residual intersections is the following: Morey and Ulrich ([13]) show that if $I$ is a linearly presented perfect ideal of height two in a polynomial ring, it’s Rees algebra $\mathcal{R}(I)$ is a geometric residual intersection when $I$ satisfies certain conditions (see Setup 3.1. For such an ideal $I$, the authors in [11] also show that $\mathcal{R}(I)_\Delta$ is Cohen Macaulay for $c, e \gg 0$.

The Koszul property of the diagonal subalgebras of the Rees algebra of an ideal has also been studied in multiple articles. In particular, for an ideal $I$, generated by a homogeneous regular sequence $f_1, f_2, f_3 \in k[x_1, \ldots, x_n]$ of the same degree, explicit lower bounds are obtained in [5] and

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such that for larger values of $c$ and $e$, $\mathcal{R}(I)_{\Delta}$ is Koszul. More generally, the authors in [10] also show that for given any standard bigraded $k$-algebra $S$, one has $S_{\Delta}$ is Koszul for $c, e \gg 0$.

Motivated by the above results, we focus on giving explicit bounds for $c, e$ for diagonal subalgebras of certain classes of geometric residual intersections to be Koszul, or Cohen-Macaulay. More specifically, in this article, given positive integers $m \geq n$, we study an ideal $J = \langle z_1, \ldots, z_m \rangle + I_n(\phi)$ in the polynomial ring $S = k[x_1, \ldots, x_n, y_1, \ldots, y_p]$, where $\phi$ is an $m \times n$ matrix, with entries linear in $y$, such that $\tilde{z} = x\phi$. Such an ideal is shown to be a geometric residual intersection in [4] Lemma 4.7, when $\text{ht}(J) \geq m$, and $\text{ht}(I_n(\phi)) \geq m - n + 1$. In the same paper, Bruns, Kustin and Miller, also construct an $S$-free resolution of $S/J$ ([4] Theorem 3.6).

Our main tool is to show that this resolution gives a bigraded resolution of $S/J$ over $S$, when $S$ is bigraded with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. Using the construction given in [4], we first compute the bigraded shifts in Remark 2.2 and use this to compute the bigraded Betti numbers, $x$-regularity and $y$-regularity of $S/J$ over $S$ in Proposition 2.3. In order to understand the Cohen-Macaulay property, we use this to give a bound on the depth of $(S/J)_{\Delta}$ in Theorem 2.5 for all $\Delta$. Moreover, we also show that it is Koszul when $e \geq n/2$, in Theorem 2.7.

We give applications of the results in Section 2 to perfect ideals $I$ of height two, which are linearly presented in a polynomial ring $R = k[x_1, \ldots, x_n]$, which further satisfy $\mu(I_p) \leq \dim(R_p)$ for every $p \in V(I)$ with $\dim(R_p) \leq p - 1$. This property can also be reformulated as the ideal $I$ being of linear type in the punctured spectrum. In Theorem 3.3, we first show that all powers of such ideals have a linear resolution, generalising a result of Romer ([14 Corollary 5.11]). We also prove that the diagonal subalgebras of the Rees algebra of $I$ are always Cohen-Macaulay, and are Koszul when $e \geq n/2$. The Koszul nature of diagonal subalgebras of the Rees algebras of such ideals has not been studied before.

Section 1 contains the definitions, basic observations, and known properties of the main objects appearing in this paper. We end the paper with an example in Section 4 to understand the construction given in Section 2 explicitly, and to apply it to understand our results.

1. Preliminaries

**Notations.**

a) $c$ and $e$ denote positive integers, $\Delta = \{(ci, ei) \mid i \in \mathbb{Z}\}$ is the $(c, e)$-diagonal of $\mathbb{Z}^2$, and for a real number $\alpha$, we let $[\alpha]$ denote the least integer greater than or equal to $\alpha$.

b) $k$ denotes a field, $x = x_1, \ldots, x_n$, $y = y_1, \ldots, y_p$ denote sets of indeterminates over $k$, $R_x = k[x]$ and $R_y = k[y]$ are polynomial rings.

c) $R$ denotes a graded $k$-algebra, i.e., $R = \bigoplus_{i \geq 0} R_i$ is a graded ring with $R_0 = k$. Let $R_+ = \bigoplus_{i \geq 1} R_i$ be the unique homogeneous maximal ideal of $R$. We say that $R$ is standard graded if $R_+$ is generated by $R_1$.

d) Given a graded $R$-module $M$ and $j \in \mathbb{Z}$, the shifted module $M(j)$ is the graded $R$-module with $i$th graded component $M(j)_i = M_{i+j}$. In particular, $R(-j)$ is the graded free $R$-module of rank one, with generator in degree $j$. 

The Segre product

Definition 1.3. Let $c, e \in \mathbb{N}$, $S$ a bigraded $k$-algebra, $W$ a bigraded $S$-module, $A$ and $B$ be graded $k$-algebras, $M$ and $N$ graded $A$ and $B$ modules respectively.

a) The Segre product of $A$ and $B$ is defined as $A \boxtimes_k B = \bigoplus_{i \geq 0} (A_i \otimes B_i)$. The Segre product of $M$ and $N$ is the $A \boxtimes_k B$-module $M \boxtimes_k N = \bigoplus_{i \geq 0} (M_i \otimes N_i)$.

b) The $c$-th Veronese subring of $A$ is the graded $k$-algebra $A^{(c)} = \bigoplus_{i \geq 0} A_{ci}$.
c) The \((c, e)\)-diagonal subalgebra of \(S\) is the graded \(k\)-algebra \(S_\Delta = \bigoplus_{i \geq 0} S_{(ci, ei)}\), and the \((c, e)\)-diagonal module corresponding to \(W\) is the graded \(S_\Delta\)-module \(W_\Delta = \bigoplus_{i \geq 0} W_{(ci, ei)}\).

Remark 1.4. Let the notation be as in the above definition. 

a) \((\_)_\Delta\) is an exact functor from the category of bigraded \(S\)-modules to the category of graded \(S_\Delta\)-modules. However, for integers \(a, b\), note that \(S(-a, -b)_\Delta = \bigoplus_{i \geq 0} S_{(a+ci, b+ei)}\) need not be a free \(S_\Delta\)-module.

b) For a bihomogeneous ideal \(J\) in \(S\), \((S/J)_\Delta\) has a natural graded \(k\)-algebra structure by definition.

c) The \(k\)-algebra \(A \otimes_k B\) is naturally bigraded and the Segre product \(A \otimes_k B\) is the \((1, 1)\)-diagonal subalgebra of \(A \otimes_k B\). More generally, the \((c, e)\)-diagonal subalgebra of \(A \otimes_k B\) is the Segre product of the Veronese subalgebras \(A^{(c)}\) and \(B^{(e)}\).

d) Let \(S = k[x_1, \ldots, x_n, y_1, \ldots, y_p]\) with bigrading \(\deg(x_i) = (1, 0)\) and \(\deg(y_j) = (0, 1)\) for all \(i\) and \(j\). By [6] Lemmas 3.1 and 3.3 we have \(\dim (S(-a, -b)_\Delta) = p + n - 1\). Moreover, \(S(-a, -b)_\Delta\) is Cohen-Macaulay if and only if the following two conditions are satisfied: 

\[
\frac{b}{e} \leq \frac{a-n}{c} < \frac{b}{e} \quad \text{and} \quad \frac{b-p}{e} < \frac{a}{c} \quad \text{or} \quad \frac{b-p}{e} \leq \frac{a}{c} \leq \frac{b}{e} \leq \frac{a}{c}.
\]

Remark 1.5. The conclusion stated in [6] Proposition 3.4(ii)] is incorrect. The authors state that \(S(-a, -b)_\Delta\) is Cohen-Macaulay if and only if \(\frac{a-n}{c} < \frac{b}{e}\) and \(\frac{b-p}{e} < \frac{a}{c}\), and claim that it follows from Lemmas 3.1 and 3.3 in their paper. However, it is the second part of Remark 1.4 which follows from Lemmas 3.1 and 3.3 in [6], and hence is the correct version of Proposition 3.4(ii) in [6].

Residual Intersections and Rees Algebras. Let \(R\) be any Cohen-Macaulay local ring, \(I \subset R\) be an ideal and \(\underline{z} = z_1, \ldots, z_m \in I\) with \(\langle \underline{z} \rangle \neq I\).

Definition 1.6. We say \(J = \langle \underline{z} \rangle : I\) is an \(m\)-residual intersection of \(I\) if \(\text{ht}(J) \geq m \geq \text{ht}(I)\). Furthermore, if \(I_p = \langle \underline{z} \rangle p\) for all \(p \in V(I)\) with \(\text{ht}(p) \leq m\), then \(J\) is a geometric residual intersection of \(I\).

Example 1.7. Suppose \(I\) is generated by a regular sequence \(u_1, \ldots, u_r\) and \(v_1, \ldots, v_r\) is another regular sequence satisfying \(\langle v_1, \ldots, v_r \rangle \subseteq \langle u_1, \ldots, u_r \rangle\). Then \(\langle v_1, \ldots, v_r \rangle : \langle u_1, \ldots, u_r \rangle\) is a geometric residual intersection of \(\langle u_1, \ldots, u_r \rangle\) [11] Example 3.4.

Definition 1.8. Let \(I = \langle f_1, \ldots, f_p \rangle\) be a homogeneous ideal in \(R_x = k[x_1, \ldots, x_n]\). We say that \(\Phi\) is a presentation matrix of \(I\) if \((k[\underline{x}])^m \xrightarrow{\Phi} (k[\underline{x}])^p \to I \to 0\) is exact. The Rees algebra of \(I\) is the subalgebra of \(k[\underline{x}, t]\) defined as \(\mathcal{R}(I) = k[\underline{x}, f_1 t, \ldots, f_p t]\).

Remark 1.9. Let \(S = R_x \otimes_k R_y = k[x_1, \ldots, x_n, y_1, \ldots, y_p]\).

a) The ring \(S\) maps naturally onto \(\mathcal{R}(I)\) by \(x_i \mapsto x_i\) and \(y_j \mapsto f_j t\). The kernel \(K\) of this map, is called the defining ideal of the Rees algebra \(\mathcal{R}(I)\).
b) Let \([z_1 \cdots z_m] = [y_1 \cdots y_p] \cdot \Phi\), where \(\Phi\) is a \(p \times m\) presentation matrix of \(I\). Then we have \(\langle z_1, \ldots, z_m \rangle \subset \mathcal{K}\). If \(\mathcal{K} = \langle z_1, \ldots, z_m \rangle\), then we say that \(I\) is an ideal of linear type.

c) \([6, \text{Lemma 1.3}]\) \(\dim (R(I)_\Delta) = n\).

**Regularity and the Koszul property.**

**Definition 1.10.** Let \(R\) be a standard graded \(k\)-algebra, and \(M\) be a finitely generated graded \(R\)-module with \((i, j)\)-th graded Betti number \(\beta^R_{ij}(M) = \dim_k (\text{Tor}^R_i(M, k)_j)\).

a) The Castelnuovo-Mumford regularity of \(M\) over \(R\) is \(\text{reg}_R(M) = \sup \{j - i \mid \beta^R_{ij}(M) \neq 0\}\).

b) Let \(I\) be a homogeneous ideal in \(R\) generated in degree \(d\). Then \(I\) has a linear resolution if \(\text{reg}_R(I) = d\), i.e., if for all \(i, \beta^R_{i,j}(I) = 0\) for \(j \neq i + d\).

c) We say that \(R\) is a Koszul algebra if \(\text{reg}_R(k) = 0\), i.e., if for all \(i\), we have \(\beta^R_{i,j}(k) = 0\) for \(j \neq i\).

**Definition 1.11.** Let \(S = k[x, y]\) with \(\deg(x_i) = (1,0)\) and \(\deg(y_j) = (0,1)\) be a bigraded polynomial ring, and \(J \subset S\) a bigraded ideal. If \(\beta^S_{i,(a,b)}(S/J) = \dim_k (\text{Tor}^S_i(S/J,k)_{(a,b)})\) denote the bigraded Betti numbers of \(S/J\), we define the \(x\)-regularity and \(y\)-regularity of \(S/J\) to be:

\[
\text{reg}_x^S(S/J) = \sup \{a - i \mid \beta^S_{i,(a,b)}(S/J) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},
\]

\[
\text{reg}_y^S(S/J) = \sup \{b - i \mid \beta^S_{i,(a,b)}(S/J) \neq 0 \text{ for some } i, a \in \mathbb{Z}\}.
\]

**Remark 1.12.** Let \(R\) be a standard graded \(k\)-algebra, and \(S = R_x \otimes_k R_y\) be naturally bigraded by setting \(\deg(x_i) = (1,0)\) and \(\deg(y_j) = (0,1)\).

a) Given an exact sequence \(0 \rightarrow N_m \rightarrow \cdots \rightarrow N_2 \rightarrow N_1 \rightarrow N_0 \rightarrow M \rightarrow 0\) of graded \(R\)-modules, repeated application of the depth lemma (e.g., \([7, \text{Corollary 18.16}]\) ) and the regularity lemma (e.g., \([7, \text{Corollary 20.19}]\)) respectively yield the following:

\[\text{depth}_R(M) \geq \min\{\text{depth}_R(N_i) - i \mid 0 \leq i \leq m\}\] and \(\text{reg}_R(M) \leq \max\{\text{reg}_R(N_i) - i \mid 0 \leq i \leq m\}\).

b) \([3, \text{Theorem 4}]\) The Segre product of two Koszul algebras are Koszul. Moreover, Veronese subrings of Koszul algebras are Koszul. In particular, \(S_\Delta = (R_x)^{(e)} \otimes (R_y)^{(e)}\) is Koszul.

c) \([6, \text{Lemma 6.5}]\)

\[
\text{reg}_{(R_x)^{(e)} \otimes (R_y)^{(e)}}(R_x(-a)^{(e)} \otimes R_y(-b)^{(e)}) = \max \left\{ \text{reg}_{R_x^{(e)}}(R_x(-a)^{(e)}), \text{reg}_{R_y^{(e)}}(R_y(-b)^{(e)}) \right\}.
\]

d) By \([11, \text{Theorem 2.1}]\), we have \(\text{reg}_{R_x^{(e)}}(R_x(-a)^{(e)}) = \left\lceil \frac{a}{c} \right\rceil\).

Hence by (c), we get \(\text{reg}_{S_\Delta}(S(-a,-b)_\Delta) = \max \left\{ \left\lceil \frac{a}{c} \right\rceil, \left\lceil \frac{b}{c} \right\rceil \right\}\).

e) \([6, \text{Lemma 6.6}]\) Let \(I\) be a homogeneous ideal in \(R\) with \(\text{reg}_R(R/I) \leq 1\). If \(R\) is Koszul, then so is \(R/I\).

f) \([14, \text{Theorem 5.3}]\) Let \(I = \langle f_1, \ldots, f_p \rangle \subset R_x\) be a graded ideal generated in degree \(d\). Write the Rees algebra of \(I\), \(\mathcal{R}(I)\) as a quotient of \(S\) (by Remark \([14, \text{a}]\), \(\mathcal{R}(I) \cong S/\mathcal{K}\)). Then \(\text{reg}_{R_x}(I^s) \leq sd + \text{reg}_x^S(\mathcal{R}(I))\). In particular, if \(\text{reg}_x(\mathcal{R}(I)) = 0\), then \(I^s\) is linear for all \(s\).

2. **Geometric Residual Intersections and their Diagonal Subalgebras**

Let \(S = R_x[y_1, \ldots, y_p]\) be a polynomial ring over \(R_x\), and \(m = \langle x \rangle\). Define a bigrading on \(S\) by setting \(\deg(x_i) = (1,0), \deg(y_j) = (0,1)\) for \(1 \leq i \leq n\) and \(1 \leq j \leq p\).
As in the definition of the Eagon-Northcott complex, let \( m \geq n \), and \( \phi \) be an \( n \times m \) matrix with linear entries in \( R_g \), and let \( z = z_1, \ldots, z_m \in S \) be given by

\[
\begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \cdot \phi.
\]

We study the bigraded \( S \)-ideals of the form \( J = \langle z \rangle + I_n(\phi) \) where \( \text{grade}(I_n(\phi)) \geq m - n + 1 \). Suppose \( \text{ht}(J) \geq m \). Since \( J \subseteq \langle z \rangle : m \) (e.g., by Cramer’s Rule), we see that \( \langle z \rangle : m \) is an \( m \)-residual intersection of \( m \). If we further assume that \( \langle z \rangle : m \) is a geometric \( m \)-residual intersection of \( m \), then it is shown in the proof of [3, Theorem 4.8] that equality holds, i.e., \( J = \langle z \rangle : m \).

With \( J \) as above, the authors in [3] construct an \( S \)-free resolution of \( S/J \). We show that this is a bigraded resolution, and begin with a brief review of the construction in [3] below.

**Bigraded resolution of \( S/J \).** With notation as above, let \( \mathbb{K}_*(x; S) \) be the Koszul complex on the sequence \( x_1, \ldots, x_n \). As in the construction of the Eagon-Northcott complex, let \( \mathbb{K}_*(x; S)_d \) be the corresponding \( d \)-th graded component with differentials \( \psi_d \). Set \( K^b_a = \ker((\psi^{n-b+1}_{a+n-b})^*) \).

**Remark 2.1.** In [3], the authors have denoted the kernel \( K^b_a \) as \( K^b_a(G) \) and the map \((\psi^{n-b+1}_{a+n-b})^*\) by \( \eta^b_a \), hence the need for the notation of \( K^b_a \) presented above. In (cf. [3, Proposition 1.13]), the authors also show that each \( K^b_a \) is a free \( R \)-module with rank\( R(K^b_a) = \binom{n + a - 1}{a} \binom{n + a}{b} \).

Consider the following bi-complex \( \mathbb{B}_* \) of free \( S \)-modules:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K^{n-1}_{m-n} \otimes S & \rightarrow & \cdots & \rightarrow & K^{n-1}_1 \otimes S^{m}_{n+1} & \rightarrow & K^{n-1}_0 \otimes S^{m}_n & \rightarrow & S^n_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \phi^{n-1} \\
0 & \rightarrow & K^{n-2}_{m-n} \otimes S & \rightarrow & \cdots & \rightarrow & K^{n-2}_1 \otimes S^{m}_{n+1} & \rightarrow & K^{n-2}_0 \otimes S^{m}_n & \rightarrow & S^{m}_{n-2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \phi^{n-2} \\
& \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \rightarrow & K^{1}_{m-n} \otimes S & \rightarrow & \cdots & \rightarrow & K^{1}_1 \otimes S^{m}_{n+1} & \rightarrow & K^{1}_0 \otimes S^{m}_n & \rightarrow & S^{m}_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \phi^1 \\
0 & \rightarrow & K^{0}_{m-n} \otimes S & \rightarrow & \cdots & \rightarrow & K^{0}_1 \otimes S^{m}_{n+1} & \rightarrow & K^{0}_0 \otimes S^{m}_n & \rightarrow & S \\
\end{array}
\]

Under the given conditions, it was proved in [3, Theorem 3.6] that the total complex \( \mathbb{T}_* \) of the bicomplex \( \mathbb{B}_* \) is a free resolution of \( S/J \). Thus, we see that

\[
\mathbb{T}_* : \quad 0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_n \rightarrow Q_{n-1} \oplus P_{n-1} \rightarrow \cdots \rightarrow Q_1 \oplus P_1 \rightarrow Q_0 \rightarrow S/J \rightarrow 0
\]
is an $S$-free resolution of $S/J$, with $Q_i = S^{(m)}_{(i)}$ for $1 \leq i \leq n - 1$, and $P_i = \bigoplus K_a^b \otimes S^{(m)}_{(n+a)}$, $1 \leq i \leq m$, where the direct sum is over all $(a, b)$ such that
\[ a + b = i - 1, \quad 0 \leq a \leq m - n, \quad 0 \leq b \leq n - 1. \tag{5} \]

We show that this is a bigraded resolution of $S/J$ over $S$ by computing the bigraded shifts in $T_*$. In order to do this, we first compute the bigraded shifts in the bicomplex $B_*$.\[ \]

Remark 2.2. The bigraded shifts appearing in $B_*$ are as follows:

a) The column on the right of the above diagram is the truncation of the Koszul complex $K_*(z; S)$. Thus the maps on the last column are the differentials $\phi^i$, which shifts the bidegree by $(-1, -1)$.

b) Using [4, Lemma 2.5] the maps appearing in the other columns are induced by $K_*(x; S)$ and hence shifts the bidegree by $(0, -1)$.

c) The horizontal map between the last two columns are the augmentation map $\epsilon^b$ as explained in [4, Corollary 2.7]. Thus this augmentation map $K_0^a \otimes S^{(m)} \to S^{(n)}$ shifts the bidegree by $(0, -(n - b))$. In particular, when $b = 0$, since $K_0^0 = R$, the map $K_0^0 \otimes S^{(m)} \to S$ is induced by the $\epsilon$ map appearing in the Eagon-Northcott complex (Definition [1.1]).

d) Using [4, Lemma 2.6] the horizontal maps appearing in the other rows shift the bidegree by $(0, -1)$.

In particular, since $K_0^0 \cong (S_a)^*$, the maps in the last row of the bicomplex $B_*$ is induced by the maps in the Eagon-Northcott complex (Definition [1.1]).

The bigraded shifts of each free module appearing in the bicomplex $B_*$ are determined by the fact that each homomorphism has bidegree $(0, 0)$. For ease of reference, we rewrite the bicomplex $B_*$ with only the bigraded shifts:

\[
\begin{array}{cccccccc}
0 & \rightarrow & (-n - 1, m) & \rightarrow & \cdots & \rightarrow & (-n - 1, n) & \rightarrow & (-n + 1, -(n - 1)) \\
& & \downarrow & & & & \downarrow & & \phi^{n - 1} \\
& & \vdots & & & & \vdots & & \\
0 & \rightarrow & (1, -m) & \rightarrow & \cdots & \rightarrow & (1, n) & \rightarrow & (1, 1) \\
& & \downarrow & & & & \downarrow & & \phi^1 \\
0 & \rightarrow & (0, -m) & \rightarrow & \cdots & \rightarrow & (0, n) & \rightarrow & (0, 0).
\end{array}
\] \tag{6}

Thus, we get the following:

Proposition 2.3. The bigraded shifts appearing in the $S$-free resolution (4) of the residual intersection $S/J$ are as follows:

\[ Q_k = S(-k, -k)^{(n)}_{(k)}, \quad 0 \leq k \leq n - 1; \tag{7} \]

\[ P_i = \bigoplus_j S(-j, -(n + (i - 1) - j))^{(i,j)}, \quad 1 \leq i \leq m. \tag{8} \]
where \( \max\{0, i - (m - n + 1)\} \leq j \leq \min\{i - 1, n - 1\} \) and
\[
r(i, j) = \binom{n + i - 2j - 2}{i - 1 - j} \binom{n + i - j}{j} \frac{m}{n - i - 1 - j}.
\]
In particular, we have \( \text{reg}_x(S/J) \) is exact.

By Remark 1.12(a) and Remark 2.4, we have
\[
\text{depth}(S/J) \geq \text{depth}(S) - \text{depth}(J).
\]

In this subsection, we retain the setup of the previous one, and identify lower bounds on the depth of \((S/J)_\Delta\), which gives sufficient conditions for it to be Cohen-Macaulay.

**Remark 2.4.** The exactness of \( (\_)_\Delta \) shows that
\[
0 \to (F_m)_\Delta \to \cdots \to (F_1)_\Delta \to (F_0)_\Delta = S \to (S/J)_\Delta \to 0
\]
is exact.

Thus, by Remark 2.4, we see that \( \text{depth}_{S_\Delta}(S/J)_\Delta \geq \min\{\text{depth}_{S_\Delta}(F_i)_\Delta - i \mid 0 \leq i \leq m\} \), and \( \text{reg}_{S_\Delta}(S/J)_\Delta \leq \max\{\text{reg}_{S_\Delta}(F_i)_\Delta - i \mid 0 \leq i \leq m\} \).

**Depth and the Cohen-Macaulay Property of \((S/J)_\Delta\).**

In this subsection, we retain the setup of the previous one, and identify lower bounds on the depth of \((S/J)_\Delta\), which gives sufficient conditions for it to be Cohen-Macaulay.

**Theorem 2.5.** Suppose \( p > m \geq n \), then \( \text{depth}(S/J)_\Delta \geq p + n - (m + 1) \) for all \( \Delta \).

**Proof.** By Remark 2.4 and Remark 2.4, we have
\[
\text{depth}_{S_\Delta}(S/J)_\Delta \geq \min\{\text{depth}_{S_\Delta}(F_i)_\Delta - i \mid 0 \leq i \leq m\}.
\]
Since \( \dim(S(-a, -b)_\Delta) = p + n - 1 \) for any pair \((-a, -b)\) by Remark 2.4, we see that \( \dim((F_i)_\Delta) = p + n - 1 \). The bigraded shifts \((-a, -b)\), described in Proposition 2.3, satisfy \( 0 \leq a < n \), and \( 0 \leq b \leq m \). Hence they satisfy the conditions in Remark 2.4, and therefore, \( S(-a, -b)_\Delta \) is Cohen-Macaulay for these shifts. This implies that \((F_i)_\Delta \) is Cohen-Macaulay for all \( i \).

Thus, \( \text{depth}((F_i)_\Delta) = p + n - 1 \) for each \( i \), and hence we see that \( \text{depth}(S/J)_\Delta \geq p + n - (m + 1) \), proving the result.

**Corollary 2.6.** Suppose \( p > m \geq n \) and \( \dim(S/J)_\Delta \leq p + n - (m + 1) \), then \((S/J)_\Delta \) is Cohen-Macaulay.
Proof. Note that $S_{\Delta}$ is Koszul by Remark [1.12](b). Using Remark [1.12](c), it is enough to prove that $\text{reg}_{S_{\Delta}}(S/J)_{\Delta} \leq 1$. Since $\text{reg}_{S_{\Delta}}(S/J)_{\Delta} \leq \max\{\text{reg}_{S_{\Delta}}(F_i)_{\Delta} − i \mid 0 \leq i \leq m\}$, we compute $\text{reg}_{S_{\Delta}}((F_i)_{\Delta})$ for all $i$.

By Remark [1.12](d),

$$\text{reg}_{S_{\Delta}}(F_i)_{\Delta} = \max \left\{ \left[ \frac{a_i^{\max}}{c} \right], \left[ \frac{b_i^{\max}}{e} \right] \right\},$$

where $a_i^{\max} = \max\{a \mid S(-a,-b) \text{ is a direct summand of } F_i\}$ and $b_i^{\max}$ is defined similarly.

Therefore we can write

$$\text{reg}_{S_{\Delta}}(S/J)_{\Delta} \leq \max \left\{ \left[ \frac{a_i^{\max}}{c} \right] − i, \left[ \frac{b_i^{\max}}{e} \right] − i \mid 0 \leq i \leq m \right\}.$$ 

Notice that $a_i^{\max} = i$ for $i = 0, 1, \ldots, (n − 2)$, and $a_i^{\max} = (n − 1)$ for $i = (n − 1), \ldots, m$. Therefore

$$\left[ \frac{a_i^{\max}}{c} \right] − i \leq \max \left\{ \left[ \frac{i}{c} \right] − i \mid 1 \leq i \leq (n − 1) \right\} \leq 1,$$

for all $c \geq 1$.

We now claim that $\left[ \frac{b_i^{\max}}{e} \right] − i \leq 1$ for all $i$. Consider the case $m − n \leq n − 1$, the calculations in the case $m − n \geq n − 1$ are similar. From Proposition [2.3] we have $b_i^{\max} = n − 1 + i$ for $i = 1, \ldots, (m − n)$ and $b_i^{\max} = m$ for $i = (m − n + 1), \ldots, m$. Therefore,

$$\left[ \frac{b_i^{\max}}{e} \right] − i \leq \max \left\{ \left[ \frac{n − 1 + i}{e} \right] − i \mid 1 \leq i \leq (m − n + 1) \right\} = \left[ \frac{n}{e} \right] − 1 \leq 1,$$

since $e \geq \frac{n}{2}$. This proves the result. \hfill \Box

**Corollary 2.8.** If $n = 2$, then the diagonal subalgebra $(S/J)_{\Delta}$ is always Koszul.

### 3. Applications to perfect ideals of height two

A natural source of geometric residual intersections are the Rees algebras of certain classes of perfect ideals of height two, which are linearly presented in a polynomial ring. We use the following:

**Setup 3.1.** Let $I$ be a homogeneous perfect ideal of height two in a polynomial ring $R_x = k[x_1, \ldots, x_n]$ with a presentation matrix $\Phi$ (Definition [1.8]). We assume that $I$ satisfies the following properties:

1. $\mu(I) = p > n$.
2. $I$ satisfies $\mu(I_p) \leq \text{ht } p$ for every $p \in V(I) \setminus \{m\}$. Equivalently, $\text{ht}(I_{p−i}(\Phi)) > i$ for $0 \leq i < n$ (e.g., [7] Proposition 20.6).
3. The presentation matrix $\Phi$ of $I$ is linear in the entries of $R_x$.

Since $I$ is a perfect ideal of height two, the presentation matrix $\Phi$ of $I$ is of size $p \times (p − 1)$ by the Hilbert-Burch theorem (e.g., [7] Theorem 20.15). Let $S = R_x[y_1, \ldots, y_p]$ with bigrading $\text{deg } x_i = (1,0)$, and $\text{deg } y_j = (0,1)$. With notation as in Remark [1.9](b), let $m = p − 1$, and $[z_1 \cdots z_{p−1}] = [y_1 \cdots y_p] \Phi$. We can rewrite this as $[z_1 \cdots z_{p−1}] = [x_1 \cdots x_n] \phi$ where $\phi$ is a $n \times (p−1)$ matrix with entries in $R_y$. Since $\Phi$ is linear in $k[x_1, \ldots, x_n]$, we have $\phi$ is linear in $R_y$. 


Remark 3.2. With the notation as in Setup 3.1 and \( \phi \) as above, we get the following from [13] Theorems 1.2, 1.3 (and their proofs): The Rees algebra of \( I \), \( \mathcal{R}(I) \cong S/J \) where \( J = \langle z_1, \ldots, z_{p-1} \rangle + I_n(\phi) \). Furthermore, \( \text{ht}(I_n(\phi)) \geq m - n + 1 \), \( J = \langle z_1, \ldots, z_{p-1} \rangle : \langle x \rangle \), and in particular, \( J \) is a geometric \( m \)-residual intersection of \( \langle x \rangle \).

Thus Proposition 2.3 gives a free resolution of \( \mathcal{R}(I) \) over \( S \). We can now apply the results from Section 2 to \( \mathcal{R}(I) \).

Theorem 3.3. With the notation as in Setup 3.1, we have the following:

a) The ideal \( I \) has a linear resolution for all \( s \in \mathbb{N} \).

b) The diagonal subalgebra \( (\mathcal{R}(I))_\Delta \) of \( \mathcal{R}(I) \) is Cohen-Macaulay for all \( \Delta = (c, e) \).

c) The diagonal subalgebra \( (\mathcal{R}(I))_\Delta \) of \( \mathcal{R}(I) \) is Koszul for all \( \Delta = (c, e) \) with \( c \geq 1 \) and \( e \geq \frac{n}{2} \).

**Proof.**

a) This follows from Remark 1.12(f), since \( \text{reg}_s(\mathcal{R}(I)) = 0 \) by Proposition 2.3.

b) Recall that \( m = p - 1 \) and, by [4, Lemma 1.2,1.3], we have \( \dim \mathcal{R}(I)_\Delta = n \). Thus, by Theorem 2.5 and Corollary 2.6, we have \( \text{depth}(\mathcal{R}(I)_\Delta) = \dim(\mathcal{R}(I)_\Delta) = n \) for all \( \Delta \).

c) This is follows immediately from Theorem 2.7. \( \square \)

Remark 3.4.

a) The special case of Theorem 3.3(a) when \( p = n + 1 \) is proved in [14, Corollary 5.11].

b) In [9, Corollary 3.13], the authors show that the \( \mathcal{R}(I)_\Delta \) is Cohen-Macaulay for large \( \Delta \).

4. Example

Let \( I \) be a in \( k[x_1, x_2, 3] \) whose presentation matrix is

\[
\Phi = \begin{bmatrix}
x_1 & 0 & 0 & 0 \\
x_2 & x_1 & 0 & 0 \\
x_3 & x_2 & x_1 & 0 \\
0 & x_3 & x_2 & x_1 \\
0 & 0 & x_3 & x_2 \\
\end{bmatrix}
\]

Notice that \( \text{ht}(I_4(\Phi)) \geq 2 \) (as \( x_1^4, x_2^4 - 3x_1^3x_2^3 \in I_4(\Phi) \)). By the Hilbert-Burch theorem (e.g., [7, Theorem 20.15]), \( I \) is a perfect ideal of height two in \( k[x_1, x_2, x_3] \). One can easily check (2) of Setup 3.1 by noticing that \( \text{ht}(I_4(\Phi)) \geq 2 \), \( \text{ht}(I_3(\Phi)) = 3 \). Since \( \Phi \) is linear in \( k[x_1, \ldots, x_n] \), all the hypothesis of Setup 3.1 is satisfied.

Now let \( S = k[y_1, \ldots, y_5, x_1, x_2, x_3] \) with bigrading \( \deg y_i = (0,1) \) and \( \deg x_i = (1,0) \). As mentioned in Remark 3.2, the Rees algebra \( \mathcal{R}(I) \cong S/J \) where \( J = \langle z_1, \ldots, z_4 \rangle + I_3(\phi) \) where \( [z_1 \cdots z_4] = [y_1 \cdots y_5] \cdot \Phi = [x_1 x_2 x_3] \cdot \phi \) and \( \phi \) can be obtained as follows

\[
[z_1 \cdots z_4] = [y_1 \cdots y_5] \cdot \Phi \\
= [x_1y_2 + x_2y_2 + x_3y_3 x_1y_2 + x_2y_3 + x_3y_4 x_1y_3 + x_2y_4 + x_3y_5 x_1y_4 + x_2y_5] \\
= [x_1 x_2 x_3] \begin{bmatrix}
y_1 & y_2 & y_3 & y_4 \\
y_2 & y_3 & y_4 & y_5 \\
y_3 & y_4 & y_5 & 0 \\
\end{bmatrix} = [x_1 x_2 x_3] \cdot \phi
\]
The discussion in Remark 3.2 shows that $J$ is a geometric residual intersection with $ht \phi \geq 2$. Using Proposition 2.3 we can compute the complete resolution of $J$ as follows:

$$0 \to S(-2, -4)^6 \to S(-2, -3)^8 \oplus S(-1, -4)^{12} \to S(-1, -3)^{12} \oplus S(0, -4)^3 \oplus S(-2, -2)^6 \to S(0, -3)^4 \oplus S(-1, -1)^4 \to S.$$  

Using the techniques in the proof of Theorem 2.7 we see that $reg_{S}(S/J)_{\Delta} \leq \max \{ \left\lceil \frac{1}{c} \right\rceil - 1, \left\lceil \frac{3}{e} \right\rceil - 1, \left\lceil \frac{2}{c} \right\rceil - 2, \left\lceil \frac{4}{e} \right\rceil - 2, \left\lceil \frac{2}{c} \right\rceil - 3, \left\lceil \frac{4}{e} \right\rceil - 3, \left\lceil \frac{2}{c} \right\rceil - 4, \left\lceil \frac{4}{e} \right\rceil - 4 \}.$$

By Theorem 3.3 we have that $R(I)_{\Delta} \cong (S/J)_{\Delta}$ is Cohen-Macaulay for all $\Delta$, and is Koszul for $e \geq \frac{3}{2}$.

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