Large butterfly Cayley graphs and digraphs

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Abstract

We present families of large undirected and directed Cayley graphs whose construction is related to butterfly networks. One approach yields, for every large \( k \) and for values of \( d \) taken from a large interval, the largest known Cayley graphs and digraphs of diameter \( k \) and degree \( d \). Another method yields, for sufficiently large \( k \) and infinitely many values of \( d \), Cayley graphs and digraphs of diameter \( k \) and degree \( d \) whose order is exponentially larger in \( k \) than any previously constructed. In the directed case, these are within a linear factor in \( k \) of the Moore bound.

1 Introduction

The goal of the degree–diameter problem is to determine the largest possible order of a graph or digraph, perhaps restricted to some special class, with given maximum (out)degree and diameter. For an overview of progress on a wide variety of approaches to this problem, see the survey by Miller & Širáň [6].

Our concern here is with large Cayley graphs and digraphs. Recall that, for a group \( G \) and a unit-free generating subset \( S \) of \( G \), the Cayley digraph of \( G \) generated by \( S \) has vertex set \( G \) and a directed edge from \( g \) to \( gs \) for all \( g \in G \) and \( s \in S \). If \( S \) is symmetric, i.e. \( S = S^{-1} \), then the corresponding undirected simple graph is the Cayley graph of \( G \) generated by \( S \). The Cayley (di)graph is thus regular of (out)degree \( |S| \) and vertex-transitive.

We are interested in graphs and digraphs of degree \( d \) and diameter \( k \), for arbitrary large \( k \) and varying \( d \). If a construction yields graphs of order \( n_{d,k} \), we say that it has asymptotic order \( f(d,k) \) if, for fixed \( k \),

\[
\lim_{d \to \infty} \frac{n_{d,k}}{f(d,k)} = 1.
\]

No graph or digraph can be larger than the corresponding Moore bound. For undirected graphs, this bound is \( M_{d,k} = 1 + \frac{d}{d-2}((d-1)^k - 1) \) if \( d > 2 \). In the directed case, it is \( DM_{d,k} = \frac{1}{d-1}(d^{k+1} - 1) \) if \( d > 1 \). In both cases, the Moore bound has asymptotic order \( d^k \).

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Previously, for arbitrary degree and diameter, the largest known directed Cayley graphs were obtained by Vetrik [7] and Abas & Vetrik [1], whose constructions have asymptotic order \( k(\frac{d}{2})^k \) for even \( k \), and \( 2k(\frac{d}{2})^k \) for odd \( k \). Our construction yields Cayley digraphs whose order is asymptotically \( kd^{k-1} \). For fixed diameter \( k \geq 8 \), these digraphs are larger than those in [7] and [1] for every value of \( d \) in a large interval. We also construct, for fixed \( k \) and infinitely many values of \( d \), Cayley digraphs whose asymptotic order is \( \frac{d^k}{e^d} \), a factor of \( \frac{2^{k-1}}{e^d} \) larger than those of Abas & Vetrik, and within a linear factor in \( k \) of the Moore bound.

The undirected case is similar. Previously, the largest known Cayley graphs were obtained by Macbeth, Šiagiová, Širáň & Vetrik [5], whose construction has asymptotic order \( k(\frac{d}{4})^k \). For \( d - k \neq 3 \) (mod 4), we construct Cayley graphs whose order is asymptotically \( k(\frac{d}{4})^{k-1} \). For sufficiently large diameter \( k \), these graphs are larger than those in [5] for every suitable value of \( d \) in a large interval. We also construct, for given \( k \) and infinitely many values of \( d \), Cayley graphs whose asymptotic order is \( \frac{1}{e^k} (\frac{d}{2})^k \), a factor of \( \frac{1}{e^{k/2}} (\frac{3}{2})^k \) larger than those in [5].

Our constructions are based on a two-parameter family of groups. For \( t \geq 2 \), let \( \mathbb{Z}_t = \mathbb{Z}/t\mathbb{Z} \) be the additive group of integers modulo \( t \), and for \( r \geq 2 \), let \( \mathbb{Z}_t^r \) denote the product \( \mathbb{Z}_t \times \ldots \times \mathbb{Z}_t \), where \( \mathbb{Z}_t \) occurs \( r \) times, considered as an additive group of vectors. Let \( \alpha \) be the automorphism of \( \mathbb{Z}_t^r \), defined by \( \alpha(v_0, \ldots, v_{r-1}) = (v_{r-1}, v_0, \ldots, v_{r-2}) \), that cyclically shifts coordinates rightwards by one, and consider the semidirect product \( G = \mathbb{Z}_t^r \times \mathbb{Z}_r \), of order \( rt^r \), with the group operation given by \( (u, s) \cdot (v, s') = (u + \alpha^s(v), s + s') \), for \( u, v \in \mathbb{Z}_t^r \) and \( s, s' \in \mathbb{Z}_r \). We write elements of \( G \) in the form \( (v_0, \ldots, v_{r-1}; s) \), where each \( v_i \in \mathbb{Z}_t \) and \( s \in \mathbb{Z}_r \). Using this notation, the group operation is

\[
\begin{align*}
(u_0, \ldots, u_{r-1}; s) \cdot (v_0, \ldots, v_{r-1}; s') &= (u_0 + v_{r-s}, \ldots, u_{s-1} + v_{r-1}, u_s + v_0, \ldots, u_{r-1} + v_{r-1-s}; s + s'),
\end{align*}
\]

arithmetic in the subscripts being performed modulo \( r \). The group \( G \) is used to create all our Cayley graphs and digraphs.

The Cayley digraph generated by elements of \( G \) of the form \( (\alpha, 0, \ldots, 0; 1) \), \( \alpha \in \mathbb{Z}_t \) is isomorphic to the base-\( t \) order-\( r \) (wrapped) butterfly network, \( B_t(r) \), so called because it is composed of \( rt^{r-1} \) edge-disjoint \( t \)-butterflies (copies of the complete bipartite graph \( K_{1,t} \)); see [2, Figure 2]. Butterfly networks are closely related to the de Bruijn graphs [3], the directed base-\( t \) order-\( r \) de Bruijn graph being a coset graph of \( B_t(r) \) [2, Theorem 4.4].

Cayley graphs and digraphs of \( G \) were used previously by Macbeth, Šiagiová, Širáň & Vetrik [5] and Vetrik [7] in the constructions mentioned above, though in neither case is the connection to the butterfly networks made explicit. Each of our results is a consequence of choosing an appropriate set of generators for \( G \). We make use of two distinct constructions.
2 The first construction

We present the directed case first, since it is slightly simpler.

**Theorem 1.** For any \( k \geq 4 \) and \( d \geq k - 1 \), there exist Cayley digraphs that have diameter \( k \), outdegree \( d \), and order \((k-1)(d-k+3)^{k-1}\).

**Proof.** Let \( r = k - 1 \) and \( t = d - k + 3 \), and let the underlying group of the Cayley digraph be \( G = \mathbb{Z}_r^* \times \mathbb{Z}_t \). The order of \( G \) is \( rt^r = (k-1)(d-k+3)^{k-1} \).

As generators for the Cayley digraph we use the \( t \) shift and add elements \((a,0,\ldots,0;1)\), for each \( a \in \mathbb{Z}_t \), together with the remaining \( r - 2 \) nonzero cyclic shift elements \((0,\ldots,0;s)\), for \( 2 \leq s \leq r - 1 \). Thus the digraph has outdegree \( t + r - 2 = d \).

It also has diameter \( r + 1 = k \). Every element is the product of \( r \) shift and add operations (establishing the vector) and possibly a single cyclic shift (to establish the final shift value if it is nonzero). On the other hand, if \( s \neq 0 \) then \((1,\ldots,1;s)\) cannot be obtained as a product of fewer than \( k \) generators.

Clearly, the butterfly network \( B_4(r) \) is a subdigraph of the Cayley digraph of Theorem 1. The additional edges in our construction, corresponding to the cyclic shift elements, consist of \( t^r \) vertex-disjoint copies of the complete digraph on \( r \) vertices with a directed \( r \)-cycle removed.

Vetrík [7] presents, for any \( k \geq 3 \) and \( d \geq 4 \), a family of Cayley digraphs of diameter \( k \), degree \( d \), and order \( k \lfloor \frac{d}{2} \rfloor^k \). For odd diameters, Abas & Vetrík [1] improve this result by a factor of two, constructing Cayley digraphs of diameter at most \( k \) and degree \( d \) of order \( 2k \lfloor \frac{d}{2} \rfloor^k \). Clearly, for large enough \( d \), these digraphs are bigger than those of Theorem 1. However, for any given diameter \( k \geq 8 \), it can be confirmed (using a computer algebra system, or otherwise) that the digraphs of Theorem 1 are larger than those of Vetrík and Abas & Vetrík if

\[
2k + 2 \ln k < d < 2^{k-1}(1 - \frac{1}{k}) - k^2.
\]

For specific values of the degree, we can do much better. If we set \( d = k^2 - 3k \), then the digraphs of Theorem 1 have orders at least \( 2^{d-3} \), within a linear factor of the Moore bound, and exceeding those of Abas & Vetrík by a factor of at least \( 2^{k-1}/ek^2 \), which exceeds 1 for \( k \geq 9 \).

For the undirected case, we simply add elements to the generating set to make it symmetric.

**Theorem 2.** For any \( k \geq 5 \) and \( d \geq k \) such that \( d - k \equiv 3 \pmod{4} \), there exist Cayley graphs that have diameter \( k \), degree \( d \), and order \((k-1)(\lfloor \frac{d-k}{2} \rfloor + 2)^{k-1}\).

**Proof.** Let \( r = k - 1 \) and \( t = \lfloor \frac{d-k}{2} \rfloor + 2 \), and let \( G = \mathbb{Z}_r^* \times \mathbb{Z}_t \). As generators for the Cayley graph of \( G \) we use the \( t \) elements \((a,0,\ldots,0;1)\), along with their inverses \((0,\ldots,0,-a;-1)\), and the remaining \( r - 3 \) nonzero elements \((0,\ldots,0;s)\) for \( 2 \leq s \leq r - 2 \). In addition, if \( d - k \equiv 1 \pmod{4} \), in which case \( t \) is even, then the involution \((0,\ldots,0,\frac{1}{2},0)\) is also included as a generator.
Thus the graph has degree $2t + r - 3 + (d - k \mod 2) = d$. As in the directed case, it has diameter $r + 1 = k$. Every element is the product of $k - 1$ shift and add operations and possibly a single cyclic shift. On the other hand, if $s \notin \{-1, 0, 1\}$ then $(1, \ldots, 1; s)$ cannot be obtained as a product of fewer than $k$ generators, and $G$ has such an element since $r \geq 4$.

Macbeth, Šiagiová, Širáň & Vetrík [5] present, for any $k \geq 3$ and $d \geq 5$, a family of Cayley graphs with diameter at most $k$, degree $d$, and order no greater than $k(\frac{d+1}{2})^k$. Their constructions also use the group $G$, with a different generating set. For large enough $d$, these graphs are bigger than those of Theorem 2. However, for $k \geq 27$, the graphs of Theorem 2 are larger than those of Macbeth, Šiagiová, Širáň & Vetrík for any $d - k \neq 3 \mod 4$ satisfying

$$3k + 6 \ln k < d < 2\left(\frac{3}{2}\right)^k \left(1 - \frac{1}{k}\right) - k^2.$$ 

For specific values of the degree, we can do much better. If we set $d = k^2 - 2k$, then the graphs of Theorem 2 have orders exceeding those in [5] by a factor of at least $\frac{2}{\ln^2} \left(\frac{3}{2}\right)^k$, which exceeds 1 for $k \geq 14$.

### 3 The second construction

In our second construction, we conceive of the vectors of length $r$ as being partitioned into $k - 1$ long blocks, each of length $\ell$, and a single short block, of length $m$.

Again, the directed case is presented first, since it is simpler.

**Theorem 3.** For any $k, \ell, t \geq 2$ and positive $m < \ell$, there exist Cayley digraphs that have diameter $k$, outdegree $t^{\ell} + (r - 1)t^{m-1}$, and order $rt^r$, where $r = (k - 1)\ell + m$.

**Proof.** As before, let $G = \mathbb{Z}_t^\ell \times \mathbb{Z}_r$, of order $rt^r$. As generators for the Cayley digraph, we use the $t^\ell$ long elements $(a_1, \ldots, a_\ell, 0, \ldots, 0; \ell)$, $a_i \in \mathbb{Z}_t$, together with the additional $(r - 1)t^{m-1} - 1$ nonzero short elements $(a_1, \ldots, a_m, 0, \ldots, 0; s)$, $a_i \in \mathbb{Z}_t$, $s \neq \ell$. Thus the digraph has outdegree $t^\ell + (r - 1)t^{m-1}$. Long elements shift by $\ell$ and modify a long block; short elements shift arbitrarily and modify a short block.

The digraph has diameter $k$. Every element is the product of a single short element (establishing $m$ components of the vector and guaranteeing the final shift value) and $k - 1$ long elements (establishing the remaining $(k - 1)\ell = r - m$ components of the vector). On the other hand, $(1, \ldots, 1; 0)$ cannot be obtained as a product of fewer than $k$ generators.

The Cayley digraph of Theorem 3 contains both of the butterfly networks $B_{t^\ell}(r)$ and $B_{t^m}(r)$ as subdigraphs. Its edges can be partitioned into $rt^{r-\ell}$ copies of the $t^{\ell}$-butterfly, from the long elements, $r(r - 2)t^{r-m}$ copies of the $t^m$-butterfly, from the short elements that have nonzero shift, and a collection of directed cycles from the short elements with zero shift.

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1 The graphs in [5] are slightly larger than those of Macbeth, Šiagiová & Širáň [4], whose order is at most $k(\frac{d+1}{2})^k - k$.  

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Given $k$, $\ell$ and $t$, for judicious choice of $m$, these digraphs are larger than those of Abas & Vetrik [1]. For example, if we let $t = 2$, then for all $k \geq 31$ and sufficiently large $\ell$, the order of our digraphs is greater than that of those in [1] if

$$\ell - k - \log_2 \ell + 2 < m < \ell - \log_2 k\ell - \frac{2}{k} (\log_2 k + 2).$$

If $m$ is chosen optimally, we can do much better than that.

**Corollary 4.** For any $k \geq 3$, there are arbitrarily large values of $d$ for which there exist Cayley digraphs that have diameter $k$, outdegree $d$, and order at least $\frac{1}{k}(\frac{k}{k+2}(d + 1))^k$.

**Proof.** We use the construction of Theorem 3. Let $t = 2$, and let $\ell$ be any sufficiently large positive integer such that $\log_2 k^2 \ell \leq \frac{3}{4} \ell$. Let $r = [k\ell - \log_2 k^2 \ell]$, and $m = r - (k - 1)\ell$, so $r = (k - 1)\ell + m$. Note that $0 < m < \ell$.

The digraph has diameter $k$ and order $r 2^\ell$, which (rounding $r$ down) is at least

$$n_0 = (k\ell - \log_2 k^2 \ell) 2^{k\ell - \log_2 k^2 \ell} = \left(1 - \frac{\log_2 k^2 \ell}{k\ell}\right) 2^{k\ell}.$$

Its degree is $d = 2\ell + (r - 1)2^m - 1$, which (substituting for $m$ and rounding $r$ up) is less than

$$d^+ = 2\ell + (k\ell - \log_2 k^2 \ell) 2^{k\ell - \log_2 k^2 \ell} 2^{k\ell + 1 - (k - 1)\ell} - 1 = \left(1 + \frac{2}{k} - \frac{2\log_2 k^2 \ell}{k\ell}\right) 2^{k\ell} - 1.$$

Let $\theta = \frac{\log_2 k^2 \ell}{k\ell}$. Note that the condition on $\ell$ implies that $\theta \leq \frac{3}{4k} \leq \frac{1}{4}$, since $k \geq 3$.

Now,

$$kn_0 \left(\frac{k}{k+2}(d^+ + 1)\right)^{-k} = (1 - \theta) \left(1 + \frac{2\theta}{k + 2 - 2\theta}\right)^k > (1 - \theta) \left(1 + \frac{2k\theta}{k + 2 - 2\theta}\right),$$

which is at least 1 if $k \geq 2$ and $0 \leq \theta \leq \frac{k - 2}{2k - 2}$. Since $k \geq 3$ and $\theta \leq \frac{1}{4}$, the result follows. □

These digraphs have asymptotic order exceeding $\frac{d^k}{e^{rt^*}}$, a factor of $\frac{2^{k-1}}{e^{rt^*}}$ larger than those of Abas & Vetřík, and within a linear factor in $k$ of the Moore bound.

It is worth briefly explaining the choice of values for $t$ and $r$ in the proof of Corollary 4. Suppose we fix $t$ and $r$ (and hence the order $rt^*$), and also fix the diameter $k$. What is the optimal choice for $\ell$, that minimises the degree $t\ell + (r - 1)tr - (k - 1)\ell - 1$? Differentiating with respect to $\ell$ and equating to zero yields $\ell = \frac{1}{r}(r + \log_4 (k - 1)(r - 1))$. Solving for $r$ then gives

$$r = \frac{1}{\ln t} W\left(\frac{t^{k-1} \ln t}{k-1}\right) + 1,$$

where $W$ is the Lambert $W$ function, defined implicitly by $W(z)e^{W(z)} = z$. Asymptotically, $W(z) = \ln z - \ln \ln z + o(1)$. Applying this approximation for $W$ then yields $r \approx k\ell - \log_4 k^2 \ell$. 

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Using this value for \( r \) results in a digraph whose order is asymptotically at least \( \frac{1}{k} \left( \frac{k}{k+t} (d+1) \right)^k \).

Setting \( t = 2 \) makes this maximal.

The results in the undirected case are similar. As before, we just add elements to the generating set to make it symmetric.

**Theorem 5.** For any \( k, \ell, t \geq 2 \) and positive \( m < \ell \), there exist Cayley graphs that have diameter \( k \), degree \( 2t^\ell + (2r-3)t^m - r \), and order \( rt^r \), where \( r = (k-1)\ell + m \).

**Proof.** Let \( G = \mathbb{Z}_t^r \times \mathbb{Z}_r \). As generators for the Cayley graph of \( G \) with these parameters, we use:

- the \( t^\ell \) long elements \( (a_1, \ldots, a_{t^\ell}, 0, \ldots, 0; a) \), \( a \in \mathbb{Z}_t \)
- their \( t^\ell \) inverses \( (0, \ldots, 0, a_1, \ldots, a_{t^\ell}; -a) \)
- the \( (r-2)(t^m - 1) \) short elements \( (a_1, \ldots, a_m, 0, \ldots, 0; s) \), \( a_i \in \mathbb{Z}_t \) not all zero, \( s \notin \{0, \ell\} \)
- their \( (r-2)(t^m - 1) \) inverses \( (0, \ldots, 0, a_1, \ldots, a_m, 0, \ldots, 0; -s) \)
- the \( t^m - 1 \) nonzero short elements \( (a_1, \ldots, a_m, 0, \ldots, 0; 0) \); this set is symmetric
- the \( r-3 \) short elements \( (0, \ldots, 0; s) \), \( s \notin \{0, \pm \ell\} \); this set is also symmetric

Thus the graph has degree \( 2t^\ell + (2r-3)t^m - r \). As in the directed case, it has order \( rt^r \) and diameter \( k \).

Given \( k, \ell, t \), for appropriate choice of \( m \), these graphs are larger than those of Macbeth, Šiagiová, Širáň & Vetrík [5]. For example, if we let \( t = 2 \), then for all \( k \geq 69 \) and sufficiently large \( \ell \), the order of our graphs is greater than that of those in [5] if

\[
\ell + k - \log_2 3^k \ell + 1 < m < \ell - \log_2 k\ell - \frac{1}{2} (\log_2 k + 2) - 1.
\]

If \( m \) is chosen optimally, we have the following.

**Corollary 6.** For any \( k \geq 3 \), there are arbitrarily large values of \( d \) for which there exist Cayley graphs that have diameter \( k \), degree \( d \), and order at least

\[
\frac{1}{k} \left( \frac{k}{2k+4} \left( d + k \log_2 \frac{d}{2} - \log_2 \log_2 d - \log_2 8k^2 \right) \right)^k.
\]

**Proof.** We use the construction of Theorem 5. As in the proof of Corollary 4, let \( t = 2 \), and let \( \ell \) be any sufficiently large positive integer such that \( \log_2 k^2 \ell \leq \frac{3}{4} \ell \). Let \( r = \lceil k\ell - \log_2 k^2 \ell \rceil \), and \( m = r - (k-1)\ell \), so \( r = (k-1)\ell + m \).

The graph has diameter \( k \) and order \( r2^r \), which is at least

\[
n_0 = (k\ell - \log_2 k^2 \ell)2^{k\ell - \log_2 k^2 \ell} = \left( \frac{1}{k} - \frac{\log_2 k^2 \ell}{k^2 \ell} \right) 2^{k\ell}.
\]
Its degree is $d = 2^{\ell+1} + (2r - 3)2^m - r$, which (substituting for $m$ and rounding $r$ up in the second term) is less than

$$2^{\ell+1} + (2k\ell - 2\log_2 k^2\ell - 1)2^{k\ell-\log_2 k^2\ell+1-(k-1)\ell} - r = \left(2 + \frac{4}{k} - \frac{1 + 4\log_2 k^2\ell}{k^2\ell}\right)2^{\ell} - r.$$

Thus, $\frac{1}{2}(d + r)$ is less than $q = \left(1 + \frac{2}{k} - \frac{2\log_2 k^2\ell}{k^2\ell}\right)2^{\ell}$, and by the argument in the proof of Corollary 4 (with $q = d^* + 1$), we know that $kn_0 > \left(\frac{ka}{k+2}\right)^k > \left(\frac{k}{2k+4}(d + r)\right)^k$.

It remains to establish the appropriate lower bound for $r$.

Now, $kn_0 < 2^{k\ell}$ and $q > \frac{d}{2}$, so $2^{\ell} > \frac{kd}{2k+4}$ and thus $\ell > \log_2 \frac{kd}{2k+4} = \log_2 \frac{d}{2} - \log_2 \left(1 + \frac{2}{k}\right)$.

Since $(1 + \frac{2}{k})^k < e^2 < 2^3$, we have $\log_2 \left(1 + \frac{2}{k}\right) < \frac{3}{k}$ and thus $\ell > \log_2 \frac{d}{2} - \frac{3}{k}$.

Now, $r \geq k\ell - \log_2 k^2\ell$, so

$$r > k\log_2 \frac{d}{2} - 3 - \log_2 k^2 - \log_2 \left(\log_2 \frac{d}{2} - \frac{3}{k}\right),$$

which is greater than $k\log_2 \frac{d}{2} - \log_2 d - \log_2 8k^2$, as required. \hfill $\Box$

These graphs have asymptotic order exceeding $\frac{1}{e_k}\left(\frac{d}{2}\right)^k$, a factor of $\frac{1}{e_k}\left(\frac{d}{2}\right)^k$ larger than those of Macbeth, Šiagiová, Širáň & Vetrík.

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