Miller Spaces and Spherical Resolvability of Finite Complexes

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Abstract

We show that if $K$ is a nilpotent finite complex, then $\Omega K$ can be built from spheres using fibrations and homotopy (inverse) limits. This is applied to show that if map$_*(X,S^n)$ is weakly contractible for all $n$, then map$_*(X,K)$ is weakly contractible for any nilpotent finite complex $K$.

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Discussion of Results

A Miller space is a CW complex $X$ with the property that the space of pointed maps from $X$ to $K$ is weakly contractible for every nilpotent finite complex $K$, written map$_*(X,K) \sim *$. They are named for Haynes Miller, who proved in [13] that the spaces $B\mathbb{Z}/p$ are all Miller spaces; in fact, he proved that map$_*(B\mathbb{Z}/p,K)$ is weakly contractible for every finite dimensional CW complex $K$.

In the stable category, one can define a Miller spectrum by requiring that the mapping spectrum $F(X,K)$ is contractible for every finite spectrum $K$. Since cofibrations and fibrations are the same in the stable category, a finite spectrum $K$ with $m$ cells is the fiber in a fibration $K \to L \to S^n$.
in which $L$ has only $m - 1$ cells; in the terminology of [3, 12], this means that $K$ is \textit{spherically resolvable with weight $m$}. An easy induction shows that $X$ is a Miller spectrum if and only if $F(X, S^n) \simeq *$ for every $n$.

Our goal is to prove the following unstable analog of this observation: if $\text{map}_*(X, S^n) \sim *$, for all $n$, then $X$ is a Miller space. The proof of the stable version is not available to us because cofibrations are not fibrations, unstably. To prove our result, it is necessary to determine the extent to which a finite complex can be constructed from spheres in a more general way, i.e., by arbitrary homotopy (inverse) limits and extensions by fibrations.

To be more precise, we require some new terminology. We call a nonempty class $\mathcal{R}$ of spaces a \textbf{resolving class} if it is closed under weak equivalences and pointed homotopy (inverse) limits (all spaces, maps and homotopy limits will be pointed). It is a \textbf{strong resolving class} if it is further closed under extensions by fibrations, i.e., if whenever $F \to E \to B$ is a fibration with $F, B \in \mathcal{R}$, then $E \in \mathcal{R}$. Resolving classes are dual to closed classes as defined in [3] and [8, p. 45].

Notice that every resolving class $\mathcal{R}$ contains the one-point space $*$ (cf. [8, p. 47]). From this, it follows that if $F \to E \to B$ is a fibration with $E, B \in \mathcal{R}$, then $F \in \mathcal{R}$. Similarly, if $A_\alpha \in \mathcal{R}$ for each $\alpha$ then the \textit{categorical product} $\Pi_\alpha A_\alpha \in \mathcal{R}$ also. The \textit{weak product} $\tilde{\Pi}_\alpha A_\alpha$ is the homotopy colimit of the finite subproducts; if for each $i$ only finitely many of the groups $\pi_i(A_\alpha)$ are nonzero, then the weak product has the same weak homotopy type as the categorical product.

Let $\mathcal{S}$ be the smallest resolving class that contains $S^n$ for each $n$, and let $\overline{\mathcal{S}}$ be the smallest strong resolving class that contains $S^n$ for each $n$. We say that a space $K$ is \textbf{spherically resolvable} if $\Omega^k K \in \overline{\mathcal{S}}$ for some $k$. This concept is related to, but not the same as, the notion of spherical resolvability described in [3, 12].

\textbf{Examples}

(a) If $f : A \to B$ is any map then the class of all $f$-local spaces is a resolving class [8, p. 5]. This includes, for example, the class of all spaces with $\pi_i(X) = 0$ for $i > n$, or all $h_*$-local spaces, where $h_*$ is a homology theory.
(b) If $P$ is a set of primes, then the class of all $P$-local spaces is a strong resolving class.

(c) If $f : W \to \ast$, then the class of all $f$-local spaces is a strong resolving class [8, p. 5]. This includes, for example, the class $\{K^+\}$, where $K^+$ denotes the Quillen plus construction on $K$ [8, p. 27].

(d) More generally, if $F$ is a covariant functor that commutes with homotopy limits (and hence with fibrations) and $\mathcal{R}$ is a (strong) resolving class, then the class $\{K \mid F(K) \in \mathcal{R}\}$ is also a (strong) resolving class. This applies, for example to the functor $F(K) = \text{map}_*(X, K)$.

(e) The class $\{K \mid K \sim \ast\}$ is a strong resolving class.

Our proofs will proceed by induction on a certain kind of cone length [1]. Let $\mathcal{F}$ denote the collection of all finite type wedges of spheres. The $\mathcal{F}$-cone length $\text{cl}_\mathcal{F}(K)$ of a space $K$ is the least integer $n$ for which there are cofibrations $S_i \to K_i \to K_{i+1}$, $0 \leq i < n$, with $K_0 \simeq \ast$, $K_n \simeq K$ and each $S_i \in \mathcal{F}$. If no such $n$ exists, then $\text{cl}_\mathcal{F}(K) = \infty$. Clearly every finite complex $K$ has $\text{cl}_\mathcal{F}(K) < \infty$.

We denote by $\Sigma \mathcal{F} \subseteq \mathcal{F}$ the subcollection of all simply-connected finite type wedges of spheres. Finally, let $\mathcal{S}^\vee$ be the smallest strong resolving class that contains $\Sigma \mathcal{F}$.

With these preliminaries in place, we can state our main result.

**Theorem 1** If $K$ is a nilpotent space with $\text{cl}_\mathcal{F}(K) = n < \infty$, then

(a) $K \in \mathcal{S}^\vee$,

(b) $\Omega K \in \mathcal{S}$, and

(c) $\Omega^n K \in \mathcal{S}$.

In particular, (b) implies that every nilpotent finite complex $K$ is spherically resolvable in our sense.

Our application to Miller spaces follows from the following more general consequence of Theorem 1.
Theorem 2  Let $\mathcal{R}$ be a strong resolving class and let $F$ be a functor that commutes with homotopy limits.

(a) Assume that $F(S^n) \in \mathcal{R}$ for each $n$. Then $F(\Omega K) \in \mathcal{R}$ for each nilpotent space $K$ with $\text{cl}_F(K) < \infty$.

(b) Assume that $F(S) \in \mathcal{R}$ for each $S \in \Sigma \mathcal{F}$. Then $F(K) \in \mathcal{R}$ for each nilpotent space $K$ with $\text{cl}_F(K) < \infty$.

To apply part (b), we require the following result of Dwyer [6].

Proposition 3  Let $F$ be a functor that commutes with homotopy limits, let $W$ be a space and let $\mathcal{R} = \{K \mid \text{map}_*(W, F(K)) \sim *\}$. If $S^n \in \mathcal{R}$ for each $n$, then $\Sigma \mathcal{F} \subseteq \mathcal{R}$.

Together, Theorem 2(b) and Proposition 3 immediately imply the desired statement about Miller spaces.

Corollary 4  If $\text{map}_*(X, S^n) \sim *$ for all $n$, then $\text{map}_*(X, K) \sim *$ for every nilpotent space $K$ with $\text{cl}_F(K) < \infty$. In other words, $X$ is a Miller space.

Corollary 4 is by no means the only corollary of interest. Other consequences are easily obtained by applying Theorem 2 to various strong resolving classes. For example, if $\text{map}_*(X, S^n)$ is $P$-local for all $n$, then $\text{map}_*(\Sigma X, K)$ is $P$-local for every nilpotent space $K$ with $\text{cl}_F(K) < \infty$.

If $X$ is simply-connected then Corollary 4 can be strengthened somewhat. If $L$ is a space with a nilpotent covering space $K$ having $\text{cl}_F(K) < \infty$, then it is easy to see that $\text{map}_*(X, L) \sim *$. We end by making the surprising observation that a (non-nilpotent, of course) finite complex can be a Miller space!

Example  Let $A$ be a connected 2-dimensional acyclic finite complex. (The classifying space of the Higman group [10] is such a space [7]; so is the space obtained by removing a point from a homology 3-sphere). Since $\pi_1(A)$ is equal to its commutator subgroup, there are no nontrivial homomorphisms from $\pi_1(A)$ to any nilpotent group. It follows that if $f : A \longrightarrow K$ with $K$ a nilpotent finite complex, then $\pi_1(f) = 0$ and so $f$ factors through $q : A \longrightarrow A/A_1 \cong \vee S^2$. Since $[A, S^2] \cong H^2(A) = 0$, we conclude $f \simeq *$. Thus $A$ is a Miller space.
This example shows that the nilpotency hypothesis on the targets in Corollary 4 cannot be entirely removed. It remains possible, however, that if $X$ is simply-connected and $\text{map}_*(X, S^n) \sim *$ for each $n$, then $\text{map}_*(X, K) \sim *$ for every finite complex $K$, or even for every finite-dimensional complex.

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1 Proof of Theorem 1

We begin with two supporting results.

**Proposition 5** Let $K$ be a connected nilpotent space, let $\mathcal{R}$ be a resolving class and let $F$ be a functor that commutes with homotopy (inverse) limits. If $F(\vee_{i=1}^m \Sigma K) \in \mathcal{R}$ for each $m$, then $F(K) \in \mathcal{R}$.

**Proof** This follows from a result of Hopkins [11, p. 222], which says that $K$ is homotopy equivalent to the homotopy (inverse) limit of a tower

$$A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_n \leftarrow A_{n+1} \leftarrow \cdots$$

of spaces, each of which is a homotopy (inverse) limit of a diagram of spaces of the form $\vee_{i=1}^m \Sigma K$. \hfill \Box

**Proposition 6** Let $A \rightarrow B \rightarrow C$ be a cofibration, and let $F$ be the homotopy fiber of $B \rightarrow C$. Then

$$\Sigma F \simeq \Sigma A \vee (\Sigma A \wedge \Omega C).$$

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Proof Convert the maps $A \to C$, $B \to C$ and $C \to C$ to fibrations. The total spaces and fibers form the commutative diagram

\begin{align*}
A \times \Omega C & \to F \\
\Omega C & \to * \\
A & \to B \\
* & \to C,
\end{align*}

in which the bottom square is a homotopy pushout. A result of V. Puppe [14] shows that the top square is also a homotopy pushout. Hence, the cofiber $\Sigma F$ of the map $F \to *$ has the same homotopy type as the cofiber of $A \times \Omega C \to \Omega C$, namely $\Sigma A \vee (\Sigma A \wedge \Omega C)$, as can be seen from the diagram

\begin{align*}
A \times \Omega C & \to \Omega C \to \Sigma F \\
\text{pushout} & \downarrow \downarrow \downarrow \downarrow \downarrow \\
A & \to A \vee \Omega C \to \Sigma A \vee (\Sigma A \wedge \Omega C).
\end{align*}

Proof of Theorem 1 Notice that the assumption on $X$ implies that $X$ is connected; we may therefore assume that $K$ is also connected.

We prove assertion (a) by induction on $\cl_F(K)$. If $\cl_F(K) = 1$, then $K$ is weakly equivalent to a connected finite type wedge of spheres. Therefore each $\bigvee_{i=1}^m \Sigma K \in \Sigma F$ and Proposition 3 proves the assertion in the initial case.

Now assume that the result is known for all nilpotent spaces with $\mathcal{F}$-cone length less than $n$, and that $K$ is nilpotent with $\cl_F(K) = n$. Two applications of Proposition 3 reveal that it is enough to show $\bigvee_{i=1}^m \Sigma^2 K \in \mathcal{S}^\vee$ for each $m$.

Write $V = \bigvee_{i=1}^m \Sigma^2 K$. Notice that $\cl_F(\bigvee_{i=1}^m K) \leq \cl_F(K)$, and the double suspension of an $\mathcal{F}$-cone decomposition of $\bigvee_{i=1}^m K$ is an $\mathcal{F}$-cone decomposition of $V$. Thus we may assume that $V$ has an $\mathcal{F}$-cone decomposition $S_i \to V_i \to V_{i+1}$, $0 \leq i < n$ with $S_i, V_i \in \Sigma \mathcal{F}$ for each $i$. Therefore, we
have a cofibration $L \rightarrow V \rightarrow W$ with $L$ simply-connected, $\text{cl}_F(L) < n$ and $W \in \Sigma F$. Let $F$ denote the homotopy fiber of $V \rightarrow W$, so

$$F \rightarrow V \rightarrow W$$

is a fibration. Since $W \in S^\vee$, it suffices to show that $F \in S^\vee$.

Now we use Proposition 6 to determine the homotopy type of $\Sigma F$:

$$\Sigma F \simeq \Sigma L \vee (L \wedge \Sigma \Omega W) \simeq L \wedge \left( \bigvee_{\alpha} S^{n\alpha} \right)$$

which is a finite type wedge of suspensions of $L$. If we smash an $\mathcal{F}$-cone length decomposition of $L$ with the space $\bigvee_{\alpha} S^{n\alpha}$ we obtain an $\mathcal{F}$-cone length decomposition for $\Sigma F$ — in other words, $\text{cl}_F(\Sigma F) < n$ and, more importantly, $\text{cl}_F(\bigvee_{i=1}^l \Sigma F) < n$ for each $l$.

By the inductive hypothesis, $\bigvee_{i=1}^l \Sigma F \in S^\vee$ for each $l$. Since $L, V$ and $W$ are each simply-connected, so is $F$, and Proposition 3 implies that $F \in S^\vee$, as desired.

To prove (b), observe that the collection $\mathcal{M}$ of all $K$ with $\Omega K \in S$ is a resolving class that contains $\Sigma F$ by the Hilton-Milnor theorem [9]. Hence $\mathcal{M}$ contains all nilpotent spaces $K$ with $\text{cl}_F(K) < \infty$ by part (a).

The proof of (c) is similar to the proof of (a). The initial case of the induction is a special case of (b). To prove the inductive step, we write $V = \bigvee_{i=1}^m \Sigma^2 K$ and show that $V \in S$. As before, we consider the cofiber sequence $L \rightarrow V \rightarrow W$ with $W \in \Sigma F$ and the corresponding fibration $F \rightarrow V \rightarrow W$. This gives us a fibration

$$\Omega^n V \rightarrow \Omega^n W \rightarrow \Omega^{n-1} F$$

with $\Omega^n W \in S$. It now suffices to prove that $\Omega^{n-1} F \in S$, which follows by induction using Proposition 3. \qed

## 2 Proof of Theorem 2 and Proposition 3

**Proof of Theorem 2**

Let $\mathcal{M}$ be the class of all spaces $K$ such that $F(K) \in \mathcal{R}$; we have already seen that $\mathcal{M}$ is a strong resolving class. For part (a), $S^n \in \mathcal{M}$ for each $n$
by assumption, so \( \mathcal{S} \subseteq \mathcal{M} \). By Theorem 3(b), \( \mathcal{M} \) contains \( \Omega K \) for every nilpotent space \( K \) with \( \text{cl}_F(K) < \infty \). In part (b), we find that \( \mathcal{S}^\vee \subseteq \mathcal{M} \), and so \( \mathcal{M} \) contains every nilpotent space \( K \) with \( \text{cl}_F(K) < \infty \). \( \square \)

**Proof of Proposition 3** Define a relation \( \prec \) on \( \Sigma \mathcal{F} \) as follows: \( S \prec T \) if either (1) the connectivity of \( S \) is greater than the connectivity of \( T \), or (2) the connectivity of \( S \) equals the connectivity of \( T \) (say both are \( (n-1) \)-connected) and the rank of \( \pi_n S \) is less than the rank of \( \pi_n T \).

The key to this proof is the following claim.

**Claim** Suppose that \( S \in \mathcal{F} \) is \( (n-1) \)-connected and has \( \pi_n S \neq 0 \). Then there is a map \( f : S \to S^n \) such that the homotopy fibre \( T \) of \( f \) belongs to \( \mathcal{F} \) and \( T < S \).

**Proof of Claim** Write \( S \sim S' \vee S^n \), and let \( f : S \to S^n \) be the map which collapses \( S' \). By [9], the homotopy fibre of \( f \) is

\[
(S' \times \Omega S^n)/(\ast \times \Omega S^n) \sim S' \wedge (\Omega S^n)_+ \sim \bigvee_{m=0}^{\infty} \Sigma^{(n-1)m} S' \in \Sigma \mathcal{F},
\]

using the James splitting of \( \Sigma \Omega S^n \).

Now let \( S = S_0 \in \Sigma \mathcal{F} \). Define \( S_{n+1} \) as the fiber of a map \( f_n : S_n \to S^{k(n)} \) as in the claim. The result is a tower of spaces

\[
S_0 \leftarrow S_1 \leftarrow \cdots \leftarrow S_n \leftarrow S_{n+1} \leftarrow \cdots
\]

with \( S_n \in \mathcal{F} \) and \( S_{n+1} < S_n \) for each \( n \). Since spaces in this tower become arbitrarily highly connected as \( n \) increases, \( \text{holim}_n S_n \sim \ast \).

The fibrations \( S_n \to S_{n+1} \to S^{k(n)} \) give rise to fibrations

\[
\text{map}_s(W, F(S_{n+1})) \to \text{map}_s(W, F(S_n)) \to \text{map}_s(W, F(S^{k(n)}))\]

It follows by induction that each map \( S_n \to S \) induces a weak equivalence \( \text{map}_s(W, F(S_n)) \sim \text{map}_s(W, F(S)) \). Finally, we compute

\[
\text{map}_s(W, F(S)) \sim \text{holim}_n \text{map}_s(W, F(S)) \sim \text{holim}_n \text{map}_s(W, F(S_n)) \sim \text{map}_s(W, \text{holim}_n F(S_n)) \sim \text{map}_s(W, \ast) \sim \ast
\]

\( \square \)
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