On functional equations for the elliptic dilogarithm

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1 Introduction

1.1 Summary

We recall that the Bloch–Wigner dilogarithm [1] is defined by the following formula:

\[ D(z) = \text{Im} \, \text{Li}_2(z) + \text{arg}(1 - z) \ln |z|. \]

This formula defines a real analytic function on \( \mathbb{C} \backslash \{0, 1\} \) which satisfies the relation

\[ D(z) = -D(z^{-1}). \]

Let \( E = \mathbb{C}/\langle 1, \tau \rangle \) be an elliptic curve over \( \mathbb{C} \). The elliptic dilogarithm was defined by Spencer Bloch in [1], see also [5]. An equivalent representation is given by the
following formula:

$$D_\tau (\xi) = \sum_{n=-\infty}^{\infty} D(\epsilon e^{2\pi i \xi + 2\pi i \tau n}).$$

This formula defines a real analytic function on $E$. From the relation $D(z) = -D(z^{-1})$ the following antisymmetry relation can be easily deduced:

$$D_\tau (\xi) + D_\tau (-\xi) = 0.$$

Starting from here we assume that $E$ is defined over an arbitrary algebraically closed field $k$. Denote by $Z[E]$ the free abelian group generated by the points of $E$. For a point $z \in E$ we denote by $[z]$ the corresponding element in the group $\mathbb{Z}[E]$. When $k = \mathbb{C}$, the elliptic dilogarithm gives a well-defined map $\widetilde{D}_\tau : \mathbb{Z}[E] \to \mathbb{C}$ defined by the formula $\widetilde{D}_\tau ([z]) = D_\tau (z)$.

For a rational function $g$ on some smooth algebraic curve, denote by $(g)$ its divisor. Let us formulate the so-called elliptic Bloch relations ([1, Theorem 9.2.1], see also [5]). Let $f$ be a rational function on $E$ of degree $n$ such that

$$(f) = \sum_{i=1}^{n} ([\alpha_i] - [\gamma_i]), \quad (1 - f) = \sum_{i=1}^{n} ([\beta_i] - [\gamma_i]).$$

Define the element $\eta_f \in \mathbb{Z}[E]$ by the following formula:

$$\eta_f = \sum_{i,j=1}^{n} ([\alpha_i - \beta_j] + [\beta_i - \gamma_j] + [\gamma_i - \alpha_j]).$$

**Definition 1.1** ([3]) Let $\mathcal{R}(E)$ be a subgroup of the group $\mathbb{Z}[E]$ generated by the following elements:

- $\eta_f$, where $f \in k(E)$,
- $[z] + [-z]$, where $z \in E$,
- $2 \cdot (z - \sum_{z \neq z'} [z'])$, where $z \in E$.

Then the quotient $\mathbb{Z}[E]/\mathcal{R}(E)$ is called the elliptic Bloch group $B_3(E)$.

According to [3], when $k = \mathbb{C}$, the map $\widetilde{D}_\tau$ is zero on $\mathcal{R}(E)$.

Consider the following geometric example (see [3, Lemma 3.29]). Let us realize the elliptic curve $E$ as a cubic plane curve in $\mathbb{P}^2$ and consider three different lines $l, m, n \subset \mathbb{P}^2(\mathbb{C})$ intersecting at a point in the complement of $E$. Let $h_l, h_m, h_n$ be homogeneous equations of these lines such that $h_m = h_n + h_l$. Denote by $\alpha_i, \beta_i, \gamma_i, i = 1, 2, 3$, the intersection points of the lines $l, n, m$ with $E$. Denote the element $\eta_f$, where $f = h_l/h_m$, by $\eta_{l,m,n}$. This element has the following form:

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Our main result is the following statement.

**Theorem 1.2** Let $E$ be an elliptic curve over an algebraically closed field of characteristic zero. For any rational function $f$ on $E$ the element $\eta_f$ can be represented as a linear combination with integer coefficients of the elements of the form $\eta_{l,m,n}$ and $[z] + [-z]$. This implies that when defining the elliptic Bloch group, one can omit the elements $\eta_f$ with $\deg f > 3$.

This theorem gives a solution to [3, Conjecture 3.30]. We remark that it does not follow directly from any of the statements mentioned in [3, 5].

Denote by $K$ the field of rational functions on $E$. There are a lot of relations between the elements of the form $\eta_f$. More precisely, the following is true:

**Proposition 1.3** Let $f, g \in K \{0, 1\}$ and $f \neq g$. Modulo elements of the form $[z] + [-z]$, the following identity is true:

$$\eta_f + \eta_{g/f} + \eta((1-f)/(1-g)) = \eta_g + \eta((1-f^{-1})/(1-g^{-1})).$$

This proposition motivates the following definition:

**Definition 1.4** (pre-Bloch group) Denote by $\mathbb{Z}[K \{0, 1\}]$ the free abelian group generated by the set $K \{0, 1\}$. Denote these generators as $[f]$, where $f \in K \{0, 1\}$. The pre-Bloch group of the field $K$ is the quotient of the group $\mathbb{Z}[K \{0, 1\}]$ by the following elements (so-called Abel five-term relations):

$$[x] - [y] + [y/x] + [(1-x)/(1-y)] - \left[ \frac{1-x^{-1}}{1-y^{-1}} \right],$$

where $x, y \in K \{0, 1\}, x \neq y$.

We recall that the degree of a non-zero rational function $f \in K$ is the number of zeros of $f$ counted with multiplicities. (In particular, the degree of any non-zero constant is equal to 0.) Theorem 1.2 is an easy consequence of the following statement:

**Theorem 1.5** Let $E$ be an elliptic curve over an algebraically closed field of characteristic 0. The group $B_2(K)$ is generated by elements of the form $[f]$ with $\deg f \leq 3$.

The corresponding statement for the projective line is the main result of [2]. It is interesting to generalise this result to curves of arbitrary genus.

**Outline of the paper**

In Sect. 2 we deduce Theorem 1.2 from Theorem 1.5.

Let us denote by $\mathcal{F}_nB_2(K)$ the subgroup of $B_2(K)$ generated by elements of the form $[f]$ with $\deg f \leq n$. In Sect. 3 we define the notion of a generic function and prove the following:
Proposition 1.6 Let $n \geq 3$. The group $T_{n+1}B_2(K)$ is generated by the subgroup $T_nB_2(K)$ and generic rational functions of degree $n + 1$.

The main result of Sect. 4 is the following:

Proposition 1.7 Let $n \geq 3$. Let $f$ be a generic function of degree $n + 1$. Then the element $\{f\}$ lies in the subgroup $T_nB_2(K)$.

Theorem 1.5 follows immediately from these two propositions.

2 Deduction of Theorem 1.2 from Theorem 1.5

Denote by $\mathbb{Z}[E]^{-}$ the quotient of $\mathbb{Z}[E]$ by the subgroup generated by elements of the form $[\xi] + [-\xi]$, where $\xi \in E$. Denote by $\pi: \mathbb{Z}[E] \to \mathbb{Z}[E]$ the natural projection. Introduce the map $\beta: \Lambda^2K^\times \to \mathbb{Z}[E]^{-}$ defined by the formula $f \otimes g \mapsto \sum_{i,j} n_i m_j \pi((x_i - y_j))$, where $\sum_i n_i [x_i], \sum_j m_j [y_j]$ are the divisors of the functions $f$ and $g$. Denote by $R_{as}$ the subgroup of $\mathbb{Z}[E]^{-}$ generated by the elements of the form $\beta(f \wedge (1 - f))$ where $f \in K(E)$. According to [4, Lemma 1.1], there is a well-defined map $\delta: B_2(K) \to \Lambda^2K^\times$ given by the formula $\delta([f]) = f \wedge (1 - f)$. (It is non-trivial to prove that this map is zero on the Abel five-term relations (1).)

By the definition, the image of the map $\beta \circ \delta: B_2(K) \to \mathbb{Z}[E]^{-}$ coincides with $R_{as}$. According to Theorem 1.5, the group $B_2(K)$ is generated by elements of the form $\{f\}$ with $\deg f \leq 3$. Thus we proved the following statement:

Corollary 2.1 The group $R_{as}$ is generated by elements of the form $\beta(f \wedge (1 - f))$ with $\deg f \leq 3$.

It is easy to see that the element $\pi(\eta_f)$ is equal to $\beta(f \wedge (1 - f))$. Now Proposition 1.3 follows from the fact that the map $\delta$ is zero on the Abel five-term relations (1).

Let us prove Theorem 1.2. By the above corollary the group $R_{as}$ is generated by elements of the form $\beta(f \wedge (1 - f))$ with $\deg f \leq 3$. It follows that the element $\eta_f$ can be represented as a sum of elements $\eta_{f'}$ with $\deg f' \leq 3$ and elements of the form $[z] + [-z]$. So it is enough to prove that the theorem holds for functions of degree $\leq 3$. If $\deg f \leq 2$ then it is easy to see that the element $\eta_f$ is a sum of elements of the form $[z] + [-z]$. Let $\deg f = 3$. Since the map $\beta$ is translation-invariant and any point on $E$ is 3-divisible, we can assume that the sum of zeros of $f$ is equal to zero. It is easy to see that in this case the element $\eta_f$ is equal to $\eta_{l,m,n}$ for some lines $l, m, n$.

3 Reduction to the generic case

Let us fix some algebraically closed field $k$ of characteristic zero. Starting from here we assume that $E$ is an arbitrary elliptic curve defined over $k$. We need the following:

Definition 3.1 A rational function $f$ on an elliptic curve $E$ is called generic if there are points $\alpha_1, \alpha_2 \in f^{-1}(0), \beta_1, \beta_2 \in f^{-1}(1), \gamma_1, \gamma_2 \in f^{-1}(\infty)$ on $E$, such that the following conditions hold:
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- If the points $\alpha_1, \alpha_2$ are equal then the multiplicity of zero of the function $f$ at the point $\alpha_1$ is at least 2. In the same way if the points $\gamma_1, \gamma_2$ are equal then the multiplicity of pole of the function $f$ at the point $\gamma_1$ is at least 2.
- The points $\beta_1, \beta_2$ are different from each other.
- The points $\alpha_1 + \alpha_2, \beta_1 + \beta_2$ and $\gamma_1 + \gamma_2$ are mutually different.
- The points $\alpha_1 + \alpha_2 - \gamma_1 - \beta_1, \alpha_1 + \alpha_2 - \gamma_1 - \beta_2$ are non-zero.

In order to prove Proposition 1.6 we need the following:

**Lemma 3.2** Let $n \geq 3$. Let $f$ be a function of degree $n + 1$ satisfying the first two conditions of Definition 3.1 and the third condition fail. Then $[f] \in \mathcal{F}_n B_2(K)$.

**Proof** Let us consider the case when $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, the other cases are similar. There is a function $h$ of degree 2 on $E$ taking the value zero at the points $\alpha_1, \alpha_2$, the value 1 at the points $\beta_1, \beta_2$ and the value infinity at the point $\gamma_1$. Let us substitute $x = h, y = f$ into (1):

$$[h] - [f] + [f/h] + [(1 - h)/(1 - f)] - \left[ \frac{1 - h^{-1}}{1 - f^{-1}} \right].$$

(2)

Let us prove that the degree of the function $f/h$ is $\leq n$. The divisors of the functions $f$ and $h$ have the following forms:

$$(f) = \sum_{i=1}^{n+1} ([\alpha_i - \gamma_i]), \quad (h) = [\alpha_1] + [\alpha_2] - [\gamma_1] - [\gamma'],$$

for some points $\alpha_i, \gamma_i, \gamma' \in E, i = 3, \ldots, n + 1$. We have

$$(f/h) = (f) - (h) = \sum_{i=1}^{n+1} ([\alpha_i - \gamma_i]) - ([\alpha_1] + [\alpha_2] - [\gamma_1] - [\gamma'])$$

$$= ([\gamma'] + \sum_{i=3}^{n+1} [\alpha_i]) - \sum_{i=2}^{n+1} [\gamma_i].$$

So the degree of the function $f/h$ is not higher than $n$. The cases of the functions $(1 - f)/(1 - h), (1 - f^{-1})/(1 - h^{-1})$ are similar. Therefore degrees of all but the second term of the relation (2) are $\leq n$. So $[f] \in \mathcal{F}_n B_2(K)$. \qed

**Proof of Proposition 1.6** Let $a \in k \setminus \{0, 1\}$. If we substitute $x = a, y = f$ into (1) we get the following relation in the group $B_2(K)$:

$$[a] - [f] + [f/a] + [(1 - a)/(1 - f)] - \left[ \frac{1 - a^{-1}}{1 - f^{-1}} \right].$$

Let $f_{1,a} = f/a, f_{2,a} = (1 - a)/(1 - f), f_{3,a} = (1 - a^{-1})/(1 - f^{-1})$. Since the field $k$ is infinite, it is enough to prove that for any $j \in \{1, 2, 3\}$ and all but finite values
of $a$ the function $f_{j,a}$ is either a generic or the element $[f_{j,a}]$ lies in the subspace $\mathcal{F}_n B_2(K)$.

Let $\alpha_1, \alpha_2$ and $\gamma_1, \gamma_2$ be as in the first condition of Definition 3.1 for the function $f$. Let $A_0 \subset \mathbb{P}^1$ be the set of critical values of the function $f$, and $A = A_0 \cup \{f(\alpha_1 + \alpha_2 - \gamma_1)\} \cup \{0, \infty\}$. Let $a \notin A$. The set $f^{-1}(a)$ has precisely $n+1$ elements. Let $\beta_1, \beta_2 \in f^{-1}(a)$ be two different elements. It easy to see that for the function $f/a$ the first, the second and the fourth conditions of Definition 3.1 hold. If the third condition also holds then $f/a$ is a generic rational function. If it does not hold then according to Lemma 3.2 the element $[f/a]$ lies in the subgroup $\mathcal{F}_n B_2(K)$. The cases of the functions $f_{2,a}$, $f_{3,a}$ are similar.

\[\square\]

4 Decreasing of the degree in the case of a generic function

Let us denote by $\wp$ the function of degree 2 on $E$, satisfying $\wp(z) = z - 2 + o(z)$ as $z \to 0$. It is easy to see that for arbitrary points $\alpha, \beta \in E$, $\alpha \neq \beta$, the function $\wp(z - \alpha) - \wp(z - \beta)$ has the following divisor:

\[
(\wp(z - \alpha) - \wp(z - \beta)) = \left( \sum_{2x \in \alpha + \beta} [x] \right) - 2[\alpha] - 2[\beta].
\]

We recall that the cross ratio of four points on $\mathbb{P}^1$ is defined by the following formula:

\[
[a, b, c, d] = \frac{a - c}{b - c} : \frac{a - d}{b - d} = \frac{(a - c)(b - d)}{(b - c)(a - d)}.
\]

Let $\alpha, \beta, \gamma, \delta$ be four mutually different points on $E$. We define the function $h_{\alpha,\beta,\gamma,\delta}$ by the following formula:

\[
h_{\alpha,\beta,\gamma,\delta}(z) = [\wp(z - \alpha), \wp(z - \beta), \wp(z - \gamma), \wp(z - \delta)].
\]

It follows from (3) that

\[
(h_{\alpha,\beta,\gamma,\delta}) = \sum_{2x \in \{\alpha + \gamma, \beta + \delta\} \setminus \{\alpha + \delta, \beta + \gamma\}} [x] - \sum_{2x \in \{\alpha + \delta, \beta + \gamma\}} [x].
\]

Since the points $\alpha, \beta, \gamma, \delta$ are mutually different, the degree of the function $h$ is equal to 8. The group $E(2) = \{z \in E \mid 2z = 0\}$ acts on $E$ by translations. Hence the divisors of the functions $h_{\alpha,\beta,\gamma,\delta}$ and $1 - h_{\alpha,\beta,\gamma,\delta} = h_{\alpha,\gamma,\beta,\delta}$ are invariant under the group $E[2]$. Therefore the function $h_{\alpha,\beta,\gamma,\delta}$ is also invariant under the group $E[2]$. We need the following:

Lemma 4.1 For any $m \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ there is $\mu \in E$, such that the following conditions hold:

- $h_{\alpha,\beta,\gamma,\delta}(\mu) = m$,
- $\mu \notin \{\alpha, \beta\}$.

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• \( 2\mu \notin \{\alpha + \gamma, \beta + \delta, \alpha + \delta, \beta + \gamma, \alpha + \beta, \delta + \gamma\} \).

**Proof** As a map between two projective curves the map \( h_{\alpha, \beta, \gamma, \delta} \) is surjective. Therefore the set \( f^{-1}(m) \) is non-empty. Since the function \( h_{\alpha, \beta, \gamma, \delta} \) is invariant under the group \( E[2] \), the set \( h_{\alpha, \beta, \gamma, \delta}^{-1}(m) \) is also invariant under the group \( E[2] \). So there are at least four points satisfying the first condition of the lemma. We can pick from them a point \( \mu \) different from \( \alpha \) and \( \beta \). So the first and the second conditions for the point \( \mu \) hold. Since the point \( m \) does not lie in the set \( \{0, 1, \infty\} \), the point \( \mu \) does not lie on the divisors of the functions \( h_{\alpha, \beta, \gamma, \delta} \), \( 1 - h_{\alpha, \beta, \gamma, \delta} = h_{\alpha, \beta, \gamma, \delta} \). Now the third statement of the lemma follows from (4). \( \square \)

**Proposition 4.2** Let \( \alpha, \beta, \gamma, \delta \) be four mutually different points on \( E \) and let \( a, b, c, d \) be four mutually different points on \( \mathbb{P}^1 \). Then there is a function \( f \) of degree 2 on \( E \) such that \( f(\alpha) = a, f(\beta) = b, f(\gamma) = c, f(\delta) = d \).

**Proof** Let \( m \) be the cross-relation of points \( a, b, c, d \) and \( \mu \) be the point given by Lemma 4.1. Let us define the function \( f \) by the following formula:

\[
 f(z) = \frac{\wp(z - \mu) - \wp(\alpha - \mu)}{\wp(z - \mu) - \wp(\beta - \mu)}.
\]

The divisor of this function is equal to \( (f) = [\alpha] + [2\mu - \alpha] - [\beta] - [2\mu - \beta] \).

It follows from the statement of the previous lemma that the function \( f \) satisfies the following two conditions:

• The degree of \( f \) is equal to 2.
• \( f(\gamma), f(\delta) \notin \{0, \infty\} \).

We have

\[
 \frac{f(\gamma)}{f(\delta)} = \frac{\wp(\gamma - \mu) - \wp(\alpha - \mu)}{\wp(\gamma - \mu) - \wp(\beta - \mu)} \cdot \frac{\wp(\delta - \mu) - \wp(\beta - \mu)}{\wp(\delta - \mu) - \wp(\alpha - \mu)} = h_{\alpha, \beta, \gamma, \delta}(\mu) = m.
\]

So the function \( \tilde{f} \) given by the formula \( \tilde{f}(z) = f(z)/f(\delta) \) satisfies \( \tilde{f}(\alpha) = 0, \tilde{f}(\gamma) = m, \tilde{f}(\delta) = 1, \tilde{f}(\beta) = \infty \). Since \( \{0, m, 1, \infty\} = \{a, c, d, b\} \), there is an element \( g \) of the group \( \text{PSL}_2(k) \) transforming the points \( 0, m, 1, \infty \) to the points \( a, c, d, b \). It is easy to see that the function \( g(\tilde{f}(z)) \) satisfies the statement of the proposition. \( \square \)

We have the following:

**Corollary 4.3** Let \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) be points on \( E \), such that the following conditions hold:

• The sets \( \{\alpha_i\}_{i=1,2}, \{\beta_i\}_{i=1,2}, \{\gamma_i\}_{i=1,2} \) do not intersect.
• The points \( \beta_1 \) and \( \beta_2 \) are different.
• The points \( \alpha_1 + \alpha_2 - \gamma_1 - \beta_1, \alpha_1 + \alpha_2 - \gamma_1 - \beta_2 \) are non-zero.
• The points \( \alpha_1 + \alpha_2, \beta_1 + \beta_2 \) and \( \gamma_1 + \gamma_2 \) are mutually different.

Then there is a rational function \( g \) on \( E \) satisfying \( g(\beta_1) = g(\beta_2) = 1 \) and one of following statements hold:
The degree of $g$ is equal to 3 and its divisor is equal to $[\alpha_1] + [\alpha_2] + [\alpha'] - [\gamma_1] - [\gamma_2] - [\gamma']$, where $\alpha', \gamma'$ are some points on $E$.

The degree of $g$ is equal to 2 and its divisor is equal to $[\alpha_1] + [\alpha_2] - [\gamma_j] - [\gamma']$, where $j \in \{1, 2\}$ and $\gamma'$ is some point on $E$.

The degree of $g$ is equal to 2 and its divisor is equal to $[\alpha_j] + [\alpha'] - [\gamma_1] - [\gamma_2]$, where $j \in \{1, 2\}$ and $\alpha'$ is some point on $E$.

(a) is the “generic” case and (b) and (c) are its degenerations.

Proof It follows from the first condition of the corollary that there is a function $g_1$ of degree 2 with the divisor equal to $[\alpha_1] + [\alpha_2] - [\gamma_1] - [\alpha_1 + \alpha_2 - \gamma_1]$. From the first and the third conditions it follows that $g_1(\beta_1) \notin \{0, \infty\}$. Let us denote by $\tilde{g}_1$ the function $g_1/g_1(\beta_1)$. From the conditions of the corollary it follows that $\tilde{g}_1(\beta_2) \notin \{0, 1, \infty\}$. According to Proposition 4.2 there is a function $g_2$ of degree 2 on $E$ taking the values $0, \infty, 1$, $g_1(\beta_2)^{-1}$ at the points $\alpha_1 + \alpha_2 - \gamma_1, \gamma_2, \beta_1, \beta_2$, resp. Let $g := \tilde{g}_1 g_2$. By the construction of the function $g$ we have $g(\beta_1) = g(\beta_2) = 1$. Denote by $[\alpha_1 + \alpha_2 - \gamma_1] + [s] - [\gamma_2] - [t]$ the divisor of the function $g_2$ for some $s, t \in E$. We have

$$g = [\alpha_1] + [\alpha_2] + [s] - [\gamma_1] - [\gamma_2] - [t].$$

There are three cases:

- $s \notin \{\gamma_1, \gamma_2\}, t \notin \{\alpha_1, \alpha_2\}$. In this case the degree of the function $g$ is equal to 3 and the first case of the proposition holds.
- $s \in \{\gamma_1, \gamma_2\}$. In this case $t \notin \{\alpha_1, \alpha_2\}$ and so the second case of the proposition holds.
- $t \in \{\alpha_1, \alpha_2\}$. In this case $s \notin \{\gamma_1, \gamma_2\}$ and so the third case of the proposition holds.□

Proof of Proposition 1.7 Let $f$ be a generic rational function of degree $n + 1$ on $E$. Denote by $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ the corresponding points from Definition 3.1. They satisfy the conditions of Corollary 4.3. Let $g$ be a function satisfying one of the statements of Corollary 4.3. Let us substitute $x = g, y = f$ into (1):

$$[g] - [f] + [f/g] + [(1 - g)/(1 - f)] - [(1 - g^{-1})/(1 - f^{-1})].$$

Similarly to the proof of Lemma 3.2 it is not difficult to show that all but the second term has degree $\leq n$. So $[f] \in \mathcal{F}_n B_2(K)$.□

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