DISCRETE AND CONTINUUM VIRASORO CONSTRAINTS
IN TWO-CUT HERMITIAN MATRIX MODELS

Waichi OGURA*

Institute of High Energy Physics
Department of Physics
Virginia Polytechnic Institute and State University
Blacksburg, Va. 24061-0435, U.S.A.

ABSTRACT

Continuum Virasoro constraints in the two-cut hermitian matrix models are derived from the discrete Ward identities by means of the mapping from the $GL(\infty)$ Toda hierarchy to the nonlinear Schrödinger (NLS) hierarchy. The invariance of the string equation under the NLS flows is worked out. Also the quantization of the integration constant $\alpha$ reported by Hollowood et al. is explained by the analyticity of the continuum limit.

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* OGURA@VTVM2.BITNET
Address after April 1, 1992: YITP Uji Research Center, Kyoto University, Uji 611, Japan.
1. Introduction

Two dimensional quantum gravity has been solved non-perturbatively by taking the double-scaling limit of the matrix model – the dual model for the 2d discretized gravity. Here one has a parameter $\kappa^2$ which determines the way this double limit is approached. The potential is fine-tuned as the size of matrix gets large, and the contributions from different 2d topologies may be obtained as a power series in $\kappa^2$ with the power of $\kappa^{-1}$ being the Euler number of the Riemann surface. Surprisingly, the scaling property (dependence on the cosmological constant) of the $\kappa^2$ expansion agrees order by order with the genus expansion in the 2d quantum gravity coupled with conformal matters.* This is the reason why $\kappa$ is called the string coupling constant. Furthermore, one can determine the scaling limit non-perturbatively by solving a characteristic non-linear ordinary differential equation (string equation) satisfied by the string susceptibility.

Even though one can find infinitely many critical points in the variety of models and their potentials, which all have double scaling limits, these continuum limits can be classified into hierarchical universality classes due to the integrability emerging at these critical points. Actually integrability is there even at the discrete level. If the size of matrix is finite, say $N$, the hermitian one matrix model can be embedded into the $GL(N)$ Toda system, with variation of the coupling constants of the matrix model forming the Toda flows [3], so that the continuum integrability may be understood by the scaling from the discrete integrability.

1.1 $GL(N)$ Toda Hierarchy

We will explain the Toda flow briefly by following refs. [1,3] and Neuberger [4]. After integrating out the degrees of freedom for the similarity transformations, one may evaluate the matrix integral

$$Z \equiv \frac{1}{N!} \int_{N \times N \text{hermitian}} dN^2 \phi \exp \left[ -\beta \text{tr} V(\phi) \right] = \langle \text{vac} | \text{vac} \rangle, \quad (1.1)$$

in terms of the $N$-fermion vacuum

$$| \text{vac} \rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} P_0(\lambda_1) & \cdots & P_0(\lambda_N) \\ \vdots & \ddots & \vdots \\ P_{N-1}(\lambda_1) & \cdots & P_{N-1}(\lambda_N) \end{vmatrix}, \quad (1.2)$$

where the $(n+1)^{th}$ state $P_n(\lambda)$ is an orthogonal monic polynomial of degree $n$ defined by

$$\int_{-\infty}^{\infty} d\lambda e^{-\beta V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{n,m}. \quad (1.3)$$

* Exact equivalence has been proved at least on a torus [2].
Thus we can find $Z = \prod_{n=0}^{N-1} h_n$, and the vacuum expectation value of $\text{tr} \phi^k$ may be calculated as
\[
\langle \frac{1}{N} \text{tr} \phi^k \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \langle n | \hat{\lambda}^k | n \rangle,
\] (1.4)
where $\hat{\lambda}$ is the operator multiplying with $\lambda$, and $| n \rangle$ is the orthonormal basis defined by $| n \rangle = \exp \left( -\frac{\beta}{2} V(\lambda) \right) P_n(\lambda)/\sqrt{h_n}$. Henceforth the r.h.s. will be simply denoted by $\langle \hat{\lambda}^k \rangle$.

Though the potential $V(\lambda) = \sum_{k=0}^\infty g^k \lambda^k$ should be fixed at one of the critical points at the double-scaling limit, at finite $N$ we are free to alter it from the critical point, and find that
\[
\delta (P_n/\sqrt{h_n}) = \sum_{m=0}^\infty O_{nm} P_m/\sqrt{h_m},
\] (1.5)
where all upper half components of $O$ are empty. It is convenient to work on the orthonormal basis, and define $Q_{nm} = \langle n | \hat{\lambda} | m \rangle$. $Q$ is a symmetric matrix, and $\delta V(Q) = O + [O]$. Thus $O$ can be identified with $[\delta V(Q)]_-$, where $[ ]_\pm$ denotes the projection such as $\frac{1}{2} [ ]_{\text{diagonal}} + [ ]_{\text{upper/lower triangular}}$. Hence the variation of $Q$ can be obtained as
\[
\delta Q = [Q, \frac{1}{2} ([\delta V(Q)]_+ - [\delta V(Q)]_-)].
\] (1.6)

Toda Hamiltonian flows defined by Adler [3] are simply
\[
\frac{\partial Q}{\partial g_k} = [Q, [Q^k]_+] = -[Q, [Q^k]_-],
\] (1.7)
and from $\delta V(\lambda) = V(\lambda - \delta \lambda) - V(\lambda)$, one also finds the discrete string equation
\[
[P, Q] = 1,
\] (1.8)
with $P = \frac{1}{2}([V'(Q)]_+ - [V'(Q)]_-)$. As proved by Martinec [3], the Toda flows are commutative, and also preserve the string equation.

In terms of the Ward identities under the Toda flows, Mironov and Morozov [5] have found the Virasoro constraints
\[
L_n Z = 0, \quad (n \geq -1)
\] (1.9)
where
\[
L_n = \sum_{l=1}^\infty l g_l \frac{\partial}{\partial g_{l+n}} + \beta^{-2} \sum_{l=0}^n \frac{\partial^2}{\partial g_l \partial g_{n-l}},
\] (1.10)
satisfies the Virasoro algebra $[L_n, L_m] = (n - m) L_{n+m}$ for $n, m \geq -1$. 


1.2 Double-Scaling Limit

The $GL(\infty)$ Toda hierarchy is transmuted into the KdV hierarchy at the double-scaling limit, and more generally, it has been shown by Douglas [6] that there appears the generalized KdV hierarchy in the one-cut hermitian multi-matrix models. The degree $n$ of the orthogonal polynomials may be described in terms of a continuous parameter $t$ at the double-scaling limit, such that $n/\beta = 1 - at$. Then the pair $Q$ and $P$, which are the matrices defined similarly for the first piece of the matrix chain, scale to the scalar Lax pair [7] as $\beta \to \infty$ and $a \to 0$ simultaneously.

In a critical multi-matrix model, for instance, one can find the Lax operator $L = \partial_t^q + q_{-2} \partial_t^{q-2} + \ldots + u_0$, and (1.8) scales to the continuum string equation

$$\left[(L^{p/q})_+, L\right] = 1,$$

(1.11)

where $u_{q-2}, \ldots, u_0$ are functions of $t$ called the scaling functions, and $(\ )_+$ denotes the ordinary differential operator part of the pseudo differential operator $L^{p/q}$. Similarly to the Toda hierarchy, the generalized KdV hierarchy has infinite number of commutative Hamiltonian flows

$$\frac{\partial L}{\partial t_p} = \left[(L^{p/q})_+, L\right] = -\left[(L^{p/q})_-, L\right],$$

(1.12)

and in the matrix model, these generalized KdV flows can be generated by the scaling operators $\sigma_k(O_\alpha)$ with $p = kq + \alpha + 1$ such that

$$\frac{\partial F}{\partial t_{k,\alpha}} = \left\langle \sigma_k(O_\alpha) \right\rangle = \int_t^\infty dt \left\langle t^L L^{p/q} | t \right\rangle,$$

(1.13)

where $F = \log Z$, and the residue $H_{k,\alpha} = \left\langle t^L L^{p/q} | t \right\rangle$ is a differential polynomial of the scaling functions satisfying certain recursion relations [8,10]. Integrating the string equation (1.11), one also finds the Virasoro and the $W_n$-constraints ($n = 3, \ldots, q$) on $Z$, and thereby $Z$ can be identified with the $\tau$-function of the generalized KdV hierarchy [9-11]. Since the integrability appearing in the continuum is merely a consequence of the integrability in the discrete, it has been conjectured that the Virasoro and the $W$-constraints can be derived from the discrete Ward identities at any scaling limit. This has been proved at least for the one-cut hermitian one matrix models [11,12].

These constraints have been already found in the 2d topological gravity coupled with the ADE series of the topological minimal models [8,13], and very recently, the same Virasoro constraints has also arisen in the Kontsevich model [14,15].

* Note that $1 - N/\beta$ is the cosmological constant.
1.3 Overview

The one-cut family of the hermitian matrix models has demonstrated their rich structures in the generalized KdV hierarchy and W-constraints. In this paper, we shall study the two-cut family of the hermitian one matrix models [16-18], and look for similar structures. The procedure developed in ref. [12], which is easier to handle and more straightforward than the one used in ref. [11], will be applied to the two-cut family; and consequently, the continuum Virasoro constraints will be derived from the discrete Ward identities (§3). The result agrees with refs. [17,18], and the meaning of the additional parameter $\alpha$ appearing in the continuum Virasoro constraints will be clarified. Eventually one can find the nonlinear Schrödinger (NLS) hierarchy for the $2 \times 2$ matrix Lax operators [19,20], and obtain the mKdV hierarchy as the reduction into the even potential models [17,18]. The invariance of the string equation under the NLS flows will be proved in §4. Hollowood et al. [18] have obtained what they call the Zakharov-Shabat (ZS) hierarchy by rotating the anti-hermitian model to the hermitian, and from the ZS hierarchy one can get both KdV and mKdV hierarchies by reduction. We will investigate this rotation further in §5.1 according to the mapping we found, and construct a rotation from the NLS hierarchy to the ZS hierarchy. Finally the quantization of $\alpha$ reported in ref. [18] will be interpreted as the analyticity of the model at the continuum limit $\epsilon \to 0$ by means of the rescaling of $\epsilon$ (§5.2).

Generally one can show that the family of matrix models is governed by two kinds of distinct integrable structures in the discrete and the continuum, and the investigation on the relation between these two structures is not just of mathematical interest, but is essential in order to understand the physical outcome from the integrability in the continuum. As we will examine later, in order to reach a general point $(t_1, t_2, \ldots)$, we must perturb the matrix model by the associated flows, which may or may not be allowed within the original model. Especially the flow connecting two different critical points must deviate from the matrix model. This is because even though the critical points are individually realized by the critical matrix models, in between one cannot find the corresponding matrix models, in other words, if we impose the physical conditions on the solution of the string equation in order to extract the physics from the formal system like the generalized KdV, the NLS, or the ZS hierarchy, those solutions generally become unstable under the associated flows [21]. Unfortunately the mapping itself is not strong enough to solve this reduction problem, but it will provide a foundation to get further insight.
2. Two-cut Models

2.1 Hierarchical Criticality

We will determine the critical potentials of the hermitian one matrix models, and classify them into hierarchies. The string equation (1.8) governs the matrix model completely, but here we use an alternative, but equivalent pair of equations

\[ \langle n - 1 \mid V'(\bar{\lambda}) \mid n \rangle = \frac{n}{\beta}, \]  
\[ \langle n \mid V'(\bar{\lambda}) \mid n \rangle = 0, \]  

which we also call the string equation. In terms of the recursion relation

\[ \lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_\lambda(\lambda) + R_n P_{n-1}(\lambda), \]  

one can show that

\[ \hat{\lambda} \overset{\text{def}}{=} \frac{1}{\sqrt{\hat{h}}} \hat{\lambda} \sqrt{\hat{h}} = \hat{\delta} + \hat{S} + \hat{\delta}^\dagger \hat{R}, \]  

where \( \hat{R} \mid n \rangle = R_n \mid n \rangle, \hat{S} \mid n \rangle = S_n \mid n \rangle, \hat{h} \mid n \rangle = h_n \mid n \rangle, \hat{\delta} \mid n \rangle = \mid n + 1 \rangle, \) and \( \hat{\delta}^\dagger \mid n \rangle = \mid n - 1 \rangle. \) Due to the hermiticity of \( \hat{\lambda}, \) (2.1a) may be rewritten as

\[ \langle n \mid 1 - \frac{1}{2} \hat{\lambda} \hat{V}'(\lambda) \mid n \rangle = 1 - \frac{n + \frac{1}{2}}{\beta}, \]  

which is more convenient for our present purpose. Now assume that \( R_n \) and \( S_n \) converge as \( n \to \beta, \) and denote those limits by \( a_2 \) and \( b, \) respectively (the positivity of \( R_n \) follows from \( R_n = h_n/h_{n-1}, \)) then (2.1c) and (2.1b) may be evaluated by the contour integrals around \( z = 0 \)

\[ \int_{C_0} \frac{dz}{z} \left[ 1 - \frac{1}{2} \lambda(z) V'(\lambda(z)) \right] = 0, \]  
\[ \int_{C_0} \frac{dz}{z} V'(\lambda(z)) = 0, \]  

with \( \lambda(z) = z + b + a_2^2/z. \) We can rewrite (2.4) further in terms of \( z = [\lambda - b - \sqrt{(\lambda - b)^2 - 4a^2}]/2 \) such that

\[ \int_{C_\infty} d\lambda \left( (\lambda - b)^2 - 4a^2 \right)^{-\frac{1}{2}} \left[ 1 - \frac{1}{2} \lambda V'(\lambda) \right] = 0, \]  
\[ \int_{C_\infty} d\lambda \left( (\lambda - b)^2 - 4a^2 \right)^{-\frac{1}{2}} V'(\lambda) = 0, \]  

(2.5)
where the contour $C_\infty$ must be large enough to enclose all singularities on the complex plane. Noticing that

$$\int_{C_\infty} d\eta (\eta^2 - 4a^2)^{-\frac{3}{2}} \eta^m = 0,$$

(2.6)

for $m$ odd integral or negative even integral, it is easy to obtain a general solution

$$1 - \frac{1}{2} \lambda V'(\lambda) = [\lambda F(\lambda)]_+, \quad F(\lambda) = \alpha(\lambda - \lambda_+)^{\frac{1}{2}} (\lambda - \lambda_-)^{\frac{1}{2}}.$$

(2.7)

Here $\lambda_\pm = b \pm 2a$, and $[\ ]_+$ denotes the principal part (including the $\lambda^0$ term) of the Laurent expansion about $\lambda = \infty$. $c(\lambda)$ is an arbitrary entire function satisfying $\text{Res}_{\lambda=\infty} F(\lambda) = -1$ in order that the $\lambda^0$ term of $\lambda F(\lambda)$ equals 1. Note that the first line in (2.7) may be expressed as $V'(\lambda) = -2 [F(\lambda)]_+$. $F(\lambda)$ controls the eigenvalue distribution of $\lambda$ through (see e.g. [22])

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \text{Im} F(\lambda - i\epsilon).$$

(2.8)

Hence $c(\lambda)$ must be negative along the support of $\rho(\lambda)$, which turns out to be the interval $(\lambda_-, \lambda_+)$ separated by the even order zeros of $c(\lambda)$ (no odd order zeros are allowed along $(\lambda_-, \lambda_+)$). One might think $c(\lambda)$ could have cuts along $(\lambda_-, \lambda_+)$, but this cannot happen because of (2.5) and the negativity of $c(\lambda)$. We will see, for the critical potential $V(\lambda)$ given by (2.7), that (2.1a) and (2.1b) scale to the continuum string equation at the double-scaling limit, and this scaling behavior is completely governed by the eigenvalue distribution at the edge, i.e. the zeros of $F(\lambda)$.

Suppose $c(\lambda)$ has a zero only at $\lambda = \lambda_+$, then $c(\lambda)$ may be expanded as

$$c(\lambda) = \sum_{k \geq 1} c_k (\lambda_+ - \lambda)^k,$$

(2.9)

where $c_k$ must satisfy $\sum_{k \geq 1} (2a)^{k+2} \frac{(2k+1)!!}{(k+2)!} c_k = -1$. This is the family of one-cut models, in which the $m^{th}$ critical potentials can be obtained by choosing $c_m < 0$ and $c_k = 0$ for $k < m$, where the negativity of $c_m$ follows from the negativity of $c(\lambda)$. More generally, $c(\lambda)$ may have zeros at both ends such as

$$c(\lambda) = \sum_{k_+ \geq 0, k_- \geq 0} c_{k_+, k_-} (\lambda_+ - \lambda)^{k_+} (\lambda - \lambda_-)^{k_-},$$

(2.10)

which is the doubling studied in ref. [16]; in particular, $k_+ = k_-$ gives the family of even one-cut models. All of these one-cut families yield the KdV hierarchy at
the double-scaling limit, whereas the mKdV hierarchy and more generally the NLS hierarchy may be obtained from the two-cut family as follows.

Suppose $c(\lambda)$ has an even order zero at $\lambda = b$

$$c(\lambda) = \sum_{k \geq 1} c_{2k} (\lambda - b)^{2k}, \quad (2.11)$$

then $c_{2k}$ must satisfy $\sum_{k \geq 1} (2a^2)^{k+1} \frac{(2k-1)!!}{(k+1)!} c_{2k} = -1$. These are the two-cut models studied in ref. [18], in which the $2m^{th}$ critical potentials* are given by $c_{2m} < 0$ and $c_{2k} = 0$ for $k < m$; in particular we call $c_{2k} = -\frac{(m+1)!}{(2a^2)^{m+1}(2m-1)!!} \delta_{k,m}$ the $2m^{th}$ critical point. More general critical model can be obtained by taking the product of these two kinds of critical models in the same manner as the ($k_+,k_-$) model has been constructed from the doubling of the $k_+^{th}$ and $k_-^{th}$ critical one-cut models in ref. [16].

2.2 String Equation

We will fix our notation for the two-cut models by following Crnković and Moore [17], and give a heuristic argument to derive the continuum string equations. The $2m^{th}$ double-scaling limit may be defined by $\epsilon \to 0$ in

$$R_n = 1 + (-1)^n 2\epsilon f(t) + \ldots,$$

$$S_n = b + (-1)^n 2\epsilon g(t) + \ldots, \quad (2.12)$$

with $x = n/\beta = 1 - \epsilon^{2m} t$, and $\kappa^2 \beta \epsilon^{2m+1} = 1$ ($a$ has been normalized to 1). For the orthogonal two component vector defined by

$$P_{2l+1}^\dagger(\lambda) = \exp\left(-\frac{\beta}{2} V(\lambda)\right) (-1)^l P_{2l}(\lambda)/\sqrt{h_{2l+1}},$$

$$P_{2l+1}^-(\lambda) = \exp\left(-\frac{\beta}{2} V(\lambda)\right) (-1)^l P_{2l+1}(\lambda)/\sqrt{h_{2l+1}}, \quad (2.13)$$

$\hat{\lambda}$ may be represented as a $2 \times 2$ matrix operator

$$\hat{\lambda} \begin{pmatrix} P_{2l}^+ \\ P_{2l+1}^+ \end{pmatrix} = \begin{pmatrix} \hat{S} & 1 - \hat{z}^\dagger \hat{R} \hat{z}^\dagger \\ -\hat{z}^2 + \hat{R} \end{pmatrix} \begin{pmatrix} P_{2l}^+ \\ P_{2l+1}^+ \end{pmatrix}. \quad (2.14)$$

At the scaling limit, $P_{n}^\pm(\lambda)$ scales to $\kappa \sqrt{\epsilon} \langle t | \lambda \rangle_\pm$ with $|\lambda\rangle$ being a two-component vector, while both $\hat{\lambda}$ and $\hat{\lambda}$ scale as

$$\langle t | \hat{\lambda} | \lambda \rangle = \langle t | bE + 2\epsilon L | \lambda \rangle. \quad (2.15)$$

* No odd critical potentials are allowed within the hermitian models.
Hence we will not distinguish \( \hat{\lambda} \) from \( \check{\lambda} \). \( E \) is the identity \( 2 \times 2 \) matrix, and \( L \) is the Lax operator \( L = -i \sigma_2 \kappa^2 \partial_t - \sigma_1 f + \sigma_3 g \) with \( \sigma_i \)'s being the Pauli matrices.

For the critical potential given by (2.11), the leading part of (2.1a) is trivially satisfied due to (2.4), and the rest of (2.1a) and (2.1b) can be combined into the string equation
\[
\langle t \mid V'(\hat{\lambda}) \mid t \rangle = -i \epsilon^{2m} t \langle t \mid t \rangle,
\]
where the factor \(-i\) is necessary to compensate for the extra \( i \) factor contained in \( \langle t \mid t \rangle \) (see §2.3). Based on the same argument as ref. [12], all \( \langle t \mid \hat{\lambda}^n \mid t \rangle \) with \( n \) negative vanish. Thus we can remove the restriction \([ \ ]_+\) from \( V'(\hat{\lambda}) \), and find that
\[
t \langle t \mid L^0 \mid t \rangle + 4^{m+1} c_{2m} \langle t \mid L^{2m} \mid t \rangle = 0.
\]
This holds for any \( 2m^{th} \) critical potential, since the higher order terms disappear as \( \epsilon \to 0 \).

### 2.3 Diagonal Kernel of the Resolvent

The explicit form of the residues for the general matrix Lax operator \( L \) can be obtained from the resolvent \( L - \lambda E \), which has been studied in ref. [20] and their theorem 6 provides us with the following results. The Laurent expansion of the diagonal kernel of the resolvent about \( \lambda = \infty \) starts from the zero-th order, and for \( k \geq -1 \)
\[
\langle t \mid L^k \mid t \rangle = \frac{i}{2} (F_k \sigma_3 + G_k \sigma_1 + H_k E),
\]
where
\[
F_{k+1} = g H_k - \frac{1}{2} G_k',
\]
\[
G_{k+1} = -f H_k + \frac{1}{2} F_k',
\]
\[
\frac{1}{2} H_k' = f F_k + g G_k,
\]
with \( ' \) denoting \( D = \kappa^2 \partial_t \). First two levels can be determined as \( F_{-1} = G_{-1} = 0 \), \( H_{-1} = 1 \), \( F_0 = g \), \( G_0 = -f \), \( H_0 = 0 \) by assuming that \( f \) and \( g \) are \( t \) independent and directly evaluating the diagonal kernel as
\[
\langle t \mid L - \lambda E \mid t \rangle = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \left( \begin{array}{cc} g - \lambda + i\epsilon & -f - ip \\ -f + ip & -g - \lambda + i\epsilon \end{array} \right)^{-1},
\]
\[
= \frac{i}{2\sqrt{\lambda^2 - f^2 - g^2}} \left( \begin{array}{cc} -g - \lambda & f \\ f & g - \lambda \end{array} \right).
\]
Higher levels may be obtained from (2.19), where the first few are

\[ F_1 = \frac{1}{2} f', \quad G_1 = \frac{1}{2} g', \quad H_1 = \frac{1}{2} (f^2 + g^2), \]
\[ F_2 = -\frac{1}{4} g'' + \frac{1}{2} g(f^2 + g^2), \quad G_2 = \frac{1}{4} f'' - \frac{1}{2} f(f^2 + g^2), \quad H_2 = \frac{1}{2} (gf' - fg'), \]
\[ F_3 = -\frac{1}{8} f''' + \frac{3}{4} f'(f^2 + g^2), \quad G_3 = -\frac{1}{8} g''' + \frac{3}{4} g'(f^2 + g^2), \]
\[ H_3 = -\frac{1}{4} f f'' + \frac{1}{8} (f')^2 - \frac{1}{4} g g'' + \frac{1}{8} (g')^2 + \frac{3}{8} (f^2 + g^2)^2. \]  

(2.21)

2.4 Reduction to the mKdV Hierarchy

For an even potential model, one finds \( S_n = 0 \) and therefore \( g = 0 \), for which (2.19) yields \( F_{2k+1} = \frac{1}{2} S'_k[f] \), \( G_{2k} = -S_k[f] \), \( H_{2k+1} = R_{k+1}[u] - \frac{1}{2} S'_k[f] = R_{k+1}[\bar{u}] + \frac{1}{2} S'_k[f] \), and otherwise zero, where \( u \) and \( \bar{u} \) are defined by the Miura mapping \( u = f^2 + f' \) and \( \bar{u} = f^2 - f' \), and \( S_k[f] \) and \( R_k[u] \) are the \( k \)th Gel'fand-Dikii differential polynomial in the mKdV and the KdV hierarchies, respectively.

The definitions are

\[ S_k[f] = [(f - \frac{1}{2} D) D^{-1} (f + \frac{1}{2} D) D]^k f = [-\frac{1}{4} D^2 + f D^{-1} f D]^k f, \]
\[ R_k[u] = [D^{-1} (f + \frac{1}{2} D) D (f - \frac{1}{2} D)]^k 1 = [-\frac{1}{4} D^2 + \frac{1}{2} u + \frac{1}{2} D^{-1} u D]^k 1. \]  

(2.22)

The \( 2m \)th string equations in the NLS hierarchy are therefore

\[ tg + 4^{m+1} c_{2m} F_{2m} = 0, \]  
\[ tf - 4^{m+1} c_{2m} G_{2m} = 0, \]  

(2.23a)
\[ (2.23b) \]

with \( c_{2m} < 0 \). (2.17) also provides the third string equation \( H_{2m} = 0 \), but since \( H'_{2m} = 0 \) follows from (2.23), and \( H_{2m} \) is odd under either \( f \to -f \) or \( g \to -g \) while any of \((\pm f, \pm g)\) satisfies (2.23), the third string equation may be obtained by integrating (2.23).*

We will conclude this section with a few comments. Since the scaling behavior does not depend on the parameter \( b \), we omit \( b \) hereafter, while keeping \( g \) general. In the sphere limit \( \kappa = 0 \), or as \( t \to \infty \), all \( t \) derivative disappear, so that (2.20) provides exact relations, such as \( F_{2m} = \frac{(2m-1)!}{2m m!} g (f^2 + g^2)^m \), \( G_{2m} = -\frac{(2m-1)!}{2m m!} f (f^2 + g^2)^m \). The string equation at the \( 2m \)th critical point is thus given by

\[ t = 2 (m + 1) (f^2 + g^2)^m. \]  

(2.24)

* The integration constant does not vanish if the odd modes are mixed in.
For an even potential, the system is reduced by \( g = 0 \), and only (2.23b) remains nontrivial, which may be written down as

\[
tf = \frac{2^{m+1}(m+1)!}{(2m-1)!!} S_m[f],
\]

(2.25)

at the \( 2m^{th} \) critical point. The same string equation has been found also in the one-cut family of the unitary matrix models in ref. [23]; furthermore it is proved that (2.25) has a unique, real, pole-free, solution consistent with the above asymptotic behavior (2.24) for \( t \rightarrow \infty \) [24].

3. Discrete and Continuum

3.1 Mapping from Toda to NLS

The partition function (1.1) is a function of coupling constants through the potential \( V(z) = \sum_{k=0}^{\infty} g_k z^k \), and satisfies the differential equation

\[
J(z) Z = \left[ \frac{1}{2} V'(z) - \langle \hat{W}(z) \rangle \right] Z.
\]

(3.1)

Here \( \langle \hat{W}(z) \rangle = \langle \beta^{-1} \text{tr} \left[ (z - \phi)^{-1} \right] \rangle \) is the generating function, and \( J(z) = \sum_{k \in \mathbb{Z}} z^{-k-1} J_k \) is defined by \( J_k = \beta^{-2} \partial / \partial g_k \) for \( k \geq 0 \), and \( J_{-k} = \frac{k}{2} g_k \) for \( k \geq 1 \) [12], so that \( J_k \) satisfies

\[
[J_k, J_l] = \frac{1}{2} \beta^{-2} k \delta_{k+l,0}.
\]

(3.2)

Re-expanding \( J(z) \) as \( J(z) = \sum_{k \in \mathbb{Z}} z^{-k} (z^2 - 4)^{-\frac{1}{2}} \tilde{J}_k \), one can find that

\[
\tilde{J}_{2k} = \int_{C_\infty} dz \frac{dz}{2\pi i} (z^2 - 4)^{\frac{1}{2}} z^{2k-1} J^{\text{even}}(z),
\]

\[
\tilde{J}_{2k+1} = \int_{C_\infty} dz \frac{dz}{2\pi i} (z^2 - 4)^{\frac{1}{2}} z^{2k} J^{\text{odd}}(z),
\]

(3.3)

where \( J(z) \) is divided into odd and even functions of \( z \), denoted as \( J^{\text{even}}(z) \) and \( J^{\text{odd}}(z) \), respectively. Note that the superscripts of \( J \) are named according to the even/odd nature of the potential, not its derivative. \( J^{\text{odd}}(z) \) disappears from \( \tilde{J}_{2k} \) because of the cancellation along \( C_\infty \), and so does \( J^{\text{even}}(z) \) from \( \tilde{J}_{2k} \). From (3.2) and (3.3), the commutation relations for \( \tilde{J}_k \) can be calculated as

\[
[J_k, \tilde{J}_l] = \beta^{-2} \left[ \frac{1}{2} k \delta_{k+l,0} - 2 (k - 1) \delta_{k+l,2} \right].
\]

(3.4)
According to (2.11), $V'$ may be expand such that
\[ V'(z) = -2 \sum_{k \geq -1} c_k \left[ z^k (z^2 - 4)^{1/2} \right]_+, \tag{3.5} \]
where the $2m^{th}$ critical point is defined by $c_k = -\frac{(m+1)!}{2m+1 (2m-1)!!} \delta_{k,2m}$. From (3.4), the non-positive modes $\tilde{\mathcal{J}}_k$; $k = 0, -1, -2, \ldots$ commute with each other and contain no differential operators except for $\partial/\partial g_0$ in $\tilde{\mathcal{J}}_0$, which may be replaced by $-x$ in terms of $\beta^{-2} \partial (\log Z)/\partial g_0 = -x$. Therefore these may be considered as a new coordinate system for the space of the coupling constants instead of $c_k$'s or $g_k$'s. These are related to $c_k$ such that
\[ \tilde{\mathcal{J}}_{-k} \overset{\text{def}}{=} 2 (k+1) p_k = 4 c_k - c_{k-2}, \quad (k \geq 0) \tag{3.6} \]
where $c_{-2} = x - 1$, since $\text{Res}_{z = \infty} \left[ \sum_{k \geq -1} c_k z^k (z^2 - 4)^{1/2} \right] = -1$. On the other hand, the positive modes $\mathcal{J}_k$, $k = 1, 2, \ldots$, are the pure differential operators w.r.t. $p_k$, namely
\[ \tilde{\mathcal{J}}_k = \beta^{-2} \left[ \frac{k}{4(k+1)} \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_{k-2}} \right], \quad (k \geq 2) \tag{3.7} \]
This is not the case if we write down $\mathcal{J}_k$'s as differential operators w.r.t. $g_k$'s, i.e.
\[ \tilde{\mathcal{J}}_k = \tilde{\mathcal{J}}^+_k + \tilde{\mathcal{J}}^-_k, \quad (k \geq 1) \tag{3.8} \]
where
\[ \tilde{\mathcal{J}}^+_k Z = - \int_{C_{\infty}} \frac{dz}{2\pi i} (z^2 - 4)^{1/2} z^{k-1} \beta^{-2} \langle \text{tr} \left( \frac{1}{z - \phi} \right) \rangle Z, \tag{3.9a} \]
and
\[ \tilde{\mathcal{J}}^-_k = \frac{1}{2} \int_{C_{\infty}} \frac{dz}{2\pi i} (z^2 - 4)^{1/2} z^{k-1} V'(z), \tag{3.9b} \]
Since $\tilde{\mathcal{J}}^-_k$ is a $t$ independent number, we can absorb $\tilde{\mathcal{J}}^-_k$ into the integration constant of the trace $\langle \rangle$ for each $\tilde{\mathcal{J}}^+_k$, and obtain
\[ \tilde{\mathcal{J}}_k Z = -\beta^{-1} \langle \hat{\lambda}^{k-1} (\hat{\lambda}^2 - 4)^{1/2} \rangle Z. \tag{3.10} \]
Defining the scaling parameters $t_k$ by
\[ 4 c_{-1} = 2 \epsilon^{2m+1} q, \]
\[ 4 c_0 - c_{-2} = \epsilon^{2m} t_0, \tag{3.11} \]
\[ 4 c_k = \epsilon^{2m-k} 2^{-k} (k+1) t_k, \quad (k \geq 1) \]
the $2m^{th}$ critical point of the two-cut model is given by $q = 0$, $t_0 = t$ and $t_k = -\frac{2^{m+1}(m+1)!}{(2m+1)!!} \delta_{k,2m}$. As $\epsilon \to 0$, $c_{k-2}; k \geq 1$ in (3.6) and $\partial/\partial p_k; k \geq 2$ in (3.7) become negligible due to the relative $\epsilon$ factor, and hence $J_k$ scales as

$$J_k = -\kappa^4 \epsilon^{2m+k} 2^{k-1} \frac{\partial}{\partial t_{k-2}}, \quad (k \geq 2)$$

$$\tilde{J}_1 = \epsilon^{2m+1} \left( 2 \alpha - \kappa^4 \frac{\partial}{\partial t_{-1}} \right),$$

$$\tilde{J}_{-k} = \epsilon^{2m-k} 2^{-k} (k + 1) t_k, \quad (k \geq 0)$$

where from (3.9) one can obtain

$$2 \alpha = q - \kappa^2 \beta \sum_{l \geq 1} c_{2l-1} \frac{2^{l+1} (2l - 1)!!}{(l+1)!},$$

$$\kappa^2 \frac{\partial}{\partial t_{-1}} = -\beta^{-1} \frac{\partial}{\partial g_1}.\quad (3.13)$$

Using (3.12) in (3.10), one can find the commutative flows in the NLS hierarchy such that

$$\kappa^2 \frac{\partial F}{\partial t_k} = 2^{-k-1} \epsilon^{-k-1} \langle \tilde{\lambda}^{k+1} (\tilde{\lambda}^2 - 4)^{\frac{1}{2}} \rangle,$$

$$= 2t \int_t^\infty dt \text{tr} \langle t | L^{k+1} | t \rangle,$$

$$= -2 \int_t^\infty dt H_{k+1},\quad (3.14)$$

for $k \geq -1$, where $F = \log Z$ and the trace on the $2 \times 2$ matrix is necessary in order to sum up the contributions from even and odd polynomials. Since $\partial F/\partial t_{-1} = 0$ follows from $H_0 = 0$, the $t_{-1}$ derivative vanishes in $\tilde{J}_1$. Note that $\alpha$ appears in these relations because it cannot be absorbed into the integration constant. $\alpha$ will be identified with the integration constant of the third string equation, and proved to be independent of $t_k$’s in §4.2.
3.2 Virasoro Constraints

Multiplying (3.1) once again with $J(z)$, one can find the discrete Ward identity [12]

$$ : J(z)^2 : Z = \frac{1}{4} \Delta(z) Z , $$

(3.15)

where $: :$ denotes the normal ordering, and $\Delta(z)$ is an entire function defined by

$$ \Delta(z) = (V'(z))^2 - 4 \left\langle \frac{V'(z) - V'(-\lambda)}{z - \lambda} \right\rangle. $$

(3.16)

Hence $Z$ satisfies

$$ \int_{C_\infty} \frac{dz}{2\pi i} z^{n+1} : J(z)^2 : Z = 0, \quad (n \geq -1) $$

(3.17)

for which one can find the discrete Virasoro constraints (1.10) by calculating the contour integral. Alternatively (3.17) may be rewritten as

$$ \begin{align*}
\int_{C_\infty} \frac{dz}{2\pi i} : J(z)^2 : Z &= 0, \\
\int_{C_\infty} \frac{dz}{2\pi i} z : J(z)^2 : Z &= 0, \\
\int_{C_\infty} \frac{dz}{2\pi i} z^{n+1} (z^2 - 4) : J(z)^2 : Z &= 0, \quad (n \geq -1)
\end{align*} $$

(3.18)

and calculating the contour integral we can replace (1.10) with

$$ \begin{align*}
\left( \sum_{k \in \mathbb{Z}} 2^{-2k} \tilde{J}_{2k} \right) \left( \sum_{l \in \mathbb{Z}} 2^{-2l-1} \tilde{J}_{2l+1} \right) Z &= 0, \\
\left[ \left( \sum_{k \in \mathbb{Z}} 2^{-2k} \tilde{J}_{2k} \right)^2 + \left( \sum_{l \in \mathbb{Z}} 2^{-2l-1} \tilde{J}_{2l+1} \right)^2 \right] Z &= 0, \\
\left( \sum_{k \in \mathbb{Z}} : \tilde{J}_{-k} \tilde{J}_{n+k-2} : \right) Z &= 0.
\end{align*} $$

(3.19)

From (3.3) and (2.6), we can get

$$ \begin{align*}
\left( \sum_{k \in \mathbb{Z}} 2^{-2k} \tilde{J}_{2k} \right) Z &= \int_{C_\infty - C_0} \frac{dz}{2\pi i} z (z^2 - 4)^{-\frac{1}{2}} J^{even}(z) Z, \\
&= \int_{C_\infty} \frac{dz}{2\pi i} (z^2 - 4)^{-\frac{1}{2}} \left[ \frac{1}{2} z V'(z) - 1 \right] Z,
\end{align*} $$

(3.20)
where the r.h.s. of both (3.20a) and (3.20b) vanish automatically, since the critical potential (3.5) always satisfies (2.5). Hence (3.19a) and (3.19b) are satisfied trivially before taking the scaling limit. In terms of the scaling of \( \tilde{J}_k \) in the vicinity of the \( 2m^{th} \) critical point, (3.19c) scales to the continuum Virasoro constraints

\[
L_n Z = 0, \quad (n \geq -1)
\]

where

\[
L_{-1} = \sum_{k=1}^{\infty} (k + 1) t_k \frac{\partial}{\partial t_{k-1}} - 2 \kappa^{-4} \alpha t,
\]

\[
L_0 = \sum_{k=0}^{\infty} (k + 1) t_k \frac{\partial}{\partial t_k} - \kappa^{-4} \alpha^2,
\]

\[
L_1 = \sum_{k=0}^{\infty} (k + 1) t_k \frac{\partial}{\partial t_{k+1}} + \alpha \frac{\partial}{\partial t},
\]

\[
L_n = \sum_{k=0}^{\infty} (k + 1) t_k \frac{\partial}{\partial t_{k+n}} - \kappa^4 \frac{1}{4} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_{k-1} \partial t_{n-k-1}} + \alpha \frac{\partial}{\partial t_{n-1}}.
\]

These constraints have been found already in refs. [17,18] by integrating the string equations

\[
\sum_{k \geq 0} (k + 1) t_k F_k = 0,
\]

\[
\sum_{k \geq 0} (k + 1) t_k G_k = 0.
\]

Finally, defining \( \langle \sigma_k \rangle = \int dt \, H_{k+1} \), we arrive at

\[
\sum_{k=1}^{\infty} (k + 1) t_k \langle \sigma_{k-1} \rangle - \kappa^{-2} \alpha t = 0,
\]

\[
\sum_{k=0}^{\infty} (k + 1) t_k \langle \sigma_k \rangle - \kappa^{-2} \frac{\alpha^2}{2} = 0,
\]

\[
\sum_{k=0}^{\infty} (k + 1) t_k \langle \sigma_{k+1} \rangle + \alpha \langle \sigma_0 \rangle = 0,
\]

\[
\sum_{k=0}^{\infty} (k + 1) t_k \langle \sigma_{k+n} \rangle + \alpha \langle \sigma_{n-1} \rangle
\]

\[
- \kappa^2 \frac{2}{2} \sum_{k=1}^{n-1} (\langle \sigma_{k-1} \sigma_{n-k-1} \rangle + \langle \sigma_{k-1} \rangle \langle \sigma_{n-k-1} \rangle) = 0.
\]
4. NLS Hierarchy

4.1 Flows and String Equations

Gel’fand and Dikii [20] have obtained the Lax pair for the general matrix differential operator $L$ in terms of the diagonal kernel $R(z) = \langle t | (L - z E)^{-1} | t \rangle$. For our first order $2 \times 2$ matrix operator $L$, their theorem 7 can be read as

$$\sum_{k \geq 0} P_k z^{-k-1} \overset{\text{def}}{=} R(z) \sigma_2 (L - z E)^{-1} = (L - z E)^{-1} \sigma_2 R(z), \quad (4.1)$$

where the last equality follows from their theorem 6. First few of $P_k$ are

$$P_0 = -\frac{i}{2} \sigma_2, \quad P_1 = -\frac{1}{2} D,$$

$$P_2 = \frac{i}{2} \sigma_2 [D^2 - \frac{1}{2} \{D, f \sigma_3 + g \sigma_1\} - \frac{1}{2} (f \sigma_3 + g \sigma_1)^2], \quad (4.2)$$

$$P_3 = \frac{1}{2} \{D^3 - \frac{3}{4} D^2, f \sigma_3 + g \sigma_1\} - \frac{3}{4} \{D, (f \sigma_3 + g \sigma_1)^2\},$$

and the $k^{th}$ equation in the NLS hierarchy may be defined as

$$\frac{\partial L}{\partial t_k} = 2[L, P_{k+1}] = 2 G_{k+1} \sigma_3 - 2 F_{k+1} \sigma_1. \quad (4.3)$$

Here the $k^{th}$ NLS flows

$$\frac{\partial f}{\partial t_k} = 2 F_{k+1},$$

$$\frac{\partial g}{\partial t_k} = 2 G_{k+1}, \quad (4.4)$$

are all commutative for $k \geq -1$.

The original definition in ref. [20] has no flows for $k = -1$ due to the extra $(k + 1)$ factor appearing in the r.h.s. of (4.4), which we have been normalized to the definition of $t_k$. Both are consistent with the previous definition based on the mapping from the Toda flows. In fact, by taking the $t$ derivative twice on (3.14), one can find that

$$\frac{\partial (f^2 + g^2)}{\partial t_k} = 2 H'_{k+1} = 4 (f F_{k+1} + g G_{k+1}), \quad (4.5)$$

for $k \geq -1$, which agrees with (4.4), while the $(-1)^{th}$ flow vanishes on $f^2 + g^2$ anyway. Actually these two definitions are equivalent due to the factorization of the $t_{-1}$ dependence. This can be done as follows. First introduce the complex
notation $\psi = f + ig$ and $U_k = F_k + iG_k$ $(k \geq -1)$; thus the $t_{-1}$ dependence of $\psi$ can be factorized as $\psi = e^{-2it_{-1}} \psi_0$ by solving the $(-1)^{th}$ equation

$$\frac{\partial f}{\partial t_{-1}} = 2g,$$

$$\frac{\partial g}{\partial t_{-1}} = -2f. \tag{4.6}$$

Rewriting (2.19) with the complex notation*

$$U_{k+1} = -i\psi H_k + \frac{i}{2} U'_k,$$

$$H'_k = \psi^* U_k + \psi U^*_k, \tag{4.7}$$

and using the induction w.r.t. $k$ starting with $U_{-1} = -i\psi$, one can find that all $U_k$ have the same $t_{-1}$ dependence as $\psi$, namely $U_k = e^{-2it_{-1}} (U_k)_0$, and $t_{-1}$ independent $\psi_0$ and $(U_k)_0$ again satisfy the same recursion relations as (4.7), where $H_k$'s are $t_{-1}$ independent by virtue of (4.5). Therefore we can conclude that the $t_{-1}$ dependence is redundant, and $\psi$ has a global phase ambiguity. We will leave the $(-1)^{th}$ flow non-vanishing for convenience.

It is noteworthy that the string equations (3.23) can be rewritten as

$$[M, L] = E, \tag{4.8}$$

by defining $M = \sum_{k \geq 0} (k+1) t_k P_k$. This is the scaling from (1.8) instead of (2.1).

4.2 Invariance of the String Equations

We will prove the invariance of the string equations under the NLS flow. Defining

$$K_{-1} = \sum_{k \geq 0} (k+1) t_k \frac{\partial}{\partial t_{k-1}}, \tag{4.9}$$

the string equations (3.23) may be expressed as

$$K_{-1} f = K_{-1} g = 0, \tag{4.10}$$

and integrating $K_{-1} (f^2 + g^2) = 0$ once w.r.t. $t$, one can obtain the third string equation

$$\sum_{k \geq 0} (k+1) t_k H_k = c, \tag{4.11}$$

* The first equation in the NLS hierarchy is $i \psi_{t_1} = -\psi''/2 + |\psi|^2 \psi$, the nonlinear Schrödinger equation.
where the integration constant $c$ may not vanish, and by taking the $t$ derivative once on $L_{-1} Z = 0$, one can identify it as $c = \kappa^{-2} \alpha$.

The string equations (4.10) and (4.11) are invariant under the NLS flows if

$$
(l + 1) F_{l} + K_{-1} F_{l+1} = 0, \\
(l + 1) G_{l} + K_{-1} G_{l+1} = 0, \\
(l + 1) H_{l} + K_{-1} H_{l+1} = 0,
$$

(4.12a)

(4.12b)

(4.12c)

hold for $l \geq -1$. Here we have used the $t_k$ independence of $\alpha$ to get (4.12c)

$$
\partial \alpha/\partial t_{-1} = \partial \alpha/\partial t_{0} = 0
$$

follows directly from the string equations, and $\partial \alpha/\partial t_{k} = 0$ ($k : \text{even}$) follows from the definition of $\alpha$, while $\partial \alpha/\partial t_{k} = 0$ ($k : \text{odd}$) is a result of the fine-tuning of the odd couplings to make $\alpha$ finite. Now we can prove (4.12) by the induction w.r.t. $l$ starting from $l = -1$, which is nothing other than the string equations (4.10). Suppose that (4.12) holds for $l \leq k - 1$, then the recursion relations yield

$$
K_{-1} F_{k+1} = g K_{-1} H_{k} - \frac{1}{2} (K_{-1} G_{k})' + \frac{1}{2} \frac{\partial}{\partial t_{-1}} G_{k},
$$

(4.13)

$$
= -(k + 1) F_{k}.
$$

(4.12b) may be proved similarly.

In the presence of (4.11), we can derive alternative string equations by taking the $t$ derivative once on (4.10); namely

$$
K_{0} f = -f, \quad K_{0} g = -g,
$$

(4.14)

where

$$
K_{0} = \sum_{k \geq 0} (k + 1) t_{k} \frac{\partial}{\partial t_{k}} - c \frac{\partial}{\partial t_{-1}}.
$$

(4.15)

Taking the $t$ derivative $n$ times on (4.14), one can find that

$$
K_{0} f^{(n)} = -(n + 1) f^{(n)},
$$

$$
K_{0} g^{(n)} = -(n + 1) g^{(n)}.
$$

(4.16)

It is convenient to define “dimensions” by $d(f) = d(g) = -1$ and $d(t_{k}) = k + 1$. Since $K_{0}$ is a first order differential operator, i.e. $K_{0} (A B) = (K_{0} A) B + A (K_{0} B)$, is therefore an operator counting the dimensions of the differential polynomial $A[f, g]$ if $f$ and $g$ are the solution of (4.14); namely

$$
K_{0} A[f, g] = d(A) A[f, g],
$$

(4.17)

where $A[f, g]$ may have explicit $t_{k}$ dependences except for $t_{-1}$. It is then easy to prove the invariance of (4.14) under the NLS flows directly.
By restricting the potential to be an even function of $\lambda$ ($b \neq 0$), we can find a constraint $g = 0$, and consequently the NLS hierarchy is reduced to the mKdV hierarchy with $t_{2k-1} = 0$ for $k \geq 0$, where the odd flows are all frozen because they vary the constraint $g = 0$. Only the second string equation $K_{-1} g = 0$ remains nontrivial in this reduction, and one can obtain the mKdV flows and the string equation

$$\frac{\partial f}{\partial t_{2k}} = S'_k[f], \quad (k \geq 0)$$

$$\sum_{k \geq 0} (2k + 1) t_{2k} S_k[f] = 0,$$

where the string susceptibility is given by $\partial^2 F = \kappa^{-2} f^2$. It is more appropriate to describe (4.19) as the once integrated form of $K_0 f = -f$, which is manifestly invariant under the mKdV flows.

5. **ZS Hierarchy**

5.1 Rotation from NLS to ZS

One can deal with the anti-hermitian matrix model in the same manner as we did with the hermitian matrix model in §2. Hollowood et al. [18] have already studied the anti-hermitian matrix model with real couplings, and found what they call the Zakharov-Shabat (ZS) hierarchy. Here we will re-investigate this model in terms of the mapping we have found. By rotating the anti-hermitian matrix $\tilde{\phi}$ to the hermitian matrix $\phi = i \tilde{\phi}$, one can find that the hermitian matrix model has a complex potential, namely $g_k = (-i)^k \tilde{g}_k$, where $\tilde{g}_k$’s are the couplings of the anti-hermitian model, and all real valued. Since the eigenvalues of $\tilde{\phi}$ distributes along the imaginary axis, the limit values of $R_n$ and $S_n$ also rotate as $\tilde{a} = -i a$ and $\tilde{b} = -i b$, where $a$ and $b$ are the limits defined previously in the hermitian model, and consequently the scaling functions are given by $\tilde{f} = -f$ and $\tilde{g} = -i g$ with $f$ and $g$ being real valued. Defining the coupling constant of the hermitian matrix model by $c_k = (-i)^{2+k} \tilde{c}_k$ ($\tilde{c}_k \in \mathbb{R}$), and the spectral parameter by $z = i \tilde{z}$ ($z \in \mathbb{R}$), one can find that

$$1 - \frac{1}{2} \tilde{z} V'(\tilde{z}) = \sum_{k \geq 1} c_k \left[(z - b)^{k+1} (z - \lambda_+)^{\frac{1}{2}} (z - \lambda_-)^{\frac{1}{2}} \right]_+,$$

in which the $k^{th}$ critical potential is either real or pure imaginary depending on $k$ even or odd as we expect. $c_{2k}$ must satisfy the constraint $\sum_{k \geq 1} \left(2 a^2 k+1 (2k-1)!! \right)^2 c_{2k} = -1$, but unlike the previous model, $c_{2k-1}$ does not have to be zero, because the pure imaginary potential does not contribute to the eigenvalue density. Now the Lax operator

$$\tilde{L} = -i L = -i \sigma_2 D + \sigma_1 f - i \sigma_3 g,$$
turns out to be complex, thus we need an \( SU(2) \) rotation in order to make it pure imaginary, \( i.e. \)
\[
L = \sigma_3 D + i \sigma_2 f + \sigma_1 g.
\]  

(5.3)

Under the rotation from \( c_k \) to \((-i)^{k+2} \tilde{c}_k\), the NLS hierarchy is rotated to the ZS hierarchy by \( t_k \to i^k \tilde{t}_k \), where we have changed \( \tilde{t}_k \) to \((-1)^{k+1} \tilde{t}_k \) for convenience. Note that \( \tilde{t}_0 = t \), and the \( 2m \)th critical point is given by \( \tilde{t}_k = \frac{(-2)^m(m+1)!}{(2m+1)!} \delta_{k,2m} \).

Defining new differential polynomials in such a way that the flow equations appear to be the same, namely
\[
F_k \to (-i)^{k+1} \tilde{F}_k, \quad G_k \to (-i)^k \tilde{G}_k, \quad H_k \to (-i)^{k+1} \tilde{H}_k,
\]
(5.4)
with \( \tilde{F}_{-1} = \tilde{G}_{-1} = 0, \tilde{H}_{-1} = 1, \tilde{F}_0 = g, \tilde{G}_0 = f, \tilde{H}_0 = 0 \); we can find the residue
\[
\langle t \mid L^k \mid t \rangle = \frac{1}{2} \left( \tilde{F}_k \sigma_1 + i \tilde{G}_k \sigma_2 + \tilde{H}_k E \right),
\]
(5.5)
with the recursion relations
\[
\tilde{F}_{k+1} = g \tilde{H}_k + \frac{1}{2} \tilde{G}'_k, \\
\tilde{G}_{k+1} = f \tilde{H}_k + \frac{1}{2} \tilde{F}'_k, \\
\frac{1}{2} \tilde{H}'_k = g \tilde{G}_k - f \tilde{F}_k.
\]
(5.6)

Now all negative signs in the Virasoro constraint (3.22) change to the positive signs because of the \( i^k \) factor, but (3.24) does not change because
\[
k^2 \frac{\partial F}{\partial \tilde{t}_k} = 2 \int t \infty dt \tilde{H}_{k+1}.
\]
(5.7)

In terms of \( \psi = f - g, \bar{\psi} = f + g, U_k = \tilde{F}_k - \tilde{G}_k \), and \( \bar{U}_k = \tilde{F}_k + \tilde{G}_k \), the ZS flows and the string equations may be rewritten as
\[
\frac{\partial \psi}{\partial \tilde{t}_k} = 2 U_{k+1}, \quad \frac{\partial \bar{\psi}}{\partial \tilde{t}_k} = 2 \bar{U}_{k+1},
\]
(5.8)
\[
\sum_{k \geq 0} (k+1) \tilde{t}_k U_k = 0, \quad \sum_{k \geq 0} (k+1) \tilde{t}_k \bar{U}_k = 0, \quad \sum_{k \geq 0} (k+1) \tilde{t}_k \tilde{H}_k = \tilde{c}.
\]
(5.9)

For the even potential, again \( g = 0 \) reduces the ZS hierarchy to the mKdV hierarchy, for which the differential polynomials are \( F_{2k+1} = (-1)^k \frac{1}{2} S_k[f], G_{2k} = \)
\(-1\)^k S_k[f], H_{2k+1} = (-1)^k \left( \frac{1}{2} S'_k[f] - R_{k+1}[f^2 + f'] \right) \), and zero otherwise, hence we can find the same mKdV flows as (4.18) and the same string equation as (4.19) by assigning \( t_{2k} = (-1)^k \tilde{t}_{2k} \), and also the string susceptibility is again \( \partial^2_t F = \kappa^{-2} f^2 \). Note that the \( 2m^{th} \) critical point, \( t_k = -\frac{2^{m+1}(m+1)!}{(2m+1)!!} \delta_{k,2m} \), is precisely same as before.

Hollowood et al. [18] have found an alternative way to freeze odd flows, namely \( \bar{\psi} = e^{2\tilde{t} - 1} \) and \( u = \psi \bar{\psi} \), for which one can find that

\[
\begin{align*}
U_{2k} &= (-1)^k \left( \frac{1}{2} D^2 - u \right) R_k[u], \\
\bar{U}_{2k} &= H_{2k-1} = (-1)^k R_k[u], \\
U_{2k+1} &= 2 H_{2k+2} = (-1)^k R'_{k+1}[u], \\
\bar{U}_{2k+1} &= 0,
\end{align*}
\]

with \( R_k[u] \) being the \( k^{th} \) Gel'fand-Dikii differential polynomials of the KdV hierarchy, therefore this is the reduction into the KdV hierarchy

where the string susceptibility is given by \( \partial^2_t F = \kappa^{-2} u \). This result agrees with the one-cut family of the hermitian models (renormalization of \( t_{2k} \) is necessary to adjust the critical points). The KdV string equation is the once integrated form of the third string equation with \( \bar{c} = \frac{1}{2} \), and is therefore manifestly invariant under the KdV flows.

5.2 Quantization of \( \bar{c} \)

The NLS hierarchy may be obtained from the two component KP hierarchy by the so-called reduction procedure, or alternatively, from the hierarchy of soliton equations associated with the \( A^{(1)}_1 \) Kac-Moody algebra, for which two different vertex operator realizations (the principal and the homogeneous pictures) respectively lead to the KdV and the NLS hierarchies [25]. In the homogeneous picture, the vertex operator has one free parameter \( \gamma \) due to the zero mode of the scalar field, where \( \gamma \) could be any element of the finite root lattice, and consequently the tau-function depends also on \( \gamma \). For \( A^{(1)}_1 \), one finds \( \gamma = m \alpha_1 \) (\( m \in \mathbb{Z} \)), and the scaling functions \( \psi \) and \( \bar{\psi} \) may be assigned to be

\[
\psi = \kappa^{2+4\bar{c}} \frac{\tau_{m+1}}{\tau_m}, \quad \bar{\psi} = \kappa^{2-4\bar{c}} \frac{\tau_{m-1}}{\tau_m}, \quad \psi \bar{\psi} = -\kappa^4 \partial^2_t \log \tau_m,
\]

for which one can get the NLS and the ZS hierarchies by taking particular real slices, \textit{i.e.} \( \bar{\psi} = \psi^* \) (resp. \( \psi, \bar{\psi} \in \mathbb{R} \)).
By following this tau-function formalism, Hollowood et al. [18] have found that the Virasoro constraints $L_n(\tilde{c}) \tau_m = 0 \quad (n \geq -1)$ produce

$$L_n(\tilde{c} \pm 1) \tau_{m \pm 1} = 0, \quad (n \geq -1) \quad (5.14)$$

which strongly suggests the quantization of $\tilde{c}$.

Actually $\tilde{c}$ must be quantized as $\tilde{c} \in \mathbb{Z}/2$ due to the following mechanism. If we rescale $\epsilon$ to $a \epsilon$, while keeping the model and $\kappa$ fixed, $\tilde{t}_k$ changes to $a^{k+1} \tilde{t}_k$. This can be seen most easily in the first line of (3.14), where $F$ and $\langle \rangle$ do not change by assumption. Since (4.14) holds as well in the ZS hierarchy, and the $\tilde{t}_{-1}$ dependence may be factorized as $\psi = e^{-2\tilde{t}_{-1}} \psi_0$ and $\bar{\psi} = e^{2\tilde{t}_{-1}} \bar{\psi}_0$, $\psi$ (and $\bar{\psi}$) has the anomalous dimension $-2\tilde{c}$ (resp. $2\tilde{c}$), i.e.

$$\sum_{k \geq 0} (k + 1) \tilde{t}_k \frac{\partial \psi}{\partial \tilde{t}_k} = -(1 + 2\tilde{c}) \psi,$$

$$\sum_{k \geq 0} (k + 1) \tilde{t}_k \frac{\partial \bar{\psi}}{\partial \tilde{t}_k} = -(1 - 2\tilde{c}) \bar{\psi}. \quad (5.15)$$

Therefore if $2\tilde{c}$ is not integral, the continuum limit $\epsilon \to 0$ becomes non-analytic with respect to $\epsilon$, and hence $\tilde{c}$ must be half-integral.

(5.15) also explains the $\kappa$ factors appearing in (5.13), because the rescaling of $\tilde{t}_0$ may be realized by the rescaling of $\kappa^2$. If the way $R_n$ and $S_n$ approach their limit values is scale invariant, $\tilde{c}$ must be zero, which is the case for the reduction to the mKdV hierarchy. On the other hand, the reduction to the KdV hierarchy has been achieved by the constraint $\bar{\psi} = e^{2\tilde{t}_{-1}}$, thus the scaling dimension of $\bar{\psi}$ must be zero, and hence (5.15) gives $\tilde{c} = 1/2$ which agrees with the previous calculation.

We have derived (4.14) from the string equations (4.10) and (4.11), therefore any solution of the string equations (5.9) satisfies (5.15) for arbitrary value of $\tilde{c}$. However, there is no guarantee that (5.15) with arbitrary $\tilde{c}$ is compatible with the constraint we require such as $\tilde{t}_k = 0 \quad (k : \text{odd})$. Above examples show that the compatibility actually breaks down. But of course, not all constraints quantize $\tilde{c}$, for instance, in the topological phase emerging at $\tilde{t}_1 \neq 0$ and $\tilde{t}_k = 0$ for $k \geq 2$, the string equations have exact solution such as

$$\psi = -\tilde{c} r \tilde{t}_1^{-\frac{1}{2} - \tilde{c}} \exp\left(-2\tilde{t}_{-1} + \frac{t^2}{2t_1}\right), \quad \bar{\psi} = \tilde{c} s \tilde{t}_1^{-\frac{1}{2} + \tilde{c}} \exp\left(2\tilde{t}_{-1} - \frac{t^2}{2t_1}\right), \quad (5.16)$$

with $r + s = 1$, which satisfy (5.15) and the third string equation for arbitrary $\tilde{c}$.  

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6. Discussions and Conclusion

6.1 Hermitian Models and the NLS Hierarchy

The $2m^{th}$ mapping we have obtained in the hermitian model is

$$2c_{-1} - \sum_{l \geq 1} \frac{2^{l+1}(2l-1)!!}{(l+1)!} c_{2l-1} = \epsilon^{2m+1} 2 \alpha,$$

$$4c_0 - c_{-2} = \epsilon^2 m t_0, \quad (t_0 = t) \quad (6.1)$$

$$4c_k = \epsilon^{2m-k} 2^{-k} (k+1) t_k, \quad (k \geq 1)$$

which maps the $2m^{th}$ critical potentials, i.e. $c_{2m} < 0$ and $c_k = 0$ ($k < 2m$), to the subspace $t_{2m} < 0$ and $t_k = 0$ ($k > 2m$). At $\epsilon = 0$, only the $2m^{th}$ critical point is admissible, because $c_k = 0$ ($k < 2m$) and $t_k = 0$ ($k > 2m$) respectively imply $t_k = 0$ ($k < 2m$) and $c_k = 0$ ($k > 2m$) at $\epsilon = 0$, thus $t_{2m} = -\frac{2^{m+1} (m+1)!}{(2m+1)!}$ from the constraint. Owing to the positivity of the density of the eigenvalue distribution, $c_k$ must vanish for $k$ odd, so that no odd critical points are allowed within the hermitian model; nevertheless we may turn on $\alpha, t_1, \ldots t_{2m-1}$, since $c_{-1} = \ldots = c_{2m-1} = 0$ hold as $\epsilon \to 0$. To this extent, we can perturb the system away from the $2m^{th}$ critical point and generalize the system to allow $c_{2m} > 0$ as well as the odd critical points. (6.1) should be understood in this sense, and also we extend it to the $(2m - 1)^{th}$ mapping, where any point of the subspace $t_{2m-1} \neq 0$ and $t_k = 0$ ($k \neq 0, 2m - 1$) may be chosen as the $(2m - 1)^{th}$ critical point.

Among these “time” parameters, the $(-1)^{th}$ parameters $\alpha$ and $t_{-1}$ are very special. $\alpha$ may be identified with the integration constant of the third string equation, and in order to get finite $\alpha$ we must fine-tune the odd couplings such that

$$\sum_{k \geq 1} \frac{2^k (2k-1)!!}{(k+1)!} c_{2k-1} = c_{-1},$$

or equivalently,

$$2 \alpha + \sum_{k \geq 1} \epsilon^{-2k} \frac{k (2k-1)!!}{2^{k-1} (k+1)!} t_{2k-1} = q;$$

however we do not have any flows associated with $\alpha$, instead the $(-1)^{th}$ flow changes the parameter $t_{-1}$, and its numerical value is independent of any of the coupling constants. The reason of the quantization of $c = \kappa^{-2} \alpha$ is therefore not because of the momentum zero-mode quantization such as $2c = -\kappa^2 \partial / \partial t_{-1}$, but because of the analyticity of the continuum limit discussed in §5.2. This is somehow analogous to the quantization of the central extension $c$ in the Kac-Moody algebra, where $c$ commutes with its dual $d$ (the zero mode of the Virasoro algebra).

The image of the generalized $m^{th}$ critical potentials consists of two disconnected components depending on $t_m$ negative or positive, which we call the negative (positive) side of the $m^{th}$ leaf. The NLS flows from the first to the $m^{th}$ order define smooth coordinates on each side of the $m^{th}$ leaf, where the even critical point is located on the negative side of the even leaf, so that the odd leaves and the positive side of even leaves are unreachable by the matrix model. The $2m^{th}$ flow cannot pass through $t_{2m} = 0$ due to the constraint we have, and because of that, if we impose the “physical” conditions to the solution of the string equations, i.e. real and
pole-free along the positive real axis and proper asymptotic behavior for $t \to \infty$, that solution has an instability at $t_{2m} = 0$.

### 6.2 Anti-Hermitian Models and the ZS Hierarchy

For the anti-hermitian model, (6.1) may be modified as

$$2 \tilde{c}_{-1} - \sum_{l \geq 1} \frac{(-2)^{l+1} (2l - 1)!!}{(l + 1)!} \tilde{c}_{2l-1} = \epsilon^{m+1} 2 \tilde{\alpha},$$

$$-4 \tilde{c}_0 - \tilde{c}_{-2} = \epsilon^m \tilde{t}_0,$$

$$( -1 )^{k+1} 4 \tilde{c}_k = \epsilon^{m-k} 2^{-k} (k + 1) \tilde{t}_k. \quad (k \geq 1)$$

(6.2) Rotating NLS to ZS, we can find that the above arguments hold perfectly good, except for the fact that the anti-hermitian model can realize the odd critical points as well, so the odd leaves are reachable by the anti-hermitian matrix models. Consequently, at the $(2m - 1)^{th}$ critical point, $\tilde{c}_k < 0$ for the first non-vanishing even coupling constant and $k$ must be equal to or higher than $2m$. Furthermore it is not possible to have only one odd coupling constant $\tilde{c}_{2m-1}$ non-vanishing in order to get finite $\tilde{\alpha}$.

### 6.3 Conclusion

We will conclude with a few comments. So far we have studied the mappings individually, but since these leaves cover the entire t-space, and also the structure of the NLS (ZS) flows is universal, we may sew them and get a universal mapping. This universal system is however highly mathematical unless one imposes the physical conditions, and among the points of the t-space, only the $2m^{th}$ critical point has a definite scaling dimension $\gamma_{str} = -1/m$. In fact, by solving the $2m^{th}$ string equation asymptotically, one can find that

$$F(t) = \sum_{h \geq 0} ( \kappa^2 t^{\gamma_{str} - 2} )^{h-1} F_h,$$

(6.3) which agrees with the scaling behavior of the continuum gravity coupled with conformal matter.

The quantization of $\tilde{c}$ is necessary to make the continuum limit analytic, and it can be realized by fixing the odd flows by $g = 0$, or $\tilde{\psi} = e^{2t_{-1}}$ with $\tilde{c} = 0$ (resp. $\tilde{c} = \frac{1}{2}$). Combining these results with (5.14), it seems that there are no other inequivalent fixings to realize the half-integer $\tilde{c}$, but this is an open problem, as is the interpretation of the quantization of $\tilde{c}$ topologically or field theoretically in the continuum quantum gravity.

The first critical point in the ZS hierarchy has been studied in ref. [17] as the topological phase, but so far no topological models have been found to be equivalent.
to this phase. Nonetheless this does not discourage us to search for new topological gravity, which may usher in a new geometry such as the topological phase of the one-cut hermitian model did in the intersection theory on the moduli space of Riemann surfaces [14,15].

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