LONG-TIME EXISTENCE FOR A WHITHAM–BOUSSINESQ SYSTEM IN TWO DIMENSIONS

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ABSTRACT. This paper is concerned with a two dimensional Whitham–Boussinesq system modeling surface waves of an inviscid incompressible fluid layer. We prove that the associated Cauchy problem is well-posed for initial data of low regularity, with existence time of scale $O\left(\mu^{3/2} - \epsilon^{-2}\right)$, where $\mu$ and $\epsilon$ are small parameters related to the level of dispersion and nonlinearity, respectively. In particular, in the KdV regime $\{\mu \sim \epsilon\}$, the existence time is of order $\epsilon^{-1/2}$. The main ingredients in the proof are frequency localised dispersive estimates and bilinear Strichartz estimates that depend on the parameter $\mu$.

1. Introduction

We consider the Cauchy problem for Whitham–Boussinesq system

$$\begin{cases}
\partial_t \eta + \nabla \cdot \mathbf{v} = -\epsilon \text{K}_\mu \nabla \cdot (\eta \mathbf{v}), \\
\partial_t \mathbf{v} + \text{K}_\mu \nabla \eta = -\epsilon \text{K}_\mu \nabla (|\mathbf{v}|^2/2), \\
(\eta, \mathbf{v})(0) = (\eta_0, \mathbf{v}_0),
\end{cases}$$

(1)

where $\eta : \mathbb{R}^{2+1} \rightarrow \mathbb{R}$, $\mathbf{v} : \mathbb{R}^{2+1} \rightarrow \mathbb{R}^2$ is a curl–free vector field, i.e., $\nabla \times \mathbf{v} = 0$, and

$$\text{K}_\mu := \text{K}_\mu(D) = \frac{\tanh(\sqrt{\mu} |D|)}{\sqrt{\mu} |D|} \quad \text{with} \quad D = -i\nabla.$$

The system (1) describes the evolution with time of surface waves of a liquid layer in the three dimensional physical space. The variables $\eta$ and $\mathbf{v}$ denote the surface elevation and the fluid velocity, respectively. The shallowness parameter $\mu$ and the nonlinearity parameter $\epsilon$ are defined by

$$\mu = (h/\lambda)^2, \quad \epsilon = a/h,$$

where $h$ denotes the mean depth of the fluid layer, $a$ is a typical amplitude of the wave, $\lambda$ a typical horizontal wavelength.

For $\mu \ll 1$, one has formally

$$\text{K}_\mu(D) = 1 + \frac{\mu}{3} \Delta + O(\mu^2),$$

and hence one can write (1) as

$$\begin{cases}
\partial_t \eta + \nabla \cdot \mathbf{v} = -\epsilon \nabla \cdot (\eta \mathbf{v}) + O(\mu^2 + \mu \epsilon), \\
\partial_t \mathbf{v} + \nabla \eta + \frac{\mu}{3} \Delta \eta = -\epsilon \nabla (|\mathbf{v}|^2/2) + O(\mu^2 + \mu \epsilon),
\end{cases}$$

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which is a perturbation of the Boussinesq system
\[
\begin{aligned}
\partial_t \eta + \nabla \cdot \mathbf{v} &= -\epsilon \nabla \cdot (\eta \mathbf{v}), \\
\partial_t \mathbf{v} + \nabla \eta + \frac{\mu}{3} \Delta \eta &= -\epsilon \nabla (|\mathbf{v}|^2/2).
\end{aligned}
\] (2)

The later system is a particular member of the (abcd) family of Boussinesq systems derived in [1] as asymptotic models for water waves in the Boussinesq regime. Unfortunately, this system is linearly ill-posed and thus cannot be used as a relevant water wave model. The system (1) can be viewed as a regularization of this ill-posed system.

The system (1) enjoys the Hamiltonian structure
\[
\partial_t (\eta, \mathbf{v})^T = J_\mu \nabla \mathcal{H}_\mu (\eta, \mathbf{v})
\]
with the skew-adjoint matrix
\[
J_\mu = \begin{pmatrix}
0 & -K_\mu \partial_{x_1} & -K_\mu \partial_{x_2} \\
-K_\mu \partial_{x_1} & 0 & 0 \\
-K_\mu \partial_{x_2} & 0 & 0
\end{pmatrix}.
\]

This in particular guarantees conservation of the energy functional
\[
\mathcal{E}_\mu (\eta, \mathbf{v}) = \frac{1}{2} \int_{\mathbb{R}^2} \left( \eta^2 + \left| \frac{1}{2} K_\mu^{-\frac{1}{2}} \mathbf{v} \right|^2 + \eta |\mathbf{v}|^2 \right) \, dx.
\] (3)

The one dimensional version of (1) that describes the evolution with time of surface waves of a liquid layer in the two dimensional physical space is written as
\[
\begin{aligned}
\partial_t \eta + \partial_x \mathbf{v} &= -\epsilon L_\mu \partial_x (\eta \mathbf{v}), \\
\partial_t \mathbf{v} + L_\mu \partial_x \eta &= -\epsilon L_\mu \partial_x (|\mathbf{v}|^2/2),
\end{aligned}
\] (4)

where \( \eta, \mathbf{v} : \mathbb{R}^{2+1} \to \mathbb{R} \) and
\[
L_\mu := L_\mu (D) = \frac{\tanh (\sqrt{\mu} D)}{\sqrt{\mu} D} \quad \text{with } D = -i \partial_x.
\]

For more details on the study of (1), (4) or related equations, we refer the reader to [2, 5, 6, 7, 4, 19].

Recently, together with Dinay and Selberg [7] we studied low regularity well-posedness of the Cauchy problems (1) and (4) for \( \mu = \epsilon = 1 \). In particular, we proved that (4) is globally well-posed for initial data \( (\eta_0, \mathbf{v}_0) \) that is small in the \( L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \)-norm. Moreover, we showed that (1) is locally well-posed for initial data \( (\eta_0, \mathbf{v}_0) \in H^s(\mathbb{R}^2) \times H^{s+1/2}(\mathbb{R}^2) \times H^{s+1/2}(\mathbb{R}^2) \) with \( s > 1/4 \).

In the present paper, we are interested in the problem of long-time existence of solution to (1) assuming that the nonlinearity parameter \( \epsilon \) is sufficiently small. In particular, by exploiting the dispersive nature of the system we prove that (1) is well-posed with existence time of scale \( O \left( \mu^{3/2} - \epsilon^{-2+} \right) \) if \( (\eta_0, \mathbf{v}_0) \in H^s(\mathbb{R}^2) \times H^{s+1/2}(\mathbb{R}^2) \times H^{s+1/2}(\mathbb{R}^2) \) with \( s > 1/4 \). This in particular recovers the local well-posedness result in [7, Theorem 2] for \( \mu = \epsilon = 1 \).

There has been several studies by J-C. Saut et al. [10, 15, 16, 17, 14] (see also [11, 12, 8]) regarding long-time existence of solutions for Whitham-Boussinesq type equations with initial data of size \( O(1) \) in some Sobolev norm, where the
time of existence depends on parameter $\epsilon$. In [16, 17, 14] the analysis is based only on symmetrization and energy techniques, and do not exploit the dispersive properties of the equations. The time of existence obtained for the equations involved is at most of scale $O(1/\epsilon)$ in the KdV regime $\{\mu \sim \epsilon\}$, but the space of resolutions are smaller. On the other hand, in [10] and [15] the dispersive nature of the systems involved is used to study the long-time existence problem. For instance, in [10] using the dispersive method the authors proved that the two dimensional dispersive Boussinesq system of the form

$$\begin{cases}
\partial_t \eta + \nabla \cdot (1 + \epsilon \Delta) v = -\epsilon \nabla \cdot (\eta \nabla), \\
\partial_t v + \nabla (1 + \epsilon \Delta) \eta = -\epsilon \nabla (|v|^2/2),
\end{cases} \tag{5}$$

is locally well-posed with existence time of scale $O(1/\sqrt{\epsilon})$ whenever $s > 3/2$.

Our main result is as follows.

**Theorem 1.** Let $s > 1/4$ and $\mu, \epsilon \in (0, 1]$. Suppose that $v_0$ is curl free, i.e., $\nabla \times v_0 = 0$ and the initial data has size

$$\left(\|\eta_0\|_{H^s(\mathbb{R}^2)} + \|v_0\|_{(H^{s+1/2}(\mathbb{R}^2))^2}\right) \sim D_0.$$

Then there is a solution

$$\begin{pmatrix} \eta, v \end{pmatrix} \in C \left([0, T]; H^s(\mathbb{R}^2) \times \left(H^{s+1/2}(\mathbb{R}^2)\right)^2 \right)$$

of the Cauchy problem (1) with existence time $T$ given by

$$T \sim D_0^{-2+\mu^3/2-\epsilon^{-2+}}.$$

Moreover, the solution is unique in some subspace of the above solution space and the solution depends continuously on the initial data.

**Remark 1.** From Theorem 1 we deduce the following:

- In the regime $\{\epsilon \ll \mu \sim 1\}$, the solution exists on a larger time scale of order $\epsilon^{-2+}$. This is due to the presence of weak nonlinearities $\{\epsilon \ll 1\}$, which is also regularized by the operator

$$K_\mu(D) \sim \langle D \rangle^{-1} \sim \langle D \rangle^{-1} \quad \text{for} \quad \mu \sim 1.$$

- In the regime $\{\mu \ll \epsilon \sim 1\}$, the time of existence is of order $\mu^{3/2-}$ and hence shrinks to 0 as $\mu \to 0+$. This is due to the presence of strong nonlinearities $\{\epsilon \sim 1\}$ which is not regularized, since

$$K_\mu(D) \sim 1 - \frac{\mu}{3} |D|^2 \quad \text{for} \quad \mu \ll 1.$$

In fact, in this regime the system (1) is a perturbation of the ill-posed system (2).

- Finally, in the KdV regime $\{\mu \sim \epsilon\}$, the existence time is of order $\epsilon^{-1/2}$.

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We use the notation $a \pm := a \pm \delta$ for sufficiently small $\delta > 0$. For any positive numbers $A$ and $B$, the notation $A \lesssim B$ stands for $A \leq CB$, where $C$ is a positive constant that is independent $\mu$, $\epsilon$ or $T$. Moreover, we denote $A \sim B$ when $A \lesssim B$ and $B \lesssim A$. 

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In what follows we diagonalize (1), and then reduce Theorem 1 to Theorem 2 below which corresponds to the diagonalized system. To this end, we define

\[ u_\pm = \frac{\eta \mp iK^{-1/2}_\mu \cdot v}{\sqrt{\mu}} \]

where \((\eta, v)\) is a solution to (1), and \(R = |D|^{-1} \nabla\) is the Riesz transform. Then we have

\[ \eta = u_+ + u_- \quad v = -i\sqrt{K_\mu} R(u_+ - u_-). \]

Set

\[ m_\mu(D) := |D| K_\mu(D). \]

The system (1) therefore transforms to

\[
\begin{cases}
(i\partial_t \mp m_\mu(D))u_\pm = \epsilon N^\pm_\mu(u_+, u_-), \\
u_\pm(0) = f_\pm,
\end{cases}
\]

where

\[ f_\pm = \frac{\eta_0 \mp iK^{-1/2}_\mu \cdot v_0}{\sqrt{\mu}} \]

and

\[
N^\pm_\mu(u_+, u_-) = 2^{-1}|D|K_\mu R \left\{ (u_+ + u_-)R \sqrt{K_\mu} (u_+ - u_-) \right\}
\pm 4^{-1}|D| \sqrt{K_\mu} \left( R \sqrt{K_\mu} \right)^2.
\]

Thus, Theorem 1 reduces to the following:

**Theorem 2.** Let \(s > 1/4, \mu, \epsilon \in (0, 1]\), and that the initial data has size

\[
\sum_\pm ||f_\pm||_{H^s} \sim D_0.
\]

Then there exists a solution

\[(u_+, u_-) \in C \left([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\right)\]

of the Cauchy problem (6–7) with existence time \(T\) as in Theorem 1. Moreover, the solution is unique in some subspace of \(C \left([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\right)\) and the solution depends continuously on the initial data.

The paper is organized as follows: In Section 2, we prove localized dispersive and Strichartz estimates for the linear propagators associated to (6). In Section 3 and 4 we prove Theorem 2 and bilinear estimates that are crucial in the proof of Theorem 2.

**Notation.** We fix an even smooth function \(\chi \in C^\infty_0(\mathbb{R})\) such that

\[ 0 \leq \chi \leq 1, \quad \chi_{|s| < 1/|s|} = 1 \quad \text{and} \quad \text{supp}(\chi) \subset [-2, 2] \]

and set

\[ \beta(s) = \chi(s) - \chi(2s). \]

For a dyadic number \(\lambda \in 2^\mathbb{Z}\), we set \(\beta_\lambda(s) := \beta(s/\lambda),\) and thus \(\text{supp} \beta_\lambda = \{s \in \mathbb{R} : \lambda/2 \leq |s| \leq 2\lambda\}.\) Now define the frequency projection \(P_\lambda\) via

\[ \widehat{P_\lambda f}(\xi) = \beta_\lambda(|\xi|) \hat{f}(\xi). \]
We sometimes write \( f_\lambda := \mathcal{P}_\lambda f \), so that
\[
f = \sum_\lambda f_\lambda,
\]
where summations throughout the paper are done over dyadic numbers in \( 2\mathbb{Z} \).

The Sobolev space \( H^s(\mathbb{R}^2) \) is defined via the norm
\[
\| f \|_{H^s} \sim \left[ \sum_\lambda \langle \lambda \rangle^{2s} \| f_\lambda \|_{L^2_x}^2 \right]^{1/2},
\]
where \( \langle \xi \rangle := (1 + |\xi|^2)^{1/2} \). If \( B \) is a space of functions on \( \mathbb{R}^2 \), \( T > 0 \) and \( 1 \leq p \leq \infty \), we define the spaces \( L^p((0,T):B) \) and \( L^p(\mathbb{R}:B) \) respectively via the norms
\[
\| f \|_{L^p((0,T):B)} = \left( \int_0^T \| f(\cdot, t) \|_B^p \, dt \right)^{1/p} \quad \text{and} \quad \| f \|_{L^p(\mathbb{R}:B)} = \left( \int_{\mathbb{R}} \| f(\cdot, t) \|_B^p \, dt \right)^{1/p},
\]
when \( 1 \leq p < \infty \), with the usual modifications when \( p = \infty \).

2. Localised dispersive and Strichartz estimates

First we derive lower bound estimates for the phase function
\[
m_\mu(r) = r \sqrt{K_\mu(r)}
\]
and its first two derivatives. Clearly, \( K_\mu(r) \sim \langle \sqrt{\mu r} \rangle^{-1} \), and hence
\[
|m_\mu(r)| \sim |r| \langle \sqrt{\mu r} \rangle^{-1/2}.
\]

**Lemma 1.** For all \( r > 0 \), we have
\[
0 < m'_\mu(r) \sim \langle \sqrt{\mu r} \rangle^{-1/2},
\]
\[
0 < -m''_\mu(r) \sim \mu r \langle \sqrt{\mu r} \rangle^{-5/2}.
\]

**Proof.** Since
\[
m_\mu(r) = \frac{1}{\sqrt{\mu}} m_1(\sqrt{\mu r})
\]
(9) and (10) reduce to proving
\[
0 < m'_1(r) \sim \langle r \rangle^{-1/2},
\]
\[
0 < -m''_1(r) \sim |r| \langle \sqrt{\mu r} \rangle^{-5/2}
\]
both of which are proved in [7, Lemma 8].

Van der Corput’s Lemma will be useful in the derivation of the dispersive estimate in Lemma 3 below.

**Lemma 2** (Van der Corput’s Lemma, [18]). Assume \( g \in C^1(a, b) \), \( \psi \in C^2(a, b) \) and \( |\psi''(r)| \geq A \) for all \( r \in (a, b) \). Then
\[
\left| \int_a^b e^{it\psi(r)} g(r) \, dr \right| \leq C(A(t))^{-1/2} \left[ |g(b)| + \int_a^b |g'(r)| \, dr \right],
\]
for some constant \( C > 0 \) that is independent of \( a, b \) and \( t \).
Now we follow a similar argument\footnote{Inequality (12) is derived in [7] in the case \( \mu = 1 \) and \( \lambda > 1 \).} as in [7, Lemma 9] to derive a frequency localized \( L^1 - L^\infty \) decay estimate for the linear propagator associated to (6):
\[
S_{m_{\mu}}(\pm t) := e^{\pm itm_{\mu}(D)}.
\]

**Lemma 3** (Localised dispersive estimate). Let \( 0 < \mu \leq 1 \) and \( \lambda \in 2\mathbb{Z} \). Then
\[
\|S_{m_{\mu}}(\pm t)f\|_{L^\infty_{\mathcal{F}}(\mathbb{R}^2)} \lesssim \min \left( \lambda^2, \mu^{-\frac{1}{2}}(\sqrt{\mu \lambda})^\frac{3}{2}t^{-1} \right) \|f\|_{L^1_{\mathcal{F}}(\mathbb{R}^2)}
\]
for all \( f \in \mathcal{S}(\mathbb{R}^2) \).

**Proof.** Without loss of generality we may assume \( t > 0 \) and \( \pm = - \). Now we can write
\[
[S_{m_{\mu}}(-t)f_{\lambda}](x) = (I_{\lambda,\mu}(\cdot,t) \ast f)(x),
\]
where
\[
I_{\lambda,\mu}(x,t) = \lambda^2 \int_{\mathbb{R}^2} e^{i\lambda x \cdot \xi + itm_{\mu}(\lambda \xi)} \beta(\xi) \, d\xi.
\]
By Young’s inequality
\[
\|S_{m_{\mu}}(-t)f_{\lambda}\|_{L^\infty_{\mathcal{F}}(\mathbb{R}^2)} \leq \|I_{\lambda,\mu}(\cdot,t)\|_{L^\infty_{\mathcal{F}}(\mathbb{R}^2)} \|f\|_{L^1_{\mathcal{F}}(\mathbb{R}^2)},
\]
and therefore, (12) reduces to proving
\[
\|I_{\lambda,\mu}(\cdot,t)\|_{L^\infty_{\mathcal{F}}(\mathbb{R}^2)} \lesssim \min \left( \lambda^2, \mu^{-\frac{1}{2}}(\sqrt{\mu \lambda})^\frac{3}{2}t^{-1} \right).
\]
But clearly,
\[
\|I_{\lambda,\mu}(\cdot,t)\|_{L^\infty_{\mathcal{F}}(\mathbb{R}^2)} \lesssim \lambda^2,
\]
and so we reduce further to proving
\[
\|I_{\lambda,\mu}(\cdot,t)\|_{L^\infty_{\mathcal{F}}(\mathbb{R}^2)} \lesssim \mu^{-\frac{1}{2}}(\sqrt{\mu \lambda})^\frac{3}{2}t^{-1}.
\]
To prove (15) first observe that \( I_{\lambda,\mu}(x,t) \) is radially symmetric w.r.t \( x \), as it is the inverse Fourier transform of the radial function \( e^{itm_{\mu}(\lambda \xi)} \beta(\xi) \), and so we may set \( x = (|x|, 0) \). Therefore, using polar coordinates we can write
\[
I_{\lambda,\mu}(x,t) = 2\pi \lambda^2 \int_{1/2}^{2} e^{itm_{\mu}(\lambda \rho)} f_0(\lambda |x|) |\rho| \beta(\rho) \, d\rho,
\]
where \( J_k(r) \) is the Bessel function:
\[
J_k(r) = \frac{(r/2)^k}{(k+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{irs} \left( 1 - s^2 \right)^{k-1/2} \, ds \quad \text{for } k > -1/2.
\]
The Bessel function \( J_k(r) \) satisfies the following properties for \( k > -1/2 \) and \( r > 0 \) (See [9, Appendix B] and [18]):
\[
J_k(r) \lesssim Cr^k, \\
J_k(r) \lesssim Cr^{-1/2}, \\
\partial_r \left[ r^{-k}J_k(r) \right] = -r^{-k}J_{k+1}(r)
\]
Moreover, we can write
\[
J_0(s) = e^{is\tilde{h}(s)} + e^{-is\tilde{h}(s)}
\]
for some function \( h \) satisfying the estimate
\[
|\partial^j_x h(r)| \leq C_j \langle r \rangle^{-1/2-j} \quad \text{for all } j \geq 0.
\] (21)

We prove (15) by treating the cases \(|x| \lesssim \lambda^{-1}\) and \(|x| \gg \lambda^{-1}\) separately.

2.0.1. Case 1: \(|x| \lesssim \lambda^{-1}\). By (17) and (19) we have for all \( r \in (1/2, 2) \) the estimate
\[
|\partial^j_x J_0(\lambda r|x|)| \lesssim 1 \quad \text{for } j = 0, 1.
\] (22)

We integrate by parts (16) to obtain
\[
I_{\lambda, \mu}(x, t) = -2\pi i \lambda t^{-1} \int_{1/2}^2 \frac{d}{dr} \left\{ e^{itm_\mu(\lambda r)} \right\} \left[ m'_\mu(\lambda r) \right]^{-1} J_0(\lambda r|x|) \tilde{\beta}(r) \, dr
\]
\[
= 2\pi i \lambda t^{-1} \int_{1/2}^2 e^{itm_\mu(\lambda r)} [m'_\mu(\lambda r)]^{-1} \partial_r [J_0(\lambda r|x|) \tilde{\beta}(r)] \, dr
\]
\[
- 2\pi i \lambda t^{-1} \int_{1/2}^2 e^{itm_\mu(\lambda r)} [m''_\mu(\lambda r)]^{-2} \lambda m''_\mu(\lambda r) J_0(\lambda r|x|) \tilde{\beta}(r) \, dr.
\]

Then applying Lemma 1 and (22) we obtain
\[
|I_{\lambda, \mu}(x, t)| \leq \lambda t^{-1} \left( \langle \sqrt{\mu} \lambda \rangle^{\frac{1}{4}} + \mu^{-\frac{1}{2}} \langle \sqrt{\mu} \lambda \rangle^{-\frac{5}{4}} \right)
\]
\[
\leq \lambda \langle \sqrt{\mu} \lambda \rangle^{\frac{1}{4}} t^{-1} \leq \mu^{-\frac{1}{2}} \langle \sqrt{\mu} \lambda \rangle^{\frac{3}{4}} t^{-1}.
\] (23)

2.0.2. Case 2: \(|x| \gg \lambda^{-1}\). Using (20) in (16) we write
\[
I_{\lambda, \mu}(x, t) = 2\pi \lambda^2 \left\{ \int_{1/2}^2 e^{it\Phi^\pm_{\lambda, \mu}(r)} h(\lambda r|x|) \tilde{\beta}(r) \, dr + \int_{1/2}^2 e^{-it\Phi^\pm_{\lambda, \mu}(r)} \tilde{H}(\lambda r|x|) \tilde{\beta}(r) \, dr \right\},
\]
where
\[
\Phi^\pm_{\lambda, \mu}(r) = \lambda r|x|/t \pm m_\mu(\lambda r).
\]

Set \( H_\lambda(|x|, r) = h(\lambda r|x|) \tilde{\beta}(r) \). In view of (21) we have
\[
|H_\lambda(|x|, r)| + |\partial_r H_\lambda(|x|, r)| \lesssim (\lambda |x|)^{-1/2}
\] (24)

for all \( r \in (1/2, 2) \), where we also used the fact \( \lambda |x| \gg 1 \).

Now we write
\[
I_{\lambda, \mu}(x, t) = I^+_x_{\lambda, \mu}(x, t) + I^-_{\lambda, \mu}(x, t),
\]
where
\[
I^+_x_{\lambda, \mu}(x, t) = \int_{1/2}^2 e^{it\Phi^+_{\lambda, \mu}(r)} H_\lambda(|x|, r) \, dr,
\]
\[
I^-_{\lambda, \mu}(x, t) = \int_{1/2}^2 e^{-it\Phi^+_{\lambda, \mu}(r)} \tilde{H}(\lambda |x|, r) \, dr.
\]

Observe that
\[
\partial_r \Phi^\pm_{\lambda, \mu}(r) = \lambda \left[ |x|/t \pm m'_\mu(\lambda r) \right], \quad \partial_r^2 \Phi^\pm_{\lambda, \mu}(r) = \pm \lambda^2 m''_\mu(\lambda r),
\]
and hence by Lemma 1,
\[
|\partial_r \Phi^\pm_{\lambda, \mu}(r)| \gtrsim \lambda (\sqrt{\mu} \lambda)^{-1/2} \quad |\partial_r^2 \Phi^\pm_{\lambda, \mu}(r)| \sim \mu \lambda^3 (\sqrt{\mu} \lambda)^{-5/2}
\] (25)

for all \( r \in (1/2, 2) \), where we also used the fact that \( m' \) is positive.
Lemma 4 (Localised Strichartz estimates). Let \( 0 < \mu \leq 1 \) and \( \lambda \in 2\mathbb{Z} \). Assume that the pair \((q, r)\) is Strichartz admissible in the sense that

\[
q > 2, \quad r \geq 2 \quad \text{and} \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{2}.
\]

Then

\[
\| \mathcal{S}_m(\pm t)f_\lambda \|_{L^q_tL^r_x(\mathbb{R}^{2+1})} \lesssim \mu^{-\frac{1}{r}}(\sqrt{\mu\lambda})^\frac{3q}{2r} \|f_\lambda\|_{L^q_1(\mathbb{R}^2)}, \tag{28}
\]

\[
\left\| \int_0^t \mathcal{S}_m(\pm(t-s))F_\lambda(s) \, ds \right\|_{L^q_tL^r_x(\mathbb{R}^{2+1})} \lesssim \mu^{-\frac{1}{2q}}(\sqrt{\mu\lambda})^\frac{3q}{2r} \|F_\lambda\|_{L^1_tL^2_x(\mathbb{R}^{2+1})}, \tag{29}
\]

for all \( f \in \mathcal{S}(\mathbb{R}^2) \) and \( F \in \mathcal{S}(\mathbb{R}^{2+1}) \).
Proof. We shall use the Hardy-Littlewood-Sobolev inequality which asserts that
\[ \|1 - \gamma \ast f\|_{L^a(R)} \lesssim \|f\|_{L^b(R)} \]  
whenever \( 1 < b < a < \infty \) and \( 0 < \gamma < 1 \) obey the scaling condition
\[ \frac{1}{b} = \frac{1}{a} + 1 - \gamma. \]

We prove only (29) since (30) follows from (29) by the standard \( TT^* \)–argument. First note that (29) holds true for the pair \((q, r) = (\infty, 2)\) as this is just the energy inequality. So we may assume \( q \in (2, \infty) \).

Let \( q' \) and \( r' \) be the conjugates of \( q \) and \( r \), respectively, i.e., \( q' = \frac{q}{q-r} \) and \( r' = \frac{r}{r-q} \). By the standard \( TT^* \)–argument, (29) is equivalent to the estimate
\[ \|TT^*F\|_{L^q_q L^{r'}_r(R^{2+1})} \lesssim \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \|F\|_{L^q_q L^{r'}_r(R^{2+1})}, \]  
where
\[ TT^*F(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi + i(1-s)m_x(\xi)B^2(\xi)} F(\xi, s) \, ds \, d\xi, \]  
and
\[ K_{\lambda,t}(x) = \int_{\mathbb{R}} e^{ix\xi + itm_x(\xi)} \beta^2(\xi) \, d\xi. \]

Since \( K_{\lambda,t} \ast g(x) = S_{m_x}(t)P_{\lambda} g(\lambda x) \), it follows from (12) that
\[ \|K_{\lambda,t} \ast g\|_{L^\infty_v(R)} \lesssim \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \|g\|_{L^q_q L^{r'}_r(R)}. \]  
On the other hand, we have by Plancherel
\[ \|K_{\lambda,t} \ast g\|_{L^2_q(R)} \lesssim \|g\|_{L^2_q(R)}. \]  
So interpolation between (34) and (35) yields
\[ \|K_{\lambda,t} \ast g\|_{L^\infty_q(R)} \lesssim \left[ \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \right]^{-\frac{1}{2}} \left[ \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \right]^{-\frac{1}{2}} \|g\|_{L^2_q(R)}, \]  
for all \( r \in [2, \infty) \).

Applying Minkowski’s inequality to (33), and then (36) and (31) with \((a, b) = (q, q')\) and \( \gamma = 1 - 2/r = 2/q \), we obtain
\[ \|TT^*F\|_{L^q_q L^{r'}_r(R^{2+1})} \lesssim \int_{\mathbb{R}} \|K_{\lambda,t} \ast g\|_{L^\infty_q(R)} \, ds \]  
\[ \lesssim \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \int_{\mathbb{R}} |t-s|^{-\frac{1}{2}} \|F(s, \cdot)\|_{L^{r'}_r(R^2)} \, ds \]  
\[ \lesssim \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \|F\|_{L^{r'}_r(R^2)} \]  
\[ = \mu^{-\frac{1}{2}}(\sqrt{\mu})^\frac{q}{r} \|F\|_{L^q_q L^{r'}_r(R^{2+1})}, \]  
which is the desired estimate (32).
3. Proof of Theorem 2

The bilinear terms in (7) can be written as
\[
\mathcal{N}_\mu^\pm(u_+, u_-) = \frac{1}{2} \sum_{\pm_1, \pm_2} \pm_2 D K_\mu R \cdot (u_{\pm_1} R \sqrt{K_\mu u_{\pm_2}})
\]
\[\pm \frac{1}{4} \sum_{\pm_1, \pm_2} (\pm_1)(\pm_2) D \sqrt{K_\mu} (R \sqrt{K_\mu u_{\pm_1}} \cdot R \sqrt{K_\mu u_{\pm_2}}),\]
where \(\pm_1\) and \(\pm_2\) are independent signs. Then the Duhamel's representation of (6) is given by
\[
u_\pm(t) = S_{m_\mu}(\pm t) f_\pm \mp \frac{i \epsilon}{2} \sum_{\pm_1, \pm_2} (\pm_2) A_\mu^\pm(u_{\pm_1}, u_{\pm_2})(t)
\]
\[\mp \frac{i \epsilon}{4} \sum_{\pm_1, \pm_2} (\pm_1)(\pm_2) B_\mu^\pm(u_{\pm_1}, u_{\pm_2})(t),\]
where
\[
A_\mu^\pm(u, v)(t) := \int_0^t S_{m_\mu}(\pm (t - t')) D K_\mu R \cdot (u R \sqrt{K_\mu v} (t')) dt',
\]
\[
B_\mu^\pm(u, v)(t) := \int_0^t S_{m_\mu}(\pm (t - t')) D \sqrt{K_\mu} (R \sqrt{K_\mu u} \cdot R \sqrt{K_\mu v})(t') dt'.
\]

Now let \((q, r)\) with \(q, r > 2\) be a Strichartz admissible pair. We define the contraction space, \(X_\gamma^T\), via the norm
\[
\|u\|_{X_\gamma^T} = \left[ \sum_{\lambda} \langle \lambda \rangle^{2s} \|P_\lambda u\|_{L_T^q L_x^r}^2 \right]^{\frac{1}{2}},
\]
where
\[
\|u\|_{X_\gamma} = \left[ \|P_\lambda u\|_{L_T^q L_x^r}^2 + \mu^{\frac{q}{4}} (\sqrt{\mu})^{-\frac{q}{4}} \|P_\lambda u\|_{L_T^q L_x^r}^2 \right]^{\frac{1}{2}}.
\]
Observe that
\[
\|P_\lambda u\|_{L_T^q L_x^r} \leq \|u\|_{X_\gamma},
\]
\[
\|P_\lambda u\|_{L_T^q L_x^r} \leq \mu^{-\frac{q}{8}} (\sqrt{\mu})^{-\frac{4}{8}} \|u\|_{X_\gamma}.\]
Moreover,
\[X_\gamma^T \subset L_T^\infty H_\gamma^s.\]

We estimate the linear part of (37) using (29) as follows:
\[
\|S_{m_\mu}(\pm t) f_\pm\|_{X_\gamma^T} \leq \left[ \sum_{\lambda} \langle \lambda \rangle^{2s} \|S_{m_\mu}(\pm t) f_\pm\|_{X_\gamma}^2 \right]^{\frac{1}{2}}
\]
\[\leq \left[ \sum_{\lambda} \langle \lambda \rangle^{2s} \|P_\lambda f_\pm\|_{L_T^q L_x^r}^2 \right]^{\frac{1}{2}} \sim \|f_\pm\|_{H_\gamma^s}.\]

So Theorem 2 reduces to proving the following bilinear estimates whose proof will be given in the next section.
Lemma 5. Let $0 < \alpha \ll 1$, $0 < \mu \leq 1$, $s \geq 1/4 + \alpha$ and $T > 0$. Then for all $u, v \in X^s_T$, we have

$$
\|A^\pm_\mu (u, v)\|_{X^s_T} \lesssim T^{\frac{1}{2} + \alpha} \mu^{-\frac{3}{2} + \frac{3}{2}} \|u\|_{X^s_T} \|v\|_{X^s_T},
$$

(41)

$$
\|B^\pm_\mu (u, v)\|_{X^s_T} \lesssim T^{\frac{1}{2} + \alpha} \mu^{-\frac{3}{2} + \frac{3}{2}} \|u\|_{X^s_T} \|v\|_{X^s_T},
$$

(42)

where $A^\pm_\mu$ and $B^\pm_\mu$ are as in (38).

Indeed, given that Lemma 5 holds, we solve the integral equations (37) by contraction mapping techniques as follows. Define the mapping

$$(u_+, u_-) \mapsto (\Phi^+_\mu (u_+, u_-), \Phi^-_\mu (u_+, u_-)),
$$

where $\Phi^\pm_\mu (u_+, u_-)$ is given by the right hand side of (37).

Now given initial data with norm

$$
\sum_\pm \|f_\pm\|_{H^s} \leq D_0,
$$

we look for a solution in the set

$$
E_T = \left\{ (u_+ \in X^s_T : \sum_\pm \|u_\pm\|_{X^s_T} \leq 2C D_0 \right\}.
$$

Then by (40) and Lemma 5 we have

$$
\sum_\pm \|\Phi^+_\mu (u_+, u_-)\|_{X^s_T} \leq C \sum_\pm \|f_\pm\|_{H^s} + CT^{\frac{1}{2} + \alpha} \mu^{-\frac{3}{2} + \frac{3}{2}} \epsilon \left( \sum_\pm \|u_\pm\|_{X^s_T} \right)^2
$$

$$
\leq CD_0 + CT^{\frac{1}{2} + \alpha} \mu^{-\frac{3}{2} + \frac{3}{2}} \epsilon (2CD_0)^2
$$

$$
\leq 2CD_0,
$$

provided that

$$
T \leq \left( 8C^2 D_0 \right)^{-2+\delta} \mu^{\frac{3}{2} - \delta} \epsilon^{-2+\delta},
$$

(43)

where $\delta = \frac{4\alpha}{1+4\alpha} \ll 1$.

Similarly, for two pair of solutions $(u_+, u_-)$ and $(v_+, v_-)$ in $E_T$ with the same data, one can derive the difference estimate

$$
\sum_\pm \|\Phi^+_\mu (u_+, u_-) \neq \Phi^+_\mu (v_+, v_-)\|_{X^s_T}
$$

$$
\leq CT^{\frac{1}{2} + \alpha} \mu^{-\frac{3}{2} + \frac{3}{2}} \epsilon \left( \sum_\pm \|u_\pm\|_{X^s_T} + \|v_\pm\|_{X^s_T} \right) \left( \sum_\pm \|u_\pm - v_\pm\|_{X^s_T} \right)
$$

$$
\leq 4C^2 T^{\frac{1}{2} + \alpha} \mu^{-\frac{3}{2} + \frac{3}{2}} \epsilon D_0 \left( \sum_\pm \|u_\pm - v_\pm\|_{X^s_T} \right)
$$

$$
\leq \frac{1}{2} \left( \sum_\pm \|u_\pm - v_\pm\|_{X^s_T} \right),
$$

where in the last inequality we used (43).
Therefore, \((\Phi^+, \Phi^-)\) is a contraction on \(E_T\) and therefore it has a unique fixed point \((u_+, u_-) \in E_T\) solving the integral equation (37)-(38) on \(\mathbb{R}^2 \times [0, T]\), where

\[
T = T_0^{-2} + \mu^{\frac{3}{2}} e^{-2\cdot}.
\]

Uniqueness in the space \(X^q_T \times X^q_T\) and continuous dependence on the initial data can be shown in a similar way, by the difference estimates. This concludes the proof of Theorems 2.

4. Proof of Lemma 5

First we prove key bilinear estimates in Lemma 6 below that will be crucial in the proof of Lemma 5. By Bernstein inequality, we have for all \(r_1, r_2 \in \mathbb{R}\) and \(p \geq 1\),

\[
\|D|^{r_1}K_{i\mu} R f^\lambda\|_{L^p} \lesssim \lambda^{r_1} (\sqrt{\mu})^{-r_2} \|f^\lambda\|_{L^p},
\]

where \(R = |D|^{-1} \nabla\) is the Riesz transform.

For \(\lambda \in 2^\mathbb{Z}\) \((j = 0, 1, 2)\), we denote

\[
\lambda = (\lambda_0, \lambda_1, \lambda_2), \quad \lambda_1 \wedge \lambda_2 = \min(\lambda_1, \lambda_2).
\]

Lemma 6. Let \(0 < \alpha \ll 1\), \(0 < \mu \ll 1\) and \(T > 0\). Then for all \(u \in X_{\lambda_1}\) and \(v \in X_{\lambda_2}\) we have

\[
\|D|K_{i\mu} P_{\lambda_0} R \cdot (u_{\lambda_1} R \sqrt{K_{i\mu} v_{\lambda_2}})\|_{L^2_{t,x}} \lesssim C_{\mu, T}(\lambda) \|u\|_{X_{\lambda_1}} \|v\|_{X_{\lambda_2}},
\]

\[
\|D|^{\frac{1}{2}} K_{i\mu} P_{\lambda_0} \left(R \sqrt{K_{i\mu} u_{\lambda_1} \cdot R \sqrt{K_{i\mu} v_{\lambda_2}}}\right)\|_{L^2_{t,x}} \lesssim \bar{C}_{\mu, T}(\lambda) \|u\|_{X_{\lambda_1}} \|v\|_{X_{\lambda_2}},
\]

where

\[
C_{\mu, T}(\lambda) = T^{\alpha} \mu^{-\frac{1}{4} + \frac{3}{2} \frac{\lambda_0 (\lambda_1 \wedge \lambda_2)}{2(\sqrt{\mu})^{\frac{3}{4}}}} \frac{2^\alpha (\sqrt{\mu})^{\frac{3}{4}}}{(\sqrt{\mu})^{\frac{3}{4}}},
\]

\[
\bar{C}_{\mu, T}(\lambda) = T^{\alpha} \mu^{-\frac{1}{4} + \frac{3}{2} \frac{\lambda_0 (\lambda_1 \wedge \lambda_2)}{2(\sqrt{\mu})^{\frac{3}{4}}}} \frac{2^\alpha (\sqrt{\mu})^{\frac{3}{4}}}{(\sqrt{\mu})^{\frac{3}{4}}}.\]

Proof. We only prove (45) since the proof for (46) is similar. By symmetry we may assume \(\lambda_1 \leq \lambda_2\). Let

\[
\frac{1}{q} = \frac{1}{2} - \alpha, \quad \frac{1}{r} = \alpha
\]

so that \((q, r)\) is Strichartz admissible. Then by Hölder, Bernstein inequality, (44) and (39) we obtain

\[
\text{LHS (45)} \lesssim \lambda_0 (\sqrt{\mu})^{-1} \|R \cdot (u_{\lambda_1} R \sqrt{K_{i\mu} v_{\lambda_2}})\|_{L^2_{t,x}}
\]

\[
\lesssim \lambda_0 (\sqrt{\mu})^{-1} (\sqrt{\mu})^{-\frac{1}{2} \cdot T^{\frac{1}{2} - \frac{1}{4} \lambda_1^2}} \|u_{\lambda_1}\|_{L^q_{t,x}} \|v_{\lambda_2}\|_{L^\infty_{t,x}}
\]

\[
\lesssim T^{\alpha} \mu^{-\frac{1}{4} + \frac{3}{2} \frac{\lambda_0^{\frac{1}{2}} (\sqrt{\mu})^{\frac{3}{4}}}{(\sqrt{\mu})^{\frac{3}{4}}} - \frac{2\alpha}{2}} \|u\|_{X_{\lambda_1}} \|v\|_{X_{\lambda_2}}
\]

which proves (45).
Now we are ready to prove Lemma 5: (41) & (42). To this end, we decompose
\[ u = \sum_\lambda u_\lambda \text{ and } v = \sum_\lambda v_\lambda. \]
Note that by denoting
\[ a_\lambda := \|u\|_{X^s_\lambda}, \quad b_\lambda := \|v\|_{X^s_\lambda} \]
we can write
\[ \|u\|_{X^s_\lambda}^2 = \|\langle \lambda \rangle^s a_\lambda\|_{L^2_\lambda}^2, \quad \|v\|_{X^s_\lambda}^2 = \|\langle \lambda \rangle^s b_\lambda\|_{L^2_\lambda}^2. \]  
(49)

We shall make a frequent use of the following dyadic summation estimate, for \( \mu, \lambda \in 2^\mathbb{Z} \) and \( c_1, c_2, p > 0 \):
\[ \sum_{\mu \sim \lambda} a_\mu \sim a_\lambda, \quad \sum_{c_1 \leq \lambda \leq c_2} \lambda^p \lesssim \begin{cases} c_2^p & \text{if } p > 0, \\ c_1^p & \text{if } p < 0. \end{cases} \]

4.1. Proof of (41). Using (30) and Hölder, we have
\[ \|A^\pm_{\mu}(u, v)\|_{X^s_\lambda}^2 \lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \|D|K_\mu R \cdot P_{\lambda_0} \left( u R^2 \sqrt{K_\mu v} \right)\|_{L^2_\lambda}^2 \]
\[ \lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \|D|K_\mu R \cdot P_{\lambda_0} \left( u R^2 \sqrt{K_\mu v} \right)\|_{L^2_\lambda}^2. \]
(50)

But the dyadic decomposition
\[ \|D|K_\mu R \cdot P_{\lambda_0} \left( u R^2 \sqrt{K_\mu v} \right)\|_{L^2_\lambda} \lesssim \sum_{\lambda_1, \lambda_2} \left\| D|K_\mu R \cdot P_{\lambda_0} \left( u_{\lambda_1} R^2 \sqrt{K_\mu v_{\lambda_2}} \right) \right\|_{L^2_\lambda}. \]
(51)

Now let \( \lambda_{\text{min}}, \lambda_{\text{med}} \) and \( \lambda_{\text{max}} \) denote the minimum, median and the maximum of \( \{\lambda_0, \lambda_1, \lambda_2\} \), respectively. By checking the support properties in Fourier space of the bilinear term on the right hand side of (50) one can see that this term vanishes unless \( \lambda = (\lambda_0, \lambda_1, \lambda_2) \in \Lambda \), where
\[ \Lambda = \{\lambda : \lambda_{\text{med}} \sim \lambda_{\text{max}}\}. \]

Thus, we have a non-trivial contribution in (50) if \( \lambda \in \cup_{j=0}^2 \Lambda_j \), where
\[ \Lambda_0 = \{\lambda : \lambda_0 \lesssim \lambda_1 \sim \lambda_2\}, \]
\[ \Lambda_1 = \{\lambda : \lambda_2 \ll \lambda_1 \sim \lambda_0\}, \]
\[ \Lambda_2 = \{\lambda : \lambda_1 \ll \lambda_2 \sim \lambda_0\}. \]

By using these facts, and applying (45) to the right hand side of (50), we get
\[ \|A^\pm_{\mu}(u, v)\|_{X^s_\lambda}^2 \lesssim T \sum_{j=0}^2 g^{(j)}_{\mu, T}, \]
where
\[ g^{(j)}_{\mu, T} = \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left( \sum_{\lambda_1, \lambda_2 : \lambda \in \Lambda_j} C_{\mu, T}(\lambda) a_{\lambda_1} b_{\lambda_2} \right)^2 \]
(51)

with \( C_{\mu, T}(\lambda) \) as in (47).

So (41) reduces to proving
\[ g^{(j)}_{\mu, T} \lesssim T^{2s} \|u\|_{X^s_\lambda}^2 \|v\|_{X^s_\lambda}^2 \quad (j = 0, 1, 2). \]  
(52)
Further, observe that \( \Lambda_j \subset \bigcup_{k=0}^{2} \Lambda_{jk} \), where
\[
\begin{align*}
\Lambda_{j0} & = \{ \lambda \in \Lambda_j : \lambda_{\text{max}} \lesssim \mu^{-1/2} \}, \\
\Lambda_{j1} & = \{ \lambda \in \Lambda_j : \lambda_{\text{min}} \lesssim \mu^{-1/2} & \& \lambda_{\text{med}} \gg \mu^{-1/2} \}, \\
\Lambda_{j2} & = \{ \lambda \in \Lambda_j : \lambda_{\text{min}} \gg \mu^{-1/2} \}.
\end{align*}
\]
So
\[
\sum_{j=0}^{2} \gamma_{jk} \lesssim \sum_{j=0}^{2} g_{jk},
\]
where
\[
g_{jk} = \sum_{\lambda_0} (\lambda_0)^2 \left[ \sum_{\lambda_1, \lambda_2 : \lambda \in \Lambda_{jk}} C_{\mu, T}(\lambda) a_{\lambda_1} b_{\lambda_2} \right]^2.
\]
Thus, (52) reduces further to proving
\[
g_{jk} \lesssim T^{2\alpha} \mu^{-\frac{3}{2} + \alpha} \|u\|_{L^\infty(\mu)}^2 \|v\|_{L^\infty(\mu)}^2 (j, k = 0, 1, 2).
\]
We prove (54) as follows.

4.1.1. Estimates for \( \gamma_{jk} \). In the case \( \lambda \in \Lambda_{00} \), we have
\[
C_{\mu, T}(\lambda) \lesssim T^\alpha \mu^{-\frac{1}{2} + \frac{\alpha}{2}} \lambda_0 \lambda_2^2/n,
\]
and hence
\[
\gamma_{00} \lesssim T^{2\alpha} \mu^{-3/2 + \alpha} \sum_{\lambda_0 \lesssim \mu^{-1/2}} \lambda_0^2 \left( \sum_{\lambda_1, \lambda_2 \lesssim \mu^{-1/2}} (\lambda_1)^n a_{\lambda_1} : (\lambda_2)^{2\alpha} b_{\lambda_2} \right)^2 \lesssim T^{2\alpha} \mu^{-3/2 + \alpha} \|u\|_{L^\infty(\mu)}^2 \|v\|_{L^\infty(\mu)}^2
\]
for all \( s \geq 2\alpha \), where to obtain the second line we used Cauchy Schwarz inequality in \( \lambda_1 \sim \lambda_2 \) and (49).

If \( \lambda \in \Lambda_{01} \), we have
\[
C_{\mu, T}(\lambda) \lesssim T^\alpha \mu^{-\frac{3}{2} + \frac{\alpha}{2}} \lambda_0 \lambda_2^{1/2 + \frac{\alpha}{2}},
\]
and hence
\[
\gamma_{01} \lesssim T^{2\alpha} \mu^{-5/4 - \frac{\alpha}{2}} \sum_{\lambda_0 \lesssim \mu^{-1/2}} \lambda_0^2 \left( \sum_{\lambda_1, \lambda_2 \gg \mu^{-1/2}} (\lambda_1)^n a_{\lambda_1} : (\lambda_2)^{1/2 + \frac{\alpha}{2}} b_{\lambda_2} \right)^2 \lesssim T^{2\alpha} \mu^{-5/4 - \frac{\alpha}{2}} \|u\|_{L^\infty(\mu)}^2 \|v\|_{L^\infty(\mu)}^2
\]
for all \( s \geq 1/4 + \alpha/4 \).

Finally, if \( \lambda \in \Lambda_{02} \), we have
\[
C_{\mu, T}(\lambda) \lesssim T^\alpha \mu^{-\frac{\alpha}{2} - \frac{\alpha}{4}} \lambda_2^{1/2 + \frac{\alpha}{2}},
\]
and hence

\[ j_{\mu,T}^{(02)} \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \frac{\alpha}{2}} \sum_{\lambda_0 \gg \mu^{-1/2}} (\lambda_0)^{\frac{3}{2} + \alpha - 2s} \left( \sum_{\lambda_1 \sim \lambda_2 \gg \mu^{-1/2}} (\lambda_1)^s a_{\lambda_1} \cdot (\lambda_2)^s b_{\lambda_2} \right)^2 \]

\[ \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \frac{\alpha}{2}} \|u\|_{X_T^1}^2 \|v\|_{X_T^1}^2 \]

for all \( s > 1/4 + \alpha/2 \).

4.1.2. Estimates for \( j_{\mu,T}^{(1k)} \) (k = 0, 1, 2). If \( \lambda \in \Lambda_{10} \), we have

\[ C_{\mu,T}(\lambda) \lesssim T^{\alpha} \mu^{-\frac{1}{4} + \frac{\alpha}{2}} \lambda_0 \lambda_2^{2\alpha}. \]

Hence

\[ j_{\mu,T}^{(10)} \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \alpha} \sum_{\lambda_0 \lesssim \mu^{-1/2}} (\lambda_0)^{2s} \lambda_0^2 \left( \sum_{\lambda_2 \ll \lambda_1 \lesssim \lambda_0 \lesssim \mu^{-1/2}} a_{\lambda_1} \cdot \lambda_2^{2\alpha} b_{\lambda_2} \right)^2 \]

\[ \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \alpha} \sum_{\lambda_0 \lesssim \mu^{-1/2}} (\lambda_0)^{2s} \lambda_0^2 \left( \sum_{\lambda_2 \lesssim \mu^{-1/2}} \lambda_2^{2\alpha} b_{\lambda_2} \right)^2 \]

\[ \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \alpha} \|u\|_{X_T^1}^2 \|v\|_{X_T^1}^2, \]

where to get the last two inequalities we used \( \sum_{\lambda_1 \sim \lambda_0} a_{\lambda_1} \sim a_{\lambda_0} \) and by Cauchy Schwarz

\[ \sum_{\lambda_2 \lesssim \mu^{-1/2}} \lambda_2^{2\alpha} b_{\lambda_2} \lesssim \left( \sum_{\lambda_2 \leq 1} + \sum_{\lambda_2 > 1} \right) \lambda_2^{2\alpha} b_{\lambda_2} \]

\[ \lesssim \|b_{\lambda_2}\|_{L^1_{\lambda_2}}^2 + \||\lambda_2)^s b_{\lambda_2}\|_{L^2_{\lambda_2}}^2 \lesssim \|v\|_{X_T^1} \]

for all \( s > 2\alpha \).

If \( \lambda \in \Lambda_{11} \), we have

\[ C_{\mu,T}(\lambda) \lesssim T^{\alpha} \mu^{-\frac{3}{4} + \frac{\alpha}{2}} \lambda_2^{2\alpha}, \]

and hence using the previous argument

\[ j_{\mu,T}^{(11)} \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \alpha} \sum_{\lambda_0 \gg \mu^{-1/2}} (\lambda_0)^{2s} \left( \sum_{\lambda_1 \sim \lambda_0 \gg \lambda_2, \lambda_2 \lesssim \mu^{-1/2}} a_{\lambda_1} \lambda_2^{2\alpha} b_{\lambda_2} \right)^2 \]

\[ \lesssim T^{2\alpha} \mu^{-\frac{3}{4} + \alpha} \|u\|_{X_T^1}^2 \|v\|_{X_T^1}^2 \]

for all \( s > 2\alpha \).

Finally, if \( \lambda \in \Lambda_{12} \), we have

\[ C_{\mu,T}(\lambda) \lesssim T^{\alpha} \mu^{-\frac{3}{8} + \frac{\alpha}{4}} \lambda_2^{1 + \frac{\alpha}{2}}, \]
and hence using the previous argument
\[
\mathcal{J}^{(12)}_{\mu,T} \lesssim T^{2\alpha} \mu^{-\frac{5}{2} - \frac{\alpha}{2}} \sum_{\lambda_0 \gg \mu^{-1/2}} \langle \lambda_0 \rangle^{2s} \left( \sum_{\lambda_1 \prec \lambda_0 \gg \mu^{-1/2}} a_{\lambda_1} \lambda_2^{\frac{1}{2} + \frac{\alpha}{2}} b_{\lambda_2} \right)^2 \]
\[
\lesssim T^{2\alpha} \mu^{-\frac{3}{2} + \alpha} \|u\|_{\dot{X}^1_T}^2 \|v\|_{\dot{X}^1_T}^2
\]
for all \( s > 1/4 + \alpha/2 \).

4.1.3. **Estimates for \( J^{(2k)}_{\mu,T} \) (k = 0, 1, 2).** If \( \lambda \in \Lambda_{20} \), we have
\[
\mathcal{C}_{\mu,T}(\lambda) \lesssim T^{\alpha} \mu^{-\frac{1}{2} + \frac{\alpha}{2}} \lambda_0^{2\alpha}
\]
and hence arguing as in the preceding subsection we obtain
\[
\mathcal{J}^{(20)}_{\mu,T} \lesssim T^{2\alpha} \mu^{-\frac{1}{2} + \alpha} \sum_{\lambda_0 \lesssim \mu^{-1/2}} \langle \lambda_0 \rangle^{2s} \lambda_0^2 \left( \sum_{\lambda_1 \ll \lambda_2, \lambda_0 \lesssim \mu^{-1/2}} \lambda_1^{2\alpha} a_{\lambda_1} b_{\lambda_2} \right)^2 \]
\[
\lesssim T^{2\alpha} \mu^{-\frac{3}{2} + \alpha} \|u\|_{\dot{X}^1_T}^2 \|v\|_{\dot{X}^1_T}^2
\]
for all \( s > 2\alpha \).

Next if \( \lambda \in \Lambda_{21} \), then
\[
\mathcal{C}_{\mu,T}(\lambda) \lesssim T^{\alpha} \mu^{-1 + \frac{\alpha}{2}} \lambda_0^{-\frac{1}{2}} \lambda_1^{2\alpha},
\]
and hence
\[
\mathcal{J}^{(21)}_{\mu,T} \lesssim T^{2\alpha} \mu^{-2 + \alpha} \sum_{\lambda_0 \gg \mu^{-1/2}} \langle \lambda_0 \rangle^{2s} \lambda_0^{-1} \left( \sum_{\lambda_1 \ll \lambda_2, \lambda_0 \ll \mu^{-1/2}} \lambda_1^{2\alpha} a_{\lambda_1} b_{\lambda_2} \right)^2 \]
\[
\lesssim T^{2\alpha} \mu^{-\frac{3}{2} + \alpha} \|u\|_{\dot{X}^1_T}^2 \|v\|_{\dot{X}^1_T}^2
\]
for all \( s > 2\alpha \).

Finally, if \( \lambda \in \Lambda_{21} \), then
\[
\mathcal{C}_{\mu,T}(\lambda) \lesssim T^{\alpha} \mu^{-\frac{5}{2} - \frac{\alpha}{2}} \lambda_1^{\frac{1}{2} + \frac{\alpha}{2}},
\]
and hence
\[
\mathcal{J}^{(22)}_{\mu,T} \lesssim T^{2\alpha} \mu^{-\frac{5}{4} - \frac{\alpha}{4}} \sum_{\lambda_0 \gg \mu^{-1/2}} \langle \lambda_0 \rangle^{2s} \left( \sum_{\lambda_2 \ll \lambda_0 \gg \mu^{-1/2}} \lambda_1^{\frac{1}{2} + \frac{\alpha}{2}} a_{\lambda_1} b_{\lambda_2} \right)^2 \]
\[
\lesssim T^{2\alpha} \mu^{-\frac{3}{2} - \frac{\alpha}{4}} \|u\|_{\dot{X}^1_T}^2 \|v\|_{\dot{X}^1_T}^2
\]
for all \( s > 1/4 + \alpha/2 \).

4.2. **Proof of (42).** Arguing as in the preceding subsection we use (30), Hölder, (46) and (48) to obtain
\[
\|B^\pm_T (u, v)\|_{\dot{X}^1_T}^2 \lesssim T \sum_{j=0}^2 \mathcal{J}^{(j)}_{\mu,T},
\]
where

\[ J^{(j)}_{\mu,T} = \sum_{\lambda_0} (\lambda_0)^{2s} \left[ \sum_{\lambda_1, \lambda_2 \in \Lambda_j} \tilde{C}_{\mu,T}(\lambda) a_{\lambda_1} b_{\lambda_2} \right]^2, \]  

(55)

with \( \tilde{C}_{\mu,T}(\lambda) \) as in (48).

By symmetry it suffices to estimate \( J^{(0)}_{\mu,T} \) and \( J^{(1)}_{\mu,T} \). Now if \( \lambda \in \Lambda_0 \) or \( \lambda \in \Lambda_1 \), then from (47)–(48) we have

\[ \tilde{C}_{\mu,T}(\lambda) \lesssim C_{\mu,T}(\lambda), \]

and hence

\[ J^{(j)}_{\mu,T} \lesssim J^{(j)}_{\mu,T} \lesssim T^{2\alpha} \mu^{-\frac{3}{2}+\alpha} \| u \|_{H^{\frac{3}{2}}}^2 \| v \|_{H^{\frac{3}{2}}}^2 \]  

(\( j = 0, 1 \)),

where to get the second inequality we used (52).

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