An upright circular cylindrical rigid tank performs a small-magnitude prescribed periodic horizontal motion, which is described by the two generalized coordinates $\eta_1(t)$ and $\eta_2(t)$ ($r_0$ is the tank radius) as shown in fig. 1. Those tank motions are relevant for bioreactors [1]. In contrast to industrial containers whose dimensions are relatively large, the bioreactors have $r_0 \approx (5-10) [\text{cm}]$ that requires accounting for the damping associated with a laminar boundary layer and the bulk viscosity.

The problem is studied in the nondimensional statement provided by the characteristic size $r_0$ and time $1/\sigma$, where $\sigma$ is the forcing frequency close to the lowest natural sloshing frequency. Whereas the undamped sloshing implies coexisting the co-directed (with forcing) and counter-directed angular progressive waves (swirling), the damping makes the counter-directed swirling impossible as the forcing orbit tends to a circle.

Keywords: sloshing, damping, steady-state waves.

An upright circular cylindrical rigid tank performs a small-magnitude prescribed periodic horizontal motion, which is described by the two generalized coordinates $\eta_1(t)$ and $\eta_2(t)$ ($r_0$ is the tank radius) as shown in fig. 1. Those tank motions are relevant for bioreactors [1]. In contrast to industrial containers whose dimensions are relatively large, the bioreactors have $r_0 \approx (5-10) [\text{cm}]$ that requires accounting for the damping associated with a laminar boundary layer and the bulk viscosity.

The problem is studied in the nondimensional statement provided by the characteristic size $r_0$ and time $1/\sigma$, where $\sigma$ is the forcing frequency close to the lowest natural sloshing frequency $\sigma_{11}$. The nondimensional forcing magnitude is small, i.e. $\eta_i(t) = O(\varepsilon), i = 1, 2$. Fig. 1 illustrates the adopted nomenclature. The unknowns, $\zeta$ and $\Phi$ (the velocity potential), are defined in the tank-fixed coordinate system and can be found from either the corresponding free-surface problem or its equivalent variational formulation. Using the Fourier-type representation (in the cylindrical coordinates)

$$
\zeta(r, \theta, t) = \sum_{M, i} \int M(k_Mr) \cos(M\theta) p_{M_i}(t) + \sum_{m, i} \int m(k_m r) \sin(m\theta) r_{m}(t)
$$

makes it possible to derive an approximate system of ordinary differential equations (nonlinear modal equations [2]) with respect to the free-surface generalized coordinates $p_{M_i}(t)$.
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and $r_m(t)$; here, $J_M(\cdot)$ is the Bessel functions of the first kind, $k_{Mi}$ are the radial wave numbers ($J'_M(k_{Mi}) = 0$), and 
\[
\sigma_{Mi} = \sqrt{[k_{Mi} \tanh(k_{Mi}h)]g / \rho_0}
\]
are the dimensional natural sloshing frequencies ($g$ is the gravity acceleration).

Furthermore, the nonlinear Narimanov—Moiseev-type modal system [2] (the infinite-dimensional system of ordinary differential equations with respect to $p_{Mi}(t)$ and $r_{mi}(t)$) is equipped with the linear damping terms 
\[
2\xi_{Mi} \sigma_{Mi} p_{Mi} \quad \text{and} \quad 2\xi_{Mi} \sigma_{Mi} r_{Mi},
\]
where the damping coefficients $\xi_{Mi}$ are taken according to the formula by Miles [3], which provides a rather accurate theoretical prediction of the logarithmic decrements of the natural sloshing modes due to the boundary layer and the bulk viscosity. The $2\pi$-periodic solutions of the modified modal system describe the resonant steady-state sloshing. To find the asymptotic steady-state solutions, we use the Bubnov—Galerkin procedure [2, 4] by posing the lowest-order components of the primary resonantly excited modes as

\[
p_{11}(t) = a \cos t + \bar{a} \sin t + O(\varepsilon), \quad r_{11}(t) = \bar{b} \cos t + b \sin t + O(\varepsilon),
\]

where the nondimensional amplitudes $a$, $\bar{a}$, $\bar{b}$, and $b$ are of $O(\varepsilon^{1/3})$. Having known these amplitudes approximates the steady-state free-surface elevations as the superposition of the two out-of-phase angular modes

\[
\zeta(r, \theta, t) = J_1(k_{11}r)[(a \cos \theta + \bar{b} \sin \theta) \cos t + (\bar{a} \cos \theta + b \sin \theta) \sin t] + O(\varepsilon^{1/3}),
\]

which implies the so-called swirling (angular progressive wave) unless $(a \cos \theta + \bar{b} \sin \theta)$ and $(\bar{a} \cos \theta + b \sin \theta)$ are congruent patterns ($\Leftrightarrow ab = \bar{a}\bar{b}$). The latter means that (3) determines a standing wave. Occurrence of swirling and standing waves was in many details discussed in [2, 4—6].

The Bubnov—Galerkin procedure leads to a necessary solvability condition with respect of $a$, $\bar{a}$, $\bar{b}$, and $b$ appearing as a system of nonlinear algebraic equations [2, 4, 5]. To describe the steady-state sloshing, we should solve the system for any $\sigma_{11} = \sigma_{11} / \sigma$ close to 1. The first Lyapunov method can be used to study the stability. The algebraic system is rederived in terms of the integral amplitudes $A, B$ (the main wave elevation components in the $Ox$ and $Oy$ directions, respectively) and the phase-lags $\psi, \phi$:

\[
A = \sqrt{a^2 + \bar{a}^2} \quad \text{and} \quad B = \sqrt{b^2 + \bar{b}^2}
\]

\[
a = A \cos \psi, \quad \bar{a} = A \sin \psi, \quad \bar{b} = B \cos \phi, \quad \bar{b} = B \sin \phi,
\]

\[
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\]

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Fig. 2. Response curves in the $(\sigma/\sigma_{11}, A, B)$ -space for the longitudinal ($\varepsilon = 0$ ) harmonic forcing in the Oxz-plane, $h/r_0 = 1.5$, the nondimensional forcing amplitude $\eta_{1a} = 0.01$ ($\eta_{2a} = 0$). The undamped sloshing ($\xi = 0$) is presented in (a) and the damped case ($\xi = 0.02$) is shown in (b). There is no stable steady-state sloshing between $E_1$ and $E_2$, where irregular (chaotic) waves are expected. Curves on (close to) the $(\sigma/\sigma_{11}, A)$-plane correspond to the (almost) planar wave regime

\[ \begin{align*}
A[\bar{\sigma}_{11}^2 - 1 + m_1 A^2 + (m_3 - F) B^2] &= \varepsilon_x \cos \psi; \quad A[DB^2 + \xi] = \varepsilon_x \sin \psi; \\
B[\bar{\sigma}_{11}^2 - 1 + m_1 B^2 + (m_3 - F) A^2] &= \varepsilon_y \sin \phi; \quad B[DA^2 - \xi] = \varepsilon_y \cos \phi; \\
F &= (m_3 - m_1) \cos^2 (\alpha) = (m_3 - m_1) / (1 + C^2), \\
D &= (m_3 - m_1) \sin (\alpha) \cos (\alpha) = (m_3 - m_1) C / (1 + C^2),
\end{align*} \]

(5a)

\[ \begin{align*}
F &= (m_3 - m_1) \cos^2 (\alpha) = (m_3 - m_1) / (1 + C^2), \\
D &= (m_3 - m_1) \sin (\alpha) \cos (\alpha) = (m_3 - m_1) C / (1 + C^2),
\end{align*} \]

(5b)

where $\alpha = \varphi - \psi$, $C = \tan \alpha$, $0 \leq \varepsilon_y \leq \varepsilon_x \neq 0$, $F(\alpha)$ and $D(\alpha)$ are $\pi$ -periodic functions of the phase-lags difference $\alpha$, and $\varepsilon_x, \varepsilon_y$ are linear functions of the forcing amplitudes $\eta_{1a}, \eta_{2a}$. The coefficients $m_1$ and $m_2$ are known functions of the liquid depth (see, [2, 4]) but $\xi = 2 \xi_{11}$ (damping rate of the two lowest natural sloshing modes). A special numerical scheme [7] was developed to solve (5), i.e. to describe how the main wave amplitude components $A$ and $B$ change versus $\sigma/\sigma_{11}$.

The undamped resonant steady-state sloshing due to longitudinal excitations along the Ox-axis ($\varepsilon_x > 0, \varepsilon_y = 0, \xi = 0$) was analyzed in [2, 4]. A planar standing wave and the swirling are identified. In terms of (4) and (5) with $\xi = 0$ these imply $B = 0, \sin \psi = 0, C = 0$, and $AB \neq 0, \sin \psi = \cos \varphi = 0$, $(C = \pm \infty)$, respectively. The swirling consists of two identical angular progressive waves occurring in either counter- or clockwise directions, they correspond to $C = +\infty$ and $-\infty$ respectively. Fig. 2, a presents the corresponding response curves. Case (b) shows the linear damping effect with $\xi = 0.02$ The branches belonging (close) to the plane $\sigma/\sigma_{11}, A$ are responsible for the (almost) planar standing wave regime. The regime is stable to the left of $E_1$ and to the right of $E_2$. It becomes unstable in a neighborhood of the primary resonance $\sigma/\sigma_{11} = 1$, where the stable swirling (to the right of $H(H_1)$) and irregular waves (the steady-state sloshing is unstable) between $E_1$ and $H(H_1)$ are predicted. The damping removes infinite points on the response curves of (a), so that the steady-state swirling branching in (b) constitutes an arc pinned
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The response curves for \( \delta = \varepsilon_y / \varepsilon_x > 0 \) in the \((\sigma/\sigma_{11}, A, B)\)-space. The steady-state resonant sloshing is due to an elliptic counterclockwise forcing with \( \eta_{1a} = 0.01, \eta_{2a} = \delta \eta_{1a}; \xi = 0.02 \). All the points on the response curves correspond to the swirling. The bold lines mark the stability.

**Fig. 3.** Response curves for \( \delta = \varepsilon_y / \varepsilon_x > 0 \) in the \((\sigma/\sigma_{11}, A, B)\)-space. The steady-state resonant sloshing is due to an elliptic counterclockwise forcing with \( \eta_{1a} = 0.01, \eta_{2a} = \delta \eta_{1a}; \xi = 0.02 \). All the points on the response curves correspond to the swirling. The bold lines mark the stability.

at \( E_2 \) and \( P \), which can be treated as bifurcation points, where the swirling emerges from the (almost) planar steady-state wave regime.

In [5], we showed that any orbital small-magnitude periodic tank motions are equivalent, to within the higher-order terms, to an artificial elliptic-type horizontal excitation with \( \varepsilon_y = \delta \varepsilon_x \), \( 0 < \delta \leq 1 \). How the response curves of the damped steady-state sloshing change with increasing \( \delta \) is shown in Fig. 3. When \( \delta \neq 0 \), all the steady-state sloshing regimes are of the swirling type. Specifically, there are no identical swirling waves with opposite directions, as it has been in the
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longitudinal case (each point on $PH_1H_2E_2$ in Fig. 2, $b$ implies the pair of these waves). The connected branching in Fig. 2, $b$ splits into the response curve $E_1H_1H_2E_2$ existing for any $\sigma / \sigma_{11}$ and $0 < \delta \leq 1$ and corresponding to the co-directed (with the counterclockwise elliptic forcing) angular progressive waves and the loop-like branch with $R_1$ and $R_2$ whose points imply the counter-directed swirling. Fig. 3 shows that the latter branch disappears, as $\delta$ increases. This is a very interesting fact, which contradicts the steady-state analysis of the undamped sloshing in [2], where both the co- and counter-directed angular progressive waves exist and can be stable in certain frequency ranges for any $0 < \delta \leq 1$.

In summary, the linear viscous damping matters for the orbitally-excited sloshing in bioreactors of an upright circular cylindrical shape. It affects qualitatively and quantitatively the steady-state sloshing and the corresponding response curves. The most interesting fact is that the damping, even being relatively small, makes the counter-directed angular progressive waves (swirling) impossible, as the forcing orbit tends to a circle. This fact contradicts the undamped steady-state analysis, but it is qualitatively consistent with model tests by M. Reclari in [1].

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ХЛЮПАННЯ ІЗ ДЕМПФУВАННЯМ
У ВЕРТИКАЛЬНОМУ ЦИЛІНДРИЧНОМУ БАКУ
ПРИ ОРБІТАЛЬНИХ ЗБУРЕННЯХ

З використанням нелінійної модальної системи Наріманова—Мойсеєва з лінійним демпфуванням вивчається затухаюче усталене хлюпання рідини у вертикальному круговому баку при заданому горизонтальному орбітальному (еліптичному) русі посудини з вимушеною частотою, близькою до власної частоти

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The damped sloshing in an upright circular tank due to an orbital forcing

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ПЛЕСКАНИЕ С ДЕМПФИРОВАНИЕМ В ВЕРТИКАЛЬНОМ ЦИЛИНДРИЧЕСКОМ БАКЕ ПРИ ОРБИТАЛЬНЫХ ВОЗБУЖДЕНИЯХ

С использованием нелинейной модальной системы Нариманова—Моисеева с линейным демпфированием изучается затухающее установившееся плескание жидкости в вертикальном круговом баке при заданном горизонтальном орбитальном (эллиптическом) движении сосуда с вынужденной частотой, близкой к собственной частоте колебаний жидкости. В то время как случай без демпфирования включает как сонаправленные (с направлением орбитального движения), так и противоположно направленные угловые прогрессивные волны, демпфирование делает невозможным существование противоположно направленной волны при возбуждениях, близких к круговым.

Ключевые слова: плескание жидкости, демпфирование, установившиеся волны.