On $\sqrt{n}$--consistency for Bayesian quantile regression based on the misspecified asymmetric Laplace likelihood

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Abstract

The asymmetric Laplace density (ALD) is used as a working likelihood for Bayesian quantile regression. Sriram et al. (2013) derived posterior consistency for Bayesian linear quantile regression based on the mispecified ALD. While their paper also argued for $\sqrt{n}$--consistency, Sriram and Ramamoorthi (2017) highlighted that the argument was only valid for $n^\alpha$ rate for $\alpha < 1/2$. However, $\sqrt{n}$--rate is necessary to carry out meaningful Bayesian inference based on the ALD. In this paper, we give sufficient conditions for $\sqrt{n}$--consistency in the more general setting of Bayesian non-linear quantile regression based on ALD. In particular, we derive $\sqrt{n}$--consistency for the Bayesian linear quantile regression. Our approach also enables an interesting extension of the linear case when number of parameters $p$ increases with $n$, where we obtain posterior consistency at the rate $n^\alpha$ for $\alpha < 1/2$.

Key words: Asymmetric Laplace density; Bayesian quantile regression; Non-linear.

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1 Introduction

The classical linear quantile regression problem (Koenker and Bassett, 1978), for estimating the $\tau^{th}$ quantile ($\tau \in (0, 1)$) of the independent responses $Y_i, i \in \{1, 2, \ldots, n\}$ conditional on explanatory variables $X_i$, involves solving the problem:

$$\min_{\beta_\tau \in \mathbb{R}^p} \sum_{i=1}^{N} \rho_\tau(Y_i - X_i^T \beta_\tau),$$

where

$$\rho_\tau(u) = u(\tau - I_{(u \leq 0)}), \quad u \in \mathbb{R}. \quad (2)$$

Here, $I_{(\cdot)}$ is the indicator function and $\mathbb{R}^p$ is the $p$-dimensional Euclidean space. Yu and Moyeed (2001) proposed a now widely used Bayesian approach to model a given quantile, by using the asymmetric Laplace density (ALD) for the response, i.e. $Y_i \sim f_\tau(y_i - Q_\tau(X_i))$, by taking $Q_\tau(X_i) = \alpha_\tau + X_i^T \beta_\tau$ and

$$f_\tau(y) = \tau(1 - \tau) \cdot e^{-\rho_\tau(y)}, \quad y \in \mathbb{R}. \quad (3)$$

This approach to Bayesian inference is motivated by the fact that obtaining a maximum likelihood estimate (MLE) based on the ALD likelihood in equation (3) is equivalent to solving problem (1). However, the ALD would seldom be the true data generating mechanism and hence is often a misspecification. Towards a formal justification of the approach Sriram et al. (2013) derived posterior consistency for the linear Bayesian quantile regression parameters based on the “misspecified” ALD model, under fairly general conditions. Yet, posterior consistency itself does not ensure asymptotically correct inference, as the posterior credible intervals turn out to be inadequate due to the model misspecification. Yang et al. (2016) and Sriram (2015) both suggested a similar correction to the posterior variance matrix so as to obtain asymptotically valid credible intervals. Of importance to us is that such a correction necessarily requires posterior consistency to hold at the rate $\sqrt{n}$, which is made explicit in Sriram (2015).

For Bayesian linear quantile regression using ALD, Sriram et al. (2013) had further argued that posterior consistency holds at the rate $\sqrt{n}$. However, in a recent correction note, Sriram and Ramamoorthi (2017) highlighted that the argument was only valid only for $n^{\alpha}$—rate for $\alpha < \frac{1}{2}$, and was flawed for $\sqrt{n}$—consistency. In this paper, we give sufficient conditions for $\sqrt{n}$—consistency in the more general setting of Bayesian non-linear quantile regression using ALD. As a special case, we derive posterior consistency at $\sqrt{n}$—rate for the Bayesian linear quantile regression using ALD. We note that Sriram et al. (2016) considered posterior consistency for non-linear Bayesian quantile regression and for joint estimation of multiple quantiles. However, their approach does not yield $\sqrt{n}$—consistency. Our approach also enables an inter-
testing extension to Bayesian linear quantile regression with the number of parameters \( p \) growing with \( n \), where we obtain posterior consistency at a rate \( n^\alpha \) for \( \alpha < 1/2 \).

2 Main Result

In this section, we give sufficient conditions for posterior consistency at the rate \( \sqrt{n} \) for nonlinear Bayesian quantile regression based on ALD. Let \( Y_{1:n} := (Y_1, Y_2, \cdots, Y_n) \) be a vector of \( n \) independent but non-identically distributed responses (i.n.i.d), and \( X_i \) be \( p \)-dimensional non-random covariate vectors. The true distribution of \( Y_i \) is denoted by \( P_{0i} \) and is assumed to depend on \( X_i \). For ease of notation, we will denote the finite product measure \( \prod_{i=1}^{n} P_{0i} \) as well as the infinite product measure \( \prod_{i=1}^{\infty} P_{0i} \) by \( P \), and the corresponding expectations by \( E[\cdot] \).

We will denote by \( Q_\tau(X_i) \), the true \( \tau \)th quantile for \( Y_i \) given \( X_i \). \( Q_\tau \) can be non-linear and is assumed to belong to a class of functions \( G \). We denote the true unknown quantile function by \( Q_{0\tau} \). The Bayesian approach to quantile regression based on ALD specifies the likelihood for the data (using equation 3) as

\[
L(Y_{1:n}|Q_\tau) = \prod_{i=1}^{n} f_\tau(Y_i - Q_\tau(X_i)). \tag{4}
\]

A proper prior \( \Pi \) is specified for \( Q_\tau \in G \) and the posterior distribution is obtained as

\[
\Pi(Q_\tau|Y_{1:n}) \propto L(Y_{1:n}|Q_\tau) \cdot \Pi(Q_\tau). \tag{5}
\]

We derive posterior consistency with respect to the empirical \( L_2 \) metric \( d_n \) given by:

\[
d_n(Q_\tau, Q_{0\tau}) := \sqrt{\frac{1}{n} \sum_{i=1}^{n} (Q_\tau(X_i) - Q_{0\tau}(X_i))^2}. \tag{6}
\]

It is natural to consider such an empirical average metric for non-linear models with i.n.i.d observations (e.g. see Ghosal and van der Vaart 2007, Sriram et al. 2016). Our aim is to show under suitable assumptions that for any positive sequence \( M_n \to \infty \) and \( \epsilon_n = \frac{M_n}{\sqrt{n}} \), there exists some constant \( J > 0 \) such that

\[
\Pi(d_n(Q, Q_0) > J\epsilon_n|Y_{1:n}) \to 0 \ \text{in probability} \ [P]. \tag{7}
\]

We define \( U_n^\epsilon := \{d_n(Q, Q_0) > J\epsilon_n\} \) and write its posterior probability as

\[
\Pi(U_n^\epsilon|Y_{1:n}) := \frac{\int_{U_n^\epsilon} \prod_{i=1}^{n} \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} d\Pi(Q_\tau)}{\int_{G} \prod_{i=1}^{n} \frac{f_\tau(Y_i - Q_{0\tau}(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} d\Pi(Q_\tau)}. \tag{8}
\]

Our first assumption specifies that the quantile function space be uniformly bounded.

3
Assumption 1: \( \exists M > 0 \) such that \( \sup_x \sup_{Q, \tau \in G} |Q_\tau(x)| \leq M \).

In many practical situations, it is reasonable to assume that the specific quantile of interest is finite and bounded. Our second assumption relates to the true underlying distribution of the data.

Assumption 2:

(2a) \( \exists C_1 > 0 \) and \( \Delta_0 > 0 \) such that for all \( 0 < \Delta < \Delta_0 \),
\[
P(0 < Y_i - Q_0(X_i) < \Delta) > C_1 \Delta \quad \text{and} \quad P(-\Delta < Y_i - Q_0(X_i) < 0) > C_1 \Delta.
\]

(2b) For any \( u, u_0 \in \mathbb{R} \) such that \( |u - u_0| \leq 2M \), for some constant \( M \), there exists \( S > 0 \) such that
\[
E|\rho_\tau(Y_i - u) - \rho_\tau(Y_i - u_0)| \leq S|u - u_0|^2, \quad \forall \ u : |u - u_0| \leq 2M.
\]

A similar assumption to Assumption 2a is made by Sriram et al. (2013). It holds when the probability density function of \( Y_i \) at the \( \tau^{th} \) quantile is continuous, strictly positive, and uniformly bounded away from 0. Assumption 2b is a technical condition we will use to prove Lemma 4. Asumption 2b will hold if the function \( h_{u_0,i}(u) := E|\rho_\tau(Y_i - u) - \rho_\tau(Y_i - u_0)| \) is smooth, with uniformly bounded second derivative. To see this, note that \( h_{u_0,i}(u_0) = 0 \) (since minimum is achieved at \( u = u_0 \)). Therefore, by Taylor’s theorem \( |h_{u_0,i}(u)| \leq |h''_{u_0,i}(\xi)|(u - u_0)^2/2 \) for some \( \xi \) between \( u \) and \( u_0 \). If the second derivative is uniformly bounded on \( |u - u_0| \leq M \), i.e., \( \exists S > 0 \) such that for all \( i, |h''_{u_0,i}(\xi)| \leq 2S \), then Assumption 2b follows. The second derivative will be uniformly bounded for example if it is a continuous function of the covariate \( X_i \), which in turn belongs to a compact set.

This leads to the following important lemma that can be used to control the numerator of the posterior probability in equation (8).

**Lemma 1.** If Assumptions 1 and 2 hold, then \( \exists \alpha \in (0, 1) \) and constant \( C_3 > 0 \), such that for any \( Q_\tau \in G \),
\[
E \left( \prod_{i=1}^n \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right) \leq e^{-\alpha C_3 n d^2(Q_\tau, Q_{0\tau})}.
\]

Proof of the lemma is provided in the appendix. Lemma helps bound the numerator of the posterior probability. Our next assumption is essentially on the positivity of prior probabilities for neighbourhoods around the true quantile function.

Assumption 3: Let \( S \) be a constant as in Assumption 2b.
(3a) $\Pi$ is a proper prior. $\exists$ a constant $L > 0$ such that for any sequence $\epsilon_n = \frac{M_n}{\sqrt{n}} \to 0$ with $M_n \to \infty$,
\[
\frac{\Pi(j\epsilon_n < d_n(Q_\tau, Q_{\tau_0}) \leq 2j\epsilon_n)}{\Pi \left( d_n(Q_\tau, Q_{\tau_0}) < \frac{M_n}{\sqrt{n}} \right)} \leq e^{Lj^2}, \quad \forall \ n, j.
\]

(3b) $\Pi$ is a proper prior. $\exists$ a sequence $\{L_n\} > 0$ such that for any sequence $\epsilon_n = \frac{M_n}{\sqrt{n}} \to 0$ with $M_n \to \infty$, for all sufficiently large $j$,
\[
\frac{\Pi(j\epsilon_n < d_n(Q_\tau, Q_{\tau_0}) \leq 2j\epsilon_n)}{\Pi \left( d_n(Q_\tau, Q_{\tau_0}) < \frac{M_n}{\sqrt{n}} \right)} \leq e^{L_nj^2}, \quad \forall \ n, j.
\]

As will be clear from the proof of our main result, Assumption 3a is crucial for achieving posterior consistency at $\sqrt{n}$-rate. Assumption 3b is a weaker condition than Assumption 3a (i.e. $3a \implies 3b$), and we utilize it for deriving posterior consistency when $p$ depends on $n$. While it is relatively easier to satisfy Assumption 3b, in that case, the rate of $\sqrt{n}$ may not be achieved.

The next lemma shows that Assumption 3 essentially relates to positivity of Kullback-Leibler neighborhoods and is in the lines of equation (2.9) of Ghosal et al. (2000).

**Lemma 2.** Define the Kullback-Leibler neighborhood $V_{\epsilon_n}^2$ as
\[
V_{\epsilon_n}^2 := \left\{ Q_\tau \in \mathcal{G} : -\frac{1}{n} \sum_{i=1}^{n} E \log \frac{f_\tau(Y_i - Q_{\tau}(X_i))}{f_\tau(Y_i - Q_{\tau_0}(X_i))} < \epsilon_n^2, \frac{1}{n} \sum_{i=1}^{n} E \left( \log \frac{f_\tau(Y_i - Q_{\tau_0}(X_i))}{f_\tau(Y_i - Q_{\tau}(X_i))} \right)^2 < \epsilon_n^2 \right\}.
\]

Suppose Assumption 2 holds. If Assumption 3a holds, then $\exists$ constant $L > 0$ (sequence $\{L_n\} > 0$), such that for every sufficiently large $j$
\[
\frac{\Pi(j\epsilon_n < d_n(Q_\tau, Q_{\tau_0}) \leq 2j\epsilon_n)}{\Pi \left( V_{\epsilon_n}^2 \right)} \leq e^{Lj^2}, \quad \forall \ n.
\]

If Assumption 3b holds, then same result holds with $L$ on the right hand side replaced by $L_n$.

Proof of the lemma is in the appendix. The following lemma relates to the denominator of the posterior probability in equation 8.

**Lemma 3.** For any given constant $D_1 > 0$ and for $V_{\epsilon_n}^2$ as in equation 9, define the set $S_n$ as
\[
S_n := \left\{ \int_{V_{\epsilon_n}^2} \prod_{i=1}^{n} \frac{f_\tau(Y_i - Q_{\tau}(X_i))}{f_\tau(Y_i - Q_{\tau_0}(X_i))} d\Pi(Q_\tau) \leq e^{-(1+D_1)n\epsilon_n^2} \right\}.
\]

Then, $P(S_n) \leq \frac{1}{D_1^2 n\epsilon_n^2}$.

The proof of the lemma is in the lines of Lemma 8.1 in Ghosal et al. (2000) and is provided in the appendix. Our next assumption specifies the sieve and entropy condition in the lines of equation (2.18) of Kleijn and van der Vaart (2006).
**Assumption 4:** Define the sieves $A_{kn} := \{Q_\tau \in \mathcal{G} : k\epsilon_n \leq d_n(Q_\tau, Q_0) < 2k\epsilon_n\}$. For any given constant $R > 0$, let $N_{kn} := \text{minimum number of open } d_n\text{-balls with radius } Rk\epsilon_n\text{, that cover } A_{kn}$. We assume that $\exists \ b > 0$ such that for any sequence $\epsilon_n = M_n \rightarrow 0$ with $M_n \rightarrow \infty$ and any constant $R > 0$, $N_{kn} \leq e^{b\epsilon_n^2}$.

Our next lemma, along with Lemma 1 and the entropy condition helps bound the numerator of the posterior probability. We provide proof of the Lemma 4 in the appendix.

**Lemma 4.** Let $M_n \rightarrow \infty$ be a sequence such that $\epsilon_n = M_n \rightarrow 0$. Let Assumptions 1, 2 and 3a hold. Suppose $\{B_{jkn}, j \in \{1, 2, \ldots, N_{kn}\}\}$ is an open cover of $A_{kn}$, where each $B_{jkn}$ is an open $d_n$-ball of radius $R\epsilon_n$, centered at some $Q_{\tau jkn} \in A_{kn}$. Then, for any $0 < \alpha < 1$, $\exists$ some constant $C_4 > 0$ such that,

$$\frac{1}{(\Pi(V_\epsilon^2))^{\alpha}} E \left( \int_{B_{jkn} \cap A_{kn}} \prod_{i=1}^n \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{\tau jkn}(X_i))} d\Pi(Q_\tau) \right)^\alpha \leq e^{\alpha k^2 R^2 C_4 \epsilon_n^2 \epsilon_n L_j^2}.
$$

If Assumption 3b holds instead of Assumption 3a, then the right hand side of the above inequality will hold with $L_n$ in place of $L$.

We now state the main result.

**Theorem 1.**

(a). $\sqrt{n}$-consistency: Suppose Assumptions 1, 2, 3a and 4. Then, there exists a constant $J$ such that for all sequences $M_n$ such that $M_n \rightarrow \infty$ and $\epsilon_n = M_n \sqrt{n} \rightarrow 0$.

$$\Pi (d_n(Q_\tau, Q_0) > J\epsilon_n | Y_{1:n}) \rightarrow 0 \text{ in probability } [P].$$

(b). $n^\alpha$-consistency: Suppose Assumptions 1, 2 and 3b hold. Let $L_n$ be sequence as in Assumption 3b. Suppose $\exists b$ such that Assumption 4 holds for all sequences $M_n$ such that $M_n \rightarrow \infty$, $\epsilon_n = M_n \sqrt{n} \rightarrow 0$, and $\lim_{n \rightarrow \infty} \frac{L_n}{M_n} = 0$. Then, there exists a constant $J$ such that for all such sequences $M_n$, $\Pi (d_n(Q_\tau, Q_0) > J\epsilon_n | Y_{1:n}) \rightarrow 0 \text{ in probability } [P].$

In particular, suppose Assumption 3b holds for $L_n = n^{1-\eta}$ for some $0 < \eta < 1/2$. Suppose for any $0 < \eta < \epsilon_n$, $\exists b$ as in Assumption 4 for $M_n = n^{1/2-\alpha}$, then $\Pi (d_n(Q_\tau, Q_0) > Jn^{-\alpha} | Y_{1:n}) \rightarrow 0 \text{ in probability } [P].$

In the interest of flow, we defer the proof of the theorem to the appendix. Here, we make a remark.
Remark 1. We note that the dimensionality of $X_i$ has no direct role to play in the proof of the theorem, since $X_i$ always appears through the function $Q_\tau$. Also, the result goes through even if $\Pi$ and $Q_\tau$ happen to depend on $n$, as long as Assumptions 1, 2, 3, and 4 continue to hold.

3 Bayesian Linear Quantile Regression

Posterior consistency for Bayesian linear quantile regression based on ALD was shown by Sriram et al. (2013). While they also discussed rates of convergence, Sriram and Ramamoorthi (2017) highlighted that the argument in the paper was valid only for a rate of convergence of $n^\alpha$ with $\alpha < 1/2$ and did not go through for $\sqrt{n}$. In this section, we first apply Theorem 1 to obtain $\sqrt{n}$-consistency for finite dimensional linear quantile regression. Then, we apply the theorem to obtain an posterior consistency for Bayesian linear quantile regression when the number of covariates $p_n$ depends on $n$.

3.1 Finite dimensional linear quantile regression

For a given $0 < \tau < 1$, we take $\mathcal{G} = \{Q_\tau : Q_\tau(x) = x^T \beta, \ \beta \in \mathbb{R}^d\}$. We will assume that the covariate space and the parameter spaces are bounded, thus satisfying Assumption 1. Suppose that the true $\tau^{th}$ quantile function is given by $Q_{0\tau}(x) = x^T \beta_0$. As noted in the paragraph following Assumption 2 in Section 2, Assumption 2a holds when the probability density function of $Y_i$ is strictly positive near the $\tau^{th}$ quantile, with the density uniformly bounded away from 0 and Assumption 2b will hold if the function $E|\rho_\tau(Y_i - u) - \rho_\tau(Y_i - u_0)|$ is smooth, with uniformly bounded second derivative.

Suppose $Q^{(1)}_\tau(x) = x^T \beta_1$ and $Q^{(2)}_\tau(x) = x^T \beta_2$. Then
\[
d_n(Q^{(1)}_\tau, Q^{(2)}_\tau) = (\beta_1 - \beta_2)^T \left( \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T \right) (\beta_1 - \beta_2). \tag{11}
\]

If we assume that for all large enough $n$, the minimum eigen value of $\frac{1}{n} \sum_{i=1}^{n} X_i X_i^T$ is greater than $\lambda_0^2$ and the maximum eigen value is less than $\lambda_1^2$, then $d_n$ and the Euclidean metric will be equivalent, more precisely:
\[
\lambda_0 \| \beta_1 - \beta_2 \| \leq d_n(Q^{(1)}_\tau, Q^{(2)}_\tau) \leq \lambda_1 \| \beta_1 - \beta_2 \|. \tag{12}
\]

To check Assumption 3a, it is enough to check that there exists an $L$ such that
\[
\frac{\Pi(j \epsilon_n < \| \beta - \beta_0 \| \leq 2j \epsilon_n)}{\Pi \left( \| \beta - \beta_0 \| < \frac{2 \epsilon_n}{\sqrt{n}} \right)} \leq e^{Lj^2}, \ \forall \ n, j.
\]
The above condition will hold if \( \Pi \) is proper and has a continuous density that is bounded away from infinity for all \( \beta \), and bounded away from zero at \( \beta_0 \), for then \( \exists \) suitable constants \( A \) and \( B \) such that

\[
\begin{align*}
\Pi(j\epsilon_n < \|\beta - \beta_0\| \leq 2j\epsilon_n) & \leq A(j\epsilon_n)^p \\
\Pi\left(\|\beta - \beta_0\| < \frac{\epsilon_n}{\sqrt{S}}\right) & \geq B \left(\frac{\epsilon_n}{\sqrt{S}}\right)^p.
\end{align*}
\]

So,

\[
\frac{\Pi(j\epsilon_n < \|\beta - \beta_0\| \leq 2j\epsilon_n)}{\Pi\left(\|\beta - \beta_0\| < \frac{\epsilon_n}{\sqrt{S}}\right)} \leq \left(\max\left(\frac{A}{B}, 1\right)\right)^{p/2} \leq e^{S_0p^2},
\]

(13)

for a suitable constant \( S_0 = \max\left(\frac{A}{B}, 1\right) \frac{S}{2} \). This gives Assumption 3a.

As for assumption 4, for large enough \( n \), for any \( \epsilon \), the set \( \{ \epsilon < d_n(Q_{\tau}, Q_{0\tau}) \leq 2\epsilon \} \subset \{ \frac{A}{B} < \|\beta - \beta_0\| \leq \frac{2\epsilon_n}{\sqrt{S}} \} \) The minimum number of Euclidean balls of radius \( R\epsilon \) required to cover this set will be \( \leq A_1^p \) for some constant \( A_1 \). So, Assumption 4 is satisfied.

Under the conditions discussed above, part a of Theorem 1 applies. Hence, we can obtain \( \sqrt{n} \) consistency for Bayesian linear quantile regression. We summarize these findings in the result below.

**Theorem 2.** Suppose the data is \( Y_{1:n} = (Y_1, Y_2, \ldots, Y_n) \). For a given \( 0 < \tau < 1 \), the \( \tau \)th quantile function for \( Y_i \) is modeled as \( Q_{\tau}(x) = x^T\beta, \beta \in \mathbb{R}^p \). Suppose the true quantile function for any given covariate value of \( x \) is obtained at \( \beta = \beta_0 \). Let \( \Pi \) be a prior on \( \beta \).

Let \( M_n > 0 \) be any sequence such that \( M_n \to \infty \) and \( \epsilon_n = \frac{M_n}{\sqrt{n}} \to 0 \). Suppose the following conditions hold (i) to (iv) hold:

(i) \( \Pi \) is proper and has a continuous density that is bounded away from infinity for all \( \beta \), and bounded away from zero at \( \beta_0 \).

(ii) Assumption 2 holds for the true underlying probability distribution of \( Y_i \).

(iii) Covariate space and the parameter space are bounded.

(iv) For some \( \lambda_0 > 0 \) and \( \lambda_1 > 0 \), for all large enough \( n \), the minimum eigen value of \( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \) is greater than \( \lambda_0^2 \) and the maximum eigen value is less than \( \lambda_1^2 \).

Then, \( \exists \ J > 0 \) such that

\[ \Pi\left(\|\beta - \beta_0\| > J\epsilon_n|Y_{1:n}|\right) \to 0 \text{ in probability } [P]. \]

**Remark 2.** Finite dimensional non-linear quantile regression

The argument for the linear regression can be easily extended to the case of parametric non-linear models. Suppose \( Q_{\tau}(x) \) is of the form \( q(x, \beta) \), where the covariate vector \( x \) and parameters
\(\beta\) are both finite dimensional and \(q(\cdot, \cdot)\) is a known smooth function in \((x, \beta)\). Suppose for any given \(x\), the true quantile is given by \(q(x, \beta_0)\). Suppose \(\|\beta\|\) and \(\|X_i\|\) for all \(i\), are bounded. Then Assumption 1 is satisfied. Based on the same conditions discussed in the previous section Assumption 2 will hold. Suppose there exist \(\lambda_0\) and \(\lambda_1\) such that equation (12) holds, then the same arguments as above yield Assumptions 3 and 4.

### 3.2 Bayesian Linear Quantile Regression when \(p\) depends on \(n\)

As noted in remark 1, Theorem 1 holds even if \(\Pi, Q, X, \) depend on \(n\), as long as the assumptions hold. This enables us obtain an extension to the case when the number of covariates vary with \(n\). Suppose the data for each \(n\) is denoted by \(Y_{1:n, n} = (Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n})\). Suppose the covariate vector is denoted by \(X_{in} = (X_{1:n}, X_{2:n}, \ldots, X_{p:n})\); so the dimensionality \(p_n\) depends on \(n\). For any \(n\), suppose the true \(\tau^{\text{th}}\) quantile is given by \(Q_{bn\tau}(X_{ni}) = X_{ni}^T \beta_{bn}\). We will assume that there exist \(\lambda_0\) and \(\lambda_1\) such that for large enough \(n\), the minimum eigen value of \(\frac{1}{n} \sum_{i=1}^{n} X_{ni}X_{ni}^T\) is greater than \(\lambda_0^2\) and the maximum eigen value is less than \(\lambda_1^2\). Similar to the previous section, it follows that

\[
d^2_n(Q_n^{(1)}, Q_n^{(2)}) = (\beta_{n1} - \beta_{n2})^T \left[ \frac{1}{n} \sum_{i=1}^{n} X_{ni}X_{ni}^T \right] (\beta_{n1} - \beta_{n2}) \leq \lambda_0 \|\beta_{n1} - \beta_{n2}\| \leq d_n(Q_n^{(1)}, Q_n^{(2)}) \leq \lambda_1 \|\beta_{n1} - \beta_{n2}\|. \tag{15}
\]

We will assume Assumption 1 holds, by assuming that the covariates \(x_n\) and \(\beta_n\) satisfy \(x_n^T \beta_n\) for all \(n\). We will assume that for each \(Y_{in}\) Assumption 2 holds, based on similar conditions discussed immediately following Assumption 2 in Section 2. We will model the quantiles as a linear function \(Q_n\tau(X_{ni}) = X_{ni}^T \beta_n\), by considering a prior \(\Pi_n\) on \(\beta_n\), such that its pdf \(\pi_n(\beta_n)\) is such that

\[0 < c_0^n < \pi_n(\beta_n) < c_1^n < \infty, \text{ for some } c_0 > 0, c_1 > 0.\]

By using arguments similar to the ones leading up to equation (13) we have

\[
\frac{\Pi(j\epsilon_n < \|\beta - \beta_0\| \leq 2j\epsilon_n)}{\Pi \left( \|\beta - \beta_0\| < \frac{\epsilon_n}{2\epsilon_n} \right)} \leq e^{S_nj^2}, \tag{16}
\]

So, Assumption 3b is satisfied with \(L_n = S_0p_n\). To apply Theorem 1 we assume \(\lim_{n \to \infty} \frac{p_n}{M_n} = 0\).

Similarly, for any \(\epsilon > 0\), the minimum number of \(p_n\)-dimensional Euclidean balls of radius \(R\) needed to cover the set \(\epsilon < \|\beta_n - \beta_{n0}\| < 2\epsilon\), will be less than or equal to \(A_{1n}^\epsilon\) for some suitable constant \(A_1\). If we assume \(\lim_{n \to \infty} \frac{p_n}{M_n} = 0\), then \(N_{kn} \leq A_{1n}^\epsilon \leq e^{b\epsilon n^2}\), for a suitable constant \(b\) and large enough \(n\). In particular, if we assume \(p_n < n^{1/2-\eta}\) for some \(\eta < 1/2\), then
for some $J$, for any $\alpha < \eta$, Theorem 1 would apply. We summarize these observations in the result below.

**Theorem 3.** Suppose for a given $0 < \tau < 1$, the $\tau$th quantile function for $Y_{in}$ is modeled as $Q_{n\tau}(x_n) = x_n^T \beta_n$. Suppose the true quantile function for any given covariate value of $x_n$ is obtained at $\beta_n = \beta_{0n}$. Let $\Pi_n$ be a prior on $\beta_n$. Let $p_n = \text{dimensionality of } \beta_n$ and let $p_n \leq n^{1/2-\eta}$ for some $0 < \eta < 1/2$. Suppose the following conditions hold

(i) $\Pi_n$ is proper and its density $\pi_n$ is such that

$$0 < c_0 < \pi_n(\beta_n) < c_1 < \infty,$$

for some $c_0 > 0, c_1 > 0$.

(ii) Assumption 2 holds for the true underlying probability distribution of $Y_{in}$ uniformly across $i$ and $n$.

(iii) For some $M$, $|X_{in}\beta_n| \leq M$ for all $i, n$.

(iv) For some $\lambda_0 > 0$ and $\lambda_1 > 0$, for all large enough $n$, the minimum eigen value of $
\frac{1}{n} \sum_{i=1}^{n} X_{in}X_{in}^T$n

is greater than $\lambda_0^2$ and the maximum eigen value is less than $\lambda_1^2$.

Then, $\exists$ $J > 0$ such that for all $\alpha < \eta$,

$$\Pi(\|\beta - \beta_0\| > Jn^{-\alpha}|Y_{1:n}) \to 0 \text{ in probability } [P].$$

### 4 Conclusion

We have obtained sufficient conditions for posterior consistency at $\sqrt{n}$-rate under a general setting of Bayesian non-linear quantile regression based on the misspecified asymmetric Laplace likelihood. $\sqrt{n}$ consistency is obtained for the linear case. The approach enables an extension to Bayesian linear quantile regression with the number of covariates depending on data size, where we obtain consistency at rate less than $\sqrt{n}$.
Appendix A  Proofs of Results

Proof of Lemma 7  Proof of this Lemma is in the lines of Lemma 4 of [Sriram and Ramamoorthi (2017)], with some minor modifications. For completeness, we provide the proof here. Define

\[ b_i := Q_\tau(X_i) - Q_{0\tau}(X_i), \]
\[ T_i := \log \left( \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right). \]

Based on Lemma 1a of [Sriram et al. (2013)], we first note the following identity:

\[ T_i = \begin{cases} 
-b_i(1 - \tau), & \text{if } Y_i \leq \min(Q_\tau(X_i), Q_{0\tau}(X_i)) \\
(Y_i - Q_{0\tau}(X_i)) - b_i(1 - \tau), & \text{if } Q_{0\tau}(X_i) < Y_i \leq Q_\tau(X_i) \\
b_i\tau - (Y_i - Q_{0\tau}(X_i)), & \text{if } Q_\tau(X_i) < Y_i \leq Q_{0\tau}(X_i) \\
b_i\tau, & \text{if } Y_i \geq \max(Q_\tau(X_i), Q_{0\tau}(X_i)). 
\end{cases} \quad (17) \]

We will consider the case where \( b_i \geq 0 \) as the argument is similar when \( b_i < 0 \). When \( Q_{0\tau}(X_i) < Y_i \leq Q_\tau(X_i) \),

\[ T_i = (Y_i - Q_{0\tau}) - b_i(1 - \tau) = Y_i - q_i, \]

where \( q_i = Q_{0\tau}(X_i)\tau + Q_\tau(X_i)(1 - \tau) \).

So, \( (Y_i - q_i) \leq \begin{cases} 
0, & \text{if } Q_\tau(X_i) < Y_i \leq q_i \\
(Q_\tau(X_i) - q_i) = b_i\tau, & \text{if } q_i < Y_i < Q_\tau(X_i). \end{cases} \)

This implies

\[ T_i \leq -b_i(1 - \tau) \times I_{Y_i \leq Q_{0\tau}(X_i)} + 0 \times I_{Q_{0\tau}(X_i) < Y_i \leq q_i} + b_i\tau \times I_{Y_i > q_i}. \quad (18) \]

Let \( \tau_i^* = P(Y_i \leq q_i) \) and note that \( P(Y_i \leq Q_{0\tau}(X_i)) = \tau \). For any \( d > 0 \),

\[ E[e^{dT_i}] \leq \tau e^{-db_i(1 - \tau)} + (\tau_i^* - \tau) + e^{db_i\tau}(1 - \tau_i^*). \quad (19) \]

Define the function

\[ g_i(t) := e^{-tb_i(1 - \tau)} + (\tau_i^* - \tau) + e^{tb_i\tau}(1 - \tau_i^*). \quad (20) \]

By Taylor’s formula,

\[ g_i(t) = 1 + g_i'(0)t + g_i''(\xi)t^2/2, \quad \text{for some } 0 < \xi < t. \quad (21) \]

In equation (21), we first note that \( g_i'(0) = -b_i\tau(\tau_i^* - \tau) \). Suppose, \( C_1, \Delta_0 \) are as in Assumption 2a, then \( P(0 < Y_i - Q_{0\tau}(X_i) < \Delta) > C_1\Delta \ \forall \ \Delta \leq \Delta_0 \). Defining \( b_i^* = \min(b_i, \Delta_0) \) and noting
that $q_i - Q_{0r}(X_i) = b_i(1 - \tau)$, we have

$$
\tau^*_i - \tau = P(Q_{0r}(X_i) < Y_i \leq q_i) = P(0 < Y_i - Q_{0r}(X_i) \leq b_i(1 - \tau)) \geq P(0 < Y_i - Q_{0r}(X_i) \leq b^*_i(1 - \tau)) > C_1 b^*_i (1 - \tau).
$$

Hence $g'_i(0) \leq -C_1 \tau(1 - \tau)b^*_i \tau^2$. (22)

Further, we note $g''_i(t) = b^2_i \times (\tau(1 - \tau)2e^{-t b_i(1-\tau)} + \tau^2(1-\tau^*_i)e^{tb_i\tau})$. By Assumption 1, $|b_i| \leq |Q_r(X_i)| + |Q_{0r}(X_i)| \leq 2M$ and hence the term within the parenthesis in the above expression can be bounded by some constant $K_1 > 0$. Hence $g''_i(t) \leq K_1 b^2_i$. Further, note by definition that $b^*_i = b_i I_{b_i \leq \Delta_0} + \Delta_0 I_{b_i > \Delta_0}$. Therefore, if we choose $K_2 > 1$, such that $K_2 \Delta_0 > 2M$, then we would have $K_2 b^*_i = (K_2 b_i I_{b_i \leq \Delta_0} + K_2 \Delta_0 I_{b_i > \Delta_0}) \geq b_i$. In other words, $\exists K_2$ such that $b_i \leq K_2 b^*_i$ or $b^*_i \geq \frac{b_i}{K_2}$. Therefore,

$$
g_i(t) \leq 1 - K_i b_i^2 \cdot t \cdot \left(\frac{C_1 \tau(1 - \tau)}{K_1 \cdot K_2^2} - t\right).
$$

So, for any $t < \min\left(\frac{1}{2}, \frac{1}{2} \frac{C_1 \tau(1 - \tau)}{K_1 \cdot K_2^2}\right)$ and $K = \frac{C_1 \tau(1 - \tau)}{2 \cdot K_2^2}$, we have

$$
g_i(t) \leq 1 - Ktb_i^2 \leq e^{-Ktb_i^2}.
$$

So, if we choose $\alpha = \frac{1}{2} \min\left(\frac{1}{2}, \frac{1}{2} \frac{C_1 \tau(1 - \tau)}{K_1 \cdot K_2^2}\right)$ then

$$
E \left(\prod_{i=1}^{n} f_r(Y_i - Q_r(X_i)) \right)^\alpha \leq E \left(\prod_{i=1}^{n} e^{\alpha T_i}\right) \leq \prod_{i=1}^{n} g_i(\alpha) = e^{-\alpha K \sum_{i=1}^{n} b_i^2} = e^{-\alpha C_3 n \Delta_2^2 (Q_r, Q_{0r})}.
$$

Last step follows by noting that $\sum_{i=1}^{n} b_i^2 = nd^2_n(Q_r, Q_{0r})$ and taking $C_3 = K$.

**Proof of Lemma** 3. Note that by Assumption 2b,

$$
- \frac{1}{n} \sum_{i=1}^{n} E \log \frac{f_r(Y_i - Q_r(X_i))}{f_r(Y_i - Q_{0r}(X_i))} = \frac{1}{n} \sum_{i=1}^{n} E(\rho_r(Y_i - Q_r(X_i)) - \rho_r(Y_i - Q_{0r}(X_i))) \leq \frac{1}{n} \sum_{i=1}^{n} S|Q_r(X_i) - Q_{0r}(X_i)|^2 = Sd^2_n(Q_r, Q_{0r}).
$$

Without loss of generality, we can assume $S \geq 1$. Also, by Lemma 1b of [Sriram et al. (2013)](https://example.com), $|\log \frac{f_r(Y_i - Q_r(X_i))}{f_r(Y_i - Q_{0r}(X_i))}| \leq |Q_r(X_i) - Q_{0r}(X_i)|$. So

$$
\frac{1}{n} \sum_{i=1}^{n} E \left(\log \frac{f_r(Y_i - Q_r(X_i))}{f_r(Y_i - Q_{0r}(X_i))}\right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} |Q_r(X_i) - Q_{0r}(X_i)|^2 \leq Sd^2_n(Q_r, Q_{0r}).
$$
It follows that
\[ (d_n(Q_\tau, Q_{\tau_0}) < \frac{\epsilon_n}{\sqrt{S}}) \implies Q_\tau \in V_{\epsilon_n^2}. \]
So, if Assumption 3a holds then,
\[ e^{L_2^j} \Pi(V_{\epsilon_n^2}) \geq e^{L_2^j} \Pi \left( d_n(Q_\tau, Q_{\tau_0}) < \frac{\epsilon_n}{\sqrt{S}} \right) \geq \Pi(j\epsilon_n < d_n(Q_\tau, Q_{\tau_0}) \leq 2j\epsilon_n), \]
which shows the result. Argument is similar if Assumption 3b holds.

**Proof of Lemma** The proof is in the lines of Lemma 8.1 in [Ghosal et al. (2000)].

By Jensen’s inequality we get
\[
\log \int_{V_{\epsilon_n^2}} \prod_{i=1}^n \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \geq \sum_{i=1}^n \int_{V_{\epsilon_n^2}} \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} = \log \left( \frac{\prod_{i=1}^n f_\tau(Y_i - Q_\tau(X_i))}{\prod_{i=1}^n f_\tau(Y_i - Q_{0\tau}(X_i))} \right). \tag{23}
\]
Further, noting that for any $Q_\tau \in V_{\epsilon_n^2}$, $-\sum_{i=1}^n E \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right) < n\epsilon_n^2$, we observe that the inequality
\[
\sum_{i=1}^n \int_{V_{\epsilon_n^2}} \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \leq -(1 + D_1)n\epsilon_n^2, \quad \text{implies}
\]
\[
\sum_{i=1}^n \int_{V_{\epsilon_n^2}} \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} - E \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right) \right) \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \leq -(1 + D_1)n\epsilon_n^2 - \int_{V_{\epsilon_n^2}} \sum_{i=1}^n E \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right) \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \leq -D_1n\epsilon_n^2.
\]
We note that $P(S_n)$ is bounded by
\[
P \left( \sum_{i=1}^n \int_{V_{\epsilon_n^2}} \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} - E \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right) \right) \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \leq -D_1n\epsilon_n^2 \right)
\]
which by Chebyshev’s inequality is
\[
\sum_{i=1}^n E \left( \int_{V_{\epsilon_n^2}} \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right) \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \right)^2 \leq \frac{\sum_{i=1}^n \int_{V_{\epsilon_n^2}} \left( \log \frac{f_\tau(Y_i - Q_\tau(X_i))}{f_\tau(Y_i - Q_{0\tau}(X_i))} \right)^2 \frac{d\Pi(Q_\tau)}{\Pi(V_{\epsilon_n^2})} \leq \frac{n\epsilon_n^2}{D_1^2 n^2 \epsilon_n^4} = \frac{1}{D_1^2 n^2 \epsilon_n^2},
\]
\[ \square \]

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Proof of Lemma \[\text{[2]}\] For simplicity of notation, we omit subscripts and define \( W_i := \log \frac{f_i(Y_i - Q_i(X_i))}{f_i(Y_i - Q_{\tau jkn}(X_i))} \) and \( b_i = Q_i(X_i) - Q_{\tau jkn}(X_i) \). First, since \( 0 < \alpha < 1 \), we note by Jensen’s inequality and Fubini’s theorem that
\[
E \left( \int_{B_{jkn}} \prod_{i=1}^{n} e^{W_i} d\Pi \right)^{\alpha} \leq \left( \int_{B_{jkn}} \prod_{i=1}^{n} E \left( e^{W_i} \right) d\Pi \right)^{\alpha}.
\]
Using Taylor’s theorem, we can write
\[
E \left( e^{W_i} \right) = 1 + E \left( |W_i| \right) + E \left( e^{\xi W_i} W_i^2 \right), \quad \text{for some } 0 < \xi < 1.
\]
By Lemma 1b of Sriram et al. (2013), \( |W_i| \leq |b_i| = |Q_i(X_i) - Q_{\tau jkn}(X_i)| \leq 2M \), with \( M \) as in Assumption 1. By Assumption 2b, \( \exists \delta \) such that the second term in the above expression \( E|W_i| \leq S|Q_i(X_i) - Q_{\tau jkn}(X_i)|^2 \). Further, the third term in the above expression \( E(e^{\xi W_i} W_i^2) \leq e^{2M} E(W_i^2) \leq e^{2M} b_i^2 = e^{2M}|Q_i(X_i) - Q_{\tau jkn}(X_i)|^2 \). It follows that for a suitable constant \( C_4 \),
\[
E \left( e^{W_i} \right) \leq 1 + (S + e^{2M})|Q_i(X_i) - Q_{\tau jkn}(X_i)|^2 \leq e^{C_4|Q_i(X_i) - Q_{\tau jkn}(X_i)|^2}.
\]
So, we have
\[
\frac{1}{(\Pi(V_{i,2}^e)^{\alpha}} E \left( \int_{B_{jkn}} \prod_{i=1}^{n} f_i(Y_i - Q_i(X_i)) d\Pi(Q_i) \right)^{\alpha},
\]
which by using Jensen’s inequality (since \( 0 < \alpha < 1 \)) and Fubini’s theorem is
\[
\leq \frac{1}{(\Pi(V_{i,2}^e)^{\alpha}} \left( \int_{B_{jkn}} \prod_{i=1}^{n} E \left( e^{W_i} \right) d\Pi(Q_i) \right)^{\alpha},
\]
\[
\leq \frac{1}{(\Pi(V_{i,2}^e)^{\alpha}} \left( \int_{B_{jkn}} e^{C_4\delta^2|Q_i - Q_{\tau jkn}|} d\Pi(Q_i) \right)^{\alpha},
\]
\[
\leq e^{\alpha k^2 R^2 C_4 \delta^2 n} \left( \frac{\Pi(B_{jkn})}{\Pi(V_{i,2}^e)^{\alpha}} \right) \leq e^{\alpha k^2 R^2 C_4 \delta^2 n} \left( \frac{\Pi(A_{jkn})}{\Pi(V_{i,2}^e)^{\alpha}} \right) \leq e^{\alpha k^2 R^2 C_4 \delta^2 n} e^{\alpha L j^2}.
\]
The last step follows by observing that \( Q_i \) belongs to an \( d_n \) ball around \( Q_{\tau jkn} \) with radius \( kR\epsilon_n \), and by using Assumption 3a along with Lemma \[\text{[2]}\]. If Assumption 3b holds instead of Assumption 3a, then we just need to use \( L_n \) in place of \( L \) in the last step. \( \square \)

Proof of Theorem \[\text{[3]}\]
The initial steps of the proof is common to both part a and part b of the theorem. Let \( 0 < \alpha < 1 \) be as in Lemma \[\text{[3]}\] and \( S_n \) be as in equation (10) for some \( D_1 > 0 \). Recall \( U_n^e := \{ d_n(Q, Q_0) > J\epsilon_n \} \). We will choose \( J \geq 1 \) and \( 0 < \alpha < 1 \) so that \( E(\Pi(U_n^e[Y_{i,n}]))^{\alpha/2} \to 0 \). By using Lemma
note that

\[
E(\Pi(U_n|Y_{1:n}))^{\alpha/2} = E\left[\Pi(U_n|Y_{1:n})^{\alpha/2} \cdot I_{S_n}\right] + E\left[(\Pi(U_n|Y_{1:n}))^{\alpha/2} \cdot I_{S_n}\right]
\]

\[
\leq P(S_n) + E\left[(\Pi(U_n|Y_{1:n}))^{\alpha/2} \cdot I_{S_n}\right] \leq \frac{1}{D_1^2 n c_n^2} + E\left[(\Pi(U_n|Y_{1:n}))^{\alpha/2} \cdot I_{S_n}\right].
\]

(24)

Consider the second term on the right hand side of inequality (24),

\[
E\left[(\Pi(U_n|Y_{1:n}))^{\alpha/2} \cdot I_{S_n}\right]
\]

\[
\leq E\left[\left(\int_{U_n}^{\infty} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2} \cdot I_{S_n}\right]
\]

\[
\leq E\left[\left(\int_{U_n}^{\infty} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2} \cdot I_{S_n}\right]
\]

\[
\leq \frac{e^{\alpha(1+D_1)n c_n^2/2}}{(\Pi(V_n^2))^{\alpha/2}} E\left[\left(\int_{U_n}^{\infty} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2} \cdot I_{S_n}\right].
\]

(25)

The right hand side of inequality (25) can be bounded as follows. We have

\[
E_1 = \frac{1}{(\Pi(V_n^2))^{\alpha/2}} E\left[\left(\int_{U_n}^{\infty} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2}\right]
\]

\[
\leq E\left[\sum_{k \geq j} \left(\int_{A_{kn}} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2}\right]
\]

\[
\leq E\left[\sum_{k \geq j} \sum_{j \leq N_{kn}} \left(\int_{B_{jkn}} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2}\right],
\]

which By using Cauchy-Schwartz inequality on each term is

\[
\leq \sum_{k \geq j} \sum_{j \leq N_{kn}} \left(E\left(\prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i))\right)^{\alpha/2}\right) \cdot \left(E\left(\int_{B_{jkn}} \prod_{i=1}^{n} f_{\tau}(Y_i - Q_{\tau}(X_i)) d\Pi(Q_{\tau})\right)^{\alpha/2}\right),
\]

(26)

Now, specifically to prove part (a) of the theorem, we assume that Assumption 3a holds. By
using Lemmas 1 and 4 for the first and second terms respectively, we have

\[
E_1 \leq \sum_{k \geq J_2} N_k n e^{-\alpha C_3 n^2} \left( Q_{\epsilon_k n} + Q_{\epsilon_k} \right) / 2 \cdot e^{\alpha k^2 R^2 C_4 n^2 / 2} e^{\alpha L k^2 / 2} \tag{27}
\]

\[
\leq \sum_{k \geq J} N_k n e^{-\alpha C_3 k^2 n^2 / 2} \cdot e^{\alpha k^2 R^2 C_4 n^2 / 2} e^{\alpha L k^2 / 2},
\]

which by Assumption 4 on entropy bound is \( \leq \sum_{k \geq J} e^{b n^2 / 2} e^{-\alpha k^2 R^2 C_3 / 2} e^{\alpha L k^2 / 2} \),

and by choosing \( R^2 = \frac{C_3}{2C_4} \) is \( \leq \sum_{k \geq J} e^{b n^2 / 2} e^{\alpha L k^2 / 2} e^{-\alpha k^2 C_3 / 4} \)

\[
\leq e^{b n^2 / 2} e^{-\alpha J^2 C_3 / 4} \sum_{k \geq 0} e^{-\alpha J^2 C_3 / 4} e^{\alpha L k / 2}
\]

\[
\leq e^{\alpha J^2 C_3 / 4} \sum_{k \geq 0} e^{-(\alpha C_3 / 4 - \alpha L / 2) k}. \tag{28}
\]

For large enough \( n \), \( (\alpha n^2 C_3 / 4 - \alpha L / 2) = (\alpha M_n C_3 / 4 - \alpha L / 2) > 0 \). Therefore, the right hand side of equation (28) is

\[
\leq e^{-\alpha n^2 C_3 / 4 - \alpha L / 2} \leq 2e^{-\alpha n^2 C_3 / 4 - \alpha L / 2} \text{, for large enough } n. \tag{29}
\]

Using inequalities (25) and (29) together, we get

\[
E \left[ \left( \Pi(U_n | Y_{1:n}) \right)^{\alpha / 2} \cdot I_{\gamma_0} \right] \leq 2e^{\alpha (1 + D_1) n^2 / 2} e^{-\alpha n^2 C_3 / 4 - \alpha L / 2}
\]

\[
\leq 2e^{-\alpha n^2 (j^2 C_3 / 4 - (1/2 + \alpha(1 + D_1)) / 2)}. \tag{30}
\]

It follows using equations (30) that we can choose a large enough integer \( J \) such that the right hand side of equation (30) goes to zero. It follows that

\[
E (\Pi(U_n | Y_{1:n}) )^{\alpha / 2} \to 0.
\]

Finally, by Markov’s inequality, we obtain the part (a) of the theorem, i.e. for any sequence \( M_n \to \infty \) such that \( \epsilon_n = \frac{M_n}{\sqrt{n}} \to 0 \),

\[
\Pi(U_n | Y_{1:n}) \to 0 \text{ in probability } [P].
\]

Now, to obtain part (b) of the theorem, first we assume Assumption 3b holds instead of Assumption 3a. Then, we note that the steps starting from equation (27) leading up to equation (28) can be repeated with \( L_n \) in place of \( L \). Then we note that for any sequence \( M_n \) such that \( \lim_{n \to \infty} \frac{L_n}{M_n} = 0 \), we would still have \( (\alpha n^2 C_3 / 4 - \alpha L_n / 2) = (\alpha M_n C_3 / 4 - \alpha L_n / 2) > 0 \) for large enough \( n \). Therefore, equation (29) and all subsequent steps follow.
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