An Axiomatics and a Combinatorial Model of Creation/Annihilation Operators

Marcelo Fiore

Abstract. A categorical axiomatic theory of creation/annihilation operators on bosonic Fock space is introduced and the combinatorial model that motivated it is presented. Commutation relations and coherent states are considered in both frameworks.

Introduction

This work is an investigation into the mathematical structure of creation/annihilation operators on (bosonic or symmetric) Fock space. My aim is two-fold: to introduce an axiomatic setting for commutation relations and coherent states, and to provide and exercise one such model of combinatorial nature. In the spirit of Paul Dirac’s credo

“One should allow oneself to be led in the direction which the mathematics suggests ... one must follow up a mathematical idea and see what its consequences are, even though one gets led to a domain which is completely foreign to what one started with ... Mathematics can lead us in a direction we would not take if we only followed up physical ideas by themselves.”

my hope is that the mathematical theories presented here, and the ideas that underly them, can be of use to physics.

Axiomatics. Section II considers the axiomatics. This is set up in the framework of category theory, which is particularly suitable for our purposes. Our starting point is the consideration of categories of spaces and
linear maps. So as to be able to accommodate Fock space, these should allow for the formation of superposed and of noninteracting systems. In Section 1.1 I respectively formalise these as compatible biproduct and symmetric monoidal structures. The linear-algebraic structure is then derived by convolution with respect to the biproduct structure. For completeness, other equivalent formalisations are also given. Of central importance to our development is the algebraic axiomatisation of biproduct structure as monoidal bialgebra structure (see Proposition 1.2 and Lemma 1.4). The resulting setting is rich enough for formalising Fock space together with creation/annihilation operators on it. Specifically, in Section 1.2 the Fock-space construction is axiomatised as a functor on the category of spaces and linear maps that transforms the biproduct (i.e. superposition) structure to the symmetric monoidal (i.e. noninteracting) structure. A fundamental aspect of this definition is that it lifts the biproduct bialgebra structure to a bialgebra structure on Fock space. This allows for a general definition of creation/annihilation operators (Definition 1.15) and embodies the essential mathematical structure of the commutation relations (Theorem 1.16). Section 1.3 considers coherent states on Fock space. To this end, however, one needs specialise the discussion to Fock-space constructions with suitable comonad structure. This additional structure plays two roles: it provides a canonical notion of annihilation operator and permits the association of coherent states in Fock space to vectors (Definition 1.21 and Theorem 1.22).

**Combinatorial model.** Section 2 puts forward a bicategorical combinatorial model. Its combinatorial nature resides in the structure being a generalisation of that of the combinatorial species of structures of Joyal [28, 29] (see [22] for details). The main consequence of this for us here is that identities, such as the commutation relations, acquire combinatorial meaning in the form of natural bijective correspondences.

The combinatorial model is based on the bicategory of profunctors (or bimodules, or distributors) as the setting for spaces and linear maps. These structures, I briefly review in Section 2.1 noting analogies with vector spaces. Combinatorial (bosonic or symmetric) Fock space is then introduced in Section 2.2. The definition mimics that of the conventional construction as a biproduct of symmetric tensor powers. After making explicit the mathematical structure of combinatorial Fock space, the commutation relation involv-
ing creation and annihilation is considered. We see here that the essence of its combinatorial content arises from the simple fact that

$$\mathcal{S}_{n+1} \cong \mathcal{S}_n \cup ([n] \times \mathcal{S}_n) \quad \text{for } [n] = \{1, \ldots, n\}$$

classifying the permutations on the set $[n+1]$ according as to whether or not they fix the element $n+1$, see (14) and (15). It is an important aspect of the theory, however, that all such calculations are done formally in the calculus of coends (within the generalized logic of Lawvere [31]). I further illustrate how the calculus can be seen diagrammatically.

Finally, Section 2.3 considers coherent states in the combinatorial model. Taking advantage of the duality structure available in it, a notion of exponential (in the form of a comonadic/monadic convolution) is introduced. The exponential of the creation operator of a vector at the vacuum state is shown, both algebraically and combinatorially, to yield the coherent state of the vector.

**Related work.** This work lies at the intersection of computer science, logic, mathematics, and physics. As such, it bears relationship with a variety of developments.

In relation to mathematical logic, the notion of comonad needed in the discussion of coherent states is as it arises in models of the linear logic of Girard [25]. The connection between the exponential modality of linear logic and the Fock-space construction of physics was recognised long ago by Panangaden (see e.g. [9, 8]). In view of recent developments, however, the connection further puts this work in the context of models of the differential linear logic of Ehrhard and Regnier [14]; and indeed the models to be found in [12, 13, 6, 27, 7] all fall within the axiomatisation here. A stronger axiomatisation (of which the combinatorial model is the motivating example [16]) leading to fully-fledged differential structure has been pursued in [21].

An axiomatics for Fock space has independently been considered by Vicary [36]. His setting, which aims at a tight correspondence with that of Fock space on Hilbert space, is stronger than the minimalist one put forward here. As acknowledged in his work, the argument used for establishing the commutation relation between creation and annihilation is based on a private communication of mine.
The combinatorial model is closely related to the *stuff-type model* of Baez and Dolan [1], see also [34], being both founded on species of structures. Roughly, their main difference resides in that the combinatorial model organises structure as presheaves, whilst the stuff-type model does so as bundles.

In connection to mathematical physics, the stuff-type model has been related to Feynman diagrams and, in connection to mathematical logic, these have been related to the proof theory of linear logic by means of the $\phi$-calculus of Blute and Panangaden [8], which, in turn, has formal syntactic structure similar to that of the calculus of the combinatorial model. These intriguing relationships are worth investigating.

**Acknowledgements.** The mathematical structure underlying the combinatorial model in the setting of generalised species of structures was developed in collaboration with Nicola Gambino, Martin Hyland, and Glynn Winskel [17, 22, 26]. The fact that it supports creation/annihilation operators, I realised shortly after giving a seminar at Oxford in 2004 on this material and the differential structure of generalised species of structures [16, 17, 19] where Prakash Panangaden raised the question. The axiomatics came later [21], and was influenced by the work of Thomas Ehrhard and Laurent Regnier on differential nets [15]. The work presented here is a write up of the talk [20], which I was invited to give by Bob Coecke. I’m grateful to them all for their part in this work.

1. Axiomatic theory

This section introduces an axiomatisation of the (bosonic or symmetric) Fock-space construction on categories of spaces and linear maps, see e.g. [24].

Spaces and linear maps are axiomatised by means of a category $S$ equipped with compatible biproduct $(O, \oplus)$ and symmetric monoidal $(I, \otimes)$ structures. Section 1.1 reviews these notions and explains the linear-algebraic structure that they embody. For a category of spaces and linear maps, the Fock-space construction is axiomatised as a strong symmetric monoidal functor $F$ mapping $(O, \oplus)$ to $(I, \otimes)$. Section 1.2 reviews this notion and explains how it supports an axiomatisation of creation/annihilation operators subject to commutation relations. For $F$ underlying a linear exponential comonad, coherent
states are considered and studied in Section 1.3.

1.1 Spaces and linear maps

**Biproduct structure.** A category with finite coproducts and finite products is said to be *bicartesian*. One typically writes 0, + for the empty and binary coproducts and 1, × for the empty and binary products.

An object that is both initial and terminal (i.e. an empty coproduct and product) is said to be a *zero object*. For a zero object O, I will write O_{A,B} for the map \( A \to B \) given by the composite \( A \to O \to B \).

**Definition 1.1.** A bicartesian category is said to have *biproducts* whenever:

1. it has a zero object O, and
2. for all objects A and B, the canonical map
   \[
   [\langle \text{id}_A, O_{A,B} \rangle, \langle O_{B,A}, \text{id}_B \rangle] : A + B \to A \times B
   \]
   is an isomorphism.

In this context, one typically writes \( \oplus \) for the binary biproduct.

The proposition below gives an algebraic presentation of biproduct structure which is crucial to our development. Recall that a *symmetric monoidal structure* \((I, \otimes, \lambda, \rho, \alpha, \sigma)\) on a category \( \mathcal{C} \) is given by an object \( I \in \mathcal{C} \), a functor \( \otimes : \mathcal{C}^2 \to \mathcal{C} \), and natural isomorphisms \( \lambda_C : I \otimes C \cong C \), \( \rho_C : C \otimes I \cong C \), \( \alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C) \), and \( \sigma_{A,B} : A \otimes B \cong B \otimes A \) subject to coherence conditions, see e.g. [32].

**Proposition 1.2.** To give a choice of biproducts in a category is equivalent to giving a symmetric monoidal structure \((O, \oplus)\) on it together with natural transformations

\[
\begin{array}{ccc}
O & \xrightarrow{u_A} & A \\
\downarrow{\Delta_A} & & \downarrow{\Delta_A} \\
A \oplus A & \xrightarrow{\Delta_A} & A \oplus A
\end{array}
\]

\hspace{1cm} (1)

such that
1. \((A, u_A, \nabla_A)\) is a commutative monoid.

\[
\begin{align*}
O \oplus A & \xrightarrow{u_A \oplus \text{id}_A} A \oplus A \xrightarrow{\text{id}_A \oplus u_A} A \oplus O \\
A \oplus A & \xrightarrow{\nabla_A \oplus \text{id}_A} A \oplus A \\
A & \xrightarrow{\nabla_A} A
\end{align*}
\]

\[
\begin{align*}
A \oplus A & \xrightarrow{\sigma_{A,A}} A \oplus A \\
A & \xrightarrow{\nabla_A} A
\end{align*}
\]

2. \((A, n_A, \Delta_A)\) is a commutative comonoid.

\[
\begin{align*}
O \oplus A & \xrightarrow{n_A \oplus \text{id}_A} A \oplus A \xrightarrow{\text{id}_A \oplus n_A} A \oplus O \\
A & \xrightarrow{\Delta_A} A \oplus A \\
A & \xrightarrow{\Delta_A} A \oplus A
\end{align*}
\]

\[
\begin{align*}
A \oplus A & \xrightarrow{\sigma_{A,A}} A \oplus A \\
A & \xrightarrow{\nabla_A} A \oplus A
\end{align*}
\]

3. \(u_{A \oplus B} = (O \cong O \oplus O \xrightarrow{u_A \oplus u_B} A \oplus B)\)

\(n_{A \oplus B} = (A \oplus B \xrightarrow{n_A \oplus n_B} O \oplus O \cong O)\)

\(\nabla_{A \oplus B} = \left( (A \oplus B) \oplus (A \oplus B) \cong (A \oplus A) \oplus (B \oplus B) \xrightarrow{\nabla_A \oplus \nabla_B} A \oplus B \right)\)

\(\Delta_{A \oplus B} = \left( (A \oplus B) \xrightarrow{\Delta_A \oplus \Delta_B} (A \oplus A) \oplus (B \oplus B) \cong (A \oplus B) \oplus (A \oplus B) \right)\)

The biproduct structure induced by (1) has coproduct diagrams

\[
\begin{align*}
A \cong A \oplus O \xrightarrow{\text{id}_A \oplus u_B} A \oplus B & \xrightarrow{u_A \oplus \text{id}_B} O \oplus B \cong B
\end{align*}
\]
and product diagrams

\[ A \cong A \oplus O \xrightarrow{\text{id}_A \oplus n_B} A \oplus B \xrightarrow{n_A \oplus \text{id}_B} O \oplus B \cong B \]

**Proposition 1.3.** In a category with biproduct structure \((O, \oplus)\), we have that

\[(A \xrightarrow{\pi_i} A \oplus A) = \begin{cases} \text{id}_A, & \text{if } i = j \\ O_{A,A}, & \text{if } i \neq j \end{cases}\]

**Lemma 1.4.** In a category with biproduct structure \((O, \oplus; u, \nabla; n, \Delta)\), the commutative monoid and comonoid structures \((u, \nabla; n, \Delta)\) form a commutative bialgebra. That is, \(u\) and \(\nabla\) are comonoid homomorphisms and, equivalently, \(n\) and \(\Delta\) are monoid homomorphisms.

\[
\begin{align*}
A & \xrightarrow{u_A} A & A \oplus A & \xrightarrow{\nabla_A} A \\
O & \xrightarrow{id_O} O & A \oplus A & \xrightarrow{\Delta_A} A \oplus A \\
\end{align*}
\]

\[
\begin{align*}
A & \xrightarrow{\nabla_A} A \oplus A & A \oplus A & \xrightarrow{\Delta_A} A \oplus A \\
\end{align*}
\]

**Linear-algebraic structure.** We examine the linear-algebraic structure of categories with biproduct structure. This I present in the language of enriched category theory [30].

Let \(\text{Mon}\) (\(\text{CMon}\)) be the symmetric monoidal category of (commutative) monoids with respect to the universal bilinear tensor product. Recall that \(\text{Mon}\)-categories (\(\text{CMon}\)-categories) are categories all of whose homs \([A, B]\) come equipped with a (commutative) monoid structure

\[0_{A,B} \in [A, B] , \quad +_{A,B} : [A, B]^2 \rightarrow [A, B]\]

such that composition is strict and bilinear; that is,
\[ 0_{B,C} f = 0_{A,C} \quad \text{and} \quad f 0_{C,A} = 0_{C,B} \]

for all \( f : A \to B \), and

\[ g (f +_{A,B} f') = g f +_{A,C} g f' \quad \text{and} \quad (g +_{B,C} g') f = g f +_{A,C} g' f \]

for all \( f, f' : A \to B \) and \( g, g' : B \to C \).

**Proposition 1.5.** The following are equivalent.

1. Categories with biproduct structure.
2. Mon-categories with (necessarily enriched) finite products.
3. CMon-categories with (necessarily enriched) finite products.

The enrichment of categories with biproduct structure \((O, \oplus; u, \nabla; n, \Delta)\) is given by convolution (see e.g. \([35]\)) as follows:

\[
0_{A,B} = (A \xrightarrow{n_A} O \xrightarrow{u_B} B) = O_{A,B}
\]

\[
f +_{A,B} g = (A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\nabla_B} B)
\]

**Proposition 1.6.** In a category with biproduct structure, \(\nabla_A = \pi_1 + \pi_2 : A \oplus A \to A\) and \(\Delta_A = \Pi_1 + \Pi_2 : A \to A \oplus A\).

We now consider biproduct structure on symmetric monoidal categories. To this end, note that in a monoidal category with tensor \(\otimes\) and binary products \(\times\) there is a natural distributive law as follows:

\[
\ell_{A,B,C} = \langle \pi_1 \otimes \text{id}_C, \pi_2 \otimes \text{id}_C \rangle : (A \times B) \otimes C \to (A \otimes C) \times (B \otimes C)
\]

**Definition 1.7.** A biproduct structure \((O, \oplus; u, \nabla; n, \Delta)\) and a symmetric monoidal structure \((I, \otimes)\) on a category are compatible whenever the following hold:
Proposition 1.5 extends to the symmetric monoidal setting. Recall that a \textit{Mon-enriched (symmetric) monoidal category} is a (symmetric) monoidal category with a \textit{Mon-enrichment} for which the tensor is strict and bilinear; that is, such that
\[
0_{X,Y} \otimes f = 0_{X \otimes A, Y \otimes B} \quad \text{and} \quad f \otimes 0_{X,Y} = 0_{A \otimes X, B \otimes Y}
\]
for all \( f : A \to B \), and
\[
g \otimes (f + f') = g \otimes f + g \otimes f' \quad \text{and} \quad (g + g') \otimes f = g \otimes f + g' \otimes f
\]
for all \( f, f' : A \to B \) and \( g, g' : X \to Y \).

\textbf{Proposition 1.8.} The following are equivalent.

1. Categories with compatible biproduct and symmetric monoidal structures.

2. \textit{Mon-enriched symmetric monoidal categories} with (necessarily enriched) finite products.

3. \textit{CMon-enriched symmetric monoidal categories} with (necessarily enriched) finite products.

\textbf{Definition 1.9.} A category with compatible biproduct and symmetric monoidal structures is referred to as a \textit{category of spaces and linear maps}.

1.2 \textbf{Fock space} \textbf{

\textbf{Strong-monoidal functorial structure.} A \textit{strong monoidal functor} \((F, \phi, \varphi) : (\mathcal{C}, I, \otimes) \to (\mathcal{C}', I', \otimes')\) between monoidal categories consists of a functor \( F : \mathcal{C} \to \mathcal{C}' \), an isomorphism \( \phi : I' \cong F(I) \), and a natural isomorphism \( \varphi_{A,B} : F A \otimes' F B \cong F(A \otimes B) \) subject to the coherence conditions below.

\[
\begin{array}{c}
\begin{array}{c}
FC \otimes' I' \xrightarrow{\rho_{FC}} FC \\
\text{id}_{FC} \otimes' \phi \\
FC \otimes' FI \xrightarrow{\varphi_{C,1}} F(C \otimes I)
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
I' \otimes' FC \xrightarrow{\lambda_{FC}} FC \\
\phi \otimes' \text{id}_{FC} \\
FI \otimes' FC \xrightarrow{\varphi_{1,C}} F(1 \otimes C)
\end{array}
\end{array}
\]
\[ (FA \otimes FB) \otimes' FC \xrightarrow{\varphi_{A,B} \otimes' \mathrm{id}_{FC}} FA \otimes' (FB \otimes' FC) \xrightarrow{\mathrm{id}_{FA} \otimes \varphi_{A,B}} FA \otimes' F(B \otimes C) \]
\[ F(A \otimes B) \otimes' FC \xrightarrow{\varphi_{A,B}} F\left((A \otimes B) \otimes C\right) \xrightarrow{F\varphi_{A,B,C}} F\left(A \otimes (B \otimes C)\right) \]

**Definition 1.10.** A strong monoidal functor \((\mathcal{S}, O, \oplus) \to (\mathcal{S}, I, \otimes)\) for a category of spaces and linear maps \(\mathcal{S}\) is referred to as a *(bosonic or symmetric)* Fock-space construction.

The Fock-space construction supports operations for *initialising* and *merging* \((i, m)\), and for *finalising* and *splitting* \((f, s)\).

**Definition 1.11.** For a Fock-space construction on a category of spaces and linear maps, set:

\[ i_A = ( I \cong FO \xrightarrow{\mathrm{Fu}_A} FA ) , \quad m_A = ( FA \otimes FA \cong F(A \oplus A) \xrightarrow{F\varphi_{A,A}} FA ) \]
\[ f_A = ( FA \xrightarrow{\mathrm{Fu}_A} FO \cong I ) , \quad s_A = ( FA \xrightarrow{FA} F(A \oplus A) \cong FA \otimes FA ) \]

The commutative bialgebra structure induced by the biproduct structure yields commutative bialgebraic structure on Fock space.

**Lemma 1.12.** For a Fock-space construction \(F\) on a category of spaces and linear maps, the natural transformations

\[ \begin{array}{c}
I & \xrightarrow{i_A} & FA \\
FA & \xrightarrow{f_A} & I \\
FA \otimes FA & \xrightarrow{m_A} & FA \otimes FA \\
FA \otimes FA & \xrightarrow{s_A} & FA \otimes FA \\
\end{array} \]

form a commutative bialgebra.
Indeed, by means of the coherence conditions of strong monoidal functors, the application of $F$ to the diagrams (2–7) yields the commutativity of the diagrams below.

**Proposition 1.13.** For a Fock-space construction $(F, \phi, \varphi)$, the isomorphism $\varphi_{A,B}$ has inverse $(F\pi_1 \otimes F\pi_2)s_{A\oplus B}$. 
Proof. Follows from the commutativity of

\[
\begin{align*}
F(A \oplus B) \otimes F(A \oplus B) & \xrightarrow{F \pi_1 \otimes F \pi_2} FA \otimes FB \\
\xrightarrow{\sim} & \\
F(A \oplus B) & \xrightarrow{F \Delta_{A \oplus B}} F\left( (A \oplus B) \oplus (A \oplus B) \right) \xrightarrow{F(\pi_1 \oplus \pi_2)} F(A \oplus B)
\end{align*}
\]

\[\square\]

Proposition 1.14. For a Fock-space construction \( F \), we have that \( F(0_{A,B}) = i_B f_A \) and that \( F(f + g) = m_B (Ff \otimes Fg) s_A : FA \rightarrow FB \) for all \( f, g : A \rightarrow B \).

1.2.1 Creation/annihilation operators

Definition 1.15. Let \( F \) be a Fock-space construction. For natural transformations \( \eta_A : A \rightarrow FA \) and \( \epsilon_A : FA \rightarrow A \), define the associated creation (or raising) natural transformation \( \overline{\eta}_A \) and annihilation (or lowering) natural transformation \( \underline{\epsilon}_A \) as

\[
\begin{align*}
\overline{\eta}_A & = (FA \xrightarrow{\eta_A \otimes id_F A} FA \otimes FA \xrightarrow{m_A} FA ) \\
\underline{\epsilon}_A & = (FA \xrightarrow{s_A} FA \otimes FA \xrightarrow{\epsilon_A \otimes id_F A} A \otimes FA )
\end{align*}
\]

The above form for creation and annihilation operators is non-standard. More commonly, see e.g. [24], the literature deals with creation operators \( \overline{\eta}_A : FA \rightarrow FA \) for vectors \( v : I \rightarrow A \) and annihilation operators \( \underline{\epsilon}_A' : FA \rightarrow FA \) for covectors \( v' : A \rightarrow I \). In the present setting, these are derived as follows:

\[
\begin{align*}
\overline{\eta}_A & = (FA \cong I \otimes FA \xrightarrow{v \otimes id_F A} A \otimes FA \xrightarrow{\overline{\eta}_A} FA ) \\
\underline{\epsilon}_A' & = (FA \xrightarrow{\underline{\epsilon}_A} A \otimes FA \xrightarrow{v' \otimes id_F A} I \otimes FA \cong FA )
\end{align*}
\]

Theorem 1.16. Let \( F \) be a Fock-space construction on a category of spaces and linear maps. For natural transformations \( \eta_A : A \rightarrow FA \) and \( \epsilon_A : FA \rightarrow A \), their associated creation and annihilation natural transformations \( \overline{\eta}_A : A \otimes FA \rightarrow FA \) and \( \underline{\epsilon}_A : FA \rightarrow A \otimes FA \) satisfy the commutation relations:
1. \( \varepsilon_{A} \eta_{A} = (\varepsilon_{A} \eta_{A} \otimes \text{id}_{FA}) + (\text{id}_{A} \otimes \eta_{A}) (\sigma_{A,A} \otimes \text{id}_{FA}) (\text{id}_{A} \otimes \varepsilon_{A}) : A \otimes FA \rightarrow A \otimes FA \)

2. \( \eta_{A} (\text{id}_{A} \otimes \eta_{A}) = \eta_{A} (\text{id}_{A} \otimes \eta_{A}) (\sigma_{A,A} \otimes \text{id}_{FA}) : A \otimes A \otimes FA \rightarrow FA \)

3. \( (\text{id}_{A} \otimes \varepsilon_{A}) \varepsilon_{A} = (\sigma_{A,A} \otimes \text{id}_{FA}) (\text{id}_{A} \otimes \varepsilon_{A}) : FA \rightarrow A \otimes A \otimes FA \)

It follows as a corollary that

\[
\varepsilon_{v'} \eta_{v} = (\varepsilon_{v'} \eta_{v} \otimes \text{id}_{FA}) (\sigma_{A,A} \otimes \text{id}_{FA}) (\text{id}_{A} \otimes \varepsilon_{v'}) : I \otimes FA \rightarrow FA
\]

\[
\eta_{v} \eta_{u} = \eta_{v} \eta_{u} (\sigma_{A,A} \otimes \text{id}_{FA}) (\text{id}_{A} \otimes \varepsilon_{u'}) : FA \rightarrow A \otimes FA
\]

\[
\varepsilon_{u'} \varepsilon_{v'} = \varepsilon_{u'} \varepsilon_{v'} (\sigma_{A,A} \otimes \text{id}_{FA}) (\text{id}_{A} \otimes \varepsilon_{v'}) : F(\Pi_{1} + \Pi_{2}) \rightarrow A \otimes A
\]

\[
\varepsilon_{A} \varepsilon_{A} = \varepsilon_{A} \varepsilon_{A} (\sigma_{A,A} \otimes \text{id}_{FA}) (\text{id}_{A} \otimes \varepsilon_{A}) : FA \rightarrow A \otimes A \otimes FA
\]

\[
f_{A} \eta_{A} = 0, \quad \eta_{A} i_{A} = 0 : I \rightarrow A \quad \text{and} \quad \varepsilon_{A} i_{A} = 0, \quad \varepsilon_{A} i_{A} = 0 : I \rightarrow A.
\]

\[
(\Pi_{1}) \eta_{A \otimes A} \Delta_{A} = \eta_{A \otimes A} (\Pi_{1} + \Pi_{2}) = \eta_{A \otimes A} \Pi_{1} + \eta_{A \otimes A} \Pi_{2} = F(\Pi_{1}) \eta_{A} + F(\Pi_{2}) \eta_{A} = (F \Pi_{1} + F \Pi_{2}) \eta_{A}.
\]

The proof of the theorem depends on the following lemma.

**Lemma 1.17.** For a Fock-space construction \( F \), the following hold for all natural transformations \( \eta_{A} : A \rightarrow FA \) and \( \varepsilon_{A} : FA \rightarrow A \).

1. \( \eta_{A \oplus A} \Delta_{A} = (F \Pi_{1} + F \Pi_{2}) \eta_{A} : A \rightarrow F(A \oplus A) \) and \( \nabla_{A} \varepsilon_{A \oplus A} = \varepsilon_{A} (F \pi_{1} + F \pi_{2}) : F(A \oplus A) \rightarrow A \).

2. \( s_{A} \eta_{A} = (A \cong A \oplus I \eta_{A} \otimes F(A \oplus A)) + (A \cong I \otimes A \eta_{A} \otimes F(A \otimes FA)) + (F \otimes FA \eta_{A} \otimes A \otimes F(A \otimes FA)) \)

   and \( \varepsilon_{A} i_{A} = (F \otimes FA \varepsilon_{A} \otimes \text{id}_{FA}) : F(A \otimes FA) \rightarrow A \cong A \).

3. \( f_{A} \eta_{A} = 0, i_{A} = 0 : A \rightarrow I \) and \( \varepsilon_{A} i_{A} = 0, i_{A} = 0 : I \rightarrow A \).

**Proof.** For the first and third items, I only detail the proof of one of the identities; the other identity being established dually.

One calculates as follows:

\[
\Pi_{1} \eta_{A \otimes A} \Delta_{A} = \eta_{A \otimes A} (\Pi_{1} + \Pi_{2}) = \eta_{A \otimes A} \Pi_{1} + \eta_{A \otimes A} \Pi_{2} = F(\Pi_{1}) \eta_{A} + F(\Pi_{2}) \eta_{A} = (F \Pi_{1} + F \Pi_{2}) \eta_{A}.
\]
\( s_A \eta_A = (F\pi_1 \circ F\pi_2) s_{A \otimes A} F(\Delta_A) \eta_A \), by definition of \( s \) and Proposition 1.13

\( = (F\pi_1 \circ F\pi_2) s_{A \otimes A} (F \Pi_1 + \Pi_2) \eta_A \), by naturality of \( \eta \) and item (1) of this lemma

\( = (F\pi_1 \circ F\pi_2) ((F \Pi_1 + \Pi_2) + (F \Pi_2 + \Pi_2)) s_A \eta_A \), by naturality of \( s \)

\( = ((id_F \otimes i_A F_A) + (i_A F_A \otimes id_F)) s_A \eta_A \), by Proposition 1.1 and the definitions of \( i \) and \( f \)

\( = (A \cong A \otimes I \circ (\eta_A \otimes \epsilon_A) \circ FA \otimes FA) + (A \cong I \otimes A \circ \epsilon_A \otimes \eta_A \circ FA \otimes FA) \), by the comonoid structure of \( (f, s) \)

\( \varepsilon_A m_A = (\pi_1 + \pi_2) \varepsilon_{A \otimes A} \varphi_A, A \), by definition of \( s \) and naturality of \( \varepsilon \)

\( = (\varepsilon_A F(\pi_1) \varphi_A, A) + (\varepsilon_A F(\pi_2) \varphi_A, A) \), by bilinearity of composition and naturality

\( = (FA \otimes FA \circ \varepsilon_A \otimes \eta_A) A \otimes I \cong A + (FA \otimes FA \circ \eta_A \otimes \varepsilon_A) I \otimes A \cong A \), by definition of \( f \) and coherence of \( F \)

\( = (A \eta_A \circ FA \circ F\eta_A \circ FO \cong I) = (A \eta_A \circ FO \cong I) \).

**Proof of Theorem 1.16** (1) By means of Lemma 1.1 and (2), the commutativity of the diagram
shows that \( \varepsilon_A, \eta_A \) equals

\[
\begin{align*}
(A \otimes FA) & \cong A \otimes I \otimes FA \quad \eta_A \otimes \sigma_{A,F} \otimes \sigma_{A,F} \quad FA \otimes FA \otimes FA \\
& \overset{id_{FA} \otimes \sigma_{FA,FA} \otimes id_{FA}}{=} FA \otimes FA \otimes FA \quad \varepsilon_A \otimes \sigma_{A,FA} \quad A \otimes I \otimes FA \cong A \otimes FA \\
& \overset{+}{=} \quad FA \otimes FA \otimes FA \quad \varepsilon_A \otimes \sigma_{A,FA} \quad A \otimes I \otimes FA \cong A \otimes FA \\
& \overset{+}{=} \quad FA \otimes FA \otimes FA \quad f_A \otimes \varepsilon_A \otimes m_A \quad I \otimes A \otimes FA \cong A \otimes FA \\
& \overset{+}{=} \quad FA \otimes FA \otimes FA \quad f_A \otimes \varepsilon_A \otimes m_A \quad I \otimes A \otimes FA \cong A \otimes FA
\end{align*}
\]

which, in turn, by the bialgebra laws and Lemma 1.17 (3), equals

\[
((\varepsilon_A \otimes \eta_A) \otimes id_{FA}) + 0_{A \otimes FA, A \otimes FA} + 0_{A \otimes FA, A \otimes FA} + ((id_A \otimes \eta_A)(\sigma_{A, A} \otimes id_{FA})(id_A \otimes \varepsilon_A))
\]

\( (2) \) & \( (3) \) The arguments crucially rely on the commutativity of the Fock-space bialgebra structure. Since the two arguments are dual of each other, I only consider one of them.

Analogously, one can establish the following laws of interaction between the creation/annihilation operators and the bialgebra structure.
Proposition 1.18. For a Fock-space construction $F$, the following hold for all natural transformations $\eta_A : A \to FA$ and $\varepsilon_A : FA \to A$.

1. $f_A \eta_A = 0_{A \otimes FA},$ and $i_A \varepsilon_A = 0_{I \otimes FA}.$
2. $s_A \eta_A = \left( (\eta_A \otimes \text{id}_{FA}) + (\text{id}_{FA} \otimes \eta_A) (\sigma_{A,FA} \otimes \text{id}_{FA}) \right) (\text{id}_A \otimes s_A) : A \otimes FA \to FA \otimes FA$ and $m_A = (\varepsilon_A \otimes \text{id}_{FA}) + (\sigma_{FA,\text{id}_{FA}})(\text{id}_{FA} \otimes \varepsilon_A) : FA \otimes FA \to A \otimes FA.$

1.3 Coherent states

Our discussion of coherent states is within the framework of categorical models of linear logic, see e.g. [33].

Definition 1.19. A linear Fock-space construction is one equipped with linear exponential comonad structure $(\varepsilon, \delta)$ in the form of natural transformations $\varepsilon_A : FA \to A$ and $\delta_A : FA \to FF_A$ such that

and subject to the coherence conditions

Definition 1.20. Let $F$ be a linear Fock-space construction. A coherent state $\gamma$ is a map $I \to FA$ such that

1. $\xi_A \gamma = (I \cong I \otimes I \otimes \gamma) A \otimes FA$ for some $v : I \to A,$
2. $f_A \gamma = \text{id}_I,$ and
3. $s_A \gamma = (I \cong I \otimes I \otimes \gamma) FA \otimes FA.$
Definition 1.21. Let $F$ be a linear Fock-space construction.

1. The **Kleisli extension** $u^# : FX \to FA$ of $u : FX \to A$ is defined as $F(u) \circ \delta_X$.

2. The extension $\tilde{v} : I \to FA$ of $v : I \to A$ is the composite

   $$I \cong FO \overset{\delta_O}{\to} FFO \cong FI \overset{Fv}{\to} FA$$

   For instance, $\tilde{0}_{I,A} = i_A : I \to FA$.

Theorem 1.22. For every $v : I \to A$, the extension $\tilde{v} : I \to FA$ is a coherent state.

The theorem arises from the following facts.

Proposition 1.23. Let $F$ be a linear Fock-space construction.

1. For $f : A \to B$, $f_B \circ F(f) = f_A : FA \to I$.

2. $f_{FA} \delta_A = f_A : FA \to I$.

3. $s_{FA} \delta_A = (\delta_A \otimes \delta_A) s_A : FA \to FAA \otimes FAA$.

4. For $u : FX \to A$, $\epsilon_A \circ u^# = (u \otimes u^#) s_X$.

5. $s_O = (FO \cong I \cong I \otimes I \cong FO \otimes FO)$.

We conclude the section by recording a property that will be useful at the end of the paper.

Proposition 1.24. Let $\eta_A : A \to FA$ be a natural transformation for a linear Fock-space construction $F$. For $v : I \to A$,

$$\left(\eta_A^v\right)^# i_A = \tilde{\eta_A} v$$

(10)
2. Combinatorial model

I introduce and study a model for Fock space with creation/annihilation operators that arises in the setting of generalised species of structures \[17, 22\]. These are a categorical generalisation of both the structural combinatorial theory of species of structures \[28, 29, 4\] and the relational model of linear logic.

Our combinatorial model conforms to the axiomatics of the previous section by being an example of its generalisation from categories to bicategories \[2\], by which I roughly mean the categorical setting where all structural identities hold up to canonical coherent isomorphism. However, we will not dwell on this here.

2.1 The bicategory of profunctors

Our setting for spaces and linear maps will be the bicategory of profunctors \( \mathbf{Prof} \), for which see e.g. \[31, 3\]. A profunctor (or bimodule, or distributor) \( \mathcal{A} \to \mathcal{B} \) between small categories \( \mathcal{A} \) and \( \mathcal{B} \) is a functor \( \mathcal{A}^\circ \times \mathcal{B} \to \mathbf{Set} \). It might be useful to think of these as category-indexed set-valued matrices.

The bicategory \( \mathbf{Prof} \) has objects given by small categories, maps given by profunctors, and 2-cells given by natural transformations. The profunctor composition \( T S : \mathcal{A} \to \mathcal{C} \) of \( S : \mathcal{A} \to \mathcal{B} \) and \( T : \mathcal{B} \to \mathcal{C} \) is given by the matrix-multiplication formula

\[
TS(a, c) = \int_{b \in \mathcal{B}} S(a, b) \times T(b, c)
\]

where \( \times \) and \( \int \) respectively denote the cartesian product and coend operations. The associated identity profunctors \( I_{\mathcal{C}} \) are the hom-set functors \( \mathcal{C}^\circ \times \mathcal{C} \to \mathbf{Set} : (c', c) \mapsto \mathcal{C}(c', c) \).

The notion of coend and its properties, for which see e.g. \[32\] Chapter X], is central to the calculus of this section. A coend is a colimit arising as a coproduct under a quotient that establishes compatibility under left and right actions. Technically, the coend \( \int_{z \in \mathcal{C}} H(z, z) \in \mathbf{Set} \) of a functor \( H : \)
$C^\circ \times C \rightarrow \text{Set}$ can be presented as the following coequaliser:

$$\begin{align*}
\coprod_{f : x \rightarrow y \in C} H(y, x) & \xrightarrow{(f : x \rightarrow y, h)} \coprod_{z \in C} H(z, z) \xrightarrow{\int_{z \in C} H(z, z)} T \in C \\
(f : x \rightarrow y, h) & \xrightarrow{(x, H(id_x, f)(h))} (y, H(id_y, f)(h))
\end{align*}$$

As for (11), then, $TS(a, c)$ consists of equivalence classes of triples in $\coprod_{b \in B} S(a, b) \times T(b, c)$ under the equivalence relation generated by identifying $(b, s, T(f, id_y)(t))$ and $(b', S(id_a, f)(s), t')$ for all $f : b \rightarrow b'$ in $B$, $s \in S(a, b)$, $t' \in T(b', c)$. Note also that, for all $P : C^\circ \rightarrow \text{Set}$, there is a canonical natural isomorphism

$$P(c) \cong \int_{z \in C} P(z) \times C(c, z) ,$$

known as the density formula [32] or Yoneda lemma [30], that essentially embodies the unit laws of profunctor composition with the identities.

The bicategory $\text{Prof}$ not only has compatible biproduct and symmetric monoidal structures but is in fact a compact closed bicategory, see [11]. The biproduct structure is given by the empty and binary coproduct of categories (i.e. $O = 0$ and $\oplus = +$), and the tensor product structure is given by the empty and binary product of categories (i.e. $I = 1$ and $\otimes = \times$).

Remark 2.1. The analogy of profunctors between categories as matrices between bases can be also phrased as an analogy between cocontinuous functors between presheaf categories and linear transformations between free vector spaces.

As it is well-known, the free small-colimit completion of a small category $C$ is the functor category $\text{Set}^{C^\circ}$ of (contravariant) presheaves on $C$ and natural transformations between them. The universal map is the Yoneda embedding $\downarrow \downarrow C \rightarrow \text{Set}^{C^\circ} : z \mapsto \langle z \rangle$ where

$$\langle z \rangle : C^\circ \rightarrow \text{Set} : c \mapsto C(c, z)$$

The use of Dirac’s ket notation in this context is justified by regarding presheaves as vectors and noticing that the isomorphism (12) above amounts to the following one

$$P \cong \int_{z \in C} P_z \cdot \langle z \rangle$$
in \( \mathbf{Set}^{C^\circ} \) expressing every presheaf as a colimit of the basis vectors (referred to as \textit{representable presheaves} in categorical terminology). Associated to this representation, the notion of linearity for transformations corresponds to that of cocontinuity (i.e. colimit preservation) for functors. Indeed, the bicategory of profunctors is biequivalent to the 2-category with objects consisting of small categories, morphisms from \( \mathbb{A} \) to \( \mathbb{B} \) given by cocontinuous functors \( \mathbf{Set}^\mathbb{A} \to \mathbf{Set}^\mathbb{B} \), and 2-cells given by natural transformations. The biequivalence associates a profunctor \( \mathbf{Pro}(T) : \mathbb{A} \to \mathbb{B} \) with the cocontinuous functor \( \mathbf{Fun}(T) : \mathbf{Set}^{\mathbb{A}} \to \mathbf{Set}^{\mathbb{B}} \): \( P \mapsto \int^{b \in \mathbb{B}} \left[ \int^{a \in \mathbb{A}} P_a \times T(a, b) \cdot | b \right] \cdot | b \rangle \sim \int^{a \in \mathbb{A}} P_a \cdot \left( \int^{b \in \mathbb{B}} F(a) \cdot | b \right) \), by cocontinuity \( \sim \mathbf{F}(P) \)

\[ \mathbf{Pro}(\mathbf{Fun}(T))(a, b) = \left( \int^{y \in \mathbb{B}} \left[ \int^{x \in \mathbb{A}} T(x, y) \right] \cdot | y \right) \cdot | y \rangle \sim \left( \int^{y \in \mathbb{B}} T(a, y) \cdot | y \right) \cdot | y \rangle \sim T(a, b) \]

\[ \mathbf{Fun}(\mathbf{Pro}(F))(P) = \int^{b \in \mathbb{B}} \left[ \int^{a \in \mathbb{A}} P_a \times F(a) \cdot | b \right] \cdot | b \rangle \sim \int^{a \in \mathbb{A}} P_a \cdot \left( \int^{b \in \mathbb{B}} F(a) \cdot | b \right) \sim F(P) \]

\[ \mathbf{Pro}(\mathbf{Fun}(T))(a, b) = \left( \int^{y \in \mathbb{B}} \left[ \int^{x \in \mathbb{A}} T(x, y) \right] \cdot | y \right) \cdot | y \rangle \sim \left( \int^{y \in \mathbb{B}} T(a, y) \cdot | y \right) \cdot | y \rangle \sim T(a, b) \]

\[ \mathbf{Fun}(\mathbf{Pro}(F))(P) = \int^{b \in \mathbb{B}} \left[ \int^{a \in \mathbb{A}} P_a \times F(a) \cdot | b \right] \cdot | b \rangle \sim \int^{a \in \mathbb{A}} P_a \cdot \left( \int^{b \in \mathbb{B}} F(a) \cdot | b \right) \sim F(P) \]

\[ \mathbf{Pro}(\mathbf{Fun}(T))(a, b) = \left( \int^{y \in \mathbb{B}} \left[ \int^{x \in \mathbb{A}} T(x, y) \right] \cdot | y \right) \cdot | y \rangle \sim \left( \int^{y \in \mathbb{B}} T(a, y) \cdot | y \right) \cdot | y \rangle \sim T(a, b) \]

\[ \mathbf{Fun}(\mathbf{Pro}(F))(P) = \int^{b \in \mathbb{B}} \left[ \int^{a \in \mathbb{A}} P_a \times F(a) \cdot | b \right] \cdot | b \rangle \sim \int^{a \in \mathbb{A}} P_a \cdot \left( \int^{b \in \mathbb{B}} F(a) \cdot | b \right) \sim F(P) \]

\[ \mathbf{Pro}(\mathbf{Fun}(T))(a, b) = \left( \int^{y \in \mathbb{B}} \left[ \int^{x \in \mathbb{A}} T(x, y) \right] \cdot | y \right) \cdot | y \rangle \sim \left( \int^{y \in \mathbb{B}} T(a, y) \cdot | y \right) \cdot | y \rangle \sim T(a, b) \]

### 2.2 Combinatorial Fock space

Let us introduce the combinatorial Fock-space construction.

**Definition 2.2.** The **combinatorial Fock space** of a small category \( C \) is the small category

\[ \mathbf{F}(C) = \coprod_{n \in \mathbb{N}} C^n\#_{\mathbb{S}_n} \]

where \( C^n\#_{\mathbb{S}_n} \) has objects given by \( n \)-tuples of objects of \( C \) and hom-sets

\[ C^n\#_{\mathbb{S}_n}(\vec{c}, \vec{z}) = \coprod_{\sigma \in \mathbb{S}_n} \prod_{1 \leq i \leq n} C(c_i, \pi_{\sigma i}) \]

It is a very important part of the general theory, for which see [17, 22], that the combinatorial Fock-space construction is the free symmetric (strict) monoidal completion; the unit and tensor product being respectively given by the empty tuple and tuple concatenation, and denoted as \( (\cdot) \) and \( \cdot \).
Proposition 2.3. Hom-sets in combinatorial Fock space satisfy the following combinatorial laws.

1. \( F_A(\vec{u} \cdot \vec{v}, \vec{x} \cdot \vec{y}) \)
   \[ \cong \int_{\vec{a}, \vec{b}, \vec{c}, \vec{d} \in F_A} F_A(\vec{u}, \vec{a} \cdot \vec{b}) \times F_A(\vec{c} \cdot \vec{d}) \times F_A(\vec{a} \cdot \vec{c}, \vec{x}) \times F_A(\vec{b} \cdot \vec{d}, \vec{y}) \]

2. \( F_A(((), ())) \cong 1 \quad , \quad F_A((a), (x)) \cong A(a, x) \)
   \( F_A(((), (a))) \cong 0 \quad , \quad F_A((a), ())) \cong 0 \)

3. \( F_A(((), \vec{x} \cdot \vec{y})) \cong F_A(((), \vec{x}) \times F_A(((), \vec{y})) \)
   \( F_A(\vec{x} \cdot \vec{y}, ()) \cong F_A(\vec{x}, ()) \times F_A(\vec{y}, ()) \)

4. \( F_A((a), \vec{x} \cdot \vec{y}) \cong (F_A((a), \vec{x}) \times F_A(((), \vec{y})) + (F_A(((), \vec{x}) \times F_A((a), \vec{y})) \)
   \( F_A(\vec{x} \cdot \vec{y}, (a)) \cong (F_A(\vec{x}, (a)) \times F_A(\vec{y}, ())) + (F_A(\vec{x}, ()) \times F_A(\vec{y}, (a))) \)

5. \( F(\mathbb{A} + \mathbb{B})(F \Pi_1(\vec{a}) \cdot F \Pi_2(\vec{b}), F \Pi_1(\vec{x}) \cdot F \Pi_2(\vec{y})) \cong F \mathbb{A}(\vec{a}, x) \times F \mathbb{B}(\vec{b}, y) \)

I propose to describe the structure of the combinatorial Fock space.

§ 2.2.1. For a profunctor \( T : \mathbb{A} \rightarrow \mathbb{B} \), the profunctor \( F T : F \mathbb{A} \rightarrow F \mathbb{B} \) is given by

\[ FT(\vec{x}, \vec{y}) = \int_{\vec{z} \in F(A \times B)} (\prod_{z_i \in \vec{z}} T z_i) \times F\mathbb{A}(\vec{x}, F\pi_1 \vec{z}) \times F\mathbb{B}(F\pi_2 \vec{z}, \vec{y}) \]

so that

\[ FT((a_1, \ldots, a_m), (b_1, \ldots, b_n)) \cong \left\{ \begin{array}{ll} \prod_{\sigma \in S_m} \prod_{1 \leq i \leq m} T(a_i, b_{\sigma_i}) & \text{if } m = n \\ 0 & \text{otherwise} \end{array} \right. \]

§ 2.2.2. There are canonical natural coherent equivalences as follows:

\( \phi : 1 \cong F 0 \) \quad , \quad \phi(\ast, () ) = 1

\( \varphi_{A,B} : F\mathbb{A} \times F\mathbb{B} \cong F(\mathbb{A} + \mathbb{B}) \) \quad , \quad \varphi_{A,B}(\vec{x}, \vec{y}, \vec{z}) = F(\mathbb{A} + \mathbb{B})(F \Pi_1(\vec{x}) \cdot F \Pi_2(\vec{y}), \vec{z}) \)
\section*{2.2.3.} The pseudo commutative bialgebra structure \((\mathfrak{B})\) consists of:
\begin{align*}
i_A : 1 & \rightarrow \mathcal{F}A, \quad i_A(*) = \mathcal{F}A((\ ), \tilde{a}) \\
m_A : \mathcal{F}A \times \mathcal{F}A & \rightarrow \mathcal{F}A, \quad m_A(\langle \tilde{x}, \tilde{y}\rangle, \tilde{z}) = \mathcal{F}A(\tilde{x} \cdot \tilde{y}, \tilde{z}) \\
f_A : \mathcal{F}A & \rightarrow 1, \quad f_A(\tilde{a}, *) = \mathcal{F}A(\tilde{a}, ( ) ) \\
s_A : \mathcal{F}A \rightarrow \mathcal{F}A \times \mathcal{F}A, \quad s_A(\tilde{z}, \langle \tilde{x}, \tilde{y}\rangle) = \mathcal{F}A(\tilde{z}, \tilde{x} \cdot \tilde{y})
\end{align*}

The bialgebra law for \(m_A s_A\) arises from the combinatorial law of Proposition \[.3\], which is a formal expression for the diagrammatic law:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$\tilde{a}$};
\node (b) at (2,0) {$\tilde{x}$};
\node (c) at (0,-2) {$\tilde{u}$};
\node (d) at (2,-2) {$\tilde{y}$};
\path (a) edge (b)
(a) edge (c)
(b) edge (d)
(c) edge (d);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$\tilde{u}$};
\node (b) at (2,0) {$\tilde{x}$};
\node (c) at (0,-2) {$\tilde{v}$};
\node (d) at (2,-2) {$\tilde{y}$};
\path (a) edge (b)
(a) edge (c)
(b) edge (d)
(c) edge (d);
\end{tikzpicture}
\end{center}

\section*{2.2.4.} The linear exponential pseudo comonad structure is given by:
\begin{align*}
\epsilon_A : \mathcal{F}A & \rightarrow A, \quad \epsilon_A(x, a) = \mathcal{F}A(\tilde{x}, (a)) \\
\delta_A : \mathcal{F}A & \rightarrow \mathcal{F}A, \quad \delta_A(\alpha, \tilde{a}) = \mathcal{F}A(\alpha \cdot \tilde{a})
\end{align*}
where \((\tilde{a}_1, \ldots, \tilde{a}_n)^* = \tilde{a}_1 \cdot \ldots \cdot \tilde{a}_n \in \mathcal{F}a\) for \(\tilde{a}_i \in \mathcal{F}a\).

The laws of Proposition \[.3\] exhibit the combinatorial context of the identities of Proposition \[.1\].

\section*{2.2.5.} The bicategory \(\mathbf{P}\text{rof}\) admits a duality, by which a small category \(\mathfrak{A}\) is mapped to its opposite category \(\mathfrak{A}^\circ\) and a profunctor \(T : \mathfrak{A} \rightarrow \mathfrak{B}\) to the profunctor \(T^\circ : \mathfrak{B}^\circ \rightarrow \mathfrak{A}^\circ\) with \(T^\circ(\tilde{y}, \tilde{x}) = T(\tilde{x}, \tilde{y})\). Thereby, the pseudo comonadic structure of the combinatorial Fock-space construction can be turned into pseudo monadic structure \((\eta, \mu)\) by setting \(\eta_A = (\epsilon_A)^\circ\) and \(\mu_A = (\delta_A)^\circ\). Specifically, we have:
\begin{align*}
\eta_A : \mathfrak{A} & \rightarrow \mathcal{F}A, \quad \eta_A(a, \tilde{x}) = \mathcal{F}A(\tilde{x}, (a)) \\
\mu_A : \mathcal{F}A & \rightarrow \mathcal{F}A, \quad \mu_A(\alpha, \tilde{a}) = \mathcal{F}A(\alpha^*, \tilde{a})
\end{align*}
2.2.6. The structure results in canonical creation and annihilation operators:

\[ \eta_A : A \times FA \to FA, \quad \eta_A((a, \vec{x}), \vec{y}) = FA(\vec{x} \cdot (a), \vec{y}) \]

\[ \epsilon_A : FA \to A \times FA, \quad \epsilon_A(\vec{x}, (a, \vec{y})) = FA(\vec{x}, (a) \cdot \vec{y}) \]

so that, for \( V : 1 \to A \) and \( V' : A \to 1 \), we have

\[ \eta_A^V(\vec{x}, \vec{y}) \cong \int_{a \in A} V_a \times FA(\vec{x} \cdot (a), \vec{y}) \quad \text{for } V_a = V(\ast, a) \]  

\[ \epsilon_A^{V'}(\vec{x}, \vec{y}) \cong \int_{a \in A} V'_a \times FA(\vec{x}, (a) \cdot \vec{y}) \quad \text{for } V'_a = V(a, \ast) \]

yielding the functorial forms

\[ (\text{Fun} \eta_A^V)(X) \cong \int_{a \in A, \vec{z} \in FA} [V_a \times X_{\vec{z}}] \cdot |\vec{z} \cdot (a)\rangle \]

\[ (\text{Fun} \epsilon_A^{V'})(X) \cong \int_{a \in A, \vec{z} \in FA} [V'_a \times X_{(a) \cdot \vec{z}}] \cdot |\vec{z}\rangle \]

Identity (9) then becomes

\[ \text{Fun}(\epsilon_A^{V'} \eta_A^V)(X) \cong \langle V, V' \rangle \cdot X + \int_{a,b \in A, \vec{z} \in FA} [V_a \times V'_b \times X(X_{(a), \vec{z}})] \cdot |\vec{z} \cdot a\rangle \]

where \( \langle V, V' \rangle = \int_{a \in A} V_a \times V'_a \).

2.2.7. In the current setting, the axiomatic proof of the commutation relation for \( \epsilon_A \eta_A \) acquires formal combinatorial content made explicit by
the following chain of isomorphisms:

\[ \mathcal{L}_A \eta_A \left( (a, \vec{x}), (b, \vec{y}) \right) \]

\[ \cong \mathcal{F}_A (\vec{x} \cdot (a), (b) \cdot \vec{y}) \]  \hspace{1cm} (14)

\[ \cong \int_{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4 \in \mathcal{F}_A} \mathcal{F}_A (\vec{x}, \vec{z}_1 \cdot \vec{z}_2) \times \mathcal{F}_A ((a), \vec{z}_3 \cdot \vec{z}_4) \times \mathcal{F}_A (\vec{z}_1 \cdot \vec{z}_3, (b)) \times \mathcal{F}_A (\vec{z}_2 \cdot \vec{z}_4, \vec{y}) \]

\[ \cong \int_{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4 \in \mathcal{F}_A} \left[ \mathcal{F}_A ((a), \vec{z}_1) \times \mathcal{F}_A (( ), \vec{z}_4) + \mathcal{F}_A (( ), \vec{z}_3) \times \mathcal{F}_A ((a), \vec{z}_4) \right] 
\times \left[ \mathcal{F}_A (\vec{z}_1, (b)) \times \mathcal{F}_A (\vec{z}_3, ( )) + \mathcal{F}_A (\vec{z}_1, ( )) \times \mathcal{F}_A (\vec{z}_3, (b)) \right] 
\times \mathcal{F}_A (\vec{z}_2 \cdot \vec{z}_4, \vec{y}) \]

\[ \cong \left[ \mathcal{F}_A (\vec{x}, (b) \cdot \vec{y}) \times \mathcal{F}_A ((a), ( )) \right] + \left[ \mathcal{F}_A (\vec{x}, \vec{y}) \times \mathcal{F}_A ((a), (b)) \right] 
+ \left[ \int_{\vec{z}_2 \in \mathcal{F}_A} \mathcal{F}_A (\vec{x}, (b) \cdot \vec{z}_2) \times \mathcal{F}_A (\vec{z}_2 \cdot (a), \vec{y}) \right] + \left[ \mathcal{F}_A (( ), (b)) \times \mathcal{F}_A (\vec{x} \cdot (a), \vec{y}) \right] \]

\[ \cong \left[ \bigwedge (a, b) \times \mathcal{F}_A (\vec{x}, \vec{y}) \right] + \left[ \int_{\vec{z} \in \mathcal{F}_A} \mathcal{F}_A (\vec{x}, (b) \cdot \vec{z}) \times \mathcal{F}_A (\vec{z} \cdot (a), \vec{y}) \right] \]  \hspace{1cm} (15)

\[ \cong I_{\bigwedge \times \mathcal{F}_A} \left( (a, \vec{x}), (b, \vec{y}) \right) 
+ \int_{\vec{z} \in \mathcal{F}_A, c, d \in \bigwedge} \mathcal{F}_A (\vec{x}, (c) \cdot \vec{z}) \times (\bigwedge \times \bigwedge) \left( (a, c), (d, b) \right) \times \mathcal{F}_A (\vec{z} \cdot (d), \vec{y}) \]

\[ \cong \left( I_{\bigwedge \times \mathcal{F}_A} + (I_{\bigwedge} \times \eta_{\mathcal{F}_A}) (\sigma_{\bigwedge, \times} I_{\mathcal{F}_A}) (I_{\bigwedge} \times \epsilon_{\mathcal{F}_A}) \right) \left( (a, \vec{x}), (b, \vec{y}) \right) \]

This formal derivation can be pictorially represented as follows:

\[ a \rightarrow \begin{array}{c} \vec{x} \end{array} \rightarrow \begin{array}{c} \vec{y} \end{array} \cong a \rightarrow \begin{array}{c} b \end{array} \rightarrow \begin{array}{c} \vec{y} \end{array} \]

\[ \begin{array}{c} \vec{x} \end{array} \Rightarrow \begin{array}{c} \vec{x} \end{array} \]
2.3 Coherent states

In this section, I will indistinguishably regard profunctors $1 \to A$ as presheaves in $\mathcal{S}et^A$, and vice versa. Thus, according to Definition $1.21(2)$, every $V \in \mathcal{S}et^A$ has a coherent state extension $\tilde{V} \in \mathcal{S}et^F_A$. A calculation shows this to be given as

$$\tilde{V} \simeq \int^{\vec{a} \in F_A} (\prod_{a_i \in \vec{a}} V_{a_i}) \cdot |\vec{a}\rangle$$

The combinatorial version of the coherent state property of Definition $1.20(1)$ enjoyed by $\tilde{V}$ according to Theorem $1.22$ yields the isomorphism

$$(\text{Fun}_{\mathcal{E}_k})(\tilde{V})_{(a,\vec{x})} \cong V_a \times \tilde{V}_{\vec{x}}$$

from which we obtain the functorial form

$$(\text{Fun}_{\mathcal{E}_k})(\tilde{V}) \cong \int^{a \in A, \vec{x} \in F_A} (V_a \times \prod_{x_i \in \vec{x}} P_{x_i}) \cdot (a, \vec{x})$$

I now proceed to introduce a notion of exponential (as parameterised by algebras) and show how, when applied to the creation operator (with respect
to the free algebra), generalises the coherent state extension. The definition of exponential is based on that given in [36, Section 4].

I have remarked in Section 2.2.5 that \((\mathcal{F}, \eta, \mu)\) is a pseudo monad on the bicategory of profunctors. Pseudo algebras for it consist of profunctors \(M : \mathcal{F}A \to A\) equipped with natural isomorphisms

\[
\begin{array}{c}
\mathcal{F}A \\
\downarrow^\eta
\end{array} \cong 
\begin{array}{c}
\mathcal{F}A \\
\downarrow^M
\end{array} 
\begin{array}{c}
A \\
\downarrow^\mu
\end{array}
\]

subject to coherence conditions, see e.g. [5]. These pseudo algebras provide the right notion of unbiased commutative promonoidal category, generalising the notion of symmetric promonoidal category [10] (viz. commutative pseudo monoids in the bicategory of profunctors) to biequivalent structures specified by \(n\)-ary operations \(M^{(n)} : A^{n} \#_{\otimes} \to A\) for all \(n \in \mathbb{N}\) that are commutative and associative with unit \(M^{(0)}\) up to coherent isomorphism. The most common examples of pseudo \(\mathcal{F}\)-algebras arise from small symmetric monoidal categories, say \((\mathcal{M}, 1, \odot)\), by letting \(M^* : \mathcal{F}\mathcal{M} \to \mathcal{M}\) be given by \(M^*((x_1, \ldots, x_n), x) = \mathcal{M}(x_1 \odot \cdots \odot x_n, x)\), so that \(M^*((), x) = \mathcal{M}(1, x)\). In particular, the free pseudo algebra \(\mu_A : \mathcal{F}A \to A\) on \(A\) is obtained by this construction on the free symmetric monoidal category \((\mathcal{F}A, (\cdot), \cdot\cdot\cdot)\) on \(A\).

Definition 2.4. Let \(M : \mathcal{F}A \to A\) be a pseudo \(\mathcal{F}\)-algebra. For \(T : \mathcal{F}X \to A\), define \(\exp_M(T) = MT^\# : \mathcal{F}X \to A\).

In particular, for \(V \in \mathcal{S}ett^A\), we have that

\[
\exp_M(V) = \int^aA \left( \int xA \left( \prod_{x_i \in x} V_{x_i} \right) \times M(x, a) \right) \cdot |a|
\]

Proposition 2.5. For a pseudo \(\mathcal{F}\)-algebra \(M : \mathcal{F}A \to A\),

\[
\exp_M(0_{A}) \cong M^{(0)}
\]

and

\[
\exp_M(S+T) = (FX \xrightarrow{spX} FX \times FX \xrightarrow{exp_M(S) \times exp_M(T)} FA \times FA \xrightarrow{exp_A} FA)
\]

for all \(S, T : \mathcal{F}X \to A\).
Note that the notion of exponential with respect to free algebras is a form of comonadic/monadic convolution, as for \( T : FX \rightarrow FA \), the definition of \( \text{exp}_{\mu_A}(T) \) amounts to the composite
\[
FX \xrightarrow{\delta_X} FFX \xrightarrow{FT} FAA \xrightarrow{\mu_A} FA
\]

\[\text{(16)}\]

**Theorem 2.6.** For \( V \in \text{Set}^A \),
\[
\text{exp}_{\mu_A}(\eta_V^A) i_A \cong \tilde{V}
\]

**Proof.** A simple algebraic proof follows:
\[
\text{exp}_{\mu_A}(\eta_V^A) i_A = \mu_A(\eta_V^A) \# i_A \cong \mu_A \eta_A \tilde{V}, \text{ by (10)}
\]
\[
\cong \mu_A F(\eta_A) \tilde{V} \cong \tilde{V}, \text{ by a monad law}
\]

I conclude the paper with a formal combinatorial proof of this result. Observe first that for the composite (16), we have:
\[
(\mu_A F(T) \delta_X) (\bar{x}, \bar{a})
\]
\[
\cong \int \xi \in FFX, \alpha \in FAA \int \bar{z} \in F(FX \times FA) \left( \prod_{z_i \in \bar{z}} T(z_i) \right) \times FFX(\xi, F\pi_1 \bar{z}) \times FAA(F\pi_2 \bar{z}, \alpha)
\]
\[
\times FX(\bar{x}, \xi) \times FA(\alpha^*, \bar{a})
\]
and hence that
\[
(\mu_A F(T) \delta_X i_X) (\bar{a})
\]
\[
\cong \int \bar{z} \in F(FX \times FA) \left( \prod_{z_i \in \bar{z}} T(z_i) \right) \times FX((), \bar{z}) \times FAA([F\pi_2 \bar{z}]^*, \bar{a})
\]
\[
\cong \int \bar{z} \in FAA \left( \prod_{z_i \in \bar{z}} T((), z_i) \right) \times FAA([\bar{z}]^*, \bar{a})
\]

Then, according to (13),
\[
(\mu_A F(\eta_V^A) \delta_A i_A) (\bar{a})
\]
\[
\cong \int \bar{z} \in FAA \left( \prod_{z_i \in \bar{z}} Fx \times FAA((x), z_i) \right) \times FAA([\bar{z}]^*, \bar{a})
\]
\[
\cong \int \bar{z} \in FAA \int x_i \in A (z_i \in \bar{z}) \left( \prod_{z_i \in \bar{z}} V_{x_i} \right) \times (\prod_{z_i \in \bar{z}} FAA((x_{z_i}), z_i) \right) \times FAA([\bar{z}]^*, \bar{a})
\]
\[
\cong \int \bar{z} \in FAA \left( \prod_{x_i \in \bar{z}} V_{x_i} \right) \times FAA([\bar{z}]^*, \bar{a})
\]
\[
\cong \prod_{x_i \in \bar{a}} V_{x_i}
\]
where, for $a_i \in A$, $\lfloor (a_1, \ldots, a_n) \rfloor = \left( (a_1), \ldots, (a_n) \right) \in \mathbb{F}A$; so that, for $\bar{a} \in \mathbb{F}A$, $\lfloor \bar{a} \rfloor^* = \bar{a}$.

References

[1] J. Baez and J. Dolan. From finite sets to Feynman diagrams. In Mathematics Unlimited - 2001 and Beyond, pages 29–50, 2001.

[2] J. Bénabou. Introduction to bicategories. Reports of the Midwest Category Seminar, Lecture Notes in Mathematics, vol. 47, pages 1–77, 1967.

[3] J. Bénabou. Distributors at work. Lecture notes by T. Streicher of a course given at TU Darmstadt, 2000.

[4] F. Bergeron, G. Labelle, and P. Leroux. Combinatorial species and tree-like structures. Cambridge University Press, 1998.

[5] R. Blackwell, G.M. Kelly, and A.J. Power. Two-dimensional monad theory. Journal of Pure and Applied Algebra 59, pages 1–41, 1989.

[6] R. Blute, J. Cockett, and R. Seely. Differential categories. Mathematical Structures in Computer Science, 16(6):1049–1083, 2006.

[7] R. Blute, T. Ehrhard, and C. Tasson. A convenient differential category Manuscript, 2010.

[8] R. Blute and P. Panangaden. Proof nets as formal Feynman diagrams. In New Structures for Physics, Lecture Notes in Physics, vol. 813, pages 437–466, 2009.

[9] R. Blute, P. Panangaden, and R. Seely. Fock space: A model of linear exponential types. Corrected version of Holomorphic models of exponential types in linear logic, in Proceedings of MFPS’93, number 802 of Lecture Notes in Computer Science, 1993.

[10] B. Day. On closed categories of functors. Reports of the Midwest Category Seminar IV, Lecture Notes in Mathematics, vol. 137, pages 1–38, 1970.
[11] B. Day and R. Street. Monoidal bicategories and Hopf algebroids. Advances in Mathematics, 129(1):99–157, 1997.

[12] T. Ehrhard. On Köthe sequence spaces and linear logic. Mathematical Structures in Computer Science, 12(5):579–623, 2002.

[13] T. Ehrhard. Finiteness spaces. Mathematical Structures in Computer Science, 15(4):615–646, 2005.

[14] T. Ehrhard and L. Regnier. The differential lambda-calculus. Theoretical Computer Science, vol. 309, pages 1–41, 2003.

[15] T. Ehrhard and L. Regnier. Differential interaction nets. Theoretical Computer Science, vol. 364, pages 166–195, 2006.

[16] M. Fiore. Generalised species of structures: Cartesian closed and differential structure. Draft, 2004.

[17] M. Fiore. Mathematical models of computational and combinatorial structures. Invited address for Foundations of Software Science and Computation Structures (FOSSACS 2005), volume 3441 of Lecture Notes in Computer Science, pages 25–46. Springer-Verlag, 2005.

[18] M. Fiore. Adjoints and Fock space in the context of profunctors. Talk given at the Cats, Kets and Cloisters Workshop (CKC in OXFORD), Computing Laboratory, University of Oxford (England), 2006.

[19] M. Fiore. Analytic functors and domain theory. Invited talk at the Symposium for Gordon Plotkin, LFCS, University of Edinburgh (Scotland), 2006.

[20] M. Fiore. An axiomatics and a combinatorial model of creation/annihilation operators and differential structure. Invited talk at the Categorical Quantum Logic Workshop (CQL), Computing Laboratory, University of Oxford (England), 2007.

[21] M. Fiore. Differential structure in models of multiplicative biadditive intuitionistic linear logic. In Typed Lambda Calculi and Applications (TLCA 2007), Lecture Notes in Computer Science 4583, pages 163-177, 2007.
[22] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. The cartesian closed bicategory of generalised species of structures. In *J. London Math. Soc.*, 77:203-220, 2008.

[23] A. Frölicher and A. Kriegl. *Linear spaces and differentiation theory*. Wiley Series in Pure and Applied Mathematics. Wiley-Interscience Publication, 1988.

[24] R. Geroch. *Mathematical Physics*. University of Chicago Press, 1985.

[25] J.-Y. Girard. *Linear logic*. *Theoretical Computer Science*, vol. 50, pages 1–101, 1987.

[26] M. Hyland. Some reasons for generalising domain theory *Mathematical Structures in Computer Science*, 20:239–265, 2010.

[27] P. Hyvernat. Interaction systems and linear logic, a different game semantics. Manuscript, 2009.

[28] A. Joyal. *Une théorie combinatoire des séries formelles*. Advances in Mathematics 42, pages 1–82, 1981.

[29] A. Joyal. Foncteurs analytiques et espèces de structures. In *Combinatoire Énumérative*, Lecture Notes in Mathematics, vol. 1234, pages 126–159, 1986.

[30] G. M. Kelly. Basic concepts of enriched category theory. *Cambridge University Press*, 1982. (Also in *Reprints in Theory and Applications of Categories*, 10:1–136, 2005.)

[31] F. W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rend. del Sem. Mat. e Fis. di Milano* vol. 43, pages 135–166, 1973. (Also in *Reprints in Theory and Applications of Categories*, 1:1–37, 2002.)

[32] S. Mac Lane. Categories for the working mathematician. Springer-Verlag, 1971. (Second edition 1998).

[33] P.-A. Melliès. Categorical semantics of linear logic. In *Interactive Models of Computation and Program Behaviour*, Panoramas et Synthèses, Société Mathématique de France, 2009.
[34] J. Morton. Categorified algebra and quantum mechanics. *Theory and Applications of Categories*, vol. 16, pages 785–854, 2006.

[35] M. Sweedler. *Hopf algebras*. Benjamin, N.Y., 1969.

[36] J. Vicary. A categorical framework for the quantum harmonic oscillator. *Int. J. Theor. Phys.*, 47:3408–3447, 2008.

Marcelo Fiore
Computer Laboratory
University of Cambridge
15 JJ Thomson Avenue
Cambridge CB3 0FD
UK
Marcelo.Fiore@cl.cam.ac.uk