Computable negativity in two mode squeezing subject to dissipation

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We study a system of two bosonic fields subject to two-mode squeezing in the presence of dissipation. We find the Lie algebra governing the dynamics of the problem and use the Wei-Norman method to determine the solutions. Using this scheme we arrive at a closed form expression for an infinitely dimensional density operator which we use to calculate the degree of entanglement (quantified by Horodeckis’ negativity) between the modes. We compare our result to the known continuous variable entanglement measures. We analyse the conditions for entanglement generation and the influence of thermal environments on the state formed. The problem is relevant, in particular, for understanding of quantum dynamics of coupled optical and/or mechanical modes in optomechanical and nanomechanical systems.

I. INTRODUCTION

Entanglement is a fascinating, unique, and classically unparallelled feature of quantum mechanics. In optical systems entanglement is produced by means of nonlinear media through a process of three- or more- wave mixing, spontaneous down-conversion or two mode squeezing [1], where in each case dissipation and the temperature play a negligible role. Modern quantum optomechanical [2, 3] and nanomechanical [4] systems equipped with elements with sufficiently strong nonlinearities could give rise to similar effective squeezing phenomena. Such bosonic systems are subject to dissipation through coupling to the classical environment, which might reduce the degree of produced entanglement. It is therefore very important to get a quantitative understanding of the squeezing versus dissipation interplay, with the aim of producing or maintaining entanglement between bosonic degrees of freedom such as photons (light or microwave quanta) or phonons (vibrational quanta).

The main obstacle in determining the environmentally induced effects on the degree of bosonic entanglement formed is twofold. For one, the Hilbert space of both of the bosonic modes is infinitely dimensional (quantified by Horodeckis’ negativity) between the modes. We compare our result to the known continuous variable entanglement measures. We analyse the conditions for entanglement generation and the influence of thermal environments on the state formed. The problem is relevant, in particular, for understanding of quantum dynamics of coupled optical and/or mechanical modes in optomechanical and nanomechanical systems.

II. NONLINEAR ENTITY

The degree of entanglement formed is twofold. For one, the Hilbert space of both of the bosonic modes is infinitely dimensional (quantified by Horodeckis’ negativity) between the modes. We compare our result to the known continuous variable entanglement measures. We analyse the conditions for entanglement generation and the influence of thermal environments on the state formed. The problem is relevant, in particular, for understanding of quantum dynamics of coupled optical and/or mechanical modes in optomechanical and nanomechanical systems.

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blad type superoperator master equation into a Lie algebra valued problem, where superoperators present in the equations are identified with Lie algebra elements. Furthermore, we present the generic form of the solution of the problem with the system initialised in a vacuum state, which allows us in Section III to determine the degree of entanglement present in the system. In Section IV, we compare this result and steaming from it separability condition to that obtained from continuous variable separability condition. Afterwards, in Section V we present the explicit solutions to the master equation and interpret the separability condition in terms of the bath temperature, proving that regardless squeezing strength and dissipation rate the state is inseparable at zero temperature. Additionally we also investigate the effects of the asymmetric modes-baths coupling strengths and deviation from resonance between the two squeezed modes and the driving mode. Finally, in Section VI we study the condition to that obtained from continuous variable separation approximation, we arrive at a Lindblad type master equation

\[ \dot{\rho} = -i \left[ \hat{H}, \rho \right] + \kappa_1 (n_{1,th} + 1) D_{g\rho} + \kappa_2 n_{1,th} D_{\rho\rho} + \kappa_2 (n_{2,th} + 1) D_{\rho\rho} + \kappa_2 n_{2,th} D_{\rho\rho}, \]

where \( D_{\Theta\rho} = \Theta \rho \Theta^\dagger - \frac{1}{2} \{ \Theta^\dagger \Theta, \rho \} \), \( \kappa_i \) is the dissipation rate and \( n_{i,th} = (e^{\omega_i/k_B T_i} - 1)^{-1} \) is the thermal occupation number in the bath at temperature \( T_i \) of bosonic mode \( i = 1, 2 \) (given by operators \( \hat{a} \) and \( \hat{b} \) respectively).

For the system initially in the vacuum state \( |00\rangle \), the equation (1) has a solution, that can be written in a form

\[ \rho(t) = \mathcal{N} \exp \left[ f_3 H_3 + f_5 H_5 + f_9 H_9 + f_{12} H_{12} \right] |00\rangle \langle 00| \] \( \mathcal{N} = (1 - x_+)(1 - x_-), \quad x_{\pm} = f_3 \pm \sqrt{f_3^2 + f_5^2 + f_{12}^2}, \]

where the numerical prefactor marks the trace-normalisation condition and where

\[ H_3 \rho = \hat{a}^\dagger \rho \hat{a} + \hat{b}^\dagger \rho \hat{b}, \quad H_5 \rho = ic^{-ir} \hat{a}^\dagger \rho \hat{b} - ic^{ir} \rho \hat{b}, \quad H_{12} \rho = e^{-ir} \hat{a}^\dagger \rho \hat{b} + e^{ir} \rho \hat{b}, \]

are four of the fifteen elements of the \( \mathfrak{so}(4,2) \) Lie algebra presented and elaborated on in the Appendix; the remaining eleven generators drop out due to the initial condition choice. Here \( \varphi = \pi - \text{Arg}(\xi) \). The solution (2), despite being written in a compact form, spans the whole of the infinite dimensional Hilbert space of both modes. Moreover, the exponents of the superoperators should be understood either in terms of Taylor expansions or in terms of matrix exponents of the matrices operating on the product space

\[ \rho = \hat{A} |n\rangle \langle m| \hat{B} \rightarrow \hat{\rho} = \hat{A} \otimes \hat{B}^\dagger |n\rangle \otimes |m\rangle. \]

In this work we will adapt the Taylor series approach.

Lastly, quite remarkably thanks to this construction one can obtain analytical expressions for moments

\[ \langle \hat{a}^{ik} \hat{b}^l \hat{b}^m \rangle = \text{Tr} \left( \hat{a}^{ik} \hat{b}^l \hat{b}^m \rho(t) \right), \]

in terms of functions \( f_i \) by skilfully differentiating with respect to \( f_3, f_5, f_9 \) and/or \( f_{12} \) and then renormalising the moment generating function

\[ \Lambda = \text{Tr} \left( \exp \left[ f_3 H_3 + f_5 H_5 + f_9 H_9 + f_{12} H_{12} \right] \right) = \mathcal{N}^{-1}, \]

for example

\[ \langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{2} \mathcal{N} \left( \partial f_3 + \partial f_9 \right) \mathcal{N}^{-1}, \]

where from the solution we can see that the only non-zero moments must be of the form \( \langle \hat{a}^{x+y-z} \hat{b}^l \hat{b}^m \rangle \) with
$x, y, z \in \mathbb{Z}$ and $x + y \geq z$. This automatically implies that the joined power of the moment must be even, and that of the simplest (quadratic) moments the only non-zero ones are $\langle a^\dagger a \rangle$, $\langle b^\dagger b \rangle$, $\langle ab \rangle$ and $\langle a^\dagger b^\dagger \rangle$. We will need these in the next section we will study the entanglement stored in this bosonic system.

The time- and system parameters-dependent real functions $f_3, f_5, f_9$ and $f_{12}$ are determined using the Wei-Norman method and obey the complicated set of first order nonlinear differential equations presented in the Appendix. We postpone the discussion about how the solutions are obtained to Section V and first, in Sections III and IV, we focus on the entanglement measures as the discussion in terms of the functions $f_i$ is more transparent.

III. ENTANGLEMENT MEASURES

The solutions in equation (2) are described by the application of exponents of creation super-operators on a two-mode vacuum state. If we wish to work with an exact solution and not truncate the Taylor expansion of the super-operator exponent we arrive at an infinitely dimensional density operator $\rho = |\psi \rangle \langle \psi|$ of a potentially entangled state $|\psi \rangle$. In such a case the finite dimensional entanglement measures no longer apply, which is a reason why here we will attempt to use negativity [18–20] which is not limited by the dimensional restrictions[26]. The result (2) could also be interpreted as a continuous variable state (CVS), where we could use the entanglement measure bounds imposed by the conditions first presented in Ref. [21, 22]. Here we will show that we can calculate the negativity explicitly, which we will later compare to the CVS separability criterion [21, 22]. Both of these measures in their core rely on the partial-transposition $pT$ operation, given by

$$\langle ij | k\ell |pT_1 = |kj | il \rangle \langle ij | k\ell |pT_2 = |il | kj \rangle,$$

e.g. to solving the equation $\det [\rho^{pT} - I\lambda] = 0$. Using the determinant of a matrix exponent – exponent of a trace relation we can write

$$\det [\rho^{pT} - I\lambda] = \det [-\lambda I] \exp \left( \sum_{j=1}^{\infty} -\frac{\text{Tr} \left( (\rho^{pT})^j \right)}{j\lambda} \right),$$

which, thanks to the property of the form of the solution (2),

$$\text{Tr} \left( (\rho^{pT})^j \right) = \frac{(1-x_+)^j (1-x_-)^j}{(1-x_+^2) (1-x_-^2)},$$
yields

$$\det [\rho^{pT} - I\lambda] = \det [-\lambda I] \prod_{p,q=0}^{\infty} \left( 1 - \frac{x_+^p x_-^q N}{\lambda} \right).$$

where $x_\pm$ were defined before, and where the eigenvalues can be directly read out. Since (as we will later show) $f_3$ is always be positive, the only negative eigenvalues will have the form $x_-^p x_+^{2q+1}$ provided that $x_- < 0$. Upon adding all of them up we obtain the negativity

$$\text{Neg} = N \sum_{p,q=0}^{\infty} x_+^p x_-^{2q+1} = \frac{x_-}{1+x_-}$$

$$= \text{Max} \left( 0, \frac{-f_3 + \sqrt{f_5^2 + f_3^2 + f_{12}^2}}{1 + f_4 - \sqrt{f_5^2 + f_3^2 + f_{12}^2}} \right), \quad (5)$$

This is the main result of this paper.

IV. CONTINUOUS VARIABLE STATES SEPARABILITY CONDITION

As first simultaneously and independently formulated by [21, 22] the continuous variable states separability criterion stems from quadratic relations of the type

$$\left( \Delta \hat{X}_{\vec{d}} \right)^2 + \left( \Delta \hat{X}_{\vec{d}} \right)^2 \geq |d_1 d'_2 - d_2 d'_1 + d_3 d'_4 - d_4 d'_3|,$$

which is the Heisenberg uncertainty relation obeyed by all states, with $d_i$ and $d'_i$ being components of the real $\vec{d}$ and $\vec{d}'$ four-vectors, $\hat{X}_{\vec{d}} = v_1 \hat{x}_1 + v_2 \hat{p}_1 + v_3 \hat{x}_2 + v_4 \hat{p}_2$, and $\Delta \hat{X} = \hat{X} - \langle \hat{X} \rangle$. Separable states, on the other hand, need to obey a more stricter inequality

$$\left( \Delta \hat{X}_{\vec{d}} \right)^2 + \left( \Delta \hat{X}_{\vec{d}} \right)^2 \geq |d_1 d'_2 - d_2 d'_1| + |d_3 d'_4 - d_4 d'_3|,$$

such that for the right combination of $\vec{d}$ and $\vec{d}'$ with $|d_i| = |d'_i| = 1 \forall i$, the first relation is bounded from below by zero, and the second one can be bounded by four.
The uncertainty on the left hand side can be expressed as

\[
\langle (\Delta X_d^2) \rangle = |\lambda_1|^2 \left(2\langle \hat{a}^\dagger \hat{a} \rangle + 1 \right) + |\lambda_2|^2 \left(2\langle \hat{b}^\dagger \hat{b} \rangle + 1 \right) + 4\text{Re} \left( \lambda_1 \lambda_2 \langle \hat{a}^\dagger \hat{b} \rangle \right),
\]

where \( \lambda_1 = (d_1 + id_2)/\sqrt{2} \) and \( \lambda_2 = (d_3 + id_4)/\sqrt{2} \), and where the other quadratic terms evaluate to zero for the state given by the equation (2).

By imposing that \( |\lambda_1| = |\lambda_2| = 1 \) we can find the optimal conditions for separability. Next, without a loss of generality we can set \( \text{Arg} (\lambda_1) = 0 \), impose the saturation of the lowest possible bound of the Heisenberg uncertainty principle by setting \( d_1d_2' - d_3d_4' + d_3d_4 - d_1d_2 = 0 \), and maximize \( |d_1d_2' - d_3d_4'| + |d_3d_4' - d_1d_2'| = 4 \), which requires \( \text{Arg} (\lambda_1') = \text{Arg} (\lambda_2) - \text{Arg} (\lambda_0') = \pi/2 \). This way we turn the separability condition (7) into

\[
4 \leq \frac{4 \left(1 - f_2^2 + |\tau|^2 + 2|\tau| \cos (\text{Arg} (\lambda_2) - \text{Arg} (\tau)) \right)}{(1 - (f_3 + |\tau|))(1 - (f_3 - |\tau|))},
\]

where \( z = e^{-i\varphi} (f_{12} + if_3) \) and where we have assumed identical baths \( \kappa_1 = \kappa_2 \) and \( n_{th,1} = n_{th,2} \), implying \( f_3 = 0 \), see discussion in the Appendix. This criterion has one left degree of freedom \( \text{Arg} (\lambda_2) - \text{Arg} (\tau) \), which when fixed to be equal to \( \pi/2 \) gives \( f_3 > |\tau| \) the same separability criterion as that obtained from the explicit negativity calculation.

In the next sections we will use this result in combination with the solutions to the equations of motion to determine the system parameters separability condition.

V. IMPLICATIONS OF THE SEPARABILITY CONDITIONS

In order to understand the separability condition in terms of the system parameters we need to first translate the master equation (1) into a set of equations for functions \( f_i(t) \) with the initial condition \( f_i(0) = 0 \). The details of the procedure are outlined in the Appendix, where we outline how an entire set of fifteen functions can be obtained. In this work so far we have focused on systems initialised in vacuum state \( \rho(0) = |00\rangle \langle 00| \), allowing us to narrow our interest to but four functions, which independent of the initial condition \( \rho(0) \), obey the following set of equations

\[
\begin{align*}
\dot{f}_3 &= -\frac{1}{2}c_{21,+}f_3 - \frac{1}{2}c_{21,-}f_0 + c_{11,-}f_3f_0 - \frac{1}{2} |\xi| f_3f_5 \\
+ \frac{1}{2}c_{21,+}f_3^2 + f_3^2 + f_3^2 + f_1^2) + c_{10,+}, \quad (8)
\end{align*}
\]

\[
\begin{align*}
\dot{f}_5 &= -\frac{1}{2}c_{21,+}f_5 + c_{11,+}f_3f_5 + c_{11,-}f_5f_0 + 2\delta f_{12} \\
+ \frac{1}{4} |\xi| (f_2^2 - f_5^2 + f_5^2 + f_2^2 + 1), \quad (9)
\end{align*}
\]

\[
\begin{align*}
\dot{f}_9 &= -\frac{1}{2}c_{21,+}f_9 - \frac{1}{2}c_{21,+}f_0 + c_{11,+}f_3f_9 - \frac{1}{2} |\xi| f_5f_9 \\
+ \frac{1}{2}c_{21,+}f_3f_9^2 + f_9^2 + f_9^2 + f_1^2) + c_{10,+}, \quad (10)
\end{align*}
\]

\[
\begin{align*}
\dot{f}_{12} &= -\frac{1}{2}c_{21,+}f_{12} + c_{11,+}f_3f_{12} + c_{11,-}f_0f_{12} - 2\delta f_5 \\
- \frac{1}{2} |\xi| f_5f_{12}, \quad (11)
\end{align*}
\]

where we have defined

\[
c_{xy,\pm} = \kappa_1 (x_{th,1} + y) \pm \kappa_2 (x_{th,2} + y).
\]

It is easy to see that for an identical baths case \( \kappa_1 = \kappa_2 = \kappa \) and \( n_{th,1} = n_{th,2} \), all \( c_{xy,\pm} = 0 \), implying that \( f_3(t) = 0 \) and completely independently in the resonant regime \( \delta = 0 \) we have \( f_{12}(t) = 0 \). Moreover in the absence of dissipation, all \( c_{xy,\pm} = 0 \) both \( f_3(t) \) and \( f_9(t) \) vanish, and for \( \xi = 0 \) we get \( f_5(t) = f_{12}(t) = 0 \). All of these are examples of parameter and Lie algebra reductions leading to significant simplifications in the equations above, to the extent that the non-linear set of equations above can be solved analytically in a resonant identical baths case, and we were able to analytically determine the steady state solutions if either the baths are identical, or the system is driven resonantly, or both with \( n_{th,1} = n_{th,2} = 0 \).

The solutions to the equations above fall into two parameter regimes with a baths populations independent boundary

\[
\Xi^2 = \left(1 - \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2} \right) \left(\delta^2 + (\kappa_1 + \kappa_2)^2 \right).
\]

as a result we define the underdamped (\( |\xi|^2 \geq \Xi^2 \)), and overdamped (\( |\xi|^2 < \Xi^2 \)) regime, which we call this way due to either unbounded or bounded expectation values \( \langle \hat{a}^\dagger \hat{a} \rangle \) and \( \langle \hat{b}^\dagger \hat{b} \rangle \) respectively.

One can verify numerically that both in the over- and the underdamped regime the equations (8)-(11) possess steady state solutions. By setting the left-hand sides of these four equations to zero one can obtain the steady state values of \( f_3, f_5, f_9 \) and \( f_{12} \) algebraically. The process yields a set of solutions larger than those obtained by considering the set of nonlinear ordinary differential equations with initial conditions \( f_i(0) = 0 \), therefore the algebraic solutions found have been verified by solving the differential equations numerically. The steady state solutions to the equations in the overdamped regime results at zero bath temperature take the compact form

\[
\begin{align*}
f_3 &= \frac{|\xi|^2 (\kappa_1^2 + \kappa_2^2)}{8\kappa_1\kappa_2 (\delta^2 + (\kappa_1 + \kappa_2)^2) - 2\kappa_1\kappa_2 |\xi|^2}, \\
f_5 &= \frac{2|\xi| (\kappa_1 + \kappa_2)}{4 (\delta^2 + (\kappa_1 + \kappa_2)^2) - |\xi|^2}, \\
f_9 &= \frac{|\xi|^2 (\kappa_1^2 - \kappa_2^2)}{8\kappa_1\kappa_2 (\delta^2 + (\kappa_1 + \kappa_2)^2) - 2\kappa_1\kappa_2 |\xi|^2}, \\
f_{12} &= \frac{-2\delta |\xi|}{4 (\delta^2 + (\kappa_1 + \kappa_2)^2) - |\xi|^2},
\end{align*}
\]

and the solutions in the \( |\xi|^2 > \Xi^2 \) regime are too incomprenhensible to present here, which is why we will also present the parameter simplified ones. The solutions in the detuned regime with identical non-zero temperature baths read

\[
\begin{align*}
f_3 &= \frac{2\kappa (2n_{th} + 1)}{\sqrt{|\xi|^2 - \delta^2 + 4\kappa (n_{th} + 1)}}, \\
|\xi| &= \sqrt{|\xi|^2 - \delta^2 + 2\kappa/ \sqrt{|\xi|^2 - \delta^2 + 4\kappa (n_{th} + 1)}}.
\end{align*}
\]
in the underdamped regime, and in the overdamped they become
\[
f_3 = \frac{\vert \xi \vert^2 + 4m_{th}(m_{th} + 1) (\delta^2 + 4\kappa^2)}{4(m_{th} + 1)^2 (\delta^2 + 4\kappa^2) - \vert \xi \vert^2} ,
\]
\[
|z| = \frac{2 \vert \xi \vert \sqrt{\delta^2 + 4\kappa^2} (2m_{th} + 1)}{4(m_{th} + 1)^2 (\delta^2 + 4\kappa^2) - \vert \xi \vert^2} .
\]

The two sets of solutions above imply that the separability condition \( f_3 > |z| \) reduces to
\[
4m_{th}\kappa > \sqrt{\vert \xi \vert^2 - \delta^2} \quad \text{for} \quad \vert \xi \vert^2 \geq 4\kappa^2 + \delta^2 , \tag{12}
\]
\[
2m_{th} \sqrt{\delta^2 + 4\kappa^2} > \vert \xi \vert \quad \text{for} \quad \vert \xi \vert^2 < 4\kappa^2 + \delta^2 , \tag{13}
\]
where the parameter regime discontinuity in this result is gone in the absence of detuning, and the same form is obeyed for both the under- and the over-damped regime, where negativity is described by a single function independent of the parameter regime.

Moreover, in this symmetric resonant regime, where one only needs to consider the solutions to the equations of motion for functions \( f_1 \) - \( f_6 \), the other ones returning \( f_{7-15} (t) = 0 \), one can solve the complete set of differential equations analytically also in the transient regime. This has to do with the fact that the master equation is described by a set of operators spanning the \( su(2, 2) \) Lie algebra, which decomposes into two sets of \( su(1, 1) \) Lie algebras, with the set of six equations decoupling into two sets of three equations which independently can be solved by the method of quadratures. Identical separability conditions and the same expression for negativity
\[
\text{Neg} = \text{Max} \left( \frac{2 \left( 1 - e^{-t (\kappa + \vert \xi \vert / 2)} \right) \left( \vert \xi \vert - 4\kappa m_{th} \right)}{e^{-t (\kappa + \vert \xi \vert / 2)} \left( \vert \xi \vert - 4\kappa m_{th} \right) + 2\kappa (2m_{th} + 1)} , 0 \right) ,
\]
\[
(14)
\]
in either under- and overdamped regimes is reflected by the fact that only one of the copies of the \( su(1, 1) \) Lie algebras determines the entanglement. Equation (14) shows that at \( t = 0 \) the state is completely separable, i.e. \( \text{Neg} = 0 \), while in the steady state it inseparable provided that \( \vert \xi \vert > 4\kappa m_{th} \), which is a parameter reduced expression (12) and (13). Moreover, at zero bath temperature i.e. \( m_{th} = 0 \) states violate the separability condition in all parameter regimes. Lastly, it is worth observing that in the absence of dissipation \( f_5 = \tanh \vert \xi \vert / 4 \) and all other \( f_i = 0 \), which not only violates the separability condition for any \( t > 0 \) and gives rise to a divergent negativity as \( t \to \infty \).

VI. SYSTEM INITIALLY IN A THERMAL STATE

Since the temperature of the bath plays an important role in the separability condition, it is worth investigating the effect of the initial state’s temperature on the steady state entanglement obtained. In the previous section we have assumed that the system is initiated in the vacuum state \( |00\rangle \), however in the presence of the environment at a non-zero temperature, this might be difficult to accomplish, and the state prior to two-mode squeeze driving should be initiated in a separable state \( \rho (0) = \rho_{a, th} \otimes \rho_{b, th} \), where denote the thermal state density operators \( \rho_{a, th} = \exp \left[ \frac{\hbar \omega_a \hat{c}^\dagger \hat{c}}{\kappa} T_b \right] / \text{Tr} \left( \exp \left[ \frac{\hbar \omega_a \hat{c}^\dagger \hat{c}}{\kappa} T_b \right] \right) \). Here we will treat the simplest case of the initial condition already in equilibrium with the environment such that \( T_a = T_b \), \( \omega_a = \omega_b = \omega \) and hence \( m_{th, 1} = m_{th, 2} = (1 - \tau)^{-1} \), with \( \tau = e^{-\beta \omega} \). The two modes still remain orthogonal, i.e. \( \hat{a} \neq \hat{b} \). The initial condition now can be written as \( \rho (0) = \rho_{a, th} \otimes \rho_{b, th} (1 - \tau)^2 e^{\hat{H}(t)|00\rangle \langle 00|} \). With the suitably chosen normal ordered solution Ansatz
\[
\rho (t) = e^{f_0(t)} e^{f_3(t)\hat{H}_3} e^{f_5(t)\hat{H}_5} e^{f_1(t)\hat{H}_1} e^{f_6(t)\hat{H}_6} e^{f_2(t)\hat{H}_2} e^{f_4(t)\hat{H}_4} \rho (0) ,
\]
where operators \( \hat{H}_{1,2,4,6} \) contain normal ordered annihilation operators. We see that since the system is no longer initiated in the vacuum, we cannot disregard a given set of exponents of operators. Using the form of the initial condition
\[
\rho (t) = \left( 1 - \tau \right)^2 e^{f_0(t)} e^{f_3(t)\hat{H}_3} e^{f_5(t)\hat{H}_5} e^{f_1(t)\hat{H}_1} e^{f_6(t)\hat{H}_6} e^{f_2(t)\hat{H}_2} e^{f_4(t)\hat{H}_4} \rho_{H_3} |00\rangle \langle 00| , \tag{15}
\]
we can now commute \( \exp [\tau \hat{H}_3] \) through exponents of operators \( \hat{H}_{1,2,4,6} \), and re-decompose using the Wei-Norman scheme as presented in the Appendix. As a result we obtain
\[
\rho (t) = \left[ (1 - g_3)^2 - g_3^2 \right] e^{g_3(t)\hat{H}_3} e^{g_5(t)\hat{H}_5} |00\rangle \langle 00| , \tag{15}
\]
where \( g_i (t) = f_i (t) + \mathcal{F}_i (t) \), and where the only two relevant \( \mathcal{F}_i \) as functions of \( f_i \) read
\[
\mathcal{F}_3 (t) = \frac{1}{2} \left( - \frac{e^{2f_3 + f_6} - e^{2f_3 - f_6}}{f_2 - f_4 - f_1 - 1} - \frac{e^{2f_3 - f_6} - e^{2f_3 + f_6}}{f_2 + f_4 - f_1 - 1} \right) ,
\]
\[
\mathcal{F}_5 (t) = \frac{1}{2} \left( - \frac{e^{2f_5 + f_6} - e^{2f_5 - f_6}}{f_2 - f_4 + f_1 - 1} - \frac{e^{2f_5 - f_6} - e^{2f_5 + f_6}}{f_2 + f_4 + f_1 - 1} \right) .
\]

The form of the later functions reflects how important it was to know the transients of all of the six functions \( f_i \) as well as the apparent simplification in the problem allowed us to determine functions \( \mathcal{F}_{3,5} \) to begin with. By virtue of the form of equation (15), we can immediately state that negativity will take the form
\[
\text{Neg} = \text{Max} \left( \frac{\vert \xi \vert - 4\kappa m_{th}}{2 (2m_{th} + 1) (\vert \xi \vert - 4\kappa m_{th}) + 2\kappa (2m_{th} + 1)} , 0 \right) .
\]

Moreover, this result in comparison to the equation (14) has the same steady state amount of entanglement \( \frac{\vert \xi \vert - 4\kappa m_{th}}{2 (2m_{th} + 1)} \), however the key difference is that when the state starts in a thermal equilibrium with the environment its negativity remains zero for a finite amount of time \( t = 2 \left( 2k + \vert \xi \vert \right)^{-1} \log (\vert \xi \vert - 4\kappa m_{th}^{1/2}) \), which only makes sense for the case of any entanglement formed, i.e. \( \vert \xi \vert > 4\kappa m_{th} \).

VII. CONCLUSIONS

In this work we have shown that, one can use the Wei-Norman method to study analytically a bosonic entanglement process subject to dissipation. The Lie algebra valued description based solution Ansatz allows one to calculate the exact expression of entanglement evolution or its steady state form as measured by negativity. Additionally, we have shown that the negativity calculated from the solution is completely compatible with the continuous variable separability condition.
Moreover, we have shown that for time independent system parameters, one can determine analytically the solutions to the equations of motion in the Wei-Norman setting in the transient and the steady state. Finally, should the bipartite state be initially in thermal equilibrium with the environment, then the steady state entanglement does not change, however there is a finite amount of time in the transient regime where the degree of entanglement is lower compared to that when the state is initialised in vacuum.

The results formulated in this paper in terms of general functions $f_i$ remain applicable for (effective) two-mode driven systems with time dependent parameters (driving strength $\xi$, dissipation rates $\kappa_i$, or detuning $\delta$), which then require using the same equations with time dependent coefficients. Moreover, this method is very well suited for investigating similar problems of more than two modes with pairwise-squeezing interaction terms. As a result such extensions can be very important in experiments investigating entanglement in continuous variable systems.

VIII. ACKNOWLEDGEMENTS

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[1] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997).
[2] M. Aspelmeyer, S. Gröblacher, K. Hammerer, and N. Kiesel, J. Opt. Soc. Am. B 27, A189 (2010).
[3] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, arXiv:1303.0733.
[4] G. Labadze, M. Dukalski, and Ya. M. Blanter, arXiv:1308.4521.
[5] G. Lindblad, Communications in Mathematical Physics 48, 119 (1976).
[6] A. Kossakowski, Reports on Mathematical Physics 3, 247 (1972).
[7] J. Wei and E. Norman, Journal of Mathematical Physics 4, 575 (1963).
[8] J. Wei and E. Norman, Proceedings of the American Mathematical Society 15, 327 (1964).
[9] M. Dukalski, and Ya.M. Blanter, *in preparation* (2014).
[10] M. Ban, Journal of Mathematical Physics 33, 3213 (1992).
[11] C. C. Gerry, Phys. Rev. A 31, 2721 (1985).
[12] C. C. Gerry, Phys. Rev. A 35, 2146 (1987).
[13] G. Dattoli, S. Solimeno, and A. Torre, Phys. Rev. A 34, 2646 (1986).
[14] G. Dattoli, P. Di Lazzaro, and A. Torre, Phys. Rev. A 35, 1582 (1987).
[15] G. Dattoli, M. Richetta, and A. Torre, Phys. Rev. A 37, 2007 (1988).
[16] J. Twamley, Phys. Rev. A 48, 2627 (1993).
[17] J. M. Cerveró and J. D. Lejarreta, Journal of Physics A: Mathematical and General 29, 7545 (1996).
[18] M. Horodecki, P. Horodecki, and R. Horodecki, Physics Letters A 223, 1 (1996).
[19] G. Vidal and R. F. Werner, *biboinfojournalPhys. Rev. A* 65, 032314 (2002).
[20] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[21] Lu-Ming Du, G. Giedke, J. I. Cirac, and P. Zoller, Phys.Rev.Lett. 84, 2722 (2000).
[22] R. Simon, Phys.Rev.Lett. 84, 2726 (2000).
[23] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[24] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306+ (2000).

[26] Other than the presence of bound entanglement in systems with dimensions greater than $2 \times 3$. As we will see in the sections to come, bound entanglement is unlikely to be present.
IX. APPENDIX

A. The Lindblad equation of motion and the underlying Lie algebra structure

The Lindblad type master equation (1) can be rewritten in the form

\[ \dot{\rho} = \sum_{i=0}^{15} \alpha_i H_i \rho, \quad (16) \]

where

\[
\begin{align*}
\alpha_1 &= -\frac{1}{4} (\kappa_1 (2n_{1,t} + 1) + \kappa_2 (2n_{2,t} + 1)), \\
\alpha_{2,8} &= \frac{1}{2} (\kappa_1 (n_{1,t} + 1) \pm \kappa_2 (n_{2,t} + 1)), \\
\alpha_7 &= -\frac{1}{2} (\kappa_1 (2n_{1,t} + 1) - \kappa_2 (2n_{2,t} + 1)), \\
\alpha_{3,9} &= \frac{1}{2} (\kappa_1 n_{1,t} \pm \kappa_2 n_{2,t}), \\
\alpha_4 &= \alpha_5 = \frac{1}{4} |\xi|, \\
\alpha_{15} &= 2 \delta,
\end{align*}
\]

with all other \( \alpha_i = 0 \) and where

\[
\begin{align*}
H_{1,7,\rho} &= \frac{1}{2} \left( \hat{a} \hat{a} \rho + \hat{a} \hat{a} \rho + \hat{a} \hat{a} \rho + \rho \hat{a} \hat{a} \right), \\
H_{2,8,\rho} &= \frac{1}{2} \left( \hat{b} \hat{b} \rho + \hat{b} \hat{b} \rho + \hat{b} \hat{b} \rho + \rho \hat{b} \hat{b} \right), \\
H_{4,11,\rho} &= e^{i(\varphi + (1 \pm 1) \pi/4)} \hat{a} \hat{b} \rho - e^{-i(\varphi + (1 \pm 1) \pi/4)} \rho \hat{a} \hat{b}, \\
H_{5,12,\rho} &= -e^{-i(\varphi - (1 \pm 1) \pi/4)} \hat{b} \hat{b} \rho - e^{i(\varphi - (1 \pm 1) \pi/4)} \rho \hat{b} \hat{b},
\end{align*}
\]

with the first (second) index corresponding to the upper (lower) signs and with \( H_0 \) being the identity superoperator, i.e. \( H_0 \rho = \rho \).

Thanks to the elementary commutation relation

\[
\left[ \Theta_i, \Theta_j \right] = \delta_{i,j} \quad \text{and} \quad [\Theta_i, \Theta_j] = 0 , \quad (17)
\]

where \( \Theta_1 = \hat{a} \) and \( \Theta_2 = \hat{b} \), the set of fifteen superoperators closes under commutation (see Table I), thus forming a Lie algebra. In what follows we define skew (anti-)symmetric matrices

\[ L_{i,j} = E_{i,j} + E_{j,i}, \quad K_{i,j} = E_{i,j} - E_{j,i}, \]

where \( E_{i,j} \) is a matrix with 1 in the \( i \)-th row and \( j \)-th column and zero elsewhere. It is easy to verify that the linear combinations of the above

\[
\begin{align*}
\mathcal{H}_{1} &= -2L_{2,3}, \quad \mathcal{H}_{6} = L_{1,5}, \quad \mathcal{H}_{7} = -L_{1,4}, \\
\mathcal{H}_{10} &= K_{4,5}, \quad \mathcal{H}_{13} = L_{4,6}, \quad \mathcal{H}_{14} = K_{5,6}, \\
\mathcal{H}_{15} &= K_{4,6}, \quad \mathcal{H}_{2,3} = K_{1,2} + L_{1,3}, \quad \mathcal{H}_{4,5} = L_{2,5} + K_{4,5}, \\
\mathcal{H}_{8,9} &= K_{3,4} + L_{2,4}, \quad \mathcal{H}_{11,12} = -K_{3,6} + L_{2,6}
\end{align*}
\]

obey the same commutation relations, and that the \( L_{i,j} \) and \( K_{i,j} \) above are elements of the \( so(4,2) \) Lie algebra. Upon a homomorphism \( H_i \rightarrow \mathcal{H}_i \), we can show that the superoperators from the master equation (1) are just a different incarnation of the \( so(4,2) \) Lie algebra.

Realising that, for time independent \( \kappa_i, n_{th,i}, \delta \) and \( \xi \), the solution to the master equation (1) in the form of (16) is simply

\[
\rho(t) = \exp \left( \sum_{i=0}^{15} \alpha_i H_i t \right) \rho(0) , \quad (18)
\]

which is simply given by a Lie group element acting on the initial state. This can be thought of as a rotation, or a movement on the surface embedded in six dimensions satisfying the equation

\[
1 = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 ,
\]

which can be understood as the trace-preservation condition of the density operator \( \rho \) [1]. The form of equation (18) however is not very useful for any purposes, and we will proceed with the so called Wei-Norman method [7, 8], to decompose the right hand side of equation (18), however this method is much more powerful and allows one to solve the equation (1) for time dependent \( \kappa_i, n_{th,i}, \delta \) and \( \xi \), allowing for studying modulated squeeze-driving and non-Markovian baths.

B. Wei-Norman method treatment and the resultant equations of motion

We take the Ansatz

\[
\rho(t) = e^{f_0(t)} \prod_i e^{f_i(t) H_i} \rho(0) ,
\]

with the ordering \( i = 3, 5, 9, 12, 1, 7, 15, 6, 14, 10, 13, 2, 8, 4, 11 \), with the exponent of \( H_3 \) acting last and the exponent of \( H_{11} \) acting first on the initial condition \( \rho(0) \). From the definition of the super-operators \( H_i \), one can see that the ordering chosen above is normal, i.e. annihilation (creation) super-operators acting first (last), and in the middle super-operators which are composed of creation and annihilation operators [2]. Lastly, it is important to note that a different ordering Ansatz will result in a different set of equations for functions \( f_i(t) \).

The set of resultant differential equations is quite complicated and non-transparent, thus outside the scope of this article. In a special, very convenient, case when \( \rho(0) = |00\rangle \langle 00| \), we see that the set of eleven right-most operators acting on the initial condition leaves it unchanged, with the exception of \( e^{f_4 H_4} \) which contributes an overall scaling factor, thus in combination with the only decoupled equation of motion \( f_0 = \frac{1}{2} (\kappa_1 + \kappa_2) \), gives

\[
\begin{align*}
\rho(t) &= e^{\frac{1}{2} (\kappa_1 + \kappa_2)^t \pi/2} \rho(0) , \\
\mathcal{H}_2,3 &= K_{1,2} + L_{1,3}, \quad \mathcal{H}_4,5 = L_{2,5} + K_{4,5}, \\
\mathcal{H}_8,9 &= K_{3,4} + L_{2,4}, \quad \mathcal{H}_{11,12} = -K_{3,6} + L_{2,6}
\end{align*}
\]

(19)
TABLE I: Commutation relation table. The smallest (middle) $6 \times 6$ $(10 \times 10)$ box encloses the $\mathfrak{so}(2, 2)$ ($\mathfrak{so}(3, 2)$) Lie algebra.

|                  | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ | $H_9$ | $H_{10}$ | $H_{11}$ | $H_{12}$ | $H_{13}$ | $H_{14}$ | $H_{15}$ |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| $H_1$            | 0     | -2 $H_2$ | 2 $H_3$ | -2 $H_4$ | 2 $H_5$ | 0     | 0     | -2 $H_8$ | 2 $H_9$ | 0         | -2 $H_{11}$ | 2 $H_{12}$ | 0         | 0         | 0         |
| $H_2$            | 2 $H_1$ | 0     | $H_1$ | 0     | 2 $H_6$ | -$H_4$ | $H_4$ | 0     | 2 $H_7$ | 0         | 0         | 2 $H_{13}$ | $H_{11}$ | 0         | 0         |
| $H_3$            | -2 $H_3$ | -$H_1$ | 0     | 2 $H_6$ | 0     | -$H_4$ | -$H_5$ | -$H_9$ | 2 $H_7$ | 0         | 0         | -2 $H_{13}$ | 0         | -$H_{12}$ | 0         |
| $H_4$            | 2 $H_4$ | 0     | -2 $H_6$ | 0     | -$H_1$ | -$H_2$ | 0     | 0     | 2 $H_{10}$ | -$H_8$ | 0         | 2 $H_{15}$ | 0         | 0         | -$H_{11}$ |
| $H_5$            | -2 $H_5$ | -2 $H_6$ | 0     | $H_1$ | 0     | -$H_3$ | 0     | -2 $H_{10}$ | 0         | $H_9$ | -2 $H_{15}$ | 0         | 0         | 0         | $H_{12}$ |
| $H_6$            | 0     | $H_4$ | $H_5$ | $H_2$ | 0     | $H_3$ | $H_{10}$ | 0     | 0         | $H_7$ | 0         | 0         | 0         | $H_{15}$ | 0         | $H_{13}$ |
| $H_7$            | 0     | -$H_8$ | $H_9$ | 0     | 0     | -$H_{10}$ | 0     | -2 $H_3$ | $H_3$ | -$H_8$ | 0         | 0         | -2 $H_{13}$ | -$H_{13}$ | 0         | 0         |
| $H_8$            | 2 $H_8$ | 0     | 2 $H_7$ | 0     | 2 $H_{10}$ | 0     | 0     | $H_2$ | 0         | $H_4$ | 0         | 2 $H_{14}$ | 0         | 0         | -$H_{11}$ |
| $H_9$            | -2 $H_9$ | -2 $H_7$ | 0     | -2 $H_{10}$ | 0     | 0     | -$H_3$ | -$H_4$ | 0     | -$H_5$ | 2 $H_{14}$ | 0         | 0         | -2 $H_{12}$ |
| $H_{10}$         | 0     | 0     | 0     | 0     | 0     | 0     | -$H_8$ | -$H_9$ | -$H_7$ | 0     | -$H_{10}$ | 0         | 0         | 0         | $H_{15}$ |

at which point the order does not matter due to the mutual commutativity of the remaining operators, and the scalar prefactor plays the role of a normalisation condition. Regardless of the initial condition, the kinetic equations for functions $f_3$, $f_5$, $f_9$, and $f_{12}$ were given in the main body of the text and the necessary equation for $f_1$ takes the form

$$f_1 = \frac{1}{2} c_{11,+} f_3 + \frac{1}{2} c_{11,-} f_9 - \frac{1}{2} \frac{\xi}{2} f_5 - \frac{1}{4} c_{11,+} .$$

Using the set of equations (8-11) and the equation above one can verify by differentiating both sides and remembering the initial condition $f_1(t = 0) = 0$, that

$$e^{\frac{\partial f_1}{\partial t} + \frac{1}{2}(\kappa_1 + \kappa_2)} \left(1 - f_3^2 - f_6^2 - f_9^2 - f_{12}^2\right) = 1 ,$$

which allows us to eliminate the scale factor in equation (20) in favour of functional dependence on functions $f_3$, $f_5$, $f_9$, and $f_{12}$. One can verify the trace-preserving nature of the evolution by taking the trace of the equation and arriving at $\partial_t \Tr(\rho) = 0$.

Additionally, the tabular display of the operators into subgroups marks the use of smaller Lie algebra equation decompositions, and surface dimensional reduction, such that:

1 $\delta = 0$ decouples operators $H_{11-15}$ from the algebra ($H_{15}$ vanishes directly from the equation of motion, and $H_{11-14}$ do not enter the dynamics due to their commutation relation properties - see Table 1).

2 $\kappa_1 = \kappa_2 = \kappa \neq 0$ and $\kappa_{10,1} = \kappa_{10,2}$, decouples operators $H_7-9$, and effectively the operator $H_{10}$, and the system reduces again to $\mathfrak{so}(3, 2) \subset \mathfrak{so}(4, 2)$.

3 $\xi = 0$, decouples operators $H_{4,5}$ and effectively $H_{6,11-14}$. The relevant operators form an $\mathfrak{so}(2, 2) \subset \mathfrak{so}(4, 2)$ Lie algebra which decomposes into two copies of $\mathfrak{su}(1, 1)$ Lie algebras acting in separate subspaces of $\hat{a}$ and $\hat{b}$ bosons.

4 $\kappa_1 = \kappa_2 = 0$, decouples operators $H_{2,3,6}$ and $H_{10}$.

The remaining operators form again an $\mathfrak{so}(2, 2) \subset \mathfrak{so}(4, 2)$ Lie algebra which decomposes into two sets of operators acting separately from the right or from the left of the density operator. This has to do with the fact that said evolution no longer needs to be described using a von-Neumann equation in superoperators, but rather a Schrödinger equation described only by right or left acting operators separately.

5 Moreover, conditions 1 and 2 combined, also lead to an $\mathfrak{so}(2, 2)$ reduction, where the decomposition into two copies of $\mathfrak{su}(1, 1)$ Lie algebras is different, and it resembles the decomposition found in [9].

The Lie-algebraic reduction described above has to do with the reduced dimensionality of the space embedding the surface, to which one can deem the evolution to be confined, such that for the $\mathfrak{so}(3, 2)$ Lie algebra case we are dealing with a five-dimensional space with a surface given by the equation $1 = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2$ and in the $\mathfrak{so}(2, 2) \sim \mathfrak{su}(1, 1) \otimes \mathfrak{su}(1, 1) \sim \mathfrak{so}(2, 1) \otimes \mathfrak{so}(2, 1)$ the evolution is confined to a product space of two hyperboloids embedded in three dimensions.

C. Computing the negativity

The solutions to the equations of motion written using the Wei-Norman conditioned on both modes initially in the vacuum state are given in equation (2). For the purposes of this proof we will rewrite this result using the property of mutual commutation of the above operators

$$\rho(t) = N e^{(f_3 + f_6) a^\dagger \hat{a}} e^{(f_3 + f_6) b^\dagger \hat{b}} \times e^{(f_{12} + f_{15}) e^{-i\phi} a^\dagger b} e^{(f_{12} + f_{15}) e^{i\phi} a b} |00\rangle \langle 00|$$

(21)
and we will rewrite the matrix in terms of a quadruple infinite sum with redefinitions \( g_z = f_z \pm f_0 \) and \( z = (f_2 + f_3) e^{-i\varphi} \)

\[
\rho = N \sum_{i,j,k,l=0}^{\infty} \frac{g_i g_j z^k \bar{z}^l}{i!j!k!l!} \left( \hat{a}^\dagger \right)^{i+k} \left( \hat{b}^\dagger \right)^{j+l} |00 \rangle \langle 00 | \hat{a}^{i+l} \hat{b}^{j+k}.
\]

\[
= N \sum_{i,j,k,l=0}^{\infty} \frac{g_i g_j z^k \bar{z}^l}{i!j!k!l!} \sqrt{(i+k)! (j+l)! (i+l)! (j+k)!} \times |i+k,j+k,i+l,j+l \rangle.
\]

In order to calculate the negativity, we need to partial transpose the matrix above, which can be done very easily

\[
\rho^{PT} = N \sum_{i,j,k,l=0}^{\infty} \frac{g_i g_j z^k \bar{z}^l}{i!j!k!l!} \sqrt{(i+k)! (j+l)! (i+l)! (j+k)!} \times |i+k,j+k,i+l,j+l \rangle.
\]

Next we use the relationship and define a shorthand \( \rho^{PT} \equiv X \) and the relationships

\[
\det \{ \exp (A) \} = \exp (\text{Tr} (A)) \Rightarrow \det [X] = \exp (\text{Tr} (\log X))
\]

to derive the characteristic equation and determine the eigenvalues we use

\[
\det [X - \lambda I] = \det [-\lambda I] \det \left[ I - \frac{X}{\lambda} \right] = \det [-\lambda I] \exp \left( \text{Tr} \left( \log \left( I - \frac{X}{\lambda} \right) \right) \right) = \det [-\lambda I] \exp \left( \text{Tr} \left( \sum_{j=1}^{\infty} \frac{X^j}{j! \lambda^j} \right) \right) = \det [-\lambda I] \exp \left( \sum_{j=1}^{\infty} - \frac{\text{Tr} (X^j)}{j! \lambda^j} \right),
\]

where the log Taylor expansion holds if the eigenvalues of \( X \) observe the condition \( |\lambda| \leq 1 \), which is the case for the eigenvalues of any (partial transposed) density operator. If we find a general form of \( \text{Tr} (X^j) \), then we can hope to find the form of this infinitely long polynomial. Already when calculating a square of \( X \) we can see a pattern,

\[
X^2 = N^2 \sum_{i,j,k,l=0}^{\infty} g_i g_j z^k \bar{z}^l \left( \hat{a}^\dagger \right)^{i+k} \left( \hat{b}^\dagger \right)^{j+l} |00 \rangle \langle 00 |
\times \hat{a}^{i+l} \hat{b}^{j+k} \left( \hat{a}^\dagger \right)^{p+s} \left( \hat{b}^\dagger \right)^{q+r} |00 \rangle \langle 00 | \hat{a}^{p+r} \hat{b}^{q+s} \frac{g^p g^q z^p \bar{z}^r}{p! q! r! s!}.
\]

Then, focusing on the ket-operator sandwich in the middle we see that

\[
\langle 00 | \hat{a}^w \hat{b}^s \left( \hat{a}^\dagger \right)^{y} \left( \hat{b}^\dagger \right)^{z} |00 \rangle = \delta_{w,y} \delta_{x,x} w! x! ,
\]

in our case implying \( p = i + l - s \) and \( q = j + k - r \), implying that \( i + l \geq s \) and that \( j + k \geq r \)

\[
X^2 = N^2 \sum_{i,j,k,l=0}^{\infty} \sum_{s=0}^{i+l} \sum_{r=0}^{j+k} g_i g_j z^k \bar{z}^l \left( \hat{a}^\dagger \right)^{i+k} \left( \hat{b}^\dagger \right)^{j+l} \left( \hat{a} \right)^{i+l-s} \left( \hat{b} \right)^{j+k-r} \frac{g_{i-l-s} g_{j-k-r} z^s \bar{z}^r}{(i+l-s)! (j+k-r)!} \hat{a}^{i+l-s} \hat{b}^{j+k-r+s}.
\]

Since the trace is unaffected by (partial) transposition, and the trace of \( X^2 / N^2 \) is the same as the trace of \( \rho / N \) with the replacement of \( f_3 \rightarrow f_3^2 + f_3^2 + f_3^2 + f_3^2, f_0 \rightarrow 2f_3f_0 \) and

\[
(f_3^2 + f_3^2 + f_3^2 + f_3^2) H_3 + 2f_3 f_0 H_0 + 2f_3 \sqrt{f_3^2 + f_0^2} \left( \sin (\theta + \varphi) H_3^{PT} + \cos (\theta + \varphi) H_0^{PT} \right) |00 \rangle \langle 00 |.
\]
\( f_2^2 + f_{12}^2 \to 4f_2^2(f_2^2 + f_{12}^2) \), giving after simplification

\[
\text{Tr} \left( \mathcal{X}^2 \right) = \frac{(1-x_+^2)(1-x_-)^2}{(1-x_+^2)(1-x_-^2)}.
\]

Following the argument above it is easy to prove in general (after some algebra) that every additional power of \( \mathcal{X} \) gives rise to the transformation \((\cdot)\rightarrow g_+(\cdot)\rightarrow z\cdot\hat{b} \rightarrow \bar{z}\cdot\hat{a}+g_-\cdot\hat{b}\). Using a proof by induction one can prove that upon tracing \( \mathcal{X} \) we get

\[
\text{Tr} \left( \mathcal{X}^j \right) = \frac{(1-x_+)^j(1-x_-)^j}{(1-x_+^j)(1-x_-^j)}.
\]

(22)

Alternatively, one can see that \( \mathcal{X} \) can always be written in the form

\[
\mathcal{X}^j = \mathcal{N}^j \exp \left[ F_3^{(j)} H_3 + F_5^{(j)} H_9 + F_5^{(j)} H_5^{(1)} + F_1^{(j)} H_1^{(1)} \right] \tag{23}
\]

with \( \text{Tr} \left( \mathcal{X}^i \right) \) in the form

\[
\text{Tr} \left( \mathcal{X}^i \right) = \frac{\mathcal{N}^i}{(1-X_+^i)(1-X_-^i)}.
\]

(24)

with

\[
X_{\pm}^{(j)} = F_{\pm}^{(j)} \pm \sqrt{(F_{\pm}^{(j)})^2 + (F_{\pm}^{(j)})^2 + (F_{\pm}^{(j)})^2}.
\]

(25)

By multiplying both sides of equation (23) by \( \mathcal{X} \) one can arrive at a set of recursive linear algebraic equations for functions \( F_3^{(j)} \)

\[
F_3^{j+1} = f_3 F_3 + f_3 F_3 + f_3 F_3 + f_3 F_3,
\]

\[
F_5^{j+1} = -iF_3 F_1 + iF_3 F_1 + iF_3 F_1 + iF_3 F_3,
\]

\[
F_5^{j+1} = iF_3 F_1 - iF_3 F_1 + iF_3 F_1 + iF_3 F_3,
\]

\[
F_1^{j+1} = iF_3 F_1 + iF_3 F_1 + iF_3 F_3,
\]

with solutions

\[
F_3^j = \frac{1}{2} \left( x_+^j + x_-^j \right), \quad F_3^j = \frac{1}{2} \left( x_+^j - x_-^j \right),
\]

\[
F_5^j = \frac{1}{2} \left( x_+^j - x_-^j \right), \quad F_1^j = \frac{1}{2} \left( x_+^j + x_-^j \right),
\]

which when substituted into the equation (24) yield again equation (22).

We will now use this result to calculate the eigenvalues of \( \mathcal{X} = e^{\rho \sigma^0 T} \) as

\[
\det \left[ \mathcal{X} - \lambda I \right] = \det \left[ -\lambda I \right] \exp \left( \sum_{j=1}^{\infty} \frac{\text{Tr} \left( \mathcal{X}^j \right)}{j\lambda^j} \right)
\]

\[
= \det \left[ -\lambda I \right] \exp \left( -\sum_{j=1}^{\infty} \frac{\left(1-x_+\right)^j}{j\lambda^j \left(1-x_-\right)^j} \right).
\]

Let us define

\[
\sigma = \frac{\lambda}{\mathcal{N}} = \frac{\lambda}{(1-x_+)(1-x_-)},
\]

then this in combination with

\[
\frac{1}{1-\tau} = \sum_{i=0}^{\infty} \tau^i
\]

gives us

\[
\det \left[ \mathcal{X} - \lambda I \right] = \det \left[ -\lambda I \right] \exp \left( -\sum_{j=1}^{\infty} \frac{1}{j\lambda^j} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} x_p^{(j)} \sum_{q=0}^{\infty} x_q^{(j)} \right)
\]

\[
= \det \left[ -\lambda I \right] \exp \left( \sum_{p,q=0}^{\infty} \sum_{j=1}^{\infty} -\frac{1}{j} \left( x_p^j \frac{x_q^j}{\sigma} \right)^j \right)
\]

\[
= \det \left[ -\lambda I \right] \exp \left( \sum_{p,q=0}^{\infty} \sum_{j=1}^{\infty} \log \left( 1 - \frac{x_p^j x_q^j}{\sigma} \right) \right)
\]

\[
= \det \left[ -\lambda I \right] \prod_{p,q=0}^{\infty} \left( 1 - \frac{x_p^j x_q^j \mathcal{N}}{\lambda} \right)
\]

so that when the above is equal to zero, it is easy to see that all of the eigenvalues \( \lambda_i \) are of the form \( x^j x_+ \mathcal{N} \). Since \( x_+ > 0 \) and \( x_- < 0 \) if \( \sqrt{f_3^2 + f_5^2 + f_1^2} > f_3 \), then the only negative eigenvalues will be present for odd powers of \( x_+ \) and any power of \( x_- \). Thus negativity takes the form given by equation (5).

D. The smallest non-trivial problem, the largest with analytically obtainable transient solutions

The master equation (1) can be solved exactly for an arbitrary initial condition under the parameter reduction 5, i.e. \( \kappa_1 = \kappa_2 = \kappa \) and \( n_{th,1} = n_{th,2} = n_{th} \), using the normal ordering solution Ansatz

\[
\rho(t) = e^{(f_0(t)) f_3(t) H_3 + f_5(t) H_5 + f_1(t) H_1 + f_6(t) H_6}
\]

\[
\times e^{f_2(t) H_2 f_4(t) H_4} \rho(0),
\]

and the Wei-Norman method [7, 8] we obtain equations

\[
\hat{f}_1 = \frac{1}{2} \left( \kappa (2n_{th} + 1) f_3 - 2n_{th} - 1 \right) - \frac{1}{4} \left( \kappa \right) f_3
\]

\[
\hat{f}_2 = \frac{1}{2} e^{2f_1} \left( \kappa (2n_{th} + 1) \cosh f_6 - \frac{1}{2} \left( \kappa \right) f_6 \right)
\]

\[
\hat{f}_3 = \frac{1}{2} \left( \kappa \right) f_3 f_5 - \kappa (2n_{th} + 1) f_3 + \kappa (n_{th} + 1) f_3 ^2
\]

\[
+ \kappa (n_{th} + 1) f_5^2 + n_{th} \right)
\]

\[
\hat{f}_4 = \frac{1}{2} e^{2f_1} \left( \frac{1}{2} \left( \kappa \right) f_6 - 2\kappa (n_{th} + 1) \cosh f_6 \right)
\]

\[
\hat{f}_5 = \kappa f_5 (2n_{th} + 1) f_3 - 2n_{th} - 1 - \frac{1}{4} \left( \kappa \right) \left( f_3^2 + f_5^2 - 1 \right)
\]

\[
\hat{f}_6 = 2\kappa (n_{th} + 1) f_5 - \frac{1}{2} \left( \kappa \right) f_5
\]

and the last one being \( \hat{f}_0(t) = \kappa \). These equations linearly decompose into two sets of equations for functions \( \{p_+, q_+, r_+\} \) and \( \{p_-, q_-, r_-\} \) such that

\[
f_1(t) = \frac{1}{4} (p_+ (t) + p_+ (t)) \quad f_4(t) = \frac{1}{2} (q_+ (t) - q_- (t))
\]

\[
f_2(t) = \frac{1}{2} (q_- (t) + q_+ (t)) \quad f_5(t) = \frac{1}{2} (r_- (t) - r_+ (t))
\]

\[
f_3(t) = \frac{1}{2} (r_- (t) + r_+ (t)) \quad f_6(t) = \frac{1}{2} (p_- (t) - p_+ (t))
\]
where

\[ p_\pm(t) = 2 \log \left( \frac{2 e^{\pm i\pi/2} (\kappa \pm |\xi|/2) (\kappa \pm |\xi|/2)}{2n_{th} \kappa + e^{\pm i\pi/2} (2(n_{th} + 1) \kappa \pm |\xi|/2) \pm |\xi|} \right) \]

\[ q_\pm(t) = \frac{1 - e^{\pm i\pi/2}}{(1 - e^{\pm i\pi/2}) (2(n_{th} + 1) \kappa \pm |\xi|/2) - 2(\kappa \pm |\xi|/2)} , \]

\[ r_\pm(t) = -2n_{th} \kappa + e^{\pm i\pi/2} (2(n_{th} + 1) \kappa \pm |\xi|/2) \pm |\xi|/2 . \]

E. Details of the initial thermal state computation

If the system is initially in the thermal state

\[ \rho(0) = (1 - e^{-\beta \omega \hat{a}^\dagger \hat{a}}) = (1 - e^{-\beta \omega}) \exp(\frac{-\beta \omega \hat{a}^\dagger \hat{a}}{2}) = (1 - \tau) e^{r^{H_{3}} \hat{0}} \] ,

then the evolution takes the form

\[ \rho = (1 - \tau)^2 e^{f_{0}(t) e^{f_{3}(t) H_{3}} e^{f_{5}(t) H_{3}} e^{f_{1}(t) H_{1}} e^{f_{6}(t) H_{6}}} \]

\[ \times e^{f_{2}(t) H_{2}} e^{f_{4}(t) H_{4}} e^{r^{H_{3}} \hat{0}} \hat{0} . \]

This can be rewritten again in the form involving only exponents of \( H_{3} \) and \( \mathcal{H}_{3} \), by means of sandwiching the last exponent in the following manner

\[ \rho = (1 - \tau)^2 e^{f_{01}(t) e^{f_{31}(t) H_{3}} e^{f_{51}(t) H_{3}} e^{f_{11}(t) H_{1}} e^{f_{61}(t) H_{6}}} \]

\[ \times e^{-f_{21}(t) H_{2}} e^{-f_{41}(t) H_{4}} e^{r^{H_{3}} \hat{0}} \hat{0} , \]

which can be brought back to an easier form by realising that

\[ e^{f_{11}(t) H_{1}} e^{f_{61}(t) H_{6}} e^{f_{21}(t) H_{2}} e^{f_{41}(t) H_{4}} e^{-f_{21}(t) H_{2}} e^{-f_{41}(t) H_{4}} e^{-f_{11}(t) H_{1}} = \prod_{i=1}^{6} A_{i} H_{i} \equiv \mathcal{J} , \]

with

\[ A_{1} = 2f_{2} , \quad A_{3} = e^{f_{1}} \cosh f_{6} , \]

\[ A_{6} = 2f_{4} , \quad A_{5} = e^{f_{1}} \sinh f_{6} , \]

and

\[ A_{2,4} = 2 \left( e^{-(f_{2} + f_{4})} (f_{2} + f_{4})^2 + e^{-(2f_{1} - f_{6})} (f_{2} - f_{4})^2 \right) , \]

and that \( e^{-A} e^{B} e^{-A} = e^{e^{B} e^{A}} \), gives

\[ e^{f_{11}(t) H_{1}} e^{f_{61}(t) H_{6}} e^{f_{21}(t) H_{2}} e^{f_{41}(t) H_{4}} e^{-f_{21}(t) H_{2}} e^{-f_{41}(t) H_{4}} e^{-f_{11}(t) H_{1}} \]

\[ = \exp \left[ \sum_{i=1}^{6} A_{i} H_{i} \right] = e^{\mathcal{J}} \]

and we set out to find \( \mathcal{F}_{i} \) such that

\[ e^{\tau \mathcal{F}} = e^{\mathcal{F}_{3}^{H_{3}} e^{\mathcal{F}_{5}^{H_{5}} e^{\mathcal{F}_{1}^{H_{1}} e^{\mathcal{F}_{6}^{H_{6}} e^{\mathcal{F}_{2}^{H_{2}} e^{\mathcal{F}_{4}^{H_{4}}}}}}}} \]

is another normal ordering decomposition Ansatz of an operator exponent. This time however it is not a decomposition based on time evolution, but rather the initial condition parameter \( \tau \) is acting like an artificial evolution operator which ranges from 0 \((k_{b}T < \hbar \omega)\) to 1 \((k_{b}T > \hbar \omega)\). We derive a set of differential equations for functions \( \mathcal{F}_{i} \) (the Wei-Norman method) based on

\[ \frac{\partial e^{\tau \mathcal{F}}}{\partial \tau} = \mathcal{J} e^{\mathcal{F}_{3}^{H_{3}} e^{\mathcal{F}_{5}^{H_{5}} e^{\mathcal{F}_{1}^{H_{1}} e^{\mathcal{F}_{6}^{H_{6}} e^{\mathcal{F}_{2}^{H_{2}} e^{\mathcal{F}_{4}^{H_{4}}}}}}}} \]

with the solutions

\[ \mathcal{P}_{\pm} = 2 \log(1 - \tau \mathcal{q}_{\pm}) , \quad \mathcal{Q}_{\pm} = \frac{\tau e^{\pm} q_{\pm}^{2}}{1 - \tau \mathcal{q}_{\pm}} \]

with

\[ \mathcal{F}_{1}(t) = \frac{1}{4} (\mathcal{P}_{-}(t) + \mathcal{P}_{+}(t)) , \quad \mathcal{F}_{4}(t) = \frac{1}{2} (\mathcal{Q}_{+}(t) - \mathcal{Q}_{-}(t)) \]

\[ \mathcal{F}_{2}(t) = \frac{1}{2} (\mathcal{Q}_{-}(t) + \mathcal{Q}_{+}(t)) , \quad \mathcal{F}_{5}(t) = \frac{1}{2} (\mathcal{R}_{-}(t) - \mathcal{R}_{+}(t)) \]

\[ \mathcal{F}_{3}(t) = \frac{1}{2} (\mathcal{R}_{-}(t) + \mathcal{R}_{+}(t)) , \quad \mathcal{F}_{6}(t) = \frac{1}{2} (\mathcal{P}_{-}(t) - \mathcal{P}_{+}(t)) \]

and then the density operator reads

\[ \rho = (1 - \tau)^2 e^{f_{01}(t) e^{f_{31}(t) H_{3}} e^{f_{51}(t) H_{3}} e^{f_{11}(t) H_{1}} e^{f_{61}(t) H_{6}}} \]

\[ \times e^{f_{21}(t) H_{2}} e^{f_{41}(t) H_{4}} e^{r^{H_{3}} \hat{0}} \hat{0} , \]

which upon the action of the annihilation operators on the vacuum state yields

\[ \rho = \left( (1 - g_{3}(t)^{2}) e^{g_{5}(t) H_{5} e^{g_{3}(t) H_{3}} e^{g_{3}(t) H_{3}} e^{g_{3}(t) H_{3}}} e^{g_{5}(t) H_{5}} e^{g_{5}(t) H_{5}} \hat{0} \hat{0} \right) , \]

where \( g_{i}(t) = f_{i}(t) + \mathcal{F}_{i}(t) \).

It is important to note that only functions \( \mathcal{F}_{i} \) carry the information about the initial thermal state stored in the variable \( \tau \), and that in the final result only \( \mathcal{F}_{3} \) and \( \mathcal{F}_{5} \) remain relevant. What is very interesting is that these two functions in the steady state vanish, i.e. \( \lim_{t \to \infty} \mathcal{F}_{3} = 0 = \lim_{t \to \infty} \mathcal{F}_{3} \).

This means that any impact of this initial state parameter \( \tau \) is completely irrelevant to the steady state entanglement of the system on both sides the parameter regimes boundary \( |\xi| = 2\kappa \).

[1] F. Salimistraro and R. Rosso, Journal of Mathematical Physics 34, 3964 (1993).

[2] The ordering of the indices might not seem very natural, however here we had to make a choice between transparent Lie sub-algebra division (discussed later) and re-naming of the superoperators such that the ordering is \( i = 1, 2, \ldots, 15 \); in this work we chose for the former.