ZEROS OF DIRICHLET $L$-FUNCTIONS NEAR THE CRITICAL LINE

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Abstract. We prove an upper bound on the density of zeros very close to the critical line of the family of Dirichlet $L$-functions of modulus $q$ at height $T$. To do this, we derive an asymptotic for the twisted second moment of Dirichlet $L$-functions uniformly in $q$ and $t$. As a second application of the asymptotic formula we prove that, for every integer $q$, at least $38.2\%$ of zeros of the primitive Dirichlet $L$-functions of modulus $q$ lie on the critical line.

1. Introduction

The Riemann zeta-function and the Dirichlet $L$-functions are objects of great importance in number theory, and are the subject of many conjectures. One such conjecture is the Density Hypothesis.

Conjecture 1 (Density Hypothesis). Let

\[ N(\sigma, T) := \{ \rho \in \mathbb{C} : \zeta(\rho) = 0, \text{Re}(\rho) \geq \sigma, |\text{Im}(\rho)| \leq T \}, \]

\[ N(\sigma, T, \chi) := \{ \rho \in \mathbb{C} : L(\rho, \chi) = 0, \text{Re}(\rho) \geq \sigma, |\text{Im}(\rho)| \leq T \}. \]

Then,

\[ N(\sigma, T) \ll T^{2(1-\sigma)} \log(T) \]

and

\[ \sum_{\chi (\text{mod } q)}^* N(\sigma, T; \chi) \ll (qT)^{2(1-\sigma)} \log(qT) \]

for $\sigma \in [1/2, 1]$, $q \geq 2$ and $T \geq 3$. Here the implied constants are absolute.

If the Density Hypothesis is true, then it could be used as a replacement for the Riemann Hypothesis for various applications to the distribution of primes. For example, see Section 10.5 of [8]. In this paper, we consider the Density Hypothesis for Dirichlet $L$-functions.

Notable progress has been made towards proving this conjecture in various ranges of $\sigma$. While some techniques are more appropriate for proving density results closer to $\sigma = 1$ (see [12] for recent results and further references), this paper is concerned with a range of $\sigma$ very close to 1/2. To date, the best density theorem in this context is by Montgomery (Theorem 12.1 in [11]) which states that

\[ \sum_{\chi (\text{mod } q)}^* N(\sigma, T, \chi) \ll (qT)^{3(1-\sigma)} \log(qT) \]

for $\sigma \in [1/2, 4/5]$, $q \geq 1$ and $T \geq 2$.

Montgomery used zero detecting polynomials, while we shall be using moments of $L$-functions to prove the following theorem.

Theorem 1.1. For all $\kappa < 69/128$ and $\epsilon > 0$,

\[ \sum_{\chi (\text{mod } q)}^* N(\sigma, T; \chi) \ll \epsilon (qT)^{2-2\sigma} \log^5(qT) + (qT)^{1+\kappa(1-2\sigma)} \log(qT)^2 \log \log(qT) \]

for all $q \geq 2$, $T \geq q^\epsilon$, and $\sigma \in [1/2, 1]$.

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This improves Montgomery’s result in the range
\[ \frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{27 \log \log(qT)}{\log(qT)}. \]

This result is proved using an asymptotic for the second moment of the Dirichlet \( L \)-functions in both the \( q \) and the \( t \)-aspect, twisted by a mollifier. As the Dirichlet \( L \)-functions have different functional equations depending on whether their associated Dirichlet character is odd or even (i.e. odd characters satisfy \( \chi(-1) = -1 \), while even satisfy \( \chi(-1) = 1 \)), we split the sum over the characters into sums over the odds and the evens. The sum over the odd characters and the sum over the even characters are denoted as

\[ \sum_{\chi \pmod{q}}^- \quad \text{and} \quad \sum_{\chi \pmod{q}}^+ \]

respectively. For the sake of simplicity, we focus on just the even sum then address the minor differences in proof needed for the odd sum in Section 2.5. In total, there are \( \phi^*(q) \) principal characters of modulus \( q \). To distinguish the principal character of modulus \( q \), we write it as \( \chi_{0,q} \).

**Theorem 1.2.** Let \( q \) be a positive integer with \( T \gg q^\epsilon \). Let \( \psi(t) \) be a smooth real valued function supported on \([1, 2]\) with \( \psi^{(j)}(t) \ll T^{-j} \). Let \( \alpha, \beta \in \mathbb{C} \) satisfy \( \alpha, \beta \ll \log(qT)/\log(qT) \).

Suppose that \( 1/2 < \kappa < 1/2 + 1/66 \). For all \( n \in \mathbb{N}, \alpha_n, \beta_n \in \mathbb{C} \) such that \( \alpha_n, \beta_n \ll n^\epsilon \),

\[
\frac{1}{\phi^*(q)T} \int \sum_{\chi \pmod{q}}^+ L(1/2 + \alpha + it, \chi)L(1/2 + \beta - it, \bar{\chi}) \sum_{\substack{a,b \leq (qT)^\kappa \\Gamma(1/2+\beta b,1/2-it)}} \frac{\alpha_a \beta_b \chi(a) \bar{\chi(b)}}{\Gamma(1/2+\beta b,1/2-it)} \psi\left(\frac{t}{T}\right) dt
= \frac{\psi(0)}{2} L(1 + \alpha + \beta, \chi_{0,q}) \sum_{\substack{a,b \leq (qT)^\kappa \\Gamma(1/2+\beta b,1/2-it)}} \frac{\alpha_ad \beta_bd}{\alpha^+ \beta_1^+ + \alpha_d} + \frac{1}{2T} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} \Gamma\left(1/2-\beta-it\right) \psi\left(\frac{t}{T}\right) dt + O_\epsilon \left(||qT||^{-\epsilon}\right).
\]

By introducing the small shifts \( \alpha \) and \( \beta \), we not only derive a more general result, but calculating the second moment (by letting the shifts tend to zero and taking the limit) is actually easier. In the case that \( \alpha = -\beta \), then the above result should be considered as a limit.

A natural choice of mollifier (and one that we shall use to prove Theorem 1.3) is \( M(s, \chi) = \sum_{n} \frac{\mu(n)f(n)^\chi(n)}{n^s} \) where \( f(n) \) is some smoothing function. In this case, it is possible to exploit the properties of the Möbius function to get a smaller error term.

**Theorem 1.3.** Suppose that the conditions of Theorem 1.2 hold, with the added assumption that \( \alpha(n) = \mu(n)f(n) \) for some smooth bounded function \( f(x) \) with \( f'(x) \ll x^{-1+\epsilon} \). Then the same result holds for \( 1/2 < \kappa < 1/2 + 5/128 \).

It is worth noting that as \( T \) grows arbitrarily large compared to \( q \) then by using similar techniques as in [4], \( \kappa \) can be increased up to a limit of \( \kappa < 4/7 \approx 0.571 \).

Asymptotics for twisted second moments that break the half barrier are not new, as Bettin, Chandee and Radziwiłł achieved this in the \( t \)-aspect for the Riemann zeta-function in [2] and Bui, Pratt, Robles and Zaharescu in the \( q \)-aspect in [3]. However, finding an asymptotic that
is uniform in both has its own challenges, mostly due to terms that are negligible in the \( q \)-aspect no longer being negligible when the \( t \)-aspect is introduced. Previous results in just the \( q \) aspect only work when \( q \) is prime, while this result applies to all positive integers \( q \).

We demonstrate a second application of Theorem 1.3 using it to prove a result on the proportion of simple zeros on the critical line. Let \( N(T, \chi) \) denote the number of zeros \( \rho = \beta + i\gamma \) of the Dirichlet \( L \)-function \( L(s, \chi) \) for a character \( \chi \) of conductor \( q \), with \( 0 < \gamma < T \). Let \( N_0(T, \chi) \) denote the number of these zeros that are simple with \( \beta = 1/2 \).

**Theorem 1.4.** Define
\[
N(T, q) = \frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* N(T, \chi), \quad \text{and} \quad N_0(T, q) = \frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* N_0(T, \chi).
\]
Then for \( \kappa < 1/2 + 5/128 \) we have
\[
\frac{N_0(T, q)}{N(T, q)} \geq 1 - \frac{1}{R} \log(c(P, Q, R)) + o(1)
\]
where
\[
c(P, Q, R) = 1 + \frac{1}{\kappa} \int_0^1 \int_0^1 e^{2Rv} \left( \frac{d}{dx} e^{Rez} P(x + u) Q(v + \kappa x) \right)_{x=0}^2 \, du \, dv,
\]
\( R > 0 \) is a positive constant, \( P(x) \) is polynomial with \( P(0) = 0, P(1) = 1 \), and \( Q(x) \) is a real linear polynomial with \( Q(0) = 1 \).

By choosing \( Q \) to be a non-linear polynomial, we would obtain a lower bound on the number of zeros on the critical line, simple or otherwise. In fact it is conjectured that all non-trivial zeros are simple. By choosing \( R, P, \) and \( Q \) optimally, we arrive at the following corollary.

**Corollary 1.1.**
\[
\liminf_{qT \to \infty} \frac{N_0(T, q)}{N(T, q)} \geq 0.382.
\]
Informally, this means that for integer \( q \) at least 38.2% of zeros up to a large height \( T \) of the primitive Dirichlet \( L \)-functions of modulus \( q \) lie on the critical line as we vary \( q \) such that \( \log(q) \ll \log(T) \).

Theorem 1.4 comes from applying Levinson’s method to Theorem 1.3. Levinson’s method is an elegant and widely used technique for determining the proportion of critical zeros of an \( L \)-function. See [5] for a nice demonstration of the method, and [16] for an elegant application of the method to the Riemann zeta-function.

Levinson’s method has been used by Conrey in just the \( t \)-aspect in [4] to show that at least 40.7% non-trivial zeros of the Riemann zeta-function are critical (this has since been improved to 41.7% in [13]), while in [5] Conrey, Iwaniec, and Soundararajan consider the \( q \)-aspect, averaged over \( q \leq Q \) to conclude that at least 56% of low-lying zeros lie on the critical line (see also [14]). In comparison, our result is uniform in \( q \) and \( t \), and does not require averaging over \( q \leq Q \).

We begin by proving Theorem 1.2 and Theorem 1.3 in Section 2. Then we focus on the applications and prove Theorem 1.1 in Section 3 and Theorem 1.4 in Section 4.

Throughout this paper we shall use the convention that \( \epsilon \) is an arbitrarily small positive constant that may change value between lines.

## 2. The Twisted Second Moment

### 2.1. Initial Manipulations
Lemma 2.1 (Approximate Functional Equation). Let $\chi$ be an even primitive character. Then we have the approximate functional equation

$$L\left(\frac{1}{2} + \alpha + it, \chi\right) L\left(\frac{1}{2} + \beta - it, \chi\bar{\chi}\right) = \sum_{m,n \geq 1} \frac{\chi(m)\chi(n)}{m^{1/2+\alpha + it}n^{1/2+\beta - it}} V_+ \left(\frac{\pi mn}{q}, t\right)$$

$$+ \left(\frac{q}{\pi}\right)^{-\alpha - \beta} \sum_{m,n \geq 1} \frac{\chi(m)\chi(n)}{m^{1/2-\beta + it}n^{1/2-\alpha + it}} V_- \left(\frac{\pi mn}{q}, t\right)$$

where

$$V_\pm(x, t) = \frac{1}{2\pi i} \int_{(c)} X_\pm(s, t) x^{-s} ds \quad \text{s}$$

and

$$X_\pm(s, t) = G(s) \frac{\Gamma\left(\frac{1/2+\alpha + it + s}{2}\right)\Gamma\left(\frac{1/2+\beta - it + s}{2}\right)}{\Gamma\left(\frac{1/2+\alpha + it}{2}\right)\Gamma\left(\frac{1/2+\beta - it}{2}\right)}$$

and $G(s)$ is a function that is even, entire, of rapid decay in any fixed strip and with $G(0) = 1$, $G(\pm(\alpha + \beta)/2) = 0$.

The proof is standard. For example, see Theorem 5.3 of [8].

Lemma 2.2. For all $i, j, C \geq 0$,

$$x^{\alpha + i\beta} \frac{\partial^{i+j}}{\partial x^i \partial t^j} V_+(x, t) \ll_{i, j, C} (1 + |x/t|)^{-C}$$

(1)

$$x^{\alpha + i\beta} \frac{\partial^{i+j}}{\partial x^i \partial t^j} V_-(x, t) \ll_{i, j, C} t^{-\Re(\alpha + \beta)} (1 + |x/t|)^{-C}.$$  

(2)

The proof is a simple application of Stirling’s approximation applied to

$$\frac{\Gamma\left(\frac{1/2+\alpha + it + s}{2}\right)\Gamma\left(\frac{1/2+\beta - it + s}{2}\right)}{\Gamma\left(\frac{1/2+\alpha + it}{2}\right)\Gamma\left(\frac{1/2+\beta - it}{2}\right)} = t^s (1 + O((1 + |t|)^{-1})).$$

Lemma 2.3 (Orthogonality). Suppose that $(m, q) = 1$, then

$$\sum_{\chi \pmod{q}}^+ \chi(m) = \frac{1}{2} \sum_{\substack{u \equiv q \pmod{q} \atop m \equiv 1 \pmod{w}}} \mu(u) \phi(w).$$

The proof of this result is standard. See for example, (3.1) and (3.2) of [9].

Applying Lemma 2.3 to the approximate functional equation gives

$$\sum_{\chi \pmod{q}}^+ L(1/2 + \alpha + it, \chi) L(1/2 + \beta - it, \chi) \sum_{a,b \leq (qT)^{\alpha}} \frac{\alpha_a \beta_b \chi(a)\chi(b)}{a^{1/2+\alpha}b^{1/2-\beta}} =$$

$$\frac{1}{2} \sum_{\substack{w \mid q \atop \chi \pmod{w}}} \mu\left(\frac{q}{w}\right) \phi(w) \left(\sum_{a,b,m,n \atop am \equiv \pm bn \pmod{q} \atop (amn, q) = 1} \frac{\alpha_a \beta_b}{(ab)^{1/2+m^{1/2+\alpha}n^{1/2+\beta}}} \frac{\partial}{\partial x^i \partial t^j} \left(\frac{bn}{am}\right)^{it} V_+ \left(\frac{\pi mn}{q}, t\right) \psi \left(\frac{t}{T}\right) dt\right)$$

$$+ \left(\frac{q}{\pi}\right)^{-\alpha - \beta} \sum_{a,b,m,n \atop am \equiv \pm bn \pmod{w} \atop (abmn, q) = 1} \frac{\alpha_a \beta_b}{(ab)^{1/2-m^{1/2-\beta}n^{1/2-\alpha}}} \frac{\partial}{\partial x^i \partial t^j} \left(\frac{bn}{am}\right)^{it} V_- \left(\frac{\pi mn}{q}, t\right) \psi \left(\frac{t}{T}\right) dt\right).$$
\[
\int \sum_{\chi \pmod{q}}^* \ L(1/2 + \alpha + it, \chi)L(1/2 + \beta - it, \bar{\chi}) \ \sum_{a, b \leq (q T)^s \atop a, b \neq (q T)^s} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2 + \alpha} b^{1/2 + \beta}} \psi \left( \frac{t}{T} \right) dt = D^+ + O^+ + \left( \frac{q}{\pi} \right)^{-\alpha - \beta} (D^- + O^-)
\]

where

\[
D^+ := \frac{\phi^+(q)}{2} \int \sum_{a, b \leq (q T)^s} \frac{\alpha_a \beta_b}{(ab)^{1/2 + \alpha} m^{1/2 + \alpha} n^{1/2 + \beta}} V_+ \left( \frac{\pi mn}{q}, t \right) \psi \left( \frac{t}{T} \right) dt
\]

and

\[
O^+ := \frac{1}{2} \sum_{w \mid q} \mu \left( \frac{q}{w} \right) \phi(w) \int \sum_{a, b, m, n \atop \chi \pmod{q}} \frac{\alpha_a \beta_b}{(ab)^{1/2 + \alpha} m^{1/2 + \alpha} n^{1/2 + \beta}} V_+ \left( \frac{\pi mn}{q}, t \right) \psi \left( \frac{t}{T} \right) dt.
\]

\(D^-\) and \(O^-\) are obtained by from \(D^+\) and \(O^+\) by substituting \(\alpha, \beta \rightarrow -\beta, -\alpha\) and replacing \(V_+\) by \(V_-\). As the \(D^+\) and \(O^+\) cases are almost identical to the \(D^-\) and \(O^-\) cases, we shall only demonstrate the former.

2.1.1. The Diagonals. As the diagonals are made up of sums over the condition \(am = bn\), we may write \(a, b, m, n = ad, bd, bn', an'\) with \((a, b) = 1\) and \((n', q) = 1\). Hence, by relabelling, \(D^+\) is

\[
\int \sum_{a, b \leq (q T)^s} \frac{\alpha_{ad} \beta_{bd}}{a^{1/2 + \alpha} b^{1/2 + \alpha} m^{1/2 + \alpha} n^{1/2 + \beta}} V_+ \left( \frac{\pi abm^2}{q}, t \right) \psi \left( \frac{t}{T} \right) dt
\]

\[
= \frac{\phi^+(q)}{2} \sum_{a, b \leq (q T)^s} \sum_{n \geq 1 \atop (a, b) = 1, (q, n) = 1} \frac{\alpha_{ad} \beta_{bd}}{a^{1/2 + \alpha} b^{1/2 + \alpha} m^{1/2 + \alpha} n^{1/2 + \beta}} \int \int_{(2)} X_+(s, t) \left( \frac{q}{abm^2 \pi} \right)^s \psi \left( \frac{t}{T} \right) ds \ dt
\]

\[
= \frac{\phi^+(q)}{2} \sum_{a, b \leq (q T)^s} \sum_{n \geq 1 \atop (a, b) = 1, (q, n) = 1} \frac{\alpha_{ad} \beta_{bd}}{a^{1/2 + \alpha} b^{1/2 + \alpha} m^{1/2 + \alpha} n^{1/2 + \beta}} \int \int_{(2)} X_+(s, t) \left( \frac{q}{ab \pi} \right)^s L(1 + \alpha + \beta + 2s, \chi_{0, q}) \psi \left( \frac{t}{T} \right) ds \ dt
\]

Note that we chose the contour of integration in the \(V\)-function to be \(Re(s) = 2\) at first so that the sum over \(n\) converges to the \(L\)-function, and then moved the contour back to \(Re(s) = \epsilon\) with the pole at \(s = -(\alpha + \beta)/2\) being cancelled by the zero coming from \(X_+(s, t)\).

Similarly

\[
D^- = \frac{\phi^+(q)}{2} \sum_{a, b, d \leq (q T)^s} \frac{\alpha_{ad} \beta_{bd}}{a^{1/2 - \alpha} b^{1/2 - \alpha} m^{1/2 - \alpha} n^{1/2 + \beta}} \int \int_{(2)} X_-(s, t) \left( \frac{q}{ab \pi} \right)^s L(1 - \alpha - \beta + 2s, \chi_{0, q}) \psi \left( \frac{t}{T} \right) ds \ dt.
\]
2.1.2. Off-diagonals. The remaining terms (i.e. when \( am \neq bn \)) are the off-diagonals. The following lemma will allow us to show that the terms in the sum with \( am \) and \( bn \) sufficiently far away from each other will contribute a negligible amount.

**Lemma 2.4.** Suppose that \( \psi : [1, 2] \to \mathbb{R} \) is a smooth function with derivative \( \frac{dK}{dt} \psi(t) \ll_{K,T} T^\varepsilon \). Then,

\[
\int V_\pm \left( \frac{\pi mn}{q}, t \right) \left( \frac{bn}{am} \right)^{it} \psi \left( \frac{t}{T} \right) dt \ll_K |\log(bn/am)|^{-K} T^{1-K+\varepsilon}
\]

and hence this integral is vanishingly small unless

\[
1 - T^{-1} < \frac{bn}{am} < 1 + T^{-1}.
\]

**Proof.** By (1), for \( t \in [T, 2T] \)

\[
\frac{\partial K}{\partial t} V_\pm(x,t) \psi \left( \frac{t}{T} \right) dt \ll_{K,C} T^{-K+\varepsilon} \left( 1 + \frac{|x|}{T} \right)^{-C}
\]

for all \( C > 0 \). This implies

\[
\int V_\pm \left( \frac{\pi mn}{q}, t \right) \left( \frac{bn}{am} \right)^{it} \psi \left( \frac{t}{T} \right) dt \ll_K |\log(bn/am)|^{-K} T^{1-K+\varepsilon}.
\]

By taking \( K \to \infty \), this becomes negligibly small unless \( |\log(bn/am)| \ll T^{-1} \). Taking the Taylor expansion of \( \log(1+x) = x + O(x^2) \) for \( |x| < 1 \) to see that the \( t \)-integral is vanishingly small unless

\[
1 - T^{-1} < \frac{bn}{am} < 1 + T^{-1}.
\]

To help restrict to these non-negligible cases, we introduce a dyadic partition of unity to the sums over \( m \) and \( n \): let \( W \) be a smooth non-negative function supported in \([1,2]\) such that

\[
\sum_M W \left( \frac{m}{M} \right) = 1,
\]

where \( M \) runs over a sequence of real numbers with \( |\{M : X^{-1} \leq M \leq X\}| \ll \log X \). By the rapid decay of \( V_\pm \), in (1) and (2) we may assume that \( MN \ll (qT)^{1+\varepsilon} \). We also split up the mollifying coefficients \( \alpha_a, \beta_n \) dyadically, supposing that \( \alpha_n(A) \) is supported on \( n \in [A, 2A] \) and \( \beta_n(B) \) is supported on \( n \in [B, 2B] \) i.e. \( \alpha_a = \sum_A \alpha_a(A) \) and by the assumptions in Theorem 1.2 \( A, B \ll (qT)^\varepsilon \). In the next section, we extract the main term from the off-diagonal terms, and bound the rest into an error term.

2.2. Main Propositions. When the mollifier is short enough, a trivial bound is sufficient to bound the contribution from the off-diagonal term. However, to break the half-barrier, a more sophisticated method is needed as the off-diagonals begin to contribute to the main term. The trivial bound shall be of use later on in the proof.

**Lemma 2.5** (Trivial bound). For pairwise co-prime \( f, h, w|q \) and \( \forall a, b \alpha_a, \beta_b \ll (qT)^\varepsilon \)

\[
\sum_{amf=\pm bnw+wr \atop a,b,m,n,\tau \ll A,B} \alpha_a \beta_b \ll \frac{AR}{w} \left( 1 + \frac{M}{f} \right) (qT(AM+R))^\varepsilon
\]

**Proof.** We bound this sum trivially by summing over \( a \) and \( r \). Then we sum over \( m \equiv af \pmod{h} \), of which there are \( \ll 1 + M/fh \) possible values of \( m \). Then we bound the sums over \( b \) and \( n \) using the divisor bound.

\( \square \)
The next proposition shows how the off-diagonals contribute to the main term.

**Proposition 2.1.** Let $T \gg q^r$ for a positive integer $q$, and $w|q$. Let $\psi(t)$ and $W(x)$ be smooth real valued functions supported on $[1, 2]$ such that for all $j \geq 0 \psi^{(j)}(t) \ll T^c$ and $W^{(j)}(x) \ll_{\epsilon} (qT)^{\epsilon}$. Let $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha, \beta \ll \log \log(qT)/\log(qT)$. Suppose that $1/2 < \kappa < 1/2 + 1/66$. Suppose that for positive constants $1 \leq A, B \leq (qT)^{1/2 + \epsilon}$ with $AM \asymp BN$, we have complex sequences $\alpha_a(A), \beta_b(B) \in \mathbb{C}$ with support on $[A, 2A]$ and $[B, 2B]$ respectively such that $\alpha_a, \beta_b \ll n^\epsilon$.

Define $S_w^+(A, B, M, N)$ as

$$
\int \sum_{a,b,m,n} \frac{\alpha_a(A)\beta_b(B)}{(ab)^{1/2}m^{1/2+\alpha}n^{1/2+\beta}} \left( \frac{bn}{am} \right)^{it} W\left( \frac{m}{M} \right) W\left( \frac{n}{N} \right) V_+ \left( \frac{\pi mn}{q}, t \right) \psi \left( \frac{t}{T} \right) dt
$$

then

$$
\frac{1}{\phi^*(q)T} \sum_{w|q} \mu \left( \frac{q}{w} \right) \phi(w) S_w^+(A, B, M, N) = \int (\frac{q}{\pi})^s \left( \frac{abx - wr}{b} \right)^{-(1/2+s+\beta-it)} \left( \frac{bx}{M} \right)^{-(1/2+\alpha+s+it)} W\left( \frac{bx}{M} \right) W\left( \frac{wr - abx}{bN} \right) \psi \left( \frac{t}{T} \right) \frac{dz ds}{s} dt
$$

and

$$
\frac{1}{\phi^*(q)T} \sum_{w|q} \mu \left( \frac{q}{w} \right) \phi(w) S_w^-(A, B, M, N) = \int (\frac{q}{\pi})^s \left( \frac{wr - abx}{b} \right)^{-(1/2+s+\beta-it)} \left( \frac{bx}{M} \right)^{-(1/2+\alpha+s+it)} W\left( \frac{bx}{M} \right) W\left( \frac{wr - abx}{bN} \right) \psi \left( \frac{t}{T} \right) \frac{dz ds}{s} dt
$$

**Proof.** We begin by writing the $am \equiv \pm bn \pmod{w}$ condition as $am = \pm bn + wr$. As $am \not\equiv bn$, $r$ must be non-zero, and by Lemma (2.1) we may assume that $|r| \leq 2AMw^{-1}T^c - 1$, so we sum over $0 < |r| \leq R/w$ where $R := 2AMT^{-1+c}$. We remove the $mn, q = 1$ condition as follows: for any smooth function $F(a, b, m, n)$ for a fixed $a, b, w$ and $r$,

$$
\sum_{am = \pm bn + wr \pmod{abmn,q} = 1} F(a, b, m, n) = \sum_{f|q} \mu(f) \sum_{am = \pm bn + wr \pmod{abmn,q} = 1} F(a, b, m, n)
$$

$$
= \sum_{f|q} \mu(f) \sum_{am = \pm wr \pmod{bf}} F\left( a, b, m, \frac{\pm (am - wr)}{b} \right)
$$

Note that if $(f, rw) > 1$ then the sum is empty as then $(am, q) > 1$. Given this, we can then relax the condition that $(m, q) = 1$ to $(m, q/f) = 1$, as $m$ must be coprime to $f$ by the residue condition $am \equiv wr \pmod{bf}$. Suppose for contradiction that $p|(f, q/f)$ then $p^2|q$ and hence...
as \( q/w \) is square free it must be the case that \( p|w \). Hence \( p \) can not divide \( f \), so \( (f, q/f) = 1 \).

So

\[
\sum_{f|q, (f, w) = 1} \mu(f) \sum_{am \equiv wr \pmod{bf}, (amq, g) = 1} \mu(f) = \sum_{f|q, (f, w) = 1} \mu(f) \sum_{am \equiv wr \pmod{bf}, (amq, g) = 1} \mu(f) = \sum_{f|q, (f, w) = 1} \mu(f) \sum_{u \in \left(\frac{z}{\nu(f)w}\right)^*} \sum_{m \equiv au \pmod{q/f}} \sum_{(abq, g) = 1}.
\]

Let \( x_1 \equiv aq/f \pmod{bf} \) and \( x_2 \equiv abf \pmod{q/f} \), so that by appealing to the Chinese remainder theorem \( m \equiv \bar{u}wr \pmod{bf} \) and \( m \equiv \bar{u}w \pmod{q/f} \) iff \( m \equiv wrx_1q/f + ux_2bf \pmod{bf} \).

Then we apply Poisson summation to find that

\[
\sum_{am = \pm bn + wr, (abmn, q) = 1} F(a, b, m, n) = \sum_{f|q} \mu(f) \sum_{u \in \left(\frac{z}{\nu(f)w}\right)^*} \sum_{g \in \mathbb{Z}} e\left(-gwrx_1q/f - quwrbf\right)\int F(a, b, bx, \mp (abx - wr)/b) e(gx/q) dx
\]

\[
= \sum_{f|q} \mu(f) \sum_{u \in \left(\frac{z}{\nu(f)w}\right)^*} \sum_{g \in \mathbb{Z}} e\left(-gwrq/f - quwbf\right)\int F(a, b, bx, \mp (abx - wr)/b) e(gx/q) dx
\]

Summing over \( u \) gives a Ramanujan sum i.e.

\[
\sum_{f|q} \mu(f) \sum_{g \in \mathbb{Z}} e\left(-gwrq/f - quwbf\right) c_g(f)/q e(gx/q) dx
\]

where \( c_g(n) = \sum_{k|q} \mu(n/k)k \). The main term comes from when \( g = 0 \) i.e.

\[
\sum_{f|q} \mu(f) \phi(f)/q e(gx/q) dx
\]

For the contributions when \( g \neq 0 \), we expand out \( c_g(f)/q \) to get

\[
\sum_{f|q} \mu(f) \sum_{k|q} \frac{q}{k} e\left(-gkwqaq/f - quwbf\right) k \int F(a, b, bx, \mp (abx - wr)/b) e(gkx/q) dx.
\]

As \( k|q \), write \( q = kf \) to get

\[
\sum_{f|q} \mu(f) \sum_{h|q} \mu(h) e\left(-gwrhbf\right) h \int F(a, b, bx, \mp (abx - wr)/b) e(gxf/h) dx.
\]

We will be able to bound the size of \( g \) by integrating by parts \( j \) times i.e.

\[
\int (bx)^{-(1/2+a+s+it)} W\left(\frac{bx}{M}\right) W\left(\frac{b(N)}{M}\right) e\left(\frac{gx}{fM}\right) dx
\]

is

\[
O_{e,j}\left((qT)^{dM^{1/2}} \left|\frac{BfM}{gM}\right|^j\right)
\]
for any fixed \( j \geq 0 \). So we may restrict the sum to \( 0 < |g| \leq Gfh/d \) where \( G = \frac{R}{M} T^\epsilon \). Hence

\[
\sum_{w|q} \mu \left( \frac{w}{q} \right) \phi(w) S_w^+(A, B, M, N)
\]

is equal to

\[
\sum_{w|q} \mu \left( \frac{w}{q} \right) \phi(w) \sum_{f|q} \sum_{(f,rw)} \mu(f) \frac{\phi(q/f)}{q} \left( \mathcal{M}_{w,f}^+(A, B, M, N) + \mathcal{M}_{w,f}^-(A, B, M, N) \right) + \mathcal{E}
\]

where

\[
\mathcal{E} = \sum_{w|q} \mu \left( \frac{w}{q} \right) \phi(w) \sum_{f|q} \sum_{h|(q/f)} \mu(h) \mathcal{E}_{w,f,h}(A, B, M, N)
\]

with

\[
\mathcal{E}_{w,f,h}(A, B, M, N) := \sum_{d \geq 1} \sum_{(d,q) = 1} \sum_{0 \leq |r| \leq R/wd} \sum_{(a,b) = 1} \sum_{(ab,q) = 1} \alpha_{da}(A) \beta_{db}(B) \left( \frac{b}{a} \right)^{it} e \left( \frac{-gwr ah}{bf} \right) 2\pi ihf
\]

\[
\int_{(\epsilon)} \int X_+(s,t) \left( \frac{q}{\pi} \right)^s \left( \frac{wr - abx}{b} \right)^{- \left( \frac{1}{2} + \frac{s}{2} + \beta - it \right)} (bx)^{- \left( \frac{1}{2} + \alpha + \frac{s}{2} + it \right)} W \left( \frac{bx}{M} \right) W \left( \frac{wr - abx}{bN} \right) \psi \left( \frac{t}{T} \right) e \left( \frac{gx}{hf} \right) ds dt.
\]

For each \( w, h \) and \( f \), we treat the error term differently depending on the size of \( hf \). In short, when \( hf \) is large compared to \( qT \), then the contribution to the error term (and the main term) can be trivially bounded to be small enough to be absorbed into the error term. When \( hf \) is small, we need a more sophisticated method which is an adaptation of Bettin and Chandee’s Theorem 1 in [1]. To this aim, define

\[
\gamma := \frac{\log(hf)}{\log(q)}.
\]

The contribution to the main term and the error term for a fixed \( w, f \) and \( h \) can be bounded trivially by reversing the Poisson summation to get the contribution

\[
\frac{\phi(w)}{\phi^*(q)T} \sum_{d \geq 1} \sum_{(d,q) = 1} \sum_{0 \leq |r| \leq R/wd} \sum_{(a,b) = 1} \sum_{(ab,q) = 1} \alpha_a(A) \beta_b(B) \left( \frac{bnf}{amh} \right)^{it} W \left( \frac{mh}{M} \right) W \left( \frac{nf}{N} \right) V_+ \left( \frac{\pi mhf t}{q}, t \right) \psi \left( \frac{t}{T} \right) dt.
\]

Using the trivial bound Lemma 2.5 we see that

\[
\frac{\mathcal{E}_{w,f,h}(A, B, M, N)}{\phi^*(q)T} \ll \epsilon \left( \frac{w AR}{q w} \left( 1 + \frac{M}{fh} \right) (qT(AM + R))^\epsilon \right)
\]

\[
\ll \epsilon \left( qT \right)^\epsilon \left( \frac{A}{qT} + \frac{AM}{fh(qT)} \right).
\]

Hence for

\[
\kappa \in \left( \frac{1}{2}, \frac{17}{33} \right), \quad \frac{\log(T)}{\log(q)} < \frac{2\gamma + 1 - 2\kappa}{2\kappa - 1}
\]
we have
\[
\frac{\mathcal{E}_{w,f,h}(A, B, M, N)\phi(w)}{\phi^*(q)T} \ll_{\epsilon} (qT)^{\kappa+\epsilon-1} + q^{\kappa-\frac{1}{2}-\gamma+\epsilon} T^{\kappa-\frac{1}{2}+\epsilon}
\]  
(5)

which is
\[
O_{\epsilon} ((qT)^{-\epsilon}) .
\]

When the trivial bound will not suffice, we use Mellin inversions to separate the variables in (4) to reduce to finding a bound for

\[
\frac{\mathcal{E}_{w,f,h}(A, B, M, N)\phi(w)}{\phi^*(q)T} \ll_{\epsilon} \sum_{d \geq 1} \frac{1}{(ABMN)^{1/2}} \frac{w(qT)^{\epsilon-1}}{hf}
\]

\[
\times \left| \int \int_{x \geq dM/B} \sum_{a \leq A/d} \sum_{b \leq B/d} \sum_{0 < |r| < R/\sqrt{d}} \sum_{(a,q)=1} \sum_{(b,q)=1} \sum_{0 < |g| \leq G/h/d} \alpha_{ad} \beta_{bd} \nu_{rg} \epsilon \left( -\frac{gwrh}{bf} + \frac{gx}{hf} \right) dxdt \right|
\]

where we may assume without loss of generality that \( f \leq h \), otherwise we take Poisson summation modulo \( ah \) instead of \( bf \). We may also factor out \((w,h)\) from both \( w \) and \( h \), so that we can assume \( w,h \) and \( f \) are all pairwise co-prime, and all divide \( q \), hence \( whf \leq q \).

By an adapted theorem of Bettin and Chandee from (1), we arrive at the conclusion that the error

\[
\frac{\mathcal{E}_{w,f,h}(A, B, M, N)\phi(w)}{\phi^*(q)T}
\]

is at most

\[
\ll_{\epsilon} \sum_{d \geq 1} \frac{1}{(ABMN)^{1/2}} \frac{Md}{B} (qT)^{\epsilon} \frac{w}{qf} \frac{ABd^{-2}}{B/d} \left( 1 + \frac{wABfbd^2}{fhdwT^2} \right)^{1/4}
\]

\[
\times \left( h^{1/4} f^{1/2} \frac{ABhf}{wT^2} (A/d)(B/d)^{3/4} + h^{3/4} f^{1/2} (ABhf) wT^2 (A/d)^{1/2} (B/d)^{5/4}
\]

\[
+1^{2/5} f^{6/5} (ABhf) wT^2 (A/d)^{6/5} (B/d)^{1/10} + h^{1/5} f^{6/5} (A/d)^{2/5} (A/d)^{6/5} (B/d)^{7/10}
\]

\[
+ h^{1/2} f^{3/5} \frac{ABhf}{wT^2} \left( A/d \right)^{5/10} (B/d)^{1/10} + h^{1/2} (A/d)(B/d)^{7/4} \right)^{1/2}
\]

\[
\ll_{\epsilon} (qT)^{\epsilon} \left( \frac{h^{1/8} f^{1/4} AB^{7/8}}{qT} + \frac{h^{3/8} f^{1/4} A^{3/8} B^{9/8}}{qT} + \frac{h^{-1/5} f^{3/5} A^{11/10} B^{11/20}}{qT}
\]

\[
+ \frac{h^{-1/5} f^{3/10} w^{3/10} A^{4/5} B^{11/20}}{qT^{7/10}} + \frac{h^{1/10} f^{3/20} w^{3/20} A^{13/20} B^{11/20}}{qT^{17/20}} + \frac{h^{1/4} A^{1/2} B^{11/8}}{qT} \right)
\]

as \( AM \asymp BN \).

For the first, second, and sixth terms substitute in \( fh = q^7 \) and \( f \leq h \Rightarrow f \leq q^{7/2} \). For the third term, write \( h^{-1/5} \leq f^{-1/5} \) then substitute in \( f^{-2/5} \leq q^{7/5} \). For the fourth and fifth terms, we use the fact that \( whf \leq q \). Hence the error is

\[
\ll_{\epsilon} (qT)^{\epsilon} \left( (qT)^{15\alpha-\frac{5}{8}} q^{\frac{1}{20}} + (qT)^{15\alpha-\frac{3}{8}} q^{\frac{1}{20}} + (qT)^{15\alpha-\frac{20}{25}} q^{\frac{1}{20}} + (qT)^{17\nu+\frac{7}{20}} + (qT)^{17\nu+\frac{17}{20}} + (qT)^{17\nu+\frac{17}{20}} .
\]

The first and sixth terms are smaller than the second, and the fourth is smaller than the fifth so

\[
\frac{\mathcal{E}_{w,f,h}(A, B, M, N)\phi(w)}{\phi^*(q)T} \ll_{\epsilon} (qT)^{15\alpha-\frac{5}{8}} + (qT)^{15\alpha-\frac{3}{8}} + (qT)^{15\alpha-\frac{20}{25}} + (qT)^{17\nu+\frac{7}{20}} + (qT)^{17\nu+\frac{17}{20}} + (qT)^{17\nu+\frac{17}{20}} .
\]

\ll_{\epsilon} (qT)^{-\epsilon}
for
\[ \kappa \in \left( \frac{1}{2}, \frac{17}{33} \right) \text{ and } \frac{\log(T)}{\log(q)} \geq \frac{2\gamma + 1 - 2\kappa}{2\kappa - 1}. \]

2.3. Proof of Theorem \([1,2]\) In this section, we manipulate the main terms from the off-diagonals into a convenient form, then combine them with the diagonal terms.

We focus on the \( \mathcal{M}^+ \) terms first. Writing \( W \) in terms of its Mellin transform, we see that
\[
\mathcal{M}_{w,f}^+(A, B, M, N) = \frac{1}{(2\pi i)^3} \int \int \int X_+(s, t) \tilde{W}(u) \tilde{W}(v) \left( \frac{q}{\pi} \right)^s \sum_{r \neq 0} \sum_{d, a, b, c, d, a, b}^{(r, f) = 1} (a, b, d, c, a, b) \int x^{-1/2+\alpha+it+s+u} (x - wr/ab)^{-1/2+\beta-it+s+v} dx \psi \left( \frac{t}{T} \right) du \frac{ds}{s} dt.
\]

Now, we calculate the \( x \) integral. If \( r > 0 \) then the integral over \( x \) is restricted to \( x > wr/ab \) and if \( r \) is negative then we have \( x > 0 \). For absolute convergence, if \( r > 0 \), we impose the condition
\[ \text{Re}(\alpha + \beta + 2s + u + v) > 0, \text{ Re}(\beta + s + v) < 1/2 \]
and if \( r < 0 \) we impose the condition
\[ \text{Re}(\alpha + \beta + 2s + u + v) > 0, \text{ Re}(\alpha + s + u) < 1/2. \]

Under these assumptions, the \( x \)-integral is equal to (see for example 17.43.21 and 17.43.22 of \([6]\))
\[
\left( \frac{w}{r} \right)^{-(\alpha + \beta + 2s + u + v)} \Gamma(\alpha + \beta + 2s + u + v) \Gamma(1/2 - \beta + it - s - v) \left( \frac{1/2 + \beta + it + s + v}{\Gamma(1/2 + \beta - it + s + v)} \right) \Gamma(1/2 - \alpha - it - s - u) \Gamma(1/2 + \alpha + it + s + u) \left( \frac{1/2 - \alpha - it - s - u}{\Gamma(1/2 - \alpha - it - s - v)} \right) \text{ if } r > 0
\]
and hence
\[
\mathcal{M}_{w,f}^+(A, B, M, N) = \frac{1}{(2\pi i)^3} \int \int \int X_+(s, t) \tilde{W}(u) \tilde{W}(v) \left( \frac{q}{\pi} \right)^s w^{-(\alpha + \beta + 2s + u + v)} a^{s+u-1+\alpha+2s+v-1+\beta} M^{\alpha} N^{\beta} H_+(s) r^{-(\alpha + \beta + 2s + u + v)} du \frac{ds}{s}.
\]

where
\[ H_+(s) = \Gamma(\alpha + \beta + 2s + u + v) \left( \frac{1/2 - \beta + it - s - v}{\Gamma(1/2 + \alpha + it + s + u)} \right) + \left( \frac{1/2 + \alpha + it - s - u}{\Gamma(1/2 - \beta + it - s + v)} \right). \]

In the \( \mathcal{M}^- \) cases, due to the extra minus sign in the \( x \)-integral, we arrive at the same result but with \( H_+(s) \) replaced by
\[ H_-(s) := \frac{\Gamma(1/2 - \alpha - it - s - u) \Gamma(1/2 - \beta + it - s - v)}{\Gamma(1 - \alpha - \beta - 2s - u - v)}. \]

Writing \( H(s) := H_+(s) + H_-(s) \), and summing over \( A, B \leq (qT)^\kappa \) in the dyadic decomposition allows us to write
\[
\frac{O^+}{\phi^*(q)T} = \frac{1}{\phi^*(q)T} \sum_{M, N} O^+(M, N) + O_\kappa ((qT)^{-\kappa}) \quad (6)
\]
\[ O^+(M, N) := \frac{1}{2} \sum_{w|q} \mu \left( \frac{q}{w} \right) \phi(w) \sum_{f/q} \mu(f) \frac{\phi(q/f)}{q} \frac{1}{(2\pi i)^3} \sum_{r \geq 1} \sum_{(r,f) = 1} \frac{\alpha_{ad} \beta_{bd}}{d} \int \int \int \ (c1) \ (c2) \ (c3) \]

\[ X_+(s, t) \tilde{W}(u) \tilde{W}(v) \left( \frac{q}{\pi} \right)^s w^{-\alpha(\beta + 2\epsilon + u + v)} \sum_{(f,w) = 1} \mu(f) \frac{\phi(q/f)}{q} \frac{1}{(2\pi i)^3} \sum_{d,a,b} \frac{\alpha_{ad} \beta_{bd}}{d} \int \int \int \ (c1) \ (c2) \ (c3) \]

Lemma 2.6. \( H(s) \) has simple poles at \( s = 1/2 - \alpha - it - u \) and \( s = 1/2 - \beta + it - v \), each of residue 2, and a zero at \( s = \frac{1 - \alpha - \beta - u - v}{2} \).

Proof. Writing (for the sake of clarity) \( x = \alpha + it + s + u \) and \( y = \beta - it + s + v \), \( H(s) \) is

\[
\frac{\Gamma(x+y)\Gamma(1/2-y)\Gamma(1/2+y) + \Gamma(1/2-x)\Gamma(1/2+x)}{\Gamma(1/2+x)\Gamma(1/2+y)} + \frac{\Gamma(1/2-x)\Gamma(1/2-y)}{\Gamma(1-x-y)}
\]

which has poles at \( x, y = 1/2 \) i.e. if \( s = 1/2 - \alpha - it - u \) or \( 1/2 - \beta + it - v \). It is easy to check that these have residue 2. Also note that if \( x + y = 1 \) then the second fraction vanishes (as there is a pole in the denominator from \( \Gamma(1-x-y) \)) and the first fraction is 0.

\[
\frac{\Gamma(1)}{\Gamma(1/2+y)\Gamma(1/2+y)} \left( \frac{\pi}{\sin(\pi/2+y)} + \frac{\pi}{\sin(\pi/2+x)} \right) = 0
\]

as \( \sin(\pi/2+y) = \sin(\pi + \pi(1/2-x)) = -\sin(\pi(1/2-x)) = -\sin(\pi(1/2+x)) \).

Returning to \( O^+(M, N) \) we move the contours to replace the \( r \)-sum with a zeta-function. Choose \( c_1 = 0 \), \( c_2 = \epsilon \) and move the \( s \)-contour to the right to \( 1/2 - \epsilon/3 \) crossing a simple pole of \( H(s) \) at \( s = 1/2 - \beta + it - v \). Write \( P^+(M, N) \) as the integral along the new line and \( R^+(M, N) \) as the residue. We can then move the \( u \) contour in the residue to \( \text{Re}(u) = 2\epsilon \) which hits no poles and allows us to replace the \( r \)-sum with a zeta function. i.e.

\[
R^+(M, N) = \frac{1}{2} \sum_{w|q} \mu \left( \frac{q}{w} \right) \phi(w) \sum_{f/q} \mu(f) \frac{\phi(q/f)}{q} \frac{1}{(2\pi i)^3} \sum_{d,a,b} \frac{\alpha_{ad} \beta_{bd}}{d} \int \int \int \ (c1) \ (c2) \ (c3) \]

\[
X_+(1/2 - \beta + it - v, t) \tilde{W}(u) \tilde{W}(v) M^u N^v \]

\[
\prod_p \left( 1 - p^{-\alpha(\beta + 2it + u + v + 1)} \right) \zeta(\alpha + \beta + 1 + 2it + u + v) \psi \left( \frac{t}{T} \right) \frac{dudv}{1/2 - \beta + it - v} dt.
\]

Using the following lemma, we can simplify \( R^+ \) and \( P^+ \).

Lemma 2.7.

\[
M := \sum_{w|q} \mu \left( \frac{q}{w} \right) \phi(w) \sum_{r \geq 1} \sum_{f/q} \mu(f) \phi(q/f) w^{-s} \prod_{p|f} (1 - p^{-s}) \phi^*(q) \prod_{p|q} (1 - p^{-s-1}) q^{-s}.
\]
Proof. If $p|f$ and $p|q/f$ then $p^2|q$ but $(f, w) = 1$ so $p^2|q/w \Rightarrow \mu(q/w) = 0$. Hence we may factorise $\phi(q/f) = \phi(q)/\phi(f)$, so

$$M = \frac{\phi(q)}{q} \sum_{w|q} \mu\left(\frac{q}{w}\right) \phi(w) \sum_{f|w} \frac{\mu(f)}{\phi(f)} w^{-s} \prod_{p|f} (1 - p^{-s})$$

$$= \frac{\phi(q)}{q} q^{-s} \sum_{w|q} \mu(w) \phi\left(\frac{q}{w}\right) w^{s} \sum_{f|w} \frac{\mu(f)}{\phi(f)} \prod_{p|f} (1 - p^{-s})$$

$$= \frac{\phi(q)}{q} q^{-s} \sum_{w|q} \mu(w) \phi\left(\frac{q}{w}\right) w^{s} \prod_{p|w} \left(1 - \frac{1}{1 - p^{-s}}\right).$$

Given that

$$\phi\left(\frac{q}{w}\right) = \frac{q}{w} \prod_{p|q/w} \left(1 - \frac{1}{p}\right) = \frac{\phi(q)}{w} \prod_{p|q/w} \left(\frac{1}{1 - \frac{1}{p}}\right) = \frac{\phi(q)}{w} \prod_{p|w} \left(1 - \frac{1}{p}\right)^{-1}$$

we see that

$$M = \frac{\phi(q)^2}{q} q^{-s} \sum_{w|q} \mu(w) w^{s-1} \prod_{p|w} \left(1 - \frac{1}{1 - p^{-s}}\right) \left(1 - \frac{1}{p}\right)^{-1}$$

$$= \frac{\phi(q)^2}{q} q^{-s} \sum_{w|q} \mu(w) w^{s-1} \prod_{p|w} \left(\frac{p^2 - 2p + p^{1-s}}{(p-1)^{2}}\right)$$

$$= \frac{\phi(q)^2}{q} q^{-s} \prod_{p|q} \left(1 - p^{s-1}\right) \prod_{p|q} \left(1 - p^{-1}\left(\frac{p^2 - 2p + p^{1-s}}{(p-1)^2}\right)\right)$$

as if $p^2|q$ and $p|w$ then either $\mu(w) = 0$ or $p|(w, q/w)$ so either the sum is empty (i.e. equal to zero) or the product is empty (equal to 1). Then rearranging gives that

$$M = \frac{\phi(q)^2}{q} q^{-s} \prod_{p^2|q} \left(1 - p^{s-1}\right) \prod_{p|q} \left(1 - p^{-s-1}\right) \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2 \left(1 - p^{s-1}\right)$$

$$= \phi^*(q) \prod_{p|q} \left(1 - p^{-s-1}\right) q^{-s}.$$

as $\phi^*(q) = q \prod_{p|q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2$.

By Lemma 2.7

$$R^+(M, N) = \frac{\phi^*(q)}{2} \sum_{d,a,b} \frac{\alpha_{ad} \beta_{bd}}{d} \frac{1}{(2\pi i)^2} \int_{c} \int_{2c} X_{+}(1/2 - \beta + it - v, t)$$

$$\tilde{W}(u) \tilde{W}(v) M^u N^v \left(\frac{d}{\pi}\right)^{1/2-\beta-it-v} q^{-(\alpha - \beta + 1 + 2it + u - v)} a^{-1/2-\beta + it + u + \alpha} b^{-1/2+it}$$
\[
\prod_{p|q} \left(1 - p^{\alpha - \beta + 2it + u + v}\right) \zeta(\alpha - \beta + 1 + 2it + u - v) \psi \left(\frac{t}{T}\right) \frac{dudv}{1/2 - \beta + it - v}. 
\]

With the \(P^+(M, N)\) term, we replace the \(r\)-sum with a zeta-function as before, apply Lemma 2.7 and shift the \(s\)-contour back to \(\text{Re}(s) = \epsilon\). This crosses the same pole at \(s = 1/2 - \beta + it - v\), while the pole from the zeta function at \(s = (1 - \alpha - \beta - u - v)/2\) is cancelled out by the zero of \(H(s)\) at this point. Denote the contribution from the first pole as \(R^+(M, N)\) and the new integral with the \(r\)-sum replaced as \(P^{\eta'}(M, N)\) so

\[
O^+(M, N) = P^{\eta'}(M, N) + R^+(M, N) - R^+(M, N)
\]

where \(P^{\eta'}(M, N)\) is

\[
\frac{\phi^*(q)}{2(2\pi i)^3} \sum_{d,a,b} \frac{\alpha_{ad} \beta_{bd}}{d} \int_0^1 \int_0^1 \int_0^1 X_+(s, t) W(u) W(v) \left(\frac{q}{\pi}\right)^s q^{-(\alpha + \beta + 2s + u + v)}
\]

\[a^{s+u-1+\alpha} b^{s-1+\beta} M^\alpha N^\beta H(s) \prod_{p|q} \left(1 - p^{\alpha + \beta - 2s + u + v}\right) \zeta(\alpha + \beta + 2s + u + v) \psi \left(\frac{t}{T}\right) ds dt. \]

The difference between the two residue terms is in the \(u\)-contour i.e. integrating over \(\text{Re}(u) = 0, 2\epsilon\). Therefore \(R^+(M, N) - R^+(M, N)\) is the residue at \(u = \beta - \alpha + v - 2it\) but this is cancelled by the zero from the \((1 - p^{\alpha + \beta + 2it + u + v})\) factors, i.e. \(R^+(M, N) - R^+(M, N) = 0\). Hence \(O^+(M, N) = P^{\eta'}(M, N)\).

By Lemma 4.3 in [3] we can remove the dyadic partition i.e.

\[
O_0^+ := \sum_{M,N} P^{\eta'}(M, N) = \frac{\phi^*(q)}{4\pi i} \sum_{d,a,b} \frac{\alpha_{ad} \beta_{bd}}{d} \int_0^1 \int_0^1 X_+(s, t) \left(\frac{q}{\pi}\right)^s q^{-(\alpha + \beta + 2s - 1 + \alpha)} b^{s-1+\beta} H(s) \prod_{p|q} \left(1 - p^{\alpha + \beta - 2s}\right) \zeta(\alpha + \beta + 2s) \psi \left(\frac{t}{T}\right) ds dt.
\]

We can now write \(H(s)\) as

\[
H(s) = \frac{\Gamma(\alpha + \beta + 2s) \Gamma(1/2 - \beta + it - s)}{\Gamma(1/2 + \alpha + it + s)} + \frac{\Gamma(\alpha + \beta + 2s) \Gamma(1/2 - \alpha - it - s)}{\Gamma(1/2 + \beta - it + s)} + \frac{\Gamma(1/2 - \alpha - it - s) \Gamma(1/2 - \beta + it - s)}{\Gamma(1 - \alpha - \beta - 2s)}
\]

which by Lemma 8.2 of [17] is equal to

\[
\pi^{1/2} \frac{\Gamma(\alpha + \beta + 2s/2) \Gamma(1/2 - \alpha - it - s/2) \Gamma(1/2 - \beta + it - s/2)}{\Gamma(1 - \alpha - \beta - 2s/2) \Gamma(1/2 + \alpha + it + s/2) \Gamma(1/2 + \beta - it + s/2)}.
\]

Hence,

\[
H(s)X_+(s, t) = \pi^{1/2} X_+(s, t) \frac{\Gamma(\alpha + \beta + 2s/2)}{\Gamma(1 - \alpha - \beta - 2s/2)}. \tag{7}
\]

Applying the functional equation

\[
\pi^{-(\alpha + \beta + 2s)/2} \Gamma(\frac{\alpha + \beta + 2s}{2}) \zeta(\alpha + \beta + 2s)
\]

\[
= \pi^{-(1 - \alpha - \beta - 2s)/2} \Gamma\left(\frac{1 - \alpha - \beta - 2s}{2}\right) \zeta(1 - \alpha - \beta - 2s)
\]
and the change of variable \( s \rightarrow -s \) gives \( O_0^+ \) as

\[
= -\frac{\phi^*(q)}{2} \left( \frac{q}{\pi} \right)^{-\alpha-\beta} \sum_{d,a,b \atop (a,b)=1 \atop (abd,q)=1} \frac{\alpha_a \beta_b}{a^{1-\alpha} b^{1-\beta} d^{2 \pi i}} \int_{(-\epsilon)} X_-(s, t) \left( \frac{\pi ab}{q} \right)^{-s} \prod_{p \mid q} (1 - p^{\alpha+\beta-1-2s})
\]

\[
\zeta(1 - \alpha - \beta + 2s) \psi \left( \frac{t}{T} \right) \frac{ds}{s} dt
\]

\[
= -\frac{\phi^*(q)}{2} \left( \frac{q}{\pi} \right)^{-\alpha-\beta} \sum_{d,a,b \atop (a,b)=1 \atop (abd,q)=1} \frac{\alpha_a \beta_b}{a^{1-\alpha} b^{1-\beta} d^{2 \pi i}} \int_{(-\epsilon)} X_-(s, t) \left( \frac{\pi ab}{q} \right)^{-s} L(1 - \alpha - \beta + 2s, \chi_{0,q}) \psi \left( \frac{t}{T} \right) \frac{ds}{s} dt.
\]

To summarise:

\[
O_0^+ = \frac{O_0^+}{\phi^*(q)T} + O_e \left( (qT)^{-\epsilon} \right).
\]

The \( O^- \) case is identical by replacing \( X_+ \) with \( X_- \) and the substitution \( \alpha, \beta \rightarrow -\beta, -\alpha \).

### 2.4. Combining the Main Terms.

We have shown that

\[
\frac{1}{\phi^*(q)T} \int \sum_{\chi \pmod{q}}^+ L(1/2 + \alpha + it, \chi)L(1/2 + \beta - it, \chi) \sum_{a,b \leq (qT)^{2 \pi i}} \frac{\alpha_a \beta_b \chi(a) \bar{\chi}(b)}{a^{1/2+it} b^{1/2-it}} \psi(t/T) dt
\]

\[
= \frac{1}{\phi^*(q)T} \left( D^+ + O_0^+ \left( \frac{q}{\pi} \right)^{-\alpha-\beta} (D^- + O_0^-) \right) + O_e \left( (qT)^{33/20 + \epsilon} \right)
\]

where for instance

\[
\left( \frac{q}{\pi} \right)^{-\alpha-\beta} D^+ + O_0^+ = \left( \frac{q}{\pi} \right)^{-\alpha-\beta} \frac{\phi^*(q)}{2} \sum_{a,b \leq (qT)^{2 \pi i}} \frac{\alpha_a \beta_b}{a^{1-\alpha} b^{1-\beta} d^{2 \pi i}}
\]

\[
\left( \int_{(-\epsilon)} - \int_{(-\epsilon)} \right) \int X_-(s, t) \left( \frac{q}{\pi ab} \right)^s L(1 - \alpha - \beta + 2s, \chi_{0,q}) \psi \left( \frac{t}{T} \right) dt \frac{ds}{s}
\]

\[
= \text{Res}_{s=0} \int X_-(s, t) \left( \frac{q}{\pi ab} \right)^s L(1 - \alpha - \beta, \chi_{0,q}) \sum_{a,b \leq (qT)^{2 \pi i}} \frac{\alpha_a \beta_b}{a^{1-\alpha} b^{1-\beta} d} \int X_-(0, t) \psi \left( \frac{t}{T} \right) dt.
\]

Note that the pole at \( s = (\alpha + \beta)/2 \) of the \( L \)-function is cancelled by the function \( G \). A similar expression holds for the sum of the other two terms, giving the result in Theorem 122 for the sum over even Dirichlet characters.

### 2.5. The Odd Characters.

The odd characters go through almost identically, but with two differences: firstly we have to redefine the functions \( V_\pm(.) \) (because of the different functional equation for odd Dirichlet characters) by altering the gamma functions, and secondly summing over the odd primitive characters gives a different sum to the even characters i.e. for \( (m, q) = 1 \),

\[
\sum_{\chi \pmod{q}}^\chi \frac{1}{2} \sum_{m \equiv 1 \mod{w}} \mu(u) \phi(w) - \frac{1}{2} \sum_{m \equiv 1 \mod{w}} \mu(u) \phi(w).
\]
This manifests itself in our definition of the function $H(s)$ at (6). In this setting we must redefine $H(s) := H_+(s) - H_-(s)$. The same method still works as our new $H(s)$ has zeros in the same positions, and no poles so there are not any residue terms to deal with. To show at (7) that

$$ H(s)X_+(s, t) = \pi^{1/2} X_-(s, t) \frac{\Gamma\left(\frac{\alpha+\beta+2\kappa}{2}\right)}{\Gamma\left(\frac{1-\alpha-\beta-2\kappa}{2}\right)} $$

we appeal to Lemma 8.4 of [17] instead of Lemma 8.2. Hence

$$ \frac{1}{\phi^*(q)T} \int \sum_{\chi \mod q}^{-} L(1/2 + \alpha + it, \chi)L(1/2 + \beta - it, \overline{\chi}) \sum_{a,b \leq (qT)\epsilon} \frac{\alpha_{ad}\beta_{bd}}{a^{1/2+it}b^{1/2-it}} \psi \left(\frac{q}{\alpha}\right) dt $$

$$ = \frac{\hat{\psi}(0)}{2} L(1 + \alpha + \beta, \chi_{0,q}) \sum_{\alpha, \beta \leq (qT)\epsilon} \frac{\alpha_{ad}\beta_{bd}}{a^{1/2+\beta}b^{1/2-\alpha}} \frac{1}{2T} \left(\frac{q}{\pi}\right)^{-\alpha-\beta} L(1 - \alpha - \beta, \chi_{0,q}) $$

$$ \times \sum_{\alpha, \beta \leq (qT)\epsilon} \frac{\alpha_{ad}\beta_{bd}}{a^{1-\alpha}b^{1-\beta}} \int \frac{\Gamma\left(\frac{3/2 - \alpha-it}{2}\right)}{\Gamma\left(\frac{3/2 + \alpha-it}{2}\right)} \frac{\Gamma\left(\frac{3/2 - \beta+it}{2}\right)}{\Gamma\left(\frac{3/2 + \beta+it}{2}\right)} \psi \left(\frac{t}{T}\right) dt + O\left((qT)^{-\epsilon}\right). $$

2.6. **Proof of Theorem [13]** This proof is the same as Theorem [12] except that we use the Vaughan identity with the Möbius function to split up $E_{w,f,h}$ in (4) into three sums, which are then bounded separately.

2.6.1. **The Vaughan Identity.** Let $U(s) = \sum_{n \leq W} \mu(n)n^{-s}$ for $W$ a constant that we shall choose later. By comparing the coefficients of

$$ \frac{1}{\zeta(s)} = \frac{1}{\zeta(s)}(1 - \zeta(s)U(s))^2 + 2U(s) - \zeta(s)U(s)^2 $$

we see that

$$ \mu(u) = c_1(u) + c_2(u) + c_3(u) $$

where

$$ c_1(u) = \sum_{a \geq W, b \geq W} \mu(c)c_4(a)c_4(b) \text{ with } c_4(a) = -\sum_{\epsilon \leq W} \mu(\epsilon) $$

$$ c_2(u) = \begin{cases} 2\mu(u) & \text{if } u \leq W \\ 0 & \text{if } u > W \end{cases} $$

$$ c_3(u) = -\sum_{a \leq W, b \leq W} \mu(a)e^{-u}. $$

Substituting $\alpha_{ad}(A) = \mu(ad)f_A(ad) = c_1(ad)f_A(ad) + c_2(ad)f_A(ad) + c_3(ad)f_A(ad)$ into (4) (where $f_A$ is $f$ multiplied by a smooth function supported on $[A, 2A]$), so that $f_A'(x) \ll x^{-1+\epsilon}$ produces

$$ E_{w,f,h}(A, B, M, N) = E_1(A, B, M, N) + E_2(A, B, M, N) + E_3(A, B, M, N) $$

where (using Mellin transforms to separate variables)

$$ \frac{E_i(A, B, M, N)\phi(w)}{\phi^*(q)T} \ll \sum_{d \geq 1} \frac{1}{(ABMN)^{1/2}} \frac{w(qT)^{-1}}{hf} $$
\[
\times \left| \int \int_{x \sim dM/B} \sum_{a \sim A/d} \sum_{b \sim B/d} \sum_{\substack{\mu(c) = \pm 1 \text{ or } \pm \mu(c) \neq 0}} c_i(ad) f_A(ad) \beta_{bd} \nu_{rg} e \left( \frac{\text{sgn}(\ell) + \frac{gx}{hf}}{bf} \right) \, dx \, dt \right|
\]

Let \( W = A^{1/4} \). This means that \( E_2(A, B, M, N) \) is an empty sum as the sequence \( \alpha_n(A) \) has support on \([A, 2A] \cap [1, A^{1/4}]\).

### 2.6.2. \( E_1 \)

To bound \( E_1(A, B, M, N) \phi(w) \) we write it as a linear combination of at most \( O_\epsilon ((qT)^\epsilon) \) sums, each of which is

\[
\ll_\epsilon \sum_{d \geq 1} 1_{d_1 d_2 = d} \sum_{a_1 \sim A/d_1} \sum_{a_2 \sim A/d_2} \sum_{a_3 \sim A/d_3} c_4(a_1 d_1)
\]

\[c_4(a_2 d_2) \mu(a_3 d_3) f_A(a_1 a_2 a_3 d) \beta_{bd} \nu_{rg} e \left( \frac{\text{sgn}(\ell) + \frac{gx}{hf}}{bf} \right) \, dx \, dt \]

with \( A_1 A_2 A_3 = A \) and where we may assume that \( a_1, a_2, a_3, d \) are all pairwise coprime and square-free due to the presence of the Möbius function. By the definition of \( c_4 \) we see that \( A_1, A_2 \gg W/d \) and without loss of generality \( A_1 \leq A_2 \). By defining

\[
c_5(a_2' d_2') = \sum_{d_2 d_2' = d_2} \mu(a_3 d_3) c_4(a_2 d_2)
\]

we change (8) into sums of the form

\[
\ll_\epsilon \sum_{d \geq 1} 1_{d_1 d_2 = d} \sum_{a_1 \sim A/d_1} \sum_{a_2 \sim A/d_2} \sum_{a_3 \sim A/d_3} c_4(a_1 d_1)
\]

\[c_5(a_2 d_2) f_A(a_1 a_2 d) \beta_{bd} \nu_{rg} e \left( \frac{\text{sgn}(\ell) + \frac{gx}{hf}}{bf} \right) \, dx \, dt \]

with \( W \ll A_1 \ll A_2 \ll A/W \) and \( A_1 A_2 = A \). Let \( A_1 = (qT)^{\kappa_1} \) and \( A_2 = (qT)^{\kappa_2} \) so that \( \kappa = \kappa_1 + \kappa_2 \). Note that we may bound \( c_4(n), c_5(n) \) by \( O_\epsilon ((qT)^\epsilon) \). By applying Lemma 6 of [7] (slightly adapted to include the extra \( h, f \) in the trilinear fraction) with

- \( U \leftrightarrow \frac{Bf}{d_1} \)
- \( K \leftrightarrow \frac{\text{sgn}(\ell) + \frac{gx}{hf}}{bf} \)
- \( S \leftrightarrow \frac{A h}{d_1} \)
- \( T \leftrightarrow \frac{A h}{d_2} \)
to bound the sums in (9) by

\[
\ll \epsilon \sum_{d \geq 1, \frac{d}{d_1} = \frac{d_2}{} \oplus d} \frac{1}{(ABMN)^{1/2}} \frac{w(qT)^{\epsilon - 1}}{h f} T d M B \frac{ABhf A_1 A_2}{B d} \frac{d}{dT} \frac{1}{(AA_1 Bhf)} \frac{1}{d_1 d_2} \left( f d_1 d_2 w T \right)^{1/4}
\]

\[
+ \left( \frac{B f w T d_1 d_2}{dABhf A_1 A_2} \right)^{1/4} \frac{d^1}{B^1/4} + \frac{d_3^1/2}{A_3^1/2} \right)
\]

\[
\ll \epsilon AB(qT)^{\epsilon - 1} \left( \left( \frac{qT}{A_1 AB} \right)^{1/4} + \left( \frac{qT}{A_2 A_2} \right)^{1/4} + B^{-1/4} + A_2^{-1/2} \right)
\]

\[
\ll \epsilon (qT)^{\frac{2a}{3} - \frac{1}{2} - \frac{1}{3}} + (qT)^{\frac{2a}{3} - \frac{1}{2} - \frac{1}{3}} + (qT)^{\frac{2a}{3} - 1} + (qT)^{\frac{2a}{3} + \frac{1}{3} - \frac{1}{3}}
\]

\[
\ll \epsilon (qT)^{\frac{2a}{3} - \frac{1}{2} - \frac{1}{3}}
\]

for \(\kappa < 4/7\) and \(\kappa_1 \leq \kappa/2\). This bound is less effective when \(A_1\) is small, so another bound is needed in this case. Using lemmas 10 and 11 from [10] with

- \(C \leftrightarrow B/d\)
- \(M \leftrightarrow A_2/d_2\)
- \(K \leftrightarrow R Ghf/wd^2 \ll Abhf(wT)^{\epsilon - 1}d^{-2}\)
- \(R \leftrightarrow A_1 h/d_1\)
- \(d \leftrightarrow w\)
- \(s \leftrightarrow f\)
- \(X_d \leftrightarrow T^{\epsilon - 1/2}\)

we may bound the sums in (9) by

\[
\ll \epsilon \sum_{d \geq 1, \frac{d}{d_1} = \frac{d_2}{} \oplus d} \frac{1}{(ABMN)^{1/2}} \frac{w(qT)^{\epsilon - 1}}{h f} T d M B \frac{ABhf A_1 A_2}{B d} \frac{d}{dT} \frac{1}{(AA_1 Bhf)} \frac{1}{d_1 d_2} \left( f d_1 d_2 w T \right)^{1/4}
\]

\[
+ \left( \frac{B f w T d_1 d_2}{dABhf A_1 A_2} \right)^{1/4} \frac{d^1}{B^1/4} + \frac{d_3^1/2}{A_3^1/2} \right)
\]

\[
\ll \epsilon (qT)^{\frac{2a}{3} - \frac{1}{2} - \frac{1}{3}} + (qT)^{\frac{2a}{3} - \frac{1}{2} - \frac{1}{3}} + (qT)^{\frac{2a}{3} - 1} + (qT)^{\frac{2a}{3} + \frac{1}{3} - \frac{1}{3}}
\]

\[
+ (qT)^{\frac{2a}{3} + \frac{1}{3} - \frac{1}{3}} + (qT)^{\frac{2a}{3} + \frac{1}{3} - \frac{1}{3}} + (qT)^{\frac{2a}{3} - \frac{1}{3} - \frac{1}{3}}
\]

\[
\ll \epsilon (qT)^{\frac{2a}{3} - \frac{1}{2} - \frac{1}{3}}
\]

for \(\kappa < 1/2 + 5/128\) and \(\kappa_1 < \kappa/2\). We use the first bound when \(\kappa_1 \leq \kappa - \frac{39}{128}\) and the second bound for when \(\kappa_1 \geq \kappa - \frac{39}{128}\), resulting in the bound

\[
\frac{E_1(A, B, M, N) \phi(w)}{\phi^*(q) T} \ll \epsilon (qT)^{\frac{2a}{3} - \frac{1}{12} + \epsilon} \ll \epsilon (qT)^{-\epsilon}.
\]
2.6.3.  $E_3(\phi(w))/\sigma(q)$ may be bounded by a sum of at most $O_\epsilon((qT)^\epsilon)$ sums of the form

$$
\ll \sum_{d_1,d_2=d} \frac{1}{(ABMN)^{1/2}} \frac{w(qT)^{\epsilon-1}}{hT} \int \int_{x=\pm M/B} \sum_{a_1 \leq A_1/d_1} \sum_{b,c \in [0,T]} \sum_{\substack{0<|r|\leq R/wd \\ 0<|g|\leq GfT/d}} c_6(a_1d_1)f_A(a_1a_2d)\beta_{bd} \nu_{rg} e \left( -\frac{grwra_1a_2h}{bf} + \frac{gx}{hf} \right) dx dt
$$

with

$$
c_6(n) = \sum_{x,y=n \mod W} \mu(x)\mu(y).
$$

This means that $A_1 \leq W^2 = A^{1/2}$. When $A_1 \gg A^{1/4}$ we use the same method as for $E_1$, but when $A_1 \ll A^{1/4}$ we shall apply the Weil bound for Kloosterman sums. This implies that

$$
\sum_{A_2/d_2 \leq a_2 \leq A_2/d_2 + x} e \left( -\frac{grwra_1a_2h}{bf} \right) \ll (bf)^{1/2+\epsilon}(grwra_1h,bf) \left( 1 + \frac{A_2}{bf}d_2 \right).
$$

By partial summation over $a_2$ we may bound the sums above by

$$
\ll \sum_{d_1,d_2=d} \frac{1}{(ABMN)^{1/2}} \frac{w(qT)^{\epsilon-1}}{hT} \frac{dM}{B} \left( \frac{Bf}{d_1} \right)^{1/2} \left( 1 + \frac{A_2d}{Bf}d_2 \right) \sum_{0<|r|\leq R/wd} \sum_{b \in [B/d]} \sum_{|g|\leq GfT/d} (rg,bf)
$$

$$
\ll (qT)^{\epsilon-1} A_1 B^{3/2} f^{1/2} \left( 1 + \frac{A_2}{bf} \right)
$$

$$
\ll f^{1/2}(qT)^{3\epsilon/4 + \epsilon - 1}
$$

$$
\ll q^{7/2}(qT)^{5\epsilon/4 - 1}.
$$

By the trivial bound in Lemma 2.5 and 3, we may assume that for $1/2 < \kappa < 1/2 + 5/128$

$$
\frac{\log(T)}{\log(q)} \\
\geq \frac{\log(q)}{2}\frac{2\gamma + 1 - 2\kappa}{2\kappa - 1}
$$

which means that

$$
\frac{E_3(A,B,M,N)\phi(w)}{\sigma(q)T^{\epsilon}} \ll q^{\frac{7\epsilon}{4} + (\gamma - 1)(1 + \frac{2\gamma + 1 - 2\kappa}{2\kappa - 1}) + \epsilon}
$$

$$
\ll q^{\gamma \frac{(\gamma - 5)}{4\epsilon - 2} + \epsilon}
$$

$$
\ll q^{-\epsilon} (qT)^{-\epsilon}.
$$

This concludes the proof of Theorem 1.3.

3. PROOF OF THEOREM 1.1

To prove Theorem 1.1 first note that for $0 \leq \sigma - \frac{1}{2} \leq \frac{1}{\log(qT)}$, $N(\sigma,T,\chi) \leq N(1/2, T, \chi) < T \log(qT)$.

Also, for $\sigma - \frac{1}{2} \geq \frac{28 \log \log(qT)}{\log(qT)}$, the theorem is true by Montgomery’s result. So it is sufficient to prove the following proposition.
Proposition 3.1. For $\frac{1}{\log(qT)} \leq \sigma - \frac{1}{2} \leq \frac{28\log\log(qT)}{\log(qT)}$ and $\kappa < 1/2 + 5/128$,

$$\sum\limits_{\chi \pmod{q}}^{*} N(\sigma, T; \chi) \ll (qT)^{2-2\sigma} \log^5(qT) + (2\sigma - 1)(qT)^{1+\kappa(1-2\sigma)} \log(qT)^3.$$ 

To prove this proposition, we rely on Littlewood’s lemma (see [16] Theorem 9.16), which reduces the problem of bounding

$$\int_T^{2T} \sum_{\chi \pmod{q}}^{*} |L(\sigma + it, \chi)M(\sigma + it, \chi) - 1|^2 \psi\left(\frac{t}{T}\right) dt$$

by

$$O \left( (qT)^{2-2\sigma} \log^4(qT) + (2\sigma - 1)(qT)^{1+\kappa(1-2\sigma)} \log(qT)^2 \right)$$

where $\psi(t)$ is a smoothing function as in Theorem 1.2. By expanding out the square in the integral, we get three terms

$$\int_T^{2T} \sum_{\chi \pmod{q}}^{*} |L(\sigma + it, \chi)M(\sigma + it, \chi)|^2 \psi\left(\frac{t}{T}\right) dt$$

$$- 2 \text{Re} \left( \int_T^{2T} \sum_{\chi \pmod{q}}^{*} L(\sigma + it, \chi)M(\sigma + it, \chi)\psi\left(\frac{t}{T}\right) dt \right) + \phi^*(qT)\hat{\psi}(0).$$

We look first at the term (10). Using methods similar to those used by Iwaniec and Sarnak in [9], if our mollifier is of the form $M(s, \chi) = \sum_{n \leq x} v(n)\chi(n)n^{-\sigma}$

then the optimal mollifier (with the normalisation that $v(1) = 1$) can be shown to be close to

$$v(n) = \frac{\mu(n)(1 - (x/n)^{1-2\sigma})}{1 - (x)^{1-2\sigma}}$$

for $1 \leq n \leq x$ and 0 otherwise. Note that

$$\lim_{\sigma \to 1/2} v(n) = \frac{\mu(n)\log(x/n)}{\log(x)}$$

which is a standard mollifier on the half-line. This choice of mollifier satisfies the conditions of Theorem 1.3 and so by defining

$$S_1(x) := \sum_{\substack{a,b,d \\mid (a,b)=1 \\mid (abd,q)=1}} \frac{v(ad)v(bd)}{(abd)^{2\sigma}}$$

and

$$S_2(x) := \sum_{\substack{a,b,d \\mid (a,b)=1 \\mid (abd,q)=1}} \frac{v(ad)v(bd)}{ab^d2\sigma}$$

then we see that by Theorem 1.3 that (10) is equal to

$$\phi^*(qT)\hat{\psi}(0)L(2\sigma, \chi_{0,q})S_1((qT)^\kappa) + O_\epsilon \left( (qT)^{2-2\sigma} L(2 - 2\sigma, \chi_{0,q})|S_2((qT)^\kappa)| + (qT)^{1-\epsilon} \right).$$

To deal with $S_1$, we will need the following lemma.
Lemma 3.1. For all \( t \geq 1 \)
\[
\sum_{\substack{a \leq t \\ (a,n)=1}} \frac{\mu(a)}{a} \ll \frac{n}{\phi(n)}.
\]

Proof. Define
\[
f_n(t) := \left| \sum_{\substack{a \leq t \\ (a,n)=1}} \frac{\mu(a)}{a} \right|
\]
and
\[
M_n := \max \left| \sum_{\substack{a \leq t \\ (a,n)=1}} \frac{\mu(a)}{a} \right|.
\]

Then for any prime \( p \) with \( p^k | n \)
\[
f_n(t) \leq M_{n/p^k} + \frac{1}{p} f_n(t/p).
\]

Then by recursion, and as \( \lim_{h \to \infty} f_n(t/p^h) = 0 \) we see that
\[
f_n(t) \leq M_1 \frac{n}{\phi(n)}
\]
but by the prime number theorem, \( \sum_a \frac{\mu(a)}{a} = 0 \), hence \( M_1 \ll 1. \)

We can now handle \( S_1 \) and \( S_2. \)

Lemma 3.2. \( S_1(x) = L^{-1}(2\sigma, \chi_q) (1 + O ((2\sigma - 1)x^{1-2\sigma} \log^2(x))) \)

Proof. We may assume that \( \sigma \) is close to 1/2 meaning that
\[
L(2\sigma, \chi_0, qn) \asymp \frac{\phi_{2\sigma}(qn)}{(qn)^{2\sigma}(2\sigma - 1)}
\]
where
\[
\phi_{2\sigma}(n) = \sum_{cd=n} \mu(c)d^{2\sigma} = n^{2\sigma} \prod_{p | n} (1 - p^{-2\sigma}).
\]

\[
S_1(x) = \sum_{a,b,d \atop (a,b)=1 \atop (abd,q)=1} \frac{v(ad)v(bd)}{(abd)^{2\sigma}} = \sum_{a,b,c,d \atop acd,bcd \leq x \atop (abcd,q)=1} \mu(c) \frac{v(acd)v(bcd)}{(abcd)^{2\sigma}}
\]
\[
= \sum_{a,b \atop (ab,q)=1 \atop (n,q)=1} \frac{v(an)v(bn)}{(abn^2)^{2\sigma}} \sum_{cd=n} \mu(c)d^{2\sigma}.
\]

Let
\[ y_n = \sum_{\sigma \leq x/n} \frac{v(an)}{(an)^{2\sigma}} \]

then

\[ S_1(x) = \sum_{n \leq x} y_n \phi(2\sigma)(n). \]

Inserting the definition of \( v(n) \) in to the definition of \( y_n \) gives

\[ y_n = \frac{\mu(n)}{n^{2\sigma}} \sum_{a \leq t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}} \left( 1 - \left( \frac{x}{an} \right)^{1-2\sigma} \right) \left( 1 - x^{1-2\sigma} \right)^{-1} \]

\[ = \frac{\mu(n)}{n^{2\sigma}} (2\sigma - 1) \left( \frac{x}{n} \right)^{1-2\sigma} (1 - x^{1-2\sigma})^{-1} \int_{1}^{x/n} \left( \sum_{a \leq t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}} \right) t^{2\sigma - 2} \, dt \]

by partial summation. As \( 2\sigma > 1 \), the sum converges so we may write

\[ \sum_{a \leq t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}} = L^{-1}(2\sigma, \chi_{0,qn}) - \sum_{a > t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}}. \]

As \( 2\sigma \) is close to 1, it is not sufficient to bound the error by \( O \left( \frac{1}{t^{2\sigma - 1}} \right) \). Instead, we write

\[ \sum_{a > t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}} = - \sum_{a \leq t \atop (a,qn)=1} \frac{\mu(a)}{a} t^{1-2\sigma} + (2\sigma - 1) \int_{t}^{\infty} \sum_{a \leq s \atop (a,qn)=1} \frac{\mu(a)}{a} s^{-2\sigma} \, ds. \]

So by Lemma 3.1 for \( q \) and \( n \) co-prime

\[ \sum_{a \leq t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}} \ll \frac{qn}{\phi(qn)} t^{1-2\sigma}. \]

This means

\[ \sum_{a \leq t \atop (a,qn)=1} \frac{\mu(a)}{a^{2\sigma}} = L^{-1}(2\sigma, \chi_{0,qn}) + O \left( \frac{qn}{\phi(qn)} t^{1-2\sigma} \right) \]

so

\[ y_n = \frac{\mu(n)}{n^{2\sigma}} (2\sigma - 1) \left( \frac{x}{n} \right)^{1-2\sigma} \int_{1}^{x/n} \left( L^{-1}(2\sigma, \chi_{0,qn}) + O \left( \frac{qn}{\phi(qn)} t^{1-2\sigma} \right) \right) t^{2\sigma - 2} \, dt \]

\[ = \frac{\mu(n)}{n^{2\sigma}} L^{-1}(2\sigma, \chi_{0,qn}) \left( 1 - \left( \frac{x}{n} \right)^{1-2\sigma} \right) + O \left( \frac{x}{n}^{1-2\sigma} \log \left( \frac{x}{n} \right) \right) \]

\[ = \frac{\mu(n)}{\phi(2\sigma)(n)} L^{-1}(2\sigma, \chi_q) \left( 1 - \left( \frac{x}{n} \right)^{1-2\sigma} \right) + O \left( \frac{x}{n}^{1-2\sigma} \log \left( \frac{x}{n} \right) \right) \]
so
\[ y_n^2 \phi_{2\sigma}(n) = \frac{\mu(n)^2}{\phi_{2\sigma}(n)} L^{-2}(2\sigma, \chi_q) \left( \left( 1 - \frac{(x/n)^{1-2\sigma}}{1 - x^{1-2\sigma}} \right)^2 + O \left( \frac{x}{n} \right)^{1-2\sigma} \log(x/n) \right). \]

Note that for square-free \( n \),
\[ \frac{\mu(n)^2}{\phi_{2\sigma}(n)} = n^{-2\sigma} \prod_{p|n} (1 - p^{-2\sigma})^{-1} = n^{-2\sigma} \prod_{p|n} (1 + p^{-2\sigma} + p^{-4\sigma} + p^{-6\sigma} + ...) = \sum_{\text{rad}(m) = n} \frac{1}{m^{2\sigma}} \]
where \( \text{rad}(m) = \prod_{p|m} p \).

Therefore
\[ \sum_{n \leq x \atop (n, q) = 1} \frac{\mu(n)^2}{\phi_{2\sigma}(n)} = \sum_{m \leq x \atop \text{rad}(m) = n} \frac{1}{m^{2\sigma}} + O \left( \sum_{m > x \atop \text{rad}(m) = 1} \frac{1}{m^{2\sigma}} \right) \]
\[ = L(2\sigma, \chi_q) + O \left( \frac{x^{1-2\sigma} \phi_{2\sigma}(q)}{(2\sigma - 1)q^{2\sigma}} \right) \]
\[ = L(2\sigma, \chi_q) + O \left( x^{1-2\sigma} L(2\sigma, \chi_q) \right) \]

So, supposing that \( f(t) \) is a differentiable function with \( f(x) = 0 \), partial summation shows that
\[ \sum_{n \leq x \atop (n, q) = 1} \frac{\mu(n)^2}{\phi_{2\sigma}(n)} f(n) = L(2\sigma, \chi_q) \left( f(1) + O \left( \int_1^x t^{1-2\sigma} f'(t) dt \right) \right) \]

hence
\[ \sum_{n \leq x \atop (n, q) = 1} \frac{\mu(n)^2}{\phi_{2\sigma}(n)} \left( \frac{1 - (x/n)^{1-2\sigma}}{1 - x^{1-2\sigma}} \right)^2 = L(2\sigma, \chi_q) \]
\[ + O \left( L(2\sigma, \chi_q) \int_1^x t^{1-2\sigma} 2(2\sigma - 1)x^{1-2\sigma} t^{2\sigma - 2} \left( 1 - \frac{x}{t} \right)^{1-2\sigma} dt \right) \]
\[ = L(2\sigma, \chi_q) \left( 1 + O \left( (2\sigma - 1)x^{1-2\sigma} \log(x) \right) \right) \]
and
\[ \sum_{n \leq x \atop (n, q) = 1} \frac{\mu(n)^2}{\phi_{2\sigma}(n)} \left( \frac{x}{n} \right)^{1-2\sigma} \log(x/n) \]
is
\[ = L(2\sigma, \chi_q) x^{1-2\sigma} \log(x) + O \left( L(2\sigma, \chi_q) x^{1-2\sigma} \int_1^x t^{1-2\sigma} ((2\sigma - 1) \log(x/t) - 1)t^{2\sigma - 2} dt \right) \]
\[ = L(2\sigma, \chi_q) \left( x^{1-2\sigma} \log(x) + O \left( (2\sigma - 1)x^{1-2\sigma} \log^2(x) \right) \right). \]

Hence,
\[ S_1(x) = \sum_{n \leq x \atop (n, q) = 1} y_n^2 \phi_{2\sigma}(n) = L^{-1}(2\sigma, \chi_q) \left( 1 + O \left( (2\sigma - 1)x^{1-2\sigma} \log^2(x) \right) \right) \]
We now bound $S_2(x)$.

**Lemma 3.3.**

$$S_2(x) \ll L(2\sigma, \chi_q) \log(x)^2$$

**Proof.** Similar to before, we see that

$$S_2(x) = \sum_{n \leq x} y_n^2 \phi_{2-2\sigma}(n)$$

with

$$y_n := \mu(n) \sum_{a \leq x/n \ (a,qn)=1} \mu(a) \left(1 - \left(\frac{x}{an}\right)^{1-2\sigma}\right)^{-1} \left(1 - x^{1-2\sigma}\right)^{-1}$$

$$\ll \frac{1}{n} \sum_{a \leq x/n} \frac{1}{a} \ll \frac{\log(x)}{n}$$

so

$$\sum_{n \leq x \ (n,q)=1} y_n^2 \phi_{2-2\sigma}(n) \ll \sum_{n \leq x \ (n,q)=1} \frac{\phi_{2-2\sigma}(n)}{n^2} \log(x)^2$$

$$\ll \sum_{n \leq x \ (n,q)=1} \frac{\log(x)^2}{n^{2\sigma}} \ll L(2\sigma, \chi_q) \log(x)^2$$

□

Hence by lemmas 3.2 and 3.3, we arrive at the conclusion that $\frac{1}{\log^{1/2}(qT)}$ is equal to

$$\phi^*(q)T^\psi(0) + O_{\epsilon} \left( (qT)^{2-2\sigma} \log^4(qT) + (2\sigma - 1)(qT)^{1+1/2-2\sigma} \log(qT)^2 + (qT)^{1-\epsilon}\right). \quad (11)$$

We turn our attention to the first moment.

**Lemma 3.4.**

$$\int \sum_{\chi \ (mod \ q)} L(\sigma+it, \chi)M(\sigma+it, \chi)\psi(t/T)dt = \phi^*(q)T^\psi(0) + O \left((qT)^{\epsilon+1/2-2\sigma}\log(qT)^2 + (qT)^{1-\epsilon}\right). \quad (12)$$

**Proof.** Suppose that $\chi$ is a primitive character of conductor $q$, $\sigma \in [1/2, 1]$, $t \in [T, 2T]$, then

Define

$$A := \sum_{n} \frac{\chi(n)}{n^{\sigma+it}} e^{-\frac{\pi}{\langle qT \rangle^2}}$$

Then, as

$$e^{-x} = \frac{1}{2\pi i} \int_{(1)} \Gamma(s)x^{-s}ds$$

$A$ may be written as

$$\frac{1}{2\pi i} \int_{(1)} (qT)^{2s}\Gamma(s)L(\sigma + it + s, \chi)ds.$$
Moving the contour of integration to have real part \(-1 + \epsilon\), we hit a pole at \(s = 0\). The integral at the new contour may be bounded by the exponential decay of the Gamma function, and by the functional equation for Dirichlet \(L\)-functions,

\[
L(\sigma - 1 + \epsilon + it, \chi) \ll (qT)^{\frac{3}{2} - \sigma - \epsilon} |L(2 - \sigma - \epsilon - it, \bar{\chi})| \ll (qT)^{1-\epsilon}.
\]

Hence

\[
L(\sigma + it, \chi) = A + O \left( (qT)^{-1+\epsilon} \right).
\]

By (13), \(\int \sum \ast \chi (mod \ q) L(\sigma + it, \chi) M(\sigma + it, \chi) \psi \left( \frac{t}{T} \right) dt\) is equal to

\[
\sum \mu \left( \frac{q}{w} \right) \phi(w) \sum_{n \leq (qT)^{2+\epsilon}} v(q)^{-\frac{n}{(an)^{\sigma}}} \int (an)^{-it} \psi \left( \frac{t}{T} \right) dt + O \left( (qT)^{\epsilon+\kappa(1-\sigma)} \right)
\]

If \(an > 1\) then by integration by parts \(K\) times

\[
\int (an)^{it} \psi(t/T) dt \ll K \log(an)^{-K} T^{1+\epsilon-K}
\]

so we may make the error term arbitrarily small. When \(an = 1\) then the integral is just \(T \hat{\psi}(0)\). Hence

\[
\int \sum \ast \chi (mod \ q) L(\sigma + it, \chi) M(\sigma + it, \chi) \psi(t/T) dt = \phi^\ast(q) T \hat{\psi}(0) + O \left( (qT)^{\epsilon+\kappa(1-\sigma)} \right).
\]

By (11) and (12), we see that

\[
\int \sum \ast \chi (mod \ q) |L(\sigma + it, \chi) M(\sigma + it, \chi) - 1|^2 \psi(t/T) dt
\]

is bounded by

\[
O \left( (qT)^{2-2\sigma} \log^4(qT) + (2\sigma - 1)(qT)^{1+\kappa(1-2\sigma)} \log(qT)^2 + (qT)^{1-\epsilon} \right)
\]

which concludes the proof of Proposition 3.1 and Theorem 1.1.

4. Proof of Theorem 1.4

Levinson’s original proof was long and allegedly had a reputation for being difficult. In this section we shall follow the elegant reformulation of the method by Young in [16], but in the context of families of Dirichlet \(L\)-functions. Assume the conditions of Theorem 1.4 and let \(L = \log(qT)\), and

\[
V_\chi(s) = Q \left( -\frac{1}{L} \frac{d}{ds} \right) L(s, \chi).
\]

Suppose that \(M(s, \chi)\) is a mollifier of the form

\[
M(s, \chi) = \sum_{a \leq X} \frac{\chi(a)\mu(a)}{a^s} P \left( \frac{\log(X/a)}{\log(X)} \right)
\]
where \( P(x) = \sum_i a_i x^i \) with \( P(0) = 0, P(1) = 1 \) and for convenience we shall write \( P[a] = P\left( \frac{\log(X/a)}{\log(X)} \right) \). Levinson’s method (see for example Corollary A of [5]) shows that

\[
\frac{N_0(T, q)}{N(T, q)} \geq 1 - \frac{1}{R} \log \left( \frac{1}{\varphi^*(q)T} \int_T^1 \sum_{\chi \cong q} V_\chi \left( \frac{1}{2} - \frac{R}{L} + it \right) M \left( \frac{1}{2} + it \right)^2 dt \right) + o(1) \quad (14)
\]

as \( qT \to \infty \). Additionally, restricting \( Q(t) \) to be a linear polynomial restricts \( N_0(T, q) \) to only counting simple zeros. Defining

\[
I(\alpha, \beta) = \sum_{\chi \cong q} \int_{\mathbb{R}} L(1/2 + \alpha + it, \chi)L(1/2 + \beta - it, \chi)|M(1/2 + it, \chi)|^2 \psi(t/T) dt
\]

for \( \alpha, \beta \ll L^{-1} \), and for a smooth function \( \psi(t) \) supported on \([T, 2T]\) we arrive at the integral in (14) by evaluating

\[
Q \left( -\frac{1}{L \, d\alpha} \right) Q \left( -\frac{1}{L \, d\beta} \right) I(\alpha, \beta)
\]

at \( \alpha = \beta = -R/L \). Applying Theorem (1.3) for \( X = (qT)^\kappa \) with \( \kappa < 69/128 \) we see that

\[
\frac{I(\alpha, \beta)}{\varphi^*(q)T} = \hat{\psi}(0)L(1 + \alpha + \beta, \chi_0, q) \sum_{(a, b) = 1} \frac{\mu(a)P[ad]\mu(b)P[bd]}{a^{1+\alpha}b^{1+\alpha}}
\]

\[
+ \hat{\psi}(0) \left( \frac{qT}{\pi} \right)^{-\alpha-\beta} (1 + O(L^{-1})) L(1 - \alpha - \beta, \chi_0, q) \sum_{(a, b) = 1} \frac{\mu(a)P[ad]\mu(b)P[bd]}{a^{1-\alpha}b^{1-\beta}}
\]

\[
+ O_\delta \left( (qT)^{-\delta} \right)
\]

for a positive constant \( \delta > 0 \), as the ratio of gamma functions in the integral is \( t^{-\alpha-\beta}(1 + O(t^{-1})) = T^{-\alpha-\beta}(1 + O(\log(T)^{-1})) \) for \( t \in [T, 2T] \) by Lemma 2.2, and by the assumption that \( T \gg q' \) we have \( \log(T) \gg \log(q) \). Let

\[
S(\alpha, \beta) = L(1 + \alpha + \beta, \chi_0, q) \sum_{(a, b) = 1} \frac{\mu(a)P[ad]\mu(b)P[bd]}{a^{1+\alpha}b^{1+\alpha}}
\]

so

\[
\frac{I(\alpha, \beta)}{\varphi^*(q)T} = \hat{\psi}(0) \left( S(\alpha, \beta) + (qT)^{-\alpha-\beta} S(-\beta, -\alpha)(1 + O(L^{-1})) \right) + O_\delta \left( (qT)^{-\delta} \right)
\]

\[
= \hat{\psi}(0) \left( S(\alpha, \beta) + (qT)^{-\alpha-\beta} S(-\beta, -\alpha) \right) + O(L^{-1})
\]

as long as \( S(-\beta, -\alpha) \ll 1 \) which we shall show is the case in the following lemma.

**Lemma 4.1.** Uniformly on any fixed annuli such that \( \alpha, \beta \asymp L^{-1}, |\alpha + \beta| \gg L^{-1} \)

\[
S(\alpha, \beta) = \frac{1}{(\alpha + \beta) \log(X)} \int_{\mathbb{R}^2} X^{ax+by} P(x + u)P(y + u) du|_{x=y=0} + O(L^{-1})
\]

**Proof.** For \( 1 \leq a \leq X \) and \( i \in \mathbb{N} \)

\[
\frac{i!}{\log(X)^i} \frac{1}{2\pi i} \int_{(1)} \frac{X^v}{a^v} \frac{dv}{v^{i+1}} = \left\{ \frac{\log(X/a)}{\log(X)} \right\}^i \text{ if } 1 \leq a \leq X
\]

\[
0 \text{ if } a > X
\]
hence $S(\alpha, \beta)$ is equal to
\[
L(1 + \alpha + \beta, \chi_{0,q}) \sum_{a,b,d \atop (a,b)=1 \atop (abd,q)=1} \frac{\mu(ad)\mu(bd)}{\alpha^{1+\beta}+\beta d^{1+u+v}} \frac{du}{u+1} \frac{dv}{v+1}.
\]

By considering Euler products,
\[
L(1 + \alpha + \beta, \chi_{0,q}) \sum_{a,b,d \atop (a,b)=1 \atop (abd,q)=1} \frac{\mu(ad)\mu(bd)}{\alpha^{1+\beta}+\beta d^{1+u+v}} = \frac{\zeta(1 + \alpha + \beta)\zeta(1 + u + v)A_{0,0}(u, v)}{\zeta(1 + \alpha + \beta + u)} \tag{15}
\]

where $A_{0,0}(u, v)$ is an absolutely convergent Euler product in some product of half planes containing the origin. If we can show that $A_{0,0}(0, 0) = 1$, then we may appeal to Lemma 7 from [16] to show that
\[
\frac{1}{(2\pi i)^2} \int(11) \frac{\zeta(1 + u + v)A_{0,0}(u, v)}{\zeta(1 + \alpha + \beta + u)} \frac{du}{u+1} \frac{dv}{v+1} = \frac{(\log(X))^{i+j-1}d^2}{ilj!} \frac{d^2}{dxdy} X^{x+y} \int_0^1 (x + u)^i(y + u)^j du \big|_{x=y=0} + O(L^{i+j-1})
\]
at which point we may sum over $i$ and $j$, and take a Taylor expansion of $\zeta(1 + \alpha + \beta)$ to obtain the desired result.

All that remains to be shown is that $A_{0,0}(0, 0) = 1$. Suppose that $\alpha = \beta = u = v = s > 0$, then by (15)
\[
A_{s,s}(s, s) = L(1 + 2s, \chi_{0,q}) \sum_{a,b,d \atop (a,b)=1 \atop (abd,q)=1} \frac{\mu(ad)\mu(bd)}{(abd)^{1+2s}} = \sum_{a,b,d,u \atop (a,b)=1 \atop (abdn,q)=1} \frac{\mu(ad)\mu(bd)}{(abdn)^{1+2s}}
\]
by the Dirichlet series of the $L$-function. Re-labelling $a = ad$, $b = bd$, $m = bn$, $n = an$, we may write the sum as
\[
\sum_{am=bn \atop (abmn,q)=1} \frac{\mu(a)\mu(b)}{(abmn)^{1/2+s}} = 1
\]
by the Möbius formula.

Hence $A_{s,s}(s, s) = 1$ for all $s > 1$. As the Euler product converges absolutely at the origin, $A_{0,0}(0, 0) = \lim_{s \to 0} A_{s,s}(s, s) = 1$. \hfill $\square$

We have now arrived at the equivalent of Lemma 6 in [16]. By precisely the same method as Young’s we may arrive at the following proposition.

**Proposition 4.1.**
\[
\frac{1}{\phi^*(q)T} \int_1^T \sum_{\chi \pmod{q}} |V_\chi(1/2 - R/L + it)M(1/2 + it)|^2 \, dt = c(P, Q, R) + O(L^{-1})
\]
where
\[
c(P, Q, R) = 1 + \frac{1}{\kappa} \int_0^1 \int_0^1 e^{2Rv} \left( \frac{d}{dx} e^{R\kappa x} P(x + u)Q(v + \kappa x) \big|_{x=0} \right)^2 \, du \, dv
\]
for some positive constant $R$. 
The next step is to choose $R, P,$ and $Q$ to maximise
\[
1 - \frac{1}{R} \log(c(P, Q, R))
\]
subject to the conditions that $R$ is a positive constant, $P(0) = 0, P(1) = 1$ and $Q(0) = 1$. We shall stipulate that $Q$ is a linear polynomial, in order to determine a lower bound on the proportion of simple zeros on the critical line. The optimisation process can be found in Section 4 of Conrey’s paper [4]. This method demonstrates that the optimal choice for $P(x)$ is of the form
\[
P(x) = e^{rx} - e^{sx}
\]
for $r, s$ constants. While this is not a polynomial, it may be uniformly approximated by real polynomials. Choosing
\[
Q(x) = 1 - 1.035x, \quad R = 1.179
\]
gives
\[
1 - \frac{1}{R} \log(c(P, Q, R)) = 0.382156
\]
and hence
\[
\frac{N_0(T, q)}{N(T, q)} \geq 0.382
\]
for large enough $qT$.

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