The Structure of a Viscoplastic Fluid Flow during Filling of a Circular Pipe/Plane Channel

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Abstract—The viscoplastic fluid flow initiated in a circular pipe/plane channel during its filling in the gravity field at the flow rate specified at the inlet section is investigated. A mathematical formulation of the problem is stated based on complete equations of motion, the continuity equation, the natural boundary conditions on a free surface, and no-slip boundary condition on the solid wall. The rheological behavior of the medium is described by the Schwedoff–Bingham law, which presupposes the existence of quasi-solid motion zones (unyielded zones) in regions of low strain rates. The numerical solution of the problem is based on the finite-difference approach including the finite volume method and SIMPLE algorithm for calculating velocity and pressure fields at the internal nodes of a staggered grid. The method of invariants is used to satisfy the boundary conditions on the free surface. To provide a thorough computation of the flow with unyielded regions, regularization of the rheological equation is implemented. The behavior of a free boundary, flow structure, and flow characteristics as a function of the main parameters is investigated. It is found that in the course of time the initially flat free boundary acquires a stationary convex shape, which remains invariant while moving through the pipe/channel at the rate-average velocity. In the flow near and far away from the free boundary, one can distinguish fountain flow zones and one-dimensional flow regions, respectively. The typical flow structures with different numbers and various locations of unyielded regions in the flow are shown. The topograms of the above-mentioned flow structures as functions of the ratio of viscous and gravity forces and plasticity in the fluid flow are plotted. The stable and unstable behavior of the free boundary shape is shown to be related to the values of the constitutive parameters.

Keywords: channel, pipe, viscoplastic fluid, free surface, filling, numerical method, fountain flow, regime, unyielded regions

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1. INTRODUCTION

Flows of non-Newtonian fluids with a free boundary occur in many branches of industry. In particular, filling plane channels and circular pipes is an integral part of the technology of forming products made of polymeric materials by injection molding. The rheology of free-flowing polymeric compounds is related to processing conditions and can vary from Newtonian rheology to the exhibition of properties typical for nonlinearly viscous and viscoelastic fluids. The Schwedoff–Bingham law [1] is widely used for describing the rheological behavior of polymer melts.

A large number of investigations in which filling of plane and axisymmetric reservoirs is considered in the approximation of the Newtonian fluid behavior have been carried out. They include those with allowance for dissipative heating and dependence of viscosity on temperature [2–8]. Processes of channel filling with a non-Newtonian fluid were considered in [9–16] where the kinematics of the flow of non-Newtonian fluids was discussed; however, the presented facts about the flow structure for viscoplastic media are very scarce and information about its relation with values of other rheological parameters is absent.
This work is aimed at studying the influence of plastic properties of a liquid medium on the formation of the flow structure during filling of a plane channel/circular pipe based on the Schwedoff–Bingham rheological equation.

2. PROBLEM FORMULATION

We consider filling of a plane channel/circular pipe with a Schwedoff–Bingham viscoplastic fluid. It is assumed that the effect of viscous dissipation is weak and the flow occurs in the isothermal regime [17]. At time \( t = 0 \), the fluid occupies only a part of the channel; the free boundary of the fluid \( \Gamma_1 \) is a plane (Fig. 1) and, to exclude the influence on the character of the flow in the vicinity of the inlet boundary, is well spaced from the opposite boundary \( \Gamma_2 \). At the boundary \( \Gamma_2 \), the velocity profile corresponding to the steady-state flow of the Schwedoff–Bingham fluid in a channel/pipe with a known constant flow rate is specified. On the solid wall \( \Gamma_3 \), no-slip boundary conditions are specified; on the line \( \Gamma_4 \), flow symmetry. At the free boundary, the shear stress is set to zero, and the normal stress is equal to the external pressure. Capillary effects are not taken into account, and the external pressure is equal to zero. The motion of the free boundary is described by the kinematic condition.

The mathematical formulation of the problem includes motion and continuity equations, which are written in the dimensionless form as follows:

\[
\begin{align*}
\text{Re} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + v \frac{\partial u}{\partial x_2} \right) &= - \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_1} \left( \eta \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \eta \frac{\partial u}{\partial x_2} \right) + \frac{\partial \eta}{\partial x_1} \frac{\partial u}{\partial x_1} + \frac{\partial \eta}{\partial x_2} \frac{\partial u}{\partial x_2} + \alpha \eta \frac{\partial}{\partial x_1} \left( \frac{u}{x_1} \right), \\
\text{Re} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + v \frac{\partial v}{\partial x_2} \right) &= - \frac{\partial p}{\partial x_2} + \frac{\partial}{\partial x_1} \left( \eta \frac{\partial v}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \eta \frac{\partial v}{\partial x_2} \right) + \frac{\partial \eta}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial \eta}{\partial x_2} \frac{\partial v}{\partial x_2} + \alpha \eta \frac{\partial}{\partial x_2} \left( \frac{v}{x_1} \right) - W, \\
\frac{\partial u}{\partial x_1} + \alpha \frac{u}{x_1} &= 0.
\end{align*}
\]

Here, \( t \) is the time; \( u, v \) are projections of the velocity vector on the coordinate axes \( x_1, x_2 \); \( p \) is the pressure; \( \eta \) is the apparent viscosity; parameter \( \alpha \) is responsible for the channel geometry and coordinate system used (\( \alpha = 0 \) corresponds to a plane channel and Cartesian coordinate system; \( \alpha = 1 \), to a round pipe and cylindrical system). The formulation of the problem includes dimensionless criteria: the Reynolds number \( \text{Re} = \rho L U \mu \) and a parameter \( W = (\rho L^2 g) / (\mu U) \) that characterizes the relation of gravity and viscous...
forces in the flow. For the scales, the following quantities are chosen: for length, \( L \) which represents the channel half-width (at \( \alpha = 0 \)) or the pipe radius (at \( \alpha = 1 \)); for velocity, the rate-average velocity in the inlet section \( U \); for time, \( L/U \); for pressure, the complex \( \mu U/L \), where \( \rho \) is the fluid density and \( g \) is the modulus of the gravity vector.

System (1) is closed by the Schwedoff–Bingham rheological relation with the apparent viscosity defined by the formula

\[
\eta = (\text{Bn} + A)/A,
\]

where \( A \) is the intensity of the strain rate tensor, \( \text{Bn} = (\tau_0 L)/\mu U \) is the Bingham number, and \( \mu, \tau_0 \) are the rheological parameters.

The boundary conditions are written in the form

\[
\Gamma_1: \quad \frac{\partial u_n}{s} + \frac{\partial u_n}{\partial n} + \alpha \frac{u}{x_i} = 0, \quad p = 2\eta \frac{\partial u_n}{\partial n};
\]

\[
\Gamma_2: \quad u = 0, \quad v = V(x_i);
\]

\[
\Gamma_3: \quad u = 0, \quad v = 0;
\]

\[
\Gamma_4: \quad u = 0, \quad \frac{\partial v}{\partial x_i} = 0,
\]

where \( u_n, u_t \) are the normal and tangential components of the velocity on the free surface, and \( V(x_i) \) is the velocity profile of the fully developed flow in the plane channel or circular pipe with a specified constant flow rate. Conditions at the boundary \( \Gamma_1 \) (see (2)) are written in the local Cartesian coordinate system (\( n, s \)) normally related to the free surface. The motion of the free boundary is described by kinematic conditions in the Lagrangian representation

\[
\frac{dx_i}{dt} = u, \quad \frac{dx_s}{dt} = v.
\]

The function \( V(x_i) \) in (3) is a solution to the steady-state problem of the viscoplastic fluid flow in an infinite channel/pipe under the action of the pressure gradient \( \delta p \) providing the unit flow rate. It has the following form:

\[
V(x_i) = \begin{cases} 
  v_0, & 0 \leq x_i \leq h, \\
  v_0 \left[ 1 - \left( \frac{x_i - h}{1 - h} \right)^2 \right], & h < x_i \leq 1,
\end{cases}
\]

where \( h = -\left(1 + \alpha \right) \text{Bn}/\delta p \) is the coordinate of the unyielded region boundary. In this region, the velocity \( v_0 \) is constant and is calculated by the formula

\[
v_0 = -\frac{\delta p}{2(1 + \alpha)} (1 - h)^2.
\]

The pressure gradient \( \delta p \) is determined from the condition implying that the dimensionless flow rate through a unit area is equal to unity. In the case of a plane channel (\( \alpha = 0 \)), it is determined by the expression

\[
\delta p = -(\text{Bn} + 2) \left\{ \cos \left[ \frac{1}{3} \arccos \left( 1 - 2 \left( \frac{\text{Bn}}{\text{Bn} + 2} \right)^3 \right) \right] + 0.5 \right\},
\]

and, in the case of a round pipe (\( \alpha = 1 \)),

\[
\delta p = -\frac{4}{3} \text{Bn} \cosh(\phi) + \frac{4}{9} (\text{Bn} + 3)^2 - 2\sqrt{A^2 + B^2} \cos \left( \frac{1}{2} \arctan \left( \frac{B}{A} \right) \right) - \frac{2(\text{Bn} + 3)}{3},
\]

where \( \phi = 2 \text{arctan} \left( \frac{B}{A} \right) \).
where

\[ \phi = \frac{1}{3} \arccosh \left( \frac{Bn + 3}{Bn} \right)^2, \quad A = -\frac{2}{3} Bn^2 \cosh(\phi) + \frac{4}{9} (Bn + 3)^2, \quad B = \frac{2}{\sqrt{3}} Bn^2 \sinh(\phi). \]

3. METHOD OF SOLUTION

The formulated problem was solved numerically. In the domain of the solution, a difference staggered grid was constructed. On this grid, using the finite volume approach and SIMPLE algorithm, the unknown variables were calculated at the internal nodes [18]. At irregular nodes in the vicinity of the free boundary, velocity and pressure values were determined by linear interpolation with the involvement of data from the free surface and regular nodes.

The free surface at the discrete level was represented by a set of markers. The velocity components at the free boundary were calculated using the invariant method [19] based on the combined writing of the equation of continuity and condition of the absence of shear stresses. The pressure at the free boundary was determined from the difference analog of the condition implying that the normal stress is equal to the external pressure. The evolution of the free surface front was calculated according to the difference analog of the kinematic condition by Euler’s scheme. The motion of the marker positioned on the contact line occurs with allowance for slip condition and the fact that the dynamic contact angle is equal to \( \pi \) [20].

The flow of a viscoplastic fluid is characterized by the formation of unyielded regions in the flow. In these regions, the stress level is lower than the yield stress and the apparent viscosity tends to infinity. To provide a thorough computation in the case of a viscoplastic medium flow without explicit distinguishing of unyielded regions, it is necessary to regularize the rheological model, which will allow one to eliminate viscosity values in the unyielded regions. This work involves the approach proposed in [21], which implies the calculation of the viscosity by the formula

\[ \eta = \frac{Bn + (A^2 + \varepsilon^2)^{3/2}}{(A^2 + \varepsilon^2)^{1/2}}, \]

where \( \varepsilon \) is the regularization parameter; when studying the process of the appearance and evolution of unyielded regions in the flow, the inequality \( \eta \lambda \leq Bn \) is used as a condition for distinguishing. This is a dimensionless analog of the condition of low strain rates referred to as the von Mises criterion for stresses.

4. RESULTS OF CALCULATIONS

Studying the process of filling of a planar channel and circular pipe with a viscoplastic medium with a given constant flow rate in the inlet section demonstrates that the initially planar free surface becomes convex with time and then moves along the channel with the rate-average velocity. This is illustrated by Fig. 2, which presents the positions of the free boundary with a time step equal to a dimensionless unity. To describe the shape of the free boundary, the parameter \( \chi \) indicating the distance between the points \( B \) and \( C \) along the \( x_2 \) axis is introduced (see Fig. 1).

Distributions of flow characteristics at time \( t = 5.5 \) are presented in Fig. 3 for the plane channel. In the vicinity of the free boundary (Figs. 3a and 3b), one can distinguish a two-dimensional flow zone; in the rest part, a one-dimensional flow zone occurs.

Figure 3d presents the distribution of the apparent viscosity. The unyielded region is colored in black. Figure 3e shows streamlines in the coordinate system moving upwards with the rate-average velocity. It is seen that the flow kinematics in the two-dimensional flow zone is of fountain behavior. The term “fountain effect” was used for the first time in [22] for describing the flow near the interface for the case when a fluid displaces another one in a capillary.
In the course of the parametric investigations of the flow during filling of the channel, three characteristic flow structures were revealed. They are presented in Fig. 4. The structures differ in the quantity and position of unyielded regions. The first structure has a single unyielded region in the symmetry line vicinity (Fig. 4a). Figure 4b shows the second structure of the flow with two unyielded regions. One of them is formed near the line of symmetry, at a distance from the free surface; the second one is adjacent to the free surface. With an increase in the Bingham number, these regions unite and form the third typical flow structure (Fig. 4c). Similar flow structures are observed in the case of a circular pipe filling.

The evolution of the free surface shape developing with time for the flow structures shown in Fig. 4 is presented in Fig. 5. In the flow structure in Fig. 4c, the unyielded region occupies the whole area of the free boundary, from the line of symmetry to the solid wall. This results in the formation of a unyielded
“plug.” The incompatibility of the no-slip condition at the solid wall and plug flow regime leads to unstable filling, which is demonstrated by the behavior of the parameter $\chi(t)$ (see curve 3, Fig. 5).

In [15], two flow structures with one and two unyielded regions were presented. Figure 6 makes it possible to compare the results obtained in this work with those from [15]. It is seen that the results are in good agreement with each other: the discrepancy in the unyielded region width in the one-dimensional flow zone does not exceed 4% of the channel width.

The flow regimes described above are determined by the value of the yield stress and the ratio of viscous and gravity forces in the flow, i.e., by the parameters $Bn$ and $W$. Figure 7 demonstrates topograms of the distinguished flow structures for a plane channel and a circular pipe where filling regimes are identified by values from the region in coordinates $W/Bn$ and $Bn$. The ratio $W/Bn$ characterizes the contribution of the force of gravity and plasticity to the formation of the flow structure. In the region above curve 1, the flow structure has a single unyielded region; below curve 3, it corresponds to unstable filling. In the region
bounded by lines 1 and 3, to the left from line 2, two unyielded regions occur in the flow; to the right, a united unyielded region is observed.

5. CONCLUSIONS

The problem of filling of a plane channel/circular pipe with a Schwedoff–Bingham viscoplastic fluid in the gravitational field has been formulated mathematically. The numerical technique of the solution has been developed based on a regularized rheological relation that excludes singularity of viscosity values in regions where the stress level is less than the yield stress. Characteristic flow structures differing in the quantity and position of unyielded regions have been revealed. Topograms of these structures have been constructed as functions of the constitutive parameters $W/Bn$ and $Bn$. 

Fig. 6. The topograms of flow structures (1) data from [15] and (2) results of this work) at $\alpha = 0$, $W = 5$, and two values of the Bingham number $Bn$: (a) 1 and (b) 1.5.

Fig. 7. The distribution of flow structures (a) in the channel ($\alpha = 0$) and (b) in the pipe ($\alpha = 1$); (○) one unyielded region, (▷) two unyielded regions, (△) united unyielded region, and (×) unstable filling.
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