Power Input to non-deterministic Subsystems via Piezoelectric Patch Actuators: Effect of the patch size and location

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Abstract. Many engineering systems such as aircraft and automotive are considered built-up structures, fabricated from components that are classified as deterministic subsystems (DS) and non-deterministic subsystems (Non-DS). The response of Non-DS is sensitive to uncertainties therefore presents problems in vibration control. Therefore, analytical solution to estimate average power delivered by a piezoelectric (PZT) patch actuator when attached to a Non-DS needs to be established. The response of Non-DS is estimated using statistical modelling technique such as statistical energy analysis (SEA), in which any external input to the subsystem must be represented in terms of power input. In this paper, ensemble average of power given by a PZT patch actuator to a simply-supported plate when subjected to structural uncertainties is studied using Lagrangian method. The effects of size and location of the PZT actuators on the power delivered to the plate are investigated. It is found that changing the patch location on the structure will not affect the average power supplied while changing the patch size will change the power magnitude proportionally but with some variations at higher frequency.

1. Introduction

Most of engineering systems are subjected to forms of excitation with frequencies that range from low to high which later determine the type of response of the system. If vibrational response characteristics such as displacement, acceleration, and stress are known precisely as functions of time, the vibration is known as deterministic vibration (DS) i.e. can be mathematically described using deterministic method. In addition, systems such as aircraft are usually subjected to excitation at high frequency. The modelling of such systems presents the problem of high-frequency vibration [1]. The physical characteristic of a high-frequency vibration is that the precise natural frequencies and shapes of high-order modes are impossible to be calculated hence causing difficulties in vibration analysis and therefore any control effort. The system response subjected to this type of vibration is also very sensitive to uncertainties due to the short wavelength deformation, in which identical structures can produce very different dynamic response properties [2]. Subsystem subjected to this type of vibration falls into non-deterministic (Non-DS) subsystem category and therefore statistical modelling method such as Statistical Energy Analysis (SEA) are commonly employed [3] [4].

PZT patch actuator has received significant attention for vibration analysis and control due to its unique direct-and-inverse PZT effect. Adding a PZT actuator on a structure is equivalent to adding an external moment to the dynamics of the structure. Conveniently, the influence of adding moment on a DS is known; mathematical model describing power supplied by PZT patch actuator is available.
However, analytical solution to estimate average power delivered by a PZT actuator when attached to a Non-DS needs to be established [5]. Such analytical solution is also required when attempting to control vibration of Non-DS as built-up structure using PZT actuator.

In this paper, ensemble average of power given by the PZT patch actuator to a plate when subjected to structural uncertainties is simulated. The effects of size and location of PZT actuator on the power delivered to the plate are investigated and presented. These two parameters are of interest considering the practicality in real application. A simply-supported plate with attached PZT patch and distributed point masses is taken as a benchmark model. The distributed point masses are used to introduce uncertainties in the response of the plate. Equation of motion for the benchmark model is derived using Lagrangian method [6].

2. Derivation of Equation of motion

2.1 A rectangular plate with distributed point masses attached with a PZT patch actuator

The constitutive equations of a general PZT material are [7]:

\[ \sigma_{kl} = c_{ijkl}^{E} \varepsilon_{ij} - e_{kij} E_k \]  \hfill (1)

\[ D_l = e_{kl} \sigma_{kl} + \xi_T^{E} E_k \]  \hfill (2)

where \( \sigma_{kl} \) and \( \varepsilon_{ij} \) are components of stress and strain tensors, respectively, \( c_{ijkl}^{E} \) is the elastic stiffness under constant electric field (Hooke’s tensor), \( e_{kij} \) is the PZT constant, \( E_k \) is the electric field, \( D_l \) is the electric displacement and \( \xi_T^{E} \) is the dielectric constant under constant strain. Consider a rectangular plate with length \( a \), width \( b \) and thickness \( h \). The plate has a PZT actuator patch located at distance \( x_1 \) along the length and distance \( y_1 \) along the width. The patch has \( x_2-x_1 \) length, \( y_2-y_1 \) width and thickness \( t_p \). A number of point masses are distributed randomly across the plate.

![Figure 1. Schematic diagram of a rectangular plate attached with PZT patch and 20 randomly-located point masses.](image)

The stresses corresponding to normal strains and the shear strain are represented as:

\[ \sigma_x = \frac{E}{1-v^2} (\varepsilon_x + v \varepsilon_y) = \frac{Ez}{1-v^2} \left( \frac{\partial^2w}{\partial x^2} + v \frac{\partial^2w}{\partial y^2} \right) \] \hfill (3)

\[ \sigma_y = \frac{E}{1-v^2} (\varepsilon_y + v \varepsilon_x) = \frac{Ez}{1-v^2} \left( \frac{\partial^2w}{\partial y^2} + v \frac{\partial^2w}{\partial x^2} \right); \quad \tau_{xy} = G_{xy} = \frac{Ez}{(1+v)} \left( \frac{\partial^2w}{\partial x \partial y} \right) \] \hfill (4)
However, with the existence of PZT patch on the plate, the cross section of the plate will not be uniform therefore stress distribution along the thickness of the plate needs to be modified [8][9]. The neutral axis of the system is shifted towards the PZT patch and this shift can be calculated by balancing the force and moment across the patch thickness:

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z - z_n)dz + \int_{\frac{h}{2}}^{h + t_p} E_{PZT}(z - z_n)dz = 0 \]  

which resulted to:

\[ z_n = \frac{E_{PZT} t_p (h + t_p)}{2(Eh + E_{PZT} t_p)} \]

where \(z_n\) is the shifted neutral axis measured from the plate centre.

![Figure 2. Cross section along xy-axes to show the shift of neutral axis due to PZT patch attachment [8]](image)

Hence, the linear strain displacement relationships are modified as [6]:

\[ \epsilon_x = -(z - z_n) \frac{\partial^2 w}{\partial x^2} \quad \epsilon_y = -(z - z_n) \frac{\partial^2 w}{\partial y^2} \quad \gamma_{xy} = -2(z - z_n) \frac{\partial^2 w}{\partial x \partial y} \]

To obtain the equation of motion of the system, Lagrangian method will be employed:

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial W_{mn}} \right) - \frac{\partial L}{\partial W_{mn}} = F_{mn}, \text{ where } L = T - U \]

where \( T \) and \( U \) are the total kinetic energy and total potential energy of the system, respectively. The total kinetic energy of the system has contributions from the plate, PZT actuator and distributed point masses:

\[ T_{total} = T_{plate} + T_{PZT} + T_{dist. point mass} \]

\[ T_{total} = \frac{1}{2} \rho h \int \left( \frac{\partial w}{\partial t} \right)^2 dA + \frac{1}{2} \rho_p t_p \int \left( \frac{\partial w}{\partial t} \right)^2 S(x, y) dA + \frac{1}{2} \int \sum_r m_r \delta(x - x_r, y - y_r) \left( \frac{\partial w}{\partial t} \right)^2 dA \]
Where \( \rho_p \) is the density of the PZT actuator, \( A_p \) is the area of the PZT actuator and \( S(x, y) = \left[ H(x - x_1) - H(x - x_2) \right] \left[ H(y - y_1) - H(y - y_2) \right] \) is a product of Heaviside function which takes care of the presence of PZT patch on the plate. The total potential energy of bending for the system has contributions from the plate and PZT actuator:

\[
U_{total} = U_{strain\; plate} + U_{strain\; PZT}
\]

\[
U_{total} = \frac{D}{2} \int \int \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] \right\} dA + \frac{D_1(x, y)}{2} \int \int \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] \right\} dA
\]

\[
+ \frac{D_2(x, y)}{2} \int \int \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu_{pe}) \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] \right\} dA + B(t) \int S(x, y) \left[ (d_{31} + \nu_{pe} d_{32}) \frac{\partial^2 w}{\partial x^2} + (d_{32} + \nu_{pe} d_{31}) \frac{\partial^2 w}{\partial y^2} \right] dA
\]

Where,

\[
D_1(x, y) = d_1 S(x, y) = \frac{E}{(1 - \nu^2)} \left( \frac{h^3}{12} + \frac{z_n^2 h}{12} \right) S(x, y)
\]

\[
D_2(x, y) = d_2 S(x, y) = \frac{E_{pe}}{(1 - \nu_{pe}^2)} \left[ \frac{t_p^3}{3} + \frac{h t_p}{4} + \frac{h t_p^2}{2} - z_n (h t_p + t_p^2) + z_n^2 t_p \right] S(x, y)
\]

\[
B(t) = \frac{E_{pe} V(t)}{2(1 - \nu_{pe}^2)} (h + t_p - 2z_n)
\]

Solving equation (8) using equations (10) and (12) and evaluating the integrations lead to the general form of equation of motion:

\[
(-\omega^2 [M] + [K])W_{mn} = [F_{mn}]
\]

Equation (16) will be used to simulate vibration response of a Non-DS rectangular plate attached with a PZT patch actuator.
3. Power delivered to a non-deterministic rectangular plate

To obtain the power delivered to the plate, the following equations for energy of the plate due to vibration are first attained [10]:

\[ E = \frac{1}{2} W_{mn}^\tau K_C W_{mn} \quad \text{or} \quad E = \frac{1}{2} \omega^2 W_{mn}^\tau MW_{mn} \]  \hspace{1cm} (17)

which uses complex modal stiffness matrix, \( K_C = K(1 + j\eta) \) and mass matrix, \( M \) respectively. \( \omega \) is frequency, \( \eta \) is modal loss factor and \( K \) is stiffness matrix. The extra term \((1 + j\eta)\) is incorporated by assuming a hysteretic damping model, introduced for proportional structural damping effect to the system [7]. Since the system is not coupled to any other subsystem (single system), power input, \( P_{in} \) is equal to power dissipated from the system.

\[ P_{in} = \eta \omega E = \frac{1}{2} \eta \omega W_{mn}^\tau K_C W_{mn} \quad \text{or} \quad P_{in} = \frac{1}{2} \eta \omega^3 W_{mn}^\tau MW_{mn} \]  \hspace{1cm} (18)

4. Simulation studies

A rectangular plate made of steel is used for simulation studies. The properties of the plate and PZT patch actuator used are tabulated in Table 1. The centre of the PZT actuator is located at 0.25 of the length and width of the plate. Twenty point masses with total mass 20% of the plate mass are located randomly on the plate surface. To simulate high-order mode response, high modal overlap (M) factor are captured by using high numbers of half-waves in the x direction and y direction; \( m=15 \) and \( n=15 \) respectively. Modal loss factor of 0.05 is used. Responses for fifty ensembles (location of point masses is randomized for each simulation) are taken, equation (18) is then used to obtain power delivered by the patch and then the responses are averaged to get the ensemble average power. Table 2 shows frequency range and its corresponding modal overlap factor for the plate.

| Properties                   | Plate (Steel)       | PZT actuator (PZT-5A) |
|------------------------------|---------------------|-----------------------|
| Young's Modulus              | 200x10^9 (Pa)       | 70x10^9 (Pa)          |
| Density                      | 7850 (kg/m^3)       | 7750 (kg/m^3)        |
| Length x Width x thickness   | 0.7x0.6x0.001 (m)   | 6x5x0.02 (cm)        |
| Poisson's ratio              | 0.33                | 0.31                 |
| Piezoelectric constant, \( d_{31} \) | -                   | -1.71x10^{-10} (V/m) |

| MOF  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| Frequency (Hz)               | 118 | 237 | 355 | 473 | 592 | 710 | 828 | 947 |

For each ensemble in Figure 3, the response exhibit small magnitude difference at \( M \) less than unity, but significant as frequency increases. When modal overlap factor, \( M \) is less than unity, the ensemble average exhibits distinct narrow peaks which indicate resonant modal response of the system. As \( M \) approaches unity, the individual modal responses begin to overlap and as it increases beyond unity, the peaks gradually become broader and indistinct i.e. resonant modal response is not clearly observable. The frequency response of multi-mode system in the high frequency range is very
sensitive to small disturbance due to the randomly located point masses in the system. Hence the detail of high frequency response is unpredictable i.e. non-deterministic.

4.1 Influence of PZT patch actuator location and size on the estimation of average power

Two case studies are considered in order to further investigate how the average power delivered by PZT patch actuator to the plate is affected when physical parameter of the system is changed.

4.1.1 Case 1: Changing PZT actuator location on plate.
Nine different grid locations on the plate, as shown in Figure 4, are taken to locate the patch and for each location, ensemble average power of 50 simulations is plotted in Figure 5. From Figure 5, it can be seen that narrow peaks are exhibited at $M<1$ to indicate resonant modal response of the system. At higher frequency range, it is observed that the peaks become broader and indistinct. In addition, the difference in magnitude between responses at higher frequency range is not significant. An important observation can be made that different PZT actuator location on structure does not affect the ensemble average power obtained.

Figure 3. Ensemble average of power delivered by a PZT actuator to plate (solid black line) for 50 ensembles.

Figure 4. Schematic figure of nine different location of PZT actuator on plate.
4.1.2 Case 2: Changing PZT actuator size.

Five different sizes of PZT actuator are taken; the original size, 0.25, 0.5, 1.5 and twice of the original size are considered separately. Ensemble average power of 50 simulations is plotted in Figure 6.

From figure 6, a general observation can be made that the magnitude of response decreases as the PZT actuator size decreases however the response shape between different PZT sizes has no
significant difference. There are some variations when the patch size is made too big or too small from the original size which indicates that mass effect becomes important at high frequency. The inertia effect tends to pull down the amplitude of the response in proportion to frequency which explains the increasing variance as patch size is made bigger.

5. Conclusions

PZT material is used widely as actuators for excitation or structural vibration control because it is compact and has wide frequency range. Understanding the way vibrational response behaves at high frequency that is by knowing the average power delivered by PZT actuator to a non-deterministic structure is a huge advantage for analysing structural response using SEA method. This paper has presented simulation studies on the ensemble average power delivered by a PZT actuator to a non-deterministic structure. Parametric studies showed that changing the patch location on the structure will not affect the average power supplied. On the contrary, changing the patch size will change the power magnitude proportionally as larger patch means larger force is applied. However, there are some variations for ensemble average which becomes more significant as the patch size is made too big or too small. This is due to the inertial effect which pulls down the amplitude of the response in proportion to frequency.

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