Two-Dimensional Compact $\mathcal{N} = (2, 2)$ Lattice Super Yang-Mills Theory with Exact Supersymmetry

Fumihiko Sugino

Okayama Institute for Quantum Physics
Kyoyama 1-9-1, Okayama 700-0015, Japan

fumihiko_sugino@pref.okayama.jp

Abstract

We construct two-dimensional $\mathcal{N} = (2, 2)$ lattice super Yang-Mills theory, where the gauge and Higgs fields are all represented by $U(N)$ compact variables, with keeping one exact supercharge along the line of the papers [1, 2, 3]. Interestingly, requirements of the exact supersymmetry as well as of the compact gauge and Higgs fields lead to the gauge group $U(N)$ rather than $SU(N)$. As a result of the perturbative renormalization argument, the model is shown to flow to the target continuum theory without any fine-tuning. Different from the case of noncompact Higgs fields, the path integral along the flat directions is well-defined in this model.
1 Introduction

Lattice formulations of supersymmetric theories have a long history and still have been vigorously investigated [4]–[14]. Recently, a formulation of super Yang-Mills (SYM) theories based on the idea of the deconstruction has been discussed [16]. Also, various supersymmetric lattice models have been constructed from the connection to topological field theory via twisted supersymmetry [1, 2, 3, 19, 20, 21].

In this paper, starting from the two-dimensional $\mathcal{N} = (2, 2)$ SYM model constructed in ref. [2], we will present a new lattice model where the Higgs scalar fields are represented by compact variables with one supercharge exactly realized. In the previous model discussed in [2], the Higgs fields are described as noncompact variables so that the flat directions (the minima of the Higgs potential) are continuously distributed over a noncompact region. It may cause some difficulty for the path integral over the configurations corresponding to the flat directions on performing actual numerical simulations. The model presented here is supersymmetric and free from the problem. In a sense, this model may be regarded as a modification of the model considered in ref. [13] so that the supersymmetry $Q$ is exactly realized on the lattice. From the theoretical point of view, in theories defined on three- or higher dimensions, each of the Higgs vacuum expectation values in the flat directions composes a superselection sector. Because the superselection sectors do not affect each other, we can say that each of the Higgs vacuum expectation values defines a theory independently. On the other hand, in two-dimensional case, because continuous symmetries are not spontaneously broken [23], we should take into account all the superselection sectors together. Namely, we have to sum over all the contributions from the flat directions. In the lattice model constructed here, the summation over the flat directions is unambiguously defined, and its dynamics can be explicitly investigated with some supersymmetry preserved. We hope it useful for a progress on lattice formulations of supersymmetric theories, although the positivity of the fermion determinant remains to be investigated for numerical computations of the model.

We consider the gauge group $G = U(N)$ instead of $SU(N)$. The case $G = SU(N)$ seems not to realize the exact supersymmetry with the compact gauge and Higgs fields in a straightforward manner. For $G = U(N)$, we impose some admissibility condition for the lattice action in order to resolve the problem of the vacuum degeneracy in the gauge terms as discussed in ref. [2]4. The action has a similar form to that of the U(1) chiral gauge theory presented by Lüscher [24]. Unfortunately, it does not allow to perform the strong coupling expansion with respect to the gauge

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1For recent reviews, see refs. [15].
2For related works, see refs. [17, 18].
3In the works [1, 2, 3, 22], we are considering nonchiral theories and using the term “two-dimensional $\mathcal{N} = 2$ or 4”, although it may be most precise to use “two-dimensional $\mathcal{N} = (2, 2)$ or (4, 4)”.
4The similar problem was encountered in ref. [7].
part. It is quite interesting to overcome this point and modify the model to make the strong coupling expansion possible.

This paper is organized as follows. In section 2, we discuss the two-dimensional \( \mathcal{N} = (2, 2) \) SYM lattice model for the gauge group \( G = U(N) \) introduced in [2]. Also, the path-integral measure is shown to be invariant under the exact supersymmetry \( Q \). The renormalization argument for the \( G = U(N) \) case is presented. In section 3, we modify the supersymmetry \( Q \) in the last section so that the Higgs fields can be represented as compact unitary variables. We construct the lattice action where all the gauge and Higgs fields are compact variables with the modified exact supersymmetry \( Q \) preserved. The invariance of the path-integral measure is also maintained. The perturbative analysis shows that the target continuum theory is obtained from the lattice model without any fine-tuning. In this action, the flat directions are compactified to a finite domain, and thus the difficulty on the numerical simulations is resolved. We summarize the results obtained so far and discuss some future directions in section 4.

2 Two-dimensional Lattice \( \mathcal{N} = (2, 2) \) SYM with Noncompact Higgs Fields

We discuss the supersymmetric lattice action of two-dimensional \( \mathcal{N} = (2, 2) \) SYM theory for \( G = U(N) \) presented in [2].

Other than the gauge link variables \( U_\mu(x) \in U(N) (\mu = 1, 2) \), the two-dimensional \( \mathcal{N} = (2, 2) \) SYM theory has noncompact complex Higgs scalars \( \phi(x) = X_3(x) + iX_4(x), \quad \bar{\phi}(x) = X_3(x) - iX_4(x) \), and fermions denoted as \( \psi_\mu(x), \chi(x), \eta(x) \). They are transformed under the exact supersymmetry \( Q \) as

\[
QU_\mu(x) = i\psi_\mu(x)U_\mu(x), \\
Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i(\phi(x) - U_\mu(x)\phi(x + \hat{\mu})U_\mu(x)^{-1}), \\
Q\phi(x) = 0, \\
Q\chi(x) = H(x), \quad QH(x) = [\phi(x), \chi(x)], \\
Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)],
\]

(2.1)

where \( H(x) \) is an auxiliary field. Bosonic fields \( H(x), X_3(x), X_4(x) \) and the fermions are \( N \times N \) hermitian matrices put on the lattice site \( x \). \( Q \) is nilpotent up to the infinitesimal gauge transformation with the complexified parameter \( \phi(x) \). In terms of \( X_3(x) \) and \( X_4(x) \), the transformation reads

\[
QX_3(x) = \frac{1}{2}\eta(x), \quad QX_4(x) = \frac{i}{2}\eta(x).
\]

(2.2)

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\(^5\)It is based on a formulation of topological field theory in continuum space-time by Witten [25].
Notice that the $Q$-transformation of $U_\mu(x)$ is a left-action of $U(N)$ group and remains the $U(N)$ Haar measure $[dU_\mu(x)]$ invariant. For the real Grassmann parameter $\varepsilon$ ($\varepsilon^2 = 0$),

$$U_\mu(x) \mapsto e^{-\varepsilon\psi_\mu(x)}U_\mu(x) = U_\mu(x) + i\varepsilon QU_\mu(x). \quad (2.3)$$

Since $\varepsilon\psi_\mu(x)$ is anti-hermitian

$$e^{-\varepsilon\psi_\mu(x)}$$

may be regarded as an element of $U(N)$.

In the expansion by a basis of the hermitian matrices \{\(T^a\)\}_{a=1,\ldots,N^2}: (field)(x) = \sum_a (\text{field})^a(x)T^a, the coefficients \(\phi^a(x), \bar{\phi}^a(x)\) are complex, and the fermionic variables \(\psi_\mu^a(x), \chi^a(x), \eta^a(x)\) may be regarded as complexified Grassmann to be compatible to the $U(1)_R$ rotations (2.13) given later. Notice that $\phi^a(x)$ and $\bar{\phi}^a(x)$ can be treated as independent variables in the path integral and that each of $H^a(x)$ is allowed to be shifted by a complex number. Thus, (2.1) is seen to be consistently closed in the path-integral expression of the theory.

The lattice action is constructed as

$$S_{\text{noncompact}}^{2D,N=2} = Q\frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i\chi(x)\hat{\Phi}(x) + \chi(x)H(x) 
+ i \sum_{\mu=1}^2 \psi_\mu(x) \left( \bar{\phi}(x) - U_\mu(x)\bar{\phi}(x + \hat{\mu})U_\mu(x)^{-1} \right) \right] \quad (2.4)$$

when \(|1 - U_{12}(x)| < \epsilon\) for \(\forall x\), and

$$S_{\text{noncompact}}^{2D,N=2} = +\infty \quad (2.5)$$

otherwise. Here $U_{\mu\nu}(x)$ are plaquette variables written as

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + \hat{\mu})U_\mu(x + \hat{\nu})^{-1}U_\nu(x)^{-1} = U_{\nu\mu}(x)^\dagger. \quad (2.6)$$

For definiteness, here and in what follows we use the following definition of the norm of a matrix $A$:

$$||A|| \equiv \left[ \text{tr} \left( AA^\dagger \right) \right]^{1/2}, \quad (2.7)$$

and $\epsilon$ is a positive number chosen as

$$0 < \epsilon < 2. \quad (2.8)$$

Also

$$\hat{\Phi}(x) = \frac{\Phi(x)}{1 - \frac{1}{4\epsilon^2}||1 - U_{12}(x)||^2}, \quad (2.9)$$

$$\Phi(x) = -i [U_{12}(x) - U_{21}(x)]. \quad (2.10)$$

$^6(\varepsilon\psi_\mu(x))^\dagger = \psi_\mu(x)^\dagger \varepsilon^\dagger = \psi_\mu(x)\varepsilon = -\varepsilon\psi_\mu(x)$. 

$^7$Note that since the action is a holomorphic function with respect to the complexified Grassmann variables and there do not appear the anti-holomorphic variables, the degrees of freedom of the fermions do not change by the complexification. In a sense, it can be regarded as a sort of analytic continuation of the Grassmann variables [26].
Explicitly, (2.4) is

\[
S_{2D N=2}^{\text{noncompact}} = \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 + H(x)^2 - iH(x)\hat{\Phi}(x) \right. \\
+ \sum_{\mu=1}^2 \left( \phi(x) - U_\mu(x)\phi(x + \hat{\mu})U_\mu(x) \right) \left( \bar{\phi}(x) - U_\mu(x)\bar{\phi}(x + \hat{\mu})U_\mu(x) \right) \\
- \frac{1}{4} \eta(x)[\phi(x), \eta(x)] - \chi(x)[\phi(x), \chi(x)] \\
- \sum_{\mu=1}^2 \psi_\mu(x)\psi_\mu(x) \left( \bar{\phi}(x) + U_\mu(x)\bar{\phi}(x + \hat{\mu})U_\mu(x) \right) \\
\left. + i\chi(x)Q\hat{\Phi}(x) - i \sum_{\mu=1}^2 \psi_\mu(x) \left( \eta(x) - U_\mu(x)\eta(x + \hat{\mu})U_\mu(x) \right) \right]. 
\]

(2.11)

After integrating out \( H(x) \), the gauge kinetic terms are induced as

\[
\frac{1}{8g_0^2} \sum_x \text{tr} \left( \hat{\Phi}(x)^2 \right) = \frac{1}{8g_0^2} \sum_x \text{tr} \left[ \frac{2 - U_{12}(x)^2 - U_{21}(x)^2}{(1 - \frac{1}{g^2}||1 - U_{12}(x)||^2)^2} \right]. 
\]

(2.12)

Although the numerator means the minima given by the configurations satisfying \( U_{12}(x)^2 = 1 \), the admissibility condition \( ||1 - U_{12}(x)|| < \epsilon \) with (2.8) allows the unique one \( U_{12}(x) = 1 \) among them. The denominator guarantees smoothness of the Boltzmann weight \( \exp \left[ -S_{2D N=2}^{\text{noncompact}} \right] \). Also, the Boltzmann weight consists of a product of local factors, leading the locality of the theory. The form of the action is somewhat similar to that of the U(1) chiral gauge theory constructed by Lüscher [24].

The action (2.4) is clearly \( Q \)-invariant from its \( Q \)-exact form. Furthermore, the invariance under the following global \( U(1)_R \) rotation holds:

\[
U_\mu(x) \rightarrow U_\mu(x), \quad \psi_\mu(x) \rightarrow e^{i\alpha} \psi_\mu(x), \\
\phi(x) \rightarrow e^{2i\alpha} \phi(x), \quad \chi(x) \rightarrow e^{-i\alpha} \chi(x), \\
\bar{\phi}(x) \rightarrow e^{-2i\alpha} \bar{\phi}(x), \quad \eta(x) \rightarrow e^{-i\alpha} \eta(x).
\]

(2.13)

The \( U(1)_R \) charge of each variable is read off from (2.13), and \( Q \) increases the charge by one unit.

We can see the \( Q \)-invariance of the path-integral measure of the model:

\[
d\mu_{2D N=2}^{\text{noncompact}} \equiv \left( \prod_{\mu} [dU_\mu(x)] \right) [dX_3(x)][dX_4(x)][dH(x)] \left( \prod_{\mu} [d\psi_\mu(x)] \right) [d\chi(x)][d\eta(x)] 
\]

(2.14)
with \([d(field)(x)] \equiv \prod_x \prod_{\alpha=1}^{N^2} d(field)_{\alpha}(x)\) for each hermitian field. If we express the hermitian fields collectively as \(\varphi_I\) with the index \(I\) running over the species of the fields, the sites and the gauge indices:

\[
\varphi_I = \{X_3^{\alpha}(x), X_4^{\alpha}(x), H^{\alpha}(x), \psi_{\mu}^{\alpha}(x), \chi^{\alpha}(x), \eta^{\alpha}(x)\},
\]

(2.15)

the response of the measure under the \(Q\)-transformation is written as

\[
d_{\mu_{2D}^{\text{noncompact}}} \to d_{\mu_{2D}^{\text{noncompact}}} \left(1 + i \varepsilon \sum_I (-1)^{|\varphi_I|} \partial(Q\varphi_I)/\partial \varphi_I\right).
\]

(2.16)

\((-1)^{|\varphi_I|}\) means the statistics of the field \(\varphi_I\): +1 (-1) for bosons (fermions). Although \(QU_{\mu}(x)\) contains the variable \(\psi_{\mu}(x)\), the effect to the measure is proportional to \(\varepsilon^2 (=0)\) because it contributes to off-diagonal parts of the Jacobian matrix. Note that \(\partial(Q\psi_{\mu}(x))/\partial \psi_{\mu}^{\alpha}(x) = 0\), since the first term of \(Q\psi_{\mu}(x)\) in (2.1) is

\[
iv_{\mu}(x)\psi_{\mu}(x) = -\frac{1}{2} \sum_{a,b,c} f_{abc} \psi_{\mu}^{a}(x)\psi_{\mu}^{b}(x)T_{\mu}^{c}
\]

(2.17)

with \(f_{abc}\) being the structure constant of \(U(N)\). Thus, it is easily seen that

\[
\sum_I (-1)^{|\varphi_I|} \partial(Q\varphi_I)/\partial \varphi_I = 0
\]

and the measure \(d_{\mu_{2D}^{\text{noncompact}}}\) is \(Q\)-invariant.

**Renormalization** It is straightforward to see that after the rescaling as

\[
\phi(x) \to a\phi(x), \quad \bar{\phi}(x) \to a\bar{\phi}(x), \quad H(x) \to a^2H(x), \quad \psi_{\mu}(x) \to a^{3/2}\psi_{\mu}(x),
\]

\[
\chi(x) \to a^{3/2}\chi(x), \quad \eta(x) \to a^{3/2}\eta(x)
\]

(2.18)

for the lattice spacing \(a\), the lattice action (2.11) reduces to the continuum action of \(\mathcal{N} = (2,2)\) SYM in the continuum limit \(a \to 0\) with \(g_2 \equiv g_0/a\) fixed. Thus the full \(\mathcal{N} = (2,2)\) supersymmetry and the rotational symmetry in two dimensions are restored in the classical sense. We will check whether the symmetry restoration persists against the quantum corrections, i.e. whether symmetries of the lattice action forbid any induced relevant or marginal operators which are to obstruct the symmetry restoration. In order to consider the quantum effects near the continuum limit, we assume the fixed point at \(g_0 = 0\), which is suggested by the asymptotic freedom of the theory, and treat the quantum fluctuations in the perturbative way around \(g_0 = 0\).

It is useful for the renormalization argument to note symmetries of the lattice action (2.4):

- lattice translational symmetry
• U(N) gauge symmetry
• supersymmetry Q
• global U(1)$_R$ internal symmetry
• reflection symmetry $x \equiv (x_1, x_2) \to \tilde{x} \equiv (x_2, x_1)$ with

$$(U_1(x), U_2(x)) \to (U_2(\tilde{x}), U_1(\tilde{x}))$$

$$(\psi_1(x), \psi_2(x)) \to (\psi_2(\tilde{x}), \psi_1(\tilde{x}))$$

$$(H(x), \chi(x)) \to (-H(\tilde{x}), -\chi(\tilde{x}))$$

$$(\phi(x), \bar{\phi}(x), \eta(x)) \to (\phi(\tilde{x}), \bar{\phi}(\tilde{x}), \eta(\tilde{x})). \quad (2.19)$$

The mass dimension of the coupling constant squared $g^2$ is two. For generic boson field $\phi$ (other than the auxiliary field) and fermion field $\psi$, the dimensions are 1 and 3/2 respectively. Thus, operators of the type $\varphi^a \partial^b \psi^{2c}$ have the dimension $p \equiv a + b + 3c$, where ‘$\partial$’ means a derivative with respect to a coordinate. From the dimensional analysis, the operators receive the following radiative corrections up to some powers of possible logarithmic factors:

$$\left( \frac{a^{p-4}}{g^2} + c_1 a^{p-2} + c_2 a^p g^2 + \cdots \right) \int d^2 x \varphi^a \partial^b \psi^{2c}, \quad (2.20)$$

where the first term in the parentheses represents the contribution from the tree level, and the term containing the coefficient $c_\ell$ comes from the $\ell$-loop contributions. It is easily seen from $g^2$ playing the same role as the Planck constant $\hbar$ in the action (2.4). From the formula (2.20), we read that operators with $p = 1, 2$ are relevant or marginal arising only at the one- and two-loop levels.

Because we know that the lattice action reduces to the desired continuum SYM action in the classical continuum limit, we should check operators with $p = 0, 1, 2$. Operators with $p \leq 4$ are listed in Table 1. The identity operator corresponding to $p = 0$ merely shifts the action by a constant, which is not interesting to us. For the cases $p = 1, 2$, the U(N) gauge invariance and the U(1)$_R$ symmetry allow the scalar mass operator $\text{tr} (\phi \bar{\phi})$ and the auxiliary field $\text{tr} H$. The former is forbidden by the supersymmetry $Q$, and the latter by the reflection symmetry (2.19). Hence, it is seen that no relevant or marginal operators except the identity are generated by the radiative corrections, and that the model flows to the desirable continuum theory without any fine-tuning. As a conclusion, the rotational symmetry and the full $\mathcal{N} = (2, 2)$ supersymmetries are restored in the continuum limit.

Since the Higgs fields $\phi(x)$ and $\bar{\phi}(x)$ are noncompact, the action has noncompact flat directions satisfying $[\phi(x), \bar{\phi}(x)] = 0$. Numerical calculations for the model may lead to divergent quantities from the integration along the flat directions. In

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8It is not anomalous because all the matter fields belong to the adjoint representation. Note that the first Chern class $\text{Tr} (\text{Adj}) F_{12}^{(\text{Adj})}$ with respect to the adjoint representation vanishes.
As indicated in the r.h.s., they reproduce all the bosonic variables except the auxiliary field $H$ sitting on the lattice site $QU$ and resolves the problem of the flat directions.

In this section, we promote the variables $X_3(x)$ and $X_4(x)$ to the $U(N)$ matrices:

$$X_3(x) \to V_3(x) \equiv e^{iaX_3(x)}, \quad X_4(x) \to V_4(x) \equiv e^{iaX_4(x)}$$

(3.1)

sitting on the lattice site $x$, and construct a supersymmetric lattice theory where all the bosonic variables except the auxiliary field $H(x)$ are represented as compact unitary matrices.

As counterparts of noncompact $\phi(x)$ and $\bar{\phi}(x)$, let us introduce

$$V(x) \equiv -i(V_3(x) + iV_4(x)) = 1 - i + a\phi(x) + O(a^2),$$

$$V(x)^\dagger = i(V_3(x)^{-1} - iV_4(x)^{-1}) = 1 + i + a\bar{\phi}(x) + O(a^2).$$

(3.2)

As indicated in the r.h.s., they reproduce $\phi(x)$, $\bar{\phi}(x)$ for small $a$ up to some unimportant additive constants. Similarly to the transformation $QU_\mu(x)$ in the last section, we would like to consider a suitable modification of the $Q$-transformation which remains the Haar measures $[dV_3(x)]$ and $[dV_4(x)]$ invariant with keeping the nilpotency (up to infinitesimal gauge transformation). Suppose that such a transformation can be written as left- or right- actions of $U(N)$ group:

$$V_3(x) \mapsto e^{i\varepsilon A(x)}V_3(x)e^{-i\varepsilon B(x)} = V_3 + i\varepsilon(A(x)V_3(x) + V_3(x)B(x)),$$

$$V_4(x) \mapsto e^{i\varepsilon E(x)}V_4(x)e^{-i\varepsilon F(x)} = V_4 + i\varepsilon(E(x)V_4(x) + V_4(x)F(x)),$$

(3.3)

with a real Grassmann parameter $\varepsilon$. Also, $A(x)$ and $B(x)$ ($E(x)$ and $F(x)$) should be Grassmann odd and independent of $V_3(x)$ ($V_4(x)$). The $Q$-transformation reads as

$$QV_3(x) = A(x)V_3(x) + V_3(x)B(x), \quad QV_4(x) = E(x)V_4(x) + V_4(x)F(x)$$

(3.4)
Here, we demand $QV(x) = 0$ corresponding to $Q\phi(x) = 0$ in (2.1), and require that (3.4) reduces to (2.2) in the $a \to 0$ limit. As a solution, we shall take a choice of $A(x), B(x), E(x), F(x)$ as\(^9\)

$$A(x) = \frac{i}{2}V_4(x)\eta(x), \quad B(x) = E(x) = 0, \quad F(x) = -\frac{1}{2}\eta(x)V_3(x). \quad (3.7)$$

Then,

$$QV_3(x)^{-1} = -V_3(x)^{-1}(QV_3(x)V_3(x)^{-1} = -\frac{i}{2}V_3(x)^{-1}V_4(x)\eta(x),$$

$$QV_4(x)^{-1} = -V_4(x)^{-1}(QV_4(x)V_4(x)^{-1} = \frac{1}{2}\eta(x)V_3(x)V_4(x)^{-1},$$

$$QV(x)^\dagger = \frac{1}{2}V_3(x)^{-1}V_4(x)\eta(x) + \frac{1}{2}\eta(x)V_3(x)V_4(x)^{-1}. \quad (3.8)$$

Note that, due to the Grassmann nature of $\varepsilon$ in (3.3), $A(x), B(x), E(x), F(x)$ cause only infinitesimal actions to the $U(N)$ group manifold. Although $\varepsilon A(x)$ and $\varepsilon F(x)$ are not hermitian in (3.7), they are expanded by a basis of $N \times N$ hermitian matrices with complex coefficients. Thus, the actions $\varepsilon A(x)$ and $\varepsilon F(x)$ can be regarded as some infinitesimal complexified translations, which is somewhat similar to the infinitesimal complexified gauge transformation with the parameter $\phi(x)$ induced from the $Q$-supersymmetry in the last section.

Here let us comment on the $G = SU(N)$ case. If we consider the case $G = SU(N)$ instead of $U(N)$, we must further impose the traceless condition on $A(x), B(x), E(x), F(x)$. It seems quite nontrivial how to do it. In this paper, we do not consider it and concentrate to the $U(N)$ case.

Next, in order to close the $Q$-algebra, we impose

$$Q^2V(x)^\dagger = [V(x), V(x)^\dagger], \quad (3.9)$$

which determines $Q\eta(x)$ as

$$Q\eta(x) = \frac{i}{2}\eta(x)V(x)\eta(x) + 2i\left(V_4(x)^{-1}V_3(x)V_4(x)V_3(x)^{-1} - 1\right). \quad (3.10)$$

\(^9\)Of course, there are other solutions for $A(x), B(x), E(x), F(x)$ satisfying the requirements. For example, we can choose

$$A(x) = 0, \quad B(x) = \frac{i}{2}\eta(x)V_4(x), \quad E(x) = -\frac{1}{2}V_3(x)\eta(x), \quad F(x) = 0 \quad (3.5)$$

instead of (3.7). The symmetrized version

$$A(x) = \frac{i}{4}V_4(x)\eta(x), \quad B(x) = \frac{i}{4}\eta(x)V_4(x), \quad E(x) = -\frac{1}{4}V_3(x)\eta(x), \quad F(x) = -\frac{1}{4}\eta(x)V_3(x) \quad (3.6)$$

is also a solution. In a later analysis, however it turns out that (3.6) does not lead to consistently closed $Q$-algebra.
Then, happily we can check the closure as\(^{10}\) \(Q^2 \eta(x) = [V(x), \eta(x)]\).

Finally, the \(Q\)-transformation of the compact model is summarized as

\[
Q U_\mu(x) = i \psi_\mu(x) U_\mu(x), \\
Q \psi_\mu(x) = i \psi_\mu(x) \psi_\mu(x) - i \{V(x) - U_\mu(x) V(x + \hat{\mu}) U_\mu(x)^{-1}\}, \\
Q V(x) = 0, \\
Q \chi(x) = H(x), \quad Q H(x) = [V(x), \chi(x)], \\
Q V(x) = \frac{1}{2} V_3(x)^{-1} V_4(x) \eta(x) + \frac{1}{2} \eta(x) V_3(x) V_4(x)^{-1}, \\
Q \eta(x) = \frac{i}{2} \eta(x) V(x) \eta(x) + 2i \left( V_4(x)^{-1} V_3(x) V_4(x) V_3(x)^{-1} - 1 \right), \quad (3.12)
\]

where

\[
Q V_3(x) = \frac{i}{2} V_4(x) \eta(x) V_3(x), \quad Q V_4(x) = -\frac{1}{2} V_4(x) \eta(x) V_3(x). \quad (3.13)
\]

It is nilpotent up to the infinitesimal gauge transformation with the complexified parameter \(V(x)\). After expanding \(V_3(x)\) and \(V_4(x)\) for small \(a\), (3.12) reduces to the noncompact case (2.1) (with the rescaling \(\phi(x) \rightarrow a^{-1} \phi(x), \bar{\phi}(x) \rightarrow a^{-1} \bar{\phi}(x)\)).

Making use of the transformation rule (3.12), we write down the action of the \(Q\)-exact form as

\[
S^{\text{compact}}_{2DN=2} = Q \frac{1}{2g^2_6} \sum_x \text{tr} \left\{ -\frac{i}{2} \eta(x) \{V_3(x) V_4(x)^{-1} V_3(x)^{-1} V_4(x) - 1\} - i \chi(x) \hat{\Phi}(x) \\
+ \chi(x) H(x) + i \sum_{\mu=1}^2 \psi_\mu(x) \{V(x)^\dagger - U_\mu(x) V(x + \hat{\mu})^\dagger U_\mu(x)^{-1}\} \right\} \quad (3.14)
\]

when \(|1 - U_{12}(x)|| < \epsilon\) for \(\forall x\), and

\[
S^{\text{compact}}_{2DN=2} = +\infty \quad (3.15)
\]

otherwise, with \(\hat{\Phi}(x)\) chosen as (2.9). After the \(Q\)-action in (3.14), the action becomes

\[
S^{\text{compact}}_{2DN=2} = \frac{1}{2g^2_6} \sum_x \text{tr} \left\{ -\frac{1}{4} \eta(x) V(x) \eta(x) \{V_3(x) V_4(x)^\dagger V_3(x)^\dagger V_4(x) + 1\} \\
+ [V_3(x), V_4(x)] [V_4(x)^\dagger, V_3(x)^\dagger] + H(x)^2 - i H(x) \hat{\Phi}(x) \right\}
\]

\(^{10}\)For the solution (3.5) we can also find the transformation rule \(Q \eta(x)\) consistent with (3.9) as

\[
Q \eta(x) = -\frac{i}{2} \eta(x) V(x) \eta(x) - 2i \left( V_3(x)^{-1} V_4(x) V_3(x) V_4(x)^{-1} - 1 \right), \\
Q^2 \eta(x) = [V(x), \eta(x)], \quad (3.11)
\]

while for (3.6) we can not.
+i\chi(x) \left( Q\hat{\Phi}(x) \right) - \chi(x)[V(x), \chi(x)]
- \sum_{\mu=1}^{2} \psi_\mu(x)\psi_\mu(x) \left( V(x)^\dagger + U_\mu(x)V(x + \hat{\mu})U_\mu(x)^\dagger \right)
+ \sum_{\mu=1}^{2} \left( V(x) - U_\mu(x)V(x + \hat{\mu})U_\mu(x)^\dagger \right) \left( V(x)^\dagger - U_\mu(x)V(x + \hat{\mu})U_\mu(x)^\dagger \right)
- \frac{i}{2} \sum_{\mu=1}^{2} \psi_\mu(x) \left( V_3(x)^\dagger V_4(x)\eta(x) + \eta(x)V_3(x)V_4(x)^\dagger \right)
\quad - U_\mu(x)V_3(x + \hat{\mu})V_4(x + \hat{\mu})\eta(x + \hat{\mu})U_\mu(x)^\dagger
\quad - U_\mu(x)\eta(x + \hat{\mu})V_3(x + \hat{\mu})V_4(x + \hat{\mu})U_\mu(x)^\dagger \right] .

\tag{3.16}

In this action, the Higgs fields are expressed by the compact variables \( V_3(x) \) and \( V_4(x) \), and thus the flat directions satisfying \([V_3(x), V_4(x)] = 0\) are compactified to a finite region with the supersymmetry \( Q \) maintained. After \( H(x) \) integrated out, \( \hat{\Phi}^2 \) term is induced:

\[
\frac{1}{8g_0^2} \sum_x \text{tr} \left( \hat{\Phi}(x)^2 \right) = \frac{1}{8g_0^2} \sum_x \frac{\text{tr} \left( 2 - U_{12}(x)^2 - U_{21}(x)^2 \right)}{(1 - \frac{1}{4x})||1 - U_{12}(x)||^2} .
\tag{3.17}
\]

We should remark that the strong coupling expansion is not possible with respect to (3.17) due to the denominator\(^{11}\). In \( \exp(-S_{2D_{N=2}}^\text{compact}) \), the zeros of the denominator are essential singularities. It is essentially same as the following fact. The function

\[
f(t) = \begin{cases} 
\frac{1}{t^2}e^{-c/t^2} & \text{for } t > 0 \\
0 & \text{for } t \leq 0 
\end{cases}
\tag{3.18}
\]

with \( c \) positive constant is smooth and differentiable at \( t = 0 \) for \( n = 0, 1, 2, \cdots \). In evaluating the integral

\[
\int_{-L}^{L} dt \ f(t) ,
\tag{3.19}
\]

however it is not allowed to expand the exponential in (3.18) and integrate term-by-term.

Interestingly, the \( U(1)_R \) charge is still consistently defined in the compact model. The charge 0 is assigned to \( U_\mu(x) \) and \( H(x) \), +1 to \( \psi_\mu(x) \), -1 to \( \chi(x) \) and \( \eta(x) \), +2 to \( V_3(x) \) and \( V_4(x) \), -2 to \( V_3(x)^{-1} \) and \( V_4(x)^{-1} \).

We shall see the \( Q \)-invariance of the path-integral measure

\[
d\mu_{2D_{N=2}}^\text{compact} = \left( \prod_\mu [dU_\mu(x)] \right) [dV_3(x)][dV_4(x)][dH(x)] \left( \prod_\mu [d\psi_\mu(x)] \right) [d\chi(x)][d\eta(x)] .
\tag{3.20}
\]

\(^{11}\)Of course, it is impossible to perform the strong coupling expansion for the \( Q \)-exact action (3.14) from the reasons both of the noncompact \( H(x) \)-integral and the denominator of \( \hat{\Phi}(x) \). It gives a way out of Neuberger’s no go theorem [27].
For the collective expression of the hermitian fields
\[
\varphi_I = \{H^a(x), \psi^a_\mu(x), \chi^a(x), \eta^a(x)\},
\]
the similar formula to (2.16) holds. Although \(QV_3(x)\) contains \(V_4(x)\), \(\eta(x)\) and \(QV_4(x)\) depends on \(V_3(x)\) and \(\eta(x)\), the effect to the measure vanishes again since \(\varepsilon^2 = 0\). The potentially dangerous part is only the first term of \(Q\eta(x)\) in (3.12).

Assuming the normalization of the basis tr \((T^aT^b) = \delta^{ab}\) and the completeness \(\sum_{a=1}^{N^2}(T^a)_{ij}(T^a)_{kl} = \delta_{il}\delta_{jk}\),
\[
\sum_a \frac{\partial(Q\eta^a(x))}{\partial \eta^a(x)} = \frac{i}{2} \sum_a \text{tr} (T^a[T^a[\eta(x),V(x)])
\]
\[
= \frac{i}{2} N \text{tr} ([\eta(x),V(x)]) = 0.
\]
Thus, \(\sum_I (-1)^{|\varphi_I|} \frac{\partial(Q\varphi_I)}{\partial \varphi_I} = 0\) leading the \(Q\)-invariance of the measure \(d\mu_{\text{compact}}^{2D N=2}\).

Clearly from the construction, the action (3.14) reduces to the noncompact case (2.4) in the \(a \to 0\) limit. As long as considering the renormalization in the perturbative framework, the ultraviolet property of the model does not change from the noncompact case. Thus, the argument goes parallel to the previous section showing that the lattice model leads to the target continuum theory without any fine-tuning.

4 Summary and Discussions

In this paper, we have presented a lattice formulation of two-dimensional \(N = (2,2)\) SYM theory, where the gauge and Higgs fields are all represented as \(U(N)\) compact variables with keeping one exact supercharge \(Q\). In this construction, the path-integral measure is shown to be \(Q\)-invariant. On the basis of the perturbative argument, it can be shown that the target continuum theory is obtained with no fine-tuning, similarly to the noncompact Higgs case. Here, we should remark that it does not exclude the possibility that the compact Higgs model may belong to a different universality class from the noncompact one due to some nonperturbative effects\(^{12}\). It will be important to perform nonperturbative investigation for both lattice models and clarify this point.

In the case of noncompact Higgs fields discussed in the previous papers [1, 2, 3], the flat directions are continuously distributed and spread over noncompact domains. It may give divergent quantities from the path integrals along the flat directions on

\(^{12}\)As such an example, the XY model on the two-dimensional lattice is well-known. The target space of the variable is a circle, and it is compared to a free scalar field taking values on an infinite real line. They exhibit the same behavior within the perturbative analysis of the XY model around the trivial vacuum. However, when taking into account the contribution from vortex configurations nonperturbatively, it turns out that the XY model takes place the Kosterlitz-Thouless phase transition at the nonzero coupling and has a different phase diagram from a free scalar field. For example, see ref. [28].
actual numerical simulations of the models. However, the model constructed in this paper has compact flat directions, and thus overcomes such obstruction with preserving the exact supersymmetry $Q$.

Somewhat interestingly, in order to realize both of the compact gauge and Higgs fields and the exact supersymmetry, we have been led to consider the gauge group $G = U(N)$ rather than $SU(N)$. For the $G = U(N)$ case, we have employed a formulation based on the admissibility condition to resolve the problem of the degenerate vacua as discussed in [2]. As a result, the strong coupling expansion is not allowed in this formulation for the gauge terms. It is certainly worth to reformulate the theory so that the strong coupling expansion is possible with the exact supersymmetry preserved. If it succeeds, it could become an important step towards to investigate the strong coupling dynamics of SYM theories in the framework of supersymmetric lattice models.

It is quite interesting to consider to extend the formulation to the other cases of two-dimensional $\mathcal{N} = (4, 4), (8, 8)$ and three-dimensional $\mathcal{N} = 4, 8$ discussed in [1, 2, 3]. In particular, for the cases of the two exact supercharges $Q_\pm$ realized, it is intriguing to find how to modify the lattice supersymmetry transformation rules so as to incorporate compact Higgs scalars. Since the models with two exact supercharges have much more rich symmetries compared to the cases of one supercharge, the realization is expected to be fruitful from both of theoretical and practical aspects of lattice field theories.

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