NODAL INTERSECTIONS AND GEOMETRIC CONTROL

JOHN A. TOTH AND STEVE ZELDITCH

Abstract. We prove that the number of nodal points on an $S$-good real analytic curve $C$ of a sequence $S$ of Laplace eigenfunctions $\varphi_j$ of eigenvalue $-\lambda_j^2$ of a real analytic Riemannian manifold $(M, g)$ is bounded above by $A_{g, C} \lambda_j$. Moreover, we prove that the codimension-two Hausdorff measure $H^{m-2}(N_{\varphi_j} \cap H)$ of nodal intersections with a connected, irreducible real analytic hypersurface $H \subset M$ is $A_{g, H} \lambda_j$. The $S$-goodness condition is that the sequence of normalized logarithms $\frac{1}{\lambda_j} \log |\varphi_j|^2$ does not tend to $-\infty$ uniformly on $C$, resp. $H$. We further show that a hypersurface satisfying a geometric control condition is $S$-good for a density one subsequence of eigenfunctions.

This article is concerned with the growth of the number $n(\varphi_j, C)$ of zeros of a sequence $S = \{\varphi_{\lambda_j}\}_{j=1}^{\infty}$ of Laplace eigenfunction $\varphi_{\lambda_j}$ of eigenvalue $-\lambda_j^2$ on a connected, irreducible real analytic curve $C$ of a real analytic Riemannian manifold $(M^m, g)$ of dimension $m$ without boundary. To rule out degenerate cases, we assume (as in [TZ]) that the pair $(C, S)$ satisfies a quantitative unique continuation condition $\|\varphi_j\|_{L^2(C)} \geq e^{-a\lambda_j}$ called $S$-goodness. (Definition 0.1). When $C$ is $S$-good, Theorem 0.2 asserts that there exists a constant $A$ depending only on $g, C$ so that

\begin{equation}
\tag{1}
n(\varphi_{\lambda_j}, C) \leq A \lambda_j, \quad (\lambda_j \in S)
\end{equation}

(see Figure 1). This bound generalizes Theorem 6 of [TZ] for Dirichlet/Neumann eigenfunctions of piecewise real analytic plane domains to any real analytic Riemannian manifold without boundary (of any dimension). Motivation to study nodal points on curves and related results are discussed in Section 0.6. It is a special case of estimating the codimension-two Hausdorff measure $H^{m-2}(N_{\varphi_j} \cap H)$ of nodal intersections with a connected, irreducible real analytic hypersurface $H \subset M$ and in Theorem 0.3 we prove this generalization.

The main ‘defect’ in Theorems 0.2, 0.3 is that the condition that $(C, S)$ be $S$-good is subtle and difficult to establish. Much of this article is devoted to providing sufficient conditions for ‘goodness’. The definition of $S$-good makes sense for any connected, irreducible analytic submanifold $H \subset M$, not only curves. One of the main results of this article (Theorems 0.6) gives a kind of geometric control condition that a $C^\infty$ hypersurface $H \subset M$ be $S$-good for a density one subsequence of an orthonormal basis of eigenfunctions. When $\dim M = 2$, the condition applies to curves and gives concrete and purely dynamical conditions under which (1) holds for a density one subsequence of eigenfunctions (Theorem 0.7).

To state our results, we need some notation. We denote by $\{\varphi_j\}_{j=0}^{\infty}$ an orthonormal basis of Laplace eigenfunctions,

$$-\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk},$$

Research of J.T. was partially supported by NSERC Discovery Grant # OGP0170280, an FRQNT Team Grant and the French National Research Agency project Gerasic-ANR-13-BS01-0007-0. Research of S.Z. was partially supported by NSF grant # DMS-1541126.
where \( \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) and where \( \langle u, v \rangle = \int_M u v dV_g \) (\( dV_g \) being the volume form). We denote a subsequence \( \{j_k\}_{k=1}^\infty \) of (indices of) eigenvalues by \( S \). By a slight abuse of notation, we also let \( S \) denote the associated sequence \( \{\lambda_j\} \) of eigenvalues or the sequence \( \{\varphi_j\} \) of eigenfunctions from the given orthonormal basis.

Let \( H \subset M \) be a connected, irreducible analytic submanifold. The assumptions that \( H \) is connected, irreducible and analytic will be made throughout the paper. Given a submanifold \( H \subset M \), we denote the restriction operator to \( H \) by \( \gamma_H f := f|_H \). To simplify notation, we also write \( \gamma_H f := f_H \). The criterion that a pair \((H, S)\) be good is stated in terms of the associated sequence \( u_j := \frac{1}{\lambda_j} \log |\varphi_j|^2 \) of normalized logarithms, and in particular their restrictions \( u^H_j := \gamma_H u_j := \frac{1}{\lambda_j} \log |\varphi^H_j|^2 \) to \( H \). We only consider the goodness of connected, irreducible, real analytic submanifolds.

**Definition 0.1.** Given a subsequence \( S := \{\varphi_{j_k}\} \), we say that a connected, irreducible real analytic submanifold \( H \subset M \) is \( S \)-good, or that \((H, S)\) is a good pair, if the sequence \( u_j \) with \( j_k \in S \) does not tend to \( -\infty \) uniformly on compact subsets of \( H \), i.e. there exists a constant \( M_S > 0 \) so that
\[
\sup_H u^H_j \geq -M_S, \quad \forall j \in S.
\]

If \( H \) is \( S \)-good when \( S \) is the entire orthonormal basis sequence, we say that \( H \) is completely good.

The opposite of a good pair \((H, S)\) is a bad pair. The terminology is not ideal, but was introduced in [TZ] and used in a number of articles (e.g. [JJ, BR12]) and so we continue to use it here. Note that the connected, irreducible assumption is made to prohibit taking unions \( H_1 \cup H_2 \) of two analytic submanifolds, one of which may be good and the other bad. By the definition above, the union would be good but the nodal bounds could be false.

We denote the nodal set of an eigenfunction \( \varphi_\lambda \) of eigenvalue \( -\lambda_\lambda \) by \( N_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\} \).

Our first result is the following

**Theorem 0.2.** Suppose that \((M^m, g)\) is a real analytic Riemannian manifold of dimension \( m \) without boundary and that \( C \subset M \) is connected, irreducible real analytic curve. If \( C \) is \( S \)-good, then there exists a constant \( A_{S,g} \) so that
\[
n(\varphi_j, C) := \#\{C \cap N_{\varphi_j}\} \leq A_{S,g} \lambda_j, \quad j \in S.
\]

Section 2 is devoted to the proof of Theorem 0.2. As in [TZ, Zint], we prove the bound on nodal points on curves of Theorem 0.2 by analytic continuation of the eigenfunctions and curves to the complexification of \( M \). Complexification is useful for upper bounds since the number \( n(\varphi^C_\lambda, C_C) \) of zeros of the complexified eigenfunction on the complexified curve is \( \geq \) the number of real zeros, i.e.
\[
n(\varphi^C_\lambda, C_C) := \#\{N^C_\lambda \cap C_C\} \geq n(\varphi_\lambda, C) := \#\{N^R_\lambda \cap C\}.
\]
The same technique was used in [Zint] to obtain lower bounds on the number of intersections of geodesics with the nodal set when the geodesic flow is ergodic. Since there is a significant overlap with [Zint, Ze16], we refer to those articles for much of the background on complexification. The special case of Theorem 0.2 where $M$ is a surface and $H$ is a $C^\omega$-curve was proved in [CT] using a somewhat different frequency function approach.

Figure 1: Nodal lines of a high energy state, $\lambda \sim 84$, in the quarter stadium.

We then generalize the theorem to real analytic hypersurfaces $H \subset M^m$ for manifolds of any dimension $m$. We separate out the statements and proofs because a new integral geometric method adapted from [Ze16] is used in higher dimensions.

**Theorem 0.3.** Let $(M^m, g)$ be a real analytic Riemannian manifold of dimension $m$ and let $H \subset M$ be a connected, irreducible, $S$-good real analytic hyperurface. Then, there exists a constant $C > 0$ depending only on $(M, g, H)$ so that

$$H^{m-2}(\mathcal{N}_{\varphi_{j_k}} \cap H) \leq C\lambda_{j_k}, \ (j_k \in S).$$

The remainder of the Introduction is concerned with criteria for goodness.

0.1. **Measures of goodness.** There are some natural parameters associated with a good pair $(H, S)$. The first is the density of $S$. We recall that the density of a set $S \subset \mathbb{N}$ is defined by by

$$D^*(S) := \lim_{X \to \infty} \frac{1}{X} |\{j \in S \mid j < X\}|,$$

when the limit exists. When the limit does not exist we refer to the lim sup as the upper density and the lim inf as the lower density. We say “almost all” when $D^*(S) = 1$ and if $(H, S)$ is a good pair with $D^*(S) = 1$ then we say that $H$ is ‘almost completely good’.

The second natural parameter is the rate of decay of $||\varphi^H_j||$ in the $L^2$-norm or sup-norm.

In [TZ, ET], a curve or other submanifold was defined to be good if there exists a constant $a > 0$ so that for all $\lambda_j$ sufficiently large,

$$||\varphi^H_j||_{L^2(H)} \geq e^{-a\lambda_j}.$$ (5)

In [ET] a ‘revised goodness’ condition was defined by the apriori stronger criterion that $||\varphi^H_j||_{L^\infty(H)} \geq e^{-a\lambda_j}$. In §1.2 we show that (5) (and the sup-norm analogue) are equivalent to Definition 0.1.

A much stronger quantitative goodness condition is a uniform lower bound $||\varphi^H_{j_k}||_{L^2(H)} \geq C_S$ for the $L^2$-norms of restricted eigenfunctions in the sequence $S$. Somewhat surprisingly, our main criterion for goodness produces subsequences of density $\geq 1 - \delta$ for any $\delta > 0$ which possess uniform lower bounds $C_\delta > 0$. 

0.2. A sufficient microlocal condition for goodness of a hypersurface. In this section, we give our main criterion for almost complete goodness of a hypersurface in the strong sense that the restrictions possess uniform lower bounds in the sense just mentioned. The criterion consists of two conditions on $H$: (i) asymmetry with respect to geodesic flow, and (ii) a full measure flow-out condition.

We begin with (i). In [TZ13], a geodesic asymmetry condition on a hypersurface was introduced which is sufficient that restrictions of quantum ergodic eigenfunctions on $M$ remain quantum ergodic on the hypersurface. It is reviewed in Definition 0.1 and is the same as Definition 1 of [TZ13] as well as [TZ12, DZ]. It turns out that the same asymmetry condition plus a flow-out condition implies that a hypersurface is good for a density one subsequence of eigenfunctions and that for any $\delta > 0$, the $L^2$ norms of the restricted eigenfunctions have a uniform lower bound $C_\delta > 0$ for a subsequence of density $1 - \delta$.

The asymmetry condition pertains to the two ‘sides’ of $H$, i.e. to the two lifts of $(y, \eta) \in B^*H$ to unit covectors $\xi^\pm(y, \eta) \in S^*_H M$ to $M$. We denote the symplectic volume measure on $B^*_H$ by $\mu_H$. We define the symmetric subset $B^*_S H$ to be the set of $(y, \eta)$ so that $G^t(\xi_+^+(y, \eta)) = G^t(\xi_-(y, \eta))$ for some $t \neq 0$.

**Definition 0.4.** $H$ is microlocally asymmetric if $\mu_H(B^*_S H) = 0$.

Next we turn to the flow-out condition (ii). It is that

$$\mu_L(FL(H)) = 1.$$  

where

$$FL(H) := \bigcup_{t \in \mathbb{R}} G^t(S^*_H M \setminus S^* H)$$

is the geodesic flowout of of the non-tangential unit cotangent vectors $S^*_H M \setminus S^* H$ along $H$. Since $H$ is a hypersurface, $S^*_H M \subset S^* M$ is also a hypersurface which is almost everywhere transverse to the geodesic flow, i.e. it is a symplectic transversal (see [TZ13]). It follows that the flowout is an invariant set of positive measure in $S^* M$. When $G^t$ is ergodic on $S^* M$, since $FL(H)$ is $G^t$-invariant with $\mu_L(FL(H)) > 0$, it follows that every hypersurface satisfies (6), but we do not assume ergodicity here. In section 9, we show that a large class of curves satisfy (6) on surfaces with completely integrable geodesic flows. These include convex surfaces of revolution and Liouville tori satisfying generic twist assumptions.

The next result is a sufficient condition that $H$ be almost completely good.

**Theorem 0.5.** Suppose that $H$ is a microlocally asymmetric hypersurface satisfying (6). Then: if $S = \{\varphi_{jk}\}$ is a sequence of eigenfunctions satisfying $||\varphi_{jk}||_{L^2(H)} = o(1)$, then the upper density $D^*(S)$ equals zero.

The following theorem gives a more quantitative version:

**Theorem 0.6.** Let $H \subset M$ be a microlocally asymmetric hypersurface satisfying (6). Then, for any $\delta > 0$, there exists a subset $S(\delta) \subset \{1, \ldots, \lambda\}$ of density $D^*(S(\delta)) \geq 1 - \delta$ such that $||\varphi_{\lambda j}||_{L^2(H)} \geq C(\delta) > 0, \ j \in S(\delta)$.

As mentioned above, the assumption $||\varphi_{jk}||_{L^2(H)} = o(1)$ is much weaker than the $S$-badness of $H$. In fact, we do not know any microlocal (or other techniques) that prove
goodness without proving the stronger positive lower bound. There do exist other non-microlocal techniques which directly prove goodness. In [JJ], J. Jung proved that geodesic distance circles and horocycles in the hyperbolic plane are good relative to eigenfunctions on compact or finite area hyperbolic surfaces. In [ET] it is proved that curves of positive geodesic curvature are good for Neumann or Dirichlet quantum ergodic eigenfunctions on a Euclidean plane domain.

0.3. The main results on counting nodal points on curves or measuring Hausdorff measures on hypersurfaces. A combination of Theorems 0.2 and 0.6 gives the main result on nodal intersections:

Theorem 0.7. Let \( C \) be an asymmetric \( C^\omega \) curve on a compact, closed, \( C^\omega \) Riemannian surface \((M^2, g)\) satisfying (6). Then, for any \( \delta > 0 \) there exists a subsequence \( S(\delta) \) with \( D^*(S(\delta)) \geq 1 - \delta \) for which \( C \) is \( S' \)-good and a constant \( A_{S,g}(\delta) > 0 \) such that

\[
\text{n}(\varphi_j, C) := \#\{C \cap N_{\varphi_j}\} \leq A_{S,g}(\delta) \lambda_j, \quad j \in S(\delta).
\]

The higher dimensional generalization is as follows:

Theorem 0.8. Let \( H \) be an asymmetric \( C^\omega \) hypersurface of a compact, closed, \( C^\omega \) Riemannian manifold \((M^m, g)\) satisfying (6). Then, for any \( \delta > 0 \) there exists a subsequence \( S(\delta) \) with \( D^*(S(\delta)) \geq 1 - \delta \) for which \( C \) is \( S' \)-good and a constant \( A_{S,g}(\delta) > 0 \) such that

\[
\mathcal{H}^{m-2}(N_{\varphi_j} \cap H) \leq A_{S,g}(\delta) \lambda_j, \quad j \in S(\delta).
\]

0.4. Relating weak* limits on \( M \) and on \( H \). The rest of the article is devoted to proving Theorems 0.5-0.6 which together with Theorem 0.2 imply Theorem 0.7. These results belong to the theory of weak* limits and geometric control theory and seem to us to have an independent interest.

We recall that an invariant measure \( d\mu \) for the geodesic flow on \( S^*M \) is called a microlocal defect (or defect measure, or quantum limit) if there exists a sequence \( \{\varphi_{jk}\} \) of eigenfunctions such that \( \langle A\varphi_{jk}, \varphi_{jk}\rangle_{L^2(M)} \to \int_{S^*M} \sigma_A d\mu \) for all pseudo-differential operators \( A \in \Psi^0(M) \).

There are analogous notions for semi-classical pseudo-differential operators. We assume familiarity with these notions and refer to [Zw] for background.

In Section 5 we relate matrix elements of eigenfunctions on \( M \) to those of their restrictions to a hypersurface \( H \). This material is largely drawn from [TZ13], and we review the necessary background in Section 4.

In Section 5 we relate matrix elements of eigenfunctions on \( M \) to those of their restrictions to a hypersurface \( H \). This material is largely drawn from [TZ13], and we review the necessary background in Section 4. There is an obvious relation between matrix elements on \( M \) and matrix elements on \( H \) given in Lemma 5.1. It involves a time average \( \nabla_{T,\epsilon}(a) \) of \( \gamma_H^*\text{Op}_h(a)\gamma_H \).

In [TZ13], \( \nabla_{T,\epsilon}(a) \) was decomposed into a pseudo-differential term \( P_{T,\epsilon} \) and a Fourier integral term \( F_{T,\epsilon} \) (see Proposition 5.2). The symbol of \( P_{T,\epsilon} \) is essentially a flow-out of \( a \) using that \( S^*_HM \) is a sort-of cross-section to the geodesic flow.

It was proved in [TZ13] (see also \[DZ\]) that for asymmetric hypersurfaces, the matrix elements of \( P_{T,\epsilon} \) tend to zero almost surely. For the sake of completeness, sketch the proof in Section 6.1 that for any \( (T, \epsilon) \) there exists a subsequence \( S_F \) of density one so that the matrix elements \( \langle F_{T,\epsilon}\varphi_{jk}, \varphi_{jk}\rangle_{jk \in S_F} \to 0 \).

1As discussed below, it is not even literally a cross section of \( FL(H) \).
Proposition 0.9. Suppose that $H$ is asymmetric. Then, for any $T, \epsilon > 0$ there exists a density-one sequence $S_F(T, \epsilon)$ such that for $a \in S^0(H)$,

$$\lim_{k \to \infty; j_k \in S_F(T, \epsilon)} \left( \langle Op_H(a(1 - \chi^H_\epsilon)) \varphi_{j_k} | H \rangle | L^2(H) - \langle P_{T, \epsilon}(a) \varphi_{j_k}, \varphi_{j_k} \rangle | L^2(M) \right) = 0.$$

To simplify notation, in the following we will simply write $S_F := S_F(T, \epsilon)$ suppressing the dependence on $T, \epsilon > 0$. It is necessary in general to remove a density zero subsequence. For instance, special sequences of Gaussian beams along a geodesic $\gamma$ blow up when restricted to $\gamma$.

The following Theorem asserts that the microlocal defect measures on $S^*M$ of typical subsequences on $M$ induce finite measures on $S^*_H M$ and $B^* H$. This cannot be true for all subsequences in general, since restrictions of subsequences of eigenfunctions to hypersurfaces can blow up in the $L^2$ norm. This happens for instance in the case of highest-weight spherical harmonics $\varphi_k(x, y, z) = c_0 k^{\frac{3}{2}} (x + iy)^k; k = 1, 2, 3, ...$ with $(x, y, z) \in S^2$ and $H = \{(x, y, z) \in S^2; z = 0\}$.

Theorem 0.10. Suppose that $H$ is a microlocally asymmetric hypersurface. Then, there exists a density-one subsequence $\tilde{S}$ with the property that to any microlocal defect measure $d\mu$ of a subsequence $S \subset \tilde{S}$ there corresponds a ‘disintegration measure’ $d\mu^H_S$ on $B^* H$ such that

$$\langle Op_H(a) \varphi_{j_k} | H \rangle \to \int_{B^* H} a \, d\mu^H_S, \quad a \in S^0(H).$$

Since $B^* H$ is diffeomorphic to $S^*_H M$, one can rewrite the integral in Theorem 0.10 over $S^*_H M$ instead of $B^* H$. The QER theorem of [TZ] is the special case where $\mu = \mu_L$ (Liouville measure) and the geodesic flow $G^t : S^* M \to S^* M$ is ergodic with respect to $\mu_L$.

The definition of $d\mu^H_S$ is given in Section 6.2 and is essentially the relation between a flow-invariant measure on $S^* M$ and its disintegration in terms of an induced invariant measure on the cross section $S^*_H M$. But as explained in Section 4.3, $S^*_H M$ is not a genuine cross-section and one cannot always express the disintegration measure as a measure on $S^*_H M$. This obstruction is responsible for the possible deletion of a zero density subsequence. We mainly use Theorem 0.10 in the case where $d\mu^H_S = 0$, which forces $d\mu^M_S = 0$. This can be compared with the possible microlocal defect measures of $S$ on $S^* M$, showing that they must have zero integrals against $\sigma_{Pr,s}$.

Remark 0.11. It would be interesting to see if the hypotheses of Theorem 0.5 (and the related results on weak* limits of restrictions) can be weakened, and if the conclusion can be strengthened. For instance, one ‘loss’ of a density zero subsequence occurs in Lemma 6.1. But it is possible that $\langle F \varphi_{j_k}, \varphi_{j_k} \rangle$ tends to zero for the entire sequence. It is then possible that the hypotheses imply $H$ is $S$-good for the entire sequence of $\varphi_j$. It is also possible that asymmetry alone is a sufficient hypothesis for the density one statement.
0.5. Pluri-subharmonic theory and goodness. It is natural to ask if the theory of PSH (pluri-subharmonic) functions can help identify good curves. As mentioned above, ‘goodness’ is a much weaker condition than possession of uniform lower $L^2$ bounds. In Section 1, we draw some rather modest conclusions from the literature of PSH functions. The weakness of the conclusions is due to the fact that they are valid for general sequences of PSH functions and do not make full use of the assumption that our sequences are log-moduli of eigenfunctions (2). What seems to be lacking is a theory of $L^1$ limits of normalized log-moduli of complexified eigenfunctions ((2) or (3)). For instance, no connection is known relating such limits to the geodesic flow. Developing a microlocal theory of such limits seems to us a fundamental problem.

Except in Sections 2 and 1, we do not employ complex analytic methods.

0.6. Related results and open problems. There are several motivations to study nodal points on curves. Nodal sets and curves have complementary dimension, so that the number of intersections is finite under a suitable transversality hypothesis. The goodness assumption gives a strong formulation of this transversality.

One motivation is that Crofton’s formula expresses the Hausdorff measure of a hypersurface $Y$ sets as the average number of intersections of $Y \subset M^m$ with a random line (or geodesic arc). When $Y = N_{\varphi_\lambda}$ is a nodal hypersurface, this method was used in [DF] to obtain upper bounds on $H^{m-1}(N_{\varphi_\lambda})$. More precisely, Crofton’s formula implies that

$$H^{m-1}(N_{\varphi_\lambda}) \leq \int_{\mathcal{L}} \#\{L \cap N_{\varphi_\lambda}\} d\mu(L)$$

where $\mathcal{L}$ is the set of unit geodesic arcs and $d\mu$ is the Crofton measure [DF, p. 164] (and [DF, p. 178]. As explained there, for polynomials of degree $\lambda$, $\#\{L \cap N_{P_\lambda}\} \leq C\lambda d\mu$-almost everywhere, and a more complicated argument establishes the integral bound for eigenfunctions. A related argument is given in [Lin, Lemma 3.2] using Crofton’s formula [Lin, (3.21)] and an upper bound on the number of zeros of a non-zero analytic function in the unit disc in terms of its frequency function. In [Ze16], the analogous sharp upper bound for nodal sets of Steklov eigenfunctions was proved using Crofton’s formula. The potential theoretic facts of Section 1 show that $\#\{L \cap N_{\varphi_\lambda}\} \leq C\lambda d\mu$-almost everywhere for eigenfunctions.

Counting zeros on curves is also the mechanism for obtaining lower bounds on numbers of nodal domains on certain surfaces (see e.g. [GRS, JJZ]). In contrast to this article, the main point is to obtain lower bounds on numbers of nodal points on special curves rather than upper bounds.

Another question raised and studied by Bourgain-Rudnick [BR12] is to characterize the possible submanifolds $Y$ on which some sequence $\mathcal{S}$ of eigenfunctions vanishes. In our language, $Y$ is nodal (Definition 1.7), which is an extreme form of $\mathcal{S}$-bad. Theorem 0.3 shows that $D^*(\mathcal{S}) = 0$ if $Y$ is an asymmetric hypersurface satisfying (6). This is non-trivial, since the standard example of odd eigenfunctions vanishing on the fixed point set of an isometric involution shows that a positive density sequence can vanish on a hypersurface. But the results of this paper do not determine whether there exists a subsequence of density zero vanishing on such a hypersurface. The Bourgain-Rudnick question can be generalized as follows:
**Problem** Characterize submanifolds $H$ which are $S$-bad for some subsequence. Moreover, characterize $H$ which are bad for a positive density subsequence $S = \{\varphi_{j_k}\}$ of eigenfunctions, that is, $||\varphi_{j_k}|_H||_{L^2(H)} \leq e^{-M\lambda_{j_k}}$ for all $M$. Must the sequence actually vanish on $H$?

On a flat torus all periodic geodesics are $S$-bad; in fact, they are nodal in the sense of Definition 1.7 (see subsection 1.3). On the other hand, if $H \subset \mathbb{R}^2/(2\pi\mathbb{Z})^2$ is a strictly convex curve, it is proved in [BR12] that

$$||\varphi_{j_k}||_{L^2(H)} \geq C_H > 0. \tag{8}$$

Consequently, any such curve $H$ is good. We note that if $H$ is strictly convex, it is not hard to show that $H$ is microlocally asymmetric in the sense of Definition 0.4 and also satisfies the flowout assumption $\mu_L(\text{FL}(H)) = 1$. Consequently, the lower bound in (8) is also a consequence of our Theorem 0.6, albeit only for an eigenfunction sequence of density arbitrarily close to one.

The methods of this article and of [TZ] are rather different, though both are based on analytic continuation. In this article we analytically continue the Poisson-wave kernel. At the present time, the analytic continuation is only known for manifolds without boundary (see [ZP], [L], [ST]). The analytic continuation is based on parametrix constructions which are not known at present for general manifolds with boundary. This is obviously an interesting problem. Parametrices are known for diffractive (concave) boundaries, and that would be a natural first step.

In [TZ] we used the analytic continuation of Euclidean layer potentials of $\mathbb{R}^2$ for bounded analytic domains as semi-classical Fourier integral operators. This construction should generalize to all dimensions and also to complete manifolds of negative curvature, where it is known that layer potentials are singular Fourier integral operators. The latter statement may hold in a suitable sense for domains in general complete Riemannian manifolds but to our knowledge this also remains an open problem.

0.7. **Acknowledgements.** We thank J. Galkowski for discussions of our geometric control condition, and Z. Rudnick for discussions of his results with Bourgain on nodal curves and hypersurfaces.

1. **Good curves and submanifolds**

The definition of ‘goodness’ in Definition 0.1 is motivated by properties of sequences of subharmonic functions, and they are used in the proof of Theorem 0.2. The sequence $u_j$ is not subharmonic on $M$ but has a natural extension to the complexification $M_C$ of $M$ as subharmonic functions. We denote the extension of (2) by

$$u_j^C := \frac{1}{\lambda_j} \log |\varphi_j^C|^2, \tag{9}$$

and their restrictions (3) to a complexified analytic submanifold $H_C$ by

$$u_j^{H,C} := \gamma_{H_C} u_j^C := \frac{1}{\lambda_j} \log |\varphi_j^C|_{H_C}|^2. \tag{10}$$

As we show in section 1.2, the Definition 0.1 of ‘good’ is equivalent to the following complex version:
Definition 1.1. Given a subsequence $S := \{ \varphi_{j_k} \}$ of eigenfunctions, we say that a connected, irreducible real analytic submanifold $H \subset M$ is $S$-good if the sequence (10) with $j_k \in S$ does not tend to $-\infty$ uniformly on compact subsets of $H_C$, i.e. there exists a constant $M_S > 0$ so that

$$\sup_{H_C} u_{j_k}^{H,C} \geq -M_S, \ \forall j \in S.$$  

Otherwise we call $H$ $S$-bad.

Here, $H_C$ refers to some Grauert tube of $H$ in $M_C$. The Definition does not depend on the specific radius, nor whether we use the intrinsic Grauert tube of $H$ or the intersection of $H_C$ with a Grauert tube $M_{\epsilon}$ of $M$.

Thus, $H$ is $S$-bad if the $u_{j_k}^{H,C} \to -\infty$ uniformly on compact subsets of $H$. If $H$ fails to be good, then there exists a sequence $S$ so that $H$ is $S$-bad and we refer to $H$ as a bad sequence for $H$. The simplest example of a bad pair $(H, S)$ is where the eigenfunctions of $S$ vanish on $H$; in this case we say that $H$ is a nodal submanifold (see Definition 1.7). Examples of nodal hypersurfaces are fixed point sets of an isometric involution, and then $H$ is $S$-bad for the sequence of odd eigenfunctions.

It is also obvious that if a real analytic arc $\beta$, or piece of a real analytic submanifold $H$, is bad then the entire analytic continuation of it $H$ is bad.

These definitions are motivated by the standard compactness Lemma for families subharmonic functions (see [LG] or [Ho2], Theorems 3.2.12-3.2.13). Let $v^*$ denote the USC (upper semi-continuous) regularization of $v$.

Lemma 1.2. For any compact connected irreducible analytic Riemannian manifold $(M, g)$, and any real analytic submanifold $H$, the family of pluri-subharmonic functions (10),

$$\mathcal{F}^H := \{ u_{j_k}^{H,C}, \ j = 1, 2, \ldots \}$$

on $H_\tau$ is precompact in $L_1^{\text{loc}}(H_\tau)$ as long as it does not converge uniformly to $-\infty$ on all compact subsets of $H_\tau$. Moreover:

- $\limsup_{k \to \infty} u_{j_k}^{H,C}(t + i\tau) \leq 2|\tau_H|$.

- Let $\{ u_{j_k}^{H,C} \}$ be any subsequence of $\{ u_{j_k}^{H,C} \}$ with a unique $L_1^{\text{loc}}$ limit $v$ on $S_\epsilon$ and let $v^*$ be its USC regularization. Then if $v^* < 2|\tau_H| - \epsilon$ on an open set $U \subset S_\epsilon$ then

  $$v^* \leq 2|\tau_H| - \epsilon \text{ for } \tilde{U} = \bigcup_{t \in \mathbb{R}} (U + t)$$

(11)  

$$\limsup_{k \to \infty} u_{j_k}^{H,C} \leq |\tau_H| - \epsilon \text{ on } \tilde{U}.$$  

The conditions of connectedness and irreducibility arise from this Lemma. The original statement in [Ho2, Theorem 3.2.12] pertains to sequences of subharmonic functions on connected open sets $U \subset \mathbb{R}^n$. Since the theorem is local it generalizes with no essential change to connected, irreducible complexified hypersurfaces of $M_C$. Clearly, connectedness is necessary: as mentioned in the introduction, unions $H_1 \cup H_2$ of a disjoint good and bad hypersurface would be good. If $H_1 \cap H_2 \neq \emptyset$ then $H_1 \cup H_2$ might be connected but $H_1 \cup H_2$ would still be good. The condition of ‘connected irreducible’ means that $H$ has only one component. Hence in taking unions, each hypersurface separately must be good.
1.1. **Bad submanifolds are polar.** In this section we review results on sequences of pluri-subharmonic functions which imply that $S$-bad sets are polar. This is not a restriction on real analytic curves or hypersurfaces (since they are necessarily polar) but is useful in proving the equivalence of different notions of ‘good’.

Let $\{u_j\}$ be the sequence (2) of pluri-subharmonic functions. Let $u \in L^1(M)$. We say that a subsequence $\{u_{j_k}\}_{j_k \in S}$ is a $u$-sequence if $u_{j_k} \to u$ in $L^1(M)$.

**Definition 1.3.** Suppose that $\{u_j\}_{j \in S}$ is a $u$-sequence. Define

$$W_S = \{ z \in M : \limsup_{j \to \infty} u_j(z) < u(z) \}.$$  

The following Proposition 1.39 from [LG] (see also Theorem 1.27 of [LG] and Theorem 3.4.14 of [Ho2]) will be relevant:

**Proposition 1.4.** If $S = \{u_j\}$ is a sequence of pluri-subharmonic functions on an open set $U$ and $u_j \to u$ in $L^1(U)$ then the set of points $W_S$ in $U$ where

$$\limsup_{j \to \infty} u_j < u$$

is pluri-polar.

**Definition 1.5.** Given a subsequence $S \subset \mathbb{N}$, we define $P_S \subset M$ be the set of points $z$ satisfying

$$\limsup_{j \in S} u_j(z) = -\infty.$$  

Thus, $P_S \subset W_S$ and $P_S$ is contained in a pluri-polar set. The Hausdorff dimension of a polar set in $\mathbb{R}^m$ is $\leq m - 2$ ([Ho2]). Since the statement is local the proof applies to $W \subset M$.

1.2. **Equivalence of different notions of goodness.** Here we prove the equivalence of the following notions of goodness for a real analytic function on a real analytic curve.

1. Goodness in the sense of Definition [0.1]
2. Goodness in the sense of Definition [1.1]
3. Goodness in the sense that $\| \varphi_j |_H \|_{L^2(H)} \geq e^{-a\lambda_j}$.
4. Goodness in the sense of $\| \varphi_j |_H \|_{L^\infty(H)} \geq e^{-a\lambda_j}$.

Each goodness criterion implies that there is a point $q \in H$ where $\lim_{j \to \infty} u_j^H(q) \geq -M$ for some $M > 0$. Hence, they all imply goodness in the sense (2) of Definition [1.1]. The main content of the equivalence is that the latter criterion (2) implies (1)-(3). This is non-obvious since these criterion only involve the behavior of $u_j$ on the real points of $H$.

**Proposition 1.6.** If $H$ is a real analytic curve, then (1)-(4) are equivalent.

**Proof.** First, consider the simplest case where $H$ is a curve such that (2) holds. Then $\{u_j^H\}$ is pre-compact in $L^1$. Proposition [1.4] then implies that the set where $u_j \to -\infty$ is polar in $H$. Since it has Hausdorff dimension 0 in $H$, it cannot contains the real curve $H$ and there must exist points such that (1) holds. In fact, such points of the real curve must have dimension 1 and so (3)-(4) also hold. That is, for any $\epsilon > 0$, there exists $M > 0$ and a
measureable subset $E \subset H$ of $H^1$-measure $\geq 1 - \epsilon$ where $\frac{1}{\lambda_j} \log |\varphi_j(z_0)| \geq -M$ on $E$. But then $||\varphi_j||_{L^1(H)} \geq ||\varphi_j||_{L^1(E)} \geq e^{-MA}\lambda |E|$.

\[ \square \]

Alternatively, one can prove the equivalences between (1), (2) and (3) using the following Hadamard three circles argument. We first treat the case where $\dim H = 1$ and $n = 1$.

**Proof.** (3) $\Rightarrow$ (2) since there must exist a point $q \in H$ at which $|\varphi_\lambda(q)| \geq e^{-C\lambda}$.

(2) $\Rightarrow$ (3) Suppose

\[ \sup_{z \in H_{e^1}} |\varphi_\lambda^C(z)| \geq e^{-C\lambda}. \]

By the Hadamard three circles theorem, with $0 < \theta < 1$,

\[ \sup_{z \in H_{e^1}} |\varphi_\lambda^C(z)| \leq \sup_{z \in H_{e^2}} |\varphi_\lambda^C(z)|^{1-\theta} \times \sup_{q \in H} |\varphi_\lambda(q)|^\theta \leq e^{2\epsilon^2(1-\theta)} \cdot \|\varphi_\lambda\|^\theta_{L^\infty(H)}. \]

In the last line we needed a sup estimate for $|\varphi_\lambda^C|$. For this, we recall that \[\text{[ZPI]}\]

\[ \|\varphi_\lambda^C\|_{L^\infty(H_{e^2})} = O(\lambda^{\frac{n-1}{2}} e^{\epsilon^2\lambda}) = O(e^{2\epsilon^2\lambda}). \]

Consequently, by the weak goodness assumption (12) and (13),

\[ \|\varphi_\lambda\|_{L^\infty(H)} \geq e^{-C\lambda}. \]

By continuity, we choose $q_0 \in H$ so that

\[ |\varphi_\lambda(q_0)| = e^{-C\lambda}. \]

Let $q : [0, L] \to H$ be the arclength parametrization with arclength parameter $s$. By the standard bound for Laplace eigenfunctions, one also has that

\[ \|\partial_s \varphi_\lambda\|_{L^\infty(H)} = O(\lambda^{n+1/2}). \]

Since by (14) the tangential derivative of $\varphi_\lambda$ along $H$ has at most polynomial growth in $\lambda$, it follows by Taylor expansion along $H$ centered at $q_0$ that there is an subinterval $I(\lambda) \subset H$ containing $q_0$ of length $e^{-C'\lambda}$ with $C' > C > 0$ such that for $q \in I(\lambda)$,

\[ |\varphi_\lambda(q)| \geq e^{-C'\lambda}. \]

Consequently,

\[ \|\varphi_\lambda\|_{L^2(H)} \geq e^{-C'\lambda} \]

and so, $H$ is good in the sense of (2).
We note that the argument above using (14) also proves that \((2) \iff (3)\).

\[\square\]

The case where \(H\) is a real analytic submanifold of dimension \(\geq 2\) is more complicated because \(H \subset H_C\) has codimension \(\geq 2\) and is not ruled out as a pluri-polar set. Rather we use that it is a totally real submanifold. The equivalence then follows from an (unpublished) theorem of B. Berndtsson, which says that if \(H \subset \Omega\) is totally real submanifold of a complex manifold \(\Omega\) and if \(\{u_j\}\) is a sequence of pluri-subharmonic functions converging in \(L^1(\Omega)\) to \(u\), then \(u_j|_H \to u|_H\) in \(L^1_{\text{loc}}(H)\) [Ber Theorem 3.3]. It follows immediately that (2) implies (1) and (3).

1.3. Nodal curves. The only known examples of bad curves are nodal curves in the following sense:

**Definition 1.7.** We say that a curve (e.g. a geodesic) \(H\) is a nodal curve (geodesic) if there exists a sequence \(\{\phi_{jk}\}\) of distinct eigenfunctions which vanish in \(H\). Similarly for submanifolds of higher dimension.

There are many examples of nodal geodesics. These include:

- **Rational radial geodesics on the unit disc or rational meridians on the unit sphere** are nodal geodesics. That is, one may fix an axis of rotation and consider real and imaginary parts of the associated basis \(Y^m_l(\theta, \phi)\) of spherical harmonics to get \(\sin m\theta P^m_l(\cos \phi)\). Here, \(\partial/\partial \theta\) is the generator of the rotations. Obviously the meridians defined by \(\sin m\theta = 0\) i.e. \(\theta = \frac{j\pi}{m}\) are nodal geodesics through the poles for the sequence with \(m\) fixed and \(\ell\) varying. Since \(m\) is arbitrary, any ‘rational meridian’ is a nodal geodesic, where rational means that the the angle to the fixed meridian \(\theta = 0\) is a rational number \(\frac{j\pi}{m}\) times \(\pi\).

- **Fixed point sets of involutions on surfaces of negative curvature** are nodal geodesics. Thus, any closed geodesic of the standard \(S^2\) is nodal with respect to its associated odd eigenfunctions.

- **Periodic geodesics on a flat torus \(\mathbb{R}^2/(2\pi \mathbb{Z})^2\)**. Given a periodic geodesic \(\gamma(t) = (mt, nt); 0 \leq t \leq 2\pi\) with \((m, n) \in \mathbb{Z}^2\), the sequence of Laplace eigenfunctions
  \[
  \varphi_k(x, y) = \sin (k(nx - my)), \quad (x, y) \in [0, 2\pi] \times [0, 2\pi], \quad k = 0, 1, 2, 3, ...
  \]
  clearly satisfies \(\varphi_k|_\gamma = 0\) and so \(\gamma\) is nodal. In [BR11] (see Theorem 1.1), Bourgain and Rudnick prove that in fact segments of periodic geodesics are the only real-analytic nodal curves on a flat torus. In higher dimensions, they prove that postively-curved hypersurfaces on the flat torus cannot be nodal.

It is not clear at present whether a geodesic \(H \subset W\) that fails to be good is necessarily a nodal geodesic. Another question is whether bad nodal curves must be geodesics and more general bad curves (if they exist) must be geodesics. In the case of ergodic eigenfunctions, this question is studied in [ET1]. Even in the case of the sphere, the characterization of nodal curves seems a rather difficult and open problem. For instance, it is unknown whether or not a circle of latitude different from the equator is nodal. The latter question is closely related to a classical conjecture of Stieltjes (see [BR11]).
2. Proof of Theorem 0.2

The proof of Theorem 0.2 is based on the analytic continuation of eigenfunctions to a Grauert tube $M_\tau$ in the complexification of $M$. We will not review the background on Grauert tubes and on analytic continuation of eigenfunctions and Poisson kernels, but refer to [Ze07, ZP, Zint, Ze16] for the necessary material.

We recall that any real analytic manifold $M$ admits a Bruhat-Whitney complexification $M_\mathbb{C}$, and that for any real analytic metric $g$ all of the eigenfunctions $\varphi_j$ extend holomorphically to a fixed open neighborhood $M_\varepsilon$ of $M$ in $M_\mathbb{C}$ called a Grauert tube of radius $\varepsilon$. Given a real analytic hypersurface $H$, we define $H_\varepsilon := H_\mathbb{C} \cap M_\varepsilon$.

We denote the holomorphic extension of an eigenfunction $\varphi_\lambda$ by $\varphi_\lambda^C$, respectively elements of an orthonormal basis by $\varphi_j^C$. The complex nodal set is denoted by

$$N_\lambda^C = \{ z \in M_\mathbb{C} : \varphi_\lambda^C(z) = 0 \}.$$  

We also denote the complexification of a real analytic submanifold $H$ by $H_\mathbb{C}$. We further denote the restriction of an eigenfunction to $H$ by $\varphi_j|_H$ or equivalently by $\gamma^*_H \varphi_j$ and the holomorphic extensions by $\varphi_j^C|_H$.

Let $\alpha_H : H \rightarrow M$ be a real analytic paramaterization of a real analytic submanifold. In the case of a curve $C$, we use the complexification of an arc-length parameterization,

$$\alpha_C: \mathbb{R} \rightarrow M$$

The parametrization extends to some strip $S_\varepsilon = \{ (t + i\tau) \in \mathbb{C} : |\tau| \leq \varepsilon \}$ as a holomorphic curve

$$\alpha_C^C: S_\varepsilon \rightarrow M_\varepsilon.$$  

We let $\tau_H$ be the maximal $\varepsilon$ for which there exists an analytic extension of $\alpha_C^C$.

The intersection points of $\alpha_H^C$ and $N_\lambda^C|_C$ correspond to the zeros of the pullback $(\alpha^C_H)^* \varphi_j^C$. When $C$ is a good curve, then $\varphi_j^C|_C$ has a discrete set of zeros, we can define the current of summation over the zero set by

$$[N_\lambda^C] = \sum_{\{t + i\tau : \varphi_j^C(\alpha_C^C(t + i\tau)) = 0\}} \delta_{t + i\tau}.$$  

Slightly modifying the definition (10), we define the sequence

$$v_j^C := \frac{1}{\lambda_j} \log \left| \alpha_C^* \varphi_j^C(t + i\tau) \right|^2$$

of subharmonic functions on the strip $S_\varepsilon \subset \mathbb{C}$.

By the Poincaré-Lelong formula,

$$[N_\lambda^C] = \frac{i}{\pi} \partial \bar{\partial}_{t + i\tau} \log \left| \alpha_C^* \varphi_j^C(t + i\tau) \right|^2.$$  

Put:

$$A_{L,}\left( \frac{1}{\lambda} dd^c \log |\varphi_j^C|^2 \right) = \frac{1}{\lambda} \int_{S_{\varepsilon, L}} dd^c t + i\tau \log |\varphi_j^C|^2(\alpha_C(t + i\tau)).$$
To prove that \( n(\varphi^C_\lambda, C_C) \leq A\lambda \), it suffices to show that there exists \( M < \infty \) so that
\[
A_{L,\epsilon}(\frac{1}{\lambda} d\epsilon \log |\varphi^C_j|^2) \leq M. \tag{22}
\]

To prove (22), we observe that since \( d\epsilon \log |\varphi^C_j|^2(\alpha_C(t + i\tau)) \) is a positive \((1, 1)\) form on the strip, the integral over \( S_\epsilon \) is only increased if we integrate against a positive smooth test function \( \chi_\epsilon \in C^\infty C(\mathbb{C}) \) which equals one on \( S_{\epsilon, L} \) and vanishes off \( S_{2\epsilon, L} \). Integrating by parts the \( d\epsilon \) onto \( \chi_\epsilon \), we have
\[
A_{L,\epsilon}(\frac{1}{\lambda} d\epsilon \log |\varphi^C_j|^2) \leq \frac{1}{\lambda} \int_{S_\epsilon} d\epsilon_{t+i\tau} \log |\varphi^C_j|^2(\alpha_C(t + i\tau)) \chi_\epsilon(t + i\tau)
\]
\[
= \frac{1}{\lambda} \int_{S_\epsilon} \log |\varphi^C_j|^2(\alpha_C(t + i\tau)) d\epsilon_{t+i\tau} \chi_\epsilon(t + i\tau).
\]

To complete the proof of (22) it suffices to prove that
\[
\limsup_{\lambda_j \to \infty} \frac{1}{\lambda_j} \log \left| \varphi^C_j(\zeta) \right|^2 \leq C, \quad \zeta \in S_\epsilon \tag{23}
\]
for some \( C > 0 \). Now write \( \log |x| = \log_+ |x| - \log_- |x| \). Here \( \log_+ |x| = \max\{0, \log |x|\} \) and \( \log_- |x| = \max\{0, -\log |x|\} \). In view of (23), we need upper bounds for
\[
\frac{1}{\lambda} \int_{S_{\epsilon}} \log_\pm |\varphi^C_j|^2(\alpha_C(t + i\tau)) d\epsilon_{t+i\tau} \chi_\epsilon(t + i\tau).
\]

For \( \log_+ \) the upper bound is an immediate consequence the global upper bound
\[
\limsup_{k \to \infty} \frac{1}{\lambda_j} \log |\varphi^C_j(\zeta)|^2 \leq 2\sqrt{\rho}(\zeta) \tag{24}
\]
proved in [ZPl] using the complexified wave (ie. Poisson operator). Here, \( \sqrt{\rho} \) is the Grauert tube function of \((M, g)\). On the complexified wave or strip, one lets \( A = \sup_{C_\tau} \sqrt{\rho} < \infty \) where \( C_\tau \) is the intrinsic Grauert tube of radius \( \tau \) of the curve, which in general is not defined by the same as the Grauert tube radius \( \sqrt{\rho} \) of \((M, g)\). The proof is valid for any \( \tau > 0 \) less than the maximal radius of analytic continuation of the curve.

For \( \log_- \) the lower bound follows from the \( S \)-good assumption that \( \log_- |\varphi^C_j| \leq A\lambda_j \). This establishes the bound in (23) and completes the proof of Theorem 0.2

3. Proof of Theorem 0.3

In higher dimensions, we use Crofton’s formula to prove that for \( S \)-good hypersurfaces, \( \mathcal{H}^{m-2}(N_{\varphi_\lambda} \cap H) \) is bounded above by a certain measure of the complexified nodal set. We closely follow [Ze16] and refer there for some of the background. The principal difference is that we let \((H, g_H)\) with \( g_H := g|_{TH} \) be the Riemannian manifold of [Ze16] instead of \((M, g)\). Thus, \( H \cap N_{\varphi_\lambda} \) is a real analytic hypersurface (i.e. a real analytic variety of codimension one) of \( H \) in the sense that it is the subset \( \{ \varphi_j|_H = 0 \} \subset H \) defined by the analytic function \( \varphi_j|_H \). The \( S \)-goodness assumption on \( H \) implies that \( H \cap N_{\varphi_\lambda} \) has codimension one since it certainly implies that the restricted analytic function is non-zero. It may have a singular set of codimension one in \( H \cap N_{\varphi_\lambda} \). In the following, we write \( N = H \cap N_{\varphi_\lambda} \). We retain \( m = \dim M \) so that \( m - 1 = \dim H \).

\[<|\text{One way to prove this is to use Whitney’s stratification theorem [K]. For hypersurfaces there is probably a simpler proof.}>\]
3.1. **Crofton formula.** The main result of this section is Proposition 3.3. To prepare for the statement and proof, we introduce some notation and make some useful observations. Most are from [Ze16] in a section on general hypersurfaces in Riemannian manifolds, and hence they apply to $\mathcal{N} \subset \mathcal{H}$ with only a change of notation. We recall some of the statements for the sake of completeness.

Let $\pi : T^* \mathcal{H} \to \mathcal{H}$ be the natural projection. We denote by $\omega$ the standard symplectic form on $T^* \mathcal{H}$ and by $\alpha$ the canonical one form. As above, we denote by $d\mu_L$ the Liouville measure on $\mathcal{S}^* \mathcal{H}$. Then $d\mu_L = \omega^{m-1} \wedge \alpha$ on $\mathcal{S}^* \mathcal{H}$. We also denote the Hamiltonian generating the geodesic flow $G^t_H$ by the Hamiltonian $|\xi|_g^H$ and its Hamilton vector field by $\Xi = \Xi^H$. Note that it is quite different from the geodesic flow of $(\mathcal{M},g^H)$.

Let $\mathcal{N} \subset \mathcal{H}$ be a smooth hypersurface in a Riemannian manifold $(\mathcal{H},g^H)$. We denote by $T^* \mathcal{N}^H$ the space of covectors with footpoint on $\mathcal{N}$ and $\mathcal{S}^* \mathcal{N}^H$ the unit covectors along $\mathcal{N}$. We introduce Fermi normal coordinates $(s,x_m)$ along $\mathcal{N} \subset \mathcal{H}$, where $s$ are coordinates on $\mathcal{H}$ and $x_m-1$ is the normal coordinate, so that $x_m-1 = 0$ is a local defining function for $\mathcal{N}$. We also let $\sigma, \xi_m-1$ be the dual symplectic Darboux coordinates. Thus the canonical symplectic form is $\omega_{T^* \mathcal{H}} = ds \wedge d\sigma + dx_{m-1} \wedge d\xi_{m-1}$.

**Lemma 3.1.** The restriction $\omega|_{\mathcal{S}^* \mathcal{N}^H}$ is symplectic on $\mathcal{S}^* \mathcal{N} \setminus \mathcal{S}^* \mathcal{N}$. Indeed, $\omega|_{\mathcal{S}^* \mathcal{N}^H}$ is symplectic on $T_y,\eta |_{\mathcal{S}^* \mathcal{N}^H}$ as long as $T_y,\eta$ is transverse to $\Xi^H$, since $\ker(\omega|_{\mathcal{S}^* \mathcal{M}}) = \mathbb{R}\Xi$.

It follows from Lemma 3.1 that the symplectic volume form of $\mathcal{S}^* \mathcal{M} \setminus \mathcal{S}^* \mathcal{N}$ is $\omega^{m-2}|_{\mathcal{S}^* \mathcal{N}}$. The following Lemma gives a useful alternative formula:

**Lemma 3.2.** Define

$$d\mu_{L,N} = \iota_\Xi d\mu_L |_{\mathcal{S}^* \mathcal{N}^H},$$

where as above, $d\mu_L$ is Liouville measure on $\mathcal{S}^* \mathcal{H}$. Then

$$d\mu_{L,N} = \omega^{m-2}|_{\mathcal{S}^* \mathcal{N}^H}.$$  

Indeed, $d\mu_L = \omega^{m-2} \wedge \alpha$, and $\iota_\Xi d\mu_L = \omega^{m-2}$. As in [Ze16, Corollary 8],

$$(25) \quad \mathcal{H}^{m-2}(N) = \frac{1}{\beta_m} \int_{\mathcal{S}^* \mathcal{N}^H} |\omega^{m-2}|.$$

As reviewed in [Ze16], a Crofton formula arises from a double fibration

$$\mathcal{T}$$

$$\pi_1 \leftarrow \Gamma \to \pi_2$$

where $\Gamma$ parametrizes a family of submanifolds $\mathcal{B}_\gamma$ of $\mathcal{B}$. The points $b \in \mathcal{B}$ then parametrize a family of submanifolds $\Gamma_b = \{ \gamma \in \Gamma : b \in \mathcal{B}_\gamma \}$ and the top space is the incidence relation in $\mathcal{B} \times \Gamma$ that $b \in \mathcal{B}_\gamma$. See [AB, AP] for background.

We would like to define $\Gamma$ as the space of geodesics of $\mathcal{H}$. This is not a Hausdorff space, so instead of we defined $\Gamma$ to be the set of $\mathcal{H}$-geodesic arcs of some fixed length $L$ (less than the injectivity radius $L_1$ of $\mathcal{H}$).

The relevant Crofton formula is the following,
Proposition 3.3. Let $N \subset H$ be a real analytic irreducible hypersurface and let $S^*_N H$ denote the unit covers to $M$ with footpoint on $N$. Then for $0 < T < L_1$,

$$H^{m-2}(N) = \frac{1}{\beta_m T} \int_{S^*_N H} \# \{ t \in [-T, T] : G^t_H(x, \omega) \in S^*_N H \} \, d\mu_L(x, \omega),$$

where $\beta_m$ is $2(m-1)!$ times the volume of the unit ball in $\mathbb{R}^{m-2}$.

Proof. We argue as in [Ze16, Proposition 9] and repeat some of the arguments there to keep the proof self-contained. We define the incidence relation

$$\mathcal{I}_T = \{ ((y, \eta), (x, \omega), t) \subset S^* H \times S^* H \times [-T, T] : (y, \eta) = G^t_H(x, \omega) \},$$

and then define $\mathcal{I}_{T, N}$ by restricting $x \in N$. We then consider the diagram,

$$\mathcal{I}_T \simeq S^* H \times [-T, T] \quad (26)$$

where

$$\pi_1(t, x, \xi) = G^t_H(x, \xi), \quad \pi_2(t, x, \xi) = (x, \xi),$$

and restrict it to $S^*_N H$ to obtain

$$\mathcal{I}_{T, N} \simeq S^*_N H \times [-T, T] \quad (27)$$

where

$$(S^*_N H)_T = \pi_1 \pi_2^{-1}(S^*_N H) = \bigcup_{|t| < T} C^t_H(S^*_N H).$$

We define the Crofton density $\varphi_T$ on $S^*_N H$ corresponding to the diagram [26] [AP] (section 4) by

$$\varphi_T = (\pi_2)_*, \pi_1^* d\mu_L. \quad (28)$$

$\varphi_T$ is a differential form of dimension $2 \dim H - 2$ on $S^* H$. Let $\chi$ be a smooth cutoff equal to 1 on $(-\frac{1}{2}, \frac{1}{2})$, and let $\chi_T(t) = \chi(t^2)$. Then a smooth version of (28) is $\pi_1^*(d\mu_L \otimes \chi_T dt)$ is a smooth density on $\mathcal{I}_{T, N}$. As in [Ze16, Lemma 10] one has,

Lemma 3.4. The Crofton density (28) is given by, $\varphi_T = T d\mu_{L, N}$

Combining Lemma 3.4 with (25) gives

$$\int_{S^*_N H} \varphi_T = \int_{\pi_2^{-1}(S^*_N H)} d\mu_L = T \beta_m H^{m-2}(N). \quad (29)$$

The same formula is true if $N$ has a singular set $\Sigma$ with $H^{m-2}(\Sigma) = 0$. 
We then relate the integral on the left side to numbers of intersections of $H$-geodesic arcs with $N$. By the co-area formula (see [Ze16, Section 3.2]),

\begin{equation}
\int_{\pi^*_1(S^*_N H)} \pi^*_1 d\mu_L = \int_{S^*_H} \# \{ t \in [0, T] : G^t(x, \omega) \in S^*_N H \} d\mu_L(x, \omega).
\end{equation}

Combining (29) and (30) gives the result stated in Proposition 3.3.

Now, we relate the integral on the left side to numbers of intersections of $H$ with $N$ since every real zero is a complex zero. It follows then from Proposition 3.3 (with $\epsilon, L$) that

\begin{equation}
\text{Combining (29) and (30) gives the result stated in Proposition 3.3.}
\end{equation}

3.2. Complexification. The next step is to complexify geodesics of $H$ and also the nodal set $N = N_{\varphi_{\lambda}}$. Here, geodesics and exponential maps always refer to geodesics of $H$.

Define

$$F : S_\epsilon \times S^* H \to H_C, \quad F(t + i \tau, x, v) = \exp_x(t + i \tau)v, \quad (|\tau| \leq \epsilon)$$

Let $\sqrt{\rho}_H$ be the Grauert tube function of $H$, which in general is distinct from the ambient Grauert tube function of $(M, g)$ denoted above by $\sqrt{\rho}$. Let $H_\tau = \{ z \in H_C : \sqrt{\rho}_H(z) < \tau \}$ be the intrinsic Grauert tube of radius $\tau$ of $H$.

For each $(x, v) \in S^* H$,

$$F_{x,v}(t + i \tau) = \exp_x(t + i \tau)v$$

is a holomorphic strip contained in $H_\tau$. Here, $S_\epsilon = \{ t + i \tau \in \mathbb{C} : |\tau| \leq \epsilon \}$. We also denote by $S_{\epsilon, L} = \{ t + i \tau \in \mathbb{C} : |\tau| \leq \epsilon, |t| \leq L \}$.

Since $F_{x,v}$ is a holomorphic function in the strip $S_{\epsilon}$, by Poincaré-Lelong,

$$F^*_{x,v}(\frac{1}{\lambda} dd^c \log |\psi^C_j|^2) = \frac{1}{\lambda} dd^c_{t + i \tau} \log |\varphi^C_j|^2(\exp_x(t + i \tau)v) = \frac{1}{\lambda} \sum_{t + i \tau : \varphi^C_j(\exp_x(t + i \tau)v) = 0} \delta_{t + i \tau}.$$}

As in (21), put

\begin{equation}
(31) \quad \mathcal{A}_{L, \epsilon}(\frac{1}{\lambda} dd^c \log |\varphi^C_j|^2) = \frac{1}{\lambda} \int_{S^* H} \int_{S_{\epsilon, L}} dd^c_{t + i \tau} \log |\varphi^C_j|^2(\exp_x(t + i \tau)v) d\mu_L(x, v).
\end{equation}

A key observation of [DF, Lin] is that (with $N_{\lambda} := N_{\varphi_{\lambda}}$) for any $(x, v) \in S^* H$,

\begin{equation}
(32) \quad \# \{ N_{\lambda}^C \cap F_{x,v}(S_{\epsilon, L}) \} \geq \# \{ N_{\lambda}^R \cap F_{x,v}(S_{0, L}) \},
\end{equation}

since every real zero is a complex zero. It follows then from Proposition 3.3 (with $N = N_{\lambda}$) that

$$\mathcal{A}_{L, \epsilon}(\frac{1}{\lambda} dd^c \log |\varphi^C_j|^2) = \frac{1}{\lambda} \int_{S^* H} \# \{ t + i \tau \in S_{\epsilon, L} ; F_{x,v}(t + i \tau) \in N_{\lambda}^C \} d\mu_L(x, v) \geq \frac{1}{\lambda} \mathcal{H}^{m-2}(N_{\lambda} \cap H).$$

Hence to obtain an upper bound on $\frac{1}{\lambda} \mathcal{H}^{m-2}(N_{\lambda} \cap H)$, it suffices to prove that there exists $M < \infty$ so that

\begin{equation}
(33) \quad \mathcal{A}_{L, \epsilon}(\frac{1}{\lambda} dd^c \log |\varphi^C_j|^2) \leq M.
\end{equation}

To prove (33), we observe that since $dd^c_{t + i \tau} \log |\psi^C_j|^2(\exp_x(t + i \tau)v)$ is a positive $(1, 1)$ form on the strip, the integral over $S_{\epsilon}$ is only increased if we integrate against a positive smooth
test function $\chi_\epsilon \in C^\infty_c(\mathbb{C})$ which equals one on $S_{\epsilon,L}$ and vanishes off $S_{2\epsilon,L}$. Integrating by parts the $dd^c$ onto $\chi_\epsilon$, we have
\[
A_{L,\epsilon}(\frac{1}{\lambda}dd^c \log |\varphi^c_j|^2) \leq \frac{1}{\lambda} \int_{S^*H} \int_C dd^c_{t+i\tau} \log |\varphi^c_j|^2(\exp_x(t+i\tau)v)\chi_\epsilon(t+i\tau) d\mu_L(x,v) \leq \frac{1}{\lambda} \int_{S^*H} \int_C \log |\varphi^c_j|^2(\exp_x(t+i\tau)v) dd^c_{t+i\tau} \chi_\epsilon(t+i\tau) d\mu_L(x,v).
\]

As in the case of curves, we need upper bounds for
\[
\frac{1}{\lambda} \int_{S^*H} \int_C \log_+ |\varphi^c_j|^2(\exp_x(t+i\tau)v) dd^c_{t+i\tau} \chi_\epsilon(t+i\tau) d\mu_L(x,v).
\]

For $\log_+$ the upper bound is an immediate consequence of (24).

For $\log_-$ we use the assumption that $H$ is a good hypersurface, which implies that for any smooth function $J$ there exists $C > 0$ so that
\[
\frac{1}{\lambda} \int_{H_\epsilon} \log |\varphi^c_\epsilon|^2 J dV \geq -C.
\]

We then rewrite (31) to show that (34) gives the same lower bound $-C$ for (31).

We use the diffeomorphism $E : B^*_\epsilon H \to H_\epsilon$ defined by $E(x,\xi) = \exp_x i\xi$. Since $B^*_\epsilon H = \bigcup_{0 \leq \tau \leq \epsilon} S^*H$ we also have that
\[
E : S_{\epsilon,L} \times S^*H \to H_\epsilon, \quad E(t+i\tau,x,v) = \exp_x(t+i\tau)v
\]
is a diffeomorphism for each fixed $t$. Hence by letting $t$ vary, $E$ is a smooth fibration with fibers given by geodesic arcs. Over a point $\zeta \in H_\epsilon$ the fiber of the map is a geodesic arc
\[
\{(t+i\tau,x,v) : \exp_x(t+i\tau)v = \zeta, \tau = \sqrt{\rho_H(\zeta)}\}.
\]

Pushing forward the measure $dd^c_{t+i\tau} \chi_\epsilon(t+i\tau) d\mu_L(x,v)$ under $E$ gives a measure $d\omega$ on $H_\epsilon$, and as in [Ze16],
\[
\omega := E_* dd^c_{t+i\tau} \chi_\epsilon(t+i\tau) d\mu_L(x,v) = \left(\int_{S^*H} \Delta_{t+i\tau} \chi_\epsilon ds\right) dV,
\]
where $dV$ is the Kähler volume form on $H_\epsilon$. In particular it is a smooth multiple $J$ of the Kähler volume form $dV$. It follows that
\[
\int_{S^*H} \int_C \log |\varphi^c_j|^2(\exp_x(t+i\tau)v) dd^c_{t+i\tau} \chi_\epsilon(t+i\tau) d\mu_L(x,v) = \int_{H_\epsilon} \log |\varphi^c_\epsilon|^2 J dV.
\]

It follows that (21) is bounded above and below, completing the proof of Theorem 0.3.

4. Background on asymmetry and the geometry of flowouts

For the remainder of the article we prove Theorems 0.5, 0.6. In this section we review the geodesic asymmetry condition of Definition 0.4. We further consider the geometry of the condition [6]. We begin with some background from [TZ13].

Let $(s,y_n)$ denote Fermi normal coordinates on $H = \{y_n = 0\}$ and let $\sigma, \eta_n$ denote the dual symplectic coordinates. Define
\[
\gamma(s,y_n,\sigma,\eta_n) = \frac{|\eta_n|}{\sqrt{|\sigma|^2 + |\eta_n|^2}} = (1 - \frac{|\sigma|^2}{r^2})^{\frac{1}{2}}, \quad (r^2 = |\sigma|^2 + |\eta_n|^2)
\]
on $T^*_H M$ and also denote by
\begin{equation}
\gamma_{B^*H} = (1 - |\sigma|^2)^{\frac{1}{2}}
\end{equation}
its restriction to $S^*_H M = \{r = 1\}$.

We denote by $G^t$ the homogeneous geodesic flow of $(M,g)$, i.e. Hamiltonian flow on $T^* M = 0$ generated by $|\xi|_g$. We then put $\exp_x t\xi = \pi \circ G^t(x,\xi)$. We further denote by
\begin{equation}
T^*_H M = \{(q,\xi) \in T^*_q M, \ q \in H\}
\end{equation}
the co-vectors to $M$ with footpoint on $H$, and by $T^*_H = \{(g,\eta) \in T^*_q H, \ q \in H\}$ the cotangent bundle of $H$. We further denote by $\pi_H : T^*_H M \to T^* H$ the restriction map,
\begin{equation}
\pi_H(x,\xi) = \xi|_{TH}.
\end{equation}
It is a linear map whose kernel is the conormal bundle $N^* H$ to $H$, i.e. the annihilator of the tangent bundle $TH$. In the presence of the metric $g$, we may identify co-vectors in $T^* M$ with vectors in $TM$ and induce a co-metric $g$ on $T^* M$. The orthogonal decomposition $T_H M = TH \oplus NH$ induces an orthogonal decomposition $T^*_H M = T^* H \oplus N^* H$, and the restriction map $\pi_H$ is equivalent modulo metric identifications to the tangential orthogonal projection (or restriction)
\begin{equation}
\pi_H : T^*_H M \to T^* H.
\end{equation}

For any orientable (embedded) hypersurface $H \subset M$, there exists two unit normal co-vector fields $\nu_\pm$ to $H$ which span half ray bundles $N_\pm = \mathbb{R}_+ \nu_\pm \subset N^* H$. Infinitesimally, they define two ‘sides’ of $H$, indeed they are the two components of $T^*_H \setminus T^* H$. We often use Fermi normal coordinates $(s, y_n)$ along $H$ with $s \in H$ and with $x = \exp_s y_n \nu$. We let $\sigma, \eta_n$ denote the dual symplectic coordinates.

We also denote by $S^*_H M$, resp. $S^* H$, the unit co-vectors in $T^*_H M$, resp. $T^* H$. We restrict to get $\pi_H : S^*_H M \to B^* H$, with where $B^* H$ is the unit coball bundle of $H$. Conversely, if $(s, \sigma) \in B^* H$, then there exist two unit co-vectors $\xi_\pm(s, \sigma) \in S^*_H M$ such that $|\xi_\pm(s, \sigma)| = 1$ and $\xi|_{T^*_s H} = \sigma$. In the above orthogonal decomposition, they are given by
\begin{equation}
\xi_\pm(s, \sigma) = \sigma \pm \gamma(s, \sigma) \nu_+(s), \ \gamma(s, \sigma) := \sqrt{1 - |\sigma|^2}.
\end{equation}

We define the reflection involution through $T^* H$ by
\begin{equation}
r_H : T^*_H M \to T^*_H M, \ r_H(s, \mu \xi_\pm(s, \sigma)) = (s, \mu \xi_\pm(s, \sigma)), \ \mu \in \mathbb{R}_+.
\end{equation}
Its fixed point set is $T^* H$.

We define the first return time $T(s, \xi)$ on $S^*_H M$ by,
\begin{equation}
T(s, \xi) = \inf \{t > 0 : G^t(s, \xi) \in S^*_H M, \ (s, \xi) \in S^*_H M\}.
\end{equation}
By definition $T(s, \xi) = +\infty$ if the trajectory through $(s, \xi)$ fails to return to $H$. We define the first return map by
\begin{equation}
\Phi : S^*_H M \to S^*_H M, \ \Phi(s, \xi) = G^{T(s,\xi)}(s, \xi)
\end{equation}
Inductively, we define the jth return time $T_j(s, \xi)$ to $S^*_H M$ and the jth return map $\Phi^j$ when the return times are finite.
We further define the ‘first impact time’ on all of $S^*M$,

$$t_1(x, \xi) = \begin{cases} 
\inf\{t \geq 0, G^t(x, \xi) \in S_H^*M\}, \\
+\infty, \text{ if no such } t \text{ exists}
\end{cases}$$

(46)

Note that $t_1$ is lower semi-continuous, so that its sublevel sets $\{t_1 \leq \alpha\}$ are closed. Similarly, define $t_j(x, \xi)$ to be the $j$th ‘impact time’, i.e. the time to the $j$th impact with $H$. By homogeneity of $G^t : T^*M \to T^*M$, for all $j \in \mathbb{Z}$,

$$t_j(x, \xi) = t_j(x, \xi |\xi|); \quad \xi \neq 0.$$ 

(47)

Obviously, $t_j(x, \xi) = t_1(x, \xi) + T_j(G^{t_1}(x, \xi))(x, \xi))$.

Define

$$\Delta_{T^*M \times T^*M} := \{(x, \xi, x, \xi) \in T^*M \times T^*M\},$$

(48)

$$\Gamma_T = \bigcup_{(s, \xi) \in T_{\mathbb{R}}M} \bigcup_{|t| < T} \{(G^t(s, \xi), G^t(r_H(s, \xi)))\}.$$

The two ‘branches’ or components intersect along the singular set

$$\Sigma_T := \bigcup_{|t| < T} (G^t \times G^t)\Delta_{T^*H \times T^*H}.$$ 

We further subscript $\Gamma_T$ with $\epsilon$ to indicate the points $\Gamma_{T, \epsilon}$ making an angle $\geq \epsilon$ with $TH$.

Since $G^t(r_H(s, \xi)) = G^t r_H G^{-t} G^t(s, \xi)$, $\Gamma_{T, \epsilon} \subset \Gamma_T \setminus \Sigma_T$ is the graph of a symplectic correspondence. More precisely, for any $\epsilon > 0$, $\Gamma_{T, \epsilon}$ is the union of a finite number $N_{T, \epsilon}$ of graphs of partially defined canonical transformations

$$\mathcal{R}_j(x, \xi) = G^{t_j(x, \xi)} r_H G^{-t_j(x, \xi)}(x, \xi).$$ 

which we term $H$-reflection maps.

4.1. Asymmetric hypersurfaces. In the following, $\mu_L$ denotes Liouville measure on $S^*M$ and $\mu_{L,H}$ is the induced hypersurface measure on $H$ satisfying $d\mu_L = d\mu_{L,H} dx_n$.

**Definition 4.1.** We say that $H$ has a positive measure of microlocal reflection symmetry if

$$\mu_{L,H} \left( \bigcup_{j \neq 0} \left\{ (s, \xi) \in S^*_H M : r_H G^{T_j(s, \xi)}(s, \xi) = G^{T_j(s, \xi)} r_H(s, \xi) \right\} \right) > 0.$$ 

Otherwise we say that $H$ is asymmetric with respect to the geodesic flow.

Thus, the return time condition is that the $+$ and $-$ trajectories return at the same time to the same point of $H$ and project to the same covector in $B^*H$ on a set of positive measure.
4.2. Filtering the flowout by return times and by tangential angle. We recall that our full-measure flowout assumption (6) is $\mu_L(\text{FL}(H)) = 1$. Since $\bigcup_{|t|<\infty}G^t(S^*H)$ has Hausdorff dimension $\leq 2n - 2$, it follows that

$$\mu_L\left(\bigcup_{|t|<\infty}G^t(S^*H)\right) = 0,$$

and so, in particular,

$$\mu_L(\{(x, \xi) \in \text{FL}(H), G^{t_1(x,\xi)}(x, \xi) \in S^*H\}) = 0.$$

Here, $t_1(x, \xi)$ is the first hitting time (46). Let

$$\Lambda = \{(x, \xi) \in S^*M, |t_1(x, \xi)| < \infty, \}$$

and let

$$\Lambda := \{(x, \xi) \in S^*M, |t_1(x, \xi)| < \infty, G^{t_1(x,\xi)}(x, \xi) \in S^*_H \setminus S^*H\}.$$

Here, $\overline{\Lambda}$ is the set of covectors whose orbits hit $H$ at some time, and $\Lambda \subset \overline{\Lambda}$ is the subset which never tangentially. Evidently, $\Lambda \subset \text{FL}(H) \subset \overline{\Lambda}$ and the differences of these sets have measure zero. Then (6) is equivalent to

$$\mu_L(\Lambda) = 1.$$

One can clearly make the decomposition

$$\Lambda = \bigcup_{R=0}^{\infty} \Lambda_R, \quad \Lambda_R := \{(x, \xi) \in \Lambda, |t_1(x, \xi)| < R\}.$$

Moreover, for all $R_1 \leq R_2 \leq R_3 \leq \ldots$, the sets $\Lambda_{R_1} \subset \Lambda_{R_2} \subset \Lambda_{R_3} \subset \ldots$ and so, by monotonicity of measure,

$$\mu_L(\Lambda_R) \nearrow 1 \quad \text{as } R \to \infty.$$

One can make a further decomposition

$$\Lambda_R = \bigcup_{\varepsilon} \Lambda_{R,\varepsilon}, \quad \Lambda_{R,\varepsilon} := \{(x, \xi) \in \Lambda_R, G^{t_1(x,\xi)}(x, \xi) \in S^*_H \setminus S^*H, |\pi_H(G^{t_1(x,\xi)}(x, \xi))| < 1 - 2\varepsilon\}.$$

Since $\Lambda_{R,\varepsilon_1} \subset \Lambda_{R,\varepsilon_2} \subset \ldots$ and $\epsilon_1 \geq \epsilon_2 \geq \ldots$ it follows again by monotonicity that

$$\mu_L(\Lambda_{R,\varepsilon}) \nearth \mu_L(\Lambda_R) \quad \text{as } \varepsilon \to 0^+.$$

Thus from (53) and (54) it follows that for any $\delta \in (0, 1/2)$ one can choose $R = R(\delta)$ and $\epsilon = \epsilon(\delta)$ such that

$$\mu_L(\Lambda_{R,\varepsilon}) \geq 1 - 2\delta.$$

We will need the following facts about $\Lambda \subset S^*M$ in (51):

**Lemma 4.2.** We have:

1. $\Lambda$ is open.
2. The first impact time $t_1|_{\Lambda}$ is $C^\infty$ on $\Lambda$. 
Remark 4.3. Openness is not obvious, since \( t_1 \) is lower semi-continuous and has open super-level sets \( \{ t_1 > \alpha \} \). This is not a contradiction, since the tangential directions are punctured out in \( \Lambda \) and they form its boundary.

Proof. Let \( \rho \in C^\infty(M) \) be a defining function for \( H \), i.e.

\[
H = \{ x \in M; \rho(x) = 0 \}, \quad d\rho(x) \neq 0, \quad x \in H.
\]

Let \( (x_0, \xi_0) \in \Lambda \subset S^*M \) and so, in particular, \( t_1(x_0, \xi_0) < \infty \). We claim that there exists an open set \( U \) around \( (x_0, \xi_0) \) so that \( U \subset \Lambda \).

Consider the map \( G : \mathbb{R} \times S^*M \to S^*M \) given by \( G(t, (x, \xi)) = G^t(x, \xi) \) and let \( \pi : S^*M \to M \) be the canonical projection. Let \( \gamma_{x, \xi}(t) = \pi G^t(x, \xi) \) and consider the sets,

where \( H \) is the Hamilton vector field and \( \pi^*H_{g_{\xi_0}}(G^t(x, \xi)) = \hat{\gamma}_{x, \xi}(t) \). Hence, (57) is non-zero for \( (t_0, x_0, \xi_0) \in C_0 \). By the implicit function theorem, there exists an open set \( U_{x_0, \xi_0} \subset S^*M \) around \( (x_0, \xi_0) \) on which there exists a \( C^\infty \) function \( \tilde{t} : U_{x_0, \xi_0} \to \mathbb{R} \) satisfying \( \tilde{t}(x_0, \xi_0) = t_0 \) and \( \rho(G^\tilde{t}(x, \xi))(x, \xi) = 0 \).

Now suppose that \( (x_0, \xi_0) \in \Lambda \). Then \( (t_1(x_0, \xi_0), x_0, \xi_0) \in C_0 \) and \( \tilde{t} = t_1 \) on \( U_{x_0, \xi_0} \). Then \( t_1 \) is \( C^\infty \) on \( U_{x_0, \xi_0} \) and in particular is finite. Hence, \( U_{x_0, \xi_0} \subset \Lambda \). 

4.3. The space of geodesics hitting \( H \) and disintegration of invariant measures.

Although \( S^*_H M \) is not literally a cross section to the geodesic flow, inasmuch as some geodesics might not hit \( H \), one might think of it as a cross section to the geodesic flow in the set \( FL(H) \). But even that is not true, because a given geodesic may intersect \( H \) multiple times, and it is also possible that a geodesic arc or a complete geodesic lies in \( S^*H \). Roughly speaking we define the space of geodesics hitting \( H \) to be \( \mathfrak{G}_H = FL(H)/\sim \) where \( \sim \) is the equivalence relation of belonging to the same orbit. Since every orbit intersects \( S^*_H M \) one also has \( \mathfrak{G}_H = S^*_H M / \sim \). One then has maps \( \pi : FL(H) \to \mathfrak{G}_H, \pi_1 : S^*_H M \to \mathfrak{G}_H \). These maps play an important role below in relating microlocal defect measures on \( S^*M \) to microlocal defect measures on \( B^*H \). To prepare for that, we consider disintegration of invariant measures.

The general disintegration theorem states the following: Let \( (Y, \mu) \) be a probability space, let \( \pi : Y \to X \) be a measurable map, and let \( \nu = \pi_*\mu \). There exist a family of measures \( \{ \mu_x \} \subset \text{Prob}(Y) \) so that \( \mu_x \) lives on the fiber \( \pi^{-1}(x) \), i.e. \( \mu_x(Y \setminus \pi^{-1}(x)) = 0 \) for \( \nu \) a.e. \( x \), and for any measurable \( f : Y \to \mathbb{R}_+ \),

\[
\int_Y f(y)d\mu(y) = \int_X \int_{\pi^{-1}(x)} f(y)d\mu_x(y)d\nu(x).
\]
In our setting, \( Y = FL(H), X = \mathfrak{G}_H \) and \( \pi : FL(H) \to \mathfrak{G}_H \) is the natural projection as above.

As defined above \( \mathfrak{G}_H \) is not a Hausdorff space since a geodesic may intersect \( S^*_H M \) in an infinite set with an accumulation point. Moreover, the ‘fibers’ (geodesics) have infinite measure. For our purposes, it is possible to avoid this problem by truncation: fix \( \delta > 0 \) and let \( FL_\delta(H) = \bigcup_{|t| < \delta} G^t(S^*_H M) \). We then let \( Y = FL_\delta(H), \mathfrak{G}_\delta = FL_\delta(H)/\sim \). This is a much simpler quotient but note that any geodesic arcs on \( H \) get collapsed to points. In particular if \( H = \gamma \) is a closed geodesic, then \( S^*_\gamma \) is a single orbit and a single point in the quotient. We thus have a map \( \pi : S^*_H M \to \mathfrak{G}_\delta \), but it may fail to be 1-1 due to tangential geodesics.

To remove the latter problem, we use a truncation from [TZ, CGT17] that punctures out a neighborhood of the tangent directions \( S^*H \) as well as in time. In terms of Fermi normal coordinates \( (x', x_n) \) with \( H = \{x_n = 0\} \), for \( \delta > 0 \), let \( S^*H(\delta) = \{(x', x_n) \in S^*_H M; |x_n| < \delta\} \) and let \( S^*_H M(\delta) = S^*_H M \setminus S^*H(\delta) \). Also let \( S^*M(H, \delta) = \{|x_n| < \delta, |x_n| > C\delta\} \) with \( C = C(H, g) > 1 \) is a sufficiently large constant. We then have a map

\[
\pi_\delta : S^*M(H, \delta) \to \bigcup_{|t| < \delta} G^t(S^*_H M(\delta))
\]

which is 1-1 for \( \delta \) sufficiently small.

Now consider a general invariant measure \( \mu \) on \( S^*M \). To apply the disintegration theorem, we first restrict \( \mu \) to \( FL(H) \), by multiplying \( \mu \) by the characteristic function \( 1_{FL(H)} \). It is equivalent to use \( FL(H) \) or \( \Lambda \). Then,

\[
\begin{align*}
(58) & \quad \left\{
\begin{array}{l}
(i) \quad \int_{FL(H)} f d\mu = \int_{\mathfrak{G}_\delta} \left( \int_{\pi^{-1}(y)} f d\mu_y \right) d\nu(y), \\
(ii) \quad \int_{S^*M(H, \delta)} f d\mu = \int_{S^*_H M(\delta)} \left( \int_{\pi^{-1}(x', \xi)} f dt \right) d\nu^H_\delta(x', \xi).
\end{array}
\right.
\end{align*}
\]

Evidently, \( d\mu_y = dt \) when \( d\mu \) is an invariant measure. Note that (i) is independent of \( \delta \) but the disintegration measure \( d\nu \) is not a measure on \( S^*_H M \). In the integral (ii), \( d\nu^H_\delta \) is a measure on \( S^*_H M \) but depends on \( \delta \). In [CGT17] the same measure is written in terms of Fermi-coordinates as

\[
(59) \quad d\mu(x, \xi) = |\xi_n|^{-1} d\nu^H_\delta(x', \xi', \xi_n) dx_n, \quad (x, \xi) \in S^*M(H, \delta),
\]

using that \( dt = |\xi_n|^{-1} dx_n \). For future reference (see Proposition 6.2) we set

\[
(60) \quad d\mu^H_\delta(x', \xi) := |\xi_n|^{-1} d\nu^H_\delta(x', \xi), \quad |\xi_n| > C\delta.
\]

The special case where \( \mu = \mu_L \) (Liouville measure) is discussed in [TZ] Lemma 13.

A natural question regarding (ii) is the behavior of the integrals as \( \delta \to 0 \). To consider an extreme case, suppose that \( d\mu \) is a periodic orbit measure \( \delta_\gamma \) along a closed geodesic \( \gamma \) of a surface \( M \) and that \( H = \gamma \). Then the left sides of either equation are \( \int_{\gamma} f ds \). On the right side of (i), \( d\nu \) is a point mass at \( \gamma \in \mathfrak{G}_\delta \). This measure cannot be represented by the right equation since it punctures out \( \gamma \subset S^*H \).

We now formulate a condition so that the integral (i) over \( \mathfrak{G}_\delta \) can be given by an integral (ii) over \( S^*_H M \). This is the case if \( \nu \)-almost every orbit in \( \mathfrak{G}_\delta(H) \) intersects \( S^*_H M \) once. For future reference, we state this as the following.
Lemma 4.4. If the disintegration measure $d\nu$ of an invariant measure $d\mu$ has the property that $\nu$-almost every orbit in $S_\delta(H)$ intersects $S^*_{H}M$ once, then there exists a Borel measure $\nu_H$ on $S^*_{H}M$ with the property that

$$
\int_{F\Lambda_i(H)} f d\mu = \int_{S^*_{H}M} \left( \int_{\pi^{-1}(y)} f d\mu_y \right) d\nu_H(y) = \int_{S^*_{H}M} \left( \int_{\pi^{-1}(y)} f dt \right) d\nu_H(y)
$$

Proof. By deleting a set of $\nu$-measure zero of $\mathcal{S}_\delta$, $\pi : S^*_{H}M \to \mathcal{S}_\delta$ is 1-1. Hence, it admits an inverse $\pi^{-1} : \mathcal{S}_\delta \to S^*_{H}M$. Then, $d\nu_H = (\pi^{-1})_* d\nu$ or equivalently $d\nu = \pi_* d\nu_H$.

Remark 4.5. Another map is $\pi_1(x,\xi) = G_{t_1(x,\xi)}(x,\xi)$, the first impact map. If $H$ is strictly convex or concave, so that geodesics can only have first order contact with $H$ then the first return time to $H$ is strictly bounded below even for tangential directions. Hence there exists $\delta > 0$ so that each orbit in $F\Lambda_i(H)$ intersects $S^*_{H}M$ exactly once.

In Section 6.2 some conditions on sequences of eigenfunctions are given so that their defect measures satisfy the hypotheses of Lemma 4.4.

5. Relating matrix elements on $H$ to matrix elements on $M$

This section reviews the relation between matrix elements on $H$ and matrix elements on $M$. The main result (Proposition 5.2) is repeated from [TZ13]. To make this article relatively self-contained we also review the background leading to its statement and proof.

Let $(M,g)$ be a compact Riemannian manifold and let $H$ be a compact embedded $C^\infty$ submanifold. We denote by $U(t) = e^{it\sqrt{-\Delta}}$ the wave group of $(M,g)$. As is well-known, it is a homogeneous unitary Fourier integral operator of order 0 whose canonical relation is the graph of the homogeneous geodesic flow at time $t$; we refer to [HoIII, HolV1] for background.

We denote by $\gamma_H$ the restriction operator $\gamma_H f = f|_H : C(M) \to C(H)$ and by $\gamma_H^*$ the adjoint of $\gamma_H$ with respect to the inner product on $L^2(M,dV)$ where $dV$ is the Riemannian volume form. Thus,

$$
\gamma_H^* f = f \delta_H, \quad \text{since } \langle \gamma_H^* f, g \rangle = \int_H f g dS,
$$

where $dS$ is the surface measure on $H$ induced by the ambient Riemannian metric. The fact that $\gamma_H^*$ does not preserve smooth functions is due to the fact that $WF'_M(\gamma_H) = N^*H$. Thus, $\gamma_H^* Op_H(a) \gamma_H$ is not a Fourier integral operator with a homogeneous canonical relations in the sense of [HoIII] because its wave front relation contains $N^*H \times 0_{T^*M} \cup 0_{T^*M} \times N^*H$ (where $0_{T^*M}$ is the zero section of $T^*M$). For this reason we need to introduce microlocal cutoffs as in [TZ13]. In the following, $\chi \in C^\infty_0(\mathbb{R}; [0,1])$ is a cutoff function with $\chi(t) = 1$ for $|t| \leq 1$ and supp $\chi \subset [-2,2]$.

Define:

$$
V(t;a) := U(-t)\gamma_H^* Op_H(a) \gamma_H U(t),
$$

$$
\bar{V}_T(a) := \frac{1}{T} \int_{-T}^{T} \chi(T^{-1}t) V(t;a) dt,
$$

Lemma 5.1. For any $a \in C^\infty_0(T^*H)$,

$$
\langle Op_H(a)\varphi_j|_H, \varphi_j|_H \rangle_{L^2(H)} = \langle \bar{V}_T(a)\varphi_j, \varphi_j \rangle_{L^2(M)},
$$
Proof. This follows from the sequence of identities,

\begin{align}
\langle Op_H(a) \varphi_j | H, \varphi_j | H \rangle_{L^2(H)} &= \langle Op_H(a) \gamma_H \varphi_j, \gamma_H \varphi_j \rangle_{L^2(H)} \\
&= \langle \gamma_H Op_H(a) \gamma_H U(t) \varphi_j, U(t) \varphi_j \rangle_{L^2(M)} \\
&= \langle V(t; a) \varphi_j, \varphi_j \rangle_{L^2(M)} \\
&= \langle \tilde{V}_T(a) \varphi_j, \varphi_j \rangle_{L^2(M)}
\end{align}

(64)

A detailed description of $\nabla T(a)$ is given in Proposition 3.2 from [TZ13]. There it is proved that, after cutting off from the tangential singular set $\Sigma_T \subset T^*M \times T^*M$ and the the conormal sets $N^*H \times \partial T^*M, \partial T^*M \times N^*H$, $\tilde{V}_T(a)$ becomes a Fourier integral operator $\tilde{V}_{T, \epsilon}(a)$ with canonical relation given by

(65)

\[ WF(\tilde{V}_{T, \epsilon}(a)) = \{ (x, \xi, x', \xi') \in T^*M \times T^*M : \exists t \in (-T, T), \]

\[ \exp_x t \xi = \exp_{x'} t \xi' = s \in H, \ G^t(x, \xi)|_{T_s H} = G^t(x', \xi')|_{T_s H}, \ |\xi| = |\xi'| \}. \]

5.1. Good cutoffs. In view of (55) and Lemma 4.2, we consider the disjoint, closed sets

\[ K_1 := S^*M \setminus \Lambda, \quad K_2 := \overline{N_{R, \epsilon}} \]

where $R = R(\delta)$ and $\epsilon = \epsilon(\delta)$ are chosen as in (55). Thus, by the $C^\infty$ Urysohn lemma, there exists a cutoff $\chi_{R, \epsilon} \in C^\infty(S^*M; [0, 1])$ with

\[ \chi_{R, \epsilon}(x, \xi) = \begin{cases} 
1 & (x, \xi) \in K_2 \\
0 & (x, \xi) \in K_1.
\end{cases} \]

We abuse notation somewhat and denote the positive homogeneous degree zero extension of $\chi_{R, \epsilon}$ to $T^*M - 0$ also by $\chi_{R, \epsilon}$ and the corresponding pseudodifferential operator by $\chi_{R, \epsilon}(x, D) \in Op(S^0(T^*M - 0))$.

At this point, as in [TZ] we introduce some further cutoff operators supported away from glancing and conormal directions to $H$. For fixed $\epsilon > 0$, let $\chi_{\epsilon}^{(\text{tan})}(x, D) = Op(\chi_{\epsilon}^{(\text{tan})}) \in Op(S^0(T^*M))$, with homogeneous symbol $\chi_{\epsilon}^{(\text{tan})}(x, \xi)$ supported in an $\epsilon$-aperture conic neighbourhood of $T^*H \subset T^*M$ with $\chi_{\epsilon}^{(\text{tan})} \equiv 1$ in an $\frac{\epsilon}{2}$-aperture subcone. The second cutoff operator $\chi_{\epsilon}^{(\text{n})}(x, D) = Op(\chi_{\epsilon}^{(\text{n})}) \in Op(S^0(T^*M))$ has its homogeneous symbol $\chi_{\epsilon}^{(\text{n})}(x, \xi)$ supported in an $\epsilon$-conic neighbourhood of $N^*H$ with $\chi_{\epsilon}^{(\text{n})} \equiv 1$ in an $\frac{\epsilon}{2}$ subcone. Both $\chi_{\epsilon}^{(\text{tan})}$ and $\chi_{\epsilon}^{(\text{n})}$ have spatial support in the set where $|x_n| < \epsilon$ (see [TZ] (5.1) and (5.2)). To simplify notation, define the total cutoff operator

(66)

\[ \chi_{\epsilon}(x, D) := \chi_{\epsilon}^{(\text{tan})}(x, D) + \chi_{\epsilon}^{(\text{n})}(x, D). \]

5.2. Cutoff of $\gamma_H^* Op_H(a) \gamma_H$ and its time average. We define

(67)

\[ (\gamma_H^* Op_H(a) \gamma_H)_{\geq \epsilon} = (I - \chi_{\frac{\epsilon}{2}}) \gamma_H^* Op_H(a) \gamma_H (I - \chi_{\epsilon}), \]
and
\begin{equation}
(\gamma_H^* Op_H(a)\gamma_H)_{\leq \epsilon} = \chi_{2\epsilon} \gamma_H^* Op_H(a)\gamma_H \chi_{\epsilon}.
\end{equation}

By a standard wave front calculation, it follows that
\begin{equation}
\gamma_H^* Op_H(a)\gamma_H(\gamma_H^* Op_H(a)\gamma_H)_{\geq \epsilon} + (\gamma_H^* Op_H(a)\gamma_H)_{\leq \epsilon} + K_\epsilon,
\end{equation}
where, \( (K_\epsilon \varphi_j; \varphi_j)_{L^2(M)} = \mathcal{O}(\lambda_j^{-\infty}) \). We then define
\begin{equation}
V_\epsilon(t; a) := U(-t)(\gamma_H^* Op_H(a)\gamma_H)_{\geq \epsilon} U(t),
\end{equation}
and
\begin{equation}
\overline{V}_{T,\epsilon}(a) := \frac{1}{T} \int_{-\infty}^{\infty} \chi(T^{-1}t) V_\epsilon(t; a) \, dt.
\end{equation}

The next proposition provides a detailed description of \( \overline{V}_{T,\epsilon}(a) \) as a Fourier integral operator with local canonical graph away from its fold set and computes its principal symbol.

**Proposition 5.2.** Fix \( T, \epsilon > 0 \) and let \( a \in S^0_0(T^*H) \) with \( a_H(s, \xi) = a(s, \xi|_H) \in S^0(T^*_H M) \). Then \( \overline{V}_{T,\epsilon}(a) \) is a Fourier integral operator with local canonical graph, and possesses the decomposition
\begin{equation}
\overline{V}_{T,\epsilon}(a) = P_{T,\epsilon}(a) + F_{T,\epsilon}(a) + R_{T,\epsilon}(a),
\end{equation}
where, (i) \( P_{T,\epsilon}(a) \in Op_{cl}(S^0_0(T^*M)) \) is a pseudo-differential operator of order zero with principal symbol
\begin{equation}
a_{T,\epsilon}(x, \xi) := \sigma(P_{T,\epsilon}(a))(x, \xi) = \frac{1}{T} \sum_{j \in \mathbb{Z}} (1 - \chi_\epsilon) (\pi^* \gamma^{-1} a_H)(G_{t_j(x, \xi)}^{t_j(x, \xi)}(x, \xi)) \chi(T^{-1}t_j(x, \xi))
\end{equation}
where, \( t_j(x, \xi) \in C^\infty(T^*M) \) are the impact times of the geodesic \( \exp_x \gamma(t\xi) \) with \( H \), and \( \gamma \) is defined by \([37]\).

(ii) \( F_{T,\epsilon}(a) \) is a Fourier integral operator of order zero with canonical relation \( \Gamma_{T,\epsilon} \).

\begin{equation}
F_{T,\epsilon}(a) = \sum_{j=1}^{N_{T,\epsilon}} F_{T,\epsilon}^{(j)}(a),
\end{equation}
where the \( F_{T,\epsilon}^{(j)}(a); j = 1, \ldots, N_{T,\epsilon} \) are zeroth-order homogeneous Fourier integral operators with
\[ WF'(F_{T,\epsilon}^{(j)}(a)) = \text{graph}(R_j) \cap \Gamma_{T,\epsilon}, \]
and symbol
\[ \sigma(F_{T,\epsilon}^{(j)}(x, \xi)) = \frac{1}{T} (\gamma^{-1} a_H)(G_{t_j(x, \xi)}^{t_j(x, \xi)}(x, \xi)) \chi(T^{-1}t_j(x, \xi)) |dx|^{\frac{1}{2}}. \]

(iii) \( R_{T,\epsilon}(a) \) is a smoothing operator.

The proof of Proposition 5.2 goes roughly as follows: we decompose \( \overline{V}_{T,\epsilon}(a) \) into a pseudodifferential and a Fourier integral part according to the dichotomy that \( (x, \xi, x', \xi') \) in \([65]\) satisfy either
\begin{enumerate}
\item[(i)] \( G^t(x, \xi) = G^t(x', \xi') \), or
\item[(ii)] \( G^t(x', \xi') = r_H G^t(x, \xi) \),
\end{enumerate}
where \( r_H \) is the reflection map of \( T^*H \) in [43]. Thus,

\[
WF(\nabla_{T,\epsilon}(a)) = \Delta_{T^*M \times T^*M} \cup \Gamma_T.
\]

The pseudo-differential part \( P_{T,\epsilon} \) of \( \nabla_{T,\epsilon}(a) \) is its microlocalization to (i) and the Fourier integral part \( F_{T,\epsilon} \) is its microlocalization to (ii). For further details, we refer the reader to Proposition 7 in [TZ].

6. Proof of Proposition 0.9 and Theorem 0.10

By (63)-(64) and by Proposition 5.2 the weak* limits of the restricted matrix elements are those of

\[
\langle \nabla_{T,\epsilon}(a) | \varphi_j, \varphi_j \rangle_{L^2(M)} = \langle P_{T,\epsilon}\varphi_j, \varphi_j \rangle_{L^2(M)} + \langle F_{T,\epsilon}\varphi_j, \varphi_j \rangle_{L^2(M)} + \langle R_{T,\epsilon}\varphi_j, \varphi_j \rangle_{L^2(M)}.
\]

It is clear that \( \langle R_{T,\epsilon}\varphi_j, \varphi_j \rangle_{L^2(M)} \to 0 \) for the entire sequence of eigenfunctions. We now argue that the \( F_{T,\epsilon} \) term is negligible for a density one subsequence.

6.1. Removing the \( F_{T,\epsilon} \) term and proof of Proposition 0.9

We now consider the Fourier integral matrix elements \( \langle F_{T,\epsilon}\varphi_j, \varphi_j \rangle_{L^2(M)} \).

**Lemma 6.1.** Suppose that \( H \) is an asymmetric hypersurface. Then for any fixed \( T > 0, \epsilon > 0 \) there exists a subsequence \( S_F(T,\epsilon) \) of the eigenfunctions of density one such that

\[
\langle F_{T,\epsilon}\varphi_j, \varphi_j \rangle_{L^2(M)} \to 0, \quad j \in S_F(T,\epsilon).
\]

**Proof.** It suffices to show that

\[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \langle F_{T,\epsilon}(a) | \varphi_j, \varphi_j \rangle_{L^2(M)} \right|^2 = 0.
\]

The proof of this Lemma is essentially identical to that in [TZ13], since it did not use ergodicity of the geodesic flow. Hence we only sketch it for the sake of completeness. First, we note that for any \( R > 0 \), we clearly have

\[
\langle F_{T,\epsilon}(a) | \varphi_j, \varphi_j \rangle_{L^2(M)} = \langle F_{R,T,\epsilon}(a) | \varphi_j, \varphi_j \rangle_{L^2(M)},
\]

where \( F_{R,T,\epsilon}(a) := \frac{1}{2R} \int_{-R}^{R} U(r)^* F_{T,\epsilon}(a) U(r) \, dr \). Then, the Weyl sum in (77) by the corresponding Weyl sum with \( F_{T,\epsilon} \) replaced with \( F_{R,T,\epsilon} \). To prove (77) we first use the Schwartz inequality

\[
\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \langle F_{R,T,\epsilon}(a) | \varphi_j, \varphi_j \rangle_{L^2(M)} \right|^2 \leq \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \langle F_{R,T,\epsilon}(a)^* F_{R,T,\epsilon}(a) | \varphi_j, \varphi_j \rangle_{L^2(M)}
\]

to bound the variance sum by a trace. We then use the local Weyl law for Fourier integral operators associated to local canonical graphs,

\[
\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \langle F | \varphi_j, \varphi_j \rangle \to \int_{ST_F \cap \Delta_{T^*M}} \sigma_{\Delta}(F) \, d\mu_L,
\]

where \( \Gamma_F \) is the canonical relation of \( F \), \( ST_F \) is the set of vectors of norm one, and \( ST_F \cap \Delta_{T^*M} \) is its intersection with the diagonal of \( T^*M \times T^*M \). Also, \( \sigma_{\Delta}(F) \) is the (scalar) symbol in
We now prove that there exists a subsquence of density one so that geodesic, or restrictions of whispering gallery modes of a convex domain to its boundary.

As suggested above, a sequence failing to have this property must blow up along a density-one sequence. In fact, the next Lemma proves more:

\[ \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j : \lambda j \leq \lambda} \left| \langle F_{R,T,\epsilon}(a) \varphi_j, \varphi_j \rangle_{L^2(M)} \right|^2 = o_{T,\epsilon}(1) \]

as \( R \to \infty \). Since the LHS in (80) equals the LHS in (77) and the latter is independent of \( R \), letting \( R \to \infty \) in (80) completes the proof of Proposition 0.9.

As in the introduction, for fixed \( \epsilon \in (0,1) \), let \( \chi^H_\epsilon \in C^\infty_0 (B^*H) \) be a cutoff with \( \chi^H_\epsilon \subset \{(s,\sigma) \in B^*H : ||\sigma||_s - 1| < \epsilon\} \). Proposition 0.9 then follows from Lemma 5.1, Proposition 5.2 and Lemma 6.1.

6.2. Proof of Theorem 0.10. To complete the proof of Theorem 0.10 we ‘disintegrate’ each microlocal defect measure \( d\mu \) of the sequence \( \mathcal{S}_F \) in the sense of Section 4.3. As discussed there, we first have to localize \( d\mu \) to \( FL(H) \), and we further localize it to \( FL_\delta(H) \) by multiplying by its characteristic function. If we apply the disintegration theorem, we obtain a measure on \( \mathcal{G}_S \). Proposition 0.10 asserts that there exists a sequence of density one so that each of its microlocal defect measures can be disintegrated to a measure on \( S^*_H M \). To prove this, we show that there exists a subsequence of density one for which Lemma 4.4 holds.

Let \( S' \subset \mathcal{S}_F \) be a subsequence corresponding to a global defect measure \( d\mu^N_{S'} \) on \( S^* M \). To simplify notation we simply write \( d\mu = d\mu^N_{S'} \) below (and similiarly, we write \( d\mu^H \) for \( d\mu^H_{S'} \)).

**Proposition 6.2.** There exists a density one set \( \mathcal{S} \subset \mathcal{S}_F \) such that, for any defect measure \( \mu \) arising from a subsequence \( \{\varphi_{j_k}\} \) with \( j_k \in S' \subset \mathcal{S} \), the corresponding Borel measures \( d\mu^H \) on \( S^* M \) defined in (60) converge weakly as \( \delta \to 0 \) to a Borel measure \( d\mu^H \) on \( S^*_H M \).

**Proof.** As suggested above, a sequence failing to have this property must blow up along \( H \). An example would be restrictions of Gaussian beams along a closed geodesic \( \gamma \) to the geodesic, or restrictions of whispering gallery modes of a convex domain to its boundary. We now prove that there exists a subsequence of density one so that

\[ \|\varphi^H_{\varphi_j}\|_{L^2(H)} = O(1). \]

In fact, the next Lemma proves more:

**Lemma 6.3.** Fix \( \delta_0 > 0 \) small and let \( H_\tau = \{x_n = \tau\} \) with \( |\tau| < \delta_0 \). Then, there is a density-one sequence \( \mathcal{S} \) such that

\[ \sup_{|\tau| \leq \delta_0} \|\varphi^H_{\varphi_j}\|_{L^2(H_\tau)} = O(1), \quad j \in \mathcal{S}, \]

as \( h \to 0 \).

**Proof.** Let \( \{\tau_n\}_{n=1}^{\infty} \) be a countable set in \([0,1]\) and let \( H_{\tau_n} \) be the corresponding sequence of hypersurfaces.

By the pointwise Weyl law, \( \frac{1}{N(h)} \sum_{h \geq h_n} \|\varphi_{\phi_j}\|_{L^2(H_{\tau_n})} \sim_{h \to 0} c_n |H_{\tau_n}| \) and consequently for each \( n \), there exists a density-one subset \( \mathcal{S}_n \) such that for \( j \in \mathcal{S}_n \),

\[ \|\varphi^H_{\varphi_j}\|_{L^2(H_{\tau_n})} \leq C, \quad j \in \mathcal{S}_n. \]
where $C > 0$ is independent of $n$. Since $\tilde{S} = \bigcap_{n \geq 1} \hat{S}_n$ is also of density-one, it follows from (82) that
\begin{equation}
\| \varphi_{H_j}^r \|_{L^2(H_{\tau_n})} \leq C, \quad j \in \tilde{S}, \quad n = 1, 2, 3, \ldots.
\end{equation}

Now consider a general $H_r$. We pick $\{\tau_n\}$ to be dense in $[0, 1]$ and to contain 0. In particular, (81) holds for $j \in \tilde{S}$. To prove that it holds for all $\tau \in [0, \delta_0]$ for some $\delta_0 > 0$ we argue by contradiction. For any $\tau$ and any $\epsilon > 0$, there exists $H_{\tau_n}$ with $d(H_{\tau_n}, H\tau) = \inf_{(x,y) \in H_{\tau_n} \times H_r} d(x, y) < \epsilon$. We then consider the functions
\[ \rho(\tau) = \limsup_{j \to \infty} \| \varphi_{h_j} \|_{L^2(H_r)}. \]

Since $\tau \to \| \varphi_{h_j} \|_{L^2(H_r)}$ is a continuous function, $\rho(\tau)$ is lower semi-continuous. If $\rho$ is not bounded on any interval $[0, \delta]$, then there exists a sequence $\{\tau_k\}_{k=1}^\infty$ with $\tau_k \to 0$ (disjoint from $\{\tau_n\}$) and so that $\| \varphi_{h_j} \|_{L^2(H_{\tau_k})} > k$. Since $\rho$ is lower semi-continuous, each superlevel set $\{\rho > k\}$ is open and non-empty. But this contradicts the fact that $\rho \leq C$ on the dense set $\{\tau_n\}$.

We now show that the disintegration measure of any defect measure arising from a subsequence in $S_F \cap \tilde{S}$ is a measure on $S_H^* M$, i.e. the measures $d\mu_{\delta}$ have a weak limit.

Suppose $0 \leq a \in C^\infty(S_H^* M)$. Then, since $\xi^2 = 1 - |\xi'|^2$, for $(x', \xi) \in S_H^* M$, stereograph projection maps $\pi^\pm : S_H^* M \setminus S^* H \to B^* H$ given by $\pi^\pm(x', \xi', \pm \sqrt{1 - |\xi'|^2}) = (x', \xi')$ are diffeomorphisms. Consequently, there exist $a^\pm \in C^\infty(B^* H)$ with $a^\pm \circ \pi^\pm = a|_{S_H^* M}$, and in terms of Fermi coordinates, $a^\pm(x', \xi') = a(x', \xi', \pm \sqrt{1 - |\xi'|^2})$, $(x', \xi') \in \tilde{B}^* H$. Next, we decompose Riemann measure and write $dx = d\sigma_{x_n=\tau} d\tau$ where $d\sigma_{x_n=\tau}$ is hypersurface measure on $H_\tau = \{x_n = \tau\}$. Then, in view of (59), one can write
\begin{align}
\int_{S^*_H M} a d\mu_{\delta}^H &= \int_{S^*_H M} a d\mu_{\delta}^H + \int_{S_H^* M} a d\mu_{\delta}^H \\
&\leq \limsup_{h \to 0} \left( \sup_{|\tau| < \delta_0} \langle a^+(x', hD') (1 - \chi_{C\delta}) (\sqrt{1 + h^2 R_{H_r}}) \varphi_{h^r}^{H_r}, \varphi_{h^r}^{H_r} \rangle_{L^2(H_r)} \right) \\
&\quad + \limsup_{h \to 0} \left( \sup_{|\tau| < \delta_0} \langle a^-(x', hD') (1 - \chi_{C\delta}) (\sqrt{1 + h^2 R_{H_r}}) \varphi_{h^r}^{H_r}, \varphi_{h^r}^{H_r} \rangle_{L^2(H_r)} \right) \\
&\leq \limsup_{h \to 0} \sup_{|\tau| < \delta_0} \langle a(x', hD') \varphi_{h^r}^{H_r}, \varphi_{h^r}^{H_r} \rangle_{L^2(H_r)} = O(1)\|a\|_{L^\infty},
\end{align}

by $L^2$-boundedness and Lemma 6.3.

Since the $O(1)$ bound on the RHS of (84) is uniform in $\delta$ and $\int_{S^*_H M} a d\mu_{H, \delta}$ is monotone non-decreasing as $\delta \to 0$, it follows that for $0 \leq a \in C^\infty(S^*_H M)$, the limit $\lim_{\delta \to 0} \int_{S^*_H M} a d\mu_{H, \delta}$ exists. A similar argument applies to $a \in C^\infty$ with $a \leq 0$. In general, for $a \in C^0(S^*_H M)$ we make the decomposition $a = a^+ - a^-$ and mollify $a^\pm$ by considering $a_{\pm, \beta} := \chi_{\beta} \ast a^\pm \in C^\infty$ with $0 \leq \chi \beta \in C^\infty$ an approximation of the identity. One then applies the above argument to $a_{\beta, \pm}$ separately and takes the $\beta \to 0$ limit at the end.
Finally, setting

\[ d\mu^H(a) := \lim_{\delta \to 0} \int_{S^*_H \delta} a \, d\mu^H_\delta, \]

it is clear that \( d\mu^H \) is linear, non-negative and from (84) satisfies \(|d\mu^H(a)| \leq C\|a\|_{L^\infty(S^*_H \delta)}\) and is consequently a measure on \( S^*_H \delta \).

**Remark 6.4.** We note that the eigenfunction subsequences with defect measures \( d\mu \) satisfying the conditions in Proposition 6.2 are precisely the ones for which the map \( \pi : S^*_H \delta \to \mathcal{E}_\delta \) in Lemma 4.4 of section 4.3 is \( \nu \) almost everywhere 1-1.

In view of Proposition 0.9, to complete the proof of Theorem 0.10, we must compute the integral of \( \sigma(P_{T,\epsilon}) \) against an invariant measure \( \mu \) using the disintegration decomposition of \( d\mu \) in (58) (ii) and Proposition 6.2. First, we recall from (58) (ii) and (60) that for \( \delta > 0 \) sufficiently small, (85)

\[ d\mu|_{S^*_H \delta} = dt d\nu^H|_{S^*_H \delta} = dx_n \, d\mu^H_\delta(x', \xi), \quad (x', \xi) \in S^*_H \delta. \]

Provided the defect measure \( d\mu \) corresponds to a density-one eigenfunction subsequence in Proposition 6.2, one can take the weak limit in (85) as \( \delta \to 0 \). The result is that

(86)

\[ d\mu(x, \xi) = dx_n \, d\mu^H(x', \xi), \quad (x, \xi) \in S^* H, \]

where \( d\mu^H \) is a Borel measure on \( S^*_H \). Provided one chooses \( T = T(\epsilon) \) small enough so that there is only one term in (72), it then follows by (72) and (86) that

\[ \int_{S^*_H} a_{T,\epsilon}(x, \xi) d\mu = \frac{1}{T} \int_{S^*_H} (1 - \chi_\epsilon)(\pi^* \gamma^{-1} a_H)(G_{\epsilon}(x, \xi)) \, d\mu \]

\[ = \frac{1}{T} \int_0^T \chi(x, \xi) dt \int_{S^*_H}(1 - \chi_\epsilon)(x', \xi) \pi^* \gamma^{-1}(x', \xi) \, a_H(x', \xi) \, d\mu^H(x', \xi) \]

\[ = \int_{S^*_H}(1 - \chi_\epsilon) a_H(x', \xi) \, d\mu^H(x', \xi). \]

In the penultimate line of (87) we have used that for \( (x', \xi) \in S^*_H, \, |\xi_\epsilon| = \pi^* \gamma^{-1} \) so that the \( \pi^* \gamma^{-1} \) factor in the symbol of \( P_{T,\epsilon} \) gets cancelled by the \( |\xi_\epsilon| = \pi^* \gamma^{-1} \) factor in the numerator coming from the disintegration of \( d\mu \).

From Proposition 0.9 it then follows that for any \( \epsilon > 0 \), there exists a density-one sequence \( \tilde{S}(\epsilon) \subset S_T(T(\epsilon), \epsilon) \) such that

(88)

\[ \langle Op_H(a(1 - \chi^H))\varphi_{jk}|H, \varphi_{jk}|H \rangle_{L^2(H)} \sim_{k \to \infty} \int_{S^*_H}(1 - \chi_\epsilon) a_H(x', \xi) \, d\mu^H(x', \xi), \quad j_k \in \tilde{S}(\epsilon). \]

Finally, choose a sequence \( \epsilon_n, n = 1, 2, 3, \ldots \) with \( \epsilon_n \to 0^+ \) in (88) and set \( \tilde{S} := \cap_{n \geq 1} \tilde{S}(\epsilon_n) \). Then, as in Theorem 0.10, it follows by taking the \( \epsilon_n \to 0 \) limit in (88), with the result that for all \( a \in S^0(H), \)

\[ \langle Op_H(a|H, \varphi_{jk}|H \rangle_{L^2(H)} \sim_{k \to \infty} \int_{S^*_H} a_H(x', \xi) \, d\mu^H(x', \xi), \quad j_k \in \tilde{S}. \]
7. Mass and microsupport: Proof of Theorem 0.5

We consider the space
\[ A_H = \bigcup_{T, \epsilon > 0} \{ P_{T, \epsilon}(a) : a \in S^0(B^*H) \} \]
of “cross-sectional pseudo-differential operators” operators acting on \( L^2(M) \) and let
\[ a_{T, \epsilon} = \sigma_{P_{T, \epsilon}}. \]

**Definition 7.1.** We define the cross-sectional symbol space \( S^0_A_H \) to be the space of zeroth-order symbols \( a_{T, \epsilon} \) of \( P_{T, \epsilon} \in A_H \).

As in [Ge], define the wave front set \( WF(S) \) of obstructions to microlocal compactness of \( S \) as follows:

**Definition 7.2.** We define the semi-classical wave front set of a sequence \( S = \{ u_n \} \) with respect to \( A_H \) such as
\[ WF_{A_H}(S) = \bigcap_{A \in A_H, A \text{ compact}} \{ \sigma_A = 0 \}, \]
where the intersection runs over all \( A \in A_H \) such that \( Au_n \) is relatively compact in \( L^2 \).

By a microlocal defect measure \( \mu \) of \( S \) we mean a probability measure on \( S^*M \) obtained as a weak* limit of the functionals \( \rho_j(A) = \langle Au_j, u_j \rangle \). In the case that \( S \) has a unique microlocal defect measure (quantum limit), a well-known result equates the wave front set with the support of the microlocal defect measure: \( WF(S) = \text{Supp} \mu \); see [Ge] for background. We define the relative analogue using the subspace \( A_H \):

**Lemma 7.3.** If \( S = \{ \varphi_j \} \) is a sequence satisfying \( ||\varphi_j||_{L^2(H)} = o(1) \), then any microlocal defect measure (quantum limit measure) \( \mu \) of \( S \) on \( S^*M \) satisfies
\[ \text{supp} \mu \subseteq \bigcap_{a, \epsilon, T} \{ \sigma_{P_{T, \epsilon}(a)} = 0 \}. \]

**Proof.** It is obvious that if \( ||\varphi_j||_{L^2(H)} = o(1) \) then \( ||Op_{h_j}(a)\varphi_j||_{L^2(H)} \rightarrow 0 \) for all \( a \in C^\infty_c(T^*H) \). Hence, all microlocal defect measures of the sequence \( \langle Op_{h_j}(a)\varphi_j, \varphi_j \rangle \) on \( B^*H \) must vanish.

Proposition 0.10 relates matrix elements on \( H \) to matrix elements on \( M \). If \( ||\varphi_j||_{L^2(H)} = o(1) \) then
\[ 0 = \lim_{k \rightarrow \infty} \langle P_{T, \epsilon}(a)\varphi_{j_k}, \varphi_{j_k} \rangle = \int_{S^*M} \sigma_{P_{T, \epsilon}(a)} d\mu. \]

**Remark 7.4.** Note that \( d\mu \) is the microlocal defect measure of \( S \) on \( S^*M \). It does not need to equal \( d\mu_S^M \) since the latter is the defect measure only relative to the subspace \( A_H \). What the Lemma asserts is that both measures must have the same integrals with respect to symbols of operators in \( A_H \).

We now want to show that \( \{ \sigma_{P_{T, \epsilon}(a)} = 0 \} \) has measure zero and that a microlocal defect measure supported in a set of measure zero must come from a zero-density subsequence.
Lemma 7.5. $\bigcap_{A \in \mathcal{A}_H} \{\sigma_A = 0\} \subset S^*M \setminus FL(H)$. That is, $\bigcap_a \{\sigma_{P_T,a} = 0\}$ is the complement of the flowout $FL(H)$.

Proof. We denote by $\sigma_{A_H}$ the set of all possible symbols of $P_{T,\epsilon} \in \mathcal{A}_H$. By Lemma 8.1 if $a_H > 0$ then $\sigma_{P_{T,\epsilon}}(x, \xi) > 0$ if $G'(x, \xi)$ intersects $S^*H_M$. Hence the set $\bigcap_a \{\sigma_{P_T,a} = 0\}$ cannot contain any points for $T$ small (depending on $\epsilon$ and $\epsilon > 0$. As a result, the zero set can only contain points $(x, \xi)$ for which the orbit never hits $H$. \qed

7.1. Spectral projections in $H_S$. We have been considering microlocal defect measures (quantum limits) of $\langle P_{T,\epsilon}(a) \varphi_j, \varphi_j \rangle$. But we may also consider microlocal defect measures of the normalized traces

$$\rho_{S,\lambda}(A) := \frac{1}{N(\lambda, S)} \text{Tr} \Pi_{S,\lambda}, \quad A \in \mathcal{A}_H,$$

where if $S = \{\varphi_j\}$ then

$$\Pi_{S,\lambda} f = \sum_{j: \lambda_j \leq \lambda} \langle f, \varphi_{j_k} \rangle \varphi_{j_k}.$$

These are states on the space $\mathcal{A}_H$.

Lemma 7.6. Let $\mu_S$ be a microlocal defect measure for the functionals $\rho_{S,\lambda}$. Then $a_{T,\epsilon} \mu_S = 0$ for all symbols in $\sigma_{A_H}$.

Proof. The argument above for individual eigenfunctions is also true for the microlocal lift of the projector $\Pi_{S,\lambda}$. Pick $\epsilon, T$ so that the complement has measure $< \delta$, the putative density of $S$.

Let $d\Phi_j$ be a the positive microlocal lift of $\varphi_j$, i.e. $\int_{S^*M} d\Phi_j = \langle Op(a) \varphi_j, \varphi_j \rangle$ where $Op(a)$ is a positive quantization (for example, a Friedrichs quantization). $S$ and its density are independent of $T, \epsilon$. But the limit $\mu_S$ of

$$\hat{\rho}_{S,\lambda} := \frac{1}{N(\lambda, S)} \sum_{j: \lambda_j \leq \lambda, \lambda_j \in S} d\Phi_j$$

must also satisfy

$$a_{T,\epsilon} \mu_S = 0. \quad \Box$$

Corollary 7.7. The defect measures $\mu_S$ of the trace functionals $\rho_{S,\lambda}$ are supported in a the complement of $FL(H)$ in $S^*M$, a set of Liouville measure zero.

Proof. The limit is a $G^t$ invariant probability measure. Hence $\mu_S$ must vanish on $FL(H)$. Thus, $\mu_S$ is supported on a closed invariant set of Liouville measure zero, i.e. $S$ is a positive sequence which ‘concentrates’ on a closed invariant set $\Gamma$ of $\mu_L$-measure zero, where $\mu_L$ is Liouville measure. \qed

To get a contradiction, we need to show that if $S$ has positive density, then the microlocal defect measures cannot all be supported in a set of measure zero.
Proposition 7.8. Suppose that $\Gamma$ is a closed invariant set of $\mu_L$-measure 0, and that $S$ is a subsequence all of whose microlocal defect measures are supported in $\Gamma$. Then $D^*(S) = 0$.

Proof. We argue by contradiction and show that if $S$ has positive density, then the ‘maximal’ microlocal defect measure cannot be supported in a set of $\mu_L$ measure zero. This maximal measure comes from the spectral projections onto the sequence $S$.

If $S$ has positive density, then there exists $A > 0$ so that

$$\limsup_{\lambda \to \infty} \frac{N(\lambda)}{N(\lambda, S)} \leq A.$$  

Let $V$ be a conic neighborhood of $\Gamma$ and let $\chi_V$ be a conic cutoff to $V$. Then for any $a$,

$$\lim_{\lambda \to \infty} \hat{\rho}_{S,L}(Op(a)) = \lim_{\lambda \to \infty} \hat{\rho}_{S,L}(Op(\chi_V a)).$$

Let

$$\hat{\rho}_\lambda := \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} d\Phi_j.$$  

Recall the local Weyl law

$$(91) \quad \hat{\rho}_\lambda(A) = \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \langle A \varphi_j, \varphi_j \rangle_{L^2(M)} \rightarrow \int_{S^*M} \sigma_A d\mu_L$$

on $M$.

For $\lambda$ sufficiently large, by (90),

$$\limsup_{\lambda \to \infty} \frac{1}{N(\lambda, S)} \sum_{j: \lambda_j \leq \lambda, \lambda_j \in S} \rho_j(Op(a)) = \limsup_{\lambda \to \infty} \frac{N(\lambda)}{N(\lambda, S)} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda, \lambda_j \in S} \rho_j(Op(\chi_V a))$$

$$\leq A \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \rho_j(Op(\chi_V a))$$

$$= A \limsup_{\lambda \to \infty} \hat{\rho}_\lambda(Op(\chi a)) \leq \mu_L(V).$$

If $\mu_L(\Gamma) = 0$ the the right side is $\leq \epsilon$ if $V$ is an $\epsilon$-neighborhood. It follows that $\lim_{\lambda \to \infty} \hat{\rho}_{S,L}(Op(a)) = 0$ for all $a$, which is absurd. This contradiction completes the proof.

8. Proof of Theorem 0.6

In this section we prove the quantitative refinement of Theorem 0.5 stated in Theorem 0.6.

Proof. To prove Theorem 0.6 we study integrals $\int_H f |\varphi_j|^2 dS$ or more general matrix elements $\langle Op_h(a) \gamma^*_H \varphi_j, \gamma^*_H \varphi_j \rangle_{L^2(H)}$. In order to prove that $H$ is good for a density one sequence of eigenfunctions, it suffices to show that the matrix elements do not tend to zero for at least one symbol $a$. Since

$$\langle (\gamma^*_H Op_H(a) \gamma_H) |_{H} \varphi_j, \varphi_j \rangle_{L^2(H)} = \langle \nabla_{\gamma_H} (a) \varphi_j, \varphi_j \rangle_{L^2(M)}$$

it follows from (76) that
(92) \[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \langle (\gamma_H^* \text{Op}_H(a) \gamma_H) \rangle_{\leq \epsilon \varphi_j, \varphi_j} \rangle_{L^2(M)} - \langle (P_{T, \epsilon}(a) + F_{T, \epsilon}(a)) \varphi_j, \varphi_j \rangle_{L^2(M)} \right|^2 = 0.
\]

8.1. Contribution of the pseudo-differential term \( P_{T, \epsilon}(a) \). In view of (71), it follows from (92) that

(93) \[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \langle (\gamma_H^* \text{Op}_H(a) \gamma_H) \rangle_{\leq \epsilon \varphi_j, \varphi_j} \rangle_{L^2(M)} - \langle P_{T, \epsilon}(a) \varphi_j, \varphi_j \rangle_{L^2(M)} \right|^2 = 0.
\]

Since we are free to choose the non-negative symbol \( a \), we henceforth put \( a(s, \sigma) := 1 \) and simply write

(94) \[
P^1_{T, \epsilon} := P_{T, \epsilon}(1)
\]

with

(95) \[
\sigma(P^1_{T, \epsilon})(x, \xi) = \frac{1}{T} \sum_{j \in \mathbb{Z}} (1 - \chi_\epsilon) \pi_H^*(\gamma^{-1})(G_{t_j(x, \xi)}(x, \xi)) \chi(T^{-1}t_j(x, \xi))
\]

8.1.1. Microlocal ellipticity of \( P^1_{T, \epsilon} \). We now observe that for fixed \( T > 0, \epsilon > 0 \), \( P^1_{T, \epsilon} \) is microlocally elliptic on the support of \( \chi_{T, \epsilon} \).

**Lemma 8.1.** We have

(96) \[
\sigma(P^1_{T, \epsilon})(x, \xi) \geq \frac{1}{T}, \quad (x, \xi) \in \text{supp} \chi_{T, \epsilon}.
\]

**Proof.** The symbol of \( P^1_{T, \epsilon} \) is

(97) \[
\sigma(P^1_{T, \epsilon})(x, \xi) = \frac{1}{T} \sum_{j \in \mathbb{Z}} (1 - \chi_\epsilon) \pi_H^*(\gamma^{-1})(G_{t_j(x, \xi)}(x, \xi)) \chi(T^{-1}t_j(x, \xi))
\]

where, \( t_j(x, \xi) \in C^\infty(T^*M) \) are the impact times of the geodesic \( \exp_x(t \xi) \) with \( H \), where \( \gamma \) is defined by (37).

By definition of the cutoff (66), it follows that for \( (x, \xi) \in \text{supp} \chi_{T, \epsilon} \), the hitting time \( |t_1(x, \xi)| < T \) and \( (1 - \chi_\epsilon)(G_{t_1(x, \xi)}(x, \xi)) = 1 \). Consequently,

\[
\frac{1}{T} \sum_{j \in \mathbb{Z}} (1 - \chi_\epsilon) \pi_H^*(\gamma^{-1})(G_{t_j(x, \xi)}(x, \xi)) \chi(T^{-1}t_j(x, \xi)) \geq \frac{1}{T} \pi_H^*(\gamma^{-1})(G_{t_1(x, \xi)}(x, \xi)) \geq \frac{1}{T}
\]

since \( \pi_H^*(\gamma^{-1})(s, \eta) \geq 1 \) for any \( (s, \eta) \in S^*_H M \).

\[\square\]

By the pointwise local Weyl law, for any \( a \in S^0(T^*H) \),

\[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \left| \langle (\gamma_H \text{Op}_H(a) \gamma_H) \rangle_{\leq \epsilon \varphi_j, \varphi_j} \rangle_{L^2(M)} \right|^2 = O(\varepsilon)
\]

and so, from (93) if follows that for any fixed \( T > 0 \),
(98) \[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j, \lambda_j \leq \lambda} \left\| \varphi_j^H \right\|_{L^2(H)}^2 - \left\langle P_{T, \epsilon}^1 \varphi_j, \varphi_j \right\rangle_{L^2(M)}^2 = O(\epsilon).
\]

We note that in (98), one is free to choose the time-average parameter \( T \). In \cite{TZ} we take \( T \to \infty \) in order to apply the mean ergodic theorem, but here we will not take \( T \to \infty \); rather, here \( T \) will be a fixed constant to be specified later on. Consequently, taking \( \liminf_{\epsilon \to 0} \) of both sides of (98), it follows that there is a density-one subset \( S \subset \{1, \ldots, \lambda \} \) such that

(99) \[
\liminf_{\epsilon \to 0} \left\| \varphi_j^H \right\|_{L^2(H)}^2 - \left\langle P_{T, \epsilon}^1 \varphi_j, \varphi_j \right\rangle_{L^2(M)}^2 = o(1), \quad \lambda_j \to \infty, \ j \in S.
\]

As a consequence of (99) it suffices to estimate \( \liminf_{\epsilon \to 0} \left\langle P_{T, \epsilon}^1 \varphi_j, \varphi_j \right\rangle_{L^2(M)} \) from below. To do this, we will need the microlocal ellipticity result in Lemma 8.1 combined with the following lemma on a priori, microlocal eigenfunction mass estimates near \( S^*_H M \) under the full-measure flowout assumption.

8.1.2. Microlocal eigenfunction mass estimates. In this section we prove that for all \( \delta > 0 \) there exists a positive lower bound \( C_\delta > 0 \) on a density \( \geq 1 - \delta \) subsequence for the matrix elements of \( \left\langle P_{T, \epsilon} \varphi_{j_k}, \varphi_{j_k} \right\rangle \) where \( \epsilon = \epsilon(\delta) \) be taken sufficiently small. Here, \( P_{T, \epsilon} \) corresponds to a positive symbol on \( B^* H \) and as above we take it to equal 1. Then, \( P_{T, \epsilon} = \chi_{T, \epsilon}(x, D_x) \)

**Lemma 8.2.** Fix \( T > 0 \). Then, for any \( 0 < \delta < 1 \), there exists a subsequence \( S(\delta) \) of density greater that \( 1 - \delta \) such that with \( \epsilon = \epsilon(\delta) > 0 \) sufficiently small, there exists \( C_\delta > 0 \),

\[
\left\langle \chi_{T, \epsilon}(\delta)(x, D_x) \varphi_j, \varphi_j \right\rangle_{L^2(M)} \geq C_\delta, \quad j \in S(\delta).
\]

**Proof.** Throughout \( T > 0 \) will be fixed and so dependence of constants on \( T \) will be suppressed. W let \( R > 0 \) be an independent parameter that we will choose sufficiently large (see (101) below). Letting \( \left\langle \chi_{T, \epsilon}(\delta) \right\rangle_R := \frac{1}{R} \int_0^R U(-t) \chi_{T, \epsilon}(\delta) U(t) dt \), it follows that

(100) \[
\left\langle \chi_{T, \epsilon}(\delta) \varphi_j, \varphi_j \right\rangle_{L^2(M)} = \left\langle \left( \chi_{T, \epsilon}(\delta) \right)_R \varphi_j, \varphi_j \right\rangle_{L^2(M)} + O(T \lambda_j^{-1}).
\]

In view of (55) and since \( \sigma((\chi_{T, \epsilon}(\delta))'_R)(x, \xi) = \frac{1}{R} \int_0^R \chi_{T, \epsilon}(\delta)(G^t(x, \xi)) dt \) can find \( R = R(\delta) \) and enlarge \( \epsilon(\delta) \) to \( \epsilon'(\delta) > \epsilon(\delta) \) so that

(101) \[
\mu_L(\text{supp}(1 - \chi_{R(\delta), \epsilon'(\delta)})) = O(\delta)
\]

and there exists \( C(\delta) > 0 \) with

(102) \[
\sigma((\chi_{T, \epsilon}(\delta))'_R)(x, \xi) \geq 2C(\delta) > 0, \quad (x, \xi) \in \text{supp} \chi_{R(\delta), \epsilon'(\delta)}.
\]

To simplify the writing, in the following we sometimes suppress the dependence of \( R, \epsilon \) and \( \epsilon' \) on \( \delta \).

By (102), \( (\chi_{T, \epsilon})_R \) is microlocally elliptic on \( \text{supp} \chi_{R, \epsilon'} \) and so by the sharp Garding inequality (cf. \cite{HoIII} Theorem 18.1.14 or \cite{Tay} Theorem 6.1, p. 20) applied to the operator \( \chi_{R, \epsilon}(x, D) = 2C(\delta) \chi_{R, \epsilon'}(x, D) \),

(103) \[
\left\langle (\chi_{T, \epsilon})_R(x, D_x) \varphi_j, \varphi_j \right\rangle_{L^2(M)} \geq 2C(\delta) \left\langle \chi_{R, \epsilon'} \varphi_j, \varphi_j \right\rangle_{L^2(M)} + O_3(\lambda_j^{-1})
\]

In view of (101) and (103) it follows that

(104) \[
\left\langle \chi_{T, \epsilon}(\delta)(x, D_x) \varphi_j, \varphi_j \right\rangle_{L^2(M)} \geq C(\delta) \left\langle \chi_{R, \epsilon'} \varphi_j, \varphi_j \right\rangle_{L^2(M)} + O_3(\lambda_j^{-1}).
\]
We recall that the sharp Garding inequality states: If $p \in S^0$ with $\Re p \geq 0$, then there exists a constant $C_0 > 0$ so that $\Re(p(x,D)u,u) \geq -C_0||u||^2_{H^{-1/2}}$.

One can write

$$\langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle = \beta(\lambda_j; R, \varepsilon) - C_\delta \langle (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle_{L^2(M)} + O_\delta(\lambda_j^{-1}),$$

where

$$\beta(\lambda_j; R, \varepsilon) := \langle (\chi_{T,\varepsilon}) R \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle + \langle (\chi_{T,\varepsilon}) R (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle + C_\delta \langle (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle.$$}

The point of isolating the $\beta(\lambda_j; R, \varepsilon)$ term on the RHS of (104) is that, as we now show, this term is uniformly bounded from below as $\lambda_j \to \infty$. The $C_\delta(1 - \chi_{R,T,\varepsilon})$-term is added in the definition of $\beta(\lambda_j; R, \varepsilon)$ to get a globally elliptic operator.

More precisely, from (103)

$$\beta(\lambda_j; R, \varepsilon) \geq C_\delta \langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle + \langle (\chi_{T,\varepsilon}) R (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle + C_\delta \langle (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle.$$}

Using the fact that $\langle (\chi_{T,\varepsilon}) R (1 - \chi_{R,T,\varepsilon}(x, \xi)) \rangle \geq 0$, it follows by application of sharp Garding in the second term on the RHS of (104) that

$$\beta(\lambda_j; R, \varepsilon) \geq C_\delta \langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle + C_\delta \langle (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle + O(\lambda_j^{-1}) \geq C_\delta + O(\lambda_j^{-1}),$$

since $\|\varphi_j\|_{L^2}^2 = 1$.

To estimate the matrix value $\langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle$, the term involving $C_\delta(1 - \chi_{R,T,\varepsilon})$ is subtracted out in (104), but in the variance sum this term gives a small contribution since $\mu_L(\text{supp} (1 - \chi_{R,T,\varepsilon}))$ is small. More precisely, since $|\langle (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle|^2 \leq \langle (I - \chi_{R,T,\varepsilon})^2 \varphi_j, \varphi_j \rangle$ it follows from the local Weyl law that

$$\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle (I - \chi_{R,T,\varepsilon}) \varphi_j, \varphi_j \rangle|^2 \leq \int_{S^* M} (1 - \chi_{R,T,\varepsilon})^2 = O(\delta),$$

from which it follows that

$$\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle - \beta(\lambda_j; R, \varepsilon) \right|^2 = O(C_\delta \delta).$$

By Chebyshev’s inequality,

$$D^*\{ j; |\langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle - \beta(\lambda_j; R, \varepsilon) | \geq \frac{C_\delta}{2} \} \leq O\left( \frac{2}{C_\delta} \right) = O(\delta),$$

and consequently,

$$D^*\{ j; |\langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle - \beta(\lambda_j; R, \varepsilon) | \leq \frac{C_\delta}{2} \} \geq 1 - C\delta,$$

where $C > 0$ is a constant independent of $\delta > 0$. For eigenfunctions $\varphi_j$ satisfying $|\langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle - \beta(\lambda_j; T, \varepsilon) | \leq \frac{C_\delta}{2}$ it follows from the lower bound $\beta(\lambda_j; R, \varepsilon) \geq C_\delta + O(\lambda_j^{-1})$ in (106) that for $\lambda_j$ sufficiently large,

$$\langle \chi_{R,T,\varepsilon} \varphi_j, \varphi_j \rangle \geq \frac{C_\delta}{2} > 0.$$
Since then
\[ D^*(\{j; \langle \chi_{R,\varepsilon} \varphi_j, \varphi_j \rangle \geq \frac{C_\delta}{2} \}) \geq 1 - C\delta, \]
that finishes the proof of Lemma 8.2.

We are now in a position to prove lower bounds for \( \langle P_{T,\varepsilon}^1 \varphi_j, \varphi_j \rangle_{L^2(M)} \) where \( j \in S(\delta) \) with \( D(S(\delta)) \geq 1 - \delta \). To do this, we use the sharp Garding inequality yet again. Recalling [94], we have \( \sigma(P_{T,\varepsilon}^1) \geq \sigma(P_{T,\varepsilon}^1 \chi_{T,\varepsilon}) \) and since \( P_{T,\varepsilon}^1 \in Op(S^0) \) and \( P_{T,\varepsilon}^1 \chi_{T,\varepsilon} \in Op(S^0) \), it follows from the sharp Garding inequality that
\[
\langle P_{T,\varepsilon}^1 \varphi_j, \varphi_j \rangle_{L^2(M)} \geq \langle P_{T,\varepsilon}^1 \chi_{T,\varepsilon} \varphi_j, \varphi_j \rangle_{L^2(M)} - C_\varepsilon \| \varphi_j \|^2_{H^{-\frac{1}{2}}(M)},
\]
and so,
\[ \langle P_{T,\varepsilon}^1 \varphi_j, \varphi_j \rangle_{L^2(M)} \geq \langle P_{T,\varepsilon}^1 \chi_{T,\varepsilon} \varphi_j, \varphi_j \rangle_{L^2(M)} + O_\varepsilon(\lambda_j^{-1}). \]

In view of (109), it is enough to bound the matrix elements \( \langle P_{T,\varepsilon}^1 \chi_{T,\varepsilon} \varphi_j, \varphi_j \rangle_{L^2(M)} \) from below. Combining the microlocal mass estimate in Lemma 8.2 and the microlocal ellipticity result in Lemma 8.1, it follows by sharp Garding that with \( \epsilon = \epsilon(\delta) \) sufficiently small,
\[ \langle P_{T,\varepsilon}^1 \chi_{T,\varepsilon} \varphi_j, \varphi_j \rangle_{L^2(M)} \geq \frac{1}{T} \langle \chi_{T,\varepsilon} \varphi_j, \varphi_j \rangle_{L^2(M)} - C_\delta \lambda_j^{-1} \geq \frac{C_\delta}{2T}, \quad j \in S(\delta), \lambda_j \geq \lambda(\delta). \]

Consequently, from (109) and (110),
\[ \langle P_{T,\varepsilon}^1 \varphi_j, \varphi_j \rangle_{L^2(M)} \geq \frac{C_\delta}{2T} + O_\varepsilon(\lambda_j^{-1}), \quad j \in S(\delta). \]

From (109) it follows that after possibly shrinking \( \varepsilon(\delta) \) further,
\[ \| \varphi_j^H \|^2_{L^2(H)} - \langle P_{T,\varepsilon}^1 \varphi_j, \varphi_j \rangle_{L^2(M)} \leq \frac{C_\delta}{4T}, \quad \lambda_j \geq \lambda(\delta), \quad j \in S. \]

Now, restricting to \( j \in S(\delta) \cap S \) in (112) and using (111) one gets
\[ \| \varphi_j^H \|^2_{L^2(H)} \geq \frac{C_\delta}{4T}, \quad \lambda_j \geq \lambda(\delta), \quad j \in S \cap S(\delta). \]

This completes the proof of Theorem 0.6.

9. Examples of hypersurfaces with \( \mu_L(FL(H)) = 1 \).

9.1. Simple convex surfaces of revolution. Let \( (M, g) \) be a strictly-convex surface of revolution with metric \( g = d\theta^2 + f(\theta) d\varphi^2 \) where \( 0 \leq \theta \leq L \) and \( \varphi \in [0, 2\pi] \) and \( f \in C^\infty([0, L], \mathbb{R}) \) with \( f(\theta) > 0 \),
\[ f'(\theta_0) = 0, \quad f''(\theta_0) < 0 \]
The Hamiltonian

\[ H(\theta, \varphi, \xi_{\theta}, \xi_{\varphi}) = \xi_{\theta}^2 + f^{-1}(\theta) \xi_{\varphi}^2 \]

is Liouville completely integrable with integral in involution \( P((\theta, \varphi, \xi_{\theta}, \xi_{\varphi}) = \xi_{\varphi}. \)

The “equator” of the surface is the periodic geodesic
\[ \gamma_0 := \{ \theta = \theta_0, \ 0 \leq \varphi \leq 2\pi \}. \]

The moment map (restricted to \( S^*M \)) is
\[ P := (1, \xi_{\varphi}) : S^*M \rightarrow \mathbb{R}^2 \]

If \( \mathcal{B}_{reg} \) denotes the regular values of the moment map, \( \mathcal{P}^{-1}(b) \) consists of (two) Lagrangian tori \([TZ03]\) (with slight abuse of notation we denote them both by \( \Lambda_b \subset S^*M \)). If \( S^*M_{reg} := \bigcup_{b \in \mathcal{B}_{reg}} \Lambda_b \) then
\[ \mu_L(S^*M_{reg}) = 1. \]

The projections \( \pi(\Lambda_b) \) with \( b \in \mathcal{B}_{reg} \) of the invariant tori are the equatorial bands
\[ (113) \quad \pi(\Lambda_b) = \{ (\theta, \varphi); \theta \in [\theta_0 - r_1(b), \theta_0 + r_1(b)], 0 \leq \varphi \leq 2\pi \} \]
where \( f(r_j(b)) = b, \ j = 1, 2. \)

On \( S^*M_{reg} \) there exist action-angle variables \((\theta, I)\) in terms of which the Hamiltonian \( H = H(I) \) and so the Hamilton equations are solvable by quadrature. Explicitly, the action variables in this case are \([TZ03]\)
\[ I_1(b) = b, \quad I_2(b) = \frac{1}{\pi} \int_{r_1(b)}^{r_2(b)} \left( 1 - \frac{b^2}{f^2(\theta)} \right)^{\frac{1}{2}} d\theta. \]

Under the twist assumption
\[ (114) \quad \nabla_I \omega(I) \neq 0, \quad \omega(I) := \nabla_I H(I) \]
on the metric, there exists a family of irrational tori (which we denote by \( \bigcup_{b \in \mathbb{Q}^c} \Lambda_b \)) such that the geodesic flow on each such torus \( \Lambda_b \) is dense. That is, for any \((\theta, I) \in \Lambda_b \) with \( b \in \mathbb{Q}^c \) we have that
\[ (115) \quad \bigcup_{t \in \mathbb{R}} \gamma(t; I, \theta) = \Lambda_b, \ b \in \mathbb{Q}^c. \]

Consequently, for the projected geodesic,
\[ (116) \quad \bigcup_{t \in \mathbb{R}} \pi \circ \gamma(t; I, \theta) = \pi(\Lambda_b), \ b \in \mathbb{Q}^c. \]

Now suppose \( H \subset M \) is an simple, closed curve with
\[ H \cap \gamma_0 \neq \emptyset. \]

Then, in view of \([116]\) and the band structure in \([113]\) it follows that for any geodesic \( \gamma(t; \theta, I) \) on \( \Lambda_b \) with \( b \in \mathbb{Q}^c \), the projection \( \pi \circ \gamma(t; \theta, I) \) must intersect \( H \) for some \( t \in \mathbb{R} \) and so,
\[ \bigcup_{t \in \mathbb{R}} \gamma(t; \theta, I) \cap S^*_H M \neq \emptyset. \]
Then since $\Lambda_b$ is $G_t$-invariant, $\bigcup_{b \in Q^c} \Lambda_b \subset \text{FL}(H)$ and so,

$$\mu_L(\text{FL}(H)) \geq \mu_L(\bigcup_{b \in Q^c} \Lambda_b) = \mu_L(S^*\text{M}_\text{reg}) = 1.$$  

Since trivially $\mu_L(\text{FL}(H)) \leq 1$ it follows that $\mu_L(\text{FL}(H)) = 1$.

To summarize: under the twist condition (114), for any simple closed curve $H$ with $H \cap \gamma_0 \neq \emptyset$, we have that $\mu_L(\text{FL}(H)) = 1$.

It is well-known [Bl] that both oblong ($a < b$) and oblate ($a > b$) ellipsoids $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ satisfy the twist condition, whereas the sphere does not. Indeed, in the latter case $\gamma_0 = \{(x, y, z) \in S^2; z = 0\}$ and so the condition $H \cap \gamma_0 \neq \emptyset$ is clearly not sufficient since for any closed curve $H$ with $\text{diam } H < \text{diam } \gamma_0$ there exist a positive measure of $2\pi$-periodic great circles that do not intersect $H$.

9.2. Liouville metrics on tori. A Liouville torus $(M, g)$ is a topological two-torus with metric

$$g = [U_1(x_1) - U_2(x_2)](dx_1^2 + dx_2^2), \quad x = (x_1, x_2) \in [0, 1] \times [0, 1].$$

Here, $U_j$ are smooth Morse functions with period 1 and are required to satisfy $U_1(x_1) - U_2(x_2) > 0$ for all $x \in [0, 1] \times [0, 1]$. Moreover, we assume that $U_1$ and $U_2$ each have one maximum and minimum and that these critical points are all distinct. The associated geodesic flow is generated by the Hamiltonian

$$H(x, \xi) = [U_1(x_1) - U_2(x_2)]^{-1}(\xi_1^2 + \xi_2^2)$$

and the integral in involution is

$$P(x, \xi) = \frac{U_2(x_2)}{[U_1(x_1) - U_2(x_2)]}\xi_1^2 + \frac{U_1(x_1)}{[U_1(x_1) - U_2(x_2)]}\xi_2^2.$$  

The restricted moment map is then $\mathcal{P} = (1, P) : S^*M \to \mathbb{R}^2$. Then [TZ03], the singular leaves of the Lagrangian foliation consist of two horizontal periodic geodesics $\gamma_{1h}^1$ and $\gamma_{1h}^2$ along with two vertical ones $\gamma_{1v}^1$ and $\gamma_{1v}^2$. The associated bands (projections onto $M$ of invariant Lagrangian tori) come in two horizontal (resp. vertical) families containing the projected geodesics $\pi \circ \gamma_{1h}^{1,2}$ (resp. $\pi \circ \gamma_{1v}^{1,2}$). (see [TZ03] for further details).

A similar argument to the one above for revolution surfaces, shows that under a twist assumption on $g$, for any simple closed curve $H \subset M$ satisfying

$$H \cap \gamma_{1h}^{1,2} \neq \emptyset, \quad H \cap \gamma_{1v}^{1,2} \neq \emptyset,$$

we have $\mu_L(\text{FL}(H)) = 1$.

References

[AB] J.C. Alvarez Paiva and G. Berck, What is wrong with the Hausdorff measure in Finsler spaces. Adv. Math. 204 (2006), no. 2, 647–663.

[AP] J. C. Alvarez Paiva and E. Fernandes, Gelfand transforms and Crofton formulas. Selecta Math. (N.S.) 13 (2007), no. 3, 369–390.

[Ber] B. Berndtsson, Restrictions of plurisubharmonic functions to submanifolds (preprint, 2016).
P. Bleher, Distribution of Energy Levels of a Quantum Free Particle on a Surface of Revolution. Duke Math. J. 74 (1994), no. 1, 45-93.

J. Bourgain and Z. Rudnick, On the nodal sets of toral eigenfunctions, Invent. Math. 185 (2011), no. 1, 199-237.

J. Bourgain and Z. Rudnick, Restriction of toral eigenfunctions to hypersurfaces and nodal sets, Geom. Funct. Anal. 22 (2012), no. 4, 878-937 (arXiv:1105.0018).

Y. Canzani, J. Galkowski and J. A. Toth, Averages of eigenfunctions over hypersurfaces (arXiv:1705.09595).

J. Bourgain and Z. Rudnick, Restriction of toral eigenfunctions to hypersurfaces and nodal sets, Geom. Funct. Anal. 22 (2012), no. 4, 878-937 (arXiv:1105.0018).

J. Bourgain and Z. Rudnick, Quantum ergodic restriction theorems for Cauchy data, Math. Res. Lett. 20 (2013), no. 3, 465-475.

H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93 (1988), 161-183, MR1039348, Zbl 0784.31006.

S. Dyatlov and M. Zworski, Quantum ergodicity for restrictions to hypersurfaces. Nonlinearity 26 (2013), no. 1, 35-52.

L. El-Hajj and J. A. Toth, Intersection bounds for nodal sets of planar Neumann eigenfunctions with interior analytic curves. J. Differential Geom. 100 (2015), no. 1, 1-53.

J. Galkowski, The $L^2$-behaviour of eigenfunctions near the glancing set. Comm. P.D.E. 41 (2016), 1619-1648.

P. Gerard, Microlocal defect measures. Comm. Partial Differential Equations 16 (1991), no. 11, 1761-1794.

A. Ghosh, A. Reznikov, and P. Sarnak, Nodal domains of Maass forms I, Geom. Funct. Anal. 23 (2013), no. 5, 1515-1568 (arXiv:1207.6625).

L. Hörmander, The analysis of linear partial differential operators. III. Pseudo-differential operators. Classics in Mathematics. Springer, Berlin, 2007.

L. Hörmander, Notions of convexity. Reprint of the 1994 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2009.

J. Jung, Sharp bounds for the intersection of nodal lines with certain curves. J. Eur. Math. Soc. (JEMS) 16 (2014), no. 2, 273-288.

J. Jung and S. Zelditch, Number of nodal domains and singular points of eigenfunctions of negatively curved surfaces with an isometric involution. J. Differential Geom. 102 (2016), no. 1, 37-66.

V. Kaloshin, A geometric proof of the existence of Whitney stratifications. Mosc. Math. J. 5 (2005), no. 1, 125-133.

G. Lebeau, The complex Poisson kernel on a compact analytic Riemannian manifold, to appear (2013).

P. Lelong and L. Gruman, Entire functions of several complex variables. Grundlehren der Mathematishen Wissenschaften 282. Springer-Verlag, Berlin, 1986

F-H. Lin, Nodal sets of solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. 44 (1991), no. 3, 287-308.

M. Stenzel, On the analytic continuation of the Poisson kernel, Manuscripta Math. 144 (2014), no. 1-2, 253-276.

M.E. Taylor, Pseudodifferential Operators Princeton University Press, Princeton, N.J., 1981.

J. A. Toth and S. Zelditch, Counting Nodal Lines Which Touch the Boundary of an Analytic Domain, Jour. Diff. Geom. 81 (2009), 649-686 (arXiv:0710.0101).

J.A. Toth and S. Zelditch, Quantum ergodic restriction theorems: manifolds without boundary. Geom. Funct. Anal. 23 (2013), no. 2, 715-775 (arXiv:1104.4531).

J.A. Toth and S. Zelditch, Quantum ergodic restriction theorems: I. Interior hypersurfaces in domains with ergodic billiards Ann. H. Poincare 13 (2012) 599-670.
[TZ03] J.A. Toth and S. Zelditch, Norms of modes and quasimodes revisited. Proc. of AMS (Harmonic Analysis at Mount Holyoke) (2003), 435-458.

[Ze07] S. Zelditch, Complex zeros of real ergodic eigenfunctions. Invent. Math. 167 (2007), no. 2, 419 - 443.

[ZPl] S. Zelditch, Pluri-potential theory on Grauert tubes of real analytic Riemannian manifolds, I Spectral geometry, 299-339, Proc. Sympos. Pure Math., 84, Amer. Math. Soc., Providence, RI, 2012. 58J50 (32U99 32V99 35P20) (arXiv:1107.0463).

[Zint] S. Zelditch, Ergodicity and intersections of nodal sets and geodesics on real analytic surfaces. J. Differential Geom. 96 (2014), no. 2, 305-351.

[Ze16] S. Zelditch, Measure of nodal sets of analytic Steklov eigenfunctions, to appear in Math. Res. Letts (arXiv:1403.0647).

[Zw] M. Zworski, Semiclassical analysis, Graduate Studies in Mathematics, 138. American Mathematical Society, Providence, RI (2012).

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL, CANADA

E-mail address: jtoth@math.mcgill.ca

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, USA

E-mail address: zelditch@math.northwestern.edu