Sterile Neutrinos with Altered Dispersion Relations as an Explanation for the MiniBooNE, LSND, Gallium and Reactor Anomalies

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Abstract

Recently the MiniBooNE Collaboration has reported an anomalous excess in $\nu^\mu \rightarrow \nu^e$ neutrino oscillation data. Combined with long-standing results from the LSND experiment this amounts to a $6.1 \sigma$ evidence for new physics beyond the Standard Model. In this paper we develop a framework with 3 active and 3 sterile neutrinos with altered dispersion relations that can explain these anomalies without being in conflict with the absence of anomalous neutrino disappearance in other neutrino oscillation experiments.

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I. INTRODUCTION

The recently reported results of the MiniBooNE collaboration [1] exhibit a 4.8 $\sigma$ excess for $(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ transitions which cannot be explained by the standard three neutrino picture. Combined with the LSND results [2], these findings even provide a significance of 6.1 $\sigma$ for new physics beyond the Standard Model. The high significance demonstrates that this excess is not simply due to statistics, but has to be systematic. Albeit it is possible that the excess is due to an underestimation of the background, it is worthwhile to explore the possibility of this excess being a sign for new physics in the neutrino oscillation regime. Moreover, several other anomalies, like the LSND- [?]\, the Reactor- [3] and Gallium anomalies [4] also provide hints for new physics or additional sterile neutrinos. At the same time atmospheric neutrino experiments [5, 6] or accelerator experiments [7–9] searching for $\nu_\mu$ disappearance do not show any deviation from the standard three neutrino picture and yield stringent constraints on additional sterile neutrinos that rule out the most simple models.

More concretely, neutrino data exhibit:

- An excess of data, interpretable as $(\nu_\mu \rightarrow \nu_e)$ oscillations in both the neutrino and the anti-neutrino channels in the LSND and MiniBooNE data.
- A resonance-like excess in MiniBooNE’s low energy data.
- Evidence for electron neutrino disappearance in experiments utilizing a Gallium source or reactor neutrinos.
- No anomalous neutrino or anti-neutrino disappearance at higher energy accelerator experiments or atmospheric neutrino experiments.

In this paper we discuss the problems associated with the conflicting results from appearance and disappearance experiments (for a recent review see [10]) and develop a simple model adopting altered dispersion relations (ADRs) for sterile neutrinos which can explain both results consistently and describe the world’s neutrino data successfully. To our knowledge, at present there exists no other model which achieves this task. The paper is organized as follows: In section 2 we review the general framework of neutrino oscillations in the presence of sterile neutrinos. Section 3 discusses the effect of altered dispersion relations and points out the phenomenological difficulties that arise when only a single sterile neutrino is assumed.
Section 4 discusses the complete $3 + 3\nu$ neutrino model and develops a phenomenological framework that can successfully describe all present neutrino data.

II. NEUTRINO OSCILLATIONS IN THE PRESENCE OF STERILE NEUTRINOS

The transition probability of neutrinos from one active flavor $\alpha$ to another flavor $\beta$ can be written as

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{k>j}^N \text{Re}(U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*) \sin^2 \left( \frac{\Delta m_{kj}^2 L}{4E} \right)$$

$$+ 2 \sum_{k>j}^N \text{Im}(U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*) \sin \left( \frac{\Delta m_{kj}^2 L}{2E} \right),$$

with $N$ being the total number of active and sterile light neutrinos, $U$ the $N \times N$ unitary mixing-matrix, $\Delta m_{kj}^2 = m_k^2 - m_j^2$ the mass-squared differences of the mass eigenvalues, $L$ the baseline and $E$ the energy of the neutrino. From here on we neglect possible CP violation for all practical purposes and consider only real elements of the mixing matrix $U$, so the last term in Eq. (1) vanishes. Also, the CPT theorem and CP invariance imply that $T$ is a good symmetry, so $\sin^2 2\theta_{\alpha\beta} = \sin^2 2\theta_{\beta\alpha}$ also pertains. Equation (1) holds for mixing with any number of additional sterile neutrinos.

As the proposed additional mass-squared difference $\Delta m_{L\text{SND}}^2$ is in the $\sim 1$ eV$^2$-region and the mass-squared differences $\Delta m_{21}^2$, $\Delta m_{31}^2$ and $\Delta m_{32}^2$ are experimentally tested to be orders of magnitude smaller, it is possible to neglect $\Delta m_{21}^2$, $\Delta m_{31}^2$ and $\Delta m_{32}^2$. Therefore the transition probability for sensible values of $L/E$ and a 3+1$\nu$ model reduces to

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_j^3 \text{Re}(U_{\alpha 4}^* U_{\beta 4} U_{\alpha j} U_{\beta j}^*) \sin^2 \left( \frac{\Delta m_{4j}^2 L}{4E} \right)$$

$$= \delta_{\alpha\beta} - 4 \sin^2 \left( \frac{\Delta m_{4j}^2 L_{\text{SND}}}{4E} \right) \sum_j^3 U_{\alpha 4} U_{\beta 4} U_{\alpha j} U_{\beta j}$$

$$- \frac{1}{4} \sin^2 2\theta_{\alpha\beta}, \text{ for } \alpha \neq \beta \text{ etc.}$$

Due to unitarity of $U$, one has $\sum_j^3 U_{\alpha j} U_{\beta j} = -U_{\alpha 4} U_{\beta 4} + \delta_{\alpha\beta}$, and the resulting effective amplitudes can be reduced to the appearance value $\sin^2 2\theta_{\alpha\beta} = 4|U_{\alpha 4}|^2 |U_{\beta 4}|^2$ and the disappearance value $\sin^2 2\theta_{\alpha\alpha} = 4|U_{\alpha 4}|^2 (1 - |U_{\alpha 4}|^2)$. It is obvious that at the appropriate mass,
energy and baselength scales a $3 + 1\nu$ model with a $\sim 1\text{eV}^2$ mass squared difference looks similar to, and can be analyzed like, a two neutrino scenario.

Appearance experiments like LSND or MiniBooNE measure the transition $(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ and therefore require a sizable value for $\sin^2 2\theta_{\mu e}$, in order to explain the observed excess. On the other hand, disappearance experiments do not observe a significant deficit in $(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu)$ or $(\bar{\nu}_e \rightarrow \bar{\nu}_e)$ oscillations, and therefore constrain the corresponding value for $\sin^2 2\theta_{\mu\mu} = 4|U_{\mu 4}|^2 (1 - |U_{\mu 4}|^2) \simeq 4|U_{\mu 4}|^2$ small, and $\sin^2 2\theta_{ee} = 4|U_{e 4}|^2 (1 - |U_{e 4}|^2) \simeq 4|U_{e 4}|^2$ for $|U_{e 4}|^2$ small. By comparing the different amplitudes, the following relation is derived:

$$\sin^2 2\theta_{\mu e} = 4|U_{\mu 4}|^2 |U_{e 4}|^2 = \frac{1}{4} (\sin^2 2\theta_{\mu\mu} + 4|U_{\mu\mu}|^4)(\sin^2 2\theta_{ee} + 4|U_{ee}|^4) \simeq \frac{1}{4} \sin^2 2\theta_{\mu\mu} \sin^2 2\theta_{ee},$$

where the exact third expression becomes the approximate result since it is known from data that $\sin^2 2\theta_{\mu s} = 4|U_{\mu s}|^2 |U_{s 4}|^2$ and $\sin^2 2\theta_{es} = 4|U_{e s}|^2 |U_{s 4}|^2$ are small.

Both disappearance probabilities $(\bar{\nu}_e \rightarrow \bar{\nu}_e)$ and $(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu)$ have to be relatively large to generate a sufficient $\sin^2 (2\theta_{\mu e})$. But large values contradict the data from reactor and disappearance accelerator experiments. This relation is well known and exhibits the basic problem of current short baseline results. The problem persists in models adopting more than one additional sterile state [11].

The above relationship holds as long as one considers the elements of the mixing matrix to be constant. If this condition is given up and the elements are allowed to become energy or baseline dependent (as is the case for CP-violating matter vs. anti-matter effects), this tension can in principle be avoided. The relevant experiments such as MINOS ($E \sim 7$ GeV, $L \sim 735$ km), typical atmospheric neutrino experiments ($E \sim 0.6-100$ GeV, $L \sim 15-13,000$ km), LSND ($E \sim 10-100$ MeV, $L \sim 30$ m) and MiniBooNE ($E \sim 0.2-3$ GeV, $L \sim 541$ m) are indeed all sensitive to a mass-squared difference of $\sim 1\text{eV}^2$ due to similar values of $L/E$, but they operate on different energy and baselength scales and so an altered dispersion relation could be useful to resolve the above mentioned problem. In this case the energy dependence has to be fairly strong, since the energy regime for MiniBooNE almost overlaps with the low-energy range of atmospheric neutrino experiments.

\footnote{The survival amplitude can also be written as $\sin^2 2\theta_{aa} = 1 - 4 (|U_{aa}|^2 - \frac{1}{2})^2$, which shows the symmetry about the midpoint $|U_{aa}|^2 = \frac{1}{2}$. Note that for a given $\sin^2 2\theta_{aa} < 1$, there are two solutions of $|U_{aa}|^2$, $|U_{aa}|^2$ and $1 - |U_{aa}|^2$.}
So to relieve the tension between appearance and disappearance experiments, a model is required which allows for a small $|U_{\mu 4}|^2$ at high energies (GeV) and a sufficiently large $|U_{e4}|^2 |U_{\mu 4}|^2$ at low energies (MeV).

III. ALTERED DISPERSION RELATIONS FOR A SINGLE STEREILE NEUTRINO

Scenarios with altered dispersion relations (ADRs) adopt additional terms in the usual relation between energy $E$ and momentum $\vec{p}$, so $E^2 \neq |\vec{p}|^2 + m^2$. As we demonstrate below, energy dependent elements of the mixing matrix and mass-squared differences can be generated by an additional effective potential in the Hamiltonian in flavor space.

A typical example for such effects are neutrino matter effects, as they arise e.g. in the MSW description of solar neutrinos [12]. Matter effects imply a new, energy-independent term in the Hamiltonian and are significant typically either for neutrinos or for anti-neutrinos, but not for both. In addition, a solution for the MiniBooNE anomaly exploiting matter effects would require unusually large couplings. The formalism developed in the following can be adapted to include matter effects, but their effects are expected to be small.

Scenarios with sterile neutrino Lorentz violation allow for different energy dependencies which typically apply to neutrinos and anti-neutrinos in the same way. This is because the application of Lorentz violation affects spacetime directly, and not particles vs. antiparticles. The model proposed in [13, 14] (see also [15–18]) adopts one additional sterile neutrino taking a shortcut via an asymmetrically warped extra dimension [19–21]. In a semi-classical picture, the sterile neutrino oscillates on its geodesic in the warped bulk surrounding the brane, and thereby a running time difference is generated between active and sterile neutrinos. This running time difference manifests itself as an additional negative potential in the Hamiltonian proportional to the relative time difference $\frac{\delta t}{t} =: \varepsilon$, the so-called shortcut parameter (always entering the Hamiltonian as multiplied by the energy $E$). Although the semi-classical picture may not be truly accurate, its predictions regarding the form of the potential are correct to leading order in the shortcut parameter [24].
The resulting Hamiltonian in flavor space can be written as

\[ H_{(F)} = \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 & 0 & 0 \\ 0 & m_2^2 & 0 & 0 \\ 0 & 0 & m_3^2 & 0 \\ 0 & 0 & 0 & m_4^2 \end{pmatrix} U^\dagger - E \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix} \approx \begin{pmatrix} V & 0 & 0 & 0 \\ V & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R_{34} & 0 \\ 0 & 0 & 0 & R_{34}^T \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix} - E \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V & 0 & 0 & 0 \\ V & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (4)

with the energy \( E \), shortcut parameter \( \varepsilon \), and

\[ U = \begin{pmatrix} 0 \\ V & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R_{34} & 0 \\ 0 & 0 & 0 & R_{34}^T \end{pmatrix}, \] (5)

being the full \( 4 \times 4 \) unitary mixing matrix. Here, \( V \) is the unitary \( 3 \times 3 \) mixing matrix corresponding to the standard \( U_{\text{PMNS}} \) and \( R_{34} \) is the rotation in the \( 3-4 \) plane generating the sterile admixture of mass eigenstate \( \nu_3 \) with the mixing angle \( \theta_{34} \):

\[ R_{34} = \begin{pmatrix} \cos \theta_{34} & \sin \theta_{34} \\ -\sin \theta_{34} & \cos \theta_{34} \end{pmatrix}. \] (6)

In the last line of Eq. (4), we have used the aforementioned approximation \( m_1^2 = m_2^2 = m_3^2 = 0 \) and \( m_4^2 = \Delta m_{\text{LSND}}^2 \).

As already calculated in [14], the eigenvalues of the Hamiltonian become

\[ \lambda_1 = \lambda_2 = 0, \quad \lambda_{\pm} = \frac{\Delta m_{\text{LSND}}^2}{4E} \left( 1 - \cos 2\theta_{34} \left( \frac{E}{E_R} \right)^2 \pm \sqrt{\sin^2 2\theta_{34} + \cos^2 2\theta_{34} \left[ 1 - \left( \frac{E}{E_R} \right)^2 \right]^2} \right) \] (7)

with the resonance energy

\[ E_R = \sqrt{\frac{\Delta m_{\text{LSND}}^2 \cos 2\theta_{34}}{2\varepsilon}}. \] (8)
Below we follow the arguments given by [14] with one exception regarding the reasoning why the probability $P_{\nu_\alpha \rightarrow \nu_\alpha}$, with $\alpha$ being an active flavor, should vanish. One has

$$P_{\nu_\alpha \rightarrow \nu_\alpha} = 1 - 4U_{\alpha 3}^2 \times \begin{cases} \sin^2 \left( \frac{L(\lambda_+ - \lambda_-)}{2} \right) \sin^2 \tilde{\theta} \cos^2 \tilde{\theta} U_{\alpha 3}^2 \\ \sin^2 \left( \frac{L(\lambda_+)}{2} \right) \sin^2 \tilde{\theta} (1 - U_{\alpha 3}^2) \\ \sin^2 \left( \frac{L(\lambda_-)}{2} \right) \cos^2 \tilde{\theta} (1 - U_{\alpha 3}^2). \end{cases}$$  \tag{9}

Here, $\tilde{\theta}$ denotes the effective mixing angle defined via

$$\sin^2 2\tilde{\theta} = \frac{\sin^2 2\theta_{34}}{\sin^2 2\theta_{34} + (\cos 2\theta_{34} - \frac{2E^2\epsilon}{\Delta m_{LSND}^2})^2}. \tag{10}$$

According to [14], while $\sin^2 \tilde{\theta}$ does not vanish, the eigenvalue $\lambda_+$ vanishes and therefore $P_{\nu_\alpha \rightarrow \nu_\alpha}$ should vanish as well. Technically, this is a correct statement which applies as long as one considers only a single experiment with a fixed base-length $L$. However, various experiments now observe neutrinos in a wide energy range above the resonance. For example, for atmospheric experiments not only the energy becomes larger than the energies at LSND or MiniBooNE, but also the base-length can be as large as 15,000 km, which results in a value of $L/E$ which is up to 4 magnitudes larger than the one probed in MiniBooNE. Therefore, the relevant quantities to be examined are the mass-squared eigenvalues rather than the Hamiltonian eigenvalues. The mass-squared eigenvalues are

$$m_{\pm}^2 = 2E \cdot \lambda_{\pm}, \tag{11}$$

which give rise to the oscillatory term $\sin^2 \left( m_{\pm}^2 \frac{L}{2E} \right)$ in the probability in Eq. (9). Adopting this more familiar form we continue to analyze the oscillations of atmospheric neutrinos. While it is true that $\lambda_+ \propto 1/E$ becomes zero for energies much larger than $E_R$, $m_+^2$ as defined herein does not:

$$\lim_{E \to \infty} m_+^2 = \Delta m_{LSND}^2 \cdot \frac{1 - \cos 2\theta_{43}}{2}. \tag{12}$$

Note that, although the effective mass-squared eigenvalue $m_+^2$ can become small for extremely small mixing angles $\theta_{34}$, the important value is still the bare (or “vacuum”) mass of the 4th eigenvalue, in this case $\Delta m_{LSND}^2$. This value is still large compared to the standard mass-and squared differences. Therefore, even above the resonance there is still a
non-vanishing oscillation mode, which becomes accessible experimentally if the oscillation length is large enough. Such is the case for atmospheric and astrophysical neutrinos.

For a better understanding we plot the effective masses for this scenario in Fig. 1. As can be seen in Fig. 1 at the resonance a level crossing occurs and the Hamiltonian eigenstates swap their flavor content. While the predominantly sterile state decouples above the resonance, the now heavier predominantly active state approaches a constant value that due to the level crossing gap is different from its initial value. This implies a large effective $\Delta m^2_{13}$ that gives rise to large and fast active-to-sterile oscillations, e.g. in atmospheric neutrinos.

![Figure 1](image)

Figure 1. Effective mass-squareds as a function of the energy $E$ in the $3 + 1\nu$ model including an effective potential due to sterile neutrino shortcuts.

This argument can be generalized to mixing with $\theta_{14}$, $\theta_{24}$, or a combination of the two. (Also, the case where all standard mass-squared differences do not vanish can be treated accordingly.)

Although the sterile neutrino decouples from the active ones above the resonance, the impact on the disappearance experiments is significant. As has been first discovered numerically by Patrick Huber, in any possible mixing pattern the atmospheric experiments or MINOS should notice a deviation from the standard three neutrino case (especially for longer baselines such as the MINOS far detector, or upward going atmospheric neutrinos). One could possibly argue that MINOS might miss the deviation from standard 3 neutrino case due to its narrow energy spectrum at around 7 GeV, but atmospheric experiments like IceCube or SuperKamiokaNDE or KM3NeT have not only high statistics but also a wide energy
spectrum and high resolution for the azimuth angle and the energy. Therefore, these experiments should be highly sensitive to this deviation. Atmospheric neutrino experiments also tested the $L/E$-dependence of the oscillation probability, finding no observable deviations from the $3\nu$ case. A simple $3 + 1\nu$ model even with an altered dispersion relation for the sterile neutrino is therefore ruled out by current data.

IV. A REALISTIC 3+3 MODEL

A. $3 + 3\nu$ with a Common Sterile Neutrino Potential

The emergence of a large mass-squared difference in the energy regime far above the resonance described above can be avoided by adding three sterile neutrinos instead of a single sterile neutrino. In the following we assume that each sterile state mixes with one of the predominantly active mass eigenstates, respectively. Therefore all mass eigenstates become affected by the common sterile potential. If all sterile neutrinos are affected by the same potentials, the mass differences among the predominantly active states will not be altered even though their masses change as the $E$-dependent potential and mixings change. This mechanism removes the ‘unwanted’ mass difference which spoiled atmospheric neutrino oscillations in the $3 + 1\nu$ case.

The resulting $6 \times 6$ mixing matrix can be parametrized as

$$U^{6\times6} = U_{12}U_{13}U_{23}U_{14}U_{25}U_{36}. \quad (13)$$

The bare masses read

$$\Delta m^2_{41} = \Delta m^2_{\text{LSND}}, \quad (14)$$
$$\Delta m^2_{51} = \Delta m^2_{\text{LSND}} + \Delta m^2_{21} \rightarrow \Delta m^2_{52} = \Delta m^2_{\text{LSND}}, \quad (15)$$
$$\Delta m^2_{61} = \Delta m^2_{\text{LSND}} + \Delta m^2_{31} \rightarrow \Delta m^2_{63} = \Delta m^2_{\text{LSND}}. \quad (16)$$

Assuming universal resonance energies ($E_R$) or shortcut parameters ($\varepsilon$) for the three sterile
neutrinos, the effective potential that results is

\[
V_{\text{eff}} = 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varepsilon E & 0 & 0 \\
0 & 0 & 0 & 0 & \varepsilon E & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon E
\end{pmatrix},
\]

(17)

Every mass eigenstate \(\nu_{1,2,3}\) has its own sterile state admixture. Still, the results from [14] remain applicable. The resulting mass eigenvalues are denoted by \(m_4^{2\pm}, m_5^{2\pm}, m_6^{2\pm}\), which correspond to the \(m_\pm^{2}\) mass eigenstates in the previous section. Considering also the non-vanishing small masses \(m_1^2, m_2^2\) and \(m_3^2\), the eigenvalues read

\[
m_4^{2\pm} = \frac{m_1^2 + m_4^2}{2} - \frac{\Delta m_{\text{LSND}}^2}{2} \left( \cos 2\theta_{14} \left( \frac{E}{E_{R,4}} \right)^2 \right) + \left[ \sin^2 2\theta_{14} \right. + \cos^2 2\theta_{14} \left. \left[ 1 - \left( \frac{E}{E_{R,4}} \right)^2 \right] \right]^2,
\]

(18)

\[
m_5^{2\pm} = \frac{m_2^2 + m_5^2}{2} - \frac{\Delta m_{\text{LSND}}^2}{2} \left( \cos 2\theta_{25} \left( \frac{E}{E_{R,5}} \right)^2 \right) + \left[ \sin^2 2\theta_{25} \right. + \cos^2 2\theta_{25} \left. \left[ 1 - \left( \frac{E}{E_{R,5}} \right)^2 \right] \right]^2,
\]

(19)

\[
m_6^{2\pm} = \frac{m_3^2 + m_6^2}{2} - \frac{\Delta m_{\text{LSND}}^2}{2} \left( \cos 2\theta_{36} \left( \frac{E}{E_{R,6}} \right)^2 \right) + \left[ \sin^2 2\theta_{36} \right. + \cos^2 2\theta_{36} \left. \left[ 1 - \left( \frac{E}{E_{R,6}} \right)^2 \right] \right]^2,
\]

(20)

with the corresponding resonance energies

\[
E_{R,4} = \sqrt{\frac{\Delta m_{\text{LSND}}^2 \cos 2\theta_{14}}{2\varepsilon}}, \quad E_{R,5} = \sqrt{\frac{\Delta m_{\text{LSND}}^2 \cos 2\theta_{25}}{2\varepsilon}}, \quad E_{R,6} = \sqrt{\frac{\Delta m_{\text{LSND}}^2 \cos 2\theta_{36}}{2\varepsilon}}.
\]

(21)

All mass-squareds \(m_i^{2\pm}\) approach minus infinity and decouple in the limit \(E \gg E_{R,i}\). The mass-squared eigenvalues of interest are again \(m_i^{2+}\), whose values in the limit \(E \gg E_{R,i}\) are

\[
\lim_{E \to \infty} m_4^{2+} = \frac{1}{2} \left( m_1^2 + m_4^2 + \Delta m_{\text{LSND}}^2 \cdot \cos 2\theta_{14} \right),
\]

(22)

\[
\lim_{E \to \infty} m_5^{2+} = \frac{1}{2} \left( m_2^2 + m_5^2 + \Delta m_{\text{LSND}}^2 \cdot \cos 2\theta_{25} \right),
\]

(23)

\[
\lim_{E \to \infty} m_6^{2+} = \frac{1}{2} \left( m_3^2 + m_6^2 + \Delta m_{\text{LSND}}^2 \cdot \cos 2\theta_{36} \right),
\]

(24)
while the relevant mass-squared differences far above the resonance become

\[ m_{5+}^2 - m_{5+}^2 = \Delta m_{21}^2 + \frac{\Delta m_{\text{LSND}}^2}{2} (\cos 2\theta_{14} - \cos 2\theta_{25}) , \]  
\[ m_{6+}^2 - m_{4+}^2 = \Delta m_{31}^2 + \frac{\Delta m_{\text{LSND}}^2}{2} (\cos 2\theta_{14} - \cos 2\theta_{36}) , \]  
\[ m_{6+}^2 - m_{5+}^2 = \Delta m_{32}^2 + \frac{\Delta m_{\text{LSND}}^2}{2} (\cos 2\theta_{25} - \cos 2\theta_{36}) . \]

If these mass-squared differences are all assumed to lie in the same region as the mass-squared difference in the $3 + 1\nu$ case, a corresponding oscillation should be measurable in atmospheric neutrino experiments. Such an “extra” oscillation is not seen.

The only way to avoid the generation of such a mass-squared difference, is by imposing a common mixing in addition to the common potential, i.e. by setting all new mixing angles to the same value: $\theta_{14} = \theta_{25} = \theta_{36} \equiv \theta$. In this case the second terms in Eqns. (25)-(27) vanish for all mass-squared differences, and one ends up with the same mass-squared differences as in a standard three neutrino scenario (see Fig. 2). Consequently, it is possible to avoid the constraints by atmospheric neutrino experiments. However, as long as the resonance energies for the three sterile neutrinos are assumed to be universal, a new problem arises in the form of a vanishing amplitude for the MiniBooNE experiment in the resonant region.

In general, the oscillation probability including the active mass-squared differences, in the case of CP conservation, is given by

\[ P_{\nu_\mu \rightarrow \nu_e} = -4 \sum_{k>j} \tilde{U}_{\mu j} \tilde{U}_{\mu k} \tilde{U}_{e j} \tilde{U}_{e k} \sin^2 \left( \frac{\Delta \tilde{m}_{kj}^2 L}{2E} \right) . \]  

At energies far below the resonance, all $\tilde{U}$’s and $\tilde{m}^2$’s can be replaced by their bare values $U$ and $m^2$. At MiniBooNE or LSND only those terms contribute where the mass-squared difference is in the $\Delta m_{\text{LSND}}^2$ region. These are the mass-squared differences ($\Delta m_{41}^2$, $\Delta m_{42}^2$, $\Delta m_{43}^2$), ($\Delta m_{51}^2$, $\Delta m_{52}^2$, $\Delta m_{53}^2$), and ($\Delta m_{61}^2$, $\Delta m_{62}^2$, $\Delta m_{63}^2$). The transition probability factorizes as:

\[ P_{\nu_\mu \rightarrow \nu_e} \sim -4 \sin^2 \left( \frac{\Delta m_{\text{LSND}}^2 L}{2E} \right) \left( \sum_{j=1,2,3} U_{\mu j} U_{e j} \right) \left( \sum_{j=4,5,6} U_{\mu j} U_{e j} \right) . \]
Figure 2. Schematic overview of mass eigenstates and their flavor content depending on the Energy $E$.

For simplicity, we define the mixing matrix as

$$U^{6 \times 6} = (U_{12}U_{13}U_{23})(U_{14}U_{25}U_{36})$$

$$U_{6 \times 6} = U_0 \cdot \begin{pmatrix} c_\theta & 0 & 0 & s_\theta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -s_\theta & 0 & 0 & c_\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_\theta & 0 & 0 & s_\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_\theta & 0 & 0 & s_\theta \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -s_\theta & 0 & 0 & c_\theta \end{pmatrix}$$

$$= U_0 \cdot \left( c_\theta \cdot \mathbb{1}_{3 \times 3} + s_\theta \cdot \mathbb{1}_{3 \times 3} \right) \cdot \left( -s_\theta \cdot \mathbb{1}_{3 \times 3} + c_\theta \cdot \mathbb{1}_{3 \times 3} \right), \quad (30)$$

and we have the common mixing angle $\theta$ for $\theta_{14} = \theta_{25} = \theta$. 

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Since $U_0$ only describes a rotation in the upper left corner, it can be written as

$$U_0 = \begin{pmatrix} A_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 1_{3\times3} \end{pmatrix},$$  \hspace{1cm} (31)$$

Any submatrix formed from rotations alone is orthogonal, and therefore unitary. So the sub-matrix $A_{3\times3}$ is unitary. This leads to the full mixing matrix

$$U_{6\times6} = \begin{pmatrix} c_{\theta} \cdot A_{3\times3} & s_{\theta} \cdot A_{3\times3} \\ -s_{\theta} \cdot 1_{3\times3} & c_{\theta} \cdot 1_{3\times3} \end{pmatrix},$$  \hspace{1cm} (32)$$

The oscillation probabilities for LSND and MiniBooNE in this model are generally

$$P_{\nu_\mu \rightarrow \nu_e} \sim -4 \sin^2 \left( \frac{\Delta m^2_{\text{LSND}} L}{2E} \right) \times$$
$$\times (U_{\mu 1} U_{e 1} + U_{\mu 2} U_{e 2} + U_{\mu 3} U_{e 3}) (U_{\mu 4} U_{e 4} + U_{\mu 5} U_{e 5} + U_{\mu 6} U_{e 6}) .$$  \hspace{1cm} (33)$$

Since the submatrix $A_{3\times3}$ itself is unitary and the unitarity conditions $\sum_k U_{\mu k} U_{ek} = 0$ as well as $\sum_k A_{\mu k} A_{ek} = 0$ hold, it is readily seen that both brackets have to vanish when all new mixing angles $\theta_{ij}$ are the same. (The first bracketed term is $\cos^2 \theta (A_{3\times3}^{(\mu \text{ row})} \cdot A_{3\times3}^{(e \text{ row})} = 0)$ and the second bracketed term is $\sin^2 \theta (A_{3\times3}^{(\mu \text{ row})} \cdot A_{3\times3}^{(e \text{ row})} = 0)$. Consequently, a $3+3\nu$ model with three additional sterile neutrinos and a common resonance energy also fails. On the one hand it is indeed possible to avoid the constraints from atmospheric neutrinos above the resonance, if all three sterile neutrinos mix with the same mixing angle. This removes the additional mass-squared differences in the considered region and thereby immunizes the model against constraints from high energy atmospheric neutrinos. On the other hand, however, the democratic mixing simultaneously implies a vanishing transition amplitude for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations at MiniBooNE or LSND and consequently invalidates the desired main feature of the model. As we next show, the issue can be resolved by assigning different resonance energies to the different sterile neutrinos.

### B. Different Effective Potentials for Different Sterile Neutrinos

#### 1. Treatment of Short Baseline and Atmospheric/Accelerator Experiments

From the previous discussion it becomes clear that both the low energy limit $E \rightarrow 0$ and the high energy limit $E \rightarrow \infty$ are independent of the specific values of the effective potentials
of the different sterile neutrinos. What matters is only that the energy is well below or well above the respective resonance energy. However, there exists the possibility of assigning different resonance energies to the sterile neutrinos (e.g. by tying each sterile neutrino to its own extra dimension). The potentials for the neutrinos do not necessarily have to be the same for each sterile neutrino. If the effective potentials differ, we still expect a resonant behavior around the resonance energy also for $\nu_\mu \rightarrow \nu_e$ transitions: In the intermediate energy region the arguments made in the previous chapter no longer hold since the effective mixing angles differ for the different sterile neutrinos as a consequence of to the different effective potentials. As long as the bare mixing angle is the same for all sterile neutrinos, we nevertheless end up with the aforementioned low and high energy behavior.

According to the current best-fit reported by MiniBooNE, we adopt the new mass-squared difference to be $\Delta m^2_{\text{LSND}} = \Delta m^2_{41} = 1.59 \text{eV}^2$, the bare mixing angle to $\sin^2 \theta \sim 0.05$ and the shortcut parameter to $\varepsilon = 5 \cdot 10^{-17}$, resulting in a resonance energy of roughly $E_R \sim 120$ MeV. The effective potential with different shortcut parameters can be written as

$$V_{\text{eff}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon E & 0 & 0 \\ 0 & 0 & 0 & \kappa \cdot \varepsilon E & 0 \\ 0 & 0 & 0 & 0 & \xi \cdot \varepsilon E \end{pmatrix}.$$  \hspace{1cm} (34)

Due to the larger effective potential for the $s_2$ and $s_3$ states, the resonance energy becomes smaller and the decoupling of these states happens at lower energies. We chose a rather large factor of $\kappa = \xi = 100$ to generate a resonance not only at MiniBooNE but also at LSND in the energy region of $\sim 20 - 50$ GeV to explain also the excess reported for LSND [2].

In the numerical analysis we adopt the best-fit values from [22] for the standard 3$\nu$ mixing angles and mass-squared differences, and assume normal ordering and vanishing CP violation. We also neglect matter effects, since they do not solve the problem we intend to address. Such matter effects are known to exist, and they make a significant difference in the few GeV realm, but the sterile neutrino is already completely decoupled above the
highest $E_R \propto 1/\sqrt{\varepsilon}$, due to the ADR potential. The best-fit values are
\[
\Delta m_{21}^2 = 7.37 \times 10^{-5} \text{eV}^2, \quad \Delta m_{31}^2 = 2.50 \times 10^{-3} \text{eV}^2, \\
\sin^2 \theta_{12} = 0.297, \quad \sin^2 \theta_{13} = 0.437, \quad \sin^2 \theta_{23} = 0.0214.
\] (35)

Numerical calculations for the oscillation probabilities are shown in Figures 3 - 5 and the effective squared masses are shown in Fig. 6. As can clearly be seen, we can achieve

Figure 3. Different Probabilities at MiniBooNE
resonant behavior in the $\nu_\mu \rightarrow \nu_e$ channel at MiniBooNE with no significant deviation from three neutrino mixing in atmospheric data: The oscillations amplitudes for $\nu_\mu \rightarrow \nu_e/\tau$ at MiniBooNE (Fig. 3 (a)/(c)) feature the same resonance pattern. One observes a resonant
enhancement of the transition probability in the sub $\sim 120$ MeV region, combined with a suppression in the energy region above $\sim 120$ MeV. The latter behavior results from the decoupling of the sterile neutrinos, as can be seen in the oscillations amplitudes for the individual sterile flavors, $\nu_\mu \rightarrow \nu_{s1,2,3}$ (Fig. 3 (d)-(f)). The disappearance oscillation probability $\nu_\mu \rightarrow \nu_\mu$ at MiniBooNE (Fig. 3 (b)) exhibits a characteristic behavior at the resonance, since the transition into sterile neutrinos dominates in this energy region. A measurement of the survival probability $\nu_\mu \rightarrow \nu_\mu$ at energies at MiniBooNE could thus provide a good test for this model: a depletion of the survival probability significantly higher than the transition to $\nu_e$ or $\nu_\tau$, would be a clear sign for a transition into sterile neutrinos (i.e. $P_{\nu_\mu \rightarrow \nu_s} \neq 0$).

Taking a look at atmospheric experiments in Fig. 4 one can observe that in the realm $E > $ GeV, the predictions of this model are exactly the same as in the standard $3$+$1$ neutrino paradigm, due to the complete decoupling of the sterile states above the highest $E_R \sim 120$ MeV. This leads to perfect satisfaction of the current fits at these kind of experiments, in contrast to the standard $3 + 1$ models. A discussion of the sub-GeV region is presented in section IV B.3

The same high energy properties as in atmospheric experiments can also be found at MINOS, where one can see the same strong convergence to the $3\nu$ probabilities. MINOS, however, has a neutrino beam with a distinct peak energy at about $7$ GeV, and so is blind to the low
energy effects to be discussed in Sec. [IVB3].

We note that the price paid is relatively high. Nine new parameters are introduced via the sterile sector: three heavy neutrino masses, three active-sterile mixing angles, and three resonant-energy values. Eq. [8] holds for each sterile neutrino, so we see that choosing the ε’s is the same as choosing these parameters.2

2. Behavior Below the Resonance

Although the transition \( \bar{\nu}_\mu \rightarrow \nu_{e/\tau} \) vanishes far below the resonance, \( \bar{\nu}_\mu \rightarrow \nu_{s1,2,3} \) does not vanish due to the bare mixing. The same is also true for \( \bar{\nu}_e \rightarrow \nu_{s1,2,3} \) and \( \bar{\nu}_\tau \rightarrow \nu_{s1,2,3} \). This is particularly interesting for reactor experiments, which usually operate in the MeV-region, since they are predicted by this model to measure a deviation in the \( \bar{\nu}_e \rightarrow \bar{\nu}_e \) channel. A good approximation for \( \bar{\nu}_e \rightarrow \bar{\nu}_e \) in the low energy region well below the resonances is given by (compare Eq. (33)):

\[
P_{\nu_e \rightarrow \nu_e} \sim 1 - 4 \sin^2 \left( \Delta m^2_{\text{LSND}} \frac{L}{2E} \right) \left( U^2_{e1} + U^2_{e2} + U^2_{e3} \right) \left( U^2_{e4} + U^2_{e5} + U^2_{e6} \right) \\
= 1 - 4 \sin^2 \left( \Delta m^2_{\text{LSND}} \frac{L}{2E} \right) \cos^2 \theta \sin^2 \theta \\
= 1 - \sin^2 \left( \Delta m^2_{\text{LSND}} \frac{L}{2E} \right) \sin^2 2\theta ,
\]

where again the unitarity conditions are used. This expression resembles a simple \( 3 + 1 \nu \) model for disappearance experiments in the low energy region, which is actually favored by the Reactor- or Gallium anomalies.

3. Open Questions

As can be seen in Fig. [4], the proposed model resembles three neutrino oscillations far above the resonance (in this case above the GeV region). However, due to the desired resonance at around 120 MeV for explaining the MiniBooNE data, this resonance will also

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2 We have chosen to vary ε alone, but Eq. [8] shows that changes in \( \Delta m^2 \)’s or \( \cos 2\theta_{ij} \)’s, or in all three of the parameter types, can produce a similar change in \( E_R \). E.g., adopting the active neutrino result that \( \Delta m^2_{32} = 3 \times 10^{-3}\text{eV}^2 \sim 30 \times (\Delta m^2_{21} = 10^{-4}\text{eV}^2) \) for the sterile sector might suggest that \( M^2_{s3} \approx 30 \times M^2_{s1} \), which in turn would give \( \sqrt{30} \sim 5.5 \) for the ratio of the third-to-first resonant energies. We have not explored this new extended parameter space, but remain content to produce the one viable, testable model we have presented herein.
have impact on the sub-GeV neutrinos at atmospheric experiments. The corresponding oscillations in this energy region are plotted in Fig. 7. As expected, the model predicts a significant deviation from the simple three neutrino model. Nevertheless, without access to the actual data it is hard to judge whether these oscillation patterns are excluded by current experiments. Previous analyses [5, 6] searched for sterile neutrinos without altered dispersion relations. Oscillation probabilities within the standard 3 neutrino scenario for the range of these searches are shown in Fig. 7 for comparison. While the difference to the probability for our proposed model is obvious, proposed exclusion limits in the literature cannot be adopted for our case. To either exclude or confirm the model proposed in this article, we recommend a reanalysis of the current sub-GeV data in atmospheric experiments.

Figure 7. Different Probabilities at atmospheric neutrino experiments that highlight the Sub-GeV to 10 GeV region.

V. COSMOLOGICAL BOUNDS

Apart from neutrino oscillation experiments, cosmology provides stringent bounds on any early-Universe population of eV-scale sterile neutrinos. Such neutrinos constitute additional radiation degrees of freedom which alter the Hubble expansion and consequently via the freeze-out temperature of electroweak interactions and the proton-to neutron ratio the cosmic abundances of light elements explained very well in Big Bang nucleosynthesis (BBN). Moreover, eV scale neutrinos are way above the cosmological neutrino mass bounds due to the effect that such heavy neutrinos efficiently suppress the formation of structure on
scales smaller than their free-streaming length. As proposed in [13] and analyzed in detail in [18, 23], altered dispersion relations suppress active-sterile neutrino mixing above the resonance and thus can prevent sterile neutrinos from being populated at high energies. While naively the resonance energies discussed in this work appear to be too high to prevent sterile neutrinos from being populated at the MeV scale relevant for BBN and neutrino decoupling, it is conceivable that the responsible Lorentz violating altered dispersion relation depends on temperature and density. This occurs for example in extra-dimensional ADR models where Einstein’s equations obtain new source terms due to the hot dense plasma in the early universe, which leads to modifications in the metric of the warped extra dimensions and therefore again alters the dispersion relation. Such considerations are strongly model dependent, and are beyond the scope of this paper. It should be stressed, however, that altered dispersion relations proposed here as a solution for the MiniBooNE and LSND anomalies may also help to evade the cosmological bounds usually applied to light sterile neutrinos.

VI. SUMMARY AND OUTLOOK

In this paper we have developed a 3+3 neutrino framework with altered dispersion relations, that can explain the LSND, MiniBooNE, Reactor and Gallium anomalies. In particular, the model features two resonances, one in the oscillation amplitude resembling the low-energy excess in the MiniBooNE data, and one at lower energy to enhance the LSND signal. To our best knowledge, this is the only model which can successfully achieve the task at hand. We have discussed in detail which constructions fail in this context, and how we arrived at the model which seems to provide a successful explanation of the world’s neutrino data. First, we have pointed out that for 3+1 neutrino models with sterile neutrino altered dispersion relations, the predominantly sterile state decouples at energies far above the resonance, which thus hides the sterile neutrino in accelerator experiments operating at high energies [19, 21]. As has been recently pointed out [24], such a scenario results as the effective field theory limit of sterile neutrinos propagating in an asymmetrically warped extra dimension. However, a level crossing occurs at the resonance energy, and the Hamiltonian eigenstates swap their flavor content. While the predominantly sterile state decouples above the resonance as discussed above, the now heavier predominantly active state approaches a constant
value that is different from its initial value because of the level crossing gap. This implies a large effective $\Delta m^2_{13}$ that gives rise to large and fast active-to-active oscillations (e.g. in atmospheric neutrinos) which rules out the 3+1$\nu$ model.

This consequence can be avoided when 3 active and 3 sterile neutrinos are introduced and the sterile neutrinos are mixed with the active ones via a common effective potential. In such a model, $E_R$ is necessarily common too. In this case all Hamiltonian states are altered in the same way by the admixture with the shortcut-taking sterile neutrino (they change in parallel and the effective $\Delta m^2$’s remain constant). While this feature solves the problem above, it also leads also to a cancellation of active-to-active oscillations over the LSND mass gap due to unitarity constraints - and thus implies there are no oscillations at all at both LSND and MiniBooNE, as long as all sterile neutrinos feature the same resonance energy.

However, in the scenario with three active and three sterile neutrinos, the high energy limit of the Hamiltonian eigenvalues for the predominantly active neutrinos is independent of both energy and the shortcut parameter parametrizing the altered dispersion relation. So once one assigns different resonance energies to the 3 sterile neutrinos, it is possible to obtain oscillations and resonances in the intermediate mass regime while an effective 3-active neutrino scenario is restored at high energies. New parameters are inevitably introduced via the sterile neutrino sector.

While it is possible that the various neutrino anomalies are due to our limited understanding of experimental backgrounds, this “Beyond the Standard Model” physics scenario is testable by the MicroBooNE and ICARUS experiments, and may reveal itself first in sub-GeV atmospheric neutrino data.

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