Model Order Selection Based on Information Theoretic Criteria: Design of the Penalty

Andrea Mariani, Member, IEEE, Andrea Giorgetti, Senior Member, IEEE, and Marco Chiani, Fellow, IEEE

Abstract—Information theoretic criteria (ITC) have been widely adopted in engineering and statistics for selecting, among an ordered set of candidate models, the one that better fits the observed sample data. The selected model minimizes a penalized likelihood metric, where the penalty is determined by the criterion adopted. While rules for choosing a penalty that guarantees a consistent estimate of the model order are known, theoretical tools for its design with finite samples have never been provided in a general setting. In this paper, we study model order selection for finite samples under a design perspective, focusing on the generalized information criterion (GIC), which embraces the most common ITC. The theory is general, and as case studies we consider: a) the problem of estimating the number of signals embedded in additive white Gaussian noise (AWGN) by using multiple sensors; b) model selection for the general linear model (GLM), which includes e.g. the problem of estimating the number of sinusoids in AWGN. The analysis reveals a trade-off between the probabilities of overestimating and underestimating the order of the model. We then propose to design the GIC penalty to minimize underestimation while keeping the overestimation probability below a specified level. For the considered problems, this method leads to analytical derivation of the optimal penalty for a given sample size. A performance comparison between the penalty optimized GIC and common AIC and BIC is provided, demonstrating the effectiveness of the proposed design strategy.

Index Terms—Akaike information criterion, Bayesian information criterion, general linear model, generalized information criterion, information theoretic criteria, model order selection.

I. INTRODUCTION

MODEL ORDER SELECTION problems occurring in engineering and statistics are often solved by means of information theoretic criteria (ITC) [1]–[3]. The selected model order minimizes a penalized likelihood metric, where the penalty is determined by the criterion adopted. The most commonly used criteria are the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), which are the forefathers of the classes of criteria derived from Kullback-Leibler (K-L) information and from Bayesian estimation, respectively [2], [4]. Despite the fact that ITC have been largely studied and adopted, there are relatively few works that address the derivation of ITC as a design problem. Most of them study the consistency of model order selection deriving the conditions under which asymptotically, for a large number of observations, the correct model order is chosen [6]–[9]. However, in practice, finite sample sizes are used, and consistency considerations are not sufficient for controlling the error probabilities. Some works empirically study how to set the penalty in specific selection problems [3], [10], [11]. For example, in [10] and [11] the values to be adopted in autoregressive model selection problems are discussed, while in [12] a modification of AIC has been proposed. Some effort has been placed on non-asymptotic penalties for some specific problems, such as Gaussian model selection [13].

In this paper, we study ITC under a design perspective, focusing on the study of the generalized information criterion (GIC), which embraces most common criteria such as AIC and BIC. The GIC performance analysis for finite sample sizes reveals a trade-off between the probabilities of overestimating and underestimating the order of the model. Thus, we propose to design the GIC penalty to minimize underestimation while keeping the overestimation probability below a specified level. As a practical case study, we focus on the classical problem of estimating the number of signals in Gaussian noise, which arises in many statistical signal processing and wireless communication applications. For example, in the context of cognitive radio, the enumeration of active transmissions is of great interest for increasing the spectrum awareness of the secondary user systems [14], [15]. The most commonly used approaches for solving this problem are the non-parametric model order estimators proposed in [16], that received a considerable attention in the past decades [12], [15]–[20]. As a second example, we consider model order selection for the general linear model (GLM), which can be used, e.g., for estimating the number of sinusoids in additive white Gaussian noise (AWGN), and for model selection in autoregressive processes [21]–[23]. In both cases it is shown that the performance for high signal-to-noise ratios (SNRs) is determined by noise distribution.

The contributions of the paper are as follows.

- We analyze the probability of correct model selection for the GIC. This study gives an insight on the performance of ITC, relating underestimation and overestimation events to the penalty adopted. This applies to the whole class of GIC, including AIC and BIC.
- We propose a design strategy for the GIC penalty. This approach minimizes underestimation while keeping the overestimation probability below a specified level.
We address the problem of estimating the number of sources in white noise applying the GIC design approach proposed. For this case, design is based on a new closed-form approximation of the probability of correct model selection for high SNR, based on the statistic of the ratio of the largest eigenvalue to the trace of a white central Wishart random matrix.

We address model selection for the GLM. In this case, being an analytical form of the correct selection probability not available, we design the penalty by means of tight performance bounds. As an application example, we focus on the problem of estimating the number of sinusoids in AWGN.

The paper is organized as follows. Model order selection is introduced in Section II. In Section III, we derive the GIC performance and propose a design approach, which is applied to the problem of estimating the number of sources in Section IV and to the GLM in Section V. Numerical results are presented in Section VI.

Throughout the paper, boldface letters denote matrices and vectors, and \( X \sim \mathcal{CN}(0, \Sigma) \) denotes a circularly symmetric complex Gaussian random vector with zero mean and covariance matrix \( \Sigma \). Also, \( X \sim \chi^2_m \) is a central chi squared distributed random variable (r.v.) with \( m \) degrees of freedom, \( X \sim \mathcal{G}(\kappa, \theta) \) is a gamma distributed r.v. with shape parameter \( \kappa \) and scale parameter \( \theta \), and \( X \sim \beta_{a,b} \) is a beta distributed r.v. with parameters \( a \) and \( b \). We denote the probability density function (p.d.f.) and cumulative distribution function (CDF) of the r.v. \( X \) with \( f_X(x) \) and \( F_X(x) \), respectively. The notation \( X \overset{d}{=} Y \) means that the distribution of the r.v. \( X \) can be approximated by the distribution of the r.v. \( Y \). Moreover, \( I_m \) represents an identity matrix of order \( m \), \( \text{tr}(A) \) is the trace of the matrix \( A \), \( (\cdot)^T \) and \( (\cdot)^H \) stand, respectively, for simple and Hermitian transposition.

II. INFORMATION THEORETIC CRITERIA FOR MODEL ORDER SELECTION

In [5] Akaike first proposed an information theoretic criterion for statistical model selection based on the observation of \( n \) independent, identically distributed (i.i.d.) samples of the \( p \) dimensional random vector \( X \), generated by the distribution \( f(X; \Theta^{(q)}) \), where \( \Theta^{(q)} \) is the vector that contains the unknown parameters of the model. The length of \( \Theta^{(q)} \) increases with the model order \( q \). Model order selection consists in identifying the model that better fits data among a set of possible models \( \left\{ f(X; \Theta^{(k)}) \right\}_{k \in \mathcal{K}} \), each one characterized by the model order \( k \) and the corresponding parameter vector \( \Theta^{(k)} \).\(^2\) Throughout the paper, we assume that the true model is included in the model set considered. The analysis of the case in which the true model is misspecified is out of the scope of the paper. We focus, in particular, on the selection problems in which the hypotheses are nested, which means that the \( i \)-th hypothesis is always contained in the \( j \)-th one, with \( i < j \). The set of the possible values assumed by \( k \) is

\[ \mathcal{K} = \{0, 1, \ldots, q_{\text{max}}\} \], where \( q_{\text{max}} \) is the maximum model order considered.

Denoting by \( x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,p})^T \) the \( i \)-th sample of \( X \), we build the \( p \times n \) observation matrix

\[ Y = (x_1 | x_2 | \cdots | x_n) \tag{1} \]

We assume that each sample \( x_i \) is composed by a signal part \( s_i \), corrupted by an additive noise component \( n_i \), i.e., \( x_i = s_i + n_i \), and we define the SNR as

\[ \text{SNR} = \frac{\mathbb{E}\{s_i^H s_i\}}{\mathbb{E}\{n_i^H n_i\}} \]

which is assumed to be independent of \( i \). According to the general formulation of ITC, the model that better fits data is the one that minimizes the metric

\[ \text{ITC}(k) = -2 \sum_{i=1}^n \ln f(x_i; \hat{\Theta}^{(k)}) + \mathcal{P}(k) \tag{2} \]

where \( \hat{\Theta}^{(k)} \) is the maximum likelihood (ML) estimate of the vector \( \Theta^{(k)} \), and \( \mathcal{P}(k) \) is the penalty.\(^3\) Thus, the model order selected is

\[ \hat{q} = \arg \min_k \text{ITC}(k) \tag{3} \]

Each criterion is defined by its particular penalty which impacts the performance and the complexity of model order selection.

Note that the formulation of the selection problem as in (2) and (3) supports the interpretation of ITC as extensions of the ML principle in the form of penalized likelihood. In fact, the ML approach performs poorly in model order selection problems, always leading to the choice with maximum number of unknown parameters [24]. The penalty is introduced in (2) as a cost to account for the increased complexity of the model, related to the presence of unknown parameters that must be estimated [5], [25]. Thus, model selection based on ITC extends the ML approach, in that it takes into account both the estimation (of the unknown parameters) and the decision (among the possible models).

A. Review of fundamental criteria

Akaike proposed to select the model which minimizes the K-L divergence from \( f(X; \Theta^{(q)}) \) to \( f(X; \Theta^{(k)}) \). In fact, since

\[ q = \arg \min_k \int f(X; \Theta^{(q)}) \ln \frac{f(X; \Theta^{(q)})}{f(X; \Theta^{(k))}} dX \tag{4} \]

the correct order is the one minimizing the cross entropy

\[ -\int f(X; \Theta^{(q)}) \ln f(X; \Theta^{(k)}) dX \tag{5} \]

for which an estimate, under the \( k \)-th hypothesis, is given by the average log-likelihood with ML estimate of the parameters

\[ -\frac{1}{n} \sum_{i=1}^n \ln f(x_i; \Theta^{(k)}) \tag{6} \]

\(^2\)We refer to the \( k \)-th model also as the \( k \)-th hypothesis.

\(^3\)Using the notation \( \mathcal{P}(k) \) we emphasize that the penalty depends on \( k \), which is important for the minimization in (3). Note that, in general, \( \mathcal{P}(k) \) could also depend on other parameters.
Akaite noted that the average log-likelihood is a biased estimate of the cross entropy, and added a penalty that asymptotically, for large $n$, compensates the estimation error. Exploiting the asymptotical chi squared distribution of the log-likelihood, he derived what is now called the AIC, that corresponds to (2) and (3) with penalty

$$\mathcal{P}_{\text{AIC}}(k) = 2 \phi(k)$$

(7)

where $\phi(k)$ is the number of free parameters in $\Theta^{(k)}$. Thus, the AIC metric aims to minimize an unbiased estimate of the K-L divergence. However, in many situations it tends to overestimate the order of the model, even asymptotically [2], [3], [6], [8], [12], [16], [18], [26]–[28].

Alternative ITC are derived adopting the Bayesian approach, which chooses the model maximizing the a posteriori probability $\mathbb{P}(\Theta^{(k)} | x_1, x_2, \ldots, x_n)$. In this context, the most simple criterion is the BIC with penalty$^4$ [24]

$$\mathcal{P}_{\text{BIC}}(k) = \phi(k) \ln n.$$  

(8)

For large enough samples, BIC coincides with the MDL criterion, which attempts to construct a model that permits the shortest description of the data [29]. It has been demonstrated in that in some cases BIC provides a consistent estimate of the model order [6], [30], [31].$^5$

More generally, a large number of ITC, including AIC and BIC, present a penalty in the form

$$\mathcal{P}_{\text{GIC}}(k) = \phi(k) \cdot \nu$$

(9)

where $\nu$ can be a constant (as in (7)) or a function of other parameters (as in (8)). We refer to this criterion as the GIC [3], [6], [32], [33]. It has been shown that consistency of GIC can be reached by properly adjusting the parameter $\nu$ [6], [16], [17], [34]. In particular, it can be demonstrated that it is required, for $n$ that goes to infinity, that $\nu/n \to 0$ to avoid underestimation and $\nu/\ln \ln n \to +\infty$ to avoid overestimation [6]. Further rules can be derived in some specific selection problems [17]. Based on these general results, different criteria have been proposed, such as in [8], where $\nu = 1 + \ln n$ is used, and in [35], where $\nu = 2 \ln n$ has been adopted. A summary of the main ITC proposed in literature can be found in [1, Section 4].

In the next section, we discuss the performance and the design of GIC for finite samples, proposing a method for setting $\nu$ given a target maximum probability of overestimation.

### III. GIC PERFORMANCE AND DESIGN

#### A. Model selection performance

The performance of model order selection is evaluated in terms of probability to correctly detect $q$, $P_c \triangleq \mathbb{P}(\hat{q} = q)$, that can be expressed as

$$P_c = P_c(q; \text{SNR}, \nu) = 1 - P_{\text{over}} - P_{\text{under}}$$

(10)

where $P_{\text{over}} \triangleq \mathbb{P}(\hat{q} > q)$, with $q \in \{0, 1, \ldots, p-2\}$, and $P_{\text{under}} \triangleq \mathbb{P}(\hat{q} < q)$, with $q \in \{1, 2, \ldots, p-1\}$, are the probabilities of overestimation and underestimation, respectively. Given (3), $P_{\text{over}}$ and $P_{\text{under}}$ can be expressed as$^6$

$$P_{\text{over}} \simeq \mathbb{P} \left( \bigcup_{i=1}^{q_{\text{max}}-q} \{ \text{ITC}(q+i) < \text{ITC}(q) \} \right)$$

(11)

$$P_{\text{under}} \simeq \mathbb{P} \left( \bigcup_{i=1}^{q} \{ \text{ITC}(q-i) < \text{ITC}(q) \} \right).$$

(12)

Considering $P_{\text{over}}$, simple upper and lower bounds, $P_{\text{over}}^{\text{UB}}$ and $P_{\text{over}}^{\text{LB}}$, are respectively given by

$$P_{\text{over}} \leq \sum_{i=1}^{q_{\text{max}}-q} \mathbb{P}(\text{ITC}(q+i) < \text{ITC}(q))$$

$$\simeq \sum_{i=1}^{\nu_{\text{max}}} \mathbb{P}(\text{ITC}(q+i) < \text{ITC}(q)) = P_{\text{over}}^{\text{UB}}$$

(13)

$$P_{\text{over}} \geq \max_{\nu \in \{1, \ldots, q_{\text{max}}-q\}} \mathbb{P}(\text{ITC}(q+i) < \text{ITC}(q))$$

$$\geq \mathbb{P}(\text{ITC}(q+1) < \text{ITC}(q)) = P_{\text{over}}^{\text{LB}}$$

(14)

where the sum in (13) is truncated to the integer value $\nu_{\text{max}}$, with $1 \leq \nu_{\text{max}} \leq q_{\text{max}} - q$. The expressions of the bounds in (13) and (14) are based on the assumption that $\mathbb{P}(\text{ITC}(q+i) < \text{ITC}(q))$ is decreasing with $i$, which is common for ITC based model order selection, as in the case studies discussed in the following sections.$^7$ Similar considerations can be applied to the analysis of $P_{\text{under}}$.

When the SNR increases it has been noted that $P_{\text{under}}$ goes to zero, while $P_{\text{over}}$ converges to a non zero value [19], [22], [27], [36], [37].$^8$ This means that in the high SNR regime an incorrect selection always consists in an overestimation and thus we can express the probability of correct model selection as

$$P_c \simeq 1 - P_{\text{over}} \quad \text{(high SNR regime)}.$$  

(15)

#### B. Design of the penalty

Theoretical and experimental results show that the probability of correct selection, $P_c$, exhibits a sigmoidal dependence on the SNR, raising from zero to a maximum value [12], [22], [27]. In particular, it has been noted that BIC does not provide overestimations, allowing to reach a probability of correct selection close to 1 high SNR. For the AIC, instead, the maximum $P_c$ is smaller, but it is reached at lower SNRs. This behavior of the AIC and BIC, reported in previous literature (see, e.g., [19], [22], [27], [36], [37]) and confirmed by numerical results in Section VI, suggests that $\mathcal{P}_{\text{AIC}}(k)$ is

$^4$These expressions of $P_{\text{over}}$ and $P_{\text{under}}$ are based on the fact that in most model order selection problems $\text{ITC}(k)$ is a concave function with a minimum that in case of correct selection corresponds to $k = q$. This occurs, for example, in the case studies considered in the paper [18], [27].

$^5$In general, different problems require a different $\nu_{\text{max}}$. For example, in the Section VI we show that for the problem of estimating the number of signals $\nu_{\text{max}} = 1$ is sufficient for approximating $P_{\text{err}}$, while for the problem of estimating the number of sinusoids at least $\nu_{\text{max}} = 2$ is required.

$^6$In Fig. 3 and Fig. 7 we show some simulation results that confirm this effect.
too low to ensure a high $P_c$, while $\mathcal{P}_{\text{BIC}}(k)$ is excessively high, providing good results only for high SNR.\textsuperscript{9}

The dependence of $P_c$ on $\nu$ can be better understood analyzing the two error probabilities, $P_{\text{over}}$ and $P_{\text{under}}$, separately. Considering that $\mathcal{P}(k)$ is always an increasing function of $k$, from the model order selection rule defined by (2) and (3) it is easy to see that by increasing the penalty the selection of a small model order is favored, and thus a higher $P_{\text{under}}$ is provided. On the other hand, when the penalty decreases a higher $P_{\text{over}}$ occurs. Thus, the choice of the penalty implies a tradeoff between $P_{\text{over}}$ and $P_{\text{under}}$. This behaviour is confirmed by the simulation results in Section VI.

Based on these considerations we propose to use GIC setting the parameter $\nu$ to minimize $P_{\text{under}}$ uniformly over all SNRs while $P_{\text{over}}$ is constrained below a maximum value $P_{\text{over}}^{\text{max}}$. Note that this approach is analogous to the Neyman-Pearson criterion in binary hypothesis testing, in which $P_{\text{under}}$ and $P_{\text{over}}$ play the role of the probability of misdetection and the probability of false alarm, respectively. Considering the performance tradeoff between underestimation and overestimation, minimizing $P_{\text{under}}$ corresponds to the maximization of $P_{\text{over}}$, and thus the optimal value of $\nu$ is given by

$$\bar{\nu} = \arg \max \nu \left\{ P_{\text{over}} \left| P_{\text{over}} \leq P_{\text{over}}^{\text{max}} \right. \right\}. \quad (16)$$

Then, since $P_{\text{over}}$ is dependent on the true model order $q$ and the SNR, we consider a worst case design where maximization with respect to these parameters is considered. Therefore the design rule becomes

$$\bar{\nu} = \arg \max_{\nu, q, \text{SNR}} \max \nu \left\{ P_{\text{over}}(q, \text{SNR}, \nu) \left| P_{\text{over}} \leq P_{\text{over}}^{\text{max}} \right. \right\} \quad (17)$$

$$= \arg \max_{\nu, q} \left\{ P_{\text{over}}(q, \infty, \nu) \left| P_{\text{over}} \leq P_{\text{over}}^{\text{max}} \right. \right\}. \quad (18)$$

Equation (18) is due to the fact that the maximum $P_{\text{over}}$ always occurs in the high SNR regime ($\text{SNR} \to \infty$). Given (15), the approach (18) is equivalent to design the GIC penalty for reaching a target probability of correct selection $P_{\text{DES}}^\nu = 1 - P_{\text{over}}^{\text{max}}$ for high SNR. Note, however, that for any SNR it is not possible to find a $\nu < \bar{\nu}$ that gives a lower $P_{\text{under}}$, while satisfying $\max P_{\text{over}} \leq P_{\text{over}}^{\text{max}}$.

If an analytical form for $P_{\text{over}}$ is not available, we can design $\nu$ considering an upper bound on the probability of overestimation, which gives

$$\bar{\nu} = \arg \max_{\nu, q, \text{SNR}} \max \nu \left\{ P_{\text{over}}^\text{UB}(q, \text{SNR}, \nu) \left| P_{\text{over}} \leq P_{\text{over}}^{\text{max}} \right. \right\} \quad (19)$$

In general, the adoption of bounds leads to a performance loss in terms of SNR, which is smaller as the bound is tighter.

In the next sections, we discuss two examples of model order selection problems with the design of the GIC penalty. In particular, considering the estimation of the number of signals, in Section IV we adopt a design based on (18), while for the GLM problem, in Section V, we adopt a design based on (19).

IV. ESTIMATING THE NUMBER OF SIGNALS

The problem of estimating the number of signals arises in many statistical signal processing and time series analysis applications [12], [15]–[19], [38], [39]. We adopt the standard model in which the observation is the output of $p$ sensors, represented, at the $i$-th time instant, by the vector

$$\mathbf{x}_i = \mathbf{H} \mathbf{z}_i + \mathbf{n}_i \quad (20)$$

where $\mathbf{z}_i \in \mathbb{C}^{n \times 1}$ is the vector of the samples of the $q$ signals present, $\mathbf{H} \in \mathbb{C}^{p \times q}$ is a deterministic unknown channel matrix, and $\mathbf{n}_i \in \mathbb{C}^{p \times 1}$ represents noise. We assume that $\mathbf{z}_i \sim \mathcal{CN}(0, \mathbf{\sigma}^2 \mathbf{I}_p)$, where $\mathbf{\sigma}^2$ is the noise power at each sensor, and that $\mathbf{z}_i \sim \mathcal{CN}(0, \mathbf{R})$. Thus, for a given $\mathbf{H}$, the vectors $\mathbf{x}_i$ are zero mean Gaussian random vectors with covariance matrix

$$\Sigma = \mathbb{E} \{ \mathbf{x}_i \mathbf{x}_i^H \} = \mathbf{H} \mathbf{R} \mathbf{H}^H + \mathbf{\sigma}^2 \mathbf{I}_p. \quad (21)$$

Assuming that $\mathbf{R}$ is non singular and that the matrix $\mathbf{H}$ is of full column rank (implying $p > q$), which means that its columns are linearly independent vectors, the rank of $\Sigma$ is $q$ and thus the smallest $p - q$ eigenvalues are all equal to $\mathbf{\sigma}^2$. In [16] the estimation of the number of signals $q$ has been posed as a model order selection problem, solved by means of ITC. In this case we have $p$ possible models, where the $k$-th corresponds to the situation in which exactly $k$ signals are present, with $k \in \{0, \ldots, p-1\}$.

In this problem the parameter vector under the $k$-th hypothesis is

$$\Theta^{(k)} = (\lambda_1, \ldots, \lambda_k, \mathbf{v}_1, \ldots, \mathbf{v}_k, \sigma^2) \quad (22)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ are the eigenvalues of $\Sigma$ and $\{\mathbf{v}_i\}_{i=1,\ldots,k}$ are the corresponding eigenvectors. Considering the orthonormality constraints on the eigenvectors, the number of free parameters in $\Theta^{(k)}$ is $\phi(k) = k(2p-k)+1$ [16]. Using the joint ML estimates of the eigenvalues and eigenvectors of $\Sigma$, obtained by [40], the ITC model order estimate (3) is

$$\hat{q} = \arg\min_k \left\{ -2 \ln \left( \prod_{l=k+1}^{p+1} \frac{l^{1/(p-k)}}{\sum_{l=k+1}^{p+1} l^{1/2}} \right)^{(p-k)n} \mathcal{P}(k) \right\} \quad (23)$$

where $l_1 \geq l_2 \geq \cdots \geq l_p$ are the eigenvalues of the sample covariance matrix (SCM), $\mathbf{S} = \frac{1}{n} \mathbf{Y} \mathbf{Y}^H$, and $\mathbf{Y}$ is defined as (1).

This approach is known to provide good selection performance for sufficiently large number of observations $n$ [26], [27]. For small sample sizes using the exact marginal distribution of the eigenvalues of the SCM (without eigenvectors) gives better results [15]. In this paper we use (23) for ease of analysis.

In the following we focus on the probability of overestimation, useful for the design approach described in Section III-B. Note that characterizing $P_{\text{over}}$ is in general a mathematically difficult problem even in the high SNR regime.

\textsuperscript{9}Note that these considerations are limited to the problems in which BIC has been proven to provide a consistent model order selection. For example, considering the estimation of the number of signals discussed in Section IV, this holds in presence of white noise, while it has been demonstrated that in presence of colored noise BIC is no more a consistent model order estimator [18], [26]. Our analysis suggests that in this case an increase in the penalty is required to compensate the dispersion of the noise eigenvalues. This problem is out of the scope of the paper and will be object of further investigations.
A. Probability of overestimation

Previous works showed that in this problem for the analysis of overestimation and underestimation it is sufficient to consider the minimum of $\text{ITC}(k)$ for $k = \{q, q + 1\}$, which is equivalent to keep just the first term in (11) and (12) [18], [19], [26], [27], [36]. Thus, a good approximation for the probability of overestimation is given by

$$P_{\text{over}} \simeq P(\text{ITC}(q + 1) < \text{ITC}(q)) = P_{\text{over}^{LB}}.$$  (24)

Substituting (23) in (24) after some manipulations we obtain the expression [18], [26], [27], [36]. Thus, a good approximation for the probability of overestimation is given by

$$P_{\text{over}} \simeq P(v(1 - v)^{p - q - 1} < \xi_q)$$  (25)

where

$$v = \frac{l_{q+1}}{\ell_i}$$  (26)

and

$$\xi_q = \frac{(p - q - 1)^{p - q - 1}}{(p - q)^{p - q}} \exp \left( \frac{\mathcal{P}(q) - \mathcal{P}(q + 1)}{2n} \right).$$  (27)

The equation $v(1 - v)^{p - q - 1} = \xi_q$ has a single real root, denoted by $v$, in $[1/(p - q), 1]$, which is the range of $v$. This root can be easily computed using standard root finding algorithms.\footnote{Note that for this problem $P_{\text{over}^{LB}}$ is a tight bound and has been often used as an approximation of $P_{\text{over}}$ [18], [19], [26], [27], [36].} Thus (25) can be expressed as

$$P_{\text{over}} = 1 - F_v(v).$$  (28)

In the following we derive an approximated form for the computation of $F_v(\cdot)$ that can be adopted for the design of $\nu$ using (18) and (28).

B. Distribution of $v$

The statistic of $v$ has been studied in [12], [18], [27], considering that asymptotically, for large $n$, the smallest $p - q$ eigenvalues of the SCM are distributed as the eigenvalues of a central Wishart matrix $W$ with covariance matrix $\sigma^2 I_p$, where $p' = p - q$. Thus, the probability of overestimation has been evaluated considering that

$$v = \frac{u}{t} \approx u$$  (29)

where $u = \ell_1/t$, $\ell_1$ and $t$ are the largest eigenvalue and the trace of $W$, respectively. In [27] an infinite series expression for the computation of $P_{\text{over}}$ has been derived, while [18] provides an upper bound. Note that (29) allows to derive an expression of $P_{\text{over}}$ that has been independent of the SNR. In [12] an approximation of the CDF of $u$ based on the Tracy-Widom distribution has been adopted. In the following we provide an approximated form of $F_u(\cdot)$ that is easily invertible and is therefore useful for the design approach described in Section IV-C. Our approximation is based on the method of moments, which consists in choosing a simple distribution model and setting its parameters to match the first exact moments [35], [41], [42].

As shown in [43], the moments of $u$ can be computed considering that, conditioned on $\ell_1$, $u$ and $t$ are independently distributed, which leads to

$$m_i = \frac{m_i^{(\ell_1)}}{m_i^{(t)}}$$  (30)

where $m_i$, $m_i^{(\ell_1)}$ and $m_i^{(t)}$ are the $i$-th moments of $u$, $\ell_1$ and $t$, respectively.

In [44] and [45] it has been shown that the p.d.f. of $\ell_1$ is a gamma mixture distribution, which can be expressed as a linear combination of gamma-shaped functions as

$$f_{\ell_1}(x) = \sum_{s=1}^{p'} \sum_j \varepsilon_{s,j} x^j e^{-sx}.$$  (31)

The evaluation of the parameters $\varepsilon_{s,j}$ can be found in [44] and [45]. Based on (31) the $i$-th moment of $\ell_1$ can be derived in closed-form as

$$m_i^{(\ell_1)} = \int_0^\infty x^i f_{\ell_1}(x)dx = \frac{\prod_{s=1}^{p'} \varepsilon_{s,j}}{s^{j+i+1}} \Gamma(j + i + 1)$$  (32)

where $\Gamma(a) \triangleq \int_0^\infty y^{a-1} e^{-y}dy$ is the gamma function. Alternative methods for computing the moments of $\ell_1$ based on integral expressions or approximations are discussed in Appendix A.

Considering that $t \sim \mathcal{G}(n p', 1)$ [43], the moments of the trace are given by

$$m_i^{(t)} = \frac{\Gamma(p'n + i)}{\Gamma(p'n)}.$$  (33)

Once the moments $m_i$ are computed using (30), (32) and (33), we approximate $u$ to a shifted gamma distributed r.v. as

$$u + \alpha \overset{d}{\approx} \mathcal{G}(\kappa, \theta)$$  (34)

where $\kappa$, $\theta$, and the shift $\alpha$ are expressed as [42]

$$\kappa = \frac{4 (m_2 - m_1^3)^3}{(m_3 - 3 m_1 m_2 + 2 m_1^3)^2}$$  (35)

$$\theta = \frac{m_3 - 3 m_1 m_2 + 2 m_1^3}{2 (m_2 - m_1^2)}$$  (36)

$$\alpha = \kappa \theta - m_1.$$  (37)

Thus the approximated CDF of $u$ is given by

$$F_u(x) \simeq \begin{cases} \gamma(\kappa, \frac{x + \alpha}{\theta}), & x > -\alpha \\ 0, & x \leq -\alpha \end{cases}$$  (38)

where $\gamma(\alpha, z) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^z y^{\alpha-1} e^{-y}dy$ is the normalized incomplete gamma function. Note that (38) can be inverted using the inverse incomplete gamma function, which is already implemented in standard mathematical software. In Fig. 1 the comparison between the simulated and approximated CDF of $u$ are reported. As can be seen, the shifted gamma approximation in (38) matches very well the simulated distribution of $u$.\footnote{Alternatively, an approximation of $v$ is given in [26], while [18] provides an asymptotic expression.}
Then the optimal unknown deterministic parameters, and in the real case, adopted e.g. in [22], can be derived as a special case.

In this case, the selection problem consists in estimating the length of $u$. Under the GLM, the observation consists in a random vector defined as $u = 8, 20, \ldots

Then the optimal $\nu$ to be used in (9) is given by

$$
\hat{\nu} = \frac{2n}{2(p - q^*)} - 1 \ln \left( \frac{(p - q^*)^{p-q^*} - 1}{(p - q^*)^{p-q^*} - 1} \right) \times \ln \left( \hat{v}_{q^*} (1 - \hat{v}_{q^*})^{p-q^*} \right).
$$

(41)

Numerical results assessing the effectiveness of this design strategy are presented in Section VI-A.

V. GENERAL LINEAR MODEL

The GLM can be applied to a large set of problems in different fields of science and engineering (see [21] and [22] for some examples). Under the GLM, the observation consists in a $n$ length random vector defined as $\mathbf{y} = \mathbf{X}^{\mathbf{H}} \mathbf{n}$

$$
\mathbf{y} = \mathbf{\beta}^{\mathbf{H}} \mathbf{H}_q + \mathbf{n}
$$

(42)

where $\mathbf{H}_q$ is a $\psi(q) \times n$ matrix of known fixed values with linearly independent columns, $\mathbf{\beta} \in \mathbb{C}^{\psi(q) \times 1}$ is a vector of unknown deterministic parameters, and $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_n)$.

In this case, the selection problem consists in estimating the length of $\beta$, $\psi(q)$, which is a function of the model order $q$. Here, ITC can be applied using (2), (3), and the fact that under the $k$-th hypothesis we have [22]

$$
-2 \sum_{i=1}^{n} \ln f \left( x_i; \hat{\Theta}^{(k)} \right) = n \ln \hat{\sigma}^2_k
$$

(43)

where

$$
\hat{\sigma}^2_k = \frac{1}{n} \mathbf{y}^{\mathbf{H}} \mathbf{P}^{-1} \mathbf{y}
$$

(44)

and

$$
\mathbf{P}^{-1}_k = \mathbf{I}_n - \mathbf{H}_k^{\mathbf{H}} \mathbf{H}_k \mathbf{P}^{-1}_k
$$

(45)

is a projection matrix.

A. Bounds on the probability of overestimation

Differently from the case in the previous section, being the derivation of an analytic expression of (11) non trivial, we adopt the bound based approach described in Section III-B. The probability $\mathbb{P}(\text{ITC}(q + i) < \text{ITC}(q))$ can be expressed, using (2), (3) and (4), as

$$
\mathbb{P}(\text{ITC}(q + i) < \text{ITC}(q)) = \mathbb{P} \left( R_i < \exp \left( \frac{-2i}{n} \right) \right)
$$

(46)

where

$$
R_i = \frac{\mathbf{y}^{\mathbf{H}} \mathbf{P}^{-1} \mathbf{y}}{\mathbf{P}^{-1} \mathbf{y}}
$$

(47)

In the Appendix B we prove that $R_i$ is a beta distributed r.v. with parameters $n - 2(q - i)$ and $2i$, and thus the terms in (13) and (14) can be expressed in closed-form as

$$
\mathbb{P}(\text{ITC}(q + i) < \text{ITC}(q)) = I_{\exp(-\nu/n)}(n - 2(q - i), 2i)
$$

(48)

where $I_x(a, b) = \frac{1}{\Gamma(a) \Gamma(b)} \int_0^x z^{a-1}(1 - z)^{b-1}dz$, with $0 \leq x \leq 1$, is the incomplete beta function and $B(a, b) = \Gamma(a) \Gamma(b) / \Gamma(a + b)$ is the beta function. Thus, by using (48), we can easily compute $\mathbb{P}_\text{over}$ in (13). Note, in particular, that the probability in (46), and thus also the bounds (13) and (14), does not depend on the SNR.

B. Design for the GLM

Based on the bound derived, the design of the penalty can be performed according to (19). In this problem the probability in (48) is a decreasing function of $q$, and thus the maximum in (19) is reached for $q = 0$, giving

$$
\hat{\nu} = \arg \max_{\nu} \left\{ \mathbb{P}_\text{over}(0, \infty, \nu) \mid \mathbb{P}_\text{over} \leq \mathbb{P}_\text{over} \right\}
$$

(49)

which can be numerically computed inverting (13). Numerical results based on (49) are presented in Section VI-B.

Note that when $q = 0$ we cannot have underestimation, and thus (15) holds in general, not only for high SNR. Moreover, note that when $q = 0$, $\mathbb{P}_\text{over}$ corresponds to the probability that model selection fails when only noise is present, i.e., the probability of false alarm in signal detection. Therefore, in this case our design strategy corresponds to the Neyman-Pearson design criterion, in which the target probability of false alarm is $\mathbb{P}_\text{over}$.

VI. NUMERICAL RESULTS

In this section we present some numerical results to prove the effectiveness of the design approach proposed.
A. Estimating the number of transmitting sources

Considering the problem described in Section IV, we focus, as an example, on the estimation of the number of transmitting sources by a multiple antenna system that arises in array signal processing and cognitive radio contexts [15], [16]. Thus in this case \( \mathbf{x}_i \) is the vector of the output samples of the sensor antennas at the \( i \)-th time instant after downconversion and sampling, \( \mathbf{s}_q \) is the vector of the samples of the \( q \) signals present, \( \mathbf{H} \) describes the gain of the radio channel between the \( q \) signal sources and the \( p \) antennas, and \( \mathbf{n}_i \) represents the thermal noise. For this problem we have \( \text{SNR} = \text{tr} \{ \mathbf{HRH}^H \} / (p \sigma^2) \).

In Fig. 2 we show \( P_c \) as function of the SNR when \( q = 4 \), \( p = 8 \) and \( n = 1000 \). We can see that the curves confirm the behaviour described in Section III. Considering AIC, we can see that it reaches a maximum \( P_c \) of about 0.9, while BIC provides probability of correct selection almost one at the expense of a loss for \( \text{SNR} < -3 \text{dB} \). By changing the GIC parameter \( \nu \) we can trade-off between the high and low SNR performance. Note that the maximum \( P_c \) is correctly predicted using (18), (28) and (38) (dotted lines). The corresponding overestimation and underestimation probabilities are shown in Fig. 3. We can see that an increase of \( \nu \) gives a lower \( P_{\text{over}} \) but a higher \( P_{\text{under}} \). Note, in particular, that by increasing the SNR \( P_{\text{under}} \) goes to zero, which supports the approximation in (15). Also note that (15) is a very favorable property in CR scenarios, implying that ITC never misdetect the presence of primary users (PUs) if the SNR is sufficiently high.

In Fig. 4 we show \( P_c \) as a function of the number of signal sources. We can see that in different situations the maximum occurs for different \( q \) and thus, in general, the maximization in (18) requires the evaluation of \( P_c \) for all the number of sources...
considered. Note, however, that for high $P_e$, e.g., greater than 0.9, which is the most interesting case in practice, the curves decrease with $q$, and thus the design can be based on $q^* = q_{\text{max}}$.

In Fig. 5 we show $P_e$ as function of the SNR considering $q_{\text{max}} = 4$, $p = 10$ and $n = 1000$. Using (41) with $P_{\text{MAX}} = 0.05$ we obtain $\bar{v} = 2.281$. Note that when $q = q_{\text{max}}$, for high SNR, $P_e$ coincides with $1 - P_{\text{MAX}}$, while when $q < q_{\text{max}}$, we reach, as expected, a higher probability of correct selection. From the comparison with Fig. 2 ($q = 4$ case) we can see that when $\text{SNR} = 0 \text{dB}$ BIC provides probability of correct selection almost one, while AIC gives $P_e \approx 0.9$. Note that the advantage of BIC is lost at lower SNRs. For example, considering $\text{SNR} = -10 \text{dB}$, BIC provides $P_e \approx 0.16$, while GIC with the design of the penalty gives $P_e \approx 0.76$.

B. Estimating the number of sinusoids in AWGN

In this section we focus, as an example of GLM, on the problem of estimating the number of sinusoids in AWGN, described in [22], [37], [46], [47]. In this case, the $i$-th element of $y$ in (42) is given by

$$x_i = \sum_{l=1}^{q} a_l e^{j(2\pi f_l i + \phi_l)} + n_i$$

(50)

that can be rewritten as

$$x_i = \sum_{l=1}^{q} a_l e^{j\phi_l} \cos(2\pi f_l i) + j a_l e^{j\phi_l} \sin(2\pi f_l i) + n_i.$$  

(51)

where $n_i$ is the $i$-th element of $n$, and $j = \sqrt{-1}$.

We assume, as in [22, Section IV-A], that the sinusoids considered are taken from a known frequency set ${f_k}_{k \in \mathbb{K}}$ and that, considering the $k$-th hypothesis, the matrix $H_k$ is given by [22]

$$H_k = (h_{1,(k)} | h_{2,(k)} | \ldots | h_{n,(k)})$$

(52)

where

$$h_{i,(k)} = (\cos(2\pi f_1 i), \sin(2\pi f_1 i), \cos(2\pi f_2 i), \sin(2\pi f_2 i), \ldots, \cos(2\pi f_k i), \sin(2\pi f_k i))^T.$$  

(53)

The vector $\beta$, that contains the information on the sinusoids amplitudes and phases, has a length $\psi(k) = 2k$ and is given by

$$\beta = (a_1 e^{j\phi_1}, j a_1 e^{j\phi_1}, a_2 e^{j\phi_2}, j a_2 e^{j\phi_2}, \ldots, a_k e^{j\phi_k}, j a_k e^{j\phi_k})^T.$$  

(54)

In this problem, the number of free parameters in the $k$-th hypothesis is $\phi(k) = 2k + 1$, accounting for the $k$ unknown amplitudes, the $k$ unknown phases, and the noise power, and the SNR is given by $\text{SNR} = \sum_{l=1}^{k} |a_l|^2 / \sigma^2$.

Differently from the case in the previous section, in the following we adopt the performance bounds described in Section III-A. Numerical simulations show that a good approximation for $P_e$ is provided when $i_{\text{max}} = 2$ in (13) and (14). Therefore we have

$$P_{\text{over}}^{\text{UB}} = I_{\text{exp}}(-\frac{4}{\nu}) (n - 2 (q + 1), 2) + I_{\text{exp}}(-\frac{4}{\nu}) (n - 2 (q + 2), 4)$$

(55)

$$P_{\text{over}}^{\text{LB}} = I_{\text{exp}}(-\frac{4}{\nu}) (n - 2 (q + 1), 2).$$

(56)

We adopt, in particular, the example proposed in [37], in which the $k$-th frequency in the considered set is $f_k = 0.2 + \ldots$
(k − 1)/n, with k = 1, . . . , q_{\text{max}}.

In Fig. 6 we show P_{c} as function of the SNR when q = 3, q_{\text{max}} = 6 and n = 1000. The three sinusoids have equal amplitude and phases 0, π/4 and π/3 rad, respectively. The corresponding error probabilities are plotted in Fig. 7. Also for this problem we can see that the curves confirm the behaviour described in Section III. We also plot the upper and lower bounds for P_{\text{over}} and the corresponding bounds for P_{c}. Note that increasing P_{c} the bounds become tighter, and thus they can be considered P_{c} approximations.

An example of penalty design for the problem of estimating the number of sinusoids is reported in Fig. 8. Choosing P_{\text{over}} = 0.05 and n = 1000, from (49) and (55) we obtain ν = 2.499. We can see that the maximum P_{c} is always above the bound 1 − P_{\text{over}}^{\text{UB}}, which, in this case, is very tight to the estimated curves. From the comparison with Fig. 6 (q = 3 case) we can see that for SNR = 0 dB BIC provides probability of correct selection almost one, while AIC gives P_{c} ≈ 0.89. Note that the advantage of BIC is lost a lower SNRs. For example, considering SNR = −15 dB, BIC provides P_{c} ≈ 0.16, while GIC with the design of the threshold gives P_{c} ≈ 0.92.

VII. CONCLUSION

In this paper, we studied model order selection based on ITC under a design perspective. We focused on the GIC, which embraces most common criteria, and we proposed a strategy for designing its penalty for finite sample sizes. This method allows to keep the probability of overestimation below a specified level. We applied this design strategy to two model selection problems. Firstly, we studied the problem of estimating the number of sources, which received considerable attention in the past decades. We provided, in particular, a new approximated form for the computation of the maximum probability of correct selection based on the ratio of the largest eigenvalue to the trace of a central white Wishart matrix. We also applied model selection to the GLM, proposing a design strategy based on the bounds of the probability of overestimation, which can be applied to any selection problem with nested hypotheses. As a particular case, we focused on the problem of estimating the number of sinusoids in AWGN.

In both case studies we showed that the high SNR performance analysis can be addressed independently on the signal adopted. The proposed design strategy aims to choose proper ITC penalties to control the model order selection performance in finite sample size problems.

APPENDIX A

In Section IV-B we provide the exact expression of the moments of ℓ_1 based on the gamma mixture distribution (31). In the following we propose two alternative approaches for simplifying their computation.

For large n and p', ℓ_1 can be approximated using simpler distributions. For instance, a well known approximation of BIC is lost a

\[ \ell_1 - \mu_{np} \approx d \sqrt{\nu} \]  

(57)

where \( \mu_{np} = \sqrt{np} \) and \( \nu = (1/\sqrt{n} + 1/\sqrt{p})^{1/3} \). In this case, the first three moments of ℓ_1 can be approximated by

\[ m_1(\ell_1) \approx \lambda_{np} + \sigma_{np} m_1^{(\Gamma)} \]  

\[ m_2(\ell_1) \approx \lambda_{np}^2 + 2\lambda_{np} \sigma_{np} m_1^{(\Gamma)} + \sigma_{np}^2 m_2^{(\Gamma)} \]  

\[ m_3(\ell_1) \approx \lambda_{np}^3 + 3\lambda_{np}^2 \sigma_{np} m_1^{(\Gamma)} + 3\lambda_{np} \sigma_{np}^2 m_2^{(\Gamma)} + \sigma_{np}^3 m_3^{(\Gamma)} \]  

(58) \quad (59) \quad (60)

where \( \lambda_{np} = \mu_{np} - \sigma_{np} \) and the moments of a gamma distributed r.v. are given by \( m_i^{(\Gamma)} = \theta^i / \Gamma(\theta) \), \( \forall i \in \mathbb{N} \).

Alternatively, when n and p' are not large, the moments can be computed using numerical integration as

\[ m_i(\ell_1) = \int_0^{\infty} \left(1 - F_{\ell_1}(x^{1/i})\right) dx \]  

(61)

using the efficient computation of the CDF of ℓ_1 proposed in [42].

APPENDIX B

Let us denote with \( S_k \) the row space of \( \mathbf{H}_k \) and with \( S_k^\perp \) the corresponding orthogonal space. Given the assumption of nested models, \( \mathbf{H}_j \) is a submatrix of \( \mathbf{H}_k \) with k > j, and thus \( S_j \subset S_k \) and \( S_k^\perp \subset S_j^\perp \). Considering (45), we can see that \( \mathbf{P}_k^\perp \) is the projection matrix on \( S_k^\perp \), and thus it is idempotent and symmetric with rank \( n - 2k \). Also note that s ∈ \( S_q \). We then have the following original theorem.

Theorem 1: Consider \( \mathbf{M}_0 \) and \( \mathbf{M}_1 \), projection matrices on the spaces \( \Omega_0 \) and \( \Omega_1 \), respectively, with \( \Omega_0 \subset \Omega_0^\perp \). Given the random row vector \( y \sim \mathcal{CN}(\mu, \sigma^2 I_n) \), with \( \mu \in \Omega_0^\perp \), the r.v.

\[ R = \frac{\mathbf{y}^\mathbf{M}_1 \mathbf{y}^\mathbf{H}}{\mathbf{y}^\mathbf{M}_0 \mathbf{y}^\mathbf{H}} \]  

(62)

is beta distributed with parameters \( r_1 \) and \( r_0 - r_1 \), where \( r_0 \) and \( r_1 \) are the ranks of \( \mathbf{M}_0 \) and \( \mathbf{M}_1 \), respectively.
Proof: Let us rewrite $yM_0y^H$ as $yM_1y^H + y(M_0 - M_1)y^H$ where $M_0 - M_1$ is the projection matrix on the orthogonal complement of $\Omega_1$ to $\Omega_0$. Given [21, Theorem 4.4.2] the quadratic forms $yM_0y^H$ and $yM_1y^H$ are chi squared distributed r.v.s with $2r_0$ and $2r_1$ degrees of freedom, respectively, and, given the assumptions, non centrality parameter $\mu M_0 \mu^H = \mu M_1 \mu^H = 0$. Similarly, we can see that $y(M_0 - M_1)y^H \sim \chi^2(2r_0 - r_1)$. Due to the properties of projection matrices and the fact that $\Omega_1 \subseteq \Omega_0$, we have $M_1(M_0 - M_1) = M_1M_0 - M_1 = M_1 - M_1 = 0$, and thus the quadratic forms $yM_0y^H$ and $yM_1y^H$ are independent thanks to [21, Theorem 4.5.3]. Then the ratio in (62) can be rewritten as a combination of independent chi squared r.v.s as

$$R = \frac{yM_1y^H}{yM_0y^H + y(M_0 - M_1)y^H}$$

and thus $R \sim \beta_{1,2,(r_0-r_1)}/2$.

To this theorem $R_i$ in (47) is a beta distributed r.v. with parameters $n - 2(q - i) + 2i$. It is easy to see that Theorem 1 can be demonstrated also in the real case, in which $R \sim \beta_{1,2,(r_0-r_1)}/2$.

REFERENCES

[1] C. R. Rao and Y. Wu, “On model selection,” IMS Lecture Notes - Monograph Series, 2001.

[2] K. Burnham and D. Anderson, Model selection and multi-model inference: a practical information-theoretic approach. Springer, 2002.

[3] P. Stoica and Y. Selen, “Model-order selection: a review of information criterion rules,” IEEE Signal Process. Mag., vol. 21, no. 4, pp. 36–47, Jul. 2004.

[4] A. Chakrabarti and J. K. Ghosh, “AIC, BIC, and recent advances in model selection,” Handbook of the philosophy of science, vol. 7, pp. 583–605, 2011.

[5] H. Akaike, “A new look at the statistical model identification,” IEEE Trans. Autom. Control, vol. 19, no. 6, pp. 716–723, Dec. 1974.

[6] R. Nishii, “Maximum likelihood principle and model selection when the true model is unspecified,” Journal of Multivariate Analysis, vol. 27, no. 2, pp. 392–403, Nov. 1988.

[7] R. V. Foutz and R. C. Srivastava, “The performance of the likelihood ratio test when the model is incorrect,” The Annals of Statistics, vol. 5, no. 6, pp. 1183–1194, Nov. 1977.

[8] H. Bozdogan, “Model selection and Akaike’s information criterion (AIC): The general theory and its analytical extensions,” Psychometrika, vol. 52, no. 3, pp. 345–370, Sep. 1987.

[9] E. Gassiat and R. Van Handel, “Consistent order estimation and minimal penalties,” IEEE Trans. Inf. Theory, vol. 59, no. 2, pp. 1115–1128, Feb. 2013.

[10] R. J. Bhansali and D. Y. Downham, “Some properties of the order of an autoregressive model selected by a generalization of Akaike’s EPF criterion,” Biometrika, vol. 64, no. 3, pp. 547–551, 1977.

[11] A. C. Atkinson, “A note on the generalized information criterion for choice of a model,” Biometrika, vol. 67, no. 2, pp. 413–418, 1980.

[12] B. Nadler, “Nonparametric detection of signals by information theoretic criteria: performance analysis and an improved estimator,” IEEE Trans. Signal Process., vol. 58, no. 5, pp. 2764–2756, May 2010.

[13] L. Birgé and P. Massart, “Minimal penalties for gaussian model selection,” Probability theory and related fields, vol. 138, no. 1-2, pp. 33–73, May 2007.

[14] S. Kandeepan and A. Giorgetti, Cognitive Radios and Enabling Technologies. Boston: Artech House Publishers, Nov. 2012.

[15] M. Chiani and M. Z. Win, “Estimating the number of signals observed by multiple sensors,” in Proc. IEEE Int. Workshop on Cognitive Inf. Process. (CIP 2010), Elba Island, Italy, Jun. 2010.

[16] M. Wax and T. Kailath, “Detection of signals by information theoretic criteria,” IEEE Trans. Acoust., Speech, Signal Process., vol. 33, pp. 387–392, Apr. 1985.

[17] L. C. Zhao, P. R. Krishnaiah, and Z. D. Bai. “On detection of the number of signals when the noise covariance matrix is arbitrary,” Journal of Multivariate Analysis, vol. 20, no. 1, pp. 26–49, Oct. 1986.

[18] W. Xu and M. Kaveh, “Analysis of the performance and sensitivity of eigendecomposition-based detectors,” IEEE Trans. Signal Process., vol. 43, no. 6, pp. 1413–1426, 1995.

[19] E. Fishler, M. Grossman, and H. Messer, “Detection of signals by information theoretic criteria: general asymptotic performance analysis,” IEEE Trans. Signal Process., vol. 50, no. 5, pp. 1027–1036, May 2002.

[20] S. Kleinman and B. Nadler, “Non-parametric detection of the number of signals: Hypothesis testing and random matrix theory,” IEEE Trans. Signal Process., vol. 57, no. 10, pp. 3930–3941, 2009.

[21] F. A. Graybill, Theory and applications of the linear model. North Scituate, MA: Duxbury Press, 1976.

[22] M. Djuric, “Asymptotic MAP criteria for model selection,” IEEE Trans. Signal Process., vol. 44, no. 10, pp. 2726–2735, 1998.

[23] B. Choi, ARMA model identification. Springer, 1992.

[24] R. Schwarz, “Estimating the dimension of a model,” The annals of statistics, vol. 6, no. 2, pp. 461–464, 1978.

[25] M. H. Hansen and B. Yu, “Model selection and the principle of minimum description length,” Journal of the American Statistical Association, vol. 96, no. 454, pp. 746–774, 2001.

[26] A. P. Liavas and P. A. Regalia, “On the behaviour of information theoretic criteria for model order selection,” IEEE Trans. Signal Process., vol. 49, no. 8, pp. 1689–1695, Aug. 2001.

[27] Q. T. Zhang, K. M. Wong, P. C. Yip, and J. P. Reilly, “Statistical analysis of the performance of information theoretic criteria in the detection of the number of signals in array processing,” IEEE Trans. Acoust., Speech, Signal Process., vol. 37, no. 10, pp. 1557–1567, Oct. 1989.

[28] R. Shibata, “Selection of the order of an autoregressive model by Akaike’s information criterion,” Biometrika, vol. 63, pp. 117–126, 1976.

[29] J. Rissanen, “An introduction to the MDL principle,” 2004. [Online]. Available: www.mdl-research.org

[30] L. C. Zhao, P. R. Krishnaiah, and Z. D. Bai. “On detection of the number of signals in presence of white noise,” Journal of Multivariate Analysis, vol. 20, no. 1, pp. 1–25, 1986.

[31] G. Casella, F. J. Girón, M. L. Martínez, and E. Moreno, “Consistency of bayesian procedures for variable selection,” The Annals of Statistics, pp. 1207–1228, 2009.

[32] R. Nishii, “Asymptotic properties of criteria for selection of variables in multiple regression,” The Annals of Statistics, vol. 12, no. 2, pp. 758–765, Jun. 1984.

[33] Y. Zhang, R. Li, and C.-L. Tsai, “Regularization parameter selections via generalized information criterion,” Journal of the American Statistical Association, vol. 105, no. 489, pp. 312–323, 2010.

[34] H. Akaike, “On newer statistical approaches to parameter estimation and structure determination,” in Int. Federation of Automatic Control, vol. 3, 1978, pp. 1877–1884.

[35] A. Giorgetti and M. Chiani, “Time-of-arrival estimation based on information theoretic criteria,” IEEE Trans. Signal Process., vol. 61, no. 8, pp. 1869–1879, Apr. 2013.

[36] M. Kaveh, H. Wang, and H. Hao. “On the theoretical performance of a class of estimators of the number of narrow-band sources,” IEEE Trans. Acoust., Speech, Signal Process., vol. 35, no. 9, pp. 1350–1352, Sep. 1987.

[37] P. M. Djuric, “A model selection rule for sinusoids in white Gaussian noise,” IEEE Trans. Signal Process., vol. 44, no. 7, pp. 1744–1751, 1996.
[46] B. Nadler and A. Kontorovich, “Model selection for sinusoids in noise: Statistical analysis and a new penalty term,” *IEEE Trans. Signal Process.*, vol. 59, no. 4, pp. 1333–1345, 2011.

[47] C. Ma and B. P. Ng, “A CFAR based model order selection criterion for complex sinusoids,” *Signal processing*, vol. 86, no. 9, 2006.

[48] I. M. Johnstone, “On the distribution of the largest eigenvalue in principal components analysis,” *The Annals of statistics*, vol. 29, no. 2, pp. 295–327, 2001.

[49] M. Zelen and N. C. Severo, “Probability functions,” in *Handbook of Mathematical Functions*, M. Abramovitz and I. A. Stegun, Eds. NBS Applied Math. Series 55. Washington, D. C.: U. S. Government Printing Office, 1964, ch. 26.