Invariant Functionals in Higher-Spin Theory

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Abstract

A new construction for gauge invariant functionals in the nonlinear higher-spin theory is proposed. Being supported by differential forms closed by virtue of the higher-spin equations, invariant functionals are associated with central elements of the higher-spin algebra. In the on-shell $AdS_4$ higher-spin theory we identify a four-form conjectured to represent the generating functional for 3d boundary correlators and a two-form argued to support charges for black-hole solutions. Two actions for 3d boundary conformal higher-spin theory are associated with the two parity invariant higher-spin models in $AdS_4$. The peculiarity of the spinorial formulation of the on-shell $AdS_3$ higher-spin theory, where the invariant functional is supported by a two-form, is conjectured to be related to the holomorphic factorization at the boundary. The nonlinear part of the star-product function $F_\ast(B)$ in the higher-spin equations is argued to be incompatible with the boundary conformal symmetry. An interpretation of the RG flow in terms of proposed construction is briefly discussed.
Contents

1 Introduction .................................................. 3

2 Unfolded equations and invariant functionals .................. 8

3 HS equations in $AdS_4$ ....................................... 11
   3.1 Original nonlinear system .................................. 11
   3.2 Fock behavior at infinity .................................. 14

4 Lagrangian extension .......................................... 16

5 Invariants of $AdS_4$ HS theory ............................... 21
   5.1 Extended system ............................................ 21
   5.2 Vacuum solution ........................................... 22
   5.3 Sketch of the first order ................................... 23
   5.4 Boundary functionals, parity, and conformal HS theory ....... 26
   5.5 Black holes .................................................. 28
   5.6 Vacuum partition ............................................ 30

6 $AdS_3$ HS theory ............................................... 32

7 Conclusion .................................................... 33
1 Introduction

Standard holographic prescription for computation of correlators [1, 2, 3, 4] is based on the bulk action evaluated on solutions of the dynamical field equations with appropriate boundary conditions. The lower-order actions for higher-spin (HS) gauge fields, that extend the Fronsdal’s quadratic actions [5, 6] to the cubic order, are known since [7, 8, 9] (see also [10, 11, 12] for recent progress and more references). These actions are however incomplete even at the cubic order, not fixing relative coupling constants of cubic vertices. The lower-order results indicate that the full nonlinear extension of the Fronsdal’s action does exist. However, unavailability of its explicit form complicates the holographic analysis of the HS theories. An interesting alternative proposal suggested in [13, 14], where the action is defined in a higher-dimensional space-time, leads, however, to unconventional actions even for lower spins and its application in the context of HS holography remains to be explored.

Despite the impressive progress on the verification of the Klebanov-Polyakov conjecture [15, 16, 17] on the holographic duality between HS gauge theories and vectorial boundary theories achieved via analysis of the HS field equations in [18] (for more references and recent developments see, e.g., [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]), it is desirable to have a direct prescription for the generating functional of boundary correlators. The situation with the AdS$_3$/CFT$_2$ HS holography [32, 33, 34] is analogous.

In this paper we propose a new approach to the construction of invariant functionals in HS theory which leads to differential forms $\mathcal{L}(\phi)$ built from dynamical fields $\phi$, that are closed,

$$d\mathcal{L}(\phi) = 0,$$

(1.1)

by virtue of the nonlinear HS field equations. To this end we suggest the extension of the nonlinear HS field equations of [35] which determines Lagrangians $\mathcal{L}(\phi)$ associated with central elements of the HS algebra. The functional

$$S = \int \mathcal{L}(\phi)$$

(1.2)

turns out to be gauge invariant.

Generally, there exist two types of unfolded systems called off-shell and on-shell. Off-shell systems describe a set of constraints that express a (usually infinite) set of auxiliary fields via derivatives of some ground fields imposing no differential restrictions on the latter. On-shell systems impose differential field equations on the ground fields called dynamical in this case. For off-shell HS systems the functional $S$ is anticipated to describe the action. For on-shell systems $S$ can be thought of as an on-shell action underlying the analysis of AdS/CFT. In this paper we focus on the on-shell spinorial HS theories in AdS$_4$ and AdS$_3$.

In the standard AdS/CFT, the generating functional of the boundary theory on $\Sigma$ is identified with [2]

$$S_\epsilon = \int_{\epsilon}^\infty d\mathbf{z} \int_\Sigma \mathcal{L}(\phi)$$

(1.3)

as a functional of appropriate boundary values of fields. Here $\mathbf{z}$ is the Poincaré coordinate integrated till the cutoff $\epsilon$. In this setup, the cutoff can break the symmetries at the boundary.
while the generating functional can depend on total derivatives in $\mathcal{L}$. In principle, the latter can be adjusted to ensure appropriate properties of the theory in spirit of, e.g., \cite{36, 37, 38}.

Alternatively, we suggest to consider the functional of the form

$$ S = \frac{1}{2\pi i} \oint_{z=0} \int_{\Sigma} \mathcal{L}(\phi) $$

resulting from the integration over a cycle on the plane of complexified $z$ encircling the infinite point $z = 0$. As explained in \cite{19} (see also below), the possibility of the integration in the plane of complex $z$ is provided by the unfolded formulation of HS field equations operating with differential forms and allowing at least locally to extend the system to a larger space including the complexified one. In this construction, $\mathcal{L}(\phi)$ remains closed in the extended space and $S$ remains invariant under all gauge symmetries of the original bulk system provided that the pushforward is well defined in some neighborhood of the infinity $z = 0$ allowing the integration around $z = 0$.

Note that if the system was off-shell in the original space, its dynamical fields will necessarily obey certain differential equations in the extended space allowing $\mathcal{L}$ be closed in a larger space.\footnote{This is somewhat analogous to the Group Manifold Approach \cite{39, 40} (see also \cite{41}) requiring so-called rheonomy conditions on the curvatures to extend the system to the higher-dimensional group manifold.} If the extension to the complex plane of $z$ exhibits branch cuts the standard definition (1.3) may be more appropriate. This is unlikely to happen in the $AdS_4$ HS model \cite{19} but remains to be investigated in other models.

Though the whole setting also applies to the standard construction (1.3), for definiteness, in the sequel we will mostly refer to the boundary functionals (1.4). As argued in Section 5.4, in certain cases functionals (1.4) describe local actions for boundary conformal HS theory that only give local contribution to the boundary correlators. To reproduce the nonlocal part of the correlators, in these cases one should either use the standard construction (1.3) or a limiting procedure explained in Section 5.4.

Let us consider the $AdS_4/CFT_3$ case in some more detail. In spinor notation with two-component spinor indices $\alpha, \beta = 1, 2, \dot{\alpha}, \dot{\beta} = 1, 2$, local coordinates of $AdS_4$ are

$$ x^{\alpha \dot{\alpha}} = (x^{\alpha \dot{\alpha}}, -\frac{i}{2} \epsilon^{\alpha \dot{\alpha}} z^{-1}) , \quad (1.5) $$

where the symmetric part of $4d$ coordinates $x^{\alpha \dot{\alpha}} = x^{\dot{\alpha} \alpha}$ is identified with coordinates of the boundary $\Sigma$ while $z$ is the radial coordinate of $AdS_4$. The appearance of $\epsilon^{\alpha \dot{\alpha}} = -\epsilon^{\dot{\alpha} \alpha}$ in the definition of $z$ breaks the $4d$ Lorentz symmetry $sp(2; \mathbb{C})$ to the $3d$ Lorentz symmetry $sp(2; \mathbb{R})$ which acts on the both types of spinor indices. In Poincaré coordinates, $AdS_4$ vierbein and Lorentz connection can be chosen in the form

$$ e^{\alpha \dot{\alpha}} = \frac{1}{2z} dx^{\alpha \dot{\alpha}} , \quad \omega^{\alpha \beta} = -\frac{i}{4z} dx^{\alpha \beta} , \quad \bar{\omega}^{\dot{\alpha} \dot{\beta}} = \frac{i}{4z} d\dot{x}^{\dot{\alpha} \dot{\beta}} . \quad (1.6) $$

Meromorphic dependence on $z$ makes it possible to complexify the Poincaré coordinate $z$. The connection (1.6) remains flat provided that all its $dz$ components are zero. Strictly
speaking, this is true everywhere except for the point $z = 0$ of infinity since $\frac{\partial}{\partial z} z \neq 0$. Hence, our analysis applies to the complexified $AdS_4$ space with removed infinity $z = 0$.

In the $AdS/CFT$ correspondence dictionary, the source term for the spin-$s$ conserved current $J_{n_1...n_s}$ is

$$S = \int dx^3 \varphi^{n_1...n_s} J_{n_1...n_s}. \quad (1.7)$$

The current conservation

$$\partial_m J^m_{n_2...n_s} = 0 \quad (1.8)$$

is dual to the gauge symmetry of the gauge field

$$\delta \varphi_{n_1...n_s} = \partial_{(n_1} \varepsilon_{n_2...n_s)} . \quad (1.9)$$

The variation of $\langle \exp -S \rangle$ over $\varphi_{n_1...n_s}$ gives correlators of currents.

In the frame-like approach to 3d boundary theory, the symmetric tensor field $\varphi_{n_1...n_s}$ is substituted by the frame-like one-form connection $\omega_{\alpha_1...\alpha_{2(s-1)}} = dx^n \omega_{n \alpha_1...\alpha_{2(s-1)}}$ where the indices $\alpha = 1, 2$ are spinorial and $n = 0, 1, 2$. As explained in [19], the role of $J_{n_1...n_s}$ is played by the so-called HS Weyl tensor $C_{\alpha_1...\alpha_{2(s-1)}}$ and its conjugate $\bar{C}_{\dot{\alpha}_1...\dot{\alpha}_{2s}}$. Though $C_{\alpha_1...\alpha_{2s}}$ is not an operator at the boundary but rather the gauge-invariant curvature tensor built in terms of $s$ derivatives of the connections $\omega_{\alpha_1...\alpha_{2(s-1)}}$, it obeys the conservation condition (which from the bulk perspective is the Bianchi identity) and is a primary field of the conformal module equivalent to that of the 3d conformal current. The counterpart of action (1.7) is

$$S(\omega) = \int \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \omega^{\alpha_1...\alpha_{2(s-1)}} e_x^{\alpha_{2s-1}} e^{\alpha_2 \gamma} (a C_{\alpha_1...\alpha_{2s}}(\omega) + \bar{a} \bar{C}_{\dot{\alpha}_1...\dot{\alpha}_{2s}}(\omega)), \quad (1.10)$$

where $e_{x \alpha \beta}$ is the boundary dreibein one-form, $e_z$ is the component of the $AdS_4$ vierbein along the Poincaré coordinate and $a, \bar{a}$ are some complex conjugate coefficients (in the sequel the wedge symbol is omitted). Depending on a model, different linear combinations of the Weyl zero-forms represent either $R_{xx}$ or $R_{xz}$ components of the HS curvatures at $z = 0$

$$R_{xx} \sim a^{loc} e_x e_x C + \bar{a}^{loc} e_x e_x \bar{C}, \quad R_{xz} \sim a^{nloc} e_z e_x C + \bar{a}^{nloc} e_z e_x \bar{C}. \quad (1.11)$$

(For explicit expressions see [19].)

At the quadratic level, the decomposition of invariant functional (1.10) into the local and nonlocal parts

$$S = S^{loc} + S^{nloc} \quad (1.12)$$

corresponds to the decomposition of (1.10) into a linear combination of the terms proportional to $R_{xx}$ and $R_{xz}$, respectively. Indeed, the part of $S$ that only contains the boundary derivatives of boundary fields describes some boundary functional. Correspondingly, the Lagrangian (1.10) with $a = a_x, \bar{a} = \bar{a}_x$ describes the boundary Chern-Simons action of conformal HS theory $a la$ 42, 43, 44, 45. These give local contribution to the correlators. In the case $a = a_z, \bar{a} = \bar{a}_z$, the Lagrangian (1.10) contains the bulk derivative hence giving a generating function for the nonlocal part of the correlators. As explained in Section 5.4,

5
for the $P$-invariant HS models the naive functional $S$ (1.4) gives rise to the local boundary conformal HS theory with $S^{\text{loc}} = 0$ while $S^{\text{nonloc}}$ can be associated with its derivative over the parameter $\eta$ in the nonlinear HS equations.

In addition to the HS conformal gauge fields, the model contains spin-zero conformal currents of different conformal dimensions

$$C_1(\mathbf{x}, z) = C(\mathbf{x}, z), \quad C_2(\mathbf{x}, z) = \partial_\eta C(\mathbf{x}, z). \quad (1.13)$$

In this sector, the action functional is

$$S(C) = \int \mathcal{L}, \quad \mathcal{L} = V_\Sigma \epsilon_4 C_1(x) C_2(x), \quad (1.14)$$

where $V_\Sigma$ is the 3d volume form. The behavior of the fields $C(\mathbf{x}, z)$ and $\omega(\mathbf{x}, z)$ at $z \to 0$ is in agreement with their conformal dimensions (for more detail see [19]). As a result, the generating functional (1.14) just singles out the conformal invariant part of $\mathcal{L}$.

Let the dynamical boundary fields which are the primary conformal components among $\omega$ and $C_1$ or $C_2$ be collectively denoted $\phi(x)$. Then the boundary correlators are conjectured to be given by

$$\langle J(x_1)J(x_2)\ldots \rangle = \left. \frac{\delta^n \exp[-S(\phi)]}{\delta \phi(x_1)\delta \phi(x_2)\ldots} \right|_{\phi=0}, \quad (1.15)$$

where

$$x_1 = (\mathbf{x}_1, z), \quad x_2 = (\mathbf{x}_2, z), \quad \ldots \quad (1.16)$$

are taken at different boundary points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ and some small $z$ inside the integration contour in the definition (1.4). (The issue of the dependence on $z$ is analogous to that [2] in the standard approach (1.2).)

Evaluation of (1.15) for $\phi = C_1$ or $\phi = C_2$ is equivalent to the evaluation of correlators with different boundary conditions in the standard approach, namely with $C_2 = 0$ or $C_1 = 0$, respectively. Analogously, one can choose the generalized Weyl tensor as an independent field, expressing connections $\omega$ in terms of $C$ by the field equations (though the resulting expressions are nonlocal and are defined modulo the gauge freedom). Combining two such exchanges with the dualization of the HS Weyl tensors should reproduce the Witten’s $SL(2, \mathbb{Z})$ duality [46] extended to higher spins by Leigh and Petkou [47].

The problem is to find a Lagrangian $\mathcal{L}$ leading to the gauge invariant functional $S$ (1.4) in the full nonlinear HS theory. In this paper we mostly focus on the general scheme which opens a new way toward solution of this problem. Main attention will be paid to the spinorial $AdS_4$ HS theory where in particular we identify the local boundary functionals which are anticipated to describe 3d conformal HS theories and are associated with the so-called $A$ and $B$ HS theories. Also we briefly consider the on-shell spinorial HS model in 2+1 dimensions. Elaboration of the detailed structure of the invariant functionals introduced in this paper requires significant technical work to be presented elsewhere [48].

Apart from invariants associated with Lagrangian forms of maximal degree, our construction gives rise the “on-shell Lagrangians” of lower degrees. In particular, the 3d and 4d
on-shell HS systems considered in this paper admit the closed two-form $L^2$. For the 4d HS system this is conjectured to describe the black hole (BH) charge as

$$Q \sim \int_{\Sigma^2} L^2 \quad \text{(1.17)}$$

integrated over a cycle $\Sigma^2$ surrounding a BH singularity. Since $L^2$ is closed, $Q$ is insensitive to local variations of $\Sigma^2$. The thermodynamical first law should relate $\delta Q$ evaluated at infinity to $\delta Q$ evaluated at the BH horizon. To make contact with the standard approach [49] one should take into account some novelties of our construction.

First of all, it may look surprising that the two-form $L^2$ exists at all since it is closed and gauge invariant up to exact forms not just for a BH solution that admits Killing vectors but for any solution including, in particular, fluctuations around the BH solution. Here it is important that $L^2$ is not a local functional of fields. Rather it is (minimally) nonlocal in the sense specified in [50], depending on all derivatives of the fields and containing inverse powers of the background curvature in the derivative expansion that, in particular, complicates a straightforward flat limit analogously to the situation with the HS actions [9]. Nonetheless $L^2$ should be well defined as a space-time closed form, i.e., (1.17) makes sense for any $\Sigma^2$.

The infinity cycle $\Sigma^2_{\infty}$ and the horizon cycle $\Sigma^2_H$ are special. At $\Sigma^2_{\infty}$, where the theory becomes asymptotically free and $L^2$ becomes asymptotically local, $Q$ should reproduce usual asymptotic charges. (It would be interesting to establish their relation to the construction of [51, 52].) The horizon $\Sigma^2_H$ is a Killing bifurcation surface. As discussed in Section 5.5 for the case of GR, from the perspective of unfolded equations this implies the trivialization of the evolution equations with respect to certain directions. So far it is not known whether or not a horizon $\Sigma^2_H$ possessing such properties exists for the HS solutions of [53, 54, 55] of the full nonlinear theory. To answer this question it should be explored whether there exists such a surface $\Sigma^2_H$ on which some of the unfolded equations trivialize in terms of the coordinates of the observer at infinity. For $L^2 \big|_H$, starting with the volume form on $H$ times a constant proportional to $\beta$, $Q$ will start with the term proportional to the area of $H$.

Hopefully, the realization of the BH charge in terms of $L^2$ ($L^{d-2}$ for higher dimensions) can help to clarify the microscopic origin of the BH entropy the profound example of which was proposed in [56]. A natural guess is to identify the Lagrangian of the microscopic system with $L^2 \big|_H$ for the restriction of the original unfolded system to the horizon $H$.

As discussed in Section 4, the 3d on-shell HS system, where the only invariant Lagrangian is a two-form $L^2$, is special. Naively the form degree two is smaller than anticipated for a Lagrangian and larger than is needed for the 3d BH charge. However, very likely this is just appropriate for the both problems with the invariant functionals of the form

$$S = \int_{S^1 \times \Sigma^1} L^2, \quad \text{(1.18)}$$

where $S^1$ is a cycle around $AdS_3$ infinity as in (1.4) while $\Sigma^1$ is either a cycle at the conformal boundary for the generating functional of boundary correlators or a cycle around the
singularity of the BTZ-like BH solutions [57], which in the HS theory were considered in [58, 59] (and references therein).

Since it is hard to consider in detail all these questions in a single paper, here we focus on the general scheme providing a starting point for the future studies. The rest of the paper is organized as follows. In Section 2 we summarize general properties of invariant functionals in the unfolded dynamics approach and related interpretation of the RG flow. The structure of the field equations of the nonlinear HS theory in $AdS_4$ is recalled in Section 3. Subtleties of the boundary limit in HS theories affecting conformal properties in their holographic interpretation are also discussed here. In particular it is shown that any nonlinear star-product function of the zero-form $B$ in the nonlinear HS equations is incompatible with the boundary conformal symmetry. General structure of the extended unfolded systems allowing to define invariant Lagrangians and its application to the $AdS_4$ HS theory are presented in Sections 4 and 5, respectively. In particular, possible application to BH physics is sketched in Section 5.5. The on-shell $AdS_3$ spinorial HS theory is considered in Section 6. Section 7 contains conclusions.

2 Unfolded equations and invariant functionals

Let $M^d$ be a $d$-dimensional manifold (space-time) with local coordinates $x^a$ ($n = 0, 1, \ldots, d - 1$). By unfolded formulation of a linear or nonlinear system of partial differential equations in $M^d$ we mean its reformulation in the first-order form [60]

$$d_x W^\Omega(x) = G^\Omega(W(x)),$$

where $d_x = dx^a \frac{\partial}{\partial x^a}$ is the exterior derivative in $M^d$, $W^\Omega(x)$ is a set of degree-$p_\Omega$ differential forms, and $G^\Omega(W)$ is some degree-$(p_\Omega + 1)$ function of $W^\Lambda$

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega_{\Lambda_1 \ldots \Lambda_n} W^{\Lambda_1} \ldots W^{\Lambda_n}$$

that satisfies the generalized Jacobi identity on the structure coefficients $f^\Omega_{\Lambda_1 \ldots \Lambda_n}$

$$G^\Lambda(W) \frac{\partial G^\Omega(W)}{\partial W^\Lambda} = 0.$$  

Strictly speaking, generalized Jacobi identities (2.3) have to be satisfied at $p_\Omega < d$ since any $(d + 1)$-form in $M^d$ is zero. Any solution of (2.3) defines a free differential algebra [11, 32, 33, 34]. A free differential algebra is universal [35, 69] if (2.3) holds independently of the space-time dimension, i.e., for abstract supercoordinates $W^\Lambda$ which are (anti)commuting for variables associated with differential forms of (odd)even degrees. All free differential algebras associated with known HS theories are universal.

Condition (2.3), which can equivalently be written as

$$Q^2 = 0, \quad Q =: G^\Omega(W) \frac{\partial}{\partial W^\Omega},$$

8
guarantees formal consistency of the unfolded system (2.1) which can be put into the Hamiltonian-like form
\[
d_x F(W(x)) = Q(F(W(x)) \tag{2.5}
\]
for any \( F(W) \). Universal equation (2.4) is invariant under the gauge transformation
\[
\delta W^\Omega = d_x \varepsilon^\Omega + \varepsilon^\Lambda \partial G^\Omega(W) / \partial W^\Lambda, \tag{2.6}
\]
where the gauge parameter \( \varepsilon^\Omega(x) \) is a \((p_\Omega - 1)\)-form. (Zero-forms have no gauge parameters.)

Dynamics of a universal unfolded system is characterized entirely by differential \( Q \) (2.4) defined on the “target space” of dynamical variables \( W^\Omega \) independently of the original space-time. In particular, invariants like actions and conserved charges are characterized by the \( Q \)-cohomology. Indeed, as shown in \[66\], a gauge invariant functional is an integral over a \( p \)-cycle \( M^p \)
\[
S = \int_{M^p} \mathcal{L}(W) \tag{2.7}
\]
of some \( Q \)-closed Lagrangian \( p \)-form \( \mathcal{L}(W) \)
\[
Q \mathcal{L} = 0 : \quad G^\Omega(W) \partial \mathcal{L}(W) / \partial W^\Omega = 0. \tag{2.8}
\]
(It is elementary to see that such \( S \) is invariant under gauge transformations (2.6).) If \( \mathcal{L} \) is \( Q \)-exact, by virtue of (2.3) it is \( d_x \)-exact giving a trivial functional up to possible boundary terms. Hence nontrivial invariant functionals represent \( Q \)-cohomology of the system in question. Analysis of invariant functionals in terms of \( Q \)-cohomology, which applies to both linear and nonlinear unfolded systems (for examples see \[66\]), is complete: any invariant functional of the universal unfolded field equations corresponds to some their \( Q \)-cohomology. However, as for any other general approach, the direct search of invariant functionals via \( Q \)-cohomology may be involved for concrete nonlinear systems.

The remarkable feature of universal unfolded equations (2.1), which has deep connection \[19\] with holographic duality, is that they can be written in space-times of different dimensions since the fact of their consistency is insensitive to the number of space-time coordinates. Whether unfolded system (2.1) is on-shell or off-shell depends in the first place on the dimension of space-time where it is considered. A typical situation is when the same unfolded system is off-shell in \( d \) dimensions and on-shell in \( d + 1 \) dimensions. If a system in \( d \) dimensions is off-shell it can only have nontrivial \( H^d(Q) \)-cohomology since in the topologically trivial case it is impossible to construct a closed local functional of (derivatives) of the ground fields not subject to any field equations, that is not exact.\footnote{Note that this analysis is local, discarding possible topological obstructions. In particular, in this setup, topological invariants like Chern classes are treated as locally exact.} Hence, actions \( S^d \) of off-shell systems in \( d \) dimensions are usually \( d \)-forms. However, the \( S^d \) remains \( Q \)-closed for the same dynamical system uplifted to higher dimensions where it becomes on-shell.
The property that $L$ is $Q$-closed suggests that formula (1.4) should have general applicability beyond HS gauge theories. Indeed, this implies that $L$ remains d-closed in a larger space with complexified $z$. As a result, $S$ (1.4) turns out to be independent of local variations of the integration contour.

Note that the distinguished rôle of the closed functionals in the unfolded dynamics may also be related to the fact that averaging over integration cycles if occurs in some underlying fundamental theory has no effect on the integrals of closed forms, giving zero for other functionals. Since in the unfolded dynamics the gauge symmetries are consequences of the $Q \sim d_x$ closure of the Lagrangian forms, it is tempting to speculate that, other way around, gauge symmetries can result from some sort of averaging over integration cycles.\(^3\)

Assuming that the infinity is the only singular point and choosing the contour around $z = 0$ to be a circle of radius $r$ on the complex $z$-plane, this implies

$$\frac{dS}{dr} = 0. \quad (2.9)$$

By virtue of (2.3) this is equivalent to

$$G^\Omega_r(W)\frac{dS(W)}{dW^\Omega} = 0, \quad (2.10)$$

where $G^\Omega_r$ is the component of $G^\Omega$ along the radial direction, i.e., discarding the terms not containing $dr$,

$$G^\Omega = drG^\Omega_r + \ldots . \quad (2.11)$$

In the perturbative analysis, the forms $W^\Omega$ are decomposed into the vacuum part $W^\omega_0$ (the index $\omega$ is different from $\Omega$ to stress that some vacuum components of $W^\Omega$ may be zero) and the fluctuational part $W^\Omega_1$. Decomposing $G^\Omega(W)$ into the vacuum and fluctuational parts

$$G^\Omega(W) = G^\omega_0(W_0) + G^{\alpha\Omega}(W_0, W_1) \quad (2.12)$$

at the condition that

$$G^{\alpha\Omega}(W_0, 0) = 0, \quad (2.13)$$

guaranteeing that the vacuum fields do not source the dynamical ones, equation (2.10) can be rewritten in the Hamiltonian-like form

$$\dot{S}(W_0, W_1) + HS(W_0, W_1) = 0, \quad (2.14)$$

where

$$\dot{F}(W_0, W_1) := G^{\omega_0}_r(W_0)\frac{\partial F(W_0, W_1)}{\partial W^\omega_0}, \quad (2.15)$$

$$H := G^{\alpha\Omega}_r(W_0, W_1)\frac{\partial}{\partial W^{\Omega_1}}. \quad (2.16)$$

\(^3\)It should be noted that even non-gauge systems like scalar field acquire gauge symmetries in their unfolded form. These are gauge symmetries of the background one-form flat connections expressing coordinate independence of the unfolded formulation.
Note that the property that $W^\omega_0$ is a solution of (2.1) at $W^\Omega = 0$ implies that $G^\Omega$ only contributes to the evolution of the fluctuations $W^\Omega$.

Using (2.1) for the vacuum solution it follows that $\dot{F}(W_0, W_1)$ indeed describes the $r$-evolution due to the dependence of $W^\omega_0$ on $r$ (e.g. $z$ evolution for the vacuum connection (1.6)). Note that, analogously to the usual Hamiltonian formalism, to proceed one should first evaluate the derivatives $\frac{\partial F(W_0, W_1)}{\partial W^\Omega_0}$ over the general vacuum connection setting $W_0$ to its particular value like (1.6) afterwards. The second term in (2.14) describes an effective Hamiltonian in agreement with [66] where it was argued that universal unfolded equations (2.5) provide a proper multidimensional generalization of the Hamiltonian dynamics.

For the functional $S_z$ (1.3) analogous analysis gives

$$\dot{S}_z(W_0, W_1) + H S_z(W_0, W_1) = \int_\Sigma L_z, \quad (2.17)$$

where $\Sigma$ is the boundary surface and $L_z$ is the component of $L$ along the (real) $z$-direction, i.e.,

$$L = dz L_z + \ldots .$$

An interesting aspect of the HS holography is the holographic interpretation of the RG flow in terms of the bulk dynamical equations with respect to the radial coordinate [67, 68, 69, 70]. In approach (1.4) the RG-like equation controls independence of $S$ of the integration contour $S^1$. This interpretation is reminiscent of the Wilsonian approach based on the independence of the scale of fields distinguishing between UV and IR regions which can be regarded as those inside and outside $S^1$, respectively. The analogy of Eqs. (2.14) and (2.17) with the Hamilton-Jacoby approach of [71] is also encouraging. It would be interesting to check more closely the relation of Eq. (2.14) to the holographic interpretation of the RG flow in the boundary duals of the HS theories.

Note that at this stage the status of boundary divergencies in HS theory and its dual is not quite clear. As explained in Section 3.2 potential divergencies can result from the collision of different Fock vacua which dominate the boundary behavior in the $z \to 0$ limit. On the other hand, it is not obvious whether they survive in the final result in presence of the infinite-dimensional HS symmetries.

### 3 HS equations in $AdS_4$

#### 3.1 Original nonlinear system

In this section we recall the formulation of the $AdS_4$ HS field equations with the emphasis on their properties relevant to the construction of invariant functionals.

HS dynamics was formulated in [35] in terms of zero-form $B(Z; Y; \mathcal{K}|x)$, space-time connection one-form $W(Z; Y; \mathcal{K}|x)$ and one-form connection $S(Z; Y; \mathcal{K}|x)$ in the $Z$-space. $W(Z; Y; \mathcal{K}|x)$ and $S(Z; Y; \mathcal{K}|x)$ can be combined into the total connection one-form

$$\mathcal{W} = \theta x + \theta A W_\Omega(Z; Y; \mathcal{K}|x) + \theta A S_A(Z; Y; \mathcal{K}|x), \quad d x = \theta x \frac{\partial}{\partial x^\Omega}, \quad (3.1)$$
where all differentials \( dZ^A \) and \( dx^\underline{a} \), denoted in the sequel \( \theta^A \) and \( \theta^\underline{a} \), respectively, are anticommuting. \( A = 1, \ldots, 4 \) and \( a = 0, \ldots, 3 \) are indices of 4d Majorana spinors and vectors, respectively. \( Z^A \) and \( Y^A \) are commuting spinorial variables. Every Majorana spinor can be represented as a pair of two-component spinors with \( A = (\alpha, \bar{\alpha}) \), e.g., \( \theta^A = (\theta^\alpha, \bar{\theta}^\bar{\alpha}) \). \( K = (k, \bar{k}) \) denotes a pair of Klein operators that reflect two-component spinor indices as

\[
\begin{align*}
k \ast w^\alpha &= -w^\alpha \ast k, & k \ast \bar{\omega}^\bar{\alpha} &= \bar{\omega}^\bar{\alpha} \ast k, & \bar{k} \ast w^\alpha &= w^\alpha \ast \bar{k}, & \bar{k} \ast \bar{\omega}^\bar{\alpha} &= -\bar{\omega}^\bar{\alpha} \ast \bar{k},
\end{align*}
\]

with \( w^\alpha = (y^\alpha, z^\alpha, \theta^\alpha) \), \( \bar{\omega}^\bar{\alpha} = (\bar{y}^{\bar{\alpha}}, \bar{z}^{\bar{\alpha}}, \bar{\theta}^{\bar{\alpha}}) \). Note that relations (3.2) provide the definition of the star product with \( k \) and \( \bar{k} \).

The nonlinear HS equations of [35] are

\[
W \ast W = -i(\theta_A \theta^A + \delta^2(\theta_z) F_s(B) \ast k \ast v + \delta^2(\bar{\theta}_z) \bar{F}_s(B) \ast \bar{k} \ast \bar{v}),
\]

\[
W \ast B = B \ast W,
\]

where

\[
\delta^2(\theta_z) = \frac{1}{2} \theta_\alpha \theta^{\bar{\alpha}}, \quad \delta^2(\bar{\theta}_z) = \frac{1}{2} \bar{\theta}_{\bar{\alpha}} \bar{\theta}^{\alpha}
\]

and \( F_s(B) \) is some star-product function of the zero-form \( B \)

\[
F_s(B) = \sum_{n=1}^{\infty} f_n \frac{B \ast B \ast \ldots \ast B}{n}.
\]

The simplest case of linear \( F_s(B) \)

\[
F_s(B) = \eta B, \quad \bar{F}_s(B) = \bar{\eta} B,
\]

where \( \eta = \exp[i \varphi] \), \( \varphi \in [0, \pi] \) (the absolute value of \( \eta \) can be absorbed into \( B \)) leads to a class of pairwise nonequivalent nonlinear HS theories. The cases of \( \eta = 1 \) and \( \eta = \exp \frac{i \pi}{2} \) are particularly interesting, corresponding to the so called \( A \) and \( B \) models that respect parity [17]. As argued in Section 3.2 nonlinear terms in \( F_s(B) \) do not correspond to any conformal theory, i.e., they have to be set to zero in HS models compatible with unbroken conformal symmetry at the boundary.

The associative HS star product \( \ast \) acts on functions of two spinor variables \( Z_A \) and \( Y_A \)

\[
(f \ast g)(Z; Y) = \frac{1}{(2\pi)^4} \int d^4U d^4V \exp[iU^A V^B C_{AB}] f(Z + U; Y + U) g(Z - V; Y + V),
\]

where \( C_{AB} = (\epsilon_{\alpha \beta}, \bar{\epsilon}_{\bar{\alpha} \bar{\beta}}) \) is the 4d charge conjugation matrix allowing to raise and lower indices

\[
Y^A = C^{AB} Y_B, \quad Y_A = Y^B C_{BA},
\]

and \( U^A, V^B \) are real integration variables. It is normalized so that 1 is the unit element, i.e., \( f \ast 1 = 1 \ast f = f \). Star product (3.8) yields a particular realization of the Weyl algebra

\[
[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2i C_{AB}, \quad [Y_A, Z_B]_* = 0, \quad [a, b]_* = a \ast b - b \ast a
\]
and possesses a supertrace operation
\[
\text{str}(f(Z,Y)) = \frac{1}{(2\pi)^4} \int d^4U d^4V \exp [-iU^A V^B C_{AB}] f(U;V)
\] (3.11)

obeying the cyclic property
\[
\text{str}(f * g) = \text{str}(g * f)
\] (3.12)

provided that, in accordance with the normal spin-statistic relation, the coefficients of the expansions of \(f(Z,Y)\) in powers of spinor variables \(Z\) and \(Y\) are (anti)commuting for \(f(Z,Y)\) (odd) even with respect to \(f(-Z,-Y) = (-1)^{\text{odd}} f(Z,Y)\) (and similarly for \(g(Z,Y)\)).

Star product (3.8) admits the inner Klein operator
\[
\Upsilon = \exp iZ_{A} Y^{A},
\] (3.13)

which obeys
\[
\Upsilon * \Upsilon = 1,
\] (3.14)
\[
\Upsilon * f(Z;Y) = f(-Z; -Y) * \Upsilon.
\] (3.15)

The left and right inner Klein operators
\[
v = \exp i \bar{z}_{\alpha} \bar{y}^{\alpha}, \quad \bar{v} = \exp i \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}},
\] (3.16)

which enter Eq. (3.3), act analogously on undotted and dotted spinors, respectively,
\[
v * f(z, \bar{z}; y, \bar{y}) = f(-z, \bar{z}; -y, \bar{y}) * v, \quad \bar{v} * f(z, \bar{z}; y, \bar{y}) = f(z, -\bar{z}; y, -\bar{y}) * \bar{v},
\] (3.17)
\[
v * v = \bar{v} * \bar{v} = 1, \quad v * \bar{v} = \bar{v} * v.
\] (3.18)

From (3.11) and (3.13) it follows that the supertrace of the inner Klein operators \(v\) and \(\bar{v}\) diverges as \(\delta^4(0)\) (for more detail see [50]). Hence, one has to be careful with the expressions defined as \(\text{str}(f)\) for \(f\) containing exponentials behaving like \(v\) and \(\bar{v}\). This fact is of key importance for the further analysis since, as explained in Section 4, nontrivial invariant functionals should have divergent supertrace. From this perspective our approach is opposite to the construction of invariants in [72, 14, 54, 22] where divergent supertraces were somehow regularized. (Consistency of such a regularization is not quite obvious to us since the star-product algebra admits a uniquely defined supertrace.)

Naively, field equations (3.3) and (3.4) leave no room for a nontrivial invariant action written as a space-time differential form built from \(W\) and \(B\). Indeed, since all space-time curvature tensors \(W \ast W\) are zero by virtue of the field equations as well as the star-commutator \([W, B]_s\), \(p\)-form Lagrangians with \(p > 1\) like, e.g., \(\text{str}(W \ast f(B) \ast W \ast g(B))\) are zero. One-form functionals \(\text{str}(W \ast f(B))\) are not gauge invariant. The zero-forms \(\text{str}(f(B))\) at a point \(x = x_0\), are the invariants considered in [72, 22]. These however are not well-defined due to divergencies of the supertrace. The trick explained in Section 4 is to consider Lagrangians which, not being of the form \(\text{str}(L)\), would be exact if the trace operation was well defined but become nontrivial just because the respective trace is divergent.
As shown in [50], perturbative analysis of the HS field equations leads to solutions valued in the HS field algebra $\mathcal{H}$. General elements of $\mathcal{H}$ have divergent supertrace. On the other hand, $\mathcal{H}$ contains a subalgebra $\mathcal{H}_{0}^{\text{loc}}$ all elements of which have finite supertrace. Therefore, to be nontrivial, $\mathcal{L}$ should be projected from $\mathcal{H}/\mathcal{H}_{0}^{\text{loc}}$. This observation will give us hints in Section 5 on the structure of the system generating the Lagrangian forms.

### 3.2 Fock behavior at infinity

In [19], it was shown that the dependence of the HS zero-forms on the Poincaré coordinate $z$ of $AdS_4$ is

$$C(y, \bar{y}|x, z) = z \exp(y_{\alpha}\bar{y}^{\alpha})T(w, \bar{w}; K|x, z),$$

where $x$ and $z$ are, respectively, the boundary and Poincaré coordinates,

$$w^{\alpha} = z^{1/2}y^{\alpha}, \quad \bar{w}^{\alpha} = z^{1/2}\bar{y}^{\alpha},$$

and $T(w, \bar{w}|x, z)$ is holomorphic in $z$. The exponential factor in (3.19) leads to the non-poly-nomiality of the star-product element in the boundary limit of the HS theory. This can lead to infinities in the local conformal limit $z \to 0$ as we discuss now.

As observed in [19], the exponential

$$F = 4 \exp y_{\alpha}\bar{y}^{\alpha}$$

provides the star-product realization of the Fock vacuum that satisfies

$$y_{\alpha}^{+}F = F y_{\alpha}^{-} = 0,$$

where

$$y_{\alpha}^{+} = \frac{1}{2}(y_{\alpha} - iy_{\alpha}), \quad y_{\alpha}^{-} = \frac{1}{2}(\bar{y}_{\alpha} - iy_{\alpha})$$

obey

$$[y_{\alpha}^{-}, y_{\beta}^{+}]_{*} = \delta_{\alpha}^{\beta}, \quad [y_{\alpha}^{+}, y_{\beta}^{-}]_{*} = 0, \quad [y_{\alpha}^{+}, y_{\beta}^{+}]_{*} = 0.$$ (3.24)

$F$ is a projector, i.e.,

$$F * F = F.$$ (3.25)

The Klein operators $k$ and $\bar{k}$ exchange $y^{+}$ and $y^{-}$

$$ky_{\alpha}^{\pm} = \mp iy_{\alpha}^{\pm} k, \quad \bar{k}y_{\alpha}^{\mp} = \pm iy_{\alpha}^{\mp} \bar{k}.$$ (3.26)

This implies that

$$\bar{F} = kFk = \bar{k}F\bar{k} = 4 \exp -y_{\alpha}\bar{y}^{\alpha}$$

obeys

$$y_{\alpha}^{+} * \bar{F} = \bar{F} * y_{\alpha}^{-} = 0.$$

$F$ is also a projector,

$$\bar{F} * \bar{F} = \bar{F}.$$ (3.29)
However, the star product of $F$ with $\bar{F}$ is ill defined, being infinite

$$F \ast \bar{F} = \infty.$$  (3.30)

This fact is insensitive to the particular form of the star product. Indeed, the relation

$$F \ast \bar{F} = \frac{1}{4} F \ast [y_\alpha^-, y_\alpha^+] \ast \bar{F} = 0 \ast \bar{F}$$  (3.31)

demands $F \ast \bar{F}$ be either zero or infinity. It is infinity for bosonic oscillators and zero for fermionic (if introduced).

Let $F(t)$ interpolate between $F$ and $\bar{F}$

$$F(t) = 4 \exp ty_\alpha\bar{y}_\alpha^0,$$  \hspace{0.5cm} $F(1) = F,$ \hspace{0.5cm} $F(-1) = \bar{F}.$  (3.32)

In accordance with [54], direct computation gives

$$F(t) \ast F(t') = \frac{4}{(1 + tt')^2} \exp \left[ \frac{t + t'}{1 + tt'}y_\alpha\bar{y}_\alpha^0 \right].$$  (3.33)

Eqs. (3.30), (3.25) and (3.29) are particular cases of this formula.

Note that formula (3.33) was used in [54] for the analysis of the HS BH solutions which turns out to be analogous to the AdS/CFT problem being based [53] on the Fock vacuum analogous to $F$ (3.21). This analogy is very intriguing and suggestive.

HS equations (3.3) contain an arbitrary star-product function $F_\ast(B)$ (3.6) introduced in [35]. In the linearized approximation, the physical component of the field $B$ is given by (3.19) with $T(w, \bar{w}; \tilde{\mathcal{K}}|x, z)$ proportional to either $k$ or $\bar{k}$. This implies that the product $B \ast B$ contains

$$F \ast k \ast F = F \ast \bar{F}k = \infty$$  (3.34)

and similarly in the $\bar{k}$ sector. As a result, such terms make no sense in the conformal limit where the behavior (3.19) is imperative. Hence, to allow a consistent interpretation of the boundary limit compatible with conformal symmetry the non-linear terms in the function $F_\ast(B)$ in (5.2) should not be permitted. Let us stress that this does not necessarily mean that a HS theory with nonlinear $F_\ast(B)$ makes no sense in the bulk. An alternative interpretation might be that it undergoes a phase transition with broken boundary conformal symmetry.

The terms linear in $C$ are free of this difficulty. A less trivial question is whether the terms linear in $\mathcal{B}$ on the r.h.s. of the nonlinear equations give rise to terms that remain meaningful in the boundary limit in the higher orders of the perturbative expansion. Analysis of this question requires systematic investigation in spirit of [50] which has not been yet completed. Instead, we give here a simple indication that this is likely to be true.

\footnote{The authors of [54] used such a regularization of their computation scheme that $F \ast \bar{F} = 0$. We do not quite see however how the whole setting should be redefined to make this compatible with the associativity of a sufficiently rich class of functions appropriate for the description of fluctuations.}
From formulae (5.29) and (3.19) it follows that the star product of the first-order contributions involves the star product of the exponentials of the following type

\[
\int_0^1 \rho_1(t_1) dt_1 \int_0^1 dt_2 \rho_2(t_2) \exp i[t_1 z_\alpha (y^\alpha + i \bar{y}^{\bar{\alpha}})] \ast \exp i[-t_2 z_\alpha (y^\alpha + i \bar{y}^{\bar{\alpha}})].
\]  

(3.35)

Evaluation of the star product yields the following integration measure in \( t_1, t_2 \)

\[
\int_0^1 \rho_1(t_1) dt_1 \int_0^1 \rho_2(t_2) dt_2 \frac{(1 - t_1)(1 - t_2)}{(1 - t_1 t_2)^2}
\]  

(3.36)

which converges since

\[
\frac{1 - t_1}{1 - t_1 t_2} \leq 1, \quad \frac{1 - t_2}{1 - t_1 t_2} \leq 1, \quad t_1, t_2 \in [0, 1].
\]  

(3.37)

Since analogous terms in \( B \ast B \) just diverge, the above check indicates that the contribution of the terms linear in \( B \) on the r.h.s. of (3.3) should make sense in the higher orders. On the other hand, if survive, divergencies of this type may be related to the field-theoretic divergencies of the boundary conformal theory which is also one of the interesting questions for the future study.

4 Lagrangian extension

Known HS equations like (3.3), (3.4) are particular examples of the following system

\[
\mathcal{W} \ast \mathcal{W} = \mathcal{F}_L(c, \mathcal{B}, \mathcal{L}), \quad \mathcal{W} \ast \mathcal{B} = \mathcal{B} \ast \mathcal{W}, \quad d_x \mathcal{L} = 0,
\]

(4.1)

where \( \mathcal{W} \) and \( \mathcal{B} \) describe, respectively, forms of odd and even degrees both in the space-time differentials \( \theta_x \) and in the twistor-like differentials \( \theta_Z \), whatever they are. More precisely \( \mathcal{W} = d_x + \mathcal{W}' \) contains the space-time de Rham derivative \( d_x \) so that the covariant derivative is equivalent to the commutator with \( \mathcal{W} \).

Known HS theories are either directly based on some associative algebra \( A \), where all fields are valued, or are their reductions. This allows an extension of the system with fields valued in the tensor product of the original associative HS algebra \( A \) with any internal associative algebra \( A_{int} \) bringing Chan-Paton-like indices carried by all fields. Note that supersymmetric HS theories result from this construction with \( A_{int} \) being a Clifford algebra \( \mathcal{C}_3, \mathcal{C}_4 \) (see \( \mathcal{C}_3, \mathcal{C}_5 \) for more detail and \( \mathcal{C}_6 \) for recent applications). In the sequel, the internal indices will be implicit with the convention that central elements of HS algebras are valued in the center of \( A_{int} \), that is in the unit matrix of \( A_{int} = Mat_n(\mathbb{C}) \).

We introduce the following conventions. The \( * \) in \( (1.1) \) is the product in \( A \). In the \( AdS_4 \) HS system, \( A \) is the star-product algebra \( \mathcal{F}_A \) of functions \( f(Z; Y; K) \). The tensor product of \( A \) with the wedge algebra of differentials \( \theta_x, \theta_Z \) will be denoted \( \Lambda A \). Analogously, \( \Lambda_x A \) and \( \Lambda_Z A \) denote the tensor products of \( A \) with the wedge algebra of differentials \( \theta_x \) and \( \theta_Z \).
respectively. $C$, $\Lambda C$, $\Lambda Z C$ and $\Lambda x C$ are, respectively, the centers of $A$, $\Lambda A$, $\Lambda Z A$ and $\Lambda x A$. Fields of the theory are $x$-dependent differential forms valued in $\Lambda A$ (i.e., sections of the respective fiber bundles).

c denotes various $x$-independent elements of $\Lambda Z C$. Central elements $c$ which appear in the original $AdS_4$ HS theory are $I$, $\theta A \theta A$, $\delta 2(\theta)k * v$ and $\delta 2(\overline{\theta})\overline{k} * \overline{v}$. They play different rōles in HS theory. For instance, $\theta A \theta A$ does not belong to the field HS algebra $H$ where it was shown that it cannot appear anywhere in the nonlinear field equations except for the first term on the r.h.s. of (3.3) if solutions of the HS equations are demanded to be minimally nonlocal, belonging to $H$. This is related to the fact that the central element $-i\theta A \theta A$ is the square of the operator

$$Q = \theta A Z_A$$  \hspace{1cm} (4.2)

which determines the $Z$-dependence of the HS fields via perturbative solution of system (3.3), (3.4) (for more detail see Section 5.3) and which also does not belong to $H$. Hence, in what follows we demand that the central element that affect interactions should belong to $\Lambda C H \subset \Lambda Z H$.

New ingredients of the construction are Lagrangians $L$ which are space-time differential forms valued in $\Lambda x C$. In the $AdS_4$ HS theory considered in Section 5 central elements $c_0 \in C H$ that belong to the $Q$-cohomology in $H$ for $Q$ (4.2) play a distinguished role. In the twistorial HS theories considered in this paper the $Q$-cohomology is represented by the zero-forms in the $\theta Z$-space, i.e., by the unit element of the star-product algebra. In this case $L = L(\theta x | x)$ is a differential form that depends only on the space-time coordinates $x$ and differentials $\theta x$.

System (4.1) is consistent with respect to further commutators with $W$ (i.e., covariant differentiation) because $B$ commutes with itself, as well as with all $c$ and $L$. It is invariant under the following gauge transformations with three types of gauge parameters $\varepsilon$, $\xi$ and $\chi$:

$$\delta W = [W, \varepsilon]_s + \xi^N \frac{\partial F \xi(c, B, L)}{\partial B^N} + \chi_i \frac{\partial F L(c, B, L)}{\partial L_i}$$  \hspace{1cm} (4.3)

$$\delta B = \{W, \xi\}_s + [B, \varepsilon]_s$$  \hspace{1cm} (4.4)

$$\delta L_i = d\chi_i$$  \hspace{1cm} (4.5)

where $N$ is the multiindex running over all components of $B$, the gauge parameters $\varepsilon$ and $\xi$ are arbitrary functions of coordinates, differentials and the generating elements of the star-product algebra while $\chi$ are functions only of the space-time coordinates and differentials. For instance, in the $AdS_4$ HS theory

$$\varepsilon = \varepsilon(\theta; Z; Y; K | x), \quad \xi = \xi(\theta; Z; Y; K | x), \quad \chi = \chi(\theta | x).$$  \hspace{1cm} (4.6)

Transformations (4.3) are usual HS gauge transformations extended to higher differential forms. Transformations (4.4) are their analogues for higher-form components of $B$. (Such transformations were considered in [13].) Gauge transformation (4.5) implies equivalence of the Lagrangians modulo exact forms. All three types of gauge transformations provide a
realization of gauge transformations (2.6) associated with various differential forms in the system.

The most important case of appearance of the Lagrangian form(s) $L$ is additive with

$$\mathcal{F}_L(c, B, \mathcal{L}) = \mathcal{F}(c, B) + \mathcal{L} \cdot c_0^i,$$

(4.7)

where $c_0^i \in C_H$. For the models considered in this paper the only option is $c_0 = I$.

In the additive case, $\mathcal{L}$ is expressed by the first equation in (4.1) in terms of the other fields, which is just appropriate for a Lagrangian. In this case $d_x \mathcal{L} = 0$ is not an independent condition but rather a consequence of the other equations in system (4.1). Note that this may not be true for those $c^i \in \Lambda H$ that do not belong to $Q$ cohomology because, containing a product with some nonzero power of $\theta_Z$, in this case the compatibility condition for system (4.3) restricts the derivative of $\mathcal{L}$ modulo terms that give zero upon multiplication by $c^i$.

The gauge transformation of the $(p_{\mathcal{L}_i} - 1)$-forms $\mathcal{W}_{\mathcal{L}_i}$ valued in the same central elements $c_0^i$ is

$$\delta \mathcal{W}_{\mathcal{L}_i} = \pi_i \left( [\mathcal{W}, \varepsilon]_s + \xi^N \frac{\partial \mathcal{F}(c, B)}{\partial B^N}\right) + \chi_i,$$

(4.8)

where $\pi_i$ is the projection to the central element $c_0^i$. For example, in the usual HS algebra, the projection to the unit element $I$ realized by $f(Z, Y) = 1$ is

$$\pi_I(f(Y, Z|x)) = f(0, 0|x).$$

(4.9)

The projection $\pi_I$ does not respect the trace property, i.e.,

$$\pi_I(f * g) \neq \pi_I(g * f).$$

(4.10)

Nevertheless, belonging to the center of the algebra, $\mathcal{L}_i$ turns out to be closed and gauge invariant modulo exact forms by virtue of the other field equations. Indeed, the $\chi$-dependent part of the gauge transformation (4.8) makes it possible to impose the canonical gauge

$$\mathcal{W}_{\mathcal{L}_i} = 0.$$  

(4.11)

To preserve this gauge under the action of the other gauge transformations the gauge parameter $\chi$ has to be adjusted as follows

$$\chi_i = -\pi_{c_i} \left( [\mathcal{W}, \varepsilon]_s + \xi^N \frac{\partial \mathcal{F}(c, B)}{\partial B^N}\right).$$

(4.12)

Eq. (4.3) with $\chi_i$ (4.12) gives the transformation law of the Lagrangian form $\mathcal{L}_i$ in the canonical gauge. Let us stress that for the additive Lagrangian systems the gauge fields $\mathcal{W}_{\mathcal{L}_i}$ from the center of the algebra $H$ become dynamically trivial, i.e., pure gauge.

Suppose now that the HS algebra $A$ possesses a supertrace obeying the cyclic property (3.12). Let $c_i^*$ be the central elements dual to $c^i$ in the sense that

$$\text{str}(c^i * c_j^*) = \delta^i_j.$$  

(4.13)
For instance, for $c = I$ it is convenient to normalize the supertrace so that $\text{str}(I) = 1$ and $I^* = I$. Setting the differentials $\theta_Z$ in $F(c, B)$ to zero yields

$$L_i = \text{str} \left( c_i^* \left( (d_x W' + W' \star W') \right) \right) \bigg|_{\theta_Z = 0}. \quad (4.14)$$

By the cyclic property (3.12), from (4.14) it seemingly follows that the second term does not contribute and hence $L_i$ is exact. There are two subtleties however.

Firstly, it may be wrong to conclude that the first term on the r.h.s. of (4.14) is exact because $W'$ has to be determined from the differential equation containing $d_x$ (for more detail on this point see Section 5.5). However, this term is zero in the canonical gauge.

Secondly, formula (4.14) is ill defined if $\text{str}(c_i^* \left( W' \star W' \right))$ is divergent. In other words, as explained in [50], the actual class of functions valued in $\mathcal{H}$ that appear in the analysis is wider than those admitting the supertrace which form the algebra $\mathcal{H}_0^{\text{loc}}$ [50].

Alternatively, it can happen that some of central elements $c$ admit no $c^*$ obeying (4.13). For instance, $c^*$ does not exist if $\text{str}(I) = 0$ which case is known to play a role in the maximally supersymmetric $N = 4$ SYM. This gives a criterion distinguishing between trivial (exact) and non-trivial Lagrangians: those $\mathcal{L}_i$ for which $\text{str}(c_i^* \left( W' \star W' \right))$ is well defined are exact while those for which it is ill defined either being divergent or because $c_i^*$ obeying (4.13) does not exist have a chance to be nontrivial.

The Lagrangians $\mathcal{L}_i$ can represent space-time differential forms of different degrees, thus providing invariants to be integrated over cycles of different dimensions. Whether the resulting invariants are non-zero or trivial depends on a particular solution of the theory. Since the forms $\mathcal{L}_i$ are closed, the result of their integration can be non-zero only for non-contractible cycles, i.e., for singular solutions. As explained in Introduction, in the case of $AdS/CFT$ the singularity is at infinity and $\mathcal{L}$ is a four-form in the complexified $AdS_4$ case. For $AdS_4$ BH solutions of [53], [54], [55] the corresponding invariants are two-forms $\mathcal{L}^2$. As argued in Section 5.3, these are anticipated to reproduce the BH charges in the gauge invariant way with the invariant functional $S$ saturated due to the BH singularity.

The construction of this section exhibits an essential difference between Lagrangian forms of even and odd degrees. The reason is that $\mathcal{L}$ appears on the r.h.s. of the first of equations (4.1) whose l.h.s. is the square of odd forms $\mathcal{W}$.

In the sequel of this section we consider a more general construction which can also lead to Lagrangian forms of odd degrees. This end system (4.1) can be modified to

$$\mathcal{W} \star \mathcal{W} = \mathcal{F}_L(c, \mathcal{B}, \mathcal{L}), \quad (4.15)$$

$$\mathcal{W} \star \mathcal{B} - \mathcal{B} \star \mathcal{W} = \mathcal{G}_L(c, \mathcal{B}, \mathcal{L}), \quad (4.16)$$

$$d_x \mathcal{L} = 0. \quad (4.17)$$

Compatibility of (4.15), (4.16) demands

$$\mathcal{W} \star \mathcal{F}_L(c, \mathcal{B}, \mathcal{L}) - \mathcal{F}_L(c, \mathcal{B}, \mathcal{L}) \star \mathcal{W} = 0, \quad \mathcal{W} \star \mathcal{G}_L(c, \mathcal{B}, \mathcal{L}) + \mathcal{G}_L(c, \mathcal{B}, \mathcal{L}) \star \mathcal{W} = 0 \quad (4.18)$$
and, hence,
\[ \mathcal{G}_L^N (c, B, \mathcal{L}) \frac{\partial \mathcal{F}_L(c, B, \mathcal{L})}{\partial B^N} = 0, \quad \mathcal{G}_L^N (c, B, \mathcal{L}) \frac{\partial G_M^N(c, B, \mathcal{L})}{\partial G_M^N} = 0. \] (4.19)

\( \mathcal{F}_L \) and \( \mathcal{G}_L \) are even and odd differential forms, respectively. In the case of \( \mathcal{G}_L = 0 \) we recover system (4.1) free of any restrictions on \( \mathcal{F}_L \).

System (4.15)-(4.17) is invariant under the gauge transformations (4.3) and (4.5) in the sector of \( \mathcal{W} \) and \( \mathcal{L} \) while the transformation law for \( B \) modifies to
\[ \delta B = \{ \mathcal{W}, \xi \} + [B, \varepsilon] + \xi N \frac{\partial \mathcal{G}_L(c, B, \mathcal{L})}{\partial B^N} + \chi_i \frac{\partial \mathcal{G}_L(c, B, \mathcal{L})}{\partial \mathcal{L}_i}. \] (4.20)

Main features of the analysis of the gauge transformations remain the same as at \( \mathcal{G} = 0 \). The novelty is that the components of the fields \( B_L \) associated with the Lagrangians \( \mathcal{L} \) in \( \mathcal{G}_L \) on the r.h.s. of (4.16) become pure gauge.

Conditions (4.19) have the following interesting interpretation. The second condition implies that the odd vector field
\[ Q := \mathcal{G}_L^N \frac{\partial}{\partial B^N} \] (4.21)
is nilpotent
\[ Q^2 = 0. \] (4.22)
The first implies that \( \mathcal{F}_L \) must be \( Q \)-closed
\[ Q \mathcal{F}_L = 0. \] (4.23)

\( Q \)-exact \( \mathcal{F}_L \)
\[ \mathcal{F}_L(c, B, \mathcal{L}) = Q f(c, B, \mathcal{L}) \] (4.24)
are dynamically trivial since they can be removed by a field redefinition of \( \mathcal{W} \) which in the infinitesimal case is
\[ \delta \mathcal{W} = f(c, B, \mathcal{L}). \] (4.25)

Thus, general HS system (4.15)-(4.17) is characterized by a nilpotent vector field \( Q \) (4.21) and some its cohomology \( \mathcal{F}_L \). Similarity of this construction with the description of unfolded systems in Section 2 is obvious. Note however that systems with \( Q = 0 \) are nontrivial and, in fact, most interesting while unfolded equations (2.8) with \( Q = 0 \) are trivial.

The general case with \( \mathcal{G}_L \neq 0 \) may also have applications. Let us note however that, to fulfill conditions (4.19), \( \mathcal{G}_L \) should contain such a combination of the \( \theta_Z \) differentials that it would give zero upon multiplication with the \( B \)-derivatives of \( \mathcal{F}_L(c, B, \mathcal{L}) \) and \( \mathcal{G}_L(c, B, \mathcal{L}) \) in (4.19). This is impossible for the Lagrangians associated with central elements \( c_0^Z \in H_0 \) which have zero degree in the differentials \( \theta_Z \). Indeed, for the twistorial 3d and 4d models considered in this paper, where the HS field algebra \( \mathcal{H} \) is known [31], we were not able to construct invariant Lagrangians of odd degrees. However, for more general models like vectorial HS models of [77] a proper generalization of the construction of [30] of the HS field
algebra remains unknown. For this case not only the structure of the HS field algebra can be changed but also the structure of cohomology of the respective generalization of $Q$. If cohomology $H^p(Q)$ with $p > 0$ has nonzero components in the center of the HS algebra, the construction of this section can lead to nontrivial invariants. It would be interesting to apply it to invariants of the vectorial HS models of \[77\].

5 Invariants of $AdS_4$ HS theory

5.1 Extended system

To construct invariant functionals of the $AdS_4$ HS theory we extend the system \((3.3), (3.4)\) as follows. $W(\theta; Z; Y; \mathcal{K}|x)$ is extended to all odd forms while $B(Z; Y; \mathcal{K}|x)$ is extended to all even forms $B(\theta; Z; Y; \mathcal{K}|x)$, i.e., $W(\theta; Z; Y; \mathcal{K}|x)$ is a polynomial of $\theta_e$ and $\theta_Z$ of total degrees $1, 3, 5, \ldots$, while $B(\theta; Z; Y; \mathcal{K}|x)$ is a polynomial of $\theta_e$ and $\theta_Z$ of total degrees $0, 2, \ldots$. (Such an extension was considered, e.g., in \[78\].) Also we introduce the Lagrangian forms of even degrees

$$L(\theta_x|x) = L^2(\theta_x|x) + L^4(\theta_x|x) + \ldots, \quad (5.1)$$

that only depend on the space-time coordinates and differentials.

The extended HS system has the form \((1.1)\) with

$$F_{\mathcal{L}}(c, \mathcal{B}, \mathcal{L}) = -i \left( \theta_A \theta^A + \delta^2(\theta_Z) F_* (\mathcal{B}) \star k \star v + \delta^2(\bar{\theta}_Z) \bar{F}_* (\mathcal{B}) \star \bar{k} \star \bar{v} \right) + \delta^4(\theta_Z) G_* (\mathcal{B}) \star k \star \bar{k} \star v \star \bar{v} + L(\theta_x|x), \quad (5.2)$$

$$G_* (\mathcal{B}) = g + g_1 \mathcal{B} + g_2 \mathcal{B} \star \mathcal{B} \ldots. \quad (5.3)$$

The simplest case of $G_* (\mathcal{B}) = g = const$ is nontrivial and, in fact, most interesting.

The extended system is chosen in this form because the additional terms should belong to the HS field algebra $\mathcal{H}$ introduced in \[50\] where it was shown that the central elements $\delta^2(\theta_Z) \star k \star v$ and $\delta^2(\bar{\theta}_Z) \star \bar{k} \star \bar{v}$ do not belong to $\mathcal{H}$ while $\delta^2(\theta_z)$, $\delta^2(\bar{\theta}_z)$ and $\delta^4(\theta_Z)$ do not. This means that, surprisingly, being in a certain sense singular, the latter operators are not allowed to appear in the HS system with the only exception for the first term in \((5.2)\) compensating the “singularity” of $Q$ \((1.2)\) yielding an exterior derivation in $\mathcal{H}$ \[70\].

Thus all $\mathcal{B}$-dependent terms in \((5.2)\) belong to $\mathcal{H}$. The perturbative analysis of Section \[5.3\] shows that the $g$-dependent four-form in the twistor space induces Lagrangians forms of degrees four or higher. According to the analysis of Section \[8\] the presence of the Klein operators in the $G$-term of \((5.2)\) gives rise to divergent traces and, hence, nontrivial Lagrangians.

$L^4$ is anticipated to give rise to the generating functional of correlators in the $AdS_4/CFT_3$ HS holography. Since the expression for $L^4$ in terms of dynamical fields turns out to be proportional to $g$, the latter acquires the meaning of the (inverse) coupling constant in front of the Lagrangian $(N$ within the $1/N$ expansion) also containing the inverse Planck constant $\hbar^{-1}$ in the generating functional for boundary correlators. The absence of such a constant in original system \((3.3), (3.4)\) complicated its holographic interpretation. Extended system \((1.1), (5.2)\) contains the missed elements appropriate for the description of the quantum
regime of the boundary theory. Note that, to account higher quantum corrections, it may be necessary to consider higher-order differential forms in $W, B$ and $L$ contributing to higher-order corrections in $g$ via terms integrated over Cartesian products of the original space-time.

As explained in more detail in Section 5.5, the two-form Lagrangian $L^2(\theta_x|x)$ supports the BH charges in the 4d HS theory. This should be saturated by nontrivial BH-like solutions \[53, 54, 55\] of the original HS system \((3.3), (3.4)\) with the BH mass being a counterpart of $g$ via the contribution of the BH solution to the r.h.s. of \((3.3)\).

5.2 Vacuum solution

As usual we consider a vacuum solution with $B = 0$. In the one-form sector it has the form

$$W_0 = d_x + W_{0,0}^{1,0} + W_{0,1}^{0,1}, \quad W_{0,0}^{1,0} = Q, \quad W_{0,1}^{0,1} = W_0(Y|x),$$

where the space-time one-form $W_0(Y|x)$ (the differentials $\theta_x$ are implicit) is some solution of the flatness equation

$$d_x W_0(Y|x) + W_0(Y|x) \ast W_0(Y|x) = 0.$$  \hspace{1cm} (5.5)

For bilinear $W_0(Y|x)$

$$W_0(Y|x) = \frac{i}{2} W_0^{AB}(x)Y_A Y_B$$  \hspace{1cm} (5.6)

(5.5) implies that the components $W_0^{AB}(x)$ describe locally AdS$_4$ geometry provided that the frame one-form $e^\alpha_{\dot{\alpha}}(x) := W_0^{\alpha\dot{\alpha}}(x)$ is nondegenerate.

As a consequence of \((3.10)\), the star-commutator with $W_0^{1,0} = Q$ \((1.2)\) is proportional to the de Rham derivative in $Z^A$

$$Q \ast f(Z; Y) - (-1)^{deg_f} f(Z; Y) \ast Q = -2id_Z f(Z; Y), \quad d_Z = \theta^A \frac{\partial}{\partial Z^A}$$  \hspace{1cm} (5.7)

where $deg_f$ is the form degree of $f$. We use notation

$$\mathcal{W} = \sum_{p,q} \mathcal{W}^{p,q},$$ \hspace{1cm} (5.8)

where $\mathcal{W}^{p,q}$ is a $p$-form in the $Z$-differentials $\theta_Z$ and a $q$-form in the $x$-differentials $\theta_x$.

Clearly, Eq. \((5.4)\) gives a solution to \((5.2)\) at $g = 0$. For $g \neq 0$ it suffices to find the deformation of \((5.4)\) linear in $g$ since higher-order terms in $g$ contribute to forms of degrees six or higher irrelevant in this paper. To this end one can use the standard homotopy formula for the de Rham derivative which is easy to check by differentiation

$$d_Z f(\theta_Z; Z; Y) = g(\theta_Z; Z; Y) \quad \Rightarrow \quad f(\theta_Z; Z; Y) = \partial_z^* g + d_z^* \varepsilon + f(0; 0; Y),$$ \hspace{1cm} (5.9)

where

$$\partial_z^* g := d^*_Z H(g), \quad H(g) := \int_0^1 dt t^{-1} g(t\theta_Z; tZ; Y), \quad d_z^* = Z^A \frac{\partial}{\partial \theta_A}.$$ \hspace{1cm} (5.10)
The term $d\varepsilon$ in Eq. (5.9) describes the freedom in exact forms while $f(0;0;Y)$ represents the de Rham cohomology. Eq. (5.9) is valid provided that the homotopy integral over $t$ converges, which, in accordance with the Poincaré lemma, is true if $g(0;0;Y) = 0$. Note that

$$\partial^*_Z\partial^*_Z = 0$$

(5.11)

since

$$d^*_Z d^*_Z = 0.$$  

(5.12)

Equipped with these formulae it is straightforward to obtain the $(M-1)$-form components $W_{p,q}^{0,0}$ for the general case of $A = 1, \ldots, M$. The final result has the concise form

$$W_M^{M-1} = \frac{ig}{2} Z^A \frac{\partial}{\partial A} \int_0^1 d\tau \tau^{M-1} \exp \left[ i \tau Z_A Y^A + (1 - \tau) W_0^{AB}(x) Z_A \frac{\partial}{\partial \theta} \right] \delta^M(\theta) k\bar{k}.$$  

(5.13)

For $M = 4$ this yields

$$W_3^{3-q,q} = \frac{ig}{2} \sum_{q=0}^3 \int_0^1 d\tau \tau^q (1 - \tau)^q \exp \left[ i \tau Z_A Y^A \right] Z^B W_0^{A_1}(Z) \cdots W_0^{A_q}(Z) \delta_{A_q \cdots A_1}(\theta Z) k\bar{k}.$$  

(5.14)

where

$$W_B^B(Z|x) = W_0^{AB}(x) Z_A, \quad \delta_{A_q \cdots A_1}^M(\theta Z) = \frac{\partial}{\partial \theta A_1} \cdots \frac{\partial}{\partial \theta A_q} \delta^M(\theta Z).$$  

(5.15)

An important property of $W_0^{0,3}$, which has to be checked separately to make sure that it obeys (5.2) with $B = 0$, is that

$$dW_0^{0,3}(Z;Y;K|x) + W_0(Y|x) * W_0^{0,3}(Z;Y;K|x) + W_0^{0,3}(Z;Y;K|x) * W_0(Y|x) = 0.$$  

(5.16)

To see this it suffices to observe that the star-commutator of the l.h.s. of (5.14) with $W_0^{1,0} = \theta^A Z_A$ is zero as a consequence of the other vacuum equations which have been already resolved, leading to (5.14) via the repeated application of homotopy formula (5.9). On the other hand, the substitution of (5.14) into (5.16) gives terms that are zero at $Z = 0$. Hence the l.h.s. of (5.14) is zero for any $Z$. Straightforward verification of (5.14) involves a partial integration over $\tau$.

### 5.3 Sketch of the first order

Let

$$\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_1 + \ldots, \quad \mathcal{B} = \mathcal{B}_1 + \ldots,$$

(5.17)

where $\mathcal{W}_1$ and $\mathcal{B}_1$ are first-order fluctuations. Linearized equations (4.1) have the form

$$d\mathcal{W}_1 + \mathcal{W}_0 * \mathcal{W}_1 + \mathcal{W}_1 * \mathcal{W}_0 = -i \left( \eta \delta^2(\theta_z) B_1 * k * v + \bar{\eta} \delta^2(\bar{\theta}_z) B_1 * \bar{k} * \bar{v} \right),$$

(5.18)

$$d\mathcal{B}_1 + \mathcal{W}_0 * \mathcal{B}_1 - \mathcal{B}_1 * \mathcal{W}_0 = 0,$$

(5.19)

$$d = d_Z + d_x.$$
Since $\mathcal{W}_0$ contains $\mathcal{W}^{1,0}_0$ proportional to $dz$ \cite{3.7} these equations express all components of $\mathcal{W}_1$ that are not $dz$ closed via the other fields. $dz$-exact fields are pure gauge with respect to the gauge transformations \cite{1.3} and \cite{1.4}. Hence, the remaining physical fields, that are neither expressed via the other fields nor pure gauge with respect to the part of the gauge transformations containing $dz$, are in the $dz$-cohomology.

By Poincaré Lemma, these are fields independent of both $Z^A$ and $\theta^A$, i.e.,

$$C(\theta_z; Y; \mathcal{K}|x) := B_1(\theta; Z; Y; \mathcal{K}|x)\bigg|_{\theta_Z = Z = 0} \tag{5.20}$$

and

$$\omega(\theta_z; Y; \mathcal{K}|x) := \mathcal{W}_1(\theta; Z; Y; \mathcal{K}|x)\bigg|_{\theta_Z = Z = 0}. \tag{5.21}$$

$C(\theta_z; Y; \mathcal{K}|x)$ and $\omega(\theta_z; Y; \mathcal{K}|x)$ contain space-time forms of even and odd degrees, respectively,

$$C(\theta_z; Y; \mathcal{K}|x) = C^0(\theta_z; Y; \mathcal{K}|x) + C^2(\theta_z; Y; \mathcal{K}|x) + \ldots, \tag{5.22}$$

$$\omega(\theta_z; Y; \mathcal{K}|x) = \omega^1(\theta_z; Y; \mathcal{K}|x) + \omega^3(\theta_z; Y; \mathcal{K}|x) + \ldots. \tag{5.23}$$

$C^0(\theta_z; Y; \mathcal{K}|x)$ and $\omega^1(\theta_z; Y; \mathcal{K}|x)$ are the HS fields of the original system. $C^p(\theta_z; Y; \mathcal{K}|x)$ and $\omega^{p+1}(\theta_z; Y; \mathcal{K}|x)$ with $p \geq 2$ are new. Note that most of components of $C^p(\theta_z; Y; \mathcal{K}|x)$ are expressed via derivatives of $\omega^{p+1}(\theta_z; Y; \mathcal{K}|x)$ by \cite{3.18}.

The situation with Lagrangians is analogous: nontrivial Lagrangians should appear in combination with those central charges $c^i_0$ in \cite{4.7} that belong to the $Q$-cohomology. Indeed, being central, $c^i$ is $Q$-closed. If it is $Q$-exact, $c^i = [Q, \chi^i]_\pm$, in the lowest order, the term with the Lagrangian can be removed by the transformation

$$\mathcal{W}_1 \rightarrow \mathcal{W}'_1 = \mathcal{W}_1 + \chi^i \mathcal{L}_i, \quad \mathcal{L}_i \rightarrow 0 \tag{5.24}$$

which is a consequence of the following perturbative symmetry with the parameter $\alpha$

$$\mathcal{W}_1 \rightarrow \mathcal{W}'_1 = \mathcal{W}_1 + \alpha \chi^i \mathcal{L}_i, \quad \mathcal{L}_i \rightarrow \mathcal{L}'_i = (1 + \alpha) \mathcal{L}_i. \tag{5.25}$$

Hence, only central elements in the $Q$-cohomology $H(Q)$ generate nontrivial Lagrangians\cite{5}.

For the de Rham derivative $Q$ \cite{4.2} acting on the freely generated functions of $Z$ this implies by Poincaré lemma that nontrivial Lagrangians are associated with the unit element of the star-product algebra as in \cite{5.2}. On the other hand, the terms

$$\delta^2(\theta_z) * k * v \mathcal{T}(\theta_z|x) + \delta^2(\bar{\theta}_z) * \bar{k} * \bar{\mathcal{T}}(\theta_z|x) \tag{5.26}$$

with conjugate Lagrangians $\mathcal{T}$ and $\bar{\mathcal{T}}$, that can also be added to the r.h.s. of \cite{5.2} provided that differential forms of higher degrees among $\mathcal{W}$ and $\mathcal{B}$ are introduced, unlikely give rise to nontrivial Lagrangians. It would be instructive to understand the condition that $c^i$ should belong to $H(Q)$ in more general terms either defining the actions in terms of certain integrals over $Z$-variables to which $Q$-exact terms do not contribute or to trace the origin of symmetry

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\footnote{I am grateful to Nikita Misuna for the illuminating discussion of this point.}

24
According to the standard analysis of the HS field equations [35, 75], generating function for all gauge invariant degrees of freedom in the system. The fields [19].

The one-form sector of (5.19) gives
\[ B_1^0(Z; Y; \mathcal{K}|x) = C^0(Y; \mathcal{K}|x) \] (5.27)
and
\[ dC^0(Y; \mathcal{K}|x) + W_0(Y|x) * C^0(Y; \mathcal{K}|x) - C^0(Y; \mathcal{K}|x) * W_0(Y|x) = 0. \] (5.28)
According to the standard analysis of the HS field equations [35, 75], \( C^0(Y; \mathcal{K}|x) \) is the generating function for all gauge invariant degrees of freedom in the system. The fields \( C_{\alpha_1...\alpha_n}(x) \) considered in Introduction are components of \( C^0(Y; \mathcal{K}|x) \) in the conformal basis [19].

The two-form sector of (5.18) gives
\[ W_1^{0,0} = \frac{1}{2} \eta \int_0^1 dt e^{itz_\alpha y^\alpha} z^\alpha \tilde{\theta}^\alpha C(-tz, \tilde{y}; \mathcal{K})k + \frac{1}{2} \eta \int_0^1 d\tilde{t} \tilde{t} \exp[i\tilde{t}z_\alpha y^\alpha] z^\alpha \tilde{\theta}^\alpha C(y, -\tilde{t}z; \mathcal{K})k \] (5.29)
and
\[ W_1^{0,1} = -i \frac{1}{2} \partial z \{ W_0, W_1^{0,0} \}, \] (5.30)
where the term with \( d_x \) in \( D_0 \) does not contribute because of (5.11). Plugging (5.30) into the \( \theta^2 \) sector of (5.18) yields the so-called First On-Shell Theorem
\[ d_x \omega^1(Y; \mathcal{K}|x) + W_0(Y) * \omega^1(Y; \mathcal{K}|x) + \omega^1(Y; \mathcal{K}|x) * W_0(Y) = \]
\[ = \eta H^{\dot{\alpha}\dot{\beta}} \frac{\partial^2}{\partial \tilde{y}^\dot{\alpha} \partial \tilde{y}^\dot{\beta}} C^0(0, \tilde{y}; \mathcal{K}|x)k + \bar{\eta} H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C^0(y, 0; \mathcal{K}|x)k, \] (5.31)
where
\[ H^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon_{\alpha\beta} \quad \text{and} \quad H^{\alpha\beta} = \epsilon^{\alpha\dot{\alpha}} \epsilon_{\dot{\alpha}\beta}, \quad \epsilon^{\alpha\dot{\alpha}} := W_0^{\alpha\dot{\alpha}}, \] (5.32)
where \( \epsilon^{\alpha\dot{\alpha}} \) is the vierbein part of the vacuum connection \( W_0^{AB} \) (5.6). The First On-Shell Theorem imposes spin \( s > 1 \) equations on the frame-like connections contained in \( \omega^1(Y; \mathcal{K}|x) \). Eq. (5.28) contains the field equations for spins \( s \leq 1 \). In addition, Eqs. (5.18), (5.28) express infinitely many auxiliary fields via derivatives of the frame-like connections and matter fields [10, 35] (see also [73]).

To find \( \mathcal{W}_1 \), which eventually determines the Lagrangian terms in (5.2) one has to find \( \mathcal{B}_1 \). Reconstruction of these fields by the homotopy formula (5.9) is straightforward but lengthy. Leaving details for [48], here we would like to stress that the multiple application of the formula (5.9) to the products of the \( g \)-dependent part of the vacuum field (5.14) with the first-order HS fields \( \omega^1 \) and \( C^0 \) reconstructs the first-order contributions to \( \mathcal{B}_1 \) and \( \mathcal{W}_1 \). So, the contributions to the higher-form connections and, eventually, to the Lagrangians are induced by the \( g \)-dependent term in (5.2), (5.3). The bilinear part of \( \mathcal{L}^4 \)
\[ \pi_1(\{ \mathcal{W}_1, \mathcal{W}_1 \}_* + \{ \mathcal{W}_2, \mathcal{W}_0 \}_*) \] (5.33)
contains both $W_1$ and the second-order part $W_2$ of $W$ which needs another involved computation. By this procedure the quadratic part of the Lagrangian $L^4$ turns out to be proportional to $g$. The full Lagrangian contains higher-order corrections which can be reconstructed order by order from (4.1).

The only subtlety of this analysis is that, apart from straightforward application of homotopy formula (5.9), to reconstruct all perturbations one has to solve the seemingly differential equations on the space-time differential forms like $C_0^0$, $C_0^1$, and $W_0^0$, $W_0^1$. These equations have an important property that at $M = 4$ they are, in fact, off-shell constraints expressing some fields in terms of derivatives of the others. In the language of unfolded machinery this is expressed by the fact that the respective $\sigma$-cohomology groups are zero [79, 80].

As explained in Section 4, the appearance of the Lagrangian forms makes the connections valued in the $Q$-closed central elements trivial, i.e., Stueckelberg. For instance, $L^2$ and $L^4$ make dynamically trivial $W_1^0$, $W_3^1$ $(0, \theta; 0; 0 | x)$. In particular the spin-one connection valued in the center of the Chan-Paton group $U(n)$ of the original HS theory can be gauge fixed to zero in presence of $L^2$. This does not mean, however, that spin-one massless modes disappear. They are still described by the zero-form $C^0$ obeying (5.28) (see also Section 5.5).

5.4 Boundary functionals, parity, and conformal HS theory

As explained in Introduction, the local and non-local parts of the boundary functionals are associated with different combinations of the coefficients in (1.10), (1.11). Although these coefficients can only be determined by the direct computation which is the subject of [48], important piece of information can be deduced from the parity properties of the theory.

From (1.5) it is clear that the parity transformation $z \rightarrow -z$, $x \rightarrow x$ is generated by the automorphism of the algebra that exchanges left and right sectors, including the respective Klein operators, i.e.,

$$\theta^\alpha, z^\alpha, y^\alpha, k \leftrightarrow \bar{\theta}^\dot{\alpha}, \bar{z}^\dot{\alpha}, \bar{y}^\dot{\alpha}, \bar{k}. \quad (5.34)$$

For general $\eta$ in (3.7), HS equations (5.2) are not $P$-invariant. However for the $A$-model with $\eta = 1$ and $B$-model with $\eta = i$ they are $P$-invariant provided that

$$P \left( B(\theta; Z; Y; Y_V | x) \right) = \eta^2 B(P(\theta); P(Z); P(Y); P(Y_V) | P(x)), \quad (5.35)$$

which implies, in particular, that the spin-zero modes of $B$ describe scalars in the $A$ model and pseudoscalars in the $B$-model [17].

Since $z^{-1}dz$ is even under $z \rightarrow -z$, the non-zero contribution to the parity invariant functional (1.4) comes from the part $S^{loc}$ in (1.12) that only contains boundary derivatives (recall that an even combination of $z$-derivatives can be expressed via boundary derivatives by virtue of the field equations). Hence, for the $A$ and $B$-models $S$ (1.4) is some gauge invariant boundary functional. Since the $g$-dependent term in (5.2) is $P$-invariant, the original bulk action is invariant under reflection of all bulk coordinates. As a result, upon the $z$ integration taking away one power of $z$, the resulting boundary functional should be odd under reflection of the boundary coordinates hence being of Chern-Simons type. The resulting gauge invariant local boundary functionals are conjectured to represent actions of
3d conformal HS theory. Interestingly, our construction predicts two different actions for 3d conformal HS theories associated with the $A$ and $B$ models. As for the bulk theory they should differ by the parity properties of the scalar field.

Naively, this consideration suggests that the nonlocal part of the boundary functional in the $A$ and $B$ models is zero. This is not quite the case as we explain now. To this end, consider the HS theory with general $\eta$. Since, being invariant under the exchange of left and right sectors, the $g$-dependent term in (5.2) is $P$-invariant, the whole setting is invariant under the $P$-transformation supplemented with $\eta \rightarrow \bar{\eta}$. For our consideration it is essential that $L$ is evaluated at $Y = Z = 0$ and that the $g$-dependent term contains an additional factor of $k^* \bar{k}$. This implies that the computation in the dotted and undotted sectors are parallel except that in the $g$-dependent contribution to the Lagrangian $k$ is replaced by $\bar{k}$ and vice versa. As a result, with first on-shell theorem (5.31), the analogue of (1.10) has the structure

$$L \sim \omega(\eta \bar{C} + \bar{\eta}C).$$

Setting schematically

$$R_{xx} \sim \eta e_x e_x C + \bar{\eta} e_x e_x \bar{C}, \quad R_{xz} \sim i\eta e_x e_x C - i\bar{\eta} e_x e_x \bar{C}$$

yields

$$C \sim \bar{\eta}(R_{xx} - iR_{xz}), \quad \bar{C} \sim \eta(R_{xx} + iR_{xz}),$$

At the linearized level this yields with $\eta = \exp i\varphi$

$$L \sim \omega(\cos(2\varphi)R_{xx} - \sin(2\varphi)R_{xz}),$$

i.e., $S^{loc}$ contains the factor of $\cos(2\varphi)$ while $S^{nloc}$ contains the factor of $\sin(2\varphi)$. In accordance with the parity analysis, $S^{nloc}$ vanishes at $\phi = 0, \frac{\pi}{2}$, i.e., for $A$ and $B$ models.

To define both local and non-local functionals for the $A$ and $B$ models it makes sense to extract the factors of $\cos(2\varphi)$ and $\sin(2\varphi)$ setting

$$S^{loc}_A = S(0), \quad S^{nloc}_A = \frac{1}{2} \left. \frac{\partial S(\varphi)}{\partial \varphi} \right|_{\varphi=0},$$

$$S^{loc}_B = S\left(\frac{\pi}{2}\right), \quad S^{nloc}_B = \frac{1}{2} \left. \frac{\partial S(\varphi)}{\partial \varphi} \right|_{\varphi=\frac{\pi}{2}}.$$  

Beyond the parity invariant HS models it is impossible to separate the local and nonlocal parts of the gauge invariant functional $S$ ([14]) since only the full action $S$ is gauge invariant. Indeed, the variation of the nonlocal part can contain local terms compensating the nonzero gauge variation of the local part. Only for the $P$-invariant $A$ and $B$ models it is possible to define the gauge invariant local boundary functionals $S^{loc}_{A,B}$ to be identified with the actions of the boundary conformal HS theory. (Note that our conclusions fit the identification of the action of the boundary conformal HS theory with the local part of the boundary functional suggested in [37] (see also [81]).) On the other hand, the nonlocal functionals $S^{nloc}_{A,B}$ ([40], [41]) are guaranteed to be gauge invariant only up to local terms resulting from the derivative of the gauge transformation of $S^{loc}(\varphi)$ over $\varphi$, i.e., the HS gauge symmetry of $S^{nloc}_{A,B}$ (and hence correlators) is respected up to local boundary terms.
5.5 Black holes

The two-form part $L^2$ of $\mathcal{L}$ in (5.2) is anticipated to support the BH charges. In presence of $L^2$, the spin-one sector of linearized Eq. (5.31) is

$$d_x \omega^1(0; 0; 0|x) = \left( \eta W^{\alpha \tilde{\alpha}} W^\alpha_{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C^0(Y; K|x) k + \bar{\eta} W^{\alpha \tilde{\alpha}} W^\alpha_{\tilde{\alpha} \beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C^0(Y; K|x) \bar{k} \right) \bigg|_{y = K = 0} + L^2,$$

where, abusing notations, we set $k|_{k=0} = 0$, $k^2|_{k=0} = 1$.

As shown in [82], a 4$d$ GR BH solution is fully characterized by a spin-one Papapetrou field [83]. In terms of components of the field $C(Y|x)$ which extend the spin-two BH solution to all other fields, the two-form field strength of the Papapetrou field $\mathcal{F}$ is

$$W^{\alpha \tilde{\alpha}} W^\alpha_{\alpha \beta} \bar{C}_{\tilde{\alpha} \beta} + W^{\alpha \tilde{\alpha}} W^\alpha_{\tilde{\alpha} \beta} C_{\alpha \beta} = M \mathcal{F},$$

where $M$ is the BH mass and zero-forms $C_{\alpha \beta}$ and $\bar{C}_{\tilde{\alpha} \beta}$ are self dual and anti-self dual components of the spin-one field strength. The Hodge dual two-form $\tilde{\mathcal{F}}$ is

$$i \left( W^{\alpha \tilde{\alpha}} W^\beta_{\beta \alpha} C_{\alpha \beta} - W^{\alpha \tilde{\alpha}} W^\beta_{\tilde{\alpha} \alpha} \bar{C}_{\tilde{\alpha} \beta} \right) = M \tilde{\mathcal{F}}.$$

The Papapetrou field obeys the sourceless Maxwell equations everywhere except for the singularity, i.e., both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are closed,

$$d_x \mathcal{F} = 0, \quad d_x \tilde{\mathcal{F}} = 0, \quad x \neq 0.$$

For $\eta = \exp [i \varphi]$, Eq. (5.42) implies in the canonical gauge $\omega^1(0; 0; 0|x) = 0$ that

$$L^2 = -M \left( \cos(\varphi) \mathcal{F} + \sin(\varphi) \tilde{\mathcal{F}} \right).$$

For the sake of simplicity in the sequel we consider the simplest case of the Schwarzschild BH in GR leaving details of the general case to [84]. The Papapetrou two-form of the Schwarzschild BH is

$$\mathcal{F} = \frac{4}{r^2} dt dr,$$

where $t$ and $r$ are the time and radial coordinates. Correspondingly,

$$\tilde{\mathcal{F}} = d\Omega,$$

where $d\Omega$ is the angular two-form. The properties of the form $M\tilde{\mathcal{F}}$ suggest that, at least at the linearized level in the HS theory, it should coincide with the two-form that supports the BH charge. Indeed, at the horizon it has the form

$$\tilde{\mathcal{F}} = (2M)^{-2} V_H,$$

where $V_H$ is the horizon volume form. This gives

$$-L^2 = \frac{\sin(\varphi)}{4M} V_H + M \cos(\varphi) \mathcal{F}.$$
For the Schwarzschild BH the second term does not contribute to the BH charge resulting from the integration over space infinity while the first gives

\[- \int_\Sigma \mathcal{L}^2 \frac{\sin(\varphi)}{4M} A_H, \quad (5.51)\]

where \(A_H\) is the horizon area. For the \(A\)-model with \(\varphi = 0\) this is zero. Analogously to the consideration of the boundary functional in Section 5.4, a proper definition is

\[Q(0) = - \frac{\partial \mathcal{L}^2(\varphi)}{\partial \varphi} \bigg|_{\varphi=0}. \quad (5.52)\]

However application of this formula to a BH solution in the nonlinear HS theory is not straightforward since exact HS BH solutions at \(\varphi \neq 0\) are not yet available.

There are several reasons why \(\mathcal{L}^2(0)\) does not contribute to the BH charge of the Schwarzschild BH while \(Q(0)\) (5.52) does. The simplest one follows from the parity analysis analogous to that of Section 5.4. Another reason is that the Papapetrou field \(\mathcal{F}\) is equivalent to the electromagnetic field of a point-wise source. This implies that equation (5.42) admits a solution with \(\mathcal{L}^2 = 0\) and some \(\omega^1(0; 0; 0|x)\) regular at infinity, which is just the Coulomb field. As a result, \(\mathcal{L}^2\) is exact at infinity and hence cannot give a nonzero charge. On the other hand, \(\tilde{\mathcal{F}}\) describes a monopole solution. In this case, due to the Dirac string, the corresponding potential \(\omega^1(0; 0; 0|x)\) is singular at infinity and formula (5.42) with \(\omega^1(0; 0; 0|x) \neq 0\) cannot be used for the computation of the charge at infinity. On the other hand, in the regular gauge \(\omega^1(0; 0; 0|x) = 0\), it determines \(\mathcal{L}^2\) in terms of the Papapetrou field. So defined \(\mathcal{L}^2\) is closed but not exact just because the respective \(\omega^1(0; 0; 0|x)\) is not regular.

The fact that the HS theory possesses a nontrivial on-shell closed form \(\mathcal{L}^2\) can look surprising since it does not rely on a Killing symmetry of a particular solution, holding for any solution. Indeed, no on-shell closed \(\mathcal{L}^2\) is expected to exist in the class of local functionals in a nonlinear on-shell theory in four dimensions. The point is that the invariant functionals \(\mathcal{L}\) in the HS theory are nonlocal (for more detail on the characterization of degree of nonlocality in HS theories see [50]). Like nonlinear HS Lagrangians of [9], \(\mathcal{L}^2\) can involve infinitely many derivatives of the dynamical fields with the coefficients containing inverse powers of the cosmological constant. (The property that the cosmological constant is nonzero is important and the flat limit of \(\mathcal{L}^2\) is obscure.) Hence, the integral \(Q = \int_{\Sigma^2} \mathcal{L}^2(\phi)\) over some surface \(\Sigma^2\) may depend on the values of fields away from \(\Sigma^2\). Nevertheless, \(\mathcal{L}^2(\phi(x))\) is well-defined as a closed space-time two-form and hence \(Q\) is independent of local variations of \(\Sigma^2\). On the other hand, \(Q\) evaluated for asymptotically free theory at infinity, where \(\mathcal{L}^2\) becomes asymptotically local, should reproduce usual asymptotic charges.

Contracting Eq. (5.28) with the time-like Killing vector \(\xi^\mu\) and using that

\[\xi^\mu \frac{\partial}{\partial x^\mu} \bigg|_H = \frac{\partial}{\partial t} \bigg|_H, \quad \xi^\mu \xi_\alpha \frac{\partial}{\partial x^\alpha} \bigg|_H = 0 \quad (5.53)\]

we observe that the generalized HS Weyl tensors in the unfolded equations for fluctuations of massless fields at the horizon of the Schwarzschild BH are \(t\)-independent. Hence, from
the point of view of the observer at infinity, $Q$ evaluated at $H$ is associated with the lower-dimensional system of $t$-independent fluctuations. The form of this system can, in principle, be derived as the reduction of system (4.1), (5.2). It is tempting to speculate that this scheme can lead to the identification of a microscopic pattern of the problem in terms of $L^2$.

5.6 Vacuum partition

Property (5.16) has the consequence that the vacuum value of the Lagrangian form $L$ is zero

$$L_0 = 0 \tag{5.54}$$

implying that the vacuum partition function is trivial, $Z_0 = \exp -S_0 = 1$. Naively, this conclusion holds for any boundary geometry consistent with the vacuum connection of the form (5.5), (5.6). In particular this is true for AdS$_3$ boundary or $S^3$ in the Euclidean case.

Here however is a subtlety. Indeed, if the cohomology $H^4$ of the bulk with complexified Poincaré coordinate is nonzero one can look for another vacuum solution with nonzero $L^4_0 \in H^4$ and appropriately adjusted vacuum two-form $B^2_0$ in (5.2). This is analogous to the BH analysis in the previous section where the closed form $L^2_0$ was supported by the Hodge dual of the Papapetrou field via (5.44) where $C_{\alpha\beta}$ and $\bar{C}_{\dot{\alpha}\dot{\beta}}$ are components of the zero-form $B^0$. Such $L^4_0$ will contribute to the vacuum partition function. Remaining arbitrary, its magnitude will affect the perturbative analysis becoming an essential parameter of the model analogous to the BH mass. A careful analysis of this issue demands in particular an appropriate reformulation of the Poincaré-type foliation of the bulk space. This is another interesting direction for the future work, being beyond the scope of this paper. As an example, we consider below a particular realization of the topological mechanism originating from the standard structure of the frame-like HS action [85].

A typical HS action [83] allowing a nonlinear deformation [3] differs from the standard Fronsdal action [5, 6] by a topological term. In the spin-two gravitational sector this is the MacDowell-Mansouri action [86]

$$S^{MM} = \frac{i}{4\kappa^2\lambda^2} \int_{M^4} (R_{\alpha\beta} R^{\alpha\beta} - R^\alpha_\alpha R^{\dot{\alpha}\dot{\beta}}), \tag{5.55}$$

where

$$R_{\alpha\beta} = R_{\alpha\beta} + \lambda^2 e^\delta_{\alpha} e^\delta_{\beta}, \quad R_{\alpha\beta} := d_x \omega_{\alpha\beta} + \omega_{\alpha}^\gamma \omega_{\beta\gamma}, \tag{5.56}$$

$$\bar{R}_{\dot{\alpha}\dot{\beta}} = \bar{R}_{\dot{\alpha}\dot{\beta}} + \lambda^2 e_{\gamma}^\delta e_{\gamma\delta}, \quad \bar{R}_{\dot{\alpha}\dot{\beta}} := d_x \bar{\omega}_{\dot{\alpha}\dot{\beta}} + \bar{\omega}_{\dot{\alpha}}^\dot{\gamma} \bar{\omega}_{\dot{\beta}\dot{\gamma}}, \tag{5.57}$$

are the Lorentz components of the Riemann tensor shifted by the cosmological term which are defined in terms of vierbein $e^a_{\alpha}$ and Lorentz connection $\omega^{\alpha\beta}, \bar{\omega}^{\dot{\alpha}\dot{\beta}}$. Together with the torsion two-form

$$R_{\alpha\beta} := d_x e_{\alpha\beta} + \omega_{\alpha}^\gamma e_{\gamma\beta} + \bar{\omega}_{\dot{\beta}}^\dot{\delta} e_{\alpha\delta}, \tag{5.58}$$

they are components of the $sp(4)$ curvature

$$R(Y|x) := d_x W(Y|x) + W(Y|x) * W(Y|x), \tag{5.59}$$

30
\[
R(Y|x) = \frac{i}{2} \left( R_{\alpha\beta} y^\alpha y^\beta + \tilde{R}_{\dot{\alpha}\dot{\beta}} \tilde{y}^\dot{\alpha} \tilde{y}^\dot{\beta} + 2 R_{\alpha\beta} y^\alpha \tilde{y}^\beta \right).
\] (5.60)

A locally AdS_4 space obeys flatness equation (5.3) \( R(W_0) = 0 \). Hence, in accordance with (5.54), the MacDowell-Mansouri action is zero on any locally AdS bulk.

To see that the MacDowell-Mansouri action differs from the Einstein-Hilbert action by the Gauss-Bonnet topological term we observe using (5.56), (5.57) that
\[
S^{MM} = S^{top} + S^{EH} + S^c,
\] (5.61)
where
\[
S^{top} = \frac{i}{4\kappa^2 \lambda^2} \int_{M^4} \left( \bar{R}_{\alpha\beta} \bar{\mathcal{R}}^{\alpha\beta} - \tilde{R}_{\dot{\alpha}\dot{\beta}} \tilde{\mathcal{R}}^{\dot{\alpha}\dot{\beta}} \right),
\] (5.62)
\[
S^{EH} = \frac{i}{2\kappa^2} \int_{M^4} \left( e_{\alpha}^{\dot{\beta}} e_{\beta}^{\dot{\gamma}} \bar{\mathcal{R}}^{\alpha\beta} - e_{\gamma}^{\dot{\alpha}} e_{\beta}^{\dot{\gamma}} \bar{\mathcal{R}}^{\dot{\alpha}\dot{\beta}} \right),
\] (5.63)
\[
S^c = \frac{i\lambda^2}{4\kappa^2} \int_{M^4} \left( e_{\alpha}^{\dot{\beta}} e_{\beta}^{\dot{\gamma}} e_{\dot{\gamma}}^{\alpha} e_{\beta}^{\dot{\gamma}} - e_{\gamma}^{\dot{\alpha}} e_{\beta}^{\dot{\gamma}} e_{\dot{\gamma}}^{\alpha} e_{\beta}^{\dot{\gamma}} \right).
\]

Upon imposing the zero-torsion condition \( R_{\alpha\dot{\alpha}} = 0 \), which is one of the field equations of the MacDowell-Mansouri action, the two-forms \( \mathcal{R}_{\alpha\beta} \) and \( \tilde{\mathcal{R}}_{\dot{\alpha}\dot{\beta}} \) describe the Riemann tensor. Hence, \( S^{EH} \) and \( S^c \) are the Einstein-Hilbert action and the cosmological term, respectively. The action \( S^{top} \) is topological describing the Euler characteristic of \( M^4 \). Its variation over \( \omega_{\alpha\beta} \) and \( \bar{\omega}_{\dot{\alpha}\dot{\beta}} \) is zero. Generally, the vacuum contribution of \( S^{top} \) is nonzero, precisely compensating that of the Einstein-Hilbert action.

Note that in the AdS_4 HS theory, the Gauss-Bonnet contribution extends to higher spins as follows
\[
S^{top} = \frac{i\alpha}{4\kappa^2 \lambda^2} \int_{M^4} \text{str} \left( R(y, 0; K) \ast R(y, 0; K) - \tilde{R}(0, \tilde{y}; K) \ast R(0, \tilde{y}; K) \right),
\] (5.65)
where
\[
R(y, \tilde{y}; K) = d\omega^1(y, \tilde{y}; K) + \omega^1(y, \tilde{y}; K) \ast \omega^1(y, \tilde{y}; K)
\]
(5.66)
is expressed in terms of the one-form HS fields (5.23).

Since the gauge invariant HS action enjoys the MacDowell-Mansouri form at least in the lowest order, this can explain the compensation of the vacuum contribution to the partition function in the gauge-invariant HS theory. More generally, possible contribution of the topological terms makes the vacuum contribution to the action undetermined.

The Gauss-Bonnet Lagrangian gives an example of a nonzero vacuum Lagrangian form \( \mathcal{L}_0^4 \). For the vacuum solution obeying (5.3) it is proportional to the volume form
\[
\mathcal{L}_0^4 = aiH_{\alpha\beta} H^{\alpha\beta} = -ai\bar{H}_{\dot{\alpha}\dot{\beta}} \bar{H}^{\dot{\alpha}\dot{\beta}}
\] (5.67)
with some coefficient \( a \) and two-forms \( H_{\alpha\beta}, \bar{H}_{\dot{\alpha}\dot{\beta}} \) (5.32). The \( \tau_I \) projection of the r.h.s. of the four-form analogue of equation (5.31) contains the terms
\[
(\eta H^{\dot{\alpha}\dot{\beta}} \frac{\partial^2}{\partial y^\dot{\alpha} \partial y^\dot{\beta}} C^2(0, \tilde{y}; K|x)\bar{k}) + (\bar{\eta} H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C^2(y, 0; K|x)\bar{k}) \bigg|_{y=0} + \mathcal{L}_0^4.
\] (5.68)
To compensate the term with $L_0^4$ (5.67) it suffices to set

$$C_0^2(Y) = \frac{ai}{4}(\bar{\eta}\dot{H}\dot{\gamma}\bar{y}_\alpha \bar{y}_\beta k - \eta H^{\alpha\beta}y_\alpha y_\beta \bar{k})$$

using that $\eta\bar{\eta} = 1$ and

$$H^{\alpha\beta} \dot{H}^{\dot{\alpha}\dot{\beta}} = 0$$

as a consequence of the relations

$$e(\alpha\dot{\alpha}e\beta\dot{\beta}e\gamma\dot{\gamma}) = 0, \quad e(\dot{\alpha}\dot{\beta}e\gamma\dot{\gamma}) = 0$$

expressing the fact that the symmetrization over, say, three undotted indices of the vierbeins implies the antisymmetrization over three dotted ones, which take only two values.

It is important that by virtue of (5.71) so defined $C^2(Y)$ is covariantly constant obeying equation analogous to (5.19) thus solving (4.1). Moreover, it cannot be represented in the exact form, i.e., as the covariant derivative of something else, thus being cohomologically nontrivial.

In the conventional definition of the boundary functional (1.3), for conformally flat $M^4$ with volume $V_{M^4}$ this gives a contribution proportional to $a^\lambda_2^\kappa_2 V_{M^4}$ which remains arbitrary. For the prescription (1.4), the integration is over $S^1 \times \Sigma^3$ where $S^1$ is the cycle around infinity and $\Sigma^3$ is the boundary surface. Though for $\Sigma^3 = S^3$ the additional contribution is likely to be zero in the gravity case since the Euler number of $S^1 \times \Sigma^3$ is zero it would be interesting to see this directly for an appropriate Poincaré-type foliation. In any case, if the outlined cohomological mechanism gives a non-zero contribution to the vacuum partition its value becomes a free parameter distinguishing between different phases of the theory.

6 AdS$_3$ HS theory

The form of nonlinear field equations of the AdS$_3$ HS theory [87] is analogous to (3.3), (3.4). The field variables $W(\theta, z; \psi_{1,2}; k|x)$, $B(z; y; \psi_{1,2}; k|x)$ and $S_\alpha(z; y; \psi_{1,2}; k|x)$ depend on the space-time coordinates $x^\alpha$ ($\alpha = 0, 1, 2$), auxiliary commuting spinors $z_\alpha$, $y_\alpha$ ($\alpha = 1, 2$), a pair of Clifford elements $\{\psi_i, \psi_j\} = 2\delta_{ij}$ ($i = 1, 2$) that commute to all other generating elements and the Klein operator $k$

$$k^2 = 1, \quad ky_\alpha = -y_\alpha k, \quad kz_\alpha = -z_\alpha k.$$  (6.1)

In terms of the one-form connection

$$W = d_x + W + S,$$  (6.2)

the 3d nonlinear field equations take the form

$$W \ast W = -i\delta^2(\theta)(1 + B \ast k \ast v),$$  (6.3)
\( W \ast B = B \ast W. \)  \hspace{1cm} (6.4)

By analogy with the \( AdS_4 \) HS equations a natural goal would be to construct a three-form Lagrangian. However, this is impossible because every three-form in the two-dimensional twistor space is zero that leaves no room for a term analogous to that with \( \delta^4(\theta) \) in (5.2). Without such a term it is not clear how to generate nontrivial higher differential forms both in the \( \theta_z \) and in the \( \theta_x \) sector which eventually would give rise to a nontrivial three-form Lagrangian. Note that a constant term proportional to \( \delta^2(\theta)k \ast u \) is contained in (6.3) as a constant part of \( B \). The respective coupling constant was shown in [87] to be related to the parameter of mass of the matter fields in the 3d HS theory.

The absence of a Lagrangian three-form in the 3d HS theory may be related to the peculiarity of two-dimensional boundary conformal theory exhibiting the holomorphic factorization. We conjecture that the appropriate invariant functional in the 3d HS theory is supported by a two-form \( L^2(\theta|x) \) in the following generalization of (6.3)

\[
W \ast W = -i\delta^2(\theta)(1 + B \ast k \ast u) + L^2(\theta|x) I, \quad d_x L^2(\theta|x) = 0. \hspace{1cm} (6.5)
\]

The part \( \mathcal{L} \) of \( L^2 \), that contributes to the generating functional (1.4), is

\[
\mathcal{L} = dz(dx L_{zx} + d\bar{x} \bar{L}_{zx}), \hspace{1cm} (6.6)
\]

where \( x \) and \( \bar{x} \) are complex coordinates of the two-dimensional boundary. The invariant functionals \( S = \int \mathcal{L}^2 \) should result from the integration over \( S^1 \times \Sigma \) where \( S^1 \) is a contour around the \( AdS_3 \) infinity \( z = 0 \) while \( \Sigma \) is a complex curve on the 2d boundary. Since \( L^2 \) is closed, the result will be independent of local variations of \( \Sigma \). Hence, for a Riemann surface \( \Sigma \), so defined \( S \) will only depend on its genus.

This conjecture can be checked using the analysis of the boundary behavior in \( AdS_3 \) of [19]. The same two-form functional \( L^2 \) integrated over a different cycle surrounding the BH singularity is anticipated to describe charges of the BTZ-like BH solutions [18, 20] in 3d HS theories. We hope to consider these problems in more detail elsewhere.

Analogously to the 4d case, the extension of the set of fields by the two-form \( L^2 \) makes the one-form \( \omega := W(0, \theta_x; 0; 0; 0; 0|x) \) dynamically trivial. The difference is that in the 3d theory the gauge fields are of Chern-Simons type hence admitting no non-zero on-shell curvatures. This implies that there is no room for the zero forms \( C \) in the 3d First On-Shell Theorem which takes the form

\[
d_x \omega(x) = L^2 + \ldots, \hspace{1cm} (6.7)
\]

where ellipses denotes nonlinear corrections. As a result, in the standard gauge \( L^2 \) starts from quadratic corrections in \( C \) which is just appropriate for the generating functional of the boundary correlators.

### 7 Conclusion

The construction of invariant functionals \( S \) in HS theory proposed in this paper associates them with central elements of the HS algebra. \( S \) are integrals of space-time differential forms
\( \mathcal{L} \) that are closed by virtue of the HS field equations and/or constraints. The Lagrangians \( \mathcal{L} \) are specific fields in the extended unfolded system of HS equations, which are expressed by this system in terms of the other fields. Since the gauge transformation of \( \mathcal{L} \) has the form
\[
\delta \mathcal{L} = d\chi,
\]
the functionals \( S \) are gauge invariant. The new element of our construction is that nontrivial functionals \( S \) are conjectured to be supported by such Lagrangians \( \mathcal{L} \) that cannot be represented in the form of supertrace of some pre-Lagrangian, i.e., \( \mathcal{L} \neq \text{str}(\mathcal{L}') \) for any \( \mathcal{L}' \) built from the HS gauge fields. In this respect our proposal differs from most of other proposals in the literature where invariant functionals are searched in the form of supertrace of a pre-Lagrangian \( \mathcal{L}' \).

The closely related property is that the invariant functionals proposed in this paper are not local, containing infinite expansions in powers of derivatives. Since the unbroken phase of the HS theories is anticipated to describe physics at ultrahigh (transPlanckian) energies, such theories should be non-local one way or another. It is important to specify the degree of nonlocality in such theories. In \([50]\) it was suggested how to distinguish between local, minimally nonlocal and strongly nonlocal functionals. The degree of nonlocality in the HS theory is minimally nonlocal.

The Lagrangian forms \( \mathcal{L}_i \) are associated with certain central elements \( c_0^i \) of the HS algebra. An interesting effect of the appearance of the Lagrangian forms \( \mathcal{L}_i \) is that the fields of the original HS theory valued in the central elements \( c_0^i \) disappear becoming Stueckelberg with respect to the gauge symmetries associated with \( \mathcal{L} \). For instance, the spin-one connection that carries no color indices disappears from the 3d HS theory due to the appearance of the Lagrangian two-form \( \mathcal{L}^2 \).

Our scheme is coordinate independent being applicable to configurations of any topology. To be nontrivial, invariant actions have to be integrated over noncontractible cycles. In the on-shell case, one option is to integrate a \( d \)-form Lagrangian \( \mathcal{L}_d \) over \( S^1 \times \Sigma^{d-1} \) where \( \Sigma^{d-1} \) is a \( (d-1) \)-dimensional boundary of the \( d \)-dimensional bulk space while \( S^1 \) is a circle around the infinite point \( z = 0 \) on the complex plane of the complexified Poincaré coordinate. The respective action \([1.4]\) is conjectured to give rise to the generating functional of correlators of the boundary theory. Usual functional \([1.3]\) can also be considered. Explicit check of whether the proposed functional properly reproduces boundary correlators will be reported in \([48]\). The analysis of this paper shows however that some of the generating functionals for nonlocal contributions to the boundary correlators in the HS theory should be associated with derivatives \( \frac{\partial \mathcal{L}^4(\phi)}{\partial \phi} \bigg|_{\phi=0} \) rather than \( \mathcal{L}^4(0) \), where \( \phi \) is the phase parameter distinguishing between different HS models. In these cases \( \mathcal{L}^4(0) \) describes the Lagrangian of the boundary conformal HS theory that only gives local contribution to the correlators.

Hopefully, the boundary functional in the form \([1.4]\) may have applicability beyond HS theories. The peculiar property that the integration is over the region beyond \( \text{AdS} \) infinity may have something to do with the classical to quantum transmutation in the \( \text{AdS/CFT} \) holography being somewhat reminiscent of quantum tunneling allowing to reach configurations unreachable in classical physics.

Another problem is to evaluate invariants associated with \( (d-2) \)-forms as integrals over lower-dimensional surfaces surrounding a BH singularity. Invariants of this class, including
derivatives \( \frac{\partial L^2(\phi)}{\partial \phi} \bigg|_{\phi=0} \), are conjectured to describe the BH charges in HS theory. It is tempting to speculate that the proposed approach may provide tools for a microscopic interpretation of the BH entropy in terms of the unfolded system associated with the pullback of the original system to the horizon. An intriguing point is that the BH problem turns out to be analogous to the \( AdS/CFT \) problem since the BH solutions in the HS theory \([53, 54]\) are based on the Fock vacua in the twistor space analogous to the Fock vacua \((3.21), (3.27)\) which determine the boundary behavior of the bulk fields \([19]\). In fact, the analysis of BH physics is in a certain sense technically simpler than of the boundary correlators since in the former case nontrivial contributions start from the first order while in the latter from the second. Specifically, as shown in Section 5.5, in the 4d HS theory \( L^2 \) identifies with the spin-one field strength of the Papapetrou field \([83]\).

It should be stressed that the existence of the form \( L^2 \) closed on the HS field equations is possible because, away from the free field limit, \( L^2 \) is a nonlocal functional of the dynamical fields. Such objects naturally appear in the HS theory formulated in the \( AdS \) space but can hardly be introduced in conventional local theories in flat Minkowski space.

In this paper we consider the on-shell HS systems in \( AdS_4 \) and \( AdS_3 \), formulated in terms of spinorial star-product algebras. An interesting peculiarity of the on-shell spinorial HS theory in \( AdS_3 \) is that the Lagrangian form of maximal degree in this theory is a two-form. From the \( AdS_3/CFT_2 \) correspondence perspective this implies that it should be integrated over a one-dimensional surface of the boundary times the circle around infinity. This picture matches holomorphicity of two-dimensional conformal theories. One of the most interesting problems for the future is to see details of this mechanism in the \( AdS_3/CFT_2 \) HS holography.

The proposed construction raises many questions for the further work. Our approach applies to both on-shell and off-shell unfolded systems. An interesting problem is to construct on-shell and off-shell invariants of the vectorial HS theories of \([77]\). This requires analysis of \( Q \)-cohomology in these theories as well as the proper extension of the construction of functional classes of \([50]\) which is more subtle because the HS algebra underlying vectorial HS theory is not freely generated, resulting from quotiening certain constraints. Another interesting problem is to work out the form of the on-shell actions in the conventional lower-spin theories. Also it is important to investigate more carefully the structure of boundary singularities associated with the Fock behavior at \( z \to 0 \) initiated in Section 3.2.

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