A Youla Operator State-Space Framework for Stably Realizable Distributed Control

Mohammad Naghnaeian, Petros G. Voulgaris, and Nicola Elia

Abstract—This paper deals with the problem of distributed control synthesis. We seek to find structured controllers that are stably realizable over the underlying network. We address the problem using an operator form of discrete-time linear systems. This allows for uniform treatment of various classes of linear systems, e.g., Linear Time Invariant (LTI), Linear Time Varying (LTV), or linear switched systems. We combine this operator representation for linear systems with the classical Youla parameterization to characterize the set of stably realizable controllers for a given network structure. Using this Youla Operator State-Space (YOSS) framework, we show that if the structure satisfies certain subspace like assumptions, then both stability and performance problems can be formulated as convex optimization and more precisely as tractable model-matching problems to any a priori accuracy. Furthermore, we show that the structured controllers found from our approach can be stably realized over the network and provide a generalized separation principle.

I. INTRODUCTION

Modern large-scale cyber-physical systems are composed of many interconnected subsystems that are usually spread over a large geographic area and communicate over a network. Many difficulties arise when designing a centralized controller for such systems due to communication delays, the structure of the underlying communication network, scalability, etc. Due to these issues, there has been a shift towards designing decentralized controllers, in which subcontrollers are designed and implemented for each subsystem and they can communicate over the network.

Decentralized, structured and distributed controller design has attracted the renewed attention of many researchers over the last 15 years or so. Several new developments occurred using state space methods (e.g., in the LMI framework [1], [2]) which suit quadratic criteria but could generally lead to suboptimal solutions. On the other hand, input-output approaches using the Youla-parametrization were found to be very powerful in providing truly optimal solutions for several classes of structured problems by reducing them to convex problems over the Youla parameter, encompassing a variety of criteria, including nonquadratic (e.g., [3], [4], [5], [6], [7], [8], [9]).

In the input-output, or transfer function, domain, the stabilizing controllers are parametrized by the so-called Youla parameter, and the search for the optimal Youla parameter is carried out over the space of stable systems. Here, the order of the Youla parameter or that of the controller is not assumed a priori. However, unlike the state-space approaches, the realizability of the controller over the underlying communication network may become an issue, if not taken directly into account as pointed out in [10], [11]. That is, although the controller transfer function structure is compatible with the underlying network communication graph, it may lead to an internally unstable realization, i.e., a non-minimal realization with unstable pole zero cancellations (e.g., [12], [13] and [14]). Certain alternative input-output approaches have recently been proposed (e.g., a system level approach in [15] and references therein) that hold the potential to handle certain optimal and stably realizable structured design, by convex programming without resorting to Youla-parametrization. A potential drawback is the need to solve an exact model-matching problem, i.e., equations that, if possible to satisfy, may require infinite support of the LTI maps involved and hence stopping criteria for the approximation by finite support may not be precisely characterized.

In this paper, we propose a unified way to synthesize stably realizable controllers with respect to any measure of performance, e.g., \( l_1 \), \( l_2 \), or \( l_\infty \) induced norms. Our approach is based on utilizing a state-space based operator form of the system and combining it with the ideas in the Youla-parametrization. This has been developed initially in the context of switching system analysis and design in [16], [17], and as it turns out, it fits well for optimally solving structured problems [18]. Our approach involves revisiting classical robust control problems, e.g., \( H_\infty \) or \( l_1 \), and proposing a new way to check for stability and to parametrize the set of stabilizing controllers. The stability check that we will develop in this paper is in the form of a model-matching problem

\[
\inf_{RT \text{ stable}} \| T_1 + T_2 RT_3 \|,
\]

where \( T_1 \), \( T_2 \), and \( T_3 \) are stable systems. Such problems are convex and efficient algorithms exist to solve them with arbitrary accuracy. As such, we depart from standard stability tests, e.g., eigenvalue or quadratic Lyapunov results. This will allow us to propose a new way to parameterize the set of all stabilizing controllers, which is especially beneficial to designing decentralized controllers with stable realization.
Another novelty of this approach is a new separation principle. Although the standard separation principle does not generally hold true in the context of decentralized control, this newly formulated separation principle holds valid. In order to present this new separation principle, first, we introduce the class of full-information-like controllers. These are controllers that map both the state and measured output to control input. That is,

$$u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$  

where $u$, $x$, and $y$ are control input, states, and measured output, respectively; $K_1$ and $K_2$ are linear systems. The proposed separation principle states that any full-information-like controller yields an output feedback controller if $x$ is replaced by its estimation $\hat{x}$, which in turn is generated by an observer from the measured output. Furthermore, although the reader can focus on the LTI case as a concrete example, these methods are general and hold for LTV, delayed, and switching systems as well.

II. Preliminaries

In this paper, $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. The set of $n$-tuples $x = \{x(k)\}_{k=0}^{n-1}$ where $x(k)$s are real numbers is denoted by $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, its $l_\infty$ and $l_p$ norms are defined as $\|x\|_\infty = \max_{x \in \mathbb{R}^n} |x(k)|$ and $\|x\|_p = \left(\sum_{k=0}^{n-1} |x(k)|^p \right)^{1/p}$, respectively. Let $g = \{g(k)\}_{k=0}^{\infty}$ be a sequence where $g(k) \in \mathbb{R}^n$. Then, the $\ell_\infty$ and $\ell_p$ norm of this sequence are defined as $\|g\|_\infty = \sup_{k \in \mathbb{Z}_N} g(k)$ and $\|g\|_p = \left(\sum_{k=0}^{\infty} |g(k)|^p \right)^{1/p}$ whenever they are finite. The set of $\mathbb{R}^n$-valued sequences whose $\ell_p$ norm ($\ell_\infty$ norm) is finite is denoted by $l_p^\infty$ ($l_\infty^\infty$). Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and a linear operator $T : X \to Y$, its induced norm is defined as $\|T\|_{X \to Y} = \sup_{f \neq 0} \|Tf\|_Y / \|f\|_X$. Whenever both vector spaces are $\mathbb{X}$, we use the notation $\|T\|$ without any subscript if the result holds for any induced norm. We will call $T$ bounded or stable if $\|T\| < \infty$.

Any linear causal map $T$, on the space of all sequences ($l_\infty, \ell_1$), can be thought of as an infinite dimensional lower triangular matrix,

$$T = \begin{bmatrix} T_{0,0} & 0 & \cdots \\ T_{1,1} & T_{1,0} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$  

(2)

Such a causal map $T$ is called strictly causal if its diagonal elements are zeros, i.e., $T_{0,0} = T_{1,1} = \ldots = 0$. We say $T$ is a square operator if it has the same number of inputs as outputs, that is, $T_{i,j}$ terms are square matrices. Given a sequence $g = \{g(k)\}_{k=0}^{\infty}$, the delay or shift operator $\Lambda$ is defined by $\Lambda^k g = \{g(k), g(k+1), \ldots\}$, and, with a slight abuse of notation, $\Lambda^{-k}g = \{g(k), g(k+1), \ldots\}$. It is easy to show that if $T$ is a causal map then $\Lambda T$ and $T\Lambda$ are strictly causal. Conversely, any strictly causal operator $T$ can be written as $T = \Lambda \tilde{T}$ where $\tilde{T}$ is causal. A linear causal map $T$ is called time-invariant if it commutes with the delay operator, i.e., $\Lambda T = T\Lambda$. If $T$ is a Linear Time-Invariant (LTI), it is fully characterize by its impulse response denoted by $\{T(k)\}_{k=0}^{\infty}$. In this case, its infinite dimensional matrix representation is given by

$$T = \begin{bmatrix} T(0) & 0 & \cdots \\ T(1) & T(0) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$  

A finite dimensional LTI system has the state-space representation of

$$G : \begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \text{ with } x(t_0) = x_0,$$

(3)

where $u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p$, and $x_0 \in \mathbb{R}^n$ are input, state, output, and the initial condition of the system and $A, B, C,$ and $D$ are matrices with appropriate dimensions for all $t \in \mathbb{Z}_N$. Given a matrix $S$, we define $\hat{S}$ to be the diagonal operator

$$\hat{S} = \begin{bmatrix} S & 0 & \cdots \\ 0 & S & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$  

(4)

Using this notation, we can define diagonal operators $\hat{A}, \hat{B}, \hat{C},$ and $\hat{D}$ and rewrite (3) as

$$G : \begin{cases} x = \hat{A}x + \hat{B}u + \hat{x}_0 \\ y = \hat{C}x + \hat{D}u \end{cases},$$  

(5)

where $\hat{x}_0 = \{x(t)\}_{t=0}^{\infty}, u = \{u(t)\}_{t=0}^{\infty}$, and $\Lambda$ is the delay operator. The above representation of $G$ is referred to as a realization in operator form. One can also write an operator realization for the time delay systems. Consider the system given by

$$H : \begin{cases} x(t+1) = \sum_{i=0}^{N} A_i x(t-i) + \sum_{i=0}^{N} B_i u(t-i) \\ y(t) = \sum_{i=0}^{N} C_i x(t-i) + \sum_{i=0}^{N} D_i u(t-i) \end{cases},$$  

(6)

with initial condition $x_0 = \{x(k)\}_{k=0}^{N}$. Define $\hat{A} = \sum_{i=0}^{N} A_i \Lambda^i$. Similarly, we define $\hat{B}, \hat{C},$ and $\hat{D}$. Then, the time-delay system can be written in the operator form as

$$H : \begin{cases} x = \hat{A}x + \hat{B}u + \hat{x}_0 \\ y = \hat{C}x + \hat{D}u \end{cases}.$$  

(7)

In this operator framework, we make a distinction between an operator and its realization as follows:

**Definition 1 (Operator Realization):** For a given linear (possibly unbounded) causal operator $T : u \to y$, we will refer to the relationship

$$T : \begin{cases} x = A_T x + B_T u \\ y = C_T x + D_T u \end{cases},$$

as an operator realization or simply a realization only if operators $A_T, B_T, C_T,$ and $D_T$ are bounded causal operators and $(I - A_T)^{-1}$ exists.
For example, realizations for LTI operators \([6]\) and delayed systems \([6]\) are given in \([5]\) and \([7]\), respectively. Also, any bounded operator \(T\) has a trivial realization with \(AT = BT = CT = 0\) and \(DT = T\). This realization, however, is not trivial for unstable operator \(T\).

Throughout this paper, we prefer to write the systems in the operator form \([1]\) as it allows for treating various classes of systems (e.g. time-delay, switching, and LTV systems \([16]\) in a unified way. Henceforth, we consider the systems that have operator forms as in \([1]\). Such a system can be seen as a mapping from \(x_0\) and \(u\) to \(x\) and \(y\). For this system, we adopt the following definitions of stability and gain.

**Definition 2:** Given two normed spaces \((U, \|\cdot\|_U)\) and \((X, \|\cdot\|_X)\), we say that the system \(H\) in \([7]\) is \(U\) to \(X\) stable if it is a bounded operator from \([\bar{x}_0^T, u^T] \in X \times U\) to \([x^T, y^T] \in X \times X\). More precisely, \(H\) is \(U\) to \(X\) stable if, for some \(\gamma_1, \gamma_2 \geq 0\), \(\|x\|_X \leq \gamma_1 \|\bar{x}_0\|_X + \gamma_2 \|u\|_U\) and \(\|y\|_X \leq \gamma_1 \|\bar{x}_0\|_X + \gamma_2 \|u\|_U\) whenever \(\|\bar{x}_0\|_X\) and \(\|u\|_U\) are finite.

**Definition 3:** Given two normed spaces \((U, \|\cdot\|_U)\) and \((X, \|\cdot\|_X)\), and a \(U\) to \(X\) stable system \(H\), its gain is defined as \(H\|\cdot\|_U \rightarrow X = \sup_{x \neq 0} \frac{\|Hx\|_X}{\|x\|_U}\).

For simplicity, we let \(X\) and \(U\) to be the same (but possibly with different dimension) \(l_p\) spaces.

Finally, when need to ensure the invertibility of certain operators on \(l_{\infty, c}\), we will appeal to the following lemma:

**Lemma 1:** The following hold:

1. Given a causal square operator \(T\) as in \([2]\), the inverse \((I - T)^{-1}\) exists if \(T_{i,0}\) is invertible for all \(i = 0, 1, 2, \ldots\).
2. Given a partitioned squared operator \(X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\), \((I - X)^{-1}\) exists if \((I - X_{11})\) and \((I - X_{22})\) are invertible.

## III. Basic Setup

A standard practice for designing a distributed controller for subsystems communicating over a given network is to aggregate all subsystems into one system \(P\) and design a controller for this system. The controller must be designed in a way so that it can be implemented as subcontrollers communicating over the given network. We consider the aggregate system with a realization

\[
P : \begin{cases} 
  x = \Lambda \bar{A}x + \Lambda \bar{B}_1 w + \Lambda \bar{B}_2 u + \bar{x}_0 \\
  z = \bar{C}_1 x + \bar{D}_{11} w + \bar{D}_{12} u \\
  y = \bar{C}_2 x + \bar{D}_{22} u,
\end{cases}
\]

where \(x, y,\) and \(z\) are the states, measurements, and the regulated output; \(w\) and \(u\) are the exogenous and control inputs; and, \(\Lambda, \bar{A}, \bar{B}_i, \bar{C}_j, \bar{D}_{ij}\), for \(i,j \in \{1, 2\}\) are bounded operators.

**Example 1:** Consider a network with \(N\) subsystems. Each subsystem is given by

\[
x_i(t + 1) = A_i x_i(t) + B_i^1 w_i(t) + B_i^2 u_i(t) + \sum_{j=1}^{N} B_{ij}^3 \eta_{ij}(t),
\]

\[
z_i(t) = C_i^1 x_i(t) + D_{i1}^1 w_i(t) + D_{i2}^1 u_i(t),
\]

\[
y_i(t) = C_i^2 x_i(t) + D_{21}^i w_i(t),
\]

\[
\nu_{ij}(t) = C_{ij}^3 x_i(t) + D_{31}^i w_j(t), \quad j = 1, 2, \ldots, N,
\]

where \(x_i, y_i,\) and \(z_i\) are the states, measured output, and regulated output of the \(i^{th}\) subsystem; \(\nu_{ij}\) is the signal that the \(i^{th}\) subsystem communicates to the \(j^{th}\) subsystem and \(\eta_{ij}\) is the signal that \(i^{th}\) subsystem receives through its communication link with the \(j^{th}\) subsystem. We let \(B_{ij}^3 = C_i^3 = D_{31}^i = 0\) if there is no communication link between \(i^{th}\) and \(j^{th}\) subsystems. Furthermore, due to the delay in the communication links, we set

\[
\eta_{ij}(t) = \nu_{ij}(t - \tau_{ij}),
\]

where \(\tau_{ij} \in \mathbb{Z}_{\geq 0}\) is delay in communication from \(j^{th}\) to \(i^{th}\) subsystem. Substituting (10) in (9), the \(i^{th}\) subsystem, in the operator form, can be written as

\[
x_i = \Lambda \bar{A} x_i + \sum_{j=1}^{3} \Lambda \bar{A}_{ij} x_j + \Lambda \bar{B}_1 w_i + \sum_{j=1}^{3} \Lambda \bar{B}_{ij} w_j + \Lambda \bar{B}_2 u_i,
\]

where \(\bar{A}_{ij} = \bar{B}_{ij} \bar{C}_j^3\) and \(\bar{B}_1 = \bar{B}_1^j \bar{D}_{31}^j\). Based on the above expression, it can be easily seen that for properly defined operators \(\bar{A}, \bar{B}_i, \bar{C}_j, \bar{D}_{ij}\), for \(i \in \{1, 2\}\), the aggregate system can be written as in \([8]\).

The structure of the network is reflected in the coefficient operators involved in \([5]\), e.g., as sparsity patterns \([14]\). Given a fixed network consisting of \(N\) nodes (subsystems) and a set of \(N\) inputs \(\xi = [\xi]\) to, and \(N\) outputs \(\zeta = [\zeta]\) from, these \(N\) nodes, let \(S\) denote the set of all input-output maps (or, transfer functions in the LTI case) \(T\) from \(\zeta\) to \(\xi\), i.e., \(\xi = T\zeta\) that can be obtained from this network. That is, the input-output aggregation of all subsystem (or, subcontroller) dynamics, interconnected via the network, form an element \(T\) in \(S\) and, conversely, any element in \(S\) can be implemented, stably or unstably (to be precisely defined later), as subsystems communicating over the given network. Consider the following example:

**Example 2:** Nested network: An example of a nested network is given in Figure 1. We adopt the notations introduced in Example 1. It can be easily verified that the aggregate system is given by \([8]\) where \(\bar{A} = \{\bar{A}(0), \bar{A}(1), 0, 0, \ldots\}\), \(\bar{B}_1 = \{\bar{B}_1(0), \bar{B}_1(1), 0, 0, \ldots\}\), \(\bar{B}_2 = \{\bar{B}_2(0), 0, \ldots\}\), \(\bar{C}_j = \{\bar{C}_j(0), 0, \ldots\}\), \(\bar{D}_{ij} = \{\bar{D}_{ij}(0), 0, \ldots\}\) with \(\bar{A}(0) = \text{diag} \{A^1, A^2, A^3\}\), \(\bar{B}_j(0) = \text{diag} \{B^1_j, B^2_j, B^3_j\}\), \(\bar{C}_j(0) = \text{diag} \{C^1_j, C^2_j, C^3_j\}\).
set indicates with its neighbors with a delay. For this network, the structure in Eq. (1) conforms with the flow of communication from subsystem terms in the impulse response of \(A\), where

\[
A(1) = \begin{bmatrix}
0 & B^3_{21} & 0 & 0 \\
0 & B^3_{23} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
B_1(1) = \begin{bmatrix}
B^3_{12} & 0 & 0 & 0 \\
B^3_{13} & 0 & 0 & 0 \\
B^3_{14} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

In this example, the structure of the network is reflected on the impulse response of the coefficient operators, e.g. \(A\). The terms in the impulse response of \(A\) are lower triangular, which conforms with the flow of communication from subsystem 1 to subsystem 2 and then to subsystem 3. And the sparsity structure in \(A(0)\) and \(A(1)\) is because each subsystem has immediate access to its own measurement signal but communicates with its neighbors with a delay. For this network, the set \(S\) is the space of all systems \(P\) whose impulse response \(\{P(k)\}_{k=0}^{\infty}\) satisfies the following conditions: \(P(k)\) is lower triangular for \(k = 2, 3, ..., P(0)\) is diagonal, and

\[
P(1) = \begin{bmatrix}
* & 0 & 0 \\
* & * & 0 \\
* & * & *
\end{bmatrix},
\]

where * stands for a possibly non-zero entry. Or, in transfer function terms,

\[
P[\lambda] = \begin{bmatrix}
* & 0 & 0 \\
\lambda^* & * & 0 \\
\lambda^2 & * & * 
\end{bmatrix},
\]

where \(P[\lambda] = \sum_{k=0}^{\infty} \lambda^k P(k)\) is the \(\lambda\)-transform. Accordingly, if \(K\) is a controller for \(P\) within the same communication network, \(K[\lambda]\) should also be of the same form, i.e., \(K \in S\).

In the sequel, in generating the control input \(u\), depending on what information is available, we face two categories of problems, full-information or output feedback. In full-information feedback, the controller has access to the entire state and measured outputs, i.e.,

\[
u = [K_1 K_2] \begin{bmatrix} x \\ y \end{bmatrix}.
\]

In output feedback, however, \(u\) must be generated only using the information available in signal \(y\), i.e.

\[
u = [0 K_2] \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Remark 1: We need to point out that there is a difference between the full-information controller that we defined in [11] and the one defined in [19]. In the latter, the controller has access to the state \(x\) and the disturbances \(w\) (and consequently \(y\)) while the full-information controller defined in [11] only has access to \(x\) and \(y\). The two definitions are equivalent if \(\tilde{D}_{21}\) is a square and invertible map.

The generalized plant

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Lambda \bar{A} & 0 \\ \bar{C}_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Lambda \bar{B}_1 w + \Lambda \bar{B}_2 u + \bar{x}_0 \\ D_{21} w \end{bmatrix},
\]

\[
z = [\bar{C}_1 0] \begin{bmatrix} x \\ y \end{bmatrix} + \bar{D}_{11} w + \bar{D}_{12} u,
\]

with the full-information feedback controller (or output feedback when \(K_1 = 0\) results in the closed-loop system

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \end{bmatrix}^{-1} \begin{bmatrix} I \\ \bar{C}_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \bar{B}_1 w + I \bar{x}_0 \end{bmatrix},
\]

\[
z = [\bar{C}_1 0] \begin{bmatrix} x \\ y \end{bmatrix} + \bar{D}_{11} w + \bar{D}_{12} u,
\]

\[
u = [K_1 K_2] \begin{bmatrix} x \\ y \end{bmatrix}.
\]

This closed-loop system can be thought of as a linear operator from \((\bar{x}_0, w)\) to signals \(x, y, z, \) and \(u\). In conjunction with Definition 2, we adopt the following definition for centralized stabilizing controllers.

**Definition 4 (Centralized Stabilizing Structured Controller):**

A full-information or output feedback structured controller \(K = [K_1 K_2] \in S^{1 \times 2}\) is said to be stabilizing in the centralized way if the closed loop system (13) is a bounded operator from \(\bar{x}_0\) and \(w\) to \(x, y, z,\) and \(u\).

It is important to note that an implementation of a controller in a distributed way over a network is carried out through implementing a realization of such controller that conforms with the network structure. However, unless \(K \in S\) is a bounded operator itself, there is no a priori guarantee that it has an operator realization conforming with the structure of the network. In the next subsection, we will discuss this issue further.

### A. Stable realizability/implementability

The set \(S\) is fully characterized by the underlying network. In this paper, given a (stabilizable and detectable in the usual sense) generalized plant \(P = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{wy} & P_{yu} \end{bmatrix}\) as in (8), we are interested in finding the controllers \(K \in S\) that are also stably realizable over the network. We should point out that even if \(K\) belongs to \(S\) and stabilizes \(P\) in the usual centralized sense (Definition 4), i.e.,

\[
[ I \ P_{yu} \\ K \ I ]
\]

has a stable inverse, it does not mean that the controller \(K\) can automatically be realized stably, although we can implement

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**Fig. 1. A simple nested network.**

\[
\begin{array}{c}
\begin{array}{c}
\text{Fig. 1. A simple nested network.}
\end{array}
\end{array}
\]
it as the interconnection of subcontrollers consistent with the network. Unless we guarantee that the implementation of $K$ does not have internal hidden unstable modes, the closed-loop system may not be stable. This is because a stabilizing controller in the centralized sense, by design, guarantees the boundedness (stability) of the measured output and control input, $y$ and $u$. Also, under the detectability assumption of the plant, the boundedness of $y$ and $u$ translates to that of $x$ and $z$ in $S$. However, when an aggregate controller is implemented in a decentralized way, new signals are introduced and should be taken into consideration. In particular, these are the signals travelling between subcontrollers and also the intrinsic noise on such signals. Therefore, in designing a controller that is implementable over the network in a stable way, one needs to identify the subcontrollers’ communication signals and guarantee that the effects of their intrinsic noise on every other signal of the system is bounded.

**Example 3:** Consider the nested network in Example 2 and let $K \in S$ be a stabilizing controller in the centralized sense as in 14. Since $K = \{K(k)\}^{\infty}_{k=0} \in S$, it can be partitioned as $K = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$. One way to implement this controller over the network is illustrated in Figure 2. By definition, the signals $y$, $z$, and $u$ are bounded when $w$ and $\bar{x}_0$ are bounded. However, as mentioned above, it is not guaranteed that the signals travelling between the subcontrollers, i.e., $\nu_{21}^K$ and $\nu_{32}^K$, are bounded. Furthermore, it is not clear if small bounded noise signals, shown as $n_{21}$ and $n_{32}$ in the figure, corrupt the subcontrollers’ communications, the stability of the system, i.e., the boundedness of other signals is preserved. Therefore, special attention should be paid to the stable implementability of structured controllers.

Accordingly, we define the set $S_K \subseteq S$ to be the set of stabilizing controllers $K$ that can be stably implemented over the network without losing stability, i.e., the subcontrollers communicate bounded signals provided that the measured outputs, $y$, and control inputs, $u$, are bounded. We will make the definition of $S_K$ concrete in what follows.

**Definition 5 (Stably Realizable Controller):** Given a structured aggregate controller $K = [K_1, K_2] \in S^{1 \times 2}$ ( $K_1 = 0$ for output feedback) that is centralized stabilizing in the sense of Definition 4, we say $K$ is stably realizable over the structure $S$, i.e., $K \in S_K$ if the following two conditions hold:

1) It has an operator realization

$$K : \begin{cases} x_K = A_K x_K + B_K \bar{y} \\ u = C_K x_K + D_K \bar{y} \end{cases},$$

(15)

where

a) for full-information feedback $\bar{y} = [x^T, y^T]^T$ and

$$A_K \in S^{m \times m}, B_K \in S^{m \times 2},$$

$$C_K \in S^{1 \times m}, D_K \in S^{1 \times 2},$$

b) and for output feedback $\bar{y} = y$,

$$A_K \in S^{m \times m}, B_K \in S^{m \times 1},$$

$$C_K \in S^{1 \times m}, D_K \in S^{1 \times 1},$$

for some positive integer $m$ and stable operators $A_K$, $B_K$, $C_K$, and $D_K$.

2) The effects of fictitious subcontrollers’ communication noise, denoted by $n_x$ and $n_u$, on the system is bounded. That is, the interconnection of “noisy controller”

$$\bar{K} : \begin{cases} x_K = A_K x_K + B_K \bar{y} + n_x \\ u = C_K x_K + D_K \bar{y} + n_u \end{cases},$$

(16)

and the plant $P$ in $\bar{S}$ results in a bounded closed-loop system, i.e., $x, y$, and $x_K$ and $u$. We refer to (15) as a stable realization or implementation of the controller $K$.

Given $S$, it is not clear that every centralized stabilizing controller (in the sense of Definition 4), $K \in S$, can be stably realized over the network structure. Therefore, we always have $S_K \subseteq S$. In this paper, given a structure $S$, we develop necessary and sufficient conditions in terms of convex problems under which it is possible to find optimal structured controllers $K \in S_K$ that are stably implementable over the network. A typical $S$ of interest consists of controllers with certain sparsity or delay patterns. Furthermore, we assume that $S$ is a subspace that satisfies the following:

**Assumption 1:** The set $S$ is delay-invariant, contains identity and zero, and is closed under addition and multiplication, i.e., $I, 0 \in S$, $AS \in S$ and for any $X, Y \in S$, $X + Y \in S$ and $XY \in S$.

**Assumption 2:** The set $S$ contains the coefficient operator $\bar{A}, \bar{B}_2$, and $\bar{C}_2$.

Under such assumptions, as we will show $S_K = S$. In addition to the above assumption, we make the following
assumption pertaining to the fact that all of the measured outputs, $y$, are corrupted by noise.

**Assumption 3**: $\hat{D}_{21}$ has a trivial left null space.

IV. **YOULLA OPERATOR STATE-SPACE PARAMETRIZATION OF STABILIZING CONTROLLERS**

In this section, we revisit a classical robust result on parametrizing the set of stabilizing controllers. Traditionally, a typical approach to find the set of stabilizing controllers is via Youla-Kucera parametrization that utilizes the doubly coprime factorization. Here, we propose a new approach in the operator framework that do not require coprime factorization. This approach is referred to as the *Youla Operator State-Space* (YOULLA) and proves to be particularly powerful in designing decentralized controllers for linear systems.

A. **YOULLA for Full-Information Feedback**

For the full-information feedback problems, we first parametrize the set of all centralized stabilizing structured controllers in the sense of Definition 1. And then, we will show how such controllers can be stably realized/implemented in a distributed way over the network structure in the sense of Definition 1. In order to state the result, we define an affine expression $E_{Q,Z}$ in terms of bounded operators $Q$ and $Z$ as follows:

$$E_{Q,Z} := \left[ \begin{array}{c} \Lambda \bar{A} 0 \\ C_2 0 \end{array} \right] + \left[ \begin{array}{c} \Lambda \bar{A} - I 0 \\ C_2 - I \end{array} \right] Q + \left[ \begin{array}{c} \Lambda \bar{B}_2 \\ 0 \end{array} \right] Z.$$  
(17)

Then, the following theorem holds:

**Theorem 1**: The following conditions are equivalent:

1) There exists a centralized stabilizing structured full-information feedback controller $K \in S : [x^T, y^T]^T \to u$ for the the plant (12).
2) There exist stable causal operators $Q$ and $Z$ and some $\varepsilon \in [0, 1]$ such that:
   a) **Model-Matching**:
   $$\|E_{Q,Z}\| \leq \varepsilon.$$  
   (18)
   
   b) **Structure**: $Q$ and $Z$ can be partitioned as
   $$Q = \left[ \begin{array}{cc} \Lambda Q_{11} & \Lambda Q_{12} \\ Q_{21} & \Lambda Q_{22} \end{array} \right], Z = \left[ \begin{array}{c} Z_1 \\ Z_2 \end{array} \right],$$
   $$Q_{i,j} \in S, Z_i \in S \text{ for } i, j = 1, 2.$$  
   (19)

3) For any $\varepsilon \in [0, 1]$, there exist stable causal operators $Q^\varepsilon$ and $Z^\varepsilon$ such that Conditions 2.a and 2.b hold for $Q = Q^\varepsilon$ and $Z = Z^\varepsilon$.

4) There exist stable causal operators $Q^0$ and $Z^0$ such that Conditions 2.a and 2.b hold for $\varepsilon = 0$, $Q = Q^0$, and $Z = Z^0$.

**Corollary 1**: Given a centralized full-information feedback controller $K \in S^{1 \times 2}$, there exists stable operators $Q$ and $Z$ such that (18) and (19) hold for some $\varepsilon \in [0, 1)$ and $K$ can be decomposed as

$$K = [K_1 K_2] = Z (I + Q)^{-1}.$$  
(20)

where the inverse $(I + Q)^{-1}$ exists by Lemma 1.

Theorem 1 provides a convex way to parametrize the set of all centralized stabilizing controllers that have the structure. Furthermore, Corollary 1 provides a factorization for the controller that can be used to obtain an operator realization for the controller. To this end, note that according to (20)

$$u = Z (I + Q)^{-1} [x^T, y^T]^T.$$  

Defining

$$x_K = (I + Q)^{-1} [x^T, y^T]^T,$$

a full-information controller can be realized as

$$K : \{ x_K = A_K x_K + B_K \bar{y} \},$$

$$u = C_K x_K + D_K \bar{y},$$  
(21)

where $\bar{y} = [x^T, y^T]^T$ and

$$A_K = -Q \in S^{2 \times 2}, B_K = I_{2 \times 2} \in S^{2 \times 2},$$

$$C_K = Z \in S^{1 \times 2}^2, D_K = 0_{1 \times 2} \in S^{1 \times 2},$$

with $Q$ and $Z$ satisfying (18) for some $\varepsilon \in [0, 1)$. This is an operator realization of the controller that conforms with the network structure. In order to show that this realization provides a stable implementation, in the sense of Definition 1, we need to show that the noisy version of such a controller with fictitious subcontrollers’ communication noise as given in (16) results in a bounded closed-loop map.

**Theorem 2**: Any centralized stabilizing full-information controller with structure can be stably realized over the network structure through realization (21). That is, $S_K = S$.

According to Theorem 1, parametrizing the set of all full-information stabilizing controllers amounts to solving an optimization problem (18) over the space of stable operators $(Q, Z)$ for some given $\varepsilon \in [0, 1)$. Such optimization problems are in the generic form $\inf R \|T_1 + T_3 R T_3\|$ where $T_1, T_2, T_3$, and $R$ are stable operators. This optimization, which is commonly referred to as a **model-matching problem** in robust control community, is convex and there are efficient methods to solve it with arbitrary accuracy. In particular, when the operator norm is taken to be the $l_\infty$ induced norm, this optimization can be cast as a linear program.

**Remark 2**: According to Theorem 1, although the model-matching (18) can be carried out with $\varepsilon = 0$, it is advantageous, for computations, to relax the condition to $\varepsilon \in [0, 1)$ as stated in the theorem. As a matter fact, if $\varepsilon$ is set to zero, there might not exist any finite-impulse response $Q$ and $Z$ that satisfy (18). For example, consider a trivial uncontrollable but stable system

$$x(t + 1) = \frac{1}{2} x(t) + w(t) + 0 u(t), y(t) = x(t) + w(t).$$

Then, it can be easily verified that $Q = \left[ \begin{array}{c} \Lambda Q_{11} \\ I + \Lambda Q_{11} \end{array} \right]$, and any stable $Z$ satisfy (18) for $\varepsilon = 0$ if and only if $Q_{11}$ has the impulse response $\left\{ \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots \right\}$. This is particularly important since the approximation with FIR will fail to provide performance guarantees.
B. YOSS for Output Feedback

The output feedback controllers form a subset of full-information feedback controllers, which were parametrized in Theorem 1 and Corollary 1. More precisely, a full-information controller (20) is an output feedback controller if \( K_1 = 0 \) or equivalently

\[
Z(1 + Q)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} = 0.
\] (22)

Therefore, we have the following theorem.

**Theorem 3:** The following conditions are equivalent:

1. There exists a centralized stabilizing output-feedback controller \( K : y \to u \).
2. There exist stable causal operators \( Q \) and \( Z \) and some \( \varepsilon \in [0, 1) \) such that (18), (19), and (22) hold.
3. For any \( \varepsilon \in [0, 1) \), there exist stable causal operators \( Q^\varepsilon \) and \( Z^\varepsilon \) such that Condition 2 holds for \( Q = Q^\varepsilon \) and \( Z = Z^\varepsilon \).
4. There exist stable causal operators \( Q^0 \) and \( Z^0 \) such that (18), (19), and (22) hold for \( Q = Q^0 \), \( Z = Z^0 \), and \( \varepsilon = 0 \).

If any of the above equivalent conditions hold, a stabilizing output-feedback controller \( K : y \to u \) is given by

\[
K = Z(1 + Q)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.
\]

Although finding the set of stabilizing full-information feedback controllers, as given in Theorem 1, is a convex and tractable problem, this is not the case for output feedback due to the non-convexity associated with (22). We emphasize that (22) is enforced to restrict the set of full-information stabilizing controllers, \( u = [K_1\ K_2] \begin{bmatrix} x \\ y \end{bmatrix} \), to the ones whose \( K_1 \) element is zero. In what follows, we will relax condition (22), in order to preserve the convexity, and will replace \( x \) with its estimation \( \hat{x} \), which is generated based on \( y \), in order to preserve the output feedback structure. More precisely, we will consider controllers of the following form

\[
u = [K_1\ K_2] \begin{bmatrix} \hat{x} \\ y \end{bmatrix},
\]

where \( \hat{x} \) is an estimation of \( x \) generated from observing \( y \) and \( u \) such that the estimation error, \( e = \hat{x} - x \), is a bounded signal. The estimation \( \hat{x} \) is generated by an state-estimator in the following generic form

\[
\dot{x} = E_1 u + E_2 y,
\] (23)

where \( E_1 \) and \( E_2 \) are causal operators. Later (in Theorem 5), we will show that there always exist bounded operators \( E_1 \) and \( E_2 \) conforming with the network structure, i.e., \( E_1, E_2 \in \mathcal{S} \).

**Theorem 4:** Suppose \( \hat{x} \) is an estimation of \( x \) generated via the generic estimator (23) such that \( E_1, E_2 \in \mathcal{S} \) and \( ||e|| \leq \delta \) for some \( \delta > 0 \), where \( e = \hat{x} - x \). Then, the following conditions are equivalent:

1. A structured output-feedback controller \( \bar{K} \in \mathcal{S} : y \to u \) is stabilizing when implemented in the centralized way.
2. There exists causal operators \( K_1, K_2 \in \mathcal{S} \) such that

\[
u = [K_1\ K_2] \begin{bmatrix} \hat{x} \\ y \end{bmatrix} = \bar{K} y.
\] (24)
3. There exist stable causal operators \( Q \) and \( Z \) and some \( \varepsilon \in [0, 1) \) such that

\[
K_1 K_2 = Z(1 + Q)^{-1},
\] (25)

and \( Q \) and \( Z \) satisfy:

a) Model-Matching (18) holds.

b) Structure constraints (19) are satisfied.

4. For any \( \varepsilon \in [0, 1) \), there exist stable causal operators \( Q^\varepsilon \) and \( Z^\varepsilon \) such that Condition 3 holds for \( Q = Q^\varepsilon \) and \( Z = Z^\varepsilon \).

In the light of this theorem, the parameterization of all stabilizing output feedback controllers can be carried out via a convex search over parameters \( (Q, Z) \) provided that an estimation \( \hat{x} \) of the states \( x \) with uniform bound, \( ||\hat{x} - x|| \leq \delta \), is available, which is the subject of next section. By uniformity we mean that the bound \( \delta \) does not depend on the control input \( u \) and is uniform with respect to \( w \) and \( \bar{x}_0 \). More precisely, the should exist \( \delta_1, \delta_2 \geq 0 \) such that \( ||e|| \leq \delta_1 ||x_0|| + \delta_2 ||w|| \). The parameterization presented here does not require the doubly coprime factorization of the plant and this is an advantage especially when designing decentralized controllers. Furthermore, this parameterization holds valid not only for LTI systems but also any other linear system, e.g., delayed, time-varying, or switching systems.

1) **State-Estimator:** Here, we will define a state-estimator that mimics the standard Luenberger observer with a difference that its observer gain is a possibly unbounded operator. This state-estimator is referred to as the Generalized Luenberger Observer and has the form

\[
\dot{x} = \Lambda \hat{x} + \Lambda B_2 u + \Lambda L \left( C_2 \hat{x} - y \right),
\] (26)

where \( L \) is the observer operator-gain which can possibly be unstable.

**Theorem 5:** If there exists a centralized stabilizing output feedback controller with structure, there always exists a generalized Luenberger observer (26) with the network structure. The observer operator-gain is given by

\[
L = (I + Q \Lambda)^{-1} Z \in \mathcal{S}
\] (27)

where, for any \( \varepsilon \in [0, 1) \), \( Q \in \mathcal{S} \) and \( Z \in \mathcal{S} \) are stable causal operators satisfying \( ||E_L|| \leq \varepsilon \), where

\[
E_L := \bar{A} + Z \bar{C} - Q \left( I - \Lambda \bar{A} \right).
\] (28)

In this case, the state-estimator is simplified to

\[
\dot{x} = R_1 \Lambda \bar{B}_2 u - R_2 y,
\] (29)

where

\[
R_1 := \left( I - \Lambda \left( \bar{A} + L \bar{C}_2 \right) \right)^{-1} \in \mathcal{S},
\] (30)

and

\[
R_2 := R_1 \Lambda L \in \mathcal{S},
\] (31)
are bounded operators. Furthermore, the estimation error \( e \) is given by
\[
e = \hat{x} - x = - \left( R_1 \Lambda \hat{B}_1 + R_2 \hat{D}_{21} \right) w = R_1 x_0.
\] (32)

A realization of the state-estimator is given by (28) and can also be rewritten as
\[
\hat{x} = \Delta \xi_1 \hat{x} + (I + \Lambda Q_1^T) \Lambda \hat{B}_2 u - \Lambda Z_2^T y.
\] (33)

2) Stable Realization of Output Feedback: The operator forms (25) and (33) provide basis for stable realization of the overall controller over the network. The controller (25) can be realized as
\[
\left[ \begin{array}{c}
\xi_1 \\
\xi_2
\end{array} \right] = - \left[ \begin{array}{cc}
\Lambda Q_{11} & \Lambda Q_{12} \\
Q_{21} & \Lambda Q_{22}
\end{array} \right] \left[ \begin{array}{c}
\xi_1 \\
\xi_2
\end{array} \right] + \left[ \begin{array}{c}
\hat{x} \\
y
\end{array} \right],
\]
\[
u = \left[ Z_1 Z_2 \right] \left[ \begin{array}{c}
\xi_1 \\
\xi_2
\end{array} \right].
\] (34)

Then, combining the realization of the state-estimator (33) with (34), we obtain the following realization for the output feedback controller \( K : y \to u \):
\[
K : \left\{ \begin{array}{l}
x_K = A_K x_K + B_K y \\
u = C_K x_K
\end{array} \right.,
\] (35)

where
\[
A_K = \left[ \begin{array}{cc}
\Delta \xi_1 \left( I + \Lambda Q_1^T \right) \Lambda \hat{B}_2 Z_1 \\
I & -\Lambda Q_{11} \\
0 & -Q_{21} & -\Lambda Q_{22}
\end{array} \right],
\]
\[
B_K = \left[ \begin{array}{c}
-\Lambda Z_2^T \\
0 \\
I
\end{array} \right], C_K = \left[ 0 Z_1 Z_2 \right].
\] (36)

In the above expression, \( x_K \) is given by \( x_K = \left[ \hat{x} \xi_1 \xi_2 \right]^T \), where \( \hat{x} \) is the state-estimator as given in (33) and \( \xi_1 \) and \( \xi_2 \) are the signals given in (34). The realization (35) is implementable over the network as operators \( A_K, B_K, \) and \( C_K \) conform with the network structure \( S \). We, however, need to show that such implementation is stable. That is, the stability of the system is preserved even if the signals traveling between the subcontrollers are subject to noise, that is, if the noisy controller
\[
\bar{K} : \left\{ \begin{array}{l}
x_K = A_K x_K + B_K y + n_x \\
u = C_K x_K + n_u
\end{array} \right.,
\] (37)
is implemented. In other words, we need to show that the closed-loop maps from subcontrollers communication noise \( n_x \) and \( n_u \) to signals \( x, y, u, \) and \( x_K \) are bounded.

Theorem 6: The realization given in (35) can be implemented stably over the given network structure \( S \). That is, the feedback interconnection of \( \bar{K} \) and the plant results in bounded signals \( x, y, x_K \) and \( u \). This is \( \bar{S}_K = \bar{S} \).

For sparsity structures, one can find the realization of each subcontroller, from the realization of the aggregate controller, using a procedure similar to that of [9].

V. OPTIMAL CONTROL SYNTHESIS

In the last section, we parametrized the set of all structured stabilizing controllers via a convex model-matching problem (18) while enforcing the network structure on components of \( Q \) and \( Z \) as in (19). These conditions are identical for full-information and output feedback controllers. Furthermore, a stable realization for stabilizing controllers was given in (21) for full-information feedback and (35) for output feedback controller. This subsection is devoted to finding the optimal controller such that the closed-loop gain from the exogenous input \( w \) to regulated output \( z \) is minimized. To this end, first, we will derive the set of all closed-loop maps from \( w \) to \( z \) when an output stabilizing controller is utilized. The set of all output stabilizing controllers is parametrized by Theorem 4. For fixed \( \varepsilon \in [0, 1] \), the set of all such controllers are given by
\[
u = Z (I + Q)^{-1} \left[ \begin{array}{c}
\hat{x} \\
y
\end{array} \right],
\] (38)

where \( (Q, Z) \) satisfies (18)-(19) and \( \hat{x} \) is an state estimation. First, we find and fix a state-estimator (33) with structure as in Theorem 5. The following holds:

Proposition 1: The generalized plant (12) with the output feedback controller (38) results in the following closed-loop map \( \Phi_{wz} \) from \( w \) to \( z \):
\[
\Phi_{wz} = H + U \left[ \begin{array}{c}
Q \\
Z
\end{array} \right] \mathcal{E} (I - \mathcal{E})^{-1} V,
\] (39)

where \( \mathcal{E} = \mathcal{E}_{Q, Z} \) given in (17).

\[
H = C_1 \Lambda \hat{B}_1 + C_1 \Lambda \hat{A} \left( R_1 \Lambda \hat{B}_1 + R_2 \hat{D}_{21} \right) + \hat{D}_{11},
\] (40)

\[
U = \left[ \left[ \hat{C} \right] 0 \right] \hat{D}_{12},
\] (41)

\[
V = \left[ \left( \Lambda \hat{B}_1 \right) \hat{D}_{21} \right] + \left[ \Lambda \hat{A} - I \right] \left( R_1 \Lambda \hat{B}_1 + R_2 \hat{D}_{21} \right),
\] (42)

with \( R_1 \) and \( R_2 \) being stable operators defined in (30)-(31).

Notice that \( \mathcal{E} \) as defined in (17) can be made arbitrarily small according to Theorem 4. Therefore, without loss of generality, one can set \( \varepsilon \) in (17) equal to zero; in which case, the closed-loop map \( \Phi_{wz} \) will be an affine function of stable operators \( Q \) and \( Z \). This is stated in the following theorem without any further proof:

Theorem 7: Given a fixed generalized Luenberger observer, as provided in Theorem 5 any stably realizable output feedback controller can be written as a mapping from state-estimator \( \hat{x} \) and measured output \( y \) to control input \( u \), i.e.,
\[
K : \left[ \begin{array}{c}
\hat{x} \\
y
\end{array} \right] \to u, \text{ in the form (38), where (Q, Z) satisfies (18)-(19) for } \varepsilon = 0. \text{ Then, the closed-loop norm from the exogenous input } w \text{ to regulated output } z \text{ is given by}
\]
\[
\Phi_{wz} (K) = H + U \left[ \begin{array}{c}
Q \\
Z
\end{array} \right] V.
\]

Furthermore, the optimal achievable closed-loop gain, i.e.,
\[
\gamma^opt := \inf_{K \in \mathcal{S}} \text{ stabilizing } \left\| \Phi_{wz} (K) \right\|,
\]
is given by
\[
\gamma^opt = \inf_{Q, Z \in \mathcal{S}} \left\| H + U \left[ \begin{array}{c}
Q \\
Z
\end{array} \right] V \right\|,
\]
subject to that $Q$ and $Z$ satisfying the model-matching (18), with $\varepsilon = 0$, i.e. forcing $\mathcal{E} = 0$ in (17), and structure constraints (19).

**Remark 3:** Given Assumptions 1-4, any stabilizable controller has a corresponding factorization $(Q, Z)$ satisfying $\mathcal{E} = 0$. In our development, however, we showed that we can work with the relaxed condition $|\mathcal{E}| < 1$ instead of $\mathcal{E} = 0$. Relaxing this condition provides a computational advantage specifically for the $l_1$ or $l_\infty$ problems, where one typically seeks for the solution amongst finite impulse response $Q$ and $Z$ operators. In such cases, satisfying $\mathcal{E} = 0$ might be more challenging. Furthermore, as we will see later, for the class of problems where Assumptions 1-4 do not hold, one can still find subset of all stabilizing controllers by enforcing the structure on $Q$ and $Z$ as before. However, in those cases, $\mathcal{E}$ cannot necessarily be made arbitrarily small.

Although $\mathcal{E}$ given in (17) can be made equal to zero, for computational purposes, we would be interested to find upper and lower bounds on the closed-loop system gain when $\mathcal{E}$ is not exactly equal to zero.

**Theorem 8:** Fix $\rho_1 \in [0, 1)$ and $\rho_2 > 0$. Then, one can bound the optimal closed-loop norm from above and below, respectively, by positive numbers $\gamma_{upper}$ and $\gamma_{lower}$ via the following convex optimizations:

**Upper Bound:** An upper bound on the optimal closed-loop norm, i.e., $\inf_K \|\Phi_{wz}(K)\| \leq \gamma_{upper}$, can be obtained via convex optimization:

$$\gamma_{upper} = \inf_{Q, Z, \varepsilon} \left\| H + U \begin{bmatrix} Q \\ Z \end{bmatrix} V \right\| + \frac{\varepsilon \rho_2}{1 - \rho_1}, \quad (43)$$

subject to

$$\left\| U \right\|\begin{bmatrix} I + Q \\ Z \end{bmatrix} \right\| \left\| V \right\| \leq \rho_2, \quad (44)$$

$$\varepsilon \leq \rho_1, \quad (45)$$

$$\left[ \Lambda \bar{A} \begin{bmatrix} 0 \\ Z \end{bmatrix} \right] \left[ \Lambda \bar{A} - I \begin{bmatrix} 0 \\ Z \end{bmatrix} \right] \leq \varepsilon. \quad (46)$$

**Lower Bound:** A lower bound on the optimal closed-loop norm, i.e., $\inf_K \|\Phi_{wz}(K)\| \geq \gamma_{lower}$, can be obtained via convex optimization:

$$\gamma_{lower} = \inf_{Q, Z, \varepsilon} \left\| H + U \begin{bmatrix} Q \\ Z \end{bmatrix} V \right\| + \frac{\varepsilon \rho_2}{1 - \rho_1}, \quad (47)$$

subject to (45) and (46).

**A. Tractable algorithm to synthesize optimal controllers with arbitrary accuracy**

**Theorem 8** quantifies upper and lower bounds on the optimal performance. These upper and lower bounds converge as value of the parameter $\rho_2$ grows large, as we will show next. Without loss of generality, one can take $\rho_2$ to be integer valued. The algorithm is as follows:

**Algorithm 1.**

**Step 1** (Initialization): Find a stabilizing output-feedback controller. To this end, find a state-estimator using Theorem 5. Then, given $\varepsilon$ and $\rho_1$ with $0 \leq \varepsilon \leq \rho_1 < 1$, use Theorem 4 to find $(Q, Z)$ such that (18)-(19) hold. Then, a stabilizing controller is given by (25). Furthermore, for this pair $(Q, Z)$, pick an initial integer value for $\rho_2$ such that (44) is satisfied.

**Step 2:** Given $\rho_2$, find an upper bound, $\rho_2^{upper}$, on the input-output gain by solving the convex optimization (43) subject to constraints (44)-(45).

**Step 3:** Given $\rho_2$, find a lower bound, $\rho_2^{lower}$, on the input-output gain by solving the convex optimization (47) subject to constraints (45) and (46).

**Step 4:** Increase $\rho_2$ by one unit, i.e., $\rho_2 = \rho_2 + 1$, and jump to Step 2. Repeat the iterations until $\rho_2^{upper} - \rho_2^{lower}$ is less than the desired value.

**Theorem 9:** Algorithm 1 converges. That is,

$$\lim_{\rho_2 \to \infty} \rho_2^{upper} = \lim_{\rho_2 \to \infty} \rho_2^{lower} = \inf_K \|\Phi_{wz}(K)\|.$$

**VI. EXTENSION TO OTHER CONTROL STRUCTURES**

Throughout this paper, Assumptions 1-2 were made on the control structure $S$. These assumptions are satisfied if the topology of subcontrollers’ network is identical to or richer than that of the subsystems. There are, however, situations that the controller’s information structure is different than that of the generalized plant. In such cases, Assumptions 1-2 are not necessarily satisfied. In this case, for a controller to be stably implementable over the structure we still need it to satisfy the conditions in Definition 3. That is, the controller needs to have an operator realization that conforms with the network structure. Therefore, we can directly enforce the structure on the realization of the full-information controller (21) or output feedback (35). Additionally, for output feedback, we need to find a state-estimator that conforms with the structure. In the rest of this section, we do not make Assumptions 1-2 and, instead, we adopt the following assumption:

**Assumption 4:** The network structure $S$ is a subspace containing the identity and zero elements and is closed under addition. Furthermore, assume $AX \in S$ for any $X \in S$.

**A. Full-Information Feedback**

For the full-information feedback problems, we first parametrize the set of all centralized stabilizing controllers in the sense of Definition 4. That is the set of controllers that conform with the structure of the network $S$ but the stability is guaranteed if they are implemented in a centralized way. Next, we will show how such controllers can be stably realized/implemented in a distributed way over the network structure in the sense of Definition 5.

**Theorem 10:** A full-information controller $K$ is stably realizable over the network structure $S$ if there exist $\varepsilon \in [0, 1)$ and stable operators $Q \in S^{2 \times 2}$ and $Z \in S^{1 \times 2}$ such that

$$\left[ \begin{array}{cc} \Lambda \bar{A} & 0 \\ \varepsilon \bar{C}_2 & 0 \end{array} \right] + \left[ \begin{array}{cc} \Lambda \bar{A} - I & 0 \\ \varepsilon \bar{C}_2 & -I \end{array} \right] Q + \left[ \begin{array}{c} \Lambda B_2 \\ 0 \end{array} \right] Z \leq \varepsilon. \quad (48)$$

In this case, a stable realization is given by

$$K : \begin{cases} x_K = -Qx_K + \frac{x}{y} \\ u = Zx_K \end{cases}.$$
Under Assumptions $[12]$, $[48]$ becomes a necessary as well as sufficient condition for stable realizability of a controller. 

**Proof.** The proof is similar to that of Theorems $[1]$ and $[2]$. 

**Remark 4:** It is important to note that, unlike Theorem $[1]$ where Assumptions $[12]$ were made, it is not necessarily the case that $\varepsilon$ in $[48]$ can be made arbitrarily small. Therefore, in the absence of Assumptions $[12]$, it is crucial to work with the relaxed model-matching $[48]$ with $\varepsilon \in [0, 1)$ as opposed to fixing $\varepsilon = 0$. 

**B. Output Feedback**

Our approach of finding output feedback controllers relies on a new separation principle which utilizes a state-estimator. A state-estimator in a generic form is given by

$$\hat{x} = E_1 u + E_2 y,$$  

(49)

where $E_1$ and $E_2$ are causal operators.

**Lemma 2:** The state-estimator (49) results in a bounded estimation error, $e = \hat{x} - x$, if and only if for any $\varepsilon_L > 0$ there exists stable causal operators $Q_L^e$ and $Z_L^e$ such that $\|\mathcal{E}_L\| < \varepsilon_L$, where

$$(I + Q_L^e) \Lambda \hat{A} + Z_L^e \hat{C}_2 - Q_L^e = \mathcal{E}_L.$$  

(50)

In this case,

$$E_1 = (I - \mathcal{E}_L)^{-1} (I + Q_L^e) \Lambda \hat{B}_2,$$

$$E_2 = - (I - \mathcal{E}_L)^{-1} Z_L^e.$$  

The above lemma characterizes the set of all state-estimators. Such state-estimators conform with the network structure $\mathcal{S}$ if and only if $E_1$ and $E_2$, which are bounded operators, conform with the network structure. When $\varepsilon_L = 0$, in Lemma $\ref{L2}$ $E_1, E_2 \in \mathcal{S}$ if and only if

$$(I + Q_L^e) \Lambda \hat{B}_2 \in \mathcal{S}, Z_L^e \in \mathcal{S}.$$  

(51)

However, when $0 < \varepsilon_L < 1$, without Assumptions $[12]$ $\mathcal{E}_L$ may not have the network structure $\mathcal{S}$ and hence (51) does not, in general, yield $E_1, E_2 \in \mathcal{S}$. Therefore, for $\varepsilon_L \neq 0$, we will consider an alternative observer

$$\hat{x}_{\text{new}} = (I - \mathcal{E}_L) \hat{x} = (I + Q_L^e) \Lambda \hat{B}_2 u - Z_L^e y.$$  

We note that (51) ensures that $\hat{x}_{\text{new}}$ has the network structure and furthermore since $\hat{x}_{\text{new}} - \hat{x} = \mathcal{E}_L \hat{x}$, the difference can be made arbitrarily small by choosing a small $\varepsilon_L$. Then, for the output feedback problem, we will use $\hat{x}_{\text{new}}$ instead of $\hat{x}$. We write the control input as

$$u = [K_1 \ K_2] \left[ \begin{array}{c} \hat{x}_{\text{new}} \\ y \end{array} \right] = \left[ \begin{array}{c} [K_1 \ K_2] + \left[ \begin{array}{c} \mathcal{E}_L \\ 0 \end{array} \right] \\ 0 \end{array} \right] \left[ \begin{array}{c} \hat{x} \\ y \end{array} \right],$$

and treat $\left[ \begin{array}{c} \mathcal{E}_L \\ 0 \end{array} \right]$ as an additive uncertainty with size $\varepsilon_L$. Then, appealing to the small-gain theorem, we have the following:

**Theorem 11:** Suppose there exists $(\varepsilon_L, Q_L^e, Z_L^e)$ satisfying (50) and (51), and there exist bounded operators $Q = [Q_{11} \ Q_{12} \ Q_{21} \ Q_{22}] \in \mathcal{S}^{2\times 2}$ and $Z = [Z_1 \ Z_2] \in \mathcal{S}^{1\times 2}$ such that

$$\|\left[ \begin{array}{c} \Lambda \hat{A} \\ C_2 \end{array} \right] 0 + \left[ \begin{array}{c} \Lambda \hat{A} - I \ 0 \\ C_2 \end{array} \right] Q + [\hat{A} \hat{B}_2] Z \| \leq \varepsilon,$$

$$\varepsilon_L \|I + Q\| < 1,$$

for some $\varepsilon \in [0, 1)$. Then the following controller is stabilizing and stably implementable over the network structure:

$$K : \begin{cases} x_K = A_K x_K + B_K y, \\ u = C_K x_K, \\ \end{cases}$$  

where $x_K = [x^T, \xi_1^T, \xi_2^T, u^T]$.

**VII. ILLUSTRATIVE EXAMPLE**

Consider a nested structure with two subsystems $P_1$ and $P_2$ given as follows:

$$P_1 : \begin{cases} x_1(t + 1) = A x_1(t) + B_1 w_1(t) + B_2 u_1(t) \\ z_1(t) = C_1 x_1(t) + D_{12} u_1(t) \\ y_1(t) = C_2 x_1(t) + w_1(t) \\ u_{21}(t) = C_3 x_1(t) \\ \end{cases},$$

$$P_2 : \begin{cases} x_2(t + 1) = A x_2(t) + B_2 u_2(t) + B_3 u_{21}(t - 1) \\ z_2(t) = C_1 x_2(t) + D_{12} u_2(t) \\ y_2(t) = C_2 x_2(t) + u_2(t) \end{cases},$$

where

$$A = \begin{bmatrix} 0.5 & 0 \\ 0.3 & 1.2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C_2 = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, D_{12} = 1, C_3 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}.$$  

We enforce a lower-triangular structure on the aggregate controller with a single step delay from subsystem 1 to 2. That is,

$$u = \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{c} K_{11} \ 0 \\ \Lambda K_{21} \ K_{22} \end{array} \right] \left[ \begin{array}{c} y_1 \\ y_2 \end{array} \right].$$

Following our developments in this paper, to parametrize the set of all stabilizing controllers we need to find one state-estimator. This state estimator is given by (33) with $\mathcal{E}_L = 0$.
and FIR $Q_L$ and $Z_L$ as follows: $Q_L = \sum_{k=0}^{2} \Lambda^k q_L (k)$, $Z_L = \sum_{k=0}^{2} \Lambda^k z_L (k)$, with

$q_L (0) = \begin{bmatrix} 0.4680 & -0.3197 & 0 & 0 \\ 0.1685 & -0.1151 & 0 & 0 \\ 0 & 0 & 0.4785 & -0.2148 \\ 0 & 0 & 0.1722 & -0.0780 \end{bmatrix}$,

$q_L (1) = \begin{bmatrix} 0.1381 & -0.3836 & 0 & 0 \\ 0.0497 & -0.1381 & 0 & 0 \\ 0.0941 & -0.9590 & 0.1570 & -0.4361 \\ -0.0048 & -0.2376 & 0.0564 & -0.1567 \end{bmatrix}$,

$q_L (2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0397 & -0.0094 & 0 & 0 \\ -0.0143 & -0.0034 & 0 & 0 \end{bmatrix}$,

$z_L (0) = \begin{bmatrix} -0.3197 \\ -1.3151 \\ 0 \\ 0 \end{bmatrix}$,

$z_L (1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$z_L (2) = \begin{bmatrix} 0.4604 \\ 0.1657 \\ 1.0957 \\ 0.2653 \\ 0.1880 \end{bmatrix}$.

We are interested in finding the $l_\infty$ induced optimal controller. For this particular performance metric, the model matching problems in Algorithm 1 can be reduced to linear programs. This is carried out by performing the optimization over the space of impulse responses of stable systems $Q$ and $Z$. Furthermore, such infinite-dimensional optimizations can be approximated from above and below by finite-dimensional optimizations over the set of FIR $Q$ and $Z$ of the following form $Q = \sum_{k=0}^{N} \Lambda^k q (k)$, $Z = \sum_{k=0}^{N} \Lambda^k z (k)$, using the scaled-Q method of [20]. The finite upper and lower finite-dimensional approximations converge to optimal solution as $N$ grows large. For large $N$, we achieve the optimal closed-loop $l_\infty$ induced norm of 1.5 as shown in Figure 3.

![Fig. 3. Upper and lower bound approximations of optimal gain](image)

**VIII. CONCLUSION**

In this paper, we proposed a framework to synthesize structured controllers that can be stably realized over the network. This framework is unifying in the sense that various linear system, e.g., LTI, LTV, and linear switched systems, can be treated analogously with respect to any measure of performance, e.g., $l_1$, $l_2$, or $l_\infty$ induced norms. Our approach is based on utilizing an operator representation of the system and combining it with the classical Youla-parameterization. We formulated the stability and performance problems as tractable model-matching convex optimization. We proposed a new separation principle which was utilized to find output feedback controllers. Furthermore, the controllers can be stably realized over the network.

**IX. APPENDIX**

In this section, we present the proofs of the results presented in the paper. We will make use of the following lemmas regarding operators on $l_{\infty,e}$ in the proofs.

**Lemma 3:** Given an invertible partitioned operator $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, its inverse is given by $X^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ where $S_{11} = (X_{11} - X_{12} X_{22}^{-1} X_{21})^{-1}$, $S_{21} = -X_{22}^{-1} X_{21}$, $S_{12} = (X_{22} - X_{21} X_{11}^{-1} X_{12})^{-1}$, and $S_{12} = -X_{11}^{-1} X_{12} X_{22}$.

**Lemma 4:** Suppose $X \in S$ with $S$ satisfying Assumptions 1 and 2. Then, $(I - \Lambda X)^{-1} \in S$.

**A. Proof of Lemmas 1, 3, 4**

The proof of Lemma 3 is by straight inspection that $X X^{-1} = X^{-1} X = I$. The proofs of 1 and 4 follow similarly to the proofs in [7] and omitted here in the interest of space.

**B. Proof of Theorem 1**

We will prove this theorem in the following order $1 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

1. $\Rightarrow$ 4) Suppose there exists a centralized stabilizing controller $K$, conforming with the network structure $S$, from $[x^T, y^T]^T$ to $u$ such that the closed-loop system is stable.

Direct calculation verifies that the interconnection of the generalized plant (12) with $K = [K_1 K_2]$ results in the closed-loop map (13). According to Definition 4, the mappings from $\bar{x}_0$ and $w$ to $x$, $y$, and $u$ need to be bounded. Under Assumption 3, this renders operators

$$Q^0 := \left[I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \right]^{-1} \bar{C}_2 - I,$$  \hspace{1cm} (53)

and

$$Z^0 := \left[K_1 K_2 \right] \left[I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \right]^{-1} \bar{C}_2 (I + Q^0)$$  \hspace{1cm} (54)

bounded. We note that the inverse in (53) exists according to Lemma 1. In terms of these operators, (13) can be rewritten as

$$x = (I + Q^0) \begin{bmatrix} \Lambda \bar{B}_1 \\ D_{21} \end{bmatrix} w + \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{x}_0,$$

$$u = Z^0 \begin{bmatrix} \Lambda \bar{B}_1 \\ D_{21} \end{bmatrix} w + \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{x}_0.$$

We need to show (18) holds for $Q^0$ and $Z^0$ given in (53)-(54). To this end, pre-multiplying (53) by

$$\left[I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \right]^{-1} \left[I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \right],$$

we obtain

$$\left[I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \right] (I + Q^0) - I = 0.$$
The above expression, using \( Z^0 = \begin{bmatrix} K_1 & K_2 \end{bmatrix} (I + Q^0) \), yields
\[
\begin{bmatrix}
I - \Lambda \bar{A} & 0 \\
-C_2 & I
\end{bmatrix}
\begin{bmatrix}
I + Q^0 \\
\Lambda \bar{B}_2 & 0
\end{bmatrix} Z^0 - I = 0,
\]
which is the same as (18) for \( \varepsilon = 0 \). We, further need to show that \( Q^0 \) and \( Z^0 \) have the required structure given in (19). This is carried out by direct inspection of \( Q^0 \) and \( Z^0 \) given by (53) and (54). According to Lemma 3, \( Q^0 = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \) where
\[
q_{11} = (I - \Lambda \bar{A} + \bar{B}_2 K_1 + \bar{B}_2 K_2 C_2)^{-1} - I, \
q_{21} = \bar{C}_2 q_{11}, \\
q_{12} = (I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1)^{-1} \Lambda \bar{B}_2 K_2 q_{22}, \\
q_{22} = (I - \bar{C}_2 - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1) \Lambda \bar{B}_2 K_2)^{-1} - I,
\]
Now, notice that \( q_{ij} \), for \( i, j = 1, 2 \), is strictly causal and furthermore by Applying Lemma 4, \( q_{ij} \in S \). This completes the proof of (1 \( \Rightarrow \) 4).

4 \( \Rightarrow \) 3) Let \( Q^* = Q^0 \) and \( Z^* = Z^0 \). Then (18) holds for any \( \varepsilon \in [0, 1) \).

3 \( \Rightarrow \) 2) Immediate!

2 \( \Rightarrow \) 1) Suppose there exist stable operators \( Q^* \) and \( Z^* \) and some \( \varepsilon \in [0, 1) \) such that (18) holds. Define the controller
\[
K : \begin{bmatrix} x^T, y^T \end{bmatrix}^T \rightarrow u \text{ by }
K = Z^* (I + Q^*)^{-1}.
\]
Then, one can verify that the following identity holds
\[
\begin{bmatrix}
I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \\
-C_2 & I
\end{bmatrix}
\begin{bmatrix}
I + Q^* \\
\Lambda \bar{B}_2 & 0
\end{bmatrix} Z^* (I + Q^*)^{-1}.
\]
Therefore, the the closed-loop system (13) reduces to
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = (I + Q^*) (I - \varepsilon Q^*, Z^*)^{-1} \begin{bmatrix}
\bar{A}_1 w + \bar{x}_0 \\
\bar{D}_2 (1 - \varepsilon Q^*)^{-1} \bar{B}_1 w
\end{bmatrix} \\
u = Z^* (I - \varepsilon Q^*, Z^*)^{-1} \begin{bmatrix}
\bar{A}_1 w + \bar{x}_0 \\
\bar{B}_1 (1 - \varepsilon Q^*)^{-1} w
\end{bmatrix},
\]
which is a stable system since \( Z^* \) and \( Q^* \) are stable operators and \( (I - \varepsilon Q^*, Z^*)^{-1} \leq \frac{1}{1 - \varepsilon} \). This completes the proof.

C. Proof of Corollary 7

The proof is identical to the first part of Theorem 1. Suppose \( K = [K_1, K_2] \) is stabilizing. Then, \( Q^0 \) is defined as in (53) and \( Z^0 \) in (54) is given by
\[
Z^0 = \begin{bmatrix} K_1 & K_2 \end{bmatrix} (I + Q^0)
\]
or equivalently
\[
\begin{bmatrix} K_1 & K_2 \end{bmatrix} = Z^0 (I + Q^0)^{-1},
\]
and this completes the proof.

D. Proof of Theorem 2

According to Definition 5 in order for (21) to be a stable realization, one needs to show that the interconnection of noisy controller (16) with the generalized plant results in a stable system. The noisy controller is given by
\[
K : \begin{cases}
x_K = -Q x_K + \begin{bmatrix} x \\ y \end{bmatrix} + n_x, \\
u = Z x_K + n_u,
\end{cases}
\]
where \( n_x \) and \( n_u \) stand for subcontrollers’ communication noise. Then
\[
u = Z (I + Q)^{-1} \begin{bmatrix} x \\ y \end{bmatrix} + n_x + n_u.
\]
The generalized plant (12) driven by such control input yields
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
\Lambda \bar{A}_0 \\
\bar{C}_2 & 0
\end{bmatrix} + \begin{bmatrix}
\Lambda \bar{B}_2 \\
0 & 0
\end{bmatrix} Z (I + Q)^{-1} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
\Lambda \bar{B}_2 \\
0 & 0
\end{bmatrix} Z (I + Q)^{-1} n_x + \bar{n},
\]
where
\[
\bar{n} = \begin{bmatrix}
\Lambda \bar{B}_1 \\
\bar{D}_2 (1 - \varepsilon Q^*)^{-1} \bar{B}_1 \\
0 & 0
\end{bmatrix} w + \begin{bmatrix}
\Lambda \bar{B}_2 \\
0 & 0
\end{bmatrix} n_u + \begin{bmatrix}
I \\
0
\end{bmatrix} \bar{x}_0,
\]
is a bounded signal. Simplifying (57), we obtain
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = (I + Q) (I - \varepsilon Q^*, Z^*)^{-1} \begin{bmatrix}
I - \Lambda \bar{A}_0 \\
-\bar{C}_2 & I
\end{bmatrix} n_x - n_x + (I + Q) (I - \varepsilon Q^*, Z^*)^{-1} \bar{n},
\]
where \( \varepsilon Q^*, Z^* \) is given by (17). Noticing that \( \| \varepsilon Q^*, Z^* \| < 1 \), by (18), guarantees that \( (I - \varepsilon Q^*, Z^*)^{-1} \) is a bounded operator. Furthermore, since \( \bar{n} \) and \( n_x \) are bounded, the boundedness of \( x \) and \( y \) are guaranteed. Combining (58) with and (55), the controller satisfies
\[
x_K = (I - \varepsilon Q^*, Z^*)^{-1} \begin{bmatrix}
I - \Lambda \bar{A}_0 \\
-\bar{C}_2 & I
\end{bmatrix} n_x + \bar{n},
\]
\[
u = Z (I - \varepsilon Q^*, Z^*)^{-1} \begin{bmatrix}
I - \Lambda \bar{A}_0 \\
-\bar{C}_2 & I
\end{bmatrix} n_x + \bar{n} + n_u,
\]
and hence both the controller state \( x_K \) and control input \( u \) are bounded signals. This completes the proof.

E. Proof of Theorem 3

The proof is immediate by noticing that an output feedback is a special case of full-information controller \( K = [K_1, K_2] \) when \( K_1 = 0 \). Since any full-information controller can be written as \( K = Z (I + Q)^{-1} \) where \( Q \) and \( Z \) satisfy any of the equivalent conditions of Theorem 1 any output feedback can also be written as such while satisfying all of those equivalent conditions. In addition to those conditions, however, output feedback needs to satisfy \( K_1 = Z (I + Q)^{-1} \begin{bmatrix}
I \\
0
\end{bmatrix} = 0.\)
F. Proof of Theorem 4

We prove this theorem in the following order 1 \Rightarrow 2 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2

Proof of 1 \Rightarrow 2) Immediate by letting \( K_1 = 0 \) and \( K_2 = \hat{K} \).

Proof of 2 \Rightarrow 1) Suppose \( u = [K_1 K_2] \begin{bmatrix} \hat{x} \\ y \end{bmatrix} \) is stabilizing. Since the estimation \( \hat{x} \) is given by (23), we have
\[
u = K_1 \hat{x} + K_2 y = K_1 E_1 u + K_1 E_2 y + K_2 y,
\]
or equivalently \( u = (I - K_1 E_1)^{-1} (K_1 E_2 + K_2) y \), if \( (I - K_1 E_1) \) is invertible. We will prove later, in Theorem 3 that \( E_1 \) can be always made strictly causal and hence, by Lemma 1 \((I - K_1 E_1)^{-1}\) exists. Define \( \hat{K} = (I - K_1 E_1)^{-1} (K_1 E_2 + K_2) \). According to Assumption 1 and Lemma 4 \((I - K_1 E_1)^{-1} (K_1 E_2 + K_2) \in S \) and this completes this part of the proof.

Proof of 2 \Rightarrow 4) Suppose \( u = [K_1 K_2] \begin{bmatrix} \hat{x} \\ y \end{bmatrix} \) is stabilizing. We will show that there exists bounded operators \( Q^0 \) and \( Z^0 \) structured as in (19) such that (18) holds for \( \varepsilon = 0 \) and hence for any other \( \varepsilon \in [0, 1) \). The control input can be also written as
\[
u = [K_1 K_2] \begin{bmatrix} x + e \\ y \end{bmatrix},
\]
where \( e = \hat{x} - x \) is a bounded signal. Then, the closed-loop system becomes
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \\ \bar{C}_2 \end{bmatrix}^{-1} \begin{bmatrix} \Lambda \bar{B}_1 \\ D_{21} \end{bmatrix} \times \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{x}_0 + [K_1 K_2] \begin{bmatrix} e \\ 0 \end{bmatrix},
\]
\[
u = [K_1 K_2] \begin{bmatrix} x \\ y \end{bmatrix}.
\]
According to Definition 4, the mappings from \( \bar{x}_0 \) and \( w \) to \( x, y \) and \( u \) need to be bounded. Making Assumption 3 similarly to the proof of Theorem 1 operators
\[
Q^0 := \begin{bmatrix} I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 - \Lambda \bar{B}_2 K_2 \\ \bar{C}_2 \end{bmatrix}^{-1} - I,
\]
and
\[
Z^0 := [K_1 K_2] (I + Q^0),
\]
are bounded. Furthermore, identical to the proof of Theorem 2 one can show that \( Q = Q^0 \) and \( Z = Z^0 \) satisfy \( \mathcal{E}_{Q^0, Z^0} = 0 \) and this completes the proof of this part.

Proof of 4 \Rightarrow 3) Immediate!

Proof of 3 \Rightarrow 2) We need to show that
\[
u = Z^\varepsilon (I + Q^\varepsilon)^{-1} \begin{bmatrix} \hat{x} \\ y \end{bmatrix}
\]
results in structured controller from \( y \) to \( u \) and stable closed-loop system, where \( Z^\varepsilon \) and \( Q^\varepsilon \) satisfy \( \| \mathcal{E}_{Q^\varepsilon, Z^\varepsilon} \| \leq \varepsilon \), for some \( \varepsilon \in [0, 1) \), and structured as in (19). Controller (61) can be rewritten as
\[
u = Z^\varepsilon (I + Q^\varepsilon)^{-1} \begin{bmatrix} x \\ y \end{bmatrix} + Z^\varepsilon (I + Q^\varepsilon)^{-1} \begin{bmatrix} e \\ 0 \end{bmatrix},
\]
where \( e = \hat{x} - x \) and \( \| e \| \leq \delta \) for some \( \delta \geq 0 \). Then the closed loop system is given by
\[
\begin{bmatrix} x \\ y \end{bmatrix} = (I + Q^\varepsilon) (I - \mathcal{E}_{Q^\varepsilon, Z^\varepsilon})^{-1} \times \begin{bmatrix} \Lambda \bar{B}_1 \\ D_{21} \end{bmatrix} \times \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{x}_0 + \begin{bmatrix} \Lambda \bar{B}_2 \\ 0 \end{bmatrix} Z^\varepsilon (I + Q^\varepsilon)^{-1} \begin{bmatrix} e \\ 0 \end{bmatrix}.
\]
From the definition of \( \mathcal{E}_{Q^\varepsilon, Z^\varepsilon} \), we have
\[
\begin{bmatrix} \Lambda \bar{B}_2 \\ 0 \end{bmatrix} Z^\varepsilon (I + Q^\varepsilon)^{-1} = - (I - \mathcal{E}_{Q^\varepsilon, Z^\varepsilon}) (I + Q^\varepsilon)^{-1} + \begin{bmatrix} I - \Lambda \bar{A} \\ -\bar{C}_2 \end{bmatrix} I.
\]
Therefore, (62) can be further simplified to
\[
\begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} e \\ 0 \end{bmatrix} + (I + Q^\varepsilon) (I - \mathcal{E}_{Q^\varepsilon, Z^\varepsilon})^{-1} \times \begin{bmatrix} \Lambda \bar{B}_1 \\ D_{21} \end{bmatrix} \times \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{x}_0 + \begin{bmatrix} I - \Lambda \bar{A} \\ -\bar{C}_2 \end{bmatrix} I \begin{bmatrix} e \\ 0 \end{bmatrix}.
\]
From this expression, it is obvious that signals \( x \) and \( y \) remain bounded. Furthermore, one can verify that
\[
u = Z^\varepsilon \begin{bmatrix} \Lambda \bar{B}_1 \\ D_{21} \end{bmatrix} \times \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{x}_0 + \begin{bmatrix} I - \Lambda \bar{A} \\ -\bar{C}_2 \end{bmatrix} I \begin{bmatrix} e \\ 0 \end{bmatrix},
\]
which implies \( u \) is a bounded signal as well.

G. Proof of Theorem 5

Suppose \( \bar{K} = [0 K_2] : \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow u \) is a centralized stabilizing output feedback with \( K_2 \in S \). Therefore, the closed-loop maps \( [I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_1 \bar{C}_2] \) and \( [I - \Lambda \bar{A} - \Lambda \bar{B}_2 K_2 \bar{C}_2] \) are bounded operators. Define
\[
L := \bar{B}_2 K_2 \in S,
\]
and bounded operators \( Q^0_L, Z^0_L \in S \)
\[
\Lambda Q^0_L := [I - \Lambda \bar{A} - \Lambda \bar{L} \bar{C}_2]^{-1} - I,
\]
\[
\Lambda Z^0_L := [I - \Lambda \bar{A} - \Lambda \bar{L} \bar{C}_2]^{-1} \Lambda \bar{B}_2 K_2.
\]
Then it can be easily verified that
\[
\Lambda L = \Lambda \bar{B}_2 K_2 = (I + \Lambda Q^0_L)^{-1} \Lambda Z^0_L.
\]
Furthermore, from (66),
\[
(I + \Lambda Q^0_L) [I - \Lambda \bar{A} - \Lambda \bar{L} \bar{C}_2] = I,
\]
or equivalently \( \Lambda Q^0_L (I - \Lambda \bar{A}) - \Lambda \bar{A} + \Lambda Z^0_L = 0 \). This proves that the observer gain as defined in (65) satisfies (28) with \( \varepsilon = 0 \). It remains to show that such a state-estimator results in a bounded estimation error. To this end, suppose there exists \( Q^\varepsilon_1 \) and \( Z^\varepsilon_1 \) such that (28) holds for some \( \varepsilon \) and \( L = (I + Q^\varepsilon_L)^{-1} Z^\varepsilon_L \). Then, by adding and subtracting the term \( \Lambda L_0 \) to plant’s state equation, we obtain
\[
u = \Lambda (\bar{A} + \bar{L} \bar{C}_2) x + \Lambda (\bar{B}_{11} + \bar{L} \bar{D}_{21}) w + \Lambda \bar{B}_2 u + \Lambda L_0 y + \bar{x}_0,
\]
or equivalently
\[
u = R_1 (\Lambda (\bar{B}_{11} + \bar{L} \bar{D}_{21}) w + \bar{x}_0 + \Lambda \bar{B}_2 u - \Lambda L_0 y),
\]
where $R_1$ is defined in (30). Also, define $R_2$ as in (31). Then, from (28), we have

$$(I + \Lambda Q_L^T) \dot{A} + Z_L^T \dot{C}_2 - \dot{Q}_L = \mathcal{E}_L,$$

and it can be shown that

$$R_1 = (I - \Lambda \mathcal{E}_L)^{-1} (I + \Lambda Q_L^T) \in \mathcal{S},$$

$$R_2 = (I - \Lambda \mathcal{E}_L)^{-1} \Lambda Z_L^T \in \mathcal{S},$$

and hence both are structured and bounded. Furthermore, (67) can be simplified to

$$x = (R_1 \Lambda \dot{B}_1 + R_2 \dot{D}_{21}) w + R_1 \dot{x}_0 + R_1 \Lambda \dot{B}_2 u - R_2 y,$$  \(68\)

and (26) is reduced to

$$\dot{x} = R_1 \Lambda \dot{B}_2 u - R_2 y.$$  \(69\)

Therefore, the estimation error is given by

$$e = \dot{x} - x = -(R_1 \Lambda \dot{B}_1 + R_2 \dot{D}_{21}) w - R_1 \dot{x}_0,$$

which is the same as (32) and completes the proof.

\[H. \ Proof \ of \ Theorem \ [4]\]

We need to show that the noisy controller (37) results in a stable closed-loop system. Assumed partitioned $n_x = [n_x^T, n_\xi^T]^T$. First, we will show that $\dot{x}$ entry of $x_K$ remains an estimation of plant states, $x$, with bounded error. From (37), we obtain

$$\dot{x} = \Lambda \mathcal{E}_L \dot{x} + (I + \Lambda Q_L^T) \Lambda \dot{B}_2 (u - n_u) - \Lambda Z_L^T y + n_\xi,$$

where we used $u = Z \xi + n_u$. Then, given the plant dynamic (68), the new estimator error, $e = \dot{x} - x$, is given by

$$e = e_0 + (I - \Lambda \mathcal{E}_L)^{-1} n_x - R_1 n_u,$$

where $e_0 = -(R_1 \Lambda \dot{B}_1 + R_2 \dot{D}_{21}) w - R_1 \dot{x}_0$. This implies that the estimation error remains bounded. Next, we will show that the plant’s states and output, i.e., signals $x$ and $y$, are bounded using the fact that the estimation error remains bounded. To this end, from (37), the control input $u$ is given by

$$\xi = -Q \xi + \left[\begin{array}{c} \dot{x} \\ y \end{array}\right] + n_\xi,$$

$$u = Z \xi + n_u.$$  \(70\)

Consequently, using $\dot{x} = x + e,$

$$\xi = (I + Q)^{-1} \left[\begin{array}{c} \dot{x} \\ y \end{array}\right] + n_1,$$  \(71\)

$$u = Z (I + Q)^{-1} \left[\begin{array}{c} \dot{x} \\ y \end{array}\right] + Z (I + Q)^{-1} n_1 + Z n_u,$$

where $n_1 = [e^T, 0]^T + n_\xi$. Then, the closed-loop plant dynamics becomes

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} [\Lambda A 0] \\ [C_2 0] \end{array}\right] \left[\begin{array}{c} \dot{x} \\ y \end{array}\right] + \left[\begin{array}{c} \Lambda \dot{B}_2 \\ 0 \end{array}\right] Z (I + Q)^{-1} \left[\begin{array}{c} \dot{x} \\ y \end{array}\right]$$

$$+ \left[\begin{array}{c} \Lambda \dot{B}_1 w + \dot{x}_0 + \Lambda \dot{B}_2 Z n_u \\ \dot{D}_{21} w \end{array}\right] + \left[\begin{array}{c} \Lambda \dot{B}_2 \\ 0 \end{array}\right] Z (I + Q)^{-1} \bar{n},$$

where $n = n_1 + (I + Q) n_u$. Using the definition of $\mathcal{E}_{Q, z}$ to simplify, we obtain

$$\left[\begin{array}{c} x \\ y \end{array}\right] = -\bar{n} + (I + Q) (I - \mathcal{E}_{Q, z})^{-1} \times$$

$$\left\{ \begin{array}{c} \Lambda \dot{B}_1 w + \dot{x}_0 + \Lambda \dot{B}_2 Z n_u \\ \dot{D}_{21} w \end{array}\right\} + \left[\begin{array}{c} I - \Lambda \bar{A} 0 \\ -\bar{C}_2 I \end{array}\right] \bar{n}.$$

The above expression implies that $x$ and $y$ are bounded signals. Furthermore, from (71), the control input can be written as simplified as follows:

$$u = Z (I + Q)^{-1} \left[\begin{array}{c} \dot{x} \\ y \end{array}\right] + Z (I + Q)^{-1} n_\xi + Z n_u$$

$$= Z (I - \mathcal{E}_{Q, z})^{-1} \times$$

$$\left\{ \begin{array}{c} [I - \Lambda \bar{A} 0] \\ -\bar{C}_2 I \end{array}\right\} \bar{n} + \left[\begin{array}{c} I \\ 0 \end{array}\right] \dot{x}_0 + \left[\begin{array}{c} \Lambda \dot{B}_1 \\ \dot{D}_{21} w \end{array}\right],$$

which implies $u$ is also a bounded signal. Given that $x, y, u,$ and $e$ are bounded signals, we will show that the states of the controller, $x_K$, are bounded. The states of the controller are $\dot{x}$ and $\xi$. Signal $\dot{x}$ is bounded since $\dot{x} = e + x$. Then, from the definition of $\mathcal{E}_{Q, z}$, we have

$$\mathcal{E}_{Q, z} \xi = \left[\begin{array}{c} \Lambda \bar{A} 0 \\ \bar{C}_2 0 \end{array}\right] (I + Q) \xi + \left[\begin{array}{c} \Lambda \dot{B}_2 \\ 0 \end{array}\right] Z \xi - Q_\xi.$$  \(72\)

Combining (72) with (71) and (70), we obtain

$$\mathcal{E}_{Q, z} \xi = \left[\begin{array}{c} \Lambda \bar{A} 0 \\ \bar{C}_2 0 \end{array}\right] \left\{ \begin{array}{c} \dot{x} \\ y \end{array}\right\} + n_1 + \left[\begin{array}{c} \Lambda \dot{B}_2 \\ 0 \end{array}\right] (u - n_u) - Q_\xi,$$  \(73\)

Using the generalized plant, one can simplify (73) to

$$\mathcal{E}_{Q, z} \xi = \left[\begin{array}{c} x \\ y \end{array}\right] - Q_\xi + n_2,$$  \(74\)

where

$$n_2 = -\left[\begin{array}{c} \Lambda \dot{B}_1 w + \dot{x}_0 \\ \dot{D}_{21} w \end{array}\right] + \left[\begin{array}{c} \Lambda \bar{A} 0 \\ \bar{C}_2 0 \end{array}\right] n_1 - \left[\begin{array}{c} \Lambda \dot{B}_2 \\ 0 \end{array}\right] n_u.$$  \(75\)

Again from (71), we have $\xi = [x^T, y^T]^T - Q_\xi + n_1$, and right hand side of (74) is simplified to yield

$$\mathcal{E}_{Q, z} \xi = \xi + n_2 - n_1,$$

and consequently $\xi = (I - \mathcal{E}_{Q, z})^{-1} (n_2 - n_1)$. Hence $\xi_1$ and $\xi_2$ are bounded signals and the proof is complete.

\[I. \ Proof \ of \ Proposition \ [7]\]

The proof is carried out by direct calculations. First, given $[x^T, y^T]^T$ and $u$ as in (63)-(64), the regulated output $z$ can be simplified to

$$z = \left\{ \begin{array}{c} [C_1 0] (I + Q) + \dot{D}_{12} Z \end{array}\right\} (I - \mathcal{E}_{Q, z})^{-1} \times$$

$$\left\{ \begin{array}{c} \Lambda \dot{B}_1 w + \dot{x}_0 + \left[\begin{array}{c} I - \Lambda \bar{A} 0 \\ -\bar{C}_2 I \end{array}\right] \bar{n} \\ \dot{D}_{21} w \end{array}\right\} + \dot{D}_{11} w - [C_1 0] \bar{e}.$$  \(76\)
From Theorem 5, the estimation error is given by (32). Combining (75) and (32), we obtain \( z = \Phi_{wz} w + \Phi_{xu} x_0 \), where

\[
\Phi_{wz} = \{ [\bar{C}_1 0] (I + Q) + \bar{D}_{12} Z \} (I - \mathcal{E}_{Q,Z})^{-1} \times \left\{ \begin{array}{c}
\Lambda \bar{B}_1 \\
\bar{C}_2 \end{array} \right\} (R_1 \Lambda \bar{B}_1 + R_2 \bar{D}_{21}) + \bar{C}_1 (R_1 \Lambda \bar{B}_1 + R_2 \bar{D}_{21}) + \bar{D}_{11}.
\]

Using identity \((I - \mathcal{E})^{-1} = I + \mathcal{E} (I - \mathcal{E})^{-1}\), and rearranging the terms, we obtain

\[
\Phi_{wz} = H + U \left[ \begin{array}{c} Q \\ Z \end{array} \right] V + U \left[ \begin{array}{c} I + Q \\ Z \end{array} \right] \mathcal{E} (I - \mathcal{E})^{-1} V,
\]

where \( H, U, \) and \( V \) are given in the theorem statement.

\( J. \) Proof of Theorem 8

1) Proof of Upper Bound:

Pick \( \rho_1 \in (0,1) \). By Theorem 1, any stabilizing controller can be written as \( K = Z (I + Q)^{-1} \) with \( \|\mathcal{E}_{Q,Z}\| \leq \varepsilon \leq \rho_1 \). Then, by Proposition 1

\[
\|\Phi_{wz} (K)\| \leq \left\| H + U \left[ \begin{array}{c} Q \\ Z \end{array} \right] V \right\| + \frac{\varepsilon}{1 - \varepsilon} \| U \| \| V \| \left\| \left[ \begin{array}{c} I + Q \\ Z \end{array} \right] \right\|.
\]

Therefore,

\[
\gamma^{opt} \leq \inf_{Q,Z,\varepsilon} \left\| H + U \left[ \begin{array}{c} Q \\ Z \end{array} \right] V \right\| + \frac{\varepsilon \rho_2}{1 - \rho_1},
\]

Restricting the set of \( Q \) and \( Z \) to those that satisfy (44), we obtain

\[
\gamma^{opt} \leq \inf_{Q,Z,\varepsilon} \left\| H + U \left[ \begin{array}{c} Q \\ Z \end{array} \right] V \right\| + \frac{\varepsilon \rho_2}{1 - \rho_1},
\]

where we used \( \frac{\varepsilon}{1 - \varepsilon} \leq \frac{\varepsilon}{1 - \rho_1} \).

2) Proof of Lower Bound:

Given a small number \( \delta > 0 \), let \( K^* \) be a stabilizing controller that results in a closed-loop performance in the \( \delta \)-ball of optimal. By Theorem 4 there exists \( Q^* \) and \( Z^* \) such that \( K^* = Z^* (I + Q^*)^{-1} \),

\[
\left[ \begin{array}{c}
\Lambda A \\
\bar{C}_2 \end{array} \right] \left[ \begin{array}{c} 0 \\
\bar{C}_2 \end{array} \right] Q^* + \left[ \begin{array}{c} \Lambda \bar{B}_2 \\
0 \end{array} \right] Z^* = 0,
\]

\[
\Phi_{wz} (K^*) = \left\| H + U \left[ \begin{array}{c} Q^* \\ Z^* \end{array} \right] V \right\| \leq \gamma^{opt} + \delta.
\]

Given \( \rho_1, \rho_2 > 0 \), notice that \((Q^*,Z^*)\) is a feasible solution for (45) and (46) with \( \varepsilon = 0 \). Therefore, we have

\[
\gamma_{lower} \leq \left\| H + U \left[ \begin{array}{c} Q^* \\ Z^* \end{array} \right] V \right\|.
\]

Hence, \( \gamma_{lower} \leq \gamma^{opt} + \delta \), for any arbitrary \( \delta \geq 0 \).

\( K. \) Proof of Algorithm 1

First, we will show that the optimization in Step 2 provide a converging upper bound on \( \gamma^{opt} \) as \( \rho_2 \) grows larger. The fact that \( \gamma_{upper}^{opt} \) is an upper bound on \( \gamma^{opt} \) comes directly from Theorem 8. Now, given a small number \( \delta > 0 \), let \( K^* \) be a stabilizing controller that results in a closed-loop performance in the \( \delta \)-ball of optimal. That is, there exists \( Q^* \) and \( Z^* \) such that \( K^* = Z^* (I + Q^*)^{-1} \) such that \( \Phi_{wz} (K^*) \leq \gamma^{opt} + \delta \), where \( Q^* \) and \( Z^* \) satisfy \( \mathcal{E} = 0 \) with \( \mathcal{E} \) defined in (17). Define, \( m^{opt} \) to be an integer such that

\[
\| U \| \left\| \left[ \begin{array}{c} I + Q^* \\ Z^* \end{array} \right] \right\| \| V \| \leq m^{opt}.
\]

Thus, it is obvious that \((Q^*,Z^*)\) together with \( \varepsilon = 0 \) is a feasible solution for the optimization in Step 2 if \( \rho_2 \geq m^{opt} \).

Since \( \delta \) was arbitrary, we have \( \lim_{\rho_2 \to \infty} \gamma_{upper}^{opt} = \gamma^{opt} \). Next, we will show that Step 3 provides a converging lower bound. Pick \( \delta > 0 \) arbitrary and \( K^* \) as before. Note that \((Q^*,Z^*)\) together with \( \varepsilon = 0 \) is a feasible solution to the optimization in Step 3, for any value of \( \rho_2 \). Therefore, we always have

\[
\gamma_{lower} = \inf_{Q,Z,\varepsilon} \left\| H + U \left[ \begin{array}{c} Q \\ Z \end{array} \right] V \right\| + \frac{\varepsilon \rho_2}{1 - \rho_1} \leq \left\| H + U \left[ \begin{array}{c} Q^* \\ Z^* \end{array} \right] V \right\| + \frac{\varepsilon \rho_2}{1 - \rho_1} = \gamma^{opt} + \delta.
\]

Therefore, \( \gamma_{lower} \leq \gamma^{opt} + \delta \), and since, this holds for any arbitrary \( \delta \) we must have \( \gamma_{lower}^{opt} \leq \gamma^{opt} \). To show that convergence of \( \gamma_{lower}^{opt} \) to \( \gamma_{upper}^{opt} \) and consequently \( \gamma^{opt} \), we note that the only difference between the optimizations in Step 2 and 3 is constraint (44). This constraint becomes inactive if \( \rho_2 \geq m^{opt} \) as given by (77). Therefore, we have \( \lim_{\rho_2 \to \infty} \gamma_{lower}^{opt} = \lim_{\rho_2 \to \infty} \gamma_{upper}^{opt} = \gamma^{opt} \) and also \( \gamma_{lower} = \gamma_{upper}^{opt} \), for \( \rho_2 \geq m^{opt} \).

\( L. \) Proof of Lemma 2

Suppose an state-estimator in the form of \( \Phi_{\epsilon} \) results in a bounded estimation error, \( e = \hat{x} - x \). Then \( e = E_1 u + E_2 y - x \), is bounded where \( x = (I - \Lambda^A)^{-1} [\Lambda \bar{B}_1 w + \Lambda \bar{B}_2 u + \tilde{x}_0] \) and \( y = \bar{C}_2 x + \bar{D}_{21} w \). Simplifying the error dynamics further, we obtain

\[
\epsilon = E_1 u + (E_2 \bar{C}_2 - I) x + E_2 \bar{D}_{21} w
\]

\[
= \left[ (E_2 \bar{C}_2 - I) (I - \Lambda^A)^{-1} \Lambda \bar{B}_1 + E_2 \bar{D}_{21} \right] u
\]

\[
+ \left[ (E_2 \bar{C}_2 - I) (I - \Lambda^A)^{-1} \Lambda \bar{B}_2 + E_1 \right] u
\]

\[
+ (E_2 \bar{C}_2 - I) (I - \Lambda^A)^{-1} \tilde{x}_0.
\]

For the error to be a bounded signal, the operators \( (E_2 \bar{C}_2 - I) (I - \Lambda^A)^{-1} \) and \( E_2 \) must be bounded. Define

\[
Q_L = - (E_2 \bar{C}_2 - I) (I - \Lambda^A)^{-1} I, Z_L = - E_2,
\]

and pick \( E_1 = (I + Q_L) \Lambda \bar{B}_2 \). Then, from (78), we have \( (I + Q_L) \Lambda A + Z_L \bar{C}_2 - Q_L = 0 \), and the estimator and error dynamics read

\[
\dot{x} = (I + Q_L) \Lambda \bar{B}_1 + Z_L \bar{D}_{21} w - (I + Q_L) \tilde{x}_0.
\]

The proof of converse is similar to the proof of Theorem 5.
REFERENCES

[1] G. E. Dullerud and R. D’Andrea, “Distributed control of heterogeneous systems,” IEEE Transactions on Automatic Control, vol. 49, no. 12, pp. 2113–2128, 2004.

[2] C. Langbort, R. S. Chandra, and R. D’Andrea, “Distributed control design for systems interconnected over an arbitrary graph,” IEEE Transactions on Automatic Control, vol. 49, no. 9, pp. 1502–1519, 2004.

[3] P. G. Voulgaris, “A convex characterization of classes of problems in control with specific interaction and communication structures,” in American Control Conference, 2001. Proceedings of the 2001, vol. 4. IEEE, 2001, pp. 3128–3133.

[4] X. Qi, M. V. Salapaka, P. G. Voulgaris, and M. Khammash, “Structured optimal and robust control with multiple criteria: A convex solution,” IEEE Transactions on Automatic Control, vol. 49, no. 10, pp. 1623–1640, 2004.

[5] P. G. Voulgaris, G. Bianchini, and B. Bamieh, “Optimal h2 controllers for spatially invariant systems with delayed communication requirements,” Systems & Control Letters, vol. 50, no. 5, pp. 347–361, 2003.

[6] B. Bamieh and P. G. Voulgaris, “A convex characterization of distributed control problems in spatially invariant systems with communication constraints,” Systems & Control Letters, vol. 54, no. 6, pp. 575–583, 2005.

[7] M. Rotkowitz and S. Lall, “A characterization of convex problems in decentralized control,” IEEE Transactions on Automatic Control, vol. 51, no. 2, pp. 274–286, 2006.

[8] Ş. Sabău, N. C. Martins, and M. C. Rotkowitz, “A convex characterization of multidimensional linear systems subject to sqi constraints,” IEEE Transactions on Automatic Control, vol. 62, no. 6, pp. 2981–2986, 2016.

[9] Ş. Sabău and N. C. Martins, “Youla-like parametrizations subject to qi subspace constraints,” IEEE Transactions on Automatic Control, vol. 59, no. 6, pp. 1411–1422, 2014.

[10] S. M. V. Andalam and N. Elia, “Design of distributed controllers realizable over arbitrary directed networks,” in Decision and Control (CDC), 2010 49th IEEE Conference on. IEEE, 2010, pp. 4795–4800.

[11] A. S. M. Vamsi and N. Elia, “A sub-optimal approach to design distributed controllers realizable over arbitrary networks,” IFAC Proceedings Volumes, vol. 43, no. 19, pp. 329–334, 2010.

[12] V. Yadav, M. V. Salapaka, and P. G. Voulgaris, “Architectures for distributed controller with sub-controller communication uncertainty,” IEEE Transactions on Automatic Control, vol. 55, no. 8, pp. 1765–1780, 2010.

[13] L. Lessard, M. Kristaly, and A. Rantzer, “On structured realizability and stabilizability of linear systems,” in American Control Conference (ACC), 2013. IEEE, 2013, pp. 5784–5790.

[14] A. S. M. Vamsi and N. Elia, “Optimal distributed controllers realizable over arbitrary networks,” IEEE Transactions on Automatic Control, vol. 61, no. 1, pp. 129–144, 2016.

[15] Y.-S. Wang, N. Matni, and J. C. Doyle, “System level parameterizations, constraints and synthesis,” in American Control Conference (ACC). IEEE, 2017.

[16] M. Naghnaeian, P. G. Voulgaris, and G. E. Dullerud, “A unified framework for Lp analysis and synthesis of linear switched systems,” in American Control Conference (ACC), 2016. IEEE, 2016, pp. 715–720.

[17] M. Naghnaeian and P. G. Voulgaris, “Characterization and optimization of L∞ gains of linear switched systems,” IEEE Transactions on Automatic Control, vol. 61, no. 8, pp. 2203–2218, 2016.

[18] M. Naghnaeian, P. G. Voulgaris, and N. Elia, “A unified framework for decentralized control synthesis,” in 2018 European Control Conference (ECC). IEEE, 2018, pp. 2482–2487.

[19] K. Zhou, J. C. Doyle, K. Glover et al., Robust and optimal control. Prentice hall New Jersey, 1996, vol. 40.

[20] M. Khammash, “A new approach to the solution of the L∞/L1 control problem: the scaled-q method,” IEEE Transactions on Automatic Control, vol. 45, no. 2, pp. 180–187, 2000.

Mohammad Naghnaeian received a Ph.D. degree, in 2016, from the University of Illinois, Urbana-Champaign, IL, USA. He is currently an assistant professor at the Department of Mechanical Engineering, Clemson University. His research interests include robust and distributed control and estimation, linear switched systems, positive systems, the security of cyber-physical systems, adaptive control, and biological systems.

Nicola Elia received the Laurea degree in Electrical Engineering from the Politecnico di Torin, Turin, Italy, in 1987, and the Ph.D. degree in Electrical Engineering and Computer Science from the Massachusetts Institute of Technology, Cambridge, MA., in 1996. He is a Fellow of IEEE and the Vincentine Hermes-Luh Chair of Electrical and Computer Engineering at University of Minnesota, Minneapolis. His research interests include computational methods for controller design, communication systems with access to feedback, control with communication constraints, and networked systems.

Petros G. Voulgaris received the Diploma in Mechanical Engineering from the National Technical University, Athens, Greece, in 1986, and the S.M. and Ph.D. degrees in Aeronautics and Astronautics from the Massachusetts Institute of Technology, Cambridge, in 1988 and 1991, respectively. Since 1991, he has been with the Department of Aerospace Engineering, University of Illinois at Urbana Champaign, where he is currently a Professor. He also holds joint appointments with the Coordinated Science Laboratory, and the department of Electrical and Computer Engineering at the same university. His research interests include robust and optimal control and estimation, communications and control, networks and control, and applications of advanced control methods to engineering practice including flight control, nano-scale control, robotics, and structural control systems. Dr. Voulgaris is a recipient of the National Science Foundation Research Initiation Award (1993), the Office of Naval Research Young Investigator Award (1995) and the UIUC Xerox Award for research. He has been an Associate Editor for the IEEE Transactions on Automatic Control and the ASME Journal of Dynamic Systems, Measurement and Control. He is also a Fellow of IEEE.