Complete Asymptotics for Solution of Singularly Perturbed Dynamical Systems with Single Well Potential

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Abstract: We consider a singularly perturbed boundary value problem \((-\varepsilon^2 \Delta + \nabla V \cdot \nabla) u_\varepsilon = 0\) in \(\Omega\), \(u_\varepsilon = f\) on \(\partial \Omega\), \(f \in C^\infty(\partial \Omega)\). The function \(V\) is supposed to be sufficiently smooth and to have the only minimum in the domain \(\Omega\). This minimum can degenerate. The potential \(V\) has no other stationary points in \(\Omega\) and its normal derivative at the boundary is non-zero. Such a problem arises in studying Brownian motion governed by overdamped Langevin dynamics in the presence of a single attracting point. It describes the distribution of the points at the boundary \(\partial \Omega\), at which the trajectories of the Brownian particle hit the boundary for the first time. Our main result is a complete asymptotic expansion for \(u_\varepsilon\) as \(\varepsilon \to +0\). This asymptotic is a sum of a term \(K_\varepsilon \Psi_\varepsilon\) and a boundary layer, where \(\Psi_\varepsilon\) is the eigenfunction associated with the lowest eigenvalue of the considered problem and \(K_\varepsilon\) is some constant. We provide complete asymptotic expansions for both \(K_\varepsilon\) and \(\Psi_\varepsilon\); the boundary layer is also an infinite asymptotic series power in \(\varepsilon\). The error term in the asymptotics for \(u_\varepsilon\) is estimated in various norms.

Keywords: exit time problem; equations with small parameter at higher derivatives; asymptotics; overdamped Langevin dynamics

MSC: 35B25; 35C20

1. Introduction

The present work is devoted to studying the asymptotic behavior of a solution to the Dirichlet problem for an elliptic equation with a small parameter at higher derivatives:

\[-\varepsilon^2 \Delta u_\varepsilon - \sum_{i=1}^n F_i(x) \frac{\partial u_\varepsilon}{\partial x_i} = 0, \quad x \in \Omega, \quad u_\varepsilon = f, \quad x \in \partial \Omega,\]

where \(F_j, j = 1, \ldots, n\), and \(f\) are some given functions. Our main motivation for considering such problem comes from stochastic processes \(Y_\varepsilon(t,a) \in \mathbb{R}^n\) defined by means of Itô stochastic differential equations ([1], Section 4.1):

\[dY_\varepsilon(t,a) = F(Y_\varepsilon(t,a))dt + \sqrt{2}\varepsilon dW, \quad Y_\varepsilon(0,a) \equiv x, \quad a \in \mathcal{A},\]

where \(F = (F_1, \ldots, F_n),\) \(W = (W_1(t,a), \ldots, W_n(t,a))\) is an \(n\)-dimensional Wiener process defined on a probability space \((\mathcal{A}, \mathcal{F}, \mathbb{P})\). Here \(\mathcal{A} = \{a\}\) is the sample space, \(\mathcal{F}\) is a \(\sigma\)-algebra, \(\mathbb{P}\) is a
probability measure. The symbol $\varepsilon$ stands for a small parameter characterizing the intensity of the perturbation $W$. We assume that the origin is an asymptotically stable equilibrium for unperturbed Equation (2) as $\varepsilon = 0$. A solution to problem (2) is a random process $Y_t(a)$, $t \geq 0$, $a \in A$, whose trajectories leave each neighbourhood $\Omega$ of the equilibrium almost surely, see ([2], Ch. 4). At that, one of the key characteristics of this process is the first time, when a trajectory $Y_t(a)$ hits the boundary of the neighbourhood $\Omega$. This first time is defined as

$$T_\varepsilon(a) = \min\{t : Y_t(a) \in \partial \Omega\}.$$ 

In terms of this time we introduce the function

$$u_\varepsilon(x) := \mathbb{E}\left(f(Y_{T_\varepsilon(a)}(a)) \mid Y_0(0, a) = x\right).$$

This function characterizes the distribution of the points $Y_t(T_\varepsilon(a), a)$ on the boundary $\partial \Omega$, at which the trajectories hit the boundary for the first time. According ([1], Section 4.4.2), the function $u_\varepsilon(x)$ solves boundary value problem (1).

Asymptotic behaviour of the solutions to such and similar problems was studied before in a series of works. In [3], the Dirichlet problem for the equation

$$\varepsilon^2 \Delta u + Au_x + Bu_y + C = 0$$

was considered in a two-dimensional domain under the assumption $A^2 + B^2 > 0$ or $C < 0$. An asymptotic expansion for the solution was constructed and a proposed way for finding its terms was rather complicated. In [4], the Dirichlet problem was considered for a multi-dimensional equation

$$\varepsilon \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x) \frac{\partial u}{\partial x_i} = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \partial \Omega.$$ 

In a non-rigorous way, basing on heuristic considerations, a formula for the leading term in the asymptotics for the solution was found. No justification was made and no estimates for the error term were established. A rigorous result was obtained in [5]. Namely, it was shown that, as $\varepsilon \to 0$, the solution converges to a certain explicitly found constant uniformly on compact subsets of the domain $\Omega$. In [6], this result was improved in the following sense: it was proved that a solution $u_\varepsilon$ of problem (1) converges to $u(0)$ exponentially fast uniformly on compact subsets of the domain $\Omega$. Various aspects of the convergence of the solutions to constants were also discussed in papers [7–10]. In [11,12], more general evolutionary problems were considered and asymptotic results were obtained by analysing the behavior of the solutions for a large time. In [13], a result of such type was obtained for a quasilinear equation.

It should be also mentioned that a close problem for an inhomogeneous equation with a homogeneous boundary condition

$$-\varepsilon^2 \Delta \nu_\varepsilon + \nabla V \cdot \nabla \nu_\varepsilon = 1, \quad x \in \Omega, \quad \nu_\varepsilon = 0, \quad x \in \partial \Omega, \quad (3)$$

was actively studied in the case, when a potential $V$ has several stationary points in $\Omega$. The solution to this problem is the mean exit time for the above-discussed process $Y_t(t, a)$ in the case $F = -\nabla V$, that is,

$$\nu_\varepsilon(x) = \mathbb{E}(T_\varepsilon(a) : Y_t(0, a) = x). \quad (4)$$

Such model describes an overdamped Langevin dynamics. Several stationary points of the potential $V$ correspond to a metastability phenomenon, when the Brownian particle can be attracted by several local minima of the potential and can leave their neighbourhoods in a finite time. In [14–19], there were constructed leading terms in the asymptotics for $u_\varepsilon$ as well as for other quantities.
characterizing this Brownian motion. In our recent work [20], the case of the potential $V$ with a single minimum in the domain $\Omega$ was considered. In this case, a complete asymptotic expansion for the solution to problem (3) was constructed. An explicit solution in the model case $F(x) = -x$ as the domain $\Omega$ is a circle was found in [21] and a complete asymptotic expansion was found on the base of this explicit representation. The problem on constructing a complete asymptotic expansion for the solution of (1) and (3) for general vector fields $F$ and potentials $V$ and for general domains $\Omega$ still remains open.

In the present work, we consider problem (1) in an arbitrary multi-dimensional domain with an infinitely differentiable boundary. We assume that $F = -\nabla V$ and the potential $V$ is some sufficiently smooth function with the only minimum at origin and having no other stationary points in $\Omega \setminus \{0\}$. The minimum of $V$ at the origin can be degenerate; that is, the function $V$ is not assumed to be a Morse one. However, we assume that some differential of $V$ of a higher order is non-zero. Under these assumptions, we construct a complete asymptotic expansion for the solution to the considered problem. The structure of this expansion is as follows:

$$u_\varepsilon = K_\varepsilon \Psi_\varepsilon + \text{boundary layer} + \text{error term},$$

where $\Psi_\varepsilon$ is the eigenfunction associated with the lowest eigenvalue of the considered problem and $K_\varepsilon$ is some constant. The boundary layer is constructed in the usual way and is employed to satisfy the boundary condition. It solves problem (1) with $F = -\nabla V$ up to an arbitrary small error. However, this boundary layer does not provide a correct asymptotics for the solution since an error term turns out to exceed the boundary layer by the order. And only after extracting the term $K_\varepsilon \Psi_\varepsilon$ the error term becomes small. This is a specific feature of the considered problem since very often a formal asymptotic solution provides a true asymptotics. In our case, the boundary layer serves as such formal asymptotic solution, but nevertheless, true asymptotics (5) involve an extra term $K_\varepsilon \Psi_\varepsilon$.

Our technique follows the ideas developed in our recent work [20]. While problem (3) is rather specific since the right-hand side in the equation is just 1, this is not the case for problem (1) with $F = -\nabla V$ in the sense that the right-hand side in the boundary condition is an arbitrary function. The approach used in [20] for constructing a complete asymptotic expansion employed essentially that the right-hand side in the equation was the constant 1 and not an arbitrary function. At the same time, as we show in the present work, after appropriate modifications, the ideas of [20] can be successfully transferred to problem (1) with $F = -\nabla V$ and with an arbitrary right-hand side in the boundary condition.

In conclusion, we outline the structure of the paper. In the next section, we describe the problem and formulate the main result. Section 3 contains several auxiliary lemmata, which are employed in the proof of the main result given in Section 4. In Section 5 we give some short concluding remarks.
We define the following functions on \( \partial \): denote the metric tensor on \( \partial \) with a given infinitely differentiable function \( \rho \). \( \partial \) functions with the zero trace on \( \Omega \). On the domain \( \Omega \), we let \( V = V(x) \) as \( x \in \Omega \). \( V(0) = 0 \), \( V(x) > 0 \) as \( x \in \Omega \). \( \partial V(0) \neq 0 \),\( |\nabla V(x)| \geq c_1 > 0 \) as \( x \in \Omega \), \( \frac{\partial V}{\partial \tau} \leq -c_2 < 0 \) as \( 0 \leq \tau \leq \tau_0 \).

\[ \nabla \cdot \partial V \] where \( \rho_1 > 0 \), \( k \in \mathbb{N} \) are some numbers, \( B_0 \) is an open ball of radius \( \rho \) centered at the origin, \( c_1, c_2 \) are some positive constant independent of \( x \). Since the function \( V \) is smooth in the vicinity of the origin, the inequality \( V(x) > 0 \) holds as \( x \neq 0 \) and \( V(0) = 0 \), we see that this function has the global minimum at the origin and therefore, \( \nabla V(0) = 0 \) and \( k \geq 2 \).

Our work is devoted to studying the following singularly perturbed problem

\[ (-\varepsilon^2 \Delta + \nabla \cdot \nabla) u_\varepsilon = 0 \quad \text{in} \quad \Omega, \quad u_\varepsilon = f \quad \text{on} \quad \partial \Omega \]

with a given infinitely differentiable function \( f \) defined on \( \partial \). Here \( \varepsilon \) is a small positive parameter. Our main aim is to construct a complete asymptotic expansion for the solution of this problem as \( \varepsilon \to +0 \); we shall show that this problem is uniquely solvable.

In order to formulate our main result, we shall make use of some auxiliary notations. By

\[ g = \begin{pmatrix} g_{11} & \cdots & g_{1, n-1} \\ \vdots & \ddots & \vdots \\ g_{n-1, 1} & \cdots & g_{n-1, n-1} \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & \cdots & b_{1, n-1} \\ \vdots & \ddots & \vdots \\ b_{n-1, 1} & \cdots & b_{n-1, n-1} \end{pmatrix}, \quad g_{ij} = g_{ij}(s), \quad b_{ij} = b_{ij}(s), \]

we denote the metric tensor on \( \partial \) and the second fundamental form on the inward side of \( \partial \). We define the following functions on \( \partial \):

\[ \theta_0(s) := V(x(s)), \quad \theta_{\min} := \min_{\partial \Omega} \theta_0(s), \quad \theta_1(s) := \left| \frac{\partial V}{\partial \tau} \right|_{\tau=0} \leq -c_2 < 0, \]

\[ \theta_2(s) := \frac{1}{2} \frac{\partial^2 V}{\partial \tau^2} \bigg|_{\tau=0}, \quad \Theta_0 := \ln \sqrt{\det g}, \quad \Theta_1 := -\sqrt{\det g} \text{Tr} b. \]

Let \( \chi_0 = \chi_0(t) \) be a differentiable cut-off function vanishing as \( t > 2 \) and equaling to one as \( t < 1 \). On the domain \( \Omega \), we introduce one more cut-off function:

\[ \chi(x) := \begin{cases} \chi_0 \left( \frac{2 \tau}{\delta} \right) & \text{as } 0 \leq \tau \leq \delta, \\ 0 & \text{as } \tau > \delta, \end{cases} \]

\[ 0 < \delta = \text{const} \leq \min \left\{ \frac{2 \tau_0}{3}, \frac{\theta_{\min}}{6c_2} \right\}. \] (8)

Given a subdomain \( \omega \subset \Omega \), we let \( V_\omega := \sup_{\overline{\omega}} V(x) \) and

\[ \|\nabla^2 u\|_{L^2(\omega)} = \sum_{i,j=1}^{2} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^2(\omega)} \quad \text{for } u \in W^2_2(\omega). \]

In the space \( L^2(\Omega) \) we consider a closed differential operator

\[ \mathcal{H}_\varepsilon := -\varepsilon^2 \Delta + \nabla V \cdot \nabla \]

on the domain \( D(\mathcal{H}_\varepsilon) := W^2_2(\Omega) \), which is a subspace of the Sobolev space \( W^2_2(\Omega) \) consisting of the functions with the zero trace on \( \partial \). This operator satisfies the identity

\[ \mathcal{H} = \mathcal{E}^{-1}_\varepsilon \mathcal{H}_\varepsilon \mathcal{E}_\varepsilon, \] (9)
where $\mathcal{E}_\varepsilon$ is the operator of multiplication by $e^{\frac{\nu}{\varepsilon^2}}$, that is, $\mathcal{E}_\varepsilon u = e^{\frac{\nu}{\varepsilon^2}} u$, and $\mathcal{H}_\varepsilon$ is the self-adjoint operator in $L_2(\Omega)$ with the differential expression

$$
\mathcal{H}_\varepsilon := -\varepsilon^2 \Delta + W_\varepsilon, \quad W_\varepsilon(x) := \frac{1}{4\varepsilon^2} |\nabla V(x)|^2 - \frac{1}{2} \Delta V(x),
$$

on the domain $\mathcal{D}(\mathcal{H}_\varepsilon) := \hat{W}_2^2(\Omega)$. The spectra of the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}'$ coincide and consist of countably many discrete eigenvalues, which we take in the ascending order counting their multiplicities [22]. The lowest eigenvalue $\lambda_\varepsilon$ is simple and positive ([20], Lm. 4.1). By $\Psi_\varepsilon$ we denote an associated eigenfunction of the operator $\mathcal{H}_\varepsilon$. The following theorem was proved in ([20], Thm. 2.1).

**Theorem 1.** Let assumptions (6) hold. The eigenfunction $\Psi_\varepsilon$ of the operator $\mathcal{H}_\varepsilon$ associated with the lowest eigenvalue $\lambda_\varepsilon$ belongs to $C^2(\overline{\Omega})$ and is non-zero at $x = 0$. Under the normalization

$$
\Psi_\varepsilon(0) = 1,
$$

it satisfies the asymptotic formula

$$
\Psi_\varepsilon(x) = 1 - \chi(x)e^{\frac{\nu_1(x)(x)}{\varepsilon}} \sum_{j=0}^{N} \frac{\varepsilon^j}{\nu_j^j} \Phi_j(\tau(x)e^{-2}s(x)) + E_{\varepsilon,N}(x) \quad (11)
$$

for all $x \in \overline{\Omega}$ and all nonnegative integer $N$. The symbols $\Phi_j = \Phi_j(\zeta, s)$ denote some polynomials in $\zeta$ of degree at most 2j with infinitely differentiable in $s$ coefficients such that $\Phi_j(0, s) = 0$, $j \geq 1$. In particular,

$$
\Phi_0(\zeta, s) \equiv 1, \quad \Phi_1(\zeta, s) = \frac{\Phi(s)}{2\theta_1(s)} \tilde{\gamma}^2 - \left( \Theta_1 + \frac{\Phi(s)}{\theta_1^2(s)} \right) \zeta, \quad \Phi := \tilde{g}^{-1} \nabla \theta_1 \cdot \nabla \tilde{\theta}_1 + 2\Theta_1 \tilde{\theta}_2.
$$

As $\varepsilon \to 0$, the error term obeys the estimates

$$
\|E_{\varepsilon,N}\|_{L_2(\omega)} = O\left(\varepsilon^{2N+3}e^{-\frac{\nu_{\min} - \nu_{\omega}}{2\varepsilon}}\right), \quad \|\nabla E_{\varepsilon,N}\|_{L_2(\omega)} = O\left(\varepsilon^{2N+1}e^{-\frac{\nu_{\min} - \nu_{\omega}}{2\varepsilon}}\right),
$$

$$
\|\tilde{\tau}_x E_{\varepsilon,N}\|_{L_2(\omega)} = O\left(\varepsilon^{2N-1}e^{-\frac{\nu_{\min}}{2\varepsilon}}\right), \quad \|E_{\varepsilon,N}\|_{C(\overline{\Omega})} = O\left(\varepsilon^{2N+2}e^{-\frac{\nu_{\min}}{2\varepsilon}}\right),
$$

for each subdomain $\omega \subset \Omega$ and

$$
\|E_{\varepsilon,N}\|_{L_2(\Omega)} = O\left(\varepsilon^{2N+3}\right), \quad \|\nabla E_{\varepsilon,N}\|_{L_2(\Omega)} = O\left(\varepsilon^{2N+1}\right),
$$

$$
\|\tilde{\tau}_x E_{\varepsilon,N}\|_{L_2(\Omega)} = O\left(\varepsilon^{2N-1}e^{-\frac{\nu_{\min}}{2\varepsilon}}\right), \quad \|\tilde{\gamma}^{-1} \nabla E_{\varepsilon,N}\|_{L_2(\Omega)} = O\left(\varepsilon^{2N+1}e^{-\frac{\nu_{\min}}{2\varepsilon}}\right),
$$

$$
\|\tilde{\tau}_x \tilde{\gamma}^{-1} \nabla E_{\varepsilon,N}\|_{L_2(\Omega)} = O\left(\varepsilon^{2N+1}e^{-\frac{\nu_{\min} - \nu_{\omega}}{2\varepsilon}}\right), \quad \|\tilde{\tau}_x e^{-\frac{\nu_{\min}}{2\varepsilon}} E_{\varepsilon,N}\|_{L_2(\Omega)} = O\left(\varepsilon^{2N+2}e^{-\frac{\nu_{\min}}{2\varepsilon}}\right).
$$

In what follows, the eigenfunction $\Psi_\varepsilon$ is supposed to be chosen according the above theorem. Now we are in position to formulate our main result.

**Theorem 2.** Let assumptions (6) hold. Problem (7) is uniquely solvable. The solution has the following complete asymptotic expansion:

$$
u_\varepsilon(x) = K_\varepsilon \Psi_\varepsilon(x) + \chi(x)e^{\frac{\nu_1(x)(x)}{\varepsilon}} \sum_{j=0}^{N} \frac{\varepsilon^j}{\nu_j^j} \Phi_j(\tau(x)e^{-2}s(x)) + Z_{\varepsilon,N}(x), \quad N \in \mathbb{N}. \quad (15)$$
Here $P_j(\zeta, s)$ are some polynomials in $\zeta$ of degree at most $2j$ with infinitely differentiable in $s$ coefficients and the initial conditions $P_0(0, s) = f$, $P_j(0, s) = 0$, $j \geq 1$, are satisfied. In particular,

$$
P_0 = f, \quad P_1 = \left(2\theta_1\theta_2 + g^{-1}\nabla_s \theta_0 \nabla \theta_1 \right) \left(\frac{1}{2} \theta_1^{-1} g^2 - \zeta \theta_1^{-2} \right) f + \left(\theta_1^{-1} g^{-1}\nabla_s \theta_0 \cdot \nabla_s f - \Theta_1 f \right) \zeta. \quad (16)
$$

The symbol $K_\varepsilon$ denotes a constant given by the formula

$$
K_\varepsilon := \frac{\int e^{-\frac{\nu}{\varepsilon}} \frac{\partial\Phi}{\partial r} f \, ds}{\int e^{-\frac{\nu}{\varepsilon}} \frac{\partial\Phi}{\partial r} \, ds}. \quad (17)
$$

The constant $K_\varepsilon$ has the following asymptotic expansion:

$$
K_\varepsilon = \frac{\eta(\varepsilon)}{\mu(\varepsilon)} + \varepsilon^2 \left( \frac{\eta_1(\varepsilon)}{\mu_0(\varepsilon)} - \frac{\mu_0(\varepsilon)\eta_1(\varepsilon)}{\mu_0^2(\varepsilon)} \right) + \varepsilon^4 \left( \frac{\eta_2(\varepsilon)}{\mu_0(\varepsilon)} - \frac{\mu_0(\varepsilon)\eta_1(\varepsilon)}{\mu_0^2(\varepsilon)} - \frac{\eta_0(\varepsilon)\mu_2(\varepsilon)}{\mu_0^3(\varepsilon)} + \frac{\eta_0(\varepsilon)\mu_1(\varepsilon)}{\mu_0^2(\varepsilon)} \right) + \sum_{j=3}^N \varepsilon^j F_j \left( \frac{\eta_0(\varepsilon)}{\mu_0(\varepsilon)}, \ldots, \frac{\eta_j(\varepsilon)}{\mu_0(\varepsilon)}, \frac{\mu_0(\varepsilon)}{\mu_0(\varepsilon)}, \ldots, \frac{\mu_j(\varepsilon)}{\mu_0(\varepsilon)} \right) + O(\varepsilon^{2N+2}),
$$

where $F_j$ are some polynomials with constant coefficients, and

$$
\eta(\varepsilon) := \int e^{-\frac{\nu}{\varepsilon}} \frac{\partial\Phi}{\partial r} \theta(s) f(s) \, ds, \quad \eta_j(\varepsilon) := \int e^{-\frac{\nu}{\varepsilon}} \frac{\partial^{\nu j}}{\partial r^j} (0, s) f(s) \, ds, \quad j \geq 1,
$$

$$
\mu_0(\varepsilon) := \int e^{-\frac{\nu}{\varepsilon}} \frac{\partial\Phi}{\partial r} \theta(s) ds, \quad \mu_j(\varepsilon) := \int e^{-\frac{\nu}{\varepsilon}} \frac{\partial^{\nu j}}{\partial r^j} (0, s) ds, \quad j \geq 1. \quad (19)
$$

The error term in (15) obeys the estimates

$$
\|Z_{e,N} \|_{L^2(\omega)} = O(\varepsilon^{2N+3} e^{-\frac{\theta_{min} - \eta_{min}}{2\varepsilon}}), \quad \|\nabla Z_{e,N} \|_{L^2(\omega)} = O(\varepsilon^{2N+1} e^{-\frac{\theta_{min} - \eta_{min}}{2\varepsilon}}), \quad \|\|Z_{e,N} \|_{L^2(\omega)} = O(\varepsilon^{2N+2} e^{-\frac{\theta_{min} - \eta_{min}}{2\varepsilon}}), \quad (20)
$$

for each subdomain $\omega \subset \Omega$ and

$$
\|Z_{e,N} \|_{L^2(\Omega)} = O(\varepsilon^{2N+3}), \quad \|\nabla Z_{e,N} \|_{L^2(\Omega)} = O(\varepsilon^{2N+1}), \quad \|\|Z_{e,N} \|_{L^2(\Omega)} = O(\varepsilon^{2N+2}), \quad (21)
$$

$$
\|e^{-\frac{\nu}{\varepsilon}} Z_{e,N} \|_{L^2(\Omega)} = O(\varepsilon^{2N+3} e^{-\frac{\theta_{min} - \eta_{min}}{2\varepsilon}}), \quad \|\nabla e^{-\frac{\nu}{\varepsilon}} Z_{e,N} \|_{L^2(\Omega)} = O(\varepsilon^{2N+1} e^{-\frac{\theta_{min} - \eta_{min}}{2\varepsilon}}), \quad \|e^{-\frac{\nu}{\varepsilon}} Z_{e,N} \|_{L^2(\Omega)} = O(\varepsilon^{2N+2} e^{-\frac{\theta_{min} - \eta_{min}}{2\varepsilon}}), \quad (22)
$$

Let us discuss the main result. It provides a complete asymptotic expansion for the solution of problem (7) as $\varepsilon$ goes to zero. This expansion consists of two main components and first we consider the second component. This is a boundary layer, which is the sum in the right-hand side in (15). It is non-zero only in a small neighbourhood of the boundary $\partial \Omega$ and decays exponentially fast as $\tau$ increases. The main aim this layer serves for is to satisfy the inhomogeneous boundary condition in (7). It is constructed in a standard way by substituting this expansion into boundary value problem (7) and equating the coefficients at the like powers of $\varepsilon$, see Section 4.1. However, as we see in (15), this boundary layer does not provide a true asymptotic expansion for the solution to problem (7) since also the first term $K_\varepsilon \Psi_\varepsilon$ is to be taken into consideration. This is a specific feature in
comparison with the usual approach to constructing asymptotic expansions for solutions of singularly perturbed problems. Namely, given a singularly perturbed problem, not necessary with a small parameter at a higher derivative, one can try to guess how an asymptotic ansatz for the solution should look like. Then the assumed ansatz is substituted into the problem and this determines somehow the terms in the ansatz. The next step is to truncate the constructed formal asymptotic series and to see whether the truncated series solves the problem up to a small error. If this is the case, very often it turns out that a constructed formal asymptotic expansion provides a true asymptotic for the solution of the considered perturbed problem, but in view of the first term $K_\varepsilon \Psi_\varepsilon$ in (15), this is not the case for the considered problem.

The matter is as follows. The constructed boundary layer does solve problem (7) up to an arbitrary small error and this problem is uniquely solvable. The latter unique solvability means that there exists an inverse operator $H_\varepsilon^{-1}$. However, the lowest eigenvalue $\lambda_\varepsilon$ of this operator is positive and exponentially small, see Lemma 1, and hence, the norm of the inverse operator $H_\varepsilon^{-1}$ is exponentially large. This is why, trying to estimate an error term for the boundary layer, we face exponentially large $\varepsilon$ estimates, while the boundary layer is power in $\varepsilon$. A way to resolve this difficulty is to select an exponentially growing part of the inverse operator $H_\varepsilon^{-1}$ and to consider it independently. This part is a projector on the eigenfunction $\Psi_\varepsilon$ (with the weight $e^{-\frac{\alpha}{\varepsilon}}$). In terms of the solution $u_\varepsilon$, this means that we need to find a projection of $u_\varepsilon$ on $\Psi_\varepsilon$ and deduct it from $u_\varepsilon$. Once we do this, a remaining part of the solution becomes orthogonal to $\Psi_\varepsilon$ and the inverse operator $H_\varepsilon^{-1}$ on the orthogonal complement to $\Psi_\varepsilon$ is bounded uniformly in $\varepsilon$. This allows us to get nice estimates for the error term. In other words, the boundary layer discussed above provides an asymptotic expansion for an orthogonal to $\Psi_\varepsilon$ part of the solution.

The aforementioned projection of the solution $u_\varepsilon$ on $\Psi_\varepsilon$ is represented in (15) by the term $K_\varepsilon \Psi_\varepsilon$. The constant $K_\varepsilon$ is not exact but provides an approximation for the true one up to an exponentially small error. Although the term $K_\varepsilon \Psi_\varepsilon$ is given in a closed and rather implicit form, it is possible to find its complete asymptotic expansion. The expansion for $K_\varepsilon$ is given by (18). This asymptotics arises by substituting (11) into formula (17) and considering then the quotient of two obtained asymptotic series

$$K_\varepsilon = \frac{\sum_{j=0}^{\infty} \varepsilon^{2j} \eta_j(\varepsilon)}{\sum_{j=0}^{\infty} \varepsilon^{2j} \mu_j(\varepsilon)}.$$

Having this formula in mind, it is easy to find all polynomials $F_j$ in (18). By applying the Laplace method, it is possible to find complete asymptotic expansions for all $\eta_j$ and $\mu_j$:

$$\eta_j(\varepsilon) = e^{-\frac{\theta_{\min}}{\varepsilon^2}} \sum_{i=n-1}^{\infty} a_{ji} \varepsilon^{\frac{2i}{p}}, \quad \mu_j(\varepsilon) = e^{-\frac{\theta_{\min}}{\varepsilon^2}} \sum_{i=n-1}^{\infty} \beta_{ji} \varepsilon^{\frac{2i}{p}},$$

where $a_{ji}$, $\beta_{ji}$ are some constants and $p \in \mathbb{N}$ is some fixed number. And if $\beta_{0i0}$ is a first non-zero coefficient in the above asymptotics for $\mu_0$, then $a_{ji} = \beta_{ji} = 0$ for all $j$ and all $i < i_0$. This means that asymptotics (18) can be rewritten in terms of the fractional powers $\varepsilon^{\frac{2i}{p}}$.

The asymptotic expansion for $\Psi_\varepsilon$ is provided in Theorem 1 in (11). This asymptotics is a sum of the constant 1 and a boundary layer of a similar structure as that in (15). The error terms in asymptotics (11) are estimated in the same norms as the error terms in (15). We shall discuss these estimates in the next paragraph. And now we just should say that by multiplying the above-discussed expansions for $K_\varepsilon$ and $\Psi_\varepsilon$ we can find find a complete asymptotic expansion for the first term in (15).

Theorem 2 provides a series of various estimates for the error term in asymptotics (15), and similar estimates are given for the error terms in asymptotics (11). Estimates (21) and (13) are given for the the uniform and integral norms over entire domain $\Omega$, while estimates (22) and (14) are given in the
same norm but with an exponential weight. This weight comes from identity (9) and the definition of the operator \(E_\varepsilon\). Estimates (20) and (14) show how small the error term is in the subdomains of \(\Omega\). Due to conditions (6), the only minimum of the function \(V\) is located at zero and the function \(V\) attains its maximum on the boundary \(\partial \Omega\). Hence, if a subdomain \(\omega\) is separated from the boundary \(\partial \Omega\) by a positive distance, according to estimates (20) and (12), the error term is exponentially small in \(\omega\). If we also substitute asymptotics (11) into (15), we see that on such subdomain the solution \(u_\varepsilon\) is exponentially close to the constant \(K_\varepsilon\). Then asymptotics (18) becomes in fact that for \(u_\varepsilon\) in the subdomain \(\omega\). This result is consistent with ones of works \([5–10]\) stating that the solution converges to a constant uniformly on compact subsets of \(\Omega\).

3. Auxiliary Statements

In the present section we provide some auxiliary statements, which will be employed later in the proof of the main result in the next section. The first auxiliary lemma is as follows, see Lemma 4.1 and Theorem 2.2 in \([20]\).

**Lemma 1.** Let assumptions (6) be satisfied. The estimates

\[
0 < \lambda_\varepsilon < c_3 \varepsilon^{-\frac{2\varepsilon}{1+2\varepsilon}} e^{-\frac{c_4}{\varepsilon^{1+2\varepsilon}}}, \quad \lambda_0^\varepsilon > c_4 \varepsilon^{2-\frac{4}{\varepsilon}}
\]

hold true, where \(\lambda_0^\varepsilon > \lambda_\varepsilon\) is the second eigenvalue of the operator \(\mathcal{H}_\varepsilon\), and \(c_3, c_4\) are some positive constants independent of \(\varepsilon\). The lowest eigenvalue \(\lambda\) of the operator \(\mathcal{H}_\varepsilon\) satisfies the identity

\[
\lambda = \frac{\varepsilon^2 \int_{\partial \Omega} e^{\frac{\varepsilon}{H} \frac{\partial \varepsilon}{\partial \varepsilon}} ds}{\int_{\Omega} e^{\frac{\varepsilon}{H} \frac{\partial \varepsilon}{\partial \varepsilon}} dx}.
\]

By \(\lfloor \cdot \rfloor\) we denote an integer part of a number. One more auxiliary statement is Lemma 4.2 from \([20]\); we reproduce it below.

**Lemma 2.** Let assumptions (6) be satisfied. For each function \(u \in C^2(\overline{\Omega}) \cap C^0(\overline{B_{r_1}}) \cap \mathcal{D}(\mathcal{H}_\varepsilon)\) and each \(\lambda \in [0, \varepsilon^4]\) the estimates hold:

\[
\|\nabla u\|_{L_2(\Omega)} \leq C \varepsilon^{-2} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{L_2(\Omega)} + \| u \|_{L_2(\Omega)} \right),
\]

\[
\|\partial_{x_1} \partial_{x_2}^2 u\|_{L_2(\Omega)} \leq C \varepsilon^{-4} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{L_2(\Omega)} + \| u \|_{L_2(\Omega)} \right),
\]

\[
\| u \|_{W^{m,1}_2(B_{\frac{1}{2}r_1})} \leq C \varepsilon^{-2m-2} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{W^{m,1}_2(B_{r_1})} + \| (\mathcal{H}_\varepsilon - \lambda) u \|_{L_2(\Omega)} + \| u \|_{L_2(\Omega)} \right), \quad m \in \mathbb{N},
\]

\[
\| u \|_{C(\overline{\Omega})} \leq C \varepsilon^{-n-2} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{W^{n,1}_2(B_{r_1})} + \| (\mathcal{H}_\varepsilon - \lambda) u \|_{C(\overline{\Omega})} + \| u \|_{L_2(\Omega)} \right),
\]

\[
\| e^{\frac{\varepsilon}{\mathcal{H}_\varepsilon}} u \|_{C(\overline{\Omega})} \leq C \varepsilon^{-n-3} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{W^{n,1}_2(B_{r_1})} + \| e^{\frac{\varepsilon}{\mathcal{H}_\varepsilon}} (\mathcal{H}_\varepsilon - \lambda) u \|_{C(\overline{\Omega})} + \| u \|_{L_2(\Omega)} \right),
\]

\[
\| \nabla e^{\frac{\varepsilon}{\mathcal{H}_\varepsilon}} u \|_{L_2(\Omega)} \leq C \varepsilon^{-n-5} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{W^{n,1}_2(B_{r_1})} + \| e^{\frac{\varepsilon}{\mathcal{H}_\varepsilon}} (\mathcal{H}_\varepsilon - \lambda) u \|_{C(\overline{\Omega})} + \| u \|_{L_2(\Omega)} \right),
\]

\[
\| \partial_{x_1} \partial_{x_2} e^{\frac{\varepsilon}{\mathcal{H}_\varepsilon}} u \|_{L_2(\Omega)} \leq C \varepsilon^{-n-6} \left( \| (\mathcal{H}_\varepsilon - \lambda) u \|_{W^{n,1}_2(B_{r_1})} + \| e^{\frac{\varepsilon}{\mathcal{H}_\varepsilon}} (\mathcal{H}_\varepsilon - \lambda) u \|_{C(\overline{\Omega})} + \| u \|_{L_2(\Omega)} \right),
\]

where \(C\) is a constant independent of \(u, \varepsilon\) and \(\lambda\).

The following lemma ensures the unique solvability of problem (7).
Lemma 3. Problem (6) is uniquely solvable for each \( f \in C^\infty(\partial \Omega) \). This solution belongs to \( C^2(\Omega) \).

Proof. Since \( f \in C^\infty(\partial \Omega) \), there exists a function \( U \in C^\infty(\Omega) \) such that \( U = f \) on \( \partial \Omega \). For instance, as \( U \) we can take a function \( U(x) = \chi(x)f(s) \). In view of identity (9), for the function \( u^\varepsilon = \varepsilon^{-1}(u_\varepsilon - U) \) we obtain the operator equation

\[
\mathcal{H}^\varepsilon u^\varepsilon = F^\varepsilon, \quad F^\varepsilon := -\varepsilon^{-1}(-\varepsilon^2\Delta + \nabla V \cdot \nabla)U, \quad u_\varepsilon = \varepsilon u^\varepsilon + U. \tag{24}
\]

This equation is uniquely solvable since the lowest eigenvalue \( \lambda_\varepsilon \) of the operator \( \mathcal{H}^\varepsilon \) is positive and hence, the operator \( \mathcal{H}^\varepsilon \) is boundedly invertible. We rewrite Equation (24) to a corresponding boundary value problem and by the smoothness improving theorems we conclude that \( u^\varepsilon \in C^2(\Omega) \). Returning back to the function \( u_\varepsilon \), we arrive at the statement of the lemma. The proof is complete. \( \square \)

4. Proof of Main Result

In the present section we prove Theorem 2. It consists of three main steps. The first is a formal construction of a boundary layer in (15) as a formal asymptotic solution to problem (7). In the second step we determine the first term in the right-hand side in formula (15). And the third step is devoted to estimating the error term in (15).

In the proof we follow the ideas developed in the proof of Theorem 2.3 in [20]. However, certain modifications are to be made. The main difference is that now we first construct a boundary layer only after we find the first term in (15), while in (3) it was done in the opposite order. The reason is that in [20], it was impossible to construct an appropriate boundary layer before determining the projection of the solution on the eigenfunction \( \Psi_\varepsilon \) and before deducting it from the solution. In the present work, we have to construct first the boundary layer and only after we determine the projection on the eigenfunction in order to make it possible to obtain nice estimates for the error term in (15). The reason is that the justification of the asymptotics is based on working with the self-adjoint operator \( \mathcal{H}^\varepsilon \), whose domain consists of the functions with the zero trace on \( \partial \Omega \). This is why, while finding the projection on the first eigenfunction, we first need to deduct from the solution some auxiliary function satisfying the boundary condition in (7). There are plenty of choices for such auxiliary function, but the optimal choice is exactly the boundary layer, which we construct first. Other choices are also possible but they make the proof of Theorem 2 more lengthy, more complicated and in some sense messier.

4.1. Boundary Layer

In this section we construct a boundary layer being a formal asymptotic solution to problem (7). The construction is made in the same way as the construction of boundary layers in [20], but since now the problem is different, we have to adapt these calculations to our case.

We construct the boundary layer as

\[
U_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^{2j}u_j(\zeta, s), \quad \zeta := \varepsilon^{-2}r. \tag{25}
\]

In view of the boundary condition in problem (7), for the functions \( u_j \) we impose the initial conditions

\[
u_0(0, s) = f, \quad u_j(0, s) = 0, \quad j \geq 1. \tag{26}\]

Since expansion (25) is a boundary layer, for the functions \( u_j \) we postulate an exponential decay at infinity:

\[
u_j(\zeta, s) = O(e^{-\gamma_j \zeta}), \quad \zeta \to +\infty, \tag{27}\]

uniformly in \( s \), where \( \gamma_j \) are some positive numbers.
Let us find the equations for the functions \( u_j \). In terms of the local variables \((s, \tau)\), the Laplace operator and the gradient are rewritten as

\[
\Delta = \frac{\partial^2}{\partial s^2} + \frac{\partial \ln f}{\partial \tau} + \text{div}_{s} L \nabla_{s} + \nabla_{s} \ln f \cdot L \nabla_{s}, \quad \nabla_{s} \cdot \nabla_{s} = \frac{\partial V}{\partial \tau} + L \nabla_{s} \cdot \nabla_{s},
\]

(28)

\[
L = L(\tau, s) := (I - \tau b)^{-1} e^{-1}(I - \tau b)^{-1}, \quad f(\tau, s) := \sqrt{\text{det} g} \text{det}(I - \tau b),
\]

where \( I \) is the unit matrix of size \((n - 1) \times (n - 1)\). The smoothness of the functions \( V, L \) and \( f \) in the vicinity of the boundary \( \partial \Omega \) ensures the Taylor series

\[
\ln f(\tau, s) = \sum_{j=0}^{\infty} \tau^j \Theta_j(s), \quad L(\tau, s) = \sum_{j=0}^{\infty} \tau^j L_j(s),
\]

(29)

\[
V(x) = \sum_{j=0}^{\infty} \tau^j \theta_j(s), \quad \theta_j(s) := \frac{1}{\tau} \frac{\partial V}{\partial \tau} |_{\partial \Omega},
\]

treated as asymptotic ones as \( \tau \to +0 \), where \( \theta_j, \Theta_j \in C^{\infty}(\partial \Omega) \) are some functions and \( L_j \in C^{\infty}(\partial \Omega) \) are some Hermitian matrices. We substitute these series and (25), (28) into the equation in (7) and pass to the variable \( \zeta \). This gives an equation for \( U_{\zeta} \):

\[
L^g U_{\zeta} = 0 \quad \text{as} \quad \zeta > 0.
\]

(30)

Here

\[
L^g := -\varepsilon^{-2} \frac{\partial^2}{\partial \zeta^2} + \varepsilon^2 \frac{\partial V}{\partial \zeta} (e^{2 \zeta}, s) \frac{\partial}{\partial \zeta} - \frac{\partial \ln f}{\partial \tau} (e^{2 \zeta}, s) \frac{\partial}{\partial \zeta} + L(e^{2 \zeta}, s) \nabla_{s} \cdot \nabla_{s}
\]

\[
- e^2 \text{div}_{s} L(e^{2 \zeta}, s) \nabla_{s} - e^2 \nabla_{s} \ln f(e^{2 \zeta}, s) \cdot L(e^{2 \zeta}, s) \nabla_{s}.
\]

We substitute the above formula and series (25), (29) into (30) and equate the coefficients at the like powers of \( \varepsilon \). This gives the equations for the functions \( u_j \):

\[
- \frac{\partial^2 u_j}{\partial \zeta^2} + \theta_1 \frac{\partial u_j}{\partial \zeta} = G_j, \quad \zeta > 0, \quad G_j(\zeta, s) := \sum_{i=0}^{j-1} L_i Q_{j-i-1},
\]

(31)

\[
L_i := \zeta^{i-1} \text{div}_{s} L_{i-1}(s) \nabla_{s} - \left( (i + 2) \zeta^{i+1} \theta_{i+2} - (i + 1) \zeta^i \Theta_{i+1} \right) \frac{\partial}{\partial \zeta}
\]

\[
- \sum_{q=0}^{i} \zeta^{q-1} L_{q}(\zeta \nabla_{s} \theta_{i-q} - \nabla_{s} \Theta_{i-q-1}) \cdot \nabla_{s}.
\]

where we let \( L_j := 0, \Theta_j := 0 \) as \( j \leq -1 \).

Equation (31) for \( u_0 \) is homogeneous. Since the function \( \theta_1 \) is strictly negative and is independent of \( \zeta \), this equation has a unique solution obeying the initial condition in (26) and decay condition (27):

\[
u_0(\zeta, s) = f(s) e^{\theta(s) \zeta}.
\]

The right-hand side \( G_1 \) in Equation (31) for \( u_1 \) is a product of the exponential \( e^{\theta(s) \zeta} \) and a polynomial in \( \zeta \) of the first degree with infinitely differentiable in \( s \) coefficients:

\[
G_1(\zeta, s) = (G_{11}(s) \zeta + G_{10}(s)) e^{\theta(s) \zeta},
\]

where

\[
G_{11} := -\left( 2 \theta_1 \theta_2 + g^{-1} \nabla_{s} \theta_0 \nabla_{s} \theta_1 \right) f, \quad G_{10} := \theta_1 \Theta_1 f - g^{-1} \nabla_{s} \theta_0 \cdot \nabla_{s} f.
\]
Problem (26), (27) and (31) for $u_1$ has a solution of a similar structure

$$u_1(\zeta, s) = P_1(\zeta, s)e^{\delta_1(s)\zeta},$$

where

$$P_1(\zeta, s) = -\frac{G_{11}(s)}{2\theta_1(s)}\zeta^2 + \left(\frac{G_{11}(s)}{\theta_1(s)} - \frac{G_{10}(s)}{\theta_1(s)}\right)\zeta.$$

Other terms of series (25) are successively determined in the same way: in each step, the right-hand in Equation (33) is satisfied. It is straightforward to check that the equation

Proof. It follows from the definition of the function $U$ implies the estimates:

$$\|e^{-\frac{\nu}{2}\tau}h_{e,N}\|_{L^2(\Omega)} = O\left(\varepsilon^{N+1}e^{-\frac{\theta_1}{2\varepsilon}}\right), \quad \|e^{-\frac{\nu}{2}\tau}h_{e,N}\|_{C(\Omega)} = O\left(\varepsilon^{2N}e^{-\frac{\theta_1}{2\varepsilon}}\right), \quad \|h_{e,N}\|_{C(\Omega)} = O(\varepsilon^{2N}).$$

Lemma 4. Problems (26), (27), and (31) are uniquely solvable. The solutions read as

$$u_j(\zeta, s) = P_j(\zeta, s)e^{\delta_1(s)\zeta},$$

where $P_j$ are some polynomials in $\zeta$ of degree at most $2j$ with infinitely differentiable in $s$ coefficients satisfying the identities $P_0(0, s) = f(s)$, $P_j(0, s) = 0$, $j \geq 1$. The functions $P_0$ and $P_1$ are given by formulae (16).

For each $N \in \mathbb{N}$ we let

$$U_{e,N}(x) := \chi(x) \sum_{j=0}^{N} e^{j\tau}u_j(\varepsilon^{-2}\tau, s).$$

(32)

The next lemma states that the introduced function is a formal asymptotic solution to problem (7).

Lemma 5. The function $U_{e,N}$ is infinitely differentiable in $\overline{\Omega}$ and satisfies the boundary value problem

$$\left(-\varepsilon^2\Delta + \nabla V \cdot \nabla\right)U_{e,N} = h_{e,N} \quad \text{in} \quad \Omega, \quad U_{e,N} = f \quad \text{on} \quad \partial\Omega$$

(33)

where $h_{e,N} \in C^\infty(\overline{\Omega})$ is a function supported in $\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \tau_0\}$ and satisfying the estimates

$$\|e^{-\frac{\nu}{2}\tau}h_{e,N}\|_{L^2(\Omega)} = O\left(\varepsilon^{N+1}e^{-\frac{\theta_1}{2\varepsilon}}\right), \quad \|e^{-\frac{\nu}{2}\tau}h_{e,N}\|_{C(\Omega)} = O\left(\varepsilon^{2N}e^{-\frac{\theta_1}{2\varepsilon}}\right), \quad \|h_{e,N}\|_{C(\Omega)} = O(\varepsilon^{2N}).$$

Proof. It follows from the definition of the function $U_{e,N}$ that it is extremely differentiable in $\overline{\Omega}$ and in view of (26), the boundary condition in (33) is satisfied. It is straightforward to check that the equation in (33) holds true with

$$h_{e,N} := -\left(\varepsilon^2\Delta - \nabla V \cdot \nabla\right)\chi(x) \sum_{j=0}^{N} e^{j\tau}u_j(\varepsilon^{-2}\tau, s)$$

$$= \chi(x)\varepsilon^2 \sum_{j=0}^{N} e^{j\tau}u_j(\zeta, s) - \sum_{j=0}^{N} e^{j\tau}\left(2\varepsilon^2\nabla \chi \cdot \nabla u_j(\varepsilon^{-2}\tau, s)\right)$$

$$+ e^{2\tau}u_j(\varepsilon^{-2}\tau, s)\Delta \chi - u_j(\varepsilon^{-2}\tau, s)\nabla V \cdot \nabla \chi(x)$$

$$=: h_{e,N}^{(1)} + h_{e,N}^{(2)}.$$

The functions $h_{e,N}^{(1)}, h_{e,N}^{(2)}, h_{e,N}$ are differentiable in $\overline{\Omega}$. In view of the presence of the cut-off function in their definition, they are supported in $\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \tau_0\}$. Equation (31) and Lemma 4 imply the estimates:

$$\|h_{e,N}^{(1)}\|_{C(\Omega)} \leq C\varepsilon^{2N}, \quad \|h_{e,N}^{(2)}\|_{C(\Omega)} \leq C\varepsilon^{-2N}e^{-\frac{\theta_1}{2\varepsilon}}.$$

Since $V = \delta_0 + \tau \theta_1 + O(\tau^2)$ as $\tau \to 0$, by Lemma 4 we obtain:
where

\[ K \]

As we shall see in the next subsection, this is an important step in determining an asymptotic expansion.

This allows us to integrate by parts as follows:

\[ \text{Hence, in view of (37),} \]

\[ K_{e,N} = K_{e} + K_{e,N}, \]

where
\[ \tilde{K}_\varepsilon := \frac{\varepsilon^2}{\lambda_\varepsilon \| e^{-\frac{V}{\varepsilon^2}} \Psi_\varepsilon \|_{L^2(\Omega)}^2} \int e^{-\frac{\theta}{\varepsilon^2}} \frac{\partial \Psi_\varepsilon}{\partial \tau} f \, ds, \]  
(39)

\[ \tilde{K}_{\varepsilon, N} = -\frac{1}{\| e^{-\frac{V}{\varepsilon^2}} \Psi_\varepsilon \|_{L^2(\Omega)}^2} \int e^{-\frac{V}{\varepsilon^2}} U_{\varepsilon, N} \Psi_\varepsilon \, dx. \]  
(40)

Let us study the asymptotic behaviour of the constant \( K_{\varepsilon, N} \). It follows from Theorem 1 and the Laplace method that

\[ \| e^{-\frac{V}{\varepsilon^2}} \Psi_\varepsilon \|_{L^2(\Omega)}^2 = \int_{\Omega} e^{-\frac{V(x)}{\varepsilon^2}} \, dx + O \left( \varepsilon^2 e^{-\frac{\theta_{\min}}{2\varepsilon^2}} \right) = \sum_{j=0}^{\infty} \frac{2j}{\pi} a_j, \]  
(41)

where the latter series is treated as an asymptotic one, \( a_j \) are some real numbers and, in particular,

\[ a_n = \int_{\mathbb{R}^n} e^{-V_0} \, dx \neq 0, \quad V_0(x) := \frac{1}{k!} \sum_{\alpha \in \mathbb{Z}^n_+} \frac{\partial^\gamma V}{\partial x^\gamma} (0) x^\gamma, \quad V_0(x) > 0, \quad x \neq 0. \]  
(42)

Taking into consideration the definition of the function \( U_{\varepsilon, N} \) and Lemma 4, we estimate the numerator in definition (40) of \( \tilde{K}_{\varepsilon, N} \):

\[ \int_{\Omega} e^{-\frac{V}{\varepsilon^2}} U_{\varepsilon, N} \Psi_\varepsilon \, dx = O \left( \varepsilon^2 e^{-\frac{\theta_{\min}}{2\varepsilon^2}} \right). \]

Together with (41) and (42) this implies that

\[ \tilde{K}_{\varepsilon, N} = O \left( \varepsilon^2 e^{-\frac{\theta_{\min}}{2\varepsilon^2}} \right). \]  
(43)

We substitute formula (23) into definition (39) of \( \tilde{K}_\varepsilon \):

\[ \tilde{K}_\varepsilon = K_\varepsilon (1 + \varphi_\varepsilon), \]  
(44)

where \( K_\varepsilon \) is given by formula (17) and

\[ \varphi_\varepsilon := \frac{1}{\| e^{-\frac{V}{\varepsilon^2}} \Psi_\varepsilon \|_{L^2(\Omega)}^2} \int e^{-\frac{V}{\varepsilon^2}} \Psi_\varepsilon (1 - \Psi_\varepsilon) \, dx. \]  
(45)

The estimates for \( \frac{\partial V}{\partial \tau} \) in (6) and definition (8) of the cut-off function \( \chi \) yield that

\[ \min_{\text{supp} \chi} V \geq \frac{5}{6} \theta_{\min}. \]

By this inequality and asymptotics (11) and (41) we get an estimate for \( \varphi_\varepsilon \):

\[ \varphi_\varepsilon = O \left( e^{-\frac{3\theta_{\min}}{8\varepsilon^2}} \right). \]  
(46)

We substitute asymptotics (11) into the numerator in formula (17) for \( K_\varepsilon \) and this gives:
where

\[ C \]

The obtained relation allows us to apply Lemma 2 to the function \( \theta_1 \). This estimate (48) and Equation (47) yield that the estimate

\[ \sum_{j=0}^{N-1} e^{2j} \int e^{-\frac{\|g\|}{2}} \partial^{j+1}_x (0,s) f(s) ds + O\left( e^{2N-1} - e^{-\frac{\theta_{\min}}{2\tau}} \right). \]

The same identity but with \( f \equiv 1 \) provides an asymptotics expansion for the denominator in (17). Calculating the asymptotics for the quotient of these two expansions, we get asymptotics (18) and (19) for \( K_\varepsilon \).

4.3. Error Terms

In this subsection we complete the proof of Theorem 2 by establishing estimates for the error term in (15). We shall do this by analyzing the second term in the right-hand side in (36).

We denote by \( L^2_\perp (\Omega) \) the orthogonal complement to \( \Psi^\varepsilon \) in \( L^2 (\Omega) \). It follows from Equation (35) and definition (37) of the constant \( K_\varepsilon \) that the function \( v^{\perp}_{\varepsilon, N} \) belongs to \( L^2_\perp (\Omega) \) and solves the equation

\[ \mathcal{H}^\varepsilon v^{\perp}_{\varepsilon, N} = g_{\varepsilon, N}, \quad g_{\varepsilon, N} := -E_\varepsilon^{-1} h_{\varepsilon, N} + \frac{(E_\varepsilon^{-1} h_{\varepsilon, N}, \Psi^\varepsilon)}{\|\Psi^\varepsilon\|^2_{L^2(\Omega)}} \Psi^\varepsilon. \]  

(47)

It is clear that the function \( g_{\varepsilon, N} \) belongs to \( L^2_\perp (\Omega) \) as well and in view of Lemma 27 it satisfies the estimate

\[ \|g_{\varepsilon, N}\|_{L^2(\Omega)} \leq \|E_\varepsilon^{-1} h_{\varepsilon, N}\|_{L^2(\Omega)} \leq C \varepsilon^{2N+1} e^{-\frac{\theta_{\min}}{2\tau}}, \]  

(48)

where \( C \) is some constant independent of \( \varepsilon \). Let \( \mathcal{H}^\varepsilon \) be the restriction of the operator \( \mathcal{H}^\varepsilon \) on \( L^2_\perp (\Omega) \). The spectrum of this restriction does not include \( \lambda_\varepsilon \) and starts from the second eigenvalue \( \lambda^2_\varepsilon \). By Lemma 1 this implies the estimate

\[ \| (\mathcal{H}^\varepsilon - \lambda)^{-1} \| \leq C \varepsilon^{\frac{1}{2}} \]  

as \( \lambda \in [0, \varepsilon^2] \).

This estimate (48) and Equation (47) yield that

\[ \|v^{\perp}_{\varepsilon, N}\|_{L^2(\Omega)} = O\left( \varepsilon^{2N+\frac{1}{2}} e^{-\frac{\theta_{\min}}{2\tau}} \right). \]

The obtained relation allows us to apply Lemma 2 to the function \( v^{\perp, \perp} \) and Equation (47). This leads us to the following estimates:

\[ \|v^{\perp}_{\varepsilon, N}\|_{C(\overline{\Omega})} = O\left( \varepsilon^{2N+2n+3} e^{-\frac{\theta_{\min}}{2\tau}} \right), \quad \|\nabla \|v^{\perp}_{\varepsilon, N}\|_{C(\overline{\Omega})} = O\left( \varepsilon^{2N-n-5} \right), \]

\[ \|\nabla \varepsilon^{2n} v^{\perp}_{\varepsilon, N}\|_{L^2(\Omega)} = O\left( \varepsilon^{2n-3} e^{-\frac{\theta_{\min}}{2\tau}} \right), \quad \|\partial^2_{x x} \varepsilon^{n} v^{\perp}_{\varepsilon, N}\|_{L^2(\Omega)} = O\left( \varepsilon^{2n-5} e^{-\frac{\theta_{\min}}{2\tau}} \right), \]

\[ \|\nabla e^{2n} v^{\perp}_{\varepsilon, N}\|_{L^2(\Omega)} = O\left( \varepsilon^{2n-n-6} \right), \quad \|\partial^2_{x x} e^{2n} v^{\perp}_{\varepsilon, N}\|_{L^2(\Omega)} = O\left( \varepsilon^{2n-n-8} \right). \]  

(49)

Given an arbitrary subdomain \( \omega \subseteq \Omega \), by the above estimate for \( \|v^{\perp}_{\varepsilon, N}\|_{C(\overline{\Omega})} \) we obtain:

\[ \|e^{2n} v^{\perp}_{\varepsilon, N}\|_{C(\overline{\omega})} \leq e^{2n} \|v^{\perp}_{\varepsilon, N}\|_{C(\overline{\Omega})} = O\left( \varepsilon^{2N-2n+3} e^{-\frac{\theta_{\min} + \theta_0}{2\tau}} \right). \]

(50)

In the same way we get:
\[\|e^\frac{\epsilon}{2\sigma} v_{\epsilon, N}\|_{L_2(\Omega)} = O\left(e^{2N-1}e^{-\min\frac{\epsilon}{2\sigma}}\right),\]
\[\|\nabla e^\frac{\epsilon}{2\sigma} v_{\epsilon, N}\|_{L_2(\Omega)} = O\left(e^{2N-3}e^{-\min\frac{\epsilon}{2\sigma}}\right),\]
\[\|\partial_{xx}^2 e^\frac{\epsilon}{2\sigma} v_{\epsilon, N}\|_{L_2(\Omega)} = O\left(e^{2N-5}e^{-\min\frac{\epsilon}{2\sigma}}\right).\]

It follows from (32), (34), (38)–(40) and (44) and Lemma 4 that identity (15) holds with
\[Z_{\epsilon, N} := e^\frac{\epsilon}{2\sigma} v_{\epsilon, N} + (\tilde{K}_{\epsilon, N} + K_{\epsilon} \psi_\epsilon)\psi_\epsilon.\]

In view of this identity, estimates (43), (45), (46), (49) and (50), Theorem 1, and asymptotics (18) imply the following relations:
\[
\|Z_{\epsilon, N}\|_{L_2(\Omega)} = O(e^{2N-n-5}), \quad \|\nabla Z_{\epsilon, N}\|_{L_2(\Omega)} = O(e^{2N-n-6}),
\]
\[
\|\partial_{xx}^2 Z_{\epsilon, N}\|_{L_2(\Omega)} = O(e^{2N-2n+3}e^{-\min\frac{\epsilon}{2\sigma}}), \quad \|\partial_{xx}^4 e^\frac{\epsilon}{2\sigma} Z_{\epsilon, N}\|_{L_2(\Omega)} = O(e^{2N-5}e^{-\min\frac{\epsilon}{2\sigma}}),
\]
\[
\|\partial_{xx}^2 Z_{\epsilon, N}\|_{L_2(\Omega)} = O(e^{2N-5}e^{-\min\frac{\epsilon}{2\sigma}}), \quad Z_{\epsilon, N}\|_{C(\overline{\Omega})} = O(e^{2N-2n+3}e^{-\min\frac{\epsilon}{2\sigma}}).
\]

Comparing formulae (15) with different \(N\), it is easy to see that
\[Z_{\epsilon, N}(x) = \chi(x)e^{\frac{\theta_j(\psi_{\epsilon}(x)\tau(x))}{\epsilon^2}} + \sum_{j=N+1}^{N+p} e^{2j} P_j(\tau(x)e^{-2}, s(x)) + Z_{\epsilon, N+p}(x)\]

for an arbitrary \(p \in \mathbb{N}\). This identity allows us to improve the above estimates for \(Z_{\epsilon, N}\). Namely, by Lemma 4, the terms in the boundary layer in (15) satisfy the inequalities
\[
\|\chi e^\frac{\epsilon}{\sigma} P_j(\tau e^{-2}, s)\|_{C(\overline{\Omega})} = O(1), \quad \|\nabla e^\frac{\epsilon}{\sigma} P_j(\tau e^{-2}, s)\|_{L_2(\Omega)} = O(\epsilon),
\]
\[
\|\nabla \chi e^\frac{\epsilon}{\sigma} P_j(\tau e^{-2}, s)\|_{L_2(\Omega)} = O(\epsilon^{-1}), \quad \|\partial_{xx}^2 \chi e^\frac{\epsilon}{\sigma} P_j(\tau e^{-2}, s)\|_{L_2(\Omega)} = O(\epsilon^{-3}),
\]
where \(C\) are some constants independent of \(\epsilon\) but depending on \(j\). Employing these estimates, identity (52) and estimates (51) with \(N\) replaced by \(N+p\), we get first four estimates in (21). Other estimates in (21) can be obtained in the same way. This completes the proof of Theorem 2.

5. Concluding Remarks

We first stress one more feature of asymptotics (15). Let \(\Gamma\) be some non-empty subset of \(\partial \Omega\) and \(f\) be supported in \(\Gamma\). Then the probabilistic interpretation given in the Introduction means that in this case, the solution \(u_\epsilon\) describes how many trajectories of the Brownian particle hit the boundary for the first time somewhere in \(\Gamma\) and how these trajectories are distributed along \(\Gamma\). In view of the construction of the boundary layer in Section 4.1, in the considered case, the functions \(P_j\) vanish on \(\partial \Omega \setminus \Gamma\). The first term \(K_{\epsilon} \psi_\epsilon\) in (15) depends on \(f\) only via the integral in (17). This means that exactly the boundary layer in (15) describes a gentle nature of the number and distribution of the trajectories of the Brownian particle hitting \(\Gamma\), while the first term \(K_{\epsilon} \psi_\epsilon\) serves only as a leading term and a rather rough approximation.
From the point of view of the singular perturbation theory, the results of this work and of [20] show that problems (3) and (7) are nice specific examples of the asymptotic problem with the above-discussed property: the formal asymptotic expansions constructed for these problems do not provide true asymptotics for their solutions. The terms \( K_\varepsilon \Psi_\varepsilon \) are to be taken into consideration and the constants \( K_\varepsilon \) is to be chosen so that it minimizes the smallness order for the error terms. These ideas first developed in [20] for problem (3) with a specific right-hand side in the equation turn out to be rather general and we adapted them to problem (7) with an arbitrary right-hand side in the boundary condition.

In comparison with previous works, in this work and in [20], we succeeded to construct the complete asymptotic expansions for the considered problems, however, assuming that the potential \( V \) has the only minimum at the origin and no other stationary points in \( \Omega \). The issue on constructing similar complete asymptotics in the case of several stationary points of \( V \) in \( \Omega \) is still open and only leading terms of the asymptotics for the solutions are known [5–10,14–19]. This is a more complicated case and our technique can not be applied directly to this case. Nevertheless, we strongly believe that at least partially, our ideas can be used also in this more complicated case.

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