UNCONDITIONAL GLOBAL WELL-POSEDNESS FOR THE 3D
GROSS-PITAЕVSKII EQUATION FOR DATA WITHOUT FINITE
ENERGY

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Abstract. The Cauchy problem for the Gross-Pitaevskii equation in three
space dimensions is shown to have an unconditionally unique global solution
for data of the form $1 + H^s$ for $5/6 < s < 1$, which do not have necessarily
finite energy. The proof uses the I-method which is complicated by the fact
that no $L^2$-conservation law holds. This shows that earlier results of Bethuel-
Saut for data of the form $1 + H^1$ and Gérard for finite energy data remain
true for this class of rough data.

1. Introduction and main results

The Cauchy problem for the Gross-Pitaevskii equation in three space dimen-
sions reads as follows

\begin{align}
\frac{i\partial v}{\partial t} - \Delta v &= v(1 - |v|^2) \\
v(x,0) &= v_0(x),
\end{align}

under the condition

\begin{equation}
v \to 1 \quad \text{as} \quad |x| \to +\infty,
\end{equation}

where $v : \mathbb{R}^{1+3} \to \mathbb{C}$.

This problem occurs in theoretical physics, e.g. Bose-Einstein condensation
and superfluidity, see [Gr], [P], [SS].

Then one has the energy conservation law (see below)

\begin{equation}
E(v(t)) = \int \left( |\nabla v(x,t)|^2 + \frac{1}{2}(|v(x,t)|^2 - 1)^2 \right) dx = E(v_0).
\end{equation}

Because the solution does not vanish at infinity the standard theory for nonlinear
Schrödinger equations is not directly applicable and it is natural to consider instead
$u = v - 1$ for a solution $v$ of (11). Then $u$ satisfies the equivalent problem

\begin{align}
\frac{i\partial u}{\partial t} - \Delta u + (1 + u)(|u|^2 + 2 \Re u) &= 0 \\
u(x,0) &= u_0(x),
\end{align}

under the condition

\begin{equation}
u \to 0 \quad \text{as} \quad |x| \to +\infty.
\end{equation}
The real part of the $L^2$-scalar product of equation (5) with $\frac{\partial u}{\partial t}$ gives
\[
\frac{\partial}{\partial t} \int |\nabla u|^2 \, dx + \frac{1}{2} \frac{\partial}{\partial t} \int (|u(t)|^2 + 2 \Re u(t))^2 \, dx = 0,
\]
because
\[
\frac{1}{2} \frac{\partial}{\partial t} ((|u|^2 + 2 \Re u)^2) = 2 \Re ((1 + u)(|u|^2 + 2 \Re u) \frac{\partial u}{\partial t}).
\]
This gives the energy conservation law
\[
E(u(t)) = \int |\nabla u(t)|^2 \, dx + \frac{1}{2} \int (|u(t)|^2 + 2 \Re u(t))^2 \, dx = E(u_0).
\]
In terms of $v$ one gets (4). Remark that no conservation of $\|u(t)\|_{L^2}$ holds (in contrast to standard problems)!

We however get a bound for $\|u(t)\|_{L^2} = \|u(t)\|$ for finite energy data, which also belong to $L^2$, in the following way. The imaginary part of the scalar product of equation (5) with $u$ gives
\[
\frac{1}{2} \frac{\partial}{\partial t} \|u(t)\|^2 - \int (|u|^2 + 2 \Re u) \Im u \, dx = 0,
\]
because
\[
\Im (1 + u)(|u|^2 + 2 \Re u) \bar{u} = -(|u|^2 + 2 \Re u) \Im u.
\]
This immediately implies
\[
\frac{\partial}{\partial t} \|u(t)\|^2 \leq 2 \int (|u(t)|^3 + 2 |u(t)|^2) \, dx.
\]
We also get
\[
\frac{\partial}{\partial t} \|u(t)\|^2 \leq 2 \left( \int (|u(t)|^2 + 2 \Re u(t))^2 \, dx \right)^{\frac{1}{2}} \|u(t)\| \leq 2 \sqrt{2E(u_0)} \|u(t)\|,
\]
which implies
\[
\|u(t)\|^2 \leq \|u_0\|^2 + \int_0^t 2 \sqrt{2E(u_0)} \|u(s)\| \, ds,
\]
thus by a Gronwall type lemma
\[
\|u(t)\| \leq \|u_0\| + \sqrt{2E(u_0)t}.
\]
For data $u_0 \in H^1(\mathbb{R}^3)$ these considerations lead directly to an a-priori-bound of $\|\nabla u(t)\|_{L^2} \leq E(u(t)) = E(u_0)$, which is finite, because $H^1 \subset L^2$ by Sobolev’s embedding theorem, and also to an a-priori bound of $\|u(t)\|_{L^2}$. Together with local well-posedness (cf. Theorem 2.4 below) this shows that our problem (4), (6), (7) (and equivalently (1), (2), (3)) has a unique global solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^3))$.

The original proof was given by Bethuel and Saut [BS], Appendix A. Later Gérard [Gé] proved global well-posedness in the larger energy space using Strichartz estimates in two and three space dimensions. Gallo [Ga] proved global well-posedness for more general nonlinearities for data with finite energy and space dimension $n \leq 4$.

In the work at hand we are now interested in global well-posedness for data without finite energy, more precisely we consider solutions $v = 1 + u$, where $u \in H^s(\mathbb{R}^3)$ for $s < 1$. We apply the so called I-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT] and successfully applied to various problems. There are two facts which complicate the problem: on one hand there is no scaling invariance and on the other hand no conservation law for the $L^2$-norm of $u$. As usual the energy conservation law is not directly applicable for $H^s$-data with $s < 1$. However there is an “almost conservation law” for the modified energy $E(Iu)$, which is well defined for $u \in H^s$ (see the definition of $I$ below). This leads to an a-priori bound of $\|\nabla Iu(t)\|_{L^2}$, if $s$ is close enough to 1, namely $s > 5/6$. 


This can be shown to be enough for an a-priori bound also for \( \|u(t)\|_{L^2} \), which together gives a bound for \( \|u(t)\|_{H^s} \). A local well-posedness result in Bourgain type spaces \( X^{s,2}+ [0, T] \subset C^0([0, T], H^s) \) with existence time dependent only on \( \|u_0\|_{H^s} \) completes the global well-posedness result in this space. We even get unconditional global well-posedness in the space \( C^0([0, T], H^s) \) using a result of Kato [K]. This leads to the following main results (cf. the definition of the \( X^{s,b} \)-spaces below):

**Theorem 1.1.** Let \( T > 0 \), \( s > 5/6 \) and \( u_0 \in H^s(\mathbb{R}^3) \). The Cauchy problem (3), (4) has a unique global solution in \( X^{s,2}+ [0, T] \). This solution belongs to \( C^0([0, T], H^s(\mathbb{R}^3)) \).

Combining this with the uniqueness inequality result of T. Kato which we prove in Proposition 1.1 below we even get

**Theorem 1.2.** Let \( T > 0 \), \( s > 5/6 \) and \( u_0 \in H^s(\mathbb{R}^3) \). The Cauchy problem (3), (4) has a unique global solution in \( C^0([0, T], H^s(\mathbb{R}^3)) \). Equivalently the Cauchy problem (7), (8) has a unique global solution in \( C^0([0, T], 1 + H^s(\mathbb{R}^3)) \) for data \( v_0 \in 1 + H^s(\mathbb{R}^3) \).

The following proposition for more general nonlinearities and arbitrary dimensions goes back to Kato [K]. We give the (short) proof in the special case of cubic polynomials as nonlinearity in three space dimensions.

**Proposition 1.1.** Assume \( u_0 \in H^s(\mathbb{R}^3) \). The Cauchy problem

\[
i \frac{\partial u}{\partial t} - \Delta u = F(u, \bar{u}), \quad u(0) = u_0,
\]

where \( F(u, \bar{u}) \) is a polynomial of degree three, has at most one solution \( u \in C^0([0, T], H^s(\mathbb{R}^3)) \) for any \( T > 0 \), provided \( s \geq 2/3 \).

**Proof.** Let \( u, v \in C^0([0, T], H^s(\mathbb{R}^3)) \) be two solutions. By Sobolev’s embedding \( u, v \in C^0([0, T], L^{12}) \) using \( s \geq 2/3 \). By the Strichartz estimates (see below) for the inhomogeneous Schrödinger equation we get (ignoring complex conjugates, which play no role here)

\[
\|u - v\|_{L^2_t L^{12}_{x}} + \|u - v\|_{L^{6/5}_t L^{12}_{x}} \\
\leq \|u^2 - v^2\|_{L^2_t L^{6/5}_{x}} + \|u^2 - v^2\|_{L^2_t L^{6/5}_{x}} + \|u - v\|_{L^2_t L^{6/5}_{x}} \\
\leq \|u - v\|_{L^2_t L^{6/5}_{x}} + \|u\|_{L^2_t L^{6/5}_{x}}^{2} + \|v\|_{L^2_t L^{6/5}_{x}}^{2} + \|u - v\|_{L^2_t L^{6/5}_{x}} \\
\leq \|u - v\|_{L^2_t L^{6/5}_{x}} + \|u - v\|_{L^2_t L^{6/5}_{x}} T^{\frac{1}{2}}(\|u\|_{L^\infty_t H^s_x}^{2} + \|v\|_{L^\infty_t H^s_x}^{2}) \\
+ \|u - v\|_{L^2_t L^{6/5}_{x}} T^{\frac{1}{2}}(\|u\|_{L^\infty_t H^s_x}^{2} + \|v\|_{L^\infty_t H^s_x}^{2} + 1) \\
\leq \frac{1}{2}(\|u - v\|_{L^2_t L^{6/5}_{x}}^{2} + \|u - v\|_{L^2_t L^{6/5}_{x}}^{2}),
\]

choosing \( T \) small enough, which shows \( u = v \). \( \square \)

We use the following notation and well-known facts: the multiplier \( I = I_N \) is for given \( s < 1 \) and \( N \geq 1 \) defined by

\[
\widehat{I_N f}(\xi) := m_N(\xi) \hat{f}(\xi),
\]

where \( \hat{\cdot} \) denotes the Fourier transform with respect to the space variables. Here \( m_N(\xi) \) is a smooth, radially symmetric, nonincreasing function of \( |\xi| \) with

\[
m_N(\xi) = \begin{cases} 
1 & |\xi| \leq N \\
 (\frac{N}{|\xi|})^{1-s} & |\xi| \geq 2N
\end{cases}
\]
We remark that $I : H^s \to H^1$ is a smoothing operator, so that especially $E(Iu)$ is well-defined for $u \in H^s(\mathbb{R}^3)$ (remark that $H^1(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)$).

We use the Bourgain type function space $X^{m,b}$ belonging to the Schrödinger equation $iu_t - \Delta u = 0$, which is defined as follows: let $\widehat{\cdot}$ or $\mathcal{F}$ denote the Fourier transform with respect to space and time and $\mathcal{F}^{-1}$ its inverse. $X^{m,b}$ is the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ with respect to

$$\|f\|_{X^{m,b}} = \|\langle \xi \rangle^m \langle \tau \rangle^b \mathcal{F}(e^{-it\Delta}f(x,t))\|_{L^2_{\xi,\tau}} = \|\langle \xi \rangle^m \langle \tau \rangle + |\xi|^2 b \hat{f}(\xi,\tau)\|_{L^2_{\xi,\tau}},$$

For a given time interval $I$ we define

$$\|f\|_{X^{m,b}(I)} := \inf_{g_i = f} \|g\|_{X^{m,b}}.$$  

For $s \geq 0$ and $1 \leq r < \infty$ we denote by $H^{s,r}$ the standard Sobolev space, i.e. the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to

$$\|f\|_{H^{s,r}} = \|\mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f}(\xi))\|_{L^r_x}.$$  

We recall the following facts about the solutions $u$ of the inhomogeneous linear Schrödinger equation (see e.g. [GTV])

$$iu_t - \Delta u = F, \quad u(0) = f.$$  

For $b' + 1 \geq b \geq 0 \geq b' > -1/2$ and $T \leq 1$ we have

$$\|u\|_{X^{s,b}[0,T]} \lesssim \|f\|_{H^{s,1+b-b'}[0,T]}\|f\|_{X^{s,b}[0,T]}.$$  

For $1/2 > b > b' \geq 0$ or $0 \geq b > b' > -1/2$,

$$\|f\|_{X^{s,b}([0,T])} \lesssim \|f\|_{X^{s,b}[0,T]}.$$  

(see e.g. [C], Lemma 1.10).

Fundamental are the following Strichartz type estimates for the solution $u$ of (9) in three space dimensions (see [CH], [KT]):

$$\|u\|_{L^q(I,L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^q(\mathbb{R}^3)} + \|F\|_{L^q(I,L^r(\mathbb{R}^3))}$$

with implicit constant independent of the interval $I \subset \mathbb{R}$ for all pairs $(q,r), (\tilde{q},\tilde{r})$ with $q,r,\tilde{q},\tilde{r} \geq 2$ and $\frac{1}{q} + \frac{3}{2r} = \frac{3}{4}$, $\frac{1}{\tilde{q}} + \frac{3}{2\tilde{r}} = \frac{3}{4}$, where $\frac{1}{q} + \frac{1}{r} = 1$ and $\frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = 1$. This implies

$$\|\hat{u}\|_{L^q(I,L^r(\mathbb{R}^3))} \lesssim \|\hat{f}\|_{X^{s,b}([0,T])}.$$  

For real numbers $a$ we denote by $a+, a+, a-,$ and $a-$ the numbers $a + \epsilon, a + 2\epsilon, a - \epsilon$ and $a - 2\epsilon$, respectively, where $\epsilon > 0$ is sufficiently small.

Of special interest is also a bilinear refinement, which goes back to Bourgain [B], namely the following frequency localized version in three dimensions:

**Lemma 1.1.** Let $u_j$ be given with $\text{supp} \hat{u}_j \subset \{\xi \sim N_j\}$ $(j = 1, 2)$, $N_1 \leq N_2$. Then the following estimates hold

$$\|u_1 u_2\|_{L^2_{\xi,\tau}} \lesssim N_1^{-1} \|u_1\|_{X^{0,\frac{1}{2}+}} \|u_2\|_{X^{0,\frac{1}{2}+}},$$

$$\|u_1 u_2\|_{L^2_{\xi,\tau}} \lesssim N_2^{-1} \|u_1\|_{X^{0,\frac{1}{2}+}} \|u_2\|_{X^{0,\frac{1}{2}+}}.$$  

**Proof.** For a proof of (10) we refer to Bourgain [B], Lemma 5 or Grünrock [C]. (11) follows by interpolation of (10) with the crude estimate

$$\|u_1 u_2\|_{L^2_{\xi,\tau}} \lesssim \|u_1\|_{L^\infty_x L^2_{\xi,\tau}} \|u_2\|_{L^2_{\xi,\tau}L^2_{\xi,-}} \lesssim N_1^{1+}\|u_1\|_{X^{0,\frac{1}{2}+}} \|u_2\|_{X^{0,\frac{1}{2}+}},$$

using $X^{0,\frac{1}{2}+} \subset L^2_{\xi}L^2_{\xi,-}$ and Sobolev’s embedding $\dot{H}^{1+} \subset L^6_x$. \qed
The paper is organized as follows: in chapter 1 we prove two versions of a local well-posedness result for (5), namely \( u \in X^{s, 1+}([0, \delta]) \) for data \( u_0 \in H^s \) with \( s > 1/2 \), and a modification where \( \nabla I u \in X^{0, 1+}([0, \delta]) \) for data \( \nabla I u_0 \in L^2 \), which is necessary in order to combine it with an almost conservation law for the modified energy \( E(Iu) \). In chapter 2 we use these local results and bounds for the modified energy given in chapter 3 in order to get the main theorem. It is namely shown that the bounds for the modified energy are enough to give also a uniform exponential bound for the \( L^2 \)-norm of \( u(t) \) and as a consequence for the \( H^s \)-norm for \( u(t) \), which in view of the local well-posedness results suffices to get a global solution. In chapter 3 we calculate \( \frac{d}{dt} E(Iu) \) for any solution of the equation \( i \frac{du}{dt} - \Delta I u + I((1 + u)(|u|^2 + 2 Re \, u)) = 0 \). The most complicated part is to estimate the time integrated terms which appear in \( \frac{d}{dt} E(Iu) \). Finally we show that these estimates control the modified energy \( E(Iu) \) uniformly on arbitrary time intervals \([0, T]\), provided \( s > 5/6 \).

2. Local well-posedness

The following local well-posedness theorem is more or less standard.

**Theorem 2.1.** Assume \( s > 1/2 \) and \( u_0 \in H^s(\mathbb{R}^3) \). Then the Cauchy problem (5), (6) is locally well-posed, i.e. there exists \( T_0 = T_0(\|u_0\|_{H^s}) \) such that there exists a unique solution \( u \in X^{s, 1+}(0, T_0) \). This solution belongs to \( C^0([0, T_0], H^s(\mathbb{R}^3)) \).

**Proof.** We have to estimate \( \|F(u)\|_{X^{s, -1+}} \), where we define

\[
F(u) = (1 + u)(|u|^2 + 2 \text{ Re } u).
\]

We want to show

\[
\|u^2\|_{X^{s, -1+}} \lesssim T^{s - \frac{1}{2}} \|u\|_{X^{s, \frac{1}{2}+}}^3,
\]

where here and in the sequel we skip the interval \([0, T]\) in the \( X^{s, \delta}[0, T]\)-spaces.
We ignore complex conjugates, because they play no role here, and use a fractional Leibniz rule and dual duality to reduce to the estimate

\[
\|u^2(D)^s u\|_{L^1_t} \lesssim T^{s - \frac{1}{2}} \|u\|_{X^{s, \frac{1}{2}+}}^3 \|\psi\|_{X^{0, -\frac{1}{2}+}}.
\]

We have

\[
\|u^2(D)^s u\|_{L^1_t} \lesssim \|u^2(D)^s u\|_{L_t^{1+} L_x^2} \|\psi\|_{L_x^\infty - L_x^2} \lesssim \|u\|_{L_t^{1+} L_x^2}^2 \|\psi\|_{X^{0, \frac{1}{2}-}} \lesssim \|u\|_{L_t^{1+} H_x^{\frac{1}{2}+}} \|\psi\|_{X^{0, \frac{1}{2}-}} \lesssim T^{s - \frac{1}{2}} \|u\|_{X^{s, \frac{1}{2}+}}^2 \|\psi\|_{X^{0, -\frac{1}{2}+}},
\]

where \( \frac{1}{q} = \frac{1}{2} + \frac{1}{4} \), so that \( H_x^{\frac{1}{2}+} \subset L^6 \), and \( X^{s, \frac{1}{2}+} \subset L_t^2 L_x^2 \) by Strichartz, and \( \frac{1}{q} = \frac{1}{2} + \frac{1}{4} \), so that \( \frac{2}{q} = 3\left(\frac{1}{2} + \frac{1}{4}\right) \), thus \( X^{s, \frac{1}{2}+} \subset L_t^2 H_x^{\frac{1}{2}+} \) by Strichartz’ estimate.

Similarly we get by Strichartz’ estimate and Sobolev’s embedding

\[
\|u(D)^s u\|_{L_t^{1+} L_x^2} \lesssim T^{s - \frac{1}{2}} \|u(D)^s u\|_{L_t^{1+} L_x^2} \lesssim T^{s - \frac{1}{2}} \|u\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 L_x^{\infty}} \lesssim T^{s - \frac{1}{2}} \|u\|_{X^{s, \frac{1}{2}+}}^2,
\]

thus

\[
\|u^2\|_{X^{s, -1+}} \lesssim T^{s - \frac{1}{2}} \|u\|_{X^{s, \frac{1}{2}+}}^2 \|\psi\|_{X^{0, -\frac{1}{2}+}}.
\]

Finally

\[
\|u\|_{X^{s, -1+}} \lesssim T^{s - \frac{1}{2}} \|u\|_{L_t^2 L_x^2} \lesssim \|u\|_{X^{s, \frac{1}{2}+}}.
\]
Similar estimates hold for the difference $\|F(u) - F(v)\|_{X^{0, -\frac{1}{s}+}}$. The standard Picard iteration shows the claimed result, where $T \leq 1$ has to be chosen that $T^{s-\frac{1}{s}+} \|u_0\|_{H^s} \lesssim 1$ and $T^{s-\frac{1}{s}+} \|u_0\|_{H^s} \gtrsim 1$. Thus the choice as claimed in the theorem is possible. □

Remarks: 1. A similar proof in spaces of the type $L^p_t L^q_x$ could also be given. This goes back to [CW], where $s = 1/2$ is included, but in this limiting case the existence time depends not only on $\|u_0\|_{H^s}$.

2. Theorem 2.1 shows that in order to get a global solution it is sufficient to have an a-priori bound of $\|u(t)\|_{H^s}$, if $s > 1/2$.

We next prove a similar local well-posedness result involving the operator $I$.

Proposition 2.1. Assume $s > 1/2$ and $\nabla I u_0 \in L^2(\mathbb{R}^3)$. Then (after application of $I$) the problem (5), (6) has a unique local solution $u$ with $\nabla I u \in X^{0, \frac{1}{s}+}(0, \delta)$ and

$$\|\nabla I u\|_{X^{0, \frac{1}{s}+}(0, \delta)} \lesssim \sqrt{2} \|\nabla I u_0\|_{L^2},$$

where $\delta \leq 1$ can be chosen such that

$$\left(\frac{\delta^{s-\frac{1}{s}}}{N^{2(1-s)}} + \frac{\delta^{2s-1}}{N} + \delta^{\frac{1}{s}}\right) \|\nabla I u_0\|_{L^2} \sim 1. \quad (13)$$

Proof. The cubic term in the nonlinearity will be estimated as follows:

$$\|\nabla I(u_1, u_2, u_3)\|_{X^{0, -\frac{1}{s}+}} \lesssim \left(\frac{\delta^{s-\frac{1}{s}}}{N^{2(1-s)}} + \frac{\delta^{2s-1}}{N} + \delta^{\frac{1}{s}}\right) \prod_{i=1}^3 \|u_i\|_{X^{0, \frac{1}{s}+}} \|\psi\|_{X^{0, \frac{1}{s}+}} \quad (14)$$

This follows from

$$A := \int_0^\delta \int_{\mathbb{R}^3} M(\xi_1, \xi_2, \xi_3) \prod_{i=1}^3 \hat{\psi}(\xi_i, t) \hat{\psi}(\xi_i, t) d\xi_1 d\xi_2 d\xi_3 dt$$

$$\lesssim \left(\frac{\delta^{s-\frac{1}{s}}}{N^{2(1-s)}} + \frac{\delta^{2s-1}}{N} + \delta^{\frac{1}{s}}\right) \prod_{i=1}^3 \|u_i\|_{X^{0, \frac{1}{s}+}} \|\psi\|_{X^{0, \frac{1}{s}+}}$$

where

$$M(\xi_1, \xi_2, \xi_3) := \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)} \frac{|\xi_1 + \xi_2 + \xi_3|}{|\xi_1||\xi_2||\xi_3|}$$

and $\ast$ denotes integration over the region $\{\sum_{i=1}^4 \xi_i = 0\}$. We assume here and in the following w.l.o.g. that the Fourier transforms are nonnegative. We also assume w.l.o.g. $|\xi_1| \geq |\xi_2| \geq |\xi_3|$, $|\xi_1| \geq |\xi_2| \geq |\xi_3| \geq N$.

Case 1: $|\xi_1| \geq |\xi_2| \geq |\xi_3| \geq N$.

We first estimate the multiplier $M$. If $|\xi_1 + \xi_2 + \xi_3| \geq N$ we get

$$M(\xi_1, \xi_2, \xi_3) \lesssim \prod_{i=1}^3 \frac{|\xi_i|^{1-s}}{N} \left(\frac{N^{1-s}}{|\xi_1 + \xi_2 + \xi_3|^{1-s}}\right) \frac{|\xi_1 + \xi_2 + \xi_3|}{|\xi_1||\xi_2||\xi_3|}$$

$$\lesssim \prod_{i=1}^3 \left(\frac{|\xi_i|}{N}\right)^{1-s} \frac{N^{1-s}|\xi_1|^s}{|\xi_1||\xi_2||\xi_3|} \lesssim \frac{1}{|\xi_2|^s|\xi_3|^sN^{2(1-s)}},$$

and if $|\xi_1 + \xi_2 + \xi_3| \leq N$ we have

$$M(\xi_1, \xi_2, \xi_3) \lesssim \prod_{i=1}^3 \left(\frac{|\xi_i|}{N}\right)^{1-s} \frac{N}{|\xi_1||\xi_2||\xi_3|} \lesssim \prod_{i=1}^3 \left(\frac{N^s}{|\xi_i|^sN^{2(1-s)}}\right) \lesssim \frac{1}{|\xi_2|^s|\xi_3|^sN^{2(1-s)}}.$$

as before. This implies by Hölder’s and Strichartz’ inequality and Sobolev’s embedding
\[
A \lesssim \frac{1}{N^{1-s}} \| \psi \|_{L^{1}_t L^{2}_x} \| u_1 \|_{L^{1}_t L^{6}_x} \| F^{-1}(\frac{u_2}{|\xi_3|}) \|_{L^{1}_t L^{6}_x} \| F^{-1}(\frac{u_3}{|\xi_3|}) \|_{L^{1}_t L^{6}_x}
\lesssim \frac{\delta^{s-\frac{1}{2}}}{N^{1-s}} \| \psi \|_{L^{1}_t L^{2}_x} \| u_1 \|_{X^{0,\frac{1}{2}+}} + \| u_2 \|_{L^{1}_t L^{6}_x} \| u_3 \|_{L^{1}_t L^{6}_x}
\lesssim \frac{\delta^{s-\frac{1}{2}}}{N^{1-s}} \| \psi \|_{X^{0,\frac{1}{2}+}} \prod_{i=1}^{3} \| u_i \|_{X^{0,\frac{1}{2}+}}
\]
where \( \frac{1}{q} = \frac{1}{2} - \frac{1}{2}, \frac{1}{q} = s - \frac{1}{2}, \) such that \( \tilde{H}^{s,r} \subset L^q \) with \( \frac{1}{p} = \frac{1}{6} + \frac{1}{r} \) and \( \frac{1}{q} = \frac{1}{2} - \frac{3}{2r} \).

**Case 2:** \( |\xi_1| \geq |\xi_2| \geq N \geq |\xi_3| \)

The multiplier \( M \) is estimated as follows: if \( |\xi_1 + \xi_2 + \xi_3| \geq N \) we get
\[
M(\xi_1, \xi_2, \xi_3) \lesssim (\frac{|\xi_1|}{N})^{1-s} (\frac{|\xi_2|}{N})^{1-s} |\xi_1 + \xi_2 + \xi_3|^{1-s} \frac{|\xi_1 + \xi_2 + \xi_3|}{|\xi_1||\xi_2||\xi_3|} \lesssim \frac{1}{|\xi_2|^{s}|\xi_3|N^{1-s}},
\]
and if \( |\xi_1 + \xi_2 + \xi_3| \leq N \) we also have
\[
M(\xi_1, \xi_2, \xi_3) \lesssim (\frac{|\xi_1|}{N})^{1-s} (\frac{|\xi_2|}{N})^{1-s} \frac{N}{|\xi_1||\xi_2||\xi_3|} \lesssim \frac{1}{|\xi_2|^{s}|\xi_3|N^{1-s}}.
\]
This implies by Hölder, Strichartz and Sobolev
\[
A \lesssim \frac{1}{N^{1-s}} \| \psi \|_{L^{1}_t L^{2}_x} \| u_1 \|_{L^{1}_t L^{6}_x} \| F^{-1}(\xi_2 \xi_3^{-1} u_2) \|_{L^{1}_t L^{6}_x} \| F^{-1}(\xi_3 \xi_1^{-1} u_3) \|_{L^{1}_t L^{6}_x}
\lesssim \frac{1}{N^{1-s}} \delta^{s-\frac{1}{2}} \| \psi \|_{X^{0,\frac{1}{2}+}} \| u_1 \|_{L^{1}_t L^{2}_x} \| u_2 \|_{L^{1}_t L^{6}_x} \| u_3 \|_{L^{1}_t L^{6}_x}
\lesssim \frac{1}{N^{1-s}} \delta^{s-\frac{1}{2}} \| \psi \|_{X^{0,\frac{1}{2}+}} \prod_{i=1}^{3} \| u_i \|_{X^{0,\frac{1}{2}+}},
\]
where \( \frac{1}{q} = \frac{1}{2} - \frac{1}{2}, \tilde{H}^{s,r} \subset L^q \) for \( \frac{1}{p} = \frac{1}{6} + \frac{1}{r} \), thus \( \frac{1}{q} = \frac{1}{2} - \frac{3}{2r} \) and \( X^{0,\frac{1}{2}+} \subset L^{1}_t L^{6}_x \).

**Case 3:** \( |\xi_1| \geq N \geq |\xi_2| \geq |\xi_3| \) and \( |\xi_1| \gg |\xi_2| \), or \( N \geq |\xi_1| \geq |\xi_2| \geq |\xi_3| \).

In these cases we have \( M(\xi_1, \xi_2, \xi_3) \lesssim \frac{1}{|\xi_2|^{s}|\xi_3|} \), thus
\[
A \lesssim \| \psi \|_{L^{1}_t L^{2}_x} \| u_1 \|_{L^{1}_t L^{6}_x} \| F^{-1}(\xi_2 \xi_3^{-1} u_2) \|_{L^{1}_t L^{6}_x} \| F^{-1}(\xi_3 \xi_1^{-1} u_3) \|_{L^{1}_t L^{6}_x}
\lesssim \delta^{s-\frac{1}{2}} \| \psi \|_{X^{0,\frac{1}{2}+}} \prod_{i=1}^{3} \| u_i \|_{X^{0,\frac{1}{2}+}}
\]
Similarly as in case 2. This implies (14).

Next we have to estimate the quadratic terms in the nonlinearity. We want to show
\[
\| \nabla I(u_1 u_2) \|_{X^{\delta,\frac{1}{2}+}} \lesssim \delta^{s-\frac{1}{2}} \| \nabla I u_1 \|_{X^{0,\frac{1}{2}+}} \| \nabla I u_2 \|_{X^{0,\frac{1}{2}+}},
\]
which follows from
\[
B := \int_0^\delta \int M(\xi_1, \xi_2) \tilde{\psi}(\xi_1, t) \tilde{\psi}(\xi_2, t) \tilde{\psi}(\xi_3, t) d\xi_1 d\xi_2 d\xi_3 dt
\lesssim \delta^{s-\frac{1}{2}} \| u_1 \|_{X^{0,\frac{1}{2}+}} \| u_2 \|_{X^{0,\frac{1}{2}+}} \| \psi \|_{X^{0,\frac{1}{2}+}},
\]
where * denotes integration over the region \( \{ \xi_1 + \xi_2 + \xi_3 = 0 \} \) and
\[
M(\xi_1, \xi_2) := \frac{m(\xi_1 + \xi_2)}{m(\xi_1)m(\xi_2)} \frac{|\xi_1 + \xi_2|}{|\xi_1||\xi_2|}.
\]
Assuming w.l.o.g. $|\xi_1| \geq |\xi_2|$ we first consider

**Case 1:** $|\xi_2| \geq N$.

If $|\xi_1 + \xi_2| \geq N$ we have

$$M(\xi_1, \xi_2) \lesssim \left( \frac{|\xi_1|}{N} \right)^{1-s} \left( \frac{|\xi_2|}{N} \right)^{1-s} \frac{|\xi_1 + \xi_2|}{|\xi_1||\xi_2|} \lesssim \frac{1}{|\xi_2|^s N^{1-s}} ,$$

and in the case $|\xi_1 + \xi_2| \leq N$ we also get

$$M(\xi_1, \xi_2) \lesssim \left( \frac{|\xi_1|}{N} \right)^{1-s} \left( \frac{|\xi_2|}{N} \right)^{1-s} \frac{N}{|\xi_1||\xi_2|} \lesssim \frac{1}{N^{2(1-s)} |\xi_2|^s |\xi_1|^s} \lesssim \frac{1}{|\xi_2|^s N^{1-s}} ,$$

so that by Strichartz’ estimate using

$$|\xi| \lesssim |\xi_1|^{1/2} |\xi_2|^{1/2} ,$$

we have

**Case 2:** $N \geq |\xi_2|$. 

If $|\xi_1| \gg |\xi_2|$ we have $M(\xi_1 + \xi_2) \sim \frac{m(\xi_1 + \xi_2)}{m(\xi_1)m(\xi_2)} \sim \frac{1}{|\xi_2|^s}$, whereas, if $|\xi_1| \sim |\xi_2|$, we have $|\xi_1 + \xi_2| \leq N$ and $m(\xi_1) \sim m(\xi_2) \sim m(\xi_1 + \xi_2) \sim 1$, which leads to the same bound for $M(\xi_1, \xi_2)$. Thus

$$B \lesssim \|\psi\|_{L^2_x} \|u_1\|_{L^2_t L^6_x} \|F^{-1}(\frac{\hat{\psi}}{|\xi_2|^s})\|_{L^\infty_x L^3_x} \lesssim \frac{\delta^{1-s}}{N^{2(1-s)}} \|\psi\|_{X^{0,\frac{1}{2}+}} \|u_1\|_{X^{0,\frac{1}{2}+}} \|u_2\|_{X^{0,\frac{1}{4}+}} .$$

Finally

$$\|\nabla Iu\|_{X^{0,\frac{1}{2}+}} \lesssim \delta^{1-s} \|\nabla Iu\|_{X^{0,0}} \lesssim \delta^{1-s} \|\nabla Iu\|_{X^{0,\frac{1}{4}+}} .$$

Similar estimates hold for the difference $\|\nabla I(F(u) - F(v))\|_{X^{0,\frac{1}{4}+}}$.

A Picard iteration leads to the desired solution in $[0, \delta]$, where $\delta \leq 1$ has to be chosen such that

$$\frac{\delta^{1-s}}{N^{2(1-s)}} \|\nabla Iu_0\|_{L^2} \lesssim 1 , \quad \delta^{\frac{1}{4}-} \|\nabla Iu_0\|_{L^2} \lesssim 1 ,$$

$$\frac{\delta^{1-s}}{N^{1-}} \|\nabla Iu_0\|_{L^2} \lesssim 1 , \quad \delta^{\frac{1}{4}-} \|\nabla Iu_0\|_{L^2} \lesssim 1 .$$

**Remark:** We want to iterate this local existence theorem with time steps of equal length until we reach a given (large) time $T$. For this we need to control

$$\|\nabla Iu(t)\|_{L^2} \leq c(T) \quad \forall 0 \leq t \leq T .$$

This is achieved for $u_0 \in H^s$ and $s > 5/6$ by giving uniform bounds of the modified energy $E(Iu(t))$, which is done in chapter 3.

3. **Proof of Theorem 1.1**

**Proof.** Let us assume for the moment that (16) holds and show that this leads to the claim of Theorem 1.1. We thus have an a-priori bound for our local solution of Proposition 2.1 on any existence interval $[0, T]$, namely of

$$\|\nabla Iu(t)\|_{L^2} \sim \|\xi|^s \hat{\rho}(\xi, t)\|_{L^2((|\xi| \leq N))} + \|\xi|^s \hat{\rho}(\xi, t)\|_{L^2((|\xi| \geq N))} N^{1-s} .$$

What remains to be given is an a-priori bound for $\|u(t)\|_{L^2}$ as a consequence of 8.
Lemma 3.1. On any existence interval \([0,T]\) of our solution \(u \in X^{s,\frac{1}{2}+}[0,T]\) we have \(\|u(t)\|_{L^2(\mathbb{R}^3)} \leq c(T)\).

Proof. We smoothly decompose \(\hat{u} = \hat{u}_1 + \hat{u}_2\) with \(\text{supp} \hat{u}_1 \subset \{\|\xi\| \leq 2\}\) and \(\text{supp} \hat{u}_2 \subset \{\|\xi\| \geq 1\}\). Then we have by Gagliardo-Nirenberg

\[
\|u\|_{L^3} \leq \|u_1\|_{L^3} + \|u_2\|_{L^3} \lesssim \|\nabla u_1\|_{L^2}^\frac{1}{2} \|u_1\|_{L^2}^\frac{1}{2} + \|D^s u_2\|_{L^2} \lesssim \|\nabla u_1\|_{L^2}^\frac{1}{2} + \|u_1\|_{L^2}^\frac{1}{2} + \|D^s u_2\|_{L^2} \lesssim \|\nabla u_1\|_{L^2}^\frac{1}{2} + \|u_1\|_{L^2}^\frac{1}{2} + \|D^s u_2\|_{L^2}^\frac{1}{2},
\]

so that by \([16]\) and \([17]\) we get on \([0,T]\):

\[
\|u(t)\|_{L^3}^2 \lesssim \|\xi\|\|\hat{u}_1(\xi,t)\|_{L^2}^2 + \|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 + \|\xi\|\|\hat{u}_2(\xi,t)\|_{L^2}^2 \lesssim c(T)(\|u(t)\|_{L^2}^2 + 1).
\]

\([8]\) gives

\[
\frac{d}{dt}\|u(t)\|_{L^2}^2 \leq c''(T)(\|u(t)\|_{L^2}^2 + 1),
\]

so that Gronwall’s lemma gives

\[
\|u(t)\|_{L^2}^2 + 1 \leq \left(\|u_0\|_{L^2}^2 + 1\right)e^{c''(T)t}
\]
on \([0,T]\).

Combining Lemma [3.1] with \([16]\) and \([17]\) we get an a-priori bound of \(\|u(t)\|_{H^s}\).

Together with Theorem 2.1 we immediately get Theorem 1.1.

\[\square\]

4. Estimates for the Modifed Energy

Application of the operator \(I\) to equation \([5]\) gives

\[
i \partial_t Iu - \Delta Iu + IF(u) = 0,
\]

with

\[
F(u) := (1 + u)|(u|^2 + 2 \text{Re } u).
\]

We define the modified energy

\[
E(Iu) = \int |\nabla Iu|^2 dx + \frac{1}{2} \int (|u|^2 + 2 \text{Re } u)^2 dx.
\]

Of course one cannot expect that it is conserved, but we want to show an almost conservation law for it. We calculate its derivative as follows:

\[
\frac{d}{dt}E(Iu) = 2 \text{Re } \langle -\Delta Iu + (|u|^2 + 2 \text{Re } Iu)(1 + Iu), Iu \rangle = 2 \text{Re } \langle F(Iu) - IF(u), Iu \rangle
\]

by replacing \(\Delta Iu\) using \([18]\). Next we replace \(Iu\) again by use of \([18]\) and get

\[
\frac{d}{dt}E(Iu) = Im \langle (\nabla (F(Iu) - IF(u)), \nabla u) + (F(Iu) - IF(u), IF(u)) \rangle \leq \|(\nabla (F(Iu) - IF(u)), \nabla u)\| + \|(F(Iu) - IF(u), IF(u))\|.
\]

In order to control the increment of \(E(Iu)\) by \([19]\) on the local existence interval \([0,\delta]\) we have to estimate several terms. We assume from now on \(s \geq 3/4\).

1. Let us first consider the first term on the right hand side of \([19]\). Here and in the following we ignore complex conjugates, because they are of no interest. We want to show

\[
\int_0^\delta \|\nabla (I(u^3) - (Iu)^3), \nabla u\| dt \lesssim N^{-1+\frac{s}{2}} \|\nabla u\|_{X^{s,\frac{1}{2}+}}^4.
\]
This proves (20) after dyadic summation over \( N \).

**Case 2:**

This gives the same bound as in case 1 (without the factor \( \frac{N_{\min}}{N_{\max}} \)), where \( N_{\min} \) and \( N_{\max} \) is the smallest and the largest of the numbers \( N_i \), respectively. \( N_{\max} \geq 1 \) can be assumed in all the cases, because otherwise our multiplier \( M \) is identically zero.

In the term at hand we also assume w.l.o.g. \( N_1 \geq N_2 \geq N_3 \) and \( N_1 \geq N \).

**Case 1:** \( N_1 \geq N_2 \geq N_3 \geq N \Rightarrow N_4 \leq N_1 \sim N_{\max} \).

We have for \( s \geq 3/4 \):

\[
M(\xi_1, \xi_2, \xi_3) \lesssim \frac{N_1}{N} \frac{N_2}{N} \frac{N_3}{N} \frac{N_1}{N_1 N_2 N_3}.
\]

Thus by the bilinear Strichartz estimate \([11]\) we get

\[
A \lesssim \left( \frac{N_1}{N} \right)^{\frac{s}{2}} \left( \frac{N_2}{N} \right)^{\frac{s}{2}} \left( \frac{N_3}{N} \right)^{\frac{s}{2}} \frac{1}{N_2 N_3} \|u_1 u_3\|_{L_{xt}^2} \|u_2\|_{L_t^{\infty} L_x^3} \|u_4\|_{L_t^{\infty} L_x^\infty} + \frac{N_3^{\frac{s}{2}} \frac{N_1^{1+}}{N_1} \frac{N_2^{0+}}{N_2} \frac{N_4^{0+}}{N_4}}{N_1^{\frac{s}{2}} N_2^{\frac{s}{2}} N_3^{\frac{s}{2}}} \prod_{i=1}^4 \|u_i\|_{X^0, \frac{s}{2}^+} + \frac{1 \wedge N_3^{0+}}{N_1^{0+}} \frac{N_1}{N_1} \prod_{i=1}^4 \|u_i\|_{X^0, \frac{s}{2}^+}.
\]

**Case 2:** \( N_1 \geq N_2 \geq N \geq N_3 \).

This gives the same bound as in case 1 (without the factor \( \frac{N_{\min}}{N_{\max}} \)).

**Case 3:** \( N_1 \geq N \geq N_2 \geq N_3 \) and \( N_1 \gg N_2 \).

By the mean value theorem we get

\[
M(\xi_1, \xi_2, \xi_3) \lesssim \frac{|\nabla m(\xi_1)\xi_2|}{m(\xi_1)} \frac{N_1}{N_1 N_2 N_3} \lesssim \frac{N_1}{N_1 N_2 N_3}.
\]

leading as in case 1 to the bound

\[
A \lesssim \frac{N_2}{N_1} \frac{N_1}{N_1 N_2 N_3} \frac{N_3^{\frac{s}{2}} \frac{N_2^{1+}}{N_2} \frac{N_3^{0+}}{N_3}}{N_1^{\frac{s}{2}} N_2^{\frac{s}{2}} N_3^{\frac{s}{2}}} \prod_{i=1}^4 \|u_i\|_{X^0, \frac{s}{2}^+} + \frac{1 \wedge N_3^{0+}}{N_1^{0+}} \frac{N_1}{N_1} \prod_{i=1}^4 \|u_i\|_{X^0, \frac{s}{2}^+}.
\]

This proves (20) after dyadic summation over \( N_1, N_2, N_3, N_4 \).

2. We next want to show

\[
\int_0^f \langle \nabla (I(u^2) - (Iu)^2), \nabla Iu \rangle dt \lesssim N^{-1+} \delta^{\frac{s}{2}} \|\nabla Iu\|^3_{X^0, \frac{s}{2}^+}.
\]
which follows from

$$B = \int_0^\infty \int \sum_{i=1}^3 \mathcal{M} \left( \xi_i, t \right) dt \lesssim N^{-1} \delta^{\frac{3}{4}} \sum_{i=1}^3 \|u_i\|_{X^{0, \frac{4}{3}}}$$

where * denotes integration over the region \{\sum_{i=1}^3 \xi_i = 0\} and

$$\mathcal{M}(\xi_1, \xi_2, \xi_3) := \frac{|m(\xi_1 + \xi_3) - m(\xi_2) m(\xi_3)| |\xi_2 + \xi_3|}{m(\xi_2) m(\xi_3) |\xi_2| |\xi_3|}.$$ 

Assume w.l.o.g. \(|\xi_2| \geq |\xi_3|\) and \(|\xi_2| \geq N \Rightarrow |\xi_1| \lesssim |\xi_2| \sim N_{\text{max}}\).

Case 1: \(N_2 \sim N_3 \sim N\).

We have

$$\mathcal{M}(\xi_1, \xi_2, \xi_3) \lesssim \frac{1}{m(\xi_2)^2 N_2} \lesssim \frac{(N_2)^{\frac{1}{4}}}{N_2}.$$ 

Using \(\dot{H}^{0,3}_\xi \subset L^2_\xi\) and \(X^{0,\frac{4}{3}} \subset L^1_t L^2_\xi\) we get

$$B \lesssim \left( \frac{N_2}{N} \right)^{\frac{1}{4}} \left\| u_1 \right\|_{L^2_\xi} \left\| u_2 \right\|_{L^2_\xi} \left\| u_3 \right\|_{L^2_\xi} \lesssim \frac{\delta^{\frac{1}{4}} \left( N_{\text{max}}^0 \right)^{N_{\text{max}}^0 + 3}}{N_{\text{max}}^0 N_{\text{max}}^{-1}} \prod_{i=1}^3 \left\| u_i \right\|_{L^1_t L^2_\xi}.$$ 

Case 2: \(N_1 \sim N_3 \Rightarrow N_2 \lesssim N_1\).

Because (see above) \(N_1 \lesssim N_2\) we have \(N_1 \sim N_2 \sim N_3\), which was already considered in case 1.

Case 3: \(|\xi_1| \sim |\xi_2| \Rightarrow |\xi_3| \lesssim |\xi_1|\) and \(|\xi_3| \ll |\xi_1| \sim |\xi_2|\).

a. If \(|\xi_3| \gtrsim N\), we get as in case 1

$$\mathcal{M}(\xi_1, \xi_2, \xi_3) \lesssim \frac{1}{m(\xi_3)|\xi_3|} \lesssim \left( \frac{N_2}{N} \right)^{\frac{1}{4}} \frac{1}{|\xi_3|} \lesssim \frac{1}{N}.$$ 

Thus by Strichartz’ estimate

$$B \lesssim \frac{1}{N} \prod_{i=1}^3 \left\| u_i \right\|_{L^2_\xi} \lesssim \frac{\delta^{\frac{1}{4}}}{N} \prod_{i=1}^3 \left\| u_i \right\|_{L^1_t L^2_\xi} \lesssim \frac{\delta^{\frac{1}{4}}}{N} \prod_{i=1}^3 \left\| u_i \right\|_{X^{0, \frac{4}{3}}}.$$ 

b. If \(|\xi_3| \lesssim N\) we estimate

$$\mathcal{M}(\xi_1, \xi_2, \xi_3) \sim \frac{|\nabla m(\xi_2) \xi_3|}{m(\xi_2)|\xi_3|} \sim \frac{1}{|\xi_2|} \lesssim \frac{1}{N}$$

leading to the same bound as in a.

This proves (21).

3. Next we consider the second term on the right hand side of (19) and want to show

$$\int_0^\delta \left( (I(u^3)) - (Iu)^3, I(u^3) \right) dt \lesssim N^{-2} \left\| \nabla Iu \right\|_{X^{0, \frac{4}{3}}}.$$ 

This means that we have to show

$$C = \left| \int_0^\delta \int \sum_{i=1}^6 \mathcal{M}(\xi_1, \ldots, \xi_6) \prod_{i=1}^6 \mathcal{M}(\xi_1, \ldots, \xi_6) dt \right| \lesssim N^{-2} \prod_{i=1}^6 \left\| u_i \right\|_{X^{0, \frac{4}{3}}}$$

where

$$\mathcal{M}(\xi_1, \ldots, \xi_6) := \frac{|m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1) m(\xi_2) m(\xi_3)| m(\xi_1 + \xi_5 + \xi_6)}{m(\xi_1) m(\xi_2) m(\xi_3) m(\xi_5) m(\xi_6) \prod_{i=1}^6 |\xi_1|^{-1}}.$$ 

Assume w.l.o.g. \(N_1 \gtrsim N_2 \gtrsim N_3\), \(N_1 \gtrsim N\) and \(N_4 \gtrsim N_5 \gtrsim N_6\).

Case 1: \(N_4 \gtrsim N_5 \gtrsim N_6 \gtrsim N\).
a. \( N_1 \geq N_2 \geq N_3 \geq N \).

We have

\[
M(\xi_1, \ldots, \xi_6) \lesssim \prod_{i=1}^{6} \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{6} N_i^{-1}.
\]

Thus by Strichartz and Sobolev:

\[
C \lesssim \prod_{i=1}^{6} \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{6} N_i^{-1} \| u_1 \|_{L_t^2 L_x^6} \| u_2 \|_{L_t^\infty L_x^3} \| u_3 \|_{L_t^\infty L_x^3} \| u_4 \|_{L_t^\infty L_x^3} \| u_5 \|_{L_t^\infty L_x^3} \| u_6 \|_{L_t^\infty L_x^6}
\]

\[
\lesssim \frac{1}{\prod_{i=1}^{6} N_i^{\frac{1}{2}}} \frac{N_2^2 N_3^2 N_6^2}{N_1^2 N_2^2 N_4 N_5 N_6^2} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +}.
\]

This can be handled similarly as case a. without the factor \( \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \). Thus

\[
C \lesssim \frac{N_2^2 N_6^2}{N_1^2 N_4 N_5 N_6^2} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +} \lesssim \frac{N_2^2}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +}.
\]

b. \( N_1 \geq N_2 \geq N \geq N_3 \).

This can be handled similarly as case a. without the factor \( \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \). Thus

\[
C \lesssim \frac{N_2^2}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +} \lesssim \frac{1}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +}.
\]

This can be handled as case b. without the factor \( \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \) leading to the bound

\[
C \lesssim \frac{N_2^2}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +} \lesssim \frac{1}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +}.
\]

Case 2: \( N_4 \geq N_5 \geq N \geq N_6 \).

a. \( N_1 \geq N_2 \geq N_3 \geq N \).

This is handled like case 1a. without the factor \( \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \) and gives

\[
C \lesssim \frac{N_2^2}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +} \lesssim \frac{1}{N_1^2} \frac{N_4}{N_2} \frac{N_6}{N_5} \prod_{i=1}^{6} \| u_i \|_{X^{\sigma, 0} +}.
\]
b. $N_1 \geq N_2 \geq N \geq N_3$.

It can be handled like case 1b. without the factor $(\frac{\Delta}{\gamma})^\frac{1}{2}$ leading to

$$
C \lesssim \frac{N_{5}^{+} N_{6}^{+}}{N_{1}^{-} N_{2}^{-} N_{4}^{-} N_{5}^{-}} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1 \wedge N_{min}^{0+}}{N_{1}^{-} N_{2}^{-} N_{4}^{-} N_{5}^{-}} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}
$$

$$
\lesssim \frac{1 \wedge N_{min}^{0+}}{N_{max}^{-} N_{2}^{-} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}}
$$

\[\text{Case 3:} \ N_4 \geq N \geq N_5 \geq N_6.\]

\[\text{a.} \ N_1 \geq N_2 \geq N_3 \geq N.\]

We have

$$
M(\xi_1, \ldots, \xi_6) \lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{6} N_i^{-1}.
$$

Thus by Strichartz and Sobolev and the bilinear Strichartz estimate we get

$$
C \lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{6} N_i^{-1} \|u_1\|_{L_t^2 L_x^2} \|u_2\|_{L_t^2 L_x^2} \|u_3 u_4\|_{L_t^2 L_x^2} \|u_5\|_{L_t^\infty x} \|u_6\|_{L_t^\infty x}
$$

$$
\lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{6} N_i^{-1} \left( N_i^3 + N_i^3 \right) \left( N_{5}^{+} + N_{5}^{-} \right) \left( N_{6}^{+} + N_{6}^{-} \right) \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}
$$

$$
\lesssim \frac{N_{3}^{+} N_{5}^{-} N_{6}^{-}}{N_{max}^{-} N_{5}^{-} N_{6}^{-} N_{2}^{-}} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}
$$

$$
\lesssim \frac{N_{3}^{+} N_{5}^{-} N_{6}^{-}}{N_{max}^{-} N_{5}^{-} N_{6}^{-} N_{2}^{-} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}}
$$

b. $N_1 \geq N_2 \geq N \geq N_3$.

Similarly as in case a. without the factor $(\frac{\Delta}{\gamma})^\frac{1}{2}$ we get

$$
C \lesssim \frac{N_{3}^{+} N_{5}^{+} + N_{5}^{-} + N_{6}^{+} + N_{6}^{-}}{N_{max}^{-} N_{5}^{-} N_{6}^{-} N_{2}^{-}} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}
$$

$$
\lesssim \frac{N_{3}^{+} N_{5}^{+} + N_{5}^{-} + N_{6}^{+} + N_{6}^{-}}{N_{max}^{-} N_{5}^{-} N_{6}^{-} N_{2}^{-} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}}
$$

$$
\lesssim \frac{1 \wedge N_{min}^{0+}}{N_{max}^{-} N_{2}^{-} \prod_{i=1}^{6} \|u_i\|_{X^{0, \frac{1}{2}+}}}
$$
c. \( N_1 \geq N \geq N_2 \geq N_3 \).

The multiplier is estimated as follows using the mean value theorem
\[
M(\xi_1, \ldots, \xi_6) \lesssim \left| \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)}{m(\xi_1)} \right| \prod_{i=1}^{6} |\xi_i|^{-1} \\
\sim \left| \frac{(\nabla m)(\xi_1)(\xi_2 + \xi_3)}{m(\xi_1)} \right| \prod_{i=1}^{6} |\xi_i|^{-1} \lesssim \frac{N_2}{N_1} \prod_{i=1}^{6} |\xi_i|^{-1}.
\]

Thus we get by Sobolev and Strichartz:
\[
C \lesssim \frac{N_2}{N_1} \prod_{i=1}^{6} \left| \xi_i \right|^{-1} \|u_4\|_{L^{\infty}L^2} \|u_2\|_{L^2L^{\infty}} \|u_3\|_{L^2L^\infty} \|u_4\|_{L^2L^{\infty}} \|u_5\|_{L^2L^2} \|u_6\|_{L^2L^{\infty}}
\]
\[
\lesssim \frac{N_2}{N_1} \prod_{i=1}^{6} \left| \xi_i \right|^{-1} N_2 N_3^2 N_5^2 N_6^2 \prod_{i=1}^{6} \|u_i\|_{X^0, \frac{3}{2}^+}
\]
\[
\lesssim \frac{N_2 N_3^2 N_5^2 N_6^2}{N_1^2} \prod_{i=1}^{6} \|u_i\|_{X^0, \frac{3}{2}^+}
\]
\[
\lesssim \frac{1 \wedge N_0\max N_2}{N_1^2} \prod_{i=1}^{6} \|u_i\|_{X^0, \frac{3}{2}^+}.
\]

Case 4: \( N \gg N_4 \geq N_5 \geq N_6 \).

a. \( N_1 \geq N_2 \geq N_3 \geq N \).

We have
\[
C \lesssim \frac{N_1^2}{N} \frac{N_2}{N} \frac{N_3}{N} \prod_{i=1}^{6} \left| \xi_i \right|^{-1} \|u_1\|_{L^\infty L^2} \|u_2\|_{L^2L^\infty} \|u_3\|_{L^2L^\infty} \prod_{i=1}^{6} \|u_i\|_{L^2L^\infty}
\]
\[
\lesssim \frac{N_1^2}{N} \frac{N_2}{N} \frac{N_3}{N} \prod_{i=1}^{6} \left| \xi_i \right|^{-1} N_2 N_3^2 N_5^2 N_6^2 \prod_{i=1}^{6} \|u_i\|_{X^0, \frac{3}{2}^+}
\]
\[
\lesssim \frac{N_3^2 N_5^2 N_6^2}{N_1^2} \prod_{i=1}^{6} \|u_i\|_{X^0, \frac{3}{2}^+}
\]
\[
\lesssim \frac{1 \wedge N_0\max N_2}{N_1^2} \prod_{i=1}^{6} \|u_i\|_{X^0, \frac{3}{2}^+}.
\]

b. \( N_1 \geq N_2 \geq N \geq N_3 \).

We get the same bound as in case a. without the factor \( \left( \frac{N}{N_1} \right)^{\frac{1}{2}} \) leading to the same estimate.

c. \( N_1 \geq N \gg N_2 \geq N_3 \).

This case cannot occur, because \( \sum_{i=1}^{6} \xi_i = 0 \).

4. Next we aim to show
\[
\int_0^5 |(I(u^3) - (Iu)^3, I(u^2))|dt \lesssim N^{-\frac{5}{2}} \|\nabla Iu\|_{X^0, \frac{3}{2}^+},
\]

which follows from
\[
D = \int_0^5 \int M(\xi_1, \ldots, \xi_5) \prod_{i=1}^{5} u_i(\xi_i, t) d\xi_1 \ldots d\xi_5 dt \lesssim N^{-\frac{5}{2}} \prod_{i=1}^{5} \|u_i\|_{X^0, \frac{3}{2}^+},
\]

where
\[
M(\xi_1, \ldots, \xi_5) := \frac{|m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)|}{m(\xi_1)m(\xi_2)m(\xi_3)} \frac{m(\xi_4 + \xi_5)}{m(\xi_4)m(\xi_5)} \prod_{i=1}^{5} |\xi_i|^{-1}.
\]
Assume w.l.o.g. $N_1 \geq N_2 \geq N_3$, $N_4 \geq N$ and $N_4 \geq N_5$.

Case 1: $N_4 \geq N_5 \geq N$.

a. $N_1 \geq N_2 \geq N_3 \geq N$.

Thus by Strichartz and Sobolev:

$$D \lesssim \prod_{i=1}^{5} \left( \frac{N_i}{N} \right) \prod_{i=1}^{5} N_i^{-1} \|u_1\|_{L_t^\infty L_x^4} \|u_2\|_{L_t^2 L_x^6} \|u_3\|_{L_t^\infty L_x^4} \|u_4\|_{L_t^2 L_x^6} \|u_5\|_{L_t^\infty L_x^4}$$

$$\lesssim \prod_{i=1}^{5} \left( \frac{N_i}{N} \right) \prod_{i=1}^{5} N_i^{-1} N_{13}^{1/2} N_3^{1/2} N_5^{1/2} \prod_{i=1}^{5} \|u_i\|_{X^{0,1/2}^i}$$

$$\lesssim \frac{N_1^{1/2} N_5^{1/2}}{N_1^{1/4} N_3^{1/4} N_5^{1/4}} \prod_{i=1}^{5} \|u_i\|_{X^{0,1/2}^i}$$

b. $N_1 \geq N_2 \geq N \geq N_3$.

As in case a, (without the factor $(\frac{N_4}{N})^{1/2}$) we get the same estimate.

c. $N_1 \geq N \geq N_2 \geq N_3$ and $N_1 \gg N_2$.

The multiplier is estimated as follows:

$$\left| \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1) m(\xi_2) m(\xi_3)}{m(\xi_1) m(\xi_2) m(\xi_3)} \right| = \left| \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)}{m(\xi_1)} \right|$$

$$\sim \left| \frac{(\nabla m)(\xi_2 + \xi_3)}{m(\xi_1)} \right| \lesssim \frac{N_2}{N_1}.$$

Thus as in case a. we get

$$D \lesssim \left( \frac{N_2}{N_1} \right) \prod_{i=1}^{5} N_i^{-1} N_{13}^{1/2} N_3^{1/2} N_5^{1/2} \prod_{i=1}^{5} \|u_i\|_{X^{0,1/2}^i}$$

$$\lesssim \frac{N_1^{1/2} N_5^{1/2}}{N_1^{1/4} N_3^{1/4} N_5^{1/4}} \prod_{i=1}^{5} \|u_i\|_{X^{0,1/2}^i}$$

$$\lesssim \frac{1 \land N_1^{0+}}{N_{\max}^{0+} N_5^{0+}} \prod_{i=1}^{5} \|u_i\|_{X^{0,1/2}^i}.$$
By Strichartz and Sobolev and the bilinear Strichartz estimate (Π) we get

\[
D \lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^p \prod_{i=1}^{5} N_i^{-1} \|u_1 u_3\|_{L_x^2} \|u_2\|_{L_x^2 L_t^{\frac{5}{3}}} \|u_4\|_{L_x^2 L_t^{\frac{5}{3}}} \|u_5\|_{L_x^2 L_t^{\frac{5}{3}}}
\]

\[
\lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^p \prod_{i=1}^{5} N_i^{-1} \frac{N_i^4 + N_i^3 N_i^4 + \prod_{i=1}^{5} \|u_i\|}{N_i^{\frac{5}{2}}} \lesssim \frac{N_3^5 + N_2^5}{N_1^{\frac{5}{2}} N_2^{\frac{5}{2}} N_3^{\frac{5}{2}}} \prod_{i=1}^{5} \|u_i\| X^{\frac{n}{4} + \frac{3}{4}}.
\]

b. \( N_1 \sim N_2 \gtrsim N \geq N_3 \).

Without the factor \( \left( \frac{N_3}{N} \right)^{\frac{5}{2}} \) we get the same estimate as in case a.

5. Next we want to show

\[
\int_0^\delta |\langle I(u^2) - (Iu)^2, I(u^3)\rangle| \, dt \lesssim N^{-\frac{7}{2}} \|\nabla Iu\|_{X^{\frac{n}{4} + \frac{3}{4}}}^5.
\]

We have to prove

\[
E = \left| \int_0^\delta \int_{\mathbb{R}} M(\xi_1, \ldots, \xi_5) \prod_{i=1}^{5} \widehat{u_i}(\xi_i, t) \, d\xi_1 \ldots d\xi_5 \, dt \right| \lesssim N^{-\frac{7}{2}} \prod_{i=1}^{5} \|u_i\|_{X^{\frac{n}{4} + \frac{3}{4}}}, \quad (24)
\]

where

\[
M(\xi_1, \ldots, \xi_5) := \frac{|m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \frac{m(\xi_3 + \xi_4 + \xi_5)}{m(\xi_3)m(\xi_4)m(\xi_5)} \prod_{i=1}^{5} |\xi_i|^{-1}.
\]

Assume w.l.o.g. \( N_3 \geq N_4 \geq N_5 \), \( N_1 \geq N_2 \) and \( N_1 \geq N \).

Case 1: \( N_3 \geq N_4 \geq N_5 \geq N \).

a. \( N_1 \geq N_2 \gtrsim N \).

This case can be handled exactly as in case 1a.

b. \( N_1 \gtrsim N \gg N_2 \).

We get by use of the mean value theorem for the first fraction and estimating similarly as in 4. case 1a. (interchanging the roles of \( u_2 \) and \( u_3 \)):

\[
E \lesssim \frac{N_3^2}{N_1^4} \prod_{i=1}^{5} \left( \frac{N_i}{N} \right)^p \prod_{i=1}^{5} N_i^{-1} N_i^4 + N_i^1 N_i^5 + \prod_{i=1}^{5} \|u_i\|_{X^{\frac{n}{4} + \frac{3}{4}}}
\]

\[
\lesssim \frac{N_3^5 + N_2^5}{N_1^{\frac{5}{2}} N_2^{\frac{5}{2}} N_3^{\frac{5}{2}}} \prod_{i=1}^{5} \|u_i\|_{X^{\frac{n}{4} + \frac{3}{4}}}
\]

\[
\lesssim \frac{1 \wedge N_3^{\frac{5}{2} + \frac{5}{2}}}{N_1^{\frac{5}{2} + \frac{5}{2}} N_2^{\frac{5}{2} + \frac{5}{2}}} \prod_{i=1}^{5} \|u_i\|_{X^{\frac{n}{4} + \frac{3}{4}}}.
\]

Case 2: \( N_3 \geq N_4 \geq N \geq N_5 \).

a. \( N_1 \geq N_2 \gtrsim N \).
By Strichartz and Sobolev we get

\[ E \leq \prod_{i=1}^{4} \left( \frac{N_i}{N} \right)^{\frac{i}{2}} \prod_{i=1}^{5} \left( N_i^{-1} \right) \| u_1 \|_{L^2_t L^2_x} \| u_2 \|_{L^2_t L^5_x} \| u_3 \|_{L^2_t L^6_x} \| u_4 \|_{L^2_t L^6_x} \| u_5 \|_{L^2_t L^6_x} \]

\[ \leq \prod_{i=1}^{4} \left( \frac{N_i}{N} \right)^{\frac{i}{2}} \prod_{i=1}^{5} N_i^{-1} N_i^{\frac{1}{2}} N_4 N_5^{1+} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{N_i^{\frac{1}{2}} N_0^+}{N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}} N_4^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{1 \wedge N_0^+}{N_{\min}^{\frac{1}{2}} N_{\max}^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} . \]

b. \( N_1 \geq N_2 \gg N_2 \).

By the mean value theorem we get as in a. (slightly modified):

\[ E \leq \frac{N_2}{N_1} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{5} N_i^{-1} N_i^{\frac{1}{2}} N_2^{1+} N_4^{1-} N_5^{1+} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{N_2^{0+} N_3^{1-} N_5^{1+}}{N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{1 \wedge N_0^+}{N_{\min}^{\frac{1}{2}} N_{\max}^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} . \]

Case 3: \( N_3 \geq N \geq N_4 \geq N_5 \).

a. \( N_1 \geq N_2 \geq N \).

The second fraction is bounded, so that as in case 2a. we get

\[ E \leq \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{5} N_i^{-1} N_i^{\frac{1}{2}} N_4 N_5^{1+} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{N_5^{1+}}{N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} N_3^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{1 \wedge N_0^+}{N_{\max}^{\frac{1}{2}} N_{\max}^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} . \]

b. \( N_1 \geq N \geq N_2 \).

Using the mean value theorem and interchanging the roles of \( u_1 \) and \( u_2 \) in case a. gives

\[ E \leq \frac{N_2}{N_1} \prod_{i=1}^{5} N_i^{-1} N_i^{\frac{1}{2}} N_4 N_5^{1+} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{N_2^{\frac{1}{2}} N_0^+}{N_1^{\frac{1}{2}} N_3^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} \]

\[ \leq \frac{1 \wedge N_0^+}{N_{\max}^{\frac{1}{2}} N_{\max}^{\frac{1}{2}}} \prod_{i=1}^{5} \| u_i \|_{X^{\frac{1}{2}, 4}} . \]
**Case 4:** $N \gg N_1 \geq N_2 \geq N_3 \Rightarrow N_1 \sim N_2 \geq N$, because $\sum_{i=1}^{5} \xi_i = 0$.

This gives by Strichartz, Sobolev and the bilinear Strichartz estimate (10) we get

$$E \lesssim \left( \frac{N_1}{N} \right)^{1+} \frac{N_2}{N} \prod_{i=1}^{5} N_i^{-1} \left\| u_1 u_3 \right\|_{L^2} \left\| u_2 \right\|_{L^s} \left\| u_4 \right\|_{L^s} \left\| u_5 \right\|_{L^s} \left\| u_5 \right\|_{L^s}$$

$$\lesssim \left( \frac{N_1}{N} \right)^{1+} \frac{N_2}{N} \prod_{i=1}^{5} N_i^{-1} \frac{N_3}{N_1} \prod_{i=1}^{5} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}}$$

$$\lesssim \frac{N^{0+}}{N_1^4 N_2^4 N_3^4} \prod_{i=1}^{5} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}}$$

$$\lesssim \frac{1 \land N^{0+}}{N_{\text{max}}^4 N^4} \prod_{i=1}^{5} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}} .$$

6. Next we want to prove

$$F = \int_0^{T} \int \left[ (I(u^3) - (Iu)^3, Iu) \right] dt \lesssim N^{-3} \left\| \nabla u \right\|_{X^{0, \frac{1}{2}+}}^4 ,$$

which follows from

$$F = \int_0^{T} \int M(\xi_1, \ldots, \xi_4) \prod_{i=1}^{4} \tilde{\omega}(\xi_i, t) d\xi_1 \ldots d\xi_4 dt \leq N^{-3} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}} , \quad (25)$$

where

$$M(\xi_1, \xi_2, \xi_3) := \frac{\left| m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1) m(\xi_2) m(\xi_3) \right|}{m(\xi_1) m(\xi_2) m(\xi_3)} \prod_{i=1}^{4} \left\| \xi_i \right\|_{X^{0, \frac{1}{2}+}} .$$

Assume w.l.o.g. $N_1 \geq N_2 \geq N_3$ and $N_1 \geq N$.

**Case 1:** $N \gg N_4$.

a. $N_1 \geq N_2 \geq N_3 \geq N$.

By the bilinear Strichartz refinement (11) we get

$$E \lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^{1+} \prod_{i=1}^{4} N_i^{-1} \left\| u_2 u_4 \right\|_{L^2} \left\| u_1 u_3 \right\|_{L^2}$$

$$\lesssim \prod_{i=1}^{3} \left( \frac{N_i}{N} \right)^{1+} \prod_{i=1}^{4} N_i^{-1} \frac{N_3^{1+} N_4^{1+}}{N_2^2 - N_1^2} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}}$$

$$\lesssim \frac{N^{1+} N^{0+}}{N_1^4 N_2^4 N_3^4} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1 \land N^{0+}}{N_{\text{max}}^4 N^4} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}} .$$

b. $N_1 \geq N_2 \gg N \geq N_3$.

Similarly as in case a. we get

$$E \lesssim \prod_{i=1}^{2} \left( \frac{N_i}{N} \right)^{1+} \prod_{i=1}^{4} N_i^{-1} \frac{N_3^{1+} N_4^{1+}}{N_2^2 - N_1^2} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}}$$

$$\lesssim \frac{N^{0+} N^{0+}}{N_1^4 N_2^4 N_3^4} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1 \land N^{0+}}{N_{\text{max}}^4 N^4} \prod_{i=1}^{4} \left\| u_i \right\|_{X^{0, \frac{1}{2}+}} .$$

c. $N_1 \geq N \gg N_2 \geq N_3$.

This case does not occur, because $\sum_{i=1}^{4} \xi_i = 0$.

**Case 2:** $N_4 \gg N$.

a. $N_1 \geq N_2 \geq N_3 \geq N$.

b. $N_1 \geq N_2 \gg N \geq N_3$ can be treated as in case 1.
c. \( N_1 \geq N \Rightarrow N_2 \geq N_3 \Rightarrow N_1 \sim N_4 \)

By the mean value theorem and the bilinear Strichartz refinement (11) we get
\[
E \lesssim \frac{N_2}{N_1} \prod_{i=1}^{4} N_i^{-1} \|u_1 u_2\|_{L^2_t} \|u_3 u_4\|_{L^2_t}
\]
\[
\lesssim \frac{N_2}{N_1} \prod_{i=1}^{4} N_i^{-1} \frac{N_2}{N_i^2} \frac{N_1^{1+}}{N_2^{1+}} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}}
\]
\[
\lesssim \frac{N_2 N_3^{1+}}{N_1^{1+} N_4} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1}{N_{\text{max}}^{1+} N_3} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}}.
\]

7. Next we prove
\[
\int_0^\delta |(I(u^2) - (Iu)^2, (Iu)^2)| dt \lesssim N^{-3+} \|\nabla Iu\|_{X^{0, \frac{1}{2}+}},
\]
which follows from
\[
F = |\int_0^\delta \int M(\xi_1, \ldots, \xi_4) \prod_{i=1}^{4} \hat{u}_i(\xi_i, t) d\xi_1 \ldots d\xi_4 dt| \lesssim N^{-3+} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}},
\]
where
\[
M(\xi_1, \xi_2, \xi_3) := \frac{|m(\xi_1 + \xi_2) - m(\xi_1) m(\xi_2)| m(\xi_3 + \xi_4) m(\xi_3) m(\xi_4)}{|m(\xi_1) m(\xi_2)|} \prod_{i=1}^{4} |\xi_i|^{-1}.
\]

Assume w.l.o.g. \( N_1 \geq N_2, N_1 \geq N \) and \( N_3 \geq N_4 \).

**Case 1:** \( N_3 \geq N_4 > N \).

**a.** \( N_1 \geq N_2 \geq N \).

By the bilinear Strichartz refinement (11) we get
\[
F \lesssim \prod_{i=1}^{4} \left( \frac{N_i}{N} \right)^{1+} \prod_{i=1}^{4} N_i^{-1} \|u_1 u_2\|_{L^2_t} \|u_3 u_4\|_{L^2_t}
\]
\[
\lesssim \prod_{i=1}^{4} \left( \frac{N_i}{N} \right)^{1+} \prod_{i=1}^{4} N_i^{-1} \frac{N_1^{1+}}{N_2^{1+}} \frac{N_2^{1+}}{N_3} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}}
\]
\[
\lesssim \frac{N_2^{1+} N_3^{1+}}{N_1^{1+} N_4} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1}{N_{\text{max}}^{1+} N_3} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}}.
\]

**b.** \( N_1 \geq N \geq N_2 \).

By the mean value theorem we get
\[
F \lesssim \frac{N_2 N_3}{N_1} \prod_{i=1}^{4} N_i^{-1} \frac{N_2^{1+}}{N_1^{1+}} \frac{N_1^{1+}}{N_3} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}}
\]
\[
\lesssim \frac{N_2^{1+} N_3^{1+}}{N_1^{1+} N_3} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}} \lesssim \frac{1}{N_{\text{max}}^{1+} N_3} \prod_{i=1}^{4} \|u_i\|_{X^{0, \frac{1}{2}+}}.
\]

**Case 2:** \( N_3 \geq N \geq N_4 \) and \( N_3 \gg N_4 \).

**a.** \( N_1 \geq N_2 \geq N \).
We get similarly as in case 1a.

\[
F \lesssim \prod_{i=1}^{2} \left( \frac{N_i}{N} \right)^{\frac{1}{2}} \prod_{i=1}^{4} \frac{N_i^{-1} N_i^{1+} N_i^{4+}}{N_i^{2+} N_i^{3+}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}
\]

\[
\lesssim \frac{N_2^{\frac{1}{2}+} N_1^{0+}}{N_1^{\frac{1}{2}+} N_3^{0+}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}
\]

\[
\lesssim \frac{1 \wedge N_0^{\min}}{N_{\max}^{0+} N^{3-}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}.
\]

b. \(N_1 \geq N \geq N_2\).

By the mean value theorem we get similarly as in case 1a:

\[
F \lesssim \frac{N_2}{N_1} \prod_{i=1}^{4} \frac{N_i^{-1} N_i^{1+} N_i^{4+}}{N_i^{2+} N_i^{3+}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}
\]

\[
\lesssim \frac{N_1^{1+} N_2^{0+}}{N_1^{\frac{1}{2}+} N_3^{\frac{1}{2}+}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}} \lesssim \frac{1 \wedge N_0^{\min}}{N_{\max}^{0+} N^{3-}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}.
\]

Case 3: \(N \gg N_3 \geq N_4 \Rightarrow N_1 \sim N_2 \gg N\), because \(\sum_{i=1}^{4} \xi_i = 0\).

We get by (11):

\[
F \lesssim \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{4}} \prod_{i=1}^{4} N_i^{-1} \|u_1 u_3\|_{L^2_{Iz}} \|u_2 u_4\|_{L^2_{Iz}}
\]

\[
\lesssim \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{4}} \prod_{i=1}^{4} N_i^{-1} \frac{N_3 - N_4^{1+}}{N_i^2 N_i^{2+}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}
\]

\[
\lesssim \frac{N_4^{0+}}{N_1^{\frac{1}{2}+} N_2^{\frac{1}{2}+}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}} \lesssim \frac{1 \wedge N_0^{\min}}{N_{\max}^{0+} N^{3-}} \prod_{i=1}^{4} \|u_i\|_{X^{0,\frac{1}{2}+}}.
\]

8. Finally we prove

\[
\int_0^\delta |(I(\xi)^2) - (Iu)^2, Iu)|dt \lesssim N^{-\frac{1}{2}+} \delta^{\frac{1}{2}} \|\nabla Iu\|_{X^{0,\frac{1}{2}+}}^3,
\]

which follows from

\[
G = \left| \int_0^\delta M(\xi_1, ..., \xi_3) \prod_{i=1}^{3} \hat{u}_i(\xi_i, t)d\xi_1d\xi_2d\xi_3dt \right| \lesssim N^{-\frac{1}{2}+} \delta^{\frac{1}{2}} \prod_{i=1}^{3} \|u_i\|_{X^{0,\frac{1}{2}+}}, \tag{27}
\]

where

\[
M(\xi_1, \xi_2, \xi_3) := \left| \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right| \prod_{i=1}^{3} |\xi_i|^{-1},
\]

Assume w.l.o.g. \(N_1 \geq N_2\) and \(N_1 \geq N\).

Case 1: \(N_3 \gtrsim N\).

a. \(N_1 \geq N_2 \gtrsim N\).
Case 2: This completes the estimates for the increment of interval \([0, T]\) has a unique solution \(u\) and (28):

By (11) we get the estimate

\[ G \lesssim \frac{N_1}{N} \left( \frac{N_2}{N} \right)^{\frac{3}{2}} \prod_{i=1}^{3} N_i^{-1} \| u_1 \|_{L^2_x} \| u_2 \|_{L^\infty_x L^2_t} \| u_3 \|_{L^\infty_x L^2_t}, \]

\[ \lesssim \left( \frac{N_1}{N} \right)^{\frac{3}{2}} \left( \frac{N_2}{N} \right)^{\frac{3}{2}} \prod_{i=1}^{3} N_i^{-N_2^\frac{3}{2}} \delta^2 \prod_{i=1}^{3} \| u_i \|_{X^0, \frac{1}{2} +}. \]

Using the mean value theorem we get estimates for the modified energy just given.

\[ N \geq N \Rightarrow N_1 \sim N_2 \gtrsim N, \text{ because } \sum_{i=1}^{3} \delta^2. \]

By (11) we get the estimate

\[ G \lesssim \frac{N_2^+}{N_1^3 N_3^2 N_2^\frac{3}{2}} \prod_{i=1}^{3} \| u_i \|_{X^0, \frac{1}{2} +}. \]

This completes the estimates for the increment of \( E(Iu) \) on the local existence interval \([0, \delta]\) in terms of the parameter \( N \).

We recall our aim to give an a-priori bound of \( \| \nabla u(t) \|_{L^2} \) (cf. (19)) on any interval \([0, T]\). We want to show this as a consequence of Proposition 2.1 and the estimates for the modified energy just given.

We assume \( N \geq 1 \) to be a number to be specified later and \( s \geq \frac{3}{4} \). Let data \( u_0 \in H^s(\mathbb{R}^3) \) be given. Then we have

\[ \| \nabla u_0 \|_{L^2_x} \lesssim \| \xi^{\frac{s}{2}} \hat{u}_0(\xi) \|_{L^2_x(\{|\xi| \leq N_1\})} + \| \nabla^{1-s} \xi^{\frac{s}{2}} \hat{u}_0(\xi) \|_{L^2_x(\{\xi \geq N_1\})}, \]

\[ \lesssim \| \nabla^{1-s} \xi^{\frac{s}{2}} \hat{u}_0(\xi) \|_{L^2_x(\mathbb{R}^3)} = N^{2(1-s)} \| u_0 \|_{H^s}, \]

This immediately implies an estimate for \( E(Iu_0) \). We namely have for \( s \geq \frac{3}{4} \):

\[ \| Iu_0 \|_{L^4_x(\mathbb{R}^3)} \lesssim \| u_0 \|_{H^s} \lesssim \| u_0 \|_{H^s}. \]

and trivially \( \| Iu_0 \|_{L^2_x} \lesssim \| u_0 \|_{L^2_x} \). Thus using the definition of the modified energy and (28):

\[ E(Iu_0) \leq c_0 N^{2(1-s)}. \]

From this we get

\[ \| \nabla u_0 \|_{L^2} \leq c_0 N^{2(1-s)}. \]

Our local existence theorem (Proposition 2.1) shows that the Cauchy problem (5), (6) has a unique solution \( u \) with \( \nabla u \in X^{0, \frac{1}{2} +}[0, \delta] \) and

\[ \| \nabla u(\delta) \|_{L^2_x} \leq \| \nabla u(t) \|_{X^{0, \frac{1}{2} +}[0, \delta]} \leq 2 \| \nabla u_0 \|_{L^2_x} \leq 2 c_0 N^{2(1-s)}. \]
The number of iteration steps to reach a given time $t$ where $c$ easily checks that the first term is the decisive one, so that
\[ \max(\delta^s - 1 - s, \delta^s - N^{1-s}, \delta^s - N^{2(1-s)}) \sim 1 \iff \delta \sim \frac{1}{N^{s(1-s)+}}. \]

In order to reapply the local existence theorem with time steps of equal length we need a uniform bound of $\|\nabla Iu(t)\|_{L^2}$ at time $t = \delta$, $t = 2\delta$ etc., which follows from a uniform control over the modified energy. The increment of the energy is controlled by (19) and the estimates of this section as follows, provided $s \geq 3/4$:
\[
|E(Iu(\delta)) - E(Iu_0)| \lesssim N^{-1+} \|\nabla Iu\|_{X^{0,\frac{1}{2}+}(0,\delta)}^4 + N^{-1+} \delta^{\frac{1}{2}+} \|\nabla Iu\|_{X^{0,\frac{1}{2}+}(0,\delta)}^3 + N^{-2+} \|\nabla Iu\|_{X^{0,\frac{1}{2}+}(0,\delta)}^6 + N^{-2+} \frac{1}{\delta} \|\nabla Iu\|_{X^{0,\frac{1}{2}+}(0,\delta)}^3
\]

The last two terms can be neglected in comparison to the others. Thus we get
\[
|E(Iu(\delta)) - E(Iu_0)| \lesssim N^{-1+} N^{4(1-s)} + N^{-1+} N^{-(1-s)} N^{3(1-s)} + N^{-2+} N^{6(1-s)} + N^{-2+} N^{5(1-s)}.
\]

One easily checks that the first term is the decisive one, so that
\[
|E(Iu(\delta)) - E(Iu_0)| \leq c_1 N^{-1+} N^{4(1-s)},
\]

where $c_1 = c_1(c_0)$. This is easily seen to be bounded by $c_0 N^{2(1-s)}$ for large $N$. The number of iteration steps to reach a given time $T$ is $\frac{T}{\delta} \sim TN^{4(1-s)+}$. This means that in order to give a uniform bound of the energy of the iterated solutions, namely by $2c_0 N^{2(1-s)}$, from the last inequality, the following condition has to be fulfilled:
\[
c_1 N^{-1+} N^{4(1-s)} TN^{4(1-s)+} < c_0 N^{2(1-s)},
\]

where $c_1 = c_1(2c_0)$ (recall here that the initial energy is bounded by $c_0 N^{2(1-s)}$). This can be fulfilled for $N$ sufficiently large, provided
\[
-1 + 4(1 - s) + 4(1 - s) < 2(1 - s) \iff s > 5/6.
\]

This gives the desired bound for $\|\nabla Iu(t)\|_{L^2}$ on any interval $[0, T]$, i.e. (10) is proved. As explained above this completes the proof of our Theorem 1.1.

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