Adaptive cyclically dominating game on co-evolving networks: numerical and analytic results

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Abstract. A co-evolving and adaptive Rock (R)–Paper (P)–Scissors (S) game (ARPS) in which an agent uses one of three cyclically dominating strategies is proposed and studied numerically and analytically. An agent takes adaptive actions to achieve a neighborhood to his advantage by rewiring a dissatisfying link with a probability \( p \) or switching strategy with a probability \( 1 - p \). Numerical results revealed two phases in the steady state. An active phase for \( p < p_0 \) has one connected network of agents using different strategies who are continually interacting and taking adaptive actions. A frozen phase for \( p > p_0 \) has three separate clusters of agents using only R, P, and S, respectively with terminated adaptive actions. A mean-field theory based on the link densities in co-evolving network is formulated and the trinomial closure scheme is applied to obtain analytical solutions. The analytic results agree with simulation results on ARPS well. In addition, the different probabilities of winning, losing, and drawing a game among the agents are identified as the origin of the small discrepancy between analytic and simulation results. As a result of the adaptive actions, agents of higher degrees are often those being taken advantage of. Agents with a smaller (larger) degree than the mean degree have a higher (smaller) probability of winning than losing. The results are informative for future attempts on formulating more accurate theories.

1 Introduction

Agent-based modelling is an important tool for studying the autonomous actions of individual entities and their interactions [1–3] in complex systems. The interactions often reflect how agents compete, especially in the context of competing games. Examples of extensively studied games are the prisoner’s dilemma (PD), snowdrift game (SG), and stag hunt (SH) game [1,4–6]. These are two-strategy games with agents having a choice of two possible options. In the present work, we focus on the Rock–Paper–Scissors (RPS) game [1,4,7], characterized by three strategies that dominate each other cyclically. Depending on the context, the strategy of an agent can be regarded as his state, character, opinion or species. The strategies are related cyclically through: Rock (R) crushes Scissors (S), Scissors (S) cuts Paper (P), and Paper (P) covers Rock (R) [1,7,8]. Despite its simplicity, many phenomena in nature can be described within the framework of RPS game. A well-known example is related to the mating practice [17] with the loser updating the strategy to be that of the winner, spatial self-organized patterns emerged. Szabó et al. studied the small-world effect on RPS game by replacing a fraction \( r \) of the links in a square lattice by links that connect two randomly selected agents [18]. It was found that two qualitatively different phases result, depending on the value of \( r \). Szolnoki and Szabó studied the RPS game in Kagome, honeycomb, triangular, cubic, and ladder-shape lattices [19]. They found that while the spatial dimension of the lattices affects the transitions between different phases strongly, the clustering coefficient does not.

An interesting question is how network structures [1,7] affect the RPS game. The focus so far has been on static networks, i.e., the links connecting two competing agents are fixed. For RPS game, agents interact in a square lattice [17] with the loser updating the strategy to be that of the winner, spatial self-organized patterns emerged. Szabó et al. studied the small-world effect on RPS game by replacing a fraction \( r \) of the links in a square lattice by links that connect two randomly selected agents [18]. It was found that two qualitatively different phases result, depending on the value of \( r \). Szolnoki and Szabó studied the RPS game in Kagome, honeycomb, triangular, cubic, and ladder-shape lattices [19]. They found that while the spatial dimension of the lattices affects the transitions between different phases strongly, the clustering coefficient does not.

Going beyond static networks, co-evolving networks have attracted much attention in recent years [1,2,7,20,21]. In co-evolving networks, an agent may switch his strategy or alter his competing neighbors so as to attain an environment that is to his advantage. Such adaptive actions
couples the dynamics of strategy selections and network evolution. Co-evolving networks invoking PD, SG, SH games have been studied [1,2,20–23]. In particular, the present work is motivated by the two-option adaptive co-evolving voter model [24] and the dissatisfied adaptive SG [22,25]. In the co-evolving voter model [24], there are two opposite opinions competing for dominance in an initially random regular network and agents prefer to be surrounded by like-opinion neighbors. When an agent interacts with a randomly chosen neighbor of the opposite opinion, he has a probability \( p \) to cut the link to the neighbor and rewire it to a randomly chosen agent of the same opinion. With a probability \( 1 - p \), the agent is convinced by the neighbor and switches to the opposite opinion. Both actions are rational in that the agents tend to pursue local consensus. Despite its simplicity, the phenomena are rich. For values of \( p \) below (above) a critical value, the system evolves into an active (a frozen) phase in which the network evolution and strategy selection continue (cease). A detailed theoretical analysis of the adaptive voter model by invoking the moment-closure approximation was given in reference [26]. Similar adaptive actions (i.e., switching strategies and rewiring the links to dissatisfying neighbors) were included in the model of dissatisfied adaptive snowdrift game (DASG) [22,25]. In DASG, adaptive actions are taken when agents become dissatisfied with non-cooperative neighbors. The resulting network is either in a disconnected, dynamically frozen, and character-segregated phase or a connected, dynamical, and character-mixed phase, depending on a payoff parameter. Analytic approaches to co-evolving networks require careful treatment of spatial correlations [26–28]. Other examples in which similar adaptive actions are invoked include a reversed opinion-formation model [29] and an inverse voter model [30,31]. Networking effects, including co-evolving networks, also pose challenging questions to formulating analytic approaches. Typically approaches such as mean field approximation and pair approximation [1,7] often only give results in qualitative agreement with simulations [1,22,24,25,28,30,31]. The reason is that the adaptive actions are sensitive to the local competing environment and thus spatial correlations are important. We have made various attempts in understanding the key factors in formulating theories that better capture spatial correlations [25,28,31–35]. In particular, an improved mean field theory was shown to give good results for DASG [25] and the inverse voter model [31], and a local configuration approximation was applied to understanding cooperative behavior in a structured population with unoccupied sites [35]. However, it does not mean that the approximations that work for these models will also be good for others, as discussed in detail in reference [26].

Here, we extend the study of adaptive co-evolving models to cyclic multiple-strategy cases. In particular, an adaptive and co-evolving RPS model, abbreviated as ARPS, is proposed and studied in detail. It should be noted that our model is different from the ARPS model studied by Demirel et al. [36]. In our model, the agents use a mechanism in which an agent who has lost in a RPS game would adopt the strategy of the winner or to seek a new neighbor of the same opinion. As a result, the adaptive mechanism is one that agents prefer to have neighbors of the same opinion. Within the context of RPS game, the mechanism is to avoid losing repeatedly to the same neighbor. The authors focused on the time evolution of the fractions of agents using the different strategies. In our model, the agents take adaptive actions to enhance their chance of winning. We focus on the different phases exhibited in the steady state and the formulation of analytic approaches. In Section 2, we define the model and present the key features as revealed by simulations. The model is parameterized by a probability \( p \) of rewiring an unfavorable link. The system evolves to two different phases for different ranges of \( p \). In Section 3, a theory based on the densities of different kinds of links connecting agents of different strategies is constructed. Results are found to be in good agreement with simulations, with small yet noticeable discrepancies. In Section 4, we point out that the small discrepancies are important hints for studying the validity of the assumptions in a theory. We analyze the dependence of the probabilities of winning and losing on different types of agents. These probabilities are found to depend on an agent’s degree as well as his role in an adaptive process. These features are usually not included in analytic approaches. Although the context of ARPS is studied, the discussions on the formalism of mean field theory and its validity are intentionally carried out in a general form. As such, the analysis here can be readily applied to other co-evolving network models with three or more options or strategies. Results are summarized in Section 5.

Consider a system of \( N \) agents. For concreteness, the agents are initially connected via a random regular graph of uniform degree \( \mu \) and each of them is assigned one of the three strategies (R, P, or S) with equal probabilities. In a time step, an agent, referred to as the active agent, is selected randomly. If there is no connected neighbor, i.e., of degree zero, there will be no action and the time step ends. Otherwise, the active agent selects a connected neighbor, referred to as the passive agent, at random. They interact via a RPS game. If the active agent wins or there is a draw, he is satisfied and no adaptive actions take place. If the active agent loses, he is dissatisfied and he will take one of the following adaptive actions: (i) with a probability \( p \) to cut the link to the passive agent and rewire it to another agent (called the rewiring target) randomly chosen from all the agents in the system who are not a neighbor, or (ii) with a probability \( (1 - p) \) to switch his strategy to the one that can defeat the passive agent. Figure 1 illustrates the possible events and adaptive actions in a time step with examples. As one active agent is picked at a time step, the interactions are asynchronous. The probability \( p \) is the only parameter in ARPS. The adaptive actions are rational in that an agent always aims to prevent losing to the same opponent by altering the local competing environment. They drive the co-evolution of strategies employed by the agents and the network connections. The process continues until the network achieves a macroscopically steady state, i.e., when macroscopic quantities become unchanged (albeit fluctuations).
as the system evolves. When $p$ is far from the transition point $p_c$, $10^3$ evolving time steps are observed. For $p$ close to $p_c$, $3 \times 10^3$ evolving time steps are observed.

2 Adaptive Rock–Paper–Scissors model and key features

The long-time behavior of the system can be characterized by a few macroscopic quantities. They include the fractions $f_R$, $f_P$, and $f_S$ of agents using the strategies R, P, and S, respectively, the fractions of undirected inert links $l_{RR}$, $l_{PP}$ and $l_{SS}$ connecting agents using the same strategy that would lead to a draw, and the fractions of undirected active links $l_{RP}$, $l_{PS}$ and $l_{SR}$ connecting agents using different strategies that would lead to a win-lose situation. It should be pointed out that ARPS can be implemented with different initial strategy assignments and initial network connections. Here, we take advantage of the simplicity provided by random initial strategy assignments and the symmetry among the three strategies so that we could focus on the discussion of the $p$-dependence of two link densities, one for inert and the other for active links.

Detailed numerical simulations were carried out for ARPS. Here we focus on an initial network of uniform degree $\mu = 4$ and $N = 10000$ agents. The results illustrated that $f_R = f_P = f_S = 1/3$, $l_{RR} = l_{PP} = l_{SS}$ and $l_{RP} = l_{PS} = l_{SR}$. These are expected as no strategy plays a more special role in RPS and in the adaptive actions. The random initial conditions make sure that all strategies are evenly present. This allows us to focus the discussion on how $l_{RR}(p)$ and $l_{RP}(p)$ behave at long time. Figure 2 shows the simulation results (symbols). Figure 2a confirms $f_R = f_P = f_S = 1/3$ for all values of $p_c$ as expected from symmetry consideration. Figure 2b shows the behavior of $l_{RR}(p)$ (squares) and $l_{RP}(p)$ (circles). These quantities reveal the two different phases classified by $p$. For $0 < p < p_c$, $l_{RR}$ increases monotonically with $p$ and approaches $l_{RR} = 1/3$ at $p = p_c$ continuously while $l_{RP}$ drops monotonically with $p$ and vanishes continuously at $p = p_c$. In the range $p_c < p < 1$, $l_{RR} = 1/3$ and $l_{RP} = 0$. We found that $p_c \approx 0.89$ for $\mu = 4$. We also studied initial networks of different values of $\mu$. The results show the same qualitative behavior, but $p_c$ increases with $\mu$. For example, $p_c \approx 0.94$ for $\mu = 8$.

For $p < p_c$, the system has active links and it is in the active phase. These active links promote agents’ interactions and adaptive actions. This is a dynamic phase as strategy switching and network rewiring persist. For $p > p_c$, the system has only inert links and it is in an inactive and frozen phase. There is no more adaptive action. The two phases also differ drastically in network structure. In the active phase, the system has a main cluster consisting of agents using the three strategies with both active and inert links connecting them. When $p$ is far from $p_c$, almost all the nodes are connected and form a cluster with dispersed active and inert links. At $p \lesssim p_c$, nodes of the same strategy (R, P, or S) form a group with a few active links connecting the three groups. In the frozen phase, the system breaks into three segregated pure-strategy clusters of equal size, with each cluster having agents using only R, P, or S. Another noticeable feature is a jump in $l_{RP}$ from $l_{RP} = 2/9$ at $p = 0$ to a larger value when $p$ increases.
from zero, as shown in Figure 2c. There is a similar discontinuity in \( t_{RR} \) from \( t_{RR} = 1/9 \) at \( p = 0 \) to a smaller value. These discontinuities will be further discussed in Section 4.

### 3 Mean-field approach for three-strategy co-evolving games

Inspired by previous analytic approaches [22,24,25,28,30,31] for co-evolving agent-based models with two strategies, we formulate a theory by tracing the expected changes in the macroscopic quantities in a time step. In principle, the system has many macroscopic variables. At the single-agent level, we have the link density \( f_R, f_P, \) and \( f_S \). At the two-agent or link level, there are the link densities \( l_{RR}, l_{RP}, l_{PS}, l_{RP}, l_{PS} \) and \( l_{SR} \). In general, steady states can be found by iterating the dynamical equations for these variables from given initial conditions to long time. For problems with symmetric properties among the variables, the formalism could be simplified to give explicit analytic results. As discussed, symmetry in ARPS implies \( l_{RR} = l_{PP} = l_{SS} \) and \( l_{RP} = l_{PS} = l_{SR} \). Therefore, we could take \( l_{RR} \) and \( l_{RP} \) as variables. Together with \( f_R = f_P = f_S = 1/3 \), the two variables obey \( l_{RR} + l_{RP} = 1/3 \). This sum rule is also demonstrated by the simulation results in Figure 2b. As a result, a single variable suffices for a theory up to the level of links. We choose \( l_{RP} \) as the variable, although other choices can also be made.

We formulate a theory in a way that can be readily generalized to other co-evolving network problems. We start by writing down an equation for the change \( \Delta l_{RP} \) in \( l_{RP} \) in a time step. Based on the adaptive actions, \( \Delta l_{RP} \) is determined by: (i) the strategy of the active agent, (ii) his local configuration including the degree \( \kappa \) and the numbers of neighbors using the different strategies, (iii) the probability of losing the RPS game, (iv) the adaptive action taken after losing, (v) the change in the number of links \( \Delta L_{RP} \) connecting an agent using strategy-R and an agent using strategy-P (called RP-links) due to the adaptive action. Table 1 gives the possible values of \( \Delta L_{RP} \) due to the adaptive actions. Schematically, the expected change in the link density \( \Delta L_{RP} \) can be expressed in terms of the probabilities of all possible local configurations, strategies and adaptive actions, and the corresponding local changes in the number of RP-links as follows:

\[
\Delta l_{RP} = \sum_{X=R,P,S} f_X \sum_{\kappa} P_X(\kappa) \sum_{\lambda_{XY},\lambda_{XZ}} Q_{X,\kappa}(\lambda_{XY},\lambda_{XZ}) \times \frac{\lambda_{XY}}{\kappa} \left[ p \Delta l_{RP}^{\text{rewire}} + (1-p) \Delta l_{RP}^{\text{switch}} \right].
\]

(1)

Here, \( P_X(\kappa) \) is the probability of an agent using strategy-X and having degree \( \kappa \). \( Y \) (Z) is the strategy which wins over (loses to) X. \( Q_{X,\kappa}(\lambda_{XY},\lambda_{XZ}) \) is the probability of an agent using strategy-X and having degree \( \kappa \) to have \( \lambda_{XY} \) XY-links and \( \lambda_{XZ} \) XZ-links, \( \Delta l_{RP}^{\text{rewire}} \) (\( \Delta l_{RP}^{\text{switch}} \)) is the local change in RP-links due to rewiring (switching) with its possible values listed in Table 1, the term \( \lambda_{XY}/\kappa \) gives the probability of randomly picking a losing link, \( L_{\text{total}} = \mu N/2 \) is the total number of links in the network, and thus the factor of the form \( \Delta L/L_{\text{total}} \) gives the change in the link density.

Equation (1) is general but hard to solve. Formally, the quantities on the right-hand side changes with time as the adaptive actions proceed, and dynamical equations tracing their variations should also be established. Fortunately, simplifications are possible when we focus only on the long time behavior when various quantities become stable in time and the equation can be closed by proper approximations. Equation (1) can be written into three terms, each of them corresponding to the active agent using X = R, P, S, respectively, i.e.,

\[
\Delta l_{RP} = \frac{2}{\mu N} \left( f_R \Delta l_{RP}^{R} + f_P \Delta l_{RP}^{P} + f_S \Delta l_{RP}^{S} \right).
\]

(2)

For given strategy-X and value of \( \kappa \), there is an expected value

\[
\langle \cdots \rangle_{X,\kappa} \equiv \sum_{\lambda_{XY},\lambda_{XZ}} Q_{X,\kappa}(\lambda_{XY},\lambda_{XZ}) \langle \cdots \rangle,
\]

(3)

for the agents using strategy-X and having exactly degree \( \kappa \) to be carried out. The notation \( \langle \cdots \rangle_{X,\kappa} \) stresses two points: (i) the average is taken over possible \( \lambda \)’s and (ii) the result is a function of X and \( \kappa \). Following similar notations, we further define an expected value over possible values of the degrees for agents using strategy-X as:

\[
\sum_{\kappa} P_X(\kappa) \langle \cdots \rangle_{\kappa,X} \equiv \langle \cdots \rangle_{X,\kappa}.
\]

(4)

with the result depending on the strategy-X. The quantities \( \Delta l_{RP}^{R}, \Delta l_{RP}^{P}, \) and \( \Delta l_{RP}^{S} \) in equation (2) can be expressed in terms of these expected values. Using Table 1, they are given explicitly as

\[
\Delta l_{RP}^{R} = -p (f_R + f_S) \left( \frac{\langle \lambda_{RP}\rangle_{X,R,\kappa}}{\kappa} \right)_{X,R} - (1-p) \left( \frac{\langle \lambda_{RP}^2 \rangle_{X,R,\kappa}}{\kappa} \right)_{X,R},
\]

\[
\Delta l_{RP}^{P} = \frac{1}{\mu N} \left( \sum_{X,Y} Q_{X,R}(\lambda_{XY},\lambda_{XZ}) \langle \lambda_{XY} \rangle_{X,R,\kappa} + \sum_{X,Z} Q_{X,R}(\lambda_{XY},\lambda_{XZ}) \langle \lambda_{XZ} \rangle_{X,R,\kappa} \right),
\]

\[
\Delta l_{RP}^{S} = \frac{1}{\mu N} \left( \sum_{X,Y} Q_{X,R}(\lambda_{XY},\lambda_{XZ}) \langle \lambda_{XY} \rangle_{X,R,\kappa} + \sum_{X,Z} Q_{X,R}(\lambda_{XY},\lambda_{XZ}) \langle \lambda_{XZ} \rangle_{X,R,\kappa} \right).
\]
\[
\Delta l^P_{RP} = p f_R \left( \frac{\langle \lambda P S \rangle_{\lambda |\rho, \kappa}}{\kappa} \right)_{\kappa | P} + (1-p) \left( \frac{\langle \lambda P S \rangle_{\lambda |\rho, \kappa}}{\kappa} \right)_{\kappa | P} \right)_{\kappa | P} + (1-p) \left( \frac{\langle \lambda P S \rangle_{\lambda |\rho, \kappa}}{\kappa} \right)_{\kappa | P} \right)
\]

\[
\Delta l^S_{RP} = (1-p) \left( \frac{\langle \lambda S R \rangle_{\lambda |\rho, \kappa}}{\kappa} \right)_{\kappa | S}
\]

where the terms proportional to \( p \) are due to rewiring and those proportional to \((1-p)\) are due to strategy switching.

To proceed, we make approximations to the expected values in order to close the equations. Firstly, the equations can be simplified by the symmetry of the three strategies. As a result, it is sufficient to consider the expected values in regard to only one of the strategies. Without loss of generality, we retain averages over agents using strategy-R. The other expected values for strategies-P and S are given by:

\[
\langle \lambda P S \rangle_{\lambda |\rho, \kappa} = \langle \lambda P R \rangle_{\lambda |\rho, \kappa} \frac{\kappa}{\kappa} \\
\langle \lambda P S \rangle_{\lambda |\rho, \kappa} = \langle \lambda P R \rangle_{\lambda |\rho, \kappa} \frac{\kappa}{\kappa} \\
\langle \lambda P S \rangle_{\lambda |\rho, \kappa} = \langle \lambda P R \rangle_{\lambda |\rho, \kappa} \frac{\kappa}{\kappa} \\
\langle \lambda P S \rangle_{\lambda |\rho, \kappa} = \langle \lambda P R \rangle_{\lambda |\rho, \kappa} \frac{\kappa}{\kappa} \\
\langle \lambda P S \rangle_{\lambda |\rho, \kappa} = \langle \lambda P R \rangle_{\lambda |\rho, \kappa} \frac{\kappa}{\kappa}
\]

Secondly, the expected values can be expressed in terms of the macroscopic quantities (link densities and fractions) that we want to solve. The expected values \( \langle \lambda X Y \rangle_{\lambda |\rho, \kappa} \) and \( \langle \kappa \rangle_{\kappa | X} \) are readily given by

\[
\langle \lambda X Y \rangle_{\lambda |\rho, \kappa} = \frac{p}{2f_X} l_{XY} \\
\langle \kappa \rangle_{\kappa | X} = \frac{p}{2f_X} (l_{XY} + l_{XZ} + 2l_{XY})
\]

The first equality follows from \( l_{XY} = \frac{2}{\mu X} n_X \langle \lambda X Y \rangle_{\lambda |X} \), where \( n_X \) is the number of agents using strategy-X. It says that the total number of XY-links is given by the product of \( n_X \) and the average number of XY-links per agent using strategy-X. The second equality relates the mean degree \( \langle \kappa \rangle_{\kappa | X} \) among agents using strategy-X to the link densities.

For agents using strategy-R of a certain degree \( \kappa \), the first moment \( \langle \lambda R P \rangle_{\lambda |\rho, \kappa} \) and the second moment \( \langle \lambda R P \rangle_{\lambda |\rho, \kappa} \) are related to the expected value and the variance of \( \lambda R P \) respectively, and the mixed moment \( \langle \lambda R P \cdot \lambda S R \rangle_{\lambda |\rho, \kappa} \) is related to the covariance of \( \lambda R P \) and \( \lambda S R \) via [37]

\[
\langle \lambda R P \rangle_{\lambda |\rho, \kappa} = E(\lambda R P) \\
\langle \lambda R P \rangle_{\lambda |\rho, \kappa} = \text{var}(\lambda R P) + \langle \lambda R P \rangle_{\lambda |\rho, \kappa}^2 \\
\langle \lambda R P \cdot \lambda S R \rangle_{\lambda |\rho, \kappa} = \text{cov}(\lambda R P, \lambda S R) + \langle \lambda R P \rangle_{\lambda |\rho, \kappa} \langle \lambda S R \rangle_{\lambda |\rho, \kappa} \\
\]

We invoke a trinomial closure scheme to handle \( E(\lambda R P) \), \( \text{var}(\lambda R P) \) and \( \text{cov}(\lambda R P, \lambda S R) \) and close the equations. It is an extension of the binomial closure scheme in two-strategy models [24,25,28,30]. The essence is to treat averages \( \langle \cdots \rangle_{\lambda |\rho, \kappa} \) that involve the sums of \( \lambda X Y, \lambda X Z \) \( Q_{X,\kappa}(\lambda X Y, \lambda X Z) \) \( \langle \cdots \rangle \) approximately. In physical terms, \( Q_{X,\kappa}(\lambda X Y, \lambda X Z) \) is the probability of having exactly \( \lambda X Y \) XY-links, \( \lambda X Z \) XZ-links and \( \kappa - \lambda X Y - \lambda X Z \) XX-links, giving an agent with \( \kappa \) neighbors using the strategy-X. This echoes the question on the distribution of exactly \( n \) from all agents using strategy-X. It is important to note that this approximation does not distinguish between \( p_i \) and \( p_j \) going XY-links pointing to the neighbors using strategy-Y.

To express all quantities in terms of the link densities, we make the further approximation

\[
\rho_Y | X, \kappa = \frac{l_{XY}}{2l_{XX} + l_{XY} + l_{XZ}}
\]

that the probability \( \rho_Y | X, \kappa \) is given by the fraction of outgoing XY-links pointing to the neighbors using strategy-Y from all agents using strategy-X. It is important to note that this approximation does not distinguish between different degrees \( \kappa \) as \( \rho_Y | X, \kappa \) is independent of \( \kappa \).

Finally, using equations (5), (9) and (10) allows us to express all the quantities in equation (2) in terms of a link density and thus close the equation. The expected value in equation (2) vanish at long time. Setting the resulting equation to zero gives the link density \( l_{RP} \) as a function of the rewiring probability \( p \). The non-trivial solution of \( l_{RP}(p) \) is found to have the closed form of

\[
l_{RP}(p) = \frac{2}{9} \left( 1 - \frac{p}{3(\mu - 1)(1-p)} \right) = \frac{2}{9} \frac{p_c - p}{1 - p}
\]

for \( p < p_c \) with

\[
p_c = \frac{3(\mu - 1)}{3\mu - 2}
\]
and \( l_{RP}(p) = 0 \) for \( p > p_c \). Results for \( l_{RR}(p) \) follow form \( l_{RR}(p) + l_{RP}(p) = 1/3 \). The analytic results in equations (11) and (12) are shown in Figure 2b (lines) for comparison for the case of \( \mu = 4 \). We note that equation (11) is different from the result in reference [36]. Here, we used a trinomial closure scheme here instead of pair-approximation and the strategy update rule, though similar, is different between the two models. It is also worth noting that our value of \( p_c \) is different from that in reference [36] and it is higher. Our theoretical results are in good agreement with the simulation results. The theory captures the two phases and the behavior of the phase transition. There are slight discrepancies near the phase transition. The theory predicts that \( p_c = 0.9 \) for \( \mu = 4 \), which is slightly larger than \( p_c \approx 0.89 \) obtained by numerical simulations. The theory also misses the jump in \( l_{RP} \) as \( p \) becomes finite, as shown in Figure 2c.

The theory also predicts a shift in \( p_c \) that separates the active and frozen phases for different mean degree \( \mu \). Figure 3 shows a phase diagram on the \( \mu - p \) plane obtained by our theory, where the boundary (line) separating the active and frozen phases obtained by equation (12). We also carried out simulations for different values of \( \mu \) and the resulting \( p_c \) are included in Figure 3 for comparison. The theory and simulation results are in good agreement, except for the case of \( \mu = 2 \). This is reasonable as the assumptions behind the closure scheme work better for larger values of \( \mu \).

\section*{4 Who’s winning and who’s losing?}

The discrepancies between analytic and simulation results, though small, reveal important information on the effects of the co-evolving mechanisms and the validity of the assumptions in the mean-field approach, as we now show. In every turn, the active agent may win, lose, or draw. We recorded the probabilities of winning, losing, and drawing of the active agents over many rounds and obtained the averages \( \langle f_{\text{win}} \rangle, \langle f_{\text{draw}} \rangle \) and \( \langle f_{\text{lose}} \rangle \) of these probabilities for the active agents. Due to the cyclic symmetry of the strategies, we could focus on any strategy for an active agent, say R, and express the three probabilities as follows:

\[
\begin{align*}
\langle f_{\text{win}} \rangle &= \left\langle \frac{\Delta S|\kappa}{\kappa} \right\rangle_{\lambda|R,\kappa} |_{\kappa} \neq |_{R} \langle \rho S | R, \kappa \rangle |_{\kappa} |_{R} \\
\langle f_{\text{draw}} \rangle &= \left\langle \frac{\Delta S|\kappa}{\kappa} \right\rangle_{\lambda|R,\kappa} |_{\kappa} \neq |_{R} \langle \rho P | R, \kappa \rangle |_{\kappa} |_{R} \\
\langle f_{\text{lose}} \rangle &= \left\langle \frac{\Delta S|\kappa}{\kappa} \right\rangle_{\lambda|R,\kappa} |_{\kappa} \neq |_{R} \langle \rho P | R, \kappa \rangle |_{\kappa} |_{R} 
\end{align*}
\]

The quantities \( \rho S | R, \kappa \), \( \rho P | R, \kappa \) and \( \rho P | R, \kappa \) were introduced in equation (9). They are the conditional probabilities of encountering a neighbor using the S, R and P strategies, respectively, given that the strategy of the active agent is R and the degree is \( \kappa \). In the present context, they are also the probabilities of winning (\( \langle f_{\text{win}} \rangle_{R,\kappa} \)), drawing (\( \langle f_{\text{draw}} \rangle_{R,\kappa} \)) and losing (\( \langle f_{\text{lose}} \rangle_{R,\kappa} \)) of an active agent of R who has a degree \( \kappa \). Due to the symmetry of the system, \( \langle f_{\text{win}} \rangle_{R,\kappa} = \langle f_{\text{win}} \rangle_{S,\kappa} / \kappa \) and thus they are defined as \( \langle f_{\text{win}} \rangle_{\kappa} \). Similarly, \( \langle f_{\text{draw}} \rangle_{R,\kappa} = \langle f_{\text{draw}} \rangle_{S,\kappa} / \kappa \) and \( \langle f_{\text{lose}} \rangle_{R,\kappa} = \langle f_{\text{lose}} \rangle_{S,\kappa} / \kappa \). Figure 4 shows the simulation results (symbols) of these probabilities as a function of \( p \) for the case of mean degree \( \mu = 4 \). These results are illuminating. At \( p = 0 \), \( \langle f_{\text{win}} \rangle = \langle f_{\text{draw}} \rangle = \langle f_{\text{lose}} \rangle = 1/3 \). A slight deviation from \( p = 0 \) immediately makes them different from 1/3 with a jump. For \( p > 0 \), these quantities also show the existence of two phases. In the active phase, \( \langle f_{\text{win}} \rangle \) and \( \langle f_{\text{lose}} \rangle \) drops monotonically with \( p \) and vanish for \( p > p_c \), while \( \langle f_{\text{draw}} \rangle \) increases monotonically with \( p \) and becomes unity for \( p > p_c \). The latter signifies the segregation into three groups with each group using one strategy and hence a situation that all encounters resulting in a draw.

The most important feature in Figure 4 is \( \langle f_{\text{win}} \rangle > \langle f_{\text{lose}} \rangle \) for active agents in the range \( p < p_c \), i.e., active agents are more likely to win on average in a game of fixed \( p \). In contrast, passive agents are more likely to lose on average. Thus, examining the numerical results of \( \langle f_{\text{win}} \rangle, \langle f_{\text{draw}} \rangle \) and \( \langle f_{\text{lose}} \rangle \) unveils a deficiency in the theory. The theory assumes \( \langle f_{\text{win}} \rangle = \langle f_{\text{lose}} \rangle \) and approximates them by \( \langle \rho S | R, \kappa \rangle |_{\kappa} \neq |_{R} \langle \rho P | R, \kappa \rangle |_{\kappa} |_{R} = 3 |_{\kappa} |_{R} / 2 \) after invoking equation (10). The analytic results of \( \langle f_{\text{win}} \rangle = \langle f_{\text{lose}} \rangle \)
and $\langle f_{\text{draw}} \rangle$ as a function of $p$ are also shown in Figure 4 (lines) for comparison. For a large part of $p$ below $p_c$, the analytic results lie between the actual $\langle f_{\text{win}} \rangle$ and $\langle f_{\text{lose}} \rangle$, except that they are slightly above actual $\langle f_{\text{win}} \rangle$ and $\langle f_{\text{lose}} \rangle$ in a small region of $p \lesssim p_c$. The analytic results are in exact agreement with the simulation results right at $p = 0$, but do not predict the jump in $\langle f_{\text{win}} \rangle$, $\langle f_{\text{draw}} \rangle$ and $\langle f_{\text{lose}} \rangle$ for any deviation from $p = 0$.

These features shed light on the validity of the mean field theory. In the theory, the quantity $\rho_{Y|X,\kappa}$, which is the fraction of links to neighbors using strategy-Y for agents using strategy-X and having $\kappa$ neighbors, is approximated by equation (10) and thus assumed to be independent of $\kappa$. It follows that $f_{\text{win}}, f_{\text{draw}}, \text{and } f_{\text{lose}}$ are also assumed to be $\kappa$-independent. At $p = 0$, there is no rewiring. The network is static and every agent indeed has the same number $\mu$ of neighbors. The fact that the theory gives the correct value at $p = 0$ but not for $p \neq 0$ implies that the spread in the values of $\kappa$ among the agents induced by the rewiring mechanism becomes important. We therefore recorded the winning and losing probabilities of the active agents according to their degree $\kappa$ in simulations. Figures 5a and 5b show the results of $f_{\text{win}}, f_{\text{draw}}, \text{and } f_{\text{lose}}$ as a function of $\kappa$ at a fixed $p = 0.3$ obtained in two different systems of $\mu = 4$ and $\mu = 8$. Similarly, we also recorded the winning and losing probabilities $w_{\text{win}}, w_{\text{draw}}, \text{and } w_{\text{lose}}$ of passive agents according to their degree $\kappa$ and showed the results. Immediately, we note that $f_{\text{win}}, f_{\text{draw}}, \text{and } f_{\text{lose}}$ do depend on $\kappa$, and so do $w_{\text{win}}, w_{\text{draw}}, \text{and } w_{\text{lose}}$. This dependence on $\kappa$, which enters through rewiring for any $p \neq 0$, causes the mean field theory to miss the jump in the probabilities as $p$ starts to take on finite values (see Figs. 2c and 4).

Closer inspection of Figures 5a and 5b further reveals the different fates of agents with $\kappa < \mu$ and $\kappa > \mu$. In particular, $f_{\text{win}}, f_{\text{draw}}, \text{and } f_{\text{lose}}$ do not depend on $\kappa$ for $\kappa < \mu$; and $f_{\text{win}}, f_{\text{draw}}, \text{and } f_{\text{lose}}$ do for $\kappa > \mu$. Although the results in Figure 5 were obtained for $p = 0.3$, we examined the range of $0 < p < p_c$ and found the same features. The results imply that an agent with a degree smaller (higher) than the mean degree $\mu$ is more likely to win than to lose (more likely to lose than to win), regardless of whether being an active or passive agent, with a crossover at $\kappa \approx \mu$. In addition, the results show that $f_{\text{win}}, f_{\text{draw}}, \text{and } f_{\text{lose}}$ do not depend on $\kappa$ for all values of $\kappa$, implying that being an active agent is more likely to win than passive agent when the degree is the same.

A qualitative understanding of the $\kappa$-dependence of the winning and losing probabilities follows from the adaptive actions taken by active agents. An agent takes actions to make his neighborhood better, i.e., to enhance his chance of winning. Switching strategy helps an active agent to win over the same opponent if they meet again (provided that their strategies are not further altered before they meet again). Rewiring dissatisfying link lowers the losing probability of an active agent when he becomes involved in a RPS game later. Generally, the neighborhood of an active (a passive) agent gets better (gets worse) as a result of adaptive actions. The probability of an agent being chosen to be an active agent in a time step is $1/N$. However, the probability of an agent being a passive agent depends on his degree $\kappa$. Ignoring spatial correlation in the network for simplicity, the probability of being a passive agent is $\kappa/\mu N$, as given by the ratio of an agent’s out-links to the total number of out-links in the network. Notice that the ratio $\kappa/\mu$ emerges. For agents with $\kappa < \mu$, they are more likely to be active agents and thus have a better chance to shape the neighborhood to his advantage. The more favorable neighborhood gives them a larger winning probability than losing, no matter which role they play in a round. Therefore, the co-evolving mechanism leads to $f_{\text{win}} > f_{\text{lose}}$, $w_{\text{win}} > w_{\text{lose}}$, and $\kappa < \mu$, as shown in Figure 5. Following a similar argument, agents with $\kappa > \mu$ are more likely to be passive agents. On one hand, they have a lower chance to make their neighborhood better. On the other hand, their neighbors’ adaptive actions make the neighborhood worse. These agents will have a higher losing probability than winning. Therefore, the co-evolving mechanism leads to $f_{\text{win}} < f_{\text{lose}}$, $w_{\text{win}} < w_{\text{lose}}$, and $\kappa > \mu$, as observed in Figure 5.

The analysis on the winning and losing probabilities in Figures 4 and 5 brings out the inadequacy of the mean field theory in capturing the spatial correlation between neighboring agents’ strategies and degrees after co-evolution takes place. An improved theory would have to account for the $\kappa$-dependence of a competing neighborhood. Such a theory necessarily invokes many more variables, e.g. those describing the neighborhood of an agent of degree $\kappa$ for different values of $\kappa$. In light of these complications, the present mean field theory has the merit of being simple and yet providing reasonable results. The analysis also provides a physical picture on the effects of

![Fig. 5. Simulation results of $f_{\text{win}}, f_{\text{lose}}, g_{\text{win}}, g_{\text{lose}}$ for agents of different degrees $\kappa$ at $p = 0.3$ for systems with (a) mean degree $\mu = 4$ and (b) $\mu = 8$, respectively. The data are obtained by averaging results of 300 independent runs in a network of $N = 10000$. The lines joining the data points serve as a guide to the eyes.](image-url)
the co-evolving mechanism. Agents with many neighbors (high $\kappa$) are those often lose to their neighbors and so the relationships are retained by their winning neighbors, while agents with fewer neighbors can protect themselves from losing and strive for higher chance of winning in the next RPS game. Although each agent has the same chance to be an active agent, an agent with a higher degree and thus a competing environment to his disadvantage is more likely to be chosen as the passive agent in a round of the game. This leads to the differentiation in the winning and losing probabilities according to the degrees and to an agent’s role as observed numerically in Figure 5.

5 Conclusion

We have proposed and studied ARPS in detail, with a focus on issues related to formulating a mean field theory for co-evolving network problems with multiple strategies. In ARPS, three cyclically dominating strategies are involved in a co-evolving network. An agent with a dissatisfied neighbor takes action to improve his competing neighborhood by rewiring the dissatisfying link with a probability $p$ or switching to a strategy that could defeat the neighbor with a probability $(1 - p)$. The network shows two different phases: an active phase for $p < p_c$ and a frozen phase for $p > p_c$. The active phase is characterized by one connected network with agents using different strategies continually interacting and taking adaptive actions. The frozen phase is characterized by three separate clusters of agents using R, P, and S, respectively and terminated adaptive actions. We have discussed in detail the formulation of a mean-field theory that starts with tracing the changes in a link density due to all possible adaptive actions as the network evolves. A trinomial closure scheme, which approximates the distribution of different types of lines that an agent carries given his strategy and degree, has been invoked to close the equation. Ignoring the dependence on the degree, the theory gives an analytic expression for the link density as a function of $p$. The results agree with simulation results reasonably well and capture the two-phase structure. The results indicate that the trinomial closure scheme is a suitable approximation for the present model. However, the diversity in the details of various adaptive models leads to different correlations in the system variables, despite the similarity in the general dynamical equations. Therefore, studying how to formulate a proper decoupling scheme to retain the necessary correlations that would give reliable results in adaptive systems is an important issue. Here, we suggested a method to study the phase transitions in adaptive and evolving games with three strategies theoretically. The approach can be extended to other problems in co-evolving games in structured populations and in other interacting systems in statistical physics.

Closer examination of the small deviations between analytic and simulation results turns out to be illuminating. We have studied the averaged probabilities of winning ($\langle f_{\text{win}} \rangle$), drawing ($\langle f_{\text{draw}} \rangle$) and losing ($\langle f_{\text{lose}} \rangle$) for active agents. It was found that $\langle f_{\text{win}} \rangle$ is always higher than $\langle f_{\text{lose}} \rangle$ in the active phase—a feature that the mean field theory does not capture. The origin has been traced to the spread in the degrees among agents due to rewiring and the dependence of the winning and losing probabilities on the degree of agents. Agents with a degree smaller (larger) than the mean degree $\mu$ have a larger (smaller) probability of winning than losing. Physically, agents of higher degrees are those being put in an adverse environment by their neighbors and they are more likely to be chosen as the passive agent in a round of the game. The results thus inform us of the necessity of including correlations between the nearest neighbors’ strategies and degrees for formulating a more accurate theory.

We close with a discussion on a few possible extensions. In the present work, we simplified the discussion by using the symmetry that comes from the cyclically dominating strategies as well as the random initial strategy assignments. It will be of interest to study the sensitivity of the steady state to different initial strategy distributions. The theory presented here can also be modified to study the problem. Here, we discussed the analytic approach not only for applying the results to ARPS, but also in a general way that could be readily modified to other co-evolving network models involving multiple strategies. These models need not be cyclically dominating and the number of strategies could be more than three. The detailed study on the reasons of the small deviation between analytic and simulation results provides useful information on how better theories can be formulated. The analytic results also provides a guide for further studies on exploring possible scaling behavior near the transition between the two phases.

Author contribution statement

C.W.C. constructed and studied the model by numerical simulations and mean-field theory, and carried out the analysis as reported in Section 4. C.X. and P.M.H. supervised the research while C.W.C. was carrying this work as part of his PhD research. C.W.C. designed the figures. C.X. and P.M.H. finalized the manuscript based on a draft prepared by C.W.C.

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