Provably Correct Controller Synthesis of Switched Stochastic Systems with Metric Temporal Logic Specifications: A Case Study on Power Systems

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Abstract—In this paper, we present a provably correct controller synthesis approach for switched stochastic control systems with metric temporal logic (MTL) specifications with provable probabilistic guarantees. We first present the stochastic control bisimulation function for switched stochastic control systems, which bounds the trajectory divergence between the switched stochastic control system and its nominal deterministic control system in a probabilistic fashion. We then develop a method to synthesize the optimal control input signals for the nominal deterministic system with calculated robustness margins, and the same input signals can be applied to the switched stochastic control system with a lower bound guarantee for satisfying the MTL specifications. We implement our robust stochastic controller synthesis approach on both a four-bus power system and a nine-bus power system under generation loss disturbances, with MTL specifications expressing requirements for the grid frequency deviations, wind turbine generator rotor speed variations and the power flow constraints at different power lines.

I. INTRODUCTION

A switched stochastic system [1], [2] consists of a set of stochastic dynamic modes and switchings between the modes triggered by external events. Many cyber-physical systems (e.g., power systems) can be modeled as switched stochastic systems and the control synthesis of such systems with formal specifications has been a challenging problem.

In this paper, we present a provably correct controller synthesis approach for switched stochastic control systems with metric temporal logic (MTL) specifications. MTL has been used as specifications in power systems [3], artificial intelligence [4], robotics [5], biology [6], etc. We first present the stochastic control bisimulation function for switched stochastic control systems, which bounds the trajectory divergence between the switched stochastic control system and its nominal deterministic control system in a probabilistic fashion. Then, we synthesize the the optimal input signals for the nominal deterministic system with calculated robustness margins, and the same input signals can be applied to the switched stochastic control system with a lower bound guarantee for satisfying the MTL specifications.

In [7], we presented a coordinated control method of wind turbine generator and energy storage system for frequency regulation under temporal logic specifications. In this paper, we extend the results in [7] to switched stochastic control systems, and generalize the predicates of the MTL specifications to include both the state and the input (e.g., so that online power constraints in power systems can be incorporated into the MTL specifications). Besides, we add an exponential term to the stochastic control bisimulation function so that both stable and unstable linear dynamics can be approached with less conservativeness.

We implement the proposed controller synthesis approach in two scenarios in power systems. The results show that the synthesized control inputs can indeed lead to satisfaction of the MTL specifications with larger empirical probabilities than the derived theoretical lower bounds for the satisfaction probability.

II. RELATED WORKS

There is a rich literature on controller synthesis with temporal logic specifications in the stochastic environment [8], [9]. For discrete-time temporal logics such as co-safe linear temporal logics (LTL), the specifications can be converted to finite state machines, then the optimal control strategy is computed in the state space augmented with the states of the constructed finite state machines [10]. For dense-time temporal logics such as MTL or signal temporal logics (STL), the specifications can be converted to timed automata [11], [12] and the optimal control strategy is computed in the state space augmented with the states of the constructed timed automata. In [13], the authors proposed a verification approach of switched stochastic systems with LTL specifications. However, as far as we know, there has been no work on controller synthesis of switched stochastic systems with (dense-time) temporal logic specifications.

III. SWITCHED STOCHASTIC CONTROL SYSTEMS

Definition 1 (Switched Stochastic Control Systems). A switched stochastic control system is a 6-tuple $T = (Q, \mathcal{X}, X_0, \mathcal{V}, F, E)$ where

- $Q = \{1, 2, \ldots, M\} \text{ is the set of indices for the modes (or subsystems);}$
- $\mathcal{X}$ is the domain of the continuous state, $x \in \mathcal{X}$ is the continuous state of the system, $X_0 \subset \mathcal{X}$ is the initial set of states;
- $\mathcal{V}$ is the domain of the input, $u \in \mathcal{V}$ is the input of the system;
- $F = \{(f_q, g_q)|q \in Q\}$ where $f_q$ describes the continuous time-invariant dynamics for the mode $dx = f_q(x, u)dt + g_q(x, u)dw$, which admits a unique solution $\xi_q(t; x_0^q, u)$, where $\xi_q$ satisfies $d\xi_q(t; x_0^q, u) = f_q(\xi_q(t; x_0^q, u), u)dt + g_q(\xi_q(t; x_0^q, u), u)dw$, and $\xi_q(0; x_0^q, u) = x_0^q$ is an initial condition in mode $q$.
• \( E \) is a subset of \( Q \times Q \) which contains the valid transitions. If a transition \( e = (q, q') \in E \) takes place, the system switches from mode \( q \) to \( q' \).

Similarly, we can define the switched nominal control system \( T^* = (Q, \mathcal{X}, X_0, \nu, F^*, \mathcal{E}) \) of \( T \), and \( T^* \) only differs from \( T \) as \( F^* = \{ f_q | q \in Q \} \), where \( dx^* = f_q(x^*, u)dt \) is the nominal deterministic version of \( dx = f_q(x, u)dt + g_q(x, u)dw \).

**Definition 2** (Trajectory). A trajectory of a stochastic switched control system \( T \) is denoted as a sequence \( \rho = \{(q^t, \xi_{q^t}(t); x_{0q}, u), (T^t)\}_{t=0}^{N_q} \) \((N_q \in \mathbb{N})\), where

- \( \forall i \geq 0, q^i \in Q, x_{0q} \in \mathcal{X} \) is the initial state at mode \( q^i \), \( x_0 = x_{0q} \in \mathcal{X}_0 \) is the initial state of the entire trajectory, \( x^{i+1} = \xi_{q^i}(T^i; x_{0q}, u) \) is the initial state at mode \( q^{i+1} \);
- \( \forall i \geq 0, T^i > 0 \) is the dwell time at mode \( q^i \), while the transition times are \( T^0, T^0 + T^1, \ldots, T^0 + T^1 + \ldots + T^{N_q-1} \);
- \( \forall i \geq 0, (q^i, q^{i+1}) \in E \).

A trajectory of a switched nominal control system \( T^* \) can similarly be denoted as a sequence \( \rho^* = \{(q^t, \xi_{q^t}(t); x_{0q}, u), (T^t)\}_{t=0}^{N_q} \) \((N_q \in \mathbb{N})\), where

\[
\begin{aligned}
\xi_{q^t}(t; x_{0q}, u), & \quad \text{if } t < T^0, \\
\xi_{q^t}(t - \sum_{k=0}^{i-1} T^k; x_{0q}), & \quad \text{if } \sum_{k=0}^{i-1} T^k \leq t < \sum_{k=0}^{i} T^k, 1 \leq i \leq N_q.
\end{aligned}
\]

The output trajectory of a trajectory \( \rho^* = \{(q^t, \xi_{q^t}(t); x_{0q}, u), (T^t)\}_{t=0}^{N_q} \) of a switched nominal control system is denoted as \( s_{\rho^*}(:,:, x_{0q}, u) \).

**IV. STOCHASTIC CONTROL BISIMULATION FUNCTION**

**A. Stochastic Control Bisimulation Function**

We consider the switched stochastic control system with the following dynamics in the mode \( q^i \):

\[
dx = f_q(x, u)dt + g_q(x, u)dw, \quad (1)
\]

where the state \( x \in \mathcal{X} \subset \mathbb{R}^n \), the input \( u \in \mathcal{U} \subset \mathbb{R}^p \), \( w \) is an \( \mathbb{R}^m \)-valued standard Brownian motion.

Note that the dynamics is essentially the same as that in [14] when the input signal \( u(\cdot) \) is given and bounded, while the existence and uniqueness of the solution of (1) can be guaranteed with the conditions given in [14].

We also consider the switched nominal control system in the mode \( q \) as the nominal deterministic version:

\[
dx^* = f_q(x^*, u)dt, \quad (2)
\]

**Definition 4.** A continuously differentiable function \( \psi_q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0} \) is a time-varying control autobisimulation function of the switched nominal control system (2) in the mode \( q \) if for any \( x, \bar{x} \in \mathcal{X} \) \((x \neq \bar{x})\) and any \( t \in T \) there exists a function \( \psi_q(x, \bar{x}, t) = 0 \) such that

\[
\psi_q(x, \bar{x}, t) > 0, \quad \psi_q(x, x, t) = 0 \quad \text{and} \quad \frac{d\psi_q(x, \bar{x}, t)}{dt} \leq 0.
\]

In the following, we extend the concept of control autobisimulation function to the stochastic setting.

**Definition 5.** A twice differentiable function \( \phi_q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0} \) is a stochastic control bisimulation function between (1) and its nominal system (2) if it satisfies [7]

\[
\phi_q(x, \bar{x}, t) > 0, \forall x, \bar{x} \in \mathcal{X}, x \neq \bar{x}, \forall t \in T,
\]

and there exist \( \alpha_q > 0 \) and a function \( u : \mathbb{R}^n \times T \to \mathbb{R}^p \) such that

\[
\frac{d\phi_q(x, \bar{x}, t)}{dt} + \frac{\partial\phi_q(x, \bar{x}, t)}{\partial x} f_q(x, u(t)) + \frac{\partial\phi_q(x, \bar{x}, t)}{\partial x} f_q(\bar{x}, u(\bar{x}, t)) + \frac{1}{2} g_q(x, u(t)) \frac{\partial^2\phi_q(x, \bar{x}, t)}{\partial x^2} g_q(x, u(t)) < \alpha_q,
\]

for any \( x, \bar{x} \in \mathcal{X} \).

The stochastic control bisimulation function establishes a bound between the trajectories of system (1) and its nominal system (2).

**B. Stochastic Control Bisimulation Function for Switched Linear Dynamics**

In this subsection, we consider the switched stochastic control system with the following linear dynamics in the mode \( q \):

\[
dx = (A_q x + B_q u)dt + \Sigma dw, \quad (5)
\]

where \( A_q \in \mathbb{R}^{n \times n}, B_q \in \mathbb{R}^{n \times p}, \Sigma \in \mathbb{R}^{n \times m} \).

We can construct a stochastic control bisimulation function of the form

\[
\phi_q(x, \bar{x}, t) = (x - \bar{x})^T M_q (x - \bar{x}) e^{A_q t},
\]

where \( M_q \) is a symmetric positive definite matrix. In order for this function to qualify as a stochastic control bisimulation function (see Definition 5), we need to have \( M_q > 0 \), and

\[
\frac{d\phi_q(x, \bar{x}, t)}{dt} (A_q x + B_q u) + \frac{\partial^2\phi_q(x, \bar{x}, t)}{\partial x^2} (A_q x + B_q u) + \frac{\partial\phi_q(x, \bar{x}, t)}{\partial x} + \frac{\partial^2\phi_q(x, \bar{x}, t)}{\partial x^2} (x - \bar{x})^T (2 M_q A_q + \mu_q M_q) (x - \bar{x}) + \frac{\partial\phi_q(x, \bar{x}, t)}{\partial x} (x - \bar{x})^T M_q (x - \bar{x}) + \partial\phi_q(x, \bar{x}, t) (x - \bar{x})^T M_q (x - \bar{x}) \leq \alpha_q,
\]

for some \( \alpha_q > 0 \). If we pick \( \alpha_q = \text{trace}(\Sigma_q^T M_q \Sigma_q) \), the above inequality becomes a linear matrix inequality (LMI)

\[
A_q^T M_q + M_q A_q + \mu_q M_q < 0. \quad (6)
\]
We denote the system trajectory starting from \( x_0 \) with the input signal \( u(\cdot) \) as \( \xi(\cdot; x_0, u) \). We can verify that \( \{z_{k,\nu}(i)\}_{i=0}^{N_q} = \{\xi_1(t; x_0^q, u), \xi_2(t; x_0^q, u)\} \) is a trajectory of the nominal system.

\[
dx^* = (A_q x^* + B_q u)dt.
\]  

(7)

**Proposition 1.** Given the dynamics of (7), \( \psi_q(x, \tilde{x}, t) = (x - \tilde{x})^T M_q (x - \tilde{x}) \) is an autostibilization function if the matrix \( M_q \) satisfies the following:

\[
M_q > 0, \quad A_q^T M_q + M_q A_q + \mu_q M_q \leq 0.
\]  

(8)

We denote the output trajectory of the nominal system starting from \( x_0 \) with the input signal \( u(\cdot) \) as \( s_{p^*}(\cdot; x_0, u) \). We can verify that \( \{z_{k,\nu}(i)\}_{i=0}^{N_q} = \{\xi_1(t; x_0^q, u), \xi_2(t; x_0^q, u)\} \) is a trajectory of the nominal system, \( \varphi_{\tilde{z}} \) is the \( \delta_{k,\nu} \)-robust modified formula of \( \varphi \), \( \tilde{z}_{k,\nu} = (\sqrt{T_q^2 + \gamma^2})/z_{k,\nu}^* \), and the \( \delta_{k,\nu} \)-robust modified formula of \( \varphi \), \( \tilde{z}_{k,\nu} \), \( \delta_{k,\nu} \) satisfies MTL specification \( \varphi \) with probability at least \( 1 - \epsilon \), i.e. \( P\{[[\varphi]](s_{p^*}(; \tilde{x}_0, u), 0) > 0\} > 1 - \epsilon \).

From Theorem 1, if we can design the input signal \( u(\cdot) \) such that the nominal trajectory \( s_{p^*}(\cdot; x_0, u) \) of the nominal system (2) satisfies the \( \delta_{k,\nu} \)-robust modified formula of \( \varphi \) (here \( \tilde{z}_{k,\nu} = (\sqrt{T_q^2 + \gamma^2})/z_{k,\nu}^* \) for each mode \( q \)), and \( \delta_{k,\nu} \) is the \( \delta_{k,\nu} \)-robust modified formula of \( \varphi \) (the predicates in \( \varphi_{\tilde{z}} \) are denoted as \( \tilde{p}^* \)).

The optimization problem to find the optimal input signal such that the nominal trajectory satisfies the \( \delta_{k,\nu} \)-robust modified formula \( \varphi_{\tilde{z}} \) is formulated as follows:

\[
\arg \min_{u(\cdot)} J(u(\cdot)) \text{ subject to } [[[\varphi_{\tilde{z}}]](s_{p^*}(; x_0^q, u), 0) \geq 0].
\]  

(14)

subject to \([[[\varphi]](s_{p^*}(; x_0^q, u), 0) \geq 0] \).

The performance measure \( J(u(\cdot)) \) can be set as the control effort \( ||u(\cdot)||^2 \) (or \( ||u(\cdot)||^2 \)). For linear systems, the above optimization problem can be converted to a mixed-integer linear (or quadratic) programming problem, which can be more efficiently solved using techniques such as McCormick’s relaxation [16], [17]. Furthermore, if the MTL formula \( \varphi_{\tilde{z}} \) consists of only conjunctions (\( \wedge \)) and the always operator (\( \square \)), the integers in the optimization problem can be eliminated [18] and the problem becomes a linear (or quadratic) programming problem.

**VI. CASE STUDY ON POWER SYSTEMS**

In this section, we implement the proposed controller synthesis approach in two scenarios in power systems via MATLAB simulations.

**A. Scenario I**

In this subsection, we implement the controller synthesis method for regulating the grid frequency of a four-bus system with a 600 MW thermal plant \( G_1 \), made up of four identical units, a wind farm \( G_2 \) consisting of 200 identical 1.5 MW Type-C wind turbine generators (WTG) and an energy storage system (ESS), as shown in Fig. 1. The configuration
parameters of each Type-C WTG can be found in Appendix B of [19] (we set $C_{opt} = 16.1985 \times 10^{-9} \frac{\text{s}^3}{\text{rad}^3}$).

![Diagram](image)

Fig. 1. The four-bus system [20] with a thermal plant, a wind farm and an energy storage system (ESS).

By linearizing the system of differential-algebraic equations at the equilibrium point, we have

$$d \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} dt + \begin{bmatrix} M_s \\ N_s \end{bmatrix} w dt + \begin{bmatrix} \Sigma_{s1} \\ \Sigma_{s2} \end{bmatrix} dw,$$

$$\Delta P_{gen} = \begin{bmatrix} E_s & F_s \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix},$$

where $w$ is standard Brownian motion representing the stochasticity of the wind, $u^w$ is a control input through which the wind turbine generator can adjust its power output, $\Delta x = [\Delta E'_{Dd}, \Delta E'_{Dq}, \Delta \omega, \Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4]^T$, $\Delta y = [\Delta P_{gen}, \Delta Q_{gen}, \Delta V_{Dr}, \Delta V_{Dq}, \Delta I_{dr}, \Delta I_{dqr}, \Delta I_{ds}, \Delta I_{dqs}, \Delta V_{Dr}, \Delta \theta_{Dd}, \Delta E_{dD}, \Delta E_{qD}, \Delta \omega_r]$ are the $d, q$ axis voltage variation and rotor speed variation of the WTG, respectively, $\Delta x_1$ to $\Delta x_4$ are variations of proportion-integral (PI) regulator induced states, $\Delta P_{gen}$ and $\Delta Q_{gen}$ are the active and reactive power variation from each WTG, $\Delta V_{Dr}, \Delta V_{Dq}, \Delta I_{dr}, \Delta I_{dqr}$ are the rotor $d, q$ axis voltage and current variation, respectively, $\Delta I_{ds}, \Delta I_{dqs}$ are the stator $d, q$ axis current variation, respectively, and $A_s, B_s, C_s, D_s, E_s,$ and $F_s$ are matrices from the linearization at the equilibrium point.

Through the Kron reduction, we have

$$d\Delta x = A_{kr} \Delta x dt + B_{kr} u^w dt + \Sigma_{kr} dw,$$

$$\Delta P_{gen} = C_{kr} \Delta x + D_{kr} u^w + E_{kr} dw dt,$$

where

$$A_{kr} = A_s - B_s D_s^{-1} C_s,$$

$$B_{kr} = B_s - B_s D_s^{-1} N_s,$$

$$C_{kr} = E_s - F_s D_s^{-1} C_s,$$

$$D_{kr} = -F_s D_s^{-1} N_s,$$

$$\Sigma_{kr} = \Sigma_{s1} - B_s D_s^{-1} \Sigma_{s2},$$

$$E_{kr} = -F_s D_s^{-1} \Sigma_{s2}.$$

We consider a disturbance of generation loss of 150 MW (loss of one unit), denoted as $\Delta P_d = 0.15$, that occurs at time 0. From 0 second to 5 seconds after the disturbance, the system frequency response model of the four-bus system is as follows (we choose base MVA as 1000MVA):

$$\begin{align*}
\Delta \dot{\omega} &= \frac{\Delta \omega}{2\pi} = \frac{\Delta P_m + u^a + \Delta P_s - \Delta P_d + 200 \Delta P_{gen}}{1000} - \frac{D_{kr}}{\omega_r} \Delta \omega, \\
\Delta \dot{P}_m &= 0; \\
\Delta \dot{P}_v &= \frac{1}{\tau_v} (\Delta P_v - \Delta P_m), \\
\Delta \dot{P}_v &= \frac{1}{\tau_g} (-\Delta P_v - \frac{1}{2\pi R} \Delta \omega),
\end{align*}$$

where $u^a$ is a control input representing the power injection from the energy storage system (ESS), $\Delta \omega$ is the grid frequency deviation, $\Delta P_m$ is the governor mechanical power variation, $\Delta P_s$ is the governor valve position variation, $\Delta P_d$ is the variation of the generator power re-dispatch, and $\Delta P_{gen}$ denotes a large disturbance. $\Delta P_{gen}$ times 200 as there are 200 WTGs, and it is divided by 1000 as the base MVA for each WTG and the power system is 1 MVA and 1000 MVA, respectively. We set $\omega_s = 2\pi \times 60$ rad/s, $D = 1$, $H = 4$s, $\tau_v = 0.3$s, $\tau_g = 0.1$s, $R = 0.05$.

From 5 seconds to 87.5 seconds after the disturbance, the generator power re-dispatch ($\Delta P_v$) starts with a ramping rate of 0.04 with the following system frequency response model.

$$\begin{align*}
\Delta \dot{\omega} &= \frac{\omega_s}{2\pi} = \frac{\Delta P_m + u^a + \Delta P_s - \Delta P_d + 200 \Delta P_{gen}}{1000} - \frac{D_{kr}}{\omega_r} \Delta \omega, \\
\Delta \dot{P}_m &= 0.04; \\
\Delta \dot{P}_m &= \frac{1}{\tau_v} (\Delta P_v - \Delta P_m), \\
\Delta \dot{P}_v &= \frac{1}{\tau_g} (-\Delta P_v - \frac{1}{2\pi R} \Delta \omega),
\end{align*}$$

At 87.5 seconds, the generation and load are balanced again. So from 87.5 seconds to 10 seconds after the disturbance, the system frequency response model is the same as that in (17).

With (16), (17) and (18), we have the following switched stochastic control system with two modes corresponding to (17) and (18) respectively:

$$d\dot{x} = (A_q \dot{x} + B_q u) dt + \dot{\Sigma}_q dw,$$

where $\dot{x} = [\Delta E'_{Dd}, \Delta E'_{Dq}, \Delta \omega_r, \Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4, \Delta \omega, \Delta P_s, \Delta P_m, \Delta P_v]^T$, the input $u = [u^w, u^a]^T$. As the matrix $A_q$ is computed as Hurwitz for both modes, the system in each mode is stable.

We use the following MTL specification for frequency regulation after the disturbance:

$$\varphi = [\square_{[0,T_{end}]} p_1 \land \square_{[2,T_{end}]} p_2],$$

$$p_1 = (-0.5Hz \leq \Delta f \leq 0.5Hz) \land (-10Hz \leq \Delta f_r \leq 10Hz),$$

$$p_2 = (-0.4Hz \leq \Delta f \leq 0.4Hz),$$

where $\Delta f = \frac{\Delta \omega}{2\pi}$, $\Delta f_r = \frac{\Delta \omega}{2\pi}$. The specification means “After a disturbance, the grid frequency deviation should never exceed ±0.5Hz, the WTG rotor speed deviation should never exceed ±10Hz, after 2 seconds the grid frequency deviation should always be within ±0.4Hz”.

We set $k_w = 1$, $T_{end} = 5$s, $\epsilon = \alpha T_{end}/\gamma = 5\%$, so $\alpha = 0.05\gamma/T_{end} = 0.01\gamma$. As $\alpha = \text{trace}(\Sigma^T M \Sigma) = k^2_w M(3,3)$, we have $\gamma = 100k^2_w M(3,3) = 100M(3,3).$ We assume that the initial state variations can be covered by $B_{q0}(\hat{x}^0_0, r)$, where $r = 4\gamma$ ($4 = 2^2$ is chosen as the initial state variations due to the time needed for running the algorithm to generate the controller, which is about twice the simulation time), $\hat{x}^0_0$ is zero in every dimension. It can be seen from (20) that the allowable variation range of the
grid frequency variation \( \Delta \omega \) is much smaller than that of the wind turbine rotor speed variation \( \Delta \omega_r \). Therefore, in order to decrease the conservativeness of the probabilistic bound as much as possible, we further optimize both \( z_{k,i} \) and the matrix \( M_{qi} \) such that the outer bounds of the stochastic robust neighbourhoods in the dimension of the grid frequency variation \( \delta_{1,1}^* (\delta_{1,2}^* = \delta_{2,1}^* = \delta_{2,2}^*) \) are much smaller than the outer bounds in the dimension of the wind turbine rotor speed variation \( \delta_{1,1}^* (\delta_{1,3}^* = \delta_{3,1}^*). \) As \( \delta_{k,i} = (\sqrt{q} + \sqrt{r})/z_{k,i} \) and \( \hat{\gamma} = 100M_{qi}(3,3) \), minimizing \( \delta_{1,1} \) can be achieved by minimizing \( M_{qi}(3,3) (i = 1,2,3) \) and maximizing \( z_{1,1} \). Therefore, to obtain both \( M^* \) and \( z_{1,1}^* \), we solve the following semidefinite programming (SDP) problem as follows.

\[
\begin{align*}
\text{min.} & \quad -(z_{1,1}^*)^2 \\
\text{s.t.} & \quad M_{qi} > 0, A_{qi}^T M_{qi} + M_{qi} \hat{A}_{qi} + M_{qi} \mu_{qi} M_{qi} \preceq 0, \tag{21} \\
& \quad c_{1}^T M_{qi} \hat{c}_1 \preceq \zeta_i, M_{qi} - (z_{1,1}^*)^2 a_{11} q_{1,1}^T \preceq 0,
\end{align*}
\]

where \( \hat{c}_1 = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0]^T \), \( \mu_{qi} = 0.1 \), \( \zeta_i \) is tuned manually to be as small as possible while the optimization problem is feasible.

With the \( M_{qi}^* \) obtained from (21), we compute the tightest outer bound in the dimension of \( \Delta \omega_r \) as follows:

\[
\begin{align*}
\text{min.} & \quad -(z_{1,3}^*)^2 \\
\text{s.t.} & \quad M_{qi}^* - (z_{1,3}^*)^2 a_{11} q_{1,3}^T \geq 0, \tag{22}
\end{align*}
\]

From (21) and (22), we obtain the \( \delta_{k,i}^* \)-robust modified formula as follows.

\[
\begin{align*}
\varphi_{\delta^*} & = \square_{[0,T_{mal}]} \hat{p}_1^* \land \square_{[2,T_{mal}]} \hat{p}_2^*, \\
\hat{p}_1 & = (-0.5Hz + 0.217e^{-0.01Hz} \leq \Delta f) \land \leq 0.5Hz - 0.217e^{-0.01Hz}) \land \leq 10Hz - 6.08e^{-0.01Hz}), \\
\hat{p}_2 & = (-0.4Hz + 0.217e^{-0.01Hz} \leq \Delta f) \land \leq 0.4Hz - 0.217e^{-0.01Hz}).
\end{align*}
\]

We perform the controller synthesis with respect to \( \varphi_{\delta^*} \). We set \( J(u(\cdot)) = \| u(u(\cdot)) \|_2 + \lambda \| u^*(\cdot) \|_2 \), where \( \lambda = 100 \) (we encourage power input from the WTGs). Fig. 2 shows the computed optimal input signals. Fig. 3 shows that all 100 trajectories (realizations) starting from \( B_{q1}(x_0^r, r) \) with the optimal input signals satisfy the specification \( \varphi \).

**B. Scenario II**

In this section, we apply the controller synthesis method on a nine-bus system as shown in Fig. 4 in the longer version [21]. The thermal plant \( G_1 \) and the wind farm \( G_2 \) are the same as those in Scenario I, with two energy storage systems (ESS) placed near them respectively. Four constant power loads are denoted as \( L_1, L_2, L_3, \) and \( L_4 \). The line data can be found in Tab. I and Tab. II in the longer version [21]. We consider a disturbance of generation loss of 150 MW (loss of one unit, \( \Delta P_d = 0.15 \)) that occurs at time 0. The switched stochastic system can be written in a similar form as in (19), with the modes transitioning at 5 seconds and 8.75 seconds, respectively.

We use the following MTL specification for frequency regulation after the disturbance (note that here in Scenario II, we use \( \varphi \) to show difference with \( \varphi \) in Scenario I):

\[
\begin{align*}
\varphi & = \square_{[0,T_{mal}]} \hat{p}_1 \land \square_{[2,T_{mal}]} \hat{p}_2 \land \square_{[2,T_{mal}]} \hat{p}_3, \\
\hat{p}_1 & = (-0.5Hz \leq \Delta f \leq 0.5Hz) \land (-10Hz \leq \Delta f_r \leq 10Hz), \\
\hat{p}_2 & = (-0.4Hz \leq \Delta f \leq 0.4Hz), \\
\hat{p}_3 & = \bigwedge_{ij \in E} (-0.25 \leq P_{ij} \leq 0.25).
\end{align*}
\]

The first two subformulas of \( \varphi \) are the same as in (20) used in Scenario I, while the third subformula \( \square_{[2,T_{mal}]} \hat{p}_3 \) specifies the real power constraints in each line. We obtain the following \( \delta_{k,i}^* \)-robust modified formula:

\[
\begin{align*}
\varphi_{\delta^*} & = \square_{[0,T_{mal}]} \hat{p}_1^* \land \square_{[2,T_{mal}]} \hat{p}_2^* \land \square_{[2,T_{mal}]} \hat{p}_3^*, \\
\hat{p}_1^* & = (-0.5Hz + 0.217e^{-0.01Hz} \leq \Delta f) \land \leq 0.5Hz - 0.217e^{-0.01Hz}) \land \leq 10Hz - 6.08e^{-0.01Hz}, \\
\hat{p}_2^* & = (-0.4Hz + 0.217e^{-0.01Hz} \leq \Delta f) \land \leq 0.4Hz - 0.217e^{-0.01Hz}), \\
\hat{p}_3^* & = \bigwedge_{ij \in E} (-0.25 \leq 0.0258e^{-0.01t} \leq P_{ij} \leq 0.25 - 0.0258e^{-0.01t}),
\end{align*}
\]

where \( E \subset N \times N \) is the set of transmission lines (\( N \) is the set of buses).

As there are 9 different lines corresponding to 9 different inequalities in the MTL specification, solving the optimization problem with all the inequality constraints could be computationally expensive. To reduce computation, we first
set an initial MTL specification and iteratively add the line power flow inequality constraints that are violated with the previous optimization. The initial MTL specification $\varphi^0_{\delta^*}$ as follows (by reducing the line power flow constraints in $\varphi^*_\delta$):

$$\varphi^0_{\delta^*} = \square_{[0,T_{\text{end}}]} P^*_1 \land \square_{[2,T_{\text{end}}]} P^*_2.$$  

We perform the controller synthesis with respect to $\varphi^0_{\delta^*}$. We set $J(u(\cdot)) = \|u(\cdot)\|_2 + \lambda \|u'(\cdot)\|_2$, where $\lambda = 100$ (larger $\lambda$ encourages power input from the wind turbine generator). After the first iteration, the line 2-8 is overloaded and thus does not satisfy the line flow constraint in $\varphi^0_{\delta^*}$ (as shown in Fig. 4). Then we add line 2-8 power specification and obtain the following MTL specification $\varphi^1_{\delta^*}$:

$$\varphi^1_{\delta^*} = \square_{[0,T_{\text{end}}]} P^*_{12} \land \square_{[2,T_{\text{end}}]} P^*_2 \land \square_{[2,T_{\text{end}}]} P^*_3,$$

$$\hat{P}^*_{31} = (-0.25 + 0.0258e^{-0.011t} \leq P_{28} \leq 0.25 - 0.0258e^{-0.011t}).$$

In the second iteration, the computed control inputs not only lead to satisfaction of $\varphi^1_{\delta^*}$ but also the satisfaction of $\varphi^0_{\delta^*}$. Thus the iteration stops and the optimal input signals are obtained (as shown in Fig. 5). Using the same $r$ as that in Scenario I, Fig. 6 shows that all 100 trajectories (realizations) of $\Delta f$ and $\Delta f_r$ starting from $B_{g^*}(x_0, r)$ with the synthesized optimal input signals satisfy the MTL specification $\varphi$.

![Fig. 4. 100 trajectories (realizations) of real power of 9 different lines with the synthesized control inputs (blue) after the first iteration in Scenario II.](image)

![Fig. 5. The synthesized optimal input signals in Scenario II.](image)

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