RETURN TO EQUILIBRIUM FOR AN ANHARMONIC OSCILLATOR COUPLED TO A HEAT BATH

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ABSTRACT. We study a $C^*$-dynamical system describing a particle coupled to an infinitely extended heat bath at positive temperature. For small coupling constant we prove return to equilibrium exponentially fast in time. The novelty in this context is to model the particle by a harmonic or anharmonic oscillator, respectively. The proof is based on explicit formulas for the time evolution of Weyl operators in the harmonic oscillator case. In the anharmonic oscillator case, a Dyson’s expansion for the dynamics is essential. Moreover, we show in the harmonic oscillator case, that $\mathbb{R}$ is the absolute continuous spectrum of the Standard Liouvillean and that zero is a unique eigenvalue.

1. INTRODUCTION

In this paper, we study an interacting system of a single particle coupled to a heat bath, which is infinitely extended and near its thermal equilibrium at inverse temperature $\beta$. The heat bath consists in infinitely many bosons with a momentum density given by Planck’s law for the black body radiation. The particle is confined by an increasing potential, which inhibits an escape to infinity. In this situation one expects that the interacting system is driven to a joint equilibrium state at inverse temperature $\beta$ as time tends to infinity. This behavior is called ‘return to equilibrium’.

The mathematical model is formulated in the framework of a quantum dynamical system. Here the observables are modeled by a Weyl algebra and the automorphism group is implemented by conjugating with a group of unitaries, which is generated by the Hamiltonian of the interacting system. In this paper the particle Hamiltonian is either a harmonic or an anharmonic oscillator and the interaction with the heat bath is given by a dipole expression.

In the last decade small systems coupled to a heat bath were subject of extensive mathematical research. In [15] and [16] an approach is established that traces back the ergodic properties of a certain $W^*$-dynamical system to the spectral characteristics of the so-called Standard Liouvillean. Moreover, the

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return to equilibrium occurs exponentially fast, the rate of decay is obtained by Fermi’s golden rule for the Standard Liouvillean. In [5] the spectral analysis of the Standard Liouvillean is studied by a renormalization group technique. The methods in [15, 16] and [5] require analytic form factor. The assumptions on the singularity of the form factors at zero are, however, less restrictive in [5].

In [10, 11] the Liouvillean is studied by means of the limiting absorption principle and the Feshbach method. In [10, 13] the positive commutator theorem and a viral theorem are applied to the Liouvillean. In all papers mentioned above the Hilbert space representing the small system is finite dimensional, and the interaction is small. In summary we call the strategy used by all authors above Liouvillean approach.

The harmonic oscillator that interacts with the heat bath by a dipole expression is a quadratic operator in annihilation and creation operators, this allows to define a *-automorphism group $\tau$ on a Weyl algebra $\mathfrak{A}$. Moreover, for any inverse temperature $\beta$ we can define a KMS state $\omega$ on the Weyl algebra $\mathfrak{A}$, see Theorem 2.1 below. This is a special property of the harmonic oscillator case. We benefit from very explicit results for the harmonic oscillator coupled to a Boson field at temperature zero, studied in [1, 2], or in [21] for a closely related model.

We prove in Theorem 2.3 that the return to equilibrium occurs exponentially fast for a large class of states and observables for small interaction. The rate of decay is related to Fermi’s golden rule for the Hamiltonian at temperature zero. This is different from the approach using Liouvilleans, where the rate of decay is deduced from Fermi’s golden rule for the Standard Liouvillean, which is larger.

We recall that the harmonic oscillator case is not covered by the methods of the Liouvillean approach, yet. However, we can formulate another model analog to the Liouvillean approach. In this model we define a Standard Liouvillean for each $\beta$, which generates the time evolution. One can show existence of a KMS state on some $W^*$-algebra $\mathfrak{M}$, see [17]. However, we prove in Corollary 3.5 that the spectrum of the Standard Liouvillean covers $\mathbb{R}$, it is absolutely continuous except in zero, and zero a unique, non-degenerate eigenvalue. Moreover, the model in the Liouvillean approach is an extension of the quantum dynamical system formulated on the Weyl algebra $\mathfrak{A}$ in the following sense:

The Weyl algebra $\mathfrak{A}$ can be embedded into $\mathfrak{M}$ and the dynamics generated by the Standard Liouvillean extends $\tau$, which is defined on the Weyl algebra. This is stated in Theorem 3.2.

Moreover, we perturb the harmonic potential by a potential $V$, which is the Fourier-transform of a complex measure. The perturbed Hamiltonian is called anharmonic oscillator. The class of perturbations is adopted from [18], where
the Langevin equation is studied, see also [21] and [12]. The model is the following, we fix an inverse temperature $\beta$, and construct a GNS representation associated with $\omega$. The theory of KMS states ensures the existence of a KMS state $\omega_Q$ for the perturbed dynamics in the Hilbert space of the GNS representation. To obtain convergence to $\omega_Q$ for large times we use Dyson’s expansion for the perturbed dynamics. In Theorem 4.2 and Corollary 4.5 return to equilibrium is proved for the anharmonic oscillator model with an exponential rate of decay for small coupling and small $V$. The strategy for the proof is based on an estimate, we learned form [18], it is based on the fact that certain integrals that occur in Dyson’s expansion decay exponential fast in $t$. Recently, in [7] a combinatorial argument was found, that relaxes this assumption on the decay. One could hope that this result can be used to prove return to return to equilibrium for the anharmonic oscillator, whether no analyticity of the form factor is assumed, c.f. Hypothesis 1.

1.1. Organization of the Paper. In the subsequent subsection we recall the formalism of second quantization. We give here the definition of the Hamiltonian in the harmonic oscillator case and formulate the Hypothesis on the form factors. Moreover, we give definitions in the context of quantum dynamical systems. In Section 2 we define the so-called analytic states and the analytic observables, for which the return to equilibrium is exponentially fast. We formulate and prove Theorem 2.1 and Theorem 2.3. Section 3 is devoted to the Liouvillean approach in the harmonic oscillator case. In this section we prove Theorem 3.2, Theorem 3.3 and Corollary 3.5. The anharmonic oscillator coupled to a heat bath is studied in Section 4, where we prove Theorem 4.2 and Corollary 4.5. The paper has two appendices: In the first we quote some definitions and results given in [1, 2], in the second we recall an estimate, which is important for the proof of Theorem 4.2, it was proved originally in [18].

1.2. Notation and Definition. The starting point is the state space $\mathcal{H}$, which represents the coupled system of a particle and the bosons at temperature zero,

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{F}_b[\mathfrak{h}].$$

The Hilbert space $L^2(\mathbb{R})$ contains the states for an isolated particle, which is for simplicity assumed to be one-dimensional. The bosonic Fock space,

$$\mathcal{F}_b[\mathfrak{h}] = \mathbb{C}\Omega_\mathfrak{h} \oplus \bigoplus_{n=1}^{\infty} S_n \otimes \mathfrak{h}$$

is modeled over the one boson space $\mathfrak{h} := L^2(\mathbb{R}^3)$. Later on we will also use a Fock space over $\mathbb{C} \oplus \mathfrak{h}$ or $\mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{h} \oplus \mathfrak{h}$. In this context $\Omega_\mathfrak{h}$ is a fixed normed vector called the vacuum and $S_n$ is the projection onto the subspace of totally symmetric tensors. The vectors in $\mathcal{F}_b[\mathfrak{h}]$ are sequences $\psi = (\psi_n)_{n=0}^{\infty}$ such that $\psi_n \in S_n \otimes \mathfrak{h}$ for $n \geq 1$ and $\psi_0 \in \mathbb{C}\Omega_\mathfrak{h}$. The annihilation and creation
operators on $\mathcal{F}_b[\mathfrak{h}]$ are denoted by $a(f), a^*(g)$ for $f, g \in \mathfrak{h}$. They satisfy the Canonical Commutator Relations (CCR),

\begin{equation}
[a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0,
\end{equation}

\[ [a(f), a^*(g)] = \langle f|g\rangle_{\mathfrak{h}} \]

and $a(f)\Omega_{\mathfrak{h}} = 0$. Furthermore, we define the field operators by

\[ \Phi(f) = \frac{1}{\sqrt{2}} (a(f) + a^*(f)). \]

$a(f), a^*(f)$ and $\Phi(f)$ are defined on the dense subspace of finite sequences $(\psi_n)_{n=0}^{\infty}$ of $\mathcal{F}_b[\mathfrak{h}]$. All three operators are closable, we denote their closures by the same symbol. As suggested in the notation $a^*(f)$ is the adjoint operator of $a(f)$. Moreover, $\Phi(f)$ is self-adjoint, the exponential $W[f] := e^{i\Phi(f)}$ is called Weyl operator. From (1.1) we deduce the CCR for Weyl operators

\begin{equation}
W[f] W[g] = e^{-i\text{Im} \langle f|g\rangle_{\mathfrak{h}}/2} W[f + g].
\end{equation}

For a definition of the formalism of second quantization, $C^*$-algebras and related topic we refer the reader to the textbooks [8, 9].

On the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ we define by $(q\psi)(q) = q \cdot \psi(q)$ the position operator and by $(p\psi)(q) = -i \frac{\partial}{\partial q}(q) \psi(q)$ the momentum operator of the particle. Annihilation and creation operators in $L^2(\mathbb{R})$ are given by

\[ A^* = \frac{1}{\sqrt{2}} (q - ip), \quad A = \frac{1}{\sqrt{2}} (q + ip). \]

They satisfy the CCR

\[ [A, A^*] = [A^*, A^*] = 0, \quad [A, A^*] = 1_{S(\mathbb{R})}, \]

in addition we have $A\Omega_{\mathbb{C}} = 0$ for $\Omega_{\mathbb{C}}(q) = \pi^{-1/4} e^{-q^2/2}$. It is known that $\text{cl}_{L^2(\mathbb{R})} \text{LH} \{ (A^*)^n \Omega_{\mathbb{C}} \in L^2(\mathbb{R}) : n \in \mathbb{N}_0 \} = L^2(\mathbb{R})$, see for instance [9]. The letter $\text{cl}_{L^2(\mathbb{R})}$ denotes the closure in the topology of $L^2(\mathbb{R})$ and LH denotes the linear hull. Thus we can identify $L^2(\mathbb{R})$ with $\mathcal{F}_b[\mathbb{C}]$. As a consequence we obtain $\mathcal{H} \cong \mathcal{F}_b[\mathbb{C} \oplus \mathfrak{h}]$ and

\[ \Omega_{\mathbb{C} \oplus \mathfrak{h}} \cong \Omega_{\mathbb{C}} \otimes \Omega_{\mathfrak{h}}, \quad a(c \oplus h) \cong cA \oplus a(h), \]

\[ a^*(c \oplus h) \cong cA^* \oplus a^*(h). \]

Moreover, we field operators $\Phi(c \oplus f)$ and Weyl operators $W[c \oplus h]$ in $\mathcal{H}$ are given by

\begin{equation}
\Phi(c \oplus h) = \frac{1}{\sqrt{2}} (a^*(c \oplus h) + a(c \oplus h)),
\end{equation}

\begin{equation}
W[c \oplus h] = W[c] \otimes W[h].
\end{equation}
For these operators the CCR in Equation (1.1) and Equation (1.2) are satisfied with $\langle \cdot | \cdot \rangle_h$ replaced by $\langle \cdot | \cdot \rangle_{BV} \oplus h$. In this model the particle is confined by the potential $(1/2)x^2$, which inhibits an escape to infinity of the particle. The harmonic oscillator is

$$H_{osc} = (1/2)(p^2 + q^2).$$

We remark, that a large class of potentials should ensure return to equilibrium, but the choice of the harmonic potential is essential for our analysis. The reason is, that we can write the Hamiltonian in the formalism of second quantization, i.e. $H_{osc} = A^*A + 1/2$. This allows us to perform calculations explicitly.

Throughout this paper the bosons are massless, being modeled in the momentum space with the dispersion relation $|k|$. The free Hamiltonian is defined by $H_f \Omega_h := 0$ and

$$(1.5) \quad (H_f \psi)_n := (\sum_{j=1}^{n} 1 \otimes \ldots \otimes h_{ph} \otimes \ldots \otimes 1) \psi_n,$$

where $h_{ph}$ is the multiplication with $|k|$ in $h$. In the following we will not introduce an extra symbol for multiplication operators, so $h_{ph}$ is just $|k|$.

The interaction operator is given by

$$H_I = \lambda q \cdot \Phi(|k|^{-1/2} \hat{\rho}),$$

where $\Phi(|k|^{-1/2} \hat{\rho}) := \Phi(0 \oplus |k|^{-1/2} \hat{\rho})$. The parameter $\lambda \neq 0$ is the coupling constant, which models the strength of the interaction. The Hamiltonian for the interacting system is

$$(1.6) \quad H_h = H_{osc} \otimes 1 + 1 \otimes H_f + H_I + Rq^2 \otimes 1,$$

where $R := (\lambda^2/2)|||k|^{-1/2} \hat{\rho}||^2$. The operator $H_I$ and the additional potential $R \cdot q^2$, $\hat{\rho} \in h$ is the coupling function, form the so-called dipole approximation, which arises from a model with a minimally coupled Hamiltonian by replacing $e^{i k y}$ by $1 + k q$ and by applying a unitary transform, see [21]. The crucial simplification is, that $H_h$ is quadratic in annihilation and creation operators. Note, that we have the representation $H_f = \int |k|a^*(k)a(k)d^3k$, where $a(k)$ and $a^*(k)$ are annihilation and creation operators for sharp momentum $k$, see [20]. Thus $H_f$ is quadratic as well.

Hypothesis 1. We assume

1. $\hat{\rho}(k) > 0, \ k \in \mathbb{R}^3$.
2. $\hat{\rho}$ is rotation invariant.
3. In polar coordinates: $0, \infty \ni r \mapsto \hat{\rho}(r)$ has an analytic continuation on $\{z \in \mathbb{C} : |Im z| \leq 2\pi \beta^{-1}\}$, also denoted by $\hat{\rho}$.
4. $\text{sup}_{|r| < 2\pi \beta^{-1}} \int |\hat{\rho}(r + is)|^2(1 + |r|^2)dr < \infty$
5. For the analytic continuation we have $\hat{\rho}(r) = \hat{\rho}(-r), \ r \in \mathbb{R}$. 

5
We write for a functions \( f \) with domain in \( \mathbb{R}^3 \), \( f(r, \Theta) \) with \( r > 0 \) and \( \Theta \in S^2 \) if it is expressed in polar coordinates. Moreover, we write \( f(r) \), if \( f \) is rotation invariant. \( S^2 \) denotes in this context the unit sphere.

The Hamiltonian in (1.6) was analyzed in [1, 2]. Among other things therein is shown, that \( \text{dom}(H_h) = \text{dom}(H_{osc} \otimes 1 + 1 \otimes H_f) \), where \( \text{dom} \) denotes the domain of an operator. An explicit formula for the asymptotic annihilation and creation operators is given. It can be deduced from [1], that up to a unitary isomorphism we have \( H_h \sim H_f + E_{gs} \), where \( E_{gs} \) is the ground state energy of \( H_h \). This unitary isomorphism is a Bogoliubov transform, that means, it can be defined by introducing new annihilation and creation operators and a new vacuum vector, see [6]. In the following, we will make use of some definitions and lemmata in [1, 2], which are reformulated in Appendix A below.

We define the dynamics in the Heisenberg picture on the algebra of observables

\[
\mathfrak{A} := \text{cl}_{\mathcal{B}(\mathcal{H})} \text{LH} \{ \mathcal{W}[c \oplus f] : c \oplus f \in C \oplus \mathfrak{f} \},
\]

with \( \mathfrak{f} := \{ f \in \mathfrak{h} : (1 + |k|^{-1/2})f \in \mathfrak{h} \} \). Note, that we can extend regular states on \( \mathfrak{A} \) to annihilation- and creation operators, which are the actual observables of interest. The time-evolution is given by

\[
A \mapsto \tau_t(A) \in \mathfrak{A}, \quad \tau_t(A) := e^{itH_h} Ae^{-itH_h}, \quad A \in \mathfrak{A}.
\]

For fixed \( t \in \mathbb{R} \) \( \tau_t \) is a map from \( \mathfrak{A} \) to \( \mathfrak{A} \), which is linear and multiplicative, and obeys \( \tau_t(A^*) = \tau_t(A)^* \). Moreover, \( \tau = (\tau_t)_{t \in \mathbb{R}} \) is a group since \( \tau_t \circ \tau_s = \tau_{t+s}, \quad \tau_0 = 1 \). In the sequel this is called a \( * \)-automorphism group.

A state is a positive, normed, linear functional on \( \mathfrak{A} \). This definition covers the vector states in \( \mathcal{H} \). Furthermore, it allows to define equilibrium states for positive temperature \( \beta^{-1} \). Since \( H_h \) has continuous spectrum, it is not possible to define a Gibbs state for \( H_h \). Thus, we follow the lines of Haag, Hugenholz and Winnink [14], who characterized equilibrium states by means of the \((\tau, \beta)\)-KMS property, see Definition 1.2 below. An extensive representation of the theory of operator algebras can be found in the textbooks [8, 9].

Next, we give definitions in the context of operator algebras, see also [4].

**Definition 1.1** (Quantum Dynamical System).  
(1) Let \( \mathcal{A} \) be a unital \( C^* \)-algebra, \( \tau \) a \( * \)-automorphism group and \( \omega \) a state. The triple \( (\mathcal{A}, \tau, \omega) \) is a quantum dynamical system, if \( \mathbb{R} \ni t \mapsto \omega(A^* \tau_t(A)) \) is continuous for every \( A \in \mathcal{A} \).

(2) Let \( \mathfrak{h} \) be any Hilbert space and \( \mathcal{M} \subset \mathcal{B}(\mathfrak{h}) \) be a unital \( C^* \)-algebra. \( \mathcal{M} \) is a \( W^* \)-algebra, if it is weakly closed. \( \mathcal{M}, \tau \) is called a \( W^* \)-dynamical system, if in addition \( \mathbb{R} \ni t \mapsto \tau_t(A), \quad A \in \mathcal{M} \) is continuous, while \( \mathcal{M} \) carries the \( \sigma \)-weak topology of \( \mathcal{B}(\mathfrak{h}) \).

The next definition fixes some dynamical properties, which are essential for the following.
Definition 1.2 (Mixing, Clustering, Equilibrium State). Let \((\mathcal{A}, \tau, \omega)\) be a quantum dynamical system.

1. \((\mathcal{A}, \tau, \omega)\) is mixing, if \(\lim_{t \to \infty} \mu(\tau_t(A)) = \omega(A)\) for any \(\omega\)-normal state \(\mu\) and any \(A \in \mathcal{A}\).

2. \((\mathcal{A}, \tau, \omega)\) is clustering, if \(\lim_{t \to \infty} \omega(A \tau_t(B)C) = \omega(AC)\omega(B)\) for \(A, B, C \in \mathcal{A}\).

3. \(\omega\) is an equilibrium state for \((\mathcal{A}, \tau)\) at inverse temperature \(\beta\), if for any \(A, B \in \mathcal{A}\) there is a function \(F_\beta(\cdot, A, B)\) being analytic in the strip \(\{z \in \mathbb{C} : 0 < \text{Im} z < \beta\}\), continuous on its closure and taking the boundary conditions

\[
F_\beta(A, B, t) = \omega(\tau_t(B)),
\]

\[
F_\beta(A, B, t + i \beta) = \omega(\tau_t(B)A).
\]

In this case \(\omega\) is also called a \((\tau, \beta)\)-KMS state.

We will say that a dynamical system has the property of return to equilibrium, if it is mixing. Let \((h_\omega, \pi_\omega, \Omega_\omega)\) be the GNS-triple, see [8]. For \(\mu\) being \(\omega\)-normal means, that there is a sequence of positive numbers \((p_n)\) with \(\sum_{n=1}^{\infty} p_n = 1\) and an orthonormal system of vectors \((\phi_n) \subset h_\omega\) satisfying \(\mu(A) = \sum_{n=1}^{\infty} p_n \langle \phi_n | \pi_\omega[A] \phi_n \rangle_h\). \(\forall A \in \mathfrak{A}\). It can be seen using an approximation argument that \((\mathcal{A}, \tau, \omega)\) is mixing if it is clustering. Let \(\eta := 1 + 2 \rho\). \(\rho(k) = (e^{\beta|k|} - 1)^{-1}\) is the momentum density in Planck’s law.

Definition 1.3 (Bose gas system). Let \(\mathfrak{A}_f := W(\mathfrak{f})\). \(\tau^f\) is defined by

\[
\tau^f_t(W) := e^{itH_f} W e^{-itH_f}, \quad W \in \mathfrak{A}_f
\]

An equilibrium state is established by

\[
\omega_f(W[f]) = \exp(-1/4\langle f | \eta f \rangle_h).
\]

We call the quantum dynamical system \((A_f, \tau^f, \omega_f)\) Bose gas system.

Note that \(\tau^f_t(W[f]) = W[e^{it|k|}f]\) and \(\tau^f_t(\Phi(f)) = \Phi(e^{it|k|}f)\).

2. Return to Equilibrium for the Harmonic Oscillator

2.1. Statements and Discussions. In the next theorem we compare the quantum dynamical system \((\mathfrak{A}, \tau, \omega)\) with the Bose gas system. We prove that both are equivalent up to Bogoliubov transform.

Theorem 2.1 (Isomorphism of the Dynamical Systems). There is a unique symplectic map \(v : \mathbb{C} \oplus \mathfrak{f} \to \mathfrak{f}\), such that

\[
\gamma : \mathfrak{A} \to \mathfrak{A}_f, \quad W[c \oplus f] \to W[v(c \oplus f)]
\]
is a $*$-isomorphism and $\omega := \omega_f \circ \gamma$ is an equilibrium state for $(\mathcal{A}, \tau)$. Moreover, we have that

\begin{equation}
\omega(\mathcal{W}[c \oplus f]) = \exp(-1/4\langle v(c \oplus f)|\eta\eta(v(c \oplus f))\rangle)
\end{equation}

The following diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\tau} & \mathcal{A} \\
\uparrow{\gamma} & & \uparrow{\omega} \\
\mathcal{A}_f & \xrightarrow{\tau_f} & \mathcal{A}_f
\end{array}
\end{equation}

is commutative. We call the quantum dynamical system $(\mathcal{A}, \tau, \omega)$ the interacting system. Moreover, there is for $t \in \mathbb{R}$ a unique symplectic map $w_t : \mathbb{C} \oplus \mathcal{f} \to \mathbb{C} \oplus \mathcal{f}$, such that $\tau_t(\mathcal{W}[c \oplus f]) = \mathcal{W}[w_t(c \oplus f)]$ and $w_{t+s} = w_t \circ w_s$.

Let $\mathfrak{k}, \mathfrak{k}'$ be complex pre-Hilbert spaces. We say that $s : \mathfrak{k} \to \mathfrak{k}'$ is symplectic, whenever $s$ is real linear and $\text{Im}(s(f)|s(f'))_\mathfrak{v} = \text{Im}(f|f')_\mathfrak{k}$, $f, f' \in \mathfrak{k}$. Of special interest is the subalgebra $\mathcal{W}(\mathbb{C} \oplus 0)$, it contains the observables for the particles. In the context of open quantum systems it is the small system. The excited states of the small system are represented as states of $\mathcal{A}$ for some $B \in \mathcal{W}(\mathbb{C} \oplus 0)$.

Let $\mathcal{A}$ be the subalgebra of analytic observables and let $\omega$ be a function from $\mathcal{A}^\ast$. For $f \in L^2(\mathbb{R}^3)$ we denote by $\tilde{f}$ a function from $\mathbb{R} \times S^2$ to $\mathbb{C}$, defined by

\begin{equation}
\tilde{f}(r, \Theta) = \begin{cases} f(r, \Theta), & r \geq 0 \\ f(-r, \Theta), & r < 0. \end{cases}
\end{equation}

Let $H^2(\kappa)$ be the real vector space of functions $f : (0, \infty) \to L^2(S^2)$, such that $\tilde{f}$ has an analytic continuation on the strip $\{ z \in \mathbb{C} : |\text{Im} z| < \kappa \}$ and obeying $\int_0^\infty \sup_{\eta \in \mathbb{R}, |\eta| < \kappa} ||\tilde{f}(r + i\eta)||^2_{L^2(S^2)}(1 + |r|^3)dr < \infty$.

$H^2(\kappa)$ is also invariant with respect to complex conjugation. We also can regard each $f \in H^2(\kappa)$ as a function in $\mathfrak{f}$, in the sense that $k \mapsto f(|k|, k \cdot |k|^{-1})$ belongs to $\mathfrak{f}$. Let $\mathcal{g}$ be the real vector space of functions $h \in \mathbb{C} \oplus \mathfrak{f}$, such that $\psi(h)$ belongs to

\begin{equation}
\mathcal{R} = \{(a |k|^{1/2} + b|k|^{-1/2})f \in \mathfrak{f} : a, b \in \mathbb{R}, f \in H^2(\kappa)\},
\end{equation}

which is dense in the norm $\|g\|_+ := \|g\|_h + \| |k|^{-1/2}g\|_h$. This is proved in Lemma [2.5] below. The positive constant $\kappa$ will be fixed in Theorem [2.3] $v$ is introduced in Theorem [2.1].

**Definition 2.2** (Analytic states and analytic observables). Let $\mathcal{A}_0 := \text{LH}\{\mathcal{W}[g] \in \mathcal{W}(\mathbb{C} \oplus \mathfrak{f}) : g \in \mathcal{g}\} \subset \mathcal{A}$ be the subalgebra of analytic observables and let $\mathcal{S}_0$ be the analytic states of $\mathcal{A}$ defined by

\[ \mathcal{S}_0 = \{ \mu : \exists B \in \mathcal{A}_0 \forall A \in \mathcal{A} \mu(A) = \omega(B^*AB) \}. \]
The next theorem states the property of return to equilibrium, and that the decay in time is exponentially fast for states in \( S_0 \) and observables in \( A_0 \). The proof is given in the subsequent subsection. Here and in what follows, \( c \) denotes a constant independent of \( t \), its values may change from one line to the other.

**Theorem 2.3 (Return to Equilibrium).** For small \( \lambda \neq 0 \) the interacting system \((\mathfrak{A}, \tau, \omega)\) is clustering. Moreover, for \( A \in \mathfrak{A}_0 \) and \( \mu \in S_0 \) one has

\[
|\mu(\tau_t(A)) - \omega(A)| \leq c e^{-\kappa t}
\]

for some constant and for \( 0 < \kappa < \hat{\kappa} \). \( \hat{\kappa} \) is a fixed decay rate \( 2\pi\beta^{-1} > \hat{\kappa} > 0 \) depending only on \( \lambda \). \( \hat{\kappa} \) is defined in Lemma 2.4 below.

### 2.2. Auxiliary Statements.

Note, that \( D_+ \) and \( D \) are defined in Definition [A.1] below. In the next lemma the zeros of the analytic continuation of \( \Re \ni r \mapsto D_+(r^2) \) on the strip around the real axis are determined. The imaginary part of the zeros gives the decay rate in Theorem 2.3. The function \( [0, \infty) \ni r \mapsto D(r^2) \) already occurs in the analysis in [1, 2] to determine the life time of resonances for the temperature zero Hamiltonian \( H_h \). The zeros are related to Fermi’s golden rule for \( H_h \).

**Lemma 2.4.** For \( \lambda \neq 0 \) small enough, there is an analytic continuation of \( \Re \ni r \mapsto D_+(r^2) \) on \( \{ z \in \mathbb{C} : |\text{Im} z| < |\text{Im} \hat{\kappa}(\lambda)| \} \) that has no zeros. Furthermore, \( \hat{\kappa}(\cdot) \) is even, analytic and

\[
\hat{\kappa}(\lambda) = 1 + \kappa_2\lambda^2 + \kappa_4\lambda^4 + \ldots,
\]

where \( \text{Im} \kappa_2 = 2\pi^2 \beta^2(1) \).

**Lemma 2.5.** \( \mathcal{R} \) is dense in \((f, \|\cdot\|)\), where \( \|g\|_+ := \|g\|_h + \|k\|^{-1/2} \|g\|_h \).

This Lemma is used to show, that the algebra \( \mathfrak{A}_0 \) and the set of states \( S_0 \) are large in the following sense:

**Lemma 2.6.** \( \mathfrak{g} \) is dense in \( \mathbb{C} \oplus \mathfrak{f} \), with respect to the norm \( \|g\|_+ := \|g\|_{\mathbb{C}\oplus \mathfrak{h}} + \|k\|^{-1/2} \|g\|_{\mathbb{C}\oplus \mathfrak{h}} \). Moreover, the image of \( \mathfrak{g} \) under \( v \) is dense in \((f, \|\cdot\|_+)\).

The following Lemma is an ingredient in the proof of Theorem 2.3, ensuring the exponential decay. The special choice of form factors in \( \mathfrak{g} \) allows to express the real- and the imaginary part of the scalar products in (2.6) and (2.7) below as an integral over \( \mathbb{R} \). The decay is obtained by shifting the contour of integration in the upper complex half plane. Since in the definition of \( f, g \in \mathfrak{g} \) the inverse of \( D_+(r^2) \) (analytically continued) occurs, the decay rate is determined by the zeros of \( D_+(r^2) \).

**Lemma 2.7.** For \( f, g \in \mathfrak{g} \) one has

\[
|\text{Re} \langle v(f) | e^{i|k|t} v(g) \rangle_h| \leq c e^{-\kappa t} \tag{2.6}
\]

\[
|\text{Im} \langle v(f) | e^{i|k|t} v(g) \rangle_h| \leq c e^{-\kappa t} \tag{2.7}
\]
\( c \) is a positive constant uniform in \( t \).

2.3. Proofs.

Proof of Theorem 2.1. The proof uses intensively results in \([1\ 2]\), quoted in Appendix \([A]\) below. From Lemma \([A.7]\) follows that \( \Phi_m(f) := \frac{1}{\sqrt{2}}(a_m(f) + a_m^*(f)) \) is equal to

\[
\Phi\left(\langle Q_+ |f\rangle_h + \langle f|Q_-\rangle_h \right) + (W_+ f + W_- f)
\]

\( a_m(f) \) and \( a_m^*(f) \) are the incoming annihilation and creation operators. \( a_m(f), a_m^*(f) \) as well as \( W_+, W_-, Q_-, Q_+ \) are defined in Appendix \([A]\). Thus we have

\[
e^{itH_h} \Phi_m(f) e^{-itH_h} = \Phi_m(e^{it|f|})
\]

Combining Lemma \([A.5.4]\) and \([A.5.5]\), such as Equation \([A.5]\), \([A.6]\) and \([A.1]\), we obtain

\[
(2.8) \quad W_+ Q_- = W_- Q_+, \quad W_- Q_+ = W_+ Q_-, \quad \|Q\|_h = 1.
\]

A simple but lengthy calculation using the Identities in \((2.8)\) and Lemma \([A.6]\) yields \( \Phi(c \oplus h) = \Phi_m(v(c \oplus h)) \). Here, \( v \) is a real linear operator from \( C \oplus \mathfrak{f} \) to \( \mathfrak{f} \) defined by

\[
(2.9) \quad v(c \oplus h) := W_+ h + e^{itH_h} - W_- h - e^{-itH_h}.
\]

By Lemma \([A.6]\) the operator \( v \) is surjective since \( h = v((\langle Q_+ |h\rangle_h + \langle Q_- |h\rangle_h) \oplus (W_+ h + W_- h)) \). Moreover, \( v \) is symplectic, since

\[
\begin{align*}
\operatorname{Im} \langle c \oplus h | c' \oplus h' \rangle_{C \oplus h} & = -i[\Phi(c \oplus h), \Phi(c' \oplus h')] \\
& = -i[\Phi_m(v(c \oplus h)), \Phi_m(v(c' \oplus h'))] \\
& = \operatorname{Im} \langle v(c \oplus h) | v(c' \oplus h') \rangle_{h}.
\end{align*}
\]

We deduce that \( v \) is injective. From the definition of \( \Phi_m \) follows that

\[
\tau_t(\Phi(c \oplus h)) = \Phi_m(e^{it|k|}v(c \oplus h)) = \Phi(w_t(c \oplus h))
\]

for the real linear, time dependent operator \( w_t \) defined by

\[
(2.10) \quad w_t(c \oplus h) = \left(\langle Q_+ |e^{it|k|}v\rangle_h + \langle e^{it|k|}v|Q_-\rangle_h \right) \\
\quad \quad \quad \oplus \left( W_+ e^{it|k|}v + W_- e^{-it|k|}v \right),
\]

where \( v := v(c \oplus h) \). Since \( \Phi_m(e^{it|k|}v(c \oplus h)) = \Phi_m(v(w_t(c \oplus h))) \) we have that

\[
(2.11) \quad e^{it|k|}v(c \oplus h) = v(w_t(c \oplus h)).
\]

Since \( v \) is symplectic, we may define a *-isomorphism \( \gamma \) by \( \gamma : \mathfrak{A} \to \mathfrak{A}_f \), \( \mathcal{W}[c \oplus h] \mapsto \mathcal{W}[v(c \oplus h)] \). By Equation \((2.11)\), Definition \([1.2]\) and Definition \([1.3]\) we conclude that \( \omega := \omega_f \circ \gamma \) is a \((\tau, \beta)\)-KMS state over \( \mathfrak{A} \).
Proof of Theorem 2.3. First, we prove the exponential decay. Let $v_i := v(f_i)$ for $f_i \in \mathfrak{g}$, $i = 1, 2, 3$. Using $\tau_i(\mathcal{W}[f_2]) = \mathcal{W}[w_i(f_2)]$, the CCR relation for Weyl operators in (1.2), such as (2.1) and (2.11), we obtain

$$
\omega(\mathcal{W}[f_1] \tau_1(\mathcal{W}[f_2]) \mathcal{W}[f_3])
= \exp \left( -1/4 \| \eta^{1/2}(v_1 + e^{i\|k\|}v_2 + v_3) \|_h^2 \right) \cdot \exp \left( -(i/2)\text{Im} \left( \langle v_1 | e^{i\|k\|}v_2 \rangle_h \right) \cdot \exp \left( -(i/2)\text{Im} \left( \langle v_1 | v_3 \rangle_h \right) \right)
$$

Because of

$$
\omega(\mathcal{W}[f_1] \mathcal{W}[f_3]) = \exp(-1/4)\| \eta^{1/2}(v_1 + v_3) \|_h^2 \cdot \exp(-i/2)\text{Im} \langle v_1 | v_3 \rangle_h
$$

we get

(2.12)

$$
\omega(\mathcal{W}[f_1]) \tau_1(\mathcal{W}[f_2]) \mathcal{W}[f_3]) = \omega(\mathcal{W}[f_1]) \mathcal{W}[f_3]) \cdot \exp \left( -(1/2)\text{Re} \langle v_1 + v_3 | \eta e^{i\|k\|}v_2 \rangle_h \right) \cdot \exp \left( -(i/2)\text{Im} \langle v_1 - v_3 | e^{i\|k\|}v_2 \rangle_h \right).
$$

Lemma 2.7 implies

(2.13)

$$
|\omega(\mathcal{W}[f_1]) \tau_1(\mathcal{W}[f_2]) \mathcal{W}[f_3]) - \omega(\mathcal{W}[f_1]) \mathcal{W}[f_3])| \leq ce^{-\alpha t}.
$$

By definition of $\mathfrak{A}_0$ and $\mathcal{S}_0$ the exponentially decay in (2.5) follows. To prove the first statement, we assume $f_1, f_2, f_3 \in \mathfrak{f}$. As before we obtain (2.12), but with $v_1, v_2, v_3 \in \mathfrak{f}$. Since $\text{w-lim}_{t \to \infty} e^{i\|k\|}v_2 = 0$, we get

$$
\lim_{t \to \infty} \omega(\mathcal{W}[f_1] \tau_1(\mathcal{W}[f_2]) \mathcal{W}[f_3]) = \omega(\mathcal{W}[f_1]) \mathcal{W}[f_3]) \omega(\mathcal{W}[f_2])
$$

Here, $\text{w-lim}_{t \to \infty}$ is the weak limit in the Hilbert space sense. From this we obtain

$$
\lim_{t \to \infty} \omega(A \tau_i(B) C) = \omega(A \mathcal{C}) \omega(B), \text{ for } A, B, C \in \text{LH}\{\mathcal{W}[f] : f \in \mathfrak{f}\}. A density argument yields, that $(\mathfrak{A}, \tau, \omega)$ is clustering, and thus mixing. □

Proof of Lemma 2.4. Let $G$ be defined for $|\text{Im} z| < \eta$ and $\lambda \in \mathbb{C}$ by

$$
G(z, \lambda) := -z^2 + 1 + ||k||^{-1} \hat{\rho}^2 \lambda^2 + 4\pi^2 i \lambda^2 \hat{\rho}^2(z)z + 2\pi \lambda \int_{-\infty}^{\infty} \frac{\hat{\rho}^2(r + i\eta)(r + i\eta)}{z - (r + i\eta)} dr
$$

11
Recall that \( \hat{\rho}(r) \) is an even function. By definition of \( D(z) \) in \( \text{A.1} \) we obtain for \( \text{Im } z > 0 \), that
\[
D(z^2) = -z^2 + 1 + \| |k|^{-1} \hat{\rho} \|^2 \lambda^2 + 2\pi \lambda^2 \int_{-\infty}^{\infty} \frac{\hat{\rho}^2(r) r}{z - r} \, dr.
\]
The residue theorem yields that \( G_\lambda(\cdot) := G(\cdot, \lambda) \) is an analytic continuation of \( D(z^2) \) on the lower half plane. By uniqueness, and since \( D(z^2) \) is even, we get
\[
G(z, \lambda) = G(-z, \lambda) = G(z, -\lambda).
\]
Let \( s \geq 0 \). We can choose \( p_\epsilon(s) \), such that \( s^2 + i \epsilon = p_\epsilon(s) \), \( \text{Re } p_\epsilon(s) \geq 0 \) and \( \text{Im } p_\epsilon(s) > 0 \), then
\[
G(s, \lambda) = \lim_{\epsilon \to 0^+} G(p_\epsilon(s), \lambda) = \lim_{\epsilon \to 0^+} D(s^2 + i \epsilon) = D_+(s^2).
\]
Thus \( G_\lambda \) is an analytic continuation of \( D_+(r^2) \). For any \( 0 < \eta' < \eta \) we have
\[
\sup_{\{z : |\text{Im } z| < \eta'\}} |G(z, 0) - G(z, \lambda)| \leq C_\eta' |\lambda|^2.
\]
Since \( \partial_z G(\pm 1, 0) = \mp 2 \) the implicit function theorem yields two analytic functions \( \kappa_{\pm 1} \) in a neighborhood of zero, with \( \kappa_{\pm 1}(0) = \pm 1 \) and
\[
G(z, \lambda) = 0 \iff z = \kappa_{\pm 1}(\lambda)
\]
for any \( z \) in a complex neighborhood of \( 1 \) or \( -1 \), respectively. By (2.14) there is a neighborhood of zero, such that (2.15) holds for all \( z \in \{ z \in \mathbb{C} : |\text{Im } z| \leq \eta' \} \) for some \( \eta' > 0 \) independent of \( \lambda \). By symmetry of \( G \) and uniqueness of \( \kappa_{\pm 1} \) we have \( \kappa_{-1}(\lambda) = -\kappa_{+1}(\lambda) \) and \( \kappa_{+1}(\lambda) = \kappa_{+1}(-\lambda) \), in particular \( \partial_\lambda^{(2n+1)} \kappa_{+1}(0) = 0 \).

For the second derivative we have
\[
\partial_\lambda^2 \kappa_{+1}(0) = - \frac{(\partial_\lambda^2 G)(1, 0)}{(\partial_z G)(1, 0)} = \| |k|^{-1} \hat{\rho} \|^2 \lambda^2 + 2\pi \mathcal{P} \int_{-\infty}^{\infty} \frac{\hat{\rho}^2(r) r}{1 - r} \, dr + 4\pi^2 \hat{\rho}^2(1) i,
\]
where \( \mathcal{P} \int_{-\infty}^{\infty} \) is the Cauchy-principal value. This completes the proof. \( \square \)

**Proof of Lemma 2.5.** We will first prove an auxiliary statement:
\( H^2(\kappa) \) is dense in \( \hat{\mathcal{H}} := L^2(\mathbb{R}^3, (|k|^{-2} + |k|)d^3k) \).

For \( f \in \hat{\mathcal{H}} \) we have that
\[
\int_0^{\infty} \int_{S^2} |f(r, \Theta)|^2 d\Theta(1 + r^3) dr < \infty.
\]
Let $\hat{\mathcal{K}} = L^2((0, \infty), (1 + r^3)dr)$. Since $\hat{\mathcal{H}} = \hat{\mathcal{K}} \otimes L^2(S^2)$ we may restrict ourselves the case where $f = f_1 \cdot f_2$. Assume first that $f_2$ is real-valued. There is a smooth function $\phi$ with compact support in $(0, \infty)$, and $\|f_1 - \phi\|_K \leq \epsilon$. Next, we define a continuation $\psi$ of $\phi$ to $\mathbb{R}$ by $\psi(0) := 0$ and $\psi(r) := \phi(-r)$ for $r < 0$. Since $\phi$ vanishes near zero, we get $\psi \in C^\infty_c(\mathbb{R}, \mathbb{C})$. The Fourier transform $\hat{\psi}$ is a real-valued Schwartz function. Thus there is a $\eta \in C^\infty_c(\mathbb{R}, \mathbb{R})$ such that

$$\left(\|\eta - \hat{\psi}\|^2\right)^2 := \sum_{\nu, \mu \in \mathbb{N}_0} \left\| (1 + |s|^2)^{\nu/2} \partial^\mu \left(\eta - \hat{\psi}\right) \right\|^2_{L^2(\mathbb{R})} \leq \epsilon^2.$$ 

For some universal constant we obtain for the inverse Fourier transform $\hat{\eta}$ that $\|\hat{\eta} - \psi\|_\infty \leq \epsilon \epsilon$. We obtain for another constant that $\|\hat{\eta} - f_1\|_K \leq \epsilon \epsilon$. It is elementary to see that $\hat{\eta} \cdot f_2$ belongs to $H^2(\kappa)$. When $-i f_2$ is real valued, we modify the proof. We choose $\phi$ as before. Next choose the continuation to $\mathbb{R}$, such that $\psi(r) = -\overline{\psi(-r)}$. Then we can choose $\eta \in C^\infty_c(\mathbb{R}, i \mathbb{R})$, thus $\eta(r) = -\overline{\eta(-r)}$. As before, $\eta \cdot f_2$ belongs to $H^2(\kappa)$. This proves the auxiliary statement.

Let $g \in C^\infty_c(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$, then $f := (a |k|^{1/2} + b |k|^{-1/2})^{-1} g \in \hat{\mathcal{H}}$. Since multiplication with $(a |k|^{1/2} + b |k|^{-1/2})$ is a continuous map from $\hat{\mathcal{H}}$ to $(f, \| \cdot \|_+)$, $C^\infty_c(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$ is dense in $(f, \| \cdot \|_+)$ and $H^2(\kappa)$ is dense in $\hat{\mathcal{H}}$ we conclude the proof.\[\square\]

**Proof of Lemma 2.6.** The definition of $v$ and $Q$ in (2.9) and Lemma [A.1] below imply that for $c = a + i b$

$$v(c \oplus 0) = a |k|^{-1/2} \overline{Q} + i b |k|^{1/2} \overline{Q} \in \mathcal{R}.$$ 

Thus $\mathcal{C} \oplus 0 \subseteq g$. Let $f \in \mathcal{J}$, we need to show that $0 \oplus f \in \text{cl}_{\| \cdot \|_+} g$. From [A.6] we deduce that $f = \mathcal{W}_+ g + W_- g$ for $g = \mathcal{W}_+ f - \mathcal{W}_- f \in \mathcal{J}$.

Since $\mathcal{R}$, defined in (2.4), is dense in $(f, \| \cdot \|_+)$, there is a sequence $(g_\nu)_\nu \subseteq \mathcal{R}$ with $g_\nu \to g$, as $\nu \to \infty$. We obtain

$$f_\nu := \mathcal{W}_+ g_\nu + W_- g_\nu \to f, \quad \text{as } \nu \to \infty.$$ 

On account of Lemma [A.6] we get $g_\nu = v(c_\nu \oplus f_\nu) \in v(g)$ for $c_\nu := (\overline{Q}_+ |g_\nu|_b + \overline{Q}_- |g_\nu|_b)$. We conclude that $c_\nu \oplus f_\nu \in g$ and $0 \oplus f_\nu \in g$. This completes the proof.\[\square\]

**Proof of Lemma 2.7.** Let

$$f := (i a |k|^{1/2} + b |k|^{-1/2}) f'$$

$$g := (i a' |k|^{1/2} + b' |k|^{-1/2}) g'.$$
and \( h \) be defined by
\[
h(r) := \int_{S^2} f'(r, \Theta) g'(r, \Theta) d\Theta, \quad r \geq 0,
\]
where \( d\Theta \) is the uniform measure on the sphere. The definition of \( f' \) and \( g' \) implies that \( \tilde{h} \) has an analytic continuation on the strip \( \{ z \in \mathbb{C} : |\text{Im} \ z| < \kappa \} \), see also (2.3). In order to calculate \( \text{Re} \langle f | \eta e^{i t |k|} | g \rangle_h \) we write the scalar product as an integral over \( \mathbb{R}^3 \). Next, we introduce polar-coordinates. After integrating over the sphere we obtain the following:
\[
\text{Re} \langle f | \eta e^{i t |k|} | g \rangle_h \\
= \text{Re} \int_0^\infty \left( aa' r^3 + bb'r + i a'br^2 - i b'ar^2 \right) \\
\cdot \coth(\beta r/2) e^{i tr} h(r) dr \\
= \frac{1}{2} \int_{\mathbb{R}} \left( aa' r^3 + bb'r + i a'br^2 - i b'ar^2 \right) \\
\cdot \coth(\beta r/2) e^{i tr} \tilde{h}(r) dr
\]
Since \( \tilde{h} \) has an analytic continuation on the strip, we may shift the contour of integration in the upper half plane. This complete the proof of (2.6). The proof of (2.7) follows analogously. □

3. COMPARISON WITH THE LIOUVILLEAN APPROACH

3.1. Liouvillean Approach. In this section we will sketch the Liouvillean approach. As mentioned in the introduction, this approach is widely used to study dynamical properties of small systems, coupled to a heat bath, see for instance [15, 16, 5, 19, 10, 11]. For this approach it is not necessary that the particle is described by a harmonic oscillator, it is applicable to a broader class of Hamiltonians \( H_{el} \). The key ingredient is the existence of a Gibbs state. For this it is sufficient and necessary that
\[
Z_\beta := \text{Tr} \{ e^{-\beta H_{el}} \} < \infty,
\]
for a fixed \( \beta > 0 \). The starting point for the model underlying the Liouvillean approach is the \( C^* \)-algebra
\[
\tilde{\mathfrak{A}} = \mathcal{B}(\mathcal{H}_{el}) \otimes W(f).
\]
The algebra \( \tilde{\mathfrak{A}} \) is taken instead of \( \mathfrak{A} \), since it is left invariant by the dynamics \( \tau^0 \) of the noninteracting system for any choice of \( H_{el} \). In general, the dynamics
\( \tau^0 \) is given by
\[
\tau^0_\tau(A) := e^{itH_0}Ae^{-itH_0}, \quad A \in \tilde{A}.
\]

\( H_0 \) is the sum of the particle Hamiltonian \( H_{el} \otimes 1 \) and the field Hamiltonian \( 1 \otimes H_f \) multiplied with a tensor-factor. More precisely, for the harmonic oscillator in the dipole approximation \( H_0 \) is obtained from \( H_h \) by setting \( H_I := 0 \) in Equation (1.6) and the particle Hamiltonian is
\[
H_{el} := \frac{1}{2}(-\Delta + (1 + \lambda^2 ||k|^{-1}\rho_{\eta}^2)q^2),
\]
acting in \( L^2(\mathbb{R}) \). We deduce from Equation (3.2), that
\[
\tau^0_\tau = \tau^el_\tau \otimes \tau^f_\tau,
\]
where \( \tau^f_\tau \) is the free dynamics, see Definition 1.3. \( \tau^el_\tau \) is dynamics for the particle system, given by
\[
\tau^el_\tau(B) = e^{iH_{el}}Be^{-iH_{el}}, \quad B \in B(H_{el}).
\]

On \( \tilde{A} \) define a state \( \omega_0 \) by
\[
\omega_0(B \otimes W[f]) := (Z^{-1} \beta \text{Tr}\{Be^{-\beta H_{el}}\}) \cdot \omega_0(W[f]).
\]

We recall, that the first factor is the Gibbs state for \( H_{el} \) at inverse temperature \( \beta \), and the second is the equilibrium state for the Bose gas system. The \((\tau^0, \beta)\)-KMS property of \( \upsilon_0 \) in the sense of Definition 1.2 can be verified by a short calculation.

In a second step, one makes an explicit GNS-construction \((\mathcal{K}, \pi_0, \Omega_0)\). Let
\[
\mathcal{K} := L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes \mathcal{F}_h[\mathbf{h} \oplus \mathbf{h}] \cong \mathcal{F}_h[\mathbb{C} \oplus \mathbb{C} \oplus \mathbf{h} \oplus \mathbf{h}]
\]
and for \( X \in B(\mathcal{K}) \)
\[
\Omega_0 = Z^{-1/2}k_{\beta/2} \otimes \Omega_{h \oplus h}, \quad \upsilon_0(X) := \langle \Omega_0 | X \Omega_0 \rangle
\]
where \( k_{\beta/2} \) is the integral kernel of \( e^{-\beta H_{el}} \) in \( L^2(\mathbb{R}^2) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \). The \(*\)-isomorphism \( \pi_0 \) is given by
\[
\pi_0[A \otimes \mathcal{W}[f]] = A \otimes 1 \otimes \mathcal{W}[(1 + \varrho)^{1/2}f \oplus \varrho^{1/2} f]
\]
Let \( \mathfrak{M} := \pi_0[\mathfrak{A}'] \) be the bicommutant of \( \pi_0[\mathfrak{A}] \), it is the weak closure of \( \pi_0[\mathfrak{A}] \) in \( B(\mathcal{K}) \). To define automorphism groups on \( \mathfrak{M} \) we introduce the operators
\[
\mathcal{L}_0 := \mathcal{L}_{el} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \mathcal{L}_f
\]
\[
\mathcal{L}_el := H_{el} \otimes 1 - 1 \otimes H_{el}
\]
\( \mathcal{L}_f \) is defined as \( H_f \) in Equation (1.5), with the difference that \( \mathbf{h} \) is replaced by \( \mathbf{h} \oplus \mathbf{h} \), and \( h_{ph} \) is given by \( |k| \oplus (-|k|) \). Since \( \Omega_0 \) is cyclic and separating for \( \mathfrak{M} \) we can define the modular conjugation by
\[
\mathcal{J} X \Omega_0 = X^* \Omega_0, \quad X \in \mathfrak{M}.
\]
As a consequence of the \((\tau^0, \beta)\)-KMS property we obtain for the commutant \( \mathfrak{M}' = \mathcal{J} \mathfrak{M} \mathcal{J} \).
In general, there is no reason that $\tau$ should leave $\tilde{A}$ invariant. However, we define a dynamics $\alpha$ on $\mathcal{M}$ by
\[ \alpha_t(X) := e^{it\mathcal{L}_h}Xe^{-it\mathcal{L}_h}, \quad X \in \mathcal{M}. \]
Here, $\mathcal{L}_h$ is the so-called Standard Liouvillian defined by
\[ (3.5) \quad \mathcal{L}_h := \mathcal{L}_0 + Q_I - \mathcal{J}Q_I\mathcal{J}. \]
$Q_I$ describes the interaction of the particle with the heat bath. In the case of the dipole approximation $Q_I$ is given by
\[ (3.6) \quad Q_I := \lambda q \otimes 1 \otimes \Phi((1 + q)^{1/2}|k|^{-1/2}\hat{\rho} \oplus \hat{g}^{1/2}|k|^{-1/2}\hat{\rho}). \]
The reason for the choice of $Q_I$ is the subsequent formal calculation
\[ (3.7) \quad \pi_0[\tau_t(B \otimes \Phi(f))] = \sum_{j=0}^{\infty} \pi_0([\ldots,[iH_t, B \otimes \Phi(f)]\ldots]) \]
\[ = \sum_{j=0}^{\infty} [\ldots,[i\mathcal{L}_t, \pi_0(B \otimes \Phi(f))]\ldots] \]
\[ = \alpha_t(\pi_0[B \otimes \Phi(f)]) \]
Note, that there is no contribution in Equation (3.7) from $JQ_IJ$, and that $\pi_0(B \otimes \Phi(f)) = B \otimes 1 \otimes \Phi((1 + q)^{1/2}f \oplus \hat{g}^{1/2}f)$. Surely, one can define $Q_I$ that is obtained from a different interaction, as long as the calculation (3.7) remains true.

Once we have $\mathcal{L}_h$, one can show self-adjointness, the invariance of $\mathcal{M}$ with respect to $\alpha_t$. Furthermore, it can be proved that
\[ \Omega = ce^{-(\beta/2)(\mathcal{L}_0 + Q_I)}\Omega_0, \]
is cyclic for $\pi_0[\tilde{A}]$, separating for $\mathcal{M}$, and normed for some $c > 0$. To show that $\Omega_0 \in \text{dom}(e^{-(\beta/2)(\mathcal{L}_0 + Q_I)})$ one needs that $|\lambda|$ is sufficiently small. For more details see [17].

Furthermore $\Omega$ is in the kernel of $\mathcal{L}_h$ and
\[ (3.8) \quad \psi(X) := \langle \Omega | X \Omega \rangle \]
and $\psi$ defines an $(\alpha, \beta)$-KMS-state on $\mathcal{M}$. It was shown in [15, 16], that the dynamical properties of $(\mathcal{M}, \alpha, \Omega)$ as the mixing property or the ergodicity are encoded in the spectrum of $\mathcal{L}_h$. But an analysis of the spectrum of $\mathcal{L}_h$ has not been done for the harmonic oscillator, yet. For a short introduction in this topic we refer the reader to [4].

However, in the case of the harmonic oscillator in the dipole approximation, the algebra is $\tilde{A}$ is left invariant by $\tau$, and the question arises for a rigorous proof of the identity in (3.7). Let $j$ be the canonical embedding of $\mathcal{A}$ into $\tilde{A}$. We will show
Lemma 3.1. The diagram

\[
\begin{array}{c}
\mathcal{A} \xrightarrow{j} \tilde{\mathcal{A}} \xrightarrow{\pi_0} \mathcal{M} \\
\downarrow \tau_t \downarrow \pi_t \\
\mathcal{A} \xrightarrow{j} \tilde{\mathcal{A}} \xrightarrow{\pi_0} \mathcal{M}
\end{array}
\]

is commutative.

This diagram illustrates, that \(\alpha\) extends the dynamics \(\tau\) on \(\mathcal{M}\). The proof is given in Subsection 3.2.

Since we have for small \(0 < |\lambda| \ll 1\) two \((\tau, \beta)\)-KMS states on \(\mathcal{A}\), namely \(\omega := \omega_f \circ \gamma\) and \(\nu \circ \pi_0 \circ j\), the question arises if both coincide. We will prove

Theorem 3.2. For small \(0 < |\lambda| \ll 1\) we have that

\[
\omega = \nu \circ \pi_0 \circ j, \quad \text{on } \mathcal{A}.
\]

This is illustrated in the next diagram

\[
\begin{array}{c}
\mathcal{A} \xrightarrow{\pi_0 \circ j} \mathcal{M} \\
\downarrow \omega \downarrow \nu \\
\mathcal{C}
\end{array}
\]

In Subsection 3.2 we will give a proof of Theorem 3.2. Next, we introduce an \(W^*\)-dynamical system for the Bose gas. This is essential to formulate our results.

Let \(K_f = \mathcal{F}_b[\hbar \oplus \hbar]\) and \(\Omega_f := \Omega_{\hbar \oplus \hbar}\). The Araki-Woods-Representation \(\pi_{AW}: \mathcal{A}_f \rightarrow \mathcal{B}(K_f)\) is given by

\[
\pi_{AW}(\mathcal{W}[g]) = \mathcal{W}[(1 + \rho)^{1/2}g \oplus \rho^{1/2}g].
\]

It can be shown that \((K_f, \pi_{AW}, \Omega_f)\) is a GNS-triple for \(\mathcal{A}_f\) and \(\omega_f\).

Let \(\mathcal{M}_f := \pi_{AW}[\mathcal{A}_f]^\prime\) and \(\nu_f(X) = \langle \Omega_f | X \Omega_f \rangle_{K_f}\), and \(\alpha_f(X) = e^{it\mathcal{L}_f} X e^{-it\mathcal{L}_f}, X \in \mathcal{M}_f\). \(\Omega_f\) is cyclic for \(\pi_{AW}[\mathcal{A}_f]\) and separating for \(\mathcal{M}_f\). The diagram is commutative

\[
\begin{array}{c}
\mathcal{A}_f \xrightarrow{\tau_f} \mathcal{A}_f \xrightarrow{\mathcal{W}[g]} \mathcal{M}_f \\
\downarrow \pi_{AW} \downarrow \pi_{AW} \\
\mathcal{M}_f \xrightarrow{\nu_f} \mathcal{C}
\end{array}
\]

in this setting. We say that \((\mathcal{M}_f, \alpha_f)\) is the \(W^*\)-dynamical system for the Bose gas. For a proof of this statements see [4].

Moreover, we have an embedding of \((\mathcal{A}, \tau, \omega)\) into \((\mathcal{M}, \alpha, \nu)\) by means of \(\pi_0 \circ j\) and an embedding into \((\mathcal{M}_f, \alpha_f, \nu_f)\) by means of \(\pi_{AW} \circ \gamma\). In fact, both embeddings are unitarily equivalent. We first show
Theorem 3.3. There is an isometric isomorphism $U : K_f \rightarrow K$, such that $U e^{it\mathcal{L}_f} = e^{it\mathcal{L}_h} U$ and $U\Omega_f = \Omega$. Let $\gamma_U : B(K_f) \rightarrow B(K)$, $\gamma_U(A) = UAU^{-1}$. Then $\gamma_U \circ \pi_{AW} \circ \gamma = \pi_0 \circ j$.

The statement of Theorem 3.3 is illustrated by

$$
\begin{array}{c}
(K_f, \Omega_f) \xrightarrow{e^{it\mathcal{L}_f}} (K_f, \Omega_f) \\
\downarrow U \hspace{1cm} \downarrow U \\
(K, \Omega) \xrightarrow{e^{it\mathcal{L}_h}} (K, \Omega) \\
\mathfrak{A} \xrightarrow{\pi_{AW}} B(K) \\
\mathfrak{A}_f \xrightarrow{\pi_{AW}} B(K_f)
\end{array}
$$

A key ingredient in the proof of Theorem 3.3 is

Lemma 3.4. $(\pi_0 \circ j)[\mathfrak{A}]'' = \mathfrak{M}$

Thus all elements of $\mathfrak{M}$ can be approximated by elements of $(\pi_0 \circ j)[\mathfrak{A}]$. From this we obtain:

Corollary 3.5. (1) The $W^*$-dynamical systems $(\mathfrak{M}, \alpha, \Omega)$ and $(\mathfrak{M}_f, \alpha_f, \Omega_f)$ are unitarily equivalent.

(2) $\mathcal{L}_h$ is unitarily equivalent to $\mathcal{L}_f$ and $\text{dom}(\mathcal{L}_h) = U \text{dom}(\mathcal{L}_f)$.

(3) $\sigma(\mathcal{L}_h) = \mathbb{R}$, $\sigma_{ac}(\mathcal{L}_h) = \emptyset$, $\sigma_{pp}(\mathcal{L}_h) = \{0\}$ and $\Omega$ is up to scalar multiples the only vector in the kernel of $\mathcal{L}_h$.

Proof. (1) Since $\gamma(\mathfrak{A}) = \mathfrak{A}_f$ and $(\pi_0 \circ j)[\mathfrak{A}] \subset \mathfrak{M}$, we have

$$
\gamma_U : \pi_{AW}[\mathfrak{A}_f] \rightarrow \mathfrak{M}.
$$

Furthermore, since $\mathfrak{M}_f$ (resp. $\mathfrak{M}$) is the $\sigma$-weak closure of $\pi_{AW}[\mathfrak{A}_f]$ (resp. $(\pi_0 \circ j)[\mathfrak{A}]$), and $\gamma_U, \gamma_{U^{-1}}$ are $\sigma$-weakly continuous, we conclude that $\gamma_U : \mathfrak{M}_f \rightarrow \mathfrak{M}$ is a spatial $*$-isomorphism.

(2) follows from $e^{it\mathcal{L}_h} = U e^{it\mathcal{L}_f} U^{-1}$.

(3) The statements are well known for $\mathcal{L}_f$ and $\Omega_f$.

3.2. Proofs. The proof of Lemma 3.1 is based on the fact that both $H_h$ and $\mathcal{L}_h$ are quadratic in the field operators. It follows that $\tau_t(\mathcal{W}[c \oplus f]) = \mathcal{W}[w_t(c \oplus f)]$ and that $\alpha_t(\mathcal{W}[c \oplus c' \oplus h \oplus \hbar]) = \mathcal{W}[\tilde{w}_t(c \oplus c' \oplus h \oplus \hbar)]$ for some real linear operator $w_t$ acting on $\mathbb{C} \oplus f$ and some real linear operator $\tilde{w}_t$ acting on $\mathbb{C} \oplus \hbar$. The proof of this follows from a theorem of Berezin for annihilation and creation operators, see [6]. In the proof of Lemma 3.1 we compare $\tilde{w}_t$ with $w_t$ on a subspace of $\mathbb{C} \oplus \mathbb{C} \oplus \hbar \oplus \hbar$.

Proof of Lemma 3.1. It is sufficient, to check that

$$
(\pi_0 \circ j)[\tau_t(\mathcal{W}[c \oplus f])] = \alpha_t((\pi_0 \circ j)[\mathcal{W}[c \oplus f]]).
$$

(3.10)
By Theorem 2.1 we have that
\begin{equation}
(\pi_0 \circ j)[\tau_t(\mathcal{W}[c \oplus f])] = \mathcal{W}(w_t(c \oplus f)^{(1)} \oplus 0
\oplus (1 + \varrho)^{1/2}w_t(c \oplus f)^{(2)} \oplus \varrho^{1/2}g_t(c \oplus f)^{(2)\nu})
\end{equation}
with \(w_t(c \oplus f) =: w_t(c \oplus f)^{(1)} \oplus w_t(c \oplus f)^{(2)}\). Since \(\pi_0 \circ j\) is a regular representation, we only need to check Equality (3.10) for field operators instead of Weyl operators. Since \(\mathcal{L}_h\) is quadratic in creation- and annihilation operators, there is a vector \(\tilde{w}_t(c \oplus c' \oplus g \oplus g')\) such that
\begin{equation}
\Phi(\tilde{w}_t(c \oplus c' \oplus g \oplus g')) = \alpha_t(\Phi(c \oplus c' \oplus g \oplus g'))
\end{equation}
Moreover, \(t \mapsto \tilde{w}_t\) is a strongly continuous one-parameter group of real linear operators in \(\mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{h} \oplus \mathfrak{h}\). The domain of its generator \(\hat{A}\) is \(\mathbb{C} \oplus \mathbb{C} \oplus \text{dom}(|k|) \oplus \text{dom}(|k|)\). Moreover,
\begin{equation}
[\mathbf{i} \mathcal{L}_h, \Phi(c \oplus c' \oplus g \oplus g')] = \Phi(\hat{A}(c \oplus c' \oplus g \oplus g')).
\end{equation}
On the other side for field operators in \(\mathcal{F}_b[\mathbb{C} \oplus \mathfrak{h}]\), \(t \mapsto w_t\) defines a strongly continuous one-parameter group of real linear operators. Let \(A\) be its generator. We have
\begin{equation}
[i H_h, \Phi(c \oplus f)] = \Phi(A(c \oplus f)),
\end{equation}
for \(c \oplus f \in \mathbb{C} \oplus \text{dom}(|k|)\). Thus it suffices to show that
\begin{equation}
\hat{A}(c \oplus 0 \oplus (1 + \varrho)^{1/2}f \oplus \varrho^{1/2}f) = A(c \oplus f)^{(1)} \oplus 0
\oplus (1 + \varrho)^{1/2}A(c \oplus f)^{(2)} \oplus \varrho^{1/2}A(c \oplus f)^{(2)}\nu)
\end{equation}
where \(A(c \oplus f) = A(c \oplus f)^{(1)} \oplus A(c \oplus f)^{(2)}\). Since \(\mathcal{J}\mathcal{Q}\mathcal{J}\) makes no contribution to Equation (3.13) for \(c' = 0\), \(g = (1 + \varrho)^{1/2}f\) and \(g' = \varrho^{1/2}f\) we can verify (3.15) using (1.6) and (3.5). □

The proof of Theorem 3.2 is the most technical in this section, while the idea is simple. Since \(\Omega\) is in the kernel of \(\mathcal{L}_h\) and by Lemma 3.1 we deduce, that
\begin{equation}
\nu \circ \pi_0 \circ j = \nu \circ \alpha_t \circ \pi_0 \circ j = \nu \circ \pi_0 \circ j \circ \tau_t.
\end{equation}
\(\nu\) is normal with respect to \(\nu_0\), by (3.8). The proof of Theorem 3.2 will be completed, if we show that we have \(\lim_{t \to \infty} \mu \circ \alpha_t \circ \pi_0 \circ j = \omega\) for all \(\nu_0\)-normal states \(\mu\). In the next statement we deduce an explicit formula for \(\nu_0 \circ (\pi_0 \circ j)\).

Lemma 3.6. For \(c \oplus f \in \mathbb{C} \oplus \mathfrak{f}\) we have that
\(\nu_0((\pi_0 \circ j)(\mathcal{W}[c \oplus f])) = \exp(-1/4\|\eta_0^{1/2}(c \oplus f)\|^2_h),\)
where \(\eta_0(c \oplus f) = (\exp(\beta \alpha) - 1)^{-1}(\text{Re}(c)\alpha^{-1/2} + \text{Im}(c)\alpha^{-1/2}) \oplus gf\) and \(\alpha^2 = 1 + \lambda^2\|k|^{-1}\hat{\rho}\|_h\).
Proof. By definition of \( \nu_0 \) it suffices to show that

\[
(\exp(\beta \alpha) - 1)Z^{-1}_\beta \text{Tr}\{\mathcal{W}[c]e^{-\beta H_{el}}\} = \left(\text{Re}(c)\alpha^{-1/2} + i \text{Im}(c)\alpha^{1/2}\right).
\]

Let us first introduce ladder operators for the harmonic oscillator

\[
B^* := \frac{\alpha^{1/2}q - i \alpha^{-1/2}p}{\sqrt{2}}, \quad B := \frac{\alpha^{1/2}q + i \alpha^{-1/2}p}{\sqrt{2}}
\]
defined on the Schwartz functions. There is up to a normalization constant unique vector \( \Omega_{\alpha} := (\alpha \pi)^{-1/4} e^{-\alpha q^2/2} \), such that \( B\Omega_{\alpha} = 0 \). We also have \( [B, B^*] = \mathbb{I}_{s(\mathbb{R})} \) and \( H_{el} := \alpha B^*B + \alpha/2 \). A complete system of eigenvectors for \( H_{el} \) is given by

\[
(n!)^{-1/2}(B^*)^n \Omega_{\alpha}, \quad n = 0, 1, 2, \ldots
\]

corresponding to the eigenvalues \( E_n = \alpha n + \alpha/2 \). Furthermore, we obtain that \( \mathcal{W}[c] = \exp(i(c'B^* + \overline{c'}/B)/\sqrt{2}) \), where \( c' := \text{Re}(c)\alpha^{-1/2} + i \text{Im}(c)\alpha^{1/2} \). The lemma follows by a general theorem about Gibbs state of operators defined by second quantization, see for instance [9, Prop. 5.28]. \( \square \)

The subsequent Lemma is essential to prove the convergence of \( \nu_0 \)-normal states to \( \omega \).

**Lemma 3.7.** We have for \( A, B \in \mathfrak{A} \) that

\[
\lim_{t \to \infty} \nu_0((\pi_0 \circ j)(B^*\tau_t(A)B)) = \nu_0((\pi_0 \circ j)(B^*B)) \cdot \omega(A)
\]

Before we will give a proof of Lemma 3.7 we show

**Lemma 3.8.** The following operators, defined as quadratic forms, on \( \mathfrak{h} \) can be extended to compact operators on \( \mathfrak{h} \)

\[
[\eta^{1/2}, W_\pm^*], \quad [\eta^{1/2}, W_\pm].
\]

Moreover, there exists compact operators \( \mathfrak{e}_1 \) and \( \mathfrak{e}_2 \) with

\[
W_\pm^*\eta W_\pm = \eta^{1/2}\mathfrak{e}_1\eta^{1/2}
\]

\[
W_\pm^*\eta W_\pm = \eta + \eta^{1/2}\mathfrak{e}_2\eta^{1/2},
\]

regarded as quadratic forms on \( \mathfrak{h} \).

**Proof.** First, assume that \( g, h \) are some sufficiently regular multiplication operators on \( L^2(\mathbb{R}^3) \). Recall the Definition of \( G_c \) and \( G \) in A.2 and A.5. On \( \mathfrak{h} \) we define the operator as a quadratic form

\[
K_c := g[\eta^{1/2}, G_c]h.
\]
This operator has the integral kernel

\[ K_\epsilon(k, k') = \frac{g(k) \eta(k)^{1/2} - \eta(k')^{1/2}}{|k|^{1/2}} h(k') - i \epsilon \frac{n(k')}{|k'|^{1/2}}. \]

By definition of \( \eta \) in Definition 1.3 we obtain

\[ \left| \frac{\eta(k)^{1/2} - \eta(k')^{1/2}}{|k| - |k'|} \right| \leq \frac{|g(k) - g(k')|}{|k| - |k'|} \eta(k)^{-1/4} \eta(k')^{-1/4} \]

\[ \leq c \eta(k)^{3/4} \eta(k')^{3/4}. \]

Thus for \( \alpha \in \{0, 1\} \) we get

\[ \int |K_\epsilon(k, k')|^2 d^3k d^3k' \leq c \left\| \frac{\eta^{3/4} \hat{g}}{|k|^{1/2}} \right\|_h^2 \cdot \left\| \frac{\eta^{3/4} \hat{h}}{|k'|^{1/2}} \right\|_h^2. \]

Thus \( K_\epsilon \) extends to a Hilbert-Schmidt operator for sufficiently regular \( g, h \).

Applying dominated convergence to the integral kernels of \( K_\epsilon \), we obtain

\[ K_\epsilon := \lim_{\epsilon \to 0} K_\epsilon \text{ in the Hilbert-Schmidt-norm } \| \cdot \|_{HS} \text{ and that } K \text{ is an extension of } g[\eta^{1/2}, G]h. \]

By the definitions of \( T^* \) and \( W^* \pm \) given in A.2 we obtain

\[ \| g[\eta^{1/2}, T^*]h \|_{HS} \leq c \left\| \frac{\eta^{3/4}}{|k|^{1/2}} \hat{g} \right\|_h \cdot \left\| \frac{\eta^{3/4}}{|k'|^{1/2}} \hat{h} \right\|_h, \]

and for \( g = h = 1 \)

\[ (3.20) \quad \| [\eta^{1/2}, W^*_\pm] \|_{HS} \leq c \left\| \frac{\eta^{3/4}}{|k|^{1/2}} \hat{\rho} \right\|_h \cdot \left\| \frac{\eta^{3/4}}{|k'|^{1/2}} \hat{\rho} \right\|_h. \]

By Hypothesis 1 we deduce that right side of (3.20) is finite. The some upper bound holds for \( W^*_\pm \) is replaced by \( W_\pm \). Since \( W^*_+ W^- \) is compact, see Lemma A.6, we obtain Equation (3.18). Equation (3.19) follows analog to Lemma A.6.

**Proof of Lemma 3.7.** By Lemma 3.6 we obtain

\[ \nu_0((\pi_0 \circ j)(W[f])) = \exp \left( -1/4 \| \eta_0^{1/2} f \|_h^2 \right) \]
for \( f \in \mathbb{C} \oplus \mathfrak{f} \). Using \( \tau_t(W[f]) = W[w_t(f)] \) and the CCR for Weyl operators we obtain

\[
v_0((\pi_0 \circ j)(W[f] \tau_t(W[g])W[h]))
= v_0((\pi_0 \circ j)(W[f]W[h])) \cdot \exp(-1/4||\eta_0^{1/2}w_t(g)||^2_h)
\cdot \exp\left(i/2\Im\langle w_t(g)|f - h\rangle_h\right)
= 1/2\Re\langle f + h|\eta_0w_t(g)\rangle_h
\]

By the Riemann-Lebesgue Lemma, we get that \( w\text{-lim}_{t \to \infty} e^{it|k|}v(g) = w\text{-lim}_{t \to \infty} \eta^{1/2} e^{it|k|}v(g) = 0 \) for \( g \in \mathbb{C} \oplus \mathfrak{f} \). From (2.10) and Lemma 3.8 follows that \( \lim_{t \to \infty} \|\langle f + h|\eta_0w_t(g)\rangle_h\| = 0 \). Moreover, for any compact operator \( \mathfrak{f} \) and any \( u \in \mathfrak{h} \) we have that \( \lim_{t \to \infty} \|\mathfrak{f} e^{-it|k|}u\|_h = 0 \). Thus Lemma 3.8 implies

\[
\lim_{t \to \infty} v_0((\pi_0 \circ j)(W[f] \tau_t(W[g])W[h])) = v_0((\pi_0 \circ j)(W[f]W[h]))
\cdot \lim_{t \to \infty} \exp\left(-\|\eta^{1/2}(\overline{W_+ e^{it|k|}v(g)} + W_- e^{-it|k|}v(g))\|^2_h/4\right)
= v_0((\pi_0 \circ j)(W[f]W[h])) \exp(-1/4\|\eta^{1/2}v(g)\|^2_h)
\]

As the linear combinations of Weyl operators are dense in \( W(\mathbb{C} \oplus \mathfrak{f}) \) we infer the convergence in Equation (3.17). \( \square \)

**Proof of Theorem 3.2.** Let \( \phi := (\pi_0 \circ j)[B]\Omega_0 \in \mathcal{K} \) with \( \|\phi\|^2_K = \omega_0(B^*B) = 1 \).

\[
|\langle \Omega|(\pi_0 \circ j)[A]\Omega\rangle_K - \langle \phi|e^{it\mathcal{L}_h}(\pi_0 \circ j)[A]e^{-it\mathcal{L}_h}\phi\rangle_K|
= |\langle \Omega|e^{it\mathcal{L}_h}(\pi_0 \circ j)[A]e^{-it\mathcal{L}_h}\Omega\rangle_K
- \langle \phi|e^{it\mathcal{L}_h}(\pi_0 \circ j)[A]e^{-it\mathcal{L}_h}\phi\rangle_K|
\leq 2\|\phi - \Omega\|_K \cdot \|A\|_A.
\]

Next, because of Lemma 3.7

\[
\lim_{t \to \infty} \langle \phi|e^{it\mathcal{L}_h}(\pi_0 \circ j)[A]e^{-it\mathcal{L}_h}\phi\rangle_K
= \lim_{t \to \infty} \omega_0(B^*\tau_t(A)B)
= \omega(A).
\]

Hence

\[
|\langle \Omega|(\pi_0 \circ j)(A)\Omega\rangle_K - \omega(A)| \leq 2\|\phi - \Omega\|_K \cdot \|A\|_A.
\]

On the other hand, by Lemma 3.4 we have

\[
\text{cl}(\pi_0 \circ j)[A]\Omega = \text{cl}(\pi_0 \circ j)[A]''\Omega = \text{cl} \mathfrak{M}\Omega = \mathcal{K}.
\]
Therefore, since \( \| \phi - \Omega \|_K \) can be chosen arbitrarily small, \( \nu((\pi_0 \circ j)(A)) = \omega(A) \) follows. \( \square \)

**Proof of Lemma 3.3.** Obviously, we have \( (\pi_0 \circ j)[\mathcal{A}]'' \subset \mathcal{M} \). We need to show that \( \mathcal{M} \) is contained in the weak closure of \( (\pi_0 \circ j)[\mathcal{A}] \).

Let \( X \in \mathcal{M}, \phi \in \mathcal{K} \) and \( \epsilon > 0 \). By the bicommutant-theorem there is a \( X' \in \pi_0[\mathcal{A}] \) such that \( \| X' \phi - X \phi \| < \epsilon \). Using linearity and density arguments we may assume that

\[
X' = A \otimes 1 \otimes (\mathcal{W}[\sqrt{1 + gf} \oplus \sqrt{gf}]), \quad A \in \mathcal{B}(\mathcal{H}_{cl})
\]

and \( \phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4 \). Since \( \mathcal{W}(\mathbb{C})'' = \mathcal{B}(\mathcal{H}_{cl}) \), the bicommutant-theorem yields the existence of \( W \in \mathcal{W}(\mathbb{C}) \) such that \( \| A\phi_1 - W\phi_1 \|_K < \epsilon \). Thus \( \| Y \phi - X \phi \| < 2\epsilon \) for \( Y := (\pi_0 \circ j)[W \otimes \mathcal{W}[f]] \).

**Proof of Theorem 4.3.** Let \( \mathcal{H}_1 := (\pi_{AW} \circ \gamma)[\mathcal{A}]\Omega_f \) and \( \mathcal{H}_2 := (\pi_0 \circ j)[\mathcal{A}]\Omega \). Since \( \Omega_f \) is separating for \( (\pi_{AW} \circ \gamma)[\mathcal{A}] \), we can define \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) by

\[
(\pi_{AW} \circ \gamma)[A]\Omega_f \mapsto (\pi_0 \circ j)[A]\Omega, \quad A \in \mathcal{A}.
\]

We observe that

\[
\begin{align*}
((\pi_0 \circ j)[A]\Omega)(\pi_0 \circ j)[B]\Omega_K &= \nu((\pi_0 \circ j)[A^*B]) \\
&= \omega(A^*B) = \omega_f((\pi_{AW} \circ \gamma)[A^*B]) \\
&= ((\pi_{AW} \circ \gamma)[A]\Omega_f)(\pi_{AW} \circ \gamma)[B]\Omega_f)_K.
\end{align*}
\]

Therefore, \( U \) is an isometric isomorphism. Moreover,

\[
\begin{align*}
e^{ilt_h}U((\pi_{AW} \circ \gamma)[A]\Omega_f) &= e^{ilt_h}(\pi_0 \circ j)[A]\Omega \\
&= e^{ilt_h}(\pi_0 \circ j)[A]e^{-ilt_h}\Omega = (\pi_0 \circ j)[\tau_t(A)]\Omega \\
&= U((\pi_{AW} \circ \gamma)[\tau_t(A)]\Omega_f) \\
&= Ue^{ilt_f}(\pi_{AW} \circ \gamma)[A]e^{-ilt_f}\Omega_f \\
&= Ue^{ilt_f}(\pi_{AW} \circ \gamma)[A]\Omega_f.
\end{align*}
\]

That is \( e^{ilt_h}U = Ue^{ilt_f} \) on \( \mathcal{H}_1 \). We can extend \( U \) to an isometric map from \( K_f \) onto \( K \), since \( \text{cl} \mathcal{H}_1 = \text{cl}(\pi_{AW} \circ \gamma)[\mathcal{A}]\Omega_f = \text{cl} \pi_{AW}[W(\mathfrak{h})]\Omega_f = K_f \) and \( \text{cl} \mathcal{H}_2 = K \).

This completes the proof. \( \square \)

4. **Anharmonic Oscillator**

The goal of this section is to prove the property of “return to equilibrium” for a slightly modified model. We will study the behavior of dynamics of an
anharmonic oscillator coupled to a heat bath for large times. The starting point
is the automorphism group
\[ t \mapsto \tau^V_t(A) := e^{itH_V}Ae^{-itH_V}, \ A \in \mathfrak{A}, \]
where \( H_V := H_h + V \). We call \( H_V \) the anharmonic Hamiltonian. \( V \) is a
bounded real-valued potential over \( \mathbb{R} \). The class of admissible potentials are
Fourier transforms of measures, that is \( V(q) = \int_{\mathbb{R}} \nu(d\mu)e^{iq\mu} \). More precisely, \( \nu \)
is a complex-valued Borel measure, such that \( \nu(A) = \nu(-A) \) for any Borel set
\( A \subset \mathbb{R} \) and \( -A = \{-a : a \in A\}. \) Let \( |\nu| \) be the absolute value of \( \nu \), we assume
that \( a_i := \int_{\mathbb{R}} |\nu|(d\mu)|\mu|^i < \infty, \ i = 0, 1, 2 \) and that
\[
(4.1) \quad \kappa > 2(a_0 + \tilde{\kappa}a_2).
\]
The number \( \tilde{\kappa} \) depends on \( \lambda \) and is given by
\[
\tilde{\kappa} := 2\pi \int_{\mathbb{R}} \frac{\lambda^2 |\hat{\rho}(r + i\kappa)|^2 |r + i\kappa|^2}{\left|G_\lambda(r + i\kappa)G_\lambda(r - i\kappa)\right|} dr.
\]
Note that Condition (4.1) only makes sense for fixed \( \lambda \neq 0 \). This inhibits to
prove Fermi’s golden rule for the time decay.

By means of Equation (1.3) we obtain \( V = \int_{\mathbb{R}} \nu(d\mu)W[\mu \oplus 0] \) in the sense
of operators.

Note that \( \tau^V_t(A) \) need not to be an element of \( \mathfrak{A} \) for \( A \in \mathfrak{A} \). So we require
an extension to a larger dynamical system. The first step in this direction is
to consider the Hamiltonian \( H_P \) and \( \tau^P_t(A) := e^{itH_P}Ae^{-itH_P}, \ A \in \mathfrak{A}P \) with
\( H_P := H_f + P \) and \( P := \int_{\mathbb{R}} \nu(d\mu)W[\nu(\mu \oplus 0)] \in \mathcal{B}(H_f) \). This definition is
motivated by the following formal calculations:

1. \( P = \gamma(V) \)
2. \( \gamma(\tau^V_t(A)) = \gamma(e^{itH_V}e^{-itH_h}) \cdot \tau^f_t(\gamma(A)) \cdot \gamma(e^{itH_h}e^{-itH_V}) \).
3. Using Dyson’s expansion we get
\[
\gamma(e^{itH_V}e^{-itH_h}) = 1 + \sum_{n=1}^{\infty} \int_0^{t} \frac{dt^1}{0 \leq t_n \leq \ldots \leq t_1 \leq t} \tau_{t_n}(V) \ldots \tau_{t_1}(V)
\]
\[
= 1 + \sum_{n=1}^{\infty} \int_0^{t} \frac{dt^1}{0 \leq t_n \leq \ldots \leq t_1 \leq t} \tau_{t_n}^f(P) \ldots \tau_{t_1}^f(P)
\]
\[
= e^{itH_P}e^{-itH_f}
\]
Likewise we obtain
\[
\gamma(e^{itH_h}e^{-itH_V}) = e^{itH_f}e^{-itH_P}.
\]
4. All together we get formally \( \gamma(\tau^V_t(A)) = \tau^P_t(\gamma(A)) \)
In a second step we define a dynamics $\alpha^Q$ on the $W^*$-algebra $\mathcal{M}_f$. The automorphism group is given by

$$\alpha^Q_t(X) := e^{itL_Q}Xe^{-itL_Q}, \quad X \in \mathcal{M}_f$$

$L_Q$ denotes the Standard Liouvillean for the dynamics $\alpha^Q$, it is given by $L_Q = L_f + Q - JQJ$, with $Q := \int_\mathbb{R} \nu(d\mu)\pi_{AW}[W[v(\mu \oplus 0)]] \in \mathcal{M}_f$. $\mathcal{J}$ is the modular conjugation corresponding to $\Omega_f$. Note that $Q \in \mathcal{M}_f$ and since $\mathcal{M}_f$ is a $W^*$-algebra, we obtain $\alpha^Q_t(X) \in \mathcal{M}_f$ for $X \in \mathcal{M}_f$. The definition of $\alpha^Q$ can be motivated in the same way as that of $\alpha$ in Section 3. Moreover, a $(\alpha^Q, \beta)$-KMS state is given by $\omega^Q$, where $\omega^Q(A) := \langle \Omega^Q | A \Omega^Q \rangle_{K_f}$, $\Omega^Q := ce^{-\beta/2(L_f + Q)}\Omega_f$. $c > 0$ is a normalization constant. We remark that $\alpha^Q_t(A) = \alpha_f^t(A) + \sum_{n=1}^{\infty} \int_0^t dt_1 \cdots dt_n \alpha_{t-t_n}^f(Q)$, $\alpha^Q_t(A) = \alpha^f_t(A) + \sum_{n=1}^{\infty} \int_0^t dt_1 \cdots dt_n \alpha_{t-t_n}^f(Q)$.

**Definition 4.1.** We define

1. The observables of the small systems are $(\pi_{AW} \circ \gamma)[W(\mathbb{C} \oplus 0)]$.
2. The analytic observables are $\mathcal{M}_a^f := (\pi_{AW} \circ \gamma)[\mathcal{A}_0]$.
3. The analytic states are given by $\mathcal{S}_a := \{\mu : \exists A \in \mathcal{M}_a^f \forall X \in \mathcal{M}_f \mu(X) = \langle A \Omega_f | X A \Omega_f \rangle\}$

We will prove the following:

**Theorem 4.2.** The $W^*$-dynamical system $(\mathcal{M}_f, \alpha^Q, \omega^Q)$ is mixing. Moreover, we have for $\mu \in \mathcal{S}_a$ and $A \in \mathcal{M}_a^f$

$$|\mu(\alpha^Q_t(A)) - \omega^Q(A)| \leq c \exp(-(\kappa - 2(a_0 + \tilde{\kappa}a_2))t),$$

where $c$ is some constant depending on $\mu, A$ and $Q$, and for $\tilde{\kappa}$ defined in (4.1).

4.1. **Auxiliary Statements.** The proof of Theorem 4.2 is splitted in three parts. The first is

**Lemma 4.3.** For $A, B, C \in \mathcal{M}_f^a$ we have

$$|v_f(A\alpha^Q_t(B)C) - v_f(AC)\tilde{v}(B)| \leq c \exp(-(\kappa - 2(a_0 + \tilde{\kappa}a_2))t),$$

where $\tilde{v}(B) := \lim_{t \to \infty} v_f(\alpha^Q_t(B))$.

The second part is

**Lemma 4.4.** $(\mathcal{M}_f^a)^\prime = \mathcal{M}_f$.
Now and in the sequel we omit the arguments of \( f \) by Lemma 2.6, there exists for every \( f \in \mathfrak{f} \) a sequence \((f_n)_n \subset v(\mathfrak{g})\), so that
\[
\sqrt{1 + \varrho f_n} + \sqrt{\varrho f_n} \to \sqrt{1 + \varrho f} + \sqrt{\varrho f}, \quad n \to \infty,
\]
where the limit is taken with respect to the norm of \( \mathfrak{h} \oplus \mathfrak{h} \). It follows, that
\[
\text{s-lim}_{n \to \infty} \mathcal{W}[\sqrt{1 + \varrho f_n} + \sqrt{\varrho f_n}] = \mathcal{W}[\sqrt{1 + \varrho f} + \sqrt{\varrho f}].
\]
We conclude that \((\mathfrak{M}_f^a)^{''} = \text{LH}\{\mathcal{W}[\sqrt{1 + \varrho f} + \sqrt{\varrho f}] : f \in \mathfrak{f}\}^{''} = \mathfrak{M}_f\). □

The third part is

**Corollary 4.5.** For every \( \omega_Q \)-normal state \( \mu \) over \( \mathfrak{M}_f \) and every \( C \in \mathfrak{M}_f \) we obtain \( \lim_{t \to \infty} \mu(\alpha_f^t(C)) = \omega_Q(C) \). Hence \( (\mathfrak{M}_f, \alpha, \omega_Q) \) is mixing.

### 4.2. Proofs

In the following \( c \) denotes a constant independent of \( t > 0 \). The value of \( c \) may change from line to line. In this subsection the norms and scalar products without subscript belong to \( \mathcal{K} \) or to \( \mathcal{B}(\mathcal{K}) \).

**Proof of Lemma 4.3.** First, we prove Estimate (4.4) for \( A := \pi_{AW}[\mathcal{W}[f]], B := \pi_{AW}[\mathcal{W}[g]], C := \pi_{AW}[\mathcal{W}[h]] \) for \( f, g, h \in v(\mathfrak{g}) \). Note, that \( e^{it\varrho q} = \mathcal{W}[\mu \otimes 0] \) and let \( v_k := v(\mu_k \otimes 0) \). Recalling the statements of Diagram (2.2) and (3.9) we get
\[
\alpha_f^{t-t_k}(P) = \int \nu(d\mu_k)\pi_{AW}[\mathcal{W}[e^{it(t-t_k)|k|}v_k]].
\]
Employing the CCR for Weyl operators we obtain
\[
[\mathcal{W}[f], \mathcal{W}[g]] = (-2i) \sin ((1/2)\text{Im} \langle f|g \rangle_h)\mathcal{W}[f + g].
\]
The commutator expression in Equation (4.3) equals
\[
c_n\pi_{AW}\left(\mathcal{W}[e^{it|k|}u_n]\right) = \left[\pi_{AW}\left(\mathcal{W}[e^{i(t-t_1)|k|}v_1]\right), \ldots \right. \ldots \left. \left[\pi_{AW}\left(\mathcal{W}[e^{i(t-t_n)|k|}v_n]\right), \pi_{AW}\left(\mathcal{W}[e^{it|k|}g]\right)\right] \ldots \right]
\]
with
\[
c_n(L, \mu) := (-2i)^n \prod_{k=1}^n \sin \left(\text{Im} \langle e^{-it_k|k|}v_k|u_{k,n}\rangle_h/2\right)
\]
\[
u_k, n(L, \mu) := \sum_{m=k+1}^n e^{-it_m|k|}v_m + g, \quad u_n, n(L, \mu) := \mu
\]
Now and in the sequel we omit the arguments of \( c_n, u_{k,n}, u_n \). Let \( \nu(d\mu) := \nu(d\mu_1) \otimes \ldots \otimes \nu(d\mu_n) \) the \( n \)-fold product measure of \( \nu \). In the following we will
use the same symbol for different $n$. The equation in (4.4) reads

\[(4.5)\quad v_f(A\alpha_t^Q(B)C) = \omega_f(W[f]\tau_f^t(W[g])W[h])
+ \sum_{n=1}^{\infty} \int_{0 \leq t_n \leq ... \leq t} dt \cdot \int \nu(d\mu)c_n \cdot \omega_f(W[f]W[e^{i|k|}u_n]W[h]).\]

Moreover, for the expectation in $\omega_f$ in the line before we find

\[(4.6)\quad \omega_f(W[f]W[e^{i|k|}u_n]W[h]) = e^{\Delta_n}\omega_f(W[f]W[h])\omega_f(W[u_n]),\]

with

\[\Delta_n(t,t,\mu) := -(1/2)\text{Re} \langle f + h|\eta e^{i|k|}u_n\rangle_h
- (i/2)\text{Im} \langle f - h|e^{i|k|}u_n\rangle_h.\]

Note, that $|e^{\Delta_n} - 1| \leq |\Delta_n| e^{\text{Re} \Delta_n}$ and

\[(4.7)\quad |\omega_f(W[f]W[h])\omega_f(W[u_n]) e^{\text{Re} \Delta_n}| \leq 1.\]

Using Equation (2.13) we obtain an exponentially fast decay with rate $\kappa$ for

\[\omega_f(W[f]\tau_f^t(W[g])W[h]) - \omega_f(W[f]W[h])\omega_f(\tau_f^t(W[g])).\]

Moreover, an explicit expression for $v_f(\alpha_t^Q(B))$ is obtained by setting $f = 0 = h$ in (4.5). Combining this with (4.6) and (4.7) we obtain for some constant

\[(4.8)\quad |v_f(A\alpha_t^Q(B)C) - v_f(AC)v_f(\alpha_t^Q(B))| \leq ce^{-\kappa t} + \sum_{n=1}^{\infty} \int_{0 \leq t_n \leq ... \leq t} dt \int |\nu|(d\mu)|c_n\Delta_n|.

Here, $|\nu|$ is the absolute value of the measure $\nu$. The constant depends only on $f, g, h$. Recalling Lemma 2.7 and the definition of $\Delta_n$ and $u_n$

\[(4.9)\quad |\Delta_n| \leq c\left(\sum_{i=1}^{n} e^{-(t-t_i)\kappa}|\mu_i| + e^{-\kappa t}\right).

for some constant depending on $f, g, h$. Furthermore, estimate 3.1 yields

\[(4.10)\quad |c_n| \leq ce^{-\kappa t_1}|\mu_1| \prod_{k=i+1}^{n} (1 + \tilde{\kappa}\mu_k^2), \quad i = 1, 2, \ldots, n\]
for some constant. Note, that \( \tilde{\kappa} \) is independent of \( f, g, h \). We insert Estimate (4.9) in (4.8), depending on the summands in (4.9) we use a different \( i \) in Estimate (4.10). Thus

\[
|v_f(AQ_t(B)C) - v_f(AC) v_f(AQ_t(B))| \\
\leq c e^{-\kappa t} \sum_{n=1}^{\infty} \int_{0 \leq t_n \leq ... \leq t_1} dt \\
\times \int |\nu|(d\mu) \left( \sum_{i=1}^{n} \mu_i^2 \prod_{k=i+1}^{n} (1 + \tilde{\kappa} \mu_k^2) + 1 \right)
\leq c e^{-\kappa t} \left( 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int |\nu|(d\mu) \prod_{k=1}^{n} (1 + \tilde{\kappa} \mu_k^2) \right)
\leq c \exp(-t(\kappa - a_0 - \tilde{\kappa} a_2))
\]

To finish the proof we need to show that the convergence \( \lim_{t \to \infty} v_f(AQ_t(B)) \) is exponentially fast. Therefore we shall compare \( v_f(AQ_t(B)) \) and \( v_f(AQ_s(B)) \) for \( 0 < t < s \). Again, recall the \( v_f(AQ_t(B)) \) is calculated in (4.5) (for \( f = 0 = h \)). Using that \( \omega_f \) is \( \tau^f \)-invariant we obtain

\[
|v_f(AQ_s(B)) - v_f(AQ_t(B))| \leq \sum_{n=1}^{\infty} \int_{0 \leq t_n \leq ... \leq t_1 \leq s} dt \\
\times 1[t_1 \geq t] \int |\nu|(d\mu)|c_n| \omega_f(W[u_n])|
\]

Since \( W[u_n] \) is a unitary we get \( |\omega_f(W[u_n])| \leq 1 \). Employing Estimate (4.10) for \( i = 1 \) we get

\[
|v_f(AQ_s(B)) - v_f(AQ_t(B))| \\
\leq \sum_{n=1}^{\infty} c 2^n \int_{0 \leq t_n \leq ... \leq t_1 \leq s} dt \times 1[t_1 \geq t] (a_0 + \tilde{\kappa} a_2)^{n-1} e^{-\kappa t_1} \\
= \sum_{n=1}^{\infty} c 2^n \int_t^{s} \frac{(r(a_0 + \tilde{\kappa} a_2))^{n-1}}{(n-1)!} e^{-\kappa r} dr \\
\leq c \int_t^{\infty} \exp(-(\kappa - 2(a_0 + \tilde{\kappa} a_2))r) dr \\
\leq c \exp(-(\kappa - 2(a_0 + \tilde{\kappa} a_2))t).
\]

We define \( \tilde{\nu}(B) := \lim_{t \to \infty} v_f(AQ_t(B)) \) using the Cauchy-criterion. \( \square \)
Proof of Corollary 4.5. Let now $\phi \in K_f$, $\|\phi\| = 1$ and $A, B \in M_f^\gamma$, so that $\|A\Omega\| = 1$. 

$$|\langle \phi | \alpha_t^Q(B) \phi \rangle - \hat{\nu}(B)| 
\leq |\langle \phi | \alpha_t^Q(B) \phi \rangle - \langle A\Omega_f | \alpha_t^Q(B) A\Omega_f \rangle| 
+ |\omega_f(A^* \alpha_t^Q(B) A) - \hat{\nu}(B)| 
\leq 2\|A\Omega_f - \phi\| \cdot \|B\| + |\omega_f(A^* \alpha_t^Q(B) A) - \hat{\nu}(B)|.$$

We obtain directly 

$$\limsup_{t \to \infty} |\langle \phi | \alpha_t^Q(B) \phi \rangle - \hat{\nu}(B)| 
\leq 2\|B\| \inf \{\|A\Omega_f - \phi\| : A \in (\pi_{AW} \circ \gamma)(\mathfrak{A})_0, \|A\Omega_f\| = 1\} = 0.$$ 

Choosing $\phi = \Omega_Q$ we obtain by time-invariance of KMS states, that $\omega_Q = \hat{\nu}$ over $M_f^\gamma$. Assume now $C \in M_f$, $(C_n)_n \in M_f^\gamma$ and $D \in M_f^\gamma$, so that $C = \operatorname{s-lim}_{n \to \infty} C_n$. Since $L_Q\Omega_Q = 0$ we have that 

$$\langle D\Omega_Q | \alpha_t^Q(C - C_n) D\Omega_Q \rangle = \langle D^* D\Omega_Q | \alpha_t^Q(C - C_n) \Omega_Q \rangle 
= \langle D^* D\Omega_Q | e^{itL_Q}(C - C_n) \Omega_Q \rangle.$$ 

Hence, 

$$\limsup_{t \to \infty} |\omega_Q(C) - \langle D\Omega_Q | \alpha_t^Q(C) D\Omega_Q \rangle| 
\leq \limsup_{t \to \infty} |\omega_Q(C_n) - \langle D\Omega_Q | \alpha_t^Q(C_n) D\Omega_Q \rangle| 
+ \limsup_{t \to \infty} |\omega_Q(C - C_n) - \langle D\Omega_Q | \alpha_t^Q(C - C_n) D\Omega_Q \rangle| 
\leq (1 + \|D^* D\|) \cdot \|(C - C_n)\Omega_Q\|.$$

For $n \to \infty$ we obtain $\omega_Q(C) = \lim_{t \to \infty} \langle D\Omega_Q | \alpha_t^Q(C) D\Omega_Q \rangle$. Since $\Omega_Q$ is separating for $M_f$, it is cyclic for $(M_f^\gamma)'$. A approximation argument yields $\omega_Q(C) = \lim_{t \to \infty} \langle \phi | \alpha_t^Q(C) \phi \rangle$, for all $\phi \in K_f$. Note, that any normal state $\mu$ over $M_f$ has a vector representative $\phi \in K_f$, since $\omega_Q$ is a KMS state. \qed

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Appendix A. A summary of results and definitions in [1] [2]

In this section we recall definitions and statements, that are made in [1] [2]. The most important result is the formula for the asymptotic creation and annihilation operators given in Lemma A.7. It is the starting point of our analysis.

Definition A.1. For $z \in \mathbb{C} \setminus [0, \infty)$ we get

\[
D(z) := -z + 1 + \lambda^2 \|k\|^{-1} \hat{\rho}^2 + \lambda^2 \int d^3 k \frac{\hat{\rho}(k)^2}{z - k^2},
\]

\[
D_\pm(r) := \lim_{\epsilon \to 0^+} D(r + i \epsilon), \quad r \in [0, \infty)
\]

\[
Q(k) := -\lambda \frac{\hat{\rho}(k)}{D_\pm(k^2)}
\]

\[
Q_\pm(k) := (1/2)(|k|^{1/2} \pm |k|^{-1/2})Q(k).
\]

Furthermore, we define the following operators on $\mathfrak{h}$.

Definition A.2.

(A.1) \[(G_\epsilon g)(k) := \int \frac{g(k')}{(|k||k'|)^{1/2}(k^2 - k'^2 + i \epsilon)} d^3 k'.\]

(A.2) \[G := \lim_{\epsilon \to 0^+} G_\epsilon.\]

(A.3) \[Tg := g + \lambda |k|^{1/2} QG |k|^{1/2} \hat{\rho} g\]

(A.4) \[T^*g := g - \lambda |k|^{1/2} \hat{\rho} G |k|^{1/2} \bar{Q} g\]

(A.5) \[W_+g := (1/2) \{|k|^{-1/2} T^* |k|^{1/2} + |k|^{1/2} T^* |k|^{-1/2}\} g\]

(A.6) \[W_-g := (1/2) \{|k|^{-1/2} T^* |k|^{1/2} - |k|^{1/2} T^* |k|^{-1/2}\} g\]

For the functions $D$ and $D_\pm$ we summarize some results in

Lemma A.3. (1) $D$ is analytic in $\mathbb{C} \setminus [0, \infty)$,

(2) $D_\pm(s) := \lim_{\epsilon \to 0^+} D(s + i \epsilon)$ exists and is continuous for $s \in [0, \infty)$,

(3) $\inf_{s \in [0, \infty)} |D_\pm(s)| > 0$,

(4) $|D(z) + z| < c_1$ and $|D(z)| > c_2$ for all $z \in \mathbb{C} \setminus [0, \infty)$ and real constants $c_1$ and $c_2$.

Let $M_\alpha(\mathbb{R}^3) = \{f : \|f\|_\alpha = \|k^\alpha f\|_\mathfrak{h} < \infty\}$, for $\alpha \in \mathbb{R}$. For the operators introduced in Definition [A.2] we have:

Lemma A.4. (1) $G_\epsilon$ is bounded on $\mathfrak{h}$, uniformly for $\epsilon > 0$.

(2) $G := \text{slim}_{\epsilon \to 0^+} G_\epsilon$ exists as an operator on $\mathfrak{h}$.

(3) $G$ is bounded on $\mathfrak{h}$ and $M_{-1/2}(\mathbb{R}^3)$.

(4) $G^* = -G$, i.e $G$ is skew-symmetric on $\mathfrak{h}$. 

30
Given a bounded operator $A$ on $\mathfrak{h}$ we denote by $\overline{A}$ an operator acting on $g \in \mathfrak{h}$ by means of $(\overline{Ag})(k) := (Ag)(k)$. The bar is of course the complex conjugation.

**Lemma A.5.**

1. $T$ and $T^*$ (see Definition A.2) are bounded on $M_\alpha(\mathbb{R}^3)$ for $\alpha = 1/2, 0, -1$.
2. $T^*$ is the adjoint of $T$.
3. For a rotation invariant function $h$ on $\mathbb{R}^3$, we have $T^* h T = \overline{T} h \overline{T}$.
4. Furthermore, if $hQ \in \mathfrak{h}$, then $T^* h Q = \overline{T} h Q$.
5. $T^* Q = 0$ and $\|Q\|_\mathfrak{h} = 1$.

The next algebraic relations ensure that the incoming creation- and annihilation operators fulfill the CCR.

**Lemma A.6.** The operators $W_+$ and $W_-$ defined in Definition A.2 are bounded on $M_\alpha(\mathbb{R}^3)$ for $\alpha = -1/2, 0$ and fulfill

$$W_+^* W_+ - W_-^* W_- + P_+ - P_- = 1,$$
$$W_+ W_+^* - \overline{W_- W_-^*} = 1,$$
$$\overline{W_+} W_- - \overline{W_-} W_+ + P_+ - P_- = 0,$$
$$W_- W_+^* - \overline{W_+ W_-^*} = 0,$$

where

$$P_\pm f = \langle Q\pm | f \rangle_\mathfrak{h} Q_\pm, \quad P_{\pm f} = \langle Q_\mp | f \rangle_\mathfrak{h} \overline{Q}_\mp.$$

Furthermore, $W_-$ is a Hilbert-Schmidt operator with integral kernel

$$W_-(k, k') = \frac{\lambda \hat{\rho}(k) Q(k')}{2(|k| |k'|)^{1/2}(|k| + |k'|)}.$$

The starting point of our work is the following result:

**Lemma A.7.** Let $H_0$ be $H_\mathfrak{h}$ with $\lambda = 0$. The asymptotic creation- and annihilation-operators $a_{\text{in}}^\#(f)$ exist for $f \in M_0(\mathbb{R}^3) \cap M_{-1/2}(\mathbb{R}^3)$,

$$a_{\text{in}}^\#(f) = \text{s-lim}_{t \to -\infty} e^{itH_0} e^{-i t H_0} a^\#(f) e^{i t H_0} e^{-i t H_0},$$

$$\text{dom}(a_{\text{in}}^\#(f)) \supset \text{dom}(H_\mathfrak{h})$$

for

$$a_{\text{in}}(f) = \langle Q_- | \overline{f} \rangle_\mathfrak{h} A^* + \langle Q_+ | \overline{f} \rangle_\mathfrak{h} A + a^*(W_- f) + a(\overline{W}_+ f),$$
$$a_{\text{in}}^*(f) = \langle \overline{f} | Q_- \rangle_\mathfrak{h} A + \langle \overline{f} | Q_+ \rangle_\mathfrak{h} A^* + a(W_- f) + a^*(\overline{W}_+ f).$$

Moreover, $a_{\text{in}}(f)$ and $a_{\text{in}}^*(g)$ fulfill the CCR for $f, g \in \mathfrak{h}$. 

31
APPENDIX B.

The next lemma was originally proved in [18].

**Lemma B.1.** Let $f, f_1, \ldots, f_n$ be vectors in a Hilbert space $\mathcal{H}$, and real numbers $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and $\alpha, \beta, \gamma > 0$, such that

\[
|\text{Im} \langle f_k | f \rangle| \leq \alpha |\lambda_k| e^{-t_k \gamma}, \quad k = 1, \ldots, n
\]

\[
|\text{Im} \langle f_k | f_j \rangle| \leq \beta |\lambda_k \lambda_j| \cdot e^{-(t_k - t_j) \gamma}, \quad n > k > j \geq 1.
\]

Then for $j = 1, \ldots, n$ we get

\[
A(j) := \left| \prod_{k=j}^{n} \sin \left( \sum_{m=k+1}^{n} \text{Im} \langle f_k | f_m \rangle + \text{Im} \langle f_k | f \rangle \right) \right|
\]

\[
\leq e^{-\gamma t_j} |\lambda_j| \prod_{k=j+1}^{n} (1 + \beta \lambda_k^2).
\]

We use the convention that $\sum_{m=n+1}^{n}(*) = 0$ and $\prod_{m=n+1}^{n}(*) = 1$.

**Proof.** First we remark that for $x, y \in \mathbb{R}$

\[
|\sin(x + y)| \leq |\sin(x)| + |\sin(y)|, \quad |\sin(x)| \leq |x|, \quad |\sin(x)| \leq 1
\]

We proceed by induction for $j$. We assume $j \leq n - 1$ and that $A(j+1), \ldots, A(n)$ obey Estimate (B.1). Since $0 \leq A(i) \leq A(i + 1) \leq 1$ for all $i = 1, \ldots, n$ we get

\[
A(j) \leq A(j + 1) \cdot \left( \sum_{m=j+1}^{n} |\sin(\text{Im} \langle f_j | f_m \rangle)| \right.
\]

\[
\left. + |\sin(\text{Im} \langle f_j | f \rangle)| \right)
\]

\[
\leq \sum_{m=j+1}^{n} \left| \sin(\text{Im} \langle f_j | f_m \rangle) \right| A(m) + \left| \sin(\text{Im} \langle f_j | f \rangle) \right|
\]

\[
\leq \sum_{m=j+1}^{n} \left( \beta |\lambda_m \lambda_j| \cdot e^{-(t_j - t_m) \gamma} \right)
\]

\[
\left( e^{-\gamma t_m} |\lambda_m| \prod_{k=m+1}^{n} (1 + \beta \lambda_k^2) \right) + \alpha |\lambda_j| e^{-t_j \gamma}.
\]

Since the right-hand side (r.h.s) of Equation (B.2) is less than the (r.h.s) of Equation (B.1), we obtain Lemma B.1. \[\square\]
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