OPTIMAL ERROR ESTIMATES FOR CHEBYSHEV APPROXIMATIONS OF FUNCTIONS WITH LIMITED REGULARITY IN FRACTIONAL SOBOLEV-TYPE SPACES

WENJIE LIU$^{1,2}$, LI-LIAN WANG$^2$ AND HUIYUAN LI$^3$

Abstract. In this paper, we introduce a new theoretical framework built upon fractional Sobolev-type spaces involving Riemann-Liouville (RL) fractional integrals/derivatives, which is naturally arisen from exact representations of Chebyshev expansion coefficients, for optimal error estimates of Chebyshev approximations to functions with limited regularity. The essential pieces of the puzzle for the error analysis include (i) fractional integration by parts (under the weakest possible conditions), and (ii) generalised Gegenbauer functions of fractional degree (GGF-Fs): a new family of special functions with notable fractional calculus properties. Under this framework, we are able to estimate the optimal decay rate of Chebyshev expansion coefficients for a large class of functions with interior and endpoint singularities, which are deemed suboptimal or complicated to characterize in existing literature. We can then derive optimal error estimates for spectral expansions and the related Chebyshev interpolation and quadrature measured in various norms, and also improve the available results in usual Sobolev spaces of integer regularity exponentials in several senses. As a by-product, this study results in some analytically perspicuous formulas particularly on GGF-Fs, which are potentially useful in spectral algorithms. The idea and analysis techniques can be extended to general Jacobi spectral approximations.

1. Introduction

It is known that polynomial approximation theory is of fundamental importance in numerical analysis and algorithm development of many computational methods, e.g., $p/hp$ finite elements or spectral/spectral-element methods (see, e.g., [16, 31, 8, 13, 21, 32] and references therein). Typically, the documented approximation results take the form

$$
\|Q_Nu - u\|_{S_1} \leq cN^{-\sigma}|u|_{S_1}, \quad \sigma \geq 0,
$$

(1.1)

where $Q_N$ is an orthogonal projection (or interpolation operator) upon the set of all polynomials of degree at most $N$, and $c$ is a positive constant independent of $N$ and $u$. In (1.1), $S_1$ is a certain Sobolev space, $B_r$ is a related Sobolev or Besov space, and $\sigma$ depends on the regularity exponentials of both $B_r$ and $S_1$. In practice, one would expect (a) the space $B_r$ should contain the classes of functions as broad as possible; and (b) the space $B_r$ can best characterise their regularity leading to optimal order of convergence. In general, the space $B_r$ is of the following types.

(i) $B_r$ is the standard weighted Sobolev space $H^m_\omega(\Omega)$ with integer $m \geq 0$ and certain weight function $\omega(x)$ on $\Omega = (-1, 1)$ (see, e.g., [8, 13, 21]). However, it could not lead to optimal order for functions with endpoint singularities (see, e.g., [8, 20]) or with interior singularities, e.g., $|x|$ (see [34]).
(ii) $B_t$ is the non-uniformly Jacobi-weighted Sobolev space (see, e.g., [10, 11, 12, 20, 18, 32]). For example, $B_t = H^{m,\beta}(\Omega)$ with integer $m \geq 0$ and $\beta > -1$, is defined as a closure of $C^\infty$-functions endowed with the weighted norm

$$\|u\|_{H^{m,\beta}(\Omega)} = \left\{ \sum_{k=0}^m \int_{-1}^1 |u^{(k)}(x)|^2 (1-x^2)^{\beta+k} dx \right\}^{1/2},$$

(1.2)

Compared with the standard Sobolev space in (i), such spaces can better describe the endpoint singularities, but still produce suboptimal estimates for $(1+x)\alpha$-type singular functions with non-integer $\alpha > 0$ (cf. [13, P. 474]). Indeed, for the Chebyshev approximation, we find that

$$u = (1+x)^\alpha \in H^{m,-1/2}(\Omega)$$

with integer $m < 2\alpha + 1/2$, and

$$\|\pi_C^u u - u\|_{L^2(\Omega)} \leq cN^{-m}\|u\|_{H^{m,-1/2}(\Omega)},$$

(1.3)

where $\pi_C^u$ is the $L^2$-orthogonal projection of $u$ (with $\omega = (1-x^2)^{-1/2}$). However, the expected optimal order is $O(N^{-2\alpha-1/2})$, so the loss of an order of the fractional part of $2\alpha + 1/2$ or one order (when $2\alpha = k + 1/2$ with $k \in \mathbb{N}_0$), is inevitable under this framework. This is due to the space $H^{m,\beta}(\Omega)$ is only defined for integer $m \geq 0$.

(iii) In a series of works [3, 5, 6], Babuška and Guo introduced the Jacobi-weighted Besov space defined by space interpolation based on the so-called K-method. One commonly used Besov space for $(1+x)^\alpha$-type corner singularities is $B^{s,\beta}_{2,2}(\Omega) = (H^{l,\beta}(\Omega), H^{m,\beta}(\Omega))_{s,2}$ with integers $l < m$ and $s = (1-\theta)l + \theta m$, $\theta \in (0,1)$, equipped with the norm

$$\|u\|_{B^{s,\beta}_{2,2}(\Omega)} = \left( \int_0^\infty t^{-2\theta} |K(t, u)|^2 dt \right)^{1/2}, \quad K(t, u) = \inf_{v = u + w} \left( \|v\|_{H^{l,\beta}(\Omega)} + t \|w\|_{H^{m,\beta}(\Omega)} \right).$$

(1.4)

However, to deal with $(1+x)^\alpha \log^\nu(1+x)$-type corner singularities, Babuška and Guo had to further modify the K-method by incorporating a log-factor into the norm.

The aforementioned framework might lead to suboptimal estimates for functions with interior singularities. For example, we consider $u(x) = |x|$ and note that $u'(x) = 2H(x) - 1$ and $u''(x) = 2\delta(x)$ (where $H, \delta$ are respectively the Heaviside function and the Dirac delta function). Since $u'' \notin L^2(\Omega)$, the Chebyshev approximation of $|x|$ has a convergence:

$$\|\pi_C^u u - u\|_{L^2(\Omega)} \leq cN^{-1}\|u\|_{H^{1,\beta}(\Omega)},$$

(1.5)

but the expected optimal order is $O(N^{-3/2})$ (cf. [34, 35]). In fact, as shown in [34, Thms 4.2-4.3] and [35 Thms 7.1-7.2] (also see Lemma 5.1 below), one should choose $B_t \subset BV(\Omega)$ (the space of functions of bounded variation) to achieve optimality (see Section 6 and refer to [34, 36, 38, 24] for more details). Unfortunately, the Sobolev spaces therein were defined through integer-order derivatives, so they could not best characterise the regularity of e.g., $u(x) = |x|^\alpha$ with non-integer $\alpha > 0$. In other words, the order of convergence can only be suboptimal.

In this paper, we intend to introduce a new framework of fractional Sobolev-type spaces that can meet the two requirements (a)-(b) and overcome the deficiencies mentioned above. We focus on the Chebyshev approximation but the analysis techniques are extendable to general Jacobi approximations. Here, we put the emphasis on estimating the decay rate of expansion coefficients for the reason that the errors of spectral expansions in various norms, and the related interpolation and quadratures can be estimated directly from the sums of the coefficients (cf. [34, 24]). The essential ideas and main contributions of this study are summarised as follows.

(i) We derive the exact representation of the Chebyshev expansion coefficients (see Theorem 4.1) by using the fractional calculus properties of GGF-Fs and fractional integration by parts (under the weakest possible conditions). This allows us to naturally define the fractional Sobolev spaces to characterise the regularity of a large class of singular functions, leading to optimal order of convergence.

(ii) When the fractional regularity exponential $s \to 1$, our results improve the existing bounds in usual Sobolev spaces (see, e.g., [34, 38, 36, 24]). In this sense, the fractional Sobolev-type space with
regularity exponential \( m + s \ (s \in (0, 1) \text{ and integer } m \geq 0) \) can be regarded as an intermediate space inbetween the spaces with regularity exponentials \( m + 1 \) and \( m \) in [35].

(iii) We provide some useful analytical formulas on fractional calculus of GGF-Fs, and the Chebyshev expansions of some specific singular functions. Some of them are new or difficult to be derived by other means (cf. [37]). They can also be useful in the design of spectral algorithms.

The paper is organised as follows. In Sections 2-3, we introduce the GGF-Fs, and present their important properties, including the uniform bounds and RL fractional integral/derivative formulas. We derive the main results in Section 4 and improve the existing estimates in Sobolev spaces with integer-order derivatives in Section 5. We discuss in Section 6 the extension of the main results to the analysis of interpolation, quadrature and endpoint singularities.

2. Generalised Gegenbauer functions of fractional degree

In this section, we collect some relevant properties of the hypergeometric functions and Gegenbauer polynomials, upon which we define the GGF-Fs and derive their relevant properties. These pave the way for the forthcoming error analysis.

2.1. Hypergeometric functions and Gegenbauer polynomials. Let \( \mathbb{Z} \) and \( \mathbb{R} \) be the sets of all integers and real numbers, respectively, and denote

\[
\mathbb{N} = \{ k \in \mathbb{Z} : k \geq 1 \}, \quad \mathbb{N}_0 := \{ 0 \} \cup \mathbb{N}, \quad \mathbb{R}^+ := \{ a \in \mathbb{R} : a > 0 \}, \quad \mathbb{R}^+_0 := \{ 0 \} \cup \mathbb{R}^+.
\]

For \( a \in \mathbb{R} \), the rising factorial in the Pochhammer symbol is defined by

\[
(a)_0 = 1; \quad (a)_j = a(a + 1) \cdots (a + j - 1), \quad \forall j \in \mathbb{N}.
\]

The hypergeometric function is a power series, defined by (cf. [3])

\[
\begin{align*}
_2F_1(a, b; c; z) &= \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} z^j = 1 + \sum_{j=1}^{\infty} \frac{a(a+1) \cdots (a+j-1) b(b+1) \cdots (b+j-1)}{1 \cdot 2 \cdots j (c+1) \cdots (c+j-1)} z^j, \\
&= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.
\end{align*}
\]

where \( a, b, c \in \mathbb{R} \) and \( -c \not\in \mathbb{N}_0 \). The series converges absolutely for all \( |z| < 1 \), and apparently, we have

\[
_2F_1(a, b; c; 0) = 1, \quad _2F_1(a, b; c; z) = _2F_1(b, a; c; z).
\]

If \( a = -n \) with \( n \in \mathbb{N}_0 \), then \( (a)_j = 0, j \geq n + 1 \), so \( _2F_1(-n, b; c; x) \) reduces to a polynomial of degree not more than \( n \).

The following properties can be found in [3, Ch.2], if not stated otherwise.

- If \( c - a - b > 0 \), the series \( (2.3) \) converges absolutely at \( z = \pm 1 \), and

\[
_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.
\]

Here, the Gamma function with negative non-integer arguments should be understood by the Euler’s reflection formula:

\[
\Gamma(1-a)\Gamma(a) = \frac{\pi}{\sin(\pi a)}, \quad a \not\in \mathbb{Z}.
\]

Note that \( \Gamma(-a) = \infty \), if \( a \in \mathbb{N} \).

- If \( -1 < c - a - b \leq 0 \), the series \( (2.3) \) converges conditionally at \( z = -1 \), but diverges at \( z = 1 \); while for \( c - (a+b) \leq -1 \), it diverges at \( z = \pm 1 \). In fact, it has the following singular behaviours at \( z = 1 \):

\[
\lim_{z \to 1^-} \frac{\_2F_1(a, b; c; z)}{-\ln(1-z)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}, \quad \text{if} \quad c = a + b,
\]

and

\[
\lim_{z \to 1^-} \frac{\_2F_1(a, b; c; z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad \text{if} \quad c < a + b.
\]
Recall the transform identity: for $a, b, c \in \mathbb{R}$ and $-c \not\in \mathbb{N}_0$,
\[ 2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z), \quad |z| < 1. \tag{2.9} \]

The hypergeometric function satisfies the differential equation (cf. [33, P. 98]):
\[ \{ z^n (1 - z)^{a+b-c-1} y' (z) \} ' = a z^{n-1} (1 - z)^{a+b-c} y(z), \quad y(z) = 2F_1(a, b; z). \tag{2.10} \]
We shall use the value at $z = 1/2$ (cf. [28, (15.4.28)]):
\[ 2F_1(a, b; \frac{a+b+1}{2}; \frac{1}{2}) = \frac{\sqrt{\pi} \Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}. \tag{2.11} \]

Many functions are associated with the hypergeometric function. For example, the Jacobi polynomial of degree $n \in \mathbb{N}_0$ with $\alpha, \beta > -1$ (cf. Szegö [33]) is defined by
\[ P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n+\alpha+\beta+1)}{n!} 2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \]
\[ = (-1)^n \frac{(\beta+1)n}{n!} 2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right), \quad x \in (-1, 1), \tag{2.12} \]
which satisfies
\[ P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x), \quad P_n^{(\alpha, \beta)}(1) = \frac{(\alpha+1)n}{n!}. \tag{2.13} \]

For $\alpha, \beta > -1$, the Jacobi polynomials are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha, \beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, namely,
\[ \int_{-1}^{1} P_n^{(\alpha, \beta)}(x) P_{n'}^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) \, dx = \gamma_n^{(\alpha, \beta)} \delta_{nn'}, \tag{2.14} \]
where $\delta_{nn'}$ is the Kronecker Delta symbol, and
\[ \gamma_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n! \Gamma(n+\alpha+\beta+1)}. \tag{2.15} \]

**Remark 2.1.** The definition (2.12) is valid for all $\alpha, \beta \in \mathbb{R}$. In fact, many properties of the classical Jacobi polynomials (e.g., (2.13)) still hold, but the orthogonality is lacking in general (cf. Szegö [33, P. 63-67]).

Throughout this paper, the Gegenbauer polynomial with $\lambda > -1/2$ is defined by
\[ G_n^{(\lambda)}(x) = \frac{P_n^{(\lambda-1/2, \lambda-1/2)}(x)}{P_n^{(\lambda-1/2, \lambda-1/2)}(1)} = 2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\right) \]
\[ = (-1)^n 2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1+x}{2}\right), \quad x \in (-1, 1), \tag{2.16} \]
which has a normalization different from that in Szegö [33]. If $\lambda = 0$, it reduces to the Chebyshev polynomial
\[ T_n(x) = G_n^{(0)}(x) = 2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) = \cos(n \arccos(x)). \tag{2.17} \]

Note that under the above normalization, we derive from (2.14)-(2.15) the orthogonality:
\[ \int_{-1}^{1} G_n^{(\lambda)}(x) G_{n'}^{(\lambda)}(x) \omega_{\lambda}(x) \, dx = \gamma_n^{(\lambda)} \delta_{nn'}, \quad \gamma_n^{(\lambda)} = \frac{2^{2\lambda-1} \Gamma^2(\lambda+1/2) n!}{(n+\lambda)! \Gamma(n+2\lambda)}, \tag{2.18} \]
where $\omega_{\lambda}(x) = (1-x^2)^{\lambda-1/2}$. In the analysis, we shall use the derivative relation derived from the generalised Rodrigues’ formula (see [33, (4.10.1)]) with $\alpha = \beta = \lambda - 1/2 > -1$ and $m = 1$:
\[ \omega_{\lambda}(x) G_n^{(\lambda)}(x) = -\frac{1}{2\lambda+1} \frac{d}{dx} \left\{ \omega_{\lambda+1}(x) G_{n+1}^{(\lambda+1)}(x) \right\}, \quad n \geq 1. \tag{2.19} \]
2.2. Generalised Gegenbauer functions of fractional degree. As an indispensable tool for the error analysis, we introduce the GGF-Fs by allowing the degree \( n \) of the Gegenbauer polynomials in (2.16) to be real.

Definition 2.1. For real \( \lambda > -1/2 \) and real \( \nu \geq 0 \), the right GGF-F of degree \( \nu \) is defined by

\[
G_{\nu}^{(\lambda)}(x) = 2F_1(-\nu, \nu + 2\lambda; \lambda + \frac{1}{2}; -\frac{x}{2}) = 1 + \sum_{j=1}^{\infty} \frac{(-\nu)_j (\nu + 2\lambda)_j}{j! (\lambda + 1/2)_j} \left( -\frac{x}{2} \right)^j, \tag{2.20}
\]

for \( x \in (1, 1); \) while the left GGF-F of degree \( \nu \) is defined by

\[
G_{\nu}^{(\lambda)}(x) = (-1)^{\nu} 2F_1(-\nu, \nu + 2\lambda; \lambda + \frac{1}{2}; \frac{1}{2} + \frac{x}{2}) = (-1)^{\nu} \left( 1 + \sum_{j=1}^{\infty} \frac{(-\nu)_j (\nu + 2\lambda)_j}{j! (\lambda + 1/2)_j} \left( \frac{1}{2} + \frac{x}{2} \right)^j \right), \tag{2.21}
\]

where \([\nu]\) is the largest integer \( \leq \nu \).

\[\square\]

Remark 2.2. For \( \lambda = 1/2 \), the right GGF-F turns to be the Legendre function (cf. [3]): \( P_\nu(x) = G_{\nu}^{(1/2)}(x) \). In Handbook ([25] (15.9.15)), \( G_{\nu}^{(1/2)}(x) \) (with a different normalisation) is defined as the Gegenbauer function. However, there is nearly no discussion on its properties.

Observe from (2.16) and Definition 2.1 that the GGF-Fs reduce to the classical Gegenbauer polynomials when \( \nu \in \mathbb{N}_0 \), but they are non-polynomials when \( \nu \) is not an integer.

Proposition 2.1. The GGF-Fs defined in Definition 2.1 satisfy

\[
G_n^{(\lambda)}(x) = G_n^{(\lambda)}(x) = G_n^{(\lambda)}(x), \quad n \in \mathbb{N}_0; \tag{2.22a}
\]

\[
G_n^{(\lambda)}(-x) = (-1)^{\nu} G_n^{(\lambda)}(x), \quad G_n^{(\lambda)}(1) = 1. \tag{2.22b}
\]

The special GGF-Fs \( \{G_n^{(1/2)}(x)\} \) and \( \{G_n^{(1/2)}(x)\} \) are closely related to the Jacobi polynomials with the parameters maybe \( \leq -1 \) (cf. Remark 2.1).

Proposition 2.2. For \( \alpha > -1 \) and \( n \geq \alpha \) with \( n \in \mathbb{N}_0 \), we have

\[
P_n^{(\alpha, -\alpha)}(x) = \frac{2}{\left( 1 + \frac{x}{2} \right)^\alpha} G_n^{(\alpha + 1/2)}(-x), \quad \frac{P_n^{(\alpha, -\alpha)}(x)}{P_n^{(\alpha, -\alpha)}(1)} = (-1)^{\alpha} \left( 1 + \frac{x}{2} \right)^\alpha G_n^{(\alpha + 1/2)}(x). \tag{2.23}
\]

\[\text{Proof.}\] Taking \( a = -n + \alpha, b = n + \alpha + 1, c = \alpha + 1 \) and \( z = (1-x)/2 \) in (2.9), we obtain from (2.4) that

\[
2F_1(-n + \alpha + 1; \alpha + 1; \frac{1}{2} - \frac{x}{2}) = \left( 1 + \frac{x}{2} \right)^\alpha 2F_1(-n, n + 1; \alpha + 1; 1 - \frac{x}{2}). \tag{2.24}
\]

By (2.12) and (2.13),

\[
2F_1(-n, n + 1; \alpha + 1; 1 - \frac{x}{2}) = \frac{P_n^{(\alpha, -\alpha)}(x)}{P_n^{(\alpha, -\alpha)}(1)},
\]

and by (2.20) (taking \( \nu = n - \alpha \)), the hypergeometric function in the left-hand side of (2.24) equals to

\[
r_n^{(\alpha + 1/2)}(x). \tag{2.24}
\]

Thus, we derive the first identity in (2.23).

Thanks to (2.12) and (2.22b), the second identity in (2.23) follows from the first one immediately. \[\square\]

Remark 2.3. If \( -1 < \alpha < 1 \), we rewrite (2.23) as

\[
G_n^{(\alpha + 1/2)}(x) = d_{n, \alpha} P_n^{(\alpha, -\alpha)}(x); \quad G_n^{(\alpha + 1/2)}(x) = (-1)^{\alpha} d_{n, \alpha} (1 - x)^{\alpha} P_n^{(\alpha, -\alpha)}(x), \tag{2.25}
\]

where \( d_{n, \alpha} = 2^\alpha / P_n^{(\alpha, -\alpha)}(1) \). From (2.14) and (2.15), we immediately obtain the orthogonality:

\[
\int_{-1}^{1} G_n^{(\alpha + 1/2)}(x) G_m^{(\alpha + 1/2)}(x) (1 - x^2)^\alpha dx
\]

\[
= d_{n, \alpha} d_{m, \alpha} \int_{-1}^{1} P_n^{(\alpha, -\alpha)}(x) P_m^{(\alpha, -\alpha)}(x) (1 - x)^{\alpha} (1 + x)^{-\alpha} dx = d_{n, \alpha}^2 \gamma^{(\alpha, -\alpha)}_{nm}, \tag{2.26}
\]
and likewise for \( \{G_{n}^{(\alpha+1/2)}(x)\} \). It is noteworthy that \( \{(1 + x)^{-\alpha} P_{n}^{(\alpha, -\alpha)}\} \) are defined as the Jacobi polyfractonomials in [39] and special generalised Jacobi functions in [19] [15], which serve as effective (singular) basis functions in accurate solutions of fractional differential equations (cf. [39] [15]). It is seen from (2.25) that they turn out to be special GGF-Fs.

It is important to point out that the GGF-Fs may be singular at \( x = \pm 1 \), and they behave differently in different ranges of \( \lambda \).

**Proposition 2.3.** Let \( \nu \in \mathbb{R}_{0}^{+} \).

(i) If \(-1/2 < \lambda < 1/2\), then

\[
\tau G_{\nu}^{(\lambda)}(-1) = \frac{\cos((\nu + \lambda)\pi)}{\cos(\lambda \pi)} = (-1)^{[\nu]} \tau G_{\nu}^{(\lambda)}(1). \tag{2.27}
\]

(ii) If \(\lambda = 1/2\) and \(\nu \notin \mathbb{N}_{0}\), then

\[
\lim_{x \to -1^+} \frac{\tau G_{\nu}^{(\lambda)}(x)}{\ln(1 + x)} = \frac{\sin(\nu\pi)}{\pi} = \lim_{x \to -1^-} \frac{(-1)^{[\nu]} \tau G_{\nu}^{(\lambda)}(x)}{\ln(1 - x)}. \tag{2.28}
\]

(iii) If \(\lambda > 1/2\) and \(\nu \notin \mathbb{N}_{0}\), then

\[
\lim_{x \to -1^+} \left(\frac{1 + x}{2}\right)^{-\lambda/2} \tau G_{\nu}^{(\lambda)}(x) = \frac{\sin(\nu\pi)}{\pi} \frac{\Gamma(\lambda - 1/2)\Gamma(\lambda + 1/2)\Gamma(\nu + 1)}{\Gamma(\nu + 2\lambda)} = \frac{(-1)^{[\nu]} \tau G_{\nu}^{(\lambda)}(x)}{2}. \tag{2.29}
\]

**Proof.** Thanks to (2.22b), it suffices to prove the results for \( \tau G_{\nu}^{(\lambda)}(x) \).

(i) By (2.5), (2.6) and (2.20),

\[
\tau G_{\nu}^{(\lambda)}(-1) = _2F_1(-\nu, \nu + 2\lambda; \lambda + 1/2; 1) = \frac{\Gamma(\lambda + 1/2)\Gamma(1/2 - \lambda)}{\Gamma(\nu + \lambda + 1/2)\Gamma(-\nu - \lambda + 1/2)} = \frac{\pi}{\sin((\nu + \lambda + 1/2)\pi)} \cdot \frac{\sin((\nu + \lambda + 1/2)\pi)}{\pi} = \frac{\cos((\nu + \lambda)\pi)}{\cos(\lambda \pi)}, \tag{2.30}
\]

which yields (2.27).

(ii) Using (2.6), (2.7) and (2.20), and noting that \( \ln((1 + x)/2)/\ln(1 + x) \to 1 \) as \( x \to -1^+ \), we obtain (2.28).

(iii) Next, taking \( a = -\nu, b = \nu + 2\lambda, c = \lambda + 1/2 \) and \( z = (1 - x)/2 \) in (2.9), and using (2.4), we obtain

\[
_{2}F_{1}\left(-\nu, \nu + 2\lambda; \lambda + 1/2; \frac{1 - x}{2}\right) = \left(\frac{2}{1 + x}\right)^{-\lambda/2} _{2}F_{1}\left(\nu + \lambda + \frac{1}{2}, -\nu - \lambda + 1/2; \lambda + 1/2; \frac{1}{2} - x\right). \tag{2.29}
\]

For \( \lambda > 1/2 \), we find from (2.5) and (2.6) that

\[
_{2}F_{1}\left(\nu + \lambda + \frac{1}{2}, -\nu - \lambda + 1/2; \lambda + 1/2; 1\right) = -\frac{\sin(\nu\pi)}{\pi} \frac{\Gamma(\lambda - 1/2)\Gamma(\lambda + 1/2)\Gamma(\nu + 1)}{\Gamma(\nu + 2\lambda)},
\]

so we derive (2.29) from (2.20) and the above. \( \square \)

As some illustrations, we depict in Figure 2.1 the right generalised Chebyshev/Legendre functions, i.e., \( \tau G_{\nu}^{(\lambda)}(x) \) with \( \lambda = 0, 1/2 \) and for various \( \nu \). Note that the left counterparts \( G_{\nu}^{(\lambda)}(x) = (-1)^{[\nu]} \tau G_{\nu}^{(\lambda)}(-x) \) (cf. [222b]). Observe that in the Legendre case (the figure on the right), \( \tau G_{\nu}^{(1/2)}(x) \) with non-integer degree has a logarithmic singularity at \( x = -1 \) (cf. (2.28)), while the generalised Chebyshev functions (left) are well defined at \( x = -1 \).
2.3. Uniform upper bounds. The uniform bounds of the GGF-Fs stated in the following two theorems are of paramount importance in the forthcoming error analysis.

**Theorem 2.1.** For \( \lambda \geq 1 \) and real \( \nu \geq 0 \), we have

\[
\max_{|x| \leq 1} \{\omega(x)|^{\nu}(\lambda)(x), \omega(x)|^{\nu}(\lambda)(x)\} \leq \kappa_{\nu}(\lambda),
\]

where \( \omega(x) = (1 - x^2)^{\lambda-1/2} \) and

\[
\kappa_{\nu}(\lambda) = \frac{\Gamma(\lambda + 1/2)}{\sqrt{\pi}} \left( \frac{\cos^2(\pi\nu/2)\Gamma^2((\nu + 1)/2)}{\Gamma^2((\nu + 1)/2 + \lambda)} + \frac{4\sin^2(\pi\nu/2)}{2\lambda - 1 + \nu(\nu + 2\lambda)} \frac{\Gamma^2(\nu/2 + 1)}{\Gamma^2(\nu/2 + \lambda)} \right)^{1/2}.
\]

**Proof.** Thanks to (2.22b), it suffices to prove the result for \( ^{\nu}(\lambda)(x) \). For notational simplicity, we denote

\[
G(x) := ^{\nu}(\lambda)(x); \quad M(x) := \omega(x)G(x); \quad H(x) := M^2(x) + \varrho^{-1}(1 - x^2)(M'(x))^2,
\]

where the constant \( \varrho := 2\lambda - 1 + \nu(\nu + 2\lambda) \).

We take three steps to complete the proof.

**Step 1:** Show that \( H(x) \) is continuous on \([-1, 1]\), that is, \( H(\pm 1) \) are well defined. It is evident that by (2.22a), \( M(1) = 0 \); and from (2.29), we find that \( M(-1) \) is a finite value, when \( \lambda \geq 1 \). Next, from (3.13a) with \( s = 1 \), we derive

\[
(1 - x^2)^{1/2}M'(x) = (1 - 2\lambda)(1 - x^2)^{\lambda-1}G_{\nu+1}^{(\lambda-1)}(x).
\]

Similarly, by (2.22b), \((1 - x^2)^{1/2}M'(x)|_{x=1} = 0 \) for \( \lambda > 1 \), and it’s finite for \( \lambda = 1 \). We now justify \((1 - x^2)^{1/2}M'(x)|_{x=-1} \) is also well defined. We infer from Proposition 2.3 that (a) if \( 1 \leq \lambda < 3/2, \ G_{\nu+1}^{(\lambda-1)}(x) \) is finite at \( x = -1 \); (b) if \( \lambda = 3/2, \ G_{\nu+1}^{(\lambda-1)}(-1) = 0 \); and (c) if \( \lambda > 3/2, \ G_{\nu+1}^{(\lambda-1)}(x) \) tends to a finite value as \( x \to -1 \). Hence, by (2.33), \( H(\pm 1) \) are well defined.

**Step 2:** Derive the identity:

\[
H'(x) = -\frac{4(\lambda - 1)x}{\varrho} (M'(x))^2, \quad x \in (-1, 1).
\]

Indeed, taking \( a = -\nu, b = \nu + 2\lambda, c = \lambda + 1/2 \) and \( \zeta = (1 \pm x)/2 \) in (2.10), we find that \( G(x) \) satisfies the Sturm-Liouville problem

\[
\{\omega_{\lambda+1}(x)G'(x)\}' + \nu(\nu + 2\lambda)\omega_{\lambda}(x)G(x) = 0.
\]

Substituting \( G(x) = \omega_{\lambda}^{-1}(x)M(x) \) into (2.36), we obtain from a direct calculation that

\[
(1 - x^2)M''(x) + (2\lambda - 3)xM'(x) + qM(x) = 0.
\]
Differentiating $H(x)$ and using (2.37), leads to
\[ H'(x) = \frac{2}{\varrho} M'(x) \left\{ (1 - x^2)M''(x) + \varrho M(x) \right\} - \frac{2x}{\varrho} (M'(x))^2 = -\frac{4(\lambda - 1)x}{\varrho} (M'(x))^2. \tag{2.38} \]

**STEP 3:** Prove the following bounds and calculate the values at $x = 0$:
\[ M^2(x) \leq H(x) \leq H(0) = M^2(0) + \varrho^{-1} (M'(0))^2, \quad \forall x \in [-1, 1]. \tag{2.39} \]

By (2.35), we have $H'(x) \equiv 0$, if $\lambda = 1$, so $H(x)$ is a constant and $H(x) = H(0)$. In other words, (2.39) is true for $\lambda = 1$.

If $\lambda > 1$, we deduce from (2.35) that the stationary points of $H(x)$ are $x = 0$ or zeros of $M'(x)$ (if any). Let $0 \neq \tilde{x} \in (-1, 1)$ be any zero of $M'(x)$ (note: $M(\tilde{x}) \neq 0$). Apparently, by (2.35), $H'(x)$ does not change sign in the neighbourhood of $\tilde{x}$, which means $\tilde{x}$ cannot be an extreme point of $H(x)$. In fact, $x = 0$ is the only extreme point in ($-1, 0$). Let $0 < x < 1$ not change sign in the neighbourhood of $\tilde{x}$, which means $\tilde{x}$ cannot be an extreme point of $H(x)$. In fact, $x = 0$ is the only extreme point in ($-1, 0$). We also see from (2.35) that $H'(x) \geq 0$ (resp. $H'(x) \leq 0$) as $x \rightarrow 1^-$ (resp. $x \rightarrow -1^+$). Note that $H(x)$ attains its maximum at $x = 0$, as $H(x)$ is ascending when $x < 0$, and is descending when $x > 0$. Therefore, we obtain (2.39) from (2.33) and the above reasoning.

Now, we calculate $H(0)$. From (2.6) and (2.11), we obtain that for $\lambda \geq 0$,
\[ M(0) = G_\nu^{(\lambda)}(0) = 2F_1 \left( -\nu, \nu + 2\lambda; \frac{1}{2}; \frac{1}{2} \right) = \frac{\sqrt{\pi} \Gamma(\lambda + 1/2)}{\Gamma(-\nu/2 + 1/2) \Gamma(\nu/2 + \lambda + 1/2)} \tag{2.40} \]
which, together with (3.13b), implies
\[ \{(1 - x^2)^{1/2} M'(x)\} |_{x=0} = (1 - 2\lambda) G_\nu^{(\lambda-1)}(0) = (1 - 2\lambda) \sin \left( \frac{\pi}{2} \right) = 2 \sin \left( \frac{\pi}{2} \right). \tag{2.41} \]

In the last step, we used the identity: $\Gamma(z+1) = z\Gamma(z)$.

Substituting (2.40)-(2.41) into (2.39), we obtain the bound in (2.31).

As a direct consequence of Theorem 2.1, we have the following bound for the Gegenbauer polynomials.

**Corollary 2.1.** For real $\lambda \geq 1$ and integer $l \geq 0$, we have
\[ \max_{|x| \leq 1} \{ \omega_\nu(x) | G_\nu^{(\lambda)}(x) | \} \leq \frac{\Gamma(\lambda + 1/2) \Gamma(l + 1/2)}{\sqrt{\pi} \Gamma(l + \lambda + 1/2)}, \tag{2.42a} \]
\[ \max_{|x| \leq 1} \{ \omega_\nu(x) | G_\nu^{(\lambda)}(x) | \} \leq \frac{2l + 1}{\sqrt{2\lambda - 1} (2l + 1)(2l + 2\lambda + 1)} \frac{\Gamma(\lambda + 1/2) \Gamma(l + 1/2)}{\sqrt{\pi} \Gamma(l + \lambda + 1/2)}. \tag{2.42b} \]

**Remark 2.4.** The bounds for Gegenbauer polynomials multiplying by a different weight function: $(1 - x^2)^{\lambda/2 - 1/4}$ can be found in [25]. To the best of our knowledge, the bounds herein are new.

The bound in Theorem 2.1 is valid for $\lambda \geq 1$. In the analysis, we also need to use the bound with $0 < \lambda < 1$. Note that in this case, we have to multiply the GGF-Fs by a different weight function, and conduct the analysis in a slightly different manner.

**Theorem 2.2.** For real $0 < \lambda < 1$ and real $\nu \geq 0$, we have
\[ \max_{|x| \leq 1} \{(1 - x^2)^{\lambda/2} | G_\nu^{(\lambda)}(x) | \} \leq \tilde{\kappa}_\nu^{(\lambda)}, \tag{2.43} \]
where
\[ \tilde{\kappa}_\nu^{(\lambda)} = \frac{\Gamma(\lambda + 1/2)}{\sqrt{\pi}} \left( \frac{\cos^2(\pi \nu/2) \Gamma^2(\nu/2 + 1/2)}{\Gamma^2((\nu + 1)/2 + \lambda)} + \frac{4 \sin^2(\pi \nu/2) \Gamma^2(\nu/2 + 1)}{\nu^2 + 2\lambda \nu + \lambda \Gamma^2(\nu/2 + \lambda)} \right)^{1/2}. \tag{2.44} \]
Proof. Once again, thanks to (2.22b), it suffices to prove the result for \( G^{(\lambda)}(x) \). Here, we denote
\[
\widehat{M}(x) := (1 - x^2)^{\lambda/2}G^{(\lambda)}(x);
\]
\[
\widehat{H}(x) := \frac{1}{\rho(x)}(\widehat{M}'(x))^2,
\]
\[
\rho(x) := ((\nu + \lambda)^2(1 - x^2) - \lambda(\nu - 1))(1 - x^2)^{-2}.
\] (2.45)

Using Proposition 2.3, we can justify as with Step 1 in the proof of Theorem 2.1 that \( \widehat{H}(x) \) is continuous on \([-1,1]\). A direct calculation from (2.36) leads to
\[
(1 - x^2)\widehat{M}''(x) - x\widehat{M}'(x) + (1 - x^2)\rho(x)\widehat{M}(x) = 0, \quad x \in (-1,1).
\] (2.46)

Like (2.35), we can show
\[
\widehat{H}'(x) = \frac{2\lambda(\lambda - 1)x}{(\lambda + \nu)^2(1 - x^2)^2 - \lambda(\nu - 1)(1 - x^2)^{-2}}(\widehat{M}'(x))^2, \quad x \in (-1,1).
\] (2.47)

For \( 0 < \lambda < 1 \), \( \widehat{H}(x) \) is increasing for \( x < 0 \), and decreasing for \( x > 0 \), so \( H(x) \) attains its maximum at \( x = 0 \). Thus,
\[
\widehat{M}'(x) \leq \widehat{H}(x) \leq \widehat{H}(0) = \widehat{M}'(0) + \rho^{-1}(0)(\widehat{M}'(0))^2, \quad \forall x \in [-1,1].
\] (2.48)

By (2.40),
\[
\widehat{M}(0) = \cos(\pi\nu/2) \frac{\Gamma(\lambda + 1/2)\Gamma(\nu/2 + 1/2)}{\sqrt{\pi}\Gamma(\nu/2 + \lambda + 1/2)}.
\] (2.49)

Recall the identity (cf. [28 (15.5.1)]):
\[
\frac{d}{dx} F(a, b; c; z) = \frac{ab}{c} F(a + 1, b + 1; c + 1; z).
\] (2.50)

From (2.20) and (2.50), we obtain
\[
\frac{d}{dx} G^{(\lambda)}(x) = \frac{d}{dx} F(-\nu, \nu + 2\lambda; \lambda + \frac{1}{2}; \frac{1 - x}{2})
\]
\[
= \frac{\nu(\nu + 2\lambda)}{2\lambda + 1} F(-\nu + 1, \nu + 2\lambda + 1; \lambda + \frac{3}{2}; \frac{1 - x}{2}).
\] (2.51)

Thanks to
\[
\widehat{M}'(x) = -\lambda x(1 - x^2)^{\lambda/2 - 1} G^{(\lambda)}(x) + (1 - x^2)^{\lambda/2} \frac{d}{dx} G^{(\lambda)}(x),
\]
we deduce from (2.6), (2.11) and (2.51) that
\[
\rho^{-1/2}(x)\widehat{M}'(x)|_{x=0} = \frac{2\sin(\pi\nu/2)}{\sqrt{\pi}} \frac{\Gamma(\nu/2 + 1)\Gamma(\lambda + 1/2)}{\sqrt{\nu^2 + 2\lambda\nu + \lambda}}.
\] (2.52)

From (2.49)-(2.49) and (2.52), we derive (2.43)-(2.44). \( \square \)

3. Fractional integral/derivative formulas of GGF-Fs

In this section, we show that GGF-Fs enjoy some remarkable fractional calculus properties, which are important pieces of the puzzle for the analysis.

3.1. Fractional integrals/derivatives and related spaces of functions. Let \( \Omega = (a, b) \subset \mathbb{R} \) be a finite open interval. For real \( p \in [1, \infty] \), let \( L^p(\Omega) \) (resp. \( W^{m,p}(\Omega) \) with \( m \in \mathbb{N} \)) be the usual \( p \)-Lebesgue space (resp. Sobolev space), equipped with the norm \( \| \cdot \|_{L^p(\Omega)} \) (resp. \( \| \cdot \|_{W^{m,p}(\Omega)} \)). Let \( C(\Omega) \) be the classical space of continuous functions on \([a, b]\). Denote by \( AC(\Omega) \) the space of absolutely continuous functions on \([a, b]\). Recall that (cf. [23, 24]): \( f(x) \in AC(\Omega) \) if and only if \( f(x) \in L^1(\Omega) \), \( f(x) \) has a derivative \( f'(x) \) almost everywhere on \([a, b]\) such that \( f'(x) \in L^1(\Omega) \), and \( f \) has the integral representation:
\[
f(x) = f(a) + \int_a^x f'(t) \, dt, \quad \forall x \in [a, b].
\] (3.1)

Note that \( AC(\Omega) = W^{1,1}(\Omega) \) (cf. [23, Sec. 7.2]). Let \( BV(\Omega) \) be the space of functions of bounded variation on \( \Omega \). It is known that every function in \( BV(\Omega) \) has at most a countable number of discontinuities, which
are either jump or removable discontinuities, so it is differentiable almost everywhere. As such, there hold
\[ W^{1,1}(\Omega) = AC(\Omega) \subset BV(\Omega) \subset L^1(\Omega). \] (3.2)

In what follows, we denote the ordinary derivatives by \( D = d/dx \) and \( D^k = d^k/dx^k \) with integer \( k \geq 2 \). Recall the definitions of the RL fractional integrals and derivatives (cf. [30, P. 33, P. 44]).

**Definition 3.1.** For any \( u \in L^1(\Omega) \), the left-sided and right-sided RL fractional integrals of order \( s \in \mathbb{R}^+ \) are defined by
\[
 a^I_s u(x) = \frac{1}{\Gamma(s)} \int_a^x \frac{u(y)}{(x-y)^{1-s}}dy, \quad x \in \Omega. \tag{3.3}
\]

A function \( u \in L^1(\Omega) \) is said to possess a left-sided (resp. right-sided) RL fractional derivative \( a^R_d u \) (resp. \( x^R_d u \)) of order \( s \in (0,1) \), if \( a^I_s u \in AC(\Omega) \) (resp. \( x^I_s u \in AC(\Omega) \)). Moreover, we have
\[
a^R_d u = D(a^I_s u), \quad x^R_d u = -D(x^I_s u), \quad x \in \Omega. \tag{3.4}
\]

Similarly, for \( s \in [k-1,k) \) with \( k \in \mathbb{N} \), the left-sided and right-sided RL fractional derivatives for \( u \in L^1(\Omega) \) satisfying \( a^I_s u \in AC^k(\Omega) \) (i.e., the space of all \( f(x) \) having continuous derivatives up to order \( k-1 \) on \( \Omega \) and \( f^{(k-1)} \in AC(\Omega) \)) are defined by
\[
a^R_d u = D^k(a^I_s u), \quad x^R_d u = (-1)^k D^k(x^I_s u). \tag{3.5}
\]

As a generalisation of (3.1), we have the following fractional integral representation, which can also be regarded as the definition of RL fractional derivatives alternative to Definition 3.1 (see [9] Prop. 3 and [30] P. 45).

**Proposition 3.1.** A function \( u \in L^1(\Omega) \) possesses a left-sided RL fractional derivative \( a^R_d u \) of order \( s \in (0,1) \), if and only if there exist \( C_a \in \mathbb{R} \) and \( \phi \in L^1(\Omega) \) such that
\[
u(x) = C_a \frac{(x-a)^{s-1}}{\Gamma(s)} + a^I_s \phi(x) \mathrm{ a.e. on } [a,b], \tag{3.6}
\]

where \( C_a = (a^I_s u)(a) \) and \( \phi(x) = a^R_d u(x) \mathrm{ a.e. on } [a,b] \).

Similarly, a function \( u \in L^1(\Omega) \) has a right-sided RL fractional derivative \( x^R_d u \) of order \( s \in (0,1) \), if and only if there exist \( C_b \in \mathbb{R} \) and \( \psi \in L^1(\Omega) \) such that
\[
u(x) = C_b \frac{(b-x)^{s-1}}{\Gamma(s)} + x^I_s \psi(x) \mathrm{ a.e. on } [a,b], \tag{3.7}
\]

where \( C_b = (x^I_s u)(b) \) and \( \psi(x) = x^R_d u(x) \mathrm{ a.e. on } [a,b] \).

**Remark 3.1.** We infer from Proposition 3.1 the equivalence of these two fractional spaces:
\[
W^{s,1}_{RL,a+}(\Omega) := \{ u \in L^1(\Omega) : a^I_s u \in AC(\Omega) \} \equiv \{ u \in L^1(\Omega) : x^I_s u \in L^1(\Omega) \}. \tag{3.8}
\]

for \( s \in (0,1) \). The inclusion “\( \subseteq \)” follows immediately from \( u \in L^1(\Omega), a^I_s u \in AC(\Omega) \) and Definition 3.1. To show the opposite inclusion “\( \supseteq \)”, we find
\[
\int_a^b |a^I_s u|dx = \frac{1}{\Gamma(1-s)} \int_a^b \int_a^x (x-y)^{-s}u(y)dy \mathrm{d}x \leq \frac{1}{\Gamma(1-s)} \int_a^b \int_a^x (x-y)^{-s}|u(y)|dy \mathrm{d}x \leq \frac{1}{\Gamma(1-s)} \int_a^b \int_a^x (x-y)^{-s}|u(y)|dy \mathrm{d}x \leq \frac{1}{\Gamma(2-s)} \int_a^b |u(y)|dy.
\]

Since \( u \in L^1(\Omega) \), we conclude \( a^I_s u \in L^1(\Omega) \). As \( x^I_s u = D(a^I_s u) \in L^1(\Omega) \), we infer that \( a^I_s u \in W^{s,1}_{RL,a+}(\Omega) \equiv AC(\Omega) \). Therefore, the equivalence in (3.8) follows. The same property for \( W^{s,1}_{RL,b-}(\Omega) \) with \( x^I_{s,b-} u, x^R_d u \) in place of \( a^I_{s,a} u, x^R_d u \), respectively, holds. We refer to [17] for insightful discussions of the relation between \( W^{s,1}_{RL,a+}(\Omega) \) and the fractional Sobolev space in the sense of Gagliardo [26].
Recall the explicit formulas (cf. \[3\]): for real \(\eta > -1\) and \(s > 0\),
\[
a_{\nu}I_{a}^{\nu}(x-a)^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+s+1)}(x-a)^{\eta+s}, \quad R_{a}D_{x}^{\nu}(x-a)^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-s+1)}(x-a)^{\eta-s}. \tag{3.9}
\]
We have similar formulas for right-sided RL fractional integral/derivative of \((b-x)^{\eta}\). In particular,
\[
a_{\nu}I_{b}^{\nu}(x-a)^{\eta-1} = \Gamma(s), \quad x_{b}I_{b}^{\nu}(b-x)^{\eta-1} = \Gamma(s), \quad s \in (0,1), \tag{3.10}
\]
which implies the boundary values \(C_{a}\) and \(C_{b}\) in Proposition 3.1 are not always zero as \(x \to a^{+}\) and \(x \to b^{-}\), respectively. On the other hand, if \(\eta - s + 1 = -n\) with \(n \in \mathbb{N}_{0}\) in the second formula of (3.9) (note: \(\Gamma(-n) = \infty\)), then
\[
R_{a}D_{x}^{\nu}(x-a)^{s-n-1} = R_{x}D_{a}^{\nu}(b-x)^{s-n-1} = 0, \quad \text{for } s > n \in \mathbb{N}_{0}. \tag{3.11}
\]
We see that the first term in the integral representations in (3.6)-(3.7) actually plays the same role as a “constant” in (3.1).

3.2. Important formulas.

**Theorem 3.1.** For real \(\nu \geq s > 0\) and real \(\lambda > -1/2\), the GGF-Fs on \((-1,1)\) satisfy the RL fractional integral formulas:
\[
x_{b}I_{b}^{\lambda}(\{\omega_{\lambda}(x)^{\nu}\}G_{\nu}(\lambda)(x)) = h_{\lambda}^{(s)}(\omega_{\lambda}(x)^{\nu})G_{\nu}(\lambda-s)(x), \tag{3.12a}
\]
\[
-1_{b}I_{b}^{\lambda}(\{\omega_{\lambda}(x)^{\nu}\}G_{\nu}(\lambda)(x)) = (-1)^{[\nu+|s-\nu|]}h_{\lambda}^{(s)}(\omega_{\lambda}(x)^{\nu})G_{\nu}(\lambda-s)(x). \tag{3.12b}
\]
For real \(\lambda > s-1/2\) and real \(\nu \geq 0\), the GGF-Fs on \((-1,1)\) satisfy the RL fractional derivative formulas:
\[
R_{b}D_{x}^{\lambda}(\{\omega_{\lambda}(x)^{\nu}\}G_{\nu}(\lambda)(x)) = h_{\lambda}^{(s)}(\omega_{\lambda}(x)^{\nu})G_{\nu}(\lambda+s)(x), \tag{3.13a}
\]
\[
-R_{b}D_{x}^{\lambda}(\{\omega_{\lambda}(x)^{\nu}\}G_{\nu}(\lambda)(x)) = (-1)^{[\nu+|s-\nu|]}h_{\lambda}^{(s)}(\omega_{\lambda}(x)^{\nu})G_{\nu}(\lambda-s)(x). \tag{3.13b}
\]
In the above, we denote
\[
\omega_{\alpha}(x) = (1-x^{2})^{\alpha-1/2}, \quad h_{\lambda}^{(s)} = \frac{2^{\beta}}{\Gamma(\lambda - \beta + 1/2)}. \tag{3.14}
\]
**Proof.** Recall the Bateman’s fractional integral formula (cf. \[3\, P.\ 313\]): for \(c, s > 0\) and \(|z| < 1\),
\[
2_{F_{1}}(a,b;c+s;z) = z^{1-(c+s)}\frac{\Gamma(c+s)}{\Gamma(c)\Gamma(s)} \int_{0}^{z} t^{c-1}(z-t)^{s-1}2_{F_{1}}(a,b;c;t) \, dt, \tag{3.15}
\]
which, together with (2.9), yields
\[
z^{c+s-1}(1-z)^{c+s-a-b}2_{F_{1}}(c-a+s,c+b-s;c+s;z) = \frac{\Gamma(c+s)}{\Gamma(c)\Gamma(s)} \int_{0}^{z} t^{c-1}(z-t)^{s-1}(1-t)^{c-a-b}2_{F_{1}}(c-a,c-b;c;t) \, dt. \tag{3.16}
\]
Applying the variable substitutions: \(z = (1-x)/2\) and \(t = (1+y)/2\) to (3.16), leads to
\[
(1-x)^{c+s-1}(1+x)^{c+s-a-b}2_{F_{1}}(c-a+s,c+b-s;c+s;1-x/2) = \frac{2^{\nu}}{\Gamma(c+s)} \int_{x}^{1} (1-y)^{c-1}(1+y)^{c-a-b}(y-x)^{s-1}2_{F_{1}}(c-a,c-b;c;1-y/2) \, dy. \tag{3.17}
\]
Taking \(a = \nu + \lambda + 1/2\), \(b = -\nu - \lambda + 1/2\) and \(c = \lambda + 1/2\) in (3.17), we obtain
\[
(1-x)^{\lambda+s+1/2}2_{F_{1}}(s-\nu,\nu+s+2\lambda;\lambda+s+1/2;1-x/2) = \frac{2^{\nu}}{\Gamma(\lambda+s+1/2)\Gamma(s)} \int_{x}^{1} (1-y)^{\lambda-1/2}(y-x)^{s-1}2_{F_{1}}(-\nu,\nu+2\lambda;\lambda+1/2;1-y/2) \, dy. \tag{3.18}
\]
From (3.3) and (2.20), we derive (3.12a) immediately.
Similarly, performing the variable substitutions: \(z = (1+x)/2\) and \(t = (1+y)/2\) to (3.16), we can obtain (3.12b) in the same manner.
Applying $\frac{d}{dx}D_1^s$ to both sides of (3.12a) and noting that $\frac{d}{dx}D_1^s I_1^r$ is an identity operator (cf. [30]), we obtain for real $\nu \geq s > 0$ and real $\lambda > -1/2$,
\[
\omega_\lambda(x) G_\nu^{(\lambda)}(x) = h_\lambda^{(s)} \frac{d}{dx} D_1^s I_1^r \{ \omega_\lambda + s(x) G_\nu^{(\lambda + s)}(x) \}.
\]  
(3.18)
Replacing $\lambda, \nu$ in the above equation by $\lambda - s, \nu + s$, and noting that
\[
(h_\lambda^{(-s)})^{-1} = \frac{2s \Gamma(\lambda + 1/2)}{\Gamma(\lambda - s + 1/2)} = h_\lambda^{(s)},
\]  
(3.19)
we obtain (3.13a). Similarly, applying $\frac{d}{dx}D_1^s$ to both sides of (3.12b), we can derive (3.13b).

4. Chebyshev approximations of functions in fractional Sobolev-type spaces

In this section, we introduce a new theoretical framework and present the main results on Chebyshev approximations. Here, we focus on the approximation of functions with interior singularities, and shall extend the estimates to deal with functions with endpoint singularities in Subsection 6.2.

4.1. Fractional Sobolev-type spaces. For a fixed $\theta \in \Omega := (-1, 1)$, we denote $\Omega^- := (-1, \theta)$ and $\Omega^+ := (\theta, 1)$. For $m \in \mathbb{N}_0$ and $s \in (0, 1)$, we define the fractional Sobolev-type space:
\[
W^{m+s}_\theta(\Omega) := \left\{ u \in L^1(\Omega) : u, u', \ldots, u^{(m-1)} \in AC(\Omega) \text{ and } x I_{\theta}^{-s} u^{(m)} \in \text{BV}(\Omega^-), \quad \theta^1 I_{\theta}^{-s} u^{(m)} \in \text{BV}(\Omega^+) \right\},
\]  
where the semi-norm is defined by
\[
\|u\|_{W^{m+s}_\theta(\Omega)} = \sum_{k=0}^{m} \|u^{(k)}\|_{L^1(\Omega)} + U^{m,s}_\theta,
\]  
(4.2)
where 

- for $m = 1, 2, \ldots$,
\[
U^{m,s}_\theta := \int_{-1}^{\theta} |x D_1^s u^{(m)}(x)| \, dx + \int_{\theta}^{1} |x D_1^s u^{(m)}(x)| \, dx + \left| \{ x I_{\theta}^{-s} u^{(m)} \}(\theta+) \right| + \left| \{ x I_{\theta}^{-s} u^{(m)} \}(\theta-) \right|,
\]  
(4.3)

- for $m = 0$ and $s \in (1/2, 1)$,
\[
U^{0,s}_\theta := \int_{-1}^{\theta} |x D_1^s u(x)| \, \omega_{s/2}(x) \, dx + \int_{\theta}^{1} |x D_1^s u(x)| \, \omega_{s/2}(x) \, dx + \left| \{ \omega_{s/2} x I_{\theta}^{-s} u \}(\theta+) \right| + \left| \{ \omega_{s/2} x I_{\theta}^{-s} u \}(\theta-) \right|.
\]  
(4.4)

Remark 4.1. Some remarks are in order.

(i) If $u \in W^{m+s}_\theta(\Omega)$, we infer from Proposition 3.2 and Remark 3.4 that $\frac{d}{dx}D_1^s u^{(m)} = \frac{d}{dx}D_1^s I_1^r u^{(m)}$ are well-defined and belong to $L^1(\Omega)$.

(ii) The parameter $\theta$ is related to the location of the singular point of $u(x)$. For example, if $u = |x|$, then $\theta = 0$. For a function of multiple interior singular points, we partition $(-1, 1)$ into multiple subintervals and introduce the same number of parameters accordingly.

(iii) The so-defined space $W^{m+s}_\theta(\Omega)$ is an intermediate fractional space in the sense that
\[
W^{1,1}(\Omega) \subseteq W^{s}_\theta(\Omega) \subseteq L^1(\Omega); \quad W^{m+1}(\Omega) \subseteq W^{m+s}_\theta(\Omega) \subseteq W^{m}(\Omega), \quad m \geq 1.
\]  
(4.5)
In particular, when $s \rightarrow 1^-$, the space $W^{m+s}_\theta(\Omega)$ reduces to
\[
W^{m+1}(\Omega) := \left\{ u \in L^1(\Omega) : u', \ldots, u^{(m-1)} \in AC(\Omega), \quad u^{(m)} \in \text{BV}(\Omega) \right\},
\]  
(4.6)
which has been used in [35, 24] for Chebyshev approximation of functions with limited regularity.
To deal with endpoint singularities, letting \( \theta \to \pm 1 \), we denote the corresponding fractional spaces by

\[
\mathbb{W}^{m,s}_+(\Omega) := \{ u \in L^1(\Omega) : u, u', \ldots, u^{(m-1)} \in AC(\Omega), x_1^{1-s} u^{(m)} \in BV(\Omega) \},
\]

\[
\mathbb{W}^{m,s}_-\omega(\Omega) := \{ u \in L^1(\Omega) : u, u', \ldots, u^{(m-1)} \in AC(\Omega), -x_1^{1-s} u^{(m)} \in BV(\Omega) \}.
\]

(4.7)

Accordingly, the semi-norm \( U^{m,s}_\omega \) (resp. \( U^{m,s}_- \)) only involves the right (resp. left) RL fractional integrals/derivatives. We remark that for \( s \in (0, 1) \), \( \mathbb{W}^s(\Omega) \supseteq \mathbb{W}^{s,1}_{RL, -1}(\Omega) \) defined in (3.8).

4.2. Exact formulas and decay rate of Chebyshev expansion coefficients. Let \( \omega(x) = (1 - x^2)^{-1/2} = \omega_0(x) \) be the Chebyshev weight function. For any \( u \in L^2(\omega_\omega) \), we expand it in Chebyshev series and denote the partial sum by

\[
u(x) = \sum_{n=0}^\infty \hat{u}_n^C T_n(x), \quad \pi_n^C u(x) = \sum_{n=0}^N \hat{u}_n^C T_n(x),
\]

where the prime denotes a sum whose first term is halved, and

\[
\hat{u}_n^C = \frac{2}{\pi} \int_{-1}^{1} u(x) \frac{T_n(x)}{\sqrt{1 - x^2}} dx = \frac{2}{\pi} \int_{0}^{\pi} u(\cos \theta) \cos(n\theta) d\theta.
\]

(4.8)

Recall the formula of integration by parts involving the Stieltjes integrals (cf. (2.12)).

**Lemma 4.1.** For any \( u, v \in BV(\Omega) \), we have

\[
\int_a^b u(x) \, dv(x) = \{ u(x) v(x) \}_{a}^{b} - \int_a^b v(x) \, du(x),
\]

where the notation \( u(x) \) stands for the right- and left-limit of \( u \) at \( x \), respectively. Here, \( u(x_\pm), v(x_\pm) \) can also be replaced by \( u(x_+), v(x+) \).

In particular, if \( u, v \in AC(\Omega) \), we have

\[
\int_a^b u(x) v'(x) dx + \int_a^b u'(x) v(x) dx = \{ u(x) v(x) \}_{a}^{b}.
\]

(4.9)

(4.10)

(4.11)

As highlighted in (2.4), the error analysis of Chebyshev expansions in various norms, and the related interpolation and quadrature errors essentially depends on estimating the decay rate of \( |\hat{u}^C_n| \). We present the main results below.

**Theorem 4.1.** Given \( \theta \in (-1, 1) \), if \( u \in \mathbb{W}^{m,s}_\omega(\Omega) \) with \( s \in (0, 1) \) and integer \( m \geq 0 \), then for \( n \geq m + s > 1/2 \),

\[
\hat{u}_n^C = \frac{1}{\sqrt{\pi}} 2^{m+s-1} \Gamma(m + s + 1/2) \left\{ (-1)^{n+s} \int_{-1}^{\theta} D_x^s u(x) \right\} \left\{ G^{(m+s)}_{n-m-s}(x) \omega_{m+s}(x) \right\} dx
\]

\[
+ (-1)^{n+s} \left\{ (\theta \Gamma_{n-s}^m(x)) \left\{ G^{(m+s)}_{n-m-s}(x) \omega_{m+s}(x) \right\} \right\} \right|_{x=\theta^+}
\]

\[
+ \int_{\theta}^{1} \theta D_x^s u(x) \right\} \left\{ G^{(m+s)}_{n-m-s}(x) \omega_{m+s}(x) \right\} dx + \left\{ (\theta \Gamma_{n-s}^m(x)) \left\{ G^{(m+s)}_{n-m-s}(x) \omega_{m+s}(x) \right\} \right\} \right|_{x=\theta^-}
\]

\[
(4.12)
\]

where \( \omega_{n}(x) = (1 - x^2)^{-1/2} \).

For \( n \geq m + s \), we have the following upper bounds:

(i) If \( m = 0 \) and \( s \in (1/2, 1) \), then we have

\[
|\hat{u}_n^C| \leq U_{\omega}^{0,s} \frac{\Gamma((n - s + 1)/2)}{\Gamma((n + s + 1)/2)} \frac{2}{\sqrt{n^2 - s^2} + s} \frac{\Gamma((n - s)/2 + 1)}{\Gamma((n + s)/2)}.
\]

(4.13)

(ii) If \( m \geq 1 \), then we have

\[
|\hat{u}_n^C| \leq U_{\omega}^{m,s} \frac{\Gamma((n - m - s + 1)/2)}{\Gamma((n + m + s + 1)/2)}.
\]

(4.14)
Proof. Substituting $n \to n - k$, $\lambda \to k$ in (2.19), leads to
\[
\omega_k(x)G_{n-k}^{(k)}(x) = -\frac{1}{2k + 1} \{\omega_{k+1}(x)G_{n-k-1}^{(k+1)}(x)\}', \quad n \geq k + 1.
\] (4.15)
For $u, u', \ldots, u^{(m-1)} \in \text{AC}(\Omega)$, using (4.15) with $k = 0, 1, \cdots, m - 1$, and the integration by parts in Lemma 4.1 we obtain that for $n \geq m$,
\[
\hat{u}_n^C = \frac{2}{\pi} \int_{-1}^{1} u(x)G_n^{(0)}(x)\omega_0(x)\,dx = -\frac{2}{\pi} \int_{-1}^{1} u(x)\{G_{n-1}^{(1)}(x)\omega_1(x)\}', \quad n \geq m.
\] (4.16)
Using the identity (cf. (28)):
\[
\Gamma(k + 1/2) = \frac{\sqrt{\pi} (2k - 1)!!}{2^k}, \quad k \in \mathbb{N}_0,
\] (4.17)
we can rewrite the expansion coefficient as
\[
\hat{u}_n^C = \frac{1}{2^m \pi^{m-1}(m + 1/2)} \int_{-1}^{1} u^{(m)}(x) G_{n-m}^{(m)}(x) \omega_m(x)\,dx.
\] (4.18)

We proceed with the proof by fractional integration by parts. Then it is necessary to use the following identities: for $m > s > 1/2$, and $n \geq m + s$,
\[
\omega_m(x)G_{n-m}^{(m)}(x) = -\frac{\Gamma(m + 1/2)}{2^s \Gamma(m + s + 1/2)} x I_1^{-s} \{\omega_{m+s}(x)G_{n-m-s}^{(m+s)}(x)\}',
\] (4.19)
To derive (4.19), we substitute $s, \lambda, \nu$ in (3.12a)-(3.12b) by $1 - s, m - s, n - m + s$, respectively, leading to
\[
\omega_m(x)G_{n-m}^{(m)}(x) = \frac{2^{1-s} \Gamma(m + 1/2)}{\Gamma(m + s - 1/2)} x I_1^{-s} \{\omega_{m+s-1}(x)G_{n-m-s+1}^{(m+s-1)}(x)\}',
\] (4.20)
Taking $s = 1, \lambda = m + s$ and $\nu = n - m - s$ in (3.13a)-(3.13b), we obtain that for $m + s > 1/2$,
\[
\omega_{m+s-1}(x)G_{n-m-s+1}^{(m+s-1)}(x) = -\frac{\Gamma(m + s - 1/2)}{2 \Gamma(m + s + 1/2)} \{\omega_{m+s}(x)G_{n-m-s}^{(m+s)}(x)\}',
\] (4.21)
Substituting (4.21) into (4.20) leads to (4.19).

For notational convenience, we denote
\[
f(x) = u^{(m)}(x), \quad g(x) = (-1)^{n-s} \omega_{m+s} x G^{(m+s)}_{n-m-s}(x), \quad h(x) = -\omega_{m+s} x G^{(m+s)}_{n-m-s}(x).
\] (4.22)
By (4.19), we can rewrite (4.18) as
\[
\hat{u}_n^C = \frac{1}{\sqrt{\pi} 2^{m-1} \Gamma(m + 1/2)} \left\{ \int_{-1}^{\theta} u^{(m)} G_{n-m}^{(m)} \omega_m\,dx + \int_{\theta}^{1} u^{(m)} G_{n-m}^{(m)} \omega_m\,dx \right\}
\] (4.23)
where in the last step, we used the identity: $\Gamma(z + 1) = z\Gamma(z)$. Next, we rewrite (4.29) as

\begin{equation}
\frac{2}{2\sqrt{\lambda}-1+\nu(\nu+2\lambda)} \leq \frac{\Gamma((\nu+1)/2)}{\Gamma((\nu+1)/2+1)}, \quad \nu \geq 0, \quad \lambda \geq 1.
\end{equation}

To prove (4.28), we use the property in [12] Corollary 2, that is, the ratio $f(z) := \frac{1}{\sqrt{\pi}} \Gamma(z+1/2)$, $z > 0$,
is decreasing. Then using the facts:

$$(\nu - 1)/2 + \lambda > 0, \quad (\nu - 1)/2 + \lambda > (\nu + 1)/2,$$

we can derive

\begin{equation}
\frac{1}{\sqrt{\nu - 1}/2 + \lambda} \leq \frac{\Gamma((\nu+1)/2+1)}{\Gamma((\nu+1)/2+1)} = \frac{\nu+1}{\nu+2} \frac{\Gamma((\nu+1)/2)}{\Gamma((\nu+1)/2+1)},
\end{equation}

where in the last step, we used the identity: $\Gamma(z + 1) = z\Gamma(z)$. Next, we rewrite (4.29) as

\begin{equation}
\frac{2}{\sqrt{\nu+2\lambda-1}(\nu+1)\Gamma(\nu/2+\lambda)} \leq \frac{\Gamma((\nu+1)/2)}{\Gamma((\nu+1)/2+1)}.
\end{equation}
Noting that
\[
\frac{2}{\sqrt{2\lambda - 1 + \nu + 2\lambda}} = \frac{2}{\sqrt{(\nu + 2\lambda - 1)(\nu + 1)}}
\]
we obtain (4.28) from (4.30) immediately. Thanks to (4.28), we derive from Theorem 2.1 that
\[
\max_{|x| \leq 1} \left\{ \omega_\lambda(x) |G_\nu^{(1)}(x)|, \omega_\lambda(x) |G_\nu^{(2)}(x)| \right\} \leq \frac{\Gamma((\lambda + 1)/2)}{\sqrt{\pi}} \frac{\Gamma((\nu + 1)/2)}{\Gamma((\nu + 1)/2 + \lambda)},
\]
so the bound in (4.14) follows from (4.12) with \( \lambda = m + s \) and \( \nu = n - m - s \) in (4.31).

4.3. \( L^\infty \) and \( L^2 \)-estimates of Chebyshev expansions. With Theorem 4.1 at our disposal, we can analyze all related orthogonal projections, interpolations and quadratures (cf. [24]). Here, we first estimate the Chebyshev expansion errors in the \( L^\infty \)-norm and \( L^2 \)-norm for functions with interior singularities. We remark that if the function is sufficiently smooth, we understand the results with \( m = 1 \), i.e., in the space \( \mathbb{W}^{m+1}(\Omega) \) defined in (4.6). We refer to Theorem 5.2 for the integer case.

**Theorem 4.2.** Given \( \theta \in (-1, 1) \), if \( u \in \mathbb{W}_\theta^{m+s}(\Omega) \) with \( s \in (0, 1) \) and integer \( m \geq 0 \), we have the following estimates.

(i) For \( 1 < m + s < N + 1 \),
\[
\|u - \pi_N^C u\|_{L^\infty(\Omega)} \leq \frac{U_{\theta}^{m,s}}{2^{m+s-2}(m+s-1)\pi} \frac{\Gamma((N - m - s)/2 + 1)}{\Gamma((N + m + s)/2)}.
\]

(ii) For \( 1/2 < m + s < N + 1 \),
\[
\|u - \pi_N^C u\|_{L^2(\Omega)} \leq \left\{ \frac{2^3}{(2m + 2s - 1)\pi} \frac{\Gamma(N - m - s + 1)}{\Gamma(N + m + s)} \right\}^{1/2} U_{\theta}^{m,s}.
\]

**Proof.** We first prove (4.32). For simplicity, we denote
\[
S_{\sigma}^n := \frac{\Gamma((n - \sigma + 1)/2)}{\Gamma((n + \sigma + 1)/2)}, \quad T_{\sigma}^n := \frac{\Gamma((n - \sigma + 1)/2)}{\Gamma((n + \sigma - 1)/2)}, \quad \sigma := m + s.
\]

A direct calculation leads to the identity:
\[
T_{\sigma}^n - T_{\sigma}^{n+2} = \frac{n + \sigma - 1}{2} \frac{\Gamma((n - \sigma + 1)/2)}{\Gamma((n + \sigma + 1)/2)} - \frac{n - \sigma + 1}{2} \frac{\Gamma((n - \sigma + 1)/2)}{\Gamma((n + \sigma + 1)/2)}\
= (\sigma - 1) \frac{\Gamma((n - \sigma + 1)/2)}{\Gamma((n + \sigma + 1)/2)} = (\sigma - 1) S_{\sigma}^n,
\]
where we used the identity \( z \Gamma(z) = \Gamma(z + 1) \). By (4.14),
\[
|u(x) - \pi_N^C u(x)| \leq \sum_{n=N+1}^{\infty} |\hat{u}_n^{(N)}| \leq \frac{U_{\theta}^{m,s}}{2^{\sigma - 1}(\sigma - 1)\pi} \sum_{n=N+1}^{\infty} S_{\sigma}^n = \frac{U_{\theta}^{m,s}}{2^{\sigma - 1}(\sigma - 1)\pi} \sum_{n=N+1}^{\infty} \left\{ T_{\sigma}^n - T_{\sigma}^{n+2} \right\}.
\]

We find from (1.1) and Theorem 10 that for \( 0 \leq a < b \), the ratio
\[
R_{\sigma}(z) := \frac{\Gamma(z + a)}{\Gamma(z + b)}, \quad z \geq 0,
\]
is decreasing with respect to \( z \). As \( \sigma - 1 > 0 \), we have
\[
T_{\sigma}^{N+2} = R_{\sigma-1}(1 + (N - \sigma + 1)/2) \leq R_{\sigma-1}(1 + (N - \sigma)/2) = T_{\sigma}^{N+1}.
\]

Therefore, the estimate (4.32) follows from (4.36) and (4.38).
We now turn to the estimate (4.33) with $1 < m + s \leq N + 1$. Similar to (4.38), we can use (4.37) to show that $S_n^\sigma \leq S_{n-1}^\sigma$. Thus, using the identity
\[
\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma(z + 1/2),
\] (4.39)
we derive
\[
(S_n^\sigma)^2 \leq S_n^\sigma S_{n-1}^\sigma = \frac{\Gamma((n - \sigma + 1)/2) \Gamma((n - \sigma)/2)}{\Gamma((n + \sigma + 1)/2) \Gamma((n + \sigma)/2)} = 2^{2\sigma} \frac{\Gamma(n - \sigma)}{\Gamma(n + \sigma)}
\] (4.40)
Then, for $\sigma > 1$,
\[
\left\|u - \pi_N^C u\right\|^2_{L^2_\omega(\Omega)} = \pi \sum_{n=N+1}^\infty \left|\hat{u}_n^C\right|^2 \leq \frac{4}{n^2 - s^2 + s} \frac{\Gamma((n - s)/2 + 1)}{\Gamma((n + s)/2)} \sum_{n=N+1}^\infty (S_n^\sigma)^2 \leq 2^{2s} \left(\frac{\Gamma(n - s)}{2s - 1} - \frac{\Gamma(n + 1 - s)}{\Gamma(n + s)}\right).
\] (4.41)
Finally, we prove (4.33) with $m = 0$ and $s \in (1/2, 1)$ by using (4.13). Note that (4.41) is valid for $m = 0$, so we have
\[
(S_n^\sigma)^2 = \frac{\Gamma((n - s)/2 + 1)}{\Gamma((n + s)/2)} \leq \frac{2^{2s}}{2s - 1} \left(\frac{\Gamma(n - s)}{\Gamma(n - 1 + s)} - \frac{\Gamma(n + 1 - s)}{\Gamma(n + s)}\right).
\] (4.42)
For the second factor in the upper bound (4.13), we also use (4.37) and (4.39)-(4.40) to show
\[
\frac{4}{n^2 - s^2 + s} \frac{\Gamma((n - s)/2 + 1)}{\Gamma((n + s)/2)} \leq \frac{4}{n^2 - s^2 + s} \frac{\Gamma((n - s)/2 + 1) \Gamma((n - s)/2 + 1/2)}{\Gamma((n + s)/2) \Gamma((n + s)/2)} = 2^{2s} \frac{\Gamma(n - s + 1)}{n^2 - s^2 + s} \frac{\Gamma(n + s)}{n^2 - s^2 + s} \frac{\Gamma(n - s)}{\Gamma(n + s)}
\] (4.43)
\[
\leq 2^{2s} \frac{\Gamma(n - s)}{2s - 1} \left(\frac{\Gamma(n - s)}{\Gamma(n - 1 + s)} - \frac{\Gamma(n + 1 - s)}{\Gamma(n + s)}\right).
\]
Thus, from (4.13), we obtain
\[
|\hat{u}_n^C|^2 \leq \frac{4}{(2s - 1)\pi^2} \left(\frac{\Gamma(n - s)}{\Gamma(n - 1 + s)} - \frac{\Gamma(n + 1 - s)}{\Gamma(n + s)}\right).
\] (4.44)
With this, we derive
\[
\left\|u - \pi_N^C u\right\|^2_{L^2_\omega(\Omega)} = \frac{\pi}{2} \sum_{n=N+1}^\infty |\hat{u}_n^C|^2 \leq \frac{4}{(2s - 1)\pi^2} \frac{\Gamma(N - s + 1)}{\Gamma(N + s)}.
\] (4.45)
This completes the proof. \[\square\]

**Remark 4.2.** Recall that (cf. [28, (5.11.13)]): for $a < b$,
\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b} + \frac{1}{2} (a-b)(a+b-1)z^{a-b-1} + O(z^{a-b-2}), \quad z \gg 1.
\] (4.46)
Thus, under the conditions of Theorem 4.3 and for fixed $m$ and large $n$ or $N$, we have
\[
|\hat{u}_n^C| \leq C n^{-(m+s)} U_\theta^{m,s}, \quad \left\|u - \pi_N^C u\right\|_{L^\infty(\Omega)} \leq C N^{1-(m+s)} U_\theta^{m,s},
\] (4.47)
\[
\left\|u - \pi_N^C u\right\|_{L^2_\omega(\Omega)} \leq C N^{\frac{1}{2}-(m+s)} U_\theta^{m,s},
\]
where $C$ is a positive constant independent of $n, N$ and $u$. 

4.4. Applications to functions with interior singularities. In what follows, we apply the main results to two typical types of singular functions, and provide numerical illustrations of the optimal convergence order.

- **Type-I:** Consider
  \[ u(x) = |x - \theta|^\alpha, \quad \alpha > -1/2, \quad x, \theta \in (-1, 1), \tag{4.48} \]
  where \( \alpha \) is not an even integer.

- **Type-II:** Consider
  \[ u(x) = |x - \theta|^\alpha \ln |x - \theta|, \quad \alpha > -1/2, \quad x, \theta \in (-1, 1). \tag{4.49} \]

4.4.1. **Type-I:** \( u(x) = |x - \theta|^\alpha \) in (4.48).

**Theorem 4.3.** Given the function in (4.48), we have that (i) if \( \alpha \) is an odd integer, then \( u \in W^{\alpha+1}(\Omega) \) (defined in (4.6)); and (ii) if \( \alpha \) is not an integer, then \( u \in W^{\alpha+1}_\theta(\Omega) \) (defined in (4.1)).

Its Chebyshev expansion coefficients can be expressed as
\[
\hat{u}_n^C = \frac{\Gamma(\alpha+1)}{2\alpha \Gamma(\alpha+3/2)\sqrt{\pi}} \left\{ \frac{\Gamma(n-\alpha+1)}{\Gamma(n+\alpha+1/2)} - (1)_{n+\alpha}^\alpha G^{(\alpha+1)}_{n-\alpha-1}(\theta) \right\} \omega_{\alpha+1}(\theta), \tag{4.50}\]
for all \( n \geq 1 \). Moreover, we have the bounds uniform for \( n \):

- (a) for \(-1/2 < \alpha < 0\),
  \[
  |\hat{u}_n^C| \leq \frac{\Gamma(\alpha+1)}{2\alpha \Gamma(\alpha+3/2)\sqrt{\pi}} \left\{ \frac{\Gamma((n-\alpha)/2)}{\Gamma((n+\alpha)/2+1)} - \frac{2}{\sqrt{n^2 - \alpha(\alpha+1)}} \frac{\Gamma((n+\alpha+1)/2)}{\Gamma((n+\alpha+1)/2)} \right\}, \tag{4.51}\]

- (b) for \( \alpha \geq 0 \),
  \[
  |\hat{u}_n^C| \leq \frac{\Gamma(\alpha+1)}{2\alpha \Gamma(\alpha+3/2)\sqrt{\pi}} \frac{\Gamma((n-\alpha)/2)}{\Gamma((n+\alpha)/2+1)}. \tag{4.52}\]

**Proof.** (i) If \( \alpha \) is an odd integer, we find
\[
\hat{u}^{(k)}_n = d_{\alpha}^k |x - \theta|^\alpha (\text{sign}(x - \theta))^k \in AC(\Omega), \quad 0 \leq k \leq \alpha - 1;
\]
\[
\hat{u}^{(\alpha)}_n = d_{\alpha}^\alpha (2H(x - \theta) - 1) \in BV(\Omega). \tag{4.53}\]

where \( \text{sign}(z), H(z), \delta(z) \) are the sign, Heaviside and Dirac Delta functions, respectively, and
\[
d_{\alpha}^\alpha := \alpha(\alpha - 1) \cdots (\alpha - k + 1) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}. \tag{4.54}\]

Thus, from (4.6), we claim \( u \in W^{\alpha+1}(\Omega) \). Moreover, by (5.8) (with \( m = \alpha \)), (4.17) and (4.53),
\[
\hat{u}_n^C = \frac{1}{(2\alpha+1)!!} \frac{2}{\pi} \int_{-1}^{1} G^{(\alpha+1)}_{n-\alpha-1}(x) \omega_{\alpha+1}(x) du^{(\alpha)}(x)
= \frac{\Gamma(\alpha+1)}{2\alpha \Gamma(\alpha+3/2)\sqrt{\pi}} G^{(\alpha+1)}_{n-\alpha-1}(\theta) \omega_{\alpha+1}(\theta),
\]
which is identical to (4.50) with \( \alpha \) being an odd integer, thanks to (2.22a).

(ii) If \( \alpha \) is not an integer, let \( m = \lceil \alpha \rceil + 1 \) and \( s = \{\alpha + 1\} \in (0, 1) \). Like (4.53), we have \( u, \cdots, u^{(m-1)} \in AC(\Omega) \). By a direct calculation, we infer from (3.9) that for \( x \in (-1, \theta) \),
\[
x^s B^{m-\alpha}_{\alpha} u^{(m)} = (1)^m d_{\alpha}^m x^s B^{m-\alpha}_{\alpha} (\theta - x)^{\alpha-\alpha - m} = (1)^m d_{\alpha}^m \Gamma(s) = (-1)^{\lceil \alpha \rceil + 1} \Gamma(\alpha+1), \tag{4.55}\]
while for \( x \in (\theta, 1) \),
\[
\theta^s B^{m-\alpha}_{\alpha} u^{(m)} = d_{\alpha}^m \theta^s B^{m-\alpha}_{\alpha} (x - \theta)^{\alpha-\alpha - m} = d_{\alpha}^m \Gamma(s) = \Gamma(\alpha+1). \tag{4.56}\]

Therefore, by the definition (4.1), we have \( u \in W^{\alpha+1}_\theta(\Omega) \).

It is clear that by (4.55)-(4.56), \( B^{m}_{\alpha} u^{(m)}(x) = B^{m}_{\alpha} u^{(m)}(x) = 0 \), so we can derive the exact formula (4.50) by using (4.12) straightforwardly.
(a) For \(-1/2 < \alpha < 0\), taking \(s = \alpha + 1\) in (4.13), leads to

\[
|\hat{u}_n^C| \leq \frac{\Gamma(\alpha + 1)}{2^{\alpha - 1/2}} \left(1 - 2^\alpha\right)^{\alpha/2} \max \left\{ \frac{\Gamma((n - \alpha)/2)}{\Gamma((n + \alpha)/2 + 1)}, \frac{2}{\sqrt{n^2 - \alpha(\alpha + 1)}} \right\},
\]

where we used the fact \(U_{\alpha,\alpha+1} = 2(1 - 2^\alpha)^{\alpha/2}\Gamma(\alpha + 1)\).

(b) Similarly, we can obtain (4.52) directly from (4.14). □

**Remark 4.3.** As a special case of (4.50) with \(\theta = 0\), we obtain from (2.22h) and (2.40) that the Chebyshev expansion coefficients of \(|x|^\alpha\) have the exact representation for each integer \(n \geq 0\),

\[
\hat{u}_n^C = ((-1)^n + 1) \frac{\Gamma(\alpha + 1)\Gamma((n - \alpha)/2)}{2^{\alpha}\pi \Gamma((n + \alpha)/2 + 1)} \sin \left(\frac{(n - \alpha)\pi}{2}\right),
\]

which implies that for integer \(k \geq 0\),

\[
\hat{u}_{2k+1}^C = 0, \quad \hat{u}_{2k}^C = (-1)^k \sin \frac{\alpha\pi}{2} \frac{\Gamma(\alpha + 1)}{2^{\alpha - 1/2}} \frac{\Gamma(k - \alpha/2)}{\Gamma(k + \alpha/2)}. \tag{4.58}
\]

It is noteworthy that the following asymptotic estimate for large \(k\) was obtained in [27, Sec. 3.11]:

\[
\hat{u}_{2k}^C \approx (-1)^k \sin \frac{\alpha\pi}{2} \frac{\Gamma(\alpha + 1)}{2^{\alpha - 1/2}} k^{\alpha - 1}, \tag{4.59}
\]

but by different means. Indeed, our approach leads to exact representations for all \(n\).

Note that we can directly apply Theorem 4.2 (also see Remark 4.2) to bound the errors of the Chebyshev expansion of the above type of singular functions. For example, if \(\alpha\) is not an integer, we know \(u \in W_{\theta}^{\alpha+1}(\Omega)\), so we have

\[
\|u - \pi u^C\|_{L^\infty(\Omega)} \leq CN^{-\alpha}, \quad \|u - \pi u^C\|_{L^2(\Omega)} \leq CN^{-\alpha - 1/2}. \tag{4.60}
\]

We tabulate in Table 4.1 the errors and convergence order of Chebyshev approximations to \(u(x) = |x - \theta|^\alpha\) with various \(\alpha\) and \(\theta = 0, 1/2\).

**Table 4.1.** Convergence order of \(u = |x - \theta|^\alpha\) with \(\theta = 0, 1/2\).

| \(N\) | \(u = |x|^\alpha\) (error in \(L^\infty\)-norm) | \(u = |x - 1/2|^\alpha\) (error in \(L^\infty\)-norm) |
|---|---|---|
| \(\alpha = 0.1\) | order | \(\alpha = 1.2\) | order | \(\alpha = 2.6\) | order | \(\alpha = 0.1\) | order | \(\alpha = 1.2\) | order | \(\alpha = 2.6\) | order |
| 2\(^2\) | 6.68e-1 | – | 8.37e-3 | – | 8.32e-5 | – | 6.60e-1 | – | 7.31e-3 | – | 6.18e-5 | – |
| 2\(^4\) | 6.24e-1 | 1.0 | 3.71e-3 | 1.17 | 1.42e-5 | 2.55 | 6.16e-1 | 1.0 | 3.15e-3 | 1.21 | 1.00e-3 | 2.63 |
| 2\(^6\) | 5.83e-1 | 1.0 | 1.63e-3 | 1.19 | 2.40e-6 | 2.57 | 5.75e-1 | 1.0 | 1.38e-3 | 1.19 | 1.68e-6 | 2.57 |
| 2\(^8\) | 5.44e-1 | 1.0 | 7.13e-4 | 1.19 | 3.99e-7 | 2.59 | 5.36e-1 | 1.0 | 6.01e-4 | 1.20 | 2.76e-7 | 2.61 |
| 2\(^{10}\) | 5.08e-1 | 1.0 | 3.14e-4 | 1.20 | 6.62e-8 | 2.59 | 5.00e-1 | 1.0 | 2.62e-4 | 1.20 | 4.58e-8 | 2.59 |
| 2\(^{12}\) | 4.74e-1 | 1.0 | 1.36e-4 | 1.20 | 1.09e-8 | 2.60 | 4.67e-1 | 1.0 | 1.14e-4 | 1.20 | 7.54e-9 | 2.60 |

| \(N\) | \(u = |x|^\alpha\) (error in \(L^2\)-norm) | \(u = |x - 1/2|^\alpha\) (error in \(L^2\)-norm) |
|---|---|---|
| \(\alpha = 0.1\) | order | \(\alpha = 1.2\) | order | \(\alpha = 2.6\) | order | \(\alpha = 0.1\) | order | \(\alpha = 1.2\) | order | \(\alpha = 2.6\) | order |
| 2\(^2\) | 1.88e-2 | – | 1.68e-3 | – | 2.68e-5 | – | 1.89e-2 | – | 1.49e-3 | – | 2.02e-5 | – |
| 2\(^4\) | 1.25e-2 | 0.59 | 5.31e-4 | 1.66 | 3.27e-6 | 3.03 | 1.24e-2 | 0.61 | 4.53e-4 | 1.72 | 2.31e-6 | 3.13 |
| 2\(^6\) | 8.29e-3 | 0.59 | 1.66e-4 | 1.68 | 3.91e-7 | 3.07 | 8.22e-3 | 0.59 | 1.41e-4 | 1.68 | 2.75e-7 | 3.07 |
| 2\(^8\) | 5.48e-3 | 0.60 | 5.13e-5 | 1.69 | 4.61e-8 | 3.08 | 5.42e-3 | 0.60 | 4.33e-5 | 1.70 | 3.19e-8 | 3.11 |
| 2\(^{10}\) | 3.61e-3 | 0.60 | 1.58e-5 | 1.70 | 5.41e-9 | 3.09 | 3.58e-3 | 0.60 | 1.34e-5 | 1.70 | 3.74e-9 | 3.09 |
| 2\(^{12}\) | 2.37e-3 | 0.61 | 4.88e-6 | 1.70 | 6.33e-10 | 3.10 | 2.36e-3 | 0.60 | 4.11e-6 | 1.70 | 4.36e-10 | 3.10 |
4.4.2. Type-II: \( u(x) = |x - \theta|^\alpha \ln |x - \theta| \) in \( (4.49) \). We first present the following useful formulas.

**Lemma 4.2.** For real \( \eta > -1, s \geq 0 \) and \( x > a \),
\[
a_x I_x^\eta \{ (x-a)^{\nu} \ln(x-a) \} = \frac{\Gamma(n+1)}{\Gamma(\eta + s + 1)} \{ \ln(x-a) + \psi(\eta + 1) - \psi(s + 1) \} (x-a)^{\eta + s},
\]
and the same formula holds for \( x I_x^\eta \{ (b-x)^{\nu} \ln(b-x) \} \) (for \( x < b \)) with \( b - x \) in place of \( x - a \). Here,
\[
\ln z - \frac{1}{2z} < \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} < \ln z - \frac{1}{z}, \quad z > 0.
\]

**Proof.** The formula \( (4.61) \) is a direct consequence of \([30, (2.50)]\). The property of the \( \psi \)-function in \( (4.62) \) can be found in \([1, (2.2)]\). Note that we can derive the formula for \( u \) and the exact formula:
\[
\hat{u}_{2k} = \frac{\Gamma(a + 1)}{2\alpha - 2\pi} \frac{\Gamma(k - \alpha/2)}{\Gamma(k + \alpha/2 + 1)} \left\{ \pi \cos \frac{\alpha\pi}{2} + \sin \frac{\alpha\pi}{2} (2\psi(a + 1) - 2 \ln 2 - \psi(k + \alpha/2 + 1)) \right\}, \quad \forall k \in \mathbb{N}_0,
\]
which enjoys the asymptotic behaviour
\[
\hat{u}_{2k} = \frac{\Gamma(a + 1)}{2\alpha - 3\pi} \left\{ \frac{\pi}{2} \cos \frac{\alpha\pi}{2} + \sin \frac{\alpha\pi}{2} (\psi(a + 1) - \ln 2 - \ln k) \right\}, \quad k \gg 1.
\]

**Theorem 4.4.** For any \( \alpha \geq 0 \) and \( \theta \in (-1, 1) \), we have
\[
u \in \mathbb{N}_0.
\]

where \( \sigma := \alpha + 1 - \epsilon \), and \( |\hat{u}_n^C| \leq Cn^{-\sigma} \) for large \( n \).
If \( \theta = 0 \), then we have \( \hat{u}_{2k+1}^C = 0 \), and the exact formula:
\[
\hat{u}_{2k} = \frac{\Gamma(a + 1)}{2\alpha - 3\pi} \left\{ \frac{\pi}{2} \cos \frac{\alpha\pi}{2} + \sin \frac{\alpha\pi}{2} (\psi(a + 1) - \ln 2 - \ln k) \right\}, \quad k \gg 1.
\]

**Proof.** Let \( m = [\alpha] + 1 \) and \( \nu = \alpha - m \). We derive from a direct calculation that
\[
u \in \mathbb{N}_0.
\]

where \( \hat{d}_n^k \) is the same as in \( (4.54) \), and
\[
\hat{f}_n^k := \sum_{j=1}^{k} (-1)^{j-1} \frac{\Gamma(k + 1)\Gamma(\alpha + 1)}{\Gamma(k - j + 1)\Gamma(\alpha - k + j + 1)}.
\]

We see that \( u \in L^1(\Omega) \) and \( u, \cdots, u^{(m-1)} \in AC(\Omega) \). Next, using Lemma 4.2 we obtain that for \( x \in (\theta, 1) \),
\[
\hat{a}_{n} I_x^{\nu+1-s} \{ \ln(x-\theta) \} = \hat{d}_n^m \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2 - s)} \psi(\nu + 1) - \psi(\nu + 2 - s) + f_{n}^m \psi(\nu + s - 1) \psi^{\nu+1-s}.
\]

Thus, if \( \nu + 1 - s > 0 \), i.e., \( s < \alpha - 1 - m \), then \( \hat{a}_{n} I_x^{\nu+1-s} \{ \ln(x-\theta) \} \in BV(\Omega_{\theta}) \). Similarly, under the same condition, we have \( \hat{a}_{n} I_x^{\nu+1-s} \{ \ln(x-\theta) \} \in BV(\Omega_{\theta}) \). By the definition \( (4.1) \), we obtain \( u \in \mathbb{W}_\theta^\mu(\Omega) \), where \( \mu = m + s < \alpha + 1 \).

This implies \( (4.63) \). The bound in \( (4.64) \) follows from \( (4.14) \) straightforwardly.

If \( \theta = 0 \), then \( u(x) \) is an even function, so \( \hat{u}_{2k+1} = 0 \). It is known that
\[
\ln z = \lim_{\epsilon \to 0} \frac{z^\epsilon - 1}{\epsilon}, \quad z > 0.
\]
Using (4.68), we derive from (4.58) that
\[
\tilde{u}_{2k}^C = \frac{2}{\pi} \int_{-1}^{1} \left\{ \lim_{\varepsilon \to 0} \frac{|x|^{\varepsilon + \alpha} - |x|^\alpha}{\varepsilon} \right\} \frac{T_{2k}(x)}{\sqrt{1 - x^2}} \, dx
\]
\[= (-1)^k \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \sin \left( (\varepsilon + \alpha)\pi \right) \frac{\Gamma(\varepsilon + \alpha + 1)}{2} \frac{\Gamma(k - (\varepsilon + \alpha)/2)}{\Gamma(k + (\varepsilon + \alpha)/2 + 1)} \right\}.
\]
(4.69)
Noting that
\[
\frac{d}{d\varepsilon} \left\{ \sin \left( (\varepsilon + \alpha)\pi \right) \frac{\Gamma(\varepsilon + \alpha + 1)}{2} \frac{\Gamma(k - (\varepsilon + \alpha)/2)}{\Gamma(k + (\varepsilon + \alpha)/2 + 1)} \right\}
\[= \frac{\Gamma(\varepsilon + \alpha + 1)}{2^\varepsilon} \frac{\Gamma(k - (\varepsilon + \alpha)/2)}{\Gamma(k + (\varepsilon + \alpha)/2 + 1)} \left\{ \frac{\pi}{2} \cos \left( (\varepsilon + \alpha)\pi \right) \frac{\Gamma(k - (\varepsilon + \alpha)/2)}{\Gamma(k + (\varepsilon + \alpha)/2 + 1)} \right\}
\times \left( \psi(\varepsilon + \alpha) - \ln 2 - \psi(\varepsilon + \alpha)/2 - \psi(k + (\varepsilon + \alpha)/2 + 1)/2 \right),
\]
we obtain (4.65) from (4.69) and the L’Hospital’s rule immediately.
Taking \( \varepsilon = k - \alpha/2 \) in (4.62), we obtain
\[
\ln(k - \alpha/2) - \frac{1}{2k - \alpha} < \psi(k - \alpha/2) < \ln(k - \alpha/2) - \frac{1}{k - \alpha/2},
\]
which implies that for \( k \gg 1 \),
\[
\psi(k - \alpha/2) = \ln k + O(k^{-1}); \quad \psi(k + \alpha/2 + 1) = \ln k + O(k^{-1}).
\]
Using (4.66) leads to
\[
\frac{\Gamma(k - \alpha/2)}{\Gamma(k + \alpha/2 + 1)} = k^{-\alpha-1}(1 + O(k^{-2})), \quad k \gg 1.
\]
(4.71)
From (4.65) and (4.70)-(4.71), we obtain (4.66). \( \square \)

**Remark 4.4.** Consider the Chebyshev expansion of \( u = |x|^\alpha \ln |x| \), we observe from (4.66) that for \( n \gg 1, |\tilde{u}_n^C| \leq C(\ln n)n^{-(\alpha+1)} \). Therefore, we obtain directly the optimal estimates:
\[
\|u - \pi_n^C u\|_{L^\infty(\Omega)} \leq \sum_{n=N+1}^{\infty} |\tilde{u}_n^C| \leq C(\ln N)N^{\alpha}; \quad \|u - \pi_n^C u\|_{L^2(\Omega)} \leq C(\ln N)N^{-\alpha-1/2}.
\]
(4.72)

However, we find from (4.63) that the space \( W_0^{\alpha+1-\epsilon}(\Omega) \) is suboptimal to characterize this type of singularity. Indeed, by Theorem 4.2, we only have \( \|u - \pi_n^C u\|_{L^\infty(\Omega)} = O(N^{\epsilon-\alpha}) \) and \( \|u - \pi_n^C u\|_{L^2(\Omega)} = O(N^{\epsilon-\alpha-1/2}) \). The situation is reminiscent of the Besov framework in [4], where the spaces of Type-I and Type-II are defined through different space interpolation. The question of how to modify the fractional space to best characterize Type-II singularity in our setting appears nontrivial and is still open.

## 5. Improving existing results

In this section, we show that the previous estimates with \( s \to 1 \) improve the existing results on Chebyshev approximations (see, e.g., [34] [35] [35] [24]).

### 5.1. Existing estimates

As in [34], let \( \| \cdot \|_{T} \) be the Chebyshev-weighted 1-norm:
\[
\|u\|_{T} = \left\| \frac{u'(x)}{\sqrt{1 - x^2}} \right\|_{1},
\]
(5.1)
which is defined via a Stieltjes integral for any \( u \) of bounded variation.
Lemma 5.1. (see [34] Thms 4.2-4.3). If \( u, u', \ldots, u^{(m-1)} \) are absolutely continuous on \([-1, 1]\), and if \( \|u^{(m)}\|_T = V_T < \infty \) with integer \( m \geq 0 \), then for each \( n \geq m + 1 \),

\[
|\hat{u}_n^C| \leq \frac{2V_T}{\pi n(n-1)\cdots(n-m)},
\]

and for integer \( m \geq 1 \), and integer \( N \geq m + 1 \),

\[
\|u - \pi_N u\|_{L^\infty(\Omega)} \leq \frac{2V_T}{\pi m (N-m)^m}.
\]

We remark that the Chebyshev weight is removed in Trefethen [35] Thms 7.1-7.2], i.e., \( V_T \) is replaced by the total variation of \( u^{(m)} \).

Following the argument of summation by certain telescoping series in [38], Majidian (cf. [24, Thm 2.1]) derived sharper bounds. For comparison, we quote the estimates therein below.

Lemma 5.2. (see [24] Thm 2.1). If \( u, u', \ldots, u^{(m-1)} \) are absolutely continuous on \([-1, 1]\), and if \( \|u^{(m)}\|_T = V_T < \infty \) with integer \( m \geq 0 \), then for each \( n \geq m + 1 \),

\[
|\hat{u}_n^C| \leq \frac{2V_T}{\pi} \prod_{j=0}^{m} \frac{1}{n - m + 2j}.
\]

5.2. Improved estimates.

Theorem 5.1. Suppose that for integer \( m \geq 0 \), \( u, u', \ldots, u^{(m-1)} \) are absolutely continuous on \([-1, 1]\), and \( u^{(m)} \) is of bounded variation with the total variation denoted by \( V_L^{(m)} \).

(i) If \( n \geq m + 1 \) and \( n - m \) is odd, then

\[
|\hat{u}_n^C| \leq \frac{2V_L^{(m)}}{\pi} \prod_{j=0}^{m} \frac{1}{n - m + 2j}.
\]

(ii) If \( n \geq m + 1 \) and \( n - m \) is even, then

\[
|\hat{u}_n^C| \leq \frac{2V_L^{(m)}}{\pi \sqrt{n^2 - m^2}} \prod_{j=0}^{m-1} \frac{1}{n - m + 2j - 1}.
\]

(iii) If \( 0 \leq n \leq m + 1 \), then

\[
|\hat{u}_n^C| \leq \frac{2V_L^{(m)}}{\pi (2n - 1)!!}.
\]

Proof. We find from (4.12) (or (4.16) with one more step of integration by parts) that for \( n \geq m + 1 \),

\[
\hat{u}_n^C = \frac{1}{(2m+1)!! \pi} \int_{-1}^{1} G_n^{(m)}(x) \omega_{m+1}(x) du^{(m)}(x).
\]

Thus, by (5.8),

\[
|\hat{u}_n^C| \leq \frac{V_L^{(m)}}{(2m+1)!! \pi} \max_{|x| \leq 1} \{ \omega_{m+1}(x) |G_n^{(m)}(x)| \}.
\]

If \( n = m + 2p + 1 \) with \( p \in \mathbb{N}_0 \), we derive from (2.42a) with \( l = p \) and \( \lambda = m + 1 \) that

\[
\max_{|x| \leq 1} \{ \omega_{m+1}(x) |G_n^{(m)}(x)| \} \leq \frac{\Gamma(m+3/2)\Gamma(p+1/2)}{\sqrt{\pi} \Gamma(m+p+3/2)} = \frac{(2m+1)!! (2p-1)!!}{(2m+2p+1)!!}.
\]

Consequently, for \( n = m + 2p + 1 \) with \( p \in \mathbb{N}_0 \), we obtain from (5.9) (5.10) that

\[
|\hat{u}_n^C| \leq \frac{2}{\pi (2p+1)!! \pi} \frac{V_L^{(m)}}{(2m+1)!! \pi} \frac{V_L^{(m)}}{(2m+1)!! \pi} = \frac{V_L^{(m)}}{\pi (n-m) \cdot (n-m+2) \cdots (n+m)}.
\]
which implies \((5.5)\).

Similarly, if \(n = m + 2p + 2\) with \(p \in \mathbb{N}_0\), we derive from \((2.42b)\) with \(l = p\) and \(\lambda = m + 1\) that
\[
\max_{|x| \leq 1} \{\omega_{m+1}(x)|G_{n-m-1}^{(m+1)}(x)|\} \leq \frac{1}{\sqrt{(2p+2)(2m+2p+1)}} \frac{(2m+1)!!(2p+1)!!}{(2m+2p+1)!!},
\]
so by \((5.9)\), we have
\[
|\hat{u}_n^C| \leq \frac{2}{\pi} \frac{1}{\sqrt{(2p+2)(2m+2p+1)}} \frac{V_L}{(2p+3)(2p+5)\cdots(2p+2m+1)}
\]
\[
= \frac{2}{\pi} \frac{1}{\sqrt{n^2-m^2}} \frac{V_L}{(n-m+1)(n-m+3)\cdots(n+m-1)}.
\]
This leads to \((5.6)\).

In case of \(0 \leq n \leq m+1\), we derive from \((5.8)\) (with \(n = m+1\)) and the factor \(G_0^{(n)}(x) \equiv 1\) that
\[
\hat{u}_n^C = \frac{1}{(2n-1)!!} \frac{2}{\pi} \int_{-1}^{1} \omega_n(x) d\omega^{(n-1)}(x).
\]
Then we obtain \((5.7)\) immediately.

Next, we unify the bounds in (i)-(ii) of Theorem \(5.1\) without loss of the rate of convergence. In fact, this relaxation leads to the estimate \((5.4)\) in [21] Thm 2.1, but with \(V_L\) in place of \(V_T\). In other words, the bounds in Theorem \(5.1\) indeed improve the best available results.

**Corollary 5.1.** Under the same conditions as in Theorem \(5.1\), we have that for all \(n \geq m + 1\),
\[
|\hat{u}_n^C| \leq \frac{2V_L}{\pi} \frac{1}{\frac{1}{n-m+2j}}.
\]

**Proof.** It is evident that by \((5.5)-(5.6)\), we only need to prove this bound for \(n-m\) being even. One verifies readily the fundamental inequality:
\[
n^2 - (p-1)^2 \geq \sqrt{(n^2-p^2)(n^2-(p-2)^2)}, \quad \text{for} \quad 2 \leq p \leq n.
\]

If \(m\) is even, we can pair up the factors and use the above inequality with \(p = m, m-2, \ldots, 2\) to derive
\[
(n-m+1)(n-m+3)\cdots(n-1)(n+1)\cdots(n+m-3)(n+m-1)
\]
\[
= (n-m-1)^2(n^2-m^2)\cdots(n^2-1)
\]
\[
\geq \sqrt{n^2-m^2} \sqrt{n^2-(m-2)^2} \sqrt{n^2-(m-2)^2} \sqrt{n^2-(m-4)^2} \cdots
\]
\[
= \sqrt{n^2-m^2} \sqrt{n^2-m^2} \cdots(n+m-2).
\]

Similarly, if \(m\) is odd, we remain the middle most factor intact and pair up the factors to derive the above. Therefore, multiplying both sides of \((5.16)\) by \(\sqrt{n^2-m^2}\), we obtain
\[
\frac{1}{\sqrt{n^2-m^2}} \prod_{j=0}^{m-1} \frac{1}{n-m+2j-1} \leq \prod_{j=0}^{m} \frac{1}{n-m+2j}.
\]

Then \((5.15)\) follows from \((5.17)\) and (i)-(ii) of Theorem \(5.1\) directly.

To show the sharpness of our improved bounds, we consider \(u = |x-\theta|, \theta \in (-1, 1)\) to compare upper bounds of \(\hat{u}_n^C\). In this case, we have \(m = 1, u'' = 2\theta(x-\theta), V_L^{(1)} = 2\) and \(V_T = 2(1-\theta^2)^{-1/2}\). Let \(\text{Ratio}_1\) and \(\text{Ratio}_2\) be the ratios of upper bounds in \([35, 21]\) (cf. \((5.2)\) with \(V_T\) being replaced by the bounded variation of \(u'\), and the bound in \((5.4)\) and our improved bound in Theorem \(5.1\)) respectively. In Figure \(5.1\) we depict two ratios against various \(n\) for two values of \(\theta\). We see that the improve bound is sharper than the existing ones, and the removal of the Chebyshev weight in \(V_T\) is also significant for the sharpness of the bounds.

To conclude this section, we state below the improved \(L^\infty\)-estimates, and remark on the improvements in Remark \(5.1\) below.
Theorem 5.2. Let $u \in W^{m+1}(\Omega)$ with integer $m \geq 0$.

(i) If $1 \leq m \leq N$, then

$$
\|u - \pi_N^{C}u\|_{L^\infty(\Omega)} \leq \frac{2}{m\pi} \left( \prod_{j=1}^{m} \frac{1}{N - m + 2j - 1} \right) V_L^{(m)}.
$$

(ii) If $m = 0$, then for all integer $N \geq 1$,

$$
\|u - \pi_N^{C}u\|_{L^\infty(\Omega)} \leq V^{(0)}_L.
$$

(iii) If $m \geq N + 1$, then

$$
\|u - \pi_N^{C}u\|_{L^\infty(\Omega)} \leq \frac{2}{(2N + 1)!! \pi} \sum_{n=N}^{m} \frac{(2N + 1)!!}{(2n + 1)!!} V_L^{(n)},
$$

where $c_n = 1$ for all $N \leq n \leq m - 1$ and $c_m = 2$.

Proof. From Theorem 4.2 with $s \to 1$ and (4.17), we obtain that for $1 \leq m \leq N + 1$,

$$
\|u - \pi_N^{C}u\|_{L^\infty(\Omega)} \leq \frac{1}{2^{m-1}m\pi} \frac{\Gamma((N - m + 1)/2)}{\Gamma((N + m + 1)/2)} V_L^{(m)}
$$

$$
= \frac{2}{m\pi} \frac{(N - m - 1)!!}{(N + m + 1)!!} V_L^{(m)} = \frac{2}{m\pi} \left( \prod_{j=1}^{m} \frac{1}{N - m + 2j - 1} \right) V_L^{(m)}.
$$

This gives (5.18). We now prove (5.19). Using integration part parts leads to

$$
(u - \pi_N^{C}u)(x) = \sum_{n=N+1}^{\infty} \hat{u}_n^C T_n(x) = \sum_{n=N+1}^{\infty} \left( \int_{0}^{\pi} u(\cos \varphi) \cos(n\varphi) d\varphi \right) \cos(n\theta)
$$

$$
= \frac{2}{\pi} \sum_{n=N+1}^{\infty} \left( \int_{0}^{\pi} u'(\cos \varphi) \sin(n\varphi) \sin(n\theta) \cos(n\varphi) d\varphi \right) \frac{\cos(n\theta)}{n} = \frac{2}{\pi} \int_{0}^{\pi} u'(\cos \varphi) \sin(n\varphi) \Psi_N^{\infty}(\varphi, \theta) d\varphi,
$$

so we have

$$
\left| (u - \pi_N^{C}u)(x) \right| \leq \frac{2}{\pi} \max_{\varphi \in [0, \pi]} \left| \Psi_N^{\infty}(\varphi, \theta) \right| \int_{0}^{\pi} \left| u'(\cos \varphi) \right| \sin \varphi d\varphi = \frac{2}{\pi} \max_{\varphi \in [0, \pi]} \left| \Psi_N^{\infty}(\varphi, \theta) \right| V^{(0)}_L,
$$

for $x = \cos \theta$, $\theta \in (0, \pi)$, where

$$
\Psi_N^{\infty}(\varphi, \theta) = \sum_{n=N+1}^{\infty} \frac{\sin(n\varphi) \cos(n\theta)}{n} = \sum_{n=N+1}^{\infty} \frac{\sin(n(\varphi + \theta)) + \sin(n(\varphi - \theta))}{2n}.
$$
We next show that for \( \vartheta \in \mathbb{R} \),
\[
\left| \sum_{n=N+1}^{\infty} \frac{\sin(n \vartheta)}{n} \right| \leq \frac{\pi}{2},
\]
(5.25)
In fact, it suffices to derive this bound for \( \vartheta \in (0, \pi) \), as the series defines an odd, 2\( \pi \)-periodic function which vanishes at \( \vartheta = 0, \pi \). It is known that
\[
\sum_{n=1}^{\infty} \frac{\sin(n \vartheta)}{n} = \frac{\pi}{2} - \vartheta, \quad \vartheta \in (0, \pi).
\]
(5.26)
According to [2], we have that for \( N \geq 2 \),
\[
0 < \sum_{n=1}^{N} \frac{\sin(n \vartheta)}{n} \leq \alpha \left( \pi - \vartheta \right), \quad \vartheta \in (0, \pi),
\]
(5.27)
with the best possible constant \( \alpha = 0.66395 \cdots \). As a direct consequence of (5.26)-(5.27), we have
\[
\left| \sum_{n=N+1}^{\infty} \frac{\sin(n \vartheta)}{n} \right| \leq \left| \sum_{n=N+1}^{\infty} \frac{\sin(n \vartheta)}{n} \right| < \frac{\pi}{2} - \vartheta; \quad \vartheta \in (0, \pi)
\]
(5.28)
for \( N \geq 2 \) and \( \vartheta \in (0, \pi) \). In fact, the bound (5.28) also holds for \( N = 1 \), as by (5.26),
\[
\sum_{n=2}^{\infty} \frac{\sin(n \vartheta)}{n} = \frac{\pi}{2} - \vartheta - \sin \vartheta < \frac{\pi}{2}.
\]
Thus, we complete the proof of (5.25). The estimate (5.19) is a direct consequence of (5.23)-(5.25).

Finally, we turn to the proof of the estimate (5.20). For \( m \geq N+1 \), we use (5.7) to bound \( \{ u_n^C \}_{n=N+1}^{m} \), and use (5.18) (with \( N \to m \)) to derive
\[
|u(x) - \pi_N^C u(x)| \leq |\pi_m^C u(x) - \pi_N^C u(x)| + |u(x) - \pi_m^C u(x)|
\]
\[
\leq \sum_{n=N+1}^{m} \frac{2}{\pi(2n-1)!!} V_L^{(n-1)} + \frac{2}{m(2m-1)!! \pi} V_L^{(m)}
\]
\[
\leq \frac{2}{\pi(2N+1)!!} \left\{ \sum_{n=N+1}^{m} \frac{(2N+1)!! V_L^{(n-1)}}{(2n-1)!! V_L} + \frac{m+1(2N+1)!! V_L^{(m)}}{m(2m+1)!! V_L} \right\}
\]
\[
\leq \frac{2}{\pi(2N+1)!!} \sum_{n=N}^{m} c_n(2N+1)!! V_L^{(n)} / (2n+1)!! V_L,
\]
(5.29)
where \( c_n = 1 \) for all \( N \leq n \leq m-1 \) and \( c_m = 2 \).

**Remark 5.1.** Taking a different route, we improve the existing bounds in the following senses.

(i) The Chebyshev-weighted 1-norm in Lemma 5.2 is replaced by the Legendre-weighted 1-norm.

(ii) Sharper bound is obtained than the best one in [24 Thm 2.1].

(iii) We obtain the “stability” result, that is, \( m = 0 \) in (5.3), and the estimate for the case \( n \leq m+1 \) in (5.7), which are new.

6. Analysis of interpolation, quadrature and endpoint singularities

In this section, we discuss the extension of the main results to the error estimates of the related interpolation, quadratures and also special types of functions with endpoint singularities. We then conclude the paper with some final remarks.
6.1. Analysis of interpolations and quadrature. As remarked in [38, 35, 24], the error analysis of several widely-used interpolations and quadrature boils down to estimating the coefficients \{\hat{u}_n^C\} and their partial sums. We refer to [24] for a list of more than six examples. Here, we just examine two cases and present sharp bounds by using our new estimates on the decay of the expansion coefficients.

(i) Interpolation and quadrature at Chebyshev-Gauss (CG) points \{x_j\}_{j=0}^N\), i.e., zeros of \(T_{N+1}(x)\)

\[
(I_N^C u)(x) = \sum_{n=0}^{N} b_n T_n(x), \quad b_n = \frac{2}{N+1} \sum_{j=0}^{N} u(x_j)T_n(x_j),
\]

and

\[
\int_{-1}^{1} u(x)(1-x^2)^{-1/2} \, dx = \frac{\pi}{N+1} \sum_{j=0}^{N} u(x_j) + \mathcal{R}_N^C[u].
\]

Then we have (cf. [38] and [29] (6))

\[
\|I_N^C u - u\|_{L^\infty(\Omega)} \leq 2 \sum_{n=N+1}^{\infty} |\hat{u}_n^C|; \quad \mathcal{R}_N^C[u] = \pi \sum_{k=1}^{\infty} (-1)^k \hat{u}_{2k(N+1)}^C.
\]

(ii) Legendre-Gauss quadrature rule at the zeros \{x_j\}_{j=0}^N\) of the Legendre polynomial \(P_{N+1}(x)\) and with quadrature weights \{\omega_j\}_{j=0}^N\) (cf. [32, 24, P. 96]):

\[
\int_{-1}^{1} u(x)\, dx = \sum_{j=0}^{N} u(x_j)\omega_j + \mathcal{R}_N^L[u].
\]

Then we have (cf. [34, 24]):

\[
|\mathcal{R}_N^L[u]| \leq \frac{32}{10} \sum_{n=N+1}^{\infty} |\hat{u}_{2n}^C|.
\]

Using Theorem 4.1 and the argument similar to Theorem 4.2 (also see Remark 4.2), we can obtain the following estimates.

**Theorem 6.1.** Given \(\theta \in (-1, 1)\), if \(u \in \mathbb{W}^{m+s}_\theta(\Omega)\) with \(s \in (0, 1)\) and integer \(m \geq 0\), then for \(m+s > 1\), we have

\[
\|u - I_N^C u\|_{L^\infty(\Omega)} \leq C N^{1-m-s} u^m_\theta; \quad \|u - I_N^C u\|_{L^2(\Omega)} \leq C N^{\frac{1}{2}-m-s} u^m_\theta;
\]

and

\[
|\mathcal{R}_N^C[u]| \leq C N^{-(m+s)} u^m_\theta; \quad |\mathcal{R}_N^L[u]| \leq C N^{-(m+s)} u^m_\theta,
\]

where \(C\) is a positive constant independent of \(N\) and \(u\).

**Proof.** We just provide the proof of the \(L^2\)-error of the CG interpolation, since the others can be proved by summing up the bounds of \{\hat{u}_n^C\} in Theorem 4.1 and Remark 4.2. Note that

\[
I_N^C u(x) - u(x) = T_N^C u(x) - \pi_N^C u + \pi_N^C u - u = \sum_{n=0}^{N} (b_n - \hat{u}_n^C) T_n(x) + \pi_N^C u - u.
\]

Hence, we obtain

\[
\|u - I_N^C u\|_{L^2(\Omega)}^2 \leq \frac{\pi}{2} \sum_{n=0}^{N} |b_n - \hat{u}_n^C|^2 + \|u - \pi_N^C u\|_{L^2(\Omega)}^2.
\]

Recall that (cf. [11] (4.56)):

\[
b_n - \hat{u}_n^C = \sum_{k=1}^{\infty} (-1)^k (\hat{u}_{2k(N+1)-n}^C + \hat{u}_{2k(N+1)+n}^C), \quad n = 0, \ldots, N.
\]
Using (4.14) and (4.46), we find that for $N \gg 1$, $\sigma = m + s > 1$ and $n = 0, \ldots, N$,

$$|b_n - \hat{u}_n^C| \leq \sum_{k=1}^{\infty} \left| \hat{u}_{2k(N+1)-n}^C \right| + |\hat{u}_{2k(N+1)+n}^C| \leq \frac{U_{\theta}^{m,s}}{2^\sigma - 1} \frac{2}{\pi} \sum_{k=1}^{\infty} \Gamma((2k(N+1) - n - \sigma + 1)/2) \leq \frac{U_{\theta}^{m,s}}{2^\sigma - 1} \sum_{k=1}^{\infty} \Gamma((2k(N+1) - n + \sigma + 1)/2)$$

(6.11)

From (6.9), we obtain from a direct calculation and Remark 4.2 the $L^2_\omega$-estimate. □

6.2. Analysis of endpoint singularities. The previous discussions were centred around the Chebyshev expansions and approximation of singular functions with interior singularities. In what follows, we extend the results to the cases with $\theta = \pm 1$, and study endpoint singularities. To fix the idea, we shall focus on the exact formulas and decay rate of the Chebyshev expansion coefficients, since it is the basis to derive many other related error bounds.

Let $\mathbb{W}^{m,s}_\pm(\Omega)$ be the fractional Sobolev-type spaces defined in (6.17). The following representation of $\hat{u}_n^C$ is a direct consequence of Theorem 4.1.

**Theorem 6.2.** If $u \in \mathbb{W}^\sigma_\pm(\Omega)$ with $\sigma := m + s$, $s \in (0, 1)$ and $m \in \mathbb{N}_0$, then for $n \geq \sigma > 1/2$,

$$\hat{u}_n^C = (-1)^{n^+ [n-s]} C_\sigma \left\{ \int_{-1}^{1} R_{1}^s x^m u^\omega(x) \omega_\nu(x) \frac{dx}{x} \right\} + \left\{ \int_{-1}^{1} R_{1}^s x^m u^\omega(x) \omega_\nu(x) \frac{dx}{x} \right\}$$

(6.12)

Similarly, if $u \in \mathbb{W}^\sigma_\pm(\Omega)$ with $\sigma := m + s$, $s \in (0, 1)$ and $m \in \mathbb{N}_0$, then for $n \geq \sigma > 1/2$,

$$\hat{u}_n^C = C_\sigma \left\{ \int_{-1}^{1} R_{1}^s x^m u^\omega(x) \omega_\nu(x) \frac{dx}{x} \right\}$$

(6.13)

Here, $\omega_\nu(x) = (1 - x^2)^{1/2}$ and $C_\sigma := (\sqrt{\pi} 2^\sigma - 1 \Gamma(\sigma + 1))^{-1}$.

We next apply the formulas to several typical types of singular functions. We first consider $u(x) = (1 + x)^\alpha$ with $\alpha > -1/2$ and $\alpha \notin \mathbb{N}_0$ (see, e.g., [36], [17]). Following the proof of Proposition 4.3, we have $u \in \mathbb{W}^{\sigma+1}_\pm(\Omega)$. Then using (6.13), one obtains the exact formula of the Chebyshev expansion coefficient. Equivalently, one can derive it by taking $\theta \to -1^+$ in (4.50). More precisely, by (2.22) and (4.50),

$$\hat{u}_n^C = \frac{\Gamma(\alpha + 1)}{2^\sigma \Gamma(\alpha + 3/2) \sqrt{\pi}} \lim_{\nu \to 1^+} \left\{ G^{(\alpha+1)}_{n-\alpha-1}(\theta) \omega_{\alpha+1}(\theta) - \frac{(-1)^{n^+[n-\alpha]} G^{(\alpha+1)}_{n-\alpha-1}(\theta) \omega_{\alpha+1}(\theta)}{\Gamma(\nu + 1)} \right\}$$

(6.14)

Using (2.29) leads to that for $\lambda > 1/2$,

$$\lim_{x \to 1^+} \omega_\nu(x) G^{(1)}_{\nu}(x) = -2^{\alpha-1} \sin(\nu \pi) \frac{\Gamma(\lambda - 1/2) \Gamma(\lambda + 1/2) \Gamma(\nu + 1)}{\Gamma(\nu + 2\lambda)}$$

(6.15)

Therefore, from (4.39) and (6.14) and (6.15), we obtain the formula:

$$\hat{u}_n^C = \frac{(-1)^{n+1} \sin(\pi \alpha) \Gamma(2\alpha + 1)}{2^{\alpha-1} \pi} \frac{\Gamma(n - \alpha)}{\Gamma(n + \alpha + 1)}, \quad n \geq \alpha + 1,$$

(6.16)

and for large $n$, we have $|\hat{u}_n^C| = O(n^{-2\alpha-1})$.

With the aid of (6.16), we next consider a more general case: $u(x) = (1 + x)^\alpha g(x)$ with $g(x)$ being a sufficiently smooth function. Here, we need to use the formula of $\hat{u}_n^C$ for $(1 + x)^\alpha$ with $n < \alpha + 1$.Taking
Compared with the interior singularities, the proposed spaces are naturally arisen from the analytic representations of the expansion coefficients.

In view of (4.62), we can the asymptotic behaviour

\[ u_n^C = \frac{(-1)^n + 1}{2} \pi^{\alpha - 1} (2\psi(2\alpha + 1) + \ln 2) + O(n^{-2\alpha - 3}). \]

Remark 6.1. With the above analysis of the expansion coefficients, we can then obtain directly the optimal estimates for the Chebyshev approximation to these specific singular functions. More precisely, for \( u(x) = (1 + x)^\alpha \ln(1 + x) \) with \( g(x) \) being a sufficiently smooth function, we have

\[ \|u - \pi_N^C u\|_{L^\infty(\Omega)} \leq CN^{-2\alpha}, \quad \|u - \pi_N^C u\|_{L_2^\infty(\Omega)} \leq CN^{-2\alpha - 1/2}, \]

(6.19)

and for \( u(x) = (1 + x)^\alpha \ln(1 + x) \), we have

\[ \|u - \pi_N^C u\|_{L^\infty(\Omega)} \leq C(\ln N)^{-2\alpha}, \quad \|u - \pi_N^C u\|_{L_2^\infty(\Omega)} \leq C(\ln N)^{-2\alpha - 1/2}. \]

(6.20)

Compared with the interior singularities (see (4.60) and (4.72)), a higher convergence order \( O(N^{-\alpha}) \) is observed which is as expected.

6.3. Concluding remarks. Broadly speaking, we position this work as our first attempt to show how the RL fractional calculus can alter the fundamental polynomial approximation theory. Some estimates and bounds herein are completely new, or significantly improve the existing results.

More precisely, we introduce a new theoretical framework of fractional Sobolev-type spaces for orthogonal polynomial approximations to functions with limited regularities (or interior/endpoint singularities). The proposed spaces are naturally arisen from the analytic representations of the expansion coefficients.
involving RL fractional integrals/derivatives and GGF-Fs. We present a collection of notable properties of the new family of GGF-Fs, and derive optimal estimates of Chebyshev approximations in various norms for a wide class of singular functions. The analysis techniques can be extended to general Jacobi approximations. We are confident that this study, together with our follow-up works, will have far-reaching impact on numerical analysis of $p$-version and $hp$-version for singular problems.

REFERENCES

[1] H. Alzer. On some inequalities for the Gamma and Psi functions. Math. Comput., 66(217):373–389, 1997.
[2] H. Alzer and S. Koumandos. Sharp inequalities for trigonometric sums. Math. Proc. Camb. Phil. Soc., 139:139–152, 2003.
[3] G.E. Andrews, R. Askey, and R. Roy. Special Functions, Encyclopedia of Mathematics and its Applications, Vol. 71. Cambridge University Press, Cambridge, 1999.
[4] I. Babuška and B.Q. Guo. Optimal estimates for lower and upper bounds of approximation errors in the $p$-version of the finite element method in two dimensions. Numer. Math., 85(2):219–255, 2000.
[5] I. Babuška and B.Q. Guo. Direct and inverse approximation theorems for the $p$-version of the finite element method in the framework of weighted Besov spaces. Part I: Approximability of functions in the weighted Besov spaces. SIAM J. Numer. Anal., 39(5):1512–1538, 2001.
[6] I. Babuška and B.Q. Guo. Direct and inverse approximation theorems for the $p$-version of the finite element method in the framework of weighted Besov spaces, Part II: Optimal rate of convergence of the $p$-version finite element solutions. Math. Models Methods Appl. Sci., 12(5):689–719, 2002.
[7] M. Bergounioux, A. Leaci, G. Nardi, and F. Tomarelli. Fractional Sobolev spaces and functions of bounded variation. Fract. Calc. Appl. Anal., 20(4):936–962, 2017.
[8] C. Bernardi and Y. Maday. Spectral Methods. In P.G. Ciarlet and J.L. Lions, editors, Handbook of Numerical Analysis, Vol. V, Part 2, pages 209–485. North-Holland, Amsterdam, 1997.
[9] L. Bourdin and D. Idczak. A fractional fundamental lemma and a fractional integration by parts formula-applications to critical points of Bolza functionals and to linear boundary value problems. Adv. Differential Equations, 20(3–4):213–232, 2015.
[10] J.P. Boyd. The asymptotic Chebyshev coefficients for functions with logarithmic endpoint singularities. Appl. Math. Comput., 29:49–67, 1989.
[11] J.P. Boyd. Chebyshev and Fourier Spectral Methods, 2nd Ed. Dover, New York, 2001.
[12] J. Bustoz and M.E.H. Ismail. On Gamma function inequalities. Math. Comput., 47(176):659–667, 1986.
[13] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. Spectral Methods: Fundamentals in Single Domains. Springer, Berlin, 2006.
[14] P. Castillo, B. Cockburn, Schötzau, and C. Schwab. Optimal a priori error estimates for the $hp$-version of the local discontinuous Galerkin method for convection-diffusion problems. Math. Comp., 71(238):455–478, 2002.
[15] S. Chen, J. Shen, and L.L. Wang. Generalized Jacobi functions and their applications to fractional differential equations. Math. Comp., 85(300):1603–1638, 2016.
[16] D. Funaro. Polynomial Approximations of Differential Equations. Springer-Verlag, Berlin, 1992.
[17] W. Gui and I. Babuška. The $h$, $p$ and $h$-$p$ versions of the finite element method in 1 dimension, Part I: The error analysis of the $p$-version. Numer. Math., 49:205–612, 1986.
[18] B.Y. Guo, J. Shen, and L.L. Wang. Optimal spectral-Galerkin methods using generalized Jacobi polynomials. J. Sci. Comput., 27(1-3):305–322, 2006.
[19] B.Y. Guo, J. Shen, and L.L. Wang. Generalized Jacobi polynomials/functions and their applications. Appl. Numer. Math., 59(5):1011–1028, 2009.
[20] B.Y. Guo and L.L. Wang. Jacobi approximations in non-uniformly Jacobi-weighted Sobolev spaces. J. Approx. Theory, 128(1):1–41, 2004.
[21] J. Hesthaven, S. Gottlieb, and D. Gottlieb. Spectral Methods for Time-Dependent Problems. Cambridge University Press, Cambridge, 2007.
[22] F.C. Klebaner. Introduction to Stochastic Calculus with Applications, 2nd Ed. Imperial College Press, London, 2005.
[23] G. Leoni. A First Course in Sobolev Spaces. Amer. Math. Soc., Providence, RI, 2009.
[24] H. Majidian. On the decay rate of Chebyshev coefficients. Appl. Numer. Math., 85(2):602–614, 1994.
[25] P. Neval, T. Erdélyi, and A.P. Magnus. Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials. SIAM J. Math. Anal., 25(2):602–614, 1994.
[26] F.W.J. Olver. Asymptotics and Special Functions. Academic Press, New York, 1974.
[27] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark. NIST Handbook of Mathematical Functions. Cambridge University Press, New York, 2010.
[28] R.D. Riess and L.W. Johnson. Estimating Gauss-Chebyshev quadrature errors. SIAM J. Numer. Anal., 6:557–559, 1969.
[30] S.G. Samko, A.A. Kilbas, and O.I. Marichev. Fractional Integrals and Derivatives, Theory and Applications. Gordon and Breach Science Publisher, New York, 1993.
[31] C. Schwab. p- and hp-FEM. Theory and Application to Solid and Fluid Mechanics. Oxford University Press, New York, 1998.
[32] J. Shen, T. Tang, and L.L. Wang. Spectral Methods: Algorithms, Analysis and Applications, volume 41 of Series in Computational Mathematics. Springer-Verlag, Berlin, Heidelberg, 2011.
[33] G. Szegő. Orthogonal Polynomials, 4th Ed. Amer. Math. Soc., Providence, RI, 1975.
[34] L.N. Trefethen. Is Gauss quadrature better than Clenshaw-Curtis? SIAM Rev., 51(1):67–87, 2008.
[35] L.N. Trefethen. Approximation Theory and Approximation Practice. SIAM, Philadelphia, 2013.
[36] P.D. Tuan and D. Elliott. Coefficients in series expansions for certain classes of functions. Math. Comp., 26:213–232, 1972.
[37] H.Y. Wang. On the convergence rate of Clenshaw-Curtis quadrature for integrals with algebraic endpoint singularities. J. Comput. Appl. Math., 333:87–98, 2018.
[38] S.H. Xiang, X.J. Chen, and H.Y. Wang. Error bounds for approximation in Chebyshev points. Numer. Math., 116:463–491, 2010.
[39] M. Zayernouri and G.E. Karniadakis. Fractional Sturm-Liouville eigen-problems: theory and numerical approximation. J. Comput. Phys., 252:495–517, 2013.