Graph hierarchy and spread of infections

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Abstract

Trophic levels and hence trophic coherence can be defined only on networks with well defined sources, trophic analysis of networks had been restricted to the ecological domain until now. Trophic coherence, a measure of a network’s hierarchical organisation, has been shown to be linked to a network’s structural and dynamical aspects. In this paper we introduce hierarchical levels, which is a generalisation of trophic levels, that can be defined on any simple graph and we interpret it as a network influence metric. We discuss how our generalisation relates to the previous definition and what new insights our generalisation shines on the topological and dynamical aspects of networks. We also show that the mean of hierarchical differences correlates strongly with the topology of the graph. Finally, we model an epidemiological dynamics and show how the statistical properties of hierarchical differences relate to the incidence rate and how it affects the spreading process in a SIS model.
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1 Introduction

Patient zero is the start of an epidemic that spreads through a city. A rumour spreads like wildfire amongst a group of friends. An accident happens on the road and the associated disturbance spreads congestion throughout the road network in the vicinity of the incident. These are just a small number of examples of real life processes involving the spread of some quantity, whether it be information or a physical, tangible quantity, across a network structure. Networks are omnipresent and they constitute many of the complex systems that underlie much of our infrastructure but also our social interactions as well as ecological and biological systems that control and regulate life.

Since the turn of the millennium there has been an explosion of research in network science [1], [2]. Understanding how signals or processes percolate through a network and what role network topology and structure plays in this, has been a key research aim [3]. How is the most critical part of the network determined when a flow is spreading? How does this affect the network resilience? How does the topology of a network affect the dynamics? These are just some of the questions present in the field of network dynamics.

One field where networks play a significant role is ecology [4]. Network tools are used to understand the complex ecosystems and food webs that are present in our environment. Ecological networks have a natural trophic structure, and researchers have defined a quantity known as trophic level to better describe the hierarchical nature of these networks [5]. The trophic level of a node is its hierarchical level in a network when taking into account its position relative to all other nodes. This partitions the network into an ordered hierarchy, known as a partial order in the mathematical literature. In ecological networks this represents the flow of energy from prey to predators.

Trophic coherence presents a measure of this ordering via the distribution of differences of trophic level among the nodes of the network which are adjacent to one another. This provides a measure of how organised a network is and how neatly the structure is defined by discrete levels or partitions. Research has shown that trophic coherence is a proxy for the stability of a food web network [6]. More work in this area also found that the lack of cycles in a network is inherently linked with the trophic coherence of a network, [7]. These ideas have also been used to analyse the spread of infections on a network, [8], and to assess robustness of rail networks, [9]. This illustrates a link between the stability and dynamics of the network, and the underlying graph structure.

But trophic levels have only been defined for networks where there are clear basal nodes, i.e. nodes with zero in-degree. Basal nodes correspond to producers like plants in a food web. In this paper we define a notion of trophic levels which is applicable to any network.

Using our definition we can apply a trophic structure to networks of any type and determine a hierarchy of the nodes present in the network. Moreover, we introduce the hierarchical incoherence parameter as a measure of the presence of discrete layers in a network. The hierarchical incoherence parameter corresponds to the trophic incoherence parameter defined by Johnson et al. in [6].
also introduce the democracy coefficient as a measure of the size of subgraphs that are not influenced by the rest of the graph. We show that the democracy coefficient correlates strongly with the topology of the graph. Finally, we study the relationship between the democracy coefficient and hierarchical incoherence parameter of a network and the diffusive properties of the network by modelling a contagion dynamics.

**Notation**

On a graph $G$ we can define trophic levels if and only if there is a directed path from any vertex to at least one basal vertex, we give a proof of this statement in Section 6.

Food webs model energy flow between species and in this context, any vertex with no in-neighbours represents a primary producer species, for example grass. These are the species that input energy in the food web. Such vertices are called basal.

We shift our point of view from energy flow to influence and we consider the following dynamics on a graph. We assign a colour to each vertex. Then, at each time step a vertex chooses uniformly at random a colour between its own colour and the colours of its in-neighbours and changes its colour to correspondingly. We will call this influence dynamics. In this dynamics a vertex with no in-neighbours is important because it stays at its original colour forever. We call such vertices influencers, see Definition 1.1. We can control the equilibrium state of influence dynamics on graph on which we can define trophic levels by choosing the initial colour of its influencer vertices.

Throughout this article we will consider only simple directed graphs and we will use $G$ to denote them. We will denote the set of all vertices of $G$ by $V(G)$. We will use $n$ to denote the number of vertices of $G$.

**Definition 1.1.** For any a simple directed graph $G$, vertex $v \in V(G)$ will be called influencer if its in-degree is 0. The set of all influencer vertices of $G$ will be denoted by $I(G)$.

**Definition 1.2.** Any simple graph $G$ will be called simply influenced if $I(G)$ is not empty and for any $v \in V(G) \setminus I(G)$ there exists $u \in I(G)$ such that there is a directed path from $u$ to $v$.

Johnson et al. [6] define the adjacency matrix of a graph as the matrix $A$, where $(A)_{ij} = 1$ if there exists a directed edge from $j$ to $i$ ($j \to i$) and $(A)_{ij} = 0$ otherwise. In this article we will follow the standard definition, i.e. $(A)_{ij} = 1$ means there is a directed edge from $i$ to $j$ ($i \to j$) and 0 otherwise. We can change from our notation to the notation in [6] by taking the transpose of the adjacency matrix.

We define $d_i = \sum_j a_{ji}$ to be the in-degree of vertex $i$, $d = (d_1, \ldots, d_n)$ the in-degree vector and $D = \text{diag}(d)$ the in-degree matrix. The in-degree Laplacian of a graph is defined to be the matrix $L = D - A$. For notational convenience we define $M = L^T$, where $L^T$ is the transpose of $L$. 
We also define $\tilde{d}_i = \max\{1, d_i\}$ to be the positive in-degree and similarly we define $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_n)$, $\tilde{D} = \text{diag}(\tilde{d})$, $\tilde{L} = \tilde{D} - A$ and $\tilde{M} = \tilde{L}^T$.

We will denote by $s_i$ the trophic level (TL) of vertex $i$ and the vector of TLs by $s = (s_1, \ldots, s_n)$. Similarly, we will denote by $\tau_i$ the hierarchical level (HL) of vertex $i$ and the vector of HLs by $\tau = (\tau_1, \ldots, \tau_n)$.

2 Trophic levels and trophic differences

The concept of trophic levels was introduced in [10] as a way to determine the hierarchy of species in a food chain. Primary producers, for example plants, have trophic level 1 and the trophic level of every other species is 1 plus the average trophic level of the species it eats. Interconnected food chains form what is called a food web. In a perfectly layered food web, all species have integer trophic levels and the difference between the trophic levels of the prey and the predator is 1. In practice this rarely happens and the notion of the trophic incoherence parameter was introduced as a way to measure how far a food web is from being perfectly layered.

2.1 Trophic levels

A food web can be visualised as a directed graph. Typically, the direction of arrows indicate the flow of energy. See Figure 1 for an example. Trophic levels are defined by the following linear equations:

$$
\begin{align*}
    s_i &= 1 + \frac{1}{d_i} \sum_j a_{ji} s_j, & \text{if } d_i \neq 0, \\
    s_i &= 1 & \text{if } d_i = 0.
\end{align*}
$$

(1)

Using our notation we can write this system of equations in a compact form: $\tilde{M} \cdot s = \tilde{d}$. This leads to the following definition.

**Definition 2.1.** Let $G$ be a simply influenced graph. Then the vector of trophic levels on $G$ is

$$
    s = \tilde{M}^{-1} \cdot \tilde{d}.
$$

2.2 Trophic differences

Trophic difference (TD) is the difference of trophic levels between two vertices connected by an edge, i.e. the TDs of a simply influenced graph $G$ is the set

$$
    \text{TD}(G) = \{ s_j - s_i | a_{ij} = 1 \}.
$$

**Lemma 2.2.** Let $G$ be a simply influenced graph $G$, then $\text{Mean}(\text{TD}(G)) = 1$. 

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Figure 1: Two graphs representing two different food webs. Trophic levels are printed in black and trophic differences in red. (a) A totally coherent graph with integer trophic levels and trophic incoherence 0. (b) A less coherent graph with non-integer trophic levels and trophic incoherence 0.322.

We give the proof of this lemma in section 6. The standard deviation of $\text{TD}(G)$ is called the trophic incoherence parameter or just trophic incoherence of a simply influenced graph. We will use $q$ for the trophic incoherence and since the mean is always 1 we have

$$q(G) = \sqrt{\frac{\sum_{ij}(s_i - s_j)^2 a_{ij}}{\sum_{ij} a_{ij}}} - 1.$$ 

In a perfectly layered food web, all TDs are 1, so $q(G) = 0$.

3 Hierarchical levels and hierarchical differences

Trophic levels can only be defined on simply influenced graphs, which severely limits potential applications. In this section we propose a generalisation of trophic levels that can be applied to any simple directed graph\textsuperscript{1}.

3.1 Hierarchical levels

In order to make the connection between trophic levels and hierarchical levels clear, we first discuss in section 3.1.1 the case of simply influenced graphs and after that in section 3.1.2 we discuss the general case.

\textsuperscript{1}If a graph is undirected we turn it to a directed graph by replacing each undirected edge by a pair of directed edges.
3.1.1 Simply influenced graphs

Let us consider linear system (1) and rewrite the first equation using \( \tau \) instead of \( s \) as the unknown. We get

\[
d_i \tau_i - \sum_j a_{ji} \tau_j = d_i
\]

which we use to define trophic levels. Notice that in this case, if \( d_i = 0 \), the equation is still defined but it is trivially satisfied as it becomes \( 0 = 0 \). Using our notation we rewrite equations (2) as

\[
M \cdot \tau = d.
\]

Because the matrix \( M \) is singular, the above linear system does not have a unique solution. For a simply influenced graph the dimension of the kernel of \( M \) equals the number of influencer vertices. This means that we can get a unique solution by choosing arbitrary values of the trophic levels of the influencer vertices. A proof of this is given in Section 6.

We can recover the original definition of trophic levels by setting the trophic levels of all influencer vertices to 1. However, using this viewpoint, we see that the choice of 1 is somewhat arbitrary and any other choice is equally valid. Instead of prescribing the trophic levels of influencer vertices we use the following definition.

**Definition 3.1.** Let \( G \) be a simply influenced graph, \( d \) the in-degree vector, \( L \) its in-Laplacian matrix and \( M = L^T \). Let \( \mathcal{T} = \{ x \in \mathbb{R}^n | M \cdot x = d \} \) be the linear vector space of solutions of the equation (3). Then the vector of hierarchical levels of \( G \) is

\[
\tau = \arg \min_{x \in \mathcal{T}} \|x\|
\]

where \( \| \cdot \| \) denotes the 2-norm.

In practice we know that the solution is unique and is given by \( \tau = M^+ \cdot d \), where \( M^+ \) is the pseudo-inverse of \( M \), see [11].

We see in Figure 2 that the HLs of influencer vertices are typically not equal. This may seem strange for a food web, however it is worth noticing that the influencer vertex with the lowest HL is the root vertex for more food chains than the other influencer vertices. This shows that the hierarchical level is a measure of influence, for example in spreading ideas or an infection.

3.1.2 Generic graphs

In the case of generic graphs, the system \( M \cdot \tau = d \) has no solution. So instead we search for \( \tau \) that has minimal length in the subspace of \( \mathbb{R}^n \) where the 2-norm \( \|M \cdot \tau - d\| \) is minimal. Formally we define:
Figure 2: Hierarchical levels and differences on the same graphs as in Figure 1. HL are printed in black and HD in red. (a) The influencer vertices do not have the same HL and its hierarchical incoherence of the graph is 0.107. (b) A less coherent graph with hierarchical incoherence 0.423.

**Definition 3.2.** Let $G$ be a directed graph with $n$ vertices, $d$ be its in-degree vector, $L$ be its in-Laplacian matrix and $M = L^T$. We define

$$T = \arg \min_{x \in \mathbb{R}^n} \| M \cdot x - d \|.$$

Then the vector of hierarchical levels of $G$ is

$$\tau = \arg \min_{x \in T} \| x \|.$$

Notice, that if the graph is simply influenced then this definition coincides with Definition 3.1. Similarly to the case of simply influenced graphs, we know that the solution is unique and given by $\tau = M^+ \cdot d$, see [11].

Hierarchical levels identify the influential vertices in a graph. The lower the hierarchical level, the more influence the vertex has. There is also a non-trivial connection between HLs and a random walker on the reversed graph, i.e. the graph we get by reversing all edges. The stationary probability distribution of such a random walker is in the kernel of $D^+ \cdot M$.

### 3.2 Hierarchical differences

Similarly to trophic differences, we define hierarchical differences (HDs). We will see that the mean of HDs has many interesting properties and strong connection with the topology of the graph. The proofs of the lemmas are given in Section 6.
Definition 3.3. The HDs of a directed graph $G$ is the set
\[ \text{HD}(G) = \{ \tau_j - \tau_i \mid a_{ij} = 1, \ i, j \in V(G) \} \]

We can also define the HDs at a vertex $i$ by considering the HDs of the incoming edges.

Definition 3.4. The hierarchical differences at vertex $j$ of the directed graph $G$ is the set
\[ \text{HD}(G, j) = \{ \tau_j - \tau_i \mid a_{ij} = 1, \ j \in V(G) \} \]

For a graph $G$, the mean of $\text{HD}(G)$ is an important metric. This leads us to the following definition.

Definition 3.5. The democracy coefficient of a directed graph $G$ is
\[ \eta(G) = 1 - \text{Mean}(\text{HD}(G)) \]

A low democracy coefficient means that the graph is controlled by a small percentage of its vertices. Moreover, we can use the hierarchical differences to define a new graph centrality measure.

Definition 3.6. The influence centrality of vertex $j$ of a graph directed $G$ is
\[ \eta(G, j) = \begin{cases} 1 - \text{Mean}(\text{HD}(G, j)) & \text{if } |\text{HD}(G, j)| > 0 \\ 1 & \text{if } |\text{HD}(G, j)| = 0 \end{cases} \]

Similarly to the trophic incoherence parameter we define the hierarchical incoherence parameter.

Definition 3.7. The hierarchical incoherence parameter, or just hierarchical incoherence, of a directed graph $G$ is
\[ \rho(G) = \sqrt{\text{Var}(\text{HD}(G))} \]

In order to interpret what hierarchical incoherence for a given graph means, we need to take into account the value of its democracy coefficient. For a graph with low democracy coefficient, low hierarchical incoherence means that there are distinct hierarchical levels. For a graph with high democracy coefficient, low hierarchical incoherence means that all the vertices have approximately the same hierarchical level.

The democracy coefficient of a graph correlates strongly with its topology. However, before we discuss its properties, we need a few more definitions.

Definition 3.8. Let $G$ be a directed graph. Then
- $G$ is strongly connected if for any pair of vertices $i$ and $j$ there exists a directed path from $i$ to $j$.
- $G$ is weakly connected if for any pair of vertices $i$ and $j$ there exists an undirected path from $i$ to $j$. 

Figure 3: An example of a hierarchically decomposable graph. (a) The original graph. The influenced vertices are marked red and the influencer subgraph is marked green. (b) Its simply influenced subgraph.

In many graphs we can identify a simply influenced subgraph, that is a subgraph whose initial state affects the state of the graph only for a finite amount of time. This can be useful and leads us to the following definitions.

Definition 3.9. Let $G$ be a weakly connected graph and $\Gamma$ a subgraph of $G$. Then $\Gamma$ is called an influencer subgraph of $G$ if there is no edge from $G \setminus \Gamma$ to $\Gamma$. An influencer subgraph is called minimal if it stops being an influencer subgraph if we remove any single vertex.

Definition 3.10. Let $G$ be a weakly connected directed graph. Then

- $G$ is called *hierarchically decomposable* if there exist minimal influencer subgraphs $\Gamma_1, \ldots, \Gamma_l$ such that $G \setminus (\bigcup \Gamma_i)$ is not empty and for any vertex $v$ in $G \setminus (\bigcup \Gamma_i)$ there is a directed path to $v$ from at least one influencer subgraph.

- $G$ is called *hierarchically indecomposable* if the only minimal influencer subgraph of $G$ is $G$ itself.

Notice that by the definitions it is not obvious that a graph is hierarchically indecomposable if and only if it is hierarchically decomposable. However, this is true and we give a proof in Section 6. An example of a hierarchically decomposable graph and its simply influenced subgraph can be seen in Figure 3b.

Definition 3.11. Let $G$ be a hierarchically decomposable graph and $\Gamma_1, \ldots,
Γ_i be its minimal influencer subgraphs, a vertex \( v \in V(G) \) is called *influencing* if \( v \in \bigcup i V(\Gamma_i) \) and \( v \) is called *influenced* if \( v \in V(G) \setminus \bigcup i V(\Gamma_i) \).

**Definition 3.12.** Let \( G \) be a hierarchically decomposable graph and \( \Gamma_1, \ldots, \Gamma_l \) be its influencer subgraphs. The subgraph that we get if we remove all edges that belong to \( \Gamma_i \)'s from \( G \) and delete all isolated vertices will be called the *simply influenced subgraph of \( G \).*

If \( G \) is hierarchically decomposable, we can calculate \( \mu_G \) by calculating only the HLs of the minimal influencer subgraphs.

**Lemma 3.13.** Let \( G \) be a hierarchically decomposable graph, \( \Gamma_1, \ldots, \Gamma_l \) be the minimal influencer subgraphs of \( G \) and \( H \) be the simply influenced subgraph of \( G \). Let \( m \) be the number of edges in \( H \) and \( k_i \) the number of edges in \( \Gamma_i \). Then

\[
\eta(G) = \frac{\sum_i \eta(\Gamma_i) k_i}{m + \sum_i k_i}.
\]

On the other hand, if we have calculated the HLs of the graph, we can easily get information about its structure.

**Lemma 3.14.** Let \( G \) be a weakly connected directed graph and \( i \in V(G) \). Then \( \eta(G, i) = 0 \) if and only if \( i \) is non-influencing.

Moreover, the democracy coefficient has the following properties.

**Lemma 3.15.** For a weakly connected directed graph \( G \), \( \eta(G) \geq 0 \). Moreover, \( \eta(G) = 0 \) if and only if \( G \) is a simply influenced graph.

**Definition 3.16.** A directed graph is called *balanced* if for any vertex its in-degree equals its out-degree.

**Lemma 3.17.** For a balanced graph \( G \), \( \eta(G) = 1 \).

We conjecture that the democracy coefficient has also the following properties.

**Conjecture 3.18.** Let \( G \) be a weakly connected directed graph. Then the following are true:

- \( \eta(G) \leq 1 \).
- \( \eta(G) = 1 \) if and only if the graph is balanced.

Moreover, we conjecture that \( \mu_G \) cannot take values arbitrarily close to 0. We will see in Section 4.2 that this can be used to differentiate between graphs that otherwise look similar.

**Conjecture 3.19.** Let \( G \) be a directed graph with \( m \) edges. Then

\[
\mu_G \notin (0, \frac{2}{m}) \cup (\frac{2}{m}, \frac{3}{m}).
\]

Moreover, if \( m > 3 \), then \( \mu_G = 2/m \) if and only if \( G \) is hierarchically decomposable and its minimal influencer subgraphs are all influencer vertices except one which is an influencer pair, i.e. a strongly connected subgraph with 2 vertices.
Figure 4: The correlation between temperature, trophic incoherence and hierarchical incoherence. (a) Scatter plot of trophic incoherence over temperature. (b) Scatter plot of hierarchical incoherence over temperature. (c) Scatter plot of hierarchical incoherence over trophic incoherence. Notice that there is some divergence between them for small values of trophic incoherence.

4 Graph generation

We use two models of graph generation. The first is the preferential prey model introduced in [6]. We use it to compare the trophic incoherence with the hierarchical incoherence. The second is non-influencer preferential preying model, which is a modification of the preferential prey model and we use it to demonstrate that hierarchical incoherence can be used to predict the spread of infection.

4.1 Preferential preying model

The preferential preying model (PPM) was introduced in [6] as a way to generate graphs that are similar to food webs. In order to generate a graph with PPM we choose the number of vertices \( N \), the number of influencer vertices \( B \), the number of edges \( E \) and the “temperature” \( T \).

The algorithm is:

1. We introduce \( B \) influencer vertices and no edges.
2. We choose uniformly at random one of the existing vertices \( i \) and we add a new vertex \( j \) and the edge \( i \rightarrow j \).
3. We repeat step 2 until we have \( N \) vertices in total.
4. We assign each vertex \( i \) its trophic level \( s_i \) according to the graph we have up to this point.
5. From all possible edges \( i \rightarrow j \) such that \( j \) is not an influencer vertex, we choose \( L - N + B \) with probability proportional to

\[
P(a_{ij} = 1) \propto \exp \left( -\frac{(s_j - s_i - 1)^2}{2T^2} \right).
\]
We generated graphs with $N = 500$, $B = 25$ and $E = 2500$ for different values of temperature and computed their trophic incoherence and hierarchical incoherence. The results are shown in Figure 4. We see that the two values are equivalent and there is a divergence only for small values of trophic incoherence.

4.2 Non-influencer preferential preying model

Figure 5: A scatter plots of hierarchical incoherence over democracy coefficient for NIPPM graphs. Two different regions are visible, one with democracy coefficient greater than 0.008 and hierarchical incoherence less than 6 and one with democracy coefficient less or equal to 0.008 and hierarchical incoherence that can take values as big as 60 or more. (a) Scatter plot with hierarchical incoherence values between 0 and 60. (b) Scatter plot with hierarchical incoherence values between 0 and 6.

We propose the following simple modification to PPM. This modification creates graphs which are not simply influenced because they lack influencer vertices, but which are similar to PPM graphs. We will call this the non-influencer preferential preying model (NIPPM).

Similarly to the PPM, in order to generate a graph with NIPPM, we choose the number of vertices $N$, the number of influencer-like vertices $B$, the number of edges $E$ and the “temperature” $T$.

The algorithm is:

1. We introduce $B$ influencer-like vertices and no edges.
2. We choose uniformly at random one of the existing vertices $i$ and we add a new vertex $j$ and the edge $i \rightarrow j$.
3. We repeat step 2 until we have $N$ vertices in total.
4. We assign each vertex $i$ its trophic level $s_i$ according to the graph we have up to this point.
5. We pick an influencer-like vertex $i$ with in-degree 0, we pick another vertex $j$ with probability proportional to $\exp(-s_j)$ and we add the edge $j \rightarrow i$. 

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6. We repeat step 6 until all influencer-like vertices have in-degree 1.

7. From all possible edges $i \rightarrow j$ such that $j$ is not an influencer-like vertex, we choose $L - N$ with probability proportional to

$$P(a_{ij} = 1) \propto \exp\left(-\frac{(s_j - s_i - 1)^2}{2T^2}\right).$$

Figure 6: Scatter plots of hierarchical incoherence and democracy coefficient over temperature for NIPPM graphs. (a) Democracy coefficient over temperature. The graphs with democracy coefficient less or equal than 0.008 form a very tight band on the top of the figure. The graphs with democracy coefficient greater than 0.008 form a much wider band and there is a clear gap between them. (b) Hierarchical incoherence over temperature of graphs with democracy coefficient greater than 0.008. (c) Hierarchical incoherence over temperature of graphs with democracy coefficient greater or equal than 0.992.

We generated graphs with $N = 500$, $B = 25$ and $E = 2500$ for different values of temperature and computed democracy coefficient and hierarchical incoherence. We see in Figure 5 that there is a clear distinction between graphs that have democracy coefficient greater or equal to 0.992 and those that have less.

Using Lemma 3.13 we see that if a graph has one influencer pair, then its democracy coefficient will be $2/500 = 0.004$. Similarly a graph with an influencer 3-cycle has democracy coefficient $3/500 = 0.006$ and a graph with 2 influencer pairs has democracy coefficient $4/500 = 0.008$. Such graphs have a very different distribution of hierarchical incoherence than the ones with larger influencer subgraphs. This difference can be seen in Figure 6.

5 Contagion dynamics

We look at how hierarchical incoherence affects the spreading of an infection by using the Susceptible-Infected-Susceptible epidemic model [12]. Following [8],

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Figure 7: Scatter plot of average incidence values from Monte Carlo simulations of the infection spreading with varying temperature $T$ and infection parameter $a$. The average is taken over 500 runs. (a) Incidence against $a$ for different values of $T$. (b) Incidence against $T$ for different values of $a$.

Figure 8: Scatter plot of average incidence values from Monte Carlo simulations of the infection spreading with varying hierarchical incoherence $\rho(G)$ and infection parameter $a$. The average is taken over an interval of hierarchical incoherence values. (a) Incidence against $a$ for different values of $\rho(G)$. (b) Incidence against $\rho(G)$ for different values of $a$. 

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(a) Incidence against $a$ and $T$. (b) Incidence against $a$ and $\rho(G)$. We define the probability that vertex $i$ is infected at time $t + 1$ to be

$$P(i \text{ is infected at time } t + 1) = f_i(t)^a,$$

where $f_i(t)$ is the fraction of $i$’s in-neighbours which are infected at time $t$ and $a$ is a positive parameter that controls the infection rate. The smaller $a$ is, the easier it is for a vertex to be infected.

We ran Monte Carlo simulation using NIPPM graphs with 500 total vertices, 25 influencer-like vertices and 2500 edges with varying $T$ and measured the incidence of the infection, i.e. the proportion of vertices that have been infected at least once.

During the simulation we generated a graph and we infected the 25 vertices with the lowest HLs. Then we run the simulation until incidence became 1 or there was no infected vertex left or we reached time step 1000. This last condition is required because with NIPPM graphs the influencer-like vertices have just 1 in-edge. This means that if the neighbour of an influencer-like vertex becomes infected, then the influencer-like vertex becomes infected at the next time step. This can cause periodic “waves of infection” that could in principle continue forever without incidence ever reaching the value 1.

In Section 4.2 we discussed how graphs with democracy coefficient 0.008 or less have different distribution than the rest. For the infection dynamics this is particularly true. If there exists an influencer pair, then the vertices of the pair will begin infected and since they are each other’s neighbour, they stay infected forever. This creates a lot of noise in the results. For this reason we have included in the results only graphs that have democracy coefficient greater than 0.008, which is the 98% of the graphs generated.

The results of the simulation are shown in Figures 7, 8 and 9. We see that
if the infection parameter is 1 or smaller then on average, every vertex becomes infected at least once. This is due to the aforementioned infection waves that appear in NIPPM graphs. In Figure 7 the average is taken over 500 runs.

However, since we cannot choose precise values for the hierarchical incoherence of a graph, in Figure 8 the average is taken over an interval of hierarchical incoherence values. This means that for common values of hierarchical incoherence, the mean is taken over more runs than for less common hierarchical incoherence values. This results in the relatively big error seen in Figure 8a for $\rho(G) = 0.8$ and in Figure 8b for $\rho(G) \leq 1.3$ or $\rho(G) \geq 2.8$. Heat maps of the average incidence for different $T$, $\rho(G)$ and $a$ can be found in Figure 9. We see that both $T$ and $\rho(G)$ can be used equally well as predictors of the spread of an infection.

6 Proofs of lemmas

In this section we provide the proofs of the lemmas that appear in Section 3.2. The proofs are not written in the order that the lemmas appear in Section 3.2, but in the order they are used in other proofs, i.e. a lemma is used in a proof only if its proof was written before.

**Lemma 6.1.** Let $G$ be a simply influenced graph with $l$ influencer vertices, we order the vertices of $G$ starting by the influencer ones. Let $d$ be its in-degree vector $A$ its adjacency matrix and $L$ be its in-degree Laplacian. Then for any real numbers $c_1, \ldots, c_l$ there exist real numbers $x_{l+1}, \ldots, x_n$ such that the vector

$$x = (c_1, \ldots, c_l, x_{l+1}, \ldots, x_n)$$

satisfies

$$L^T \cdot x = d. \quad (4)$$

Moreover, let $D$ the set of differences of $x$ defined by

$$D = \{x_j - x_i \mid (A)_{ij} = 1, i, j \in V(G)\}.$$ 

Then $\text{Mean}(D) = 1$.

**Proof.** We define $a_{ij} = (A)_{ij}$. Since $G$ is trophic with $l$ influencer vertices, we know from [13] that the dimension of $\ker(L)$ is $l$. We write the linear system (4) as

$$d_i x_i - \sum_j a_{ji} x_j = d_i.$$ 

The first $l$ equations correspond to influencer vertices and become $0 = 0$. This means that we can choose any value for $x_i, i \in \{1, \ldots, l\}$. Moreover, since the dimension of $\ker(L)$ is $l$, the rest of the equations can be solved. So we conclude that such $x$ exists.
We have

\[
\text{Mean}(\mathcal{D}) = \frac{\sum_i \sum_j a_{ji} (x_i - x_j)}{\sum_i \sum_j a_{ji}} = \frac{\sum_i (\sum_j a_{ji}x_i - \sum_j a_{ji}x_j)}{\sum_i d_i} = \frac{\sum_i (d_i x_i - \sum_j a_{ji}x_j)}{\sum_i d_i} = \frac{\sum_i d_i}{\sum_i d_i} = 1.
\]

Lemma 2.2 is a straightforward corollary.

**Proof of Lemma 3.13.** Let \(\chi_G\) be the sum of TDs of graph \(G\). Trivially it is true that

\[
\chi_G = \chi_H + \sum_i \chi_{\Gamma_i}.
\]

From Lemma 6.1 we know that no matter what are the values of the influencer vertices of \(H\), the mean of differences will be 1. This means that \(\chi_H = m\). We have \(\chi_{\Gamma_i} = (1 - \eta(\Gamma_i))k_i\). From this we get

\[
\chi_G = \chi_h + \sum_i \chi_{\Gamma_i} = m + \sum_i (1 - \eta(\Gamma_i))k_i.
\]

Then we have

\[
\eta(G) = 1 - \frac{\chi_G}{m + \sum_i k_i} = \frac{m + \sum_i k_i - \chi_G}{m + \sum_i k_i} = \frac{m + \sum_i k_i - m - \sum_i (1 - \eta(\Gamma_i))k_i}{m + \sum_i k_i} = \frac{\sum_i \eta(\Gamma_i)k_i}{m + \sum_i k_i}.
\]

**Lemma 6.2.** Let \(G\) be a weakly connected directed graph. Then \(G\) is not hierarchically decomposable if and only if \(G\) is hierarchically indecomposable.

**Proof.** For one direction we assume that \(G\) is hierarchically decomposable. Then there exist minimal influencer subgraphs \(\Gamma_1, \ldots, \Gamma_l\) such that \(G \setminus (\bigcup_i \Gamma_i)\) is not empty. This implies that \(G\) is not a minimal influencer subgraph of itself.

For the other direction, assume that \(G\) is not a minimal influencer subgraph of itself. Let \(\Gamma_1, \ldots, \Gamma_l\), where \(l \geq 1\), be the minimal subgraphs of \(G\). If \(l = 1\), \(G \setminus \Gamma_1\) cannot be empty and this implies that \(G\) is hierarchically indecomposable.

Assume that \(l \geq 2\) and that \(G \setminus (\bigcup_i \Gamma_i)\) is empty. We pick any two subgraph \(\Gamma_i\) and \(\Gamma_j\) and since both are influencer subgraphs, there cannot be an edge from one to the other. This implies that \(G\) is not connected, which is a contradiction. \(\square\)
Lemma 6.3. Let $G$ be a hierarchically indecomposable weakly connected graph. Then $G$ is strongly connected, i.e. for any two vertices $i$ and $j$ there exists a directed path from $i$ to $j$.

Proof. Assume that there exist vertices $i$ and $j$ such that there is no directed path from $i$ to $j$. We define $V$ to be the set of all vertices from which there is a directed path to $j$. We accept paths of length 0 so that $j \in V$. We define $W$ to be the complement of $V$, i.e. the set of all vertices from which there is no directed path to $j$. By definition $i \in W$.

If there exists a directed edge from a vertex $w \in W$ to a vertex $v \in V$, then there exists a directed path from $w$ to $j$, so $w \notin W$. This means that $W$ is an influencer subgraph of $G$, which is a contradiction.

Lemma 6.4. Let $G$ be a hierarchically indecomposable weakly connected graph. Then ker($L$) is spanned by a positive integer vector.

Proof. Lemma 6.3 shows that $G$ is strongly connected. Then the kernel of $L$ is 1-dimensional, see [13]. Moreover, Proposition 4.1 in [14] shows that there exists a positive integer vector that belongs to ker($L$). These two facts prove the lemma.

Lemma 6.5. Let $G$ be a hierarchically decomposable directed graph and let $\Gamma_1, \ldots, \Gamma_l$ be its minimal influencer subgraphs. Let $d$ be its in-degree vector, $L$ be its in-degree Laplacian and $L_i$ be the in-degree Laplacian of $\Gamma_i$. Then

1. ker($L_i$) is spanned by a positive vector $\kappa_i$.

2. ker($L$) is spanned by the vectors $k_i = (0, \ldots, 0, \kappa_i, 0, \ldots, 0)$, where $i \in \{i, \ldots, l\}$ and the position of $\kappa_i$ in $k_i$ corresponds to the position of $L_i$ in $L$.

3. $d \cdot k_i = 0$ if $\Gamma_i$ is just a single vertex and $d \cdot k_i > 0$ otherwise.

Proof.

1. Since $\Gamma_i$ is a minimal influencer subgraph, it is hierarchically indecomposable and by virtue of Lemma 6.4, ker($L_i$) is spanned by a positive vector $\kappa_i$.

2. It follows that the dimension of ker($L$) is $l$, see [13]. The Laplacian $L$ has the form

$$L = \begin{pmatrix}
L_1 & 0 & \cdots & 0 & C_1 \\
0 & L_2 & \cdots & 0 & C_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & L_l & C_l \\
0 & 0 & \cdots & 0 & C_{l+1}
\end{pmatrix}$$

It is straightforward to check that the vector $k_i = (0, \ldots, 0, \kappa_i, 0, \ldots, 0)$ is in ker($L$). Since we can construct $l$ such vectors and by construction they are orthogonal, they form a basis of ker($L$).
3. Without loss of generality we will consider $\Gamma_1$. If $\Gamma_1$ is a single vertex then $L_1$ is just the $1 \times 1$ zero matrix. This means that $k_1 = (1, 0, \ldots, 0)$ and $d = (0, d_2, \ldots, 0)$, thus $k_1 \cdot d = 0$. If $\Gamma_1$ is a graph with $m$ vertices, then $k_1$ is a positive $m$-vector and the in-degree vector has the form $d = (d_1, \ldots, d_m, \ldots, d_n)$. Since $\Gamma_1$ is strongly connected, by Lemma 6.3, $d_i > 0$ for all $i \in \{1, \ldots, m\}$, so $k_1 \cdot d > 0$.

\textbf{Lemma 6.6.} Let $G$ be a weakly connected directed graph, $L$ be its in-degree Laplacian and $d$ be its in-degree vector. Then a vector $x$ that satisfies $L^T \cdot x = d$ exists if and only if $G$ is simply influenced.

\textit{Proof.} Lemma 6.1 states that if a graph is simply influenced, then the system can be solved.

For the converse we recall from linear algebra that $x$ exists if and only if the orthogonal projection of $d$ onto $\ker(L)$ is the 0 vector. We assume that there exists a vector $x$ that satisfies $L^T \cdot x = d$.

If $G$ is hierarchically indecomposable, by Lemma 6.4 we know that the projection of $d$ onto $\ker(L)$ cannot be 0. So $G$ has to be hierarchically decomposable. Let $\Gamma_1, \ldots, \Gamma_l$ be its minimal influence subgraphs.

Let $k_i$, where $i \in \{1, \ldots, l\}$, be the vectors that span $\ker(L)$. Since the vector $x$ exists, this means that $d \cdot k_i = 0$ for all $i \in \{1, \ldots, l\}$. Then by virtue of Lemma 6.5 $\Gamma_i$ has to be a single vertex for all $i$, thus $G$ is simply influenced. \hfill \Box

\textbf{Lemma 6.7.} Let $G$ be a simple directed graph, let $d$ be its in-degree vector and let $L$ be its in-degree Laplacian. Let $\beta$ be the orthogonal projection of $d$ onto $\ker(L)$. Then $\beta$ is a non-negative vector and

$$
\eta(G) = \frac{\sum_i \beta_i}{\sum_i d_i}
$$

\textit{Proof.} Let $\tau$ be the vector of HLs of $G$. Then we have

$$
\eta(G) = 1 - \frac{\sum_i \sum_j a_{ij} (\tau_i - \tau_j)}{\sum_j a_{ji}}.
$$

We define

$$
\beta_i = d_i - \left( d_i \tau_i - \sum_j a_{ij} \tau_j \right)
$$

and we set $\beta = (\beta_1, \ldots, \beta_n)$. This means that

$$
\beta = d - M \tau = d - MM^+ d = (I - MM^+) d.
$$

The matrix $I - MM^+$ is the orthogonal projector onto the kernel of $M^T = L$, see [15]. So $\beta$ is indeed the orthogonal projection of $d$ onto $\ker(L)$. Lemma 6.5 shows that the kernel of $L$ is spanned by non-negative vectors. Since $d$ is also
a non-negative vector, the projection of $d$ onto $\text{ker}(L)$ is a non-negative vector, so $\sum_i \beta_i \geq 0$.

Using the definitions of Section 3.2 we have

$$\eta(G) = \frac{\sum_i \sum_j a_{ji} - \sum_i \sum_j a_{ji}(\tau_i - \tau_j)}{\sum_i \sum_j a_{ji}}$$

$$= \frac{\sum_i (d_i - d_i \tau_i + \sum_j a_{ji} \tau_j)}{\sum_i d_i}$$

$$= \frac{\sum_i \beta_i}{\sum_i d_i}.$$

\[\square\]

**Proof of Lemma 3.15.** We know from Lemma 6.7 that $\eta(G) = \sum_i \beta_i / \sum_i d_i$ and $\sum_i \beta_i > 0$. These prove the first assertion of the lemma.

The second assertion will be proved in two steps. Let $\tau$ be the vector of HLs of $G$. First assume that $G$ is simply influenced. This means that the hierarchical levels vector $\tau$ satisfies the equation $M \cdot \tau = d$, i.e. $d_i \tau_i - \sum_j a_{ji} \tau_j = d_i$ for all $i$. This gives

$$\eta(G) = 1 - \frac{\sum_i \sum_j a_{ji}(\tau_i - \tau_j)}{\sum_i \sum_j a_{ji}}$$

$$= 1 - \frac{\sum_i (d_i \tau_i - \sum_j a_{ji} \tau_j)}{\sum_i d_i}$$

$$= 1 - \frac{\sum_i d_i}{\sum_i d_i} = 0.$$

Now we assume that $G$ is a weakly connected graph with $\eta(G) = 0$, thus $\sum_i \beta_i = 0$. Since $\beta$ is a non-negative vector, this implies that $\beta = 0$. This implies that the projection of $d$ onto the kernel of $M^T$ is 0 and that $d$ is in the range of $M$. From this we deduce that the linear system $M \cdot \tau = d$ can be solved and we use Lemma 6.6 to deduce that $G$ is simply influenced. \[\square\]

**Proof of Lemma 3.17.** First we prove that any balanced graph is strongly connected. Similarly to the proof of Lemma 6.3, we assume that the graph is not strongly connected and we separate $G$ into an influencer subgraph $\Gamma$ and its complement $G \setminus \Gamma$. We know that there cannot be a directed edge from $G \setminus \Gamma$ to $\Gamma$, but there has to be at least one directed edge from $\Gamma$ to $G \setminus \Gamma$. However, since the sum of in-degrees in $\Gamma$ equals the sum of out-degrees, this is impossible, so $G$ is strongly connected.

Let $L$ be the in-degree Laplacian of $G$. Because $G$ is balanced, every row and every column of $L$ sums to 0. From this we deduce that the vector $1 = (1, \ldots, 1)$ is in the kernel of both $L$ and $L^T$. Since $G$ is strongly connected, the kernel of $L$ is 1-dimensional, see [13]. So $1$ spans both $\text{ker}(L)$ and $\text{ker}(L^T)$.

The projection of $d$ onto $\text{ker}(L)$ is

$$\beta = \frac{d \cdot 1}{1 \cdot 1} = \frac{\sum_i d_i}{n}.$$
This means that $\sum_i \beta_i = \sum_i d_i$. Then by Lemma 6.7 we get $\eta(g) = 1$. □

Proof of Lemma 3.14. If $d_i = 0$ then the vertex $i$ is influencer subgraph and by definition $\eta(G, i) = 1$.

Assume that $d_i > 0$. We use the vector $\beta$ that was defined in the proof of Lemma 6.7. We have

$$\eta(G, i) = 1 - \frac{\sum_j a_{ji}(\tau_i - \tau_j)}{\sum_j a_{ji}} = 1 - \frac{\sum_j a_{ji}\tau_i - \sum_j a_{ji}\tau_j}{d_i} = 1 - \frac{d_i\tau_i - \sum_j a_{ji}\tau_j}{d_i} = 1 - \frac{d_i - \beta_i}{d_i} = \frac{\beta_i}{d_i}.$$ 

Recall that $\beta$ is the orthogonal projection of $d$ on $\ker(L)$. We use Lemma 6.5 and we see that for any $i$ with $d_i > 0$, $\beta_i = 0$ if and only if $i \in G \setminus (\cap_i \Gamma_i)$. This concludes the proof. □

Contributions

The idea of hierarchical levels was conceived by CS and developed by GM and CS. The claims were proved by GM.

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