The length of $\text{PU}(2, 1)$ relative to special elliptic isometries with fixed parameter

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Abstract

Generalizing the involution length of the complex hyperbolic plane, we obtain that the $\alpha$-length of $\text{PU}(2, 1)$ is 4, that is, every element of $\text{PU}(2, 1)$ can be decomposed as the product of at most 4 special elliptic isometries with parameter $\alpha$. We also describe the isometries that can be written as the product of 2 or 3 such special elliptic isometries.

1 Introduction

This work concerns the decomposition of isometries of the complex hyperbolic plane $\mathbb{H}^2$ in the product of special elliptic isometries, also known as complex reflections. Special elliptic isometries (see Subsection 2.1) can be seen as rotations either around a point or around a complex geodesic. They have a center (a nonisotropic point) and a parameter $\alpha$ (a unit complex number that is not a cube root of unity); a special elliptic isometry with center $p$ and parameter $\alpha$ is denoted by $R_p^\alpha$. The isometry $R_p^\alpha$ acts on $\mathbb{H}^2$ as a rotation around $p$ (if $p \in \mathbb{H}^2$) by the angle $\text{Arg} \alpha^3$ or around a complex geodesic (whose pole point is the positive point $p$) by the angle $\text{Arg} \alpha^{-3}$.

Here we approach the problem of finding the smallest number $m$ such that every element of $\text{PU}(2, 1)$, the group of orientation-preserving isometries of $\mathbb{H}^2$, admits a decomposition as the product of at most $m$ special elliptic isometries with parameter $\alpha$, for a given $\alpha$. We call this number the $\alpha$-length of $\text{PU}(2, 1)$ (see Definition 3.3). Analogously we can consider the $\alpha$-length of a given isometry and, in this way, the $\alpha$-length of $\text{PU}(2, 1)$ is the maximum of the $\alpha$-lengths of all of its elements.

As orientation-preserving involutions of $\mathbb{H}^2$ are special elliptic isometries with parameter satisfying $\alpha^3 = -1$, the $\alpha$-length is closely related to the idea of involution length: for a symmetric Riemannian space $X$, the involution length is defined similarly to the $\alpha$-length, but considering decompositions into the products of involutions of $\text{Isom}(X)$. In [13] Will and Paupert obtained that the (orientation-preserving) involution length of $\text{PU}(2, 1)$ is 4 and that the (orientation-preserving) involution length of $\text{PU}(n, 1)$, $n \geq 2$, is at most 8. Allowing orientation-reversing involutions, Falbel and Zocca obtained in [5] that the involution length of $\text{PU}(2, 1)$ is 2, and it turns out that this is also true for $\text{PU}(n, 1)$, $n \geq 2$ (a proof of this fact can be found in [8]). The involution length of symmetric spaces of constant curvature was obtained by Basmajian and Maskit in [3].

In this context, for fixed parameters $\alpha_1$ and $\alpha_2$, we determine every isometry of $\mathbb{H}^2$ that admits a decomposition as the product of two special elliptic isometries, one with parameter $\alpha_1$ and the other with parameter $\alpha_2$ (Propositions 5.6, and 5.8). (In $\text{SU}(2, 1)$, such a decomposition is of the form $F = \delta R_{\alpha_2}^\alpha R_{\alpha_1}^\alpha$, where $\delta$ is a cube root of unity.) The isometries admitting such a decomposition are related to lines tangent to Goldman’s deltoid (see Subsection 4.4). Moreover, deciding if an elliptic isometry admits such a decomposition is more involved and gives rise to the study of the interaction between these tangent lines and the unfolded trace, introduced in Subsection 4.1 — a trace-like function that, unlike the usual trace, can distinguish the classes of regular elliptic isometries. This interaction is also central to obtain our main result, Theorem 1.1. The unit complex numbers in the following theorem are considered up to a cube root of unity (see Corollary 3.6).

1.1. Theorem. The $\alpha$-length of $\text{PU}(2, 1)$ is 4, for any parameter $\alpha$. Moreover, writing $\alpha = e^{a_i}$, if $0 < a \leq \frac{2\pi}{3}$ or $\frac{4\pi}{3} \leq a < \frac{2\pi}{3}$, then every isometry, except possibly 2-step unipotent ones (see Remark 6.8), admits a decomposition as the product of three special elliptic isometries with parameter $\alpha$. When $\frac{2\pi}{3} < a < \frac{4\pi}{3}$, there exist regular and special elliptic isometries that do not
admit such a decomposition. Finally, if $\alpha^3 = \pm 1$, i.e., $a = \frac{2\alpha}{\alpha^2 - 1}$, then every 2-step unipotent isometry is the product of three special elliptic isometries with parameter $\alpha$.

The general strategy to prove this theorem follows [6] and [13]. In [6] it was described every (generic) relation of length at most 4 between special elliptic isometries and the space $S_{\alpha, \sigma, \tau}$ of (strongly regular) triples $p_1, p_2, p_3$ of nonisotropic points that satisfy $\text{tr} R_{\alpha}^2 R_{\sigma}^2 R_{\tau}^1 = \tau$ for a fixed triple of parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, a fixed triple of signs $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ (the signatures $\sigma_i$ of the points $p_i$), and fixed $\tau \in C$. Here we prove that, given a triple of parameters $\alpha$ and $\tau \in C$, there exists a triple of signs $\sigma$ such that the space $S_{\alpha, \sigma, \tau}$ is nonempty (proof of Proposition 6.3). From the nature of the space of PU(2,1)-conjugacy classes (see Subsection 3.1), this is sufficient to prove that the $\alpha$-length of regular parabolic (ellipto-parabolic or 3-step unipotent, see Definition 2.2) and loxodromic isometries is at most 3, for any parameter $\alpha$ (Proposition 6.3), but this is not sufficient for the general case (particularly, for elliptic isometries).

To deal with the decomposition of the remaining elliptic isometries, following [13], we use the product map $\tilde{\mu} : C_1 \times C_2 \rightarrow G$, where $G$ denotes the space of all PU(2,1)-conjugacy classes, $C_i$ denotes semisimple conjugacy classes, $i = 1, 2$, and $\tilde{\mu}(A, B) := [AB]$, where $[F]$ stands for the conjugacy class of $F$. (As will be clear in Subsection 6.9, we actually consider the projection $\pi$ of the product map over the maximal Hausdorff quotient $c(G)$ of $G$.)

The idea is the following: suppose that $C_1$ is the class of a special elliptic isometry with parameter $\alpha$ and $C_2$ is the class of a semisimple isometry that admits a decomposition as the product of 2 special elliptic isometries with parameter $\alpha$. Then any isometry $F$ with $[F] \in \tilde{\mu}(C_1 \times C_2)$ admits a decomposition as the product of 3 special elliptic isometries. In this way, to obtain all elliptic isometries admitting such length 3 decomposition, it remains to describe the union of all possible images of the product map, varying the semisimple classes $C_1$ and $C_2$. Properties of the product map $\tilde{\mu}$ and its image (and mainly its intersection with $\rho(\mathcal{E})$, where $\mathcal{E}$ is the space of elliptic conjugacy classes) were described in [4, 12] and we list some of this properties in Subsection 6.9. If at most one of the classes $C_1$ and $C_2$ is the class of a special elliptic isometry, then the image $\tilde{\mu}(C_1 \times C_2)$ is the union of closed chambers bounded by reducible walls; the intersection of the reducible walls with $\rho(\mathcal{E})$ is composed by finitely many line segments of slopes $-1, \frac{1}{2}, 2$. The interaction between tangential lines to Goldman’s deltoid and the unfolded trace help us describe the union of the images mentioned above and cast light over the nature of the segments of slopes $-1, \frac{1}{2}, 2$ in $\rho(\mathcal{E})$: the image under the unfolded trace of segments of slopes $-1, \frac{1}{2}, 2$ in $\mathcal{E}$ are subsegments of lines tangent to the deltoid.

The last ingredient to prove the main theorem are bending relations (see Subsection 6.12 and [6, Section 4]). Such relations can be used to deform a given decomposition of the form $F = \delta R_{\alpha}^m \cdots R_{\alpha}^1$, where $\delta$ is a cube root of unity, and they appear as natural coordinates of the space $S_{\alpha, \sigma, \tau}$ (see [6, Theorem 5.4]). In the context of Theorem 1.1, bending the decomposition $F = \delta R_{\alpha}^m R_{\sigma}^2 R_{\tau}^1$, we can decide whether $[F]$ is an interior point in the union of all images of $\tilde{\mu}$ or not. This allows us to avoid directly obtaining such a union, as is done in [13], and leads to the description in Proposition 6.17 which, together with the results in Subsection 6.18, proves our main theorem.

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## 2 Complex hyperbolic geometry

In this section, following [1, 2, 7], we briefly describe the complex hyperbolic plane and its isometries.

Let $V$ be a 3-dimensional $\mathbb{C}$-linear space equipped with a Hermitian form of signature $++-$. We consider the projectivization $\mathbb{P}V := \mathbb{P}_2 V$ divided into negative, positive and isotropic points:

$$BV := \{ p \in \mathbb{P}V \mid \langle p, p \rangle < 0 \}, \quad SV := \{ p \in \mathbb{P}V \mid \langle p, p \rangle = 0 \}, \quad EV := \{ p \in \mathbb{P}V \mid \langle p, p \rangle > 0 \}.$$  

Here and throughout this paper we denote a point in $\mathbb{P}V$ and a representable of it in $V$ by the same letter, but no confusion should arise. We denote the signature of a point $p \in \mathbb{P}V$ by $\sigma p$, i.e., $\sigma p$ is respectively $-1, 0, 1$ if $p$ is negative, isotropic, or positive.
If \( p \in \mathbb{P}V \) is a nonisotropic point, we have the identification \( T_p \mathbb{P}V \simeq \text{Lin}_2(\mathbb{C}p, p^\perp) \), where \( p^\perp \) is the linear subspace of \( V \) orthogonal to \( p \). Through this identification we can define a Hermitian metric in \( BV \) and \( EV \) by
\[
\langle t_1, t_2 \rangle := -\frac{(t_1(p), t_2(p))}{(p, p)},
\]
where \( t_1, t_2 \in \text{Lin}_2(\mathbb{C}p, p^\perp) \) are tangent vectors to \( \mathbb{P}V \) at \( p \). This metric is positive definite in \( BV \) and has signature \(-+\) in \( EV \). Thus, the real part of this metric defines a Riemannian metric in \( BV \) and a pseudo-Riemannian metric in \( EV \). The 4-ball \( BV \) equipped with such Riemannian metric is the complex hyperbolic plane \( \mathbb{H}^2_2 \). Its ideal boundary, also called absolute, is the 3-sphere \( SV \).

The projectivization \( \mathbb{P}W \) of a 2-dimensional complex subspace \( W \leq V \) is called a complex line. Given a complex line \( L \), the point \( c \in \mathbb{P}V \) such that \( L = \mathbb{P}c^\perp \) is the polar point of \( L \). We say that a complex line \( L = \mathbb{P}c^\perp \) is hyperbolic, spherical, Euclidean if \( c \in EV \), \( c \in BV \), \( c \in SV \), respectively.

A complex geodesic is a set of the form \( L \cap BV \), where \( L \) is a hyperbolic complex line. For distinct points \( p_1, p_2 \in \mathbb{P}V \), we denote by \( L(p_1, p_2) \) the complex line \( \mathbb{P}(\mathbb{C}p_1 + \mathbb{C}p_2) \).

The tangent between two nonisotropic points \( p_1, p_2 \in \mathbb{P}V \) \( \setminus SV \) is given by
\[
\tau(p_1, p_2) := \frac{(p_1, p_2)(p_2, p_1)}{(p_1, p_1)(p_2, p_2)}.
\]
By Sylvester’s criterion, the line \( L(p_1, p_2) \), with \( p_1, p_2 \) distinct nonisotropic points in \( \mathbb{P}V \), is hyperbolic if \( \tau(p_1, p_2) > 1 \) or \( \tau(p_1, p_2) < 0 \); spherical if \( 0 < \tau(p_1, p_2) < 1 \); and Euclidean if \( \tau(p_1, p_2) = 1 \).

2.1. Isometries of the complex hyperbolic plane. Consider the special unitary group \( SU(2, 1) \) given by the elements in \( GLV \) that preserve the Hermitian form of \( V \) and have determinant 1. The group of orientation-preserving isometries of \( \mathbb{H}^2_2 \) is the projectivization \( PU(2, 1) \) of \( SU(2, 1) \), i.e., \( PU(2, 1) = SU(2, 1)/\{1, \omega, \omega^2\} \), where \( \omega := e^{2\pi i/3} \). We also refer to elements of \( SU(2, 1) \) as isometries.

We say that a nonidentical isometry in \( PU(2, 1) \) is elliptic if it fixes a point in \( BV \), parabolic if it fixes exactly one point in \( SV \), and loxodromic if it fixes exactly two points in \( SV \). Elliptic and parabolic isometries are further divided into subtypes as follows.

Let \( I \in SU(2, 1) \) be an elliptic isometry and let \( c \in BV \) be an \( I \)-fixed point. Thus, the spherical complex line \( \mathbb{P}c^\perp \) is \( I \)-stable and \( I \) fixes another point \( p \) in this line. Clearly, \( I \) must also fix the point \( \tilde{p} \in \mathbb{P}c^\perp \) orthogonal to \( p \). We obtain an orthogonal basis \( \{c, p, \tilde{p}\} \) for \( V \) given by eigenvectors of \( I \). If \( \mu_1, \mu_2, \mu_3 \in \mathbb{C} \) are the eigenvalues of \( c, p, \tilde{p} \), respectively, we have \( \mu_1 \mu_2 \mu_3 = 1 \) and \( |\mu_i| = 1 \). We say that the elliptic isometry \( I \) is regular if the eigenvalues \( \mu_i \) are pairwise distinct; otherwise, we say that it is special.

Throughout the paper we will denote by \( S^1 \) the set of unit complex numbers and by \( \Omega := \{1, \omega, \omega^2\} \), where \( \omega := e^{2\pi i/3} \), the set of cubic roots of unity. Every special elliptic isometry can be written in the form (see [9])
\[
R^p_\alpha : x \mapsto (\alpha^{-2} - \alpha)\langle x, p \rangle \langle p, p \rangle p + \alpha x
\]
for some \( p \in PV \setminus SV \) and \( \alpha \in S^1 \setminus \Omega \). We say that \( p \) is the center and that \( \alpha \) is the parameter of \( R^p_\alpha \).

Parabolic isometries are divided into three subtypes. We say that a parabolic isometry is unipotent if it lifts to a unipotent element of \( SU(2, 1) \). Unipotent isometries can be either 2-step or 3-step unipotent; 2-step unipotent isometries fix an isotropic point and pointwise fix its Euclidean polar complex line, and the 3-step ones fix an isotropic point and no other point (so, they do not preserve any hyperbolic complex line). Parabolic isometries that are not unipotent are called ellipto-parabolic; they fix an isotropic point and also fix a positive point in the Euclidean stable line.

Consider the function \( f : \mathbb{C} \to \mathbb{R} \) given by
\[
f(z) := |z|^4 - 8 \Re(z^3) + 18|z|^2 - 27,
\]
and denote $\Delta := \{ z \in \mathbb{C} \mid f(z) \leq 0 \}$, $\Delta^\circ := \{ z \in \mathbb{C} \mid f(z) < 0 \}$, and $\partial \Delta := \{ z \in \mathbb{C} \mid f(z) = 0 \}$.

Given a nonidentical isometry $I \in \mathrm{SU}(2, 1)$, we have:

- $I$ is regular elliptic if $\mathrm{tr} \ I \in \Delta^\circ$;
- $I$ is loxodromic if $I \in \mathbb{C} \setminus \Delta$;
- if $I$ is elliptic, then it is special elliptic if $\mathrm{tr} \ I \in \partial \Delta$;
- $I$ is parabolic if $I$ is not elliptic and $\mathrm{tr} \ I \in \partial \Delta$. Moreover, if $\mathrm{tr} \ I \notin \{3, 3\omega, 3\omega^2\}$, then $I$ is ellipto-parabolic;
- $I$ is unipotent if $I \in \{3, 3\omega, 3\omega^2\}$.

2.2. Definition. An isometry is regular if its eigenspaces have dimension 1, i.e., it does not pointwise fix a complex line. (This definition coincides with the one in [14].)

In other words, a nonidentical isometry in $\mathrm{PU}(2, 1)$ is regular if it is neither special elliptic nor 2-step unipotent.

3 Conjugacy classes and the product map

In this section we describe the space of $\mathrm{PU}(2, 1)$-conjugacy classes. Differently from the case of the Poincaré disk, the trace of an isometry does not determine its conjugacy class (but, as we will see here, it ‘almost’ does).

The trace determines the $\mathrm{SU}(2, 1)$-conjugacy class of loxodromic isometries, i.e., two loxodromic isometries in $\mathrm{SU}(2, 1)$ with the same trace $\tau \in \mathbb{C} \setminus \Delta$ are $\mathrm{SU}(2, 1)$-conjugated. Now, this is not true for values of trace in $\Delta$.

Given two elliptic isometries $F_1, F_2 \in \mathrm{SU}(2, 1)$ with same trace or, equivalently, same eigenvalues (see [7, Proof of Lemma 6.2.5]), $F_1$ and $F_2$ are $\mathrm{SU}(2, 1)$-conjugated iff their negative fixed points have the same eigenvalue. Therefore, elliptic classes are distinguished by the types of its eigenvalues: we say that an eigenvalue $\alpha \in S^1$ of an isometry $F \in \mathrm{SU}(2, 1)$ is of negative type if there exists a negative eigenvector associated with $\alpha$, i.e., if there exists $v \in V$ with $Fv = \alpha v$ and $(v, v) < 0$. As discussed above, two elliptic isometries with same trace and same negative type eigenvalue are conjugated. Since regular elliptic isometries have three distinct eigenvalues (three possible values for the negative type eigenvalue), for each $\tau \in \Delta^\circ$, there exists three distinct $\mathrm{SU}(2, 1)$-conjugacy classes of trace $\tau$.

Now, for $\tau \in \partial \Delta \setminus \{3, 3\omega, 3\omega^2\}$ (remember that $\omega := e^{2\pi i/3}$), there exists three distinct $\mathrm{SU}(2, 1)$-conjugacy classes of isometries with trace $\tau$. Two distinct classes of special elliptic isometries (distinguished as above by their negative type eigenvalue or, equivalently, by the signature of their centers), and one class of ellipto-parabolic isometries.

Finally, for $\tau \in \{3, 3\omega, 3\omega^2\}$, we have three distinct nonidentical $\mathrm{SU}(2, 1)$-conjugacy classes of isometries with trace $\tau$: two classes of 2-step unipotent isometries, and one of 3-step unipotent isometries.

Now we focus our attention on $\mathrm{PU}(2, 1)$-conjugacy classes. Note that, given $\tau \in \mathbb{C} \setminus \Delta$, the three loxodromic $\mathrm{SU}(2, 1)$-conjugacy classes of traces $\tau$, $\omega \tau$, $\omega^2 \tau$ determine the same loxodromic $\mathrm{SU}(2, 1)$-conjugacy class, as they differ (up to conjugacy) by an element of $\Omega$. The three nonidentical distinct $\mathrm{SU}(2, 1)$-conjugacy classes of trace $\tau \in \Delta$, determine three distinct $\mathrm{PU}(2, 1)$-classes when $\tau \neq 0$, and these classes coincide with the ones determined by the traces $\omega \tau$ and $\omega^2 \tau$; however, for $\tau = 0 = 1 + \omega + \omega^2$, they determine the same $\mathrm{PU}(2, 1)$-conjugacy class.

3.1. The space of $\mathrm{PU}(2, 1)$-conjugacy classes. Our approach here follows [4] and [13]. Let $\mathcal{G}$ be the space of all $\mathrm{PU}(2, 1)$-conjugacy classes, i.e., the quotient of $\mathrm{PU}(2, 1)$ by the action of $\mathrm{PU}(2, 1)$ on itself by conjugacy, equipped with the quotient topology.

Let $\mathcal{G}^{\text{reg}}$ be the space of classes of regular isometries (see Definition 2.2), let $\mathcal{E} \subset \mathcal{G}$ be the space of elliptic conjugacy classes (including the identical one), let $\mathcal{B}$ be the space of boundary classes (classes of parabolic or special elliptic isometries), and let $\mathcal{L} \subset \mathcal{G}$ be the space of loxodromic conjugacy classes. We denote by $\mathcal{E}^{\text{reg}} \subset \mathcal{G}^{\text{reg}}$ the space of regular elliptic conjugacy classes. Note that $\mathcal{G}^{\text{reg}} = \mathcal{E}^{\text{reg}} \cup \mathcal{L}$.

An elliptic isometry $F \in \mathrm{PU}(2, 1)$ stabilizes two orthogonal complex geodesics in $\mathbb{H}^2$, acting as a rotation by $\theta_1$ on one of them and as a rotation by $\theta_2$ on the other. We call the (nonoriented) pair $\{\theta_1, \theta_2\}$ the angle pair of $F$. Two elliptic isometries with same angle pair are $\mathrm{PU}(2, 1)$-conjugated.
Hence, $\mathcal{E}$ is the space of nonoriented angle pairs, and it can be seen as the triangular region $T := \{(\theta_1, \theta_2) \in \mathbb{R}^2 \mid 0 \leq \theta_2 \leq \theta_1 \leq 2\pi\}$ quotiented by the identification $(\theta, 0) \simeq (2\pi, \theta)$. Clearly, every point in $\mathbb{R}^2$ has a representative in $\mathcal{E}$. In what follows, if we write an element of $\mathcal{E}$ as $(\theta_1, \theta_2)$, with parenthesis, we are assuming that $0 \leq \theta_2 \leq \theta_1 \leq 2\pi$. Note that $\mathcal{E}^{\text{reg}}$ is homeomorphic to the interior of $T$. Also, points of the form $(\theta, \theta)$, for $0 < \theta < 2\pi$, correspond to classes of special elliptic isometries with negative center, while those of the form $(0, \theta) \simeq (2\pi, \theta)$ correspond to classes of special elliptic isometries with positive center.

As two loxodromic isometries admitting lifts with same trace are $\text{PU}(2,1)$-conjugated, $\mathcal{L}$ is homeomorphic to $\mathbb{C} \setminus \Delta$ quotiented by the action of the subgroup $\Omega \subset \mathbb{C}$, i.e., it is homeomorphic the cylinder $S^1 \times \mathbb{R}_{>0}$.

Note that the space $\mathcal{G}$ is not Hausdorff as, for example, any neighborhood of a unipotent class always intersect a neighborhood of the identical class. So, in order to use standard topological arguments, it is useful to consider the maximal Hausdorff quotient $c(\mathcal{G})$ of $\mathcal{G}$ and its natural projection $\rho: \mathcal{G} \to c(\mathcal{G})$.

Since the space $\mathcal{E} \cup \mathcal{B}$ is compact in $\mathcal{G}$, if follows that $\rho(\mathcal{E}) = \rho(\mathcal{E} \cup \mathcal{B})$ is compact in $c(\mathcal{G})$. (Note that $\mathcal{E}$ is not compact in $\mathcal{G}$, since there are sequences of regular elliptic isometries converging to parabolic ones.) Moreover, $\rho(\mathcal{E})$ is homeomorphic to the quotient $T/\simeq$. Such homeomorphism is given by the description of $\mathcal{E}$ as the space of angle pairs, together with $\rho$.

In order to understand how $\rho(\mathcal{B})$ is identified with the sides of $T/\simeq$, we observe that ellipto-parabolic isometries also have their angle pairs: if $F$ is a parabolic isometry with repeated eigenvalue $e^{i\theta} \in S^1$, then $e^{-2i\theta}$ is also an eigenvalue of $F$ and it is associated to the positive fixed point of $F$. Such an isometry stabilizes a complex geodesic, where it acts as a parabolic isometry in the sense of the geometry of the Poincaré disk, and rotates points around this line by the angle $-3\theta$. Therefore, the angle pair of $F$ is $\{-3\theta, 0\}$. Two parabolic isometries with the same angle pair are $\text{PU}(2,1)$-conjugated.

So, we identify $\rho(\mathcal{E})$ with the quotient $T/\simeq$ by seeing points in $\rho(\mathcal{E})$ as angles pairs corresponding either to regular elliptic classes or to special elliptic and parabolic classes. Furthermore, we identify the set $\mathcal{E}^{\text{reg}}$ with $\rho(\mathcal{E}^{\text{reg}})$ and frequently consider $\mathcal{E}^{\text{reg}} = T^\circ \subset \rho(\mathcal{E})$. Using these identifications, we will consider the space $\rho(\mathcal{E}) \subset c(\mathcal{G})$ as having two sides and one vertex determined by the sides and vertices of $T$: the sides and the vertex of $\rho(\mathcal{E})$ constitute the set $\rho(\mathcal{B})$. The points $(0,0)$, $(2\pi,0)$, $(2\pi,2\pi)$ of $T$ are identified in the vertex of $\rho(\mathcal{E})$. The diagonal (resp. the nondiagonal) side of $\rho(\mathcal{E})$ is given by angle pairs of the form $(\theta,\theta)$ (resp. $(\theta,0) \simeq (2\pi,\theta)$), with $0 < \theta < 2\pi$.

Hence, the fiber of $\rho$ over an angle pair in the nondiagonal side of $\rho(\mathcal{E})$ contains, besides the mentioned special elliptic class with positive center, the class of ellipto-parabolic isometries with this angle pair. Furthermore, the fiber of $\rho$ over a point in the diagonal side of $\rho(\mathcal{E})$ corresponds only to the class of a special elliptic isometry with negative center. Finally, the fiber of $\rho$ over the vertex of $\rho(\mathcal{E})$ has 4 elements: the identical class, the two classes of 2-step unipotent isometries, and the class of 3-step unipotent isometries.

### 3.2. Decomposing isometries. Here we discuss what it means to decompose, in $\text{PU}(2,1)$ and $\text{SU}(2,1)$, an isometry as the product of special elliptic isometries and introduce some notation.

#### 3.3. Definition. Given parameters $\alpha_1, \ldots, \alpha_n \in S^1 \setminus \Omega$, we say that an isometry in $\text{PU}(2,1)$ admits an $(\alpha_1, \ldots, \alpha_n)$-decomposition if it has a lift $F \in \text{SU}(2,1)$ such that $F = R^{p_n}_{\alpha_n} \cdots R^{p_1}_{\alpha_1}$, for points $p_1, \ldots, p_n \in \mathbb{P}V \setminus S V$. We say that an isometry admits an $\alpha^{(n)}$-decomposition, for $\alpha \in S^1 \setminus \Omega$, if it admits an $(\alpha, \ldots, \alpha)$-decomposition (with $n$-terms).
Given an isometry $F \in \text{PU}(2,1)$, the smallest number $n \in \mathbb{N}$ such that $F$ admits an $\alpha^{(n)}$-decomposition is the $\alpha$-length of $F$. The maximum of all $\alpha$-lengths over isometries in $\text{PU}(2,1)$ is the $\alpha$-length of $\text{PU}(2,1)$.

3.4. Proposition. Given parameters $\alpha_i \in S^1 \setminus \Omega$, the following statements hold:

(i) If an isometry in $\text{PU}(2,1)$ admits an $(\alpha_1, \ldots, \alpha_n)$-decomposition, then every isometry in its $\text{PU}(2,1)$-conjugacy class also does.

(ii) If an isometry in $\text{PU}(2,1)$ admits an $(\alpha_1, \ldots, \alpha_n)$-decomposition, then it also admits a $(\delta_1 \alpha_1, \ldots, \delta_n \alpha_n)$-decomposition, for any $\delta_1, \ldots, \delta_n \in \Omega$.

(iii) If an isometry in $\text{PU}(2,1)$ admits an $(\alpha_1, \ldots, \alpha_n)$-decomposition, then it also admits a $(\beta_1, \ldots, \beta_n)$-decomposition, where $(\beta_1, \ldots, \beta_n)$ is a cyclic permutation of $(\alpha_1, \ldots, \alpha_n)$.

Proof. Let $F \in \text{SU}(2,1)$ be an isometry that can be written as $F = R_{\alpha_1}^{p_1} \cdots R_{\alpha_n}^{p_n}$.

For any $I \in \text{SU}(2,1)$, we have $IFI^{-1} = R_{\alpha_1}^{ip_1} \cdots R_{\alpha_n}^{ip_n}$ since $I R_{\alpha_i}^{p_i} I^{-1} = R_{\alpha_i}^{ip_i}$. This proves (i).

Since $R_{\alpha_i}^{p_i} = \delta R_{\alpha_i}^{p_i}$ for any special elliptic isometry $R_{\alpha_i}$ and any $\delta \in \Omega$, given $\delta_1, \ldots, \delta_n \in \Omega$ we have $F = \delta R_{\alpha_1}^{p_1} \cdots R_{\alpha_n}^{p_n}$, where $\delta := \prod_{i=1}^{n} \delta_i \in \Omega$, which proves (ii). Finally, note that $R_{\alpha_1}^{n_1} F R_{\alpha_1}^{-n_1} = R_{\alpha_1}^{n_1} R_{\alpha_1}^{ip_1} \cdots R_{\alpha_n}^{ip_n}$. Using (i), we obtain (iii). \hfill \Box

Using item (i) of Proposition 3.4, we can say that a conjugacy class admits an $(\alpha_1, \ldots, \alpha_n)$-decomposition, for given parameters $\alpha_1, \ldots, \alpha_n \in S^1 \setminus \Omega$; this means that one (and therefore every) isometry in such a class admits an $(\alpha_1, \ldots, \alpha_n)$-decomposition. In the same way, we can consider the $\alpha$-length of conjugacy classes in $\Omega$.

3.5. Notation. We denote by $G_{\alpha_1, \ldots, \alpha_n} \subset \mathcal{J}$ (see Subsection 3.1) the set of semisimple classes admitting an $(\alpha_1, \ldots, \alpha_n)$-decomposition. We also denote by $E_{\alpha_1, \ldots, \alpha_n} \subset \mathcal{S}(\mathcal{J})$ the projection under $\rho$ of the set of classes in $\mathcal{E} \cup \mathcal{B}$ admitting an $(\alpha_1, \ldots, \alpha_n)$-decomposition (note that $E_{\alpha_1, \ldots, \alpha_n} \subset \mathcal{S}(\mathcal{J})$).

Following Definition 3.3, we use $G_{\alpha_1, \ldots, \alpha_n}$ and $E_{\alpha_1, \ldots, \alpha_n}$ when $\alpha_1 = \cdots = \alpha_n := \alpha$. Moreover, we denote $S_{\alpha}$ the set composed by the two classes of special elliptic isometries of parameter $\alpha$.

The next result is a direct consequence of item (ii) of Proposition 3.4 and implies that we can focus our attention in finding the $\alpha$-length of $\text{PU}(2,1)$ for parameters $\alpha = e^{ai}$ with $0 < a < 2\pi/3$.

3.6. Corollary. The $\alpha$-length of an isometry in $\text{PU}(2,1)$ is equal to its $\delta \alpha$-length of $\text{PU}(2,1)$, for any $\delta \in \Omega$.

Therefore, for any cube root of unity $\delta \in \Omega$, the $\alpha$-length and the $\delta \alpha$-length of $\text{PU}(2,1)$ coincide.

4 The unfolded trace and lines tangent to the deltoid

In this section we introduce the tools that will compose, together with the product map (see Subsection 6.9), the technique used to obtain the $\alpha$-length of $\text{PU}(2,1)$. The main idea is the interaction between lines tangent to the deltoid $\partial \Delta$ (and their relation to the eigenvalues of isometries in $\text{SU}(2,1)$) and the unfolded trace — a trace-like function that can distinguish regular elliptic $\text{PU}(2,1)$-conjugacy classes.

4.1. Unfolding the trace of elliptic isometries. As seen in Section 3, there are three distinct $\text{PU}(2,1)$-conjugacy classes corresponding to the three traces $\tau, \omega \tau, \omega^2 \tau$, where $\omega := e^{2\pi i/3}$ and $\tau \in \Delta \setminus \{0\}$. So, we have enough space to ‘unfold’ the trace of $\text{SU}(2,1)$ into a function that (while coinciding with the trace in some sense) distinguishes regular elliptic conjugacy classes. Such a function should, for any $\tau \in \Delta^0$, continuously send each of the three classes determined by $\tau$ to distinct values in $\{\tau, \omega \tau, \omega^2 \tau\}$. Here, we introduce a function that does just that.

Given $(\theta_1, \theta_2) \in \mathbb{T}$, put (see [13, Subsection 3.3.3])

$$E_{\theta_1, \theta_2} := \begin{bmatrix} e^{\frac{2\pi i - \theta_1}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i - \theta_2}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i + \theta_1}{3}} \end{bmatrix}$$.
and define $utr : T \to \mathbb{C}$ by $utr(\theta_1, \theta_2) = tr E_{\theta_1, \theta_2}$. This function does not descend to a well defined function on $\mathcal{E}$ since $tr E_{(\theta, 0)} \neq tr E_{(2\pi, \theta)}$ for every $0 \leq \theta \leq 2\pi$, but it clearly well defines a function $utr : \mathcal{E}^{\text{reg}} \to \mathbb{C}$. Abusing notation, we also evaluate $utr$ directly on regular elliptic isometries by considering $utr F = utr(\theta_1, \theta_2)$, where $(\theta_1, \theta_2) \in \mathcal{E}^{\text{reg}}$ is the angle pair of $F$.

![Figure 1: How utr maps the boundary of $T$ onto $\partial \Delta$. (Colors are visible in the online version)](image)

**4.2. Proposition.** The function $utr$ maps $T$ bijectively onto $\Delta$.

**Proof.** Take $(\theta_1, \theta_2) \in \mathbb{T}$ and let $F_1 \in \text{SU}(2, 1)$ be an elliptic isometry with angle pair $(\theta_1, \theta_2)$ and eigenvalues $e^{\frac{2\pi i}{3} + \frac{2\pi i}{3}}$, $e^{\frac{2\pi i}{3} + \frac{2\pi i}{3}}$ and $e^{\frac{2\pi i}{3} + \frac{2\pi i}{3}}$, with $e^{\frac{2\pi i}{3} + \frac{2\pi i}{3}}$ being of negative type.

Now, consider $F_2 \in \text{SU}(2, 1)$ such that $tr F_1 = tr F_2$ (in particular $F_1$ and $F_2$, have the same eigenvalues, see [7, Proof of Lemma 6.2.5]) and such that $e^{\frac{2\pi i}{3} + \frac{2\pi i}{3}}$ is its negative type eigenvalue. It follows that the angle pair of $F_2$ is $(-\theta_1, \theta_1 + \theta_2)$. Using the fact that $0 \leq \theta_2 \leq \theta_1 \leq 2\pi$, we obtain that such angle pair projects to the point $(2\pi - \theta_1 + \theta_2, 2\pi - \theta_1) \in T$. This implies that $utr F_2 = utr(2\pi - \theta_1, 2\pi - \theta_1 + \theta_2) = \omega \cdot utr F_1$. Therefore, if $tr F_1 \neq 0$, $utr F_1 \neq utr F_2$. The argument follows analogously if we assume that the negative type eigenvalue of $F_2$ is $e^{\frac{2\pi i}{3} + \frac{2\pi i}{3}}$.

It remains to prove that $utr$ is surjective. Given $\tau \in \Delta^\circ$, $\tau \neq 0$, there are three distinct $\text{PU}(2, 1)$-conjugacy classes that admit a lift with trace $\tau$ (see Section 3). Since $utr$ is injective, it sends each of these classes to a distinct value, and the values it can assume lie in the set $\{\tau, \omega \tau, \omega^2 \tau\}$. For $\tau = 0$, there is only one conjugacy class with such trace. Finally, $utr$ sends the boundary of $T$ onto $\partial \Delta$.

In fact,

$$utr(\theta, \theta) = 2e^{\frac{\theta}{2}i} + e^{-\frac{\theta}{3}i}, \quad utr(0, \theta) = 2e^{-\frac{\theta}{2}i} + e^{\frac{\theta}{2}i}, \quad \text{and} \quad utr(\theta, 0) = 2e^{-\frac{\theta}{2}i} + e^{\frac{\theta}{2}i},$$

for $\theta \in [0, 2\pi]$. It follows that each side of the boundary of $T$ is mapped onto a distinct side of $\partial \Delta$ (see Figure 1).

**4.3. Corollary.** Let $F_1, F_2 \in \text{PU}(2, 1)$ be regular elliptic isometries. If $utr F_1 = utr F_2$, then $F_1$ and $F_2$ are $\text{PU}(2, 1)$-conjugated.

**4.4. Lines tangent to the deltoid.** Given $\alpha_1, \alpha_2 \in \mathbb{S}^1$, consider the function $\tau_{\alpha_1, \alpha_2} : \mathbb{R} \to \mathbb{C}$ defined by

$$\tau_{\alpha_1, \alpha_2}(t) := \alpha_1 \alpha_2 + \alpha_1^{-1} \alpha_2^{-1} + (\alpha_1^{-2} - \alpha_1)(\alpha_2^{-2} - \alpha_2)t. \quad (2)$$

From [6, Section 6] we have the following lemma.

**4.5. Lemma.** If $p_1, p_2 \in \mathbb{P}V \setminus S V$ are such that $ta(p_1, p_2) = t$ (see Section 2), then $tr R^{p_2}_{p_1} R^{p_1}_{p_2} = \tau_{\alpha_1, \alpha_2}(t)$.

Moreover, by [6, Lemma 6.17], $\tau_{\alpha_1, \alpha_2}(\mathbb{R})$ is a line tangent to $\partial \Delta$ at $\tau_{\alpha_1, \alpha_2}(1)$, and it only depends on the unit complex number $\alpha_1 \alpha_2$. We denote this line by $t_\alpha$ where $\alpha = \alpha_1 \alpha_2$.

**4.6. Proposition.** Let $F \in \text{SU}(2, 1)$ and let $\alpha \in \mathbb{S}^1$. Then $tr F \in t_\alpha$ iff $\alpha$ is an eigenvalue of $F$.

**Proof.** Suppose that $tr F \in t_\alpha$, and let $\alpha_1, \alpha_2 \in \mathbb{S}^1 \setminus \Omega$ be such that $\alpha_1 \alpha_2 = \alpha$. Then, $\ell_\alpha = \ell_{\alpha_1 \alpha_2}$ which implies that there exists $t \in \mathbb{R}$ such that $\tau_{\alpha_1, \alpha_2}(t) = tr F$. Taking $p_1, p_2 \in \mathbb{P}V \setminus S V$ with $ta(p_1, p_2) = t$, by Lemma 4.5, we have $tr R^{p_2}_{p_1} R^{p_1}_{p_2} = tr F$ and, since trace determines eigenvalues and $\alpha_1 \alpha_2$ is an eigenvalue of $R^{p_2}_{p_1} R^{p_1}_{p_2}$, $\alpha$ is an eigenvalue of $F$. 

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Conversely, suppose that $\alpha \in S^1$ is an eigenvalue of $F$.

Assume that $F$ is not loxodromic. Then $\text{tr} F = \alpha + \beta + \overline{\alpha \beta}$ for some $\beta \in S^1$. In fact, if $F$ is not loxodromic, then it can be triangularized with unit norm eigenvalues (see [10, Section 3.2]). Take $\alpha_1, \alpha_2 \in S^1 \setminus \Omega$ with $\alpha_1 \alpha_2 = \alpha$. Note that,

$$z = \frac{\beta + \alpha_1^{-1} \alpha_2^{-1} \beta^{-1} - \alpha_1^{-2} \alpha_2 - \alpha_1 \alpha_2^{-2}}{(\alpha_1^{-2} - 1)(\alpha_2^{-2} - 1)} \in \mathbb{R}.$$

In fact, it is easy to see that $z - \tau = 0$. Hence, the equation $\tau_{\alpha_1, \alpha_2}(t) = \text{tr} F$ has a real solution in $t$; therefore $\text{tr} F \in \ell_{\alpha_1 \alpha_2} = \ell_\alpha$.

Now, assume that $F$ is loxodromic. Let $\ell_\beta$ be a line tangent to the deltoid and passing through $\text{tr} F$. Thus, given parameters $\alpha_1, \alpha_2 \in S^1 \setminus \Omega$ with $\alpha_1 \alpha_2 = \beta$, there exist points $p_1, p_2 \in \mathbb{P}V \setminus S V$ such that $\text{tr} R_{p_2} R_{p_1}^\tau = \text{tr} F$, and $\beta$ is an eigenvalue of $F$. Since a loxodromic isometry cannot have two unit norm eigenvalues (see [10, Lemma 3.2]), we conclude that $\alpha = \beta$ and $\text{tr} F \in \ell_\alpha$.

It follows that, given $\tau \in \Delta^\circ$, there exists exactly three lines through $\tau$ that are tangent to $\partial \Delta$. Writing these lines as $\ell_\alpha$, $\ell_\beta$, $\ell_\gamma$, we have $\gamma = \overline{\alpha \beta}$, $\tau = \alpha + \beta + \overline{\alpha \beta}$ and the eigenvalues of any isometry with trace $\tau$ are $\alpha, \beta, \overline{\alpha \beta}$. In the case where $\tau \in \partial \Delta$, there are exactly two tangent lines to $\partial \Delta$ that contain $\tau$; if one of the lines is $\ell_\alpha$, the other is $\ell_{\alpha - 2}$.

Furthermore, two distinct lines $\ell_\alpha$ and $\ell_\beta$ intersect in $\Delta$. In fact, they cannot be parallel since there exists an isometry with trace $\tau := \alpha + \beta + \overline{\alpha \beta}$ and $\tau \in \ell_\alpha \cap \ell_\beta$ by the previous proposition. Moreover, if they intersect at a point $\tau \in \mathbb{C} \setminus \Delta$, then $\tau$ is the trace of a loxodromic isometry that have $\alpha$ and $\beta$ as eigenvalues, what cannot happen as loxodromic isometries have a single unitary eigenvalue.

### 4.7. Tangent lines and the unfolded trace

Here we describe the inverse image under $\text{utr}$ of lines tangent to the deltoid; it will be composed by line segments of slopes $-1, \frac{1}{2}, 2$ in $T$.

#### 4.8. Lemma

Given $\alpha \in S^1$, the inverse image $\text{utr}^{-1}(\ell_\alpha \cup \ell_{\alpha \omega} \cup \ell_{\alpha \omega^2})$ is given by the projection on $T$ of the lines in $\mathbb{R}^2$ defined by the equations $y = -x - 3a$ and $y = 2x - 3 \alpha$, where $0 \leq a < 2\pi$ is such that $\alpha = e^{ia}$. More precisely, $\text{utr}^{-1}(\ell_\alpha \cup \ell_{\alpha \omega} \cup \ell_{\alpha \omega^2})$ is given by the union of the segments:

- $\{(\alpha, 0), (3a, 3a)\}, \{(3a, 3a), (2\pi - 3a, 2\pi - 3a)\}, \{(2\pi - 3a, 2\pi - 3a), (2\pi - 3a, 2\pi - 3a)\}, \{(2\pi - 3a, 0), (2\pi, 2\pi - 3a)\}$, if $0 \leq a \leq \frac{\pi}{2}$;
- $\{(2\pi - 3a, 0), (3a - 2\pi, 3a - 2\pi), (2\pi, 4a - 3a), (\frac{3a}{2}, 0)\}, \{(2\pi, 4a - 3a), (\frac{3a}{2}, 0)\}, \{(4a - 3a, 0), (2\pi - 3a, 2\pi - 3a)\}$, if $\frac{\pi}{2} \leq a \leq \frac{3\pi}{4}$;
- $\{(3a - 3a, 3a - 3a), (2\pi - 3a, 2\pi - 3a)\}, \{(2\pi, 6a - 3a), (\frac{3a}{2}, 2\pi - 3a)\}$, if $\frac{3\pi}{4} \leq a < 2\pi$.

Proof. Let $\tau \in \ell_\alpha$. By Proposition 4.6, every isometry with trace $\tau$ has $\alpha$ as an eigenvalue. We write $\alpha = e^{ia}$, $0 \leq a < 2\pi$, and write the other eigenvalues as $e^{it}$ and $e^{-(a + it)}$, $0 \leq t < 2\pi$; this is well defined since the trace of an element of $SU(2, 1)$ determines its eigenvalues. It follows that the (nonoriented) angle pairs of the isometries in $PU(2, 1)$ with trace $\tau$ are given by $(t - a, -t - 2a)$ and $(t + 2a, 2t + a)$, with $t$ varying. The projection of this pair in $T$ coincides with the projection of the lines defined in the proposition.

#### 4.9. Corollary

Given $\alpha \in S^1$, the inverse image under $\text{utr}$ of the line $\ell_\alpha$ is given by

- $\{(\alpha, 0), (3a, 3a)\}, \{(3a, 3a), (2\pi, 2\pi - 3a)\}$, if $0 \leq a < 2\pi / 3$;
- $\{(3\pi - 3a, 3a - 3a), (2\pi, 4a - 3a), (\frac{3a}{2}, 0)\}$, if $2\pi / 3 \leq a \leq 4\pi / 3$;
- $\{(2\pi, 2\pi - 2a), (6\pi - 3a, 0)\} \cup \{(6\pi - 3a, 0), (3\pi - 2\pi, 3\pi - 2\pi)\}$, if $4\pi / 3 \leq a < 2\pi$.

where $0 \leq a < 2\pi$ is such that $\alpha = e^{ia}$.

Proof. By Lemma 4.8, $\text{utr}^{-1}(\ell_\alpha \cup \ell_{\alpha \omega} \cup \ell_{\alpha \omega^2})$ is either the union of three connected curves, each being the union of two line segments with a vertex in the boundary of $T$, if $a \neq 0, 2\pi / 3, 4\pi / 3$, or a median of $T$, if $a = 0, 2\pi / 3, 4\pi / 3$. Since we know how $\text{utr}$ sends the boundary of $T$ onto $\partial \Delta$ (see Figure 1), we determine which of these three curves is $\text{utr}^{-1}(\ell_\alpha)$.
Lemma 4.8 and Corollary 4.9. The inverse of utr sends each line tangent to \( \partial \Delta \) to a curve that is the union of two line segments with a vertex in the boundary of the triangle \( T \); it also sends each line tangent to one of the vertices of \( \partial \Delta \) to a median of \( T \). The color in Figure 1 shows us how to color Figure 2 (see online version). Pictures relating Goldman’s deltoid and the triangle \( T \) have appeared before (see, for example, [11, Figure 1]).

5 Length 2 decompositions

In this section, using the tools introduced in Section 4, we study decompositions of isometries as the product of two special elliptic isometries. First, in Subsection 5.1, we prove that every isometry that is not 2-step unipotent admits an \((\alpha_1, \alpha_2)\)-decomposition, for some parameters \( \alpha_1, \alpha_2 \).

In Subsection 5.5, for given parameters \( \alpha_1, \alpha_2 \), we obtain all isometries that admit an \((\alpha_1, \alpha_2)\)-decomposition. In particular, we determine which isometries have \( \alpha \)-length 2 for a given parameter \( \alpha \).

5.1. Generic decompositions. The following lemma characterizes the product of two special elliptic isometries that have their centers generating an Euclidean complex line.

5.2. Lemma. Let \( p_1, p_2 \in \mathbb{PV} \setminus SV \) be distinct nonisotropic points and let \( \alpha_1, \alpha_2 \in S^1 \setminus \Omega \) be parameters. If the complex line \( L(p_1, p_2) \) is Euclidean, then the isometry \( R := R_{\alpha_2}^p R_{\alpha_1}^p \) is parabolic. Moreover, either \( \alpha_1 \alpha_2 \notin \Omega \) and \( R \) is ellipto-parabolic, or \( \alpha_1 \alpha_2 \in \Omega \) and \( R \) is 3-step unipotent.

Proof. By hypothesis, \( \text{ta}(p_1, p_2) = 1 \) (see Section 2) and, by Lemma 4.5, \( \text{tr} R = \tau_{\alpha_1, \alpha_2}(1) = 2 \alpha_1 \alpha_2 + (\alpha_1 \alpha_2)^{-2} \). Then, the isometry \( R \) is either special elliptic or parabolic (not necessarily special elliptic since \( p_1 \neq p_2 \)). Let \( v \) be the polar point of the line \( L(p_1, p_2) \); it follows that \( v \) is an isotropic fixed point of \( R \). Suppose that \( R \) stabilizes a hyperbolic line \( L \) through \( v \). Using (1) to solve the equation \( R_{\alpha_2}^p R_{\alpha_1}^p x = \alpha_1 \alpha_2 x \), we obtain that a point \( x \in L \) is \( R \)-fixed with eigenvalue \( \alpha_1 \alpha_2 \) if \( \langle x, p_1 \rangle = \langle x, p_2 \rangle = 0 \), which implies \( x = v \). Therefore, \( R \) is parabolic, and it is unipotent if \( \alpha_1 \alpha_2 = \delta \) for some \( \delta \in \Omega \).

5.3. Remark. (See [6, Section 6]) Let \( p_1, p_2 \in \mathbb{PV} \setminus SV \) be distinct points such that the line \( L := L(p_1, p_2) \) is hyperbolic and let \( \alpha_1, \alpha_2 \in S^1 \setminus \Omega \) be parameters.

Suppose that the isometry \( R := R_{\alpha_2}^p R_{\alpha_1}^p \) is regular elliptic. If \( p_1, p_2 \) are respectively the points in \( L \) orthogonal to \( p_1, p_2 \), then \( \text{ta}(p_1, p_2) = \text{ta}(p_1, p_2) \). Moreover the isometries \( R \) and \( \tilde{R} := R_{\alpha_2}^p R_{\alpha_1}^p \) have the same trace but lie in distinct SU(2, 1)-conjugacy classes.

If the isometry \( R \) is regular parabolic or loxodromic, an analogous process produces an isometry \( \tilde{R} \) that lies in the same SU(2, 1)-conjugacy class as \( R \). This implies that there exists a relation of the form \( R_{\alpha_2}^p R_{\alpha_1}^p = R_{\alpha_2}^{p_1} R_{\alpha_1}^{p_1} \) with \( \sigma p_1 = -\sigma p_1 \) (\( \sigma p \) stands for the signature of a point \( p \)). Relations obtained in this way are called simultaneous change of signs.

5.4. Proposition. Every isometry that is not 2-step unipotent admits an \((\alpha_1, \alpha_2)\)-decomposition, for some parameters \( \alpha_1, \alpha_2 \in S^1 \setminus \Omega \). Moreover, 2-step unipotent isometries do not admit an \((\alpha_1, \alpha_2)\)-decomposition, for any parameters \( \alpha_1, \alpha_2 \in S^1 \setminus \Omega \).
Proof. Since $IR_{\alpha}^{2}I^{-1} = R_{\alpha}^{2}$ for any isometry $I \in SU(2,1)$, in order to prove that a given isometry $F \in SU(2,1)$ admits an $(\alpha_1, \alpha_2)$-decomposition, it suffices to prove that $F$ is in the same $SU(2,1)$-conjugacy class as an isometry of the form $R_{\alpha_2}^{2}R_{\alpha_1}^{1}$, for some $p_1, p_2 \in PV \setminus SV$ and $\alpha_1, \alpha_2 \in S^1 \setminus \Omega$.

If $F = R_{\alpha_1}^{1}$, we just take $\alpha_1, \alpha_2 \in S^1 \setminus \Omega$ with $\alpha_1 \alpha_2 = \beta$ and we have $F = R_{\alpha_1}^{2}R_{\alpha_2}^{1}$.

Assume that $F$ is loxodromic. Let $\alpha \in C$ be the unit norm eigenvalue of $F$. Then, by Proposition 4.6, $tr F \in \ell_\alpha$. If $\alpha, \alpha_2 \in S^1 \setminus \Omega$ are such that $\alpha \alpha_2 = \alpha$, then $\ell_\alpha = \ell_{\alpha \alpha_2}$ and, by Lemma 4.5, $tr F = tr R_{\alpha_2}^{2}R_{\alpha_1}^{1}$ for some $p_1, p_2 \in PV \setminus SV$. Since two loxodromic isometries in $SU(2,1)$ with the same trace are conjugated (see Section 3), the result follows.

Now suppose that $F$ is regular elliptic, let $\alpha$ and $\beta$ be two distinct eigenvalues of $F$ with $\alpha, \beta \in S^1 \setminus \Omega$. Let $\alpha_i, \beta_i \in S^1 \setminus (\Omega \cup -\Omega)$ be parameters such that $\alpha_1 \alpha_2 = \alpha$, $\beta_1 \beta_2 = \beta$ (here, $-\Omega := \{ -1, -\omega, -\omega^2 \}$). Then, by Lemma 4.5, there exist $s, t \in \mathbb{R}$ satisfying $\tau_{\alpha_1, \alpha_2}(s) = \tau_{\beta_1, \beta_2}(t) = tr F$. Taking $-\alpha_i$ (resp., $-\beta_i$) in place of $\alpha_i$ (resp., $\beta_i$) if necessary, we can assume that $s, t \notin [0, 1]$; this follows from the fact that $\tau_{\alpha_1, \alpha_2}$ and $\tau_{-\alpha_1, -\alpha_2}$ parametrize $\ell_\alpha$ in opposite directions (see [6, Lemma 6.19]). Now, let $p_1, q_i \in PV \setminus SV$, $i = 1, 2$, be points with $ta(p_1, p_2) = s$ and $ta(q_1, q_2) = t$; it follows that $tr R_{\alpha_2}^{2}R_{\alpha_1}^{1} = tr R_{\beta_2}^{2}R_{\beta_1}^{1}$. Denote by $\bar{p}_i$ the point in $P_1 := L(p_1, p_2)$ orthogonal to $p_1$, and by $\bar{q}_i$ the point in $L_2 := L(q_1, q_2)$ orthogonal to $q_i$. Note that since $s, t \notin [0, 1]$, the lines $L_1$ and $L_2$ are hyperbolic, thus $\sigma_{\bar{p}_i} = -\sigma_{p_1}$ and $\sigma_{\bar{q}_i} = -\sigma_{q_i}$. By Remark 5.3, $R_1 := R_{\alpha_2}^{2}R_{\alpha_1}^{1}$ and $R_1 := R_{\beta_2}^{2}R_{\beta_1}^{1}$ are isometries of same trace (equal to $tr F$) but living in distinct $SU(2,1)$-conjugacy classes; the same is true for the isometries $R_2 := R_{\alpha_2}^{2}R_{\alpha_1}^{1}$ and $R_2 := R_{\beta_2}^{2}R_{\beta_1}^{1}$. Therefore, since we have three $SU(2,1)$-conjugacy classes for each trace, one of the isometries $R_1, \bar{R}_1, R_2, \bar{R}_2$ must lie in the $SU(2,1)$-conjugacy class of $F$.

Finally, the case where $F$ is elliptic-parabolic or 3-step unipotent follows from Lemma 5.2: if $\alpha$ and $\alpha^2$ are the eigenvalues of $F$ we just need to take parameters $\alpha_1, \alpha_2 \in S^1 \setminus \Omega$ with $\alpha_1 \alpha_2 = \alpha$ and points $p_1, p_2 \in PV \setminus SV$ that generate a Euclidean line. If follows that $F$ is conjugated to $R_{\alpha_2}^{2}R_{\alpha_1}^{1}$. (If $F$ is elliptic-parabolic, proceeding as above and taking $\alpha_1 \alpha_2 = \alpha^2$, we obtain $p_1, p_2$ such that $L(p_1, p_2)$ is noneuclidean.)

The second part of the proposition follows from the fact that, if $p_1 \neq p_2$ the isometry $R_{\alpha_2}^{2}R_{\alpha_1}^{1}$ is regular (see Definition 2.2, Lemma 5.2, and [6, Lemma 4.2]).

5.5. Decompositions with fixed parameters. Here we are interested in the case where the parameters $\alpha_1, \alpha_2$ are given. We start determining which regular parabolic or loxodromic isometries admit an $(\alpha_1, \alpha_2)$-decomposition.

5.6. Proposition. Let $\alpha_1, \alpha_2 \in S^1 \setminus \Omega$ be parameters. The following statements hold:

- a loxodromic isometry admits an $(\alpha_1, \alpha_2)$-decomposition iff it is lifted to an isometry $F \in SU(2,1)$ with $tr F \in \ell_{\alpha_1, \alpha_2}$ (see Subsection 4.4);
- a regular parabolic isometry (Definition 2.2) admits an $(\alpha_1, \alpha_2)$-decomposition iff it is lifted to an isometry $F \in SU(2,1)$ with $tr F \in \ell_{\alpha_1, \alpha_2}$ and $tr F \neq \tau_{\alpha_1, \alpha_2}(0)$.

Proof. First, note that, by Proposition 4.6, if an isometry admits an $(\alpha_1, \alpha_2)$-decomposition, then it lifts to $SU(2,1)$ to an isometry with trace lying in $\ell_{\alpha_1, \alpha_2}$. Moreover, if $p_1, p_2 \in PV \setminus SV$ are orthogonal points (i.e., $ta(p_1, p_2) = 0$), then $\delta R_{\alpha_2}^{2}R_{\alpha_1}^{1}$ is an elliptic isometry, for any $\delta \in \Omega$.

Suppose that $F \in SU(2,1)$ is loxodromic and $tr F \in \ell_{\alpha_1, \alpha_2}$. If $t_0 \in \mathbb{R}$ is such that $tr F = \tau_{\alpha_1, \alpha_2}(t_0)$, and $p_1, p_2 \in PV \setminus SV$ satisfy $ta(p_1, p_2) = t_0$, then $tr F = tr R_{\alpha_2}^{2}R_{\alpha_1}^{1}$, and, since two loxodromic isometries with the same trace are $SU(2,1)$-conjugated, the result follows.

Now, suppose that $F \in SU(2,1)$ is regular parabolic with $tr F \in \ell_{\alpha_1, \alpha_2}$ and $tr F \neq \tau_{\alpha_1, \alpha_2}(0)$. Parametrize the line $\ell_{\alpha_1, \alpha_2}$ by $\tau_{\alpha_1, \alpha_2}(t)$ and let $t_0 \in \mathbb{R}$ be such that $\tau_{\alpha_1, \alpha_2}(t_0) = tr F$. By hypothesis, $t_0 \neq 0$. If $t_0 = 1$, then $F$ is conjugated to $R := R_{\alpha_2}^{2}R_{\alpha_1}^{1}$, where $p_1, p_2 \in PV \setminus SV$ are any points with $ta(p_1, p_2) = t_0$. (This follows from $tr F = tr R$ together with the fact that $R$ is not special elliptic, since $t_0 \neq 0$.1) If $t_0 = 1$, let $p_1, p_2 \in EV$ be points such that the line $L(p_1, p_2)$ is Euclidean and let $R$ be defined as above. Then, $tr F = tr R$ and, by Lemma 5.2, the isometry $R$ is either ellipto-parabolic or 3-step unipotent. Therefore, $R$ and $F$ are conjugated.

Now, to obtain the remaining isometries (regular or special elliptic) admitting an $(\alpha_1, \alpha_2)$-decomposition, we determine $E_{\alpha_1, \alpha_2}$ (see Notation 3.5) and the classes of its intersection with the boundary of $\rho(\hat{E})$. We write $E_{\alpha_1, \alpha_2}$ as the union of sets $E_{\alpha_1, \alpha_2}$ that are defined as follows.
5.7. Definition. We denote by \( E_{\sigma_1,\sigma_2} \), the set composed by \((\text{the projection on } \rho(E) \text{ of})\) classes of elliptic or parabolic isometries admitting a decomposition of the form \( F = R_{\sigma_1}^2 R_{\sigma_2}^1 \), with \( \rho_1 = \sigma_1, \rho_2 = \sigma_2 \), \( i = 1, 2 \). (To simplify the notation, the signs \( \sigma_1, \sigma_2 \) are taken as one of the symbols \(-, +\) instead of values in \( \{-1, 1\} \).)

5.8. Proposition. Let \( \alpha_1 = e^{\alpha_1 i} \) and \( \alpha_2 = e^{\alpha_2 i} \) be parameters with \( 0 < \alpha_j < 2\pi/3, j = 1, 2 \). Then \( E_{\alpha_1,\alpha_2} \) is given by the union of the sets \( E_{\alpha_1,\alpha_2} \) where:

- \( E_{\alpha_1,\alpha_2} \) is a single line segment (possibly, a single point) given by either \([3(\alpha_1 + 3\alpha_2, 3\alpha_1 + 3\alpha_2), (2\pi, \pi + 3(\alpha_1 + \alpha_2)/2)]\), if \( 0 < \alpha_1 + \alpha_2 \leq 2\pi/3 \), or \([3(\alpha_1 + \alpha_2) - 2\pi, 3(\alpha_1 + \alpha_2) - 2\pi), (3(\alpha_1 + \alpha_2)/2 - \pi, 0)]\), if \( \alpha_1 + \alpha_2 > 2\pi/3 \). If \( E_{\alpha_3,\alpha_2} \) is a single point, then it is the class of the identical isometry. Otherwise, the vertex of \( E_{\alpha_1,\alpha_2} \) in the diagonal side of \( \rho(E) \), corresponds to a special elliptic isometry with negative center, and the vertex of \( E_{\alpha_1,\alpha_2} \) lying in the nondiagonal side of \( \rho(E) \) corresponds to an ellipto-parabolic isometry.

- \( E_{\alpha_1,\alpha_2}^{\perp} \) is the union of two line segments with a common vertex, one being of slope \(-1\) and the other of slope \( \pm 1/2 \) (possibly, a single point), given by

\[
\left[ (2\pi - 3\alpha_1, 2\pi - 3\alpha_2), (2\pi, 2\pi - 3(\alpha_1 + \alpha_2)) \right] \cup \left[ (2\pi, 2\pi - 3(\alpha_1 + \alpha_2)), (\pi + 3(\alpha_1 + \alpha_2)/2, 0) \right],
\]

if \( \alpha_1 \leq \alpha_2 \) and \( 0 < \alpha_1 + \alpha_2 \leq 2\pi/3 \); by

\[
\left[ (2\pi - 3\alpha_2, 2\pi - 3\alpha_1), (2\pi, 2\pi - 3(\alpha_1 + \alpha_2)) \right] \cup \left[ (2\pi, 2\pi - 3(\alpha_1 + \alpha_2)), (\pi + 3(\alpha_1 + \alpha_2)/2, 0) \right],
\]

if \( \alpha_1 \geq \alpha_2 \) and \( 0 < \alpha_1 + \alpha_2 \leq 2\pi/3 \); by

\[
\left[ (2\pi, 3(\alpha_1 + \alpha_2)/2 - \pi), (4\pi - 3(\alpha_1 + \alpha_2), 0) \right] \cup \left[ (4\pi - 3(\alpha_1 + \alpha_2), 0), (2\pi - 3\alpha_1, 2\pi - 3\alpha_2) \right],
\]

if \( \alpha_1 \leq \alpha_2 \) and \( \alpha_1 + \alpha_2 > 2\pi/3 \); or by

\[
\left[ (2\pi, 3(\alpha_1 + \alpha_2)/2 - \pi), (4\pi - 3(\alpha_1 + \alpha_2), 0) \right] \cup \left[ (4\pi - 3(\alpha_1 + \alpha_2), 0), (2\pi - 3\alpha_2, 2\pi - 3\alpha_1) \right],
\]

if \( \alpha_1 \geq \alpha_2 \) and \( \alpha_1 + \alpha_2 > 2\pi/3 \). If the second segment is a single point, it corresponds to both the identical class and the 3-step unipotent class. Otherwise, the common vertex of the segments corresponds to both a class of a special elliptic isometry with positive center and an ellipto-parabolic class; the other vertex of the segment of slope \(-1\) corresponds to a (possibly special) elliptic isometry; the remaining vertex corresponds to an ellipto-parabolic isometry.

- \( E_{\alpha_1,\alpha_2}^{\perp} \) is a single line segment (possibly, a point) given by \([3(\alpha_1, 3\alpha_2 - 3\alpha_1), (3(\alpha_1 + \alpha_2)/2, 0)]\), if \( \alpha_1 \leq \alpha_2 \); or by \([2\pi + 3\alpha_1 - 3\alpha_2, 3\alpha_1, 3\alpha_2), (2\pi, 3(\alpha_1 + \alpha_2)/2)]\), if \( \alpha_1 \geq \alpha_2 \). Furthermore, \( E_{\alpha_1,\alpha_2}^{\perp} \) is a single point corresponding to the class of a special elliptic isometry with positive center iff \( \alpha_1 = \alpha_2 \); otherwise it has a vertex in \( E_{\alpha_1}^{\rho E} \) and the other vertex correspond to an ellipto-parabolic class.

- \( E_{\alpha_1,\alpha_2}^{\perp} \) is a single line segment (possibly, a point) given by \([2\pi + 3\alpha_1 - 3\alpha_2, 3\alpha_1, 3\alpha_2), (2\pi, 3(\alpha_1 + \alpha_2)/2)]\), if \( \alpha_1 \leq \alpha_2 \); or by \([3(\alpha_1, 3\alpha_2 - 3\alpha_1), (3(\alpha_1 + \alpha_2)/2, 0)]\), if \( \alpha_1 \geq \alpha_2 \). Furthermore, \( E_{\alpha_1,\alpha_2}^{\perp} \) is a single point corresponding to the class of a special elliptic isometry with positive center iff \( \alpha_1 = \alpha_2 \); otherwise it has a vertex in \( E_{\alpha_1}^{\rho E} \) and the other vertex correspond to an ellipto-parabolic class.

Proof. Denote \( \beta_j := \omega^{j-1}\alpha_1, j = 1, 2, 3 \). By Proposition 4.6, \( E_{\alpha_1,\alpha_2} \) is contained in the subset of \( \rho(E) \) determined by \( ut^{\perp}(\ell_{\beta_1,\alpha_2} \cup \ell_{\beta_2,\alpha_2} \cup \ell_{\beta_3,\alpha_2}) \subseteq T \). We consider the lines \( \ell_{\beta_j,\alpha_2} \) parametrized by \( \tau_{\beta_j,\alpha_2}(t), j = 1, 2, 3 \), (see Subsection 4.4). Define \( \chi_1 := \Im(\frac{\chi_1}{\alpha_1}, \chi_2 := \Im(\frac{\chi_2}{\alpha_2}), \alpha_2) \), and

\[
t_{\pm} := \frac{1 + 4\chi_1\chi_2 \pm \sqrt{(1 + 4\chi_1^2)(1 + 4\chi_2^2)}}{2}.
\]

By [6, Corollary 5.8], \( \tau_{\beta_j,\alpha_2}(t_{\pm}) \) lie in \( \partial \Delta \), and \( t_- < 0 < 1 \leq t_+ \). Moreover, for \( j = 1, 2, 3 \),

(A) \( \tau_{\beta_j,\alpha_2}(t) = \omega^{j-1}(2e^{-\alpha_j + \alpha_2} + \alpha_j + \alpha_2) \);

(B) \( \tau_{\beta_j,\alpha_2}(0) = \omega^{j-1}(\alpha_j + \alpha_2) + \alpha_j - 2\alpha_2 \).

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implies that its angle pair is \( \{ \gamma \} \). The solid points correspond to elliptic or identical classes, while the punctured ones correspond to regular parabolic classes.

\[
(C) \quad \tau_{\beta_1, \alpha_2}(1) = \omega^{j-1}(2e^{(\alpha_1+\alpha_2)i} + e^{-(\alpha_1+\alpha_2)i});
\]
\[
(D) \quad \tau_{\beta_1, \alpha_2}(t) = \omega^{j-1}((2e^{\alpha_1+i} + e^{\alpha_2}));
\]

Note that the points in \((C)\) are the ones where the lines \( \ell_{\beta_1, \alpha_2} \) are tangent to \( \partial \Delta \), and that if \( \alpha_1 = \alpha_2 \), then \( t_* = 0 \) and the points in \((A)\) coincide with the one in \((B)\).

\((E_{\alpha_1, \alpha_2}^-)\) Let \( p_1 \in B V \) and let \( L \) be a hyperbolic complex line through \( p_1 \). Consider a curve \( \gamma : [1, t_*] \to BV \) such that \( \gamma(t) \in L \) and \( \tau(p_1, \gamma(t)) = t \) for all \( t \in [1, t_*] \); in particular \( \gamma(1) = p_1 \). Note that every elliptic or regular parabolic isometry \( F \in SU(2, 1) \) that admits a decomposition \( R_{a_2}^R R_{a_1}^R \) with \( q_1, q_2 \in BV \) is in the same \( SU(2, 1) \)-conjugacy class of the isometry \( R_{a_2}^{(ta(q_1, q_2))} R_{a_1}^{(1)} \).

Thus, the curve \( \tilde{\gamma} \) in \( S \) defined by \( \tilde{\gamma}(t) = [R_{a_2}^{(t)} R_{a_1}^{(1)}] \) is such that the projection on \( c(S) \) of its image is \( E_{\alpha_1, \alpha_2}^- \). Moreover, \( R_{a_2}^{(1)} R_{a_1}^{(1)} \) is special elliptic with angle pair \( \{ 3a_1 + 3a_2, 3a_1 + 3a_2 \} \), and \( R_{a_2}^{(t)} R_{a_1}^{(1)} \) is parabolic with one of the numbers in \( (D) \) as trace, which implies that its angle pair is \( \{ \pi + \frac{3(a_1+a_2)}{2}, 0 \} \).

\((E_{\alpha_1, \alpha_2}^+)\) Let \( p_1 \in E V \) and consider a spherical line \( L_1 \) and a hyperbolic line \( L_2 \) both through \( p_1 \). Consider a (continuous) curve \( \gamma : [0, t_*] \to EV \) such that \( \gamma(t) \in L_1 \) for every \( t \in [0, 1] \), \( \gamma(t) \in L_2 \) for every \( [1, t_*] \), and \( \tau(p_1, \gamma(t)) = t \) for every \( t \); in particular \( \gamma(1) = p_1 \). Clearly, every elliptic isometry of the form \( R_{a_2}^R R_{a_1}^R \) with \( q_1, q_2 \in EV \), is \( SU(2, 1) \)-conjugated to the isometry \( R_{a_2}^{(ta(q_1, q_2))} R_{a_1}^{(1)} \).

Moreover, if \( q_1, q_2 \in EV \) are such that the line \( L(q_1, q_2) \) is Euclidean, then \( R_{a_2}^R R_{a_1}^R \) is regular parabolic with the same angle pair as \( R_{a_2}^{(1)} R_{a_1}^{(1)} \), and if \( q_1, q_2 \in EV \) are distinct points such that \( R_{a_2}^R R_{a_1}^R \) is parabolic, then such isometry is conjugated to \( R_{a_2}^{(t_*)} R_{a_1}^{(1)} \). Therefore, the projection on \( c(S) \) of the image of the curve \( \tilde{\gamma} : [0, t_*] \to S \), defined as in the previous case, is \( E_{\alpha_1, \alpha_2}^+ \). The result follows from the fact that \( R_{a_2}^{(0)} R_{a_1}^{(1)} \) is an elliptic isometry with angle pair \( \{ -3a_1, -3a_2 \} \); and \( R_{a_2}^{(1)} R_{a_1}^{(1)} \) is a special elliptic isometry with angle pair \( \{ -3(a_1 + a_2), 0 \} \), and \( R_{a_2}^{(t_*)} R_{a_1}^{(1)} \) is a parabolic isometry having one of the points in \( (D) \) as trace, which implies that its angle pair is \( \{ \pi + \frac{3(a_1+a_2)}{2}, 0 \} \).

\((E_{\alpha_1, \alpha_2}^-)\) Let \( p_1 \in E V \) and let \( L \) be a hyperbolic line through \( p_1 \). Consider a curve \( \gamma : [t_-, 0] \to BV \) such that \( \gamma(t) \in L \) and \( \tau(p_1, \gamma(t)) = t \) for all \( t \in [t_-, 0] \). In particular \( \{p_1, \gamma(0)\} = 0 \), i.e., \( \gamma(0) \) is the point in \( L \) orthogonal to \( p_1 \). As before, this defines a curve \( \tilde{\gamma} \in S \) whose image, projected on \( c(S) \), is \( E_{\alpha_1, \alpha_2}^- \). Note that \( R_{a_2}^{(t)} R_{a_1}^{(1)} \) is a parabolic (if \( \alpha_1 \neq \alpha_2 \)) or special elliptic (if \( \alpha_1 = \alpha_2 \)) isometry with trace being one of the points in \( (A) \) (that coincides with \( (B) \) if \( \alpha_1 = \alpha_2 \)), which implies that its angle pair is \( \{ \frac{3(a_1+a_2)}{2}, 0 \} \). Also, \( R_{a_2}^{(0)} R_{a_1}^{(1)} \) is an elliptic isometry with angle pair \( \{ 3a_2, 3a_2 - 3a_1 \} \), and the result follows.

\((E_{\alpha_1, \alpha_2}^+)\) This case is analogous to the one above, considering \( p_1 \in B V \) and \( \gamma \) as a curve in \( E V \).

5.9. Remark. The following observations are direct consequences of Proposition 5.8 and its proof. Figure 3 might be useful as an illustration.
(1) $E_{\alpha_1, \alpha_2}^+$ is the only subset of $E_{\alpha_1, \alpha_2}$ containing a segment of slope $-1$, and it is composed by classes of elliptic isometries that admit a decomposition $F = R_{\alpha_1}^\ell R_{\alpha_2}^p$ with spherical line $L(p_1, p_2)$.

(2) Define $R_{\alpha_1, \alpha_2}^{\sigma_1, \sigma_2} := E_{\alpha_1, \alpha_2} \cap E^{reg}$ (see Subsection 3.1). There exist $\delta_0, \delta_1, \delta_2 \in \Omega$ such that $\text{utr}(R_{\alpha_1, \alpha_2}^{\sigma_1, \sigma_2}) \subset \ell_{\delta_0, \alpha_1, \alpha_2}$,

$$\left(\delta_1 \text{utr}(R_{\alpha_1, \alpha_2}^{\sigma_1, \sigma_2}) \cup \text{utr}(R_{\alpha_1, \alpha_2}^{\sigma_1, \sigma_2})\right) \cup \text{utr}(R_{\alpha_1, \alpha_2}^{\sigma_1, \sigma_2}) = \ell_{\delta_0, \alpha_1, \alpha_2} \cap \Delta^\sigma.$$

(3) By item (iii) of Proposition 3.4, $E_{\alpha_1, \alpha_2} = E_{\alpha_2, \alpha_1}$. Moreover, since $R_{\alpha_1}^R_{\alpha_2} = R_{\alpha_1}^R_{\alpha_2} R_{\alpha_2}^1 R_{\alpha_1}^R_{\alpha_2}$ for any points $p_1, p_2 \in PV \setminus SV$, we have $E_{\alpha_1, \alpha_2} = E_{\alpha_2, \alpha_1}$. Therefore, if $\alpha_1 = \alpha_2 =: \alpha$, then $E_{\alpha_1, \alpha_2} = E_{\alpha, \alpha}$ is a single point in a nondiagonal side of $\rho(L)$. It follows from item (2) above that there exists $\delta \in \Omega$ such that $\text{utr}(R_{\alpha_1, \alpha_2}^{\alpha_1, \alpha_2}) = \ell_{\delta_0, \alpha_2} \cap \Delta^\sigma$.

5.10. Corollary. Let $\alpha := \alpha^\Omega$ be a parameter with $0 < a < 2\pi/3$. Then $E_{\alpha_1, \alpha_2}$ is given as the union of its subsets $E_{\alpha_1, \alpha_2}^\alpha$ where

- $E_{\alpha_1, \alpha_2}^\alpha = \{(6a, 6a, 2\pi, \pi + 3\alpha)\}$ if $0 < a \leq \pi/3$, or $E_{\alpha_1, \alpha_2}^\alpha = \{(6a - 2a, 6a - 2a, 3a, \pi - 0)\}$ if $\pi/3 < a < 2\pi/3$;
- $E_{\alpha_1, \alpha_2}^{\alpha_1, \alpha_2} = \{(2\pi - 3a, 2\pi - 3a), (2\pi - 2\pi - 6a), (2\pi - 3a, 0)\}$ if $0 < a \leq \pi/3$, or $E_{\alpha_1, \alpha_2}^{\alpha_1, \alpha_2} = \{(2\pi, 3a - \pi), (4\pi - 6a, 0)\}$ if $\pi/3 < a < 2\pi/3$;
- $E_{\alpha_1, \alpha_2}^{\alpha_1, \alpha_2} = \{(2\pi, 3a, 0)\}$ if $\pi/3 < a < 2\pi/3$.

Figure 4 illustrates the set $E_{\alpha_1, \alpha_2}$ for some parameters $\alpha$. Summarizing, the space of classes with $\alpha$-length equal to 2 is given by theloxodromic isometries that have a lift with trace lying in $\ell_{\alpha, \alpha}$, and by the space $E_{\alpha_1, \alpha_2}$, considering the description of the classes of its intersection with the boundary of $\rho(L)$ obtained in Proposition 5.8.

6 Length 3 decomposition

In this section we describe all isometries in $PU(2, 1)$, that are not 2-step unipotent, admitting an $\alpha^{(3)}$-decomposition, i.e., those with $\alpha$-length equal to 3, for a given parameter $\alpha$. We start by proving that allloxodromic and regular parabolic isometries admit an $\alpha^{(3)}$-decomposition, for any parameter $\alpha \in S^1 \setminus \Omega$. Then, using the results ofSection 5, we obtain that this is also true for special elliptic isometries with positive center.

In order to obtain the remaining isometries admitting an $\alpha^{(3)}$-decomposition, we describe $E_{\alpha_1, \alpha_2}$ (see Notation 3.5) using the properties of the product map $\pi$, that are summarized in Subsection 6.9. The set $E_{\alpha_1, \alpha_2}$ is the union of closed chambers in $\rho(L)$ delimited by the inverse image under $\text{utr}$ of the tangent lines $\ell_{\alpha, \alpha}, \ell_{\alpha, \alpha}, \ell_{\alpha, \alpha}$; each of these chambers is either full or empty (Proposition 6.17). In Subsection 6.12, we decide whether each such chamber is full or empty.

6.1. Decomposing regular parabolic andloxodromic isometries. Given a triple of parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, a triple of signs $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1 \in \{0, 1, 2\}$ and at most one of them is positive, and $\tau \in \mathbb{C}$, we say that a triple of points $p_1, p_2, p_3 \in PV \setminus SV$ is strongly regular with respect to $\alpha, \sigma, \tau$ if: $p_1, p_2, p_3$ are pairwise distinct; $p_2$ is neither orthogonal to $p_1$ nor to $p_3$; $p_1, p_2, p_3$ do not lie in a same complex line; $\text{tr} R_{\alpha_1}^R_{\alpha_2} R_{\alpha_2}^R_{\alpha_1} = \tau$; and the isometry $R_{\alpha_1}^R_{\alpha_2} R_{\alpha_2}^R_{\alpha_1}$ is regular (Definition 2.2). We denote by $S_{\alpha, \sigma, \tau}$ the space of strongly regular triples with respect to $\alpha, \sigma, \tau$.

In the case where $\alpha := (\alpha, \alpha, \alpha)$ for a given parameter $\alpha \in S^1 \setminus \Omega$, considering the function $\kappa_\alpha : \mathbb{C} \to \mathbb{C}$ defined by

$$\kappa_\alpha(\tau) := \frac{\tau - 3}{(\alpha^2 - \alpha)^3},$$

and denoting

$$t_1 := \text{ta}(p_1, p_2), \quad t_2 := \text{ta}(p_2, p_3), \quad t := \Re \left(\frac{p_1, p_2, p_3}{p_1, p_2, p_3} \cdot \frac{p_2, p_3, p_1}{p_2, p_3, p_1}\right),$$

and $t_1 := \text{ta}(p_1, p_2)$, $t_2 := \text{ta}(p_2, p_3)$, $t := \Re \left(\frac{p_1, p_2, p_3}{p_1, p_2, p_3} \cdot \frac{p_2, p_3, p_1}{p_2, p_3, p_1}\right)$.
we obtain that $S_{\sigma,\tau,\tau}$ is the real semialgebraic surface in $\mathbb{R}^3(t_1,t_2,t)$ given by the equation (see [6, Theorem 5.2])

$$t_1^2t_2 + t_1t_2^2 - 2t_1t_2t + d_1t^2 + d_2t + d_3 = 0,$$

and by the inequalities

$$\sigma_1\sigma_2 t_1 > 0, \quad \sigma_1\sigma_2 t_1 > \sigma_1\sigma_2, \quad \sigma_2\sigma_3 t_2 > 0, \quad \sigma_2\sigma_3 t_2 > \sigma_2\sigma_3, \quad \sigma_1\sigma_2\sigma_3(2 \text{Re} \kappa_\alpha(\tau) + 1) < 0, \quad (4)$$

where $d_1 := 1 + 4\chi^2$, $d_2 := -4\chi(2\text{Re} \kappa_\alpha(\tau) + \text{Im} \kappa_\alpha(\tau))$, $d_3 := (2\chi \text{Re} \kappa_\alpha(\tau) + \text{Im} \kappa_\alpha(\tau))^2$, and

$$\chi := \text{Im} \left( \frac{2}{\pi - 2\alpha} \right).$$
Note that if $\tau \in \mathbb{C}$ is such that $2 \Re \kappa_\alpha(\tau) + 1 = 0$, then $S_{\alpha, \sigma, \tau}$ is empty. But $\det [g_{3}] = \sigma_1 \sigma_2 \sigma_3 (2 \Re \kappa_\alpha(\tau) + 1)$, where $[g_{3}]$ is the Gram matrix of the points $p_1, p_2, p_3$. Therefore, if $2 \Re \kappa_\alpha(\tau) + 1 = 0$, a triple of pairwise distinct and pairwise nonorthogonal points $p_1, p_2, p_3$ satisfying $\tr R_\alpha R_\sigma R_\tau = \tau$ is collinear, i.e., $p_1, p_2, p_3$ lie in the same complex line. (The space of such triples, with respect to $\alpha, \sigma, \tau$, is also parametrized by (3) and satisfies inequalities obtained by changing the last inequality in (4) by $\sigma_1 \sigma_2 \sigma_3 (2 \Re \kappa_\alpha(\tau) + 1) \leq 0$.)

6.2. Lemma. Given $\alpha \in S^1$ and $\tau \in \mathbb{C}$, we have $2 \Re (\kappa_\alpha(\tau)) + 1 = 0$ iff $\tau \in \ell_\alpha$.

Proof. Suppose that $2 \Re (\kappa_\alpha(\tau)) + 1 = 0$. As discussed above, if $p_1, p_2, p_3 \in \mathbb{P} V \setminus \mathbb{S} V$ is a triple of pairwise distinct, pairwise nonorthogonal points such that $\tr R_\alpha R_\sigma R_\tau = \tau$, then $p_1, p_2, p_3$ lie in a same complex line $L$ and, in this case, $R_\alpha R_\sigma R_\tau$ has a fixed point with eigenvalue $\alpha^3$, namely the polar point of the line $L$. By Proposition 4.6, $\tau \in \ell_\alpha$.

On the other hand, given $\tau \in \ell_\alpha$, we have $\tau = \tau_{\alpha, \sigma, \tau}(t)$ (see Subsection 4.4), for some $t \in \mathbb{R}$, and

$$2 \Re (\kappa_\alpha(\tau_{\alpha, \sigma, \tau}(t))) + 1 = 2 \Re \left( \frac{t(1-\alpha^6)}{(\alpha^3-1)^2} - \frac{\alpha^3}{\alpha^3-1} \right) + 1.$$ 

But, $\Re \left( \frac{\alpha^4}{\alpha^3-1} \right) = \frac{1}{2}$ and $\Re\left( \frac{1-\alpha^6}{(\alpha^3-1)^2} \right) = 0$. In fact, by straightforward calculations, we have

$$\frac{\alpha^4}{\alpha^3-1} + \frac{\alpha^{-3}}{\alpha^{-3}-1} = 1 \quad \text{and} \quad \frac{1-\alpha^6}{(\alpha^3-1)^2} - \frac{1-\alpha^{-6}}{(\alpha^{-3}-1)^2} = 0.$$

Therefore, $2 \Re (\kappa_\alpha(\tau_{\alpha, \sigma, \tau}(t))) + 1 = 0$. \qed

6.3. Proposition. Every regular parabolic or loxodromic isometry in $\PU(2,1)$ admits an $\alpha^{(3)}$-decomposition, for any parameter $\alpha \in S^1 \setminus \Omega$.

Proof. Let $F \in \SU(2,1)$ be a regular parabolic or loxodromic isometry and let $\tau := \tr F$. If there exists a triple of points $p_1, p_2, p_3$ such that $\tr R_\alpha R_\sigma R_\tau = \tau$, then $F$ admits an $\alpha^{(3)}$-decomposition since there exists a single regular parabolic or loxodromic class corresponding to $\tau$ (see Section 3). In this way, we prove that the space given, in $\mathbb{R}^3(t_1, t_2, t)$, by equation (3) and by inequalities obtained by substituting the last one in (4) by $\sigma_1 \sigma_2 \sigma_3 (2 \Re \kappa_\alpha(\tau) + 1) \leq 0$ is nonempty for some choice of signs $\sigma_i$.

For fixed values of $t_1, t_2$ we have a quadratic equation in $t$ (as $d_1 \neq 0$) with discriminant $(d_2 - 2t_1 t_2)^2 - 4d_1(t_1^2 t_2 + t_1 t_2^2 + d_3)$. Thus, the equation (3) has a solution for the given values of $t_1, t_2$ iff

$$d_2^2 - 4d_2 t_1 t_2 + 4t_1^2 t_2^2 \geq 4d_1(t_1^2 t_2 + t_1 t_2^2 + d_3).$$

Since $d_1 > 0$, this inequality holds if $t_1, t_2 \ll 0$. By the same reason, for a fixed value of $t_1 < 0$, there exists $t_2 > 1$ satisfying the inequality above. In this way, we prove that there are solutions for any choice of signs satisfying $\sigma_1 = -\sigma_2 = \sigma_3$ and $-\sigma_1 = \sigma_2 = \sigma_3$. \qed

6.4. Remark. From the proof of Proposition 6.3, it follows that if $F$ is an elliptic isometry with $\tr F = 0$, then $F$ admits an $\alpha^{(3)}$-decomposition for any parameter $\alpha \in S^1 \setminus \Omega$. In fact, it follows from the mentioned proof that there exist $p_1, p_2, p_3 \in \mathbb{P} V \setminus \mathbb{S} V$ and $\delta \in \Omega$ such that $F = \delta R_\alpha R_\sigma R_\tau$. Since two isometries with trace $\tau = 0$ are conjugated (see Section 3), the result follows.

6.5. Decomposing nonregular isometries. Since, for any parameter $\alpha \in S^1 \setminus \Omega$ and any triple $p_1, p_2, p_3 \in \mathbb{P} V \setminus \mathbb{S} V$ of pairwise orthogonal points, we have $R_\alpha R_\sigma R_\tau = 1$, the identical class admits an $\alpha^{(3)}$-decomposition. This implies, together with Proposition 6.3, that $E_{\alpha, \sigma, \tau}$ contains the nondiagonal side and the vertex of $\rho(E)$. Using the sets $E_{\alpha, \sigma, \tau}$ described in Proposition 5.8, we obtain in the next proposition that the special elliptic classes in the nondiagonal side of $\rho(E)$, i.e., those with positive center, also admit an $\alpha^{(3)}$-decomposition.

6.6. Proposition. Every special elliptic isometry with positive center admits an $\alpha^{(3)}$-decomposition, for any parameter $\alpha \in S^1 \setminus \Omega$.
Proposition. If $\alpha^3 \in \Omega \cup -\Omega$, every 2-step unipotent isometry admits an $\alpha^{(3)}$-decomposition.

Proof. Suppose that $\alpha^3 \in \Omega$, i.e., $\alpha^3 = \omega^j$ for some $j = 1, 2$. Let $U \in SU(2,1)$ be a 2-step unipotent isometry fixing an isotropic point $v \in SV$ with eigenvalue 1. If $p \in Pu^+$ is a nonisotropic point, then the isometry $R := R_u^pU$ fixes $v$ with eigenvalue $\tau\beta$ and fixes $p$ with eigenvalue $\alpha^3$. Since $\alpha^3 \neq 1$, $R$ is ellipto-parabolic with tr $R = 2\tau\beta + \alpha^3 = \tau_{\alpha,\alpha}(0)$. But $\alpha^3 = \omega^j$ implies $\tau_{\alpha,\alpha}(0) = \omega^j \tau_{\alpha,\alpha}(1)$. 


Therefore, by Proposition 5.2, the isometry \( R \) admits an \( \alpha^{(2)} \)-decomposition, implying that \( U \) admits an \( \alpha^{(3)} \)-decomposition.

If \( \alpha^3 \in -\Omega \) we are in the case of involutions and the proposition follows directly from [13, Proposition 16]. \( \square \)

6.8. Remark. It remains an open problem to determine if a 2-step unipotent isometry admits an \( \alpha^{(3)} \)-decomposition for parameters satisfying \( \alpha^3 \not\in \Omega \cup -\Omega \).

We now determine which regular elliptic isometries and special elliptic isometries with negative center admit an \( \alpha^{(3)} \)-decomposition. In order to do so, it suffices to describe \( \mathcal{E}_{\alpha^{(3)}} \), since its intersection with the diagonal side of \( \rho(\mathcal{E}) \) corresponds uniquely to the classes of special elliptic isometries with negative center. Such description is based on the product map.

6.9. The product map. In this subsection we briefly summarize some definitions and results that can be found in [4, 12, 13]. Consider the spaces \( \mathfrak{g}, c(\mathfrak{g}) \) and the projection \( \rho : \mathfrak{c} \to c(\mathfrak{g}) \) as defined in Subsection 3.1. Given two semisimple \( \text{PU}(2,1) \)-conjugacy classes \( C_1, C_2 \in c(\mathfrak{g}) \) the product map, with respect to the given classes, is the function \( \tilde{\mu} : C_1 \times C_2 \to \mathfrak{g} \) defined by \( \tilde{\mu}(A, B) = [AB] \), where \([I]\) denotes the conjugacy class of the isometry \( I \). In what follows, we will mainly consider the function (which we also refer as product map) \( \overline{\mu} : C_1 \times C_2 \to c(\mathfrak{g}) \), defined by \( \overline{\mu} := \rho \circ \tilde{\mu} \).

6.10. Definition. We say that a subgroup \( \Gamma \) of \( \text{PU}(2,1) \) is reducible if it fixes a point in \( \mathbb{P}V \). Given isometries \( A, B \in \text{PU}(2,1) \), we say that the pair \( (A, B) \) is reducible if it generates a reducible group. If a subgroup is not reducible we say that it is irreducible. Given two semisimple conjugacy classes \( C_1 \) and \( C_2 \), the image under \( \overline{\mu} : C_1 \times C_2 \to c(\mathfrak{g}) \) of reducible pairs is called reducible walls.

In terms of the above definition, we have the following properties of the product map.

6.11. Proposition. Given two semisimple conjugacy classes \( C_1 \) and \( C_2 \) we have:

• \( \overline{\mu} \) is proper; in particular the image \( \overline{\mu}(C_1 \times C_2) \) is closed in \( c(\mathfrak{g}) \);

• the image of an irreducible pair in \( C_1 \times C_2 \) under \( \overline{\mu} \) is an interior point of \( \overline{\mu}(C_1 \times C_2) \);

• the reducible walls, divide \( c(\mathfrak{g}) \) in closed chambers. Each of these chambers is either full or empty;

• the intersection of the reducible walls of \( \overline{\mu}(C_1 \times C_2) \) with \( \rho(\mathcal{E}) \) is given by the union of finitely many line segments of slopes \(-1, \frac{1}{2}, 2\).

6.12. Dividing \( \mathcal{E}_{\alpha^{(3)}} \) into chambers. We prove (see Corollary 6.15) that \( \mathcal{E}_{\alpha^{(3)}} \) is the intersection of \( \rho(\mathcal{E}) \) with the union of all images \( \overline{\mu}(C_1 \times C_2) \), where \( C_1 \) is the class of a special elliptic isometry with parameter \( \alpha \), and \( C_2 \) is a semisimple class admitting an \( \alpha^{(2)} \)-decomposition. It is quite hard to directly obtain such union, writing down every image (as done in [13] for the case of involutions.

So, we do this indirectly using bendings (see [6]).

Given a product \( R := R_{p_2}^{q_2} R_{p_1}^{q_1} \), where \( p_1, p_2 \) are nonisotropic points and \( \alpha_1, \alpha_2 \in \mathfrak{s} \not\in \Omega \) are parameters, if \( C \in \text{SU}(2,1) \) is an isometry in the centralizer of \( R \), we have \( R_{C_2}^{q_2} R_{C_1}^{q_1} = R_{C_2}^{q_2} R_{C_1}^{q_1} \). These are called bending relations. Moreover, by [6, Proposition 4.3], there exists a one-parameter subgroup \( B : \mathbb{R} \to \text{SU}(2,1) \) such that \( B(s) \) is in the centralizer of \( R \) for every \( s \in \mathbb{R} \) and for every isometry \( C \) that commutes with \( R \), there exists \( s \in \mathbb{R} \) with \( C p_i = B(s) p_i, i = 1, 2 \). We say that \( B(s) \) is a bending of \( R \). Bendings act on \( p_1, p_2 \) by moving these points over metric circles, hypercycles, horocycles, contained in the line \( L(p_1, p_2) \), depending on the nature of the isometry \( R \).

6.13. Remark. Let \( \alpha \in \mathfrak{s} \not\in \Omega \) be a parameter and let \( p_1, p_2, p_3 \in \mathbb{P}V \setminus S \). Consider the isometry \( F := R^{p_2}_{p_1} R^{q_2}_{p_3} \). If \( p_1 \) and \( p_2 \) are distinct nonorthogonal points and \( L := L(p_1, p_2) \) is noneucidean, then a nontrivial bending \( B(s) \) of \( L \) satisfies \( B(s) p_2 = B(s) p_3 \) iff either \( p_3 \) is orthogonal to a fixed point of \( L \) and \( p_3 \notin L(p_1, p_2) \) or \( p_2 \in L(p_1, p_2) \). If \( L \) is Euclidean, then \( B(s) p_2 = B(s) p_3 \) iff either \( p_3 \notin L \) and \( p_3 \) is orthogonal to an \( R_1 \)-fixed point or \( p_3 \in L \).

In what follows, for an isometry \( I \in \text{SU}(2,1) \), \([I]\) denotes the \( \text{PU}(2,1) \)-conjugacy class of the corresponding isometry in \( \text{PU}(2,1) \).

6.14. Lemma. Let \( F \in \text{SU}(2,1) \) be an elliptic isometry admitting a decomposition of the form \( F = \delta R_{p_2}^{q_2} R_{p_1}^{q_1} \), where \( p_1, p_2, p_3 \in \mathbb{P}V \setminus S \) and \( \delta \in \Omega \), such that \( R_{p_2}^{q_2} R_{p_1}^{q_1} \) is parabolic. If \( [F] \neq [R_{p_2}^{q_2}] \), \( q \in \mathbb{E} \), then there exist \( q_1, q_2, q_3 \in \mathbb{P}V \setminus S \) such that \( F = \delta R_{q_2}^{q_1} R_{p_1}^{q_3} \) and \( R_{p_2}^{q_2} R_{q_3}^{q_1} \) is either regular elliptic or loxodromic.
Proof. Denote $R_1 := R^p_\alpha R^p_\alpha$, $R_2 = R^q_\alpha R^q_\alpha$, $R_3 = R^b_\alpha R^b_\alpha$, $L_1 := L(p_1, p_2)$, $L_2 := L(p_2, p_3)$, and $L_3 := L(p_1, p_3)$. If $R_2$ or $R_3$ is regular elliptic or loxodromic, since $[F] = [R^p_\alpha R^p_\alpha R^p_\alpha] = [R^q_\alpha R^q_\alpha R^q_\alpha]$, the result follows. So, we can assume that the points $p_1, p_2, p_3$ have the same signature. We can also assume that $p_1, p_2, p_3$ do not lie in a same Euclidean line, otherwise the isometry $F$ is not elliptic.

The idea now is, by bending $R_2$, to obtain new centers $B(s)p_2, B(s)p_3$ such that $\text{ta}(p_1, B(s)p_2) \neq \text{ta}(p_1, p_2)$, which implies that the isometry $R_\alpha^B(s)p_2 R_\alpha^B$ is either regular elliptic or loxodromic (see [6, Corollary 5.10]). Note that this approach does not work if $\langle p_2, p_3 \rangle = 0$ or if $p_1$ is orthogonal to a fixed point of $R_2$.

The fact that $R_1$ is parabolic implies that $p_1, p_2$ are distinct nonorthogonal points of same signature. Hence, if $p_3$ is in $L_1$, $p_3$ cannot be an $R_1$-fixed point (otherwise $L_1 = L_2 = \text{Euclidean}$) and thus, by bending $R_1$ if necessary, we can assume that $p_2, p_3$ are also distinct and nonorthogonal. So, by bending $R_2$, the result follows (see Remark 6.13).

Suppose that $\langle p_2, p_3 \rangle = 0$. By what was discussed above, we can assume that $L_1$ is Euclidean and $L_2$ is spherical. In this case, $p_1$ is not orthogonal to any fixed point of $R_2$; hence, there is a bending $B(s)$ of $R_2$ such that $B(s)p_2$ and $p_3$ are nonorthogonal points and the result follows.

It remains to consider the case where $p_1 \in \text{EV}, L_1$ is Euclidean, and $p_1$ is orthogonal to a fixed point of $R_{i+1}$ (indices modulo 3). By Lemma 5.2, for each $i$, there exists $c_i \in L_1$ such that $c_i$ is a fixed point of $R_i$ with eigenvalue $\delta \alpha^{-3}$. Therefore, $F = \delta R_\alpha^q$, for some $q \in \text{EV} \setminus \text{SV}$. If $q \in \text{EV}$, applying a simultaneous change of signs in $R^q_\alpha R^q_\alpha$ (see Remark 5.3) we obtain a relation of the form $R^q_\alpha R^q_\alpha R^q_\alpha = \delta R_\alpha^q$, where $q_3 \in \text{BV}$ and $\bar{q} \in \text{EV}$, in which case $R^q_\alpha R^q_\alpha$ is loxodromic and the result follows as above.

6.15. Corollary. For any parameter $\alpha \in S^1 \setminus \Omega$, we have

$$E_{\alpha, (3)} = \bigcup_{c_1 \in S_\alpha \cup C_2 \in G_{\alpha, (2)}} \overline{\text{pr}(C_1 \times C_2)} \cap \rho(\mathcal{E}).$$

Proof. By its definition (see Notation 3.5), $E_{\alpha, (3)}$ contains the right side of (5).

Conversely, by Propositions 6.3 and 6.6, we need to prove that given a regular elliptic isometry or a special elliptic isometry with negative center $F \in \text{SU}(2, 1)$ admitting a decomposition of the form $F = \delta R_\alpha^p R^p_\alpha R^p_\alpha$, where $R_1 := R^p_\alpha R^p_\alpha$ is parabolic and $\delta \in \Omega$, then the $\text{PU}(2, 1)$-conjugacy class of $F$ lies in an image $\overline{\text{pr}(C_1 \times C_2)}$, for some $C_1 \in S_\alpha$ and $C_2 \in G_{\alpha, (2)}$. This follows directly from Lemma 6.14.

6.16. Lemma. Let $\alpha \in S^1 \setminus \Omega$ be a parameter and consider a reducible pair $(R^p_\alpha, R^q_\alpha R^p_\alpha)$ such that $R^q_\alpha R^q_\alpha R^p_\alpha$ is regular elliptic and the points $p_1, p_2, p_3 \in \mathbb{PV} \setminus \text{SV}$ do not lie in a same complex line. Then the image under $\overline{\mathcal{E}}$ of such pair is an interior point of $E_{\alpha, (3)}$.

Proof. Define $L_1 := L(p_1, p_2)$, $R_1 := R^p_\alpha R^p_\alpha$, $R_2 := R^q_\alpha R^p_\alpha$, and $F := R^p_\alpha R^p_\alpha R^q_\alpha$. Suppose that the pair $(R^p_\alpha, R_1)$ is reducible and that $p_3 \notin L_1$. Then $p_3$ is orthogonal to a fixed point of $R_1$.

Note that $R^p_\alpha F R^p_\alpha$ $\overline{\text{pr}} = R^q_\alpha R^p_\alpha R^p_\alpha$. Then if $p_3$ is not orthogonal to a fixed point of $R_2$, the image under $\overline{\mathcal{E}}$ of the irreducible pair $(R^p_\alpha, R_2)$, which by Proposition 6.11 is an interior point in the image of $\overline{\mathcal{E}}$, coincides with the image (under a distinct $\overline{\mathcal{E}}$) of the pair $(R^p_\alpha, R_1)$. Hence, we can also assume that $p_1$ is orthogonal to a fixed point of $R_2$.

Suppose that $p_3$ is not orthogonal to $p_2$. We will prove that there exists a triple $q_1, q_2, q_3 \in \mathbb{PV} \setminus \text{SV}$ of points not lying in a same complex line such that $F = R^p_\alpha R^q_\alpha R^p_\alpha$ and $q_3$ is not orthogonal to a fixed point of $R_1$, i.e., $(R^p_\alpha, R^q_\alpha R^p_\alpha)$ is an irreducible pair. Note that if at most one of the points $p_i$ is positive, then the triple $p_1, p_2, p_3$ is strongly regular with respect to $\alpha := (\alpha, \alpha, \alpha)$, $\sigma := (\sigma p_1, \sigma p_2, \sigma p_3)$, and $\tau := \text{tr} F$, so the result follows from [6, Lemma 5.5]. Thus, we can assume that at least two of the points $p_1, p_2, p_3$ is positive. Suppose that one of the isometries $R_1, R_2$ is loxodromic. In this case, using simultaneous change of signs (together, if necessary, with a conjugation that cyclic permutes the points $p_1, p_2, p_3$ as above) we obtain a strongly regular triple $q_1, q_2, q_3$ with $F = R^p_\alpha R^q_\alpha R^p_\alpha$ and the result follows as in the previous case. Finally, we assume that the points $p_1, p_2, p_3$ are positive. Note that neither $R_1$ nor $R_2$ can be parabolic, since $F$ is regular elliptic. Then $R_1$ and $R_2$ are regular elliptic and, if $a$ is the $R_1$-fixed point orthogonal to $p_3$ and $b$ is the $R_2$-fixed point orthogonal to $p_1$, then $a$ and $b$ are nonisotropic points with $(a, b) = 0$. 18
Denote \( L := L(p_1, \alpha) \); then \( L = \mathbb{P}b^+ \) and \( L = L_1 \). It follows that \( p_2 \) is orthogonal to \( b \) and either \( p_3 \in L_1 \) or \( L = \mathbb{P}p_2^+ \), both cases contradicting the hypothesis.

Now, suppose that \((p_3, p_2) = 0 \) and \((p_3, p_1) \neq 0 \). If follows that \( p_2 \) is a fixed point of \( R_2 \) which implies that the triple \( p_1, p_2, p_3 \) is pairwise orthogonal and the isometry \( F \) is not regular elliptic, a contradiction. (We obtain the same contradiction supposing that \((p_3, p_1) = 0 \) and \((p_3, p_2) \neq 0 \).)

Finally, assume \((p_3, p_2) = (p_3, p_1) = 0 \), i.e., \( p_3 \) is the polar point of the line \( L_1 \). By Proposition \([6, \text{Corollary 3.7}] \), since \( F \) is regular elliptic, the points \( p_1, p_2, p_3 \) cannot be pairwise orthogonal. In this case, we cannot bend the decomposition of \( F \). But, if \( q \) is any point in the line \( L(p_2, p_3) \) with \( \sigma q = \sigma p_2 \), and \( \tilde{q} \) is the point in \( L(p_2, p_3) \) orthogonal to \( q_2 \), we have \( R_0^2 R_0^3 = R_0^2 R_0^3 = R_{\tilde{q}p_2} \), where \( c \) is the polar point of the line \( L(p_2, p_3) \). So, we can without loss of generality assume that \((q, p_1) \neq 0 \), and proceed as in the previous paragraph.

### 6.17. Proposition

Let \( \alpha \in S^1 \setminus \Omega \) be a parameter. The set \( E_{\alpha^{(3)}} \) is given by the union of closed chambers in \( \rho(\mathcal{E}) \) delimited by \( \cup_{\ell_{(\alpha^3)} \cup \ell_{\omega_{\alpha^3}} \cup \ell_{\omega_{\alpha^2}}} \subset T \). Each of these chambers is either full or empty.

**Proof.** We will show that every convergent sequence \( x_n \in \rho(\mathcal{E}) \) such that, for all \( n \), \( x_n \) lies in a reducible wall of \( \mathcal{C}(C_1 \times C_2, n) \), for some sequences of chambers \( C_1, n \in S_\alpha \) and \( C_2, n \in G_{\alpha^{(2)}} \), converges either to an interior point of \( E_{\alpha^{(3)}} \) or to a point lying in a reducible wall of \( \mathcal{C}(C_1 \times C_2) \), for some semisimple classes \( C_1 \in S_\alpha \) and \( C_2 \in G_{\alpha^{(2)}} \). Hence, the first part of the result follows from Lemma 6.16.

By Corollary 6.15 (and remembering that \( S_\alpha \) consists of only two elements), by considering a subsequence if necessary, we can assume that there exists \( p_3 \in P \setminus S_\alpha \) such that \( C_1, n = [R_{p_3}^\alpha] =: C_1 \), for all \( n \).

We can also assume that \( p(C_2, n) \in \rho(\mathcal{E}) \), for all \( n \). In fact, let \( F \in SU(2, 1) \) be a regular elliptic isometry that admits a decomposition \( F = R_{p_3}^\alpha R_{p_2}^\alpha R_{p_1}^\alpha \), for points \( p_1, p_2, p_3 \in \mathbb{P} \setminus S_\alpha \), such that \( A := R_{p_1}^\alpha R_{p_2}^\alpha \) is loxodromic and the pair \( (R_{p_2}^\alpha, A) \) is reducible. Then either \( p_1 \in L(p_1, p_2) \) or \( p_3 \) is orthogonal to an isotropic fixed point \( v \) of \( A \). But, if \( (p_3, v) = 0 \), then \( R_{p_3}^\alpha \) also fixes \( v \) and, therefore, \( F \) is not regular elliptic. Hence, \( p_3 \in L(p_1, p_2) \) and, by Lemma 6.2, \( \cup_{\ell_{(\alpha^3)}} \subset T \) for some \( \delta \in \Omega \).

So, we have a convergent sequence of points \( x_n \in E_{\alpha^{(3)}} \subset \rho(\mathcal{E}) \), each one lying in a reducible wall of \( \mathcal{C}(C_1 \times C_2, n) \), where \( C_2, n \in G_{\alpha^{(2)}} \) is such that \( p(C_2, n) \in \rho(\mathcal{E}) \). Since \( \rho(\mathcal{E}) \) is compact in \( c(\mathcal{S}) \), the sequence \( x_n \) converges to a point \( x \in \rho(\mathcal{E}) \). As \( \mathcal{E} \cup \mathcal{B} \) is compact in \( \mathcal{G} \), the sequence \( C_2, n \) has a subsequence converging to a class \( C_2 \in \mathcal{E} \cup \mathcal{B} \). If \( C_2 \notin G_{\alpha^{(2)}} \) (note that \( G_{\alpha^{(2)}} \) is not closed in \( \mathcal{G} \), see for instance Proposition 5.8), then \( C_2 \) corresponds to a parabolic class admitting a \( \alpha^{(2)} \)-decomposition. In this case, by Lemma 6.14, every point in \( \mathcal{C}(C_1 \times C_2) \) is either interior in \( E_{\alpha^{(3)}} \) or lie in \( \cup_{\ell_{(\alpha^3)} \cup \ell_{\omega_{\alpha^3}} \cup \ell_{\omega_{\alpha^2}}} \subset \rho(\mathcal{E}) \). If \( C_2 \in G_{\alpha^{(2)}} \), then \( x \) lies in a reducible wall of \( \mathcal{C}(C_1 \times C_2) \).

The second part follows from Proposition 6.11.

By the above proposition, if we consider \( \rho(\mathcal{E}) \) divided into chambers by the set given by \( \cup_{\ell_{(\alpha^3)} \cup \ell_{\omega_{\alpha^3}} \cup \ell_{\omega_{\alpha^2}}} \), i.e., \( \rho(\mathcal{E}) \) divided by the segments given in Lemma 4.8, we obtain the region \( E_{\alpha^{(3)}} \) by finding which of these chambers are full/empty. Figure 2 gives us an idea of how these chambers may look like.

### 6.18. Deciding which chambers are full.

Here we present tools to determine which of the chambers of \( E_{\alpha^{(3)}} \) are full/empty, for a given parameter \( \alpha \in S^1 \setminus \Omega \).

### 6.19. Proposition

Let \( \alpha \in S^1 \setminus \Omega \) be a parameter and consider \( E_{\alpha^{(3)}} \) decomposed in the union of chambers defined by \( \cup_{\ell_{(\alpha^3)} \cup \ell_{\omega_{\alpha^3}} \cup \ell_{\omega_{\alpha^2}}} \subset T \). Then, every chamber that contain an open segment of the non diagonal side of \( \rho(\mathcal{E}) \) in its closure is full.

**Proof.** As the non diagonal side of \( \rho(\mathcal{E}) \) corresponds to the classes of special elliptic isometries with positive center and to the classes of eilmpo-parabolic isometries, the result follows from Propositions 6.3, 6.6 and 6.17.

### 6.20. Proposition

Let \( 0 \leq \theta < 2\pi \) be such that \((\theta, \theta)\) does not lie in \( \cup_{\ell_{(\alpha^3)} \cup \ell_{\omega_{\alpha^3}} \cup \ell_{\omega_{\alpha^2}}} \), and let \( \beta := e^{i\theta} \). Then the chamber of \( E_{\alpha^{(3)}} \) containing \((\theta, \theta)\) in its closure is full iff \( E_{\beta, \alpha} \cup E_{\beta, \alpha} \cap E_{\beta, \alpha} \cap E_{\beta, \alpha} \).
Proof. If \( E_{-\alpha,\alpha}^- \cup E_{+\alpha,\alpha}^+ \) intersects \( E_{-\beta,\beta}^- \cup E_{+\beta,\beta}^+ \), arguing as in the proof of Proposition 6.6, we obtain that there exists a relation of the form \( R_{p_3}^\delta \ell_{\omega_3} = \delta R_{p_3}^\delta R_{p_3}^e \) where \( \delta \in \Omega \) and \( p_3 \in BV \). Thus, the isometry \( R_{p_3}^p \), with angle pair \((\theta, \theta)\), admits a \( \alpha^{(3)} \)-decomposition and, since \((\theta, \theta)\) does not lie in \( utr^{-1}(\ell_{\omega_3} \cup \ell_{\omega_3} \cup \ell_{\omega_3})\), by Proposition 6.19, the chamber containing \((\theta, \theta)\) in its closure is full.

Conversely, if the chamber containing \((\theta, \theta)\) in its closure is full, then any isometry with angle pair \((\theta, \theta)\) admits an \( \alpha^{(3)} \)-decomposition which implies that \( E_{-\alpha,\alpha}^- \cup E_{+\alpha,\alpha}^+ \) intersects \( E_{-\beta,\beta}^- \cup E_{+\beta,\beta}^+ \). \( \square \)

Figure 5: Walls of the \( \alpha^{(3)} \)-decomposition, for \( \alpha = e^{\hat{\pi}i} \) (left) and \( \alpha = e^{\hat{\pi}i} \) (right).

6.21. Example. We apply the previous propositions to obtain the polygonal region \( E_{\alpha^{(3)}} \) in the case \( \alpha = e^{\alpha_i}, \alpha = \pi/9 \). By Lemma 4.8, \( utr^{-1}(\ell_{\alpha^{(3)}}) \) is given by the two segments connecting the points \((\pi/2, 0), (\pi, \pi), (2\pi, 3\pi/2)\); \( utr^{-1}(\ell_{\omega^{(3)}}) \) is given by the segments connecting \((3\pi/2, 2\pi), (2\pi, \pi), (3\pi/2, 0)\); and \( utr^{-1}(\ell_{\omega^{(3)}}) \) is given by the segments connecting \((2\pi, \pi/2), (\pi, 0), (\pi/2, \pi/2)\). These sets are represented in the Figure 5 in blue, green, and red, respectively (see online version). The central hexagonal chamber is full since it contains the angle pair \((\pi/3, \pi/3)\) corresponding to the class where the trace vanishes (see Remark 6.4). Moreover, Proposition 6.19 implies that all but two chambers are full; the remaining chambers we must check to be full or not are those that contain an open segment of the diagonal but don’t intersect nondiagonal sides.

We take two points in the diagonal, one in each of these regions and not lying one of the walls, and apply Proposition 6.20. The points \((3\pi/4, 3\pi/4)\) and \((5\pi/4, 5\pi/4)\) satisfy this condition, and are marked in Figure 5.

Figure 6: The sets \( E_{-\beta,\beta}^- \cup E_{+\beta,\beta}^+ \) and \( E_{-\alpha,\alpha}^- \cup E_{+\alpha,\alpha}^+ \) for \( \alpha = e^{\hat{\pi}i}, \beta = e^{\hat{\pi}i} \) (left) and \( \beta = e^{\hat{\pi}i} \) (right).

So, for \( \beta := e^{\hat{\pi}i}, \) where \( \theta = 3\pi/4 \) or \( \theta = 5\pi/4 \), we need to verify whether the sets \( E_{-\beta,\beta}^- \cup E_{-\beta,\beta}^+ \) and \( E_{-\alpha,\alpha}^- \cup E_{+\alpha,\alpha}^+ \) intersect or not. In Figure 6 we picture these sets: \( E_{-\beta,\beta}^- \cup E_{+\beta,\beta}^+ \) is in light blue/orange, and the \( E_{-\alpha,\alpha}^- \cup E_{+\alpha,\alpha}^+ \) is in dark blue/dark green (following the color scheme for each pair of signs as in Figure 3; see online version). We see that in both cases these sets intersect and both chambers are full.

In the case where \( a = \pi/3 \), the walls are given by the same set of segments as in the previous case (see Figure 5) but with a cyclic permutation of colors. Note that this case is the one of involutions and Figure 5 and the corresponding figure in 6.22 are [13, Figure 11]. Taking the same points in the diagonal side of \( E \), the sets \( E_{-\beta,\beta}^- \cup E_{+\beta,\beta}^+ \) are now given by Figure 7 and we obtain that both corresponding chambers are empty.

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6.22. Elliptic isometries that admit an $\alpha^{(3)}$-decomposition. Proceeding as in the example above, we determine which chambers are full/empty for the cases where $a$ is an integral multiple of $\pi/27$, $0 < a < 2\pi/3$. (This choice will be clear in the proof of the main theorem.) The result is given by the following pictures. For each value of $\alpha$, the chambers in gray are full and the chambers in white are empty. The union of the chambers in gray is the region $E_{\alpha}^{(3)}$ for the given value of $\alpha$. 

Figure 7: The sets $E_{\beta,\pi}^{-} \cup E_{\beta,\pi}^{+}$ and $E_{\alpha,\alpha}^{-} \cup E_{\alpha,\alpha}^{+}$ for $\alpha = e^{\pi i}$, $\beta = e^{\pi i}$ (left) and $\beta = e^{2\pi i}$ (right)
\[ \alpha = e^{\frac{10\pi}{27}i} \]

\[ \alpha = e^{\frac{11\pi}{27}i} \]

\[ \alpha = e^{\frac{14\pi}{27}i} \]

\[ \alpha = e^{\frac{17\pi}{27}i} \]

7 The \( \alpha \)-length

We are now able to prove our main theorem. We do so using the hypothesis that the parameter \( \alpha \in S^1 \setminus \Omega \) is such that \( \alpha = e^{a\pi i} \) for some \( 0 < a < \frac{2\pi}{3} \) which, by Corollary 3.6, can be assumed without loss of generality.

**Proof of Theorem 1.1.** We start by proving that every isometry that does not admit an \( \alpha^{(3)} \)-decomposition, admits an \( \alpha^{(4)} \)-decomposition. By the results in the previous sections, we only need to prove this to those elliptic isometries whose conjugacy classes lie in empty chambers and for 2-step unipotent isometries.

First, let \( F \in SU(2,1) \) be an elliptic isometry representing a PU(2,1)-conjugacy class whose projection under does not lie in \( E_{\alpha^{(3)}} \). There exists a point \( p \in PV \setminus SV \) such that \( R_{p\alpha}F \) is loxodromic. In the case where \( F \) is special elliptic, this follows from [6, Corollary 5.10]. Suppose \( F \) is regular elliptic. Let \( c \in BV \) be the negative \( F \)-fixed point and let \( L \) be an \( F \)-stable complex line through \( c \). Denote by \( \theta \) the angle in which \( F \) rotates points in \( L \) around \( c \). Given a nonisotropic point \( p \in L \cap BV \), consider the isometry \( R := R_{c\alpha}F \). As \( R \) acts over \( L \) as an isometry of the Poincaré disk, the action of \( R \) over \( L \) can be decomposed as the product \( r_2r_1 \) of reflections \( r_1, r_2 \) over geodesics \( G_1, G_2 \) through points \( c, p \), respectively. If the (dis)tance between \( c \) and \( p \) is big enough, the geodesics \( G_1, G_2 \) are ultraparallel (do not intersect, not even in the absolute \( SV \)) and \( R \) acts on \( L \) as a hyperbolic isometry of the Poincaré disk; thus \( R \) is loxodromic. By Proposition 6.3, \( R \) admits an \( \alpha^{(3)} \)-decomposition, which implies that \( F \) admits an \( \alpha^{(4)} \)-decomposition.

Now, consider a 2-step unipotent isometry \( U \in SU(2,1) \). Let \( v \) be the isotropic fixed point of \( U \); then for any nonisotropic point \( q \in PV^+ \), the isometry \( R_{q\alpha}U \) is ellipto parabolic. It follows
from Proposition 6.3 that the isometry $U$ admits an $α^{(4)}$-decomposition.

To prove the second part of the theorem, we use Subsection 6.22. Note that, since the lines $ℓ_{α^2}$, $ℓ_{ωα^3}$, and $ℓ_{ω^2α^3}$ vary continuously with $α$ and $utr$ is a homeomorphism, it follows that the chambers of $E_{α}$ vary continuously with $α$. Moreover, since the line segments that compose $E_{α}$, $α$ vary continuously with $α_1$, $α_2$ (see Proposition 5.8), the criteria to determine if a chamber is full/empty (Propositions 6.19 and 6.20) is also continuous, i.e., if a chamber of $E_{α}$ is full/empty for a given $α$, it continues to be full/empty for parameters sufficiently close to $α$. It follows that if a chamber is full/empty, it continues to be full/empty until it disappears. So, we need to determine the transition parameters (the parameters where chambers appear or disappear).

The transition parameters are those in which the lines $ℓ_{α^3}$, $ℓ_{ωα^3}$, $ℓ_{ω^2α^3}$ either pairwise intersect at a point where one of them is tangent to $∂Δ$ or all intersect at the same point (in this case such point must be $0 ∈ C$ and the lines are tangent to a vertex of $∂Δ$). In the first case, by Proposition 4.6, we must have $α^3 = ωα^6$ which implies that $a = 0 (mod \frac{27π}{4})$. In the second case, we must have $a = 0 (mod \frac{27}{4})$. Thus, the transition angles are those satisfying $a ≠ 0 (mod \frac{27}{4})$. In other words, if we are looking at the chambers of $E_{α}$ while continuously increasing the value of $α$ (remember that $α = e^{ai}$), chambers appear or disappear while passing through a parameter such that $a = 0 (mod \frac{27}{4})$.

Therefore, the second part follows from the cases we obtained in Subsection 6.22; the last part is just Proposition 6.7.

As the isometries that contribute to the $α$-length, $α = e^{ai}$, not being 3 when $0 < a < \frac{47π}{27}$ are only, possibly, the 2-step unipotent ones (see Remark 6.8), which are not semisimple, we have the following result (see Subsection 3.1 for definitions).

7.1. Corollary. The $α$-length of the space of semisimple conjugacy classes in $G$ is

- 3, if $0 < a < \frac{47π}{27}$ or $\frac{14π}{27} < a < \frac{47π}{27}$;
- 4, if $\frac{14π}{27} ≤ a ≤ \frac{47π}{27}$.

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