Abstract

In this paper, we give the solution of the three dimensional quantum stationary Hamilton-Jacobi Equation (3D-QSHJE) for a general form of the potential. We present the quantum coordinates transformation with which the 3D-QSHJE takes its classical form. Then, we derived the 3D quantum law of motion.

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1- Introduction

The deterministic interpretation of the quantum mechanics takes none contesting progress during the last twenties years, and many theoretical physicists contribute to this progress. Floyd took up the quantum stationary Hamilton-Jacobi equation (QSHJE) already established by Bohm and de Broglie.

\[
\frac{1}{2m_0} \left( \nabla S_0 \right)^2 - \frac{\hbar^2}{2m_0} \frac{\Delta R}{R} + (E - V) = 0 , \quad (a)
\]

\[
\nabla \cdot \left( R^2 \nabla S_0 \right) = 0 . \quad (b)
\]

These equations arise from the 3D Schrödinger equation, after writing the wave function \( \Psi \) in the form

\[
\Psi(\vec{r}) = R(\vec{r}) \left( \alpha e^{i S_0} + \beta e^{-i S_0} \right) . \quad (2)
\]

\( R \) and \( S_0 \) are real functions. In the literature of Hamilton-Jacobi formalism, \( S_0 \) represents the reduced action of the studied system. Floyd focused his work on the one dimensional (1D) motions and write the 1D QSHJE as

\[
\frac{1}{2m_0} \left( \frac{dS_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \left[ \frac{3}{2} \left( \frac{dS_0}{dx} \right)^2 - \frac{\left( \frac{d^2 S_0}{dx^2} \right)^2}{\left( \frac{dS_0}{dx} \right)^2} - \frac{\left( \frac{d^3 S_0}{dx^3} \right)^2}{\left( \frac{dS_0}{dx} \right)^4} \right] + V(x) = E . \quad (3)
\]

He established the solution of Eq. (3) in Ref. and used the Jacobi’s theorem with its classical version in order to plot the quantum trajectories.
Recently, Djama and Bouda had criticized the use of the classical version of Jacobi’s theorem [6] and proposed a new dynamical and deterministic approach to study the motions of quantum particles [6, 7, 8, 9]. This new approach consists on the introduction of a quantum Lagrangian from which we deduce the fundamental expression of the quantum conjugate momentum

\[
\dot{x} \frac{dS_0}{dx} = 2 \left( E - V(x) \right). \tag{4}
\]

We also derived the first integral of the quantum Newton’s Law (FIQNL) [6] and plotted the quantum trajectories [7]. The Floyd quantum trajectories and ours are established each either in one dimension, while the real motions are in 3D spaces. So, a generalization in 3D of the deterministic approach of quantum mechanics presented in Refs. [6, 7, 8] is necessary.

The aim of the present paper is the resolution of the 3D-QSHJE and deduce the 3D quantum law of motion. In this order, in Sec. 2, we propose a solution of the 3D-QSHJE (Eqs. (1)). In Sec. 3, we introduce a quantum coordinates transformation with which the 3D-QSHJE takes the form of the classical Hamilton-Jacobi equation (CSHJE). Then, in Sec. 4, we derive the quantum law of 3D motion. Finally, in Sec. 5, we present a conclusion.

2- The solution of the 3D QSHJE

First, let us investigate Eq. (1.b). We can easily check that it can be solved to give [2, 10]

\[
R^2(\vec{r}) \nabla S_0 = k \left( \phi \nabla \theta - \theta \nabla \phi \right), \tag{5}
\]

where \( k \) is a real constant. \( \theta \) and \( \phi \) are two real and independent solutions
of the 3D Schrödinger equation
\[ -\frac{\hbar^2}{2m_0} \Delta \psi + V(\vec{r}) = E. \]  \hspace{1cm} (6)

In fact, we can check that Eq. (5) is a consequence of the writing the wave function in the form given by Eq. (2). In addition, Let us write the reduced action \( S_0 \) in the form
\[ S_0(\vec{r}) = \hbar \arctan \left( a \frac{\theta(\vec{r})}{\phi(\vec{r})} + b \right). \]  \hspace{1cm} (7)

Replacing Eq. (7) into Eq. (5), we get
\[ R(\vec{r}) = \sqrt{\frac{k}{\hbar a}} \left[ (a \theta + b \phi)^2 + \phi^2 \right]^{\frac{1}{2}}. \]  \hspace{1cm} (8)

Expressions given by Eq. (7) and (8) are solutions of Eq. (1.b). Are they solutions of Eq. (1.a)? In order to check this, we write the reduced action \( S_0 \) in the form
\[ S_0(\vec{r}) = \hbar \arctan \left( \frac{\theta'(\vec{r})}{\phi(\vec{r})} \right), \]  \hspace{1cm} (9)
where \( \theta'(\vec{r}) = a \theta(\vec{r}) + b \phi(\vec{r}) \). Then, \( R \) will be written as
\[ R(\vec{r}) = \sqrt{\frac{k}{\hbar a}} \left[ \theta^2 + \phi^2 \right]^{\frac{1}{2}}. \]  \hspace{1cm} (10)

Now, Taking the usual derivative of \( S_0 \) and \( R \) from Eqs. (9) and (10), and replacing them into Eq. (1.a), we find
\[ \frac{\hbar^2}{2m_0} \left( \phi \vec{\nabla} \theta' - \theta' \vec{\nabla} \phi \right)^2 - \frac{\hbar^2}{2m_0} \left( \phi \vec{\nabla} \theta - \theta \vec{\nabla} \phi \right)^2 \]
\[ -\frac{\hbar^2}{2m_0} \frac{\theta' \Delta \theta' + \phi \Delta \phi}{\theta^2 + \phi^2} + V(\vec{r}) - E = 0, \]  \hspace{1cm} (11)
which reduces to
\[ \frac{\theta'}{\theta^2 + \phi^2} \left[ -\frac{\hbar^2}{2m_0} \Delta \theta' + (V(\vec{r}) - E) \theta' \right] + \]
\[ \frac{\phi}{\theta^2 + \phi^2} \left[ -\frac{\hbar^2}{2m_0} \Delta \phi + (V(\vec{r}) - E) \phi \right] = 0. \]  \hspace{1cm} (12)
Because it is a linear combination of $\theta$ and $\phi$, $\theta'$ is a solution of the Schrödinger equation (Eq. (6)). This means that Eq. (12) is automatically satisfied. So, expressions (7) and (8) of $S_0$ and $R$ are solutions of the 3D-QSHJE (Eqs. (1)).

3- The quantum coordinates and the classical form of the 3D-QSHJE

In order to write the 1D-QSHJE with a similar form of the classical Hamilton-Jacobi equation (CSHJE), Faraggi and Matone have introduced the quantum coordinate $\hat{x}$ defined as

$$\left(\frac{dx}{d\hat{x}}\right)^2 = 1 - \frac{\hbar^2}{2} \left(\frac{dS_0}{dx}\right)^2 \{S_0, x\}$$

(13)

After setting $\hat{S}_0 = S_0$ and $\hat{V} = V$ they write the QSHJE as

$$\frac{1}{2m_0} \left(\frac{d\hat{S}_0}{d\hat{x}}\right)^2 + \hat{V}(\hat{x}) = E.$$  

(14)

That means that, in 3D, such a transformation, if it exist, will permit to reduce the 3D-QSHJE in the form of the CSHJE. Before going more, let us review some basic formula of the coordinates transformation under a curved space.

a- coordinates transformation under curved space

Let us consider a curved space with the spatial metric (here, we do not consider time)

$$ds^2 = a_{\mu\nu}dx^\mu dx^\nu.$$  

(15)
\( a_{\mu\nu} \) being the metric tensor, \( x^\mu \) is the coordinate with respect to the \( \vec{e}_\mu \) direction. The well known CSHJE can be written in such a space as

\[
a^{\mu\nu}(x)(\partial_\mu S_0)(\partial_\nu S_0) + V(x) = E ,
\]

where \( a^{\mu\nu} \) is the inverse tensor of \( a_{\mu\nu} \), and \( \partial_\mu \) is the partial derivative with respect to \( x^\mu \). \( S_0 \) still represents the reduced action of the particle.

Now, if we apply a coordinates transformation

\[
x^\mu \rightarrow x'^\mu ,
\]

the components of the metric tensor \( a_{\mu\nu} \) and \( a^{\mu\nu} \) will transform as

\[
\begin{align*}
\left\{ \begin{array}{l}
a'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} a_{\alpha\beta}(x) , \\
a'^{\mu\nu}(x') = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\nu}{\partial x^\alpha} a^{\alpha\beta}(x) .
\end{array} \right.
\end{align*}
\]

And the partial derivatives transform as

\[
\partial'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} \partial_\alpha .
\]

Under this transformation, and after setting \( S_0'(x') = S_0(x) \), the CSHJE (Eq. (16)) takes the form

\[
a'^{\mu\nu}(x')(\partial'_\mu S_0)(\partial'_\nu S_0) + V(x(x')) = E .
\]

In what follows, such transformation will be studied. It is the transformation which reduce the 3D-QSHJE into its classical form.
Now, let us return to the 3D-QSHJE. We are looking for a coordinate transformation

\[
\begin{align*}
  x & \rightarrow \hat{x}(x, y, z) \\
y & \rightarrow \hat{y}(x, y, z) \\
z & \rightarrow \hat{z}(x, y, z)
\end{align*}
\] (20)

by which and after setting

\[
\begin{align*}
  \hat{S}_0(\hat{x}, \hat{y}, \hat{z}) &= S_0[\vec{r}(\hat{x}, \hat{y}, \hat{z})] \\
  \hat{V}(\hat{x}, \hat{y}, \hat{z}) &= V[\vec{r}(\hat{x}, \hat{y}, \hat{z})]
\end{align*}
\]

the 3D-QSHJE will be written in the classical form

\[
\frac{1}{2m_0} \left( \frac{\partial \hat{S}_0}{\partial \hat{x}} \right)^2 + \frac{1}{2m_0} \left( \frac{\partial \hat{S}_0}{\partial \hat{y}} \right)^2 + \frac{1}{2m_0} \left( \frac{\partial \hat{S}_0}{\partial \hat{z}} \right)^2 + \hat{V}(\hat{x}, \hat{y}, \hat{z}) = E .
\] (21)

For this form, the corresponding metric tensor is the Euclidean one

\[
\hat{a}^{\mu\nu} = 1 , \quad \hat{a}^{\mu\nu} = 0 \quad \mu, \nu = 1, 2, 3 ,
\] (22)

with 1, 2 and 3 correspond to the directions of \( \hat{x}, \hat{y} \) and \( \hat{z} \). In what follows, we demonstrate the existence of the transformation (20). So, using relations (18), we write

\[
\begin{align*}
  \frac{\partial}{\partial \hat{x}} &= \frac{\partial}{\partial x} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \hat{x}}{\partial y} + \frac{\partial}{\partial z} \frac{\partial \hat{x}}{\partial z} \\
  \frac{\partial}{\partial \hat{y}} &= \frac{\partial}{\partial x} \frac{\partial \hat{y}}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \hat{y}}{\partial y} + \frac{\partial}{\partial z} \frac{\partial \hat{y}}{\partial z} \\
  \frac{\partial}{\partial \hat{z}} &= \frac{\partial}{\partial x} \frac{\partial \hat{z}}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \hat{z}}{\partial y} + \frac{\partial}{\partial z} \frac{\partial \hat{z}}{\partial z} .
\end{align*}
\] (23)

Replacing these last relations into Eq. (21), we find
\[
\frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 a^{xx} + \frac{1}{2m_0} \left( \frac{\partial S_0}{\partial y} \right)^2 a^{yy} + \frac{1}{2m_0} \left( \frac{\partial S_0}{\partial z} \right)^2 a^{zz} + \\
\frac{1}{m_0} \frac{\partial S_0}{\partial x} \frac{\partial S_0}{\partial y} a^{xy} + \frac{1}{m_0} \frac{\partial S_0}{\partial x} \frac{\partial S_0}{\partial z} a^{xz} + \frac{1}{m_0} \frac{\partial S_0}{\partial z} \frac{\partial S_0}{\partial y} a^{yz} + V(r) = E, \quad (24)
\]

where

\[
\begin{align*}
 a^{xx} &= \left( \frac{\partial x}{\partial x} \right)^2 + \left( \frac{\partial x}{\partial y} \right)^2 + \left( \frac{\partial x}{\partial z} \right)^2 \\
 a^{yy} &= \left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial y} \right)^2 + \left( \frac{\partial y}{\partial z} \right)^2 \\
 a^{zz} &= \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + \left( \frac{\partial z}{\partial z} \right)^2 \\
 a^{xy} &= a^{yx} = \frac{\partial x}{\partial x} \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial x}{\partial z} \frac{\partial y}{\partial z} + \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \\
 a^{xz} &= a^{zx} = \frac{\partial x}{\partial x} \frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} \frac{\partial z}{\partial y} + \frac{\partial x}{\partial z} \frac{\partial z}{\partial z} \\
 a^{yz} &= a^{zy} = \frac{\partial y}{\partial x} \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} \frac{\partial z}{\partial y} + \frac{\partial y}{\partial z} \frac{\partial z}{\partial z} + \frac{\partial y}{\partial y} \frac{\partial z}{\partial z} \quad (25)
\end{align*}
\]

The quantities \(a^{xy}\) are the components of the metric tensor of the quantum curved space, where the 3D-QSHJE is described.

Eq. (24) represents the 3D-QSHJE written with the coordinates \(x, y\) and \(z\), then it must be equivalent to Eq. (1.a). A comparison of Eqs. (24) and (1.a) permit to define the quantities \(\partial x^\mu/\partial \hat{x}^\nu\). Indeed, if we write Eq. (1.a) as

\[
\frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 \left[ 1 - \hbar^2 \left( \frac{\partial S_0}{\partial x} \right)^2 \frac{\partial^2 R/\partial x^2}{R} \right] + \\
\frac{1}{2m_0} \left( \frac{\partial S_0}{\partial y} \right)^2 \left[ 1 - \hbar^2 \left( \frac{\partial S_0}{\partial y} \right)^2 \frac{\partial^2 R/\partial y^2}{R} \right] + \\
\frac{1}{2m_0} \left( \frac{\partial S_0}{\partial z} \right)^2 \left[ 1 - \hbar^2 \left( \frac{\partial S_0}{\partial z} \right)^2 \frac{\partial^2 R/\partial z^2}{R} \right] + V(\vec{r}) = E \quad (26)
\]
and compare it with Eq. (26), we find

\[
\begin{align*}
  a^{xx} &= \left( \frac{\partial x}{\partial \mu} \right)^2 + \left( \frac{\partial y}{\partial \nu} \right)^2 + \left( \frac{\partial z}{\partial \nu} \right)^2 = 1 - \hbar^2 \left( \frac{\partial S_0}{\partial x} \right)^{-2} \frac{\partial^2 R/\partial x^2}{R} \\
  a^{yy} &= \left( \frac{\partial y}{\partial \mu} \right)^2 + \left( \frac{\partial y}{\partial \nu} \right)^2 + \left( \frac{\partial z}{\partial \nu} \right)^2 = 1 - \hbar^2 \left( \frac{\partial S_0}{\partial y} \right)^{-2} \frac{\partial^2 R/\partial y^2}{R} \\
  a^{zz} &= \left( \frac{\partial z}{\partial \mu} \right)^2 + \left( \frac{\partial z}{\partial \nu} \right)^2 + \left( \frac{\partial z}{\partial \nu} \right)^2 = 1 - \hbar^2 \left( \frac{\partial S_0}{\partial z} \right)^{-2} \frac{\partial^2 R/\partial z^2}{R} \\
  a^{xy} &= a^{yx} = \frac{\partial x}{\partial \mu} \frac{\partial y}{\partial \nu} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} = 0 \\
  a^{xz} &= a^{zx} = \frac{\partial x}{\partial \mu} \frac{\partial z}{\partial \nu} + \frac{\partial x}{\partial \nu} \frac{\partial z}{\partial \nu} + \frac{\partial x}{\partial \nu} \frac{\partial z}{\partial \nu} = 0 \\
  a^{yz} &= a^{zy} = \frac{\partial y}{\partial \mu} \frac{\partial z}{\partial \nu} + \frac{\partial y}{\partial \nu} \frac{\partial z}{\partial \nu} + \frac{\partial y}{\partial \nu} \frac{\partial z}{\partial \nu} = 0.
\end{align*}
\]

The set (27) of equations permit to define partially the quantum transformation that we are searching for. Indeed, to define these transformation, we must determine the nine quantities \( \partial x^\mu / \partial \tilde{\nu}^\nu \). However, Eq. (25) contains six equalities and nine unknown quantities \( \partial x^\mu / \partial \tilde{\nu}^\nu \). Then, from Eq. (27), we can just determine six of these unknown. In order to determine the three other quantities, we use the tensor transformation relations given in Eq. (17) and taking into account the relations (22), we write six equations containing three additive unknown quantities which are the components \( a_{\mu\nu} (\mu = 1, 2, 3) \) \((a_{\mu\nu} = 0 (\mu \neq \nu))\)

\[
\begin{align*}
  \left( \frac{\partial x}{\partial \mu} \right)^2 a_{xx} + \left( \frac{\partial y}{\partial \nu} \right)^2 a_{yy} + \left( \frac{\partial z}{\partial \nu} \right)^2 a_{zz} &= 1, \\
  \left( \frac{\partial x}{\partial \mu} \right)^2 a_{xx} + \left( \frac{\partial y}{\partial \nu} \right)^2 a_{yy} + \left( \frac{\partial z}{\partial \nu} \right)^2 a_{zz} &= 1, \\
  \left( \frac{\partial x}{\partial \mu} \right)^2 a_{xx} + \left( \frac{\partial y}{\partial \nu} \right)^2 a_{yy} + \left( \frac{\partial z}{\partial \nu} \right)^2 a_{zz} &= 1, \\
  \frac{\partial x}{\partial \mu} \frac{\partial y}{\partial \nu} a_{xx} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} a_{yy} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} a_{zz} &= 0, \\
  \frac{\partial x}{\partial \mu} \frac{\partial y}{\partial \nu} a_{xx} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} a_{yy} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} a_{zz} &= 0, \\
  \frac{\partial x}{\partial \mu} \frac{\partial y}{\partial \nu} a_{xx} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} a_{yy} + \frac{\partial x}{\partial \nu} \frac{\partial y}{\partial \nu} a_{zz} &= 0.
\end{align*}
\]

(28)
Finally, we have twelve equations containing twelve unknown quantities. Thus, the quantum transformation, that we searching for really exist and is deduced from Eqs. (27) and (28). In conclusion, we stress that this transformation reduces the 3D-QSHJE into its classical form (Eq. (21)).

4- The quantum Lagrangian and the quantum law of motion

The Use of the quantum coordinates bring the idea that the quantum Lagrangian must have the same form as the classical one, but written with the 3D quantum coordinates. So, we write

$$L_q = \frac{m_0}{2} \dot{x}^2 (x, y, z) + \frac{m_0}{2} \dot{y}^2 (x, y, z) + \frac{m_0}{2} \dot{z}^2 (x, y, z) - \hat{V}(\hat{x}, \hat{y}, \hat{z})$$ (29)

The total differentials $d\hat{x}$, $d\hat{y}$ and $d\hat{z}$ might be expressed with the partial derivatives $\partial \hat{x}^\mu / \partial x^\nu$ and the total differentials $dx$, $dy$ and $dz$ as

$$d\hat{x}^\mu = \frac{\partial \hat{x}^\mu}{\partial x^\nu} dx^\nu \quad \mu, \nu = 1, 2, 3.$$

Replacing the last relations into Eq. (29), we find

$$L_q(x^\mu, \dot{x}^\mu) = \frac{m_0}{2} \dot{x}^2 a_{xx}(\vec{r}) + \frac{m_0}{2} \dot{y}^2 a_{yy}(\vec{r}) + \frac{m_0}{2} \dot{z}^2 a_{zz}(\vec{r}) - V(\vec{r}) ,$$ (30)

where $a_{\mu\nu}$ are the diagonal components of the inverse of the metric tensor ($a_{\mu\nu}(\mu \neq \nu) = 0$). They satisfy the following relations

$$\begin{cases} a_{xx} = \frac{1}{a^{xx}} = (\frac{\partial \hat{x}}{\partial x})^2 + (\frac{\partial \hat{y}}{\partial x})^2 + (\frac{\partial \hat{z}}{\partial x})^2 \\ a_{yy} = \frac{1}{a^{yy}} = (\frac{\partial \hat{x}}{\partial y})^2 + (\frac{\partial \hat{y}}{\partial y})^2 + (\frac{\partial \hat{z}}{\partial y})^2 \\ a_{zz} = \frac{1}{a^{zz}} = (\frac{\partial \hat{x}}{\partial z})^2 + (\frac{\partial \hat{y}}{\partial z})^2 + (\frac{\partial \hat{z}}{\partial z})^2 \end{cases}$$ (31)

Expression (30) of the 3D quantum Lagrangian is analogous to the one exhibited in Ref. [6] for the 1D cases where the function $f(x, a, b)$ plays the
same role as the components of the metric tensor of the quantum curved space.

Now, using the expression (30) of the Lagrangian, the least action principle leads to

\[
\begin{cases}
  m_0 \ddot{x} a_{xx} + m_0 \dot{x} \frac{d a_{xx}}{d t} - \frac{m_0}{2} \dot{x}^2 \frac{\partial a_{xx}}{\partial x} - \frac{m_0}{2} \dot{y}^2 \frac{\partial a_{yy}}{\partial x} - \frac{m_0}{2} \dot{z}^2 \frac{\partial a_{zz}}{\partial x} + \frac{\partial V}{\partial x} = 0, \\
  m_0 \ddot{y} a_{yy} + m_0 \dot{y} \frac{d a_{yy}}{d t} - \frac{m_0}{2} \dot{x}^2 \frac{\partial a_{xx}}{\partial y} - \frac{m_0}{2} \dot{y}^2 \frac{\partial a_{yy}}{\partial y} - \frac{m_0}{2} \dot{z}^2 \frac{\partial a_{zz}}{\partial y} + \frac{\partial V}{\partial y} = 0, \\
  m_0 \ddot{z} a_{zz} + m_0 \dot{z} \frac{d a_{zz}}{d t} - \frac{m_0}{2} \dot{x}^2 \frac{\partial a_{xx}}{\partial z} - \frac{m_0}{2} \dot{y}^2 \frac{\partial a_{yy}}{\partial z} - \frac{m_0}{2} \dot{z}^2 \frac{\partial a_{zz}}{\partial z} + \frac{\partial V}{\partial z} = 0.
\end{cases}
\tag{32}
\]

Summing Eqs. (32), we get after calculation

\[
\frac{m_0}{2} d(\dot{x}^2) a_{xx} + \frac{m_0}{2} d(\dot{y}^2) a_{yy} + \frac{m_0}{2} d(\dot{z}^2) a_{zz} + \frac{m_0}{2} \dot{x}^2 da_{xx} + \frac{m_0}{2} \dot{y}^2 da_{yy} + \frac{m_0}{2} \dot{z}^2 da_{zz} + dV = 0, \tag{33}
\]

which, after integrating, reduces to

\[
\frac{m_0}{2} [\dot{x}^2 a_{xx} + \dot{y}^2 a_{yy} + \dot{z}^2 a_{zz}] + V(\vec{r}) = E. \tag{34}
\]

\(E\) being an integration constant representing the energy of the particle. Eq. (34) represents the 3D quantum conservation equation. Comparing Eqs. (26) and (34), and taking into account of Eqs. (27), we deduce

\[
\begin{align*}
  a_{xx} &= \frac{\partial S_0 / \partial x}{m_0 x}, \\
  a_{yy} &= \frac{\partial S_0 / \partial y}{m_0 y}, \\
  a_{zz} &= \frac{\partial S_0 / \partial z}{m_0 z}.
\end{align*}
\tag{35}
\]

Replacing Eqs. (35) into Eq. (34), we find

\[
\dot{x} \frac{\partial S_0}{\partial x} + \dot{y} \frac{\partial S_0}{\partial y} + \dot{z} \frac{\partial S_0}{\partial z} = 2 [E - V(\vec{r})]. \tag{36}
\]
Eq. (36) expresses the relation between the quantum conjugate momenta and the components of the speed of the particle. It can be written, with vectors as

$$\vec{v} \cdot \nabla S_0 = 2 [E - V(\vec{r})]$$

(37)

where $\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$ is the 3D speed of the particle and

$$\nabla S_0 = \frac{\partial S_0}{\partial x} \vec{i} + \frac{\partial S_0}{\partial y} \vec{j} + \frac{\partial S_0}{\partial z} \vec{k}$$

is the 3D quantum conjugate momentum.

From Eqs. (36) and (37), and, knowing the two independent solutions of the Schrödinger equation, one can plot the quantum trajectories in 3D. So, Eqs. (36) and (37) might be considered as a fundamental law of 3D quantum motions. The 3D quantum Newton’s Law can be derived from Eqs. (36) and (37), after using the 3D-QSHJE, with the same manner as it is done in Ref. [6] for the 1D quantum Newton’s Law. That will be investigated in a next paper. Remark that at the classical limit ($\hbar \to 0$), $a_{\mu\mu}$ and $a^{\mu\mu}$ reduces to 1 (see Eqs. (27)), then, from Eqs. (35), we deduce that

$$\frac{\partial S_0}{\partial x} = m_0\dot{x} \quad \frac{\partial S_0}{\partial y} = m_0\dot{y} \quad \text{and} \quad \frac{\partial S_0}{\partial z} = m_0\dot{z}.$$ 

Taking these last relations into Eq. (36), we find the well known classical conservation equation

$$\frac{m_0}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(\vec{r}) = E$$

Another important remark is that, when the particle have not motions along $y$ and $z$ axis, Eqs. (36) and (37) reduce to the expression (4) of the 1D conjugate momentum.
5- Conclusion

The resolution of the quantum stationary Hamilton-Jacobi equation in three dimensions (Eqs. (1)) is a very important realization in the frame of the construction of a deterministic approach of quantum mechanics. The reduced action $S_0(\vec{r})$ that we proposed as solution of the 3D-QSHJE has the same form as the one dimensional reduced action already established by Floyd [4] and Faraggi and Matone [11].

The introduction of a quantum coordinates transformation reducing the 3D-QSHJE into its classical form constitute a hopeful step to construct a dynamical approach of quantum mechanics. This transformation give us the idea about the manner with which the dynamical approach of quantum motions in three dimension must be constructed. Indeed, in 1D, we have constructed in Ref. [6] a quantum Lagrangian and Hamiltonian from which we established the expression of the quantum conjugate momentum, and derived the Quantum Newton’s Law. For 3D cases, a similar construction is possible now. That is what we investigate in this paper.

The quantum law of 3D motions that we exhibited in Eqs. (36) and (37) is the generalization to the 3D spaces of the quantum Law of motion that we have already established with Bouda in Ref. [6]. For this deterministic and causal approach of quantum mechanics such a generalization represents an important advancement of theory, since all possible physical phenomena happen in 3D spaces. The fundamental relation expressed in Eqs. (36) and (37) contains, in the left hand side, the scalar product of the conjugate momentum and the velocity, and in the right hand side, the kinetic energy of the particle up a multiplicative constant.

Finally, we stress that the generalization into 3D problems of the deter-
ministic approach, that we have already introduced in Ref. [6], can be seen as the first step to the investigation of the quantum gravity in the framework of a causal quantum theory.

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