GROTHENDIECK TOPOLOGIES AND SHEAF THEORY FOR ČECH CLOSURE SPACES

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Abstract. We introduce a new approach to the construction of sheaf theory on discrete and combinatorial spaces by constructing sheaves on Čech closure spaces, a category which contains the major classes of spaces of most interest to applications: topological spaces, graphs, and metric spaces with a privileged scale. In the process, we construct a new Grothendieck topology on Čech closure spaces, and we show that sheaves on Čech closure spaces are isomorphic to the sheaf of sections of an appropriately defined étale space.

1. Introduction

Starting from its origins in algebraic topology, sheaf theory has become an indispensable part of homological algebra, and it has many important applications in fields as diverse as algebraic geometry and partial differential equations. In the past decade, particularly with the rise of topological data analysis, there has been increased interest in extending the reach of sheaf theory to a number of scientific and engineering applications, and a number of intriguing efforts have been made in this direction [4, 8, 10, 14, 22–24]. The main difficulty which must be resolved when using sheaves to address computational problems is to construct the theory on combinatorially defined objects in order to compute the resulting sheaf cohomology algorithmically. The most common solution currently found in the literature involves studying sheaves on a combinatorially defined space, such as a simplicial complex, with a $T_0$ topology, either in the version on cell complexes first developed in [24] and revived in [9], where the topology need not be used explicitly to develop much of the theory, or else on spaces of posets as in [23], where the $T_0$ topology takes on a more central role. Related constructions of sheaf theory on simplicial complexes can also be found in Section 8.1 of [17].

In this article, we introduce a new approach to the construction of sheaf theory on discrete and combinatorial spaces by constructing sheaves on Čech closure spaces, a category which contains the major classes of spaces of most interest to applications: topological spaces, graphs, and metric spaces with a privileged scale. Unlike in the theories of sheaves built from a $T_0$ topology, the technique developed here allows one to define sheaves directly on point clouds, rather than needing to pass to an auxiliary simplicial complex in a non-functorial manner. A similar advantage appears when working with graphs, where the sheaf cohomology constructed in this article is a

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priori non-trivial in dimensions greater than one, which is the desired behavior in situations where one would like to approximate a metric space by a graph built from samples of the space. As in the case of point clouds, the sheaf cohomology introduced in this article can be built directly on the graphs themselves, without explicitly constructing an associated simplicial complex.

To the best of our knowledge, the algebraic topology of Čech closure spaces was first used to construct new invariants of graphs in [11, 12]. This idea was taken up again by the current author in [21], where we continued the development of homotopy theory on Čech closure spaces and showed that the framework also covers the case of metric spaces which are endowed with a preferred scale, the scale indicating the minimal size of a neighborhood. The closure space perspective has also led to a characterization of discrete homotopy theories in [5], where it was shown that nearly all discrete homotopy theories on graphs currently known [2, 3, 13, 16] can be expressed as different homotopy theories on Čech closure spaces, simply by changing the cylinder functor and the product used to define homotopy. Although not mentioned in [6], both digital homotopy [1, 5] and the discrete homotopies studied in [20] may be expressed in terms of homotopies on appropriate closure spaces as well.

There are two major obstacles which must be overcome in order to apply the essential ideas and constructions of sheaf theory to Čech closure spaces. First, in most interesting closure spaces, there are simply not enough open sets for the classical sheaf cohomology to provide much information. We see this already with the closure space $(V, c_G)$ induced by a connected graph $G = (V, E)$. In this case, the topological modification is simply the indiscrete topology on the vertex set, i.e. the topology consisting only of $\{\emptyset, V\}$, and, like the topology, the resulting sheaf theory is trivial. The notion of a (non-open) set with non-empty interior, with which we may form interior covers, provides us with an apparent solution to this problem; however, this leads to a second, more serious issue, which is that the category $\mathcal{N}(X, c)$ of $(X, c)$ of neighborhoods of subsets of $(X, c)$ is not closed under finite intersections, and without this, the classical theory of sheaves on topological spaces cannot be adapted to this case.

In order to address this second problem, we observe that the category in which we should work is not $\mathcal{N}(X, c)$, but rather the category which contains $\mathcal{N}(X, c)$ together with all of the finite intersections of its elements, which we denote by $\mathcal{L}(X, c)$. While the objects of $\mathcal{L}(X, c)$ now form a topology on $X$, this topology is often too fine to be interesting, and if we build sheaves using open covers of $X$ in the topology $\mathcal{L}(X, c)$, then much of the structure we wish to capture may be lost. We solve this second problem by constructing a special Grothendieck topology on $\mathcal{L}(X, c)$. We first construct a new class of covers, called $i$-covers, for arbitrary subsets $U \subset X$ which, for the entire space $X$, are exactly the interior covers of $X$, and we then generate the Grothendieck topology with the $i$-covers. This procedure effectively limits our attention to interior covers on $(X, c)$, while using arbitrary sets in $\mathcal{L}(X, c)$, including those with empty interior, to define matching conditions for the sheaves. Once this is accomplished, the entire machinery of Grothendieck topologies becomes available, and the general results of sheaf theory on sites may be immediately applied.
While Čech closure spaces are more general than topological spaces, they also share many important features, and, after constructing the Grothendieck topology on \( \mathcal{L}(X, c) \), we then turn our attention to aspects of the theory which are specific to Čech closure spaces, and which directly generalize the topological case. In particular, we construct a Godement resolution and show that a sheaf on \((X, c)\) is isomorphic to the sheaf of sections of an étale space. Finally, in the last section, we give a construction of Čech cohomology with coefficients in a presheaf, and we use the resulting relationship between Čech cohomology and sheaf cohomology to produce a non-topological closure space with non-trivial sheaf cohomology.

2. Čech Closure Spaces

**Definition 2.1.** Let \( X \) be a set, and let \( c : \mathcal{P}(X) \to \mathcal{P}(X) \) be a map on the power set of \( X \) which satisfies

1. \( c(\emptyset) = \emptyset \)
2. \( A \subset c(A) \) for all \( A \subset X \)
3. \( c(A \cup B) = c(A) \cup c(B) \) for all \( A, B \subset X \)

The map \( c \) is called a Čech closure operator (or closure operator) on \( X \), and the pair \((X, c)\) is called a Čech closure space (or closure space).

**Example 2.2.** (1) Let \((X, \tau)\) be a topological space with topology \( \tau \). For any \( A \subset X \), denote by \( \bar{A} \) the (topological) closure of \( A \). Then \( c_\tau(A) = \bar{A} \) is a Čech closure operator. Note that, in this case, \( c_\tau^2(A) = c_\tau(A) \). Closure operators \( c \) with the property that \( c^2 = c \) are called Kuratowski or topological closure operators, and it can be shown that, for these closure operators, the collection

\[ \mathcal{O} := \{ X \setminus c(A) \mid A \subset X \} \]

forms the open sets of a topology on \( X \). (See [7], Theorem 15.A.2(a) for a proof.)

(2) Let \( G = (V, E) \) be a graph with vertices \( V \) and edges \( E \). We define a closure operator \( c_G : \mathcal{P}(V) \to \mathcal{P}(V) \) in the following way. First, let \( c : V \to \mathcal{P}(V) \) be the map

\[ c(v) := \{ v \} \cup \{ v' \in V \mid (v, v') \in E \}. \]

For an arbitrary \( A \subset V \), we now define the operator \( c_G \) by

\[ c_G(A) = \cup_{v \in A} c(v). \]

Then \( c_G \) is a closure operator, and \((V, c_G)\) is a closure space, which we call the closure space induced by the graph \( G \).

(3) Let \((X, d)\) be a metric space, and \( r \geq 0 \) a non-negative real number. For any \( A \subset X \), define

\[ c_r(A) := \{ x \in X \mid d(x, A) \leq r \}. \]

Then \( c_r \) is a closure operator on \( X \). For \( r = 0 \), \((X, c_0)\) is topological by the discussion in Example 2.2(1), and if \( r > 0 \), we call \((X, c_r)\) a mesoscopic space.

**Definition 2.3.** Let \((X, c)\) be a closure space. For any \( A \subset X \), we define the interior of \( A \) by

\[ i(A) := X - c(X - A), \]
and we say that \( U \subset X \) is a neighborhood of \( A \) iff \( A \subset i(U) \). We will call \( i : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) the interior operator on \((X, c)\).

The next two propositions enumerate the essential properties of interior operators.

**Proposition 2.4** ([14], 14 A.11). Let \((X, c)\) be a closure space. The interior operator \( i : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) satisfies the following

1. \( i(X) = X \)
2. \( i(A) \subset A \) for all \( A \subset X \)
3. \( i(A \cap B) = i(A) \cap i(B) \) for all \( A, B \subset X \)

Item (3) above implies

**Proposition 2.5.** Let \((X, c)\) be a Čech closure space, and suppose that \( B \subset A \subset X \).

Then \( i(B) \subset i(A) \).

**Definition 2.6** (Covers). Let \((X, c)\) be a closure space.

1. We say that a collection \( U \) of subsets of \( X \) is a cover of \((X, c)\) iff \( X = \bigcup_{\alpha \in U} U_\alpha \).

2. We say that a collection \( U \) of subsets of \( X \) is an interior cover of \((X, c)\) iff \( X = \bigcup_{\alpha \in U} i(U_\alpha) \).

3. Let \( i : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) be the interior operator on \((X, c)\). For any subset \( U \subset X \), we say that a collection \( U \) of subsets of \( U \) is an \( i \)-cover of \( U \) iff

\[
U = \bigcup_{\alpha \in U} U_\alpha \\
i(U) = \bigcup_{\alpha \in U} i(U_\alpha).
\]

If \( U = \{U_\alpha\}_{\alpha \in A} \) is an \( i \)-cover of \( U \subset X \) and for every \( \alpha \in A, U_\alpha \in \text{Ob} \mathcal{L}(X, c) \), then we call \( U \) an \( i \)-cover of \( U \) in \( \mathcal{L}(X, c) \).

**Remark 2.7.** Note that, in Equation (2.1) in the above definition, we take the interior operator \( i \) which is induced by the closure operator \( c \) on \( X \) when considering any subset \( U \subset X \), and not the interior operator which is induced by the subspace closure operator \( c_U \) on \( U \). Also, since \( i(X) = X \), a collection of sets \( U \) is an \( i \)-cover of the entire space \( X \) iff \( U \) is an interior cover of \((X, c)\). Conversely, if \( i(U) = \emptyset \), then an \( i \)-cover of \( U \) is simply a cover.

**Proposition 2.8.** Let \((X, c)\) be a Čech closure space. For any two neighborhoods \( U \) and \( V \) of a point \( x \in X \), the intersection \( U \cap V \) is also a neighborhood of \( x \).
Proof. By hypothesis, \( x \in i(U) \cap i(V) \), and by Item (3) of Proposition 2.4, \( i(U) \cap i(V) = i(U \cap V) \).

Corollary 2.9. Suppose that \( \mathcal{U} \) and \( \mathcal{V} \) are interior covers of a Čech closure space \((X, c)\). Then the collection

\[ \mathcal{U} \cap \mathcal{V} := \{ U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V} \} \]

is an interior cover of \((X, c)\).

Proof. Let \( x \in X \). Since \( \mathcal{U} \) and \( \mathcal{V} \) are interior covers of \((X, c)\), there exist \( U_x \in \mathcal{U} \) and \( V_x \in \mathcal{V} \) which are both neighborhoods of \( x \). By Proposition 2.8, \( U_x \cap V_x \) is a neighborhood of \( x \) as well. Since \( U_x \cap V_x \in \mathcal{U} \cap \mathcal{V} \) and \( x \in X \) is arbitrary, the conclusion follows.

Corollary 2.10. Let \((X, c)\) be a closure space, and suppose that \( \mathcal{U} \) and \( \mathcal{V} \) are \( i \)-covers of a subset \( U \subset X \). Then the collection

\[ \mathcal{U} \cap \mathcal{V} := \{ U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V} \} \]

is an \( i \)-cover of \( U \subset X \).

Let \( x \in i(U) \). Since \( \mathcal{U} \) and \( \mathcal{V} \) are \( i \)-covers of \( U \subset X \), there exist \( U_x \in \mathcal{U} \) and \( V_x \in \mathcal{V} \) which are both neighborhoods of \( x \). By Proposition 2.8, \( U_x \cap V_x \) is a neighborhood of \( x \) as well. Since \( U_x \cap V_x \in \mathcal{U} \cap \mathcal{V} \) and \( x \in i(U) \) is arbitrary, we have that

\[ i(U) \subset \bigcup_{U_\alpha \in \mathcal{U}} i(U_\alpha). \]

Since \( U_\alpha \subset U \) for all \( U_\alpha \in \mathcal{U} \), it follows that \( i(U_\alpha) \subset i(U) \) for any \( U_\alpha \in \mathcal{U} \) by the properties

\[ i(U) = \bigcup_{U_\alpha \in \mathcal{U}} i(U_\alpha). \]

Now suppose that \( x \in U \). Since \( \mathcal{U} \) and \( \mathcal{V} \) are \( i \)-covers of \( U \subset X \), there exist \( U'_x \in \mathcal{U} \) and \( V'_x \in \mathcal{V} \) which contain \( x \). Then \( x \in U'_x \cap V'_x \) and therefore

\[ U = \bigcup_{U_\alpha \in \mathcal{U}} U_\alpha. \]

It follows that \( \mathcal{U} \cap \mathcal{V} \) is an \( i \)-cover of \( U \).

Remark 2.11. We note that Corollary 2.9 implies that the collection \( \mathcal{I} \) of interior covers of \((X, c)\) forms a directed set with partial order \( \mathcal{V} < \mathcal{U} \) iff \( \mathcal{U} \) refines \( \mathcal{V} \).

We end this section by recalling the construction and basic facts about inductively generated closure structures, i.e. closure structures on a set \( Y \) which are generated by a set of maps \( \{ f_a : (X, c_X) \to Y \} \).

Definition 2.12 ([7], Definition 33 A.1). A closure operation \( c_Y \) on a set \( Y \) is said to be inductively generated by a family of mappings \( \{ f_a : (X, c_a) \to Y \mid a \in A \} \) iff \( c_Y \) is the finest closure structure on \( Y \) such that all of the mappings \( f_a : (X, c_a) \to (Y, c_Y) \) are continuous.
Theorem 2.13 ([7], Theorem 33 A.11). Every non-empty family of mappings \( \{ f_a : (X, c_a) \to Y \mid a \in A \} \) inductively generates exactly one closure structure \( c_Y \) on \( Y \). If a closure \( c_Y \) for a set \( Y \) is inductively generated by a family of mappings \( \{ f_a : (X, c_X) \to Y \mid a \in A \} \), then
\[
c_Y(V) = V \cup \left( \bigcup_{a \in A} f_a(c_X(f_a^{-1}(V))) \right)
\]
for each \( V \subset Y \).

Theorem 2.14 ([7], Theorem 33 A.11). Let \( (Y, c_Y) \) be a closure space inductively generated by a family of mappings \( \{ f_a : (X, c_X) \to Y \} \). A subset \( V \subset Y \) is a neighborhood of a point \( y \in Y \) iff \( y \in V \) and \( f^{-1}(V) \) is a neighborhood of \( f^{-1}(x) \) in \( (X, c_X) \).

3. Sheaf Theory

3.1. Grothendieck Topologies. In this section, we review the construction of a Grothendieck topology from a basis, following the presentations in [18] and [19]. We then continue our discussion by constructing a Grothendieck topology on the category \( \mathcal{L}(X, c) \) using the basis consisting of all \( i \)-covers of sets in \( \mathcal{L}(X, c) \).

Let \( \mathcal{C} \) be a category, and for any \( U \in \text{Ob}(\mathcal{C}) \), let \( \mathcal{C}_U \) denote the slice category over \( U \).

Definition 3.1. Let \( U \in \text{Ob}(\mathcal{C}) \). A sieve \( S \) over \( U \) is a subset of \( \text{Ob}(\mathcal{C}_U) \) such that, if \( V \to U \in S \), then the composition \( W \to V \to U \in S \) for any \( W \to V \in \text{Hom}_{\mathcal{C}}(W, V) \).

Definition 3.2. A Grothendieck topology \( T = \{ S(U) \}_{U \in \text{Ob}(\mathcal{C})} \) on a category \( \mathcal{C} \) is a collection of sieves \( S(U) \) for each object \( U \in \text{Ob}(\mathcal{C}) \), such that

1. The maximal sieve on \( U \) is in \( S(U) \), i.e. \( \text{Ob}(\mathcal{C}_U) \in S(U) \)
2. \( S_1 \in S(U), S_1 \subset S_2 \) implies \( S_2 \in S(U) \)
3. Let \( V \to U \in \text{Ob}(\mathcal{C}_U) \). If \( S \in S(U) \), then \( S \times_U V \in S(V) \), where we define
\[
S \times_U V := \{ W \to V \mid \text{the composition } W \to V \to U \in S(U) \}
\]
4. Let \( S \) and \( S' \) be sieves over \( U \). Assume that \( S' \in S(V) \) and that \( S \times_U U \in S(U) \) for any \( U \to V \in S' \). Then \( S \in S(V) \).

A sieve \( S \) over \( U \) is called a covering sieve if \( S \in S(U) \). A site is a pair \( (\mathcal{C}, T) \) consisting of a category \( \mathcal{C} \) and a Grothendieck topology on \( \mathcal{C} \).

Definition 3.3. Let \( \mathcal{C} \) be a category with pullbacks. A basis for a Grothendieck topology on \( \mathcal{C} \) is a function \( K \) which assigns to every \( U \in \mathcal{C} \) a collection of families of objects in \( \mathcal{C}_U \) which satisfies

1. For any isomorphism \( U' \to U \), the family consisting of the single morphism \( \{ U' \to U \} \) is a member of \( K(U) \).
(2) If \( \{ f_\alpha : U_\alpha \to U \mid \alpha \in A \} \in K(U) \), then for any morphism \( g : V \to U \), the family \( \{ U_\alpha \times_U V \mid \alpha \in A \} \in K(V) \).

(3) If \( \{ f_\alpha : U_\alpha \to U \mid \alpha \in A \} \in K(U) \) and, for every \( \alpha \in A \), there is a family \( \{ g_{\alpha \beta} : V_{\alpha \beta} \to U_\alpha \mid \beta \in B_\alpha \} \in K(U_\alpha) \), then the family \( \{ f_\alpha \circ g_{\alpha \beta} : V_{\alpha \beta} \to U \mid \alpha \in A, \beta \in B_\alpha \} \in K(U) \).

Given a basis \( K \) on a category with pullbacks \( C \), we define a Grothendieck topology on \( C \) by
\[
S \in T(U) \iff \exists R \in K(U), R \subset S.
\]
We refer the reader to [13, 19] for further details, including the proof that this forms a Grothendieck topology on \( C \).

**Definition 3.4.** A site is a pair \((C, J)\), where \( C \) is a category and \( J \) is a Grothendieck topology on \( C \).

**Theorem 3.5.** Let \((X, c)\) be a Čech closure space with interior operator \( i \).

1. If \( \{ U_\alpha \}_{\alpha \in A} \) is an \( i \)-cover of \( U \in \mathcal{L}(X, c) \) and \( V \in \mathcal{L}(X, c) \), \( V \subset U \), then \( \{ V \cap U_\alpha \}_{\alpha \in A} \) is an \( i \)-cover of \( V \).

2. Let \( \{ U_\alpha \}_{\alpha \in A} \) be an \( i \)-cover of \( U \in \mathcal{L}(X) \), and, for each \( \alpha \in A \), let \( \{ V_{\beta \alpha} \}_{\beta \in B_\alpha} \) be an \( i \)-cover of \( U_\alpha \). Then \( \{ V_{\beta \alpha} \}_{\alpha \in A, \beta \in B} \) is an \( i \)-cover of \( U \).

**Proof.** (1) It follows from Proposition 2.4, Item 3 that
\[
\bigcup_{\alpha \in A} i(V \cap U_\alpha) = \bigcup_{\alpha \in A} (i(V) \cap i(U_\alpha))
= i(V) \cap \left( \bigcup_{\alpha \in A} i(U_\alpha) \right).
\]
Since \( \{ U_\alpha \} \) is an \( i \)-cover of \( U \) by hypothesis,
\[
\bigcup_{\alpha \in A} i(U_\alpha) = i(U),
\]
and therefore
\[
i(V) \cap \left( \bigcup_{\alpha \in A} i(U_\alpha) \right) = i(V) \cap i(U)
= i(V),
\]
where the last inequality follows from the monotonicity of the interior operator and the hypothesis that \( V \subset U \).

(2) Since \( \{ V_{\beta \alpha} \}_{\beta \in B} \) is an \( i \)-cover for each \( U_\alpha \), and \( \{ U_\alpha \}_{\alpha \in A} \) is an \( i \)-cover of \( U \), we have
\[
\bigcup_{\alpha \in A} \bigcup_{\beta \in B} i(V_{\beta \alpha}) = \bigcup_{\alpha \in A} i(U_\alpha) = i(U),
\]
which completes the proof. \( \square \)
It follows that the \( i \)-covers form a basis for a Grothendieck topology on \( \mathcal{L}(X,c) \), as shown in the following corollary.

**Corollary 3.6.** For each \( U \in \text{Ob}(\mathcal{L}(X,c)) \), define
\[
K(U) := \{ U \mid U \text{ is an } i\text{-cover of } U \}.
\]
Then \( \{ K(U) \}_{U \in \text{Ob}(\mathcal{L}(X,c))} \) is a basis for a Grothendieck topology on \( \mathcal{L}(X,c) \).

**Proof.** Denote by \( K(X,c) \) the collection of \( i \)-covers \( \{ K(U) \}_{U \in \text{Ob}(\mathcal{L}(X,c))} \). Since any homeomorphism \( U' \to U \) is an \( i \)-cover, \( \{ U' \to U \} \in K \) for every \( U \in \text{Ob}(\mathcal{L}(X,c)) \), and Item 1 is satisfied.

To see Item 2, first note that \( U_\alpha \times_U V = U_\alpha \cap V \). The conclusion now follows from Theorem 3.5, Item (2).

Finally, Item 3 follows from Theorem 3.5, Item (2). \( \square \)

### 3.2. Sheaves on \( Č \)ech closure spaces.

**Definition 3.7.** Let \( (X,c) \) be a \( Č \)ech closure space, and \( C \) a category. A **presheaf on** \( (X,c) \) is a contravariant functor \( F : \mathcal{L}(X,c)^{\text{op}} \to C \). If \( F : \mathcal{L}(X,c)^{\text{op}} \to C \) is a presheaf on \( (X,c) \) with restriction maps \( \rho_{U,V}^{\alpha} : F(V) \to F(W) \) for any \( V, W \in \text{Ob}(\mathcal{L}(X,c)) \), \( W \subseteq V \), then we say that \( F \) is a **sheaf** on \( (X,c) \) iff \( F \) satisfies

1. For any \( U \in \mathcal{L}(X,c) \), if \( s \in F(U) \) and there is an \( i \)-cover \( \{ U_\alpha \}_{\alpha \in A} \) of \( U \) in \( \mathcal{L}(X,c) \) where \( \rho_{U_\alpha}^U(s) = 0 \) for all \( \alpha \in A \), then \( s = 0 \).
2. If \( U_\alpha \subseteq I \) is an \( i \)-cover of \( U \) in \( \mathcal{L}(X,c) \) and the family \( (s_\alpha)_{\alpha \in I}, s_\alpha \in F(U_\alpha) \) is such that
\[
\rho_{U_{\alpha \cap \beta}}^U_{U_\alpha}(s_\alpha) = \rho_{U_{\alpha \cap \beta}}^U(s_\beta)
\]
for all \( \alpha, \beta \in I \), then there exists an \( s \in F(U) \) such that \( \rho_{U_\alpha}^U(s) = s_\alpha \) for all \( \alpha \in I \).

**Proposition 3.8.** A presheaf \( F \) on the \( Č \)ech closure space \( (X,c) \) is a sheaf iff, for any \( U \in \mathcal{L}(X,c) \) and \( i \)-cover \( \{ U_\alpha \}_{\alpha \in A} \) of \( U \) in \( \mathcal{L}(X,c) \), the diagram
\[
F(U) \to \prod_{\alpha \in A} F(U_\alpha) \Rightarrow \prod_{\alpha, \beta \in A} F(U_\alpha \cap U_\beta)
\]
is an equalizer.

**Proof.** Let \( f \) and \( g \) denote the top and bottom maps in
\[
\prod_{\alpha \in A} F(U_\alpha) \Rightarrow \prod_{\alpha, \beta \in A} F(U_\alpha \cap U_\beta).
\]
Since \( F \) is a presheaf, it takes values in an abelian category by definition, and we may define \( h = f - g \). Then the diagram above is an equalizer iff the diagram
\[
0 \to F(U) \to \prod_{\alpha \in A} F(U_\alpha) \xrightarrow{h} \prod_{\alpha, \beta \in A} F(U_\alpha \cap U_\beta)
\]
is left exact. However, this is equivalent to the condition that \( F \) is a sheaf, as condition (1) in the definition of a sheaf is satisfied iff the first map is injective, and
condition (2) in the definition of a sheaf is satisfied iff the diagram \( \prod_{\alpha \in A} F(U_\alpha) \) is exact at \( \prod_{\alpha \in A} F(U_\alpha) \).

As in the topological case, morphisms between sheaves are the corresponding natural transformations between the underlying presheaves. We therefore have an inclusion of categories \( \iota : \text{Sh}(X,c) \to \text{PSh}(X,c) \), where \( \text{PSh}(X,c) \) and \( \text{Sh}(X,c) \) are the categories of abelian presheaves and sheaves on \( (X,c) \), respectively. Conversely, Theorem 17.4.1(iii) in [18] or, equivalently, Theorem III.5.1 in [19], give

Theorem 3.9. The functor \( \iota \) has a right adjoint \( \sigma : \text{PSh}(X,c) \to \text{Sh}(X,c) \).

We call \( \sigma(F) \) the sheafification of the presheaf \( F \).

3.3. Sheaves as Étale Spaces. In the topological case, it is well known that any sheaf of sets is isomorphic to the sheaf of sections of an étale space (see, e.g. [15], Theorem 1.2.1). While it is unclear if sheaves of sets over closure spaces have such a characterization, we show in this section that abelian sheaves over closure spaces may indeed be described this way. We begin with a few definitions.

Definition 3.10. We say that \( ((E,c_E),p) \) is an étale space over the closure space \( (X,c) \) if \( (E,c_E) \) is a closure space and \( p : (E,c_E) \to (X,c_X) \) is a local homeomorphism. We will often write \( (E,p) \) when the closure structure \( c_E \) is understood.

Definition 3.11. For any \( x \in X \), we define

\[
\mathcal{N}(x) = \{ U \in \mathcal{N}(X,c) \mid x \in i(U) \}
\]

\[
\mathcal{L}(x) = \{ V \in \mathcal{L}(X,c) \mid x \in V \}
\]

\[
\mathcal{M}(x) = \{ V \in \mathcal{L}(X,c) \mid x \in V \setminus i(V) \}.
\]

Definition 3.12. Let \( F \) be an abelian presheaf over the closure space \( (X,c) \). We define the neighborhood stalk \( F_{\mathcal{N}}(x) \), the lattice stalk \( F_{\mathcal{L}}(x) \), and the stalk \( F(x) \) of \( F \) at \( x \) by

\[
F_{\mathcal{N}}(x) := \begin{cases} 
\lim_{U \in \mathcal{N}(x)} F(U), & \text{if } \emptyset \neq \mathcal{N}(X,c) \\
F(\emptyset), & \text{if } \emptyset = \mathcal{N}(X,c)
\end{cases}
\]

\[
F_{\mathcal{L}}(x) := \lim_{V \in \mathcal{L}(x)} F(V)
\]

\[
F(x) := F_{\mathcal{N}}(x) \oplus F_{\mathcal{L}}(x)
\]

Note that if \( F \) is a sheaf, then the sheaf conditions imply that \( F(\emptyset) = 0 \).

Theorem 3.13. Let \( F \) be an abelian presheaf over a closure space \( (X,c) \). Then \( F \) is isomorphic to the sheaf of sections of an étale space \( (E,p) \) iff \( F \) is a sheaf, where \( (E,p) \) is unique up to isomorphism.
Proof. For any \( U, V \in \mathcal{L}(X, c), V \subseteq U \), let \( \rho^U_x : F(U) \to F(V) \) denote the restriction map of the presheaf \( F \). We also define the maps

\[
\psi^U_V = \begin{cases} 
\rho^U_x & U, V \in \mathcal{N}(X, c), i(V) \subseteq i(U) \\
\rho^U_0 & \text{Otherwise}
\end{cases}
\]

\[
\phi^U_V = \rho^U_0.
\]

Denote by \( \tilde{F} \) the disjoint union \( \bigsqcup_{x \in X} F(x) \), and let \( p : \tilde{F} \to X \) be the map which sends each \( f \in F(x) \) to \( x \). Let \( x \in U \). Note that, for each \( U \in \mathcal{N}(x), V \in \mathcal{L}(x) \), the definition of the direct limit gives the canonical maps

\[
\psi^U_x : F(U) \to F_X(x)
\]

\[
\phi^U_x : F(V) \to F_\mathcal{C}(x),
\]

where \( \psi^U_x \) sends \( s \in F(U) \) to its image in \( F_X(x) \) and \( \phi^U_x \) sends \( s \in F(V) \) to its image in \( F_\mathcal{C}(x) \). For \( U \in \mathcal{M}(x) \), we define \( \psi^U_x : F(U) \to F(\emptyset) \) by \( \psi^U_x(s) = \rho^U_0(s) \).

We let \( \rho^U_x : F(U) \to F(x) \) be the map

\[
(3.2) \quad \rho^U_x(s) = \psi^U_x(s) + \phi^U_x(s).
\]

Note that, for any \( V \in \mathcal{M}(x) \), the definitions give us \( \rho^V_x(s) = \rho^V_0(s) + \phi^V_x(s) \).

Given \( U \in \mathcal{L}(x) \) and a section \( s \in F(U) \), we denote by \( s(x) \) the image of \( s \) under \( \rho^U_x \). Furthermore, to each section \( s \in F(U) \), we have an induced map \( \tilde{s} : U \to \tilde{F} \) given by \( x \mapsto s(x) \), which additionally verifies \( p(\tilde{s}(x)) = p(s(x)) = x \).

Suppose that \( U, V \in \mathcal{L}(X, c), V \subseteq U \), and that either \( V \in \mathcal{N}(x) \) or \( U \in \mathcal{M}(x) \) (so either \( U \in \mathcal{N}(x) \) or \( V \in \mathcal{M}(x) \) as well, respectively). Then, again, by the definition of the direct limit, we have \( \rho^U_x = \rho^V_x \circ \rho^U_V \). Defining \( t : V \to \mathcal{F} \) by \( t := \rho^U_V(s) \), the map \( t \) is the restriction to \( V \) of the map \( \tilde{s} : U \to \tilde{F} \).

We endow \( \mathcal{F} \) with closure structure \( \tilde{c} \) inductively generated by the maps \( \tilde{s} : U \to \tilde{F} \) induced by the sections \( s \in \mathcal{F}(U) \). Note that, by construction, \( \tilde{c} \) is the finest closure structure such that for each \( x \in X, U \in \mathcal{L}(x) \), and \( s \in \mathcal{F}(U) \), the map \( \tilde{s} : U \to \tilde{F} \) is continuous.

By Theorem 2.14, a subset \( G \subseteq \tilde{F} \) is a neighborhood in \( \tilde{F} \) iff, for every \( U \in \mathcal{N}(X, c) \) and every \( s \in F(U) \), the set

\[
\{ x \in U \mid \tilde{s}(x) \in G \} \in \mathcal{N}(X, c)
\]

is a neighborhood in \( X \).

The map \( p : (\tilde{F}, \tilde{c}) \to (X, c) \) is therefore continuous, and, by definition, for every \( U \in \mathcal{N}(X, c) \) and every \( s \in \mathcal{F}(U) \), the set \( \tilde{s}(U) \in \mathcal{N}(\tilde{F}, \tilde{c}) \). Since

\[
\tilde{s}(U) \cong U = p(\tilde{s}(U)),
\]

\( p \) is a local homeomorphism, and we have now shown that \( (\tilde{F}, p) \) is an étale space.

Let \( U \in \mathcal{L}(X, c) \), and denote by \( \tilde{F}(U) \) the space of continuous sections of \( \tilde{F} \) above \( U \). The canonical maps

\[
\rho : F(U) \to \tilde{F}(U)
\]

form a morphism of presheaves. We wish to show that \( \rho \) is bijective iff \( \mathcal{F} \) is a sheaf.
We first show that $\rho$ is injective iff $F$ is a satisfies Part 1 of Definition 3.7. First, suppose that $F$ satisfies Part 1 of the Definition 3.7 and further suppose that $s', s'' \in \mathcal{F}(U)$ define the same section $\tilde{s} = s' = s'' \in \mathcal{F}$. If $x \in U$, then $\tilde{s'}(x) = \tilde{s''}(x)$ implies that there exists a neighborhood $U_x \in \mathcal{N}(x)$ such that $U_x \subset U$ and $s'|_{U_x} = s''|_{U_x}$. Similarly, if $x \not\in U \setminus i(U)$, then there exists a $V_x \in \mathcal{L}(X, c)$ such that $V_x \subset U$ and $s'|_{V_x} = s''|_{V_x}$. It follows that the collection

$$
\{U_x \}_{x \in i(U)} \cup \{V_x \}_{x \not\in U \setminus i(V)}
$$

of such subsets forms an $i$-cover of $U$. Part 1 of Definition 3.7 now implies that $s' = s''$.

Now suppose that $\rho$ is injective, let $s', s'' \in F(U)$, and suppose that there exists an $i$-cover $\{U_\alpha\}_{\alpha \in A}$ of $U$ such that $s|_{U_\alpha} = s'|_{U_\alpha}$ for all $\alpha \in A$. It follows that $\tilde{s'}(x) = \tilde{s''}(x)$ for all $x \in U$. However, since $\phi$ is injective, this implies that $s' = s'' \in \mathcal{F}$, as desired.

It remains to show that $\rho$ is surjective iff $F$ is a sheaf. First, we suppose that $F$ is a sheaf. Consider a section $f \in \mathcal{F}(U)$. By construction, any two sections of $\mathcal{F}$ which are equal at a point $x \in i(U)$ coincide in a neighborhood $U' \subset U$ of $x$, and any two sections of $\mathcal{F}$ which are equal at a point in $x \in U \setminus i(U)$ coincide in some element $V' \in \mathcal{L}(X, c), V' \subset U$. Therefore, we may find an $i$-cover $U = \{U_\alpha\}_{\alpha \in A}$ of $U$ and elements $s_\alpha \in F(U_\alpha)$ such that $\tilde{s_\alpha} = f$ on each $U_\alpha \subset U$.

Consider two sets $U_\alpha, U_\beta \in U$, and the associated $s_\alpha \in F(U_\alpha), s_\beta \in F(U_\beta)$ constructed above. Then $\rho_{U_\alpha \cap U_\beta}(\tilde{s_\alpha}) = \rho_{U_\alpha \cap U_\beta}(\tilde{s_\beta})$ on $U_\alpha \cap U_\beta$. Using Part 1 of Definition 3.7 on $U_\alpha \cap U_\beta$ and the argument for injection above, we see that the restrictions of $s_\alpha$ and $s_\beta$ to $U_\alpha \cap U_\beta$ are equal as well. Since $F$ is a sheaf, by Part 2 of Definition 3.7 there exists a section $s \in F(U)$ such that $s|_{U_\alpha} = s_\alpha$, and therefore $\tilde{s}(x) = f(x)$ for all $x \in U$. Since $f$ was arbitrary, $\rho$ is surjective.

Now suppose that $\rho$ is bijective and that there exists an $i$-cover $U = \{U_\alpha\}_{\alpha \in A}$ and sections $s_\alpha \in F(U_\alpha)$ such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$. Since $U$ is an $i$-cover and $\mathcal{F}(U)$ is the sheaf of sections above $U$, there exists an $f \in \mathcal{F}(U)$ such that $\tilde{s_\alpha} = f(x)$ for all $x \in U_\alpha$ and all $\alpha \in A$. Since $\rho$ is surjective, there exists a section $s \in F(U)$ such that $\rho(s) = f$. Since $\rho$ is injective by assumption, Part 1 of Definition 3.7 holds, and we see that $s|_{U_\alpha} = s_\alpha$ for all $\alpha \in A$. Therefore $F$ is a sheaf.

3.4. Sheaf Cohomology. For the remainder of the article, we assume that all sheaves take values in an abelian category with enough injectives.

The cohomology of a closure space $(X, c)$ with coefficients in a sheaf $\mathcal{A}$ with values in an abelian category with enough injectives is now defined in the usual way, as the right derived functor of the global section functor $\Gamma_X$.

**Definition 3.14.** Let $\text{Sh}(X, c)$ denote the category of abelian sheaves on $(X, c)$, let $\mathcal{C}$ be an abelian category, and let $\mathcal{A} : \mathcal{L}(X, c)^{op} \rightarrow \mathcal{C}$ be a sheaf on $(X, c)$. Denote the global section functor on $(X, c)$ by $\Gamma_X : \text{Sh}(X, c) \rightarrow \mathcal{C}$. We define the
$n$-th cohomology group of $(X, c)$ with coefficients in $\mathcal{A}$ to be the functor

$$H^n(X; \mathcal{A}) := R^n\Gamma_X(\mathcal{A}).$$

As in the topological case, it follows from general results of homological algebra that if the cohomology groups $H^*(X; \mathcal{A})$ may be computed from $\Gamma(\mathcal{L}^*)$, where $\mathcal{L}^*$ is an acyclic resolution of $\mathcal{A}$.

**Definition 3.15.** Let $\mathcal{F}$ be a sheaf of modules over a closure space $(X, c)$. We say that $\mathcal{F}$ is flabby iff, for every $U \in \mathcal{L}(X, c)$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

**Proposition 3.16.** Flabby sheaves are acyclic.

**Proof.** The result follows from [15], Theorem 3.1.3. □

The subject of other classes of acyclic sheaves on closure spaces will be addressed in a future work.

### 3.5. The Godement Resolution

In this section, we construct the canonical flabby resolution for a sheaf $\mathcal{A}$ on a Čech closure space. It proceeds as in the topological case, given the appropriate changes required by the construction of the étale space for general closure spaces. We follow [15].

Let $\mathcal{A}$ be a sheaf on $(X, c)$. We denote by $\mathcal{C}^0(X; \mathcal{A})$ the space of germs of not necessarily continuous sections of $\mathcal{A}$, i.e., germs of maps $s : U \to \mathcal{A}$, where the only requirement on the maps $s$ is that $p(s(x)) = x$. Using Theorem 3.13 to identify the sheaf $\mathcal{A}$ with the sheaf of germs of continuous sections $U \to \mathcal{A}$, we see that there is an injection $\mathcal{A} \hookrightarrow \mathcal{C}^0(X; \mathcal{A})$. We now define the spaces

$$\mathcal{L}^1(X; \mathcal{A}) := \mathcal{C}^0(X; \mathcal{A})/\mathcal{A}$$
$$\mathcal{L}^2(X; \mathcal{A}) := \mathcal{C}^1(X; \mathcal{A})/\mathcal{L}^1(X; \mathcal{A})$$
$$\vdots$$
$$\mathcal{L}^{n+1}(X; \mathcal{A}) := \mathcal{C}^n(X; \mathcal{A})/\mathcal{L}^n(X; \mathcal{A})$$
$$\mathcal{C}^{n+1}(X; \mathcal{A}) := \mathcal{C}^0(X; \mathcal{L}^{n+1}(X; \mathcal{A}))$$

It follows from the definitions of the spaces involved that the sequence

$$0 \to \mathcal{L}^n(X; \mathcal{A}) \to \mathcal{C}^n(X; \mathcal{A}) \to \mathcal{L}^{n+1}(X; \mathcal{A}) \to 0$$

is exact. Furthermore, defining the maps $\delta_n : \mathcal{C}^n(X; \mathcal{A}) \to \mathcal{C}^{n+1}(X; \mathcal{A})$ by the composition $\mathcal{C}^n(X; \mathcal{A}) \to \mathcal{L}^{n+1}(X; \mathcal{A}) \to \mathcal{C}^n(X; \mathcal{A})$, we have that $\delta^2 = 0$, and therefore...
3.5.1. Čech Cohomology. We now recall the definition of Čech cohomology with coefficients in a presheaf from [26], given a basis for a Grothendieck topology. We then recall the construction of the spectral sequence relating Čech and sheaf cohomology, and we use this to give examples of non-topological closure spaces for which the sheaf cohomology of the constant sheaf \( \mathbb{Z} \) is non-trivial.

**Definition 3.17.** Let \((X, c)\) be a Čech closure space, \(F\) a presheaf on \((X, c)\), and \(\mathcal{U}\) an interior cover of \((X, c)\). We define \(H^0(\mathcal{U}, F) := \ker \left( \prod_{U \in \mathcal{U}} F(U) \to \prod_{U, V, \in \mathcal{U}} F(U \times_X V) \right)\).

For each \(q > 0\), we define the \(q\)-th Čech cohomology group of the cover \(\mathcal{U}\) of \((X, c)\) with coefficients in \(F\) by \(H^q(\mathcal{U}; F) := R^q H^0(\mathcal{U}, F)\), the \(q\)-th right derived functor of \(H^0(\mathcal{U}, \cdot)\) applied to \(F\).

We now recall that the Čech cohomology groups of an interior cover \(\mathcal{U}\) may be identified with the cohomology of the following complex.

**Definition 3.18.** Let \((X, c)\) be a Čech closure space, \(\mathcal{U} = \{U_\alpha\}_{\alpha \in A}\) be an interior cover on \((X, c)\), and suppose that \(F\) is a full presheaf on \((X, c)\). Let \(U_{\alpha_0,\ldots,\alpha_q} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}\).

For each integer \(q \geq 0\), we define \(C^q(\mathcal{U}, F) := \prod_{(\alpha_0,\ldots,\alpha_q) \in A^{q+1}} F(U_{\alpha_0,\ldots,\alpha_q})\).

We additionally define the codifferential \(d^q : C^q(\mathcal{U}, F) \to C^{q+1}(\mathcal{U}, F)\) by \((d^q s)_{i_0,\ldots,i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k F(U_{i_0,\ldots,i_k,\ldots,i_{q+1}})\).

Since \(d^2 = 0\), \(C^*(\mathcal{U}, F)\) is a cochain complex. Its homology is given by the following theorem.

**Theorem 3.19** ([26], Theorem 2.2.3). For every abelian presheaf \(F\) on a Čech closure space \((X, c)\), and for every interior cover \(\mathcal{U}\) of \((X, c)\), the group \(H^q(\mathcal{U}, F)\) is canonically isomorphic to the \(q\)-th cohomology group of the complex \(C^*(\mathcal{U}, F)\).

The interior covers of \((X, c)\) form a directed set, where we write \(\mathcal{U} < \mathcal{V}\) iff \(\mathcal{V}\) are interior covers of \((X, c)\) and \(\mathcal{V}\) refines \(\mathcal{U}\). Furthermore, if \(\mathcal{U} < \mathcal{V}\), there exists a well-defined homomorphism \(H^*(\mathcal{U}; F) \to H^*(\mathcal{V}; F)\), and we may therefore make the following definition.

**Definition 3.20.** \(\check{H}^q(X; F) := \lim_{\to} H^q(\mathcal{U}; F)\), where the limit is taken over the interior covers \(\mathcal{U}\) of \(X\). \(\check{H}^*(X; F)\) is called the Čech cohomology of \((X, c)\) with coefficients in the presheaf \(F\).
Theorem 3.21 ([26], Theorem 3.4.4(ii)). Let \( \mathcal{U} \) be an interior covering of a closure space \((X,c)\), and let \( F \) be a sheaf on \((X,c)\). Then there is a spectral sequence

\[
E^{pq}_2 = \tilde{H}^p(X; \mathcal{H}^q(F))
\]

which converges to the sheaf cohomology \( H^{p+q}(X,F) \), where \( \mathcal{H}^*(F) \) denotes the sheaf defined by \( \mathcal{H}^q(F)(U) := H^q(U,F) \).

Corollary 3.22 ([26], Corollary 3.4.5). Let \( \mathcal{U} \) be an interior cover of the closure space \((X,c)\), and let \( F \) be an abelian sheaf.

From the edge morphisms \( \tilde{H}^p(X,F) \to H^p(X,F) \) we have the following corollary

Corollary 3.23 ([26], Corollary 3.4.6). For all abelian sheaves \( F \) on a closure space \((X,c)\), the homomorphism

\[
\tilde{H}^p(X,F) \to H^p(X,F)
\]

is a bijection for \( p = 0, 1 \) and an injection for \( p = 2 \).

Example 3.24. Consider the closure space \((X,c_G)\) generated by the cyclic graph \( G = (V,E) \), where \( X = V = \{0,\ldots,n\} \) and \( E \) consists of all pairs of the form \((k,k+1)\) or \((k,k-1)\), where all entries are taken \( \text{mod} \ (n+1) \). We see that \( \mathcal{U} = \{(k-1,k,k+1) \mid k = 0,\ldots,n\} \), forms an interior cover of \((X,c_G)\), and it is the maximal interior cover with respect to the partial order given by refinement. It follows that \( \tilde{H}^p(X,F) = H^p(\mathcal{U},F) \). When \( F \) is the constant sheaf \( F = \mathbb{Z} \), the latter is the simplicial cohomology of the nerve of the cover \( \mathcal{U} \), and therefore the sheaf cohomology \( H^p(X,\mathbb{Z}) \cong H^p(S^1,\mathbb{Z}) \), for \( p = 0, 1 \) by Corollary 3.23.

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