AUTOMORPHISM RELATED PARAMETERS OF GRAPH ASSOCIATED TO A FINITE VECTOR SPACE

HIRA BENISH, IMRAN JAVAID*, M. MURTAZA

Abstract. In this paper, we discuss automorphism related parameters of a graph associated to a finite vector space. The fixing neighborhood of a pair \((u, v)\) of vertices of a graph \(G\) is the set of all those vertices \(w\) of \(G\), such that the orbits of \(u\) and \(v\) under the action of stabilizer of \(w\) are not equal. The fixed number of a graph is the minimum number \(k\) such that every subset of vertices of \(G\) of cardinality \(k\) is a fixing set of \(G\). We study some properties of automorphisms of a graph associated to finite vector space and find the fixing neighborhood of pair of vertices of the graph. We also find the fixed number of the graph. It is shown that, for every positive integer \(N\), there exists a graph \(G\) with \(f_{xd}(G) - f_{ix}(G) \geq N\), where \(f_{xd}(G)\) is the fixed number and \(f_{ix}(G)\) is the fixing number of \(G\).

1. Preliminaries

The notion of fixing set of graph has its origin in the idea of symmetry breaking which was introduced by Albertson and Collins [1]. Erwin and Harary [18] introduced the fixing number of a graph \(G\). Fixing sets have been studied extensively to destroy the automorphisms of various graphs [9, 10, 11, 21, 22].

Fixing number and metric dimension are two closely related invariants. The two invariants coincides on many families of graph like path graph, cycle graph etc. The difference of two invariant on families of graphs is studied in [7, 11, 18] in terms of the order of graph. Boutin [9] studied fixing sets in connection with distance determining set (she used the name for resolving sets). Arumugam et al. [3] defined resolving neighborhood to study the fractional metric dimension of graph. We defined the fixing neighborhood of a pair of vertices of a graph in [6] in order to study all those vertices of the graph whose fixing can destroy an automorphism that maps the two vertices of the pair on each other. Fixing neighborhood of a pair of vertices contains all such vertices of graph that destroy all automorphisms between those two vertices of graph. Motivated by the definition of resolving number of a graph, Javaid et al. [24] defined the fixed number of a graph. The authors found characterization and realizable results based on fixed number.

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* Corresponding author: imran.javaid@bzu.edu.pk.
A new area of research in graph theory is associating graphs with various algebraic structures. Beck [5] initiated the study of zero divisor graph of a commutative ring with unity to address coloring problem. Bondy et al. [4] studied commuting graphs associated to groups. Power graphs for groups and semigroups were discussed in [12, 15, 26]. In [23, 28] intersection graphs were associated to vector spaces. In [16], Das assigned a non-zero component graph to finite dimensional vector spaces. In [12, 15, 26] dimension and partition dimension of non-zero component graph are studied in [2]. In this paper we study some properties of automorphisms of non zero component graph as defined by Das. Fazil studied its fixing number in [19]. Murtaza et al. studied locating-dominating sets and identifying codes of non-zero component graph [27]. In this paper we study the author discussed edge-connectivity and chromatic number of the graph. Metric dimension and partition dimension of non-zero component graph are studied in [2]. We find the fixing neighborhood of pairs of vertices of the graph and the fixed number of the graph.

Now we define some graph related terminology which is used in the article: Let \( G \) be a graph with the vertex set \( V(G) \) and the edge set \( E(G) \). Two vertices \( u \) and \( v \) are adjacent, if they share an edge, otherwise they are called non-adjacent. The number of adjacent vertices of \( v \) is called the degree of \( v \) in \( G \). For a graph \( G \), an automorphism of \( G \) is a bijective mapping \( f \) on \( V(G) \) such that \( f(u)f(v) \in E(G) \) if and only if \( uv \in E(G) \). The set of all automorphisms of \( G \) forms a group, denoted by \( \Gamma(G) \), under the operation of composition. For a vertex \( v \) of \( G \), the set \( \{ f(v) : f \in \Gamma(G) \} \) is the orbit of \( v \), denoted by \( \mathcal{O}(v) \). If two vertices \( u, v \) are mapped on each other under the action of an automorphism \( g \in \Gamma(G) \), then we write it \( u \sim g v \). An automorphism \( g \in \Gamma(G) \) is said to fix a vertex \( v \in V(G) \) if \( v \sim g v \). The stabilizer of a vertex \( v \) is the set of all automorphisms that fix \( v \) and it is denoted by \( \Gamma_v(G) \). Also, \( \Gamma_v(G) \) is a subgroup of \( \Gamma(G) \). Let us consider sets \( S(G) = \{ v \in V(G) : |\mathcal{O}(v)| \geq 2 \} \) and \( V_s(G) = \{ (u, v) \in S(G) \times S(G) : u \neq v \text{ and } \mathcal{O}(u) = \mathcal{O}(v) \} \). If \( G \) is a rigid graph (i.e., a graph with \( \Gamma(G) = \text{id} \)), then \( V_s(G) = \emptyset \). For \( v \in V(G) \), the subgroup \( \Gamma_v(G) \) has a natural action on \( V(G) \) and the orbit of \( u \) under this action is denoted by \( \mathcal{O}_v(u) \) i.e., \( \mathcal{O}_v(u) = \{ g(u) : g \in \Gamma_v(G) \} \). An automorphism \( g \in \Gamma(G) \) is said to fix a set \( D \subseteq V(G) \) if for all \( v \in D, v \sim g v \). The set of automorphisms that fix \( D \), denoted by \( \Gamma_D(G) \), is a subgroup of \( \Gamma(G) \) and \( \Gamma_D(G) = \bigcap_{v \in D} \Gamma_v(G) \). If \( D \) is a set of vertices for which \( \Gamma_D(G) = \{ \text{id} \} \), then we say that \( D \) is a fixing set of \( G \). The fixing number of a graph \( G \) is defined as the minimum cardinality of a fixing set, denoted by \( \text{fix}(G) \).

Throughout the paper, \( V \) denotes a vector space of dimension \( n \) over the field of \( q \) elements and \( \{ b_1, b_2, ..., b_n \} \) be a basis of \( V \). The non-zero component graph of \( V \) [16], denoted by \( G(V) \), is a graph whose vertex set consists of the non-zero vectors of \( V \) and two vertices are joined by an edge if they share at least one \( b_i \) with non-zero
coefficient in their unique linear combination with respect to \( \{b_1, b_2, \ldots, b_n\} \). It is proved in [16] that \( G(\mathbb{V}) \) is independent of the choice of basis, i.e., isomorphic non-zero component graphs are obtained for two different bases. In [16], Das studied automorphisms of \( G(\mathbb{V}) \). It is shown that an automorphism maps basis of \( G(\mathbb{V}) \) to a basis of a special type, namely non-zero scalar multiples of a permutation of basis vectors.

**Theorem 1.1.** [16] Let \( \varphi : G(\mathbb{V}) \rightarrow G(\mathbb{V}) \) be a graph automorphism. Then, \( \varphi \) maps a basis \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) of \( \mathbb{V} \) to another basis \( \{\beta_1, \beta_2, \ldots, \beta_n\} \) such that there exists a permutation \( \sigma \) from the symmetric group on \( n \) elements, where each \( \beta_i \) is of the form \( c_i\alpha_{\sigma(i)} \) and each \( c_i \)'s are non-zero.

The skeleton of a vertex \( u \in V(G(\mathbb{V})) \) denoted by \( S_u \), is the set of all those basis vectors of \( \mathbb{V} \) which have non-zero coefficients in the representation of \( u \) as the linear combination of basis vectors. In [27], we partition the vertex set of \( G(\mathbb{V}) \) into \( n \) classes \( T_i \), \( (1 \leq i \leq n) \), where \( T_i = \{v \in \mathbb{V} : |S_v| = i\} \). For example, if \( n = 4 \) and \( q = 2 \), then \( T_3 = \{b_1 + b_2 + b_3, b_1 + b_2 + b_3, b_1 + b_3 + b_4, b_2 + b_3 + b_4\} \).

The section wise break up of the article is as follows: In Section 2, we study the properties of automorphisms of non-zero component graph. We discuss the relation between vertices and their images in terms of their skeletons. In Section 3, we discuss some properties and the cardinality of fixing neighborhood of pair of vertices of non-zero component graph. The last section is devoted to the study of the fixed number of graphs. We find the fixed number of non-zero component graph. We give a realizable result about the existence of a graph \( G \), for every positive integer \( N \), such that \( fxd(G) - fix(G) \geq N \), where \( fxd(G) \) is the fixed number of graph.

## 2. Automorphisms of non-zero component graph

In this section, \( \mathbb{V} \) is a vector space of dimension \( n \geq 3 \) over the field of 2 element and \( G(\mathbb{V}) \) is the corresponding non-zero component graph.

**Lemma 2.1.** [27] If \( v \in T_s \) for \( s \) \( (1 \leq s \leq n) \), then \( \deg(v) = (2^s - 1)2^{n-s} - 1 \).

**Lemma 2.2.** Let \( u, v \in V(G(\mathbb{V})) \) such that \( u \in T_r \) and \( v \in T_s \) where \( r \neq s \) and \( 1 \leq r, s \leq n \), then \( u \not\sim^g v \) for all \( g \in \Gamma(G(\mathbb{V})) \).

**Proof.** Since \( r \neq s \Rightarrow (2^r - 1)2^{n-r} - 1 \neq (2^s - 1)2^{n-s} - 1 \). Thus \( \deg(u) \neq \deg(v) \) by Lemma 2.1 and hence \( u \not\sim^g v \).

From Lemma 2.2, we have the following straightforward remarks.

**Remark 2.3.** Since \( \sum_{i=1}^{n} b_i \in T_n \) is the only element in \( T_n \). Therefore, by Lemma 2.2 \( g(\sum_{i=1}^{n} b_i) = \sum_{i=1}^{n} b_i \) for all \( g \in \Gamma(G(\mathbb{V})) \).
Remark 2.4. Let $u, v \in V(G(\mathbb{V}))$, then $(u, v) \in S(G(\mathbb{V})) \times S(G(\mathbb{V}))$ if and only if both $u, v \in T_i$ for some $i$, $1 \leq i \leq n - 1$.

Lemma 2.5. Let $b_l \in T_i$ be a basis vector and $g \in \Gamma_{b_l}$. Let $u \in V(G(\mathbb{V}))$, then $b_l \in S_u$ if and only if $b_l \in S_{g(u)}$.

Proof. Let $b_l \in S_u$, then $u$ is adjacent to $b_l$. Suppose on contrary $b_l \not\in S_{g(u)}$, then $g(u)$ is not adjacent to $b_l$, a contradiction. Conversely, let $b_l \in S_{g(u)}$, then $b_l$ is adjacent to $g(u)$. Suppose on contrary $b_l \not\in S_u$, then $u$ is not adjacent to $b_l = g(b_l)$, a contradiction.

Lemma 2.6. Let $b_l, b_m \in T_i$ be two distinct basis vectors of $\mathbb{V}$ and $g \in \Gamma(G(\mathbb{V}))$ be an automorphism such that $b_l \sim^g b_m$. Let $u \in V(G(\mathbb{V}))$, then we have:

(i) If $b_l \in S_u$ and $b_m \not\in S_u$, then $b_l \not\in S_{g(u)}$ and $b_m \in S_{g(u)}$.

(ii) $b_l, b_m \in S_u$ if and only if $b_l, b_m \in S_{g(u)}$.

Proof. (i) If $b_l \in S_u$ and $b_m \not\in S_u$, then $u$ is adjacent to $b_l$ and non-adjacent to $b_m$.

We discuss four possible cases:

1. If both $b_l, b_m \not\in S_{g(u)}$, then $g(u)$ is non-adjacent $b_m = g(b_l)$, a contradiction.
2. If both $b_l, b_m \in S_{g(u)}$, then $g(u)$ is adjacent to $b_l = g(b_m)$, a contradiction.
3. If $b_l \in S_{g(u)}$ and $b_m \not\in S_{g(u)}$, then $g(u)$ is adjacent to $b_l = g(b_m)$, a contradiction.
4. If $b_l \not\in S_{g(u)}$ and $b_m \in S_{g(u)}$, then $g(u)$ is adjacent to $b_m = g(b_l)$ and non-adjacent to $b_l = g(b_m)$.

As in Case (4), $g$ is preserving the relation of adjacency and non-adjacency, hence proved.

(ii) Let $b_l, b_m \in S_u$, then $u$ is adjacent to both $b_l$ and $b_m$. We discuss four possible cases:

1. If both $b_l, b_m \not\in S_{g(u)}$, then $g(u)$ is non-adjacent $b_m = g(b_l)$, a contradiction.
2. If both $b_l, b_m \in S_{g(u)}$, then $g(u)$ is adjacent to $b_l = g(b_m)$ and $b_m = g(b_l)$.
3. If $b_l \in S_{g(u)}$ and $b_m \not\in S_{g(u)}$, then $g(u)$ is non-adjacent to $b_m = g(b_l)$, a contradiction.
4. If $b_l \not\in S_{g(u)}$ and $b_m \in S_{g(u)}$, then $g(u)$ is non-adjacent to $b_l = g(b_m)$, a contradiction.

Since, $g$ preserves the relation of adjacency in Case (2), therefore $b_l, b_m \in S_{g(u)}$. Converse part can be proved by the similar arguments.

Lemma 2.7. Let $u, v \in T_i$ for some $i$ ($1 \leq i \leq n - 1$) and $u \sim^g v$ for some $g \in \Gamma(G(\mathbb{V}))$. The following statements hold:

(i) If $b \in S_u \cap S_v$, then $g(b) \in S_u \cap S_v$.

(ii) If $b \in S_u - S_v$, then $g(b) \in S_v - S_u$. 


Proof. (i) Since \( b \in S_u \cap S_v \), therefore \( b \) is adjacent to both \( u \) and \( v \). Suppose on contrary \( g(u) \not\in S_u \cap S_v \), then \( g(u) \) is non-adjacent to either \( u = g(v) \) or \( v = g(u) \), a contradiction.

(ii) Let \( b \in S_u - S_v \), then \( b \) is adjacent to \( u \) and non-adjacent to \( v \). Since \( g \) is an automorphism, therefore \( g(b) \) must be adjacent to \( g(u) = v \) and non-adjacent to \( g(v) = u \). Hence, \( g(b) \in S_v - S_u \). \( \square \)

Lemma 2.8. Let \( u \in V(G(\mathbb{V})) \) and \( b \in S_u \) (or \( b \not\in S_u \)). If \( g \in \Gamma_u \), then \( g(b) \in S_u \) (or \( g(b) \not\in S_u \)).

Proof. Proof follows from the fact that \( b \) and \( u \) are adjacent (non-adjacent), therefore \( g(b) \) must be adjacent (non-adjacent) to \( g(u) = u \). \( \square \)

Lemma 2.9. Let \( \mathbb{V} \) be a vector space of dimension \( n \geq 4 \). Let \( b_l, b_m \in T_1 \) be any two basis vectors where \( m \not= l \) and \( 1 \leq m, l \leq n \). Let \( b_l \sim^g b_m \) for some \( g \in \Gamma(G(\mathbb{V})) \).

(i) Let \( u \in T_2 \) be such that \( S_u = \{b_l, b_m\} \), then \( g \in \Gamma_u \).

(ii) Let \( u \in T_{n-2} \) be such that \( S_u = T_1 - \{b_l, b_m\} \), then \( g \in \Gamma_u \).

(iii) Let \( u, v \in T_{n-1} \) be such that \( S_u = T_1 - \{b_l\} \) and \( S_v = T_1 - \{b_m\} \), then \( u \sim^g v \).

Proof. (i) As \( b_l, b_m \in S_u \), therefore by Lemma 2.6(ii), \( b_l, b_m \in S_g(u) \). Since, \( u \in T_2 \) is the only element of \( T_2 \) which have both \( b_l \) and \( b_m \) in its skeleton, therefor \( g(u) = u \).

(ii) As \( b_l, b_m \not\in S_u \), therefore by Lemma 2.6(ii), \( b_l, b_m \not\in S_g(u) \). Since \( u \) is the only element of \( T_{n-2} \), which does not have both \( b_l \) and \( b_m \) in its skeleton. Therefore, \( g(u) = u \).

(iii) As \( b_m \in S_u \) and \( b_l \not\in S_u \), therefore by Lemma 2.6(i), \( b_m \not\in S_g(u) \) and \( b_l \in S_g(u) \). Since \( v \) is the only element of \( T_{n-1} \), which have \( b_l \) and does not have \( b_m \) in its skeleton, therefore \( u \sim^g v \). \( \square \)

3. The Fixing neighborhood of non-zero component graph

A vertex \( x \in S(G) \) is said to fix a pair \((u, v) \in S(G) \times S(G)\), if \( O_x(u) \neq O_x(v) \) in \( G \). For \((u, v) \in S(G) \times S(G)\), the set \( fix(u, v) = \{ x \in S(G) : O_x(u) \neq O_x(v) \} \) is called the fixing neighborhood of pair \((u, v) \). For any two distinct vertices \( u \) and \( v \) in \( G \) with \( O(u) \neq O(v) \), \( fix(u, v) = \emptyset \).

Lemma 3.1. Let \( b_l, b_m \in T_1 \) be two distinct basis vectors of a vector space \( \mathbb{V} \) of dimension \( n \geq 3 \) over the field of 2 elements and \( u \in V(G(\mathbb{V})) \) be such that either \( b_l \in S_u \) or \( b_m \in S_u \). If \( g \in \Gamma_u(G(\mathbb{V})) \), then \( b_l \not\sim^g b_m \). Moreover, \( fix(b_l, b_m) = \{ u \in S(G(\mathbb{V})) : either b_l \in S_u or b_m \in S_u \} \).

Proof. Without loss of generality, assume \( b_l \in S_u \) and \( b_m \not\in S_u \), then \( u = g(u) \) is adjacent to \( b_l \) and non-adjacent \( b_m \). Suppose on contrary \( b_l \sim^g b_m \), then \( b_m = g(b_l) \) must be adjacent to \( u = g(u) \), a contradiction. Hence \( b_l \not\sim^g b_m \) and consequently, \( fix(b_l, b_m) = \{ u \in S(G(\mathbb{V})) : either b_l \in S_u or b_m \in S_u \} \). \( \square \)
Theorem 3.2. Let $\mathbb{V}$ be a vector space of dimension $n \geq 3$ over the field of 2 elements and $G(\mathbb{V})$ be its non-zero component graph. Let $u, v \in T_i$, for some $i$ $(1 \leq i \leq n-1)$, then $\text{fix}(u, v) = \{w \in S(G(\mathbb{V})) : |S_w \cap S_u| \neq |S_w \cap S_v|\}$.

Proof. Let $w \in S(G(\mathbb{V}))$ be such that $|S_w \cap S_u| \neq |S_w \cap S_v|$, then there are two possible cases:

Case 1: Either $S_w \cap S_u = \emptyset$ or $S_w \cap S_u = \emptyset$. Without loss of generality, consider $S_w \cap S_u = \emptyset$. Let $g \in \Gamma_w$. We claim that $O_w(u) \neq O_w(v)$ i.e., $u \not\sim^g v$. Suppose on contrary $u \sim^g v$. Since $S_w \cap S_v \neq \emptyset$, therefore let $b \in S_w \cap S_v$. Then $b \in S_v - S_u$. By Lemma 2.7(ii), $g(b) \in S_u - S_v$. Also $g(b) \in S_w$ by Lemma 2.8. Hence $g(b) \in S_u \cap S_w$, a contradiction that $S_w \cap S_u = \emptyset$.

Case 2: Both $S_w \cap S_u \neq \emptyset$ and $S_w \cap S_v \neq \emptyset$. Since $|S_u| = |S_v|$ and $|S_u \cap S_v| \neq |S_w \cap S_v|$, therefore $|S_u \cup S_w| \neq |S_v \cup S_w|$ and hence $|S_u - (S_u \cup S_w)| \neq |S_v - (S_v \cup S_w)|$. Moreover, $S_u - \{S_u \cup S_w\}$ and $S_v - \{S_v \cup S_w\}$ are disjoint sets of basis vectors and at least one of them is non-empty. Without loss of generality assume $S_u - \{S_u \cup S_w\}$ is non-empty and let $b \in S_u - \{S_u \cup S_w\}$. For $g \in \Gamma_w$, we claim that $O_w(u) \neq O_w(v)$ i.e., $u \not\sim^g v$. Suppose on contrary $u \sim^g v$. Since $b \in S_v$ and $b \notin S_u \cup S_w$, therefore $b \in S_v - S_u$ and $b \notin S_w$. By Lemma 2.7(ii), $g(b) \in S_u - S_v$ and by Lemma 2.8 $g(b) \notin S_w$. Thus $g(b) \in S_u - \{S_v \cup S_w\}$. Thus, if $b \in S_v - \{S_u \cup S_w\}$, then $g(b) \in S_u - \{S_v \cup S_w\}$, which is a contradiction as $S_u - \{S_v \cup S_w\} \cap S_v = \emptyset$ and $S_u - \{S_v \cup S_w\}$ are disjoint sets with $|S_u - \{S_v \cup S_w\}| \neq |S_u - \{S_v \cup S_w\}|$ and $g$ is a bijective function. \qed

Lemma 3.3. Let $G(\mathbb{V})$ be the non-zero component graph of a vector space $\mathbb{V}$ of dimension $n \geq 3$ over the field of 2 elements. Let $u, v \in T_i$ for some $i$ $(1 \leq i \leq n-1)$, then we have:

(i) $\text{fix}(u, v) \cap \{S_u \cap S_v\} = \emptyset$

(ii) $\text{fix}(u, v) \cap T_i = \{S_u \cup S_v\} - \{S_u \cap S_v\}$.

Proof. (i) Let $b \in \{S_u \cap S_v\}$, then $S_b = \{b\} \subset \{S_u \cap S_v\}$. Since $|S_u| = |S_v|$, therefore $|S_b \cap S_u| = |S_b \cap S_v|$. Hence by Theorem 3.2 $b \notin \text{fix}(u, v)$. Thus $\{S_u \cap S_v\} \notin \text{fix}(u, v)$.

Conversely, let $b \in \text{fix}(u, v)$ is a basis vector. Since $|S_u| = |S_v|$ and by Theorem 3.2 $|S_b \cap S_u| \neq |S_b \cap S_v|$ implies that $\{b\} = S_b \not\subset \{S_u \cap S_v\}$, thus $b \not\in \{S_u \cap S_v\}$. Therefore, $\text{fix}(u, v) \not\subset \{S_u \cap S_v\}$. Hence $\text{fix}(u, v) \not\subset \{S_u \cap S_v\}$. Therefore $\text{fix}(u, v) \not\subset \{S_u \cap S_v\}$. Hence $\text{fix}(u, v) \not\subset \{S_u \cap S_v\}$.

(ii) Let $b \in \text{fix}(u, v) \cap T_i$, then by Theorem 3.2 $|S_b \cap S_u| \neq |S_b \cap S_v|$, also $|S_u| = |S_v|$, which implies that at least one of $S_b \cap S_u$ and $S_b \cap S_v$ is non-empty. Also by (i), $b \notin S_u \cap S_v$. Thus either $b \in S_u - S_v$ or $b \in S_v - S_u$. Hence, $b \in \{S_u \cup S_v\} - \{S_u \cap S_v\}$. Conversely, let $b \in \{S_u \cup S_v\} - \{S_u \cap S_v\}$. Then either $S_b \subset S_u$ or $S_b \subset S_v$. In both cases $|S_b \cap S_u| \neq |S_b \cap S_v|$ as $|S_u| = |S_v|$. Hence by Theorem 3.2 $b \in \text{fix}(u, v)$. Since $b$ is basis vector, therefore $b \notin \text{fix}(u, v) \cap T_i$. \qed

Theorem 3.4. Let $G(\mathbb{V})$ be the non-zero component graph of a vector space $\mathbb{V}$ of dimension $n \geq 3$ over the field of 2 elements. Let $u, v \in T_i$ for some $i$ $(1 \leq i \leq n-1)$,
then

\[ fix(u, v) = fix(u - \sum_{b \in S_u \cap S_v} b, v - \sum_{b \in S_u \cap S_v} b). \]

**Proof.** Let \( w \in fix(u, v) \), then by Theorem 3.2

\[ |S_u \cap S_w| \neq |S_v \cap S_w| \iff S_u \cap S_w \neq S_v \cap S_w \]

\[ \iff \{S_u \cap S_w\} - \{S_u \cap S_v\} \neq \{S_v \cap S_w\} - \{S_u \cap S_v\} \]

\[ \iff \{S_u - (S_u \cap S_w)\} \cup \{S_w - (S_u \cap S_v)\} \neq \{S_v - (S_u \cap S_v)\} \cap \{S_w - (S_u \cap S_v)\}. \]

Since \( fix(u, v) \cap \{S_u \cap S_v\} = \emptyset \) by Lemma 3.3(i). Therefore,

\[ \iff \{S_u - (S_u \cap S_v)\} \cap S_w \neq \{S_v - (S_u \cap S_v)\} \cap S_w \]

\[ \iff |\{S_u - (S_u \cap S_v)\} \cap S_w| \neq |\{S_v - (S_u \cap S_v)\} \cap S_w|. \] Hence by Theorem 3.2

\[ \iff w \in fix(u - \sum_{b \in S_u \cap S_v} b, v - \sum_{b \in S_u \cap S_v} b). \]

\[ \square \]

**Theorem 3.5.** Let \( G(\mathbb{V}) \) be the non-zero component graph of a finite dimensional vector space \( \mathbb{V} \) of dimension \( n \geq 3 \) over the field of 2 elements. Let \( u, v \in T_i \) for some \( i' \), \( 1 \leq i' \leq n - 1 \), such that \( S_u \cap S_v = \emptyset \). Then

\[ |fix(u, v) \cap T_i| = \binom{n}{i} - \sum_{0 \leq j \leq i', \, 0 \leq k \leq n-2i'} \binom{i'}{j} \binom{n-2i'}{k} - \binom{n-2i'}{i}, \]

where \( j, k \) are two numbers such that \( i = 2j + k \).

**Proof.** First we count the vertices in \( T_i - fix(u, v) \). Let \( w \in T_i - fix(u, v) \) for some \( i \), \( 1 \leq i \leq n - 1 \), then by Theorem 3.2, \( |S_w \cap S_u| = |S_w \cap S_v| \). Since \( |S_u| = |S_v| = i' \), therefore \( |S_w \cap S_u| = |S_w \cap S_v| = j \), where \((0 \leq j \leq i')\). If \( j = 0 \), then \( S_w \cap S_u = S_w \cap S_v = \emptyset \), and \( S_w \) contains \( i \) basis vectors of \( T_1 - \{S_u \cup S_v\} \). Since \( i \) elements out of \( n - 2i' \) elements of \( T_1 - \{S_u \cup S_v\} \) can be chosen in \( \binom{n-i}{i} \) ways, therefore there are \( \binom{n-2i'}{i} \) elements \( w \) in each \( T_i \), \((1 \leq i \leq n-2i')\) such that \( S_w \cap S_u = S_w \cap S_v = \emptyset \) and hence these elements belong to \( T_i - fix(u, v) \). If \( 1 \leq j \leq i' \), then the set \( S_w \) of cardinality \( i \) contains \( j \) vertices from each of two disjoint sets \( S_u \) and \( S_v \) of cardinality \( i' \) if and only if \( i = 2j + k \), where \( k = |S_w - \{S_u \cup S_v\}| \). As \( j \) vertices can be chosen out of \( i' \) vertices of \( S_u \) in \( \binom{i'}{j} \) ways, \( j \) vertices can be chosen out \( i' \) vertices of \( S_w \) in \( \binom{i'}{j} \) ways and the remaining \( k \) vertices of \( S_w \) can be chosen out of \( n - 2i' \) vertices of \( T_1 - \{S_u \cup S_v\} \) in \( \binom{n-2i'}{k} \) ways. Thus, by the fundamental principle of counting, there are \( \binom{i'}{j} \binom{n-2i'}{k} \) vertices \( w \) in each \( T_i \) for each possibility of the numbers \( j \) and \( k \), such that \( |S_w \cap S_u| = |S_w \cap S_v| = j \). Therefore, the number of vertices in each \( T_i \) such that \( |S_w \cap S_u| = |S_w \cap S_v| \neq 0 \) is \( \sum_{0 \leq j \leq i', \, 0 \leq k \leq n-2i'} \binom{i'}{j} \binom{n-2i'}{k} \). Hence, the proof follows by the fact that the cardinality of \( T_i \) for each \( i \), \( 1 \leq i \leq n - 1 \) is \( \binom{n}{i}. \) \( \square \)
Corollary 3.6. Let $G(\mathbb{V})$ be the non-zero component graph of a finite dimensional vector space $\mathbb{V}$ of dimension $n \geq 3$ over the field of 2 elements. Let $u, v \in T_r$ for some $i', (2 \leq i' \leq n - 1)$, such that $S_u \cap S_v \neq \emptyset$. If $r = i' - |S_u \cap S_v|$, then

$$|\text{fix}(u, v) \cap T_i| = \binom{n}{i} - \sum_{0 \leq j \leq r} \sum_{0 \leq k \leq n-2r} \binom{r}{j} \binom{n-2r}{k} - \binom{n-2r}{i},$$

where $j, k$ are two numbers such that $i = 2j + k$.

Proof. Since the skeletons of $u - \sum_{b \in S_u \cap S_e} b$ and $v - \sum_{b \in S_u \cap S_e} b$ are disjoint and $u - \sum_{b \in S_u \cap S_e} b, v - \sum_{b \in S_u \cap S_e} b \in T_r$. Therefore by Theorem 3.5

$$|\text{fix}(u - \sum_{b \in S_u \cap S_e} b, v - \sum_{b \in S_u \cap S_e} b) \cap T_i| = \binom{n}{i} - \sum_{0 \leq j \leq r} \sum_{0 \leq k \leq n-2r} \binom{r}{j} \binom{n-2r}{k} - \binom{n-2r}{i}. \quad \text{Also by Theorem 3.4}$$

$$|\text{fix}(u, v) \cap T_i| = |\text{fix}(u - \sum_{b \in S_u \cap S_e} b, v - \sum_{b \in S_u \cap S_e} b) \cap T_i|. \quad \text{Hence proved.}$$

4. The fixed number of graph

In this section, we give a realizable result about the existence of a graph in terms of the difference of fixed number and fixing number of the graph. We find an upper bound on the size of fixing graph of a graph $G$. We also find the fixed number of non-zero component graph.

The open neighbourhood of a vertex $u$ in a graph $G$ is $N_G(u) = \{v \in V(G) : v$ is adjacent to $u$ in $G\}$. The fixed number of a graph $G$, $\text{fxd}(G)$, is the minimum number $k$ such that every subset of vertices of $G$ of cardinality $k$ is a fixing set of $G$. Note that $0 \leq \text{fxd}(G) \leq \text{fxd}(G) \leq |V(G)| - 1$.

In the next theorem, we will prove the existence of a graph $G$ for a given positive integer $N$ such that $\text{fxd}(G) - \text{fix}(G) \geq N$.

Theorem 4.1. For every positive integer $N$, there exists a graph $G$ such that $\text{fxd}(G) - \text{fix}(G) \geq N$.

Proof. We choose $k \geq \max\{3, \frac{N+3}{2}\}$. Let $G$ be a bipartite graph and $V(G) = U \cup W$, where $U = \{u_1, ..., u_{2^k-2}\}$ and ordered set $W = \{w_1, w_2, ..., w_{k-1}\}$ and both $U$ and $W$ are disjoint. Before defining adjacency, we assign coordinates to each vertex of $U$ by expressing each integer $j$ ($1 \leq j \leq 2^k - 2$) in its base 2 (binary) representation. We assign each $u_j$ ($1 \leq j \leq 2^k - 2$) the coordinates $(a_{k-1}, a_{k-2}, ..., a_0)$ where $a_m$ ($0 \leq m \leq k - 1$) is the value in the $2^m$ position of binary representation of $j$. For integers $i$ ($1 \leq i \leq k - 1$) and $j$ ($1 \leq j \leq 2^k - 2$), we join $w_i$ and $u_j(a_{k-1}, a_{k-2}, ..., a_0)$ if and only if $i = \sum_{m=0}^{k-1} a_m$. This completes the construction of graph $G$.

Next we will prove that $\text{fix}(G) = 2^k - (k + 1)$. We denote $N(w_i) = \{u_j \in U : u_j$ is adjacent to $w_i, (1 \leq j \leq 2^k - 2)\}$ and it is obvious to see that $N(w_i) \cap N(w_j) = \emptyset$ as
if \( u(a_{k-1}, a_{k-2}, \ldots, a_0) \in N(w_i) \cap N(w_j) \), then \( i = \sum_{m=0}^{k-1} a_m = j \). Number of vertices in each \( N(w_i) \) is the permutation of \( k \) digits in which digit 1 is appears \( i \) times and digit 0 appears \( (k-i) \) times, hence \( |N(w_i)| = \binom{k}{i} \). As \( N(w_i) \cap N(w_j) = \emptyset \), so a fixing set \( D \) with the minimum cardinality must have \( \binom{k}{i} - 1 \) vertices from each \( N(w_i) \) \((1 \leq i \leq k-1)\), for otherwise if \( u, v \in N(w_i) \) and \( u, v \notin D \) for some \( i \), then there exists an automorphism \( g \in \Gamma(G) \) such that \( g(u) = v \) because \( u \) and \( v \) have only one common neighbor \( w_i \), which is a contradiction that \( D \) is a fixing set. Moreover \( D \subseteq U \) as each \( w_i \) \((0 \leq i \leq k-1)\), is fixed while fixing at least \( \binom{k}{i} - 1 \) vertices in each \( N(w_i) \). Hence,

\[
fix(G) = \sum_{i=1}^{k-1} \binom{k}{i} - (k - 1) = 2^k - (k + 1)
\]

Next we will find \( fxd(G) \). As order of \( G \) is \( 2^k + k - 3 \) and set of all vertices of \( G \) except one vertex forms a fixing set of \( G \). It can be seen that \( fxd(G) = 2^k + k - 4 \), for otherwise if \( fxd(G) < 2^k + k - 4 \) and \( u, v \in N(w_i) \) for some \( i \), then the set \( E = W \cup U \setminus \{u, v\} \) consisting of \( 2^k + k - 5 \) is not a fixing set, which implies that \( fxd(G) = 2^k + k - 4 \). Hence for the graph \( G \), we have \( fxd(G) - fix(G) = 2k - 3 \geq N \) as required.

Javaid et al. [24] defined the fixing graph of a graph \( G \). The authors gave a method for finding fixing number of a graph with the help of its fixing graph. The authors found a lower bound on the size of fixing graph of a graph for which \( fxd(G) = fix(G) \). We give an upper bound on the size of fixing graph of such a graph. The fixing graph, \( F(G) \), of a graph \( G \) is a bipartite graph with bipartition \((S(G), V_s(G))\) and a vertex \( x \in S(G) \) is adjacent to a pair \((u, v) \in V_s(G) \) if \( x \in fix(u, v) \). For a set \( D \subseteq S(G) \), \( N_{F(G)}(D) = \{(u, v) \in V_s(G) : x \in fix(u, v) \text{ for some } x \in D \} \). In the fixing graph, \( F(G) \), the minimum cardinality of a subset \( D \) of \( V(G) \) such that \( N_{F(G)}(D) = V_s(G) \) is the fixing number of \( G \).

**Theorem 4.2.** Let \( G \) be a graph of order \( n \geq 2 \) such that \( fix(G) = fxd(G) = k \), then

\[
|E(F(G))| \leq n(\binom{n}{2} - k + 1).
\]

**Proof.** Let \( |S(G)| = r \) and \( |V_s(G)| = s \), then \( r \leq n \) and \( s \leq \binom{n}{2} \leq \binom{n}{2} \). Let \( v \in S(G) \). We will prove that \( \deg_{F(G)}(v) \leq s - k + 1 \). Suppose \( \deg_{F(G)}(v) \geq s - k + 2 \), then there are at most \( k - 2 \) pairs in \( V_s(G) \) which are not adjacent to \( v \). Let \( V_s(G) \setminus N_{F(G)}(v) = \{(u_1, v_1), (u_2, v_2), \ldots, (u_t, v_t)\} \), where \( t \leq k - 2 \). Note that, \( u_i \) fixes \( (u_i, v_i) \) for each \( i \) with \( 1 \leq i \leq t \). Hence, \( u_i \) is adjacent to pair \((u_i, v_i) \) in \( F(G) \) for each \( i, 1 \leq i \leq t \) and so \( N_{F(G)}(\{v, u_1, u_2, \ldots, u_t\}) = V_s(G) \). Hence, \( fix(G) \leq t + 1 \leq k - 1 \), which is a contradiction. Thus, \( \deg_{F(G)}(v) \leq s - k + 1 \leq \binom{n}{2} - k + 1 \), and consequently, \( |E(F(G))| \leq n(\binom{n}{2} - k + 1) \).
A set of vertices \(A \subset V(G)\) is referred as non-fixing set if \(\Gamma_A(G) \setminus \{id\} \neq \emptyset\). The following remark is useful in finding the fixed number of non-zero component graph of a graph \(G\).

**Remark 4.3.** [24] Let \(G\) be a graph of order \(n\). If \(r, (1 \leq r \leq n - 2)\) is the largest cardinality of a non-fixing subset of \(G\), then \(f_xd(G) = r + 1\).

The following result gives the fixed number of non-zero component graph.

**Theorem 4.4.** Let \(G(\mathbb{V})\) be the non-zero component graph of a finite dimensional vector space \(\mathbb{V}\) of dimension \(n \geq 3\) over the field of 2 elements, then \(f_xd(G(\mathbb{V})) = 2^{n-1}\).

**Proof.** We choose a pair of basis vectors \(b_1, b_m \in T_1\) and define a set \(A = \{x \in V(G(\mathbb{V})) : \text{either } b_1, b_m \in S_x \text{ or } b_1, b_m \notin S_x\}\). Take \(B = V(G(\mathbb{V})) - A = \{y \in V(G(\mathbb{V})) : \text{either } b_1 \notin S_y, b_m \notin S_y \text{ or } b_1 \notin S_y, b_m \in S_y\}\). We choose two vertices \(u, v \in B \cap T_i\) for some \(i (1 \leq i \leq n)\), such that \(S_u - S_v = \{b_1\}\) and \(S_v - S_u = \{b_m\}\). This implies \(\{S_u - S_v\} \cup \{b_m\} = \{S_v - S_u\} \cup \{b_1\}\). We prove that \(A\) is a non-fixing set of \(G(\mathbb{V})\) by proving that there exist a non-trivial \(g \in \Gamma_A\) such that \(u \sim^g v\). Now let \(w \in A\) be an arbitrary vertex of set \(A\). We discuss two cases of \(w\).

**Case 1:** If \(b_1, b_m \in S_w\), then \(b_1 \in S_w, b_m \in S_w\). Also \(\{S_w - S_u\} \cup \{b_m\} = \{S_w - S_u\} \cup \{b_1\}\) implies \(|\{S_w - S_u\} \cup \{b_m\} - \{S_w - S_u\} \cup \{b_1\}| = \emptyset\). This implies \(|S_w - S_u| = |S_w - S_v|\). Therefore by Theorem 3.2, \(w \notin fix(u, v)\) and \(u \sim^g v\) for some \(g \in \Gamma_w\).

**Case 2:** If \(b_1, b_m \notin S_w\), then \(S_w - S_u = S_w - S_v\) implies \(|S_w - S_u| = |S_w - S_v|\). Therefore by Theorem 3.2, \(w \notin fix(u, v)\) and \(u \sim^g v\) for some \(g \in \Gamma_w\).

Since the choice of \(w\) is arbitrary in \(A\). Therefore, \(\Gamma_A - \{id\} \neq \emptyset\). Hence, \(A\) is a non-fixing set of \(G(\mathbb{V})\).

Next we prove that any superset of set \(A\) is a fixing set of \(G(\mathbb{V})\). Let \(z \in B\) be an arbitrary vertex and \(A' = A \cup \{z\}\). Without loss of generality assume \(b_1 \in S_z\) and \(b_m \notin S_z\). We choose \(u, v \in B \setminus \{z\}\) such that \(S_u - S_v = \{b_1\}\) and \(S_v - S_u = \{b_m\}\), then \(b_1 \in S_z, b_m \notin S_z\). This implies \(|S_z - S_u| = |S_z - S_v|\). Therefore by Theorem 3.2, \(z \in fix(b_1, b_m)\). In particular \(z \in fix(b_1, b_m)\), thus \(b_1 \sim^g b_m\) where \(g \in \Gamma_{A'}\). Hence \(\Gamma_{A'}\) is trivial and \(A'\) is a fixing set of \(G(\mathbb{V})\). Thus \(A\) is non-fixing set of \(G(\mathbb{V})\) with the largest cardinality.

Since \(T_i\) for each \(i (2 \leq i \leq n)\), contains \(\binom{n-2}{i-2}\) elements which have both \(b_i\) and \(b_m\) in their skeletons. Also \(T_i\) for each \(i (1 \leq i \leq n - 2)\), contains \(\binom{n-2}{i}\) elements which do not have both \(b_i\) and \(b_m\) in their skeletons. Therefore, \(|A| = \sum_{i=1}^{n-2} \binom{n-2}{i} + \sum_{i=2}^{n} \binom{n-2}{i-2} = 2^{n-1} - 1 + 2^{n-2} = 2^{n-1} - 1\). Hence by Remark 4.3, \(f_xd(G(\mathbb{V})) = 2^{n-1}\).

**Theorem 4.5.** [24] Let \(G\) be a connected graph of order \(n\). Then \(f_xd(G) = n - 1\) if and only if \(N(v) \setminus \{u\} = N(u) \setminus \{v\}\) for some \(u, v \in V(G)\).
Let $\mathbb{V}$ be a vector space of dimension $n$ over the field of $q \geq 3$ elements and $G(\mathbb{V})$ be its non-zero component graph. Then $T_i$, for each $i$ ($1 \leq i \leq n$), has $\binom{n}{i}$ sets of vertices each of cardinality $(q-1)^i$ whose skeletons are same. Hence their neighborhoods are also same. Thus each of these $\binom{n}{i}$ sets of vertices form twin classes of vertices. Thus from Theorem 4.5, we have the following result:

**Theorem 4.6.** Let $G(\mathbb{V})$ be a non-zero component graph of a vector space $\mathbb{V}$ of dimension $n$ over the field of $q \geq 3$ elements, then $f\text{xd}(G(\mathbb{V})) = |G(\mathbb{V})| - 1$.

**Proof.** Since $G(\mathbb{V})$ has twin vertices, therefor result follows by Theorem 4.5. $\square$

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Centre for advanced studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan

Email: hira_benish@yahoo.com, imran.javaid@bzu.edu.pk, mahru830@gmail.com.