A geometric $q$-character formula for snake modules

Bing Duan and Ralf Schiffler

Dedicated to Bernard Leclerc on the occasion of his 60th birthday.

Abstract

Let $C$ be the category of finite-dimensional modules over the quantum affine algebra $U_q(\hat{g})$ of a simple complex Lie algebra $g$. Let $C^-$ be the subcategory introduced by Hernandez and Leclerc. We prove the geometric $q$-character formula conjectured by Hernandez and Leclerc in types $A$ and $B$ for a class of simple modules called snake modules introduced by Mukhin and Young. Moreover, we give a combinatorial formula for the $F$-polynomial of the generic kernel associated to the snake module. As an application, we show that snake modules correspond to cluster monomials with square-free denominators and we show that snake modules are real modules. We also show that the cluster algebras of the category $C_1$ are factorial for Dynkin types $A, D, E$.

1. Introduction

Let $g$ be a simple complex Lie algebra and let $U_q(\hat{g})$ be the corresponding quantum affine algebra with quantum parameter $q \in \mathbb{C}^\times$ not a root of unity. Denote by $C$ the category of finite-dimensional $U_q(\hat{g})$-modules. The simple modules in $C$ have been classified in [4, 5] by Chari and Pressley in terms of Drinfeld polynomials. In [14], Frenkel and Reshetikhin attached a $q$-character to every module in $C$ and showed that the simple modules are determined up to isomorphism by their $q$-characters. Moreover, the simple modules are parametrized by the highest dominant monomials in their $q$-characters.

Cluster algebras were introduced in [11] by Fomin and Zelevinsky as a tool for studying canonical bases in Lie theory. A cluster algebra is a commutative algebra with a distinguished set of generators, the cluster variables. These cluster variables are constructed by a recursive method called mutation, which is determined by the choice of a quiver $Q$ without loops and 2-cycles. Given a cluster algebra $\mathcal{A}(Q)$, every cluster variable can be expressed as a Laurent polynomial with integer coefficients with respect to any given cluster [11] and this Laurent polynomial has positive coefficients [23]. A cluster monomial is a product of cluster variables from the same cluster. It was proved in [2] that the set of all cluster monomials is linearly independent.

1.1. Category $C_1$

A connection between representations of quantum affine algebras and cluster algebras was discovered by Hernandez and Leclerc in [18], where a monoidal categorification of certain cluster algebras was given. One significant aspect of a monoidal categorification is that, if it
exists, it implies the positivity of the cluster variables and the linear independence of the cluster monomials, see [18, Proposition 2.2].

The monoidal categorification is realized as a subcategory of the category $\mathcal{C}$ as follows. Let $I$ be the vertex set of the Dynkin diagram of $\mathfrak{g}$ for Dynkin types $\Delta, \Delta, \Delta$ and $I = I_0 \cup I_1$ be a partition of $I$ such that every edge connects a vertex of $I_0$ with a vertex of $I_1$. Let $\mathcal{C}_0$, $\mathcal{C}_1$, $\ell \geq 0$, be the full subcategory of $\mathcal{C}$ whose objects $V$ satisfy the following property. For any composition factor $S$ of $V$ and every $i \in I$, the roots of the $i$th Drinfeld polynomial of $S$ belong to $\{q^{-2k-\xi_i} | 0 \leq k \leq \ell \}$, where $\xi_i = 1$ if $i \in I_1$ and $\xi_i = 0$ if $i \in I_0$.

For Dynkin types $\Delta$ and $\Delta_4$, it has been shown in [18] that the category $\mathcal{C}_1$ is a monoidal categorification of a cluster algebra of the same Dynkin type. This result was extended to Dynkin types $\Delta, \Delta, \Delta$ by Nakajima in [27], see also [19]. In [31], Qin proved that every cluster monomial corresponds to a simple module in $\mathcal{C}$ for Dynkin types $\Delta, \Delta, \Delta$.

A simple module $M$ in $\mathcal{C}$ is said to be real if $M \otimes M$ is simple [21], and $M$ is said to be prime if it cannot be written as a non-trivial tensor product of modules [7].

1.2. Category $\mathcal{C}^-$

In [20], Hernandez and Leclerc considered a much larger subcategory $\mathcal{C}^-$ of $\mathcal{C}$ which contains, up to spectral shifts, all the simple finite-dimensional $U_q(\mathfrak{g})$-modules. They showed that the Grothendieck ring of $\mathcal{C}^-$ has a cluster algebra structure [20, Theorem 5.1], and they proposed two conjectures.

**Conjecture 1.1** [20, Conjecture 5.2]. The cluster monomials of the cluster algebra are in bijection with the isomorphism classes of real simple objects in $\mathcal{C}^-$. The second conjecture uses the theory of quivers with potentials developed in [8, 9]. In [20, Section 5.2.2], Hernandez and Leclerc associated to every simple $U_q(\mathfrak{g})$-module $M$ a so-called *generic kernel* $K(M)$, which is a module over the Jacobian algebra of the quiver with potential. They showed that, up to normalization, the truncated $q$-character of a Kirillov–Reshetikhin module is equal to the $F$-polynomial of the associated generic kernel, and they conjectured the following generalization.

**Conjecture 1.2** [20, Conjecture 5.3]. Up to normalization, the truncated $q$-character of a real simple module in $\mathcal{C}^-$ is equal to the $F$-polynomial of the associated generic kernel.

1.3. Snake modules

In this paper, we prove both conjectures in Dynkin types $\Delta$ and $\Delta$ for snake modules, a class of simple $U_q(\mathfrak{g})$-modules introduced by Mukhin and Young in [24, 25]. In [24], they introduced a purely combinatorial method to compute $q$-characters for snake modules of types $\Delta$ and $\Delta$, and in [25], they used snake modules to construct extended $T$-systems for types $\Delta$ and $\Delta$. In [10], it was shown that all prime snake modules are real and that they correspond to some cluster variables in the cluster algebra constructed by Hernandez and Leclerc.

Our first main theorem is the following.

**Theorem 1.3** (Theorem 3.2 and Remark 3.4). Let $L(m)$ be a prime snake module in $\mathcal{C}^-$. Then up to normalization, the truncated $q$-character of $L(m)$ is equal to the $F$-polynomial of the associated generic kernel $K(m)$. More precisely,

$$\chi_q(L(m)) = mF_{K(m)}.$$ 

Replacing the module $K(m)$ by a direct sum, we obtain a similar geometric character formula for arbitrary snake module of types $\Delta$ and $\Delta$. 

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This proves Conjecture 1.2 for snake modules and gives a geometric algorithm for the truncated $q$-characters. As a slight generalization of Theorem 3.4 of [10], we show in Theorem 4.2 that snake modules are real modules.

Then in Theorem 3.9 and Remark 3.10 (3), we give a combinatorial formula for the $F$-polynomial of the generic kernel $K(m)$ associated to a snake module $L(m)$ as a sum over certain non-overlapping paths in a subset of the $\mathbb{Z} \times \mathbb{Z}$-grid determined by $m$. This result uses the model of Mukhin and Young [24, 25]. As a consequence, we also obtain a combinatorial method to find the dimension vector of $K(m)$ as well as all its submodules. Furthermore, we show that $K(m)$ is always rigid and it is indecomposable if the snake module $L(m)$ is prime.

As an application, we prove Conjecture 1.1 for snake modules in the first part of the following theorem.

Theorem 1.4 (Theorems 4.1 and 4.4). The truncated $q$-character of a snake module $L(m)$ is a cluster monomial. Moreover the denominator of this cluster monomial is square free and is parametrized by the support of $K(m)$ as a representation of the quiver with potential.

It is natural to ask whether all cluster variables with square-free denominators correspond to snake modules. This is not the case. However, the only counter examples we found are modules whose truncated $q$-characters are not equal to their ordinary $q$-characters.

Finally, the study of square-free denominators led us to questions of factorization in the cluster algebra. Applying the results of [16], we include a proof that the $\mathcal{C}_1$ cluster algebras of Hernandez–Leclerc are factorial for Dynkin types $A, D, E$.

This paper is organized as follows. In Section 2, we briefly review basic materials on cluster algebras, quantum affine algebras, snake modules, and Hernandez and Leclerc’s results. Section 3 is on our geometric character formula for snake modules and Section 4 is devoted to the study of denominator vectors of cluster monomials corresponding to snake modules. In the last section, we prove that the $\mathcal{C}_1$ cluster algebras are factorial for Dynkin types $A, D, E$.

2. Preliminaries

2.1. Quivers and cluster algebras

We recall the definition of the cluster algebras introduced in [20]. Let $C = (c_{ij})_{i,j \in I}$ be an indecomposable Cartan matrix of finite type. Then there exists a diagonal matrix $D = \text{diag}(d_i \mid i \in I)$ with positive entries such that $B = DC = (b_{ij})_{i,j \in I}$ is symmetric. Following [20], one normalizes $D$ such that $\min\{d_i \mid i \in I\} = 1$. Let $t = \max\{d_i \mid i \in I\}$. Thus

$$t = \begin{cases} 1 & \text{if } C \text{ is of type } A_n, D_n, E_6, E_7, E_8, \\ 2 & \text{if } C \text{ is of type } B_n, C_n \text{ or } F_4, \\ 3 & \text{if } C \text{ is of type } G_2. \end{cases}$$

Let $\tilde{G}$ be the infinite quiver with vertex set $\tilde{G}_0 = I \times \mathbb{Z}$ and arrows $(i, r) \to (j, s)$ if $b_{ij} \neq 0$ and $s = r + b_{ij} + d_j - d_i$. It needs to be pointed out that $\tilde{G}$ has two isomorphic connected components, see [20, Lemma 2.2]. We pick one of the two components and denote it by $G$ with vertex set $G_0$. We consider the full subquiver $G^-$ of $G$ with vertex set $G_0^- = G_0 \cap (I \times \mathbb{Z}_{\leq 0})$, see Figure 1.

Let $z = \{z_{i,r} \mid (i, r) \in G_0^-\}$ and let $\mathcal{A}$ be the cluster algebra defined by the initial seed $(z, G^-)$. The cluster algebra $\mathcal{A}$ is a cluster algebra of infinite rank.

Let $Y^- = \{Y_{i,r}^{-1} \mid (i, r) \in G_0^-\}$ be a new set of indeterminates over $\mathbb{Q}$. For $(i, r) \in G_0^-$, one defines $k_{i,r}$ to be the unique positive integer $k$ satisfying

$$0 < kb_{ii} - |r| \leq b_{ii}.$$
In other words, \((i, r)\) is the \(k\)th vertex in its column, counting from the top.

For \((i, r) \in G_{0}^{-}\), we perform the substitution

\[
 z_{i,r} = \prod_{j=0}^{k_{i,r}-1} Y_{i,r+jb_{ii}}. 
\] (2.1)

Note that

\[
 \frac{z_{i,r}}{z_{i,r+b_{ii}}} = Y_{i,r}
\]

for \((i, r + b_{ii}) \in G_{0}^{-}\).

Let \(\Gamma\) be the same quiver as \(G\) but with vertex set \(\Gamma_{0} = \{(i, r - d_{i}) : (i, r) \in G_{0}\}\). Let \(\Gamma^{-}\) be the full subquiver of \(\Gamma\) with vertex set \(\Gamma_{0}^{-} = \{(i, r - d_{i}) : (i, r) \in G_{0}^{-}\}\), see Figure 2.

In this paper, we let \(g\) be of type \(A\) or \(B\). We work in the full subcategory \(\mathcal{C}^{-}\) of \(\mathcal{C}\) whose objects have all their composition factors of the form \(L(m)\), where \(m\) is a monomial in the variables \(Y_{i,r} \in Y^{-}\).

### 2.2. Quantum affine algebras

Let \(g\) be a simple complex Lie algebra whose Dynkin diagram has vertex set \(I\) and \(h^\vee\) be the dual Coxeter number of \(g\), see Table 1. Let \(\hat{g}\) be the corresponding untwisted affine Lie algebra which is realized as a central extension of the loop algebra \(g \otimes \mathbb{C}[t, t^{-1}]\). Let \(U_q(\hat{g})\) be

| \(g\)  | \(A_n\) | \(B_n\) | \(C_n\) | \(D_n\) | \(E_6\) | \(E_7\) | \(E_8\) | \(F_4\) | \(G_2\) |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(t\) | \(1\)  | \(2\)  | \(2\)  | \(1\)  | \(1\)  | \(1\)  | \(2\)  | \(3\)  |
| \(h^\vee\) | \(n + 1\) | \(2n - 1\) | \(n + 1\) | \(2n - 2\) | 12 | 18 | 30 | 9 | 4 |
the Drinfeld–Jimbo quantum enveloping algebra (quantum affine algebra for short) of $\hat{\mathfrak{g}}$ with parameter $q \in \mathbb{C}^*$ not a root of unity, see [5].

Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra. Recall that a $U_q(\mathfrak{g})$-module $V$ is of type 1 if it is a direct sum of its weight subspaces. A $U_q(\mathfrak{g})$-module $V$ is said to be of type 1 if the central element $c^{1/2}$ acts as the identity on $V$, a nd $V$ is of type 1 as a $U_q(\mathfrak{g})$-module. Let $\mathcal{C}$ be the (abelian tensor) category of finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules of type 1. Every finite-dimensional simple $U_q(\hat{\mathfrak{g}})$-module can be obtained from a type 1 module by twisting with an automorphism of $U_q(\hat{\mathfrak{g}})$, see [5, 6].

Let $K_0(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$. Let $P$ be the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_i^\pm, a, (i, r)) \in \mathbb{C}^\times$. The $q$-character of an object $M$ in $\mathcal{C}$ is defined as an injective ring homomorphism $\chi_q$ from $K_0(\mathcal{C})$ to the ring $\mathbb{Z}P$ of Laurent polynomial in infinitely many formal variables.

In this paper, we will be concerned only with polynomials involving the subset of variables $Y_{i,aq^r}$, $a \in \mathbb{C}^\times$, $(i, r) \in G_0$. For simplicity of notation, we write $Y_{i,r}$ for $Y_{i,aq^r}$.

A monomial in $\mathbb{Z}P$ is called dominant (respectively, anti-dominant) if it does not contain a factor $Y_{i,r}^{-1}$ (respectively, $Y_{i,r}$) with $(i, r) \in \Gamma_0$. Following [14], for $(i, r) \in \Gamma_0$, define

$$v_{i,r} := A_{i,r}^{-1} = Y_{i,r-d_i}^{-1} Y_{i,r+d_i}^{-1} \prod_{j:c_{ij}=-1} Y_{j,r} \prod_{j:c_{ij}=-2} Y_{j,r-1} Y_{j,r+1} \prod_{j:c_{ij}=-3} Y_{j,r-2} Y_{j,r} Y_{j,r+2}, \quad (2.2)$$

where the $c_{ij}$ are the entries of the Cartan matrix. It follows that $A_{i,r}$ is a Laurent monomial in the variables $Y_{j,s}$ with $(j, s) \in G_0$, see [20, Section 2.3.2].

For any simple object $V$ in $\mathcal{C}$, it was shown by Frenkel and Mukhin [13] that the $q$-character can be expressed as

$$\chi_q(V) = m_+ \left( 1 + \sum_p M_p \right),$$

where $m_+ \in \mathbb{Z}P$ is a monomial in the variables $Y_{i,r}$, $(i, r) \in G_0$, with positive powers, hence $m_+$ is a dominant monomial, and each $M_p$ is a product of factors $A_{i,r}^{-1}$, $(i, r) \in \Gamma_0$. The monomial

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The quiver $\Gamma^-$ in type $\mathfrak{A}_3$ (left) and the quiver $\Gamma^-$ in type $\mathfrak{B}_2$ (right).}
\end{figure}
Each step between them can either go up one unit or go down one unit. So the module has no dimension greater than 1.

Every simple object in \( \mathcal{C} \) can be parametrized by the highest weight monomial occurring in its \( q \)-character \([4, 14]\). The highest weight monomial is dominant, but in general the highest weight monomial is not the only dominant monomial occurring in \( q \)-characters. Given a dominant monomial \( m \), one can construct the corresponding simple module \( L(m) \).

A simple module \( L(m) \) is called special or minuscule if \( m \) is the only dominant monomial occurring in \( \chi_q((L(m))) \), see [26, Definition 10.1] or [18, Section 5.2.2]. It is anti-special if there is exactly one anti-dominant monomial occurring in its \( q \)-character. Clearly, a special or anti-special module must be simple. A simple module is called thin if any weight space of the simple module has no dimension greater than 1.

Following [18], define the truncated \( q \)-character \( \chi_q^-((L(m))) \) to be the Laurent polynomial obtained from \( \chi_q(L(m)) \) by deleting all the monomials involving variables \( Y_{i,r} \not\in Y^- \). In other words, \( \chi_q^-((L(m))) \in \mathbb{Z}[Y_{i,r}^\pm | (i,r) \in G_0^-] \). By [20, Proposition 3.10], \( \chi_q^- \) is an injective ring homomorphism from the Grothendieck ring of \( \mathcal{C}^- \) to \( \mathbb{Z}[Y_{i,r}^\pm | (i,r) \in G_0^-] \).

2.3. Paths

Define a subset \( \mathcal{X} \subset I \times \mathbb{Z} \) and an injective mapping \( \iota : \mathcal{X} \to \mathbb{Z} \times \mathbb{Z} \) as follows.

Type \( \mathbb{A}_n \) : Let \( \mathcal{X} := \{(i,k) \in I \times \mathbb{Z} : i - k \equiv 0 \pmod{2}\} \) and \( \iota(i,k) = (i,k) \).

Type \( \mathbb{B}_n \) : Let \( \mathcal{X} := \{(n,2k) : k \in \mathbb{Z}\} \cup \{(i,k) \in I \times \mathbb{Z} : i < n \text{ and } k \equiv 1 \pmod{2}\} \) and

\[
\iota(i,k) = \begin{cases} 
(2i,k), & \text{if } i < n \text{ and } 2n + k - 2i \equiv 1 \pmod{4}, \\
(4n - 2 - 2i,k), & \text{if } i < n \text{ and } 2n + k - 2i \equiv 3 \pmod{4}, \\
(2n - 1,k), & \text{if } i = n.
\end{cases}
\]

Following [24, 25], for every \((i,k) \in \mathcal{X}\), a set \( \mathcal{P}_{i,k} \) of paths is defined as follows. Here a path is a finite sequence of points in the plane \( \mathbb{R}^2 \). We write \((j,\ell) \in p \) if \((j,\ell) \) is a point of the path \( p \). In our diagrams, we connect consecutive points of a path by line segments for illustrative purposes only.

The following is the case of type \( \mathbb{A}_n \). For all \((i,k) \in I \times \mathbb{Z}\), let

\[
\mathcal{P}_{i,k} = \{(0,y_0),(1,y_1),\ldots,(n+1,y_{n+1}) : y_0 = i + k, \\
y_{n+1} = n + 1 - i + k, \text{ and } y_{j+1} - y_j \in \{1,-1\}, 0 \leq j \leq n\}.
\]

In other words, a path in \( \mathcal{P}_{i,k} \) must start at \((0,i+k)\) and end at \((n+1,n+1-i+k)\) and each step between them can either go up one unit or go down one unit. So \(|\mathcal{P}_{i,k}| = \binom{n+1}{2}\).

Note that the cardinality of \( \mathcal{P}_{i,k} \) is equal to the number of Young tableaux that fit in an \( i \times (n+1-i) \) rectangle.

The sets \( C_p^- \) of upper and lower corners of a path \( p = ((r,y_r))_{0 \leq r \leq n+1} \in \mathcal{P}_{i,k} \) are defined as follows (see Figure 3):

\[
C_p^+ = \{(r,y_r) \in p : r \in I, \ y_{r-1} = y_r + 1 = y_{r+1}\},
\]

\[
C_p^- = \{(r,y_r) \in p : r \in I, \ y_{r-1} = y_r - 1 = y_{r+1}\}.
\]

The following is the case of type \( \mathbb{B}_n \). Fix an \( \varepsilon \) such that \( 0 < \varepsilon < 1/2 \), for all \( \ell \in 2\mathbb{Z} \), the set \( \mathcal{P}_{n,\ell} \) is defined as follows.
The sets \( \mathcal{C}_+^{i,k} \) of upper and lower corners of a path \( p = ((j_r, \ell_r))_{0 \leq r \leq |p|-1} \in \mathcal{P}_{i,k} \), where \(|p|\) is the number of points in the path \( p \), are defined as follows:

\[
\mathcal{C}_+^{i,k} = \ell^{-1}\{(j_r, \ell_r) \in p : j_r \notin \{0, 2n-1, 4n-2\}, \ell_{r-1} > \ell_r, \ell_{r+1} > \ell_r\}
\]

\[
\cup \{(n, \ell) \in X : (2n-1, \ell - \epsilon) \notin p \text{ and } (2n-1, \ell + \epsilon) \notin p\},
\]

\[
\mathcal{C}_-^{i,k} = \ell^{-1}\{(j_r, \ell_r) \in p : j_r \notin \{0, 2n-1, 4n-2\}, \ell_{r-1} < \ell_r, \ell_{r+1} < \ell_r\}
\]

\[
\cup \{(n, \ell) \in X : (2n-1, \ell - \epsilon) \notin p \text{ and } (2n-1, \ell + \epsilon) \notin p\}.
\]

These definitions are illustrated in Figures 4 and 5.

In order to subsequently describe our \( F \)-polynomials, it is helpful to define the notion of \textit{cell}. We call a region in \( \mathcal{P}_{i,k} \) a cell if it is a minimal region enclosed by paths. For every cell, we define the coordinate of the cell as follows. If the cell is a square or a square missing a corner, its coordinate is defined as the coordinate of the intersection of two diagonals. If the cell is a right triangle, its coordinate is defined as the coordinate of the midpoint of its hypotenuse. It is obvious that for any \((i, k) \in \mathcal{P}_{i,k}\), our cell coordinate is an element of \( \Gamma_0 \).

We also need the following notations in this paper. For all \((i, k) \in X\), let \( p_{i,k}^+ \) be the highest path which is the unique path in \( \mathcal{P}_{i,k} \) with no lower corners and \( p_{i,k}^- \) the lowest path which
is the unique path in $\mathcal{P}_{i,k}$ with no upper corners. Let $p, p'$ be paths. We say that $p$ is strictly above $p'$ or $p'$ is strictly below $p$ if

$$(x, y) \in p \text{ and } (x, z) \in p' \Rightarrow y < z.$$  

2.4. **Snake modules**

A simple module $L(m)$ is called a Kirillov–Reshetikhin module if $m$ is of the form

$$m = \prod_{j=0}^{k-1} Y_{i,r+jb_{i}}, \quad (i \in I, r \in \mathbb{Z}, k \geq 1),$$  

and is usually denoted by $W_{k,r}^{(i)}$. 

**Figure 4.** In type $B_3$: left, $\mathcal{P}_{3,2}$; right, $\mathcal{P}_{3,0}$. 

**Figure 5.** In type $B_3$: left, $\mathcal{P}_{1,1}$; right, $\mathcal{P}_{2,1}$. 
For completeness we recall the definition of snake module introduced by Mukhin and Young in [24, 25]. Let \((i, k) \in \mathcal{X}\). A point \((i', k') \in \mathcal{X}\) is said to be in snake position with respect to \((i, k)\) if

\[
\text{Type } \Lambda_n : \quad k' - k \geq |i' - i| + 2 \text{ and } k' - k \equiv |i' - i| \pmod{2}.
\]

\[
\text{Type } \mathbb{B}_n : \quad i = i' = n : \quad k' - k \geq 2 \text{ and } k' - k \equiv 2 \pmod{4},
\]

\[
i \neq i' = n \text{ or } i' \neq i = n : \quad k' - k \geq 2|i' - i| + 3 \text{ and } k' - k \equiv 2|i' - i| - 1 \pmod{4},
\]

\[
i < n \text{ and } i' < n : \quad k' - k \geq 2|i' - i| + 4 \text{ and } k' - k \equiv 2|i' - i| \pmod{4}.
\]

The point \((i', k')\) is in minimal snake position to \((i, k)\) if \(k' - k\) is equal to the given lower bound. The point \((i', k')\) is in prime snake position to \((i, k)\) if

\[
\text{Type } \Lambda_n : \quad \min\{2n + 2 - i - i', i + i'\} \geq k' - k \geq |i' - i| + 2 \text{ and } k' - k \equiv |i' - i| \pmod{2}.
\]

\[
\text{Type } \mathbb{B}_n : \quad i = i' = n : \quad 4n - 2 \geq k' - k \geq 2 \text{ and } k' - k \equiv 2 \pmod{4},
\]

\[
i \neq i' = n \text{ or } i' \neq i = n : \quad 2i' + 2i - 1 \geq k' - k \geq 2|i' - i| + 3 \text{ and }
\]

\[
k' - k \equiv 2|i' - i| - 1 \pmod{4},
\]

\[
i < n \text{ and } i' < n : \quad 2i' + 2i \geq k' - k \geq 2|i' - i| + 4 \text{ and } k' - k \equiv 2|i' - i| \pmod{4}.
\]

A finite sequence \((i_t, k_t)\), \(1 \leq t \leq T, T \in \mathbb{Z}_{\geq 0}\), of points in \(\mathcal{X}\) is called a snake (respectively, prime snake, minimal snake) if for all \(2 \leq t \leq T\), the point \((i_t, k_t)\) is in snake position (respectively, prime snake position, minimal snake position) with respect to \((i_{t-1}, k_{t-1})\) [24, 25].

The simple module \(L(m)\) is called a snake module (respectively, prime snake module, minimal snake module) if \(m = \prod_{t=1}^{T} Y_{i_t, k_t}\) for some snake (respectively, prime snake, minimal snake) \((i_t, k_t)_{1 \leq t \leq T}\) [24, 25]. In this case, we say that \((i_t, k_t)_{1 \leq t \leq T}\) is the snake of \(L(m)\).

**Theorem 2.1** [10, Section 4.1]. The snake modules of type \(\Lambda_n\) or \(\mathbb{B}_n\) are precisely the \(U_q(\hat{\mathfrak{g}})\)-modules \(L(m)\) with highest weight monomial

\[
m = \prod_{j=1}^{N} \left( \prod_{s=0}^{k_j - 1} Y_{i_{j}, r + b_{i_{j}, s} + \sum_{t=1}^{j-1} n_{t}} \right),
\]

where \(r \in \mathbb{Z}\), \(i_j \in I\), \(k_j \geq 1\) for \(1 \leq j \leq N\); furthermore, \(n_{\ell} = b_{i_{\ell}, i_{\ell}}(k_{\ell} - 1) + 2 t + t|\ell_{t+1} - i_{\ell}| + 2 t_{\ell} + \varepsilon_{i_{\ell}, i_{\ell+1}}\), where \(j_{\ell} \in \mathbb{Z}_{\geq 0}\) for \(1 \leq \ell \leq N - 1\),

\[
\varepsilon_{i_{j}, j} = \begin{cases} 0 & \text{if } g \text{ is of type } \Lambda_n, \\ -\delta_{i_{j}, j} \delta_{j, n} & \text{if } g \text{ is of type } \mathbb{B}_n, \\ 1 & \text{if } g \text{ is of type } \Lambda_n, \\ 2 & \text{if } g \text{ is of type } \mathbb{B}_n, \\ \end{cases}
\]

where \(\delta_{i, j}\) is the Kronecker delta, and we use the convention \(\sum_{t=1}^{0} n_{t} = 0\). In particular, \(L(m)\) is a prime snake module if \(j_{\ell}\) satisfies the following bounded conditions:

\[
\text{Type } \Lambda_n : \quad 0 \leq j_{\ell} \leq \begin{cases} n - \max\{i_{\ell}, i_{\ell+1}\} & \text{if } i_{\ell} + i_{\ell+1} \geq n + 1; \\
\min\{i_{\ell}, i_{\ell+1}\} - 1 & \text{if } i_{\ell} + i_{\ell+1} < n + 1.
\end{cases}
\]

\[
\text{Type } \mathbb{B}_n : \quad 0 \leq j_{\ell} \leq \min\{i_{\ell}, i_{\ell+1}\} - 1.
\]

**Example 2.2.** (1) Every Kirillov–Reshetikhin module is a snake module, by taking \(N = 1\) in Theorem 2.1.
Every snake module can be uniquely written as a tensor product of prime snake modules (see also [32]). Moreover, the module \( L_p \) mentioned in Lemma 5.8. We say that a non-overlapping property is satisfied by the mapping \( m \) given by a set of \( (p, ..., p_T) \) is non-overlapping if \( p_s \) is strictly above \( p_t \) for all \( 1 \leq s < t \leq T \). Let

\[
\mathcal{P}_{(i_t, k_t)} = \{(p_1, \ldots, p_T) : p_t \in \mathcal{P}_{i_t, k_t}, 1 \leq t \leq T, (p_1, \ldots, p_T) \text{ is non-overlapping}\}.
\]

Mukhin and Young have proved the following theorem.

**Theorem 2.3** [24, Theorem 6.1; 25, Theorem 6.5]. Let \( (i_t, k_t) \in \mathcal{X}, 1 \leq t \leq T \), be a snake of length \( T \in \mathbb{Z}_{\geq 1} \). Then

\[
\chi_q \left( L \left( \prod_{i=1}^{T} Y_{i_t, k_t} \right) \right) = \sum_{(p_1, \ldots, p_T) \in \mathcal{P}_{(i_t, k_t)} \mid 1 \leq t \leq T} \prod_{t=1}^{T} m(p_t),
\]

where the mapping \( m \) is defined by

\[
m : \bigsqcup_{(i, k) \in \mathcal{X}} \mathcal{P}_{i, k} \to \mathbb{Z}[Y^\pm_{j, t}]_{(j, t) \in \mathcal{X}}
\]

\[
p \mapsto m(p) = \prod_{(j, t) \in C^+_{p}} Y_{j, t} \prod_{(j, t) \in C^-_{p}} Y_{j, t}^{-1}.
\]

Moreover, the module \( L(\prod_{t=1}^{T} Y_{i_t, k_t}) \) is thin, special and anti-special.

**Remark 2.4.** A snake module \( L(\prod_{t=1}^{T} Y_{i_t, k_t}) \) is prime if and only if for all \( 2 \leq t \leq T \) the paths \( p_{i_t, k_t} \) and \( p_{i_{t-1}, k_{t-1}} \) are overlapping.

In view of Theorem 2.3, the \( q \)-characters of snake modules of types \( A_n \) and \( B_n \) with length \( T \) are given by a set of \( T \)-tuples of non-overlapping paths, the path in each \( T \)-tuple is non-overlapping. This property is called the non-overlapping property.

For any two paths \( p_1, p_2 \in \mathcal{P}_{i, k}, p_1 \) can be obtained from \( p_2 \) by a sequence of moves, see [24, Lemma 5.8]. We say that \( p_1 \preceq p_2 \) if \( m(p_1) \leq m(p_2) \).

In the following we list some known facts about snake modules.

**Theorem 2.5** [25, Proposition 3.1]. A snake module is prime if and only if its snake is prime. Every snake module can be uniquely written as a tensor product of prime snake modules (up to permutation).

**Theorem 2.6** [10, Theorems 3.4 and 5.9]. Prime snake modules are real and they correspond to some cluster variables in the cluster algebra \( \mathcal{A} \).

Moreover, in [25, Theorem 4.1], Mukhin and Young introduced a set of 3-term recurrence relations satisfied by \( q \)-characters of prime snake modules, called extended T-system, which
generalizes the usual $T$-system. Moreover, in [10, Theorem 4.1], the authors introduced a system of equations satisfied by $q$-characters of prime snake modules, called $S$-system, which contains the usual $T$-system. In fact, the equations in the $S$-system can be interpreted as cluster transformations in the cluster algebra $\mathcal{A}$ where the initial cluster variables correspond to certain Kirillov–Reshetikhin modules.

2.5. Quivers with potentials

Following [20], for every $i \neq j$ with $c_{ij} \neq 0$, and every $(i, r) \in \Gamma_0^-$, we have in $\Gamma^-$ an oriented cycle with length $2 + |c_{ij}|$:

A potential $S$ is defined as the formal (infinite) sum for all these oriented cycles up to cyclic permutations, see [8, Section 3]. Hence in $\Gamma^-$, all the cyclic derivatives of $S$, introduced in [8, Definition 3.1], are finite sums of paths. Indeed, a given arrow of $\Gamma^-$ can only occur in a finite number of summands.

Let $R$ be the set of all cyclic derivatives of $S$. Let $J$ be the two-sided ideal of the path algebra $CG^-\to$ generated by $R$. Following [9, 20], one defines the Jacobian algebra $A = CG^-\to/J$. Then $A$ is an infinite-dimensional $\mathbb{C}$-algebra.

Let $M$ be a finite-dimensional $A$-module, and $e \in \mathbb{N}^{\Gamma_0^-}$ be a dimension vector. Let $Gr_e(M)$ be the quiver Grassmannian of $M$. Thus $Gr_e(M)$ is the variety of submodules of $M$ with dimension vector $e$. This is a projective complex variety. Denote by $\chi(Gr_e(M))$ its Euler characteristic. Following [9, 20], define the $F$-polynomial of $M$ as a polynomial in the indeterminates $v_{i,r}$, $(i, r) \in \Gamma_0^-$, as follows:

$$F_M = \sum_{e \in \mathbb{N}^{\Gamma_0^-}} \chi(Gr_e(M)) \prod_{(i, r) \in \Gamma_0^-} v_{i,r}^{e_{i,r}}.$$  

It was shown in [9] that for any finite-dimensional $M$, $F_M$ is a monic polynomial with constant term equal to 1.

Following [20, Section 4.5.2], let $\ell \in \mathbb{Z}_{<0}$ and let $\Gamma_{\ell}$ be the full subquiver of $\Gamma^-$ with vertex set

$$(\Gamma_0^-)_{\ell} := \{(i, m) \in \Gamma_0^- \mid m \geq \ell\}.$$  

Let $S_{\ell}$ be the sum of all cycles in the potential $S$ which only involve vertices of $(\Gamma_0^-)_{\ell}$, called a truncation of $S$. Let $J_{\ell}$ be the two-sided ideal of $CG_{\ell}^-\to$ generated by all cyclic derivatives of $S_{\ell}$ and let

$$A_{\ell} = CG_{\ell}^-\to/J_{\ell}$$  

be the truncated Jacobian algebra at height $\ell$. Denote by $\pi : CG_{\ell}^-\to \to A_{\ell}$ the natural projection.
It has been shown in [20, Proposition 4.17] that for any \( \ell \), \( A_\ell \) is finite-dimensional and the quiver with potential \( (\Gamma^\ell, J_\ell) \) is rigid, namely, every cycle is cyclically equivalent to an element of \( J_\ell \).

2.6. \( q \)-characters and \( F \)-polynomials

Let \( m \) be a dominant monomial in the variables \( Y_{i,r} \) for \( (i, r) \in G_0^- \). Following [12, 20], for each \( (i, r) \in G_0^- \), define

\[
\hat{y}_{i,r} = \prod_{(i,r)\to (j,s)} z_{j,s} \prod_{(j,s)\to (i,r)} z_{j,s}^{-1}.
\]

It was shown in [20, Lemma 4.15] that \( \hat{y}_{i,r} = A_{i,r-d_i}^{-1} \) for \( (i, r) \in G_0^- \), so \( \hat{y}_{i,r} \) is a monomial in the variables \( Y_{i,s} \), \( (i,s) \in G_0^- \) by (2.2).

Using [12, Corollary 6.3], Hernandez and Leclerc gave the following formula for a cluster variable in terms of its \( F \)-polynomial and \( g \)-vector. Every cluster variable \( x \) of \( A \) has the following form

\[
x = z^{g_x} F_x(\hat{y}).
\]  

(2.6)

On the other hand, in [13], the truncated \( q \)-character \( \chi_q^- (L(m)) \) is expressed as

\[
\chi_q^- (L(m)) = m P_m, \tag{2.7}
\]

where \( P_m \) is a polynomial with integer coefficients in the variables \( \{ A_{i,r-d_i}^{-1} \mid (i, r) \in G_0^- \} \) and has constant term 1. Thus, by [20], if \( L(m) \) corresponds to a cluster variable of \( A \), then \( m = z^{g(m)} \), where the integer vector \( g(m) \in \mathbb{Z}^{G_0^-} \) is the \( g \)-vector of \( L(m) \).

Let \( I_{i,r} \) be the indecomposable injective \( A \)-module at vertex \( (i, r) \in G_0^- \). Motivated by quivers with potentials [9] and cluster character [28, 29], Hernandez and Leclerc defined the following notion of generic kernel.

DEFINITION 2.7 [20, Definition 4.5, Section 5.2.2]. Let \( K(m) \) be the kernel of a generic \( A \)-module homomorphism from the injective \( A \)-module \( I(m)^- \) to the injective \( A \)-module \( I(m)^+ \), where

\[
I(m)^+ = \bigoplus_{g_{i,r}(m) > 0} I_{i,r-d_i}^{\oplus g_{i,r}(m)}, \quad I(m)^- = \bigoplus_{g_{i,r}(m) < 0} I_{i,r-d_i}^{\oplus |g_{i,r}(m)|}.
\]

The support of \( K(m) \) is the collection of all points \( (j, s) \in G_0^- \) such that the \( (j, s) \)-component of \( K(m) \) is nonzero. We denote by \( \text{Supp}(K(m)) \) the support of \( K(m) \).

In [20], Hernandez and Leclerc proposed the following conjecture.

CONJECTURE 2.8 [20, Conjecture 5.3]. Let \( L(m) \) be a real simple \( U_q(\mathfrak{g}) \)-module in \( \mathcal{C}^- \). Then up to normalization, the truncated \( q \)-character of \( L(m) \) is equal to the \( F \)-polynomial of the associated generic kernel. More precisely,

\[
\chi_q^- (L(m)) = m F_{K(m)},
\]

where the variables \( v_{i,r} \) of the \( F \)-polynomial are evaluated as in (2.2).

In [20, Theorem 4.8], Hernandez and Leclerc proved Conjecture 2.8 for Kirillov–Reshetikhin modules, that is, up to renormalizing, the truncated \( q \)-character of the Kirillov–Reshetikhin module \( W_{(i)}^{(k,r-d_i(2k-1))} \) is equal to the \( F \)-polynomial of the generic kernel \( F_{K_{(i)}^{(k,r)}} \), where \( K_{(i)}^{(k,r)} \) is the
kernel of a generic $A$-module homomorphism from $I_{i,r}$ to $I_{i,r-kb_i}$. We will prove Conjecture 2.8 for snake modules in Theorem 3.2.

2.7. A formula for the lowest weight monomial
Recall that $t = \max\{d_i \mid i \in I\}$ as defined in Section 2.1. As a generalization of [20, Remark 4.14], we can calculate the dimension vectors of the $A$-module $K_{k,r}^{(i)}$ for $r \leq d_i(2k-1) - th^\vee$. Indeed, by [13, Lemma 6.8, Corollary 6.9], the lowest monomial of $\chi_q(\prod_{s=1}^k Y_{i,r-d_i(2s-1)})$ is equal to $\prod_{s=1}^k Y_{v(i),r-d_i(2s-1)+th^\vee}$, where $v$ is the involution of $I$ defined by $w_0(\alpha_i) = -\alpha_v(i)$, where $w_0$ is the longest element in the Weyl group of $g$. Using [20, Theorem 4.8], we can calculate the lowest monomial, which corresponds to the term in the $F$-polynomial for the trivial submodule $K_{k,r}^{(i)} \subset K_{k,r}^{(i)}$. Thus

$$
\prod_{s=1}^k Y_{v(i),r-d_i(2s-1)+th^\vee} = \left( \prod_{s=1}^k Y_{i,r-d_i(2s-1)} \right) \prod_{(j,s) \in \Gamma^i_0} d_{j,s}(K_{k,r}^{(i)})^{-v_j,s(K_{k,r}^{(i)})},
$$

where $(d_{j,s}(K_{k,r}^{(i)}))_{(j,s) \in [r]^n}$ is the dimension vector of $K_{k,r}^{(i)}$.

In the next section, we will introduce a combinatorial method to calculate the dimension vector of the $A$-module $K(m)$ associated to the snake module $L(m)$.

3. A geometric character formula for snake modules
In this section, we show that the geometric character formula conjectured by Hernandez and Leclerc holds for snake modules of types $A$ and $B$. We give a combinatorial formula for the $F$-polynomial of the generic kernel $K(m)$ associated to the snake module $L(m)$. As a consequence, we obtain a combinatorial method to compute the dimension vector of $K(m)$ as well as all its submodules.

3.1. A geometric character formula for snake modules
We first give a description of the $g$-vector $g(m) := (g_{i,s})_{(i,s) \in G_0^-}$ for arbitrary prime snake module $L(m)$.

**PROPOSITION 3.1.** Let $L(m)$ be a prime snake module with highest weight monomial $m$ of the form (2.4). Then we can rewrite

$$
m = z^{g(m)} := \prod_{(i,s) \in G_0^-} z_{i,s}^{g_{i,s}(m)},
$$

where

$$
g_{i,s}(m) = \begin{cases} 
1 & \text{if } (i,s) = (i_j, r + \sum_{\ell=1}^{j-1} n_{i\ell}) \text{ and } r + \sum_{\ell=1}^{j-1} n_{i\ell} \leq 0, \\
-1 & \text{if } (i,s) = (i_j, r + \sum_{\ell=1}^{j-1} n_{i\ell} + b_{i,j} k_j) \text{ and } r + \sum_{\ell=1}^{j-1} n_{i\ell} + b_{i,j} k_j \leq 0, \\
0 & \text{otherwise.}
\end{cases}
$$

Here $j = 1, \ldots, N$ as in (2.4).

**Proof.** From Theorem 2.1, it follows that every prime snake module $L(m)$ is a $U_q(\mathfrak{g})$-module with highest weight monomial $m$ of the form (2.4). Thus $m$ is a product of terms of the form

$$
\prod_{s=0}^{k_j-1} Y_{i_j,r+\sum_{\ell=1}^{j-1} n_{i\ell}+b_{i,j} k_j}.
$$
Because of (2.3), for any \( 1 \leq j \leq N \),

\[
L \left( \prod_{s=0}^{k_j-1} Y_{i_j,r^s} \right)
\]

is a Kirillov–Reshitikhin module. Now the result follows from Theorem 2.6 and [20, Proposition 4.16].

We are now ready for the main result of this section. The following theorem gives a positive answer to the Hernandez–Leclerc Conjecture (Conjecture 2.8) for snake modules.

**Theorem 3.2.** Let \( L(m) \) be a prime snake module in \( \mathcal{C}^- \). Then up to normalization, the truncated \( q \)-character of \( L(m) \) is equal to the \( F \)-polynomial of the associated generic kernel \( K(m) \). More precisely,

\[
\chi_q^- (L(m)) = mF_{K(m)},
\]

where \( F_{K(m)} \) is a polynomial in the variables (2.2).

**Proof.** Recall that \( \mathcal{A} \) is the cluster algebra defined in Section 2.1. We use the characterization of \( m \) from Theorem 2.1. The fact that \( L(m) \in \mathcal{C}^- \) implies that each index of \( Y \) in the formula (2.4) is a vertex in \( G_0^- \). This implies that for some integer \( N \),

\[
\left( i, r + \sum_{\ell=1}^{N-1} n_\ell + b_{iN} (k_N - 1) \right) \in G_0^- . \tag{3.1}
\]

In particular, the second coordinate of (3.1) is non-positive. Thus \( r + \sum_{\ell=1}^{j-1} n_\ell + b_{ij} (k_j - 1) \leq 0 \) for all \( j = 1, \ldots, N \).

By Theorem 2.6, the truncated \( q \)-character \( \chi_q^- (L(m)) \) is a cluster variable \( x \) of \( \mathcal{A} \). By Proposition 3.1, the \( g \)-vector of \( x \) is given by

\[
g_{i,s}(m) = \begin{cases} 1 & \text{if } (i, s) = \left( i, r + \sum_{\ell=1}^{j-1} n_\ell \right), \\ -1 & \text{if } (i, s) = \left( i, r + \sum_{\ell=1}^{j-1} n_\ell + b_{ij} k_j \right) \text{ and } r + \sum_{\ell=1}^{j-1} n_\ell + b_{ij} k_j \leq 0, \\ 0 & \text{otherwise}, \end{cases}
\]

where we use that \( r + \sum_{\ell=1}^{j-1} n_\ell \leq 0 \), because of (3.1).

For \( \ell \leq 0 \), let \( (G_0^-)_\ell := \{ (i, r + d) : (i, r) \in (G_0^-) \} \) and \( z_\ell^- = \{ z_{i,r} \mid (i, r) \in (G_0^-) \} \). We denote by \( G_{\ell}^- \) the same quiver as \( \Gamma_{\ell}^- \), but with vertices labeled by \( (G_0^-)_\ell \). Clearly, the cluster variable \( x \) is a Laurent polynomial in the variables of \( z_\ell^- \) for some \( \ell \leq 0 \), and can be regarded as a cluster variable of the cluster algebra \( \mathcal{A} \) defined by the initial seed \( (z_0^-, G_0^-) \).

The rest of the proof is similar to the proof of [20, Theorem 4.8]. Since the quiver with potential \( (\Gamma_{\ell}^-, J_{\ell}) \) is rigid, we can apply the theory of [8, 9] and deduce that the \( F \)-polynomial of \( x \) coincides with the polynomial \( F_M \) associated with a certain \( A_\ell \)-module \( M \). Furthermore, \( M \) is rigid by [1, 15].

By [30, Remark 4.1], \( M \) is the kernel of a generic element of the homomorphism space between two injective \( A_\ell \)-modules corresponding to the negative and positive components of the \( g \)-vector of \( x \). More precisely, let \( I_{i,m}^\ell \) be the injective \( A_\ell \)-module at vertex \( (i,m) \), then \( M \) is the kernel of a generic element of \( \text{Hom}_{A}(I_{i,m}^\ell, I_{i,m}^\ell) \), where

\[
I_{i,m}^\ell = \bigoplus_{g_{i,s}(m) > 0} I_{i,s-di}^\ell \oplus g_{i,s}(m), \quad I_{i,m}^\ell = \bigoplus_{g_{i,s}(m) < 0} I_{i,s-di}^\ell \oplus g_{i,s}(m) .
\]
It was shown in [20] that our $A_\ell$-module $M$ does not change when $\ell$ increases and that in the direct limit
\[ A = \lim_{\ell \to -\infty} A_\ell. \]
The $A$-module $M$ is the kernel of a generic element of $\text{Hom}_A(I(m)^-, I(m)^+)$. Thus $M = K(m)$.

From the proof of Theorem 3.2, we obtain the following corollary.

**Corollary 3.3.** Let $L(m)$ be a prime snake module in $\mathcal{C}^-$. Then the generic kernel $K(m)$ is rigid and indecomposable.

**Remark 3.4.** By Theorem 2.5, every snake module of type $\mathbb{A}_n$ or type $\mathbb{B}_n$ is isomorphic to a tensor product of prime snake modules defined uniquely up to permutation. On the other hand, if $M$ and $N$ are two finite-dimensional $A$-modules, then by [9, Proposition 3.2], we have $F_{M \oplus N} = F_M F_N$. Therefore, replacing the module $K(m)$ in Theorem 3.2 by a direct sum, we obtain a similar geometric character formula for arbitrary snake module of types $\mathbb{A}_n$ and $\mathbb{B}_n$.

We present several examples to illustrate Theorem 3.2.

**Example 3.5.** In type $\mathbb{A}_3$, let $N = 3$, $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $i_1 = 1$, $i_2 = 3$, $i_3 = 2$, $r = 15$, $j_1 = j_2 = 0$, $n_1 = 4$, and $n_2 = 5$. Then $m = Y_{1,-15} Y_{3,-11} Y_{3,-9} Y_{2,-6}$. We get
\[ g_{i,s}(m) = \begin{cases} 1 & \text{if } (i,s) = (1,-15), (3,-11), \text{ or } (2,-6); \\ -1 & \text{if } (i,s) = (1,-13), (3,-7), \text{ or } (2,-4); \\ 0 & \text{otherwise.} \end{cases} \]
Thus by Definition 2.7
\[ I(m)^+ = I_{1,-16} \oplus I_{3,-12} \oplus I_{2,-7}, \quad I(m)^- = I_{1,-14} \oplus I_{3,-8} \oplus I_{2,-5}. \]

The module $K(m)$ has dimension 13 and is displayed in Figure 6. In Figure 6, all vertices carry a vector space of dimension 1. Applying Theorem 3.2, we can compute its $q$-character as follows. There are 160 submodules in $K(m)$.

\[ \chi_q^- (L(m)) = m((1 + v_{1,-14} + v_{1,-14} v_{2,-13}) (v_{3,-8} + v_{3,-10} v_{3,-8} + v_{2,-5} v_{2,-7} v_{3,-8} + v_{2,-5} v_{2,-7} v_{3,-8}) + v_{2,-5} v_{2,-7} v_{3,-8} v_{3,-10} + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{3,-8} + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{3,-8} v_{3,-10} + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{2,-9} v_{3,-8} v_{3,-10} + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{2,-9} v_{3,-8} v_{3,-10}) + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{2,-9} v_{3,-8} v_{3,-10} + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{2,-9} + v_{1,-4} v_{1,-6} v_{2,-5} v_{2,-7} v_{2,-9}) + (1 + v_{1,-14} + v_{1,-14} v_{2,-13} + v_{1,-14} v_{2,-13} v_{3,-12} v_{3,-10} v_{3,-8}) (1 + v_{2,-5} + v_{1,-4} v_{2,-5} + v_{1,-4} v_{2,-5} v_{2,-5} + v_{1,-4} v_{2,-5} v_{2,-5} v_{2,-5}) + (1 + v_{1,-14} + v_{1,-14} v_{2,-13}) ((v_{3,-8} + v_{3,-10} v_{3,-8})(v_{2,-5} + v_{1,-4} v_{2,-5} v_{2,-5} v_{2,-5} v_{2,-5}) v_{3,-8} + v_{3,-10} v_{3,-8}) v_{3,-8} + v_{3,-10} v_{3,-8}) (v_{2,-5} + v_{1,-4} v_{2,-5} v_{2,-5} v_{2,-5} v_{2,-5}).
The cluster variable corresponding to \( k \) carries a vector space of dimension 1. Applying Theorem 3.2, we can compute its \( q \)-character as follows. There are 35 submodules in \( K(m) \).

Starting from the initial seed \((z, G^-)\), the following sequence of mutations produces (in the last step) the cluster variable corresponding to \( L(m) \).

\[
(3, -3), (2, -2), (1, -3), (2, -4), (1, -5), (2, -6), (3, -7),
(1, -7), (2, -8), (3, -9), (3, -11), (2, -12), (1, -13).
\]

**Example 3.6.** In type \( A_3 \), let \( N = 2, k_1 = 1, k_2 = 1, i_1 = 2, i_2 = 2, r = -10, j_1 = 1, \) and \( n_1 = 4 \). Then \( m = Y_{2,-10} Y_{2,-6} \). We get

\[
g_{i,s}(m) = \begin{cases} 
1 & \text{if } (i, s) = (2, -10), \text{ or } (2, -6); \\
-1 & \text{if } (i, s) = (2, -8), \text{ or } (2, -4); \\
0 & \text{otherwise}.
\end{cases}
\]

Thus by Definition 2.7

\[ I(m)^+ = I_{2,-11} \oplus I_{2,-7}, \quad I(m)^- = I_{2,-9} \oplus I_{2,-5}. \]

The module \( K(m) \) has dimension 8 and is displayed in Figure 7. In Figure 7, all vertices carry a vector space of dimension 1. Applying Theorem 3.2, we can compute its \( q \)-character as follows. There are 35 submodules in \( K(m) \).

\[
\chi_q^-(L(m)) = m(1 + v_{2,-9} + v_{2,-9}v_{1,-8} + v_{2,-9}v_{3,-8} + v_{2,-9}v_{1,-8}v_{3,-8}
+ v_{2,-9}v_{1,-8}v_{3,-8}v_{2,-7}v_{2,-5} + (v_{2,-5} + v_{1,-4}v_{2,-5} + v_{3,-4}v_{2,-5})
+ v_{1,-4}v_{3,-4}v_{2,-5} + v_{1,-4}v_{3,-4}v_{2,-5}v_{2,-3})(1 + v_{2,-9}
+ v_{2,-9}v_{1,-8} + v_{2,-9}v_{3,-8} + v_{2,-9}v_{1,-8}v_{3,-8})
+ v_{2,-9}v_{1,-8}v_{3,-8}v_{2,-7}v_{2,-5}(v_{1,-4} + v_{3,-4} + v_{1,-4}v_{3,-4} + v_{1,-4}v_{3,-4}v_{2,-3})�.
\]

Starting from the initial seed \((z, G^-)\), the following sequence of mutations produces (in the last step) the cluster variable corresponding to \( L(m) \).

\[
(3, -3), (2, -2), (1, -3), (2, -4), (3, -7), (2, -6), (1, -7), (2, -8).
\]
Figure 6. The support of the $A$-module $K(Y_{1,-15}Y_{3,-11}Y_{3,-9}Y_{2,-6})$ in type $A_3$.

Figure 7. The support of the $A$-module $K(Y_{2,-10}Y_{2,-6})$ in type $A_3$. 
Example 3.7. In type $\mathbb{B}_2$, let $N = 2$, $k_1 = 1$, $k_2 = 1$, $i_1 = 2$, $i_2 = 2$, $r = -12$, $j_1 = 1$, and $n_1 = 6$. Then $m = Y_{2, -12} Y_{2, -6}$. We get

$$g_{i,s}(m) = \begin{cases} 1 & \text{if } (i,s) = (2,-12), \text{ or } (2,-6); \\ -1 & \text{if } (i,s) = (2,-10), \text{ or } (2,-4); \\ 0 & \text{otherwise}. \end{cases}$$

Thus by Definition 2.7

$$I(m)^+ = I_{2, -13} \oplus I_{2, -7}, \quad I(m)^- = I_{2, -11} \oplus I_{2, -5}.$$
We define the height monomial $h(p)$ of a path $p \in \mathcal{P}_{i,k}$ by
\[
h(p) = \prod_{(i,r) \in (p \ominus p_{i,k}^+)} v_{i,r},
\]
where $(i,r) \in \Gamma_0$ runs over all the cell coordinates in $p \ominus p_{i,k}^+$ and we use the convention: $v_{i,r} = 0$ if $(i,r) \not\in \Gamma_0$. In particular, $h(p_{i,k}^+) = 1$.

Recall that for any snake $(i_t, k_t)$, $1 \leq t \leq T \in \mathbb{Z}_{\geq 1}$,
\[
\mathcal{F}_{(i_t, k_t)_{1 \leq t \leq T}} = \{(p_1, \ldots, p_T) : p_t \in \mathcal{P}_{i_t, k_t}, 1 \leq t \leq T, (p_1, \ldots, p_T) \text{ is non-overlapping}\}.
\]

**Theorem 3.9.** Let $L(m) = L(\prod_{t=1}^{T} Y_{i_t, k_t})$ be a prime snake module and $K(m)$ be the generic kernel associated to $L(m)$. Then
\[
F_{K(m)} = \sum_{(p_1, \ldots, p_T) \in \mathcal{F}_{(i_t, k_t)_{1 \leq t \leq T}}} \prod_{t=1}^{T} h(p_t).
\]

**Proof.** By Theorem 3.2, we have
\[
F_{K(m)} = \frac{\chi_q(L(m))}{m},
\]
and Theorem 2.3 gives a formula for $\chi_q(L(m))$ in terms of paths
\[
\chi_q(L(m)) = \sum_{(p_1, \ldots, p_T) \in \mathcal{F}_{(i_t, k_t)_{1 \leq t \leq T}}} \prod_{t=1}^{T} m(p_t).
\]

Note that equation (3.2) uses the truncated $q$-character $\chi_q^-(L(m))$, whereas equation (3.3) uses the complete $q$-character $\chi_q(L(m))$. First we prove the statement in the case where $\chi_q^-(L(m)) = \chi_q(L(m))$.

Applying [24, Lemma 5.10] and using induction, we have
\[
\prod_{t=1}^{T} m(p_t) = \prod_{t=1}^{T} m(p_{i_t,k_t}^+) \prod_{r=1}^{R} A_{j_r \ell_r}^{-1},
\]
where $(j_r, \ell_r)$, $1 \leq r \leq R \in \mathbb{Z}_{\geq 0}$, is a sequence of cell coordinates determined by $p_t \ominus p_{i_t,k_t}^+$, $1 \leq t \leq T$. Therefore
\[
\prod_{t=1}^{T} h(p_t) = \prod_{r=1}^{R} A_{j_r \ell_r}^{-1}.
\]

Moreover, since $p_{i_t,k_t}^+$ is the highest path in $\mathcal{P}_{i_t,k_t}$, Theorem 2.3 implies that $\prod_{t=1}^{T} m(p_{i_t,k_t}^+)$ contains no negative powers. Since $L(m)$ is special, its highest weight monomial $m$ is the unique dominant monomial in $\chi_q(L(m))$, and thus
\[
m = \prod_{t=1}^{T} m(p_{i_t,k_t}^+).
\]

Thus equations (3.2)–(3.6) imply
\[
mF_{K(m)} = \chi_q(L(m)) = \sum_{(p_1, \ldots, p_T) \in \mathcal{F}_{(i_t, k_t)_{1 \leq t \leq T}}} m \prod_{t=1}^{T} A_{j_t \ell_t}^{-1} = \sum_{(p_1, \ldots, p_T) \in \mathcal{F}_{(i_t, k_t)_{1 \leq t \leq T}}} m \prod_{t=1}^{T} h(p_t).
\]
Now suppose $\chi_q^- (L(m)) \neq \chi_q (L(m))$. Then we have to modify the above argument as follows. Equation (3.3) is replaced by

$$\chi_q^- (L(m)) = \sum_{(p_1, \ldots, p_T) \in \mathcal{P}_{(i_t, k_t)}: 1 \leq t \leq T} \prod_{t=1}^{T} m(p_t).$$

(3.7)

In other words, we require that for each path $p_t$ the upper and lower corners $C_{p_t}^+, C_{p_t}^-$ lie in $G^-$. Moreover, in equation (3.5), we replace $A^{-1}_{j_r, \ell_r}$ by $A'^{-1}_{j_r, \ell_r}$ where

$$A'^{-1}_{j_r, \ell_r} = \begin{cases} A^{-1}_{j_r, \ell_r} & \text{if } (j_r, \ell_r) \in \Gamma^-_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$mF_{K(m)} = \chi_q^- (L(m)) = \sum_{(p_1, \ldots, p_T) \in \mathcal{P}_{(i_t, k_t)}: 1 \leq t \leq T} \prod_{t=1}^{T} h(p_t).$$

Remark 3.10. (1) Theorem 3.9 allows us to calculate the dimension vector $(d_{i,r}(K(m)))_{(i,r) \in \Gamma^-_0}$ of the $A$-module $K(m)$ in a combinatorial way using all $T$-tuples of non-overlapping paths. We will explain this in the next section.

(2) Theorem 3.9 provides a combinatorial approach to find all submodules of $K(m)$, see Examples 3.14–3.16.

(3) Using [9, Proposition 3.2], for any two finite-dimensional $A$-modules $M$ and $N$, we have

$$F_M \oplus N = F_M F_N.$$ 

Replacing the $A$-module $K(m)$ in Theorem 3.9 by a direct sum of such modules, we obtain a similar combinatorial formula for arbitrary snake modules.

Corollary 3.11. If $L(m) = L(\prod_{t=1}^{T} Y_{i_t, k_t})$ is a snake module and $K(m)$ is the associated generic kernel, then for all dimension vectors $e$, we have

$$\chi(Gr_e (K(m))) = 0 \text{ or } 1.$$ 

Proof. Using Theorem 3.9 and the definition of the $F$-polynomial, it suffices to show that for any two $T$-tuples $(p_1, \ldots, p_T) \neq (p'_1, \ldots, p'_T) \in \mathcal{P}_{(i_t, k_t)}: 1 \leq t \leq T$ of non-overlapping paths, we have $\prod_{t=1}^{T} h(p_t) \neq \prod_{t=1}^{T} h(p'_t)$. This holds because $(p_1, \ldots, p_T)$ are disjoint paths and each $p_t$ is determined by $p_t \oplus p^+_t$. \qed

Remark 3.12. Corollary 3.11 holds for any thin and real module if [18, Conjecture 13.2] or [20, Conjecture 5.2] or [22, Conjecture 9.1] holds.
3.3. Generic kernel

Recall that \( \mathcal{P}_{i,k} \) is a collection of paths defined in Section 2.3. Let

\[ \mathcal{P}'_{i,k} = \{ p \in \mathcal{P}_{i,k} \mid C_p^\pm \subset G_0 \} \subset \mathcal{P}_{i,k}. \]

Let \( \mathcal{P}'_{(i_t,k_t)}_{1 \leq t \leq T} \) be a collection of paths associated to a snake module \( L(m) \) of the form (2.4) in \( \mathcal{C}^- \). For any snake \( (i_t, k_t), 1 \leq t \leq T \in \mathbb{Z}_{\geq 1} \), let

\[ \mathcal{F}'_{(i_t,k_t)}_{1 \leq t \leq T} = \{ (p'_1, \ldots, p'_T) : p'_t \in \mathcal{P}'_{i_t,k_t}, \ 1 \leq t \leq T, (p'_1, \ldots, p'_T) \text{ is non-overlapping} \}. \]

Let \( V_m \) be the set of all the cell coordinates in the set \( \bigcup_{1 \leq t \leq T} (p'^-_{i_t,k_t} \oplus p^+_{i_t,k_t}) \) where \( p'^-_{i_t,k_t} \) is a minimal path in \( \mathcal{P}'_{i_t,k_t} \) for each \( 1 \leq t \leq T \) and \( (p'^-_{i_t,k_t}, \ldots, p'^-_{i_T,k_T}) \in \mathcal{F}'_{(i_t,k_t)}_{1 \leq t \leq T} \).

Note that when \( \chi_q^{-1}(L(\prod_{t=1}^T Y_{i_t,k_t})) = \chi_q(L(\prod_{t=1}^T Y_{i_t,k_t})) \), the set \( V_m \) is the set of all the cell coordinates in the set \( \bigcup_{1 \leq t \leq T} (p'^-_{i_t,k_t} \oplus p^+_{i_t,k_t}) \).

**Definition 3.13.** Let \( Q(m) \) be the full subquiver of \( \Gamma^- \) with vertex set \( V_m \).

If we assign a vector space whose dimension is equal to the multiplicity of cells with coordinate \((i, r)\) occurring in the multiset \( \bigcup_{1 \leq t \leq T} (p'^-_{i_t,k_t} \oplus p^+_{i_t,k_t}) \) to every point \((i, r) \in V_m\), then by Theorem 3.9, the generic kernel \( K(m) \) is a representation of \( Q(m) \). In general, \( K(m) \) is not unique, not even up to isomorphism, but its \( F \)-polynomial is unique. In particular, the linear maps associated with arrows satisfy relations in the Jacobian ideal \( J \).

The following several examples hold that \( \chi_q^{-1}(L(\prod_{t=1}^T Y_{i_t,k_t})) = \chi_q(L(\prod_{t=1}^T Y_{i_t,k_t})) \).

**Example 3.14.** In type \( \mathbb{A}_3 \), let \( m = Y_{1,-15} Y_{3,-11} Y_{3,-9} Y_{2,-6} \). Then \( K(m) \) is displayed in Figure 9 (Here \( K(m) \) is drawn opposite as Figure 6, because of the definition of paths). For each vertex \((i, r) \in V_m\), we find it convenient to always label the dimension of the vector space at the vertex \((i, r)\). The dimension associated with a vertex \((i, r) \in V_m\) is the multiplicity of cells with coordinate \((i, r)\) occurring in the multiset

\[ (p^-_{1,-15} \ominus p^+_{1,-15}) \cup (p^-_{3,-11} \ominus p^+_{3,-11}) \cup (p^-_{3,-9} \ominus p^+_{3,-9}) \cup (p^-_{2,-6} \ominus p^+_{2,-6}). \]

The maps associated with arrows are \((\pm 1)\), whose sign is deduced from the defining relations of the Jacobian algebra \( A \).

In the sense of Theorem 3.9, finding all possible submodules of \( K(m) \) is equivalent to finding all 4-tuple sets of non-overlapping paths in \( \mathcal{P}_{1,-15} \times \mathcal{P}_{3,-11} \times \mathcal{P}_{3,-9} \times \mathcal{P}_{2,-6} \).

**Example 3.15.** In type \( \mathbb{A}_3 \), let \( m = Y_{2,-10} Y_{2,-6} \). Then \( K(m) \) is displayed in Figure 10 (Here \( K(m) \) is drawn opposite as Figure 7, because of the definition of paths). For each vertex \((i, r) \in V_m\), we label the dimension of the vector space at the vertex \((i, r)\). The dimension associated with a vertex \((i, r) \in V_m\) is the multiplicity of cells with coordinate \((i, r)\) occurring in the multiset

\[ (p^-_{2,-10} \ominus p^+_{2,-10}) \cup (p^-_{2,-6} \ominus p^+_{2,-6}). \]

The maps associated with arrows are \((\pm 1)\), whose sign is deduced from the defining relations of the Jacobian algebra \( A \).

In the sense of Theorem 3.9, finding all possible submodules of \( K(m) \) is equivalent to finding all pairs of non-overlapping paths in \( \mathcal{P}_{2,-10} \times \mathcal{P}_{2,-6} \).

**Example 3.16.** In type \( \mathbb{B}_2 \), let \( m = Y_{2,-12} Y_{2,-6} \). Then \( K(m) \) is displayed in Figure 11 (Here \( K(m) \) is drawn opposite as Figure 8, because of the definition of paths). For each vertex \((i, r) \in V_m\), we label the dimension of the vector space at the vertex \((i, r)\). The dimension
Figure 9 (colour online). The module $K(Y_{1,-15}Y_{3,-11}Y_{3,-9}Y_{2,-6})$ in type $A_3$. The index of $Y_{i,r}$ corresponds to the top vertex of the associated path rectangle $\mathcal{P}_{i,r}$. The highest (respectively, lowest) path $p_{i,r}^+$ (respectively, $p_{i,r}^-$) in $\mathcal{P}_{i,r}$ is drawn in blue (respectively, green) color and the cells are these square regions in $p_{i,r}^- \ominus p_{i,r}^+$. The dimension is 1 at each cell coordinate that lies in the interior of these cells; see the online version for colors.

associated with a vertex $(i, r) \in V_m$ is the multiplicity of cells with coordinate $(i, r)$ occurring in the multiset

$$(p_{2,-12}^- \ominus p_{2,-12}^+) \cup (p_{2,-6}^- \ominus p_{2,-6}^+).$$

The maps associated with arrows are $(\pm 1)$, whose sign is deduced from the defining relations of the Jacobian algebra $A$.

In the sense of Theorem 3.9, finding all possible submodules of $K(m)$ is equivalent to finding all pairs of non-overlapping paths in $\mathcal{P}_{2,-12} \times \mathcal{P}_{2,-6}$.

The following is an example where the dimensions of $K(m)$ are larger than 1.
Figure 10 (colour online). The module $K(Y_{2,-10}Y_{2,-6})$ in type $A_3$. The index of $Y_{i,r}$ corresponds to the top vertex of the associated path rectangle $\mathcal{P}_{i,r}$. The dimension is 1 at those vertices of $\Gamma^-$ that lie in the interior of these rectangles.

Example 3.17. In type $A_3$, let $m = Y_{2,-8}Y_{2,-6}$. Then $L(m)$ is a Kirillov–Reshetikhin module and $K(m)$ is displayed in Figure 12. For each vertex $(i, r) \in V_m$, we label the dimension of the vector space at the vertex $(i, r)$. The dimension associated with a vertex $(i, r) \in V_m$ is the multiplicity of cells with coordinate $(i, r)$ occurring in the multiset $(p_{2,-8}^r \oplus p_{2,-8}^l) \cup (p_{2,-6}^r \oplus p_{2,-6}^l)$. In Figure 12, almost all vertices carry a vector space of dimension 1, except the vertex $(2, -5)$ which carries a vector space of dimension 2.

Starting from the initial seed $(z, G^-)$, the following sequence of mutations produces (in the last step) the cluster variable corresponding to $L(m)$.

$$(2, -2), (2, -4), (2, -6), (1, -3), (1, -5), (3, -3), (3, -5), (2, -2), (2, -4).$$

Remark 3.18. The dimension of $K(m)$ at a vertex $(i, r)$ can be arbitrary large in the sense that given any integer $\alpha$ there is a snake module $L(m)$ and a vertex $(i, r)$ such that the generic kernel $K(m)$ is of dimension at least $\alpha$ at $(i, r)$. Therefore Corollary 3.11 is non-trivial.

The following is an example that $\chi_q(L(\prod_{t=1}^T Y_{i_t,k_t})) \neq \chi_q(L(\prod_{t=1}^T Y_{i_t,k_t}))$.

Example 3.19. In type $A_3$, let $m = Y_{2,-4}Y_{2,-2}$. Then $L(m)$ is a Kirillov–Reshetikhin module and $K(m)$ is displayed in Figure 13. By definition, we have

$$V_m = ((p_{2,-4}^r \oplus p_{2,-4}^l) \cup (p_{2,-2}^r \oplus p_{2,-2}^l)) \cap \Gamma_0^- = \{(2, -3), (2, -1)\},$$

where $p_{2,-2}^r = \{(0, 0), (1, -1), (2, 0), (3, -1), (4, 0)\}$ and

$$p_{2,-4}^r = \{(0, -2), (1, -3), (2, -2), (3, -3), (4, -2)\}.$$
Figure 11 (colour online). The module $K(Y_{2,-12}Y_{2,-6})$ in type $B_2$. The index of $Y_{i,r}$ corresponds to the top vertex of the associated path triangle $\mathcal{P}_{i,r}$. The dimension is 1 at those vertices of $\Gamma^-$ that lie in the interior of these triangles.

Figure 12 (colour online). The module $K(Y_{2,-8}Y_{2,-6})$ in type $A_3$. The index of $Y_{i,r}$ corresponds to the top vertex of the associated path rectangle $\mathcal{P}_{i,r}$. The dimension is 1 at those vertices of $\Gamma^-$ that lie in the interior of these rectangles, except that the vertex $(2,-5)$ has a vector space of dimension 2.

Note that $p_{2,-2}^-$ is a path in the set $\mathcal{P}_{2,-2}$, so it cannot go through points $(i,r)$ with $r > 0$.

For each vertex $(i,r) \in V_m$, we label the dimension of the vector space at the vertex $(i,r)$. The dimension associated with a vertex $(i,r) \in V_m$ is the multiplicity of cells with coordinate $(i,r)$ occurring in the multiset $(p_{2,-4}^- \ominus p_{2,-4}^+) \cup (p_{2,-2}^- \ominus p_{2,-2}^+)$. In Figure 13, all vertices carry a vector space of dimension 1.

Starting from the initial seed $(z,G^-)$, the sequence $((2,0),(2,-2))$ of mutations produces (in the last step) the cluster variable corresponding to $L(m)$. 
Figure 13 (colour online). The module $K(Y_{2,-4}Y_{2,-2})$ in type $A_3$. The index of $Y_{i,r}$ corresponds to the top vertex of the associated path rectangle $\mathcal{P}_{i,r}$. The dimension is 1 at vertices $(2,-3)$ and $(2,-1)$.

4. Denominator vector

In this section, we show that every snake module corresponds to a cluster monomial with square-free denominator in the cluster algebra $\mathcal{A}$ and that snake modules are real modules.

**Theorem 4.1.** Let $L(m)$ be an arbitrary snake module. Then the truncated $q$-character $\chi_q(L(m))$ is a cluster monomial in $\mathcal{A}$, and its denominator is square free as a monomial in the initial cluster variables $z_{i,r}, (i,r) \in G_0^{-}$.

**Proof.** By Theorem 2.5, we can write $L(m)$ as a tensor product $L(m) \cong L(m_1) \otimes \cdots \otimes L(m_n)$ of prime snake modules. Let $K(m_i)$ be the generic kernel associated to $L(m_i)$ and let $Q(m_i)$ be the full subquiver of $\Gamma^-$ whose vertices are in the support of $K(m_i)$. Thus $K(m) = \bigoplus_{i=1}^n K(m_i)$ is the generic kernel associated to $L(m)$. To show that $L(m)$ corresponds to a cluster monomial we need to prove that $K(m)$ is a rigid object in the cluster category $[1, 15]$.

Let $\mathcal{P}_{(i,m_i),k_i(m_i)}$, $i \leq j \leq T_i$, be the set of paths associated to $L(m_i)$. By Theorem 2.3 and Remark 2.4, we know that for any $1 \leq i \neq j \leq n$, the sets $\mathcal{P}_{(i,m_i),k_i(m_i)}$ and $\mathcal{P}_{(j,m_j),k_j(m_j)}$ are non-overlapping. By our construction in Section 3.3, this implies that the quivers $Q(m_i)$ and $Q(m_j)$ are disjoint and there are no arrows in $\Gamma^-$ which connect $Q(m_i)$ and $Q(m_j)$.

Definition 2.7 and Proposition 3.1 imply that for the prime snake module $L(m_j)$, we have

$$I(m_j)^+ = \bigoplus_{g_{e,s}(m_j)=1} I_{\ell,s-d_\ell}, \quad I(m_j)^- = \bigoplus_{g_{e,s}(m_j)=-1} I_{\ell,s-d_\ell}.$$ 

By Section 3.3, the support of $K(m_i)$ is contained in $\mathcal{P}_{(i,m_i),k_i(m_i)}$, $i \leq j \leq T_i$. Thus the socle points $(\ell, s-d_\ell)$ in $I(m_j)^+$ cannot be in the support of $K(m_i)$. Otherwise, the set $\mathcal{P}_{\ell,s}$ for $L(m_j)$ and $\mathcal{P}_{(i,m_i),k_i(m_i)}$, $i \leq j \leq T_i$, would be overlapping (there is at least a common vertex $(\ell, s)$). This is a contradiction to the fact that $L(m_i) \otimes L(m_j)$ is not prime, see Remark 2.4.
Therefore

\[
\text{Hom}_A(K(m_i), I(m_j)^+) = \text{Hom}_A\left(K(m_i), \bigoplus_{g_{\ell,s}(m_j) = 1} I_{\ell,s-d_{\ell}}\right) \\
\cong \bigoplus_{g_{\ell,s}(m_j) = 1} (K(m_i))_{\ell,s-d_{\ell}} = 0.
\]

(4.1)

Similarly, \(\text{Hom}_A(K(m_j), I(m_i)^+) = 0\).

Consider the injective resolution

\[
0 \longrightarrow K(m_j) \xrightarrow{i_0} I(m_j) \xrightarrow{i_1} I(m_j)^+ \xrightarrow{i_2} \cdots.
\]

Then \(\text{Ext}_A^1(K(m_i), K(m_j))\) is a quotient of \(\{ f \in \text{Hom}_A(K(m_i), I(m_j)^+) \mid i_2f = 0 \}\) which is zero by (4.1). Thus

\[
\text{Ext}_A^1(K(m_i), K(m_j)) = 0.
\]

Similarly, \(\text{Ext}_A^1(K(m_j), K(m_i)) = 0\).

By Corollary 3.3, we have that \(\text{Ext}_A^1(K(m_i), K(m_i)) = 0\) for any \(1 \leq i \leq n\). In \([1]\), \(K(m_i)\) and \(K(m_j)\) are compatible if and only if

\[
\text{Ext}_A^1(K(m_i), K(m_j)) = 0,
\]

where \(C\) is the (generalized) cluster category of the Jacobian algebra \(A\).

Applying

\[
\text{Ext}_A^1(K(m_i), K(m_j)) \cong \text{Ext}_A^1(K(m_i), K(m_i)) \bigoplus \text{Ext}_A^1(K(m_j), K(m_i)),
\]

we see that \(K(m_i)\) and \(K(m_j)\) are compatible for all \(i, j = 1, \ldots, n\), and hence snake modules are cluster monomials.

Next we prove the statement about square-free denominators using the Mukhin–Young’s formulas in Theorem 2.3. By Theorem 3.9 and its proof, we have

\[
\chi_q^- (L(m)) = mF_{K(m)} = \sum_{(p_1, \ldots, p_T) \in \mathcal{P}(it,kt)_{1 \leq t \leq T}} m \prod_{t=1}^{T} h(p_t).
\]

For any \(T\)-tuple \((p_1, \ldots, p_T)\) of non-overlapping paths, either \(m \prod_{t=1}^{T} h(p_t) = 0\) or by Theorem 2.3,

\[
m \prod_{t=1}^{T} h(p_t) = \prod_{t=1}^{T} m(p_t) = \prod_{t=1}^{T} \left( \prod_{(j,\ell) \in C_{it}^+} Y_{j,\ell} \prod_{(j,\ell) \in C_{it}^-} Y_{j,\ell}^{-1} \right)
\]

\[
= \prod_{t=1}^{T} \left( \prod_{(j,\ell) \in C_{it}^+} \frac{z_{j,\ell}}{z_{j,\ell+b_{ij}}} \prod_{(j,\ell) \in C_{it}^-} \frac{z_{j,\ell+b_{ij}}}{z_{j,\ell}} \right),
\]

(4.2)

where the last equation is obtained by performing the change of variables (2.1). It is obvious that

\[
\prod_{(j,\ell) \in C_{it}^+} \frac{z_{j,\ell}}{z_{j,\ell+b_{ij}}} \prod_{(j,\ell) \in C_{it}^-} \frac{z_{j,\ell+b_{ij}}}{z_{j,\ell}}
\]

is a fraction with square-free denominator in the initial cluster variables \(z_{i,r}, (i, r) \in G^-_{0}\).
For any $1 \leq t_1 \neq t_2 \leq T$, the expression
\[
\left( \prod_{(j,\ell)\in C_{p_1}^+} \frac{z_{j,\ell}}{z_{j,\ell+b_{ij}}} \prod_{(j,\ell)\in C_{p_1}^-} \frac{z_{j,\ell+b_{ij}}}{z_{j,\ell}} \right) \left( \prod_{(j,\ell)\in C_{p_2}^+} \frac{z_{j,\ell}}{z_{j,\ell+b_{ij}}} \prod_{(j,\ell)\in C_{p_2}^-} \frac{z_{j,\ell+b_{ij}}}{z_{j,\ell}} \right)
\]
(4.3)
is still a fraction with square-free denominator. Otherwise either $z_{j,\ell+b_{ij}}$ for some $(j,\ell) \in C_{p_1}^+$ or $z_{j,\ell}$ for some $(j,\ell) \in C_{p_1}^-$ in the first term also appear in the denominator of the second term. If $(j,\ell) \in C_{p_2}^+$, then $p_{t_1}$ and $p_{t_2}$ overlap at least at the vertex $(j,\ell)$. If $(j,\ell+b_{ij}) \in C_{p_2}^-$, then $p_{t_1}$ and $p_{t_2}$ overlap at least at a vertex $(i,r)$, where $\ell^{-1}(i,r) = (j \pm 1, \ell + 1)$ or $(j \pm 1, \ell + 2)$. This is a contradiction. Similarly we deal with $(j,\ell) \in C_{p_1}^-$. Therefore (by induction) $\chi_q^{-1}(L(m))$ is a Laurent polynomial with square-free denominator in the initial cluster variables. 

Recall that prime snake modules are prime, real modules, see Theorem 2.5. As a slight generalization, we have the following theorem.

**Theorem 4.2.** Snake modules are real simple modules.

**Proof.** We assume that $L(m)$ is a snake module. Then $L(m) = L(m_1) \otimes \cdots \otimes L(m_n)$ with $L(m_i)$ prime by Theorem 2.5. We only need to show that snake modules are real.

Using the fact that $\chi_q$ is a ring homomorphism, we have
\[
\chi_q(L(m) \otimes L(m)) = \chi_q(L(m))\chi_q(L(m))
\]
\[
= \chi_q(L(m_1)) \cdots \chi_q(L(m_n))\chi_q(L(m_1)) \cdots \chi_q(L(m_n))
\]
\[
= \chi_q(L(m_1) \otimes L(m_1)) \cdots \chi_q(L(m_n) \otimes L(m_n)).
\]

By [10, Theorem 3.4], we have the fact that for every $1 \leq \ell \leq n$, $\chi_q(L(m_\ell) \otimes L(m_\ell))$ has only one dominant monomial $m_\ell^2$.

Using Theorem 2.3 and Remark 2.4, for any $1 \leq i \neq j \leq n$, we see that
\[
\mathcal{P}(i(m_i),k_i(m_i))_{1 \leq i \leq T_i} \text{ and } \mathcal{P}(i(m_j),k_i(m_j))_{1 \leq i \leq T_j}
\]
are non-overlapping. So monomials with negative exponents occurring in $\chi_q(L(m_i) \otimes L(m_i))$ cannot be canceled by any monomial occurring in $\chi_q(L(m_j) \otimes L(m_j))$. Thus
\[
\chi_q(L(m))\chi_q(L(m)) = \chi_q(L(m) \otimes L(m))
\]
has only one dominant monomial $m^2$. This shows that $L(m) \otimes L(m)$ is simple, and thus $L(m)$ is real. 

**Remark 4.3.** (1) From Proposition 3.1, it follows that for different snake modules, the corresponding cluster monomials have different $g$-vectors with respect to a given initial seed.

(2) Combining Theorem 4.1 and Theorem 4.2, we give a partial answer of Conjecture 1.1.

Recall that $\mathcal{A}$ is the cluster algebra introduced by Hernandez and Leclerc in [20], also see Section 2.1. In [20], Hernandez and Leclerc applied the method of cluster mutations to give an algorithm for computing the $q$-characters of Kirillov–Reshetikhin modules by successive approximations.

By Theorem 4.1, given a snake module $L(m)$, the truncated $q$-character $\chi_q^{-1}(L(m))$ is a cluster monomial in $\mathcal{A}$. So we may regard $\chi_q^{-1}$ as a map from rigid modules (the generic kernel) over the Jacobian algebra to cluster monomials similar to the cluster character [28, 29]. Following
\[\chi_q^{-}(L(m))\] can be expressed as a Laurent polynomial with integer coefficients with respect to a given initial cluster. However, an explicit formula of the expansion for \(\chi_q^{-}(L(m))\) is usually very complicated. In the following theorem, we give explicitly the denominator of the cluster monomial associated to a snake module \(L(m)\).

**Theorem 4.4.** Suppose that \(L(m)\) is a snake module. Then the denominator of the cluster monomial associated to \(L(m)\) is

\[
d(\chi_q^{-}(L(m))) = \prod_{(i,r-d_i) \in \text{Supp}(K(m))} z_{i,r}. \tag{4.4}
\]

**Proof.** We first show that for any \((i,r-d_i) \in \text{Supp}(K(m))\), the variable \(z_{i,r}\) appears in the denominator of the cluster monomial associated to \(L(m)\).

We assume without loss of generality that \(m = \prod_{t=1}^{T} Y_{i_t,k_t}\), for some snake \((i_t,k_t)_{1 \leq t \leq T}\). The cluster monomial of \(L(m)\) is given by the truncated \(q\)-character \(\chi_q^{-}(L(m))\) after the change of variables (2.1). For any \((i,r)\) such that \((i,r-d_i) \in \text{Supp}(K(m))\), there exists a path \(p_t \in \mathcal{P}_{i_t,k_t}\) such that \((i,r)\) is the unique lower corner of \(p_t\). We choose \(p_t\) such that \(t\) is maximal. Define \(p_t \in \mathcal{P}_{i_t,k_t}\) by

\[
p_t = \begin{cases} p_{t}^{i_t,k_t} & 1 \leq \ell < t, \\ p_{t}^{i_t,k_t} & t < \ell \leq T. \end{cases}
\]

Then the \(T\)-tuple \((p_1, \ldots, p_T)\) is a set of non-overlapping paths, because \(p_t\) is a path between \(p_{t-1}^{i_t,k_t}\) and \(p_{t}^{-1}^{i_t,k_t}\). Using equation (3.7), we have

\[
\chi_q^{-}(L(m)) = \sum_{(p_1, \ldots, p_T) \in \mathcal{P}_{i,k} \cap G_0} \prod_{t=1}^{T} m(p_t).
\]

After performing the change of variables (2.1), the variable \(z_{i,r}\) appears in the denominator of \(\prod_{t=1}^{T} m(p_t)\). Thus the product (4.4) appears in the denominator of the cluster monomial \(\chi_q^{-}(L(m))\).

On the other hand, for any \(T\)-tuple \((p_1, \ldots, p_T)\) of non-overlapping paths, we have the following: If \(z_{i,r} = (i,r) \in G_0\), appears in the denominator of \(\prod_{t=1}^{T} m(p_t)\), then \((i,r) \in C_{p_t}^{-}\) or \((i,r-b_{ii}) \in C_{p_t}^{+}\) for some \(t\) by equation (4.2). For the lower corner \((i,r) \in C_{p_t}^{-}\), we have \((i,r-d_i) \in \text{Supp}(K(m))\), by Remark (3.10) (1). For the upper corner \((i,r-b_{ii}) \in C_{p_t}^{+}\), we need to modify our path \(p_t\) by replacing the point \((i,r-b_{ii})\) by the point \((i,r)\). Note that \((i,r) \in G_0\), so the modified path is still in \(\mathcal{P}_{i,k}\). Now Remark (3.10) (1) implies \((i,r-d_i) \in \text{Supp}(K(m))\). \(\square\)

We illustrate this result in our five running examples. Note that for each vertex \((i,r-d_i)\) in the support of \(K(m)\), as shown in Figures 9–13, we have a contribution \(z_{i,r}\) in the denominator. Here \(d_1 = 1\) in type \(A\) and \(d_1 = 2\) and \(d_2 = 1\) in type \(B_2\).

**Example 4.5.** In type \(A_3\), let \(m = Y_{1,-15}Y_{3,-11}Y_{3,-9}Y_{2,-6}\). Then by Theorem 4.4,

\[
d(\chi_q^{-}(L(m))) = z_{2,-2}z_{1,-3}z_{3,-3}z_{2,-4}z_{1,-5}z_{2,-6}z_{3,-7}z_{1,-7}z_{2,-8}z_{3,-9}z_{2,-11}z_{2,-12}z_{1,-13}.
\]

**Example 4.6.** In type \(A_3\), let \(m = Y_{2,-10}Y_{2,-6}\). Then by Theorem 4.4,

\[
d(\chi_q^{-}(L(m))) = z_{2,-2}z_{1,-3}z_{3,-3}z_{2,-4}z_{2,-6}z_{1,-7}z_{3,-7}z_{2,-8}.
\]
Example 4.7. In type $B_2$, let $m = Y_{2,-12}Y_{2,-6}$. Then by Theorem 4.4, 
$$d(\chi_q^-(L(m))) = z_{2,0}z_{1,-1}z_{2,-4}z_{2,-6}z_{1,-7}z_{2,-10}.$$  

Example 4.8. In type $A_3$, let $m = Y_{2,-8}Y_{2,-6}$. Then by Theorem 4.4, 
$$d(\chi_q^-(L(m))) = z_{2,-2}z_{1,-3}z_{3,-3}z_{2,-4}z_{1,-5}z_{3,-5}z_{2,-6}.$$  

Example 4.9. In type $A_3$, let $m = Y_{2,-4}Y_{2,-2}$. Then by Theorem 4.4, 
$$d(\chi_q^-(L(m))) = z_{2,0}z_{2,-2}.$$  

It is natural to ask whether all cluster variables with square-free denominator are always prime snake modules. The answer is No. The following example shows that there exists a module that is not a snake module and such that its truncated $q$-character corresponds to a cluster variable with square-free denominator.  

Example 4.10. In type $A_3$, let $m = Y_{1,-3}Y_{2,0}Y_{3,-3}$. This is not a snake module. Because the second coordinates in the indices do not form an increasing sequence. By [18, Example 12.2], we have 
$$[L(Y_{1,-3}Y_{2,0}Y_{3,-3})][L(Y_{2,0})] = [L(Y_{1,-3}Y_{2,0})][L(Y_{3,-3}Y_{2,0})] + [L(Y_{2,-2}Y_{2,0})].$$  

Thus by [20, Theorem 5.1] 
$$\chi_q^-(L(m)) = \frac{\chi_q(L(Y_{1,-3}Y_{2,0}))\chi_q(L(Y_{3,-3}Y_{2,0})) + \chi_q(L(Y_{2,-2}Y_{2,0}))}{\chi_q(L(Y_{2,0}))}.$$  

On the right-hand side of the equation, every truncated $q$-character is known by the Frenkel–Mukhin algorithm, so 
$$\chi_q^-(L(m)) = m(1 + A_{1,-2}^{-1} + A_{3,-2}^{-1} + A_{1,-2}^{-1}A_{3,-2}^{-1} + A_{1,-2}^{-1}A_{3,-2}^{-1}A_{2,-1}^{-1}).$$  

The corresponding cluster variable $x_m$ is 
$$x_m = \frac{(z_{1,-1}z_{3,-1} + z_{2,0}z_{1,-3})z_{2,-2} + z_{2,-2}^2 + (z_{2,0}z_{1,-3} + z_{2,0}z_{2,-2})z_{3,-3}}{z_{1,-1}z_{2,0}z_{3,-1}},$$  

which has a square-free denominator.  

Finally, we point out that there exists a module beyond snake modules in $\mathcal{C}^-$ for which Hernandez and Leclerc’s conjectural geometric formula holds.  

Example 4.11. In type $A_3$, let $m = Y_{1,-7}Y_{2,-4}Y_{3,-7}$. By [18], we know that $L(m)$ corresponds to a cluster variable in $\mathcal{A}$ (up to scalar), equivalently, its truncated $q$-character is a cluster variable in $\mathcal{A}$. Let 
$$I(m)^+ = I_{1,-8} \oplus I_{2,-5} \oplus I_{3,-8}, \quad I(m)^- = I_{1,-6} \oplus I_{2,-3} \oplus I_{3,-6}.$$  

By [18, Example 12.2], we have 
$$[L(Y_{1,-7}Y_{2,-4}Y_{3,-7})][L(Y_{2,-4})] = [L(Y_{1,-7}Y_{2,-4})][L(Y_{3,-7}Y_{2,-4})] + [L(Y_{2,-6}Y_{2,-4})].$$  

With the exception of $L(Y_{1,-7}Y_{2,-4}Y_{3,-7})$, those modules are minimal affinizations [3], and we can compute their $q$-characters by the Frenkel–Mukhin algorithm.  

On the other hand, the formula in Theorem 3.2 holds for $L(Y_{1,-7}Y_{2,-4}Y_{3,-7})$. The module $K(m)$ has dimension 10 and is displayed in Figure 14. In Figure 14, almost all vertices carry a
There are 70 submodules in $K_Y$, whose sign is easily deduced from the defining relations of the Jacobian algebra $A$. There are 70 submodules in $K(m)$.

Then

$$\chi_{\gamma}^{-}(L(m)) = m((1 + v_{1,-6} + v_{3,-6} + v_{1,-6v_{3},-6})(1 + v_{2,-3} + v_{2,-3v_{1},-2} + v_{2,-3v_{3},-2}$$

$$+ v_{1,-2v_{2},-3v_{3},-2} + v_{2,-3v_{1},-2v_{3},-2v_{2},-1})$$

$$+ (v_{1,-6v_{2},-5 + v_{3,-6v_{2},-5})(v_{2,-3} + v_{2,-3v_{1},-2} + v_{2,-3v_{3},-2} + v_{1,-2v_{2},-3v_{3},-2}$$

$$+ v_{2,-3v_{1},-2v_{3},-2v_{2},-1})$$

$$+ v_{1,-6v_{2},-3v_{3},-2v_{3},-4(1 + v_{1,-2} + v_{1,-2v_{2},-1})}$$

$$+ v_{3,-6v_{2},-3v_{3},-2v_{1},-4(1 + v_{3,-2} + v_{3,-2v_{2},-1})}$$

$$+ v_{1,-6v_{2},-3v_{3},-2v_{3},-4v_{3},-6(1 + v_{1,-2} + v_{1,-2v_{2},-1})}$$

$$+ v_{3,-6v_{2},-3v_{3},-2v_{1},-4v_{1},-6(1 + v_{3,-2} + v_{3,-2v_{2},-1})}$$

$$+ v_{1,-6v_{2},-3v_{3},-6v_{3},-6 + 2v_{1,-6v_{2},-3v_{2},-5v_{3},-6} + 2v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{3},-6v_{1},-2}$$

$$+ 2v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{3},-2} + 2v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{1},-2v_{3},-2}$$

$$+ 2v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{1},-2v_{3},-2v_{2},-1}$$

$$+ v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{3},-2v_{2},-5(1 + v_{1,-2} + v_{3,-2} + v_{1,-2v_{3},-2} + v_{1,-2v_{3},-2v_{2},-1})$$

$$+ v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{2},-5v_{1},-2v_{1},-4(1 + v_{3,-2} + v_{3,-2v_{2},-1})}$$

$$+ v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{2},-5v_{3},-2v_{3},-4(1 + v_{1,-2} + v_{1,-2v_{2},-1})}$$

$$+ v_{1,-6v_{2},-3v_{2},-5v_{3},-6v_{2},-5v_{1},-2v_{1},-4v_{3},-2v_{3},-4$$

$$+ v_{2,-3v_{3},-2v_{1},-2v_{3},-6v_{2},-5v_{1},-4v_{1},-6v_{2},-5v_{3},-4v_{2},-1),$$
in agreement with Hernandez and Leclerc’s conjectural geometric formula and its denominator is not square free. Indeed, there exists a submodule of \( K(m) \) whose support at vertices \((2, -3), (1, -6), (3, -6)\) such that

\[
mv_{1, -6}v_{2, -3}v_{3, -6} = mA_{1, -6}^{-1}A_{2, -3}A_{3, -6}^{-1} = \frac{z_5^2 z_3^2 z_6^2}{z_{1, -1} z_{1, -5} z_2^2 - z_{2, -4} z_3^2 - z_{3, -1}}.
\]

Moreover, the simple module \( L(Y_{1, -7}Y_{2, -4}Y_{3, -7}) \) is not special, because there are two dominant monomials \( m = Y_{1, -7}Y_{2, -4}Y_{3, -7} \) and \( mv_{1, -6}v_{2, -3}v_{3, -6} = Y_{2, -6} \) in \( \chi_q(L(Y_{1, -7}Y_{2, -4}Y_{3, -7})) \). The simple module \( L(Y_{1, -7}Y_{2, -4}Y_{3, -7}) \) is not thin, because some terms in the \( q \)-character have coefficient \( 2 > 1 \).

Using the method introduced in Section 3.3, we obtain the dimension vector \( (d_{j, s}(K(m)))_{(j, s) \in \mathbb{N}_0} \) of \( K(m) \) as follows.

\[
d_{j, s}(K(m)) = \begin{cases} 1 & (j, s) = (2, -1), (1, -2), (3, -2), (2, -3), (1, -4), (3, -4), (1, -6), (3, -6), \\ 2 & (j, s) = (2, -5), \\ 0 & \text{otherwise.} \end{cases}
\]

Comparing Example 4.10 with Example 4.11, we reformulate the following classification question.

**Question 4.12.** Is it true that the \( q \)-character of a simple module is a cluster variable (cluster monomial) with square-free denominator if and only if the module is a prime snake module (snake module).

**Remark 4.13.** It is important to notice that in the question we use the \( q \)-character and not the truncated \( q \)-character. For the truncated \( q \)-character, Example 4.10 would be a counter example.

5. **Factorial cluster algebras**

In this section, we apply the results of [16] to show that the \( \mathcal{C}_1 \) cluster algebra is factorial for Dynkin types \( \mathbb{A}, \mathbb{D}, \mathbb{E} \).

Following [18, Section 4.2], let \( Q \) be a quiver with vertex set \( \{1, \ldots, n, 1', \ldots, n'\} \) subject to the following two conditions.

1. The full subquiver on \( \{1, \ldots, n\} \) is an orientation of the associated Dynkin diagram \( \Delta \) of type \( \mathbb{A}, \mathbb{D} \) or \( \mathbb{E} \), oriented in such a way that every vertex in \( I_0 \) is a source and every vertex of \( I_1 \) is a sink.

2. For every \( i \in I \), one adds a frozen vertex \( i' \) and an arrow \( i' \rightarrow i \) if \( i \in I_0 \) and an arrow \( i \rightarrow i' \) if \( i \in I_1 \).

Obviously, the defining quiver \( Q \) is an acyclic quiver. Let \( \mathcal{A}(Q) \) be the cluster algebra defined by the initial seed \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\}, Q \). Then \( \mathcal{A}(Q) \) is the \( \mathcal{C}_1 \) cluster algebra of type \( \Delta \) in [18]. Let \( x_1', \ldots, x_n' \) be the \( n \) cluster variables obtained from the initial seed by one single mutation. Then, for each \( i \), we have \( x_i x_i' = f_i \), where \( f_i \) is a binomial in the initial seed. Recall from [16] that two vertices \( i, j \in \{1, 2, \ldots, n\} \) are called partners if \( f_i \) and \( f_j \) have a non-trivial common factor. Partnership is an equivalence relation and the equivalence classes are called partner sets.

**Theorem 5.1.** The \( \mathcal{C}_1 \) cluster algebra is factorial for Dynkin types \( \mathbb{A}, \mathbb{D}, \mathbb{E} \).
Proof. By [16, Corollary 5.2], we only need to show that every partner set in \( Q \) is a singleton. This holds because for every \( i \in I \) the variable \( y_i \) appears in the polynomial \( f_i \). \[ \square \]

We give an example to explain Theorem 5.1.

Example 5.2. Let \( g \) be of type \( A_3 \). We choose \( I_0 = \{1, 3\} \) and \( I_1 = \{2\} \). The quiver \( Q \) is as follows.

Here the vertices with boxes are frozen vertices and its associated exchange matrix is

\[
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The exchange polynomials are

\[
f_1 = x_2 + y_1,\\n f_2 = x_1 x_3 + y_2,\\n f_3 = x_2 + y_3.
\]

The polynomials \( f_1, f_2, f_3 \) are pairwise coprime and hence every partner set in \( Q \) is a singleton.

Remark 5.3. Geiss, Leclerc, and Schröer proved in [17] that the cluster algebra \( \mathcal{A} \) associated to Dynkin type \( A_1 \) is a factorial cluster algebra. They also showed that the cluster variables in a factorial cluster algebra are prime elements. In [16, Theorem 3.10], it was shown that if \( \mathcal{A} \) is a factorial cluster algebra and \( x \) is a non-initial cluster variable, then the associated \( F \)-polynomial \( F_x \) is prime.

It is natural to ask whether the \( \mathcal{C}_\ell \) cluster algebras, with \( \ell > 1 \), are factorial. The argument in the proof of Theorem 5.1 does not work in this case, because we do not know whether these cluster algebras are of acyclic type.

Acknowledgement. We would like to thank A. Garcia Elsener for explaining the results of [16] to us.

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Bing Duan
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000
P. R. China
duan890818@163.com

Ralf Schiffler
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs, CT 06269
USA
schiffler@math.uconn.edu

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