A polynomial time log barrier method for problems with nonconvex constraints

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Abstract

Interior point methods (IPMs) that handle nonconvex constraints such as IPOPT, KNITRO and LOQO have had enormous practical success. Unfortunately, all known analyses of log barrier methods with general constraints (implicitly) prove guarantees with exponential dependencies on $1/\mu$ where $\mu$ is the barrier penalty parameter. This paper provides an IPM that finds a $\mu$-approximate Fritz John point in $O(\mu^{-7/4})$ iterations when the objective and constraints have Lipschitz first and second derivatives. For this setup, the results represent both the first polynomial time dependence on $1/\mu$ for a log barrier method and the best-known guarantee for finding Fritz John points. We also show that, given convexity and regularity conditions, our algorithm finds an $\epsilon$-optimal point in at most $O(\epsilon^{-2/3})$ iterations. The algorithm that we study in this paper, although naive, provides inspiration for our practical one-phase IPM [16].

1 Introduction

This paper is concerned with the following problem:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{such that} \quad a(x) \geq 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $a : \mathbb{R}^n \to \mathbb{R}^m$ have Lipschitz continuous first and second derivatives. Since finding the global optimum to this problem has an exponential worst-case runtime [25] we instead seek a Fritz John point [17], a necessary condition for local optimality, defined as a point $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$

$$t, a(x), y \geq 0$$
$$y_i a_i(x) = 0 \quad \forall i \in \{1, \ldots, m\}$$
$$t \nabla f(x) - \nabla a(x)^T y = 0$$

where $y$ are dual variables, $t$ is a scalar which is equal to one in the KKT conditions, and $(y, t) \neq 0$. When the Magenerian Fromovitz constraint qualification [20] holds all Fritz John points are KKT points. Since it is not possible to find an exact Fritz John point we require a notion of an approximate Fritz John point. For our purposes, an approximate Fritz John point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfies

$$a(x), y > 0$$
$$y_i a_i(x) \in [\mu/2, 3\mu/2] \quad \forall i \in \{1, \ldots, m\}$$
$$\|\nabla x \mathcal{L}(x, y)\|_2 \leq \mu \sqrt{\|y\|_1 + 1},$$

where the Lagrangian is $\mathcal{L}(x, y) := f(x) - y^T a(x)$, and $\mu > 0$ a parameter measuring the accuracy of our approximation with small $\mu$ desirable. There are many other possible definitions for an
approximate Fritz John point, we chose this definition because it is the most natural condition for our interior point method to satisfy.

Our approach is loosely inspired by feasible start IPMs \[19, 21, 23, 31\] and trust region algorithms \[12, 34\]. To guide our trust region method we use the log barrier,

$$
\psi_\mu(x) := f(x) - \mu \sum_{i=1}^m \log(a_i(x)),
$$

with some parameter $\mu > 0$ and start from a strictly feasible point. The log barrier penalizes points too close to the boundary, this allows us to use unconstrained methods to solve a constrained problem. Typically, if $f$ and each $a_i$ were linear we would apply Newton’s method to the log barrier. However, since we allow $a_i$ to be nonlinear, $\nabla^2 \psi_\mu$ could be singular or (if $a_i$ is not concave) indefinite. To avoid this issue, we use a trust region method, to generate our next direction we solve problems of the form

$$
d_x \in \arg\min_{u \in B_r(0)} M_x^{\psi_\mu}(u)
$$

with

$$
M_x^{\psi_\mu}(u) := \frac{1}{2} u^T \nabla^2 \psi_\mu(x) u + \nabla \psi_\mu(x)^T u
$$

$$
B_r(v) := \{ x \in \mathbb{R}^n : \| x - v \|_2 \leq r \}.
$$

Critically, we vary the size of the radius $r$ to scale with the size of the current dual iterates. From one iteration to the other this radius size can vary dramatically due to large swings in the size of the dual iterates.

We now briefly overview our results, omitting Lipschitz constants and higher-order terms for cleanliness. Our main results assume that we are given a feasible starting point, i.e.,

$$
x^{(0)} \in \mathcal{X} := \{ x \in \mathbb{R}^n : a(x) > 0 \}.
$$

This assumption is removed in Section 7 where we use a two-phase algorithm: first minimizing the constraint violation to obtain a feasible point and then minimizing the objective subject to the constraint violation. The first main result is Theorem 1 this result assumes the function $a_i$ and $f$ have Lipschitz first and second derivatives on the set $\mathcal{X}$. Under these conditions Theorem 1 states that after at most

$$
O \left( \left( \psi_\mu(x^{(0)}) - \inf_{z \in \mathcal{X}} \psi_\mu(z) \right) \mu^{-7/4} \right)
$$

trust region subproblem solves we find a $\mu$-approximate Fritz John point, i.e., a point satisfying (1).

Our second main result is Theorem 2. This result assumes the function $a_i$ and $f$ have Lipschitz first and second derivatives. In addition, it assumes that the constraints are concave functions (implying the feasible region is convex) and regularity condition hold to ensure Fritz John points are KKT points. Under these assumptions Theorem 2 states that after at most

$$
O( (m^{1/3} \epsilon^{-2/3} + m^2 )
$$

trust region subproblem solves we find a $\epsilon$-optimal solution, i.e., a point $x$ with $f(x) - \inf_{z \in \mathcal{X}} f(z) \leq \epsilon$.

We proceed as follows. The remainder of the introduction gives definitions and overviews related work. Section 2 analyzes gradient descent applied to the log barrier and explains why previous analyses gave (implicitly) exponential runtime bounds in $\mu$. Section 3 introduces our main algorithm, a trust region IPM. Section 4 gives a series of useful Lemmas for the analysis. Section 5 proves Theorem 1 and Section 6 proves Theorem 2. Section 7 compares the runtime our IPM achieves with existing runtime bounds for problems with nonconvex constraints [3, 8, 9].
1.1 Definitions

Let $\text{diag}(v)$ be a diagonal matrix with entries comprising of the vector $v$. Let $\mathbf{R}$ denote the set of real numbers, $\mathbf{R}_+$ denote the set of nonnegative real numbers and $\mathbf{R}_{++}$ the set of strictly positive real numbers. Let $\text{Convex}\{x, y\} = \{\alpha x + (1 - \alpha) y : \alpha \in [0, 1]\}$. Let $\lambda_{\min}(\cdot)$ denote the minimum eigenvalue of a matrix.

**Definition 1.** Let $L_p \in (0, \infty)$ be a constant and $p$ a nonnegative integer. A function is $w : \mathbf{R}^n \rightarrow \mathbf{R}$ has $L_p$-Lipschitz $p$th derivatives on the set $S \subseteq \mathbf{R}^n$ if for any $x \in S$ and $v \in \mathbf{B}_1(0)$ the one-dimensional function $g : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$g(\theta) := w(x + v\theta)$$

satisfies $|g^{(p+1)}(0)| \leq L_p$.

Taylor’s theorem states that given a one-dimensional function $g : \mathbf{R} \rightarrow \mathbf{R}$ with $L_p$-Lipschitz $p$th derivatives on the set $[0, \theta]$ then for all $q \in \{0, \ldots, p\}$ one has

$$\left| \sum_{i=0}^{p-q} \frac{\theta^i g^{(q+i)}(0)}{i!} - g(q)(\theta) \right| \leq \frac{L_p|\theta|^{1+p-q}}{(1 + p - q)!}.$$  

See [32, Theorem 50.3] for a proof of the remainder version of this theorem.

We will often refer to the function $a : \mathbf{R}^n \rightarrow \mathbf{R}^m$ as having $L_p$-Lipschitz $p$th derivatives. In this case, we mean that each component function $a_i$ has $L_p$-Lipschitz $p$th derivatives. Finally $\nabla a(x)$ is the $m \times n$ Jacobian of $a(x)$.

1.2 Related work and motivation

Interior point methods (IPMs) have excellent practical performance in linear [22], conic [35], general convex [1] and nonconvex optimization [5, 37, 40]. The theoretical performance of IPMs for linear [18, 31, 41, 45, 46] and conic [27] optimization is well-studied. The main result in this area is that it takes at most $O(\sqrt{v} \log(1/\epsilon))$ iterations to find an $\epsilon$-global optima where $v$ is the self-concordance parameter (for linear programming $v = m + n$). Each iteration consists of a Newton step, i.e., one linear system solve, applied to an unconstrained optimization problem. Unfortunately, this approach only works for convex cones with tractable self-concordant barriers functions.

While self-concordance theory is designed for structured convex problems, there is a rich literature on the minimization of general blackbox unconstrained objectives, particularly if the objective is convex [23, 28]. Here we briefly review results in nonconvex optimization. In unconstrained nonconvex optimization the measure of local optimality is usually whether $\|\nabla f(x)\|_2 \leq \epsilon$, known as an $\epsilon$-stationary point. A fundamental result is that gradient descent needs at most $O(\epsilon^{-2})$ iterations to find an $\epsilon$-stationary point if the function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ has Lipschitz continuous first derivatives. Nesterov and Polyak [28] showed that cubic regularized Newton has a better iteration guarantee of $O(\epsilon^{-3/2})$ for finding finding stationary points. The same runtime bound can be extended to trust region methods [13, 14]. These $O(\epsilon^{-2})$ and $O(\epsilon^{-3/2})$ runtimes match the black box lower bounds for functions with Lipschitz continuous first and second derivatives respectively [6, 7].

However, there is relatively little theory studying nonconvex optimization with constraints. An important contribution in this area are the work of Ye [43], Bian et al. [2], Haeser et al. [15] who consider an affine scaling technique for general objectives with linear inequality constraints, i.e., $a_i$ are linear. At each iteration they solve problems of the form:

$$d_x \in \arg\min_{u \in \mathbf{R}^n : \|S^{-1} \nabla a(x) + u\|_2 \leq \epsilon} \mathcal{M}_{\mu}^S(u)$$

with $S = \text{diag}(a(x))$. In this context [15] give an algorithm with an $O(\mu^{-3/2})$ runtime for finding KKT points. This work is pertinent to ours, but the addition of nonconvex constraints and the use of a trust region method instead of affine scaling distinguish this paper.
A motivation for our work is trying to understand the performance of practical interior point methods — most of which tend to use an approach similar to our paper. To see this relationship observe that if we are at a feasible solution and set the dual variables to exactly satisfy perturbed complementarity \((y = \mu S^{-1}1)\) then LOQO \([37]\) and the one-phase IPM \([16]\) both generate directions of the form

\[
d_x \in \arg\min_{u \in \mathbb{R}^n} \mathcal{M}_x^{\psi_{\mu}}(u) + \delta\|u\|_2^2
\]

for some \(\delta > 0\) chosen such that \(\nabla^2 \psi_{\mu}(x) + \delta I \succ 0\). There is a well-known duality between this modified Newton approach and the trust-region approach. In particular, for any \(\delta > 0\) there exists some \(r > 0\) such that the direction generated by (5) satisfies

\[
d_x \in \arg\min_{u \in B_r(x)} \mathcal{M}_x^{\psi_{\mu}}(u).
\]

The reverse statement holds except in the hard case \([29, \text{Chapter 4}.]\). Therefore, our algorithm can be heuristically viewed as a simplified variant of LOQO, the one-phase IPM or IPOPT \([40]\). There are major differences between our paper and practical methods: we ignore feasibility issues, our method is not primal-dual, we use a trust-region instead of adding \(\delta I\) to the Hessian, and our algorithm require knowledge of Lipschitz constants. However, these differences should be viewed in context of our goal: to develop a simple algorithm that captures the essence of practical nonconvex interior point codes.

While, there has been theoretical work studying these practically successful log barrier methods with nonconvex constraints, most of this work tends to only show that algorithm eventually converges \([4, 10, 11, 14, 16, 39]\) without giving runtime bounds, or focuses on superlinear convergence in regions close to local minima \([36, 38]\). However, there has been analysis of other methods for optimization with nonconvex constraints using methods other than IPMs \([3, 8, 9]\). We compare with these results in Section 7.

There is a vast body of literature analyzing the convergence of unconstrained optimization methods on self-concordant functions or functions with Lipschitz derivatives. Unfortunately, with general constraints one cannot assume that the log barrier is self-concordant nor that the derivatives are Lipschitz (even if the derivatives of the constraints are Lipschitz). Therefore this paper develops a new approach. To help the reader understand the crux of this problem, we begin by analyzing the worst-case performance of gradient descent on the log barrier.

\section{A warm-up: gradient descent on the log barrier}

This sections explains how naive theoretical analysis (which is often used to analyze IPM with nonlinear constraints) can give exponential runtime bounds for gradient descent for finding stationary points of the log barrier. At the end of this section we provide a simple fix to the analysis of gradient descent on the log barrier. Hence the exponential runtime bounds are a flaw of the analysis — not the algorithm. The goal of this section is to get the reader into the correct mindset for analyzing the more challenging trust region IPM that are the focus of this paper.

The log barrier does not have Lipschitz continuous derivatives. However, typical analysis of interior point methods in the nonlinear programming community is as follows:

\begin{enumerate}[A.]
\item Observe that if we apply a descent method to the log barrier all iterates remain in the set \(S := \{x \in \mathbb{R}^n : \psi_{\mu}(x) \leq \psi_{\mu}(x^{(0)})\}\) where \(x^{(0)}\) is the starting point.
\item Show \(p\)th derivatives are \(L_p\)-Lipschitz continuous on the set \(S\). This is usually done by showing there exists some \(\varepsilon > 0\) such that if \(x \in S\) then \(a(x) > \varepsilon\) and using the assumption that the objective and constraints have Lipschitz derivatives.
\item Prove that for sufficiently small steps the line segment between the current and new iterates remains in \(S\). Apply generic bounds from cubic regularization/gradient descent to give the runtime.
\end{enumerate}
For examples of this style of analysis see [4, 10, 11, 16]. Turning this into a polynomial time proof requires showing that the constant $L_p$ is a polynomial function of the desired tolerance. However $L_p$, can have an exponentially large value in $\mu$ because the bound on $\varepsilon$ is exponentially small in $\mu$. This can occur even when the constraints are linear. For example, consider the log barrier arising from the linear program $\min x \text{ s.t. } 0 \leq x \leq 2$, 

$$\psi_\mu(x) := x - \mu (\log(x) + \log(2-x))$$

with $\mu \in (0, 1)$. Let us assume $x^{(0)} = 1$ and $\exp(-1/\mu) \leq 1$. We will show under these assumptions that the Lipschitz constants for the first and second derivatives are exponentially large in $1/\mu$ on the set $S := \{ x \in \mathbb{R}^n : \psi_\mu(x) \leq \psi_\mu(x^{(0)}) \}$. Observe that $\psi_\mu(x^{(0)}) = 1$ and at the point $x = \exp(-1/\mu) \leq \psi_\mu(x^{(0)})$ we have $\nabla^2 \psi_\mu(x) = \mu \left( \frac{1}{x^2} + \frac{1}{(2-x)^2} \right) \geq \mu \exp(2/\mu)$ and $\nabla^3 \psi_\mu(x) = 2 \mu \left( -\frac{1}{x^2} + \frac{1}{(2-x)^2} \right) \leq -\mu \exp(3/\mu)$. This is illustrated in Figure 1.

The methods [4, 10, 11, 16] use line searches to choose the step sizes rather than fixed step sizes. Line search methods have many benefits over constant step size methods, including removing the need to do hyperparameter searches over Lipschitz constants and faster convergence in practice. However, the (A)-(C) argument where we prove a uniform bound on the Lipschitz constant of $\nabla \psi_\mu$ is roughly equivalent to proving a runtime bound on a constant step size algorithm and then arguing that an adaptive step size algorithm is faster than the constant step size algorithm. Therefore the adaptive step size method inherits the worst-case runtime bound of the constant step size algorithm. While in some situations this argument gives a good worst-case runtime bound, as we soon show, there exists problem classes where the worst-case runtime bound of the constant step size method is exponentially worse than an adaptive method.

Claim 1 shows that gradient descent with a fixed step size $\alpha \in (0, \infty)$, i.e.,

$$x^{(k+1)} \leftarrow x^{(k)} - \alpha \nabla \psi_\mu(x^{(k)}) \quad (7)$$

cannot efficiently minimize a log barrier.

**Claim 1.** Let $\psi_\mu(x) := x - \mu (\log(x) + \log(2-x))$, $\psi_\mu^* = \inf_z \psi_\mu(z)$, $\mu \in (0, 1/5]$ and $C \in [10, \mu \exp(1/\mu)]$. Consider the set $S_C = \{ x \in \mathbb{R} : \psi_\mu(x) \leq \psi_\mu^* + C \}$. Fix $\alpha \in (0, \infty)$. Suppose the $x^{(k)}$ iterates satisfy [1] and remain in the interval $[0, 2]$ for any starting point $x^{(0)} \in S_C$. Then for the starting point $x^{(0)} = 1 \in S_C$ and for all $k \leq 16C \mu \exp(1/\mu)$ we have $\| \nabla \psi_\mu(x^{(k)}) \| \geq \mu$. 

Figure 1: Why a traditional nonlinear programming analysis of IPMs will not give polynomial time bound in $1/\mu$. In this example $\mu = 0.5$. 

Derivatives are moving very quickly and have exponentially large Lipshitz constant in $\mu$. 

Region iterates must lie in:

$$S = \{ x : \psi_\mu(x) \leq \psi_\mu(1) \}$$

initial point $x^{(0)} = 1$.
The proof appears in Appendix A and involves first arguing the step size \( \alpha \) must be tiny, otherwise, if we initialize close to the boundary the iterates will leave the feasible region. Furthermore, if the step size \( \alpha \) is tiny then if we initialize away from the boundary the algorithm will converge very slowly.

An astute reader might observe that Claim 1 is dependent on allowing a starting point close to the boundary. However, any constant step size algorithm that circumvents this issue must show that all of its iterates do not get too close to the boundary. This requires an innovation on the (A)-(C) argument. Moreover, the fact that the log barrier does not have Lipschitz continuous derivatives causes the same issues for cubic regularized Newton with a fixed regularization parameter or trust region methods with a fixed trust region radius. Implicitly when using the analysis (A)-(C) we are arguing our algorithm cannot do worse than a constant step size algorithm. Unfortunately, as we have seen in Claim 1 constant step size algorithms can be very poor benchmarks.

This is the insight of the polynomial time IPM analysis for linear programming—it circumvents these issues using the self-concordant properties of the barrier function \([27]\). However, the function
\[-\mu \log(a(x))\]
in general is not self-concordant. While we do not expect to obtain an algorithm with a polynomial dependent on \( \log(1/\mu) \), can we still obtain a polynomial time algorithm in the desired tolerance \( 1/\mu \)? As a warm up we show gradient descent with the adaptive step size routine, defined as
\[
y^{(k)}_i \leftarrow \frac{\mu}{a_i(x^{(k)})} \quad \forall i \in \{1, \ldots, m\} \tag{8a}
\]
\[
d^{(k)} \leftarrow -\nabla \psi_\mu(x^{(k)}) \tag{8b}
\]
\[
x^{(k+1)} \leftarrow x^{(k)} + \alpha^{(k)} d^{(k)} \tag{8c}
\]
This procedure does not tell us how to choose \( \alpha^{(k)} \). One approach is to pick,
\[
\alpha^{(k)} \leftarrow \min \left\{ \frac{\min_i a_i(x^{(k)})}{2L_0 \|d^{(k)}\|_2^2}, \frac{1}{\ell_1(x^{(k)})} \right\} \tag{9}
\]
where the term \( \frac{\min_i a_i(x^{(k)})}{2L_0 \|d^{(k)}\|_2^2} \) represents the step size that guarantees \( a_i(x^{(k+1)}) > 0 \) and
\[
\ell_1(x) := L_1 (1 + 2 \|y\|_1) + \frac{4L_0^2 \|y\|_2^2}{\mu} \quad \text{with} \quad y_i = \frac{\mu}{a_i(x)} \quad \text{for} \quad i \in \{1, \ldots, m\}
\]
represents the ‘local’ Lipschitz constant of \( \nabla \psi_\mu \) at the point \( x \). See Figure 2 for intuitive justification for this scheme. To prove our results we require the following assumptions.

**Assumption 1.** (Lipschitz function and first derivatives) Assume that each \( a_i : \mathbb{R}^n \rightarrow \mathbb{R} \) for \( i \in \{1, \ldots, m\} \) is a continuous function on \( \mathbb{R}^n \). Let \( L_0, L_1 \in (0, \infty) \). Assume that, on the set \( \mathcal{X} \), each \( a_i \) is \( L_0 \)-Lipschitz continuous with \( L_1 \)-Lipschitz continuous derivatives. Also assume the first derivatives of \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) are \( L_1 \)-Lipschitz continuous on the set \( \mathcal{X} \).

We assume that each \( a_i \) is continuous on \( \mathbb{R}^n \) because if we removed this assumption then a function such as
\[
a_i(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}
\]
would satisfy assumption 1 with \( L_0 \) and \( L_1 \) arbitrarily small. The discontinuity of the function at \( x = 0 \) with \( a_i(x) = 1 \) would mean we could not guarantee that the next iterate was feasible, even when the step size taken was arbitrarily small.

Assumption 1 is quite general since if \( \mathcal{X} \) is a bounded set, and if \( f \) and each \( a_i \) are differentiable functions on \( \mathbb{R}^n \) then \( f \) and \( a_i \) are Lipschitz functions with Lipschitz first derivatives. Of course, this does not give an explicit value for these Lipschitz constants which could be arbitrarily big depending on the functions \( f \) and \( a_i \).
Derivatives are moving very quickly. Take small gradient descent steps sizes.

Derivatives are moving slowly. Take large gradient descent steps.

Figure 2: Explanation of adaptive step sizes.

Claim 2. Let $\tau_1, \mu \in (0, \infty)$. Suppose assumption [2] holds. Let $x^{(0)} \in \mathcal{X}$ be the initial point. Then after at most

$$k = O \left( 1 + (\psi_{\mu}(x^{(0)}) - \inf_{z \in \mathcal{X}} \psi_{\mu}(z))(L_0 \tau_1^{-1} \mu^{-2} + L_0^2 \tau_1^{-2} \mu^{-3} + L_1 \tau_1^{-2} \mu^{-2}) \right)$$

iterations the procedure (8) finds a point $(x^{(k)}, y^{(k)})$ with

$$\|\nabla \psi_{\mu}(x^{(k)})\|_2 \leq \tau \mu(1 + \|y^{(k)}\|_1).$$

Claim 2 is a consequence of the following Lemma which proves that indeed $\ell_1(x)$ represents the local Lipschitz constant for $\nabla \psi_{\mu}$.

Lemma 1. Let $v \in B_1(0)$ and $x \in \mathcal{X}$. Let the assumptions of Claim 2 hold and $g(\theta) := \psi_{\mu}(x + \theta v)$. For all $\theta \in \left[0, \frac{\min a_i(x)}{2L_0}\right]$ we have $\frac{a_i(x + \theta v)}{a_i(x)} \in [1/2, 3/2]$ for all $i = 1, \ldots, m$ and

$$\left| g^{(2)}(\theta) \right| \leq \ell_1(x).$$

Proof of Lemma 1: To obtain a contradiction assume $|a_i(x + \theta v) - a_i(x)| > a_i(x)/2$ for some $\theta \in \left[0, \frac{\min a_i(x)}{2L_0}\right]$ and $i \in \{1, \ldots, m\}$. Define $q(\theta) := \sup_{\tilde{\theta} \in [0, \theta]} \left| a_i(x + v\tilde{\theta}) - a_i(x) \right|$. Since $a_i$ is continuous $q$ is continuous and by the intermediate value theorem there exists some $\tilde{\theta} \in [0, \theta]$ such that $q(\tilde{\theta}) = a_i(x)/2$. By assumption $a_i(x)$ is Lipschitz continuous on the set $\mathcal{X}$ and $a_i(x + \theta v) \geq \frac{a_i(x)}{2}$ for all $\theta \in \left[0, \tilde{\theta}\right]$ we have

$$\left| a_i(x) - a_i(x + \theta v) \right| = q(\theta) \leq L_0 \tilde{\theta} < \frac{a_i(x)}{2},$$

obtaining our required contradiction. Since $\frac{a_i(x + \theta v) - a_i(x)}{a_i(x)} \leq \frac{1}{2} \Rightarrow a_i(x + \theta v) \leq \frac{3}{2}$ and $\frac{a_i(x) - a_i(x + \theta v)}{a_i(x)} \leq \frac{1}{2} \Rightarrow a_i(x + \theta v) \geq \frac{1}{2}$, we have established $\frac{a_i(x + \theta v)}{a_i(x)} \in [1/2, 3/2]$.

Let $x^+ = x + \theta v$. Now,

$$\nabla^2 \psi_{\mu}(x^+) = \nabla^2 f(x^+) + \mu \sum_{i=1}^{m} \left( \frac{\nabla^2 a_i(x^+)}{a_i(x^+)} + \frac{\nabla a_i(x^+) \nabla a_i(x^+)^T}{a_i(x^+)^2} \right).$$
Using \( \frac{2i(x^+)}{n_i(x)} \in [1/2, 3/2] \) and \( y_i = \frac{\mu}{n_i(x)} \) it follows that
\[
\left| g^{(2)}(\theta) \right| = \left| v^T \nabla^2 \psi_\mu(x + \theta v)v \right| \leq L_1 + \mu \sum_{i=1}^m \left( \frac{2L_1}{a_i(x)} + \frac{4L_0}{a_i(x)^2} \right) = L_1(1+2\|y\|_1)+\frac{4L_0\|y\|_2^2}{\mu} = \ell_1(x).
\]

With Lemma 1 in hand we can now prove Claim 2.

**Proof of Claim 2** At each iteration of (8) with \( \|\nabla \psi_\mu(x^{(k)})\|_2 \geq \gamma_\mu(\|y^{(k)}\|_1 + 1) \) we have
\[
\psi_\mu(x^{(k)}) - \psi_\mu(x^{(k+1)}) \geq \alpha^{(k)} \left( \nabla \psi_\mu(x^{(k)})^T d^{(k)} + \frac{1}{2} \ell_1(x^{(k)}) \alpha^{(k)} \|y^{(k)}\|_2^2 \right)
\]
\[
= \alpha^{(k)} \left( \|\nabla \psi_\mu(x^{(k)})\|_2^2 \right) \left( 1 - \frac{1}{2} \alpha^{(k)} \ell_1(x^{(k)}) \right)
\]
\[
\geq \frac{\alpha^{(k)}}{2} \left( \frac{\|\nabla \psi_\mu(x^{(k)})\|_2^2}{\min_i a_i(x^{(k)})} \right) \left( 1 \frac{\ell_1(x^{(k)})}{\|\nabla \psi_\mu(x^{(k)})\|_2^2} \right)
\]
\[
\geq \frac{1}{2} \min \left\{ \frac{\|\nabla \psi_\mu(x^{(k)})\|_2^2}{\min_i a_i(x^{(k)})} \right\} \frac{1}{\ell_1(x^{(k)})} \frac{\ell_1(x^{(k)})}{\|\nabla \psi_\mu(x^{(k)})\|_2^2}
\]
\[
= \frac{1}{2} \min \left\{ \frac{\tau_\mu^2 \mu^2}{\tau_\mu^2 \mu^2 (1+\|y^{(k)}\|_1)^2} \right\}
\]
\[
\geq \frac{1}{2} \min \left\{ \frac{\tau_\mu^2 \mu^2}{\tau_\mu^2 \mu^2 (1+\|y^{(k)}\|_1)^2} \right\}
\]
\[
\geq \frac{1}{2} \frac{\tau_\mu^2 \mu^2}{\tau_\mu^2 \mu^2 (1+\|y^{(k)}\|_1)^2} \frac{\ell_1(x^{(k)})}{\|\nabla \psi_\mu(x^{(k)})\|_2^2}
\]
\[
\geq \frac{1}{\mu} \frac{\|\nabla \psi_\mu(x^{(k)})\|_2}{\mu} \frac{\|\nabla \psi_\mu(x^{(k)})\|_2^2}{\mu}
\]
where the first inequality uses Lemma 1 and Taylor’s theorem, the second and third inequality uses (9), the fourth inequality uses \( \|\nabla \psi_\mu(x^{(k)})\|_2 \geq \gamma_\mu(\|y^{(k)}\|_1 + 1) \) and \( \min_i a_i(x) \geq \mu/\|y^{(k)}\|_1 \).

Therefore if \( \|\nabla \psi_\mu(x^{(k)})\|_2 \geq \gamma_\mu(\|y^{(k)}\|_1 + 1) \) for \( k = 0, \ldots, K \)
\[
\psi_\mu(x^{(0)}) - \inf_{z \in \mathcal{X}} \psi_\mu(x) \geq \psi_\mu(x^{(0)}) - \psi_\mu(x^{(K)}) = \sum_{k=0}^{K} (\psi_\mu(x^{(k)}) - \psi_\mu(x^{(k+1)})) \geq \frac{K}{2} \min \left\{ \frac{\tau_\mu^2 \mu^2}{\tau_\mu^2 \mu^2} \frac{\ell_1(x^{(k)})}{\|\nabla \psi_\mu(x^{(k)})\|_2^2} \right\}
\]
rearranging this expression to upperbound \( K \) gives the result.

This section demonstrated that gradient descent with a constant step sizes applied to the log barrier requires an exponential amount of time to find a Fritz John point whereas gradient descent with adaptive step sizes requires a polynomial amount of time. While it is well-known that constant step size algorithms are practically slower than adaptive step size algorithms, all known theoretical results both in convex and nonconvex optimization, show no difference in the worst-case performance of these methods. Therefore we have demonstrated that adaptive step sizes can improve worst-case performance guarantees.

Finally, we remark that the algorithms in this paper are not practical, for example, they require knowledge of unknown Lipschitz constants. Therefore our primary contributions are theoretical. It remains a subject of further inquiry to develop practical methods with similar worst-case guarantees. One possibility is to replace (9) with a backtracking line search using the Armijo rule, i.e., starting with the trial step size \( \alpha^{(k)} = 1 \) (assuming \( L_1 > 1 \)) and backtracking until satisfying
\[
\psi_\mu(x^{(k)}) + \alpha^{(k)} d^{(k)} \leq \psi_\mu(x^{(k)}) + c \alpha^{(k)} \nabla \psi_\mu(x^{(k)})^T d^{(k)}
\]
for some constant \( c \in (0, 1) \). This approach also obtains the same iteration bound as Claim 2 but requires \( O(B^2/\mu^2) \) backtracking steps at each iteration where \( B \geq \psi_\mu(x^{(0)}) - \psi_\mu(x) + \mu \log(a_i(x)) \) for all \( i \in \{1, \ldots, m\} \) and \( x \in \mathcal{X} \). This bound on the number of backtracking steps comes from the fact that \( a_i(x) \geq \exp(-B/\mu) \) for all \( x \in \{ z \in \mathcal{X} : \psi_\mu(z) \leq \psi_\mu(x^{(0)}) \} \).
3 Our trust region IPM

This section introduces our trust region IPM (Algorithm 1). First, we develop some notation to help describe the algorithm. Recall that

\[ M_x^{\psi_\mu}(u) := \frac{1}{2}u^T \nabla^2 \psi_\mu(x)u + \nabla \psi_\mu(x)^Tu. \]

The function \( M_x^{\psi_\mu}(u) \) is a second-order Taylor series local approximation of the function \( \psi_\mu(x) \) at the point \( x \). It predicts how much the function \( \psi_\mu \) changes as we change move from \( x \) to \( x + u \). A naive algorithm we could use is

\[
\begin{align*}
    d^{(k+1)}_x & \leftarrow x^{(k)} + d^{(k)}_x \\
    x^{(k+1)} & \leftarrow x^{(k)} - \frac{\mu}{L} d^{(k)}_x
\end{align*}
\]

for some fixed constant \( r \in (0, \infty) \). If the function \( \nabla^2 \psi_\mu \) is \( L_2 \)-Lipschitz then one can show a convergence to an \( \epsilon \)-approximate stationary point of \( \psi_\mu \) in \( O(L_2^2 \epsilon^{-3/2}) \) iterations [28]. However, as we described in Section 2 this method will struggle because the log barrier ensures the effective Lipschitz constant of \( \nabla \psi_\mu \) is exponentially large in \( \mu \). Instead, as per line 7 of Algorithm 1 we adaptively choose the trust region radius using the formula

\[
r \leftarrow \eta_x \sqrt{\frac{\mu}{L_1(1 + \|y\|_1)}},
\]

this ensures that for constant \( \eta_x \in (0, \infty) \) the trust region radius becomes smaller as the dual variable size increases.

However, what if the predicted progress \( M_{x^{(k)}}^{\psi_\mu}(d_x) \) from our model is small? In this case we would like to find an approximate Fritz John point. To do this we need a method for selecting the dual variable \( y^+ \). An instinctive solution would be to pick \( y^+ \) such that \( y^+ = \mu(S^+)^{-1}1 \) with \( S^+ = \text{diag}(a(x^+)) \), i.e., a typical primal barrier update. Unfortunately, using this method it is unclear how to construct efficient bounds on \( \|\nabla_x \mathcal{L}(x^+, y^+))\|_2 \). Instead we pick \( y^+ \) using a typical primal-dual step, i.e,

\[
y^+ \leftarrow y + d_y
\]

where \( d_y \) satisfies

\[
Sd_y + Yd_s + Sy = \mu 1
\]

with \( y = \mu S^{-1}1 \) and \( d_s = \nabla a(x)d_x \). We remark that because \( y = \mu S^{-1}1 \) this can be simplified to \( y^+ \leftarrow \mu S^{-1}1 - \mu S^{-2}d_s \). Hence, Algorithm 1 is a hybrid between a traditional primal-dual method and a pure primal method. We remark that one could develop a pure primal-dual version of our interior method. However, to keep our proofs as simple as possible we decided to use this hybrid algorithm. To further understand how our algorithm generates its direction note that the direction \( d_x \) satisfies

\[
(\nabla^2 \psi_\mu(x) + \mu \delta)d_x = -\nabla \psi_\mu(x)
\]

for some \( \delta \geq 0 \). Using \( Sd_y + Yd_s + Sy = \mu 1 \) and substituting \( d_s = \nabla a(x)d_x \) into \( (\nabla^2 \psi_\mu(x) + \mu \delta)d_x = -\nabla \psi_\mu(x) \) we deduce

\[
\begin{bmatrix}
\nabla_{xx} \mathcal{L}(x, y) + \delta \mathbf{1} & -\nabla a(x)^T \\
\n\nabla a(x) & 0 \\
0 & S & Y
\end{bmatrix}
\begin{bmatrix}
d_x \\
d_y \\
d_s
\end{bmatrix}
= -
\begin{bmatrix}
\nabla_x \mathcal{L}(x, y) \\
0 \\
Sy - \mu \mathbf{1}
\end{bmatrix}.
\]

We remark that Algorithm 1 has two places where there is adaptivity. Firstly, on line 7 the trust region size \( r \) which gets smaller as the dual variable gets bigger. Secondly, the step size
\( \alpha \in [0, 1] \) on line 9 is chosen to ensure \( \alpha \| S^{-1} d_s \|_2 \) is sufficiently small. We need this adaptivity in both places to prove our results.

Algorithm terminates when it reaches an approximate second-order Fritz John point which is defined in the following paragraphs by (FJ1) and (FJ2).

A \( (\mu, \tau_l, \tau_c) \)-approximate first-order Fritz John point is a point \( (x^+, y^+) \) defined by

\[
\begin{align*}
a(x^+), y^+ &> 0 & \text{(FJ1.a)} \\
y^+_i a_i(x^+) &\in \tau_c \mu \times [1/2, 3/2] & \forall i \in \{1, \ldots, m\} \quad \text{(FJ1.b)} \\
\| \nabla_x \mathcal{L}(x^+, y^+) \|_2 &\leq \tau_l \mu \sqrt{\| y^+ \|_1 + 1}. & \text{(FJ1.c)}
\end{align*}
\]

One should interpret (FJ1) thinking of \( \mu \in (0, \infty) \) becoming arbitrarily small, and \( \tau_l \in (0, \infty) \) as a fixed constant which allows us to trade off how small we want \( \| \nabla_x \mathcal{L}(x, y) \|_2 \) relative to \( y_i a_i(x) \). Similarly, \( \tau_c \in (0, 1) \) defines how tightly we want perturbed complementarity to hold.

A \( (\mu, \tau_l, \tau_c) \)-approximate second-order Fritz John point is a point \( (x^+, y^+) \) that satisfies equation (FJ1) and

\[
\nabla^2 \psi_\mu (x^+) \succeq -40 \sqrt{\tau_l} (1 + \| y^+ \|_1) I. \quad \text{(FJ2)}
\]

Note that (FJ1.b) and (FJ2) imply

\[
\nabla_{xx} \mathcal{L}(x^+, y^+) + 2 \nabla a(x^+)^T Y^+ (S^+)^{-1} \nabla a(x^+) \succeq -(40 \sqrt{\tau_l} + L_1 \tau_c) (1 + \| y^+ \|_1) I
\]

with \( S^+ = \text{diag}(s^+) \) and \( Y^+ = \text{diag}(y^+) \).

This is an approximate version of the second-order necessary conditions which state that \( \nabla_{xx} \mathcal{L}(x^+, y^+) \) is positive semidefinite projected onto the nullspace of the Jacobian of the active constraints. See [29, Section 12.4] for an explanation of the second-order necessary conditions.
Algorithm 1: Adaptive trust region interior point algorithm with fixed $\mu$

1: function $\text{Trust-IPM}(f, a, \mu, \tau_l, L_1, \eta_s, \eta_x, x^{(0)})$
2: Input: $\nabla f$ and $\nabla a$ are $L_1$-Lipschitz. The parameters $\eta_s \in (0,1)$, $\eta_x \in (0,1)$ are selected using different formulas depending on whether the problem is convex or nonconvex. Always $x^{(0)} \in \mathcal{X}$.

3: $x \leftarrow x^{(0)}$
4: for $k = 0, \ldots, \infty$ do
5:  $S \leftarrow \text{diag}(a(x))$
6:  $y \leftarrow \mu S^{-1} 1$  \text{Primal update of dual variables.}
7:  $r \leftarrow \eta_x \sqrt{\frac{\mu}{L_1(1+\|y\|_1)}}$  \text{Trust region radius gets smaller as the dual variables get larger.}
8:  $(d_x, d_s, d_y) \leftarrow \text{Trust-region-direction}(f, a, \mu, x, r)$
9:  $\alpha \leftarrow \min \left\{ \frac{\eta_s \|S^{-1} d_s\|_2}{2}, 1 \right\}$ \text{Pick a step size $\alpha \in (0,1]$ to guarantee $x^+ \in \mathcal{X}$.}
10:  $x^+ \leftarrow x + \alpha d_x$
11:  $y^+ \leftarrow y + \alpha d_y$
12: if $(x^+, y^+)$ that satisfies (FJ1) and (FJ2) then
13:  return $(x^+, y^+)$ \text{Termination criterion met.}
14: else
15:  $x \leftarrow x^+$ \text{Only update primal variables, throw away new dual variable $y^+$.}
16: end if
17: end for
18: end function

function $\text{Trust-region-direction}(f, a, \mu, x, r)$

20:  $d_x \in \text{argmin}_{u \in B_r(0)} \mathcal{M}_x^{(\mu)}(u)$
21:  $d_s \leftarrow \nabla a(x) d_x$
22:  $S \leftarrow \text{diag}(a(x))$
23:  $d_y \leftarrow -\mu S^{-2} d_s$
24:  return $(d_x, d_s, d_y)$
25: end function

Algorithm [1] operates with $\mu$ fixed. Practically log barrier methods solve a sequence of problems with decreasing $\mu$. Algorithm [2] (Section 5) which we use to prove Theorem 2 decreases $\mu$. However, we only present Algorithm 1 here since for Theorem 1 (Section 5) fixed $\mu$ suffices.

The rough intuition for Algorithm 1 is as follows. At each iteration the radius $r$ is selected sufficiently small such that the error on the Taylor series approximations are small, i.e.,

\[
|f(x) + \nabla f(x) d_x - f(x + d_x)| \approx 0
\]

\[
|a_i(x) + \nabla a_i(x) d_x - a_i(x + d_x)| \approx 0
\]

\[
\|\nabla_x L(x, y) - \nabla_x L(x + d_x, y + d_y) - \delta d_x\|_2 \approx 0.
\]

This does not guarantee that the point $x + d_x$ is feasible (even if these terms were equal to zero, for example, if we were solving a linear program). Therefore, following the direction computation, we pick $\alpha$ small enough that we remain feasible and decrease the log barrier. This $\alpha$ selection is similar to the step size selection in a long step interior point method for quadratic programming. This paper provided intuition for the design of our practical one-phase IPM code [16]. The stabilization steps of the one-phase IPM, where one attempts to minimize a log barrier, is most strongly related to [Trust-IPM]. Similarities during these stabilization steps include:
A. Maintaining iterates that are exactly feasible using nonlinear slack variable updates \((s^+ = a(x^+))\).

B. Adaptive step size and trust region/regularization parameter choice.

There are significant differences between the algorithms. Differences include that the one-phase IPM unlike Trust-IPM is a primal-dual IPM, does not need a strictly feasible initial point, and does not need to know any Lipschitz constants. Since the algorithm presented in this paper is not practical, it remains an open problem to develop a practical IPM with a polynomial worst-case runtime bound.

We have omitted the details on how to solve the trust-region problem to solving a linear system. However, the matrix \(\nabla^2 \psi_\mu(x)\) and vector \(\nabla \psi_\mu(x)\) may contain components that are exponentially large in \(1/\mu\). While we omit details of this issue from the paper, this can be resolved using the results of [42] which provide a \(O(\log \log(1/\epsilon))\) runtime for solving the trust region problem.

4 Lemmas on local approximations and directions sizes

We develop some useful Lemmas in Section 4.1 to predict the quality of our local approximations as a function of the direction sizes. In Section 4.2, we prove a key lemma, which bounds the directions size in terms of predicted progress. To prove our main results we need the following assumption.

**Assumption 2. (Lipschitz derivatives)** Assume that each \(a_i : \mathbb{R}^n \to \mathbb{R}\) for \(i \in \{1, \ldots, m\}\) is a continuous function on \(\mathbb{R}^n\). Let \(L_1, L_2 \in (0, \infty)\). The functions \(f : \mathbb{R}^n \to \mathbb{R}\) and \(a_i : \mathbb{R}^n \to \mathbb{R}\) have \(L_1\)-Lipschitz first derivatives and \(L_2\)-Lipschitz second derivatives on the set \(X\).

4.1 The accuracy of local approximations

Recall that \(x^+\) and \(y^+\) are the next iterates given by Algorithm 1. In this section, as a function of the direction sizes \(\|d_x\|_2\), \(\|Y^{-1}d_y\|_2\) and \(\|S^{-1}d_s\|_2\), we bound the following.

A. The gap between the predicted reduction and the actual reduction of the log barrier as given by Lemma 3. This allows us to convert predicted reduction \(M_\psi^2(d_x)\) into a reduction in the log barrier.

B. Perturbed complementarity \(|a_i(x^+)^T y_i^+ - \mu|\) as given by Lemma 4. This allows us to establish when (FJ1.b) holds.

C. The norm of the gradient of the Lagrangian as given by Lemma 5. This allows us to establish when (FJ1.c) holds. Therefore Lemma 4 and 5 allow us to reason about when we are at an approximate Fritz John point.

**Lemma 2.** Suppose the function \(g : \mathbb{R} \to \mathbb{R}\) has \(L_1\)-Lipschitz first derivatives, \(L_2\)-Lipschitz second derivatives on the set \([0, \theta]\). Also assume \(g(0) > 0\). For all \(\theta \in [0, \theta]\) which the following inequality holds:

\[
\frac{|g'(0)|}{g(0)} + \frac{L_1\theta^2}{2g(0)} \leq \beta \leq 1/2
\]

with \(\beta \in \mathbb{R}_+\), we have \(\frac{g(0)}{g(\theta)} \in \left[\frac{1}{2}, \frac{3}{2}\right]\) and

\[
\left|\frac{\partial^3 \log(g(\theta))}{\partial^3 \theta}\right| \leq 2\frac{L_2\theta^3 + 6L_1\theta^2 \beta}{g(0)} + 8\beta^3.
\]

The proof of Lemma 2 is given in Section 4.1. Globally the log barrier does not have Lipschitz second derivatives. One interpretation of Lemma 2 is that it provides a bound on the Lipschitz constant of second derivatives of \(\log(g(\theta))\) in a neighborhood of the current point.
Furthermore, there is a relationship between Lemma \ref{lem:log-barrier-bound} and the bounds used for interior point methods for linear programming. In particular, with \( L_1 = L_2 = 0 \), i.e., \( g \) is linear, the bound from Lemma \ref{lem:log-barrier-bound} becomes
\[
\theta^3 \left| \frac{\partial^3 \log(g(\theta))}{\partial^3 \theta} \right| \leq 8 \left( \frac{\left| \theta g'(0) \right|}{g(0)} \right)^3 = 8 \left( - \frac{\partial^2 \log(g(\theta))}{\partial \theta^2} \theta^2 \right)^{3/2},
\]
which is the statement that the log barrier of a linear function is self-concordant.

Lemma \ref{lem:log-barrier-bound} only gives us a bound on the local Lipschitz constant for the second derivatives of \( \log(g(\theta)) \) with \( g \) is univariate. By applying Lemma \ref{lem:log-barrier-bound} with \( g(\theta) := a_i(x + \theta v) \), \( v = \frac{d}{dx} g(0) \), we can bound the difference between the actual and predicted progress on the log barrier function. This bound is given in Lemma \ref{lem:log-barrier-bound-lin}.

**Lemma 3.** Suppose assumption \ref{ass:convexity} holds (Lipschitz derivatives). Let \( x \in X \) and \( S = \text{diag}(a(x)) \). Consider any \( d_x \in \mathbb{R}^n \). Suppose the following inequality holds:
\[
\|S^{-1}d_x\|_2 + \frac{L_1(1 + \|y\|_1)\|d_x\|_2^2}{2\mu} \leq \kappa \leq 1/2
\] (11)
where \( \kappa \in \mathbb{R}_+, S = \text{diag}(a(x)) \), \( d_x = \nabla a(x)d_x \) and \( y = \mu S^{-1}1 \). Then \( \frac{a_i(x)}{a_i(x)} \in [1/2, 3/2] \) for all \( i \in \{1, \ldots, m\} \) and
\[
|\psi_\mu(x) + \mathcal{M}^{\psi_\mu}(d_x) - \psi_\mu(x + d_x)| \leq \frac{L_2}{6} (1 + 2\|y\|_1)\|d_x\|_2^2 + 3L_1\|d_x\|_2\|y\|_{2k} + 3\mu \kappa^3.
\]

The proof of Lemma \ref{lem:log-barrier-bound-lin} is given in Section \ref{sec:proof-lem:log-barrier-bound-lin}. One can see from (11) if we wish to use Lemma \ref{lem:log-barrier-bound-lin} to guarantee we remain feasible we must select \( |d_x| \) such that \( \frac{L_1(1 + \|y\|_1)\|d_x\|_2^2}{\mu} = \mathcal{O}(1) \). Hence \( r = \mathcal{O} \left( \sqrt{\frac{\mu}{1 + \|y\|_1}} \right) \) in line 7 of Algorithm \ref{alg:interior-point}. Also, observe that if (11) holds for some \( x \in X \) and \( d_x \) then (11) holds for any damped direction \( ad_x \) with \( \alpha \in [0, 1] \), i.e., \( \text{CONVEX}\{x, x + d_x\} \subseteq X \). This observation ensures we can use Lemma \ref{lem:log-barrier-bound-lin} to establish the premises of Lemma \ref{lem:log-barrier-bound-lin} and \ref{lem:log-barrier-bound-lin-lin} which require \( \text{CONVEX}\{x, x^+\} \subseteq X \).

**Lemma 4.** Suppose assumption \ref{ass:convexity} holds. Let \( \text{CONVEX}\{x, x^+\} \subseteq X \), \( s = a(x), s^+ = a(x^+) \), \( S = \text{diag}(a(x)) \), \( Y = \text{diag}(y), y^+ \in \mathbb{R}^m \), and \( Y^+ = \text{diag}(y^+) \). Also, let \( d_x = x^+ - x \) and \( d_y = y^+ - y \), and \( d_s = \nabla a(x)d_x \). If the following equation holds:
\[
Sy + Sd_y + Yd_s = \mu 1,
\]
then:
\[
\|Y^{-1}d_y\|_2 \leq \|S^{-1}d_x\|_2 + \|\mu(SY)^{-1}1 - 1\|_2
\] (12)
\[
\|Y^+s^+ - \mu 1\|_2 \leq \|Sy\|_\infty \|S^{-1}d_x\|_2 \|Y^{-1}d_y\|_2 + \frac{L_1}{2}\|y\|_2(1 + \|Y^{-1}d_y\|_2)\|d_x\|_2^2.
\] (13)

Furthermore, if \( \|Y^+s^+ - \mu 1\|_\infty < \mu \) and \( \|Y^{-1}d_y\|_\infty \leq 1 \) then \( s^+, y^+ \in \mathbb{R}^m_{+} \).

We give the proof of Lemma \ref{lem:log-barrier-bound-lin} in Section \ref{sec:proof-lem:log-barrier-bound-lin-lin}. Lemma \ref{lem:log-barrier-bound-lin} will allow us to guarantee \( (x^+, y^+) \) satisfies (FJ1.a) and (FJ1.b) when we take a primal-dual step in Algorithm \ref{alg:interior-point}. This a typical Lemma used for interior point methods in linear programming except that the nonlinearity of the constraints creates the additional \( \frac{L_2}{2}\|y\|_2(1 + \|Y^{-1}d_y\|_2)\|d_x\|_2^2 \) term in (13).

**Lemma 5.** Suppose assumption \ref{ass:convexity} holds. Let \( y, y^+ \in \mathbb{R}^m \) and \( \text{CONVEX}\{x, x^+\} \subseteq X \). Then the following inequality holds:
\[
\|\nabla_x \mathcal{L}(x, y) + \nabla_{xx} \mathcal{L}(x, y)^T d_x - d_y^T \nabla_x a(x) - \nabla_x \mathcal{L}(x^+, y^+)\|_2 \leq L_1\|y\|_2\|d_x\|_2 \|Y^{-1}d_y\|_2 + \frac{L_2}{2}(\|y\|_1 + 1)\|d_x\|_2^2
\]
with \( d_x = x^+ - x \) and \( d_y = y^+ - y \).

The proof of Lemma \ref{lem:log-barrier-bound-lin} is given in Section \ref{sec:proof-lem:log-barrier-bound-lin-lin}. Lemma \ref{lem:log-barrier-bound-lin} allows us to guarantee that (FJ1.c) holds at \( (x^+, y^+) \) when \( \|d_x\|_2 \) and \( \|Y^{-1}d_y\|_2 \) are small. The introduction of the \( L_1\|y\|_2\|d_x\|_2 \|Y^{-1}d_y\|_2 \) term is the key reason that the analysis of [2] [7] [12] for affine scaling does not automatically extend to nonlinear constraints because this method does not efficiently bound \( \|Y^{-1}d_y\|_2 \).
4.2 Bounding the direction size of the slack variables

This section presents Lemma 7 which allows us to bound the direction size of the slack variables. Before proving Lemma 7 we state Lemma 6 which contains some basic and well-known facts about trust region subproblems that we will find useful. The proof is given in Section B.5 for completeness.

**Lemma 6.** Consider $H \in \mathbb{R}^{n \times n}$ and $g \in \mathbb{R}^n$. Define $\Delta(u) := \frac{1}{2}u^THu + g^Tu$ where $\Delta : \mathbb{R}^n \rightarrow \mathbb{R}$ and let $u^* \in \text{argmin}_{u \in B_r(0)} \Delta(u)$ be an optimal solution to the trust region subproblem for some $r \geq 0$. Then there exists some $\delta \geq 0$ such that:

$$\delta(\|u^*\|_2 - r) = 0, \quad (H + \delta I)u^* = -g, \quad \text{and} \quad H + \delta I \succeq 0. \quad (14)$$

Conversely, if $u^*$ satisfies (14) then $u^* \in \text{argmin}_{u \in B_r(0)} \Delta(u)$. Let $\sigma(r) := \min_{u \in B_r(0)} \Delta(u)$, then for all $r \in [0, \infty)$ we have

$$\sigma(r) \leq -\frac{\delta r^2}{2} \quad (15a)$$

$$\sigma(r) \leq \sigma(ar) \leq \alpha^2 \sigma(r) \quad \forall \alpha \in [0, 1]. \quad (15b)$$

Furthermore, the function $\sigma(r)$ is monotone decreasing and continuous.

Lemma 7 which follows is key to our result, because it allows us to bound the size of $\|S^{-1}d_x\|_2$ (recall $d_x = \nabla a(x)d_x$). We remark that often in linear programming one shows $\|S^{-1}d_x\|_2 = O(1)$ to prove an $O(\sqrt{n \log(1/\mu)})$ iteration bound. Combining Lemma 7 with the Lemmas from Section 4.1 allows us to give concrete bounds on the reduction of the log barrier at each iteration. This underpins our main results in Section 5.

**Lemma 7.** Consider $H \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{m \times n}$ and $g \in \mathbb{R}^n$. Define $\Delta(u) := \frac{1}{2}u^T(H + AT)A + g^Tu$ where $\Delta : \mathbb{R}^n \rightarrow \mathbb{R}$ and let $d_x \in \text{argmin}_{u \in B_r(0)} \Delta(u)$ for some $r \geq 0$. Then

$$\|Ad_x\|_2 \leq \sqrt{-d_x^THd_x - 2\Delta(d_x)}. \quad (16)$$

**Proof.** Observe that

$$\Delta(d_x) = \frac{1}{2}d_x^T(H + AT)\text{A}d_x + g^Td_x$$

$$= \frac{1}{2}d_x^T(H + AT)\text{A}d_x - d_x^T(H + AT)\text{A}d_x + \frac{1}{2}d_x^T(H + AT)\text{A}d_x - \delta \|d_x\|_2^2$$

where the second transition use the fact from Lemma 6 that there exists some $\delta$ such that $(H + AT)\text{A}d_x = -g$. Rearranging this expression and using $\delta \|d_x\|_2^2 \geq 0$ yields

$$\|Ad_x\|_2^2 \leq -d_x^THd_x - 2\Delta(d_x). \quad (17)$$

This concludes the proof of Lemma 7. \qed

Now, if we set $H = \nabla^2_{xx}\mathcal{L}(x, y)$, $A = \frac{1}{\sqrt{n}}S^{-1}\nabla a(x)$, $S = \text{diag}(a(x))$, and $d_x = \nabla a(x)d_x$ then we deduce from Lemma 7 that

$$\|S^{-1}d_x\|_2 \leq \sqrt{-d_x^T\nabla^2_{xx}\mathcal{L}(x, y)d_x - 2\mathcal{M}_s^{\nabla a}(d_x)}$$

which if we assume $\nabla^2_{xx}\mathcal{L}(x, y)$ is positive definite we deduce that

$$\|S^{-1}d_x\|_2 \leq \sqrt{-2\mathcal{M}_s^{\nabla a}(d_x)} \quad (18)$$
Alternately, in the nonconvex case if \( \|\nabla^2 f(x)\|_2 \leq L_1 \) and \( \|\nabla^2 a_i(x)\|_2 \leq L_1 \) then
\[
\|S^{-1}d_s\|_2 \leq \sqrt{\frac{L_1(1 + \|y\|_1)\|d_x\|_2^2 - 2\mathcal{M}_s^{\mu}(d_x)}{\mu}}. \tag{19}
\]

We emphasize that Lemma 7 is unusual because the bound on \( \|S^{-1}d_s\|_2 \) is dependent on the amount of predicted progress, i.e., \( \mathcal{M}_s^{\mu}(d_x) \). This explains why it is critical in Algorithm 1 that we use an adaptive step step size. In particular, if \( \|S^{-1}d_s\|_2 = \|S^{-1}\nabla a(x)d_x\|_2 \) is large then it is possible \( a(x + d_x) \neq 0 \). However, since \( \|S^{-1}d_s\|_2 \) is large and \( L_1(1 + \|y\|_1)\|d_x\|_2^2 \) bounded (as per line 7) then we know the predicted reduction in the log barrier function must be large. Therefore the direction \((ad_x, ad_s, ad_y)\) for the small value of the scalar \( \alpha \), as specified in line 9 will still reduce the log barrier merit function sufficiently.

5 Runtime to find Fritz John points

This section outlines the proof of our main result, a bound on the number of iterations Trust-IPM algorithm takes to find a Fritz John point. Section 5.1 gives a general bound for the runtime to find a Fritz John point, i.e., proves Theorem 1. Section 5.2 gives a tighter bound in the case that \( f \) is convex and each \( a_i \) is concave.

5.1 Runtime to find Fritz John points in the nonconvex case

In this section we prove our main result, Theorem 1 which bounds the runtime of Algorithm 1 to find a Fritz John point by \( \mathcal{O}(\mu^{-7/4}) \). On a high level this proof is similar to typical cubic regularization/trust region arguments: we argue that if the termination conditions are not satisfied at the next iterate then we have reduced the log barrier function by at least \( \Omega(\mu^{-7/4}) \). Before proving Theorem 1 we prove the auxiliary Lemmas 8 and 9. Lemma 8 shows we reduce the barrier merit function if the predicted progress at each iteration is large; Lemma 9 allows us to reason about when the algorithm will terminate.

Lemma 8 provides a bound on the progress as a function of the parameter \( \eta_x \in [0,1] \) which controls the step size. This will allow us to ensure that we will be able to reduce the barrier function during Algorithm 1 if the predicted progress from solving the trust region subproblem \( \mathcal{M}_s^{\mu}(d_x) \) is sufficiently large. Recall that algorithm 1 computes steps via
\[
S = \text{diag}(a(x)), \quad y = \mu S^{-1}1 \quad \text{(ITRS.a)}
\]
\[
r = \eta_x \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)}} \quad \text{(ITRS.b)}
\]
\[
d_x \in \arg\min_{u \in B_r(0)} \mathcal{M}_s^{\mu}(u) \quad d_s \leftarrow \nabla a(x)d_x \quad d_y \leftarrow \mu S^{-2}d_s \quad \text{(ITRS.c)}
\]
\[
x^+ = x + ad_x \quad y^+ = y + ad_y \quad \text{(ITRS.d)}
\]
where \( \text{ITRS} \) stands for interior trust region subproblem.

Recall that \( \tau_1, \tau_c \) and \( \mu \) are all parameters for our termination criterion (FJ1). To simplify the analysis assume \( \mu \) is small enough such that the following assumptions holds.

Assumption 3 (Sufficiently small \( \mu \)). Let
\[
\frac{\mu}{\tau_1^2} \in \left(0, \frac{L_1}{\tau_1^2}\right], \quad \mu \in \left(0, \frac{L_1^2}{L_2}\right] \quad \text{and we pick } \tau_c \text{ such that }
\]
\[
\frac{\tau_1^2 \mu}{L_1} \frac{1}{\tau_1^2} \quad \text{(A2.}\tau_c \text{)}
\]
Note that by (A2.\(\mu\).a) we know \(\tau_c \in (0, 1]\).

**Lemma 8.** Suppose assumptions 2 and 3 hold (Lipschitz derivatives, and sufficiently small \(\mu\)). Also assume \(x \in X, \eta_s \in (0, 1/4]\). Let \(\text{(ITRS)}\) hold with \(\eta_c = \frac{\eta_s}{2}\). Let \(\alpha = \min \left\{1, \frac{\eta_s}{\|S - d_x\|_2} \right\}\). Then \(x^+ \in X\) and

\[
\psi_\mu(x^+) - \psi_\mu(x) \leq 2\mu\eta_s^3 + \max \left\{M_\psi\mu(d_x), -\frac{\eta_s^2\mu}{3} \right\}.
\]

(20)

The proof of Lemma 8 is given in Section C.1. Lemma 8 allows us to guarantee how much the log barrier will actually be reduced if we take a step, given the reduction predicted by \(M_\psi\mu(d_x)\).

**Lemma 9.** Suppose \(\text{(ITRS), assumption 2 and 3 hold (direction selection, Lipschitz derivatives, and sufficiently small } \mu\). Let \(x \in X, \eta_x \in (0, \frac{1}{6}(\frac{\mu \tau^2}{L_1})^{1/4}]\). Further assume \(M_\psi\mu(d_x) \geq -\frac{\tau_l \mu r \sqrt{1 + \|y\|_1^3}}{3}\) and \(\alpha = 1\). Under these assumptions, \((x^+, y^+)\) is satisfies \(\text{(FJ1)}\) and \(\nabla^2 \psi_\mu(x) \succeq -\sqrt{\tau_l (1 + \|y\|_1)} \sqrt{\frac{\tau_l}{\tau_c} I}\).

The proof of Lemma 9 is given in section C.2. Lemma 9 shows that if the predicted progress, \(M_\psi\mu(d_x)\), from the trust region step is small then the algorithm must terminate at the next iterate. With Lemma 8 and 9 in hand we are now ready to prove our main result, Theorem 1.

**Theorem 1.** Suppose assumptions 2 and 3 hold (Lipschitz derivatives, and sufficiently small \(\mu\)). Then \(\text{TRUST-IPM}(f, a, \mu, \tau_l, \eta_s, \eta_x, x^{(0)})\) with \(x^{(0)} \in X\) and

\[
\eta_s = \frac{1}{40} \left(\frac{\tau_l^2 \mu}{L_1}\right)^{1/4}, \quad \eta_x = \frac{\eta_s}{2},
\]

(η-1)

takes at most

\[
\mathcal{O} \left(1 + \frac{\psi_\mu(x^{(0)}) - \inf_{x \in X} \psi_\mu(x)}{\mu} \left(\frac{L_1}{\mu \tau_l^2}\right)^{3/4} \right)
\]

iterations to terminate with a \((\mu, \tau_l, \tau_c)\)-approximate second-order Fritz John point \((x^+, y^+)\), i.e., \(\text{FJ1}\) and \(\text{FJ2}\) hold.

The proof is given in Section C.3.

5.2 Runtime to find Fritz John points in the convex case

To obtain our results in this section we will assume that the function \(f\) is convex and each function \(a_i\) is concave. The result, Lemma 10, only gives the runtime bound to find a Fritz John point. In the subsequence section we use this Lemma to prove Theorem 2 which gives a runtime bound for finding an \(\epsilon\)-optimal solution.

Similar, to assumption 4 given in Section 5.1 we use assumption 4 to require that \(\mu\) is small to simplify the analysis and final bound.

**Assumption 4 (Sufficiently small \(\mu\)).** Let

\[
\mu \in \left(0, \frac{L_1}{\tau_l^2}\right], \quad \mu \in \left(0, \frac{L_1^4 \tau_l}{L_3^2}\right], \quad \text{(A4.}\mu\text{.a, b)}
\]

and we let

\[
\tau_c = \left(\frac{\tau_l^2 \mu}{L_1}\right)^{1/3}, \quad \text{(A4.}\tau_c\text{)}
\]
Lemma 10. Suppose assumptions $\mathcal{A}$ and $\mathcal{F}$ hold (Lipschitz derivatives, and sufficiently small $\mu$). Let $f$ be convex and each $a_i$ concave. Then $\text{TRUST-IPM}(f, a, \mu, \tau, L_1, \eta, x(0))$ with $x(0) \in \mathcal{X}$ and

$$\eta_z = \theta \left( \frac{\tau_i^2 \mu}{L_1} \right)^{1/6}, \quad \eta_s = \theta \left( \frac{\tau_i^2 \mu}{L_1} \right)^{1/3} \quad \theta = 1/30.$$  \hspace{1cm} (\eta-2)

takes at most

$$O \left( 1 + \frac{\psi_\mu(x(0)) - \inf_z \psi_\mu(z)}{\mu} \left( \frac{L_1}{\tau_i^2 \mu} \right)^{2/3} \right)$$

iterations to terminate with a $(\mu, \tau, \tau_c)$-approximate first-order Fritz John point $(x^+, y^+)$, i.e., $\mathcal{FJ}$. holds.

The proof of Lemma 10 is similar to Theorem 1 and is given in Section D. For this result we only need to prove that we have found an approximate first-order Fritz John rather than an approximate second-order Fritz John point (by the assumption $f$ is convex and $a_i$ is concave we trivially have $\nabla^2 \psi_\mu(x) \geq 0$). The key to improving the runtime bound given in Theorem 1 is that $f$ is convex and $a_i$ concave so we can apply (18) to bound $\|S^{-1}d_s\|_2$ instead of (19).

6 Optimality guarantees with convexity and regularity condition

While Lemma 10 specialized our guarantees to when $f$ is convex and $a_i$ is concave, it only made a statement on how long it takes to find a Fritz John point. However, one would hope to give optimality guarantees. This is the purpose of this section. We begin with a simple lemma showing finding an approximate KKT point implies approximate optimality. We use this lemma to convert algorithms that find approximate KKT points of the log barrier to algorithms that find approximate optimal solutions. Finally the main result (Theorem 2) is that under a regularity assumption on our algorithm, when applied to a sequence of subproblems with decreasing $\mu$, takes at most $O(\epsilon^{-2/3})$ trust region subproblem solves to find an $\epsilon$-optimal solution.

Lemma 11. Let $f: \mathbb{R}^n \to \mathbb{R}$ and $a: \mathbb{R}^n \to \mathbb{R}^m$. Let $\|X\|_2 \leq R$. If $(x, y) \in \mathcal{X} \times \mathbb{R}^m_+$ and $a_i(x)y_i \geq \mu$ for $i \in \{1, \ldots, m\}$ then

$$\psi_\mu(x) - \inf_{z \in \mathcal{X}} \psi_\mu(z) \leq \|\nabla_x \mathcal{L}(x, y)\|_2 R + \sum_{i=1}^{m} (a_i(x)y_i - \mu).$$

Proof: Let $S = \text{diag}(a(x))$. Define $\tilde{y} = y - \mu S^{-1}{1}$. By $a_i(x)y_i \geq \mu$ we have $\tilde{y}_i \geq 0$. Let

$$q(z) := \psi_\mu(z) - a(z)^T \tilde{y}.$$  

Now,

$$\inf_{z \in \mathcal{X}} \psi_\mu(z) \geq \inf_{z \in \mathcal{X}} q(z) \geq \|q(x)\|_2 R = \psi_\mu(x) - \|\nabla q(x)\|_2 R - a(x)^T \tilde{y},$$

where the first inequality uses $a(z)^T \tilde{y} \geq 0$, the second inequality the convexity of $q$, and the final inequality the definition of $q$. The result follows by $\nabla q(x) = \nabla_x \mathcal{L}(x, y)$. \hspace{1cm} \[\square\]

So far we have presented TRUST-IPM which only minimizes the log barrier with $\mu$ fixed. However, log barrier methods traditionally solve a sequence of subproblems with $\mu$ tending toward zero as described in Algorithm 2.
Algorithm 2 IPM with decreasing $\mu$

```plaintext
function ANNEALED-IPM($f, a, \mu^{(0)}, x^{(0)}, \epsilon$)
    for $j = 0, \ldots, \infty$ do
        \begin{align*}
        (x^{(j+1)}, y^{(j+1)}) &\leftarrow \text{GENERIC-IPM}(f, a, \mu^{(j)}, x^{(j)}) \\
        \mu^{(j+1)} &\leftarrow \mu^{(j)}/2 \\
        \text{if } 2\mu^{(j)}m &\leq \epsilon \text{ then return } x^{(j+1)}
        \end{align*}
    end if
end for
end function
```

In Algorithm 2 we write GENERIC-IPM as a placeholder for any algorithm that finds a Fritz John point. The precise properties we need GENERIC-IPM to satisfy are given in assumption 5. For this paper will use GENERIC-IPM = \text{TRUST-IPM} but any other method satisfying assumption 5 would suffice. Then, as we show in Lemma 12 it is possible to give a runtime for the algorithm to find a $\epsilon$-optimal point.

**Assumption 5.** Let $\|X\|_2 \leq R$. Suppose that for any $\mu \in (0, \infty), x \in X$ that GENERIC-IPM($f, a, \mu, x$) finds a point $(x^+, y^+)$ with $\|\nabla_x L(x^+, y^+)\|_2 \leq \frac{\mu m}{2}$ and $|a(x^+)y_i^+ - \mu| \leq \mu/2$ in at most $O(1) + \frac{\psi_{\mu}(x) - \inf_{z \in X} \psi_{\mu}(z)}{\mu} w(\mu)$ unit operations, where the function $w : \mathbb{R} \rightarrow \mathbb{R}$ is monotone decreasing.

**Lemma 12.** Let $f$ be convex and each $a_i$ concave. Suppose that assumption 5 holds. Let $x^{(0)} \in X$, $\Delta = f(x^{(0)}) - \inf_{z \in X} f(z)$ and $\epsilon \in (0, \Delta)$. Then ANNEALED-IPM($f, a, \mu^{(0)}, x^{(0)}, \epsilon$) takes at most

$$
O(1) + 6m \times w\left(\frac{\epsilon}{3m}\right) \log_2\left(\frac{3m\mu^{(0)}}{\epsilon}\right) + \frac{\psi_{\mu^{(0)}}(x^{(0)}) - \inf_{z \in X} \psi_{\mu^{(0)}}(z)}{\mu^{(0)}} w(\mu^{(0)})
$$

unit operations to return an $\epsilon$-optimal solution, where $\log_2^+(x) = \max\{\log_2(x), 1\}$.

The proof of Lemma 12 appears in Section E.2. Next, we present a regularity assumption which enables us to convert a Fritz John point into a KKT point and thereby enable \text{TRUST-IPM} to satisfy assumption 6.

**Assumption 6 (Regularity conditions).** Assume there exists some $\zeta > 1$ that if (FJ1) holds then $\|y^+\|_1 + 1 \leq \zeta$.

One sufficient condition for assumption 6 to hold is Slater’s condition, i.e., there exists some point $x \in X$ and $\gamma > 0$ with $a(x) > \gamma 1$. We show this formally in Section E.1.

Next, we present the main result of this section, Theorem 2, which combines Lemma 10 and Lemma 12. To satisfy the premises of these Lemmas we make the following assumption.

**Assumption 7 (Sufficiently small $\mu^{(0)}$).** Let

$$
\eta_I = \frac{m}{R\zeta^{1/2}} \quad \text{(A7.}\eta_I) \\
\tau_c = \left(\frac{T^2 \mu}{L_1}\right)^{1/3} \quad \text{(A7.}\tau_c) \\
\mu^{(0)} = \min\left\{\frac{L_1 R^2 \zeta}{m^2}, \frac{L_1^4 m}{RL^2 \sqrt{\zeta}}\right\} \quad \text{(A7.}\mu^{(0)})
$$

where $\mu^{(0)}$ represents the initial $\mu$ value of ANNEALED-IPM.
**Theorem 2.** Suppose assumptions 3, 6, and 7 hold (Lipschitz derivatives, regularity conditions, and sufficiently small \( \mu^{(0)} \)). Let \( f \) be convex and each \( a_i \) concave. Let \( x^{(0)} \in \mathcal{X}, \|x\|_2 \leq R, \Delta = f(x^{(0)}) - \inf_{z \in \mathcal{X}} f(z), \) and \( \epsilon \in (0, \Delta) \). Define \( \eta_s, \eta_c \) by (7.2) and set

\[
\text{GENERIC-IPM}(f, a, \mu, x) := \text{TRUST-IPM}(f, a, \mu, \tau_i, L_1, \eta_s, \eta_c, x, \epsilon)
\]

inside \( \text{ANNEALED-IPM}. \) Then \( \text{ANNEALED-IPM}(f, a, \mu^{(0)}, x^{(0)}, \epsilon) \) takes at most

\[
O \left( \left( m^{1/3} \left( \frac{L_1 R^2 \xi}{\epsilon} \right)^{2/3} + 1 \right) \log \left( \frac{m \mu^{(0)}}{\epsilon} \right) + \frac{\psi_{\mu^{(0)}}(x^{(0)}) - \inf_{z \in \mathcal{X}} \psi_{\mu^{(0)}}(z)}{\mu^{(0)}} \left( \frac{L_1 R^2 \xi}{m^2 \mu^{(0)}} \right)^{2/3} \right)
\]

iterations to return an \( \epsilon \)-optimal solution, where \( \log^+(x) = \max\{\log(x), 1\} \).

The proof of Theorem 2 appears in Section 5.3. Notice that the runtime bound given in Theorem 2 comprises of two terms. The first term is dependent on \( \epsilon \) and corresponds to the total number of inner iterations used during the \( 1, \ldots, j \) iterations of \( \text{ANNEALED-IPM}. \) The second term corresponds to the number of inner iterations required in the first iteration of \( \text{ANNEALED-IPM}. \) This second term has no \( \epsilon \) dependence, and by substituting the value of \( \mu^{(0)} \) given by (7.2) we observe this term is bounded by

\[
O \left( \Delta f \left( \frac{m^2}{L_1 R^2 \xi} + \frac{R^3 L^2 \xi^{7/6}}{m^3 L^6} \right) + \log(c) \left( \frac{m + L_3 R^2 \xi^{1/3}}{L^3 \xi^{1/3} m^{1/3}} \right) \right)
\]

where \( c \) is some constant such that \( \frac{a(x)}{a(x^{(0)})} \leq c1 \) for all \( x \in \mathcal{X}. \)

## 7 Comparison with existing results

### 7.1 Nonconvex comparisons

One difficulty with nonconvex optimization is that there are many choices termination criterion and this choice affects runtime bounds. The results of Birgin et al. [3] guarantee to find an unscaled KKT points or a certificate of local infeasibility. Their criterion is different from our Fritz John termination criterion. Therefore for the sake of comparison we now introduce a new pair of termination criterion similar to the criterion they presented. Our own definition of an unscaled KKT point is

\[
a(x) \geq -\varepsilon_{\text{opt}} 1 \quad \text{(KKT.a)}
\]

\[
\|\nabla_x L(x, y)\|_2 \leq \varepsilon_{\text{opt}} \quad \text{(KKT.b)}
\]

\[
y \geq 0 \quad \text{(KKT.c)}
\]

\[
a_i(x)y_i \leq \varepsilon_{\text{opt}} \quad \text{(KKT.d)}
\]

Let us contrast this definition with the definition of an unscaled KKT point given in Birgin et al. [3]. The most important difference is how complementarity is measured. In particular, in Birgin et al. [3] their termination criterion replaces (KKT.d) of our criterion with \( \min \{a_i(x), y_i\} \leq \varepsilon_{\text{opt}} \). In this respect the termination criterion of Birgin et al. [3] is stronger than (KKT). To detect infeasibility we consider the following termination criterion.

\[
\min_i a_i(x) < -\varepsilon_{\text{opt}}/2 \quad \text{(INF1.a)}
\]

\[
a(x) + t1 \geq 0 \quad \text{(INF1.b)}
\]

\[
\|\nabla a(x)^T y\|_2 \leq \varepsilon_{\text{inf}} \quad \text{(INF1.c)}
\]

\[
\|y\|_1 = 1 \quad \text{(INF1.d)}
\]

\[
y \geq 0 \quad \text{(INF1.e)}
\]

\[
(a_i(x) + t)y_i \leq \varepsilon_{\text{inf\text{-}opt}} \quad \text{(INF1.f)}
\]
System (INF1) finds an approximate KKT point for the problem of minimizing the infinity norm of the constraint violation. In contrast, Birgin et al. find a stationary point for the Euclidean norm of the constraint violation squared which they denote by \( \theta(x) \). However, this is a weak measure of infeasibility since if \( \theta(x) \leq \varepsilon_{\text{opt}}^2 \) then automatically \( \|\nabla \theta(x)\|_2 \leq \varepsilon_{\text{opt}} \). The natural termination criterion corresponding to (INF1) is an approximate KKT point for the problem of minimizing the Euclidean norm of the constraint violation. This can be written as

\[
\begin{align*}
\min_{i} a_i(x) &< -\varepsilon_{\text{opt}}/2 \quad \text{(INF2.a)} \\
\|\nabla a(x)^T y\|_2 &\leq \varepsilon_{\text{inf}} \quad \text{(INF2.b)} \\
y &= \frac{z}{\|z\|_2} \quad \text{(INF2.c)} \\
a(x) + z &\geq 0 \quad \text{(INF2.d)} \\
(a_i(x) + z_i)y_i &= 0 \quad \text{(INF2.e)} \\
y &\geq 0. \quad \text{(INF2.f)}
\end{align*}
\]

To find a point satisfying (INF2) they require \( \|\nabla \theta(x)\|_2 \leq \varepsilon_{\text{opt}}\varepsilon_{\text{inf}} \). If this condition holds then \( z = \min\{a(x), 0\}, y = \frac{z}{\|z\|_2} \) satisfies (INF2). Finally, notice that both (INF1) and (INF2) find points with

\[
\begin{align*}
\min_{i} a_i(x) &< -\varepsilon_{\text{opt}}/2 \\
a_i(x)y_i &\leq \varepsilon_{\text{inf}} \\
\|\nabla a(x)^T y\|_2 &\leq \varepsilon_{\text{inf}} \\
y &\geq 0,
\end{align*}
\]

which proves infeasibility in ball of radius \( R \) if \( \varepsilon_{\text{inf}} = \mathcal{O}(\varepsilon_{\text{opt}}/(1 + R)) \), \( f \) is convex and \( a_i \) is concave [16, Observation 1].

To obtain our algorithm that finds a point satisfying either (KKT) or (INF1), we apply Trust-IPM in two-phases (see Two-Phase-IPM in Appendix F.1).

Let \( x^{(0)} \in \mathbb{R}^n \) be our starting point and define

\[
t^{(0)} := \frac{\varepsilon_{\text{opt}}}{2} + \max\{\min_{i} -a_i(x^{(0)}), 0\}. \quad (21)
\]

Phase-one applies Algorithm 1 to minimize the infinity norm of the constraint violation, i.e., we find a Fritz John point of

\[
\begin{align*}
\min_{x,t} f^{P_1}(x,t) &:= t \quad \text{(PI.a)} \\
a^{P_1}(x,t) := \left( \begin{array}{c}
a(x) + t1 \\
t \\
\frac{\varepsilon_{\text{opt}}}{2} + t^{(0)} - t \end{array} \right) &\geq 0. \quad \text{(PI.b)}
\end{align*}
\]

Let \( (x^{(P_1)}, t^{(P_1)}) \) be the solution obtained. Starting from \( x^{(P_1)} \), phase-two minimizes the objective subject to the \((\varepsilon_{\text{opt}}\text{-relaxed})\) constraints, i.e., we find a Fritz John point of

\[
\begin{align*}
\min_{x} f(x) & \quad \text{(PII.a)} \\
a^{P_2}(x) := a(x) + \varepsilon_{\text{opt}}1 &\geq 0 \quad \text{(PII.b)}
\end{align*}
\]

starting from the point obtained in phase-one.

All feasible solutions to (PI) and (PII) satisfy \( a(x) \geq -(t^{(0)} + \varepsilon_{\text{opt}})1 \). Therefore, we replace assumption 2 with assumption 8 where \( X \) replaced with two sets, corresponding to phase-one and phase-two respectively:

\[
\begin{align*}
\tilde{X}^{(P_1)} := \{x \in \mathbb{R}^n : a(x) \geq -(\varepsilon_{\text{opt}}/2 + t^{(0)})1\} \\
\tilde{X}^{(P_2)} := \{x \in \mathbb{R}^n : a(x) \geq -\varepsilon_{\text{opt}}1\}.
\end{align*}
\]

By the definition of \( t^{(0)} \) we have \( \tilde{X}^{(P_2)} \subseteq \tilde{X}^{(P_1)} \).
Assumption 8. Assume that each $a_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, \ldots, m\}$ is a continuous function on $\mathbb{R}^n$. Let $L_1, L_2 \in (0, \infty)$. The functions $a_i : \mathbb{R}^n \to \mathbb{R}$ have $L_1$-Lipschitz first derivatives and $L_2$-Lipschitz second derivatives on the set $\mathcal{X}^{(P1)}$. The function $f : \mathbb{R}^n \to \mathbb{R}$ and $a_i : \mathbb{R}^n \to \mathbb{R}$ has $L_1$-Lipschitz first derivatives and $L_2$-Lipschitz second derivatives on the set $\mathcal{X}^{(P2)}$.

Before presenting Claim 3 let us introduce non-negative scalars $c$, $\Delta_f$, and $\Delta_a$ chosen as follows.

$$c \geq \sup_{x \in \mathcal{X}^{(P1)}} \max_{i \in \{1, \ldots, m\}} a_i(x)$$

$$\Delta_f \geq \sup_{z \in \mathcal{X}^{(P2)}} f(z) - \inf_{z \in \mathcal{X}^{(P2)}} f(z)$$

$$\Delta_a \geq \min_{i \in \{1, \ldots, m\}} \max\{-a_i(x^{(0)}), 0\}.$$  \hspace{1cm} (22a, 22b, 22c)

Claim 3. Let $x^{(0)} \in \mathbb{R}^n$. Suppose assumption 8 and 22 holds. Let $f$ be $L_0$-Lipschitz. Assume $c, \Delta_a, \Delta_f, L_1, L_0 \geq 1$, $\varepsilon_{opt} \in \left(0, \frac{1}{m \log^2(\varepsilon_{opt})} \right]$, and $\varepsilon_{inf} \in (0, 1]$. Then TWO-PHASE-IPM$(f, a, \varepsilon_{opt}, \varepsilon_{inf}, L_0, L_1, x^{(0)})$ takes at most

$$O \left( \frac{L_1^{3/4}}{\varepsilon_{inf}^{3/4} \varepsilon_{opt}} + \frac{1}{\varepsilon_{inf} \varepsilon_{opt}} \right) + \frac{\Delta_f}{\varepsilon_{opt}} \left( \frac{L_1 L_0}{\varepsilon_{opt} \varepsilon_{inf}} \right)^{3/4}.$$  \hspace{1cm} (23)

trust region subproblem solves to return a point that satisfies either KKT or INF1.

The definition of TWO-PHASE-IPM appears in Section 4.1 and the proof of Claim 3 appears in Section 4.2. The proof is primarily devoted to analyzing phase-two when we minimize the objective while appropriately satisfying the constraints. We argue that when we terminate with a Fritz John point in phase-two then either the dual variables are small enough that this is a KKT point or if the dual variables are large the scaled dual variables give an infeasibility certificate. If we add the assumption that $\varepsilon_{opt} \in (0, \varepsilon_{inf}]$ the runtime bound of Claim 3 can be even more simply stated as

$$O \left( \frac{\Delta_a + \Delta_f}{\varepsilon_{opt}} \left( \frac{L_1 L_0}{\varepsilon_{opt} \varepsilon_{inf}} \right)^{3/4} \right).$$  \hspace{1cm} (24)

We can now compare with the results of 3 in Table 1.

---

**Table 1** This table compares runtime bounds under the setup of (23). It only includes dependencies on $\varepsilon_{opt}$ and $\varepsilon_{inf}$. CRN stands for cubic regularized Newton [28].

| algorithm           | # iteration | iteration subproblem                          | evaluates |
|---------------------|-------------|-----------------------------------------------|-----------|
| Birgin et al. [3]   | $O\left(\varepsilon_{opt}^{-3} \varepsilon_{inf}^{-2}\right)$ | gradient computation                     | $\triangledown$ |
| Birgin et al. [3]   | $O\left(\varepsilon_{opt}^{-2} \varepsilon_{inf}^{-3/2}\right)$ | CRN with **non-negativity constraint**    | $\triangledown, \triangledown^2$ |
| IPM (this paper)    | $O\left(\varepsilon_{opt}^{-7/4} \varepsilon_{inf}^{-3/4}\right)$ | trust-region                              | $\triangledown, \triangledown^2$ |

The results of [3] find KKT points of a sequence of quadratic penalty subproblems of the form

$$\min_{(x, r, s) \in \mathbb{R}^n \times [3]} \Phi_t(x, r, s) := (f(x) - t + r)^2 + \|a(x) + s\|_2^2 \quad \text{such that} \quad r \geq 0, \quad s \geq 0.$$  \hspace{1cm} (24)

To solve this subproblem method [3] suggested using $p$th order regularization with non-negativity constraints. For $p = 2$ this reduces to cubic regularization Newton’s method with non-negativity.


\[ \text{minimize} \quad \frac{1}{2} d^T \nabla^2 \Phi_t(x, r, s) d + \nabla \Phi_t(x, r, s)^T d + \rho \| d \|_2^3 \quad \text{such that} \quad r + d_r \geq 0, \quad s + d_s \geq 0. \]

Solving this subproblem might be computationally expensive. It is well-known that checking if a point is a local optimum of (25) is in general NP-hard \[30\]. It is possible to find an approximate KKT point using projected gradient descent or an interior point method for solving nonconvex quadratic program \[43\]. However, both these approaches are likely to result in a computation runtime worse than \( O(\varepsilon^{-2} \log(1/\varepsilon)) \). We speculate that one might also be able to apply the interior point method of Haeser et al. \[15\] as the unconstrained minimization algorithm for solving (24) and potentially obtain the runtime bound of \( O(\varepsilon^{-2} \log(1/\varepsilon)) \) given by \[3\], although further analysis is needed to confirm this.

Finally, the work of Cartis et al. \[8, 9\] show that one requires \( O(\varepsilon^{-2} \log(1/\varepsilon)) \) iterations to find a scaled KKT point:

\[ \| \nabla x L(x, y) \|_2 \leq \varepsilon_{\text{opt}}(\| y \|_2 + 1) \quad y \geq 0 \quad a(x) \geq -\varepsilon_{\text{opt}} 1 \quad a_i(x) y_i \leq \varepsilon_{\text{opt}} (1 + \| y \|_2) \]

or certificate of infeasibility. Their method only requires computation of first-derivatives but has the disadvantage that it requires solving a linear program at each iteration.

### 7.2 Convex comparisons

Since there has been relatively little work with general convex constraints we generate a set of simple baselines for comparison using existing methods for unconstrained optimization. To simplify these comparisons consider the weaker problem of finding an \( \varepsilon \)-optimal solution to the problem of

\[ \max_{x \in \mathbb{R}^n} \min_{i \in \{1, \ldots, m\}} a_i(x), \]

and we assume optimal objective value of (26) is zero. Note that one approach to solve this problem is to minimize

\[ \rho_p(x) := \sum_{i=1}^{m} \max\{-a_i(x), 0\}^{p+1} \]

using a method that only requires the \( p \)th order derivative to be Lipschitz. It is easy to see to find a point satisfying \( a(x) \geq -\varepsilon 1 \) we need to find a point with \( \rho_p(x) \leq \varepsilon^{p+1} \). The results in Table 2 immediately follow by substituting the optimality tolerance of \( \varepsilon^{p+1} \) and a Lipschitz constant of \( \Theta(m) \) into each method’s runtime bounds.

| Algorithm            | # iteration | Iteration cost | Evaluates |
|----------------------|-------------|----------------|-----------|
| SG on (26)           | \( O(\varepsilon^{-2}) \) | matrix vector product | \( \nabla \) |
| CRN on \( \rho_2 \)  | \( O(m^{1/2} \varepsilon^{-3/2}) \) | linear system solve* | \( \nabla, \nabla^2 \) |
| AGD on \( \rho_1 \)  | \( O(m^{1/2} \varepsilon^{-1}) \) | matrix vector product | \( \nabla \) |
| ACRN on \( \rho_2 \) | \( O(m^{2/7} \varepsilon^{-6/7}) \) | linear system solve* | \( \nabla, \nabla^2 \) |
| IPM (this paper)     | \( O(m^{1/3} \varepsilon^{-2/3}) \) | linear system solve* | \( \nabla, \nabla^2 \) |
| cutting plane        | \( O(n \log(1/\varepsilon)) \) | centre of polytope  | \( \nabla \) |
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where the last inequality uses that $\mu$.

Let $\psi_\mu(x) := x - \mu (\log(x) + \log(2-x))$, $\psi_\mu^* = \inf_z \psi_\mu(z)$, $\mu \in (0,1/5]$ and $C \in [10, 2 \exp(1/\mu)]$. Consider the set $S_C = \{ x\in \mathbb{R} : \psi_\mu(x) \leq \psi_\mu^* + C \}$. Fix $\alpha \in (0, \infty)$. Suppose the $x(k)$ iterates satisfy $\psi_\mu^* \geq 0$ and remain in the interval $[0,2]$ for any starting point $x(0) \in S_C$. Then for the starting point $x(0) = 1 \in S_C$ and for all $k \leq 16C\mu\exp(1/\mu)$ we have $\| \nabla \psi_\mu(x(k)) \| \geq \mu$.

**Proof.** First note that $\psi_\mu^* \geq 0$. Suppose $x(0) = \exp(-C/(2\mu))$. Note that

$$\psi_\mu(x(0)) = x(0) - \mu \left( \log(x(0)) + \log(2 - x(0)) \right) \leq x(0) + C/2 \leq C \Rightarrow x(0) \in S_C.$$

The first inequality uses that $\log(x(0)) = \log(\exp(-C/(2\mu))) = -C/(2\mu) \leq -C/2$ and $\log(2 - x(0)) \leq \log(1) = 0$. The second inequality $x(0) \leq \exp(-25) \leq C$. Furthermore,

$$\nabla \psi_\mu(x(0)) = x(0) - \mu \left( \frac{1}{x(0)} + \frac{1}{2 - x(0)} \right) \leq \exp(-25) - \mu (\exp(C/(2\mu)) - 1) \leq -\mu/2 \exp(\exp(C/(2\mu))).$$

where the last inequality uses that $\frac{\mu}{2} \exp(C/(2\mu)) = \frac{\mu}{2} \exp((C - 2)/(2\mu) + 1/\mu + \log(\mu)) \geq \frac{\mu}{4} \exp((C - 2)/(2\mu)) \geq \exp(-25)$ and $\frac{\mu}{4} \exp(C/(2\mu)) \geq \frac{\mu}{4} \exp(25) \geq 1$. Therefore we conclude to guarantee that $x(1) \leq 2 \mu$ we need $\alpha \leq 4 \mu \exp(C/(2\mu)).$

Furthermore, if $x(k) \in [1/2,1]$ then $\nabla \psi_\mu(x(k)) \in [0,2]$. By induction if $x(0) = 1$ it will take at least $16\mu \exp(C/(2\mu))$ iterations until $x(k) \not\in [1/2,1]$. 

### A Proof of Claim 1

**Claim 1.** Let $\psi_\mu(x) := x - \mu (\log(x) + \log(2-x))$, $\psi_\mu^* = \inf_z \psi_\mu(z)$, $\mu \in (0,1/5]$ and $C \in [10, 2 \exp(1/\mu)]$. Consider the set $S_C = \{ x\in \mathbb{R} : \psi_\mu(x) \leq \psi_\mu^* + C \}$. Fix $\alpha \in (0, \infty)$. Suppose the $x(k)$ iterates satisfy $\psi_\mu^* \geq 0$ and remain in the interval $[0,2]$ for any starting point $x(0) \in S_C$. Then for the starting point $x(0) = 1 \in S_C$ and for all $k \leq 16C\mu\exp(1/\mu)$ we have $\| \nabla \psi_\mu(x(k)) \| \geq \mu$.

**Proof.** First note that $\psi_\mu^* \geq 0$. Suppose $x(0) = \exp(-C/(2\mu))$. Note that

$$\psi_\mu(x(0)) = x(0) - \mu \left( \log(x(0)) + \log(2 - x(0)) \right) \leq x(0) + C/2 \leq C \Rightarrow x(0) \in S_C.$$

The first inequality uses that $\log(x(0)) = \log(\exp(-C/(2\mu))) = -C/(2\mu) \leq -C/2$ and $\log(2 - x(0)) \leq \log(1) = 0$. The second inequality $x(0) \leq \exp(-25) \leq C$. Furthermore,

$$\nabla \psi_\mu(x(0)) = x(0) - \mu \left( \frac{1}{x(0)} + \frac{1}{2 - x(0)} \right) \leq \exp(-25) - \mu (\exp(C/(2\mu)) - 1) \leq -\mu/2 \exp(\exp(C/(2\mu))).$$

where the last inequality uses that $\frac{\mu}{2} \exp(C/(2\mu)) = \frac{\mu}{2} \exp((C - 2)/(2\mu) + 1/\mu + \log(\mu)) \geq \frac{\mu}{4} \exp((C - 2)/(2\mu)) \geq \exp(-25)$ and $\frac{\mu}{4} \exp(C/(2\mu)) \geq \frac{\mu}{4} \exp(25) \geq 1$. Therefore we conclude to guarantee that $x(1) \leq 2 \mu$ we need $\alpha \leq 4 \mu \exp(C/(2\mu))$.

Furthermore, if $x(k) \in [1/2,1]$ then $\nabla \psi_\mu(x(k)) \in [0,2]$. By induction if $x(0) = 1$ it will take at least $16\mu \exp(C/(2\mu))$ iterations until $x(k) \not\in [1/2,1]$.

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B Proofs from Section 4.1

B.1 Proof of Lemma 2

Lemma 2. Suppose the function \( g : \mathbb{R} \to \mathbb{R} \) has \( L_1 \)-Lipschitz first derivatives, \( L_2 \)-Lipschitz second derivatives on the set \([0, \vartheta]\). Also assume \( g(0) > 0 \). For all \( \theta \in [0, \vartheta] \) which the following inequality holds:

\[
\frac{|\theta g'(0)|}{g(0)} + \frac{L_1 \theta^2}{2g(0)} \leq \beta \leq 1/2
\]

with \( \beta \in \mathbb{R}_+ \), we have \( \frac{g(0)}{g(0)} \in \left[\frac{1}{2}, \frac{3}{2}\right] \) and

\[
\theta^3 \left| \frac{\partial^3 \log(g(\theta))}{\partial \theta^3} \right| \leq 2 \frac{L_2 \theta^3 + 6L_1 \theta^2 \beta}{g(0)} + 8\beta^3.
\]

Proof We have

\[
\frac{|g(0) - g(\theta)|}{g(0)} \leq \frac{|\theta g'(0)|}{g(0)} + \frac{L_1 \theta^2}{2g(0)} \leq \frac{1}{2}.
\]

The first inequality uses that \( |g(0) + g'(0)\theta - g(\theta)| \leq \frac{L_1 \theta^2}{2} \) since \( g \) has \( L_1 \)-Lipschitz derivatives on \([0, \vartheta] \subseteq [0, \vartheta]\), the triangle inequality and \( g(0) > 0 \). The second and third inequality is simply the assumed bound in the theorem statement. Therefore we have established \( \frac{g(0)}{g(0)} \in [1/2, 3/2] \).

We turn to proving our bound on the third derivatives of \( \log(g(\theta)) \). For any function \( g : \mathbb{R} \to \mathbb{R} \) we have

\[
\frac{\partial \log(g(\theta))}{\partial \theta} = \frac{g'(\theta)}{g(\theta)},
\]

\[
\frac{\partial^2 \log(g(\theta))}{\partial \theta^2} = \frac{g''(\theta)}{g(\theta)} - \frac{g'(\theta)^2}{g(\theta)^2},
\]

\[
\frac{\partial^3 \log(g(\theta))}{\partial \theta^3} = \frac{g'''(\theta)}{g(\theta)} - \frac{3g'(\theta)g''(\theta)}{g(\theta)^2} + \frac{g'(\theta)^3}{g(\theta)^3}.
\]

By (27), \( \frac{g(0)}{g(0)} \in [1/2, 3/2] \), \( |g'''(\theta)| \leq L_2 \), and \( |g''(\theta)| \leq L_1 \) we have

\[
\left| \frac{\partial^3 \log(g(\theta))}{\partial \theta^3} \right| \leq 2 \frac{L_2 g(\theta)}{g(\theta)} + 12 \frac{L_1 |g'(\theta)|}{g(\theta)^2} + 8 \frac{|g'(\theta)|^3}{g(\theta)^3}.
\]

Now, since

\[
g'(\theta) \leq g'(0) + L_1 \theta
\]

we have

\[
\theta^3 \left| \frac{\partial^3 \log(g(\theta))}{\partial \theta^3} \right| \leq \left( 2 \frac{L_2 \theta^3}{g(0)} + 12 \frac{L_1 \theta^2 (g'(0) + L_1 \theta^2)}{g(0)^2} + 8 \frac{(\theta g'(0) + L_1 \theta^2)^3}{g(0)^3} \right) \leq 2 \frac{L_2 \theta^3 + 6L_1 \theta^2 \beta}{g(0)} + 8\beta^3.
\]

\[\square\]

B.2 Proof of Lemma 3

Lemma 3. Suppose assumption 3 holds (Lipschitz derivatives). Let \( x \in X \) and \( S = \text{diag}(a(x)) \). Consider any \( dx \in \mathbb{R}^n \). Suppose the following inequality holds:

\[
\|S^{-1}dx\|_2 + \frac{L_1 (1 + \|y\|_1)}{2\mu} \|dx\|_2^2 \leq \kappa \leq 1/2
\]

where \( \kappa \in \mathbb{R}_+ \), \( S = \text{diag}(a(x)) \), \( dx = \nabla a(x)dx \) and \( y = \mu S^{-1}1 \). Then \( \frac{a(x+dx)}{a(x)} \in [1/2, 3/2] \) for all \( i \in \{1, \ldots, m\} \) and

\[
|\psi_\mu(x) + M_x^{\psi_\mu}(dx) - \psi_\mu(x + dx)| \leq \frac{L_2}{6} (1 + 2\|y\|_1) \|dx\|_2^2 + 3L_1 \|dx\|_2^2 \|y\|_2 \kappa + 3\mu \kappa^3.
\]
Proof First we show $a_i(x + d_x) \in [1/2, 3/2]$. To obtain a contradiction assume $|a_i(x + \theta v) - a_i(x)| > a_i(x)/2$ with $v = d_x/\|d_x\|_2$ for some $\theta \in [0, \|d_x\|_2]$ and $i \in \{1, \ldots, m\}$. Define $q(\theta) := \sup_{\theta \in [0, \theta]} |a_i(x + \theta v) - a_i(x)|$. Since $a_i$ is continuous $q$ is continuous and by the intermediate value theorem there exists some $\tilde{\theta} \in [0, \theta]$ such that $a_i(x)/2 < q(\tilde{\theta}) < a_i(x)$. By Lemma 2 we have $q(\tilde{\theta}) \leq a_i(x)/2$ which is a contradiction.

Define the vector $\beta$ to be

$$
\beta_i = \frac{\nabla a_i(x)^T d_x}{a_i(x)} + \frac{L_1\|d_x\|^2}{2a_i(0)},
$$

for all $i \in \{1, \ldots, m\}$.

Then we have,

$$
\|\beta\|^2 = \sum_{i=1}^{m} \left( \frac{\nabla a_i(x)^T d_x}{a_i(x)} + \frac{L_1\|d_x\|^2 y_i}{2\mu} \right)^2,
$$

$$
\leq 2 \sum_{i=1}^{m} \left( \frac{\nabla a_i(x)^T d_x}{a_i(x)} \right)^2 + \left( \frac{L_1\|d_x\|^2 y_i}{2\mu} \right)^2,
$$

$$
= 2\|S^{-1}d_x\|_2^2 + 2\left( \frac{L_1\|d_x\|^2 y_i}{2\mu} \right)^2\|y\|_2^2,
$$

$$
\leq 2\|S^{-1}d_x\|_2^2 + 2\left( \frac{L_1\|d_x\|^2 y_i}{2\mu} \right)^2(\|y\|_1 + 1)^2,
$$

$$
\leq 2\left( \|S^{-1}d_x\|_2 + \frac{L_1(\|y\|_1 + 1)\|d_x\|^2}{2\mu} \right)^2 = 2\kappa^2
$$

where the first inequality uses $1/a_i(x) = y_i/\mu$, the second inequality uses the fact that $(a + b)^2 \leq 2(a^2 + b^2)$, and the final inequality uses $a^2 + b^2 \leq (a + b)^2$. Hence,

$$
\sum_{i=1}^{m} \beta_i^3 \leq \|\beta\|^2 \max_i \{\beta_i\} \leq 2\kappa^3
$$

and by Cauchy-Schwarz,

$$
\sum_{i=1}^{m} \beta_i y_i \leq \|\beta\|_2 \|y\|_2 \leq \sqrt{2\kappa} \|y\|_2.
$$

Observe, also by Taylor’s Theorem and the fact that $f$ is Lipschitz on $X$ that

$$
\left| f(x) + \frac{1}{2} d_x \nabla^2 f(x) d_x + \nabla f(x)^T d_x - f(x + d_x) \right| \leq \frac{L_2}{6}\|d_x\|_2^3.
$$

Using Lemma 2 and Taylor’s Theorem with $h(\theta) := \log(g(\theta))$, $g(\theta) := a_i(x + \theta v)$, $v = d_x/\|d_x\|_2$ we get

$$
\left| h(0) + \theta h'(0) + \frac{\theta^2}{2} h''(0) - h(\theta) \right| \leq \frac{\theta^3}{6} \sup_{\theta \in [0, \theta]} h''(\tilde{\theta}) \leq \frac{1}{6} \left( \frac{2L_2\theta^3 + 6L_1\theta^2\beta_i + 8\beta_i^3}{g(0)} \right)
$$

We can now bound the quality of a second-order Taylor series expansion of $\psi_\mu$ as

$$
|\psi_\mu(x + M_\mu^2 (d_x) - \psi_\mu(x + d_x)| \leq \frac{L_2}{6}\|d_x\|_2^3 + \mu \sum_{i=1}^{m} \left( \frac{L_2\|d_x\|_2^3 + 6L_1\|d_x\|_2\beta_i}{3a_i(x)} + \frac{4\beta_i^3}{3} \right)
$$

$$
\leq \frac{L_2}{6}\|d_x\|_2^3 + \mu \sum_{i=1}^{m} \left( \frac{L_2\|d_x\|_2^3}{3} + 2L_1\|d_x\|_2\beta_i + \frac{4\beta_i^3}{3} \right)
$$

$$
\leq \frac{L_2}{6}(1 + 2\|y\|_1)\|d_x\|_2^3 + 3L_1\|d_x\|_2\|y\|_2\kappa + 3\mu\kappa^3.
$$
The first inequality uses (30) and (31). The second inequality uses $1/a_i(x) = y_i/\mu$. The third inequality uses (28) and (29). □

B.3 Proof of Lemma 4

Lemma 4. Suppose assumption 3 holds. Let \( \text{CONVEX}(x, x^+) \subseteq \mathcal{X} \), \( s = a(x) \), \( s^+ = a(x^+) \), \( S = \text{diag}(a(x)) \), \( Y = \text{diag}(y) \), \( y^+ \in \mathbb{R}^m \), and \( Y^+ = \text{diag}(y^+) \). Also, let \( d_x = x^+ - x \) and \( d_y = y^+ - y \), and \( d_s = \nabla a(x)d_x \). If the following equation holds:

\[
Sy + Sd_y + Yd_s = \mu 1,
\]

then:

\begin{align*}
\|Y^{-1}d_y\|_2 & \leq \|S^{-1}d_s\|_2 + \|\mu(SY)^{-1}1 - 1\|_2 \tag{12} \\
\|Y^+s^+ - \mu 1\|_2 & \leq \|Sy\|_\infty\|S^{-1}d_s\|_2\|Y^{-1}d_y\|_2 + \frac{L_1}{2}\|y\|_2(1 + \|Y^{-1}d_y\|_2)\|d_x\|_2^2. \tag{13}
\end{align*}

Furthermore, if \( \|Y^+s^+ - \mu 1\|_\infty < \mu \) and \( \|Y^{-1}d_y\|_\infty \leq 1 \) then \( s^+, y^+ \in \mathbb{R}^m_{++} \).

Proof To show (12) notice that multiplying \( Sy + Sd_y + Yd_s = \mu 1 \) by \( (SY)^{-1} \) and rearranging yields \( Y^{-1}d_y = -S^{-1}d_s + ((SY)^{-1}1 - 1) \).

Next, we show (13). Observe that

\[
s_i^+(y_i^+) - \mu = a_i(x + d_x)(y_i + d_{y_i}) - \mu \\
= (d_{s_i} + a_i(x))(y_i + d_{y_i}) + (a_i(x + d_x) - (d_{s_i} + a_i(x)))(y_i + d_{y_i}) - \mu \\
= d_{s_i}d_{y_i} + (a_i(x + d_x) - (d_{s_i} + a_i(x)))(y_i + d_{y_i}), \tag{32}
\]

where the first transition is by definition of \( s_i^+ \), the second transition comes from adding and subtracting \( (d_{s_i}, a_i(x))(y_i + d_{y_i}) \) and the third transition by substituting \( \mu = s_iy_i + s_id_{y_i} + y_id_s = a_i(x)y_i + a_i(x)d_{y_i} + y_id_s \). Furthermore, since \( \nabla a_i \) is \( L_1 \)-Lipschitz continuous

\[
|a_i(x + d_x) - (d_{s_i} + a_i(x))| = |a_i(x + d_x) - (\nabla a_i(x)d_x + a_i(x))| \leq \frac{L_1}{2}\|d_x\|_2^2,
\]

combining this equality with (32) yields

\[
|s_i^+(y_i^+) - \mu| \leq |d_{s_i}d_{y_i}| + \frac{L_1}{2}\|d_x\|_2^2 \leq |s_iy_i||s_i^{-1}d_{s_i}|\|y_i^{-1}d_{y_i}\| + \frac{L_1}{2}y_i(1 + y_i^{-1}d_{y_i})\|d_x\|_2^2.
\]

We deduce (13) by Cauchy-Schwarz. The fact that \( y^+ \in \mathbb{R}^m_{++} \) follows from \( \|Y^{-1}d_y\|_\infty \leq 1 \); \( s^+ \in \mathbb{R}^m_{++} \) follows from \( y^+ \in \mathbb{R}^m_{++} \) and \( \|S^{+}y^+ - \mu\|_\infty < \mu \). □

B.4 Proof of Lemma 5

Lemma 5. Suppose assumption 3 holds. Let \( y, y^+ \in \mathbb{R}^m \) and \( \text{CONVEX}(x, x^+) \subseteq \mathcal{X} \). Then the following inequality holds:

\[
\|\nabla_x L(x, y) + \nabla_{xx} L(x, y)^T d_x - d_y^T \nabla_x a(x) - \nabla_x L(x^+, y^+)\|_2 \leq L_1\|y\|_2\|d_x\|_2\|Y^{-1}d_y\|_2 + \frac{L_2}{2}(\|y\|_1 + 1)\|d_x\|_2^2
\]

with \( d_x = x^+ - x \) and \( d_y = y^+ - y \).

Proof Observe that:

\[
\begin{align*}
&\left\| \sum_i (y_i \nabla a_i(x) + y_i \nabla^2 a_i(x)d_x - d_{y_i} \nabla a_i(x) - y_i^+ \nabla a_i(x^+)) \right\|_2 \\
&\leq \left\| \sum_i (y_i \nabla a_i(x) + y_i \nabla^2 a_i(x)d_x + d_{y_i} \nabla a_i(x) - y_i^+ \nabla a_i(x^+)) \right\|_2 \\
&\leq \|y\|_1 \left\| \nabla a_i(x) + \nabla^2 a_i(x)d_x - \nabla a_i(x^+) \right\|_2 + \|d_y\|_1 \left\| \nabla a_i(x) - \nabla a_i(x^+) \right\|_2 \\
&\leq \frac{L_2}{2}\|y\|_1\|d_x\|_2^2 + L_1\|d_y\|_1\|d_x\|_2,
\end{align*}
\]
where the first and second transition hold by the triangle inequality, the third transition using \( \delta \) with the Lipschitz continuity of \( \nabla a \) and \( \nabla^2 a \). Next, by the triangle inequality, the inequality we just established, and Taylor’s theorem with Lipschitz continuity of \( \nabla f \) we get

\[
\| \nabla_x \mathcal{L}(x, y) + \nabla_{xx} \mathcal{L}(x, y)^T d_x - d_y \nabla_x a(x) - \nabla_x \mathcal{L}(x^+, y^+) \|_2 \\
\leq \left\| \nabla f(x) + \nabla^2 f(x) d_x - \nabla f(x^+) \right\|_2 + \left\| \sum_i (y_i \nabla a_i(x) + y_i \nabla^2 a_i(x) d_x + d_y_i \nabla a_i(x) - y_i^+ \nabla a_i(x^+)) \right\|_2 \\
\leq \frac{L_2}{2}(\|y\|_1 + 1)\|d_x\|_2 + L_1\|d_y\|_1\|d_x\|_2.
\]

(33)

\[ \Box \]

\section*{B.5 Proof of Lemma \ref{lem:delta_u}}

\textbf{Lemma 6.} \textit{Consider} \( H \in \mathbb{R}^{n \times n} \) \textit{and} \( g \in \mathbb{R}^n \). \textit{Define} \( \Delta(u) := \frac{1}{2} u^T H u + g^T u \) \textit{where} \( \Delta : \mathbb{R}^n \rightarrow \mathbb{R} \) \textit{and let} \( u^* \in \text{argmin}_{u \in \mathbb{R}^n} \Delta(u) \) \textit{be an optimal solution to the trust region subproblem for some} \( r \geq 0 \). \textit{Then there exists some} \( \delta \geq 0 \) \textit{such that:}

\[ \delta(\|u^*\|_2 - r) = 0, \quad (H + \delta I)u^* = -g, \quad \text{and} \quad H + \delta I \succeq 0. \tag{14} \]

Conversely, \textit{if} \( u^* \) \textit{satisfies} \textit{(14)} \textit{then} \( u^* \in \text{argmin}_{u \in \mathbb{R}^n} \Delta(u) \). \textit{Let} \( \sigma(\tau) := \min_{u \in \mathbb{R}^n} \Delta(u) \), \textit{then for all} \( r \in [0, \infty) \) \textit{we have}

\[ \sigma(\tau) \leq -\frac{\delta \tau^2}{2} \tag{15a} \]

\[ \sigma(\tau) \leq \sigma(\alpha \tau) \leq \alpha^2 \sigma(\tau) \quad \forall \alpha \in [0, 1]. \tag{15b} \]

Furthermore, \textit{the function} \( \sigma(\tau) \) \textit{is monotone decreasing and continuous.}

\textit{Proof} \ Equation \ [(14)] \ follows \ from \ the \ KKT \ conditions, \ see \ Sorensen \ [34, \ Lemma 2.4.], \ Conn \ et \ al. \ [12, \ Corollary 7.2.2] \ or \ Nocedal \ and \ Wright \ [29, \ Theorem 4.3.]. \ We \ now \ show \ (15a). \ Substituting \ (H + \delta I)u^* = -g \ into \ \frac{1}{2}(u^*)^T H u^* + g^T u^* \ yields \ \sigma(\tau) = \Delta(u^*) = 1/2 g^T u^* - \delta / 2 \|u^*\|_2^2 \leq -\delta / 2 \|u^*\|_2^2 \ where \ the \ last \ inequality \ follows \ from \ g^T u^* = -g^T (H + \delta I)^{-1} g \leq 0. \ Since \ (14) \ states \ that \ either \ \delta = 0 \ or \ \|u^*\|_2 = r \ \we \ conclude \ (15a) \ holds. \ The \ inequality \ \sigma(\alpha \tau) \leq \alpha^2 \sigma(\tau) \ holds \ since \ \sigma(\alpha \tau) \leq \Delta(\alpha u^*) = \frac{1}{2} \alpha^2 (u^*)^T H u^* + \alpha g^T u^* \leq \frac{1}{2} \alpha^2 (u^*)^T H u^* + \alpha^2 g^T u^* = \alpha^2 \sigma(\tau) \ where \ the \ inequality \ uses \ g^T u^* \leq 0. \ The \ inequality \ \sigma(\tau) \leq \sigma(\alpha \tau) \ holds \ since \ any \ solution \ to \ \|u\|_2 \leq r \ is \ feasible \ to \ \|u\|_2 \leq \alpha \tau. \ The \ fact \ that \ \sigma(\tau) \ is \ monotone \ decreasing \ and \ continuous \ follows \ from \ (15b). \ \Box}

\section*{C Proofs of results in Section \ref{sec:5.1}}

\subsection*{C.1 Proof of Lemma \ref{lem:psi}}

\textbf{Lemma 8.} \textit{Suppose assumption \ref{assum:3} \ and \ref{assum:4} \ hold (Lipschitz derivatives, and sufficiently small} \( \mu \). \textit{Also assume} \( x \in \mathcal{X} \), \( \eta_s \in [0, 1/4] \). \textit{Let} \( \text{TRG} \) \textit{hold with} \( \eta_x = \frac{\eta_s}{2} \). \textit{Let} \( \alpha = \min \left\{ 1, \frac{\eta_s}{\|S + \delta_i \|_2} \right\} \). \textit{Then} \( x^+ \in \mathcal{X} \) \textit{and}

\[ \psi_\mu(x^+) - \psi_\mu(x) \leq 2 \mu \eta_s^2 + \max \left\{ \mathcal{M}_x^{\psi_\mu}(d_x), -\frac{\eta_s^2 \mu}{3} \right\}. \tag{20} \]

\textit{Proof} \ First \ we \ show \ for \ all \ \( \alpha \in (0, 1) \) \ that

\[ \mathcal{M}_x^{\psi_\mu}(\alpha d_x) \leq \max \left\{ -\frac{\eta_s^2 \mu}{3}, \mathcal{M}_x^{\psi_\mu}(d_x) \right\}. \tag{34} \]
Note (34) trivially holds if \( \alpha = 1 \). Therefore let us consider the case \( \alpha \in (0, 1) \). In this case,

\[
\alpha = \frac{\eta_s}{\|S^{-1}d_s\|_2} \geq \eta_s \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)\|d_x\|_2^2 - 2\mathcal{M}_x^{\psi}(d_x)}} \geq \eta_s \sqrt{\frac{\mu}{\eta_s^2 \mu / 4 - 2\mathcal{M}_x^{\psi}(d_x)}},
\]

where the first inequality uses (19) and the second \( \|d_x\|_2 \leq \frac{\eta_s}{2} \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)} \cdot \frac{1}{\alpha d}} \) Furthermore, if \( \mathcal{M}_x^{\psi}(d_x) \in \left[-\frac{\eta_s^2 \mu}{4}, 0\right] \) from (35) we get \( \alpha \geq 4/3 > 1 \); by contradiction we conclude \( \mathcal{M}_x^{\psi}(d_x) \not\in \left[-\frac{\eta_s^2 \mu}{4}, 0\right] \). Furthermore, since \( \mathcal{M}_x^{\psi}(d_x) = \min \mathcal{M}_x^{\psi}(u) \leq \mathcal{M}_x^{\psi}(0) = 0 \) we have \( \mathcal{M}_x^{\psi}(d_x) < -\frac{\eta_s^2 \mu}{4} \). Combining this inequality with (35) yields \( \alpha \geq \eta_s \sqrt{\frac{\mu}{-3\mathcal{M}_x^{\psi}(d_x)}} \).

Therefore,

\[
\mathcal{M}_x^{\psi}(\alpha d_x) = \alpha^2 \frac{1}{2} d_x^T \nabla^2 \psi_{\mu}(x) d_x + \alpha \nabla \psi_{\mu}(x)^T d_x \leq \alpha^2 \mathcal{M}_x^{\psi}(d_x) \leq \frac{\eta_s^2 \mu}{3}
\]

where the first inequality follows by \( \nabla \psi_{\mu}(x)^T d_x \leq 0 \) as implied by (14) and the second by \( \alpha \geq \eta_s \sqrt{\frac{\mu}{-3\mathcal{M}_x^{\psi}(d_x)}} \). Therefore have proven (34).

It remains to bound the accuracy of the predicted decrease \( \mathcal{M}_x^{\psi}(\alpha d_x) \). Let us bound the constant \( \kappa \) from Lemma [3]

\[
\alpha \|S^{-1}d_s\|_2 + \frac{L_1 \|\alpha d_x\|_2^2 (1 + \|y\|_1)}{2\mu} \leq \eta_s + \frac{\eta_s^2}{8} \leq (33/32) \eta_s = \kappa
\]

where the second inequality comes from \( \alpha \|S^{-1}d_s\|_2 \leq \eta_s \) and

\[
\|\alpha d_x\|_2 \leq \|d_x\|_2 \leq \frac{\eta_s}{2} \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)}} = \eta_s \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)}}.
\]

the third inequality from \( \eta_s \in [0, 1/4] \). Since \( \eta_s \in [0, 1/4] \) we deduce \( \kappa \leq 1/2 \) so the conditions of Lemma [3] hold. Therefore \( x^+ \in \mathcal{X} \). From Lemma [3]

\[
|\psi_{\mu}(x) + \mathcal{M}_x^{\psi}(\alpha d_x) - \psi_{\mu}(x + \alpha d_x)| \leq \frac{L_2}{6} \left(1 + 2\|y\|_1\right) \|\alpha d_x\|_2^3 + \frac{3L_1}{6} \|\alpha d_x\|_2^2 \|y\|_2 \kappa + 3\mu \kappa^3
\]
\[
\leq \frac{L_2^{3/2} \mu^{-1/2}}{6} \left(1 + 2\|y\|_1\right) \|\alpha d_x\|_2^3 + \frac{3L_1 \|\alpha d_x\|_2^2}{2} \|y\|_2 \kappa + 3\mu \kappa^3
\]
\[
\leq \frac{2\mu \eta_s^3}{6 \times 2^3} + 3(1/2^2)(33/32)^2 \mu \eta_s^3 + \frac{3\mu \eta_s^3}{2^3} \times (33/32)^3 \leq 2\mu \eta_s^3
\]

where the second inequality uses \( \frac{L_2^{3/2} \mu^{-1/2}}{L_1^{3/2} 2^3} \in (0, 1) \), the final inequality uses our bound on \( \kappa \) and \( \|\alpha d_x\|_2 \), i.e. (36) and (37). Combining (34) and (38) gives (20). \( \square \)

C.2 Proof of Lemma 9

Lemma 9. Suppose (TTRS), assumption [3] and [3] hold (direction selection, Lipschitz derivatives, and sufficiently small \( \mu \)). Let \( x \in \mathcal{X}, \eta_s \in (0, \frac{1}{5}(\frac{\eta_s^4}{4\nu L_1})^{1/4}) \). Further assume \( \mathcal{M}_x^{\psi}(d_x) \geq -\frac{\eta_s^4 \mu}{3\nu L_1} \) and \( \alpha = 1 \). Under these assumptions, \( (x^+, y^+) \) satisfies (FJ1) and \( \nabla^2 \psi_{\mu}(x) \succeq -\sqrt{\eta_s}(1 + \|y\|_1) \sqrt{\frac{\eta_s \mu}{\eta_s^4 \nu L_1}} I \).
Proof First, let us bound $\|S^{-1}d_x\|_2$:

$$
\|S^{-1}d_x\|_2 \leq \sqrt{\frac{L_1(\|y\|_1 + 1)\|d_x\|_2^2 - 2\mathcal{M}_\mu^n(d_x)}{\mu}} \leq \sqrt{\eta_x^2 + \frac{2}{3} \eta_x \left( \frac{\mu r^2}{L_1} \right)^{1/2}}
$$

$$
\leq \sqrt{\frac{1}{6^2} \left( \frac{\mu r^2}{L_1} \right)^{1/2} + \frac{2}{18} \left( \frac{\mu r^2}{L_1} \right)^{3/4}}
$$

$$
\leq \frac{4}{10} \left( \frac{\mu r^2}{L_1} \right)^{1/4},
$$

where the first inequality uses (19), the second $r = \eta_x \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)}}$ and $\mathcal{M}_\mu^n(d_x) \geq -\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3}$, the third inequality uses $\eta_x \in (0, \frac{1}{6} \left( \frac{\mu r^2}{L_1} \right)^{1/4}]$, and the final inequality uses $\frac{\mu r^2}{L_1} \in (0, 1]$.

Let us compute $\kappa$ from Lemma 3 $\|S^{-1}d_x\|_2 + \frac{L_1(1+\|y\|_1)}{2\mu} \|d_x\|_2^2 \leq \frac{4}{10} \left( \frac{\mu r^2}{L_1} \right)^{1/4} + \frac{\eta_x}{2} \leq \frac{1}{2} = \kappa$.

It follows that $\text{CONVEX} \{x, x^+\} \subseteq \mathcal{X}$.

Furthermore, by Lemma 4 and the fact $y = \mu S^{-1}1$ we have

$$
\|Y^{-1}d_y\|_2 \leq \|S^{-1}d_x\|_2 \leq \frac{1}{2} \left( \frac{\mu r^2}{L_1} \right)^{1/4}.
$$

Therefore,

$$
\|\delta d_x - \nabla_x \mathcal{L}(x^+, y^+)\|_2 \leq L_1\|d_x\|_2\|y\|_2\|Y^{-1}d_y\|_2 + \frac{L_2}{2} (\|y\|_1 + 1)\|d_x\|_2^2
$$

$$
\leq L_1\|d_x\|_2\|y\|_2\|Y^{-1}d_y\|_2 + \frac{L_2^{1/2}}{2} \mu^{-1/2} (\|y\|_1 + 1)\|d_x\|_2^2
$$

$$
\leq \frac{\tau_1 \mu \sqrt{\|y\|_2^2}}{2} \left( \frac{\mu r^2}{L_1} \right)^{-1/4} \eta_x + \frac{\mu r^2}{12} \frac{1}{2}
$$

$$
\leq \frac{\mu r^2}{10} \sqrt{1 + \|y\|_2^2},
$$

where the first inequality follows from Lemma 5, the second by $\frac{L_1^{1/2}L_2}{L_1^{1/2}} \in (0, 1]$, the third using the bounds $\|d_x\|_2 \leq \eta_x \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)}}$ and $\|Y^{-1}d_y\|_2 \leq \frac{1}{2} \left( \frac{\mu r^2}{L_1} \right)^{1/4}$ that we proved, and the fourth inequality using $\eta_x \in (0, \frac{1}{6} \left( \frac{\mu r^2}{L_1} \right)^{1/4}]$.

Next, we bound $\|d_x\|_2$. By (10) there exists some $\delta \geq 0$ such that $\nabla_x \mathcal{L}(x, y) + \nabla_x \mathcal{L}(x, y)^T d_x - d^T_y \nabla_x a(x) = \delta d_x$. Moreover, $-\frac{\tau_1 \mu \sqrt{1 + \|y\|_1}}{3} \leq \mathcal{M}_\mu^n(u) \leq -\frac{\delta^2}{2}$ by (15a), so

$$
\delta \|d_x\|_2 \leq \delta r \leq \frac{2}{3} \tau_1 \mu \sqrt{1 + \|y\|_1}.
$$

Therefore using the bounds on $\delta \|d_x\|_2$ and $\|\delta d_x - \nabla_x \mathcal{L}(x^+, y^+)\|_2$ that we proved,

$$
\|\nabla_x \mathcal{L}(x^+, y^+)\|_2 \leq \|\delta d_x - \nabla_x \mathcal{L}(x^+, y^+)\|_2 + \delta \|d_x\|_2
$$

$$
\leq \frac{2\tau_1 \mu}{3} \sqrt{1 + \|y\|_1} + \frac{\tau_1 \mu}{10} \sqrt{1 + \|y\|_1}
$$

$$
\leq \tau_1 \mu \sqrt{1 + \|y\|_1}.$$
This shows (FJ1.c) holds. It remains to show (FJ1.a) and (FJ1.b). From Lemma 4 we get
\[
\|S^+y^+ - \mu 1\|_2 \leq \mu \|S^{-1} d_s\|_2 \|Y^{-1} d_y\|_2 + \frac{L_1}{2}\|y\|_2(1 + \|Y^{-1} d_y\|_2)\|d_x\|_2^2
\]
\[
\leq \mu \left( \frac{\mu \tau r}{L_1} \right)^{1/2} + \mu \eta_x^2 \leq \frac{\mu}{2} \left( \frac{\mu \tau r}{L_1} \right)^{1/2} \leq \frac{\mu}{2},
\]
where the second inequality uses \(\|Y^{-1} d_y\|_2 \leq \|S^{-1} d_s\|_2 \leq \frac{1}{2} \left( \frac{\mu \tau r}{L_1} \right)^{1/2} \leq 1\) and \(\|d_x\|_2 \leq r = \eta_x \sqrt{\frac{\mu}{L_1(\|y\|_2 + 1)}}\), the third inequality \(\eta_x \in (0, \frac{1}{8} \left( \frac{\mu \tau r}{L_1} \right)^{1/4}]\), and the final inequality uses \(\frac{\mu \tau r}{L_1} \in (0, 1]\).

Therefore (FJ1) holds.

Let \(v_{\min}\) be the eigenvector of \(\nabla^2 \psi_\mu(x)\) corresponding to the minimum eigenvalue of \(\nabla^2 \psi_\mu(x)\).

Note that
\[
-\frac{\tau \mu r \sqrt{1 + \|y\|_2}}{3} \leq \mathcal{M}_{\psi_\mu}(d_x) \leq \min \{ \mathcal{M}_{\psi_\mu}(rv_{\min}), \mathcal{M}_{\psi_\mu}(-rv_{\min}) \} \leq \frac{\lambda_{\min}(\nabla^2 \psi_\mu(x)) r^2}{2}
\]
where \(\lambda_{\min}(\cdot)\) denotes the minimum eigenvalue. Therefore
\[
\lambda_{\min}(\nabla^2 \psi_\mu(x)) \geq -\frac{2\tau \mu r \sqrt{1 + \|y\|_2}}{3r} = -\frac{2}{3} \sqrt{\tau \mu r (1 + \|y\|_2)} \frac{\tau \mu}{\eta_x^2 L_1}.
\]

\[
\eta_s = \frac{1}{40} \left( \frac{\tau \mu r}{L_1} \right)^{1/4} \quad \eta_x = \frac{\eta_s}{2}, \tag{η-1}
\]

\[
\mathcal{O} \left( 1 + \frac{\psi_\mu(x^{(0)}) - \inf_{x \in X} \psi_\mu(x)}{\mu} \left( \frac{L_1}{\mu \tau r} \right)^{3/4} \right)
\]

C.3 Proof of Theorem 1

**Theorem 1.** Suppose assumptions 2 and 3 hold (Lipschitz derivatives, and sufficiently small \(\mu\)). Then \(\text{Trust-IPM}(f, a, \mu, \tau, L_1, \eta_s, \eta_x, x^{(0)})\) with \(x^{(0)} \in X\) and
\[
\eta_s = \frac{1}{40} \left( \frac{\tau \mu r}{L_1} \right)^{1/4} \quad \eta_x = \frac{\eta_s}{2}, \tag{η-1}
\]

\[
\mathcal{O} \left( 1 + \frac{\psi_\mu(x^{(0)}) - \inf_{x \in X} \psi_\mu(x)}{\mu} \left( \frac{L_1}{\mu \tau r} \right)^{3/4} \right)
\]

iterations to terminate with a \((\mu, \tau, \tau_c)\)-approximate second-order Fritz John point \((x^+, y^+)\), i.e., (FJ1) and (FJ2) hold.

**Proof.** Let \(x \in X\) be some iterate of the algorithm with corresponding direction \(d_x\). If
\[
-\frac{\tau \mu r \sqrt{1 + \|y\|_2}}{3} \geq \mathcal{M}_{\psi_\mu}(d_x)
\]
then
\[
\psi_\mu(x + \alpha d_x) - \psi_\mu(x) \leq 2\mu \eta_s^3 + \max \left\{ \mathcal{M}_{\psi_\mu}(d_x), -\frac{\eta_s^2 \mu}{3} \right\}
\]
\[
\leq \mu \eta_s \max \left\{ 2\eta_s^2 - \frac{1}{6} \sqrt{\frac{\tau \mu r}{L_1}}, 2\eta_s^2 - \frac{\eta_s}{3} \right\} \tag{39}
\]
\[
= \mu \left( \frac{\tau \mu r}{L_1} \right)^{1/2} \max \left\{ \frac{2}{40^3} - \frac{1}{6 \times 40}, \left( \frac{\tau \mu r}{L_1} \right)^{1/4}, \frac{2}{40^3} \left( \frac{\tau \mu r}{L_1} \right)^{1/4} - \frac{1}{3 \times 40^2} \right\}
\]
\[
\leq \mu \left( \frac{\tau \mu r}{L_1} \right)^{3/4} \left[ \frac{2}{40^3} - \frac{1}{3 \times 40^2} \right]
\]
\[
= -\frac{17\mu}{60 \times 40^2} \left( \frac{\tau \mu r}{L_1} \right)^{3/4} \tag{40}
\]
where the first transition uses Lemma 8, the second transition uses $M_\psi^\psi(d_x) \geq -\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3}$, the third transition uses $\eta_x = \frac{1}{30} \left( \frac{\tau \mu r}{L_1} \right)^{1/4}$, and the fourth transition uses $\frac{\tau}{L_1} \in (0, 1]$.

Let $(x, d_x)$ denote the current primal iterate and direction. Let $(x^+, d_{x^+})$ denote the subsequent primal iterate and direction. By Lemma 9, if $\frac{\tau}{L_1} \leq M_\psi^\psi(d_x)$ then (FJ1) holds. Also, by Lemma 10 if $-\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3} \leq M_\psi^\psi(d_x^+) \leq M_\psi^\psi(d_x)$, then $\psi^2 \psi(x^+) \geq -\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3}$, i.e., (FJ2) holds. Therefore if both $-\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3} \leq M_\psi^\psi(d_x)$ and $\frac{\tau}{L_1} \leq M_\psi^\psi(d_x^+)$ the algorithm terminates.

It remains to show that if either $-\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3} > M_\psi^\psi(d_x)$ or $-\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3} > M_\psi^\psi(d_x^+)$ then over these two iterations the function value by a constant quantity. First note that even if $M_\psi^\psi(d_x) \geq -\frac{\tau \mu r \sqrt{1 + \|y\|_1}}{3}$ we by $M_\psi^\psi(d_x) \leq 0$ we still have

$$\psi(x + \alpha d_x) - \psi(x) \leq 2 \mu \eta_x^3 + \max \left\{ M_x^\psi(d_x), -\frac{\tau^2 \mu}{3} \right\}$$

$$\leq 2 \mu \eta_x^3 = \frac{2 \mu}{40^3} \left( \frac{\tau^2 \mu}{L_1} \right)^{3/4}$$

(41)

where the first inequality follows from Lemma 8. The same equation applies replacing $(x, d_x)$ with $(x^+, d_{x^+})$. By applying (40) and (41) we can see that if over these two iterations the algorithm did not terminate then $\psi^2 \psi^2 \psi$ must have been reduced by at least

$$\mu \left( \frac{17}{60 \times 40^2} - \frac{2}{40^3} \right) \left( \frac{\tau^2 \mu}{L_1} \right)^{3/4} = \frac{7 \mu}{30 \times 40^2} \left( \frac{\tau^2 \mu}{L_1} \right)^{3/4}.$$ 

The result follows by sum the progress across iterations, telescoping and rearranging. \qed

## D Proof of results in Section 5.2

The main purpose of this section is to prove the following result

**Lemma 10.** Suppose assumption 2 and 3 hold (Lipschitz derivatives, and sufficiently small $\mu$). Let $f$ be convex and each $a_i$ concave. Then TRUST-IPM$(f, a, \mu, \tau, L_1, \eta_x, x^{(0)})$ with $x^{(0)} \in \mathcal{X}$ and

$$\eta_x = \theta \left( \frac{\tau^2 \mu}{L_1} \right)^{1/6}$$

$$\eta_s = \theta \left( \frac{\tau^2 \mu}{L_1} \right)^{1/3} \quad \theta = 1/30.$$  

(\eta-2)

takes at most

$$O \left( \left( 1 + \frac{\psi^2 \psi(x^{(0)}) - \inf_z \psi^2 \psi(z)}{\mu} \right) \frac{L_1}{\tau^2 \mu} \right)^{2/3}$$

iterations to terminate with a $(\mu, \tau, \tau_c)$-approximate first-order Fritz John point $(x^+, y^+)$, i.e., (FJ1) holds.

Before we prove this result in Section D.3 we prove two auxiliary Lemmas. Lemma 13 is the convex version of Lemma 8 and Lemma 14 is the convex version of Lemma 8

### D.1 Proof of Lemma 13

**Lemma 13.** Suppose (ITRS), assumption 2 and 3 hold (direction selection, Lipschitz derivatives, and sufficiently small $\mu$). Let $f$ be convex and each $a_i$ concave. Also assume $x \in \mathcal{X}$,

...
\[ \eta_s \in [0, 1/4]. \] Let \( \alpha = \min \left\{ 1, \frac{\eta_s}{\|S^{-1}d_s\|_2} \right\}, \eta_x = \theta \left( \frac{\mu^2}{L_1} \right)^{1/6}, \eta_h = \theta \left( \frac{\mu^2}{L_1} \right)^{1/3} \text{ and } \theta \in [0, 1/4]. \]

Then \( x^+ \in \mathcal{X} \) and

\[ \psi_\mu(x + \alpha d_x) - \psi_\mu(x) \leq 8\mu \theta^3 \left( \frac{\mu^2}{L_1} \right)^{2/3} + \max \left\{ \mathcal{M}_x^\psi(d_x), -\frac{\theta^2 \mu}{2} \left( \frac{\mu^2}{L_1} \right)^{2/3} \right\}. \]

Proof First we show

\[ \mathcal{M}_x^\psi(\alpha d_x) \leq \max \left\{ \mathcal{M}_x^\psi(d_x), -\frac{\eta_s^2 \mu}{2} \right\}, \]

which trivially holds if \( \alpha = 1 \). Therefore let us consider the case \( \alpha \in (0, 1) \). In this case, by \eqref{eq:18}, we have

\[ \alpha = \frac{\eta_s}{\|S^{-1}d_s\|_2} \geq \eta_h \sqrt{\frac{\mu}{-2 \mathcal{M}_x^\psi(d_x)}}, \]

Therefore,

\[ \mathcal{M}_x^\psi(\alpha d_x) = \alpha^2 \frac{1}{2} d_x^T \nabla^2 \psi_\mu(x) d_x + \alpha \nabla \psi_\mu(x)^T d_x \leq \alpha^2 \mathcal{M}_x^\psi(d_x) \leq -\frac{\eta_s^2 \mu}{2} \]

where the first inequality follows by \( \nabla \psi_\mu(x)^T d_x \leq 0 \) \( \text{Lemma} \[6\] \), the second by \( \alpha \geq \eta_h \sqrt{\frac{\mu}{-2 \mathcal{M}_x^\psi(d_x)}} \), and the third inequality \eqref{A4.1}. From these two cases we conclude \eqref{eq:42} holds.

It remains to bound the accuracy of the predicted decrease \( \mathcal{M}_x^\psi(\alpha d_x) \). Let us bound the constant \( \kappa \text{ from Lemma} [3] \)

\[ \alpha\|S^{-1}d_s\|_2 + \frac{L_1 \|\alpha d_x\|^2 (1 + \|y\|_1)}{2\mu} \leq \theta \left( \frac{\mu^2}{L_1} \right)^{1/3} + \frac{\theta^2 \left( \frac{\mu^2}{L_1} \right)^{1/3}}{2} \leq (9/8) \theta \left( \frac{\mu^2}{L_1} \right)^{1/3} = \kappa \]

where the second inequality comes from \( \alpha\|S^{-1}d_s\|_2 \leq \eta_s = \theta \left( \frac{\mu^2}{L_1} \right)^{1/3} \) and

\[ \|\alpha d_x\|_2 \leq \theta \left( \frac{\mu^2}{L_1} \right)^{1/6} \sqrt{\frac{\mu}{L_1 (\|y\|_1 + 1)}}, \]

the third inequality from \( \theta \in [0, 1/4] \). Since \( \theta \in [0, 1/4] \) and \( \tau_1^2 \mu / L_1 \in (0, 1] \) we deduce \( \kappa \leq 1/2 \) so the conditions of Lemma \[3\] hold. Therefore \( x^+ \in \mathcal{X} \). From Lemma \[3\]

\[ |\psi_\mu(x) + \mathcal{M}_x^\psi(\alpha d_x) - \psi_\mu(x + \alpha d_x)| \leq L_2 \left( 1 + 2 \|y\|_1 \right) \|\alpha d_x\|^3 + 3 L_1 \|\alpha d_x\|^2 \|y\|_2 \kappa + 3 \mu \kappa^3 \]

\[ \leq L_1^{4/3} \tau_1^{1/3} \mu^{-1/3} \left( 6 + (1/3) \|y\|_1 \right) \|\alpha d_x\|^3 + 3 L_1 \|\alpha d_x\|^2 \|y\|_2 \kappa + 3 \mu \kappa^3 \]

\[ \leq \frac{1}{3} \mu^3 \left( \tau_1^2 \mu / L_1 \right)^{2/3} + 3 \mu \theta^3 \left( \tau_1^2 \mu / L_1 \right)^{2/3} \]

\[ \leq 8 \mu \theta^3 \left( \tau_1^2 \mu / L_1 \right)^{2/3} \]

where the second inequality uses \eqref{A4.1.1}, the third inequality uses our bound on \( \alpha d_x \) and \( \kappa \). Combining \eqref{eq:42} and \eqref{eq:46} gives the result. \( \square \)

**D.2 Proof of Lemma [14]**

**Lemma 14.** Suppose assumptions \[3\] and \[4\] hold (Lipschitz derivatives, and sufficiently small \( \mu \)). Let \( f \) be convex and each \( a_i \) concave. Let \( x \in \mathcal{X}, \eta_x \in (0, \frac{1}{5} \left( \frac{\mu^2}{L_1} \right)^{1/6}), \) and \( \alpha = 1 \). Under these assumptions, if \( \mathcal{M}_x^\psi(d_x) \geq -\frac{\tau_1 \mu \sqrt{1 + \|y\|_1}}{3} \) then \( (x^+, y^+) \) is an \((\mu, \tau_1, \tau_c)\)-approximate first-order Fritz John point.
Proof By (18), and our assumed bound on \( \|d_x\|_2 \) and \( M_x^{\psi a}(d_x) \) we get
\[
\|S^{-1}d_x\|_2 \leq \sqrt{-2M_x^{\psi a}(d_x) / \mu} \leq \sqrt{\tau_1 \sqrt{1 + \|y\|_1}}
\]
\[
\leq \frac{1}{2} \left( \frac{\mu \tau_1^2}{L_1} \right)^{2/3}
\]
\[
\leq \frac{1}{2} \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/3},
\]
where the third inequality uses uses (A4.\( \mu.a \)).

Let us compute \( \kappa \) from Lemma 3, \( \|S^{-1}d_x\|_2 + \frac{L_1(1 + \|y\|_1)\|d_x\|_2^2}{2\mu} \leq \frac{4}{10} \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/4} + \eta_x \leq \frac{1}{2} = \kappa. \)
It follows that \( \text{Convex}\{x, x^+\} \subseteq X. \)

By Lemma 4 and the fact that \( y = \mu S^{-1}1 \) we have
\[
\|Y^{-1}d_y\|_2 \leq \|S^{-1}d_x\|_2 \leq \frac{1}{2} \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/3}.
\]
(47)

Therefore,
\[
\|\delta d_x - \nabla_x L(x^+, y^+)\|_2 \leq L_1\|d_x\|_2\|y\|_2\|Y^{-1}d_y\|_2 + \frac{L_2}{2}(\|y\|_1 + 1)\|d_x\|_2^2
\]
\[
\leq L_1\|d_x\|_2\|y\|_2\|Y^{-1}d_y\|_2 + \frac{L_1^{4/3} \mu^{-1/3} \tau_1^{1/3}}{2}(\|y\|_1 + 1)\|d_x\|_2^2
\]
\[
\leq \frac{\tau_1 \mu \sqrt{\|y\|_2}}{2} \left( \frac{\mu \tau_1^2}{L_1} \right)^{-1/6} \eta_x + \frac{L_1^{4/3} \mu^{2/3} \tau_1^{1/3}}{2} \eta_x^2
\]
\[
\leq \frac{\tau_1 \mu \sqrt{\|y\|_2}}{10} + \frac{\mu \tau_1}{25}
\]
\[
\leq \frac{\mu \tau_1}{7} \sqrt{1 + \|y\|_2},
\]
where the first inequality follows from Lemma 3, the second by A4.\( \mu.a \), the second using \( \|d_x\|_2 \leq \eta_x \sqrt{\frac{\mu}{L_1(\|y\|_1 + 1)}} \) and \( \|Y^{-1}d_y\|_2 \leq \frac{1}{2} \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/4} \), and the fourth inequality using \( \eta_x \in (0, \frac{1}{5} \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/4}] \).

Now, by (10), there exists some \( \delta \geq 0 \) such that \( \nabla_x L(x, y) + \nabla_x L(x, y)^T d_x - d_y^T \nabla_x a(x) = \delta d_x. \) Moreover, \( \frac{\tau_1 \mu \sqrt{1 + \|y\|_1}}{3} \leq M_x^{\psi a}(u) \leq -\frac{\delta \tau_1}{2} \) by (15a). Therefore
\[
\delta \|d_x\|_2 \leq \delta r \leq \frac{2\tau_1 \mu \tau_1 \sqrt{1 + \|y\|_1}}{3r} = \frac{2}{3} \gamma \mu \sqrt{1 + \|y\|_1}.
\]

Therefore using the bounds on \( \delta \|d_x\|_2 \) and \( \|\delta d_x - \nabla_x L(x^+, y^+)\|_2 \) that we proved,
\[
\|\nabla x L(x^+, y^+)\|_2 \leq \|\delta d_x - \nabla_x L(x^+, y^+)\|_2 + \delta \|d_x\|_2
\]
\[
\leq \frac{2\tau_1 \mu}{3} \sqrt{1 + \|y\|_1} + \frac{\tau_1 \mu}{7} \sqrt{1 + \|y\|_1}
\]
\[
\leq \frac{\tau_1 \mu \sqrt{1 + \|y\|_1}}{7}.
\]

This shows (FJ1.a) holds. It remains to show (FJ1.a) and (FJ1.b). From Lemma 4 we get
\[
\|S^+ y - \mu 1\|_2 \leq \mu \|S^{-1}d_x\|_2\|Y^{-1}d_y\|_2 + \frac{L_1}{2}\|y\|_2(1 + \|Y^{-1}d_y\|_2)\|d_x\|_2^2
\]
\[
\leq \mu \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/2} + \mu \eta_x^2 \leq \frac{\mu}{2} \left( \frac{\mu \tau_1^2}{L_1} \right)^{1/3} \leq \frac{\mu}{2},
\]
where the second inequality uses \(\|Y^{-1}d_y\|_2 \leq \|S^{-1}d_s\|_2 \leq \frac{1}{2} \left( \frac{\mu \tau}{L_1} \right)^{1/2} < 1\) and our assumption on \(r\), the third inequality \(\eta_e \in (0, \frac{1}{3} \left( \frac{\mu \tau^2}{L_1} \right)^{1/4}]\), and the final inequality uses \([A4, \mu, a]\). Therefore \([FJ1]\) holds.

### D.3 Proof of Lemma 10

**Lemma 10.** Suppose assumptions 2 and 4 hold (Lipschitz derivatives, and sufficiently small \(\mu\)). Let \(f\) be convex and each \(a_i\) concave. Then \(\text{TRUST-IPM}(f, a, \mu, \tau, L_1, \eta_e, x^{(0)})\) with \(x^{(0)} \in X\) and

\[
\eta_e = \theta \left( \frac{\tau^2 \mu}{L_1} \right)^{1/6}, \quad \eta_s = \theta \left( \frac{\tau^2 \mu}{L_1} \right)^{1/3}, \quad \theta = 1/30.
\]

takes at most

\[
O \left( 1 + \frac{\psi_{\mu}(x^{(0)}) - \inf_{x \in X} \psi_{\mu}(z)}{\mu} \left( \frac{L_1}{\tau \mu} \right)^{2/3} \right)
\]

iterations to terminate with a \((\mu, \tau, \gamma, \alpha)\)-approximate first-order Fritz John point \((x^+, y^+)\), i.e., \([FJ1]\) holds.

**Proof** Let \(x \in X\) be some iterate of the algorithm with corresponding direction \(d_x\). If \(\mathcal{M}_{x, \psi_{\mu}}(d_x) \geq -\frac{\tau_1 \mu \sqrt{1 + \|z\|}}{3}\) then the algorithm terminates at the next iteration by Lemma 14. Therefore consider the case that \(-\frac{\tau_1 \mu \sqrt{1 + \|z\|}}{3} < \mathcal{M}_{x, \psi_{\mu}}(d_x)\). By Lemma 13 we have \(x^+ \in X\).

Furthermore,

\[
\psi_{\mu}(x + \alpha d_x) - \psi_{\mu}(x) \leq 8\mu \theta^3 \left( \frac{\tau^2 \mu}{L_1} \right)^{2/3} + \max \left\{ \mathcal{M}_{x, \psi_{\mu}}(d_x), \frac{\theta^2 \mu}{2} \left( \frac{\tau^2 \mu}{L_1} \right)^{2/3} \right\} \\
\leq \mu \theta \left( \frac{\tau^2 \mu}{L_1} \right)^{2/3} \max \left\{ 8\theta^2 - \frac{1}{3}, 8\theta^2 - \frac{\theta}{2} \right\} \\
\leq \mu \theta^2 \left( \frac{\tau^2 \mu}{L_1} \right)^{2/3} \left( 8\theta - \frac{1}{2} \right) \\
\leq -\frac{\mu}{5} \times 30^2 \left( \frac{\tau^2 \mu}{L_1} \right)^{2/3}
\]

where the first inequality uses Lemma 13, the second inequality uses \(\mathcal{M}_{x, \psi_{\mu}}(d_x) \geq -\frac{\tau_1 \mu \sqrt{1 + \|z\|}}{3} = -\mu \theta \left( \frac{\tau^2 \mu}{L_1} \right)^{2/3}\), the third inequality uses \(\theta = 1/30\), the final inequality comes from substituting in the value of \(\theta\). The result follows.

### E Proof of results in Section 6

#### E.1 Proof of Lemma 15

**Assumption 9** (Slater’s condition). Suppose that there exists some \(R > 0\) such that \(\|X\|_2 \leq R\) and there exists some \(z \in X\), \(\gamma \in R_{++}\) such that \(a(z) \geq \gamma 1\). Further assume there exists some constant \(L_0 > 0\) such that \(\|\nabla f(x)\|_2 \leq L_0\) for all \(x \in X\).

Furthermore, in order to apply Slater’s condition we need \(\mu\) to be sufficiently small:

\[
\mu \leq \frac{\gamma}{2\tau_1 R}.
\]
Lemma 15. Suppose that $f$ is convex, $a_i$ is concave and assumption [FJ] holds. If $(x^+, y^+)$ is a Fritz John point (i.e., [FJ] holds) and (48) holds then

$$\|y^+\|_1 \leq 1 + \frac{3m\mu + 2L_0R}{\gamma}.$$  

Proof Observe that if [FJ] holds then

$$\|y^+\|_1 \leq \frac{a(z)^Ty^+}{\gamma} \leq \frac{(a(x^+) + \nabla a(x^+)(z - x^+))^Ty^+}{\gamma} \leq \frac{a(x^+)^Ty^+ + \|\nabla a(x^+)^Ty^+\|_2R}{\gamma} \leq \frac{2m\mu + (L_0 + \tau\mu\sqrt{\|y^+\|_1 + 1})R}{\gamma}$$

where the first inequality uses assumption [9] which implies $a(z)/\gamma \geq 1$, the second inequality uses that $a_i$ is concave, the third inequality uses $\|X\|_2 \leq R$, and the fourth inequality uses [FJ] and assumption [9]. Using (48) we deduce the result. 

E.2 Proof of Lemma 12

Lemma 12. Let $f$ be convex and each $a_i$ concave. Suppose that assumption [4] holds. Let $x^{(0)} \in \mathcal{X}$, $\Delta = f(x^{(0)}) - \inf_{z \in \mathcal{X}} f(z)$ and $\epsilon \in (0, \Delta)$. Then Annealed-IPM$(f, a, x^{(0)}, \epsilon)$ takes at most

$$\left(\mathcal{O} \left(1 + 6m \times \left(\frac{\epsilon}{3m}\right)\right) \log \frac{3m\mu^{(0)}}{\epsilon} + \psi_{\mu^{(0)}}(x^{(0)}) - \inf_{z \in \mathcal{X}} \psi_{\mu^{(0)}}(z)\right) w(\mu^{(0)})$$

unit operations to return an $\epsilon$-optimal solution, where $\log^+\left(\frac{3m\mu^{(0)}}{\epsilon}\right)$.

Proof Let $J = \left[\log^+\left(\frac{3m\mu^{(0)}}{\epsilon}\right)\right]$. At this point if we apply Lemma 11 with $\mu = 0$, we obtain

$$f(x^{(J)}) - \inf_{z \in \mathcal{X}} f(z) \leq 3\sum_{i=1}^{m} a_i(x^{(j)})y_i^{(j)} \leq (1 + 2\mu)(1 + 2)\mu m = 3\mu^{(j)m}m = 3\mu^{(j-1)}2 - \epsilon.$$ 

Hence after $J$ iterations we have found an $\epsilon$-optimal solution. By Lemma 11

$$\psi_{\mu^{(j)}}(x^{(j-1)}) - \inf_{z \in \mathcal{X}} \psi_{\mu^{(j)}}(z) \leq 3\sum_{i=1}^{m} a_i(x^{(j-1)})y_i^{(j-1)} - \mu^{(j-1)} \leq 3\frac{\mu^{(j-1)}}{2\mu^{(j)}}.$$ 

Applying this bound in assumption [5] we bound deduce the computational cost of each iteration $j > 0$ by

$$\mathcal{O} \left(1 \times \frac{\psi_{\mu^{(j)}}(x^{(j-1)}) - \inf_{z \in \mathcal{X}} \psi_{\mu^{(j)}}(z)}{\mu^{(j)}}\right) w(\mu^{(j)}) \leq O \left(1 \times \frac{3\mu^{(j-1)}m}{2\mu^{(j)}}\right) w(\mu^{(j)}) \leq \mathcal{O} \left(1 + \frac{3\mu^{(j-1)}m}{2\mu^{(j)}}\right) w(\mu^{(j)}).$$

The second inequality uses $\mu^{(j)} = 2\mu^{(j-1)}$. The final inequality uses that $w$ is monotone decreasing by assumption [5].

E.3 Proof of Theorem 2

Theorem 2. Suppose assumption [3] [6] and [8] hold (Lipschitz derivatives, regularity conditions, and sufficiently small $\mu^{(0)}$). Let $f$ be convex and each $a_i$ concave. Let $x^{(0)} \in \mathcal{X}$, $\|X\|_2 \leq R$, $\Delta = f(x^{(0)}) - \inf_{z \in \mathcal{X}} f(z)$, and $\epsilon \in (0, \Delta)$. Define $\eta_x, \eta_x$ by (42) and set

Generic-IPM$(f, a, \mu, x) :=$ Trust-IPM$(f, a, \mu, \tau_1, L_1, \eta_x, \eta_x, x)$

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inside **Annealed-IPM**. Then \( \text{Annealed-IPM}(f,a,\mu(0),x(0),\epsilon) \) takes at most

\[
O \left( \left( m^{1/3} \left( \frac{L_1 R^2 \zeta}{\epsilon} \right)^{2/3} + 1 \right) \log^{+} \left( \frac{m \mu(0)}{\epsilon} \right) + \frac{\psi_{\mu(0)}(x(0)) - \inf_{z \in X} \psi_{\mu(0)}(z)}{\mu(0)} \left( \frac{L_1 R^2 \zeta}{m^2 \mu(0)} \right)^{2/3} \right)
\]

iterations to return an \( \epsilon \)-optimal solution, where \( \log^{+}(x) = \max\{\log(x), 1\} \).

**Proof** Our first goal is to show for any \( \mu \in (0, \mu(0)] \) that (A7.\( \tau_l \)), i.e., \( \mu(0) = \min \left\{ \frac{L_1 R^2 \zeta}{m^2}, \frac{L_4 m}{RL_2 \sqrt{\zeta}} \right\} \) implies the assumptions of Lemma 10, and Lemma 12 are met.

We begin by showing the assumptions of Lemma 10 are met. Recall that (A7.\( \tau_l \)) states that \( \tau_l = \frac{m}{R \zeta^{1/2}} \). In particular, (A4.\( \mu.a \)) holds by \( \mu(0) \leq \frac{L_1 R^2 \zeta}{m^2} = \frac{L_4}{\tau_l} \) and (A4.\( \mu.b \)) holds by \( \mu(0) \leq \frac{L_1 L_4}{RL_2 \sqrt{\zeta}} \).

It follows that

\[
\| \nabla_x L(x(j), y(j)) \|_2 \leq \mu \tau_l \sqrt{1 + \| y(j) \|_1} \leq \mu \tau_l \zeta^{1/2} \leq \frac{m \mu}{R}
\]

where the first inequality uses (FJ1), the second inequality uses assumption 6 and the final inequality by (A7.\( \tau_l \)). Therefore assumption 5 holds. Hence the assumptions of Lemma 12 are met.

We conclude the assumptions of Lemma 10 and Lemma 12 are met. By Lemma 10 we have

\[
w(\mu) = O \left( \left( \frac{\tau_l^2 \mu}{L_1} \right)^{-2/3} \right) = O \left( \left( \frac{m^2 \mu}{L_1 R^2 \zeta} \right)^{-2/3} \right)
\]

where the second equality uses \( \tau_l = \frac{m}{R \zeta^{1/2}} \). Substituting this into Lemma 12 yields the runtime bound. \( \square \)
F A two-phase method to find unscaled KKT points

F.1 Algorithm definition

Algorithm 3 Two-phase IPM

function TWO-PHASE-IPM(f, a, εopt, εinf, L0, L1, x(0))

Output: A status (KKT if (KKT) holds and INF if (INF) holds) and a point (x, t, y).

Phase-one.
Let \( \mu^{(P1)} = \frac{\varepsilon_{\text{opt}}}{2}, \quad \tau^{(P1)} = \min \left\{ \frac{1}{\varepsilon_{\text{opt}}}, \sqrt{\frac{L_1}{2\varepsilon_{\text{inf}}}} \right\}, \quad t(0) = \frac{\varepsilon_{\text{opt}}}{2} + \max\{\min_i -a_i(x(0)), 0\}, \) and \( \eta \) satisfy (η-1).
if \( t(0) \leq \frac{\varepsilon_{\text{opt}}}{2} \) then
\( x(P1) \leftarrow x(0) \)
else
\( (x^{(P1)}, t^{(P1)}, y^{(P1)}, \lambda^{(P1)}, \gamma^{(P1)}) \leftarrow \text{TRUST-IPM}(f, a^{(P1)}, \mu^{(P1)}, \tau^{(P1)}, L_1, \eta, \eta, (x(0), t(0))) \)
if \( \min_i a_i(x^{(P1)}) < -\varepsilon_{\text{opt}}/2 \) then
\( (x, t, y) \leftarrow (x^{(P1)}, t^{(P1)}, y^{(P1)}/\|y^{(P1)}\|_1) \)
return INF, (x, t, y)
end if
end if

Phase-two.
Let \( \mu^{(P2)} = \frac{\varepsilon_{\text{opt}}}{4}, \quad \tau^{(P2)} = \sqrt{\frac{\varepsilon_{\text{inf}}}{2(L_0+1)}}, \) and \( \eta \) satisfy (η-1).
\( (x^{(P2)}, y^{(P2)}) \leftarrow \text{TRUST-IPM}(f, a^{(P2)}, \mu^{(P2)}, \tau^{(P2)}, L_1, \eta, \eta, x^{(P1)}) \)
if \( \|y^{(P2)}\|_1 > 1/\varepsilon_{\text{inf}} \) then
\( (x, t, y) \leftarrow (x^{(P2)}, \varepsilon_{\text{opt}}, y^{(P2)}/\|y^{(P2)}\|_1) \)
return INF, (x, t, y)
else
\( (x, t, y) \leftarrow (x^{(P2)}, \emptyset, y^{(P2)}) \).
return KKT, (x, t, y)
end if
end function

F.2 Proof of Claim 3

Claim 3. Let \( x(0) \in \mathbb{R}^n \). Suppose assumption \( \emptyset \) and \( \emptyset \) holds. Let \( f \) be \( L_0 \)-Lipschitz. Assume \( c, \Delta_0, \Delta_f, L_1, L_0 \geq 1, \varepsilon_{\text{opt}} \in \left( 0, \frac{1}{m \log_{c/\varepsilon_{\text{opt}}}} \right], \) and \( \varepsilon_{\text{inf}} \in (0, 1] \). Then \( \text{Two-Phase-IPM}(f, a, \varepsilon_{\text{opt}}, \varepsilon_{\text{inf}}, L_0, L_1, x(0)) \) takes at most
\[
O\left( \Delta_0 \left( \frac{L_1^{3/4}}{\varepsilon_{\text{inf}}^{1/4} \varepsilon_{\text{opt}}} + \frac{1}{\varepsilon_{\text{inf}} \varepsilon_{\text{opt}}} \right) + \frac{\Delta_f}{\varepsilon_{\text{opt}} \varepsilon_{\text{inf}}} \left( \frac{L_1 L_0}{\varepsilon_{\text{opt}} \varepsilon_{\text{inf}}} \right)^{3/4} \right)
\]
trust region subproblem solves to return a point that satisfies either (KKT) or (INF).

Proof Recall Theorem 1 gives a bound on the iteration count of \( \text{Trust-IPM} \) of
\[
O\left( 1 + \frac{\psi_\mu(x(0)) - \inf_{z \in \mathcal{X}} \psi_\mu(z)}{\mu} \left( \frac{L_1}{\mu \tau^{(P1)}} \right)^{3/4} \right).
\]
Let \( \psi_{\mu(P;1)}^1 \) and \( \psi_{\mu(P;2)}^2 \) denote the log barrier for problems (P1) and (P2) respectively. Now, using \( \mu(P;1) = \frac{1}{12} \varepsilon \log \varepsilon_{\text{opt}} \) we get

\[
\psi_{\mu(P;1)}^1(x(0)) = \inf_{z \in \mathcal{X}(P;1)} \psi_{\mu(P;1)}^1(z) = O \left( \min \{ -a_i(x(0)), 0 \} + \mu(P;1) \log^+(c/\varepsilon_{\text{opt}}) \right) = O(\Delta_a). 
\]

Similarly, using \( \mu(P;2) = \frac{\varepsilon_{\text{opt}}}{4}, \varepsilon_{\text{opt}} \in \left( \frac{1}{m \log^+(c/\varepsilon_{\text{opt}})} \right) \) and \( \Delta_f \geq 1 \) we get

\[
\psi_{\mu(P;2)}^2(x(1)) = \inf_{z \in \mathcal{X}(P;2)} \psi_{\mu(P;2)}^2(z) = O \left( f(x(1)) + \inf_{z \in \mathcal{X}(P;2)} f(z) + \mu(P;2) \log^+(c/\varepsilon_{\text{opt}}) \right) = O(\Delta_f).
\]

Substituting the appropriate values of \( \tau_i \) and \( \mu \) from Algorithm 3 yields a bound of

\[
O \left( \Delta_a \left( \frac{L_1^3}{\varepsilon_{\text{inf}}^4 \varepsilon_{\text{opt}}^2} + \frac{1}{\varepsilon_{\text{inf}}^2 \varepsilon_{\text{opt}}} + \frac{\Delta_f}{\varepsilon_{\text{opt}}} \right) \right)
\]

trust region subproblem solves for Two-Phase-IPM.

It remains to show either (KKT) or (INF1) is satisfied. Observe that after calling Trust-IPM in phase-one we find a point satisfying the Fritz John conditions for the problem of minimizing the infinity norm of the constraint violation, i.e.,

\[
\begin{align*}
\left\| \nabla a(x(P;1))^T y(P;1) \right\|_2 & \leq \frac{\varepsilon_{\text{inf}}}{12} \left( \left\| y(P;1) \right\|_1 + 1 \right) \\
\left\| 1^T y(P;1) - 1 + \lambda(P;1) - \gamma(P;1) \right\|_2 & \leq \frac{\varepsilon_{\text{inf}}}{12} \left( \left\| y(P;1) \right\|_1 + 2 \varepsilon_{\text{inf}} + 1 + 2 \right) \\
\left\| \frac{\nabla a(x(P;1))^T y(P;1)}{y(P;1)} \right\|_2 & \leq \varepsilon_{\text{inf}} \left( \frac{(a_i(x(P;1)) + t(P;1))y_i(P;1)}{y(P;1)} \right) \leq \varepsilon_{\text{inf}}.
\end{align*}
\]

Consider the case where in phase-one the status is INF, in which case min \( a_i(x(P;1)) \leq -\varepsilon_{\text{opt}}/2 \). Consequently, \( t(P;1) > \varepsilon_{\text{opt}}/2 \) by (50) and (51). Using \( t(P;1) > \varepsilon_{\text{opt}}/2 \) and (53) we deduce \( \lambda(P;1) \leq 2\varepsilon_{\text{inf}} \). Therefore using (49), \( \varepsilon_{\text{inf}} \in (0,1) \) and we deduce

\[
\left\| \nabla a(x(P;1))^T y(P;1) \right\|_2 \leq \frac{\varepsilon_{\text{inf}}}{12} \left( \left\| y(P;1) \right\|_1 + 2 \varepsilon_{\text{inf}} + 1 + 2 \right) \leq \frac{\varepsilon_{\text{inf}}}{12} \left( \left\| y(P;1) \right\|_1 + 4 \right).
\]

If \( \left\| y(P;1) \right\|_1 \leq \left\| y(P;1) \right\|_1 + 1/2 \) then using (55) we deduce \( 1/2 < 1 - \left\| y(P;1) \right\|_1 \geq 1/2 \). Using \( \left\| y(P;1) \right\|_1 \geq 1/2 \), (55), and (52) we deduce

\[
\left\| \nabla a(x(P;1))^T y(P;1) \right\|_2 \leq \varepsilon_{\text{inf}} \left( \frac{(a_i(x(P;1)) + t(P;1))y_i(P;1)}{y(P;1)} \right) \leq \varepsilon_{\text{inf}} \varepsilon_{\text{opt}}.
\]

Observe that after calling Trust-IPM in phase-two we find a point satisfying

\[
\begin{align*}
a(x(P;2)) > -\varepsilon_{\text{opt}} 1 \\
y_i(P;2)(a_i(x(P;2)) + \varepsilon_{\text{opt}}) \leq \frac{1}{2} \varepsilon_{\text{opt}} \quad \forall i \in \{1, \ldots, m\} \\
\left\| \nabla x \mathcal{L}(x(P;2), y(P;2)) \right\|_2 & \leq \frac{\varepsilon_{\text{opt}}}{4} \sqrt{\frac{\varepsilon_{\text{inf}}}{2(L_0 + 1)} \left\| y(P;2) \right\|_1 + 1} \\
y(P;2) & > 0.
\end{align*}
\]
If $\|y^{(P_2)}\|_1 < \frac{\varepsilon_{\text{opt}}^2}{\varepsilon_{\text{inf}}^2} + \frac{L_0 + 1}{\varepsilon_{\text{inf}}} \varepsilon_{\text{inf}}$ then using the fact that $\varepsilon_{\text{opt}} \in (0, 1]$ and $L_0 \geq 1$ we get

$$\left\| \nabla_x \mathcal{L}(x^{(P_2)}, y^{(P_2)}) \right\|_2 \leq \frac{\varepsilon_{\text{opt}}}{4} \sqrt{\frac{\varepsilon_{\text{inf}}}{2(L_0 + 1)} \left( \frac{\varepsilon_{\text{opt}}^2}{\varepsilon_{\text{inf}}^2} + \frac{L_0 + 1}{\varepsilon_{\text{inf}}} \varepsilon_{\text{inf}} \right)} \leq \frac{\varepsilon_{\text{opt}}}{4}.$$  

Therefore clearly \((\text{KKT})\) is satisfied. Otherwise if $\|y^{(P_2)}\|_1 \geq \frac{\varepsilon_{\text{opt}}^2}{\varepsilon_{\text{inf}}^2} + \frac{L_0 + 1}{\varepsilon_{\text{inf}}} \varepsilon_{\text{inf}}$ then

$$\frac{\|\nabla a_i(x^{(P_2)})^T y^{(P_2)}\|_2}{\|y^{(P_2)}\|_1} \leq \frac{\|\nabla_x \mathcal{L}(x^{(P_2)}, y^{(P_2)})\|_2 + \|\nabla f(x^{(P_2)})\|_2}{\|y^{(P_2)}\|_1} \leq \frac{\varepsilon_{\text{opt}}}{\|y^{(P_2)}\|_1^{1/2}} + \frac{\varepsilon_{\text{opt}}}{\|y^{(P_2)}\|_1} + \frac{L_0}{\|y^{(P_2)}\|_1} \leq \varepsilon_{\text{inf}}$$

and

$$\frac{(a_i(x^{(P_2)}) + \varepsilon_{\text{opt}})y_i^{(P_2)}}{\|y^{(P_2)}\|_1} \leq \varepsilon_{\text{inf}} \varepsilon_{\text{opt}}.$$  

Finally note that since $y_i^{(P_2)}(a_i(x^{(P_2)}) + \varepsilon_{\text{opt}}) \leq \frac{1}{2} \varepsilon_{\text{opt}}$ and $\|y^{(P_2)}\|_1 \geq \frac{\varepsilon_{\text{opt}}^2}{\varepsilon_{\text{inf}}^2} + \frac{L_0 + 1}{\varepsilon_{\text{inf}}} \geq m$ we deduce $\min_i a_i(x^{(P_2)}) \leq \varepsilon_{\text{opt}} \min_i \left( \frac{1}{2y_i^{(P_2)} + 1} \right) \leq -\varepsilon_{\text{opt}}/2$. Hence \((\text{INF1})\) is satisfied with $(x, t, y) = \left( x^{(P_2)}, \varepsilon_{\text{opt}}, \frac{y^{(P_2)}}{\|y^{(P_2)}\|_1} \right)$. \qed