SPIN(9)-STRUCTURES AND CONNECTIONS WITH TOTALLY
SKEW-SYMMETRIC TORSION

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Abstract. We study Spin(9)-structures on 16-dimensional Riemannian manifolds and characterize the geometric types admitting a connection with totally skew-symmetric torsion.

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1. Introduction

The basic model in type II string theory is a 6-tuple \((M^n, g, \nabla, T, \Phi, \Psi)\) consisting of a Riemannian metric \(g\), a metric connection \(\nabla\) with totally skew-symmetric torsion form \(T\), a dilation function \(\Phi\) and a spinor field \(\Psi\). If the dilation function is constant, the string equations can be written in the following form (see [Stro] and [IP, FI1]):

\[
\text{Ric}^{\nabla} = 0, \quad \delta^g(T) = 0, \quad \nabla \Psi = 0, \quad T \cdot \Psi = 0.
\]

Therefore, an interesting problem is the investigation of metric connections with totally skew-symmetric torsion. In [FI1] we proved that several non-integrable geometric structures (almost contact metric structures, almost complex structures, G_2-structures) admit a unique connection \(\nabla\) preserving it with totally skew-symmetric torsion. Moreover, we computed the corresponding torsion form \(T\) and we studied the integrability condition for \(\nabla\)-parallel spinors as well as the Ricci tensor \(\text{Ric}^{\nabla}\). In particular, we constructed 7-dimensional solutions of the string equations related to non-integrable G_2-structures. The 5-dimensional case and its link with contact geometry was investigated in more details in the paper [FI2]. Similar results concerning 8-dimensional manifolds with a Spin(7)-structure are contained in the paper [K], the hyperkähler case was investigated in the papers [DoFi], [MI] and [Ver]. Homogeneous models and the relation to Kostant’s cubic Dirac operators were discussed in [Agr]. The aim of this note is to work out the case of 16-dimensional Riemannian manifolds with a non-integrable Spin(9)-structure. Alfred Gray (see [Gray]) has pointed out that this special geometry may occur as a geometry with a weak holonomy group. Only recently we once again revisited the special Spin(9)-geometries in dimension sixteen and, in particular, we proved that there are 4 basic classes (see [Fri1]). Here we will study the problem which of these classes admit a connection \(\nabla\) with totally skew-symmetric torsion.

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2. THE GEOMETRY OF Spin(9)-STRUCTURES

The geometric types of Spin(9)-structures on 16-dimensional oriented Riemannian manifolds were investigated in the paper [Fri]. We summarize the basic facts defining this special geometry. Let us consider the 16-dimensional oriented Euclidean space \( \mathbb{R}^{16} \). This space is the real spin representation of the group Spin(9) and, therefore, there exist nine linear operators \( I_\alpha : \mathbb{R}^{16} \to \mathbb{R}^{16} \) such that the following relations hold:

\[
I_\alpha^2 = I_\alpha, \quad I_\alpha \cdot I_\beta + I_\beta \cdot I_\alpha = 0 \quad (\alpha \neq \beta), \quad \text{Tr}(I_\alpha) = 0.
\]

The subgroup Spin(9) \( \subset \text{SO}(16) \) can be defined as the group of all automorphisms of \( \mathbb{R}^{16} \) preserving, under conjugation, the 9-dimensional subspace \( \mathbb{R}^9 := \text{Lin}\{I_1, \ldots, I_9\} \subset \text{End}(\mathbb{R}^{16}) \),

\[
\text{Spin}(9) := \{ g \in \text{SO}(16) : g \cdot \mathbb{R}^9 \cdot g^{-1} = \mathbb{R}^9 \}.
\]

The decomposition of the Lie algebra \( \mathfrak{so}(16) = \mathfrak{so}(9) \oplus \mathfrak{m} \) is explicitly given by

\[
\mathfrak{so}(9) := \text{Lin}\{I_\alpha \cdot I_\beta : \alpha < \beta\} = \Lambda^2(\mathbb{R}^9), \quad \mathfrak{m} := \text{Lin}\{I_\alpha \cdot I_\beta \cdot I_\gamma : \alpha < \beta < \gamma\} = \Lambda^3(\mathbb{R}^9).
\]

The operators \( I_\alpha \cdot I_\beta \) and \( I_\alpha \cdot I_\beta \cdot I_\gamma \) are skew-symmetric and, consequently, they define two systems of 2-forms \( \omega_{\alpha\beta} \) and \( \sigma_{\alpha\beta\gamma} \).

Let \((M^{16},g)\) be an oriented, 16-dimensional Riemannian manifold. A Spin(9)-structure is a 9-dimensional subbundle \( V^9 \subset \text{End}(TM^{16}) \) of endomorphisms which is locally generated by sections \( I_\alpha \) satisfying the algebraic relations described before. Denote by \( \mathcal{F}(M^{16}) \) the frame bundle of the oriented Riemannian manifold. Equivalently, a Spin(9)-structure is a reduction \( \mathcal{R} \subset \mathcal{F}(M^{16}) \) of the principal fibre bundle to the subgroup Spin(9). The Levi-Civita connection is a 1-form on \( \mathcal{F}(M^{16}) \) with values in the Lie algebra \( \mathfrak{so}(16) \),

\[
Z : T(\mathcal{F}(M^{16})) \longrightarrow \mathfrak{so}(16).
\]

We restrict the Levi-Civita connection to a fixed Spin(9)-structure \( \mathcal{R} \) and decompose it with respect to the decomposition of the Lie algebra \( \mathfrak{so}(16) \):

\[
Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.
\]

Then, \( Z^* \) is a connection in the principal Spin(9)-bundle \( \mathcal{R} \) and \( \Gamma \) is a torsional 1-form of type \( \text{Ad} \), i.e., a 1-form on \( M^{16} \) with values in the associated bundle

\[
\mathcal{R} \times_{\text{Spin}(9)} \mathfrak{m} = \mathcal{R} \times_{\text{Spin}(9)} \Lambda^3(\mathbb{R}^9) = \Lambda^3(V^9).
\]

The Spin(9)-representation \( \mathbb{R}^{16} \otimes \mathfrak{m} = \mathbb{R}^{16} \otimes \Lambda^3(\mathbb{R}^9) \) splits into four irreducible components,

\[
\mathbb{R}^{16} \otimes \mathfrak{m} = \mathbb{R}^{16} \oplus \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9) \oplus \mathcal{P}_3(\mathbb{R}^9),
\]

and, therefore, we obtain a similar decomposition of the bundle \( \Lambda^1(M^{16}) \otimes \Lambda^3(V^9) \). The representation \( \mathcal{P}_1(\mathbb{R}^9) \) has dimension 128. It is the restriction of the half spin representation \( \Delta_{16} \) of \( \text{Spin}(16) \) to the subgroup \( \text{Spin}(9) \). The dimensions of the irreducible representations \( \mathcal{P}_2(\mathbb{R}^9) \) and \( \mathcal{P}_3(\mathbb{R}^9) \) are 432 and 768, respectively.

The decomposition of the section \( \Gamma \) yields the classification of all geometric types of Spin(9)-structures. In particular, there are four basic classes (see [Fri]). We remark that the sum \( \mathcal{P}_1 \oplus \mathcal{P}_2 \) is isomorphic to the bundle of 3-forms on \( M^{16} \),

\[
\Lambda^3(M^{16}) = \mathcal{P}_1(V^9) \oplus \mathcal{P}_2(V^9).
\]

In order to fix the normalization, let us describe the embeddings \( \Lambda^i(M^{16}) \longrightarrow \Lambda^1(M^{16}) \otimes \Lambda^3(V^9), i = 1, 3 \), by explicit formulas. If \( \mu^1 \in \Lambda^1(M^{16}) \) is a (co-)vector, then the 1-form on \( M^{16} \) with values in the bundle \( \Lambda^3(V^9) \) is given by

\[
\mu^1 \longrightarrow \frac{1}{8} \sum_{\alpha < \beta < \gamma} I_\alpha I_\beta I_\gamma(\mu^1) \otimes I_\alpha \cdot I_\beta \cdot I_\gamma.
\]
Similarly, if \( \mu^{3} \in \Lambda^{3}(M^{16}) \) is a 3-form, we define
\[
\mu^{3} \mapsto \frac{1}{8} \sum_{\alpha < \beta < \gamma} (\sigma_{\alpha \beta \gamma} \wedge \mu^{3}) \otimes I_{\alpha} \cdot I_{\beta} \cdot I_{\gamma},
\]
where \( \sigma_{\alpha \beta \gamma} \wedge \mu^{3} \) denotes the inner product of the 2-forms \( \sigma_{\alpha \beta \gamma} \) by \( \mu^{3} \).

3. \textbf{Spin}(9)-\textbf{connections with totally skew-symmetric torsion}

We introduce the following equivariant maps:
\[
\Phi: \mathbb{R}^{16} \otimes \text{spin}(9) \to \mathbb{R}^{16} \otimes S^{2}(\mathbb{R}^{16}), \quad \Phi(\Sigma)(X, Y, Z) := g(\Sigma(Z)(X), Y) + g(\Sigma(Y)(X), Z),
\]
\[
\Psi: \mathbb{R}^{16} \otimes m \to \mathbb{R}^{16} \otimes S^{2}(\mathbb{R}^{16}), \quad \Psi(\Gamma)(X, Y, Z) := g(\Gamma(Y)(X), Z) + g(\Gamma(Z)(X), Y).
\]
It is well known (see [FH]) that a geometric Spin(9)-structure admits a connection \( \nabla \) with totally skew-symmetric torsion if and only if \( \Psi(\Gamma) \) is contained in the image of the homomorphism \( \Phi \). The representation \( \mathbb{R}^{16} \otimes \text{spin}(9) \) splits into
\[
\mathbb{R}^{16} \otimes \text{spin}(9) = \mathbb{R}^{16} \oplus \mathcal{P}_{1}(\mathbb{R}^{9}) \oplus \mathcal{P}_{2}(\mathbb{R}^{9}).
\]

Consequently, if a Spin(9)-structure admits a connection \( \nabla \) with totally skew-symmetric torsion, then the \( \mathcal{P}_{3} \)-part of the form \( \Gamma \) must vanish. We split the Spin(9)-representation \( \mathbb{R}^{16} \otimes S^{2}(\mathbb{R}^{16}) \) into irreducible components. Since the symmetric linear maps \( I_{\alpha} \) are traceless, the representation \( \mathbb{R}^{9} \) is contained in \( S_{2}^{D} (\mathbb{R}^{16}) \) and we obtain the decomposition (see [FH])
\[
\mathbb{R}^{16} \otimes S^{2}(\mathbb{R}^{16}) = \mathbb{R}^{16} \oplus \mathbb{R}^{16} \otimes (\mathbb{R}^{9} \oplus D^{126}) = 2 \cdot \mathbb{R}^{16} \oplus \mathcal{P}_{1}(\mathbb{R}^{9}) \oplus \mathbb{R}^{16} \otimes D^{126},
\]
where \( D^{126} := \Lambda^{4}(\mathbb{R}^{9}) \) is the unique irreducible representation of Spin(9) in dimension 126. Denote by \( D^{672} \) the unique irreducible Spin(9)-representation of dimension 672. Its highest weight is the 4-tuple (3/2, 3/2, 3/2, 3/2).

**Lemma 3.1.** *The Spin(9)-representation \( \mathbb{R}^{16} \otimes S^{2}(\mathbb{R}^{16}) \) splits into the irreducible components
\[
\mathbb{R}^{16} \otimes S^{2}(\mathbb{R}^{16}) = 3 \cdot \mathbb{R}^{16} \oplus 2 \cdot \mathcal{P}_{1}(\mathbb{R}^{9}) \oplus \mathcal{P}_{2}(\mathbb{R}^{9}) \oplus \mathcal{P}_{3}(\mathbb{R}^{9}) \oplus D^{672}.
\]

**Proof.** Since \( \mathbb{R}^{16} \otimes m \) contains the representations \( \mathcal{P}_{2}(\mathbb{R}^{9}), \mathcal{P}_{3}(\mathbb{R}^{9}) \) and \( \Psi \) is nontrivial, the tensor product \( \mathbb{R}^{16} \otimes D^{126} \) contains the two representations, too. Moreover, the highest weights of \( \mathbb{R}^{16} \) and \( D^{126} \) are (1/2, 1/2, 1/2, 1/2) and (1, 1, 1, 1), respectively. Then the tensor product \( \mathbb{R}^{16} \otimes D^{126} \) contains the representation \( D^{672} \) of highest weight (3/2, 3/2, 3/2, 3/2) (see [FH], page 425). Consequently, we obtain
\[
\mathbb{R}^{16} \otimes D^{126} = \mathcal{P}_{2}(\mathbb{R}^{9}) \oplus \mathcal{P}_{3}(\mathbb{R}^{9}) \oplus D^{672} \oplus S,
\]
where the dimension of the rest equals \( \dim(S) = 144 \). The representation \( S \) is not an SO(9)-representation. The list of small-dimensional Spin(9)-representations yields that \( S = \mathbb{R}^{16} \oplus \mathcal{P}_{1}(\mathbb{R}^{9}) \), the final result. The decomposition of \( \mathbb{R}^{16} \otimes D^{126} \) can be computed by a suitable computer program, too. \( \square \)

**Lemma 3.2.** *For any two vectors \( X, Y \in \mathbb{R}^{16} \) the following identity holds:
\[
\sum_{\alpha < \beta} \omega_{\alpha \beta}(X, Y) \cdot \omega_{\alpha \beta} + \sum_{\alpha < \beta < \gamma} \sigma_{\alpha \beta \gamma}(X, Y) \cdot \sigma_{\alpha \beta \gamma} = 8 \cdot X \wedge Y.
\]

**Proof.** The 2-forms \( \omega_{\alpha \beta} \) and \( \sigma_{\alpha \beta \gamma} \) constitute a basis of the space \( \Lambda^{2}(\mathbb{R}^{16}) \) of all 2-forms in sixteen variables. Therefore, the identity is simply the decomposition of the 2-form \( X \wedge Y \) with respect to this basis. Remark that the length of the basic forms \( \omega_{\alpha \beta} \) and \( \sigma_{\alpha \beta \gamma} \) equals \( 2 \cdot \sqrt{2} \). \( \square \)

**Theorem 3.1.** *A Spin(9)-structure on a 16-dimensional Riemannian manifold \( M^{16} \) admits a connection \( \nabla \) with totally skew-symmetric torsion if and only if the \( (\mathbb{R}^{16} \oplus \mathcal{P}_{3}) \)-part of the form \( \Gamma \) vanishes. In this case \( \Gamma \) is a usual 3-form on the manifold \( M^{16} \), the connection \( \nabla \) is unique and its torsion form \( T \) is given by the formula \( T = -2 \cdot \Gamma \). \( \square \)
Proof. For a fixed vector $\Gamma \in \mathbb{R}^{16}$ the tensor $\Psi(\Gamma)(X,Y,Y)$ is given by the formula
\[
\Psi(\Gamma)(X,Y,Y) = \frac{1}{4} \sum_{\alpha<\beta<\gamma} \sigma_{\alpha\beta\gamma}(\Gamma,Y) \cdot \sigma_{\alpha\beta\gamma}(X,Y).
\]
Since the multiplicity of $\mathbb{R}^{16}$ in the representation $\mathbb{R}^{16} \otimes \text{spin}(9)$ equals one, any Spin(9)-equivariant map $\Sigma: \mathbb{R}^{16} \rightarrow \mathbb{R}^{16} \otimes \text{spin}(9)$ is a multiple of
\[
\Sigma(\Gamma) = \sum_{\alpha<\beta} I_{\alpha\beta}(\Gamma) \otimes I_{\alpha\beta}.
\]
Consequently, if $\Psi(\Gamma)$ is in the image of $\Phi$, there exists a constant $c$ such that
\[
\sum_{\alpha<\beta<\gamma} \sigma_{\alpha\beta\gamma}(\Gamma,Y) \cdot \sigma_{\alpha\beta\gamma}(X,Y) = c \cdot \sum_{\alpha<\beta} \omega_{\alpha\beta}(\Gamma,Y) \cdot \omega_{\alpha\beta}(X,Y).
\]
For $\Gamma = X = e_{16}$ we compute the corresponding quadratic forms in the variables $y_1, \ldots, y_{16}$:
\[
\Psi(e_{16}) = \sum_{i=1}^{8} y_i^2 + 4 \cdot \sum_{j=9}^{15} y_j^2, \quad \Phi(\Sigma(e_{16})) = 7 \cdot \sum_{i=1}^{8} y_i^2 + 4 \cdot \sum_{j=9}^{15} y_j^2,
\]
a contradiction. Next consider the case that $\Gamma \in \Lambda^3(\mathbb{R}^{16})$ is a $3$-form. By Lemma 3.2 we have
\[
\Psi(\Gamma)(X,Y,Y) = \frac{1}{4} \sum_{\alpha<\beta<\gamma} \Gamma(\sigma_{\alpha\beta\gamma}(\Gamma,Y) \cdot \sigma_{\alpha\beta\gamma}(X,Y) = -\frac{1}{4} \sum_{\alpha<\beta} \Gamma(\omega_{\alpha\beta}(\Gamma,Y) \cdot \omega_{\alpha\beta}(X,Y)) + 2 \Gamma(X,Y,Y).
\]
Since $\Gamma$ is a $3$-form, the term $\Gamma(X,Y,Y)$ vanishes. Let us introduce
\[
\Sigma(\Gamma) := -\frac{1}{8} \sum_{\alpha<\beta}(\omega_{\alpha\beta} \mathcal{J} \Gamma) \otimes \omega_{\alpha\beta}.
\]
Then $\Sigma(\Gamma)$ belongs to the space $\mathbb{R}^{16} \otimes \text{spin}(9)$ and we have $\Phi(\Sigma(\Gamma)) = \Psi(\Gamma)$. Consequently, in case $\Gamma$ is a $3$-form on $M^{16}$, there exists a unique connection $\nabla$ preserving the Spin(9)-structure with totally skew-symmetric torsion. Its torsion form $T$ is basically given by the difference $\Gamma(X) - \Sigma(\Gamma)(X)$ (see [FI1]) and we obtain the formula $T = -2 \cdot \Gamma$. $\square$

Let us characterize Spin(9)-structures of type $\mathcal{P}_1 \oplus \mathcal{P}_2$ using the Riemannian covariant derivatives $\nabla I_\alpha$ of the symmetric endomorphisms describing the structure. For an arbitrary $2$-form $S$ we introduce the symmetric forms by the formula
\[
S_\alpha(Y, Z) := -S(I_\alpha(Y), Z) + S(Y, I_\alpha(Z)), \quad \alpha = 1, \ldots, 9.
\]
The connection $\nabla$ preserves the 9-dimensional bundle of endomorphisms $I_\alpha$ and therefore there exist $1$-forms $M_{\alpha\beta}$ such that $\nabla I_\alpha = \sum_{\beta=1}^{9} M_{\alpha\beta} \cdot I_\beta$. Since $\nabla X Y = \nabla^g_X Y + \frac{1}{2} (X \mathcal{J} T)_\alpha$ we obtain the following formula for the Riemannian covariant derivative of the endomorphisms $I_\alpha$
\[
\nabla^g_X I_\alpha = \sum_{\beta=1}^{9} M_{\alpha\beta}(X) \cdot I_\beta + \frac{1}{2} (X \mathcal{J} T)_\alpha,
\]
where $T$ is a $3$-form. The latter equation characterizes Spin(9)-structures of type $\mathcal{P}_1 \oplus \mathcal{P}_2$. 


4. Homogeneous Spin(9)-structures

Consider a Lie group $G$, a subgroup $H$ and suppose that the homogeneous space $G/H$ is naturally reductive of dimension 16. We fix a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad [\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}, \quad \mathfrak{n} = \mathbb{R}^{16}$$

as well as a scalar product $(\ , \ )_n$ such that for all $X, Y, Z \in \mathfrak{n}$

$$([X, Y]_n, Z)_n + (Y, [X, Z]_n)_n = 0$$

holds, where $[X, Y]_n$ denotes the $n$-part of the commutator. Moreover, suppose that the isotropy representation leaves a Spin(9)-structure in the vector space $\mathfrak{n}$ invariant. Then $G/H$ admits a homogeneous Spin(9)-structure. Indeed, the frame bundle is an associated bundle,

$$\mathcal{F}(G/H) = G \times_{\text{Ad}} \text{SO}(\mathfrak{n}),$$

and $\mathcal{R} := G \to \mathcal{F}(G/H)$ is a reduction to the subgroup $H$ contained in Spin(9). The canonical connection $\nabla^{\text{can}}$ of the reductive space preserves the Spin(9)-structure and has totally skew-symmetric torsion,

$$T^{\nabla^{\text{can}}}(X, Y, Z) = -([X, Y]_n, Z)_n. $$

Consequently, any homogeneous Spin(9)-structure admits an affine connection with totally skew-symmetric torsion, i.e., it is of type $\mathcal{P}_1 \oplus \mathcal{P}_2$.

**Corollary 4.1.** Any homogeneous Spin(9)-structure on a naturally reductive space $M^{16} = G/H$ is of type $\mathcal{P}_1 \oplus \mathcal{P}_2$.

**Remark 4.1.** In particular, for any homogeneous Spin(9)-structure the difference $\Gamma$ between the Levi-Civita connection and the canonical connection is a 3-form. Indeed, the Levi-Civita connection of a reductive space is given by the map $\mathfrak{n} \to \text{End}(\mathfrak{n})$

$$X \mapsto \frac{1}{2} \cdot [X, \cdot]_n.$$

Then we obtain

$$\Gamma(X) = \frac{1}{2} \cdot \text{pr}_m ([X, \cdot]_n) = \frac{1}{32} \sum_{i,j=1}^{16} \sum_{\alpha<\beta<\gamma} ([X, e_i]_n, e_j)_n \cdot \sigma_{\alpha\beta\gamma}(e_i, e_j) \cdot \sigma_{\alpha\beta\gamma}. $$

We write the latter equation in the following form

$$\Gamma(X) = -\frac{1}{16} \sum_{\alpha<\beta<\gamma} (\sigma_{\alpha\beta\gamma} \cdot T^{\nabla^{\text{can}}})(X) \cdot \sigma_{\alpha\beta\gamma} = -\frac{1}{2} \cdot T^{\nabla^{\text{can}}}(X, \cdot, \cdot),$$

i.e., $\Gamma$ is proportional to the torsion of the canonical connection,

$$\Gamma(X)(Y, Z) = -\frac{1}{2} \cdot T^{\nabla^{\text{can}}}(X, Y, Z).$$

There are homogeneous Spin(9)-structures on different reductive spaces (see [Fri1]).

**Example 4.1.** The group Spin(9) acts transitively on the sphere $S^{15}$, the isotropy group is isomorphic to Spin(7) and the isotropic representation of the reductive space $S^1 \times S^1 = (S^1 \times \text{Spin}(9))/\text{Spin}(7)$ is contained in Spin(9).

**Example 4.2.** The space $S^1 \times S^1 \times (\text{SO}(8)/G_2)$ admits a homogeneous Spin(9)-structure.

**Example 4.3.** The space $SU(5)/SU(3)$ admits a homogeneous Spin(9)-structure.
5. G-connections with totally skew-symmetric torsion

The class of Spin(9)-structures corresponding to the representation \( \mathbb{R}^{16} \subset \mathbb{R}^{16} \otimes \mathfrak{m} \) is related with conformal changes of the metric. Indeed, if \((M^{16}, g, V^9)\) is a Riemannian manifold with a fixed Spin(9)-structure \( V^9 \subset \text{End}(TM^{16}) \) and \( g^* = e^{2f} \cdot g \) is a conformal change of the metric, then the triple \((M^{16}, g^*, V^9)\) is a Riemannian manifold with a Spin(9)-structure, too. The fact that the 16-dimensional class of Spin(9)-structures corresponding to \( \mathbb{R}^{16} \) is not admissible in Theorem 5.1 means that the existence of a connection with totally skew-symmetric torsion and preserving a Spin(9)-structure is not invariant under conformal transformations of the metric. From this point of view the behavior of Spin(9)-structures is different from the behavior of \( G \)-structures, Spin(7)-structures, quaternionic Kähler structures or contact structures (see \([\text{FI}2, \text{FI}3, \text{Iv}, \text{IM}]\)). We will explain this effect in a more general context.

Let \( G \subset \text{SO}(n) \) be a closed subgroup of the orthogonal group and decompose the Lie algebra

\[
\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}.
\]

A \( G \)-structure of a Riemannian manifolds \( M^n \) is a reduction \( \mathcal{R} \subset \mathcal{F}(M^n) \) of the frame bundle to the subgroup \( G \). The Levi-Civita connection is a 1-form \( \Theta \) on \( \mathcal{F}(M^n) \) with values in the Lie algebra \( \mathfrak{so}(n) \). We restrict the Levi-Civita connection to a fixed \( G \)-structure \( \mathcal{R} \) and decompose it with respect to the decomposition of the Lie algebra \( \mathfrak{so}(n) \):

\[
Z|_{\mathcal{T}(\mathcal{R})} := Z^* \oplus \Gamma.
\]

Then, \( Z^* \) is a connection in the principal \( G \)-bundle \( \mathcal{R} \) and \( \Gamma \) is a tensorial 1-form of type \( \text{Ad} \), i.e., a 1-form on \( M^n \) with values in the associated bundle \( \mathcal{R} \times_G \mathfrak{m} \). The \( G \)-representation \( \mathbb{R}^n \otimes \mathfrak{m} \) splits into irreducible components and the corresponding decomposition of \( \Gamma \) characterizes the different non-integrable \( G \)-structures. We introduce the equivariant maps:

\[
\Phi : \mathbb{R}^n \otimes \mathfrak{g} \rightarrow \mathbb{R}^n \otimes S^2(\mathbb{R}^n), \quad \Phi(\Sigma)(X, Y, Z) := g(\Sigma(Z)(X), Y) + g(\Sigma(Y)(X), Z),
\]

\[
\Psi : \mathbb{R}^n \otimes \mathfrak{m} \rightarrow \mathbb{R}^n \otimes S^2(\mathbb{R}^n), \quad \Psi(\Gamma)(X, Y, Z) := g(\Gamma(Y)(X), Z) + g(\Gamma(Z)(X), Y).
\]

It is well known (see \([\text{FI}1]\)) that a geometric \( G \)-structure admits a connection \( \nabla \) with totally skew-symmetric torsion if and only if \( \Psi(\Gamma) \) is contained in the image of the homomorphism \( \Phi \). There is an equivalent formulation of this condition. Indeed, let us introduce the maps

\[
\Theta_1 : \Lambda^3(\mathbb{R}^n) \rightarrow \mathbb{R}^n \otimes \mathfrak{g}, \quad \Theta_2 : \Lambda^3(\mathbb{R}^n) \rightarrow \mathbb{R}^n \otimes \mathfrak{m}
\]

given by the formulas

\[
\Theta_1(T) := \sum \sigma_i \cdot J T) \otimes \sigma_i, \quad \Theta_2(T) := \sum \mu_j \cdot J T) \otimes \mu_j
\]

where \( \sigma_i \) is an orthonormal basis in \( \mathfrak{m} \) and \( \mu_j \) is an orthonormal basis in \( \mathfrak{g} \). Observe that the kernel of the map \((\Psi \oplus \Phi) : \mathbb{R}^n \otimes \mathfrak{so}(n) \rightarrow \mathbb{R}^n \otimes S^2(\mathbb{R}^n)\) coincides with the image of the map \((\Theta_1 \oplus \Theta_2) : \Lambda^3(\mathbb{R}^n) \rightarrow \mathbb{R}^n \otimes \mathfrak{so}(n)\). Consequently, for any element \( \Gamma \in \mathbb{R}^n \otimes \mathfrak{m} \), the condition \( \Psi(\Gamma) \in \text{Image}(\Phi) \) is equivalent to \( \Gamma \in \text{Image}(\Theta_1) \).

**Theorem 5.1.** A \( G \)-structure \( \mathcal{R} \subset \mathcal{F}(M^n) \) of a Riemannian manifold admits a connection \( \nabla \) with totally skew-symmetric torsion if and only if the 1-form \( \Gamma \) belongs to the image of \( \Theta_1 \), \( \Gamma = \Theta_1(T) \). In this case the 3-form \( -2 \cdot T \) is the torsion form of the connection.

Consequently, only such geometric types (i.e. irreducible components of \( \mathbb{R}^n \otimes \mathfrak{m} \)) are admissible which occur in the \( G \)-decomposition of \( \Lambda^3(\mathbb{R}^n) \). This explains the different behavior of \( G \)-structures with respect to conformal transformations.

**Example 5.1.** In case of \( G = \text{Spin}(9) \) we have

\[
\mathbb{R}^{16} \otimes \mathfrak{m} = \mathbb{R}^{16} \oplus \Lambda^3(\mathbb{R}^{16}) \oplus \mathcal{P}_3(\mathbb{R}^9)
\]
and the \( \mathbb{R}^{16} \)-component is not contained in \( \Lambda^3(\mathbb{R}^{16}) \), i.e., a conformal change of a Spin(9)-structure does not preserve the property that the structure admits a connection with totally skew-symmetric torsion.

**Example 5.2.** In case of a 7-dimensional \( G_2 \)-structure the situation is different. Indeed, we decompose the \( G_2 \)-representation (see [FI1])

\[
\Lambda^3(\mathbb{R}^7) = \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^3_{27}, \quad \mathbb{R}^7 \otimes m = \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^3_{14} \oplus \Lambda^3_{27}
\]

and, consequently, a conformal change of a \( G_2 \)-structure preserves the property that the structure admits a connection with totally skew-symmetric torsion.

**Example 5.3.** Let us consider Spin(7)-structures on 8-dimensional Riemannian manifolds. The subgroup Spin(7) \( \subset \text{SO}(8) \) is the real Spin(7)-representation \( \Delta_7 = \mathbb{R}^8 \). The complement \( m = \mathbb{R}^7 \) is the standard 7-dimensional representation and the Spin(7)-structures on an 8-dimensional Riemannian manifold \( M^8 \) correspond to the irreducible components of the tensor product

\[
\mathbb{R}^8 \otimes m = \mathbb{R}^8 \otimes \mathbb{R}^7 = \Delta_7 \otimes \mathbb{R}^7 = \Delta_7 \oplus K,
\]

where \( K \) denotes the kernel of the Clifford multiplication \( \Delta_7 \otimes \mathbb{R}^7 \to \Delta_7 \). It is well known that \( K \) is an irreducible Spin-representation. Therefore, there are only two basic types of Spin(7)-structures (see [Fer]). On the other hand, the map \( \Lambda^3(\mathbb{R}^8) \to \mathbb{R}^8 \otimes m \) is injective and the Spin(7)-representation \( \Lambda^3(\mathbb{R}^8) = \Lambda^3(\Delta_7) \) splits again into the irreducible components

\[
\Lambda^3(\Delta_7) = \Delta_7 \oplus K,
\]

i.e., \( \Lambda^3(\mathbb{R}^8) \to \mathbb{R}^8 \otimes m \) is an isomorphism. Theorem 5.1 yields immediately that any Spin(7)-structure on an 8-dimensional Riemannian manifold admits a connection with totally skew-symmetric torsion (see [Fer]). We remark that \( n = 8 \) is the smallest dimension where this effect can occur. Indeed, let \( G \subset \text{SO}(n) \) be a subgroup of dimension \( g \) and suppose that any \( G \)-structure admits a connection with totally skew-symmetric torsion, i.e., the map \( \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes m \) is surjective. On the other side, the isotropy representation \( G \to \text{SO}(m) \) of the compact Riemannian manifold \( \text{SO}(n)/G \) is injective. Consequently, we obtain the inequalities

\[
\frac{1}{3}(n^2 - 1) \leq g \leq \frac{1}{2}(n^2 - 3n + 2).
\]

The minimal pair satisfying this condition is \( n = 8, g = 21 \). Using not only the dimension of the \( G \)-representation one can exclude other dimensions, for example \( n = 9 \).

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