Homotopical Computations in Quantum Fields Theory

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Abstract. This paper is a mathematical study of quantum correlation functions in quantum field theory within a homotopy algebraic framework motivated from the BV quantization scheme. We characterize quantum correlation functions by algebraic homotopy theoretical methods which circumvent gauge fixing and perturbative Feynman diagrams. We show that there is a universal algebraic structure, closely related with that of the WDVV equation, governing quantum correlation functions of every quantum field theory in our framework up to a certain ambiguity. The algebraic structure is independent of the details of the quantum expectation, other than its existence with the prescribed symmetry, and comes with a concrete algorithm for explicit computations. We will also make proposals for the precise natures of quantum expectation and physical equivalence of quantum field theories.

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1. Introduction

1.1. Background

The Batalin-Vilkovisky (BV) quantization scheme of classical field theory is a versatile framework in dealing with a complete tower of infinitesimal classical symmetries — the (gauge) symmetry of a classical action $S_{cl}$, symmetries of the symmetry, etc.— which provides physicists with not only a general method of gauge fixing but also a criterion for independence of path integrals from gauge choices, so that one may compute quantum correlation functions by enumerating Feynman diagrams after a suitable gauge choice [3]. But ultimately, the scheme is a homological algebraic implementation of an infinitesimal symmetry of the quantum expectation that satisfies a Schwinger–Dyson type equation [10,21,12,17]: that is, the (non-existing) path integral measure is translation invariant before being twisted by $e^{-\frac{1}{\hbar}S_{cl}}$.

Exploiting such a symmetry systematically, we attempt to characterize quantum correlation functions by algebraic homotopy theoretical methods which circumvent gauge fixing and perturbative Feynman diagrams. We show that there is a universal algebraic structure, closely related with that of the WDVV equation, governing quantum correlation functions of every quantum field theory in our framework up to a certain ambiguity. The algebraic structure is independent of the details of the quantum expectation, other than its existence with the prescribed symmetry, and comes with a concrete algorithm for explicit computations. We will also make proposals for the precise natures of quantum expectation and physical equivalence of quantum field theories. We will treat the Planck constant $\hbar$ as a formal parameter but our main results on the universal algebra governing quantum correlations will be exact.

Algebraically, working over a ground field $k$ of characteristic zero, we regard a BV quantization of a classical field theory as a blackbox producing:

- a unital $\mathbb{Z}$-graded commutative and associative algebra $(\mathcal{E}, 1_{\mathcal{E}}, \cdot)$ over $k$,
- an odd second order algebraic differential operator $\Delta$, satisfying $\Delta^2 = \Delta 1_{\mathcal{E}} = 0$, whose failure to be a derivation of the product $\cdot$ defines an odd Poisson bracket $(\cdot, \cdot)_{BV}$, and
- a quantum master action $S = S + \hbar S^{(1)} + \hbar^2 S^{(2)} + \cdots \in \mathcal{E}[[\hbar]]$, which is a solution to the quantum master equation:

$$\Delta e^{-\frac{1}{\hbar}S} = 0 \iff -\hbar \Delta S + \frac{1}{2} (S, S)_{BV} = 0,$$
and whose classical limit $S$ incorporates both the classical action $S_{cl}$ and a complete
tower of infinitesimal classical symmetries.

We can organize those outcomes as a tuple $\left( \mathcal{C}[\hbar], 1_q, \cdot, \mathbf{K} \right)$, where $\left( \mathcal{C}[\hbar], 1_q, \cdot \right)$
is the topologically-free $k[[\hbar]]$-algebra generated by $\left( \mathcal{C}, 1_q, \cdot \right)$ and $\mathbf{K} : \mathcal{C}[\hbar] \to \mathcal{C}[\hbar]$ is a $k[[\hbar]]$-linear differential defined by $\mathbf{K} := -\hbar \Delta + (S,)_\mathbf{BV}$ which satisfies $K^2 = K1_q = 0$ and, for all homogeneous $x_1, x_2 \in \mathcal{C}[\hbar]$,

$$K(x_1 \cdot x_2) - Kx_1 \cdot x_2 - \langle x_1 \rangle x_2 (\hbar)(x_1, x_2)_\mathbf{BV},$$

while $(\cdot,)_\mathbf{BV}$ is a derivation of the product. To summarize all those properties, we say that the tuple $\left( \mathcal{C}[\hbar], 1_q, \cdot, K \right)$ is a BV-QFT algebra with quantum descendant $\left( \mathcal{C}[\hbar], 1_q, K, (\cdot,)_\mathbf{BV} \right)$, which is a topologically-free unital sDGLA over $k[[\hbar]].$ Then a quantum expectation can be interpreted as a cochain map $\mathcal{C} : \mathcal{C}[\hbar] \to k[[\hbar]]$ from the pointed cochain complex $(\mathcal{C}[\hbar], 1_q, K)$ to the pointed cochain complex $(k[[\hbar]], 1, 0)$ with zero differential, both over $k[[\hbar]]$. The condition $c(1_q) = 1$ is a normalization and the condition $c \circ K = 0$ is the homological algebraic condition defining an infinitesimal symmetry of the quantum expectation. Another quantum expectation $\tilde{c}$ is physically equivalent to $c$ if they are homotopic $c \sim \tilde{c}$ as pointed cochain maps or, equivalently, if they have the same cochain homotopy type $[c] = [\tilde{c}]$. Such a variation of quantum expectation within its homotopy type corresponds to a change of gauge fixing in the BV quantization scheme and every quantum correlation function should be invariant of the homotopy type of the quantum expectation.

For example, we can consider a quantum observable as a homogeneous element $O \in \mathcal{C}[\hbar]$ satisfying $KO = 0$ so that its expectation value $\langle O \rangle_{\tilde{c}} := c(O)$ depends only on the cochain homotopy type of the quantum expectation $c$, i.e., $\langle O \rangle_{\tilde{c}} = \langle O \rangle_{\hat{c}}$ whenever $c \sim \tilde{c}$. Let $O$ and $\hat{O}$ be quantum observables that belong to the same $K$-cohomology class. Then, they have the same quantum expectation value $\langle O \rangle_{\hat{c}} = \langle \hat{O} \rangle_{\hat{c}}$. Therefore, we may consider the space of equivalence classes of quantum observables as the cohomology $H$ of the cochain complex $\left( \mathcal{C}[\hbar], K \right)$ on which the homotopy class of quantum expectation $c$ induces uniquely a $k[[\hbar]]$-linear map $\mathfrak{u} : H \to k[[\hbar]]$ such that $\mathfrak{u}(1_H) = 1$, where $1_H$ is the cohomology class of $1_q$. Then, we might try to introduce an algebraic structure on $H$ governing quantum correlation functions between and

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1. Given a classical field theory such as Yang–Mills theory, setting up and working out its BV quantization is important mathematical challenges, for which we refer to Costello’s book [5].

2. The acronym sDGLA stands for shifted differential graded Lie algebra; in such an algebra, the Lie bracket has degree 1. This is a special case of an $sL_{\infty}$-algebra — a homotopy Lie algebra with degrees shifted by one.
among quantum observables thought of as pairs, triples, and so on. However, we immediately run into an interesting conundrum.

To wit, the products $O \cdot O$ and $\tilde{O} \cdot \tilde{O}$ of quantum observables $O$ and $\tilde{O}$, which are assumed to have the same $K$-cohomology class, are not quantum observables in general and, even if they happen to be quantum observables by some accident, they may belong to different $K$-cohomology classes, i.e., in general $\langle O \cdot O \rangle_\epsilon \neq \langle \tilde{O} \cdot \tilde{O} \rangle_\epsilon$. Therefore the naive definition of $n$-fold quantum correlation functions among quantum observables as the quantum expectation value of $n$-fold products of quantum observables is inadequate. These problems in general originate from the nature of the differential $K$ which is not a derivation of the product. Rather the failure of $K$ to be a derivation of the product $\cdot$ is divisible by $\hbar$. This property, on the other hand, is crucial in capturing quantum correlations: Assume that $K$ is a derivation of the product and $O - \langle O \rangle_\epsilon \cdot 1_{\epsilon} = K\lambda$. Then, the variance $\left( \langle O - \langle O \rangle_\epsilon \cdot 1_{\epsilon} \rangle_\epsilon \right)^2 = \langle O \cdot O \rangle_\epsilon - \langle O \rangle_\epsilon^2$ of any quantum observable $O$ should always be zero. But we expect that the variance vanishes identically only in the classical limit in the presence of quantum correlation.

A partial resolution of the above problems can be accomplished by adopting the notion of a homotopical family of quantum observables. We can regard a non-empty set $\{ O_a \}_{a \in I}$ of quantum observables as the image of a basis $\{ e_a \}_{a \in I}$ of a $\mathbb{Z}$-graded vector space $V$ under a cochain map $\varphi_1 : (V[[\hbar]], 0) \to (\mathcal{C}[[\hbar]], K)$, i.e., $\varphi_1 : V[[\hbar]] \to \mathcal{C}[[\hbar]]$ is a degree zero map satisfying $K \circ \varphi_1 = 0$ and $O_a = \varphi_1(e_a)$, for all $a \in I$. On the other hand, a homotopical family of quantum observables is defined to be an $sL_\infty$-morphism $\varphi : (V[[\hbar]], 0) \longrightarrow (\mathcal{C}[[\hbar]], K, ( , )_{BV})$, where $\varphi = \varphi_1, \varphi_2, \ldots$ and $\varphi_n = \varphi_n^{(0)} + \hbar \varphi_n^{(1)} + \ldots$ is a family of degree preserving $k$-linear maps, parametrized by $\hbar$, from the $n$th (super)symmetric power $S^nV$ of $V$ to $\mathcal{C}$, i.e., $\varphi \in \text{Hom}(S^nV, \mathcal{C})^{0[[\hbar]]}$. Then, there is an associated family $\Pi^\varphi_n = \Pi^\varphi_1, \Pi^\varphi_2, \ldots$ of quantum correlators, where $\Pi^\varphi_n \in \text{Hom}(S^nV, \mathcal{C})^{0[[\hbar]]}$ satisfies $K \circ \Pi^\varphi_n = 0$, and the family of (joint) quantum moments can be defined as the following:

$$\left\{ \langle \Pi^\varphi_n(e_{a_1}, \ldots, e_{a_n}) \rangle_\epsilon \mid n \geq 1; a_1, \ldots, a_n \in I \right\}.$$

This family is an invariant of the homotopy type of the quantum expectation $\epsilon$. We also have $\langle \Pi^\varphi_n(e_{a_1}, \ldots, e_{a_n}) \rangle_\epsilon = \langle \Pi^\varphi_n(e_{a_1}, \ldots, e_{a_n}) \rangle_\epsilon$ for all $n \geq 1$ and $a_1, \ldots, a_n \in I$ whenever the two corresponding $sL_\infty$-morphisms $\varphi$ and $\tilde{\varphi}$ are homotopic. We can also view the homotopical family of quantum observables as the simultaneous quantum correction $\Pi^\varphi_n(e_{a_1}, \ldots, e_{a_n})$ of every multiple product $O_{a_1} \cdot \ldots \cdot O_{a_n}$ of quantum
observables in the set \( \{ O_a \}_{a \in I} \). For example, we have

\[
\Pi^2_3(\epsilon_{a_1}, \epsilon_{a_2}) = O_{a_1} \cdot O_{a_2} - \hbar O_{a_1 a_2},
\]

\[
\Pi^3_3(\epsilon_{a_1}, \epsilon_{a_2}, \epsilon_{a_3}) = O_{a_1} \cdot O_{a_2} \cdot O_{a_3} - \hbar O_{a_1 a_2} \cdot O_{a_3} - \hbar O_{a_1} \cdot O_{a_2 a_3} - \hbar (-1)^{I_{a_1} || O_{a_2} | O_{a_2} \cdot O_{a_1 a_3}} + \hbar^2 O_{a_1 a_2 a_3},
\]

e etc., where \( O_{a_1 \cdots a_n} := \Phi_{a}(\epsilon_{a_1}, \ldots, \epsilon_{a_n}) \). A homotopical family of quantum observables also corresponds to a formal deformation of the quantum master action \( S \) to a new quantum master action \( S + \Theta^\varphi \), where \( \Theta^\varphi = t^a \varphi_1(\epsilon_a) + \frac{1}{2!} t^{a_1} t^{a_2} \varphi_2(\epsilon_{a_1}, \epsilon_{a_2}) + \ldots \) and \( \{ t^a \}_{a \in I} \) are variables dual to \( \{ \epsilon_a \}_{a \in I} \). In the deformation we have \( \Delta e^{-\frac{1}{\hbar} S + \Theta^\varphi} = 0 \) and homotopic \( sL_\infty \)-morphisms produce equivalent deformations. Then, we have the following generating function for the quantum moments — depending only on the homotopy types of \( \varphi \) and \( \varphi \) —

\[
\epsilon \left( e^{-\frac{1}{\hbar} \Theta^\varphi} \right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \left( \Pi^3_n(\epsilon_{a_1}, \ldots, \epsilon_{a_n}) \right) \epsilon.
\]

However, the above resolution is not entirely satisfactory since there are numerous inequivalent ways of extending a cochain map \( \varphi_1 : (V[\hbar], 0) \to (\mathcal{C}[\hbar], K) \) into non-homotopic \( sL_\infty \)-morphisms. This means that quantum corrections to the products \( O_{a_1} \cdots O_{a_n} \) can be arbitrary so that the interpretation of \( \{ \Pi^3_3(\epsilon_{a_1}, \ldots, \epsilon_{a_n}) \} \) as the family of joint quantum correlation functions among the set \( \{ O_a \}_{a \in I} \) of quantum observables is ambiguous. The second approach, which is eventually our resolution to this problem adopts a kind of complimentary principle, swinging back and forth between the classical and quantum vantage points and will reveal to us, arguably, the true nature of quantum correlations.

The classical limit \( (\mathcal{C}, 1, \mathcal{C}, \cdots, K) \) of the BV-QFT algebra \( (\mathcal{C}[\hbar], 1, \mathcal{C}, \cdots, K) \) is a unital differential graded commutative algebra (CDGA) over \( k \), where \( K := (S, )_{BV} \) and \( S = S_0 + \ldots \) is the classical limit of the quantum master action \( S \) so that \( S \) satisfies the classical BV master equation \( (S, S)_{BV} = 0 \). In contrast to the differential \( K = K - \hbar (\Delta - (S^{(1)}, )_{BV}) + \ldots \), the classical differential \( K \) is a derivation of the product. We say an element \( O \in \mathcal{C} \) is an off-shell classical observable if \( KO = 0 \) and two off-shell classical observables are equivalent if they have the same \( K \)-cohomology class. Then any product \( O_1 \cdots O_n \) of off-shell classical observables \( O_1, \ldots, O_n \) is an off-shell classical observable, whose equivalence class depends only on the equivalence classes of \( O_1, \ldots, O_n \). In fact, it is the cohomology \( (H, 1_H, 0) \) of the classical pointed cochain

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3 We use Einstein summation convention throughout this paper.
complex \((\mathcal{C}, 1, 1, K)\) that is the arena of classical physics, where the differential \(K\) can be viewed as an odd vector field on an appropriate classical fields space whose vanishing loci is the classical equation motion space — the moduli space defined by classical equations of motion modulo classical symmetry, and \(H\) corresponds to the space of \(k\)-valued function(als) on the classical equation motion space. Therefore, it is appropriate to define a classical observable as a homogeneous element in \(H\) — an equivalence class of off-shell classical observables.

Then we will propose precise notions for a quantization of off-shell classical observables and corresponding quantum correlators among classical observables, which will lead us to a universal and exactly computable algebraic structure governing every quantum correlation.

1.2. The Results

We will work with binary QFT algebras, which are a natural generalization of BV-QFT algebras which share the same relevant universal properties. A binary QFT algebra is a tuple \(\mathcal{A}[\hbar] = (\mathcal{A}[\hbar], 1, \cdot, K)\), where \((\mathcal{A}[\hbar], 1, \cdot)\) is a topologically-free unital \(\mathbb{Z}\)-graded commutative and associative algebra over \(k[\hbar]\) and \((\mathcal{A}[\hbar], 1, \cdot, K)\) is a pointed and topologically-free cochain complex over \(k[\hbar]\), which satisfies a sequence of \(\hbar\)-compatibility axioms between the product \(\cdot\) and the differential \(K\) — namely the failure of \(K\) to be a derivation of \(\cdot\) is divisible by \(\hbar\) and the \(n\)th iterated failure is divisible by \(\hbar^n\). Then the role of the topologically-free sDGLA associated to a BV-QFT algebra is played by a topologically-free unital \(sL_\infty\)-algebra \((\mathcal{C}[\hbar], 1, \cdot, \mathcal{L} = K, \ell_2, \ell_3, \cdots)\) over \(k[\hbar]\), called the quantum descendant of \(\mathcal{A}[\hbar]_{BQFT}\).

The classical limit \((\mathcal{C}, 1, \cdot, K)\) of the binary QFT algebra \(\mathcal{A}[\hbar]_{BQFT}\) is still a unital CDGA over \(k\). Therefore, the underlying pointed cochain complex \((\mathcal{A}, 1, \cdot, K)\) over \(k\) is homotopy equivalent to its cohomology \((H, 1_H, 0)\), which is regarded as the arena of classical physics. We fix a homotopy equivalence \((f, h, s)\), where \(s : \mathcal{C} \to \mathcal{C}\) is a splitting and both \(f : H \to \mathcal{C}\) and \(h : \mathcal{C} \to H\) are pointed cochain quasi-isomorphisms. In particular, \(f\) is a \(k\)-linear choice of a set of off-shell representatives of all classical observables. A quantization of every off-shell classical observable is defined as a deformation \((f, h, s)\) of \((f, h, s)\) such that \((f, h, s)\) is a homotopy equivalence between \((H[\hbar], 1_H, 0)\) and \((\mathcal{A}[\hbar], 1, \cdot, K)\) as pointed cochain complexes over \(k[\hbar]\).

In general, there are obstructions order by order in \(\hbar\) to such a deformation. These obstructions can be organized into a differential \(\kappa = hX^{(1)} + h^2X^{(2)} + \cdots\) in such a way that \((f, h, s)\) is a homotopy equivalence between \((H[\hbar], 1_H, \kappa)\) and \((\mathcal{A}[\hbar], 1, \cdot, K)\)
as pointed cochain complexes over $k[[h]]$. This construction, which is an application of standard homological perturbation theory, is not unique but is as canonical as possible in the sense that everything depends at most on the choice of the splitting $s$ and, in particular, the condition $\kappa = 0$ is independent of this choice. We say a binary QFT algebra is anomaly-free if $\kappa = 0$, so that any off-shell classical observable admits a quantization.

We will develop a general theory including the anomalous case but here we state the main theorems of this paper restricted to the anomaly-free case, which has a more straightforward physical interpretation.

**Theorem 1.1.** Let $\mathcal{C}_{\text{BFQFTA}}$ be an anomaly-free binary QFT algebra with associated quantum descendant unital $sL_\infty$-algebra $(\mathcal{C}[[h]], 1, \ell)$. Then, there is a distinguished unital $sL_\infty$-quasi-isomorphism

$$\phi : (H[[h]], 1_H, \mathcal{L}) \longrightarrow (\mathcal{C}[[h]], 1, \ell)$$

with the following factorization and $\hbar$-finiteness property.

To state the property, define the associated family $\Pi = \Pi_1, \Pi_2, \ldots$ of quantum correlators, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in H$, as

$$\Pi_n(v_1, \ldots, v_n) := \sum_{p \in P(n)} (-\hbar)^{|p|} \varepsilon(p) \phi(B_1) \cdots \phi(B_{|p|}).$$

The condition is then that for any quantum expectation $c$ we have a $k[[\hbar]]$-linear functional $\iota : H[[\hbar]] \rightarrow k[[\hbar]]$ such that the $n$-fold quantum correlation function $c \circ \Pi_n$ admits a factorization

$$c \circ \Pi_n = c \circ \hat{\pi}_n : S^n H[[\hbar]] \longrightarrow k[[\hbar]],$$

where $\hat{\pi}_1$ is the identity $\id_H$ map on $H$ and, for all $n \geq 2$,

- $\hat{\pi}_n$ has polynomial dependence on $\hbar$ of degree not higher than $n - 2$:

$$\hat{\pi}_n = \hat{\pi}^{(0)}_n + (-\hbar) \hat{\pi}^{(1)}_n + \cdots + (-\hbar)^{n-2} \hat{\pi}^{(n-2)}_n,$$

where $\hat{\pi}^{(j)}_n : S^n H \rightarrow H$ for $j = 0, 1, \ldots, n - 2$ and

- $\hat{\pi}_n(v_1, \ldots, v_{n-1}, 1_H) = \hat{\pi}_{n-1}(v_1, \ldots, v_{n-1})$. 


The first part of our theorem says that $H[[\hbar]]$, viewed as an unital $sL_\infty$-algebra over $k[[\hbar]]$ with the trivial $sL_\infty$-structure $(H[[\hbar]], 1, \mu, \eta)$, is quasi-isomorphic to the unital $sL_\infty$-algebra $(\mathcal{C}[[[\hbar]], 1, \varphi, \zeta)$, whenever $\kappa = 0$. It also says that there is a distinguished unital $sL_\infty$-quasi-isomorphism $\phi$ between them. The rest of the theorem explicates the consequences of this.

For a quantum expectation $\epsilon : \mathcal{C}[[\hbar]] \to k[[\hbar]]$, we call $\iota := \epsilon \circ f : H[[\hbar]] \to k[[\hbar]]$ the on-shell quantum expectation. We regard the on-shell quantum expectation as a family of $k$-linear maps parametrized by $\hbar$ from $H$ to $k$: the quantum expectation value of a classical observable $\nu \in H$ is $(\iota(\nu)) = \iota^{(0)}(\nu) + \hbar \iota^{(1)}(\nu) + \hbar^2 \iota^{(2)}(\nu) + \ldots \in k[[\hbar]]$. We call $\epsilon \circ \Pi_n$ the $n$-fold quantum correlation function, and regard it as a family of $k$-linear maps parametrized by $\hbar$ from $S^n H$ to $k$. The family of joint quantum moments of a non-empty set of classical observables $\{v_1, \ldots, v_k\} \subset H$ is defined to be the family

$$\left\{ (\Pi_n(v_{j_1}, \ldots, v_{j_n}))_{\epsilon} = \epsilon \circ \Pi_n(v_{j_1}, \ldots, v_{j_n}) \in k[[\hbar]] \right\}$$

of invariants of the homotopy type of quantum expectation $\epsilon$.

The factorization property $\epsilon \circ \Pi_n = \iota \circ \Pi_n$ says that $n$-fold quantum correlation functions are determined by the on-shell quantum expectation $\iota$ and $\Pi_n$, which has the distinguished property that it only has polynomial dependence in $\hbar$ of degree at most $n - 2$ for all $n \geq 2$. This pushes convergence issues, when $\hbar$ is no-longer treated as a formal variable, for quantum correlation functions to the on-shell quantum expectation, about which this paper takes an agnostic viewpoint.

There can be infinitely many different unital $sL_\infty$-quasi-isomorphisms from $H[[\hbar]]$ to $\mathcal{C}[[\hbar]]$. For each such quasi-isomorphism $\Phi$, we can define an associated family $\Pi^\Phi_n$ of quantum correlators and corresponding quantum correlation functions $\epsilon \circ \Pi^\Phi_n$ so that $\epsilon \circ \Pi^\Phi_n = \iota \circ \Pi^\Phi_n$, where $\Pi^\Phi_n := h \circ \Pi^\Phi_n$. However, if $\Phi$ is not homotopic to the distinguished morphism $\phi$ as a unital $sL_\infty$-morphism, then $\Pi^\Phi_n$ is only a formal power series in $\hbar$ for all $n \geq 2$ in general, rather than a polynomial. We call the $sL_\infty$-homotopy type of $\phi$ a quantum structure on $\mathcal{C}[[\hbar]]_{BQFTA}$.

The proof of Theorem 1.1 will be constructive: there is a concrete algorithm to determine the family $\hat{\Pi}_n = \hat{\Pi}_{n, 1}, \hat{\Pi}_{n, 2}, \ldots$. The next main theorem is about the internal structure of the family $\hat{\Pi}_n$: that it is determined by the components $\hat{\Pi}^{(n-2)}_{n,n}$ for $n \geq 2$. 


Theorem 1.2. There is a family \( \hat{\mathfrak{m}} = \hat{m}_2, \hat{m}_3, \ldots \), which determines \( \hat{\pi}_2, \hat{\pi}_3, \ldots \) by the following recursive formula: for all \( n \geq 2 \) and homogeneous \( v_1, \ldots, v_n \in H \),

\[
\hat{\pi}_n(v_1, \ldots, v_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \epsilon(p) \hat{\pi}_{|p|}(v_{B_1}, \ldots, v_{B_{|p|}}, \hat{m}(v_{B_{|p|}})).
\]

Moreover, the family \( \hat{\mathfrak{m}} \) has the following properties:

- symmetry: \( \hat{m}_n \) is a \( \hbar \)-linear map of degree 0 from \( S^nH \) to \( H \) for all \( n \geq 2 \);
- unity: \( \hat{m}_2(1_H, v_1) = v_1 \), while \( \hat{m}_n(1_H, v_1, \ldots, v_{n-1}) = 0 \) for all \( n \geq 3 \);

- generalized associativity: for all \( n \geq 0 \),

\[
\sum_{\varsigma \subseteq [n]} \epsilon(\varsigma \sqcup \varsigma^c) \hat{m}(v_{\varsigma} \otimes \hat{m}(v_{\varsigma^c} \otimes w_1 \otimes w_2) \otimes w_3)
= \sum_{\varsigma \subseteq [n]} \epsilon(\varsigma \sqcup \varsigma^c)(-1)^{|v_{\varsigma^c}|+|w_1|} \hat{m}(v_{\varsigma} \otimes w_1 \otimes \hat{m}(v_{\varsigma^c} \otimes w_2 \otimes w_3)).
\]

See Theorems 6.2 and 6.3 for the full details.

We call the tuple \((H, 1_H, \hat{\mathfrak{m}})\) the on-shell quantum correlation algebra of the anomaly-free binary QFT algebra \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \) and call \( \hat{\pi} \) the family of iterated quantum correlation products generated by \( \hat{\mathfrak{m}} \).

Here are some explicit expressions for \( \hat{\pi} \) in terms of \( \hat{\mathfrak{m}} \) as the notation for these interesting relations are yet to be introduced:

\[
\hat{\pi}_2(v_1, v_2) = \hat{m}_2(v_1, v_2),
\hat{\pi}_3(v_1, v_2, v_3) = \hat{m}_2(v_1, \hat{m}_2(v_2, v_3)) - \hbar \hat{m}_3(v_1, v_2, v_3),
\hat{\pi}_4(v_1, v_2, v_3, v_4) = \hat{m}_2(v_1, \hat{m}_2(v_2, \hat{m}_2(v_3, v_4))) - \hbar \hat{m}_2(v_1, \hat{m}_2(v_2, v_3, v_4)) - \hbar(-1)^{|v_2||v_3|} \hat{m}_2(v_2, \hat{m}_3(v_1, v_3, v_4)) - \hbar \hat{m}_3(v_1, v_2, \hat{m}_2(v_3, v_4)) + \hbar^2 \hat{m}_4(v_1, v_2, v_3, v_4).
\]

The following theorem will provide us an algorithm to determine the family \( \hat{\mathfrak{m}} \) directly by a certain sequence of classical cohomology computations:
Theorem 1.3. There is a family $\phi^{-1} = \phi_2^{-1}, \phi_3^{-1}, \ldots$ of $k$-linear maps $\phi^{-1}_n$ from $S^{n-2} H \otimes S^2 H$ to $\mathcal{C}$ of degree $-1$ so that $\hat{m}_n = h \circ M_n$ for all $n \geq 2$, where

$$M_n(v_1, \ldots, v_n) := \sum_{p \in P(n)} e(p) \phi(B_p) \cdot \phi(v_{B_p}) - \sum_{p \in P(n)} e(p) \phi|_{p}(v_{B_p}, \ldots, v_{B_{|p|-1}}, \hat{m}(v_{B_{|p|}}))$$

$$+ \sum_{p \in P(n)} e(p) \ell|_{p}(\phi(J v_{B_1}), \ldots, \phi(J v_{B_{|p|-1}}), \phi^{-1}(v_{B_{|p|}})),$$

and $\phi$ is the classical limit of the distinguished unital $sL_{\infty}$-quasi-morphism $\Phi$.

Note the classical limit $(\mathcal{C}, 1_{\mathcal{C}}, \hat{E})$ of the quantum descendant algebra $(\mathcal{C}[\hbar], 1_{\mathcal{C}}, \hat{E})$ is a unital $sL_{\infty}$-algebra over $k$. Then the classical limit $\Phi$ of the distinguished unital $sL_{\infty}$-quasi-isomorphism $\Phi$ in Theorem 1.1 is a distinguished unital $sL_{\infty}$-quasi-isomorphism $\Phi : (H, 1_H, 0) \overset{\Phi}{\longrightarrow} (\mathcal{C}, 1_{\mathcal{C}}, \hat{E})$. If we assume that $H$ is a finite dimensional $\mathbb{Z}$-graded vector space, it follows that the deformation functor defined by the Maurer–Cartan equation of the unital $sL_{\infty}$-algebra $(\mathcal{C}, 1_{\mathcal{C}}, \hat{E})$ is pro-representable — by the completed symmetric algebra $\hat{S}(H^*)$ of the dual vector space $H^*$ of $H$ — so that we can associate to it a based smooth formal super-moduli space, where the homotopy type of a unital $sL_{\infty}$-quasi-isomorphism from $H$ to $\mathcal{C}$ can be viewed as affine coordinates. (See [15][11].) Therefore we can associate a based smooth formal super-moduli space $\mathcal{M}_0$ to an anomaly-free binary QFT algebra with finite dimensional classical cohomology $H$ and equip this moduli space with distinguished affine coordinates from the homotopy type of $\Phi$ which we call quantum coordinates as they came from a quantum structure.

The notion of quantum coordinates turns out to be closely related with that of special coordinates in the moduli space of type IIB topological string theory [5][2][11] or the flat coordinates in the moduli space of universal unfoldings of an isolated singularity [18], which is a crucial concept in mirror symmetry. Moreover, the on-shell quantum correlation algebra $(H, 1_H, \hat{m})$ is equivalent to a formal Frobenius manifold without the flat metric. That is, $\hat{m}$ satisfies the Witten–Dijkraaf–Verlinde–Verlinde (WDDV) equation [8][24].

Consider Theorem 1.1 and assume that $H$ is a finite dimensional $\mathbb{Z}$-graded vector space over $k$. It is convenient to introduce homogeneous coordinates $t_H = \{t^n\}$ on $H$.
so that \( \{ \partial_a = \partial / \partial t^a \} \) form a homogeneous basis of \( H \) with the distinguished element \( \partial_0 = 1_H \). Then we extend \( \partial_a \) as a derivation on \( k[[t_H]] \). From \( \bar{\Phi} \) and \( \bar{\zeta} \), define

\[
\Theta := \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \bar{\Phi}_n(\partial_{a_1}, \ldots, \partial_{a_n}) \in \left( \mathcal{C}[\![[t_H]] \right) \left[ [H] \right],
\]

\[
\mathcal{T}^\gamma := \mathcal{T}^\gamma + \sum_{n=2}^{\infty} \frac{1}{n!(-\hbar)^n} t^{a_n} \cdots t^{a_1} \pi_{a_1 \cdots a_n}^\gamma \in k[[t_H]][[H^{-1}]],
\]

where \( \{ \pi_{a_1 \cdots a_n}^\gamma \} \) is the set of structure constants defined for the operators \( \pi_n \) as \( \pi_n(\partial_{a_1}, \ldots, \partial_{a_n}) = \pi_{a_1 \cdots a_n}^\gamma \partial_\gamma \). Then, we have \( \partial_0 \Theta = 1_\mathcal{C} \) and

\[
K e^{-\frac{1}{\hbar} \Theta} = 0 \iff K \Theta + \frac{1}{2!} \mathcal{L}_2(\Theta, \Theta) + \frac{1}{3!} \mathcal{L}_3(\Theta, \Theta, \Theta) + \ldots = 0,
\]

so that \( \Theta \) is a universal solution to the Maurer–Cartan equation of the unital sL\( \infty \)-algebra \( (\mathcal{C}[\![[t_H]] \right), 1_\mathcal{C}, K, L_2, L_3, \ldots \) corresponding to a distinguished choice of affine coordinates on the associated \( \mathcal{Z} \)-graded formal based moduli space \( M_o \) with tangent space \( H \). We also have the generating function \( Z_\epsilon \) of all quantum correlation functions with respect to a quantum expectation \( \epsilon \) defined as follows:

\[
Z_\epsilon := \left( \epsilon^{-\frac{1}{\hbar} \Theta} \right)_\epsilon = 1 + \sum_{n=1}^{\infty} \frac{1}{n!(-\hbar)^n} t^{a_n} \cdots t^{a_1} \left( \Pi_n(\partial_{a_1}, \ldots, \partial_{a_n}) \right)_\epsilon.
\]

From the factorization property, we obtain the identity

\[
Z_\epsilon = 1 - \frac{1}{\hbar} \mathcal{T}^\gamma \langle \mathcal{F}(\partial_\gamma) \rangle_\epsilon
\]

so that the quantum expectation values \( \{ \langle \mathcal{F}(\partial_\gamma) \rangle_\epsilon \} \) and \( \{ \mathcal{T}^\gamma \} \) determine every quantum correlation function.

Now the unital sL\( \infty \)-quasi-isomorphism \( \bar{\Phi}^0 \) being distinguished means that \( \{ \mathcal{T}^\gamma \} \) is a formal power series in \( \hbar^{-1} \) since \( \pi_n \) has at most degree \( n-2 \) polynomial dependence on \( \hbar \). If we consider the structure constants \( \{ \mathcal{m}_{a_1 \cdots a_n}^\gamma \} \) of \( \mathcal{m} \) in Theorem 1.2, i.e., \( \mathcal{m}_n(\partial_{a_1}, \ldots, \partial_{a_n}) = \mathcal{m}_{a_1 \cdots a_n}^\gamma \partial_\gamma \), and define

\[
\mathcal{A}_{ab}^\gamma = \mathcal{m}_{ab}^\gamma + \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \mathcal{m}_{\rho_1 \cdots \rho_n a b}^\gamma \in k[[t_H]],
\]

it can be checked that \( \{ \mathcal{T}^\gamma \} \) is the unique solution in formal power series in \( \hbar^{-1} \) to the following system of formal differential equations:

\[
\hbar \partial_a \partial_b \mathcal{T}^\gamma + \mathcal{A}_{ab}^\gamma \partial_\gamma \mathcal{T}^\gamma = 0, \quad \partial_0 \mathcal{T}^\gamma = \delta_0^\gamma - \frac{1}{\hbar} \mathcal{T}^\gamma,
\]
with the boundary conditions \( T^{\gamma} |_{t_0 = 0} = \partial_\beta T^{\gamma} |_{t_0 = 0} - \delta^{\gamma}_\beta = 0 \), where \( \delta^{\gamma}_\beta \) denotes the Kronecker delta. Finally, the properties of \( n_{\beta}^{(\gamma)} \) in Theorem 1.2 imply that \( \{ \hat{A}_{\alpha \beta} \} \) satisfy the following relations:

- unity: \( \hat{A}_{0 \beta}^{\gamma} = \delta^{\gamma}_\beta \),
- super-commutativity: \( \hat{A}_{\alpha \beta}^{\gamma} = (-1)^{|t^\alpha||t^\beta|} \hat{A}_{\beta \alpha}^{\gamma} \) and \( \partial_\alpha \hat{A}_{\beta \gamma}^{\sigma} = (-1)^{|t^\alpha||t^\beta|} \partial_\beta \hat{A}_{\alpha \gamma}^{\sigma} \),
- associativity: \( \hat{A}_{\alpha \beta}^{\gamma} \hat{A}_{\rho \gamma}^{\sigma} = \hat{A}_{\beta \gamma}^{\rho} \hat{A}_{\alpha \rho}^{\sigma} \),

so that the triple \( (H \otimes k[[\hbar]], \partial_0, \ast) \) is a unital super-commutative associative algebra over \( k[[\hbar]] \), where \( \partial_\alpha \ast \partial_\beta := \hat{A}_{\alpha \beta} \partial \gamma \).

The proof of our main theorems involves solutions to what we call the master equations for the families of quantum correlators at levels 0 and 1, where the family \( \Pi \) of quantum correlators in Theorem 1.1 is the level 0 case. In the due course, it will become clear that there should be a tower of quantum correlators for all levels, \( n \geq 0 \). We shall need another new and more versatile framework, beyond the scope of this paper, to characterize this total structure. Here we will use the master equation for level 1 quantum correlators as an auxiliary device in proving Theorems 1.2 and 1.3.

We will also define morphisms and homotopy types of morphisms of binary QFT algebras to form the category \( \text{BQFTA}(k) \) and the homotopy category \( \text{hoBQFTA}(k) \) of binary QFT algebras, and the (homotopy) category of BV-QFT algebras will arise as a full subcategory of \( \text{hoBQFTA}(k) \).

A binary QFT algebra is both a topologically-free unital \( \mathbb{Z} \)-graded commutative and associative algebra over \( k[[\hbar]] \) and a pointed and topologically-free cochain complex over \( k[[\hbar]] \) together with a set of \( \hbar \)-compatibility conditions between the differential and the product as was mentioned before. A morphism of binary QFT algebras is defined similarly as a pointed cochain map, satisfying a corresponding set of \( \hbar \)-compatibility conditions involving the products in the source and target binary QFT algebras, so that the failure of being an algebra homomorphism is divisible by \( \hbar \) and the \( n \)th iterated failure is divisible by \( \hbar^n \) (the classical limit of a morphism of binary QFT algebras is a morphism of unital CDGAs over \( k \)). These \( \hbar \)-compatibility conditions are crucial in organizing quantum correlations algebraically and can themselves be assembled to give a functor \( \mathcal{R} : \text{BQFTA}(k) \to \text{UsL}_\infty(k[[\hbar]]) \), called the quantum descendant functor, to the category \( \text{UsL}_\infty(k[[\hbar]]) \) of unital \( sL_\infty \)-algebras over \( k[[\hbar]] \).

We define homotopy classes of morphisms of binary QFT algebras so that \( \mathcal{R} \) is a ho-
motopy functor, i.e., it induces a well-defined functor from the homotopy category $\text{hoBQFTA}(k)$ to the homotopy category $\text{hoUsL}_\infty(k[[\hbar]])$ of unital $sL_\infty$-algebras.

An impetus for these definitions is to have the following:

**Theorem 1.4.** A homotopy equivalence of anomaly-free binary QFT algebras induces an isomorphism of on-shell quantum correlation algebras and, in particular, sends a flat structure to a flat structure.

We then define a binary QFT as a diagram

![Diagram](image.png)

in the category $\text{BQFTA}(k)$, where $k[[\hbar]]$ is regarded as a binary QFT algebra concentrated in degree zero and is actually an initial object in the category. We call the binary QFT algebra morphism $\epsilon$ a strong quantum expectation, as it is not only a pointed cochain map but also satisfies the $\hbar$-compatibility condition between the products in $C$ and $k$. Namely, let $\mathfrak{R}(\epsilon) = \mathfrak{z} = z_1, z_2, \ldots$, then we have $z_1 = \epsilon$ and $(-\hbar)z_2(x_1, x_2) = z_1(x_1 \cdot x_2) - z_1(x_1)z_1(x_2)$ etc. In particular, $z_1(x_1) = \epsilon(x_1) \equiv (x_1)_\epsilon$ is the quantum expectation value of $x_1$ and $(-\hbar)z_2(x_1, x_2) = (x_1 \cdot x_2)_\epsilon - (x_1)_\epsilon(x_2)_\epsilon$ is the covariance between $x_1$ and $x_2$, which should be non-zero in general due to quantum correlations but vanishes in the classical limit. We may call $\mathfrak{z}$ the family of quantum cumulants, or the family of connected quantum correlation functions, measuring the strength and depth of quantum correlations between and among events thought of as singletons, pairs, triples, and so on. The family of quantum cumulants has the structure of a unital $sL_\infty$-morphism $\mathfrak{z} : (\epsilon[[\hbar]], 1, \ell) \rightarrow (k[[\hbar]], 1, 0)$ between the quantum descendant unital $sL_\infty$-algebras. It is natural to declare that two binary QFTs $\mathfrak{e} : \epsilon[[\hbar]] \rightarrow k[[\hbar]]$ and $\mathfrak{e} : \epsilon'[[\hbar]] \rightarrow k[[\hbar]]$ are physically equivalent if the following diagram in the category $\text{BQFTA}(k)$ of binary QFT algebras is commutative up to homotopy

![Diagram](image.png)

and $f$ is a homotopy equivalence of binary QFT algebras — Physically equivalent binary QFTs have isomorphic quantum correlation functions.

### 1.3. Organization

This paper is organized as follows:
In Sect. 2, we define the (homotopy) category $(\text{ho})\mathcal{BQFTA}(k)$ of binary QFT algebras together with the homotopy functor $\mathcal{F} : \mathcal{BQFTA}(k) \rightarrow \mathcal{UsL}_\infty(k[[\hbar]])$.

In Sect. 3, we define the notion of a homotopical family of quantum observables and discuss several consequences of the definition. We show that homotopy equivalent binary QFT algebras have isomorphic sets of homotopical families of quantum observables.

In Sect. 4, we study quantization of the classical off-to-on-shell retract and prove an important technical lemma called homotopy $\hbar$-divisibility. We also comment on the dependence of quantization of classical observables on the data of the classical off-to-on-shell retract.

In Sect. 5, we define the levels zero and one master equations for quantum correlators and find canonical solutions without assuming the anomaly-free condition. This section is the technical core of this paper.

In Sect. 6, we specialize to the anomaly-free case, leading to Theorems 1.1, 1.2, 1.3 and 1.4 together with some physical interpretations. We further specialize to the case that $H$ is finite dimensional to discuss relationships between the universal algebraic structure governing quantum correlations and the WDDV equation and some simple but historically important examples.

We have a pedagogical Appendix on the theme of $sL_\infty$-algebras. The first part is a self-contained review on the category and homotopy category of $sL_\infty$-algebras. The second part is a sketch of the recipe for constructing a classical BV master action out of a classical action incorporating both the complete tower of classical symmetries and subsequent gauge fixing. The key point is that every notion in an off-shell formalism for classical physics should be defined modulo the classical equation of motion and that the concomitant coherence issues are naturally resolved using the language of $sL_\infty$-morphisms.

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The homotopy category of binary QFT algebras

The main purpose of this section is to define the category $\text{BQFTA}(k)$ and homotopy category $\text{hoBQFTA}(k)$ of binary QFT algebras over $k$. A BV-QFT algebra is a special kind of binary QFT algebra and the category $\text{BV-QFTA}(k)$ and homotopy category $\text{hoBV-QFTA}(k)$ category of BV QFT algebras shall be defined as full subcategories of $\text{BQFTA}(k)$ and $\text{hoBQFTA}(k)$, respectively. We also consider briefly the (homotopy) category $\text{hoBCFTA}(k)$ of binary CFT algebras, where a binary CFT algebra appears as as the combined classical limit of a binary QFT algebra and its quantum descendant.⁴

2. Notation

Fix a ground field $k$ of characteristic zero, usually $\mathbb{R}$ or $\mathbb{C}$.

Let $V$ be a $\mathbb{Z}$-graded $k$-vector space. We shall call the $\mathbb{Z}$-grading the ghost number and denote it $\text{gh}$, i.e., we have a decomposition $V = \bigoplus_{j \in \mathbb{Z}} V^j$ and an element $v \in V^j$ is said to have ghost number $j = \text{gh}(v)$. We often use the notation $(-1)^{|v|}$ instead of $(-1)^{\text{gh}(v)}$ for the parity $\pm 1$ of $v$ as well as the notation $J_v = (-1)^{|v|}v$.

We use the notation $\otimes$ for the tensor product over $k$, the notation $T_n V$ for the $n$th tensor power of $V$ and the notation $T(V) = \bigoplus_{n=1}^{\infty} T^n V$ for the reduced tensor module generated by $V$. We treat the Planck constant $\hbar$ as a formal parameter with $\text{gh}(\hbar) = 0$. A $\mathbb{Z}$-graded $k[[\hbar]]$-module $V = \bigoplus_{j \in \mathbb{Z}} V^j$ is topologically free if it is isomorphic to $V[[\hbar]]$.

⁴ The acronyms QFT and CFT stand wishfully for Q(uantum) F(ield) T(heory) and C(lassical) F(ield) T(heory), respectively.
The completed tensor product $V \otimes W$ of $V \otimes_{k[[\hbar]]} W$, where $V$ and $W$ are $k[[\hbar]]$-modules, is defined to be the inverse limit \[ \lim_{\longrightarrow} \left( V \otimes_{k[[\hbar]]} W \right)/\hbar^n \cdot \left( V \otimes_{k[[\hbar]]} W \right) \]. The corresponding completed symmetric product is denoted by $V \circledast W$. If $V \cong V[[\hbar]]$ and $W \cong W[[\hbar]]$ are topologically free, then both $V \otimes W$ and $V \circledast W$ are also topologically free: we have $V \otimes W \cong (V \otimes W)[[\hbar]]$ and $V \circledast W \cong (V \otimes W)[[\hbar]]$. Therefore we use the notation $T^n V[[\hbar]]$ for $V[[\hbar]] \circledast \ldots \circledast V[[\hbar]]$, and the notation $S^n V[[\hbar]]$ for $V[[\hbar]] \otimes \ldots \otimes V[[\hbar]]$, etc.

The space of $k$-linear maps between $\mathbb{Z}$-graded vector spaces $V$ and $W$ is denoted by $\text{Hom}(V,W) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}(V,W)^j$, where $\text{Hom}(V,W)^j$ is the space of $k$-linear maps increasing the ghost number by $j$. Let $\mathbf{p} = p_0 + \hbar p_1 + \hbar^2 p_2 + \ldots$ be a family parametrized by $\hbar$, with $p_n \in \text{Hom}(V,W)^j$. Then $\mathbf{p}$ determines a $k[[\hbar]]$-linear map, denoted by the same symbol, between the topologically-free modules $V[[\hbar]]$ and $W[[\hbar]]$ increasing the ghost number by $j$ by $\hbar$-adic continuity: for all $v = v^{(0)} + \hbar v^{(1)} + \ldots \in V[[\hbar]]$ we have

$$ p(v) = \sum_{n=0}^{\infty} \hbar^n \sum_{i=0}^{n} p^{(n-i)}(v^{(i)}), $$

and the converse is also true. In other words, a $k[[\hbar]]$-linear map $\mathbf{p} : V[[\hbar]] \rightarrow W[[\hbar]]$ is determined by its restriction to $V$. Accordingly, the space of all $k[[\hbar]]$-linear maps from $V[[\hbar]]$ to $W[[\hbar]]$ will be denoted by

$$ \text{Hom}(V,W)[[\hbar]] = \bigoplus_{j \in \mathbb{Z}} \text{Hom}(V,W)^j[[\hbar]]. $$

The projection $p_0$ of $\mathbf{p} \in \text{Hom}(V,W)[[\hbar]]$ to $\text{Hom}(V,W)$ is said to be the classical limit of $\mathbf{p}$, and we often use $\mathbf{p}$ as shorthand notation for $p_0$. Composition of $\mathbf{p} \in \text{Hom}(U,V)[[\hbar]]$ and $\gamma \in \text{Hom}(V,W)[[\hbar]]$ is defined in a similar fashion:

$$ \gamma \circ \mathbf{p} = \sum_{n=0}^{\infty} \hbar^n \sum_{i=0}^{n} \gamma^{(n-i)} \circ p_i \in \text{Hom}(U,W)^{j+k}[[\hbar]]. $$

We often use the notation $\mathbf{p}_n(v_1, \ldots, v_n)$ in place of either $\mathbf{p}_n(v_1 \otimes \ldots \otimes v_n)$ for $\mathbf{p}_n \in \text{Hom}(T^n V,W)[[\hbar]]$, or $\mathbf{p}_n(v_1 \otimes \ldots \otimes v_n)$ for $\mathbf{p}_n \in \text{Hom}(S^n V,W)[[\hbar]]$. It is obvious that
a family \( \underline{\beta} = \beta_1, \beta_2, \ldots \) of \( \beta_n \in \text{Hom}(S^n V, W) \) for \( n \geq 1 \) determine uniquely a \( \beta \in \text{Hom}(S(V), W) \) and vice versa such that \( \beta_n = \beta \circ \text{eb}_{S^n V} \), for all \( n \geq 1 \), where \( \text{eb}_{S^n V} : S^n V \rightarrow S(V) \) is the natural embedding. We use the notation \( \underline{\beta} \) and \( \beta \) interchangeably.

We shall denote an element of \( V[[h]] \) by a bold letter, i.e., \( v \in V[[h]] \), and an element of \( V \) by an italic letter, i.e., \( v \in V \), and will write the formal power series expansion of \( v \in V[[h]] \) as \( v = v^{(0)} + hv^{(1)} + h^2v^{(2)} + \cdots \). We shall often denote \( v^{(0)} \) by \( v \) and say that \( v \) is the classical limit of \( v \).

A partition of the ordered set \([n] = \{1, 2, \ldots, n\}\) is a decomposition \( p = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_{|p|} \) into pairwise disjoint and non-empty subsets \( B_i \) called blocks. We denote the number of blocks in the partition \( p \) by \( |p| \) and the size of a block \( B \) by \( |B| \). We shall use the strictly ordered representation for a partition. That is, blocks are ordered by the maximum element of each block and each block is ordered via the ordering induced from the natural numbers. We denote the set of all partitions of \([n]\) by \( P(n) \). For example, we have \( P(1) = \{1\} \), \( P(2) = \{1, 2\}, \{1\} \sqcup \{2\} \) and

\[
P(3) = \{1, 2, 3\}, \{1, 2\} \sqcup \{3\}, \{2\} \sqcup \{1, 3\}, \{1\} \sqcup \{2, 3\}, \{1\} \sqcup \{2\} \sqcup \{3\}.
\]

For \( k, k' \) in \([n]\), we use the notation \( k \sim_p k' \) if both \( k \) and \( k' \) belong to the same block in the partition \( p \) and the notation \( k \neq_p k' \) otherwise. For a given set of \( n \) homogeneous elements \( v_1, \ldots, v_n \in V[[h]] \), the Koszul sign \( \varepsilon(p) \) for a partition \( p = B_1 \sqcup \cdots \sqcup B_{|p|} \in P(n) \) means the Koszul sign \( \varepsilon(\sigma) \) associated with the permutation \( \sigma \in \text{Perm}_n \), i.e. the unique permutation \( \sigma \) that satisfies the equation

\[
v_{B_1} \otimes v_{B_2} \otimes \cdots \otimes v_{B_{|p|}} = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},
\]

where \( v_B = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_{|B|}} \) if \( B = \{j_1, j_2, \cdots, j_{|B|}\} \). For a family \( \underline{\beta} = \beta_1, \beta_2, \ldots \) of \( \beta_n \in \text{Hom}(S^n V, W) \), for all \( n \geq 1 \), and a block \( B = \{j_1, \ldots, j_r\} \) of a partition \( p \in P(n) \), the notation \( \beta_B(v_B) \) is taken to mean \( \beta_r(v_{j_1}, \ldots, v_{j_r}) \).

2.2. Binary QFT algebras

Let \( \mathcal{C} \) be a \( \mathbb{Z}\)-graded vector space over \( k \). A structure of a binary QFT algebra on \( \mathcal{C} \) will consist of the structure of a QFT complex and the structure of a unital super-commutative and associative algebra satisfying an \( h \)-compatibility condition.
Definition 2.1. The structure of a QFT complex on \( \mathcal{C} \) is a tuple \( (\mathcal{C}[\hbar], 1_{\mathcal{C}}, K) \), where \( 1_{\mathcal{C}} \in \mathcal{C}^0 \) and \( K = K(0) + \hbar K(1) + \hbar^2 K(2) + \ldots \in \text{Hom}(\mathcal{C}, \mathcal{C})[[\hbar]] \), with the properties that \( K_1 \mathcal{C} = K \circ K = 0 \) and \( 1_{\mathcal{C}} \neq K^{(0)} x \) for all \( x \in \mathcal{C} \).

We remark that the conditions \( K_1 \mathcal{C} = K \circ K = 0 \) are equivalent to the set of conditions that \( K^{(n)} 1_{\mathcal{C}} = \sum_{j=0}^{n} K^{(j)} \circ K^{(n-j)} = 0 \) for all \( n \geq 0 \), and the condition that \( 1_{\mathcal{C}} \neq K^{(0)} x \) for all \( x \in \mathcal{C} \) implies that \( 1_{\mathcal{C}} \neq K x \) for all \( x \in \mathcal{C}[\hbar] \) — hence the \( K \)-cohomology class of \( 1_{\mathcal{C}} \) is non-trivial. We shall usually denote \( K^{(0)} \) by \( K \), so that \( K \in \text{Hom}(\mathcal{C}, \mathcal{C}) \) and \( K \circ K = K_1 \mathcal{C} = 0 \). Therefore, the tuple \( (\mathcal{C}, 1_{\mathcal{C}}, K) \) is a pointed cochain complex over \( k \) such that the \( K \)-cohomology class of the cocycle \( 1_{\mathcal{C}} \) is non-trivial. We call \( (\mathcal{C}, 1_{\mathcal{C}}, K) \) the classical limit of the QFT complex \( (\mathcal{C}[\hbar], 1_{\mathcal{C}}, K) \).

In the geometrical picture of the BV quantization scheme, the classical limit \( K \) of \( K \) corresponds to an odd nilpotent vector field on the space of all fields and anti-fields whose vanishing locus in the space of all classical fields is the solution space of the classical (on-shell) equations of motion. Then the \( K \)-cohomology is the space of functions on this on-shell motion space modulo classical symmetry (the space of classical observables modulo classical equivalence). This motivates the following definition.

Definition 2.2. A QFT complex is on-shell if the differential vanishes in the classical limit.

A unital \( \mathbb{Z} \)-graded commutative associative algebra on \( \mathcal{C} \) is a tuple \( (\mathcal{C}, 1_{\mathcal{C}}, \cdot) \), where \( 1_{\mathcal{C}} \in \mathcal{C}^0 \) and \( \cdot \) is a \( k \)-bilinear product on \( \mathcal{C} \), so that we have \( gh(x \cdot y) = gh(x)+gh(y) \), \( x \cdot 1_{\mathcal{C}} = x \), \( x \cdot y = (-1)^{|x||y|} y \cdot x \) and \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \) for all homogeneous elements \( x, y, z \in \mathcal{C} \). Then, we have the canonical structure \( (\mathcal{C}[\hbar], 1_{\mathcal{C}}, \cdot) \) of a unital \( \mathbb{Z} \)-graded commutative associative algebra over \( k[[\hbar]] \) on the topologically-free module \( \mathcal{C}[\hbar] \) with the product \( x \cdot y := \sum_{n=0}^{\infty} \hbar^n \sum_{i=0}^{n} x^{(i)} \cdot y^{(n-i)} \) for all \( x = x^{(0)} + \hbar x^{(1)} + \ldots \in \mathcal{C}[\hbar] \) and \( y = y^{(0)} + \hbar y^{(1)} + \ldots \in \mathcal{C}[\hbar] \).

Definition 2.3. A QFT complex \( (\mathcal{C}[\hbar], 1_{\mathcal{C}}, \cdot, K) \) structure on \( \mathcal{C} \) is \( \hbar \)-compatible with a unital \( \mathbb{Z} \)-graded commutative associative algebra structure \( (\mathcal{C}, 1_{\mathcal{C}}, \cdot) \) if the family \( \mathcal{L} = \ell_1, \ell_2, \ldots \) defined recursively for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[\hbar] \) via the equation
\[
K(x_1 \cdots x_n) = \sum_{p \in P(n), \sum_{|B_i|=n-|p|+1} \cdot \ell_1 \cdot \ell_2 \cdot \cdots \cdot \ell_{|B_i|} \cdot x_1 \cdots x_n}, \tag{2.1}
\]
has the property that \( \ell_n \in \text{Hom}(\mathcal{S}^n \mathcal{C}, \mathcal{C})[[\hbar]] \) for all \( n \geq 1 \) (a priori there should be negative powers of \( \hbar \)). Then,
we call the tuple \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} := \left( \mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \cdot, K \right) \) a binary QFT algebra structure on \( \mathcal{C} \), and

we call the tuple \( \mathfrak{R}(\mathcal{C}[[\hbar]]_{\text{BQFTA}}) := \left( \mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \ell \right) \) the quantum descendant of the binary QFT algebra \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \).

**Remark 2.1.** Note that the recursive formula eq. (2.1) for the family \( \ell \) can be rewritten as follows:

\[
(-\hbar)^{n-1} \ell_n(x_1, \ldots, x_n) = K(x_1 \cdot \ldots \cdot x_n)
- \sum_{p \in P(n)} (-\hbar)^{n-|p|} \varepsilon(p) Jx_{B_1} \cdot \ldots \cdot Jx_{B_{|p|-1}} \cdot \ell(x_{B_1}) \cdot x_{B_{|p|-1}} \cdot \ldots \cdot x_{B_{|p|}},
\]

where the right-hand-side of the above depends only on \( \ell_1, \ldots, \ell_{n-1} \). Therefore, the formula determine the family \( \ell \) uniquely. Note also that the condition \( |B_i| = n-|p|+1 \) implies that the blocks \( B_1, \ldots, B_{|p|-1}, B_{|p|+1}, \ldots, B_{|p|} \) are singletons. For example, we have \( \ell_1 = K \) and

\[
(-\hbar)\ell_2(x_1, x_2) = K(x_1 \cdot x_2) - Kx_1 \cdot x_2 - Jx_1 \cdot Kx_2,
\]

\[
(-\hbar)^2\ell_3(x_1, x_2, x_3) = K(x_1 \cdot x_2 \cdot x_3) - Kx_1 \cdot x_2 \cdot x_3 - Jx_1 \cdot Kx_2 \cdot x_3 - Jx_1 \cdot x_2 \cdot Kx_3
- (-\hbar) \left( \ell_2(x_1, x_2) \cdot x_3 + Jx_1 \cdot \ell_2(x_2, x_3) + (-1)^{|x_1||x_2|} Jx_2 \cdot \ell_2(x_1, x_3) \right).
\]

Again, without imposing any compatibility, the definition of \( \ell \) would only imply that \( (-\hbar)^{n-1} \ell_n \) was in \( \text{Hom}(S^n \mathcal{C}, \mathcal{C})[[\hbar]] \). The \( \hbar \)-compatibility condition requires that in fact \( \ell_n \) is in \( \text{Hom}(S^n \mathcal{C}, \mathcal{C})[[\hbar]] \) for all \( n \geq 1 \), imposing a set of non-trivial relations between the product \( \cdot \) and the differential \( K \).

**Lemma 2.1.** The quantum descendant \( \mathfrak{R}(\mathcal{C}[[\hbar]]_{\text{BQFTA}}) = \left( \mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \ell \right) \) of a binary QFT algebra \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \) is a topologically-free unital sL_\infty-algebra over \( k[[\hbar]] \), i.e., for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \), we have \( \ell_n \in \text{Hom}(S^n \mathcal{C}, \mathcal{C})[[\hbar]] \) and

\[
\ell_n(x_1, \ldots, x_{n-1}, 1_{\mathcal{C}}) = 0,
\]

\[
\sum_{|p| \in P(n)} \varepsilon(p) \ell_p(Jx_{B_1}, \ldots, Jx_{B_{|p|-1}}, \ell(x_{B_1}), x_{B_{|p|-1}}, \ldots, x_{B_{|p|}}) = 0.
\]

**Proof.** By definition, we have \( \ell_n \in \text{Hom}(S^n \mathcal{C}, \mathcal{C})[[\hbar]] \) for all \( n \geq 1 \). Since \( \ell_1 = K \) and \( K1_{\mathcal{C}} = 0 \), we have \( \ell_1(1_{\mathcal{C}}) = 0 \). It can be checked that \( \ell_n(x_1, \ldots, x_{n-1}, 1_{\mathcal{C}}) = 0 \)
for all \( n \geq 1 \) and \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \) by an easy induction. It remains to show the second set of relations in eq. (2.2).

From the family \( \ell = \ell_1, \ell_2, \ldots \) of \( \ell_n \in \text{Hom}(S^n, \mathcal{C})[[\hbar]] \), we define a \( k[[\hbar]] \)-linear operator \( \delta_{\ell} : \overline{S}(\mathcal{C})[[\hbar]] \to \overline{S}(\mathcal{C})[[\hbar]] \) with \( gh(\delta_{\ell}) = 1 \) by defining, for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \),

\[
\delta_{\ell}(x_1 \otimes \cdots \otimes x_n) := \sum_{p \in P(n)} (-\hbar)^{n-|p|} \epsilon(p) Jx_{B_1} \otimes \cdots \otimes Jx_{B_{i-1}} \otimes \ell(x_{B_i}) \otimes x_{B_{i+1}} \otimes \cdots \otimes x_{B_{|p|}}.
\]  

(2.3)

It is straightforward to check that for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \),

\[
\text{pr}_{\mathcal{C}[[\hbar]]} \circ \delta_{\ell} \circ \delta_{\ell}(x_1 \otimes \cdots \otimes x_n) = (-\hbar)^{n-1} \sum_{|p| \in P(n)} \epsilon(p) \ell_{|p|} \left( Jx_{B_1}, \ldots, Jx_{B_{i-1}}, \ell(x_{B_i}), x_{B_{i+1}}, \ldots, x_{B_{|p|}} \right).
\]  

(2.4)

Therefore, all that remains is to establish that \( \text{pr}_{\mathcal{C}[[\hbar]]} \circ \delta_{\ell} \circ \delta_{\ell} = 0 \).

From the \( \mathbb{Z} \)-graded commutative and associative product \( \cdot \) of \( \mathcal{C}[[\hbar]] \), we define an operator \( \pi \in \text{Hom}(\overline{S}(\mathcal{C}), \mathcal{C})^0[[\hbar]] \) for all \( n \geq 1 \) and \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \) as

\[
\pi(x_1 \otimes \cdots \otimes x_n) := x_1 \cdot \cdots \cdot x_n.
\]  

(2.5)

Let \( \pi_n := \pi \circ \text{eb}_{S^n \mathcal{C}[[\hbar]]} \), for all \( n \geq 1 \). We note that \( \pi_1 = \mathbb{1}\mathcal{C}[[\hbar]] \), the identity map on \( \mathcal{C}[[\hbar]] \).

Now the set of relations in eq. (2.1) defining the family \( \ell \) can be rewritten as follows:

\[
K \circ \pi = \pi \circ \delta_{\ell},
\]  

(2.6)

which implies that \( \pi \circ \delta_{\ell} \circ \delta_{\ell} = 0 \), since \( (\pi \circ \delta_{\ell}) \circ \delta_{\ell} = K \circ (\pi \circ \delta_{\ell}) = (K \circ K) \circ \pi \) and \( K \circ K = 0 \). From \( \pi_1 = \mathbb{1}_{\mathcal{C}[[\hbar]]} \), we conclude that \( \text{pr}_{\mathcal{C}[[\hbar]]} \circ \delta_{\ell} \circ \delta_{\ell} = 0 \).

Remark 2.2. Consider the reduced symmetric coalgebra \( \overline{S}^0(\mathcal{C}) = (\overline{S}(\mathcal{C})[[\hbar]], \overline{\cdot}) \) co-generated by \( \mathcal{C} \) (see Appendix [A.3]) The k-linear coproduct

\[
\overline{\cdot} : \overline{S}(\mathcal{C}) \to \overline{S}(\mathcal{C}) \otimes \overline{S}(\mathcal{C})
\]

uniquely induces a \( k[[\hbar]] \)-linear coproduct, denoted by the same symbol,

\[
\overline{\cdot} : \overline{S}(\mathcal{C})[[\hbar]] \to \overline{S}(\mathcal{C})[[\hbar]] \otimes \overline{S}(\mathcal{C})[[\hbar]]
\]
by \( \hbar \)-adic continuity so that \( \overline{\mathcal{S}(\mathcal{C})[[\hbar]]} = (\mathcal{S}(\mathcal{C})[[\hbar]], \Delta) \) is a (topologically-free) \( \mathbb{Z} \)-graded cocommutative coalgebra over \( k[[\hbar]] \).

Define \( \tilde{\ell} \in \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C})[[\hbar]] \) as \( \tilde{\ell}_n := (-\hbar)^{n-1} \ell_n \), where \( \tilde{\ell} \circ \text{eb}_{S^2 \mathcal{C}}[[\hbar]] \). Consider the unique extension \( \mathcal{D}(\tilde{\ell}) \) of \( \tilde{\ell} \) as a coderivation on \( \overline{\mathcal{S}(\mathcal{C})[[\hbar]]} \): we have, for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \),

\[
\mathcal{D}(\tilde{\ell})(x_1 \otimes \cdots \otimes x_n) = \sum_{p \in P(n)} \varepsilon(p) Jx_{B_1} \otimes \cdots \otimes Jx_{B_{n-1}} \otimes \tilde{\ell}(x_{B_j}) \otimes x_{B_{j+1}} \otimes \cdots \otimes x_{B_{n-j}},
\]

(See Lemma [A.5] in Appendix A.3.) We note that \( \partial \tilde{\ell} = \mathcal{D}(\tilde{\ell}) \). Also note that the condition \( \text{pr}_{\mathcal{C}[[\hbar]]} \circ \mathcal{D}(\tilde{\ell}) \circ \mathcal{D}(\tilde{\ell}) = 0 \) actually implies that \( \mathcal{D}(\tilde{\ell}) \circ \mathcal{D}(\tilde{\ell}) = 0 \). Therefore, we have \( \delta \tilde{\ell} = \delta \tilde{\ell} = 0 \).

**Example 2.1.** The ground field \( k \) has the obvious structure \( (k[[\hbar]], 1, 0) \) of a binary QFT algebra, denoted by \( k[[\hbar]] \), with the zero differential 0. The quantum descendant of this binary QFT algebra is the unital \( sL_{\infty} \)-algebra \( (k[[\hbar]], 1, 0) \), i.e., the zero \( sL_{\infty} \)-structure \( \mathcal{Q} \) on \( k[[\hbar]] \).

**Definition 2.4.** A binary QFT algebra \( (\mathcal{C}[[\hbar]], 1, \cdot, K) \) is a BV-QFT algebra if the quantum descendant \( (\mathcal{C}[[\hbar]], 1, \cdot, \ell) \) is a unital sDGLA, i.e., \( \ell_n = 0 \) for all \( n \geq 3 \), and \( \ell_2 \) does not depend on \( \hbar \), i.e., \( \ell_2 = \ell_2^{(0)} = (.,_.)_{BV} \in \text{Hom}(S^2 \mathcal{C}, \mathcal{C}) \).

Fix a binary QFT algebra \( (\mathcal{C}[[\hbar]], 1, \cdot, K) \) and let \( (\mathcal{C}[[\hbar]], 1, \cdot, \ell) \) be its quantum descendant unital \( sL_{\infty} \)-algebra. It is straightforward to check the following lemmas:

**Lemma 2.2.** For any \( \gamma \in \mathcal{C}[[\hbar]] \) and nilpotent \( \varnothing \in \mathcal{C}^0[[\hbar]] \), we have

\[
\mathcal{K}(\gamma, e^{-\frac{1}{\hbar} \varnothing}) = \left( \mathcal{K} + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \ell_n(\varnothing, \varnothing, \gamma) \right) \cdot e^{-\frac{1}{\hbar} \varnothing} - \frac{1}{\hbar} \left( \mathcal{K} \cdot e^{-\frac{1}{\hbar} \varnothing} \right).
\]

where \( e^{-\frac{1}{\hbar} \varnothing} := 1_{\mathcal{C}} - \frac{1}{\hbar} \varnothing + \frac{1}{2! \hbar^2} \varnothing \cdot \varnothing - \frac{1}{3! \hbar^3} \varnothing \cdot \varnothing \cdot \varnothing + \cdots \in \mathcal{C}[[\hbar]] \).

**Lemma 2.3.** We have \( \ell_1 = \mathcal{K} \) and, for all \( n \geq 2 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \),

\[
-\hbar \ell_n(x_1, \ldots, x_n) = \ell_{n-1}(x_1, \ldots, x_{n-2}, x_{n-1} \cdot x_n) - \ell_{n-1}(x_1, \ldots, x_{n-2}, x_{n-1}) \cdot x_n - (-1)^{|x_{n-1}|(|x_1| + \cdots + |x_{n-2}|)} \mathcal{J} x_{n-1} \cdot \ell_{n-1}(x_1, \ldots, x_{n-2}, x_n),
\]

so that the failure of \( \ell_n \) being a derivation of the product is divisible by \( \hbar \).
2.3. Morphisms of binary QFT algebras

A morphism of binary QFT algebras will be defined as a morphism of the underlying QFT complexes satisfying a $\hbar$-compatibility condition.

**Definition 2.5.** Let $\left( \mathcal{C}[[\hbar]], 1_{\psi}, K \right)$ and $\left( \mathcal{C}'[[\hbar]], 1_{\psi'}, K' \right)$ be QFT complexes. A morphism between them is a map $f = f^{(0)} + \hbar f^{(1)} + \hbar^2 f^{(2)} + \ldots \in \text{Hom}(\mathcal{C}, \mathcal{C}')[[\hbar]]$ satisfying the conditions $f(1_{\psi}) = 1_{\psi'}$ and $K' \circ f = f \circ K$. A morphism of QFT complexes is a quasi-isomorphism if it induces an isomorphism of $k[[\hbar]]$-modules on cohomology.

**Remark 2.3.** A QFT complex is a pointed cochain complex on a topologically-free $k[[\hbar]]$-module. The cohomology of a QFT complex is a $k[[\hbar]]$-module but not necessarily a topologically-free $k[[\hbar]]$-module. The classical limit $f^{(0)} \in \text{Hom}(\mathcal{C}, \mathcal{C}')[[\hbar]]$ of a morphism $f$ of QFT complexes is a pointed cochain map between the pointed cochain complexes $\left( \mathcal{C}, 1_{\psi}, K \right)$ and $\left( \mathcal{C}', 1_{\psi'}, K' \right)$, i.e., $f(1_{\psi}) = 1_{\psi'}$ and $K' \circ f^{(0)} = f^{(0)} \circ K$, and $f^{(0)}$ is a quasi-isomorphism whenever $f$ is a quasi-isomorphism.

Consider binary QFT algebras $\mathcal{C}[[\hbar]]_{BQFTA} = (\mathcal{C}[[\hbar]], 1_{\psi}, \cdot, K)$ and $\mathcal{C}'[[\hbar]]_{BQFTA} = (\mathcal{C}'[[\hbar]], 1_{\psi'}, \cdot, K')$.

**Definition 2.6.** A morphism of QFT complexes $f : (\mathcal{C}[[\hbar]], 1_{\psi}, K) \to (\mathcal{C}'[[\hbar]], 1_{\psi'}, K')$ is $\hbar$-compatible (with the products) up to homotopy if the family $\psi^f = \psi^f_1, \psi^f_2, \ldots$ defined recursively for all $n \geq 1$ and homogeneous elements by the equation $x_1, \ldots, x_n \in \mathcal{C}'[[\hbar]]$,

$$f(x_1 \cdot \ldots \cdot x_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|} \epsilon(p) \psi^f(x_{B_1}) \cdot \ldots \cdot \psi^f(x_{B_{|p|}}),$$

has the property that $\psi^f_n \in \text{Hom}(S^n \mathcal{C}', \mathcal{C}')[[\hbar]]$ for all $n \geq 1$ (a priori it should have a negative power of $[[\hbar]]$). Then,

- we call $f$ a morphism of binary QFT algebras from $\mathcal{C}[[\hbar]]_{BQFTA}$ to $\mathcal{C}'[[\hbar]]_{BQFTA}$, and
- we call the family $\psi^f$ the quantum descendant of $(f)$, and denote it by $\mathcal{R}(f) = \psi^f$.

**Remark 2.4.** We shall usually denote $\psi^f$ by $\psi$ when the context is clear. The recursive formula for $\psi = \mathcal{R}(f)$ in the above definition can be rewritten as follows:

$$(-\hbar)^{n-1} \psi_n(x_1, \ldots, x_n) = f(x_1 \cdot \ldots \cdot x_n) - \sum_{p \in P(n)} (-\hbar)^{n-|p|} \epsilon(p) \psi(x_{B_1}) \cdot \ldots \cdot \psi(x_{B_{|p|}}),$$
where the right hand side of the equation above depends only on \( \psi_1, \ldots, \psi_{n-1} \). Therefore, the formula determines the family \( \psi \) uniquely. For example, we have \( \psi_1 = f \) and

\[
(-h)\psi_2(x_1, x_2) = \psi_1(x_1 \cdot x_2) - \psi_1(x_1) \cdot \psi_1(x_2),
\]

\[
(-h)^2\psi_3(x_1, x_2, x_3) = \psi_1(x_1 \cdot x_2 \cdot x_3) - \psi_1(x_1) \cdot \psi_1(x_2) \cdot \psi_1(x_3)
- (-h)(\psi_1(x_1) \cdot \psi_2(x_2, x_3) + (-1)^{|x_1||x_2|}\psi_1(x_2) \cdot \psi_2(x_1, x_3)
+ \psi_2(x_1, x_2) \cdot \psi_1(x_3))).
\]

Again, it is only a priori true that \((-h)^{n-1}\psi_n \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')^0[[h]]\), while the \( h \)-compatibility condition imposes the further demand that \( \psi_n \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')^0[[h]]\) for all \( n \geq 1 \) imposing a set of non-trivial restrictions on \( f \).

Let \( (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell') = \mathcal{R}(\mathcal{C}', \mathcal{C}', \mathcal{C}', \mathcal{C}', \mathcal{C}', \mathcal{C}')_{\text{BQFTA}} \) and \( (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell') = \mathcal{R}(\mathcal{C}', \mathcal{C}', \mathcal{C}', \mathcal{C}', \mathcal{C}', \mathcal{C}')_{\text{BQFTA}} \) be the quantum descendants, which are unital \( sL_{\infty} \)-algebras by Lemma 2.7.

**Lemma 2.4.** The quantum descendant \( \Psi = \mathcal{R}(f) \) of a binary QFT algebra morphism \( \mathcal{C}'[[h]]_{\text{BQFTA}} \xrightarrow{f} \mathcal{C}'[[h]]_{\text{BQFTA}} \) is a morphism of the descendant unital \( sL_{\infty} \)-algebras \( (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell') \xrightarrow{\Psi} (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell') \). That is, for all \( n \geq 1 \) and homogeneous elements \( x_1, \ldots, x_n \in \mathcal{C}'[[h]] \), we have \( \Psi_n \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')^0[[h]] \) and

\[
\Psi_n(x_1, \ldots, x_{n-1}, 1_{\mathcal{C}'}) - 1_{\mathcal{C}'} \cdot \delta_{n,1} = 0,
\]

\[
\sum_{|p| \in P(n)} \mathcal{e}(p)\ell_{|p|} \left( \Psi(x_{B_1}), \ldots, \Psi(x_{B_{|p|}}) \right) - \sum_{|p| \in P(n)} \mathcal{e}(p)\mathcal{e}|p| \left( Jx_{B_1}, \ldots, Jx_{B_{n-|p|}}, \ell(x_{B_1}), x_{B_{n-|p|}+1}, \ldots, x_{B_{|p|}} \right) = 0.
\]

**Proof.** By definition, \( \Psi_n \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')^0[[h]] \) for all \( n \geq 1 \). We have \( \Psi_1(1_{\mathcal{C}'}) = 1_{\mathcal{C}'} \) since \( \psi_1 = f \) and \( f(1_{\mathcal{C}'}) = 1_{\mathcal{C}'} \). It can be checked that \( \Psi_n(x_1, \ldots, x_{n-1}, 1_{\mathcal{C}'}) = 0 \) for all \( n \geq 2 \) by a straightforward induction. It remains to show the second set of relations in eq. \( (2.7) \).

From the family \( \Psi \), we define a \( \mathbb{k}[[h]] \)-linear map \( \Psi_\ell \in \text{Hom}(S(\mathcal{C}'), S(\mathcal{C}'))^0[[h]] \) such that, for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}'[[h]] \),

\[
\Psi_\ell(x_1 \otimes \cdots \otimes x_n) := \sum_{p \in P(n)} (-h)^{-|p|}\mathcal{e}(p)\Psi(x_{B_1}) \otimes \cdots \otimes \Psi(x_{B_{|p|}}).
\]
Therefore, the equality in eq. (2.12) reduces to 

\[
\text{pr}_{\mathcal{E}'}[[h]] \circ (\delta_{E'} \circ \Psi_{\psi} - \Psi_{\psi} \circ \delta_{l}) (x_1 \otimes \ldots \otimes x_n) = \left(\mathcal{S}(\mathcal{E})[[h]] \to \mathcal{S}('\mathcal{E}'')[[h]]\right) \text{,}
\]

Comparing the above with the desired relations in eq. (2.2), all that remains is to establish that \(\text{pr}_{\mathcal{E}'}[[h]] \circ \delta_{E'} \circ \Psi_{\psi} - \Psi_{\psi} \circ \delta_{l} \circ \Psi_{\psi} = 0\). It is straightforward to see that, for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_n \in \mathcal{E}[[h]]\),

\[
\text{pr}_{\mathcal{E}'}[[h]] \circ \left(\delta_{E'} \circ \Psi_{\psi} - \Psi_{\psi} \circ \delta_{l}\right) (x_1 \otimes \ldots \otimes x_n) = (-h)^{n-1} \sum_{|p| \in P(n)} \epsilon(p) \ell'_{|\pi|} \left(\psi(x_{B_1}), \ldots, \psi(x_{B_{|p|}})\right) \left(J x_{B_1}, \ldots, \ell (x_{B_1}), x_{B_{i+1}}, \ldots, x_{B_{|p|}}\right) - (-h)^{n-1} \sum_{|p| \in P(n)} \epsilon(p) \psi_{|\pi|} \left(J x_{B_1}, \ldots, \ell (x_{B_1}), x_{B_{i+1}}, \ldots, x_{B_{|p|}}\right).
\]

Define \(\pi \in \text{Hom} \left(\mathcal{S}(\mathcal{E}''), '\mathcal{E}'\right) \circ [[h]]\) and \(\pi' \in \text{Hom} \left(\mathcal{S}(\mathcal{E}''), '\mathcal{E}'\right) \circ [[h]]\) as follows:

\[
\pi \left(x_1 \otimes \ldots \otimes x_n\right) := x_1 \cdot \ldots \cdot x_n \quad \text{for all } n \geq 1 \text{ and } x_1, \ldots, x_n \in \mathcal{E}[[h]],
\]

\[
\pi' \left(x_1' \otimes \ldots \otimes x_n'\right) := x_1' \cdot \ldots \cdot x_n' \quad \text{for all } n \geq 1 \text{ and } x_1', \ldots, x_n' \in '\mathcal{E}\right) [[h]]\).
\]

Then, from the definition of the quantum descendant algebra, we have the following relations: (See eq. (2.6) and Remark 2.2)

\[
K \circ \pi = \pi \circ \delta_{l}, \quad K' \circ \pi' = \pi' \circ \delta_{l'}, \quad \delta_{l} \circ \delta_{l} = \delta_{l'} \circ \delta_{l'} = 0. \tag{2.10}
\]

Now we note that the system of equations for the quantum descendant \(\Psi = \mathfrak{R}(f)\) in Definition 2.6 can be rewritten as follows:

\[
f \circ \pi = \pi' \circ \Psi_{\psi}. \tag{2.11}
\]

Applying \(K'\) to the above, we obtain that

\[
K' \circ f \circ \pi = K' \circ \pi' \circ \Psi_{\psi} \tag{2.12}
\]

whose left-hand side is

\[
( K' \circ f ) \circ \pi = f \circ ( K \circ \pi ) = ( f \circ \pi ) \circ \delta_{l} = \pi' \circ \Psi_{\psi} \circ \delta_{l},
\]

and whose right-hand side is

\[
( K' \circ \pi' ) \circ \Psi_{\psi} - K' \circ \eta \circ \delta_{l} = \pi' \circ \delta_{l'} \circ \Psi_{\psi}.
\]

Therefore, the equality in eq. (2.12) reduces to \(\pi' \circ ( \delta_{l'} \circ \Psi_{\psi} - \Psi_{\psi} \circ \delta_{l} ) = 0\). From \(\pi' \circ e_{\mathcal{E}'}[[h]] = I_{\mathcal{E}'}[[h]]\), we conclude that \(\text{pr}_{\mathcal{E}'}[[h]] \circ (\delta_{l'} \circ \Psi_{\psi} - \Psi_{\psi} \circ \delta_{l}) = 0\).
Remark 2.5. Consider the topologically-free reduced symmetric coalgebras $S^{0\ell}(\mathcal{C})[[h]]$ and $S^{0\ell}(\mathcal{C}')[[[h]]]$. Define $\tilde{\psi} \in \text{Hom}(S(\mathcal{C}'), \mathcal{C}')_0[[h]]$ as $\tilde{\psi}_n := (-\hbar)^{n-1}\psi_n$, where $\tilde{\psi} \circ \text{eb}_{S^{0\ell}(\mathcal{C}')}[[h]]$. Consider the unique extension $\tilde{\delta}(\tilde{\psi})$ of $\tilde{\psi}$ as a coalgebra map from $S^{0\ell}(\mathcal{C}')[[h]]$ to $S^{0\ell}(\mathcal{C}'')[[h]]$: we have, for all $n \geq 1$ and homogeneous $x_1, \ldots, x_n \in \mathcal{C}'[[h]]$,

$$\tilde{\delta}(\tilde{\psi})(x_1 \odot \ldots \odot x_n) = \sum_{\rho \in \mathfrak{P}(n)} \rho(\tilde{\psi})(x_{i_1}) \odot \ldots \odot \tilde{\psi}(x_{i_\rho}),$$

and $\text{pr}_{\mathcal{C}'[[h]]} \circ \tilde{\delta}(\tilde{\psi}) = \tilde{\psi}$. (See Lemma A.3 in Appendix A.3) We note that $\Psi_{\psi_1} = \tilde{\delta}(\tilde{\psi})$. Also note that the condition $\text{pr}_{\mathcal{C}'[[h]]} \circ (\delta(\ell') \circ \tilde{\delta}(\tilde{\psi}) - \tilde{\delta}(\tilde{\psi}) \circ \delta(\ell)) = 0$ actually implies that $\delta(\ell') \circ \tilde{\delta}(\tilde{\psi}) - \tilde{\delta}(\tilde{\psi}) \circ \delta(\ell) = 0$. Therefore, we have $\delta(\ell') \circ \Psi_{\psi_1} = \Psi_{\psi_1} \circ \delta(\ell)$.

Definition 2.7. A morphism $\circ$ of binary QFT algebras is a quasi-isomorphism if $\circ$ is a quasi-isomorphism as a morphism of underlying QFT complexes.

Recall that a unital $sL_{\infty}$-morphism $\psi = \psi_1, \psi_2, \ldots$ is a quasi-isomorphism if $\psi_1$ is a quasi-isomorphism of the underlying pointed cochain complexes and that $\psi_1 = \circ$ if $\psi = \mathfrak{R}(\circ)$ is the quantum descendant of a binary QFT algebra morphism $\circ$. Therefore, the quantum descendant of a binary QFT algebra quasi-isomorphism is a unital $sL_{\infty}$-quasi-isomorphism.

Theorem 2.1. The composition $\circ_{\circ}$ of consecutive morphisms of binary QFT algebras

$$\mathcal{C}[[h]]_{\text{BQFTA}} \xrightarrow{\circ} \mathcal{C}'[[h]]_{\text{BQFTA}} \xrightarrow{\circ} \mathcal{C}''[[h]]_{\text{BQFTA}}$$

as pointed cochain maps is a morphism of binary QFT algebras $\mathcal{C}[[h]]_{\text{BQFTA}} \xrightarrow{\circ_{\circ}} \mathcal{C}'[[h]]_{\text{BQFTA}}$. Moreover,

- equipped with morphisms of binary QFT algebras and this composition, binary QFT algebras over $k$ form a category $\text{BQFTA}(k)$, and
- the assignment to each binary QFT algebra its quantum descendant algebra and to each morphism of binary QFT algebras its quantum descendant morphism is a functor

$$\mathfrak{R} : \text{BQFTA}(k) \rightarrow UsL_{\infty}(k[[h]])$$

from the category $\text{BQFTA}(k)$ of binary QFT algebras to the category $UsL_{\infty}(k[[h]])$ of unital $sL_{\infty}$-algebras.

Proof. It is obvious that $f'' \circ f$ is a pointed cochain map from $\mathcal{C}[[h]]$ to $\mathcal{C}'[[h]]$. It remains to show that $f'' \circ f$ also satisfies the $k$-compatibility condition and that the
quantum descendant $\mathcal{R}(f' \circ f)$ of $f' \circ f$ is the composition $\mathcal{R}(f') \circ \mathcal{R}(f)$ of the quantum descendants $\mathcal{R}(f)$ of $f$ and $\mathcal{R}(f)$ of $f'$ as morphisms of unital $sL_\infty$-algebras.

Let $(\mathcal{C}[[h]], 1_\mathcal{C}, \mathcal{L})$, $(\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \mathcal{L}')$ and $(\mathcal{C}''[[h]], 1_{\mathcal{C}''}, \mathcal{L}'')$ be the quantum descendants of $\mathcal{C}[[h]]_{\text{BQFTA}}$, $\mathcal{C}'[[h]]_{\text{BQFTA}}$ and $\mathcal{C}''[[h]]_{\text{BQFTA}}$, respectively. By hypothesis, we have the following unital $sL_\infty$-morphisms between quantum descendant algebras:

$\psi^f := \mathcal{R}(f) : (\mathcal{C}[[h]], 1_\mathcal{C}, \mathcal{L}) \longrightarrow (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \mathcal{L}')$,
$\psi^{f'} := \mathcal{R}(f') : (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \mathcal{L}') \longrightarrow (\mathcal{C}''[[h]], 1_{\mathcal{C}''}, \mathcal{L}'')$,

where

$f \circ \pi = \pi' \circ \Psi_{\psi^f}, \quad f' \circ \pi' = \pi'' \circ \Psi_{\psi^{f'}}. \tag{2.13}$

Applying $f'$ to the first identity above and using the second identity, we have $f' \circ f \circ \pi = f' \circ \pi' \circ \Psi_{\psi^f} = \pi'' \circ \Psi_{\psi^{f'}} \circ \Psi_{\psi^f}$. Therefore we obtain

$$(f' \circ f) \circ \pi = \pi'' \circ \Psi_{\psi^{f'}} \circ \Psi_{\psi^f}. \tag{2.14}$$

Recall that the composition $\psi^{f'} \cdot \psi^f$ of the unital $sL_\infty$-morphisms $\psi^{f'}$ and $\psi^f$ is defined for all $n \geq 1$ and homogeneous $x_1, \ldots, x_n \in \mathcal{C}[[h]]$, via

$$(\psi^{f'} \cdot \psi^f)(x_1 \otimes \cdots \otimes x_n) \equiv (\psi^{f'} \cdot \psi^f)_n(x_1, \ldots, x_n) = \sum_{p \in P(n)} e(p) \psi^{f'}_{|p|}(\psi^f(x_{B_1}), \ldots, \psi(x_{B_n})), \quad f' \circ \pi' = \pi'' \circ \Psi_{\psi^{f'}} \circ \Psi_{\psi^f}.$$
Therefore, eq. (2.14) is equivalent to the identity \((f' \circ f) \circ \pi = \pi' \circ \Psi_{\theta'} \cdot \Psi_f\), which implies that \(f' \circ f : \mathcal{C}[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) is a morphism of binary QFT algebras whose quantum descendant \(\Psi_{(f' \circ f)}\) is \(\Psi' \cdot \Psi_f\), i.e.,

\[
\mathfrak{A}(f' \circ f) = \mathfrak{A}(f') \cdot \mathfrak{A}(f). \tag{2.15}
\]

It is trivial that \(\mathbb{I}_{\mathcal{C}[[h]]} : \mathcal{C}[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) is the identity morphism for every binary QFT algebra \(\mathcal{C}'[[h]]_{BQFTA}\), and that its quantum descendant is the identity morphism on the quantum descendant unital \(sL_{\infty}\)-algebra.

A morphism of binary QFT algebras has some easily checkable but illuminating properties summarized by the following two lemmas.

Let \(\mathcal{C}'[[h]]_{BQFTA} \xrightarrow{\Phi} \mathcal{C}'[[h]]_{BQFTA}\) be a morphism of binary QFT algebras whose quantum descendant is \((\mathcal{C}'[[h]]), 1_{\mathcal{C}'}, \ell\) \(\xrightarrow{\Phi} (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell')\). Then,

**Lemma 2.5.** For any \(\gamma \in \mathcal{C}'[[h]]\) and nilpotent \(\theta \in \mathcal{C}'[[h]]\), there are \(\gamma' \in \mathcal{C}'[[h]]\) and \(\theta' \in \mathcal{C}'[[h]]\) given by

\[
\gamma' = \psi_1(\gamma) + \sum_{n=1}^{\infty} \frac{1}{n!} \psi_{n+1}(\gamma, \theta, \ldots, \theta), \quad \theta' = \sum_{n=1}^{\infty} \frac{1}{n!} \psi_{n}(\theta, \ldots, \theta)
\]

such that \(\Phi(\gamma' \cdot e^{-\frac{1}{h} \theta'}) = \gamma' \cdot e^{-\frac{1}{h} \theta'}\).

**Lemma 2.6.** We have \(\psi_1 = \Phi\) and for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_{n+1} \in \mathcal{C}'[[h]]\)

\[
(-h) \psi_{n+1}(x_1, \ldots, x_{n+1}) = \psi_n(x_1, \ldots, x_{n-1}, x_n \cdot x_{n+1}) - \sum_{\substack{p \in \mathbb{P}(n+1) \mid |p| = 2 \atop n = n + 1}} \psi(x_{B_1}) \cdot \psi(x_{B_2}).
\]

**Remark 2.6.** Consider Lemma 2.5. If we regard \(\gamma \cdot e^{-\frac{1}{h} \theta}\) as a wave function it can be called a physical wave function if \(K(\gamma \cdot e^{-\frac{1}{h} \theta}) = 0\). This condition is equivalent to the following by Lemma 2.2

\[
K \theta + \sum_{n=2}^{\infty} \frac{1}{n!} \ell_n(\theta, \ldots, \theta) = 0,
\]

\[
K \gamma + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \ell_n(\theta, \ldots, \theta, \gamma) = 0.
\]
Note that $K'\left(\gamma' \cdot e^{-\frac{1}{\hbar} \theta}\right) = \tilde{f} \left(K \left(\gamma \cdot e^{-\frac{1}{\hbar} \theta}\right)\right)$ since $K' \circ \tilde{f} = \tilde{f} \circ K$. Therefore, a morphism of binary QFT algebras sends physical wave functions to physical wave functions.

Remark 2.7. Consider the first three examples of the relations in Lemma 2.6 We have $\psi_1 = \tilde{f}$ and

$$ (-\hbar)\psi_2(x_1, x_2) = \psi_1(x_1 \cdot x_2) - \psi_1(x_1) \cdot \psi_1(x_2), $$
$$ (-\hbar)\psi_3(x_1, x_2, x_3) = \psi_2(x_1, x_2 \cdot x_3) - \psi_2(x_1, x_2) \cdot \psi_1(x_3) $$
$$ - (-1)^{x_1|x_2} \psi_1(x_2) \cdot \psi_1(x_1, x_3), $$

The first non-trivial $\hbar$-compatibility condition for a pointed cochain map $\tilde{f}$ to be a morphism of binary QFT algebras is that the failure of $\tilde{f}$ being an algebra homomorphism is $(-\hbar)\psi_2$, which divisible by $\hbar$. Then the second non-trivial $\hbar$-compatibility condition is that the failure of $\psi_2 \in \text{Hom}(\mathbb{S}^2 \mathcal{C}, \mathcal{C}')^0[\mathbb{h}]$ being an “algebra homomorphism” is $(-\hbar)\psi_3$, which divisible by $\hbar$, and so on.

2.4. The homotopy category of binary QFT algebras

In this subsection, we introduce the notion of homotopy of morphisms of binary QFT algebras to define the homotopy category $ho\text{BQFTA}(k)$ of binary QFT algebras and show that the quantum descendant functor $\mathcal{R} : \text{BQFTA}(k) \to \text{UsL}_\infty(k[[\hbar]])$ is a homotopy functor—that it induces a functor $ho\mathcal{R} : ho\text{BQFTA}(k) \to ho\text{UsL}_\infty(k[[\hbar]])$ from the homotopy category $ho\text{BQFTA}(k)$ of binary QFT algebras to the homotopy category $ho\text{UsL}_\infty(k[[\hbar]])$ of unital $sL_\infty$-algebras over $k[[\hbar]]$.

We begin with introducing the notion of a homotopy pair of binary QFT algebras after fixing some notation that will be useful later in the section.

Let $\mathcal{U}[\hbar]$ and $\mathcal{U}'[\hbar]$ be $\mathbb{Z}$-graded topologically-free $k[[\hbar]]$-modules. Introduce a formal time parameter $\tau$ and denote by $\text{Hom}(\mathcal{U}, \mathcal{U}')^k[\hbar][\tau]$ the space of $k[[\hbar]]$-linear maps from $\mathcal{U}[\hbar]$ to $\mathcal{U}'[\hbar]$ of ghost number $k$ parametrized by $\tau$ with polynomial dependence: $a(\tau) \in \text{Hom}(\mathcal{U}, \mathcal{U}')^k[\hbar][\tau]$ if $a(\tau) = a_0 + a_1 \tau + \ldots + a_n \tau^n$, $a_j \in \text{Hom}(\mathcal{U}, \mathcal{U}')^k[\hbar]$ for $j = 0, 1, \ldots, n$. Let $\alpha'(\tau) = a'_0 + a'_1 \tau + \ldots + a'_n \tau^n \in \text{Hom}(\mathcal{U}', \mathcal{U}')^k[\hbar][\tau]$, where $\mathcal{U}'[\hbar]$ is another $\mathbb{Z}$-graded topologically-free $k[[\hbar]]$-module, then we define

$$ \alpha'(\tau) \circ \alpha(\tau) := \sum_{i=0}^{n+1} \left(\alpha'_i \circ a_{i-j}\right) \tau^i \in \text{Hom}(\mathcal{U}, \mathcal{U}')^{k+i}[\hbar][\tau] $$
Fix two binary QFT algebras as follows:

\[ \mathcal{C}[[h]]_{BQFTA} = (\mathcal{C}[[h]], 1, \cdot, K), \quad \mathcal{A}(\mathcal{C}[[h]]_{BQFTA}) = (\mathcal{C}[[h]], 1, \cdot, \ell), \]
\[ \mathcal{C}'[[h]]_{BQFTA} = (\mathcal{C}'[[h]], 1, \cdot, \cdot', K'), \quad \mathcal{A}(\mathcal{C}'[[h]]_{BQFTA}) = (\mathcal{C}'[[h]], 1, \cdot, \ell'). \]

Define \( \pi \in \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C})^0[[h]] \) and \( \pi' \in \text{Hom}(\mathcal{S}(\mathcal{C}'), \mathcal{C}')^0[[h]] \) as in the proof of Lemma 2.4:

\[ \pi(x_1 \circ \cdots \circ x_n) := x_1 \cdots x_n \quad \text{for all } n \geq 1 \text{ and } x_1, \ldots, x_n \in \mathcal{C}[[h]], \]
\[ \pi'(x_1' \circ \cdots \circ x'_n) := x_1' \cdots x'_n \quad \text{for all } n \geq 1 \text{ and } x_1', \ldots, x'_n \in \mathcal{C}'[[h]]. \]

Define \( \delta \in \text{Hom}(\mathcal{S}(\mathcal{C}'), \mathcal{C})^1[[h]] \) and \( \delta' \in \text{Hom}(\mathcal{S}(\mathcal{C}'), \mathcal{C}')^1[[h]] \) via \( \delta_n = (-h)^{n-1} \delta_n \) and \( \delta'_n = (-h)^{n-1} \delta'_n \) for all \( n \geq 1 \). Let \( \mathcal{D}(\delta) : \mathcal{S}(\mathcal{C})[[h]] \to \mathcal{S}(\mathcal{C})[[h]] \) and \( \mathcal{D}(\delta') : \mathcal{S}(\mathcal{C}')[[h]] \to \mathcal{S}(\mathcal{C}')[[h]] \) be the unique coderivations determined by \( \delta \) and \( \delta' \), respectively. Then, we have the following relations:

\[ K \circ \pi = \pi \circ \mathcal{D}(\delta), \quad K' \circ \pi' = \pi' \circ \mathcal{D}(\delta'), \quad \mathcal{D}(\delta) \circ \mathcal{D}(\delta) = \mathcal{D}(\delta') \circ \mathcal{D}(\delta'). \]

Remember or notice that \( \mathcal{D}(\delta) = \delta_k \) and \( \mathcal{D}(\delta') = \delta_{k'} \).

Now we are ready to introduce the notion of a homotopy pair. For each pair \( (f(\tau)|\xi(\tau)) \) in \( \text{Hom}(\mathcal{C}, \mathcal{C})^0[[h]] \oplus \text{Hom}(\mathcal{C}, \mathcal{C})^{-1}[[h]] \), we can associate another pair

\[ (\tilde{\psi}(\tau)|\tilde{\lambda}(\tau)) \in \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C})^0[[h]][[\tau]] \oplus \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C})^{-1}[[h]][[\tau]] \]

implicitly defined by the equations

\[ f(\tau) \circ \pi = \pi' \circ \mathcal{S}(\tilde{\psi}(\tau)), \quad \xi(\tau) \circ \pi = \pi' \circ \mathcal{S}(\tilde{\psi}(\tau), \tilde{\lambda}(\tau)), \tag{2.16} \]

where for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[h]] \), we write

\[ \mathcal{S}(\tilde{\psi}(\tau))(x_1 \circ \cdots \circ x_n) := \sum_{\vert p \vert \in \mathcal{P}(n)} \epsilon(p) \tilde{\psi}(\tau)(x_{B_1}) \circ \cdots \circ \tilde{\psi}(\tau)(x_{B_{\vert p \vert}}), \]
\[ \Lambda(\tilde{\psi}(\tau), \tilde{\lambda}(\tau))(x_1 \circ \cdots \circ x_n) := \sum_{\vert p \vert \in \mathcal{P}(n)} \sum_{i=1}^{\vert p \vert} \epsilon(p) \tilde{\psi}(\tau)(Jx_{B_1}) \circ \cdots \circ \tilde{\psi}(\tau)(Jx_{B_{i-1}}) \circ \tilde{\lambda}(\tau)(x_{B_i}) \circ \cdots \circ \tilde{\lambda}(\tau)(x_{B_{\vert p \vert}}). \]

From the condition \( \pi'_0 = \mathbb{I}_{\mathcal{C}[[h]]} \), we can check that eq. \( (2.16) \) actually determines the pair \( (\tilde{\psi}(\tau)|\tilde{\lambda}(\tau)) \) uniquely from the pair \( (f(\tau)|\xi(\tau)) \). For example, we have \( \tilde{\psi}_1(\tau) = f(\tau) \) and \( \tilde{\lambda}_1(\tau) = \xi(\tau) \) at the first level and

\[ \tilde{\psi}_2(x_1, x_2) := \tilde{\psi}_1(x_1 \cdot x_2) - \tilde{\psi}_1(x_1) \cdot \tilde{\psi}_1(x_2), \quad \tilde{\lambda}_2(x_1, x_2) := \xi(x_1 \cdot x_2) - \tilde{\lambda}_1(x_1) \cdot \tilde{\lambda}_1(x_2). \]
at the second level.

Then, we say that \( (\tilde{\psi}(\tau)|\tilde{\lambda}(\tau)) \) is the descendant of \((f(\tau)|\xi(\tau))\), and denote this relationship \( \tilde{\lambda}(f(\tau)|\xi(\tau)) = (\tilde{\psi}(\tau)|\tilde{\lambda}(\tau)) \).

**Definition 2.8.** A pair \((\tilde{f}(\tau)|\tilde{\xi}(\tau)) \in \text{Hom}(\mathcal{C}, \mathcal{C}^0[\hbar][\tau]) \oplus \text{Hom}(\mathcal{C}, \mathcal{C}^{-1}[\hbar][\tau])\) with the descendant \((\tilde{\psi}(\tau)|\tilde{\lambda}(\tau)) = \tilde{\lambda}(\tilde{f}(\tau)|\tilde{\xi}(\tau))\) is called a homotopy pair of binary QFT algebras from \(\mathcal{C}[\hbar]_{\text{BQFTA}}\) to \(\mathcal{C}'[\hbar]_{\text{BQFTA}}\), and denote this

\[
\mathcal{C}[\hbar]_{\text{BQFTA}} \xrightarrow{(\tilde{f}(\tau)|\tilde{\xi}(\tau))} \mathcal{C}'[\hbar]_{\text{BQFTA}},
\]

if it satisfies the following axioms:

- the unital homotopy flow equation that
  \[
  \xi(\tau)(1_\mathcal{C}) = 0, \quad \frac{d}{d\tau} f(\tau) = K' \circ \xi(\tau) + \xi(\tau) \circ K \tag{2.17}
  \]
  and

- the \(\hbar\)-condition that \(\frac{1}{(-\hbar)^{n-1}} \tilde{\lambda}_n(\tau) \in \text{Hom}(S^n\mathcal{C}, \mathcal{C}^{-1}[\hbar][\tau])\) for all \(n \geq 1\).

**Proposition 2.1.** Consider a homotopy pair \(\mathcal{C}[\hbar]_{\text{BQFTA}} \xrightarrow{(\tilde{f}(\tau)|\tilde{\xi}(\tau))} \mathcal{C}'[\hbar]_{\text{BQFTA}}\) of binary QFT algebras whose descendant is \((\tilde{\psi}(\tau)|\tilde{\lambda}(\tau))\). Assume that \(f(0)\) is a morphism of binary QFT algebras. Then, we have

1. \(f(\tau): \mathcal{C}[\hbar]_{\text{BQFTA}} \to \mathcal{C}'[\hbar]_{\text{BQFTA}}\) is a (uniquely defined) family of binary QFT algebra morphisms, and

2. \((\tilde{\psi}(\tau), \tilde{\lambda}(\tau)) : (\mathcal{C}[\hbar], 1_\mathcal{C}, \lambda) \xrightarrow{\text{def}} (\mathcal{C}'[\hbar], 1_\mathcal{C}', \lambda')\) is a homotopy pair of unital \(sL_{\infty}\)-algebras, where \(\tilde{\psi}_n(\tau) := \frac{1}{(-\hbar)^{n-1}} \tilde{\psi}_n(\tau), \lambda_n(\tau) := \frac{1}{(-\hbar)^{n-1}} \tilde{\lambda}_n(\tau)\) for \(n \geq 1\), and \(\tilde{\psi}(0) = \tilde{\lambda}(f(0))\).

**Proof.** Note that the unital homotopy flow equation eq. (2.17) has the following unique solution with initial condition \(f(0)\):

\[
f(\tau) = f(0) + K' \circ \left( \int_0^\tau \xi(\sigma)d\sigma \right) + \left( \int_0^\tau \xi(\sigma)d\sigma \right) \circ K,
\]

satisfying \(f(\tau)(1_\mathcal{C}) = 1_\mathcal{C}'\). It follows that \(f(\tau)\) is a family of pointed cochain maps from \((\mathcal{C}[\hbar], 1_\mathcal{C}, K)\) to \((\mathcal{C}'[\hbar], 1_\mathcal{C}', K')\). We need to check that \(f(\tau)\) also satisfies the \(\hbar\)-compatibility condition.
We first show that \((\tilde{\psi}(\tau), \tilde{\Lambda}(\tau)) : (C[[h]], 1_{c}, \tilde{\ell}) \rightleftharpoons (C'[\hbar], 1_{c}', \tilde{\ell}')\) is a homotopy pair of unital \(sL_\infty\)-algebras — see Definition [A.4] in Appendix [A.1]. Then, we show that \((\tilde{\psi}(\tau), \Lambda(\tau)) : (C[[h]], 1_{c}, \ell) \rightleftharpoons (C'[\hbar], 1_{c}', \ell)\) is a homotopy pair between the quantum descendant unital \(sL_\infty\)-algebras — this implies that \(f(\tau)\) is a one-parameter family of binary QFT morphisms such that \(\tilde{\psi}(\tau) = \Lambda(f(\tau))\).

From the homotopy flow equations in eq. (2.17), we obtain that
\[
\frac{d}{dt} f(\tau) \circ \pi = K' \circ \hat{\pi}(\tau) \circ \pi + \hat{\pi}(\tau) \circ K \circ \pi = K' \circ \hat{\pi}(\tau) \circ \pi + \hat{\pi}(\tau) \circ \pi \circ D(\ell),
\]
where we have used \(K \circ \pi = \pi \circ D(\ell)\), Therefore, we have the following identity:
\[
\frac{d}{dt} f(\tau) \circ \pi - K' \circ \hat{\pi}(\tau) \circ \pi = \hat{\pi}(\tau) \circ \pi \circ D(\ell) = 0. \tag{2.18}
\]
Substituting for \(f(\tau) \circ \pi\) and \(\hat{\pi}(\tau) \circ \pi\) with eq. (2.16) and using \(K' \circ \pi' = \pi' \circ D(\ell')\), we obtain that
\[
\pi' \circ \left(\frac{d}{dt} \hat{\pi}(\tau) - D(\ell') \circ A(\tilde{\psi}(\tau), \hat{\Lambda}(\tau)) = A(\tilde{\psi}(\tau), \hat{\Lambda}(\tau)) \circ D(\ell')\right) = 0. \tag{2.19}
\]
From \(\pi'_1 = \Pi_{c'}[[h]]\), we have
\[
pr_{c'[[h]]} \circ \left(\frac{d}{dt} \hat{\pi}(\tau) - D(\ell') \circ A(\tilde{\psi}(\tau), \hat{\Lambda}(\tau)) = A(\tilde{\psi}(\tau), \hat{\Lambda}(\tau)) \circ D(\ell')\right) = 0.
\]
In components, this condition means that, for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_n \in C[[h]]\), we have
\[
\frac{d}{dt} \tilde{\psi}(\tau)(x_1, \ldots, x_n) = \sum_{p \in P(n)} \sum_{i=1}^{n} \epsilon(p) \tilde{\ell}'(\tilde{\psi}(\tau)(Jx_{B_1}) \ldots, \tilde{\psi}(\tau)(Jx_{B_i}), \tilde{\Lambda}(\lambda_1)(x_{B_1}), \tilde{\psi}(\tau)(x_{B_i}), \ldots, \tilde{\psi}(\tau)(x_{B_{n-1}}))
\]
\[
+ \sum_{p \in P(n)} \epsilon(p) \tilde{\ell'(\lambda_1)(Jx_{B_1}, \ldots, Jx_{B_{n-1}}, \tilde{\ell}(x_{B_1}), x_{B_1}, \ldots, x_{B_{n-1}})}.
\]
It is straightforward to check that \(\hat{\Lambda}_n(x_1, \ldots, x_{n-1}, 1_{c'}) = 0\) for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_{n-1} \in C[[h]]\). Therefore,
\[
(\tilde{\psi}(\tau), \tilde{\Lambda}(\tau)) : (C[[h]], 1_{c}, \tilde{\ell}) \rightleftharpoons (C'[\hbar], 1_{c}', \tilde{\ell}')
\]
is a homotopy pair of unital \(sL_\infty\)-algebras.
Now set $\tilde{\psi}_n(\tau) = \frac{1}{(-\hbar)^{n-1}} \tilde{\psi}_n(\tau)$, for all $n \geq 1$. If $\tilde{\psi}_n(\tau)$ is in $\text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$ for all $n \geq 1$, then the first relation in eq. (2.16) becomes

$$f(\tau) \circ \pi = \pi' \circ \tilde{\psi}(\tau),$$

which implies that $f(\tau)$ is a family of morphisms of binary QFT algebras and $\tilde{\psi}(\tau) = \mathcal{A}(\tilde{f}(\tau))$. We are going to show that $\tilde{\psi}_n(\tau)$ is indeed in $\text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$ for all $n \geq 1$.

From the assumption that $\tilde{f}(0)$ is a morphism of binary QFT algebras, it follows that $\tilde{\psi}(0) = \mathcal{A}(\tilde{f}(0))$ is a unital $sL_\infty$-morphism from $(\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \mathbf{0})$ to $(\mathcal{C}'[[\hbar]], 1_{\mathcal{C}'}, \mathbf{0}')$.

In particular we have $\tilde{\psi}_n(0) \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]]$ for all $n \geq 1$. From $\tilde{\psi}_1(\tau) = f(\tau) \in \text{Hom}(\mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$, we have $\tilde{\psi}_1(\tau) \in \text{Hom}(\mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$. Fix $n > 1$ and assume that $\tilde{\psi}_k(\tau) \in \text{Hom}(S^k \mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$ for all $k < n$ and all $\tau$. Note that $\tilde{\lambda}_n(\tau) := \frac{1}{(-\hbar)^{n-1}} \tilde{\psi}_n(\tau) \in \text{Hom}(S^k \mathcal{C}, \mathcal{C}')^{-1}[[\hbar]]$ for all $n \geq 1$ by $\hbar$-compatibility. From eq. (2.20), we have, for homogeneous $x_1, \ldots, x_n \in \mathcal{C}[[\hbar]]$,

$$\frac{d}{d\tau} \tilde{\psi}_n(\tau)(x_1, \ldots, x_n)$$

$$= (-\hbar)^{n-1} \sum_{p \in P(n)} \sum_{i=1}^{\left|p\right|} \varepsilon(p) \ell'_p(\psi(\tau)(Jx_{B_1}), \ldots, \psi(\tau)(x_{B_1}), \ldots, \psi(\tau)(x_{B_p}))$$

$$+ (-\hbar)^{n-1} \sum_{|B_i| = n-|p|+1} \varepsilon(p) \tilde{\psi}_n(\tau)(Jx_{B_1}, \ldots, Jx_{B_{|p|-1}}, \ell(x_{B_1}), \ldots, x_{B_{|p|}}),$$

which implies that $\frac{d}{d\tau} \tilde{\psi}_n(\tau) \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$. Combined with the condition $\tilde{\psi}_n(0) \in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]]$, we can conclude that $\tilde{\psi}_n(\tau)$ is in fact contained in $\text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$, as desired. It follows by induction that $\tilde{\psi}_n(\tau)$ is contained in $\in \text{Hom}(S^n \mathcal{C}, \mathcal{C}')[[\hbar]][\tau]$ for all $n \geq 1$. Therefore, we have, for all $n \geq 1$ and homogeneous $x_1, \ldots, x_n \in \mathcal{C}[[\hbar]]$,

$$\frac{d}{d\tau} \psi_n(\tau)(x_1, \ldots, x_n)$$

$$= \sum_{p \in P(n)} \sum_{i=1}^{\left|p\right|} \varepsilon(p) \ell'_p(\psi(\tau)(Jx_{B_1}), \ldots, \psi(\tau)(x_{B_1}), \ldots, \psi(\tau)(x_{B_p}))$$

$$+ \sum_{|B_i| = n-|p|+1} \varepsilon(p) \psi_n(\tau)(Jx_{B_1}, \ldots, Jx_{B_{|p|-1}}, \ell(x_{B_1}), \ldots, x_{B_{|p|}}),$$
which means that \((\bar{\psi}(\tau), \bar{\lambda}(\tau)) : (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell') \Rightarrow (\mathcal{C}'[[h]], 1_{\mathcal{C}'}, \ell')\) is a homotopy pair between the quantum descendant unital \(sL_{\infty}\)-algebras. Now eq. (2.21) implies that \(\bar{\psi}(\tau) = \mathcal{A}(f(\tau))\), so that \(f(\tau) : \mathcal{C}'[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) is a uniquely defined 1-parameter family of morphisms of binary QFT algebras.

**Remark 2.8.** It can be checked that the identity eq. (2.19) implies that

\[
\frac{d}{d\tau} \mathcal{A}(\bar{\psi}(\tau)) = \mathcal{D}(\ell') \circ \Lambda(\bar{\psi}(\tau), \bar{\lambda}(\tau)) + \Lambda(\bar{\psi}(\tau), \bar{\lambda}(\tau)) \circ \mathcal{D}(\ell). \tag{2.22}
\]

**Definition 2.9.** Two morphisms \(f, \tilde{f} : \mathcal{C}'[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) of binary QFT algebras are homotopic, which we denote by \(f \sim_{h} \tilde{f}\) if there is a homotopy pair \((f(\tau), \xi(\tau))\) of binary QFT algebras such that \(f(0) = f\) and \(f(1) = \tilde{f}\).

It is clear that \(\sim_{h}\) is an equivalence relation. We say that two morphisms of binary QFT algebras have the same homotopy type if they are homotopic to each other.

**Lemma 2.7.** Quantum descendants of homotopic morphisms of binary QFT algebras are homotopic as morphisms of the quantum descendant unital \(sL_{\infty}\)-algebras.

**Proof.** Assume that \(f \sim_{h} \tilde{f} : \mathcal{C}'[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) are homotopic morphisms of binary QFT algebras. Then, by definition, there is a homotopy pair \((f(\tau), \xi(\tau))\) of binary QFT algebras such that \(f(0) = f\) and \(f(1) = \tilde{f}\). From Proposition 2.1, it follows that there is a corresponding homotopy pair \((\bar{\psi}(\tau), \bar{\lambda}(\tau))\) between the quantum descendant unital \(sL_{\infty}\)-algebras such that \(\bar{\psi}(0) = \mathcal{A}(f)\) and \(\bar{\psi}(1) = \mathcal{A}(\tilde{f})\). Therefore \(\bar{\psi}(0)\) and \(\bar{\psi}(1)\) are homotopic as morphisms of unital \(sL_{\infty}\)-algebras.

**Definition 2.10.** A morphism \(f : \mathcal{C}'[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) of binary QFT algebras is a homotopy equivalence if there is a morphism \(f' : \mathcal{C}'[[h]]_{BQFTA} \rightarrow \mathcal{C}'[[h]]_{BQFTA}\) of binary QFT algebras such that \(f \circ f' \sim_{h} \mathbb{1}_{\mathcal{C}'[[h]]}\) and \(f \circ f' \sim_{h} \mathbb{1}_{\mathcal{C}'[[h]]}\).

A homotopy equivalence of binary QFT algebras is automatically a quasi-isomorphism of binary QFT algebras. It is also obvious that the quantum descendant of a homotopy equivalence of binary QFT algebras is a homotopy equivalence of the quantum descendant unital \(sL_{\infty}\)-algebras. We are going to define the homotopy category of binary QFT algebras as the category whose objects are binary QFT algebras and whose morphisms are the homotopy types of morphisms of binary QFT algebras. Then the homotopy type of a homotopy equivalence is an isomorphism in the homotopy category.
Theorem 2.2. There is a homotopy category $\text{hoBQFTA}(k)$ whose objects are binary QFT algebras over $k$ and whose morphisms are the homotopy types of morphisms of binary QFT algebras.

The above theorem is a corollary of the following proposition:

Proposition 2.2. Consider the following diagram in the category $\text{BQFTA}(k)$ of binary QFT algebras:

\[
\begin{array}{cccc}
\mathcal{C}[\hbar]_{\text{BQFTA}} & \overset{f}{\longrightarrow} & \mathcal{C}[\hbar]_{\text{BQFTA}} & \overset{f'}{\longrightarrow} & \mathcal{C}[\hbar]_{\text{BQFTA}} \\
\end{array}
\]

and assume that $f \sim_h \tilde{f}$ and $f' \sim_h \tilde{f}'$. Then, we have $f' \circ f \sim_h \tilde{f}' \circ \tilde{f}$ as morphisms of binary QFT algebra from $\mathcal{C}[\hbar]_{\text{BQFTA}}$ to $\mathcal{C}[\hbar]_{\text{BQFTA}}$, and the homotopy type of $f' \circ f$ depends only on the homotopy types of $f'$ and $f$.

Proof. From the assumption in the proposition, we have a sequence of homotopy pairs of binary QFT algebras

\[
\begin{array}{cccc}
\mathcal{C}[\hbar]_{\text{BQFTA}} & \overset{(f(\tau)|\xi(\tau))}{\longrightarrow} & \mathcal{C}[\hbar]_{\text{BQFTA}} & \overset{(f'(\tau)|\xi'(\tau))}{\longrightarrow} & \mathcal{C}[\hbar]_{\text{BQFTA}} \\
\end{array}
\]

such that

\[
\begin{align*}
\begin{cases}
f(0) = f, & f'(0) = f', \\
f(1) = \tilde{f}, & f'(1) = \tilde{f}'.
\end{cases}
\end{align*}
\]

Let

\[
(\tilde{\psi}(\tau), \tilde{\lambda}(\tau)) = \tilde{\mathcal{R}}(f(\tau)|\xi(\tau)), \quad (\tilde{\psi}'(\tau), \tilde{\lambda}'(\tau)) = \tilde{\mathcal{R}}(f'(\tau)|\xi'(\tau)).
\]

We define the composition of the given homotopy pairs of binary QFT algebras as follows:

\[
(f'(\tau)|\xi'(\tau)) \circ (f(\tau)|\xi(\tau)) := (f''(\tau)|\xi''(\tau)),
\]

where

\[
\begin{align*}
\begin{cases}
f''(\tau) := f'(\tau) \circ f(\tau), \\
\xi''(\tau) := f'(\tau) \circ \xi(\tau) + \xi'(\tau) \circ f(\tau),
\end{cases}
\end{align*}
\]

Note that

\[
f''(0) = f \circ f', \quad f''(1) = \tilde{f} \circ \tilde{f}'.
\]

Then what we need to show is that $(f''(\tau)|\xi''(\tau))$ is a homotopy pair of binary QFT algebras from $\mathcal{C}[\hbar]_{\text{BQFTA}}$ to $\mathcal{C}[\hbar]_{\text{BQFTA}}$. It can be checked easily that $(f''(\tau)|\xi''(\tau))$
satisfies the unital homotopy flow equation: it is obvious that $\xi^\varepsilon(\tau)(1_\varphi) = 0$, and

$$\frac{d}{d\tau} f^\varepsilon(\tau) = \frac{d}{d\tau} f'(\tau) \circ f'(\tau) + \frac{d}{d\tau} f'(\tau),$$

so it is trivial that $\frac{d}{d\tau} f^\varepsilon(\tau) = K^\varepsilon \circ \xi^\varepsilon(\tau) + \xi^\varepsilon(\tau) \circ K$. What remains is to check is the $h$-condition.

Consider the descendant $(\bar{\psi}^\varepsilon(\tau), \bar{\lambda}^\varepsilon(\tau))$ of $(f^\varepsilon(\tau), \xi^\varepsilon(\tau))$:

$$\begin{align*}
\begin{cases}
\bar{f}^\varepsilon(\tau) \circ \pi = \pi^\varepsilon \circ \mathcal{H}(\bar{\psi}^\varepsilon(\tau)), \\
\bar{\xi}^\varepsilon(\tau) \circ \pi = \pi^\varepsilon \circ \Lambda(\bar{\psi}^\varepsilon(\tau), \bar{\lambda}^\varepsilon(\tau)),
\end{cases}
\end{align*}
(2.23)
$$

From

$$\begin{align*}
\begin{cases}
\bar{f}'(\tau) \circ \pi' = \pi'^\varepsilon \circ \mathcal{H}(\bar{\psi}'(\tau)), \\
\bar{\xi}'(\tau) \circ \pi' = \pi'^\varepsilon \circ \Lambda(\bar{\psi}'(\tau), \bar{\lambda}'(\tau))
\end{cases}
\end{align*}
$$

we obtain that

$$\begin{align*}
\begin{cases}
\bar{f}'(\tau) \circ \pi = \pi'^\varepsilon \circ \mathcal{H}(\bar{\psi}'(\tau)) \circ \mathcal{H}(\bar{\psi}(\tau)), \\
\bar{\xi}'(\tau) \circ \pi = \pi'^\varepsilon \circ (\mathcal{H}(\bar{\psi}'(\tau)) \circ \Lambda(\bar{\psi}(\tau), \bar{\lambda}(\tau)) + \Lambda(\bar{\psi}'(\tau), \bar{\lambda}'(\tau)) \circ \mathcal{H}(\bar{\psi}'(\tau)))
\end{cases}
\end{align*}
(2.24)
$$

Comparing this with eq. (2.23), we obtain that

$$\begin{align*}
\text{pr}_{\varphi^\varepsilon(h)} \circ \mathcal{H}(\bar{\psi}'(\tau)) &= \text{pr}_{\varphi^\varepsilon(h)} \circ \mathcal{H}(\bar{\psi}'(\tau)) \circ \mathcal{H}(\bar{\psi}(\tau)), \\
\text{pr}_{\varphi^\varepsilon(h)} \circ \Lambda(\bar{\psi}'(\tau), \bar{\lambda}'(\tau)) &= \text{pr}_{\varphi^\varepsilon(h)} \circ \mathcal{H}(\bar{\psi}'(\tau)) \circ \Lambda(\bar{\psi}(\tau), \bar{\lambda}(\tau)) + \text{pr}_{\varphi^\varepsilon(h)} \circ \Lambda(\bar{\psi}'(\tau), \bar{\lambda}'(\tau)) \circ \mathcal{H}(\bar{\psi}'(\tau)),
\end{align*}
$$
which implies the following:

- from $\bar{\psi}^\varepsilon(\tau) = \text{pr}_{\varphi^\varepsilon(h)} \circ \mathcal{H}(\bar{\psi}(\tau))$, we have

$$\bar{\psi}^\varepsilon(\tau) = \bar{\psi}'(\tau) \circ \mathcal{H}(\bar{\psi}(\tau)) \equiv \bar{\psi}'(\tau) \bullet \bar{\psi}(\tau);$$

- from $\bar{\lambda}^\varepsilon(\tau) = \text{pr}_{\varphi^\varepsilon(h)} \circ \Lambda(\bar{\psi}'(\tau), \bar{\lambda}'(\tau))$, we have

$$\bar{\lambda}^\varepsilon(\tau) = \bar{\psi}'(\tau) \circ \Lambda(\bar{\psi}(\tau), \bar{\lambda}(\tau)) + \bar{\lambda}'(\tau) \circ \mathcal{H}(\bar{\psi}'(\tau)).$$
Recall that for all $n \geq 1$, we have

$$\lambda'(\tau)_n := \frac{1}{(-\hbar)^{n-1}} \tilde{\lambda}'(\tau)_n \in \text{Hom}(S^n\mathcal{E}', \mathcal{E}')^{-1}[\hbar],$$

$$\lambda(\tau)_n := \frac{1}{(-\hbar)^{n-1}} \tilde{\lambda}(\tau)_n \in \text{Hom}(S^n\mathcal{E}', \mathcal{E}')^{-1}[\hbar],$$

$$\psi'(\tau)_n := \frac{1}{(-\hbar)^{n-1}} \tilde{\psi}'(\tau)_n \in \text{Hom}(S^n\mathcal{E}', \mathcal{E}')^{0}[\hbar],$$

$$\psi(\tau)_n := \frac{1}{(-\hbar)^{n-1}} \tilde{\psi}(\tau)_n \in \text{Hom}(S^n\mathcal{E}', \mathcal{E}')^{0}[\hbar],$$

From eq. (2.26), we can check that $\lambda^s(\tau)_n := \frac{1}{(-\hbar)^{n-1}} \tilde{\lambda}^s(\tau)_n \in \text{Hom}(S^n\mathcal{E}', \mathcal{E}')^{-1}[\hbar]$. Therefore, we conclude that $\left(\tilde{f}^s(\tau)\right)\tilde{\xi}^s(\tau)$ is a homotopy pair of binary QFT algebras from $\mathcal{E}[\hbar][BQFTA]$ to $\mathcal{E}'[\hbar][BQFTA]$.

Note that $\left(\tilde{\psi}^s(\tau)\tilde{\lambda}^s(\tau)\right)$ is the descendant of $\left(\tilde{f}^s(\tau)\tilde{\xi}^s(\tau)\right)$ and $\left(\tilde{\psi}^s(\tau)\tilde{\lambda}^s(\tau)\right)$ is the homotopy pair of the quantum descendant unital $sL_{\infty}$-algebras from $\left(\mathcal{E}[\hbar][1_\mathcal{E}, \ell]\right)$ to $\left(\mathcal{E}'[\hbar][1_\mathcal{E}, \ell]\right)$ with $\tilde{\psi}^s(0) = \tilde{\psi}' \circ \tilde{\psi}$ and $\tilde{\psi}^s(1) = \tilde{\psi}' \circ \tilde{\psi}$, where $\tilde{\psi}' = \tilde{\tau}(\tilde{f})$, $\tilde{\psi} = \tilde{\tau}(\tilde{f})$, $\tilde{\psi}' = \tilde{\tau}(\tilde{f})$, and $\tilde{\psi} = \tilde{\tau}(\tilde{f})$. Combined with Lemma 2.7, the above proposition also implies the following.

**Theorem 2.3.** The quantum descendant functor $\mathcal{K} : BQFTA(\mathbb{k}) \rightarrow UsL_{\infty}(\mathbb{k}[\hbar])$ is a homotopy functor, i.e., it induces a functor $\text{ho}\mathcal{K} : 
hoBQFTA(\mathbb{k}) \rightarrow 
hoUsL_{\infty}(\mathbb{k}[\hbar])$ between the respective homotopy categories.

2.5. **The category and homotopy category of binary CFT algebras**

We obtain the notion of binary CFT (classical field theory) algebra over $\mathbb{k}$ by taking the classical limit $\left(\mathcal{E}', 1_\mathcal{E}', \cdot, \ell\right)$ of the tuple $\left(\mathcal{E}[\hbar], 1_\mathcal{E}, \cdot, \ell\right)$, which combines the structure of a binary QFT algebra $\left(\mathcal{E}[\hbar], 1_\mathcal{E}, \cdot, K\right)$ with its quantum descendant $\left(\mathcal{E}'[\hbar], 1_\mathcal{E}, \ell\right)$. The classical limit of the quantum descendant of a morphism of binary QFT algebras gives rise naturally to a morphism of the associated binary CFT algebras, etc., and yields the following definition of the category BCFTA($\mathbb{k}$) of binary CFT algebras over $\mathbb{k}$.

**Definition 2.11 (Category of binary CFT algebras).** A binary CFT algebra over $\mathbb{k}$ is a tuple $\mathcal{E}_{BCFTA} = \left(\mathcal{E}, 1_\mathcal{E}, \cdot, \ell\right)$, where $\left(\mathcal{E}, 1_\mathcal{E}, \cdot\right)$ is a unital $\mathbb{Z}$-graded commutative and
associative algebra and \((\mathcal{C}, 1_\mathcal{C}, \mathcal{L})\) is a unital \(sL_\infty\)-algebra, satisfying the compatibility condition that, for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_{n+1} \in \mathcal{C}\), we have
\[
\ell_n(x_1, \ldots, x_{n-1}, x_n \cdot x_{n+1}) = \ell_n(x_1, \ldots, x_{n-1}, x_n) \cdot x_{n+1} + (-1)^{|x_n|(|x_1| + \ldots + |x_{n-1}|)} x_{n+1} \cdot \ell_n(x_1, \ldots, x_{n-2}, x_{n+1}).
\]
A morphism \(\psi : \mathcal{C}_{BCFTA} \rightarrow \mathcal{C'}_{BCFTA}\) of binary CFT algebras is a morphism of the underlying unital \(sL_\infty\)-algebras such that for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_{n+1} \in \mathcal{C}\), we have
\[
\psi_n(x_1, \ldots, x_{n-1}, x_n \cdot x_{n+1}) = \sum_{p \in P(n+1)} \frac{1}{|p|=2} \psi(B_{x_1}) \cdot \psi(B_{x_{n+1}}).
\]
The composition of two consecutive morphisms of binary CFT algebras is the composition as morphisms of the underlying unital \(sL_\infty\)-algebras.

It can be checked that the composition of two consecutive morphisms of binary CFT algebras is a morphism of binary CFT algebras so that binary CFT algebras over \(\mathbb{k}\) form a category BCFTA(\(\mathbb{k}\)).

Now we turn to the homotopy category hoBCFTA(\(\mathbb{k}\)) of binary CFT algebras over \(\mathbb{k}\).

**Definition 2.12.** A homotopy pair \((\psi(\tau)|\lambda(\tau)) : \mathcal{C}_{BCFTA} \rightarrow \mathcal{C'}_{BCFTA}\) of binary CFT algebras is a homotopy pair of the underlying unital \(sL_\infty\)-algebras with the following set of additional conditions: for all \(n \geq 1\) and homogeneous \(x_1, \ldots, x_n \in \mathcal{C}\),
\[
\lambda_n(\tau)(x_1, \ldots, x_{n-1}, x_n \cdot x_{n+1}) = \sum_{p \in P(n+1)} \frac{1}{|p|=2} \lambda(\tau)(x_{B_1}) \cdot \psi(\tau)(x_{B_{n+1}}) + \psi(\tau)(Jx_{B_1}) \cdot \lambda(\tau)(x_{B_{n+1}}).
\]
Recall that \((\psi(\tau)|\lambda(\tau))\) is a homotopy pair of the underlying unital \(sL_\infty\)-algebras from \((\mathcal{C}, 1_\mathcal{C}, \mathcal{L})\) to \((\mathcal{C'}, 1_{\mathcal{C'}}, \mathcal{L'})\) if
\[
(\psi(\tau)|\lambda(\tau)) \in \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C'})^{0}[\tau] \oplus \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C'})^{-1}[\tau]
\]
and the data satisfies both the unit condition that \(\lambda(\tau)(x_1 \odot \ldots x_{n-1} \odot 1_\mathcal{C}) = 0\) for all \(n \geq 1\) and \(x_1, \ldots, x_n \in \mathcal{C}\), and the homotopy flow equation generated by \(\lambda(\tau)\), namely
\[
\frac{d}{dt}\mathcal{S}(\psi(\tau)) = \mathcal{O}(\tau) \circ \lambda(\psi(\tau), \lambda(\tau)) + \lambda(\psi(\tau), \lambda(\tau)) \circ \mathcal{O}(\tau).
\]
It follows that $\psi(\tau)$ is a uniquely defined family of unital $sL_\infty$-morphisms whenever $\psi(0)$ is a unital $sL_\infty$-morphism. The additional set of conditions in Definition 2.12 for a homotopy pair $(\psi(\tau)\lambda(\tau))$ of binary CFT algebras implies that $\psi(\tau)$ is a uniquely defined family of binary CFT algebra morphisms whenever $\psi(0)$ is a binary CFT algebra morphism. Therefore, we say that two morphisms $\psi$ and $\tilde{\psi}$ of binary CFT algebras are homotopic and have the same homotopy type if there is a homotopy pair $(\psi(\tau)\lambda(\tau))$ of binary CFT algebras such that $\psi(0) = \tilde{\psi}$ and $\psi(1) = \tilde{\psi}$.

It can be checked that the homotopy type of the composition of two consecutive morphisms of binary CFT algebras depends only on the homotopy types of the constituents. Therefore, we can form the homotopy category $\text{hoBCFTA}(k)$ of binary CFT algebras, whose objects are binary CFT algebras and whose morphisms are the homotopy types of morphisms of binary CFT algebras.

**Remark 2.9.** From Lemma 2.3, it follows that the combined classical limit of a binary QFT algebra and its quantum descendant is a binary CFT algebra. From Lemma 2.6, it follows that the classical limit of the quantum descendant $\psi$ of a morphism $f$ of binary QFT algebras (recalling that $\psi_1 = f$) is a morphism of the associated binary CFT algebras. Considering those relationships and the properties of the quantum descendant functor, it should be obvious that the composition of two consecutive morphisms of binary CFT algebras is a morphism of binary CFT algebra so that binary CFT algebras form a category BCFTA$(k)$. From the definitions of the homotopy category $\text{hoBQFTA}(k)$ of binary QFT algebras and the homotopy functoriality of the quantum descendant functor in Sect. 2.4, one can check that the classical limits of the quantum descendants of homotopic morphisms of binary QFT algebras are homotopic as morphisms of the associated binary CFT algebras. This poses the interesting problem of whether one can setup a mathematical framework for a quantization of a binary CFT algebra $\mathcal{C}_B$ to a binary QFT algebra $\mathcal{C}[\hbar][BQFTA]$, whose classical limit in the above sense gives us back $\mathcal{C}_B$ with appropriate functoriality. Such a framework may shed new light on the passage from classical field theory to quantum field theory, but is beyond the scope of this paper.

### 3. Binary QFTs

The binary QFT algebra $k[[\hbar]]$ is initial in the category $BQFTA(k)$ and represents an initial object in the homotopy category $\text{hoBQFTA}(k)$. A binary QFT is a slice over the
initial object in the homotopy category \( \text{hoBQFTA}(k) \), i.e., a diagram of the form

\[
\mathcal{C}[[\hbar]]_{\text{BQFTA}} \xrightarrow{[\mathcal{c}]} k[[\hbar]],
\]
where \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} = (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \ldots, K) \) is a binary QFT algebra and \([\mathcal{c}]\) is a homotopy type of binary QFT algebra morphisms to \( k[[\hbar]] \). In practice, we choose a representative, say \( \mathcal{c} \), of \([\mathcal{c}]\) and regard a binary QFT as a diagram \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \xrightarrow{\mathcal{c}} k[[\hbar]] \) in the category \( \text{BQFTA}(k) \) and consider only those notions and quantities that are invariants of the homotopy type of \( \mathcal{c} \). We shall call \( \mathcal{c} \) a strong quantum expectation to contrast with a quantum expectation which is merely a pointed cochain map.

### 3.1. Homotopical families of quantum observables

Fix a binary QFT \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \xrightarrow{\mathcal{c}} k[[\hbar]] \) and let \( (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, L) \) be the quantum descendant of \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \). Then, \( (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, L) \) is a unital \( sL_\infty \)-algebra. Let \( \chi = \mathfrak{h}(\mathcal{c}) \) be the quantum descendant of \( \mathcal{c} \) so that for all \( n \geq 1 \) and homogeneous \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \),

\[
\mathcal{c}(x_1 \cdot \ldots \cdot x_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|} \varepsilon(p) \chi(x_{B_1}) \ldots \chi(x_{B_p}),
\]

has the property that \( \chi_n \in \text{Hom}\left(S^n\mathcal{C}, k\right)^0[[\hbar]] \) for all \( n \geq 1 \). Equivalently, we have

\[
\mathcal{c} \circ \pi = \pi^{k[[\hbar]]} \circ \Psi_x
\]

where \( \pi(x_1 \odot \ldots \odot x_n) := x_1 \cdot \ldots \cdot x_n \) for all \( n \geq 1 \) and \( x_1, \ldots, x_n \in \mathcal{C}[[\hbar]] \) and \( \pi^{k[[\hbar]]}(a_1 \odot \ldots \odot a_n) := a_1 \cdot \ldots \cdot a_n \) for all \( n \geq 1 \) and \( a_1, \ldots, a_n \in k[[\hbar]] \). Recall that \( K \circ \pi = \pi \circ \delta_\ell \) and \( \chi : (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, L) \xrightarrow{\chi} (k[[\hbar]], 1, 0) \) is a unital \( sL_\infty \)-morphism, i.e., \( \Psi_x \circ \delta_\ell = 0 \).

Now we introduce the notion of a homotopical family of quantum observables.

For any \( \mathbb{Z} \)-graded vector space \( V \) over \( k \), we can regard \( V[[\hbar]] \) as a topologically-free \( sL_\infty \)-algebra \( (V[[\hbar]], 0) \) over \( k[[\hbar]] \) with the zero \( sL_\infty \)-structure 0.

**Definition 3.1.** A homotopical family of quantum observables is a pair \( \mathcal{V} = (V; [\varphi]) \) where \( V \) is a \( \mathbb{Z} \)-graded vector space over \( k \) and \([\varphi]\) is the homotopy type of an \( sL_\infty \)-morphism from \((V[[\hbar]], 0)\) to \((\mathcal{C}[[\hbar]], L)\).
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In practice we work with a representative \( \varphi \) of \( [ \varphi ] \) and consider only those notions and quantities that are invariants of the homotopy type. The above definition is based on the following observation:

**Lemma 3.1.** Let \( \varphi : (V[[\hbar]], 0) \longrightarrow (\mathcal{C}[\hbar], \ell) \) be a morphism of \( sL_\infty \)-algebras and define \( \Pi^\varphi \in \text{Hom}(\mathcal{S}(V), \mathcal{C})^0[[\hbar]] \) so that, for all \( n \geq 1 \) and homogeneous \( v_1, \ldots, v_n \in V, \)

\[
\Pi^\varphi(v_1 \otimes \cdots \otimes v_n) = \sum_{\mathbf{p} \in P(n)} (-\hbar)^{|p|} e(p) \varphi(v_{B_1}) \cdots \varphi(v_{B_{|p|}}).
\]

Then, we have

1. \( \Pi^\varphi : (\mathcal{S}V[[\hbar]], 0) \rightarrow (\mathcal{C}[\hbar], K) \) is a cochain map, i.e., \( K \circ \Pi^\varphi = 0; \)
2. \( \Pi^\varphi - \Pi^\mathcal{S} = K\Sigma \) for some \( \Sigma \in \text{Hom}(\mathcal{S}(V), \mathcal{C})^{-1}[[\hbar]] \) whenever \( \mathcal{S} \sim \varphi \) as \( sL_\infty \)-morphisms.

**Proof.** For an \( sL_\infty \)-morphism \( \varphi : (V[[\hbar]], 0) \longrightarrow (\mathcal{C}[\hbar], \ell) \), define \( \Psi_\varphi \) so that for all \( n \geq 1 \) and homogeneous \( v_1, \ldots, v_n \in V, \)

\[
\Psi_\varphi(v_1 \otimes \cdots \otimes v_n) = \sum_{\mathbf{p} \in P(n)} (-\hbar)^{|p|} e(p) \varphi(v_{B_1}) \cdots \varphi(v_{B_{|p|}}).
\]

Then, we have the identities \( \delta_\ell \circ \Psi_\varphi = 0 \) and \( \Pi^\varphi = \pi \circ \Psi_\varphi \). It follows that \( K \circ \Pi^\varphi = K \circ \pi \circ \Psi_\varphi = \pi \circ \delta_\ell \circ \Psi_\varphi = 0. \) Therefore, we have the first property that \( \Pi^\varphi : (\mathcal{S}V[[\hbar]], 0) \rightarrow (\mathcal{C}[\hbar], K) \) is a cochain map whenever \( \varphi \) is an \( sL_\infty \)-morphism. For the second property, consider two homotopic \( sL_\infty \)-morphisms \( \varphi \) and \( \tilde{\varphi} \). Then there is a homotopy pair \( (\varphi(\tau), \Lambda(\tau)) : (V[[\hbar]], 0) \longrightarrow (\mathcal{C}[\hbar], \ell) \) of \( sL_\infty \)-algebras such that \( \varphi(0) = \varphi \) and \( \varphi(1) = \tilde{\varphi} \). Set \( \tilde{\varphi}_n(\tau) = (-\hbar)^{n-1} \varphi_n(\tau) \) and \( \Lambda_n(\tau) = (-\hbar)^{n-1} \lambda_n(\tau), \) for all \( n \geq 1. \) Then \( (\tilde{\varphi}(\tau), \Lambda(\tau)) : (V[[\hbar]], 0) \longrightarrow (\mathcal{C}[\hbar], \ell) \) is a homotopy pair of \( sL_\infty \)-algebras such that \( \tilde{\varphi}(0) = \tilde{\varphi} \) and \( \tilde{\varphi}(1) = \tilde{\varphi}. \) (Recall that \( D(\ell) = \delta_\ell \) and \( \tilde{\varphi}(\varphi) = \Psi_\varphi \).) Therefore, we have

\[
\frac{d}{d\tau} \tilde{\varphi}(\tau) = D(\ell) \circ \Lambda(\varphi, \Lambda(\tau))
\]

which implies that

\[
\tilde{\varphi}(\tau) = \tilde{\varphi}(\varphi) + D(\ell) \circ \int_0^\tau \Lambda(\varphi(\sigma), \Lambda(\sigma)) d\sigma.
\]
It follows that
\[ \pi \circ \mathfrak{g}(\bar{\phi}(\tau)) = \pi \circ \mathfrak{g}(\phi) + \pi \circ \mathcal{D}(\bar{\ell}) \circ \int_0^\tau \Lambda(\phi(\sigma), \bar{\lambda}(\sigma)) d\sigma \]
\[ = \Pi^\phi + K \circ \pi \circ \int_0^1 \Lambda(\phi(\tau), \bar{\lambda}(\tau)) d\tau, \]
which implies that \( \Pi^\phi = \Pi^\phi + K \circ \pi \circ \int_0^1 \Lambda(\phi(\tau), \bar{\lambda}(\tau)) d\tau. \)

We sometimes refer to an \( sL_\infty \)-morphism \( (V[[\hbar]], 0) \to (\mathcal{C}[[\hbar]], \mathcal{L}) \) as a homotopical family of quantum observables and call \( \Pi^\phi \in \text{Hom}(S(V), \mathcal{C})^0[[\hbar]] \) the associated quantum correlator. The \( n \)-th component \( \Pi^\phi_n = \Pi^\phi \circ \text{Hom}_{s^{n}V, \mathcal{C}} \in \text{Hom}(s^nV, \mathcal{C})^0[[\hbar]] \) of \( \Pi^\phi \) is called the \( n \)-fold quantum correlator. For example, we have
\[ \Pi^\phi_0(v_1) = \phi_0(v_1), \]
\[ \Pi^\phi_1(v_1, v_2) = \phi(1) \cdot \phi(2) + (-\hbar)\phi_1(1) \cdot \phi_2(2), \]
\[ \Pi^\phi_2(v_1, v_2, v_3) = \phi(1) \cdot \phi(2) \cdot \phi(3) + (-\hbar)\phi_1(1) \cdot \phi_2(2) \cdot \phi_3(3) \]
\[ + (-\hbar)(-1)^{|v_1||v_2|}\phi_1(1) \cdot \phi_2(2) \cdot \phi_1(3) + (-\hbar)^3\phi_3(1) \cdot \phi_2(2) \cdot \phi_1(3) \]
\[ + (-\hbar)^2\phi_3(1) \cdot \phi_2(2) \cdot \phi_3(3). \]

Consider a homotopical family of quantum observables \( \mathcal{V} = (V; [\mathfrak{g}]) \) and let \( \mathfrak{g} \) be a representative of \( [\mathfrak{g}] \). Then we have the following diagram:

\[ \begin{array}{ccc}
(V[[\hbar]], 0) & \xrightarrow{\phi} & (\mathcal{C}[[\hbar]], \mathcal{L}) \\
\Pi^\phi \circ \phi & & \mathcal{K} \\
(k[[\hbar]], 0) \end{array} \]

\[ \begin{array}{ccc}
(S(V)[[\hbar]], 0) & \xrightarrow{\phi} & (\mathcal{C}[[\hbar]], \mathcal{L}) \\
\Pi^\phi & & \mathcal{K} \\
(k[[\hbar]], 0) \end{array} \]

where the first line is in the category of cochain complexes over \( k[[\hbar]] \), while the second line is in the category of topologically-free \( sL_\infty \)-algebras over \( k[[\hbar]] \).

Note that the composition \( \mathcal{K} \circ \Pi^\phi \) is an invariant of the homotopy type \( [\mathfrak{g}] \) of the \( sL_\infty \)-morphism \( \phi \), since for any \( \mathfrak{g} \sim \mathfrak{g} \) we have \( \mathcal{K} \circ \Pi^\phi = \mathcal{K} \circ \Pi^\phi = 0 \), so that this gives an intrinsic notion attached to \( \mathcal{V} \). Note also that \( \mathcal{K} \circ \Pi^\phi \) is an invariant of the homotopy type \( [\mathcal{K}] \) of the quantum expectation \( \mathcal{K} \), since for any \( \mathcal{K} \sim \mathcal{K} \) we have...
Theorem 3.1. Let $\mu^\gamma = c \circ \Pi^\gamma$ and $\chi^\gamma = \chi \circ \varphi$ be the quantum moment and the quantum cumulant, respectively, of a homotopical family $\gamma$ of quantum observables. Then, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$, we have

$$
\mu^\gamma(v_1 \otimes \cdots \otimes v_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|} c(\varphi_p) \chi^\gamma(v_{B_1}) \cdots \chi^\gamma(v_{B_n}).
$$

Proof. From $c \circ \pi = \pi^k[\hbar] \circ \Psi_\gamma$ and $\Pi^\gamma = \pi \circ \varphi$, we obtain that

$$
\mu^\gamma = c \circ \Pi^\gamma = c \circ \pi \circ \psi = \pi^k[\hbar] \circ \Psi_\gamma \circ \psi = \pi^k[\hbar] \circ \Psi^\gamma,
$$

where we have used $\Psi_\gamma \circ \varphi = \Psi_\gamma \circ \varphi$ and $\chi^\gamma = \chi \circ \varphi$ for the last equality. Recall that

$$
\Psi^\gamma(v_1 \otimes \cdots \otimes v_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|} \epsilon(\varphi_p) \chi^\gamma(v_{B_1}) \otimes \cdots \otimes \chi^\gamma(v_{B_n}).
$$

Therefore the identity $\mu^\gamma = \pi^k[\hbar] \circ \Psi^\gamma$ is the desired formula in the theorem.

Remark 3.1. Fix a homotopical family $\gamma = (V, [\varphi])$ of quantum observables and assume that $V$ is a finite dimensional $\mathbb{Z}$-graded vector space. Choose a homogeneous basis $e_V = \{e_a\}$ and let $t_V = \{t^a\}$ be the dual basis. Consider the $\mathbb{Z}$-graded supercommutative $\mathbb{k}$-algebra $k[[t_1]]$, where $t_1 t = (-1)^{\delta h(e_1) h(e_2)} t_1 t_0^a$. Then, it is straightforward to check the following:
1. Let $\varphi$ be a representative of $[\varphi]$ and let

$$\Theta^\varphi := \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \varphi_n(e_{a_1}, \ldots, e_{a_n}) \in (k[[t_V]] \otimes \mathcal{E})^0[[\hbar]].$$

Then, we have $\mathbf{k} e^{-\frac{1}{\hbar} \Theta^\varphi} = 0$ and

$$e^{-\frac{1}{\hbar} \Theta^\varphi} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \varphi_n(e_{a_1}, \ldots, e_{a_n}).$$

2. Let $\mu^\varphi := c \circ \Pi^\varphi$ and $\chi^\varphi = \chi \cdot \varphi$, and define the following generating series

$$Z^\varphi := 1 + \sum_{n=1}^{\infty} \frac{1}{(-\hbar)^n} t^{a_n} \cdots t^{a_1} \mu^\varphi_n(e_{a_1}, \ldots, e_{a_n}) \in k[[t_V]]((\hbar))^0,$$

$$F^\varphi := \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \chi^\varphi_n(e_{a_1}, \ldots, e_{a_n}) \in k[[t_V]][[\hbar]].$$

Then we have the identity $Z^\varphi = e^{-\frac{1}{\hbar} \Theta^\varphi}$. Note also that $Z^\varphi \equiv \langle e^{-\frac{1}{\hbar} \Theta^\varphi} \rangle_c.$

3.2. When are two binary QFTs physically equivalent?

Let $\mathcal{C}[[\hbar]]_{\text{BQFTA}} \xrightarrow{\epsilon} k[[\hbar]]$ and $\mathcal{C}''[[\hbar]]_{\text{BQFTA}} \xrightarrow{\epsilon'} k[[\hbar]]$, be binary QFTs and assume that there is a homotopy equivalence $\mathcal{C}[[\hbar]]_{\text{BQFTA}} \xrightarrow{f} \mathcal{C}''[[\hbar]]_{\text{BQFTA}}$ of binary QFT algebras, i.e., both $f$ and $f'$ are morphisms of binary QFT algebras such that $f' \circ f$ is homotopic to the identity map on $\mathcal{C}[[\hbar]]$ and $f \circ f'$ is homotopic to the identity map on $\mathcal{C}''[[\hbar]]$. Note that $f$ and $f'$ are quasi-isomorphisms. Assume also that the following diagram in the category of binary QFT algebras is commutative up to homotopy:

$$\begin{array}{ccc}
\mathcal{C}[[\hbar]]_{\text{BQFTA}} & \xrightarrow{\epsilon} & k[[\hbar]] \\
\downarrow f & & \downarrow f' \\
\mathcal{C}''[[\hbar]]_{\text{BQFTA}} & \xrightarrow{\epsilon'} & k[[\hbar]]
\end{array}$$

Then, we may say that the two binary QFTs are physically equivalent. In the remaining part of this subsection, we check that such a physical equivalence induces an isomorphism of the homotopical families of quantum observables that preserves the quantum moment of every homotopical family of quantum observables.
Let $\mathcal{C}[\mathbb{H}]_{BQFTA} = (\mathcal{C}[\mathbb{H}], 1_\psi, \cdot, K)$ and $\mathcal{C}'[\mathbb{H}]_{BQFTA} = (\mathcal{C}'[\mathbb{H}], 1_{\psi'}, \cdot, K')$, and let $(\mathcal{C}[\mathbb{H}], 1_\psi, L)$ and $(\mathcal{C}'[\mathbb{H}], 1_{\psi'}, L')$ be the corresponding quantum descendant algebras. Let $\psi = \mathcal{R}(f)$ and $\psi' = \mathcal{R}(f')$ be the quantum descendants of the two homotopy inverses $f$ and $f'$, respectively. Then, we have the following homotopy equivalence in the category $\text{UsL}_\infty(k[\mathbb{H}])$:

$$\begin{align*}
\begin{pmatrix}
\psi \\
\psi'
\end{pmatrix} : 
\begin{pmatrix}
\mathcal{C}[\mathbb{H}], 1_\psi, L \\
\mathcal{C}'[\mathbb{H}], 1_{\psi'}, L'
\end{pmatrix} 
\end{align*}
$$

Recall that

$$f \circ \pi = \pi' \circ \Psi_\psi, \quad f' \circ \pi' = \pi \circ \Psi_{\psi'}, \quad (3.1)$$

where

$$\begin{align*}
\pi(x_1 \otimes \ldots \otimes x_n) &= x_1 \cdot \ldots \cdot x_n \quad \text{for all } n \geq 1 \text{ and } x_1, \ldots, x_n \in \mathcal{C}[\mathbb{H}],
\pi'(x_1' \otimes \ldots \otimes x_n') &= x_1' \cdot \ldots \cdot x_n' \quad \text{for all } n \geq 1 \text{ and } x_1', \ldots, x_n' \in \mathcal{C}'[\mathbb{H}].
\end{align*}$$

Let $\mathcal{V} = (V, [\varphi])$ be a homotopical family of quantum observables in $\mathcal{C}[\mathbb{H}]_{BQFTA}$ and let $\varphi$ be a representative of $[\varphi]$. Then, $\varphi' := \psi \cdot \varphi : (V[\mathbb{H}], \Omega) \longrightarrow (\mathcal{C}'[\mathbb{H}], \ell)$ is an $sL_\infty$-morphism whose homotopy type depends on $\varphi$ only via its homotopy type $[\varphi]$. Therefore $\mathcal{V}' = (V, [\varphi'])$ is a a homotopical family of quantum observables in $\mathcal{C}'[\mathbb{H}]_{BQFTA}$. Note also that $\psi' \cdot \varphi' = \psi' \cdot \psi \cdot \varphi \sim \varphi'$ if $\varphi' = \psi \cdot \varphi$ so that we have $[\psi' \cdot \varphi') = [\varphi']$. Obviously, the converse is also true. Therefore, the homotopy equivalence

$$\mathcal{C}[\mathbb{H}]_{BQFTA} \xrightarrow{f} \mathcal{C}'[\mathbb{H}]_{BQFTA}$$

induces an isomorphism of homotopical families of quantum observables.

Consider homotopical families of quantum observables $\mathcal{V}$ in $\mathcal{C}[\mathbb{H}]_{BQFTA}$ and $\mathcal{V}'$ in $\mathcal{C}'[\mathbb{H}]_{BQFTA}$ as defined above. Recall that

$$\mu^\mathcal{V} := \mathcal{C} \circ \Pi^\mathcal{V} = \mathcal{C} \circ \pi \circ \mathcal{R}(\varphi), \quad \mu'^{\mathcal{V}'} := \mathcal{C}' \circ \Pi^{'\mathcal{V}'} = \mathcal{C}' \circ \pi' \circ \mathcal{R}(\varphi').$$

From $\mathcal{C} \sim h \mathcal{C}' \circ f$ and $f \circ \pi = \pi' \circ \Psi_\varphi$ we have

$$\mu^\mathcal{V} = \mathcal{C}' \circ f \circ \pi \circ \Psi_\varphi = \mathcal{C}' \circ \pi' \circ \Psi_\varphi \circ \Psi_\varphi = \mathcal{C}' \circ \pi' \circ \Psi_{\varphi'} = \mathcal{C}' \circ \Pi^{'\mathcal{V}'} = \mu'^{\mathcal{V}'}$$

where we have used $\Psi_\varphi \circ \Psi_\varphi = \Psi_{\varphi' \varphi}$ for the third equality, and the fact that $\Pi^{\psi \circ \varphi} = \pi' \circ \Psi_{\varphi' \varphi}$ is cochain homotopic to $\Pi^{'\mathcal{V}'}$ since $\varphi' \sim \psi \cdot \varphi$ as $sL_\infty$-morphisms for the
fourth equality. Therefore, the induced isomorphism of homotopical families of quantum observables also preserves the quantum moment — as well as the quantum cumulant by Theorem 3.1 — of every homotopical family of quantum observables.

4. Quantizations of classical observables

Fix the structure \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} = (\mathcal{C}[[\hbar]], 1, \cdot, K) \) of a binary QFT algebra on \( \mathcal{C} \) and consider the underlying QFT complex \( (\mathcal{C}[[\hbar]], 1, \cdot, K) \), which shall be called the off-shell QFT complex of \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \). The classical limit \( (\mathcal{C}, 1, \cdot, K) \) of the off-shell QFT complex is a pointed cochain complex over \( k \), whose cohomology \( H \) is regarded as the space of equivalence classes of off-shell classical observables. There is a distinguished element \( 1_H \in H \), which is the \( K \)-cohomology class of \( 1 \) and \( (H, 1_H, 0) \) is a pointed cochain complex over \( k \) with zero differential. The main purpose of this subsection is to construct the structure of an on-shell QFT complex on \( H \) that is homotopy equivalent to the off-shell QFT complex. This will be the obstruction theory of quantization of classical observables.

**Theorem 4.1 (Definition).** There is an on-shell QFT complex structure \( (H[[\hbar]], 1_H, \kappa) \) on the classical cohomology \( H \) of a binary QFT algebra of \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \) which is homotopy equivalent to the off-shell QFT complex \( (\mathcal{C}[[\hbar]], 1, \cdot, K) \) of \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \). We call \( \mathcal{C}[[\hbar]]_{\text{BQFTA}} \) anomaly-free if \( \kappa = 0 \).

**Remark 4.1.** A QFT complex \( (\mathcal{C}[[\hbar]], 1, \cdot, K) \) is by definition a pointed cochain complex over \( k[[\hbar]] \). However, we will rarely work with its cohomology \( (H, 1_H, 0) \), where \( 1_H \) is the \( K \)-cohomology class of \( 1 \), but almost exclusively with the on-shell QFT complex \( (H[[\hbar]], 1_H, \kappa) \) on the classical cohomology \( H \), which is quasi-isomorphic to \( (H, 1_H, 0) \). There are at least two reasons for this choice:

1. We do not expect, in general, that the \( K \)-cohomology group \( H \) is a topologically free \( k[[\hbar]] \)-module, while both \( H[[\hbar]] \) and \( \mathcal{C}[[\hbar]] \) will always be topologically free. Here is a simple demonstration: Let \( \nu \in H \subset H[[\hbar]] \) satisfy \( \kappa \nu = 0 \). Then \( \nu \) can not be \( \kappa \)-exact, since \( \kappa = h\chi^{(1)} + \cdots \), so that it gives a cohomology class in \( H \). Consider \( h\nu \in H[[\hbar]] \), which always belongs to \( \ker \kappa \). Now \( h\nu \) can be \( \kappa \)-exact. For example, there may be some \( \eta \in H \) such that \( \chi^{(n)} \eta = 0 \) for all \( n \geq 2 \), while \( \chi^{(1)} \eta = \nu \) so that \( h\eta = \kappa \eta = h\chi^{(1)} \eta \). We remark that \( H \) is a free \( k[[\hbar]] \)-module if \( \kappa = 0 \) since the cohomology of \( (H[[\hbar]], \kappa = 0) \) is \( H[[\hbar]] \), which is isomorphic to \( H \) as topologically free \( k[[\hbar]] \)-modules. In general we may regard the on-shell QFT complex \( (H[[\hbar]], 1_H, \kappa) \) as a topologically-free resolution of \( (H, 1_H, 0) \).
2. The arena of classical physics is the classical cohomology $H$ (the on-shell QFT complex) and we think of ourselves as mere mortals contemplating the quantum world from this classical vantage point.

The classical complex $(\mathcal{C}, 1_{\mathcal{C}}, K)$ is a pointed cochain complex over a field $k$ where every quasi-isomorphism splits. Then the classical complex is homotopy equivalent to $(H, 1_H, 0)$. We choose all the data of such a homotopy equivalence

$$
\begin{align*}
(H, 1_H, 0) & \xrightarrow{\eta} (\mathcal{C}, 1_{\mathcal{C}}, K), \\
\eta & = \left\{ h \circ f = 1_H, \\
\quad f \circ h = h_{\mathcal{C}} - K \circ s - s \circ K, \right.
\end{align*}
$$

(4.1)

where both $f \in \text{Hom}(H, \mathcal{C})^0$ and $h \in \text{Hom}(\mathcal{C}, H)^0$ are quasi-isomorphisms of pointed cochain complexes satisfying $f(1_H) = 1_{\mathcal{C}}, K \circ f = 0, h(1_{\mathcal{C}}) = 1_H$ and $h \circ K = 0$, and $s$ is an element in $\text{Hom}(\mathcal{C}, \mathcal{C})^-$ satisfying $K = K \circ s \circ K$ and $s(1_{\mathcal{C}}) = 0$.

We can and will choose a splitting $s$ such that it satisfies the following side condition:

$$s \circ s = s \circ f = h \circ s = 0.$$  

(4.2)

We call such a trio $(f, h, s)$ a classical off-to-on-shell retract of the classical complex $(\mathcal{C}, 1_{\mathcal{C}}, K)$ — the official name for such a trio is strong deformation retract with the side conditions in the homological perturbation theory.

4.1. A quantization

Associated to each classical off-to-on-shell retract $(f, h, s)$, we can construct an on-shell QFT complex structure $(H[[h]], 1_H, \kappa)$ on $H$ and a quantized off-to-on-shell retract $(f, h, s)$, whose classical limit is limit $(f, h, s)$. That is, this data should satisfy

$$
\begin{align*}
(H[[h]], 1_H, \kappa) & \xrightarrow{\tilde{\eta}} (\mathcal{C}[[h]], 1_{\mathcal{C}}, K), \\
\tilde{\eta} & = \left\{ h \circ f = I_H[[h]], \\
\quad f \circ h = I_{\mathcal{C}}[[h]] - K \circ s - s \circ K, \right.
\end{align*}
$$

(4.3)

where $f \in \text{Hom}(H, \mathcal{C})^0[[h]]$ and $h \in \text{Hom}(\mathcal{C}, H)^0[[h]]$ are (quasi-isom)orphisms of QFT complexes, i.e.,

$$
f(1_H) = 1_{\mathcal{C}}, \quad K \circ f = f \circ \kappa, \quad h(1_{\mathcal{C}}) = 1_H, \quad h \circ K = \kappa \circ h,
$$

and $s \in \text{Hom}(\mathcal{C}, \mathcal{C})^{-1}[[h]]$ satisfies $s(1_{\mathcal{C}}) = 0$ as well as the following side conditions:

$$
s \circ s = s \circ f = h \circ s = 0, \quad s \circ s = s \circ f = h \circ s = 0,
$$

(4.4)
Proposition 4.1. Fix a classical off-to-on-shell retract \((f, h, s)\) and define
\[
\begin{align*}
\kappa &= h x^{(1)} + h^2 x^{(2)} + \ldots, \\
f &= f + hf^{(1)} + h^2 f^{(2)} + \ldots, \\
h &= h + hh^{(1)} + h^2 h^{(2)} + \ldots, \\
s &= s + hs^{(1)} + h^2 s^{(2)} + \ldots,
\end{align*}
\]
where, for all \(n \geq 1\),
\[
\begin{align*}
\chi^{(n)} &= h \circ g^{(n)}, \\
f^{(n)} &= -s \circ g^{(n)},
\end{align*}
\]
where
\[
g^{(n)} := \sum_{j=0}^{n-1} K^{(n-j)} \circ f^{(j)} - \sum_{j=1}^{n-1} f^{(n-j)} \circ \chi^{(j)},
\]
\[
h^{(n)} = -u^{(n)} \circ s, \quad \text{where} \quad u^{(n)} := \sum_{j=0}^{n-1} h^{(j)} \circ K^{(n-j)} - \sum_{j=1}^{n-1} \chi^{(j)} \circ h^{(n-j)},
\]
and
\[
s^{(n)} = -\sum_{j=0}^{n-1} s \circ K^{(n-j)} \circ s^{(j)} = -\sum_{j=0}^{n-1} s^{(j)} \circ K^{(n-j)} \circ s.
\]
Then \((H[[h]], 1_H, \kappa)\) is an on-shell QFT complex and \((f, h, s)\) is a quantization of \((f, h, s)\).

Proof. 1. From the side condition eq. (4.2) and by definition, the trio \((f, h, s)\) satisfies eq. (4.4).

2. We check that \(\kappa^2 = \kappa 1_H = 0\) and \(f(1_H) = 1_{\varphi}\) and \(K \circ f = f \circ \kappa\) as follows: It is obvious that \(\kappa^2 = 0 \mod \hbar\) and \(K \circ f = f \circ \kappa\) mod \(\hbar\), since \(\chi^{(0)} = K \circ f = 0\). Fix \(n \geq 2\) and assume that \(\kappa^2 = 0 \mod \hbar^n\) and \(K \circ f = f \circ \kappa\) mod \(\hbar^n\). Then, it can be checked that
\[
K \circ g^{(n)} = -\sum_{t=1}^{n-1} f \circ \chi^{(n-t)} \circ \chi^{(t)}.
\]
Applying \(h \circ \) from the left to the above identity, we obtain that
\[
\sum_{t=1}^{n-1} \chi^{(n-t)} \circ \chi^{(t)} = 0,
\]
and \(K \circ g^{(n)} = 0\). Let \(\chi^{(n)} := h \circ g^{(n)}\) and \(f^{(n)} := -s \circ g^{(n)}\). Then we obtain that
\[
f \circ \chi^{(n)} = f \circ h \circ g^{(n)} \equiv g^{(n)} - K \circ s \circ g^{(n)} - s \circ K \circ g^{(n)} = g^{(n)} + K \circ f^{(n)}.
\]
This relation is equivalent to the following:
\[
\sum_{\ell=0}^{n} K^{(n-\ell)} \circ f^{(\ell)} = \sum_{\ell=1}^{n} f^{(n-\ell)} \circ \chi^{(\ell)}. \tag{4.10}
\]

Combining our assumption with eq. (4.9) and eq. (4.10), we have \( \kappa^2 = 0 \mod h^{n+1} \) and \( K \circ f = f \circ \kappa \mod h^{n+1} \). By induction we conclude that \( \kappa^2 = 0 \) and \( K \circ f = f \circ \kappa \). It is straightforward to check that \( \kappa 1_H = 0 \) and \( f(1_H) = 1 \).

3. We check that \( h \circ K = \kappa \circ h \) and \( h(1_H) = 1 \) as follows: Note that \( h \circ K = \kappa \circ h \mod h \), since \( h \circ K = \chi^{(0)} = 0 \). Fix \( n \geq 2 \) and assume that \( h \circ K = \kappa \circ h \mod h^n \).

Then, it can be checked that
\[
u^{(n)} \circ K = 0, \quad \chi^{(n)} \circ h = u^{(n)} \circ f \circ h. \tag{4.11}
\]

Let \( \nu^{(n)} := -u^{(n)} \circ s \). Then, we have
\[
u^{(n)} = u^{(n)} \circ f \circ h + u^{(n)} \circ K \circ s + u^{(n)} \circ s \circ K
= u^{(n)} \circ f \circ h + u^{(n)} \circ s \circ K
= \chi^{(n)} \circ h - h^{(n)} \circ K,
\]

which is equivalent to the following relation:
\[
\sum_{j=0}^{n} h^{(j)} \circ K^{(n-j)} = \sum_{j=1}^{n} \chi^{(j)} \circ h^{(n-j)}. \tag{4.12}
\]

Combining our assumption with eq. (4.12), we have \( h \circ K = \kappa \circ h \mod h^{n+1} \). By induction we conclude that \( h \circ K = \kappa \circ h \). It is straightforward to check that \( h(1_H) = 1_H \).

4. From
\[
f = f - \sum_{n=1}^{\infty} h^n s \circ g^{(n)}, \quad h = h - \sum_{n=1}^{\infty} h^n u^{(n)} \circ s,
\]

and \( s \circ s = h \circ s = s \circ f = 0 \), it is obvious that \( h \circ f = h \circ f = \|H\| \).

5. It remains to check that \( f \circ h = I_{\mathcal{E}[h]} - K \circ s - s \circ K \). Let \( \xi = f \circ h + K \circ s + s \circ K \in Hom(\mathcal{E}', \mathcal{E})[[h]] \). Consider the expansion \( \xi = \xi^{(0)} + h \xi^{(1)} + h^2 \xi^{(2)} + \ldots \), where \( \xi^{(n)} = (f \circ h)^{(n)} + (K \circ s)^{(n)} + (s \circ K)^{(n)} \in Hom(\mathcal{E}', \mathcal{E}) \). Note that \( \xi^{(0)} = 0 \) since \( f \circ h = I_{\mathcal{E}'}, K \circ s = s \circ K \). We need to show that \( \xi^{(n)} = 0 \) for all \( n \geq 1 \). A direct computation using the definitions of \( (f, h, s) \) show that, for all \( n \geq 1 \),
\[
(f \circ h)^{(n)} = \sum_{k=0}^{n-1} f^{(k)} \circ h^{(n-k)} + f^{(n)} \circ h = -\sum_{k=0}^{n-1} (f \circ h)^{(k)} \circ K^{(n-k)} \circ s + f^{(n)} \circ h
\]
\[
= -\sum_{k=0}^{n-1} (f \circ h)^{(k)} \circ K^{(n-k)} \circ s + \sum_{k=0}^{n-1} s^{(k)} \circ K^{(n-k)} \circ K^{(0)} \circ s - \sum_{k=0}^{n} s^{(k)} \circ K^{(n-k)}.
\]
For \( n = 1 \), we have \( \xi^{(1)} = 0 \):

\[
(f \circ h)^{(1)} = -f \circ h \circ K^{(1)} \circ s + s \circ K^{(1)} \circ K \circ s - s \circ K^{(1)} - s^{(1)} \circ K
\]

\[
= -K^{(1)} \circ s + K \circ s \circ K^{(1)} \circ s + s \circ (K \circ K^{(1)} + K^{(1)} \circ K) \circ s - (s \circ K)^{(1)}
\]

\[
= -(K \circ s)^{(1)} - (s \circ K)^{(1)}.
\]

Fix \( n \geq 2 \) and assume that \( \xi^{(1)} = \ldots = \xi^{(n-1)} = 0 \). Then, we have

\[
(f \circ h)^{(n)} = -K^{(n)} \circ s + \sum_{k=0}^{n-1} (K \circ s)^{(k)} \circ K^{(n-k)} \circ s
\]

\[
+ \sum_{k=0}^{n-1} (s \circ K)^{(k)} \circ K^{(n-k)} \circ s + \sum_{k=0}^{n-1} s^{(k)} \circ K^{(n-k)} \circ K^{(0)} \circ s
\]

\[
- \sum_{k=0}^{n} s^{(k)} \circ K^{(n-k)}
\]

\[
= -(K \circ s)^{(n)} - (s \circ K)^{(n)} + \sum_{k=1}^{n} s^{(n-k)} \circ (K \circ K)^{(k)} \circ s
\]

\[
= -(K \circ s)^{(n)} - (s \circ K)^{(n)}.
\]

Therefore \( \xi^{(n)} = 0 \), so that \( \xi^{(n)} = 0 \) for all \( n \geq 1 \) by induction.

\[
\]

### 4.2. Variations of quantization

Recall that the differential \( \kappa \) and the quantized off-to-on-shell retract \((f, h, s)\) in Proposition 4.1 were obtained in terms of a fixed classical off-to-on-shell retract \((f, h, s)\). The following proposition show that these quantizations depend only up to an automorphism and homotopy on the choice of splitting data of the classical complex. We will only consider the variations of \( \kappa \) and \( f \).

**Proposition 4.2.** Let \((f, \kappa)\) and \((f', \kappa')\) be the duos associated with off-to-on-shell retracts \((f, h, s)\) and \((f', h', s')\), respectively. Then, there is a duo \((\xi, \lambda)\) such that

\[
\kappa' = \xi^{-1} \circ \kappa \circ \xi, \quad f' = f \circ \xi + K \circ \lambda + \lambda \circ \kappa',
\]

where \( \xi = 1_H + h \xi^{(1)} + h^2 \xi^{(2)} + \ldots \in \text{Aut}(H[[h]]) \) and

\[
\lambda(1_H) = 0, \quad \lambda = \lambda^{(0)} + h \lambda^{(1)} + h^2 \lambda^{(2)} + \ldots \in \text{Hom}(H, \mathcal{C})^{-1}[[h]].
\]
Proof. Recall that $f^{(0)} := f$, $f^{(0)} := f'$ and, for all $n \geq 1$,
\[
\begin{align*}
    f^{(n)} &= -s \circ g^{(n)}, \\
    x^{(n)} &= h \circ g^{(n)},
\end{align*}
\]  
\[
\begin{align*}
    f'^{(n)} &= -s' \circ g'^{(n)}, \\
    x'^{(n)} &= h' \circ g'^{(n)},
\end{align*}
\]  
where
\[
\begin{align*}
    g^{(n)} := & \sum_{\ell=0}^{n-1} K^{(n-\ell)} \circ f^{(\ell)} - \sum_{\ell=1}^{n-1} f^{(n-\ell)} \circ x^{(\ell)}, \\
    g'^{(n)} := & \sum_{\ell=0}^{n-1} K^{(n-\ell)} \circ f'^{(\ell)} - \sum_{\ell=1}^{n-1} f'^{(n-\ell)} \circ x'^{(\ell)}.
\end{align*}
\]  
Recall also that $K \circ g^{(n)} = K \circ g'^{(n)} = 0$ and
\[
\begin{align*}
    g^{(n)} &= f \circ x^{(n)} - K \circ f^{(n)}, \\
    g'^{(n)} &= f' \circ x'^{(n)} - K \circ f'^{(n)}.
\end{align*}
\]  
We are going to construct $\xi$ and $\lambda$ inductively so that $\xi^{(0)} = 1_H$, $\lambda^{(0)} = s \circ (f' - f)$ and, for all $n \geq 1$,
\[
\begin{align*}
    \xi^{(n)} &= h \circ w^{(n)}, \\
    \lambda^{(n)} &= -s \circ w^{(n)},
\end{align*}
\]  
where
\[
\begin{align*}
    w^{(n)} := f'^{(n)} - f^{(n)} - & \sum_{\ell=1}^{n-1} f^{(n-\ell)} \circ \xi^{(\ell)} - \sum_{\ell=0}^{n-1} K^{(n-\ell)} \circ \lambda^{(\ell)} - \sum_{\ell=1}^{n} \lambda^{(n-\ell)} \circ x'^{(\ell)}.
\end{align*}
\]  
1. In this step, we construct $\lambda$ mod $h^2$ and $\xi$ mod $h^2$ and prove the theorem modulo $h^2$. It is obvious, by definition, that $x^{(1)} = x^{(1)}$. Consider the identities $K \circ f = f \circ \kappa$ and $K \circ f' = f' \circ \kappa'$ modulo $h^2$:
\[
\begin{align*}
    K \circ f &= 0, \\
    K \circ f' &= 0, \\
    K^{(1)} \circ f + K \circ f^{(1)} &= f \circ x^{(1)}, \\
    K^{(1)} \circ f' + K \circ f'^{(1)} &= f' \circ x^{(1)}.
\end{align*}
\]  
Note that both $f'$ and $f$ are assumed to induce the identity map on the cohomology $H$, so that $h \circ (f' - f) = 0$. It follows that $f' - f = -K \circ s \circ (f' - f)$. Once we define $\lambda^{(0)} := -s \circ (f' - f)$, we conclude that $f' = f + K \circ \lambda^{(0)}$. From the equations in the second line of eq. (4.18), we obtain that $(f' - f) \circ x^{(1)} = K^{(1)} \circ (f' - f) + K \circ (f'^{(1)} - f^{(1)})$, which implies that $Kw^{(1)} = 0$, where
\[
\begin{align*}
    w^{(1)} := f'^{(1)} - f^{(1)} - K^{(1)} \circ \lambda^{(0)} - \lambda^{(0)} \circ x^{(1)} \in \text{Hom}(H, \mathcal{G})^0.
\end{align*}
\]
Hence we obtain a well-defined linear map \( \xi^{(1)} := h \omega^{(1)} \in \text{Hom}(H, H)^0 \) satisfying the following relation:

\[
\begin{align*}
    f \circ \xi^{(1)} &= f \circ h \circ \omega^{(1)} = \omega^{(1)} - K \circ s \circ \omega^{(1)} \\
    &= f^{(1)} - f^{(1)} - K^{(1)} \circ \lambda^{(0)} - \lambda^{(0)} \circ \chi^{(1)} + K \circ \lambda^{(1)} ,
\end{align*}
\]

where we have defined \( \lambda^{(1)} := -s \circ \omega^{(1)} \in \text{Hom}(H, \mathcal{G})^{-1} \). Therefore, we conclude that

\[
\chi^{(1)} = \chi^{(1)}, \quad \left\{ \begin{array}{ll}
    f' &= f + K \circ \lambda^{(0)}, \\
    f^{(1)} &= f^{(1)} + f \circ \xi^{(1)} + K \circ \lambda^{(1)} + K \circ \lambda^{(0)} + \lambda^{(0)} \circ \chi^{(1)}.
\end{array} \right.
\]

(4.19)

Let \( \xi := \mathbb{I}_H + h \xi^{(1)} \mod h^2 \) and \( \lambda := \lambda^{(0)} + h \lambda^{(1)} \mod h^2 \). Then the relations in eq. (4.19) are equivalent to the following;

\[
\lambda(1_H) = 0 \mod h^2, \quad \kappa' = \xi^{-1} \circ \kappa \circ \xi \mod h^2,
\]

\[
\xi(1_H) = 1_H \mod h^2, \quad f' = f \circ \xi + K \circ \lambda + \lambda \circ \kappa' \mod h^2.
\]

2. Fix \( n > 2 \). Assume that there are some \( \xi := I_H + h \xi^{(1)} + \cdots + h^{n-1} \xi^{(n-1)} \mod h^n \) and \( \lambda := \lambda^{(0)} + h \lambda^{(1)} + \cdots + h^{n-1} \lambda^{(n-1)} \mod h^n \) such that

\[
\lambda(1_H) = 0 \mod h^n, \quad \kappa' = \xi^{-1} \circ \kappa \circ \xi \mod h^n,
\]

\[
\xi(1_H) = 1_H \mod h^n, \quad f' = f \circ \xi + K \circ \lambda + \lambda \circ \kappa' \mod h^n.
\]

3. We claim that the following is a consequence of the assumptions in step (2):

\[
g''^{(n)} - g^{(n)} = -K \left( \sum_{\ell=1}^{n-1} f^{(n-\ell)} \circ \xi^{(\ell)} + \sum_{\ell=0}^{n-1} K^{(n-\ell)} \circ \lambda^{(\ell)} + \sum_{\ell=1}^{n-1} \lambda^{(n-\ell)} \circ \chi^{(\ell)} \right)
\]

\[
+ f(\xi^{-1} \circ \kappa \circ \xi)^{(n)} - f \circ \chi^{(n)}.
\]

(4.20)

Applying \( h \) to the above identity, we have

\[
h \circ g''^{(n)} - h \circ g^{(n)} = (\xi^{-1} \circ \kappa \circ \xi)^{(n)} - \chi^{(n)}.
\]

Using \( h \circ g^{(n)} \equiv h' \circ g^{(n)} = \chi^{(n)} \), since \( h = h' \) on \( \ker K \), and \( h \circ g^{(n)} = \chi^{(n)} \), we obtain the relation

\[
\chi^{(n)} = (\xi^{-1} \circ \kappa \circ \xi)^{(n)}.
\]

(4.21)

It follows that

\[
g''^{(n)} - g^{(n)} = -K \left( \sum_{\ell=1}^{n-1} f^{(n-\ell)} \circ \xi^{(\ell)} + \sum_{\ell=0}^{n-1} K^{(n-\ell)} \circ \lambda^{(\ell)} + \sum_{\ell=1}^{n-1} \lambda^{(n-\ell)} \circ \chi^{(\ell)} \right)
\]

\[
+ f \circ \chi^{(n)} - f \circ \chi^{(n)}.
\]
Using $f' = f + K\lambda(0)$, the above identity can be rewritten as follows:

\[
g''(n) - g^n(n) = -K \circ \left( \sum_{t=1}^{n-1} f(n-t) \circ \xi(t) + \sum_{t=0}^{n-1} K(n-t) \circ \lambda(t) + \sum_{t=1}^n \lambda(n-t) \circ \chi'(t) \right) + f' \circ \chi''(n) - f \circ \chi(n).
\]

On the other hand, the identities in eq. (4.15) imply that

\[
g''(n) - g^n(n) = f' \circ \chi''(n) - f \circ \chi(n) - K \circ f''(n) + K \circ f(n).
\]

By combining the two identities above, we conclude that

\[
K \circ w(n) = 0,
\]

\[
w'(n) := f''(n) - f(n) - \sum_{t=1}^{n-1} f(n-t) \circ \xi(t) - \sum_{t=0}^{n-1} K(n-t) \circ \lambda(t) - \sum_{t=1}^n \lambda(n-t) \circ \chi'(t).
\]

Note that $w(n) \in \text{Hom}(H, \mathcal{C})^0$ and $w(n)(1_H) = 0$. Let $\xi(n) := h \circ w(1) \in \text{Hom}(H, H)^0$ and $\lambda(n) := -s \circ w(n) \in \text{Hom}(H, H)^{-1}$. Then, we obtain that $f \circ \xi(n) = f \circ h \circ w(n) = w'(n) + K \circ \lambda(n)$ and conclude that

\[
\xi(n)(1_H) = \lambda(n)(1_H) = 0,
\]

\[
f''(n) = f(n) + \sum_{t=1}^n f(n-t) \xi(t) + \sum_{t=0}^n K(n-t) \lambda(t) + \sum_{t=1}^n \lambda(n-t) \chi'(t).
\]

Let

\[
\xi := 1 + h \xi(1) + \cdots + h^n \xi(n) \mod h^{n+1},
\]

\[
\lambda := \lambda(0) + h \lambda(1) + \cdots + h^n \lambda(n) \mod h^{n+1}.
\]

Then the relations in eq. (4.22) together with our assumptions in step (2) are equivalent to the following:

\[
\lambda(1_H) = 0 \mod h^{n+1}, \quad \kappa' = \xi^{-1} \circ \kappa \circ \xi \mod h^{n+1},
\]

\[
\xi(1_H) = 1_H \mod h^{n+1}, \quad f' = f \circ \xi + K \circ \lambda + \lambda \circ \kappa' \mod h^{n+1}.
\]

so we are done.

\[\square\]

**Corollary 4.1.** Suppose $\kappa = h^n \chi(n) + h^{n+1} \chi(n+1) + \cdots$ has $\chi(n) \neq 0$ for some $n \geq 1$. Then $\kappa' = h^n \chi'(n) + h^{n+1} \chi'(n+1) + \cdots$ and $\chi'(n) = \chi(n)$. 
4.3. Homotopy $\hbar$-divisibility

The purpose of this subsection is to define a $k$-linear operator

$$\nabla_{(-\hbar)^{-1}} : \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar]) \rightarrow \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar]),$$

which will be used to prove a key technical Lemma 4.1 for constructing a distinguished homotopical family of quantum observables in the next section.

Fix a classical off-to-on-shell retract $(f, h, s)$ and consider the associated deformation quantization $(H[\hbar], 1_H, \kappa) \mapsto (\mathcal{C}[\hbar], 1_\mathcal{C}, K)$ of $(H, 1_H, 0) \mapsto (\mathcal{C}, 1_\mathcal{C}, K)$. Introduce the following $k[H]$-linear operators

$$K_{H^C} : \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar]) \rightarrow \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar]),$$

$$K_{H} : \text{Hom}(\overline{T}(H), H[[\hbar]]) \rightarrow \text{Hom}(\overline{T}(H), H[[\hbar]]),$$

defined for all $n \geq 1$ and $v_1, \ldots, v_n \in H$ via

$$(K_{H^C}\Omega)(v_1, \ldots, v_n) := K\Omega(v_1, \ldots, v_n) - (-1)^{|H|} \sum_{j=1}^{n} \Omega(Jv_1, \ldots, Jv_{j-1}, 1_H, \kappa v_j, v_{j+1}, \ldots, v_n),$$

$$(K_{H}\omega)(v_1, \ldots, v_n) := K\omega(v_1, \ldots, v_n) - (-1)^{|\omega|} \sum_{j=1}^{n} \omega(Jv_1, \ldots, Jv_{j-1}, 1_H, \kappa v_j, v_{j+1}, \ldots, v_n),$$

where $\Omega \in \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar])$ and $\omega \in \text{Hom}(\overline{T}(H), H[[\hbar]])$. It is trivial to check that $K^2_{H^C} = K^2_H = 0$. Note also that $f \circ \omega \in \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar])$ and $K_{H^C}(f \circ \omega) = f \circ (K_{H^C}\omega)$ since $K_{H^C}(f \circ \omega) = K \circ f \circ \omega - (-1)^{|\omega|} f \circ (K \circ \omega - (-1)^{|\omega|} \omega \circ K)$. For a homogeneous element $\Omega \in \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar])$, we define $\nabla_{(-\hbar)^{-1}}\Omega$ via

$$(-\hbar)\nabla_{(-\hbar)^{-1}}\Omega := \Omega - f \circ h \circ \Omega - K_{H^C}(s \circ \Omega) - s \circ K \circ \Omega. \quad (4.23)$$

Note that the right hand side vanishes is divisible by $\hbar$ because it vanishes in the classical limit:

$$(\mathcal{C} - f \circ h - K \circ s - s \circ K) \circ \Omega = 0,$$

Therefore $\nabla_{-\hbar}\Omega \in \text{Hom}(\overline{T}(H), \mathcal{C}[\hbar])$. If the classical limit of $\Omega$ vanishes, we have

$$\nabla_{(-\hbar)^{-1}}\Omega = \frac{1}{(-\hbar)}\Omega.$$
Proposition 4.3. Fix $k \in \mathbb{Z}$ with $k \geq 1$ and assume that there is a trio $(\Omega, \Xi, \omega)$, where $\Omega$ and $\Xi$ are in $\text{Hom}(\overline{T}(H), \mathcal{C})[[\hbar]]$ and $\omega$ is in $\text{Hom}(\overline{T}(H), H)[[\hbar]]$, which satisfies the following relation:

$$K_{H \Phi} \Omega = (-\hbar)^k \Xi - f \circ \omega.$$ 

Define, for $j = 0, 1, \ldots, k$,

$$\Omega^{[j]} := \left(\nabla_{(-\hbar)^{-1}}\right)^j \Omega,$$

$$(-\hbar)^j \omega^{[j]} := \omega + \sum_{i=0}^{j-1} (-\hbar)^i \kappa_{BH} (h \circ \Omega^{[i]}),$$

where $\Omega^{[j]}$ denotes the classical limit of $\Omega^{[j]}$. Then, we have

1. for $j = 0, 1, \ldots, k$,

$$K_{H \Phi} \Omega^{[j]} = (-\hbar)^{k-j} \Xi - f \circ \omega^{[j]}$$

and

$$\begin{cases} \Omega^{[j]} \in \text{Hom}(T(H), \mathcal{C})[[\hbar]], \\ \omega^{[j]} \in \text{Hom}(T(H), H)[[\hbar]], \end{cases}$$

2. for $i = 0, 1, \ldots, k-1$,

$$K \circ \Omega^{[j]} = 0.$$

Proof. Recall that $\Omega^{[j]} \in \text{Hom}(T(H), \mathcal{C})[[\hbar]]$ for all $j = 0, 1, \ldots, k$. Note that $\Omega^{[j]}$ and $\omega^{[j]}$, $j = 0, 1, \ldots, k$, can be also defined recursively by $\Omega^{[0]} := \Omega$, $\omega^{[0]} = \omega$ and, for $1 \leq j \leq k$,

$$\Omega^{[j]} := \nabla_{(-\hbar)^{-1}} \Omega^{[j-1]}, \\ (-\hbar)^j \omega^{[j]} := \omega^{[j-1]} + \kappa_{BH} (h \circ \Omega^{[j-1]}).$$

From $\Omega^{[0]} = \Omega$ and $\omega^{[0]} = \omega$, the relation we have assumed is

$$K_{H \Phi} \Omega^{[0]} = (-\hbar)^k \Xi - f \circ \omega^{[0]}. \quad (4.24)$$

From the condition that $k \geq 1$, the classical limit of this equation is

$$K \circ \Omega^{[0]} = -f \circ \omega^{[0]},$$

where $\omega^{[0]} \in \text{Hom}(\overline{T}(H), H)$ is the classical limit of $\omega^{[0]}$. Applying $h$ to this classical limit equation, we have $0 = h \circ f \circ \omega^{[0]}$ since $h \circ K = 0$. From $h \circ f = I_H$, we conclude that

$$K \circ \Omega^{[0]} = 0, \quad \omega^{[0]} = 0. \quad (4.25)$$

From, $\Omega^{[1]} := \nabla_{(-\hbar)^{-1}} \Omega$ and $K \circ \Omega^{[0]} = 0$, it follows that

$$(-\hbar) \Omega^{[1]} = \Omega^{[0]} - f \circ (h \circ \Omega^{[0]}) - K_{H \Phi} \left(s \circ \Omega^{[0]}\right).$$

(4.26)
Applying $K_{\mathcal{H}\mathcal{E}}$ to the above and using $K_{\mathcal{H}\mathcal{E}}^2 = 0$ and $K_{\mathcal{H}\mathcal{E}} f = 0$, we obtain that
\[ (-h)K_{\mathcal{H}\mathcal{E}} \Omega^{[1]} = K_{\mathcal{H}\mathcal{E}} \Omega^{[0]} - f \circ \kappa_{\mathcal{H}\mathcal{H}} (h \circ \Omega^{[0]}) . \]

From eq. (4.24), the above relation gives
\[ (-h)K_{\mathcal{H}\mathcal{E}} \Omega^{[1]} = (-h)^k \Xi - (-h) f \circ \omega^{[1]} , \quad (4.27) \]
where
\[ (-h)\omega^{[1]} := \omega^{[0]} + \kappa_{\mathcal{H}\mathcal{H}} (h \circ \Omega^{[0]}) . \]

Note that the right-hand-side of the above equation vanishes in the classical limit since both $\omega^{[0]}$ and $\kappa_{\mathcal{H}\mathcal{H}}$ vanish in the classical limit. Therefore we have
\[ \omega^{[1]} \in \text{Hom}(\mathcal{T}(H), H)[[h]]. \quad (4.28) \]

Dividing eq. (4.27) by $(-h)$, we conclude that
\[ K_{\mathcal{H}\mathcal{E}} \Omega^{[1]} = (-h)^{k-1} \Xi - f \circ \omega^{[1]} . \quad (4.29) \]

Therefore, we are done if $k = 1$.

For $k \geq 2$, we work inductively as follows: Fix $n$ such that $1 \leq n \leq k - 1$ and assume that, for all $i = 0, 1, \ldots, n$,

1. $\omega^{[i]} \in \text{Hom}(\mathcal{T}(H), H)[[h]]$;
2. $K_{\mathcal{H}\mathcal{E}} \Omega^{[i]} = (-h)^{k-1} \Xi - f \circ \omega^{[i]}$;
3. $\omega^{[i]} = K \circ \Omega^{[i]} = 0$, where $\omega^{[i]}$ is the classical limit of $\omega^{[i]}$.

From $(-h)\omega^{[n+1]} := \omega^{[n]} + \kappa_{\mathcal{H}\mathcal{H}} (h \circ \Omega^{[n]})$ we conclude that
\[ \omega^{[n+1]} \in \text{Hom}(\mathcal{T}(H), H)[[h]], \quad (4.30) \]
since both the classical limits of $\omega^{[n]}$ and $\kappa_{\mathcal{H}\mathcal{H}}$ vanish. Note that we have
\[ (-h)^{n+1} \Omega^{[n+1]} = \Omega - f \circ \left( \sum_{j=0}^{n} (-h)^j h \circ \Omega^{[j]} \right) - K_{\mathcal{H}\mathcal{E}} \left( \sum_{j=0}^{n} (-h)^j s \circ \Omega^{[j]} \right) , \]
where we have used $\Omega^{[i]} = (\nabla_{(-h)^{-1}})^i \Omega$ and the assumption that $K \circ \Omega^{[i]} = 0$, for all $i = 0, 1, \ldots, n$. Applying $K_{\mathcal{H}\mathcal{E}}$ to the above, we have
\[ (-h)^{n+1} K_{\mathcal{H}\mathcal{E}} \Omega^{[n+1]} = K_{\mathcal{H}\mathcal{E}} \Omega - f \circ \left( \sum_{j=0}^{n} (-h)^j \kappa_{\mathcal{H}\mathcal{H}} (h \circ \Omega^{[j]}) \right) \]
\[ = (-h)^k \Xi - f \circ \left( \omega + \sum_{j=0}^{n} (-h)^j \kappa_{\mathcal{H}\mathcal{H}} (h \circ \Omega^{[j]}) \right) \]
\[ = (-h)^k \Xi - (-h)^n f \circ (\omega^{[n]} + \kappa_{\mathcal{H}\mathcal{H}} (h \circ \Omega^{[n]})) . \]
Using \((-\hbar)\omega^{[n+1]} := \omega^{[n]} + \kappa_{HH}(h \circ \Omega^{[n]})\), the above identity becomes

\[ (-\hbar)^{n+1} K_{HH} \Omega^{[n+1]} = (-\hbar)^{k} \Xi - (-\hbar)^{n+1} f \circ \omega^{[n+1]} \]

Since \(n + 1 \leq k\), the above equation gives

\[ K_{HH} \Omega^{[n+1]} = (-\hbar)^{k-n-1} \Xi - f \circ \omega^{[n+1]}, \quad (4.31) \]

For \(n + 1 < k\), the classical limit of this equation is \(K \circ \Omega^{[n+1]} = -f \circ \omega^{[n+1]}\), which in turn implies that

\[ \begin{align*}
  K \circ \Omega^{[n+1]} &= 0, \\
  \omega^{[n+1]} &= 0.
\end{align*} \quad (4.32) \]

For \(n + 1 = k\), we have

\[ K_{HH} \Omega^{[k]} = \Xi - f \circ \omega^{[k]}, \quad (4.33) \]

and we are done by induction.

As a corollary of Proposition 4.3, we conclude the following.

**Lemma 4.1.** For a trio \((\Omega, \Xi, \omega)\) satisfying, for \(k \geq 1\);

\[ K_{HH} \Omega = (-\hbar)^{k} \Xi - f \circ \omega, \quad \left\{ \begin{array}{l}
  \Omega, \Xi \in \text{Hom}(\overline{T}(H), \mathcal{C}[h]), \\
  \omega \in \text{Hom}(\overline{T}(H), H)[h],
\end{array} \right. \]

we have

\[ K_{HH} \Omega^{[k]} = \Xi - f \circ \omega^{[k]} \] and

\[ \left\{ \begin{array}{l}
  \Omega^{[k]} \in \text{Hom}(\overline{T}(H), \mathcal{C}[h]), \\
  \omega^{[k]} \in \text{Hom}(\overline{T}(H), H)[h],
\end{array} \right. \] \quad (4.34)

where

\[ \begin{align*}
  (-\hbar)^{k} \Omega^{[k]} := \Omega - f \circ \left( \sum_{i=0}^{k-1} (-\hbar)^{i} h \circ \Omega^{[i]} \right) - K_{HH} \left( \sum_{i=0}^{k-1} (-\hbar)^{i} s \circ \Omega^{[i]} \right), \\
  (-\hbar)^{k} \omega^{[k]} := \omega + \sum_{i=0}^{k-1} (-\hbar)^{i} \kappa_{HH} (h \circ \Omega^{[i]}),
\end{align*} \quad (4.35) \]
5. Mastering quantum correlations

The purpose of this section is to introduce the master equations for level 0 and 1 quantum correlators and present canonical solutions. Throughout this section, we fix the following data.

1. a binary QFT algebra \(\mathcal{C}[[\hbar]]_{QFTA} = (\mathcal{C}[[\hbar]], 1, \cdot, K)\), where
   (a) the tuple \((\mathcal{C}[[\hbar]], 1, \cdot, L)\), is the quantum descendant unital \(sL_{\infty}\)-algebra and
   (b) the tuple \((H[[\hbar]], 1_{H}, \kappa)\) is the on-shell QFT complex,
2. a classical off-to-on-shell retraction \((f, h, s)\) and a quantization \((f, h, s)\) of it.

5.1. Master equation for the level zero quantum correlators

In this subsection, we define the master equation governing the level zero quantum correlators and find a canonical solution.

**Definition 5.1.** The level zero quantum master equation is a system of equations for a tuple \(\{\hat{\pi}^0, \eta^{-1}, \hat{\ell}, \phi^0\}\), where

\[
\hat{\pi}^0 \in \text{Hom}(\hat{S}(H), H)^0 [[\hbar]], \quad \eta^{-1} \in \text{Hom}(\hat{S}(H), \mathcal{C})^{-1} [[\hbar]],
\]

\[
\hat{\ell} \in \text{Hom}(\hat{S}(H), H)^1 [[\hbar]], \quad \phi^0 \in \text{Hom}(\hat{S}(H), \mathcal{C})^0 [[\hbar]],
\]

are defined recursively for all \(n \geq 1\) and homogeneous \(v_1, \ldots, v_n \in H\) by the equations

\[
f(\hat{\pi}^0_n(v_1, \ldots, v_n)) = \sum_{p \in P(n)} (-\hbar)^{n-|p|} \epsilon(p) \phi^0(v_{B_1}) \cdots \phi^0(v_{B_{|p|}}) \]
\[
- \kappa \eta^{-1}_n(v_1, \ldots, v_n)
\]
\[
- \sum_{p \in P(n)} \left(\sum_{|B_1|=n-|p|+1} (-\hbar)^{n-|p|} \eta^{-1}_n(J\nu_{B_1}, \ldots, J\nu_{B_{|p|}}, \hat{\ell}_n(v_{B_1}), v_{B_{|p|}}, \ldots, v_{B_{|p|}})\right),
\]

\[
\kappa \hat{\pi}^0_n(v_1, \ldots, v_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|} \epsilon(p) \hat{\pi}^0_n(J\nu_{B_1}, \ldots, J\nu_{B_{|p|}}, \hat{\ell}_n(v_{B_1}), v_{B_{|p|}}, \ldots, v_{B_{|p|}}),
\]

with the following initial conditions:

\[
\hat{\pi}^0_1 = \mathbb{I}_{H[[\hbar]]}, \quad \eta^{-1}_1 = 0, \quad \phi^0_1 = f, \quad \hat{\ell}_1 = \kappa.
\]
Remark 5.1. For \( n = 1 \), the (level zero) quantum master equation is

\[
\phi \circ \pi_1^0 = \eta^{-1}_1 - \eta^{-1}_1 \circ \ell_1,
\]

so that the initial conditions solve the equation for \( n = 1 \).

The following proposition may be regarded as the appropriate integrability condition for the level zero quantum master equation.

**Proposition 5.1.** Let \( \{ \hat{\lambda}^0, \eta^{-1}, \hat{\ell}, \phi^0 \} \) be a solution to the level zero quantum master equation. Then, we have

- The \( L_\infty \) structure \( \hat{\ell} = 0 \) whenever \( \kappa = 0 \).
- The tuple \( (H[[\hbar]], 1_H, \hat{\ell}) \) is a topologically-free unital \( sL_\infty \)-algebra over \( k[[\hbar]] \), i.e., for all \( n \geq 1 \) and homogeneous \( v_1, \ldots, v_n \in H \), we have
  \[
  \sum_{p \in \mathcal{P}(n)} \epsilon(p) \hat{\ell}_{|p|}(Jv_{B_1}, \ldots, Jv_{B_{n-1}}, v_{B_n}, v_{B_{n+1}}, \ldots, v_{B_{2n}})) = 0,
  \]

  \[
  \hat{\ell}_n(v_1, \ldots, v_{n-1}, 1_H) = 0.
  \]

- The map \( \phi^0 \) satisfies the conditions to be a quasi-isomorphism of topologically-free unital \( sL_\infty \)-algebras \( \phi^0 : (H[[\hbar]], 1_H, \hat{\ell}) \longrightarrow (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \hat{\ell}) \), i.e., for all \( n \geq 1 \) and homogeneous \( v_1, \ldots, v_n \in H \), we have
  \[
  \sum_{p \in \mathcal{P}(n)} \epsilon(p) \phi^0_{|p|}(v_{B_1}, \ldots, v_{B_{n-1}}, \phi(v_{B_n})), \phi(v_{B_{n+1}}), \ldots, v_{B_{2n}})) = 0.
  \]

  \[
  \phi^0_n(v_1, \ldots, v_{n-1}, 1_H) = 1_{\mathcal{C}} \cdot \delta_{n,1},
  \]

  where \( \delta_{1,1} = 1 \) and \( \delta_{n,1} = 0 \) if \( n \neq 1 \), and \( \phi^0 : (H[[\hbar]], 1_H, \hat{\ell}) \rightarrow (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \hat{\ell}_1) \) is a pointed cochain quasi-isomorphism.

**Proof.** It is obvious that \( \hat{\ell}_n = 0 \) for all \( n \geq 1 \) whenever \( \kappa = 0 \). It is also straightforward to check the following properties:

a. (unitality of structure) \( \hat{\ell}_n(v_1, \ldots, v_{n-1}, 1_H) = 0 \) for all \( n \geq 1 \) and \( v_1, \ldots, v_{n-1} \in H \).

b. (unitality of morphism) \( \phi^0_1(1_H) = 1_{\mathcal{C}} \), and \( \phi^0_{n+1}(v_1, \ldots, v_n, 1_H) = 0 \) for all \( n \geq 1 \) and \( v_1, \ldots, v_{n-1} \in H \).
Notation 5.2. Consider the reduced symmetric coalgebra $\mathcal{S}^\omega(\mathcal{C})[[h]]$ which is cogenerated by the topologically-free $k[[h]]$-module $\mathcal{C}[[h]]$. We may extend the $h$-shifted version of $\mathfrak{l}$ to a coderivation $\delta_\mathfrak{l}$ on $\mathcal{S}^\omega(\mathcal{C})[[h]]$. This coderivation is characterized for all $n \geq 1$ and homogeneous $x_1, \ldots, x_n \in \mathcal{C}[[h]]$ by the formula

$$\delta_\mathfrak{l}(x_1 \otimes \ldots \otimes x_n) := \sum_{p \in P(n)} (-h)^{-|p|} \varepsilon(p) Jx_{B_1} \otimes \ldots \otimes Jx_{B_{n-1}} \otimes \ell(x_{B_1}) \otimes x_{B_{n-1}} \otimes \ldots \otimes x_{B_{|p|}}.$$  

Define $\pi^0 \in \text{Hom}(\mathcal{S}(\mathcal{C}), \mathcal{C})^0[[h]]$ for all $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{C}[[h]]$ to be

$$\pi^0(x_1 \otimes \ldots \otimes x_n) = x_1 \cdot \ldots \cdot x_n.$$  

Then, by the definition of the quantum descendant, we have $K \circ \pi^0 = \pi^0 \circ \delta_\mathfrak{l}$, which implies that $\delta_\mathfrak{l} \circ \delta_\mathfrak{l} = 0$.

Similarly, consider the reduced symmetric coalgebra $\mathcal{S}^\omega(H)[[h]]$ which is cogenerated by the topologically-free $k[[h]]$-module $H[[h]]$. Define a coderivation $\delta_H$ on $\mathcal{S}^\omega(H)[[h]]$ (as before) and a coalgebra map $\Psi^{\phi}: \mathcal{S}^\omega(H)[[h]] \rightarrow \mathcal{S}^\omega(\mathcal{C})[[h]]$ characterized by the following equations for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in H[[h]]$:

$$\delta_H(v_1 \otimes \ldots \otimes v_n) := \sum_{p \in P(n)} (-h)^{-|p|} \varepsilon(p) Jv_{B_1} \otimes \ldots \otimes Jv_{B_{n-1}} \otimes \ell(v_{B_1}) \otimes v_{B_{n-1}} \otimes \ldots \otimes v_{B_{|p|}},$$

$$\Psi^{\phi}(v_1 \otimes \ldots \otimes v_n) := \sum_{p \in P(n)} (-h)^{-|p|} \varepsilon(p) \phi^0(v_{B_1}) \otimes \ldots \otimes \phi^0(v_{B_{|p|}}).$$

Using this notation the level zero quantum master equation can be written as follows:

$$f \circ \pi^0 = \pi^0 \circ \Psi^{\phi} - K \circ \eta^{-1} - \eta^{-1} \circ \delta_\mathfrak{l}, \quad (5.1)$$

$$K \circ \pi^0 = \pi^0 \circ \delta_H.$$  

Applying $K \circ$ to the second equation above and using $K \circ K = 0$, we obtain the equation

$$0 = K \circ \pi^0 \circ \delta_H = \pi^0 \circ \delta_H \circ \delta_H.$$  

From $\pi^0 = 1_H$, we have

$$\text{pr}_{H[[h]]} \circ \delta_H \circ \delta_H = 0 \iff \delta_\mathfrak{l} \circ \delta_\mathfrak{l} = 0. \quad (5.2)$$

In components, we have, for all $n \geq 1$,

$$0 = \text{pr}_{H[[h]]} \circ \delta_\mathfrak{l} \circ \delta_\mathfrak{l}(v_1 \otimes \ldots \otimes v_n)$$

$$\equiv (-h)^{n-1} \sum_{p \in P(n)} \varepsilon(p) \ell(Jv_{B_1} \otimes \ldots \otimes Jv_{B_{n-1}} \otimes \ell(v_{B_1}) \otimes v_{B_{n-1}} \otimes \ldots \otimes v_{B_{|p|}}).$$
Therefore \((H[[\hbar]], \hat{l})\) is an \(sL_\infty\)-algebra over \(k[[\hbar]]\). Then, property (ii) implies that \((H[[\hbar]], 1_H, \hat{1})\) is a unital \(sL_\infty\)-algebra over \(k[[\hbar]]\).

Applying \(K \circ \) to the first part of eq. (5.1) and using \(K \circ K = 0\), we obtain that

\[
K \circ f \circ \pi^0 + K \circ \eta^{-1} \circ \hat{\delta}_L = K \circ \pi^0 \circ \Psi_{\phi^0}. \tag{5.3}
\]

Consider the left hand side of this equation:

\[
K \circ f \circ \pi^0 + K \circ \eta^{-1} \circ \hat{\delta}_L = f \circ K \circ \pi^0 + K \circ \eta^{-1} \circ \hat{\delta}_L
\]

\[
= \pi^0 \circ \Psi_{\phi^0} \circ \hat{\delta}_L - K \circ \eta^{-1} \circ \hat{\delta}_L
\]

\[
= \pi^0 \circ \Psi_{\phi^0} \circ \hat{\delta}_L,
\]

where we have used \(K \circ f = f \circ K\) for the first equality, the second part of eq. (5.1) for the second equality, the first part eq. (5.1) for the third equality, and eq. (5.2) for the last equality. From \(K \circ \pi^0 = \pi^0 \circ \hat{\delta}_L\), the right hand side of eq. (5.3) is \(K \circ \pi^0 \circ \Psi_{\phi^0} = \pi \circ \hat{\delta}_L \circ \Psi_{\phi^0}\). Therefore, eq. (5.3) is equivalent to the following:

\[
\pi^0 \circ (\hat{\delta}_L \circ \Psi_{\phi^0} - \Psi_{\phi^0} \circ \hat{\delta}_L) = 0.
\]

From \(\pi^0_1 = I_{\phi^i}\), we have \(pr_{\phi^i[[\hbar]]}(\hat{\delta}_L \circ \Psi_{\phi^0} - \Psi_{\phi^0} \circ \hat{\delta}_L) = 0\). In components, we have, for all \(n \geq 1\),

\[
0 = pr_{\phi^i[[\hbar]]}(\hat{\delta}_L \circ \Psi_{\phi^0} - \Psi_{\phi^0} \circ \hat{\delta}_L)(v_1 \circ \ldots \circ v_n)
\]

\[
= (-\hbar)^{n-1} \sum_{p \in P(n)} \epsilon(p) \ell(\phi^0(v_{B_1}) \circ \ldots \circ \phi^0(v_{B_p}))
\]

\[
- (-\hbar)^{n-1} \sum_{p \in P(n)} \epsilon(p) \phi^0(f_{B_1} \circ \ldots \circ \hat{\ell}(v_{B_1}) \circ \ldots \circ v_{B_p})
\]

Therefore, \(\phi^0 : (\mathcal{C}[[\hbar]], \hat{l}) \to (H[[\hbar]], \hat{1})\) is also an \(sL_\infty\)-morphism. Combined with property (ii), we conclude that \(\phi^0 : (H[[\hbar]], 1_H, \hat{1}) \to (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, \hat{l})\) is a unital \(sL_\infty\)-morphism.

Finally, we recall that \(l_1 = K\), \(\hat{l}_1 = \kappa\), and \(\phi_1 = f\). Therefore \(\phi^0\) is a unital \(sL_\infty\)-quasi-isomorphism since \(f : (H[[\hbar]], 1_H, \kappa) \to (\mathcal{C}[[\hbar]], 1_{\mathcal{C}}, K)\) is a pointed cochain quasi-isomorphism.
Now we explain a strategy to solve the level zero quantum master equation.

For \( n = 1 \), the initial conditions solve the quantum master equation:

\[
\dot{\pi}_1^0 = I_H \cdot \Omega[\hbar], \quad \eta_1^{-1} = 0, \quad \dot{\ell}_1 = \kappa, \quad \phi_1^{-1} = f.
\]

It follows that \( K_{H^0} \phi_1^0 = 0 \). Consider the quantum master equation for \( n = 2 \):

\[
f(\tilde{\pi}_2^0(v_1, v_2)) = \phi_1^0(v_1) \cdot \phi_1^0(v_2) + (-\hbar)\phi_2^0(v_1, v_2) - K_\eta^{-1}_2(v_1, v_2) - (J v_1, v_2) - (\hbar)\ell_2(v_1, v_2),
\]

\[
K_{H^0} \tilde{\pi}_2^0(v_1, v_2) = \tilde{\pi}_2^0(\dot{\ell}_1(v_1), v_2) + \dot{\pi}_2^0(J v_1, \dot{\ell}_1(v_2)) + (-\hbar)\ell_2(v_1, v_2).
\]

which can be written in the following form:

\[
K_{H^0} \eta^{-1}_2 + f \circ \tilde{\pi}_2^0 - \Omega_2^0 = (-\hbar)\phi_2^0,
\]

\[
K_{H^0} \tilde{\pi}_2^0 = (-\hbar)\ell_2,
\]

where \( \Omega_2^0 \in \text{Hom}(S^2H, \mathcal{C})^0[[\hbar]] \) is defined by

\[
\Omega_2^0(v_1, v_2) := \phi_1^0(v_1) \cdot \phi_1^0(v_2).
\]

Then all we need is to find \( \tilde{\pi}_2^0 \in \text{Hom}(S^2H, H)^0[[\hbar]] \) and \( \eta^{-1}_2 \in \text{Hom}(S^2H, \mathcal{C})^{-1}[[\hbar]] \)

so that

\[
\phi_2^0 := \frac{1}{(-\hbar)} \left( K_{H^0} \eta^{-1}_2 + f \circ \tilde{\pi}_2^0 - \Omega_2^0 \right) \in \text{Hom}(S^2H, \mathcal{C})^0[[\hbar]].
\]

Then we will also have \( \dot{\ell}_2 := \frac{1}{(-\hbar)} K_{H^0} \tilde{\pi}_2^0 \in \text{Hom}(S^2H, \mathcal{C})^1[[\hbar]] \) since \( K_{H^0} \) is divisible by \( \hbar \). It is straightforward to check that

\[
K_{H^0} \Omega_2^0(v_1, v_2) = (-\hbar)\ell_2(\phi_1(v_1), \phi_1(v_2)),
\]

so that the classical limit \( \Omega_2^0 \in \text{Hom}(S^2H, \mathcal{C})^0 \) of \( \Omega_2^0 \) satisfies \( K \circ \Omega_2^0 = 0 \). It follows that

\[
\nabla_{(-\hbar)^{-1}} \Omega_2^0 = \frac{1}{(-\hbar)} \left( \Omega_2^0 - f \circ \Omega_2^0 - K_{H^0} (s \circ \Omega_2^0) \right) \in \text{Hom}(S^2H, \mathcal{C})^0[[\hbar]],
\]

where \( \nabla_{(-\hbar)^{-1}} \) is the operator defined in eq. (4.23). Therefore, the following is a solution to the level zero quantum master equation for \( n = 2 \):

\[
\tilde{\pi}_2^0 := h \circ \Omega_2^0, \quad \eta_2^{-1} := s \circ \Omega_2^0, \quad \phi_2^0 := -\nabla_{(-\hbar)^{-1}} \Omega_2^0, \quad \dot{\ell}_2 := \frac{1}{(-\hbar)} K_{H^0} \tilde{\pi}_2^0.
\]
In general, for \( n \geq 2 \), the level zero quantum master equation can be written in the following form:

\[
\mathbf{K}_H \epsilon \mathbf{\eta}^{-1}_n + \mathbf{f} \circ \mathbf{\xi}^0_n - \mathbf{\Omega}^0_n = (-\hbar)^{n-1} \mathbf{\Phi}^0_n,
\]

\[
\kappa_{HH} \mathbf{\xi}^0_n + \mathbf{\vartheta}^1_n = (-\hbar)^{n-1} \mathbf{\dot{\ell}}_n,
\]

where \( \mathbf{\Omega}^0_n \in \text{Hom}\left( \mathcal{S}^n H, \mathcal{C}^0 \left[ \left[ H \right] \right] \right) \) and \( \mathbf{\vartheta}^1_n \in \text{Hom}\left( \mathcal{S}^n H, H \right) \left[ \left[ H \right] \right] \) are defined for homogeneous \( v_1, \ldots, v_n \in H \) as

\[
\mathbf{\Omega}^0_n(v_1, \ldots, v_n) := \sum_{p \in \mathcal{P}(n)} (-\hbar)^{n-|p|} \epsilon(p) \mathbf{\phi}^0(v_{B_1}) \cdots \mathbf{\phi}^0(v_{B_{|p|}})
\]

\[
- \sum_{p \in \mathcal{P}(n)} (-\hbar)^{n-|p|} \mathbf{\eta}_{|p|}^{-1}(Jv_{B_1}, \ldots, Jv_{B_{|p|-1}}, \mathbf{\dot{\ell}}(v_{B_1}), v_{B_{|p|}}, \ldots, v_{B_{|p|}}),
\]

\[
\mathbf{\vartheta}^1_n(v_1, \ldots, v_n) := - \sum_{p \in \mathcal{P}(n)} (-\hbar)^{n-|p|} \mathbf{\xi}^0_{|p|}(Jv_{B_1}, \ldots, Jv_{B_{|p|-1}}, \mathbf{\dot{\ell}}(v_{B_1}), v_{B_{|p|}}, \ldots, v_{B_{|p|}}).
\]

Note that \( \mathbf{\vartheta}^1_2 = 0 \). Note also that \( \mathbf{\Omega}^0_n \) and \( \mathbf{\vartheta}^1_n \) depend only on \( \{ \mathbf{\xi}^0_k, \mathbf{\eta}^{-1}_k, \mathbf{\dot{\ell}}_k, \mathbf{\phi}^0_k \} \) for \( k < n \).

Therefore, all we need is to find appropriate \( \mathbf{\xi}^0_n \in \text{Hom}\left( \mathcal{S}^n H, H \right) \left[ \left[ H \right] \right] \) and \( \mathbf{\eta}^{-1}_n \in \text{Hom}\left( \mathcal{S}^n H, \mathcal{C}^{-1}\left[ \left[ H \right] \right] \right) \) such that

\[
\frac{1}{(-\hbar)^{n-1}} \left( \mathbf{K}_H \epsilon \mathbf{\eta}^{-1}_n + \mathbf{f} \circ \mathbf{\xi}^0_n - \mathbf{\Omega}^0_n \right) \in \text{Hom}\left( \mathcal{S}^n H, \mathcal{C}^0 \left[ \left[ H \right] \right] \right)
\]

\[
\frac{1}{(-\hbar)^{n-1}} \left( \kappa_{HH} \mathbf{\xi}^0_n + \mathbf{\vartheta}^1_n \right) \in \text{Hom}\left( \mathcal{S}^n H, \mathcal{C}^{-1}\left[ \left[ H \right] \right] \right).
\]

**Theorem 5.1.** There is a canonical solution \( \{ \mathbf{\xi}^0_n, \mathbf{\eta}^{-1}_n, \mathbf{\dot{\ell}}, \mathbf{\phi}^0 \} \) to the level zero quantum master equation of Definition 5.1 with

\[
\mathbf{\xi}_1 = \mathbb{1}_H, \quad \mathbf{\eta}^{-1}_1 = 0, \quad \mathbf{\phi}^0_1 = \mathbf{f}, \quad \mathbf{\dot{\ell}}_1 = \kappa, \quad (5.4)
\]

and for all \( n \geq 2 \)

\[
\mathbf{\xi}^0_n = \sum_{i=0}^{n-2} (-\hbar)^i \mathbf{f} \circ \mathbf{\Omega}^0_{n-i}, \quad \mathbf{\phi}^0_n = - (\nabla (-\hbar)^{-1})^1 \mathbf{\Omega}^0_n,
\]

\[
\mathbf{\eta}^{-1}_n = \sum_{i=0}^{n-2} (-\hbar)^i \mathbf{f} \circ \mathbf{\Omega}^0_{n-i}, \quad \mathbf{\dot{\ell}}_n = \frac{1}{(-\hbar)^{n-1}} \left( \mathbf{\vartheta}^1_n + \kappa_{HH} \mathbf{\xi}^0_n \right),
\]

where \( \mathbf{\Omega}^0_n \) and \( \mathbf{\vartheta}^1_n \) are defined in the previous section.
where \((\Omega^0_n)^{(i)} \in \text{Hom}(S^n H, \mathcal{C})\) is the classical limit of \((\Omega^0_n)^{(i)} := (\nabla_{\mathcal{P}})^{i} \Omega^0_n\) for \(0 \leq i \leq n-2\).

For \(v_1, \ldots, v_{n-1} \in H\), this solution satisfies

\[
\hat{\omega}^0_n(v_1, \ldots, v_{n-1}, 1) = \hat{\omega}^0_{n-1}(v_1, \ldots, v_{n-1}),
\]

\[
\eta^{-1}_n(v_1, \ldots, v_{n-1}, 1) = \eta^{-1}_{n-1}(v_1, \ldots, v_{n-1}),
\]

**Remark 5.2.** The conditions of eq. (5.4) are mostly redundant; all except the first of them are explicitly required by Definition 5.1.

**Proof.** Assume that

\[
\{ \hat{\omega}^0_1, \ldots, \hat{\omega}^0_{n-1} \}, \quad \{ 0, \eta^{-1}_2, \ldots, \eta^{-1}_{n-1} \}, \quad \{ \hat{l}_1, \ldots, \hat{l}_{n-1} \}, \quad \{ \psi^0_1, \ldots, \psi^0_{n-1} \},
\]

is such a canonical solution to the master equation up to order \(n-1\). It can be checked by an easy induction that, for all \(v_1, \ldots, v_{n-1} \in H\),

\[
\Omega^0_n(v_1, \ldots, v_{n-1}, 1) = \Omega^0_{n-1}(v_1, \ldots, v_{n-1}).
\]  

(5.5)

It can be also checked by a tedious but straightforward computation, similar to that in the proof of Proposition 5.1, that

\[
K_{H^0} : \Omega^0_n = (-h)^{n-1} L_n - f \circ \Phi^1_n,
\]

(5.6)

where \(L_n \in \text{Hom}(S^n H, \mathcal{C})[[h]]\) is defined such that, for homogeneous \(v_1, \ldots, v_n\),

\[
L_n(v_1, \ldots, v_n) := \sum_{p \in P(n)} \epsilon(p) l^p([\phi^0(v_{B_p}), \ldots, \phi^0(v_{B_p})])
\]

\[
- \sum_{p \in P(n)} \epsilon(p) \phi^0_p(J v_{B_1}, \ldots, J v_{B_{n-1}}, \hat{l}(v_{B_{n-1}}), v_{B_{n-1}}, \ldots, v_{B_{n-1}})).
\]

Therefore, we can apply Lemma 4.4 to obtain that

\[
(-h)^{n-1} \Omega^0_n = \Omega_n - f \circ \left( \sum_{i=0}^{n-2} (-h)^{i} h \circ (\Omega^0_n)^{(i)} \right) - K_{H^0} \left( \sum_{i=0}^{n-2} (-h)^{i} s \circ (\Omega^0_n)^{(i)} \right),
\]

\[
(-h)^{n-1} (\omega^0_n)^{(n-1)} = \omega^0_n + \sum_{i=0}^{n-2} (-h)^{i} \kappaHH \left( h \circ (\Omega^0_n)^{(i)} \right),
\]

\[
K_{H^0} \left( \Omega^0_n \right)^{(n-1)} = L_n - f \circ (\omega^0_n)^{(n-1)},
\]

(5.7)
where \((\Omega_n^0)^{(j)} := (\nabla(-\hbar^{-1}))^j \Omega_n^0 \in \text{Hom}(S^n\mathcal{H}, \mathcal{G})^0[[\hbar]]\) and \((\Omega_n^0)^{(j)} \in \text{Hom}(S^n\mathcal{H}, \mathcal{G})^0\) denotes the classical limit of \((\Omega_n^0)^{(j)}\). Setting

\[
\hat{\pi}_n := \sum_{j=0}^{n-2} (-\hbar)^j \phi \circ (\Omega_n^0)^{(j)} \in \text{Hom}(S^n\mathcal{H}, \mathcal{G})^0[[\hbar]],
\]

\[
\tilde{\eta}_n := \sum_{j=0}^{n-2} (-\hbar)^j \psi \circ (\Omega_n^0)^{(j)} \in \text{Hom}(S^n\mathcal{H}, \mathcal{G})^{-1}[[\hbar]],
\]

\[
\phi_n^0 := - (\Omega_n^0)^{(n-1)} \in \text{Hom}(S^n\mathcal{H}, \mathcal{G})^0[[\hbar]], \quad \text{and}
\]

\[
\hat{\ell}_n := (\phi_n^0)^{(n-1)} \in \text{Hom}(S^n\mathcal{H}, \mathcal{G})[[\hbar]],
\]

we have

\[
K_{H\mathcal{G}} \hat{\eta}_n^{-1} + \hat{\phi}_n = \hat{\eta}_n^0 \phi_n^0,
\]

\[
\kappa_{H\mathcal{G}} \hat{\eta}_n^0 + \phi_n^0 = \hbar^{n-1} \phi_n^0,
\]

\[
\kappa_{H\mathcal{G}} \phi_n^0 + \mathcal{L}_n = \hat{\phi}_n.
\]

and we are done. From eq. \((5.5)\), we also obtain that, for all \(v_1, \ldots, v_{n-1} \in \mathcal{H}\),

\[
\hat{\pi}_n^0(v_1, \ldots, v_{n-1}, 1_H) = \hat{\pi}_n^0(v_1, \ldots, v_{n-1}),
\]

\[
\eta_n^{-1}(v_1, \ldots, v_{n-1}, 1_H) = \eta_n^{-1}(v_1, \ldots, v_{n-1}).
\]

From the relations in eq. \((5.8)\), eq. \((5.9)\) and eq. \((5.10)\) together with the assumption, we have shown that

\[
\{\hat{\pi}_1^0, \ldots, \hat{\pi}_n^0\}, \quad \{0, \eta_2^{-1}, \ldots, \eta_n^{-1}\}, \quad \{\hat{\ell}_1, \ldots, \hat{\ell}_n\}, \quad \{\phi_1^0, \ldots, \phi_n^0\},
\]

constitute a canonical solution to the master equation up to order \(n\). Therefore, we are done by induction.

From now on \(\{\hat{\pi}_n^0, \eta_n^{-1}, \hat{\ell}, \phi^0\}\) will denote the canonical solution to the master equation for the level zero quantum correlators.

### 5.2. Homotopy \(\hbar\)-divisibility II

Consider the unital \(sL_\infty\)-algebra \((H[[\hbar]], 1_H, \hat{1})\). It is convenient to introduce a \(k[[\hbar]]\)-linear operator \(\partial_i : (S(H) \otimes S^jH)[[\hbar]] \to (S(H) \otimes S^jH)[[\hbar]]\) defined as follows:

\[
\partial_i(v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_j) = \delta_i(v_1 \otimes \ldots \otimes v_{n}) \otimes w_1 \otimes \ldots \otimes w_j
\]

\[
+ \sum_{j=1}^{n} \sum_{\varsigma \subset [n]} (-\hbar)^{n-|\varsigma|-1} \epsilon(\varsigma) \epsilon(i, \omega) v_\varsigma \otimes \hat{1}(v_\varsigma \otimes w_1) \otimes w_1 \otimes \ldots \otimes \tilde{w}_i \otimes \ldots \otimes w_j,
\]

where \(\epsilon(\varsigma) \epsilon(i, \omega) v_\varsigma \) is the classical limit of \(\epsilon(\varsigma) \epsilon(i, \omega) v_\varsigma \).
where \( \varepsilon(i, w) \) is the sign \((-1)^{|w_i|(|w_1|+\ldots+|w_{i-1}|)} \) and \( \delta_k : S(H)[[\hbar]] \to S(H)[[\hbar]] \) is as in Notation 5.2 and satisfies \( \delta_k \circ \delta_k = 0 \). Then it is also straightforward to show that \( \delta_k \circ \delta_k = 0 \).

**Remark 5.3.** Let \( P \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \) for a \( \mathbb{Z} \)-graded vector space \( W \). Then, we have, for \( n \geq j \),

\[
(\rho \circ \delta_k)(v_1, \ldots, v_n) = \sum_{p \in P(n)} (-h)^{|p|} \varepsilon(p) \hat{\rho}_p \left( Jv_{B_1}, \ldots, Jv_{B_{n-1}}, \hat{\iota}(v_{B_1}), v_{B_{n-1}}, \ldots, v_{B_{n-1+i}}, \ldots, v_n \right),
\]

where \( \rho_n(v_1, \ldots, v_n) = \rho(v_1 \otimes \ldots \otimes v_{n-j} \otimes v_{n-j+1} \otimes \ldots \otimes v_n) \).

Define \( k[[\hbar]] \)-linear operators

\[
K^\infty_{H^c} : \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \to \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]],
\]

\[
K^\infty_{H^1} : \text{Hom}(S(H) \otimes S^1 H, H)[[\hbar]] \to \text{Hom}(S(H) \otimes S^1 H, H)[[\hbar]],
\]

for all \( n \geq 1 \) and \( v_1, \ldots, v_n \in H \) via the equations

\[
K^\infty_{H^c} \varepsilon := K \circ \varepsilon - (-1)^{|\varepsilon|} \varepsilon \circ \delta_k,
\]

\[
K^\infty_{H^1} \xi := K \circ \xi - (-1)^{|\xi|} \omega \circ \delta_k,
\]

where \( \varepsilon \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \) and \( \xi \in \text{Hom}(S(H) \otimes S^1 H, H)[[\hbar]] \). It is trivial that \( K^\infty_{H^c} \circ K^\infty_{H^c} = K^\infty_{H^c} \circ K^\infty_{H^c} = 0 \). Note that \( f \circ \xi \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \) and \( K^\infty_{H^c}(f \circ \xi) = f \circ (K^\infty_{H^c} \circ \xi) \) since \( K \circ f = f \circ K \).

We define a \( k \)-linear operator \( \nabla^\infty_{(-h)^{-1}} \) on \( \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \) as follows: given \( \varepsilon \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \),

\[
(-h)^{-1} \nabla^\infty_{(-h)^{-1}} \varepsilon := -f \circ \varepsilon - K^\infty_{H^c}(s \circ \varepsilon) - s \circ K \circ \varepsilon,
\]

where \( \varepsilon \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V}) \) is the classical limit of \( \varepsilon \). The classical limit of the right hand side of Eq. (5.11) is \( \varepsilon - f \circ \varepsilon - K \circ \varepsilon - s \circ K \circ \varepsilon = 0 \) so it is divisible by \( \hbar \). Therefore, we have \( \nabla^\infty_{(-h)^{-1}} \varepsilon \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \). Assume that the classical limit of \( \varepsilon \) vanishes. Then we have \( \nabla^\infty_{(-h)^{-1}} \varepsilon = \frac{1}{(-h)^{-1}} \varepsilon \). The following is direct.

**Lemma 5.1.** For a triple \( (\varepsilon, M, \xi) \) satisfying, for \( k \geq 1 \);

\[
K^\infty_{H^c} \varepsilon = (-h)^k M - f \circ \xi,
\]

where \( \varepsilon, M \in \text{Hom}(S(H) \otimes S^1 H, \mathcal{V})[[\hbar]] \), \( \xi \in \text{Hom}(S(H) \otimes S^1 H, H)[[\hbar]] \),
we have

\[ K_{H^0}^{\infty} \Xi^{[k]} = \mathbf{M} - f \circ \xi^{[k]} \quad \text{and} \quad \begin{cases} \Xi^{[k]} \in \text{Hom}(S(H) \otimes S^1H, \mathcal{C})[\hbar], \\ \xi^{[k]} \in \text{Hom}(S(H) \otimes S^1H, H)[\hbar], \end{cases} \]  

(5.12)

where

\[ (-h)^{k} \Xi^{[k]} := \Xi - f \left( \sum_{i=0}^{k-1} (-h)^i \circ \Xi^{[1]} \right) \]

\[ \xi^{[k]} := \xi + \sum_{i=0}^{k-1} (-h)^i \kappa_{H^0}^{\infty} \left( h \circ \Omega^{[i]} \right). \]  

(5.13)

5.3. The master equation for level one quantum correlators

Let \( (\hat{\pi}^0, \eta^{-1}, \hat{t}, \phi^0) \) be the canonical solution to the level zero quantum master equation, which can be written in the following simpler form:

\[ \Pi^0 = f \circ \hat{\pi}^0 + K_{H^0}^{\infty} \eta^{-1}, \quad \kappa_{H^0}^{\infty} \hat{\pi}^0 = 0, \]  

(5.14)

where \( \Pi^0 \in \text{Hom}(S(H), \mathcal{C})[\hbar] \) is defined by the formula:

\[ \Pi^0(v_1 \otimes \cdots \otimes v_n) = \sum_{p \in P(n)} (-h)^{n-|p|} \iota(p) \phi^0(v_{p_1}) \cdots \phi^0(v_{p_n}). \]  

(5.15)

Note also that \( K_{H^0}^{\infty} \Pi^0 = 0 \) since \( K_{H^0}^{\infty} (f \circ \hat{\pi}^0) = f \circ K_{H^0}^{\infty} \hat{\pi}^0 = 0. \)

**Definition 5.3.** The level one quantum master equation is a system of equations for a tuple \( \{ \hat{\pi}^{-1}, \eta^{-2}, \hat{m}^0, \phi^{-1} \} \), where

\[ \hat{\pi}^{-1} \in \text{Hom}(S(H) \otimes S^2H, H)^{-1}[\hbar], \quad \eta^{-2} \in \text{Hom}(S(H) \otimes S^2H, \mathcal{C})^{-2}[\hbar], \]

\[ \hat{m}^0 \in \text{Hom}(S(H) \otimes S^2H, H)^0[\hbar], \quad \phi^{-1} \in \text{Hom}(S(H) \otimes S^2H, \mathcal{C})^{-1}[\hbar], \]
are defined recursively for \( n \geq 2 \) by the equations

\[
\mathfrak{f}(\tau_n^{-1}(v_1, \ldots, v_n)) = \eta_n^{-1}(v_1, \ldots, v_n) - K_{\mathcal{H}^\infty} \eta_n^{-2}(v_1, \ldots, v_n) - \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \Phi^0(Jv_{B_1}) \cdots \Phi^0(Jv_{B_{|p|-1}}) \cdot \Phi^{-1}(v_{B_{|p|}})
\]

\[
\sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \eta_{|p|}^{-1} \left( v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}^0(v_{B_{|p|}}) \right),
\]

\[
\hat{\xi}_n^0(v_1, \ldots, v_n) = \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \hat{\xi}_n^0 \left( v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}^0(v_{B_{|p|}}) \right)
\]

\[
- K_{\mathcal{H}^\infty} \tau_n^{-1}(v_1, \ldots, v_n),
\]

with the initial conditions

\[
\hat{\xi}_2^{-1} = 0, \quad \eta_2^{-2} = 0, \quad \hat{m}_2 = \xi_2^0, \quad \phi_2^{-1} = \eta_2^{-1}.
\]

Remark 5.4. The leading, \( n = 2 \), level one quantum master equation is

\[
\mathfrak{f}(\hat{\xi}_2^{-1}(v_1, v_2)) = \eta_2^{-1}(v_1, v_2) - \phi_2^{-1}(v_1, v_2) - K_{\mathcal{H}^\infty} \eta_2^{-2}(v_1, v_2),
\]

\[
\hat{\xi}_2^0(v_1, v_2) = \hat{m}_2^0(v_1, v_2) - K_{\mathcal{H}^\infty} \phi_2^{-2}(v_1, v_2),
\]

which is solved by the initial conditions.

Remark 5.5. Define

\[
\Omega_n^{-1}(v_1, \ldots, v_n) := \eta_n^{-1}(v_1, \ldots, v_n)
\]

\[
- \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \eta_{|p|}^{-1} \left( v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}^0(v_{B_{|p|}}) \right)
\]

\[
- \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \phi^0(Jv_{B_1}) \cdots \phi^0(Jv_{B_{|p|-1}}) \cdot \phi^{-1}(v_{B_{|p|}}),
\]

\[
(5.16)
\]
and

\[
\tilde{\omega}_n^0(v_1, \ldots, v_n) := \tilde{\pi}_n^0(v_1, \ldots, v_n)
- \sum_{p \in P(n) \atop |B_p| = n-|p|+1} (-\hbar)^{n-|p|-1} \epsilon(p) \tilde{\pi}_p^0(v_{B_1}, \ldots, v_{B_{|p|-1}}, m^0(v_{B_0})) \tag{5.17}
\]

Note that \(\Omega_n^{-1}\) is in \(\text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-1}[[\hbar]]\), that \(\tilde{\omega}_n^0\) is in \(\text{Hom}(S^{n-2}H \otimes S^2H, H)^0[[\hbar]]\), and both depend only on the families

\[
\{\tilde{\pi}_2^{-1}, \ldots, \tilde{\pi}_{n-1}^{-1}\}, \quad \{\eta_2^{-2}, \ldots, \eta_{n-2}^{-2}\}, \quad \{\phi_2^{-1}, \ldots, \phi_{n-1}^{-1}\}, \quad \{m_0^0, \ldots, m_{n-1}^0\},
\]

and the canonical solution to the level zero quantum master equation. Then the level one quantum master equation can be redefined as

\[
(-\hbar)^{n-2}\phi_n^{-1} = \Omega_n^{-1} - f \circ \tilde{\pi}_n^{-1} - K_H^\infty \eta_n^{-2},
\]

\[
(-\hbar)^{n-2}\tilde{m}_n = \tilde{\omega}_n^0 + \kappa_H^\infty \tilde{\pi}_n^{-1},
\]

Thus the key to solving these equations is to show that there is a pair \((\tilde{\pi}_n^{-1}, \eta_n^{-2})\) so that the expressions on the right hand side of Eq. (5.18) are divisible by \(\hbar^{n-2}\).

**Proposition 5.2 (Integrability of Definition 5.3).** Let \(\{\tilde{\pi}_n^{-1}, \eta_n^{-2}, m_0^0, \phi_n^{-1}\}\) be a solution to the level one quantum master equation. Then for all \(n \geq 2\) and \(v_1, \ldots, v_n \in H\),

\[
(\hbar)^{\phi_n^0(v_1, \ldots, v_n) = \sum_{p \in P(n) \atop |B_p| = n-|p|+1} \epsilon(p) \phi^0_p(v_{B_1}, \ldots, v_{B_{|p|-1}}, m^0(v_{B_0}))}
- \sum_{p \in P(n) \atop |B_p| = n-|p|+1} \epsilon(p) \phi^0(v_{B_1}) \cdot \phi^0(v_{B_2})
+ \sum_{p \in P(n) \atop |B_p| = n-|p|+1} \epsilon(p) \phi^0_{|p|-1}(Jv_{B_1}, \ldots, Jv_{B_{|p|-1}}, \phi^{-1}(v_{B_0}))
+ \sum_{p \in P(n) \atop |B_p| = n-|p|+1} \epsilon(p) \phi^0_{|p|-1}(Jv_{B_1}, \ldots, Jv_{B_{|p|-1}}, \tilde{\ell}(v_{B_1}), v_{B_2}, \ldots, v_{B_{|p|-1}}).}
\]
Here is our strategy to prove the above proposition. Consider the level one quantum master equation in the form of eq. (5.18). Applying $K_{H^0}^\infty$ to the first equation in eq. (5.18) and using the second equation in eq. (5.18), we obtain that

$$K_{H^0}^\infty \Omega_n^{-1} + f \circ \hat{\omega}^0_n = (-\hbar)^{n-2} \left( f \circ \hat{m}^0_n + K_{H^0}^\infty \phi_n^{-1} \right).$$

Therefore, there must be some $M^0_n \in Hom(S^{n-2}H \otimes S^2H, \mathcal{C})$, for $n \geq 2$, such that $K_{H^0}^\infty \Omega_n^{-1} + f \circ \hat{\omega}^0_n = (-\hbar)^{n-2} M^0_n$, which implies that

$$M^0_n = f \circ \hat{m}^0_n + K_{H^0}^\infty \phi_n^{-1}.$$

(5.19)

We claim that, for all $n \geq 2$,

$$M^0_n(v_1, \ldots, v_n) = (-\hbar) \phi^0_n(v_1, \ldots, v_n) + \sum_{p \in \mathcal{P}(n)} \varepsilon(p) \phi^0(v_{B_1}) \cdot \phi^0(v_{B_2})$$

$$- \sum_{p \in \mathcal{P}(n)} \varepsilon(p) \phi^0_{|p|} \left( v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}^0_{|p|}(v_{B_{|p|}}) \right)$$

$$+ \sum_{p \in \mathcal{P}(n)} \varepsilon(p) \ell_{|p|} \left( \phi^0(Jv_{B_1}), \ldots, \phi^0(Jv_{B_{|p|-1}}), \phi^{-1}(v_{B_{|p|}}) \right),$$

(5.20)

so that the relation eq. (5.19) is equivalent to the above proposition. Therefore, it suffices to show that $K_{H^0}^\infty \Omega_n^{-1} = (-\hbar)^{n-2} M^0_n - f \circ \hat{\omega}^0_n$ for all $n \geq 2$.

**Lemma 5.2.** Let $\{ \tilde{\pi}^{-1}_{n-1}, \ldots, \tilde{\pi}^{-1}_2 | \hat{\eta}^{-2}_{n-1}, \ldots, \hat{\eta}^{-2}_2, \phi^{-1}_{n-1}, \ldots, \phi^{-1}_2, \hat{m}^0_{n-1}, \ldots, \hat{m}^0_2 \}$ be a solution to the level one quantum master equation for $k = 2, \ldots, n - 1$. Then, we have

$$K_{H^0}^\infty \Omega_n^{-1} = (-\hbar)^{n-2} M^0_n - f \circ \hat{\omega}^0_n.$$

We remark that the triple $(\Omega_n^{-1}, M^0_n, \hat{\omega}^0_n)$ depends only on

$$\{ \tilde{\pi}^{-1}_{n-1}, \ldots, \tilde{\pi}^{-1}_2 | \hat{\eta}^{-2}_{n-1}, \ldots, \hat{\eta}^{-2}_2, \phi^{-1}_{n-1}, \ldots, \phi^{-1}_2, \hat{m}^0_{n-1}, \ldots, \hat{m}^0_2 \}$$. 

along with the canonical solution to the level zero quantum master equation. The above lemma will be the key to solving the level one quantum master equation, but its proof is purely technical.
Proof. For \( n = 2 \), we have \( M_2^0 = \Pi_2^0, \Omega_n^{-1} = \eta_2^{-1} \), and \( \hat{m}_2 = \hat{\pi}_2 \). Therefore the level zero quantum master equation \( f \circ \hat{\pi}_2 = \Pi_2^0 - K_{H^0}^\infty \eta_2^{-1} \) for \( n = 2 \) implies that \( K_{H^0}^\infty \Omega_2^{-1} = M_2^0 - f \circ \hat{m}_2^0 \).

For \( n \geq 3 \) assume that \( K_{H^0}^\infty \Omega_k^{-1} = (-\hbar)^{k-2} M_k^0 - f \circ \hat{m}_k^0 \) for \( k = 2, \ldots, n - 1 \), which implies that

\[
K_{H^0}^\infty \phi_k^{-1} = M_k^0 - f \circ \hat{m}_k^0, \quad 2 \leq k \leq n - 1. \tag{5.21}
\]

It remains only to check that \( K_{H^0}^\infty \Omega_n^{-1} = (-\hbar)^n - M_n^0 - f \circ \hat{m}_n^0 \). Rewrite \( \Omega_n^{-1} \) as follows:

\[
\Omega_n^{-1}(v_1, \ldots, v_n) = \eta_n^{-1}(v_1, \ldots, v_n)
- \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \eta_{|p|}^{-1} (v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}_0(v_{B_{|p|}}))
- \sum_{p \in P(n)} (-\hbar)^{|B_2|-2} \varepsilon(p) \Pi^0(Jv_{B_1}) \cdot \phi^{-1}(v_{B_2}), \tag{5.22}
\]

where we have used the identity

\[
\sum_{p \in P(n)} (-\hbar)^{n-|p|-1} \varepsilon(p) \phi^0(Jv_{B_1}) \cdots \phi^0(Jv_{B_{|p|-1}}) \cdot \phi^{-1}(v_{B_{|p|}})
= \sum_{p \in P(n)} (-\hbar)^{|B_2|-2} \varepsilon(p) \Pi^0(Jv_{B_1}) \cdot \phi^{-1}(v_{B_2}).
\]

It will also be convenient to decompose \( M_k^0 = X_k + Y_k + Z_k \), for \( 2 \leq k \leq n - 1 \), where

\[
X_k(v_1, \ldots, v_k) := (-\hbar) \phi_k(v_1, \ldots, v_k) + \sum_{p \in P(k)} \varepsilon(p) \phi^0(v_{B_1}) \cdots \phi^0(v_{B_{|p|-1}}) \cdot \phi^{-1}(v_{B_{|p|}}),
\]

\[
Y_k(v_1, \ldots, v_k) := \sum_{p \in P(k)} \varepsilon(p) \phi^0(v_{B_1}) \cdots \phi^0(v_{B_{|p|-1}}),
\]

\[
Z_k(v_1, \ldots, v_k) := \sum_{p \in P(k)} \varepsilon(p) \phi^{-1}(v_{B_1}) \cdots \phi^{-1}(v_{B_{|p|-1}}).
\]
Applying $K_{H^{\infty}}^\infty$ to eq. (5.22) and using $K_{H^{\infty}}^\infty \Pi^0 = 0$, we have

$$K_{H^{\infty}}^\infty \Omega_n^{-1} (v_1, \ldots, v_n) = K_{H^{\infty}}^\infty \eta_n^{-1} (v_1, \ldots, v_n)$$

$$- \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} e(p) K_{H^{\infty}}^\infty \eta_{|p|}^{-1} (v_{B_1}, \ldots, v_{B_{p-1}}, \tilde{m}^0 (v_{B_p}))$$

$$- \sum_{p \in P(n)} (-\hbar)^{|B_2|} e(p) \Pi^0 (v_{B_1}) \cdot K_{H^{\infty}}^\infty \phi^{-1} (v_{B_2})$$

$$- \sum_{p \in P(n)} (-\hbar)^{|B_2|} e(p) \ell_2 \left( \Pi^0 (J v_{B_1}), \phi^{-1} (v_{B_2}) \right)$$

Using eq. (5.14) and eq. (5.21), we obtain the decomposition

$$K_{H^{\infty}}^\infty \Omega_n^{-1} (v_1, \ldots, v_n) = R_1 + R_2 + R_3 + R_4,$$

where

$$R_1 = - \tilde{f} \left( \tilde{A}_n^0 (v_1, \ldots, v_n) - \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} e(p) \tilde{A}_{|p|}^0 \left( v_{B_1}, \ldots, v_{B_{p-1}}, \tilde{m}^0 (v_{B_p}) \right) \right),$$

$$R_2 = + \Pi_n^0 (v_1, \ldots, v_n) - \sum_{p \in P(n)} (-\hbar)^{|B_2|} e(p) \Pi (v_{B_1}) \cdot X (v_{B_2}),$$

$$R_3 = - \sum_{p \in P(n)} (-\hbar)^{n-|p|-1} e(p) \Pi_{|p|}^0 \left( v_{B_1}, \ldots, v_{B_{p-1}}, \tilde{m}^0 (v_{B_p}) \right)$$

$$+ \sum_{p \in P(n)} (-\hbar)^{|B_2|} e(p) \Pi^0 (v_{B_1}) \cdot \Phi_{|p|}^0 \left( \tilde{m}^0 (v_{B_2}) \right)$$

$$- \sum_{p \in P(n)} (-\hbar)^{|B_2|} e(p) \Pi^0 (v_{B_1}) \cdot Y (v_{B_2}),$$

$$R_4$$
\[ R_4 = - \sum_{p \in P(n)} (-\hbar)^{|B_2| - 2} e(p) \mathfrak{t}_2(\Pi^0(J v_{B_1}), \phi^{-1}(v_{B_2})) \]

From the definition of \( \check{\Omega}^0_n \) in eq. (5.17), we have
\[ R_1 = -f \circ \check{\Omega}^0_n(v_1, \ldots, v_n). \]

From the definition of \( \Pi^0 \) in eq. (5.15), we obtain the following identity:
\[ \Pi^0_n(v_1, \ldots, v_n) = (-\hbar)^{n-2} X_n(v_1, \ldots, v_n) + \sum_{p \in P(n)} (-\hbar)^{|B_2| - 2} e(p) \Pi^0(v_{B_1}) \cdot X(v_{B_2}), \]
which implies that
\[ R_2 = (-\hbar)^{n-2} X_n(v_1, \ldots, v_n). \]

Finally, the formula in Remark 2.3 and the definition of \( Z_k \) for \( 2 \leq k \leq n - 1 \), imply that
\[ R_4 = (-\hbar)^{n-2} Z_n(v_1, \ldots, v_n). \]

Therefore we have \( K_{\mathcal{H}^P}^\infty \Omega^{-1}_n = (-\hbar)^{n-2} \mathcal{M}_n^{0} - f \circ \check{\Omega}^0_n. \]

Theorem 5.2. There is a canonical solution \( \{ \hat{\pi}^{-1}, \eta^{-2}, \hat{m}, \phi^{-1} \} \) to the level one quantum master equation in Definition 5.3 with
\[ \hat{\pi}^{-1}_2 = 0, \quad \eta^{-2}_2 = 0, \quad \hat{m}_2 = \hat{\pi}_2, \quad \phi^{-1}_2 = \eta^{-2}_2 \quad (5.23) \]
and, for all \( n \geq 3 \),
\[ \hat{\pi}^{-1}_n := \sum_{i=0}^{n-3} (-\hbar)^i \hat{h} \circ (\Omega_{n-1}^{-1})^{[i]}, \quad \hat{m}_n = \frac{1}{(-\hbar)^n-2} (\check{\Omega}_n + \kappa_n^{\mathcal{H}^P} \check{\pi}_n^{-1}), \]
\[ \eta^{-2}_n := \sum_{i=0}^{n-3} (-\hbar)^i \hat{s} \circ (\Omega_{n-1}^{-1})^{[i]}, \quad \phi^{-1}_n = (\Omega_{n-1}^{-1})^{[n-2]}, \]
where \( (\Omega_{n-1}^{-1})^{[i]} := (\nabla (\hbar^{-1}))^i \Omega_{n-1}^{-1} \) and \( (\Omega_{n-1}^{-1})^{[i]} \) is the classical limit of \( (\Omega_{n-1}^{-1})^{[i]} \) for \( 0 \leq i \leq n - 3 \).
Remark 5.6. Again, the conditions of eq. (5.23) are mostly redundant; all except the third of them are explicitly required by Definition 5.3.

Proof. Fix \( n \geq 3 \) and let \( \{ \hat{\pi}_2^{-1}, \ldots, \hat{\pi}_{n-1}^{-1} | \eta_2^{-1}, \ldots, \eta_{n-1}^{-1} | \phi_2^{-1}, \ldots, \phi_{n-1}^{-1} | m_2^0, \ldots, m_{n-1}^0 \} \) be a solution to the level one quantum master equation for \( k = 2, \ldots, n - 1 \) with all the properties in Theorem 5.2. Then, we have (via Lemma 5.2)

\[
K_{H^c}^\infty \Omega_n^{-1} = (-h)^{n-2} M_n^0 - f \circ \omega_n^0.
\]

Therefore, we can apply Lemma 5.1 to obtain that

\[
(-h)^{n-2} (\Omega_n^{-1})^{[n-2]} = \Omega_n^{-1} - f \circ \left( \sum_{i=0}^{n-3} (-h)^i h \circ (\Omega_n^{-1})^{[i]} \right) - K_{H^c}^\infty \left( \sum_{i=0}^{n-3} (-h)^i \kappa_{HH} \left( h \circ (\Omega_n^{-1})^{[i]} \right) \right),
\]

\[
(-h)^{n-2} (\omega_n^0)^{[n-2]} = \omega_n^0 + \sum_{i=0}^{n-3} (-h)^i \kappa_{HH} \left( h \circ (\Omega_n^{-1})^{[i]} \right),
\]

\[
K_{H^c}^\infty (\Omega_n^{-1})^{[n-2]} = M_n^0 - f \circ (\omega_n^0)^{[n-2]}.
\]

Recall that \((\Omega_n^{-1})^{[j]} = \nabla_{(-h)^{-1}}^{[j]} \Omega_n^{-1}\) is in \( \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-1}[[h]] \) for \( j \) in the range from 0 to \( n-2 \), and \((\omega_n^0)^{[n-2]} \in \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^0[[h]] \). Setting

\[
\hat{\pi}_n^{-1} := \sum_{i=0}^{n-3} (-h)^i h \circ (\Omega_n^{-1})^{[i]} \quad \text{in } \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-1},
\]

\[
\eta_n^{-2} := \sum_{i=0}^{n-3} (-h)^i \kappa_{HH} \left( h \circ (\Omega_n^{-1})^{[i]} \right) \quad \text{in } \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-2},
\]

\[
\phi_n^{-1} := (\Omega_n^{-1})^{[n-2]} \quad \text{in } \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-1}[[h]],
\]

\[
m_n^0 := (\omega_n^0)^{[n-2]} \quad \text{in } \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^0[[h]],
\]

we obtain that

\[
(-h)^{n-2} \phi_n^{-1} = \Omega_n^{-1} - f \circ \hat{\pi}_n^{-1} - K_{H^c}^\infty \eta_n^{-2},
\]

\[
(-h)^{n-2} m_n^0 = \omega_n^0 + K_{H^c}^\infty \hat{\pi}_n^{-1},
\]

so we are done by induction. \( \square \)
Now we specialize to the anomaly-free case that $\kappa = 0$. Then the following lemma shows that the family $\bar{m} = \bar{m}_2^0, \bar{m}_3^0, \ldots$ does not depend on $\hbar$.

**Lemma 5.3.** Let $\kappa = 0$. Then, we have $\bar{m}_1^0 = \bar{m}_n^0 = \bar{\pi}_n^{0(n-2)} \in \text{Hom}(S^n H, H)^0$, for all $n \geq 2$, so that the family $\bar{m}_2^0, \bar{m}_3^0, \ldots$ determines $\bar{\pi}_n^0$ as follows: for all $n \geq 2$ and homogeneous elements $v_1, \ldots, v_n \in H$,

$$\bar{\pi}_n^0(v_1, \ldots, v_n) = \sum_{\substack{p \in P(n) \\mid B_{[p]} = n-|p|+1 \\mid n-1 \varepsilon n}} (-\hbar)^{n-|p|-1} \varepsilon(p) \bar{\pi}_p^0(1) v_{B_1}, \ldots, v_{B_{p-1}}, \bar{m}_0^0(v_{B_p}). \tag{5.24}$$

**Proof.** From the condition $\kappa = 0$, we have $\kappa_{HH}^\infty = 0$ and

$$\bar{\pi}_n^0(v_1, \ldots, v_n) = \sum_{\substack{p \in P(n) \\mid B_{[p]} = n-|p|+1 \\mid n-1 \varepsilon n}} (-\hbar)^{n-|p|-1} \varepsilon(p) \bar{\pi}_p^0(1) v_{B_1}, \ldots, v_{B_{p-1}}, \bar{m}_0^0(v_{B_p}). \tag{5.24}$$

Recall that $\bar{\pi}_n^0 = \mathbb{I}_H$ and $\bar{\pi}_n^0 = \sum_{j=0}^{n-2} (-\hbar) \bar{\pi}_n^{0(j)}$, where $\bar{\pi}_n^{0(j)} \in \text{Hom}(S^n H, H)^0$, for all $n \geq 2$. For $n = 2$, we have $\bar{\pi}_2^0 = \bar{m}_2^0$. It follows that $\bar{m}_1^0 = \bar{m}_2^0 = \bar{\pi}_2^{0(0)} \in \text{Hom}(S^2 H, H)^0$ since $\bar{\pi}_2^0 = \mathbb{I}_H \in \text{Hom}(S^2 H, H)^0$. Fix $n \geq 3$ and assume that $\bar{m}_k^0 = \bar{\pi}_k^{0(k-2)} = \bar{m}_k^0 \in \text{Hom}(S^k H, H)^0$ for $2 \leq k < n$. Then relation eq. (5.24) can be rewritten as follows:

$$(-\hbar)^{n-2} \bar{m}_n^0(1, \ldots, v_n) = (-\hbar)^{n-2} \bar{\pi}_n^{0(n-2)}(1, \ldots, v_n)\tag{5.24}$$

$$+ \sum_{j=0}^{n-3} (-\hbar)^{n-3} \bar{\pi}_n^{0(j)}(1, \ldots, v_n)\tag{5.24}$$

$$- \sum_{\substack{p \in P(n) \\mid B_{[p]} = n-|p|+1 \\mid n-1 \varepsilon n}} (-\hbar)^{n-|p|-1} \varepsilon(p) \bar{\pi}_p^0(1) v_{B_1}, \ldots, v_{B_{p-1}}, \bar{m}_0^0(v_{B_p}). \tag{5.24}$$

The term on the right hand side of the above equality with the highest power in $\hbar$ is $(-\hbar)^{n-2} \bar{\pi}_n^{0(n-2)}$. Note also that the right hand side must be divisible by $(-\hbar)^{n-2}$ since $\bar{m}_n^0 \in \text{Hom}(S^n H \otimes S^2 H, H)^0 \otimes [\hbar \cdot 1]$. It follows that $\bar{m}_n^0 = \bar{\pi}_n^{0(n-2)} = \bar{m}_n^0$. Therefore, by induction, we have $\bar{m}_n^0 = \bar{m}_n^0 \in \text{Hom}(S^n H, H)^0$, for all $n \geq 2$. \hfill \Box

### 6. Applications to anomaly-free theory

In this subsection, we specialized to an anomaly-free binary QFT algebra, leading to Theorems 1.1, 1.2 and 1.3 together with some physical interpretations and two algo-
gorithms to compute the universal algebraic structure governing quantum correlations. We further specialize to the case that $H$ is finite dimensional to discuss relationships with the notion of special coordinates and the WDDV equation in topological string theory.

6.1. The algebra governing quantum correlation functions

In this subsection, we consider an anomaly-free binary QFT algebra $\mathcal{B}_{\text{QFTA}} = \mathcal{B}_{\text{QFTA}} \left( \mathcal{C}[\mathbb{H}], 1, \cdot, K \right)$ with quantum descendant $\mathcal{C}[\mathbb{H}], 1, \cdot, L$. Fix a classical off-to-on-shell retraction $(f, h, s)$ and let $(f, h, s)$ be a quantization of it. Then $(f, h, s)$ is the data of a homotopy equivalence between $(\mathcal{B}_{\text{QFTA}}, 1, H_0)$ and $(\mathcal{C}[\mathbb{H}], 1, \cdot, K)$. Let $v_1, \ldots, v_n$ be homogeneous elements in $H$.

The results in Section 5.1 specialized to the case $\kappa = 0$ can be presented as follows:

**Theorem 6.1.** There is a distinguished unital $sL_\infty$-quasi-isomorphism

$$\phi^0 : (H[\mathbb{H}], 1_H, 0) \longrightarrow (\mathcal{C}[\mathbb{H}], 1, \cdot, L)$$

such that $\phi_1^0 = f$ and, for all $n \geq 1$,

$$\Pi^0_n = f \circ \Pi^0_n + K \circ \eta^{-1}_n,$$  \hspace{1cm} (6.1)

where $\Pi^0_n \in \text{Hom}(S^n H, \mathcal{C}[\mathbb{H}])$ is defined by

$$\Pi^0_n(v_1, \ldots, v_n) := \sum_{p \in \mathbb{P}(n)} (-1)^{\mathbb{P}(p)} \phi^0(v_{B_1} \cdots \cdot \phi^0(v_{B_1})),$$  \hspace{1cm} (6.2)

and the families $\Pi^0$ and $\eta^{-1}$ have the following properties:

- $\Pi_1 = I_H$ and, for all $n \geq 2$,

$$\Pi^0_n = \sum_{j=0}^{n-2} (-h)^j \Pi^0_n,$$

where $\Pi^0_n \in \text{Hom}(S^n H, H)$,

and $\Pi^0_{n+1}(v_1, \ldots, v_n, 1_H) = \Pi^0_n(v_1, \ldots, v_n)$, for all $n \geq 1$;

- $\eta^{-1}_1 = 0$ and, for all $n \geq 2$,

$$\eta^{-1}_n = \sum_{j=0}^{n-2} (-h)^j \eta^{-1}_n,$$

where $\eta^{-1}_n \in \text{Hom}(S^n H, \mathcal{C})^{-1}$

and $\eta^{-1}_{n+1}(v_1, \ldots, v_n, 1_H) = \eta^{-1}_n(v_1, \ldots, v_n)$, for all $n \geq 1$. 
Remark 6.1. The first part of the above theorem establishes that the classical cohomology $H$ of an anomaly-free binary QFT algebra, viewed as a topologically-free unital $sL_\infty$-algebra $(H[[\hbar]], 1_H, \Omega)$ with zero $sL_\infty$-structure $\Omega$ is quasi-isomorphic to the quantum descendant unital unital $sL_\infty$-algebra $(\mathcal{C}[[\hbar]], 1, \mathcal{C})$. Moreover, we have constructed a particular quasi-isomorphism $\varphi$ and the rest of the theorem is about its distinguished properties: let $\varphi : (H[[\hbar]], 1_H, \Omega) \longrightarrow (\mathcal{C}[[\hbar]], 1, \mathcal{C})$ be an arbitrary unital $sL_\infty$-quasi-isomorphism and $\Pi^\varphi$ be the associated family of quantum correlators (which satisfy $K \circ \Pi_n^\varphi = 0$, for all $n \geq 1$). It follows that $\Pi_n^\varphi = f \circ \pi_n^\varphi + K \circ \eta_n^\varphi$, for all $n \geq 1$, where $\pi_n^\varphi := h \circ \Pi_n^\varphi \in \text{Hom}(\overline{\mathcal{S}(H)}^\varphi, H)^\varphi[[\hbar]]$ and $\eta_n^\varphi := s \circ \Pi_n^\varphi \in \text{Hom}(\overline{\mathcal{S}(H)}^\varphi, H)^\varphi[[\hbar]]$. However, the families $\pi_n^\varphi$ and $\eta_n^\varphi$ do not, in general, have the properties of the families $\pi_0^\varphi$ and $\eta_0^\varphi$.

Remark 6.2. We reproduce the algorithm to determine the families $\Phi_n^\varphi, \bar{\pi}_n^\varphi$ and $\eta_n^\varphi$, which becomes much simpler due to the condition $\kappa = 0$. Note that the operator $\nabla_{-1/\hbar} : \text{Hom}(\overline{\mathcal{T}(H)}, \mathcal{C})[[\hbar]] \rightarrow \text{Hom}(\overline{\mathcal{T}(H)}, \mathcal{C})[[\hbar]]$ defined by eq. (4.23) reduces in the following manner: for any $\Omega \in \text{Hom}(\overline{\mathcal{T}(H)}, \mathcal{C})[[\hbar]]$ whose classical limit $\Omega$ satisfies $K \circ \Omega = 0$, we have $(-\hbar) \nabla_{-1/\hbar} \Omega = \Omega - f \circ h \circ \Omega - K \circ s \circ \Omega$.

To begin with, we set $\Phi_0^\varphi = f$, $\bar{\pi}_0^\varphi = I_H$, $\eta_1^{-1} = 0$. (6.3)

Note that $\Pi_1^0 = \Phi_1^\varphi$. Therefore, we have

$$\Pi_1^0 = f \circ \pi_1^0 + K \circ \eta_1^{-1} \implies K \circ \Phi_1^0 = 0. \quad (6.4)$$

Assume that we have $\{\phi_1^0, \ldots, \phi_{n-1}^0, \pi_1^0, \ldots, \pi_{n-1}^0, \eta_1^{-1}, \ldots, \eta_{n-1}^{-1}\}$ for $n \geq 2$, satisfying the initial conditions eq. (6.3) and, for all $k = 1, \ldots, n-1$,

$$\Pi_k^0 = f \circ \pi_k^0 + K \eta_k^{-1} \implies \sum_{p \in \mathcal{P}(k)} e_p(\ell_p)(\phi_0(v_{B_1}), \ldots, \phi(\phi_0(y_{B_p}))) = 0. \quad (6.5)$$

Define $\Omega_n^0 \in \text{Hom}(S^nH, \mathcal{C})[[\hbar]]$ and $L_n \in \text{Hom}(S^nH, \mathcal{C})[[\hbar]]$ as follows:

$$\Omega_n^0(v_1, \ldots, v_n) := \sum_{p \in \mathcal{P}(n)} (-\hbar)^{-|p|} e_p(\ell_p)(\phi_0(v_{B_1}), \ldots, \phi(\phi_0(y_{B_p}))),$$

$$L_n(v_1, \ldots, v_n) := \sum_{p \in \mathcal{P}(n)} e_p(\ell_p)(\phi_0(v_{B_1}), \ldots, \phi(v_{B_p})).$$

From eq. (6.5) and the definition of $\ell_1, \ldots, \ell_n$, we obtain that $K \circ \Omega_n^0 = (-\hbar)^{n-1} L_n$. 

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Define \((\Omega^{0}_n)^{[j]} \in \text{Hom}(S^n H, \mathcal{E})^0[ [\hbar] ]\) for \(j = 0, 1, \ldots, n - 1\), by starting with \((\Omega^{0}_n)^{[0]} = \Omega^{0}_n\) and then recursively defining \((\Omega^{0}_n)^{[i+1]} = \nabla_{-1/\hbar} (\Omega^{0}_n)^{[i]}\), \(i = 0, \ldots, n - 2\), so that we have
\[
(-\hbar) (\Omega^{0}_n)^{[i+1]} = (\Omega^{0}_n)^{[i]} - f \circ \pi_n^0(i) - K \circ \eta_n^{-1}(i),
\]
where
\[
\pi_n^0(i) := h \circ (\Omega^{0}_n)^{[i]} \in \text{Hom}(S^n H, H)^0,
\]
\[
\eta_n^{-1}(i) := s \circ (\Omega^{0}_n)^{[i]} \in \text{Hom}(S^n H, \mathcal{E})^{-1},
\]
and \((\Omega^{0}_n)^{[i]} \in \text{Hom}(S^n H, \mathcal{E})^0\) denotes the classical limit of \((\Omega^{0}_n)^{[i]}\). We remark that \(K \circ (\Omega^{0}_n)^{[j]} = (-\hbar)^{n-1-j} L_n\) for \(j = 0, \ldots, n - 1\), and \(K \circ (\Omega^{0}_n)^{[i]} = 0\) for \(i = 0, \ldots, n - 2\).
It follows that
\[
(-\hbar)^{n-1} (\Omega^{0}_n)^{[n-1]} = \Omega^{0}_n - \sum_{i=0}^{n-2} (-\hbar)^i f \circ \pi_n^0(i) - \sum_{i=0}^{n-2} (-\hbar)^i K \circ \eta_n^{-1}(i). \tag{6.6}
\]
Finally, we set
\[
\phi_n^0 = - (\Omega^{0}_n)^{[n-1]}, \quad \pi_n^0 = \sum_{i=0}^{n-2} (-\hbar)^i \pi_n^0(i), \quad \eta_n^{-1} = \sum_{i=0}^{n-2} (-\hbar)^i \eta_n^{-1}(i), \tag{6.7}
\]
and obtain
\[
-(-\hbar)^{n-1} \phi_n^0 = \Omega^{0}_n - f \circ \pi_n^0 - K \circ \eta_n^{-1} \implies K \circ \phi_n^0 + L_n = 0. \tag{6.8}
\]
From \(\Omega^{0}_n = \Omega^{0}_n + (-\hbar)^{n-1} \phi_n^0\), it follows that
\[
\Omega^{0}_n = f \circ \pi_n^0 + K \eta_n^{-1} \implies \sum_{p \in \mathcal{P}(n)} \varepsilon(p) \ell_{[p]}(\phi^0(v_{B_1}), \ldots, \phi(v_{B_p})) = 0. \tag{6.8}
\]
This concludes the construction of our well-defined algorithm to determine the families \(\phi^0, \pi^0, \text{and } \eta^{-1}\).

Now we turn to some physical interpretations of Theorem 6.1.

Due to the condition \(\kappa = 0\), we now call a homogeneous element \(v \in H\) simply an observable and \(\{\tilde{f}(v)\}_\epsilon := c(\tilde{f}(v)) \in \mathbb{k}[ [ \hbar ] ]\) the quantum expectation value of the observable \(v\) with respect to the quantum expectation \(c\) — a pointed cochain map from \((\mathcal{E}[ [ \hbar ] ], 1_\mathcal{E}, K)\) to \((\mathbb{k}[ [ \hbar ] ], 1_\mathbb{k}, 0)\). From \(K \circ f = 0\), it follows that \(c(\tilde{f}(v))\) depends only on the homotopy type of \(c\) for all \(v \in H\). We also call \(c \circ f \in \text{Hom}(H, \mathbb{k})^0[ [ \hbar ] ]\) the on-shell quantum expectation with respect to \(c\).
We call \( \Pi^0 \) the family of level 0 quantum correlators. For example,

\[
\Pi^0_1 = \Phi^0_1,
\]

\[
\Pi^0_2(v_1, v_2) = \Phi^0_1(v_1) \cdot \Phi^0_1(v_2) + (-h) \Phi^0_2(v_1, v_2),
\]

\[
\Pi^0_3(v_1, v_2, v_3) = \Phi^0_1(v_1) \cdot \Phi^0_1(v_2) \cdot \Phi^0_1(v_3) + (-h) \Phi^0_2(v_1, v_2) \cdot \Phi^0_2(v_1, v_3)
\]

\[
+ (-h)(-1)^{|v_1|v_2}|v_3| \Phi^0_1(v_1) \cdot \Phi^0_2(v_2, v_3) + (-h) \Phi^0_2(v_1, v_2) \cdot \Phi^0_2(v_1, v_3)
\]

\[
+ (-h)^2 \Phi^0_3(v_1, v_2, v_3).
\]

We call \( c \circ \Pi^0 \) the family of level 0 quantum correlation functions with respect to \( c \). From eq. (6.1), we have \( K \circ \Pi^0_n = 0 \) so that \( c \circ \Pi^0_n \in \text{Hom}(S^n H, k)[[h]] \) depends only on the homotopy type of \( c \). This also implies that, for all \( n \geq 1 \),

\[
c \circ \Pi^0_n = c \circ f \circ \hat{\pi}^0_n. \tag{6.9}
\]

Therefore the whole family \( c \circ \Pi^0 \) of level 0 quantum correlation functions is determined by the on-shell quantum expectation \( c \circ f \) and the family \( \hat{\pi}^0_n \). Note that \( c \circ f \) is a formal power series in \( h \) so that there may be divergence issues if we try to evaluate \( h \) away from zero. On the other hand \( \hat{\pi}_n \) is at most an order \( n - 2 \) polynomial in \( h \) for \( n \geq 2 \), and it is the family \( \hat{\pi}^0_n \) that governs quantum correlations.

For any set of observables \( \{v_1, \ldots, v_k\} \), we can define the associated family of joint quantum moments (at level zero) as follows:

\[
\{ (\Pi^0_n(v_{j_1}, \ldots, v_{j_n}))_\epsilon \} = c \circ \Pi^0_n(v_{j_1}, \ldots, v_{j_n}) | n \geq 1; 1 \leq j_1, \ldots, j_n \leq k. \tag{6.10}
\]

Then, the joint quantum distribution of the set \( \{v_1, \ldots, v_k\} \) of observables can be defined as a \( k \)-linear map

\[
\hat{\mu}_\epsilon : k[t_1, \ldots, t_k] \to k[[h]]
\]

as follows. For any monomial \( t_{j_1} \cdots t_{j_n} \) in \( k[t_1, \ldots, t_k] \) satisfying \( 1 \leq j_1, \ldots, j_n \leq k \), we have

\[
\hat{\mu}_\epsilon(t_{j_1} \cdots t_{j_n}) = (\Pi^0_n(v_{j_1}, \ldots, v_{j_n}))_\epsilon.
\]

It is clear that \( \hat{\mu}_\epsilon \) is determined completely by the family of joint quantum moments which is conveniently described by its generating function

\[
Z(t_1, \ldots, t_k) = \left( e^{-\frac{1}{h} \Theta(\gamma)} \right)_\epsilon = 1 + \sum_{n=1}^{\infty} \frac{1}{n!(-h)^n} (\Pi^0_n(\gamma, \ldots, \gamma))_\epsilon \in k[[t_1, \ldots, t_k]]((h)),
\]
where \( \gamma = \sum_{i=1}^{k} t_i v_i \) and
\[
\Theta(\gamma) = \sum_{n=1}^{\infty} \frac{1}{n!} \Phi_n(\gamma, \ldots, \gamma) = \sum_{n=1}^{\infty} \sum_{1 \leq j_1, \ldots, j_n \leq k} \frac{1}{n!} t_{j_1} \cdots t_{j_n} \Phi_n(v_{j_1}, \ldots, v_{j_n}).
\]

Then, we have
\[
\hat{\mu}(t_{j_1} \cdots t_{j_n}) = (-\hbar)^n \frac{\partial^n Z(t_1, \ldots, t_k)}{\partial t_{j_1} \cdots \partial t_{j_n}} \bigg|_{t_1, \ldots, t_k = 0}.
\]

An important part of the results in Section 5.2, specialized to the case \( \kappa = 0 \), can now be presented as follows.

**Theorem 6.2.** Let \( \hat{m}_n^0 := \hat{\pi}_n^0(n-2) \in \text{Hom}(S^n H, H) \) for \( n \geq 2 \). Then the family \( \hat{m}_n^0 = \hat{m}_2^0, \hat{m}_3^0, \ldots \) determines the family \( \hat{\pi}_n^0 \) recursively with initial condition \( \hat{\pi}_1^0 = 1_H \) along with the following equation for \( n \geq 2 \):
\[
\hat{\pi}_n^0(v_1, \ldots, v_n) = \sum_{\substack{p \in \mathcal{P}(n) \\mid |p| = n-|p|+1 \\text{and} \ n-1 \sim_p n}} (-\hbar)^{|p|-1} \varepsilon(p) \hat{\pi}_p^0(v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}_0^0(v_{B_p})).
\]  

(6.11)

Moreover, there is a distinguished family \( \phi^{-1} = \phi_2^{-1}, \phi_3^{-1}, \ldots \) of \( \phi_2^{-1} \) in \( \text{Hom}(S^{n-2} \otimes S^2 H, \mathcal{C})^{-1}[\hbar] \) so that \( \phi_2^{-1} = \eta_2^{-1} \) and, for all \( n \geq 2 \),
\[
\mathcal{M}_n^0 = f \circ \hat{m}_n^0 + K \circ \phi_n^{-1},
\]  

(6.12)

where \( \mathcal{M}_n^0 \in \text{Hom}(S^{n-2} H \otimes S^2 H, \mathcal{C})^0[\hbar] \) is defined as follows:
\[
\mathcal{M}_n^0(v_1, \ldots, v_n) = (-\hbar)\phi_n^0(v_1, \ldots, v_n) + \sum_{\substack{p \in \mathcal{P}(n) \\mid |p| = 2 \\text{and} \ n-1 \sim_p n \\text{and} \ |p| \neq 1}} \varepsilon(p) \phi_p^0(v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}_0^0(v_{B_p}))
\]
\[
- \sum_{\substack{p \in \mathcal{P}(n) \\mid |B_p| = n-|p|+1 \\text{and} \ n-1 \sim_p n \\text{and} \ |p| \neq 1}} \varepsilon(p) \mathcal{M}_p(v_{B_1}, \ldots, v_{B_{|p|-1}}, \phi^0(J v_{B_p})).
\]  

(6.13)
Remark 6.3. One significance to this theorem is that it suggests a different method than the one in Remark 6.2 to compute the family $\mathbf{H}^0$.

For $n \geq 1$ let $\ell_n$ be the classical limit of $\ell_n$ — note that $\ell_1 = K$. Then $(\mathcal{C}, 1_\mathcal{C}, \ell)$ is a unital $sL_\infty$-algebra over $k$. Let $\phi_n^0$ be the classical limit of $\phi_n^0$ for $n \geq 1$ — note that $\phi_1^0 = f$. Then $\phi_n^0 : (H_1, 1_H, 0) \longrightarrow (\mathcal{C}, 1_\mathcal{C}, \ell)$ is a distinguished unital $sL_\infty$-quasi-isomorphism. Let $\phi_n^{-1}$ be the classical limit of $\phi_n^{-1}$ for $n \geq 2$. Then, from the classical limit of eq. (6.12), we obtain

$$M_n^0 = f \circ \hat{m}_n^0 + K \circ \phi_n^{-1},$$

for $n \geq 2$, where

$$M_n^0(v_1, \ldots, v_n) = \sum_{p \in P(n)} \epsilon(p) \phi(v_{B_1}) \cdot \phi(v_{B_2}) - \sum_{p \in P(n)} \epsilon(p) \phi(v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}_n^0(v_{B_{|p|}}))$$

$$+ \sum_{p \in P(n)} \epsilon(p) \ell_{|p|} \left( \phi(J v_{B_1}), \ldots, \phi(J v_{B_{|p|-1}}), \phi^{-1}(v_{B_{|p|}}) \right).$$

The relation eq. (6.14) implies that $K \circ M_n = 0$ and $\hat{m}_n^0 = h \circ M_n$ for all $n \geq 2$. Therefore we can determine the family $\hat{m}_n^0$, and hence the family $\mathbf{H}^0$, by taking the classical $K$ cohomology class of the family $M_n^0$. Note that $M_n \in \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^0$, while $\hat{m}_n^0 = h \circ M_n \in \text{Hom}(S^nH, H)^0$.

Remark 6.4. Note that the recursive definition of $M_n$ depends only on the three families $\{\phi_1^0, \ldots, \phi_{n-1}^0\}$, $\{\phi_1^{-1}, \ldots, \phi_{n-1}^{-1}\}$, and $\{\hat{m}_2^0, \ldots, \hat{m}_{n-1}^0\}$. For example, we have

$$M_2^0(v_1, v_2) = \phi_1^0(v_1) \cdot \phi_1^0(v_2),$$

$$M_3^0(v_1, v_2, v_3) = \phi_2^0(v_1, v_2) \cdot \phi_2^0(v_3) + (-1)^{|v_1||v_2|} \phi_1^0(v_2) \cdot \phi_1^0(v_1, v_3) - \phi_2^0(v_1, \hat{m}_2^0(v_2, v_3)) + \ell_2(J v_1, \phi_1^{-1}(v_2, v_3)).$$

etc. Recall that the combined classical limit $(\mathcal{C}, 1_\mathcal{C}, \cdot \cdot \cdot \cdot \ell)$ of a binary QFT algebra and its quantum descendant unital $sL_\infty$-algebra is called a binary CFT algebra. (See Lemma 2.1.) The unital CDGA part $(\mathcal{C}, 1_\mathcal{C}, \cdot K)$ induces the structure of a unital $\mathbb{Z}$-graded commutative and associative algebra on $H$, whose product is exactly $\hat{m}_2^0$. 
— from \( \phi_1 = f \), we have \( M^0_2(v_1, v_2) = f(v_1) \cdot f(v_2) \) and \( \hat{m}^0_2 = h \circ M^0_2 \) defines the \( \mathbb{Z} \)-graded commutative and associative algebra \((H, 1_H, \hat{m}^0_2)\).

Note that \( f = \phi^0_1 : (H, 1_H, \hat{m}^0_2) \to (\mathcal{E}, 1_H, \cdot, K) \) is both a pointed cochain quasi-isomorphism and an algebra homomorphism up to the homotopy \( \phi^{-1}_2 \) by eq. (6.14):

\[
\phi^0_1(\hat{m}^0_2(v_1, v_2)) - \phi^0_1(v_1) \cdot \phi^0_1(v_2) = -K\phi^{-1}_2(v_1, v_2) \tag{6.15}
\]

Note also that \((H, 1_H, 0)\) is a unital \( sL_\infty \)-algebra with zero \( sL_\infty \)-structure \( 0 \), and

\[
\ell_2(\phi^0_1(v_1), \phi^0_1(v_2)) = -K\phi^0_2(v_1, v_2). \tag{6.16}
\]

Finally, note that \( \ell_2 \) is a derivation of the product \( \cdot \). Therefore, we have the following identity:

\[
\ell_2(\phi^0_1(v_1), \phi^0_1(v_2) \cdot \phi^0_1(v_3)) = \ell_2(\phi^0_1(v_1), \phi^0_1(v_2)) \cdot \phi^0_1(v_3) + (-1)^{|v_1||v_2|} \phi^0_1(v_2) \cdot \ell_2(\phi^0_1(Jv_1), \phi^0_1(v_3)).
\]

Substituting eq. (6.15) and eq. (6.16) into this equation, we obtain that \( K \circ M_3 = 0 \). Therefore, we may call \( \hat{m}^0_3 = h \circ M_3 \) the compatibility class between the product \( \cdot \) and the bracket \( \ell_2 \).

\[
\text{Theorem 6.3. The family } \hat{m}^0 = \hat{m}^0_2, \hat{m}^0_3, \ldots \text{ has the following properties:}
\]

- (symmetry): \( \hat{m}^0_n \in \text{Hom}(S^n H, H)^0 \), for all \( n \geq 2 \);
- (unity): \( \hat{m}^0_n(1_H, v_1) = v_1 \), while \( \hat{m}^0_{n+1}(1_H, v_1, \ldots, v_n) = 0 \) for all \( n \geq 2 \);
- (generalized associativity): for all \( n \geq 0 \) and homogeneous \( v_1, \ldots, v_n, w_1, w_2, w_3 \in H \), we have

\[
\sum_{\zeta \subseteq [n]} \epsilon(\zeta \cup \zeta^c) \hat{m}^0_n(v_{\zeta} \circ \hat{m}^0_n(v_{\zeta^c} \circ w_1 \circ w_2) \circ w_3)
\]

\[
= \sum_{\zeta \subseteq [n]} \epsilon(\zeta \cup \zeta^c)(-1)^{|v_{\zeta^c}|} \hat{m}^0_n(v_{\zeta} \circ w_1 \circ \hat{m}^0_n(v_{\zeta^c} \circ w_2 \circ w_3)),
\]

where the sums are over all subsets \( \zeta \subseteq [n] \) of the set \( [n] = \{1, 2, \ldots, n\} \), as ordered sets, we use \( \zeta^c \) to denote the complement of \( \zeta \), we use \( |\zeta| \) for the number of elements in \( \zeta \), we write \( v_{\zeta} = v_{j_1} \otimes \ldots \otimes v_{j_{|\zeta|}} \) if \( \zeta = \{j_1, \ldots, j_{|\zeta|}\}, j_1 < \ldots < j_{|\zeta|} \), and \( \epsilon(\zeta \cup \zeta^c) \) is notation for the Koszul sign for the permutation \( \hat{\sigma} \) of \( \{v_1 \otimes \ldots \otimes v_n\} = v_{\zeta} \otimes v_{\zeta^c} \).

\[
\text{Proof. The symmetry is obvious since } \hat{m}^0_n = \hat{m}^0_n(n-2) \in \text{Hom}(S^n H, H)^0. \text{ We can also deduce that } \hat{m}^0_2(v, 1_H) = v \text{ and } \hat{m}^0_{n+1}(v_1, \ldots, v_n, 1_H) = 0 \text{ if } n \geq 2 \text{ from the property}
\]
that \( \pi^0_{n+1}(v_1, \ldots, v_n, 1_H) = \pi^0_n(v_1, \ldots, v_n) \), for all \( n \geq 1 \). It remains to show generalized associativity, which follows from \( \bar{\pi}^0 \in \text{Hom}(\mathcal{F}(H), H)^0[[h]] \). Note that the set of relations in eq. (6.1) is equivalent to \( \bar{m}^0_2 = m^0_2 \) and, for all \( n \geq 0 \),

\[
\bar{\pi}^0_{n+3}(v_1, \ldots, v_n, w_1, w_2, w_3) = \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-h)^{\varsigma^c} \bar{\pi}^0(v_\varsigma \odot m^0(v_\varsigma \odot w_1 \odot w_2 \odot w_3)) + \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_1||v_\varsigma|} (-h)^{\varsigma^c} \bar{\pi}^0(v_\varsigma \odot w_1 \odot m^0(v_\varsigma \odot w_2 \odot w_3)).
\]

Then, from \( \bar{\pi}^0_{n+3}(v_1, \ldots, v_n, w_1, w_2, w_3) = (-1)^{|w_1||w_2|} \bar{\pi}^0_{n+3}(v_1, \ldots, v_n, w_2, w_1, w_3) \) and the symmetry of \( m^0 \), we obtain that

\[
\sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_1||v_\varsigma|} (-h)^{\varsigma^c} \bar{\pi}^0(v_\varsigma \odot w_1 \odot m^0(v_\varsigma \odot w_2 \odot w_3)) = (-1)^{|w_1||w_2|} \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_2||v_\varsigma|} (-h)^{\varsigma^c} \bar{\pi}^0(v_\varsigma \odot w_2 \odot m^0(v_\varsigma \odot w_1 \odot w_3)).
\]

Then the relation \( \bar{\pi}^0_{n+2} = \bar{m}^0_n \) implies that

\[
\sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_1||v_\varsigma|} m^0(v_\varsigma \odot w_1 \odot m^0(v_\varsigma \odot w_2 \odot w_3)) = (-1)^{|w_1||w_2|} \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_2||v_\varsigma|} m^0(v_\varsigma \odot w_2 \odot m^0(v_\varsigma \odot w_1 \odot w_3)).
\]

The right hand side of the above equality is equivalent to

\[
(-1)^{|w_1||w_2|+|w_3|} \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_2||v_\varsigma|} m^0(v_\varsigma \odot w_2 \odot m^0(v_\varsigma \odot w_3 \odot w_1)) = (-1)^{|w_1||w_2|+|w_3|+|w_2||w_1|} \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_3||v_\varsigma|} m^0(v_\varsigma \odot w_3 \odot m^0(v_\varsigma \odot w_2 \odot w_1)) = (-1)^{|w_1|+|w_2||w_3|} \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) (-1)^{|w_3||v_\varsigma|} m^0(v_\varsigma \odot w_3 \odot m^0(v_\varsigma \odot w_1 \odot w_2)) = \sum_{\varsigma \in [n]} e(\varsigma \sqcup \varsigma^c) m^0(v_\varsigma \odot m^0(v_\varsigma \odot w_1 \odot w_2) \odot w_3),
\]

so that we have generalized associativity.

\[\]

**Definition 6.1.** We call the triple \((H, 1_H, \bar{m}^0)\) the on-shell quantum correlation algebra of the anomaly-free binary QFT algebra \( \mathcal{F}[[h]] \) of QFTA and call \( \bar{\pi}^0 \) the family of iterated quantum correlation products generated by \( \bar{m}^0 \).
A large portion of Section 5.2 can be viewed as an algorithm to determine the distinguished family $\hat{\Phi}^{-1}$ in Theorem 6.2 if we specialize to the case $\kappa = 0$.

**Theorem 6.4.** There are families $\hat{\Pi}^{-1} = \hat{\Pi}^{-1}_2, \hat{\Pi}^{-1}_3, \ldots$ and $\eta^{-2} = \eta^{-2}_2, \eta^{-2}_3, \ldots$ such that, for all $n \geq 3$, \[ \Pi_n^{-1} = f \circ \hat{\Pi}_n^{-1} + K \circ \eta_n^{-2}, \tag{6.17} \]

where \[
\Pi_n^{-1}(v_1, \ldots, v_n) := \eta_n^{-1}(v_1, \ldots, v_n) \\
\quad - \sum_{p \in P(n)} (-h)^{n-|p|-1} \epsilon(p) \Phi^0(J v_B^{1}) \cdots \Phi^0(J v_B^{p-1}) \cdot \Phi^{-1}(v_B^{p}) \\
\quad - \sum_{p \in P(n)} (-h)^{n-|p|-1} \epsilon(p) \eta_n^{-1}(v_B^{1}, \ldots, v_B^{p-1}, \Pi^0(v_B^{p})) \tag{6.18} \]

and

- we have $\hat{\Pi}_2^{-1} = 0$ and, for all $n \geq 3$, \[ \hat{\Pi}_n^{-1} = \sum_{j=0}^{n-3} (-h)^j \hat{\Pi}_n^{-1}(j), \text{ where } \hat{\Pi}_n^{-1}(j) \in \text{Hom}(S^{n-2}H \otimes S^2H, H)^0 \]

and $\hat{\Pi}_n^{-1}(v_1, \ldots, v_n) = 0$ whenever $v_i = 1_H$;

- we have $\eta_2^{-2} = 0$ and, for all $n \geq 3$, \[ \eta_n^{-2} = \sum_{j=0}^{n-3} (-h)^j \eta_n^{-2}(j), \text{ where } \eta_n^{-2}(j) \in \text{Hom}(S^nH \otimes S^2H, \mathcal{C})^{-2} \]

and $\eta_n^{-2}(v_1, \ldots, v_n) = 0$ whenever $v_i = 1_H$.

Note that $\Pi_n^{-1} \in \text{Hom}(S^nH \otimes S^2H, \mathcal{C})^{-1}[[h]]$, for all $n \geq 2$. The relation in eq. (6.17) implies that $K \circ \Pi_n^{-1} = 0$, for all $n \geq 3$. It follows that $c \circ \Pi_n^{-1}$ depends only on the homotopy type of the quantum expectation $c$. We call the family $\Pi_n^{-1} = \Pi_2^{-1}, \Pi_3^{-1}, \ldots$ the level one quantum correlators and $c \circ \Pi^{-1}$ the family of level one quantum correlation functions with respect to $c$. The relation of eq. (6.17) also implies that, for all $n \geq 2$, \[ c \circ \Pi_n^{-1} = c \circ f \circ \hat{\Pi}_n^{-1}, \tag{6.19} \]

so that the $n$-fold level one quantum correlation function $c \circ \Pi_n^{-1}$ is determined by the on-shell quantum expectation $c \circ f$ and $\hat{\Pi}_n^{-1}$, which is at most a degree $n-3$ polynomial.
in $\hbar$. The physical interpretation of $c \circ \Pi^{-1}$ remains elusive but we have presented an algorithm to determine the family $\pi_\omega^{-1}$. With some effort, one can obtain Theorem 6.2 as the appropriate integrability condition of this theorem.

Remark 6.5. We reproduce the algorithm to determine the families $\phi^{-1}, \pi^{-1}$ and $\eta^{-2}$, which become much simpler due to the condition $\kappa = 0$.

Set

$$\phi_2^{-1} = \eta_2^{-1}, \quad \pi_2^{-1} = 0, \quad \eta_2^{-2} = 0. \quad (6.20)$$

Note that $\Pi_2^{-1} = \eta_2^{-1} - \phi_2^{-1} = 0$. Therefore, we have

$$\begin{cases}
\Pi_2^{-1} = f \circ \pi_2^{-1} + K \circ \eta_2^{-2}, \\
M_2^0 = f \circ m_2^0 + K \circ \phi_2^{-1}, \\
\pi_2^0 = m_2^0,
\end{cases} \quad (6.21)$$

where we used $\pi_2^0 = m_2$ and $M_2^0(v_1, v_2) := (-\hbar)\phi_2^0(v_1, v_2) + \phi_1^0(v_1) \cdot \phi_1^0(v_2) = \Pi_2^0(v_1, v_2)$.

Assume that we have $\{\phi_2^{-1}, \ldots, \phi_{n-1}^{-1}, \pi_2^{-1}, \ldots, \pi_{n-1}^{-1}, \eta_2^{-2}, \ldots, \eta_{n-1}^{-2}\}$ for $n \geq 3$ satisfying the initial conditions of eq. (6.20) and, for all $k = 2, \ldots, n - 1,$

$$\begin{align*}
\Pi_k^{-1} &= f \circ \pi_k^{-1} + K \eta_k^{-2}, \\
M_k^0 &= f \circ m_k^0 + K \circ \phi_k^{-1}, \\
\pi_k(v_1, \ldots, v_k) &= \sum_{p \in P_k} \binom{k}{p-1} (-\hbar)^k \epsilon(p) \pi_{|p|} \left( v_{B_1}, \ldots, v_{B_{p-1}}, m(v_{B_p}) \right) . \quad (6.22)
\end{align*}$$
Define $\Omega_n^{-1} \in \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-1}[\hbar]$ and $\varpi_n^0 \in \text{Hom}(S^{n-2}H \otimes S^2H, H)^0[\hbar]$ as follows:

$$
\Omega_n^{-1}(v_1, \ldots, v_n) := \eta_n^{-1}(v_1, \ldots, v_n) - \sum_{p \in P(n)} \frac{(-\hbar)^{n-|p|-1} \varepsilon(p) \phi^0(Jv_{B_1}) \cdots \phi^0(Jv_{B_{|p|-1}}) \cdot \phi^{-1}(v_{B|p|})}{|p| \neq 1} \sum_{|\mathfrak{R}_p| = n-|p|+1} \eta_{|p|}^{-1}(v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}^0(v_{B|p|})),
$$

$$
\varpi_n^0(v_1, \ldots, v_n) := \hat{\pi}_n^0(v_1, \ldots, v_n) - \sum_{p \in P(n)} \frac{(-\hbar)^{n-|p|-1} \varepsilon(p) \hat{\pi}_{|p|}^0(v_{B_1}, \ldots, v_{B_{|p|-1}}, \hat{m}^0(v_{B|p|})).}{|p| \neq 1}
$$

Note that we have

$$
\varpi_n^0 = \sum_{i=0}^{n-2} (-\hbar)^i \varpi_n^{0(i)}, \quad \text{where} \quad \varpi_n^{0(i)} \in \text{Hom}(S^{n-2}H \otimes S^2H, H)^0,
$$

and $\varpi_n^{0(n-2)} = \hat{\pi}_n^0 = \hat{m}^0$.

From eq. (6.22) and the definition of $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$, we obtain that

$$
K \circ \Omega_n^{-1} = (-\hbar)^{n-2} M_n^0 - f \circ \varpi_n^0
$$

(6.23)

Note that the classical limit of this equation is $K \circ \Omega_n^{-1} = -f \circ \varpi_n^{0(0)}$, which implies that $K \circ \Omega_n^{-1} = 0$ and $\varpi_n^{0(0)} = 0$. Therefore, we have

$$
K \circ \Omega_n^{-1} = (-\hbar)^{n-2} M_n^0 - \sum_{|\mathfrak{R}_p| = n-|p|+1} \eta_{|p|}^{-1} \varpi_n^{0(i)}. \quad (6.24)
$$

Set $(\Omega_n^{-1})^{[0]} := \Omega_n^{-1}$ and $(\Omega_n^{-1})^{[1]} := \nabla_{-1/\hbar} (\Omega_n^{-1})^{[0]} \in \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-1}[\hbar]$, so that we have

$$
(-\hbar) (\Omega_n^{-1})^{[1]} = (\Omega_n^{-1})^{[0]} - f \circ \hat{\pi}_n^{1(0)} - K \circ \eta_n^{-2(0)}, \quad \text{where} \quad \hat{\pi}_n^{-1(0)} := h \circ (\Omega_n^{-1})^{[0]}, \quad \eta_n^{-2(0)} := s \circ (\Omega_n^{-1})^{[0]}.
$$

From eq. (6.24), we have

$$
K \circ (\Omega_n^{-1})^{[1]} = (-\hbar)^{n-3} M_n^0 - \sum_{i=1}^{n-2} (-\hbar)^{i-1} f \circ \varpi_n^{0(i)}, \quad (6.25)
$$
and the classical limit for \( n > 3 \) is then \( K \circ (\Omega_n^{-1})^{[i]} = -\mathbf{f} \circ \hat{\phi}_n^{0(1)} \), which implies that \( K \circ (\Omega_n^{-1})^{[1]} = 0 \). Working inductively after setting \( (\Omega_n^{-1})^{[i+1]} := \nabla_{-\hbar} (\Omega_n^{-1})^{[i]} \), we can check that \( K \circ (\Omega_n^{-1})^{[i]} = 0 \) for \( i = 0, \ldots, n-3 \). It follows that

\[
(-\hbar)^{n-2} (\Omega_n^{-1})^{[n-2]} = \Omega_n^{-1} - \sum_{i=0}^{n-3} (-\hbar)^i \mathbf{f} \circ \hat{\phi}_n^{1(1)} - \sum_{i=0}^{n-3} (-\hbar)^i K \circ \eta_n^{2(1)},
\]

where

\[
\hat{\phi}_n^{1(1)} := \eta (\Omega_n^{-1})^{[i]} \quad \text{in} \quad \text{Hom}(S^{n-2}H \otimes S^2H, H)^{-1},
\]

\[
\eta_n^{2(1)} := \sigma (\Omega_n^{-1})^{[i]} \quad \text{in} \quad \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{-2},
\]

and \( (\Omega_n^{0})^{[i]} \in \text{Hom}(S^{n-2}H \otimes S^2H, \mathcal{C})^{0} \) denotes the classical limit of \( (\Omega_n^{0})^{[i]} \).

Finally, we set

\[
\phi_n^{-1} = (\Omega_n^{-1})^{[n-2]}, \quad \hat{\phi}_n^{-1} = \sum_{i=0}^{n-3} (-\hbar)^{i} \hat{\phi}_n^{1(1)}, \quad \eta_n^{-2} = \sum_{i=0}^{n-3} (-\hbar)^{i} \eta_n^{2(1)},
\]

and note that \( \Pi_n^{-1} = \Omega_n^{-1} + (-\hbar)^{n-2} \phi_n^{-1} \). Then, from eq. (6.26), we conclude that

\[
\Pi_n^{-1} = \mathbf{f} \circ \hat{\phi}_n^{-1} + K \eta_n^{-2},
\]

\[
\mathbf{M}_n^0 = \mathbf{f} \circ m_n^0 + K \circ \phi_n^{-1},
\]

\[
\hat{\phi}_n(v_1, \ldots, v_n) = \sum_{p \in \mathcal{P}(n)} (-\hbar)^{n-|p|-1} \epsilon(p) \hat{\phi}_n^{|p|} \left( v_{B_1}, \ldots, v_{B_{|p|-1}}, m(v_{B_{|p|}}) \right).
\]

Therefore, we have a well-defined algorithm to determine the families \( \phi_n^{-1}, \hat{\phi}_n^{-1} \) and \( \eta_n^{-2} \).

Before leaving this subsection, we prove Theorem 1.4.

Consider another binary QFT algebra \( \mathcal{C}'[\hbar]_{\text{BQFTA}} = (\mathcal{C}'[\hbar], 1_{\mathcal{C}'}, , K') \) which is homotopy equivalent to our binary QFT algebra \( \mathcal{C}[\hbar]_{\text{BQFTA}} \). Realize this equivalence by a quasi-isomorphism \( \mathbf{f} : \mathcal{C}[\hbar]_{\text{BQFTA}} \rightarrow \mathcal{C}'[\hbar]_{\text{BQFTA}} \) of binary QFT algebras. It is obvious that \( \mathcal{C}'[\hbar]_{\text{BQFTA}} \) is also anomaly-free, and we will use the induced isomorphism \( H(\mathbf{f}^{(0)}) : H \rightarrow H' \) to identify \( H' \) with \( H \). Write \( (\mathcal{C}'[\hbar], 1_{\mathcal{C}'}, \mathbf{t}') \) for the quantum
quasi-isomorphism \( \phi \) that the definition of the associated family \( \Pi \) isomorphism from \( \mathcal{L} \) be the quantum descendant of \( f \) which is a quasi-isomorphism of the unital \( sL_\infty \)-algebras. Then, by definition, we have

\[
\tag{6.28}
\pi \circ f = \pi' \circ \Psi \phi,
\]

where \( \pi(x_1 \otimes \ldots \otimes x_n) := x_1 \cdot \ldots \cdot x_n \) and \( \pi'(x'_1 \otimes \ldots \otimes x'_n) := x'_1 \cdot \ldots \cdot x'_n \) for all \( n \geq 1 \), \( x_1, \ldots, x_n \in \mathcal{C}[H] \) and \( x'_1, \ldots, x'_n \in \mathcal{C}[h] \). Consider the distinguished \( sL_\infty \)-quasi-isomorphism \( \Phi^0 : (H[[H]], 1_H, 0) \longrightarrow (\mathcal{C}[[H]], 1_{\mathcal{C}}, L) \) of Theorem 6.1. Recall that the definition of the associated family \( \Pi^0 \) of level zero quantum correlators is equivalent to \( \Pi^0 := \pi \circ \Psi \phi^0 \in \text{Hom} \left( \mathcal{S}(H)^0, [H] \right) \) and, from eq. (6.1), we have

\[
\pi \circ \Psi \phi^0 = f \circ \pi^0 + K \circ \eta^{-1}. \tag{6.29}
\]

Now we consider the following unital \( sL_\infty \)-quasi-isomorphism

\[
\Phi^0 := \Psi \cdot \Phi^0 : (H[[H]], 1_H, 0) \longrightarrow (\mathcal{C}[[H]], 1_{\mathcal{C}}, L)
\]

and the associated family \( \Pi^0 \) of level zero quantum correlators, whose definition is equivalent to \( \Pi^0 := \pi' \circ \Psi \phi^0 \in \text{Hom} \left( \mathcal{S}(H)^0, [H] \right) \). From the combination of the equation \( \Psi \phi^0 = \Psi \phi \circ \Psi \phi^0 = \Psi \phi \circ \Psi \phi^0 \), eq. (6.28), eq. (6.29), and \( f \circ K = K' \circ f \), we obtain the following:

\[
\Pi^0 = \pi' \circ \Psi \phi \circ \Psi \phi^0 = f \circ \pi \circ \Psi \phi \circ \Psi \phi^0 = f \circ (f \circ \pi + K \circ \eta^{-1}) = f \circ f \circ \pi + K' \circ f \circ \eta^{-1}
\]

\[
= f' \circ \pi + K' \circ \eta^{-1},
\]

where \( f' := f \circ f \) and \( \eta^{-1} := f \circ \eta^{-1} \). Therefore, under the isomorphism \( H[[H]] \cong H'[[[H]]] \), we can identify \( \Phi^0 := \Psi \cdot \Phi^0 \) and \( \Pi^0 \) with a distinguished unital \( sL_\infty \)-quasi-isomorphism from \( (H'[[[H]], 1_{H'}, 0) \to (\mathcal{C}'[[H]], 1_{\mathcal{C}'}, L') \) and the associated family of level zero quantum correlators. It follows that a homotopy equivalence of anomaly-free binary QFT algebras induces an isomorphism of on-shell quantum correlation algebras and, in particular, sends a quantum structure to a quantum structure.

6.2. Relations with the WDDV equation

In the remaining part of this subsection, we further specialize to the case that \( H \) is finite dimensional as a \( \mathbb{Z} \)-graded vector space. We shall see some interesting aspects of our solutions.
Assume that \( H \) is finite dimensional. Choose homogeneous coordinates \( t_H = \{ t^a \} \) so that \( \{ \partial_a = \partial / \partial t^a \} \) form a basis of \( H \) and \( \partial_0 = 1_H \). Extend \( \{ \partial_a \} \) as a derivation of \( k[[t_H]] \cong \hat{S}(H^*) \), which is the completed symmetric algebra generated by the dual \( \mathbb{Z} \)-graded vector space \( H^* \).

Using the unital \( sL_\infty \)-quasi-morphism \( \Phi^0 : (H[[h]], 1_H, 0) \longrightarrow (\mathcal{C}[[h]], 1_\mathcal{C}, \mathcal{L}) \) of Theorem 6.1, we define

\[
\Theta := \sum_{n=1}^{\infty} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \Phi^0_n(\partial_{a_1}, \ldots, \partial_{a_n}) \in (k[[t_H]][\otimes\mathcal{C}])^0[[h]].
\]

Then, by the property that \( \Phi^0 \) is a unital \( sL_\infty \)-morphism, we obtain that

\[
K\Theta + \sum_{n \geq 1} \frac{1}{n!} t^{a_n} \cdots t^{a_1} \Phi^0_n(\partial_{a_1}, \ldots, \partial_{a_n}) = 0 \iff K e^{-\frac{1}{\hbar}\Theta} = 0,
\]

(6.30)

\[
\partial_0 \Theta = 1_\mathcal{C} \iff (-\hbar)\partial_0 e^{-\frac{1}{\hbar}\Theta} = e^{-\frac{1}{\hbar}\Theta}.
\]

From the property that \( \Phi_1 : (H[[h]], 1_H, 0) \rightarrow (\mathcal{C}[[h]], 1_\mathcal{C}, K) \) is a cochain quasi-isomorphism, we see that \( \Theta \) is a universal solution to the Maurer–Cartan equation of the unital \( sL_\infty \)-algebra \( (\mathcal{C}[[h]], 1_\mathcal{C}, \mathcal{L}) \). We also have the following identity:

\[
e^{-\frac{1}{\hbar}\Theta} = 1_\mathcal{C} + \sum_{n=1}^{\infty} \frac{1}{(-\hbar)^n n!} t^{\rho_n} \cdots t^{\rho_1} \Pi^0_n(\partial_{\rho_1}, \ldots, \partial_{\rho_n}).
\]

(6.31)

Therefore \( e^{-\frac{1}{\hbar}\Theta} \) is a generating function of the family \( \Pi^0 \) of level 0 quantum correlators. From the families \( \Pi^0 \) and \( \eta^{-1} \) in Theorem 6.1 define

\[
\bar{\gamma} := t^\gamma + \sum_{n=2}^{\infty} \frac{1}{n!(-\hbar)^{n-1}} t^{\rho_n} \cdots t^{\rho_1} \Pi^0_n(\partial_{\rho_1}, \ldots, \partial_{\rho_n}) \in k[[t_H]][[\hat{S}(H^{-1})]],
\]

\[
\Sigma := \sum_{n=2}^{\infty} \frac{1}{n!(-\hbar)^{n-1}} t^{\rho_n} \cdots t^{\rho_1} \eta^{-1}(\partial_{\rho_1}, \ldots, \partial_{\rho_n}) \in (k[[t_H]][\otimes\mathcal{C}][[\hat{S}(H^{-1})]])^{-1},
\]

where \( \{ \Pi^0_{\alpha_1 \cdots \alpha_n} \} \in k[[h]] \) are structure constants, i.e., \( \Pi^0_{\alpha_1 \cdots \alpha_n}(\partial_{\alpha_1}, \ldots, \partial_{\alpha_n}) = \Pi^0_{\alpha_1 \cdots \alpha_n} = \delta_{\gamma} \partial_{\gamma} \) and \( t^\rho = (-\hbar)^{\delta h(\partial_{\rho})} t^\rho \). It is easy to check that the relation of eq. (6.1) is equivalent to the following identity:

\[
e^{-\frac{1}{\hbar}\Theta} = 1_\mathcal{C} + \frac{1}{(-\hbar)} \bar{\gamma} \Pi^0(\partial_{\gamma}) + \frac{1}{(-\hbar)} K \Sigma.
\]

(6.32)
From the property that $\Pi^0_n(v_1, \ldots, v_n, 1_H) = \Pi^0_n(v_1, \ldots, v_n)$ for all $n \geq 1$, we also have
\[
\partial_0 \hat{T}^\gamma = \delta_0^\gamma - \frac{1}{\hbar} \hat{T}^\gamma.
\] (6.33)

Note that eq. (6.32) and eq. (6.33) imply the relations in eq. (6.30).

We emphasize that both $\hat{T}^\gamma$ and $\Sigma$ are formal power series in $\hbar^{-1} = 1/\hbar$, since both $\Pi^0_n$ and $\eta_n^{-1}$ are polynomials in $\hbar$ with degree at most $n-2$ for $n \geq 2$. These both follow from $\Phi^0$ being a distinguished $sL_{\infty}$-quasi-morphism.

**Remark 6.6.** Let $\varphi : (H[[\hbar]], 1_H, 0) \to (\mathcal{C}[[\hbar]], 1_\mathcal{C}, \mathcal{L})$, be an arbitrarily chosen $sL_{\infty}$-quasi-isomorphism. We may regard $\varphi$ as a universal homotopical family of quantum observables — let $\Pi^\varphi$ be the associated family of quantum correlators. Then we also have a universal solution to the Maurer–Cartan equation of the unital $sL_{\infty}$-algebra $(\mathcal{C}[[\hbar]], 1_\mathcal{C}, \mathcal{L})$ given by $\Theta^\varphi := \sum_{n=1}^{\infty} \frac{1}{n!} t^{\rho_n} \cdots t^{\rho_1} \Pi^\varphi_n (\partial_{\rho_1}, \ldots, \partial_{\rho_n})$, and $e^{-\frac{1}{\hbar} \Theta^\varphi}$ is a generating function of $\Pi^\varphi$:
\[
e^{-\frac{1}{\hbar} \Theta^\varphi} = 1_\varphi + \sum_{n=1}^{\infty} \frac{1}{(-\hbar)^n} \frac{1}{n!} t^{\rho_n} \cdots t^{\rho_1} \Pi^\varphi_n (\partial_{\rho_1}, \ldots, \partial_{\rho_n}).
\]

From $K \circ \Pi^\varphi = 0$ for all $n \geq 1$, we have $\Pi^\varphi_n (\partial_{\rho_1}, \ldots, \partial_{\rho_n}) = \pi^\varphi_{\rho_1 \cdots \rho_n} \varphi_1 (\partial_{\gamma}) + K \eta^\varphi_{\rho_1 \cdots \rho_n}$ for some $\pi^\varphi_{\rho_1 \cdots \rho_n} \in \mathcal{C}[[\hbar]]$ and $\eta^\varphi_{\rho_1 \cdots \rho_n} \in \mathcal{C}[[\hbar]]$. Let
\[
T^\varphi := t^\gamma + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{1}{(-\hbar)^{n-1}} t^{\rho_n} \cdots t^{\rho_1} \pi^\varphi_{\rho_1 \cdots \rho_n},
\]
\[
\Sigma^\varphi := \sum_{n=2}^{\infty} \frac{1}{n!} \frac{1}{(-\hbar)^{n-1}} t^{\rho_n} \cdots t^{\rho_1} \eta^\varphi_{\rho_1 \cdots \rho_n} (\partial_{\rho_1}, \ldots, \partial_{\rho_n}).
\]

Then we have the identity $e^{-\frac{1}{\hbar} \Theta^\varphi} = 1_\varphi + \frac{1}{(-\hbar)} T^\varphi \varphi_1 (c_\gamma) + \frac{1}{(-\hbar)} K \Sigma^\varphi$. On the other hand, $T^\varphi_\varphi$ is a formal Laurent series in $\hbar$ in general.

Let $\check{\varphi} \sim \varphi$ be another unital $sL_{\infty}$-quasi-morphism homotopic to $\varphi$. Then $\Theta^\check{\varphi}$ is another universal solution to the Maurer–Cartan equation but is gauge equivalent to $\Theta^\varphi$ and we have $T^\varphi = T^\check{\varphi}$. A gauge equivalence class of such universal solutions can be viewed as a choice of affine coordinates on the based formal moduli space $\mathcal{M}_o$ defined by the Maurer–Cartan functor of the unital $sL_{\infty}$-algebra. Therefore, our distinguished unital $sL_{\infty}$-quasi-morphism $\Phi^0$ defines a distinguished choice of affine coordinates on $\mathcal{M}_o$, which we call the quantum coordinates.
From the family $\hat{m}^0$ in Theorem 6.2 we define
\[
\hat{A}_{\alpha\beta}^\gamma = \hat{m}^0_{\alpha\beta} + \sum_{n=1}^{\infty} \frac{1}{n!} t^{\rho_n} \cdots t^{\rho_1} \hat{m}^0_{\alpha \rho_1 \cdots \rho_n} \in k[[t_H]],
\]
where $\{\hat{m}^0_{\alpha_1 \cdots \alpha_n}\} \in k$ is defined by $\hat{m}^0_n(\partial_{\alpha_1}, \ldots, \partial_{\alpha_n}) = \hat{m}^0_{\alpha_1 \cdots \alpha_n} \partial_{\gamma}$. Then, we can check that the formula of eq. (6.11), relating $\hat{m}^0$ and $\hat{m}^0$, implies the following identity:
\[
\hbar \partial_\alpha \partial_\beta \hat{T}^\gamma - \hat{A}_{\alpha\beta}^\gamma \partial_\gamma \hat{T}^\gamma = 0. \tag{6.34}
\]
From the family $\hat{\Phi}^{-1}$ in Theorem 6.2 we define
\[
\Lambda_{\rho\gamma} := \sum_{n=0}^{\infty} \frac{1}{n!} t^{\sigma_n} \cdots t^{\sigma_1} \hat{\Phi}_{n+2}^{-1}(\partial_\sigma, \partial_\beta, \partial_{\rho_1}, \ldots, \partial_{\rho_n}) \in (k[[t_H]] \otimes \mathcal{O})^{\hbar(\partial_\sigma + \gamma(\partial_\rho))^{-1}}[[t_H]].
\]
Then, it can be checked that the relation of eq. (6.12) is equivalent to the following identity:
\[
(-\hbar)^2 \partial_\alpha \partial_\beta e^{\frac{1}{\hbar} \Phi} = (-\hbar)\hat{A}_{\alpha\beta}^\gamma \partial_\gamma e^{\frac{1}{\hbar} \Phi} + K \left( e^{\frac{1}{\hbar} \Phi} \cdot \Lambda_{\alpha\beta} \right). \tag{6.35}
\]
From the property of the family $\hat{m}^0$ in Theorem 6.3 we obtain that $\{\hat{A}_{\alpha\beta}^\gamma\}$ has the following properties:

- unity: $\hat{A}_{0\beta}^\gamma = \delta_{0\beta}^\gamma$,
- symmetry: $\hat{A}_{\alpha\beta}^\gamma = (-1)^{[\alpha][\beta]} \hat{A}_{\beta\alpha}^\gamma$ and $\partial_\sigma \hat{A}_{\beta\gamma}^\sigma = (-1)^{[\sigma][\beta]} \partial_\gamma \hat{A}_{\alpha\gamma}^\sigma$,
- generalized associativity: $\hat{A}_{\alpha\beta}^\gamma \hat{A}_{\gamma\rho}^\sigma = \hat{A}_{\beta\rho}^\sigma \hat{A}_{\alpha\gamma}^\sigma$,

so that $(k[[t_H]], \partial_\alpha, \cdot)$, where $\partial_\alpha = \partial_\beta$, is a unital super-commutative associative algebra over $k[[t_H]]$. This kind of structure is related to that of a Frobenius manifold. The notion of a formal Frobenius super-manifold [9,14] originated in Saito’s flat structure [18] in the context of singularity theory and the WDVV equation [8,24] associated with topological string theories. The homology $H$ doesn’t quite have the structure of a formal Frobenius super-manifold, but rather a formal $F$-manifold, which is the same thing as a formal Frobenius super-manifold without the invariant metric or inner product [13].

Example 6.1. Let $\mathcal{L} = \mathbb{C}[x_0, \ldots, x_n]$ and $S_{cl} \in \mathcal{L}$. Introduce $\eta_0, \ldots, \eta_n$ with $gh = -1$ such that $\eta_j \cdot x_j = x_j \cdot \eta_i$ and $\eta_1 \cdot \eta_i = -\eta_i \cdot \eta_i$. Let $\mathcal{O} = k[x_0, \ldots, x_n, \eta_1, \ldots, \eta_n]$ which is a unital $\mathbb{Z}$-graded commutative associative algebra $(\mathcal{O}, 1_{\mathcal{O}}, \cdot)$ with $1_{\mathcal{O}} = 1$. Note
that $\mathcal{E}^0 = \mathcal{L}$ and the ghost numbers of $\mathcal{E}$ are concentrated in non-positive integers. Define the following $k$-linear operators of ghost number 1:

$$\Delta := \sum_{i=0}^{N} \frac{\partial^2}{\partial \eta_i \partial x_i}, \quad K := \sum_{i=0}^{N} \frac{\partial S_d}{\partial x_i} \frac{\partial}{\partial \eta_i}.$$ 

Then it is trivial that $\Delta S_d = 0$ and $K_d \circ K_d = K_d \circ \Delta + \Delta \circ K_d = \Delta \circ \Delta = 0$. Let $K = -\hbar \Delta + K_d$. Then, $\mathcal{E}_{BQFTA} = \left(\mathcal{E}[\hbar], 1_{\mathcal{E}}, \cdot, K\right)$ is a BV-QFT algebra over $\mathbb{C}$ with quantum descendant sDGLA $((\mathcal{E}[\hbar], 1_{\mathcal{E}}, K, (-,-)_{BV}),$ where $(\alpha_1, \alpha_2)_{BV} = \Delta(\alpha_1 \cdot \alpha_2) - \Delta \alpha \cdot \alpha_2 - (-1)^{|\alpha_1|} \alpha_1 \cdot \Delta \alpha_2, \alpha_1, \alpha_2 \in \mathcal{E}$. Assume that $S_d \in \mathcal{L}$ has isolated singularities, which implies that

$$H = H^0 \cong k[x_0, \ldots, x_n] \left( \frac{\partial S_d}{\partial x_i} \right)$$

and $H^0$ is finite dimensional. Note that $1_{\mathcal{E}} = 1$ is non-trivial in cohomology. We denote its cohomology class by $1_H$. Let $\{e_\alpha\}$ be a basis of $H$ such that $e_0 = 1_H$. Choose a representative $f(e_\alpha) \in \mathcal{E}^0 \subset \mathcal{E}$ such that $f(e_0) = 1$ and extend linearly over $H = H^0$. Then $f : (H, 1_H, 0) \to (\mathcal{E}, 1_{\mathcal{E}}, K)$ is a quasi-isomorphism. Note that $\text{Im} \ f \subset \mathcal{E}^0$, where $\Delta$ vanishes. Therefore we have $K \circ f = \Delta \circ f = 0$. Hence $K \circ f = 0$ and $f = f : (H[[\hbar]], 1_H, 0) \to (\mathcal{E}[\hbar], 1_{\mathcal{E}}, K)$ is a quasi-isomorphism. Therefore the BV-QFT algebra $\mathcal{E}_{BQFTA}$ is anomaly-free with finite dimensional classical cohomology. Then our construction reproduces the $F$-manifold structure on $H = H^0$, the space of the universal unfolding of the isolated singularities, equivalent to that of $K$. Saito after forgetting the flat metric. His theory also contains integrals over the vanishing cycles, which correspond to quantum expectations.

**Example 6.2.** Now we consider the construction in [11]. Let $(X, \omega^{n,0})$ be a complex $n$-dimensional Calabi–Yau manifold. Let $T_X$ be the holomorphic tangent bundle to $X$, $\overline{T_X}$ be the anti-holomorphic cotangent bundle to $X$, and

$$\mathcal{E} = \bigoplus_{k=-n}^{n} \mathcal{E}^k, \quad \mathcal{E}^k = \bigoplus_{q-p=k, q,p=0, \ldots, n} \Gamma\left( \wedge^p T_X \otimes \wedge^q \overline{T_X} \right)$$

Note that $(\mathcal{E}, 1, \wedge, \overline{\partial})$ is a unital CDGA over $\mathbb{C}$. From the differential $\partial$ and $\omega^{n,0}$ define $\Delta : \mathcal{E}^* \to \mathcal{E}^{*+1}$ by the formula $(\Delta \gamma) \mapsto \omega^{n,0} = \partial(\partial \gamma \mapsto \omega^{n,0})$. Then, we have $\Delta \circ \Delta = \Delta \circ \overline{\partial} + \overline{\partial} \circ \Delta = 0$ so that $\mathcal{E}[[\hbar]]_{BQFTA} = (\mathcal{E}[[\hbar]], 1, \wedge, K = -\hbar \Delta + \overline{\partial})$ is a BV QFT algebra with quantum descendant unital sDGLA $((\mathcal{E}[[\hbar]], 1_{\mathcal{E}}, K, (-,-)_{SN}),$
Here \((\gamma_1, \gamma_2)_{SN} = \Delta(\gamma_1 \wedge \gamma_2) - \Delta \gamma \wedge \gamma_2 - (-1)^{|\gamma_1|} \gamma_1 \wedge \Delta \gamma_2\) and is equivalent to the holomorphic Schoutens–Nijenhuis bracket.\[\footnote{We remark that our grading conventions differ from those in \cite{[11]; the differences are not important.}

Note that the classical cohomology \(H\) is the \(\overline{\partial}\)-cohomology. Then the \(\overline{\partial}\overline{\partial}\)-lemma for Kähler manifolds \cite{[17]} implies that every \(\overline{\partial}\)-cohomology class has a unique representative belonging to the kernel of \(\Delta\). Choose a basis \(\{e_a\}\) of \(H\) such that \(e_0 = 1_H\), and choose representatives \(f(e_a)\) satisfying \(\Delta f(e_a) = 0\). Then we have \(K \circ f = \Delta \circ f = 0\). Thus \(K \circ f = 0\) and \(f = f : (H[[h]], 1_H, 0) \to (\mathcal{C}[[h]], 1_{\mathcal{C}}, K)\) is a quasi-isomorphism. Therefore the BV-QFT algebra \(\mathcal{C}[[h]]_{BQFTA}\) is anomaly-free with finite dimensional classical cohomology \(H\). Then our construction reproduces the super \(F\)-manifold structure on \(H\), isomorphic to the Dolbeault cohomology of \(X\), equivalent to the formal Frobenius supermanifold of Barannikov–Kontsevich after forgetting the invariant flat metric. Their theory also contains period integrals over the middle dimensional cycles on \(X\), which correspond to quantum expectations. \[\footnote{We remark that our grading conventions differ from those in \cite{[11]; the differences are not important.}

Remark 6.7. One can check that eq. (6.34), viewed as a formal differential equation for \(\{T_\gamma\} \in \mathbb{k}\langle t_H \rangle[[h^{-1}]]\), has a unique solution with the initial conditions \(T_\gamma|_{t_H=0} = 0\) and \(\partial_\beta T_\gamma|_{t_H=0} = \delta_\beta^\gamma\). The equation eq. (6.33) has the following integrability condition:

\[
\left( h \left( \partial_\sigma \dot{A}_\beta^\gamma - (-1)^{|\epsilon|}|t_\beta| \partial_\sigma \dot{A}_\alpha^\gamma \right) + \dot{A}_\beta^\rho \dot{A}_\alpha^\sigma - \dot{A}_\alpha^\rho \dot{A}_\beta^\sigma \right) \partial_\sigma T_\gamma = 0.
\]

Note that the condition \(\partial_\beta T_\gamma|_{t_H=0} = \delta_\beta^\gamma\) implies that the matrix \(\mathcal{G}\) with entries \(\dot{G}_\beta^\gamma = \partial_\beta T_\gamma\) is invertible, so that we have

\[
h \left( \partial_\sigma \dot{A}_\beta^\gamma - (-1)^{|\epsilon|}|t_\beta| \partial_\sigma \dot{A}_\alpha^\gamma \right) + \dot{A}_\beta^\rho \dot{A}_\alpha^\sigma - (-1)^{|\epsilon|}|t_\beta| \dot{A}_\alpha^\rho \dot{A}_\beta^\sigma = 0.
\]

It follows that \(\partial_\sigma \dot{A}_\beta^\gamma = (-1)^{|\epsilon|}|t_\beta| \partial_\sigma \dot{A}_\alpha^\gamma\) and

\[
\dot{A}_\beta^\rho \dot{A}_\alpha^\sigma - (-1)^{|\epsilon|}|t_\beta| \dot{A}_\alpha^\rho \dot{A}_\beta^\sigma = 0 \iff \dot{A}_\alpha^\rho \dot{A}_\beta^\sigma - \dot{A}_\beta^\rho \dot{A}_\alpha^\sigma = 0,
\]

since \(\dot{A}_\alpha^\gamma\) does not depend on \(h\). \[\footnote{We remark that our grading conventions differ from those in \cite{[11]; the differences are not important.}

From the families \(\hat{n}^{-1}\) and \(\eta^{-2}\) in Theorem [6.4], we define

\[
\hat{U}_{\alpha \beta}^\gamma := \sum_{n=1}^{\infty} \frac{1}{n!(-h)^n} t_1^{\hat{n}^{-1}} \cdots t_n^{\hat{n}^{-1}} \hat{n}^{-1} \cdots \hat{n}^{-1} \dot{A}_\alpha^\rho \dot{A}_\beta^\sigma \dot{A}_\beta^\rho \dot{A}_\alpha^\sigma,
\]
in \(k[[t_H]][[h^{-1}]])[−1] and
\[
Ξ_{\alpha\beta} := \sum_{n=1}^{\infty} \frac{1}{(-h)^{n+1}} t^\rho_n \cdots t^\rho_1 \eta_{n+2}(\partial_\alpha, \partial_\beta, \partial_\rho_1, \ldots, \partial_\rho_n)
\]
in \((k[[t_H]][[h^{-1}]])[\hat{\phi}(\hat{\phi})][h^{-1}]]\), where \(\{\hat{A}_{a_1,\ldots,a_n}^{-1}\} \in k[[h]]\) is defined for \(n \geq 3\) by \(\hat{A}_{a_1,\ldots,a_n}^{-1}(\partial_{a_1}, \ldots, \partial_{a_n}) = \hat{A}_{a_1,\ldots,a_n}^{-1} \partial_{a_1} \cdots \partial_{a_n}\). We emphasize that both \(\hat{U}_{\alpha\beta} \gamma\) and \(Ξ_{\alpha\beta}\) are formal power series in \(h^{-1} = 1/h\). Then, the relation of eq. (6.17) is equivalent to the following identity:
\[
(-h)\partial_\alpha \partial_\beta \Sigma - e^{-\hat{\Phi}_0} \cdot A_{\alpha\beta} - \hat{A}_{\alpha\beta} \rho \partial_\rho \Sigma = \hat{U}_{\alpha\beta} \gamma f(e_\gamma) + K Ξ_{\alpha\beta}.
\] (6.36)

Remark 6.8. Apply \(K\) to eq. (6.36) to obtain that
\[
((-h)\partial_\alpha \partial_\beta - \hat{A}_{\alpha\beta} \rho \partial_\rho) K \Sigma = K \left( e^{-\hat{\Phi}_0} \cdot A_{\alpha\beta} \right).
\]
From eq. (6.32) we know that \(K \Sigma = (-h)e^{-\hat{\Phi}_0} - (-h)1_\phi - \hat{T}\gamma f(e_\gamma)\), so we have
\[
(-h)^2 \partial_\alpha \partial_\beta e^{-\hat{\Phi}_0} - (-h)\hat{A}_{\alpha\beta} \rho \partial_\rho e^{-\hat{\Phi}_0} - K \left( e^{-\hat{\Phi}_0} \cdot A_{\alpha\beta} \right)
= ((-h)\partial_\alpha \partial_\beta \hat{T}\gamma - \hat{A}_{\alpha\beta} \rho \partial_\rho \hat{T}\gamma)f(e_\gamma),
\]
which implies that the relations eq. (6.34) and eq. (6.35) arise as integrability conditions of the relation eq. (6.36).

Remark 6.9. For any quantum expectation \(\phi\), define the following generating series of the level zero quantum correlation functions:
\[
Z_\phi := \phi \left( e^{-\hat{\Phi}_0} \right) = \phi \left( e^{-\hat{\Phi}_0} \right) = 1 + \sum_{n=1}^{\infty} \frac{1}{(-h)^n n!} t^a_n \cdots t^a_1 \langle \eta^a_n(\partial_{a_n}, \ldots, \partial_{a_n}) \rangle_\phi.
\]
From eq. (6.32), eq. (6.33) and eq. (6.34), we obtain that
\[
Z_\phi = 1 + \frac{1}{(-h)} T^\gamma f(e_\gamma)_\phi, \quad -h \partial_\alpha Z_\phi = Z_\phi, \quad -h \partial_\alpha Z_\phi = -h \partial_\alpha Z_\phi = \hat{A}_{\alpha\beta} \rho \partial_\rho Z_\phi.
\]
We remind the reader that \(Z_\phi \in k[[t_H]][[h]]^0\). Assume that the above quantum expectation \(\phi\) is not just a pointed cochain map from \((\hat{\phi}[[h]], 1_\phi, K)\) to \(k[[h]], 1, 0\) but also a morphism of binary QFT algebras. Let \(\bar{Z} = \hat{R}(\phi)\) be the quantum descendant of \(\phi\). Define \(\mathcal{F}_\phi \in k[[t_H]][[h]]^0\) to be defined by
\[
\mathcal{F}_\phi := \sum_{n=1}^{\infty} \frac{1}{n!} t^a_n \cdots t^a_1 (\bar{Z} \cdot \Phi^0)_n(\partial_{a_n}, \ldots, \partial_{a_n}).
\]
Then, we have the identity \( e^\frac{1}{\hbar} \mathcal{F} = \mathcal{Z} \).

Finally, we briefly return to a general binary QFT algebra without assuming either the anomaly-free condition or the finite dimensionality of \( H \).

Consider the binary QFT algebra \((\mathcal{G}[\hbar], 1, \cdot, \cdot, \mathbf{K})\) along with its quantum descendant \((\mathcal{G}[\hbar], 1, \cdot, \cdot, \mathbf{L})\) as in Sect. 5. Let \( \{ \hat{\pi}_0^0, \eta_0^{-1}, \hat{t}_0, \phi_0^0 | \hat{\pi}_0^{-1}, \eta_0^{-1}, \hat{m}_0, \phi_0^{-1} \} \) be the canonical solutions of the levels zero and one quantum master equations given in Theorems 5.1 and 5.2.

**Definition 6.2.** A superselection sector of the binary QFT algebra is a finite-dimensional pointed subspace \( W \) of \( \mathcal{H} \) such that

1. (unitality) \( 1_W := 1_H \in W \),
2. (\( \kappa \)-triviality) \( \kappa_W = 0 \) for all \( w \in W \),
3. (level zero higher triviality) \( \hat{\pi}_0^0(w_1, \ldots, w_n) \in W \) for all \( n \geq 2 \) and \( w_1, \ldots, w_n \in W \),
4. (level one higher triviality) \( \hat{\pi}_0^{-1}(w_1, \ldots, w_n) \in W \) for all \( n \geq 3 \) and \( w_1, \ldots, w_n \in W \).

From properties (iii) and (iv) of \( W \), we have \( \hat{t}_n(w_1, \ldots, w_n) = 0 \) for all \( n \geq 1 \) and \( w_1, \ldots, w_n \in W \) so that \( \hat{\phi}_0^0 : (W[\hbar], 1_W, 0) \longrightarrow (\mathcal{G}[\hbar], 1, \cdot, \cdot, \mathbf{L}) \) is a unital \( sL_{\infty} \)-morphism. From properties (ii) and (iv) of \( W \), we obtain that, for all \( n \geq 2 \) and homogeneous \( w_1, \ldots, w_n \in W \),

\[
\hat{\pi}_0^0(w_1, \ldots, w_n) = \sum_{p \in \mathbb{P}(n)} (-\hbar)^{n-|p|-1} \epsilon(p) \hat{\pi}_0^0\left( w_{B_1}, \ldots, w_{B_{|p|}-1}, \hat{m}_0(\nu_{|p|}) \right).
\]

It follows that \( \hat{m}_0^0(w_1, \ldots, w_n) = \hat{\pi}_0^{(n-2)}(w_1, \ldots, w_n) \) for all \( n \geq 2 \) and homogeneous \( w_1, \ldots, w_n \in W \). Define the family \( \hat{m}_0^0 = \hat{m}_0^0, \hat{m}_0^2, \hat{m}_0^3, \ldots \) of \( \hat{m}_0^0 \in \text{Hom}(S^0W, W)^0 \) for all \( n \geq 2 \) to be \( \hat{m}_0^0(w_1, \ldots, w_n) := \hat{\pi}_0^{(n-2)}(w_1, \ldots, w_n) \). It is clear that \( (W, 1_W, \hat{m}_0^0) \) is unital, symmetric, and satisfies generalized associativity.

Introduce homogeneous coordinates \( t_W = \{ t^a \} \) on \( W \) so that \( \{ \partial_a = \partial / \partial t^a \} \) form a homogeneous basis of \( W \) with distinguished element \( \partial_0 = 1_W \). Then extend \( \partial_a \) as a derivation on \( k[[t_W]] \). Consider the structure constants \( \{ \hat{m}_{a_1 \cdots a_n}^0 \} \) and \( \{ \hat{\pi}_{a_1 \cdots a_n}^0 \} \),
where \( \hat{m}_a^0(\partial_1, \ldots, \partial_n) = \hat{m}_{a_1 \cdots a_n}^0 \partial_c \) and \( \hat{\pi}_a^0(\partial_1, \ldots, \partial_n) = \hat{\pi}_{a_1 \cdots a_n}^0 \partial_c \), and define

\[
\Theta_W = \sum_{n=1}^{\infty} \frac{1}{n!} \hat{m}_a^0 \cdots \hat{m}_a^0 \partial_1^c \cdots \partial_n^c \in \mathcal{E}[\mathcal{T}_W]^{0}\mathcal{H}[h],
\]

\[
\hat{T}^c = c^c + \sum_{n=2}^{\infty} \frac{1}{n!} \hat{m}_a^0 \cdots \hat{m}_a^0 \partial_1^c \cdots \partial_n^c \in \mathcal{K}[\mathcal{T}_W][h^{-1}],
\]

\[
\hat{A}_{ab}^c = \hat{m}_{ab}^0 c + \sum_{n=1}^{\infty} \frac{1}{n!} \hat{m}_{ab}^0 \cdots \hat{m}_{ab}^0 \partial_1^c \cdots \partial_n^c \in \mathcal{K}[\mathcal{T}_W].
\]

Then, we have \( \textbf{K} \frac{1}{h} \Theta_W = 0 \) and the generating function \( Z_W^c \) of quantum correlation functions on the superselection sector \( W \) with respect to a quantum expectation \( c \) can be defined as follows:

\[
Z_W^c := \left< \frac{1}{h} \Theta_W \right>_c = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-h)^n \hat{m}_a^0 \cdots \hat{m}_a^0 \hat{T}_W^c (\partial_1^c, \ldots, \partial_n^c). \]

It follows that \( Z_W^c = 1 - \frac{1}{h} \hat{T}^a (\mathcal{F}(\partial_a))_c \) so that the quantum expectation values (namely \( \left< \mathcal{F}(\partial_a) \right>_c \) and \( \left< \hat{T}^a \right>_c \)) determine every quantum correlation function in the superselection sector \( W \). Moreover, \( \left< \hat{T}^a \right>_c \) is the unique solution in formal power series in \( h^{-1} \) to the following system of formal differential equations:

\[
\frac{\hbar}{h} \partial_b^c \partial_c \hat{T}^a + \hat{A}_{bc}^e \partial_e \hat{T}^a = 0, \quad \partial_0 \hat{T}^a = \delta_0^a - \frac{1}{h} \hat{T}^a,
\]

with the boundary condition \( \partial_b^c \hat{T}^a \big|_{t_W=0} = \delta_b^a \). Finally, \( \{\hat{A}_{ab}^c\} \) has the requisite compatibilities to make \( \hat{W} \) the super-commutative associative algebra over \( \mathcal{K}[\mathcal{T}_W] \) — hence \( W \) is a formal super F-manifold. Therefore, we conclude the following.

**Lemma 6.1.** Every finite dimensional super-selection sector of a binary QFT algebra is a formal super F-manifold and comes with a well-defined quantum distribution.

**A. On homotopy Lie algebras**

This appendix is intended as a self-contained introduction to the homotopy category of \( sL_{\infty} \)-algebras and a homotopy functorial description of the bar construction of \( sL_{\infty} \)-algebras.
An $sL_\infty$-algebra is a version of a homotopy Lie algebra, also known as an $L_\infty$-algebra, with its degree shifted by one. The notion of an $L_\infty$-algebra first arose in the rational homotopy theory of Sullivan [23] in disguise before its explicit form was used in the deformation theoretic approach [22][19] to rational homotopy theory. For our purposes the shifted version, $sL_\infty$-algebras, are more natural. There is no conceptual originality to the contents of this appendix, and we refer to [15] for the formal supergeometric aspects of $L_\infty$-algebras.

Throughout this appendix $R$ is a fixed commutative ground ring of characteristic zero with unit $1_R$. (In the body of the paper $R$ is either a field $k$ of characteristic zero or $k[\hbar]$.)

Let $V = \bigoplus_{k \in \mathbb{Z}} V^k$ be a $\mathbb{Z}$-graded $R$-module. An element of $V$ is homogeneous if it lies in fixed degree. For a homogeneous element $v \in V$, we use the notation $|v|$ for its degree and write $Jv$ for $(-1)^{|v|}v$. We denote by $\text{Hom}_R(V, V')^j$ the space of $R$-module homomorphisms from $V$ to $V'$ of degree $j$, and by $\text{Hom}_R(V, V')$ the sum $\bigoplus_{j \in \mathbb{Z}} \text{Hom}_R(V, V')^j$. The tensor product over $R$ is denoted by $\otimes$.

The reduced free tensor module generated by a $\mathbb{Z}$-graded $R$-module $V$ is $\overline{T}(V) = \bigoplus_{n=1}^{\infty} T^nV$, where $T^nV = V^{\otimes n}$, which has the induced structure of a $\mathbb{Z}$-graded $R$-module. Let $\text{Perm}_n$ be the group of permutations of the set $[n] = \{1, \ldots, n\}$. For each $\sigma \in \text{Perm}_n$, we define the map $\tilde{\sigma} : T^nV \to T^nV$ by specifying, for homogeneous elements $v_1, \ldots, v_n \in V$,

$$\tilde{\sigma}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \epsilon(\sigma)v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

In this equation $\epsilon(\sigma) = \pm 1$ is the Koszul sign determined by decomposing $\sigma$ as composition of transpositions $\tilde{\tau} : v_1 \otimes v_2 \to (-1)^{|v_1||v_2|}v_2 \otimes v_1$. We denote by $S^nV$ is the submodule of $T^nV$ that is fixed by $\tilde{\sigma}$. Elements in $S^nV$ are generated by elements of the form $v_1 \otimes \cdots \otimes v_n$, which is a weighted sum over the orbit of $v_1 \otimes \cdots \otimes v_n$. The reduced free symmetric module generated by $V$ is $\overline{S}(V) = \bigoplus_{n=1}^{\infty} S^nV$. We denote by $e_{S^nV} : S^nV \to \overline{S}(V)$ and $\text{pr}_{S^nV} : \overline{S}(V) \to S^nV$ for $n \geq 1$ the canonical embedding and projection.

We say $L_n \in \text{Hom}_R(T^nV, V')^j$ descends to a $R$-linear map from $S^nV$ to $V'$, and denote this map by $L_n \in \text{Hom}_R(S^nV, V')^j$, if $L_n(v_1 \otimes \cdots \otimes v_n) = \epsilon(\sigma)L_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$. A family $L = L_1, L_2, \ldots$ of $L_n \in \text{Hom}(S^nV, V')^j$ for all $n \geq 1$ determines $L \in \text{Hom}(\overline{S}(V), V')^j$ by the convention that $L(v_1 \otimes \cdots \otimes v_n) = L_n(v_1 \otimes \cdots \otimes v_n)$, for all $n \geq 1$, and vice versa. That is, $L_n = L e_{S^nV}$. We use the notation $\overline{L}$ and $L$ interchangeably. We also use the notation $L(v_1 \otimes \cdots \otimes v_n) = L_n(v_1 \otimes \cdots \otimes v_n) = L_n(v_1, \ldots, v_n)$. 
A.1. The homotopy category of $sL_\infty$-algebras

We begin with generator-relation definitions of $sL_\infty$-algebras, morphisms and homotopy types of morphisms.

**Definition A.1.** An $sL_\infty$-algebra over $R$ is a tuple $(V, l)$, where $V$ is a $\mathbb{Z}$-graded $R$-module and $l = l_1, l_2, \ldots$ is a family of operations $l_k \in \text{Hom}_R(S^k V, V)$ for $k \geq 1$, such that, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,

$$\sum_{|p| \in P(n)} e(p) l_p \left( J v_{B_1}, \ldots, J v_{B_{n-1}}, l(v_{B_n}), v_{B_{n+1}}, \ldots, v_{B_{|p|}} \right) = 0.$$

The sum in the above formula is over all classical partitions of $[n]$ satisfying the condition that every block in the partition $p$ has a single element except possibly one block which has $n - |p| + 1$ elements. For example, we have

$$d^2(v_1) = 0,$$

$$dl_2(v_1, v_2) + l_2(dv_1, v_2) + l_2(Jv_1, dv_2) = 0,$$

$$dl_3(v_1, v_2, v_3) + l_2(dv_1, v_2, v_3) + l_3(Jv_1, dv_2, v_3) + l_2(Jv_1, Jv_2, dv_3) + l_3(Jv_1, Jv_2, Jv_3) + (-1)^{|v_1||v_2|} l_2(Jv_2, l_2(v_1, v_3)) = 0,$$

etc., where $d = l_1$. Note that $(V, d)$ is a cochain complex over $R$ whose cohomology $H$ is called the cohomology of the $sL_\infty$-algebra $(V, l)$. An $sL_\infty$-algebra $(V, l)$ is called minimal if $l_1 = 0$.

**Definition A.2.** A morphism of $sL_\infty$-algebras from $(V, l)$ to $(V', l')$ is a family $\varphi = \varphi_1, \varphi_2, \ldots$ of $\varphi_k \in \text{Hom}_R(S^k V, V')$ for $k \geq 1$, such that, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,

$$\sum_{|p| \in P(n)} e(p) l'_p \left( \varphi(v_{B_1}), \ldots, \varphi(v_{B_{|p|}}) \right) = \sum_{|p| \in P(n)} e(p) \varphi_{|p|} \left( J v_{B_1}, \ldots, J v_{B_{n-1}}, l(x_{B_n}), v_{B_{n+1}}, \ldots, v_{B_{|p|}} \right).$$

For example, we have

$$d' \varphi_1(v_1) = \varphi_1(dv_1),$$

$$\varphi_1(l_2(v_1, v_2)) - l'_2(\varphi_1(v_1), \varphi_1(v_2)) = d' \varphi_2(v_1, v_2) - \varphi_2(dv_1, v_2) - \varphi_2(Jv_1, dv_2).$$
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Note that $\phi_1$ is a cochain map from $(V, d)$ to $(V', d')$. Recall that a cochain map is a cochain quasi-isomorphism if it induces an isomorphism on cohomology. An $s L_\infty$ morphism $\phi$ a quasi-isomorphism if $\phi_1$ is a cochain quasi-isomorphism between the underlying cochain complexes.

**Definition A.3 (Lemma).** Let $(V, l) \xrightarrow{\phi} (V', l') \xrightarrow{\phi'} (V'', l'')$ be consecutive $s L_\infty$-morphisms. Then, the composition $\phi' \circ \phi$ defined by the following equation for all $n \geq 1$ and $v_1, \ldots, v_n \in V$:

$$
(\phi' \circ \phi)_n(v_1, \ldots, v_n) := \sum_{|p| \in P(n)} \epsilon(p) \phi'_{|p|}(\phi(v_{B_1}), \ldots, \phi(v_{B_{|p|}})),
$$

is an $s L_\infty$-morphism from $(V, l)$ to $(V'', l'')$. The operation $\circ$ is associative.

The category of $s L_\infty$-algebras over $R$ is the category $sL_\infty(R)$ whose objects are $s L_\infty$-algebras over $R$ and whose morphisms are $s L_\infty$-morphisms with composition operation $\circ$.

Now we turn to the homotopy category of $s L_\infty$-algebras, which will require a bit of preparation.

**Definition A.4.** A homotopy pair of $s L_\infty$-algebras from $(V, l)$ to $(V', l')$ is a pair

$$
(\varphi(\tau), \lambda(\tau)) \in \text{Hom}_R(S(V), V')^{[\tau]} \oplus \text{Hom}_R(S(V), V')^{-1}[\tau]
$$

satisfying the following system of equations: for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,

$$
\frac{d^1}{d\tau} \varphi(\tau)(v_1 \circ \ldots \circ v_n) = \sum_{p \in P(n)} \epsilon(p) \lambda(\tau) \left(Jv_{B_1} \circ \ldots \circ Jv_{B_{|p|}} \circ l(x_{B_i}) \circ v_{B_{i+1}} \circ \ldots \circ v_{B_{|p|}}\right)
$$

$$
+ \sum_{p \in P(n)} \sum_{i=1}^{|p|} \epsilon(p) l' \left(\varphi(\tau)(Jv_{B_i}) \circ \ldots \circ \varphi(\tau)(Jv_{B_{i-1}}) \circ \lambda(\tau)(v_{B_i}) \circ \varphi(\tau)(v_{B_{i+1}}) \circ \ldots \circ \varphi(\tau)(v_{B_{|p|}})\right).
$$
The first two equations of this system are

\[
\frac{d}{d\tau} \varphi_1(\tau)(v_1) = \lambda_1(\tau)(dv_1) + d^2\lambda_1(\tau)(v_1),
\]

\[
\frac{d}{d\tau} \varphi_2(\tau)(v_1, v_2) = \lambda_2(\tau)(dv_1, v_2) + \lambda_2(\tau)(Jv_1, dv_2) + \lambda_1(\tau)(l_2(v_1, v_2)) + d^2\lambda_2(\tau)(v_1, v_2) + l_2'(\lambda_1(\tau)(v_1), \varphi_1(v_1)) + l_2(\varphi_1(Jv_1), \lambda_1(\tau)(v_2)).
\]

Working recursively from \( n = 1 \), it is obvious that the system of equations in Definition A.4 has a unique solution \( \varphi(\tau) \), modulo an initial condition \( \varphi(0) \), with respect to \( \lambda(\tau) \).

**Lemma A.1.** Let \( (\varphi(\tau), \lambda(\tau)) \) be a homotopy pair of \( sL_\infty \)-algebras such that \( \varphi(0) \) is an \( sL_\infty \)-morphism. Then \( \varphi(\tau) \) is a (uniquely defined) family of \( sL_\infty \)-morphisms.

Now we are ready to define homotopy types of \( sL_\infty \)-morphisms.

**Definition A.5.** Two \( sL_\infty \)-morphisms \( \varphi \) and \( \tilde{\varphi} \) are homotopic, which we denote \( \varphi \sim \tilde{\varphi} \), or have the same homotopy type, denoted \( [\varphi] = [\tilde{\varphi}] \), if there is a \( sL_\infty \)-homotopy pair \((\varphi(\tau), \lambda(\tau))\) such that \( \varphi = \varphi(0) \) and \( \tilde{\varphi} = \varphi(1) \).

It is clear that \( \sim \) is an equivalence relation. The homotopy category of \( sL_\infty \)-algebras over \( R \) is the category \( \text{hos}L_\infty(R) \), whose objects are \( sL_\infty \)-algebras over \( R \) and morphisms are homotopy types of \( sL_\infty \)-morphisms. It remains is to check that \( \text{hos}L_\infty(R) \) is indeed a category.

**Lemma A.2.** Given “composable” \( sL_\infty \)-homotopy pairs

\[
(V, l) = (\varphi(\tau), \lambda(\tau)) \quad \text{and} \quad (V', l') = (\varphi'(\tau), \lambda'(\tau))
\]

the composition \( (\varphi'(\tau), \lambda'(\tau)) \circ (\varphi(\tau), \lambda(\tau)) = (\varphi''(\tau), \lambda''(\tau)) \) defined for all \( n \geq 1 \) and homogeneous \( v_1, \ldots, v_n \in V \) by the equations

\[
\varphi''(\tau)(v_1 \odot \ldots \odot v_n) := \sum_{[p] \in P(n)} e(p)\varphi'(\tau)(\varphi(\tau)(v_{B_1}) \odot \ldots \odot \varphi(\tau)(v_{B_i}))
\]

\[
\lambda''(\tau)(v_1 \odot \ldots \odot v_n) := \sum_{[p] \in P(n)} e(p)\lambda'(\tau)(\varphi(\tau)(v_{B_1}) \odot \ldots \odot \varphi(\tau)(v_{B_i}))
\]

\[
+ \sum_{[p] \in P(n)} e(p) \sum_{i=1}^{[p]} \varphi'(\tau)(\varphi(\tau)(J_{v_{B_{i}}} \odot \ldots \odot \lambda(\tau)(v_{B_{i}}) \odot \ldots \odot \varphi(\tau)(v_{B_{i}})))
\]

is an \( sL_\infty \)-homotopy pair from \( (V, l) \) to \( (V'', l'') \). The operation \( \circ \) is associative.
Consider $sL_\infty$-morphisms $(V,L) \Rightarrow (V',L') \Rightarrow (V'',L'')$ and assume that $\varphi \sim \tilde{\varphi}$ and $\varphi' \sim \tilde{\varphi}'$. Then, there are corresponding $sL_\infty$-homotopy pairs as follows:

- $(\varphi(\tau), \lambda(\tau))$ such that $\varphi(0) = \varphi$ and $\varphi(1) = \tilde{\varphi}$;
- $(\varphi'(\tau), \lambda'(\tau))$ such that $\varphi'(0) = \varphi'$ and $\varphi'(1) = \tilde{\varphi}'$.

By Lemma A.2, the composition $(\varphi''(\tau), \lambda''(\tau)) = (\varphi'(\tau), \lambda'(\tau)) \bullet (\varphi(\tau), \lambda(\tau))$ is an $sL_\infty$-homotopy pair from $(V,L)$ to $(V'',L'')$ such that

$$\varphi''(0) = \varphi' \bullet \varphi, \quad \varphi''(1) = \tilde{\varphi}' \bullet \tilde{\varphi}.$$ 

It follows that $\varphi' \bullet \varphi \sim \tilde{\varphi}' \bullet \tilde{\varphi}$ or, equivalently, $[\varphi' \bullet \varphi] = [\tilde{\varphi}' \bullet \tilde{\varphi}]$ whenever $\varphi \sim \tilde{\varphi}$ and $\varphi' \sim \tilde{\varphi}'$ so that the homotopy type $[\varphi' \bullet \varphi]$ of $\varphi' \bullet \varphi$ depends only on the homotopy types $[\varphi]$ and $[\varphi']$ of $\varphi$ and $\varphi'$, respectively. Therefore the composition which takes $[\varphi]$ and $[\varphi']$ to $[\varphi'] \cdot_h [\varphi] := [\varphi' \bullet \varphi]$ is well-defined. It is obvious that $\cdot_h$ is associative.

**Definition A.6.** The homotopy category of $sL_\infty$-algebras over $R$ is the category $\text{hos} sL_\infty(R)$, whose objects are $sL_\infty$-algebras over $R$ and whose morphisms are homotopy types of $sL_\infty$-morphisms with composition $\cdot_h$.

In this paper, we shall work primarily with the category and homotopy category of unital $sL_\infty$-algebras.

**Definition A.7.** - A unital $sL_\infty$-algebra over $R$ is a tuple $(V,1_V,L)$, where $(V,L)$ is an $sL_\infty$-algebra over $R$ and $1_V$ is an element of $V^0$ such that, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_{n-1} \in V$,

$$l_n(v_1, \ldots, v_{n-1}, 1_V) = 0.$$ 

- A unital $sL_\infty$-morphism from $(V,1_V,L)$ to $(V',1_{V'},L')$ is an $sL_\infty$-morphism $\varphi$ from $(V,L)$ to $(V',L')$ such that, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_{n-1} \in V$,

$$\varphi_n(v_1, \ldots, v_{n-1}, 1_V) = \delta_{n,1} \times 1_{V'},$$

where $\delta_{n,1}$ is the Kronecker delta: 1 for $n = 1$ and 0 otherwise.

The composition of two unital $sL_\infty$-morphisms, as $sL_\infty$-morphisms, is a unital $sL_\infty$-morphism.
Notation A.8. We denote by $\text{UsL}_\infty(R)$ the category of unital $sL_\infty$-algebras whose objects are unital $sL_\infty$-algebras over $R$ and whose morphisms are unital $sL_\infty$-morphisms.

Definition A.9. A unital $sL_\infty$-homotopy pair is $sL_\infty$-homotopy pair $(\varphi(\tau), \lambda(\tau))$ such that for all $n \geq 1$ and homogeneous $v_1, \ldots, v_{n-1} \in V$,

$$\lambda(\tau)(v_1 \otimes \cdots \otimes v_{n-1} \otimes 1_v) = 0.$$ 

Then, $\varphi(\tau)$ is a smooth 1-parameter family of unital $sL_\infty$-morphisms if $\varphi(0)$ is a unital $sL_\infty$-morphism, so that we can define homotopy of unital $sL_\infty$-morphisms accordingly.

Notation A.10. We denote by $\text{hoUsL}_\infty(R)$ the homotopy category of unital $sL_\infty$-algebras whose objects are unital $sL_\infty$-algebras over $R$ and whose morphisms are homotopy types of unital $sL_\infty$-morphisms.

The following important lemma is well-known:

Lemma A.3. On the cohomology $H$ of an $sL_\infty$-algebra $(W, \ell)$ over a field $k$ of characteristic zero, there is the structure of a minimal $sL_\infty$-algebra $H(\ell)$ and a quasi-isomorphism $\varphi : (H, \ell) \to (W, \ell)$.

Proof. Choose the data of a strong deformation retract $(f, h, s)$ between the cochain complex $(W, d := \ell_1)$ over $k$ and its homology $(H, 0)$. So $s \in \text{Hom}_k(W, W)^{-1}$ and $f \in \text{Hom}_k(H, W)^0$ and $h \in \text{Hom}_k(W, H)^0$ satisfy $f \circ h = \text{Id}_W - d \circ s - s \circ d$ and $h \circ f = \text{Id}_H$.

Define $\varphi := \varphi_1, \varphi_2, \ldots$ and $\ell := \ell_1, \ell_2, \ldots$ recursively as follows: $\varphi_1 = f$ and $\ell_1 = 0$, while $\ell_n := h \circ L_n$ and $\varphi_n := -s \circ L_n$ for all $n \geq 2$, where $L_n \in \text{Hom}(S^n H, W)$ is given for homogeneous $v_1, \ldots, v_n \in H$ by

$$L_n(v_1, \ldots, v_n) := \sum_{|p| \in P(n)} \epsilon(p) \ell_{|p|}(\varphi(v_{B_1}), \ldots, \varphi(v_{B_{|p|}}))$$

$$- \sum_{|p| \in P(n)} \epsilon(p) \varphi_{|p|}(J v_{B_1}, \ldots, J v_{B_{|p|-1}}, \ell(x_{B_1}), \ldots, v_{B_{|p|}}),$$

Note that $L_n$ depends only on $\varphi_1, \ldots, \varphi_{n-1}$ and $\ell_2, \ldots, \ell_{n-1}$. For $n \geq 2$, let

$$F_n(v_1, \ldots, v_n) := \sum_{|p| \in P(n)} \epsilon(p) \ell_{|p|}(J v_{B_1}, \ldots, J v_{B_{n-1}}, l(v_{B_n}), \ldots, v_{B_{|p|}}),$$

$$|B_i| = -|p| + 1$$

$$|p| \neq n, 1$$
which depends only on $\ell_2, \ldots, \ell_{n-1}$. From $L_2(v_1, v_2) = (\varphi_1(v_1), \varphi_1(v_2))$, we have $d \circ L_2 = 0$ since $d \circ \varphi_1 = 0$ and $d$ is a derivation of the bracket $(, )$. From $\ell_2 = h \circ L_2$ and $\varphi_2 = -s \circ L_2$, we obtain that $L_2 = \varphi_1 \circ \ell_2 - d \circ \varphi_2$. Note that $F_2 = 0$. Fix $n \geq 3$ and assume that $L_k = \varphi_1 \circ \ell_k - d \circ \varphi_k$ and $F_k = 0$ for all $k = 2, \ldots, n-1$. Then it is straightforward to check that $d \circ L_n = f \circ F_n$, which implies that $h \circ d \circ L_n = h \circ f \circ F_n = F_n = 0$ and $d \circ L_n = 0$. From $\ell_n := h \circ L_n$ and $\varphi_n := -s \circ L_n$, we obtain that $L_n = \varphi_1 \circ \ell_n - d \circ \varphi_n$. Therefore we have proved that $d \circ \varphi_1 = \ell_1 = 0$ and $L_n + d \circ \varphi_n - \varphi_1 \circ \ell_n = F_n = 0$ for all $n \geq 2$, which are exactly the conditions for $(H, \hat{\ell})$ to be a minimal $sL_{\infty}$-algebra over $k$ and for $\varphi : (H, \hat{\ell}) \to (W, \hat{\ell})$ to be an $sL_{\infty}$-morphism, which is a quasi-isomorphism since $\varphi_1 \circ f : (H, 0) \to (W, d)$ is a cochain quasi-isomorphism.

A.2. The homotopy category of dg coalgebras

A dg coalgebra over $R$ is a tuple $(C, \Delta_C, d_C)$, where
- $(C, d_C)$ is a cochain complex over $R$, i.e., $d_C \in \text{Hom}(C, C)^1$ and $d_C \circ d_C = 0$, and
- $(C, \Delta_C)$ is a $\mathbb{Z}$-graded coassociative coalgebra over $R$, i.e.,

$$\Delta_C \in \text{Hom}(C, C \otimes C)^0, \quad (\Delta_C \otimes \mathbb{I}_C) \circ \Delta_C = (\mathbb{I}_C \otimes \Delta_C) \circ \Delta_C,$$

such that $d_C$ is a coderivation of $\Delta_C$, i.e., $\Delta_C \circ d_C = (d_C \otimes \mathbb{I}_C + \mathbb{I}_C \otimes d_C) \circ \Delta_C$.

A morphism of dg coalgebras from $(C, \Delta_C, d_C)$ to $(C', \Delta_{C'}, d_{C'})$ is both a cochain map and coalgebra map, i.e.,

$$F \in \text{Hom}(C, C')^0, \quad d_{C'} \circ F = F \circ d_C, \quad \Delta_{C'} \circ F = (F \otimes F) \circ \Delta_C.$$

It is straightforward to check that the composition $F' \circ F$ of dg coalgebra morphisms as $R$-linear maps is a dg coalgebra morphism. Therefore, we have the category $\text{dgC}(R)$ of dg coalgebras over $R$.

**Definition A.11.** A homotopy pair $(\tilde{\mathcal{H}}(\tau), \Lambda(\tau)) : (C, \Delta_C, d_C) \rightleftharpoons (C', \Delta_{C'}, d_{C'})$ of dg coalgebras is a pair $\tilde{\mathcal{H}}(\tau) \oplus \Lambda(\tau) : [0, 1] \to \text{Hom}(C, C')^0[\tau] \oplus \text{Hom}(C, C')^{-1}[\tau]$ such
that the following relations are satisfied:
\[
\begin{align*}
\frac{d}{d\tau} \tilde{\mathcal{H}}(\tau) &= d_{\mathcal{C}'} \circ \Lambda(\tau) + \Lambda(\tau) \circ d_{\mathcal{C}}, \\
\Delta_{\mathcal{C}'} \circ \tilde{\mathcal{H}}(\tau) &= (\tilde{\mathcal{H}}(\tau) \otimes \tilde{\mathcal{H}}(\tau)) \circ \Delta_{\mathcal{C}}, \\
\Delta_{\mathcal{C}'} \circ \Lambda(\tau) &= (\tilde{\mathcal{H}}(\tau) \otimes \Lambda(\tau) + \Lambda(\tau) \otimes \tilde{\mathcal{H}}(\tau)) \circ \Delta_{\mathcal{C}}.
\end{align*}
\]

Then, it is straightforward to show that \( \tilde{\mathcal{H}}(\tau) \) is determined uniquely with respect to \( \Lambda(\tau) \) for a given initial condition \( \tilde{\mathcal{H}}(0) \) and is a smooth family of dg coalgebra morphisms if \( \tilde{\mathcal{H}}(0) \) is a dg coalgebra morphism.

**Definition A.12.** Two dg coalgebra morphisms \( F \) and \( \tilde{F} \) are homotopic, denoted \( F \sim \tilde{F} \), or have the same homotopy type, denoted by \( [F] = [\tilde{F}] \), if there is a dg coalgebra homotopy pair \((\tilde{\mathcal{H}}(\tau), \Lambda(\tau))\) such that \( \tilde{\mathcal{H}}(0) = F \) and \( \tilde{\mathcal{H}}(1) = \tilde{F} \).

It is clear that \( \sim \) is an equivalence relation. The homotopy category of dg coalgebras over \( R \) shall be a category \( h\text{odgC}(R) \), whose objects are dg coalgebras over \( R \) and whose morphisms are homotopy types of dg coalgebra morphisms. We check that \( h\text{odgC}(R) \) is indeed a category in the following Lemma:

**Lemma A.4.** Given “composable” homotopy pairs of dg coalgebras
\[
\begin{align*}
(C, \Delta_{\mathcal{C}}, d_{\mathcal{C}}) \xrightarrow{\tilde{\mathcal{H}}(\tau), \Lambda(\tau)} (C', \Delta_{\mathcal{C}'}, d_{\mathcal{C}'}) \xrightarrow{\tilde{\mathcal{H}}'(\tau), \Lambda'(\tau)} (C'', \Delta_{\mathcal{C}''}, d_{\mathcal{C}''})
\end{align*}
\]
the composition \((\tilde{\mathcal{H}}'(\tau), \Lambda'(\tau)) \circ (\tilde{\mathcal{H}}(\tau), \Lambda(\tau))\), defined by the formulas
\[
\begin{align*}
\tilde{\mathcal{H}}'(\tau) &= \tilde{\mathcal{H}}'(\tau) \circ \tilde{\mathcal{H}}(\tau), \\
\Lambda'(\tau) &= \tilde{\mathcal{H}}'(\tau) \circ \Lambda(\tau) + \Lambda'(\tau) \circ \tilde{\mathcal{H}}(\tau),
\end{align*}
\]
is a homotopy pair of dg coalgebras from \((C, \Delta_{\mathcal{C}}, d_{\mathcal{C}})\) to \((C'', \Delta_{\mathcal{C}''}, d_{\mathcal{C}''})\) and \( \circ \) is associative.

Consider the following diagram in \( \text{dgC}(R) \)
\[
\begin{array}{ccc}
(C, \Delta_{\mathcal{C}}, d_{\mathcal{C}}) & \xrightarrow{F} & (C', \Delta_{\mathcal{C}'}, d_{\mathcal{C}'}) \\
& \xrightarrow{\tilde{F}} & (C'', \Delta_{\mathcal{C}''}, d_{\mathcal{C}''})
\end{array}
\]
and assume that \( F \sim \tilde{F} \) and \( F' \sim \tilde{F}' \). Then there are homotopy pairs
- \((\tilde{S}(\tau), \Lambda(\tau))\) such that \(\tilde{S}(0) = F\) and \(\tilde{S}(1) = \tilde{F}\);
- \((\tilde{S}'(\tau), \Lambda'(\tau))\) such that \(\tilde{S}'(0) = F'\) and \(\tilde{S}'(1) = \tilde{F}'\).

By Lemma A.4, their composition \((\tilde{S}'(\tau), \Lambda'(\tau))\) is a homotopy pair such that \(\tilde{S}'(0) = F' \circ F = \tilde{F}' \circ \tilde{F}\). It follows that \(F' \circ F = \tilde{F}' \circ \tilde{F}\) whenever \(F \sim \tilde{F}\) and \(F' \sim \tilde{F}'\) and the homotopy type \([F' \circ F] = [\tilde{F}' \circ \tilde{F}]\).

**Definition A.13.** The homotopy category of dg coalgebras over \(R\) is the category \(\text{ho} \text{dgC}(R)\) whose objects are dg coalgebras over \(R\) and whose morphisms are homotopy types of dg coalgebra morphisms with composition \(\circ_h\).

A dg coalgebra \((C, \Delta_C, d_C)\) is cocommutative if \(\Delta_C = \tilde{\epsilon} \circ \Delta_C\). We use the notation \(\text{cocdgC}(R)\) and \(\text{ho} \text{cocdgC}(R)\) for the category and homotopy category of cocommutative dg coalgebras over \(R\), which are full subcategories of \(\text{dgC}(R)\) and \(\text{ho} \text{dgC}(R)\), respectively.

### A.3. The bar functor.

The bar construction of \(sL_\infty\)-algebras is a homotopy functor \(\mathcal{B}\) from the category \(sL_\infty(R)\) of \(sL_\infty\)-algebras to the category \(\text{cocdgC}(R)\) of cocommutative dg-coalgebras.

The reduced symmetric module \(\overline{S}(V)\) generated by a \(Z\)-graded \(R\)-module \(V\) has the structure of a \(Z\)-graded cocommutative and coassociative coalgebra \(\overline{S}^{\sigma}(V) = (\overline{S}(V), \overline{\Delta})\) over \(R\) called the reduced symmetric coalgebra cogenerated by \(V\), where the coproduct \(\overline{\Delta} : \overline{S}(V) \to \overline{S}(V) \otimes \overline{S}(V)\) is defined for all \(n \geq 1\) and homogeneous \(v_1, \ldots, v_n \in V\) to be

\[
\overline{\Delta}(v_1 \otimes \ldots \otimes v_n) = \sum_{r=1}^{n-1} \sum_{\sigma \in Sh(r, n-r)} e(\sigma)v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)} \otimes v_{\sigma(r+1)} \otimes \ldots \otimes v_{\sigma(n)}.
\]

Here the second sum is over all \((r, n-r)\)-shuffles—these are those permutations of \(n\) such that \(\sigma(1) < \ldots < \sigma(r)\), and \(\sigma(r+1) < \ldots < \sigma(n)\).

The reduced symmetric coalgebras have the following properties:
Lemma A.5. For any \( l \in \text{Hom}_R(\mathcal{S}(V), V) \), define \( \mathcal{D}(l) \in \text{Hom}_R(\mathcal{S}(V), \mathcal{S}(V)) \) for all \( n \geq 1 \) and homogeneous element \( v_1, \ldots, v_n \in V \) by the equation

\[
\mathcal{D}(l)(v_1 \odot \cdots \odot v_n) = \sum_{|p| \in \mathcal{P}(n)} \epsilon(p) l(v_{B_1}) \odot \cdots \odot l(v_{B_{|p|-1}}) \odot v_{B_{|p|}}.
\]

Then, \( \mathcal{D}(l) \) is the unique coderivation of \( \mathcal{S}^\circ(V) \) with the property \( \text{pr}_V \circ \mathcal{D} = l \), where \( \text{pr}_V : \mathcal{S}(V) \to V \) is the natural projection. Conversely, any degree 1 coderivation \( \mathcal{D} \) of \( \mathcal{S}^\circ(V) \), \( \mathcal{T} \circ \mathcal{D} = \left( \mathcal{D} \otimes 1 + 1 \otimes \mathcal{D} \right) \circ \mathcal{T} \), is in the form \( \mathcal{D}(l) \), where \( l = \text{pr}_V \circ \mathcal{D} \in \text{Hom}(\mathcal{S}(V), V) \).

Lemma A.6. For any pair \( (\varphi, \lambda) \in \text{Hom}_R(\mathcal{S}(V), V')^0 \oplus \text{Hom}_R(\mathcal{S}(V), V')^{-1} \), define a pair \( (\mathfrak{H}(\varphi), \Lambda(\varphi, \lambda)) \in \text{Hom}_R(\mathcal{S}(V), \mathcal{S}(V))^0 \oplus \text{Hom}_R(\mathcal{S}(V), \mathcal{S}(V))^{-1} \) for all \( n \geq 1 \) and homogeneous element \( v_1, \ldots, v_n \in V \) via the equations

\[
\mathfrak{H}(\varphi)(v_1 \odot \cdots \odot v_n) := \sum_{|p| \in \mathcal{P}(n)} \epsilon(p) \varphi(v_{B_1}) \odot \cdots \odot \varphi(v_{B_{|p|-1}}),
\]

\[
\Lambda(\varphi, \lambda)(v_1 \odot \cdots \odot v_n) := \sum_{|p| \in \mathcal{P}(n)} \epsilon(p) \sum_{i=1}^n \varphi(J v_{B_i}) \odot \cdots \odot \varphi(J v_{B_{i-1}}) \odot \lambda(v_{B_i}) \odot \varphi(J v_{B_{i+1}}) \odot \cdots \odot \varphi(v_{B_{|p|-1}}).
\]

Then, we have

\[
\mathfrak{T}' \circ \mathfrak{H}(\varphi) = \left( \mathfrak{H}(\varphi) \otimes \mathfrak{H}(\varphi) \right) \circ \mathfrak{T}, \quad \text{pr}_V \circ \mathfrak{H}(\varphi) = \varphi,
\]

\[
\mathfrak{T}' \circ \Lambda(\varphi, \lambda) = \left( \Lambda(\varphi, \lambda) \otimes \mathfrak{H}(\varphi) + \mathfrak{H}(\varphi) \otimes \Lambda(\varphi, \lambda) \right) \circ \mathfrak{T}, \quad \text{pr}_V \circ \Lambda(\varphi, \lambda) = \lambda.
\]

Conversely, any pair \( (\mathfrak{H}, \Lambda) \) of \( \mathfrak{H} \in \text{Hom}_R(\mathcal{S}(V), \mathcal{S}(V))^0 \) and \( \Lambda \in \text{Hom}(\mathcal{S}(V), \mathcal{S}(V))^{-1} \) satisfying \( \mathfrak{T}' \circ \mathfrak{H} = \left( \mathfrak{H} \otimes \mathfrak{H} \right) \circ \mathfrak{T} \) and \( \mathfrak{T}' \circ \Lambda = \left( \Lambda \otimes \mathfrak{H} + \mathfrak{H} \otimes \Lambda \right) \circ \mathfrak{T} \) is in the form \( (\mathfrak{H}(\varphi), \Lambda(\varphi, \lambda)) \), where \( \varphi = \text{pr}_V \circ \mathfrak{H} \in \text{Hom}_R(\mathcal{S}(V), V')^0 \) and \( \lambda = \text{pr}_V \circ \Lambda \in \text{Hom}_R(\mathcal{S}(V), V')^{-1} \).

Definition A.14. The bar construction of an \( sL_\infty \)-algebra \( (V, l) \) is the cocommutative dg-coalgebra defined as follows:

\[
\mathfrak{B}(V, l) = (\mathcal{S}^\circ(V), \mathcal{D}(l)).
\]

The bar construction of an \( sL_\infty \)-morphism \( (V, l) \overset{\varphi}{\longrightarrow} (V', l') \) is \( \mathfrak{B}(\varphi) = \mathfrak{H}(\varphi) \).

Lemma A.7. The bar construction \( \mathfrak{B} \) is a functor from \( sL_\infty(R) \) to \( \text{co} \text{cdgC}(R) \).
Lemma A.8. For each $sL\infty$-homotopy pair $(\varphi(\tau), \lambda(\tau))$ from $(V, l) \to (V', l')$, define
\[ \mathcal{B}((\varphi(\tau), \lambda(\tau))) := (\mathcal{F}(\varphi(\tau), \Lambda(\varphi(\tau), \lambda(\tau))). \]

Proof. Lemma A.5 implies that, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,
\[
\left( \text{pr}_V \circ \mathcal{D}(l) \circ \mathcal{D}(l) \right)(v_1 \otimes \cdots \otimes v_n) = \sum_{|p| \in P(n)} \varepsilon(p) l(J v_{B_1} \otimes \cdots \otimes J v_{B_{n-1}} \otimes l(x_{B_i} \circ v_{B_{n+1}} \otimes \cdots \otimes v_{B_n})).
\]

From Definition A.1 of an $sL\infty$-algebra, the coderivation $\mathcal{D}(l)$ satisfies the condition $\text{pr}_V \circ \mathcal{D}(l) \circ \mathcal{D}(l) = 0$, which can be checked, by a straightforward induction, to be equivalent to the condition that $\mathcal{D}(l) \circ \mathcal{D}(l) = 0$. Therefore $\mathcal{B}(V, l) = (\mathcal{F}(V), \mathcal{D}(l))$ is a cocommutative dg-coalgebra.

Lemma A.6 implies that, for all $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,
\[
\left( \text{pr}_V \circ (\mathcal{D}(l') \circ \mathcal{F}(\varphi) - \mathcal{F}(\varphi) \circ \mathcal{D}(l)) \right)(v_1 \otimes \cdots \otimes v_n) = \sum_{|p| \in P(n)} \varepsilon(p) l'(\varphi(v_{B_1}) \otimes \cdots \otimes \varphi(v_{B_{n-1}})) - \sum_{|p| \in P(n)} \varepsilon(p) l(J v_{B_1} \otimes \cdots \otimes J v_{B_{n-1}} \otimes l(x_{B_i} \circ v_{B_{n+1}} \otimes \cdots \otimes v_{B_n})).
\]

From Definition A.2 of $sL\infty$-morphism, we have $\text{pr}_V \circ (\mathcal{D}(l') \circ \mathcal{F}(\varphi) - \mathcal{F}(\varphi) \circ \mathcal{D}(l)) = 0$, which can be checked to be equivalent to the condition that $\mathcal{D}(l') \circ \mathcal{F}(\varphi) = \mathcal{F}(\varphi) \circ \mathcal{D}(l)$. Therefore $\mathcal{B}(\varphi) = \mathcal{F}(\varphi)$ is a morphism of cocommutative dg-coalgebras from $\mathcal{B}(V, l) = (\mathcal{F}(V), \mathcal{D}(l))$ to $\mathcal{B}(V', l') = (\mathcal{F}(V'), \mathcal{D}(l')).$

Consider the sequence of $sL\infty$-morphisms $(V, l) \xrightarrow{\varphi'} (V', l') \xrightarrow{\varphi} (V', l')$. Then $\varphi'' := \varphi' \circ \varphi$ is an $sL\infty$-morphism from $(V, l)$ to $(V''', l''')$. Now Lemma A.6 and Definition A.2 imply that $\text{pr}_V \circ (\mathcal{F}(\varphi') - \mathcal{F}(\varphi') \circ \mathcal{F}(\varphi)) = 0$, which can be checked to be equivalent to the condition: $\mathcal{F}(\varphi'') = \mathcal{F}(\varphi') \circ \mathcal{F}(\varphi)$. Therefore, we have $\mathcal{B}(\varphi' \circ \varphi) = \mathcal{B}(\varphi') \circ \mathcal{B}(\varphi)$, so that $\mathcal{B} : sL\infty(R) \Rightarrow \text{cogdgC}(R)$ is a functor.

The next lemma implies that $\mathcal{B} : sL\infty(R) \Rightarrow \text{cogdgC}(R)$ is a homotopy functor, so that it induces a functor $\text{ho}\mathcal{B} : \text{hosL}\infty(R) \Rightarrow \text{hocogdgC}(R)$.

Lemma A.8. For each $sL\infty$-homotopy pair $(\varphi(\tau), \lambda(\tau))$ from $(V, l) \to (V', l')$, define
\[ \mathcal{B}((\varphi(\tau), \lambda(\tau))) := (\mathcal{F}(\varphi(\tau), \Lambda(\varphi(\tau), \lambda(\tau))). \]
Then \( \mathcal{B}(\varphi(\tau), \lambda(\tau)) \) is a homotopy pair of cochain coalgebras from \( \mathcal{B}(V, L) \) to \( \mathcal{B}(V', L') \). Furthermore, for composable \( sL_{\infty} \)-homotopy pairs

\[
(V, L) \stackrel{\varphi(\tau), \lambda(\tau)}{\longrightarrow} (V', L') \stackrel{\varphi'(\tau), \lambda'(\tau)}{\longrightarrow} (V'', L'')
\]

we have \( \mathcal{B}((\varphi'(\tau), \lambda'(\tau)) \circ (\varphi(\tau), \lambda(\tau))) = \mathcal{B}((\varphi'(\tau), \lambda'(\tau))) \circ \mathcal{B}((\varphi(\tau), \lambda(\tau))) \).

**Proof.** Set \( (\varphi''(\tau), \lambda''(\tau)) = (\varphi'(\tau), \lambda'(\tau)) \circ (\varphi(\tau), \lambda(\tau)) \) and

\[
\mathcal{B}(\varphi(\tau)) = \mathcal{B}(\varphi'(\tau)), \quad \mathcal{B}(\lambda(\tau)) = \mathcal{B}(\lambda'(\tau)), \quad \mathcal{B}(\lambda''(\tau)) = \mathcal{B}(\lambda''(\tau)),
\]

\[
\Lambda(\tau) = \Lambda(\varphi(\tau), \lambda(\tau)), \quad \Lambda'(\tau) = \Lambda(\varphi'(\tau), \lambda'(\tau)), \quad \Lambda''(\tau) = \Lambda(\varphi''(\tau), \lambda''(\tau)).
\]

From Lemma [A.6], we have

\[
\Delta' \circ \mathcal{B}(\tau) - (\Delta(\tau) \otimes \mathcal{B}(\tau)) \circ \Delta = 0,
\]

\[
\Delta \circ \Lambda(\tau) - (\Lambda(\tau) \otimes \mathcal{B}(\tau) + \mathcal{B}(\tau) \otimes \Lambda(\tau)) \circ \Delta = 0.
\]

Note that the system of equations given in Definition [A.4] for the \( sL_{\infty} \)-homotopy pair \( (\varphi(\tau), \lambda(\tau)) \) is equivalent to \( \text{pr}_{V'} \circ \left( \frac{d}{d\tau} \mathcal{B}(\tau) - \mathcal{B}(l') \circ \Lambda(\tau) - \Lambda(\tau) \circ \mathcal{B}(l) \right) = 0 \), which implies that

\[
\frac{d}{d\tau} \mathcal{B}(\tau) = \mathcal{B}(l') \circ \Lambda(\tau) + \Lambda(\tau) \circ \mathcal{B}(l).
\]

Therefore \( \mathcal{B}((\varphi(\tau), \lambda(\tau))) = (\mathcal{B}(\varphi(\tau), \lambda(\tau))) \) is a homotopy pair of cdg-coalgebras from \( \mathcal{B}(V, L) \) to \( \mathcal{B}(V', L') \). From Lemma [A.2] and Lemma [A.6] we have

\[
\text{pr}_{V'} \circ (\mathcal{B}(\Lambda''(\tau) - \mathcal{B}(\tau) \circ \mathcal{B}(\tau)) = 0,
\]

\[
\text{pr}_{V'} \circ (\Lambda''(\tau) - \mathcal{B}(\tau) \circ \Lambda(\tau) - \Lambda'(\tau) \circ \mathcal{B}(\tau)) = 0.
\]

Working inductively, it is straightforward to show that the above conditions imply that

\[
\begin{align*}
\mathcal{B}(\varphi''(\tau)) &= \mathcal{B}(\varphi'(\tau)) \circ \mathcal{B}(\varphi(\tau)) \\
\Lambda''(\tau) &= \mathcal{B}(\varphi'(\tau)) \circ \Lambda(\tau) + \Lambda'(\tau) \circ \mathcal{B}(\varphi(\tau))
\end{align*}
\]

so that

\[
(\mathcal{B}(\varphi''(\tau), \Lambda''(\tau)) = (\mathcal{B}(\varphi'(\tau), \Lambda'(\tau)) \circ (\mathcal{B}(\varphi(\tau), \Lambda(\tau))).
\]

Restoring the original notation, we have the desired composition relation.
Consider $sL_\infty$-morphisms as follows

\[ (V, l) \xrightarrow{\varphi} (V', l') \xrightarrow{\varphi'} (V'', l''), \]

and assume that $\varphi \sim \tilde{\varphi}$ and $\varphi' \sim \tilde{\varphi}'$. Then the first part of Lemma A.8 implies that $\mathcal{B}(\varphi)$ and $\mathcal{B}(\tilde{\varphi})$ are homotopic morphisms of coddg-algebras from $\mathcal{B}(V, l)$ to $\mathcal{B}(V', l')$, so that the homotopy type $[\mathcal{B}(\varphi)]$ of $\mathcal{B}(\varphi)$ depends only the homotopy type $[\varphi]$ of $\varphi$. Define $ho\mathcal{B}([\varphi]) := [\mathcal{B}(\varphi)]$. Now the second part of Lemma A.8 implies that $\mathcal{B}(\varphi' \cdot \varphi) = \mathcal{B}(\varphi') \circ \mathcal{B}(\varphi)$ is homotopic to $\mathcal{B}(\tilde{\varphi}' \cdot \tilde{\varphi}) = \mathcal{B}(\tilde{\varphi}') \circ \mathcal{B}(\tilde{\varphi})$ as morphisms of coddg-algebras from $\mathcal{B}(V, l)$ to $\mathcal{B}(V'', l'')$, so that the homotopy type $[\mathcal{B}(\varphi' \cdot \varphi)]$ of $\mathcal{B}(\varphi' \cdot \varphi)$ depends only on the homotopy types $[\varphi']$ and $[\varphi]$. Combining everything, we have $ho\mathcal{B}([\varphi] \cdot h[\varphi]) = ho\mathcal{B}([\varphi]) \circ ho\mathcal{B}([\varphi])$. Therefore, we conclude the following.

**Lemma A.9.** $\mathcal{B} : sL_\infty(R) \Rightarrow coddgC(R)$ is a homotopy functor: it induces a functor $ho\mathcal{B} : hosL_\infty(R) \Rightarrow hocoddgC(R)$.

### A.4. On complete towers of classical symmetries and the classical BV master action

We explain the notion of a complete tower of infinitesimal classical symmetries as well as a recipe to construct a classical BV master action and subsequent gauge fixing. The key point is that every notion in off-shell classical physics should be defined modulo the classical equation of motion and coherence issues are naturally resolved using the language of $sL_\infty$-algebras.

**Definition A.15.** A BV-CFT algebra is a tuple $(\mathcal{C}, 1_{\mathcal{C}}, \cdot, K, (\ , ))$, where $(\mathcal{C}, 1, \cdot, K)$ is a unital CDGA and $(\mathcal{C}, 1_{\mathcal{C}}, K, (\ , ))$ is a unital sDGLA, such that the degree 1 bracket is a derivation of the product.

A typical example of BV-CFT algebra with geometric origin can be built from a smooth manifold with a distinguished function on it. We present a dictionary for classical field theory.

Let $\mathcal{L}_d$ be the space of smooth functions on a smooth manifold $\mathcal{L}_d$ with a distinguished element $S_d \in \mathcal{L}_d$. Let $\mathcal{C}_d = \cdots \oplus \mathcal{C}_d^{-2} \oplus \mathcal{C}_d^{-1} \oplus \mathcal{C}_d^0$ be the $\mathbb{Z}$-graded space
of smooth poly-vector fields on \( \mathcal{L}_{cl} \), where \( \mathcal{C}_{cl}^{-k} = \Gamma(\mathcal{L}_{cl}, \Lambda^k T_{cl}) \). We regard a \( k \)-polyvector field as an element of ghost number \(-k\) in \( \mathcal{C}_{cl} \). Note that \( \mathcal{C}_{cl}^0 = \mathcal{L}_{cl} \). Then \( \mathcal{C}_{cl} \) has the structure \( \left( \mathcal{C}_{cl}, 1, \{,\}, K_{cl}, (,\) \) \) of a BV-CFT algebra, where the bracket \( (,\) \) is the Schoutens–Nijenhuis bracket, the differential is defined by \( K_{cl} = (S_{cl}, )_{cl} \), the product \( \cdot \) is the exterior product and the unit \( 1_{\mathcal{C}_{cl}} \) is the constant function on \( \mathcal{L}_{cl} \) with the value 1. Recall that the Schoutens–Nijenhuis bracket is the unique extension of the Lie bracket to the vector fields on \( \mathcal{L}_{cl} \) as a derivation of the exterior product of poly-vector fields. It follows that \( K_{cl} \circ K_{cl} = 0 \), since \( (S_{cl}, S_{cl})_{cl} = 0 \), and \( K_{cl} \) is a derivation of both the product and the bracket. Equivalently, we may regard \( \mathcal{C}_{cl} \) as the space of smooth functions on the total space \( T^*[1] \mathcal{L}_{cl} \) of the cotangent bundle to \( \mathcal{L}_{cl} \) after twisting the fiber by ghost number \(-1\). We remark that \( T^*[1] \mathcal{L}_{cl} \) has a canonical odd symplectic structure of ghost number \(-1\), whose associated odd Poisson bracket of ghost number 1 is the bracket \( (,\) \).

We regard \( \mathcal{L}_{cl} \) as the space of classical fields and \( S_{cl} \) as the classical action of a classical field theory. Choose Darboux coordinates \( \{z^I|z^*_I\} \) of \( T^*[1] \mathcal{L}_{cl} \) with \( gh(z^I) = 0 \) and \( gh(z^*_I) = -1 \), and call \( \{z^I\} \) the classical fields and \( \{z^*_I\} \) the anti-fields for the classical fields. Then, the differential \( K_{cl} \) can be expressed as

\[
K_{cl} = \left( \frac{\partial S_{cl}}{\partial z^I} \right) \frac{\partial}{\partial z^*_I}, \tag{A.1}
\]

where we use deWitt–Einstein notation. Regarded as an odd vector field on \( T^*[1] \mathcal{L}_{cl} \), the vanishing loci of \( K_{cl} \) is the solution space \( \mathcal{L}_{onshell} \subset \mathcal{L}_{cl} \) of the classical equation of motion:

\[
\frac{\partial S_{cl}}{\partial z^I} = 0, \quad \forall I. \tag{A.2}
\]

In general, any element in \( \text{Im} K_{cl} \cap \mathcal{C}_{cl} \) vanishes by the classical equation of motion.

Consider the cohomology \( H_{cl} \) of the cochain complex \( \left( \mathcal{C}_{cl}, K_{cl} \right) \). Then, by Lemma \( \text{A.3} \) there is a minimal \( sL_\infty \)-structure \( (H_{cl}, 0, \hat{\ell}_2, \hat{\ell}_3, \ldots ) \) on \( H_{cl} \) and an \( sL_\infty \)-quasi-isomorphism \( \varphi : (H_{cl}, 0, \hat{\ell}_2, \hat{\ell}_3, \ldots ) \longrightarrow (\mathcal{C}_{cl}, K_{cl}, (,\) \) \). This is closely related with the notion of a complete tower of infinitesimal classical symmetries.

We begin with a physical interpretation of the cohomology \( H_{cl} = \cdots \oplus H^{-2}_{cl} \oplus H^{-1}_{cl} \oplus H^0_{cl} \).

- We say two elements \( \mathcal{O}_{cl} \) and \( \tilde{\mathcal{O}}_{cl} \) of \( \mathcal{L}_{cl} \) are equivalent if \( \tilde{\mathcal{O}}_{cl} - \mathcal{O}_{cl} = \lambda(z)^I \frac{\partial S_{cl}}{\partial z^I} \) for some \( \lambda(z)^I \in \mathcal{L}_{cl} \) — they induce the same function on \( \mathcal{L}_{onshell} \). Then, \( H^0_{cl} \) is exactly the set of such equivalence classes: every element in \( \mathcal{C}_{cl}^0 = \mathcal{L}_{cl} \) belongs to \( \text{ker} K_{cl} \), and any \( \Lambda \in \mathcal{C}_{cl}^{-1} \) is in the form \( \Lambda = \lambda(z)^I z^*_I \) and \( K_{cl} \Lambda = \lambda(z)^I \frac{\partial S_{cl}}{\partial z^I} \). Note
that \((\mathcal{E}_{cl}, 1_{\mathcal{E}_{cl}}, \ldots)\) is a unital CDGA, which induces the structure \((\mathcal{H}_{cl}^0, 1_{\mathcal{H}_{cl}}, \tilde{m}_2)\) of a unital commutative and associative algebra on \(\mathcal{H}_{cl}^0\), which is viewed as the algebra \(\mathcal{L}_{onshell}\) of functions on the solution space \(\mathcal{L}_{onshell}\) of the classical equation of motion.

- An element \(R = R(z)^I z_I^\bullet \in \mathcal{E}_{cl}^{-1}\) is called an infinitesimal symmetry vector field (for the classical action \(S_{cl}\)) if \(K_{cl} R \equiv R(z)^I \left(\frac{\delta S_{cl}}{\delta z_I^J}\right) = 0\), i.e., \(R \in \text{Ker} K_{cl} \cap \mathcal{E}_{cl}^{-1}\). Two infinitesimal symmetry vector fields \(R\) and \(\tilde{R}\) are equivalent if \(\tilde{R} - R = \lambda(z)^I J \left(\frac{\delta S_{cl}}{\delta z_J^I}\right)\), \(\lambda(z)^I J = -\lambda(z)^J I\) — they induce the same vector fields on \(\mathcal{L}_{onshell}\). Then the cohomology group \(H_{cl}^{-1}\) is exactly the set of such equivalence classes: any \(\Lambda \in \mathcal{E}_{cl}^{-2}\) is in the form \(\Lambda = \frac{1}{2} \lambda(z)^I J z_I^\bullet z_J^\bullet\) and \(K_{cl} \Lambda = \lambda(z)^I J \left(\frac{\delta S_{cl}}{\delta z_J^I}\right) z_I^\bullet z_J^\bullet\). Note that \((H_{cl}^{-1}, \lambda(z)^I J)\) is a Lie algebra, which should be the Lie algebra of gauge symmetries of \(S_{cl}\). It is straightforward to check that \(\mathcal{H}_{cl}^0\) is a module of the Lie algebra \(H_{cl}^{-1}\) with the action \(\tilde{\gamma} : H_{cl}^{-1} \times \mathcal{H}_{cl}^0 \to \mathcal{H}_{cl}^0\) given by \((\xi, \nu) \mapsto \tilde{\gamma}(\xi, \nu) = \tilde{\gamma} \xi, \nu := \ell_{\xi}(\xi, \nu)\). In fact, the entire space \(H_{cl}^0\) is a \(\mathbb{Z}\)-graded module over the Lie algebra \(H_{cl}^{-1}\) with a similarly defined action.

In general, we call \(\mathcal{H}_{cl}^0\) the space of on-shell classical observables, \(H_{cl}^{-1}\) the space of gauge symmetries, \(\mathcal{H}_{cl}^{-2}\) the space of symmetries of the gauge symmetry, etc.

**Definition A.16.** A tower of infinitesimal classical symmetries of a BV-CFT algebra is a minimal \(sL_{\infty}\)-algebra \((\mathfrak{g}, \ell^8, \ell^8, \ldots)\) together with an \(sL_{\infty}\)-morphism \(\rho = \rho_1, \rho_2, \ldots\) to \((\mathcal{E}_{cl}, K_{cl}, (\ , )_{cl})\). Such a tower is complete if \(\rho_1\) induces an isomorphism from \(\mathfrak{g}\) to \(\overline{H}_{cl} := \ldots \oplus \mathcal{H}_{cl}^{-2} \oplus \mathcal{H}_{cl}^{-1}\).

Assume, for demonstrative purposes that \(\mathfrak{g}\) is concentrated in degree \(-1\) so that \(\mathfrak{g}\) is a Lie algebra with bracket \(\ell^8\).

- The first condition for \(\rho : (\mathfrak{g}, \ell^8) \to (\mathcal{E}_{cl}, K_{cl}, (\ , )_{cl})\) to be an \(sL_{\infty}\)-morphism is

\[
K_{cl} \circ \rho_1 = 0,
\]

and the image of \(\rho_1 : \mathfrak{g} \to \mathcal{E}_{cl}\) lies on \(\mathcal{E}_{cl}^{-1}\). The condition \(K_{cl} \circ \rho_1 = 0\) is equivalent to the condition that \(\rho_1(S_{cl}) = (\rho_1(\mathfrak{g}), S_{cl}) = 0\). Therefore, \(\rho_1(\mathfrak{g}) \in \text{Ker} K_{cl} \cap \mathcal{E}_{cl}^{-1}\) is an infinitesimal symmetry vector field of the classical action \(S_{cl}\) for all \(g \in \mathfrak{g}\).

Recall that \(\mathcal{E}_{cl}^{-1}\) is the space of vector fields on \(\mathcal{L}_{cl}\), which is equivalent to the space \(\text{Der}(\mathcal{L}_{cl}) \subset \text{End}(\mathcal{L}_{cl})\) of derivations of \(\mathcal{L}_{cl}\). Therefore \(\rho_1\) induces a linear
map \( \rho : g \to \text{End}(L) \) defined for all \( g \in g \) and \( O_d \in L \) by the equation
\[
\rho(g)(O_d) := (\rho_1(g), O_d)_{cl}.
\]
(A.4)

- The second condition for \( \rho = \rho_1, \rho_2, \ldots \) to be an \( sL_\infty \)-morphism is that, for all \( g_1, g_2 \in g \),
\[
\rho_1(\ell^g_2(g_1, g_2)) - \left( \rho_1(g_1), \rho_1(g_2) \right)_{cl} = K_d \rho_2(g_1, g_2).
\]
(A.5)

The image of \( \rho_2 : S^2 g \to \mathcal{C}_d \) lies on \( \mathcal{C}_d^{-1} \) and \( K_d \rho_2(g_1, g_2) \) vanishes by the classical equation of motion. Therefore, the linear map \( \rho : g \to \text{End}(L) \) is almost a representation of the Lie algebra \( g \) whose failure vanishes by the classical equation of motion. It follows that \( \rho \) induces a representation \( \hat{\rho} : g \to \text{End}(L_{\text{onshell}}) \) of the Lie algebra \( g \), and this is exactly what is relevant for classical physics.

- Note that the relations in eq. (A.5) come with a coherence issue. From the Jacobi-identities of \( \ell^g_2 \) and \( (\ , \ , )_d \), it can be checked that \( K_d \circ P_3 = 0 \), where \( P_3 \) is the element of \( \text{Hom}(S^3 g, \mathcal{C}_d)^1 \) defined to be the sum over cyclic permutations of the indices of the expression
\[
\rho_2(\ell^g_2(g_1, g_2), g_3)_{cl} - \left( \rho_2(g_1, g_2), \rho_1(g_3) \right)_{cl}.
\]

Choose the data of a strong deformation retract \((H_d, 0) \xrightarrow{\zeta_3} (\mathcal{C}_d, K_d)\). Define \( \zeta_3 \) to be \( h \circ P_3 \in \text{Hom}(S^3 g, H_d)^1 \) and \( \rho_3 \) to be \( s \circ P_3 \in \text{Hom}(S^3 g, H_d)^0 \). Then we have \( f \circ \zeta_3 = P_3 - K_d \circ \rho_3 \). Provided that \( \zeta_3 = 0 \), we have \( P_3 = K_d \circ \rho_3 \), which is precisely the third condition for \( \rho = \rho_1, \rho_2, \rho_3, \ldots \) to be an \( sL_\infty \)-morphism. In general, we have \( \zeta_3 \neq 0 \) so that the coherence issue can not be resolved within the Lie algebra.

To summarize, the correct notion of an infinitesimal classical symmetry is supposed to be an action of a Lie algebra \( g \) on the solution space \( L_{\text{onshell}} \) of the classical equation of motion or, equivalently, a representation \( \hat{\rho} : g \to \text{End}(L_{\text{onshell}}) \) of the Lie algebra \( g \).

Any lifting of such a representation to an off-shell \( L_{cl} \) should allow a weaker notion of representation \( \rho : g \to \text{End}(L) \) modulo the classical equation of motion, which introduces a potentially infinite sequence of coherence issues.

Resolution of all those coherence issues can be achieved, leading to the notion of a complete tower of infinitesimal symmetries as a representative of the maximal symmetry modulo equivalence. To describe this solution, we consider the reduced cochain
complex \((\mathcal{C}_{cl}, K_d)\), where \(\mathcal{C}_{cl} = \cdots \oplus \mathcal{C}_{cl}^{-2} \oplus \mathcal{C}_{cl}^{-1} \cap \text{Ker} K_d\). It follows that the cohomology of the reduced cochain complex is isomorphic to \(\mathcal{H}_{cl} = \cdots \oplus \mathcal{H}_{cl}^{-2} \oplus \mathcal{H}_{cl}^{-1}\). From the properties that the bracket \((\cdot, \cdot)_d\) has ghost number 1 and that \(K_d\) is a derivation of the bracket, it follows that \((\mathcal{C}_{cl}, K_d, (\cdot, \cdot)_d)\) is also an sDGLA. Therefore \(\mathcal{H}_{cl}\) also admits the structure \((\mathcal{H}_{cl}, \hat{\ell}_2, \hat{\ell}_3, \cdots)\) of a minimal \(sL_\infty\)-algebra together with an \(sL_\infty\)-quasi-morphism \(\varphi : (\mathcal{H}_{cl}, \hat{\ell}_2, \hat{\ell}_3, \cdots) \to (\mathcal{C}_{cl}, K_d, (\cdot, \cdot)_d)\). Then this data constitutes a complete tower of infinitesimal classical symmetries, which is supposed to be encoded by a classical BV master action \(S = S_d + \cdots\).

Choose a homogeneous basis \(\{e_a\}_{a \in \mathfrak{g}}\) of \(\mathcal{H}_{cl}\) and work out the set of structure constants \(\{C_{a_1 a_2}^{b}, C_{a_1 a_2 a_3}^{b}, \cdots\\}\) so that \(\hat{\ell}_n(e_{a_1}, \cdots, e_{a_n}) = C_{a_1 \cdots a_n}^{b} e_b\), where \(n \geq 2\) and \(a_1, \ldots, a_n, b \in \mathfrak{g}\). Then, we consider the following super-manifold:

\[
\mathcal{L} = \mathcal{L}_{cl} \times \mathcal{H}_{cl} \times \mathcal{H}_{cl} \times \mathcal{H}_{cl}[1],
\]

with homogeneous affine coordinate system \(\{q^A\} = \{z^i, \eta^a, \bar{\eta}_a, \lambda_a\}\), where \(gh(z^i) = 0\), \(gh(\eta^a) = -gh(e_a)\), \(gh(\bar{\eta}_a) = -gh(\bar{\eta}_a)\) and \(gh(\lambda_a) = gh(\bar{\eta}_a) + 1\). Then the algebra \(\mathcal{L}\) of functions on \(\mathcal{L}\) is isomorphic to \(\mathcal{L}/[[\eta^a, \bar{\eta}_a, \lambda_a]]\). In physics terminology, we call \(\eta^a\) the ghost fields, \(\bar{\eta}_a\) auxiliary or Lagrangian multiplier fields, and \(\lambda_a\) anti-ghost fields, so \(\{\eta^a, \bar{\eta}_a, \lambda_a\}\) is the full set of Faddeev–Popov ghosts, \(\{\eta^a, \bar{\eta}_a\}\) the ghosts of the Faddeev–Popov ghosts, etc. Collectively, one calls \(\{q^A\}\) the fields.

Now we consider \(T^*[−1]\mathcal{L}\), which has the canonical symplectic structure \(\Omega_{BV}\) with ghost number −1, whose expression in terms of Darboux coordinates

\[
\{q^A \vert q^*_A\} = \{z^i, \eta^a, \bar{\eta}_a, \lambda_a, e^*_i, \eta^*_a, \bar{\eta}_a^*, \lambda^*_a\}, \quad gh(q^*_A) = -gh(q^A) - 1,
\]

is \(\Omega = dz^i \wedge d\eta^a \wedge d\lambda_a + d\bar{\eta}_a \wedge d\bar{\eta}_a + d\lambda_a \wedge d\lambda_a\). We call \(q^*_A\) the antifield for the field \(q_A\). Then, the space \(\mathcal{E} \cong \mathcal{L}/[[\eta^a, \bar{\eta}_a, \lambda_a, \eta^*_a, \bar{\eta}_a^*, \lambda^*_a]]\) of functions on \(T^*[−1]\mathcal{L}\) has the structure of BV-CFT algebra \((\mathcal{E}, 1_\mathcal{E}, (\cdot, \cdot)_\mathcal{E})\) with zero differential, where \((\cdot, \cdot)_\mathcal{E}\) is the odd Poisson bracket associated with \(\Omega_{BV}\) and \(1_\mathcal{E} = 1_{\mathcal{E}}\).

Now we define a classical BV master action \(S \in \mathcal{E}^0\) as follows:

\[
S = S_{cl} + \sum_{n \geq 2} \frac{1}{n!} \eta^{a_1} \cdots \eta^{a_n} C_{a_1 \cdots a_n} \hspace{0.5cm} b \hspace{0.5cm} e_b + \sum_{n \geq 1} \frac{1}{n!} \eta^{a_1} \cdots \eta^{a_n} \varphi_a e_{a_1, \ldots, e_{a_n}} + \lambda_a \bar{\eta}_a^\dagger. \quad (A.6)
\]

Then, it is straightforward to check that \(S\) satisfies the so called classical BV master equation:

\[
(S, S)_{\mathcal{E}} = 0, \quad S|_{\mathcal{L}} = S_{cl}. \quad (A.7)
\]
Define $K := (S,-)_{BV}$. It follows that $\left( \mathcal{L}, 1_{\mathcal{L}}, \cdot, K, ( , )_{BV} \right)$ is a BV-CFT algebra.

We remark that $\varphi_n(a_1, \ldots, a_n) \in \mathcal{C}_d$ and $\text{gh}(\varphi_n(a_1, \ldots, a_n)) = \text{gh}(a_1) + \ldots + \text{gh}(a_n) \leq -1$ for all $n \geq 1$ and $a_1, \ldots, a_n \in \overline{H}_d$. Therefore, we have

$$\varphi_n(a_1, \ldots, a_n) = R(z)_{a_1 \ldots a_n} l_1 \ldots l_k z^*_{l_1} \ldots z^*_{l_k},$$

where $k = - (\text{gh}(a_1) + \ldots + \text{gh}(a_n)) \geq 1$, and we obtain the following more explicit form of $S$ as defined in eq. (A.6):

$$S = \sum_{n \geq 2} \frac{1}{n!} \eta^{a_1} \ldots \eta^{a_n} C_{a_1 \ldots a_n} \eta^b + \sum_{n \geq 1} \frac{1}{n!} \eta^{a_1} \ldots \eta^{a_n} R(z)_{a_1 \ldots a_n} l_1 \ldots l_k z^*_{l_1} \ldots z^*_{l_k} + \lambda_a \eta^a.$$

Encoding a complete tower of infinitesimal classical symmetries by the classical BV master action $S$, we turn to a general gauge fixing procedure.

**Definition A.17.** A gauge fermion is an element $\psi \in \mathcal{L}^{-1}$ and the gauge fixed classical action $S^\psi_{cl} \in \mathcal{L}^0$ with respect to the gauge fermion $\psi$ is

$$S^\psi_{cl} := S_{cl} + \sum_{n \geq 1} \frac{1}{n!} \nu_n(\psi, \ldots, \psi)$$

$$= S_{cl} + \lambda_a \frac{\delta \psi}{\delta \eta^a} + \sum_{n \geq 2} \frac{1}{n!} \eta^{a_1} \ldots \eta^{a_n} C_{a_1 \ldots a_n} \eta^b \frac{\delta \psi}{\delta \eta^b}$$

$$+ \sum_{n \geq 1} \frac{1}{n!} \eta^{a_1} \ldots \eta^{a_n} R(z)_{a_1 \ldots a_n} l_1 \ldots l_k \left( \frac{\delta \psi}{\delta z^*_{l_1}} \right) \ldots \left( \frac{\delta \psi}{\delta z^*_{l_k}} \right).$$

**Remark A.1.** Define the family $\nu = \nu_0, \nu_1, \nu_2, \ldots$ by declaring that $\nu_0 = S_{cl}$ and, for all $n \geq 1$ and homogeneous $\gamma_1, \ldots, \gamma_n \in \mathcal{L}$,

$$\nu_n(\gamma_1, \ldots, \gamma_n) := \left( (S, \gamma_1)_{BV}, \gamma_2)_{BV}, \ldots, \gamma_n)_{BV} \right|_{\mathcal{L}}.$$

Then, we have

$$S^\psi_{cl} = \nu_0 + \nu_1(\psi) + \frac{1}{2!} \nu_2(\psi, \psi) + \frac{1}{3!} \nu_3(\psi, \psi, \psi) + \ldots.$$

In fact, it is easy to check that $\left( \mathcal{L}, \nu \right)$ is a weakly homotopy Lie algebra, also known as a curved $L_{\infty}$-algebra — for example the differential $\nu_1 : \mathcal{L} \to \mathcal{L}$, which is often called a BRST operator, does not satisfy $\nu_1 \circ \nu_1 = 0$ strictly but only modulo the classical equation of motion.
Alternatively, we consider an element $\Psi \in \mathcal{C}^{-1}$ and the canonical transformation generated by $\Psi$. Then, we have

$$S \to S^\Psi = S + (S, \Psi)_{BV} + \frac{1}{2!}((S, \Psi)_{BV}, \Psi)_{BV} + \ldots \in \mathcal{C}^0$$

Let $\psi = \Psi|_{\nu}$. Then, from eq. (A.7) and by definitions, we obtain that

$$(S^\psi, S^\psi)_{BV} = 0,
S^\psi|_{\nu} = S^\psi_{cl}.$$  

Equivalently, we can interpret the canonical transformation generated by $\Psi = \psi$ as a deformation of the Lagrangian subspace $\mathcal{L}$ so that $S^\psi|_{\nu} = S_{cl}|_{\nu}$, i.e., from the zeros of the section $q_A^*$ to the zeros of the section $q_A^* - \frac{\delta S^\psi}{\delta q^A}$.  

**Example A.1.** Now we consider a simple case, where the standard BRST-FP method [1114] can be used. Assume that $H_d = H_d^{-1}$. Then, the reduced sDGLA $(\overline{\mathcal{C}}, \mathcal{K}, (\cdot, \cdot)_{BV})$ is formal for degree reasons. Then the minimal $sL_{\infty}$-structure on $H_d$ is $(H_d^{-1}, \delta_2)$, which is just a Lie algebra $\mathfrak{g}$ — which we assume to be the Lie algebra of a simply connected Lie group $\mathfrak{g}$. Also assume that we have an $sL_{\infty}$-quasi-isomorphism $\varphi$ to the sDGLA such that $\varphi_0 = 0$, for all $n \geq 2$. Note that $\varphi_1(e_u) = R(z)_a^I \eta^I \eta^a \in \mathcal{C}^{-1}$. Then the classical BV master action $S$ is reduced to the following simple form

$$S = S_{cl} + \lambda_a \eta^a + \frac{1}{2} \eta^{a_1} \eta^{a_2} f_{a_1 a_2}^b \eta^b + \eta^a R(z)^I_a^I \eta^I.$$  

where $gh(\eta^a) = 1, gh(\lambda_a) = -1$ and $gh(\lambda_a) = 0$. Then we have $(\mathcal{L}, \psi)$ where $\nu_1 = S_{cl}$, $\nu_k = \delta_{BRST}$ and $\nu_k = 0$ for all $k \geq 3$, where $\delta_{BRST}$ is the so-called BRST operator given by

$$\delta_{BRST} = \delta_{Lie} = \frac{1}{2} \eta^{a_1} \eta^{a_2} f_{a_1 a_2}^b \delta \eta^b + \eta^a R(z)^I_a^I \delta \eta^I.$$  

Then we have $\delta_{BRST} \circ \delta_{BRST} = \delta_{Lie} \circ \delta_{Lie} = 0$ and $\delta_{BRST} S_{cl} = \delta_{Lie} S_{cl} = 0$. Note that $\delta_{Lie}$ is the differential for the standard Lie algebra cohomology. In coordinates, any gauge fermion can be written $\psi = \overline{\mathcal{N}}_d G^a(z)$, It follows that

$$S^\psi_{cl} = S_{cl} + \delta_{BRST} \psi = S_{cl} + \lambda_a G^a(z) + \eta^a R(z)^I_a^I \delta G^a(z)^b \eta^I \eta^b,$$

which is precisely the gauge fixed action according to the Fadeev–Popov procedure: the integral over $\{\lambda_a\}$ imposes the constraint $\{G(z)^a = 0\}$ — a gauge fixing — and the integral over $\{\eta^a, \overline{\mathcal{N}}_d\}$ induces the so called Fadeev–Popov determinant, which are the two ingredients for constructing a quotient measure on $\mathcal{L}_{cl}/\mathfrak{g}$.  

Finally, we refer to [20] for the geometry of quantum BV master action and [6] for many subtleties in dealing with infinite dimensional spaces of fields from both the classical and quantum perspectives. For a passage from the idea of infinitesimal symmetry of quantum expectation to binary QFT algebra, we refer to Section 5 in [17], where $\hbar = 1$. 
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