On the Dimension of The Virtually Cyclic Classifying Space of a Crystallographic Group

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February 6, 2018

Abstract

In this paper we construct a model for the classifying space, $B_{VC}\Gamma$, of a crystallographic group $\Gamma$ of rank $n$ relative to the family $\mathcal{VC}$ of virtually-cyclic subgroups of $\Gamma$. The model is used to show that there exists no other model for the virtually-cyclic classifying space of $\Gamma$ with dimension less than $vcd(\Gamma)+1$, where $vcd(\Gamma)$ denotes the virtual cohomological dimension of $\Gamma$. In addition, the dimension of our construction realizes this limit.

1 Introduction. Statement of Results

Let $\Gamma$ be a discrete group. Its classifying space, $B\Gamma = E\Gamma/\Gamma$, is a polyhedron for which the $\Gamma$-space $E\Gamma$ is a terminal object in the category of free $\Gamma$-CW-complexes and homotopy classes of $\Gamma$-maps. $B\Gamma$ is therefore unique up to homotopy type.

In recent years topologists and others (e.g. Brown [5], Farrell and Jones [9], Lueck [10] and Serre [11]), have found great use for the more general notion of a classifying space of a discrete group $\Gamma$, relative to a family $\mathcal{F}$, of subgroups of $\Gamma$. This is a space $B_{\mathcal{F}}\Gamma = E_{\mathcal{F}}\Gamma/\Gamma$, where $E_{\mathcal{F}}\Gamma$ is a $\Gamma$-CW complex which is a terminal object in the category of homotopy classes of $\Gamma$-maps between $\Gamma$-CW-complexes whose isotropy groups are in $\mathcal{F}$.

According to Eilenberg and Ganea [7], if $\Gamma$ has cohomological dimension $\leq n$, then one can construct a model of $B\Gamma$ of dimension $\leq \max(n, 3)$. There are similar results showing $hdim(B_{\mathcal{F}_{\mathcal{IN}}}\Gamma) \leq \max(n, 3)$, in many cases (see
[6], [10]), but not all (see [4]), whenever $vcd(\Gamma) \leq n$. Here $\mathcal{FI}_N$ denotes the family of finite subgroups of $\Gamma$, and we write

$$hdim(Y) = \min \{dim(X) | X \text{ is a CW-complex homotopy equivalent to } Y\}$$

A remarkable paper of F.T. Farrell and L. Jones [9] introduces the family

$$\mathcal{VC} = \{H | H \text{ is a virtually cyclic subgroup of } \Gamma\}$$

(A group is virtually cyclic if it contains a cyclic subgroup of finite index). They conjecture there that the $K$ or $L$ theory of a group $\Gamma$ can be computed from the homology of $B_{\mathcal{VC}}\Gamma$, taken with stratified coefficients in the $K$ or $L$ theory of the virtually cyclic subgroups of $\Gamma$. The $K$-theoretic version of this conjecture has been proved, for many groups, by Bartels and Reich [2].

Farrell and Jones [9] also give two constructions of this space $B_{\mathcal{VC}}\Gamma$, one of which is a finite dimensional CW-complex when $\Gamma$ is a discrete subgroup of a Lie Group.

The goal of this note is to provide a geometrically simple construction of $B_{\mathcal{VC}}\Gamma$, as an $n + 1$-dimensional CW-complex, when $\Gamma$ is a crystallographic group of rank $n$ (see Section 3). We use this construction to prove the following result:

**Theorem 1.1.** Let $\Gamma$ be a crystallographic group of rank $n \geq 2$. Then:

$$hdim(B_{\mathcal{VC}}\Gamma) = n + 1$$

Note that a crystallographic group $\Gamma$ of rank 1 is virtually-cyclic. So in this case $hdim(B_{\mathcal{VC}}\Gamma) = 0$.

## 2 Basic Ideas.

A crystallographic group is a discrete co-compact subgroup of $\text{Iso}(\mathbb{R}^n)$, the group of isometries of $\mathbb{R}^n$. In this paper, the translation subgroup of a crystallographic group $\Gamma$ will be denoted

$$A = \Gamma \cap \text{Trans}(\mathbb{R}^n),$$
where $\text{Trans}(\mathbb{R}^n)$ denotes the subgroup of $\text{Iso}(\mathbb{R}^n)$ consisting of translations. $A$ is normal in $\Gamma$. The point group (or holonomy group) of $\Gamma$ will be written $G = \Gamma/A$.

The following theorem of Bieberbach [3] shows that $G$ is a finite group.

**Theorem 2.1.** Let $\Gamma$ be a crystallographic group of rank $n$. Then, $A$ is a finitely generated, free abelian group of rank $n$ with finite index in $\Gamma$.

A collection of subgroups $\mathcal{F}$ of a group $\Gamma$ is a called a family if $\mathcal{F}$ is closed under taking subgroups and under conjugation in $\Gamma$.

**Definition 2.2.** Let $\mathcal{F}$ be a family of subgroups of $\Gamma$. A $\Gamma$-space $E$ is called $\mathcal{F}$-universal if it satisfies the following conditions:

i.) $E^H$ is contractible. $\forall H \in \mathcal{F}$

ii.) $E^H = \emptyset, \forall H \notin \mathcal{F}$

iii.) $E$ is a $\Gamma$-$CW$-complex

where $E^H$ denotes the fixed set of a subgroup $H \subset \Gamma$ in $E$.

Then, we say $E/\Gamma$ is a classifying-space for $\Gamma$ relative to $\mathcal{F}$. By [10] this specifies $E/\Gamma$ uniquely up to homotopy type.

We will mainly be interested in the family:

$\mathcal{VC} = \{ H \subset \Gamma \mid H$ a virtually-cyclic subgroup $\}$.

Later, our model for a $\mathcal{VC}$-universal space of a crystallographic group $\Gamma$ will be denoted $E_{\mathcal{VC}}\Gamma$. The classifying space for $\Gamma$ relative to $\mathcal{VC}$ will be written $B_{\mathcal{VC}}\Gamma = E_{\mathcal{VC}}\Gamma/\Gamma$.

Finally, let $\mathcal{C} = \{ H \subset A \mid H$ is a maximal cyclic subgroup of $A \}$ be the set of maximal cyclic subgroups of $A$. For each subgroup $C \in \mathcal{C}$ we define:

$$\mathbb{R}^{n-1}(C) = \{ l \subset \mathbb{R}^n \mid l \text{ a line, } C \cdot l = l \}.$$  

This is the set of lines in $\mathbb{R}^n$ left invariant under the group action of $C$. The quotient map $\pi_C : \mathbb{R}^n \to \mathbb{R}^{n-1}(C)$ is the map:

$$\pi_C(x) = \text{ the unique line } l \in \mathbb{R}^{n-1}(C) \text{ containing } x \forall x \in \mathbb{R}^n.$$  

The Hausdorff metric on the non-empty closed sets of $\mathbb{R}^n$ restricts to a metric on $\mathbb{R}^{n-1}(C)$. In addition, the quotient topology on $\mathbb{R}^{n-1}(C)$ coincides with the metric topology on $\mathbb{R}^{n-1}(C)$. Note that $\mathbb{R}^{n-1}(C)$ is isometric to $\mathbb{R}^{n-1}$ in such a way that $\pi_C$ is a linear map.
3 Construction of $E_{\mathcal{V}\mathcal{C}}\Gamma$

Let $\Gamma$ be a crystallographic group of rank $n$, with holonomy group $G$. Our model for $B_{\mathcal{V}\mathcal{C}}\Gamma$ will be the orbit space of a $G$-action on an infinite union of solid $n+1$-tori sharing a common boundary. $E_{\mathcal{V}\mathcal{C}}\Gamma$ will be a similar union of mapping cylinders sharing a common source, one cylinder for each $C \in \mathcal{C}$.

Equip $\mathcal{C}$ with the discrete topology. Define an equivalence relation $\sim$ on $\mathbb{R}^n \times I \times \mathcal{C}$ as follows: $(x, t, C) \sim (x', t', C')$ if

\begin{enumerate}[(i.)]
\item $0 < t = t' < 1$, $C = C'$ and $x = x'$, or
\item $t = t' = 1$, $C = C'$ and $\pi_C(x) = \pi_C(x')$, or
\item $t = t' = 0$ and $x = x'$.
\end{enumerate}

We define:

$$E = (\mathbb{R}^n \times I \times \mathcal{C})/\sim \quad (3.1)$$

with the quotient topology. The equivalence class of an element $(x, t, C)$ is written $[x, t, C]$. For each $C \in \mathcal{C}$ we write

$$Cyl(\pi_C) = (\mathbb{R}^n \times I \times \{C\})/\sim.$$ 

Note this subspace of $E$ is just the mapping cylinder of $\pi_C : \mathbb{R}^n \to \mathbb{R}^{n-1}(C)$, and $E$ is the union of these subspaces $Cyl(\pi_C)$.

The action of $\Gamma$ on $E$ is defined by:

$$\gamma \cdot [x, t, C] = [\gamma \cdot x, t, \gamma C \gamma^{-1}] \quad \forall \gamma \in \Gamma. \quad (3.2)$$

In the following sections we prove:

Proposition 3.1. The $\Gamma$-space $E$ constructed in (3.1) and (3.2) is a $\mathcal{V}\mathcal{C}$-universal $\Gamma$-space (see Definition 2.2).

4 $E$ is a $\Gamma$-CW-Complex

In this section, we show that $E$ has the structure of a $\Gamma$-CW-complex. Choose any $\Gamma$-CW-complex structure on $\mathbb{R}^n$, called $X$, whose cells, $e \in Cell(X)$ are all convex polytopes. Here, $Cell(X)$ is the collection of cells of $X$. It is enough to show that for each $C \in \mathcal{C}$, this CW-structure extends
equivalently to a CW-structure on $Cyl(\pi_C)$. This would be obvious if $\pi_C: X \to \mathbb{R}^{n-1}(C)$ were cellular with respect to a CW-complex structure $Y$ on $\mathbb{R}^{n-1}(C)$, for we would then use the resulting CW-complex structure on $Cyl(\pi_C)$. It is nearly as obvious if there is a subdivision $X'$ of $X$ (that is, each cell of $X'$ will be a subset of a cell of $X$) relative to which $\pi_C: X' \to Y$ is cellular, since $X \times I$ can be subdivided by replacing only the cells of $X \times \{1\}$ with the cells of $X' \times \{1\}$ and leaving the other cells unchanged. This means that it suffices to show that $\mathbb{R}^{n-1}(C)$ has the structure of an $N_\Gamma(C)$-CW-complex so that $\pi_C$ is cellular relative to the subdivision $X'$ of $X$.

First note $\{\pi_C(e) \mid e \in \text{Cell}(X)\}$ is a locally finite collection in $\mathbb{R}^{n-1}(C)$, because $\text{Cell}(X)$ is a locally finite collection in $\mathbb{R}^n$. We note that a polytope generated by $n$ points is the convex hull of those $n$ points, and we choose $X$ so that $\text{Cell}(X)$ is generated by convex polytopes. For each $y \in \mathbb{R}^{n-1}(C)$ define:

$$f_y = \bigcap \{\pi_C(e) \mid y \text{ is contained in the interior of the convex polytope } \pi_C(e)\}.$$ 

This intersection is finite, so we have

$$\text{Int}(f_y) = \bigcap \{\text{Int}(\pi_C(e)) \mid y \in \text{Int}(\pi_C(e))\}.$$ 

Note $y \in \text{Int}(f_y)$ so each point of $\mathbb{R}^{n-1}(C)$ is in the interior of exactly one of the convex polytopes $f_y$. Also, $\{f_y \mid y \in \mathbb{R}^{n-1}(C)\}$ is a locally finite collection. To see that these form an $N_\Gamma(C)$-CW-complex we show that if $\dim(f_y) = d$, then $\hat{f}_y$ is a union of cells of dimension $< d$ (here, $\hat{f}_y$ denotes the boundary of $f_y$). Let $y' \in \hat{f}_y$. Then $y' \in \pi_C(e)$ for some $e$ such that $y \in \text{Int}(\pi_C(e))$. Note that $\pi_C(e)'$ is a subcomplex of $\pi_C(e)$. But $\dim(\pi_C(e)) = d$, so $\pi_C(e)'$ is a union of cells $\pi_C(e')$ of dimension $\leq d - 1$. Hence, there is a cell $\pi_C(e')$ with $y' \in \text{Int}(\pi_C(e'))$ such that $\pi_C(e') \subset \pi_C(e)'$. Therefore, $\hat{f}_y$ is a union of cells of dimension $< d$. This shows that $\text{Cell}(Y) := \{f_y \mid y \in \mathbb{R}^{n-1}(C)\}$ is the collection of cells of an $N_\Gamma(C)$-CW-complex structure on $\mathbb{R}^{n-1}(C)$ denoted $Y$. The cells of $X'$ are now easily defined. They are the nonempty sets of the form $e \cap \pi_C^{-1}(f)$ with $e \in \text{Cell}(X)$ and $f \in \text{Cell}(Y)$. Note that $e \cap \pi_C^{-1}(f)$ is a convex polytope contained in the cell $e$. Since $\pi_C$ takes each cell into a cell linearly, it follows that $\pi_C: X' \to Y$ is cellular. The above observations show that $E$ admits the structure of a $\Gamma$-CW-complex.
5 Contractibility of Fixed Sets

In this section we prove:

Lemma 5.1. Let $E$ be the $\Gamma$-space constructed in (3.1). For each subgroup $H \subset \Gamma$, $E^H$ is contractible if $H \in \mathcal{VC}$ and $E^H$ is empty if $H \notin \mathcal{VC}$.

Proof. Let $H$ be a subgroup of $\Gamma$. Then,

$$\left( \mathbb{R}^n \right)^H \begin{cases} \text{contractible} & \text{if } |H| < \infty \\ \emptyset & \text{if } |H| = \infty. \end{cases} \quad (5.1)$$

The $\Gamma$-action of (3.2) defines an action of $N_\Gamma(C)/C$ on $\mathbb{R}^{n-1}(C)$. This gives a map $j_C : N_\Gamma(C)/C \to \text{Iso}(\mathbb{R}^{n-1}(C))$. $\text{Ker}(j_C)$ has at most two elements. Recall $\mathbb{R}^{n-1}(C)$ is isometric to $\mathbb{R}^{n-1}$. The image of $j_C$ is a crystallographic group.

Assume $H$ is a subgroup of $N_\Gamma(C)$ for some $C \in \mathcal{C}$. By the above,

$$\left( \mathbb{R}^{n-1}(C) \right)^H \begin{cases} \text{contractible} & \text{if } |HC/C| < \infty \\ \emptyset & \text{if } |HC/C| = \infty. \end{cases} \quad (5.2)$$

Claim 5.2. Let $H \subset \Gamma$ satisfy $|H| = \infty$. Then, there exists at most one $C \in \mathcal{C}$ satisfying $E^H \cap \text{Cyl}(\pi_C) \neq \emptyset$.

Proof. Suppose there are two subgroups $C, C' \in \mathcal{C}$ satisfying $E^H \cap \text{Cyl}(\pi_C) \neq \emptyset$ and $E^H \cap \text{Cyl}(\pi_{C'}) \neq \emptyset$. By (3.2) this implies $H \subset N_\Gamma(C)$ and $H \subset N_\Gamma(C')$. Then, by (5.1) and (5.2) we know $|HC/C| < \infty$ and $|HC'/C'| < \infty$. Therefore, the relation $HC/C \cong H/(H \cap C)$ shows that $H \cap C$ and $H \cap C'$ have finite index in $H$. So $H \cap C \cap C'$ has finite index in $H$. But $H \cap C \cap C' = \{e\}$ and $|H| = \infty$, a contradiction. \hfill \Box

Claim 5.3. Let $H \subset \Gamma$ be an infinite subgroup. Then $E^H$ is contractible if $H \in \mathcal{VC}$ and $E^H$ is empty if $H \notin \mathcal{VC}$.

Proof. If $H \in \mathcal{VC}$ there exists precisely one $C \in \mathcal{C}$ satisfying $E^H \cap \text{Cyl}(\pi_C) \neq \emptyset$ (namely, the unique element of $\mathcal{C}$ containing $H \cap A$). For this $C$, (5.1) implies $E^H = \left( \mathbb{R}^{n-1}(C) \right)^H \times \{1\} \times \{C\}$. This space is contractible by (5.2). Conversely, suppose there exists a unique $C \in \mathcal{C}$ satisfying $E^H \cap \text{Cyl}(\pi_C) \neq \emptyset$. Then, by (5.1) and (5.2) we know $H \in \mathcal{VC}$ because $|H/(H \cap C)| < \infty$. Therefore, $E^H$ is contractible. Finally, if there exists no $C \in \mathcal{C}$ satisfying $E^H \cap \text{Cyl}(\pi_C) \neq \emptyset$ then $E^H = \emptyset$. \hfill \Box
Thus, for a subgroup $H \subset \Gamma$ the set $E^H$ is the following:

$$E^H = \begin{cases} (\mathbb{R}^n)^H \cup (\bigcup \{ Cyl(\pi^H_C) \mid H \subset N_T(C) \}) & |H| < \infty \\ (\mathbb{R}^{n-1}(C))^H & |H| = \infty, \ H \in \mathcal{VC}, \ H \subset N_T(C) \\ \emptyset & |H| = \infty, \ H \notin \mathcal{VC}. \end{cases}$$ (5.3)

If $H \in \mathcal{VC}$ and $H$ is infinite then $E^H$ is contractible by Claim 5.3. If $H$ is finite, then $\pi^H_C$ is a homotopy equivalence for each $C \in \mathcal{C}$ for which $H \subset N_T(C)$ (here, $\pi^H_C$ is the restriction of $\pi_C$ to $(\mathbb{R}^n)^H$). Therefore the subcomplex $(\mathbb{R}^n)^H$ is a strong deformation retract of the CW-complex $Cyl(\pi^H_C)$, and these deformations coalesce to make $(\mathbb{R}^n)^H$ a strong deformation retract of $E^H$. This and (5.1) show that $E^H$ is contractible.

**Proof. (of Proposition 3.1):** This is clear from Section 4 and Lemma 5.1. □

Henceforth we denote the $\mathcal{VC}$-universal $\Gamma$-space $E$ by:

$$E = E_{\mathcal{VC}} \Gamma$$

and

$$E/\Gamma = B_{\mathcal{VC}} \Gamma.$$

### 6 Geometric Properties of $B_{\mathcal{VC}} \Gamma$

Let $\Gamma$, $A$ and $G$ be as in Section 2. In this section we will show that $B_{\mathcal{VC}} \Gamma$ is the orbit space of a $G$-action on an infinite union of solid $n+1$-tori sharing a common boundary.

First we exhibit a homeomorphism

$$f: Cyl(\pi_C)/A \to D^2 \times T^{n-1}. \quad (6.1)$$

This will show $Cyl(\pi_C)/A$ is homeomorphic to the solid $n+1$-torus $D^2 \times T^{n-1}$ for each $C \in \mathcal{C}$. Choose a basis $\{a_1, ..., a_n\}$ for $A$ so that $a_1$ generates $C$. Set $v_i = a_i(0)$. Then $\{v_1, ..., v_n\}$ is a basis for $\mathbb{R}^n$. If $p = x_1v_1 + ... + x_nv_n$, we will write $p = <x_1, ..., x_n>$.

Let $f: Cyl(\pi_C)/A \to D^2 \times T^{n-1}$ be the map:

$$f([p, t, C]) = (1-t)e^{2\pi i x_1} \times e^{2\pi i x_2} \times \cdots \times e^{2\pi i x_n},$$
where $p = <x_1, ..., x_n>$ and $[p, t, C]$ denotes the orbit, in $Cyl(\pi_C)/A$, of a point $[p, t, C] \in Cyl(\pi_C)$. The map $f$ is continuous and bijective. It is a homeomorphism as $Cyl(\pi_C)/A$ is compact.

Because $E_{VC}\Gamma/A = \bigcup \{Cyl(\pi_c)/A \mid C \in C\}$ the space $E_{VC}\Gamma/A$ is homeomorphic to an infinite union of solid $n+1$-tori sharing the common boundary $\mathbb{R}^n/A$. Note that $\mathbb{R}^n/A$ is homeomorphic to $T^n$.

The relation $E_{VC}\Gamma/\Gamma \cong (E_{VC}\Gamma/A)/(\Gamma/A)$ proves

$$B_{VC}\Gamma = (E_{VC}\Gamma/A)/G.$$  

$B_{VC}\Gamma$ is therefore the orbit space of a $G$-action on a union of solid $n+1$-tori sharing a common boundary, one torus for each $C \in \mathcal{C}$.

7 Computation of $hdim(B_{VC}\Gamma)$

In this section we prove Theorem 1.1.

Proof. Let $i_C: Cyl(\pi_C)/A \to E_{VC}\Gamma/A$ be inclusion, $\tau: E_{VC}\Gamma/A \to B_{VC}\Gamma$ be the obvious projection and $\iota_C = \tau \circ i_C: Cyl(\pi_C)/A \to B_{VC}\Gamma$. We will make a computation in the $n+1$st homology group $H_{n+1}(B_{VC}\Gamma)$ using the cellular chain complex, $C_*(B_{VC}\Gamma)$. By (6.1), for each $C \in \mathcal{C}$ there exists an $n+1$ chain, $\chi_C \in C_{n+1}(Cyl(\pi_C)/A)$ such that $\partial_{n+1}(\chi_C) = [T_n]$ (where $\partial_{n+1}$ is the boundary map and $[T_n]$ is the fundamental cycle of $\mathbb{R}^n/A$). Choose $C, C' \in \mathcal{C}$ such that $C$ and $C'$ are not conjugate in $\Gamma$ (here is where we use that $\text{rank}(\Gamma) \geq 2$). Denote $\iota_{C\#}$ and $\iota_{C'\#}$ as the chain maps induced from $\iota_C$ and $\iota_{C'}$ respectively. Then, in $C_{n+1}(B_{VC}\Gamma)$:

$$z = \iota_{C\#}(\chi_C) - \iota_{C'\#}(\chi_{C'}) \neq 0.$$  

and $\partial_{n+1}(z) = 0$.

Indeed,

$$\partial_{n+1}(z) = \partial_{n+1}\iota_{C\#}(\chi_C) - \partial_{n+1}\iota_{C'\#}(\chi_{C'}) = \iota_{C\#}\partial_{n+1}(\chi_C) - \iota_{C'\#}\partial_{n+1}(\chi_{C'}) = \iota_{C\#}[T_n] - \iota_{C'\#}[T_n] = 0$$

as the restrictions of $\iota_C$ and $\iota_{C'}$ to $\mathbb{R}^n/A$ are the same. Since $C_{n+2}(B_{VC}\Gamma)$ is trivial, we conclude that $H_{n+1}(B_{VC}\Gamma)$ is nontrivial, implying that $hdim(B_{VC}\Gamma) \geq n+1$. But $B_{VC}\Gamma$ has dimension $n+1$. Therefore $hdim(B_{VC}\Gamma) = n+1$. □
8 Conclusion and an Open Question

In this paper, we were able to construct a model for both $E_{\mathcal{V}C}\Gamma$ and $B_{\mathcal{V}C}\Gamma$ for a rank-$n$ crystallographic group $\Gamma$. Despite the fairly straightforward constructions of our models, they are not locally finite CW-complexes. Therefore they can not be imbedded in any Euclidian space as they are not metrizable.

One question that is still unsolved is the following:

Conjecture 8.1. Suppose $\Gamma$ is a discrete subgroup of a Lie Group and $\Gamma$ is not virtually-cyclic. Then, $h\dim(B_{\mathcal{V}C}\Gamma) = vcd(\Gamma) + 1$.

We have shown this is so if $\Gamma$ is a crystallographic group, of rank $n \geq 2$. 
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