The covariant and on-shell statistics in $\kappa$-deformed spacetime

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It has been a long-standing issue to construct the statistics of identical particles in $\kappa$-deformed spacetime. In this letter, we investigate different ideas on this problem. Following the ideas of Young and Zegers, we obtain the covariant and on shell kappa two-particle state in $1+1 \ D$ in a simpler way. Finally, a procedure to get such state in higher dimension is proposed.

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I. INTRODUCTION

It is possible that geometry is noncommutative at planck scale because of the quantum gravitational effects. Noncommutative models, such as canonical and Lie algebra type noncommutativity, have been studied extensively. In particular, the $\kappa$-Minkowski spacetime [1] drew great interest due to its connection with double special relativity [2,3].

Recently, field theory in $\kappa$-Minkowski spacetime was extensively studied in [4-11] with some encouraging progress. [6,12] succeeded to obtain Noether charges associated with $\kappa$-Poincare symmetries of classical fields, and [4,7] generalized the results to the quantum level and got the finite vacuum energy. Besides, [13,14,15] suggested that the $\kappa$-Poincare symmetry might emerge in the low-energy limit of $1+2$ dimensional quantum gravity. However, the $\kappa$-field theory is difficult to be quantized because of its singular statistics. Although many investigations have been made on this problem [4,5,16-21], it is yet not able to construct a $\kappa$-multi-particle state that is both covariant and on shell. As emphasized in [17], to keep the notion of identical particles same in all frames, the covariant condition is necessary. It seems ridiculous that the bosons in one frame become fermions in another frame. The other condition “on-shell” is also necessary, otherwise as we will show in Sec.III, there will be infinite covariant solutions.

In this paper we prove that the twisted statistics is both covariant and on shell in the noncommutative spacetime constructed from twisted Poincare algebra, but it is often off shell in a more general noncommutative spacetime. We
get the covariant and on-shell $\kappa$-statistics in $1+1 \, D$ in a different way from [18]. Our method is simpler and applicable to all the $\kappa$-Poincare basis.

The paper is organized as follows. In Sec.II, after a brief review of the $\kappa$-Poincare Hopf algebra and twisted Poincare Hopf algebra, we build a frame to solve the statistics issue in $\kappa$-deformed spacetime. In Sec.III we compare and study previous ideas on this problem. In Sec.IV we investigate the covariant and on-shell statistics in $\kappa$-deformed spacetime in $1+1 \, D$. Finally we conclude the paper with some discussions.

II. PRELIMINARIES

A. $\kappa$-Poincare algebra

Symmetries in $\kappa$-Minkowski spacetime are described by $\kappa$-Poincare Hopf algebra $P_\kappa$, which is a deformation of usual Poincare Hopf algebra $P$. For example, the $\kappa$-statistics must be invariant under the action of $P_\kappa$. In this section, we first review the $\kappa$-Poincare Hopf algebra in the bicrossproduct basis [22], then show the difficulties to construct the corresponding $\kappa$-statistics.

$\kappa$-Poincare Hopf algebra is as follows,

(a) algebra sector

\[ [N_i, P_0] = iP_i \]
\[ [M_i, P_j] = i\epsilon_{ijk}P_k, \quad [M_i, P_0] = 0 \] (1)
\[ [M_i, M_j] = i\epsilon_{ijk}M_k \]
\[ [M_i, N_j] = i\epsilon_{ijk}N_k, \quad [N_i, N_j] = -i\epsilon_{ijk}M_k \] (2)
\[ [N_i, P_j] = i\delta_{ij} \left[ \frac{k}{2} \left( 1 - e^{-\frac{2\pi}{k}} \right) + \frac{1}{2k}P^2 \right] - \frac{i}{k}P_iP_j \] (3)

(b) coalgebra

\[ \Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \]
\[ \Delta M_i = 1 \otimes M_i + M_i \otimes 1 \] (4)
\[ \Delta P_i = P_i \otimes 1 + e^{-\frac{P_0}{2}} \otimes P_i \]  
(5)

\[ \Delta N_i = N_i \otimes 1 + e^{-\frac{P_0}{2}} \otimes N_i + \frac{1}{k} \epsilon_{ijk} P_j \otimes M_k \]  
(6)

(c) antipodes

\[ S(P_0) = -P_0, \quad S(P_i) = -P_i e^{\frac{P_0}{2k}} \]  
(7)

\[ S(M_i) = -M_i, \quad S(N_i) = -N_i e^{\frac{P_0}{2k}} + \frac{1}{k} \epsilon_{ijk} P_j e^{\frac{P_0}{2k}} M_k \]  
(8)

(d) co-units

\[ \epsilon(P_u) = \epsilon(M_j) = \epsilon(N_i) = 0 \]

where the Casimir operator \( C_k \) is

\[ C_k = (2k \sinh \frac{P_0}{2k})^2 - \frac{P^2 e^{P_0}}{2k} \]  
(9)

It is worth noting that the coproduct \( \Delta \) is deformed, which brings many new features in \( \kappa \)-Minkowski spacetime.

For example, the addition law becomes nonlinear and non-abelian:

\[ p \oplus k = (p_0 + k_0, \ p_i + k_i e^{-\frac{P_0}{2k}}) \]

\[ k \oplus p = (k_0 + p_0, \ k_i + p_i e^{-\frac{P_0}{2k}}) \]  
(10)

The oddness is due to the asymmetrical coproduct (5)

\[ P_i \triangleright (e^{ipx} \otimes e^{ikx}) \]

\[ = \Delta(P_i)(e^{ipx} \otimes e^{ikx}) \]

\[ = (P_i \triangleright e^{ipx}) \otimes \epsilon^{ikx} + (e^{-\frac{P_0}{2k}} \triangleright e^{ipx}) \otimes (P_i \triangleright e^{ikx}) \]

\[ = (p_i + k_i e^{-\frac{P_0}{2k}})(e^{ipx} \otimes e^{ikx}) \]  
(11)

While the asymmetrical coproduct is due to the noncommutative spacetime. Assume the common inner product<
\[ x_u, P_v = -i \eta_{uv}, \text{ one can obtain that,} \]

\[
< [x_i, x_0], P_k > \\
= < x_i \otimes x_0, \Delta(P_k) > - < x_0 \otimes x_i, \Delta(P_k) > \\
= < x_i, P_k > < x_0, e^{-i \frac{P}{k}} > - < x_0, 1 > < x_i, P_k > \\
= < i \frac{1}{k} x_i, P_k > \\
\Rightarrow [x_i, x_0] = i \frac{x_i}{k} \tag{12}
\]

As we expect, it is just the \( \kappa \)-Minkowski spacetime.

For the non-abelian addition law between momentum (10), the usual statistics

\[
|p > \otimes |q > \pm |q > \otimes |p > \tag{13}
\]

fail in \( \kappa \)-Minkowski spacetime. Because it is no longer the eigenstates of momentum. This is one of the difficulties which one encounters in obtaining the statistics in \( \kappa \)-deformed spacetime.

**B. Twisted Poincare algebra**

From the Poincare Hopf algebra, one can construct a new Hopf algebra if there exists a twist element \( F \in U(g) \otimes U(g) \), which satisfies the countital 2-cocyle condition.

\[
(F \otimes 1)(\Delta \otimes id)F = (1 \otimes F)(id \otimes \Delta)F \tag{14}
\]

\[
(\epsilon \otimes id)F = 1 = (id \otimes \epsilon)F \tag{15}
\]

It is known that \( F \) does not modify the counit and the algebra part, but changes the coproducts and the antipodes.

\[
\Delta_F(g) = F \Delta_0(g) F^{-1} \tag{16}
\]

\[
S_F(g) = U S_0(g) U^{-1} \tag{17}
\]

\[
U = \sum F(1) S(F_2) \tag{18}
\]

Let \( A \) be the algebra on which \( U(g) \) acts, then one can define the star product

\[
f \ast g = m_0(F^{-1} \triangleright (f \otimes g)) \tag{19}
\]
for \( f, g \in A \).

It is interesting that the canonical noncommutativity can be constructed from twisted Poincare Hopf algebra with the twist element

\[
F_\theta = \exp\left(\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta\right)
\]  

(20)

From (16)-(20), one can get:

\[
[x_u, x_v] = i \theta_{uv}
\]

\[
\Delta_\theta(P_u) = \Delta_0(P_u)
\]

\[
\Delta_\theta(M_{uv}) = \Delta_0(M_{uv}) - \frac{i}{2} \theta^{\alpha\beta} \left((\eta_{\alpha u} P_v - \eta_{\alpha v} P_u) \otimes P_\beta + (\alpha \leftrightarrow \beta)\right)
\]

(21)

It is necessary to point out that one is not able to derive the \( \kappa \)-Poincare algebra from twisted Poincare Hopf algebra. Instead, one need a larger algebra, for example, the Hopf algebra \( ifl(n, R) \) [23]. And the corresponding twist element is

\[
F_k = \exp\left[\frac{i}{2k} (P_0 \otimes D - D \otimes P_0)\right]
\]

(22)

where \( D = x_i P_i \) stands for dilation.

C. The frame to solve \( \kappa \)-Statistics

To get the covariant and on-shell statistics, we just need to construct a proper flip operator \( \tau \).

\[
\tau : | p > \otimes | q > \rightarrow | \bar{q} > \otimes | \bar{p} >
\]

(23)

Then the two-particle state becomes

\[
| p, q > = \frac{1}{\sqrt{2}} (1 \pm \tau) | p > \otimes | q >
\]

(24)

with \((+)\) for bosons (fermions).

Take the commutative case as an example, the proper flip operator \( \tau_0 \) must satisfy:

\[
[\tau_0, \Delta(g)] = 0
\]

(25)
\[ [\tau_0, C_0 \otimes 1] = [\tau_0, 1 \otimes C_0] = 0 \] (26)

\[ \tau_0^2 = 1 \] (27)

for \( \forall g \in P \). Condition (25) means that the two-particle state (24) is the momentum eigenstate and the statistics is covariant. \( C_0 \) is the Casimir operator, so condition (26) means that all the identical particles are on shell. While the requirement (27) leads to

\[ \tau \mid p, q > = \pm | p, q > \] (28)

Obviously, the flip operator \( \tau_0 \) that satisfies the above conditions is just the common exchange:

\[ \tau_0 \mid p > \otimes | q > = | q > \otimes | p > \] (29)

However, as is mentioned in Sec.II.A, \( \tau_0 \) is no longer the solution of \( \kappa \)-statistics since (13) is not the momentum eigenstate. So we have to search for a new \( \tau_\kappa \) over again with the similar conditions,

\[ [\tau_\kappa, \triangle_\kappa(g)] = 0 \] (30)

\[ [\tau_\kappa, C_\kappa \otimes 1] = [\tau_\kappa, 1 \otimes C_\kappa] = 0 \] (31)

\[ \tau_\kappa^2 = 1 \] (32)

Before we begin to seek for \( \tau_\kappa \), let us first simplify these conditions as much as possible. From the equations (1)(2) and identity \( \triangle(ab) = \triangle(a)\triangle(b) \), we can derive

\[ [\triangle(N_i), \triangle(N_j)] = -i\epsilon_{ijk}\triangle(M_k) \] (33)

\[ [\triangle(N_i), \triangle(P_0)] = i\triangle(P_i) \] (34)

So instead of the conditions (30) for all the generators of \( \kappa \)-Poincare algebra, we only need

\[ [\tau_k, \triangle_k(N_i)] = 0 \] (35)

\[ [\tau_k, \triangle_k(P_0)] = 0 \] (36)
Once we have found the solution of $\tau_k$, we can construct the $\kappa$-Fock space. However, in this paper we only focus on the two-particle Hilbert space:

$$\left| p, q \right> = \frac{1}{\sqrt{2}}(1 \pm \tau_k) \left| p \right> \otimes \left| q \right> \quad (37)$$

As to the whole Fock space, please refer to [18,19,21] for more details.

From (37), one can get the oscillator algebras in $\kappa$-Minkowski spacetime.

$$[a_p^+, a_q^+]_\kappa = a_p^+ a_q^+ - \tau_\kappa (a_p^+ a_q^+)$$

$$= a_p^+ a_q^+ - a_q^+ a_p^+ = 0 \quad (38)$$

The definition of $\tilde{q}$ and $\tilde{p}$ is as follows:

$$\tau_k \left| p \right> \otimes \left| q \right> = \left| \tilde{q} \right> \otimes \left| \tilde{p} \right> \quad (39)$$

As to the other algebra relations, we only need to replace $a_p$ with $a_S(p)^+$. Then we are able to calculate the Feynman propagator, Pauli-Jordan commutator function and so on.

III. VARIOUS IDEAS ABOUT $\kappa$-STATISTICS

A. Twisted Statistics in $\kappa$-Minkowski spacetime

Statistics in canonical noncommutative spacetime have been solved ideally by the twisted method. And this method is generalized to $\kappa$-Minkowski spacetime in [16]. We first review the main ideas and results in [16]. Then we prove that the twisted statistics in [16] is off shell and it is a general conclusion for the noncommutativity that can not be derived from twisted Poincare algebra.

Consider a system with symmetry group G in commutative spacetime. Let $\Lambda$ be an element of G which acts with some representation D. The action of G on the two-particle Hilbert space is described by the coproduct $\Delta_0$:

$$\Delta_0 : \lambda \rightarrow \lambda \otimes \lambda$$

$$f \otimes g \rightarrow (D \otimes D)\Delta_0(\lambda)f \otimes g \quad (40)$$

To be consistent with the usual multiplication map $m_0$, the coproduct must satisfy the condition

$$m_0((D \otimes D)\Delta_0(\lambda)f \otimes g) = D(\lambda)m_0(f \otimes g) \quad (41)$$
Because the multiplication map $m_0$ is abelian, the coproduct $\Delta_0$ must be symmetrical. Thus to satisfy the covariant condition

$$[\Delta_0, \tau_0] = 0 \quad (42)$$

one only need to define $\tau_0$ as the usual exchange:

$$\tau_0(f \otimes g) = g \otimes f \quad (43)$$

And one can easily check that the conditions (25)-(27) are satisfied automatically.

In the noncommutative spacetime, the non-abelian star product is defined with the twist element $F$ as

$$f * g = m_0(F f \otimes g) = m_k(f \otimes g), m_k = m_0 F \quad (44)$$

For example, for any $\varphi$ ordering [24] in $\alpha$-Minkowski spacetime, the corresponding twist element $F_\varphi$ is

$$F_\varphi = \exp(N_x[\ln(\phi(A_x + A_y) - \ln(\phi(A_x))] + (x \leftrightarrow y)) \quad (45)$$

where $N_x = x_i \partial / \partial x_i$ and similarly for $N_y, A_x = ia \partial / \partial x_i$ and similarly for $A_y$ [16].

For the non-abelian multiplication $m_\varphi = m_0 F_\varphi$, one has to redefine the coproduct $\Delta_\varphi$ as

$$\Delta_\varphi = F^{-1}_\varphi \Delta_0 F_\varphi \quad (46)$$

in order to satisfy the equation

$$m_\varphi[(D \otimes D)\Delta_\varphi(\lambda)f \otimes g] = D(\lambda)m_\varphi(f \otimes g) \quad (47)$$

However, it turns out that the flip operator $\tau_0$ does not commute with the twisted coproduct. One must find a new flip operator $\tau_\varphi$ to be consistent with the covariant condition.

$$[\Delta_\varphi, \tau_\varphi] = 0 \quad (48)$$

The simplest solution of $\tau_\varphi$ is

$$\tau_\varphi = F^{-1}_\varphi \tau_0 F_\varphi \quad (49)$$

From (45)-(49), the authors of [16] found that for the particular class of $\varphi$ realizations, the twisted flip operator is independent of the choice of ordering

$$\tau_\varphi = \exp[i(x_i P_i \otimes A - A \otimes x_i P_i)]\tau_0 \quad (50)$$
Now let us do some discussions about the main result (50) in [16]. First of all, one may note that (49) is a sufficient but not a necessary condition of (48). In fact, for any non-degenerate operator \( F \) which satisfies the following condition

\[
[\triangle \varphi, F] = F
\]

(51)

\[
\tau'_\varphi = F^{-1}\tau_\varphi F
\]

(52)

\( \tau'_\varphi \) is also one of the solutions of (48). One can not simply choose (49).

Secondly, we find that the flip operator (50) is off shell.

Proof:

\[
N = x_i \partial_i, \quad [\partial_i, x_j] = \delta^i_j, \quad [A, N] = 0
\]

(53)

The symbol \( : \) is the normal ordering with all \( x_i \) coming to the left of all \( \partial_i \). According to the appendix of [24], we have

\[
:e^N(e^A-1): = e^{NA}
\]

(54)

so

\[
\tau_\varphi |p > \otimes |k > = e^{iN\otimes A}\tau_0 e^{-iN\otimes A}|p > \otimes |k > = e^{iN\otimes A}\tau_0 :e^{N(e^{-1}\otimes A)-1} :|p > \otimes |k > = e^{iN\otimes A}\tau_0 |p_0, p_i e^{-ak_0} > \otimes |k > = e^{iN\otimes A}|k > \otimes |p_0, p_i e^{-ak_0} > = |k_0, k_i e^{ap_0} > \otimes |p_0, k_i e^{-ak_0} >
\]

(55)

In the above calculations we use \( a = 1/\kappa \) and the equation

\[
e^{-iA\otimes N}\tau_0 = \tau_0 e^{-iN\otimes A}
\]

(56)

Because \((k_0, k_i)\) and \((p_0, p_i)\) are both on shell, and \( k \) and \( p \) are irrelevant, so \((k_0, k_i e^{ap_0})\) and \((p_0, p_i e^{-ak_0})\) must be both off shell. Now the proof is completed.

Last but not least, let us seek for the deep reason why the twisted statistics succeed in canonical noncommutative spacetime but fail in \( \kappa \)-Minkowski spacetime. As is mentioned in section II.C, the key lies in the fact that the canonical noncommutativity can be constructed from twisted Poincare Hopf algebra but \( \kappa \)-Minkowski spacetime can not.
For the canonical case, $C_\theta(P) = C_0(P) = P_\mu P^\mu$

$$F_\theta = \exp \frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta$$

$$[C_\theta(P) \otimes 1, F_\theta] = [1 \otimes C_\theta(P), F_\theta] = 0$$

(57)

so it is easy to derive the on-shell condition

$$[C_\theta(P) \otimes 1, \tau_\theta] = [1 \otimes C_\theta(P), \tau_\theta] = 0$$

(58)

It implies that for all kinds of noncommutativities which can be constructed from twisted Poincare algebra, the twisted statistics is both covariant and on shell. That is because

$$C_F(P) = C_0(P) = P_\mu P^\mu$$

$$[C_F(P) \otimes 1, F] = [1 \otimes C_F(P), F] = 0$$

(59)

As to the case in $\kappa$-Minkowski spacetime, the twisted element $F_k$ of a bigger algebra must contain the dilation operator $D$. Notice that

$$\forall f(P) \neq 0, [f(P), D] = \frac{\partial f(P)}{\partial P_\mu} P_\mu \neq 0$$

(60)

so the twisted statistics must be off shell.

$$[C_k(P) \otimes 1, F_k] \neq 0$$

$$[C_k(P) \otimes 1, \tau_k] \neq 0$$

(61)

It implies that for the most general kinds of noncommutativities which can not be derived from the twisted Poincare algebra, the twisted statistics will fail if the twisted element $F$ contains generator that dose not commutate with $C(P)$.

B. Rainbow Statistics

In a recent paper [21], the authors named the $\kappa$-statistics an romantic name—"Rainbow statistics." In [21], they treat the massive and massless case differently. For massive fields they got the covariant but off-shell $\kappa$-statistics by the twisted method, while for massless fields [4,21] they got the on-shell statistics using the $\tau_k$, 

$$\tau_\kappa |p> \otimes |q> = |q> \otimes |\bar{p}>$$
\[ \vec{q} = (\vec{q}_0, q_i e^{-\frac{p_0}{\kappa}}), \vec{p} = (\vec{p}_0, p_i e^{\frac{q_0}{\kappa}}) \]

\[ \vec{q}_0 = -\kappa n(1 - \frac{|\vec{q}|}{\kappa} e^{-\frac{2p_0}{\kappa}}), \vec{p}_0 = -\kappa n(1 - \frac{|\vec{p}|}{\kappa} e^{\frac{2q_0}{\kappa}}) \] (62)

where \( \vec{q} \) and \( \vec{p} \) are both on shell, and the Casimir operator is (9). We can easily check that

\[ \tau_k^2 = 1, [\tau_k, \triangle(P_{\mu})] = 0 \] (63)

But we find that the \( \kappa \)-statistics constructed from \( \tau_k \) (62) is not covariant. Now we give a simple proof. As discussed in Section II.B, we only need to check if \( \triangle(N_i) \) commute with \( \tau_k \). For simplicity, we consider an infinitesimal Lorentz transformation in \( 1 + 1 \) \( D \).

\[ 1 + i\alpha N_1, \ \alpha \to 0 \]

with algebra sector and coproduct as,

\[ [P_0, N_1] = -i P_1, [P_1, N_1] = -i \kappa^2 (1 - e^{-\frac{2p_0}{\kappa}}) - \frac{1}{2\kappa} P_1^2 \]

\[ \triangle N_1 = N_1 \otimes 1 + e^{-\frac{p_0}{\kappa}} \otimes N_1 \] (64)

Following the method in [17], we get

\[ (1 \otimes 1 + \alpha \triangle N_1)[p_0, p_1] > \otimes |q_0, q_1 > \]

\[ = |p_0 - \alpha p_1, p_1 - \alpha \kappa^2 (1 - e^{-\frac{2p_0}{\kappa}}) - \frac{1}{2\kappa} p_1 p_1)| > \]

\[ \otimes |q_0 - \alpha e^{-\frac{p_0}{\kappa}} q_1, q_1 - \alpha e^{-\frac{p_0}{\kappa}} \kappa^2 (1 - e^{-\frac{2p_0}{\kappa}}) - \frac{1}{2\kappa} q_1 q_1| > \] (65)

Define

\[ |Q_0, Q_1 > \otimes |P_0, P_1 > \]

\[ = (1 \otimes 1 + \alpha \triangle N_1) \tau_k |p_0, p_1 > \otimes |q_0, q_1 > \]

\[ = \tau_k (1 \otimes 1 + \alpha \triangle N_1) |p_0, p_1 > \otimes |q_0, q_1 > \] (66)
then after complicated calculations, we get

\[
Q_1 - \overline{Q_1} = q_1 e^{-\frac{p_0}{k}} - \alpha|q_1| e^{-\frac{2p_0}{k}} - q_1 e^{-\frac{2p_0}{k}} + O(\alpha^2)
\]

\[
- \alpha \frac{|q_1||p_1|}{k} e^{-\frac{p_0}{k}} + \frac{\alpha k}{2} e^{-\frac{2p_0}{k}} (1 - e^{-\frac{2p_0}{k}}) - \frac{\alpha q_1^2}{2k} e^{-\frac{2p_0}{k}}
\]

\[
= -\alpha \left( \frac{|q_1||p_1|}{k} + \frac{q_1p_1}{k} + \frac{p_1^2|q_1|}{k^2} + \frac{p_1|p_1||q_1|}{k^2} \right) + O(\alpha^2)
\]

(67)

Obviously, when \(q_1p_1 > 0\), \(Q_1 - \overline{Q_1} \neq 0\). Only in the case \(q_1p_1 < 0\), \(Q_1 - \overline{Q_1} = 0\). So the massless Rainbow Statistics in [4, 21] is not covariant.

**C. Other \(\kappa\)-Statistics**

There are also some other attempts to construct \(\kappa\)-statistics in \(\kappa\)-deformed spacetime, now we only give a brief introduction. In [5, 20], the authors introduced a new \(\kappa\)-star product and the corresponding oscillator algebra in order to get the full Fock space of \(\kappa\)-quantum field theory. It is interesting that the classical four-momentum conservation law is satisfied in their scenario. However, they modified the on-shell conditions, and did not consider the covariant conditions. In [17], the authors succeeded to get the covariant and on-shell \(\kappa\)-statistics to the third order in \(1/\kappa\). Then in the following work [18], they obtained the exact solution in \(1+1\) \(D\). For more details of these papers, one can refer to the references [4, 21, 5, 10, 17, 18].

**IV. THE COVARIANT AND ON-SHELL \(\kappa\)-STATISTICS IN 1+1 \(D\)**

It is argued in [18] that the covariant and on-shell \(\kappa\)-statistics in \(1+1\) \(D\) has been found. It is also argued that for the case of two-particle state, their realization is unique. However, the elliptic functions are contained in their result. In this section, we solve the same problem in a quite different way, our method is simpler and our result contains only elementary functions.

As is emphasized in Section II.C, our goal is to find a proper \(\tau_k\) which satisfies the equations:

\[
\tau_k|p_0, p_1 > \otimes |q_0, q_1 > \rightarrow |\tilde{q}_0, \tilde{q}_1 > \otimes |\tilde{p}_0, \tilde{p}_1 >
\]

\[
[\tau_k, \triangle(N_1)] = 0, \quad [\tau_k, \triangle(p_0)] = 0
\]
\[ [C_k \otimes 1, \tau_k] = [1 \otimes C_k, \tau_k] = 0 \]

\[
\tau_k^2 = 1
\]  \hspace{1cm} (68)

As we will show later, the condition \( \tau_k^2 = 1 \) is satisfied automatically. Now there are four unknown quantities with four independent equations. So the solution of \( \tau_k \) in \( 1 + 1 \) is unique.

In view of the uniqueness, then we can replace the condition \([\tau_k, \triangle(N_1)] = 0\) with \([\tau_k, \triangle(p_1)] = 0\). Now the equations (68) become

\[
p_0 + q_0 = \tilde{q}_0 + \tilde{p}_0
\]

\[
p_1 + q_1 e^{-\frac{p_0}{\kappa}} = \tilde{q}_1 + \tilde{p}_1 e^{-\frac{\tilde{q}_0}{\kappa}}
\]

\[
C_k(q_0, q_1) = C_k(p_0, p_1) = C_k(\tilde{q}_0, \tilde{q}_1) = C_k(\tilde{p}_0, \tilde{p}_1) = m^2
\]  \hspace{1cm} (69)

where \( C_k \) is the Casimir operator (9). Simplify the above equations, one can easily get the relations between \( \tilde{p}_1 \) and \( \tilde{q}_1 \):

\[
p_1 + q_1(M - \sqrt{M^2 - 1 + \frac{p_1^2}{\kappa^2}})
\]

\[= \tilde{q}_1 + \tilde{p}_1(M - \sqrt{M^2 - 1 + \frac{1}{\kappa^2}})\]  \hspace{1cm} (70)

\[
(M - \sqrt{M^2 - 1 + \frac{p_1^2}{\kappa^2}})(M - \sqrt{M^2 - 1 + \tilde{q}_1^2})
\]

\[= (M - \sqrt{M^2 - 1 + \tilde{q}_1^2})(M - \sqrt{M^2 - 1 + \frac{1}{\kappa^2}})\]  \hspace{1cm} (71)

where \( M = 1 + m^2/2\kappa^2 \). Obviously, one solution of (70)(71) is

\[
\tilde{q}_1 = p_1, \quad \tilde{p}_1 = q_1
\]  \hspace{1cm} (72)

However, it is not the solution we want. Eliminating \( \tilde{p}_1 \) in (70)(71), we find that the equation of \( \tilde{q}_1 \) is just an quadratic equation. Having known one of the solution, we can easily get the other one:

\[
\tilde{q}_1 = (b - p_1) - \frac{b(a^2 - 1)(M^2 - 1)}{M^2(a - 1)^2 - \frac{p_0^2}{\kappa^2}}
\]

\[= q_1 e^{-\frac{p_0}{\kappa}} - \frac{b(a^2 - 1)(M^2 - 1)}{M^2(a - 1)^2 - \frac{p_0^2}{\kappa^2}}\]

\[
\tilde{p}_1 = (b - \tilde{q}_1) e^{\frac{\tilde{q}_0}{\kappa}}
\]  \hspace{1cm} (73)
with

\[ b = p_1 + q_1(M - \sqrt{M^2 - 1 + \frac{p_1^2}{\kappa^2}}) \]

\[ a = (M - \sqrt{M^2 - 1 + \frac{p_1^2}{\kappa^2}})(M - \sqrt{M^2 - 1 + \frac{q_1^2}{\kappa^2}}) \]

Now let us do some discussions about the above results.

First of all, it is worth noting that the flip operator \( \tau_k \) is just the exchange between the two solutions (72) and (73) of the equations (70)(71). So the condition \( \tau_k^2 = 1 \) comes to be true automatically.

Secondly, the \( \tau_k \) we get in (73) must be covariant. In general, the equations (69) are only the necessary conditions of (68). But in our case, there is only one solution which satisfy (68). So the conditions (68) and (69) must be equivalent to each other.

Finally, notice that the first part of \( \tilde{q}_1 \) in (73) is just the result got in [4,21] for massless fields. In the massless limit \( (M \to 1) \), our result becomes

\[ \tilde{q}_1 = q_1 e^{-\frac{p_0}{\kappa}} = q_1 (1 - \frac{|p_1|}{\kappa}), \quad p_1 q_1 < 0 \] (74)

\[ \tilde{q}_1 = q_1 (1 - \frac{|p_1|}{\kappa}) - \frac{c}{d}, \quad p_1 q_1 > 0 \]

where

\[ \begin{align*}
  c &= (p_1 + q_1 - \frac{q_1 |p_1|}{\kappa})(2 - \frac{|p_1|}{\kappa} - \frac{|q_1|}{\kappa} + \frac{|p_1 q_1|}{\kappa^2}) \\
  d &= (2 - \frac{2 |p_1|}{\kappa} - \frac{|q_1|}{|q_1|} - \frac{|p_1|}{|q_1|} + \frac{|q_1 p_1|}{q_1 \kappa^2})
\end{align*} \] (75)

For the case (74), we have proved in Section III.B, it is covariant. While for the other case (75), following the same program, after complex calculation we find exactly that it is indeed covariant too.

V. SUMMARY AND OUTLOOK

In this paper, we have investigated the statistics problem in \( \kappa \)-Minkowski spacetime. Focusing on the two-particle state, we have obtained the covariant and on-shell \( \kappa \) two-particle states in \( 1+1 \) \( D \) using a simpler method than [18]. Our result contains only elementary functions and applicable to all the \( \kappa \)-Poincare basis. Now, we suggest a scheme to construct the covariant and on-shell \( \kappa \)-two particle states in higher dimension. Notice that one can rotate an arbitrary \( \kappa \)-two particle state in \( (1+n) \) \( D \) into \( (1+1) \) \( D \) . One idea quickly comes to mind. One may first rotate the
κ two-particle state in \((1 + n) \, D\) into \((1 + 1) \, D\), then after the action of \(\tau_k^{(1)}\) which have been obtained, one rotate it back into \((1 + n) \, D\). Whether this method is workable and how to carry out the scheme explicitly are open problems.

We end the paper with some problems. Now there are too many different kinds of noncommutativity, but there is only one world. So we must search for some principle to restrict the possible form of noncommutativity. Just like in the field theory, the gauge invariance can largely restrain the form of interaction. Of course, which kind of noncommutativity is right depends on the experiments. But if noncommutativity is a correct and also beautiful theory, it must be able to give some predictions in theory. Besides, there may exist duality between different kinds of noncommutativity. Whether it does exist and how to seek such duality is also an open problem. Hopefully that some insights can emerge in the future.

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