Split Edge Geodetic Domination Number of a Graph

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Abstract: In this paper, we introduce a new graph theoretic parameter, split edge
g eo de tic domination number of a connected graph as follows. A set \(S \subseteq V(G)\) is
said to be a split edge geodetic dominating set of \(G\) if \(S\) is both a split edge geodetic
set and a dominating set of \(G\) ( \(< V-S >\) is disconnected). The minimum
cardinality of the split edge geodetic dominating set of \(G\) is called split edge geodetic
domination number of \(G\) and is denoted by \(\gamma_{sgs}(G)\). It is shown that for any 3 positive integers \(m, f\) and \(n\) with \(2 \leq m \leq f \leq n-2\), there exists a
connected graph \(G\) of order \(n\) such that \(g_1(G) = m\) and \(\gamma_{sgs}(G) = f\). For every pair
\(l, n\) of integers with \(2 \leq l \leq n-2\), there exists a connected graph \(G\) of order \(n\)
such that \(\gamma_{sgs}(G) = l\).

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1 Introduction

We consider undirected finite graph without loops and multiple edges. The graph considered here
have at least one component which is not complete ,wheel and star graph. The order and size of \(G\) are
denoted by \(n\) and \(m\) respectively. For basic graph theoretic terminology we refer to Harary [6]. Let \(v\) be a
point in \(V\). The open neighborhood of \(v\) is the set \(N(v)\) consisting of all points \(u\) which are adjacent with
\(v\) and \(N[v] = N(v) \cup \{v\}\) is the closed neighborhood of a point \(v\) in \(G\). The distance \(d(u, v)\) between
two points \(u\) and \(v\) in a connected graph \(G\) is the length of a shortest \(u-v\) path in \(G\). An \(u-v\) path of
length \(d(u, v)\) is called an \(u-v\) geodesic. The closed interval \([x, y]\) consists of all points lying on some
\(x-y\) geodesic of \(G\) and for a non empty subset \(S\) of \(V(G)\), \(I(S) = \bigcup_{x,y \in S} [x,y]\). An edge geodetic
cover of \(G\) is a set \(S \subseteq V(G)\) such that every edge of \(G\) is contained in a geodesic joining some pair of
points in \(S\). The edge geodetic number \(g_1(G)\) of \(G\) is the minimum order of its edge geodetic covers and
any edge geodetic cover of order \(g_1(G)\) is an edge geodetic basis of \(G\) [7].

A point \(v\) is a simplicial point of a graph \(G\) if \(< N(v) \rangle\) is complete. A point in a graph \(G\)
dominate s itself and its neighbours. A set of points \(D\) in a graph \(G\) is a dominating set if each point of \(G\) is
dominated by some point of \(D\). The domination number \(\gamma(G)\) of \(G\) is the minimum cardinality of a
dominating set of \(G\). An edge geodetic set \(S \subseteq V\) is said to be a split edge geodetic set if \(< V-S >\) is
disconnected. The minimum cardinality of a split edge geodetic set of \(G\) is the split edge geodetic number
and is denoted by \(g_{1s}(G)\) [9].
The following theorems are used in the sequel.

Theorem 1.1. [1] For cycle \(C_n\) of order \(n \geq 6\), \(\gamma_{sgs}(C_n) = \lceil n/3 \rceil\).

Theorem 1.2. [1] For any two integers \(m, n \geq 2\), \(\gamma_{sgs}(K_{mn}) = \min\{m, n\}\).
Theorem 1.3. [1] For path $P_n, n \geq 5$, $\gamma_{gs}(P_n) = \left\lceil \frac{n^2}{3} \right\rceil$.

Theorem 1.4. [7] Each extreme point of $G$ belongs to every edge geodetic cover of $G$. In particular, each end point of $G$ belongs to every edge geodetic cover of $G$.

2 Split Edge Geodetic Domination Number of a Graph

Definition 2.1. A set $S \subseteq V(G)$ is said to be a split edge geodetic dominating set of $G$ if $S$ is both a split edge geodetic set and a dominating set of $G$ ($< V - S >$ is disconnected). The minimum cardinality of the split edge geodetic dominating set of $G$ is called the split edge geodetic domination number of $G$ and is denoted by $\gamma_{sge}(G)$.

Example 2.2. For the graph $G$ given in Figure 1, $S = \{1, 2, 3, 4, 5, 7, 9\}$ is a minimum split edge geodetic dominating set of $G$ so that $\gamma_{sge}(G) = 7$.

![Figure 1: Graph $G$.](image)

Remark 2.3. For the graph $G$ given in Figure 1, $S_1 = \{1, 7, 9\}$ is a split geodetic dominating set of $G$ so that $\gamma_{sge}(G) = 3$. Thus the split geodetic domination number and the split edge geodetic domination number are different.

Theorem 2.4. For any connected graph of order $n$ ($n \geq 4$), $2 \leq \gamma_{sge}(G) \leq n - 2$.

Proof. A split edge geodetic dominating set needs at least two points and so $\gamma_{sge} \geq 2$. Suppose that $\gamma_{sge}(G) \geq n - 1$. Consider $\gamma_{sge}(G) = n - 1$. Let $v$ be a point of $G$ and let $S = V(G) - \{v\}$ be a split edge geodetic dominating set of $G$. Then $< V - S >$ is connected, which is a contradiction to our Definition 2.1. Therefore, $\gamma_{sge}(G) \leq n - 2$.

Theorem 2.5. Each extreme point belongs to every split edge geodetic dominating set of $G$. Moreover each end point of $G$ belongs to every split edge geodetic dominating set of $G$.

Proof. Since every split edge geodetic dominating set is a edge geodetic set, the result follows from Theorem 1.4.

Theorem 2.6. For any connected graph $G$ with $k$ extreme points, $\max\{2, k\} \leq \gamma_{sge}(G) \leq n - 2$.

Proof. This follows from Theorem 2.4. and 2.5.

Theorem 2.7. Let $G$ be a connected graph of order $n$, then there is no graph with $\gamma_{sge}(G) = n - 1$. 


Proof. Assume that $\gamma_{1 gs}(G) = n - 1$. Now consider a point $v$, which is different from split edge geodetic dominating set. By our assumption $S = V(G) - \{v\}$ is a split edge geodetic dominating set of $G$. Then $< V - S >$ gives an isolated point and $< V - S >$ is connected, which is ontradiction to our Definition 2.1. Thus there is no graph with $\gamma_{1 gs}(G) = n - 1$.

**Theorem 2.8.** If $\gamma_{1 gs}(G) = 2$ iff $G = C_4$.

Proof. Suppose, $\gamma_{1 gs}(G) = 2$ then obviously $G = C_4$. Conversely let $G = C_4$. Let $V(C_4) = \{s, t, u, v\}$. Suppose $\gamma_{1 gs}(G) = 3$. Then $V(C_4) - \{s\}$ is not a split edge geodetic dominating set because $< V - S >$ is connected. Therefore $\gamma_{1 gs}(C_4) = 2$.

**Theorem 2.9.** For cycle $C_n$ of order $n \geq 6$, $\gamma_{1 gs}(C_n) = \lceil \frac{n}{3} \rceil$.

Proof. The proof is obvious from Theorem 1.1.

**Theorem 2.10.** For any two integers $m, n \geq 2$, $\gamma_{1 gs}(K_{m,n}) = \min\{m, n\}$.

Proof. The proof is obvious from Theorem 1.2.

**Theorem 2.11.** For path $P_n$, $n \geq 5$, $\gamma_{1 gs}(P_n) = \lceil \frac{n+2}{3} \rceil$.

Proof. The proof is obvious from Theorem 1.3.

**Theorem 2.12.** If tree $T$ has $k$ end points with at least 3 internal points, then $\gamma_{1 gs}(G) > k$.

Proof. Let $S$ be the set of all end points of $T$ which is the minimum edge geodetic set of $T$. But $S$ is not a split edge geodetic dominating set. Because $< V - S >$ is connected which is a contradiction to our Definition 2.1. Then $S$ should have at least one internal point more than $k$ end points to get split edge geodetic dominating set. Thus $\gamma_{1 gs}(G) > k$.

**Definition 2.13.**[3] The crown graph $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

**Theorem 2.14.** Let $G = H_{n,n}$, be a crown graph of order $n \geq 4$, then $\gamma_{1 gs}(G) = n$.

Proof. For $n \geq 4$: Figure 2: Graph $G$. 

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Clearly from Figure 2, $S = \{h_1, l_1\}$ is the minimum edge geodetic and dominating set of $G$. Clearly $I[S] = V [G]$ and $S$ dominates all the points of $G$. But $<V - S>$ is connected. Therefore $S$ is not a split edge geodetic dominating set of $G$. Hence, the set $S_1 = \{h_1, h_2, \ldots, h_n\}$ or $\{l_1, l_2, \ldots, l_n\}$ is the minimum split edge geodetic dominating set. Thus, $\gamma_{1gs}(G) = n$.

Definition 2.15. [8] The triangular snake graph $T_n$ is obtained from a path $P_n$ by joining $j_i$ and $j_i+1$ to a new point $g_i$.

Theorem 2.16. Let $G = T_n$ be a triangular snake graph with $(n > 4)$, then $\gamma_{1gs}(G) = n + 2$.

Proof. Let $V = \{h_1, h_2, h_3, \ldots, h_{n-2}, h_{n-1}, j_1, j_2, \ldots, j_n\}$ be the points of triangular snake graph.

The set $S = \{j_1, h_1, h_2, \ldots, h_{n-1}, j_n\}$ is the minimum edge geodetic and dominating set of $G$ and it is clear that $S$ is not a split edge geodetic dominating set. Let $S' = S \cup \{j_k\}$ ,where $3 \leq k \leq n - 2$. Then $<V - S'>$ is disconnected. Thus $S'$ is the minimum split edge geodetic dominating set of $G$. Therefore, $\gamma_{1gs}(G) = n + 2$.

3 Realization Results

In this section, we give the realization results for the split edge geodetic domination number of a graph $G$.

Theorem 3.1 For any 3 positive integers $m, f$ and $n$ with $2 \leq m \leq f \leq n - 2$, there exists a connected graph $G$ of order $n$ such that $g_1(G) = m$ and $\gamma_{1gs}(G) = f$.

Proof. Let $P_{3(f-m+1)+1} : h_0, h_1, \ldots, h_{3(f-m+1)}$ be a path of order $3(f - m + 1) + 1$. Let $H$ be a graph obtained from $P_{3(f-m+1)+1}$ by adding $m - 2$ new points $d_1, d_2, \ldots, d_{m-2}$ and join each $d_i$ $(1 \leq i \leq m - 2)$ to $h_1$. Let $G$ be a graph obtained from $H$ by adding $n - 3f + 2m - 2$ new points $a_1, a_2, \ldots, a_{n-3f+2m-2}$ and join each $a_i$ $(1 \leq i \leq n - 3f + 2m - 2)$ to both $h_1$ and $h_3$. The resulting graph $G$ is given in Figure 4.
The set $S = \{h_0, d_1, d_2, \ldots, d_{m-2}, h_{3(f-m+1)}\}$ is the set of all end points of $G$. By Theorem 1.4, every edge geodetic set contains $S$. Now, it is easily seen that $S$ is a minimum edge geodetic set of $G$ so that $g_1(G) = m$. Clearly $S$ is not a split edge geodetic dominating set of $G$. Let $S_1 = S \cup \{h_3, h_6, \ldots, h_{3(f-m)}\}$. It is clear that $S_1$ is the minimum split edge geodetic dominating set of $G$, so that $\gamma_{1gs}(G) = f$.

Theorem 3.2. For every pair $l, n$ of integers with $2 \leq l \leq n - 2$, there exists a connected graph $G$ of order $n$ such that $\gamma_{1gs}(G) = l$.

Proof. Let $C_4 : o, p, q, r, o$ be a cycle of order 4. Let $H_1$ be the graph obtained from $C_4$ by adding $n - l - 2$ new points $w_1, w_2, \ldots, w_{n-l-2}$ and join each $w_i$ $(1 \leq i \leq n - l - 2)$ with $o$ and $q$. Let $H_2$ be a copy of $K_{1,l-2}$ with $l - 2$ leaves $d_1, d_2, \ldots, d_{l-2}$ and the support point $y$. Let $G$ be the graph obtained from $H_1$ and $H_2$ by adding an edge $yo$. The resulting graph $G$ is given in Figure 5.
The set \( S = \{d_1, d_2, \ldots, d_{t-2}\} \) is the set of all extreme points of \( G \). By Theorem 2.5, every split edge geodetic dominating set contains \( S \). But \( S \) is not an edge geodetic set of \( G \). Let \( S_1 = S \cup \{q\} \). It is clear that \( S_1 \) is the minimum edge geodetic set of \( G \). Clearly \( S_1 \) is not a split edge geodetic set of \( G \). Let \( S_2 = S_1 \cup \{o\} \). Clearly \( S_2 \) is a split edge geodetic set and a dominating set of \( G \). Thus \( S_2 \) is the minimum split edge geodetic dominating set of \( G \), so that \( \gamma_{sgs}(G) = l \).

**Conclusion**

In this paper, we define the split edge geodetic domination number of a graph. This work can be extended to find upper split geodetic domination number, upper split edge geodetic domination number, etc. The findings united in this paper would support to the readers to develop various useful applications to Science and Technology.

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