The quark masses and meson spectrum: A holographic approach

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Abstract

The spectrum of radially excited unflavored vector mesons is relatively well measured, especially in the heavy-quark sector. This provides a unique opportunity to observe the behavior of the hadron spectrum at fixed quantum numbers as a function of the quark mass. The experimental data suggests the approximately Regge form for the radial spectrum, \( M_n^2 = An + B \), where \( A \) and \( B \) are growing functions of the quark mass. We use the bottom-up holographic approach to find the functions \( A \) and \( B \). The obtained result shows a good agreement with the phenomenology and consistency with some predictions of the Veneziano-like dual amplitudes.

1 Introduction

The bottom-up holographic approach to QCD [1,2] turned out to be very interesting and fruitful laboratory for the theoretical study of the phenomenology of the strong interactions. Traditionally this approach is applied to the spectroscopy of the light hadrons and to the description of the related physics (the low-energy physics, hadron formfactors, finite-temperature effects etc., see, e.g., Refs. [3,4] for references). Up to now not much efforts have been invested in the holographic description of the heavy-quark sector. In particular, we are aware of only one attempt [4] to describe analytically the excited spectrum of heavy hadrons as a function of the quark masses within the framework of the holographic approach. The purpose of the present work is to address this problem in the case of the unflavored vector mesons.

Our choice of the hadron states is driven by the fact that, for the heavy mesons, only in the unflavored vector case a reach experimental spectrum of the radial excitations is available [5]. Since the radial excitations emerge naturally in the 5D holographic models — they are identified with the Kaluza-Klein modes — the chosen sector can be tested phenomenologically. In addition, the holographic description of the vector mesons is relatively simple [1,2] and looks most naturally as one deals with the conserved currents.
Table 1: The masses of known \(\omega\), \(\psi\) and \(\Upsilon\) mesons (in MeV) \(^5\). The experimental error is not displayed if it is less than 1 MeV. The following least reliable states are omitted: \(\omega(2330)\) (and another candidate \(\omega(2290)\)) \(^5\), all \(D\)-wave \(\psi\)-mesons \(^5\) and also \(\Upsilon(11023)\) \(^5\) (the last resonance has a small coupling to the \(e^+e^-\)-annihilation in comparison with \(\Upsilon(10860)\) — this suggests a strong \(D\)-wave admixture in this resonance).

| \(n\) | 0     | 1     | 2     | 3     | 4     |
|------|-------|-------|-------|-------|-------|
| \(M_\omega\) | 783   | 1425 ± 25 | 1670 ± 30 | 1960 ± 25 | 2205 ± 30 |
| \(M_\psi\)  | 3097  | 3686  | 4039 ± 1 | 4421 ± 4  | —     |
| \(M_\Upsilon\)| 9460  | 10023 | 10355  | 10579 ± 1 | 10865 ± 8 |

Table 2: The radial Regge trajectories \(^1\) (in GeV\(^2\)) for the data from Table 1 (see text).

| \(M_n^2\) | \(An + B\) |
|----------|------------|
| \(M_\omega^2\) | 0.95\(n\) + 0.99 |
| \(M_\psi^2\)  | 2.98\(n\) + 10.5  |
| \(M_\Upsilon^2\) | 5.75\(n\) + 95.1 |

The \(S\)-wave unflavored vector mesons are intensively produced in the \(e^+e^-\)-annihilation. The mechanism of resonance formation for such states is expected to be universal at all available energy scales. We will assume that in the relativistic picture\(^1\) the contribution to the mesons masses stemming from the gluon interactions is flavor-independent within the accuracy of the holographic approach to be used. The approximate value of this contribution is given by the spectrum of the \(\omega\)-mesons in which the quark masses can be set to zero. In other words, the spectrum of unflavored vector mesons is assumed to depend on the quark masses and the other contributions are encoded in the universal coefficients of the corresponding mass formula.

The spectrum of states we are going to describe is given in Table 1. The Figs. (1a)–(1c) show that the radial spectrum reveal a universal Regge-like behavior \(M_n^2 \sim n\), where \(n\) is the radial quantum number. There exists another universal effect: The ground states lie systematically below the linear trajectory. Probably some universal dynamics causes this effect. Likely the

\(^1\)Here the crucial point is that one works with the boson masses squared. Passing to the linear masses (the non-relativistic picture) a flavor-dependent "binding energy" will appear.
given effect is related with a closer location of the valent quarks in the ground states than in the excited ones. In the language of the non-relativistic potential models, this could mean that the confinement potential is strongly distorted at typical sizes of the ground states by the Coulomb part, by the spin-spin or other interactions. We are not aware of any discussions on this point in the literature. Irrespectively of the physical reason behind the effect in question, we find reasonable to exclude the ground states in fitting the corresponding linear trajectories which we are going to describe holographically.

The fits made with the linear ansatz

$$M^2_n = An + B,$$

are displayed in Table 2. It is clearly seen that both the slope $A$ and the intercept $B$ grow in response to increasing the quark mass. But the rate of this growth is quite different — the intercept grows much faster.

Anticipating some possible questions, an important remark should be made. Usually the linear Regge-like formula (1) is applied to the interpolation of the spectra of the light mesons since such a behavior is expected from the
semiclassical QCD string considerations with massless quarks. It is rather surprising that this linearity holds also for the heavy vector mesons, with the accuracy of the interpolation (1) being the same or even better. In addition, the last two states on the trajectory of Fig. (1a) need confirmation, while all states on Figs. (1b) and (1c) are well established [5]. Basing on these observations, we believe that if one uses the existing data to motivate the spectrum (1) for the light vector mesons then, staying within the same or even better accuracy, one should accept the Regge-like interpolating formula (1) for the heavy vector mesons as well.

Our aim is to find holographically the functions \( A(m) \) and \( B(m) \) (\( m \) is the quark mass) in the assumption of the linear spectrum (1).

The paper is organized as follows. In Sect. 2, we give a sketchy view of the soft-wall (SW) model [6] — a bottom-up holographic model which is accommodated for the description of the Regee-like spectrum (1). We briefly discuss also some attempts to apply this model to the heavy quarkonia. In Sect. 3, we reformulate the SW model in such a way that the dependence on the quark mass can be easily incorporated and derive our result. The phenomenological tests are discussed in Sect. 4. A particular example of the proposed class of models is given in Sect. 5. We conclude in Sect. 6.

## 2 Soft-wall model

The simplest SW holographic model of Ref. [6] describing the unflavored vector mesons is given by the action

\[
S = \int d^4x \, dz \sqrt{g} e^{-az^2} \left( -\frac{1}{4g_5^2} F_{MN} F^{MN} \right),
\]

where

\[
g = \left| \det g_{MN} \right|, \quad F_{MN} = \partial_M V_N - \partial_N V_M,
\]

and \( M, N = 0, 1, 2, 3, 4 \). The action (2) is defined in the AdS\(_5\) background space, the commonly used parametrization of its metrics reads

\[
g_{MN} dx^M dx^N = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2).
\]

Here \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) and \( R \) denotes the AdS\(_5\) radius. Below we set \( R = 1 \) for simplicity. The holographic coordinate \( z > 0 \) has the physical interpretation of inverse energy scale. The UV boundary \( z = 0 \) in (1) represents the 4D Minkowski space. The 5D vector field \( V_M(x, z) \) is dual to the 4D conserved vector current \( j_\mu = \bar{\psi}_q \gamma_\mu \psi_q \) for any quark flavor \( q \). Here the precise
sense of duality consists in the identification of the boundary value $V_M(x, 0)$ with the source for the operator $J_\mu$ \[7,8]. According to the AdS/CFT prescriptions \[7,8], the 5D mass of the field $V_M$ is $m_5^2 = (i - j)(i + j - 4) = 0$, where $j = 1$ denotes, in the given case, the spin and $i = 3$ means the canonical dimension of $J_\mu$. The ensuing gauge invariance of the action (2) allows to choose a convenient gauge for calculating the mass spectrum. This is the axial gauge

$$V_z = 0.$$ \hspace{1cm} (5)

The mass spectrum can be found either by calculating the vector correlator following the AdS/CFT dictionary \[7,8] or by finding the normalizable solutions of the equation of motion. For our purposes, the latter method is more convenient. The corresponding equation is

$$\partial_z \left( \frac{e^{-a z^2}}{z} \partial_z v_n \right) + M_n^2 \frac{e^{-a z^2}}{z} v_n = 0,$$ \hspace{1cm} (6)

where the eigenfunctions $v_n(z)$ stem from the Kaluza-Klein decomposition

$$V_\mu(x, z) = \sum_{n=0}^{\infty} V_\mu^{(n)}(x) v_n(z).$$ \hspace{1cm} (7)

The transverse part of $V_\mu^{(n)}(x)$ correspond to the 4D physical vector fields, the index $n$ is identified with the radial number. Writing the equation of motion for the 4D Fourier transform $V_\mu(q, z)$, one finds the mass spectrum as the eigenvalues $q_n^2 = M_n^2$. The substitution

$$v_n = \sqrt{z} e^{a z^2/2} \psi_n,$$ \hspace{1cm} (8)

transforms Eq. (6) into the Schrödinger form

$$-\partial_z^2 \psi_n + U(z) \psi_n = M_n^2 \psi_n,$$ \hspace{1cm} (9)

$$U(z) = \frac{3}{4z^2} + a^2 z^2,$$ \hspace{1cm} (10)

The form of the "potential" (10) is a consequence of the choice of the 5D exponential background in the action (2). This choice leads to a particularly simple Regge-like spectrum

$$M_n^2 = 4|a|(n + 1).$$ \hspace{1cm} (11)

In spite of its simplicity, the predicted equality of the slope and intercept is very close to our fit for the linear $\omega$-meson trajectory in Table 2.
The spectrum (11) does not depend on the 5D coupling $g_5$. But this coupling enters the expression for the electromagnetic decay constant $F^2_n = \frac{2\sqrt{2}|a|}{g_5}$. (12)

After calculation of the leading contribution to the two-point correlator of the vector currents the coupling $g_5$ can be fixed from matching to the corresponding QCD result [1, 2],

$$g_5^2 = \frac{12\pi^2}{N_c}. \quad (13)$$

We mention some attempts to accommodate the SW model for the description of the charmonium. They can be classified as the "shifted" models [9] and the "rescaled" modes [10,11]. A bit simplifying the matter, in the "shifted" models, one adds a constant $c^2$ to the "potential" (10) which leads to the shift of the spectrum (11),

$$M^2 = 4|a|(n+1) + c^2. \quad (14)$$

In the "rescaled" models, one just rescales the slope parameter $a$ in (11): $a \rightarrow a'$. The mass of the ground $\psi$-meson can be reproduced by choosing the parameter $c$ or $a'$. But as is clearly seen from Table 2, such simplistic models fail to describe correctly the radial spectrum since the slope and the intercept must grow simultaneously with increasing the quark mass. In those papers, however, this circumstance was not considered as a drawback because the aim of the proposed models was to describe holographically the finite-temperature effects on the $J/\psi$ meson. A much more complicated model of Ref. [4] is aimed at the description of the whole meson spectroscopy. This model can be referred to as "shifted" one since it results in an analytic expression for the shift $c^2$ in (14) as a function of the quark masses and the binding energy, with the slope being fixed. We will construct a quite different model, in which both the slope and the binding energy represent growing functions of the quark mass.

3 No-wall approach and quark masses

The $z$-dependent exponential background of the SW model (2) was inserted by hands with the aim of providing the Regge-like mass spectrum (11). There is an alternative way to achieve this goal — the so-called no-wall approach [12].

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In Ref. [1], the constant $F$ has the dimension of mass squared. We prefer another normalization — $F$ in (12) has the dimension of mass.
Here one starts from the pure $\text{AdS}_5$ space and tries to restore the UV contributions to the 5D Lagrangian using some QCD inputs in the form of various QCD operators. Then one guesses a IR continuation of the introduced contributions. This step replaces guessing the 5D background. At the end, the full contribution in the $0 < z < \infty$ range can be, in principle, effectively rewritten as a $z$-dependent background by a certain transformation of the 5D fields \[13\].

As the starting point for our analysis we consider the no-wall approach \[12\]. This approach is simpler for our purposes than the original SW one because the equations of motion for the scalar fields considered below have simple polynomial solutions in the pure $\text{AdS}_5$ space. In the case of the SW model, the solutions would be expressed via some trigonometric functions, although the final qualitative conclusions would be the same. The holographic action of the no-wall model is defined by

$$S = \int d^4 x \sqrt{g} \left\{ \sum_i \left( |D X_i|^2 - m_i^2 |X_i|^2 \right) - \frac{1}{4 g_5^2} F_{MN}^2 \right\}, \quad (15)$$

where $F_{MN}$ is given in \[3\] and the covariant derivative is

$$D_M X_i = \partial_M X_i - i V_M X_i. \quad (16)$$

On the UV boundary, $z = 0$, the scalar fields $X_i$ are identified with sources of various QCD operators with canonical dimension $i$. The corresponding 5D masses are \[7, 8\]

$$m_i^2 = i(i - 4). \quad (17)$$

By assumption, the fields $X_i$ acquire the $z$-dependent vacuum expectation values $\langle X_i \rangle$ which represent the $x$-independent solutions of the equation of motion,

$$\partial_z \left( \frac{1}{z^3} \partial_z X_i \right) = \frac{m_i^2}{z^5} X_i, \quad (18)$$

with the UV boundary condition

$$\langle X_i \rangle|_{z=0} = 0. \quad (19)$$

The analogue of Eq. \[6\] for the vector physical modes is

$$\partial_z \left( \frac{1}{z^3} \partial_z v_n \right) + \frac{M_n^2}{z} v_n = \frac{2 g_5^2}{z^3} v_n \sum_i \langle X_i \rangle^2. \quad (20)$$

The change of variables $v_n = \sqrt{z} \psi_n$ brings this equation into the Schrödinger form

$$- \partial_z^2 \psi_n + \left( \frac{3}{4 z^2} + 2 g_5^2 f(z) \right) \psi_n = M_n^2 \psi_n, \quad (21)$$

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where
\[ f(z) = \frac{1}{z^2} \sum_i \langle X_i \rangle^2 \] (22)
determines the holographic "potential".

Let us consider the dimension-two operator, \( i = 2 \), and neglect all others. The solution of Eq. (18) satisfying (19) reads
\[ \langle X_2 \rangle = C_2^{(1)} z^2 + C_2^{(2)} z^2 \ln z. \] (23)

If we set \( C_2^{(2)} = 0 \), the equation (21) coincide with (22), i.e. such a no-wall model looks equivalent to the SW model. This equivalence can be explicitly shown by a redefinition of the vector field [13] (see also [4]).

It should be emphasized once more that the potential in Eq. (21) is written near the AdS boundary, \( z = 0 \), where, by assumption, the holographic correspondence allows to use the QCD inputs. In order to obtain the mass spectrum we need to continue the function \( f(z) \) to the infrared domain, \( z \to \infty \). The linear spectrum of the kind (11) can be obtained only if
\[ f(z) \big|_{z \to \infty} = a^2 z^2. \] (24)

The fact that the UV asymptotics of the 5D field dual to the dim2 operator provides automatically the correct IR asymptotics is a lucky coincidence taking place for the simplest SW model. In the general case, these asymptotics are different even for the field \( X_2 \), see Appendix. In principle, the sum in (22) may lead to the asymptotics (24) even in the absence of the dim2 operator. But the matter looks as if the dim2 operator were dual to the sum (22) in the infrared: Whether we introduce the dim2 operator and neglect all others or we deal with the whole sum (22) assuming the asymptotics (24). We remind the reader that the dim2 operator can be built in QCD — this is the gauge non-invariant gluonic operator \( A_\mu A^\mu \). Its vacuum expectation value (v.e.v.) \( \langle A_\mu A^\mu \rangle \) often serves (in the Landau gauge where it is minimal) for a parametrization of some important non-perturbative effects [14]. There are arguments [15] that \( \langle A_\mu A^\mu \rangle \) should emerge from a resummation of perturbative corrections to the unit operator in the Operator Product Expansion (OPE) of the correlation functions [16], i.e. one either deals with the infinite sum of these corrections or with \( \langle A_\mu A^\mu \rangle \). In some sense, we have a holographic analogue for such kind of duality.

Our aim is to extract the dependence of the linear mass spectrum on the quark mass. Such a dependence can appear only from the v.e.v. \( \langle X_3 \rangle \) since the field \( X_3 \) is dual to the quark bilinear operator \( \bar{q}q \). The quark mass emerges from the AdS/CFT prescription derived in Ref. [17] which states that the
solution of classical equation of motion for a scalar field $\Phi$ corresponding to
an operator $O$ of canonical dimension $i$ has the following form near the 4D
boundary $z \to 0$,

$$\Phi(x, z) \to z^{4-i} \left[ \Phi_0(x) + O(z^2) \right] + z^i \left[ \frac{\langle O(x) \rangle}{2i} + O(z^2) \right], \quad (25)$$

where $\Phi_0(x)$ acts as a source for $O(x)$ and $\langle O(x) \rangle$ denotes the corresponding
condensate. In QCD, the quark mass $m$ acts as the source for the operator $\bar{q}q$.

For the canonical dimension $i = 3$, the solution of Eq. (18) satisfying (19)
is $\langle X_3 \rangle = C_3^{(1)} z + C_3^{(2)} z^3$. According to the prescription (25),
this solution can be rewritten in terms of the physical quantities. In our isospin-zero case,
the corresponding expression is

$$\langle X_3 \rangle = \xi mz + \frac{\sigma}{2\xi} z^3, \quad (26)$$

where $\sigma$ denotes the quark condensate $\langle \bar{q}q \rangle$ and the normalization factor $\xi$
was calculated in Ref. [18],

$$\xi^2 = \frac{N_c}{4\pi^2}. \quad (27)$$

If we take into account the field $X_3$ and neglect all other scalar fields,
the action of the model will coincide with the action of the original bottom-up
models [1,2] in the vector sector. The field $X_3$ is usually exploited for
the holographic description of the chiral symmetry breaking in QCD. This
phenomenon is important in the axial-vector sector which we do not consider.
For our purposes, $X_3$ is crucial to derive the dependence of the vector
spectrum on the current quark masses.

The relation (26) allows to extract explicitly the $m$-dependent terms in (22),

$$f(z) \to \xi^2 m^2 + m\sigma z^2 + \tilde{f}(z), \quad (28)$$

where the contribution $\frac{\sigma^2}{4\xi^2} z^6$ is absorbed into the new sum $\tilde{f}(z)$. In order
to reproduce the Regge form of the spectrum the first two terms of the IR
asymptotics of $\tilde{f}(z)$ must be given by

$$\tilde{f}(z) \bigg|_{z \to \infty} = a^2 z^2 + \delta. \quad (29)$$

If we knew explicitly all coefficient in the UV expansion of $\tilde{f}(z)$ the summation
would lead to a definite function with the IR asymptotics (29). An
illustrative example is given in Appendix. In the real situation, we need to
exploit some interpolation scheme. We make use of the following simplification: The function \( f(z) \) is replaced by its IR asymptotics \( (29) \). The UV asymptotics of the holographic potential in \( (21) \) is controlled by the first term \( \frac{3}{4z^2} \). The UV asymptotics and the behavior at intermediate \( z \) of \( f(z) \) is interpolated by the constant \( \delta \) in \( (29) \). The example of a solvable model in Appendix demonstrates that such a simplification should not modify strongly the final spectrum. This partly justifies our simplification.

Thus, we arrive at the following equation on the mass spectrum,

\[
-\partial_z^2 \psi_n + \left[ \frac{3}{4z^2} + 2g_5^2(\sigma m + a^2)z^2 + 2g_5^2\xi^2 m^2 + 2g_5^2\delta \right] \psi_n = M_n^2 \psi_n. \tag{30}
\]

The factor at \( z^2 \) in \( (30) \) determines the slope of the linear trajectory. It has two contributions and one of them depends on the quark mass. This situation correlates with the QCD sum rules \([16,19]\) in the large-\( N_c \) limit \([20]\), in which the slope is determined by the v.e.v.’s of the \( \text{dim} 4 \) operators. The OPE contains two such v.e.v.’s \([16]\): \( m\langle \bar{q}q \rangle \) and \( \frac{\alpha_s}{\pi} G_{\mu\nu}^2 \). In the light quark sector, the gluon condensate dominates while in the heavy quark one, the first v.e.v. is dominant.

The spectrum given by Eq. \( (30) \) is

\[
M_n^2 = 4\sqrt{2} g_5 \sqrt{\sigma m + a^2}(n + 1) + 2g_5^2\xi^2 m^2 + 2g_5^2\delta. \tag{31}
\]

This expression shows the parametric dependence of the linear spectrum on the quark mass \( m \). After obvious redefinition of the constants, the spectrum \( (31) \) may be written in a more compact form,

\[
M_n^2 = \sqrt{\alpha + \beta m(n + 1) + \gamma m^2 + \delta}. \tag{32}
\]

The value of the constant \( \gamma \) follows directly from the relations \( (13) \), \( (27) \) and \( (31) \).

\[
\gamma = 2g_5^2\xi^2 = 6. \tag{33}
\]

In the next Section, we will use this value for some phenomenological estimates.

### 4 Phenomenological tests

In the limit \( m \to 0 \), the parameters \( \alpha \) and \( \delta \) in the spectrum \( (32) \) can be fixed from the \( \omega \)-meson trajectory (Table 2), \( \sqrt{\alpha} \approx 1 \text{ GeV}^2 \), \( \delta \approx 0 \). The remaining parameter \( \beta \) and the quark masses may be estimated from the fits in Table 2. The charmonium trajectory gives \( \beta \approx 7 \text{ GeV}^3 \), \( m_c \approx 1.1 \text{ GeV} \).
The bottomonium one leads to $\beta \approx 8$ GeV$^3$, $m_b \approx 3.9$ GeV. At the scale 2 GeV, the Particle Data cites the current quark masses in the $\overline{\text{MS}}$ scheme $m_c = 1.27$ GeV, $m_b = 4.18$ GeV [3]. It should be noted that the $b$-quark mass is expected to be lower in the real excited bottomonia since the energy scale is about 5 GeV per quark. The renormalization group running of the quark masses [5] predict that (at the one-loop level) the cited value for $m_b$ must be then divided by a factor of 1.2. In any case, taking into account the rough approximations which we have used, the overall agreement with the experimental data is not bad. This means that the predicted value of the parameter $\gamma$ (33) — the rate of the squared meson mass dependence on $m^2$ — lies close to the phenomenologically acceptable range. A global fit of the data gives a value of $\gamma$ in the range $5 \lesssim \gamma \lesssim 7$ depending on the assumptions and inputs.

Considering the limit $m \to 0$ analytically, the mass relation (31) results in a certain linear in $m$ correction to the slope,

$$4\sqrt{2} g_5 \sqrt{\sigma m + a^2} = 4\sqrt{2} g_5 a + \frac{2\sqrt{2} g_5}{a} \sigma m + O(m^2).$$

Assuming the linear form of the spectrum at $m = 0$, the consistency of QCD sum rules in the large-$N_c$ limit (the so-called planar sum rules) leads to the slope [20]

$$4\sqrt{2} g_5 a = \frac{48\pi^2}{N_c} f_\pi^2 = 2m_\rho^2,$$

(35)

where $f_\pi$ is the weak pion decay constant, $f_\pi = 92.4$ MeV [5], and $m_\rho$ denotes the $\rho$ or $\omega$ meson mass. The expansion (31) combined with (35) yields the slope

$$4\sqrt{2} g_5 a = \frac{48\pi^2}{N_c} f_\pi^2 + \frac{4\sigma m}{N_c} + O(m^2) = 2m_\rho^2 - 2m_\pi^2 + O(m^2),$$

(36)

where we have used the Gell-Mann–Oakes–Renner (GOR) relation $m_\pi^2 f_\pi^2 = -2m_\sigma$. The dual Veneziano-like amplitudes [21] predict the same correction (36) to the slope when the pion mass is taken into account. In this sense, our holographic model passes one more test.

It should be remarked that all relations obtained within the planar QCD sum rules can be derived in the bottom-up holographic models since, in some sense, the bottom-up approach represents just a compact 5D language for expressing the phenomenology of these sum rules [3]. We have taken (35).
as an external input in order not to complicate the matter. The same can be said about the GOR relation which can be reproduced in the bottom-up models if the axial-vector field is introduced [12].

The possible additional tests may follow from the calculation of the vector correlator. Its high-momentum euclidean asymptotics can be matched with the OPE in QCD [16]. In our case, some specific polynomial and logarithmic contributions will appear due to the quark masses. Similar contributions emerge in OPE if the non-zero quark masses in the fermion loops are taken into account. Unfortunately, we could not find the corresponding trustful results in the literature. In particular, the calculations of the quark mass contributions to the unit operator (the leading logarithm) presented in [19] and [22] do not match.

The non-relativistic potential models give their relations for the meson masses in the form $M = m_1 + m_2 + E$, where $m_1$, $m_2$ are the quark masses and $E$ is the binding energy. Our result (32) shows that the binding energy grows linearly with the quark mass in the heavy quark limit, $m \to \infty$, due to (33) (where $\gamma > 4$). This behavior is in a qualitative agreement with the experimental data [5]. For example, in the case of the ground states one has $M_\psi - 2m_c \approx 0.56$ GeV and $M_\Upsilon - 2m_b \approx 1.1$ GeV if $m_{c,b}$ are taken at 2 GeV (taking $m_b$ at 5 GeV leads to $M_\Upsilon - 2m_b \approx 2.5$ GeV). The given effect can be easily interpreted: When a non-relativistic quark of mass $m$ is created and moves with the velocity $v$, the binding energy should compensate its kinetic energy $\frac{mv^2}{2}$.

5 A gauge non-invariant example

The mass relation (31) (or (32)) is not just a result of some particular model, rather it represents a result given by a class of holographic models. Within this class, the difference in the final expression for the mass spectrum competes with (or exceeds) the accuracy of the method, so that the predictions can be regarded as equivalent. This class can be extended if we do not impose the requirement of the 5D gauge invariance and consider the relation (5) as a part of the definition for the 4D physical modes. Then it is easy to construct such a fine-tuning that the sum $f(z)$ (22) contains a finite number of terms. This is equivalent to considering only a few of operators of the lowest dimensions in the OPE [16] for the calculations of the hadron masses (the standard approximation in the QCD sum rules [16,19]). In the given Section, we demonstrate a typical example for this kind of models.
The action of the model is
\[ S = \int d^4x \sqrt{g} \{ \mathcal{L}_V + \mathcal{L}_S + \mathcal{L}_{\text{int}} \}, \]  
(37)
where
\[ \mathcal{L}_V = -\frac{1}{4g_5^2} F_{MN} F^{MN}, \quad F_{MN} = \partial_M V_N - \partial_N V_M; \]  
(38)
\[ \mathcal{L}_S = \frac{1}{2} \sum_{k=1}^{3} (\partial_M \varphi_k \partial^M \varphi_k - m_i^2 \varphi_k^2); \]  
(39)
\[ \mathcal{L}_{\text{int}} = \frac{1}{2} V_M V^M (g_1 \varphi_1 + g_2 \varphi_2^2 + g_3 \varphi_3). \]  
(40)

We choose the following correspondence between the 5D fields and the operators in QCD,
\[ \varphi_1 \leftrightarrow G_{\mu
u}^2, \quad \varphi_2 \leftrightarrow \bar{q}q, \quad \varphi_3 \leftrightarrow (\bar{q}q)^2. \]  
(41)
The canonical dimensions of the operators in (41) are
\[ i_1 = 4, \quad i_2 = 3, \quad i_3 = 6, \]  
(42)
which according to (17) dictate the masses
\[ m_1^2 = 0, \quad m_2^2 = -3, \quad m_3^2 = 12. \]  
(43)
The solutions of Eq. (18) satisfying the boundary condition (19) read
\[ \varphi_1(z) = C_1 z^4, \quad \varphi_2(z) = C_{21} z + C_{22} z^3, \quad \varphi_3(z) = C_{33} z^6. \]  
(44)
Writing \( \varphi_2 \) in the form (20) we arrive at the following \( z \)-dependent vector mass term
\[ m_V^2(z) = g_5^2 \left[ g_2 \xi^2 m^2 z^2 + (g_1 C_1 + g_2 \sigma m) z^4 + \left( \frac{g_2^2 \sigma^2}{4 \xi^2} + g_3 C_3 \right) z^6 \right]. \]  
(45)
The analogue of Eq. (21) becomes
\[ -\partial^2 \psi_n + \left[ \frac{3}{4z^2} + \frac{m_V^2}{z^2} \right] \psi_n = M^2 \psi_n. \]  
(46)
In order to reproduce the linear spectrum we must set to zero the coefficient at \( z^6 \) in \( m_V^2(z) \) (45). Then the "potential" of Eq. (46) can be reparametrized as
\[ U(z) = \frac{3}{4z^2} + \frac{1}{4}(\alpha + \beta m) z^2 + \gamma m^2. \]  
(47)
This parametrization leads to the mass spectrum (32) with \( \delta = 0 \).
6 Conclusions

The experimental data on the excited heavy and light unflavored vector quarkonia strongly suggests that the spectrum follows the Regge behavior $M^2_n = An + B$, here $n$ means the radial quantum number. Both the slope $A$ and the intercept $B$ grow rapidly with increasing the quark mass. In the existing bottom-up holographic models, this behavior is not reproduced since either $A$ or $B$ is independent of the quark mass. We have constructed a bottom-up holographic model in which both $A$ and $B$ represent growing functions of the quark mass. Our model passes qualitatively several phenomenological tests.

i) In the heavy-quark limit, the behavior is $A \sim \sqrt{m}$, $B = \gamma m^2 + O(\sqrt{m})$, where $m$ is the mass of quarks constituting the unflavored vector meson. This explains qualitatively (and even semiquantitatively) why the intercept grows with $m$ much faster than the slope.

ii) The considered holographic models predict $\gamma = 6$ in the asymptotics for $B$. The given value agrees well with the phenomenology. In addition, it predicts growing the quarkonium binding energy with the quark mass. This effect is clearly seen in the experimental data.

iii) In the limit $m \to 0$, the correction to the linear spectrum due to $m \neq 0$ is consistent with the Veneziano-like dual amplitudes.

The presented approach can be applied to various physical problems. For instance, it is possible to analyze the impact of the finite-temperature effects on the whole excited spectrum of the heavy vector mesons.

Acknowledgments

The work was partially supported by the Saint Petersburg State University grants 11.38.660.2013 and 11.48.1447.2012, by the RFBR grant 13-02-00127-a and by the Dynasty Foundation.

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Appendix

In this Appendix, we give an illustrative example for some statements made in Sect. 3. The first statement was that in the 5D holographic action, the UV part of asymptotics restored by the method of Sect. 3 generically does not coincide with the corresponding IR asymptotics even for the scalar field dual to the dim2 operator.

Consider the SW model (2). It leads to the linear spectrum (11). If we wish to have an arbitrary intercept,

$$M_n^2 = 4|a|(n + 1 + b)$$

(A.1)

where $b$ is a free intercept parameter (this form is more convenient than (14)), the dilaton background of the action (2) must be modified in the following way [23],

$$S = \int d^3x \sqrt{g} e^{-az^2} U^2(b, 0; az^2) \left( -\frac{1}{4g_5^2} F_{MN} F^{MN} \right).$$

(A.2)

Here $U$ denotes the Tricomi confluent hypergeometric function. The change of the vector field

$$V_M = e^{az^2/2} U^{-1}(b, 0; az^2) \tilde{V}_M$$

(A.3)

transforms the dilaton background into an effective $z$-dependent mass term,

$$S = \int d^3x \sqrt{g} \left\{ -\frac{1}{4g_5^2} F_{MN} F^{MN} +$$

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The UV asymptotics of the mass term results from the Taylor expansion

\[
\frac{a^2}{2g^2} \left[ z^2 + 2 \frac{b}{a} - \frac{2U(b - 1, 0; az^2)}{aU(b, 0; az^2)} \right]^2 \tilde{V}_M \tilde{V}^M \].
\] (A.4)

The UV asymptotics of the mass term results from the Taylor expansion

\[
z^2 + 2 \frac{b}{a} - \frac{2U(b - 1, 0; az^2)}{aU(b, 0; az^2)} \bigg|_{z \to 0} = z^2 \left[ 1 + 2b \left( \ln(az^2) + c \right) \right],
\] (A.5)

\[c = 2\gamma - 1 + \psi(b) + b \left[ \psi(b + 1) - \psi(b) \right],\]

where \(\psi\) denotes the digamma function and \(\gamma\) is the Euler constant. The expression (A.5) shows that the UV asymptotics can be reproduced by the method of Sect. 3 if for the contribution \(\langle X^2 \rangle\) in (22) the whole solution (23) is used. Setting \(C^{(2)} = 0\) is equivalent to setting \(b = 0\). Exactly this case was considered in Ref. [13] where the no-wall holographic model was proposed.

The IR asymptotics follows from

\[
z^2 + 2 \frac{b}{a} - \frac{2U(b - 1, 0; az^2)}{aU(b, 0; az^2)} \bigg|_{z \to \infty} = - \left( z^2 + 2 \frac{b}{a} \right) + \mathcal{O}(z^{-2}).
\] (A.6)

We see that if \(b \neq 0\) the UV and IR asymptotics of the effective mass term are different. This proves our statement.

If the IR asymptotics (A.6) is substituted to (A.4) and only the two leading terms are retained, the spectrum of the ensuing model will coincide with (A.1). This illustrates the reason why in (29) only the two leading contributions have been considered.

The expansion of the r.h.s. of (A.5) models the UV contributions restored by the method of Sect. 3. It is clear that if all coefficients in the UV part are known, the UV contributions can be summed into a certain function — here the l.h.s. of (A.5) — and then continued to the IR region, \(z \to \infty\). But if we know the form of the spectrum to be obtained, it is enough to guess the leading and next-to-leading IR contributions. This property we have exploited in Sect. 3.