Scalar and mean curvature comparison via $\mu$-bubbles

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Abstract
Following ideas of Gromov we prove scalar and mean curvature comparison results for Riemannian bands with lower scalar curvature bounds in dimension $n \leq 7$. The model spaces we use are warped products over scalar-flat manifolds with log-concave warping functions.

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1 Introduction

This article is concerned with a class of manifolds called bands. While Gromov gives a very general definition in [19, Section 2], the following is enough for our purposes:

Definition 1.1 A band is a connected compact manifold $X$ together with a decomposition
$$\partial X = \partial_- X \sqcup \partial_+ X,$$
where $\partial_\pm X$ are (non-empty) unions of boundary components. If $X$ is equipped with a Riemannian metric $g$, we call $(X, g)$ a Riemannian band and denote by $\text{width}(X, g)$ the distance (with respect to $g$) between $\partial_- X$ and $\partial_+ X$.

Remark 1.2 The standard example of a band is the cylinder $Y \times [-1, 1]$, where $Y$ is a closed manifold. Such bands are called trivial in this article.

Definition 1.3 A continuous map $f : X \to X'$ between two bands is called a band map if it maps $\partial_- X$ to $\partial_- X'$ and $\partial_+ X$ to $\partial_+ X'$.

Let $X \geq 2$ be band and $\sigma$ be any real number. By Gromov’s $h$-principle, there is a Riemannian metric $g$ on $X$ with scalar curvature $\text{Sc}(X, g) \geq \sigma$ and the space of all such metrics is contractible. To encounter interesting phenomena, similar to the closed case, one needs to impose boundary conditions for the metric, which are typically phrased in terms of mean curvature comparison.
curvature of the boundary. For an overview and some recent developments we refer to [3, 5, 8].

**Remark 1.4** Let \((X, g)\) be a Riemannian manifold with boundary \(\partial X \neq \emptyset\). In this article \(H(\partial X, g)\) denotes the trace of the second fundamental form of \(\partial X\) with respect to the inner unit normal vector field. With this convention the unit sphere \(S^{n-1} \subset \mathbb{R}^n\) has mean curvature \((n-1)\).

We recall three results for the trivial band \(T^{n-1} \times [-1, 1]\). The first one is due to Gromov and Lawson and follows directly from [24, Theorem A, Theorem 5.7].

**Theorem 1.5** ([24]) Let \(X = T^{n-1} \times [-1, 1]\) and \(g\) be a Riemannian metric on \(X\) with \(\text{Sc}(X, g) \geq \sigma > 0\). If \(H(\partial X, g)\) denotes the mean curvature of the boundary, then 
\[
\inf_{x \in \partial X} H(\partial X, g)(x) < 0.
\]

The proof is based on the fact that one could produce an impossible metric with positive scalar curvature on the doubled manifold \(T^{n-1} \times S^1 = T^n\), if the mean curvature were nonnegative. The second result is due to Gromov [19], who realized that the distance between the two boundary components in \(T^{n-1} \times [-1, 1]\) is bounded from above in terms of a lower scalar curvature bound.

**Theorem 1.6** ([9, 19, 48]) Let \(X = T^{n-1} \times [-1, 1]\). If \(g\) is a Riemannian metric on \(X\) with \(\text{Sc}(X, g) \geq \sigma > 0\), then
\[
\text{width}(X, g) = \text{dist}_g \left( T^{n-1} \times (-1), T^{n-1} \times \{1\} \right) \leq 2\pi \sqrt{\frac{n-1}{\sigma n}}.
\]

Gromov proved Theorem 1.6 in [19, Section 2] for \(n \leq 7\) using the minimal hypersurface approach of Schoen and Yau [38] combined with a symmetrization argument inspired by Fischer-Colbrie and Schoen [15]. Subsequently Zeidler [48, Theorem 1.4] and Cecchini [9, Theorem D] proved Theorem 1.6 in all dimensions using Dirac operator methods. The estimate is sharp but equality cannot be attained.

Based on ideas of Gromov [21, Section 5.5], Cecchini and Zeidler [11, Theorem 7.6] showed that Theorems 1.5 and 1.6 are connected by a scalar and mean curvature comparison principle for Riemannian metrics on \(T^{n-1} \times [-1, 1]\).

**Theorem 1.7** ([11]) For \(n\) odd let \(X = T^{n-1} \times [-1, 1]\) and \(g\) be a Riemannian metric on \(X\). If
\[
\triangleright \text{Sc}(X, g) \geq n(n-1),
\]
\[
\triangleright H(\partial_{\pm} X, g) \geq \mp(n-1) \tan \left( \frac{n\ell_{\pm}}{2} \right) \text{ for some } -\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n},
\]
then 
\[
\text{width}(X, g) \leq \ell_+ - \ell_-.
\]

**Remark 1.8** Let \(g_0\) be a scalar flat Riemannian metric on \(T^{n-1}\). Consider the warped product 
\[
(M, g_\varphi) = \left( T^{n-1} \times [\ell_- , \ell_+], \varphi^2(t) g_0 + dt^2 \right),
\]
where \(\varphi(t) = \cos\left( \frac{n t}{2} \right) \). Standard results for warped products (see 3.1) imply \(\text{Sc}(M, g_\varphi) = n(n-1)\) and \(H(\partial_{\pm} M, g_\varphi) = \mp(n-1) \tan \left( \frac{n\ell_{\pm}}{2} \right) \). For \(n\) odd let \(X = T^{n-1} \times [-1, 1]\) and \(g\) be a Riemannian metric on \(X\). Theorem 1.7 is a comparison result in the following sense: If \(\text{Sc}(X, g) \geq \text{Sc}(M, g_\varphi)\) and \(H(\partial_{\pm} X, g) \geq H(\partial_{\pm} M, g_\varphi)\), then \(\text{width}(X, g) \leq \text{width}(M, g_\varphi)\).

We quickly show how Theorems 1.5 and 1.6 are implied by Theorem 1.7 for \(n\) odd.
Proof of Theorem 1.5 Suppose $H(\partial X, g) \geq 0$. By rescaling the metric we can assume that $\text{Sc}(X, g) \geq n(n - 1)$. Since $X$ is compact, width$(X, g)$ is a positive real number. Let $0 = \ell_{-} < \ell_{+} < \text{width}(X, g)$. By Theorem 1.7 we conclude that width$(X, g) \leq \ell_{+} - \ell_{-} = \ell_{+} < \text{width}(X, g)$. This is a contradiction. □

Proof of Theorem 1.6 By rescaling the metric we can assume that $\text{Sc}(X, g) \geq n(n - 1)$. Since $X$ is compact, there are constants $c_{-}$ and $c_{+}$ such that $H(\partial_{\pm} X, g) \geq c_{\pm}$. Since $(n - 1) \tan \left( \frac{nt}{2} \right) \to -\infty$ as $t \to -\pi/n$ and $-(n - 1) \tan \left( \frac{nt}{2} \right) \to -\infty$ as $t \to \pi/n$ we can find $-\pi/n < \ell_{-} < \ell_{+} < \pi/n$ with
\[
c_{\pm} \geq \mp(n - 1) \tan \left( \frac{n\ell_{\pm}}{2} \right).
\]
By Theorem 1.7 we conclude that width$(X, g) \leq \ell_{+} - \ell_{-} < \frac{2\pi}{n}$. □

In this article we establish some general results (see Sect. 2) regarding scalar and mean curvature comparison of Riemannian bands with warped products over closed Riemannian manifolds. Our work is inspired by [11], where Cecchini and Zeidler prove a number of such comparison results for certain classes of spin bands using Dirac operator methods. However, instead of relying on the Dirac operator, we follow ideas of Gromov, which already appear in [18, Section 5] and are developed further in [19, Section 9] and [21, Section 5], and use a version of the minimal surface approach involving so called $\mu$-bubbles (see Sect. 4).

To ensure regularity of these $\mu$-bubbles our results are limited to dimension $n \leq 7$. Apart from that we have the advantage that we do not need to assume our bands to be spin and odd dimensional, as is done throughout [11]. Consequently we can work in full generality and establish optimal results towards Gromov’s band width conjecture and Rosenberg’s $S^1$-stability conjecture [34, Conjecture 1.24] in dimensions $n \in \{2, 3, 4, 6, 7\}$ (see Corollary 2.25 and Remark 2.26).

Furthermore, the comparison principle we present provides a general framework, in the context of which the connections between previous results, regarding Riemannian bands with lower scalar curvature bounds, become apparent.

2 Comparison results

In the following we formulate our main result concerning scalar and mean curvature comparison of Riemannian bands with warped products.

Definition 2.1 Let $X$ be a band. We say that a closed embedded hypersurface $\Sigma \subset X$ separates $\partial_{-} X$ and $\partial_{+} X$ if no connected component of $X \setminus \Sigma$ contains a path $\gamma : [0, 1] \to X$ with $\gamma(0) \in \partial_{-} X$ and $\gamma(1) \in \partial_{+} X$. Furthermore $\Sigma$ properly separates $\partial_{-} X$ and $\partial_{+} X$ if every connected component of $\Sigma$ can be connected to both $\partial_{+} X$ and $\partial_{-} X$ inside $X \setminus \Sigma$.

Definition 2.2 A smooth function $\varphi : [a, b] \to \mathbb{R}_{+}$ is called log-concave if
\[
\frac{d^{2}}{dt^{2}} \log(\varphi)(t) = \left( \frac{\varphi'(t)}{\varphi(t)} \right)' \leq 0
\]
for all $t \in [a, b]$. If the inequality is strict we say that $\varphi$ is strictly log-concave. In case of equality we say that $\varphi$ is log-affine.
Definition 2.3 Let \((N, g_N)\) be a closed Riemannian manifold with constant scalar curvature. A warped product

\[
(M, g_\varphi) = (N \times [a, b], \varphi^2(t)g_N + dt^2)
\]

with warping function \(\varphi : [a, b] \to \mathbb{R}_+\) is called a model space if \(\text{Sc}(M, g_\varphi)\) is constant and \(\varphi\) is strictly log-concave or log-affine.

Main Theorem Let \(n \leq 7\) and \(X^n\) be an oriented band with the property that no closed embedded hypersurface \(\Sigma\), which separates \(\partial_- X\) and \(\partial_+ X\), admits a metric with positive scalar curvature. Let \(g\) be a Riemannian metric on \(X\) and \((M^n, g_\varphi)\) be a model space over a scalar flat base with warping function \(\varphi : [a, b] \to \mathbb{R}_+\). If

\[
\begin{align*}
\triangledown \text{Sc}(X, g) & \geq \text{Sc}(M, g_\varphi), \\
\triangledown H(\partial_\pm X, g) & \geq H(\partial_\pm M, g_\varphi),
\end{align*}
\]

we distinguish two cases:

(A) If \(\varphi\) is strictly log-concave, then \(\text{width}(X, g) \leq \text{width}(M, g_\varphi)\).

(B) If \(\varphi\) is log-affine, then \((X, g)\) is isometric to a warped product

\[
\left(\hat{N} \times [c, d], \varphi^2 g_{\hat{N}} + dt^2\right),
\]

where \((\hat{N}, g_{\hat{N}})\) is a closed Ricci flat Riemannian manifold.

Remark 2.4 It is expected that part (A) of the Main Theorem is rigid as well i.e for \(\varphi\) strictly log-concave we have \(\text{width}(X, g) = \text{width}(M, g_\varphi)\) if and only if \((X, g)\) is isometric to a warped product

\[
\left(\hat{N} \times [a, b], \varphi^2 g_{\hat{N}} + dt^2\right),
\]

where \((\hat{N}, g_{\hat{N}})\) is a closed Ricci flat Riemannian manifold. For spin bands with \(\hat{A}(\partial_- X) \neq 0\) this holds true by work of Cecchini and Zeidler [11, Theorem 8.3, Theorem 9.1]. In our case there are some obstacles yet to be overcome (see Remark 3.9). On the other hand the log-affine case i.e part (B) of the Main Theorem is not treated in [11].

Remark 2.5 Even if rigidity in (A) can be established, the two parts of the Main Theorem have to be treated separately, as the width of the band plays a role only in the strictly log-concave case. We point out that we have no control over the width of the band in part (B) i.e the log-affine case and \(X\) can be isometric to a model space of arbitrary finite width.

Remark 2.6 It turns out that the Main Theorem holds true for any oriented band \(X\) in dimension \(n = 2\), where the condition that no closed embedded hypersurface admits a positive scalar curvature metric is vacuous. This will become apparent in Sect. 5 (see Remark 5.6).

2.1 Model spaces and applications

For this subsection let \(X^n\) be an oriented band with the property that no closed embedded hypersurface \(\Sigma\), which separates \(\partial_- X\) and \(\partial_+ X\), admits a metric with positive scalar curvature and \((N^{n-1}, g_N)\) be a closed scalar flat Riemannian manifold. To understand the type of results we can deduce from our Main Theorem we consider five exemplary model spaces.
For $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$ and the $\varphi(t) = \cos\left(\frac{n\ell}{2}\right)^{\frac{3}{2}}$ (strictly log-concave), the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t)g_N + dt^2)$$

is a model space with scalar curvature $n(n-1)$. Plugging this into part (A) yields:

**Theorem 2.7** Let $n \leq 7$ and $g$ be a Riemannian metric on $X$. If

- $\text{Sc}(X, g) \geq n(n-1)$
- $H(\partial_\pm X, g) \geq \mp(n-1)\tan\left(\frac{n\ell_\pm}{2}\right)$ for some $-\frac{\pi}{n} < \ell_- < \ell_+ < \frac{\pi}{n}$,

then width$(X, g) \leq \ell_+ - \ell_-$. 

Theorem 2.7 is a general version of Theorem 1.7, as it applies to $X = T^{n-1} \times [-1, 1]$ (see 2.2). As Theorem 1.7 implied Theorem 1.6, Theorem 2.7 implies the following result due to Gromov, which appears in [21, Section 3.6]. Note that we upgrade his result to strict inequality.

**Theorem 2.8** ([21, Section 3.6]) If $n \leq 7$ and $g$ is a Riemannian metric on $X$ with $\text{Sc}(X, g) \geq n(n-1)$, then width$(X, g) < 2\frac{\pi}{n}$.

Furthermore we generalize Theorem 1.5:

**Corollary 2.9** If $n \leq 7$ and $g$ is a Riemannian metric on $X$ with $\text{Sc}(X, g) \geq n(n-1)$, then

$$\inf_{x \in \partial_+ X} H(\partial_+ X, x) + \inf_{x \in \partial_- X} H(\partial_- X, x) < 0.$$ 

For $0 < \ell_- < \ell_+ < \infty$ and $\varphi(t) = t^{\frac{3}{2}}$ (strictly log-concave) the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t)g_N + dt^2)$$

is a scalar flat model space. Plugging this into part (A) yields:

**Theorem 2.10** Let $n \leq 7$ and $g$ be a Riemannian metric on $X$. If

- $\text{Sc}(X, g) \geq 0$ 
- $H(\partial_\pm X, g) \geq \pm \frac{2(n-1)}{nt_\pm}$ for some $0 < \ell_- < \ell_+ < \infty$,

then width$(X, g) \leq \ell_+ - \ell_-$. 

**Corollary 2.11** If $n \leq 7$ and $g$ is a Riemannian metric on $X$ with $\text{Sc}(X, g) \geq 0$. If $H(\partial_+ X) > 0$, then

$$\text{width}(X, g) < \frac{2(n-1)}{n \left(\inf_{x \in \partial_+ X} H(\partial_+ X, x)\right)}.$$ 

For $0 < \ell_- < \ell_+ < \infty$ and $\varphi(t) = \sinh^{\frac{n}{2}}\left(\frac{n\ell}{2}\right)^{\frac{3}{2}}$ (strictly log-concave) the warped product

$$(N \times [\ell_-, \ell_+], \varphi^2(t)g_N + dt^2)$$

is a model space with scalar curvature $-n(n-1)$. Plugging this into part (A) yields:

**Theorem 2.12** Let $n \leq 7$ and $g$ be a Riemannian metric on $X$. If

- $\text{Sc}(X, g) \geq 0$
\[ \Delta H(\partial_X) \geq \pm(n - 1) \cosh \left( \frac{\ell \pm}{2} \right) \text{ for some } 0 < \ell_- < \ell_+ < \infty, \]

then width \((X, g) \leq \ell_+ - \ell_-\).

For \(X = T^{n-1} \times [-1, 1]\) the following had already been observed by Gromov [20, Section 4]:

**Corollary 2.13** Let \(n \leq 7\) and \(g\) be a Riemannian metric on \(X\) with \(\text{Sc}(X, g) \geq -n(n - 1)\). If \(H(\partial_X) > n - 1\), then

\[
\text{width}(X, g) \leq \frac{2}{n} \arcoth \left( \frac{1}{n - 1} \inf_{x \in \partial_X} H(\partial_X, x) \right).
\]

For \(-\infty < \ell_- < \ell_+ < \infty\) and \(\varphi(t) = 1\) (log-affine) the warped product

\[(N \times [\ell_-, \ell_+], g_N + dt^2)\]

is a scalar flat model space. Plugging this into part (B) yields the following result, which is probably well known to experts although we were not able to find a reference in the literature.

**Theorem 2.14** Let \(n \leq 7\) and \(g\) be a Riemannian metric on \(X\). If

\[
\Delta \text{Sc}(X, g) \geq 0,
\]
\[
\Delta H(\partial_X, g) \geq 0,
\]

then \((X, g)\) is isometric to a product \((\hat{N} \times [c, d], g_\hat{N} + dt^2)\), where \((\hat{N}, g_\hat{N})\) is a closed Ricci flat Riemannian manifold.

For \(-\infty < \ell_- < \ell_+ < \infty\) and \(\varphi(t) = \exp(t)\) (log-affine) the warped product

\[(N \times [\ell_-, \ell_+], \varphi^2(t)g_N + dt^2)\]

is a model space with scalar curvature \(-n(n - 1)\). Plugging this into part (B) yields:

**Theorem 2.15** Let \(n \leq 7\) and \(g\) be a Riemannian metric on \(X\). If

\[
\Delta \text{Sc}(X, g) \geq -n(n - 1),
\]
\[
\Delta H(\partial_X, g) \geq \pm(n - 1),
\]

then \((X, g)\) is isometric to a warped product \((\hat{N} \times [c, d], \exp(2t)g_\hat{N} + dt^2)\), where \((\hat{N}, g_\hat{N})\) is a closed Ricci flat Riemannian manifold.

**Remark 2.16** Special versions of Theorem 2.15 already appear in [18, Section 5, pp. 57–58] and in [19, Section 9], where its relation to the hyperbolic positive mass theorem is explained. Li proved a cubical version of Theorem 2.15 in [29, Theorem 1.3].

### 2.2 Topological results

The results of the previous subsection apply to oriented bands \(X\) with the property that no closed embedded hypersurface \(\Sigma\) which separates \(\partial_-X\) and \(\partial_+X\) admits a metric with positive scalar curvature. Gromov provides a list of examples for such bands in [21, Section 3.6], which we expand significantly. In particular we prove the following optimal result for trivial bands in dimension \(n \geq 6\) (see Sect. 6).
Proposition 2.17 Let $n \geq 6$ and $Y^{n-1}$ be a closed connected oriented manifold which does not admit a metric with positive scalar curvature. If $X = Y \times [-1, 1]$, then no closed embedded hypersurface $\Sigma$ which separates $\partial_- X$ and $\partial_+ X$ admits a metric with positive scalar curvature.

In the spin setting we recall an observation by Zeidler [47, 48].

Proposition 2.18 ([47, 48]) Let $n \geq 2$ and $Y^{n-1}$ be a closed connected oriented spin manifold with Rosenberg index $\alpha(Y) \neq 0 \in KO_{n-1}(C^* \pi_1 Y)$. If $X = Y \times [-1, 1]$, then no closed embedded hypersurface $\Sigma$ which separates $\partial_- X$ and $\partial_+ X$ admits a metric with positive scalar curvature.

Furthermore we consider a class of bands which are not necessarily trivial.

Definition 2.19 A closed connected oriented manifold $Y^{n-1}$ is called NPSC$^+$ if it can not be dominated by a manifold which admits a metric with positive scalar curvature. In other words: if $Z^{n-1}$ is a closed oriented manifold and there exists a continuous map $f : Z \to Y$ with $\operatorname{deg}(f) \neq 0$, then $Z$ does not admit a metric with positive scalar curvature.

Definition 2.20 A connected oriented band $X^n$ is called over-NPSC$^+$ if there is a NPSC$^+$-manifold $Y^{n-1}$ and a band map $f : X \to Y \times [-1, 1]$ with $\operatorname{deg}(f) \neq 0$.

Proposition 2.21 Let $n \geq 3$ and $X$ be a connected oriented over-NPSC$^+$ band. Then no closed embedded hypersurface $\Sigma$ which separates $\partial_- X$ and $\partial_+ X$ admits a metric with positive scalar curvature.

Remark 2.22 The two classical examples of NPSC$^+$-manifolds one should have in mind are enlargeable manifolds (compare [25, Theorem 5.8], [10, Theorem A] and [25, Proposition 5.7]) as well as Schoen-Yau-Schick manifolds (compare [38, Theorem 1], [36] and [12, Definition 23]).

Chodosh and Li [12, Theorem 2] and Gromov [22, Section 7] used $\mu$-bubbles to prove that closed aspherical manifolds of dimension $\leq 5$ do not admit metrics with positive scalar curvature. We implement Gromov’s approach from [21] in the language of [12] and present a proof of the following in Sect. 7

Theorem 2.23 All closed connected oriented aspherical 4-manifolds are NPSC$^+$.

Remark 2.24 In the first arXiv version of this article there was a mistake in the proof of Theorem 2.23, which was pointed out to us by Otis Chodosh and Chao Li (the missing piece was Proposition 7.10). In subsequent joint work with Yevgeny Liokumovich they classified sufficiently connected manifolds in dimension 4 and 5 which admits a positive scalar curvature metric. Their result implies Theorem 2.23 as well (see [13, Theorem 3]).

Combining Proposition 2.17 and Proposition 2.18 with Theorem 2.7 or Theorem 2.8 we establish the following two results towards Gromov’s band width conjecture [19, 11.12, Conjecture C].

Corollary 2.25 Let $n \in \{2, 3, 4, 6, 7\}$ and $Y^{n-1}$ be a closed connected oriented manifold which does not admit a metric with positive scalar curvature. If $X = Y \times [-1, 1]$ and $g$ is a Riemannian metric on $X$ with $\operatorname{Sc}(X, g) \geq n(n-1)$, then $\operatorname{width}(X, g) < \frac{2\pi}{n}$.  

\[ \text{Springer} \]
Remark 2.26 We point out that Corollary 2.26 implies the $S^1$-stability conjecture of Rosenberg [34, Conjecture 1.24] for closed connected orientable manifolds of dimension $n - 1 \in \{1, 2, 3, 5, 6\}$.

Corollary 2.27 Let $Y^4$ be a closed connected aspherical manifold. If $X = Y \times [-1, 1]$ and $g$ is a Riemannian metric on $X$ with $Sc(X, g) \geq n(n - 1)$, then $\text{width}(X, g) < \frac{2\pi}{n}$.

Remark 2.28 In Corollary 2.27 the manifold $Y$ can be nonorientable. In that case we simply pass to the orientable double cover, which is a closed connected aspherical manifold as well.

3 Warped products

In this section we recall some facts about warped products and develop a general framework for scalar and mean curvature comparison of Riemannian bands.

3.1 Basics

The following definitions and formulas are standard knowledge.

Definition 3.1 Let $(N, g_N)$ be a closed Riemannian manifold and $\varphi : (a, b) \rightarrow \mathbb{R}_+$ be a smooth positive function. The warped product over $(N, g_N)$ with warping function $\varphi$ is

$$(M, g_\varphi) := \left( N \times (a, b), \varphi^2 g_N + dt^2 \right).$$

The scalar curvature of $(M, g_\varphi)$ is determined by the scalar curvature of $(N, g_N)$ and the warping function $\varphi$. The following formula

$$Sc(M, g_\varphi)(p, t) = \frac{1}{\varphi^2(t)} Sc(N, g_N)(p) - 2(n - 1) \frac{\varphi''(t)}{\varphi(t)} - (n - 1)(n - 2) \left( \frac{\varphi'(t)}{\varphi(t)} \right)^2$$  \hspace{1cm} (1)

is obtained by a straightforward calculation (see also [21, Section 2.4]).

If we denote $N_t := N \times \{t\}$ for $t \in (a, b)$ and consider $N_t$ as the boundary of $N \times (a, t]$, then its second fundamental form with respect to the inner unit normal vector field is a diagonal matrix whose entries are all equal to

$$\frac{d}{dt} \log(\varphi)(t) = \frac{\varphi'(t)}{\varphi(t)}.$$

It follows that $N_t$ is what is called an umbilic hypersurface and its mean curvature is given by

$$H(N_t) = (n - 1) \frac{\varphi'(t)}{\varphi(t)} =: h_\varphi(t).$$  \hspace{1cm} (2)

Finally, we rearrange the formula (1) for the scalar curvature of a warped product in terms of $h_\varphi$:

$$Sc(M, g_\varphi)(p, t) + \frac{n}{n - 1} h_\varphi(t)^2 + 2h_\varphi(t) = \frac{1}{\varphi^2(t)} Sc(N, g_N)(p).$$  \hspace{1cm} (3)

This formula, which combines information on scalar and mean curvature of a warped product, is the basis on which we build a comparison principle.
3.2 Comparison of two warped products

As a first step towards a scalar and mean curvature comparison principle for Riemannian bands, we compare two warped products \((M_1, g_{\varphi_1})\) and \((M_2, g_{\varphi_2})\) over the same base manifold \((N, g_N)\). The results in this subsection are purely motivational and do not factor into the proof of the Main Theorem.

**Proposition 3.2** Let \((N, g_N)\) be a closed scalar flat Riemannian manifold. Let \(\varphi_1 : [a, b] \to \mathbb{R}_+\) and \(\varphi_2 : [a, b] \to \mathbb{R}_+\) be two positive functions. If
\[
\begin{align*}
\varphi(x) &\geq \mathbf{S}(\mathbf{M}, g_{\varphi_1}) \geq \mathbf{S}(\mathbf{M}, g_{\varphi_2}), \\
H(\partial_{\pm} \mathbf{M}, g_{\varphi_1}) &\geq H(\partial_{\pm} \mathbf{M}, g_{\varphi_2}),
\end{align*}
\]
then \(h_{\varphi_1} = h_{\varphi_2}\) if equality holds in both conditions.

**Lemma 3.3** Let \(\varphi_1 : [a, b] \to \mathbb{R}_+\) and \(\varphi_2 : [a, b] \to \mathbb{R}_+\) be two smooth positive functions. Then \(h_{\varphi_1} = h_{\varphi_2}\) if and only if
\[
\begin{align*}
\frac{n}{n-1} h_{\varphi_1}^2 + 2 h_{\varphi_1} ' \leq \frac{n}{n-1} h_{\varphi_2}^2 + 2 h_{\varphi_2} ' , \\
h_{\varphi_1}(a) \leq h_{\varphi_2}(a) \text{ and } h_{\varphi_1}(b) \geq h_{\varphi_2}(b) .
\end{align*}
\]

**Proof** The idea is to reduce the statement to a comparison result for the Riccati equation which can be found in [2, Lemma 4.1]. Consider \(\hat{\varphi}_i(t) = \varphi_i \left(\frac{n}{n-1} t \right)^{\frac{2}{n}}\) as functions \([\hat{a}, \hat{b}] \to \mathbb{R}_+\) where \(\hat{a} := \frac{a}{2^{\frac{n}{n-1}}}\) and \(\hat{b} := \frac{b}{2^{\frac{n}{n-1}}}\). We denote
\[
\begin{align*}
\hat{h}_1(t) &:= \frac{\hat{\varphi}_1'(t)}{\hat{\varphi}_1(t)} = \sqrt{n \frac{n-1}{n-1}} \left(\varphi_i \left(\frac{n}{n-1} t \right)^{\frac{2}{n}}\right) \left(\frac{n-2}{n-1} t \right) = \sqrt{n \frac{n-1}{n-1}} h_{\varphi_1} \left(\frac{n}{n-1} t \right) , \\
\hat{h}_2(t) &:= \frac{n}{n-1} h_{\varphi_1}^2 \left(\frac{n}{n-1} t \right) + 2 h_{\varphi_1} ' \left(\frac{n}{n-1} t \right) .
\end{align*}
\]
Then
\[
\hat{h}_1(t) + \hat{h}_2(t) = \frac{n}{n-1} h_{\varphi_1}^2 \left(\frac{n}{n-1} t \right) + 2 h_{\varphi_1} ' \left(\frac{n}{n-1} t \right) .
\]

Furthermore, if we denote \(\kappa_i := -\hat{h}_2(t) - \hat{h}_1(t)\), we see that \(\kappa_2 \leq \kappa_1\) and
\[
\hat{\varphi}_i(t) + \kappa_i \hat{\varphi}_i(t) = 0 .
\]

At this point we are in the situation where we can apply [2, Lemma 4.1] to conclude.

For the convenience of the reader, we repeat the proof here. Hence
\[
0 = \int_{\hat{a}}^{\hat{b}} \hat{\varphi}_1 \hat{\varphi}_1 '' + \kappa_2 \hat{\varphi}_2 - (\hat{\varphi}_1 '' + \kappa_1 \hat{\varphi}_1) \hat{\varphi}_2
\]
\[
= (\hat{\varphi}_1 \hat{\varphi}_2 - \hat{\varphi}_1' \hat{\varphi}_2) \bigg|_{\hat{a}}^{\hat{b}} + \int_{\hat{a}}^{\hat{b}} (\kappa_2 - \kappa_1) \hat{\varphi}_1 \hat{\varphi}_2
\]
and therefore
\[
\hat{\varphi}_1(t) \hat{\varphi}_2(t) - \hat{\varphi}_1'(t) \hat{\varphi}_2(t) = \hat{\varphi}_1(\hat{a}) \hat{\varphi}_2(\hat{a}) - \hat{\varphi}_1'(\hat{a}) \hat{\varphi}_2(\hat{a}) + \int_{\hat{a}}^{\hat{b}} (\kappa_1 - \kappa_2) \hat{\varphi}_1 \hat{\varphi}_2 .
\]
Now \( \hat{\phi}_1(\hat{a}) \hat{\phi}'_2(\hat{a}) - \hat{\phi}_i(\hat{a}) \hat{\phi}_2(\hat{a}) \geq 0 \) since \( \hat{h}_1(\hat{a}) \leq \hat{h}_2(\hat{a}) \) and the second term on the right hand side is nonnegative since \( \kappa_2 \leq \kappa_1 \) and \( \hat{\phi}_1 \hat{\phi}_2 > 0 \). It follows that

\[
\hat{\phi}_1(t) \hat{\phi}'_2(t) - \hat{\phi}_1(t) \hat{\phi}_2(t) \geq 0 \Leftrightarrow \frac{\hat{\phi}_1(t)}{\hat{\phi}_1(t)} \leq \frac{\hat{\phi}'_2(t)}{\hat{\phi}_2(t)} \Leftrightarrow \hat{h}_1(t) \leq \hat{h}_2(t)
\]

for all \( t \in [\hat{a}, \hat{b}] \). By (4) \( \hat{h}_1(\hat{t}) = \hat{h}_2(\hat{t}) \) if equality holds at \( \hat{t} \). We can replace \( \hat{a} \) by any \( t_0 \in [\hat{a}, \hat{t}] \) in the argument above since \( \hat{h}_1(t_0) \leq \hat{h}_2(t_0) \). Hence \( \hat{h}_1 = \hat{h}_2 \) on \( [\hat{a}, \hat{t}] \) if equality holds at \( \hat{t} \). Since \( \hat{h}_1(\hat{b}) \geq \hat{h}_2(\hat{b}) \) by assumption and \( \hat{h}_1(\hat{b}) \leq \hat{h}_2(\hat{b}) \) by (5), equality holds at \( \hat{b} \). Hence \( \hat{h}_1 = \hat{h}_2 \) on \( [\hat{a}, \hat{b}] \) and therefore \( h_{\varphi_1} = h_{\varphi_2} \) on \( [a, b] \). \( \square \)

**Proof of Proposition 3.2** Since \((N, g_N)\) is scalar flat (3) implies

\[
\frac{n}{n-1} h_{\varphi_1}^2(t) + 2h_{\varphi_1}'(t) = -\text{Sc}(M, g_{\varphi_1})(p, t)
\]

\[
\leq -\text{Sc}(M, g_{\varphi_2})(p, t) = \frac{n}{n-1} h_{\varphi_2}^2(t) + 2h_{\varphi_2}'(t).
\]

Furthermore we have

\[
h_{\varphi_1}(a) = -H(\partial_- M, g_{\varphi_1}) \leq -H(\partial_- M, g_{\varphi_2}) = h_{\varphi_2}(a)
\]

and

\[
h_{\varphi_1}(b) = H(\partial_+ M, g_{\varphi_1}) \geq H(\partial_+ M, g_{\varphi_2}) = h_{\varphi_2}(a)
\]

by (2). Thus \( h_{\varphi_1} = h_{\varphi_2} \) by Lemma 3.3. \( \square \)

Next, we allow the warping functions to have different domains. Let \((N, g_N)\) be a closed scalar flat Riemannian manifold and \( \varphi_1 : [a, b] \to \mathbb{R}_+ \) and \( \varphi_2 : [c, d] \to \mathbb{R}_+ \) two positive functions.

To compare the scalar curvature of the warped products \((M_1, g_{\varphi_1})\) and \((M_2, g_{\varphi_2})\) pointwise, we need to choose a band map \( \Phi : M_1 \to M_2 \). In this setting the canonical choice is \( \Phi = \text{id}_N \times \Phi \), where \( \Phi : [a, b] \to [c, d] \) is given by \( t \mapsto (\frac{d-c}{b-a})(t-a) + c \).

To prove a comparison result like Proposition 3.2 we want to apply Lemma 3.3 to the functions \( h_{\varphi_1} \) and \( h_{\varphi_2} = h_{\varphi_2} \circ \Phi = h_{\widetilde{\varphi}_2} \) where

\[
\widetilde{\varphi}_2 : [a, b] \to \mathbb{R}_+ \quad t \mapsto \varphi_2(\Phi(t))^{\frac{b-a}{d-c}}.
\]

Hence we need to ensure that \( \text{Sc}(M_1, g_{\varphi_1})(p, t) \geq \text{Sc}(M_2, g_{\varphi_2})(p, \Phi(t)) \) implies

\[
\frac{n}{n-1} h_{\varphi_1}^2(t) + 2h_{\varphi_1}'(t) \leq \frac{n}{n-1} h_{\varphi_2}^2(t) + 2h_{\varphi_2}'(t).
\]

for all \( t \in [a, b] \). By (3) this works if

\[
h_{\varphi_2}'(\Phi(t)) \leq \widetilde{h}_{\varphi_2}'(t) = h_{\varphi_2}'(\Phi(t))\Phi'(t),
\]

which in turn holds true if \( h_{\varphi_2}'(\Phi(t)) = 0 \) or \( h_{\varphi_2}'(\Phi(t)) < 0 \) and \( \Phi'(t) \leq 1 \) i.e \( b - a \geq d - c \).

For this reason we consider strictly log-concave or log-affine warping functions in our comparison results.

**Proposition 3.4** Let \((N, g_N)\) be a scalar flat Riemannian manifold. Let \( \varphi_1 : [a, b] \to \mathbb{R}_+ \) and \( \varphi_2 : [c, d] \to \mathbb{R}_+ \) be two positive functions. Consider the warped products \((M_1, g_{\varphi_1})\) and \((M_2, g_{\varphi_2})\) and the map \( \Phi : [a, b] \to [c, d] \) given by \( t \mapsto (\frac{d-c}{b-a})(t-a) + c \). If \( \varphi_2 \) is log-affine,
\[ \nabla \Phi_1 \text{ is constant on } \partial \Phi_1 \text{ pointwise, one has to choose a band map} \]

\[ \text{Proposition 3.5} \]

\[ \text{Proof of Proposition 3.4} \]

Lipschitz, the last inequality in (6) would be strict since \( h'_\varphi \) is strictly 1-
space affine \( \varphi \). Hence

\[ \frac{n}{n-1} h'^2_{\varphi_1} (t) + 2 h'_{\varphi_1} (t) \leq \frac{n}{n-1} \tilde{h}'^2_{\varphi_2} (t) + 2 \tilde{h}'_{\varphi_2} (t). \]

Furthermore \( h'_{\varphi_1} (a) \leq \tilde{h}'_{\varphi_2} (a) \) and \( h'_{\varphi_1} (b) \geq \tilde{h}'_{\varphi_2} (b) \) (this follows from (2) and the assumption on mean curvature). Now Lemma 3.3 implies \( h'_{\varphi_1} = \tilde{h}'_{\varphi_2}. \)

\[ \text{Proposition 3.5} \]

Let \((N, g_N)\) be a scalar flat Riemannian manifold. Let \( \varphi_1 : [a, b] \to \mathbb{R}_+ \) and \( \varphi_2 : [c, d] \to \mathbb{R}_+ \) be two positive functions. Consider the warped products \((M_1, g_{\varphi_1})\) and \((M_2, g_{\varphi_2})\) and the map \( \phi : [a, b] \to [c, d] \) given by \( t \mapsto \frac{d-c}{b-a} (t-a) + c \). If \( \varphi_2 \) is strictly log-concave,

\[ \text{Proposition 3.5} \]

\[ \text{Proof of Proposition 3.4} \]

Denote \( \tilde{h}'_{\varphi_2} = h_{\varphi_2} \circ \phi : [a, b] \to \mathbb{R} \). As before:

\[ \frac{n}{n-1} h'^2_{\varphi_1} (t) + 2 h'_{\varphi_1} (t) \leq \frac{n}{n-1} \tilde{h}'^2_{\varphi_2} (t) + 2 \tilde{h}'_{\varphi_2} (t), \]

where we used that \( \varphi_2 \) is log-concave and \( \phi \) is 1-Lipschitz for the last inequality.

Furthermore \( h'_{\varphi_1} (a) \leq \tilde{h}'_{\varphi_2} (a) \) and \( h'_{\varphi_1} (b) \geq \tilde{h}'_{\varphi_2} (b) \). If \( b-a > d-c \) and \( \varphi_2 \) is strictly log-concave, the last inequality in (6) would be strict since \( \tilde{h}'_{\varphi_2} < 0 \). This is impossible because Lemma 3.3 implies \( h'_{\varphi_1} = \tilde{h}'_{\varphi_2}. \)

In the following we want to generalize Proposition 3.4 and Proposition 3.5, the prototypes for the two parts of the Main Theorem, to allow for the comparison of Riemannian bands with warped products over closed Riemannian manifolds with constant scalar curvature.

\[ \text{3.3 Structural maps} \]

To compare the scalar and mean curvature of two Riemannian bands \((X, g)\) and \((V, \tau)\) pointwise, one has to choose a band map \( \Phi : X \to V \). If \( \text{Sc}(V, \tau) \) is constant and \( H(\partial V, \tau) \) is constant on \( \partial_- V \) resp. \( \partial_+ V \), the outcome does not depend on the choice of \( \Phi \).
If \((V, \tau)\) is a warped product \((M, g_\varphi)\) over a closed Riemannian manifold \((N, g_N)\) with warping function \(\varphi : [a, b] \to \mathbb{R}_+\), the second condition is always satisfied as the mean curvature of \(\partial M = \partial_-M \sqcup \partial_+M\) with respect to \(g_\varphi\) is constant equal to \(\pm h_\varphi(a)\) on \(\partial_\pm(M)\) (see (2)).

Furthermore, if \(\text{Sc}(N, g_N)\) is constant, then \(\text{Sc}(M, g_\varphi)(p, t)\) only depends on the \(t\)-coordinate (see (3)) and therefore the scalar curvature comparison between \((X, g)\) and \((M, g_\varphi)\) only depends on \(\varphi := \text{pr}_{[a,b]} \circ \phi : X \to [a, b]\).

This is the situation we focus on for the rest of this chapter, i.e., \((X^n, g)\) will be a Riemannian band, \((N^{n-1}, g_N)\) will be a closed Riemannian manifold with constant scalar curvature and \((M^n, g_\varphi)\) will be a warped product over \((N, g_N)\) with warping function \(\varphi : [a, b] \to \mathbb{R}_+\).

To compare \((X, g)\) and \((M, g_\varphi)\) we fix a point \(p_0 \in N\), choose a band map \(\phi : X \to [a, b]\) and define \(\Phi : X \to M\) by \(x \mapsto (p_0, \phi(x))\).

While every choice of \(\phi\) enables us to compare the scalar and mean curvature of \((X, g)\) and \((M, g_\varphi)\) pointwise, we will need \(\phi\) to preserve some geometric structure to prove comparison results like Proposition 3.4 or Proposition 3.5. We denote \(h = h_\varphi \circ \phi : X \to \mathbb{R}\) and consider a ‘pullback’ version of Eq. (3) on \((X, g)\).

**Definition 3.6** A band map \(\phi : X \to [a, b]\), which is used to compare \((X, g)\) and \((M, g_\varphi)\), is called **structural** if it is smooth and for any closed embedded hypersurface \(\Sigma\) with outward unit normal field \(\nu\) which separates \(\partial_-X\) and \(\partial_+X\) the inequality

\[
\text{Sc}(X, g)(x) + \frac{n}{n-1} h^2(x) + 2g(\nabla h(x), \nu(x)) \geq \frac{1}{\varphi^2(\phi(x))} \text{Sc}(N, g_N)
\]

holds at all points \(x \in \Sigma\).

**Remark 3.7** In Proposition 3.4 and Proposition 3.5 we compared two warped products \((M, g_{\varphi_1})\) and \((M, g_{\varphi_2})\). Considering \(\phi : [a, b] \to [c, d]\) and \(h_{\varphi_2} = h_{\varphi_2} \circ \phi\) we showed

\[
\frac{n}{n-1} h_{\varphi_1}(t) + 2h'_{\varphi_1}(t) \leq \frac{n}{n-1} h^2_{\varphi_2}(t) + 2h'_{\varphi_2}(t) \Leftrightarrow \text{Sc}(M, g_{\varphi_1})(p, t) + \frac{n}{n-1} h^2_{\varphi_2}(t) + 2h'_{\varphi_2}(t) \geq 0.
\]

Hence \(\phi\) was structural and together with the boundary conditions and Lemma 3.3 we were able to conclude.

In Sect. 4 we will use \(\mu\)-bubbles to prove the following Proposition:

**Proposition 3.8** Let \(n \leq 7\) and \((X, g)\) be an oriented Riemannian band. Let \((N, g_N)\) be a closed oriented Riemannian manifold with constant scalar curvature and \((M, g_\varphi)\) the warped product over \((N, g_N)\) with warping function \(\varphi : [a, b] \to \mathbb{R}_+.\) If there is a structural band map \(\phi : X \to [a, b]\) and

\[
H(\partial_\pm X, g) > H(\partial_\pm M, g_\varphi),
\]

there is a hypersurface \(\Sigma \subset X\), which separates \(\partial_-X\) and \(\partial_+X\) in \(X\) with:

\[
-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) \geq \frac{1}{2\varphi^2(\phi)} \text{Sc}(N, g_N).
\]

**Remark 3.9** From a conceptual perspective one should be able to relax the assumption \(H(\partial_\pm X, g) > H(\partial_\pm M, g_\varphi)\) in Proposition 3.8 to \(H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi)\) if \(\varphi\) is log-concave. However, we are only able to do so whenever \(\varphi\) is log-affine (see Sect. 4.2). If \(\varphi\) is
strictly log-concave we can work around certain aspects of the problem but fall short of the desired result. The reason for this discrepancy is the lack of a strong maximum principle for \( \mu \)-bubbles.

In light of Proposition 3.8 we try to identify situations where there are structural maps to compare \((X, g)\) and \((M, g_\varphi)\). As in Proposition 3.4 and Proposition 3.5 we assume \(\varphi\) to be strictly log-concave or log-affine.

**Lemma 3.10** Let \((X, g)\) be a Riemannian band, \((N, g_N)\) be a closed Riemannian manifold with constant scalar curvature and \((M, g_\varphi)\) be a warped product over \((N, g_N)\) with warping function \(\varphi : [a, b] \to \mathbb{R}_+\). If \(\varphi\) is log-affine, then any smooth band map \(\phi : X \to [a, b]\) such that \(\text{Sc}(X, g)(x) \geq \text{Sc}(M, g_\varphi)(p_0, \phi(x))\) is structural.

**Proof** Let \(\Sigma\) be a hypersurface which separates \(\partial_- X\) and \(\partial_+ X\) in \(X\). If \(\phi : X \to [a, b]\) is smooth, and \(\text{Sc}(X, g)(x) \geq \text{Sc}(M, g_\varphi)(p_0, \phi(x))\), then

\[
\text{Sc}(X, g)(x) + \frac{n}{n-1} h^2(x) + 2g(\nabla h(x), v(x)) = \text{Sc}(X, g)(x) + \frac{n}{n-1} h^2(x)
\]

since \(\nabla h = 0\) (\(\varphi\) is log-affine) and

\[
\text{Sc}(X, g)(x) + \frac{n}{n-1} h^2(x) \geq \text{Sc}(M, g_\varphi)(p_0, \phi(x)) + \frac{n}{n-1} h_\varphi^2(\phi(x))
\]

\[= \frac{1}{\varphi^2(\phi(x))} \text{Sc}(N, g_N),\]

which implies that \(\phi\) is structural. \(\Box\)

**Lemma 3.11** Let \((X, g)\) be a Riemannian band, \((N, g_N)\) be a closed Riemannian manifold with constant scalar curvature and \((M, g_\varphi)\) be a warped product over \((N, g_N)\) with warping function \(\varphi : [a, b] \to \mathbb{R}_+\). If

\(\triangleright \phi : X \to [a, b]\) is a smooth 1-Lipschitz band map,

\(\triangleright \varphi\) is log-concave and

\(\triangleright \text{Sc}(X, g)(x) \geq \text{Sc}(M, g_\varphi)(p_0, \phi(x)),\)

then \(\phi\) is structural.

**Proof** Let \(\Sigma\) be a hypersurface which separates \(\partial_- X\) and \(\partial_+ X\) in \(X\). Since \(\varphi\) is log-concave \(h_\varphi^2 \leq 0\) and \(\phi\) is 1-Lipschitz we have

\[
\frac{n}{n-1} h^2(x) + 2g(\nabla h(x), v(x)) \geq \frac{n}{n-1} h_\varphi^2(\phi(x)) + 2h_\varphi(\phi(x))|\nabla \phi|
\]

\[\geq \frac{n}{n-1} h_\varphi^2(\phi(x)) + 2h_\varphi(\phi(x)).\]

Together with \(\text{Sc}(X, g)(x) \geq \text{Sc}(M, g_\varphi)(p_0, \phi(x))\) and (3) it follows that \(\phi\) is structural. \(\Box\)

The following can also be found in [51, Lemma 4.1] and [11, Lemma 7.2].

**Lemma 3.12** Let \((X, g)\) be a Riemannian band. If width\((X, g) > a - b\), there is a smooth band map \(\phi : (X, g) \to [a, b]\) with \(\text{Lip}(\phi) < 1\).

A combination of Lemma 3.11 and Lemma 3.12 yields:
Lemma 3.13 Let \((X, g)\) be a Riemannian band, \((N, g_N)\) be a closed Riemannian manifold with constant scalar curvature and \((M, g_\varphi)\) be a warped product over \((N, g_N)\) with warping function \(\varphi : [a, b] \to \mathbb{R}_+\). If \(\varphi\) is log-concave, \(\text{Sc}(M, g_\varphi)\) is constant, and the following holds true

\[\Delta \text{Sc}(X, g) \geq \text{Sc}(M, g_\varphi),\]
\[\text{width}(X, g) > \text{width}(M, g_\varphi),\]

there exists a structural band map \(\phi : X \to [a, b].\)

\[\blacksquare\]

Remark 3.14 If \(\text{Lip}(\phi) < 1\) in Lemma 3.11 we get strict inequality in (7). This observation is important as it allows us to obtain strict inequality for the operator in Proposition 3.8 later (see Remark 4.5). In particular this applies to the band map \(\phi\) we get from Lemma 3.13.

3.4 Model spaces

The notion of a model space (compare Definition 2.3 for scalar and mean curvature comparison of Riemannian bands is motivated by our observations so far. In addition to those we introduced in Sect. 2.1 one considers Annuli in simply connected space forms.

Let \((S^n, g_1)\setminus\{p_1, p_2\}\) be the unit \(n\)-sphere with two antipodal points removed. This has constant scalar curvature equal to \(n(n-1)\) and can be written as a warped product

\[\left(S^{n-1} \times (-\frac{\pi}{2}, \frac{\pi}{2}), \cos^2(t)g_1 + dt^2\right),\]

where \((S^{n-1}, g_1)\) is the unit sphere in one dimension less. Since \(\cos(t)\) is strictly log-concave we see that for \(-\frac{\pi}{2} < \ell_- < \ell_+ < \frac{\pi}{2}\) the warped product

\[\left(S^{n-1} \times [\ell_-, \ell_+], \cos^2(t)g_1 + dt^2\right)\]

is a model space.

Let \((\mathbb{R}^n, g_{\text{std}})\setminus\{0\}\) be euclidean space with the origin removed. This is scalar flat and can be written as a warped product

\[\left(S^{n-1} \times (0, \infty), r^2g_1 + dt^2\right)\]

Since \(t\) is strictly log-concave we see that for \(0 < \ell_- < \ell_+ < \infty\) the warped product

\[\left(S^{n-1} \times [\ell_-, \ell_+], r^2g_1 + dt^2\right)\]

is a model space.

Let \((\mathbb{H}^n, g_{-1})\setminus\{p\}\) be hyperbolic space with a point removed. This has constant scalar curvature equal to \(-n(n-1)\) and can be written as a warped product

\[\left(S^{n-1} \times (0, \infty), \sinh^2(t)g_1 + dt^2\right)\]

Since \(\sinh(t)\) is strictly log-concave we see that for \(0 < \ell_- < \ell_+ < \infty\) the warped product

\[\left(S^{n-1} \times [\ell_-, \ell_+], \sinh^2(t)g_1 + dt^2\right)\]

is a model space.

Remark 3.15 Let \((X^n, g)\) be a Riemannian spin band and \((M, g_\varphi)\) one of the above model spaces. Let \(\Phi : X \to M\) be a smooth 1-Lipschitz band map with degree non zero. In [11, Corollary 10.4] Cecchini and Zeidler prove the following: If \(\text{Sc}(X, g) \geq \text{Sc}(M, g)\) and \(H(\partial_\pm X, g) \geq H(\partial_\pm M, g)\), then \(\Phi\) is an isometry.
As is explained in [21, Section 5.5] one can recreate similar results using \( \mu \)-bubbles and a stabilized version of Llarull’s theorem [30] in dimension \( 3 \leq n \leq 7 \) (one does not need to assume that \( n \) is odd).

However, as rigidity for strictly log-concave warping functions remains problematic in our setting (see Remark 2.4) the best result we could obtain at this moment is: If \( \text{Sc}(X, g) \geq \text{Sc}(M, g) \) and \( H(\partial X, g) \geq H(\partial M, g) \), there is no smooth band map \( \Phi : X \to M \) with degree non-zero and \( \text{Lip}(\Phi) < 1 \).

\section{4 \( \mu \)-bubbles}

We briefly introduce the most important definitions and results (see [12, Section 3], [21, Section 5.1] and [51, Section 2]) concerning \( \mu \)-bubbles. As a good reference for the theory of Caccioppoli sets, which will be used freely throughout the rest of this article, we recommend [16, Chapter 1].

\subsection{4.1 Basics}

Let \( (X, g) \) be an oriented Riemannian band and \( h \) be a smooth function on \( X \). Denote by \( \mathcal{C}(X) \) the set of all Caccioppoli sets in \( X \) which contain an open neighborhood of \( \partial_- X \) and are disjoint from \( \partial_+ X \).

For \( \hat{\Omega} \in \mathcal{C}(X) \) consider the functional

\[
A_h(\hat{\Omega}) = |\hat{\Omega}|^{n-1} (\partial^* \hat{\Omega} \cap \hat{X}) - \int_{\hat{\Omega}} h|d\hat{n}|
\]

where \( \partial^* \hat{\Omega} \) is the reduced boundary [16, Chapter 3,4] of \( \hat{\Omega} \).

We denote

\[
\mathcal{I} := \inf \{ A_h(\hat{\Omega}) \mid \hat{\Omega} \in \mathcal{C}(X) \}
\]

and call a Caccioppoli set \( \Omega \in \mathcal{C}(X) \) a \( \mu \)-bubble if \( A_h(\Omega) = \mathcal{I} \) i.e \( \Omega \) minimizes the \( A_h \)-functional among all Caccioppoli sets in \( X \), which contain a neighborhood of \( \partial_- X \) and are disjoint from \( \partial_+ X \).

\begin{remark}
To preempt any confusion we remind reader of our mean curvature convention in Remark 1.4, according to which \( H(\partial_- X) \) is the trace of the second fundamental form with respect to the inner unit normal field.

However, if \( \hat{\Omega} \) is a smooth Caccioppoli set which contains an open neighborhood of \( \partial_- X \) and \( \hat{\Sigma} \) is a connected component of \( \partial \hat{\Omega} \cap \hat{X} \), then the mean curvature \( H(\hat{\Sigma}) \) is the trace of the second fundamental form with respect to the unit normal field pointing into \( \hat{\Omega} \). Hence, if \( \hat{\Sigma} \) approaches \( \partial_- X \) then \( H(\hat{\Sigma}) \) approaches \( -H(\partial_- X) \) and if \( \hat{\Sigma} \) approaches \( \partial_+ X \), then \( H(\hat{\Sigma}) \) approaches \( H(\partial_+ X) \).
\end{remark}

\begin{lemma}
(see [21, Section 5.1]) If \( n \leq 7 \) and \( H(\partial X) > \pm h \) on \( \partial X \), there is a smooth \( \mu \)-bubble \( \Omega \) i.e a smooth Caccioppoli set \( \Omega \in \mathcal{C}(X) \), with \( A_h(\Omega) = \mathcal{I} \).
\end{lemma}

\begin{proof}
We adapt the proofs of [51, Proposition 2.1] and [12, Proposition 12]. For \( t > 0 \) denote by \( \Omega^t_{\pm} \) the -neighborhoods of \( \partial X \). Since \( \partial X \) is smooth \( \Omega^t_{\pm} \) has a foliation by smooth equidistant hypersurfaces \( \Sigma^t_{\pm} \) for \( t \) small enough.
\end{proof}
Denote by $v^\perp_\pm$ the unit normal vector field to $\Sigma^\perp_\pm$ pointing in the direction of $\partial_\pm X$ and by $H(\Sigma^\perp_\pm)$ the trace of the second fundamental form of $\Sigma^\perp_\pm$ with respect to $-v^\perp_\pm$. By possibly making $t$ even smaller we can guarantee

$$\mathop{\text{div}}(v^\perp_\pm) = H(\Sigma^\perp_\pm) < h(x) \text{ on } \Omega^\prime_\pm \quad \text{and} \quad \mathop{\text{div}}(v^\perp_\pm) = H(\Sigma^\perp_\pm) > h(x) \text{ on } \Omega^\prime_+.$$

Let $\hat{\Omega}$ be any Caccioppoli set with $\partial_- X \subset \hat{\Omega}$ and $\partial_+ X \cap \hat{\Omega} = \emptyset$.

We want to see the following: if we add $\Omega^\prime_- \cup \hat{\Omega}$ and by $\partial^\prime \hat{\Omega} \cap \partial \Omega^\prime_+$ from $\hat{\Omega}$ we do not increase the value of $A_h$.

$$\begin{align*}
A_h((\hat{\Omega} \cup \Omega^\prime_-) \setminus \Omega^\prime_+) - A_h(\hat{\Omega}) &= H^n - (\partial \Omega^\prime_- \setminus \hat{\Omega}) - H^n - (\partial^\prime \hat{\Omega} \cap \Omega^\prime_-) + H^n - (\partial \Omega^\prime_+ \cap \hat{\Omega}) \\
&= H^n - (\partial \Omega^\prime_- \setminus \hat{\Omega}) - H^n - (\partial^\prime \hat{\Omega} \cap \Omega^\prime_-) + H^n - (\partial \Omega^\prime_+ \cap \hat{\Omega}).
\end{align*}$$

The divergence theorem and our assumption on $h$ implies

$$\int_{\Omega^\prime_- \setminus \hat{\Omega}} hH^n > \int_{\Omega^\prime_- \setminus \hat{\Omega}} \mathop{\text{div}} v^\perp_- dH^n = \int_{\partial^\prime (\Omega^\prime_- \setminus \hat{\Omega})} \langle v^\perp_-, v \rangle dH^{n-1} \geq H^n - (\partial \Omega^\prime_- \setminus \hat{\Omega}) - H^n - (\partial^\prime \hat{\Omega} \cap \Omega^\prime_-)$$

and

$$\int_{\Omega^\prime_+ \cap \hat{\Omega}} hH^n < \int_{\Omega^\prime_+ \cap \hat{\Omega}} \mathop{\text{div}} v^\perp_+ dH^n = \int_{\partial^\prime (\Omega^\prime_+ \cap \hat{\Omega})} \langle v^\perp_+, v \rangle dH^{n-1} \leq -H^n - (\partial \Omega^\prime_+ \setminus \hat{\Omega}) + H^n - (\partial^\prime \hat{\Omega} \cap \Omega^\prime_+).$$

We conclude that

$$A_h((\hat{\Omega} \cup \Omega^\prime_-) \setminus \Omega^\prime_+) - A_h(\hat{\Omega}) < 0,$$

which implies that it is enough to search for a minimizer among all Caccioppoli sets in $X$ with $\Omega^\prime_- \subset \hat{\Omega}$ and $\Omega^\prime_+ \cap \hat{\Omega} = \emptyset$.

If $C$ is a constant such that $|h| < C$ on $X$, then for any such Caccioppoli set we have $A_h(\hat{\Omega}) > -C H^n(X) > -\infty$. We choose a minimizing sequence $\hat{\Omega}_k$. By compactness for Caccioppoli sets (compare [16, Theorems 1.19 & 1.20]) $\hat{\Omega}_k$ subconverges to a minimizing Caccioppoli set $\Omega$ which contains an open neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$.

Smoothness of $\Omega$ follows from the regularity theorem [50, Theorem 2.2].

If $\hat{\Omega} \in C(X)$ is smooth and $\hat{\Sigma}$ is a connected component of $\partial \hat{\Omega} \setminus \partial_- X$, we denote by $\nu$ the outwards pointing unit normal vector field, by $A$ the second fundamental form with respect to $-\nu$ and by $H$ the trace of $A$.

**Lemma 4.3** (First variation formula) For any smooth function $\psi$ on $\hat{\Sigma}$ let $V_\psi$ be a vector field on $X$, which vanishes outside a small neighborhood of $\hat{\Sigma}$ and agrees with $\psi \nu$ on $\hat{\Sigma}$. If we denote by $\Phi_t$ the flow generated by $V_\psi$, then

$$\frac{d}{dt} \bigg|_{t=0} A_h(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} (H - h) \psi dH^{n-1}.\quad (9)$$
Lemma 4.4 (Second variation formula) For any smooth function $\psi$ on $\hat{\Sigma}$ let $V_\psi$ be a vector field on $X$, which vanishes outside a small neighborhood of $\hat{\Sigma}$ and agrees with $\psi v$ on $\hat{\Sigma}$. If we denote by $\Phi_t$ the flow generated by $V_\psi$, then

$$
\frac{d^2}{dt^2} \bigg|_{t=0} A_h(\Phi_t(\hat{\Sigma})) = \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 + (H^2 - \text{Ric}(v, v) - |A|^2 - H h - g(\nabla_X h, v))\psi^2,
$$

which is equal to

$$
\int_{\Sigma} |\nabla_{\Sigma} \psi|^2 - \frac{1}{2} (\text{Sc}(X, g) - \text{Sc}(\hat{\Sigma}, g) - H^2 + |A|^2)\psi^2 - (H h + g(\nabla_X (h), v))\psi^2.
$$

(10)

Proof We differentiate the first variation employing the following Leibniz rule: If $f$ is a smooth function on $X$, then

$$
\frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} f \psi d\mathcal{H}^{n-1} = \int_{\Sigma} (H f + g(\nabla_X f, v))\psi^2 d\mathcal{H}^{n-1}.
$$

Furthermore we use the formula

$$
\int_{\Sigma} g(\nabla_X H_{\Sigma}, v)\psi^2 d\mathcal{H}^{n-1} = \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 - (\text{Ric}(v, v) + |A|^2)\psi^2,
$$

and the standard trick to rewrite $\text{Ric}(v, v)$ from [38, p. 165]. □

If $\Omega$ is the $\mu$-bubble we get from Lemma 4.2, then the mean curvature $H$ of $\Sigma$ is equal to $h$ by Lemma 4.3 and by stability and Lemma 4.4, we see that

$$
0 \leq \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 - \frac{1}{2} (\text{Sc}(X, g) - \text{Sc}(\hat{\Sigma}, g) - H^2 + |A|^2)\psi^2 - (H h + g(\nabla_X h, v))\psi^2
$$

$$
= \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 - \frac{1}{2} (\text{Sc}(X, g) - \text{Sc}(\hat{\Sigma}, g) + H^2 + |A|^2)\psi^2 - g(\nabla_X h, v)\psi^2
$$

$$
\leq \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 - \frac{1}{2} (\text{Sc}(X, g) - \text{Sc}(\hat{\Sigma}, g) + \frac{n}{n-1} h^2 + 2 g(\nabla_X h, v))\psi^2,
$$

where we used $|A|^2 \geq \frac{H^2}{n-1}$ for the last inequality. By rearranging terms, we conclude

$$
\int_{\Sigma} |\nabla_{\Sigma} \psi|^2 + \frac{1}{2} \text{Sc}(\Sigma, g)\psi^2 d\mathcal{H}^{n-1} \geq \int_{\Sigma} \frac{1}{2} (\text{Sc}(X, g) + \frac{n}{n-1} h^2 + 2 g(\nabla_X h, v))\psi^2 d\mathcal{H}^{n-1}.
$$

(11)

We are now ready to prove Proposition 3.8:

Proof of Proposition 3.8 By assumption there is a structural band map $\phi : X \to [a, b]$. Thus $h = h_\phi \circ \phi$ is a smooth function on $X$ and by assumption $H(\partial_\pm X) > \pm h$. Since $n \leq 7$ Lemma 4.2 yields a smooth minimizer $\hat{\Omega}$ for the $A_h$-functional, which contains a neighborhood of $\partial_- X$ and is disjoint from $\partial_+ X$. Hence $\Sigma = \partial \hat{\Omega}$ separates $\partial_- X$ and $\partial_+ X$ in $X$. Furthermore by (11)

$$
\int_{\Sigma} |\nabla_{\Sigma} \psi|^2 + \frac{1}{2} \text{Sc}(\Sigma, g)\psi^2 d\mathcal{H}^{n-1} \geq \int_{\Sigma} \frac{1}{2} (\text{Sc}(X, g) + \frac{n}{n-1} h^2 + 2 g(\nabla_X h, v))\psi^2 d\mathcal{H}^{n-1}
$$

for any $\psi \in C^\infty(\Sigma)$. Since $\phi$ is structural we have

$$
\int_{\Sigma} |\nabla_{\Sigma} \psi|^2 + \frac{1}{2} \text{Sc}(\Sigma, g)\psi^2 d\mathcal{H}^{n-1} \geq \int_{\Sigma} \frac{1}{2 \varphi^2(\phi)} \text{Sc}(N, gn)\psi^2
$$
and hence
\[-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) \geq \frac{1}{2\phi^2(\phi)} \text{Sc}(N, g_N).\]

**Remark 4.5** Let \((X, g)\) be an oriented Riemannian band, \((M, g_\varphi)\) a warped product over \((N, g_N)\) and \(\phi : X \to [a, b]\) a smooth band map. If \(\varphi\) is strictly log-concave, \(\text{Sc}(X, g)(x) \geq \text{Sc}(M, g_\varphi(p_0, \phi(x)))\) and \(\phi\) has \(\text{Lip}(\phi) < 1\), then \(\phi\) is structural by Lemma 3.11. As was observed in Remark 3.14 we even get strict inequality in (7) in this case as \(g(\nabla_X h, \nu) > h'_\varphi\). Hence the argument above yields
\[-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) > \frac{1}{2\phi^2(\phi)} \text{Sc}(N, g_N).\]

### 4.2 Constant mean curvature surfaces

If \(\Omega\) is a smooth minimizer for the \(A_h\) functional, then \(\Sigma = \partial \Omega \cap \hat{X}\) is often called a prescribed mean curvature (or short PMC) surface in the literature (see for example [50]). This terminology is based on the observation that \(H(\Sigma) = h|_\Sigma\) by the first variation formula.

In the following we assume \(h\) to be a constant function. In this case \(\Sigma\) is called a constant mean curvature (or short CMC) surface and our main goal is to understand what happens if we relax the strict boundary condition \(H(\partial_{\pm} X) > \pm h\) to \(H(\partial_{\pm} X) \geq \pm h\) in Lemma 4.2. In the proof of Lemma 4.2 the assumption \(H(\partial_{\pm} X) > \pm h\) was used to show that there is a minimizing sequence \(\hat{\Omega}_k\) in \(C(X)\) which converges to a Caccioppoli set \(\Omega \subset C(X)\).

This fails if we relax to \(H(\partial_{\pm} X) \geq \pm h\), since it might happen that the limit \(\Omega\) of any minimizing sequence \(\hat{\Omega}_k\) in \(C(X)\) contains points of \(\partial_+ X\) or does not contain a neighborhood of \(\partial_- X\) any more. However, for \(h\) constant, we can use a strong maximum principle to address this issue.

To make this precise we slightly change our set up. Let \((X, g)\) be an oriented Riemannian band and \(h\) be constant function on \(X\). Without loss of generality we can assume \(h\) to be nonnegative (otherwise we just change the roles of \(\partial_- X\) and \(\partial_+ X\)). For some \(\delta > 0\) we glue on collars \(\partial_- X \times (-\delta, 0]\) and \(\partial_+ X \times [0, \delta)\) on both sides of \(X\) and extend the metric \(g\) smoothly to produce a Riemannian manifold \((X_\delta, g_\delta)\). This can be done in such a way that \(\text{vol}(X_\delta, g_\delta) < \text{vol}(X, g) + \delta\).

Let \(C(X_\delta)\) be the set of all Caccioppoli sets in \(X_\delta\), which contain \(\partial_- X \times (-\delta, 0]\) and are disjoint from \(\partial_+ X \times (0, \delta)\). We replace \(\mathcal{H}^{n-1}(\partial^* \hat{\Omega} \cap \hat{X})\) by \(\mathcal{H}^{n-1}(\partial^* \hat{\Omega})\) in the \(A_h\)-functional and define \(I_{\delta} := \inf \{A_h(\hat{\Omega}) : \hat{\Omega} \in C(X_\delta)\}\).

**Proposition 4.6** Let \(h \geq 0\) be constant and \(n \leq 7\). If \(H(\partial_{\pm} X) \geq \pm h\) and \(\Omega \in C(X_\delta)\) is a minimizer ie \(A_h(\Omega) = I_{\delta}\), then any connected component of \(\partial \Omega\) is either contained in \(\hat{X}\) or agrees with a connected component of \(\partial_- X\) resp. \(\partial_+ X\).

**Proof** Before we start we invoke a regularity result from [26, Theorem 1.3] which in turn is based on [41]. It implies that \(\partial \Omega\) is a \(C^{1,\frac{1}{2}}\) submanifold of \(X\). Hence \(\partial^* \Omega = \partial \Omega\) and \(\Omega\) has a \(C^{0,\frac{1}{2}}\) outer unit normal vector field \(\nu\). We assume that \(\partial \Omega\) is connected. Otherwise we treat each connected component separately.

For \(h = 0\) the statement follows directly from the strong maximum principle for minimal hypersurfaces (see [46, Theorem 4] or [39]) applied to \(\partial^* \Omega\). If \(h > 0\), we can apply the strong maximum principle for hypersurfaces of bounded variation [46, Theorem 7] to see...
that \( \partial \Omega \) can only touch a component of \( \partial_+ X \) if they agree. Since these result hold for general varifolds we did not yet use the a priori regularity of \( \partial \Omega \).

To see that \( \partial \Omega \) can only touch a component of \( \partial_- X \) if they agree, we follow the standard recipe for proving a strong maximum principle as it is explained in [39] and [46]. First we assume that there is a point \( p \in \partial_- X \) with \( H(\partial_- X, g)(p) > \eta > -h \). By [46, Theorem 2] there is a compactly supported vector field \( V \) on \( X_\delta \) such that \( V(p) \) is a nonzero normal to \( \partial_- X \) and

\[
\int_{\partial \Omega} \text{div}_\Omega V dH^{n-1} + \int_{\partial \Omega} \eta |V| dH^{n-1} \leq 0.
\]

If we choose the support of \( V \) small enough and assume that \( \partial \Omega \) touches \( \partial_- X \) at \( p \) i.e. \( p \in \partial \Omega \) and the normal vectors coincide, then

\[
0 \leq \int_{\partial \Omega} \text{div}_\Omega V dH^{n-1} - \int_{\partial \Omega} h g(V, V) dH^{n-1} < \int_{\partial \Omega} \text{div}_\Omega V dH^{n-1}
\]

\[
+ \int_{\partial^* \Omega} \eta |V| dH^{n-1} \leq 0,
\]

which is a contradiction. Hence \( p \notin \partial \Omega \).

Now let \( p \in \partial_- X \) be arbitrary and assume that \( \partial \Omega \) touches \( \partial_- X \) at \( p \) but does not coincide with \( \partial_- X \) in any neighborhood of \( p \). We proceed as in [39, Step 1, p.687] (see also the comments at the end of [39]) and use the implicit function theorem (see [45, Appendix]) to foliate a neighborhood of \( p \) in \( X_\delta \) by hypersurfaces with controlled mean curvature. By the Hopf maximum principle there is such a hypersurface with mean curvature strictly bigger than \(-h\), which lies on the same side of \( \partial \Omega \) as \( \partial_- X \) and touches \( \partial \Omega \) but they do not coincide. This leads to a contradiction as we have seen before. Hence \( \partial \Omega \) and \( \partial_- X \) agree on a neighborhood of \( p \). Using a standard open-closed-connected argument we conclude that \( \partial \Omega = \partial_- X \) and \( H(\partial_- X, g) = -h \). \( \square \)

**Lemma 4.7** If \( h \) is constant, \( n \leq 7 \) and \( H(\partial_{\pm} X) \geq \pm h \) on \( \partial_{\pm} X \), there is a smooth Caccioppoli set \( \Omega \subset C(X_\delta) \), with \( A_h(\Omega) = \mathcal{I}_\delta \).

**Proof** Since \( \text{vol}(X_\delta, g_\delta) < \text{vol}(X, g) + \delta \) the \( A_h \)-functional is bounded from below on \( C(X_\delta) \).

By compactness for Caccioppoli sets there is a minimizer \( \Omega \subset C(X_\delta) \). By Proposition 4.6 any connected component of \( \partial \Omega \) is either contained in \( \hat{X} \) and hence smooth by the regularity theorem [50, Theorem 2.2] or agrees with a connected component of \( \partial_- X \) resp. \( \partial_+ X \). \( \square \)

Let \( \Omega \) be the minimizer from Lemma 4.7 and \( \Sigma = \partial \Omega \). It is important to note that \( \Omega \) is only stationary and stable for variations which preserve \( X \) which is the case if and only if the variation vector field has nonnegative scalar product with the interior normal vector fields to \( \partial\pm X \).

Let \( \Sigma_0 \subset \Sigma \) be a connected component. If \( \Sigma_0 \subset \hat{X} \) all variation vector fields \( V_\psi \) are admissible and we conclude \( H(\Sigma_0) = h|_{\Sigma_0} \) by the first variation formula. By the second variation formula and stability (11) holds for all \( \psi \in C^\infty(\Sigma_0) \).

If \( \Sigma_0 \) agrees with a component of \( \partial_- X \) (the case \( \Sigma_0 \subset \partial_+ X \) follows in analogous fashion), we only consider variation vector fields \( V_\psi \) with \( \psi \geq 0 \). By the first variation formula

\[
\int_{\Sigma} (H - h) \psi dH^{n-1} \geq 0
\]

for all nonnegative \( \psi \in C^\infty(\Sigma_0) \).
Since \((H - h)\) is nonpositive on \(\partial_- X\) by assumption (remember Remark 4.1 and Proposition 2.1) this implies \(H(\Sigma_0) = h|_{\Sigma_0}\). By stability and the second variation formula, holds for all \(\psi \in C^\infty(\Sigma_0)\) with \(\psi \geq 0\). Since the first eigenfunction of the operator
\[
-\Delta \Sigma_0 + \frac{1}{2} \text{Sc}(\Sigma_0, g) - \frac{1}{2}(\text{Sc}(X, g) + \frac{n}{n-1}h^2 + 2g(\nabla_X h, \nu))
\]
does not change sign (follows from elliptic regularity and the Hopf maximum principle), this implies that the operator is nonnegative.

**Remark 4.8** With Lemma 4.7 and the argument above we can prove Proposition 3.8 for constant \(h\) with the weakened boundary condition \(H(\partial_X) \geq \pm h\).

### 4.3 Warped \(\mu\)-bubbles

The following version of \(\mu\)-bubbles was introduced in [12, Section 3]. The results of this subsection will be used exclusively in Sect. 7.

Let \((X, g)\) be an oriented Riemannian band. Let \(u > 0\) be a smooth function on \(X\) and \(h\) be a smooth function on \(X\) respectively \(X\). We fix a Caccioppoli set \(\Omega_0\) with smooth boundary, which contains an open neighborhood of \(\partial_- X\) and is disjoint from \(\partial_+ X\). Hence all components of \(\partial_\Omega_0\), which are not part of \(\partial_- X\) are contained in \(X\). We consider
\[
A^u_h(\hat{\Omega}) = \int_{\partial^* \hat{\Omega}} ud\mathcal{H}^{n-1} - \int_X (\chi_{\hat{\Omega}} - \chi_{\Omega_0}) hud\mathcal{H}^n
\]
for all Caccioppoli sets \(\hat{\Omega}\) with \(\hat{\Omega} \Delta \Omega_0\) contained in the interior of \(X\) (this implies in particular, that \(\hat{\Omega}\) contains an open neighborhood of \(\partial_- X\) and is disjoint from \(\partial_+ X\)).

A Caccioppoli set, which is minimizing \(A^u_h\) in this class, is called a **warped \(\mu\)-bubble**. The following existence and regularity result is [51, Proposition 2.1] and [12, Proposition 12].

**Lemma 4.9** If \(n \leq 7\) and \(h(x) \to \pm \infty\) as \(x \to \partial_\pm X\), there exists a smooth minimizer \(\Omega\) for \(A^u_h\), such that \(\hat{\Omega} \Delta \Omega_0\) is contained in the interior of \(X\).

The first and second variation formulas for the \(A^u_h\)-functional are given in [12, Lemmas 13 & 14]. One can obtain the second variation formula from the first variation formula in the same way we obtained Lemma 4.4 from 4.3. In doing so we reorder the terms in a slightly different way than it is stated in [12, Lemma 14].

**Lemma 4.10** (Warped first variation formula) For any smooth function \(\psi\) on \(\hat{\Sigma}\) let \(V_\psi\) be a vector field on \(X\), which vanishes outside a small neighborhood of \(\hat{\Sigma}\) and agrees with \(\psi v\) on \(\hat{\Sigma}\). If we denote by \(\Phi_t\) the flow generated by \(V_\psi\), then
\[
\frac{d}{dt} \bigg|_{t=0} A^u_h(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} (Hu + g(\nabla_X u, v) - hu)\psi d\mathcal{H}^{n-1}. \tag{12}
\]

**Lemma 4.11** (Warped second variation formula) For any smooth function \(\psi\) on \(\hat{\Sigma}\) let \(V_\psi\) be a vector field on \(X\), which vanishes outside a small neighborhood of \(\hat{\Sigma}\) and agrees with \(\psi v\) on \(\hat{\Sigma}\). If we denote by \(\Phi_t\) the flow generated by \(V_\psi\), then
\[
\frac{d^2}{dt^2} \bigg|_{t=0} A^u_h(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} |\nabla_X \psi|^2 u + (H^2 - Ric(v, v) - |A|^2)\psi^2 u + (2Hg(\nabla_X u, v) + \frac{d^2 u}{dv^2} - Hu - g(\nabla_X (hu), \nu))\psi^2.
\]
which is equal to
\[
\int_{\Sigma} |\nabla_\Sigma \psi|^2 u - \frac{1}{2} (\text{Sc}(X, g) - \text{Sc}(\hat{\Sigma}, g) - H^2 + |A|^2) \psi^2 u + (2Hg(\nabla_X u, v) + \frac{d^2 u}{dv^2} - Hu - g(\nabla_X (hu), v)) \psi^2.
\]

(13)

5 Proof of the main theorem

In this section we prove parts (A) and (B) of our Main Theorem. We even establish the following slightly more general statements using the techniques from Sect. 3 and 4.

**Theorem 5.1** Let \( n \leq 7 \) and \( (X^n, g) \) be an oriented Riemannian band with the property that no closed embedded hypersurface \( \Sigma \) which separates \( \partial_- X \) and \( \partial_+ X \) has \(-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) > 0\). Let \( (M, g_\varphi) \) be a model space over a scalar flat base with warping function \( \varphi : [a, b] \rightarrow \mathbb{R}_+. \) If \( \varphi \) is strictly log-concave,

\[
\varphi \geq \text{Sc}(M, g_\varphi),
\]

\[
H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi),
\]

then \( \text{width}(X, g) \leq \text{width}(M, g_\varphi). \)

**Proof** If we assume for a contradiction that \( \text{width}(X, g) > \text{width}(M, g_\varphi) = b - a \), there is a small \( \varepsilon > 0 \), such that \( \text{width}(X, g) > b - a + 2\varepsilon \). Let \( (M_\varepsilon, g_\varphi^\varepsilon) \) be the model space

\[
(N \times [a - \varepsilon, b + \varepsilon], \varphi^2 g_N + dt^2).
\]

We compare \( (X, g) \) and \( (M_\varepsilon, g_\varphi^\varepsilon) \). According to Lemma 3.13 there is a structural map \( \varphi : (X, g) \rightarrow [a - \varepsilon, b + \varepsilon] \) with \( \text{Lip}(\varphi) < 1 \). Since \( \varphi \) is strictly log-concave and \( H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\varphi) \), we have \( H(\partial_\pm X, g) > H(M_\varepsilon, g_\varphi^\varepsilon) \). Proposition 3.8, together with Remark 3.14 and Remark 4.5, implies the existence of a hypersurface \( \Sigma \), which separates \( \partial_- X \) and \( \partial_+ X \) and has \(-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) > 0\). This is a contradiction. \( \square \)

Theorem 5.1 implies part (A) of Theorem 2 with the help of the following classical result of Kazdan–Warner [28] and Schoen–Yau [38]:

**Lemma 5.2** Let \( (\Sigma^{n \geq 2}, g) \) be a closed connected oriented manifold. If \(-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) \) is positive, then \( \Sigma \) admits a metric with positive scalar curvature.

**Proof** The proof is standard so we only recall the main ideas. Since the operator is positive

\[
\int_{\Sigma} -\psi \Delta_\Sigma \psi + \frac{1}{2} \text{Sc}(\Sigma, g) \psi^2 > 0
\]

for all \( \psi \in C^2(\Sigma) \). If \( n = 2 \) we choose \( \psi \equiv 1 \) and use Gauss-Bonnet to see that

\[
0 < \int_{\Sigma} \frac{1}{2} \text{Sc}(\Sigma, g) = 2\pi \chi(\Sigma).
\]

It follows that \( \Sigma \) is a 2-sphere and hence admits a metric with positive scalar curvature.

If \( n \geq 3 \), we consider the conformal Laplacian \( L_g = -\Delta_\Sigma + \frac{n-2}{4(n-1)} \text{Sc}(\Sigma, g) \). It is easy to see that this operator is positive as well. Hence the first eigenvalue \( \lambda_1(L_g) \) is positive. It follows from elliptic regularity and the strong maximum principle that the first eigenfunction \( u \in C^\infty(\Sigma) \) can be chosen positive.
We then use this function for a conformal change of metric ie \( \hat{g} = u^{\frac{4}{n-2}} g \). We conclude

\[
\text{Sc}(\Sigma, \hat{g}) = u^{-\frac{n+2}{n-2}} \frac{4(n-1)}{n-2} L_g u > 0
\]

using the standard formula for scalar curvature under a conformal change of metric. \( \square \)

Regarding part (B) of Theorem 2 we establish:

**Theorem 5.3** Let \( n \leq 7 \) and \((X^n, g)\) be an oriented Riemannian band with the property that no closed embedded hypersurface which separates \( \partial_- X \) and \( \partial_+ X \) admits a metric with positive scalar curvature. Let \((M, g_\phi)\) be a model space over a scalar flat base with warping function \( \phi : [a, b] \to \mathbb{R}_+ \). If \( \phi \) is log-affine,

- \( \text{Sc}(X, g) \geq \text{Sc}(M, g_\phi) \),
- \( H(\partial_\pm X, g) \geq H(\partial_\pm M, g_\phi) \),

then \((X, g)\) is isometric to a warped product

\[
\left( \hat{N} \times [c, d], \varphi^2 g_{\hat{N}} + dt^2 \right),
\]

where \((\hat{N}, g_{\hat{N}})\) is a closed scalar flat Riemannian manifold.

**Proof** The following proof is an adaptation of the rigidity arguments presented in [1, Section 2] and [51, Section 3]. Let \( \phi : X \to [a, b] \) be a band map. According to Lemma 3.10 \( \phi \) is structural. Following the proof Proposition 3.8, together with Remark 4.8, we see that there is a hypersurface \( \Sigma \), which separates \( \partial_- X \) and \( \partial_+ X \) and has

\[
-\Delta \Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) \geq \frac{1}{2} \left( \text{Sc}(X, g) + h^2 + |A|^2 \right) \geq 0,
\]

where \( h = h_\phi \circ \phi \). We start off by proving an infinitesimal splitting result.

**Claim 1** (cf. [1, Proposition 2.2]) For any connected component \( \Sigma_0 \subset \Sigma \) which does not admit a metric with positive scalar curvature, the following holds true:

- \( \Sigma_0 \) is umbilic; all principal curvatures of \( \Sigma_0 \) are equal to \( \frac{h}{n-1} \),
- \( \text{Sc}(\Sigma_0, g) = 0 \) and \( \text{Sc}(X, g) = \text{Sc}(M, g_\phi) \) along \( \Sigma_0 \).

**Proof of Claim** Let \( \Sigma_0 \subset \Sigma \) be a connected component which does not admit a metric with positive scalar curvature. Considering Lemma 5.2, we conclude that the first eigenvalue of \(-\Delta \Sigma_0 + \frac{1}{2} \text{Sc}(\Sigma_0, g)\) is equal to zero.

If \( w_1 \) is the corresponding positive first eigenfunction, then

\[
\int_{\Sigma_0} \frac{1}{2} \left( \text{Sc}(X, g) + h^2 + |A|^2 \right) w_1^2 = 0.
\]

Consequently \( \text{Sc}(X, g) + h^2 + |A|^2 = 0 \) which is equivalent to \(-h^2 - |A|^2 = \text{Sc}(X, g) \).

On the other hand \( \text{Sc}(X, g) \geq \text{Sc}(M, g_\phi) = -\frac{n}{n-1} h_\phi^2 = -\frac{n}{n-1} h^2 \) by (3). Since \( |A|^2 \geq \frac{H^2}{n-1} = \frac{h^2}{n-1} \), we conclude that \( \text{Sc}(X, g) = \text{Sc}(M, g_\phi) \) along \( \Sigma_0 \) and \( |A|^2 = \frac{H^2}{n-1} = \frac{h^2}{n-1} \).

At every point \( p \in \Sigma_0 \) the last equality forces \( A \) to be a diagonal matrix with all entries equal to \( \frac{h}{n-1} \) with respect to any orthonormal basis at \( p \). Let \( \Sigma_0 \) is umbilic with all principal curvatures equal to \( \frac{h}{n-1} \).

\( \square \) Springer
Regarding $\text{Sc}(\Sigma_0, g)$ we distinguish three cases: If $n = 2$, the term $\text{Sc}(\Sigma_0, g)$ does not appear. If $n = 3$, we choose $\psi \equiv 1$ in

$$\int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 + \frac{1}{2} \text{Sc}(\Sigma_0, g) \psi^2 \geq 0.$$ 

By Gauss-Bonnet $\Sigma_0$ is a torus and $\text{Sc}(\Sigma_0, g) = 0$.

If $n > 3$ we proceed as in [38, p. 166] and consider the first positive eigenfunction $w_2 \in C^\infty(\Sigma_0)$ corresponding to the first eigenvalue $\lambda_0$ of the conformal Laplacian

$$L_g = -\Delta_{\Sigma_0} + \frac{(n-3)}{4(n-2)} \text{Sc}(\Sigma_0, g).$$

If $\lambda_0$ were positive, one could use $w_2$ for a conformal change of metric which would result in a metric with positive scalar curvature on $\Sigma_0$ (compare the proof of Lemma 5.2). Since this is impossible we conclude that $\lambda_0 \leq 0$. Hence

$$\frac{2(n-2)}{n-3} \int_{\Sigma_0} |\nabla_{\Sigma_0} w_2|^2 = -\int_{\Sigma_0} \frac{1}{2} \text{Sc}(\Sigma_0, g) w_2^2 + \frac{2\lambda_0(n-2)}{n-3} \int_{\Sigma_0} w_2^2 \leq \int_{\Sigma_0} |\nabla_{\Sigma_0} w_2|^2.$$ 

Since $\frac{2(n-2)}{n-3} > 1$ we see that $\lambda_0 = 0$ and $w_2$ is a constant function. Consequently $\text{Sc}(\Sigma_0, g)$ is constant as well. Since the first eigenvalue of $-\Delta_{\Sigma_0} + \frac{1}{2} \text{Sc}(\Sigma_0, g)$ is equal to zero we conclude $\text{Sc}(\Sigma_0, g) = 0$.

Furthermore, we observe that the Jacobi operator associated to $\Sigma_0$ is

$$-\Delta_{\Sigma_0} - (\text{Ric}(v, v) + |A|^2) = -\Delta_{\Sigma_0} - \frac{1}{2} (\text{Sc}(X, g) - \text{Sc}(\Sigma_0, g) + H^2 + |A|^2) = -\Delta_{\Sigma_0}.$$

Before we continue we point out the following topological fact:

**Claim 2** There is a component $\Sigma_0 \subset \Sigma$ such that $\Sigma_0 \times [-1, 1]$ has the property that no closed embedded hypersurface which separates the two end of the band admits a metric with positive scalar curvature.

**Proof of Claim** If this were not the case, we could replace each component of $\Sigma$ inside its tubular neighborhood by a closed embedded hypersurface which admits a metric with positive scalar curvature. The union of all these components would then be a closed embedded hypersurface which separates $\partial_-X$ and $\partial_+X$ and admits a metric with positive scalar curvature. This is impossible, since $X$ is assumed to have the property that no closed embedded hypersurface which separates the two end of the band admits a metric with positive scalar curvature. 

Next, we use the infinitesimal splitting to establish that the desired warped product splitting of $(X, g)$ can be found locally around suitable components of $\Sigma$.

**Claim 3** [cf. [1, Theorem 2.3]] There is a connected component $\Sigma_0 \subset \Sigma$ and a tubular neighborhood $U$ of $\Sigma_0$ which is isometric to the warped product

$$\left(\Sigma_0 \times (-\varepsilon, \varepsilon), \exp(2s \frac{h}{n-1})g_0 + ds^2\right)$$

for some small $\varepsilon > 0$, where $g_0$ denotes the metric on $\Sigma_0$ induced by $g$. 

$\square$ Springer
Proof of Claim} Since $X$ has the property that no closed embedded hypersurface which separates the two end of the band admits a metric with positive scalar curvature, there is a connected component $\Sigma_0 \subset \Sigma$ which does not admit a metric with positive scalar curvature and such that $\Sigma_0 \times [-1, 1]$ has the same property (see Claim 2). The conclusions of Claim 1 hold for $\Sigma_0$.

We follow the proof of [1, Theorem 2.3] respectively [51, Lemma 3.4] and use the implicit function theorem to show that there is a foliation $\{ \Sigma_s \}_{−δ<s<δ}$ around $\Sigma_0$ such that

1. each $\Sigma_s$ is a graph over $\Sigma_0$ with graph function $u_s$ along the outward unit normal field $\nu$ with

$$\left. \frac{d}{ds} \right|_{s=0} u_s = 1 \quad \text{and} \quad \int_{\Sigma_0} u_s d\mathcal{H}^{n-1} = s;$$

2. $H_s = H(\Sigma_s) - h$ is a constant function on $\Sigma_s$.

For $s \in [0, \delta)$ let $\Omega_s$ be the union of $\Omega$ and the region bounded by $\Sigma_0$ and $\Sigma_s$. By choosing $\delta$ small enough, we can guarantee that the region bounded by $\Sigma_0$ and $\Sigma_s$ does not intersect $\Sigma \setminus \Sigma_0$.

Since $\Omega$ minimizes the $A_h$ functional, there is a $0 < \delta' < \delta$ such that $H_s - h \geq 0$ for all $s \in [0, \delta')$. There are two possibilities: either $H_s - h > 0$ for some $s \in [0, \delta')$ or $H_s - h = 0$ for all $s \in [0, \delta')$.

In the first case we choose a constant $0 \leq h < \hat{h} < H_s$ and consider the $A_{\hat{h}}$-functional on the Riemannian band $\hat{X}$ bounded by $\Sigma_0$ and $\Sigma_s$, which is diffeomorphic to $\Sigma_0 \times [-1, 1]$ and hence has the property that no closed embedded hypersurface which separates the two end of the band admits a metric with positive scalar curvature. By Lemma 4.2 there is a smooth hypersurface $\hat{\Sigma}$ which separates $\Sigma_0$ and $\Sigma_s$ with $H(\hat{\Sigma}) = \hat{h}$ and by stability and the second variation formula we see

$$-\Delta_{\hat{\Sigma}} + \frac{1}{2} \text{Sc}(\hat{\Sigma}, g) \geq \frac{1}{2} \left( \text{Sc}(\hat{X}, g) + \frac{n}{n-1} \hat{h}^2 \right) > 0.$$
By an analogous argument for negative values of \( s \), we see that there is some \( 0 < \delta'' < \delta \) such that the above holds true for all \( \Sigma_s \) with \( s \in (-\delta'', 0] \). We choose some small \( 0 < \varepsilon \leq \min(\delta', \delta'') \).

With the foliation we can write the metric as \( g = g_s + f_s^2 ds^2 \), where \( g_s = g|_{\Sigma_s} \). As the lapse function \( f_s \) satisfies the Jacobi equation \([26, \text{Equation (1.2)}]\), which reduces to \( \Delta_{\Sigma_s} f_s = 0 \), we see that \( f_s \) is constant. By rescaling the \( s \)-coordinate, if necessary, we can assume \( f_s = 1 \) and hence \( \Sigma_s \) is \( s \)-equidistant to \( \Sigma_0 \). Since \( \Sigma_s \) is umbilic for all \( s \in (-\varepsilon, \varepsilon) \) we conclude that the map

\[
S : \left( \Sigma_0 \times (-\varepsilon, \varepsilon), \exp(2s \frac{h}{n-1}) g_0 + ds^2 \right) \rightarrow (X, g)
\]

which is defined by

\[
(p, s) \mapsto \exp_p(s v_0)
\]
is an isometry onto its image, which we denote by \( U \).

Let \( \Sigma_0 \) be the component we get from Claim 3. We want to show that there is a maximal interval \([c, d]\) containing \((-\varepsilon, \varepsilon)\) such that

\[
S : \left( \Sigma_0 \times [c, d], \exp(2s \frac{h}{n-1}) g_0 + ds^2 \right) \rightarrow (X, g)
\]
is an isometry. Note that if the map \( S \) is defined on \( \Sigma_0 \times (c', d') \) it is also defined on \( \Sigma_0 \times [c', d'] \) since the normal geodesic can always be extended to times \( c' \) resp. \( d' \).

**Claim 4** Assume that for some real number \( 0 < d' \) the map

\[
S : \left( \Sigma_0 \times [0, d'), \exp(2s \frac{h}{n-1}) g_0 + ds^2 \right) \rightarrow (X, g)
\]

which is defined by

\[
(p, s) \mapsto \exp_p(s v_0)
\]
is an isometry onto its image. Assume further that \( S \left( \Sigma_0 \times [0, d') \right) \) does not intersect \( \Sigma \setminus \Sigma_0 \). Then the following holds true:

\[\triangleright\] \( S(\Sigma_0 \times \{d'\}) \) does not intersect \( \Sigma \setminus \Sigma_0 \),

\[\triangleright\] if \( S(\Sigma_0 \times \{d'\}) \) intersects \( \partial_+ X \), it coincides with a component of \( \partial_+ X \),

\[\triangleright\] \( S : \left( \Sigma_0 \times [0, d'), \exp(2s \frac{h}{n-1}) g_0 + ds^2 \right) \rightarrow (X, g) \) is an isometry onto its image,

**Proof of Claim** Consider an increasing sequence \( s_k \rightarrow d' \) in \([0, d')\) and the corresponding sequence \( \Sigma_{s_k} = S(\Sigma_0 \times \{s_k\}) \) of embeddings of \( \Sigma_0 \). We denote by \( \Omega_{s_k} \) the union of \( \Omega \) and the region bounded by \( \Sigma_0 \) and \( \Sigma_{s_k} \). As we have seen before \( \Omega_{s_k} \) minimizes the \( A_h \)-functional for any \( k \in \mathbb{N} \).

Furthermore \( |A_{s_k}|^2 = \frac{\kappa^2}{n-1} = \text{const} \) for all \( k \) and \( H^{n-1}(\Sigma_{s_k}) \leq I + h \text{vol}(X, g) < \infty \). By \([6, \text{Theorem 1.1}]\) (and the comments thereafter) and the compactness theorem \([49, \text{Theorem 2.11}]\) for stable CMC-hypersurface, the limit \( S(\Sigma_0 \times \{d'\}) \) of these embeddings is an immersion. In fact \( S(\Sigma_0 \times \{d'\}) \) is an almost embedded \([49, \text{Definition 2.3}]\) stable CMC-surface with mean curvature equal to \( h \).

To show that \( S(\Sigma_0 \times \{d'\}) \), which we will denote by \( \Sigma_{d'} \), does not intersect \( \Sigma \setminus \Sigma_0 \), we distinguish two cases. If \( \Sigma_{d'} \) is embedded and coincides with a component \( \Sigma' \) of \( \Sigma \setminus \Sigma_0 \) (with
the opposite orientation), then \( h = 0 \) (the normal vector fields to \( \Sigma' \) and \( \Sigma_{d'} \) are inverse to each other and both have constant mean curvature equal to \( h \)). If we consider the minimizing sequence \( \Omega_{s_{k}} \) for the \( A_{h} \)-functional, we see that it converges to an open set \( \mathcal{S}' \) with boundary \( \Sigma \setminus (\Sigma_{0} \cup \Sigma') \) (\( \Sigma' \) and \( \Sigma_{d'} \) cancel each other out). Hence \( A_{h}(\Omega') < A_{h}(\Omega) \) which contradicts the regularity result \[50, \text{Theorem 2.2}\]. Hence \( \Sigma_{d'} \) does not intersect \( \Sigma \setminus \Sigma_{0} \).

If \( \Sigma_{d'} \) intersects a component of \( \Sigma \setminus \Sigma_{0} \) but they do not coincide, then the minimizing sequence \( \Omega_{s_{k}} \) converges to an open set which is minimizing the \( A_{h} \)-functional but has non-smooth boundary. This contradicts the regularity result \[50, \text{Theorem 2.2}\]. Hence \( \Sigma_{d'} \) does not intersect \( \Sigma \setminus \Sigma_{0} \).

We denote by \( \Omega_{d'} \), the region in \( X \) which is bounded by \( \Sigma_{d'} \) together with \( \Sigma \setminus \Sigma_{0} \) and \( \partial_{-}X \). Of course \( \Omega_{d'} \) is a minimizer for the \( A_{h} \)-functional, as the limit of the minimizing sequence \( \Omega_{s_{k}} \). If \( \Sigma_{d'} \) touches \( \partial_{+}X \) it coincides with a connected component of \( \partial_{+}X \) by the strong maximum principle Proposition \[4.6\]. If \( \Sigma_{d'} \) does not touch \( \partial_{+}X \) it is embedded as a boundary component of the minimizer \( \Omega_{d'} \) by the regularity result \[50, \text{Theorem 2.2}\]. In both cases \( \Sigma_{d'} \) is embedded and the smooth limit of the embeddings \( \Sigma_{s_{k}} \).

It is important to point out that, since the image \( S(\Sigma_{0} \times (0, d')) \) does not intersect \( \Sigma \setminus \Sigma_{0} \), it is contained in \( X \setminus \Omega \). Hence \( \Sigma_{d'} \) is disjoint from \( \Sigma_{0} \) (the normal geodesics to \( \Sigma_{0} \) do not close up).

We conclude that \( S : \left( \Sigma_{0} \times [0, d'], \exp(2s \frac{h}{n-1})g_{0} + ds^{2} \right) \rightarrow (X, g) \) is an isometry (and not just a local isometry) onto its image.

Let \( s_{\text{max}} \geq 0 \) be maximal with the property that

\[ S : \left( \Sigma_{0} \times [0, s_{\text{max}}], \exp(2s \frac{h}{n-1})g_{0} + ds^{2} \right) \rightarrow (X, g) \]

is an isometry onto its image and that this image does not intersect \( \Sigma \setminus \Sigma_{0} \). We know that \( s_{\text{max}} > 0 \). We show that \( s_{\text{max}} \) is finite and that \( S(\Sigma_{0} \times \{s_{\text{max}}\}) \) touches \( \partial_{+}X \).

If \( s_{\text{max}} = \infty \), then a connected component of \( X \setminus \Sigma \) is isometric to

\[ \left( \Sigma_{0} \times [0, \infty), \exp(2s \frac{h}{n-1})g_{0} + ds^{2} \right) \]

which is impossible since \( X \) is compact.

Hence \( s_{\text{max}} \) is finite. By Claim 4 the map

\[ S : \left( \Sigma_{0} \times [0, s_{\text{max}}], \exp(2s \frac{h}{n-1})g_{0} + ds^{2} \right) \rightarrow (X, g) \]

is an isometry onto its image. We denote \( S_{s_{\text{max}}} := S(\Sigma_{0} \times \{s_{\text{max}}\}) \). As before, let \( \Omega_{s_{\text{max}}} \) be the union of \( \Omega \) and \( S(\Sigma \times \{0, s_{\text{max}}\}) \). Since \( \Omega_{s_{\text{max}}} \) minimizes the \( A_{h} \)-functional, the metric splits infinitesimally around \( \Sigma_{s_{\text{max}}} \) by Claim 1.

Assume for a contradiction that \( \Sigma_{s_{\text{max}}} \) does not touch \( \partial_{+}X \). Then \( \Sigma_{s_{\text{max}}} \) is contained in \( \hat{X} \) (remember that \( \Sigma_{s_{\text{max}}} \) can not touch \( \partial_{-}X \) since the normal geodesics never cross \( \Sigma \setminus \Sigma_{0} \) and therefore a tubular neighborhood of \( \Sigma_{s_{\text{max}}} \) is contained in \( \hat{X} \).

Since \( \Sigma_{0} \times [-1, 1] \) has the property that no closed embedded hypersurface which separates the two end of the band admits a metric with positive scalar curvature, we can repeat the proof of Claim 3 to show that the metric splits locally around \( \Sigma_{s_{\text{max}}} \). Thus, there is some \( q > 0 \) such that \( S : \left( \Sigma_{0} \times [0, s_{\text{max}} + q], \exp(2s \frac{h}{n-1})g_{0} + ds^{2} \right) \rightarrow (X, g) \) is an isometry onto its image and such that the image does not intersect \( \Sigma \setminus \Sigma_{0} \). This contradicts the assumed maximality of \( s_{\text{max}} \).
We conclude that $S(\Sigma \times \{d\})$ touches $\partial_X$. In this case we have already seen in Claim 4 that $S(\Sigma \times \{d\})$ coincides with a component of $\partial_+X$ by the strong maximum principle Proposition 4.6. We define $d := s_{\text{max}}$.

Using a version of Claim 4 for negative values of $s$, and by an analogous argument involving a minimal value $s_{\text{min}}$, we see that for $c := s_{\text{min}}$ the map

$$S : \left(\Sigma_0 \times [c, d], \exp(2s \frac{h}{n-1}g_0 + ds^2)\right) \to (X, g)$$

is an isometry onto its image and that $S(\Sigma_0 \times \{c\})$ resp. $S(\Sigma_0 \times \{d\})$ are components of $\partial_-X$ resp. $\partial_+X$. Hence the image is open and closed in $X$. Since $X$ is connected, we conclude that $(X, g)$ is isometric to

$$\left(\Sigma_0 \times [c, d], \exp(2s \frac{h}{n-1}g_0 + ds^2)\right),$$

where $g_0$ is a scalar flat metric on $\Sigma_0$.

Since $(n-1)(\log \varphi(s))' = h$, we see that

$$\varphi(s) = \exp \left(s \frac{h}{n-1} + C\right) = \exp \left(s \frac{h}{n-1}\right) \exp(C)$$

for some constant $C \in \mathbb{R}$. We define $(\tilde{N}, g_\tilde{N})$ to be $(\Sigma_0, \exp(-2C)g_0)$. □

Theorem 5.3 implies part (B) of Theorem 2. The last ingredient we need is the following observation that appears in [24, Theorem 2.3] and is attributed to J. P. Bourguignon:

Proposition 5.4 Let $\Sigma$ be closed connected Riemannian manifold. If $\Sigma$ does not admit a metric with positive scalar curvature and $g$ is a Riemannian metric on $\Sigma$ with $\text{Sc}(\Sigma, g) \geq 0$, then $(\Sigma, g)$ is Ricci flat.

Remark 5.5 The rigidity analysis in the Proof of Theorem 5.3 could be adapted to prove rigidity of part (A) of Theorem 2 in case $\text{width}(X, g) = \text{width}(M, g_\varphi)$ if there was a way to guarantee the existence of a $\mu$-bubble in this situation. This is connected to Remark 2.4 and Remark 3.9.

Remark 5.6 As we already alluded to in Remark 2.6, Theorem 5.1 as well as Theorem 5.3 apply to any oriented band $X$ in dimension $n = 2$, as for any closed hypersurface $\Sigma$ (a collection of circles), which separates $\partial_\pm X$, the operator $-\Delta_\Sigma + \frac{1}{2} \text{Sc}(\Sigma, g) = -\Delta_\Sigma$ has first eigenvalue equal to zero.

6 Concerning separating hypersurfaces

Lemma 6.1 Let $Y^{n-1}$ be a closed connected oriented manifold and $X = Y \times [-1, 1]$. If $\Sigma$ is a closed embedded hypersurface in $X$, which separates $\partial_-X$ and $\partial_+X$, there is one connected component $\Sigma_0$ of $\Sigma$ that separates $\partial_-X$ and $\partial_+X$.

Proof Without loss of generality $\Sigma$ can be assumed to be oriented, since nonorientable components are non-separating. Furthermore we can assume that $\Sigma \subset \bar{X}$ (otherwise we isotope $\Sigma$ by flowing along the interior unit normal vector field to $\partial X$ for a short time).

The relative homology group $H_1(X, \partial X)$ is generated by paths $\gamma : [0, 1] \to X$ with $\gamma(0) \in \partial_-X$ and $\gamma(1) \in \partial_+X$. Since the hypersurface $\Sigma$ separates $\partial_-X$ and $\partial_+X$ it has
nonzero algebraic intersection with every such path \( \gamma \). It follows by Lefschetz duality that 
\[ \{ \Sigma \} \neq 0 \in H_{n-1}(X) \cong H_{n-1}(Y) = \mathbb{Z}. \]

Of course \( \{ \Sigma \} \) is nothing but \( \{ \Sigma_0 \} + \ldots + \{ \Sigma_m \} \), where \( \Sigma_i \) are the connected components of \( \Sigma \). Since \( \{ \Sigma \} \neq 0 \) it follows that \( \{ \Sigma_i \} \neq 0 \) for some \( i \in \{ 0, \ldots, m \} \) (w.l.o.g. we can assume \( \{ \Sigma_0 \} \neq 0 \)). Going back, by Lefschetz duality, \( \Sigma_0 \) has nonzero algebraic intersection with any path \( \gamma \) which connects \( \partial_- X \) and \( \partial_+ X \) and therefore it separates \( \partial_- X \) and \( \partial_+ X \). \( \square \)

**Lemma 6.2** Let \( \Sigma \subset X \) be a separating hypersurface in a band \( X \). Then there exists a union of components of \( \Sigma \) which is a properly separating hypersurface in \( X \).

**Proof** Suppose that \( \Sigma \) is a separating hypersurface that contains a component not connected to both \( \partial_- X \) and \( \partial_+ X \) inside \( X \setminus \Sigma \). Then the hypersurface \( \Sigma' \) obtained from \( \Sigma \) by deleting this component is still a separating hypersurface. This shows that there is a minimal collection of components of \( \Sigma \) such that its union is still separating yields the desired properly separating hypersurface. \( \square \)

**Lemma 6.3** Let \( X^n \) be a connected oriented band and \( X' = Y^{n-1} \times [-1, 1] \), where \( Y \) is a closed connected oriented manifold. Let \( f : X \rightarrow X' \) be a band map with \( \deg(f) = d \neq 0 \) and \( \Sigma'^{n-1} \) be a closed embedded hypersurface in \( X \), which separates \( \partial_- X \) and \( \partial_+ X \). Then there is one connected component \( \Sigma_0 \) of \( \Sigma \) such that the map \( \pr_Y \circ f : \Sigma_0 \rightarrow Y \) has nonzero degree.

**Proof** Without loss of generality \( \Sigma \) can be assumed to be oriented, since nonorientable components are non-separating. Furthermore we can assume that \( \Sigma \subset \hat{X} \) (otherwise we isotope \( \Sigma \) by flowing along the interior unit normal vector field to \( \partial X \) for a short time).

By Lemma 6.2 there is a union of components of \( \Sigma \) which is a properly separating hypersurface. We denote this union of components by \( \Sigma' \). By construction every path \( \gamma : [0, 1] \rightarrow X \) with \( \gamma(0) \in \partial_- X \) and \( \gamma(1) \in \partial_+ X \) has algebraic intersection number equal to one with \( \Sigma' \).

Since \( f \) is a band map \( f \circ \gamma \) connects \( \partial_- X' \) and \( \partial_+ X' \). It follows that \( \{ \Sigma' \} \) is Lefschetz dual to \( f^* \alpha \), where \( \alpha \) is the generator of \( H^1(X', \partial X') \cong \mathbb{Z} \).

Consider the diagram:

\[
\begin{array}{ccc}
H^1(X, \partial X; \mathbb{Z}) & \xrightarrow{\cap[X, \partial X]} & H_{n-1}(X; \mathbb{Z}) \\
f^* \uparrow & & \downarrow f_* \\
H^1(X', \partial X'; \mathbb{Z}) & \xrightarrow{\cap[d(X', \partial X'), \partial Y]} & H_{n-1}(X'; \mathbb{Z}) & \xrightarrow{\pr_Y} & H_{n-1}(Y; \mathbb{Z}).
\end{array}
\]

We conclude that \( (\pr \circ f)_*[\Sigma'] = d[Y] \). Hence there is one connected component \( \Sigma_0 \) of \( \Sigma' \) with \( (\pr \circ f)_*[\Sigma'] \neq 0 \). By construction \( \Sigma_0 \) is also a component of \( \Sigma \). \( \square \)

**Proposition 6.4** Let \( Y \) be a closed connected oriented manifold of dimension \( n - 1 \geq 5 \) and \( X = Y \times [-1, 1] \). Let \( \Sigma_0 \) be a closed connected oriented hypersurface separating \( \partial_- X \) and \( \partial_+ X \) in \( X \). If \( \Sigma_0 \) admits a metric with positive scalar curvature, then so does \( Y \).

**Proof** The proof uses standard results and ideas from high dimensional topology. We can assume that \( \Sigma_0 \subset \hat{X} \) (otherwise we isotope \( \Sigma_0 \) by flowing along the interior normal vector field to \( \partial X \) for a short time).

We want to see that \( Y \) can be obtained from \( \Sigma_0 \) by a finite sequence of surgeries in codimension \( \geq 3 \) and hence, by the well known argument of Gromov and Lawson [23,
Theorem A], a positive scalar curvature metric on $\Sigma_0$ can be transported to $Y$. See [14] for full details of the proof of [23, Theorem A].

We denote by $W$ the connected component of $X \setminus \Sigma_0$ which contains $\partial X$. Then $W$ is a cobordism $W : Y \leadsto \Sigma_0$. We restrict the projection $X \to Y$ to $W$ and obtain a retract map $r : W \to Y$.

**Claim** The cobordism $W : Y \leadsto \Sigma_0$ and the retract map $r : W \to Y$ can be improved via surgery in the interior of $W$ to a cobordism $W_2 : Y \leadsto \Sigma_0$ with a retract map $r_2 : W_2 \to Y$, which is 3-connected. The inclusion $i : Y \hookrightarrow W_2$ will be 2-connected since $i \circ r_2 = id_{M'}$.

**Proof of Claim** The main idea is to use surgery below the middle dimension, as it is explained in [44, Chapter 1], on the retract map $r$. This is possible since $r$ is what is usually called a *normal* map i.e if $\nu(Y)$ denotes the stable normal bundle of $Y$ (the stable vector bundle on $Y$ represented by the normal bundle of an embedding of $Y$ into some high dimensional euclidean space), there is a stable trivialization of $r^*\nu(Y) \oplus TW$. It is important to point out that the result of surgery below the middle dimension on a normal map is normal as well (see [44, Chapter 1]).

Since $r$ is a retract map the induced map $\pi_1(r) : \pi_1(W) \to \pi_1(Y)$ is already surjective and its kernel is finitely generated as a normal subgroup of $\pi_1(W)$, since $\pi_1(W)$ is finitely generated and $\pi_1(Y)$ is finitely presented (see [37, Lemma 3.2]). Let $\alpha$ be a generator of $\ker(\pi_1(r))$. Since $1 < n/2$ we can represent $\alpha$ by an embedding $S^1 \hookrightarrow W$. Since $W$ is oriented the normal bundle of this embedding is trivial and hence we can kill $\alpha$ by surgery in the interior of $W$. We obtain a cobordism $W_\alpha : Y \leadsto \Sigma_0$ and a retract map $r_\alpha : W_\alpha \to Y$.

After repeating this step finitely many times we end up with $W_1 : Y \leadsto \Sigma_0$ and a retract map $r_1 : W_1 \to Y$, which is 2-connected.

Next we need to kill the kernel of $\pi_2(r_1) : \pi_2(W_1) \to \pi_2(Y)$. In order to do so one has to argue that this is possible with finitely many surgeries along elements of $\ker(\pi_2(r_1))$. We proceed similarly as in the proof of [37, Proposition 3.1], which in turn is based on [40, Lemma 5.6] and [42, Lemma 1.1]. Since $Y$ and $W_1$ are compact manifolds, if one starts with a handle decomposition of $W_1$ relative to $Y$ one can use handle cancellation [43] and the fact that $Y \hookrightarrow W_1$ induces an isomorphism on $\pi_0$ and $\pi_1$ to get rid of 0-handles and 1-handles. All the (finitely many) 2-handles in this new handlebody are attached to $Y$ via contractible maps (otherwise they would kill elements in $\pi_1$). Hence the 2-skeleton $(W_1, Y)^{(2)}$ arising from this new handlebody is homotopy equivalent to $Y \vee (\bigvee_{j \in J} S^2)$. The 2-spheres in this wedge product finitely generate $\ker(\pi_2(r_1))$ as a $\mathbb{Z}[\pi_1(Y)]$-module over the common fundamental group $\pi_1(Y) = \pi_1(W_1)$.

Since $2 < n/2$ we can represent each of those generators by an embedding $f : S^2 \hookrightarrow W_1$. Since $r \circ f(S^2)$ is contractible, there is a map $g : D^3 \to Y$ such that the following diagram commutes:

$$
\begin{array}{ccc}
S^2 & \overset{i}{\longrightarrow} & D^3 \\
\downarrow f & & \downarrow g \\
W_1 & \overset{r_1}{\longrightarrow} & Y.
\end{array}
$$

The stable trivialization of $r^*_1 \nu(Y) \oplus TW_1$ induces a stable trivialization of $f^*r^*_1 \nu(Y) \oplus f^*TW_1 = i^*g^*\nu(Y) \oplus f^*TW_1$. But $i^*g^*\nu(Y)$ is trivial since $D^3$ is contractible and hence $f^*TW_1 \cong \nu(S^2, W_1) \oplus TS^2$ is stably trivial. Since $TS^2$ is stably trivial it follows that $\nu(S^2, W_1)$ is stably trivial and since $2 < (n - 1)/2$, we conclude that $\nu(S^2, W_1)$ is trivial. For the last step we refer to [31, Section 3.4.1, page 73] where it is pointed out that, if $2 \leq (n - 1)/2$, the natural inclusion of the classifying spaces $BO(n - 2) \to BO(n - 2 + a)$...
is 3-connected for any $a \geq 0$. Hence $\nu(S^2, W_1)$ is trivial since $\nu(S^2, W_1) \oplus \mathbb{R}^a$ is trivial for some $a \geq 0$. Even more details can be found in [32, Lemma 4].

Hence we can kill $\text{ker}(\pi_2(r_1))$ in finitely many surgery steps. We end up with a cobordism $W_2 : Y \sim \Sigma$ and a retract map $r_2 : W_2 \to Y$ which is 3-connected. Consequently the inclusion $\iota : Y \hookrightarrow W_2$ is 2-connected. □

If we start with a handle decomposition of $W_2$ with respect to $Y$ we can use handle cancellation [43] to get rid of all the 0-, 1- or 2-handles since the inclusion $\iota : Y \hookrightarrow W_2$ is 2-connected. Turning this upside down this handle decomposition can be interpreted as a handle decomposition of $W_2$ with respect to $\Sigma_0$. In this interpretation the dimension of each handle becomes its codimension.

Consequently $W_2$ can be obtained from $\Sigma_0 \times [-1, 1]$ by attaching handles of codimension $\geq 3$ and $Y$ can be obtained from $\Sigma_0$ by a finite sequence of surgeries in codimension $\geq 3$. Thus, by [23, Theorem A], $Y$ admits a metric of positive scalar curvature if $\Sigma_0$ does. □

We have all the ingredients to prove the main results of Sect. 2.2. Proposition 2.17 follows directly from Lemma 6.1 and Proposition 6.4. Further Proposition 2.21 follows directly from Lemma 6.1 and Definition 2.19. For the convenience of the reader we also include a proof of Proposition 2.18, which heavily draws on the work of Zeidler [47, 48].

**Proof of Proposition 2.18** By Lemma 6.1 there is one connected component $\Sigma_0$ of $\Sigma$, which separates $\partial_- X$ and $\partial_+ X$. We can assume that $\Sigma_0 \subset X$ (otherwise we isotope $\Sigma_0$ by flowing along the interior normal vector field to $\partial X$ for a short time). We consider the real Miščenko bundle $L_Y \to Y$, which is the flat bundle of finitely generated projective Hilbert-$C^*\pi_1 Y$-modules associated to the representation of $\pi_1 Y$ on $C^*\pi_1 Y$ left multiplication. Recall (see for example [47, Section 2]) that the Rosenberg index $\alpha(Y) \in KO_{n-1}(C^*\pi_1 Y)$ is then the (K-theoretic) index of the Dirac operator on the spinor bundle of $Y$ twisted with $L_Y$. We pull back $L_Y$ to $X$ via the projection $X \to Y$ and restrict this pullback bundle to the connected component $W$ of $X$ which is bounded by $\partial_- X$ and $\Sigma_0$. We denote the resulting bundle by $E \to W$.

Since $Y$ is spin, so are $W$ and $\Sigma_0$. If we restrict $E$ to $\Sigma_0$, the index of the Dirac operator on the spinor bundle of $\Sigma_0$ twisted with the restriction of $E$ is an element in $KO_{n-1}(C^*\pi_1 Y)$ which we denote by $\alpha_E(\Sigma_0)$. By bordism invariance of the index $\alpha_E(\Sigma_0) = \alpha(Y) \neq 0$ and hence $\Sigma_0$ does not admit a metric with positive scalar curvature as $E$ is a flat bundle and by the usual argument involving the Lichnerowicz–Weitzenböck formula. □

## 7 A codimension two obstruction

In this section we present a detailed proof of Theorem 2.23. To do so we implement the ideas of [22, Section 7, Main Theorem] using the techniques developed in [12].

**Remark 7.1** To unburden the notation in this section we will denote the scalar curvature of a Riemannian manifold $(M, g)$ by $R_M$. If $B \subset M$ is an embedded submanifold we will denote the scalar curvature of the induced metric $g|_B$ by $R_B$.

**Definition 7.2** Let $M^n$ be a band. Let $\alpha \neq 0 \in H_{n-2}(M; \mathbb{Z})$ be a non torsion homology class. We say that $\alpha$ is a band class if there are $\alpha^+ \in H_{n-2}(\partial_+ M; \mathbb{Z})$ and $\alpha^- \in H_{n-2}(\partial_- M; \mathbb{Z})$ with $\alpha = \iota_*(\alpha^\pm)$, where $\iota : \partial M \to M$ denotes the inclusion of the boundary.

The main analytical tool we need to develop is the following proposition, which is reminiscent of what Gromov, in [22, Section 3], calls Richard’s Lemma in reference to [33]. The proof, however, follows in the line of [12, Sections 6.1 & 6.2].
Proposition 7.3 Let \((M^4, g)\) be a Riemannian band and \(\alpha \in H_2(M; \mathbb{Z})\) be a band class. If 
\(R_M > \sigma > 0\) and \(\text{width}(M, g) > \frac{2\pi}{\sqrt{\sigma}}\), there is a smooth oriented submanifold \(\Sigma\) which represents \(\alpha\) and each connected component \(\Sigma_0\) of \(\Sigma\) is homeomorphic to a 2-sphere with 
\[
\text{diam}(\Sigma_0, g|_{\Sigma_0}) \leq \sqrt{\frac{2}{\inf R_M - \sigma}}.
\]

\textbf{Proof} Denote \(\ell = \frac{\pi}{\sqrt{\sigma}}\). Let \(\beta \in H_3(M, \partial M; \mathbb{Z})\) be the relative class with \(\partial\beta = \alpha^+ - \alpha^-\). Let \(B^3\) be a free boundary minimal hypersurface in the class \(\beta\).

One can obtain \(B^3\) by directly minimizing area in \(\beta\), as it is done in [25, Proof of Theorem 12.1, pp. 398–399] and later [19, Induction Step, p.652]. However as it is noted there, while the minimizer is smooth in \(\hat{M}\), it might not be smooth at the points where it intersects the boundary \(\partial M\). This can be overcome by shaving off an arbitrarily small collar neighborhood of \(\partial M\). The strict inequality \(\text{width}(M, g) > \frac{2\pi}{\sqrt{\sigma}}\) can be preserved in this process.

By stability of \(\hat{B}\) and the classical second variation formula for the area functional we see that 
\[
\int_B |\nabla_B \psi|^2 - \frac{1}{2}(R_M - R_B + |A|^2)\psi^2 d\mathcal{H}^3 \geq 0,
\]
for all \(\psi \in C^1_0(B)\). As such, there is a function \(u \in C_0^\infty(B)\) with \(u > 0\) on \(\hat{B}\) and
\[
\Delta_B u \leq -\frac{1}{2}(R_M - R_B + |A|^2)u. \tag{15}
\]
Furthermore \((B, g|_B)\) is a Riemannian band with \(\text{width}(B, g|_B) > 2\ell\). We can shrink \(B\) a little bit from both sides such that \(\text{width} > 2\ell\) remains true but \(\partial B \subset \hat{M}\) (this guarantees \(u > 0\) on \(B\)). Let \(\phi : B \to [-\ell, \ell]\) be the map produced by Lemma 3.12 and set \(h(x) = -\frac{\pi}{\ell} \tan(\frac{\pi}{2\ell} \phi(x))\).

We define \(\Omega_0 = \phi^{-1}[-\ell, 0]\) and consider the functional 
\[
A(\hat{\Omega}) = \int_{\partial \hat{\Omega}} ud\mathcal{H}^2 - \int_B (\chi_{\hat{\Omega}} - \chi_{\Omega_0}) hud\mathcal{H}^3
\]
for all Caccioppoli sets \(\hat{\Omega}\) with \(\hat{\Omega} \cap \Omega_0\) contained in the interior of \(B\). By Lemma 4.9 we find a \(\mu\)-bubble \(\Omega \subset B\) with smooth boundary \(\Sigma = (\partial \Omega \setminus \partial B\), which represents the class \(\alpha \in H_2(M; \mathbb{Z})\). Stability of \(\Omega\) and Lemma 4.11 imply that for each connected component \(\Sigma_0\) of \(\Sigma\) we have:
\[
\int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \frac{1}{2}(R_B - R_{\Sigma_0} - H^2 + |A|^2) \psi^2 u 
+ \left(2H \langle \nabla_B u, v \rangle + \frac{d^2 u}{dv^2} - Hu - \langle \nabla_B (hu), v \rangle \right) \psi^2 
\geq 0
\]
We use:
\[
\frac{d^2 u}{dv^2} + H \langle \nabla_B u, v \rangle = \Delta_B u - \Delta_{\Sigma_0} u \leq -\frac{1}{2}(R_M - R_B + |A|^2)u - \Delta_{\Sigma_0} u.
\]
and as a consequence of Lemma 4.10
\[
Hu = H^2 u + H \langle \nabla_B u, v \rangle,
\]
and
\[ \frac{1}{2} H^2 \psi^2 u = \frac{1}{2} u^{-1} (\nabla_B u, v)^2 \psi^2 - h(\nabla_B u, v) + \frac{1}{2} h^2 \psi^2 u. \]

Hence
\[
0 \leq \int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \frac{1}{2} (R_M - R_{\Sigma_0} + H^2 + 2|A|^2) \psi^2 u - (\Delta_{\Sigma_0} u + h(\nabla_B u, v) + u(\nabla_B h, v)) \psi^2 u
\]
\[
= \int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \frac{1}{2} (R_M - R_{\Sigma_0} + 2|A|^2) \psi^2 u - (\Delta_{\Sigma_0} u) \psi^2 - \frac{1}{2} (h^2 + 2(\nabla_B h, v)) \psi^2 u
\]
\[
\leq \int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) \psi^2 u - (\Delta_{\Sigma_0} u) \psi^2 - \frac{1}{2} (\sigma + h^2 + 2(\nabla_B h, v)) \psi^2 u
\]

and since, using \( \ell = \frac{\pi}{\sqrt{\sigma}} \), we can estimate
\[
\sigma + h^2 + 2(\nabla_B h, v) \geq \sigma + h^2 - 2|\nabla_B h| = 0.
\]

We conclude
\[
0 < \int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) \psi^2 u - (\Delta_{\Sigma_0} u) \psi^2. \tag{16}
\]

If we choose \( \psi = u^{-\frac{1}{2}} \), this yields
\[
0 < \int_{\Sigma_0} |\nabla_{\Sigma_0} u^{-\frac{1}{2}}|^2 u - \frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) - (\Delta_{\Sigma_0} u) u^{-1}
\]
\[
= \int_{\Sigma_0} -\frac{3}{4} u^{-2} |\nabla_{\Sigma_0} u|^2 - \frac{1}{2} (R_M - \sigma - R_{\Sigma_0})
\]
\[
\leq \int_{\Sigma_0} -\frac{1}{2} (R_M - \sigma - R_{\Sigma_0}),
\]
or equivalently
\[
\frac{1}{2} \int_{\Sigma_0} R_M - \sigma \leq \frac{1}{2} \int_{\Sigma_0} R_{\Sigma_0} = 2\pi \chi(\Sigma_0),
\]

which implies that \( \Sigma_0 \) is a sphere with area(\( \Sigma_0 \)) \( \leq \frac{8\pi}{R_M - \sigma} \).

To finish the proof we return to (16) and see that it implies the existence of a positive function \( w \in C^\infty(\Sigma_0) \), with
\[
\text{div}_{\Sigma_0} (u \nabla_{\Sigma_0} w) \leq -\frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) wu - (\Delta_{\Sigma_0} u) w.
\]

If we set \( \lambda = uw \), then
\[
\Delta_{\Sigma_0} \lambda = \text{div}_{\Sigma_0} (u \nabla_{\Sigma_0} w) + \text{div}_{\Sigma_0} (w \nabla_{\Sigma_0} u)
\]
\[
\leq -\frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) \lambda - (\Delta_{\Sigma_0} u) w + \text{div}_{\Sigma_0} (w \nabla_{\Sigma_0} u)
\]
\[
\leq -\frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) \lambda + (\nabla_{\Sigma_0} u, \nabla_{\Sigma_0} w)
\]
\[
\leq -\frac{1}{2} (R_M - \sigma - R_{\Sigma_0}) \lambda + \frac{1}{2} \lambda^{-1} |\nabla_{\Sigma_0} \lambda|^2.
\]
Now \( \text{diam}(\Sigma_0, g|\Sigma_0) \leq \pi \sqrt{\frac{2}{\inf R_M - \sigma}} \) follows directly from the next lemma.

\[ \square \]

**Lemma 7.4** ([12, Lemma 16]) Let \((N^2, g)\) be a closed 2-dimensional Riemannian manifold. If there is a smooth function \( \lambda > 0 \) on \( \Sigma_0 \) with

\[ \Delta_N \lambda \leq \frac{1}{2}(C - R_N)\lambda + \frac{1}{2}\lambda^{-1}|\nabla_N \lambda|^2 \]

for some \( C > 0 \), then \( \text{diam}(N, g) \leq \sqrt{\frac{2}{C}} \pi \).

**Proof** Let \( \ell = \sqrt{\frac{2}{C}} \pi \). Assume for a contradiction, that there are \( p, q \in N \) with \( \text{dist}_g(p, q) > \ell \). For \( \varepsilon > 0 \) small enough \( M = N \setminus (B_\varepsilon(p) \cup B_\varepsilon(q)) \) is a band with \( \text{width}(M, g) > \ell \). Let \( \phi : M \to [-\frac{\ell}{2}, \frac{\ell}{2}] \) be the map we get from Lemma 3.12 and define \( h(x) = -\frac{2\pi}{\ell} \tan(\frac{\pi}{\ell} \phi(x)) \).

Let \( \Omega_0 = \phi^{-1}[\frac{-\ell}{2}, 0] \) (w.l.o.g 0 is a regular value of \( \phi \)) and consider the functional

\[ A(\hat{\Omega}) = \int_{\partial^+ \hat{\Omega}} \lambda - \int_M (\chi_{\hat{\Omega}} - \chi_{\Omega_0}) h \lambda d\mathcal{H}^2 \]

for all Caccioppoli sets \( \hat{\Omega} \) with \( \hat{\Omega} \Delta \Omega_0 \) contained in the interior of \( M \).

By Lemma 4.9 there is a minimizer \( \Omega \) with smooth boundary \( (\partial \Omega \setminus \partial_{-}M) \) and by Lemma 4.10 every connected component \( \Sigma \) satisfies

\[ H = -\lambda^{-1} \langle \nabla_M \lambda, v \rangle + h. \]

Note that in this case \( H \) is the geodesic curvature.

By stability and Lemma 4.11 we see

\[
0 \leq \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 \lambda - \frac{1}{2}(R_M - H^2 + |A|^2)\psi^2 \lambda + (2H \langle \nabla_M \lambda, v \rangle
\]

\[ + \frac{d^2 \lambda}{dv^2} - H h \lambda - \langle \nabla_M (h \lambda), v \rangle \psi^2 \]

for all \( \psi \in C^1(\Sigma) \). We use:

\[ \frac{d^2 \lambda}{dv^2} + H \langle \nabla_M u, v \rangle = \Delta_M \lambda - \Delta_{\Sigma} \lambda \leq -\Delta_{\Sigma} \lambda - \frac{1}{2}(C - R_M)\lambda + \frac{1}{2}\lambda^{-1}|\nabla_M \lambda|^2, \]

as a consequence of Lemma 4.10

\[ H h \lambda = H^2 \lambda + H \langle \nabla_M \lambda, v \rangle, \]

and

\[ \frac{1}{2} H^2 \psi^2 \lambda = \frac{1}{2}\lambda^{-1} \langle \nabla_M \lambda, v \rangle^2 \psi^2 - \lambda \langle \nabla_M \lambda, v \rangle + \frac{1}{2} h^2 \psi^2 \lambda. \]

It follows that

\[
0 \leq \int_{\Sigma} |\nabla_{\Sigma} \psi|^2 \lambda + \frac{1}{2}\lambda^{-1}|\nabla_M \lambda|^2 \psi^2 - \frac{1}{2}\lambda^{-1} \langle \nabla_M \lambda, v \rangle^2 \psi^2 - (\Delta_{\Sigma} \lambda) \psi^2
\]

\[ - \frac{1}{2}(C + h^2 + 2\langle \nabla_M h, v \rangle) \psi^2 \lambda \]
and since $C + h^2 + 2(\nabla_M h, v) > 0$ (remember that $\text{Lip}(\phi) < 1$) by construction of $h$, we see

$$0 < \int \nabla_{\Sigma} \psi^2 + \frac{1}{2} \lambda^{-1} |\nabla_M \lambda|^2 \psi^2 - \frac{1}{2} \lambda^{-1} (\nabla_M \lambda, v)^2 \psi^2 - (\Delta_{\Sigma} \lambda) \psi^2$$

If we choose $\psi = \lambda^{-\frac{1}{2}}$ this yields

$$0 < \int -\frac{3}{4} \lambda^{-2} |\nabla_{\Sigma} \lambda|^2 + \frac{1}{2} \lambda^{-2} |\nabla_M \lambda|^2 - \frac{1}{2} \lambda^{-2} (\nabla_M \lambda, v)^2$$

$$= \int -\frac{1}{4} \lambda^{-2} |\nabla_{\Sigma} \lambda|^2,$$

which is a contradiction. □

**Definition 7.5** Let $(X^n, g)$ be a complete oriented Riemannian manifold and $c$ a locally finite singular $k$-cycle. Then the the filling radius of $c$ in $(X, g)$ is defined to be

$$\text{FillRad}_\mathbb{Z}(c, X) = \inf\{r > 0 | [c] = 0 \in H^k_{lf}(U_r(c); \mathbb{Z})\},$$

where $U_r(c)$ denotes the open $r$-neighborhood of $c$ in $(X, g)$. If one replaces $\mathbb{Z}$ by $\mathbb{Q}$ the same definition yields the rational filling radius.

**Remark 7.6** In [17, Section 1] Gromov generalizes the above and defines the filling radius $\text{FillRad}_\mathbb{Z}(X, g)$ of a complete oriented Riemannian manifold $(X, g)$, with respect to the Kuratowski embedding of $(X, d_g)$. Furthermore he proves two results we will need in the following:

(A) $\text{FillRad}_\mathbb{Z}(\mathbb{R}, g_{std}) = \infty$ (see [17, Section 4.4.C])
(B) if $X$ is isometrically embedded in a Metric space $(S, d)$, then $\text{FillRad}_\mathbb{Z}(X, S) \geq \text{FillRad}_\mathbb{Z}(X, g)$ (see [17, Section 1]).

Both points remain true with rational coefficients.

**Lemma 7.7** ([27, Theorem IX.4.7]) Let $X^n$ be an oriented manifold. Let $C \subset X$ be a closed subset such that $\partial C = C \setminus \dot{C}$ is a smooth submanifold. Then $H^1_{lf}(C; \mathbb{Z}) \cong H^{n-1}(X, X \setminus C; \mathbb{Z})$.

**Lemma 7.8** (Codim 2 Linking Lemma [21, Lemma 4.G]) Let $Y^n$ be a closed aspherical manifold and $g$ a Riemannian metric on $Y$. For every $\sigma > 0$ there is a compact band $M$ in the universal cover $(\tilde{Y}, \tilde{g})$ such that: width$(M, \tilde{g}) > \frac{2\pi}{\sqrt{\sigma}}$, there is a band class $\alpha \in H_{n-2}(M; \mathbb{Z})$ and for every cycle $c \subset M$ representing a nonzero multiple of $\alpha$ we have $\text{FillRad}_\mathbb{Z}(c, \tilde{Y}) > \frac{2\pi}{\sqrt{\sigma}}$.

**Proof** Let $\sigma > 0$ be arbitrary. By [12, Lemma 6] there is a geodesic line

$$\gamma : \mathbb{R} \to (\tilde{Y}, \tilde{g}).$$

Since $\gamma$ is an isometric embedding of the real line,

$$\text{FillRad}_\mathbb{Z}(\gamma, \tilde{Y}) \geq \text{FillRad}_\mathbb{Z}(\mathbb{R}, g_{std}) = \infty$$

(see Remark 7.6), hence for all $r > 0$ the line $\gamma$ represents a non-zero class in $H^1_{lf}(U_r(\gamma); \mathbb{Z})$, where $U_r(\gamma)$ denotes the open $r$-neighborhood of $\gamma$ in $(\tilde{Y}, \tilde{g})$. Since

$$\text{FillRad}_\mathbb{Q}(\gamma, \tilde{Y}) \geq \text{FillRad}_\mathbb{Q}(\mathbb{R}, g_{std}) = \infty$$

$\square$ Springer
we see by the same argument, that $[\gamma]$ is non torsion in $H_{1}^{f}(U_{\tau}(\gamma); \mathbb{Z})$.

For some $\epsilon > 0$ let $\rho$ be a smooth approximation of $\text{dist}(\gamma, \cdot)$ which is $\epsilon$-close. There is a sequence $(r_{k})_{k \in \mathbb{N}}$ of regular values of $\rho$ with $r_{k} \to \infty$ and the property that for all $k \in \mathbb{N}$ we have $r_{k} > 2\epsilon$ and $r_{k+1} - r_{k} > 2\epsilon$.

Denote $\overline{U}_{k} = \rho^{-1}[0, r_{k}]$. By construction $[\gamma] \neq 0 \in H_{1}^{f}(\overline{U}_{k}; \mathbb{Z})$ for all $k$ and the class is non torsion. Thus by Lemma 7.7, the fact that $\overline{Y}$ is contractible we see:

$$0 \neq [\gamma] \in H_{1}^{f}(\overline{U}_{k}; \mathbb{Z}) \cong H^{n-1}(\overline{Y}, \overline{Y} \setminus \overline{U}_{k}; \mathbb{Z}) \cong H^{n-2}(\overline{\gamma}, \overline{\gamma} \setminus \overline{U}_{k}; \mathbb{Z}).$$

Furthermore, since $[\gamma]$ is non-torsion its image in $H^{n-2}(\overline{\gamma}, \overline{\gamma} \setminus \overline{U}_{k}; \mathbb{Z})$ corresponds by the UCT to an element $\alpha_{k} \neq 0 \in H_{n-2}(\overline{\gamma}, \overline{\gamma} \setminus \overline{U}_{k}; \mathbb{Z})$. If we represent $\alpha_{k}$ by a closed smooth submanifold $N_{k} \subset \overline{\gamma} \setminus \overline{U}_{k}$, then $N_{k}$ is linked with $\gamma$ and $\text{dist}(\gamma, N_{k}) > r_{k} - \epsilon > r_{k+1} - \epsilon$. That $N_{k}$ is linked with $\gamma$ means that $[N_{k}] \neq 0 \in H_{n-2}(\overline{\gamma}; \mathbb{Z})$ is every fill-in of $N_{k}$ in $\overline{\gamma}$, which exists because $\overline{Y}$ is contractible, intersects $\gamma$.

Let $V_{k}$ be a smoothed version (as before) of the closed $(r_{k+1}/2)$-neighborhood of $N_{k}$. Then $0 \neq \alpha_{k} \in H_{n-1}(V_{k}; \mathbb{Z})$, since $N_{k}$ is linked with $\gamma$ and $V_{k} \subset (\overline{\gamma} \setminus \overline{U}_{k})$. Furthermore any cycle $c \in V_{k}$, which represents a band class in $H_{n}(\overline{\gamma}; \mathbb{Z})$ vanishes since $\text{dist}(\gamma, c) \geq (r_{k+1}/2)$. We want to see that there is a class $\alpha_{k}^{+} \in H_{n-1}(\partial V_{k}; \mathbb{Z})$ with $i(\alpha_{k}^{+}) = \alpha_{k}$, where $i : \partial V_{k} \to V_{k}$ denotes the inclusion of the boundary.

Consider the $\epsilon$-neighborhood $U_{\epsilon}(\gamma)$ of $\gamma$. If we choose $\epsilon$ small enough, there is a closed smooth submanifold $N_{0} \subset U_{\epsilon}(\gamma)$ such that $[N_{0}] = [N_{k}] \in H_{n-2}(\overline{\gamma}; \mathbb{Z})$ (this $N_{0}$ will be the image under the exponential map of small sphere around the origin in the fiber over $\gamma(0)$ in the normal bundle of $\gamma$). Hence we can find a smooth oriented submanifold $B^{n-1}$ with $\partial B = N_{0} \cup N_{k}$ and by possibly deforming $B$ a little bit we can ensure that $B$ intersects $\partial V_{k}$ transversely in a closed $(n-2)$-dimensional submanifold $N_{k}^{+}$. Then $[N_{k}^{+}]$ is homologous to $[N_{k}]$ in $H_{n-2}(V_{k}; \mathbb{Z})$ and we can set $\alpha_{k}^{+} = [N_{k}^{+}] \in H_{n-2}(\partial V_{k}; \mathbb{Z})$.

Finally, for $\delta > 0$ small enough, $U_{\delta}(N_{k})$ is isometric via the exponential map to the normal $\delta$-disc bundle $D_{\delta}(N_{k})$. It follows that $\partial U_{\delta}(N_{k}) = N_{k} \times S^{1}$ and we set $\alpha_{k}^{-} = [N_{k} \times \{0\}] \in H_{n-2}(N_{k} \times S^{1}; \mathbb{Z})$. We conclude that $M_{k} = V_{k} \cup U_{\delta}(N_{k})$ is a compact band with boundary $\partial_{-} M_{k} = \partial V_{k}$ and $\partial_{+} M_{k} = \partial U_{\delta}(N_{k}) = N_{k} \times S^{1}$, which has width $(M_{k}, \overline{g}) \geq \frac{r_{k+1}}{2} - \epsilon - \delta$. Furthermore the triple $(\alpha_{-}, \alpha_{k}^{+}, \alpha_{k}^{-})$ represents a band class in $M_{k}$. For $k$ big enough width $M_{k}, \overline{g}) \geq \frac{r_{k+1}}{2} - \epsilon - \delta > 2\frac{\epsilon}{\sqrt{\alpha}}$. \hfill \Box

**Lemma 7.9** Let $M'$ and $M$ be two smooth connected oriented $n$-dimensional bands and $f : M' \to M$ be a map of degree $d \neq 0$ with $f(\partial_{\pm} M') = \partial_{\pm} M$. If $\alpha \in H_{n-2}(M; \mathbb{Z})$ is a band class, there is a band class $\alpha' \in H_{n-2}(M'; \mathbb{Z})$ with $f_{*}\alpha(\alpha') = d\alpha$.

**Proof** Consider the following diagram:

$$\begin{array}{ccc}
H_{n-2}(M'; \mathbb{Z}) \cap [M', \partial M'] & \cong & H^{2}(M', \partial M'; \mathbb{Z}) \\
\downarrow f_{*} & & \uparrow f^{*}
\end{array}$$

$$H_{n-2}(M; \mathbb{Z}) \cap [M, \partial M] \cong H^{2}(M, \partial M; \mathbb{Z}).$$

The diagram commutes since $f_{*}[M', \partial M'] = d[M, \partial M]$. By Lefschetz duality there is a unique cohomology class $\eta \in H^{2}(M, \partial M; \mathbb{Z})$ with $\eta \cap d[M, \partial M] = d\alpha \neq 0$ since $\alpha$ is non torsion. Then $\alpha' := f^{*}\eta \cap [M', \partial M']$ is such that $f_{*}\alpha(\alpha') = d\alpha$.

The (co)homology of $\partial M$ and $\partial M'$ splits as the direct sum of the (co)homology of the components $\partial_{-} M$ and $\partial_{+} M$ i. e. $H_{*}(\partial M) = H_{*}(\partial_{-} M) \oplus H_{*}(\partial_{+} M)$ and $H^{*}(\partial M) = \oplus$ Springer
$H^*(\partial_- M) \oplus H^*(\partial_+ M)$. The induced maps $f_*$ and $f^*$ split into components as well. By comparing the components of

$$f_*([\partial_\pm M']) - f_*([\partial_- M']) = \partial f_*[M', \partial M'] = \partial d[M, \partial M] = d[\partial_+ M] - d[\partial_- M],$$

we conclude that $f_*([\partial_\pm M']) = d[\partial_\pm M]$ and $f_*(\partial_- M')] = d[\partial_- M]$, so the restricted maps between the boundary components have degree $d$ as well.

For both boundary components we separately write down a diagram as above

$$
\begin{array}{ccc}
H_{n-2}(\partial_\pm M'; \mathbb{Z}) & \xrightarrow{\cap[d,\pm M]} & H^1(\partial_\pm M'; \mathbb{Z}) \\
\downarrow f_* & & \downarrow f^* \\
H_{n-2}(\partial_\pm M; \mathbb{Z}) & \xrightarrow{\cap[d,\pm M]} & H^1(\partial_\pm M; \mathbb{Z})
\end{array}
$$

and we find unique cohomology classes $\eta^\pm \in H^1(\partial_\pm M; \mathbb{Z})$ with $\eta^\pm \cap d[\partial_\pm M] = \pm d\alpha^\pm$.

We then consider

$$
\begin{array}{ccc}
H^1(\partial_- M; \mathbb{Z}) \oplus H^1(\partial_+ M; \mathbb{Z}) & \xrightarrow{\cap[d,M]} & H_{n-2}(\partial_- M; \mathbb{Z}) \oplus H_{n-2}(\partial_+ M; \mathbb{Z}) \\
\downarrow & & \downarrow \iota_\ast \\
H^2(M, \partial M; \mathbb{Z}) & \xrightarrow{\cap[M]} & H_{n-2}(M; \mathbb{Z})
\end{array}
$$

By comparing components we see that $\eta^-$ and $\eta^+$ map to $\eta$ under the connecting homomorphism $H^1(\partial M; \mathbb{Z}) \to H^2(M, \partial M; \mathbb{Z})$.

Thus we define classes $\alpha^\pm := f^*\eta^\pm \cap [\partial_\pm M']$. Finally

$$
\begin{array}{ccc}
H^1(\partial M; \mathbb{Z}) & \xrightarrow{f^*} & H^1(\partial M'; \mathbb{Z}) \xrightarrow{\cap[d,M']} H_{n-2}(\partial M'; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^2(M, \partial M; \mathbb{Z}) & \xrightarrow{f^*} & H^2(M', \partial M'; \mathbb{Z}) \xrightarrow{\cap[M]} H_{n-2}(M'; \mathbb{Z})
\end{array}
$$

implies that $\iota_\ast(\alpha^\pm) = \alpha'$.

The following Proposition is well known. A detailed proof can be found in [13, Section 4].

**Proposition 7.10** Let $Y^n$ and $Z^n$ be closed connected oriented manifolds and $f : Z \to Y$ a smooth map with $\deg(f) \neq 0$. The pullback $pr : \hat{Z} \to Z$ of the universal covering $\hat{Y} \to Y$ has the following properties:

- $\hat{Z}$ is non compact,
- the map $f \circ pr : \hat{Z} \to Y$ can be lifted to a map $\hat{f} : \hat{Z} \to \hat{Y}$,
- $\hat{f}$ is proper and $\deg(\hat{f}) = \deg(f)$.

**Proof of Theorem 2.23** Let $Y^4$ be a closed oriented aspherical manifold. Let $Z^4$ be a closed oriented manifold and $f : Z \to Y$ a continuous map with $\deg(f) \neq 0$.

We remind the reader that for any metric on $Y$ the universal cover, equipped with the pullback metric, is uniformly contractible (see for example [17, Section 4.5.D]) i.e. for every radius $R > 0$ there is a radius $C(R) > 0$ such that for every point $y \in \tilde{Y}$ the ball $B_R(y)$ is contractible within $B_{C(R)}(y)$.

Assume for a contradiction that $Z$ admits a Riemannian metric $g_1$ with $Sc(Z, g) = R_Z > \sigma > 0$. By possibly replacing it with a homotopic map, we can assume that $f : Z \to Y$ is
smooth. Let \( g_2 \) be a Riemannian metric on \( Y \). Then \( f : (Z; g_1) \to (Y, g_2) \) is a Lipschitz map and by possibly rescaling \( g_2 \), we can assume that it is distance decreasing.

By Proposition 7.10 there is a covering space \( \hat{Z} \) of \( Z \) and a lift \( \hat{f} : \hat{Z} \to \hat{Y} \) such that \( \hat{f} \) is proper and \( \deg(\hat{f}) = \deg(f) \). By Lemma 7.8 we can find a compact band \( M \) in \((\hat{Y}, \hat{g}_2)\) with width\((M, \hat{g}_2) > \frac{2\pi}{\sqrt{\sigma}}\) and a band class \( \alpha \in H_{n-2}(M; \mathbb{Z}) \), such that for every cycle \( c \subset M \) representing a nonzero multiple of \( \alpha \) we have \( \text{FillRad}_Z(c, \hat{Y}) > \frac{2\pi}{\sqrt{\sigma}} \). By transversality we can deform the map \( \hat{f} \) by an arbitrarily small amount to make it transverse to \( M_k \) as well as \( \partial M_k = \partial_+ M \cup \partial_- M \), while remaining distance decreasing.

Then \( M' = \hat{f}^{-1}(M) \) is a compact band in \( \hat{Z} \) with smooth boundary
\[
\partial M' = \hat{f}^{-1}(\partial M) = \hat{f}^{-1}(\partial_+ M) \cup \hat{f}^{-1}(\partial_- M) =: \partial_+ M' \cup \partial_- M'
\]
and since \( \hat{f} \) is distance decreasing \( \text{width}(M', \hat{g}_1) > \frac{2\pi}{\sqrt{\sigma}} \). Furthermore \( \hat{f} \) restricts to a map \( M' \to M \) of degree non-zero. By Lemma 7.9, there is a band class \( \alpha' \in H_{n-2}(M'; \mathbb{Z}) \) with \( \hat{f}_* \alpha' = \deg(\hat{f}) \alpha \neq 0 \).

By Proposition 7.3 there is a smooth oriented submanifold \( \Sigma \) which represents \( \alpha' \) and each connected component \( \Sigma_0 \) of \( \Sigma \) is a 2-sphere with
\[
\text{diam}(\Sigma_0, \hat{g}|_{\Sigma_0}) \leq \sqrt{\frac{2}{\inf R_Z - \sigma}} \pi.
\]

Then \( \hat{f}(\Sigma) \) is a cycle \( c_0 \) in \( M \), which represents \( \deg(\hat{f}) \alpha \) and \( \text{FillRad}_Z(c_0, \hat{Y}) \leq C\left(\sqrt{\frac{2}{\inf R_Z - \sigma}} \pi\right) \), since \( \hat{f} \) is distance decreasing and \((\hat{Y}, \hat{g})\) is uniformly contractible. For \( \sigma > 0 \) small enough this yields
\[
\text{FillRad}_Z(c_0, \hat{Y}) > \frac{2\pi}{\sqrt{\sigma}} > C\left(\sqrt{\frac{2}{\inf R_Z - \sigma}} \pi\right) \geq \text{FillRad}_Z(c_0, \hat{Y}), \tag{17}
\]
which is a contradiction. \( \square \)

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Declarations

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