The Decay of Saffman and Batchelor Turbulence
Subject to Rotation, Stratification or an Imposed Magnetic Field

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Abstract. We consider unforced, statistically-axisymmetric turbulence evolving in the presence of a background rotation, an imposed stratification, or a uniform magnetic field. We focus on two canonical cases: Saffman turbulence, in which \( E(k \to 0) \sim k^2 \), and Batchelor turbulence, in which \( E(k \to 0) \sim k^4 \). It has recently been shown that, provided the large scales evolve in a self-similar manner, then \( u^2 \ell^2 / u^2 \ell^2 = \) constant in Saffman turbulence and \( u^2 \ell^2 / u^2 \ell^2 = \) constant in Batchelor turbulence (Davidson, 2009, 2010). Here the subscripts \( \perp \) and \( \parallel \) indicate directions perpendicular and parallel to the axis of symmetry, and \( \ell, \ell', \) and \( u \) are suitably defined integral scales. These constraints on the integral scales allow us to make simple, testable predictions for the temporal evolution of \( \ell, \ell', \) and \( u \) in rotating, stratified and MHD turbulence.

1. Introduction: the behavior of the large scales in the absence of body forces

In order to place the study in context, we first discuss homogeneous turbulence in the absence of body forces, starting with isotropic turbulence.

1.1. \( E \sim k^2 \) versus \( E \sim k^4 \) spectra in isotropic turbulence

Consider isotropic turbulence in which the Reynolds number is high, \( \Re = u\ell/\nu >> 1 \), where \( u \) and \( \ell \) are suitably defined integral scales. (We shall define \( u \) via \( u^2 = \langle u^2 \rangle \) in isotropic turbulence.) Provided the two-point velocity correlations, \( \langle u_i(x)u_j(x + r) \rangle = \langle u_i u_j' \rangle \), decay sufficiently rapidly with separation \( r = |r| = |x' - x| \), the energy spectrum,

\[
E(k) = \frac{1}{\pi} \int_0^\infty \langle u \cdot u' \rangle kr \sin(kr) dr
\]

(1)
takes the form

\[
E(k \to 0) = \frac{Lk^2}{4\pi^2} + \frac{Ik^4}{24\pi^2} + \cdots
\]

(2)

where

\[
L = \int \langle u \cdot u' \rangle dr, \quad I = -\int r^2 \langle u \cdot u' \rangle dr.
\]

(3)
The condition under which expansion (2) is valid is \( (\mathbf{u} \cdot \mathbf{u'})_\infty \leq O(r^{-6}) \), where the subscript \( \infty \) indicates \( r \to \infty \). The integrals \( L \) and \( I \) are known as the Saffman and Loitsyansky integrals respectively. Turbulence in which \( L \) is non-zero, and hence \( E(k \to 0) \sim Lk^2 \), is usually known as Saffman turbulence, following Saffman (1967). Conversely, in those cases where \( L = 0 \) we have \( E(k \to 0) \sim Ik^4 \), which was the generally accepted form for \( E(k) \) prior to 1967. Such spectra are sometimes called Batchelor spectra after Batchelor & Proudman (1956).

If \( (\mathbf{u} \cdot \mathbf{u'})_\infty \) decays more slowly than \( r^{-6} \) then other forms of spectra may be observed, in particular \( E(k \to 0) \sim k^n, n < 4 \). However, these non-classical spectra require very special initial conditions in order to be realized, as discussed and in Davidson (2011).

Returning to \( E \sim Lk^2 \) and \( E \sim Ik^4 \) turbulence, we note that both may be generated in computer simulations and it is the initial condition which dictates which is seen. If \( L = 0 \) at \( t = 0 \) we get Batchelor turbulence, whereas a finite value of \( L \) at \( t = 0 \) ensures Saffman turbulence. The experimental data for grid turbulence also suggests that both classes of flow may be realised. For example, Bennett & Corrsin (1978) show energy decay exponents consistent with \( E \sim Ik^4 \) and incompatible with \( E \sim Lk^2 \), whereas the recent data of Krogstad & Davidson (2010) provide the first clear-cut evidence of a Saffman spectrum in decaying grid turbulence. \( L \) and \( I \) are invariants of fully-developed, freely-decaying turbulence. The invariance of \( L \) can be confirmed by integrating the Karman-Howarth equation in the form

\[
\frac{\partial}{\partial t}(\mathbf{u} \cdot \mathbf{u'}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r^4 u^3 K) \right) + 2 \nu \nabla^2 (\mathbf{u} \cdot \mathbf{u'})
\]

where \( u^3 K(r) = \langle u^2 (x) u (x + r \hat{e}_x) \rangle \) is the usual longitudinal triple correlation. This yields

\[
\frac{dL}{dt} = 4\pi \left[ \frac{1}{r} \frac{\partial}{\partial r} (r^4 u^3 K) \right]_\infty,
\]

where, as before, the subscript \( \infty \) indicates \( r \to \infty \). Evidently \( L \) is constant, since \( K_\infty \leq O(r^{-4}) \), as demonstrated by Saffman (1967). Integrating the second moment of (4) gives

\[
\frac{dI}{dt} = 8\pi \left[ u^3 r^4 K \right]_\infty,
\]

which suggests that \( I \) is time dependent. However, numerical simulations show that (to within the accuracy of the simulation) \( I \approx \) constant in fully-developed turbulence, though not in the initial stages of development. Hence, after a transient, \( [r^4 K]_\infty \approx 0 \) (Ishida, Davidson & Kaneda, 2006).

The physical significance of \( L \) is discussed in Saffman (1967). Noting that volume averages are equivalent to ensemble averages, we have

\[
L = \lim_{V \to \infty} \frac{1}{V} \left( \int_V \mathbf{u} dV \right)^2 = \lim_{V \to \infty} \frac{1}{V} \langle \mathbf{P}^2 \rangle,
\]

where \( V \) is some large control volume. Thus \( L \) is a measure of the linear momentum, \( \mathbf{P} = \int_V \mathbf{u} dV \), held in the volume \( V \). \( \mathbf{P} \) is, in turn, proportional to the sum of the linear impulses of the eddies in \( V \) (Davidson, 2009). Thus we expect that \( L \) will be finite and non-zero when a typical eddy possesses a significant amount of linear impulse. If a typical eddy has negligible linear impulse, on the other hand, then \( \mathbf{P} \) and \( L \) will be zero and we revert to \( E \sim k^4 \) turbulence.

The physical reason for the invariance of \( L \) is also discussed in Saffman (1967), where it is shown to be a consequence of the principle of linear momentum conservation. Let \( V_R \) be a large sphere of radius \( R \) and let \( S_R \) be its bounding surface. The momentum in \( V_R \), \( \mathbf{P} = \int_V \mathbf{u} dV \), may change due to pressure forces acting on \( S_R \), or else due to the flux of linear momentum across this surface. However, these are both random processes and so we might expect their
cumulative effect over $S_R$ scales as $O \left( S_R^{1/2} \right)$, i.e. $O \left( V_R^{1/3} \right)$. This is too weak to influence $I$ in the limit of $V_R \to \infty$, since the r.m.s. value of $P$ scales as $O \left( V_R^{1/2} \right)$.

There is an equivalent explanation for the invariance of $I$ when $[r^4K]_∞ \approx 0$, i.e. in fully-developed turbulence. It turns out that $I$ is related to the angular momentum held in $V_R$, and $I \approx constant$ is a consequence of angular momentum conservation. This was first pointed out by Landau & Lifshitz (1959), though their analysis is valid only for inhomogeneous turbulence evolving in a closed domain. The extension to homogeneous turbulence is provided in Davidson (2009).

We note in passing that, in isotropic turbulence, $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$ can be related to the longitudinal correlation function, $u^2 f(r)$, by $\langle \mathbf{u} \cdot \mathbf{u}' \rangle = r^{-2} \partial (r^3 u^2 f)/\partial r$, and so $L = 4\pi \{r^3 u^2 f\}_∞$. Thus $f_∞ \sim r^{-3}$ in Saffman turbulence. This suggests that $I$ diverges in such cases, though this need not necessarily be the case as it is possible to construct kinematically admissible fields of isotropic turbulence in which $L$ is non-zero, yet $\langle \mathbf{u} \cdot \mathbf{u}' \rangle_∞ \sim \exp(-r^2/\ell^2)$ (see Davidson, 2011.)

The fact that Saffman and Batchelor turbulence possess different invariants means that they have different decay characteristics, as pointed out by Kolmogorov (1941) and Saffman (1967). The idea is the following. The large scales in fully-developed freely-decaying, isotropic turbulence are self-similar when normalized by the integral scales, and it is these large eddies which dominate the integral moments of the cumulants of the vorticity field are all convergent. This provides $I ≈ constant$, which may be combined with the empirical, but well established, relationship

$$\frac{d u^2}{dt} = -\alpha \frac{u^3}{\ell}, \quad \alpha = constant,$$

(8) to give the Saffman decay laws:

$$\frac{u^2}{u^2_0} = \left[ 1 + \frac{5\alpha}{6} \left( \frac{u_{0t}}{\ell_0} \right) \right]^{-6/5}, \quad \frac{\ell}{\ell_0} = \left[ 1 + \frac{5\alpha}{6} \left( \frac{u_{0t}}{\ell_0} \right) \right]^{2/5}.$$  

(9) Here $u_0$ and $\ell_0$ are the initial values of $u$ and $\ell$. The equivalent laws for fully-developed Batchelor turbulence, where conservation of $I$ requires $u^2 \ell^5 \approx constant$, are

$$\frac{u^2}{u^2_0} = \left[ 1 + \frac{7\alpha}{10} \left( \frac{u_{0t}}{\ell_0} \right) \right]^{-10/7}, \quad \frac{\ell}{\ell_0} = \left[ 1 + \frac{7\alpha}{10} \left( \frac{u_{0t}}{\ell_0} \right) \right]^{2/7}.$$  

(Kolmogorov, 1941). Decay laws (9) are observed in, for example, the grid turbulence experiments of Krogstad & Davidson (2010), while (10) are realized in the numerical simulations of Ishida, Davidson & Kaneda (2006).

1.2. Saffman’s analysis of anisotropic turbulence

Saffman considered homogeneous turbulence which emerges from an initial condition in which the integral moments of the cumulants of the vorticity field are all convergent. This provides a generalization of the earlier study by Batchelor & Proudman (1956) of anisotropic, $E \sim k^4$ turbulence. The key kinematic findings are as follows. The spectral tensor, $\Phi_{ij}(k)$, takes the form

$$\Phi_{ij}(k \to 0) = \left\{ \delta_{\alpha\alpha} - \frac{k_i k_\alpha}{k^2} \right\} \left\{ \delta_{j\beta} - \frac{k_j k_\beta}{k^2} \right\} M_{\alpha\beta} + 0(k),$$  

(11)
which give \( E(k) = \frac{4}{3} \pi M_{\alpha\alpha} k^2 + O(k^3) \), where \( M_{\alpha\beta} \) is symmetric and independent of \( k \). For isotropic turbulence \( M_{\alpha\beta} = (L/16\pi^3) \delta_{\alpha\beta} \). The corresponding expression for \( \langle u_i u'_j \rangle \) is also uniquely determined by the \( M_{\alpha\beta} \) and takes the form \( \langle u_i u'_j \rangle_{\infty} \sim O(r^{-3}) \). This means that integrals of the form \( L_{ij} = \int_V \langle u_i u'_j \rangle dV, V \to \infty, \) are convergent (in the sense that they are independent of the size of \( V \)) and uniquely determined by \( M_{ij} \), since \( L_{ij} \) may be rewritten as a surface integral whose integrand is \( r_j \langle u_i u'_j \rangle_{\infty} \):

\[
L_{ij} = \lim_{V \to \infty} \int_V \langle u_i u'_j \rangle dV = \lim_{S \to \infty} \oint S \langle u_i u'_j \rangle_{\infty} r_j dS_k. \tag{12}
\]

However, the convergence is only conditional and so the value of \( L_{ij} \) will depend on the shape of \( V \) (i.e., sphere, cylinder, cube, etc.). Saffman also pointed out that the \( L_{ij} \), and hence the \( M_{ij} \), should be invariants, for much the same reason as \( L \) is an invariant in isotropic turbulence, i.e., linear momentum conservation applied to a large but finite control volume. Actually, Saffman’s analysis is not rigorous, but a formal proof may be found in Davidson (2010).

### 1.3. Axisymmetric Saffman turbulence and the implications of self-similarity

Of particular interest is the case of statistically axisymmetric turbulence. Let the subscripts \( \perp \) and \( // \) indicate directions perpendicular and parallel to the axis of symmetry, and let \( \ell_\perp, \ell_\perp, u_\perp \) and \( u_/// \) be the integral scales.

\[
\ell_\perp = \frac{1}{\langle u_\perp^2 \rangle} \int \langle u_\perp(x) \cdot u_\perp(x + r\hat{e}_x) \rangle dr, \quad u_\perp = \left( \frac{1}{2} \langle u_\perp^2 \rangle \right)^{1/2}, \tag{13}
\]

\[
\ell_/// = \frac{1}{\langle u_///^2 \rangle} \int \langle u_///(x) \cdot u_///(x + r\hat{e}_z) \rangle dr, \quad u_/// = \left( \langle u_///^2 \rangle \right)^{1/2}, \tag{14}
\]

Axial symmetry (with or without reflectional symmetry) applied to \( (11) \) tells us that \( M_{ij} = 0 \) if \( i \neq j \), and its independent components are \( M_/// \) and \( M_\perp = M_{xx} + M_{yy} \). Similarly, the only non-zero components of \( L_{ij} \) are \( L_\perp \) and \( L_{xx} = L_{yy} = \frac{1}{2} L_\perp \). The constraint \( L_{ij} = \) constant now reduces to (Davidson, 2010)

\[
L_/// = \int \langle u_/// u'_/// \rangle dV = \text{constant}, \quad L_\perp = \int \langle u_\perp \cdot u'_\perp \rangle dV = \text{constant}. \tag{15}
\]

Now let us now assume that the large scales (scales of order \( \ell_\perp \) and \( \ell_/// \)) are self-similar when normalized by the integral scales, i.e. \( \langle u_/// u'_/// \rangle / u_///^2 \) and \( \langle u_\perp \cdot u'_\perp \rangle / u_\perp^2 \) are functions of \( r_/// / \ell_/// \) and \( r_\perp / \ell_\perp \). The integrals in (15) are clearly dominated by the large scales and so self-similarity demands \( L_\perp = c_\perp u_\perp^2 \ell_\perp \ell_/// = \) constant and \( L_/// = c_/// u_///^2 \ell_/// \ell_/// = \) constant, where \( c_\perp \) and \( c_/// \) are dimensionless coefficients. In such a situation we have

\[
u_///^2 \ell_/// = \text{constant}, \quad u_/// \ell_/// \ell_/// = \text{constant}, \tag{16}
\]

from which, \( u_///^2 / u_///^2 = \text{constant} \). (Actually, the derivation of (16) is more subtle than suggested above, but a rigorous proof may be found in Davidson, 2010.) The equivalent result for axisymmetric Batchelor turbulence is (Davidson, 2009)

\[
I = - \int r_\perp^2 \langle u_\perp \cdot u'_\perp \rangle dV \sim u_\perp^2 \ell_\perp \ell_/// \approx \text{constant}. \tag{17}
\]
2. Invariants of Stratified, Rotating and MHD Turbulence

This analysis has been generalised to unforced, homogeneous, axisymmetric turbulence subject to: (i) a constant background rotation, $\Omega$; (ii) a uniform background stratification in the Boussinesq approximation; and (iii) an imposed, uniform magnetic field, $B_0$, in which the magnetic Reynolds number is small, $R_m = u\ell/\lambda \ll 1$, $\lambda$ being the magnetic diffusivity. The case of Saffman turbulence is considered in Davidson (2010) and for all three systems we find

$$u^2 \ell^2 \ell_{//} = \text{constant}, \quad \text{(Saffman turbulence)}, \quad (18)$$

while Batchelor turbulence is considered in Davidson (2009), where again all three systems have an invariant, this time

$$u^2 \ell^4 \ell_{//} \approx \text{constant}, \quad \text{(Batchelor turbulence)}. \quad (19)$$

We now explore the consequences of (18) and (19) for the temporal evolution of the integral scales in all three cases.

3. The Decay of Stratified Turbulence

Consider strongly stratified turbulence in which $Fr = u_\perp/N\ell_\perp \ll 1$ and $\Re \gg 1$, where $N$ is the Brunt-Väisälä frequency. In such flows the kinetic energy, which is dominated by $\langle u^2_\perp \rangle$, is observed to decay as

$$\frac{du^2_\perp}{dt} = -\alpha \frac{u^3_\perp}{\ell_\perp}, \quad \alpha \sim 1, \quad (20)$$

where $\alpha$ is a coefficient of order one. Moreover, the need for inertia to balance the buoyancy force yields,

$$u_\perp/N\ell_{//} = C, \quad (21)$$

where $C$ is yet another dimensionless coefficient of order unity. Both (20) and (21) are discussed in, for example, Brethouwer et al, 2007, where they are shown to hold provided $\Re Fr^2 \gg 1$. Combining (18) with (20) and (21), treating $\alpha$ and $C$ as constants, and integrating, yields, for Saffman turbulence

$$\frac{u^2_\perp}{u_0^2} = \left[ 1 + \frac{5\alpha u_0 t}{4 \ell_0} \right]^{-4/5}, \quad (22)$$

$$\frac{\ell_\perp}{\ell_0} = \left[ 1 + \frac{5\alpha u_0 t}{4 \ell_0} \right]^{3/5}, \quad \frac{\ell_{//}}{\ell_0} = \frac{1}{C N\ell_0} \left[ 1 + \frac{5\alpha u_0 t}{4 \ell_0} \right]^{-2/5} \quad (23)$$

where $u_0$ and $\ell_0$ are the initial values of $u_\perp$ and $\ell_\perp$ (Davidson, 2010). Note that $\ell_\perp$ grows as $\ell_\perp \sim t^{3/5}$, while $\ell_{//}$ falls at the rate $\ell_{//} \sim t^{-2/5}$. Thus anisotropy continually increases as $\ell_\perp/\ell_{//} \sim t$.

The equivalent results for Batchelor turbulence, where $E(k \to 0) \sim k^4$, are given in Davidson (2009), and these turn out to be

$$\frac{u^2_\perp}{u_0^2} = \left[ 1 + \frac{7\alpha u_0 t}{8 \ell_0} \right]^{-8/7}, \quad (24)$$

$$\frac{\ell_\perp}{\ell_0} = \left[ 1 + \frac{7\alpha u_0 t}{8 \ell_0} \right]^{3/7}, \quad \frac{\ell_{//}}{\ell_0} = \frac{1}{C N\ell_0} \left[ 1 + \frac{7\alpha u_0 t}{8 \ell_0} \right]^{-4/7}. \quad (25)$$
Note that the energy decay rates for the two systems are different, with $u^2_\perp \sim t^{-0.8}$ in $E \sim k^2$ turbulence and $u^2_\perp \sim t^{-1.14}$ in $E \sim k^4$ turbulence. This difference is possibly large enough to distinguish between the two cases in numerical simulations, though to date the simulations of freely-decaying, stratified turbulence have been performed in domains of relatively modest size and so it is difficult to get estimates of the decay exponent which are independent of the boundary conditions. Never-the-less, the numerical estimate of by Staquet & Godeferd (1998) lies tantalisingly between the two theoretical predictions given above.

4. The Decay of Rapidly-Rotating Turbulence

We now turn to rotating turbulence in which $\mathcal{R} \gg 1$. In laboratory experiments of such turbulence the initial Rossby number, $Ro = u/\Omega \ell$, is usually chosen to be large, thus ensuring that no inertial waves come off the grid used to generate the turbulence. However, as the turbulence decays, $Ro$ falls, and once $Ro \approx 1$, it is observed that the large scales in the turbulence start to generate their own inertial waves. These waves have a profound influence on the evolution of the turbulence. Large-scale eddies are observed to form columnar structures which grow along the $\Omega$-axis at the rate $\ell_{//} \sim \Omega \ell_0 t$, and so we have

$$\ell_{//} = \ell_0 (1 + \kappa_1 \Omega t),$$

(26)

where $\kappa_1$ is a constant of order one and $\ell_0$ is the value of $\ell_{//}$ and $\ell_{\perp}$ at the instant when the waves first appear (i.e. $t = 0$ in equation 26). The experiments of Staplehurst, Davidson & Dalziel (2008) and Jacquine et al (1990) both support (26).

Now the flux of energy to the small scales in such turbulence presumably depends on the integral scales $u_{\perp}$ and $\ell_{\perp}$, as well as on $\Omega$, and dimensional analysis then requires

$$\frac{du_{\perp}^2}{dt} = -G \left( \frac{u_{\perp}}{\Omega \ell_{\perp}} \right) \frac{u_{\perp}^3}{\ell_{\perp}},$$

(27)

where $G$ is some unknown function of $u_{\perp}/\Omega \ell_{\perp}$. Experiments show that rotation suppresses the energy dissipation for $Ro \leq 0(1)$, and so $G$ must be an increasing function of $u_{\perp}/\Omega \ell_{\perp}$ in this range. Indeed Squires et al (1994) argue that (27) must take the form $du_{\perp}^2/dt \sim \Omega^{-1}$ for small $Ro$. This then requires

$$\frac{du_{\perp}^2}{dt} = -\alpha \frac{u_{\perp}^4}{\Omega^2 \ell_{\perp}^2}, \quad Ro \leq 1,$$

(28)

for some constant $\alpha$ of order unity. Combining (18) with (26) and (28), and integrating from approximately isotropic initial conditions in which $u_0/\Omega \ell_0 \sim 1$, yields, for Saffman turbulence

$$\frac{u_{\perp}^2}{\Omega^2 \ell_0^2} = R_0 \sqrt{\kappa_1/\alpha} \left[ \left\{ \frac{\kappa_1}{\alpha R_0^2} - 1 \right\} + \hat{t}^2 \right]^{-1/2},$$

(29)

where $\hat{t} = 1 + \kappa_1 \Omega t$ is a scaled time, and $R_0 = u_0/\Omega \ell_0 \sim 1$. For large $\Omega t$ this reduces to

$$u_{\perp}^2 \sim \frac{\Omega^2 \ell_0^2}{1 + \kappa_1 \Omega t}.$$

(30)

For Batchelor turbulence, where $E(k \to 0) \sim k^4$, the equivalent result is

$$\frac{u_{\perp}^2}{\Omega^2 \ell_0^2} = R_0^{2/3} (\kappa_1/\alpha)^{2/3} \left[ \left\{ \frac{\kappa_1}{\alpha R_0^2} - 1 \right\} + \hat{t}^{3/2} \right]^{-2/3},$$

(31)

which also reduces to (30) for large $\Omega t$. In either case, then, we expect $u_{\perp}^2 \sim t^{-1}$. 


The \( u^2_\perp \sim t^{-1} \) scaling is consistent with the experimental data of Staplehurst, Davidson & Dalziel (2008), as discussed in Davidson (2010), and with the data of Jacquine et al (1990), who found \( u^2_\perp \sim t^{-n} \) where \( n \approx 0.81 \rightarrow 1.08 \) for \( \text{Ro} \sim 1 \). Moreover, Teitelbaum & Mininni (2009) report \( u^2_\perp \sim t^{-1} \) in numerical simulations of rotating turbulence.

### 5. The Decay of Low-\( R_m \) MHD Turbulence

Finally we turn to turbulence in a conducting fluid which is threaded by a uniform, imposed magnetic field, \( \mathbf{B}_0 \). The effect of this field is to induce anisotropy, with \( \ell_{//} > \ell_\perp \), as Alfvén waves propagate along the field lines. In the case of low \( R_m \) these waves are highly damped and spread diffusively. This is discussed in Davidson (2004) where it is suggested that the energy decay equation, which now includes Joule dissipation as well as viscous dissipation,

\[
\frac{d}{dt} \frac{1}{2} \langle u^2 \rangle = -\nu \langle \omega^2 \rangle - \frac{\langle \mathbf{J}^2 \rangle}{\rho \sigma},
\]

(32)

should be modeled as

\[
\frac{du^2}{dt} = -\alpha \frac{u^3}{\ell_\perp} - \beta \left( \frac{\ell_\perp}{\ell_{//}} \right)^2 \frac{u^2}{\tau}.
\]

(33)

Here \( \sigma \) is the electrical conductivity, \( \mathbf{J} \) the current density, \( \omega \) the vorticity, \( u^2 = \frac{1}{3} \langle u^2 \rangle \), \( \tau = (\sigma B^2/\rho)^{-1} \) the Joule damping time, and \( \alpha \) and \( \beta \) are dimensionless coefficients of order unity.

The model equation (33) has been tested against direct numerical simulations for \( E \sim k^4 \) turbulence in very large periodic domains by Okamoto, Davidson & Kaneda (2009), and found to be a good approximation in fully-developed turbulence. It is likely, therefore, that it is also a good approximation in \( E \sim k^2 \) turbulence. If we now combine this model equation with (18), and integrate, we find that for Saffman turbulence (Davidson, 2010)

\[
u^2 / u_0^2 = \hat{t}^{-1/2} \left[ 1 + (5\alpha/9\beta) \left( \hat{t}^{3/4} - 1 \right) N_0^{-1} \right]^{-6/5},
\]

(34)

\[
\ell_{//} / \ell_0 = \hat{t}^{1/2} \left[ 1 + (5\alpha/9\beta) \left( \hat{t}^{3/4} - 1 \right) N_0^{-1} \right]^{2/5},
\]

(35)

where \( N_0 = u_0 / \ell_0 \tau, \hat{t} = 1 + 2\beta(t/\tau) \), and we have integrated from isotropic initial conditions. These equations reduce to Saffmans decay laws \( N_0 \ll 1 \), as they must. For \( N_0 \sim 1 \) they yield \( u^2 \sim t^{-7/5}, \ell_\perp \sim t^{3/10} \) and \( \ell_{//} \sim t^{4/5} \).

The equivalent results for \( E \sim k^4 \) Batchelor turbulence, where (19) replaces (18), are given in Davidson (2004), and they are

\[
u^2 / u_0^2 = \hat{t}^{-1/2} \left[ 1 + (7\alpha/15\beta) \left( \hat{t}^{3/4} - 1 \right) N_0^{-1} \right]^{-10/7},
\]

(36)

\[
\ell_{//} / \ell_0 = \hat{t}^{1/2} \left[ 1 + (7\alpha/15\beta) \left( \hat{t}^{3/4} - 1 \right) N_0^{-1} \right]^{2/7}.
\]

(37)

The latter predictions were compared against numerical simulations by Okamoto, Davidson & Kaneda (2009), who found that they provide good estimates of \( u^2, \ell_\perp \) and \( \ell_{//} \) in fully-developed turbulence.
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