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Empowerment — An Introduction

Christoph Salge, Cornelius Glackin and Daniel Polani

University of Hertfordshire

4.1 Introduction

Is it better for you to own a corkscrew or not? If asked, you as a human being would likely say “yes”, but more importantly, you are somehow able to make this decision. You are able to decide this, even if your current acute problems or task do not include opening a wine bottle. Similarly, it is also unlikely that you evaluated several possible trajectories your life could take and looked at them with and without a corkscrew, and then measured your survival or reproductive fitness in each. When you, as a human cognitive agent, made this decision, you were likely relying on a behavioural “proxy”, an internal motivation that abstracts the problem of evaluating a decision impact on your overall life, but evaluating it in regard to some simple fitness function. One example would be the idea of curiosity, urging you to act so that your experience new sensations and learn about the environment. On average, this should lead to better and richer models of the world, which give you a better chance of reaching your ultimate goals of survival and reproduction.

But how about questions such as, would you rather be rich than poor, sick or healthy, imprisoned or free? While each options offers some interesting new experience, there seems to be a consensus that rich, healthy and free is a preferable choice. We think that all these examples, in addition to the question of tool ownership above, share a common element of preparedness. Everything else being equal it is preferable to be prepared, to keep ones options open or to be in a state where ones actions have the greatest influence on ones direct environment.

The concept of Empowerment, in a nutshell, is an attempt at formalizing and quantifying these degrees of freedom (or options) that an organism or agent has as a proxy for “preparedness”: preparedness, in turn, is considered a proxy for prospective fitness via the hypothesis that preparedness would be a good indicator to distinguish promising from less promising regions in the prospective fitness landscape, without actually having to evaluate the full fitness landscape.
Empowerment aims to reformulate the options or degrees of freedom that an agent has as the agent’s control over its environment; and not only of its control — to be reproducible, the agent needs to be aware of its control influence and sense it. Thus, empowerment is a measure of both the control an agent has over its environment, as well as its ability to sense this control. Note that this already hints at two different perspectives to evaluate the empowerment of an agent. From the agent perspective empowerment can be a tool for decision making, serving as a behavioural proxy for the agent. This empowerment value can be skewed by the quality of the agent world model, so it should be more accurately described as the agent’s approximation of its own empowerment, based on its world model. The actual empowerment depends both on the agent’s embodiment, and the world the agent is situated in. More precisely, there is a specific empowerment value for the current state of the world (the agent’s current empowerment), and there is an averaged value over all possible states of the environment, weighted by their probability (the agent’s average empowerment).

Empowerment, as introduced by Klyubin et al. (2005a,b), aims to formalize the combined notion of an agent controlling its environment and sensing this control in the language of information theory. The idea behind this is that this should provide us with a utility function that is inherently local, universal and task-independent.

1. **Local** means that the knowledge of the local dynamics of the agent is enough to compute it, and that it is not necessary to know the whole system to determine one’s empowerment. Ideally, the information that the agent itself can acquire should be enough.

2. **Universal** means that it should be possible to apply empowerment “universally” to every possible agent-world interaction. This is achieved by expressing it in the language of information theory and thus making it applicable for any system that can be probabilistically expressed. For instance, even if an agent completely changes its morphology, it is still possible to compute a comparable empowerment value. Klyubin et al. (2005b) gave the examples of money in a bank account, of social status in a group of chimpanzees, and of sugar concentration around a bacterium as different scenarios, all as examples which would be treated uniformly by the empowerment formalism.

3. **Task-independent** means that empowerment is not evaluated in regard to a specific goal or external reward state. Instead, empowerment is determined by the agent’s embodiment in the world. In particular, apart from minor niche-dependent parameters, the empowerment formalism should have the very same structure in most situations.

More concretely, the proposed formulation of empowerment defines it via the concept of potential information flow, or channel capacity, between an agent’s actuator state at earlier times and their sensor state at a later time. The idea behind this is that empowerment would quantify how much an agent can reliably and perceptibly influence the world.
4.1.1 Overview

Since its original inception by Klyubin et al. (2005a,b) in 2005, several papers have been published about empowerment, both further developing the formalism, and demonstrating a variety of behaviours in different scenarios. Our aim here is to both present an overview of what has been done so far, and to provide readers new to empowerment with an easy entry point to the current state-of-the-art in the field. Due to the amount of content, some ideas and results are only reported in abstract form, and we would refer interested reader to the cited papers, where models and experiments are explained in greater detail.

Throughout the text we also tried to identify the open problems and questions that we currently see, and we put a certain emphasis on the parameters that affect empowerment. While empowerment is defined in a generic and general way, the review of the literature shows that there are still several choices one can take on how to exactly apply empowerment, and which can affect the outcome of the computation.

The remaining paper is structured as follows. First, after briefly outlining the related work previous to empowerment, we will spell out the different empowerment hypotheses motivating the research in empowerment. This will allow us to locate empowerment in relation to different fields, and also makes it easier to see how and where insights from the empowerment formalism apply to other areas.

The next section then focusses on discrete empowerment, first, in Sec. 4.4 introducing the formalism, and then, in Sec. 4.5 describing several different examples, showcasing the genericity of the approach.

Section 4.6 then deals with empowerment in continuous settings, which is currently not as far developed and sees vigorous activity. Here we will discuss the necessity for suitable approximations, and outline the current technical challenges to provide good but fast approximations for empowerment in the continuous domain.

4.2 Related Work

Empowerment is based on and connects to several fields of scientific inquiry. One foundational idea for empowerment is to apply information theory to living, biological systems. (Gibson James 1979) points out the importance of information in embodied cognition, and earlier work (Barlow 1959; Attneave 1954) investigates the informational redundancy in an agent’s sensors. Later research (Atick 1992) based on this identifies the importance of information bottlenecks for the compression of redundancies, which are later formalized in information theoretic terms (Tishby et al. 1999).

Furthermore, it was also demonstrated that informational efficiency can be used to make sense of an agent’s sensor input (Olsson et al. 2005; Lungarella et al. 2005). The general trend observed in these works seems to be that nature optimizes the information processing in organisms in terms of efficiency (Polani 2009). Empowerment is, in this context, another of these efficiency principles.

Empowerment also relies heavily on the notion that cognition has to be understood as an immediate relationship of a situated and embodied agent with its surroundings.
This goes back to the “Umwelt” principle by (von Uexküll 1909), which also provides us with an early depiction of what is now commonly referred to as the perception action loop. This idea was also at the center of a paradigm shift in artificial intelligence towards enactivism (Varela et al. 1992; Almeida e Costa and Rocha 2005), which postulates that the human mind organizes itself by interacting with its environment. Embodied robotics (Pfeifer et al. 2007) is an approach trying to replicate these processes “in silico”.

4.2.1 Intrinsic Motivation

Central to this body of work is the desire to understand how an organism makes sense of the world and decides to act from its internal perspective. Ultimately all behaviour is connected to an organism’s survival, but most natural organisms do not have the cognitive capacity to determine this connection themselves. So, if an animal gets burned by fire, it will not consider the fire’s negative effect on its health and potential risk of death and then move away. Instead, it will feel pain via its sensors and react accordingly. The ability to feel pain and act upon it is an adaptation that acts as a proxy criterion for survival, while it still offers a certain level of abstraction from concrete hard-wired reactions. We could say the animal is motivated to avoid pain. Having an abstract motivation allows an agent a certain amount of adaptability; instead of acting like a stimulus-response look-up table the agent can evaluate actions in different situations according to how rewarding they are regarding its motivations.

Examining nature also reveals that not all motivations are based on external rewards, e.g. a well-fed and pain-free agent might be driven by an urge to explore or learn. In the following we discuss related work covering different approaches to specify and quantify such intrinsic motivations. The purpose of these models is both to better understand nature, as well as to replicate the ability of natural organism to react to a wide range of stimuli in models for artificial systems.

An evolution-based view of intrinsic motivations uses assumptions about preexisting saliency sensors to generate intrinsic motivations (Singh et al. 2005, 2010). However, where one does not want to assume such pre-evolved saliency sensors, one needs to identify other criteria that can operate with unspecialized generic sensors.

One such family of intrinsic motivation mechanisms focusses on evaluating the learning process. Artificial curiosity (Schmidhuber 2002, 1991) is one of the earlier models, where an agent receives an internal reward depending on how “boring” the environment is which it currently tries to learn. This causes the agent to avoid situations that are at either of the extremes: fully predictable or unpredictably random.

The autotelic principle by Steels (2004) tries to formalize the concept of “Flow” (Csikszentmihalyi 2000): an agent tries to maintain a state were learning is challenging, but not overwhelming (see also Gordon and Ahissar 2012). Another approach (Kaplan and Oudeyer 2004) aims to maximise the learning progress of different classical learning approaches by introducing rewards for better predictions of future states. A related idea is behind the homeokinesis approach, which can be considered a dynamic version of homoeostasis. The basic principle here is to act in a way which can be well predicted by a adaptive model of the world dynamics (Der et al. 1999).
is a tendency of such mechanisms to place the agent in stable, easily predictable environments. For this reason, to retain a significant richness of behaviors additional provisions need to be taken so that notwithstanding the predictability of the future, the current states carry potential for instability.

The ideas of homeokinesis are originally based on dynamical system theory. Further studies have transferred them into the realm of information-theoretical approaches (Ay et al. 2008). The basic idea here is to maximise the predictive information, the information the past states of the world have about the future. Here, also, predictability is desired, but predictive information will only be large if the predictions about the future are decoded from a rich past, which captures very similar ideas to the dynamical systems view of homeokinesis.

The empowerment measure which is the main concept under discussion in the present paper, also provides a universal, task-independent motivation dynamics. However, it focuses on a different niche. It is not designed to explore the environment, as most of the above measures are, but rather aims at identify preferred states in the environment, once the local dynamics are known; if not much is known about the environment, but empowerment is high, this is perfectly satisfactory for the empowerment model, but not for the earlier curiosity-based methods. Therefore, empowerment is better described as a complement to the aforementioned methods, rather than a direct competitor.

Empowerment has been motivated by a set of biological hypotheses, all related to informational sensorimotor efficiency, the ability to react to the environment and similar. However, it would be interesting to identify whether there may be a route stemming from the underlying physical principles which would ultimately lead to such a principle (or a precursor thereof). For some time, the "Maximum Entropy Production Principle" (MEPP) has been postulated as arising from first thermodynamic principles (Dewar 2003, 2005). However, unfortunately, and according to current knowledge, the derivation from first principles still remains elusive and the current attempts at doing so unsuccessful (Grinstein and Linsker 2007). If, however, one should be able to derive the MEPP from first principles, then (Wissner-Gross and Freer 2013) show that this would allow a principle to emerge on the physical (sub-biological) level which acts as a simpler proto-empowerment which shares to some extent several of the self-organizing properties with empowerment, even if in a less specific way and without reference to the "bubble of autonomy" which would accompany a cognitive agent. Nevertheless, if successful, such a line may provide a route to how a full-fledged empowerment principle could emerge from physical principles.

### 4.3 Empowerment Hypotheses

In this section we want to introduce the main hypotheses which motivated the development of empowerment. Neither the work presented here in this chapter, nor the work on empowerment in general is yet a conclusive argument for either of the three main hypotheses, but they should, nevertheless, be helpful to outline what empowerment can be used for, and to what different domains empowerment can be applied. Furthermore,
it should also be noted, that the hypotheses are stated in a generic form which might be unsuitable for experimental testing, but this can be alleviated on a case by case basis by applying a hypothesis to a specific scenario or task.

There are two main motivations for introducing the concept of empowerment: one is, of course, the desire to come up with methods to allow artificial agents to flexibly decide what to do generically, without having a specific task designed into them in every situation. This is closely related to the idea of creating a general AI. The other is to search for candidate proxies of prospective fitness, which could be detected and driven towards during the lifetime of an organism to improve its future reproductive success.

From these two starting points, several implicit and explicit claims have been made about empowerment and how it would relate to phenomena in biology. In the following section we structure these claims into three main hypotheses which we would consider as driving the “empowerment program”. This should make it easier for the reader to understand what the simulations in the later chapters should actually demonstrate.

4.3.1 Behavioural Empowerment Hypothesis

The adaptation brought about by natural evolution produced organisms that in absence of specific goals behave as if they were maximising their empowerment.

Klyubin et al. (2005a,b) argue that the direct feedback provided by selection in evolution is relatively sparse, and therefore it would be infeasible to assume that evolution adapts the behaviour of organisms specifically for every possible situation. Instead they suggest that organisms might be equipped with local, task-independent utility detectors, which allows them to react well to different situations. Such generic adaptation might have arisen as a solution to a specific problem, and then persisted as a solution to other problems, as well. This also illustrates why such a utility function should be universal: namely, because it should be possible to retain the essential structure of the utility model, even if the morphology, sensor or actuators of the organism change through evolution.

This is also based on our understanding of humans and other organisms. We seem to be, at least in part, adapted to learn, explore and reason, rather than to only have hard-coded reactions to specific stimuli. As these abilities also usually generate actions, such a drive is sometimes called intrinsic motivation. Different approaches have been proposed (see Sec. 4.2) to formalize motivation that would generate actions that are not caused by an explicit external reward. Empowerment does not consider the learning process or the agent trajectory through the world, but instead operates as a pseudo-utility which assigns a value (its empowerment) to each state in the world. Highly empowered states are preferred, and the core hypothesis states that an agent or organism attempts to reach states with high empowerment. Empowerment measures

\[\text{Here we mostly adopt an “objective” perspective in that the objective states of the world are known and their empowerment computed. However, truly subjective versions of empowerment are easily definable and will be discussed in Sec. 4.4.4 as context-dependent empowerment.}\]
the ability of the agent to potentially change its future (it does not mean that it is actually doing that). The lowest value for empowerment is 0, which means that an agent has no influence on the world that it can perceive. From the empowerment perspective, vanishing empowerment is equivalent to the agent’s death, and the empowerment maximization hypothesis provides a natural drive for death aversion.

The behaviour empowerment hypothesis now assumes that evolution has come up with a solution that produces similar behaviour. To support this hypothesis, the first step would be to demonstrate that empowerment can produce behaviour which is similar to biological organisms in analogous situations. In turn, it should also be possible to anticipate behaviour of biological organisms by considering how it would affect their empowerment. If we follow this idea further and assume that humans use empowerment-like criteria to inform their introspection, then one would expect that those states identified by humans as preferable would also be more likely to have high empowerment.

For the hypothesis to be plausible, it would also be good to ensure that empowerment is indeed local and can be computed from the information available to the agent. Similarly, it should also be universally applicable to different kinds of organisms; we would expect organisms which have undergone small changes to their sensory-motor set-up to still produce comparable empowerment values, and for organisms that discover new modalities of interaction that this is then reflected in the empowerment landscape.

So far, we have discussed a weak version of the behavioural empowerment hypothesis. A stronger version of the hypothesis would argue that an agent actually computes empowerment. While this can be easily checked for artificial agents, in a biological scenario, it becomes necessary to explain how empowerment could actually be computed by the agent. The weak version of the hypothesis, instead, says that the agents just act “as if” driven by empowerment, or are using a suitable approximation. The hypothesis then states that natural behaviours favour highly empowered behaviour routes.

4.3.2 Evolutionary Empowerment Hypothesis

The adaptation brought about by natural evolution increases the empowerment of the resulting organism.

Due to its universality, empowerment can in principle, be used to compare the average empowerment of different organisms. For instance, today, we could look at a digital organism, and then come back later after several generations of simulated adaptation, asking whether the organisms are now more empowered? Did that new sensor (and/or actuator) increase the agents empowerment? The hypothesis put forward, e.g. by Polani (2009), is that the adaptation in nature, on average, increases an agent’s empowerment. He argues that (Shannon) information operates as a “currency of life”, which imposes an inherent cost onto an organism, and, for that reason, a well-adapted

\[^{2}\] We do not actually put forward this stronger version for the biological realm, but mention it for completeness, and because of its relevance for empowerment in artificial agents.
organism should have efficient information processing. On the one hand, there is some relevant information (Polani et al. 2006) that needs to be acquired by an agent to perform at a given level, but any additional information processing would be superfluous, and should be avoided, as it creates unnecessary costs. Taking a look at agent morphologies, this also means that agents should be adapted to efficiently use their sensors and actuators. For example, a fish population living in perpetual darkness does not have a need for highly developed eyes (Jeffery 2005), and it is expected that adaptation will reduce the functionality and cognitive investment (i.e. brain operation) related to vision. On the other hand, in the dark the detection of sound could be useful; this perceptual channel could be made even more effective by actively generating sound that is then reflected from objects and could then be detected by the organism. The core question is: how can such potential advantageous gradients in the space of sensorimotoric endowment be detected?

Empowerment is the channel capacity from an agent’s actuators to its sensors, and as such, measures the efficiency of that channel. Having actuators whose effect on the environment cannot be perceived, or sensors which detect no change relevant to the current actions is inefficient, and should be selected against. In short, this adaptation would be attained by an increase of the agent’s average empowerment.

A test for this hypothesis would be to evolve agents in regard to other objectives, and then check how their empowerment develops over the course of the simulated evolution, similar to studies about complexity growth under evolutionary pressures (Yaeger 2009). Another salient effect of this hypothesis would be an adaptation of an agent morphology based on empowerment should produce sensor layouts and actions which are to some degree “sensible” and perhaps could also be compared to those found in nature.

4.3.3 AI Empowerment Hypothesis

Empowerment provides a task-independent motivation that generate AI behaviour which is beneficial for a range of goal-oriented behaviour.

In existing work, it was demonstrated that empowerment can address quite a selection of AI problems successfully (see the remaining chapter for a selection); amongst these are pole balancing, maze centrality and others. However, a clear contraindication exists for its use: if an externally desired goal state is not highly empowered, then an empowerment-maximising algorithm is not going to seek it out. Opposed to that, such tasks are the standard domain of operation for traditional AI algorithms.

However, in the realm or robotics there have been developments to design robots that are not driven by specific goals, but motivated by exploring their own morphology or other forms of intrinsic motivation. The idea is to build robots that learn and explore, rather than engineer solutions for specific problems determined externally and in advance. Here, empowerment offers itself as another alternative. While empowerment is not designed to explicitly favour exploration, it has an inbuilt incentive to avoid behaviour that leads to a robot being stuck. Having no options available to an agent is bad for empowerment. Non-robotic AI could also benefit from this approach, but since
empowerment is defined over the agent world dynamics, there needs to be a clear interface between an agent and the world over which it can be computed: in this case, there needs to be some kind of substitute for embodiment or situatedness. On the other hand, for the robotics domain it is also important that empowerment can be computed in real time and be applied to continuous variables.

The concrete and relevant question would be under which circumstances empowerment would provide a good solution, both in robotic and non-robotic settings? Furthermore, in what situations would maximising empowerment be helpful for a later to be specified task? To approach this question it is helpful to apply empowerment to a wider range of AI problems and inspect its operation in the different scenarios. The remaining chapter will showcase several such examples and discuss the insights gained from these.

In the robotic domain, one faces additional challenges, most prominently the necessity to handle empowerment in continuous spaces. This is discussed in Sec. 4.6. Note, however, that there is still very little current experience on deploying empowerment on real robots, with exception of a basic proof-of-principle context reconstruction example on an AIBO robot (Klyubin et al. 2008).

4.4 Formalism

Empowerment is formalized using terms from information theory, first introduced by Shannon (Shannon 1948). To define a consistent notation, we begin by introducing several standard notions. Entropy is defined as

$$H(X) = - \sum_{x \in X} p(x) \log p(x)$$

(4.1)

where $X$ is a discrete random variable with values $x \in X$, and $p(x)$ is the probability mass function such that $p(x) = Pr\{X = x\}$. Throughout this paper base 2 logarithms are used by convention, and therefore the resulting units are in bits. Entropy can be understood as a quantification of uncertainty about the outcome of $X$ before it is observed, or as the average surprise at the observation of $X$. Introducing another random variable $Y$ jointly distributed with $X$, enables the definition of conditional entropy as

$$H(X|Y) = - \sum_{x \in X} p(y) \sum_{y \in Y} p(x|y) \log p(x|y).$$

(4.2)

This measures the remaining uncertainty about $X$ when $Y$ is known. Since Eq. (4.1) is the general uncertainty of $X$, and Eq. (4.2), is the remaining uncertainty once $Y$ has been observed, their difference, called mutual information, quantifies the average information one can gain about $X$ by observing $Y$. Mutual information is defined as

$$I(X;Y) = H(Y) - H(Y|X).$$

(4.3)

The mutual information is symmetric (see (Cover and Thomas 1991)), and it holds that
\[ I(X; Y) = H(Y) - H(Y|X) = H(X) - H(X|Y). \] (4.4)

Finally, a quantity which is used in communication over a noisy channel to determine the maximum information rate that can be reliably transmitted, is given by the channel capacity:

\[ C = \max_{p(x)} I(X; Y). \] (4.5)

These concepts are fundamental measures in classical information theory.

Now, for the purpose of formalizing empowerment, we will now reinterpret the latter quantity in a causal context, and specialize the channel we are considering to the actuation-perception channel.

4.4.1 The Causal Interpretation of Empowerment

Core to the empowerment formalism is now the potential causal influence of one variable (or set of variables: the actuators) on another variable (or set of variables: the sensors). Further below, we will define the framework to define this in full generality; for now, we just state that we need to quantify the potential causal effect that one variable has on the other.

When we speak about causal effect, we specifically consider the interventionist notion of causality in the sense of Pearl (2000) and the notion of causal information flow based upon it (Ay and Polani 2008). We sketch this principle very briefly and refer the reader to the original literature for details.

To determine the causal information flow \( \Phi(X \rightarrow Y) \) one cannot simply consider the observed distribution \( p(x, y) \), but has to probe the distribution by actively intervening in \( X \). The change resulting from the intervention in \( X \) (which we denote by \( \hat{X} \)) is then observed in the system and used to construct the interventional conditional \( p(y|\hat{x}) \). This interventional condition will then be used as the causal channel of interest. While (causal) information flow according to (Ay and Polani 2008) has been defined as the mutual information over that channel for an independent interventional input distribution, empowerment considers the maximal potential information flow, i.e. it is not based on the actual distribution of the input variable \( X \) (with or without intervention), but considers the maximal information flow that could possibly be induced by a suitable choice of \( X \). This, however, is nothing other than the channel capacity

\[ C(X \rightarrow Y) = \max_{p(\hat{x})} I(\hat{X}; Y). \] (4.6)

for the channel defined by \( p(y|\hat{x}) \), where by the hat we indicate that this is a channel where we intervene in \( X \).

There is a well-developed literature on how to determine the conditional probability distribution \( p(y|\hat{x}) \) necessary to compute empowerment, for some approaches, see Pearl (2000), Ay and Polani (2008). This interventional conditional probability distribution can then be treated as the channel; and the channel capacity, or empowerment, can be computed with established methods, such as the Blahut-Arimoto algorithm (Blahut 1972, Arimoto 1972).
For the present discussion, it shall suffice to say that empowerment can be computed from the conditional probability distribution of observed actuation/sensing data, as long as we can ensure that the channel is a causal pair, meaning we can rule out any common cause, and any reverse influence from $y$ onto $x$.

### 4.4.2 Empowerment in the Perception Action Loop

The basic idea behind empowerment is to measure the influence of an agent on its environment, and how much of this influence can be perceived by the agent. In analogy to control theory, it is essentially a combined measure of controllability (influence on the world) and observability (perception by the agent), but, unlike in the control-theoretic context, where controllability and observability denote the dimensionality of the respective vector spaces or manifolds, empowerment is a fully information-theoretic quantity: This has two consequences: the values it can assume are not confined to integer dimensionalities, but can range over the continuum of non-negative real numbers; and, secondly, it is not limited to linear subspaces or even manifolds, but can, in principle, be used in all spaces for which one can define a probability mass measure.

We formalize the concept of empowerment, as stated earlier, as the channel capacity between an agent’s actions at a number of times and its sensoric stimuli at a later time. To understand this in detail, let us first take a step back and see how to model an agent’s interaction with the environment as a causal Bayesian network (CBN). In general we are looking at a time-discrete model where an agent interacts with the world. This can be expressed as a perception-action loop, where an agent chooses an action for the next time step based on its sensor input in the current time step. This influences the state of the world (in the next time step), which in turn influences the sensor input of the agent at that time step. The cycle then repeats itself, with the agent choosing another action. Note that this choice of action might also be influenced by some internal state of the agent which carries information about the agent’s past. To model this, we define the following four random variables:

- $A$: the agent’s actuator, which takes values $a \in A$
- $S$: the agent’s sensor, which takes values $s \in S$
- $M$: the agent’s internal state (or memory), which takes values $m \in M$
- $R$: the state of the environment, which takes values $r \in R$

Their relationship can be expressed as a time-unrolled CBN, as seen in Fig. 1(a).

Empowerment is then defined as the channel capacity between the agent’s actuators $A$ and its own sensors $S$ at a later point in time, here, for simplicity, we assume the next time step:

$$E := C(A_t \rightarrow S_{t+1}) \equiv \max_{p(a_t)} I(S_{t+1}; A_t).$$

(4.7)

This is the general empowerment of the agent. In the following text we will use $E$ as a shorthand for the causal channel capacity from the sensors to the actuators.

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3Saying actuator implicitly includes the case of multiple actuators. In fact, it is the most general case. Multiple actuators (which can be independent of each other) can always be written as being incorporated into one single actuator variable.
Note that the maximization implies that it is calculated under the assumption that the controller which chooses the action $A$ is free to act, and is not bound by possible behaviour strategy $p(a|s, m)$. Importantly, the distribution $p^*(a)$ that achieves the channel capacity is different from the one that defines the actions of an empowerment-driven agent. Empowerment considers only the potential information flow, so the agent will only calculate how it could affect the world, rather than actually carry out its potential.

### 4.4.3 $n$-step empowerment

In Sec. 4.4.2 we considered empowerment as a consequence of a single action taken and the sensor being read out in the subsequent state. However, empowerment, as a measure of the sensorimotor efficiency, may start distinguishing the characteristics of the agent-environment interaction only after several steps. Therefore, a common generalization of the concept is the $n$-step empowerment. In this case we consider not a single action variable, but actually a sequence of action variables for the next $n$ time steps: $(A_{t+1}, \ldots, A_{t+n})$. We we will sometimes condense these into a single action variable $A$ for notational convenience. The sensor variable is the resulting sensor state in the following timestep $S_{t+n+1}$, again sometimes denoted by $S$. Though it is not the most general treatment possible, here we will consider only “open-loop” action sequences, i.e. action sequences which are selected in advance and then carried out without referring to a sensor observation until the final observation $S_{t+n+1}$. This
drastically simplifies both computations and theoretical considerations, as the different possible action sequences $A$ can be treated as if they were separate atomic actions with no inner structure.

As mentioned, $A$ can typically contain actuator variables from several time steps and can also incorporate several variables per time step. $S$ is typically chosen to contain variables that are strictly temporally downstream from all variables in $A$, to ensure a clean causal interpretation of the effect of $A$ on $S$. However, the less studied concept of interleaved empowerment has been mentioned in (Klyubin et al. 2008), where $S$ contains sensor variables that lie before some variables in $A$.

### 4.4.4 Context-dependent Empowerment

Until now, we have considered empowerment as a generic characterization of the information efficiency of the perception-action loop. Now we go a step further and resolve this informational efficiency in more detail; specifically, we are going to consider empowerment when the system (e.g. agent and environment) is in different states. Assuming that the state of the system is given by $r$, it will in general affect the effect of the actions on the later sensor states, so that one now considers $p(s|a, r)$ and defines empowerment for the world being in state $r$ as

$$
E(r) = \max_{p(a)} I(S; A|r),
$$

which is referred to as state-dependent empowerment. This also allows us to define the average state-dependent empowerment for an agent that knows what state the world is in as

$$
E(R) = \sum_{r \in R} p(r) E(r)
$$

Note that this is different from the general empowerment: the general empowerment in Sec. 4.4.2 does not distinguish between different states. If different perception-action loop characteristics $p(s|a)$ are not resolved, the general empowerment can be vanishing, while average state-dependent empowerment is non-zero. In other words, empowerment can depend on being able to resolve states which affect the actuation-sensing channel.

In general, an agent will not be able to resolve all states in the environment, and will operate using a limited sensoric resolution of the world state. When we assume this, the agent might still be able to recognize that the world is in a certain context $k \in K$, based on memory and sensor input. So, an agent might not know its precise state in the world, but may be able to identify some coarse position, e.g. that it might be north or

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4Future work will investigate the effect of feedback, i.e. closed-loop sequences. However, the current hypothesis is that there will be little qualitative and quantitative difference for most scenarios, with significantly increased computational complexity.

5The interpretation of interleaved empowerment is slightly subtle and still subject to study, as in this case $S$ is then capturing rather an aspect of the richness of the action sequences and the corresponding action history, in addition to the state dynamics of the system.
south of some distinct location. [Klyubin et al. (2008)] demonstrate an example of how such a context can be created from data. Based on this context, it is then possible to define the marginal conditional distribution \( p(s|a, k) \), which then allows us to compute the (averaged) contextual empowerment for \( K \) as

\[
\mathcal{E}(K) = \sum_{k \in K} p(k) \mathcal{E}(k)
\]  

(4.10)

In comparison, context free empowerment \( \mathcal{E}_{\text{free}} \) has no assumption about the world and is based on the marginal distribution \( p(s|a) = \sum_r p(s|a, r) p(r) \) of all world states. This is the empowerment that an agent would calculate which has no information about the current world state. It can be shown (Capdepuy 2010) with Jensen’s Inequality that

\[
\mathcal{E}_{\text{free}} \leq \mathcal{E}(K) \leq \mathcal{E}(R)
\]  

(4.11)

This implies (see also Klyubin et al. 2008) that there is a (not necessarily unique) minimal optimal context \( K_{\text{opt}} \) that best characterizes the world in relation to how the agent’s actions affect the world, defined by:

\[
K_{\text{opt}} = \arg \min_{K} H(K).
\]  

(4.12)

Such a context \( K_{\text{opt}} \) is one which leads to the maximal increase in contextual empowerment. Klyubin et al. (2008) argues that such an agent internal measure could be useful to develop internal contexts which are purely intrinsic and based on the agent sensory-motor capacity, and thereby allow developing an understanding of the world based on the way they are able to interact with it.

### 4.4.5 Open vs. Closed-Loop Empowerment

An important distinction to make is the one between open- and closed-loop empowerment. Open-loop empowerment treats the perception-action loop like a unidirectional communication channel, and assumes that all inputs are chosen ahead of time and without getting any feedback about their source. Closed-loop empowerment is computed under the assumption that some of the later actions in \( n \)-step empowerment can change in reaction to the current sensor state.

In most of the existing work, empowerment calculations have been performed with open-loop empowerment only. The framework for this simplest of cases of communication theory is well developed and long known. For the more intricate cases using feedback, Capdepuy (2010) pointed out that directed information (Massey 1990) could be used to simplify the computation of closed loop empowerment, and demonstrated for an example how feedback increases empowerment.

### 4.4.6 Discrete Deterministic Empowerment

A deterministic world is one where each action leads to one specific outcome, i.e. for every \( a \in A \) there is exactly one \( s_a \in S \) with the property that
Since here every action only has one outcome, it is clear that the conditional uncertainty of $S$ given $A$ is zero, i.e., $H(S|A) = 0$. From Eq. (4.4) it follows then that

$$E = \max_{p(a)} H(S) = \max_{p(a)} H(S).$$

(4.14)

Since the entropy is maximal for a uniform distribution, $S$ can be maximised by choosing any input distribution $p(a)$ which results in a uniform distribution over the set of all reachable states of $S$, i.e over the set $S_A = \{ s \in S | \exists a \in A : p(s|a) \geq 0 \}$. As a result, empowerment for the discrete deterministic case reduces to

$$E = - \sum_{s \in S_A} \frac{1}{|S_A|} \log \left( \frac{1}{|S_A|} \right) = \log(|S_A|).$$

(4.15)

The bottom line is that in a discrete deterministic world empowerment reduces to the logarithm of the number of sensor states reachable with the available actions. This means empowerment, in the deterministic case, is fully determined by how many distinguishable states the agent can reach.

### 4.4.7 Non-deterministic Empowerment Calculation

If noise is present in the system, an action sequence $a$ will lead to multiple outcomes $s$, and thus, we have to consider an actual output distribution $p(s|a)$. In this case, the optimizing distribution needs to be determined using the standard Blahut-Arimoto (BA) algorithm (Blahut 1972; Arimoto 1972) which is an expectation maximization-type algorithm for computing the channel capacity. BA iterates over distributions $p_k(a)$, where $k$ is the iteration counter, converging towards the distribution that maximises channel capacity, and thereby towards the empowerment value defined in Eq. (4.8). Since the action variable $A$ is discrete and finite we are dealing with a finite number of actions $a_v \in A$, with $v = 1, ..., n$. Therefore $p_k(a)$ in the $k$-th iteration can be compactly represented by a vector $p_k(a) \equiv (p_1^k, ..., p_n^k)$, with $p_v^k \equiv \Pr(A = a_v)$, the probability that the action $A$ attains the value $a_v$. Furthermore, let $s \in S$ be the possible future states of the sensor input as a result of selecting the various actions with respect to which empowerment is being calculated, and $r \in R$ is the current state of the environment. If we assume that $S$ is continuous we can follow the general outline from (Jung et al. 2011), and define, for notational convenience, the variable $d_{v,k}$ as:

$$d_{v,k} := \int_S p(s|r, a_v) \log \left( \frac{p(s|r, a_v)}{\sum_{i=1}^n p(s|r, a_i)p_i^k} \right) ds.$$

(4.16)

While this is the more general case, this integral is difficult to evaluate for arbitrary distributions of $S$. We will later discuss, in Sec. 4.6.6 how this integral can be approximated, but even the approximations are very computationally expensive. If we are dealing with discrete and finite $S$ we can simply define $d_{v,k}$ with a sum as
\[ d_{v,k} := \sum_{s \in S} p(s|r, a_v) \log \left( \frac{p(s|r, a_v)}{\sum_{i=1}^{n} p(s|r, a_i)p^i_k} \right). \]  

(4.17)

The definition of \( d_{v,k} \) encapsulates the differences between a continuous and discrete \( S \). Therefore, the following parts of the BA algorithm are identical for both cases. The BA begins with initialising \( p_0(a) \) to be (e.g.) uniformly distributed \( ^6 \) by simply setting \( p_{v0} = \frac{1}{n} \) for all actions \( v = 1, \ldots, n \). At each iteration \( k \geq 1 \), the new approximation for the probability distribution \( p_k(a) \) is obtained from the old one \( p_{k-1}(a) \) using

\[ p^v_k := \frac{1}{z_k} p^v_{k-1} \exp(d_{v,k-1}) \]  

(4.18)

where \( z_k \) is a normalisation parameter ensuring that the approximation for the probability distribution \( p_k(a) \) sum to one for all actions \( v = 1, \ldots, n \), and is defined as

\[ z_k := \sum_{v=1}^{n} p^v_{k-1} \exp(d_{v,k-1}). \]  

(4.19)

Thus \( p_k(a) \) is calculated for iteration step \( k \), it can then be used to obtain an estimate \( \mathcal{E}_k(r) \) for the empowerment \( \mathcal{E}(r) \) using

\[ \mathcal{E}_k(r) = \sum_{v=1}^{n} p^v_k \cdot d_{v,k}. \]  

(4.20)

The algorithm can be iterated over a fixed number of times or until the absolute difference \( |\mathcal{E}_k(r) - \mathcal{E}_{k-1}(r)| \) drops below an arbitrary chosen threshold \( \epsilon \).

### 4.5 Discrete Examples

#### 4.5.1 Maze

Historically, the first scenario used to illustrate the properties of empowerment was a maze setting introduced in (Klyubin et al. 2005a). Here, the agent is located in a two-dimensional grid world. The agent has five different actions; it can move to the adjacent squares north, east, south and west of it, or do nothing. An outer boundary and internal walls block the agents movement. If an agent chooses the action to move against a wall, it will not move.

The states of the agent’s action variable \( A \) for \( n \)-step empowerment are constituted by all \( 5^n \) action sequences that contain \( n \) consecutive actions. The resulting sensor value \( S \) consists of the location of the agent at time step \( t_{n+1} \), after the last action was executed. Since we are dealing with a discrete and deterministic world, empowerment can be calculated as in Eq. (4.15) in Sec. 4.4.6 by taking the logarithm of all states reachable in \( n \) steps.

\(^6\)In principle, any distribution can be selected, provided none of the initial probabilities is 0, as the BA-algorithm cannot turn a vanishing probability into a finite one.
Fig. 4.2. The graph depicts the empowerment values for 5 step action sequences for the different positions in a $10 \times 10$ maze. Walls are shown in white, and cells are shaded according to empowerment. As the key suggests empowerment values are in the range $[3.46, 5.52]$ bits. This figure demonstrates that by simply assessing its options (in terms of movement possibilities) reflected in its empowerment, the agent can discover various features of the world. The most empowered cells in the labyrinth are those that can reliably reach the most positions within the next 5 steps. The graph is a reproduction of the results in (Klyubin et al., 2005a).

4.5.2 Average Distance vs. Empowerment

In this maze example, empowerment is directly related to how many states an agent can reach within the next $n$ steps. Now, note that, via the agent’s actions, a Finsler metric-like (Wilkens, 1995; López and Martínez, 2000) structure is implied on the maze, namely the minimum number of action steps necessary to move from one given position in the maze to a target position. Calculating $n$-step empowerment for the current location in the maze then is simply the logarithm of all states with a distance of $n$ or less to the current state.

Although this $n$-step horizon provides empowerment with an essentially local “cone of view”, (Klyubin et al., 2005a) showed in the maze example that empowerment of a location is negatively correlated with the average distance of that location to all other locations in the maze. The first is a local, the latter, however, a global property. This indicates that the local property of $n$-step reachability (essentially $n$-step empowerment) would relate to a global property, namely that of average distance.

It is a current study objective to which extent this local/global relation might be true, and under which conditions. Wherever it applies, the empowerment of an agent (which can be determined from knowledge of the local dynamics, i.e. how are my next $n$-steps going to affect the world) could then be used as a proxy for certain global properties of the world, such as the average distance to all other states. It is clear that this cannot, in general, be true, as outside of the empowerment horizon $n$, an environment could change its characteristics drastically, unseen to the “cone of view” of the agent’s local empowerment. However, many relevant scenarios have some regularity pervading the whole system which has the opportunity to be detected by empowerment.
This motif was further investigated by Anthony et al. (2008), who studied in more
detail the relationship between graph centrality and empowerment. The first chosen
model was a two-dimensional grid world that contained a pushable box, similar to
(Klyubin et al. 2005a). The agent could take five actions; move north, south, west,
east, or do nothing. If the agent moves into the location with the box, the box would be
pushed into the next square. The state space, the set of possible world configurations,
included the position of the agent, and also the position of the box.

The complete system can be modelled as a directed labelled graph, where each
node represents a different state of the world and the directed edges, labelled with
actions, represent the transitions from one state to another under a specific action. For
an agent with 5 possible actions, all nodes have 5 edges leading away from them. This
is a generic representation of any discrete and deterministic model. The advantage of
this representation is that it provides a core characterization of the system in graph-
theoretic language which is abstracted away from a physical representation. As before,
the distance from one state to another depends on how many actions an agent needs
to move from the first to the second state. In general, this defines a Finsler metric-like
structure (see Sec. 4.5.2), and is not necessarily tied to physical distance.

Anthony et al. (2008) then studied the correlation between closeness centrality and
empowerment, both for the previously described box pushing scenario. In addition,
he considered a different scenario, namely scale-free random networks as transition
graphs. As before, one can consider closeness centrality (which is a global property),
and empowerment (which can be calculated from a local subset of the graph). Anthony
et al. (2008) find that:

“these results show a strong indication of certain global aspects of various
worlds being ‘coded’ at a local level, and an appropriate sensory configuration
can not only detect this information, but can also use it...”

It is, however, currently unknown how generally and under which circumstances this
observation holds. As mentioned before, it is possible to construct counterexamples.
A natural example is the one that Anthony et al. note in their discussion, namely that
the relationship breaks down for the box pushing example when the agents horizon
does not extend to the box; in this case, the agent is too far away for n-step empower-
ment to be affected by the box. This might indicate that a certain degree of structural
homogeneity throughout the world is necessary for this relation to hold, and that the
existence of different “pockets” in the state space with different local rules would limit
the ability of empowerment to estimate global properties. After all, if there is a part
of the world that is radically different from the one the agent is in, and the agent is
not able to observe it in the near future, the current situation may not be able to be
informative concerning that remote part of the world.

At present, however, it remains an open question how empowerment relates to
global properties, such as in the example of graph centrality or average distance. No
full or even partial characterization of scenarios where empowerment correlates to
global values is currently known.
4.5.3 Sensor and Actuator Selection

An agent’s empowerment is not only affected by the state of the world, i.e., the context of the agent, but also depends on what the agent’s sensors and actions are. This was illustrated by Klyubin et al. (2005a) by variation of the previously mentioned box-pushing example. In all scenarios we are dealing with a two dimensional grid world where the agent has five different actions. The center of the world contains a box. In Fig. 4.3 we see the 5-step empowerment values for the agent’s starting position in four different scenarios. The scenarios differ depending on

1. whether the agent can perceive the box and
2. whether the agent can push the box.

![Empowerment maps for 5-step empowerment in a 2 dimensional grid world, containing a box in the center. The scenarios differ by whether the box can be pushed by the agent or not, and whether the agent can perceive the box. Black indicates the highest empowerment. Figure reproduced from Klyubin et al. (2005a)](image)

In Fig 4.3b the agent can push the box but cannot sense it. The box neither influences the agent’s outcome, nor is the agent able to perceive it. Basically, this is just like a scenario without a box. Consequently, the empowerment map of the world is flat, i.e., all states have the same empowerment. For empowerment applications this is typically the least interesting case, as it provides no gradient for action selection (see also the comment on the “Tragedy of the Greek Gods” towards the end of Sec. 4.5.4).

Fig. 4.3d shows the empowerment map for an agent which can perceive the box, the agent’s sensor input is both its own position and the position of the box. This different sensor configuration changes the empowerment map of the world. Being close
to the box to affect it now allows the agent to “reach” more different outcomes, because different paths that lead to the same final agent location might affect the box differently, thereby resulting in different final states. This results in higher empowerment closer to the box. Note that, comparing this to the previous scenario where the box was not visible, the agent’s actions are not suddenly able to create a larger number of resulting world states. Rather, the only change is that the agent is now able to discriminate between different world states that where present all along.

Figures 4.3.a and 4.3.c show the empowerment map for an non-pushable box, so when the agent moves into the box’s square, its movement fails. As opposed to the earlier cases, here we see that the empowerment around the box is lowered, because the box is blocking the agents way, thereby reducing the number of states that the agent can reach with its 5-step action sequence. We also see that the empowerment maps in Fig. 4.3.a and 4.3.c are identical, and that it does not matter if the agent can perceive the box or not. This connects back to our earlier arguments that empowerment is about influencing the world one can perceive. As it is not possible for the agent to affect the box’s positions, it is also not beneficial or relevant, from an empowerment perspective, to perceive the box position. This also relates back to earlier arguments about sensor and motor co-evolution. Once an agent loses it ability to affect the box, it might just as well lose it ability to sense the box.

One important insight that is demonstrated by this experiment is how different sensor and actuator configurations can lead to significantly different values for the state-dependent empowerment maps. Thus, which state has the highest empowerment might depend on an agent-sensor configuration (and not only on the world dynamics). This can be helpful when using empowerment to define an action policy. If an agent chooses its actions based on expected empowerment gain, then this method is a candidate for causing an agent to change its behaviour by only calculating empowerment for partial sensor input. For example, to drive an agent to focus on changing its location, then selecting a corresponding location sensor might be a good strategy.

4.5.4 Horizon Extension

Extending the horizon, i.e., using a larger $n$ in $n$-step empowerment, is another way to change the actions under consideration. Since the $n$-step action sequences can be treated just like atomic actions, lengthening the considered sequences creates more distinct actions to consider, which usually also have a bigger effect on the environment. Returning to the previous maze example, Fig. 4.4 illustrates how the empowerment map changes for action sequences of different length.

The short-term, 1-step empowerment only takes into account its immediate local surroundings. All that matters are if there are walls immediately next to the agent. In general, an agent locked in a room with walls just one step away would have the same empowerment as an agent on an open field. Also, this map only realizes 5 different empowerment values because the world is deterministic, and there can be maximally 5 different outcome states.

With more steps, the empowerment map starts to reflect the immediate surrounding of the agent and measures, as discussed by [Anthony et al.] (2008), how “central”
Fig. 4.4. The n-step empowerment map for the same maze with different horizons. Figure based on [Klyubin et al. 2005a]

An agent is located on the graph of possible states. But, as discussed earlier, the world could be shaped in a way that something just beyond the horizon of the agent’s empowerment calculation could change this completely. A possible solution would be to further extend the horizon of the agent. One problem, which we will address in the next section is that of computational feasibility.

Another problem is that the agent needs the sensor capacity to adequately reflect an increase in possible actions. Consider the following case: computing, say, 100-step empowerment, then the agent could reach every square from every other square, creating a flat empowerment landscape with an empowerment of $\log(100)$ everywhere. Since the agent itself is very (indeed maximally) powerful now, being able to reach every state of the world, its empowerment landscape is meaningless, as empowerment is incapable of distinguishing states via the number of options they offer. In principle, an analogous phenomenon can be created by massively extending the sensor capacity. Imagine an agent would not only be able to sense its current position, but also sense every action it has taken in the past. Now the agent could differentiate between every possible action sequence, as every one is reflected as a different sensor state. This again leads to a flat empowerment landscape, with empowerment being the logarithm of all possible actions.

So, in short, one has to be careful when the state-space of either actions or sensors is much larger than the other. In this case it is possible that the channel capacity becomes the maximal entropy of the smaller variable for all possible contexts, thereby creating a flat empowerment landscape. This phenomenon can be subsumed under the plastic notion of the “Tragedy of the Greek Gods”: all-knowing, all-powerful agents see no salient structure in the world and need to resort to avatars of limited knowledge and power (in analogy to the intervention of the Greek gods with the human fighters in the Trojan War) to attain any structured and meaningful interaction. In short, for meaningful interaction to emerge from a method such as an empowerment landscape, limitations in sensing and acting need to be present. The selection of appropriate levels of power and resolution is a current research question.
4.5.5 Impoverished Empowerment

While seeking the right resolution for actions and sensors can be an issue in worlds of limited complexity, a much more imminent challenge is the fact that as the empowerment horizon grows, the number of action sequences one needs to consider grows exponentially with the horizon. Especially when noise is involved, this becomes quickly infeasible.

To address this dilemma, [Anthony et al. (2011)] suggest a modified technique that allows for the approximation of empowerment-like quantities for longer action sequences, arguing, among other, that this will bring the empowerment approach in principle closer to what is cognitively plausible.

The basic idea of the **impoverished empowerment** approach is to consider all \( n \)-step action sequences (as in the simple empowerment computation), but then to select only a limited amount of sequences from these, namely those which contribute the most to the empowerment at this state. From the endpoint of this “impoverished” action sequence skeleton, this process is then repeated for another \( n \)-step sequence, thereby iteratively building up longer action sequences.

In the deterministic case, the selection is done so that the collection of action sequences has the highest possible empowerment. So, if several action sequences would lead to the same end state, only one of them would be chosen.

Interestingly, a small amount of noise is useful for this process, as it favours selecting action sequences which are further apart, because their end states overlap less. If no noise is present, then two action sequences which would end in neighbouring locations would be just as valid as two that lead to completely different locations, but the latter is more desirable as it spans a wider space of potential behaviours.

4.5.6 Sensor and Actuator Evolution

Since empowerment can be influenced by the choice of sensors, it is possible to ask what choice of sensors is maximising an agent’s empowerment. [Klyubin et al. (2005b, 2008)] addressed this question by using a Genetic Algorithm-based optimization for a scenario in which sensors are being evolved to maximize an agent’s empowerment. An agent is located in an infinite two-dimensional grid world. On each turn it can take one of five different actions which are to move in one of four directions, or to do nothing. Each location now has a value representing the concentration of a marker substance which is inversely proportional to the distance of the current location to the center at location \((0, 0)\).

In this scenario, the agents’ sensors can change, both in positioning and number. A sensor configuration is defined by where each of the \( n \) sensors of the agent is located relative to the agent. The sensor value has \( n \) states, and represents which of the \( n \) sensors detects the highest concentration value of the marker.

[Klyubin et al. (2005b, 2008)] then evolved the agents’ sensor configuration to maximize empowerment for different starting locations with respect to the centre. So, for example, they evolved the sensor configurations to achieve the highest empowerment when the agent starts its movement at location \((0, 0)\). To avoid degeneracy, a slight
cost factor for the number of sensors was added. In this way the adaptation has to evaluate if the added cost of further sensors are worth the increase in empowerment. The resulting sensor configurations for a 4-step empowerment calculation can be seen in Figure 4.5.

Fig. 4.5. The Figures show what sensor configurations empowerment evolves for different starting positions. The first number indicates how many spaces east of the center the agent starts, and the second number is the resulting empowerment value of the sensor configuration. Figure taken from (Klyubin et al. 2008)

The result was unsurprisingly that different starting positions would lead to different sensor layouts. More interestingly, they realized that the space of possible solutions can be more constrained in some places, so there is only one good solution, while other locations offer several different, nearly equally empowered solutions. More importantly is the observation that empowerment agnostically selects modalities which are most appropriate for the various starting locations. Consider, for instance, Fig. 4.5 which shows how the sensors are placed relative to the agent as the agent moves increasingly away from the center of the world, and to the right of it. The first images show the sensor placement when the agent is at the center of the world. The sensors are placed with more-or-less precision around the center, and there is some indifference as to their exact placement.

In the second row, when the agent has been moved seven and more fields to the right of the centre, a more prominent “blob” is placed at around the location of the centre (the diagram shows the relative placement of the sensors with respect to the agent, so a blob of black dots is covering roughly the location at which the centre of the world will be with respect to the agent.

Finally, as the agent moves further to the right (end of second and last row in Fig. 4.5), a striking effect takes place: the blob sensor, which roughly determines a two-dimensional location of the centre, collapses into a “heading” sensor which is no longer a two-dimensional blob, but rather has 1-dimensional character. This demon-
strates that empowerment is able to switch to different modalities (or, in this case, from a 2-dimensional to a 1-dimensional sensor). Because of its information-theoretic nature, empowerment is not explicitly using any assumptions about modality or dimensionality of sensors. The resulting morphologies are purely a result of the selection pressure via empowerment in interaction with the dynamics and structure of the world under consideration.

Another result of the evolutionary scenario involved the evolution of actuators. Without repeating the full details that can be found in (Klyubin et al. 2008), we would like to mention one important result, namely that the placement of actuators via empowerment-driven evolution, unlike the sensors, was extremely unspecific. Many configurations led to maximum empowerment solutions. The authors suggest that this results as a consequence of the agent being unable to choose what form the ‘information’ takes, that it has to extract from the environment. Hence, the sensors have to adapt to the information structure available in the environment, leaving the agent free to choose its actions. Therefore many different actuator settings can be used as the agent can utilize each of them to full effect by generation of suitable action sequences. This is an indicator that an agent’s action choices should be a more valuable and “concentrated” source of information than the information extracted from the environment, as every action choice is significant, while sensoric information needs to be “scopped” in on a wide front to capture some relevant features. This insight has been taken onboard in later work in form of the concept of digested information (Salge and Polani 2011) where agents observe other agents because their actions are more informationally dense than other aspects of the environment. The core idea of digested information is that relevant information (as defined in (Polani et al. 2006)) is often spread out in the environment, but since an agent needs to act upon the information it obtains, the same information is also present in the agent’s actions. Because the agent’s action state-space is usually much smaller than the state-space of the environment, the agent “concentrates” the relevant information in its actions. From the perspective of another, similar agent this basically means that the agent digests the relevant information and then provides it in a more compact format. It should be noted that all structure in the above example emerges purely from informational considerations; no other cost structure (such as e.g. energy costs) have been taken into account to shape the resulting features.

4.5.7 Multi-Agent Empowerment

If two or more agents share an environment, so that their actions all influence the state of the world, then their empowerment becomes intertwined. Capdepuy (2010); Capdepuy et al. (2007, 2012) investigate this phenomenon in detail. Here, due to lack of space, we will limit ourselves to briefly outline his results.

If both agents selfishly optimize their empowerment, then the outcome depends heavily on the scenario. A fully formal categorization is still outstanding, but the qualitative phenomenon can be described in terms similar to different game solution types in game theory. One finds situations that are analogous to zero-sum games where the empowerment of one agent can only be raised to the detriment of the other. In other sit-
uations, selfish empowerment maximisation leads to overall high empowerment, and, finally, there are scenarios where agent’s strategies converge onto the analoga of intricate equilibria reminiscent of the Nash equilibria in games.

An interesting aspect in relation to biology is Capdepuy’s work on the emergence of structure from selfish empowerment maximisation (Capdepuy et al. 2007). The model consists of a two-dimensional grid world where agents are equipped with sensors that measure the density of other agents in the directions around them. In this case, there is a tension between achieving proximity to other agents (to attain any variation in sensor input, as empty space does not provide any) and being sufficiently distant (as to attain sufficient freedom for action and not to be stuck without ability to move); this tension, in turn, provides an incentive to produce nontrivial dynamical structures. Some examples of agent populations evolved for greedy empowerment maximization and some of the better empowered structures resulting from this process can be seen in Fig. 4.6 Capdepuy et al. (2007).

![Fig. 4.6. Structures resulting from agent behaviour that was evolved to maximise the agents' individual empowerment. Each black dot in the figure represents an agent in one of the empowerment-maximizing scenarios. Agents are equipped with directional density sensors, measuring the number of other agents present in that particular direction. Creating structures becomes beneficial for the agents, as it gives features to the environment that allow different resulting sensor inputs. The different structures are high empowered solutions of the artificial evolution. Figure taken from Capdepuy (2010).](image)
4.6 Continuous Empowerment

The empowerment computations that we considered earlier were all operating in discrete spaces. But if we want to apply empowerment to the real world we need to consider that many problems, especially those related to motion or motor control, are continuous in nature. We could apply naive discretizations with finer and finer resolutions, but this will quickly lead to large state and actions spaces, with a forbidding number of options where direct computation of empowerment become very computationally expensive (Klyubin et al. 2008); therefore, different approaches need to be taken to deal with continuous dynamics effectively.

In this section, we will take a closer look at empowerment for continuous actuator and sensor variables. Compared to the discrete case, while channel capacity is still well defined for continuous input/output spaces, there are some important conceptual differences to be considered as compared to the discrete case.

One problem, as we shall illustrate, is that the continuous channel capacity could — in theory — be infinite. The reason for this is as follows: if there is no noise, and arbitrary continuous actions can be selected, these actions now allow to inject continuous, i.e. real-valued quantities (or vectors) into the world state. Reading in their (again) noiseless effect through real-valued sensors means that the full precision of a real number can be used in such a case. As arbitrary amounts of information can be stored in an infinite precision — noiseless — real number, this implies (in nondegenerate cases) an infinite channel capacity. Of course, such a situation is not realistic; in particular, relevant real-world systems always have noise and therefore the channel capacity will be limited.

However, when modeling a deterministic system with floating-point precision in simulation, there is no natural noise level. In a nondegenerate system, empowerment can be made as large as the logarithm of the number of actions (action sequences) available. This is, of course, meaningless. To be meaningful, one needs to endow the system with additional assumptions (such as an appropriate noise level) which are not required in the deterministic case.

But the main problem in the continuous case is that there is at the time of this review no known analytic solution to determine the channel capacity for a general continuous channel. To address this problem, a number of methods to approximate continuous channel capacity have been introduced. We will discuss them and how they can be used to compute empowerment.

We will briefly discuss naive binning, then the Monte Carlo Integration method developed by Jung et al. (2011), and then focus mostly on the quasi-linear Gaussian approximation, which is fast to compute.

4.6.1 Continuous Information Theory

The analogy to discrete entropy is rigorously defined for continuous random variables as differential entropy

\[ h(X) = -\int_X p(x) \log(p(x)) \, dx , \tag{4.21} \]
where \( p(x) \) now denotes not the probability, but the probability density function of \( X \), defined over a support set of \( X \subset \mathbb{R} \). Similarly, the \textit{conditional differential entropy} is defined as
\[
h(X|Y) = - \int_{y} p(y) \int_{X} p(x|y) \log(p(x|y)) \, dx \, dy.
\] (4.22)

The differential entropies cannot be directly interpreted in the same way as discrete entropies: they can become infinite or even negative. However, without delving too much into their individual interpretation, we will just state here that the difference of two differential entropy terms again can be interpreted as a proper mutual information:
\[
I(X;Y) := h(X) - h(X|Y),
\]
which shares essentially all characteristics of the discrete mutual information. Thus, consequently, the channel capacity is again defined by maximising the mutual information for the input probability density function
\[
\mathcal{C} = C(A \rightarrow S) = \max_{p(a)} I(A;S).
\] (4.23)

We will still be dealing with discrete time steps. Just like in the discrete case, we will use the notation \( A_t \) and \( S_t \) not just for single, but also for compound random variables. So, for each time \( t \), both variables \( A_t \) and \( S_t \) can consist of vectors of multiple random variables. The variables \( A \) and (where relevant) \( S \) itself are then again a selection of actuator and sensor variables at different times \( t \), so for example, the actuator input for \( n \)-step empowerment might be written compactly as \( A = (A_t, ..., A_{t+n-1}) \).

### 4.6.2 Infinite Channel Capacity

As mentioned above, in contrast to the discrete case, the continuous channel capacity can be infinite for some \( p(s|a) \). Formally, this results from the fact that differential entropy \textit{can} become negative. For instance, it becomes negative infinity for a Dirac \( \delta_x(.) \) “distribution”. The Dirac “distribution” is a probability measure concentrated on a single point: it can be mathematically defined in a precise fashion, but for the following discussion, the intuition is sufficient that \( \delta_x(.) \) is normalized (the integral over this “distribution” is 1), and is 0 everywhere with exception of the one point \( x \) at which it is concentrated, where it assumes an infinite value.

To illustrate, imagine that the channel \( p(s|a) \) exactly reproduces the real-valued input value of \( a \in \mathbb{R} \), i.e. that it implements \( s = a \), i.e. \( p(s|a) \equiv \delta_a(s) \). Every input \( a \) precisely determines the output \( s \), so \( h(S|a) = -\infty \). This remains negative infinity when we integrate over all possible inputs, so \( h(S|A) = -\infty \). If we now choose for \( p(a) \) the uniform input distribution between 0 and 1, which has a differential entropy of 0, we then get the following mutual information
\[I(A;S) = \infty.\]

\(^{7}\) One exception is that the continuous version of mutual information can become infinite in the continuum. This, however, is perfectly consistent with the ability to store infinite amount of information in continuous variables and does not change anything substantial in the interpretation.

\(^{8}\) Strictly spoken, we should denote this quantity as \textit{differential} mutual information, but unlike the differential entropy, this term retains the same interpretation in the continuous as in the discrete case, and therefore we will not especially qualify it by terming it “differential”.

\[ I(A; S) = h(S) - h(S|A) = h(A) - (-\infty) = \infty. \quad (4.24) \]

It holds \( H(S) = H(A) \), because the channel just copies the input distribution to the output. Since this is the largest possible value, this is also the channel capacity.

### 4.6.3 Continuous Empowerment Approximation

While channel capacity is well defined for any relationship between \( S \) and \( A \), it can only be computed for a subset of all possible scenarios. We will here approximate the model of the world with one for which empowerment can be computed. The following section discusses different approaches for doing so.

### 4.6.4 Binning

The most straightforward and naive approximation for continuous empowerment is to discretize all involved continuous variables and then compute the channel as described in the discrete empowerment section.

However, there are different ways to bin real-valued numbers and, as Olsson et al. (2005) demonstrated, they clearly affect the resulting informational values. Uniform binning considers the support of a real-valued random variable (i.e. the set of values of \( x \) for which \( p(x) > 0 \)), splits it into equally sized intervals and assigns to each real number the bin it falls into. Of course, this does not necessarily result in the same number of events in each bin and, furthermore, many bins can be left empty or with very few events while others contain many events. This unevenness can mean that significant “information” (in the colloquial sense) in the data is being discarded. The response is to choose the binning in a not necessarily equally spaced way, that ensures that all bins are used, and that the events are well distributed. This is achieved by Max-Entropy binning where one adaptively resizes the bins so the resulting distribution has the highest entropy, which usually results in bins containing the approximately same number of events Olsson et al. (2005).

There are two caveats for this case: If adaptive binning is chosen, one needs to take care that the informational values of different measurements are comparable, and that the binning is the same throughout the same context of use. Therefore, it is important to choose the binning in advance, say, adapted only to the overall, context-free channel, and not adapt to each state-dependent channel separately. The second caveat is that, while adaptive binning distributes the events more-or-less evenly over the bins, this can thin out the sampling very considerably and cause the bins to be almost empty or containing very few elements each. This can induce the appearance of nonzero mutual information which, however, is spurious. In this case, it is better to choose a binning that is wide enough to ensure a sufficient number of events per bin. Both approaches require the availability of actual samples, so if the channel in question is only specified as a continuous conditional probability, it is necessary to generate random samples based on \( p(s|a) \).

A final note on information estimation: much more robust approaches for mutual information estimation are known, such as the Kraskov-Stögbauer-Grassberger (KSG)
estimator \cite{Kraskov2004}. Unfortunately, this method is not suitable for use with empowerment, as it requires the full joint distribution of the variables to be given in advance. When computing empowerment, however, one iteratively selects an input distribution, computes a joint distribution and then applies the information estimator. This means that if one uses the KSG-estimator, it affects the joint distributions and hence its own estimates of mutual information at later iterations of the process, and thus the conditions for correct operation of KSG cease to hold\footnote{The authors thank Tobias Jung for this information (private communication)}.

### 4.6.5 Evaluation of Binning

One problem with this approach is that it can introduce binning artefacts. Consider the following example: imagine one bins by proper rounding to integers. In this case, outcomes such as, say, 0.6 and 1.4 become the same state, while 1.4 and 1.6 are considered different. If now an agent which moves along the real valued line by an amount of 0.2 at each time step, this binning would make the agent appear to be more empowered at 1.5 then it would be at 1.0, because it could move to two different resulting states from 1.5. If the binning would reflect true sensoric resolution of the agent, this would conform with the empowerment model of being able to resolve the corresponding states; however, in our example, we did not imply anything like that — the underlying continuous structure is completely uniform, and we did not introduce any special sensoric structure. Thus, the difference in empowerment is a pure artefact introduced by the binning itself.

Another problem that emerges with the use of a binning approach is the right choice of granularity. If too few bins are chosen, then, while one has a good number of samples in the bins, interesting structural effect and correlations are lost. If too many bins are chosen, then many (or all bins) contain very few samples, perhaps as few as only one or even none. Such a sparse sampling can significantly overestimate the mutual information of the involved variables. Another problem, specifically in conjunction with empowerment, is that such a sparse sampling is often likely to cause one action to produce exactly one distinguishable sensoric outcome. This means that empowerment reaches its maximum \( \log |A| \) for every context \( r \) depriving it of any meaning. However, if the resolution is high enough and sufficiently many samples are collected, binning can produce a quickly implemented (but typically slow to compute) approximation for empowerment. Examples of its application to the simple pendulum can be seen in \cite{Klyubin2008}.

### 4.6.6 Jung's Monte Carlo Integration

Another approximation to compute empowerment which can still deal with any kind of \( p(s|a) \) is Monte Carlo Integration \cite{Jung2011}. It is computed by sampling the outcomes of applying a representative set of available action sequences.

Assume that you have a model, so for a state \( r \) you can take actions \( a_v \), with \( v = 1, \ldots, n \), and draw \( N_{MC} \) samples, which will result in sensor states \( s_{v,j} \), with
This method then approximates the term \( d_{v,k} \) from Eq. (4.16) in the BA by

\[
d_{v,k} \approx \frac{1}{N_{MC}} \sum_{j=1}^{N_{MC}} \log \left( \frac{p(s_{v,j} | r, a_v)}{\sum_{i=1}^{N} p(s_{v,j} | r, a_i) p_k^i} \right). \tag{4.25}
\]

To compute this the model needs to provide a way to compute how probable it is that the outcome of one action was produced by another. The necessary noise in the model basically introduces a “distance measure” that indicates how hard it is to distinguish two different actions.

One simple model is to assume that \( p(s | r, a_v) \) is a multivariate Gaussian (dependent on the current state of the world \( r \)), or can be reasonably well-approximated by it, i.e.,

\[
s | r, a_v \sim \mathcal{N}(\mu_v, \Sigma_v) \tag{4.26}
\]

where \( \mu_v = (\mu_{v,1}, \ldots, \mu_{v,n})^T \) is the mean of the Gaussian and the covariance matrix is given by \( \Sigma_v = \text{diag}(\sigma_{v,1}^2, \ldots, \sigma_{v,n}^2) \). The mean and covariance will depend upon the action \( a_v \) and the state \( r \). Samples from the distribution will be denoted \( \tilde{s}_v \) and can be generated using standard algorithms.

The following algorithm summarises how to approximate the empowerment \( E(r) \) given a state \( r \in \mathcal{R} \) and transition model \( p(s | r, a_v) \):

1. **Input:**
   a) Specify state \( r \) whose empowerment is to be calculated.
   b) For every action \( a_v \) with \( v = 1, \ldots, n \), define a (Gaussian) state transition model \( p(s | r, a_v) \), which is fully specified by its mean \( \mu_v \) and covariance \( \Sigma_v \).

2. **Initialise:**
   a) \( p_0(a_v) := 1/n \) for \( v = 1, \ldots, n \).
   b) Draw \( N_{MC} \) samples \( \tilde{s}_{v,i} \) each, according to distribution density \( p(s | r, a_v) = \mathcal{N}(\mu_v, \Sigma_v) \) for \( v = 1, \ldots, n \).
   c) Evaluate \( p(\tilde{s}_{v,i} | r, a_v) \) for all \( v = 1, \ldots, n \); \( \mu = 1, \ldots, n \); and sample \( i = 1, \ldots, N_{MC} \).

3. **Iterate** the following variables for \( k = 1, 2, \ldots \) until \( |E_k - E_{k-1}| < \epsilon \) or the maximum number of iterations is reached:
   a) \( z_k := 0 \), \( E_{k-1} := 0 \)
   b) For \( v = 1, \ldots, n \)
      i. \( d_{v,k} := \frac{1}{N_{MC}} \sum_{j=1}^{N_{MC}} \log \left( \frac{p(\tilde{s}_{v,j} | r, a_v)}{\sum_{i=1}^{N} p(\tilde{s}_{v,j} | r, a_i) p_k^i} \right) \)
      ii. \( E_k := E_{k-1} + p_{k-1}(a_v) \cdot d_{v,k-1} \)
      iii. \( p_k := p_{k-1}(a_v) \cdot \exp(d_{v,k-1}) \)
      iv. \( z_k := z_{k-1} + p_k(a_v) \)
   c) For \( v = 1, \ldots, n \)
      i. \( p_k(a_v) := p_k(a_v) \cdot z_k^{-1} \)

4. **Output:**
   a) Empowerment \( E(r) \approx E_{k-1} \) (estimated).
   b) Distribution \( p(a) \) achieving the maximum mutual information.
4.6.7 Evaluation of Monte Carlo Integration

Monte Carlo Integration can still deal with the same generic distributions $p(s|a)$ as the binning approach, and it removes the artefacts caused by the arbitrary boundaries of the bins. On the downside, it requires a model with a noise assumption. In the solution suggested by Jung et al. (2011) this lead to the assumption of Gaussian Noise.

The other problem is computability. For good approximations the number of selected representative action sequences should be high, but this also leads to a quick growth of computation time. The several applications showcased in Jung et al. (2011) all had to be computed off-line, which makes them infeasible for robotic applications.

4.6.8 Quasi-Linear Gaussian Approximation

In the previous section we saw that Jung’s Monte Carlo Integration method could deal with the rather general case where the relationship between actuators and sensor can be characterized by $s = f(r, a) + Z$, where $f$ is a deterministic mapping, and $Z$ is some form of added noise. The noise is necessary to limit the channel capacity, and an integral part of the Monte Carlo Integration in Eq. 4.25. While the noise can have different distributions, Jung’s example assumed it to be Gaussian.

We will now outline how the assumption of Gaussian noise, together with an assumption regarding the nature of $f$, will allow us to accelerate the empowerment approximation. Consider now actuation-sensing mappings of the form $s = f(r, a) + N(0, N_r)$, i.e. which can be described by a deterministic mapping $f$ on which Gaussian noise (which may depend on $r$) is superimposed.\[10\]

In principle, if the actions $A$ were distributed in an arbitrarily small neighbourhood around 0, one would need $f$ to be differentiable in $a$ with the derivative $D_a f$ depending continuously on $r$. In practice, that neighbourhood will not be arbitrarily small, so the mapping from $a$ to $s$ needs to be “sufficiently well” approximated at all states $r$ by an affine (or shifted linear) function in $f_r(a)$ for the allowed distributions of actions $p(a)$. To limit the channel capacity there has to be some constraint on the possible action distributions, and the linear approximation has to be sufficiently good for the actions that $A$ can actually attain.\[11\]

In other words, assuming the channel can be adequately approximated by a linear transformation applied to $A$ with added Gaussian noise, then it is possible to speed up the empowerment calculation significantly by reducing the general problem of continuous channel capacity to parallel Gaussian channels which can be solved with well-established algorithms. This provides us with the quasi-linear Gaussian approximation for empowerment which will now be presented in detail.

---

\[10\]We will treat this as centred noise, with a mean of 0, but this is not necessary, as any non-zero mean would just shift the resulting distribution, which would leave the differential entropies and mutual information unaffected.

\[11\]We will not make this notion more precise or derive any error bounds at this point; we just informally assume that the Gaussian action distribution $A$ is concentrated well enough for $f_r$ to appear linear in $a$. 
Let $S$ be a multi-dimensional, continuous random variable defined over the vector space $\mathbb{R}^n$. Let $A$ be a multidimensional random variable defined over $\mathbb{R}^m$. As in the discrete, $A$ is the action variable, and $S$ the perception variable. According to the quasi-linear Gaussian approximation assumption, we assume that there is a linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ that allows us to express the relation between these variables via

$$S = TA + Z.$$  \hfill (4.27)

$Z$ is a suitable multi-dimensional, Gaussian variable defined over $\mathbb{R}^n$, modelling the combined acting/sensing noise in the system and is assumed to be independent of $A$ and $S$.

Consider first the simpler white noise case. Here we assume that the noise in each dimension $q \leq n$ of $Z$ is independent of the noise in all other dimensions, and has a normal distribution with $Z_q \sim \mathcal{N}(0, N_q)$ for each dimension (where $N_q$ depends on the dimension). This particular form of noise can be interpreted as having $n$ sensoric channels where each channel $q$ is subject to a source of independent Gaussian noise.

We now further introduce a limit to the power $P$ available to the actions $A$, i.e. we are going to consider only action distributions $A$ with $E(A^2) \leq P$. The reason for that is that without this constraint, the amplitude of $A$ could be made arbitrarily large and this again would render all outcomes distinguishable and thus empowerment infinite. \(^{12}\) The actual mean of the distributions is irrelevant for our purpose, as a constant shift does not affect the differential entropies. However, we need to ensure that the actuation range considered does not extend the size for which our linearity assumption holds.

It is plausible to consider this limitation as a physical power constraint. \(^{13}\) Under these constraints, the quantity of interest now becomes

$$\mathcal{E} = \max_{p(a): E(A^2) \leq P} I(S; A)$$  \hfill (4.28)

and the maximum being attained for normally distributed $A$ (thus we only need to consider Gaussian distributions for $A$ in the first place).

### 4.6.9 MIMO channel capacity

Now, assume for a moment that, in addition to our assumption of independent noise, the variance of the noise $Z$ in each dimension has the same value, namely 1, then the problem becomes equivalent to computing the channel capacity for a linear, Multiple-Input/Multiple-Output channel with additive and isotropic Gaussian noise. Though the

\(^{12}\)This specific power limit also implies that the optimal input distributions for the channel capacity results is Gaussian (Cover and Thomas [1991]).

\(^{13}\)This point is subtle: throughout the text, we had made a point that empowerment is determined by the structure of the actuation-perception loop, but otherwise purely informational. In particular, we did not include any further assumptions about the physics of the system. In the quasi-linear Gaussian case, the choice of a “physics-like” quadratic form of power limitation is only owed to the fact that it makes the problem tractable. Other constraints are likely to be more appropriate for a realistic robotic actuator model, but need to be addressed in future work.
methods to compute this quantity are well established in the literature, for reasons of self-containedness, we reiterate them here.

The MIMO problem can be solved by standard methods (Telatar 1999), namely by applying a Singular Value Decomposition (SVD) to the transformation matrix $T$, which decomposes $T$ as

$$T = U \Sigma V^T$$

(4.29)

where $U$ and $V$ are unitary matrices and $\Sigma$ is a diagonal matrix with non-negative real values on the diagonal. This allow us to transform Eq. (4.27) to

$$U^T S = \Sigma V^T A + U^T Z.$$  

(4.30)

It can be shown that each dimension of the resulting vectorial variables $U^T S$, $\Sigma V^T A$ and $U^T Z$ can be treated as an independent channel (see (Telatar 1999)), and thus reducing the computation of the overall channel capacity to computing the channel capacity for linear, parallel channels with added Gaussian noise, as in (Cover and Thomas 1991),

$$C = \max_{P_i} \sum_i \frac{1}{2} \log \left( 1 + \frac{\sigma_i P_i}{E[(U^T Z)_i^2]} \right) = \max_{P_i} \sum_i \frac{1}{2} \log(1 + \sigma_i P_i)$$

(4.31)

where $\sigma_i$ are the singular values of $\Sigma$, and $P_i$ is the average power used in the $i$-th channel, following the constraint that

$$\sum_i P_i \leq P.$$  

(4.32)

The simplification in the last step of Eq. (4.31) is based on the assumption of isotropic noise. Because the expected value for the noise is 1.0 and the unitary matrix applied to $Z$ does not scale, but only rotates $Z$, the noise retains its original value of 1.0.

We remind that the channel capacity achieving distribution for a simple linear channel with added Gaussian Noise is Gaussian (Cover and Thomas 1991). In particular, the optimal input distribution for each subchannel is a Gaussian with a variance of $P_i$. The optimal power distribution which maximizes Eq. (4.31) can then be found with the water-filling algorithm (Cover and Thomas 1991). The basic idea is to first assign power to the channel with the lowest amount of noise. This has an effect that could be described as one of “diminishing returns”: once a certain power level is reached, where adding more power to that channel has the same return as adding to the next best channel, additional power is now allocated to the two best channels. This is iterated to the next critical level and so on, until all power is allocated. Depending on the available total power, not all channels necessarily get power assigned to them.

We can also see, directly from the formula in Eq. (4.31), that since we divide by the variance of the noise $Z$, this value needs to be larger than zero. For vanishing noise, the channel capacity becomes infinite. Only the presence of noise induces an “overlap” of outcome states that allows one to obtain meaningful empowerment values. However, this is not a significant limitation in practice, as virtually all applications need to take into account actuator, system and/or sensor noise.
4.6.10 Coloured Noise

In a more general model, the Gaussian noise added to the multi-inputs, multi-output channel might also be coloured, meaning that the noise distributions in the different sensor dimensions are not independent. Let us assume that the noise is given by $Z \sim \mathcal{N}(0, K_s)$, where $K_s$ is the covariance matrix of the noise. As above, we assume that the distribution has a mean of zero, which is without loss of generality since translations are information invariant. The relationship between $S$ and $A$ is again expressed as

$$S = T'A + Z', \quad (4.33)$$

Conveniently, this can also be reduced to a channel with i.i.d. noise. For this, note that rotation, translation and scaling operators do not affect the mutual information $I(S; A)$. We start by expressing $Z'$ as

$$Z' = U\sqrt{\Sigma}ZV^T, \quad (4.34)$$

where $Z \sim \mathcal{N}(0, I)$ is isotropic noise with a variance of 1, and $U\Sigma V^T = K_s$ is the SVD of $K_s$. $U$ and $T$ are orthogonal matrices, and $\Sigma$ contains the singular values. Note that all singular values have to be strictly larger than zero, otherwise there would be a channel in the system without noise, which would allow the empowerment maximizer to inject all power into the zero-noise component of the channel and to achieve infinite channel capacity. $\sqrt{\Sigma}$ is a matrix that contains the square roots of the singular values, which should scale the variance of the isotropic noise to the singular values. The orthogonal matrices then rotate the distributions, so that they resemble $Z'$.

If we consider $\sqrt{\Sigma}^{-1}$, a diagonal matrix whose entries are the inverse of the square root of the singular values in $\Sigma$, this allows us to reformulate:

$$S = TA + U\sqrt{\Sigma}ZV^T \quad (4.35)$$

$$U^T SV = U^TTAV + \sqrt{\Sigma}Z \quad (4.36)$$

$$\sqrt{\Sigma}^{-1}U^T SV = \sqrt{\Sigma}^{-1}U^TTAV + Z \quad (4.37)$$

$$\sqrt{\Sigma}^{-1}U^T S = \sqrt{\Sigma}^{-1}U^TTA + ZV^T \quad (4.38)$$

$$\sqrt{\Sigma}^{-1}U^T S = \sqrt{\Sigma}^{-1}U^TTA + Z \quad (4.39)$$

The last step follows from the fact that the rotation of isotropic Gaussian noise remains isotropic Gaussian noise. This reduces the whole problem to a MIMO channel with isotropic noise and with the same channel capacity. We simply redefine the transformation matrix $T$ as

$$T = \sqrt{\Sigma}^{-1}U^T, \quad (4.40)$$

and solve the channel capacity for $S = TA + Z$, as outlined in section 4.6.9.

4.6.11 Evaluation of QLG Empowerment

The advantage of the quasi-linear Gaussian approximation is that it is quick to compute, the computational bottleneck being the calculation of a singular value decomposition that has the same dimensions as the sensors and actuators.
The drawbacks are its introduction of several assumptions. Like Jung’s integration, the approximation forces us to assume Gaussian noise. However, a more aggressive assumption than Jung’s approximation is that the QLG approximation also needs a locally linear model. So it is not possible to represent locally non-linear relationships between the actions and sensors. In particular the abrupt emergence of novel degrees of freedom which the empowerment formalism is so apt at discovering (see above, e.g. box pushing in Sec. 4.5.3) becomes softened by the Gaussian bell of the agent’s actuations.

Finally, the quasi-linear Gaussian approximation also introduces a new free parameter, the power-constraint $P$ which will be discussed in a later example. A more detailed examination of QLG empowerment can be found in (Salge et al. 2012).

4.7 Continuous Examples

We are aware of currently only two publications dealing with continuous empowerment. The first, by [Jung et al. (2011)], provides a good technical tutorial, and introduces the Monte Carlo Integration technique. Furthermore, it demonstrates that those states generally chosen as goals have high state-dependent empowerment, and that an empowerment-driven controller will tend to drive the system into them, even when initialized from a far away starting point. So, for example, the simple pendulum swings up, and stabilizes in the upright position, even when multiple swing-up phases are required; unlike traditional Reinforcement Learning, there is no value function that needs to be learnt over the whole phase space, but only the transition dynamics, and that needs only to be determined around the actual path taken. In principle, the algorithm does not need to visit any states but those in the neighbourhood of the path taken by the empowerment-driven controller. The empowerment-driven control method can be applied also to other, quite more intricate models, such as bicycle riding or the acrobot scenario (double-pendulum hanging from the top joint and driven by a motor at the middle joint).

The second paper (Salge et al. 2012) discusses the quasi-linear Gaussian method as a faster approximation for empowerment, and focusses on the pendulum; both to compare the QLG method with previous approximations, and to investigate how different parameters affect the empowerment map. In the following section we will use the simple pendulum from the second paper to outline some of the challenges in applying continuous empowerment.

4.7.1 Pendulum

The scenario we will focus on is that of a simple pendulum, because it incarnates many features typical for the continuous empowerment scenarios. First we will produce an empowerment map, which assigns an empowerment value for each state the pendulum can be in. Then we demonstrate empowerment-driven control; an algorithm that generates actions for the pendulum by greedily maximising its expected empowerment in the following step.
We start by observing that the pendulum’s current state at the time $t$ is completely characterized by its angle $\phi$ and its angular velocity $\dot{\phi}$.

For the model we time-discretize the input. So, the actuator variable $A$ contains real values $a_t$, which represents the external acceleration applied to the pendulum. So, at time $t$, the motor acceleration is set to $a_t$, and this acceleration is then applied for the duration $\Delta t$. At the end of $\Delta t$, we will consider the system to be in time $t + \Delta t$, and the next value is applied.

### 4.7.2 Action Selection

In general, just having a state-dependent utility function, which assigns a utility to each state (such as empowerment) does not immediately provide a control strategy. One way to address this is to implement a greedy action selection strategy, where each action is chosen based on the immediate expected gain in empowerment. Note that empowerment is not a true value function, i.e. following its maximum local gradient does not necessarily correspond to optimizing some cumulated reward.

For the discrete and deterministic case, implementing a greedy control is simple. Since we have local model knowledge, we know what state each action $a$ will lead to. We can then evaluate the empowerment for each action $a$ that can be taken in the current state, and pick that action that leads to the subsequent state with the largest empowerment. This basically provides a gradient ascent approach (modulated by the effect of the action on the dynamics) on the empowerment landscape, with all its benefits and drawbacks.

If we are dealing with a discrete but noisy system, one needs to specify in more detail what a “greedy” action selection should look like, since empowerment is not a utility function in the strict sense of utility theory, and the average empowerment over the successor states is not the same as the empowerment of the averaged dynamics. This means that one has different ways of selecting the desired action for the next step.

However, the most straightforward way remains, of course, the selection of the highest average empowerment when a particular action is selected. Assume that, given an action $a$, and a fixed starting state which we do not denote separately, one has the probability $p(s|a)$ of getting into a subsequent state $s$. Each of these successor states $s$ has an associated empowerment value $\mathcal{E}(s)$. Thus, the expected empowerment for carrying out the action $a$ is given by

$$E[\mathcal{E}(S)|a] = \sum_{s \in S} \mathcal{E}(s)p(s|a)$$

and one selects the action with the highest expected empowerment.

The necessity of distinction of deterministic and noisy cases becomes even more prominent in the continuous case, where the situation is more complicated. As we have to treat the continuum as a noisy system, there is usually no unique resulting state for an action $a$, but rather a continuous distribution density of states $p(s|a)$. Ideally, one would integrate the empowerment values over this distribution, similar to Eq. (4.41).

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14To simplify the argument, we consider here only fully observed states $s$. 
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but since empowerment cannot be expressed as a simple, integrable function, this is
not practicable. One solution is to simply look at the mean of a sampled distribution
over the resulting states and average their empowerment. At this point, however, no
bounds have been derived on how well this value represents the empowerment values
in the distribution of output states.

The continuity of actions creates another problem. Even if we can compute the
expected empowerment for a given action, then we still need to select for which actions
we want to evaluate their subsequent empowerment. Again, one possible option is to
sample several actions \(a\), distributed in a regular fashion; for example, one could look
at the resulting states for maximal positive acceleration, for no acceleration at all, and
for maximal negative acceleration, and then select the best. This may miss the action
\(a\) with the highest expected empowerment which might fall somewhere between these
sample points. Potential for future work would lie in developing an efficient method to
avoid expensive searches for the highest-valued successive expected empowerment.

4.7.3 Resulting Control

In Fig. 4.7 we can see a empowerment map for the pendulum, and the resulting trajec-
tory generated by greedy empowerment control. The controller sampled over 5 pos-
sible actuation choices, and chose the one where the resulting state had the highest
expected empowerment.

In this specific case, the pendulum swings up and comes to rest in the upright po-
sition. This solution, while typical, is not unique. Varying the parameters for time step
length and power constraint can produce different behaviour, such as cyclic oscillation
and resting in the lower position. We will discuss these cases further below.

One interesting observation to note here is that the empowerment of the pendu-
lum is not strictly increasing over the run, even though the control chooses the action
that leads to the most empowered successor state. If one considers the trajectory, it is
possible to see that the pendulum passes through regions where the empowerment low-
ers again. This can be seen in Fig. 4.7 where the trajectory passes through the darker
regions of lower empowerment after already being in much lighter regions of the em-
powerment map. This is due to the specific dynamics of the system, in which one can
only control the acceleration of the pendulum, but, of course, not its position change,
which is mediated by the current velocity. So while the controller chooses the high-
est empowered future state, all future states have lower empowerment than the current
state.

Contrast this with the discrete maze case: in the latter, the agent could maintain any
state of the environment, i.e. it position, indefinitely, by doing nothing. Greedy control
in the maze therefore moves the agent to increasingly higher empowered states, until
it would reach a local optimum, and then remain there.

Strikingly, local empowerment maxima seem to be less of a problem in the pendu-
lum model (which is, in this respect, very similar to the mountain-car problem (Sutton
and Barto 1998)). One reason turns out to be that the pendulum cannot maintain cer-
tain positions. If the pendulum has a non-zero speed, then its next position will be a
different one, because the system cannot maintain both the speed and position of the
pendulum at the same time. This sometimes forces the pendulum to enter states that are of lower empowerment than its current state. In the pendulum example this works out well in traversing the low empowered regions; and the continued local optimization of empowerment happens to lead to later, even higher empowered regions.

It is an open question to characterize actuation-perception structures which would be particularly amenable for the local empowerment optimization to actually achieve global empowerment optimization or at least a good approximation of global empowerment optimization. At this point, it is clear that sharp changes in the empowerment landscape (e.g. discovery of new degrees of freedom, e.g. because of the presence of a new manipulable object) need to be inside the local exploration range of the action.
sequences used to compute empowerment. However, in the case of the pendulum, the maximally empowered point of the upright pendulum seems to “radiate” its basin of attraction into sufficiently far regions of the state space for the local greedy optimization to pick this up. The characterization of the properties that the dynamics of the system needs to have for this to be successful is a completely open question at this point. Given the examples studied in (Jung et al. 2011), a cautious hypothesis may suggest that dynamic scenarios are good candidates for such a phenomenon.

4.7.4 Power Constraint

A closer look at the different underlying empowerment landscapes of the quasi-linear approximation in Fig. 4.8 shows their changes in regard to power constraint $P$ and time step length $\Delta t$.

How the change in the time step duration $\Delta t$ affects the empowerment, and also how it leads to worse approximations is studied in greater detail in (Salge et al. 2012). In general, it is not surprising that empowerment is indeed affected by it, in particular as the time step duration is closely related to the horizon length. The basic insight is, however, that a greater time step length allows a further look-ahead into the future, at the cost of a worsening approximation with the local linear model.

A more interesting effect in regard to the general applicability of the fast QLG method is the varying power constraint $P$. In general, an increase in power will result in an increase in empowerment, no matter where in the state space the system is. This is not immediately visible in the figures shown, since the colouring of the graphs is normalized, so the black and white correspond to the lowest and highest empowerment value in the respective subgraph.

A more unexpected effect, however, is a potential inversion of the empowerment landscape as seen in Fig. 4.8. Inversion means that for two specific points in the state space it might be that for one power level the first has a higher empowerment than the other, but for a different power level this relationship is reversed, and now the second point has a higher empowerment. For example, in Fig. 4.8 we can consider the row of landscapes for a $\Delta t$ of 0.7. With increasing power there appears a new ridge of local maximal empowerment around the lower rest position of the pendulum.

This slightly counterintuitive effect is a result of how the capacity is distributed on the separate parallel channels. Be reminded, each channel $i$ contributes its own amount to the overall capacity

$$C = \max_{P_i} \sum_i \frac{1}{2} \log(1 + \sigma_i P_i)$$

(4.42)

subject to the total power constraint $P$. Depending on the different values for $\sigma_i$, power is first allocated to the channel with the highest amplification value $\sigma_i$, up to a point where the return in capacity for the invested power diminishes so much that adding power to a different channel yields more capacity. From that point on the overall system acts as if it was one channel of bigger capacity.

In other words, for low power the factor that determines the channel capacity is the value of the largest $\sigma$ alone. Once the power increases, the values of both the $\sigma$
become important. It is therefore possible that for low power, a point with one large \( \sigma \) has comparatively high empowerment, while for a higher power level, another point has a higher empowerment, because the combination of all the \( \sigma \) is better. This is what actually happens in the pendulum example and causes the pendulum to remain in the lower rest position in the examples with higher power.

This indicates that the empowerment-induced dynamics is sensitive to the given power constraint. One interpretation is that agents with weak actuators need to fine-tune their dynamics to achieve high-empowered states. However, agents with strong actuators can afford to stay in the potential minimum of the system, as their engine is strong enough to reach all relevant points without complicated strategems (“if in doubt, use a bigger hammer”). The inversion phenomenon is a special case for a more generic principle that force may be used to change the landscape in which the agent finds itself.

Another observation emerging from the inversion phenomenon is the general question of whether the Gaussian choice for the input distribution is appropriate. We know that some form of constraint must be applied, otherwise one could just choose input distributions that are spaced so far apart that they would fully compensate for the noise, giving rise to an (unrealistic) infinite channel capacity. Not only is this unhelpful, but also, as realistic actuations will be usually limited. In the current model, inspired by well-established channel capacity applications in communication theory, the power constraints reflect how limited amount of energies are allotted to broadcast a signal. But if we instead look, for example, at the acceleration which a robot could apply to its arm, then for instance an interval constraint would be much more natural to apply. For instance, an action \( a \) the robot could choose would lie, for example, between \(-4.0\) and \(+10.0\) m/s\(^2\); a servo-based system may, instead specify a particular location instead, but still constrained by a hard-bounded interval. As consequence, it might be better to have a model where, instead of a general power constraint \( P \), a hard upper and lower limit for each dimension of the actuator input \( A \) is imposed. At present, we are not aware of a method to directly compute the channel capacity for a multiple input, multiple output channel with coloured Gaussian noise that uses such a constraint.

### 4.7.5 Model Acquisition

Before we end this overview we will at least briefly address the problem of model acquisition or model learning. As mentioned, empowerment needs the model \( p(s|a, r) \) for its computation. Strictly spoken, the acquisition or adaptation of this model is not part of the empowerment formalism. It is external to it, the model being either given in advance, or being acquired by one of many candidate techniques. However, given that empowerment will be used in scenarios where the model is not known and has to be learnt at the same time as the empowerment gradient is to be followed, model acquisition needs to be treated alongside the empowerment formalism itself.

As mentioned, empowerment only needs a local model of the dynamics from the agent’s actuators to the agent’s sensor in the current state of the world, but this local model is essential to compute empowerment.
Fig. 4.8. A visualization of the different empowerment landscapes resulting from computation with different parameters for time step length $\Delta t$ and power constraint $P$. The graphs plot empowerment for the two dimensional state space (angular speed, angular position) of the pendulum. White areas indicate the highest empowerment, black areas the lowest possible empowerment. The lower rest position is in the middle of the plots, and has low empowerment for less powered scenarios. The upper rest position is high empowered in all cases, it is located in the middle of the right or left edge of the plots. The areas of high empowerment close to the upright angle are those were the angular speed moves the pendulum towards the upper rest position. Figure is taken from (Salge et al. 2012).
Much of the earlier empowerment work operates under the assumption that the agent in question has somehow obtained or is given a sufficiently accurate model $p(s|a,r)$. Without addressing the “how”, this acknowledges the fact that an agent-centric, intrinsic motivation mechanism needs to have this forward model available within the agent.

The earliest work to touch on this [Klyubin et al., 2008] deals with context-dependent empowerment. To model the relationship between an AIBO’s discrete actions, and some discrete camera inputs, regular motions of the head are performed to sample the environment. These were then used to construct joint probability distributions and select an appropriate separation of all states of $R$ into different contexts. The choice of context itself was also a decision on how to internally represent the world in an internal model, especially if there is only limited “resources” available to model the world. By grouping together states that behave similarly, the agent gets a good approximation of the world dynamics, and its internal empowerment computation results in high-empowered states. If the agent groups states with different behaviour together, then the resulting contexts have higher levels of uncertainty, and result in comparatively lower empowerment values (from the agent’s perspective).

In general, it is clear that the quality of the model will affect the internal evaluation of empowerment. If the dynamics of a state are modelled with a great degree of uncertainty, then this noise will also reflect negatively in the empowerment value for this state. The interesting question here is then how to distinguish between those states that are truly random, and those where the action model is just currently not well known. This also indicates another field of future research. The hypothesis is that, if we would model how exploration or learning would affect our internal model, then the maximisation of (internally computed) empowerment could also lead to exploration and learning behaviour.

In the continuous case, we have to deal with the additional question on how to best represent the conditional probability distributions, since, unlike the discrete case, there is no general and exact way of doing so. Jung et al. [2011] uses Gaussian Processes to store the dynamics of the world. This also offers a good interface between the use of a Gaussian Process Learner and the Monte Carlo integration with assumed Gaussian noise. The faster quasi-linear Gaussian approximation [Salge et al., 2012] also interact well with representation, and, conveniently, the covariance metric used for the coloured noise can be directly derived from the GP. In general, one would assume that other methods and algorithms to acquire a world model could be similarly combined with empowerment. It remains an open question which of these models are well suited, not just as approximations of the world dynamics in general, but in regard to how well they represent those aspects of the world dynamics that are relevant to attain high empowerment values.

4.8 Conclusion

The different scenarios presented here, and in the literature on empowerment in general, are highlighting an important aspect of the empowerment flavour of intrinsic mo-
tivation algorithms, namely its universality. The same principle that organizes a swarm of agents into a pattern can also swing the pendulum into an upright position, seek out a central location in a maze, be driven towards a manipulable object, or drive the evolution of sensors.

The task-independent nature reflected in this list can be both a blessing and a curse. In many cases the resulting solution, such as swinging the pendulum into the upright position, is the goal implied by default by a human observer. However, if indeed a goal is desired that differs from this default, then empowerment will not be the best solution. At present, the question of how to integrate explicit non-default goals into empowerment is fully open.

Another strong assumption that comes with the use of empowerment is its local character. On the upside, it simplifies the computation and makes the associated model acquisition much cheaper as only a very small part of the state space ever needs to be explored; the assumption of the usefulness of empowerment as a proxy principle for other implicit and less accessible optimization principles depends heavily on how well the local structure of the system dynamics will reflect its global structure. The precise nature of this phenomenon is not fully understood in the successful scenarios, but is believed to have to do with the regularity (e.g. continuity/smoothness) of the system dynamics. Of course, if any qualitative changes in the dynamics happen just outside of the empowerment horizon, the locality of empowerment will prevent them from being seen. This could be due to some disastrous “cliff”, or something harmless like the discovery of an object that can be manipulated. Once, however, the change enters the empowerment horizon, and assuming that one can obtain a model of how it will affect the dynamics without losing the agent, empowerment will provide the gradients appropriate to the change.

Another central problem that, in the past, has reappeared across different applications is the computational feasibility. Empowerment quickly becomes infeasible to compute, which is a problem for both the behavioural empowerment hypothesis, and the application of empowerment to real-time AI or robotics problems. Newer methods address both the case for continuous empowerment (such as the QLG), and deeper empowerment horizons (such as the “impoverished” versions of empowerment). They, of course, come with additional assumptions and parameters, and provide only approximate solutions, but maintain the general character of the full solutions, allowing to export empowerment-like characteristics into domains that were hitherto inaccessible.

Let us conclude with a remark regarding the biological empowerment hypotheses in general: the fact that the default behaviours produced by empowerment seem often to match what intuitive expectations concerning default behaviour seem to imply, there is some relevance in investigating whether some of these behaviours are indeed approximating default behaviours observed in nature. A number of arguments in favour of why empowerment maximizing or similar behaviour could be relevant in biology have been made in (Klyubin et al. 2008), of which in this review we mainly highlighted its role as a measure of sensorimotor efficiency and the advantages that an evolutionary process would confer to more informationally efficient perception-action configurations.
Together with other intrinsic motivation measures, empowerment is thus a candidate measure which may help bridge the gap between understanding how organisms may be able to carry out default adaptations into their niche in an effective manner, and methods which would also allow artificial devices to try and copy the success that biological organisms have in doing so.

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