Nonintegrability of the two-body problem in constant curvature spaces II.

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Abstract

We consider the reduced two-body problem with a central potential on the sphere $S^2$ and the hyperbolic plane $H^2$. For two potentials different from the Newton and the oscillator ones we prove the nonexistence of an additional meromorphic integral for the complexified dynamic systems.

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1 Introduction

The study of classical mechanics in constant curvature spaces was begun in the second half of the 19th century in works of Lipschitz [1], Killing [2] and Neumann [3]. They dealt with the one-body problem in a central potential on spheres $S^2$ and $S^3$. This problem revealed much similarity with its Euclidean analogue: the existence of the Newton-like potential (known already to Lobachevski [4]) and three Kepler laws for it.

At the beginning of the 20th century these results were transferred onto the hyperbolic space by Liebmann [5], [6], who also generalized the Bertrand theorem [7] for constant curvature spaces [8].

After the rise of special and general relativity, these results were almost completely forgotten (see nevertheless [9]) and rediscovered many times in the framework of the theory of integrable dynamical systems (see [10], section 6.4).

The two-body problem with a central interaction in constant curvature spaces $S^n$ and $H^n$ considerably differs from its Euclidean analogue. The variable separation for the latter problem is trivial, while for the former one no central potentials are known that admit a variable separation.

The reduction of the two-body problem in constant curvature spaces to a Hamiltonian system with two degrees of freedom was carried out using different approaches in [11] and [12] (see also [10]).

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The Morales-Ramis theory \cite{13} contains a straightforward method for proving the nonintegrability of Hamiltonian dynamical systems. To do this one needs:

1. select a particular explicit solution of a system;
2. write a system of normal variation equations (NVE) in a neighborhood of this solution;
3. prove that the identity component of the differential Galois group (\cite{14}, \cite{15}) of this NVE is not abelian.

Note, that the direct algorithm for the third step exists only for linear differential equations with rational coefficients.

This method (steps 1-3) was realized for many dynamical systems, see, for example, \cite{16}, \cite{17} and references therein.

The meromorphic nonintegrability of the reduced two-body problem on the sphere $S^2$ and the hyperbolic plane $H^2$ with the Newton- and oscillator-like potentials was proved in \cite{18}. The particular explicit solution used there is a motion of both bodies along a common geodesic. The extension of this result onto the case of $S^3$ and $H^3$ (most general for the two-body problem on constant curvature spaces \cite{11}) is hampered by the absence of a known explicit solution, different from the mentioned above, in the case of arbitrary masses.

Here, we shall prove the nonexistence of an additional meromorphic first integral for the restricted two-body problem on $S^2$ and $H^2$ with two other central potentials, admitting the reduction of a corresponding NVE to a second order differential equation with rational coefficients. A more general description of a Hamiltonian reduction of systems under consideration can be found in \cite{10}.

## 2 Reduced two-body systems on $H^2$, $S^2$ and NVE

### 2.1 The reduced two-body problem on the hyperbolic plane $H^2$

After the excluding of the diagonal (corresponding to the collision of particles), the configuration space $H^2 \times H^2$ can be represented as a direct product

\[ Q := (H^2 \times H^2) \setminus \text{diag} \cong I \times O_0(1, 2), \]

where $I := (0, \infty)$ and the identity component $O_0(1, 2)$ of the isometry group for $H^2$ acts only onto the second factor. Therefore, the phase space

\[ M := T^*Q = T^*I \times T^* (O_0(1, 2)) \]

is reduced to $T^*I \times \mathcal{O}$, where $\mathcal{O}$ is a coadjoint orbit of $O_0(1, 2)$.

After a time rescaling the Hamiltonian function can be represented in the form

\[ h_h = \frac{1}{2\mu} \left( p_\theta^2 + \frac{p_2^2}{\sinh^2 \theta} \right) + p_0 p_0 + p_2^2 + p_1 p_2 \coth \theta + V(\theta), \quad (2.1) \]
2 Reduced two-body systems

where \( \mu := m_1/(m_1 + m_2) \neq 0 \) for bodies’ masses \( m_1, m_2 \) and the Poisson brackets for variables \( \theta, p_\theta, p_0, p_1, p_2 \) are as follows:

\[
\begin{align*}
\{ \theta, p_\theta \} &= 1, \\
\{ p_0, p_1 \} &= p_2, \\
\{ p_1, p_2 \} &= -p_0, \\
\{ p_0, p_2 \} &= p_1, \\
\{ \theta, p_i \} &= 0, \\
\{ p_\theta, p_i \} &= 0, \quad i = 0, 1, 2.
\end{align*}
\]

Here \( p_\theta \) is the momentum, conjugated to \( \theta \in I \), and \( p_0^2 + p_1^2 - p_2^2 = \gamma = \text{const} \in \mathbb{R} \) on \( O \).

The motion of bodies along a common geodesic with a total nonzero momentum \( \gamma \) corresponds to \( p_0 \equiv p_\theta = \text{const} \neq 0, p_1 \equiv p_2 \equiv 0 \) for \( \gamma = p^2 > 0 \). It is described by the Hamiltonian function

\[
h_0 = \frac{1}{2\mu} p_\theta^2 + pp_\theta + V(\theta) = \frac{1}{2\mu} (p_\theta + \mu p)^2 + V(\theta) - \frac{\mu}{2} p^2. \tag{2.2}
\]

Let \( p_\theta = p_\theta(t), \theta = \theta(t) \) be a solution of the Hamiltonian system with the Hamiltonian function \( (2.2) \). The normal variational equations in a neighborhood of this solution are (see \[18\])

\[
\begin{align*}
\frac{dp_1}{dt} &= -p \coth \theta p_1 - \left( 2p + p_\theta + \frac{p}{\mu \sinh^2 \theta} \right) p_2, \\
\frac{dp_2}{dt} &= -p_\theta p_1 + p \coth \theta p_2.
\end{align*}
\]

**Theorem 2.1.** The complexified Hamiltonian system with Hamiltonian function \( (2.1) \), potentials \( V(\theta) = \alpha \tanh \theta \) and \( V(\theta) = \alpha \sinh^{-1} \theta, \alpha \in \mathbb{R} \) does not admit an additional meromorphic first integral in the case \( m_1 m_2 \alpha \neq 0, \gamma > 0 \).

The proof of this theorem is similar to the proof of theorem \[2.2\] below.

2.2 The reduced two-body problem on the sphere \( S^2 \)

To represent the configuration space \( S^2 \times S^2 \) as a direct product \( I \times \text{SO}(3) \) one should exclude not only the diagonal, but also the set of antipodal points \( Q_{op} \cong S^2 \). This leads to

\( Q_{\text{ess}} := (S^2 \times S^2) \setminus (\text{diag} \cup Q_{op}) \cong I \times \text{SO}(3) \).

Since

\[
dim (S^2 \times S^2) - \dim (\text{diag} \cup Q_{op}) = 4 - 2 = 2 > 1,
\]

the typical trajectory does not intersect \( \text{diag} \cup Q_{op} \) and one can study the nonintegrability of a Hamiltonian system on \( S^2 \times S^2 \) using its restriction onto \( Q_{\text{ess}} \).

Thus, the corresponding phase space

\[
M := T^*Q_{\text{ess}} = T^*I \times T^*(\text{SO}(3))
\]

is reduced to \( T^*I \times O \), where \( O \) is a coadjoint orbit of \( \text{SO}(3) \) and \( I := (0, \pi) \).
Transformation of NVE’s

After a time rescaling the Hamiltonian function can be represented in the form

$$h_s = \frac{1}{2\mu} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) + p_\theta p_0 - p_2^2 + p_1 p_2 \cot \theta + V(\theta), \quad (2.3)$$

where the Poisson brackets for variables $\theta \in I, p_\theta, p_0, p_1, p_2$ are as follows:

$$\{\theta, p_\theta\} = 1, \{p_0, p_1\} = -p_2, \{p_1, p_2\} = -p_0, \{p_2, p_0\} = -p_1, \{\theta, p_i\} = 0, \{p_\theta, p_i\} = 0, i = 0, 1, 2$$

and $p_0^2 + p_1^2 + p_2^2 = \gamma = \text{const} \geq 0$ on $\mathcal{O}$.

Again, the motion $p_\theta = p_\theta(t), \theta = \theta(t)$ of bodies along a common geodesic with a total nonzero momentum $\gamma$ corresponds to $p_0 \equiv p = \text{const} \neq 0, p_1 \equiv p_2 \equiv 0$ for $\gamma = p^2 > 0$ and is described by Hamiltonian function (2.2).

The normal variational equations in a neighborhood of this solution are (see [18])

$$\frac{dp_1}{dt} = -p \cot \theta p_1 + \left( 2p + p_\theta - \frac{p}{\mu \sin^2 \theta} \right) p_2,$$

$$\frac{dp_2}{dt} = -p_\theta p_1 + p \cot \theta p_2, \quad (2.4)$$

Below, we shall prove

**Theorem 2.2.** The complexified Hamiltonian system with Hamiltonian function (2.3), potentials $V(\theta) = \alpha \tan \theta$ and $V(\theta) = -\alpha \sin^{-1} \theta$, $\alpha \in \mathbb{R}$ does not admit an additional meromorphic first integral in the case $m_1 m_2 \alpha \neq 0, \gamma > 0$.

### 3 Transformation of NVE’s

Here we shall transform system (2.4) to the second order differential equation

$$y''(z) = r(z)y(z). \quad (3.1)$$

#### 3.1 Potential $V(\theta) = \alpha \tan \theta$

Consider system (2.4) for the potential $V(\theta) = \alpha \tan \theta$. With respect to the new independent variable $z := (p_\theta + 2p)/\alpha$ it can be written as

$$p_1'(z) = A(z)p_1 + B(z)p_2$$

$$p_2'(z) = C(z)p_1 - A(z)p_2 \quad (3.2)$$

for

$$A(z) = \frac{p}{f(z)(1 + f^2(z))}, \quad B(z) = \frac{p}{\mu f^2(z)} + \frac{(\mu - 2)p - \alpha z}{1 + f^2(z)}.$$
\[ C(z) = \frac{\alpha z - \mu p}{1 + f^2(z)}, \quad f(z) = \varepsilon - \frac{\alpha z^2}{2\mu}, \]

where \( \alpha \varepsilon - \mu p^2/2 \) is a constant energy level. Let

\[ \kappa := \sqrt{\frac{2\mu \varepsilon}{\alpha}}, \quad \eta := \sqrt{\frac{2\mu}{\alpha}(\varepsilon - i)}, \quad \lambda := \sqrt{\frac{2\mu}{\alpha}(\varepsilon + i)}, \quad q := \frac{p\mu}{\alpha}; \]

One can transform (3.2) into the linear differential equation for \( p_2(z) \) of the second order

\[ p_2''(z) = \frac{C'}{C} p_2' + \left( \frac{C'}{C} A + A^2 + CB - A' \right) p_2 \]

and then into equation (3.1) for the function \( y(z) := \frac{p_2(z)}{\sqrt{C}} \), where

\[ r(z) = \frac{C'}{C} A + A^2 + CB - A' - \frac{1}{2} \left( \frac{C'}{C} \right)' + \frac{1}{4} \left( \frac{C'}{C} \right)^2. \]

For evaluation of the function \( r(z) \) one can use computer analytical calculations, which lead to

\[ r(z) = \sum_{j=1}^{7} \left( \frac{\alpha_j}{(z - z_j)^2} + \frac{\beta_j}{z - z_j} \right) = \frac{3}{4z^2} + O\left(\frac{1}{z^3}\right) \quad \text{as} \quad z \to \infty, \quad (3.4) \]

where \( z_1 = q, \ z_2, 3 = \pm \lambda, \ z_4, 5 = \pm \eta, \ z_6, 7 = \pm \kappa, \)

\[
\begin{align*}
\alpha_1 &= \frac{3}{4}, \quad \alpha_{6,7} = 0, \quad \beta_1 = -\frac{4q}{2q^2 - \lambda^2 - \eta^2}, \quad \beta_6 = \frac{q}{\kappa(q - \kappa)}, \quad \beta_7 = \frac{-q}{\kappa(q + \kappa)}, \\
\alpha_2 &= \frac{\mu - 1}{4\kappa^2} (q - \lambda) (q(\mu - 1) - \lambda(\mu + 1)), \quad \alpha_3 = \frac{\mu - 1}{4\kappa^2} (q + \lambda) (q(\mu - 1) + \lambda(\mu + 1)), \\
\alpha_4 &= \frac{\mu - 1}{4\eta^2} (q - \eta) (q(\mu - 1) - \eta(\mu + 1)), \quad \alpha_5 = \frac{\mu - 1}{4\eta^2} (q + \eta) (q(\mu - 1) + \eta(\mu + 1)), \\
\beta_2 &= \frac{\mu - 1}{4(\lambda^2 - \eta^2)^3} \left( -(\mu + 1)x^2\eta^2 + (\mu - 1)\eta^2q^2 - 3(\mu + 1)\lambda^4 - 5(\mu - 1)q^2\lambda^2 + 8\lambda^3 q\mu \right), \\
\beta_3 &= \frac{\mu - 1}{4(\lambda^2 - \eta^2)^3} \left( (\mu + 1)x^2\eta^2 - (\mu - 1)\eta^2q^2 + 3(\mu + 1)\lambda^4 + 5(\mu - 1)q^2\lambda^2 + 8\lambda^3 q\mu \right), \\
\beta_4 &= \frac{\mu - 1}{4(\lambda^2 - \eta^2)^3} \left( (\mu + 1)x^2\eta^2 - (\mu - 1)\lambda^2q^2 + 3(\mu + 1)\eta^4 - 5(\mu - 1)q^2\eta^2 - 8\eta^3 q\mu \right), \\
\beta_5 &= \frac{\mu - 1}{4(\lambda^2 - \eta^2)^3} \left( -(\mu + 1)x^2\eta^2 + (\mu - 1)\lambda^2q^2 - 3(\mu + 1)\eta^4 - 5(\mu - 1)q^2\eta^2 - 8\eta^3 q\mu \right).
\end{align*}
\]

Note that for \( \mu = 1 \) the expression for \( r(z) \) is considerably simplified since this case corresponds to \( m_2 = 0 \) and thus to an integrable one-body system.

**Lemma 3.1.** Suppose that \( \alpha, \varepsilon, \mu, p \in \mathbb{R}, \mu \neq 0, 1, p \neq 0 \) and

\[ (\sqrt{\varepsilon^2 + 1} - \varepsilon)(\varepsilon^2 + 1) \neq \frac{(\mu - 1)p^2}{4\alpha \mu}, \quad (3.6) \]

then \( \alpha_i \notin \mathbb{R} \) for \( i = 2, 3, 4, 5 \).
Proof. By direct calculations one can find
\[
\text{Im} \alpha_2 = \frac{q(\mu - 1)}{4} \left( \frac{(1 - \mu)q\alpha}{2\mu(\bar{\varepsilon}^2 + 1)} \pm \frac{\sqrt{\alpha\mu}}{\sqrt{\bar{\varepsilon}^2 + 1}} \sqrt{\bar{\varepsilon}^2 + 1 - \varepsilon} \right),
\]
therefore (3.6) implies \( \text{Im} \alpha_2 \neq 0 \). Considerations for \( \alpha_3, \alpha_4 \) and \( \alpha_5 \) are similar.

### 3.2 Potential \( V(\theta) = -\alpha \sin^{-1} \theta \)

Now, consider system (2.4) for the potential \( V(\theta) = -\alpha \sin^{-1} \theta \). Again, w. r. t. the same variable \( z \), one gets
\[
p'_1(z) = A(z)p_1 + B(z)\sqrt{f(z)}p_2
\]
\[
p'_2(z) = C(z)\sqrt{f(z)}p_1 - A(z)p_2,
\]
where
\[
A(z) = \frac{p}{\varphi(z)}, \quad B(z) = \frac{p\varphi^2(z)/\mu + (\mu - 2)p - \alpha z}{\varphi(z)(\varphi^2(z) - 1)},
\]
\[
C(z) = \frac{\alpha z - \mu p}{\varphi(z)(\varphi^2(z) - 1)}, \quad \varphi(z) = \frac{\alpha z^2}{2\mu} - \varepsilon, \quad f(z) = \varphi^2(z) - 1
\]
and \( \bar{\varepsilon} \) is the same as above. Let
\[
\kappa := \sqrt{\frac{2\mu\varepsilon}{\alpha}}, \quad \eta := \sqrt{\frac{2\mu}{\alpha}(\varepsilon - 1)}, \quad \lambda := \sqrt{\frac{2\mu}{\alpha}(\varepsilon + 1)}, \quad q := \frac{p\mu}{\alpha};
\]
then one can reduce (3.7) to equation
\[
p''_2(z) = \left( \frac{C'}{C} + \frac{f'}{2f} \right)p'_2 + \left( \left( \frac{C'}{C} + \frac{f'}{2f} \right) A + A^2 + CBf - A' \right)p_2,
\]
which can be reduced to equation (3.1) by the substitution \( p_2(z) = \sqrt{C'(z)}(f(z))^{1/4} \).

Now it holds
\[
z_1 = q, \quad z_{2,3} = \pm \lambda, \quad z_{4,5} = \pm \eta, \quad z_{6,7} = \pm \kappa, \quad \alpha_1 = 3/4, \quad \alpha_j = -3/16, \quad j = 2, 3, 4, 5,
\]
\[
\alpha_6 = \frac{3}{4} + (\mu - 1) \left( 1 - \frac{q}{\kappa} \right) \left( 1 + \mu + (1 - \mu) \frac{q}{\kappa} \right),
\]
\[
\alpha_7 = \frac{3}{4} + (\mu - 1) \left( 1 + \frac{q}{\kappa} \right) \left( 1 + \mu - (1 - \mu) \frac{q}{\kappa} \right).
\]

Expressions for \( \beta_j, j = 1, \ldots, 7 \) are omitted since they will not be used below.

**Lemma 3.2.** Suppose that \( \alpha, \varepsilon, \mu, p \in \mathbb{R}, \mu \neq 0, 1, p \neq 0 \) and \( \varepsilon < 0 \); then \( \alpha_{6,7} \notin \mathbb{R} \).

**Proof.** One has \( \kappa \in i\mathbb{R}\setminus\{0\} \) and therefore
\[
i \text{Im} \alpha_6 = -i \text{Im} \alpha_7 = -2\mu(\mu - 1)\frac{q}{\kappa} \neq 0.
\]
4 Proof of nonintegrability

In this section we shall use some facts concerning equation (3.1) that were collected in the appendix of [18]. For brevity, we shall cite these facts, using the letter A.

Lemma 4.1. Suppose that assumptions of lemma 3.1 are valid. Then the identity component $G_0$ of the Galois group for equation (3.1) with $r(z)$ given by (3.4) and (3.5) is not Abelian.

Proof. Here, equation (3.1) has eight regular singular points $\infty, z_i, i = 1, \ldots, 7$ and differences $\Delta_j$ of exponents at these points are as follows: $\Delta_1 = \Delta_\infty = 2, \Delta_{6,7} = 1, \Delta_j = \sqrt{1 + 4\alpha_j} \notin \mathbb{R}$ for $j = 2, 3, 4, 5$ (due to lemma 3.1). Therefore the third case from lemma A.1 is impossible.

Consider the first case of lemma A.1. Reasoning as in [18] (the proof of lemma 5.1), one gets one of two possibilities:

1. there are two linear independent solutions $y_k(z), k = 1, 2$ of (3.1) such that $y_k'/y_k \in \mathbb{C}(z)$ and the function $v(z) = y_1y_2$ satisfies to the equation
\[ s(z) := v''' - 4rv - 2r'v \equiv 0; \]

2. there is a unique (up to a constant factor) solution $y_1(z)$ of (3.1) such that $y_1'/y_1 \in \mathbb{C}(z)$ and then due to lemma A.2 $y_1^m(z) \in \mathbb{C}(z)$ for some $m \in \mathbb{Z}$ or the identity component $G_0$ of the differential Galois group for (3.1) is nonabelian.

For the first possibility an analysis of exponents $\rho_j^\pm = (1 \pm \Delta_j)/2$ of equation (3.1) at points $\infty, z_j, j = 1, \ldots, 7$ and the inclusion $v(z) \in \mathbb{C}(z)$ lead to
\[ v(z) = \frac{c}{z - q} \left( \prod_{j=2}^5 (z - z_j) \right), \quad c = \text{const} \neq 0. \]

Now direct calculations show that
\[ s(z) = \frac{cP_6(z)}{4(z - q)^3(z^2 - \eta^2)(z^2 - \lambda^2)(z^2 - \kappa^2)^2}, \]
where $P_6(z)$ is a degree six polynomial with leading two terms
\[ -64\frac{q\mu^2}{\alpha^2}(4\mu + 1)z^6 + 128\frac{q^2\mu^2}{\alpha^2}(4\mu - 3)z^5 \]
and thus $s(z) \neq 0$. For the second possibility we conclude, due to the complex values of exponents at $z_j, j = 2, 3, 4, 5$, that $G_0$ is nonabelian.

Now we check if (3.1) has a solution of the form $\exp(\int \omega(z)dz)$, where $\omega(z)$ is an algebraic function over $\mathbb{C}(z)$ of degree 2. To do this we apply the Kovacic algorithm (see the appendix in [18] for notations).
One gets $E_{z_1} = E_\infty = (-2, 2, 6)$, $E_{z_6} = E_{z_7} = (4)$, $E_{z_j} = (2)$, $j = 2, \ldots, 5$. Thus, there are no positive numbers $d(e)$ and therefore (3.1) has no solutions of the form $\exp \left( \int \omega(z) dz \right)$.

This implies that the second case of lemma A.1 can not realize. The forth and the last case of this lemma is $G_0 = G = SL_2(\mathbb{C})$. Hence, in all possible cases the group $G_0$ for equation (3.1) with $r(z)$ given by (3.4) and (3.5) is nonabelian.

Lemma 4.2. Suppose that assumptions of lemma 3.2 are valid. Then the identity component $G_0$ of the Galois group for equation (3.1) with $r(z)$ given by (3.4) and (3.8) is not Abelian.

Proof. Now, equation (3.1) again has eight regular singular points $\infty, z_i, i = 1, \ldots, 7$ with $\Delta_1 = \Delta_\infty = 2$, $\Delta_j = 1/2$, $j = 2, 3, 4, 5$, $\Delta_{6, 7} = \sqrt{1 + 4\alpha_j} \notin \mathbb{R}$ (due to lemma 3.2). Thus, the third case from lemma A.1 is impossible.

The first case of lemma A.1 leads to one of two possibilities (see the proof of lemma 4.1). For the first possibility, the analysis of exponents $\rho_j^\pm = (1 \pm \Delta_j)/2$ of equation (3.1) at $z_j$, $j = 1, \ldots, 7$ implies

$$v(z) = \frac{P(z)}{z - q} \prod_{j=2}^7 (z - z_j)$$

for a polynomial $P(z) \neq 0$, but since $\rho_\infty \in (-1/2, 3/2)$ the growth of $v(z)$ at infinity can not be faster than $z^3$, which leads to the contradiction. Thus, the first possibility can not realize and the second one corresponds to a nonabelian $G_0$ due to $\Delta_{6, 7} \notin \mathbb{R}$.

Check the second case of lemma A.1 using the Kovacic algorithm (see again the appendix in [18]). Now $E_{z_1} = E_\infty = (-2, 2, 6)$, $E_{z_6} = E_{z_7} = (2)$, $E_{z_j} = (1, 2, 3)$, $j = 2, \ldots, 5$ and the maximal value for $d(e)$ equals 0. Thus, one should verify if the function

$$\Theta(z) = \frac{1}{2} \left( -\frac{2}{z - z_1} + \sum_{j=2}^5 \frac{1}{z - z_j} + \sum_{j=6}^7 \frac{2}{z - z_j} \right)$$

satisfies to the equation

$$\Xi(z) := \Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r' = 0.$$ 

But computer calculations shows that

$$\Xi(z) = \frac{-12qz_6^2 + 24q^2z^5 + 12qz^2z_6^4 - 48q^2z^2z^3 + \ldots}{(z - q)^2(z^2 - \lambda^2)(z^2 - \eta^2)(z^2 - \xi^2)^2}.$$

This implies that the second case of lemma A.1 can not realize and as in the proof of the preceding lemma we conclude that the group $G_0$ for equation (3.1) with $r(z)$ given by (3.4) and (3.8) is nonabelian.

Proof of theorem 2.2. Let the base field for systems (3.2) and (3.7) be $\mathbb{C}(z)$. All transformations, made while reducing these systems to equation (3.1), are linear with
coefficients from finite extensions of $\mathbb{C}(z)$. Therefore they do not change the identity components of the corresponding differential Galois groups. Now theorem 2.2 follows from lemmas 4.1, 4.2 and the Morales-Ramis theory (see section 1).

The proof of theorem 2.1 is completely similar.

References

[1] Lipschitz R. Extension of the planet-problem to a space of $n$ dimensions and constant integral curvature, The Quaterly Journal of pure and applied mathematics, V. 12 (1873), pp. 349-370.

[2] Killing W. Die mechanik in den nicht-Euklidischen raumformen, J. Reine Angew. Math., Bd. 98 (1885), S. 1-48.

[3] Neumann C. Ausdehnung der Keppler’schen Gesetze auf der Fall, dass die Bewegung auf einer Kugelfläche stattfindet, Berichte der Königl. Sächsischen Gesellschaft der Wissenschaft, Math. Phys. Klasse, Bd. 38 (1886), S. 1-2.

[4] Lobachevskij N.I. The new foundations of geometry with full theory of parallels [in Russian], 1835-1838, In Collected Works, V. 2, GITTL, Moscow, 1949, p. 159.

[5] Liebmann H. Die Kegelschnitte und die Planetenbewegung im nichteuklidischen Raum, Berichte der Königl. Sächsischen Gesellschaft der Wissenschaft, Math. Phys. Klasse, Bd. 54, (1902), S. 393-423.

[6] Liebmann H. Nichteuklidische geometrie. G.J. Göschens, Leipzig, 1905; 2-nd ed. 1912; 3-rd ed. Walter de Gruyter, Berlin, Leipzig, 1923.

[7] Bertrand J. Théorem relatif au mouvement d’un point attiré vers un centre fixe, C. R. Acad. Sci. Paris, V. 77 (1873), pp. 849-853.

[8] Liebmann H. Über die Zentralbewegung in der nichteuklidische Geometrie, Berichte der Königl. Sächsischen Gesellschaft der Wissenschaft, Math. Phys. Klasse, Bd. 55 (1903), S. 146-153.

[9] Dombrowski P., Zitterbarth J. On the planetary motion in the three dimensional standart spaces $M_3^\kappa$ of constant curvature $\kappa \in \mathbb{R}$, Demonstratio Mathematica, V. 24 (1991), pp.375-458.

[10] Shchepetilov A.V. Calculus and Mechanics on Two-Point Homogenous Riemannian Spaces. Lecture Notes in Physics, Vol. 707, Springer Verlag, 2006.

[11] Shchepetilov A. V. Reduction of the two-body problem with central interaction on simply connected spaces of constant sectional curvature, J. Phys. A: Math. Gen., V.31 (1998), pp. 6279-6291; Corrigendum: V.32 (1999), p. 1531.

[12] Shchepetilov A.V. Two-body problem on spaces of constant curvature: I. Dependence of the Hamiltonian on the symmetry group and the reduction of the classical system, Theor. Math. Phys., V.124 (2000), pp. 1068-1081. Corrected version is available at [math-ph/0501015]
[13] Morales-Ruiz J.J. Differential Galois theory and nonintegrability of Hamiltonian systems, Birkhäuser Verlag, Basel, 1999.

[14] Kaplansky I. An introduction to differential algebra, Hermann, Paris, 1957.

[15] van der Put M., Singer M. Galois theory of linear differential equations. Springer-Verlag, Berlin, 2003.

[16] Maciejewski A.J., Strelcyn J.-M., Szydłowski M. Nonintegrability of Bianchi VIII Hamiltonian system, J. Math. Phys., V. 42 (2001), pp. 1728-1743.

[17] Maciejewski A.J., Przybylska M. Non-integrability of restricted two body problem in constant curvature spaces, Reg. Chaot. Dyn., V. 8 (2003), pp. 413-430.

[18] A.V. Shchepetilov. Nonintegrability of the two-body problem in constant curvature spaces // J. Phys. A: Math. Gen., V. 39 (2006), pp. 5787-5806; corrected version is available at math.DS/0601382.