SKEINS ON BRANES

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Abstract. We study 1-parameter families of holomorphic curves with Lagrangian boundary in Calabi-Yau 3-folds. We show that the expected codimension-1 phenomena can be organized to match the HOMFLYPT skein relations from quantum topology. It follows that counting holomorphic curves by the class of their boundaries in the skein module of the Lagrangian gives a deformation invariant result. This is a mathematically rigorous incarnation of Witten’s assertion that boundaries of open topological strings create line defects in Chern-Simons theory [38].

Using this theory, we rigorously establish the following prediction of Ooguri and Vafa: the coefficients of the HOMFLYPT polynomial of a link in the three-sphere count the holomorphic curves in the resolved conifold, with boundary on (a pushoff of) the link conormal.

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The skein relations above serve to define and calculate the HOMFLYPT invariant of knots and links [13, 30]. Said relations have celebrated interpretations as encoding the expected properties of Wilson lines in the Chern-Simons quantum field theory [37], or correspondingly of traces on braid group representations arising from the study of von Neumann algebras [22, 23] or quantum groups [31, 35]. Our purpose here is to show that the same skein relations emerge from the study of holomorphic curves with Lagrangian boundary conditions in Calabi-Yau 3-folds, and, consequently, resolve certain long-standing problems regarding the definition and computation of counts of such curves.

1.1. Skein relations from moduli of holomorphic maps. Let \( X \) be a symplectic 6-manifold with trivial first Chern class, and \( L \subset X \) a Lagrangian with vanishing Maslov class. Fix a compatible almost complex structure \( J \) which is standard near \( L \).

The moduli space of \( J \)-holomorphic curves in \((X, L)\) has expected dimension zero, so one might expect to be able to define a count of such curves. However, moduli spaces of holomorphic curves with boundary themselves have codimension one boundaries that appear in 1-parameter families, so one naively does not expect deformation invariance of counts of curves in \((X, L)\), but rather some wall-and-chamber structure with counts locally constant in the chambers but changing across walls [14]. In fact the situation is even worse: achieving transversality of moduli depends on some choice of perturbation, and for the same reason one would expect the resulting counts to depend on the choice of perturbation by some wall and chamber structure.

The driving observation of the present article is that one can set up this counting problem so that the possible wall crossings precisely match the HOMFLYPT skein relations.

Let us sketch how the skein relations appear. The generic or expected boundary point of moduli is a map

\[ u_0: (\Sigma_0, \partial \Sigma_0) \to (X, L), \]

where the domain curve \( \Sigma_0 \) has a single boundary node. Such a node is called ‘elliptic’ or ‘hyperbolic’, according as whether the nearby \( u_\epsilon(\partial \Sigma_\epsilon) \) looks like \( x^2 + y^2 = \epsilon \) or \( x^2 - y^2 = \epsilon \).
Consider a (generic) 1-parameter moduli space whose boundary corresponds to a curve with hyperbolic node, see Figure 2. Then $u_0(\partial \Sigma_0)$ is some singular link, and $u_\epsilon(\partial \Sigma_\epsilon)$ is its resolution appearing on the right hand side of the first skein relation in Figure 1. On the other hand, if we write $\tilde{u}_0: (\tilde{\Sigma}_0, \partial \tilde{\Sigma}_0) \to (X, L)$ for the map from the normalization, then $[\tilde{u}_0]$ is an interior point in a 1-parameter moduli space. If we write $\tilde{u}_\pm: (\tilde{\Sigma}_\pm, \partial \tilde{\Sigma}_\pm) \to (X, L)$ for nearby points on either side, then one finds that $\tilde{u}_\pm(\partial \tilde{\Sigma}_\pm)$ provide the two knots on the left hand side of the first skein relation in Figure 1. We will count curves weighted by $z^{-\chi}$ where $\chi$ is the Euler characteristic of the curve obtained by smoothing all nodes; in this sense, $\chi(\Sigma_\pm) = \chi(\Sigma_0) + 1$. Thus considering together the families $u_t$ and $\tilde{u}_t$, assuming transversality and possibly adjusting the sign of $z$, the curve count plausibly jumps by some multiple of the first skein relation.

The moduli spaces in the case of an elliptic node are similar. The boundaries of the normalization family of curves satisfy $u_+(\partial \Sigma_+) \simeq u_-(\partial \Sigma_-)$. Meanwhile $u_0(\partial \Sigma_0)$ is isotopic to a union of $u_+(\partial \Sigma_+)$ with the single point where the elliptic node appears. In $u_\epsilon(\partial \Sigma_\epsilon)$, this point expands to an unknot. Thus the wall crossing would plausibly match the second HOMFLYPT skein relation in Figure 1 if we count the maps weighted by $a^{u_\epsilon L}$, where $u_\epsilon L$ is some quantity satisfying

$$u_- \phi L + 1 = u_\epsilon \phi L = u_+ \phi L - 1$$

Observe that the 1-parameter family connecting $u_+(\Sigma_+)$ and $u_-(\Sigma_-)$ is a family of 2-folds in a 6-fold, which at one moment (namely $\tilde{u}_0$) meets the 3-fold $L$. Thus we may expect to be able to distinguish $u_+(\Sigma_+)$ and $u_-(\Sigma_-)$ by their linking number with $L$. Since we want in particular $u_+ \phi L - u_- \phi L = 2$, we would like to choose a 4-chain $C$ with $\partial C = 2L$, and
define \( u A := u \cap C \).

However, the quantity \( u \cap C \) cannot be expected to be invariant under deformations. Indeed, \( u_+(\Sigma_+) \) and \( u_-(\Sigma_-) \) may have boundary on \( L \), and \( u_\epsilon(\Sigma_\epsilon) \) necessarily does. Thus, the intersection points \( u \cap C \) may travel to the boundary of the domain curve, and then disappear. We correct for this as follows. We demand that the 4-chain \( C \) is smooth near the boundary and that there is a vector field \( \vec{v} \) on \( L \) such that the inward normal of \( C \) along \( L \) equals \( C = \pm J\vec{v} \). Since \( u \) is \( J \)-holomorphic, this ensures that points of \( \partial u \cap C \neq 0 \) are precisely those points where \( \vec{v} \) is parallel to \( \partial C \). Thus if we use \( \vec{v} \) to frame \( u(\partial \Sigma) \), the moments when \( a^{u \cap C} \) changes are precisely those when the framing of \( u(\partial \Sigma) \) changes. This is the third HOMFLYPT skein relation in Figure 1.

1.2. Skein-valued curve counting. Let \( \text{Sk}(L) \) be the skein module of \( L \), i.e. the formal \( \mathbb{Z}[a^\pm, z^\pm] \)-linear span of framed links in \( L \), modulo the HOMFLYPT skein relations. For a map \( u: (\Sigma, \partial \Sigma) \to (X, L) \) such that \( u|_{\partial \Sigma} \) is an embedding, there is a well defined class \([\partial u] \in \text{Sk}(L)\). We write \( \widehat{\text{Sk}}(L) \) for the completion in \( z^{-1} \). The above arguments suggest that the following curve count is deformation invariant:

\[
\sum_{u \in \mathcal{M}} z^{-\chi(u)} \cdot a^{u \cap L} \cdot [\partial u] \in \widehat{\text{Sk}}(L),
\]

where \( \mathcal{M} \) is a suitable moduli space of curves.

Let us discuss the constraints that this skein valued counting proposal imposes on the maps we may and must count.

- We must count only maps such that \( u|_{\partial \Sigma} \) is an embedding, in order that they define links in \( L \). Similarly, when we consider 1-parameter families, any appearing \( u|_{\partial \Sigma} \) should be either an embedding or a link ‘with one transverse singularities’, or an immersion with one general position double point.
- We must count maps from possibly disconnected curves of arbitrary genus and arbitrary number of boundary components (including zero), since neither genus nor number of boundary components are preserved by the skein relations.
- We may restrict attention to maps of a fixed class in \( H_2(X, L) \), since this is preserved by the skein relations.

Of course, as usual for curve counting, we also need our moduli of maps to be suitably compact and and transversely cut out.

Let us focus attention on the condition that \( u|_{\partial \Sigma} \) is an embedding or immersion. On the one hand, a naive dimension counting indicates this is reasonable to expect. Indeed, the standard analytic techniques suffice to show that, for generic almost complex structure \( J \), all desired conditions hold on the locus of maps which are somewhere injective on each component of the domain.

A \( J \)-holomorphic map fails to be somewhere injective if some component either maps by a multiple cover, or to a constant.

Let us first fix attention on some situation where multiple covers are excluded a priori, e.g. by fixing some class in \( H_2(X, L) \) where multiple covers cannot occur for topological reasons. We consider the remaining difficulty: maps which collapse some components (‘ghost bubbles’). We term maps without ghost bubbles ‘bare’. However, in fact a limit of bare maps can only develop a ghost bubble if the image of the non-contracted component has a
singularity worse than an ordinary node [10] (see also [21, 39, 27, 5]). Since such singularities do not appear for generic J or generic paths of J, the locus of bare curves is already compact.

In the present article, we detail the above reasoning, and deduce that (1.1) is well defined and invariant under deformations when multiple covers are excluded a priori. More precisely, in Theorem 5.2, we axiomatize the properties that a system of perturbations would need to have for the invariant to be defined, and in Theorem 6.2, we show that when multiple covers are excluded, the choice of a generic almost complex structure already provides such a perturbation.

In [9], we use the techniques of the abstract perturbation theory of [19, 18] to construct in full generality (i.e. in the presence of multiple covers) a system of perturbations satisfying the axiomatics of Theorem 5.2, thereby establishing the well-definedness and invariance of (1.1) in full generality.

In fact, as we will see presently, there are already striking applications of our setup in situations where no multiple covers can occur. Of course, the full strength of the theory leads to even more applications [11, 8, 33, 32, 20, 7].

Remark 1.1. The skein-valued curve counting is a mathematical shadow of Witten’s proposal that Lagrangian branes in the topological A-model carry the Chern-Simons gauge theory [38]. Witten reasoned at the level of the action functional, by the cubic vertex of the open string field theory to the Chern-Simons functional, and then further argued directly that open strings contribute Wilson line operators. By contrast, what we are doing (in more physical terms) is showing that if the open topological A-model is to have the expected background independence, then the contributions of the open strings must satisfy the same relations as do the Chern-Simons Wilson lines — without ever computing what the contribution of such a string actually is, or indeed discussing to what it should contribute.

1.3. The Ooguri-Vafa conjecture. Using our setup, we will establish a proposal from the string theory literature [16, 17, 29]: the coefficients of the HOMFLYPT invariant are counts of certain holomorphic curves. (We mention in passing a related family of works concerning the case of knots which arise as links of plane curve singularities [28, 4, 3, 26].)

Let us recall some relevant notions to state this precisely. The cotangent bundle $T^*S^3$ and the total space $X$ of the bundle $O(-1) \oplus O(-1) \to \mathbb{C}P^1$ are respectively a deformation and a resolution of a quadric cone in $\mathbb{C}^4$. The cone itself is understood as the limit where respectively the radius of $S^3$ or the area of $\mathbb{C}P^1$ tends to zero. The space $X$ is called the resolved conifold. Passing between $T^*S^3$ and $X$ is the conifold transition.

Let $K$ be a link in $S^3$, and let $L_K \subset T^*S^3$ be the conormal of $K$. Shifting $L_K$ along a closed 1-form dual to the tangent vector of $K$, by some $\epsilon > 0$, gives a non-exact Lagrangian $\tilde{L}_K$ disjoint from the 0-section. Topologically, $\tilde{L}_K$ is one copy of $S^1 \times \mathbb{R}^2$ for each knot component of $K$. The shift of $\tilde{L}_K$ determines a positive generator $\beta_j$ of $H_1(\tilde{L}_K)$ for each component $K_j$ of $K$. Write $\beta = \sum_j \beta_j \in H_1(\tilde{L}_K)$ for the sum of these generators. Using the conifold transition, we identify $\tilde{L}_K$ with a Lagrangian in $X$.

For area$(\mathbb{C}P^1) > 0$ sufficiently small compared to $\epsilon$, there exist almost complex structures tamed by the symplectic form of $X$ that agree with a standard integrable complex structure in an $\frac{1}{2}\epsilon$-neighborhood of $K = S^1 \times 0 \subset L_K \subset X$. Fix any generic such almost complex structure $J$ which is standard near $\tilde{L}_K$.

Theorem 1.2. Let $\mathcal{M}$ be the moduli of bare disconnected $J$-holomorphic curves in $X$ such that each component has boundary and the total boundary is $\beta$. Then $\mathcal{M}$ is a transversely cut
out compact oriented 0-manifold, and the HOMFLYPT polynomial $H_K(a, z)$ of $K$ is given by the following count of holomorphic curves:

$$H_K(a, z) = \sum_{u \in \mathcal{M}} (-1)^{o(u)} a^{2 \deg(u)} z^{-\chi(u)}$$

where $(-1)^{o(u)}$ is the orientation sign of $[u] \in \mathcal{M}$.

In (1.2), $\chi(u)$ is the topological Euler characteristic of the domain of $u$ and $\deg(u)$ is the homological degree of the curve $u$. Since $u$ has boundary, the definition of $\deg(u)$ requires certain choices, and correspondingly the definition of the HOMFLYPT polynomial also requires a choice of framing on the link. We explain in Section 9 how compatible choices are naturally induced by any choice of vector field on $S^3$ transverse to the link.

Note that the skein-valued curve counting (1.1) does not explicitly appear anywhere in the statement of Theorem 1.2. However, our proof (given in Section 9) will be an application of existence and invariance of the skein-valued curve counting, plus a neck stretching argument. The argument does not require consideration of multiply covered curves, so our proof of Theorem 1.2 does not depend on the further constructions of [9].

Remark 1.3. Ooguri and Vafa arrived at this formula via a ‘large $N$ duality’ relating the topological string theory on $T^*S^3$ with $N$ branes on the zero section, and topological string theory in the resolved conifold. Here the number $N$ of branes in $X_\epsilon$ is supposed to be related by a change of variables to a parameter measuring the area of the $\mathbb{C}P^1$. In our setup, we do not explicitly have a ‘number of branes’. Instead we have the 4-chain, and the resulting Lagrangian linking number it determines.

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2. Linking numbers of open curves and Lagrangians

In this section we will define a linking number between a 3-dimensional Lagrangian submanifold $L$ and an open holomorphic curve $\Sigma$ which ends on it, inside a 6-dimensional symplectic manifold. While the dimensions are appropriate, the objects are not transverse and one of them is not closed. Nevertheless the special geometry, plus certain additional choices, will allow us to define this number. Similar constructions were used before in the context of real algebraic geometry, see [36, 6, 1].

Let us first recall the definition of linking number in the usual sense. Suppose $M$ is an oriented manifold of dimension $n$ and $X, Y \subset M$ are disjoint oriented cycles of dimensions $x, y$ with $x + y = n - 1$. Assuming $Y$ bounds, then one defines the linking number $X \cdot Y$ by choosing a chain $C$ with $\partial C = Y$, and setting $X \cdot Y := X \cdot C$. A different choice $C'$ of

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\footnote{This is ultimately because neighborhood of $K \subset L_K \subset X$ can be identified with $S^1 \times (-\epsilon, \epsilon) \times B$, where $B \subset \mathbb{C}^2$ is a small ball and where $L_K$ corresponds to $S^1 \times 0 \times (B \cap \mathbb{R}^2)$. For almost complex structures on $X$ that are standard in this neighborhood, a monotonicity argument, see Lemma 8.1, shows that the projection of any holomorphic curve in a basic homology class must agree with the basic cylinders $(S^1 \times [0, \frac{1}{2} \epsilon])$ in a neighborhood of its boundary. For this reason, the curves we will count are all somewhere injective. That allows us to achieve transversality and prove Theorem 1.2, without abstract perturbation methods.}
chain will have $\partial(C' - C) = 0$, and so changing the chain affects the linking number by the intersection of $X$ with the appropriate element of $H_{y+1}(M)$. In particular, if $H_{y+1}(M) = 0$, then the linking number is well defined. If $M$ is noncompact, then it is natural to allow the chain $C$ to have closed but not necessarily compact support, hence the above homology groups should be replaced by Borel-Moore homology.

We return to our situation of interest. Let $L$ be a smooth orientable $n$-dimensional manifold, $n$ odd. Let $g$ be a metric on $L$. Fix a vector field $\xi$ on $L$ with transverse zeros and with corresponding metric-dual 1-form $\xi^*$. Consider the $\epsilon$-disk neighborhood $D_\epsilon^*L$ of $L$ in $T^*L$. Let $Z(\xi)$ denote the zeros of $\xi$. For $q \notin Z(\xi)$, let $\tilde{\xi}^*(q) = \frac{1}{[\xi^*(q)]}\xi^*(q)$ denote the normalization of $\xi^*$. For $q \in Z(\xi)$, let $(-1)^{i(q)}D_\epsilon(q)$ denote the $\epsilon$-disk in the cotangent fiber $T_q^*L$ oriented according to the index of $\xi$ at $q$. Define $\Delta_\xi \subset T^*L$ as

$$\Delta_\xi = \left\{(q, p) : p = t\tilde{\xi}^*(q), 0 \leq t \leq \epsilon\right\} \cup \bigcup_{q \in Z(\xi)} (-1)^{i(q)}D_\epsilon(q).$$

Note that $\Delta_\xi$ is a $(n+1)$-chain with singular boundary

$$\partial(\Delta_\xi) = L - \Gamma'_{\xi} - \sum_{q \in C(f)} (-1)^{i(q)}\partial D_\epsilon(q),$$

where $\Gamma'_{\xi}$ denotes the graph of $\tilde{\xi}$ over $L - Z(\xi)$. Consider next the chain

$$\Delta_{\pm \xi} = \Delta_\xi + \Delta_{-\xi}$$

and note that its boundary is

$$\partial\Delta_{\pm \xi} = 2 \cdot L - \Gamma'_{\xi} - \Gamma'_{-\xi},$$

since the fibers over critical points cancel.

Now let $X$ be any symplectic manifold of dimension $2n$, and $L \subset X$ a Lagrangian submanifold with a transverse vector field $\xi$. Identify some neighborhood of $L$ in $X$ with $D_\epsilon^*L$.

**Definition 2.1.** An $L$-bounding chain compatible with $\xi$ is a singular chain $C$ with $\partial C = 2L$, and which is sum of $\Delta_{\pm \xi}$ and a chain transverse to $L$.

**Remark 2.2.** One can produce such $C$ by taking a chain $C_0$ with $\partial C_0 = \Gamma'_{\xi} + \Gamma'_{-\xi}$, and perturb slightly. Such a chain exists whenever $2L$ is null-homologous in $X$.

**Definition 2.3.** A Lagrangian brane is a tuple $(L, \xi, g, C)$ where $L$ is a Lagrangian, $g$ a Riemannian metric on $L$, $\xi$ is a vector field with transverse zeros, and $C$ is an $L$-bounding chain compatible with $\xi$. We often denote the tuple just by $L$.

In the situation of the definition we may consider the 1-chain $\gamma := (C - \Delta_{\pm \xi}) \cap L \subset L$. Then $\gamma$ is a cycle. If $\gamma = 0 \in H_1(L, \mathbb{Z})$ then we say that $C$ is admissible.

**Definition 2.4.** An admissible Lagrangian brane is a Lagrangian brane together with a choice of chain $\sigma$ with $\partial \sigma = \gamma$.

We now specialize to dim $L = 3$.\(^2\)

\(^2\)In other dimensions, the same data should allow to define linking numbers for appropriate dimensional parametric families of holomorphic maps.
Definition 2.5. Let $L = (L, \xi, g, C)$ be a 3-dimensional Lagrangian brane. Let $u: (\Sigma, \partial \Sigma) \to (X, L)$ be a smooth map which is symplectic in some neighborhood of the boundary. We say $u$ is transverse to $L$ if:

1. $u_*(T\partial \Sigma)$ is everywhere linearly independent from $\xi$.
2. $u(\text{int}(\Sigma))$ is disjoint from $L$ and transverse to $C \setminus L$.
3. $u(\partial \Sigma)$ is disjoint from $\gamma$.

Remark 2.6. Note that the hypothesis that $u_*(T\partial \Sigma)$ is everywhere linearly independent from $\xi$ implies in particular that $u_*(T\partial \Sigma)$ is nowhere zero, i.e., $u$ is an immersion of the boundary, and that $u(\partial \Sigma)$ is disjoint from $Z(\xi)$.

Definition 2.7. Let $L$ be an admissible 3-dimensional Lagrangian brane, and let $u: (\Sigma, \partial \Sigma) \to (X, L)$ be transverse to $L$. Inside $u_*(\partial \Sigma)T L$, let $\nu$ be the (positive) unit normal to the plane spanned by the ordered basis of the tangent vector to $\partial u$ and $\xi$. Extend $\nu$ by multiplying by a cut off function with values in $[0,1]$ to a section of $u^*TX$ supported in a collar neighborhood of $\partial \Sigma$ in $\Sigma$. Shift $u$ slightly along $J\nu$, and denote the resulting curve $u_{J\nu}$. Then the boundaries of $u_{J\nu}$ and $C$ are disjoint and we define

$$u_\phi L := u_{J\nu} \cdot C + \partial u_\phi \gamma$$

Here, $\partial u_\phi \gamma := \partial u \cdot \sigma$ is the ordinary linking number in $L$.

Lemma 2.8. Let $u: (\Sigma, \partial \Sigma) \to (X, L)$ be an embedding transverse to $L$ with $u(T\Sigma)$ everywhere linearly independent from $\xi$ and disjoint from $\gamma$. If $I(u, C)$ denotes the algebraic number of interior intersection points between $u(\Sigma)$ and $C$ then $u_{J\nu} \cdot C = I(u, C)$ and consequently

$$u_\phi L = I(u, C) + \partial u_\phi \gamma.$$  

Proof. This is immediate, since $\nu$ is independent of $\xi$ there are no intersections contributing to $u_{J\nu} \cdot C$ near the boundary. \hfill $\Box$

The term $\partial u_\phi \gamma$ is a sort of self-linking correction, which is present to ensure the following invariance:

Lemma 2.9. Let $u_t: (\Sigma, \partial \Sigma) \to (X, L)$ be a 1-parameter family of holomorphic curves satisfying the hypotheses of Definition 2.5 save with the third replaced by $u_t(\partial \Sigma)$ is transverse to $\gamma'$. Then $u_t_\phi L$, which is defined away from a discrete set of $t$, is constant.

Proof. Evidently this quantity is locally constant where the conditions of Definition 2.5 hold. It remains to check moments when $u_t(\partial \Sigma)$ crosses $\gamma$ generically.

Since the curve is symplectic near the boundary we have the following local model. The ambient space is $\mathbb{C}^3$ with coordinates $(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$. The Lagrangian $L$ is $\mathbb{R}^3 \subset \mathbb{C}^3$ and the 4-chain $C$ is the subset given by $\{x_1 = y_2, x_2 = y_1\}$. Then $\gamma$ is the subset of $L$ given by $x_1 = x_2 = 0$ and we take the bounding 2-chain $\sigma$ to be $\{x_1 \leq 0, x_2 = 0\}$. The family of holomorphic curves is $u_t(\xi + i\eta) = (t, \xi + i\eta, 0)$, for $\xi + i\eta$ in the upper half plane. Then for $t < 0$ there is an intersection point contributing to $\partial u \cdot \sigma$ at $(t, 0, 0)$ and no intersection point in $u_t \cap C$, whereas for $t > 0$ there is no intersection point in $\partial u \cap \sigma$ but the point $(t, it, 0)$ contributes to $u_t \cdot C$.

We conclude that the change in linking $u_t(\partial \Sigma)_\phi \gamma$ is exactly compensated by the appearance or disappearance of an intersection point in $u_{J\nu} \cdot C$. \hfill $\Box$
Another possible degeneration in a 1-parameter family is that at some moment, the tangent vector to $u(\partial \Sigma)$ becomes a multiple of $\xi$ at one point of the boundary. The generic way this can happen is if, under the inverse of the exponential map followed by orthogonal projection along $\xi$, the families of curve boundaries gives a versal deformation of a semi-cubical cusp. Recall that we used the linear independence of $\xi$ and the boundary tangent to define the framing of $u(\partial \Sigma)$; consequently we term a degeneration of this type a transverse framing degeneration. Indeed at such a moment the framing will change by $\pm i^3$.

We used the same linear independence for another reason: to define the pushoff $u_{J^\nu}$, which entered into our linking formula.

**Lemma 2.10.** In a 1-parameter family $u: (\Sigma, \partial \Sigma) \to (X, L)$ satisfying the hypotheses of Definition 2.5 save at transverse framing degenerations, the framing plus $u_{J^\nu} \cdot L$ remains constant.

**Proof.** We will show that the framing change is canceled by a change in the term $u_{J^\nu} \cdot C$. This is a straightforward local check. Consider local coordinates

$$(z_1, z_2, z_3) = (x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) \in \mathbb{C}^3$$

on $X$ with $L$ corresponding to $\mathbb{R}^3$. Assume that the vector field $\theta$ is $\theta f = \partial x_3$. Then the 4-chain $C$ is locally given by $C = C^+ + C^-$, where

$$C^\pm = \{y_2 = y_3 = 0, \pm y_3 > 0\}.$$ 

A generic family of holomorphic curves with a tangency with $\partial x_3$ is given by the map $u_{\pm}: H \to \mathbb{C}^3$ ($H$ is the upper half plane),

$$u_{\pm}(z) = (z^2, z(z^2 + s), \pm z).$$

For $s < 0$ the projection of $u|_{\partial H}$ to the $(x_1, x_2)$ has a double point at $z = \pm \sqrt{-s}$ that contributes to linking according to the sign of $u_{\pm}$. At $s = 0$ the boundary has a tangency with $\partial x_3$ and at $s > 0$, $u_{\pm}(H)$ intersects $C^\pm$ at $u(i\sqrt{s})$ with the sign that agrees with the liking sign before the tangency.

**Lemma 2.11.** For a map $u: (\Sigma, \partial \Sigma) \to (X, L)$, the framing of $u(\partial \Sigma)$ plus $u_{J^\nu} \cdot L$ is independent of the choice of $\xi$ and metric $g$.

**Proof.** Any two vector fields can be connected by a family of vector fields such that the zeros of the vector fields never meet $u(\partial \Sigma)$. The remaining possible failures of the conditions of Definition 2.5 look exactly like those already considered in Lemma 2.9 and Lemma 2.10, save that we are moving the vector field rather than moving the map.

In case $L$ is disconnected, we can sometimes separate out the self-linking correction $\partial u_{J^\nu} \cdot \gamma$ into individual contributions from the separate branes. Indeed if $L = \bigcup L_i$ and correspondingly $C = \bigcup C_i$, we may consider the 1-chains $\gamma_{ij} = (C_i - \Delta_{\pm \xi}) \cdot L_j$. As the Lagrangians $L_i$ are disjoint, each $\gamma_{ij}$ is a cycle. Admissibility means that $\gamma = \sum_{i,j} \gamma_{ij}$ is a boundary; separating out into components, it means that $\sum_j \gamma_{ij}$ is a boundary. However we could ask that in fact each $\gamma_{ij}$ was already a boundary. In this case it makes sense to write

$$u_{J^\nu} \cdot \sum_j \partial u_{J^\nu} \cdot \gamma_{ij}$$

(2.3) $u_{J^\nu} \cdot \sum_j \partial u_{J^\nu} \cdot \gamma_{ij}$

\[3\text{Recall that framing is a } \mathbb{Z}\text{-torsor, i.e. it makes sense to add integers to it, with the meaning that the ribbon of the knot undergoes more or less twists.}\]
The virtue of this convention is that \( u \Phi L = \sum u \Phi L_i \). (A defect of it is that if one viewed \( (L_i, f|L_i, g|L_i, C_i) \) itself as a Lagrangian brane, then one would have the incompatible formula \( u \Phi L_i = u f_i \cdot C_i + \partial u \Phi \gamma_{ii} \). Nevertheless no confusion should arise.) The argument of Lemma 2.9 applies also to these numbers.

**Remark 2.12.** We will consider both compact and non-compact Lagrangians and 4-chains. In the non-compact cases we consider, outside a compact set our geometric objects are products. \( 2.12 \) Remark 2.9 applies also to these numbers.

The question of whether \( L \) and \( \gamma \) bound concerns Borel-Moore, rather than ordinary, homology. Recall that Borel-Moore homology can be calculated by the sequence

\[
0 \to \text{Ext}^1(H^i_c(X, \mathbb{Z})) \to H^B_{BM}(X, \mathbb{Z}) \to \text{Hom}(H^i_c(X, \mathbb{Z}), \mathbb{Z}) \to 0.
\]

In particular if \( L \) is a handlebody, then \( H^B_{BM}(L, \mathbb{Z}) = 0 \), so a bounding chain for \( \gamma \) can always be found. The difference between two choices of bounding chains lives in \( H^B_{BM}(L, \mathbb{Z}) \cong \mathbb{Z}^g \), which is generated e.g. by disks filling an appropriate disjoint collection of circles on \( \partial L \).

**Remark 2.13.** In [33], it is explained how the theory developed in the present article can be generalized to Lagrangians not satisfying the admissibility condition by incorporating ‘linking lines’ into the skein.

### 3. Local models for crossings and nodal curves

In this section we present the relevant local models for the wall-crossing instances in generic 1-parameter families. For a general discussion of maps from curves with boundary and their moduli, see [25] and the references therein.

We use coordinates \((z_1, z_2, z_3) = (x_1 + iy_1, x_2 + iy_2, x_3)\) on \( \mathbb{C}^3 \). The Lagrangian \( L \) corresponds to \( \mathbb{R}^3 = \{y_1 = y_2 = y_3 = 0\} \), the vector field \( \xi = \partial_{x_3} \), and the 4-chain \( C \) is

\[
C = \{(z_1, z_2, z_3) : y_1 = y_2 = 0, y_3 \geq 0\} \cup \{(z_1, z_2, z_3) : y_1 = y_2 = 0, y_3 \leq 0\},
\]

where the two pieces are both oriented so that they induce the positive orientation on \( \mathbb{R}^3 \).

#### 3.1. Hyperbolic boundary.

Consider the two 1-parameter families of maps from a strip: \( u^j_t : \mathbb{R} \times [0, \pi] \to \mathbb{C}^3, j = 1, 2 \), given by

\[
u^1_t(\zeta_1) = (e^{t\zeta_1}, 0, 0), \quad u^2_t(\zeta_2) = (0, e^{-\zeta_2}, t), \quad t \in (-\epsilon, \epsilon).
\]

The first is constant in time, and the second has image moving along the \( x_3 \)-coordinate. The first and the second have disjoint image except when \( t = 0 \). We write \( u_t = u^1_t \sqcup u^2_t \) for the map from the disjoint union of these source curves. Note that the knot diagram of the boundary restriction of of these maps are exactly the \( x_1 \)-axis and the \( x_2 \)-axis shifted \( t \) units in the \( x_3 \)-direction, just like in the left hand side of the top equation in Figure 1.

Consider the following additional family of maps from a strip: \( v_\rho : \mathbb{R} \times [0, \pi] \to \mathbb{C}^3 \)

\[
v_\rho(\zeta) = (e^{\zeta - \rho}, e^{-(\zeta + \rho)}, 0), \quad \rho \in [\rho_0, \infty), \text{for} \rho_0 > 0.
\]

This is a family limiting to a rescaled map from a nodal curve with the same image as \( u_0 \) in the following sense. Translating the domain by \(-\rho\) the reparametrized map becomes

\[
v_{\rho - \rho}(\zeta) = (e^{\zeta}, e^{-\rho} e^{-(\zeta + \rho)}, 0)
\]

which converges uniformly to \( u^1_0 \) on any bounded subset in the domain. Likewise after translation by \( \rho \) the map converges instead to \( u^2_0 \). This allows us to associate the nodal
curve \( v_\infty \) at \( \rho = \infty \) with the crossing instance at \( t = 0 \) in the family \( u_t \). We write \( v_t \) for the reparameterized family with \( t = 1/\rho \). Note that the restriction of the \( v_t \) to the boundary is a curve in the \( x_1 x_2 \)-plane just like in the right hand side of the top equation in Figure 1.

### 3.2. Elliptic boundary.

Consider the family of maps from an annulus \( u_t: \mathbb{R} \times S^1 \to \mathbb{C}^3 \) given by

\[
\begin{equation}
  u_t(\zeta) = (e^\zeta, -ie^\zeta, it), \quad t \in (-\epsilon, \epsilon).
\end{equation}
\]

The intersection \( u_t \cap C \) is the point \((0,0, it)\).

Consider also the family \( v_\rho: [-\rho, \infty) \times S^1 \to \mathbb{C}^3 \) given by

\[
\begin{equation}
  v_\rho(\zeta) = \left(e^\zeta + e^{-\zeta-2\rho}, i(e^\zeta - e^{-\zeta-2\rho}), 0\right).
\end{equation}
\]

Note that \( v(\{-\rho\} \times S^1) \) is a circle of radius \( \sqrt{2} e^{-\rho} \) in the \( x_1 x_2 \)-plane and that the interior of \( v_\rho \) is disjoint from the 4-chain. Furthermore, as \( \rho \to \infty \) the map \( v_\rho \) converges to \( u_0 \). This allows us to associate the nodal curve \( v_\infty \) with the smooth curve \( u_0 \) at the instance of crossing with \( L \) in the family \( u_t \) at \( t = 0 \). We again write \( v_t \) for the reparameterized family with \( t = 1/\rho \). Note then that the curves \( u_{\pm \epsilon} \) gives the curves in the left hand side of the second equation in Figure 1 and that boundary restriction of the curve \( v_\rho \) gives the corresponding right hand side.

### 3.3. Locally standard 1-parameter families.

We will ask that our families of maps are locally modelled on the above families in the following sense. Let \( a_t \) for \( t \in (-\epsilon, \epsilon) \) and \( b_t \) for \( t \in [0, \epsilon) \) be two families of prestable maps to \((X, L)\), such that \( b_0 \) is a map from a domain with a hyperbolic (resp. elliptic) node with the node mapping to \( p \in L \), and that \( a_0 \) is the corresponding map from the normalization of \( b_0 \).

**Definition 3.1.** The pair \((a_t, b_t)\) is a standard hyperbolic (resp. elliptic) degeneration if there exists a neighborhood \( U \) around \( p \) that can be identified with a neighborhood of the origin in \( \mathbb{C}^3 \), with the Lagrangian corresponding to \( \mathbb{R}^3 \) and an explicit 4-chain such that there is a diffeomorphism \( U \to \mathbb{C}^3 \) that respects the Lagrangian and the 4-chain and carries the intersections of the curves in the family with \( U \) to the curves in the above model family \((u_t, v_t)\). If instead \( b_t \) is a nodal family for \( t \in (-\epsilon, 0]\), we say the pair is a standard hyperbolic (resp. elliptic) degeneration if there is an identification as above with time reversed.

### 4. Axiomatics

Let \( X \) be a 3-dimensional Calabi-Yau manifold and let \( L \) be a Maslov zero (and therefore necessarily orientable and spin) Lagrangian, with a brane structure in the sense of Definition 2.3. Let \( A \in H_2(X, L) \) be a relative homology class. We are interested in stable maps \( u: (\Sigma, \partial\Sigma) \to (X, L) \). Our domains are holomorphic curves which are possibly disconnected, possibly nodal, and possibly with boundary components. By stable map, we mean that any component whose image has zero symplectic area must be a stable curve in the usual sense. For \( J \)-holomorphic maps, this is the usual notion of stability.

Recall that we call a curve \( u: (\Sigma, \partial\Sigma) \to (X, L) \) bare if it has no components of zero symplectic area.

We recall that a boundary node can be of two kinds: one modeled on half of \( x^2 + y^2 = 0 \), and the other modeled on half of \( x^2 - y^2 = 0 \). We term the former elliptic and the latter hyperbolic. The elliptic node is the stable limit of disk with a small open subdisk removed as the radius of the removed disk shrinks to zero.
Definition 4.1. A perturbation scheme for \((X, L)\) and \(A \in H_2(X, L)\) consists of the following data:

- A path-connected topological space \(\mathcal{P}\)
- For each \(\lambda\) in \(\mathcal{P}\), a topological space \(\mathcal{M}(\lambda)\) parameterizing maps \(u : (\Sigma, \partial \Sigma) \rightarrow (X, L)\).
- For each \(\lambda : [0, 1] \rightarrow \mathcal{P}\), a topological space \(\mathcal{M}(\lambda)\) equipped with a map \(\mathcal{M}(\lambda) \rightarrow [0, 1]\) such that the fiber over \(t\) is \(\mathcal{M}(\lambda(t))\), which we often denote as just \(\mathcal{M}(t)\).
- \(\mathcal{M}(\lambda)\) decomposes as a union of open and closed components \(\mathcal{M}_{g,h}(\lambda)\) of maps from domains with arithmetic genus \(g\) and \(h\) boundary components.

If \(u : (\Sigma, \partial \Sigma) \rightarrow (X, L)\) is a map, we write \(\tilde{u} : (\Sigma, \partial \Sigma) \rightarrow (X, L)\) for the induced map from the normalization.

Definition 4.2. We say that a perturbation scheme is adequate if it has the following properties:

1. Coherence. For any \(\lambda \in \mathcal{P}\), we have \(u \in \mathcal{M}(\lambda)\) if and only if \(\tilde{u} \in \mathcal{M}(\lambda)\).
2. Codimension zero transversality. There is a subset \(\mathcal{P}^\circ \subset \mathcal{P}\) such that for \(\lambda \in \mathcal{P}^\circ\),
   a. \(\mathcal{M}_{g,h}(\lambda)\) is compact.
   b. \(\mathcal{M}(\lambda)\) is a collection of points, with a weighting \(w : \mathcal{M}(\lambda) \rightarrow \mathbb{Q}\).
   c. Any map \(u\) corresponding to a point in \(\mathcal{M}(\lambda)\) is an embedding of a smooth curve and is transverse to \(L\) in the sense of Definition 2.5. In particular, \(u\) is bare.
   d. For any path \(\lambda : [0, 1] \rightarrow \mathcal{P}^\circ\), the projection \(\mathcal{M}(\lambda) \rightarrow [0, 1]\) is a proper cover of (weighted branched) manifolds.
3. Codimension one transversality. Any two points \(\lambda_0, \lambda_1 \in \mathcal{P}^\circ\) can be connected by a path \(\lambda : [0, 1] \rightarrow \mathcal{P}\) such that:
   a. \(\mathcal{M}_{g,h}(\lambda)\) is compact.
   b. If \(\{u\} \in \mathcal{M}(t_0)\) is a map from a smooth domain, then it admits a neighborhood \(U \subset \mathcal{M}(\lambda)\) which is a weighted (branched) oriented 1-manifold, and such that the projection \(U \rightarrow [0, 1]\) is proper over its image. For \(t \neq t_0\) the weight on \(U\) agrees with the pointwise weights of (2b) up to a sign given by the relative orientation of the projection \(U \rightarrow [0, 1]\).
   c. For any \(t_0 \in [0, 1]\), the neighborhoods \(U\) may be chosen so that assertions of Property (2) hold in all but at most one. If (2d) fails at \(\{u\}\), we term it a critical point; if (2c) fails, we term it a crossing, and we demand that at most one of these failures occurs.
   d. For any given topological type \((g, h)\), the \(t \in [0, 1]\) for which there is a crossing or critical point of type \((g, h)\) are isolated.
   e. The universal map over a neighborhood of a crossing \([u] \in U \subset \mathcal{M}[0, 1]\) takes one of the following forms:
      i. Hyperbolic crossing. The map \(u\) is an immersion everywhere and an embedding save at two points along the boundary which are identified by the map. We require that the images of the two boundary tangent vectors are linearly independent from each other and from \(\xi\), and that they together with the first order variation of the 1-parameter family span the tangent space of \(L\).
      ii. Elliptic crossing. The map \(u\) is an embedding, but some interior point of the domain is mapped to \(L\). The map at this point is transverse to the 4-chain \(C\). At the crossing moment the tangent space of the curve and the
tangent space of $L$ together with the first order variation of 1-parameter family span the tangent space of $X$.

(iii) **Framing change**: $u$ is an embedding, but $\partial u$ becomes tangent to $\xi$ or intersects $\gamma$ generically (as in Lemmas 2.9 or 2.10).

(4) **Gluing.** Let $[u] \in \mathcal{M}(t_0)$ be an elliptic or hyperbolic crossing as above. Let $v$ be the map with the same image which is an embedding (of a nodal curve). Let $\mathcal{M}_u$ and $\mathcal{M}_v$ be small neighborhoods of $[u]$ and $[v]$ in $\mathcal{M}(t_0 - \epsilon, t_0 + \epsilon)$, and let $u_t, v_t$ be the corresponding families of maps. Then $(u_t, v_t)$ is a standard hyperbolic or elliptic degeneration, in the sense of Definition 3.1. Moreover, the weights associated to the general members of $\mathcal{M}_u$ and $\mathcal{M}_v$ are the same.

We say the perturbation scheme is *integral* if all weights can be taken as integers.

Remark 4.3. In a typical implementation, each $\lambda \in \mathcal{P}$ is a system of choices of perturbations of the Cauchy-Riemann equation. E.g., $\mathcal{P}$ is a space of multi-sections of a certain bundle over the space of (sufficiently regular) maps realizing the homology class $A$, which are stable and ‘topologically symplectic’ in the sense that every component has non-negative symplectic area. The fiber $\mathcal{M}(\lambda)$ over some $\lambda \in \mathcal{P}$ will be a subset of moduli space of maps which are perturbed holomorphic for the perturbation $\lambda$.

Remark 4.4. Note that (3) only puts explicit conditions on maps from smooth domains. Of course, this indirectly imposes some conditions on maps from nodal domains because of (1); we directly impose further conditions in (4).

Remark 4.5. The perturbation schemes we actually deal with in this article will be integral, and there will be no need of branched manifolds. We thus do not spell out in detail what is a ‘branched cobordism’ or ‘branched covering map’. In any case, all that is needed from such notions is they behave appropriately with respect to fundamental classes, e.g. as for a usual cobordism, the fundamental class of the boundary is trivial.

Remark 4.6. If $\mathcal{P}$ is presented as an inductive limit of spaces $\mathcal{P}_{g,h}$, then ‘path connected’ may be relaxed to ‘ind-path-connected’.

Remark 4.7. To simultaneously achieve (2a) and (2c) will require the ghost bubble censorship principle [10].

Strictly speaking, the axioms are agnostic regarding whether, for $\lambda \in \mathcal{P} \setminus \mathcal{P}^0$, the moduli space $\mathcal{M}(\lambda)$ should contain non-bare curves. This is because we will never actually count curves for such $\lambda$. However, whether or not you include them, you will need ghost bubble censorship: if you don’t include them, then for the compactness property (3a), or if you do include them, then to limit the possible degenerations to the list in (3e).

5. **Skein valued curve counts**

In this section we show that given an adequate perturbation scheme (Definition 4.2), we can define skein valued curve counts which are invariant under deformations. The arguments are essentially formal.

5.1. **Definition.** Let $X$ be a 3-dimensional Calabi-Yau manifold and let $L = \bigsqcup_{j=1}^k L_j$ be a Maslov zero Lagrangian, with a brane structure in the sense of Definition 2.3. Fix $d \in H_2(X, L)$ and assume given an adequate perturbation setup for $(X, L, d)$ in the sense of Definition 4.2. Let $\text{Sk}(L)$ be the skein of $L$, in variables $(a, z)$, $a = (a_1, \ldots, a_k)$. 
Definition 5.1. For \( \lambda \in \mathcal{P}^\circ(X, L) \), we define

\[
\langle\langle L \rangle\rangle_{X,d,\lambda} = \sum_{u \in \mathcal{M}(X,L,d,\lambda)} w(u) \cdot z^{-\chi(u)} \cdot a^u \cdot \partial u \in \text{Sk}(L)
\]

The sum is over a finite set of points for each coefficient of \( z \) by Properties (2a), (2b), with the second of these providing the function \( w \). By Property (2c), \( \partial u \) is a framed link in \( L \), so we may regard it as an element of the skein. Note that in any given homology class, the Euler characteristic of (even a disconnected) representative is bounded above, so the invariant is a Laurent series in \( z \).

We emphasize that we count only bare maps from possibly disconnected curves. That is, the image of each irreducible component has nonzero symplectic area.

5.2. Invariance.

Theorem 5.2. For an adequate perturbation scheme \( \mathcal{P} \), \( \langle\langle L \rangle\rangle_{X,d,\lambda} \) is independent of \( \lambda \in \mathcal{P}^\circ(X, L) \).

Proof. It follows from Properties (2d) and (3b) that \( \langle\langle L \rangle\rangle_{X,d,\lambda} \) is locally constant in \( \mathcal{P}^\circ(X, L) \). Consider now any two points \( \lambda_0, \lambda_1 \in \mathcal{P}^\circ \), and a path \( \lambda_t \) connecting them satisfying Property (3). We want to study the \( t \) with \( \lambda_t \notin \mathcal{P}^\circ \), in order to show that also the count \( \langle\langle L \rangle\rangle_{X,d,\lambda} \) does not change when passing these.

Note that to prove the desired constancy of \( \langle\langle L \rangle\rangle_{X,d,\lambda} \), it is enough to work order by order in \( z \). Bounding the order \( z \) bounds the genus and holes, thus the set of walls \( t_0 \) we must consider is finite by Property (3d). Fix one such \( t_0 \).

As \( t \to t_0 \), by (2d), the curve counts are locally constant, and the moduli space is just a disjoint union of intervals. By compactness (3a), each such interval has a limit at \( t_0 \). Consider all components for which the limiting curve remains smooth, embedded, transverse to \( L \); i.e., (2c) remains true at the limit. Then, even if (2d) fails, nevertheless (3b) suffices to ensure that these components of moduli have the same contribution on both sides of \( t_0 \).

Let us consider the remaining components, i.e., those for which the limiting curve fails (2c). Taking the normalization gives a map from a smooth domain, which is strictly greater Euler characteristic, so among those we are considering at the current order in \( z \). Recall there is a unique such map, as per (3c), and that it must take one of the forms classified by Property (3e). Note that since (2c) fails, we have demanded that (2d) does not; i.e. the component of moduli containing this map from a smooth domain locally projects to \([0,1]\) by an isomorphism.

For crossings of ‘framing change’ type, by applying Lemmas 2.9 and 2.10, and comparing to the framing change skein relation, we see that the term \( a^u \cdot \langle \partial u \rangle \) is itself invariant.

We now turn to hyperbolic and elliptic crossings. First, the hyperbolic case. Let \( u \) be the map with smooth domain and nodal image, and \( u_\pm \) the corresponding embedding of a nodal curve. We will write \( u_{\pm \epsilon} \) for the curves immediately before and after \( u \) in its family, and \( u_{\cdot \epsilon} \) for the deformation of \( u_\cdot \) which appears on one side (either before or after). We write \( w(u_\cdot) \) for the degree of the 0-chain on the corresponding moduli space.

According to Property (3c), curves unrelated to \( u \) undergo no critical moments. The 4-chain intersection and \( H_2(X, L) \) of all the \( u_\cdot \) are the same. Thus the total change in \( \langle\langle L \rangle\rangle_{X,L,\lambda} \) is a multiple of

\[
w(u_\epsilon)z^{-\chi(u_\epsilon)\langle \partial u_\epsilon \rangle} - w(u_{-\epsilon})z^{-\chi(u_{-\epsilon})\langle \partial u_{-\epsilon} \rangle} \pm w(u_{\cdot \epsilon})z^{-\chi(u_{\cdot \epsilon})\langle \partial u_{\cdot \epsilon} \rangle}.
\]
By Properties (3b) and (2d), \( w(u_\epsilon) = w(u_{-\epsilon}) \). By Property (4) this quantity is also equal to \( w(u_i) \). In addition, note that \( \chi(u_\epsilon) = \chi(u_{-\epsilon}) = \chi(u_i) + 1 \). Thus the above discrepancy is a multiple of

\[
(5.1) \quad \langle \partial u_\epsilon \rangle - \langle \partial u_{-\epsilon} \rangle \pm z \langle \partial u_i \rangle,
\]

Recall that Property (4) asks that the crossing moment is locally given by the model in Section 3, in the sense of Definition 3.1. As explained in Section 3.1, the images of the boundary under \( u_{\pm \epsilon} \) differ precisely by a crossing change \( \chi \leftrightarrow \chi \), whereas the boundary of \( u_i \) is correspondingly the \( \chi \).

Now let us consider the elliptic crossing. We use similar notations as above, save writing \( u_\bigcirc \) instead of \( u_i \). The same considerations apply, with one exception: \( u_{\pm \epsilon} \) have one more or one less intersection with the 4-chain than \( u_0 \) does. Thus now the relation is

\[
a^{\pm 1} \langle \partial u_\epsilon \rangle - a^{\mp 1} \langle \partial u_{-\epsilon} \rangle \pm z \langle \partial u_\bigcirc \rangle.
\]

As explained in Section 3.2, the difference between the boundaries is that \( \partial u_\bigcirc \) has an extra unknot as compared to \( \partial u_{\pm \epsilon} \), the latter two being isotopic to each other. Thus, the above expression is a multiple of

\[
(5.2) \quad a - a^{-1} \pm z \langle \bigcirc \rangle.
\]

It remains to fix the signs on \( z \) in the formulas (5.1) and (5.2). See Lemma A.3 for details on orientations and Lemma A.9 which shows that orientations near skein bifurcations are determined in local models.

\textbf{Figure 3.} Relation between hyperbolic and elliptic nodes near a cusp.
Consider the 2-parameter family of holomorphic maps defined for \( \zeta \) in a neighborhood of the origin in the upper half plane \( \{ z = s + it \in \mathbb{C} : y \geq 0 \} \), see Figure 3, upper part:

\[
z \mapsto \left( \zeta^2, \zeta(\zeta^2 + \lambda), \mu \zeta \right),
\]

where \( (\lambda, \mu) \) lies in a neighborhood of the origin in \( \mathbb{R}^2 \). For \( \mu = 0, \lambda < 0 \) the holomorphic map has a hyperbolic boundary crossing and for \( \mu = 0, \lambda > 0 \) the curve has an elliptic boundary crossing.

Just as in the 1-parameter case, the orientation of this 2-dimensional space of curves is determined by the index bundle over the central curve with a cusp together with an orientation of the 2-dimensional \( (\lambda, \mu) \)-parameter space. Consider the 1-parameter family of curves with \( \mu \neq 0 \) fixed and \( \lambda \) changing sign. Such a family corresponds to a framing change and we know that the curve count in the framed skein module does not change.

Consider next the lower part of Figure 3. Here we depict the \( \lambda \)-axis as the boundary of another family of holomorphic curves of Euler characteristic one smaller. For generic \( \lambda \) and \( \rho \) the boundary is as depicted. For \( \lambda < 0 \) the distance between the components shrinks faster than the radius of the circle, for \( \lambda > 0 \) the circle shrinks faster, and at \( \lambda = 0 \) they shrink at the same rate and we limit to a cuspidal curve.

We conclude from these pictures and the orientation on the 2-parameter family that \( \pm z \) times the curve at \( (\lambda, \rho) \), \( \rho > 0 \) equals both the elliptic and the hyperbolic crossing differences. Thus, changing \( z \) to \( -z \) in our original conventions if necessary, we may ensure that the sign in formula (5.1) is + and in formula (5.2) is −. With this choice, formulas (5.1) and (5.2) are simply the skein relations, hence zero in the skein.

It follows immediately from Theorem 5.2 that if we have an adequate perturbation setup for all \( d \) then we have invariance of

\[
Z_{X,L,\lambda} := 1 + \sum_{d>0} \langle \langle L \rangle \rangle_{X,d,\lambda} \cdot Q^d \in Sk(L) \otimes \mathbb{Q}[H_2(X,L)]
\]

Remark 5.3. The more conventional approach to counting holomorphic curves is to perturb the Cauchy-Riemann equation to \( \partial u = \epsilon \) also on curves with stable domain but zero symplectic area, thus ensuring that no curve will literally map to a point, and then count the resulting solutions.

Let us explain why such an approach would not directly yield the skein relations as above. Consider a 1-parameter family of maps \( u_t : S_t \to X \) defined over \([0, \epsilon]\), which say at \( t > 0 \) are embeddings of smooth curves, while \( S_0 \) is a nodal curve with components some \( S_+ \) and an annulus \( A \), glued at a boundary point, with \( A \) having zero symplectic area. In a setup where we have perturbed the constant curves, we may expect to find such degenerations.

In order to see the skein relation as above, we would want to know that there was another family where the central fiber is a map from the normalization \( S_+ \sqcup A \). However, \( A \) is symplectic area zero and unstable, and maps are not considered in the standard approaches to Gromov-Witten theory.

Said differently, in order to see the skein relations from a more standard approach to Gromov-Witten theory, it appears that one must confront, or in an organized way avoid, the contributions of unstable maps.

5.3. **Closed and open invariants.** Let us consider the empty Lagrangian. Now \( Z_X \) is just the usual closed Gromov-Witten count of disconnected curves (which is the exponential of the series most commonly considered in the mathematics literature), re-expressed as count...
of bare curves. We can define a reduced invariant $Z_{X,L}/Z_X$; it is independent of choices because both numerator and denominator are. However, it does not generally have a direct and invariant enumerative interpretation: it may happen that closed curves have nonvanishing intersection number with the 4-chain for $L$, hence contribute differently to the numerator and denominator. However:

**Lemma 5.4.** Assume $\lambda \in P^\circ(X, L)$ is such that every map in $M(\lambda)$ has zero intersection number zero with the 4-chain for $L$. Then

$$Z_{X,L,\lambda}/Z_{X,\lambda} = 1 + \sum w(u) \cdot z^{-\chi(u)} \cdot Q^u[\Sigma] \cdot a^u \cdot L \cdot \langle \partial u \rangle \in \text{Sk}(L)[[Q]],$$

where the sum is taken only over bare curves $u \in M(X, L)$ such that every component of the domain is a curve with boundary.

**Proof.** Immediate from the definitions. $\square$

**Remark 5.5.** Absent the hypothesis of Lemma 5.4, the contributions of closed curves which intersect the 4-chain cannot be invariantly separated from those of open curves with contractible boundary. For example, suppose that we have found generic perturbation data such that there are no curves with boundary on $L$ and that no closed curve intersects the 4-chain. Then $Z_{X,L}/Z_X$ counts only actual curves with boundary. There are no such, so $Z_{X,L}/Z_X = 1$. Now deform the data, e.g. the 4-chain, in such a way that there is an instance when the 4-chain becomes tangent to one of the closed curves. After this instance that closed curve intersects the 4-chain in two points with opposite intersection signs. After further deformation one of these intersection points moves to the boundary $L$ of the 4-chain and at an elliptic crossing a new holomorphic curve with boundary on $L$ is born. After this moment there is a unique nondegenerate curve with boundary on $L$. Nevertheless, its contribution is cancelled by that of closed curves, and it remains the case that $Z_{X,L}/Z_X = 1$.

6. Adequate Perturbations in the Absence of Multiple Covers

We now show that if there is a space of almost complex structures $\mathcal{J}$ such that any $J$-holomorphic curve under consideration is somewhere injective then we can take $P = \mathcal{J}$ to construct a system of perturbations satisfying Definition 4.2. Besides standard facts about transversality for somewhere injective curves (we review the relevant ones in Appendix A), the key ingredient is the following result:

**Theorem 6.1.** [10] Let $u_\alpha: (S_\alpha, \partial S_\alpha) \to (X, L)$ be a sequence of immersed $J_\alpha$-holomorphic curves. Suppose the sequence converges to a $J$-holomorphic $u: (S, \partial S) \to (X, L)$, with a collapsed component $S_0$, i.e., $u(S_0)$ is a point. Let the bare (positive symplectic area) part of $u$ be $u_+: (S_+, \partial S_) \to (X, L)$.

Then the map $u_+$ has a triple point, or a double point with linearly dependent tangents, or there is a point $\zeta \in S_+$ where $S_0$ is attached and where the differential of $u_+$ at $\zeta$ vanishes, $du_+(\zeta) = \partial_J u_+(\zeta) = 0$.

Using this we establish:

**Theorem 6.2.** Let $(X, L)$ be as above and let $A \in H_2(X, L)$ be a homology class. Suppose $\mathcal{J}$ is an open set of almost complex structures such that for all $J \in \mathcal{J}$, every bare $J$-holomorphic curve in homology class $A$ is somewhere injective. Then we get an integral adequate perturbation scheme by taking $P = \mathcal{J}$ and defining $M(J)$ for $J \in \mathcal{J}$ to be the moduli space of bare $J$-holomorphic curves.
Proof. We take \( P = J \), and for \( J \in J \), we take \( M(J) \) to be the moduli of \( J \)-holomorphic bare curves. We must verify that the properties in Definition 4.2 are satisfied for generic points and paths.

(1) Holds trivially at all \( J \in J \).

(2) Define \( P^\circ = J^\circ \) as the subset of generic \( J \in J \) where every bare solution is transversely cut out, is embedded and in particular has boundary an embedded link, is transverse to the 4-chain \( C \), and has interior disjoint from \( L \). Then \( J^\circ \) is open and dense by Lemmas A.5, A.6, and A.7. By transversality our solution space is an oriented 0-manifold. We define the weights to be given by the orientation, so \( w = \pm 1 \). Evidently Properties (2b) is satisfied. Property (2c) holds by request, and Property (2d) by transversality.

We turn to property (2a). It is \textit{not} simply a consequence of Gromov compactness, since we have taken \( M(J) \) the moduli space of bare curves, and not all curves. That is, by Gromov we know that a sequence of bare curves must converge to some stable map; it remains to exclude the possibility that this limit is a non-bare curve. But by Theorem 6.1, such a limit would have underlying bare curve a singular \( J \)-holomorphic curve, which contradicts property (2c).

(3) Lemma A.10 classifies the degenerations possible in a generic one parameter family. Properties (3b) - (3e) follow immediately.

We turn to Property (3a). It is \textit{not} simply a consequence of Gromov compactness, since we have taken \( M(J) \) the moduli space of bare curves, and not all curves. By Gromov, a 1-parameter family will have a limit, which may however be a non-bare curve. However, from Theorem 6.1, a limiting non-bare curve will either have a non-immersed point or a triple point in the image of its bare component. By coherence (property (1)), this means that at this \( J \), we have a map from a smooth domain with either a triple point or a non-immersed point. This contradicts property (3e).

(4) This follows from Lemma A.8, Lemma A.6, and coherence.

\( \square \)

7. SFT and the conifold transition

In this section we compare curve counts in \( T^*S^3 \), in \( T^*S^3 \setminus S^3 \), and in the resolved conifold \( X \). The key ingredient is SFT compactness and the stretching arguments it makes possible.

7.1. The conifold transition. Let us review the conifold transition. Consider the locus \( X_0 = \{ w^2 + x^2 + y^2 + z^2 = 0 \} \) in \( \mathbb{C}^4 \). On the one hand, this locus is the image of the total space of the bundle \( X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) over \( \mathbb{C}P^1 \), under a map which collapses the \( \mathbb{C}P^1 \) and is elsewhere an embedding. On the other hand, there is the deformation \( X_\epsilon = \{ w^2 + x^2 + y^2 + z^2 = \epsilon \} \), which is symplectomorphic to \( T^*S^3 \). The conifold transition means we replace one with the other.

From a symplectic point of view, \( X \) provides an exact symplectic filling of the contact cosphere bundle \( S^*S^3 \): outside of a compact set, \( X \) agrees with the the positive symplectization \([0, \infty) \times S^*S^3 \) with its standard symplectic form \( d(e^t pdq) \), where \( pdq \) is the action form restricted to \( S^*S^3 \) and \( t \) a coordinate on \([0, \infty)\). The resolved conifold is not quite a symplectic filling of \( S^*S^3 \). However, far from the central \( \mathbb{C}P^1 \), also \( X \) looks topologically like \([0, \infty) \times ST^*S^3 \) and the symplectic form is deformation equivalent to \( d(e^t pdq) + \omega \), where \( \omega \)
is independent of \( t \). For large \( t \) this can be viewed as a small perturbation that vanishes in the limit \( t \to \infty \).

We say that \( X \) is an asymptotic symplectic filling of \( S^*S^3 \). More generally, a symplectic manifold with positive end which is analogously asymptotic to a symplectization of a contact manifold is said to be asymptotically convex at infinity. In this case, we say that a Lagrangian \( L \subset X \) is asymptotically Legendrian at infinity if it is, near infinity, obtained by fixed \non-\exact\ deformation of the Lagrangian cylinder \( \mathbb{R} \times \Lambda \). The fixed amount by which we deform is exponentially small with respect to the symplectic form \( \omega \). For large \( t \) this can be viewed as a small perturbation that vanishes in \( \mathbb{R} \)-invariant symplectization almost complex structure (for instance, given by conjugation by a map taking a Lagrangian to its shift) making the original \( \mathbb{R} \)-invariant curves holomorphic with the non-exact boundary condition, see also [24] for basic results about holomorphic curves in this setting.

7.2. SFT stretching near Lagrangian submanifolds. In this section we review SFT-stretching in the context we will require it. Let \( L \subset X \) be a Lagrangian. We consider a family of almost complex structures \( J_s, s \in [0, \infty) \) on \( X \) as follows. Recall that a neighborhood \( N \) of \( L \) can be symplectically identified with a neighborhood of the 0-section \( L \) inside \( T^*L \). We think of this neighborhood \( N \) as the unit disk cotangent bundle \( D_1T^*L \). For \( s \in [0, \infty) \), we consider the following three symplectic manifolds:

\[
\begin{align*}
X_s^+ &= (X - N) \cup_{\partial N} ((-s, 0] \times ST^*L_K), \\
X_s^0 &= [-s, s] \times ST^*L, \\
X_s^- &= N \cup \partial N([0, s] \times ST^*L),
\end{align*}
\]

where we equip the symplectization pieces with the same \( \mathbb{R} \)-invariant almost complex structure and denote it \( J'_s \). Define

\[
X_s = X_s^+ \cup_{\tau_{2s}} X_s^-,
\]

where \( \tau_{2s}(\sigma, y) = (\sigma + 2s, y) \) is the conformally symplectic map that intertwines the symplectic structure \( d(e^\sigma \alpha) \) with \( d(e^{\sigma + 2s} \alpha) \). Shrinking the neighborhood \( N \) slightly we can identify \( X \) with \( X_s \) for small \( \epsilon > 0 \) and define diffeomorphisms \( \phi_s: X_s \to X \) by shrinking \( X_s^0 \) to \( X_s^0 \). Via these diffeomorphisms we can transport \( J'_s \) to a family \( J_s \) of almost complex structures on \( X \), all compatible with the symplectic structure \( \omega: J_s = d\phi_s \circ J'_s \circ d\phi_s^{-1} \).

We next state the specialization of the SFT-compactness theorem to our situation. Consider a Lagrangian \( L' \subset X \) disjoint from a small neighborhood of \( L \). Let \( u_s, s \to \infty \) be a sequence of \( J_s \)-holomorphic curves with boundary on \( L' \) and of uniformly bounded area.

**Theorem 7.1.** [2] There exists a subsequence \( u_s \) that converges to a holomorphic building in \( X^- \), \( X^0 \), and \( X^+ \), where the levels are joined at Reeb orbits in \( S^*L \). Furthermore, the total action of the Reeb orbits at the negative ends of the curves in \( X^0 \) is bounded in terms of the symplectic area of the curves. If the action of Reeb orbits in \( S^*L \) is uniformly bounded from below\(^4\) it follows in particular that the holomorphic building is finite. \( \square \)

Consider SFT stretching around \( L \). Consider a Riemannian metric \( g \) on \( L \) and the induced contact 1-form \( \alpha_g \) on the unit cotangent bundle \( ST^*L \). The Reeb flow of \( \alpha_g \) is the lift of the geodesic flow of \( g \) and the Conley-Zehnder index of a Reeb orbit equals the Morse index of the corresponding geodesic. The relevant cases for the present article are the following.

\(^4\)We allow noncompact \( L \) so this is not automatic.
Example 7.2. If \( L \approx S^3 \) and we equip \( S^3 \) with the standard round metric then the closed geodesics come in Bott families, of which the minimal index is 2. It follows that after a small perturbation, still the minimal index of a geodesic is 2.

Example 7.3. If \( L \approx S^1 \times \mathbb{R}^2 \) then there is a metric with a unique geodesic loop that goes once around the generator of \( \pi_1(S^1 \times \mathbb{R}^2) \). All other geodesic loops are multiples of this basic loop, and all closed Reeb orbits have Conley-Zehnder index 0.

We next consider consequences of SFT stretching. Assume that the negative end of \( X^+_\infty \) is modeled on \( S^*S^3 \) with contact form corresponding to the round metric. We require

**Lemma 7.4.** Let \( L \subset X^+_\infty \) be a Lagrangian and let \( \gamma \) be a collection of Reeb orbits in \( S^*S^3 \). Then in generic 1-parameter families there is no holomorphic curve in \( X^+_\infty \) with negative asymptotics at \( \gamma \). Assume that there is an adequate perturbation scheme which, in addition to the properties in Definition 4.2, has the property that curves are (unperturbed) holomorphic in some neighborhood of the negative end.

Then skein valued curve counts for \((X^+_\infty, L)\) can be defined exactly as in Definition 5.1, and Theorem 5.2 holds.

**Proof.** The dimension of such a curve would be the negative of the index of \( \gamma \). By transversality the corresponding moduli space is then empty over generic 1-parameter families. It then follows that the degeneracies in 1-parameter families are exactly as in the case without negative end. \( \square \)

**Remark 7.5.** In the above argument, transversality for curves in an SFT setup was used to argue that moduli spaces with negative expected codimension are empty. We will apply this Lemma in a setting where our choices force somewhere-injectivity, guaranteeing transversality. However, to make the analogous argument in the context of curves in more general homology classes (or if one does not want to impose our requirements on the complex structure near \( L \)), one needs a perturbation setup which ensures the relevant SFT transversality.

With Lemma 7.4 established we first relate curve counts in \((T^*S^3, S^3)\) and \(T^*S^3 \setminus S^3\) and then compare curve counts in \(T^*S^3 \setminus S^3\) and the resolved conifold \( X \). We have the following results.

We say that \( L \) is a Lagrangian brane in \( T^*S^3 \setminus S^3 \) to mean that the 4-chain can be chosen to lie entirely in \( T^*S^3 \setminus S^3 \).

**Theorem 7.6.** Let \( L \) be a Lagrangian brane in \( T^*S^3 \setminus S^3 \). Assume the existence of adequate perturbation schemes in \( T^*S^3 \) and \( T^*S^3 \setminus S^3 \). Then:

\[
Z_{T^*S^3, S^3 \cup L}(a, a_L, z) = Z_{T^*S^3 \setminus S^3, L}(Q, a_L, z)|_{Q=a^2} \otimes [\partial S^3] \in \text{Sk}(S^3)
\]

**Proof.** It suffices to establish this equality order by order in \( z \) and thus we may fix area and Euler characteristic bounds. Write \( N(r) \) for an radius \( r \)-neighborhood of \( S^3 \subset T^*S^3 \).

Consider the curves contributing to \( Z_{T^*S^3, S^3 \cup L} \). Let \( \epsilon > 0 \) be sufficiently small so that \( L \) lies outside \( N(10\epsilon) \). We apply SFT-stretching near \( \partial N(5\epsilon) \). By Lemma 7.4 for sufficiently large stretching parameter \( \rho \) we find an almost complex structure \( J_{\rho} \) such that all curves with boundary on \( L \) lies outside \( N(5\epsilon) \). Furthermore, all closed \( J_{\rho} \)-holomorphic curves have zero symplectic area and are therefore constant.

That is, we may deform the complex structure for \( T^*S^3 \) so that all curves leave a neighborhood of the \( S^3 \). Now the curves contributing to the invariant are literally the same for
(T^*S^3, S^3 \cup L) and (T^*S^3 \setminus S^3, L)$ (see Lemma 7.4 for the latter count); it remains only to examine how we count them. In the former case we must account for the intersection with the 4-chain; in the latter case, for the class in $H_2(T^*S^3 \setminus S^3)$. These match under the substitution $Q = a^2$, since generating homology class in $H_2(T^*S^3 \setminus S^3)$ is dual to half of the 4-chain, viewed as an element of $H_4^{BM}(T^*S^3 \setminus S^3)$. □

\textbf{Figure 4.} Top picture shows the unique annulus with boundary on $L_K$ and along $K \subset S^3$, before SFT-stretching. As we SFT-stretch, the boundary on $S^3$ undergoes skein moves until, as illustrated in the bottom picture, all curves left an $\epsilon$-neighborhood of $S^3$.

\textbf{Theorem 7.7.} Let $L$ be a Lagrangian brane in $T^*S^3 \setminus S^3$. Then we consider $L$ as a brane also in $X$. Assume the existence of adequate perturbation schemes in $T^*S^3$, $T^*S^3 \setminus S^3$ and $X$. Then:

\begin{equation}
Z_{T^*S^3 \setminus S^3, L}(Q, a_L, z) = \frac{Z_{X,L}(Q, a_L, z)}{Z_X(Q, z)}
\end{equation}

\textbf{Proof.} Again it suffices to establish this equality order by order in $Q$ and $z$, so we may fix area and Euler characteristic bounds. Basically the point will be to apply Lemma 5.4 on the right hand side, which we may do once we can ensure the closed curves avoid the 4-chain. We will also need to relate the complex structure appropriate to the non-exact symplectic form on $eX$ to one appropriate to the exact symplectic form on $T^*S^3 \setminus S^3$.

We next construct almost complex structures on $\tilde{X}$ which give arbitrarily small perturbations of $J_\rho$ on $T^*S^3 \setminus N(5\epsilon)$ as follows. Write $\tilde{X}_\delta$ for $\tilde{X}$ with central $\mathbb{C}P^1$ or area $\delta > 0$. Note that for the standard almost complex structure all holomorphic curves in $\tilde{X}$ maps to the central sphere. Therefore, all perturbed holomorphic curves maps to an arbitrarily small neighborhood of it.
Let $B$ be a radius $1$ neighborhood of $\mathbb{CP}^1$ in $\widetilde{X}_\delta$. For $\delta = 0$, $\widetilde{X}_\delta$ is a cone and its symplectic structure near $\partial B$ can after scaling be identified with that in $T^*S^3$ near $\partial N(\epsilon)$. For small $\delta$ we can use the same identification, however, to make the symplectic forms match we must add a $\delta$-multiple of the closed 2-form dual to the fiber.

Note next that the almost complex structure $J_\rho$ is not compatible with this new symplectic form. We change it to a compatible almost complex structure $J_{\rho,\delta}$ by adding a linear operator of size $\delta$. By transversality all (perturbed, bare) $J_{\rho}$-holomorphic curves in $T^*S^3$ with boundary on $L$ are transversely cut out. Hence for $\delta > 0$ sufficiently small all $J_{\rho,\delta}$-holomorphic curves are transversely cut out as well and moreover canonically identified with the $J_{\rho}$-holomorphic curves.

Consider small $\delta > 0$ so that this identification of curves applies. Note that both the standard almost complex structure on $B$ and $J_{\rho,\delta}$ give almost complex structures on $N(5\epsilon) \setminus N(4\epsilon)$ that are compatible with the symplectic form on $\widetilde{X}$. Interpolating between the two gives a compatible almost complex structure $J_\delta$ on $\widetilde{X}$.

We claim that for all sufficiently small $\delta > 0$ all (perturbed) $J_\delta$-holomorphic curves that contributes to the numerator in the right hand side of (7.2) are either $J_\delta$-holomorphic curves in $T^*S^3 \setminus N(5\epsilon)$ with boundary on $L$, or holomorphic curves contained in a small neighborhood of the central $\mathbb{CP}^1$, or constant. To see this, consider first a closed curve. Since the minimal area of a cycle in the homology class $[\mathbb{CP}^1]$ outside a suitable small neighborhood of $\mathbb{CP}^1$ is $> 2\epsilon$ and the area of any holomorphic curve equals its symplectic area there can be no non-constant holomorphic curves outside a small neighborhood of $\mathbb{CP}^1$. Assume next that there exist a $J_\delta$-holomorphic curve with boundary on $L$ that intersects $N(5\epsilon)$-neighborhood. Consider a sequence of such curves as $\delta \to 0$ and $\rho \to \infty$. By SFT-compactness the level of the limiting curve with boundary on $L$ would be a holomorphic curve in $\mathbb{R} \times ST^*S^3$ with negative punctures at a Reeb orbit. As in Lemma 7.4 such a curve would have negative dimension and therefore does not exist by transversality.

We have thus shown that for $\rho > 0$ and $\delta > 0$ all closed (perturbed) $J_\delta$-holomorphic curves in $\widetilde{X}$ have image in a small neighborhood of the middle $\mathbb{CP}^1$, and in particular avoid the 4-chain associated to $L$. \hfill \Box

We have written Theorems 7.6 and 7.7 collecting together all $H_2(X, L)$ classes, but really they are making independent statements for each. To fully justify the statement for a particular class, it suffices to construct an adequate perturbation scheme for that particular homology class. We will later apply these results in the case

8. Somewhere injectivity for curves in basic homology classes

In this section we construct almost complex structures on $T^*S^3$ and the resolved conifold for which all bare curves in basic homology classes are somewhere injective.

Pick a real analytic representative of $K$. Then there are holomorphic coordinates $S^1 \times (-\eta_1, \eta_1) \times B(\eta_2)$, where $B(\eta_2) \subset \mathbb{C}^2$ is an $\eta_2$-ball around the origin, on a neighborhood $V_K$ of $K$ in $T^*S^3$. Here $S^3$ corresponds to $S^1 \times \{0\} \times (B(\eta_2) \cap \mathbb{R}^2)$ and $L_K$ to $S^1 \times \{0\} \times (B(\eta_2) \cap i\mathbb{R}^2)$. Also, the $\epsilon$-shifted $L_K$ corresponds to $S^1 \times \{\epsilon\} \times (B(\eta_2) \cap i\mathbb{R}^2)$, if $\epsilon < \eta_1$.

Consider almost complex structures which agrees with the complex structure in $S^1 \times (\frac{\epsilon}{2}, \frac{1}{2}) \eta_1 \times B(\frac{1}{2} \eta_2)$ inherited from the standard complex structure on $S^1 \times \mathbb{R} \times \mathbb{C}^2$. We call such almost complex structures $(\eta_1, \eta_2)$-normalized.
If \( u: (\Sigma, \partial \Sigma) \rightarrow (T^*S^3, L_K) \) is a bare holomorphic curve then recall we call its homology class basic if its boundary represents the generator \( H_1(L_K) \).

**Lemma 8.1.** Let \( J \) be any \((\eta_1, \eta_2)\)-normalized almost complex structure. For all sufficiently small shifts \( \epsilon > 0 \), any \( J \)-holomorphic curve in a basic homology class is somewhere injective and its boundary is framed isotopic and close to a constant shift of the central curve \( S^1 \times \{ \epsilon \} \).

**Proof.** The area of a curve in a basic homology class is \( C \epsilon \) for some constant \( C > 0 \). Monotonicity shows that any curve with boundary in \( L_K \) either outside an \( \frac{1}{2} \eta_1 \)-neighborhood of \( K \) or with boundary points both outside an \( \frac{1}{2} \eta_2 \)-neighborhood of \( K \) has area \( > K \eta_2^2 \) for some \( K > 0 \). It follows that the boundary must lie inside an \( \frac{1}{2} \eta_2 \) neighborhood, if \( \epsilon > 0 \) is sufficiently small.

Consider a boundary component of the curve mapping to \( L_K \). If we compose the curve near the boundary with the projection to the first coordinate we get a holomorphic map into \((\frac{1}{2} \epsilon, \eta_1) \times S^1\). Since the local degree of this map is non-negative it follows that if the projection is not contained in \((\frac{1}{2} \epsilon, \epsilon) \times S^1\) (i.e., if the curve does not lie on one side of the projection of the shifted \( L_K \)) then it must contain points \( \{ \frac{1}{2} \eta_1 \} \times S^1 \). This however contradicts the area bound as above. It follows by Taylor expansion at boundary points that the projection of any boundary component must map to \( \{ \epsilon \} \times S^1 \) as a degree one immersion. Then, since the total homology class is one we find that the curve is injective near the boundary. Similarly, projecting in the other direction we see by monotonicity that if this projection of the boundary curve has length at most \( O(\sqrt{\epsilon}) \) and hence the boundary is within distance \( O(\sqrt{\epsilon}) \) of a constant shift of the central circle. Finally, points not in a neighborhood of the boundary cannot map to points on the boundary again by monotonicity. The lemma follows. \( \square \)

We next consider the case when the target space is the resolved conifold \( X \) instead of \( T^*S^3 \). Fixing a neighborhood of the conormal \( L_K \) we have exactly the same notion of \((\eta_1, \eta_2)\)-normalized almost complex structure. We will alter the neighborhood slightly to \( S^1 \times (\eta_1', \eta_1) \times B(\eta_2) \) allowing for a smaller half interval in the direction of the shift. The injectivity result we get requires us to fix the homology class of the map. Fix a splitting \( H_2(X, L_K) \rightarrow H_2(X) \), e.g. by picking capping disks for generators of \( H_1(L_K) \). Then any holomorphic curve \( u \) in the basic homology class determines a closed 2-cycle \( k(u) \cdot [\mathbb{C}P^1] \in H_2(X) \). Let \( t \) denote the symplectic area of \( \mathbb{C}P^1 \).

**Lemma 8.2.** Let \( J \) be any \((\eta_1, \eta_2)\)-normalized almost complex structure and assume that \( L_K \) is shifted a small distance so that Lemma 8.1 holds. For any \( k_0 \) there exists \( t(k_0) > 0 \) such that any basic \( J \)-holomorphic curve \( u \) with \( k(u) < k_0 \) is somewhere injective.

**Proof.** The proof of Lemma 8.1 with \( \frac{1}{2} \epsilon \) applies if we take \( t(k_0) \) such that \( t(k_0) \cdot k_0 < \frac{1}{2} \epsilon \). \( \square \)

**Remark 8.3.** Note that \( t \) depends on the curve class in \( H_2(X, L) \). This class will anyway be fixed in our later applications.

**Proposition 8.4.** Let \( \mathcal{J} \) denote the space of \((\eta_1, \eta_2)\)-normalized almost complex structures and let \( \epsilon > 0 \) be sufficiently small for Lemma 8.1 to hold.

- In \( T^*S^3 \), \( \mathcal{P} = \mathcal{J} \) gives an adequate perturbation scheme for basic curves in \( T^*S^3 \).
- In \( X \), for any \( k_0 > 0 \) there is \( t_0 > 0 \) such that taking \( \mathcal{P} = \mathcal{J} \) gives an adequate perturbation scheme for basic curves \( u \) in \( X \) with \( k(u) < k_0 \).

**Proof.** Follows from Lemmas 8.1 and 8.2 together with Proposition 6.2. \( \square \)
9. Enumerative meaning of the HOMFLYPT polynomial

Let \( K = K_1 \cup \cdots \cup K_k \subset S^3 \) be a link, and let \( L_K \) its conormal Lagrangian, shifted off the 0-section.

9.1. A convenient choice of 4-chain. Since the conormal is disjoint from a cotangent fiber, it is null-homologous in \( T^*S^3 \setminus S^3 \), hence bounds a 4-chain there. Changing the 4-chain affects the invariants by monomial factors and monomial changes of variable.

We will now make a particular choice of 4-chain for \( S^3 \) in order to avoid framing ambiguities in the statements of theorems. Fix an \((\eta_1, \eta_2)\)-adapted almost complex structure and consider the conormal \( L_K \) shifted along a closed 1-form dual to the tangent vector of the knot. Let \( u \) denote the basic annuli stretching between \( L_K \) and \( S^3 \) which in local coordinates near \( K \) have connected components \( S^1 \times [0, \epsilon] \times \{0\} \). Fix \( \xi \) a non-zero vector field on \( S^3 \) which is nowhere tangent to the link \( K \). Consider the 4-chain \( C_\xi \) which is the union of the positive and negative half-lines in the fiber of the conormal of \( K \) at the point \( p \) in direction of the co-vector dual to \( \xi \). For the shifted conormal the intersection \( \gamma \) is a smoothing of the curve \( \gamma' \) that is disjoint from the central copy of \( K \). We see in particular that \( \gamma \) has (ideal) boundary and constructing a bounding chains correspond to paths at infinity closing \( \gamma \) up. Changing the closing path by a suitably oriented meridian changes \( \partial u \phi \gamma \) by \( \pm 1 \). It follows that there is a choice such that

\[
\partial u \phi \gamma = 0.
\]

Lemma 2.8, (9.1), and (9.2) implies that for \( C_\xi \) there is a choice of bounding chain for \( \gamma \) such that

\[
u \phi S^3 = 0,
\]

where \( u \) is the basic annulus stretching between \( L_K \) and \( S^3 \).

9.2. The count in \( S^3 \). We compute \( Z_{T^*S^3, L_K \cup S^3} \). This invariant takes values in a tensor product of skein modules of \( S^3 \) and solid tori \( S^1 \times \mathbb{R}^2 \). Recall the skein of \( S^3 \) is just the free rank one module over the coefficient ring, generated by the class of the empty knot \( \emptyset \). This assertion is essentially equivalent to the existence of the HOMFLYPT polynomial \( \langle K \rangle_{S^3} \in \mathbb{Z}[a^\pm, z^\pm] \) of links \( K \):

\[
K = \langle K \rangle_{S^3} \cdot \emptyset \in \text{Sk}(S^3).
\]

For the solid torus \( S^1 \times \mathbb{R}^2 \), we recall from [34] that \( \text{Sk}(S^1 \times \mathbb{R}^2) \) is a free polynomial algebra over \( \mathbb{Z}[a^\pm, z^\pm] \) with generators indexed by the integers.

\[
\text{Sk}(S^1 \times \mathbb{R}^2) \cong \mathbb{Z}[a^\pm, z^\pm][\ell_0, \ell_1, \ell_2, \ldots].
\]
The algebra can be viewed as a $Z$-graded, with $\text{deg}(\ell_i) = i$; the grading records the class in $H_1(S^1 \times \mathbb{R}^2)$ determined by the skein representative. We write $\text{Sk}^+(S^1 \times \mathbb{R}^2)$ and $\text{Sk}^-(S^1 \times \mathbb{R}^2)$ for the subalgebras on the positive and negative generators; $\text{Sk}(S^1 \times \mathbb{R}^2) = \text{Sk}^+(S^1 \times \mathbb{R}^2) \otimes \text{Sk}^-(S^1 \times \mathbb{R}^2)$. Then $\text{Sk}^+(S^1 \times \mathbb{R}^2)$ is a free $\mathbb{Z}[a^\pm, z^\pm]$ module on a basis naturally indexed by partitions. In fact there is a natural identification of $\text{Sk}^1 S^1(T^* S^3, S^3 \cup L_K)$.

In particular, if we fix a basis for the skein of the solid torus, e.g. monomials in the $\ell_i$, then we may extract the corresponding coefficients of $\mathbb{Z}$ values in $\text{Sk}_c S^1(T^* S^3, S^3 \cup L_K)$; said coefficient will take values in $\text{Sk}(S^3)$. Here we will focus just on the simplest term. We fix the generator $\ell = \ell_1$ represented by $S^1 \times \{0\}$ and write $1_k \in H_1(L_K) = H_2(T^* S^3, S^3 \cup L_K)$ for the homology class which is the sum of these generators for all components.

Fix 4-chain $C_\ell$ and bounding chain for $\gamma$ as in Section 9.1.

Let $J$ be an $(\eta_1, \eta_2)$-normalized almost complex structure on $T^* S^3 \setminus S^3$. Note that by Lemma 8.1, we ensure somewhere-injectivity of curves in class $1_k$; hence by Theorem 6.2 the choice of generic such $J$ gives an adequate perturbation scheme, so by Theorem 5.2, we have a well defined skein-valued curve count in this class. We now compute it:

**Theorem 9.1.** We have the following equality in $\text{Sk}(S^3 \cup L_K)$:

\begin{equation}
\langle\langle S^3 \cup L_K\rangle\rangle_{T^* S^3,1^n} = K \otimes \ell^n
\end{equation}

*Proof.* Recall from the definition of $(\eta_1, \eta_2)$-normalized complex structures, see Section 8, that there is an obvious holomorphic cylinder given in local coordinates by $S^1 \times [-\epsilon, 0] \times \{0\}$. We claim that it is unique in the basic homotopy class.

Assume that there is a boundary point of the curve passing through a point $p \in S^3 \cup L_K$ outside $N(\frac{1}{2}r)$, then, provided $r$ is sufficiently small, there is ball of radius $\frac{1}{4}r$ around $p$ that intersects $L$ in a disk. By monotonicity, it then follows that the area of the curve is larger than $Cr^2$ for some constant $C$, but the area of the curve is bounded by $C'\delta$ so for sufficiently small $\delta > 0$ the boundary of the curve lies inside $N(\frac{1}{2}r)$. Assume now that there is an interior point that maps to a point $q$ outside $N(\frac{1}{2}r)$. Then we find a ball of radius $\frac{1}{4}r$ around $q$ that contains no boundary point of the curve. Montonicity then implies that the area of the curve is $> Cr^2$ which again gives a contradiction for small enough $\delta$. We conclude that any curve lies inside $N(r)$. Let $u$ be such a curve then we compose $u$ with the local coordinate projection to the $(z_1, z_2)$-plane. The boundary of the projection then maps to $\mathbb{R} \times \mathbb{R}$ and we find that the area of the projection is either constant or at least $\frac{1}{2} \pi \rho^2$, where $\rho$ is the radius of coordinate disk contained in our neighborhood. It follows that the projection of the curve to the $(z_1, z_2)$-plane is constant. It is easy to see that the curve must then be the standard annulus $A$ in the first coordinate.

It remains to check that $A$ is transversely cut out. The linearization of the $\bar{\partial}$-operator at $A$ is simply the standard $\bar{\partial}$-operator with boundary condition in $\mathbb{R} \times \mathbb{R}^2$ along on boundary component and $\mathbb{R} \times i\mathbb{R}^2$ along the other, there is a constant solution in the $\mathbb{R}$-direction (the linearization of rotations along $A$) and no solutions in the other directions. It follows that the curve is transversely cut out.

It follows that the cylinders $A$ for each knot component of $K$ are the only curves in the class of the generator. Thus the coefficient of $\ell \otimes \ell \otimes \cdots \otimes \ell$ is just the count of the union of these singly covered cylinders. A cylinder has Euler characteristic zero, so there are no constant curve contributions. Furthermore, by our choice of $C_\ell$ and bounding chain for $\gamma$, we will not need it here.
9.3. **Proof of Theorem 1.2.** We use Lemma 8.1 to guarantee the necessary somewhere-injectivity needed by Theorem 6.2 to ensure that the choice of generic \( J \) (within the classes identified in said Lemma) gives an adequate perturbation scheme. Now applying Theorem 7.6 to Theorem 9.1 yields the following equality in \( \text{Sk}(L_K) \).

\[
\sum_d \langle L_K \rangle_{T^*S^3 \setminus S^3 \cup \ell^\otimes n} = \langle K \rangle_{S^3} \cdot \ell^\otimes n.
\]

Now Theorem 1.2 follows by applying Theorem 7.7 (using Lemma 8.2 to guarantee the necessary somewhere-injectivity). □

**Remark 9.2.** The curve counts on the left hand sides of Equations 9.4 and 9.5 can be reinterpreted as counts of curves without boundaries as follows. Consider a Riemannian metric on the \( S^1 \times \mathbb{R}^2 \)-components of \( L_K \) with a single minimizing geodesic at \( S^1 \times \{0\} \) and SFT-stretch around \( L_K \) equipped with this metric. It is straightforward to check that the only curves in connected components of \( T^*L_K \) which have boundary on the zero section and which are asymptotic to the Reeb orbit \( \gamma \) that corresponds to the geodesic which is a positive homology generator are the cylinders over this geodesic. Since this SFT-stretching is a modification of the almost complex structure near \( L_K \), the counts on the RHS of Equations 9.4 and 9.5 can be identified with the count of punctured curves in \( T^*S^3 \setminus (S^3 \cup L_K) \). Here \( \ell^\otimes n \) should be substituted by \( \gamma^\otimes n \), i.e., the curves counted have negative punctures where it is asymptotic to the simple positive Reeb orbits in the components of \( L_K \).

However, the derivation of the (rather nontrivial) Equation 9.5 from the (trivial seeming) Equation 9.4 used crucially the invariance of the skein-valued curve counting.

**Appendix A. Review of properties of somewhere-injective curves**

A map \( u: (\Sigma, \partial \Sigma) \rightarrow (X, L) \) is called **somewhere injective** if there is an open subset \( V \subset \Sigma \) such that \( u|_V \) is a smooth embedding and such that \( u^{-1}(u(V)) = V \). In this section, we assume that \( J \) is an open subset of complex structures with the property that all curves under consideration are somewhere injective. We review the classical Fredholm arguments establishing codimension estimates and gluing for somewhere-injective curves.

**A.1. General Fredholm properties.** We start with a well-known property of somewhere-injective curves. We discuss the general setup here and give details in the particular cases where we use this lemma below. Let \( J \) have properties as stated above. Let \( \mathcal{W} \) be a configuration space of maps \( u: (\Sigma, \partial \Sigma) \rightarrow (X, L) \). We will model this on Sobolev spaces with three derivatives in \( L^2 \) below so that derivatives are continuous. More concretely this means that \( \mathcal{W} \) is a bundle of Sobolev spaces where we use weights that limit to exponential weights in cylinders and strips near the boundary of the space of curves. When studying evaluations we will sometimes consider \( \mathcal{W} \) as a bundle over the product of the space of curves and some jet-bundle over the target space. We consider non-linear Fredholm problems \( F: \mathcal{W} \times J \rightarrow \mathcal{Y} \), where \( \mathcal{Y} \) is a bundle of Sobolev spaces of complex anti-linear bundle maps \( T\Sigma \rightarrow TX \) which sometimes is multiplied by an auxiliary finite dimensional space. The Fredholm problems have the form

\[
F(u, s) = (\bar{\partial}_J u, t(u, s)),
\]
where \( s \) is a point in the finite dimensional component of the source and \( t(u, s) \) a point in the finite dimensional component of the target. For fixed \( J \in \mathcal{J} \) the linearization \( d_{(u, s)}F \) of \( F \) is a Fredholm operator of index
\[
\text{index}(d_{(u, s)}F) = \text{index}(\partial J) + \dim(s) - \dim(t),
\]
where \( \dim(s) \) and \( \dim(t) \) are the dimensions of the finite dimensional factors in the source and target respectively. This equality also holds at the level of index bundles in the sense that the index bundle of \( F \) is given by adding the bundle difference \( TS \otimes TT \), where \( S \) is the finite dimensional space added to the source and \( T \) that added to the target, to the index bundle of \( \partial J \). In particular, orientations on the index bundle of \( \partial J \), on \( S \), and on \( T \) together induce an orientation on the index bundle of \( F \).

We next discuss the basic property of the linearized equation at somewhere injective curves. Write \( dF \) for the full linearization of \( F \),
\[
dF = d_{(u, s)}F \cdot \delta(u, s) + d_J F \cdot \delta J.
\]

**Lemma A.1.** If \( d_{(u, s)}t \) is surjective at \((u, s, J)\) then \( dF \) is surjective.

**Proof.** Transversality for Fredholm problems at somewhere injective curves is well-known. We recall the argument. If \( u \) is a \( J \)-holomorphic curve then the linearization \( L_u \partial J \) of the \( \partial J \)-operator at \( u \) is an elliptic operator. We consider the full linearization \( L_u \partial J + d_J F \), where the second term corresponds to variations of \( J \in \mathcal{J} \). If the full linearization is not surjective then by a partial integration argument a co-kernel element gives a solution \( v \) of the dual elliptic equation which is orthogonal to all variations \( d_J F \cdot \delta J \). By somewhere injectivity this means \( v \) vanishes on the open set \( V \) where \( u \) is injective, and by unique continuation for the dual operator this implies \( v \) vanishes identically. It follows that the full linearization is surjective. \( \square \)

Together with the Sard-Smale theorem, Lemma A.1 implies that if \( \Delta \) is a \( d \)-dimensional manifold and \( b: \Delta \to \mathcal{J} \) is a smooth map then after arbitrarily small homotopy,
\[
\mathcal{M}(\Delta, t_0) = \{(u, s, \delta): u \in \mathcal{M}(b(\delta)), \; t(u, s) = t_0\}
\]
is a transversely cut out manifold of dimension \( \text{index}(d_{(u, s)}F) \).

**Remark A.2.** The Fredholm theory described above applies directly to maps \( u: (\Sigma, \partial \Sigma) \to (X, L) \) with stable domains. Stable maps with unstable domains (disks and annuli) must first be stabilized. We use Lemma 8.1 to stabilize domains of disks and annuli as follows:

Fix distinct points \( p_j, \; j = 1, 2, 3 \) on the the central circle \( S^1 \times \{\epsilon\} \times 0 \subset L_K \) and consider the disk fibers \( \{p_j\} \times i \mathbb{R}^2 \cap B(\eta) \) through these points. Add boundary marked points at the intersections of the curve and these hypersurfaces. This gives stable domains for all curves and we use Sobolev spaces of maps with two derivatives in \( L^2 \) on these domains to give a functional analytic neighborhood of our holomorphic curves. It is straightforward to check, compare [12], that changing the location of the points give smooth coordinate changes on resulting spaces of solutions.

**A.2. Basic orientations.** To orient the space of solutions to the equation \( F(u, s) = 0 \) in the case when the finite dimensional spaces added to the source and target are oriented we must orient the index bundle of \( \partial J \). In the case of closed holomorphic curves this is straightforward since the index bundle is a complex bundle. As shown in [15] in the case of maps from the disk into a symplectic manifold \( X \) with Lagrangian boundary condition \( L \subset X \), the situation in the open case is different and involves the following relative spin condition on the Lagrangian: the second Stiefel Whitney class of the tangent bundle \( TL \)
to \( L \), \( w_2(L) \in H^2(L; \mathbb{Z}_2) \) should be the restriction of a class \( st \in H^2(X) \). In [15, Theorem 8.1.1] it is shown that the index bundle \( \text{index}(\bar{\partial}_J) \) over the space of maps from the disk is orientable if \( L \) is relatively spin and that a choice of a relative spin structure on \( L \) determines an orientation.

In the case under consideration here, orientable Lagrangians in Calabi-Yau threefolds, the relative spin condition is trivially met: an orientable 3-manifold has trivial tangent bundle. Note that an orientation and a spin structure on \( L \) is the same thing as a trivialization of \( T L \) up to homotopy. The generalization of the orientation results from [15] is straightforward. We give a brief discussion. Let \( \text{index}_{\chi,h}(\bar{\partial}_J) \) denote the index bundle over the space of maps \( u: (S, \partial S) \to (X, L) \) where \( S \) is a connected Riemann surface of Euler characteristic \( \chi \) with \( h \) boundary components.

**Lemma A.3.** The index bundle \( \text{index}_{\chi,h}(\bar{\partial}_J) \) is orientable and the choice of an orientation and spin structure of \( L \) determines an orientation.

**Proof.** We follow [15, Chapter 8]. The linearized operator \( L\bar{\partial}_J \) over \( S \) can be homotoped to a linearized operator \( L'\bar{\partial}_J \) over a closed surface \( S' \) with \( h \) disks \( D_1, \ldots, D_h \) attached, where the bundle and boundary condition are trivialized on each \( D_k \) to \((\mathbb{C}^3, \mathbb{R}^3)\) and the operator agrees with the standard \( \bar{\partial} \)-operator. (This is where the trivialization of \( TL \) is used.) Now \( \text{index}(L'\bar{\partial}_J) \) is oriented as a complex bundle and \( \text{index}(\bar{\partial}) = \det(\mathbb{R}^3) \) over each disk \( D_k \). To obtain \( \text{index}(L\bar{\partial}_J) \) we need to glue the bundles at the points where \( D_k \) are attached to \( S' \). This uses a standard linearized gluing argument, see e.g., [12, Lemma 7.2]. Introduce small positive exponential weights and explicit solutions at the marked points viewed as punctures and glue. Here the twist parameter in the gluing extends over the disk and taking this automorphism into account we find that the index space \( \text{index}'(\bar{\partial}) \) of the disks have dimension \( 2 = 3 - 1 \). We then have

\[
\text{index}_{\chi,h}(\bar{\partial}_J) = \text{index}(L'\bar{\partial}_J) \otimes \bigotimes_{k=1}^{h} (\text{index}'(\bar{\partial}) \otimes \det(\mathbb{C}^3)),
\]

where the last factors are the gluing data at each puncture: require the values of sections to agree where glued and take out the twist automorphism.

The first factor is complex and factors in the last tensor product are even dimensional, so the orientation is independent on the ordering of boundary components. The lemma follows. \( \square \)

**A.3. Jet extensions and evaluation maps.** With the basic Fredholm properties established we now describe how to show that generic holomorphic curves have the geometric properties needed to define the skein valued Gromov-Witten invariant. The ideas used are standard. We use evaluation maps of various kind and add finite dimensional spaces to the target and source of our maps.

We describe the setup for showing that the boundary of our holomorphic curves give framed links in the Lagrangian (that thus give elements in the framed skein module). The mapping spaces we use are Sobolev spaces of maps \( u: (\Sigma, \partial \Sigma) \to (X, L) \) with three derivatives in \( L^2 \). The smoothness assumption means that the restriction \( \partial u: \partial \Sigma \to L \) has \( \frac{5}{2} \) derivatives in \( L^2 \) and, in particular, the derivative \( \frac{d}{dt}(\partial u) \) is continuous and we consider the 1-jet evaluation along the boundary

\[
j^1(\partial u): \partial \Sigma \to J^1(\partial \Sigma, L).
\]
In the language above this means that we add the boundary of the surface to the source space of the Fredholm operator \( s \in \partial \Sigma \), and the 1-jet space of maps \( \partial \Sigma \to L \) to the target space, \( t \in J^1(\partial \Sigma, L) \).

**Lemma A.4.** For generic 1-parameter families in \( \mathcal{J} \) all bare holomorphic curves are immersions on the boundary.

*Proof.* Formal maps of vanishing differential forms a codimension 3 subvariety of \( J^1(\partial \Sigma, L) \). Thus adding a boundary marked point and the 1-jet extension to our Fredholm problem as described above, \( (A.1) \) implies that maps with vanishing derivative somewhere along the boundary does not appear in generic 1-parameter families (and at isolated points in generic 2-parameter families). \( \square \)

**Lemma A.5.** For generic \( J \in \mathcal{J} \) the boundary of any bare holomorphic curve is nowhere tangent to \( \xi \). For generic 1-parameter families in \( \mathcal{J} \) there is a finite set of points where the tangent vector to the boundary curve of a bare holomorphic curve is tangent to the reference vector field \( \xi \). In a neighborhood of any such instances the 1-parameter family is conjugate to the standard tangency family of Lemma 2.10.

*Proof.* Outside the zeros \( Z(\xi) \) of the vector field \( \xi \), \( \xi \) determines a codimension two subvariety \( N_\xi \subset J^1(\partial \Sigma, L) \) where the formal differential has image in the line spanned by \( \xi \). Let \( N_\xi \subset J^1(\partial \Sigma, L) \) denote the union of the fibers over the zeros of \( \xi \) and \( N'_\xi \). As above, extending the Fredholm problem with the 1-jet extension of the evaluation map \( j^1(\partial u) \), we find the following. In generic 0-parameter families the tangent vector of \( \partial u \) is everywhere linearly independent from \( \xi \). In generic 1-parameter families there are isolated instances where the 1-jet extension intersects \( N_\xi \) transversely in its smooth top dimensional stratum. Such instances correspond to the versal deformation of a tangency with \( \xi \), see the local model in the proof of Lemma 2.10. \( \square \)

In order to prove that boundaries are embedded we consider the multi-jet extension

\[
j^0 \times j^0(\partial u) : \partial \Sigma \times \partial \Sigma \to J^0(\partial \Sigma \times \partial \Sigma, L \times L),
\]

adding \( \partial \Sigma \times \partial \Sigma \) to the source and \( J^0(\partial \Sigma \times \partial \Sigma, L \times L) \) to the target.

**Lemma A.6.** For generic \( J \) all bare holomorphic curves in \( \mathcal{M}(J) \) are embeddings when restricted to the boundary. For generic 1-parameter families in \( \mathcal{J} \) there is a finite sets where the boundary curves have double points and the 1-parameter family gives a versal deformation of the double point.

*Proof.* Note that the diagonal \( \Delta_L \subset L \times L \) gives a codimension 3 subvariety of \( J^0(\partial \Sigma \times \partial \Sigma, L \times L) \) and that the immersion condition implies that there is a neighborhood \( U \subset \partial \Sigma \times \partial \Sigma \) such that \( (j^0 \times j^0)^{-1}(U \setminus \Delta_{\partial \Sigma}) = \emptyset \). Transversality of this evaluation map than implies that \( (j^0 \times j^0)^{-1}(\Delta_L \setminus \Delta_{\partial \Sigma}) \) is empty for generic 0-parameter families, and consists of isolated instances of generic crossings corresponding to the versal deformation of the intersection point where the tangent vectors are linearly independent. \( \square \)

Finally, we consider the interior of the curve in a similar way.

**Lemma A.7.** For generic \( J \) all bare holomorphic curves are embeddings transverse to the 4-chain \( C \). For generic 1-parameter families in \( \mathcal{J} \) this holds except at isolated instances where some holomorphic curve is quadratically tangent to the 4-chain or some curve intersect \( L \) at an interior point. Near such instances the 1-parameter family is a versal deformation.
Proof. The proof is similar to the arguments above. We add an interior marked point to the source and the 1-jet space \( J^1(\Sigma, X) \) to the target to find that solutions are immersions and have transversality properties with respect to \( C \) and \( L \) as stated. To see the embedding property we then add \( \Sigma \times \Sigma \) to the source and \( J^0(\Sigma \times \Sigma, X \times X) \) to the target. □

A.4. Nodal curves at the boundary of 1-parameter families – gluing. We next consider the nodes that actually appear in generic 1-parameter families. We show that they are given by the standard models; this will be an instance of Floer gluing.

Lemma A.8. If \( u \) is an hyperbolic or elliptic nodal curve at an instance \( t_0 \) in a generic 1-parameter family in \( \lambda: [0, 1] \to \mathcal{J} \) then there is a neighborhood of \( t_0 \) such that the family of solutions in the connected component of \( \mathcal{M}(t_0 - \epsilon, t_0 + \epsilon) \) is conjugate to the standard hyperbolic or elliptic family (see Section 3) respectively.

Proof. Consider first the hyperbolic case. By coherence of our perturbation setup we find at a hyperbolic node an associated smooth curve with a generic crossing in our 1-parameter family. Consider half-strip coordinates \([0, \infty) \times [0, \pi] \) around the preimages of the intersection points. We construct a pre-glued curve by cutting these neighborhoods at large \( \rho_0 \) and interpolating to a cut-off version of the standard model of the hyperbolic node (3.1). At this pre-glued curve we invert the differential on the complement of the linearized variation of the 1-parameter family and then establish the quadratic estimate needed for Floer’s Newton iteration argument using standard arguments, compare e.g. [12, Section 6]. It then follows that for sufficiently large \( \rho \) there are uniquely determined solutions in a neighborhood of the pre-glued curve and arbitrarily close to it in the functional norm that we use (three derivatives in \( L^2 \)). Here the solution will in particular be \( C^1 \)-close to the preglued curve.

The argument is similar in the elliptic case. By coherence of our perturbation setup we find at an elliptic node an associated smooth curve with a generic crossing with \( L \) in our 1-parameter family. Consider half-cylinder coordinates \([0, \infty) \times S^1 \) around the preimage of the intersection point. We construct a pre-glued curve by cutting the neighborhoods at large \( \rho_0 \) and interpolating to a cut-off version of the standard model of the elliptic node (3.2). At this pre-glued curve we invert the differential on the complement of the linearized variation of the 1-parameter family and then again establish the quadratic estimate needed for Floer’s Newton iteration argument. Again, for sufficiently large \( \rho \) there are uniquely determined solutions in a neighborhood of the pre-glued curve and arbitrarily close to it in the functional norm that we use which here implies that the solution will be \( C^1 \)-close to the preglued curve.

This gives the desired gluing result for the moduli spaces involved: the 1-dimensional moduli space of the nodal curve is obtained by gluing from at the degenerate instance in the 1-parameter family of the smooth moduli space. Since the preglued family is clearly conjugate to the standard family and the glued family is arbitrarily \( C^1 \)-close it is also conjugate to the standard family. □

We next show that relations between orientations of moduli spaces at elliptic and hyperbolic nodal curves and respective normalizations are locally determined.

Lemma A.9. Relations of orientations of moduli spaces in a family conjugate to the standard elliptic or hyperbolic family agrees with the model up to an over all sign.

Proof. To see this we consider determinant bundles of the \( \bar{\partial}_J \) operator. We start in the less complicated elliptic case. Here we degenerate the linearized operator over the curve with an
extra boundary to the closed curve with a sphere attached which further has a disk attach to it. Here the sphere and the disk constitutes a local model glued to the original closed curve. Marked points and gluing constraints are all even dimensional and the rotation parameters in the disk and sphere are coupled. It follows that the relevant index bundle is expressed as a product of the index bundle over the original curve and an index bundle in the model.

In the case of the hyperbolic node we argue similarly. Introduce two boundary marked points where we attach two disks each with a further marked point where the maps are required to agree. Now the boundary marked points shifts the index by 1 each and the orientation depends on the order. The gluing locus is the inverse image of the diagonal under the product of evaluation maps that also depends on the order, switching the order switches orientation twice. Hence the result is independent of order and the relevant index bundle is expressed as a product of the index bundle of the original curve and the index bundle in the model. □

Lemma A.10.

(a) For generic $J \in \mathcal{J}$ the moduli space $\mathcal{M}(J)$ is a transversely cut out compact oriented 0-manifold. For a generic path $\lambda: [0, 1] \to \mathcal{J}$ the solution space $\mathcal{M}(0, 1)$ is a compact oriented 1-manifold with boundary. The boundary consists of $\mathcal{M}(0) \cup \mathcal{M}(1)$ and a finite number of nodal curves $\mathcal{M}(t_j)$ for $0 < t_1 < \cdots < t_k < 1$. The projection $\pi: \mathcal{M}(0, 1) \to [0, 1]$ is a Morse function with interior critical point at non-transverse solutions in $\mathcal{M}(t'_s)$ where the dimension of the kernel of the linearized Cauchy-Riemann operator has dimension 1.

(b) The nodal boundary curves are either hyperbolic or elliptic and the 1-parameter family is locally as in Lemma A.8. The tangent vector to $\mathcal{M}(0, 1)$ at a nodal curve on the boundary lies in the kernel of the projection $\pi$, but the second derivative of the projection is non-zero.

Proof. The first two statements in (a) are direct consequences of (A.1). Consider the boundary of the 1-manifold $\mathcal{M}(0, 1)$. The boundary of the moduli space corresponds to either going to the boundary in the parameter space or going to the boundary in the space of domains. At the boundary in the space of domains, we find at hyperbolic nodes a curve with non-embedded boundary and at elliptic nodes a curve with an interior point mapping to $L$. The fact that there are isolated points where the kernel of the linearized operator equals 1 and that the 1-parameter family meets the locus of such curves transversely follows again from (A.1). This establishes (a).

We next prove (b). Lemma A.8 gives the desired normal form of the moduli spaces. The quadratic behavior of the projection map follows from the gluing argument where we invert the differential uniformly on the complement of the gluing parameter and where the constant in the total elliptic estimate blows up at a linear rate in the gluing parameter. This means that the projection map from the moduli space has a quadratic degeneration at its endpoint, see Figure 2. □

A.5. Orientation twisting and different skein relations. Our proof of the skein relation implicitly fixes choices of the orientation of the Lagrangian and a spin structure. It is easy to check that the skein relation is invariant under these choices. For example for the orientation of the Lagrangian, $a$ should be replaced by $a^{-1}$ and crossing signs change which means that the left hand sides of the skein relations change sign. The mod 2 number of boundary components differ between the left and right hand side in the skein equations and that by
definition of the Fukaya orientation there is then an additional orientation sign difference between the left and right hand side and thus the skein relations are preserved.

One orientation choice does affect the Skein relation: the choice of orientation of the complex plane itself, which induces the orientation on the moduli space of curves. Here we first note that this reversal of orientation does not change the orientation input from gluing a disk in the Fukaya orientation: the change of orientation of the disk changes its boundary orientation but then the orientation of the quotient of the orientation changes complex gluing parameter by the orientation changed rotation along the boundary remains unchanged. For the closed curve component the orientation change is \((-1)^{1-g}\). Now at an elliptic boundary crossing the genus \(g\) stays constant and the number of boundary components change by +1 and there is no sign difference. In the case of hyperbolic boundary splitting there are two cases, when the crossing branches belong to the same boundary component the number of boundary components increase and the genus remains the same which means no sign change. However, in the case when the crossing branches belong to different components the number of boundary components decrease and the genus increases by 1 which introduces a sign. Thus when the orientation of the complex plane underlying the space of stable domains is changes we get new Skein relations, the elliptic case is the same but the hyperbolic version splits into two different relations that differ by a sign in the right hand side depending on whether the number of boundary components increase (+) or decrease (−).

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