Type-II Superstrings and New Spacetime Superalgebras

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Abstract

We present a geometric formulation of type-IIA and -IIB superstring theories in which the Wess-Zumino term is second order in the supersymmetric currents. The currents are constructed using supergroup manifolds corresponding to superalgebras: the IIA superalgebra derived from M-algebra and the IIB superalgebra obtained by a T-duality transformation of the IIA superalgebra. We find that a slight modification of the IIB superalgebra is needed to describe D-string theories, in which the U(1) gauge field on the worldsheet is explicitly constructed in terms of D-string charges. A unification of the superalgebras in a (10 + 1)-dimensional $N = 2$ superalgebra is discussed too.
1 Introduction

It is now widely appreciated that super $p$-branes play an important role in non-perturbative string physics. The dynamics of the super $p$-branes is generally awfully difficult, but it does possess some algebraic properties. One of these is a modification of the Poincaré superalgebra in the presence of super $p$-branes.

Siegel [1] found a manifestly supersymmetric formulation of the Green-Schwarz superstring, based on a superalgebra discovered earlier by Green [2]. This superalgebra is a generalization of super Poincaré algebra, in which a new fermionic generator is contained and translations do not commute with the supercharges. He constructed a suitable set of supercurrents on the corresponding supergroup manifold and wrote down the Wess-Zumino term of the Green-Schwarz action in a manifestly supersymmetric form, without having to go to one higher dimension. Bergshoeff and Sezgin showed that Siegel’s formulation generalizes to higher super $p$-branes [3]. They introduced a set of new spacetime superalgebras. By introducing the new coordinates corresponding to the new generators of the underlying superalgebra and constructing the supercurrents on the supergroup manifolds, they were able to write the Wess-Zumino terms for super $p$-branes, which are $(p + 1)$-th order in the supercurrents and which equal the usual Wess-Zumino terms up to the total derivative terms.

It is not known whether the formulation generalizes to type-II branes: type-II superstrings, NS5-branes and D $p$-branes ($p =$odd for the IIB superstring theory and $p =$even for the IIA superstring theory). In this paper, we show that the formulation generalizes to type-II superstrings and D-strings. To do this, we introduce a set of new spacetime superalgebras: the IIA superalgebra derived from the M-algebra which was discovered by Sezgin [4] and the IIB superalgebra obtained by a T-duality transformation of the IIA superalgebra. Using the new superalgebras, we construct supercurrents on the supergroup manifolds corresponding to the superalgebras. In terms of these currents, we write down the Wess-Zumino terms, which are second order in the supercurrents. We find that one needs a slight modification of the IIB superalgebra in order to describe D-string theories, in which the U(1) gauge field on the worldsheet is parametrized by coordinates associated with the D-string charges. The modified IIB superalgebra, which corresponds to the description of the IIB superstring and the D-string on an equal footing, is not related by the T-duality to the IIA superalgebra obtained from the M-algebra. As a trial to relate these superalgebras, we consider a unification of the superalgebras in a $(10 + 1)$-dimensional $N = 2$ superalgebra.

This paper is organized as follows. We first present a review of the technology used in this paper and Siegel’s formulation. In sec.3, we derive the IIA superalgebra from the M-algebra and show that the superalgebra corresponds to the IIA superstring theories. Next, in sec.4, performing a T-duality transformation, we obtain the IIB superalgebra. The algebra is shown to correspond to the IIB superstrings. In sec.5, D-strings are found to be described by modifying the IIB superalgebra, in which the U(1) gauge field is parametrized by the coordinates corresponding to the D-string charges. In sec.6, considering the identities, we show that the modified IIB superalgebra can not be related
to the IIA superalgebra by T-duality transformations. A unification of these algebras in a (10 + 1)-dimensional $N = 2$ superalgebra is discussed in sec.7. The last section is devoted to a summary and discussions.

2 Superalgebra and Siegel’s Formulation

Let us denote the generators of an algebra collectively as $T_A$. The algebra can be written as

$$[T_A, T_B] = f_{AB}^C T_C.$$  \hspace{1cm} (2.1)

In the dual basis, the Maurer-Cartan super 1-forms are defined by

$$e^A = dZ^M L_M^A,$$ \hspace{1cm} (2.2)

where $dZ^M$ are the differentials on the group manifold. The left-invariant group vielbeins $L_M^A$ and the pullbacks $L_i^A$ are obtained by the left-invariant Maurer-Cartan form,

$$U^{-1} \partial_i U = \partial_i Z^M L_M^A T_A = L_i^A T_A,$$ \hspace{1cm} (2.3)

where $U$ is a supergroup element. The Maurer-Cartan equations, which are expressed in terms of the dual forms,

$$de^C = -\frac{1}{2} e^B \wedge e^A f_{AB}^C,$$ \hspace{1cm} (2.4)

contain equivalent information about the algebra. Given a super $q$-form $G$, the exterior derivative acts as follows: $d(F \wedge G) = F \wedge dG + (-1)^q dF \wedge G$. The Jacobi identities are satisfied iff the integrability conditions, $d^2 e^A = 0$, hold.

Similarly, the right-invariant group vielbeins $R_M^A$ are obtained by the right-invariant Maurer-Cartan form,

$$\partial_i U U^{-1} = \partial_i Z^M R_M^A T_A = R_i^A T_A.$$ \hspace{1cm} (2.5)

Using the the right-invariant group vielbeins $R_M^A$, supersymmetry transformations are obtained as follows. An infinitesimal transformation is written as $U' = (1 + \epsilon)U$, where $\epsilon = e^A T_A$ is the transformation parameter. This implies that $\epsilon = \delta U U^{-1} = \delta Z^B R_B^A T_A$, one then finds that an infinitesimal transformation can be expressed as

$$\delta Z^A = e^B \hat{R}_B^A,$$ \hspace{1cm} (2.6)

where $\hat{R}_B^A$ is defined by $R_M^A \hat{R}_A^N = \delta^N_M$. The transformation parameter $\epsilon^\alpha$ associated with the supercharge $Q_\alpha$ can be interpreted as a rigid supersymmetry transformation parameter.
Useful in calculating the left-/right-invariant Maurer-Cartan equations are the Zumino’s formulae:

\[ e^{-\phi}de^\phi = \left(1 - e^{-\phi}\right) \wedge d\phi, \quad (2.7) \]
\[ de^\phi e^{-\phi} = \left(e^\phi - 1\right) \wedge d\phi, \quad (2.8) \]
\[ e^{-\phi}\beta e^\phi = e^{-\phi} \wedge \beta, \quad (2.9) \]
\[ e^\phi \beta e^{-\phi} = e^\phi \wedge \beta, \quad (2.10) \]

where the wedge denotes a compact expression of the commutation relation: \( \phi \wedge \psi = [\phi, \psi] \), \( \phi^2 \wedge \psi = [\phi, [\phi, \psi]] \) and \( 1 \wedge \psi = \psi \).

Siegell found a manifestly supersymmetric formulation of the Green-Schwarz superstring based on a superalgebra [2],

\[ \{Q_\alpha, Q_\beta\} = (\gamma^\alpha)_{\alpha\beta}P_\alpha, \quad [Q_\alpha, P_\alpha] = (\gamma^\alpha)_{\alpha\beta}\Sigma^\beta, \quad (2.11) \]

in which the translation does not commute with the supercharge. Parametrizing the supergroup manifold as

\[ U = e^{\phi_\alpha \Sigma^\alpha} e^{x_a P^a} e^{\theta^\alpha Q_\alpha}, \quad (2.12) \]

where coordinates \( Z^A = (\phi_\alpha, x_a, \theta^\alpha) \) associate to the generators \( T_A = (\Sigma^\alpha, P^a, Q_\alpha) \), the superstring action is written in terms of left-invariant pullback vielbeins,

\[ I = \int d^2\xi \left[-\frac{1}{2}\sqrt{-g}g^{ij}L_i^a L_j^b \eta_{ab} - \frac{1}{2}e^{ij}L_i^a L_j^a\right], \quad (2.13) \]

where \( g_{ij} \) and \( \eta_{ab} \) are the worldsheet and spacetime metric, respectively and \( g = \det g_{ij} \). A nontrivial feature of this new action is that the new coordinates \( \phi_\alpha \) only occur as a total derivative term. Up to this total derivative term the above action is identical to the standard Green-Schwarz superstring action. Furthermore, the Wess-Zumino term in the above action is manifestly supersymmetric, while in the usual Green-Schwarz formulation the supersymmetry is up to a total derivative term. These transformations involve \( L_i^a \) and \( L_i^\alpha \), which remain unchanged by the presence of the new coordinate \( \phi_\alpha \). The coefficient of Wess-Zumino term is fixed so that the action is also invariant under the usual \( \kappa \)-symmetry transformations. Following the same line, the authors of ref. [3] showed that \( p \)-brane actions can be constructed by using new superalgebras.

### 3 IIA Superstring

We derive the IIA superalgebra by a dimensional reduction of the M-algebra. For later use in sec. [4], we include D0-brane and D2-brane charges in addition to supertranslation and superstring charges. We then show that the IIA superstring action can be constructed using the obtained IIA superalgebra.
3.1 IIA superalgebra from M-algebra

The M-algebra found by Sezgin \[4\] is characterized by generators: supertranslation \(Q_M\), M2-brane \(Z^{MN}\), “superstring” \(Z^M\) and M5-brane \(Z^{MNOPQ}\), where 11-dimensional spacetime indices, \(\mu, \nu, \cdots\) and Majorana spinor indices, \(\alpha, \beta, \cdots\) are collectively denoted as \(M, N, \cdots\), so that \(Q_M = (P_\mu, Q_\alpha)\), \(Z^{MN} = (Z^{\mu\nu}, Z^{\mu\alpha}, Z^{\alpha\beta})\) etc. The Maurer-Cartan equations, containing equivalent information about the algebra, are described in terms of the dual basis: \(e^M, e_{MN}, e_M\) and \(e_{MNOPQ}\), respectively. It is sufficient for our present purpose to consider the former three: \(e^M, e_{MN}\) and \(e_M\). The corresponding part of the M-algebra is as follows (we call this algebra M-algebra for simplicity throughout this paper):

\[
\begin{align*}
\de^\mu &= -\frac{1}{2} e^\alpha \wedge e^\beta (\gamma^\mu)_{\alpha\beta}, \\
de_\mu &= -\frac{1}{2} e^\alpha \wedge e^\beta (\gamma_\mu)_{\alpha\beta}, \\
de_\alpha &= -e^\beta \wedge e^{\mu}(\gamma_\mu)_{\alpha\beta} + (1 - \lambda)e^\beta \wedge e^{\mu'}(\gamma^\mu)_{\alpha\beta} - \frac{\lambda}{10} e^\beta \wedge e_{\mu\nu}(\gamma^{\mu\nu})_{\alpha\beta}, \\
de_{\mu\nu} &= -\frac{1}{2} e^\alpha \wedge e^\beta (\gamma_{\mu\nu})_{\alpha\beta}, \\
de_{\mu\alpha} &= -e^\beta \wedge e^{\mu}(\gamma_{\mu\nu})_{\alpha\beta} - e^\beta \wedge e_{\mu\nu}(\gamma^{\mu})_{\alpha\beta}, \\
de_{\alpha\beta} &= \frac{1}{2} e^\mu \wedge e^{\mu'}(\gamma_{\mu\nu})_{\alpha\beta} - \frac{1}{2} e_{\mu\nu} \wedge e^{\mu}(\gamma^{\mu})_{\alpha\beta} - \frac{1}{4} e_\mu \gamma \wedge e^{\gamma}(\gamma^\mu)_{\alpha\beta} - 2e_{\mu\alpha} \wedge e^{\gamma}(\gamma^\mu)_{\beta\gamma},
\end{align*}
\]

where the Jacobi identities are satisfied due to the identities in \((10 + 1)\)-dimensions:

\[
\begin{align*}
(\gamma_\mu)_{\alpha\beta}(\gamma^{\mu})_{\gamma\delta} &= 0, \\
(\gamma_\mu)_{\alpha\beta}(\gamma^{\mu})_{\gamma\delta} + \frac{1}{10}(\gamma_{\mu\nu})_{\alpha\beta}(\gamma^{\mu\nu})_{\gamma\delta} &= 0.
\end{align*}
\]

Throughout this paper we use a notation where a given spinor always has an upper or a lower spinor-index, and never raise or lower a spinor index using the charge-conjugation matrix.

The IIA superalgebra is obtained by a dimensional reduction of the 11-th direction, say \(x^5\). The obtained IIA superalgebra is characterized by generators: supertranslation \(Q_A\), superstring \(Z^A\), D0-brane \(\Sigma\), D2-brane \(\Sigma^{AB}\) and \(Z^A\) and \(Z^B\) originated from “superstring” \(Z^M\) in 11-dimensions, where the indices \(A, B, \cdots\) collectively denote 10-dimensional spacetime indices, \(a, b, \cdots\) and Majorana-Weyl spinor indices, \(\alpha, \beta, \cdots\) and \(\dot{\alpha}, \dot{\beta}, \cdots\) with positive and negative chirality, respectively. We find that the dual forms of the generators of the IIA superalgebra are defined in terms of those of the generators of the M-algebra, as follows:

\[
\begin{align*}
\text{Supertranslation } Q_A: & \quad e^A = (e^a, e^\alpha, e^{\dot{\alpha}}), \\
\text{Superstring } Z^A: & \quad e_A = (e_a, e_\alpha, e^{\dot{\alpha}}), \\
\text{D0-brane } \Sigma: & \quad e = (e^5), \\
\text{D2-brane } \Sigma^{AB}: & \quad e_{AB} = (e_ab, e_\alpha\dot{\alpha}, e_a\dot{\alpha}, e_{\alpha\beta}, e_{\dot{\alpha}\dot{\beta}}), \\
Z^A': & \quad e_A' = (e_{a'}, e_\alpha', e^{\dot{\alpha}'}), \\
Z^B': & \quad e' = (e'_{\dot{\alpha}}),
\end{align*}
\]

\footnote{This was discussed in \[11\] and \[12\].}
which is consistent with the fact that superstring and D2-brane consist of wrapped M2-brane and M2-brane, respectively. The Maurer-Cartan equations of the IIA superalgebra are found to be

\[
\begin{align*}
\text{de}^a &= -\frac{1}{2} e^\alpha \wedge e^\beta (\gamma^a)_{\alpha\beta} - \frac{1}{2} e^\delta \wedge e^\gamma (\gamma^a)_{\delta\gamma}, \\
\text{de}_a &= -\frac{1}{2} e^\alpha \wedge e^\beta (\gamma_a)_{\alpha\beta} + \frac{1}{2} e^\alpha \wedge e^\beta (\gamma_a)_{\alpha\beta}, \\
\text{de}_a &= -e^\beta \wedge e^a (\gamma_a)_{\alpha\beta} - e^\beta \wedge e_a (\gamma_a)_{\alpha\beta}, \\
\text{de}_a &= +e^\beta \wedge e^a (\gamma_a)_{\alpha\beta} - e^\beta \wedge e_a (\gamma_a)_{\alpha\beta}, \\
\text{de} &= -e^\alpha \wedge e^\beta (1)_{\alpha\beta}, \\
\text{de}_{ab} &= -e^\alpha \wedge e^\beta (\gamma_{ab})_{\alpha\beta}, \\
\text{de}_{aa} &= -e^\beta \wedge e^b (\gamma_{ab})_{\alpha\beta} + e^\beta \wedge e (\gamma_b)_{\alpha\beta} - e^\beta \wedge e_a (1)_{\alpha\beta}, \\
\text{de}_{\alpha\dot{\beta}} &= -e^\beta \wedge e^b (\gamma_{ab})_{\alpha\dot{\beta}} - e^\beta \wedge e (\gamma_b)_{\alpha\dot{\beta}} - e^\beta \wedge e_a (1)_{\alpha\dot{\beta}}, \\
\text{de}_{\alpha\dot{\beta}} &= +e \wedge e^a (\gamma_a)_{\alpha\dot{\beta}} - \frac{1}{2} e_a \wedge e^a (\gamma^b)_{\alpha\dot{\beta}} - \frac{1}{2} e_b \wedge e (\gamma^b)_{\alpha\dot{\beta}} - \frac{1}{4} e_{a\gamma} \wedge e^\gamma (\gamma^a)_{\alpha\dot{\beta}}, \\
\text{de}_{\alpha\dot{\beta}} &= -e \wedge e^a (\gamma_a)_{\alpha\dot{\beta}} - \frac{1}{2} e_a \wedge e^a (\gamma^b)_{\alpha\dot{\beta}} - \frac{1}{2} e_b \wedge e (\gamma^b)_{\alpha\dot{\beta}} - \frac{1}{4} e_{a\gamma} \wedge e^\gamma (\gamma^a)_{\alpha\dot{\beta}}, \\
\text{de}_{a'} &= -\frac{1}{2} e^\alpha \wedge e^\beta (\gamma_a)_{\alpha\beta} - \frac{1}{2} e^\alpha \wedge e^\beta (\gamma_a)_{\alpha\beta}, \\
\text{de'} &= -e^\beta \wedge e^a (\gamma_a)_{\alpha\beta} - e^\beta \wedge e (\gamma_a)_{\alpha\beta} + (1 - \lambda) e^\beta \wedge e_a (\gamma^a)_{\alpha\beta} + \frac{1}{10} e^\beta \wedge e_a (\gamma^a)_{\alpha\beta}, \\
\text{de}_\alpha &= -e^\beta \wedge e^a (\gamma_a)_{\alpha\beta} + e^\beta \wedge e (\gamma_a)_{\alpha\beta} + (1 - \lambda) e^\beta \wedge e_a (\gamma^a)_{\alpha\beta} - \frac{1}{10} e^\beta \wedge e_a (\gamma^a)_{\alpha\beta}, \\
\text{de'} &= -e^\alpha \wedge e^\beta (1)_{\alpha\dot{\beta}},
\end{align*}
\]

where we used the following relations \[ (1)_{\alpha\dot{\beta}} = -(1)_{\dot{\beta}\alpha} \text{ and } (\gamma_{ab})_{\alpha\dot{\beta}} = +(\gamma_{ab})_{\dot{\beta}\alpha}. \] The IIA superalgebra is closed due to the following identities:

\[
\begin{align*}
(\gamma_a)_{\alpha(\gamma^a)_{\gamma\delta}} &= 0, \\
(1)_{\alpha(\gamma^a)_{\gamma\delta}} + (\gamma^a)_{\alpha(\gamma^b)_{\gamma\delta}} &= 0, \\
-(1)_{\alpha(\gamma^a)_{\gamma\delta}} + 2(\gamma_a)_{\alpha\delta} (\gamma^a)_{\gamma\dot{\delta}} + \frac{1}{10} (\gamma_{ab})_{\alpha\delta} (\gamma^a)_{\gamma\dot{\delta}} &= 0,
\end{align*}
\]

and those in which the undotted spinor indices are exchanged for the dotted ones. The last identity \[(3.27)\] is needed to satisfy the Jacobi identity for primed dual forms originating from “superstring” in the M-algebra.
3.2 IIA superstring

We now consider a subalgebra of the IIA superalgebra generated by the following generators: \( T_A = \{P_a, Q_\alpha, Q_{\dot{\alpha}}, Z^a, Z^{\dot{a}}\} \). The algebra turns out to be

\[
\begin{align*}
\{Q_\alpha, Q_\beta\} &= (\gamma^a)_{\alpha\beta} P_a + (\gamma_a)_{\alpha\beta} Z^a, \\
\{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} &= (\gamma^a)_{\dot{\alpha}\dot{\beta}} P_a - (\gamma_a)_{\dot{\alpha}\dot{\beta}} Z^a, \\
[P_a, Q_\beta] &= 2(\gamma_b)_{\alpha\beta} Z^a, \quad [Z^b, Q_\beta] = 2(\gamma^b)_{\alpha\beta} Z^a, \\
[P_a, Q_{\dot{\beta}}] &= -2(\gamma_b)_{\dot{\alpha}\dot{\beta}} Z^{\dot{a}}, \quad [Z^{\dot{b}}, Q_{\dot{\beta}}] = 2(\gamma^{\dot{b}})_{\dot{\alpha}\dot{\beta}} Z^{\dot{a}},
\end{align*}
\]

which is closed due to the identity (3.25), and those undotted indices replaced by dotted spinor ones.

We show that the IIA superstring action can be constructed from the above algebra. In this sense, we refer to the algebra as IIA superstring algebra. The super group manifold is parametrized as

\[
U = e^{a Z^a} e^{\dot{a} \dot{Z}^{\dot{a}}} e^{\alpha Z^\alpha} e^{\dot{\alpha} \dot{Z}^{\dot{\alpha}}} P_a e^{\theta a} Q_\alpha e^{\dot{\theta} a} Q_{\dot{\alpha}},
\]

and the left-invariant pullback supergroup vielbeins are obtained as

\[
\begin{align*}
L_i^\alpha &= \partial_i \theta^\alpha, \quad L_i^{\dot{\alpha}} = \partial_i \dot{\theta}^{\dot{\alpha}}, \\
L_i^a &= \partial_i x^a + \frac{1}{2} (\bar{\gamma}^a \partial_i \dot{\theta}^\alpha) + \frac{1}{2} (\bar{\gamma}^a \partial_i \dot{\theta}^{\dot{\alpha}}), \\
L_{ia} &= \partial_i z_a - \frac{1}{2} (\bar{\gamma} a \partial_i \dot{\theta}^\alpha) + \frac{1}{2} (\bar{\gamma} a \partial_i \dot{\theta}^{\dot{\alpha}}), \\
L_{i\alpha} &= \partial_i \zeta_\alpha + 2 \partial_i z_a (\bar{\gamma}^a \gamma_\alpha) + 2 \partial_i x^a (\bar{\gamma} a \gamma_\alpha) + \frac{2}{3} (\bar{\gamma} a \partial_i \dot{\theta}^\alpha) (\bar{\gamma} a \gamma_\alpha), \\
L_{i\dot{\alpha}} &= \partial_i \zeta_{\dot{\alpha}} + 2 \partial_i z_a (\bar{\gamma}^{\dot{a}} \gamma_{\dot{\alpha}}) - 2 \partial_i x^a (\bar{\gamma}^a \gamma_{\dot{\alpha}}) - \frac{2}{3} (\bar{\gamma} a \partial_i \dot{\theta}^\alpha) (\bar{\gamma} a \gamma_{\dot{\alpha}}),
\end{align*}
\]

which are invariant under the following supersymmetry transformations:

\[
\begin{align*}
\delta \theta^\alpha &= \epsilon^\alpha, \quad \delta \dot{\theta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}}, \\
\delta x^a &= -\frac{1}{2} (\bar{\epsilon} \gamma^a \theta) - \frac{1}{2} (\bar{\epsilon} \gamma^a \dot{\theta}), \\
\delta z_a &= -\frac{1}{2} (\bar{\epsilon} \gamma^a \theta) + \frac{1}{2} (\bar{\epsilon} \gamma^a \dot{\theta}), \\
\delta \zeta_\alpha &= -2 x^a (\bar{\epsilon} \gamma_\alpha) - 2 z_a (\bar{\epsilon} \gamma^a) + 2 z_a (\bar{\epsilon} \gamma_\alpha) (\bar{\theta} \gamma^a), \\
\delta \zeta_{\dot{\alpha}} &= +2 x^a (\bar{\epsilon} \gamma_{\dot{\alpha}}) - 2 z_a (\bar{\epsilon} \gamma^a) - 2 z_a (\bar{\epsilon} \gamma_{\dot{\alpha}}) (\bar{\theta} \gamma^a).
\end{align*}
\]

In the case where we do not denote the spinor indices explicitly, it is always understood that they have their standard position, e.g. \((\gamma_a \theta)_\alpha = (\gamma_a)_{\alpha\beta} \theta^\beta, \quad (\bar{\theta} \gamma_a \partial_i \theta) = \theta^a (\gamma_a)_{\alpha\beta} \partial_i \theta^\beta, \) etc.

We find that the IIA superstring action can be constructed as

\[
I = \int d^2 \xi \left[ -\frac{1}{2} \sqrt{-g} g^{ij} L_i^a L_j b \eta_{ab} - \frac{1}{2} \epsilon^{ij} (L_i^a L_j a + \frac{1}{4} L_i^a L_j a + \frac{1}{4} L_i^a L_j a) \right],
\]

where \( g_{ij} \) and \( \eta_{ab} \) are the worldsheet and spacetime metric, respectively, and \( g = \text{det} g_{ij} \). The last three terms in the action constitute the manifestly supersymmetric form of the Wess-Zumino term. The
coefficient of the Wess-Zumino term is determined so that the action enjoys fermionic \( \kappa \)-symmetry,

\[
\delta_{\kappa} \theta^\alpha = (1 + \Gamma)(1 - \Gamma) \beta^\kappa, \quad \delta_{\kappa} \theta^\alpha = (1 - \Gamma)\beta^\kappa, \quad \delta_{\kappa} x^a = \frac{1}{2}(\delta_{\kappa} \theta^\alpha \theta) + \frac{1}{2}(\delta_{\kappa} \gamma^a \dot{\theta}),
\]

(3.37)

where \( \Gamma = \frac{1}{2\sqrt{g}} \epsilon^{ij} L_i L_j \beta_{ab} \).

The two-form \( b \) is defined from the Wess-Zumino term in the action as

\[
b = -\frac{1}{2}(L^a \wedge L_a + \frac{1}{4} L^\alpha \wedge L^\alpha + \frac{1}{4} L^{\dot{\alpha}} \wedge L^{\dot{\alpha}}).
\]

(3.38)

The three-form \( h = db \) is calculated as

\[
h = -\frac{1}{2} L^a \wedge L^\alpha \wedge L^{\beta}(\gamma_a)_{\alpha\beta} + \frac{1}{2} L^\alpha \wedge L^{\dot{\alpha}} \wedge L^{\dot{\beta}}(\gamma_a)^{\dot{\alpha}\dot{\beta}}.
\]

(3.39)

Note that all of the dependence on the new fermionic coordinates has dropped out from the expression for \( h \). In fact, the anti-symmetric part of the action is the well-known Wess-Zumino term of the IIA superstrings up to total derivative terms,

\[
\int \epsilon^{ij} b_{ij} = \int \frac{1}{2} \epsilon^{ij} \{ \partial_i x^a((\bar{\theta}\gamma_a \partial_j \theta) - (\bar{\theta}\gamma_a \partial_j \dot{\theta})) + \frac{1}{2}(\bar{\theta}\gamma^a \partial_i \dot{\theta})(\bar{\theta}\gamma_a \partial_j \theta) \}.
\]

(3.40)

As a result, we conclude that the IIA superstring action is constructed from the IIA superstring algebra \[3.28\].

4 IIB Superstring

In this section, the IIB superalgebra is constructed as the T-dual of the IIA superalgebra. The IIB superalgebra includes D-string charges as well as superstring charges. We show that the IIB superstring action is constructed by means of the IIB superalgebra.

4.1 IIB superalgebra and T-duality

In order to obtain IIB superalgebra, we consider a T-duality transformation of the IIA superalgebra. We denote the 9-th spacelike direction as \( x^5 \) with respect to which T-duality is performed. The IIB superalgebra is generated by supertranslations \( Q_A \), superstring \( Z^A \) and D-string \( \Sigma^A \), which can be expressed in terms of generators of the IIA superalgebra. Since the IIB superalgebra is generated by generators with undotted spinor indices, the chirality of the fermionic generators with dotted spinor indices \( \tilde{\alpha} \) in the IIA superalgebra are flipped by multiplying \( \gamma^{\tilde{\alpha}} \), as was done in ref. \[3\]. We denote chirality flipped spinor indices as \( \tilde{\alpha} \) and \( (\alpha, \alpha, \tilde{\alpha}) \) as \( A \) collectively.

It turns out to be easy to perform the T-duality transformation by using the Maurer-Cartan equations. We found that the dual forms \( \hat{e}^A, \hat{e}_A \) and \( \hat{e}'_A \) of generators \( Q_A, Z^A \) and \( \Sigma^A \), respectively, can be written in terms of those of the generators of the IIA superalgebra as follows:

Supertranslation \( Q_A \): \( \hat{e}^A = (\hat{e}^i, e^z), \quad \hat{e}^\alpha = e^\alpha, \quad \hat{e}^{\tilde{\alpha}} = \gamma^z e^{\tilde{\alpha}}, \)

Superstring \( Z^A \): \( \hat{e}_A = (e_i, \tilde{e}^z), \quad \hat{e}_\alpha = e_\alpha, \quad \hat{e}_{\tilde{\alpha}} = \gamma^z e^{\tilde{\alpha}}, \)

D-string \( \Sigma^A \): \( \hat{e}'_A = (\hat{e}'_i, -e), \quad \hat{e}'_\alpha = e_{\tilde{\alpha}^z}, \quad \hat{e}'_{\tilde{\alpha}} = \gamma^z e_{\tilde{\alpha}}, \)

(4.1)
where \( i \) runs, except for \( \sharp \). We use a notation in which the primed dual form corresponds to a dual form for D-string charges. Since the dual forms \( e_{\alpha\beta}, e_{\alpha\tilde{\beta}} \) and \( e_{\tilde{\alpha}\beta} \) of the IIA superalgebra are parts of the IIB D3-brane charges, we neglect them here. The resulting Maurer-Cartan equations for the IIB superalgebra are found to be (dropping hats)

\[
de a^a = -\frac{1}{2} e^a \wedge \gamma^\alpha \epsilon_{\alpha\beta} - \frac{1}{2} e^{\tilde{\alpha}} \wedge \gamma^\alpha \epsilon_{\tilde{\alpha}\tilde{\beta}}, \tag{4.2}
\]

\[
de e^a = -\frac{1}{2} e^a \wedge \gamma^\alpha \epsilon_{\alpha\beta} + \frac{1}{2} e^{\tilde{\alpha}} \wedge \gamma^\alpha \epsilon_{\tilde{\alpha}\tilde{\beta}}, \tag{4.3}
\]

\[
de c = -e^\beta \wedge e^a(\gamma^a)_{\alpha\beta} - e^\beta \wedge e_a(\gamma^a)_{\alpha\beta}, \tag{4.4}
\]

\[
de c_{\tilde{\alpha}} = +e^\beta \wedge e^a(\gamma^a)_{\tilde{\alpha}\beta} - e^\beta \wedge e_a(\gamma^a)_{\tilde{\alpha}\beta}, \tag{4.5}
\]

\[
de e'_a = -e^\alpha \wedge e^\tilde{\beta}(\gamma^a)_{\alpha\beta}, \tag{4.6}
\]

\[
de e'_{\tilde{\alpha}} = -e^\beta \wedge e^a(\gamma^a)_{\tilde{\alpha}\beta} - e^\beta \wedge e'_a(\gamma^a)_{\tilde{\alpha}\beta}, \tag{4.7}
\]

\[
de e'_{\tilde{\alpha}} = -e^\beta \wedge e^a(\gamma^a)_{\tilde{\alpha}\beta} - e^\beta \wedge e'_a(\gamma^a)_{\tilde{\alpha}\beta}, \tag{4.8}
\]

where the Jacobi identities are satisfied due to the well-known identity \( (\gamma_a)_{\alpha(\beta}(\gamma^a)_{\gamma\delta)} = 0 \). Here the tildes on the spinor indices of \( \gamma \)-matrices are not written because the \( \gamma \)-matrices do not see the tilded-ness of the spinors. But if one wants to see how the identity is used, one finds that used are the following identities:

\[
(\gamma_a)_{\alpha(\beta}(\gamma^a)_{\gamma\delta)} = 0, \quad (\gamma_a)_{\tilde{\alpha}(\tilde{\beta}}(\gamma^a)_{\gamma\delta)} = 0, \tag{4.9}
\]

and those in which the tilded spinor indices are exchanged for the untilded spinor indices. Note that the numbers of tildes in the identities are 0, 4, 1, 3 and the identity with two tilded spinor indices is absent. Since the T-dual of the Maurer-Cartan equations (3.21) ~ (3.24) can not be rearranged in a covariant form, we drop them here. We return to this point later in sec.4.

### 4.2 IIB superstring

We show that the IIB superstring action can be constructed from the IIB superalgebra obtained in sec.4.1. We start with writing down the IIB superalgebra:

\[
\{Q_\alpha, Q_\beta\} = (\gamma^a)_{\alpha\beta} P_a + (\gamma_a)_{\alpha\beta} Z^a;
\]

\[
\{Q_{\tilde{\alpha}}, Q_\beta\} = (\gamma^a)_{\tilde{\alpha}\beta} P_a - (\gamma_a)_{\tilde{\alpha}\beta} Z^a;
\]

\[
\{Q_\alpha, Q_{\tilde{\beta}}\} = (\gamma_a)_{\alpha\tilde{\beta}} \Sigma^a;
\]

\[
[P_a, Q_\beta] = 2(\gamma_a)_{\alpha\beta} Z^\alpha + 2(\gamma_a)_{\tilde{\alpha}\beta} \Sigma^\tilde{\alpha}, \tag{4.10}
\]

\[
[P_a, Q_{\tilde{\beta}}] = -2(\gamma_a)_{\tilde{\alpha}\beta} Z^\tilde{\alpha} + 2(\gamma_a)_{\alpha\beta} \Sigma^\alpha;
\]

\[
[Z^a, Q_\beta] = 2(\gamma^a)_{\alpha\beta} Z^\alpha, \quad [Z^a, Q_{\tilde{\beta}}] = 2(\gamma^a)_{\tilde{\alpha}\beta} Z^\tilde{\alpha},
\]

\[
[\Sigma^a, Q_\beta] = 2(\gamma^a)_{\alpha\beta} \Sigma^\alpha, \quad [\Sigma^a, Q_{\tilde{\beta}}] = 2(\gamma^a)_{\tilde{\alpha}\beta} \Sigma^\tilde{\alpha}.
\]

Parametrizing the supergroup manifold by

\[
U = e^{x_a Z^a} e^{\epsilon_\alpha Z^\alpha} e^{\phi^\alpha \Sigma^\alpha} e^{\phi_{\tilde{\alpha}} \Sigma^\tilde{\alpha}} e^{e^a P_a} e^{\theta^\alpha Q_a} e^{\theta_{\tilde{\alpha}} Q_{\tilde{\alpha}}}, \tag{4.11}
\]
we obtain the pullback vielbeins of the left-invariant supergroup as follows:

\[ L_i^\alpha = \partial_i \theta^\alpha, \quad \hat{L}_i^\alpha = \partial_i \hat{\theta}^\alpha, \quad (4.12) \]

\[ L_i^a = \partial_i x^a + \frac{1}{2} (\bar{\theta}^\alpha \partial_i \theta) + \frac{1}{2} (\bar{\theta}^\alpha \partial_i \hat{\theta}), \quad (4.13) \]

\[ L_{ia} = \partial_i z_a + \frac{1}{2} (\bar{\theta}^\alpha \partial_i \theta) - \frac{1}{2} (\bar{\theta}^\alpha \partial_i \hat{\theta}), \quad (4.14) \]

\[ L_{i\alpha} = \partial_i \zeta_\alpha + 2 \partial_i z_a (\bar{\theta}^\alpha)_{\alpha} + 2 \partial_i x^a (\bar{\theta}^\alpha)_{\alpha} + \frac{2}{3} (\bar{\theta}^\alpha \partial_i \theta) (\bar{\theta}^\alpha)_{\alpha}, \quad (4.15) \]

\[ L_{i\hat{\alpha}} = \partial_i \zeta_{\hat{\alpha}} + 2 \partial_i z_a (\bar{\theta}^\alpha)_{\hat{\alpha}} - 2 \partial_i x^a (\bar{\theta}^\alpha)_{\hat{\alpha}} - \frac{2}{3} (\bar{\theta}^\alpha \partial_i \hat{\theta}) (\bar{\theta}^\alpha)_{\hat{\alpha}}, \quad (4.16) \]

\[ L_{ia'} = \partial_i y_a + (\bar{\theta}^\alpha \partial_i \theta), \quad (4.17) \]

\[ L_{ia'} = \partial_i \phi_a + 2 \partial_i x^a (\bar{\theta}^\alpha)_{\alpha} + 2 \partial_i y_a (\bar{\theta}^\alpha)_{\alpha} + (\bar{\theta}^\alpha \partial_i \theta) (\bar{\theta}^\alpha)_{\alpha} + \frac{1}{3} (\bar{\theta}^\alpha \partial_i \hat{\theta}) (\bar{\theta}^\alpha)_{\alpha}, \quad (4.18) \]

\[ L_{i\hat{\alpha}'} = \partial_i \phi_{\hat{\alpha}} + 2 \partial_i x^a (\bar{\theta}^\alpha)_{\hat{\alpha}} + 2 \partial_i y_a (\bar{\theta}^\alpha)_{\hat{\alpha}} + (\bar{\theta}^\alpha \partial_i \theta) (\bar{\theta}^\alpha)_{\hat{\alpha}} + \frac{1}{3} (\bar{\theta}^\alpha \partial_i \hat{\theta}) (\bar{\theta}^\alpha)_{\hat{\alpha}}. \quad (4.19) \]

The supersymmetry transformations are found to be

\[ \delta \theta^\alpha = \epsilon^\alpha, \quad \delta \hat{\theta}^\alpha = \hat{\epsilon}^\alpha, \quad (4.20) \]

\[ \delta x^a = - \frac{1}{2} (\bar{\epsilon}^\alpha \partial_i \theta) - \frac{1}{2} (\bar{\epsilon}^\alpha \partial_i \hat{\theta}), \quad (4.21) \]

\[ \delta z_a = - \frac{1}{2} (\bar{\epsilon}^\alpha \partial_i \theta) + \frac{1}{2} (\bar{\epsilon}^\alpha \partial_i \hat{\theta}), \quad (4.22) \]

\[ \delta \zeta_\alpha = - 2 z_a (\bar{\epsilon}^\alpha)_{\alpha} - 2 x^a (\bar{\epsilon}^\alpha)_{\alpha} + \frac{2}{3} (\bar{\epsilon}^\alpha \partial_i \theta) (\bar{\epsilon}^\alpha)_{\alpha}, \quad (4.23) \]

\[ \delta \zeta_{\hat{\alpha}} = - 2 z_a (\bar{\epsilon}^\alpha)_{\hat{\alpha}} + 2 x^a (\bar{\epsilon}^\alpha)_{\hat{\alpha}} - \frac{2}{3} (\bar{\epsilon}^\alpha \partial_i \hat{\theta}) (\bar{\epsilon}^\alpha)_{\hat{\alpha}}, \quad (4.24) \]

\[ \delta y_a = - (\bar{\epsilon}^\alpha \theta), \quad (4.25) \]

\[ \delta \phi_a = - 2 x^a (\bar{\epsilon}^\alpha)_{\alpha} - 2 y_a (\bar{\epsilon}^\alpha)_{\alpha} + (\bar{\epsilon}^\alpha \theta) (\bar{\epsilon}^\alpha)_{\alpha} + \frac{1}{3} (\bar{\epsilon}^\alpha \partial_i \theta) (\bar{\epsilon}^\alpha)_{\alpha}, \quad (4.26) \]

\[ \delta \phi_{\hat{\alpha}} = - 2 x^a (\bar{\epsilon}^\alpha)_{\hat{\alpha}} - 2 y_a (\bar{\epsilon}^\alpha)_{\hat{\alpha}} + (\bar{\epsilon}^\alpha \partial_i \hat{\theta}) (\bar{\epsilon}^\alpha)_{\hat{\alpha}} + \frac{1}{3} (\bar{\epsilon}^\alpha \theta) (\bar{\epsilon}^\alpha)_{\hat{\alpha}}. \quad (4.27) \]

We find that the IIB superstring action is constructed as

\[ I = \int d^2 \xi [ - \frac{1}{2} \sqrt{-g} g^{ij} L_i^a L_j^b \eta_{ab} - \frac{1}{2} \epsilon^{ij} (L_i^\alpha L_j^\alpha + \frac{1}{4} L_i^\alpha L_j^\alpha + \frac{1}{4} L_i^{\hat{\alpha}} L_j^{\hat{\alpha}}) ], \quad (4.28) \]

where \( g_{ij} \) and \( \eta_{ab} \) are the worldsheet and the spacetime metric, respectively, and \( g = det g_{ij} \). The last three terms in the action constitute a manifestly supersymmetric form of the Wess-Zumino term. The two-form \( b \) is defined in terms of the Wess-Zumino term in the action as

\[ b = - \frac{1}{2} (L^a \wedge L_a + \frac{1}{4} L^\alpha \wedge L_\alpha + \frac{1}{4} L^{\hat{\alpha}} \wedge L_{\hat{\alpha}}), \quad (4.29) \]

and the-three form \( h = db \) is calculated as

\[ h = - \frac{1}{2} L^a \wedge L^\alpha \wedge L^{\beta} (\gamma_a)_{\alpha \beta} + \frac{1}{2} L^a \wedge L^{\hat{\alpha}} \wedge L^{\tilde{\beta}} (\gamma_a)_{\hat{\alpha} \tilde{\beta}}. \quad (4.30) \]

Note that all of the dependence on the new fermionic coordinates has dropped out from the expression of \( h \). In fact, the anti-symmetric part of the action is the well-known Wess-Zumino term of the IIB
superstring up to total derivative terms,

$$\int \epsilon^{ij} b_{ij} = \int \frac{1}{2} \epsilon^{ij} [\partial_i x^a (\bar{\theta} \gamma_a \partial_j \theta) - (\bar{\theta} \gamma_a \partial_x \partial_j \theta)] + \frac{1}{2} (\bar{\theta} \gamma_a \partial_x \partial_j \theta) (\bar{\theta} \gamma^a \partial_j \theta).$$

(4.31)

The coefficient of the Wess-Zumino term is chosen so that the action is invariant under the $\kappa$-symmetry transformations

$$\delta_\kappa \theta^\alpha = (1 + \Gamma) \gamma^\beta \kappa_\beta, \quad \delta_\kappa \tilde{\theta}^{\tilde{\alpha}} = (1 - \Gamma) \gamma^\beta \tilde{\kappa}_\beta, \quad \delta_\kappa x^a = -\frac{1}{2} (\bar{\theta} \gamma^a \delta_\kappa \tilde{\theta}) - \frac{1}{2} (\bar{\theta} \gamma^a \delta_\kappa \theta),$$

(4.32)

where $\Gamma = \frac{1}{2\sqrt{-g}} \epsilon^{ij} L_i^a L_j^b \gamma_{ab}$.

In this way, we construct the IIB superstring action from the IIB superalgebra. Note that by constructing the IIB superstring action, the D-string charges, $\Sigma^A$, play no role. This is consistent with $\Sigma^A$ representing D-string charges. In this sense, we refer to the algebra corresponding to the Maurer-Cartan equations $(4.2) \sim (4.5)$ as the IIB superstring algebra.

5 IIB D-string

The D-string action characterized by left-invariant vielbeins correspond to D-string charges can not be constructed by means of the IIB superalgebra. We show that the Wess-Zumino term and the modified 2-form field strength, $F = F - b$, where $F$ is the 2-form field strength and $b$ is the pullback to the worldsheet of the $R \otimes R$ two-form gauge potential, can be constructed in terms of the left-invariant vielbeins corresponding to D-string charges if we start with a superalgebra obtained by modifying the IIB superalgebra.

We start with the modified IIB superalgebra: $(4.2) \sim (4.6)$ and

$$de'_\alpha = -e^\beta \wedge e^{a(\gamma_a)_{\alpha\beta}} - e^\tilde{\beta} \wedge e'_a(\gamma^a)_{\alpha\tilde{\beta}},$$

$$de'_\tilde{\alpha} = -e^\tilde{\beta} \wedge e^{a(\gamma_a)_{\tilde{\alpha}\tilde{\beta}}} - e^\beta \wedge e'_a(\gamma^a)_{\tilde{\alpha}\beta},$$

(5.33)

(5.34)

which is closed due to the well-known identity $(\gamma_a)_{\alpha(\beta(\gamma^a)_{\gamma\delta})} = 0$. As in sec.4.1, if one writes tildes on the spinor indices of the $\gamma$-matrices, one obtains the following identities:

$$(\gamma_a)_{\alpha(\beta(\gamma^a)_{\gamma\delta})} = 0, \quad (\gamma_a)_{\alpha}(\gamma^a)_{\gamma\delta} = 0,$$

(5.35)

and those in which the tilded spinor indices are exchanged for the untilded spinor indices. The numbers of tildes in these identities are 0, 4, 2 in contrast to the identities in the IIB superalgebra, where the numbers of tildes were 0, 4, 1, 3. The difference in the number of tildes tells us that the modified IIB superalgebra is not related to the IIA superalgebra by the T-duality transformation, as shown in sec.6. The modification is simply to interchange $\Sigma^a$ with $\Sigma^{\tilde{a}}$. This is trivial from the IIB perspective, because the generators, $\Sigma^a$ and $\Sigma^{\tilde{a}}$, are not distinguished inherently. From the IIA perspective, however, this is nontrivial, since a spinor index with a tilde corresponds to one with a different chirality under T-duality.
The left-invariant vielbeins are found to be (4.12)–(4.17) and

\[ L_{α}' = \partial_α φ_α + 2∂_α x^a (θ_α)_a + 2∂_α y_a (θ_α)_a + (θ_α) (θ_α)_a + \frac{1}{3} (θ_α) (θ_α)_a, \]

\[ L_{\tilde{α}}' = \partial_{\tilde{α}} φ_{\tilde{α}} + 2∂_{\tilde{α}} x^a (θ_{\tilde{α}})_a + 2∂_{\tilde{α}} y_a (θ_{\tilde{α}})_a + (θ_{\tilde{α}}) (θ_{\tilde{α}})_a + \frac{1}{3} (θ_{\tilde{α}}) (θ_{\tilde{α}})_a. \]

The supersymmetry transformations are obtained as (4.20)–(4.25) and

\[ \delta φ_α = -2x^a (θ_α)_a - 2y_a (θ_α)_a + \frac{1}{3} (θ_α) (θ_α)_a, \]

\[ \delta φ_{\tilde{α}} = -2x^a (θ_{\tilde{α}})_a - 2y_a (θ_{\tilde{α}})_a + \frac{1}{3} (θ_{\tilde{α}}) (θ_{\tilde{α}})_a. \]

The two-form \( b \) is defined as

\[ b = -\frac{1}{4} (L^α ∧ L_{α}' - L^\tilde{α} ∧ L_{\tilde{α}}'). \]

The primed vielbeins correspond to the D-string charges \( Σ^A \), and the action constructed with this Wess-Zumino term can be regarded as a “gauge-fixed” D-string action. In fact, the three-form \( h = db \) turns out to be (4.30) obtained for the IIB superstring. All of the dependence on the new fermionic coordinates has dropped out from the expression of \( h \).

Can the total (gauge-unfixed) D-string action be constructed? To this end, we must first determine a superinvariant modified 2-form field strength, \( F = dA - b \), where \( A \) is a U(1) gauge field on the world-sheet. The \( b \) is the conventional 2-form which is read off from the integrand of the r.h.s of (4.31). If we obtain the two-form \( F \), the D-string antion can be constructed as in ref. [8] or in ref. [8]. We observe that

\[ \frac{1}{2} \epsilon^{ij} (L_i^α L_{j\alpha}' - L_i^\tilde{α} L_{j\tilde{α}}') = -2\epsilon^{ij} (F_{ij} - b_{ij}), \]

where we introduce \( F_{ij} \) as

\[ \epsilon^{ij} F_{ij} = -\frac{1}{2} \epsilon^{ij} [∂_i y_a ∂_j (θ_α)_a + \frac{1}{2} ∂_i φ_α ∂_j θ^α + \frac{1}{2} ∂_i φ_\tilde{α} ∂_j θ^\tilde{α}]. \]

The r.h.s. of (5.42) is a total derivative term, and then we can regard \( F_{ij} \) as the field strength of a U(1) gauge field \( A_i \),

\[ A_i = -\frac{1}{2} [y_a ∂_i (θ_α)_a + \frac{1}{2} φ_α ∂_i θ^α - \frac{1}{2} φ_\tilde{α} ∂_i θ^\tilde{α}]. \]

Note that this is parametrized by D-string coordinates associated to D-string charges \( Σ^A \). The supersymmetry transformation is found to be

\[ \delta A_i = -\frac{1}{2} [-x^a (θ_α) ∂_i θ^a + \frac{1}{6} (θ_α) (θ_α) ∂_i θ^a] - (θ → \tilde{θ}, \epsilon → \tilde{ε}) \] \[ -\frac{1}{4} ∂_i (θ_α) (θ_α). \]

In order to see the relation to the well-known supersymmetry transformation of the U(1) gauge field in ref. [8], we consider a U(1) gauge transformation and obtain an alternative form,

\[ A_i = \frac{1}{2} [∂_i y_a (θ_α)_a + \frac{1}{2} ∂_i φ_α θ^α - \frac{1}{2} ∂_i φ_\tilde{α} θ^\tilde{α}]. \]
This transforms under the supersymmetry transformation as

\[ \delta A_i = -\frac{1}{2} [\partial_i x^a (\bar{\epsilon} \gamma_a \theta) + \frac{1}{6} (\bar{\epsilon} \gamma_a \theta)(\bar{\theta} \gamma^a \partial_i \theta) - (\theta \to \tilde{\theta}, \epsilon \to \tilde{\epsilon}) - \frac{1}{4} (\partial_i \phi_a \epsilon^a - \partial_i \phi_{\tilde{a}} \tilde{\epsilon}^a), \] (5.46)

which is similar to the well-known form except for the last two terms. These two terms are the total derivative terms, and the supersymmetry transformation \( \delta F_{ij} \) is nothing but the one obtained there. We conclude that the U(1) gauge field on the worldsheet can be constructed in this way. It is interesting that the U(1) gauge field is constructed explicitly in terms of the D-string charges. Using the modified field strength \( F \), one constructs the D-string action, as in ref. or in . In this sense, we refer to the superalgebra corresponding to the Maurer-Cartan equations (4.2), (4.6), (5.33) and (5.34) as D-string superalgebra. Hence, the modified IIB superalgebra describes IIB superstrings and D-strings on an equal footing.

We now comment on the existence of a U(1) gauge field in type-II superstrings. We observe that the modified field strength for IIB superstrings is constructed by observing that

\[ \epsilon^{ij}(L_i a L_j a + \frac{1}{4} L_i \alpha L_j a + \frac{1}{4} L_i \tilde{\alpha} L_j \tilde{a}) = -2 \epsilon^{ij}(F_{ij} - b_{ij}), \] (5.47)

where \( F_{ij} \) is introduced as

\[ \epsilon^{ij} F_{ij} = -\frac{1}{2} \epsilon^{ij} [\partial_i x^a \partial_i z_a + \frac{1}{4} \partial_i \zeta_\alpha \partial_j \theta^\alpha + \frac{1}{4} \partial_i \zeta_{\tilde{\alpha}} \partial_j \tilde{\theta}^\tilde{\alpha}], \] (5.48)

and the \( b_{ij} \) is the conventional 2-form defined by the integrand of the r.h.s. of (4.31). Thus the U(1) gauge field can be defined as

\[ A_i = \frac{1}{2} [z_a \partial_i x^a - \frac{1}{4} \zeta_\alpha \partial_i \theta^\alpha - \frac{1}{4} \zeta_{\tilde{\alpha}} \partial_i \tilde{\theta}^\tilde{\alpha}] \] (5.49)

and is parametrized by coordinates corresponding to IIB superstring charges. The supersymmetry transformation is found to be

\[ \delta A_i = -\frac{1}{2} [\partial_i x^a (\bar{\epsilon} \gamma_a \theta) + \frac{1}{6} (\bar{\epsilon} \gamma_a \theta)(\bar{\theta} \gamma^a \partial_i \theta) - \frac{1}{2} (\theta \to \tilde{\theta}, \epsilon \to \tilde{\epsilon})], \] (5.50)

which is identical to (5.46) up to a total derivative term. In this way, we can construct a U(1) gauge field for IIB superstrings. For IIA superstrings, one can construct a U(1) gauge field parametrized by coordinates corresponding to IIA superstring charges in a similar way. The fact that a U(1) gauge field can be constructed in type-II superstring theories is consistent with the space-time scale-invariant formulation of \( p \)-branes and type-II superstrings . For \( p \)-brane theories, the same procedure presented above will produce not only the supersymmetry transformation of the \( p \)-form gauge field, but also the explicit form of the \( p \)-form gauge field in terms of the \( p \)-brane charges.

6 T-duality and Identities

The modified IIB superalgebra will not be related by the T-duality to the IIA superalgebra. This is seen by recognizing the fact that identities characterizing the IIA superalgebra can not be written in a covariant form after the T-duality transformation.
For completeness, we start with identities in the M-algebra and reduce to ones in the IIA superalgebra. Then, performing the T-duality transformation, we determine whether the resulting identities are rearranged in a covariant form or not.

One of the two identities characterizing the M-algebra is (3.7)

\[ \left( \gamma^\mu \right)_{\alpha\beta} \left( \gamma^{\mu\nu} \right)_{\gamma\delta} = 0. \] (6.51)

We reduce this identity to ones of the IIA superalgebra as follows. For \( \nu = b \neq a \), one obtains the identities

\begin{align*}
(1) & \delta(\beta)(\gamma^a)_{\gamma\delta} + (\gamma^a b) \delta(\beta)(\gamma^b)_{\gamma\delta} = 0, \\
(1) & \alpha(\beta)(\gamma^a)_{\gamma\delta} + (\gamma^a b) \alpha(\beta)(\gamma^b)_{\gamma\delta} = 0,
\end{align*}

which are the characteristic identities in the presence of D0 and D2-branes. As for \( \nu = a \), the well-known identities

\[ \left( \gamma^a \right)_{\alpha(\beta)(\gamma^a)_{\gamma\delta}} = 0, \quad \left( \gamma^a \right)_{\delta(\beta)(\gamma^a)_{\gamma\delta}} = 0 \] (6.54)

are obtained. We next consider the T-duality transformation. By the procedure explained in sect.4.1, we obtain the T-dual of the identity (6.52) for \( a = i \neq \# \)

\[ \left( \gamma^i \right)_{\alpha(\beta)(\gamma^i)_{\gamma\delta}} - \left( \gamma^i \right)_{\delta(\beta)(\gamma^i)_{\gamma\delta}} + (\gamma^i j) \alpha(\beta)(\gamma^j)_{\gamma\delta} = 0, \] (6.55)

which is rewritten in a covariant form as

\[ \left( \gamma^i \right)_{\alpha(\beta)(\gamma^i)_{\gamma\delta}} + \frac{1}{2} \left( \gamma^i c \right) \alpha(\beta)(\gamma^c)_{\gamma\delta} = 0. \] (6.56)

This is a characteristic identity in the presence of D3-branes. In turn, for \( a = \# \), one finds the identity

\[ \left( \gamma^a \right)_{\alpha(\beta)(\gamma^a)_{\gamma\delta}} = 0, \] (6.57)

which is used in satisfying the Jacobi identities for the IIB superalgebra. The T-dual of the identity (6.53) is found to be identities (6.56) and (6.57) with exchanging the tilded spinor indices for the untilded spinor indices. Identities (6.54) are transformed into

\[ \left( \gamma^a \right)_{\alpha(\beta)(\gamma^a)_{\gamma\delta}} = 0, \quad \left( \gamma^a \right)_{\delta(\beta)(\gamma^a)_{\gamma\delta}} = 0, \] (6.58)

respectively.

The second identity of the M-algebra is (3.8)

\[ (\gamma^\mu)_{\alpha\beta(\gamma^\mu)_{\gamma\delta}} + \frac{1}{10} (\gamma^\mu\nu)_{\alpha\beta(\gamma^\mu\nu)_{\gamma\delta}} = 0, \] (6.59)

which reduces to the identity in the IIA superalgebra,

\[ (\bar{\phi}) \delta(\phi) + \frac{2}{5} (\bar{\phi} \gamma_a \phi) (\bar{\phi} \gamma^a \phi) + \frac{1}{10} (\bar{\phi} \gamma_{ab} \phi) (\bar{\phi} \gamma^{ab} \phi) = 0, \] (6.60)
where $\phi$ and $\dot{\phi}$ are Grassmann-even spinors with opposite chirality each other. This is characteristic identity for the Maurer-Cartan equations (3.22) and (3.23) for primed dual forms. Under the T-duality transformation, this transforms into an identity which is not rewritten in a covariant form,

$$
(\bar{\phi}\gamma^{s}\tilde{\phi})(\bar{\phi\gamma^{s}\tilde{\phi}}) + \frac{2}{5}(\bar{\phi}\gamma^{s}\tilde{\phi})(\bar{\phi\gamma^{s}\tilde{\phi}}) + \frac{1}{10}(\bar{\phi}\gamma_{ij}\tilde{\phi})(\bar{\phi}\gamma_{ij}\tilde{\phi}) + \frac{1}{5}(\bar{\phi}\gamma_{i}\tilde{\phi})(\bar{\phi}\gamma^{i}\tilde{\phi}) = 0. 
$$

(6.61)

This was the reason why the “superstring” charges in the IIA superalgebra are discarded in performing the T-duality transformation in sec.4.1.

Let us consider, conversely, the identity in the modified IIB superalgebra,

$$
(\gamma_{a})_{\alpha\beta}(\gamma_{b})_{\gamma\delta} = 0. 
$$

(6.62)

This is transformed into an identity,

$$
(\gamma_{i})_{\alpha\beta}(\gamma^{i})_{\gamma\delta} - (\gamma_{i})_{\alpha\beta}(\gamma^{i})_{\gamma\delta} + (\gamma_{i})_{\alpha\gamma}(\gamma^{i})_{\beta\delta} + (\gamma_{i})_{\alpha\delta}(\gamma^{i})_{\beta\gamma} + (1)_{\alpha\gamma}(1)_{\beta\delta} + (1)_{\alpha\delta}(1)_{\beta\gamma} = 0, 
$$

(6.63)

which is not rewritten in a covariant form.

From these observation, the modified IIB superalgebra is not rewritten as a T-dual of the IIA superalgebra. In the next section, we try to relate the IIA superalgebra and the modified IIB superalgebra.

### 7 Unification of Type-II Superalgebras

We encountered two IIB superalgebras. One is the IIB superalgebra, which is the T-dual of the IIA superalgebra; the other is the modified IIB superalgebra, which describes the IIB superstring and D-string on an equal footing. The IIA superalgebra is related to the IIB superalgebra by the T-duality, but not to the modified IIB superalgebra.

As a trial to relate the IIA superalgebra to the modified IIB superalgebra, we examine a unification in a $(10 + 1)$-dimensional $N = 2$ superalgebra of the modified IIB superalgebra and the M-algebra (hence, the IIA superalgebra). This is motivated by the fact that the identity in $N = 2 \, D = 10 + 1$,

$$
(\gamma_{\mu})_{\alpha\beta}(\gamma^{\mu})_{\gamma\delta} = 0, 
$$

(7.1)

is projected into the identity (6.62) in the modified IIB superalgebra.

We first present the relevant part of $(10 + 1)$-dimensional $N = 2$ superalgebra, and then consider the relations to the M-algebra and the modified IIB superalgebra. We begin with the $N = 2 \, (10 + 1)$-dimensional superalgebra generated by bosonic charges:

$$
de^{\mu} = -\frac{1}{2}e^{\alpha} \wedge e^{\beta}(\gamma^{\mu})_{\alpha\beta} - \frac{1}{2}e^{\tilde{\alpha}} \wedge e^{\tilde{\beta}}(\gamma^{\mu})_{\tilde{\alpha}\tilde{\beta}}, 
$$

(7.2)

$$
de_{\mu} = -\frac{1}{2}e^{\alpha} \wedge e^{\beta}(\gamma_{\mu})_{\alpha\beta} + \frac{1}{2}e^{\tilde{\alpha}} \wedge e^{\tilde{\beta}}(\gamma_{\mu})_{\tilde{\alpha}\tilde{\beta}}, 
$$

(7.3)

$$
de_{\mu\nu} = -\frac{1}{2}e^{\alpha} \wedge e^{\beta}(\gamma_{\mu\nu})_{\alpha\beta} - \frac{1}{2}e^{\tilde{\alpha}} \wedge e^{\tilde{\beta}}(\gamma_{\mu\nu})_{\tilde{\alpha}\tilde{\beta}}, 
$$

(7.4)
In addition to these equations we consider the following equations:

\begin{align}
    d\epsilon_{\mu
u} &= -\frac{1}{2}\epsilon^\alpha \wedge \epsilon^\beta (\gamma_{\mu
u})_{\alpha\beta} + \frac{1}{2}\epsilon^\tilde{\alpha} \wedge \epsilon^\tilde{\beta} (\gamma_{\mu
u})_{\tilde{\alpha}\tilde{\beta}}, \\
    d\epsilon'_{\mu
u} &= -\epsilon^\alpha \wedge \epsilon^\beta (\gamma_{\mu
u})_{\alpha\beta}, \\
    d\epsilon'_{\mu} &= -\epsilon^\alpha \wedge \epsilon^\beta (\gamma_{\mu})_{\alpha\beta}.
\end{align}

(7.5)
(7.6)
(7.7)

The Jacobi identity for the dual form $d\epsilon_{\mu\nu}$ is satisfied such that the exterior derivatives of the first and second lines in (7.10) vanish separately. However, the Jacobi identity for the dual form $d\epsilon_{\alpha\beta}$ requires both lines in (7.10).

Let us consider the relation to the M-algebra. We discard the dual forms with spinor indices with a tilde, since we need only $N=1$ supersymmetry. In addition, we discard the primed dual forms. These procedures cause the following to occur: eqns.(7.9) and (7.11) decouple from our analysis; eqn.(7.8), with identifying $\epsilon'_{\mu\nu}$ with $\epsilon_{\mu\nu}$ turns out to be (7.12); eqns.(7.10) and (7.12) result in (8.3) and (8.4), respectively. We thus find that the N=2 superalgebra includes the M-algebra in this way.

We next consider the relation to the modified IIB superalgebra. We project the spinor indices to ones with the same chirality and perform a dimensional reduction of the 11-th dimension $x^5$. The dual forms $e_{2\alpha}$, $e'^{2\alpha}$ and $e'_{2\alpha}$ are identified with $e^\alpha$, $e_{\alpha}$ and $e'_{\alpha}$, respectively. We find that if we set $\lambda = 2$, eqn.(7.8) turns to (8.4) after a trivial overall scaling. Eqn.(7.9) is found to become (1.3). Eqns.(7.10) and (7.11) with $\mu = 5$, after a trivial overall rescaling, turn out to be

\begin{align}
    d\epsilon'_{\alpha} &= -\epsilon^\tilde{\beta} \wedge e^\alpha (\gamma_{\alpha})_{\tilde{\alpha}\tilde{\beta}} - \epsilon^\tilde{\beta} \wedge e'^\alpha (\gamma_{\alpha})_{\alpha\tilde{\beta}} - \epsilon^\beta \wedge e^\alpha (\gamma_{\alpha})_{\alpha\beta} - \epsilon^\beta \wedge e'^\alpha (\gamma_{\alpha})_{\alpha\beta}, \\
    d\epsilon'_{\tilde{\alpha}} &= -\epsilon^\beta \wedge e^\alpha (\gamma_{\alpha})_{\tilde{\alpha}\tilde{\beta}} - \epsilon^\beta \wedge e'^\alpha (\gamma_{\alpha})_{\alpha\tilde{\beta}} - \epsilon^\tilde{\beta} \wedge e^\alpha (\gamma_{\alpha})_{\tilde{\alpha}\beta} - \epsilon^\tilde{\beta} \wedge e'^\alpha (\gamma_{\alpha})_{\tilde{\alpha}\beta}.
\end{align}

(7.13)
(7.14)

Note that the r.h.s. of these equations are constructed by adding the r.h.s. of (4.7) and (4.8) of the IIB superalgebra and the r.h.s. of (5.33) and (5.34) of the modified IIB superalgebra. The rest of the equations will be a part of D3-brane charges.
In summary, we find that the $N = 2$ superalgebra includes the M-algebra and the free parameter $\lambda$ in the M-algebra has to be 2 in order for the IIB superstring algebra to be included. These imply that (5.33) and (5.34) for the D-string superalgebra naturally arise as well as (4.7) and (4.8) of the IIB superalgebra. Note that the commutators of the D-string superalgebra, which are not related to the IIA superalgebra by the T-duality, emerge by considering the (10+1)-dimensional $N = 2$ superalgebra. However, D-strings do not correspond to the superalgebra obtained from the $N = 2$ superalgebra, since the pullback vielbeins, $L_{i\alpha}'$ and $L_{i\tilde{\alpha}}'$, contain the r.h.s. of (4.18) and (4.19). It is interesting to seek a unified superalgebra from which one can construct IIA superstrings, IIB superstrings and D-strings.

8 Summary and Discussions

We have presented a set of new spacetime superalgebras: the IIA superstring superalgebra, the IIB superstring superalgebra and the D-string superalgebra. Using the new superalgebras, we have shown that Siegel’s formulation generalizes to type-II superstrings and D-strings. Namely, we have constructed supercurrents on the supergroup manifolds corresponding to the superalgebras. We then wrote down the Wess-Zumino terms, which are second order in the supercurrents. The modified 2-form field strength for D-strings was identified with a second-order expression of the supercurrents. From this expression, the U(1) gauge field on the worldsheet of D-strings was obtained explicitly in terms of coordinates corresponding to D-string charges, including the new fermionic charges.

We succeeded in constructing the pullback to the D-string worldsheet of the $R \otimes R$ 2-form potential from the modified IIB superalgebra in sec.5. We now comment on the relation to D-strings of the IIB superalgebra, which was obtained in sec.4 by a T-duality transformation of the IIA superalgebra. We observe that using the pullbacks of left-invariant vielbeins $\sim (4.12)$ corresponding to the IIB superalgebra, the second-order expression, $-\frac{1}{2}(L^a \wedge L^a' + \frac{1}{4}L^\alpha \wedge L^\alpha' + \frac{1}{4}L^\tilde{\alpha} \wedge L^\tilde{\alpha}')$, is calculated to be, up to total derivative terms,

$$-\frac{1}{2}\left(dx^a \wedge \left((\bar{\theta}^a d\bar{\theta}) + (\bar{\theta}^a d\bar{\theta})\right) + \frac{1}{3}(\bar{\theta}^a d\bar{\theta}) \wedge (\bar{\theta}^a d\bar{\theta}) + \frac{1}{3}(\bar{\theta}^a d\bar{\theta}) \wedge (\bar{\theta}^a d\bar{\theta})\right),$$

(8.1)

which corresponds to the pullback to the D-string worldsheet of the $NS \otimes NS$ 2-form potential. Thus we find that interchanging new fermionic generators $\Sigma^\alpha$ and $\Sigma^\tilde{\alpha}$ results in exchanging $R \otimes R$ gauge potential for $NS \otimes NS$ one, since the modification was simply interchanging $\Sigma^\alpha$ and $\Sigma^\tilde{\alpha}$.

In turn, the pullback to the F-string worldsheet of the $R \otimes R$ 2-form potential can be obtained from the pullback to the D-string worldsheet of the $R \otimes R$ 2-form potential. Using the resulting expressions, we can construct $(p, q)$-superstring actions in a manifestly supersymmetric form. We hope to report on this issue in the future. In addition, it is interesting to see whether the formulation can be generalized to the other type-II-branes: NS5-branes and D $p$-branes ($p=$odd for the IIB superstring theory and $p=$even for the IIA superstring theory). Especially, now that we have the IIA superalgebra, including the D2-brane charges, the generalization to the D2-branes has to be examined. We leave this to the future.
We found a modified IIB superalgebra which includes the IIB superstring and D-string superalgebras as subalgebras, and describes IIB superstrings and D-strings on an equal footing. However, this is not related by the T-duality transformation to the IIA superalgebra derived from the M-algebra. In order to relate these superalgebras, we considered a unification in a \((10 + 1)\)-dimensional \(\mathcal{N} = 2\) superalgebra. The unification implies that the free parameter in the M-algebra is fixed as \(\lambda = 2\). By considering the unification, the commutation relations of D-strings, which was not obtained by the T-duality transformation of the IIA superalgebra, are found to emerge. However, unnecessary relations are also generated in addition to the preferred D-string superalgebra, and the obtained superalgebra does not correspond to D-strings. It is interesting to consider the other unification which unifies the M-algebra and the modified IIB superalgebra.

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**References**

1. W. Siegel, “Randomizing the superstring”, Phys. Rev. **D50** (1994) 2799.

2. Michael B. Green, “Super-translations, Superstrings and Chern-Simons Forms”, Phys. Lett. **B223** (1989) 157.

3. E. Bergshoeff and E. Sezgin, “Super p-brane theories and new spacetime superalgebra”, Phys. Lett. **B354** (1995) 256.

4. E. Sezgin, “The M-algebra”, Phys. Lett. **B392** (1997) 323.

5. Mina Aganagic, Costin Popescu and John H. Schwarz, “Gauge-invariant and gauge-fixed D-brane actions”, Nucl. Phys. **B495** (1997) 99.

6. P. K. Townsend, “M-theory From Its Superalgebra”, (Cargese Lectures 1997), [hep-th/9712004](https://arxiv.org/abs/hep-th/9712004).

7. P. K. Townsend, “World Sheet Electromagnetism and The Superstring Tension”, Phys. Lett. **B277** (1992) 285-288.

8. E. Bergshoeff, L. A. J. London and P. K. Townsend, “Spacetime Scale-Invariance and the Super p-Brane”, Class.Quant.Grav. **9** (1992) 2545-2556, [hep-th/9206026](https://arxiv.org/abs/hep-th/9206026).

9. T. Kugo and P. K. Townsend, “Supersymmetry And The Division Algebra”, Nucl. Phys. **B221** (1983) 357.

10. M. Sakaguchi, in preparation.
[11] T. Curtright, “Are There Any Superstring in Eleven Dimensions?”, Phys. Rev. Lett. 60 (1988) 393.

[12] E. Sezgin, “Super p-form Charges and a Reformulation of the Supermembrane Action in Eleven Dimensions”, in Leuven Notes in Mathematical Physics. Series B. vol 6, [hep-th/9512082](http://arxiv.org/abs/hep-th/9512082).