Black Holes in 2 + 1 Teleparallel Theories of Gravity

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We apply the Hamiltonian formulation of teleparallel theories of gravity in 2 + 1 dimensions to a circularly symmetric geometry. We find a family of one-parameter black hole solutions. The BTZ solution fixes the unique free parameter of the theory. The resulting field equations coincide with the teleparallel equivalent of Einstein’s three-dimensional equations. We calculate the gravitational energy of the black holes by means of the simple expression that arises in the Hamiltonian formulation and conclude that the resulting value is identical to that calculated by means of the Brown-York method.

§1. Introduction

Gravity theories in three dimensions have gained considerable attention in the recent years.1,2) The expectation is that the study of lower-dimensional theories will provide relevant information about the corresponding theory in four dimensions. In 2+1 dimensions, the Riemann and Ricci tensor have the same number of components, and consequently the imposition of Einstein’s equations in vacuum implies that the curvature tensor also vanishes. Therefore the space-time in the absence of sources is flat, and the existence of a black hole would be precluded.3) Moreover, we know that Einstein’s gravitational theory in 2 + 1 dimensions does not have a Newtonian limit.3,4) Therefore it would be desirable to establish a theoretical formulation that would possess the two important features: a Newtonian limit and a black hole solution.

In general relativity, it is generally believed that the gravitational energy cannot be localized. However, in the framework of teleparallel theories of gravity, as for instance the teleparallel equivalent of general relativity (TEGR),5) it is possible to make precise statements about the energy and momentum densities of the gravitational field. In the 3 + 1 formulation of the TEGR, it is found that the Hamiltonian and vector constraints contain divergencies in the form of scalar and vector densities, respectively. They can be identified as the energy and momentum densities of the gravitational field.6) Therefore the Hamiltonian and vector constraints are regarded as energy-momentum equations. This identification has proved to be consistent, and it has been demonstrated that teleparallel theories provide a natural instrument for the investigation of gravitational energy. Several applications have been presented in the literature, among which we mention analysis of the gravitational energy of

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rotating black holes\textsuperscript{7)} and of Bondi’s radiative metric.\textsuperscript{8)}

In the definition of the gravitational energy in 3+1 dimensions, a condition on the triads is necessary: If we require the physical parameters of the metric tensor such as the mass and angular momentum to vanish, we must have a vanishing torsion tensor, $T_{(i)jk} = 0$. Triads that yield a vanishing torsion tensor in any coordinate system, after requiring the physical parameters of the metric tensor to vanish, are called ‘reference space triads’.\textsuperscript{7)}

A family of three-parameter teleparallel theories in 2+1 dimensions was proposed by Kawai, and a Lagrangian formulation was developed.\textsuperscript{9)} With particular conditions on the parameters of the theory, black hole solutions were found.\textsuperscript{9)} These solutions are quite different from the Schwarzschild and Kerr black holes in 3 + 1 dimensions. We also remark that black hole-like solutions are discussed in Ref. 10).

In a previous paper, we considered the canonical formulation of this three-parameter theory,\textsuperscript{11)} starting from the Lagrangian density of Ref. 9). Following Dirac’s method\textsuperscript{12)} and employing Schwinger’s time gauge condition,\textsuperscript{13)} we found that the three-parameter family is reduced to a one-parameter family. This unique family describes a theory with a first class algebra of constraints. We concluded that the Legendre transform is well defined only if certain conditions on the parameters are satisfied. One of these conditions is $\frac{3}{4}c_1 + c_2 = 0$.\textsuperscript{11)} Kawai’s conditions $[(3c_1 + 4c_2) \neq 0$ and $c_1(3c_1 + 4c_2) < 0]$ rule out the existence of a Newtonian limit and of a general class of black holes in the Hamiltonian formulation previously investigated. If the value of the remaining free parameter is fixed as $c_1 = -\frac{2}{3}$, the final theory agrees with the three-dimensional Einstein theory in vacuum.\textsuperscript{11)} However, it is possible to fix the value of $c_1$ by means of an alternative physical requirement.

In this article we apply the canonical analysis developed in Ref. 11) to a static, circularly symmetric geometry. We introduce a cosmological constant and find the well-known BTZ (Banádós, Teitelboim, Zanelli) solution.\textsuperscript{14)} This black hole solution has been used to study quantum models in 2+1 dimensions, because the corresponding models in 3 + 1 dimensions are very complicated.\textsuperscript{15)}

We also investigate the gravitational energy of the BTZ solution in a region of radius $r_0$, and compare it with the result obtained by means of the Brown-York method.\textsuperscript{16)} We find that the results are identical. However, it is noted that the result obtained by means of our method is much simpler.

The article is divided as follows. In §2, we review the Lagrangian and Hamiltonian formulation of the three-dimensional gravitational teleparallelism. In §3 we apply the formalism to arbitrary gravitational fields. In §4 we define the energy of the gravitational field in 2+1 dimensions and apply it to the BTZ solution. Finally, in §5 we present our conclusions.

We employ the following notation. Space-time indices $\mu, \nu, \cdots$ and global $SO(2,1)$ indices $a, b, \cdots$ run from 0 to 2. In the $2 + 1$ decomposition, Latin indices indicate space indices according to $\mu = 0, i$ and $a = (0), (i)$. The flat space-time metric is fixed for $\eta_{(0)(0)} = -1$. 
§2. Hamiltonian formulation of gravitational teleparallelism in 2 + 1 dimensions

We begin by introducing the three basic postulates that the Lagrangian density of the gravitational field in empty space, in the teleparallel geometry, must satisfy: It must be invariant under (i) coordinate transformations, (ii) global \([SO(2,1)]\) Lorentz’s transformations, and (iii) parity transformations. In the present formulation, we add a cosmological constant \(\pm \Lambda = \mp \frac{2}{l^2}\) to the Lagrangian density, where \(\Lambda = -\frac{2}{l^2}\) is a negative cosmological constant. The most general Lagrangian density quadratic in the torsion tensor is written

\[
L_0 = e(c_1 t^{abc} t_{abc} + c_2 v^a v_a + c_3 a_{abc} a^{abc}) ,
\]

(2.1)

where \(c_1, c_2\) and \(c_3\) are constants, \(e = \text{det}(e^a_{\mu})\), and

\[
t_{abc} = \frac{1}{2} (T_{abc} + T_{bac}) + \frac{1}{4} (\eta_{ca} v_b + \eta_{cb} v_a) - \frac{1}{2} \eta_{ab} v_c ,
\]

(2.2)

\[
v_a = T^b_{ba} = T_a ,
\]

(2.3)

\[
a_{abc} = \frac{1}{3} (T_{abc} + T_{cab} + T_{bca}) ,
\]

(2.4)

\[
T_{abc} = e^\mu_{\nu} e^\nu_{\rho} T_{a\mu\nu} .
\]

(2.5)

The definitions given above correspond to the irreducible components of the torsion tensor.\(^9\) With the purpose of obtaining the Hamiltonian formulation, we need to rewrite the three terms of \(L_0\) in such a form that the torsion tensor is factorized. Thus we rewrite \(L_0\) as

\[
L_0 = e \left( c_1 X^{abc} + c_2 Y^{abc} + c_3 Z^{abc} \right) T_{abc} ,
\]

(2.6)

where

\[
X^{abc} = \frac{1}{2} T^{abc} + \frac{1}{4} T^{bac} - \frac{1}{4} T^{cab} + \frac{3}{8} \left( \eta^{ca} v^b - \eta^{cb} v^c \right) ,
\]

(2.7a)

\[
Y^{abc} = \frac{1}{2} \left( \eta^{ab} v^c - \eta^{ac} v^b \right) ,
\]

(2.7b)

\[
Z^{abc} = \frac{1}{3} \left( T^{abc} + T^{bca} + T^{cab} \right) .
\]

(2.7c)

From the definitions above, we have

\[
X^{abc} = -X^{acb} , \quad Y^{abc} = -Y^{acb} , \quad Z^{abc} = -Z^{acb} .
\]

We define the quantity \(\Sigma^{abc}\) according to

\[
\Sigma^{abc} = c_1 X^{abc} + c_2 Y^{abc} + c_3 Z^{abc} ,
\]

which allows us rewrite \(L_0\) as

\[
L_0 = e \Sigma^{abc} T_{abc} .
\]
The Hamiltonian formulation is obtained by writing the Lagrangian density in first-order differential form. We consider the space-time to be empty. In order to carry out the Legendre transform, we carry out a 2 + 1 decomposition of the space-time triads \( 3^e_{\mu} \),11) Such a decomposition projects \( 3^e_{a\mu} \) onto the two-dimensional spacelike hypersurface. The Legendre transform is implemented provided that we eliminate some velocity components.\(^{11} \) We find that the following two conditions are necessary to implement the Legendre transform:

\[
\begin{align*}
c_2 + \frac{3}{4}c_1 &= 0, \\
c_1 &= -\frac{8c_3}{3}.
\end{align*}
\]

With the addition of a cosmological constant, the Lagrangian density can be written

\[
L = P^{(j)\dot{i}} + N^kC_k + NC - \partial_i \left[ N_kP^{ki} + N(3c_1eT^i) \right] + \lambda^{ij}P_{[ij]},
\]

where \( P^{(j)\dot{i}} = \delta L/\delta \dot{e}^{(i)\dot{j}} \), \( P^{ki} = e^{(j)k}P^{(j)\dot{i}} \), \( T^i = g^{ik}T_k = g^{ik}e^{(j)m}T_{(j)mk} \). Therefore, the Hamiltonian density is given by

\[
H(e^{(j)i},P^{(j)i}) = -N^kC_k + NC - \partial_i \left[ N_kP^{ki} + N(3c_1eT^i) \right] - \lambda^{ij}P_{[ij]},
\]

where

\[
C_k = e^{(j)k}\partial_iP^{(j)i} + P^{(j)i}T_{(j)ik}
\]

and

\[
C = \frac{1}{6ec_1} \left( P^{ij}P_{ji} - P^2 \right) + cT^{ij}\Sigma_{ikj} - \partial_k \left[ 3c_1eT^k \right] \pm \frac{2}{l^2},
\]

are the vector and Hamiltonian constraints, respectively. The quantities \( N^k \), \( N \) and \( \lambda^{ij} = -\lambda_{ji} \) are Lagrange multipliers, and \( \pm \frac{2}{l^2} = \mp \Lambda \). The addition of \( \Lambda \) to the Lagrangian density affects only the Hamiltonian constraint \( C \). It is thus seen that the Legendre transform reduces the three-parameter family of theories to a one-parameter family.

A consistent implementation of the Legendre transform is a necessary condition for the Hamiltonian formulation to be possible. However, it is not sufficient. It is also necessary to verify whether the constraints are first class, i.e., if the algebra of constraints “closes”. Such analysis is investigated in Ref. 11). The conclusion reached there is that all constraints of the theory are indeed first class.

\section{Applications of the Hamiltonian formulation in 2 + 1 dimensions}

\subsection{The Newtonian limit}

The field equations for the Lagrangian density (2.1) are obtained in Ref. 9). It has been found that there that these field equations are equivalent to the three-
dimensional Einstein equations if the parameters are fixed according to
\[ c_1 + \frac{2}{3} = 0, \quad (3.1) \]
\[ c_2 - \frac{1}{2} = 0, \quad (3.2) \]
\[ c_3 - \frac{1}{4} = 0. \quad (3.3) \]

We know that the three-dimensional Einstein equations have neither a Newtonian limit nor black hole solutions. In Ref. 9), an attempt is made to obtain the Newtonian limit of the field equations, considering static circularly symmetric fields. For this purpose, the following relations between the parameters \( c_1 \) and \( c_2 \) (in our notation) were determined:
\[ 3c_1 + 4c_2 = -6c_1c_2, \quad (3.4a) \]
\[ c_1c_2 \neq 0. \quad (3.4b) \]

Condition (2.8a), \( c_2 + \frac{3}{4}c_1 = 0 \), obtained in the Hamiltonian formulation violates the condition above. Therefore a Newtonian limit cannot exist in the theory defined by (2.9) and (2.10).

In the investigation of Kawai,\(^9\) a condition on the parameters of the theory that yields a black hole solution was found. This condition is \( c_1(3c_1 + 4c_2) < 0 \), and it corresponds to a particular solution [Eq. (5.15) of Ref. 9)] of the theory. The conditions \( c_2 + \frac{3}{4}c_1 = 0 \) and \( c_1 = -\frac{8c_3}{3} \) for the Legendre transform preclude black hole solutions. However, the possibility that different solutions of Eq. (5.15) in Ref. 9) yield black hole solutions that violate the condition \( c_1(3c_1 + 4c_2) < 0 \) is not excluded.

The Hamiltonian formulation above yields equations that are precisely equivalent to the three-dimensional Einstein equations in vacuum, as can be verified by fixing \( c_1 = -\frac{2}{3} \) in (2.10) and (2.12). However, it is worth investigating whether another value of \( c_1 \) is possible in a gravitational theory, as there does not exist a compelling physical reason for fixing \( c_1 = -\frac{2}{3} \). We address this issue in the next section.

3.2. Black holes in a circularly symmetric geometry

In this section, we investigate the fixing of the free parameter in the framework of an exact vacuum solution. We carry out a symmetry reduction and consider the constraints in a circularly symmetric geometry. We consider the space-time metric
\[ ds^2 = -N^2dt^2 + f^{-2}dr^2 + r^2(N^\phi dt + d\phi)^2, \]
which describes a rotating source, where \( f, N \) and \( N^\phi \) are a priori independent functions of \( r \) and \( t \). The diads for the spatial sector of the metric tensor above are given by
\[ (e_{(k)i}) = \begin{pmatrix} f(r,t)^{-1} & 0 \\ 0 & r \end{pmatrix}, \quad (3.5) \]
where \((k)\) and \(i\) are the row and column indices, respectively.
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The two-dimensional metric $g_{ij}$ reads

$$
(g_{ij}) = \begin{pmatrix}
  f(r,t)^{-2} & 0 \\
  0 & \frac{r^2}{2}
\end{pmatrix},
$$

(3.6)

and $\epsilon = \det (e_{(k)i}) = r f(r,t)^{-1}$.

The nonzero components of the torsion tensor in the spacelike section are $T_{(2)12} = -T_{(2)21} = 1$. With the help of the inverse metric tensor $g^{ij}$, we can write the inverse diads $e_{(k)i}$ as

$$
(e_{(k)i}) = \begin{pmatrix}
  f(r,t) & 0 \\
  0 & \frac{1}{r}
\end{pmatrix}.
$$

It follows that

$$
T^{(2)12} = g^{11} g^{22} T^{(2)12} = \frac{f(r,t)^2}{r^2},
$$

$$
T^{(2)21} = g^{11} g^{22} T^{(2)12} = -\frac{f(r,t)^2}{r^2}.
$$

Next, we calculate the term $\epsilon \Sigma^{kij} T_{kij} = \epsilon \Sigma_{(k)(i)(j)} T^{(k)(i)(j)}$ of the Hamiltonian constraint. It is not difficult to obtain

$$
\epsilon \Sigma_{kij} T^{kij} = \epsilon \Sigma_{(k)ij} T^{(k)ij} = 2 \epsilon \Sigma_{(2)12} T^{(2)12},
$$

where

$$
\Sigma_{abc} = c_1 X_{abc} + c_2 Y_{abc} + c_3 Z_{abc}
$$

and

$$
e^b_{\phantom{b}i} e^c_{\phantom{c}j} \Sigma_{abc} = \Sigma_{aij}.
$$

The latter quantity can be written in terms of the torsion tensor $T_{aij}$ as

$$
\Sigma_{aij} = \left(\frac{c_1}{2} + \frac{c_3}{3}\right) T_{aij} + \left(\frac{c_1}{4} - \frac{c_3}{3}\right) \left(e^{(k)}_{\phantom{k}i} e^a_{\phantom{a}l} T_{(k)lj} - e^{(k)}_{\phantom{k}j} e^a_{\phantom{a}l} T_{(k)li}\right)
$$

$$
+ \left(\frac{3}{16} c_1 - \frac{c_2}{2}\right) (e_{aj} T_i - e_{ai} T_j),
$$

where

$$
T_i = T^b_{\phantom{b}bi} = e^m_{\phantom{m}b} T^b_{\phantom{b}mi} = e_{(k)}^m T^{(k)}_{\phantom{k}mi}.
$$

After a long calculation, we obtain

$$
\epsilon \Sigma_{kij} T^{kij} = \epsilon \Sigma_{(k)ij} T^{(k)ij} = \epsilon \left(c_2 + \frac{3}{4} c_1\right) = 0.
$$

(3.7)

From the momentum $P^{(k)j}$, canonically conjugate to $e_{(k)j}$, it is possible to construct the tensor density $P^{ij} = e_{(k)}^i P^{(k)j}$ of weight +1. We determine the most
general form of the tensor $M_{ij} = e^{-1}P_{ij}$ restricted to a circularly symmetric geometry. For this purpose, we consider the Killing vector

$$\left(\xi_i\right) = (0, b r^2),$$

where $b$ is an arbitrary constant, and require that the Lie derivative of $M_{ij}$ vanish:

$$L_\xi \left(e^{-1}P_{ij}\right) = 0.$$  

We find that the most general matrix for $P_{ij}$ is given by

$$(P_{ij}) = \begin{pmatrix} rf(r, t)^{-1}A(r, t) & P_{12}(r, t) \\ P_{21}(r, t) & rf(r, t)^{-1}B(r, t) \end{pmatrix},$$

where $A(r, t)$ and $B(r, t)$ are arbitrary functions.

We can now obtain the momentum components $P^{(k)}_{ij} = e^{(k)}_i P^{ij}$ in the time gauge. They read

$$P^{(1)1} = rf(r, t)^2A(r, t),$$
$$P^{(2)1} = rP^{21},$$
$$P^{(1)2} = f(r, t)^{-1}P^{12},$$
$$P^{(2)2} = \frac{f(r, t)^{-1}}{r^2}B(r, t).$$

We proceed to obtain the Hamiltonian in terms of the quantities above. The term $P_{ij}P^{ji}$ can be written

$$P_{ij}P^{ji} = r^2f^2A^2 + f^{-2} \frac{1}{r^2}B^2 + 2P^{21}P_{21},$$

where we have considered $P^{21} = P_{12}$. The scalar density $P$ reads

$$P = frA + f^{-1} \frac{1}{r}B.$$  

Therefore,

$$P_{ij}P^{ji} - P^2 = 2P^{21}P_{21},$$

where we have set $B(r, t) = 0$, because there is no momentum canonically conjugate to $g_{\theta\theta}$. We also obtain

$$\partial_k \left(e^{T_k}\right) = -\partial_r f(r, t).$$

The vector constraints are easily calculated as

$$C_1 = C_r = -f^{-1}\partial_r \left(f^2P\right),$$
$$C_2 = C_\theta = -r\partial_r \left(rP^{21}\right) - rP^{21},$$
where $\bar{P} = -f^{-1}P$, and the Hamiltonian constraint is reduced to

$$\bar{C}(r, t) = -\frac{f(r, t)}{6\epsilon c_1} \left[ 2P^{21}(r, t)P_{21}(r, t) \right] - 3c_1 \partial_r f(r, t) \mp \frac{2}{l^2} f(r, t)^{-1}r,$$

(3.11)

where the minus sign in the last term of the above expression reflects a negative cosmological constant. Finally, we note that $\lambda_{ij}P^{[ij]} = 0$, since the matrix $P^{ij}$ is symmetric in the circularly symmetric geometry. We can now construct the action integral. It is given by

$$I = \int dt dr d\theta L = 2\pi \int dt dr \left( \bar{P} \dot{f} - N^i C_i - NC \right).$$

(3.12)

At $t = 0$, the equation $\bar{C}(r, 0) = 0$, that is,

$$r \partial_r \left[ rP^{21}(r, 0) \right] + rP^{21}(r, 0) = 0,$$

(3.13)

has the simple solution

$$P^{21}(r, 0) = \frac{k}{r^2},$$

(3.14)

where $k$ is a constant of integration. We also have

$$P_{21}(r, 0) = g_{22}g_{11}P^{21}(r, 0) = k f(r, 0)^{-2},$$

and

$$P^1_2 = g_{22}P^{21}(r, 0) = k = P^r_\theta,$$

where $P^r_\theta$ is the canonical variable conjugate to the cyclic variable $g_{12} = g_{r\theta}$. This is a conserved quantity, because

$$\dot{P}^{21}(r, t) = -\frac{\delta H}{\delta g_{21}(r, t)} = 0.$$

Therefore at any time $t > 0$, we have

$$P^{21}(r, t) = \frac{k}{r^2}.$$

(3.15)

By substituting (3.15) into the Hamiltonian constraint, we arrive at

$$\bar{C}(r, t) = -\frac{f^{-1}(r, t)}{3c_1 r^3} k^2 - 3c_1 \partial_r f(r, t) \mp \frac{2}{l^2} f(r, t)^{-1}r.$$

(3.16)

Let us now consider the equation $\bar{C}(r, 0) = 0$. This implies

$$-3c_1 \partial_r f(r, 0) - \frac{2}{l^2} f(r, 0)^{-1}r - \frac{f^{-1}(r, 0)}{3c_1 r^3} k^2 = 0, \quad \Lambda < 0,$$

(3.17)

$$-3c_1 \partial_r f(r, 0) + \frac{2}{l^2} f(r, 0)^{-1}r - \frac{f^{-1}(r, 0)}{3c_1 r^3} k^2 = 0, \quad \Lambda > 0,$$

(3.18)
where $\Lambda = \pm \frac{2}{l^2}$. The solutions of these equations are

\[ f(r,0)^2 = m - \frac{2}{3c_1} \frac{r^2}{l^2} + \frac{1}{9c_1^2} \frac{k^2}{r^2}, \quad \Lambda < 0, \]

\[ f(r,0)^2 = m + \frac{2}{3c_1} \frac{r^2}{l^2} + \frac{1}{9c_1^2} \frac{k^2}{r^2}, \quad \Lambda > 0, \]

where $m$ is a constant of integration. It is not difficult to show that these solutions are independent of time. The time evolution of $f(r,t)$ is given by

\[ \dot{f}(r,t) = \frac{\delta H}{\delta \bar{P}(r,t)} = 0, \]

where we have fixed the coordinate system such that $N_1(r,t) = 0$. Therefore, at any later time $t$ we have

\[ f(r)^2 = m - \frac{2}{3c_1} \frac{r^2}{l^2} + \frac{1}{9c_1^2} \frac{k^2}{r^2}, \quad \Lambda < 0, \quad (3.19)\]

\[ f(r)^2 = m + \frac{2}{3c_1} \frac{r^2}{l^2} + \frac{1}{9c_1^2} \frac{k^2}{r^2}, \quad \Lambda > 0. \quad (3.20)\]

If $c_1 = -\frac{2}{3}$, (3.19) becomes

\[ f(r)^2 = m + \frac{r^2}{l^2} + \frac{k^2}{4r^2}, \quad \Lambda < 0. \quad (3.21)\]

Identifying $m$ with $-M$ and $k$ with $J$, this solution becomes what is known in the literature as the rotational black hole solution in the three-dimensional space-time of Banados, Teitelboim and Zanelli (BTZ),\(^{14}\) with negative cosmological constant. In Ref. 14), Banados et al. used a time independent metric. At this point, we likewise restrict our consideration to a static field configuration. Since the space-like sector of the space-time triads given by (3.5) is time independent, the restriction to a static geometry amounts to further requiring the lapse and shift functions to be time independent. (Note that we have already required $N^1 = 0$.) If the space-time geometry is static, then we must have

\[ \dot{\bar{P}} = -\frac{\delta H}{\delta f(r)} = 0. \quad (3.22)\]

Consequently, we have

\[ -3c_1 \partial_r N(r) - \frac{2}{l^2} f(r)^{-2} r N(r) - \frac{f(r)^{-2}}{3c_1 r^3} N(r) k^2 = 0, \quad (3.23)\]

for a negative cosmological constant. Making use of (3.19), we arrive at

\[ -3c_1 \partial_r N(r) - \frac{1}{\left(m - \frac{2}{3c_1} \frac{r^2}{l^2} + \frac{1}{9c_1^2} \frac{k^2}{r^2}\right)} \left(\frac{2}{l^2} r + \frac{k^2}{3c_1 r^3}\right) N(r) = 0, \]
from which we obtain a solution for $N(r)$,

$$N(r) = K \sqrt{\left( m - \frac{2}{3 c_1} r^2 + \frac{1}{9 c_1^2} \frac{k^2}{r^2} \right)} = f(r),$$

(3.24)

where $K$ is an arbitrary constant. This result is identical to the corresponding result of Ref. 14), provided that we set $K = 1$ and $c_1 = -\frac{2}{3}$ and identify $k$ with $J$. We require the constant $K$ to satisfy $K = 1$, which amounts to a redefinition of the time scale of the theory. (This redefinition also implies $K = 1$ in (3.25) and (3.26) below.)

For a positive cosmological constant, we have

$$N(r) = \sqrt{\left( m + \frac{2}{3 c_1} \frac{r^2}{l^2} + \frac{1}{9 c_1^2} \frac{k^2}{r^2} \right)}.$$

(3.25)

We now investigate the time evolution of $g_{21}$. Substituting the quantity $P_{21} = r^2 f^{-2} P^{21}$ into (3.11), we find

$$C(r) = -3 c_1 \partial_r f(r) \mp \frac{2}{l^2} f(r)^{-1} r - \frac{rf(r)^{-1}}{3 c_1} \left[ P^{21}(r) \right]^2,$$

from which we obtain

$$\dot{g}_{21}(r, t) = \frac{\delta H}{\delta P^{21}(r)} = \frac{\delta}{\delta P^{21}(r)} \int dr' \left[ N^\theta(r') \left( -r' \partial_{r'} \left[ r' P^{21}(r') \right] ight) 
- r' P^{21}(r') + N(r') \left( -3 c_1 \partial_{r'} f(r') \right) \right.
\left. \pm \frac{2}{l^2} f(r')^{-1} r' - \frac{rf(r')^{-1}}{3 c_1} \left[ P^{21}(r') \right]^2 \right].$$

Taking into account the facts that the solution $P^{21} = \frac{k}{r^2}$ is time independent and that $\dot{g}_{21}(r, t) = 0$, we arrive at

$$-\partial_r N^\theta(r) r^3 - \frac{2}{3 c_1} k = 0,$$

whose solution is given by

$$N^\theta(r) = \frac{1}{3 c_1 r^2} k.$$

(3.26)

We thus again obtain a result that is identical with the corresponding result in Ref. 14), after setting $c_1 = -\frac{2}{3}$ and identifying $k$ with $J$.

The event horizon of a black hole is determined by a surface of constant radius $r = r_H$. The latter follows from the condition

$$g_{00} = 0 = N(r) = f(r),$$
which implies
\[ r^4 - \frac{3}{2}c_1 t^2 mr^2 - \frac{1}{6c_1} t^2 k^2 = 0. \]

The solution is given by
\[ r_{\pm} = l \sqrt{\frac{3}{4} c_1 m} \left[ 1 \pm \left( 1 + \frac{8}{27} \frac{1}{c_1^2} \left( \frac{k}{ml} \right)^2 \right)^{1/2} \right]^{1/2}. \] (3.27)

Setting \( c_1 = -\frac{2}{3} \), we obtain
\[ r_{\pm} = l \sqrt{-\frac{m}{2}} \left[ 1 \pm \left( 1 - \left( \frac{k}{ml} \right)^2 \right)^{1/2} \right]^{1/2}, \] (3.28)
which coincide with the radii of the event horizons of the BTZ black hole. The event horizons \( r_{\pm} \) require a further condition, namely, that \( m < 0 \), or \( m = -M \), where \( M > 0 \) is the mass of the black hole. Real solutions are obtained by requiring
\[ |k| \leq |ml|, \] (3.29)
\[ |J| \leq |Ml|. \] (3.30)

Therefore we have
\[ M > 0 \implies \text{black hole solution}, \]
\[ M = 0, \ J = 0 \implies \text{no black hole}, \]
\[ M = -1, \ J = 0 \implies \text{anti-de Sitter solution}. \]

\section*{§4. Gravitational energy in 2 + 1 dimensions}

A natural application of the present formalism is the calculation of the gravitational energy of black holes in 2 + 1 dimensions. The Hamiltonian constraint (2.12) contains a total divergence. In analogy to the 3 + 1 case, the total divergence given by \(-\partial_k (3c_1 eT^k)\) (apart from a multiplicative constant) is interpreted as the gravitational energy density, and the integral form of the Hamiltonian constraint is likewise interpreted as an energy equation for the gravitational field. Therefore, the gravitational energy is defined as a surface integral,
\[ E = -\frac{1}{16\pi G} \oint_C dS_k (3c_1 eT^k) - E_0, \] (4.1)
where \( C \) is, in the present case, a one-dimensional closed contour, and \( E_0 \) is a suitable reference energy value.

By requiring \( C \), defined by the condition \( r = r_0 \), to be a closed contour of a surface \( S \), we find that in the present case the energy contained within the interior
of \( S \) is given by

\[
E = -\frac{1}{16\pi G} \int_C d\theta (3c_1 e T^1) - E_0
\]

\[
= \frac{3c_1}{2\pi} \int_C d\theta f(r) - E_0
\]

\[
= \frac{3c_1}{2\pi} \int_C d\theta \sqrt{\left( m - \frac{2}{3c_1} \frac{r_0^2}{l^2} + \frac{1}{9c_1^2} \frac{k^2}{r_0^2} \right)} - E_0 ,
\]

where we have set \( G = \frac{1}{8} \). We remark that in previous analyses of the gravitational energy of black holes,\(^6,7\) the latter expressions were evaluated by means of a surface integral. This approach circumvents the problem of dealing with imaginary field quantities. Moreover, we note that the coordinate system employed above is not well defined inside the event horizon. (A similar situation exists in the Schwarzschild and Kerr space-times, described by the usual spherical coordinates.\(^6,7\))

We now consider the metric corresponding to a rotational black hole with \( \Lambda < 0 \) and compare it with the result appearing in the literature. For this purpose, we fix the value of \( c_1 \) as \( c_1 = -\frac{2}{3} \). The energy contained within the circle of constant radius \( r_0 \) is given by

\[
E = -\frac{1}{\pi} \int_0^{2\pi} d\theta f(r) - E_0 = -2 \left( \sqrt{-M + \frac{r_0^2}{l^2} + \frac{1}{4} \frac{J^2}{r_0^2}} - \frac{r_0}{l} \right) ,
\]

where we have set \( E_0 = -2(r_0/l) \), \( m = -M \) and \( k = J \). We follow the interpretation of Bambiados et al., according to which the vacuum state is obtained by requiring the mass of the black hole to vanish, which, by (3.30), implies the vanishing of the angular momentum, \( J = 0 \). Therefore, for the vacuum energy, we have \( E = 0 \). Note that the black hole energy also vanishes if we extend the integration to the entire bi-dimensional space. The energy value \( E = 0 \) corresponds to a space-time whose line element is given by

\[
ds_{\text{vac}}^2 = -\left( \frac{r}{T} \right)^2 dt^2 + \left( \frac{r}{T} \right)^{-2} dr^2 + r^2 d\theta^2 .
\]

Let us now show that we can obtain the above result using different diads. We consider the diads

\[
e^{(k)i} = \begin{pmatrix}
\alpha \cos \theta & -r \sin \theta \\
\alpha \sin \theta & r \cos \theta
\end{pmatrix},
\]

where \( f(r)^{-1} = \alpha = \left( -M + \frac{r^2}{l^2} + \frac{1}{4} \frac{J^2}{r_0^2} \right)^{-1/2} \).

The bi-dimensional metric tensor \( g_{ij} \) has the nonvanishing components \( g_{11} = \alpha^2 \), \( g_{22} = r^2 \), and \( e = \det (e_{(k)i}) = r\alpha \). The nonzero components of torsion tensor are

\[
T_{(1)12} = (\alpha - 1) \sin \theta ,
\]

\[
T_{(2)12} = -(\alpha - 1) \cos \theta .
\]
Note that these components vanish if we set $\alpha = 1$. After some computations, we obtain

$$eT^1 = (1 - \alpha^{-1}), \quad eT^2 = 0.$$  

Hence

$$\partial_i (eT^i) = \partial_r (1 - \alpha^{-1}). \quad (4.6)$$

Again requiring $c_1 = -\frac{2}{3}$ and setting $G = 1/8$, we find

$$E = -\frac{1}{16\pi G} \int_M dS_k \left( 3c_1 eT^k \right) - E_0 = \frac{1}{\pi} \int_0^{2\pi} d\theta \left( 1 - \alpha^{-1} \right) - E_0$$

$$= 2 \left( 1 - \alpha^{-1} \right) - E_0 = -2 \left[ \left( -M + \frac{r_0^2}{l^2} + \frac{1}{4} \frac{J^2}{r_0^2} \right)^{1/2} - \frac{r_0}{l} \right], \quad (4.7)$$

where $E_0 = 2(1 - r_0/l)$ has been adjusted to obtain $E = 0$ when setting $M = J = 0$.

The value above for the gravitational energy of the BTZ black hole is identical to the value obtained by means of the Brown-York method, according to Refs. 16) and 17),

$$E_{BY} = -2 \left( \sqrt{-M + \frac{r_0^2}{l^2} + \frac{1}{4} \frac{J^2}{r_0^2} - \varepsilon_0} \right), \quad (4.8)$$

where the reference energy $\varepsilon_0$ is, in principle, arbitrarily chosen. In Refs. 16) and 17), the reference energy is taken to be the energy of the Anti-de Sitter space-time,

$$\varepsilon_0 = \sqrt{1 + \frac{r_0^2}{l^2}},$$

for which $M = -1$ and $J = 0$. In this case, the vacuum line element is taken to be

$$ds^2_{\text{vac}} = - \left[ 1 + \left( \frac{r}{l} \right)^2 \right] dt^2 + \left[ 1 + \left( \frac{r}{l} \right)^2 \right]^{-1} dr^2 + r^2 d\theta^2.$$

In Ref. 16), a reference energy that corresponds to the absence of black holes, $M = J = 0$, is also adopted:

$$\varepsilon_0 = \frac{r_0}{l}.$$  

It is possible to obtain the reference energies above in our expression (4.7), because we can conveniently adjust the constant of integration.

It is observed that we can establish an alternative possibility for the reference energy by taking the reference space-time to be the Minkowsky space-time. In this case, we must have $M = -1$, $J = 0$ and $1/l^2 \to 0$, or setting $\alpha = 1$,

$$E_0 = -2.$$
With the values given above for $M, J$ and $1/l^2$, the line element reduces to

$$ds^2_{\text{vac}} = -dt^2 + dr^2 + r^2d\theta^2.$$ 

Therefore we conclude that the energy expressions (4.3) and (4.7) obtained in the context of the 2 + 1 teleparallel formulation represent consistent results that were obtained previously in the literature.

§5. Conclusions

In this work we have investigated the existence of black holes in 2 + 1 teleparallel theories of gravity. The analysis has been carried out in the context of the Hamiltonian formulation of arbitrary, quadratic teleparallel theories with negative cosmological constants. We considered gravitational fields with circularly symmetric geometries and obtained a black hole solution of the Hamiltonian field equations that generalizes the BTZ black hole. The latter is obtained by fixing the free parameter of the theory. We have also investigated the gravitational energy of this solution by means of a simple expression first analyzed in the context of 3+1 teleparallel theories. Energy expressions were obtained in a straightforward way, and they are found to be consistent with the results obtained previously by means of the Brown-York method. We observed that there are various possibilities for fixing the reference space-time. The simplicity of the mathematical structure of the 2 + 1 dimensional formulation motivates us to investigate the concept of gravitational angular momentum in this framework. This issue is currently being studied.

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