REMARKS ON THE KODAIRA DIMENSION OF BASE SPACES OF
FAMILIES OF MANIFOLDS

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Abstract. We prove that the variation in a smooth projective family of varieties admitting a good minimal model forms a lower bound for the Kodaira dimension of the base, if the dimension of the base is at most five and its Kodaira dimension is non-negative. This gives an affirmative answer to the conjecture of Kebekus and Kovács for base spaces of dimension at most five.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. A conjecture of Shafarevich [Sha63] and Viehweg [Vie01] predicted that if the geometric general fiber of a smooth projective family \( f_U : U \to V \) is canonically polarized, then \( \kappa(V) = \dim(V) \), assuming that \( f_U \) has maximal variation. The problem was subsequently generalized to the case of fibers with good minimal models, cf. [VZ01] and [PS17]. This conjecture is usually referred to as the Viehweg Hyperbolicity Conjecture and has been recently settled in full generality.

Generalizing Viehweg’s conjecture, in their groundbreaking series of papers, Kebekus and Kovács predicted that variation in the smooth family \( f_U \), which we denote by \( \text{Var}(f_U) \), should be closely connected to \( \kappa(V) \), even when it is not maximal.

Conjecture 1.1 (Kebekus-Kovács Hyperbolicity Conjecture. I, cf. [KK08, Conj. 1.6]). Let \( f_U : U \to V \) be a smooth projective family whose general fiber admits a good minimal model \(^1\). Then, either

\[
1.1.1 \quad \kappa(V) = -\infty \quad \text{and} \quad \text{Var}(f_U) < \dim V, \quad \text{or}
\]
\[
1.1.2 \quad \kappa(V) \geq 0 \quad \text{and} \quad \text{Var}(f_U) \leq \kappa(V).
\]

Once the family \( f_U \) arises from a moduli functor with an algebraic coarse moduli scheme, then an even stronger version of Conjecture 1.1, that is due to Campana, can be verified (see Section 5). However, the main focus of this paper is to study Conjecture 1.1, when the family \( f_U \) is not associated with a well-behaved moduli functor, even after running a relative minimal model program. Most families of higher dimensional projective manifolds that are not of general type but have pseudo-effective canonical bundle belong to this category.

The following theorem is the main result of this paper.

Theorem 1.2. Conjecture 1.1 holds when \( \dim(V) \leq 5 \).

\(^1\)The original conjecture of Kebekus and Kovács was formulated for the case of canonically polarized fibers.
When dimension of the base and fibers are equal to one, Viehweg’s hyperbolicity conjecture (or equivalently Conjecture 1.1) was proved by Parshin [Par68], in the compact case, and in general by Arakelov [Ara71]. For higher dimensional fibers and assuming that \( \text{dim}(V) = 1 \), this conjecture was confirmed by Kovács [Kov00], in the canonically polarized case (see also [Kov02]), and by Viehweg and Zuo [VZ01] in general. Over Abelian varieties Viehweg’s conjecture was solved by Kovács [Kov97]. When \( \text{dim}(V) = 2 \) or \( 3 \), it was resolved by Kebekus and Kovács, cf. [KK08] and [KK10]. In the compact case it was settled by Patakfalvi [Pat12]. Viehweg’s conjecture was finally solved in complete generality by the fundamental work of Campana and Păun [CP15] and more recently by Popa and Schnell [PS17]. For the more analytic counterparts of these results please see [VZ03], [Sch12], [TY15], [BPW17], [TY16], [PTW18] and [Den18].

By using the solution of Viehweg’s hyperbolicity conjecture, one can reformulate Conjecture 1.1 as follows.

**Conjecture 1.3** (Kebekus-Kovács Hyperbolicity Conjecture. II). Let \( f_U : U \to V \) be a smooth family of projective varieties and \((X, D)\) a smooth compactification of \( V \) with \( V \cong X \smallsetminus D \). Assume that the geometric general fiber of \( f_U \) has a good minimal model. Then, the inequality

\[
\text{Var}(f_U) \leq \kappa(X, D)
\]

holds, if \( \kappa(X, D) \geq 0 \).

When \( f_U \) is canonically polarized, Conjecture 1.3 was confirmed by Kebekus and Kovács in [KK08], assuming that \( \text{dim}(X) = 2 \), in [KK10], if \( \text{dim}(X) = 3 \), and as a consequence of [Taj16] in full generality. The latter result establishes an independent, but closely related, conjecture of Campana; the so-called Isotriviality Conjecture.

Campana’s conjecture (Conjecture 5.1) predicts that once the fibers of \( f_U \) are canonically polarized (or more generally have semi-ample canonical bundle, assuming that \( f_U \) is polarized with a fixed Hilbert polynomial), then \( f_U \) is isotrivial, if \( V \) is special. We refer to Section 5 for more details on the notion of special varieties and various other particular cases where the Isotriviality Conjecture and Conjecture 1.3 can be confirmed.

### 1.2. Brief review of the strategy of the proof

A key tool in proving Conjecture 1.3, in the canonically polarized case, is the celebrated result of Viehweg and Zuo [VZ01, Thm. 1.4.(i)] on the existence of an invertible subsheaf \( \mathcal{L} \subseteq \Omega_X^{\otimes i} \log(D) \), for some \( i \in \mathbb{N} \), whose Kodaira dimension verifies the inequality:

\[
\kappa(X, \mathcal{L}) \geq \text{Var}(f_U).
\]

The sheaf \( \mathcal{L} \) is usually referred to as a Viehweg-Zuo subsheaf. In general, once the canonically polarized condition is dropped, in the absence of a well-behaved moduli functor associated to the family, the approach of [VZ01] cannot be directly applied. Nevertheless, we show that one can still construct a subsheaf of \( \Omega_X^{\otimes i} \log(D) \) arising from the variation in \( f_U \), as long as the geometric general fiber of \( f_U \) has a good minimal model. But in this more general context the sheaf \( \mathcal{L} \) injects into \( \Omega_X^{\otimes i} \log(D) \) only after it is twisted by some pseudo-effective line bundle \( \mathcal{B} \). As such we can no longer guarantee that this (twisted) subsheaf verifies the inequality (1.3.1).
**Theorem 1.4** (Existence of pseudo-effective Viehweg-Zuo subsheaves). Let \( f_U : U \to V \) be a smooth non-isotrivial family whose general fiber admits a good minimal model. Let \((X, D)\) a smooth compactification of \( V \). There exist a positive integer \( i \), a line bundle \( L \) on \( X \), with \( \kappa(X, L) \geq \text{Var}(f_U) \), and a pseudo-effective line bundle \( B \) with an inclusion \( L \otimes B \subseteq \Omega^i_X \log(D) \).

To mark the difference between the two cases we refer to these newly constructed sheaves as pseudo-effective Viehweg-Zuo subsheaves.

For the proof of Theorem 1.4 we follow the general strategy of \( [VZ02] \) and more recently \( [PS17] \), where one combines the positivity properties of direct images of relative dualizing sheaves with the negativity of curvature of Hodge metrics along certain subsheaves of variation of Hodge structures to construct subsheaves of \( \Omega^i_X \log(D) \) that are sensitive to the variation in the birational structure of the members of the family. We note that, in our constructions, we do not make use of the theory of Hodge modules. In particular Saito’s decomposition theorem will not be used.

It is worth pointing out that when \( \dim(X) = \text{Var}(f_U) \), \( L \) in Theorem 1.4 is big and we have \( \kappa(X, L \otimes B) = \dim(X) \). Thus, in this case, Theorem 1.4 coincides with the result of \( [PS17, \text{Thm. B}] \) and \( [VZ02, \text{Thm. 1.4.(i)}] \) (the latter in the canonically polarized case), while providing a simpler proof.

When variation is not maximal, Theorem 1.4 is notably different from—and in some sense weaker than—the theorem of Viehweg and Zuo \( [VZ02, \text{Thm. 1.4.(i)}] \) in the canonically polarized case. The reason for this difference is due to the fact that without a reasonable functor associated to the family, the existence of Viehweg-Zuo subsheaves can no longer be extracted from their construction at the level of moduli stacks, where the variation is maximal. We refer the reader to Section 5 for more details and a brief review of the cases where this difficulty can be overcome.

Once Theorem 1.4 is established, it remains to trace a connection between \( \kappa(L) \) and \( \kappa(X, D) \). In the maximal variation case, Viehweg-Zuo subsheaves are guaranteed to be big (as soon as the geometric general fiber has a good minimal model). In this case a key result of Campana and Păun then implies that \( \kappa(X, D) = \dim(X) \), cf. \( [CP15, \text{Thm 7.11}] \). But when \( \text{Var}(f_U) < \dim(X) \), as \( L \) is not big, this strategy can no longer be applied. Instead, our proof of Theorem 1.2 relies on the following vanishing result.

**Theorem 1.5** (Vanishing for twisted logarithmic pluri-differential forms). Let \((X, D)\) be a pair consisting of a smooth projective variety \( X \) of dimension at most equal to 5 and a reduced effective divisor \( D \) with simple normal crossing support. If \( \kappa(X, D) \geq 0 \), then the equality

\[
H^0\left(X, \Omega^i_X \log(D) \otimes (L \otimes B)^{-1}\right) = 0
\]

holds, assuming that \( B \) is pseudo-effective and \( \kappa(L) > \kappa(X, D) \).

Theorem 1.2 is now an immediate consequence of Theorems 1.5 and 1.4.

In the presence of a flat Kähler-Einstein metric, or assuming that the main conjectures of the minimal model program hold, various analogues of Theorem 1.5 can be verified. When \( D = 0 \) and \( c_1(X) = 0 \), and \( B \cong \mathcal{O}_X \), Theorem 1.5 follows, in all dimensions, directly from Yau’s solution to Calabi’s conjecture \( [Yau77] \) and the Bochner formula. When \( B \cong \mathcal{O}_X \) and \( D = 0 \), the vanishing in Theorem 1.5 was conjectured by Campana in \( [Cam95] \) where he proved that it holds for an \( n \)-dimensional variety \( X \), if the Abundance Conjecture holds
in dimension $n$. As was shown by Campana, such vanishing results are closely related to compactness properties of the universal cover of algebraic varieties. We refer to [Cam95] for details of this very interesting subject (see also the book of Kollár [Kol95] and [Kol93]). We also invite the reader to consult [CP15, Thm. 7.3] where the authors successfully deal with a similar problem with $\kappa$ replaced by another invariant $\nu$; the latter being the numerical Kodaira dimension.

1.3. **Structure of the paper.** In Section 2 we provide the preliminary constructions needed for the proof of Theorem 1.4. The proof of Theorem 1.4 appears in Section 3. In Section 4 we prove the vanishing result; Theorem 1.5. Section 5 is devoted to further results and related problems, including a discussion on the connection between Conjecture 1.3 and a conjecture of Campana.

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2. Preliminary results and constructions

Our aim in this section is to establish two key background results that we will need in order to construct the pseudo-effective Viehweg-Zuo subsheaves in the proceeding section.

2.1. Positivity of direct images of relative dualizing sheaves. In [Kaw85, Thm 1.1] Kawamata shows that, assuming that the general fiber admits a good minimal model, for any algebraic fiber space $f : Y \to X$ of smooth projective varieties $Y$ and $X$, the inequality

$$\kappa\left( X, \left( \det(f_*\omega^m_{Y/X}) \right)^{**} \right) \geq \text{Var}(f),$$

holds, for all sufficiently large integers $m \geq 1$.

One can use (2.0.1) to extract positivity results for $f_*\omega^m_{Y/X}$. We refer the reader to [VZ03], [VZ02] and [PS17] for the case where $\text{Var}(f) = \text{dim}(X)$ (see also [Kaw85] and [Vie83]). The key ingredient is the fiber product trick of Viehweg, where, given $f : Y \to X$ as above, one considers the $r$-fold fiber product

$$Y^r := \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{r \text{ times}}.$$

We denote by $Y^{(r)}$ a desingularization of $Y^r$ and the resulting morphism by $f^{(r)} : Y^{(r)} \to X$. The next proposition is the extension of the arguments of [PS17, pp. 708–709] to the case where variation is not maximal.

**Proposition 2.1.** Let $f : Y \to X$ be a fiber space of smooth projective varieties $Y$ and $X$. If the general geometric fiber admits a good minimal model, then, for every sufficiently large $m \geq 1$, there exists $r := r(m) \in \mathbb{N}$ and a line bundle $\mathcal{L}$ on $X$, with $\kappa(X, \mathcal{L}) \geq \text{Var}(f)$, and an inclusion

$$\mathcal{L}^m \subseteq f_*^{(r)}\left( \omega^m_{Y^{(r)}/X} \right)$$

which holds in codimension one.
Proof. According to [Vie83, Prop. 6.1] there is a be a finite, flat and Galois morphism \( \gamma : X_1 \to X \), with \( G := \text{Gal}(X_1/X) \), such that the induced morphism \( f_1 : Y_1 \to X_1 \) from the \( G \)-equivariant resolution \( Y_1 \) of the fiber product \( Y \times X_1, X \) is semistable in codimension one:

\[
\begin{array}{ccc}
Y_1 & \longrightarrow & Y \times X_1, X \\
\gamma \downarrow & & \downarrow \gamma \\
X_1 & \longrightarrow & X.
\end{array}
\]

By [Vie83, Sect. 3] (see also [Mor87, Thm 4.10]), for any \( m \in \mathbb{N} \), there is an inclusion

\[
(f_1)_* \omega_{Y_1/X_1}^m \subseteq \gamma^*(f_1)_* \omega_{Y/X}^m
\]

and thus \( \det((f_1)_* \omega_{Y_1/X_1}^m) \subseteq \det(\gamma^*(f_1)_* \omega_{Y/X}^m)) \). Let us define

\[
\mathcal{B}_1 := \det((f_1)_* \omega_{Y_1/X_1}^m) \quad \text{and} \quad \mathcal{B} := \det(\gamma^*(f_1)_* \omega_{Y/X}^m)).
\]

We can allow ourselves to remove codimension two subsets from \( X \). In particular we may assume that \( \mathcal{B}_1 \) and \( \mathcal{B} \) are locally free and that

\[
(2.1.2) \quad \mathcal{B}_1 \subseteq \gamma^*(\mathcal{B}).
\]

Next, we observe that, as \( \Omega^1_{X_1} \) and \( \Omega^1_{Y_1} \) are naturally equipped with the structure of \( G \)-sheaves (or linearized sheaves), \( \omega_{Y_1/X_1}^m \) is also a \( G \)-sheaf. It follows that \( (f_1)_* \omega_{Y_1/X_1}^m \) is a \( G \)-sheaf. The inclusion (2.1.2) then implies that \( \mathcal{B}_1 \) descends, that is there is a line bundle \( \mathcal{L} \) on \( X \) such that \( \mathcal{B}_1 \cong \gamma^*(\mathcal{L}) \), cf. [HL10, Thm. 4.2.15]. Moreover, as \( \gamma \) is Galois, it follows that \( \kappa(X_1, \mathcal{B}_1) = \kappa(X, \mathcal{L}) \). Note that we also have \( \kappa(\mathcal{B}_1) \geq \text{Var}(f) \), again by using [Kaw85, Thm 1.1].

Our aim is now to show that \( \mathcal{L} \) is the line bundle admitting the inclusion (2.1.1). To this end, let \( t := \text{rank}((f_1)_* \omega_{Y_1/X_1}^m) \) and consider the injection

\[
\mathcal{B}_1^m \hookrightarrow (f_1)_* \omega_{Y_1/X_1}^m.
\]

Set \( r := (t+m) \). Let \( Y_1^{(r)} \) be a desingularization of the \( r \)-fold fiber product \( Y_1 \times X_1 \ldots \times X_1 \), such that the resulting morphism from \( Y_1^{(r)} \) to \( Y^r \) factors through the desingularization map \( Y^{(r)} \to Y^r \). Let \( f_1^{(r)} : Y_1^{(r)} \to X_1 \) be the induced map. Using the fact that \( f_1 \) is semistable in codimension one (and remembering that we are arguing in codimension one), thanks to [Vie83, Lem. 3.5], we have

\[
(f_1^{(r)})_* (\omega_{Y_1^{(r)}/X_1}^m) = \bigotimes (f_1)_* (\omega_{Y_1/X_1}^m).
\]

On the other hand, since \( \gamma \) is flat, according to [Vie83, Lem. 3.2], we have

\[
(f_1^{(r)})_* (\omega_{Y_1^{(r)}/X_1}^m) \subseteq \gamma^*(f_1^{(r)})_*(\omega_{Y^{(r)}/X}^m),
\]

where \( Y^{(r)} \) is a desingularization of \( Y^r \) with the induced map \( f^{(r)} : Y^{(r)} \to X \). Therefore, there is an injection
\[ \mathcal{F}_m^m = \gamma^*(\Omega^m) \longrightarrow \gamma^*\left(f^*(r)\omega_{Y/r/X}^m\right). \]

By applying the \( G \)-invariant section functor \( \gamma^* \left( \cdot \right)^G \) to both sides we find the required injection

\[ \mathcal{L}_m^m \hookrightarrow f^*(r)\omega_{Y/r/X}^m. \]

\[ \square \]

2.2. Hodge theoretic constructions. Let \( f : Y \to X \) be a surjective, projective morphism of smooth quasi-projective varieties \( Y \) and \( X \) of relative dimension \( n \) and let \( D = \text{disc}(f) \) be the discriminant locus of \( f \). Set \( \Delta := f^*(D) \) and assume that the support of \( D \) and \( \Delta \) is simple normal crossing.

**Definition 2.2 (Systems with \( \mathcal{W} \)-valued operators).** Let \( \mathcal{W} \) be a coherent sheaf on \( X \). We call the graded torsion free sheaf \( F = \bigoplus F^\bullet \) a system with \( \mathcal{W} \)-valued operator, if it can be equipped with a sheaf morphism \( \tau : F \to F \otimes \mathcal{W} \) satisfying the Griffiths transversality condition \( \tau|_{F^\bullet} : F^\bullet \to \mathcal{F}^\bullet_{r} \otimes \mathcal{W} \).

Throughout the rest of this section we will allow ourselves to discard closed subsets of \( X \) of codimension \( \geq 2 \), whenever necessary.

Our goal is to construct a system \((F, \tau)\) with \( \Omega^1_X \log(D) \)-operator \( \tau \) which can be equipped with compatible maps to a system of Hodge bundles underlying the variation of Hodge structures of a second family that arises from \( f \) via certain covering constructions.

To this end, let \( \mathcal{M} \) be a line bundle on \( Y \) with \( H^0(Y, \mathcal{M}^m) \neq 0 \). Let \( \psi : Z \to Y \) be a desingularization of the finite cyclic covering associated to taking roots of a non-zero section \( s \in H^0(Y, \mathcal{M}^m) \) so that

\[ (2.2.1) \quad H^0(Z, \psi^*\mathcal{M}) \neq 0, \]

cf. [Laz04, Prop. 4.1.6].

Now, consider the exact sequence of relative differential forms

\[ (2.2.2) \quad 0 \longrightarrow f^*(\Omega^1_X \log(D)) \longrightarrow \Omega^1_Y \log(\Delta) \longrightarrow \Omega^1_{Y/X} \log(\Delta) \longrightarrow 0, \]

which is locally free over \( X \) (after removing a codimension two subset of \( X \)). Define \( h := f \circ \psi \). Using the pullback of \( (2.2.2) \) the bundle \( \psi^*(\Omega^i_X \log(\Delta)) \) can be filtered by a decreasing filtration \( F^\bullet_i, i \geq 0 \), with \( F^\bullet_i/F^\bullet_{i+1} \cong \psi^*(\Omega^i_{Y/X} \log(\Delta)) \otimes h^*(\Omega^1_X \log(D)) \). After taking the quotient by \( F^\bullet_2 \), the exact sequence corresponding to \( F^\bullet_0/F^\bullet_1 \cong \psi^*(\Omega^i_{Y/X} \log(\Delta)) \)

reads as

\[ (2.2.3) \quad 0 \longrightarrow \psi^*(\Omega^{i-1}_{Y/X} \log(\Delta)) \otimes h^*(\Omega^1_X \log(D)) \longrightarrow F^\bullet_0/F^\bullet_2 \longrightarrow \psi^*(\Omega^i_{Y/X} \log(\Delta)) \longrightarrow 0, \]

where \( \psi^*(\Omega^{i-1}_{Y/X} \log(\Delta)) \otimes h^*(\Omega^1_X \log(D)) \cong F^\bullet_1/F^\bullet_2 \) and

\[ (2.2.4) \quad F^\bullet_0/F^\bullet_2 \cong \psi^*(\Omega^i_{Y/X} \log(\Delta))/(h^*\Omega_X^2 \log(D) \otimes \psi^*\Omega^i_{Y/X} \log(\Delta)). \]
We tensor the sequence (2.2.3) by $\psi^*(\mathcal{M}^{-1})$ to get the exact sequence

$$A_* : 0 \to \psi^*(\Omega^-_{Y/X} \otimes \mathcal{M}^{-1}) \otimes h^*(\Omega^1_X \log(D)) \to (F^0_0/F^0_2) \otimes \psi^*(\mathcal{M}^{-1}) \to \psi^*(\Omega^1_{Y/X} \log(\Delta) \otimes \mathcal{M}^{-1}) \to 0.$$ 

Now, define a $\Omega^1_X \log(D)$-valued system $(\mathcal{F} = \bigoplus \mathcal{F}_*, \tau)$ of weight $n$ on $X$ by

$$(2.2.5) \quad \mathcal{F}_* := R^* h_* (\psi^*(\Omega^{n-\bullet}_{Y/X} \log(\Delta) \otimes \mathcal{M}^{-1}))/\text{torsion},$$

whose map $\tau$ is determined by $\tau|_{\mathcal{F}_*} : \mathcal{F}_{n+1} \otimes \Omega^1_X \log(D)$ being equal to the corresponding morphism in the relative log complex $R^* h_*(A_{n-\bullet}).$

Next, let $\text{disc}(h) = (D + S)$. By removing a subset of $X$ of codim $X \geq 2$ we may assume that $(D + S)$ has simple normal crossing support. We further assume that, after replacing $Z$ by a higher birational model, if necessary, the divisor $h^*(D + S)$ has simple normal crossing support, which we denote by $\Delta'$.

Similar to the above construction, we can consider the exact sequence

$$(2.2.6) \quad G^*_{0}/G^*_{2} \cong \Omega^*_{Z/X} \log(\Delta')/(h^* \Omega^2_{X} \log(D + S) \otimes \Omega^*_{Z/X} \log(\Delta')).$$

Now, let $(\mathcal{E}, \theta)$ with $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X \log(D + S)$ be the logarithmic Higgs bundles underlying the canonical extension of the local system $R^n h_*(\mathcal{C}_{Z-\Delta'})$, whose residues have eigenvalues contained in $[0, 1)$, cf. [Del70, Prop. I.5.4]. According to [Ste76, Thm. 2.18] the associated graded module of the Hodge filtration induces a structure of a Hodge bundle on $(\mathcal{E}, \theta)$ whose gradings $\mathcal{E}_*$ are given by

$$(2.2.7) \quad \mathcal{E}_* = R^* h_* (\Omega^{n-\bullet}_{Z/X} \log(\Delta')),$$

with morphisms $\theta : \mathcal{E}_* \to \mathcal{E}_{n+1} \otimes \Omega^1_X \log(D + S)$ determined by those in the complex $R h_*(B_{n-\bullet}).$

Our aim is now to construct a morphism from $(\mathcal{F}, \tau)$ to $(\mathcal{E}, \theta)$ that is compatible with $\tau$ and $\theta$. To this end, we observe that the non-vanishing (2.2.1) implies that there is a natural injection

$$\psi^*(\Omega^{n-\bullet}_{Y/X} \log(\Delta) \otimes \mathcal{M}^{-1}) \hookrightarrow \Omega^{n-\bullet}_{Z/X} \log(\Delta').$$

Together with the two isomorphisms (2.2.4) and (2.2.6) it then follows that the induced map

$$\frac{F^0_0}{F^0_2} \otimes \psi^*(\mathcal{M}^{-1}) \to \frac{G^0_0}{G^0_2}.$$
is an injection. Therefore, the sequence defined by $A_{n-\ldots}$ is a subsequence of $B_{n-\ldots}$, inducing a morphism of complexes between $\mathbf{R} h_*(A_{n-\ldots})$ and $\mathbf{R} h_*(B_{n-\ldots})$. In particular there is a sheaf morphism

$$\Phi_* : \mathbf{R}^n h_*(\psi^*(\Omega^n_{Y/X} \log(\Delta)) \otimes \mathcal{M}^{-1})/\text{torsion} \rightarrow \mathbf{R}^n h_*(\Omega^n_{Z/X} \log(\Delta')).$$

The compatibility of $\Phi_*$ with respect to $\tau$ and $\theta$, follows from the fact that, by construction, each $\tau|_{\mathcal{E}_*}$ is defined by the corresponding morphism in the complex $\mathbf{R} h_*(A_{n-\ldots})$:

$$\begin{align*}
\mathbf{R} h_*(A_{n-\ldots})/\text{torsion} : & \rightarrow \mathcal{F}_* \rightarrow \mathcal{F}_{*+1} \otimes \Omega^1_X \log(D) \rightarrow \ldots \\
\mathbf{R} h_*(B_{n-\ldots}) : & \rightarrow \mathcal{E}_* \rightarrow \mathcal{E}_{*+1} \otimes \Omega^1_X \log(D + S) \rightarrow \ldots.
\end{align*}$$

3. Constructing pseudo-effective Viehweg-Zuo subsheaves

In the current section we will prove Theorem 1.4. We will be working in the context of the following set-up.

Set-up 3.1. Let $f : Y \rightarrow X$ be a smooth compactification of the smooth projective family $f_U : U \rightarrow V$ whose general fiber has a good minimal model and $\text{Var}(f) \neq \emptyset$. Set $D$ to be the divisor defined by $X \setminus D \cong V$ and $\Delta = \text{Supp}(f^*D)$ (both $D$ and $\Delta$ are assumed to have simple normal crossing support).

Proposition 3.2. In the situation of Set-up 3.1, after removing a subset of $X$ of codim $X \geq 2$, the following constructions and properties can be verified.

(3.2.1) There exists a system of logarithmic Hodge bundles $(\mathcal{E} = \bigoplus \mathcal{E}_*, \theta)$ on $X$ with $\theta : \mathcal{E}_* \rightarrow \mathcal{E}_{*+1} \otimes \Omega^1_X \log(D + S)$.

(3.2.2) The torsion free sheaf $\ker(\theta|_{\mathcal{E}_*})$ is seminegatively curved.

(3.2.3) There exists a subsystem $(\mathcal{F} = \bigoplus \mathcal{F}_*, \theta)$ of $(\mathcal{E}, \theta)$ such that $\theta(\mathcal{F}_*) \subseteq \mathcal{F}_{*+1} \otimes \Omega^1_X \log(D)$.

(3.2.4) There is a line bundle $\mathcal{L}$ on $X$, with $\kappa(X, \mathcal{L}) \geq \text{Var}(f)$, equipped with an injection $\mathcal{L} \hookrightarrow \mathcal{G}_0$.

Here, and following the terminology introduced in [BP08, p. 357] by Berndtsson and Pāun, we say that a torsion free sheaf $\mathcal{N}$ is seminegatively curved if it carries a seminegatively curved (possibly singular) metric $h$ over its smooth locus $X^{S}$ (where $\mathcal{N}$ is locally free). We refer to [BP08] for more details (see also the survey paper [Pāu16]). An immediate consequence of this property is the fact that, once it is satisfied, then $\det(\mathcal{N})$ extends to a semi-pseudo-effective line bundle on the projective variety $X$ and this is all that is needed for our purposes in the current paper.

Granting Proposition 3.2 for the moment let us proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $X^o \subseteq X$ be the open subset over which Items (3.2.1)–(3.2.4) in Proposition 3.2 are valid. By iterating the morphism

$$\theta \otimes \text{id} : \mathcal{G}_j \otimes \Omega^j_X^{\log}(D) \rightarrow \mathcal{G}_{j+1} \otimes \Omega^{j+1}_X^{\log}(D),$$

for every $j \in \mathbb{N}$, we can construct a map
\[ \theta^j : \mathcal{G}_0 \longrightarrow \mathcal{G}_j \otimes \Omega^j_X \log(D). \]

Note that \( \theta(\mathcal{G}_0) \neq 0. \) Otherwise, there is an injection
\[ \mathcal{L}|_{X \cdot} \hookrightarrow \ker(\theta|_{\mathcal{G}_0}). \]

On the other hand, by construction we have \( \ker(\theta|_{\mathcal{G}_0}) \subseteq \ker(\theta|_{\mathcal{E}_0}) \) and according to Item (3.2.2) \( \ker(\theta|_{\mathcal{E}_0}) \) is seminegatively curved. This implies that \( \mathcal{L} \) is anti-pseudo-effective on \( X \) and therefore \( \kappa(X, \mathcal{L}) \leq 0, \) contradicting our assumption on \( \operatorname{Var}(f) \) not being equal to zero.

Now, let \( k \) be the positive integer defined by \( k = \max\{j \in \mathbb{N} \mid \theta^j(\mathcal{G}_0) \neq 0\}, \) so that
\[ \theta^k(\mathcal{G}_0) \subset \ker(\theta|_{\mathcal{G}_k}) \otimes \Omega^k_X \log(D). \]

From Item (3.2.4) it follows that there is a non-trivial morphism
\[ \mathcal{L} \longrightarrow \mathcal{N}_k \otimes \Omega^k_X \log(D), \]
which implies the existence of a non-zero map
\[ (3.2.5) \quad \mathcal{L} \otimes \big( \det(\mathcal{N}_k) \big)^{-1} \longrightarrow \Omega^i_X \log(D), \]
for some \( i \in \mathbb{N}. \) Now, let \( \mathcal{R} \) be the line bundle on \( X \) defined by the extension of \( \det(\mathcal{N}_k)^{-1} \) so that (3.2.5) extends to the injection
\[ \mathcal{L} \otimes \mathcal{R} \hookrightarrow \Omega^i_X \log(D). \]

It remains to verify that \( \mathcal{R} \) is pseudo-effective. But again, according to Item (3.2.2), the torsion free sheaf \( \mathcal{N}_k \subseteq \ker(\theta|_{\mathcal{E}_0}) \) is seminegatively curved and therefore so is \( \det(\mathcal{N}_k) \) and thus extends to a anti-pseudo-effective line bundle \( \mathcal{R}^{-1} \) on \( X. \)

**Proof of Proposition 3.2.** The proof is a consequence of Proposition 2.1 combined with the Hodge theoretic constructions in Subsection 2.2. To lighten the notation we will replace the initial family \( f : Y \rightarrow X \) in Set-up 3.1 by \( f^{(r)} : Y^{(r)} \rightarrow X, \) which was constructed in Proposition 2.1.

After removing a codimension two subset from \( X \) over which the inclusion (2.1.1) holds, define the line bundle
\[ \mathcal{M} = \omega_{Y/X}(\Delta) \otimes f^*(\mathcal{L}^{-1}) \]
so that \( H^0(Y, \mathcal{M}^m) \neq 0. \) The arguments in Subsection 2.2 can now be used to construct the two systems \( \langle \mathcal{E}, \theta \rangle \) and \( \langle \mathcal{F}, \tau \rangle \) defined in (2.2.5) and (2.2.7), with logarithmic poles along \( (D + S) \) and \( D, \) respectively. Here after deleting a codimension two subset we have assumed that \( \operatorname{Supp}(D + S) \) is simple normal crossing.

Item (3.2.2) follows from Zuo’s result [Zuo00]—based on Cattani, Kaplan and Schmid’s work on asymptotic behaviour of Hodge metrics, cf. [CKS86] and [PTW18, Lem 3.2]—and Brunbarbe [Bru15], and more generally from Fujino and Fujisawa [FF17]. The reader may wish to consult Simpson [Sim90] and [Bru17] where these problems are dealt with in the more general setting of tame harmonic metrics.

For (3.2.3), define \( \langle \mathcal{G} = \oplus \mathcal{G}_0, \theta \rangle \subset \langle \mathcal{E}, \theta \rangle \) to be the image of the system \( \langle \mathcal{F} = \oplus \mathcal{F}_0, \tau \rangle \) under the morphism \( \Phi_0, \) constructed in Subsection 2.2.
It remains to verify (3.2.4). According to the definition of $\Phi$, we have

$$\Phi_0 : R^0 h_*(\psi^*(\omega_{Y/X}(\Delta) \otimes \mathcal{M}^{-1})) \rightarrow R^0 h_*(\omega_{Z/X}(\Delta')),$$

which is an injection; for the map

$$\psi^*(\omega_{Y/X}(\Delta) \otimes \mathcal{M}^{-1}) \rightarrow \omega_{Z/X}(\Delta')$$

is injective. Item (3.2.4) now follows from the isomorphism

$$h_*(\psi^*(\omega_{Y/X}(\Delta) \otimes \omega_{Y/X}^{-1}(\Delta) \otimes f^*\mathcal{L})) \cong \mathcal{L}'.$$

4. Vanishing results

In this section we will prove Theorem 1.5. The methods will heavily rely on birational techniques and results in the minimal model program. For an in-depth discussion of preliminary notions and background we refer the reader to the book of Kollár and Mori [KM98] and the references therein.

Not surprisingly an important construction that we will repeatedly make use of is the Iitaka fibration. Please see [Mor87, Sect. 1] and [Laz04, Sect. 2.1.C] for the definition and a review of the basic properties.

**Notation 4.1.** Let $L$ be a line bundle on a normal projective variety $X$ with $\kappa(X, L) > 0$. By $\phi(I) : X(I) \rightarrow Y(I)$ we denote the Iitaka fibration of $L$ with an induced birational morphism $\pi(I) : X(I) \rightarrow X$.

**4.1. Proof of Theorem 1.5.** We begin by stating the following lemma concerning the behaviour of the Kodaira dimension on fibers of the Iitaka fibration. The proof follows from standard arguments; see for example [Laz04, pp. 136–137].

**Lemma 4.2.** Let $f : X \rightarrow Z$ be a fiber space of normal projective varieties with a flattening $f' : X' \rightarrow Z'$ as above. Let $A$ be a $f$-nef and effective $\mathbb{Q}$-divisor and assume that $A |_{F'} \equiv 0$, where $F$ is the general fiber of $f$. There exists $A_{Z'} \in \text{Div}(Z')_\mathbb{Q}$ such that $\sigma^*(A) \sim_\mathbb{Q} (f')^*(A_{Z'})$.

The next lemma is the final technical background that we need before we can proceed to the proof of Theorem 1.5. It relies on the so-called flattening lemma, due to Gruson and Raynaud, which we recall below.

Let $f : X \rightarrow Z$ be an algebraic fiber space of normal (quasi-)projective varieties. There exists an equidimensional fiber space $f' : X' \rightarrow Z'$ of normal varieties that is birationally equivalent to $f$ through birational morphisms $\sigma : X' \rightarrow X$ and $\tau : Z' \rightarrow Z$. We call $f'$ the flattening of $f$.

**Lemma 4.3.** Let $f : X \rightarrow Z$ be a fiber space of normal projective varieties with a flattening $f' : X' \rightarrow Z'$ as above. Let $A$ be a $f$-nef and effective $\mathbb{Q}$-divisor and assume that $A |_{F'} \equiv 0$, where $F$ is the general fiber of $f$. There exists $A_{Z'} \in \text{Div}(Z')_\mathbb{Q}$ such that $\sigma^*(A) \sim_\mathbb{Q} (f')^*(A_{Z'})$. 
Proof. It suffices to show that $σ^*(A)$ is $f'$-numerically trivial. Aiming for a contradiction, assume that there exists a fiber $F_{i'}$ of $f'$ containing an irreducible contractible curve $C$ such that $σ^*(A) · C ≠ 0$. Let $A_1, ..., A_{d-1}$ and $A_d$ be a collection of ample divisors in $X'$ such that $(H_1 · · · H_d) ∩ F_{i'}$ defines the numerical cycle of $(C + \overline{C})$, where $\overline{C}$ is an effective 1-cycle in $F_{i'}$. Since $σ^*(A)|_{F_{i'}} ≡ 0$, where $F'$ is a general fiber of $f'$, we have

\[(4.3.1) \quad σ^*(A) · (C + C') = 0.\]

On the other hand, $σ^*(A)$ is $f'$-nef. Combined with (4.3.1), this implies that $σ^*(A) · C = 0$, which contradicts our initial assumption.

\[\square\]

Proof of Theorem 1.5. (Preparation). Let $L$ and $B$ be two line bundles on $X$, with $B$ being pseudo-effective, such that, for some $i ∈ N$, there is a non-trivial morphism

\[L ⊗ B → Ω^i_X \log(D).\]

Claim 4.4. The line bundle $ω^i_X(D) ⊗ L^{-1}$ is pseudo-effective.

Proof of Claim 4.4. Let $F$ be the saturation of the image of the non-trivial morphism

\[L ⊗ B → Ω^i_X \log(D)\]

and $Ω$ the torsion-free quotient resulting in the exact sequence

\[0 → F → Ω^i_X \log(D) → Ω → 0.\]

After taking determinants we find that

\[(4.4.1) \quad ω^i_X(D) ⊗ F^{-1} ≅ \det(Ω).\]

Thanks to [CP15, Thm. 1.3] the right-hand side of (4.4.1) is pseudo-effective and thus so is the left-hand side. This implies that $ω^i_X(D) ⊗ (L ⊗ B)^{-1} ∈ NE^i(X)$. But $B$ is also assumed to be pseudo-effective and therefore $ω^i_X(D) ⊗ L^{-1} ∈ NE^i(X)$. This finishes the proof of the claim.

Before we proceed further, notice that we may assume that $(i(K_X + D) + L)$ is not big, where $L = O_X(L)$. Otherwise the divisor $K_X + D$ is big, as it can be written as the sum of a pseudo-effective and big divisors:

\[K_X + D = \frac{1}{2i}((i · (K_X + D) + L) + (i · (K_X + D) − L)).\]

We can also assume that $κ(L) ≥ 1$.

The first step in proving the theorem consists of replacing $X$ by a birational model $Y$ where establishing the vanishing in Theorem 1.5 proves to be easier.

Claim 4.5. There exists a birational morphism $π : Y → X$ from a smooth projective variety $Y$ that can be equipped with a fiber space $f : Y → Z$ over a smooth projective variety $Z$. Furthermore, $Y$ contains a reduced divisor $D_Y$ such that $(Y, D_Y)$ is log-smooth and a divisor $L_Y$ which satisfy the following properties.

\[(4.5.1) \quad \dim(Z) = κ(i(K_X + D) + L),\]

\[(4.5.2) \quad O_Y(L_Y) ⊗ π^∗(B) ⊆ Ω^*Y \log(D_Y),\]

\[(4.5.3) \quad κ(K_Y + D_Y) = κ(K_X + D),\]
(4.5.4) $\kappa(L_Y) = \kappa(L)$,
(4.5.5) $\kappa(F, (i(K_Y + D_Y) + L_Y)|_F) = 0$ and $\kappa(F, (K_Y + D_Y)|_F) = 0$, where $F$ is a very general fiber of $f$,
(4.5.6) $(i(K_Y + D_Y) - L_Y) \in \mathbb{NE}^1(Y)$, and
(4.5.7) $L_Y \sim_{Q} f^*(L_Z)$, for some $L_Z \in \text{Div}_Q(Z)$.

**Proof of Claim 4.5.** Let $\psi : X \rightarrow Z$ be the rational mapping associated to the linear system $\{m \cdot (i(K_X + D) + L)\}$, with $m$ being sufficiently large so that $\text{dim}(Z) = \kappa(i(K_X + D) + L)$. Note that as $\kappa(L) \geq 1$, we have $\text{dim}(Z) \geq 1$. Let $\psi_1 : X_1 \rightarrow Z_1$ be the Iitaka fibration of $(i(K_X + D) + L)$ resulting in the commutative diagram

\[
\begin{array}{c}
X_1 \xymatrix{ \ar[r]^\psi & Z_1 } \\
\ar[d]_{\pi_1} & \\
X \ar[r]_{\psi} & \ast \\
\end{array}
\]

where $\pi_1 : X_1 \rightarrow X$ is a birational morphism. Define $L_1 := \pi_1^*(L)$. Let $E_1$ and $E_2$ be two effective and exceptional divisors such that

$$K_{X_1} + \pi_1^*(D) + E_2 \sim \pi_1^*(K_X + D) + E_1,$$

where $\pi_1^*(D)$ is the birational transform of $D$. Note that Claim 4.4 implies that $(i(K_{X_1} + D_1) - L_1) \in \mathbb{NE}^1(X_1)$. Furthermore, by Lemma 4.2 we have $\kappa(F_1, (i(K_{X_1} + D_1) + L_1)|_{F_1}) = 0$, where $F_1$ is a very general fibre of $\psi_1$. On the other hand, we have

$$\kappa(F_1, L_1|_{F_1}) \leq \kappa(F_1, (i(K_{X_1} + D_1) + L_1)|_{F_1}) = 0.$$

Therefore, we find that $\kappa(F_1, L_1|_{F_1}) = 0$. As a result, and thanks to [Mor87, Def–Thm. 1.11], the Iitaka fibration $\psi_2 : X_2 \rightarrow Y$ of $L_1$ factors through the fiber space $\psi_1 : X_1 \rightarrow Z_1$ via a birational morphism $\pi_2 : X_2 \rightarrow X_1$ and a rational map $\nu : Z_1 \dashrightarrow Y$ (see Diagram 4.5.8 below). Let $\tilde{\nu} : Z_2 \rightarrow Z_2$ be a desingularization of $\nu$ through the birational morphism $\mu : Z_2 \rightarrow Z_1$. Finally, let $\psi_3 : X_3 \rightarrow Z_2$ be a desingularization of the rational map $X_2 \dashrightarrow Z_2$, defined by the composition of $\pi_2, \psi_1$ and $\mu^{-1}$ (where it is defined), via the birational morphism $\pi_3 : X_3 \rightarrow X_2$.

\[
\begin{array}{c}
X_3 \xymatrix{ \ar[r]^{\pi_3}_{\pi_3} & X_2 \xymatrix{ \ar[r]^{\psi_2}_{\psi_2} & Y } \\
\ar[d]_{\pi_2} & \\
X_1 \xymatrix{ \ar[r]^{\psi_1}_{\psi_1} & Z_1 } \\
\ar[r]_{\mu} & Z_2 \xymatrix{ \ar[r]_{\tilde{\nu}}_{\tilde{\nu}} & \ast } \\
\end{array}
\]

By construction, there is an effective $\mathbb{Q}$-divisor $E \subset X_2$ and a very ample $\mathbb{Q}$-divisor $L_Y$ in $Y$ such that

$$\pi_2^*(L_1) - E \sim_{Q} \psi_2^*(L_Y).$$
Define $L_4 := \pi_3^*\left(\pi_2^*(L_1) - E\right)$. Let $E_3$ and $E_4$ be two effective exceptional divisors for which we have

$$K_{X_3} + \overline{D}_3 + E_3 \sim \pi_3^*\left(\pi_2^*(K_{X_1} + D_1)\right) + E_4,$$

where $\overline{D}_3$ is the birational transform of $D_1$.

We now claim that the two divisors $(i \cdot (K_{X_3} + D_3))$ and $L_3$ together with the fiber space $\psi_3 : X_3 \rightarrow Z_2$ satisfy the properties listed in Claim 4.5 for $Y$, $D$, $L_Y$, $Z$ and $f$.

To see this, first note that $\kappa(L_3) = \kappa(L)$ and that

$$L_3 \sim \psi_3^*\left(\overline{\nu}^*(L_Y)\right),$$

thanks to the commutativity of Diagram 4.5.8.

Next, to verify Property (4.5.5), let $F_3$ to be a very general fiber of $\psi_3$ and note that

$$0 \leq \kappa(F_3, (K_{X_3} + D_3)|_{F_3}) = \kappa(F_3, (i(K_{X_3} + D_3) + L_3)|_{F_3}),$$

On the other hand we have

$$\kappa(F_3, (i(K_{X_3} + D_3) + L_3)|_{F_3}) \leq \kappa(F_3, (i(K_{X_3} + D_3) + \pi_3^*(\pi_2^*L_1))|_{F_3}).$$

The two equalities in (4.5.5) now follow form the fact that the right-hand side of (4.5.9) is less than or equal to zero, cf. Lemma 4.2.

The pseudo-effectivity of $i \cdot (K_{X_3} + D_3) - L_3$ (Property (4.5.6)) follows from the relation

$$i \cdot (K_{X_3} + D_3) - L_3 \sim \pi_3^*\left(\pi_2^*(i \cdot (K_{X_1} + D_1) - L_1)\right) + \left(i \cdot E_4 + \pi_3^*(E)\right)$$

and the fact that $(i \cdot (K_{X_1} + D_1) - L_1) \in \overline{\NE}(X_1)$. The remaining properties hold by construction. This concludes the proof of Claim 4.5.

To lighten the notation, from now on we will assume that $i = 1$.

Proof of Theorem 1.5. After fixing the dimension of $X$, our proof will be based on induction on $d := \kappa(K_X + D + L)$, assuming that $\kappa(L) > 0$. The next claim provides the base case.

Claim 4.6. If $d = 1$, then $\kappa(\mathcal{L}) = \kappa(\omega_X(D))$.

Proof of Claim 4.6. Using (4.5.5) we can see that the Iitaka fibration $\phi^{(I)} : Y^{(I)} \rightarrow Z^{(I)}$ of $(K_Y + D_Y + L_Y)$ factors through $f : Y \rightarrow Z$ via a birational morphism $\pi^{(I)} : Y^{(I)} \rightarrow Y$ and a finite morphism $\nu : Z \rightarrow Z^{(I)}$. As both $(\pi^{(I)} \circ f)$ and $\phi^{(I)}$ are fiber spaces, the finite map $\nu$ must be trivial, that is the two maps $\phi^{(I)}$ and $f$ coincide. In particular we have $(K_Y + D_Y + L_Y) \sim_Q f^*(B_Z)$, for some very ample divisor $B_Z$ in $Z$. By using (4.5.7) it then follows that

$$(K_Y + D_Y) \sim_Q \frac{1}{2}\left(f^*(B_Z - 2 \cdot L_Z) + f^*(B_Z)\right).$$

Now, as $(B_Z - 2 \cdot L_Z) \in \overline{\NE}(Z)$ by (4.5.6), we conclude that the right-hand side of (4.6.1) is ample in $Z$. Therefore $\kappa(K_Y + D_Y) = \kappa(L_Y) = 1$, which establishes the claim.
Inductive step. We assume that Theorem 1.5 holds for any line bundle $\mathscr{A}$ on a smooth projective variety $W$ (having the same dimension as $X$) that satisfies the following two properties.

(4.6.2) There is a reduced divisor $D_W$ such that $(W, D_W)$ is log-smooth and $(\mathscr{A} \otimes \mathcal{M}) \subseteq \Omega^0_W \log(D_W)$, for some pseudo-effective line bundle $\mathcal{M}$.

(4.6.3) $\kappa(W, \omega_W(D_W) \otimes \mathscr{A}) < d$.

Let $(Y, D_Y), L_Y$ and $f : Y \to Z$ be as in the setting of Claim 4.5. By the inductive step we may assume that $\dim(Z) = \kappa(K_Y + D_Y + L_Y)$. Furthermore, we can use Claim 4.6 to exclude the possibility that $\kappa(K_Y + D_Y + L_Y) = 1$.

We treat the case where the dimension of the fibers of $f$ is equal to 3 (i.e. $\dim(Y) = 5$ and $\kappa(K_Y + D_Y + L_Y) = 2$). The case of lower dimensional fibers can be dealt with similarly.

Let $g : (Y, D_Y) \dashrightarrow (Y_n, D_{Y_n})$ be the birational map associated to a relative minimal model program for $(Y, D_Y)$ over $Z$, consisting of $n$ number of divisorial and flipping contractions

$$g_j : (Y_j, D_{Y_j}) \dashrightarrow (Y_{D_{Y_j+1}}, D_{Y_{j+1}}),$$

cf. [Kol92, Chapt. 4]. Here we have set $(Y_0, D_{Y_0}) := (Y, D_Y)$. For each $1 \leq j \leq n$, let $f_j : Y_{D_j} \to Z$ be the induced morphism resulting in the following diagram.

$$
\begin{array}{c}
\vdots \\
\downarrow g_1 \\
\vdots \\
Y \xrightarrow{g} Y_1 \xrightarrow{g_2} \ldots \xrightarrow{g_{n-1}} Y_{n-1} \xrightarrow{g_n} Y_n \\
\downarrow f_1 \\
\downarrow f_2 \\
\ldots \\
\downarrow f_{n-1} \\
\downarrow f_n \\
Z \\
\end{array}
$$

Claim 4.7. $\kappa(Y_n, K_{Y_n} + D_{Y_n} + f_n^*L_Z) = \kappa(Y, K_Y + D_Y + L_Y)$.

Claim 4.8. $(\kappa(Y_n, D_{Y_n}) - (f_n^*L_Z)) \in \overline{NE}(Y_n)$.

Let us for the moment assume that Claims 4.7 and 4.8 hold and proceed with the proof of Theorem 1.5. Let $f_n' : Y_n' \to Z'$ be a flattening of $f_n$ (see the discussion preceding Lemma 4.3), with induced birational maps $\tau : Z' \to Z$ and $\sigma : Y_n' \to Y_n$ leading to the following commutative diagram.

$$
\begin{array}{ccc}
Y_n' & \xrightarrow{\sigma} & Y_n \\
\downarrow f_n' & & \downarrow f_n \\
Z' & \xrightarrow{\tau} & Z
\end{array}
$$

Thanks to the solution of the relative log-abundance problem in $\dim(X) = 3$ by Keel, Matsuki and McKernan [KMM04], using (4.5.5), we have $(K_{Y_n} + D_{Y_n})|_{F_z} \equiv 0$, where $F_z$ is the general fiber of $f_n$. Therefore, by Lemma 4.3, there exists a $\mathbb{Q}$-divisor $A_{Z'}$ in $Z'$ such that $\sigma^*(K_{Y_n} + D_{Y_n}) \sim_{\mathbb{Q}} (f_n')^*(A_{Z'})$ so that

$$\sigma^*((K_{Y_n} + D_{Y_n}) \pm f_n^*L_Z) \sim_{\mathbb{Q}} (f_n')^*(A_{Z'} \pm \tau^*L_Z).$$
By Claim 4.7 it thus follows that $\kappa(Z', A_{Z'} + \tau^*(L_Z)) = \kappa(Y_n, K_{Y_n} + D_{Y_n} + f_n^*L_Z) = \dim(Z)$. Moreover, by using Claim 4.8, we find $(A_{Z'} - \tau^*(L_Z)) \in \NE(Z')$. Therefore $A_{Z'}$ is big in $Z'$ and we have

\begin{equation}
(4.8.1) \quad \kappa(Z', A_{Z'}) \geq \kappa(Z', \tau^*(L_Z)) = \kappa(Z, L_Z).
\end{equation}

On the other hand, by using the negativity lemma, we have

\begin{equation}
(4.8.2) \quad \kappa(Z', A_{Z'}) = \kappa(Y_n, K_{Y_n} + D_{Y_n}) = \kappa(Y, K_Y + D_Y).
\end{equation}

By combining (4.8.1) and (4.8.2) we reach the inequality

$$\kappa(Y, L_Y) \leq \kappa(Y, K_Y + D_Y).$$

We now turn to proving Claims 4.7 and 4.8.

**Proof of Claim 4.7.** Let $(\tilde{Y}, \tilde{D}_{Y})$ be a common log-smooth higher birational model for $(Y, D_Y)$ and $(Y_n, D_{Y_n})$, with birational morphisms $\mu : \tilde{Y} \to Y$ and $\mu_n : \tilde{Y} \to Y_n$. According to [KM98, Lem. 3.38] there is an effective $\mu$-exceptional divisor $E_{\mu}$ and an effective $\mu_n$-exceptional divisor $E_{\mu_n}$ such that

$$\mu^*(K_Y + D_Y) + E_{\mu} \sim Q (\mu_n)^*(K_{Y_n} + D_{Y_n}) + E_{\mu_n}.$$

Therefore, we have

$$\mu^*((K_Y + D_Y) + f^*L_Z) + E_{\mu} \sim Q (\mu_n)^*(K_{Y_n} + D_{Y_n} + f_n^*L_Z) + E_{\mu_n},$$

which establishes the claim.

**Proof of Claim 4.8.** The proof will be based on induction on $n$. Assume that

$$(K_{Y_{n-1}} + D_{Y_{n-1}} - f_{n-1}^*(L_Z)) \in \NE(Y_{n-1}).$$

Now, the map $g_n : Y_{n-1} \to Y_n$ is either a divisorial contraction or a flip over $Z$. In the case of the latter the claim is easy to check, so let us assume that $g_n$ is a divisorial contraction and let $E_1$ and $E_2$ be two effective exceptional divisors such that the equivalence

\begin{equation}
(4.8.3) \quad K_{Y_{n-1}} + D_{Y_{n-1}} + E_1 \sim Q g_n^*(K_{Y_n} + D_{Y_n}) + E_2
\end{equation}

holds. After subtracting $f_{n-1}^*(L_Z) = g_n^*(f_n^*L_Z)$ from both sides of (4.8.3) and by using the inductive hypothesis we find that

$$\left(g_n^*(K_{Y_n} + D_{Y_n} - f_n^*L_Z) + E_2\right) \in \NE(Y_{n-1}),$$

which implies $\left(K_{Y_n} + D_{Y_n} - f_n^*(L_Z)\right) \in \NE(Y_n)$. This finishes the proof of Claim 4.8. \(\square\)

**Remark 4.9.** Theorem 1.5 is naturally related to a conjecture of Campana and Peternell, where the authors predict that over a smooth projective variety $X$, if $K_X \sim Q L + B$, where $L$ is effective and $B$ is pseudo-effective, then $\kappa(X) \geq \kappa(L)$, cf. [CP11]. They also provide a proof to this conjecture when $B$ is numerically trivial [CP11, Thm. 0.3]. (See also [CKP12] for the generalization to the logarithmic setting.)
5. Concluding remarks and further questions

We recall that a quasi-projective variety $V$ of dimension $n$ is said to be special if, for every invertible subsheaf $\mathcal{L} \subseteq \Omega^p_X \log(D)$, the inequality $\kappa(X, \mathcal{L}) < p$ holds, for all $1 \leq p \leq n$. Here, by $(X, D)$ we denote a log-smooth compactification of $V$. We refer to [Cam04] for an in-depth discussion of this notion. We note that while varieties of Kodaira dimension zero form an important class of special varieties [Cam04, Thm 5.1], there are special varieties of every possible (but not maximal) Kodaira dimension.

A conjecture of Campana predicts that a smooth projective family of canonically polarized manifolds $f_U : U \rightarrow V$ parametrized by a special quasi-projective variety $V$ is isotrivial. One can naturally extend this conjecture to the following setting.

**Conjecture 5.1.** Let $f_U : U \rightarrow V$ be a smooth projective family, where $V$ is equipped with a morphism $\mu : V \rightarrow P_h$ to the coarse moduli scheme $P_h$ associated with the moduli functor $\mathcal{P}_h(\cdot)$ of polarized projective manifolds with semi-ample canonical bundle and fixed Hilbert polynomial $h$. If $V$ is special, then $f_V$ is isotrivial.

By using the refinement of [VZ02], due to Jabbusch and Kebekus [JK11], and the main result of [CP15b], in the canonically polarized case, Conjecture 5.1 was established in [Taj16]. We invite the reader to also consult [CKT16], Claudon [Cla15], [CP15] and Schnell [Sch17].

After a close inspection one can observe that the strategy of [Taj16] can be extended to establish 5.1 in its full generality. More precisely, existence of the functor $\mathcal{P}_h$ with its associated algebraic coarse moduli scheme gives us a twofold advantage. It ensures the existence of “effective” Viehweg-Zuo subsheaves $\mathcal{L} \subseteq \Omega^p_X \log(D)$ whose Kodaira dimension verifies $\kappa(\mathcal{L}) \geq \text{Var}(f_U)$, cf. [VZ02] and is constructed at the level of moduli stacks, cf. [JK11]. After imposing some additional orbifold structures naturally arising from the induced moduli map $\mu : V \rightarrow P_h$, these two properties allow us to essentially reduce the problem to the case where $\mathcal{L}$ is big, cf. [Taj16, Thm. 4.3].

**Theorem 5.2.** Conjecture 5.1 holds.

Campana has kindly informed the author that, extending their current result [AC17], in a joint work with Amerik, they have established Theorem 5.2 in a more general context of projective families with orbifold base.

Using the results of [Cam04], it is not difficult to trace a connection between the two Conjectures 5.1 and 1.3; a solution to Conjecture 5.1 leads to a solution for Conjecture 1.3. We record this observation in the following theorem.

**Theorem 5.3.** For any quasi-projective variety $V$ equipped with $\mu_V : V \rightarrow P_h$, induced by a family $f_U : U \rightarrow V$ of polarized manifolds, we have

\[(5.3.1) \quad \text{Var}(f_U) \leq \kappa(X, D),\]

$(X, D)$ being a log-smooth compactification of $V$.

Before proceeding to the proof of Theorem 5.3, let us recall the notion of the core map defined by Campana. Given a smooth quasi-projective variety $V$ that is not of log-general type, the core map is a rational map $c_X : X \dashrightarrow Z$ satisfying the following two key properties.

\[(5.3.2) \quad c_X \text{ is almost holomorphic with special geometric fibers.}\]
(5.3.3) $c_X$ is birationally equivalent to a fiber space $c_{\tilde{X}} : (\tilde{X}, \tilde{D}) \to (\tilde{Z}, \Delta_{\tilde{Z}})$, where $\Delta_{\tilde{Z}} \in \text{Div}_Q(\tilde{Z})$ and the pair $(\tilde{Z}, \Delta_{\tilde{Z}})$ is a log-smooth orbifold base for $C_{\tilde{X}}$ and is of log-general type.

Proof. Assume that $f_U$ is not isotrivial and $V$ is not of log-general type. Then, by Item (5.3.2) and Theorem 5.2, the compactification $\mu_V : X \to \overline{T}_h$ of $\mu_V$ factors through the core map $c_X : X \to Z$ with positive dimensional fibers. In particular we have

$$\text{Var}(f_U) \leq \dim(Z)$$

(5.3.4)

The theorem then follows from Campana’s orbifold $C_{n,m}$ theorem for any orbifold, log-general type fibration; an example of which is $c_{\tilde{X}}$. More precisely, by using (5.3.3) and [Cam04, Thm. 4.2] we can conclude that the inequality

$$\kappa(\tilde{X}, \tilde{D}) \geq \kappa(\tilde{Z}, \Delta_{\tilde{Z}})$$

holds, which, together with (5.3.4) and the inequality $\kappa(X, D) \geq \kappa(\tilde{X}, \tilde{D})$, establishes the theorem. \qed

When the fibers are of general type, it is conceivable that one may be able to use the result of Birkar, Cascini, Hacon and McKernan [BCHM10] on the existence of good minimal models for varieties of general type (and relative base point freeness theorem) to reduce to the case of maximal variation. More precisely, the arguments of [VZ02, Lem. 2.8] combined with Hodge theoretic construction of [PS17], or the ones in Section 3 of the current paper, may allow for the construction of a Viehweg-Zuo subsheaf $\mathcal{L}$ over a new base where variation of the pulled back family is maximal. If so, in this case the discussion prior to Theorem 5.2 again applies and Theorem 5.2 and consequently Theorem 5.3 hold. But as pointed out from the outset, the main remaining difficulty is solving the isotriviality problem in the absence of a well-behaved functor, such as $\mathcal{O}_h$, detecting the variation in the family (after running the minimal model program, if necessary). At the moment, it is not clear to the author that one can expect Conjecture 5.1 to hold for all smooth projective families whose geometric fiber has semi-ample canonical bundle or—even more generally—admits a good minimal model.

**Question 5.4.** Let $f_U : U \to V$ be a smooth projective family of manifolds admitting a good minimal model. If $V$ is special, (apart from the cases discussed above) is it true that $f_U$ is isotrivial?

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