Balanced metrics on some Hartogs type domains over bounded symmetric domains

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**Abstract** The definition of balanced metrics was originally given by Donaldson in the case of a compact polarized Kähler manifold in 2001, who also established the existence of such metrics on any compact projective Kähler manifold with constant scalar curvature. Currently, the only noncompact manifolds on which balanced metrics are known to exist are homogeneous domains. The generalized Cartan-Hartogs domain \((\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}\) is defined as the Hartogs type domain constructed over the product \(\prod_{j=1}^{k} \Omega_j\) of irreducible bounded symmetric domains \(\Omega_j\) \((1 \leq j \leq k)\), with the fiber over each point \((z_1, \ldots, z_k) \in \prod_{j=1}^{k} \Omega_j\) being a ball in \(\mathbb{C}^{d_0}\) of the radius \(\prod_{j=1}^{k} \mu_j^{2}\) of the product of positive powers of their generic norms. Any such domain \((\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}\) \((k \geq 2)\) is a bounded nonhomogeneous domain. The purpose of this paper is to obtain necessary and sufficient conditions for the metric \(\alpha g(\mu)\) \((\alpha > 0)\) on the domain \((\prod_{j=1}^{k} \Omega_j)^{B_{\mu}}\) to be a balanced metric, where \(g(\mu)\) is its canonical metric. As the main contribution of this paper, we obtain the existence of balanced metrics for a class of such bounded nonhomogeneous domains.

**Key words:** Balanced metrics · Bergman kernels · Bounded symmetric domains · Cartan-Hartogs domains · Kähler metrics

**Mathematics Subject Classification (2010):** 32A25 · 32M15 · 32Q15

1 Induction

The expansion of the Bergman kernel has received a lot of attention recently, due to the influential work of Donaldson, see e.g. [7], about the existence and uniqueness of constant scalar curvature Kähler metrics (cscK metrics). Donaldson used the asymptotics of the Bergman kernel proved by Catlin [5] and Zelditch [43] and the calculation of Lu [29] of the first coefficient in the expansion to give conditions for the existence of cscK metrics. This work inspired many papers on the subject since then. For the reference of the expansion of the Bergman kernel, see also Engliš [10], Loi [23], Ma-Marinescu [31, 32, 33], Xu [39] and references therein.

Assume that \(D\) is a bounded domain in \(\mathbb{C}^n\) and \(\varphi\) is a strictly plurisubharmonic function on \(D\).

Let \(g\) be a Kähler metric on \(D\) associated to the Kähler form \(\omega = \sum_{j=1}^{n} \partial \bar{\partial} \varphi\). For \(\alpha > 0\), let \(\mathcal{H}_\alpha\) be the weighted Hilbert space of square integrable holomorphic functions on \((D, g)\) with the weight \(\exp\{-\alpha \varphi\}\), that is,

\[ \mathcal{H}_\alpha := \left\{ f \in \text{Hol}(D) : \int_D |f|^2 \exp\{-\alpha \varphi\} \frac{\omega^n}{n!} < +\infty \right\}, \]

where \(\text{Hol}(D)\) denotes the space of holomorphic functions on \(D\). Let \(K_\alpha(z, \bar{z})\) be the Bergman kernel (namely, the reproducing kernel) of the Hilbert space \(\mathcal{H}_\alpha\) if \(\mathcal{H}_\alpha \neq \{0\}\). The Rawnsley’s \(\varepsilon\)-function on \(D\) associated to the metric \(g\) is defined by

\[ \varepsilon_\alpha(z) := \exp\{-\alpha \varphi(z)\} K_\alpha(z, \bar{z}), \quad z \in D. \quad (1.1) \]
Note the Rawnsley’s \( \varepsilon \)-function depends only on the metric \( g \) and not on the choice of the Kähler potential \( \varphi \) (which is defined up to an addition with the real part of a holomorphic function on \( D \)). The function \( \varepsilon_\alpha(z) \) has appeared in the literature under different names. The earliest one was probably the \( \eta \)-function of Rawnsley [35] (later renamed to \( \varepsilon \)-function in Cahen-Gutt-Rawnsley [4]) defined for arbitrary Kähler manifolds, followed by the distortion function of Kempf [20] and Ji [19] for the special case of Abelian varieties, and of Zhang [44] for complex projective varieties.

The asymptotics of the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_\alpha \) was expressed in terms of the parameter \( \alpha \) for compact manifolds by Catlin [5] and Zelditch [43] (for \( \alpha \in \mathbb{N} \)) and for non-compact manifolds by Ma-Marinescu [31, 32]. In some particular case it was also proved by Engliš [9, 10].

**Definition 1.1.** The metric \( \alpha g \) on \( D \) is balanced if the Rawnsley’s \( \varepsilon \)-function \( \varepsilon_\alpha(z) \) (\( z \in D \)) is a positive constant on \( D \).

The metric for which the function \( \varepsilon_\alpha(z) \) is constant was also called critical metric in Zhang [44]. The definition of balanced metrics was originally given by Donaldson [7] in the case of a compact polarized Kähler manifold \( (M, g) \) in 2001, who also established the existence of such metrics on any (compact) projective Kähler manifold with constant scalar curvature. The definition of balanced metrics was generalized in Arezzo-Loi [1] and Engliš [11] to the noncompact case. We give only the definition for those Kähler metrics which admit globally defined potentials in this paper.

Let \( g \) be a Kähler metric on a bounded domain \( D \) in \( \mathbb{C}^n \) associated to the Kähler form \( \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \varphi \) for a strictly plurisubharmonic function \( \varphi \) on \( D \). Let \( \mathcal{H}_\alpha \) \((\alpha > 0)\) be the weighted Hilbert space of square integrable holomorphic functions on \( (D, g) \) with the weight \( \exp\{-\alpha \varphi\} \). Let \( K_\alpha(z, \bar{z}) \) be the reproducing kernel of \( \mathcal{H}_\alpha \) if \( \mathcal{H}_\alpha \neq \{0\} \). Let \( h \) be a Kähler metric on \( D \) associated to the Kähler form \( \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln K_\alpha(z, \bar{z}) \). It is again readily seen that \( \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln K_\alpha(z, \bar{z}) \) is independent of the choice of the potential \( \varphi \) and thus is uniquely determined by the original metric \( g \). So, by definition, we have

\[
\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln \varepsilon_\alpha(z) + \alpha \cdot \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \varphi(z) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln K_\alpha(z, \bar{z}), \quad z \in D.
\]

Therefore, if the metric \( \alpha g \) is balanced on \( D \), then we have \( \alpha g = h \) on \( D \).

Recall that the Fubini-Study metric \( d_{FS} \) on the \( N \)-dimensional complex projective space \( P^N(\mathbb{C}) \) (\( 1 \leq N \leq +\infty \)) is this metric whose Kähler form \( \omega_{FS} \) in homogeneous coordinates \( Z_0, Z_1, Z_2, \cdots \) is given by \( \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln(\sum_{j=0}^N |Z_j|^2) \). A balanced metric \( \alpha g \) on \( D \) can be also viewed as a particular projectively induced Kähler metric for which the Kähler immersion

\[
f : D \to P^N(\mathbb{C}), \quad z \mapsto [f_0(z), f_1(z), \cdots, f_j(z), \cdots],
\]

where \( N + 1 \) \((0 \leq N \leq +\infty)\) denotes the complex dimension of \( \mathcal{H}_\alpha \), is given by an orthonormal basis \( \{f_j\} \) of the Hilbert space \( \mathcal{H}_\alpha \) if \( \mathcal{H}_\alpha \neq \{0\} \). Indeed, the map \( f : D \to P^N(\mathbb{C}) \) is well-defined since \( \varepsilon_\alpha \) is a positive constant and hence for any \( z \in D \) there exists some \( f_j \) such that \( f_j(z) \neq 0 \). Moreover,

\[
f^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln \sum_{j=0}^N |f_j(z)|^2
\]

\[= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln K_\alpha(z, \bar{z})
\]

\[= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \ln \varepsilon_\alpha(z) + \alpha \cdot \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \varphi(z).
\]

Hence, if the metric \( \alpha g \) on \( D \) is balanced, then the map \( f \) is a holomorphically isometric mapping from \( (D, \alpha g) \) into \( (P^N(\mathbb{C}), d_{FS}) \) (Note a projectively induced metric is not always balanced, e.g., see Example 1 in Loi-Zedda [27]). The map \( f \) is called in Cahen-Gutt-Rawnsley [4] the coherent states map. Balanced metric plays a fundamental role in the geometric quantization and quantization by
deformation of a Kähler manifold (e.g., see Berezin [3], Cahen-Gutt-Rawnsley [4], Engliš [8] and Luč [30]). It also related to the Bergman kernel expansion (e.g., see Loi [23] and references therein).

We remark that in Donaldson’s study (e.g., see Donaldson [7]) of asymptotic stability for polarized algebraic manifolds, balanced metrics play a central role when the polarized algebraic manifolds admit Kähler metrics of constant scalar curvature. The balanced metrics are also of significance in some questions concerning (semi) stability of projective algebraic varieties (see e.g., Mabuchi [34] and Zhang [44]). For the study of the balanced metrics, see also Cuccu-Loi [6], Engliš [8, 11], Feng-Tu [16], Greco-Loi [17], Loi [24], Loi-Mossa [25], Loi-Zedda [26, 27], Loi-Zedda-Zuddas [28] and Zedda [42].

However, very little seems to be known about the existence of balanced metrics on general noncompact manifolds or even domains in \( \mathbb{C}^n \) (e.g., see Engliš [11] and Loi-Mossa [25]). Currently, the only noncompact manifolds on which balanced metrics are known to exist are the homogeneous spaces. In the nonhomogeneous setting, the problem of existence of balanced metrics seems to be open.

In this paper we will obtain the existence of balanced metrics on a class of bounded nonhomogeneous domains.

Every Hermitian symmetric manifold of noncompact type can be realized as a bounded symmetric domain in some \( \mathbb{C}^d \) by the Harish-Chandra embedding theorem. Then, the classification of Hermitian symmetric manifolds of noncompact type coincides with the classification of bounded symmetric domains. In 1935, E. Cartan proved that there exist only six types of irreducible bounded symmetric domains. They are four types of classical bounded symmetric domains and two exceptional domains.

Let \( M_{m,n} \) be the set of all \( m \times n \) matrices \( z = (z_{ij}) \) with complex entries. Let \( \overline{z} \) be the complex conjugate of the matrix \( z \) and let \( z^t \) be the transpose of the matrix \( z \). \( I \) denotes the identity matrix.

If a square matrix \( z \) is positive definite, then we write \( z > 0 \). For each irreducible bounded symmetric domain \( \Omega \) (refer to Hua [18]), we list the generic norm \( N_{\Omega}(z, \overline{z}) \) of \( \Omega \) according to its type as following.

(i) If \( \Omega = \Omega_I(m, n) := \{ z \in M_{m,n} : I - z\overline{z} > 0 \} \subset \mathbb{C}^d \) (the classical domains of type \( I \)), then \( N_{\Omega}(z, \overline{z}) = \det(I - z\overline{z}) \). Specially, when \( m = 1 \), then \( \Omega \) is the unit ball \( \mathbb{B}^n \) in \( \mathbb{C}^n \) and \( N_{\Omega}(z, \overline{z}) = 1 - \| z \|^2 \).

(ii) If \( \Omega = \Omega_{II}(n) := \{ z \in M_{n,n} : z^t = -z, I - z\overline{z} > 0 \} \subset \mathbb{C}^d \) (the classical domains of type \( II \)), then \( N_{\Omega}(z, \overline{z}) = \det(I - z\overline{z})^{1/2} \).

(iii) If \( \Omega = \Omega_{III}(n) := \{ z \in M_{n,n} : z = z^t = z, I - z\overline{z} > 0 \} \subset \mathbb{C}^d \) (the classical domains of type \( III \)), then \( N_{\Omega}(z, \overline{z}) = \det(I - z\overline{z}) \).

(iv) If \( \Omega = \Omega_{IV}(n) := \{ z \in \mathbb{C}^n : 1 - 2z\overline{z} + |z|^2 > 0, z\overline{z} < 1 \} \) (the classical domains of type \( IV \)), then \( N_{\Omega}(z, \overline{z}) = 1 - 2z\overline{z} + |z|^2 \).

There are two more exceptional domains \( \Omega_{V}(16) \) and \( \Omega_{VI}(27) \) of dimension 16 and 27 respectively. We call them Type V and Type VI, respectively. For the precise definitions of these exceptional domains, see Part V in [13].

Now we introduce the numerical invariants: the rank \( r \), the multiplicities \( a \) and \( b \), and the genus \( p = 2 + a(r - 1) + b \) for an irreducible bounded symmetric domain. The list of numerical invariants of each irreducible bounded symmetric domain of six types is the following:

(1) For \( \Omega_I(m, n) \) (\( 1 \leq m \leq n \)), its rank \( r = m \), its multiplicities \( a = 2 \) and \( b = n - m \), and its genus \( p = m + n \).

(2) For \( \Omega_{II}(2n) \) (\( n \geq 3 \)), its rank \( r = n \), its multiplicities \( a = 4 \) and \( b = 0 \), and its genus \( p = 2(2n - 1) \).

(3) For \( \Omega_{III}(n) \) (\( n \geq 2 \)), its rank \( r = n \), its multiplicities \( a = 1 \) and \( b = 0 \), and its genus \( p = n + 1 \).

(4) For \( \Omega_{IV}(n) \) (\( n \geq 5 \)), its rank \( r = 2 \), its multiplicities \( a = n - 2 \) and \( b = 0 \), and its genus \( p = n \).

(5) For \( \Omega_{V}(16) \), its rank \( r = 2 \), its multiplicities \( a = 6 \) and \( b = 4 \), and its genus \( p = 12 \).

(6) For \( \Omega_{VI}(27) \), its rank \( r = 3 \), its multiplicities \( a = 8 \) and \( b = 0 \), and its genus \( p = 18 \).

Remark that \( \Omega_{II}(2), \Omega_{II}(1) \) and \( \Omega_{IV}(1) \) are biholomorphically equivalent to \( \Omega_I(1, 1) \); \( \Omega_{II}(3) \) is biholomorphically equivalent to \( \Omega_I(3, 1) \); \( \Omega_{II}(4) \) is biholomorphically equivalent to \( \Omega_{IV}(6) \); \( \Omega_{IV}(3) \) is biholomorphically equivalent to \( \Omega_{II}(2, 2) \), and \( \Omega_{IV}(4) \) is biholomorphically equivalent to \( \Omega_I(2, 2) \). But
\( \Omega_{IV}(2) \) is not an irreducible bounded symmetric domain, since \( \Omega_{IV}(2) \) is biholomorphically equivalent to the bidisc \( \mathbb{B} \times \mathbb{B} \).

Throughout this paper, each irreducible bounded symmetric domain is always one of irreducible bounded symmetric domains of six types.

Let \( \Omega \) be an irreducible bounded symmetric domain in \( \mathbb{C}^d \) of genus \( p \) in its Harish-Chandra realization. Since \( \Omega \) is a bounded circular domain and contains the origin, there is a homogeneous holomorphic polynomial set

\[
\left\{ \frac{1}{\sqrt{V(\Omega)}}, h_1(z), h_2(z), \cdots \right\},
\]

such that it is an orthonormal basis of the Hilbert space \( A^2(\Omega) \) of square-integrable holomorphic functions on \( \Omega \), where \( V(\Omega) \) is the Euclidean volume of \( \Omega \) in \( \mathbb{C}^d \) and \( \| \Omega \| \) is defined by

\[
\| \Omega \| = \frac{1}{\sqrt{V(\Omega)}} + h_1(z)h_1(\xi) + h_2(z)h_2(\xi) + \cdots
\]

for all \( z, \xi \in \Omega \). Obviously, \( V(\Omega)K_0(0, 0) = 1 \) and \( 1 \leq V(\Omega)K_\Omega(z, \bar{\xi}) < +\infty \) for all \( z \in \Omega \). The generic norm of \( \Omega \) is defined by

\[
N_\Omega(z, \xi) := (V(\Omega)K_\Omega(z, \xi))^{-\frac{1}{p}} (z, \xi \in \Omega),
\]

where \( (V(\Omega)K_\Omega(z, \bar{\xi}))^{-\frac{1}{p}} := \exp(-\frac{1}{p} \ln(V(\Omega)K_\Omega(z, \bar{\xi}))) \), in which \( \ln \) denotes the principal branch of logarithm (note \( K_\Omega(z, \xi) \neq 0 \) for all \( z, \xi \in \Omega \)). Thus \( N_\Omega(0, 0) = 1, 0 < N_\Omega(z, \bar{\xi}) \leq 1 \) for all \( z \in \Omega \) and \( N_\Omega(z, \bar{\xi}) = 0 \) on the boundary of \( \Omega \).

Let \( \Omega_i \subset \mathbb{C}^{d_i} \) be an irreducible bounded symmetric domain \((1 \leq i \leq k)\). For given positive integer \( d_0 \), positive real numbers \( \mu_i (1 \leq i \leq k) \), the generalized Cartan-Hartogs domain \((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)\) is defined by

\[
\left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu) := \left\{ (z, w) \in \prod_{j=1}^k \Omega_j \times \mathbb{B}^{d_0} : \|w\|^2 < \sum_{j=1}^k N_{\Omega_j}(z_j, \overline{x_j})^{\mu_j} \right\}, \tag{1.2}
\]

where \( \mu = (\mu_1, \ldots, \mu_k) \in (\mathbb{R}_+)^k \), \( z = (z_1, \ldots, z_k) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k} \), \( \| \cdot \| \) is the standard Hermitian norm in \( \mathbb{C}^{d_0} \), \( N_{\Omega_j}(z_j, \overline{x_j}) \) is the generic norm of \( \Omega_j \) \((1 \leq i \leq k)\) and \( \mathbb{B}^{d_0} := \{ w \in \mathbb{C}^{d_0} : \|w\|^2 < 1 \} \). Note \( \prod_{j=1}^k N_{\Omega_j}(0, 0)^{\mu_j} = 1, 0 < \prod_{j=1}^k N_{\Omega_j}(z_j, \overline{x_j})^{\mu_j} \leq 1 \) on \( \prod_{j=1}^k \Omega_j \) and \( \prod_{j=1}^k N_{\Omega_j}(z_j, \overline{x_j})^{\mu_j} = 0 \) on the boundary \( \partial(\prod_{j=1}^k \Omega_j) \). Thus \( \partial(\prod_{j=1}^k \Omega_j) \subset \partial((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)) \). For the reference of the generalized Cartan-Hartogs domains, see Ahn-Park [2], Tu-Wang [37] and Wang-Hao [38].

When \( k \geq 2 \), then any generalized Cartan-Hartogs domain \((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)\) is a bounded non-homogeneous domain in \( \mathbb{C}^{d_0+\cdots+d_k} \). In fact, for \((z_1, \ldots, z_k, w) \in \prod_{j=1}^k \Omega_j \times (\mathbb{C}^{d_0} \setminus \{0\})\), we have

\[
\ln \|w\|^2 - \ln\left( \prod_{j=1}^k N_{\Omega_j}(z_j, \overline{x_j})^{\mu_j} \right) = \ln\|w\|^2 + \sum_{j=1}^m \frac{H_j}{P_j} \ln(V(\Omega_j)K_{\Omega_j}(z_j, \overline{x_j}))
\]

is real analytic on \( \prod_{j=1}^k \Omega_j \times (\mathbb{C}^{d_0} \setminus \{0\}) \) and each \( \frac{H_j}{P_j} \ln(V(\Omega_j)K_{\Omega_j}(z_j, \overline{x_j})) \) is a real analytic strictly plurisubharmonic function on \( \Omega_j \) \((1 \leq j \leq k)\). Thus, the generalized Cartan-Hartogs domain
\((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\) is strongly pseudoconvex at the boundary part

\[
\left\{(z, w) \in \prod_{j=1}^{k} \Omega_j \times \mathbb{C}^{d_0} : \ln\|w\|^2 + \sum_{j=1}^{m} \frac{\mu_j}{\nu_j} \ln(V(\Omega_j)K_{\Omega_j}(z_j, \overline{z}_j)) = 0, w \neq 0 \right\}
\]

\[
= \left\{(z, w) \in \prod_{j=1}^{k} \Omega_j \times \mathbb{C}^{d_0} : \|w\|^2 = \prod_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z}_j)^{\mu_j}, w \neq 0 \right\}
\]

Therefore, if a generalized Cartan-Hartogs domain \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\) \((k \geq 2)\) is homogeneous, then the generalized Cartan-Hartogs domain must be biholomorphic to the unit ball by the Wong-Rosay theorem (see Rudin [36], Theorem 15.5.10 and its Corollary), and thus, it is a strongly pseudoconvex domain. From the boundary

\[
b\left(\prod_{j=1}^{k} \Omega_j\right)^{B^{d_0}}(\mu) \supset b\left(\prod_{j=1}^{k} \Omega_j\right),
\]

we have that the boundary \(b\left(\prod_{j=1}^{k} \Omega_j\right)^{B^{d_0}}(\mu)\) \((k \geq 2)\) contains a positive-dimensional complex submanifold. This is impossible, since, in our case, the generalized Cartan-Hartogs domain is strongly pseudoconvex. So any generalized Cartan-Hartogs domain \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\) \((k \geq 2)\) is a bounded nonhomogeneous domain.

For the generalized Cartan-Hartogs domain \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\), define

\[
\Phi(z, w) := -\ln\left(\prod_{j=1}^{k} N_{\Omega_j}(z_j, \overline{z}_j)^{\mu_j} - \|w\|^2\right).
\]

The Kähler form \(\omega(\mu)\) on \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\) is defined by

\[
\omega(\mu) := \frac{\sqrt{\pi}}{2\nu} \partial \overline{\partial} \Phi.
\]

The canonical metric \(g(\mu)\) on \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\) associated to \(\omega(\mu)\) is given by

\[
ds^2 = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial Z_i \partial \overline{Z}_j} dZ_i \otimes d\overline{Z}_j.
\]

where

\[
n = \sum_{j=0}^{k} d_j, \ Z = (Z_1, \ldots, Z_n) := (z, w).
\]

Note that \(\Phi(z, w)\) defined by (1.3) is also a real analytic strictly plurisubharmonic exhaustion function for the generalized Cartan-Hartogs domain \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\). Thus \((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu)\) is a bounded pseudoconvex domain in \(\mathbb{C}^{d_0+\cdots+d_k}\).

For the generalized Cartan-Hartogs domain \(\left((\prod_{j=1}^{k} \Omega_j)^{B^{d_0}}(\mu), g(\mu)\right)\), we have (see Theorem 3.1 in this paper) that the Rawnsley’s \(\varepsilon\)-function admits the expansion:

\[
\varepsilon_\alpha(z, w) = \sum_{j=0}^{n} a_j(z, w)\alpha^{n-j}, \ (z, w) \in \left(\prod_{j=1}^{k} \Omega_j\right)^{B^{d_0}}(\mu),
\]

(1.5)
where \( n = \sum_{j=0}^{k} d_{j} \). By Th. 1.1 of Lu [29], Th. 4.1.2 and Th. 6.1.1 of Ma-Marinescu [31], Th. 3.11 of Ma-Marinescu [32] and Th. 0.1 of Ma-Marinescu [33], see also Th. 3.3 of Xu [39], we have

\[
\begin{align*}
\begin{cases}
a_0 &= 1, \\
a_1 &= \frac{1}{2}k_g, \\
a_2 &= \frac{1}{2}\alpha g - \frac{1}{2}\triangle k_g + \frac{1}{27}|R|^2 - \frac{1}{6}|Ric|^2 + \frac{27}{8}k_g^2,
\end{cases}
\end{align*}
\]

where \( k_g, \triangle, R \) and \( Ric \) denote the scalar curvature, the Laplace, the curvature tensor and the Ricci curvature associated to the canonical metric \( g(\mu) \), respectively.

In 2012, Loi-Zedda [27] described balanced metrics on an irreducible bounded symmetric domain \( \Omega \) as follows.

**Theorem 1.1.** (Loi-Zedda [27]) Let \( \Omega \) be an irreducible bounded symmetric domain of genus \( p \) equipped with its Bergman metric \( g_B \). Then the metric \( \alpha g_B (\alpha > 0) \) is balanced if and only if \( \alpha > \frac{p-1}{p} \).

Let \( \mathbb{B}^d \) be the unit ball in \( \mathbb{C}^d \) and let the metric \( g_{hyp} \) on \( \mathbb{B}^d \) be given by

\[
ds^2 = -\sum_{i,j=1}^{d} \frac{\partial^2 \ln(1 - \|z\|^2)}{\partial z_i \overline{\partial z_j}} dz_i \otimes d\overline{z_j}.
\]

We call \( (\mathbb{B}^d, g_{hyp}) \) the complex hyperbolic space. Note \( g_{hyp} = \frac{1}{\alpha^2} g_B \) on \( \mathbb{B}^d \) and the genus \( p = d + 1 \) for \( \mathbb{B}^d \). Thus, by Theorem 1.1, we have that \( \alpha g_{hyp} (\alpha > 0) \) is a balanced metric on \( \mathbb{B}^d \) if and only if \( \alpha > d \).

In the special case of \( k = 1 \), the generalized Cartan-Hartogs domain \( \Omega^{B^d} (\mu) \) is also called the Cartan-Hartogs domain. When \( \Omega = \mathbb{B}^d, \mu = 1 \), we have \( \Omega^{B^d} (\mu) = \mathbb{B}^{d+d_0} \). In 2012, Loi-Zedda [27] gave a characterization of the complex hyperbolic space among the Cartan-Hartogs domains in terms of balanced metrics as follows.

**Theorem 1.2.** (Loi-Zedda [27]) Let \( (\Omega^{B^d} (\mu), g(\mu)) \) be a Cartan-Hartogs domain over the irreducible bounded symmetric domain \( \Omega \) in \( \mathbb{C}^d \) with the canonical metric \( g(\mu) \). Then the metric \( \alpha g(\mu) (\alpha > 0) \) on \( \Omega^{B^d} (\mu) \) is balanced if and only if \( \alpha > d + d_0 \) and \( (\Omega^{B^d} (\mu), g(\mu)) \) is biholomorphically isometric to the complex hyperbolic space \( (\mathbb{B}^{d+d_0}, g_{hyp}) \) (i.e., \( \Omega = \mathbb{B}^d, \mu = 1 \)).

In 2014, Feng-Tu [16] obtained the following improvement of Theorems 1.2.

**Theorem 1.3.** (Feng-Tu [16]) Let \( (\Omega^{B^d} (\mu), g(\mu)) \) be a Cartan-Hartogs domain over the irreducible bounded symmetric domain \( \Omega \) in \( \mathbb{C}^d \) with the canonical metric \( g(\mu) \). Then the coefficient \( a_2 \) (see (1.5)) of the Râmansley’s \( \varepsilon \)-function expansion is a constant on \( \Omega^{B^d} (\mu) \) if and only if \( (\Omega^{B^d} (\mu), g(\mu)) \) is biholomorphically isometric to the complex hyperbolic space \( (\mathbb{B}^{d+d_0}, g_{hyp}) \) (i.e., \( \Omega = \mathbb{B}^d, \mu = 1 \)).

By Ligocka [22], the Bergman kernel of the Hartogs type domain \( (\prod_{j=1}^{k} \Omega_{j})^{B^d} (\mu) \) can be expressed as infinite sum in terms of the weighted Bergman kernels of the base space \( \prod_{j=1}^{k} \Omega_{j} \) with weights \( \prod_{j=1}^{k} N_{j}(z_{j}, \overline{z}_{j})^{\mu_{j}} \) \( (0 \leq m < +\infty) \). Ahn-Park [2] use the technique in Ligocka [22] to obtain an explicit form of the Bergman kernel of the Hilbert space of square integrable holomorphic functions on the generalized Cartan-Hartogs domain \( (\prod_{j=1}^{k} \Omega_{j})^{B^d} (\mu) \) and determine the condition that their Bergman kernel functions have zeros in 2012.

By studying the explicit solutions of a class of complex Monge-Ampère equations on generalized Cartan-Hartogs domain \( (\prod_{j=1}^{k} \Omega_{j})^{B^d} (\mu) \), Wang-Hao [38] described Kähler-Einstein metrics on such domain in 2014.

Each generalized Cartan-Hartogs domain \( (\prod_{j=1}^{k} \Omega_{j})^{B^d} (\mu) \) \( (k \geq 2) \) is a bounded nonhomogeneous domain. The purpose of this paper is to obtain the existence of balanced metrics for such bounded
nonhomogeneous domains. In this paper, we obtain an explicit formula for the Bergman kernel of
the weighted Hilbert space $H_\alpha$ of square integrable holomorphic functions on $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu), g(\mu))$
with the weight $\exp(-\alpha\Phi)$ (where $\Phi$ is a globally defined Kähler potential for the canonical metric $g(\mu)$) for $\alpha > 0$. Furthermore, we give an explicit expression of the Rawnsley’s $\varepsilon$-function expansion for
$((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu), g(\mu))$, and, use the asymptotics of the Rawnsley’s $\varepsilon$-function to draw necessary and
sufficient conditions for the metric $ag(\mu)$ on the generalized Cartan-Hartogs domain $(\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$
to be balanced as follows.

**Theorem 1.4.** Let $\Omega_i \subset \mathbb{C}^{d_i}$ be an irreducible bounded symmetric domain, and denote the rank $r_i$, the
cr{characteristic multiplicities} $a_i, b_i$, the dimension $d_i$ and the genus $p_i$ for $\Omega_i$ $(1 \leq i \leq k)$. For
\begin{itemize}
\item [(i)] given positive integer $d_0$, positive real numbers $\mu_i (1 \leq i \leq k)$, let $g(\mu)$ be the canonical metric on the
\begin{itemize}
\item [generalized Cartan-Hartogs domain $(\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$.
\item [Then we have the results as follows.
\begin{itemize}
\item [(i)] The metric $ag(\mu)$ on the generalized Cartan-Hartogs domain $(\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$ is balanced if and only if
$$\alpha > \max \left\{ \frac{d_0}{\mu_1}, \ldots, \frac{p_k - 1}{\mu_k} \right\}$$
\item [and]
$$\prod_{i=1}^k \prod_{j=1}^{r_i} \left( \mu_i x - p_i + 1 + (j - 1)\frac{a_i}{2} \right)_{1+b_i+(r_i-j)a_i} = \prod_{j=1}^k \mu_j^{d_j} \cdot \prod_{j=1}^d (x-j), \quad (1.7)$$
\item [where $d = \sum_{j=1}^k d_j$ and $(x)m = \frac{\Gamma(x+m)}{\Gamma(x)} = x(x+1)(x+2)\cdots(x+m-1)$.
\item [(ii)] If the metric $ag(\mu)$ on the domain $(\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$ is balanced, then each $\Omega_j$ $(1 \leq j \leq k)$ must be biholomorphic to
$$\Omega_I(1,n) \equiv \mathbb{B}^n := \{ z \in \mathbb{C}^n : \|z\|^2 < 1 \}, \quad \Omega_{III}(2) := \{ z \in \mathcal{M}_{2,2} : z^t = z, I - z\overline{z} > 0 \},$$
\item [or]
$$\Omega_{IV}(m) := \{ z \in \mathbb{C}^m : 1 - 2z\overline{z} + |z\overline{z}|^2 > 0, \|z\|^2 < 1 \} \quad (m \geq 5 \text{ and } m \text{ are odd}).$$
\item [(iii)] For $\alpha > \max \left\{ d_0 + d_1 + d_2, \frac{p_1 - 1}{\mu_1}, \frac{p_2 - 1}{\mu_2} \right\}$, then the metric $ag(\mu)$ on the domain $(\Omega_1 \times \Omega_2)^{\mathbb{B}^{d_0}}(\mu)$
is balanced if and only if $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu), g(\mu))$ is biholomorphically isometric to
$$((\mathbb{B}^d \times \mathbb{B})^{\mathbb{B}^{d_0}}(1, \frac{1}{d+1}), g(1, \frac{1}{d+1})) \quad \text{or} \quad ((\mathbb{B} \times \Omega_{III}(2))^{\mathbb{B}^{d_0}}(1, \frac{1}{2}), g(1, \frac{1}{2})).$$
\end{itemize}
\end{itemize}

For example, for $\Omega_{IV}(5) \times \prod_{j=1}^3 \mathbb{B}$ and $\mu = (\frac{1}{2}, 1, \frac{1}{3}, \frac{1}{7})$, since
$$\prod_{i=1}^4 \prod_{j=1}^{r_i} \left( \mu_i x - p_i + 1 + (j - 1)\frac{a_i}{2} \right)_{1+b_i+(r_i-j)a_i} = \prod_{j=1}^4 \left( \mu_1 x - j \right) \cdot \prod_{j=2}^4 \left( \mu_2 x - j \right),$$
by Theorem 1.4(i), when $\alpha > 8 + d_0$, the metric $ag(\mu)$ on the generalized Cartan-Hartogs domain
$$((\Omega_{IV}(5) \times \prod_{j=1}^3 \mathbb{B})^{\mathbb{B}^{d_0}}(1, \frac{1}{2}, 1, \frac{1}{3}, \frac{1}{7}), g(1, \frac{1}{2})).$$
is balanced, where $g(\mu)$ is the canonical metric. So the example complements Theorem 1.4(ii).

As a supplement to our main result, we will prove the following result.

**Theorem 1.5.** Let $\Omega_i \subset \mathbb{C}^{d_i}$ be an irreducible bounded symmetric domain ($i = 1, 2$). For any positive integer $d_0$, then there exist positive numbers $\mu_1, \mu_2$ such that the coefficient $a_2$ (see (1.5)) of the Rawnsley’s $\varepsilon$-function expansion of

$$
((\Omega_1 \times \Omega_2)^{\mathbb{B}^{d_0}}(\mu_1, \mu_2), g(\mu_1, \mu_2))
$$

is a constant.

**Remark 1.1.** By Theorem 1.4(iii) and Theorem 1.5, there exists generalized Cartan-Hartogs domain

$$
((\Omega_1 \times \Omega_2)^{\mathbb{B}^{d_0}}(\mu_1, \mu_2), g(\mu_1, \mu_2))
$$

such that the coefficient $a_2$ of its Rawnsley’s $\varepsilon$-function expansion is constant, but the metric $a(\mu_1, \mu_2)$ on

$$(\Omega_1 \times \Omega_2)^{\mathbb{B}^{d_0}}(\mu_1, \mu_2)$$

is not balanced for all $\alpha > 0$ (cf. Theorem 1.3).

For $k \geq 2$, each generalized Cartan-Hartogs domain $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_j}}(\mu), g(\mu))$ is a noncompact, complete (the argument is analogous to [41]) and nonhomogeneous Kähler manifold. By studying the explicit solutions of a class of complex Monge-Ampère equations on generalized Cartan-Hartogs domains, Wang-Hao [38] described Kähler-Einstein metrics on such domains in 2014. From the proof in Wang-Hao [38] and (2.14) in this paper, one can state the result as follows.

**Theorem 1.6.** (Wang-Hao [38]) Let $\Omega_i \subset \mathbb{C}^{d_i}$ be an irreducible bounded symmetric domain and let $p_i$ and $d_i$ be the genus and the dimension respectively for $\Omega_i$, $1 \leq i \leq k$. Then $\mu_i = \frac{p_i}{\sum_{j=1}^k d_j + 1} (1 \leq i \leq k)$ if and only if the canonical metric $g(\mu)$ on the domain $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$ is Kähler-Einstein.

**Remark 1.2.** When $\alpha > 8 + d_0$, the metric $a(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2})$ on the generalized Cartan-Hartogs domain

$$(\Omega_{IV}(5) \times \prod_{j=1}^3 \mathbb{B})^{\mathbb{B}^{d_0}}(\frac{1}{2}, 1, \frac{1}{3}, \frac{1}{7})$$

is balanced by Theorem 1.4(i), but is not Kähler-Einstein for all $\alpha > 0$ by Theorem 1.6, where $g(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2})$ is the canonical metric.

**Remark 1.3.** The metric $a(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3})$ ($\alpha > 0$) on the generalized Cartan-Hartogs domain

$$(\mathbb{B} \times \mathbb{B})^{\mathbb{B}}(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$$

is Kähler-Einstein by Theorem 1.6, but is not balanced for all $\alpha > 0$ by Theorem 1.4(iii), where $g(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ is the canonical metric.

The paper is organized as follows. In Section 2, we obtain an explicit formula for the Bergman kernel $K_{\alpha}$ of the weighted Hilbert space $H_{\alpha}$ of square integrable holomorphic functions on a generalized Cartan-Hartogs domain $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_j}}(\mu), g(\mu))$ with the weight $\exp\{-\alpha \Phi\}$, in terms of ranks, Hua polynomials and generic norms of $\mathbb{B}^{d_0}$ and $\Omega_i$ ($1 \leq i \leq k$). In Section 3, using results in Section 2, we give the explicit expansion of the Rawnsley’s $\varepsilon$-function of $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_j}}(\mu), g(\mu))$. In Section 4, using the explicit expression of the Rawnsley’s $\varepsilon$-function expansion, we obtain the necessary and sufficient conditions (Theorem 1.4 in the paper) for the metric $a(\mu)$ on the generalized Cartan-Hartogs domain $((\prod_{j=1}^k \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$ to be a balanced metric. In Section 5, we will give the proof of Theorem 1.5.
2 The reproducing kernel of $\mathcal{H}_\alpha$ for $(\prod_{j=1}^k \Omega_j)^{B^d_\alpha}(\mu)$ with the canonical metric $g(\mu)$

Let $\Omega$ be an irreducible bounded symmetric domain in $\mathbb{C}^d$ in its Harish-Chandra realization. Thus $\Omega$ is the open unit ball of a Banach space which admits the structure of a $JB^*$-triple. We denote the rank $r$, the characteristic multiplicities $a, b$, the dimension $d$, the genus $p$, and the generic norm $N_\Omega(z, \overline{\varpi})$ for $\Omega$. Thus

$$d = \frac{r(r-1)}{2}a + rb + r, \quad p = (r-1)a + b + 2.$$  \hspace{1cm} (2.1)

For any $s > -1$, the value of the Hua integral $\int_\Omega N_\Omega(z, \overline{\varpi})^s dm(z)$ is given by

$$\int_\Omega N_\Omega(z, \overline{\varpi})^s dm(z) = \frac{\chi(0)}{\chi(s)} \int_\Omega dm(z),$$  \hspace{1cm} (2.2)

where $dm(z)$ denotes the Euclidean measure on $\mathbb{C}^d$, $\chi$ is the Hua polynomial

$$\chi(s) := \prod_{j=1}^r \left( s + 1 + \left( j - 1 \right) \frac{a}{2} \right),$$  \hspace{1cm} (2.3)

in which, for a non-negative integer $m$, $(s)_m$ denotes the raising factorial

$$(s)_m := \frac{\Gamma(s + m)}{\Gamma(s)} = s(s+1) \cdots (s + m - 1).$$

Let $G$ stand for the identity connected component of the group of biholomorphic self-maps of $\Omega$, and $K$ for the stabilizer of the origin in $G$. Under the action $f \mapsto f \circ k (k \in K)$ of $K$, the space $P$ of holomorphic polynomials on $\mathbb{C}^d$ admits the Peter-Weyl decomposition

$$P = \bigoplus_\lambda P_\lambda,$$

where the summation is taken over all partitions $\lambda$, i.e., $r$-tuples $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$, and the spaces $P_\lambda$ are $K$-invariant and irreducible. For each $\lambda$, $P_\lambda \subset P_{|\lambda|}$, where $|\lambda|$ denotes the weight of partition $\lambda$, i.e., $|\lambda| := \sum_{j=1}^r \lambda_j$, and $P_{|\lambda|}$ is the space of homogeneous holomorphic polynomials of degree $|\lambda|$. Let

$$\langle f, g \rangle_F := \int_{\mathbb{C}^d} f(z) \overline{g(z)} dm_F(z)$$  \hspace{1cm} (2.4)

be the Fock-Fischer inner product on the space $P$ of holomorphic polynomials on $\mathbb{C}^d$, where

$$dm_F(z) := \exp\{-m(z, \overline{\varpi})\} \left( \frac{\chi^{-1}}{\pi} \partial \overline{\partial} m(z, \overline{\varpi}) \right)^d$$  \hspace{1.5cm} (2.5)

and $m(z, \overline{\varpi}) := -\frac{\partial \ln N_\Omega(tz, \overline{\varpi})}{\partial t}_{t=0} = -\frac{\partial N_\Omega(tz, \overline{\varpi})}{\partial t}_{t=0}$.

For every partition $\lambda$, let $K_\lambda(z_1, \overline{z_2})$ be the reproducing kernel of $P_\lambda$ with respect to (2.4). The weighted Bergman kernel of the weighted Hilbert space $A^2(\mathbb{C}^d, \rho_F)$ of square-integrable holomorphic functions on $\mathbb{C}^d$ with the measure $dm_F$ is

$$K(z_1, \overline{z_2}) := \sum_\lambda K_\lambda(z_1, \overline{z_2}).$$  \hspace{1cm} (2.6)
The kernels $K_{\lambda}(z_1, \overline{z}_2)$ are related to the generic norm $N_{\Omega}(z_1, \overline{z}_2)$ by the Faraut-Korányi formula

$$N_{\Omega}(z_1, \overline{z}_2)^{-s} = \sum_\lambda (s)_\lambda K_{\lambda}(z_1, \overline{z}_2),$$

(2.7)

where the series converges uniformly on compact subsets of $\Omega \times \Omega$, $s \in \mathbb{C}$, in which $(s)_\lambda$ denote the generalized Pochhammer symbol

$$(s)_\lambda := \prod_{j=1}^r (s - j - 1/2)a_j.$$

(2.8)

For the proofs of above facts and additional details, we refer, e.g., to [12], [13] and [40].

**Lemma 2.1.** Let $\Omega_i$ be an irreducible bounded symmetric domain in $\mathbb{C}^{d_i}$ in its Harish-Chandra realization, and denote the generic norm $N_{\Omega_i}$ and the genus $p_i$ for $\Omega_i$ ($1 \leq i \leq k$). For $z_i^0 \in \Omega_i$, let $\phi_i$ be an automorphism of $\Omega_i$ such that $\phi_i(z_i^0) = 0$, $1 \leq i \leq k$. By [38], the function

$$\psi(z_1, \ldots, z_k) := \prod_{i=1}^k \frac{N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})}{N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i}}$$

(2.9)

satisfies

$$|\psi(z_1, \ldots, z_k)|^2 = \prod_{i=1}^k \left( \frac{N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})}{N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i}} \right)^{\mu_i}.$$  \hspace{1cm} (2.10)

Define the mapping

$$F : \left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu_1, \ldots, \mu_k) \rightarrow \left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu_1, \ldots, \mu_k),$$

$$(z_1, \ldots, z_k, w) \mapsto (\phi_1(z_1), \ldots, \phi_k(z_k), \psi(z_1, \ldots, z_k)w).$$

(2.11)

Then $F$ is an isometric automorphism of $\left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu_1, \ldots, \mu_k)$, that is

$$\partial \overline{\partial}(F(z_1, \ldots, z_k, w))) = \partial \overline{\partial}(\Phi(z_1, \ldots, z_k, w)),$$

(2.12)

where $\Phi(z_1, \ldots, z_k, w) := -\ln \left( \prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i} - \|w\|^2 \right)$ (see (1.3)).

**Proof.** It is easy to see that $F$ is an automorphism of $\left( \prod_{j=1}^k \Omega_j \right)^{\mathbb{B}^{d_0}}(\mu_1, \ldots, \mu_k)$, and

$$N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})^{\mu_i} = J\phi_i(z_i)N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i}J\phi_i(z_i),$$

(2.13)

where $J\phi_i(z_i)$ is the holomorphic Jacobian of the automorphism $\phi_i$ of $\Omega_i$, $1 \leq i \leq k$.

By (2.10) and (2.13), we have

$$\prod_{i=1}^k N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})^{\mu_i} - \|\psi(z_1, \ldots, z_k)w\|^2$$

$$= \prod_{i=1}^k N_{\Omega_i}(\phi_i(z_i), \overline{\phi_i(z_i)})^{\mu_i} \left( 1 - \frac{\|w\|^2}{\prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i}} \right)$$

$$= \prod_{i=1}^k |J\phi_i(z_i)|^{2\mu_i} \left( \prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z_i})^{\mu_i} - \|w\|^2 \right),$$

which implies (2.12). □
Lemma 2.2. Let $\Omega_i$ be an irreducible bounded symmetric domain in $\mathbb{C}^d_i$ in its Harish-Chandra realization, and denote the generic norm $N_{\Omega_i}(z_i, \overline{z}_i)$, the dimension $d_i$ and the genus $p_i$ for $\Omega_i$ ($1 \leq i \leq k$). Then we have

\[
(\partial \Phi)^n = \frac{\prod_{i=1}^k \left( \mu_i^d_i C_{\Omega_i} N_{\Omega_i}(z_i, \overline{z}_i)^\mu_i (\sum_{j=1}^k d_j + 1) - p_i \right)}{\left( \prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i} - \|w\|^2 \right)^{n+1}} \left( \sum_{j=1}^n dZ_j \wedge \overline{dZ_j} \right)^n,
\]

where

\[
\Phi(z_1, \ldots, z_k, w) = -\ln \left( \prod_{i=1}^k N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i} - \|w\|^2 \right),
\]

\[
C_{\Omega_i} = \det \left( -\frac{\partial^2 \ln N_{\Omega_i}(z_i, \overline{z}_i)}{\partial z_i^t \partial \overline{z}_i} \right) \bigg|_{z_i=0},
\]

\[
n = \sum_{j=0}^k d_j, \quad Z = (Z_1, \ldots, Z_n) = (z_1, \ldots, z_k, w).
\]

Proof. It is well known that

\[
\frac{(\overline{\omega_{2\pi}^n} \partial \Phi)^n}{n!} = \det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right) \omega_0^n, \tag{2.15}
\]

where $\omega_0 = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^n dZ_j \wedge \overline{dZ_j}$, $\frac{\partial}{\partial Z_j} = (\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n})^t$, $\frac{\partial}{\partial Z} = (\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n})$ and $\frac{\partial^2}{\partial Z^t \partial Z} = \frac{\partial}{\partial Z} \frac{\partial}{\partial Z}$.

From (2.12) and (2.15), we get

\[
\det \left( \frac{\partial^2 \Phi(F)}{\partial Z^t \partial Z} \right) = \det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right). \tag{2.16}
\]

By the identity

\[
\frac{\partial^2 \Phi(F)}{\partial Z^t \partial Z} = \frac{\partial F}{\partial Z} \frac{\partial^2 \Phi}{\partial Z^t \partial Z} (F(Z)) \left( \frac{\partial F}{\partial Z^t} \right)^t, \tag{2.17}
\]

and (2.16), we deduce

\[
\det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right)(Z) = |JF(Z)|^2 \det \left( \frac{\partial^2 \Phi}{\partial Z^t \partial Z} \right)(F(Z)), \tag{2.18}
\]

where

\[
F := (F_1, F_2, \ldots, F_n), \quad \frac{\partial F}{\partial Z} := \left( \frac{\partial F_1}{\partial Z^1}, \frac{\partial F_2}{\partial Z^2}, \ldots, \frac{\partial F_n}{\partial Z^n} \right)
\]

and

\[
JF(Z) := \det \left( \frac{\partial F}{\partial Z} \right)(Z).
\]

Let $Z^0 = (z_1^0, \ldots, z_k^0, w^0) \in \left( \prod_{j=1}^k \Omega_j \right)^{B_{d_0}}$ ($\mu_1, \ldots, \mu_k$), $\tilde{Z}^0 := (\tilde{z}_1^0, \ldots, \tilde{z}_k^0, \tilde{w}^0) = F(Z^0)$. By (2.9) and (2.11), then

\[
\tilde{Z}^0 = \left( 0, \ldots, 0, \frac{w^0}{\prod_{i=1}^k N_{\Omega_i}(z_i^0, \overline{z}_i^0)^{\mu_i}} \right)
\]

and

\[
|JF(Z^0)|^2 = \prod_{i=1}^k |J\phi_i(z_i^0)|^2 \cdot |\psi(\tilde{z}_1^0, \ldots, \tilde{z}_k^0)|^{2d_0}. \tag{2.19}
\]
Using $N_{\Omega_i}(0, z_i) = 1$, (2.9), (2.13), (2.19) and (2.18), we have
\[ |JF(Z^0)|^2 = \prod_{i=1}^{k} \frac{1}{N_{\Omega_i}(z_i^0, z_i^0)^{\mu_i + \mu_i d_0}}, \] (2.20)
and
\[ \det \left( \frac{\partial^2 \Phi}{\partial Z^i \partial Z^j} \right)(Z^0) = \prod_{i=1}^{k} \frac{1}{N_{\Omega_i}(z_i^0, z_i^0)^{\mu_i + \mu_i d_0}} \det \left( \frac{\partial^2 \Phi}{\partial Z^i \partial Z^j} \right)(Z^0). \] (2.21)

A direct calculation gives
\[ \frac{\partial^2 \Phi}{\partial Z^i \partial Z^j}(0, \ldots, 0, w) = \begin{pmatrix} \mu_1 \frac{1}{1-ww^*} C_{d_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\mu_k}{1-ww^*} C_{d_k} & 0 \\ 0 & \cdots & 0 & \frac{1}{1-ww^*} I_{d_0} + \frac{1}{1-ww^*} \overline{w}w \end{pmatrix}, \] (2.22)
where $I_{d_0}$ denotes the $d_0 \times d_0$ identity matrix, $\overline{w}$ is the complex conjugate transpose of the row vector $w = (w_1, w_2, \ldots, w_{d_0})$, and $C_{d_0} = -\frac{\partial^2 \ln N_{\Omega_i}}{\partial z^i \partial z^j} \Big|_{z_i = 0}$.

From (2.22), we have
\[ \det \left( \frac{\partial^2 \Phi}{\partial Z^i \partial Z^j} \right)(0, \ldots, 0, w) = \prod_{i=1}^{k} \frac{\mu_i^{d_i} \det C_{d_i}}{(1 - \|w\|^2)^{\sum_{j=0}^{\alpha_i} d_j + 1}}. \] (2.23)

Finally, combining (2.21) and (2.23), we have (2.14). □

**Theorem 2.3.** Let $\Omega_i$ be an irreducible bounded symmetric domain in $\mathbb{C}^{d_i}$ in its Harish-Chandra realization, and denote the generic norm $N_{\Omega_i}$, the genus $p_i$, the dimension $d_i$, and the Hua polynomial $\chi_i$ (see (2.3)) for $\Omega_i$ (1 $\leq i \leq k$). Let the generalized Cartan-Hartogs domain $\big( \prod_{j=1}^{k} \Omega_j \big)^{B_{\Phi_0}(\mu)}$ be endowed with the canonical metric $g(\mu)$. Set $n = \sum_{j=0}^{d_i} d_j$. For $\alpha > \max\{n, \frac{p_i-1}{\mu_i}, \ldots, \frac{p_k-1}{\mu_k} \}$, then the Bergman kernel $K_\alpha(Z; \overline{Z})$ of the weighted Hilbert space
\[ \mathcal{H}_\alpha = \left\{ f \in \operatorname{Hol}\left( \big( \prod_{j=1}^{k} \Omega_j \big)^{B_{\Phi_0}(\mu)} \right) : \int_{\big( \prod_{j=1}^{k} \Omega_j \big)^{B_{\Phi_0}(\mu)}} |f|^2 \exp\{-\alpha \Phi_0(\mu)\} \frac{\omega(\mu)^n}{n!} < +\infty \right\} \]
can be written as
\[ K_\alpha(Z; \overline{Z}) = \frac{(\alpha - n)_{d_0}}{\prod_{i=1}^{k} \mu_i^{d_i}} \prod_{i=1}^{k} \frac{1}{N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i + 1}} \prod_{i=1}^{k} \chi_i(1 + t \frac{d}{dt} - \mu_i)^{d_i} \left| \frac{1}{1 - t \frac{d}{dt} \frac{\|w\|^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}}} \right|^{\alpha - d} \bigg|_{t=1} \] (2.24)
where $Z = (z_1, \ldots, z_k, w)$, $d = \sum_{j=1}^{k} d_j$.

**Proof.** By (2.14), the inner product on $\mathcal{H}_\alpha$ is given by
\[ (f, g) = \frac{\prod_{i=1}^{k} \mu_i^{d_i} C_{d_i}}{\pi^n} \int_{\big( \prod_{j=1}^{k} \Omega_j \big)^{B_{\Phi_0}(\mu)}} f(Z)g(Z) \]
\[ \times \prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i(\alpha - d_0) - p_i} \left( 1 - \frac{\|w\|^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right)^{\alpha-n-1} \, dm(Z), \]
where $dm$ denotes the Euclidean measure.

For convenience, we set $\Omega_0 = \mathbb{B}^{d_i}$, $z_0 = w$. Denote the rank $r_i$, the characteristic multiplicities $a_i, b_i$, the dimension $d_i$, the genus $p_i$, the Hua polynomial $\chi_i$, the generalized Pochhammer symbol $(s)_{\lambda}^{(i)}$, the generic norm $N_{\Omega_i}$, and the Euclidean volume $V(\Omega_i)$ for the irreducible bounded symmetric domain $\Omega_i$, $0 \leq i \leq k$.

Let $G_i$ stand for the identity connected components of groups of biholomorphic self-maps of $\Omega_i \subset \mathbb{C}^{d_i}$, and $K_i$ for stabilizers of the origin in $G_i$, respectively. For any $u = (u_0, \ldots, u_k) \in \mathcal{K} := \mathcal{K}_0 \times \cdots \times \mathcal{K}_k$, we define the action

$$\pi(u)f(z_1, \ldots, z_k, w) \equiv f \circ u(z_1, \ldots, z_k, w) := f(u_1 \circ z_1, \ldots, u_k \circ z_k, u_0 \circ w)$$

of $\mathcal{K}$, then the space $\mathcal{P}$ of holomorphic polynomials on $\prod_{j=0}^{k} \mathbb{C}^{d_j}$ admits the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\ell(\lambda_i) \leq r_i, 0 \leq i \leq k} \mathcal{P}_{\lambda_i}^{(0)} \otimes \cdots \otimes \mathcal{P}_{\lambda_k}^{(k)},$$

where the space $\mathcal{P}_{\lambda_i}^{(i)}$ is $K_i$-invariant and irreducible subspace of the space of holomorphic polynomials on $\mathbb{C}^{d_i}$, and $\ell(\lambda_i)$ denotes the length of partition $\lambda_i$ ($0 \leq i \leq k$).

Since $\mathcal{H}_{\alpha}$ is invariant under the action of $\mathcal{K}$, namely, $\forall u \in \mathcal{K}, (\pi(u)f, \pi(u)g) = (f, g)$, $\mathcal{H}_{\alpha}$ admits an irreducible decomposition (see [14])

$$\mathcal{H}_{\alpha} = \bigoplus_{\ell(\lambda_i) \leq r_i, 0 \leq i \leq k} \mathcal{P}_{\lambda_i}^{(0)} \otimes \cdots \otimes \mathcal{P}_{\lambda_k}^{(k)},$$

where $\bigoplus$ denotes the orthogonal direct sum.

For given partition $\lambda_i$ of length $\leq r_i$, let $K_{\lambda_i}^{(i)}(z_i; \overline{z}_i)$ be the reproducing kernel of $\mathcal{P}_{\lambda_i}^{(i)}$ with respect to (2.4). By Schur’s lemma, there exists a positive constant $c_{\lambda_0 \ldots \lambda_k}$ such that $c_{\lambda_0 \ldots \lambda_k} \prod_{j=0}^{k} K_{\lambda_j}^{(j)}(z_j; \overline{z}_j)$ is the reproducing kernel of $\mathcal{P}_{\lambda_0}^{(0)} \otimes \cdots \otimes \mathcal{P}_{\lambda_k}^{(k)}$ with respect to the above inner product $(\cdot, \cdot)$. From the definition of the reproducing kernel, we have

$$\prod_{i=1}^{k} \left( \mu_i^{d_i} C_{\Omega_i} \right) \int_{(\prod_{j=1}^{k} \Omega_j)^{wt_0}} c_{\lambda_0 \ldots \lambda_k} \prod_{j=0}^{k} K_{\lambda_j}^{(j)}(z_j; \overline{z}_j) \times \prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i(a-d_{a_i})-p_i(1 - \frac{\|w\|^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}})} \alpha_{a_i-1}^{(k)} \prod_{j=0}^{k} dm(z_j)$$

$$= \prod_{i=0}^{k} \dim \mathcal{P}_{\lambda_i}^{(i)}.$$

Therefore, the Bergman kernel of $\mathcal{H}_{\alpha}$ can be written as

$$K_{\alpha}(Z; \overline{Z}) = \sum_{\ell(\lambda_i) \leq r_i, 0 \leq i \leq k} \frac{\prod_{i=0}^{k} \dim \mathcal{P}_{\lambda_i}^{(i)}}{\prod_{j=0}^{k} K_{\lambda_j}^{(j)}(z_j; \overline{z}_j)} \prod_{j=0}^{k} K_{\lambda_j}^{(j)}(z_j; \overline{z}_j),$$

(2.25)

where $< f >$ denotes integral

$$< f > = \prod_{i=1}^{k} \left( \mu_i^{d_i} C_{\Omega_i} \right) \int_{(\prod_{j=1}^{k} \Omega_j)^{wt_0}} f(Z) \times \prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i(a-d_{a_i})-p_i(1 - \frac{\|w\|^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}})} \alpha_{a_i-1}^{(k)} dm(Z).$$
If \( \mu_i \alpha - p_i > -1 \) and \( \alpha - n - 1 > -1 \), namely \( \alpha > \max \{ n, \frac{p-1}{\mu_1}, \ldots, \frac{p-1}{\mu_k} \} \), combining (see [15])

\[
\int_\Omega K_\lambda(z, \bar{z}) N_\Omega(z, \bar{z})^s dm(z) = \frac{\dim P_\lambda}{(p+s)_\lambda} \int_\Omega N_\Omega(z, \bar{z})^s dm(z)
\]

(2.26)

for \( s > -1 \) and (2.2), we have

\[
< \prod_{j=0}^k K_{\lambda_j}^{(j)}(z_j; \bar{z}_j) > = \prod_{i=1}^k (\mu_i^{d_i} C_{\Omega_i}) \prod_{i=1}^k \int_{\Omega_i} K_{\lambda_i}^{(i)}(z_i; \bar{z}_i) N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i(\alpha+|\lambda_0|)-p_i} dm(z_i)
\]

\[
\times \int_{\mathbb{B}^{d_0}} K_{\lambda_0}^{(0)}(w; \bar{w})(1 - ||w||^2)^{\alpha-n-1} dm(w)
\]

\[
= \prod_{i=1}^k (\mu_i^{d_i} C_{\Omega_i} \chi_i(0) V(\Omega_i)) \cdot V(\mathbb{B}^{d_0}) \chi_0(0) \prod_{i=1}^k \dim P_{\lambda_i}^{(i)} \prod_{i=1}^k (\mu_i(\alpha+|\lambda_0|))^{\lambda_i},
\]

(2.27)

where we use the fact \( p_0 = d_0 + 1 \).

Combining (2.25), (2.27) and (2.7), we get

\[
K_\lambda(Z; \bar{Z}) = \sum_{t(\lambda_0) \leq \lambda_0} c \left( \prod_{i=1}^k \chi_i(\mu_i(\alpha+|\lambda_0|) - p_i)(\mu_i(\alpha+|\lambda_0|))_{\lambda_i}^{(i)} R_{\lambda_i}^{(j)}(z_j; \bar{z}_j) \right) (\alpha - d)^{\lambda_0} K_{\lambda_0}^{(0)}(w; \bar{w})
\]

\[
= c \prod_{\lambda_0 \leq \lambda_0} \prod_{i=1}^k \chi_i(\mu_i(\alpha+|\lambda_0|) - p_i) \frac{1}{N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i(\alpha+|\lambda_0|)}} \left( (\alpha - d)^{\lambda_0} K_{\lambda_0}^{(0)}(w; \bar{w}) \right)
\]

\[
= c \prod_{\lambda_0 \leq \lambda_0} \prod_{i=1}^k \chi_i(\mu_i(\alpha+|\lambda_0|) - p_i) \left( (\alpha - d)^{\lambda_0} K_{\lambda_0}^{(0)} \left( \frac{t w}{\prod_{i=1}^k N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i}} \right) \right)
\]

\[
\left| \prod_{i=1}^k \chi_i(\mu_i(\alpha+|\lambda_0|) - p_i) \frac{1}{\prod_{i=1}^k N_{\Omega_i}(z_i, \bar{z}_i)^{\mu_i}} \right|_{(i=1)}
\]

where

\[
c = \frac{\pi^n \chi_0(\alpha - n - 1)}{\prod_{i=1}^k (\mu_i^{d_i} C_{\Omega_i} \chi_i(0) V(\Omega_i)) V(\mathbb{B}^{d_0}) \chi_0(0)}.
\]

Combining

\[
V(\Omega_i) = \frac{\pi^{d_i}}{C_{\Omega_i} \chi_i(0)}, \quad V(\mathbb{B}^{d_0}) = \frac{\pi^{d_0}}{\chi_0(0)} \quad \text{(refer to [18, 21])}
\]

\[
\text{and} \quad \chi_0(x) = (x + 1)_{d_0},
\]

we obtain (2.24). \( \square \)

In order to simplify (2.24), we need Lemma 2.4 below.

**Lemma 2.4.** (see [15]) Let \( \varphi(x) \) be a polynomial in \( x \) of degree \( n \) and let \( Z \) be a matrix of order \( m \). Assume \( t \) is a real variable such that \( ||tZ|| < 1 \), where \( ||Z|| \) denotes the norm of \( Z \). For a real number \( n_0 \), take \( x_0 = -mn_0 \). Then we have

\[
\varphi(t \frac{d}{dt} \frac{1}{\det(I - tZ)^{n_0}} = \frac{1}{\det(I - tZ)^{n_0}} \sum_{j=0}^{n} \frac{\partial_j \varphi(x_0)}{j!} \sum_{|\lambda| = j} \frac{|\lambda|!}{z_\lambda!} p_\lambda \left( \frac{1}{I - tZ} \right),
\]

(2.28)
where
\[ \lambda = (1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots), \quad m_i(\lambda) \geq 0, \]
\[ |\lambda| := \sum_i im_i(\lambda), \quad \ell(\lambda) := \sum_i m_i(\lambda), \quad z_\lambda := \prod_i \epsilon^{m_i(\lambda)}m_i(\lambda)!, \]
\[ p_\lambda(Z) := \prod_i ((\text{Tr} Z)^{m_\lambda(\lambda)}, \quad D^j \varphi(x_0) = \sum_{l=0}^{j} \binom{j}{l} (-1)^l \varphi(x_0 - l). \]

Combining Theorem 2.3 and Lemma 2.4, we obtain the explicit expression of the Bergman kernel \( K_\alpha \) of the weighted Hilbert space \( \mathcal{H}_\alpha \) as follows.

**Theorem 2.5.** Assume
\[ \tilde{\chi}(x) := \prod_{i=1}^{k} \chi_i(\mu_i x - p_i) = \prod_{i=1}^{k} \prod_{j=1}^{r_i} \left( \mu_i x - p_i + 1 + (j - 1) \frac{\alpha_i}{2} \right)^{1+b_i+(r_i-j)\alpha_i}, \quad (2.29) \]
Let \( D^j \tilde{\chi}(x) \) be the \( j \)-order difference of \( \tilde{\chi} \) at \( x \), that is
\[ D^j \tilde{\chi}(x) = \sum_{l=0}^{j} \binom{j}{l} (-1)^l \tilde{\chi}(x - l). \quad (2.30) \]

Then (2.24) can be rewritten as
\[ K_\alpha(z_1, \ldots, z_k, w; \bar{z}_1, \ldots, \bar{z}_k, \bar{w}) = \frac{1}{\prod_{i=1}^{k} \mu_i^d \prod_{i=1}^{k} \left( \mathcal{N}_{\Omega_i(z_i, \bar{z}_i)} \right)^{1/2}} \sum_{j=0}^{d} D^j \tilde{\chi}(d) \frac{\alpha - n + d}{j!} \left( \frac{1}{1 - \frac{\|w\|^2}{\prod_{i=1}^{k} \mathcal{N}_{\Omega_i(z_i, \bar{z}_i)}^{r_i}}} \right)^{\alpha - d - j}. \quad (2.31) \]

**Proof.** Let \( x_0 = d - \alpha, \varphi(x) = \prod_{i=1}^{k} \chi_i(\mu_i (\alpha + x) - p_i). \) From
\[ \prod_{i=1}^{k} \chi_i(\mu_i (\alpha + x) - p_i)|_{x=x_0-l} = \prod_{i=1}^{k} \chi_i(\mu_i (d - j) - p_i) = \tilde{\chi}(d - l), \]
we have
\[ D^j \varphi(x)|_{x=x_0} = D^j \tilde{\chi}(d). \]
Using (2.28) and
\[ (x)_j = \sum_{|\lambda|=j} \frac{|\lambda|!}{z_\lambda} x^{\ell(\lambda)}, \]
we have
\[ \varphi(t \frac{d}{dt}) \frac{1}{(1 - tz)^{\alpha - d}} = \frac{1}{(1 - tz)^{\alpha - d}} \sum_{j=0}^{d} D^j \tilde{\chi}(d) \frac{(\alpha - d)_j}{j!} \left( \frac{1}{1 - t\bar{z}} \right)^{\alpha - d - j}. \quad (2.32) \]
Combining
\[ (\alpha - n)_{d_0} (\alpha - d)_j = (\alpha - n)_{d_0 + j} \quad (2.33) \]
and (2.32), we get (2.31). \( \square \)
3 The Rawnsley’s $\varepsilon$-function for $(\prod_{j=1}^{k} \Omega_j)^{\mathbb{B}^{d_0}}(\mu)$ with the canonical metric $g(\mu)$

In this section we give the explicit expression of the Rawnsley’s $\varepsilon$-function and the coefficients $a_1, a_2$ of its expansion for the generalized Cartan-Hartogs domain $((\prod_{j=1}^{k} \Omega_j)^{\mathbb{B}^{d_0}}(\mu), g(\mu))$ with the canonical metric $g(\mu)$.

**Theorem 3.1.** Let $\Omega_i$ be an irreducible bounded symmetric domain in $\mathbb{C}^{d_i}$ in its Harish-Chandra realization, and denote the generic norm $N_{\Omega_i}(z_i, \overline{z}_i)$, the dimension $d_i$ and the genus $p_i$ for $\Omega_i$ ($1 \leq i \leq k$). Let $n = \sum_{j=0}^{k} d_j$, $d = \sum_{j=1}^{k} d_j$ and $\alpha = \max\{n, \frac{p_1-1}{\mu_1}, \ldots, \frac{p_k-1}{\mu_k}\}$. Then the Rawnsley’s $\varepsilon$-function associated to $((\prod_{j=1}^{k} \Omega_j)^{\mathbb{B}^{d_0}}(\mu), g(\mu))$ can be written as

$$\varepsilon_{\alpha}(z_1, \ldots, z_k, w) = \frac{1}{\prod_{i=1}^{k} \mu_i} \sum_{j=0}^{d} \frac{D_j^1 \tilde{\chi}(d)}{j!} \left(1 - \frac{||w||^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right)^{d-j} (\alpha - n)_{d_0}$$

(see (2.29) and (2.30) for the definition of the functions $\tilde{\chi}(x)$ and $D_j^1 \tilde{\chi}(x)$ respectively).

**Proof.** By (2.31) and

$$\varepsilon_{\alpha}(z_1, \ldots, z_k, w) := e^{-\alpha \Phi(z_1, \ldots, z_k, w)} K_{\alpha}(z_1, \ldots, z_k, w; \overline{z}_1, \ldots, \overline{z}_k, \overline{w}),$$

we obtain (3.1). $\square$

**Corollary 3.2.** For $k \geq 2$, the coefficients $a_1$ and $a_2$ of the expansion of the Rawnsley’s $\varepsilon$-function $\varepsilon_{\alpha}$, that is, the coefficients of $\alpha^{n-1}$ and $\alpha^{n-2}$ in (3.1) respectively, are given by

$$a_1(z_1, \ldots, z_k, w) = \frac{1}{\prod_{i=1}^{k} \mu_i} \frac{D^{d-1} \tilde{\chi}(d)}{(d-1)!} \left(1 - \frac{||w||^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right) - \frac{n(n+1)}{2},$$

$$a_2(z_1, \ldots, z_k, w) = \frac{1}{\prod_{i=1}^{k} \mu_i} \frac{D^{d-2} \tilde{\chi}(d)}{(d-2)!} \left(1 - \frac{||w||^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right)^2$$

$$- \frac{1}{\prod_{i=1}^{k} \mu_i} \frac{D^{d-1} \tilde{\chi}(d)}{(d-1)!} \left(1 - \frac{||w||^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right)^{d-j} (\alpha - n)_{d_0}$$

$$+ \frac{1}{24} (n-1)(n+1)(3n+2).$$

**Proof.** Write

$$(\alpha - n)_{d_0} = \sum_{j=0}^{d_0} c_{d_0+j} \alpha^j.$$

Substituting (3.4) into (3.1), we obtain

$$\varepsilon_{\alpha}(z_1, \ldots, z_k, w) = \sum_{j=0}^{n} \alpha^j \sum_{l=\max(j-d_0,0)}^{d} \frac{c_{d_0+l} \cdot D_l^1 \tilde{\chi}(d)}{l!} \left(1 - \frac{||w||^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right)^{d-l},$$

which implies

$$a_j(z_1, \ldots, z_k, w) = \sum_{l=\max(d-j,0)}^{d} \frac{c_{d_0+l} \cdot D_l^1 \tilde{\chi}(d)}{l!} \left(1 - \frac{||w||^2}{\prod_{i=1}^{k} N_{\Omega_i}(z_i, \overline{z}_i)^{\mu_i}} \right)^{d-l}.$$
From
\[
(\alpha - n)_n = \prod_{k=1}^{n} (\alpha - k), \\
(\alpha - n)_{n-1} = \prod_{k=2}^{n} (\alpha - k), \\
(\alpha - n)_{n-2} = \prod_{k=3}^{n} (\alpha - k),
\]
we have
\[
c_{n-1,n-1} = c_{n-2,n-2} = 1, \quad (3.7)
\]
\[
c_{n,n-1} = -\sum_{k=1}^{n} k = -\frac{n(n+1)}{2}, \quad (3.8)
\]
\[
c_{n-1,n-2} = -\sum_{k=2}^{n} k = -\frac{n(n+1)}{2} + 1, \quad (3.9)
\]
\[
c_{n,n-2} = \sum_{1 \leq i < j \leq n} ij = \frac{1}{2}\left\{ \left( \sum_{k=1}^{n} k \right)^2 - \sum_{k=1}^{n} k^2 \right\} \]
\[
= \frac{1}{24} (n-1)n(n+1)(3n+2). \quad (3.10)
\]
Substituting (3.7), (3.8), (3.9) and (3.10) into (3.6), we obtain (3.2) and (3.3).

In order to calculate \( D^{d-1}\widetilde{\chi} \) and \( D^{d-2}\widetilde{\chi} \), we need Lemma 3.3 and 3.4 below.

**Lemma 3.3.** (see [16]) Write \( \prod_{j=1}^{r} (\mu x - p + 1 + (j-1) \frac{a}{2})_{1+b+(r-j)a} = \sum_{j=0}^{d} c_{j} x^{d-j} \). Then
\[
c_0 = \mu^d, \quad (3.11)
\]
\[
c_1 = -\frac{1}{2} \mu^{d-1} dp, \quad (3.12)
\]
\[
c_2 = \frac{1}{2} \mu^{d-2} R(\Omega), \quad (3.13)
\]
where
\[
R(\Omega) = \frac{d^2 p^2}{4} - \frac{r(p-1)p(2p-1)}{6} + \frac{r(r-1)a(3p^2 - 3p + 1)}{12} - \frac{(r-1)r(2r-1)a^2(p-1)}{24} + \frac{r^2(r-1)^2 a^3}{48}. \quad (3.14)
\]

**Lemma 3.4.** (see [16]) For any polynomial \( f(x) \) in real variable \( x \), take \( Df(x) := f(x) - f(x-1) \). Let \( A_d = D^{d-1} x^d, B_d = D^{d-2} x^d \). Then we have
\[
A_d = \frac{d!}{2} (2x - d + 1) \quad (d \geq 1), \quad (3.15)
\]
\[
B_d = \frac{d!}{24} \left\{ 12x^2 - 12(d-2)x + 3d^2 - 11d + 10 \right\} \quad (d \geq 2). \quad (3.16)
\]

Lemma 3.3 and Lemma 3.4 imply the following results.
Lemma 3.5. Suppose that $k = 2$, $d = d_1 + d_2$, $D^{d-1} \tilde{\chi}(d)$ and $D^{d-2} \tilde{\chi}(d)$ are defined by (2.29) and (2.30). Then we have

$$\frac{1}{\mu_1^{d_1} \mu_2^{d_2}} \frac{D^{d-1} \tilde{\chi}(d)}{(d-1)!} = \frac{1}{2} \left\{ d(d + 1) - \left( \frac{d_1 p_1}{\mu_1} + \frac{d_2 p_2}{\mu_2} \right) \right\},$$

(3.17)

$$\frac{1}{\mu_1^{d_1} \mu_2^{d_2}} \frac{D^{d-2} \tilde{\chi}(d)}{(d-2)!} = \frac{1}{4} \left\{ \frac{1}{6} (d-1)(d+1)(3d+10) - (d-1)(d+2) \left( \frac{d_1 p_1}{\mu_1} + \frac{d_2 p_2}{\mu_2} \right) + 2 \frac{R(\Omega_1)}{\mu_1^2} + \frac{d_1 d_2 p_1 p_2}{\mu_1 \mu_2} + 2 \frac{R(\Omega_2)}{\mu_2^2} \right\},$$

(3.18)

where

$$R(\Omega_i) = \frac{d_1^2 p_1^2}{4} r_i(p_i - 1)p_i(2p_i - 1) + \frac{r_i(r_i - 1)a_i(3p_i^2 - 3p_i + 1)}{12} - \frac{(r_i - 1)r_i(2r_i - 1)a_i^2(p_i - 1)}{24} + \frac{r_i^2 (r_i - 1)^2 a_i^3}{48} (1 \leq i \leq 2).$$

(3.19)

Proof. Let $\tilde{\chi}(x) = c_0 x^d + c_1 x^{d-1} + c_2 x^{d-2} + \cdots + c_n$. From Lemma 3.3, we get

$$c_0 = \mu_1^{d_1} \mu_2^{d_2},$$

(3.20)

$$c_1 = -\frac{1}{2} \mu_1^{d_1} \mu_2^{d_2} \left( \frac{d_1 p_1}{\mu_1} + \frac{d_2 p_2}{\mu_2} \right),$$

(3.21)

$$c_2 = \frac{1}{4} \mu_1^{d_1} \mu_2^{d_2} \left( 2 \frac{R(\Omega_1)}{\mu_1^2} + \frac{d_1 d_2 p_1 p_2}{\mu_1 \mu_2} + 2 \frac{R(\Omega_2)}{\mu_2^2} \right),$$

(3.22)

where $R(\Omega_i)$ is defined by (3.19) $(1 \leq i \leq 2)$. By

$$\begin{cases}
D^{d-1} \tilde{\chi}(x) = c_0 A_d(x) + c_1 (d - 1)!, \\
D^{d-2} \tilde{\chi}(x) = c_0 B_d(x) + c_1 A_{d-1}(x) + c_2 (d - 2)!,
\end{cases}$$

(3.23)

letting $x = d$ and substituting (3.20), (3.21), (3.22), (3.15) and (3.16) into (3.23), we obtain (3.17) and (3.18). □

4 The proof of Theorem 1.4

(i) From the proof of Theorem 2.3, we know that the space $\mathcal{H}_\alpha \neq \{0\}$ if and only if

$$\alpha > \max\{n, \frac{p_1 - 1}{\mu_1}, \ldots, \frac{p_k - 1}{\mu_k}\}.$$

Using (3.1), we obtain that $\varepsilon_\alpha(z_1, \ldots, z_k, w)$ is a constant with respect to $(z_1, \ldots, z_k, w)$ if and only if

$$D^j \tilde{\chi}(d) = 0 \ (0 \leq j \leq d - 1).$$

This is equivalent to

$$\tilde{\chi}(x) = \sum_{j=0}^d \frac{D^j \tilde{\chi}(d)}{j!} (x - d)_j = \frac{D^d \tilde{\chi}(d)}{d!} (x - d)_d = \prod_{j=1}^k p_j^j \cdot \prod_{j=1}^d (x - j),$$

which implies (1.7) by the definition of $\tilde{\chi}(x)$ (see (2.29)).
(ii) Since there is no multiple divisor for the polynomial \( \prod_{j=1}^{d} (x - j) \), by (1.7), we obtain that each polynomial
\[
\chi_i(\mu_i x - p_i) = \prod_{j=1}^{r_i} \left( \mu_i x - p_i + 1 + (j - 1) \frac{a_i}{2} \right)_{1+b_i+(r_i-j)a_i},
\]
has no multiple divisor, namely each \( \chi_i(x) \) has no multiple divisor.

(1) For the irreducible bounded symmetric domain \( \Omega_I(m,n) \) (1 \( \leq m \leq n \)), its rank \( r = m \), the characteristic multiplicities \( a = 2, b = n - m \). Since
\[
\chi(x) = \prod_{j=1}^{m} (x + j)_{m+n+1-2j},
\]
(4.1) it is easy to see that \( \chi(x) \) has no multiple divisor iff \( r = m = 1 \).

(2) For the irreducible bounded symmetric domain \( \Omega_{II}(2n) \) (\( n \geq 3 \)), its rank \( r = n \), the characteristic multiplicities \( a = 4, b = 0 \). The polynomial
\[
\chi(x) = \prod_{j=1}^{n} (x - 1 + 2j)_{4n+1-4j}
\]
(4.2) has multiple divisors \( x + 3 \).

For the irreducible bounded symmetric domain \( \Omega_{II}(2n+1) \) (\( n \geq 2 \)), its rank \( r = n \), the characteristic multiplicities \( a = 4, b = 2 \). The polynomial
\[
\chi(x) = \prod_{j=1}^{n} (x - 1 + 2j)_{4n+3-4j}
\]
(4.3) has multiple divisors \( x + 3 \).

(3) For the irreducible bounded symmetric domain \( \Omega_{III}(n) \) (\( n \geq 2 \)), its rank \( r = n \), the characteristic multiplicities \( a = 1, b = 0 \). When \( n \geq 3 \),
\[
\chi(x) = \prod_{j=1}^{n} \left( x + \frac{1+j}{2} \right)_{n+1-j}
\]
(4.4) has multiple divisors \( x + 2 \). When \( n = 2 \),
\[
\chi(x) = (x + 1)(x + 2) \left( x + \frac{3}{2} \right)
\]
(4.5) has not any multiple divisor.

(4) For the irreducible bounded symmetric domain \( \Omega_{IV}(n) \) (\( n \geq 5 \)), its rank \( r = 2 \), the characteristic multiplicities \( a = n - 2, b = 0 \). From
\[
\chi(x) = (x + 1)_{n-1} \left( x + \frac{n}{2} \right),
\]
(4.6) we obtain that \( \chi(x) \) has no multiple divisor if and only if \( n \) is odd.

(5) For the irreducible bounded symmetric domain \( \Omega_V(16) \), its rank \( r = 2 \), the characteristic multiplicities \( a = 6, b = 4 \). By
\[
\chi(x) = (x + 1)_{11} (x + 4)_{5},
\]
(4.7) we get that \( \chi(x) \) has multiple divisors.

(6) For the irreducible bounded symmetric domain \( \Omega_{VI}(27) \), its rank \( r = 3 \), the characteristic multiplicities \( a = 8, b = 0 \). The polynomial
\[
\chi(x) = (x + 1)_{17} (x + 5)_{9} (x + 9)
\]
(4.8)
has multiple divisors.

In summary, we have that if the metric $\alpha g(\mu)$ on $\left(\prod_{j=1}^{k} \Omega_j\right)^{\mathbb{R}^{6O}} (\mu)$ is balanced, then $\Omega_i$ must be biholomorphic to one of $\Omega_f(1,n)$, $\Omega_{III}(2)$ or $\Omega_{IV}(m)$ ($m \geq 5$ and $m$ are odd).

(iii) Using Theorem 1.4(i), it is easy to show that the metrics $\alpha g(\mu)$ on

$$
\left(\mathbb{B}^d \times \mathbb{B}\right)^{\mathbb{R}^{6O}} \left(\frac{1}{d+1}\right)
$$

and

$$
\left(\mathbb{B} \times \Omega_{III}(2)\right)^{\mathbb{R}^{6O}} \left(\frac{1}{2}\right)
$$

are balanced.

Now suppose that $\alpha g(\mu)$ on $(\Omega_1 \times \Omega_2)^{\mathbb{R}^{6O}} (\mu)$ is balanced, by Theorem 1.4(ii), we have that $\Omega_j$ ($j = 1, 2$) must be biholomorphic to one of $\Omega_f(1,n)$, $\Omega_{III}(2)$ or $\Omega_{IV}(m)$ ($m \geq 5$ and $m$ are odd).

(1) When $(\Omega_1, \Omega_2) = (\mathbb{B}^{d_1}, \mathbb{B}^{d_2})$, using (1.7), we get

$$
\prod_{j=1}^{d_1} \left(x - \frac{j}{\mu_1}\right) \cdot \prod_{j=1}^{d_2} \left(x - \frac{j}{\mu_2}\right) = \prod_{j=1}^{d_1+d_2} \left(x - j\right).
$$

This means that $\frac{1}{\mu_1}, \frac{1}{\mu_2}$ (1 $\leq j \leq d_1, 1 \leq l \leq d_2, j, l \in \mathbb{N}$) are zeros of the polynomial $\prod_{j=1}^{d_1+d_2} (x - j)$. Thus $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ are integers. Assume $d_1 \geq d_2$. Since $\frac{d_1}{\mu_1}$ is a zero of the polynomial $\prod_{j=1}^{d_1+d_2} (x - j)$, we get that $\frac{d_1}{\mu_1} \leq d_1 + d_2 \leq 2d_1$, and thus we have $1 \leq \frac{1}{\mu_1} \leq 2$.

If $\frac{1}{\mu_1} = 1$, by (4.9), it follows that

$$
\frac{d_2}{\mu_2} = d_1 + d_2, \quad \frac{1}{\mu_2} = d_1 + 1.
$$

So $(d_1 - 2)(d_2 - 1) = 0$. If $d_2 \geq 2$, then $d_1 = d_2 = 2$ and $\frac{1}{\mu_2} = 3$. But, in this case, the equation (4.9) is not valid. Therefore, if $\frac{1}{\mu_1} = 1$, then $d_2 = 1$ and $\frac{1}{\mu_2} = d_1 + 1$.

If $\frac{1}{\mu_1} = 2$, since the number of even zeros of the left side of (4.9) is larger than or equal to $\left\lfloor \frac{d_1+d_2}{2}\right\rfloor$ for the right side of (4.9), where $\lfloor n \rfloor$ denotes the greatest integer which is less than or equal to $n$, we get

$$
\frac{d_1 + d_2}{2} \geq \left\lfloor \frac{d_1 + d_2}{2}\right\rfloor \geq d_1.
$$

This yields $d_2 \geq d_1$. From the assumption $d_1 \geq d_2$, we have $d_1 = d_2$. If $d_1 = d_2 > 1$, since numbers of odd zeros of both sides of (4.9) are not equal, this leads to a contradiction. If $d_1 = d_2 = 1$, using (4.9), we obtain $\frac{1}{\mu_2} = 1$.

(2) When $(\Omega_1, \Omega_2) = (\mathbb{B}^{d_1}, \Omega_{III}(2))$, applying (1.7), we obtain

$$
\prod_{j=1}^{d_1} \left(x - \frac{j}{\mu_1}\right) \cdot \left(x - \frac{2}{\mu_2}\right) = \prod_{j=1}^{d_1+3} \left(x - j\right).
$$

This means that $\frac{1}{\mu_1}(1 \leq j \leq d_1)$, $\frac{1}{\mu_2}$ and $\frac{2}{\mu_2}$ are zeros of the polynomial $\prod_{j=1}^{d_1+3} (x - j)$. So $\frac{1}{\mu_1}, \frac{1}{\mu_2}$ and $\frac{2}{\mu_2}$ are integers, and thus $\frac{1}{\mu_2} = 2t$ (i.e., $\frac{1}{\mu_2}$ is even). Since $\prod_{j=1}^{d_1} \left(x - \frac{j}{\mu_1}\right)$ is divisible by $x - 1$, we get $\frac{1}{\mu_1} = 1$ and so

$$(x - 2t)(x - 3t)(x - 4t) = (x - d_1 - 1)(x - d_1 - 2)(x - d_1 - 3).$$
Thus we have $t = 1$, $d_1 = 1$.

(3) When $(\Omega_1, \Omega_2) = (\mathbb{P}^{d_1}, \Omega_{IV}(d_2))$ (where $d_2 \geq 5$ are odd), (1.7) implies

$$\prod_{j=1}^{d_1} \left( x - \frac{j}{\mu_1} \right) \prod_{j=1}^{d_2 - 1} \left( x - \frac{j}{\mu_2} \right) \prod_{j=1}^{d_1 + d_2} \left( x - \frac{d_2}{2\mu_2} \right) = \prod_{j=1}^{d_1 + d_2} (x - j). \quad (4.11)$$

This implies that $\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{3}{2\mu_1}$, and $\frac{3}{2\mu_2}$ are integers (where $d_2 \geq 5$ are odd). So $\frac{1}{\mu_1}$ is an integer and $\frac{1}{\mu_2}$ is even. From the number of even zeros of the left side of (4.11) is greater than or equal to $\left[ \frac{d_1 + d_2}{2} \right] + d_2 - 1$ and the number of even zeros is equal to $\left[ \frac{d_1 + d_2}{2} \right]$ for the right side of (4.11), we have

$$\left[ \frac{d_1 + d_2}{2} \right] \geq \left[ \frac{d_1}{2} \right] + d_2 - 1.$$

Since

$$\frac{d_1 + d_2}{2} - \left( \frac{d_1}{2} - 1 \right) \geq \left[ \frac{d_1 + d_2}{2} \right] - \left[ \frac{d_1}{2} \right],$$

we have $d_2 \leq 4$, this conflicts with $d_2 \geq 5$.

(4) When $(\Omega_1, \Omega_2) = (\Omega_{III}(2), \Omega_{IV}(d_2))$, by (1.7), we obtain

$$\left( x - \frac{1}{\mu_1} \right) \left( x - \frac{2}{\mu_1} \right) \left( x - \frac{3}{2\mu_1} \right) \left( x - \frac{1}{\mu_2} \right) \left( x - \frac{2}{\mu_2} \right) \left( x - \frac{3}{2\mu_2} \right) = \prod_{j=1}^{d_1 + d_2} (x - j). \quad (4.12)$$

This implies that $\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{3}{2\mu_1}$, and $\frac{3}{2\mu_2}$ are integers. So $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ are even. Since the number of even zeros of the left side of (4.12) is greater than or equal to 4 and the number of even zeros of the right side of (4.12) is equal to 3, this leads to a contradiction.

(5) When $(\Omega_1, \Omega_2) = (\Omega_{III}(2), \Omega_{IV}(d_2))$, $d_2 \geq 5$ and $d_2$ is odd, by (1.7), we have

$$\left( x - \frac{1}{\mu_1} \right) \left( x - \frac{2}{\mu_1} \right) \left( x - \frac{3}{2\mu_1} \right) \prod_{j=1}^{d_1 - 1} \left( x - \frac{j}{\mu_1} \right) \prod_{j=1}^{d_1 - 1} \left( x - \frac{j}{\mu_2} \right) \prod_{j=1}^{d_2} \left( x - \frac{d_2}{2\mu_2} \right) = \prod_{j=1}^{3 + d_2} (x - j). \quad (4.13)$$

Then $\frac{1}{\mu_1}, \frac{3}{2\mu_1}$, $\frac{3}{2\mu_2}$, and $\frac{d_2}{2\mu_2}$ are zeros of $\prod_{j=1}^{3 + d_2} (x - j)$. So $\frac{1}{\mu_1}$ and $\frac{3}{2\mu_1}$ are even. In view of the number of even zeros of the left side of (4.13) is greater than or equal to $d_2 + 1$ and the number of even zeros of the right side of (4.13) is equal to $\left[ \frac{3 + d_2}{2} \right]$, we get

$$\frac{3 + d_2}{2} \geq \left[ \frac{3 + d_2}{2} \right] \geq d_2 + 1,$$

which means $d_2 \leq 1$, which conflicts with $d_2 \geq 5$.

(6) When $(\Omega_1, \Omega_2) = (\Omega_{IV}(d_1), \Omega_{IV}(d_2))$, $d_1 \geq 5$, $d_2 \geq 5$ and $d_1, d_2$ are odd, by (1.7), we have

$$\prod_{j=1}^{d_1 - 1} \left( x - \frac{j}{\mu_1} \right) \left( x - \frac{d_1}{2\mu_1} \right) \prod_{j=1}^{d_2 - 1} \left( x - \frac{j}{\mu_2} \right) \left( x - \frac{d_2}{2\mu_2} \right) = \prod_{j=1}^{d_1 + d_2} (x - j). \quad (4.14)$$

Then $\frac{1}{\mu_1}, \frac{d_1}{2\mu_1}, \frac{1}{\mu_2}$, and $\frac{d_2}{2\mu_2}$ are zeros of $\prod_{j=1}^{d_1 + d_2} (x - j)$. So $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ are even. Since the number of even zeros of the left side of (4.14) is greater than or equal to $d_1 + d_2 - 2$ and the number of even zeros of the right side of (4.14) is equal to $\left[ \frac{d_1 + d_2}{2} \right]$, we have

$$\frac{d_1 + d_2}{2} \geq \left[ \frac{d_1 + d_2}{2} \right] \geq d_1 + d_2 - 2.$$
Therefore \( d_1 + d_2 \leq 4 \), which conflicts with \( d_1 \geq 5 \) and \( d_2 \geq 5 \).

Combing the above results (1)-(6), we obtain that if \( \alpha g(\mu) \) on \( (\Omega_1 \times \Omega_2)^{B_{d_0}}(\mu) \) is balanced, then \( (\Omega_1 \times \Omega_2)^{B_{d_0}}(\mu) \) is biholomorphic to

\[
(\mathbb{B}^d \times \mathbb{B})^{B_{d_0}} \left( 1, \frac{1}{d+1} \right) \text{ or } (\mathbb{B} \times \Omega_{III}(2))^{B_{d_0}} \left( \frac{1}{2}, \frac{1}{2} \right).
\]

5 The proof of Theorem 1.5

In order to proof Theorem 1.5, we need Lemma 5.1 and 5.2 below.

**Lemma 5.1.** Let \( \Omega \) be an irreducible bounded symmetric domain in \( \mathbb{C}^d \) in its Harish-Chandra realization, and denote the rank \( r \), the characteristic multiplicities \( a, b \), the dimension \( d \), the genus \( p \), and the Hua polynomial \( \chi \) of \( \Omega \). Then

\[
\frac{1}{2} - \frac{8}{3(4d+1)} < S(\Omega) \leq \frac{1}{2} - \frac{1}{2d}, \tag{5.1}
\]

where \( S(\Omega) = \frac{2R(\Omega)}{d^2p^2} \) (see (3.14) for \( R(\Omega) \)).

**Proof.** Let \( x_1, x_2, \ldots, x_d \) be zeros of the polynomial \( \chi(x - p) \). By (2.3), we know that \( x_1, x_2, \ldots, x_d \) are real numbers. From Lemma 3.3, we have

\[
\sum_{i=1}^{d} x_i = \frac{1}{2} dp, \tag{5.2}
\]

\[
\sum_{i=1}^{d} x_i^2 = \frac{r(p-1)p(2p-1)}{6} - \frac{r(r-1)a(3p^2 - 3p + 1)}{12} + \frac{r(r-1)r(2r-1)a^2(p-1)}{24} - \frac{r^2(r-1)^2a^3}{48}, \tag{5.3}
\]

and

\[
S(\Omega) = \frac{1}{2} - \frac{2}{d^2p^2} \sum_{i=1}^{d} x_i^2. \tag{5.4}
\]

From (5.2) and

\[
\left( \sum_{i=1}^{d} x_i \right)^2 \leq d \sum_{i=1}^{d} x_i^2,
\]

we obtain

\[
\frac{2}{d^2p^2} \sum_{i=1}^{d} x_i^2 \geq \frac{1}{2d}, \tag{5.5}
\]

which implies

\[
S(\Omega) \leq \frac{1}{2} - \frac{1}{2d}.
\]

Now we show

\[
\frac{1}{2} - \frac{8}{3(4d+1)} < S(\Omega).
\]

This is equivalent to

\[
3(4d+1) \left\{ \frac{r(p-1)p(2p-1)}{6} - \frac{r(r-1)a(3p^2 - 3p + 1)}{12} + \frac{r(r-1)r(2r-1)a^2(p-1)}{24} - \frac{r^2(r-1)^2a^3}{48} \right\} - 4d^2p^2 < 0. \tag{5.6}
\]
We now calculate the left side of (5.6) by using the classification of irreducible bounded symmetric domains.

(1) For the irreducible bounded symmetric domain $\Omega_I(m,n)$ ($1 \leq m \leq n$), its rank $r = m$, the characteristic multiplicities $a = 2, b = n - m$, the dimension $d = mn$, the genus $p = m + n$.

$L.H.S$ of (5.6) $= -\frac{1}{2} mn(4m^2n^2 - 2m^2 - 2n^2 + mn + 1) \leq -\frac{1}{2} mn.$ \hfill (5.7)

(2) For the irreducible bounded symmetric domain $\Omega_{II}(2n)$ ($n \geq 3$), its rank $r = n$, the characteristic multiplicities $a = 4, b = 0$, the dimension $d = n(2n - 1)$, the genus $p = 2(2n - 1)$.

$L.H.S$ of (5.6) $= -n(32n^5 - 80n^4 + 60n^3 - 4n^2 - 9n + 2) \leq -16n^5.$ \hfill (5.8)

For the irreducible bounded symmetric domain $\Omega_{III}(2n+1)$ ($n \geq 2$), its rank $r = n$, the characteristic multiplicities $a = 4, b = 2$, the dimension $d = n(2n + 1)$, the genus $p = 4n$.

$L.H.S$ of (5.6) $= -n(2n + 1)(16n^4 - 10n^2 + 3n + 1) \leq -6n^5(n + 1).$ \hfill (5.9)

(3) For the irreducible bounded symmetric domain $\Omega_{IV}(n)$ ($n \geq 2$), its rank $r = n$, the characteristic multiplicities $a = 1, b = 0$, the dimension $d = n(n+1)/2$, the genus $p = n + 1$.

$L.H.S$ of (5.6) $= -\frac{1}{16} n(n+1)(2n^4 + 8n^3 + 7n^2 - 5n - 4) \leq -\frac{1}{8} n^3(n+1).$ \hfill (5.10)

(4) For the irreducible bounded symmetric domain $\Omega_{V}(n)$ ($n \geq 5$), its rank $r = 2$, the characteristic multiplicities $a = n - 2, b = 0$, the dimension $d = n$, the genus $p = n$.

$L.H.S$ of (5.6) $= -\frac{1}{4} n(8n^2 - 5n - 2) \leq -\frac{1}{4} n^3.$ \hfill (5.11)

(5) For the irreducible bounded symmetric domain $\Omega_{VI}(16)$, its rank $r = 2$, the characteristic multiplicities $a = 6, b = 4$, the dimension $d = 16$, the genus $p = 12$.

$L.H.S$ of (5.6) $= -11736.$ \hfill (5.12)

(6) For the irreducible bounded symmetric domain $\Omega_{VI}(17)$, its rank $r = 3$, the characteristic multiplicities $a = 8, b = 0$, the dimension $d = 27$, the genus $p = 18$.

$L.H.S$ of (5.6) $= -76599.$ \hfill (5.13)

Combing the above (1)-(7), it follows that (5.6) holds. This completes the proof. \hfill \Box

**Lemma 5.2.** Let $\Omega_i$ be an irreducible bounded symmetric domain in $\mathbb{C}^{d_i}$ in its Harish-Chandra realization, and denote the rank $r_i$, the characteristic multiplicities $a_i, b_i$, the dimension $d_i$ and the genus $p_i$ of $\Omega_i$, $1 \leq i \leq 2$. Assume that

$$d := d_1 + d_2, \quad A := d(d + 1), \quad B := d(d + 1) \left\{ (d - 1)(d + 2) - \frac{1}{6}(d - 1)(3d + 10) \right\}$$ \hfill (5.14)

and

$$S(\Omega_1) := \frac{2R(\Omega_1)}{d_1^2 p_1^2}, \quad S(\Omega_2) := \frac{2R(\Omega_2)}{d_2^2 p_2^2}.$$ \hfill (5.15)

Then

$$S(\Omega_1) + S(\Omega_2) < \frac{2B}{A^2} < 1$$ \hfill (5.16)

and

$$\frac{1}{1 - 2S(\Omega_1)} + \frac{1}{1 - 2S(\Omega_2)} > \frac{1}{1 - \frac{2B}{A^2}}.$$ \hfill (5.17)
Proof. By

\[ d^2 = (d_1 + d_2)^2 \geq 4d_1d_2, \]

we get

\[ \frac{1}{2} \left( \frac{1}{d_1} + \frac{1}{d_2} \right) > 2 \frac{2d + 1}{3(d+1)}. \]  

(5.18)

Using (5.1) and

\[ \frac{B}{A^2} = \frac{1}{2} - \frac{2d + 1}{3d(d+1)}, \]

we obtain

\[ S(\Omega_1) + S(\Omega_2) \leq 1 - \frac{1}{2} \left( \frac{1}{d_1} + \frac{1}{d_2} \right) < 1 - \frac{2d + 1}{3(d+1)} = \frac{2B}{A^2} < 1. \]

From (5.19), we have

\[ \frac{1}{1 - 2S(\Omega_1)} + \frac{1}{1 - 2S(\Omega_2)} > \left( \frac{3}{4} d_1 + \frac{3}{16} \right) + \left( \frac{3}{4} d_1 + \frac{3}{16} \right) = \frac{3}{4} d + \frac{3}{8} > \frac{1}{1 - \frac{2B}{A^2}}. \]

This completes the proof. \[ \Box \]

The proof of Theorem 1.5. It follows from (3.3) that the coefficient \( a_2 \) of the expansion of the function \( \varepsilon_\alpha \) associated to \( ((\Omega_1 \times \Omega_2) \boxplus \alpha, (\mu_1, \mu_2), g(\mu_1, \mu_2)) \) is constant if and only if

\[ \frac{D^{d-1} \chi(d)}{(d-1)!} = \frac{D^{d-2} \chi(d)}{(d-2)!} = 0. \]  

(5.20)

From (3.17), (3.18) and (3.19), we get that \( a_2 \) is constant if and only if

\[ \begin{cases} \frac{d_1 p_1}{\mu_1} + \frac{d_2 p_2}{\mu_2} = d(d + 1), \\ 2\frac{R(\Omega_1)}{\mu_1^2} + \frac{d_3 d_2 p_1 p_2}{\mu_1 \mu_2} + 2\frac{R(\Omega_2)}{\mu_2^2} = d(d + 1) \left\{ (d - 1)(d + 2) - \frac{1}{6} (d - 1)(3d + 10) \right\} \end{cases}. \]  

(5.21)

Let

\[ x_1 = \frac{d_1 p_1}{\mu_1}, \quad x_2 = \frac{d_2 p_2}{\mu_2}. \]  

(5.22)

Then there exist positive solutions \( \mu_1, \mu_2 \) for (5.21) if only if there exist positive solutions \( x_1, x_2 \) for (5.23) below.

\[ \begin{cases} x_1 + x_2 = 1, \\ S(\Omega_1)x_1^2 + x_1 x_2 + S(\Omega_2)x_2^2 = \frac{B}{A^2}. \end{cases} \]  

(5.23)

Solving the system of equations (5.23), we have

\[ \begin{cases} x_1 = \frac{1 - 2S(\Omega_1) + \sqrt{\Delta}}{2(1 - S(\Omega_1) - S(\Omega_2))}, \\ x_2 = \frac{1 - 2S(\Omega_1) - \sqrt{\Delta}}{2(1 - S(\Omega_1) - S(\Omega_2))}, \end{cases} \]  

(5.24)

or

\[ \begin{cases} x_1 = \frac{1 - 2S(\Omega_2) - \sqrt{\Delta}}{2(1 - S(\Omega_1) - S(\Omega_2))}, \\ x_2 = \frac{1 - 2S(\Omega_2) + \sqrt{\Delta}}{2(1 - S(\Omega_1) - S(\Omega_2))}, \end{cases} \]  

(5.25)
where

\[
\Delta = (1 - 2S(\Omega_1))^2 - 4(1 - S(\Omega_1) - S(\Omega_2)) \left( \frac{B}{A^2} - S(\Omega_1) \right)
\]
\[
= (1 - 2S(\Omega_2))^2 - 4(1 - S(\Omega_1) - S(\Omega_2)) \left( \frac{B}{A^2} - S(\Omega_2) \right)
\]
\[
= 1 - \frac{4B}{A^2} (1 - S(\Omega_1) - S(\Omega_2)) - 4S(\Omega_1)S(\Omega_2).
\]

(5.26)

From (5.1) and (5.16), we have that
\[1 - 2S(\Omega_1) > 0, \quad 1 - 2S(\Omega_2) > 0\]
and
\[1 - S(\Omega_1) - S(\Omega_2) > 0.
\]
So there exist positive numbers \(x_1, x_2\) satisfying (5.23) if only and if

\[
\Delta \geq 0, \quad 1 - 2S(\Omega_1) - \sqrt{\Delta} > 0 \quad \text{or} \quad \Delta \geq 0, \quad 1 - 2S(\Omega_2) - \sqrt{\Delta} > 0.
\]

(5.27)

That is

\[
\frac{1}{1 - 2S(\Omega_1)} + \frac{1}{1 - 2S(\Omega_2)} \geq \frac{1}{1 - \frac{2B}{A^2}} \quad \text{and} \quad S(\Omega_1) < \frac{B}{A^2}
\]

(5.28)

or

\[
\frac{1}{1 - 2S(\Omega_1)} + \frac{1}{1 - 2S(\Omega_2)} \geq \frac{1}{1 - \frac{2B}{A^2}} \quad \text{and} \quad S(\Omega_2) < \frac{B}{A^2}.
\]

(5.29)

By (5.16) and (5.17), we get (5.28) and (5.29). Thus there exist positive numbers \(\mu_1, \mu_2\) such that (5.21) holds. Therefore there exist positive numbers \(\mu_1, \mu_2\) such that the coefficient \(a_2\) of the Rawnsley’s \(\varepsilon\)-function expansion of

\[
\left( (\Omega_1 \times \Omega_2)^{\mathbb{R}^6}(\mu_1, \mu_2), g(\mu_1, \mu_2) \right)
\]

is a constant.

\[
\square
\]

Acknowledgments The first author was supported by the Scientific Research Fund of Sichuan Provincial Education Department (No.11ZA156), and the second author was supported by the National Natural Science Foundation of China (No.11271291).

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