Dynamical Bridges: The Electromagnetic Case

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Abstract. We present the main line of the Dynamical Bridge method applied to electromagnetic (EM) fields. In this framework, we show the equivalence between the Born-Infeld theory written in a given curved space to the Maxwell’s one written in the flat space. In the limit of weak EM fields, we obtain a geometrical contribution to the anomalous magnetic moment of the leptons and, finally, we compare it with experimental data.

1. Introduction
Recently, we have studied a mathematical map called “Dynamical Bridge” (DB) between distinct dynamical equations written in different background metrics (see Ref. [1] and references therein). This map corresponds to a new way of describing dynamical equations for fundamental fields, because it gives more than one representation of the same thing. In other words, the approach changes the space-time metric and the differential operator acting upon the dynamical field such that it is kept unchanged. Therefore, we are able to hide nonlinear terms into the curvature of the space-time.

Following Poincaré’s viewpoint [2] concerning the relationship between Geometry and Physics, we consider a curved metric without any gravitational character. Namely, the curved geometry introduced here is seen only as a way to describe geometrically nonlinear interacting terms. In particular, we specify the geometries such that the metric and its inverse assume the binomial form

\[ \hat{g}_{\mu\nu}(x^\alpha) = A(x^\alpha) \eta_{\mu\nu} + B(x^\alpha) \Phi_{\mu\nu}, \quad \hat{g}^{\mu\nu}(x^\alpha) = \alpha(x^\alpha) \eta^{\mu\nu} + \beta(x^\alpha) \Phi^{\mu\nu}, \]  

where \( \mu, \nu \) are space-time indexes running from 0 to 3 and \( x^\alpha \) represents space-time points. The functions \( A, B, \alpha \) and \( \beta \), the Minkowski metric \( \eta_{\mu\nu} \) and the symmetric tensor \( \Phi_{\mu\nu} \) are subjected to

\[ \Phi_{\mu\nu} \Phi^{\nu\lambda} = m \delta^\lambda_\mu + n \Phi^\lambda_\mu, \]  

where \( \delta^\lambda_\mu \) is the Kronecker delta. Note that \( A \) and \( B \) are completely determined if \( m, n, \alpha \) and \( \beta \) are known. This kind of metric can be seen as a disformal transformation [3] of the reference metric, which was chosen to be the Minkowski one.

The applications of this type of transformation lead us to alternative explanations of several phenomena for different fields which are not yet understood from the standard model viewpoint. For instance, we construct from scalar fields a geometric scalar theory of gravity [4] with \( \Phi_{\mu\nu} \) given in terms of gradients of the fundamental field; for massless spinor fields we could explain...
the chiral symmetry breaking [5] assuming $\Phi_{\mu \nu}$ as a tensorial product of linear combinations of the spinorial currents; for EM fields we derive a geometrical contribution to the magnetic moment of leptons [1], where $\Phi_{\mu \nu}$ is given in terms of Faraday tensor.

We will focus here only on the EM case. We start with an application in the kinematical context, explaining how the DB can be used to describe accelerated motions in a given space as geodesics in another space. Then, we move to the dynamical context and we construct the DB for EM fields in a very simple way, where the physical aspects can be discussed with more details. In the last part of the paper we introduce a comparison with experimental data.

2. Kinematical case

We present an extension of previous works [6, 7] concerning the DB in the kinematical context. We find a general and compact formula for the main theorem, which is valid now for any background (not only the Minkowski one) and it is independent of the functional form of $\hat{q}_{\mu \nu}$:

**Theorem:** Consider a congruence of curves represented by the vector field $V_\mu$ in a given space-time and parameterized by the affine parameter $\lambda$. This vector follows along a geodesic in the metric $\hat{q}_{\mu \nu}$ if the condition is satisfied

$$\frac{1}{2} \hat{N}_\mu + V_{[\mu, \nu]} \hat{q}^{\alpha \nu} V_\alpha - p(\lambda) V_\mu = 0,$$

where comma means partial derivative w.r.t the coordinates $x^\mu$ and $V_{[\mu, \nu]} \equiv V_{\mu, \nu} - V_{\nu, \mu}$. The norm $\hat{N}$ of $V_\mu$ is calculated with the metric $\hat{q}_{\mu \nu}$ and $p(\lambda)$ represents the parametrization chosen.

Applying this for a special set of metrics obtained from the disformal transformations [3], which depend on the background and the normalized field $V_\mu$ under consideration, that is

$$\hat{q}^{\mu \nu} = \alpha \eta^{\mu \nu} + \beta V_\mu V_\nu, \quad \text{and inverse} \quad \hat{q}_{\mu \nu} = \frac{1}{\alpha} \eta_{\mu \nu} - \frac{\beta}{\alpha(\alpha + \beta)} V_\mu V_\nu,$$

one can describe in terms of geodesics the motion produced by any conservative force. For instance, suppose that $V_\mu$ has an acceleration $a_\mu \equiv \partial_\mu \Phi$ in the Minkowski space, where $\Phi$ is a scalar potential. The condition for $V_\mu$ to follow along a geodesic in the metric (4) is

$$a_\mu = -\frac{1}{2} (\alpha + \beta)_{,\mu}. \quad (5)$$

It implies that $\Phi$, $\alpha$ and $\beta$ are related through

$$\alpha + \beta = e^{-2\Phi}.$$

3. Dynamical Context: Application to Vector Fields

The electromagnetic DB we present here corresponds to the equivalence between the Born-Infeld (BI) theory written in a curved space and the Maxwell theory expressed in flat space.

Let us start considering the BI Lagrangian [8]

$$L = \beta^2 \left( 1 - \sqrt{\hat{U}} \right),$$

where $\hat{U} \equiv 1 + \hat{F}/(2\beta^2) - \hat{G}^2/(16\beta^4)$ and $\hat{F} \equiv F_{\mu \nu} F^{\alpha \beta} \varepsilon^{\mu \alpha \nu \beta}$ and $\hat{G} \equiv F_{\mu \nu} F^{* \alpha \beta} \varepsilon^{\mu \alpha \nu \beta}$ are the two independent scalars constructed with the ‘electromagnetic’ metric $\hat{e}^{\mu \nu}$ and the Faraday...
tensor $F_{\mu \nu}$. The free parameter $\beta$ corresponds to the critical field of the BI theory. The curved space-time is assumed to be

$$\hat{e}_{\mu\nu} = a \eta_{\mu\nu} + b \Phi_{\mu\nu}, \quad \text{with} \quad \Phi_{\mu\nu} \equiv F_{\mu\alpha} F^{\alpha\nu}. \quad (7)$$

The variational principle applied to (6) yields the following dynamical equations

$$\frac{1}{\sqrt{-\hat{e}}} \partial_{\nu} \left[ \sqrt{-\hat{e}} \left( \hat{F}^{\mu\nu} - \frac{1}{4 \beta^2} \hat{G} \hat{F}^{*\mu\nu} \right) \right] = 0, \quad (8)$$

where $\hat{e} = \det(\hat{e}_{\mu\nu})$. We also consider the Maxwell Lagrangian defined in the Minkowski space

$$L = -\frac{1}{4} F, \quad (9)$$

with $F \equiv F_{\mu\nu} F_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta}$. The dynamical equations in this case are

$$\frac{1}{\sqrt{-\eta}} \partial_{\nu} \left( \sqrt{-\eta} F^{\mu\nu} \right) = 0. \quad (10)$$

where $\eta = \det(\eta_{\mu\nu})$. To demonstrate their equivalence is necessary to know how $\hat{F}^{\mu\nu}$ and its dual are related to the Maxwell’s fields. Their correspondence is expressed in terms of two algebraic equations

$$n - \epsilon F_m + \frac{\hat{G}}{2 \beta^2} e m = -\frac{Q}{4}, \quad \text{and} \quad -\epsilon G_m + \frac{\hat{G}}{2 \beta^2} = 0, \quad (11)$$

where $\epsilon = b/a$ and $n = 1 + \frac{1}{16} \epsilon^2 G^2$, $m = 1 - \frac{1}{4} \epsilon F$ and $Q = 1 - \frac{1}{2} \epsilon F - \frac{1}{16} \epsilon^2 G^2$.

The general solution of these equations exists, but it is very complicated. Thus, we shall consider the case of weak EM fields, i.e., only first order corrections in $F \ll \beta^2$. To simplify the second Eq. of (11), we assume the particular configuration for the EM field in which $G = 0$. Then, without loss of generality, we choose $a = 1$.

In this regime, the coefficient $\epsilon$ has a unique physical solution

$$\epsilon = \frac{2}{F} \left( 1 - \frac{1}{\sqrt{1 - F/2\beta^2}} \right)$$

and, finally, the EM metric becomes

$$\hat{e}_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2 \beta^2} \Phi_{\mu\nu}. \quad (12)$$

Now, we are ready to introduce an interaction between the EM field and leptons in the curved space, looking for nontrivial results in the flat space. In this context, the interaction is guided by some principles

(i) *Universality*: all particles (charged or not) interact with $\hat{e}_{\mu\nu}$.

(ii) The interaction is done through the *minimal coupling principle*.

Of course the Clifford algebra, the intrinsic algebra of spinors, must be preserved in both space-times. Then, from assumption (i) and Eq. (12) we obtain [1]

$$\hat{\gamma}^\mu = \gamma^\mu - \frac{1}{4 \beta^2} \Phi^{\mu\alpha} \gamma^\alpha. \quad (13)$$

where $\gamma^\mu$ are the Dirac matrices. It should be remarked that the first hypothesis (*universality*), commonly used in geometric theories of gravity, implies here a coupling between uncharged particles and the EM field. This will be crucial for coupling with neutrinos as we shall see later.

$^2$ This condition implies that field configurations like plane waves are solutions of both dynamics.
4. Geometric magnetic moment for leptons

In the curved space the dynamics of a spinor $\Psi$ is

$$i\hbar c\gamma^\mu \nabla_\mu \Psi - mc^2 \Psi = 0,$$

(14)

where $\nabla_\mu \equiv \partial_\mu - \dot{\Gamma}^F_\mu I - \dot{\nabla}_\mu$, $\dot{\Gamma}^F_\mu I$ is the Fock-Ivanenko connection and $\dot{\nabla}_\mu$ is an arbitrary element of the algebra given by

$$\dot{\nabla}_\mu = i\frac{mc}{\hbar} F^\mu_{\nu\beta} \gamma^\nu \gamma_5,$$

(15)

where $\gamma_5 = (i/4!) \epsilon^{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu$ and $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor. This yields $\nabla_\alpha \dot{\gamma}^\nu = [\dot{\nabla}_\mu, \dot{\gamma}^\nu]$, but keeps the metricity condition $\nabla_\alpha \dot{e}^{\mu\nu} = 0$ and preserves the current $\nabla_\mu \dot{J}^\mu = 0$ (see details in Ref. [1]).

Using the expressions for $\dot{e}^{\mu\nu}$, $\dot{\gamma}^\mu$ and $\dot{\nabla}_\mu$, we obtain the following equation for $\Psi$ in Minkowski space

$$i\hbar c\gamma^\mu \partial_\mu \Psi + \frac{mc^2}{2\beta} F^\mu_{\nu\beta} [\gamma^\mu, \gamma^\nu] \gamma_5 \Psi - mc^2 \Psi = 0.$$

(16)

Note that this equation provides a magnetic moment for $\Psi$, whose intensity is

$$\mu^G = \frac{mc^2}{\beta}.$$

(17)

We name this contribution as geometric magnetic moment.

Charged particles have a classical source for the magnetic moment $\mu$. For instance, electrons have the Bohr magneton $\mu_B = e\hbar/2m_e$. Therefore, the total value of the electron magnetic moment should be read as

$$\mu_e = \mu_B + \frac{m_e c^2}{\beta} + \text{quantum corrections}.$$

The first term is classical, the second one is the geometric magnetic moment and the last contribution represents the quantum corrections due to the presence of charge.

5. Comparison with experiments

The anomaly $a_l$ of the magnetic moment $\mu_l$ of a given particle $l$ corresponds to a measurable discrepancy between the theoretical prediction and the experimental result. It is defined as

$$a_l = \frac{g_l - 2}{2} = \frac{m_l}{m_e} \frac{\mu_l}{\mu_B} - 1,$$

(18)

where $l = (e^-, \mu, \tau)$, $g_l$ is the Landé factor and $m_l$ is the mass. There are measurements of this anomaly\(^3\) for the electron $e^-$ (the most precise measurement) and for the muon $\mu$ (test of the consistency of the standard model). For $\tau$'s, the anomaly is unobservable due to its short mean-life ($\sim 2.9 \times 10^{-13}$ s) but predictions indicate that it is less than $1.3 \times 10^{-2}$.

In the $\mu$-case, due to the particular role played by the anomaly of muons, the difference $\Delta a$ between the $a_l$-experimental and $a_l$-theoretical is assumed to provide the upper limit in which the geometric magnetic moment $\mu^G$ can contribute. So, if the effects of modifications caused by the existence of $\dot{e}^{\mu\nu}$ appear at this order, then we get

$$\mu^G_\mu = \Delta a_\mu \frac{e\hbar}{2m_\mu}.$$

(19)

\(^3\) All experimental data used here were taken from Particle Data Group [9]. In particular, in this section we use $\mu_B = 5.788 \times 10^{-11}$ MeV T$^{-1}$. 


On the other hand, the formula of $\mu^G$ is given by

$$\mu^G \mu = \frac{m\mu c^2}{\beta}. \quad (20)$$

Assuming that $\mu^G$ corresponds to the remaining value of muon’s anomaly, then we can estimate the critical field as $\beta \approx 1.31 \times 10^{23} T$.

$e^-$-case: Assuming this value for $\beta$, the geometric contribution for the electron is

$$\mu^G_e = 6.74 \times 10^{-14} \mu_B$$

and the discrepancy measured between theory and experiments is

$$\Delta a_e = -0.40 \times 10^{-12}. \quad (22)$$

Note that theory and experiments are in agreement up to few $10^{-12}$. For smaller scales the error bars are important, so that the geometric correction $\mu^G_e/\mu_B \sim 10^{-14}$ is not observable for $e^-$. $\nu$-case: Since neutrinos have zero charge, it is expected that their magnetic moment $\mu_\nu$ and the corresponding quantum corrections are zero. However, the extended standard model suggests that $\mu_\nu \neq 0$ and $\mu_\nu \propto m_\nu$. With this in mind, it seems natural to propose that $\mu_\nu$ has only the geometrical origin which would be

$$\mu_\nu \equiv \frac{m_\nu c^2}{\beta} = 1.32 \times 10^{-19} \mu_B. \quad (23)$$

This value is in agreement with phenomenology [10, 11, 12] and close to the extended standard model prediction ($\sim 10^{-19} \mu_B$).

6. Conclusion
The geometrical description of the nonlinearity of the Born-Infeld theory allowed us to explain, through the universality of the metric and the minimal coupling principle, part of the anomalous magnetic moment of leptons and, in particular, of neutrinos. The comparison to experimental data indicates that our proposal is in agreement with observations. So, this model should be deeply investigated in order to improve our understanding about the role of curved geometries in Physics and not only in the gravitational context. This is currently under analysis.

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