The Benefits of Over-parameterization at Initialization in Deep ReLU Networks

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Abstract
It has been noted in existing literature that over-parameterization in ReLU networks generally leads to better performance. While there could be several reasons for this, we investigate desirable network properties at initialization which may be enjoyed by ReLU networks. Without making any assumption, we derive a lower bound on the layer width of deep ReLU networks and an optimal initialization for the parameters, such that with high probability, i) the norm of hidden activation of all layers are roughly equal to the norm of the input, and, ii) the norm of parameter gradient for all the layers are roughly the same. In this way, sufficiently wide deep ReLU nets with appropriate initialization can inherently preserve the forward flow of information and avoid the gradient exploding/vanishing problem. We further show that these results hold for an infinite number of data samples, in which case the finite lower bound depends on the input dimensionality and the depth of the network. In the case of deep ReLU networks with weight normalization, we derive an initialization required to tap the aforementioned benefits from over-parameterization without which network fails to learn for large depth.

1. Introduction
Parameter initialization is an important aspect of deep network optimization and plays a crucial role in determining the quality of the final model. Too large or too small parameter scale leads to exploding or vanishing gradient problem across hidden layers. As such, some parameters can get initialized in plateaus and others along steep valleys and optimization becomes unstable. We will specifically focus on this problem for deep ReLU networks because of their popularity and success in the various applications of deep learning. There have been a number of papers that have studied initialization in deep ReLU networks previously. We contrast between our contribution and existing work in section 2.

Our analysis of the initialization aspect for deep ReLU networks centers around the claim that over-parameterization in terms of network width (number of channels) avoids the exploding and vanishing gradient problem in the backward pass, and forms a norm preserving mapping across hidden layers in the forward pass. Our findings/contributions are as follows:

1. We derive an initialization condition for which deep ReLU networks are norm preserving maps, i.e., the norm of hidden activations is approximately equal to the norm of the input vectors.
2. The same initialization also guarantees that the exploding and vanishing gradient problem does not happen in the sense that the norm of parameter gradients are equal across all layers.
3. We do not make any assumption on the data distribution as done in a number of previous papers that study initialization. Our results hold for an infinite stream of data.
4. We derive a finite lower bound on the width of the hidden layers for which the above results hold (i.e., the network needs to be sufficiently over-parameterized) in contrast to a number of previous papers that assume infinitely wide layers.
5. We show how to initialize deep ReLU networks whose weight vectors are normalized by their norm (as done in (Salimans & Kingma, 2016; Arpit et al., 2016)) so that properties 1 and 2 above hold given the network is wide enough. To the best of our knowledge, we are the first to study initialization conditions for weight normalized deep ReLU networks.
6. Finally, we derive the initialization conditions for residual networks which ensures the activation norms are preserved when the network is sufficiently wide (see appendix A).

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2. Relation with Existing Work

The seminal work of Glorot & Bengio (2010) studied for the first time a principled way to initialize deep networks to avoid gradient explosion/vanishing problem. Their analysis however is done for deep linear networks. The analysis by He et al. (2015) follows the derivation strategy of Glorot & Bengio (2010) except they tailor their derivation for deep ReLU networks. However, both these papers make a strong assumption that the dimensions of the input are statistically independent and that the network width is infinite. Our results do not make these assumptions.

(Saxe et al., 2013) introduce the notion of dynamical isometry which is achieved when all the singular values of the input-output Jacobian of the network is 1. They show that deep linear networks achieve dynamical isometry when initialized using orthogonal weights and this property allows fast learning in such networks.

Poole et al. (2016) study how the norm of hidden activations evolve when propagating an input through the network. Pennington et al. (2017; 2018) study the exploding and vanishing gradient problem in deep ReLU networks using tools from free probability theory. Under the assumption of an infinitely wide network, they show that the average squared singular value of the input-output Jacobian for deep ReLU network is 1 when initialized appropriately. Our paper on the other hand shows that deep ReLU networks are norm preserving maps at appropriate initialization. Further, there exists a finite lower bound on the width of the network for which the Frobenius norm of the hidden layer-output Jacobian (equivalently the sum of its squared singular values) are equal across all hidden layers.

(Hanin & Rolnick, 2018) show that for a fixed input, the variance of the squared norm of hidden layer activations are bounded from above and below for deep ReLU networks to be near the squared norm of the input such that the bound depends on the sum of reciprocal of layer widths of the network. Our paper shows a similar result in a PAC bound sense but as an important difference, we show that these results hold even for an infinite stream of data by making the bound depend on the dimensionality of the input.

(Hanin, 2018) show that sufficiently wide deep ReLU networks with appropriately initialized weights prevent exploding/vanishing gradient problem (EVGP) in the sense that the fluctuation between the elements of the input-output Jacobian matrix of the network is small. This avoids EVGP because a large fluctuation between the elements of the input-output Jacobian implies a large variation in its singular values. Our paper shows that sufficiently wide deep ReLU networks avoid EVGP in the sense that the norm of the gradient for the weights of each layer is roughly equal to a fixed quantity that depends on the input and target.

Arpit et al. (2016) introduced weight normalized deep networks in which the pre and post activations are scaled/summed with constants depending on the activation function, ensuring that the hidden activations have 0 mean and unit variance, especially at initialization. Their work makes assumptions on the distribution of input and pre-activations of the hidden layers. Weight normalization (Salimans & Kingma, 2016) offers a simpler alternative to Arpit et al. (2016) in which no scaling/summing constants are used aside from normalizing the weights by their norm. Our work focuses on the initialization conditions for normalized deep ReLU networks and shows that over-parameterization along with appropriate initialization ensures the activation and parameter gradient norms are preserved without making any such assumptions.

Over-parameterization in deep networks has previously been shown to have advantages. Neyshabur et al. (2014); Arpit et al. (2017) show empirically that wider networks train faster (number of epochs) and have better generalization performance. From a theoretical view point, (Neyshabur et al., 2018) derive a generalization bound for a two layer ReLU network where they show that a wider network has a lower complexity. Lee et al. (2017) show that infinitely wide deep networks act as a Gaussian process. (Arora et al., 2018) show that over-parameterization in deep linear networks acts as a conditioning on the gradient leading to faster convergence, although in this case over-parameterization in terms of depth is studied. Our analysis complements this line of work by showing another advantage of over-parameterization in deep ReLU networks.

Random projection is a popular method for dimensionality reduction based on the Johnson-Lindenstrauss (JL) lemma (Johnson & Lindenstrauss, 1984) that involves projecting a vector onto a properly constructed random matrix. This lemma states that the $\ell_2$ norm of a randomly projected vector is approximately equal to the $\ell_2$ norm of the original vector. In this work, we show that a linear transformation followed by a point-wise ReLU also preserves the norm of the input vector in the following cases: i) each element of the projection matrix is sampled i.i.d. from an isotropic distribution; ii) each row vector is i.i.d. sampled from an isotropic distribution and has unit length.

3. Un-normalized Deep ReLU Networks

Let $\mathcal{D} = \{x_i, y_i\}_{i=1}^N$ be $N$ training sample pairs of inputs vectors $x_i \in \mathbb{R}^{n_o}$ and target vectors $y_i^{\mathcal{R}}$. Define a $L$ layer deep ReLU network $f_{\theta}(x) = h_L^{l}$ with the $l^{th}$ hidden layer's activation given by,

$$h_l := ReLU(a_l)$$

$$a_l := W_l h_{l-1} + b_l \quad l \in \{1, 2, \cdots L\}$$

(1)
where $h^l \in \mathbb{R}^{n_l}$ are the hidden activations, $h^L$ is the input to the network and can be one of the input vectors $x_i$, $W^l \in \mathbb{R}^{n_l \times n_{l-1}}$ are the weight matrices, $b \in \mathbb{R}^{n_l}$ are the bias vectors which are initialized as 0s, $a^l$ are the pre-activations and $\theta = \{(W^l, b^l)\}_{l=1}^L$.

Define a loss on the deep network function for any given training data sample $(x, y)$ as,

$$\ell(f_{\theta}(x), y)$$

(2)

where $\ell(.)$ is any desired loss function. For instance, $\ell(.)$ can be log loss for a classification problem, in which case $f_{\theta}(x)$ is transformed using a weight matrix to have dimensions equal to the number of classes and the softmax activation is applied to yield class probabilities (i.e., a logistic regression like model on top of $f_{\theta}(x)$). However for our purpose, we do not need to restrict $\ell(.)$ to a specific choice, we only need it to be differentiable. We will make use of the notation,

$$\delta(x, y) := \frac{\partial \ell(f_{\theta}(x), y)}{\partial a_L}$$

(3)

We first derive the norm preservation property for finite datasets and then extend these results to an infinite data stream.

### 3.1. Activation Norm Preservation

Consider an $L$ layer deep ReLU network and a data samples $x$ from a fixed dataset $\mathcal{D}$. Then we essentially show in this section that for a sufficiently wide ReLU network, the norm of hidden layer activation of any layer is roughly equal to the norm of the input at initialization if the network weights are initialized appropriately. Specifically we show $\forall l \in [L],

$$\|h^l\|_2 \approx \|x\|_2 \quad \forall x \in \mathcal{D}$$

(4)

An important step to achieve this goal is to show that in expectation, the non-linear transformation in each layer preserves the norm of its corresponding input. Evaluating this expectation also helps determining the scale of the random initialization that leads to norm preservation.

**Lemma 1** Let $v = ReLU(Ru)$, where $u \in \mathbb{R}^n$, $R \in \mathbb{R}^{m \times n}$. If $R_{ij} \sim \mathcal{N}(0, \frac{2}{m})$, then for any fixed vector $u$,

$$E[\|v\|^2] = \|u\|^2.$$  

The above result shows that for each layer, initializing its weights from an i.i.d. Gaussian distribution with 0 mean and $2/n$ fan-out variance preserves the norm of its input in expectation. We now show that this property holds for a finite width network.

**Lemma 2** Let $v = ReLU(Ru)$, where $u \in \mathbb{R}^n$, $R \in \mathbb{R}^{m \times n}$. If $R_{ij} \sim \mathcal{N}(0, \frac{2}{m})$, and $\epsilon \in [0, 1)$, then for any fixed vector $u$,

$$\Pr (\|v\|^2 - \|u\|^2 \leq \epsilon \|u\|^2) \geq 1 - 2 \exp \left(-m \frac{\epsilon}{2} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right)$$

(5)

We note that (He et al., 2015) also find this initialization appropriate for ReLU networks (We contrast with their findings in section 2). We finally extend this result to show that all hidden layers and the output of a deep ReLU network also preserve the norm of the input for a finite width network and a finite dataset.

**Theorem 1** Let $D$ be a fixed dataset with $N$ samples and define a $L$ layer ReLU network $f_{\theta}(\cdot)$ as shown in Eq. 1 such that each weight matrix $W^l \in \mathbb{R}^{n_l \times n_{l-1}}$ has its elements sampled as $W_{ij} \sim \mathcal{N}(0, \frac{2}{n_l})$ and biases $b^l$ are set to zeros. Then for any sample $(x, y) \in D$ and $\epsilon \in [0, 1)$, we have that,

$$\Pr ((1 - \epsilon)^L \|x\|^2 \leq \|f_{\theta}(x)\|^2 \leq (1 + \epsilon)^L \|x\|^2)$$

$$\geq 1 - \sum_{l=1}^{L} 2N \exp \left(-n_l \frac{\epsilon}{2} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right)$$

(6)

If the weights are sampled from a Gaussian with variance $\epsilon$ larger (smaller) than $2/n_l$, the hidden activation norm will explode (vanish) exponentially with depth.

### 3.2. Gradient Norm Preservation

Consider any given loss function $\ell(\cdot)$ and a data sample $(x, y)$, we will show in this section that the norm of gradient for the parameter $W^l$ of the $l$th layer will roughly be equal to each other at initialization if the network weights are initialized appropriately. More specifically, we will show that for a wide enough network, the following holds at initialization for all $l \in \{1, 2, \ldots, L\}$ and $\forall x \in D$,

$$\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial W^l}\|_F \approx \|\delta(x, y)\|_2 \cdot \|x\|_2 \quad \forall l$$

(7)

As a first step, we note that the gradient for a parameter $W^l$ for a sample $(x, y)$ is given,

$$\frac{\partial \ell(f_{\theta}(x), y)}{\partial W^l} = diag \left( \frac{\partial \ell(f_{\theta}(x), y)}{\partial a^l} \right) \cdot M_{n_l}(h_l^{-1})$$

(8)

where $M_{n_l}(h_l^{-1})$ is a matrix of size $n_l \times n_{l-1}$ such that each row is the vector $h_l^{-1}$. Therefore, a simple algebraic manipulation shows that,

$$\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial W^l}\|_F = \left\| \frac{\partial \ell(f_{\theta}(x), y)}{\partial a^l} \right\|_2 \cdot \|h_l^{-1}\|_2$$

(9)
In the previous section, we showed that for a sufficiently wide network, \( \| \mathbf{h}_l \|_2 \approx \| x \|_2 \forall l \) with high probability. To show that gradient norms of parameters are preserved in the sense shown in Eq. (7), we essentially show that \( \| \partial \ell(f_\theta(x), y) \|_2 \approx \| \delta(x, y) \|_2 \forall l \) with high probability for sufficiently wide networks.

Note that \( \| \partial \ell(f_\theta(x), y) \|_2 = \| \delta(x, y) \|_2 \) by definition. To show the norm is preserved for all layers, we begin by noting that,

\[
\frac{\partial \ell(f_\theta(x), y)}{\partial \mathbf{a}^l} = \frac{\partial \mathbf{h}^l}{\partial \mathbf{a}^l} \odot \left( \frac{\partial \mathbf{a}^{l+1}^T}{\partial \mathbf{h}^l} \frac{\partial \ell(f_\theta(x), y)}{\partial \mathbf{a}^{l+1}} \right) \\
= \mathbb{I}(\mathbf{a}^l) \odot \left( \mathbf{W}^{l+1} \frac{\partial \ell(f_\theta(x), y)}{\partial \mathbf{a}^{l+1}} \right)
\]

where \( \odot \) is the point-wise product (or Hadamard product) and \( \mathbb{I}(\cdot) \) is the heaviside step function. The following proposition shows that \( \mathbb{I}(\cdot) \) follows a Bernoulli distribution w.r.t. the weights given any fixed input.

**Proposition 1** If network weights are sampled i.i.d. from a Gaussian distribution with mean 0 and biases are 0 at initialization, then conditioned on \( \mathbf{h}^{l-1} \), each dimension of \( \mathbb{I}(\mathbf{a}^l) \) follows an i.i.d. Bernoulli distribution with probability 0.5 at initialization.

Given this property of \( \mathbb{I}(\mathbf{a}^l) \), we show below that the transformation of type shown in Eq. (10) is norm preserving in expectation.

**Lemma 3** Let \( \nu = (\mathbf{R} \mathbf{u}) \odot \mathbf{z} \), where \( \mathbf{u} \in \mathbb{R}^n \), \( \mathbf{R} \in \mathbb{R}^{m \times n} \) and \( \mathbf{z} \in \mathbb{R}^m \). If \( \mathbf{R}_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{m^2}) \) and \( \mathbf{z}_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{m}) \), Bernoulli(p), then for any fixed vector \( \mathbf{u} \), \( \mathbb{E}[\| \nu \|_2^2] = \| \mathbf{u} \|_2^2 \).

The above lemma reveals the variance of the 0 mean Gaussian from which the weights must be sampled in order for the vector norm to be preserved in expectation. Since \( \mathbb{I}(\mathbf{a}^l) \) is sampled from a 0.5 probability Bernoulli, we have that the weights must be sampled from a Gaussian with variance \( \frac{2}{m} \). We now show this property holds for a finite width network.

**Lemma 4** Let \( \nu = (\mathbf{R} \mathbf{u}) \odot \mathbf{z} \), where \( \mathbf{u} \in \mathbb{R}^n \), \( \mathbf{z} \in \mathbb{R}^m \), and \( \mathbf{R} \in \mathbb{R}^{m \times n} \), \( \mathbf{R}_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{0.5m^2}) \), \( \mathbf{z}_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{m}) \), Bernoulli(0.5) and \( \epsilon \in [0, 1) \), then for any fixed vector \( \mathbf{u} \),

\[
\Pr (\| \nu \|_2^2 - \| \mathbf{u} \|_2^2 \leq \epsilon \| \mathbf{u} \|_2^2) \\
\geq 1 - 2 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)
\]

Having shown the finite width case holds, we now note that we need to apply this result to Eq. (10). In this case, we must substitute the matrix \( \mathbf{R} \) in the above lemma with the network’s weight matrix \( \mathbf{W}^{l+1} \). In the previous subsection, we showed that each element of the matrix \( \mathbf{W}^{l+1} \) must be sampled from \( \mathcal{N}(0, 2/n_{l+1}) \) in order for the norm of the input vector to be preserved. However, in order for the Jacobian norm to be preserved, we require \( \mathbf{W}^{l+1} \) to be sampled from \( \mathcal{N}(0, 2/n_l) \) as per the above lemma. This suggests that if we want the norms to be preserved in the forward and backward pass for a single layer simultaneously, it is beneficial for the width of the network to be close to uniform. The reason we want them to simultaneously hold is because as shown in Eq. (9), in order for the parameter gradient norm to be same for all layers, we need the norm of both the Jacobian \( \| \partial \ell(f_\theta(x), y) \|_2 \) as well as the hidden activation \( \| \mathbf{h}_l \|_2 \) to be preserved throughout the hidden layers. Therefore, assuming the network has a uniform width, we now prove that deep ReLU networks with the mentioned initialization prevent the exploding/vanishing gradient problem.

**Theorem 1** Let \( \mathcal{D} \) be a fixed dataset with \( N \) samples and define a L layer ReLU network as shown in Eq. (1) such that each weight matrix \( \mathbf{W}^l \in \mathbb{R}^{n \times n} \) has its elements sampled as \( \mathbf{W}_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{2}{n_l}) \) and biases \( \mathbf{b}^l \) are set to zeros. Then for any sample \( (x, y) \in \mathcal{D}, \epsilon \in [0, 1) \), and for all \( l \in \{1, 2, \ldots, L\} \) with probability at least,

\[
1 - 4NL \exp \left( -n \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)
\]

the following hold true,

\[
(1 - \epsilon)^L \| x \|_2 \cdot \| \delta(x, y) \|_2 \leq \| \frac{\partial \ell(f_\theta(x), y)}{\partial \mathbf{W}^l} \|_2 \\
\leq (1 + \epsilon)^L \| x \|_2 \cdot \| \delta(x, y) \|_2
\]

and

\[
(1 - \epsilon)^4 \| x \|_2 \leq \| \mathbf{h}_l \|_2 \leq (1 + \epsilon)^4 \| x \|_2
\]

The corollary below shows a lower bound on the network width which simultaneously ensures parameter gradients do not explode/vanish and activation norms are preserved.

**Corollary 1** Suppose all the hidden layers of the L layer deep ReLU network \( f_\theta(.) \) have the same width \( n \), and let the following hold with probability at least \( 1 - \delta \)

\[
\| f_\theta(x) \|_2 - \| x \|_2 \leq \epsilon \| x \|_2
\]

and,

\[
\frac{\partial \ell(f_\theta(x), y)}{\partial \mathbf{W}^{l+1}} \text{ in independent from } \mathbb{I}(\mathbf{a}^l) \text{ and } \mathbf{W}^{l+1} \text{ at initialization. We get rid of this assumption in the next subsection.}
\]
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\[ \left\| \frac{\partial f_0(x, y)}{\partial W^j} \right\|^2 \leq \epsilon \|x\|^2 \|\delta(x, y)\|^2 \]

hold for \( N \) fixed training samples \((x, y) \in D\), then,

\[ n \geq \frac{1}{0.25\epsilon' - \log(0.5(1 + \sqrt{1 + \epsilon'})) \log \frac{4NL}{\delta}} \tag{15} \]

where \( \epsilon' = (1 + \epsilon)^{1/L} - 1 \).

3.3. Infinite Data Stream

So far, we have shown for sufficiently wide deep ReLU network with appropriate initialization that they avoid the exploding/vanishing gradient problem and preserve the norm of hidden activations for, i) a finite dataset, ii) the assumption shown in theorem 2. We now show that the norm-preserving property of a ReLU layer can hold for an infinite stream of data if the layer width is larger than the lower bound shown below which depends on the input dimensionality.

**Theorem 3** Let \( X \) be a \( d \leq n \) dimensional subspace of \( \mathbb{R}^n \) and \( R \in \mathbb{R}^{m \times n} \). If \( R_{ij} \overset{i.i.d.}{\sim} N(0, \frac{2}{m}) \), \( \epsilon \in [0, 1) \), and,

\[ m \geq \frac{1}{1/12 - \log(0.5(1 + \sqrt{1 + \epsilon})) \log \frac{2d}{\Delta} + \log \frac{4}{\delta}} \tag{16} \]

then with probability at least \( 1 - \delta \),

\[ \|ReLU(Ru)\| - \|u\| \leq \epsilon \|u\| \quad \forall u \in X \tag{17} \]

where \( \Delta := \min\{\frac{\epsilon}{\sqrt{d}}, \frac{\sqrt{\epsilon}}{\sqrt{d}}\} \).

In order to apply theorem 3 recursively to multiple layers, the proof strategy (identical to that used in theorem 1) is to consider the representations of all the samples up until the layer below to be fixed, and then apply the above theorem to the fixed inputs of the current layer. Also notice that if the input to the network lies on a \( d \) dimensional subspace, the output of the first hidden layer with a fixed weight matrix and ReLU non-linearity will also lie on \( d \) dimensional subspace. Therefore all layers will have to deal with input lying on a \( d \) dimensional subspace.

A similar result to extend the result of lemma 4 to show norm preservation for the backward pass in the infinite data stream case is derived below.

**Theorem 4** Let \( X \) be a \( d \leq n \) dimensional subspace of \( \mathbb{R}^n \), \( z \in \mathbb{R}^m \), and \( R \in \mathbb{R}^{m \times n} \). If \( R_{ij} \overset{i.i.d.}{\sim} N(0, \frac{2}{m}) \), \( \epsilon \in [0, 1) \), \( z_i \overset{i.i.d.}{\sim} Bernoulli(0.5) \), \( \epsilon \in [0, 1) \), and,

\[ m \geq \frac{1}{\epsilon/12 - \log(0.5(1 + \sqrt{1 + \epsilon})) \log \frac{2d}{\Delta} + \log \frac{4}{\delta}} \tag{18} \]

then for all vectors \( u \in X \), with probability at least \( 1 - \delta \),

\[ \|Ru\| - \|u\| \leq \epsilon \|u\| \tag{19} \]

where \( \Delta := \min\{\frac{\epsilon}{\sqrt{d}}, \frac{\sqrt{\epsilon}}{\sqrt{d}}\} \).

The statement of theorem 2 will hold for an infinite stream of data as well if the network width is wide enough depending as \( \mathcal{O}(d \log d) \) on input manifold dimensionality \( d \) and \( \mathcal{O}(\log L) \) on depth \( L \) (of course the lower bound on width in that case needs to be re-calculated using the above two theorems). Finally note that the assumption made in theorem 2 becomes irrelevant once the network width is larger than the above mentioned bound because this assumption is required in order to apply lemma 4 to Eq. (10). But since the infinite data version of lemma 4, viz, theorem 4, applies to all possible vectors, this assumption is not necessary.

4. Weight Normalized Deep ReLU Networks

We now analyze deep ReLU networks whose weight vectors are normalized at every layer. We define a \( L \) layer normalized deep ReLU network \( f_0(x) = h^L \) with the \( l \)th hidden layer’s activation given by,

\[ h^l := ReLU(a^l) \]

\[ a^l := k_l W^l h^{l-1} + b^l \quad l \in \{1, 2, \cdots L\} \tag{20} \]

where \( k_l \) is a multiplicative factor (which we will show is important), and the notation \( W^l \) implies that each row vector of \( W^l \) has unit norm, i.e.,

\[ \frac{W_i^l}{\|W_i^l\|_2} \quad \forall i \tag{21} \]

The rest of the symbols in Eq. (20) have same definition as in Eq. (1). For any vector \( u \), we also define the notation \( \hat{u} := \frac{u}{\|u\|_2} \). The definition of \( \delta(x, y) \) is the same as that in Eq. (3).

We find that with an appropriate initialization and sufficiently wide layers, both the activation norms and parameter gradient norms are preserved in such deep ReLU networks. In contrast with the analysis in the previous section, we will only study normalized ReLU networks in terms of expectation and not extend the results to derive PAC bounds. We will instead resort to the argument of the law of large numbers due to which one can expect that if the width of the layers are large enough, the results for the finite width case approaches the results that hold in expectation. Throughout this section, take note of the distinction between the notations \( R \) and \( \hat{R} \) for any matrix \( R \).
4.1. Activation Norm Preservation

The theorem below shows that in expectation, the transformation of any hidden layer of a normalized ReLU network preserves the norm of the input.

**Theorem 5** Let \( \mathbf{v} = \text{ReLU} \left( \sqrt{\frac{2n}{m}} \cdot \mathbf{R} \mathbf{u} \right) \), where \( \mathbf{u} \in \mathbb{R}^n \) and \( \mathbf{R} \in \mathbb{R}^{m \times n} \). If \( \mathbf{R} \sim \text{i.i.d.} \ P \) where \( P \) is any isotropic distribution in \( \mathbb{R}^n \), then for any fixed vector \( \mathbf{u} \), \( \mathbb{E}[\|\mathbf{v}\|^2] = K_n \cdot \|\mathbf{u}\|^2 \) where,

\[
K_n = \begin{cases} 
\frac{2^{n-1}}{2^n} \cdot \left( \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{n-2}{n-3} \right) & \text{if } n \text{ is even} \\
\frac{2^{n-1}}{2^n} \cdot \left( \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{n-2}{n-3} \right) \cdot \frac{2}{3} & \text{otherwise}
\end{cases}
\]

and \( S_n \) is the surface area of a unit \( n \)-dimensional sphere.

The constant \( K_n \) seems hard to evaluate analytically, but remarkably, we empirically find that \( K_n = 1 \) for \( n > 1 \). This implies that in expectation, the non-linear transformation shown in theorem 5 is norm preserving for practical cases where input dimension is larger than 1. Hence, we can extend this argument to a normalized deep ReLU network and have that the norm of the output of the network is approximately equal to the norm of the input for wide networks. We summarize this statement in the following informal corollary.

**Corollary 2** (informal) For a sufficiently wide normalized deep ReLU network \( f_{\theta_i}(\cdot) \) defined in Eq. (20) with \( k_l = \sqrt{\frac{2n_i-1}{m}} \), the following holds for any fixed input \( \mathbf{x} \) at initialization,

\[
\|f_{\theta}(\mathbf{x})\| \approx \|\mathbf{x}\|
\]

4.2. Gradient Norm Preservation

When analyzing the parameter gradients of a normalized deep ReLU network, there is a slight ambiguity. In such a network (described by Eq. (20)), the output is invariant to the scale of the weights. Hence, it is only the direction of the weight vectors that matter, in which case we may study the gradient w.r.t. the normalized weights. However, in practice, we update the un-normalized weights in which case the gradient back-propagates through the normalization term as well. Therefore, we study the gradient norm for both these cases at the time of initialization.

**Gradient w.r.t. \( \hat{W} \):** In this case, we show that for a sufficiently wide network \( \forall \ell \),

\[
\left\| \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \hat{W}^\ell} \right\|_F \approx \sqrt{\frac{2n_{\ell-1}}{n_{\ell}}} \cdot \left\| \delta(\mathbf{x}, y) \right\|_2 \cdot \left\| \mathbf{x} \right\|_2
\]

The steps for showing this are very similar to those in section 3.2 with minor differences. We note that,

\[
\frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \hat{W}^\ell} = \sqrt{\frac{2n_{\ell-1}}{n_{\ell}}} \cdot \text{diag} \left( \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{a}^\ell} \right) \cdot M_{n_{\ell}}(\mathbf{h}^{\ell-1})
\]

where \( M_{n_{\ell}}(\mathbf{h}^{\ell-1}) \) is a matrix of size \( n_{\ell} \times n_{\ell-1} \) such that each row is the vector \( \mathbf{h}^{\ell-1} \). Therefore for all \( l \),

\[
\left\| \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{W}^l} \right\|_F = \sqrt{\frac{2n_{l-1}}{n_{l}}} \cdot \left\| \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{a}^l} \right\|_2 \cdot \left\| \mathbf{h}^{l-1} \right\|_2
\]

Since we have already stated in corollary 2 that for a sufficiently wide network the norm of any hidden layer’s activations is approximately equal to the norm of the input, we only need to show that \( \left\| \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{a}^l} \right\|_2 \approx \left\| \delta(\mathbf{x}, y) \right\|_2 \). To this end, we write,

\[
\frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{a}^l} = \sqrt{\frac{2n_{l-1}}{n_{l}}} \cdot \mathbb{I}(\mathbf{a}^l) \odot \left( \hat{W}^{l+1} \cdot \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{a}^{l+1}} \right)
\]

To avoid redundancy, we note that similar to proposition 1, each dimension of \( \mathbb{I}(\mathbf{a}^l) \) in the above equation follows an i.i.d. sampling from Bernoulli distribution with probability 0.5 at initialization. The following proposition then shows that transformations of type shown in the above equation is norm preserving.

**Proposition 2** Let \( \mathbf{v} = \sqrt{\frac{2n}{m}} \cdot (\hat{R} \mathbf{u}) \odot \mathbf{z} \), where \( \mathbf{u} \in \mathbb{R}^n \), \( \mathbf{R} \in \mathbb{R}^{m \times n} \) and \( \mathbf{z} \in \mathbb{R}^m \). If \( \mathbf{R} \sim \text{i.i.d.} \ P \) where \( P \) is any isotropic distribution in \( \mathbb{R}^n \) and \( \mathbf{z} \sim \text{i.i.d.} \ \text{Bernoulli}(0.5) \), then for any fixed vector \( \mathbf{u} \), \( \mathbb{E}[\|\mathbf{v}\|^2] = K_n \cdot \|\mathbf{u}\|^2 \), where \( K_n \) is same as defined in theorem 5.

The above proposition when applied to Eq. (27) shows that \( \| \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{W}^l} \|_F \approx \left\| \delta(\mathbf{x}, y) \right\|_2 \) for a sufficiently wide network due to the law of large numbers. Substituting this result along with the statement of corollary 2 into Eq. (26) leads us to the conclusion that Eq. (24) holds true.

**Gradient w.r.t. \( \mathbf{W} \):** Now we consider the gradient norm of un-normalized parameters \( \mathbf{W} \) which are the ones that are updated in practice instead of \( \hat{W} \). In this case, we show for a sufficiently wide network, \( \forall \ell \),

\[
\left\| \frac{\partial \ell(f_{\theta}(\mathbf{x}), y)}{\partial \mathbf{W}^l} \right\|_F \approx \sqrt{\frac{2n_{l-1}}{n_l}} \cdot \left\| \delta(\mathbf{x}, y) \right\|_2 \cdot \left\| \mathbf{x} \right\|_2
\]
where we have assumed for each $l$, $\|W_l\|^2$ is the same for all $i$. To begin, notice,

$$\frac{\partial \ell(f_{\theta}(x), y)}{\partial W_i} = \frac{2n_{l-1}}{n_l} \cdot \text{diag} \left( \frac{\partial \ell(f_{\theta}(x), y)}{\partial a^l} \right) \cdot M$$

where the $i$th row of the matrix $M$ is,

$$M_i = \frac{1}{\|W_i\|} \cdot \left( I - \frac{W_i W_i^T}{\|W_i\|^2} \right) \cdot h^{l-1} \quad (30)$$

In the previous two cases (Eq. (7) and Eq. (25)), we were able to show that parameter gradient decomposes into two separate terms: $\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial \theta}\|_2$ and $\|h^{l-1}\|_2$. This decomposition is not possible directly in this case. So we jointly compute the expectation of the gradient norm $\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial W_i}\|_2^2$.

### Proposition 3

Consider a matrix $M = \sqrt{\frac{2n}{m}} \cdot \text{diag}(z) \cdot H$ such that the $i$th row $H_i := \frac{1}{\|z\|} \cdot (I - R_i R_i^T) \cdot u$, where $u \in \mathbb{R}^n$, $R_i \in \mathbb{R}^{m \times n}$ and $z \in \mathbb{R}^m$. If $R_i \sim i.i.d. P$ where $P$ is any isotropic distribution in $\mathbb{R}^n$ such that $\|R_i\| = c \forall i$ for some fixed $c$ and $z$ follows any distribution independent of $R$, then for any fixed vector $u$, $E[\|M\|^2] = \frac{2n}{m^2} \cdot (1 - \frac{K_n}{m})$. 

The proposition applied to Eq. (29) shows that,

$$E[\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial W_i}\|_2^2] = \frac{2n_{l-1}}{n_l \|W_i\|^2} \cdot (1 - \frac{K_{l-1}}{n_{l-1}}) \cdot E[\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial a^l}\|_2^2] \cdot \|h^{l-1}\|_2^2 \quad (31)$$

For wide enough network, $(1 - \frac{K_{l-1}}{n_{l-1}}) \approx 1$. Further, proposition 2 applied to Eq. (27) already shows that $E[\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial a^l}\|_2^2] \approx \|\ell(x, y)\|_2^2$. Combining these arguments, we have shown that Eq. (28) holds true. Thus an interesting aspect of normalized networks is that the weight scale has no role in gradient exploding/vanishing problem.

### 4.3. Batch Normalization

Batch normalization (Ioffe & Szegedy, 2015) is designed in a way that the distribution of pre-activation in each mini-batch has zero mean and unit variance for each feature dimension. Therefore, when feed-forwarding a batch of input through hidden layers, the norm of activations must not vanish or explode. Hence the scaling factor for pre-activations needed for weights normalization may not be required in the case of batch normalization. The properties of gradients and the loss curvature have been studied by Santurkar et al. (2018), and we point the reader to this reference for a detailed analysis. We leave a thorough analysis of initialization conditions for batch norm as future work.

### 5. Experiments

We show three experiments in the main paper (see appendix for additional experiments). In the first set of experiments, we verify the hidden activations have the same norm as input norm $\|x\|_2^2 \approx 1$ (Eq. 4), and the parameter gradient norm approximately equals the product of input norm and output error norm $\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial W_i}\|_2^2 \approx 1$ (Eq. 7) for all layer indices $i$ for sufficiently wide un-normalized deep ReLU networks. For this experiment we choose a 10 layer network with 2000 randomly generated input samples in $\mathbb{R}^{1000}$ and randomly generated target labels in $\mathbb{R}^{20}$ and cross-entropy loss. We add a linear layer along with softmax activation to the ReLU network’s outputs to make the final output in $\mathbb{R}^{20}$. According to corollary 1, the network width should be at least 4060 to preserve the activation norm and gradient norm simultaneously with maximum error margin $\epsilon = 0.15$ and failure probability $\delta = 0.05$. We plot the aforementioned ratios for width 4060 and smaller widths (100, 500, 2000) for comparison. We show results for both He initialization (He et al., 2015) which we theoretically show is optimal, as well as Glorot initialization (Glorot & Bengio, 2010) which is not optimal for deep ReLU nets. As can be seen in figure 1, the mean ratio of hidden activation norm to the input norm over the dataset is roughly 1 with a small standard deviation for He initialization. This approximation becomes better with larger width. On the other hand, Glorot initialization fails at preserving activation norm for deep ReLU nets. A similar result can be seen for parameter gradients norms (figure 2), where we find He initialization circumvents the gradient exploding/vanishing problem for all layers.

In the second experiment, we make the same evaluations as above for weight normalized deep ReLU network. We compute gradient w.r.t. un-normalized weights (see section 4.2), in which case, as we theoretically showed, $\|\frac{\partial \ell(f_{\theta}(x), y)}{\partial \theta}\|_2^2 \approx 1$ (Eq. 28). Here we sample the rows of each layer’s weight matrix from an isotropic Gaussian distribution, and then normalize each row to have unit norm. We run experiments with the proposed initialization (see corollary 2; $k_i = \sqrt{\frac{2n_{l-1}}{n_l}}$), as well as traditional weight normalization (Salimans & Kingma (2016), $k_i = 1$) for network widths 100, 500 and 1000. As can be seen in figures 3 and 4, our proposed initialization preserves activation norm and prevents gradient explosion/vanishing problem as we showed theoretically, while the traditional weight normalization does not. Further, these approximations get better with larger width.

Finally, we show results on the MNIST dataset (LeCun & Cortes) with fully connected deep ReLU networks with width 500. The network is trained using SGD with batch-size 100, momentum 0.9, weight decay 0.0005. Learning
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Figure 1. He initialization preserves activation norm at initialization for sufficiently wide un-normalized deep ReLU networks.

Figure 2. He init preserves parameter gradient norm at initialization for sufficiently wide un-normalized deep ReLU networks.

rate is drop by a factor of 0.5 every 10 epochs. We tried base learning rates from the set \{0.001, 0.005, 0.01, 0.05, 0.1\} for both traditional weight normalization and our proposed initialization and finally use the learning rate which yields the best validation accuracy. The convergence plots are shown in figure 5. For depth 2 and 5 both methods converge, but for depth 10, the traditional weight normalization is unable to escape the bad initialization while our proposal trains normally.

6. Summary of Findings

We rigorously derived the conditions needed for the preservation of activation norm and prevention of gradient explosion/vanishing problems in both un-normalized and weight normalized deep ReLU networks without making any assumption and verified our predictions empirically. In general we showed that over-parameterization in terms of width is crucial for avoiding these problems, especially for deeper networks. Another useful recommendation is to keep network width as uniform as possible, especially when the width is small; for large widths the uniformity is less important. Other practical recommendations that help avoiding this problem are as follows,

1. Un-normalized deep ReLU networks: Initialize each element of each layer’s weight matrix from \(\mathcal{N}(0, 2/\text{fan-out})\), and biases to 0s (also called He initialization He et al. (2015)).

2. Weight normalized deep ReLU networks: The network layers should be designed as,

\[
h^l_i := \text{ReLU} \left( \sqrt{\frac{2\text{fan-in}}{\text{fan-out}}} \|W^l_i\|_2^{-1} h^{l-1} + b^l_i \right)
\]

where \(l\) denotes the layer index, \(i\) denotes the \(i^{th}\) unit. For each weight matrix, each element should be sampled from \(\mathcal{N}(0, 1/\text{fan-in})\) as it ensures the row norms are 1 in expectation. The biases should be initialized to 0s. We refer to the proposed scaling and way of sampling weights collectively as our initialization strategy. If using the \(\gamma\) and \(\beta\) parameters of weight normalization (Salimans & Kingma, 2016), each element of \(\gamma\) can be initialized to \(\sqrt{\frac{2\text{fan-in}}{\text{fan-out}}}\) instead of using the above parameterization.

3. Residual networks: See appendix A.

Figure 3. Proposed init preserves activation norm at initialization for sufficiently wide weight normalized deep ReLU networks.

Figure 4. Proposed init preserves gradient norm at initialization for sufficiently wide weight normalized deep ReLU networks.

Figure 5. Proposed initialization for weight normalized deep ReLU networks prevents gradient explosion/vanishing problem and hence facilitates training on MNIST for deeper networks.
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Appendix

A. Residual Networks

Residual networks (He et al., 2016) are a popular choice of architecture in various deep learning applications. Here we derive an initialization strategy for residual networks that preserves activation norm when the network is sufficiently wide.

We define a residual network with $B$ residual blocks as $f_\theta(x) = h^B$, where $h^b := h^b + \gamma F_b(h^b)$ for $b \in \{1, 2, \cdots, B - 1\}$ (33)

where $h^0 = x$ (input), $h^b$ denotes the hidden representation after applying $b$ residual blocks, each $F_b(\cdot)$ is a residual block which can be a fully connected feed-forward deep ReLU network as defined in Eq. 1 or Eq. 32, and $\gamma$ is a scalar that scales the output of the residual blocks. In practice when using residual networks, $\gamma$ is usually set to 1. The theorem below states an initialization strategy for residual networks (with and without weight normalization) for which network activations at all layers are preserved in expectation, and hence would also be preserved for a particular instance of an initialized network if it is sufficiently wide.

**Theorem 6** Let $f_\theta(\cdot)$ be a residual network as defined in Eq. 33. If the network weights are un-normalized, let each residual block be of the form shown in Eq. 1 and the element of weight matrices sampled from $\mathcal{N}(0, 2/fan-out)$. If the network weights are weight normalized, let each residual block be of the form shown in Eq. 32 such that for any weight matrix, each row of the weight matrix can be sampled from any isotropic distribution, but its norm should be re-scaled to 1. Finally, set $\gamma = 1/B$ and assume $K_n = 1$ for $n > 1$. Then in the limit of $B \to \infty$ and an infinitely wide network,

$$\frac{\|x\|^2}{e^2} \leq \|f_\theta(x)\|^2 \leq e^2 \cdot \|x\|^2$$ (34)

Note in the theorem we have assumed $K_n = 1$ for $n > 1$ in the case of normalized network which (as discussed in the main text) we empirically found to be true but did not prove analytically in theorem 5. Also, the exponential symbol in the inequality of the theorem arises due to the fact that

$$\lim_{B \to \infty} \left(1 - \frac{1}{B}\right)^B = \frac{1}{e}$$ (35)

$$\lim_{B \to \infty} \left(1 + \frac{1}{B}\right)^B = e$$ (36)

In summary, the initialization for ReLU residual networks can be done in the same way as that for fully connected deep ReLU networks depending on whether or not the weights are normalized or not. The only additional requirement for residual networks is to scale the output of each residual block by $1/B$, where $B$ denotes the total number of residual blocks in the network. Note our result will also hold in the case when a linear layer exists after the input and/or "shortcut connections" are used in residual network. These changes only require minor changes in the proof which we do not consider for the sake of simplicity. Finally, since we show that the norm of the output is approximately equal to the norm of the input for a sufficiently wide network, it automatically follows that the norm of the hidden activations will also approximately equal the norm of input as long as these activations are at a sufficient depth.

B. Discussion

We found an important distinguishing factor while investigating the initialization strategies for un-normalized deep ReLU networks and weight normalized deep ReLU networks. For un-normalized networks, we show that the scale of weights govern whether the network faces the gradient exploding/vanishing problem. Specifically, each element of each layer’s weight matrix should be sampled from $\mathcal{N}(0, 2/fan-out)$. However, in contrast, for weight normalized networks, the scale of weights do not govern this aspect. In such networks, they only scale the gradient (Eq. (28)). The presence of gradient explosion/vanishing problem is instead decided by the scaling factor used to multiply the normalized weights and we derive the scaling factor that prevents this problem (Eq. (32)).

Another novel aspect of our paper is that we establish that deep ReLU networks (un-normalized, weight normalized and with and without residual connections) are norm preserving mappings at initialization if our proposed initialization strategy
is used. This means that the norm of all hidden layers and output approximately (in PAC sense) equals the norm of the input. We also show that a similar result holds for the parameter gradients, viz, the norm of gradient w.r.t. each weight matrix is approximately equal to the product of the norm of the input and the norm of the derivative of loss w.r.t. the network output. For residual networks (both with and without normalization), we additionally need to multiply the output of each residual block with $1/B$ where $B$ is the total number of residual blocks in the network for the property to hold (theorem 6). Of course all the above results hold when the network is sufficiently over-parameterized.

For all scenarios considered in the paper (with and without weight normalization, and with and without residual connections), we call the proposed strategy for weight initialization and scaling factor (in the required case) collectively as our initialization strategy even though the scaling factor is a part of the network parameterization as it leads to an initialization condition that enjoys the aforementioned benefits.

C. Additional Experiments

C.1. Tightness of Bound

In the following experiment we verify the tightness of the bound in lemma 2. To do so, we vary the network width of a one hidden layer ReLU transformation from 500 to 4000, and feed 2000 randomly sampled inputs $x$ through it. For each sample we measure the distortion $\epsilon$ defined as,

$$\epsilon := |1 - \frac{\|h\|}{\|x\|}| \quad (37)$$

where $h$ is the output of the one hidden layer ReLU transformation. We compute the mean value of $\epsilon$ for the 2000 examples and plot them against the network width used. We call this the empirical estimate. We simultaneously plot the values of $\epsilon$ predicted by lemma 2 for failure probability $\delta = 0.05$. We call this the theoretical value. The plots are shown in figure 6 (left). As can be seen, our lower bound on width is an over-estimation but becomes tighter for smaller values of $\epsilon$. A similar result can be seen for lemma 4 in figure 6 (right). Thus our proposed bounds can be improved and we leave that as future work.

C.2. Residual Networks Norm Preservation

In theorem 6 we showed that in expectation, the norm of the output of ResNets is roughly equal to the norm of the input when initialized using our strategy and the network width and number of residual blocks tend to infinity. We now show experiments on ResNets with varying width of residual blocks and varying number of residual blocks to empirically evaluate what width and number of resblocks are sufficient for norm preservation in practice.

**Without weight normalization**: Here we use a residual network without any normalization. We initialize weights using He initialization. The only difference between the traditional way of initializing ResNets and our strategy is that we scale each residual block by $1/B$ where $B$ is the total number of residual blocks in the network.

In the first experiment, we vary the number of residual blocks while keeping the width of each block fixed to 100 neurons. We then feed-forward 2000 input samples randomly generated from a standard normal distribution in $\mathbb{R}^{3000}$. For each value
of number of resblocks used in the network, we plot the mean and standard deviation (across samples) of the ratio of norm of the network output to the norm of the input. The plot is shown in figure 7 (left). For the proposed initialization, the average ratio is $\sim 1$ for number of residual blocks larger than 20, while the output norm is not preserved for network without the proposed scaling factor.

In the second experiment, we vary the width of each residual block while keeping the number of residual blocks fixed to 10 neurons. We then feed-forward 2000 input samples randomly generated from a standard normal distribution in $\mathbb{R}^{300}$. For each value of width of resblocks used in the network, we plot the mean and standard deviation of the ratio of norm of the network output to the norm of the input. The plot is shown in figure 7 (right). For the proposed initialization, the ratio is $\sim 1$ for all values of width used and the standard deviation reduces as the network width is increased. On the other hand the activation norm explodes as the width of resblocks is increased for network without the proposed scaling factor.

**With weight normalization:** Here we use weight normalized residual networks and He initialization, but other isometric initializations with any scale yield the same result (as suggested by theory).

In the first experiment, we vary the number of residual blocks while keeping the width of each block fixed to 100 neurons. We then feed-forward 2000 input samples randomly generated from a standard normal distribution in $\mathbb{R}^{300}$. For each value of number of resblocks used in the network, we plot the mean and standard deviation (across samples) of the ratio of norm of the network output to the norm of the input. The plot is shown in figure 8 (left). For the proposed initialization, the ratio is $\sim 1$ for number of residual blocks larger than 20, while for traditional weight normalization, the activation norm is not preserved but fixed.

In the second experiment, we vary the width of each residual block while keeping the number of residual blocks fixed to 10 neurons. We then feed-forward 2000 input samples randomly generated from a standard normal distribution in $\mathbb{R}^{300}$. For each value of width of resblocks used in the network, we plot the mean and standard deviation of the ratio of norm of the network output to the norm of the input. The plot is shown in figure 8 (right). For the proposed initialization, the ratio is $\sim 1$ for all values of width used and the standard deviation reduces as the network width is increased. On the other hand the activation norm explodes as the width of resblocks is increased for network without the proposed scaling factor.
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Figure 9. Proposed initialization for weight normalized deep ReLU Residual networks prevents gradient explosion/vanishing problem and facilitates training on CIFAR-10 especially for deeper ResNets. For ReNet-500, learning rate larger than 0.001 causes divergence for traditional WN while the default learning rate of 0.1 works for the proposed initialization scheme. See text for more details.

network output to the norm of the input. The plot is shown in figure 8 (right). For the proposed initialization, the ratio is $\sim 1$ for all values of width used, while the activation norm explodes as the width resblocks is increased for traditional weight normalization.

C.3. Residual Networks on CIFAR-10

We train residual networks on the CIFAR-10 dataset (Krizhevsky, 2009) with 56 and 500 residual blocks, and with traditional weight normalization and our proposed initialization (see appendix A). For both set of experiments we train for 182 epochs with SGD with batch size 100 and momentum 0.9. We drop the learning rate by a factor of 10 at epochs 91 and 136. For ResNet-56, we use a base learning rate of 0.1 and weight decay of 0.0002 for both cases. Validation accuracy curves are shown in figure 9 (left). Both traditional and proposed initialization schemes work similarly. For ResNet-500, we use a base weight decay of 0.0001 for both cases. The validation curve is shown in figure 9 (right). While for the proposed initialization, a learning rate of 0.1 trains the network normally, the traditional weight normalization training diverges for learning rate 0.1 and 0.01, in which case we had to use 0.001.

D. Proofs

Lemma 1 Let $v = ReLU(Ru)$, where $u \in \mathbb{R}^n$ and $R \in \mathbb{R}^{m \times n}$. If $R_{ij} \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{2}{m})$, then for any fixed vector $u$, $E[||v||^2] = ||u||^2$.

Proof: Define $a_i = R^T_i u$, where $R_i$ denotes the $i^{th}$ row of $R$. Since each element $R_{ij}$ is an independent sample from Gaussian distribution, each $a_i$ is essentially a weighted sum of these independent random variables. Thus, each $a_i \sim \mathcal{N}(0, \frac{2}{m} ||u||^2)$ and independent from one another. Thus each element $v_i = ReLU(a_i) \sim \mathcal{N}^R(0, \frac{2}{m} ||u||^2)$ where $\mathcal{N}^R$ denotes the rectified Normal distribution. Our goal is to compute,

$$E[||v||^2] = \sum_{i=1}^{m} v_i^2$$

(38)

$$= m E[v_i^2]$$

(39)

From the definition of $v_i$,

$$E[v_i] = \frac{1}{2} \cdot 0 + \frac{1}{2} E[Z]$$

(40)

where $Z$ follows a half-Normal distribution corresponding to the Normal distribution $\mathcal{N}(0, \frac{2}{m} ||u||^2)$. Thus

$$E[Z] = \sqrt{\frac{2m||u||^2}{m \pi}} \cdot \sqrt{\frac{m}{2}} = 2 \sqrt{\frac{||u||^2}{m \pi}}.$$  Similarly,

$$E[v_i^2] = 0.5E[Z^2]$$

(41)

$$= 0.5(var(Z) + E[Z]^2)$$

(42)
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Since \( \text{var}(Z) = \frac{2}{m} \|u\|^2 (1 - \frac{2}{\pi}) \), we get,

\[
\mathbb{E}[v_i^2] = 0.5 \left( \frac{2}{m} \|u\|^2 (1 - \frac{2}{\pi}) + (2 \sqrt{\frac{\|u\|^2}{m\pi}})^2 \right)
\]

(43)

Thus,

\[
\frac{m \mathbb{E}[v_i^2]}{m} = \|u\|^2
\]

(44)

which proves the claim. □

**Lemma 2** Let \( \mathbf{v} = \text{ReLU} (\mathbf{R}u) \), where \( u \in \mathbb{R}^n \), \( R \in \mathbb{R}^{m \times n} \). If \( R_{ij} \sim \text{i.i.d.} N(0, \frac{2}{m}) \), and \( \epsilon \in [0,1) \), then for any fixed vector \( u \),

\[
\text{Pr} \left( \|v\|^2 - \|u\|^2 \leq \epsilon \|u\|^2 \right) \geq 1 - 2 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)
\]

(46)

**Proof:** Define \( \tilde{\mathbf{v}} = \frac{\sqrt{0.5m}}{\|u\|} \mathbf{v} \). Then we have that each element \( \tilde{v}_i \sim N^R(0,1) \) and independent from one another since \( v_i = \text{ReLU}(a_i) \sim N^R(0, \frac{2}{m} \|u\|^2) \) where \( N^R \) denotes the rectified Normal distribution. Thus to bound the probability of failure for the R.H.S.,

\[
\text{Pr} \left( \|v\|^2 \geq (1 + \epsilon) \|u\|^2 \right) = \text{Pr} \left( \|v\|^2 \geq (1 + \epsilon) \|u\|^2 \right)
\]

(47)

Using Chernoff’s bound, we get for any \( \lambda > 0 \),

\[
\text{Pr} \left( \|\tilde{v}\|^2 \geq 0.5m(1 + \epsilon) \right) = \text{Pr} \left( \exp(\lambda \|v\|^2) \geq \exp(\lambda 0.5m(1 + \epsilon)) \right)
\]

(49)

\[
\leq \frac{\mathbb{E}[\exp(\lambda \|v\|^2)]}{\exp(0.5m \lambda (1 + \epsilon))}
\]

(50)

\[
= \frac{\mathbb{E}[\exp(\sum_{i=1}^{m} \lambda \tilde{v}_i^2)]}{\exp(0.5m \lambda (1 + \epsilon))}
\]

(51)

\[
= \prod_{i=1}^{m} \frac{\mathbb{E}[\exp(\lambda \tilde{v}_i^2)]}{\exp(0.5m \lambda (1 + \epsilon))}
\]

(52)

\[
= \left( \frac{\mathbb{E}[\exp(\lambda \tilde{v}_i^2)]}{\exp(0.5 \lambda (1 + \epsilon))} \right)^m
\]

(53)

Denote \( p(\tilde{v}_i) \) as the probability distribution of the rectified Normal random variable \( \tilde{v}_i \). Then,

\[
\mathbb{E}[\exp(\lambda \tilde{v}_i^2)] = \int_{-\infty}^{\infty} \exp(\lambda \tilde{v}_i^2) p(\tilde{v}_i)
\]

(54)

We know that the mass at \( v_i = 0 \) is 0.5 and the density between \( v_i = 0 \) and \( v_i = \infty \) follows the Normal distribution. Thus,

\[
\mathbb{E}[\exp(\lambda \tilde{v}_i^2)] = 0.5 \exp(0) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\lambda \tilde{v}_i^2 - \tilde{v}_i^2/2)
\]

(55)

\[
= 0.5 + \frac{1}{2\sqrt{\pi/(1-2\lambda)}} \sqrt{2} \int_{0}^{\infty} \exp(-\frac{\tilde{v}_i^2}{2}(1 - 2\lambda))
\]

(56)

Note that \( \int_{0}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi/(1-2\lambda)}} \int_{0}^{\infty} \exp(-\frac{\tilde{v}_i^2}{2}(1 - 2\lambda)) \) is the integral of a half Normal distribution corresponding to the Normal distribution \( N(0,1/(1-2\lambda)) \). Thus,

\[
\mathbb{E}[\exp(\lambda \tilde{v}_i^2)] = 0.5 + \frac{1}{2\sqrt{(1-2\lambda)}}
\]

(57)
Thus, we get,

\[
\Pr \left( \| \tilde{v} \|^2 \geq 0.5m(1 + \epsilon) \right) \leq \left( 0.5 \left( 1 + \frac{1}{\sqrt{(1 - 2\lambda)}} \right) \exp(-0.5\lambda(1 + \epsilon)) \right)^m
\]  

(58)

The above failure probability can be bounded to be smaller by finding an appropriate value of \( \lambda \). We find that \( \lambda \approx \frac{0.5\epsilon}{1 + \epsilon} \) approximately minimizes the above bound. Substituting this value of \( \lambda \) above, we get,

\[
\Pr \left( \| \tilde{v} \|^2 \geq 0.5m(1 + \epsilon) \right) \leq \left( 0.5 \left( 1 + \sqrt{1 + \epsilon} \right) \exp\left( -\frac{\epsilon}{4} \right) \right)^m
\]

\[
= \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)
\]

(60)

Thus,

\[
\Pr \left( \| v \|^2 \leq (1 + \epsilon)\| u \|^2 \right) \geq 1 - \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)
\]

(61)

Similarly, to prove the L.H.S. by bounding the probability of failure from the other side,

\[
\Pr \left( \| v \|^2 \leq (1 - \epsilon)\| u \|^2 \right) = \Pr \left( -\| v \|^2 \geq -(1 - \epsilon)\| u \|^2 \right)
\]

\[
= \Pr \left( -\frac{\| u \|^2}{0.5m} \| \tilde{v} \|^2 \geq -(1 - \epsilon)\| u \|^2 \right)
\]

\[
= \Pr \left( -\| \tilde{v} \|^2 \geq -0.5m(1 - \epsilon) \right)
\]

(64)

Using Chernoff’s bound, we get for any \( \lambda > 0 \),

\[
\Pr \left( -\| \tilde{v} \|^2 \geq -0.5m(1 - \epsilon) \right) = \Pr \left( \exp(-\lambda\| \tilde{v} \|^2) \geq \exp(-\lambda 0.5m(1 - \epsilon)) \right)
\]

\[
\leq \frac{\mathbb{E}[\exp(-\lambda\| \tilde{v} \|^2)]}{\exp(-0.5m\lambda(1 - \epsilon))}
\]

\[
= \frac{\mathbb{E}[\exp(-\sum_{i=1}^m \lambda \tilde{v}_i^2)]}{\exp(-0.5m\lambda(1 - \epsilon))}
\]

\[
= \Pi_{i=1}^m \mathbb{E}[\exp(-\lambda \tilde{v}_i^2)]
\]

\[
\frac{\exp(-0.5m\lambda(1 - \epsilon))}{\exp(-0.5\lambda\sum_{i=1}^m \tilde{v}_i^2)}
\]

\[
= \left( \frac{\mathbb{E}[\exp(-\lambda \tilde{v}_i^2)]}{\exp(-0.5\lambda\sum_{i=1}^m \tilde{v}_i^2)} \right)^m
\]

(69)

Performing computations similar to those above to compute the expectation term, we get,

\[
\mathbb{E}[\exp(-\lambda \tilde{v}_i^2)] = 0.5 + \frac{1}{2\sqrt{(1 + 2\lambda)}}
\]

(70)

Hence, we get,

\[
\Pr \left( \| \tilde{v} \|^2 \leq 0.5m(1 - \epsilon) \right) \leq \left( 0.5 \left( 1 + \frac{1}{\sqrt{(1 + 2\lambda)}} \right) \exp(0.5\lambda(1 - \epsilon)) \right)^m
\]

(71)

Similar to the R.H.S. case, we find that \( \lambda \approx \frac{0.5\epsilon}{1 - \epsilon} \) approximately minimizes the failure probability,

\[
\Pr \left( \| \tilde{v} \|^2 \leq 0.5m(1 - \epsilon) \right) \leq \left( 0.5 \left( 1 + \sqrt{1 - \epsilon} \right) \exp\left( \frac{\epsilon}{4} \right) \right)^m
\]

\[
= \exp \left( m \left( \frac{\epsilon}{4} - \log \frac{2}{1 + \sqrt{1 - \epsilon}} \right) \right)
\]

(73)
Finally, we note that the following hold true with the above probability,

\[ \Pr (\|v\|^2 \geq (1 - \epsilon)\|u\|^2) \geq 1 - \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

Thus,

\[ \Pr (\|v\|^2 \geq (1 - \epsilon)\|u\|^2) \geq 1 - \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

Using union bound, Eq. (61) and (75) hold together with probability,

\[ \Pr ((1 - \epsilon)\|u\|^2 \leq \|v\|^2 \leq (1 + \epsilon)\|u\|^2) \geq 1 - 2 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

This proves the claim. □

**Theorem 1** Let \( \mathcal{D} \) be a fixed dataset with \( N \) samples and define a \( L \) layer ReLU network as shown in Eq. 1 such that each weight matrix \( W^l \in \mathbb{R}^{n_l \times n_{l-1}} \) has its elements sampled as \( W_{ij} \sim N(0, \frac{2}{n_i}) \) and biases \( b^l \) are set to zeros. Then for any sample \( (x, y) \in \mathcal{D} \) and \( \epsilon \in [0, 1) \), we have that,

\[ \Pr ((1 - \epsilon)x^2 \leq \|f_\theta(x)\|^2 \leq (1 + \epsilon)x^2) \geq 1 - 2 \sum_{l=1}^{L} 2N \exp \left( -n^l \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

**Proof:** When feed-forwarding a fixed input through the layers of a deep ReLU network, each hidden layer’s activation corresponding to the given input is also fixed because the network is deterministic. Thus applying lemma 2, on each layer’s transformation, the following holds true for each \( l \in \{1, 2, \cdots, L\} \),

\[ \Pr ((1 - \epsilon)\|h^{l-1}\|^2 \leq \|h^l\|^2 \leq (1 + \epsilon)\|h^{l-1}\|^2) \geq 1 - 2 \exp \left( -n^l \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

Thus, using union bound, we have the lengths of all the layers until layer \( l \) are simultaneously preserved with probability at least,

\[ 1 - \sum_{l=1}^{L} 2 \exp \left( -n^l \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

Applying union bound again, all the lengths until layer \( l \) are preserved simultaneously for \( N \) inputs with probability,

\[ 1 - \sum_{l=1}^{L} 2N \exp \left( -n^l \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]  

Finally, we note that the following hold true with the above probability,

\[ (1 - \epsilon)x^2 \leq \|h^l\|^2 \leq (1 + \epsilon)x^2 \]  

\[ (1 - \epsilon)\|h^l\|^2 \leq \|h^l\|^2 \leq (1 + \epsilon)\|h^l\|^2 \]  

Substituting \( \|h^l\|^2 \leq (1 + \epsilon)\|x\|^2 \) in the R.H.S. of the last equation, and \( (1 - \epsilon)x^2 \leq \|h^l\|^2 \) in the L.H.S. of the last equation, we get,

\[ (1 - \epsilon)^2x^2 \leq \|h^l\|^2 \leq (1 + \epsilon)^2x^2 \]  

Performing substitutions for higher layers similarly yields the claim. □
Proposition 1 If network weights are sampled i.i.d. from a Gaussian distribution with mean 0 and biases are 0 at initialization, then conditioned on $h^{l−1}$, each dimension of $\mathbf{1}(a^l)$ follows an i.i.d. Bernoulli distribution with probability 0.5 at initialization.

Proof: Note that $a^l := W^l h^{l−1}$ at initialization (biases are 0) and $W^l$ are sampled i.i.d. from a random distribution with mean 0. Therefore, each dimension $a^l_i$ is simply a weighted sum of i.i.d. zero mean Gaussian, which is also a 0 mean Gaussian random variable.

To prove the claim, note that the indicator operator applied on a random variable with 0 mean and symmetric distribution will have equal probability mass on both sides of 0, which is the same as a Bernoulli distributed random variable with probability 0.5. Finally, each dimension of $a^l$ is i.i.d. simply because all the elements of $W^l$ are sampled i.i.d., and hence each dimension of $a^l$ is a weighted sum of a different set of i.i.d. random variables. □

Lemma 3 Let $\mathbf{v} = (R\mathbf{u}) \odot \mathbf{z}$, where $\mathbf{u} \in \mathbb{R}^n$, $R \in \mathbb{R}^{m \times n}$ and $\mathbf{z} \in \mathbb{R}^m$. If $R_{ij} \sim i.i.d. \mathcal{N}(0, \frac{1}{pm})$ and $z_i \sim i.i.d. \text{Bernoulli}(p)$, then for any fixed vector $\mathbf{u}$, $\mathbb{E}[\|\mathbf{v}\|^2] = \|\mathbf{u}\|^2$.

Proof: Define $a_i = R_{ij}^T \mathbf{u}$, where $R_{ij}$ denotes the $i^{th}$ row of $R$. Since each element $R_{ij}$ is an independent sample from Gaussian distribution, each $a_i$ is essentially a weighted sum of these independent random variables. Thus, each $a_i \sim \mathcal{N}(0, \frac{1}{pm}\|\mathbf{u}\|^2)$ and independent from one another.

Our goal is to compute,

$$
\mathbb{E}[\|\mathbf{v}\|^2] = \sum_{i=1}^m \mathbb{E}[(a_i z_i)^2]
$$

(84)

$$
= \sum_{i=1}^m \mathbb{E}[a_i^2] \mathbb{E}[z_i^2]
$$

(85)

$$
= m \mathbb{E}[a_i^2] \mathbb{E}[z_i^2]
$$

(86)

$$
= mp(\text{var}(a_i) + \mathbb{E}[a_i]^2)
$$

(87)

$$
= \|\mathbf{u}\|^2
$$

(88)

which proves the claim. □

Lemma 4 Let $\mathbf{v} = (R\mathbf{u}) \odot \mathbf{z}$, where $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^m$, and $R \in \mathbb{R}^{m \times n}$. If $R_{ij} \sim i.i.d. \mathcal{N}(0, \frac{1}{0.5m})$, $z_i \sim i.i.d. \text{Bernoulli}(0.5)$ and $\epsilon \in [0, 1)$, then for any fixed vector $\mathbf{u}$,

$$
\Pr \left( \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \leq \epsilon \|\mathbf{u}\|^2 \right) \geq 1 - 2 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)
$$

(89)

Proof: Define $a_i = R_{ij}^T \mathbf{u}$, where $R_{ij}$ denotes the $i^{th}$ row of $R$. Then, each $a_i \sim \mathcal{N}(0, \frac{1}{0.5m}\|\mathbf{u}\|^2)$ and independent from one another. Define $\hat{a} = \frac{\sqrt{0.5m}}{\|\mathbf{u}\|} a$. Then we have that each element $\hat{a}_i \sim \mathcal{N}(0, 1)$.

Define $\hat{v}_i$ such that $\hat{v}_i = \hat{a}_i z_i$. Thus to bound the probability of failure for the R.H.S.,

$$
\Pr \left( \|\mathbf{v}\|^2 \geq (1 + \epsilon)\|\mathbf{u}\|^2 \right) = \Pr \left( \|\mathbf{u}\|^2\|\hat{v}\|^2 \geq (1 + \epsilon)\|\mathbf{u}\|^2 \right)
$$

(90)

$$
= \Pr \left( \|\hat{v}\|^2 \geq 0.5m(1 + \epsilon) \right)
$$

(91)
The Benefits of Over-parameterization at Initialization in Deep ReLU Networks

Using Chernoff’s bound, we get for any $\lambda > 0$,

$$\Pr \left( \|\tilde{v}\|^2 \geq 0.5m(1 + \epsilon) \right) = \Pr \left( \exp(\lambda \|\tilde{v}\|^2) \geq \exp(\lambda 0.5m(1 + \epsilon)) \right)$$

$$\leq \frac{\mathbb{E}[\exp(\lambda \|\tilde{v}\|^2)]}{\exp(0.5m\lambda(1 + \epsilon))}$$

$$= \frac{\mathbb{E}[\exp(\sum_{i=1}^{m} \lambda \tilde{e}_i^2)]}{\exp(0.5m\lambda(1 + \epsilon))}$$

$$= \prod_{i=1}^{m} \frac{\mathbb{E}[\exp(\lambda \tilde{e}_i^2)]}{\exp(0.5m\lambda(1 + \epsilon))}$$

$$= \left( \frac{\mathbb{E}[\exp(\lambda \tilde{e}_i^2)]}{\exp(0.5m\lambda(1 + \epsilon))} \right)^m$$

Denote $p(\tilde{a}_i)$ and $p(z_i)$ as the probability distribution of the random variables $\tilde{a}_i$ and $z_i$ respectively. Then,

$$\mathbb{E}[\exp(\lambda \tilde{e}_i^2)] = \sum_{z_i} p(z_i) \int_{\tilde{a}_i} p(\tilde{a}_i) \exp(\lambda \tilde{a}_i^2 z_i^2)$$

Substituting $p(\tilde{a}_i)$ with a standard Normal distribution, we get,

$$\mathbb{E}[\exp(\lambda \tilde{e}_i^2)] = \sum_{z_i} p(z_i) \int_{\tilde{a}_i} \frac{1}{\sqrt{2\pi}} \exp(-\tilde{a}_i^2 z_i^2 - \frac{\tilde{a}_i^2}{2})$$

$$= \sum_{z_i} p(z_i) \int_{\tilde{a}_i} \frac{1}{\sqrt{2\pi}} \exp(-\tilde{a}_i^2 (1 - 2\lambda z_i^2))$$

$$= \sum_{z_i} p(z_i) \int_{\tilde{a}_i} \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{1 - 2\lambda z_i^2}{1 - 2\lambda z_i^2}} \exp(-\tilde{a}_i^2 (1 - 2\lambda z_i^2))$$

$$= \sum_{z_i} p(z_i) \cdot \frac{1}{\sqrt{1 - 2\lambda z_i^2}}$$

where the last equality holds because the integral of a Gaussian distribution over its domain is 1. Finally, summing over the Bernoulli random variable $z_i$, we get,

$$\mathbb{E}[\exp(\lambda \tilde{e}_i^2)] = (1 - 0.5) + \frac{1}{\sqrt{1 - 2\lambda}}$$

Hence, we get,

$$\Pr \left( \|\tilde{v}\|^2 \geq 0.5m(1 + \epsilon) \right) \leq \left( 0.5 \left( 1 + \frac{0.5}{\sqrt{(1 - 2\lambda)}} \right) \exp(-0.5\lambda(1 + \epsilon)) \right)^m$$

$$\leq \left( 0.5 \left( 1 + \frac{1}{\sqrt{(1 - 2\lambda)}} \right) \exp(-0.5\lambda(1 + \epsilon)) \right)^m$$

We find that the above inequality is identical to that in Eq. (58). Thus $\lambda \approx \frac{0.5\epsilon}{1 + \epsilon}$ approximately minimizes the above bound as before. Substituting this value of $\lambda$ above, we get,

$$\Pr \left( \|\tilde{v}\|^2 \geq 0.5m(1 + \epsilon) \right) \leq \left( 0.5 \left( 1 + \sqrt{1 + \epsilon} \right) \exp(-\frac{\epsilon}{4}) \right)^m$$

$$= \exp \left(-m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)$$

Thus,

$$\Pr \left( \|v\|^2 \leq (1 + \epsilon)\|u\|^2 \right) \geq 1 - \exp \left(-m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)$$
Similarly, to prove the L.H.S. by bounding the probability of failure from the other side,

\[
\text{Pr} \left( \|v\|^2 \leq (1 - \epsilon) \|u\|^2 \right) = \text{Pr} \left( -\|v\|^2 \geq -(1 - \epsilon) \|u\|^2 \right) 
\]

(108)

\[
= \text{Pr} \left( \frac{\|v\|^2}{0.5m} \geq -(1 - \epsilon) \|u\|^2 \right) 
\]

(109)

\[
= \text{Pr} \left( -\|v\|^2 \geq -0.5m(1 - \epsilon) \right) 
\]

(110)

Using Chernoff’s bound, we get for any \( \lambda > 0 \),

\[
\text{Pr} \left( -\|\tilde{v}\|^2 \geq -0.5m(1 - \epsilon) \right) \leq \text{Pr} \left( \exp(-\lambda \|\tilde{v}\|^2) \geq \exp(-0.5m\lambda(1 - \epsilon)) \right) 
\]

(111)

\[
\leq \frac{\text{E}[\exp(-\lambda \|\tilde{v}\|^2)]}{\exp(-0.5m\lambda(1 - \epsilon))} 
\]

(112)

\[
= \frac{\text{E}[\exp(-\sum_{i=1}^{m} \lambda \tilde{v}_i^2)]}{\exp(-0.5m\lambda(1 - \epsilon))} 
\]

(113)

\[
= \left( \frac{\text{E}[\exp(-\lambda \tilde{v}_i^2)]}{\exp(-0.5\lambda(1 - \epsilon))} \right)^m 
\]

(115)

Performing computations similar to those above to compute the expectation term, we get,

\[
\text{E}[\exp(-\lambda \tilde{v}_i^2)] = 0.5 + \frac{1}{\sqrt{1 + 2\lambda}} 
\]

(116)

Hence, we get,

\[
\text{Pr} \left( -\|\tilde{v}\|^2 \geq 0.5(1 - \epsilon) \right) \leq \left( 0.5 \left( \frac{0.5}{\sqrt{1 + 2\lambda}} \right) \exp(0.5\lambda(1 - \epsilon)) \right)^m 
\]

(117)

\[
\leq \left( 0.5 \left( \frac{1}{\sqrt{1 + 2\lambda}} \right) \exp(0.5\lambda(1 - \epsilon)) \right)^m 
\]

(118)

Similar to the R.H.S. case, we find that \( \lambda \approx \frac{0.5\epsilon}{1 + \sqrt{1 - \epsilon}} \) approximately minimizes the failure probability,

\[
\text{Pr} \left( -\|\tilde{v}\|^2 \geq 0.5(1 - \epsilon) \right) \leq \left( 0.5 \left( 1 + \sqrt{1 - \epsilon} \right) \exp(\frac{\epsilon}{4}) \right)^m 
\]

(119)

\[
= \exp \left( m \left( \frac{\epsilon}{4} - \log \frac{2}{1 + \sqrt{1 - \epsilon}} \right) \right) 
\]

(120)

It can be shown that,

\[
\exp \left( m \left( \frac{\epsilon}{4} - \log \frac{2}{1 + \sqrt{1 - \epsilon}} \right) \right) \leq \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) 
\]

(121)

Thus,

\[
\text{Pr} \left( -\|v\|^2 \geq (1 - \epsilon) \|u\|^2 \right) \geq 1 - \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) 
\]

(122)

Using union bound, Eq. (107) and (122) hold together with probability,

\[
\text{Pr} \left( (1 - \epsilon) \|u\|^2 \leq \|v\|^2 \leq (1 + \epsilon) \|u\|^2 \right) \geq 1 - 2 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) 
\]

(123)

This proves the claim. □
Applying union bound on $N$
From proposition 1, we know that each element of $O$.
On the other hand, we have that, $Pr$
Extending this to all $L$

\begin{align*}
\text{Theorem 2} & \quad \text{Let } D \text{ be a fixed dataset with } N \text{ samples and define a } L \text{ layer ReLU network as shown in Eq. 1 such that each weight matrix } W^l \in \mathbb{R}^{n \times n} \text{ has its elements sampled as } W_{ij} \sim \mathcal{N}(0, \frac{2}{n}) \text{ and biases } b^l \text{ are set to zeros. Then for any sample } (x, y) \in D, \epsilon \in [0, 1), \text{ and for all } l \in \{1, 2, \ldots, L\} \text{ with probability at least,} \\
1 - 4NL \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) & \quad (124)
\end{align*}

the following hold true,
\begin{align*}
(1 - \epsilon)^L \|x\|^2 \cdot \|\delta(x, y)\|^2 & \leq \| \frac{\partial \ell(f_\theta(x), y)}{\partial W^l} \|^2 \leq (1 + \epsilon)^L \|x\|^2 \cdot \|\delta(x, y)\|^2 \\
& \quad (125)
\end{align*}

and
\begin{align*}
(1 - \epsilon)^l \|x\|^2 & \leq \|h^l\|^2 \leq (1 + \epsilon)^l \|x\|^2 \\
& \quad (126)
\end{align*}

\textbf{Proof:} From theorem 1, we know that the following holds for all $l$,
\begin{align*}
\Pr \left( (1 - \epsilon)^l \|x\|^2 \leq \|h^l\|^2 \leq (1 + \epsilon)^l \|x\|^2 \right) & \geq 1 - 2NL \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \quad (127)
\end{align*}

On the other hand, we have that,
\begin{align*}
\frac{\partial \ell(f_\theta(x), y)}{\partial a^{L-1}} = 1(a^{L-1}) \odot \left(W^{L} \delta(x, y)\right) \\
& \quad (128)
\end{align*}

From proposition 1, we know that each element of $1(a^{L-1})$ follows a Bernoulli distribution with probability 0.5. Thus applying lemma 4 to the above equation (under the assumption that $\delta(x, y)$ and $1(a^l)$ are statistically independent), the following holds for a fixed data sample $(x, y)$,
\begin{align*}
\Pr \left( (1 - \epsilon)\|\delta(x, y)\|^2 \leq \left\| \frac{\partial \ell(f_\theta(x), y)}{\partial a^{L-1}} \right\|^2 \leq (1 + \epsilon)\|\delta(x, y)\|^2 \right) & \geq 1 - 2 \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \quad (129)
\end{align*}

Applying union bound on $N$ fixed samples, the following holds for all $N$ samples,
\begin{align*}
\Pr \left( (1 - \epsilon)\|\delta(x, y)\|^2 \leq \| \frac{\partial \ell(f_\theta(x), y)}{\partial a^{L-1}} \|^2 \leq (1 + \epsilon)\|\delta(x, y)\|^2 \right) & \geq 1 - 2N \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \quad (130)
\end{align*}

Similarly,
\begin{align*}
\Pr \left( (1 - \epsilon)\left\| \frac{\partial \ell(f_\theta(x), y)}{\partial a^{L-2}} \right\|^2 \leq (1 + \epsilon)\left\| \frac{\partial \ell(f_\theta(x), y)}{\partial a^{L-1}} \right\|^2 \right) & \geq 1 - 2N \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \quad (131)
\end{align*}

Combining the the above two results and applying union bound, we get,
\begin{align*}
\Pr \left( (1 - \epsilon)^2\|\delta(x, y)\|^2 \leq \| \frac{\partial \ell(f_\theta(x), y)}{\partial a^{L-2}} \|^2 \leq (1 + \epsilon)^2\|\delta(x, y)\|^2 \right) & \geq 1 - 4N \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \quad (132)
\end{align*}

Extending this to all $L$ layers, we have for all $l$ that,
\begin{align*}
\Pr \left( (1 - \epsilon)^{L-l}\|\delta(x, y)\|^2 \leq \left\| \frac{\partial \ell(f_\theta(x), y)}{\partial a^{l}} \right\|^2 \leq (1 + \epsilon)^{L-l}\|\delta(x, y)\|^2 \right) & \geq 1 - 2NL \exp \left(-n \left(\frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \quad (133)
\end{align*}
Combining the above result with Eq. (127) using union bound, we get for all \( l \),
\[
\Pr \left( (1 - \epsilon)^{L-1} \|\delta(x, y)\|^2 \|x\|^2 \leq \frac{\|\partial f(y)/\partial \mathbf{a}\|}{\|a\|}\|h^{l-1}\|^2 \leq (1 + \epsilon)^{L-1} \|\delta(x, y)\|^2 \|x\|^2 \right) \\
\geq 1 - 4NL \exp \left( -n \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \tag{134}
\]
Since,
\[
\frac{\|\partial f(y)/\partial \mathbf{a}\|}{\|a\|}\|h^{l-1}\| = \frac{\|\partial f(y)/\partial \mathbf{a}\|}{\|a\|}\|h^{l-1}\|_2 \forall l
\]
we have proved the claim. \( \square \)

**Corollary 1** Suppose all the hidden layers of the \( L \) layer deep ReLU network \( f_0(.) \) have the same width \( n \), and let the following hold with probability at least \( 1 - \delta \)
\[
\|f_0(x)\|^2 - \|x\|^2 \leq \epsilon \|x\|^2
\]
and,
\[
\|\partial f_0(x)/\partial \mathbf{a}\|^2 - \|x\|^2 \cdot \|\delta(x, y)\|^2 \leq \epsilon \|x\|^2 \cdot \|\delta(x, y)\|^2
\]
hold for \( N \) fixed training samples \((x, y) \in \mathcal{D}\), then,
\[
n \geq \frac{1}{0.25\epsilon' - \log(0.5(1 + \sqrt{1 + \epsilon}))} \log \frac{4NL}{\delta} \tag{136}
\]
where \( \epsilon' = (1 + \epsilon)^{1/L} - 1 \).

**Proof:** Theorem 2 states that with probability at least,
\[
1 - 4NL \exp \left( -n \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \tag{137}
\]
the following hold true,
\[
(1 - \epsilon)^L \|x\|^2 \cdot \|\delta(x, y)\|^2 \leq \frac{\|\partial f_0(x)/\partial \mathbf{a}\|^2}{\|a\|}\|h^{l-1}\| \leq (1 + \epsilon)^L \|x\|^2 \cdot \|\delta(x, y)\|^2 \tag{138}
\]
and
\[
(1 - \epsilon)^L \|x\|^2 \leq \|f_0(x)\|^2 \leq (1 + \epsilon)^L \|x\|^2 \tag{139}
\]
We further lower bound the success probability by \( 1 - \delta \) as,
\[
1 - 4NL \exp \left( -n \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \geq 1 - \delta \tag{140}
\]
Rearranging terms to compute the lower bound on \( n \) gives us,
\[
n \geq \frac{1}{0.25\epsilon - \log(0.5(1 + \sqrt{1 + \epsilon}))} \log \frac{4NL}{\delta} \tag{141}
\]
Next, to achieve norm bound of the form stated in the corollary claim, we desire an \( \epsilon' \) such that,
\[
1 - \epsilon' \leq (1 - \epsilon)^L < (1 + \epsilon)^L \leq 1 + \epsilon' \tag{142}
\]
This holds true if \( \epsilon \leq \min\{1 - (1 - \epsilon')^{1/L}, (1 + \epsilon')^{1/L} - 1\} \). It can be shown that the minimum of the two is \( (1 + \epsilon')^{1/L} - 1 \). Replacing \( \epsilon \) in the above lower bound with \( (1 + \epsilon')^{1/L} - 1 \) proves the claim. \( \square \)
We now prove the R.H.S. of the bound in the claim. If we consider any unit vector \( \hat{r} \), also note that since any unit vector \( \hat{u} \) such that every unit vector \( u \) in the \( d \) dimensional subspace \( X \) of \( \mathbb{R}^n \). This is because for any arbitrary length vector \( u \), \( \| ReLU(Ru) \| = \| u \| : \| ReLU(Ru) \| \). The idea then is to define a grid of finite points over \( X \) such that every unit vector \( \hat{u} \) in \( X \) is close enough to one of the grid points. Then, if we choose the width of the layer to be large enough to approximately preserve the length of the finite number of grid points, we essentially guarantee that the length of any arbitrary vector approximately remains preserved.

To this end, we define a grid \( G \) on \([-1, 1]^d \) with interval of size \( \Delta := \min \{ \epsilon / \sqrt{d}, \sqrt{\epsilon} / d \} \). Note the number of points on this grid is \( \left( \frac{2}{\Delta} \right)^d \). Also, let column vectors of \( B \in \mathbb{R}^{n \times d} \) be the orthonormal basis of \( X \).

We now prove the R.H.S. of the bound in the claim. If we consider any unit vector \( \hat{u} \) in \( X \), we can find a point \( g \) on the grid \( G \) such that \( \| g \| \leq 1 \), and it is closest to \( \hat{u} \) in \( \ell^2 \) norm, and define \( r' := \hat{u} - g \). Thus the vector \( \hat{u} \) can essentially be decomposed as,

\[
\hat{u} = g + r'
\]

Also note that since \( r' \) lies in the span of \( X \), we can represent \( r' := Br \) for some vector \( r \).

Now consider the norm of the vector \( \hat{u} \) after the ReLU transformation given by \( \| ReLU(R\hat{u}) \| \). Then we have,

\[
\| ReLU(R\hat{u}) \| = \| ReLU(R(g + r')) \| \
\leq \| ReLU(Rg) + ReLU(Rr') \| \
\leq \| ReLU(Rg) \| + \| ReLU(Rr') \| 
\]

Similarly, we have,

\[
\| ReLU(Rg) \| = \| ReLU(R(g + \hat{u} - \hat{u})) \| \
\leq \| ReLU(R\hat{u}) + ReLU(R(g - \hat{u})) \| \
\leq \| ReLU(R\hat{u}) \| + \| ReLU(-Rr') \| 
\]

Therefore,

\[
\| ReLU(Rg) \| - \| Rr' \| \leq \| ReLU(R\hat{u}) \| \leq \| ReLU(Rg) \| + \| Rr' \| 
\]
Applying union bound on all the points in \( G \), from lemma 2, we know that with probability at least \( 1 - (\frac{2}{\Delta})^d \exp \left(-m \left( \frac{\xi}{4} + \log \frac{2}{1+\sqrt{1+\rho}} \right) \right) \),

\[
\| \text{ReLU}(Rg) \|^2 \leq (1 + \epsilon) \| g \|^2 \leq 1 + \epsilon \\
\leq (1 + \epsilon)^2
\]

(156) (157)

This can be substituted in the R.H.S. of Eq. (155). Now we only need to upper bound \( \| Rr' \| \). To this end, we rewrite \( \| Rr' \| = \| RBr \| \). Then,

\[
\| RBr \|^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} < RB_i r_i, RB_j r_j > 
\]

(158)

\[
\leq 2 \sum_{i=1}^{d} \sum_{j=1}^{d} |r_i| \cdot |r_j| < \frac{1}{\sqrt{2}} \| RB_i \| \cdot \frac{1}{\sqrt{2}} \| RB_j \|
\]

(159)

Note that \( \frac{1}{\sqrt{2}} R \) is a matrix whose entries are sampled from \( \mathcal{N}(0, 1) \). Invoking lemma 5 on the \( d^2 \) terms in the above sum, we have that with probability at least \( 1 - 2d^2 \exp \left(-\frac{m}{4} (\epsilon^2 - \epsilon^3) \right) \).

\[
2 \sum_{i=1}^{d} \sum_{j=1}^{d} |r_i| \cdot |r_j| < \frac{1}{\sqrt{2}} \| RB_i \| \cdot \frac{1}{\sqrt{2}} \| RB_j \|
\]

(160)

\[
= 2 \sum_{i=1}^{d} r_i^2 \| B_i \|^2 + 2 \sum_{i=1}^{d} |r_i| \cdot |r_j| \cdot \epsilon
\]

(161)

\[
= 2 \| r \|^2 + 2\epsilon \| r \|_1^2
\]

(162)

Since \( r' \), and hence \( r \) is a point inside one of the grid cells containing the origin, its length can be at most the length of the main diagonal of the grid cell. Formally, \( \| r \| \leq \sqrt{d} \Delta \leq \epsilon \), and \( \| r \|_1 \leq d \Delta \leq \sqrt{\epsilon} \). Substituting these inequalities in the above equations, we get,

\[
\| RBr \|^2 \leq 4\epsilon^2
\]

(163)

Looking back at the R.H.S. of Eq. (155), we have that with probability at least \( 1 - (\frac{2}{\Delta})^d \exp \left(-m \left( \frac{\xi}{4} + \log \frac{2}{1+\sqrt{1+\rho}} \right) \right) - 2d^2 \exp \left(-\frac{m}{4} (\epsilon^2 - \epsilon^3) \right) \),

\[
\| \text{ReLU}(R\hat{u}) \| \leq 1 + \epsilon + 2\epsilon
\]

(164)

\[
= 1 + 3\epsilon
\]

(165)

To prove the L.H.S. of the claimed bound, we can similarly find a point \( g \) on the grid \( G \) such that \( \| g \| \geq 1 \), and it is closest to \( \hat{u} \) in \( \ell^2 \) norm, and define \( r' := \hat{u} - g \). Then invoking lemma 2, we know that with probability at least \( 1 - (\frac{2}{\Delta})^d \exp \left(-m \left( \frac{\xi}{4} + \log \frac{2}{1+\sqrt{1+\rho}} \right) \right) \),

\[
\| \text{ReLU}(Rg) \|^2 \geq (1 - \epsilon) \| g \|^2 \\
\geq 1 - \epsilon \\
\geq (1 - \epsilon)^2
\]

(166) (167)

This can be substituted in the L.H.S. of Eq. (155). We then substitute the previously computed upper bound of \( \| RBr \|^2 \) once again and have that with probability at least \( 1 - 2(\frac{2}{\Delta})^d \exp \left(-m \left( \frac{\xi}{4} + \log \frac{2}{1+\sqrt{1+\rho}} \right) \right) - 2d^2 \exp \left(-\frac{m}{4} (\epsilon^2 - \epsilon^3) \right) \),

\[
1 - 3\epsilon \leq \| \text{ReLU}(R\hat{u}) \| \leq 1 + 3\epsilon
\]

(168)
Scaling $\hat{u}$ arbitrarily, we equivalently have,

$$(1 - 3\epsilon)\|u\| \leq \|\text{ReLU}(Ru)\| \leq (1 + 3\epsilon)\|u\| \tag{169}$$

Finally, since,

$$\left(\frac{2}{\Delta}\right)^d \exp\left(-m\left(\frac{\epsilon}{4} + \log\frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \geq d^2 \exp\left(-\frac{m}{4}(\epsilon^2 - \epsilon^3)\right) \tag{170}$$

We can further lower bound the success probability of Eq. (169) for mathematical ease as,

$$1 - 4\left(\frac{2}{\Delta}\right)^d \exp\left(-m\left(\frac{\epsilon}{4} + \log\frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \tag{171}$$

Therefore to guarantee a success probability of at least $1 - \delta$, we bound,

$$1 - 4\left(\frac{2}{\Delta}\right)^d \exp\left(-m\left(\frac{\epsilon}{4} + \log\frac{2}{1 + \sqrt{1 + \epsilon}}\right)\right) \geq 1 - \delta \tag{172}$$

Rearranging the terms in the equality to get a lower bound on $m$ and rescaling $\epsilon$ proves the claim. □

**Theorem 4** Let $\mathcal{X}$ be a $d \leq n$ dimensional subspace of $\mathbb{R}^n$, $z \in \mathbb{R}^m$, and $R \in \mathbb{R}^{m \times n}$. If $R_{ij} \sim \text{i.i.d.} \mathcal{N}(0, \frac{2}{m}), \epsilon \in [0, 1)$, $z_i \sim \text{Bernoulli}(0.5)$, $\epsilon \in [0, 1)$, and,

$$m \geq \frac{1}{\epsilon/12 - \log(0.5(1 + \sqrt{1 + \epsilon}/3))} \cdot \left(d\log\frac{2}{\Delta} + \log\frac{4}{\delta}\right) \tag{173}$$

then for all vectors $u \in \mathcal{X}$, with probability at least $1 - \delta$,

$$\left\|z \odot (Ru)\right\| - \|u\| \leq \epsilon\|u\| \tag{174}$$

where $\Delta := \min\{\frac{\epsilon}{3\sqrt{d}}, \frac{\sqrt{\epsilon}}{\sqrt{d}}\}$.

**Proof:** The proof is is very similar to that of theorem 3 with minor adjustments. Without any loss of generality, we will show the norm preserving property for any unit vector $u$ in the $d$ dimensional subspace $\mathcal{X}$ of $\mathbb{R}^n$. This is because for any arbitrary length vector $u$, $\|z \odot (Ru)\| = \|u\| \cdot \|z \odot (\hat{u})\|$. The idea then is to define a grid of finite points over $\mathcal{X}$ such that every unit vector $\hat{u}$ in $\mathcal{X}$ is close enough to one of the grid points. Then, if we choose the width of the layer to be large enough to approximately preserve the length of the finite number of grid points, we essentially guarantee that the length of any arbitrary vector approximately remains preserved.

To this end, we define a grid $G$ on $[-1, 1]^d$ with interval of size $\Delta := \min\{\epsilon/\sqrt{d}, \sqrt{\epsilon}/d\}$. Note the number of points on this grid is $(\frac{2}{\Delta})^d$. Also, let column vectors of $B \in \mathbb{R}^{n \times d}$ be the orthonormal basis of $\mathcal{X}$.

We now prove the R.H.S. of the bound in the claim. If we consider any unit vector $\hat{u}$ in $\mathcal{X}$, we can find a point $g$ on the grid $G$ such that $\|g\| \leq 1$, and it is closest to $\hat{u}$ in $l^2$ norm, and define $r' := \hat{u} - g$. Thus the vector $\hat{u}$ can essentially be decomposed as,

$$\hat{u} = g + r' \tag{175}$$

Also note that since $r'$ lies in the span of $\mathcal{X}$, we can represent $r' := Br$ for some vector $r$.

Now consider the norm of the vector $\hat{u}$ after the transformation give by $\|z \odot (\hat{u})\|$. Then we have,

$$\|z \odot (\hat{u})\| = \|z \odot (R(g + r'))\| \tag{176}$$

$$= \|z \odot (Rg) + z \odot (Rr')\| \tag{177}$$

$$\leq \|z \odot (Rg)\| + \|z \odot (Rr')\| \tag{178}$$

$$\leq \|z \odot (Ru)\| - \|u\| \leq \epsilon\|u\| \tag{179}$$
Similarly, we have,
\[
\|z \odot (Rg)\| = \|z \odot (R(g + \hat{u} - \hat{u}))\|
\]
\[
= \|z \odot (R\hat{u}) + z \odot (R(g - \hat{u})))\|
\]
\[
\leq \|z \odot (R\hat{u})\| + \|z \odot (Rr')\|
\]
(180) \hspace{1cm} (181) \hspace{1cm} (182) \hspace{1cm} (183)

Therefore,
\[
\|z \odot (Rg)\| - \|z \odot Rr'\| \leq \|z \odot (R\hat{u})\| \leq \|z \odot (Rg)\| + \|z \odot Rr'\|
\]
(184)

Applying union bound on all the points in $G$, from lemma 4, we know that with probability at least $1 - \left(\frac{2}{N}\right)^d \exp\left(-m \left(\frac{\epsilon}{4} + \log \frac{2}{1+\sqrt{1+\epsilon}}\right)\right)$,
\[
\|z \odot (Rg)\|^2 \leq (1 + \epsilon)\|g\|^2
\]
\[
\leq 1 + \epsilon \leq (1 + \epsilon)^2
\]
(185) \hspace{1cm} (186)

This can be substituted in the R.H.S. of Eq. (184). Now we only need to upper bound $\|z \odot Rr'\|$. To this end, we rewrite $\|z \odot Rr'\| = \|z \odot RB\|$. Then,
\[
\|z \odot RB\|^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} z \odot RB_i r_i \odot z \odot RB_j r_j >
\]
\[
\leq 2 \sum_{i=1}^{d} \sum_{j=1}^{d} |r_i| \cdot |r_j| < \frac{1}{\sqrt{2}} z \odot RB_i \odot \frac{1}{\sqrt{2}} z \odot RB_j
\]
(187) \hspace{1cm} (188)

Note that $\frac{1}{\sqrt{2}} R$ is a matrix whose entries are sampled from $\mathcal{N}(0, 1)$. Invoking lemma 6 on the $d^2$ terms in the above sum, we have that with probability at least $1 - 4d^2 \exp\left(-m \left(\frac{\epsilon}{4} + \log \frac{2}{1+\sqrt{1+\epsilon}}\right)\right)$,
\[
2 \sum_{i=1}^{d} \sum_{j=1}^{d} |r_i| \cdot |r_j| < \frac{1}{\sqrt{2}} z \odot RB_i \odot \frac{1}{\sqrt{2}} z \odot RB_j \leq 2 \sum_{i=1}^{d} \sum_{j=1}^{d} |r_i| \cdot |r_j| \cdot (B_i \odot B_j > + \epsilon)
\]
\[
= 2 \sum_{i=1}^{d} r_i^2 \|B_i\|^2 + 2 \sum_{i=1}^{d} \sum_{j=1}^{d} |r_i| \cdot |r_j| \cdot \epsilon
\]
\[
= 2 \|r\|^2 + 2\epsilon \|r\|^2
\]
(189) \hspace{1cm} (190) \hspace{1cm} (191)

Since $r'$, and hence $r$ is a point inside one of the grid cells containing the origin, its length can be at most the length of the main diagonal of the grid cell. Formally, $\|r\| \leq \sqrt{d}\Delta \leq \epsilon$, and $\|r\|_1 \leq d\Delta \leq \sqrt{\epsilon}$. Substituting these inequalities in the above equations, we get,
\[
\|RB\|^2 \leq 4\epsilon^2
\]
(192)

Looking back at the R.H.S. of Eq. (184), we have that with probability at least $1 - \left(\frac{2}{N}\right)^d \exp\left(-m \left(\frac{\epsilon}{4} + \log \frac{2}{1+\sqrt{1+\epsilon}}\right)\right) - 4d^2 \exp\left(-\frac{m}{4} (\epsilon^2 - \epsilon^2)\right)$,
\[
\|z \odot (R\hat{u})\| \leq 1 + \epsilon + 2\epsilon
\]
\[
= 1 + 3\epsilon
\]
(193) \hspace{1cm} (194)

To prove the L.H.S. of the claimed bound, we can similarly find a point $g$ on the grid $G$ such that $\|g\| \geq 1$, and it is closest to $\hat{u}$ in $\ell^2$ norm, and define $r' := \hat{u} - g$. Then invoking lemma 4, we know that with probability at least
\[ 1 - \left( \frac{2}{\Delta} \right)^d \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right), \]

\[ \| z \odot (Rg) \|^2 \geq (1 - \epsilon) \| g \|^2 \]

(195)

\[ \geq 1 - \epsilon \]

(196)

This can be substituted in the L.H.S. of Eq. (184). We then substitute the previously computed upper bound of \( \| z \odot RBr \| \) once again and have that with probability at least \( 1 - 2 \left( \frac{2}{\Delta} \right)^d \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) - 4d^2 \exp \left( - \frac{m}{4} \left( \epsilon^2 - \epsilon^3 \right) \right), \]

\[ 1 - 3\epsilon \leq \| z \odot (R\hat{u}) \| \leq 1 + 3\epsilon \]

(197)

Scaling \( \hat{u} \) arbitrarily, we equivalently have,

\[ (1 - 3\epsilon) \| u \| \leq \| z \odot (Ru) \| \leq (1 + 3\epsilon) \| u \| \]

(198)

Finally, since,

\[ \left( \frac{2}{\Delta} \right)^d \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \geq 2d^2 \exp \left( - \frac{m}{4} \left( \epsilon^2 - \epsilon^3 \right) \right) \]

(199)

We can further lower bound the success probability of Eq. (198) for mathematical ease as,

\[ 1 - 4 \left( \frac{2}{\Delta} \right)^d \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \]

(200)

Therefore to guarantee a success probability of at least \( 1 - \delta \), we bound,

\[ 1 - 4 \left( \frac{2}{\Delta} \right)^d \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \geq 1 - \delta \]

(201)

Rearranging the terms in the equality to get a lower bound on \( m \) and rescaling \( \epsilon \) proves the claim. \( \Box \)

**Theorem 5** Let \( v = ReLU \left( \sqrt{\frac{2n}{m}} \cdot \hat{R}u \right) \), where \( u \in \mathbb{R}^n \) and \( \hat{R} \in \mathbb{R}^{m \times n} \). If \( R_i \) \( i.i.d. \ P \) where \( P \) is any isotropic distribution in \( \mathbb{R}^n \), then for any fixed vector \( u \), \( E[\| v \|^2] = K_n \cdot \| u \|^2 \) where,

\[ K_n = \begin{cases} \frac{2S_{n-1}}{S_n} \cdot \left( \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{n-2}{n-1} \right) & \text{if } n \text{ is odd} \\ \frac{2S_{n-1}}{S_n} \cdot \left( \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{n-2}{n-1} \right) \cdot \frac{\pi}{2} & \text{otherwise} \end{cases} \]

(202)

and \( S_n \) is the surface area of a unit \( n \)-dimensional sphere.

**Proof:** During the proof, take note of the distinction between the notations \( \hat{R}_i \) and \( R_i \). Our goal is to compute,

\[ E[\| v \|^2] = E[\sum_{i=1}^{m} v_i^2] \]

(203)

\[ = m E[v_i^2] \]

(204)

where the last equality holds because each row of the matrix \( \hat{R} \) is sampled i.i.d.

We have,

\[ E[v_i^2] = E[\max(0, \sqrt{\frac{2n}{m}} \cdot \hat{R}_i^T u)^2] \]

(205)

\[ = \int_{R_i} p(R_i) \max(0, \sqrt{\frac{2n}{m}} \cdot \| u \| \cos \theta)^2 \]

(206)
where \( p(\mathbf{R}_i) \) denotes the probability distribution of the random variable \( \mathbf{R}_i \), and \( \theta \) is the angle between vectors \( \hat{\mathbf{R}}_i \) and \( \mathbf{u} \). Hence \( \theta \) is a function of \( \mathbf{R}_i \). Since \( \mathbf{R}_i \) is sampled from an isotropic distribution, the direction and scale of \( \mathbf{R}_i \) are independent. Thus,

\[
\int p(\mathbf{R}_i) \max(0, \sqrt{\frac{2n}{m}} \cdot \|\mathbf{u}\| \cos \theta)^2 = \int p(\|\mathbf{R}_i\|) \int_{\mathbb{R}_i} p(\mathbf{R}_i) \max(0, \sqrt{\frac{2n}{m}} \cdot \|\mathbf{u}\| \cos \theta)^2
\]

\[
= \int p(\mathbf{R}_i) \max(0, \sqrt{\frac{2n}{m}} \cdot \|\mathbf{u}\| \cos \theta)^2
\]

\[
= \frac{2n}{m} \cdot \|\mathbf{u}\|^2 \int_{\mathbb{R}_i} p(\mathbf{R}_i) \max(0, \cos \theta)^2
\]  

Since \( P \) is an isotropic distribution in \( \mathbb{R}^n \), the likelihood of all directions is uniform. It essentially means that \( p(\hat{\mathbf{R}}_i) \) can be seen as a uniform distribution over the surface area of a unit \( n \)-dimensional sphere. We can therefore re-parameterize \( p(\hat{\mathbf{R}}_i) \) in terms of \( \theta \) by aggregating the density \( p(\hat{\mathbf{R}}_i) \) over all points on this \( n \)-dimensional sphere at a fixed angle \( \theta \) from the vector \( \mathbf{u} \). This is similar to the idea of Lebesgue integral. To achieve this, we note that all the points on the \( n \)-dimensional sphere at a constant angle \( \theta \) from \( \mathbf{u} \) lie on an \((n-1)\)-dimensional sphere of radius \( \sin \theta \). Thus, the aggregate density at an angle \( \theta \) from \( \mathbf{u} \) is the ratio of the surface area of the \((n-1)\)-dimensional sphere of radius \( \sin \theta \) and the surface area of the unit \((n)\)-dimensional sphere. Therefore,

\[
\int_{\mathbb{R}_i} p(\hat{\mathbf{R}}_i) \max(0, \cos \theta)^2 = \int_0^{\pi} \frac{S_{n-1}}{S_n} \cdot |\sin^{n-1} \theta| \cdot \max(0, \cos \theta)^2
\]

\[
= \frac{S_{n-1}}{S_n} \int_0^{\pi/2} \sin^{n-1} \theta \cos^2 \theta
\]

\[
= \frac{S_{n-1}}{S_n} \int_0^{\pi/2} \sin^{n-1} \theta (1 - \sin^2 \theta)
\]

\[
= \frac{S_{n-1}}{S_n} \int_0^{\pi/2} \sin^{n-1} \theta - \sin^{n+1} \theta
\]

Now we use a known result in existing literature that uses integration by parts to evaluate the integral of exponentiated sine function, which states,

\[
\int \sin^n \theta = -\frac{1}{n} \sin^{n-1} \theta \cos \theta + \frac{n-1}{n} \int \sin^{n-2} \theta
\]

Since our integration is between the limits 0 and \( \pi/2 \), we find that the first term on the R.H.S. in the above expression is 0. Recursively expanding the \( n-2^{th} \) power sine term, we can similarly eliminate all such terms until we are left with the integral of \( \sin \theta \) or \( \sin^n \theta \) depending on whether \( n \) is odd or even. For the case when \( n \) is odd, we get,

\[
\int_0^{\pi/2} \sin^n \theta = \left(\frac{2}{3} \cdot \frac{4}{5} \cdots \frac{n-1}{n}\right) \int_0^{\pi/2} \sin \theta
\]

\[
= -\left(\frac{2}{3} \cdot \frac{4}{5} \cdots \frac{n-1}{n}\right) \cos \theta |_0^{\pi/2}
\]

\[
= \left(\frac{2}{3} \cdot \frac{4}{5} \cdots \frac{n-1}{n}\right)
\]

For the case when \( n \) is even, we similarly get,

\[
\int_0^{\pi/2} \sin^n \theta = \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{n-1}{n}\right) \int_0^{\pi/2} \sin^0 \theta
\]

\[
= \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{n-1}{n}\right) \int_0^{\pi/2} 1
\]

\[
= \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{n-1}{n}\right) \cdot \frac{\pi}{2}
\]
Thus,
\[ \int_0^{\pi/2} \sin^{n-1} \theta - \sin^{n+1} \theta = \begin{cases} \frac{1}{n} \cdot \left( \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{n-2}{n-1} \right) & \text{if } n \text{ is odd} \\ \frac{1}{n} \cdot \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n-2}{n-1} \right) \cdot \frac{n}{2} & \text{otherwise} \end{cases} \] (221)

Define,
\[ K_n = \begin{cases} \frac{2^{n-1}}{S_n} \cdot \left( \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{n-2}{n-1} \right) & \text{if } n \text{ is odd} \\ \frac{2^{n-1}}{S_n} \cdot \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n-2}{n-1} \right) \cdot \frac{n}{2} & \text{otherwise} \end{cases} \] (222)

Then,
\[ \int \hat{R}_i p(\hat{R}_i) \max(0, \cos \theta)^2 = 0.5K_n \] (223)

Thus,
\[ E[\|v\|^2] = mE[v_i^2] \]
\[ = m \cdot \frac{2n}{m} \cdot \|u\|^2 \cdot \frac{0.5K_n}{n} \]
\[ = K_n \cdot \|u\|^2 \] (226)

which proves the claim. \(\square\)

**Proposition 2** Let \( v = \sqrt{\frac{2n}{m}} \cdot \left( \hat{R}_i u \right) \odot z \), where \( u \in \mathbb{R}^n \), \( \hat{R}_i \in \mathbb{R}^{m \times n} \) and \( z \in \mathbb{R}^m \). If \( \hat{R}_i \overset{i.i.d.}{\sim} P \) where \( P \) is any isotropic distribution in \( \mathbb{R}^n \) and \( z_i \overset{i.i.d.}{\sim} \text{Bernoulli}(0.5) \), then for any fixed vector \( u \), \( E[\|v\|^2] = K_n \cdot \|u\|^2 \), where \( K_n \) is same as defined in theorem 5.

**Proof:** Our goal is to compute,
\[ E[\|v\|^2] = mE[v_i^2] \]
\[ = mE[v_i^2] \] (227)
\[ = mE[v_i^2] \] (228)

where the last equality holds because each row of the matrix \( \hat{R} \) is sampled i.i.d. Thus we have,
\[ E[v_i^2] = E[\frac{2n}{m} \cdot (\hat{R}_i u)^2 \cdot z_i^2] \]
\[ = \frac{2n}{m} \cdot \|u\|^2 \cdot \text{E}[z_i^2] \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \]
\[ = \frac{n}{m} \cdot \|u\|^2 \cdot \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \] (230)
\[ = \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \]
\[ = \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \] (231)

where \( p(\hat{R}_i) \) denotes the probability distribution of the random variable \( \hat{R}_i \), and \( \theta \) is the angle between vectors \( \hat{R}_i \) and \( u \). Hence \( \theta \) is a function of \( \hat{R}_i \). Since \( \hat{R}_i \) is sampled from an isotropic distribution, the direction and scale of \( \hat{R}_i \) are independent. Thus,
\[ \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta = \int_{\hat{R}_i} p(\|\hat{R}_i\|) \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \]
\[ = \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \] (232)
\[ = \int_{\hat{R}_i} p(\hat{R}_i) \cos^2 \theta \] (233)
The Benefits of Over-parameterization at Initialization in Deep ReLU Networks

Similar to the proof of theorem 5,
\[
\int_{\mathbf{R}_i} p(\hat{\mathbf{R}}_i) \cos^2 \theta = \int_0^\pi \frac{S_{n-1}}{S_n} \cdot |\sin^{n-1} \theta| \cdot \cos^2 \theta = 2 \int_0^{\pi/2} \frac{S_{n-1}}{S_n} (\sin^{n-1} \theta) \cos^2 \theta
\]
(234)

We have shown in the proof of theorem 5 that,
\[
\int_0^{\pi/2} \frac{S_{n-1}}{S_n} (\sin^{n-1} \theta) \cos^2 \theta = 0.5K_n \quad \text{(235)}
\]

Thus,
\[
\int_{\mathbf{R}_i} p(\hat{\mathbf{R}}_i) \cos^2 \theta = \frac{K_n}{n}
\]
(237)

Therefore,
\[
\mathbb{E}[\|v\|^2] = m\mathbb{E}[v_i^2] = m \cdot \frac{n}{m} \cdot \|u\|^2 \cdot \frac{K_n}{n}
\]
(238)

\[
= K_n \cdot \|u\|^2
\]
(240)

which proves the claim. □

**Proposition 3** Consider a matrix \(\mathbf{M} = \sqrt{\frac{2n}{m}} \text{diag}(\mathbf{z}) \cdot \mathbf{H}\) such that the \(i^{th}\) row \(\mathbf{H}_i := \frac{1}{\|\mathbf{R}_i\|} (\mathbf{I} - \hat{\mathbf{R}}_i \hat{\mathbf{R}}_i^T) \cdot \mathbf{u}\), where \(\mathbf{u} \in \mathbb{R}^n\), \(\mathbf{R} \in \mathbb{R}^{m \times n}\) and \(\mathbf{z} \in \mathbb{R}^m\). If \(\mathbf{R}_i \sim P\) where \(P\) is any isotropic distribution in \(\mathbb{R}^n\) such that \(\|\mathbf{R}_i\| = c\) for some fixed \(c\) and \(\mathbf{z}\) follows any distribution independent of \(\mathbf{R}\), then for any fixed vector \(\mathbf{u}\), \(\mathbb{E}[\|\mathbf{M}\|^2] = \frac{2n}{mc^2} \cdot (1 - \frac{K_n}{n}) \cdot \mathbb{E}[\|\mathbf{z}\|^2] \cdot \|\mathbf{u}\|^2\), where \(K_n\) is same as defined in theorem 5.

**Proof**: Our goal is to compute,
\[
\mathbb{E}[\|\mathbf{M}\|^2] = \mathbb{E}\left[\sum_{i=1}^{m} M_i^2\right]
\]
(241)

\[
= \frac{2n}{m} \cdot \mathbb{E}\left[\sum_{i=1}^{m} |\mathbf{z}_i^2|\|\mathbf{H}_i\|^2\right]
\]
(242)

\[
= \frac{2n}{m} \cdot \sum_{i=1}^{m} \mathbb{E}[\mathbf{z}_i^2 \cdot \frac{1}{\|\mathbf{R}_i\|^2} \cdot \mathbf{u}^T (\mathbf{I} - \hat{\mathbf{R}}_i \hat{\mathbf{R}}_i^T) (\mathbf{I} - \hat{\mathbf{R}}_i \hat{\mathbf{R}}_i^T) \mathbf{u}]
\]
(243)

\[
= \frac{2n}{mc^2} \cdot \sum_{i=1}^{m} \mathbb{E}[\mathbf{z}_i^2 \mathbf{u}^T (\mathbf{I} - \hat{\mathbf{R}}_i \hat{\mathbf{R}}_i^T) \mathbf{u}]
\]
(244)

\[
= \frac{2n}{mc^2} \cdot \sum_{i=1}^{m} \mathbb{E}[\mathbf{z}_i^2 \cdot (\|\mathbf{u}\|^2 - (\hat{\mathbf{R}}_i \mathbf{u})^2)]
\]
(245)

\[
= \frac{2n}{mc^2} \cdot \sum_{i=1}^{m} \mathbb{E}[\mathbf{z}_i^2 \cdot (\|\mathbf{u}\|^2 - (\hat{\mathbf{R}}_i \mathbf{u})^2)]
\]
(246)

\[
= \frac{2n}{mc^2} \cdot \|\mathbf{u}\|^2 \cdot \sum_{i=1}^{m} \mathbb{E}[\mathbf{z}_i^2 \cdot (1 - \cos^2 \theta_i)]
\]
(247)

where \(\theta_i\) is the angle between \(\hat{\mathbf{R}}_i\) and \(\mathbf{u}\).
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We know from the proof of proposition 2 that $E[\cos^2 \theta_i] = K_n / n$. Therefore,

$$E[||M||^2_F] = \frac{2n}{mc^2} \cdot ||u||^2 \cdot \sum_{i=1}^m E[z_i^2] \cdot \left(1 - \frac{K_n}{n}\right)$$

$$= \frac{2n}{mc^2} \cdot ||u||^2 \cdot \left(1 - \frac{K_n}{n}\right)E[||z||^2]$$

which proves the claim. □

**Theorem 6** Let $f_\theta(.)$ be a residual network as defined in Eq. 33. If the network weights are un-normalized, let each residual block be of the form shown in Eq. 1 and the element of weight matrices sampled from $\mathcal{N}(0, 2/fan-out)$. If the network weights are weight normalized, let each residual block be of the form shown in Eq. 32 such that for any weight matrix, each row of the weight matrix can be sampled from any isotropic distribution, but its norm should be re-scaled to 1. Finally, set $\gamma = 1/B$ and assume $K_n = 1$ for $n > 1$. Then in the limit of $B \to \infty$ and an infinitely wide network,

$$\frac{||x||^2}{e^2} \leq ||f_\theta(x)||^2 \leq e^2 \cdot ||x||^2$$

**Proof:** We will prove the result for the case when the residual blocks have normalized weights. The proof of the un-normalized case is trivially similar to this case.

Consider the first hidden state $h^1$ given by,

$$h^1 := x + \gamma F_1(x)$$

Then the squared norm of $h^1$ is given by,

$$||h^1||^2 = ||x + \gamma F_1(x)||^2$$

$$= ||x||^2 + \gamma^2 ||F_1(x)||^2 + 2\gamma < x, F_1(x) >$$

Since $E[||h^1||^2] = ||x||^2$ due to theorem 5, in the limit of an infinitely wide network, we have that $||h^1||^2 = ||x||^2$.

Then due to the boundedness of cosine function, we have,

$$||x||^2 \cdot (1 - \gamma)^2 \leq ||h^1||^2 \leq ||x||^2 \cdot (1 + \gamma)^2$$

Similarly,

$$h^2 := h^1 + \gamma F_2(h^1)$$

Thus,

$$||h^2||^2 = ||h^1||^2 + \gamma^2 ||F_2(h^1)||^2 + 2\gamma < h^1, F_2(h^1) >$$

Then the boundedness of cosine function yields,

$$||h^1||^2 \cdot (1 - \gamma)^2 \leq ||h^2||^2 \leq ||h^1||^2 \cdot (1 + \gamma)^2$$

$$||x||^2 \cdot (1 - \gamma)^4 \leq ||h^2||^2 \leq ||x||^2 \cdot (1 + \gamma)^4$$

Extending such inequalities to the $B^{th}$ residual block, we get,

$$||x||^2 \cdot (1 - \gamma)^{2B} \leq ||h^B||^2 \leq ||x||^2 \cdot (1 + \gamma)^{2B}$$

Setting $\gamma = 1/B$, we get,

$$||x||^2 \cdot \left(1 - \frac{1}{B}\right)^{2B} \leq ||h^B||^2 \leq ||x||^2 \cdot \left(1 + \frac{1}{B}\right)^{2B}$$
Now we use the well known results,

\[
\lim_{B \to \infty} \left( 1 - \frac{1}{B} \right)^B = \frac{1}{e} \tag{261}
\]

\[
\lim_{B \to \infty} \left( 1 + \frac{1}{B} \right)^B = e \tag{262}
\]

Thus we get,

\[
\frac{\|x\|^2}{e^2} \leq \|h^B\|^2 \leq e^2 \cdot \|x\|^2 \tag{263}
\]

Since \(f_\theta(.) = h^B\) by definition, we have proved the claim. \(\square\)

**Lemma 6** Let \(v_1 = (Ru_1) \odot z\) and \(v_2 = (Ru_2) \odot z\), where \(u_1, u_2 \in \mathbb{R}^n\), \(z \in \mathbb{R}^m\), and \(R \in \mathbb{R}^{m \times n}\). If \(R_{ij} \sim \text{i.i.d. } \mathcal{N}(0, \frac{1}{0.5m})\), \(z_i \sim \text{i.i.d. } \text{Bernoulli}(0.5)\) and \(\epsilon \in [0, 1)\), then for any fixed vectors \(u_1\) and \(u_2\) s.t. \(\|u_1\| \leq 1\) and \(\|u_2\| \leq 1\),

\[
\Pr (|<v_1, v_2>-<u_1, u_2>| \leq \epsilon) \geq 1 - 4 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right) \tag{264}
\]

**Proof:** Applying lemma 4 to vectors \(u_1 + u_2\) and \(u_1 - u_2\), we have with probability at least \(1 - 4 \exp \left( -m \left( \frac{\epsilon}{4} + \log \frac{2}{1 + \sqrt{1 + \epsilon}} \right) \right)\),

\[
(1 - \epsilon) \cdot \|u_1 + u_2\|^2 \leq \|z \odot Ru_1 + z \odot Ru_2\|^2 \leq (1 + \epsilon) \cdot \|u_1 + u_2\|^2 \tag{265}
\]

\[
(1 - \epsilon) \cdot \|u_1 - u_2\|^2 \leq \|z \odot Ru_1 - z \odot Ru_2\|^2 \leq (1 + \epsilon) \cdot \|u_1 - u_2\|^2 \tag{266}
\]

Then notice,

\[
4 <v_1, v_2> = 4 <z \odot Ru_1, z \odot Ru_2> \tag{267}
\]

\[
= \|z \odot Ru_1 + z \odot Ru_2\|^2 - \|z \odot Ru_1 - z \odot Ru_2\|^2 \tag{268}
\]

\[
\geq (1 - \epsilon) \cdot \|u_1 + u_2\|^2 - (1 + \epsilon) \cdot \|u_1 - u_2\|^2 \tag{269}
\]

\[
= 4 \cdot <u_1, u_2> - 2\epsilon \cdot (\|u_1\|^2 + \|u_2\|^2) \tag{270}
\]

\[
\geq 4 \cdot <u_1, u_2> - 4\epsilon \tag{271}
\]

Equivalently,

\[
<u_1, u_2> - <v_1, v_2> \leq \epsilon \tag{272}
\]

The other side of the claim can be proved similarly.