Eddy mixing of momentum and heat in stably stratified boundary layers: Numerical study

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Abstract. The paper analyzes the features of turbulent momentum and heat transfer in a stably stratified atmospheric boundary layer (ABL), in the upper troposphere and lower stratosphere, and the possibility of taking them into account in RANS (Reynolds Average Navier Stokes) turbulence models. Such features, for example, include the transfer of momentum (but not heat) by internal gravitational waves under conditions of strong stability, the formation of a low-level jet. Models for the coefficients of eddy diffusion of momentum and heat were obtained by writing the differential equations for the Reynolds stresses and the turbulent heat flux vector in the weakly equilibrium approximation, in which the advection and diffusion of some dimensionless quantities are neglected. In the considered version of the algebraic model built on the physical principles of the RANS approximation for stratified turbulence, three prognostic equations are used: for the kinetic energy of turbulence, the rate of its spectral consumption and the dispersion of temperature fluctuations. It is shown that the profile of the vertical diffusivity for momentum, which was calculated using three-parametric turbulent model, is in agreement with data of direct measurements either within a stable stratified atmospheric boundary layer or beyond this layer in free atmosphere.

1. Features of eddy mixing in the stably stratified atmospheric boundary layers

The processes of turbulent mixing in geophysical flows are conventionally parameterized using the single point schemes of turbulence closure [1, 2]. Using the equations of transport for the third-order moments for a complete description of nonlocal transport complicates the practical use of a RANS method especially for complex flows in the planetary boundary layers. Therefore, nonlocal models of turbulence that include transport for the moments of higher orders will not be considered below. If the dependence of the vertical turbulent transport of temperature on the Brunt-Väisälä frequency is taken into account [3], it is possible to formulate the anisotropic algebraic parameterizations of the eddy diffusivities of momentum and heat for stably stratified flows, which correctly take into account anisotropy and the effect of buoyancy on the vertical turbulent transport. Such parameterizations are a physically more significant alternative to the constant eddy coefficients of scalar diffusion which are used in almost all models of atmospheric and oceanic circulation as the ‘basic’ coefficients of mixing and in the cases of the strong stratification (when the eddy diffusivities of momentum and heat become too small) they are simply assumed to be equal to zero [4].
1.1. RANS approach for the stratified turbulence: eddy fluxes of momentum and heat

The governing system of the RANS equations (averaged over time or over an ensemble of the realizations of fluid dynamics equations) for stratified flows in the Boussinesq approximation has a standard form [5]. Below, it is sufficient to consider equations for the Reynolds stresses, turbulent kinetic energy, turbulent heat flux, and dispersion of temperature fluctuations.

The equation for the anisotropy tensor of turbulent stresses $b_{ij} = \langle u_i u_j \rangle - 2/3 E \delta_{ij}$ is

$$\frac{D}{Dt} b_{ij} + D_y = -\frac{4}{3} ES_y - \Sigma_y - Z_y - B_y - \Pi_y,$$  

where

$$S_y = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad R_y = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right),$$

$$\Sigma_y = b_u S_y + S_u b_y - \frac{2}{3} \delta_{ij} S_{ui} S_{uj},$$

$$Z_y = R_u b_y - b_u R_y,$$

$$B_y = \beta_y h_j + \beta_y h_i - \frac{2}{3} \delta_{ij} \beta_y h_k,$$

$$D_y \equiv \frac{\partial}{\partial x_k} \left( u_i u_j - \frac{1}{3} u_i u_k \delta_{ij} \right) u_k.$$  

Here, $S_y$ and $R_y$ are tensors of mean shear and mean vorticity.

Transport equation for turbulent kinetic energy (TKE)

$$\frac{DE}{Dt} + \frac{1}{2} D_y = -\tau_y \frac{\partial U_i}{\partial x_j} + \beta_y h_i - \varepsilon$$  

Transport equation for turbulent heat flux $h_i$

$$\frac{D}{Dt} h_i + D^b_i = -h_i \frac{\partial U_i}{\partial x_j} - \tau_y \frac{\partial \Theta}{\partial x_j} + \beta_i \langle \theta^2 \rangle - \Pi^b_i$$  

Transport equation for the variance of temperature $\langle \theta^2 \rangle$

$$\frac{D}{Dt} \left\langle \theta^2 \right\rangle + D_\theta = -2 h_i \frac{\partial \Theta}{\partial x_i} - \frac{1}{R} E \left\langle \theta^2 \right\rangle,$$  

Equation of TKE spectral consumption (TKE dissipation rate $\varepsilon$)

$$\frac{D\varepsilon}{Dt} + D_\varepsilon = -\frac{\varepsilon}{\tau} \Psi,$$  

where

$$\Psi = \Psi_0 + \Psi_1 \frac{\partial U_i}{\partial x_j} + \Psi_2 \frac{\partial \theta}{\partial x_i} + \Psi_3 \frac{2 E}{E} \langle \theta u_i \rangle \frac{\partial U_i}{\partial x_j},$$

$$R = 0.6; \quad \Psi_0 = 3.8; \quad \Psi_1 = \Psi_2 = 2.4; \quad \Psi_3 = 0.3.$$
Below, the explicit form of the diffusion terms ($D$) in the equations (1)-(5) will not be required. Let us consider the correlations between the pressure fluctuations in (1) and (3), which have the form

$$\Pi_{ij} = \langle u_i p_{j} \rangle + \langle u_j p_{i} \rangle - \frac{2}{3} \delta_{ij} \langle u_k p_{k} \rangle, \quad H^\theta_{ii} = \langle \theta p_{i} \rangle. \quad (6)$$

The parameterizations of the “relaxation” parts of correlations (6) have the form of the linear dependences on turbulence values [6, 7]

$$\Pi_{ij}^{(1)} \sim b_{ij} / \tau, \quad H^\theta_{ii}^{(1)} \sim h_{i} / \tau_{\theta^0}. \quad (7)$$

The principle of generalized linearity (which is based on the linearity of the exact scalar conservation equation [8]) proposed to model turbulent flows may serve as an argument in support of the fact that the correlation $H^\theta_{ii}$ is bound to be the linear function of scalar field. Thus, the approximation for the slow part of the correlation $H^\theta_{ii}$ is usually formulated as

$$H^\theta_{ii}^{(1)} = \langle \theta \frac{\partial \theta}{\partial x_i} \rangle \equiv - \frac{c_{10}}{\tau_{\theta^0}} h_i, \quad (8)$$

where $\tau_{\theta^0}$ is the time scale of the scalar field and $c_{10}$ is the numerical coefficient. Expression (7) was first proposed by Monin [9] for flows without the average gradients of either velocity or scalar fields under the assumption that

$$\tau_{\theta^0} - \tau = E / \varepsilon. \quad (9)$$

Although assumption (9) was widely used earlier in different schemes of second-order turbulence closure, it is not necessary that this assumption will be also applied to study stratified flows. A direct way to prove the validity of expressions (8) and (9) could be to compare them with the LES data for the correlations with pressure fluctuations in atmospheric flows. However, such data are apparently still unavailable. Therefore, one has to use other data on stably stratified flows. As was noted above, the results of studies of stably stratified turbulence [3] show that the time scale $\tau_{\theta^0}$ is bound to be the function of the Brunt–Väisälä frequency $N$

$$\tau_{\theta^0} = \tau_{\theta^0}(N) \quad (10)$$

with a specific functional dependence of the form

$$\tau_{\theta^0} = \tau / (1 + a \tau^2 N^2). \quad (11)$$

In (11) $a = 0.16$ if $N^2 > 0$ and $a = 0$ if $N^2 \leq 0$. A physical argument in support of the damping factor in the denominator of (11) can be based on the fact that, in stably stratified flows, eddies work against buoyancy forces and lose their kinetic energy, which transforms into potential energy.

With consideration for mean gradients, the two remaining parts of the correlation $H^\theta_{ii}$ have the form [10]

$$H^\theta_{ii}^{(2)} = -c_{20} h_j \frac{\partial U_j}{\partial x_i}, \quad H^\theta_{ii}^{(3)} = c_{30} \beta_i \langle \theta^2 \rangle \quad (12)$$

The coefficients in (8) and (12) have the numerical values: $c_{10} = 3.28$ and $c_{20} = c_{30} = 0.5$. Remarks on the calculation of these coefficients can be found also in [11].
1.2. Algebraic models for turbulent momentum and heat fluxes

Algebraic models for turbulent momentum and heat fluxes may be developed on the basis of equations (1) and (3) as an approximation of weakly equilibrium turbulence [5]. If such an approximation is used, there is no need to simulate the third-order moments (the terms $D$ in equations (1)-(5)). Note only that the system of algebraic equations for turbulent momentum and heat fluxes has an implicit form, and its analytical solution with the aid of a code of symbolic algebra (Maple 10) yields the following expressions for turbulent momentum and heat fluxes:

$$\langle u'w' \rangle = -K_m \left( \frac{\partial U}{\partial z}, \frac{\partial V}{\partial z} \right),$$

$$\langle w'\theta \rangle = -K_h \frac{\partial \Theta}{\partial z} + \gamma_c,$$

$$K_m = \left( \frac{E^2}{\varepsilon} \right) f_m, \quad K_h = \left( \frac{E^2}{\varepsilon} \right) f_h,$$

$$\gamma_c = \left( \frac{g}{T_0} \frac{E\left(\theta^2\right)}{\varepsilon} \right) f_c,$$

where $f_m$, $f_h$ and $f_c$ are the structural functions. Expression (14) for the vertical turbulent heat flux $\langle w'\theta \rangle$ contains the countergradient term $\gamma_c$ and, in this case, model (13)-(16) can be treated as a nonlocal model. The countergradient term takes into account the contribution made by large eddies to the vertical heat transport and appears in this model as result of a successive simplification of closed transport equations for turbulent momentum and heat fluxes as an approximation of weakly equilibrium turbulence without using any arguments of a heuristic character. A detail description of the functions $f_m$, $f_h$ and $f_c$ can be found in [10].

2. Features of turbulent transport in the stably stratified atmospheric boundary layers

In this section (as an example of the possibilities of the formulated three-parameter anisotropic algebraic model of turbulent momentum and scalar fluxes) we offer an analysis of behaviour of the turbulent Prandtl number along the height within stably stratified planetary boundary layer, and behaviour of coefficients of eddy diffusion of momentum and heat as a function of Richardson number.

2.1. Computational experiment: initial and boundary conditions

The calculation experiment is based (as in [5]) on the results of measurements in the nocturnal stable boundary layer over the surface in the Arctic zone and on the results of a simulation of the stable atmospheric boundary layer (ABL) with the method of isolating large eddies (the so-called LES method). The initial state and the forcing used are based on 1994 observations (BASE data-bank). The obtained vertical profiles of wind velocity and temperature are compared with observational data and with the results of LES simulation to verify the quantitative validity. The boundary layer is formed under the influence of barotropic geostrophic wind and a specified rate of surface cooling. The upper boundary of the calculation region is located at a height of 400 m. The calculations were performed using a vertically biased grid with a step of 6.25 m (64 vertical intervals) and a time step of 2.5 s. Such a choice made it possible to obtain a grid-independent solution. The initial wind profile was assumed to be equal to the value of geostrophic wind (8 m/s in the x-axis direction) throughout the entire layer depth. At the lower boundary, the conditions were specified for the first calculation layer as in [5]. At the upper boundary, the velocity of wind was equal to its geostrophic value. The initial fields of turbulent kinetic energy, its dissipation rate, and rms temperature fluctuations were specified in the form of “background” values as in [5], and the roughness parameter $z = 0.1$ m was taken similarly to [12, 13]. The initial profile of potential temperature had a constant temperature of 265 K in the layer 100 m from the surface and a
superimposed weak inversion above this height (with the rate of increase 0.01 K/m) up to the upper boundary of the integration region, where the temperature reached 268 K. The boundary layer was treated as dry-adiabatic.

Figure 1 shows the vertical wind velocity profile after ten-hour integration. A comparison can be made with the results [13] obtained with LES method with high resolution (a vertical step of 3.25 m). In figure 1 these data are denoted by black squares and other LES data are denoted by horizontal-line segments with an indicated standard deviation from the mean. The solid line shows the wind velocity profile calculated with the three-parameter turbulence model. The formation of an air current at a height of approximately 175 m is clearly seen. Figure 2 shows the vertical profile of the coefficients of turbulent momentum – heat exchange $K_m$ and $K_h$ for the formed jet wind velocity profile (figure 1).

![Figure 1. Profiles of the total wind velocity in a stably stratified atmospheric boundary layer. The solid line denotes the results of calculations with the improved three parameter model of turbulence after ten hour integration, the black squares correspond to the LES results from [13], and the horizontal lines correspond to other LES results with standard deviations indicated for means.](image1)

![Figure 2. Calculation profiles of the coefficients of turbulent momentum and heat transport (1) $K_m$ and (2) $K_h$ in a stably stratified atmospheric boundary layer.](image2)

2.2. Dependence of eddy diffusivities of momentum and heat on local stability

The dependence of the mechanisms of turbulent transport of momentum and heat on stability is different within the atmospheric boundary layer and, in its upper part, the layer of strongly heterogeneous temperature inversion.

Under strong stability, turbulent eddies are transported from the surface layer, suppressed by stratification, and partially turn into internal gravity waves. Transport of momentum and heat by turbulent eddies is decreasing, but the heat transport decreases more significantly. This can be seen in figures 3a and 3b, which show the vertical profile of the turbulent Prandtl number, $Pr_r = \frac{K_m}{K_h}$, for a quasi-steady strongly stable ABL, with surface forcing in the form of a constant turbulent heat flux. The solid curve in figure 3a is result of calculation using a three–parameter RANS scheme of turbulence (13)-(16) modified for taking into account the contribution of internal gravity waves. The dashed line is result of calculation with no contribution of internal gravity wave. In this case, number $Pr_r$ remains almost unchanged across the entire height of the ABL, which disagrees with the LES results [14] obtained with same surface forcing, as well as with DNS result [15] (figure 3b).
In the upper part of the ABL, in the layer of heterogeneous temperature inversion, turbulence in conditions of strong stability is suppressed by stratification and the transport of momentum and heat by turbulent eddies significantly decreases. The eddy diffusivity of heat decreases far more than the eddy diffusivity of momentum, which hardly varies at all when the local gradient Richardson number increases (figure 4 in [16]) because of the activity of gravity waves, which are efficient in transporting momentum rather than heat.

Figure 3. (a) Vertical profile of the turbulent Prandtl number $Pr_T = K_m / K_h$ in a quasi-steady stable ABL from the three-parameter RANS turbulence scheme with effect of internal waves (solid curve) and without the effect of internal waves (dashed curve) and (b) from LES scheme [14] (solid curve) in comparison with DNS results [15] (crosses).

There is a different picture of the eddy transport of momentum and heat in the upper troposphere and lower stratosphere, where turbulent eddies are generated sporadically by shear instability and by a decaying gravity wave. Both the momentum and heat are synchronously transported by these eddy motions vertically.

3. Eddy diffusivities of momentum and heat in upper troposphere and lower stratosphere

In this section the applicability of the three-parameter RANS turbulence approach (13)–(16), including the impact of gravity waves on the momentum maintenance at the strong stratification, is examined. For this purpose are used the vertical profiles of eddy diffusivities of momentum $K_m$ in the upper troposphere and lower stratosphere during clear-air conditions which were derived in [17] from direct measurements of the Reynolds stress and vertical gradient of mean wind velocity measured by the Doppler radar. Eddy diffusivity $K_h$ of heat below 8 km was determined from measurements of temperature fluctuations by the Radio Acoustic Sounding System (RASS) attached to the Doppler radar. The eddy diffusivity of momentum was on the order of $10 \text{ m}^2\text{s}^{-1}$ in upper troposphere and decreased gradually in the stratosphere by an order of magnitude or more. Estimates of eddy diffusivity from the radar echo power spectral width give fairly good values compared with the direct measurement of $K_m$.

The observation values of turbulent kinetic energy $E$ and turbulent energy dissipation rate $\varepsilon$ together with atmospheric stability observations from Rawinsonde data here also are used.

Results of direct measurement of eddy diffusivity $K_m$ are shown in figure 4. Eddy diffusivity values were about $10 \text{ m}^2\text{s}^{-1}$ in the upper troposphere. In the region above the axis of the subtropical jets, eddy diffusivity values decreased by one or two orders of magnitude, except the case I, when an exceptionally strong winter jet with a speed of 78 m s$^{-1}$ was observed (it is noted by the shaped circle in figure 4). Eddy diffusivity for momentum has been estimated by the three-parameter RANS turbulence scheme.
(13)–(16) and compared with the observed one in Figure 4. In this analysis, observed values from [17] are used for turbulent kinetic energy and the turbulent energy dissipation rate.

The eddy diffusivity of heat, calculated by the three-parameter RANS turbulence scheme (a thick dashed line), is compared in figure 5 with the same coefficient $K_h$ (a thick solid line), which is estimated in [17] using the observed data. The measured vertical eddy diffusivity of heat $K_h$ (figure 5 for case (I)) is depicted with a thin line, making the emphasis on uncertain error in coefficient measurements. In general, we see that the algebraic three-parameter RANS scheme of turbulence (13)–(16) offers estimates for eddy diffusivity of momentum and heat, which are in good compliance with measurement data.

3.1. Stability dependence of eddy diffusivities of momentum and heat in the upper troposphere and the lower stratosphere

The dependence of the eddy transport of momentum and heat on stability in the atmospheric layer from 4 to 8 km is shown in figure 6a. The values of $K_h$ do not differ much from those of $K_m$, and $K_h = O(K_m)$, where $K_h$ is slightly less than $K_m$ (in figure 6a solid circles are the three-parameter RANS turbulence scheme calculation and open circles are the results of measurements [17]). The dependence of the inverse turbulent Prandtl number $Pr = K_m / K_h$ in the lower stratosphere (14 to 18 km) is shown in figure 6b. Open circles are the measurements [17] and solid circles are results from the three-parameter RANS turbulence scheme [5]. These data fit the assumption that eddy diffusivities of momentum $K_m$ and heat $K_h$ are approximately equal.
Figure 5. Vertical eddy diffusivity coefficient of heat $K_h (m^2/s)$: solid curve – measured with a Doppler radar, dashed curve – estimated from the three-parameter RANS scheme of turbulence; $z$ is altitude above the underling surface.

Figure 6. Stability dependence of $K_h / K_m$ on the gradient Richardson number in the atmosphere: (a) the upper troposphere (atmospheric layer from 4 to 8 km) and (b) lower stratosphere (14 to 18 km). Solid circles are calculations by three-parameter RANS scheme [5] and open circles are observations [17].

The calculations by the RANS turbulence scheme give the better result than the observation data [17]. Perhaps the measurements presented are affected by the abovementioned errors in the estimate of the heat eddy diffusivity in [17]. As noticed in [17], there are many error sources, including
instrumental ones from the Doppler radar, RASS, and Rawinsonde, and data processing error sources such as the calculation of vertical gradients of mean velocity and temperature, therefore, it is difficult to estimate experimental errors.

The comparative analyses of vertical coefficients of momentum and heat diffusivity for the situation of direct measurements in the upper troposphere and lower stratosphere with analogous result calculated by three-parametric RANS scheme of turbulence have demonstrated a reasonable agreement between these sets of data. The calculated vertical coefficient of momentum diffusivity in the upper troposphere reaches value of 10 m^2 s^{-1} and then gradually declines by one order in the lower stratosphere at the altitude of 18 km. For the upper troposphere and lower stratosphere (atmospheric layers from 4 to 8 km and from 14 to 18 km, respectively), the eddy diffusivity coefficients of heat and momentum are of the same order of magnitude $K_h \sim K_m$. This is a strikingly contrast with the outer part of the stable ABL, where the ratio $K_m / K_h$ at strongly stratification may reach values of 0.1– 0.02 [16].

Acknowledgments
This work was carried out as part state assignment of the Institute of Computational Mathematics and Mathematical Geophysics SB RAS No. 0315-2019-0004 and with partial financial support from the Russian Foundation for Basic Research under project 20-01-00560 A.

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