A relative of Hadwiger’s conjecture

Katherine Edwards\textsuperscript{1}, Princeton University, Princeton, NJ 08544, USA
Dong Yeap Kang\textsuperscript{2}, KAIST, Daejeon, 305-701 Republic of Korea
Jaehoon Kim\textsuperscript{2}, KAIST, Daejeon, 305-701 Republic of Korea
Sang-il Oum\textsuperscript{2}, KAIST, Daejeon, 305-701 Republic of Korea
Paul Seymour\textsuperscript{3}, Princeton University, Princeton, NJ 08544, USA

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Abstract

Hadwiger’s conjecture asserts that if a simple graph $G$ has no $K_{t+1}$ minor, then its vertex set $V(G)$ can be partitioned into $t$ stable sets. This is still open, but we prove under the same hypotheses that $V(G)$ can be partitioned into $t$ sets $X_1, \ldots, X_t$, such that for $1 \leq i \leq t$, the subgraph induced on $X_i$ has maximum degree at most a function of $t$. This is sharp, in that the conclusion becomes false if we ask for a partition into $t - 1$ sets with the same property.
1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A graph $H$ is a minor of a graph $G$ if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by edge-contraction. In 1943, Hadwiger [3] proposed the following, perhaps the most famous open problem in graph theory:

1.1 (Hadwiger’s Conjecture.) For all integers $t \geq 0$, and every graph $G$, if $K_{t+1}$ is not a minor of $G$, then the chromatic number of $G$ is at most $t$; that is, $V(G)$ can be partitioned into $t$ stable sets.

This remains open, although it has been proved for all $t \leq 5$ (see [5]). It is best possible in that the result becomes false if we ask for a partition into $t-1$ stable sets.

In this paper we prove a much weaker relative, the following. If $G$ is a graph, $\Delta(G)$ denotes the maximum degree of $G$, and if $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced on $X$.

1.2 For all integers $t \geq 0$ there is an integer $s$, such that for every graph $G$, if $K_{t+1}$ is not a minor of $G$, then $V(G)$ can be partitioned into $t$ sets $X_1, \ldots, X_t$, such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t$.

One might view this as supporting evidence for Hadwiger’s conjecture. However, it is to the same degree “supporting evidence” for the false conjecture of Hajós [1,4], that every graph that contains no subdivision of $K_{t+1}$ is $t$-colourable; because we could replace the hypothesis of 1.2 that $G$ has no $K_{t+1}$ minor by the weaker hypothesis that no subgraph of $G$ is a subdivision of $K_{t+1}$, and the same proof (using an appropriate modification of [2]) still works.

Such partitions (into subgraphs with bounded maximum degree) are called “defective colourings” in the literature – see for instance [2]. In particular, Kawarabayashi and Mohar [5] proved the following, which is quite close to our result:

1.3 For all integers $t \geq 0$ there is an integer $s$, such that for every graph $G$, if $K_{t+1}$ is not a minor of $G$, then $V(G)$ can be partitioned into $n$ sets $X_1, \ldots, X_n$, where $n = \lceil 15.5(t+1) \rceil$, such that every component of $G[X_i]$ has at most $s$ vertices, for $1 \leq i \leq n$.

A reason for interest in 1.2 is that, despite being much weaker than the original conjecture of Hadwiger, it is still best possible in the same sense; if we ask for a partition into $t-1$ subgraphs each with bounded maximum degree, the result becomes false. Let us first see the latter assertion:

1.4 For all integers $t \geq 1$ and $s \geq 0$, there is a graph $G = G(t,s)$, such that $K_{t+1}$ is not a minor of $G$, and there is no partition $X_1, \ldots, X_{t-1}$ of $V(G)$ into $t-1$ sets such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t-1$.

Proof. If $t = 1$ we may take $G(t,s)$ to be a one-vertex graph. For $t \geq 2$, we proceed by induction on $t$. Take the disjoint union of $s$ copies $H_1, \ldots, H_s$ of $G(t-1,s)$, and add one new vertex $v$ adjacent to every other vertex, forming $G$. It follows that $G$ has no $K_{t+1}$ minor, since each $H_i$ has no $K_t$ minor. Assume that $X_1, \ldots, X_{t-1}$ is a partition of $V(G)$ into $t-1$ sets such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t-1$. We may assume that $v \in X_{t-1}$. If $X_{t-1} \cap V(H_i) \neq \emptyset$ for all $i \in \{1, \ldots, s\}$, then the degree of $v$ is at least $s$ in $G|X_{t-1}$, a contradiction; so we may assume that $X_{t-1} \cap V(H_1) = \emptyset$ say. Let $Y_i = X_i \cap V(H_1)$ for $1 \leq i \leq t-2$. Then $Y_1, \ldots, Y_{t-2}$ provides a partition of $V(H_1)$ into $t-2$ sets; and since $H_1$ is isomorphic to $G(t-1,s)$, it follows that $\Delta(H_1|Y_i) > s$ for some $i \in \{1, \ldots, t-2\}$, a contradiction. Thus there is no such partition $X_1, \ldots, X_{t-1}$. This proves 1.4. □
2 The proof

To prove 1.2 we use the following lemma, due to Kostochka [6, 7] and Thomason [9, 10].

2.1 There exists $C > 0$ such that for all integers $t \geq 0$ and all graphs $G$, if $K_{t+1}$ is not a minor of $G$ then $G$ has at most $C(t+1)\log((t+1))^\frac{1}{t} |V(G)|$ edges.

We use that to prove two more lemmas:

2.2 Let $t \geq 0$ be an integer, let $C$ be as in 2.1 and let $r \geq C(t+1)\log((t+1))^\frac{1}{t}$. Let $G$ be a graph such that $K_{t+1}$ is not a minor of $G$, and let $A \subseteq V(G)$ be a stable set of vertices each of degree at least $t$. Then

$$|E(G \setminus A)| + |A| \leq r|V(G \setminus A)|.$$  

Proof. We proceed by induction on $|A|$. By 2.1 we may assume that $A \neq \emptyset$. Let $v \in A$. Since $v$ has degree at least $t$ and $G$ has no $K_{t+1}$ subgraph, $v$ has two neighbours $x, y$ which are non-adjacent to each other. Let $G' = (G \setminus v) + xy$ and $A' = A \setminus \{v\}$. Since $G'$ is a minor of $G$ and so $K_{t+1}$ is not a minor of $G'$, it follows from the inductive hypothesis that $|E(G' \setminus A')| + |A'| \leq r|V(G' \setminus A')| = r|V(G \setminus A)|$. But $|E(G' \setminus A')| = |E(G \setminus A)| + 1$ and $|A'| = |A| - 1$. This proves 2.2.

2.3 Let $t \geq 0$ be an integer, let $C$ be as in 2.1 and let $r \geq C(t+1)\log((t+1))^\frac{1}{t}$ and $r > t/2$. Let $s$ be the least integer greater than $r(2r - t + 2)$. Let $G$ be a non-null graph, such that $K_{t+1}$ is not a minor of $G$. Then either

- some vertex has degree less than $t$, or
- there are two adjacent vertices, both with degree at most $s$.

Proof. We may assume that $t \geq 2$, for if $t \leq 1$ the result is trivially true. Let $A$ be the set of all vertices with degree less than $s$, and $B = V(G) \setminus A$. We may assume that every vertex in $A$ has degree at least $t$, for otherwise the first outcome holds. Consequently, by summing all the degrees, we deduce that $2|E(G)| \geq t|A| + s|B|$. On the other hand, by 2.1 $|E(G)| \leq r(|A| + |B|)$. It follows that $t|A| + s|B| \leq 2r(|A| + |B|)$, that is,

$$|A| \geq \frac{s - 2r}{2r - t}|B|,$$

since $2r > t$. But by 2.2 $|A| \leq r|B|$ and so $r \geq (s - 2r)/(2r - t)$, that is, $s \leq r(2r - t + 2)$, a contradiction. This proves 2.3.

Now we prove 1.2 in the following sharpened form.

2.4 For all integers $t \geq 0$, let $s$ be as in 2.3. For every graph $G$, if $K_{t+1}$ is not a minor of $G$, then $V(G)$ can be partitioned into $t$ sets $X_1, \ldots, X_t$, such that $\Delta(G|X_i) \leq s$ for $1 \leq i \leq t$.

Proof. We proceed by induction on $|V(G)| + |E(G)|$. If some vertex $v$ of $G$ has degree less than $t$, the result follows from the inductive hypothesis by deleting $v$ (find a partition by induction and add $v$ to some set $X_i$ that contains no neighbour of $v$). If some edge $e$ has both ends of degree at most $s$, then the result follows from the inductive hypothesis by deleting $e$ (find a partition by induction, and note that replacing $e$ will not cause either of the ends of $e$ to have degree too large). Thus the result follows from 2.3. This proves 2.4 and hence 1.2.
References

[1] P. A. Catlin (1979), “Hajós’s graph-colouring conjecture: variations and counterexamples”, J. Combinatorial Theory, Ser. B, 26 (1979), 268–274.

[2] L. Cowen, W. Goddard and C. E. Jesurum, “Defective coloring revisited”, J. Graph Theory, 24 (1997), 205–219.

[3] H. Hadwiger, “Über eine Klassifikation der Streckencomplexe”, Vierteljahrsschrift der naturforschenden Gesellschaft in Zurich, 88 (1943), 133–142.

[4] G. Hajós, “Über eine Konstruktion nicht n-färbbarer Graphen”, Wiss. Z. Martin-Luther Univ., Halle-Wittenberg Math.-Naturwiss. Reihe 10 (1961), 116–117.

[5] K. Kawarabayashi and B. Mohar, “A relaxed Hadwiger’s conjecture for list colorings”, J. Combinatorial Theory, Ser. B, 97 (2007), 647–651.

[6] A. V. Kostochka, “The minimum Hadwiger number for graphs with a given mean degree of vertices”, Metody Diskret. Anal., 38 (1982), 37–58.

[7] A. V. Kostochka, “Lower bound for the Hadwiger number for graphs by their average degree”, Combinatorica, 4 (1984), 307–316.

[8] N. Robertson, P. D. Seymour, and R. Thomas, “Hadwiger’s conjecture for $K_6$-free graphs”, Combinatorica, 13 (1993), 279–361.

[9] A. Thomason, “An extremal function for contractions of graphs”, Math. Proc. Cambridge Phil. Soc. 95 (1984), 261–265.

[10] A. Thomason, “The extremal function for complete minors”, J. Combinatorial Theory, Ser. B, 81 (2001), 318–338.