A Faithful Representation of the Singular Braid Monoid on Three Strands

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Abstract

We show that a certain linear representation of the singular braid monoid $SB_3$ is faithful. Furthermore we will give a second - group theoretically motivated - solution to the word problem in $SB_3$.

1 Preliminaries

While knots are important for our daily life - ties, shoelaces etc. - singular knots are not. However, since the theory of Vassiliev invariants started (see e.g. [BL93]) mathematicians became more and more interested in singular knotted objects, i.e. objects having a finite number of transversal self intersections. One of these objects are singular braids which form the singular braid monoid $SB_n$, i.e. the monoid generated by the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group $B_n$ plus the additional singular generators $\tau_1, \ldots, \tau_{n-1}$.

A presentation for the singular braid monoid in terms of these generators, build up from the usual choice of a presentation for the braid group is not hard to deduce (cf. [Bir93]).

The theory of Vassiliev invariants suggests a homomorphism from $SB_n$ into the integral group ring $\mathbb{Z}B_n$ of the braid group. This homomorphism is given by mapping a singular generator $\tau_i$ to $\sigma_i - \sigma_i^{-1}$ and $\sigma_i$ to itself (cf. [Bir93]). A famous conjecture of Joan Birman ([Bir93], [FRZ96]) asserts that this homomorphism is injective.

As it appeared to be very useful for knot theorist to have a complete understanding of the braid group $B_3$ (see e.g. [BM93]) the purpose of this paper is to give some analogous results for the braid monoid $SB_3$.

By results of Fenn et al. [FKR96] the monoid $SB_n$ embeds in a group $SG_n$. Using this result we will prove that an extension to $SB_n$ of the Burau representation for $B_n$ that was defined in [Gem97a] is faithful for $n = 3$. Since the Burau representation of $B_n$ is known to be unfaithful

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for \( n \geq 6 \) [LP93] this result cannot carry over at least for \( n \geq 6 \). So we will give a second solution to the word problem for \( SB_3 \) by using the group theoretical structure of \( SG_3 \).

It is worth mentioning that the faithfulness of the singular Burau representation yields an algorithm for the solution of the word problem for \( SB_3 \) that has a time complexity which is linear in the length of the word.

This paper was written during a visit of the first author at the Columbia University. He would like to thank Columbia for the warm hospitality and especially Joan Birman for many discussions and fruitful comments on an earlier version of this text.

The second author would like to thank Wilhelm Singhof for many useful suggestions and remarks.

Furthermore both authors are very grateful to E. Mail for her invaluable help.

### 2 A new presentation for the singular braid monoid

We recall the well-known presentation for the monoid of singular braids on \( n \) strands:

**Proposition 2.1 (Baez [Bae92], Birman [Bir93])** The monoid \( SB_n \) is generated by the elements

\[
\sigma_i^{\pm 1}, \ i = 1, \ldots, n - 1, \ \tau_i, \ i = 1, \ldots, n - 1
\]

(see Figure 1) satisfying the following relations:

\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= 1 \text{ for all } i \quad (1) \\
\sigma_i \sigma_{i+1} &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{and} \quad \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } j > i + 1 \quad (2) \\
\tau_{i+1} \sigma_i \sigma_{i+1} &= \sigma_i \sigma_{i+1} \tau_i \quad (3) \\
\sigma_{i+1} \sigma_i \tau_{i+1} &= \tau_i \sigma_i \sigma_{i+1} \quad (4) \\
\sigma_i \tau_i &= \tau_i \sigma_i \quad (5) \\
\sigma_i \tau_j &= \tau_j \sigma_i \quad \text{for } j > i + 1 \quad (6) \\
\sigma_j \tau_i &= \tau_i \sigma_j \quad \text{for } j > i + 1 \quad (7) \\
\tau_i \tau_j &= \tau_j \tau_i \quad \text{for } j > i + 1 \quad (8)
\end{align*}
\]

We use this proposition to derive a new presentation for the singular braid monoid which is more suitable for our purposes:

\begin{figure}[h]
\centering
\begin{tabular}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{tabular}
\begin{tabular}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{tabular}
\begin{tabular}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}
\end{tabular}
\end{figure}

Figure 1: \( \sigma_i, \sigma_i^{-1} \) and \( \tau_i \)

**Proposition 2.2** The monoid \( SB_n \) is generated by the elements

\[
\sigma_i^{\pm 1}, \ i = 1, \ldots, n - 1, \text{ and } \tau
\]
satisfying the following relations:

\[
\begin{align*}
\sigma_i \sigma_i^{-1} &= 1 \text{ for all } i \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{and} \quad \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } j > i+1 \\
(\sigma_2 \sigma_1)^3 \tau &= \tau (\sigma_2 \sigma_1)^3 \\
\tau \sigma_i &= \tau \sigma_i \quad \text{for } i \neq 2 \\
\sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \tau &= \tau \sigma_2 \sigma_3 \sigma_1 \sigma_2 \tau \sigma_2 \sigma_3 \sigma_1 \sigma_2 \quad \text{for } n > 3.
\end{align*}
\]

**Proof** We will see that the new relations are included in the old ones. So we will concentrate ourselves to show that all other relations can be derived from our relations.

First we note that we have

\[
\tau_{i+1} = \sigma_i \sigma_{i+1} \tau_i \sigma_{i+1}^{-1} \sigma_i^{-1}.
\]

1. **Relations of the form** \( \sigma_i \tau_i = \tau_i \sigma_i \)

   We will show that these relations can be deduced from \( \sigma_1 \tau_1 = \tau_1 \sigma_1 \), Relation (14) and the relations in the braid group (9) - (10). We will proceed by induction on \( i \):

\[
\begin{align*}
\sigma_i \tau_i &= \sigma_i \sigma_{i-1} \tau_{i-1} \sigma_{i-1}^{-1} \sigma_i^{-1} \\
&= \sigma_{i-1} \sigma_i \sigma_{i-1} \tau_{i-1} \sigma_{i-1}^{-1} \sigma_i^{-1} \\
&= \sigma_{i-1} \sigma_i \tau_{i-1} \sigma_i^{-1} \sigma_i^{-1} \\
&= \sigma_{i-1} \sigma_i \tau_{i-1} \sigma_i^{-1} \sigma_i^{-1} \\
&= \tau_i \sigma_i
\end{align*}
\]

2. **Relations of the form** \( \sigma_i \tau_j = \tau_j \sigma_i, \ |i - j| > 1 \)

   We will show that all relations of this form can be derived from the relations \( \sigma_i \tau_1 = \tau_1 \sigma_i, \ i \neq 2 \) as well as (14) and the braid group relations (9) - (10).

   **Case 1:** If \( 1 < j < i - 1 \) then we have

\[
\begin{align*}
\sigma_i \tau_j &= \sigma_i \sigma_{j-1} \tau_{j-1} \sigma_{j-1}^{-1} \sigma_j^{-1} \sigma_{j-1}^{-1} \\
&= \sigma_{j-1} \sigma_j \tau_{j-1} \sigma_j^{-1} \sigma_{j-1}^{-1} \sigma_i \\
&= \tau_j \sigma_i
\end{align*}
\]

where we use (14), (10) and induction on \( j \).

   **Case 2:** In the braid group with \( w := \sigma_{i+1} \sigma_i + 2 \sigma_i \sigma_{i+1} \) it holds:

\[
\sigma_i w = w \sigma_{i+2}.
\]

Thus, for \( j = i + 2 \):

\[
\begin{align*}
\sigma_i \tau_{i+2} &= \sigma_i w \tau_i w^{-1} \\
&= w \sigma_{i+2} \tau_i w^{-1} \\
&= w \tau_i \sigma_{i+2} w^{-1} \\
&= w \tau_i w^{-1} \sigma_i \\
&= \tau_{i+2} \sigma_i
\end{align*}
\]

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where we use (14), (15) and Case 1.

**Case 3:** If \( j \geq i + 3 \) then we can proceed as in Case 1 to show that

\[
\sigma_i \tau_j = \tau_j \sigma_i, \quad i > 1 + j
\]

follows from our relations and Case 2.

3. **Relations of the form** \( \tau_i \tau_j = \tau_j \tau_i, \, i < j, \, j - i > 1 \)

Our aim is to reduce these relations to \( \tau_1 \tau_3 = \tau_3 \tau_1 \) with the help of (14) and the relation \( \sigma_i \tau_j = \tau_j \sigma_i \), for \(|j - i| \neq 1\). We use induction on the pair \((i, j)\) that is ordered lexicographically.

If \( i > 1 \) then

\[
\tau_i \tau_j = \sigma_{i-1} \sigma_i \tau_{i-1} \sigma_i^{-1} \sigma_{i-1}^{-1} \tau_j = \tau_j \sigma_{i-1} \sigma_i \tau_{i-1} \sigma_i^{-1} \sigma_{i-1}^{-1} = \tau_j \tau_i.
\]

In the same way one can deduce \( \tau_1 \tau_j = \tau_j \tau_1 \) for \( j > 3 \).

4. **Relations of the form** \( \sigma_{i+1} \sigma_i \tau_{i+1} = \tau_i \sigma_{i+1} \sigma_i \)

We will show that these relations follow from Relation (14) together with the relations in the braid group and the initial relation:

\[
\sigma_2 \sigma_1 \tau_2 = \tau_1 \sigma_2 \sigma_1 \quad (16)
\]

We will need the following two relations in the braid group which can be easily tested:

\[
\sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} = \sigma_{i-1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i-1}^{-1} \sigma_i, \quad (17)
\]

and

\[
\sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} = \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_i^{-1} \quad (18)
\]

Now we are left with the braid group relations, Relation (12) as well as Relation (14) and

\[
\tau_2 = \sigma_1^{-1} \sigma_2^{-1} \tau_1 \sigma_2 \sigma_1 \quad (19)
\]

\[
\tau_1 \tau_3 = \tau_3 \tau_1. \quad (20)
\]

Relation (20) is equivalent to Relation (13).
Relation (19) is equivalent to:

\[ \sigma_2 \sigma_1 \sigma_2 \tau_1 = \tau_1 \sigma_2 \sigma_1 \sigma_2 \]

\[ \iff \sigma_2 \sigma_1 \sigma_2 \sigma_1^{-2} \tau_1 = \tau_1 \sigma_2 \sigma_1 \sigma_2 \]

\[ \iff (\sigma_2 \sigma_1)^3 \tau_1 = \tau_1 (\sigma_2 \sigma_1)^3. \]

Finally, we can skip the Relations (14) because the \( \tau_j, j > 1 \), only occur in these relations and we set \( \tau := \tau_1. \)

Remark 2.3 As the braid group \( B_n \) admits a presentation with two generators \( \sigma_1 \) and \( A := \sigma_1 \cdots \sigma_{n-1} \) for every \( n \) (cf. Artin’s initial paper [Art25]) we can rewrite the presentation for the singular braid monoid in terms of three generators \( \sigma_1, A \) and \( \tau_1 \). We will omit the details here.

Corollary 2.4 The monoid \( SB_3 \) is generated by the elements \( \sigma_1^\pm 1, \sigma_2^\pm 1, \tau_1 \) satisfying the following relations:

1. \( \sigma_1 \sigma_1^{-1} = \sigma_2 \sigma_2^{-1} = 1 \).
2. \( \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \)
3. \( \tau_1 (\sigma_2 \sigma_1)^3 = (\sigma_2 \sigma_1)^3 \tau_1 \)
4. \( \sigma_1 \tau_1 = \tau_1 \sigma_1 \)

The following theorem of Fenn, Keyman and Rourke [FKR96] will make our arguments on \( SB_n \) much easier:

Theorem 2.5 Let \( SG_n \) be the group given by the monoid presentation of \( SB_n \) considered as a group presentation. Then the natural homomorphism of \( SB_n \) into \( SG_n \) is an embedding.

3 A faithful representation of \( SB_3 \)

One can find a representation of the singular braid groups which is an extension of the famous Burau representation of the braid groups itself (cf. [Gem97a]). For \( SB_3 \) this representation looks like:

Proposition 3.1 The map \( \beta_s \) given by

\[ \sigma_1 \mapsto \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}, \quad \tau_1 \mapsto \begin{pmatrix} 1 - y - ty & y \\ 0 & 1 \end{pmatrix} \]

yields a representation of the singular braid monoid into a matrix ring:

\[ \beta_s : B_3 \to M_2(\mathbb{Z}[t, t^{-1}, y]). \]

We will show that this representation is faithful. For this purpose we need the following easy consequence (cf. [Pin80]) of a theorem of P.M. Cohn [Coh68]:

\begin{itemize}
  \item The matrix ring \( M_n(\mathbb{Z}[t, t^{-1}, y]) \) is faithful for all \( n \).
\end{itemize}

We will show that this representation is faithful. For this purpose we need the following easy consequence (cf. [Pin80]) of a theorem of P.M. Cohn [Coh68]:
Theorem 3.2 Let $d \neq 1, 2, 3, 7, 11$ be square-free, i.e., $d$ is not divisible by the square of an integer, and $\omega := \sqrt{-d}$ if $d \equiv 1$ or 2 modulo 4 or $\omega := (1 + \sqrt{-d})/2$ for $d \equiv 3$ modulo 4 and let $O_d = \mathbb{Z} + \omega \mathbb{Z}$ be the imaginary quadratic integers in $\mathbb{Q}[\omega]$. Furthermore, let $A, B$ and $C$ be the following elements in $\text{PSL}_2(O_d)$:

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$ 

The subgroup $\text{PE}_2(O_d)$ of $\text{PSL}_2(O_d)$ generated by all matrices of the forms

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

for $x$ and $y$ in $O_d$, has the presentation:

$$\text{PE}_2(O_d) = \langle A, B, C | B^2 = (AB)^3 = [A, C] = 1 \rangle.$$ 

With $\Sigma_1 := A, T_1 := C$ and

$$\Sigma_2 := (ABA)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

we get - after some easy transformations - the presentation:

$$\text{PE}_2(O_d) = \langle \Sigma_1, \Sigma_2, T_1 | (\Sigma_1 \Sigma_2 \Sigma_1)^2 = (\Sigma_1 \Sigma_2)^3 = 1, \Sigma_1 T_1 = T_1 \Sigma_1 \rangle.$$ 

Now we have all the tools to show that

Theorem 3.3 The singular Burau representation $\beta_s : \text{SB}_3 \to M_2(\mathbb{Z}[t, t^{-1}, y])$ is faithful.

Proof The arguments are essentially the same as in the proof of Magnus and Peluso (cf. [Bir74]) of the faithfulness of the Burau representation for 3-string braids. While in this proof the well-known presentation of the group $\text{PSL}_2(\mathbb{Z})$ is used we will make use of the presentation of $\text{PE}_2(\mathbb{Z}[\omega])$ of Theorem 3.2 for a suitable ring of integers $\mathbb{Z}[\omega]$. For reasons of convenience we choose $d = 5$, so $\omega := \sqrt{-5}$.

We will denote the extension of $\beta_s$ to $SG_3$ also by $\beta_s$. So

$$\beta_s : SG_3 \to M_2(\mathbb{Z}[t, t^{-1}, y, 1/(1 - y - ty)]).$$

The image of $\beta_s$ in this matrix group is - of course - a quotient of $SG_3$. Furthermore by setting $t := -1$ and $y := \omega$ it naturally maps onto $\text{PE}_2(\mathbb{Z}[\omega])$.

Thus we have a homomorphism from $SG_3$ onto $\text{PE}_2(\mathbb{Z}[\omega])$ given by $\sigma_1 \mapsto \Sigma_1, \sigma_2 \mapsto \Sigma_2$ and $\tau_1 \mapsto T_1$ where $\Sigma_1, \Sigma_2$ and $T_1$ are as in Theorem 3.2. Moreover by comparing the presentations of $SG_3$ and $\text{PE}_2(\mathbb{Z}[\omega])$ we see that the kernel of this homomorphism is the normal closure of the element $(\sigma_1 \sigma_2 \sigma_1)^2$, which is - as is easy to see - a central element in $SG_3$. Hence, the kernel is cyclic.

Thus, the image of $SG_3$ under $\beta_s$ is isomorphic to $SG_3$ modulo a power of $\beta_s((\sigma_1 \sigma_2 \sigma_1)^2)$. Since $(\sigma_1 \sigma_2 \sigma_1)^2$ is mapped by $\beta_s$ to the element

$$\begin{pmatrix} t^3 & 0 \\ 0 & t^3 \end{pmatrix}$$

of infinite order, we see that $SG_3$ must be isomorphic to its image under $\beta_s$.

Because of the Embedding Theorem [2.3] the theorem follows.
3.1 The Birman-Conjecture for $SB_3$

Let $\eta : SB_n \to \mathbb{Z}B_n$ be the Birman homomorphism which maps $\tau_1 \mapsto \sigma_1 - \sigma_1^{-1}$ and $\sigma_j$ to itself. In [Gem97b] the following result is proved:

**Theorem 3.4** Let $b$ and $b'$ be two braids in $SB_n$ with $\eta(b) = \eta(b')$. Then we have $\beta_s(b) = \beta_s(b')$.

Because the singular Burau representation is faithful for $n = 3$ this means that the Birman homomorphism is also faithful, i.e. the Birman conjecture is valid for $n = 3$. We learned that this result was already obtained by Antal Járai [Jr.]. It might be interesting, however, for the reader to see how the linearity of $SB_3$ and the injectivity of $\eta$ relate. Therefore we will give a sketchy and informal proof of Theorem 3.4. For full details the reader is referred to [Gem97b].

For notational reasons we will consider the special case $SB_3$. However, the argument holds also for higher $n$. Since the determinant of the Burau matrix of a given braid with $m$ singularities equals $t^k \cdot (1 - y - ty)^m$ for some $k \in \mathbb{Z}$, two braids with a different number of singularities cannot map to the same matrix under $\beta_s$. The same holds for the Birman homomorphism. Therefore we may restrict ourselves to braids with a fixed number $m$ of singularities. The set of all braids with exactly $m$ singularities will be denoted by $SB_3^{(m)}$ in the sequel.

The main idea of the proof is to imitate the Birman homomorphism on the level of Burau matrices. If we deal with braids having exactly one singularity, this can be done easily: Substituting $y$ by 1 corresponds to a right-handed resolution of the singularity, substituting $y$ by $x^{-1}$ corresponds to a left-handed resolution of the singularity. This induces a well defined homomorphism from the matrix ring $M_2(\mathbb{Z}[t, t^{-1}, y])$ into the group ring $\mathbb{Z}[M_2(\mathbb{Z}[t, t^{-1}])]$ imitating the Birman homomorphism.

If the number of singularities is greater than one, we cannot proceed in the same way. The two indicated substitutions would correspond to a right-handed (resp. left-handed) resolution of all the singularities. Unfortunately, we also have to consider cases where some singularities are resolved in a right-handed way while others are resolved in a left-handed way.

Therefore we change our point of view slightly: First, we number the singularities of our braid $b$ from 1 to $m$. Afterwards we assign to the $j$-th singularity of the braid the matrix

$$
\begin{pmatrix}
1 - y_j - ty_j & y_j \\
0 & 1
\end{pmatrix}
$$

rather than the usual matrix

$$
\begin{pmatrix}
1 - y - ty & y \\
0 & 1
\end{pmatrix}.
$$

In this way we define a modification of the Burau matrix of $b$. Its entries take values in the polynomial ring $\mathbb{Z}[t, t^{-1}, y_1, \ldots, y_m]$. Of course, the numbering of the singularities was somewhat arbitrary. Therefore we shall regard this modified matrices only up to permutation of the indices of the $y_i$. (To be more precise, we let the symmetric group $\Sigma_m$ act on $M_2(\mathbb{Z}[t, t^{-1}, y_1, \ldots, y_m])$ and consider the orbits. By abuse of notation we shall denote the set of these orbits also with $M_2(\mathbb{Z}[t, t^{-1}, y_1, \ldots, y_m])$.) It is obvious that we obtain the Burau representation of $b$ out of its modified Burau matrix by the projection $p$ which sends all the $y_i$ to $y$.

We have introduced the modified Burau matrix in order to compute the (regular) Burau matrices of all possible resolutions. In fact, define a matrix resolution

$$
r : M_2(\mathbb{Z}[t, t^{-1}, y_1, \ldots, y_m]) \to M_2(\mathbb{Z}[t, t^{-1}])
$$
as a projection where, in addition, any \( y_i \) is mapped either to 1 or to \( t^{-1} \). The index \( \mu(r) \) is defined to be the number of \( y_i \) which are sent to \( t^{-1} \). Clearly, a given resolution is not well defined on our orbits. However, taking formal sums over all possible resolutions gives a well defined map \( \rho \). So, if \( M \) is a modified Burau matrix, then

\[
\rho(M) = \bigoplus_{r} (-1)^{\mu(r)} \cdot r(M).
\]

Note that the sum in the formula is a formal sum in the group ring \( \mathbb{Z}[M_2(\mathbb{Z}[t, t^{-1}])] \).

Easy calculations show that the application \( \rho \) corresponds to the Birman homomorphism \( \eta \) on the level of matrices. In fact, we get the following commutative diagram:

\[
\begin{array}{ccc}
SB_3^{(m)} & \xrightarrow{\beta} & M_2(t, t^{-1}, y) \\
\downarrow{\beta} & & \downarrow{p} \\
\mathbb{Z}[\beta] & \xrightarrow{\rho} & \mathbb{Z}[M_2(t, t^{-1})]
\end{array}
\]

Here \( \tilde{\beta} \) denotes the application which maps a braid to its corresponding modified Burau matrix. With \( \mathbb{Z}[\beta] \) we denote the extension of the usual (regular) Burau homomorphism to the group rings.

We now claim:

**Claim 1** Let \( M, M' \in M_2(x^{\pm 1}, y_1, \ldots, y_m) \) be two elements in the image of \( \tilde{\beta} \) with \( \rho(M) = \rho(M') \). Then we have \( p(M) = p(M') \).

Let us assume that the claim is true. Then the proof of Theorem 3.4 becomes easy diagram chasing:

Let \( b, b' \) be elements of \( SB_3^{(m)} \) and suppose that \( \eta(b) = \eta(b') \). It follows that \( (\mathbb{Z}[\beta] \circ \eta)(b) = (\mathbb{Z}[\beta] \circ \eta)(b') \) and by commutativity of the diagram that \( (\rho \circ \tilde{\beta})(b) = (\rho \circ \tilde{\beta})(b') \). Using Claim 1 we get \( (p \circ \tilde{\beta})(b) = (p \circ \tilde{\beta})(b') \) and - again by commutativity of the diagram - \( \beta(b) = \beta(b') \).

Thus, we only have to show that Claim 1 holds. This is the most technical part of the proof. In fact, we have to figure out in how far the matrices of the formal sum \( \rho(M) \) determine the matrix \( M \). This leads to one system of linear equations for each entry of the matrix, which may be solved after having observed the following two facts:

1. If \( M \) is in the image of \( \tilde{\beta} \), then each \( y_i \) cannot appear in the matrix with powers greater than 1.

2. We may use the determinant of the matrices in our formal sum in order to compute the index of the resolution which has produced them. This fact is important when solving the equations.

With these two observations and some tedious computations, we derive that two matrices \( M \) and \( M' \) are mapped to the same formal sum under \( \rho \) if their entries differ by permutations of the indices of the \( y_i \). Hence, they vanish under the projection \( p \).
4 A second solution to the word problem in $SB_3$

To give a second solution to the word problem for $SB_3$, i.e. the problem whether two elements in $SB_3$ are equivalent, we will need Britton’s Lemma that can be applied to the group $SG_3$.

**Lemma 4.1 (Britton [Bri63])** Let $H = \langle S \mid R \rangle$ be a presentation of the group $H$ with a set of generators $S$ and relations $R$ in these generators.

Furthermore let $G$ be a HNN-extension of $H$ of the following form:

$$G = \langle S, t \mid R, t^{-1}X_it = X_i, i \in I \rangle$$

for some index set $I$, where $X_i$ are words over $S$.

Let $W$ be a word in the generators of $G$ which involves $t$.

If $W = 1$ in $G$ then $W$ contains a subword $t^{-1}Ct$ or $tCt^{-1}$ where $C$ is a word in $S$, and $C$, regarded as an element of the group $H$, belongs to the subgroup $X$ of $H$ generated by the $X_i$.

We will rather solve the word problem for $SG_3$ than for $SB_3$. By Corollary 2.4 $SG_3$ has a presentation as in Britton’s Lemma. So to solve the word problem in $SG_3$ we have to decide whether a given word in the generators of $B_3$ is element of the subgroup $H_3$ generated by the elements with which $\gamma_1$ commutes: $\sigma_1$ and $(\sigma_2\sigma_1)^3$.

This decision problem, called membership problem, would not be hard to solve with the help of the Burau representation of $B_3$. However we promised to give a puristic proof which gives more hope for a generalization to braids and singular braids with more than three strands.

So we will choose the approach of Xu [Xu92] for the word problem for $B_3$ - which was generalized most recently to arbitrary $B_n$ by Birman, Ko and Lee [BKL] - to solve the membership problem for the subgroup $H_3$.

We briefly recall this approach using the notation of Birman, Ko and Lee. The first step is to rewrite the presentation of $B_3$ in terms of the new generators: $a_{21} := \sigma_1, a_{32} := \sigma_2$ and $a_{31} := \sigma_2\sigma_1\sigma_2^{-1}$.

So we get a new presentation

$$B_3 = \langle a_{21}, a_{32}, a_{31} \mid a_{32}a_{21} = a_{31}a_{32} = a_{21}a_{31} \rangle.$$ 

Using the element $\delta := a_{32}a_{21}$ one can show now that every element of $B_3$ can be brought into a unique normal form $\delta^kP$ for some $k$ with $P$ a positive word, i.e. only positive exponents occurs, in the generators $a_{21}, a_{32}$ and $a_{31}$, such that none of the subwords $a_{32}a_{21}$, $a_{31}a_{32}$ or $a_{21}a_{31}$ appear in $P$.

**Lemma 4.2** The membership problem for the subgroup $H_3$ of $B_3$ generated by the elements $\sigma_1$ and $(\sigma_2\sigma_1)^3$ can be solved.

**Proof** First of all we see that $H_3$ is abelian and $(\sigma_2\sigma_1)^3 = \delta^3$. Therefore if we want to bring a word into the normal form we only have to look for $\sigma_1^k$ for $k \in \mathbb{Z}$. If $k$ is not negative, then the normal form for an element $(\sigma_2\sigma_1)^3\sigma_1^k$ is simply $\delta^3a_{21}^k$.

If $k \leq 0$ then the following identities are easy to see:

$$\begin{align*}
(\sigma_2\sigma_1)^3\sigma_1^k &= \delta^3 + 3k(a_{31}a_{21}a_{32})^{-k} \\
(\sigma_2\sigma_1)^3\sigma_1^{3k-1} &= \delta^3 + 3k-1a_{32}(a_{31}a_{21}a_{32})^{-k} \\
(\sigma_2\sigma_1)^3\sigma_1^{3k-2} &= \delta^3 + 3k-2a_{21}a_{32}(a_{31}a_{21}a_{32})^{-k}.
\end{align*}$$
Therefore for every word \( w \) in the braid group \( B_3 \) we can bring it into its unique normal form and compare this form with the normal forms for the elements in \( H_3 \). Hence, the membership problem for \( H_3 \) is solvable. □

Thus we have proved:

**Theorem 4.3** Given two words \( w_1 = b_1 \tau_1 b_2 \tau_1 \cdots \tau_1 b_m \) and \( w_2 = c_1 \tau_1 c_2 \tau_1 \cdots \tau_1 c_l \) in the generators \( \sigma_1, \sigma_2 \) and \( \tau_1 \) in \( SB_3 \), where the \( b_j \) and \( c_i \) are words in \( B_3 \).

Then \( w_1 \) and \( w_2 \) are equal in \( SB_3 \) if and only if \( b_m c_l^{-1} \) is in the subgroup \( H_3 \) of \( B_3 \) that is generated by \( \sigma_1 \) and \( (\sigma_2 \sigma_1)^3 \) and \( b_1 \tau_1 b_2 \tau_1 \cdots b_{m-1} b_m \) and \( c_1 \tau_1 c_2 \tau_1 \cdots c_{l-1} c_l \) are equal in \( SB_3 \).

This gives a solution to the word problem in \( SB_3 \) because the membership problem for \( H \) is solvable by Lemma 4.2 and the word problem in \( B_3 \) is solvable - as mentioned above - as well.

### 4.1 An algebraic proof of the embedding theorem for \( SB_3 \) into \( SG_3 \)

Actually - as a Corollary of our approach - one can get a purely algebraic proof of the Embedding Theorem 2.5 of [FKR96] for the special case \( n = 3 \):

**Corollary 4.4** \( SB_3 \) embeds into \( SG_3 \).

**Proof** We have to show that if two elements \( w_1 \) and \( w_2 \) in \( SB_3 \) are different then their images in \( SG_3 \) are different. By the HNN-structure of \( SG_3 \) the subgroup \( B_3 \) embeds in it. Furthermore two elements \( w_1 \) and \( w_2 \) that map to the same element in \( SG_3 \) must have the same number of singular points.

Now let \( w_1 = b_1 \tau_1 b_2 \tau_1 \cdots \tau_1 b_m \) and \( w_2 = c_1 \tau_1 c_2 \tau_1 \cdots \tau_1 c_l \), \( b_j \) and \( c_i \in B_3 \), be two different elements of \( SB_3 \) that map to the same element in \( SG_3 \), by slight abuse of notation also denoted by the same word. We assume that \( w_1 \) and \( w_2 \) are minimal examples with respect to the number of singular points. Since \( w_1 \beta \neq w_2 \beta \iff w_1 \neq w_2 \) for an element \( \beta \in B_3 \) we further may assume that \( c_m = 1 \). Then by Britton’s Lemma \( b_m \) must lie in the subgroup \( H_3 \) defined above. Since all the elements of \( H_3 \) commute with \( \tau_1 \) we have \( \tau_1 b_m = b_m \tau_1 \) both in \( SB_3 \) and \( SG_3 \).

So \( w_1 \) is equal within \( SB_3 \) to \( w_1 = b_1 \tau_1 b_2 \tau_1 \cdots b_{m-1} b_m \tau_1 \). Now consider the two word \( w_1' = b_1 \tau_1 b_2 \tau_1 \cdots b_{m-1} b_m \) and \( w_2' = c_1 \tau_1 c_2 \tau_1 \cdots c_{l-1} c_l \) in \( SB_3 \). These two words represent different elements in \( SB_3 \) - otherwise we would have \( w_1 = w_1' \tau_1 = w_2' \tau_1 = w_2 \) - but map to the same element in \( SG_3 \). This contradicts our assumption. □

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