Introduction. The standard cosmological model, the so-called ΛCDM scenario, provides an excellent fit to the high-precision astrophysical and cosmological observations performed over the last few decades, in particular the temperature and polarisation anisotropies of the Cosmic Microwave Background (CMB), the Large-Scale Structure (LSS) of the universe, the relative abundance of light elements and the expansion dynamics of the universe. Its three main ingredients are: (1) the use of General Relativity and the Cosmological Principle, (2) the assumption that the universe is made of baryonic matter, dark matter, radiation and some sort of dark energy, with initial abundances that are among the free parameters of the model and (3) initial density fluctuations with quasi scale-invariant, Gaussian statistics.

However, as expected for any simple model aiming to explain all different aspects of cosmic history in a unique framework, hints for a few cracks start to emerge at different stages, in the form of moderate statistical tensions in parameter inference, e.g. the local expansion rate [1, 2], or via the existence of “extreme” objects or outliers, which are more frequently observed than what ΛCDM predicts. Those may be associated with either extremely low value of the density field (such as the Eridanus supervoid [3, 4], which seems to have a direct connection with the CMB cold spot [5]), or with extremely large values of the density field (such as massive galaxy clusters like El Gordo [6] – see however Ref. [7] for a recent smaller estimate of its mass – and the presence of galaxies and Quasi-Stellar Objects at extremely high redshifts, where according to standard ΛCDM there should not be any [8, 9]). In addition to the early structure formation issues, there are late time miss-matches at small scales such as the substructure problems [10], the too-big-to-fail and the core-cusp problems [11], which could be alleviated by incorporating baryonic physics [12].

While most attempts to reconcile those potential issues focus on relaxing either the first or the second assumption mentioned above (i.e. modifying the laws of gravity, or invoking the existence of an additional component in the universe content), a natural strategy to accommodate the existence of extreme objects within the ΛCDM paradigm would be to question the third assumption, namely that the statistics of the primordial density fluctuations are Gaussian. The reason is twofold: from the experimental perspective, as mentioned above, there are more extreme objects than what Gaussian tails suggest, pointing towards the existence of heavier tails; from the theoretical perspective, the mechanisms usually invoked to produce primordial cosmological perturbations anyway lead to non-Gaussian tails.

In the early universe indeed, vacuum quantum fluctuations are amplified by gravitational instability and stretched to large distances, giving rise to classical fluctuations in the density field, that later collapse into cosmological structures. Note that this mechanism is mostly studied in the context of inflation, but it also operates in most of its alternatives [13]. When implemented at leading order in cosmological perturbation theory, it gives rise to Gaussian statistics for the perturbations, which is in good agreement with astrophysical observations, such as measurements of the CMB that tightly constrain the amount of non-Gaussianities [14]. However, those bounds
only apply to large scales and leave small scales mostly unconstrained at their primordial stage. Moreover, even at large scales, they restrict the statistics of the most likely fluctuations only, i.e. they only allow one to reconstruct the neighborhood of the maximum of the underlying distribution functions, and say little about their tails.

Nonetheless, beyond the linear order, those tails are expected to be non-Gaussian. The difficulty when trying to characterise the statistics of those tails is that they require non-perturbative techniques. Perturbative approaches, such as calculations of the bi- or tri-spectrum at higher order in cosmological perturbation theory, and the use of \( f_{\text{NL}} \)-like parametrisations, are tailored to describe the small deviations from Gaussian statistics encountered in the vicinity of the maximum of the primordial density function, and cannot account for their tail.

Quantum diffusion and non-Gaussian tails. Recently, non-perturbative techniques have been developed to study how quantum diffusion, the presence of which is inevitable in scenarios where cosmological perturbations have a quantum origin, modifies the expansion dynamics of the universe and thus affects the statistics of density fluctuations. This can be done by combining three approaches to describe the dynamics of super-Hubble degrees of freedom. First, the separate-universe picture [15–18] (the validity of which has recently been shown to extend beyond slow-roll [19, 20]), according to which spatial gradients can be neglected on super-Hubble scales, and each spatial point evolves independently along the dynamics of an unperturbed universe. Second, stochastic inflation [21, 22], which allows one to describe quantum fluctuations as stochastic noise acting on the classical, background evolution of each of these separate universes. Third, the \( \delta N \) formalism [18, 23–25], which states that, in each of these separate universes, the local fluctuation in the amount of expansion realised between an initial flat hypersurface and a final hypersurface of uniform energy density is nothing but the curvature perturbation. This gives rise to the so-called stochastic-\( \delta N \) formalism [26–31], which provides a non-perturbative scheme to compute the statistics of curvature perturbations on super-Hubble scales: using stochastic calculus, the statistics of first-passage times through the end-of-inflation surface can be computed in stochastic inflation, and interpreted as the primordial distribution of curvature perturbations. These methods were recently extended to the calculation of the statistics of the density contrast and the compaction function in Ref. [32].

While these techniques recover quasi-Gaussian distributions close to their maximum, with \( f_{\text{NL}} \)-type corrections, they also reveal the existence of systematic exponential tails [32–39], which strongly deviate from the Gaussian profile. More precisely, it is shown that the distribution function of the first-passage time \( \mathcal{N} \) can be expanded as

\[
P(\mathcal{N}) = \sum_{n \geq 0} a_n(\Phi)e^{-\Lambda_n \mathcal{N}},
\]

in which \( \Lambda_n \) are the eigenvalues of the adjoint Fokker-Planck operator associated with the stochastic problem under consideration, and \( a_n(\Phi) \) are coefficients that depend on the initial configuration in field space (here denoted as \( \Phi \)). Far on the tail, the smallest eigen-value dominates, \( P(\mathcal{N}) \propto e^{-\Lambda_0 \mathcal{N}} \), which implies that large perturbations are much more likely than what a Gaussian behaviour, \( P_G \propto e^{-\mathcal{A} \mathcal{N}^2} \), would suggest. In practice, these exponential tails are more important in models where quantum diffusion dominates at some stages of the inflationary dynamics (leading to smaller values of \( \Lambda_0 \), hence heavier tails). Depending on the time at which this happens, they affect structures at different scales. If the fluctuations are large enough, they may even collapse into black holes upon horizon re-entry after inflation. This is why non-Gaussian tails have been mostly studied in the context of primordial-black-hole production (see e.g. Refs. [32, 33, 36, 38, 40–45]).

Nonetheless, as we argue here, these heavy tails may also play a key role in the formation of the LSS and point towards potential solutions to some of the problems of \( \Lambda \text{CDM} \). Let us stress that, while the parameters \( \Lambda_n \) and \( \Lambda_0 \) depend on the details of the model under consideration, the existence of these exponential tails is ubiquitous and arises in any model where quantum diffusion is at play. In this sense, they are already embedded in the \( \Lambda \text{CDM} \) scenario. Therefore, our approach does not rely on extending the standard cosmological model to solve the above-mentioned issues: our goal is rather to point out that it may already contain the ingredients needed to explain those “anomalous” observations, provided we carefully compute the primordial statistics beyond the perturbative level.

It is also worth mentioning that heavy tails in the form of lognormal distributions are already known to develop on sub-Hubble scales after inflation, due to gravitational collapse [46–48]. However, the effect we are considering here is different: it leads to primordial heavy tails, which are present even before Hubble re-entry.

Exponential tails in the primordial statistics of perturbations. The details of the stochastic distribution associated with primordial perturbations depend on the specifics of the inflationary model that generates those perturbations (the number and the nature of the fields, their potential, their kinetic coupling etc.). In order to describe the amplitude of fluctuations coarsely-grained at a certain scale, one has to convolve the first-passage time distributions against backward distributions of the field value [32, 37] (since, in the stochastic picture, there is no longer a one-to-one correspondence between a scale \( k \) and the field configuration \( \Phi \) at which \( k \) crosses out the Hubble radius). Moreover, one must account for the non-linear mapping between the curvature perturbation and the density contrast [49], which further modifies distribution functions and can also introduce heavy tails [50, 51]. In this work, we do not aim at deriving predictions for specific models, but rather wish to explore generic consequences arising from the presence of heavy tails. This is why, in practice, we consider two
normalized templates for the distribution function of the density contrast in comoving threading \( \delta \),

\[
P_2(\delta_k) = -\frac{\pi}{2\mu^2} \vartheta_2 \left( \frac{\pi\alpha_k}{2}, e^{-\frac{\pi^2}{\alpha_k^2} \delta_k} \right),
\]

\[
P_4(\delta_k) = \frac{\pi}{2\mu^2 \alpha_k} \vartheta_4 \left( \frac{\pi\alpha_k}{2}, e^{-\frac{\pi^2}{\alpha_k^2} \delta_k} \right).
\]

In these expressions, \( \delta_k \) denotes the Fourier mode of the density contrast, which is related to the positive variable \( D_k \) through the relation \( \delta_k = D_k - \langle D_k \rangle \) where the mean value is taken with respect to the distribution function in question. These distributions depend on two parameters, \( \alpha_k \) and \( \mu \), the latter being scale independent to reflect the fact that the eigenvalues \( \Lambda_n \) mentioned above do not depend on the field configuration, hence on the scale \([34, 35]\). Finally, \( \vartheta_2 \) and \( \vartheta_4 \) are the derivatives of the elliptic theta functions of the second and fourth kind respectively \([52]\). In what follows, they are refereed to as the “elliptic 2” and “elliptic 4” templates respectively. Such functions are often found in toy models of quantum diffusion \([33, 34]\).

We stress that our goal is not to constrain these two toy models in particular, but rather to test the robustness of our results against changing the precise way by which the exponential tail is realised. This will lead us to draw conclusions that seem independent of those details, but one should bear in mind that if a particular model were to be tested, a more careful derivation of the distribution function of the density contrast would be required, along the lines mentioned above.

The two distributions are displayed in Fig. 1 as a function of \( \delta/\sigma \) where hereafter \( \sigma \) denotes the standard deviation of the distribution under consideration, and where they are compared with a Gaussian distribution, a local \( f_{NL} \) distribution and a lognormal distribution. The free parameters of those distributions are set such that they all share the same value of \( \sigma \) around the maximum and are given by \( \alpha = 0.5, \mu = \pi \) and \( \sigma = \sqrt{2} \alpha^2 \), with the same \( \alpha \) and \( \mu \) for both elliptic functions.

### Implications for the Large-Scale Structure

The simplest statistics to be extracted from the primordial density fluctuations is the one-point function, i.e. the number of collapsed objects. Following the Press-Schechter formalism \([53]\), this is given by the probability that the density is above a given threshold value \( \delta_c \):

\[
\beta = P(\delta > \delta_c) = 2 \int_{\delta_c}^{\infty} P(\delta) d\delta. \tag{4}
\]

The threshold of collapse depends on the time of re-entry of the fluctuations and has been extensively explored in the literature \([54-57]\). For our purposes it will be enough to fix it to \( \delta_c = 1.68 \), as predicted by linear theory of spherical collapse \([58]\). From Fig. 1, it is clear that as \( \nu = \delta_c/\sigma \) increases, the number of collapsed objects is larger in heavy-tailed models than in the Gaussian case.

More precisely, let us study how structures distribute across different masses. This can be achieved with the

\[\text{Note that the local } f_{\text{NL}} \text{ in Eq. (2) is assumed to be constant, and thus cannot reproduce the scale-dependence of the amplitude of the tails that comes from the fact that quantum diffusion has a different effect at different points of the inflationary evolution. Other formulations of primordial non-Gaussianity take into account the “scale-dependence” of } f_{\text{NL}} \text{ through calculations of the full bispectrum, but they would still be perturbative.}\]
Halo Mass Function (HMF), which is defined from the mass fraction \( \beta \) as

\[
\frac{dn}{d\ln M} = \frac{\rho_m}{\rho_c} \frac{d\beta}{d\ln M} = \frac{\rho_m}{M} \frac{d\ln \sigma^{-1}}{d\ln M} \nu \beta'(\nu),
\]

where \( M \) is the mass of the halo, \( \rho_m \) is the energy density of matter and a prime denotes derivation with respect to \( \nu \). Previous works in the literature have proposed to test \( f_{\text{NL}} \) with the HMF, see e.g. \[59–63\]. Here we extend those results by exploring initial density perturbations with non-Gaussian tails.

Since present observations show a good agreement with the Gaussian hypothesis at galactic scales, as mentioned above we tune the free parameters of all considered distributions such that they peak at the same value and share the same standard deviation. As a consequence, the only difference in Eq. (5) comes from the term \( \beta'(\nu) \). In a Gaussian distribution, one has \( \beta'_G(\nu) = -2e^{-\nu^2/2}/\sqrt{\pi} \), and similar expressions can be obtained for the other distributions.

Let us now study the redshift evolution of the number of halos. We can describe the HMF (5) as a function of redshift by writing \( \rho_m(a) = \Omega_m(a) \rho_c \) with \( \Omega_m(a) = \Omega_m/\alpha^3 \) and \( \sigma(a) = \sigma(1) D(a) \), with the growth function \( D(a) = \delta(a)/\delta(1) \) given by [64]

\[
D(a) \propto \text{2F1} \left[ \begin{array}{c} w-1 \ 3w \ 6w-5 \ \Omega_m(a) -1 \ \\ 2w \ 6w \ \Omega_m(a) \end{array} \right], \quad \alpha,
\]

where \( w \) is the equation-of-state parameter of Dark Energy, which we take to be \( w = -1 \) in the following and \( \alpha \) is the scale factor. By comparing the HMF at different redshifts with the abundance of massive clusters, we can estimate whether e.g. “El Gordo” is a typical cluster or not, at a given redshift.

Our main results are presented in Fig. 2, where we display the HMF for the four distributions under consideration at three redshifts, \( z = 0, 1, 7 \). The bottom panels display the ratio of the HMF with respect to the Gaussian case, with the normalization fixed to the Gaussian at \( M = 10^{11}h^{-1}M_\odot \). In the HMF at present time we observe two main effects from the exponential tails: an increase in the number of clusters (\( M > 10^{13}M_\odot \)) and a decrease in the number of substructures (\( M < 10^9M_\odot \)). Interestingly, the \( f_{\text{NL}} \) reconstruction can partially mimic the increase in clusters, but not the decrease in substructures. This makes the predictions of quantum diffusion falsifiable. Equally important, the predictions of the elliptic HMF can potentially alleviate the shortcomings of ΛCDM. Moreover, we also observe that the redshift evolution of the HMF is a key discriminator of the nature of the primordial perturbations. An elliptic HMF predicts that more massive objects formed earlier, in agreement with the recent detection of massive, high-redshift objects (see e.g. [65] for a recent census of the age of young quasars).

In addition to the number of halos per unit mass, it is
interesting to compute the number of clusters as a function of redshift. This can be probed directly for example with CMB data using the Sunyaev-Zel’dovich (SZ) effect measured by Planck [66], the South Pole Telescope [67] or the Atacama Cosmology Telescope [68], and the optically selected clusters from the Dark Energy Survey [69]. Our results are presented in Fig. 3, focusing on clusters with $M > 10^{15} M_{\odot}$. One can clearly see that, already beyond $z \sim 1$, the number of clusters is much enhanced when initial perturbations have heavy tails. This, again, shows the potential of this method to constrain the very early universe physics.

Finally, it is important to note that in most cases we do not have direct access to the HMF, but rather to the amount of luminous matter. One thus needs to take into account the astrophysical systematics connecting halo mass to luminous mass. Recently, constraints on $f_{NL}$ have been derived using UV galaxy luminosity functions that marginalize over those systematics [70]. A natural extension of this work would thus be to constrain the heavy tails from quantum diffusion with this data.

**Future prospects.** The standard cosmological model, $\Lambda$CDM, relies on the assumption of Gaussian initial conditions. Although CMB observations tightly constrain the amount of non-Gaussianities at large scales, little is known about the primordial fluctuations at smaller scales. Several processes in the early universe could lead to non-Gaussian distributions. Notably, an inevitable exponential tail arises due to quantum diffusion during inflation. In this work we have studied the imprints these heavy tails leave in the number of halos and their mass function. We have found that they enhance the number of heavy clusters and deplete the number of sub-halos, and that this difference with respect to the standard Gaussian initial conditions becomes more important at high redshift, depending on the strength of quantum diffusion. This could be compared with current SZ catalogs (e.g., Fig. 18 in [68]) that did not find clusters of $M > 10^{16} M_{\odot}$ and has a redshift distribution peaking at $z < 1$. However, there are outstanding clusters like El Gordo with $M \sim 3 \cdot 10^{15} M_{\odot}$ at $z = 0.87$ [6], and deep voids like the Eridanus supervoid [5], and many more will soon be discovered with Euclid [71] and the Vera Rubin Observatory [72].

It is interesting to note that the effect of quantum diffusion is similar to having a lognormal initial density distribution. Such lognormal profiles are indeed typically obtained in $N$-body simulations due to non-linear evolution. In fact, to speed up the computation, many codes already start their evolution with a lognormal distribution [73, 74]. Here we find that such behavior has a primordial origin. In other words, non-linear growth occurs at much earlier times, as soon as these non-Gaussian tails are present.

Using the halo abundance to probe primordial universe physics requires further developments on various fronts. From the observational side, we need to understand the systematics behind high-mass supergalactic structures at low and high redshifts, which Euclid [71] and JWST [75] observations may help to alleviate, as well as issues with baryonic physics and Halo Occupation Distributions. From the theoretical side, we need to implement realistic physical models in our pipeline, and go beyond the simple phenomenological prescription adopted here (although this is not expected to alter our qualitative conclusions, it matters for quantitative details). Finally, on the numerical side, it would be necessary to run $N$-body simulations with non-Gaussian initial conditions of the type described above.

Altogether we have demonstrated the impact of quantum diffusion on the large-scale structure of the universe and how the halo mass function and cluster abundances can be used to probe early-universe physics. More importantly, we have shown that, within the standard cosmological model itself, quantum diffusion is inevitable during inflation, and some of the current tensions can be alleviated thanks to the non-Gaussian nature of the tails of primordial perturbations.

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moments, one obtains or from integrating Eq. (3) directly. For the three first expectation value of Eq. (2) taken to some integer power, distribution can be calculated either from evaluating the of both profiles coincide. The first moments of the $f$ parameters, hence the $f$ distribution that coincides. The first moments of the $X$ are 0, the second branch

\[ P_{\text{NL}}(\xi) \simeq \frac{1}{\sqrt{2\pi\Delta}} \exp\left[-\frac{(\sqrt{\Delta}-1)^2}{2\sigma^2}\right], \quad (10) \]

where we have defined $\xi = \delta/\sigma_G$ and $\sigma = 6f_{\text{NL}}\sigma_G/5$. The maximum of this distribution is at $2\sqrt{\Delta_{\text{max}}} = 1 + \sqrt{1 - 4\sigma^2}$, hence

\[ \xi_{\text{max}} = \frac{\Delta_{\text{max}} - 1 - \sigma^2}{2\sigma}. \quad (11) \]

This allows one to fix the $f_{\text{NL}}$ parameter such that the $f_{\text{NL}}$ distribution peaks at a given value of $\xi$. In Fig. 1, we apply this procedure and display several $f_{\text{NL}}$ distributions (corresponding to several values of $\sigma_G$) that share the same maximum location with the elliptic profiles. In order to accommodate the heavy tail, one can see that a large value of $f_{\text{NL}}\sigma$ needs to be used. However, when increasing $f_{\text{NL}}\sigma_G$, the agreement at $\xi < \xi_{\text{max}}$ becomes worse, and the point where the $f_{\text{NL}}$ distribution diverges gets dangerously too close to the maximum of the distribution. From this, one concludes that there is no $f_{\text{NL}}$ distribution that provides a reliable approximation of the elliptic profile both close to its maximum and along its tail.

In order to fix $\sigma_G$, one can further expand Eq. (10) around $\xi = \xi_{\text{max}}$, and one finds

\[ P_{\text{NL}}(\xi) = P_{\text{NL}}(\xi_{\text{max}}) \left[ 1 - \frac{(\xi - \xi_{\text{max}})^2}{2\sigma_{\text{max}}^2} + \ldots \right], \quad (12) \]

where

\[ \sigma_{\text{max}} = \frac{1 + \sqrt{1 - 4\sigma^2}}{2\sqrt{2\left[1 - 4\sigma^2 + \sqrt{1 - 4\sigma^2}\right]}}. \quad (13) \]

This can be used to match the curvature of the PDF around its maximum with a given reference distribution. It leads to the $f_{\text{NL}}$ profile with $f_{\text{NL}}\sigma_G = 0.1$ in Fig. 1.

II. GAUSSIAN AND LOGNORMAL DISTRIBUTIONS

Similarly to what was done above, the elliptic profiles (1) can be approximated by a lognormal (LN)
a Gaussian (G) distribution,

\[ P(x) = A \ln(x, \rho, \sigma) = A e^{-\frac{x^2}{\sigma^2}} G(x, \rho, \sigma_G), \]  

where \( x = \frac{\pi^2 D}{\mu^2} \),

\[ \ln(x, \rho, \sigma) = \frac{1}{\rho \sigma \sqrt{2\pi}} \exp \left[ -\frac{\ln(x/\rho)^2}{2\sigma^2} - \frac{\sigma^2}{2} \right] \]

\[ G(x, \rho, \sigma_G) = \frac{1}{\sigma_G \sqrt{2\pi}} \exp \left[ -(x - \rho)^2 \right] \]

and \( \sigma_G = \rho \sigma \). In these parametrizations, \( \rho \) stands for the value of \( x \) where the PDF is maximal. It is the solution of the equations

\[ \sum_{n=0}^{\infty} (2n+1)^3 e^{-\rho n(n+1)} \sin \left[ (2n+1)\frac{\pi}{2} \alpha \right] = 0, \]

for the elliptic 2 distribution, and

\[ \sum_{n=0}^{\infty} (-1)^n n^3 e^{-\rho^2} \sin (n \pi \alpha) = 0 \]

for the elliptic 4 distribution. When \( \alpha \) is small, those equations have approximate solutions \( \rho_2(\alpha) = \pi^2 \alpha^2 / 6 \) for the elliptic 2 distribution and \( \rho_4(\alpha) = \pi^2 (1 - \alpha)^2 / 6 \) for the elliptic 4 distribution (these approximations turn out to be reliable up until \( \alpha \approx 0.6 \)), otherwise those equations have to be solved numerically. From here, the height of the elliptic distributions at their maxima can be inferred, \( P_{\text{max}}(\alpha) = P_{2/4} \left[ \rho_2(\alpha), \alpha \right] \), and equated with \( A e^{-\sigma^2/2(\sigma_G \sqrt{2\pi})} \).

The curvature of the elliptic distributions around their maximum can also be computed according to

\[ \sigma_G^{(2)}(\alpha) = \left[ -\frac{\partial^2 \ln P_2}{\partial x^2} \bigg|_{x=\rho_2(\alpha)} \right]^{-1/2}, \]

\[ \sigma_G^{(4)}(\alpha) = \left[ -\frac{\partial^2 \ln P_4}{\partial x^2} \bigg|_{x=\rho_4(\alpha)} \right]^{-1/2}. \]

In the same small-\( \alpha \) limit as above, they boil down to \( \sigma_G^{(2)}(\alpha) \approx \sqrt{2} \alpha^2 \) and \( \sigma_G^{(4)}(\alpha) \approx \sqrt{2} (1 - \alpha)^2 \). This allows one to set the value of \( \rho, \sigma_G \) (hence \( \sigma \)) and \( A \).

This procedure is performed in Fig. 1, where one can check that the agreement around the maximum is indeed excellent, but that the agreement between the lognormal and the elliptic profiles on the tail is also reasonable.

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