GRADIENT ESTIMATES AND THE FUNDAMENTAL
SOLUTION FOR HIGHER-ORDER ELLIPTIC SYSTEMS WITH
LOWER-ORDER TERMS

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Abstract. We establish the Caccioppoli inequality, a reverse Hölder inequality in the spirit of the classic estimate of Meyers, and construct the fundamental solution for linear elliptic differential equations of order $2m$ with certain lower order terms.

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1. Introduction

There is at present a very extensive theory for second order linear elliptic differential operators without lower order terms. Such an operator \( L \) may be written as

\[
(L\bar{u})_j = -\sum_{k=1}^N \sum_{a=1}^d \sum_{b=1}^d \partial_a (A_{a,b}^{j,k} \partial_b u_k)
\]

where \( \bar{u} \) is a function defined on a subset of \( \mathbb{R}^d \). Two important generalizations are higher order operators

\[
(L\bar{u})_j = \sum_{k=1}^N \sum_{|\alpha|=|\beta|=m} (-1)^m \partial^{\alpha} (A_{\alpha,\beta}^{j,k} \partial^{\beta} u_k)
\]

and operators with lower order terms

\[
(L\bar{u})_j = \sum_{k=1}^N \left( A_{0,0}^{j,k} u_k + \sum_{b=1}^d A_{0,b}^{j,k} \partial_b u_k - \sum_{a=1}^d \partial_a (A_{a,0}^{j,k} u_k) - \sum_{a=1}^d \sum_{b=1}^d \partial_a (A_{a,b}^{j,k} \partial_b u_k) \right)
\]

\[
= \sum_{k=1}^N \sum_{0 \leq |\alpha| \leq 1} \sum_{0 \leq |\beta| \leq 1} (-1)^{|\alpha|} \partial^{\alpha} (A_{\alpha,\beta}^{j,k} \partial^{\beta} u_k)
\]

where \( \alpha \) and \( \beta \) denote multiindices.

Operators of higher order \( (2) \) with variable coefficients \( A_{\alpha,\beta}^{j,k} \) have been investigated in many recent papers, including \cite{MMS10, CMY16, CMY17, Tol18, NSX18, NX19, Zat20, XN21}, and the first author’s papers with Hofmann and Mayboroda \cite{Bar16, Bar17, BHM17, BHM19a, BHM19b, BHM20, Bara, Barb}. (The theory of higher order operators with constant coefficients is older and more developed; we refer the interested reader to the references in the above papers or to the survey paper \cite{BM16a} for more details.) Harmonic analysis of second order operators with general lower order terms \( (3) \) has been done in a number of recent papers, including \cite{CMY17, DHM18, MP19, KS19, Sak19, Dav20, DW20, Bai21, BMR21, Sak21, Mon, BHLG+21, DI}.

In this paper we will combine the two approaches and investigate operators \( L \) of order \( 2m \geq 2 \) with certain lower order terms

\[
(L\bar{u})_j = \sum_{k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} (-1)^{|\alpha|} \partial^{\alpha} (A_{\alpha,\beta}^{j,k} \partial^{\beta} u_k).
\]

Specifically, three of the foundational results of the theory of elliptic operators of the form \( (1) \), which have all received considerable study in the cases of operators of the forms \( (2) \) and \( (3) \), are Caccioppoli’s inequality, Meyers’s reverse Hölder inequality for gradients, and the fundamental solution. In this paper we investigate these three topics in the case of operators of the form \( (4) \) under certain assumptions on the coefficients.
For operators (1) or (2) without lower order terms, it is usual to require that all coefficients be bounded. Applying Hölder’s inequality yields the bound

$$\|L\bar{\psi}\| = \left| \sum_{j,k=1}^{N} \sum_{a=|\alpha|=m}^{m} \sum_{b=|\beta|=m}^{m} \int_{\mathbb{R}^d} \partial^\alpha \phi_j A_{\alpha,\beta}^{k} \partial^\beta u_k \right| \leq \|A\|_{L^\infty(\mathbb{R}^d)} \|\nabla^m \phi\|_{L^p(\mathbb{R}^d)} \|\nabla^m \bar{u}\|_{L^p(\mathbb{R}^d)}$$

for any $1 \leq p \leq \infty$. Thus, under these assumptions $L$ is a bounded linear operator from the Sobolev space $W^{m,p}(\mathbb{R}^d)$ (with norm $\|\bar{u}\|_{W^{m,p}(\mathbb{R}^d)} = \|\nabla^{m} \bar{u}\|_{L^p(\mathbb{R}^d)}$) to the dual space $W^{-m,p}(\mathbb{R}^d) = (W^{m,p}(\mathbb{R}^d))'$ for any $1 \leq p \leq \infty$. This is a useful property we would like to preserve.

Observe that elements of $W^{m,p}(\mathbb{R}^d)$ are, strictly equivalent, equivalence classes of functions with the same $m$th order gradient. Their lower order derivatives may differ by polynomials. In investigating operators with lower order terms (3) and (4), the spaces $W^{m,p}(\mathbb{R}^d)$ are not satisfactory; we will need the lower order derivatives of functions in the domain of $L$ to be well defined.

The Gagliardo-Nirenberg-Sobolev inequality gives a natural normalization condition on $W^{1,p}(\mathbb{R}^d)$ if $p < d$. Specifically, if $p < d$ then every element (equivalence class of functions) in $W^{1,p}(\mathbb{R}^d)$ contains a representative that lies in a Lebesgue space $L^{p'}(\mathbb{R}^d)$ for a certain $p'$ with $p < p' < \infty$. This representative is unique as a $L^{p'}$ function (that is, up to sets of measure zero).

An induction argument shows that, if $u \in W^{m,p}(\mathbb{R}^d)$, then there is a representative of $u$ such that $\partial^\alpha u$ lies in a Lebesgue space for all $\alpha$ with $m - d/p < |\alpha| \leq m$. This representative is unique (as a locally integrable function) up to adding polynomials of degree at most $m - d/p$. (Specifically, $\partial^\alpha u \in L^{p_{m,d,\alpha}}(\mathbb{R}^d)$, where $p_{m,d,\alpha}$ is given by formula (24) below.)

We define the $Y^{m,p}(\mathbb{R}^d)$ norm by

$$\|u\|_{Y^{m,p}(\mathbb{R}^d)} := \sum_{m-d/p < |\alpha| \leq m} \|\partial^\alpha u\|_{L^{p_{m,d,\alpha}}(\mathbb{R}^d)}.$$  

$Y^{m,p}(\mathbb{R}^d)$ is thus a space of equivalence classes of functions up to adding polynomials of degree at most $m - d/p$. The Gagliardo-Nirenberg-Sobolev inequality gives a natural isomorphism between $Y^{m,p}(\mathbb{R}^d)$ and the space $W^{m,p}(\mathbb{R}^d)$.

We will consider operators that satisfy, for all suitable test functions $\bar{\psi}$ and $\psi$, the Gårding inequality (or ellipticity or coercivity condition)

$$\text{Re} \sum_{j,k=1}^{N} \sum_{a=|\alpha|=m}^{m} \sum_{b=|\beta|=m}^{m} \int_{\mathbb{R}^d} \partial^\alpha \phi_j A_{\alpha,\beta}^{k} \partial^\beta \phi_k \geq \lambda \|\psi\|^2_{Y^{m,p}(\mathbb{R}^d)}$$

and the bound

$$\int_{\mathbb{R}^d} \sum_{j,k=1}^{N} \sum_{a=|\alpha|=m}^{m} \sum_{b=|\beta|=m}^{m} \partial^\alpha \phi_j A_{\alpha,\beta}^{k} \partial^\beta \psi_k \leq \Lambda(p) \|\phi\|_{Y^{m,p}(\mathbb{R}^d)} \|\psi\|_{Y^{m,p}(\mathbb{R}^d)}$$

for a range of $p$ near 2.

(In Section 4 following [AQ00], we will consider operators satisfying a slightly weaker form (23) of the Gårding inequality (5).)

Note that if $d = 2$ and $p \geq 2$, then $m - d/p \geq m - 1$ and so $\|u\|_{Y^{m,p}(\mathbb{R}^d)} = \|u\|_{W^{m,p}(\mathbb{R}^d)}$. In this case the Gagliardo-Nirenberg-Sobolev inequality provides no
normalization and so the bound (3), for $p = 2$, can only be expected to hold if $a = b = m$. Thus, in dimension 2, the results of the present paper do not represent a generalization of previous results such as [Barb]. We will include the case $d = 2$ in our results, but only for the sake of completeness and ease of reference.

There are many possible conditions that can be imposed on the coefficients $A^{j,k}_{\alpha,\beta}$ that yield the bound (6). Following (or modifying) [DHM18, KS19, Sak19, IDW20, BMR21, Sak21, BHLG21], we will focus our attention on the case

$$\max_{a \leq |\alpha| \leq m, b \leq |\beta| \leq m} \|A^{j,k}_{\alpha,\beta}\|_{L^2 a, \beta(R^d)} \leq \Lambda, \quad a, b > m - \frac{d}{2}, \quad 2\alpha, \beta = \frac{d}{2m - |\alpha| - |\beta|}.$$  

For all $p$ in a certain range including 2, the bound (6) follows immediately from the bound (7), Hölder’s inequality, and the Gagliardo-Nirenberg-Sobolev inequality. See Lemma [54] for further discussion. Note that if $2m = 2$ and $d \geq 3$, the condition $a, b > m - \frac{d}{2}$ holds for $a = b = 0$ and so we may ignore this condition.

We will also consider coefficients satisfying Bochner norm estimates

$$\max_{a \leq |\alpha| \leq m, b \leq |\beta| \leq m} \|A^{j,k}_{\alpha,\beta}\|_{L^p a, \beta(R^d)} \leq \Lambda, \quad a, b > m - \frac{d - 1}{2}, \quad 2\alpha, \beta = \frac{d - 1}{2m - |\alpha| - |\beta|}.$$  

Again, for second order operators ($2m = 2$), if $d \geq 4$ then we may take $a = b = 0$. For example, this includes the case where coefficients are constant in a specified direction, that is, where $A^{j,k}_{\alpha,\beta}(x, t) = a^{j,k}_{\alpha,\beta}(x)$ for all $x \in \mathbb{R}^{d-1}$, $t \in \mathbb{R}$, and some function $a^{j,k}_{\alpha,\beta} \in L^2 A, \beta(R^{d-1})$. This is the case studied in [BHLG21]. Operators of the form (11) and (13) that satisfy $A^{j,k}_{\alpha,\beta}(x, t) = a^{j,k}_{\alpha,\beta}(x)$ (for $|\alpha| = |\beta| = m$) have been studied in the higher order case in [BHM17, BHM19a, BHM19b, BHM18, BHM20, Barb, Barb], and in the second order case in many papers, including but not limited to [JK81, KP93, KKP100, Rul07, AAM08, KR09, AAM10, Axe10, AAA11, Ros13, AM14, HKMP15b, HMM15a, AS16, BM16b, MM17, AA18, AM19, AE20, HZ21]. Nontrivial coefficients constant in a specified direction cannot lie in $L^p(R^d)$ for any $p < \infty$, but can easily lie in Bochner spaces.

Like the condition (7), the condition and (5) implies the bound (6) for a range of $p$ including 2; see Lemma [54] below.

We note that the conditions (7) and (8) differ from those of [CMY17, Wan20], in which the authors investigate the system (4) or (1) for coefficients $A^{j,k}_{\alpha,\beta} \in L^\infty(R^d)$ for all $\alpha$ and $\beta$. (Our conditions imply $A^{j,k}_{\alpha,\beta} \in L^\infty(R^d)$ only for $|\alpha| = |\beta| = m$.)

### 1.1. The Caccioppoli inequality and Meyers’s reverse Hölder inequality.

The Caccioppoli inequality (established in the early twentieth century) is valid for all operators $L$ of the form (1) where the coefficients $A^{j,k}_{\alpha,\beta}$ are bounded and satisfy the Gårding inequality (5), and is often written as

$$\int_{B(X_0, r)} |\nabla \tilde{u}|^2 \leq \frac{C}{r^2} \int_{B(X_0, 2r)} |\tilde{u}|^2$$  

whenever $L\tilde{u} = 0$ in $B(X_0, 2r)$.

It can be generalized to the case $L\tilde{u} \neq 0$ by adding an appropriate term on the right hand side; a very general form is

$$\int_{B(X_0, r)} |\nabla \tilde{u}|^2 \leq \frac{C}{r^2} \int_{B(X_0, 2r)} |\tilde{u}|^2 + C\|L\tilde{u}\|_{W^{-1,2}(B(X_0, 2r))}$$  

where $C$ is a constant depending on $\alpha, \beta$ and the coefficients of $L$. This is the fundamental solution to the Caccioppoli inequality.

The reverse Hölder inequality states that if $u \in L^p$ and $\lambda u \in L^q$ for some $p > q > 1$, then $\lambda u \in L^{p_1}$ for some $p_1 < p$. This implies that if $u$ is a solution to $Lu = 0$ in $B(X_0, r)$, then $\lambda u \in L^{p_1}$ for some $p_1 < p$.
where $W^{-1,2}(B(X_0, 2r))$ is the dual space to $W^{1,2}_0(B(X_0, 2r))$, the closure in $W^{1,2}(B(X_0, 2r))$ of the set of smooth functions compactly supported in $B(X_0, 2r)$. By the Poincaré or Gagliardo-Nirenberg-Sobolev inequality, this is equal (with equivalence of norms) to the closure in $W^{1,2}_0(B(X_0, 2r))$ or $Y^{1,2}_0(B(X_0, 2r))$.

**Remark 9.** It is common to formulate the Caccioppoli inequality (and Meyers’s reverse Hölder inequality below) for solutions to $Lu = \hat{f} - \div \hat{F}$ (that is, $(L\bar{u})_j = f_j - \sum_{a=1}^{d} \partial_a F_{a,j}$). This is equivalent to our formulation in terms of operator norms of $L\bar{u}$ if appropriate norms on $\hat{f}$, $\hat{F}$ are used.

Specifically, if $L\bar{u} = -\div \hat{F}$, then $|\langle L\bar{u}, \varphi \rangle| = |\langle \hat{F}, \nabla \varphi \rangle|$ for all test functions $\varphi \in W^{1,2}_0(B(X_0, 2r))$, and so by Hölder’s inequality, $\|L\bar{u}\|_{W^{-1,2}(B(X_0, 2r))} \leq \|\hat{F}\|_{L^2(B(X_0, 2r))}$. By the Gagliardo-Nirenberg-Sobolev inequality, if $d \geq 3$ and $p = 2d/(d - 2)$ then $\|\varphi\|_{L^p(B(X_0, 2r))} \leq C\|\nabla \varphi\|_{L^2(B(X_0, 2r))}$ for all $\varphi \in W^{1,2}_0(B(X_0, 2r))$, and so if $L\bar{u} = \hat{f}$ then $\|L\bar{u}\|_{W^{-1,2}(B(X_0, 2r))} \leq C\|\hat{f}\|_{L^p(B(X_0, 2r))}$.

Conversely, if $L\bar{u} \in W^{-1,2}(B(X_0, 2r))$, then by the Hahn-Banach theorem there is some $\hat{F} \in L^2(B(X_0, 2r))$ with $\|\hat{F}\|_{L^2(B(X_0, 2r))} \approx \|L\bar{u}\|_{W^{-1,2}(B(X_0, 2r))}$ such that $L\bar{u} = \div \hat{F}$.

**Remark 10.** In the case of equations ($N = 1$) with real-valued coefficients, a Caccioppoli inequality can also be established for subsolutions; that is, instead of a norm $\|Lu\|$ appearing on the right hand side, it is required that $Lu \geq 0$ in $B(X_0, 2r)$. See, for example, [Mon, Section 3]. This approach is not available in the case of systems or complex coefficients, and has received little study in the case of higher order equations.

The Caccioppoli inequality has been generalized to operators of the form (2) (higher order equations without lower order terms) in [Cam80] and with some refinements in [AQ00, Bar16]. It has been extended to operators of the form (3) (second order operators with lower order terms) in [DHM18] (see also [BHLG17]). In the case of higher order operators with lower order terms of the form (2), a parabolic Caccioppoli inequality was established in [CMY17] under the assumption that all coefficients (including the lower order coefficients) are bounded; this is different from the assumptions of this paper.

In [Mey63], Meyers established a reverse Hölder estimate. Specifically, he established that for equations $(N = 1)$ with bounded and elliptic coefficients, for all $p$ and $q$ sufficiently close to 2 (and, in particular, for some $p > 2$ and $q \leq 2$) we have the estimate

$$\left( \int_{B(X_0,r)} |\nabla u|^p \right)^{1/p} \leq C r^{d/p - d/q} \left( \int_{B(X_0,r)} |\nabla u|^q \right)^{1/q} + C \|Lu\|_{W^{-1,p}(B(X_0,r))}.$$

The exponent $q$ on the right hand side can be lowered if desired; see [FS72, Section 9, Lemma 2] in the case of harmonic functions, and [Bar16, Lemma 33] for more general functions. Meyers’s results have been generalized to second order systems (even nonlinear systems) without lower order terms (see [Gia83, Chapter V]), and to higher order equations without lower order terms (see [Cam80, AQ00, Bar16]).

Caccioppoli’s inequality is still valid for systems of the form (4), that is, higher order equations with lower order terms. The argument is largely that of [Cam80, Bar16] and is presented in Section 4.
The obvious generalization of Meyers’s reverse Hölder inequality is not valid in the case of operators (even second order operators) with lower order terms. That is, for any given positive integers \( m, d \) and nonnegative integers \( a \in \{m - d/2, m\}, b \in (m - d/2, m) \), there exists an operator \( L \) of the form

\[ Lu = \sum_{a \leq |\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (A_{\alpha, \beta} \partial^\beta u) \]

with coefficients satisfying the conditions (5) and (7), and a function \( u : Q_0 \rightarrow \mathbb{R} \) with \( Lu = 0 \) in \( Q_0 \), such that for any \( p > 2 \) and any natural number \( k \), there is a ball \( B(X_k, r_k) \) with \( B(X_k, 2r_k) \subset Q_0 \) and with

\[ \left( \int_{B(X_k, r_k)} |\nabla^m \bar{u}|^p \right)^{1/p} \geq k \left( \int_{B(X_k, 2r_k)} |\nabla^m \bar{u}|^2 \right)^{1/2} \]

and, indeed, the stronger bound

\[ \left( \int_{B(X_k, r_k)} |\nabla^m \bar{u}|^p \right)^{1/p} \geq k \sum_{i = b + 1}^m r_k^{d/p - d/2 - (m - i)} \left( \int_{B(X_k, 2r_k)} |\nabla^i \bar{u}|^2 \right)^{1/2}. \]

See Section 6.2.

Weaker generalizations have been investigated in \([BHLG]\) and the argument of Section 6 takes many ideas therefrom. The following theorem is the first main result of this paper. It will be proven in Sections 4 (the case \( p = q = \mu = 2 \)) and Section 9 (the general case), and represents a simultaneous statement of the Caccioppoli and Meyers inequalities for systems of the form (4).

**Theorem 12.** Let \( m \geq 1 \) and \( d \geq 2 \) be integers. Let \( L \) be an operator of the form (4) for some coefficients \( A \) that satisfy the ellipticity condition (5) and one of the bounds (7) or (8).

Then there is a \( \delta > 0 \) depending on \( m, d \) and the constants \( \lambda \) and \( \Lambda \) in the bounds (5) and (7) or (8) with the following significance.

Let \( p \in [2, 2 + \delta), \mu \in (2 - \delta, 2 + \delta), \) and let \( 0 < q \leq \infty \). Let \( j \) and \( \omega \) be integers with \( 0 \leq j \leq m \) and \( 0 \leq \omega \leq \min(j, b) \). If \( p = 2 \), we impose the additional requirement that either \( q \geq 2 \) or \( \omega \geq 1 \) (and thus \( j \geq 1 \)).

Let \( Q \subset \mathbb{R}^d \) be a cube with sides parallel to the coordinate axes. Let \( \bar{u} \in Y^{m, \mu}(\theta Q) \) be such that \( \|L \bar{u}\|_{W^{m, \mu}(\theta Q)} < \infty \).

Then \( \nabla^j \bar{u} \in L^p(Q) \), and there exist positive constants \( \kappa \) and \( C \) depending on \( p, q, m, d, \lambda, \) and \( \Lambda \) such that

\[ \frac{1}{|Q|^{(m-j)/d}} \|\nabla^j \bar{u}\|_{L^p(Q)} \leq \frac{C}{(\theta - 1)^\kappa} \|L \bar{u}\|_{Y^{m, \mu}(\theta Q)} + \frac{C |Q|^{1/p - 1/q - (m - \omega)/d}}{(\theta - 1)^\kappa} \|\nabla^\omega \bar{u}\|_{L^q(\theta Q \setminus Q)} \]

for all \( 1 < \theta \leq 2 \).

Here \( \theta Q \) is the cube concentric to \( Q \) with volume \( |\theta Q| = \theta^d |Q| \). Note that the condition \( \bar{u} \in Y^{m, \mu}(\theta Q) \) is stronger than the condition \( \nabla^\omega \bar{u} \in L^q(\theta Q \setminus Q) \), that is, that the right hand side of the given bound be finite. The assumption \( \nabla^m \bar{u} \in L^\mu(\theta Q) \) implies that \( L \bar{u} \) is a bounded linear operator on \( W^{0, m, \mu}(\theta Q) = \{ \bar{u} : \nabla^m \bar{u} \in L^\mu(\mathbb{R}^d); \bar{u} = 0 \text{ in } \mathbb{R}^d \setminus \theta Q \} \); we require \( L \bar{u} \) to be a bounded linear operator
on $\tilde{W}^{m,p'}(\theta Q)$ (or, more precisely, on the space $\tilde{W}_0^{m,p'}(\theta Q) \cap \dot{W}_0^{m,p'}(\theta Q)$ equipped with the $W^{m,p'}$-norm).

If $L\tilde{u} \in \tilde{W}^{-m,p}(\theta Q)$ for some $p < 2$ but sufficiently close to 2, a weaker result is still available; see Theorem \ref{main_result} below.

1.2. The fundamental solution.} The fundamental solution $\tilde{E}_{X,j}^L \delta X_e$ for the operator $L$ is, formally, the solution to $L \tilde{E}_{X,j}^L \delta X_e = \delta X_e$, where $\delta X$ denotes the Dirac mass at $X$. The fundamental solution has proven to be a very useful tool in the theory of differential equations without lower order terms (that is, of the forms \ref{1} and \ref{2}). By definition, integrating against the fundamental solution allows one to solve the Poisson problem $L\tilde{u} = f$ in $\mathbb{R}^d$. The fundamental solution is also used in the theory of layer potentials, an essential tool in the theory of boundary value problems; for example, layer potentials based on the fundamental solution for certain variable coefficient operators of the form \ref{1} were used in \cite{KR09, Ru07, Agr09, AS16, AM18} and of the form \ref{2} in \cite{BHM18, BHM20, Bar16, Bar18}. To still available; see Theorem 64 below.

This approach, with some attention to the details and use of the Caccioppoli and Meyers inequalities, yields the following theorem. This theorem is the second main result of the present paper.

\textbf{Theorem 13.} Let $L$ be an operator of order $2m$ of the form \ref{1} that satisfies the ellipticity condition \ref{ellipticity} and one of the bounds \ref{1} or \ref{2}.

Then there exists a number $\delta > 0$ and an array of functions $E_{j,k}^L$ for pairs of integers $j, k$ in $[1, N]$ and defined on $\mathbb{R}^d \times \mathbb{R}^d$ with the following properties. This array of functions is unique up to adding functions $P_{j,k}$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfy $\partial X_e \partial X_e P_{j,k}(Y, X) = 0$ whenever $m - d/2 \leq |\xi| \leq m$, $m - d/2 \leq |\xi| \leq m$, and $(|\xi|, |\xi|) \notin (m - d/2, m - d/2)$.
Suppose that α and β are two multiindices with $m - d/2 \leq |\alpha| \leq m$, $m - d/2 \leq |\beta| \leq m$, and $(|\alpha|, |\beta|) \neq (m - d/2, m - d/2)$.

Suppose further that $Q$ and $\Gamma$ are two cubes in $\mathbb{R}^d$ with $|Q| = |\Gamma|$ and $\Gamma \subset 8Q \setminus Q$. Then the partial derivative $\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X)$ exists as a $L^2(Q \times \Gamma)$ function and satisfies the bounds

$$\int_Q \int_\Gamma |\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X)|^2 \, dX \, dY \leq C|Q|^{(4m - 2|\alpha| - 2|\beta|)/d}. \tag{14}$$

If $2 - \delta < p < 2 + \delta$, and if $p < 2$ or $|\beta| > m - d/2$, then

$$\int_\Gamma \left( \int_Q |\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X)|^{p\beta} \, dY \right)^{2/p\beta} \, dX \leq C|Q|^{2m/d - 1 + 2/p - 2|\alpha|}. \tag{15}$$

where $\frac{1}{p\beta} = \frac{1}{p} - \frac{m - |\beta|}{d}$.

Furthermore, we have the symmetry property

$$\partial^\alpha_X \partial^\beta_Y E^L_{j,k}(Y, X) = \partial^\beta_Y \partial^\alpha_X E^L_{k,j}(X, Y). \tag{16}$$

Finally, suppose that $2 - \delta < q < 2 + \delta$ and that $m - d/q < |\xi| \leq m$. Let $F \in L^{(q,\ell)}(\mathbb{R}^d)$ be compactly supported, where $\frac{1}{q\ell} = 1 - \frac{1}{q} + \frac{m - |\xi|}{d}$. Let $1 \leq \ell \leq N$. For each β with $m \geq |\beta| > m - d/q'$ and each $1 \leq k \leq N$, let

$$(u_{\beta})_k(X) = \int_{\mathbb{R}^d} \partial^\beta_X \partial^\alpha_Y E^L_{k,j}(X, Y) F(Y) \, dY. \tag{17}$$

The integral converges absolutely for almost every $X \in \mathbb{R}^d$ \setminus supp $F$ for all such $\beta$ and $\xi$; if $|\beta| < m$ or $|\xi| < m$ then the integral converges absolutely for almost every $X \in \mathbb{R}^d$.

Then there is a function $\bar{u} \in Y^{m,q}(\mathbb{R}^d)$ with $\partial^\beta \bar{u} = (u_{\beta})_k$ for all such $\beta$ almost everywhere (if $|\beta| + |\xi| < 2m$) or almost everywhere in $\mathbb{R}^d$ \setminus supp $F$ (otherwise) and such that

$$\int_{\mathbb{R}^d} \partial^\xi \varphi_\ell F = \sum_{k,j=1}^N \sum_{a \leq |\alpha| \leq m} \int_{\mathbb{R}^d} \partial^\alpha \varphi_\ell A_{\alpha,k}^{i,j} \partial^\beta u_k$$

for all $\varphi \in Y^{m,q'}(\mathbb{R}^d)$.

Many assumptions on the coefficients other than (7) and (8) are reasonable. We construct the fundamental solution in Section 7. In that section, we will not explicitly use the assumptions (7) and (8); instead we will use their consequences, the Caccioppoli and Meyers inequalities, for the operator $\bar{L} = \Delta^M \bar{L} \Delta^M$. The results in Section 7 and in particular Theorem 119 will allow the interested reader to construct the fundamental solution for other classes of coefficients once a suitable higher order Caccioppoli inequality has been established.

1.2.1. Other approaches. The approach of this paper and of [Bar16] uses higher order operators, and in particular the higher order Caccioppoli and Meyers inequalities, to construct the fundamental solution, and as such has only been available since the development of a strong theory of higher order operators. The fundamental solution for second order operators has been of interest for a long time and other approaches to its construction have been used.

If $d \geq 2$, then $\delta_X e_j$ is not an element of $Y^{-1,2}(\mathbb{R}^d)$. Specifically, elements of $Y^{1,2}(\mathbb{R}^d)$ are elements of Lebesgue spaces (or of $BMO$) and so their value at a single
point is not well defined. In some special cases (discussed above), $L$ is invertible from $Y_0^{1,p}(B)$ to $Y^{-1,p}(B)$ for open balls $B$ and $p$ large enough to apply Morrey’s inequality, and so the fundamental solution can be constructed using the approach discussed above and some attention to the behavior outside of $B$. However, this approach is not available in other cases.

In some cases, solutions to $L\bar{u} = 0$ may be locally Hölder continuous even if general $Y^{1,2}$ functions are not. In this case, the fundamental solution may be constructed as a limit of $L^{-1}T_\rho$, where $T_\rho \to \delta_X e_j$ as $\rho \to 0^+$ and each $T_\rho$ is in $Y^{-1,2}(\mathbb{R}^d)$. Careful application of the Caccioppoli inequality, the local Hölder continuity, and other arguments yields that $L^{-1}T_\rho$ converges to a fundamental solution.

This was done in for operators of the form (1) ([GW82, HK07]) and (3) ([DHM18]) in dimension $d \geq 3$ under the assumption that solutions are locally Hölder continuous. Green’s functions in domains (rather than in all of $\mathbb{R}^d$) were constructed using this method in [KK10, KS19, Sak21, Mou].

A different approach involving kernels for the heat semigroup $e^{-tL}$ was used in [AMT98] to construct the fundamental solution in dimension 2; as observed in [DK09] their approach is valid for systems of the form (1) with $N \geq 1$ and with complex nonsymmetric coefficients. The papers [DK09, CDK12] establishes results analogous to those of [AMT98] for the Green’s function of a domain rather than all of $\mathbb{R}^2$.

Considerably more work must be expended to apply the semigroup approach in dimension $d \geq 3$; heat semigroups were used in [MP19] to construct the fundamental solution for the magnetic Schrödinger operator, and a different form of semigroup was used in [Ros13] to construct the fundamental solution assuming only local boundedness, not local Hölder continuity.

This approach does require the Di Giorgi-Nash property of elliptic operators, or a condition, such as real coefficients, that implies this property. However, this approach often yields stronger estimates than those of the present paper, and indeed stronger estimates than those true of the fundamental solution for the Laplace operator. See, for example, [She99, MP19, DI].

1.3. Outline. The outline of this paper is as follows. In Section 2 we will define our terminology. We will give some results concerning function spaces (in particular, Sobolev spaces) in Section 3.

We will prove the Caccioppoli inequality in Section 4. We will prove our generalization of Meyers’s reverse Hölder inequality in Section 6.1 and construct the counterexample of the inequality (1) in Section 6.2.

We will construct the fundamental solution in Section 7.

Some results concerning invertibility of the operator $L$ between certain function spaces will be used both in Section 6 and Section 7; we will give these results in Section 5.

2. Definitions

2.1. Basic notation. We consider divergence-form elliptic systems of $N$ partial differential equations of order $2m$ in $d$-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$.

When $\Omega \subset \mathbb{R}^d$ is a set of finite measure, we let $\int_\Omega f = \frac{1}{|\Omega|} \int f$, where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. 

As mentioned in Theorem 12, if $Q$ is a cube in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$ and $\theta > 0$ is a positive real number, we let $\theta Q$ denote the concentric cube with $|\theta Q| = \theta^d |Q|$ (so the side length of $\theta Q$ is $\theta$ times the side length of $Q$).

We employ the use of multiindices in $(\mathbb{N}_0)^d$. We will define $|\gamma| = \sum_{i=1}^{d} \gamma_i$ and $\gamma! = \gamma_1! \cdot \gamma_2! \cdot \ldots \cdot \gamma_d!$ for any multiindex $\gamma = (\gamma_1, \ldots, \gamma_d)$. When $\delta$ is another multiindex in $\mathbb{N}_0^d$ we say that $\delta \leq \gamma$ if $\delta_i \leq \gamma_i$ for each $1 \leq i \leq d$. Furthermore, we say $\delta < \gamma$ if $\delta_i < \gamma_i$ for at least one such $i$.

We will use the Leibniz Rule for multiindices, that is, that for all suitably differentiable functions $u$ and $v$ and a multiindex $\alpha$, we have that

$$\partial^\alpha (uv) = \sum_{\gamma \leq \alpha} \frac{\alpha!}{\gamma!(\alpha - \gamma)!} \partial^\gamma u \partial^{\alpha - \gamma} v.$$ 

2.2. Function spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a domain. We denote by $L^p(\Omega)$ and $L^\infty(\Omega)$ the standard Lebesgue spaces with respect to Lebesgue measure, with norms given by

$$\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(\Omega)} = \text{ess sup}_{\Omega} |u|.$$ 

If $1 \leq p \leq \infty$, we let $p'$ be the extended real number that satisfies $1/p + 1/p' = 1$.

If $t \in \mathbb{R}$, let $[\Omega]^t = \{ x \in \mathbb{R}^{d-1} : (x, t) \in \Omega \}$. We define the Bochner norm $L^q_{t}L^p_x(\Omega)$ by

$$\|u\|_{L^q_tL^p_x(\Omega)} = \left( \int_{-\infty}^{\infty} \left( \int_{[\Omega]^t} |u(x,t)|^p \, dx \right)^{q/p} \, dt \right)^{1/q}$$

with a suitable modification in the case $p = \infty$ or $q = \infty$.

We define the inhomogeneous Sobolev norm as

$$\|\vec{u}\|_{W^{k,p}(\Omega)} = \sum_{j=0}^{k} \|
abla^j \vec{u}\|_{L^p(\Omega)}$$

where derivatives are required to exist in the weak sense. We then define the homogeneous Sobolev norm as

$$\|\vec{u}\|_{\dot{W}^{k,p}(\Omega)} = \|
abla^k \vec{u}\|_{L^p(\Omega)}.$$ 

Observe that by the Poincaré inequality, if $\vec{u} \in \dot{W}^{k,p}(\Omega)$ and $\Omega$ is bounded, then $\nabla^j u \in L^p(\Omega)$ for all $0 \leq j < k$; however, the Poincaré inequality does not yield finiteness of $\|\nabla^j u\|_{L^p(\Omega)}$ in the case where $\Omega$ is unbounded.

The Sobolev spaces are then the spaces of equivalence classes of functions whose Sobolev norm is finite, with the equivalence relation $\vec{u} \sim \vec{v}$ if $\|\vec{u} - \vec{v}\| = 0$. Observe that elements of inhomogeneous Sobolev spaces, like elements of Lebesgue spaces, are defined up to sets of measure zero, while elements of homogeneous Sobolev spaces (in connected domains) are defined up to sets of measure zero and also up to adding polynomials of degree at most $k - 1$. 
Recall that for $1 \leq p < d$, the Sobolev conjugate of $p$ is defined to be

\[ p^* = \frac{dp}{d-p} \]

See, for example, [Eva98 Section 5.6]. Notice that

\[ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \]

We will now generalize equation (20). Let $k$ be an integer so that $m - \frac{d}{p} < k \leq m$. We then define

\[ p_{m,d,k} \]

so that

\[ \frac{1}{p_{m,d,k}} = \frac{1}{p} - \frac{m-k}{d}. \]

When considering elliptic operators of order $2m$ in dimension $d$, and the numbers $m$ and $d$ are clear from context, we will let $p_k = p_{m,d,k}$. If $\alpha$ is a multiindex, we will let $p_\alpha = p_{m,d,|\alpha|} = p_{m,d,|\alpha|}$. Notice that when $|\alpha| = m$ we have that $2_\alpha = 2$, when $|\alpha| = m-1$ then $2_\alpha = 2$, and so on. This definition for $2_\alpha$ will help keep the notation throughout this paper relatively clean and help us to avoid backwards summation.

If $\Omega \subseteq \mathbb{R}^d$ is a domain, $m \geq 1$ is an integer, and $1 \leq p \leq \infty$, we define the $Y^{m,p}(\Omega)$ norm as

\[ \|u\|_{Y^{m,p}(\Omega)} := \sum_{m-d/p < |\alpha| \leq m} \|\partial^\alpha u\|_{L^{p_{m,d,\alpha}}(\Omega)}. \]

We then define $Y^{m,p}(\Omega)$ analogously to $\dot{W}^{m,p}(\Omega)$. Observe that elements of $Y^{m,p}(\Omega)$ are defined up to adding polynomials of degree at most $m - d/p$. We let

\[ Y^{m,p}_0(\Omega) = \{ \varphi \in Y^{m,p}(\mathbb{R}^d) : \varphi = 0 \text{ outside } \Omega \}. \]

Then $Y^{m,p}_0(\Omega)$ is the space of functions in $Y^{m,p}(\Omega)$ which are zero near the boundary in an appropriate sense. Note that $Y^{m,p}_0(\mathbb{R}^d) = Y^{m,p}(\mathbb{R}^d)$. Conversely, if $\mathbb{R}^d \setminus \Omega$ has nonempty interior, then elements of $Y^{m,p}(\Omega)$ have a natural normalization condition (that is, nonzero polynomials are not representatives of elements of $Y^{m,p}_0(\Omega)$).

We will generally write bounded linear operators on $Y^{m,p}_0(\Omega)$ as $\langle T, \cdot \rangle_\Omega$; if $\Omega = \mathbb{R}^d$ we will omit the $\Omega$ subscript. We define the antidual space $Y^{-m,p}(\Omega) = (Y^{m,p}_0(\Omega))^t$, for $1/p + 1/p' = 1$, by

\[ \langle T, \cdot \rangle_\Omega \text{ is a bounded linear operator on } Y^{m,p}_0(\Omega) \text{ if and only if } T \in Y^{-m,p}(\Omega). \]

Note that if $\alpha \in \mathbb{C}$ then $\langle \alpha T, \varphi \rangle_\Omega = \overline{\alpha} \langle T, \varphi \rangle_\Omega$.

2.3. Elliptic operators. Let $m$ be a positive integer. Let $A = (A_{\alpha,\beta}^{j,k})$ be an array of measurable real or complex coefficients defined on $\mathbb{R}^d$ indexed by integers $j$ and $k$ such that $1 \leq j \leq N$ and $1 \leq k \leq N$ and multiindices $\alpha$ and $\beta$ with $|\alpha| \leq m$ and $|\beta| \leq m$.

We define the differential operator $L$ with coefficients $A$ as follows. If $\tilde{u}$ is a Sobolev function, we let $\langle L\tilde{u}, \cdot \rangle_\Omega$ be the linear operator that satisfies

\[ \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_\Omega \partial^\alpha \varphi_j A_{\alpha,\beta}^{j,k} \partial^\beta u_k = \langle L\tilde{u}, \varphi \rangle_\Omega \]

for all appropriate test functions $\varphi$. 

Remark 25. If $A$, $\bar{u}$, and $\bar{\varphi}$ are sufficiently smooth and decay sufficiently rapidly at infinity, we may integrate by parts to see that

$$\langle L\bar{u}, \bar{\varphi} \rangle = \sum_{j=1}^{N} \int_{\mathbb{R}^{d}} \varphi_{j} \sum_{k=1}^{N} \sum_{|\alpha| \leq m, |\beta| \leq m} (-1)^{|\alpha|} \partial^{\alpha} (A^{j}_{\alpha,\beta} \partial^{\beta} u_{k})$$

Thus, in this case we may write

$$\langle L\bar{u}, \bar{\varphi} \rangle = \sum_{k=1}^{N} \sum_{|\alpha| \leq m, |\beta| \leq m} (-1)^{|\alpha|} \partial^{\alpha} (A^{j}_{\alpha,\beta} \partial^{\beta} u_{k})$$

as a classically defined linear differential operator; this coincides with formula (24) if $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ denotes the usual (complex) inner product in $L^{2}(\mathbb{R}^{d}, \mathbb{C}^{N})$.

We define

$$(26) \quad a = a_{L} = \min \{|\alpha| : A^{j}_{\alpha,\beta} (X) \neq 0 \text{ for some } j, k, \beta, X\},$$

$$(27) \quad b = b_{L} = \min \{|\beta| : A^{j}_{\alpha,\beta} (X) \neq 0 \text{ for some } j, k, \alpha, X\}.$$

Definition 28. We let $\Pi_{L}$ be the largest interval with

$$\Pi_{L} \subseteq (1, \infty) \cap \left\{ p : \frac{m-b}{d} < \frac{1}{p} < \frac{d-m+a}{d} \right\}$$

and such that if $p \in \Pi_{L}$, then there is a $\Lambda(p) \in [0, \infty)$ such that the bound (29) is valid, that is, such that

$$(29) \quad \int_{\mathbb{R}^{d}} \sum_{j,k=1}^{N} \sum_{a \leq |\alpha| \leq m} \sum_{b \leq |\beta| \leq m} \partial^{\alpha} \varphi_{j} A^{j}_{\alpha,\beta} \partial^{\beta} \psi_{k} \leq \Lambda(p) \|\varphi\|_{Y^{m,p} (\mathbb{R}^{d})} \|\psi\|_{Y^{-m,p} (\mathbb{R}^{d})}$$

for all $\varphi \in Y^{m,p'} (\mathbb{R}^{d})$, $\psi \in Y^{-m,p} (\mathbb{R}^{d})$.

We consider singleton sets to be intervals, so $\{2\} = [2,2]$ is a possible value of $\Pi_{L}$. We will usually assume that $2 \in \Pi_{L}$; in particular, this implies that $a$, $b > m - d/2$.

Remark 30. If $p \in \Pi_{L}$, then $|\langle L\bar{u}, \bar{\varphi} \rangle| \leq \Lambda(p) \|\varphi\|_{Y^{m,p} (\mathbb{R}^{d})} \|\bar{u}\|_{Y^{-m,p} (\mathbb{R}^{d})}$ and the integral in the definition of $\langle L\bar{u}, \bar{\varphi} \rangle$ converges absolutely for such $\bar{u}$ and $\bar{\varphi}$; thus, if $\bar{u} \in Y^{m,p} (\mathbb{R}^{d})$ then the given integral is a linear operator on $Y_{0}^{m,p} (\mathbb{R}^{d})$, and so $L\bar{u} \in Y^{-m,p} (\mathbb{R}^{d})$. Our conventions for $Y^{-m,p}$ yield that $L$ is a bounded linear operator (and not a conjugate linear operator) from $Y^{m,p} (\mathbb{R}^{d})$ to $Y^{-m,p} (\mathbb{R}^{d})$.

Remark 31. The condition $d/(d+a-m) < p < d/(m-b)$ ensures that the derivatives $\partial^{\alpha} \bar{\varphi}$, $\partial^{\beta} \bar{\psi}$ appearing in the bound (29) satisfy $|\alpha| > m - d/p'$ and $|\beta| > m - d/p$. By the definition (22) of $Y^{m,p} (\mathbb{R}^{d})$, this means that $\partial^{\alpha} \bar{\varphi} \in \mathcal{L}^{p_{0}} (\mathbb{R}^{d})$, $\partial^{\beta} \bar{\psi} \in \mathcal{L}^{p_{0}} (\mathbb{R}^{d})$. Derivatives of $Y^{m,p} (\mathbb{R}^{d})$ or $Y^{m,p'} (\mathbb{R}^{d})$ functions of lower order are defined only up to adding constants or polynomials, which would preclude validity of the bound (29). It might be possible to consider the case $a \leq m - d/2$ or $b \leq m - d/2$ by considering more delicate cancellation conditions or Hilbert spaces other than $Y^{m,2} (\mathbb{R}^{d})$, but such constructions are beyond the scope of this paper.

As noted in the introduction, if $m = 1$ and $d \geq 3$, then the condition $a, b > m - d/2$ is vacuous, as $m - d/2 < 0$ and there are no multiindices $\alpha \in \{0\}^{d}$ with $|\alpha| \leq m - d/2$. Conversely, if $d = 2$, then $A_{\alpha,\beta} \neq 0$ only in the case when
|α| = |β| = m, and so the present paper does not represent a generalization of previous results such as [Cam80, AQ00, DHM18, Bar16].

We will consider coefficients which satisfy the Garding inequality (5). In [AQ00], Auscher and Qafsaoui consider higher order elliptic systems in divergence form in which ellipticity is in the sense of the following weaker Garding inequality

\[ \text{Re} \sum_{j,k=1}^{N} a_{j,k} A_{\alpha,\beta} \partial_j \partial_k \varphi \geq \lambda \| \nabla^m \varphi \|_{L^2(\mathbb{R}^d)}^2 - \delta \| \varphi \|_{L^2(\mathbb{R}^d)}^2 \]

where \( \lambda > 0 \) and \( \delta > 0 \) are real numbers, for all \( \varphi \) which are smooth and compactly supported in \( \mathbb{R}^d \). The standard Garding inequality (5) is thus the weak inequality (32) with \( \delta = 0 \). In Section 4, we will prove results in the generality of the bound (32) instead of (5).

Throughout we will let \( C \) denote a positive constant whose value may change from line to line, but that depends only on the dimension \( d \), the order \( 2m \) of our differential operators, the size \( N \) of our system of equations, the constant \( \lambda \) in the bound (5) (or (32)), and the constant \( \Lambda(2) \) in the bound (6). A constant depending on a number \( p \in \Pi_L \) may also depend on \( \Lambda(p) \).

A standard argument involving the Lax-Milgram lemma (see Lemma 56 below) shows that if \( L \) satisfies the condition (5) and \( 2 \in \Pi_L \), then \( L \) is not only bounded but invertible \( Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d) \).

**Definition 33.** If \( L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d) \) is bounded and invertible, then we define

\[ Y_L = \{ p : L \text{ is bounded and compatibly invertible } Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d) \} \]

By compatibly invertible, we mean that \( L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d) \) is invertible with bounded inverse and that if \( T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,2}(\mathbb{R}^d) \) then \( L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,2}(\mathbb{R}^d) \). (Thus, \( L^{-1}T \) has the same value whether we regard \( L \) as an operator on \( Y^{m,p}(\mathbb{R}^d) \) or \( Y^{m,p}(\mathbb{R}^d) \).

Compatibility is not automatically true; see [Axe10] for an example of operators which are invertible, but not compatibly invertible, in some sense.

We will conclude this section by reminding the reader that our main focus is on coefficients that satisfy the bound (7), that is,

\[
\begin{align*}
A_{\alpha,\beta}^j &= 0 \\
\| A_{\alpha,\beta}^j \|_{L^{2m,\beta}(\mathbb{R}^d)} &\leq \Lambda \\
\text{if } m \geq |\alpha| > m - \frac{d}{2} \text{ and } m \geq |\beta| > m - \frac{d}{2},
\end{align*}
\]

or the bound (8), that is,

\[
\begin{align*}
\| A_{\alpha,\beta}^j \|_{L^{2m,\beta}(\mathbb{R}^d)} &\leq \Lambda \\
\text{if } m \geq |\alpha| > m - \frac{d+1}{2} \text{ and } m \geq |\beta| > m - \frac{d+1}{2},
\end{align*}
\]

where

\[ 2_{\alpha,\beta} = \frac{d}{2m - |\alpha| - |\beta|}, \quad 3_{\alpha,\beta} = \frac{d - 1}{2m - |\alpha| - |\beta|} \]

Elementary computations involving Hölder’s inequality (see Lemma 57) shows that the conditions (7) and (8) both imply that \( \Pi_L \) contains an interval around 2 whose radius depends only on the dimension \( d \).
3. The Gagliardo-Nirenberg-Sobolev and Poincaré inequalities and their consequences

In this section we will collect some results regarding Sobolev functions that will be useful throughout the paper. These results are mainly consequences of the Gagliardo-Nirenberg-Sobolev inequality and induction arguments.

We will begin with Section 3.1 in which we will consider the global function spaces $W^{m,p}(\mathbb{R}^d)$ and $Y^{m,p}(\mathbb{R}^d)$. In Section 3.2 we will study $Y^{m,p}(Q)$ for a cube $Q$.

We will often wish to consider the behavior of functions in thin annuli. Thus, in Section 3.3 we will establish results in (possibly thin) annuli rather than cubes. We will sometimes need different forms of estimates, and so will also investigate the Poincaré inequality in thin annuli.

Finally, in Section 3.4 we will investigate the behavior of Sobolev functions when multiplied by cutoff functions; since our standard cutoff functions have gradients supported in an annulus, this will build on the results of Section 3.3.

3.1. Global Sobolev spaces. In this section we will establish some basic properties of the spaces $W^{m,p}(\mathbb{R}^d)$ and $Y^{m,p}(\mathbb{R}^d)$. We begin by citing the Gagliardo-Nirenberg-Sobolev inequality; a proof may be found in (for example) [Eva98, Section 5.6.1, Theorem 1].

**Theorem 35** (The Gagliardo-Nirenberg-Sobolev inequality). Assume $1 \leq p < d$. Then there is a constant $C$ which depends only on $p$ and $d$ so that

$$
\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}
$$

for all $u \in C^1_c(\mathbb{R}^d)$.

Here $p^*$ is as in formula (20).

**Remark 36.** If $u$ is a representative of an element of $\dot{W}^{1,p}(\mathbb{R}^d)$ for $p < d$, (that is, if $\nabla u \in L^p(\mathbb{R}^d)$), a standard argument involving the Poincaré inequality in an annulus shows that even if $u \notin C^1_c(\mathbb{R}^d)$, there is a unique constant $c$ such that $u - c \in L^{p^*}(\mathbb{R}^d)$ and

$$
\|u - c\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}.
$$

**Corollary 37.** Suppose that $m \geq 1$, $d \geq 2$ are integers and that $1 \leq p < \infty$. Then there exists a constant $c$ depending only on $d$, $m$ and $p$ with the following significance. Suppose $\bar{u}$ is a representative of an element of $W^{m,p}(\mathbb{R}^d)$. Then there is a polynomial $\bar{P}$ of order at most $m - 1$, unique up to adding polynomials of order at most $m - d/p$, such that

$$
\|u - P\|_{Y^{m,p}(\mathbb{R}^d)} \leq c \|u\|_{W^{m,p}(\mathbb{R}^d)}.
$$

In particular $\|u - P\|_{Y^{m,p}(\mathbb{R}^d)}$ is finite.

**Proof.** Recall the definition (21) of $p_{m,d,k}$. Because $(p_{m,d,k+1})^* = p_{m,d,k}$, if $m - d/p < k < m$, the bound

$$
\|\nabla^k (u - P)\|_{L^{p_{m,d,k}}(\mathbb{R}^d)} \leq C_k \|\nabla^{k+1} u\|_{L^{p_{m,d,k+1}}(\mathbb{R}^d)}
$$

for some $C_k$ follows from Remark 36. By induction, and because $p_{m,d,m} = p$,

$$
\|\nabla^k (u - P)\|_{L^{p_{m,d,m}}(\mathbb{R}^d)} \leq C_k \|\nabla^m u\|_{L^p(\mathbb{R}^d)}.
$$

Applying the definitions (19) and (22) of $W^{m,p}(\mathbb{R}^d)$ and $Y^{m,p}(\mathbb{R}^d)$ completes the proof. \(\square\)
We will now establish a bound on the Bochner norm of elements of $Y^{m,p}(\mathbb{R}^d)$.

**Corollary 38.** Let $m \in \mathbb{N}$, $p \in [1, d-1)$. Let $k \in \mathbb{N}_0$ satisfy $m-(d-1)/p < k < m$. Let $u$ be a representative of an element of $W^{m,p}(\mathbb{R}^d)$ and let $P$ be the polynomial in Corollary 37. Then
\[
\|\nabla^k (u - P)\|_{L^p_x L^m_{x,t}(\mathbb{R}^d)} \leq C \|\nabla^m u\|_{L^p(\mathbb{R}^d)}.
\]
In particular, if $u \in Y^{m,p}(\mathbb{R}^d)$ then this bound is valid with $P = 0$.

**Proof.** By definition,
\[
\|\nabla^m u\|_{L^p(\mathbb{R}^d)} = \left( \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^{d-1}} |\nabla^m u(x,t)|^p \, dx \right)^{1/p} \, dt \right)^{1/p}.
\]
Let $0 \leq j \leq k$. Applying Corollary 37 in $\mathbb{R}^{d-1}$ with $d$ replaced by $d-1$ yields that, for some polynomial $P_{j,t}$ defined on $\mathbb{R}^{d-1}$,
\[
\|\nabla^{k-j} \partial_t u(\cdot, t) - \nabla^{k-j} P_{j,t}\|_{L^p_x L^{m-d+1-k}_{x,t}(\mathbb{R}^{d-1})} \leq C \|\nabla^{m-j} \partial_t u(\cdot, t)\|_{L^p_x L^{m-1-k}_{x,t}(\mathbb{R}^{d-1})} \leq C \|\nabla^m u(\cdot, t)\|_{L^p_x L^1_{x,t}(\mathbb{R}^{d-1})}
\]
where $\nabla_x$ denotes the gradient taken strictly in the horizontal variables.

For almost every $t \in \mathbb{R}$, by Corollary 37 (in $\mathbb{R}^d$) we have that
\[
\|\nabla^{k-j} \partial_t^i u(\cdot, t) - \nabla^{k-j} \partial_t^i P(\cdot, t)\|_{L^p_x L^{m-d+k}_{x,t}(\mathbb{R}^{d-1})} < \infty.
\]
Because $\partial_t^i P(x,t)$ and $P_{j,t}(x)$ are polynomials in $x$, we must have that for almost every $t \in \mathbb{R}$, $\nabla^{k-j} \partial_t^i P(x,t) = \nabla^{k-j} P_{j,t}(x)$ for all $x \in \mathbb{R}^{d-1}$. This completes the proof. \hfill \square

### 3.2. Sobolev functions in cubes

In this section we will establish analogues to Corollaries 37 and 38 in cubes (rather than in all of Euclidean space).

**Lemma 39.** Let $m, d \in \mathbb{N}$, $d \geq 2$, $p \in [1, \infty)$, and let $j, k \in \mathbb{N}_0$ satisfy $0 \leq j \leq k$ and $m - d/p < k \leq m$. Let $p_k = p_{m,d,k}$. Then there is a constant $C$ depending only on $p$, $d$, and $m$ such that if $Q \subset \mathbb{R}^d$ is a cube and $u \in W^{m,p}(Q)$, then
\[
\|\nabla^j u\|_{L^{p_k}(Q)} \leq C \sum_{i=j}^{m-k+j} |Q|^{(i-j+k-m)/d} \|\nabla^i u\|_{L^{p}(Q)}.
\]

**Proof.** Suppose first that $|Q| = 1$. By the Gagliardo-Nirenberg-Sobolev inequality in bounded domains (see, for example, [Eva98] Section 5.6.1, Theorem 2) and the definition (21) of $p_k$, we have that
\[
\|w\|_{L^{p_k}(Q)} \leq C \|w\|_{W^{1,p_{k+1}}(Q)} = C \sum_{i=0}^{1} \|\nabla^i w\|_{L^{p_{k+1}}(Q)}
\]
for any function $w \in W^{1,p_{k+1}}(Q)$. Taking $w = \nabla^j u$, we see that
\[
\|\nabla^j u\|_{L^{p_k}(Q)} \leq C \sum_{i=j}^{j+1} \|\nabla^i u\|_{L^{p_{k+1}}(Q)}
\]
Iterating this argument with $w = \nabla^i u$ and recalling that $p = p_m$ yields the $|Q| = 1$ case of the lemma. A change of variables establishes the case for general $Q$. \hfill \square
We may also control Bochner norms; this is very useful in the case that the coefficients satisfy the condition \((\ref{cond0})\).

**Lemma 40.** Let \(m, d \in \mathbb{N}, d \geq 2, p \in [1, \infty),\) and let \(j, k \in \mathbb{N}_0\) satisfy \(0 \leq j \leq k\) and \(m - (d - 1)/p < k \leq m\). Let \(p_{k} = p_{m,d,k}\). There is a constant \(C\) depending only on \(p, d,\) and \(m\) such that if \(Q \subset \mathbb{R}^d\) is a cube with sides parallel to the coordinate axes and \(u \in W^{m,p}(Q)\), then

\[
\|\nabla^j u\|_{L^p_t L^{ar{p}_k}(Q)} \leq C \sum_{i=j}^{m-k+j} |Q|^{(i-j+k-m)/d} \|\nabla^i u\|_{L^p(Q)}.
\]

**Proof.** Let \(Q = \Delta \times [t_0, t_0 + R]\), where \(\Delta \subset \mathbb{R}^{d-1}\) is a cube, \(t_0 \in \mathbb{R}\), and \(R = |Q|^{1/d}\). Recall that

\[
\|\nabla^j u\|_{L^p_t L^{ar{p}_k}(Q)} = \left( \int_{t_0}^{t_0 + R} \left( \int_{\Delta} |\nabla^j u(x, t)|^{ar{p}_k} \, dx \right)^{p/\bar{p}_k} \, dt \right)^{1/p}.
\]

Applying Lemma 39 in dimension \(d - 1\), we see that

\[
\left( \int_{\Delta} |\nabla^j u(x, t)|^{ar{p}_k} \, dx \right)^{1/\bar{p}_k} \leq C \sum_{i=j}^{m-k+j} R^{i-j+k-m} \|\nabla^i u(\cdot, t)\|_{L^p(\Delta)}.
\]

Integrating in \(t\) completes the proof. \(\square\)

### 3.3. Sobolev functions in annuli

We will now establish analogues to Lemmas 39 and 40 in annuli.

**Lemma 41.** Let \(m, d \in \mathbb{N}, d \geq 2, p \in [1, \infty),\) and let \(j, k \in \mathbb{N}_0\) satisfy \(0 \leq j \leq k\) and \(m - (d - 1)/p < k \leq m\). Let \(p_{k} = p_{m,d,k}\). Let \(1 < \theta \leq 2\).

Then there is a constant \(C\) depending only on \(p, d,\) and \(m\) such that if \(Q \subset \mathbb{R}^d\) is a cube with sides parallel to the coordinate axes and \(u \in W^{m,p}(\theta Q \setminus Q)\), then

\[
\|\nabla^j u\|_{L^p(\theta Q \setminus Q)} \leq C \sum_{i=j}^{m-k+j} \frac{|\theta - 1| |Q|^{1/d} |m-k+i-1|}{((\theta - 1) |Q|^{1/d})^{m-k+i-1}} \|\nabla^i u\|_{L^p(\theta Q \setminus Q)}.
\]

If in addition \(k > m - (d - 1)/p\), then

\[
\|\nabla^j u\|_{L^p_t L^{ar{p}_k}(\theta Q \setminus Q)} \leq C \sum_{i=j}^{m-k+j} \frac{C}{((\theta - 1) |Q|^{1/d})^{m-k+i-1}} \|\nabla^i u\|_{L^p(\theta Q \setminus Q)}.
\]
Proof. Observe that there exists an integer \( n \geq 2 \) with \( \frac{1}{n} \leq \frac{\theta - 1}{2} < \frac{1}{n - 1} \leq \frac{2}{n} \). Without loss of generality we assume that \( Q \) is open. Let \( I_1, \ldots, I_d \) be the \( d \) open intervals that satisfy \( Q = I_1 \times \cdots \times I_d \). If \( I_k = (a_k, b_k) \), and \( r = b_k - a_k = |Q|^{1/d} \), define the \( d(n + 2) \) intervals \( I_{k,j} \) by
\[
I_{k,0} = (a_k - \frac{\theta - 1}{2} r, a_k), \quad I_{k,n+1} = (b_k, b_k + \frac{\theta - 1}{2} r), \\
I_{k,j} = (a_k + \frac{j - 1}{n} r, a_k + \frac{j}{n} r) \quad \text{if } 1 \leq j \leq n.
\]

Let \( G = \{I_{1,j_1} \times I_{2,j_2} \times \cdots \times I_{d,j_d} : j_k \in \{0, 1, \ldots, n + 1\} \} \), and let \( H \subset G \) be given by \( H = \{I_{1,j_1} \times I_{2,j_2} \times \cdots \times I_{d,j_d} : j_k \in \{1, \ldots, n\} \} \). Up to a set of measure zero, \( \theta Q = \bigcup_{R \in G} R \), \( Q = \bigcup_{R \in H} R \).

Furthermore, the rectangles in \( G \) are pairwise disjoint. If \( R \in G \) then the shortest side of \( R \) is at least \( r/n \) and the longest side is at most \( (\theta - 1)r/2 < 2r/n \). A change of variables argument shows that Lemmas \ref{lem:1} and \ref{lem:2} are valid in \( R \) with uniformly bounded constants.

Suppose \( m - (d - 1)/p < k < m \). If \( \Omega \subseteq \mathbb{R}^d \), recall that \( [\Omega]^t = \{x \in \mathbb{R}^{d-1} : (x, t) \in \Omega \} \). Then
\[
\|\nabla^j u\|_{L^p_t L^\frac{p}{p-k} (\theta Q \setminus Q)} = \left( \int_{-\infty}^{\infty} \left( \int_{[\theta Q \setminus Q]^t} \|\nabla^j u(x, t)\|^{p/p_k} dx \right)^{\frac{1}{p}} dt \right)^{1/p}
\]
\[
= \left( \int_{-\infty}^{\infty} \left( \sum_{R \in G \setminus H} \int_{[R]^t} \|\nabla^j u(x, t)\|^{p/p_k} dx \right)^{\frac{1}{p}} dt \right)^{1/p}.
\]

Because \( p/p_k \leq 1 \), we have that
\[
\|\nabla^j u\|_{L^p_t L^\frac{p}{p-k} (\theta Q \setminus Q)} \leq \left( \sum_{R \in G \setminus H} \left( \sum_{i=j}^{m-j} \left( \frac{C}{((\theta - 1)p)^{m-k+j-i}} \|\nabla^i u\|_{L^p(R)} \right)^p \right)^{1/p} \right).
\]

By Lemma \ref{lem:2} in rectangles,
\[
\|\nabla^j u\|_{L^p_t L^\frac{p}{p-k} (\theta Q \setminus Q)} \leq \left( \sum_{R \in G \setminus H} \left( \sum_{i=j}^{m-j} \left( \frac{C}{((\theta - 1)p)^{m-k+j-i}} \|\nabla^i u\|_{L^p(R)} \right)^p \right)^{1/p} \right).
\]

By the triangle inequality in the sequence space \( L^p \),
\[
\|\nabla^j u\|_{L^p_t L^\frac{p}{p-k} (\theta Q \setminus Q)} \leq \frac{C}{((\theta - 1)p)^{m-k+j-i}} \sum_{i=j}^{m-j} \left( \sum_{R \in G \setminus H} \|\nabla^i u\|_{L^p(R)} \right)^p \|\nabla^i u\|_{L^p(\theta Q \setminus Q)}.
\]

A similar (and simpler) argument establishes the bound on \( \|\nabla^j u\|_{L^p_t L^\frac{p}{p-k} (\theta Q \setminus Q)} \). \( \square \)
Lemma 42. Let \( d \geq 2 \) be an integer and let \( 1 \leq p < \infty \). There is a constant \( C = C_{d,p} \) depending only on \( d \) and \( p \) such that if \( Q \subset \mathbb{R}^d \) is a cube, \( 1 < \theta \leq 2 \), and \( u \in W^{1,p}(\partial Q \setminus Q) \), then

\[
\int_{\partial Q \setminus Q} |u - f_{\partial Q \setminus Q} u|^p \leq C_{d,p}|Q|^{p/d} \int_{\partial Q \setminus Q} |\nabla u|^p.
\]

**Proof.** We restrict to the case \( |Q| = 1 \) and where the midpoint of \( Q \) is the origin (that is, the case \( Q = (-1/2, 1/2)^d \)); rescaling and translating yields the general case.

Let \( \rho(X) = 2 \max\{X_1, \ldots, X_d\} \). Thus, if \( X \in \mathbb{R}^d \), then \( \rho(X) \) is the unique real number with \( X \in \partial(\rho(X) Q) \). Observe that \( \rho \) is a Lipschitz function with \( |\nabla \rho| = 2 \) almost everywhere and with \( \rho(X) \leq 2|X| \leq \sqrt{d} \rho(X) \). Define

\[
r(t) = \left( \frac{\theta^d - 1}{2^d - 1} t^d + \frac{2^d - \theta^d}{2^d - 1} \right)^{1/d}.
\]

Observe that \( r(1) = 1 \), \( r(2) = \theta \), \( r \) is increasing, \( r(t)/t \) is decreasing, and \( r(t)^{d-1} r'(t) = \frac{\theta^d - 1}{2^d - 1} t^{d-1} \). In particular, if \( 1 \leq t \leq 2 \) then \( 0 < r'(t) \leq \theta^d - 1 \).

Let \( \psi(X) = X r(\rho(X))/\rho(X) \). Then \( \psi \) is a bilipschitz change of variables \( \psi : 2Q \setminus Q \to \theta Q \setminus Q \).

If \( f \in L^1(2Q \setminus Q) \), then

\[
\int_{2Q \setminus Q} f = \frac{1}{2} \int_1^2 \int_{\partial (tQ)} f(X) \, d\sigma(X) \, dt
\]

where \( \sigma \) denotes \( d - 1 \)-dimensional Hausdorff measure (that is, surface measure on the boundary of the cube \( tQ \)). In particular, letting \( f = g \circ \psi \) and making the change of variables \( X = tY \) in the inner integral, we have that

\[
\int_{2Q \setminus Q} g \circ \psi = \frac{1}{2} \int_1^2 t^{d-1} \int_{\partial Q} g \circ \psi(tY) \, d\sigma(Y) \, dt.
\]

If \( Y \in \partial Q \), then \( \rho(tY) = t \) and so \( \psi(tY) = r(t) Y \). Thus

\[
\int_{2Q \setminus Q} g \circ \psi = \frac{1}{2} \int_1^2 \int_{\partial Q} g(r(t)Y) \, d\sigma(Y) \, t^{d-1} \, dt.
\]

Applying our above formula for \( r'(t) \),

\[
\int_{2Q \setminus Q} g \circ \psi = \frac{2^d - 1}{2(\theta^d - 1)} \int_1^2 \int_{\partial Q} g(r(t)X) \, d\sigma(X) \, r(t)^{d-1} \, r'(t) \, dt.
\]
Using the chain rule of single variable calculus and reversing our above arguments,
\[
\int_{2Q \setminus Q} g \circ \psi = \frac{2^d - 1}{2(2^d - 1)} \int_{\partial Q} g(rX) d\sigma(X) r^{d-1} dr
= \frac{2^d - 1}{2(2^d - 1)} \int_{\partial(rQ)} g(X) d\sigma(X) dr
= \frac{2^d - 1}{2^d - 1} \int_{\partial Q \setminus Q} g.
\]

We will apply this argument to \( g = u \) and to \( g = |u|^p \). In particular,
\[
\int_{\partial Q \setminus Q} u = \frac{1}{|\partial Q \setminus Q|} \int_{\partial Q \setminus Q} u = \frac{\theta^d - 1}{(2^d - 1)|\partial Q \setminus Q|} \int_{2Q \setminus Q} u \circ \psi = \int_{2Q \setminus Q} u \circ \psi.
\]

We also need to integrate the gradient. Let \( J_\psi \) be the Jacobian matrix for the change of variables \( \psi \), so that \( \nabla (u \circ \psi) = (J_\psi \nabla u) \circ \psi \). If \( X \in 2Q \setminus Q \), then
\[
\left| \frac{\partial u}{\partial X_k} \right| = \left| \frac{r(\rho(X))}{\rho(X)} \delta_{jk} + X_j \frac{r'(\rho(X))\rho(X) - r(\rho(X))}{\rho(X)^2} \partial_k \rho(X) \right| \leq 1 + 2(\theta^d - 1)
\]
and so \( J_\psi \) is a bounded matrix. Thus,
\[
\left( \int_{2Q \setminus Q} |\nabla (u \circ \psi)|^p \right)^{1/p} \leq C_d \left( \int_{2Q \setminus Q} |(J_\psi \nabla u) \circ \psi|^p \right)^{1/p}
\]
\[
\leq C_d \left( \int_{2Q \setminus Q} |\nabla u|^p \right)^{1/p}.
\]

Now,
\[
\int_{\partial Q \setminus Q} |u - f_{\partial Q \setminus Q}|^p u| \leq \frac{\theta^d - 1}{2^d - 1} \int_{2Q \setminus Q} |u \circ \psi - f_{2Q \setminus Q} u \circ \psi|^p
\]
\[
\leq C_{d,p} \frac{\theta^d - 1}{2^d - 1} \int_{2Q \setminus Q} |\nabla (u \circ \psi)|^p
\]
\[
\leq C_{d,p} \frac{\theta^d - 1}{2^d - 1} \int_{2Q \setminus Q} |\nabla u|^p = C_{d,p} \int_{\partial Q \setminus Q} |\nabla u|^p.
\]

Thus the Poincaré inequality holds in an annulus with constant independent of \( \theta \).

\[
3.4. \text{ Sobolev norms and cutoff functions.} \text{ A particular application of Lemmas 41 and 42 is the following result concerning smooth cutoff functions.}
\]

**Lemma 43.** Let \( m, d \in \mathbb{N}, d \geq 2, \) and let \( 1 \leq p < \infty \). There is a constant \( C \) depending on \( m, d \) and \( p \) with the following significance.

Let \( Q \subset \mathbb{R}^d \) be a cube and let \( 1 < \theta \leq 2 \). Let \( \chi \in C_c^\infty(\mathbb{R}^d) \) be a test function supported in \( \theta Q \) and identically equal to 1 in \( Q \), with \( 0 \leq \chi \leq 1 \). Define \( X = \max_{1 \leq i \leq d} (\theta - 1)|Q|^{1/d} \left\| \nabla^i \chi \right\|_{L^\infty(Q)} \).

If \( u \in W^{m,p}(\theta Q) \) \((\text{equivalently, if } u \in W^{m,p}(\theta Q)\)), and if we extend \( u \chi \) by zero outside of \( \theta Q \), then \( u \chi \in W^{m,p}(\mathbb{R}^d) \) and
\[
\left\| u \chi \right\|_{Y^{m,p}(\mathbb{R}^d)} \leq \left\| u \right\|_{Y^{m,p}(\theta Q)} + \sum_{i=0}^{m-1} \frac{CX}{((\theta - 1)|Q|^{1/d})^{m-i}} \left\| \nabla^i u \right\|_{L^p(\partial Q \setminus Q)}.
\]
Proof. We begin by using the definition of the $Y^{m,p}$-norm and the Leibniz rule.

$$
\|u\chi\|_{Y^{m,p}(R^d)} = \sum_{m-d/p<k\leq m} \|\nabla^k(u\chi)\|_{L^p(R^d)}
$$

$$
\leq \sum_{m-d/p<k\leq m} \left( \int_{\mathbb{R}^d} \left( \sum_{j=0}^k C_{j,k} |\nabla^{k-j}\chi| |\nabla^j u| \right)^{pk} \right)^{1/pk}.
$$

Observe that $C_{k,k} = 1$. By definition of $X$ and isolating the $j = k$ terms,

$$
\|u\chi\|_{Y^{m,p}(R^d)} \leq \sum_{m-d/p<k\leq m} \left( \int_{Q_{\theta}} |\nabla^k u|^{pk} \right)^{1/pk}
$$

$$
+ C \sum_{m-d/p<k\leq m} \left( \int_{Q_{\theta}} \left( \sum_{j=0}^{k-1} X(\theta - 1)^{j-k} |Q^{(j-k)/d}| |\nabla^j u| \right)^{pk} \right)^{1/pk}
$$

$$
\leq \|u\|_{Y^{m,p}(\theta Q)} + \sum_{m-d/p<k\leq m} \sum_{j=0}^{k-1} \frac{CX}{((\theta - 1)|Q|^{1/d})^{k-j}} \left( \int_{Q_{\theta}\setminus Q} |\nabla^j u|^{pk} \right)^{1/pk}.
$$

By Lemma 41

$$
\|u\chi\|_{Y^{m,p}(R^d)} \leq \|u\|_{Y^{m,p}(\theta Q)} + \sum_{i=0}^{m-1} \frac{CX}{((\theta - 1)|Q|^{1/d})^{m-i}} \|\nabla^i u\|_{L^p(\theta Q\setminus Q)}.
$$

This completes the proof. \qed

4. The Caccioppoli inequality

The Caccioppoli inequality was established first by Caccioppoli in the early twentieth century and is a foundational result used throughout the theory of second order divergence form equations. It has been generalized to the case of second order operators with lower order terms in [DHM18], and of higher order equations (without lower order terms) first in [Cam80], and later with some refinements in [AQ00, Bar16].

We now generalize these results to the case of higher order equations with lower order terms. We will follow [AQ00] and derive a Caccioppoli inequality for equations that satisfy the weak Gårding inequality (32) (and not necessarily the stronger Gårding inequality (3)). We will follow [Cam80] and establish the Caccioppoli inequality for solutions $\bar{u}$ to inhomogeneous equations $L\bar{u} = T$ for a (possibly nonzero) element $T$ of $Y^{-m,p}$.

We begin with the following lemma. This lemma was proven first in [Cam80] for operators of order $2m$ without lower order terms.

**Lemma 44.** Let $L$ be an operator of order $2m$ of the form (24) associated to coefficients $A$ that satisfy the weak Gårding inequality (32) and either the bound (7) or the bound (5).

Let $Q \subset \mathbb{R}^d$ be an open cube with sides parallel to the coordinate axes, and let $1 < \theta \leq 2$. Let $\bar{u} \in W^{m,2}(\theta Q)$. Let $T \in Y^{-m,2}(\theta Q)$. Suppose that $L\bar{u} = T$ in $\theta Q$ in the sense of formula (24).
Then we have that
\[
\int_Q |\nabla^m \bar{u}|^2 \leq \sum_{k=0}^{m-1} \frac{C}{((\theta - 1)|Q|^{1/d})^{2m-2k}} \int_{\theta Q \setminus Q} |\nabla^k \bar{u}|^2 + C\delta \int_{\theta Q} |\bar{u}|^2 + C\|T\|^2
\]
where \(C\) is a constant depending on the dimension \(d\), the order \(2m\) of \(L\), the number \(\lambda\) in the bound (52), and the number \(\Lambda\) in the bound (7) or (8). Here \(\|T\| = \|T\|_{Y_{m-2} \cap (\theta Q)}\) is the operator norm, that is, the smallest number such that \(|(\psi, T)| \leq \|\psi\|_{Y_{m-2} \cap (\theta Q)} \|T\|\) for all \(\psi \in Y_{m-2}^{m,2}(\theta Q)\).

**Proof.** Let \(\rho = ((\theta - 1)/2)|Q|^{1/d}\) be the distance from \(Q\) to \(\mathbb{R}^d \setminus \theta Q\). Let \(\varphi\) be a smooth, real valued test function with \(0 \leq \varphi \leq 1\), supported in \(\theta Q\) and identically equal to 1 on \(Q\). We require also that \(|\nabla^k \varphi| \leq C_k \rho^{-k}\) for any integer \(k \geq 0\).

Define \(\bar{\psi} = \varphi^m \bar{u}\). Notice that by Lemma 43, \(\bar{\psi} \in Y_{m,2}^{m}(\theta Q)\). Furthermore, by formula (24),

\[
\sum_{j,k=1}^{N} \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k = \langle T, \varphi^m \bar{u} \rangle_{\theta Q}.
\]

We first consider the left hand side of formula (45). By the Leibniz rule, and separating out the \(\gamma = \alpha\) terms, we see the following.

\[
\int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k = \int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \varphi^m \partial^\beta u_k
\]

\[+ \int_{\theta Q} \sum_{\gamma < \alpha} \frac{\alpha!}{\gamma!(\alpha - \gamma)!} \partial^{\alpha - \gamma}(\varphi^m) \partial^{\gamma}(\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k.
\]

Now as in Bar16, we write

\[
\sum_{\gamma < \alpha} \frac{\alpha!}{\gamma!(\alpha - \gamma)!} \partial^{\alpha - \gamma}(\varphi^m) \partial^{\gamma}(\varphi^m \pi_j) = \sum_{\zeta < \alpha} \varphi^{2m} \Phi_{\alpha,\zeta} \partial^{\zeta} \pi_j
\]

for some functions \(\Phi_{\alpha,\zeta}\) which are supported in \(\theta Q \setminus Q\) and satisfy \(|\Phi_{\alpha,\zeta}| \leq C\rho^{\zeta - |\alpha|}\). Thus we have

\[
\int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k = \int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \varphi^m \partial^\beta u_k
\]

\[+ \int_{\theta Q} \sum_{\zeta < \alpha} \Phi_{\alpha,\zeta} \partial^{\zeta} \pi_j A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k.
\]

It is desirable to have our final term in terms of \(\partial^{\beta} (\varphi^m u_k)\) rather than \(\varphi^{2m} \partial^\beta u_k\), so after one more application of the Leibniz rule, and writing as in formula (46), we have for some functions \(\Psi_{\beta,\xi}\) which are supported in \(\theta Q \setminus Q\) and satisfy \(|\Psi_{\beta,\xi}| \leq C\rho^{\xi - |\beta|}\),

\[
\int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k = \int_{\theta Q} \partial^\alpha (\varphi^m \pi_j) A^{j,k}_{\alpha,\beta} \varphi^m \partial^\beta u_k
\]

\[+ \int_{\theta Q} \sum_{\zeta < \alpha} \Phi_{\alpha,\zeta} \partial^{\zeta} \pi_j A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k
\]

\[+ \int_{\theta Q} \sum_{\zeta < \alpha} \Phi_{\alpha,\zeta} \partial^{\zeta} \pi_j A^{j,k}_{\alpha,\beta} \varphi^{2m} \Psi_{\beta,\xi} \partial^\xi u_k.
\]
By the condition (32), we have that

\[ \int_{\theta Q} \partial^n (\varphi^{2m} \bar{\Pi}) A^{j,k}_{\alpha,\beta} \partial^\beta (\varphi^{2m} u_k) = \int_{\theta Q} \partial^n (\varphi^{2m} \bar{\Pi}) A^{j,k}_{\alpha,\beta} \sum_{\xi<\beta} \varphi^m \Psi_{\beta,\xi} \partial^\xi u_k + \int_{\theta Q} \partial^n (\varphi^{2m} \bar{\Pi}) A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k. \]

Thus combining the previous two equations and reintroducing summation, we see that

\[ \sum_{j,k=1}^N \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{\theta Q} \partial^n (\varphi^{2m} \bar{\Pi}) A^{j,k}_{\alpha,\beta} \partial^\beta (\varphi^{2m} u_k) = \sum_{j,k=1}^N \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{\theta Q} \partial^n (\varphi^{4m} \bar{\Pi}) A^{j,k}_{\alpha,\beta} \partial^\beta u_k \]

\[ - \sum_{j,k=1}^N \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{\theta Q} \sum_{\xi<\alpha} \Phi_{\alpha,\xi} \partial^\xi \bar{\Pi} A^{j,k}_{\alpha,\beta} \partial^\beta (\varphi^{2m} u_k) \]

\[ + \sum_{j,k=1}^N \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{\theta Q} \sum_{\xi<\alpha} \Phi_{\alpha,\xi} \partial^\xi \bar{\Pi} A^{j,k}_{\alpha,\beta} \sum_{\xi<\beta} \varphi^m \Psi_{\beta,\xi} \partial^\xi u_k \]

\[ + \sum_{j,k=1}^N \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{\theta Q} \partial^n (\varphi^{2m} \bar{\Pi}) A^{j,k}_{\alpha,\beta} \sum_{\xi<\beta} \varphi^m \Psi_{\beta,\xi} \partial^\xi u_k. \]

We write this as I=II+III+IV+V. Observe that by formula (45),

\[ (47) \quad II = \langle T, \varphi^{4m} \bar{u} \rangle_{\theta Q}. \]

By the condition (65), we have that

\[ \lambda \| \nabla^m (\varphi^{2m} \bar{u}) \|_{L^2(\theta Q)}^2 \leq \text{Re I} + \delta \| \varphi^{2m} \bar{u} \|_{L^2(\theta Q)}. \]

Suppose that the condition (3) is true. By Hölder’s inequality and properties of \( \Phi_{\alpha,\zeta} \),

\[ |\text{III}| \leq \sum_{m-(d-1)/2 < |\alpha| \leq m} \sum_{m-(d-1)/2 < |\beta| \leq m} \frac{C \Lambda}{\rho^{||\alpha||-|\zeta|}} \| \partial^\zeta \bar{u} \|_{L^2_{\alpha} L^{\frac{2}{d}}_{\beta} (\theta Q)} \| \partial^\beta (\varphi^{2m} \bar{u}) \|_{L^2_{\alpha} L^{\frac{2}{d}}_{\beta} (\theta Q)}. \]

Recall that \( \varphi^{2m} \bar{u} \in Y^{m,2}_0 (\bar{\theta} Q) \) and so may be extended by zero to a \( Y^{m,2} (\mathbb{R}^d) \)-function. By Corollary (38) we have that

\[ \| \partial^\beta (\varphi^{2m} \bar{u}) \|_{L^2_{\alpha} L^{\frac{2}{d}}_{\beta} (\mathbb{R}^d)} = \| \partial^\beta (\varphi^{2m} \bar{u}) \|_{L^2_{\alpha} L^{\frac{2}{d}}_{\beta} (\mathbb{R}^d)} \leq C \| \nabla^m (\varphi^{2m} \bar{u}) \|_{L^2(\mathbb{R}^d)} = C \| \nabla^m (\varphi^{2m} \bar{u}) \|_{L^2(\theta Q)}. \]

Summing, we see that

\[ |\text{III}| \leq \sum_{m-d/2 < |\alpha| \leq m} \sum_{m-d/2 < |\beta| \leq m} \frac{C \Lambda}{\rho^{||\alpha||-|\zeta|}} \| \partial^\zeta \bar{u} \|_{L^2_{\alpha} L^{\frac{2}{d}}_{\beta} (\theta Q)} \| \varphi^{2m} \bar{u} \|_{W^{2,2}(\theta Q)}. \]
By Lemma 41
\[
\| \partial^C \bar{u} \|_{L^2_t L^2_x (\partial \Omega \setminus \Omega)} \leq \sum_{i=|\zeta|}^{m-(|\alpha|-|\zeta|)} \frac{C}{\rho^{m-|\alpha|+|\zeta|-|\gamma|}} \| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}.
\]

So
\[
|\text{III}| \leq \sum_{i=0}^{m-1} \frac{C}{\rho^{2m-2i}} \| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)} \| \varphi^{2m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)}.
\]

Applying Young’s inequality, we see that
\[
|\text{III}| \leq \sum_{i=0}^{m-1} \frac{C}{\rho^{2m-2i}} \| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2 + \frac{\lambda}{4} \| \varphi^{2m} u_k \|_{W^{m,2}(\partial \Omega \setminus \Omega)}.
\]

A similar argument with the roles of \( \alpha, \zeta \) and \( \beta, \xi \) reversed yields the same bound on \( V \), while an even simpler argument yields the bound
\[
|\text{IV}| \leq \sum_{i=0}^{m-1} \frac{C}{\rho^{2m-2i}} \| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2.
\]

The argument in the case that the condition (7) is true is similar.

We thus have that
\[
\lambda \| \varphi^{2m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)}^2 \leq \Re \text{I} + \delta \| \varphi^{2m} \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2
\]
\[
\leq |\text{II}| + |\text{III}| + |\text{IV}| + |V| + \delta \| \varphi^{2m} \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2
\]
\[
\leq |\text{II}| + C \sum_{i=0}^{m-1} \frac{\| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2}{\rho^{2m-2i}} + \delta \| \varphi^{2m} \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2
\]
\[
+ \frac{\lambda}{2} \| \varphi^{2m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)}^2.
\]

Subtracting the final term and applying formula (47) yields that
\[
\frac{\lambda}{2} \| \varphi^{2m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)}^2 \leq |\langle T, \varphi^{4m} \bar{u} \rangle_{\partial \Omega \setminus \Omega}| + C \sum_{i=0}^{m-1} \frac{\| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2}{\rho^{2m-2i}} + \delta \| \varphi^{2m} \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}^2.
\]

By definition of \( \| T \| \),
\[
|\langle T, \varphi^{4m} \bar{u} \rangle_{\partial \Omega \setminus \Omega}| \leq \| T \| \| \varphi^{4m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)}.
\]

By Lemma 43 with \( \chi = \varphi^{2m} \),
\[
\| \varphi^{4m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)} \leq \| \varphi^{2m} \bar{u} \|_{W^{m,2}(\partial \Omega \setminus \Omega)} + \sum_{i=0}^{m-1} \frac{C}{\rho^{m-1}} \| \nabla^i (\varphi^{2m} u) \|_{L^2 (\partial \Omega \setminus \Omega)}.
\]

Using the Leibniz rule and arguing as before,
\[
\| \varphi^{4m} \bar{u} \|_{Y^{m,2}(\mathbb{R}^d)} \leq \| \varphi^{2m} \bar{u} \|_{Y^{m,2}(\partial \Omega \setminus \Omega)} + C \sum_{i=0}^{m-1} \frac{C}{\rho^{m-1}} \| \nabla^i \bar{u} \|_{L^2 (\partial \Omega \setminus \Omega)}.
\]
By Corollary \[\text{(37)}\] \(\|\varphi^{2m}\tilde{u}\|_{Y^{m,2}(\theta Q)} \leq C\|\varphi^{2m}\tilde{u}\|_{W^{m,2}(\theta Q)}\). By Young’s inequality and formula \[\text{(48)}\] we have
\[
\frac{1}{2}\|\varphi^{2m}\tilde{u}\|^2_{W^{m,2}(\theta Q)} \leq C\|T\|^2 + \frac{1}{4}\|\varphi^{2m}\tilde{u}\|^2_{W^{m,2}(\theta Q)} + C\sum_{i=0}^{m-1}\|\nabla^i\tilde{u}\|^2_{L^{2}(\theta Q)} + \delta\|\varphi^{2m}\tilde{u}\|^2_{L^{2}(\theta Q)}.
\]
Subtracting the second term on the right hand side and observing that \(\|\nabla^m\tilde{u}\|^2_{L^2(\theta Q)} \leq \|\varphi^{2m}\tilde{u}\|^2_{W^{m,2}(\theta Q)}\) completes the proof.

We wish to improve the Caccioppoli inequality by removing the intermediate derivatives (that is, \(\nabla^k\tilde{u}\) for \(1 \leq k \leq m - 1\)). The following theorem was proven in \[\text{[Bar16, Theorem 18]}\] in the case of balls rather than cubes; the proof in \[\text{[Bar16]}\] carries through with the obvious modifications.

**Theorem 49.** Let \(Q \subset \mathbb{R}^d\) be a cube with sides parallel to the coordinate axes. Let \(1 < \theta \leq 2\). Suppose that \(\tilde{u} \in W^{m,2}(\theta Q)\) is a function that satisfies the inequality
\[
\int_{\partial Q} |\nabla^m\tilde{u}|^2 \leq \sum_{k=0}^{m-1} \frac{C_0}{((\mu - \theta)|Q|^{1/d})^{2m-2k}} \int_{\partial Q \setminus \partial Q} |\nabla^k\tilde{u}|^2 + F
\]
whenever \(0 < \vartheta < \mu < \theta\), for some \(F > 0\).

Then \(\tilde{u}\) satisfies the stronger inequality
\[
\int_{Q} |\nabla^m\tilde{u}|^2 \leq \frac{C}{((\theta - 1)|Q|^{1/d})^{2m}} \int_{\partial Q \setminus \partial Q} |\tilde{u}|^2 + CF
\]
for some constant \(C\) depending only on \(m\), the dimension \(d\), and the constant \(C_0\).

Furthermore, if \(0 \leq j \leq m\), then \(\tilde{u}\) satisfies
\[
\int_{Q} |\nabla^j\tilde{u}|^2 \leq \frac{C}{((\theta - 1)|Q|^{1/d})^{2j}} \int_{\partial Q} |\tilde{u}|^2 + C|Q|^{(m-2)/d}F.
\]

Now if we combine Lemma \[\text{[39]}\] and Theorem \[\text{[19]}\] we obtain the desired Caccioppoli inequality in which we bound \(|\nabla^m\tilde{u}|^2\) without the intermediate gradient terms, as stated in the following corollary.

**Corollary 51.** Let \(L\) be an operator of order \(2m\) of the form \[\text{(24)}\] associated to coefficients \(A\) that satisfy the weak Gårding inequality \[\text{(32)}\] and either the bound \[\text{(7)}\] or the bound \[\text{(5)}\].

Let \(Q \subset \mathbb{R}^d\) be an open cube with sides parallel to the coordinate axes, and let \(1 < \theta \leq 2\). Let \(\tilde{u} \in Y^{m,2}(\theta Q)\). Let \(T \in Y^{-m,2}(\theta Q)\). Suppose that \(L\tilde{u} = T\) in \(\theta Q\) in the sense of formula \[\text{(24)}\].

Then we have that
\[
\int_{Q} |\nabla^m\tilde{u}|^2 \leq \frac{C}{((\theta - 1)|Q|^{1/d})^{2m}} \int_{\partial Q} |\tilde{u}|^2 + C\delta \int_{\partial Q} |\tilde{u}|^2 + C\|T\|^2
\]
and for all \(j\) with \(1 \leq j \leq m - 1\) we have that
\[
\frac{1}{|Q|^{(2m-2)/d}} \int_{\partial Q} |\nabla^j\tilde{u}|^2 \leq \frac{C}{((\theta - 1)^{2j}|Q|^{2m})} \int_{\partial Q} |\tilde{u}|^2 + C\delta \int_{\partial Q} |\tilde{u}|^2 + C\|T\|^2
\]
where \(C\) is a constant depending on the dimension \(d\), the order \(2m\) of \(L\), the number \(\lambda\) in the bound \[\text{(32)}\], and the number \(\Lambda\) in the bound \[\text{(7)}\] or \[\text{(5)}\]. Here \(|T| = \).
\( \|T\|_{Y^{-m,2}(\partial \Omega)} \) is the operator norm, that is, the smallest number such that \( (\psi, T) \leq \|\psi\|_{Y^{-m,2}(\partial \Omega)} \|T\| \) for all \( \psi \in Y_{0}^{m,2}(\partial \Omega) \).

**Remark 53.** If \( m - d/2 < j < m \) and \( \delta = 0 \), then we can replace the term \( \int_{\partial \Omega} |\bar{u}|^{2} \) in the bound (22) by \( \int_{\partial \Omega_{\delta}} |\bar{u}|^{2} \) at a cost of some additional negative powers of \( \theta - 1 \). See Section 6.

5. Invertibility of \( L \)

In this section we will investigate boundedness and invertibility of the operator \( L : Y^{m,p}(\mathbb{R}^{d}) \rightarrow Y^{-m,p}(\mathbb{R}^{d}) \). The argument for invertibility parallels that used in [BHLG+ Lemma 3.4] in the second order case.

We remark that invertibility requires the Gårding inequality \([B] \), and not only the weaker Gårding inequality (22) of Section 2 and [AQ00]5; thus, for the remainder of this paper, we will always assume the strong Gårding inequality (5).

We will begin with boundedness of \( L \) for a range of \( p \).

**Lemma 54.** Let \( L \) be an operator of the form (24) associated to coefficients \( A \) that satisfy either the bound (7) or the bound (8).

If \( A \) satisfies the bound (7) then \( (\frac{d}{d+1}, \frac{2d}{d-1}) \subseteq \Pi_{L} = (\frac{d}{d+1}, \frac{d}{d-b}) \). If \( A \) satisfies the bound (8) then \( (\frac{2d}{d+1}, \frac{2d}{d-1}) \subseteq (\frac{d-1}{d+1-a-m}, \frac{d}{d-b}) \subseteq \Pi_{L} \), where \( \Pi_{L} \) is as in Definition (25).

If \( p \in \Pi_{L} \) then the constants \( \Lambda(p) \) in the bound (6) depend only on \( p, d, m \) and the constant \( \Lambda \) in the bound (7) or (8).

**Proof.** If \( L \) satisfies the condition (7) then \( m \geq a > m - d/2 \) and \( m \geq b > m - d/2 \). Observe that \( m, d \) and \( a, b \) are integers, and so \( m \geq a \geq m - d/2 + 1/2, m \geq b \geq m - d/2 + 1/2 \). A straightforward computation yields that

\[
\left( \frac{2d}{d+1}, \frac{2d}{d-1} \right) \subseteq \left( \frac{d}{d+a-m}, \frac{d}{d-b} \right).
\]

Similarly, if \( L \) satisfies the condition (8) then \( m \geq a \geq m - (d - 1)/2 + 1/2 \) and \( m \geq b \geq m - (d - 1)/2 + 1/2 \). Thus

\[
\left( \frac{2d}{d+1}, \frac{2d}{d-1} \right) \subseteq \left( \frac{2(d-1)}{d}, \frac{2(d-1)}{d-2} \right) \subseteq \left( \frac{d-1}{d-1+a-m}, \frac{d}{d-b} \right) \subseteq \left( \frac{d}{d+a-m}, \frac{d}{d-b} \right).
\]

Suppose that \( L \) satisfies the condition (7). If \( p \in (\frac{d}{d+a-m}, \frac{d}{d-b}) \) then \( a > m - d/p' \), \( b > m - d/p \), and so if \( a \leq |\alpha| \leq m \) and \( b \leq |\beta| \leq m \) then \( p'_{a} \) and \( p_{\beta} \) exist and are finite. By formulas (24) and (7),

\[
\frac{1}{p_{\beta}} + \frac{1}{(p')_{\alpha}} + \frac{1}{2a_{\alpha,\beta}} = 1.
\]

Thus by Hölder's inequality, for such \( p, \alpha, \) and \( \beta \),

\[
\int_{\mathbb{R}^{d}} |\partial^{\alpha} \varphi_{j} A_{\alpha,\beta}^{j,k} \partial^{\beta} u_{k}| \leq \| \partial^{\alpha} \varphi_{j} \|_{L^{p'}(\mathbb{R}^{d})} \| \partial^{\beta} u_{k} \|_{L^{p}(\mathbb{R}^{d})} \| A_{\alpha,\beta}^{j,k} \|_{L^{p}(\mathbb{R}^{d})}
\]

which by the condition (7) and the definition (22) of \( Y^{m,p}(\mathbb{R}^{d}) \) satisfies

\[
\int_{\mathbb{R}^{d}} |\partial^{\alpha} \varphi_{j} A_{\alpha,\beta}^{j,k} \partial^{\beta} u_{k}| \leq \Lambda \| \varphi \|_{Y^{m,p}(\mathbb{R}^{d})} \| \psi \|_{Y^{m,p}(\mathbb{R}^{d})}.
\]
Summing over \( \alpha, \beta, j \) and \( k \) and using Definition \([28]\) completes the proof.

Now suppose that \( L \) satisfies the condition \([8]\). If \( p \in \left( \frac{d-1}{d-m-1}, \frac{d-1}{m-1} \right) \) then \( a > m - (d-1)/p', \ b > m - (d-1)/p \), and so if \( a \leq |\alpha| \leq m \) and \( b \leq |\beta| \leq m \) then \( \tilde{p}'_\alpha \) and \( \tilde{p}_\beta \) exist and are finite. Again

\[
\frac{1}{p_\beta} + \frac{1}{(p')_\alpha} + \frac{1}{2} > 1.
\]

Observe that

\[
\int_{\mathbb{R}^d} |\partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} |\partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k| \, dx \, dt.
\]

Applying Hölder’s inequality first in \( \mathbb{R}^{d-1} \) and then in \( \mathbb{R} \) yields that

\[
\int_{\mathbb{R}^d} |\partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta \psi_k| \leq \Lambda \|\partial^\alpha \varphi_j\|_{L^p L^q(\mathbb{R}^d)} \|\partial^\beta \psi_k\|_{L^p L^q(\mathbb{R}^d)}.
\]

Applying Corollary \([38]\) and summing completes the proof. \( \square \)

We now establish invertibility of \( L \) for \( p = 2 \). The main tool in the proof is the complex valued Lax-Milgram lemma, which we now state.

**Theorem 55.** \([\text{Babb71}]\) Theorem 2.1] Let \( H_1 \) and \( H_2 \) be two Hilbert spaces, and let \( B \) be a bounded sesquilinear form on \( H_1 \times H_2 \) that is coercive in the sense that

\[
\sup_{w \in H_1 \setminus \{0\}} \frac{|B(u,v)|}{\|w\|_{H_1}} \geq \lambda \|v\|_{H_2}, \quad \sup_{w \in H_2 \setminus \{0\}} \frac{|B(u,w)|}{\|w\|_{H_2}} \geq \lambda \|u\|_{H_1}
\]

for every \( u \in H_1 \), and \( v \in H_2 \), for some fixed \( \lambda > 0 \). Then for every linear functional \( T \) defined on \( H_2 \) there is a unique \( u_T \in H_1 \) such that \( B(v,u_T) = \langle T, v \rangle \). Furthermore \( \|u_T\|_{H_1} \leq \frac{1}{\lambda} \|T\|_{H_2'} \).

**Lemma 56.** Let \( L \) be an operator of the form \([24]\) order \( 2m \) which satisfies the ellipticity condition \([5]\) and such that \( 2 \in \Pi_L \), where \( \Pi_L \) is as in Definition \([28]\). Then \( L \) is invertible with bounded inverse \( Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d) \).

**Proof.** Let \( B(u, v) \) be the form given by

\[
(57) \quad B(u, v) = \sum_{j,k=1}^N \sum_{|\alpha| \leq m, |\beta| \leq m} \int_{\mathbb{R}^d} \partial^\alpha u_j A^{j,k}_{\alpha,\beta} \partial^\beta v_k.
\]

Notice that by formula \([5]\) \( B \) is a coercive sesquilinear operator on \( Y^{m,2}(\mathbb{R}^d) \times Y^{m,2}(\mathbb{R}^d) \) in the sense of Theorem 55 while by Definition \([28]\) \( B \) is bounded on \( Y^{m,2}(\mathbb{R}^d) \times Y^{m,2}(\mathbb{R}^d) \) with the bound

\[
(58) \quad |B(u, v)| \leq \Lambda (2) \|u\|_{Y^{m,2}(\mathbb{R}^d)} \|v\|_{Y^{m,2}(\mathbb{R}^d)}.
\]

Let \( T \) be an element of \( Y^{-m,2}(\mathbb{R}^d) \). Recall that we write bounded linear operators on \( Y^{m,2}(\mathbb{R}^d) \) as \( \langle T, \cdot \rangle \). Let \( \tilde{u}_T \in Y^{m,2}(\mathbb{R}^d) \) be the unique element of \( Y^{m,2}(\mathbb{R}^d) \) given by the Lax-Milgram lemma, so

\[
(59) \quad \sum_{j,k=1}^N \sum_{|\alpha| \leq m, |\beta| \leq m} \int_{\mathbb{R}^d} \partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta (u_T)_k = \langle T, \varphi \rangle
\]

for all \( \varphi \in Y^{m,2}(\mathbb{R}^d) \). Observe that by formula \([24]\), \( L \tilde{u}_T = T \). By the boundedness property of the Lax-Milgram lemma, \( \|\tilde{u}_T\|_{Y^{m,2}(\mathbb{R}^d)} \leq \frac{1}{\lambda} \|T\|_{Y^{-m,2}(\mathbb{R}^d)} \), and by the uniqueness property in the Lax-Milgram lemma, \( \tilde{u} = \tilde{u}_T \) is the only element of
Y^{m,2}(\mathbb{R}^d)$ with $Lu = T$. Thus the operator $T \mapsto \tilde{u}_T$ is well defined, bounded, linear, and an inverse to $L$. \hfill \Box

We conclude this section by establishing invertibility of $L$ for a range of $p$. In this case the main tool is Šneiberg’s lemma.

**Lemma 60.** (Šneiberg’s lemma \cite{ABES19} Theorem A.1) Let $\overline{X} = (X_0, X_1)$ and $\overline{Z} = (Z_0, Z_1)$ be interpolation couples, and $T \in \mathcal{B}(\overline{X}, \overline{Z})$. Suppose that for some $\theta^* \in (0, 1)$ and some $\kappa > 0$, the lower bound $\|Tx\|_{Z_{\theta^*}} \geq \kappa \|x\|_{X_{\theta^*}}$ holds for all $x \in X_{\theta^*}$. Then the following are true.

(i) Given $0 < \epsilon < 1/4$, the lower bound $\|Tx\|_{Z_{\theta^*}} \geq \epsilon \kappa \|x\|_{X_{\theta}}$ holds for all $x \in X_{\theta}$, provided that $|\theta - \theta^*| \leq \frac{\kappa(1-\epsilon)}{\min \{\theta^*, 1-\theta^*\}}$, where $M = \max_{j=0,1} \|T\|_{X_j \to Z_j}$.

(ii) If $T : X_{\theta^*} \to Z_{\theta^*}$ is invertible, then the same is true for $T : X_{\theta} \to Z_{\theta}$ if $\theta$ is as in (i). The inverse mappings agree on $Z_{\theta} \cap Z_{\theta^*}$ and their norms are bounded by $\frac{1}{\epsilon \kappa}$.

**Lemma 61.** Let $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$ be bounded and invertible, and suppose that $L$ extends by density to a bounded operator $L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$ for all $p$ in an open neighborhood of 2.

Let $\Upsilon_L$ be as in Definition 33 that is, the set of all $p$ such that $L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$ is bounded and compatibly invertible.

Then $\Upsilon_L$ is an interval, and there is a $\delta > 0$ such that if $2 - \delta < p < 2 + \delta$ then $p \in \Upsilon_L$.

In particular, these conditions are satisfied if $L$ is an operator of the form (24) that satisfies the ellipticity condition (iii) and such that $\Pi_L$ as given by Definition 28 contains an open neighborhood of 2. In this case $\delta$ depends only on $\Pi_L$ and the standard parameters.

**Proof.** By assumption or by Lemma 60 $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$ is invertible. Thus $2 \in \Upsilon_L$.

By [Tri95], $W^{m,p}(\mathbb{R}^d)$ forms a complex interpolation scale. The map which sends an element of $W^{m,p}(\mathbb{R}^d)$ to its unique representative in $Y^{m,p}(\mathbb{R}^d)$ is a retract [KMM07] Lemma 7.11, and so we have that $Y^{m,p}(\mathbb{R}^d)$ forms a complex interpolation scale. Next, we have from [BL76] Theorem 4.5.1 that the antidual space $Y^{-m,p}(\mathbb{R}^d)$ also forms a complex interpolation scale.

A straightforward interpolation argument shows that if $L$ is bounded and compatibly invertible $Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$ then $L$ is bounded and compatibly invertible $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ whenever $q$ is between $p$ and 2, and so $\Upsilon_L$ is an interval.

Finally, by Šneiberg’s lemma, $L$ is invertible $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ whenever $2 - \delta < q < 2 + \delta$, where $\delta$ is as dictated by (i) from Šneiberg’s lemma. This completes the proof. \hfill \Box

6. $L^p$ Bounds on Solutions and Their Gradients

In [Mey63], Meyers established a reverse Hölder estimate; in the notation of the present paper, he established that if $L = -\nabla \cdot A \nabla$ is a second order divergence form operator without lower order terms, and if $Q$ is a cube, then for all $p$ and $q$ sufficiently close to 2 (and, in particular, for some $p > 2$ and $q < 2$) we have the estimate

$$\|\nabla u\|_{L^p(Q)} \leq C|Q|^{1/p-1/q}\|\nabla u\|_{L^q(2Q)} + C\|Lu\|_{Y^{-1,p}(2Q)}$$
for all suitable functions \( u \). The exponent \( q \) on the right hand side can be lowered if desired; see [FS72, Section 9, Lemma 2] in the case of harmonic functions, and [Bar16, Lemma 33] for more general functions. Meyers’s results can be generalized to second order systems (even nonlinear systems) without lower order terms (see [Gio83, Chapter V]), or to higher order equations without lower order terms (see [Cam80, AQ00, Bar16]).

Theorem 64 represents a generalization to the case of operators with lower order terms. It follows immediately from the next theorem and Lemma 61. We remark that the \( m = 1 \) case of this theorem was essentially established in \( \text{[BHLG}^+ \text{]} \). We refer to Section 3.1 and that the higher order case uses many of the same arguments.

**Theorem 62.** Let \( m \geq 1 \) and \( d \geq 2 \) be integers. Let \( L \) be an operator of order \( 2m \) of the form (24) associated to coefficients \( A \) that satisfy the Gårding inequality (10) and either the bound (7) or the bound (8).

Let \( \Pi_L \) and \( L_L \) be as in Definitions 25 and 33. Let \( p, \mu \in \mathcal{Y}_L \cap \Pi_L \) with \( p \geq 2 \) and let \( 0 < q \leq \infty \). Let \( j \) and \( \varpi \) be integers with \( 0 \leq j \leq m \) and \( 0 \leq \varpi \leq \min(j, b) \).

If \( p = 2 \), we impose the additional requirement that either \( q \geq 2 \) or \( \varpi \geq 1 \).

Let \( Q \subset \mathbb{R}^d \) be a cube with sides parallel to the coordinate axes. Let \( 1 < \theta \leq 2 \).

Suppose that \( \tilde{u} \in Y^{m,\mu}(\theta Q) \) and that \( L\tilde{u} \in Y^{-m,p}(\theta Q) \) (in the sense that if \( \psi \in Y^{m,p'}_0(\theta Q) \cap \mathcal{Y}^{m,p'}_0(\theta Q) \) then \( \|L\tilde{u},\psi\|_{\theta Q} \leq C\|\tilde{u}\|_{Y^{m,p}(\theta Q)} \).

Then \( \nabla^j u \in L^p(Q) \), and there exist positive constants \( k \) and \( C \) depending on \( p, q, \) and the standard parameters such that

\[
\frac{1}{|Q|^{(m-j)/d}} \|\nabla^j u\|_{L^p(Q)} \leq \frac{C}{(\theta - 1)^n} \|L\tilde{u}\|_{Y^{-m,p}(\theta Q)} + \frac{C|Q|^{1/p - 1/q - (m - \varpi)/d}}{(\theta - 1)^n} \|\nabla^\varpi u\|_{L^p(\theta Q \setminus Q)}.
\]

Here \( b \) is as in Definition 25 that is, \( b = \min\{|\beta| : A_{\alpha,\beta}(X) \neq 0 \text{ for some } \alpha, j, k, \text{ and } X \} \).

**Remark 63.** If \( j > m - d/p \), we may of course immediately apply the Gagliardo-Nirenberg-Sobolev inequality (Lemma 39) to bound \( \|\nabla^j u\|_{L^p(Q)} \); if \( j < m - d/p \), then improved estimates on \( \nabla^j \tilde{u} \), such as local Hölder continuity, may be derived from further Sobolev space results such as Morrey’s inequality.

In the case of operators without lower order terms (in which case \( b = m \)), we may take \( j = \varpi = m \); Theorem 62 then yields the same bounds as the classical inequality of Meyers (and the generalizations of \( \text{[Cam80, AQ00, Bar16]} \)).

We will also establish an estimate for functions \( \tilde{u} \) with \( L\tilde{u} \in Y^{-m,p}(\theta Q) \) for \( p < 2 \) sufficiently close to 2.

**Theorem 64.** Let \( m \geq 1 \) and \( d \geq 2 \) be integers. Let \( L \) be an operator of order \( 2m \) of the form (24) associated to coefficients \( A \) that satisfy the Gårding inequality (10) and either the bound (7) or the bound (8).

Let \( \Pi_L \) and \( L_L \) be as in Definitions 25 and 33. Let \( p, \mu \in \mathcal{Y}_L \cap \Pi_L \) and let \( 0 < q \leq \infty \). Let \( j \) be an integer with \( 0 \leq j \leq m \).

Let \( Q \subset \mathbb{R}^d \) be a cube with sides parallel to the coordinate axes. Let \( 1 < \theta \leq 2 \).

Suppose that \( \tilde{u} \in Y^{m,\mu}(\theta Q) \) and that \( L\tilde{u} \in Y^{-m,p}(\theta Q) \).
Then $\nabla^j u \in L^p(Q)$, and there exist positive constants $\kappa$ and $C$ depending on $p$, $q$, and the standard parameters such that
\[
\frac{1}{|Q|^{(m-j)/d}} \|\nabla^j u\|_{L^p(Q)} \leq \frac{C}{(\theta - 1)^{\kappa}} \|L\tilde{u}\|_{Y^{-m,p}(\partial Q)} + \frac{C|Q|^{1/p - 1/q}}{(\theta - 1)^{\kappa}} \sum_{i=\min(j,b)}^m \frac{1}{|Q|^{(j-i)/d}} \|\nabla^i \tilde{u}\|_{L^p(\partial Q \setminus Q)}.
\]

Given operators with lower order terms, Theorem 62 cannot be strengthened, as shown in the following example.

**Theorem 65.** Let $d \geq 3$, $m \geq 1$, $\alpha \in (m - d/2, m]$, and $\beta \in (m - d/2, m]$ be nonnegative integers, and let $\varepsilon > 0$.

Let $Q_0 \subset \mathbb{R}^d$ be the cube of volume 1 centered at the origin. Let $\tilde{A}_{\alpha,\beta}$ be real nonnegative constant coefficients such that
\[
(-\Delta)^m = (-1)^m \sum_{|\alpha| = |\beta| = m} \tilde{A}_{\alpha,\beta} \partial^\alpha \partial^\beta, \quad \tilde{A}_{\alpha,\beta} = 0 \text{ if } |\alpha| < m \text{ or } |\beta| < m.
\]

Then there exists a linear operator $L$ of the form (24) with $N = 1$ associated to coefficients $A_{\alpha,\beta} = \tilde{A}_{\alpha,\beta}$ and a function $u$ such that
- $\|A_{\alpha,\beta} - \tilde{A}_{\alpha,\beta}\|_{L^\infty(Q)} \leq \varepsilon$ for all $|\alpha| \leq m$ and $|\beta| \leq m$.
- The numbers $\alpha$ and $\beta$ chosen above also satisfy the conditions (26)–(27) given in Definition 22.
- $Lu = 0$ in $Q$.
- If $\tilde{C} > 0$ and $2 < p < \infty$ then there is a cube $Q \subseteq Q_0$ with
\[
\|\nabla^m u\|_{L^p(Q)} \geq \tilde{C} \sum_{i=b+1}^m |Q|^{1/p - 1/2 - (m-i)/d} \|\nabla^i \tilde{u}\|_{L^p(\partial Q \setminus Q)}.
\]

Constant coefficient operators without lower order terms such as $(-\Delta)^m$ clearly satisfy the bounds (7) and (8) for some $\Lambda > 0$. Extending $A_{\alpha,\beta}$ by zero, we see that by taking $\varepsilon$ small enough, we may ensure that $L$ satisfies the bound (7) with constant $\Lambda$ arbitrarily close to that of $(-\Delta)^m$.

By an elementary (and very well known) argument using the Fourier transform, the operator $(-\Delta)^m$ satisfies the bound (6) for some $\Lambda > 0$. By Corollary 37 and again by taking $\varepsilon$ small enough, the operator $L$ satisfies the bound (5) with constant $\Lambda$ arbitrarily close to that of $(-\Delta)^m$.

We will prove Theorems 62 and 64 in Section 6.4 and prove Theorem 65 in Section 6.2.

6.1. **Proof of Theorems 62 and 64** We begin with the following variant of Lemmas 39, 40, and 41 in the case where the exponents on each side are different.

**Lemma 66.** Let $m$, $d \in \mathbb{N}$, $d \geq 2$, $p \in [1, \infty)$, and let $j$, $k \in \mathbb{N}_0$ satisfy $0 \leq j \leq k - 1$ and $m - d/p < k \leq m$. Let $p_k = p_{m,d,k}$. Let $1 < \theta \leq 2$. Let $\mu$ satisfy $0 < 1/\mu \leq \min(1, 1/p + 1/d)$.
Then there is a constant $C$ depending only on $p$, $d$, and $m$ such that if $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes and $u \in W^{m,p}(\theta Q)$, then

$$
\| \nabla^j u \|_{L^p(\theta Q)} \leq \sum_{i=j}^m C \left| Q \right|^{1/p-1} \| \nabla^i u \|_{L^p(\theta Q)},
$$

$$
\| \nabla^j u \|_{L^p(\theta Q \setminus Q)} \leq \sum_{i=j}^m C \left| Q \right|^{1/p-1} \| \nabla^i u \|_{L^p(\theta Q \setminus Q)}.
$$

If in addition $k > m - (d - 1)/p$, then

$$
\| \nabla^j u \|_{L^p \cap L^p(\theta Q)} \leq \sum_{i=j}^m C \left| Q \right|^{1/p-1} \| \nabla^i u \|_{L^p(\theta Q)},
$$

$$
\| \nabla^j u \|_{L^p \cap L^p(\theta Q \setminus Q)} \leq \sum_{i=j}^m C \left| Q \right|^{1/p-1} \| \nabla^i u \|_{L^p(\theta Q \setminus Q)}.
$$

Proof. By Hölder’s inequality, it suffices to establish the listed bounds for the endpoint value $1/\mu = \min(1, 1/p + 1/d)$. We will establish the last of the listed bounds; the arguments for the three preceding bounds are similar (in the first two cases with Lemmas 39 or 40 in place of Lemma 41).

By Lemma 41 and because $k - j \geq 1$, we have that

$$
\| \nabla^j u_k \|_{L^p(\theta Q \setminus Q)} \leq \sum_{i=j}^{m-1} \frac{C}{(\theta - 1) \left| Q \right|^{1/d}} \| \nabla^i u \|_{L^p(\theta Q \setminus Q)}.
$$

Recall that we have taken $\mu$ to satisfy $1/\mu = \min(1, 1/p + 1/d)$. Because $d \geq 2$, we have that

$$
0 < \frac{1}{\mu_{m-1}} = \frac{1}{\mu} - \frac{1}{d} \leq \frac{1}{p}
$$

(in particular, $\mu_{m-1}$ exists) and so by Hölder’s inequality,

$$
\| \nabla^j u_k \|_{L^p(\theta Q \setminus Q)} \leq \sum_{i=j}^{m-1} C \left| Q \right|^{1/p-1} \| \nabla^i u \|_{L^{p-1}(\theta Q \setminus Q)}.
$$

Another application of Lemma 41 yields

$$
\| \nabla^j u_k \|_{L^p(\theta Q \setminus Q)} \leq \sum_{i=j}^{m-1} \frac{C}{(\theta - 1) \left| Q \right|^{1/d}} \| \nabla^i u \|_{L^p(\theta Q \setminus Q)}
$$

as desired. \hfill \Box

Now, recall from Lemma 43 that if $u \in Y^{m,\mu}(\theta Q)$ then $u \chi \in Y^{m,\mu}(\theta Q)$ for all $\chi \in C_0^\infty(\theta Q)$. By Definition 28 if $\mu \in \Pi_L$ then $L(u \chi) \in Y^{-m,\mu}(\mathbb{R}^d)$. We now show that under some circumstances, $L(u \chi)$ is also in $Y^{-m,p}(\mathbb{R}^d)$.

**Lemma 67.** Let $m \geq 1$ and $d \geq 2$ be integers. Let $L$ be an operator of the form (24) for some coefficients $A$ that satisfy either the bound (71) or the bound (58).

If $A$ satisfies the bound (71), let $p$, $\mu \in (\frac{d}{d-1}, \frac{d}{d-1} - m, \frac{d}{d-1})$. If $A$ satisfies the bound (58), let $p$, $\mu \in (\frac{d}{d-1} + m, \frac{d}{d-1} - m, \frac{d}{d-1})$. By Lemma 72 these ranges include $(\frac{2d}{d+1}, \frac{2d}{d-1})$. In either case we additionally require that $1/\mu \leq 1/p + 1/d$. 

Let $Q \subset \mathbb{R}^d$ be a cube with sides parallel to the coordinate axes. Let $1 < \theta \leq 2$. Let $u \in Y^{m,\mu}(\theta Q)$ be such that $L u \in Y^{-m,p}(\theta Q)$ (in the sense that if $\bar{v} \in Y^{m,p'}(\theta Q) \cap Y^{-m,\mu}(\theta Q)$ then $\langle [L u, \bar{v}]_{\theta Q}\rangle \leq C \|\bar{v}\|_{Y^{m,p'}(\theta Q)}$).

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a test function with $0 \leq \chi \leq 1$ such that $\chi = 1$ in $Q$ and $\chi = 0$ outside $\theta Q$. We extend $u \chi$ by 0 outside of $\theta Q$. Then $L(u \chi)$ extends to a bounded operator on $Y^{m,p'}(\mathbb{R}^d)$.

Furthermore, if $0 \leq \varpi \leq b$, then there is a polynomial $\bar{P}$ of degree less than $\varpi$ and positive constants $C$ and $\kappa$ depending on the standard parameters such that

$$\|L((\bar{u} - \bar{P})\chi)\|_{Y^{-m,p}(\mathbb{R}^d)} \leq \frac{C}{(\theta - 1)^m} \|L u\|_{Y^{-m,p}(\theta Q)} + \frac{C|\Lambda|Q|1/p-1/\mu|}{(\theta - 1)^\alpha} \sum_{i=\infty}^m \frac{1}{|Q|^{(m-i)/\mu}} \|\nabla^i u\|_{L^\mu(\theta Q\setminus \theta Q)}$$

where $X = \max_{1 \leq i \leq d}(\theta - 1)^i |Q|^{i/d} \|\nabla^i \chi\|_{L^\infty(\mathbb{Q})}$.

We follow the convention that the zero function is a polynomial of negative degree; thus, if $\varpi = 0$ then $P = 0$. For any $p \in \left(\frac{d}{d+a_m}, \frac{d}{m-b}\right)$ or $\left(\frac{d-1}{d-1+a_m}, \frac{d-1}{m-b}\right)$, there is a $\mu$ in the same range with $\mu < p$ and with $1/\mu \leq 1/p + 1/d$.

**Proof of Lemma 67.** Let $\bar{P}$ be the polynomial of degree less than $\varpi$ with $\int_{\theta Q\setminus \theta Q} \partial^\gamma(\bar{u} - \bar{P}) = 0$ for all $|\gamma| < \varpi$. Because $\varpi \leq b$ and by definition of $b$, $L \bar{P} = 0$. The function $\chi \bar{P}$ is smooth and compactly supported and so $L(\chi \bar{P}) \in Y^{-m,p}(\mathbb{R}^d)$. Thus, we need only show that $L((\bar{u} - \bar{P})\chi) \in Y^{-m,p}(\mathbb{R}^d)$ and establish an appropriate bound on its norm. For notational convenience we will take $\bar{P} = 0$.

Recall that $Y^{-m,p}(\mathbb{R}^d)$ is the antidual space to $Y^{m,p'}(\mathbb{R}^d)$. So to show that $L(\chi \bar{u}) \in Y^{-m,p}(\mathbb{R}^d)$, we need only bound $\langle L(\chi \bar{u}), \varphi \rangle$ for all $\varphi$ in $Y^{m,p'}(\mathbb{R}^d)$. By density we may assume that $\varphi \in Y^{m,\mu'}(\mathbb{R}^d)$, so by Lemma 63 $\langle L(\bar{u} \chi), \varphi \rangle$ represents an absolutely convergent integral.

Let $\bar{\varphi}$ be (a representative of) an element of $Y^{m,p'}(\mathbb{R}^d) \cap Y^{m,\mu'}(\mathbb{R}^d)$. By the weak definition (24) of $L$,

$$\langle L(\bar{u} \chi), \varphi \rangle = \int_{\theta Q} \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \partial^\alpha \varphi_j A^{j,k}_{\alpha,\beta} \partial^\beta(\chi u_k).$$

Let $\bar{\psi} = \bar{\varphi} - \bar{\bar{R}}$, where $\bar{\bar{R}}$ is the polynomial of degree less than $a$ with $\int_{\theta Q} \partial^\gamma(\bar{\varphi} - \bar{\bar{R}}) = 0$ for all $|\gamma| < a$. Then $L^* \bar{\bar{R}} = 0$. Therefore,

$$\langle L(\bar{u} \chi), \varphi \rangle = \langle L(\bar{u} \chi), \bar{\varphi} - \bar{\bar{R}} \rangle = \langle L(\bar{u} \chi), \bar{\psi} \rangle = \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\theta Q} \partial^\alpha \psi_j A^{j,k}_{\alpha,\beta} \partial^\beta(\chi u_k).$$

We remark on the symmetry of our situation: $\bar{\psi} \in Y^{m,p'}(\theta Q)$, $\bar{u} \in Y^{m,\mu}(\theta Q)$, $\int_{\theta Q} \partial^\gamma \bar{u} = 0$ if $|\gamma| < a$, and $\int_{\theta Q\setminus \theta Q} \partial^\gamma u = 0$ if $|\beta| < \varpi$. 
By the Leibniz rule,
\[
\langle L(\bar{\phi} \chi), \varphi \rangle = \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\theta Q} \frac{\partial^\alpha (\psi_j \chi)}{\partial^\alpha} A_{\alpha,\beta}^{j,k} \partial^\beta u_k
\]
\[
+ \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \sum_{\gamma < \beta} \int_{\theta Q \setminus Q} \frac{\beta!}{\gamma!(\beta - \gamma)!} \frac{\partial^\alpha \psi_j A_{\alpha,\beta}^{j,k} \partial^\gamma u_k}{\partial^\alpha} \partial^\beta - \gamma \chi
\]
\[
- \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \sum_{\delta < \alpha} \int_{\theta Q \setminus Q} \frac{\alpha!}{\delta!(\alpha - \delta)!} \frac{\partial^\alpha \psi_j A_{\alpha,\beta}^{j,k} \partial^\delta u_k}{\partial^\alpha} \partial^\beta - \delta \chi.
\]
Recall from Lemma 13 that $\bar{\chi} \bar{\psi} \in Y_0^{m,p'}(\theta Q) \cap Y_0^{m,p'}(Q)$. By the weak definition (24) of $L$, we have that
\[
\sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\theta Q} \frac{\partial^\alpha (\psi_j \chi)}{\partial^\alpha} A_{\alpha,\beta}^{j,k} \partial^\beta u_k = \langle L \bar{\psi}, \bar{\psi} \rangle_{\theta Q}.
\]
By definition of $Y^{-m,p}$,
\[
\|\bar{\chi} \bar{\psi}\|_{Y_0^{m,p'}(\theta Q)} \leq \|L \bar{\psi}\|_{Y^{-m,p}(\theta Q)} \|\bar{\psi}\|_{Y_0^{m,p'}(\theta Q)}.
\]
By Lemmas 13 and 39,
\[
\|\bar{\chi} \bar{\psi}\|_{Y_0^{m,p'}(\theta Q)} \leq \sum_{i=0}^m \frac{CX}{(\theta - 1)^{m-i}} \frac{1}{Q^{(m-i)/d}} \|\nabla^i \bar{\psi}\|_{L^{p'}(\theta Q)}.
\]
By the Poincaré inequality, and because $\nabla^i \bar{\psi} = \nabla^i \bar{\varphi}$ for all $i \geq a$,
\[
\|\bar{\chi} \bar{\psi}\|_{Y_0^{m,p'}(\theta Q)} \leq \sum_{i=a}^m \frac{CX}{(\theta - 1)^{m}} \frac{1}{Q^{(m-i)/d}} \|\nabla^i \bar{\varphi}\|_{L^{p'}(\theta Q)}.
\]
Recall that $p > \frac{d}{d-m+a}$. If $i \geq a$, then $1/p' - (m - i)/d > 0$ and so by formula (21) $(p')_i$ is well defined and finite. Thus by Hölder's inequality
\[
\|\bar{\chi} \bar{\psi}\|_{Y_0^{m,p'}(\theta Q)} \leq \frac{CX}{(\theta - 1)^{m}} \|\bar{\varphi}\|_{Y^{m,p'}(\theta Q)}.
\]
Thus
\[
|\langle L(\bar{\psi} \chi), \varphi \rangle| \leq \frac{C}{(\theta - 1)^m} \|L \bar{\psi}\|_{Y^{-m,p}(\theta Q)} \|\bar{\varphi}\|_{Y^{m,p'}(\theta Q)}
\]
\[
+ \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \sum_{\gamma < \beta} \frac{\beta!}{\gamma!(\beta - \gamma)!} \frac{\partial^\alpha \psi_j A_{\alpha,\beta}^{j,k} \partial^\gamma u_k}{\partial^\alpha} \partial^\beta - \gamma \chi
\]
\[
+ \sum_{j,k=1}^N \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \sum_{\delta < \alpha} \frac{\alpha!}{\delta!(\alpha - \delta)!} \frac{\partial^\alpha \psi_j A_{\alpha,\beta}^{j,k} \partial^\delta u_k}{\partial^\alpha} \partial^\beta - \delta \chi.
\]
We will now bound the integrals over $\theta Q \setminus Q$.
Suppose that the coefficients $A$ satisfy the condition (32). Let $\alpha$ and $\beta$ be such that $A_{\alpha,\beta}^{j,k}$ is not identically equal to zero. By assumption on $\mu$ and $p$, this means
that $\tilde{\mu}, \tilde{\nu}, (\tilde{p'})_{\alpha}$, and $\tilde{\mu}'_{\alpha}$ exist and are finite. By Hölder's inequality in $\mathbb{R}^{d-1}$ and then in $\mathbb{R}$,

$$\left| \int_{\theta Q} \partial^\alpha \psi \partial A^{i,j} \partial^\gamma \nu \partial^\beta u \right|$$

$$\leq \| \partial^\alpha \psi \|_{L^p(\theta Q)} \| \partial^\gamma \nu \|_{L^{\infty}(\theta Q)} \| A^{i,j} \|_{L^{p,\beta}(\theta Q)} \| \partial^\beta u \|_{L^{p,\beta}(\theta Q)}$$

and

$$\left| \int_{\theta Q} \partial^\alpha \psi \partial A^{i,j} \partial^\gamma \nu \partial^\beta u \right|$$

$$\leq \| \partial^\alpha \psi \|_{L^p(\theta Q)} \| \partial^\gamma \nu \|_{L^{\infty}(\theta Q)} \| A^{i,j} \|_{L^{p,\beta}(\theta Q)} \| \partial^\beta u \|_{L^{p,\beta}(\theta Q)}.$$

Because $|\alpha| \geq a$ we have that $\partial^\alpha \tilde{\psi} = \partial^\alpha \tilde{\varphi}$. By Lemma 40 with $j = k = |\alpha|$, the definitions 22 and 21 of $Y_{m,p'}$ and $p_{\alpha}$, and Hölder's inequality,

$$\| \partial^\alpha \psi \|_{L^p(\theta Q)} \leq C \| \tilde{\varphi} \|_{Y_{m,p'}(\theta Q)}.$$

By Lemma 41 with $j = k = |\beta|$, 

$$\| \partial^\beta u \|_{L^p(\theta Q)} \leq \sum_{i=|\beta|} C \| \nabla^i u \|_{L^p(\theta Q)}.$$

By Lemma 46 with $j = |\gamma| < k = |\beta|$, 

$$\| \partial^\gamma u \|_{L^p(\theta Q)} \leq \sum_{i=|\gamma|} \frac{C \| \nabla^i u \|_{L^p(\theta Q)}}{\theta - 1}.$$

and by Lemma 42

$$\| \partial^\gamma u \|_{L^p(\theta Q)} \leq \sum_{i=|\gamma|} \frac{C \| \nabla^i u \|_{L^p(\theta Q)}}{\theta - 1}.$$

Observe that $1/p' \leq 1/\mu' + 1/d$; thus, by Lemma 46 and the Poincaré inequality with $j = |\delta| < k = |\alpha|$, and with $p, \mu, u$ replaced by $\mu', p', \psi$, we have that

$$\| \partial^\delta \psi \|_{L^p(\theta Q)} \leq \sum_{i=|\delta|} \frac{C \| \nabla^i \psi \|_{L^{p'}(\theta Q)}}{\theta - 1}.$$

Because $\nabla^i \tilde{\psi} = \nabla^i \tilde{\varphi}$ for all $i \geq a$, and by Hölder's inequality, we have that

$$\| \partial^\delta \psi \|_{L^p(\theta Q)} \leq \sum_{i=|\delta|} \frac{C \| \nabla^i \psi \|_{L^{p'}(\theta Q)}}{\theta - 1}.$$

Combining all of the above estimates and the definitions of $X$ and $\Lambda$, we see that

$$\| \langle L(u \chi), \tilde{\varphi} \rangle \| \leq \frac{C}{\theta - 1} \| L \hat{u} \|_{Y_{m,p}(\theta Q)} \| \tilde{\varphi} \|_{Y_{m,p'}(\theta Q)}$$

$$+ \| \tilde{\varphi} \|_{Y_{m,p'}(\theta Q)} \sum_{i=|\delta|} \frac{C \| \nabla^i \psi \|_{L^{p'}(\theta Q)}}{\theta - 1}.$$

This completes the proof in the case where $A$ satisfies the condition 3.
If instead $A$ satisfies the condition (7), a similar argument with Lemma 39 in place of Lemma 40 establishes the same bound.

From Lemma 64 we have a bound on $L(\tilde{u}_1)$). We may now prove the following result; this is Theorem 64 in the case $q = \mu$.

**Lemma 68.** Let $m, d, L, p, \mu, Q, \theta, u$, and $\varpi$ be as in Lemma 64.

Suppose in addition that $p, \mu \in \Psi_L \cap \Pi_L$, where $\Pi_L$ and $\Psi_L$ are as in Definitions 28 and 33.

Then there is a constant $C$ depending only on $p$ and $L$ such that, for all $j$ with $\varpi \leq j \leq m$, we have that

$$
\frac{1}{|Q|^{(m-j)/d}} \| \nabla^j \tilde{u} \|_{L^p(Q)} \leq \frac{C}{(\theta - 1)^m} \| L \tilde{u} \|_{Y^{-m,p}(\theta Q)} + \frac{CA|Q|^{1/p - 1/\mu}}{(\theta - 1)^{m \cdot \beta}} \sum_{i=\varpi}^m \frac{1}{|Q|^{(m-i)/d}} \| \nabla^i \tilde{u} \|_{L^p(\theta Q \setminus Q)}.
$$

If $2 - \delta < p < 2 + \delta$, where $\delta$ is the number in Lemma 64, then $C$ may be taken depending only on $p$ and the standard parameters.

**Proof.** Let $\chi \in C_c^\infty(\theta Q)$ be as in Lemma 64; we may require that the parameter $X$ be bounded depending only on $m$ and $d$. We extend $(\tilde{u} - \bar{P})\chi$ by zero, where $P$ is the polynomial in Lemma 64.

By the definition of $\Psi_L$, $L$ is invertible $Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$, $Y^{m,\mu}(\mathbb{R}^d) \to Y^{-m,\mu}(\mathbb{R}^d)$, and $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$.

Furthermore, if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,2}(\mathbb{R}^d)$, then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,2}(\mathbb{R}^d)$.

Observe that we may approximate elements of $Y^{-m,p}(\mathbb{R}^d)$ by elements of $Y^{-m,\mu}(\mathbb{R}^d)$, thus, by density, if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,\mu}(\mathbb{R}^d)$, then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,\mu}(\mathbb{R}^d)$ (even if $T \notin Y^{-m,2}(\mathbb{R}^d)$).

Thus, because $(\chi(\tilde{u} - \bar{P})) \in Y^{m,\mu}(\mathbb{R}^d)$, we have that

$$
\chi(\tilde{u} - \bar{P}) = L^{-1}(L(\chi(\tilde{u} - \bar{P}))).
$$

Since $L(\chi(\tilde{u} - \bar{P})) \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,\mu}(\mathbb{R}^d)$, we have that $\chi(\tilde{u} - \bar{P}) \in Y^{m,p}(\mathbb{R}^d)$.

By boundedness of $L^{-1} : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$, we have that

$$
\| \chi(\tilde{u} - \bar{P}) \|_{Y^{m,p}(\mathbb{R}^d)} \leq C(p, L) \| L(\chi(\tilde{u} - \bar{P})) \|_{Y^{-m,p}(\mathbb{R}^d)}.
$$

By Lemma 64

$$
\| \chi(\tilde{u} - \bar{P}) \|_{Y^{m,p}(\mathbb{R}^d)} \leq \frac{C}{(\theta - 1)^m} \| L \tilde{u} \|_{Y^{-m,p}(\theta Q)} + \sum_{i=\varpi}^m \frac{CA|Q|^{1/p - 1/\mu}}{(\theta - 1)^{m \cdot \beta}} \| \nabla^i u \|_{L^p(\theta Q \setminus Q)}.
$$

If $j > m - d/p$, then $p_j$ exists and by Hölder’s inequality

$$
|Q|^{(j-m)/d} \| \nabla^j (\chi(\tilde{u} - \bar{P})) \|_{L^p(Q)} \leq \| \nabla^j (\chi(\tilde{u} - \bar{P})) \|_{L^{p_j}(Q)} \leq \| \chi(\tilde{u} - \bar{P}) \|_{Y^{m,p}(Q)}.
$$

If $\varpi \leq j \leq m - d/p$, recall that $\chi(\tilde{u} - \bar{P})$ is compactly supported; by the Poincaré inequality, we again have that

$$
|Q|^{(j-m)/d} \| \nabla^j (\chi(\tilde{u} - \bar{P})) \|_{L^p(Q)} \leq \| \nabla^m (\chi(\tilde{u} - \bar{P})) \|_{L^p(Q)} \leq \| \chi(\tilde{u} - \bar{P}) \|_{Y^{m,p}(Q)}.
$$
Therefore,
\[ |Q^{(j-m)/d}| \|\nabla^j (\chi(\vec{u} - \vec{P}))\|_{L^p(Q)} \leq \|u\|_{Y^{m,p}(Q)} \leq \|\chi(\vec{u} - \vec{P})\|_{Y^{m,p}(\mathbb{R}^d)}. \]

Because \( j \geq \varpi \), \( \nabla^j \vec{u} = \nabla^j (\chi(\vec{u} - \vec{P})) \) in \( Q \) and the proof is complete. \( \square \)

We may combine Lemma \ref{lem:68} with the Caccioppoli inequality (Lemma \ref{lem:51}) to prove Theorem \ref{thm:62} in the case \( q = 2 \).

**Lemma 69.** Let \( m, d, L, p, \mu, Q, \theta, u, \varpi, \) and \( j \) be as in Lemma \ref{lem:68} that is, that they are as in Lemma \ref{lem:67} with \( p, \mu \in Y_L \cap \Pi_L \) and \( \varpi \leq j \leq m \).

Suppose in addition that \( p \geq 2 \).

Then there is a positive constant \( \kappa \) depending only on the standard parameters and a positive constant \( C \) depending on \( p \) and \( L \) such that, if \( 0 \leq j \leq m \), then
\[
\frac{1}{|Q|^{(m-j)/d}} \|\nabla^j \vec{u}\|_{L^p(Q)} \leq C \left( \frac{\mu}{\theta - 1} \right)^\kappa \|L\vec{u}\|_{Y^{m,p}(\theta Q)} + C |Q|^{1/p - 1/2 - (m-\varpi)/d} \left( \frac{\mu}{\theta - 1} \right)^\kappa \|(\nabla^\varpi \vec{u})\|_{L^2(\theta Q \setminus Q)}.
\]

If \( 2 \leq p < 2 + \delta \), where \( \delta \) is the number in Lemma \ref{lem:61}, then \( C \) may be taken depending only on \( p \) and the standard parameters.

**Proof.** Let \( \theta_0 = 1, \theta_3 = \theta \), and \( \theta_3 - \theta_2 = \theta_2 - \theta_1 = \theta_1 - \theta_0 = (\theta - 1)/3 \). Choose \( \mu = 2 \). Lemma \ref{lem:68} yields that
\[
\frac{1}{|Q|^{(m-j)/d}} \|\nabla^j \vec{u}\|_{L^p(\theta_1 Q)} \leq C \left( \frac{\mu}{\theta - 1} \right)^m \|L\vec{u}\|_{Y^{m,p}(\theta_2 Q)} + C A |Q|^{1/p - 1/2 - (m-\varpi)/d} \left( \frac{\mu}{\theta - 1} \right)^\kappa \sum_{i=m}^m \frac{1}{|Q|^{(m-i)/d}} \|\nabla^i \vec{u}\|_{L^2(\theta_2 Q, \theta_1 Q)}.
\]

Let \( \vec{P} \) be a polynomial of degree less than \( \varpi \) \( \leq \min(j, b) \) such that \( f_{\theta Q \setminus Q} \vec{P}(\vec{u} - \vec{P}) = 0 \) for all \( |\gamma| < \varpi \). Observe that \( L\vec{u} = L(\vec{u} - \vec{P}) \) and \( \nabla^j \vec{u} = \nabla^j (\vec{u} - \vec{P}) \). Applying Corollary \ref{cor:51} to \( \vec{u} - \vec{P} \) and a covering argument yields that
\[
\frac{1}{|Q|^{(m-j)/d}} \|\nabla^j \vec{u}\|_{L^p(Q)} \leq C \left( \frac{\mu}{\theta - 1} \right)^m \|L\vec{u}\|_{Y^{m,p}(\theta Q)} + C |Q|^{1/p - 1/2 - m/d} \left( \frac{\mu}{\theta - 1} \right)^\kappa \|\nabla^\varpi \vec{u}\|_{L^2(\theta Q \setminus Q)}.
\]

Because \( p \geq 2 \), by Hölder’s inequality \( |Q|^{1/p - 1/2} \|L\vec{u}\|_{Y^{m,p}(\theta Q)} \leq C \|L\vec{u}\|_{Y^{m,p}(\theta Q)} \).

By Lemma \ref{lem:72} we may replace \( \|\nabla^\varpi \vec{u}\|_{L^2(\theta Q \setminus Q)} \) by \( |Q|^{\varpi/d} \|\nabla^\varpi \vec{u}\|_{L^2(\theta Q \setminus Q)} \). Repeating \( \kappa \) completes the proof. \( \square \)

**Remark 70.** If \( p = 2 \), Lemma \ref{lem:69} still represents an improvement over the Caccioppoli inequality (Corollary \ref{cor:51}) in that, if \( m - d/2 < j < m \), then we can bound \( \|\nabla^j \vec{u}\|_{L^2(Q)} \) by \( \|\vec{u}\|_{L^2(\theta Q \setminus Q)} \) and not \( \|\vec{u}\|_{L^2(\theta Q)} \).

**Remark 71.** If \( p = 2 \) and \( \varpi \geq 1 \), then by Lemmas \ref{lem:69} and \ref{lem:41}
\[
\frac{1}{|Q|^{(m-j)/d}} \|\nabla^j \vec{u}\|_{L^p(Q)} \leq C \left( \frac{\mu}{\theta - 1} \right)^\kappa \|L\vec{u}\|_{Y^{m,p}(\theta Q)} + C |Q|^{1/p - 1/2 - (m-\varpi+1)/d} \left( \frac{\mu}{\theta - 1} \right)^\kappa \|\nabla^\varpi \vec{u}\|_{L^2(\theta Q \setminus Q)} \leq C \left( \frac{\mu}{\theta - 1} \right)^\kappa \|L\vec{u}\|_{Y^{m,p}(\theta Q)} + C |Q|^{1/p - 1/2 - (m-\varpi)/d} \left( \frac{\mu}{\theta - 1} \right)^{\kappa+1} \|\nabla^\varpi \vec{u}\|_{L^2(\theta Q \setminus Q)} \]
for some nonnegative constants \( c \).

We have now established that Theorem 54 is valid if \( q = \mu \), and that Theorem 52 is valid if \( q = 2 \) or if \( \varpi \geq 1 \) and \( q \) takes a specific value less than 2. In particular, these theorems are valid for at least one \( q < p \). By Hölder’s inequality, these theorems are valid for all \( q \geq p \). The following lemma will complete the proof by establishing validity for all positive but smaller \( q \).

**Lemma 72.** Let \( d \geq 2 \) and \( 0 \leq \varpi \leq n \leq m \) be integers. Let \( Q \subset \mathbb{R}^d \) be a cube and let \( 1 < \theta \leq 2 \).

For each \( i \) with \( \varpi \leq i \leq m \), let \( p_i, u_i \) satisfy \( 0 < p_i \leq \infty \) and \( u_i \in L^{p_i}(\theta Q) \); if in addition \( \varpi \leq i \leq n \), let \( \tilde{q}_i \) satisfy \( 0 < \tilde{q}_i < p_i \).

Suppose that, whenever \( 1 \leq \vartheta < \zeta \leq \theta \), we have the bound

\[
\sum_{j=1}^m \|u_j\|_{L^{p_j}(\vartheta Q)} \leq \frac{F}{(\zeta - \vartheta)\kappa} + \frac{c_0}{(\zeta - \vartheta)\kappa} \sum_{i=\varpi}^n \|u_i\|_{L^{\tilde{q}_i}(\tau Q \setminus \theta Q)}
\]

for some nonnegative constants \( c_0, \kappa \) and \( F \) independent of \( \zeta \) and \( \vartheta \).

Then for every set of numbers \( q_i \), with \( 0 < q_i \leq \tilde{q}_i \), there are some constants \( C \) and \( \kappa \), depending only on the \( q_i, q_i, p_i, c_0 \), and \( \kappa \), such that

\[
\sum_{j=1}^m \|u_j\|_{L^{p_j}(\vartheta Q)} \leq \frac{C}{(\theta - 1)\kappa} \left( F + \sum_{i=\varpi}^n \|u_i\|_{L^{\tilde{q}_i}(\theta Q \setminus \vartheta Q)} \right).
\]

**Proof.** If \( c_0 = 0 \) we are done, so throughout we may assume \( c_0 > 0 \). We are also done if \( q_i = \tilde{q}_i \) for all \( i \); we will consider the case where \( q_i < \tilde{q}_i \) for at least one \( i \).

In the present paper we will only need the case where \( q_i = q, \tilde{q}_i = \tilde{q} \) for some \( q, \tilde{q} \) independent of \( i \), but for completeness we present the general case.

Let \( 1 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \ldots \) for some \( \vartheta_\ell \in [1, \theta) \) to be chosen momentarily, and let \( Q_\ell = Q_{\ell+1} \setminus Q_\ell \). Let \( A_\ell = Q_{\ell+1} \setminus Q_\ell \). If \( \varpi \leq i \leq n \), let

\[
\tau_i = \frac{1}{\tilde{q}_i - p_i}, \quad \tau \equiv \frac{1}{\tilde{q}_i - p_i} - \frac{q_i}{\tilde{q}_i - p_i} = \frac{q_i}{\tilde{q}_i - p_i}.
\]

If \( 0 < q_i < \tilde{q}_i < p_i \), we have that \( 0 < \tau_i < 1 \). Thus

\[
\sum_{i=\varpi}^n \|u_i\|_{L^{\tilde{q}_i}(A_\ell)} = \sum_{i=\varpi}^n \left( \int_{A_\ell} |u_i|^\tau_i \tilde{q}_i (1 - \tau_i) \tilde{q}_i \right)^{1/\tilde{q}_i}.
\]

We compute that

\[
\frac{q_i}{\tau_i \tilde{q}_i} = \frac{p_i - q_i}{p_i - \tilde{q}_i} \in (1, \infty), \quad \left( \frac{q_i}{\tau_i \tilde{q}_i} \right)'(1 - \tau_i) \tilde{q}_i = p_i.
\]

So we may apply Hölder’s inequality to see that

\[
\sum_{i=\varpi}^n \|u_i\|_{L^{\tilde{q}_i}(A_\ell)} \leq \sum_{i=\varpi}^n \|u_i\|_{L^{\tilde{q}_i}(A_\ell)}^{\tau_i} \|u_i\|_{L^{\tilde{q}_i}(A_\ell)}^{1-\tau_i}.
\]

By Young’s inequality,

\[
\sum_{i=\varpi}^n \|u_i\|_{L^{\tilde{q}_i}(A_\ell)} \leq \sum_{i=\varpi}^n \tau_i \left( \frac{c_0}{(\vartheta_{\ell+1} - \vartheta_\ell)\kappa} \right)^{(1-\tau_i)/\tau_i} \|u_i\|_{L^{\tilde{q}_i}(A_\ell)}
\]

\[
+ \sum_{i=\varpi}^n (1 - \tau_i) \left( \frac{(\vartheta_{\ell+1} - \vartheta_\ell)\kappa}{c_0} \right)^{\tau_i} \|u_i\|_{L^{p_i}(A_\ell)}
\]
If \( q_i = \tilde{q}_i \) and so \( \tau_i = 1 \), this bound is still true. By the bound (73),

\[
\sum_{j=\infty}^{m} \| u_j \|_{L^{p_j}(Q_\ell)} \leq \frac{F}{(\theta \ell + 1 - \theta \ell)\kappa} + \frac{c_0}{(\theta \ell + 1 - \theta \ell)\kappa} \sum_{i=\infty}^{n} \| u_i \|_{L^{q_i}(A_\ell)}
\]

\[
\leq \frac{F}{(\theta \ell + 1 - \theta \ell)\kappa} + \sum_{i=\infty}^{n} \tau_i \left( \frac{c_0}{(\theta \ell + 1 - \theta \ell)\kappa} \right)^{1/\tau_i} \| u_i \|_{L^{q_i}(A_\ell)}
\]

\[
+ \sum_{i=\infty}^{n} (1 - \tau_i) \| u_i \|_{L^{q_i}(A_\ell)}.
\]

Recall that \( \theta_0 = 1 \). We now let \( \theta_{\ell+1} = \theta \ell + (\theta - 1)(1 - \sigma)\sigma^{\ell} \) for some constant \( \sigma \in (0, 1) \) to be chosen momentarily. Notice that \( \lim_{\ell \to \infty} \theta_\ell = \theta \). Recall that \( A_\ell \subset Q_{\ell+1} \). Then

\[
\sum_{j=\infty}^{m} \| u_j \|_{L^{p_j}(Q_\ell)} \leq \frac{F}{(\theta - 1)\kappa(1 - \sigma)\kappa \sigma^{\ell}} + \sum_{i=\infty}^{n} \frac{\tau_i^{1/\tau_i}}{(\theta - 1)\kappa/\tau_i(1 - \sigma)\kappa^{\ell/\tau_i}} \| u_i \|_{L^{q_i}(A_\ell)}
\]

\[
+ \sum_{i=\infty}^{n} (1 - \tau_i) \| u_i \|_{L^{p_i}(Q_{\ell+1})}.
\]

Let \( \tau = \min_i \tau_i \). If \( \tau = 1 \) then \( q_i = \tilde{q}_i \) for all \( i \) and there is nothing to prove; otherwise, \( \tau \in (0, 1) \). Recall that \( \omega = n \leq m \). Iterating, we see that if \( K \geq 0 \) is an integer, then

\[
\sum_{j=\infty}^{m} \| u_j \|_{L^{p_j}(Q_0)} \leq \sum_{\ell=0}^{K} (1 - \tau)^{\ell} \cdot \frac{F}{(\theta - 1)\kappa(1 - \sigma)\kappa \sigma^{\ell}}
\]

\[
+ \sum_{i=\infty}^{n} \frac{\tau_i^{1/\tau_i}}{(\theta - 1)\kappa/\tau_i(1 - \sigma)\kappa^{\ell/\tau_i}} \| u_i \|_{L^{q_i}(A_\ell)}
\]

\[
+ \sum_{j=\infty}^{m} (1 - \tau)^{K+1} \| u_j \|_{L^{p_j}(Q_{\ell+1})}.
\]

Recall that \( Q_0 = Q \) and \( Q_\ell \subset Q \), \( A_\ell \subset \theta Q \setminus Q \) for all \( \ell \geq 0 \). Changing the order of summation, we see that

\[
\sum_{j=\infty}^{m} \| u_j \|_{L^{p_j}(Q)} \leq \frac{F}{(\theta - 1)\kappa(1 - \sigma)\kappa} \sum_{\ell=0}^{K} \left( \frac{1 - \tau}{\sigma^{\kappa}} \right)^{\ell}
\]

\[
+ \sum_{i=\infty}^{n} \frac{\tau_i^{1/\tau_i}}{(\theta - 1)\kappa/\tau_i(1 - \sigma)\kappa^{\ell/\tau_i}} \| u_i \|_{L^{q_i}(Q \setminus \theta Q)} \sum_{\ell=0}^{K} \left( \frac{1 - \tau}{\sigma^{\kappa/\tau_i}} \right)^{\ell}
\]

\[
+ (1 - \tau)^{K+1} \sum_{j=\infty}^{m} \| u_j \|_{L^{p_j}(Q \setminus \theta Q)}.
\]

Choose \( \sigma \in (0, 1) \) such that \( 1 - \tau < \sigma^{\kappa/\tau_i} \); since \( \tau \in (0, 1) \), this implies \( 1 - \tau < \sigma^{\kappa} \).

Taking the limit as \( K \to \infty \), we have that the geometric series converge and the final term approaches zero, and so

\[
\sum_{j=\infty}^{m} \| u_j \|_{L^{p_j}(Q)} \leq C \frac{F}{(\theta - 1)\kappa} + C \sum_{i=\infty}^{n} \frac{1}{(\theta - 1)\kappa/\tau_i} \| u_i \|_{L^{q_i}(Q \setminus \theta Q)}
\]
as desired. \qed

6.2. A counterexample. In this section we will prove Theorem [65]

Let \( a, b \) and \( \varepsilon \) be as in the theorem statement. Without loss of generality we may require \( 0 < \varepsilon \leq 1 \). Fix a multiindex \( \zeta \) with \( |\zeta| = b \).

Define \( w(X) = (1 + |X|^2)^{-d/2} \). We may easily compute that \( \nabla^m w \in L^p(\mathbb{R}^d) \) for any \( p > d/(2d + m) \) (in particular, for all \( p \geq 2 \)).

Let \( \{Q_k\}_{k=1}^{\infty} \) be a sequence of pairwise-disjoint cubes contained in \( Q \) (whose volumes necessarily tend to zero). Let \( \psi \) be a smooth cutoff function with \( \psi \) supported in \( Q \) and \( \psi = 1 \) in \( \frac{1}{2}Q \), and let \( \varphi_k(X) = \varphi((X - X_k)/\ell_k) \), where \( X_k \) is the midpoint of \( Q_k \) and \( \ell_k = |Q_k|^{1/d} \) is the side length of \( Q_k \). Then \( \varphi_k \) is a smooth cutoff function supported in \( Q_k \) and identically 1 in \( \frac{1}{2}Q_k \).

Let \( \{n_k\}_{k=1}^{\infty} \) be a sequence of positive numbers such that \( n_k\ell_k \to \infty \) and \( n_k\ell_k \geq 1 \) for all \( k \). Notice that \( \ell_k < 1 \) so \( n_k > 1 \) for all \( k \). Define

\[
\partial^\gamma u(X) = \partial^\gamma X^\zeta + \varepsilon C_0 \sum_{k=1}^{\infty} \varphi_k(X) \frac{1}{n_k^{2m}} w(n_k(X - X_k))
\]

for a positive constant \( C_0 \) to be chosen momentarily. We may easily compute that if \( X \in \frac{1}{2}Q_k \) and \( \gamma \) is a multiindex, then

\[
\partial^\gamma u(X) = \partial^\gamma X^\zeta + \varepsilon C_0 \frac{1}{n_k^{2m - |\gamma|}} (\partial^\gamma w)(n_k(X - X_k)).
\]

Furthermore, if \( X \in Q_k \) and \( 0 \leq |\gamma| \leq 2m \), then

\[
|\partial^\gamma u(X) - \partial^\gamma X^\zeta| \leq \varepsilon C_0 (|\gamma|, \varphi, \, d) n_k^{m - |\gamma| - 2m} \leq \varepsilon C_0 (|\gamma|, \varphi, \, d).
\]

We choose \( C_0 \geq 2C(\zeta, \varphi, \, d) \); this ensures that

\[
|\partial^\gamma u - \zeta| = |\partial^\gamma u - \partial^\gamma X^\zeta| \leq \frac{1}{2} \leq \frac{1}{2} \zeta
\]

and so \( |\partial^\gamma u(X)| \geq \frac{1}{2} \) for all \( X \).

Recall that \( \widetilde{A}_{\alpha,\beta} \) is a set of real nonnegative constants that satisfies

\[
(-\Delta)^m = (-1)^m \sum_{|\alpha|=m} \sum_{|\beta|=m} \widetilde{A}_{\alpha,\beta} \partial^{\alpha+\beta}.
\]

(Many possible families of such constants exist.) Similarly, for any \( a \leq m \), there exist families of constants \( \widetilde{B}_{a,\gamma} \) such that

\[
(-\Delta)^m = (-1)^m \sum_{|\alpha|=a} \sum_{|\gamma|=2m-a} \widetilde{B}_{a,\gamma} \partial^{\alpha+\gamma}.
\]

Choose some such family.

Define the coefficients \( A_{\alpha, \beta} = A_{\alpha, \beta}^{1,1} \) as follows.

- If \( |\alpha| = |\beta| = m \), let \( A_{\alpha, \beta} = \widetilde{A}_{\alpha, \beta} \).
- If \( |\alpha| = a \) and \( \beta = \zeta \), let

\[
A_{\alpha, \zeta} = - \sum_{|\gamma|=2m-a} \widetilde{B}_{a,\gamma} \frac{\partial^\gamma u}{\partial u}.
\]

- Otherwise, let \( A_{\alpha, \beta} = 0 \).
Because $|\zeta| = \beta < m$, $A_{\alpha,\beta}$ is well defined.

If $L$ is as given by formula (24), a straightforward computation yields that $Lu = 0$, formulas (26) and (27) are valid, and if $C_0$ is large enough then $|A_{\alpha,\beta}(X) - \tilde{A}_{\alpha,\beta}(X)| < \varepsilon$ for all $X$, $\alpha$, and $\beta$. It remains only to establish a lower bound on $\int_{\frac{1}{2}Q_k} |\nabla^m u|$. If $p \geq 2$ and $b + 1 \leq j \leq m$, then by definition of $u$ and a change of variables,

$$
\left( \int_{\frac{1}{2}Q_k} |\nabla^j u|^p \right)^{1/p} = \varepsilon \frac{C_0 n_k^{2m-j}}{C_0 n_k^{2m-j}(n_k \ell_k)^{2d+j}} \left( \int_{\frac{1}{2}Q_k} |\nabla^j w(X)|^p dX \right)^{1/p}.
$$

Thus, recalling that $n_k \ell_k \geq 1$, we have that

$$
\left( \int_{\frac{1}{2}Q_k} |\nabla^m u|^p \right)^{1/p} \geq \frac{1}{n_k^{m-j}(n_k \ell_k)^{d+j}} \left( \int_{\frac{1}{2}Q_k} |\nabla^m w(X)|^p dX \right)^{1/p} \geq \frac{c_1}{n_k^{m-j}(n_k \ell_k)^{d+j}}.
$$

Thus,

$$
\frac{1}{n_k^{m-j}} \left( \int_{Q_k} |\nabla^j u|^2 \right)^{1/2} \leq \frac{1}{n_k^{m-j}} \left( 2^{-d} \int_{\frac{1}{2}Q_k} |\nabla^j u|^2 + \frac{C_\varepsilon}{C_0 n_k^{2m-j}(n_k \ell_k)^{2d+j}} \right)^{1/2} \leq \varepsilon \frac{1}{n_k^{m-j}} \left( \int_{\mathbb{R}^d} |\nabla^j u|^2 + \frac{C}{(n_k \ell_k)^{d+2}} \right)^{1/2}.
$$

Again using the fact that $n_k \ell_k \geq 1$ and the fact that $\nabla^j w \in L^2(\mathbb{R}^d)$ for any $j \geq 0$, we have that

$$
\sum_{j=b+1}^m \frac{1}{n_k^{m-j}} \left( \int_{Q_k} |\nabla^j u|^2 \right)^{1/2} \leq \frac{C_2}{n_k^{m-j}(n_k \ell_k)^{d+j/2}}.
$$

If $p > 2$, then because $n_k \ell_k \to \infty$, there is some $k$ large enough that

$$
\tilde{c} \frac{C_2}{n_k^{m-j}(n_k \ell_k)^{d+j/2}} \leq \frac{c_1}{n_k^{m-j}(n_k \ell_k)^{d+j/2}}
$$

as desired. This completes the proof of Theorem 65.

7. The fundamental solution

In this section we will construct the fundamental solution. We will begin in Section 7.1 with local estimates on functions in $Y^{m,p}(\mathbb{R}^d)$ for $m$ large enough. Using these estimates, in Section 7.2 we will construct a preliminary version of the fundamental solution in the case $2m > d$. We will investigate the properties of this fundamental solution in Sections 7.3, 7.4, 7.5. We will slightly modify our definition in
Section ???. In Section 7.4 we will construct the fundamental solution in the case $2m \leq d$, and will address uniqueness in Section 7.7.

7.1. Preliminaries for operators of high order. Recall from the definition (22) of $Y^{m,q}(\mathbb{R}^d)$ that if $u \in Y^{m,q}(\mathbb{R}^d)$, then the derivatives $\partial^\gamma u$ of $u$ are defined as locally integrable functions if $|\gamma| > m - d/q$, and are defined only up to adding polynomials if $|\gamma| \leq m - d/q$. We will now wish to fix a family of normalizations of functions in $Y^{m,q}(\mathbb{R}^d)$ and investigate their properties.

If $d/m < q < \infty$, let $s_{m,d,q}$ be the number of multiindices $\gamma \in (\mathbb{N}_0)^d$ so that $|\gamma| \leq m - d/q$. Observe that $s_{m,d,q}$ is nonnegative, nondecreasing in $q$ and that if $q < \infty$ then $s_{m,d,q} \leq s_{m,d,d}$. Choose distinct points $H_1, H_2, \ldots, H_{s_{m,d,d}}$ in $B(0,1) \setminus B(0,1/2)$ (so $1/2 < |H_i| < 1$ for all $1 \leq i \leq s_{m,d,d}$). If the points $H_i$ are chosen appropriately (see [GS00] for a survey on polynomial interpolation in several variables) then for any $q$ with $d/m < q < \infty$ and any numbers $a_i$ there is a unique polynomial

$$P(X) = \sum_{|\gamma| \leq m - d/q} p_\gamma X^\gamma$$

such that $P(H_i) = a_i$ for all $1 \leq i \leq s_{m,d,q}$.

(We emphasize that if $q < d$ then we cannot specify the values of $P(H_i)$ for $s_{m,d,q} < i \leq s_{m,d,d}$) Also there is some constant $h < \infty$ depending only on $H_i$ such that

$$\sup_{|\gamma| \leq m - d/q} |p_\gamma| \leq h \sup_{1 \leq i \leq s_{m,d,q}} |a_i|.$$

We now show that this gives a normalization in $Y^{m,q}(\mathbb{R}^d)$. We will need some additional properties of this normalization.

Lemma 75. Let $m, d \in \mathbb{N}$ with $d \geq 2$, let $r > 0$, and let $Z_0 \in \mathbb{R}^d$. Let $\max(d/m) < q < \infty$. Let $U$ satisfy $\|U\|_{Y^{m,\mu}(\mathbb{R}^d)} < \infty$.

Then there is a unique function $U_{Z_0,r,q}$ that is continuous and satisfies

$$U_{Z_0,r,q}(Z_0 + rH_i) = 0, \quad \partial^\gamma U = \partial^\gamma U_{Z_0,r,q} \text{ almost everywhere}$$

for all $1 \leq i \leq s_{m,d,q}$ and all multiindices $\zeta$ with $m - d/q < |\zeta| \leq m$. In particular, if $q = \mu$ then $U$ and $U_{Z_0,r,q}$ are representatives of the same element of $Y^{m,\mu}(\mathbb{R}^d)$.

Furthermore, if $X, Y \in \mathbb{R}^d$, $R = r + |X - Z_0|$, $|X - Y| \leq \frac{1}{2}R$, and $|\gamma| < m - d/\mu$, then we have the bounds

$$|\partial^\gamma U_{Z_0,r,q}(X)| \leq C_\mu R^{m-d/\mu - |\gamma|} \left(\frac{R}{r}\right)^{\omega_q - 1} \|U\|_{Y^{m,\mu}(\mathbb{R}^d)},$$

$$|\partial^\gamma U_{Z_0,r,q}(X) - \partial^\gamma U_{Z_0,r,q}(Y)| \leq C_\mu R^{m-d/\mu - |\gamma|} \left(\frac{R}{r}\right)^{\omega_q - 1} \|U\|_{Y^{m,\mu}(\mathbb{R}^d)} \left(\frac{|X - Y|}{R}\right)^\varepsilon,$$

where $C_\mu$ and $\varepsilon > 0$ depend on $d, m$, and $\mu$, and $\omega_q$ is the smallest (necessarily positive) integer with $m - d/\mu < \omega_q$.

Proof. Fix $X \in \mathbb{R}^d$. Let $Q$ be a cube centered at $Z_0$ of side length $4R$. Observe that $\|U\|_{Y^{m,\mu}(Q)} \leq \|U\|_{Y^{m,\mu}(\mathbb{R}^d)} < \infty$. Then $\nabla^m U \in L^\mu(Q)$, so by the Poincaré inequality, we have that $\nabla^\gamma U \in L^\mu(Q)$ (and thus is integrable) for any $0 \leq i \leq m$.

Let $V = U + P$, where $P$ is a polynomial of degree at most $m - d/\mu$ so that $\int_Q \partial^\gamma V = 0$ for all $\gamma$ with $|\gamma| \leq m - d/\mu$ (that is, all $\gamma$ with $|\gamma| < \omega_q$). Observe that $\|U - V\|_{Y^{m,\mu}(Q)} = 0$, so $\|V\|_{Y^{m,\mu}(Q)} = \|U\|_{Y^{m,\mu}(Q)} < \infty$. 

If \( d/\mu \) is not an integer, let \( \theta = \mu \). Otherwise, let \( \theta \) satisfy \( d/\theta = d/\mu + 1/2 \). In either case, \( d/\theta \) is not an integer and \( \theta \leq \mu \). Since \( m > d/\mu + |\gamma| \), if \( d/\mu \) is an integer then \( m \geq d/\mu + |\gamma| + 1 \) and so \( m > d/\theta + |\gamma| \). Because \( \mu > 1 \) we have that \( d > d/\mu \), so similarly \( d > d/\theta \) and so \( \theta > 1 \).

Let \( k \) be the unique integer such that \( m - d/\theta < k < m \) and so \( m \leq k < m + 1 \). Thus

\[
|\gamma| < m - d/\theta < k < |\gamma| + 1 \leq k \leq m.
\]

By Lemma 39,

\[
\|\nabla \partial^\gamma V\|_{L^\mu(Q)} \leq C_\mu \sum_{i=1}^{m-k+1} R^{i-1+k-m} \|\nabla^i \partial^\gamma V\|_{L^\mu(Q)}
\]

and by Hölder’s inequality and because \( k \geq 1 + |\gamma| \),

\[
\|\nabla \partial^\gamma V\|_{L^\mu(Q)} \leq C_\mu \sum_{i=1}^{m} R^{i-1-|\gamma|+k-m+d/\theta-d/\mu} \|\nabla^i V\|_{L^\mu(Q)}.
\]

By formula (21),

\[
\frac{\partial}{\partial_k} = 1 - \frac{m - k}{d} \in \left( 0, \frac{1}{d} \right).
\]

By Morrey’s inequality (see [Eva98, Section 5.6.2]), we may redefine the weak derivative \( \partial^\gamma V \) of \( V \) on a set of measure zero in a unique way so that it is continuous (thus defined pointwise everywhere) and, if \( \tilde{X} \in \frac{1}{2}Q \) and \( |\tilde{X} - Y| < R/2 \), then

\[
|\partial^\gamma V(\tilde{X}) - \partial^\gamma V(Y)| \leq C_\mu |\tilde{X} - Y|^{1-\theta/\mu} \|\nabla \partial^\gamma V\|_{L^\mu(Q)}.
\]

Let \( \varepsilon = 1 - d/\theta_k = 1 - d/\theta + m - k \). Observe that \( 0 < \varepsilon < 1 \). Then

\[
|\partial^\gamma V(\tilde{X}) - \partial^\gamma V(Y)| \leq C_\mu \frac{|\tilde{X} - Y|^{\varepsilon}}{R^{|\gamma|+d/\mu}} \sum_{i=1}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)}.
\]

Averaging \( |\partial^\gamma V(\tilde{X})| \leq |\partial^\gamma V(\tilde{X}) - \partial^\gamma V(Y)| + |\partial^\gamma V(Y)| \) over \( Y \in B(\tilde{X}, R/2) \) we have that

\[
|\partial^\gamma V(\tilde{X})| \leq \frac{C_\mu}{R^{|\gamma|+d/\mu}} \sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)}.
\]

We will consider the cases \( \tilde{X} = X \) and \( \tilde{X} = Z_0 + rH_j \).

We may write

\[
\sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)} = \sum_{i=0}^{\omega_\mu - 1} R^i \left( \int_Q |\nabla^i V|^\mu \right)^{1/\mu} + \sum_{i=\omega_\mu}^{m} R^i \left( \int_Q |\nabla^i V|^\mu \right)^{1/\mu}.
\]

Recall that \( V \) satisfies \( \int_Q \nabla^i V = 0 \) for all \( 0 \leq i \leq \omega_\mu - 1 \). We may apply the Poincaré inequality in the first sum so that

\[
R^i \left( \int_Q |\nabla^i V|^\mu \right)^{1/\mu} \leq C_\mu R^{\omega_\mu} \left( \int_Q |\nabla^{\omega_\mu} V|^\mu \right)^{1/\mu}.
\]

Thus

\[
\sum_{i=0}^{m} R^i \|\nabla^i V\|_{L^\mu(Q)} \leq C_\mu \sum_{i=\omega_\mu}^{m} R^i \left( \int_Q |\nabla^i V|^\mu \right)^{1/\mu}.
\]
By Hölder’s inequality, we have that
\[
\sum_{i=0}^{m} R_i \| \nabla^i V \|_{L^\mu(Q)} \leq C_\mu \sum_{i=\omega_\mu}^{m} R_i^{i+d/\mu-d/\mu_i} \left( \int_Q |\nabla^i V|^\mu_i \right)^{1/\mu_i}.
\]

By formula (21) we have that \( i + d/\mu - d/\mu_i = m \). Thus, by the definition (22) of the norm on \( Y^{m,\mu}(Q) \), we have that
\[
(78) \quad \sum_{i=0}^{m} R_i \| \nabla^i V \|_{L^\mu(Q)} \leq C_\mu R_m \| U \|_{Y^{m,\mu}(Q)}.
\]

Let \( P_1 \) be the (unique) polynomial of degree at most \( m - d/q \) with \( P_1(Z_0 + r H_j) = V(Z_0 + r H_j) \) for each \( 1 \leq j \leq s_{m,d,q} \), and let \( U_{Z_0, r,q} = V - P_1 \). Then \( U_{Z_0, r,q} \) is the unique continuous function with \( U_{Z_0, r,q}(Z_0 + r H_j) = 0 \) for all \( 1 \leq j \leq \mu \) and with \( \partial^\gamma U_{Z_0, r,q} = \partial^\gamma V = \partial^\gamma U \) almost everywhere for all \( |\gamma| > m - d/q \). Thus the specified function \( U_{Z_0, r,q} \) is constructed; we need only establish the desired bounds on \( U_{Z_0, r,q} \).

We now take \( \tilde{X} = Z_0 + r H_j \) for some \( j \). By formulas (77) and (78),
\[
|P_1(Z_0 + r H_j)| = |V(Z_0 + r H_j)| \leq C_\mu \sum_{i=0}^{m} R_i^{i-d/\mu} \| \nabla^i V \|_{L^\mu(Q)} \leq C_\mu R_m^{m-d/\mu} \| U \|_{Y^{m,\mu}(Q)}.
\]

Let \( P_1(Z) = P_2((Z-Z_0)/r) \) so that \( P_2(H_i) = P_1(Z_0 + r H_i) \) and \( P_2(Z) = \sum_{|\gamma| \leq \omega - 1} p_\gamma Z^\gamma \) for some \( p_\gamma \) where \( |p_\gamma| \leq h \sup_{|\gamma| \leq \omega} |P_2(Z_0 + r H_j)| \leq C_\mu R_m^{m-d/\mu} \| U \|_{Y^{m,\mu}(Q)} \). We then have that \( P_1(Z) = \sum_{|\gamma| \leq \omega - 1} p_\gamma r^{-|\gamma|}(Z - Z_0)^\gamma \). We may then compute that if \( Z \in Q \) and \( 0 \leq i \leq \omega - 1 \), then
\[
|\nabla^i P_1(Z)| \leq C_\mu R_m^{m-d/\mu-i} \| U \|_{Y^{m,\mu}(Q)} (R/r)^{\omega-1}.
\]

Combining these pointwise bounds on \( P_1 \) with the bound (78) yields that
\[
(79) \quad \sum_{i=0}^{m} R_i \| \nabla^i U_{Z_0, r,q} \|_{L^\mu(Q)} \leq C_\mu R_m \| U \|_{Y^{m,\mu}(Q)} (R/r)^{\omega-1}.
\]

Combining this bound with the bounds (74), (77) with \( \tilde{X} = X \) completes the proof. \( \square \)

**Remark 80.** We observe that if \( U \in Y^{m,\mu}(\mathbb{R}^d) \), then \( \partial^\gamma U \in L^{\mu_\gamma}(\mathbb{R}^d) \) is defined up to sets of measure zero whenever \( |\gamma| > m - d/\mu \), while \( \partial^\gamma U_{Z_0, r,q} \) is continuous and satisfies the bounds given by Lemma (73) whenever \( q \geq \mu \) and \( |\gamma| < m - d/\mu \).

Suppose \( |\gamma| = m - d/\mu \). If \( k = |\gamma| + 1 \), then by formula (21) \( \mu_k = d \) and so \( \nabla \partial^\gamma U \in L^d(\mathbb{R}^d) \). By [Eva98 Section 5.8.1], we have that \( \partial^\gamma U \) lies in the space \( BMO \) of bounded mean oscillation with \( \| \partial^\gamma U \|_{BMO} \leq C_\mu \| U \|_{Y^{m,\mu}(\mathbb{R}^d)} \). By the John-Nirenberg inequality (see, for example, [Ste93]), we have that if \( 1 \leq p < \infty \) and \( Q \) is any cube then
\[
\left( \int_Q |\partial^\gamma U - f_Q \partial^\gamma U|^p \right)^{1/p} \leq C_{p,\mu} \| U \|_{Y^{m,\mu}(\mathbb{R}^d)}.
\]

Let \( Z_0, r, \) and \( U_{Z_0, r,q} \) be as in Lemma (75). Observe that \( \partial^\gamma U = \partial^\gamma U_{Z_0, r,q} \) for all \( |\gamma| > |\gamma| \), and so \( \partial^\gamma U \) differs from \( \partial^\gamma U_{Z_0, r,q} \) by a constant. Thus
\[
\left( \int_Q |\partial^\gamma U_{Z_0, r,q} - f_Q \partial^\gamma U_{Z_0, r,q}|^p \right)^{1/p} \leq C_{p,\mu} \| U \|_{Y^{m,\mu}(\mathbb{R}^d)}.
\]
By the bound (79) and Hölder’s inequality, if $Q$ is a cube centered at $Z_0$ of side length $4R > 4r$, then
\[ |\int_Q \nabla^n U_{Z_0, r, q}| \leq |Q|^{-1/\mu} \|\nabla^n U_{Z_0, r, q}\|_{L^\mu(Q)} \leq C_\mu R^{m-n}(R/r)^{m/q-1} \|U\|_{Y^{m, \mu}(Q)} \]
and so
\[ \left( \int_Q |\nabla^n U_{Z_0, r, q}|^p \right)^{1/p} \leq C_{p, \mu} R^{d/p}(R/r)^{m/q-1} \|U\|_{Y^{m, \mu}(\mathbb{R}^d)}. \]

7.2. The fundamental solution for operators of high order. We now define a preliminary version of our fundamental solution for operators of high order. If $d$ is odd, we will use this definition throughout; if $d$ is even then we will modify the definition somewhat in Section ??.

We will consider operators of lower order in Section ??.

**Definition 81.** Let $m$ and $d$ be integers with $2m > d \geq 2$. Let $L$ be a bounded and invertible linear operator $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ for some $q$ with $1 < q < \infty$ and $1 - m/d < 1/q < m/d$. Let $Z_0 \in \mathbb{R}^d$, let $r > 0$, and let $1 \leq j \leq N$.

Let $T_{X,j,Z_0,r,q}$ be given by
\[ (T_{X,j,Z_0,r,q}, \Phi) = (\Phi_j)_{Z_0, r, q'}(X) \]
where $1/q + 1/q' = 1$. By Lemma (55), this is a well defined bounded linear operator on $Y^{m,q'}(\mathbb{R}^d)$; that is, $T_{X,j,Z_0,r,q} \in Y^{-m,q'}(\mathbb{R}^d)$.

We define the fundamental solution $E_{X,j,Z_0,r,q}$ by
\[ E_{X,j,Z_0,r,q} = (L^{-1}T_{X,j,Z_0,r,q})_{Z_0, r, q}. \]

**Remark 82.** If $L$ is bounded and invertible $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$, and if $L$ is defined and bounded $Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ for all $q$ in an open neighborhood of 2, then by Lemma (61) $q$ satisfies the conditions of Definition (81) for all $q$ in a (possibly smaller) neighborhood of 2.

**Remark 83.** Since $Y^{-m,q}(\mathbb{R}^d)$ is by definition the dual space to $Y^{m,q'}(\mathbb{R}^d)$, by standard function theoretic arguments $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ is bounded and invertible if and only if its adjoint operator $L^* : Y^{m,q'}(\mathbb{R}^d) \to Y^{-m,q'}(\mathbb{R}^d)$ is bounded and invertible. Furthermore, $(L^{-1})^* = (L^*)^{-1}$. Also observe that $\max(0,1-m/d) < 1/q < \min(1,m/d)$ if and only if $\max(0,1-m/d) < 1/q' < \min(1,m/d)$. Thus, $L$ and $q$ satisfy the conditions of Definition (81) if and only if $L^*$ and $q'$ satisfy those conditions.

That is, $E_{X,j,Z_0,r,q}$ exists (for all $X$, $j$, $Z_0$, $r$) if and only if $E_{Y,k,Z_0,r,q'}^*$ exists (for all $Y$, $k$, $Z_0$, and $r$).

In the remainder of this subsection we will establish some basic properties of the fundamental solution; we will establish further properties in Sections ??, ??, ??, ??, and ??, We will begin with a symmetry property for the operators $L$ and $L^*$; we will use this property to establish certain symmetries of the fundamental solution.

**Theorem 84.** Let $L$ and $q$ satisfy the conditions of Definition (81). Let $Z_0 \in \mathbb{R}^d$, let $r > 0$, and let $j$, $k$ be integers in $[1, N]$.

For all $X$, $Y \in \mathbb{R}^d$ we have that
\[ (E_{X,j,Z_0,r,q})_k(Y) = (E_{Y,k,Z_0,r,q}^*)_j(X). \]
For every $S \in Y^{-m,q'}(\mathbb{R}^d)$ and every $X \in \mathbb{R}^d$ we have that
\begin{equation}
\langle S, \tilde{E}^{L}_{X,j,Z_0,r,q} \rangle = (((L^*)^{-1}S)_j)_{Z_0,r,q}(X).
\end{equation}
(86)
Finally, if we let
\begin{equation}
E^L_{j,k,Z_0,r,q}(X,Y) = (\tilde{E}^{L}_{Y,k,Z_0,r,q})_j(X) = (\tilde{E}^{L,*}_{X,j,Z_0,r,q})_k(Y),
\end{equation}
then $E^L_{j,k,Z_0,r,q}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

Proof. That $\tilde{E}^L_{Y,k,Z_0,r,q'}$ exists is Remark 83.
If $X, Y \in \mathbb{R}^d$ and $1 \leq j \leq N$, $1 \leq k \leq N$, then by Definition 33 and Remark 83
\begin{equation}
(\tilde{E}^{L}_{X,j,Z_0,r,q})_k(Y) = \langle T_{Y,k,Z_0,r,q}, \tilde{E}^{L}_{X,j,Z_0,r,q} \rangle = \langle T_{Y,k,Z_0,r,q}, L^{-1}T_{X,j,Z_0,r,q} \rangle
= \langle T_{X,j,Z_0,r,q}, (L^*)^{-1}T_{Y,k,Z_0,r,q} \rangle = \langle T_{X,j,Z_0,r,q}, \tilde{E}^{L,*}_{Y,k,Z_0,r,q} \rangle
= (\tilde{E}^{L,*}_{Y,k,Z_0,r,q})_j(X).
\end{equation}
In particular, observe that by Lemma 76 $\tilde{E}^{L}_{Y,k,Z_0,r,q}(X)$ is locally uniformly continuous in both $X$ and $Y$, and so $E^L_{j,k,Z_0,r,q}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

Similarly, we have that if $S \in Y^{-m,q'}(\mathbb{R}^d)$, then
\begin{equation}
\langle S, \tilde{E}^{L}_{X,j,Z_0,r,q} \rangle = \langle S, L^{-1}T_{X,j,Z_0,r,q} \rangle = \langle T_{X,j,Z_0,r,q}, (L^*)^{-1}S \rangle = (((L^*)^{-1}S)_j)_{Z_0,r,q}(X).
\end{equation}
This establishes formula (86). \hfill \Box

We will conclude this section with a preliminary bound on the derivatives of the function $\tilde{E}^L_{X,j,Z_0,r,q}$.

Theorem 87. Let $L$ and $q$ satisfy the conditions of Definition 33. Let $1 < p \leq 2q$. Suppose that $L$ also satisfies the conditions of Definition 37 with $q$ replaced by $p$, and that the inverses are compatible in the sense of Definition 33 that is, if $T \in Y^{-m,p}(\mathbb{R}^d) \cap Y^{-m,q}(\mathbb{R}^d)$ then $L^{-1}T \in Y^{m,p}(\mathbb{R}^d) \cap Y^{m,q}(\mathbb{R}^d)$.

Suppose that $\beta$ is a multindex with $0 \leq |\beta| \leq m$. Let $Q \subset \mathbb{R}^d$ be a cube. Then we have the bound
\begin{equation}
\left( \int_Q |\partial^\beta \tilde{E}^L_{X,j,Z_0,r,q}|^p \right)^{1/p} \leq CR^{q^* - m - d/p - |\beta|} \left( \frac{R}{r} \right)^{\kappa},
\end{equation}
where $R = \max\{r, |X - Z_0|, \text{dist}(Z_0, Q) + \text{diam}(Q)\}$, and where $C$ and $\kappa$ are positive constants depending on $q$, $p$, the norms of $L^{-1} : Y^{-m,q}(\mathbb{R}^d) \rightarrow Y^{m,q}(\mathbb{R}^d)$ and $L^{-1} : Y^{m,p}(\mathbb{R}^d) \rightarrow Y^{-m,p}(\mathbb{R}^d)$, and the standard parameters.

Recall from Definition 33 that $\Upsilon_L$ is the set of all $q$ such that $L^{-1}$ is compatible between $Y^{m,2}(\mathbb{R}^d)$ and $Y^{m,q}(\mathbb{R}^d)$. By density, if $p, q \in \Upsilon_L$, then $L^{-1}$ is compatible between $Y^{m,p}(\mathbb{R}^d)$ and $Y^{m,q}(\mathbb{R}^d)$, as required by the lemma.

Proof of Theorem 87. By Lemma 75 if $T_{X,j,Z_0,r,q}$ is as in Definition 81 then
\begin{equation}
\|T_{X,j,Z_0,r,q}\|_{Y^{-m,q}(\mathbb{R}^d)} \leq C_q R^{m - d/q} \left( \frac{R}{r} \right)^{\omega q'} - 1
\end{equation}
and so by invertibility of $L$,
\begin{equation}
\|\tilde{E}^L_{X,j,Z_0,r,q}\|_{Y^{m,q}(\mathbb{R}^d)} \leq CR^{q^* - m - d/p - |\beta|} \left( \frac{R}{r} \right)^{\kappa}.
\end{equation}
(89)
By Lemma 75 if $|\beta| < m - d/q$ and $|Y - Z_0| < R$ then

$$|\partial^a X, j, Z_0, r, q(Y)| \leq CR^{2m - d - |\beta|} \left( \frac{R}{r} \right) \kappa.$$  

Integration yields the bound (88) in this case (for all $p \in [1, \infty]$).

By Remark 80 if $|\beta| = m - d/q$ and $|\bar{Q}| = 4R$ with $\bar{Q}$ centered at $Z_0$, then

$$\left( \int_{\bar{Q}} |\partial^a X, j, Z_0, r, q(Y)|^p dY \right)^{1/p} \leq CR^{d/p - m - d'/q'} \left( \frac{R}{r} \right) \kappa.$$  

Because $|\beta| = m - d/q = m - d + d'$, the bound (88) is valid in this case (for all $p \in [1, \infty]$).

We are left with the case $|\beta| > m - d/q$. If $q \geq p$ and $m - d/q < |\beta|$, or if $q < p$ and $m - d/q < p \leq m - d/q + d/p$, then by formula (21) we have that $p \leq q_\beta < \infty$. By the bound (88) and Hölder’s inequality,

$$\left( \int_{\bar{Q}} |\partial^a X, j, Z_0, r, q|^p dY \right)^{1/p} \leq CR^{m - d + d/p - |\beta|} \left( \frac{R}{r} \right) \kappa.$$  

Finally, suppose that $q < p$ and that $m - d/q + d/p < |\beta| \leq m$. If $q < p$ then $q' > p'$, and so by Lemma 75

$$\|T_{X, j, Z_0, r, q}\|_{Y - m, p(\mathbb{R}^d)} \leq CR^{m - d/p} \left( \frac{R}{r} \right) \kappa.$$  

By compatible invertibility of $L : Y^{m, p}(\mathbb{R}^d) \rightarrow Y^{m, p}(\mathbb{R}^d)$, we have that

$$\|E_{X, j, Z_0, r, q}\|_{Y - m, p(\mathbb{R}^d)} \leq CR^{m + d/p} \left( \frac{R}{r} \right) \kappa.$$  

If $|\beta| > m - d/q + d/p$ and $p \leq 2q$ then $|\beta| > m - d/p$ and so this provides a Lebesgue space bound on $\partial^a E_{X, j, Z_0, r, q}$. By Hölder’s inequality,

$$\left( \int_{\bar{Q}} |\partial^a E_{X, j, Z_0, r, q}|^p dY \right)^{1/p} \leq CR^{2m - d + d/p - |\beta|} \left( \frac{R}{r} \right) \kappa$$  

which is the bound (88).

In any case, the bound (88) holds. □

7.3. Mixed derivatives of the fundamental solution. Recall that $E_{X, j, Z_0, r, q}(Y)$ is a function of both $X$ and $Y$. We may control derivatives in $Y$ using Theorem 77 and derivatives in $X$ using formula (85) and Theorem 87 applied to $E_{X, j, Z_0, r, q'}$. We will also wish to control mixed derivatives, that is, derivatives in both $X$ and $Y$. This subsection will consist of the following theorem and its proof.

**Theorem 91.** Let $L$ be an operator of the form (24) with $2 \in \mathfrak{Y}_L \cap \Pi_L$, and let $q \in \mathfrak{Y}_L \cap \Pi_L$ with $1 - m/d < 1/q < m/d$, where $\Pi_L$ and $\mathfrak{Y}_L$ are as in Definitions 25 and 36. Then $L$ and $q$ satisfy the conditions of Definition 87 and Theorem 87 for all $p \in \mathfrak{Y}_L \cap \Pi_L \cap (1, 2q]$ with $1 - m/d < 1/p < m/d$. 


Let $p \in \Upsilon_L \cap \Pi_L \cap (1, 2q)$ with $1 - m/d < 1/p < m/d$. Suppose that the Caccioppoli-Meyers inequality

$$\sum_{j=0}^{m} |Q|^{j/d} \left( \int_{Q} |\nabla u|^{p} \right)^{1/p} \leq C |Q|^{1/p-1/2} \left( \int_{2Q} |\tilde{u}|^{2} \right)^{1/2} + C |Q|^{m/d} \|L\tilde{u}\|_{Y-m,p(2Q)}$$

holds whenever $Q \subset \mathbb{R}^{d}$ is a cube with sides parallel to the coordinate axes and whenever $\tilde{u}$ is a representative of an element of $Y^{m,p}(2Q)$, with $C$ independent of $\tilde{u}$ and $Q$. Suppose in addition this statement is valid with $p$ replaced by $2$.

Suppose that $\alpha$ is a multiindex with $0 \leq |\alpha| \leq m$.

Then for every compact set $K \subseteq \mathbb{R}^{d}$, the function $\partial^{\alpha}_{\gamma} \tilde{E}_{X,j,Z_{0},r,q}(Y)$ is in $Y^{m,p}(K)$ for almost every $X \in \mathbb{R}^{d} \setminus K$. If $|\alpha| < \min(m - d/p, m - d/2)$ then $\partial^{\alpha}_{\gamma} \tilde{E}_{X,j,Z_{0},r,q}(Y)$ is a representative of an element of $Y^{m,p}(K)$ for almost every $X \in \mathbb{R}^{d}$. Furthermore, we have the bound

$$\int_{K} \|\partial^{\alpha}_{\gamma} \tilde{E}_{X,j,Z_{0},r,q}(Y)\|_{Y^{m,p}(Q)}^{2} dX \leq CR^{2m - d + 2d/p - 2|\alpha|} \left( \frac{R}{\min(|Q|^{1/d}, r)} \right)^{\kappa} R^{4m - 2|\alpha| - 2|\beta|}$$

whenever $\Gamma$ and $Q$ are cubes with $|\Gamma| = |Q|$, $\Gamma \subset 8Q$, and either $\Gamma \subset 8Q \setminus 4Q$ or $|\alpha| < m - d/p'$. Here $R = \max(r, |Q|^{1/d}, \text{dist}(\bar{Z}_{0}, Q))$ and $\kappa$ is a positive constant depending on the standard parameters.

In particular, if the Caccioppoli inequality (92) is valid for $p = 2$, then for all multiindices $\beta$ with $0 \leq |\beta| \leq m$, the mixed partial derivative $\partial^{\alpha}_{\gamma} \partial^{\beta}_{\delta} \tilde{E}_{X,j,Z_{0},r,q}(Y)$ exists as a locally $L^{2}$ function defined on $\mathbb{R}^{d} \times \mathbb{R}^{d} \setminus \{(X, X) : X \in \mathbb{R}^{d}\}$. Furthermore, if $Q, \Gamma \subset \mathbb{R}^{d}$ are two cubes with $|Q| = |\Gamma|$ and $\Gamma \subset 8Q \setminus 4Q$, then

$$\int_{Q} \int_{\Gamma} \|\partial^{\alpha}_{\gamma} \partial^{\beta}_{\delta} \tilde{E}_{X,j,Z_{0},r,q}(Y)\|_{Y^{m,p}(Q)}^{2} dY dX \leq C \left( \frac{R}{\min(|Q|^{1/d}, r)} \right)^{\kappa} R^{4m - 2|\alpha| - 2|\beta|}$$

where $R = \max(r, |Q|^{1/d}, \text{dist}(\bar{Z}_{0}, Q))$ and $\kappa$ is a positive constant depending on the standard parameters. If the Caccioppoli inequality is valid for $L^{\ast}$, that is, if the bound (92) is valid with $p = 2$ and $L$ replaced by $L^{\ast}$, then the bound (95) is valid whenever $|\beta| < m - d/2$ even if $m - d/2 > |\alpha|$. Let $\alpha$ be a multiindex with $|\alpha| \leq m$. Let $1 \leq j \leq N$.

Let $\eta$ be a nonnegative real-valued smooth cutoff function supported in $B(0, 1)$ and integrating to $1$ and define $\eta_{\varepsilon}(X') = \frac{1}{\varepsilon} \eta\left(\frac{1}{\varepsilon} X'\right)$ for $\varepsilon > 0$. Define

$$\bar{u}_{\varepsilon, \alpha, X}(Y) = \int_{B(X, \varepsilon)} \partial^{\alpha}_{\gamma} \eta_{\varepsilon}(X - X') \tilde{E}_{X', j, Z_{0}, r, q}(Y) dX'.$$
By the weak definition of derivative and the symmetry relation (85),
\[
(\tilde{u}_{\varepsilon,\alpha,X}(Y))_k = \eta_\varepsilon \ast (\partial^n \tilde{E}_{\varepsilon,k,Z_0,r,q}^L)(X).
\]  
We now investigate \(\tilde{u}_{\varepsilon,\alpha,X}\).

**Lemma 98.** With the above construction and under the conditions of Theorem 87, if \(Q \subset \mathbb{R}^d\) is a cube, then \(\tilde{u}_{\varepsilon,\alpha,X} \in W^{m,p}(2Q)\), and if \(|\beta| \leq m\) then
\[
\partial^\beta u_{\varepsilon,\alpha,X}(Y) = \int \partial^\alpha \eta_\varepsilon(X - X') \partial^\beta \tilde{E}_{X',j,Z_0,r,q}^L(Y) dX'.
\]

**Proof.** If \(0 \leq |\beta| \leq m\) and \(\tilde{\varphi} \in C_0^\infty(B(Y_0,2\rho))\), then
\[
\int \partial^\beta \tilde{\varphi} \cdot \tilde{u}_{\varepsilon,\alpha,X} = \int \partial^\beta \tilde{\varphi}(Y) \cdot \int \partial^\alpha \eta_\varepsilon(X - X') \tilde{E}_{X',j,Z_0,r,q}^L(Y) dY dX'.
\]
By Theorem 87, \(\tilde{E}_{X',j,Z_0,r,q}^L\) and \(\partial^\beta \tilde{E}_{X',j,Z_0,r,q}^L\) are locally square integrable and thus locally integrable. By Fubini’s theorem and the definition of weak derivative,
\[
\int \partial^\beta \tilde{\varphi} \cdot \tilde{u}_{\varepsilon,\alpha,X} = \int \partial^\alpha \eta_\varepsilon(X - X') \int \partial^\beta \tilde{\varphi}(Y) \cdot \tilde{E}_{X',j,Z_0,r,q}^L(Y) dY dX' \\
= (-1)^{|\beta|} \int \partial^\alpha \eta_\varepsilon(X - X') \int \tilde{\varphi}(Y) \cdot \partial^\beta \tilde{E}_{X',j,Z_0,r,q}^L(Y) dY dX' \\
= (-1)^{|\beta|} \int \tilde{\varphi}(Y) \cdot \int \partial^\alpha \eta_\varepsilon(X - X') \partial^\beta \tilde{E}_{X',j,Z_0,r,q}^L(Y) dX' dY.
\]
This is true for all test functions \(\tilde{\varphi}\), and so we have that formula (99) is valid. Another application of Fubini’s theorem yields that
\[
\int \partial^\beta \tilde{\varphi} \cdot \tilde{u}_{\varepsilon,\alpha,X} = (-1)^{|\beta|} \int \partial^\alpha \eta_\varepsilon(X - X') \int \tilde{\varphi}(Y) \cdot \partial^\beta \tilde{E}_{X',j,Z_0,r,q}^L(Y) dY dX'.
\]
By the bound (85), \(\partial^\beta \tilde{E}_{X',j,Z_0,r,q}^L \in L^p(2Q)\) and so there is some large constant \(K_\varepsilon\) depending on \(\varepsilon, r, R\), and other parameters such that
\[
\left| \int \partial^\beta \tilde{\varphi} \cdot \tilde{u}_{\varepsilon,\alpha,X} \right| \leq K_\varepsilon \|\tilde{\varphi}\|_{L^p(2Q)}.
\]
So by the Hahn-Banach theorem and the Riesz representation theorem there is a function \(\tilde{U} \in L^p(2Q)\) with
\[
\int_{B(Y_0,\rho)} (-1)^{|\beta|} \partial^\beta \tilde{\varphi}(Y) \cdot \tilde{u}_{\varepsilon,\alpha,X}(Y) dY = \int_{B(Y_0,\rho)} \tilde{\varphi}(Y) \cdot \tilde{U}(Y) dY
\]
for all \(\tilde{\varphi} \in C_0^\infty(2Q)\). By the weak definition of derivative, \(\tilde{U} = \partial^\alpha \tilde{u}_{\varepsilon,\alpha,X}\), so \(\tilde{u}_{\varepsilon,\alpha,X} \in W^{m,p}(2Q)\).

We will need a better bound on \(\|\partial^\beta \tilde{u}_{\varepsilon,\alpha,X}\|_{L^2(Q)}\). We seek to apply the Caccioppoli (and Meyers) inequalities; we will need to compute \(L\tilde{u}_{\varepsilon,\alpha,X}\).

**Lemma 100.** With the above construction, if \(L\) is of the form (24) and \(q \in \Pi_L\) with \(1 - m/d < 1/q < m/d\) and with \(L: Y^{m,q}(\mathbb{R}^d) \rightarrow Y^{-m,q}(\mathbb{R}^d)\) invertible, and if \(Q \subset \mathbb{R}^d\) is a cube, then for all \(\varepsilon \in (0, \frac{1}{2}|Q|^{1/d})\) and all \(p\) with \(1 > 1/p > \max(0, 1 - m/d)\), if \(X \in 9Q\) and either \(X \notin 3Q\) or \(|\alpha| < m - d/p'\), then
\[
\|L\tilde{u}_{\varepsilon,\alpha,X}\|_{Y^{-m,p}(2Q)} \leq CR^{-d/p' - |\alpha|} \left( \frac{R}{\varepsilon} \right)^k
\]
where \( R = \max(r, |Q|^{1/d}, \operatorname{dist}(Z_0, Q)) \) and \( C \) and \( \kappa \) are constants depending on the standard parameters.

**Proof.** Let \( \Phi \in C_0^\infty(2Q) \). By the bound (39) and the definition of \( \Pi_L \), \( \langle L^E \hat{\Phi}_L^{K_{i,j}, Z_0, r, q} \rangle \) denotes an absolutely convergent integral whenever \( \hat{\Phi} \in Y^{m,q'}(\mathbb{R}^d) \), and furthermore, the integrand has uniform \( L^1 \) norm. Thus we may apply Fubini’s theorem to the integral

\[
\int \partial^\alpha \eta(X - X') \langle L^E \hat{\Phi}_L^{K_{i,j}, Z_0, r, q} \rangle \, dX'
\]

and compute that

\[
\langle L\hat{u}_{\varepsilon, \alpha, X}, \hat{\Phi} \rangle = \int \partial^\alpha \eta(X - X') \langle L^E \hat{\Phi}_L^{K_{i,j}, Z_0, r, q} \rangle \, dX'.
\]

By formula (30),

\[
\langle L^E \hat{\Phi}_L^{K_{i,j}, Z_0, r, q} \rangle = \langle L^* \hat{\Phi}_L^{K_{i,j}, Z_0, r, q} \rangle = \langle ((1 + L^*)^{-1} L^* \hat{\Phi}) \rangle_{Z_0, r, q}(X) = \langle \Phi_j \rangle_{Z_0, r, q}(X) - \langle \Phi_j \rangle_{Z_0, r, q}(X').
\]

Thus,

\[
\langle L\hat{u}_{\varepsilon, \alpha, X}, \hat{\Phi} \rangle = \eta \ast (\partial^\alpha \langle \Phi_j \rangle_{Z_0, r, q})(X).
\]

Recall that \( \langle \Phi_j \rangle_{Z_0, r, q} = \Phi_j + P \) for some polynomial \( P \) of degree at most \( m - d/q \) satisfying \( P(Z_0 + r H_i) = -\Phi_j(Z_0 + r H_i) \). As in the proof of Lemma (75) if \( \|P(X) = \sum_{|\alpha| \leq m - d/q} P_\gamma (X - Z_0)^\alpha \| \), then

\[
|p_\gamma| \leq h \sup \left| \sum_{|\gamma| \leq m - d/q} p_\gamma H_i^\gamma \right| = h \sup_i |P(Z_0 + r H_i)| = h \sup_i |\Phi_j(Z_0 + r H_i)|.
\]

Because \( \hat{\Phi} \in C_0^\infty(2Q) \), we have that \( \hat{\Phi} = 0 \) outside of \( B(Z_0, (1 + 2\sqrt{d})R) \). Thus, \( \hat{\Phi} = \hat{\Phi}_{Z_0, C, R, q} = \hat{\Phi}_{Z_0, C, R, p'} \) because \( |H_i| > 1/2 \) for all \( i \). Thus

\[
|p_\gamma| \leq h \sup_i |\hat{\Phi}_{Z_0, C, R, p'}(Z_0 + r H_i)|
\]

and by Lemma (76) since \( p' > d/m \),

\[
|p_\gamma| \leq CR^{m - d/p'} \| \hat{\Phi} \|_{Y^{m,p'}(\mathbb{R}^d)}.
\]

Thus

\[
|\langle L\hat{u}_{\varepsilon, \alpha, X}, \hat{\Phi} \rangle| = |\eta \ast (\partial^\alpha P)(X) + \eta \ast (\partial^\alpha \Phi_j)(X)|
\leq |\eta \ast (\partial^\alpha \Phi_j)(X)| + CR^{m - d/p' - |\alpha|} \left( \frac{R}{r} \right)^\kappa \| \hat{\Phi} \|_{Y^{m,p'}(\mathbb{R}^d)}.
\]

If \( X \notin 3Q \) and \( 0 < \varepsilon < \frac{1}{2}|Q|^{1/d} = \operatorname{dist}(2Q, \mathbb{R}^d \setminus 3Q) \), then \( \eta \ast (\partial^\alpha \Phi_j)(X) = 0 \). If \( |\alpha| < m - d/p' \), then again by Lemma (75) applied to \( \Phi_j = (\Phi_j)_{Z_0, C, R, p'} \), if \( 0 < \varepsilon < R \) then

\[
|\langle L\hat{u}_{\varepsilon, \alpha, X}, \hat{\Phi} \rangle| \leq CR^{m - d/p' - |\alpha|} \left( \frac{R}{r} \right)^\kappa \| \hat{\Phi} \|_{Y^{m,p'}(\mathbb{R}^d)}.
\]

This completes the proof. \( \square \)

We have established that \( \hat{u}_{\varepsilon, \alpha, X} \in W^{m,p}(2Q) \) and have a bound on \( L\hat{u}_{\varepsilon, \alpha, X} \). We will now bound the derivatives of \( \hat{u}_{\varepsilon, \alpha, X} \).
Lemma 102. Let $L$, $q$, and $p$ satisfy the conditions of Theorem 87. Suppose in addition that the conclusion (101) of Lemma 101 is valid (under the given conditions on $\varepsilon$, $X$ and $\alpha$). Let $Q \subset \mathbb{R}^d$ be a cube. Suppose further that the Caccioppoli-Meyers estimate (92) is valid in $Q$ for all $\bar{u} \in Y^{m,p}(2Q)$. Let $\Gamma \subset Q$ be a cube with $|\Gamma| = |Q|$. Then for all $\varepsilon \in (0, \frac{1}{2}|Q|^{1/d})$, if either $\Gamma \subset Q \setminus 4Q$ or $|\alpha| < m - d/p'$, then
\[
\int_{\Gamma} \|\vec{u}_{\varepsilon,\alpha,X}\|_{Y^{m,p}(Q)}^2 dX \leq C|Q|^{2/|2\varepsilon| - 2m/d} R^{4m-2|\alpha|} \left(\frac{R}{r}\right)^\kappa
\]
where $R = \max(r, |Q|^{1/d})$ and $C$ and $\kappa$ are constants depending on the standard parameters.

In particular, if $p = 2$ then for all $\beta$ with $|\beta| \leq m$ we have that
\[
\int_{\Gamma} \int_{Q} |\partial^\beta \vec{u}_{\varepsilon,\alpha,X}(Y)|^2 dY dX \leq C|Q|^{-|\beta|/d} R^{4m-2|\alpha|} \left(\frac{R}{r}\right)^\kappa.
\]

Proof. Applying the bounds (101) and (92) and Lemma 39 to $\bar{u} = \vec{u}_{\varepsilon,\alpha,X}$ yields
\[
\|\vec{u}_{\varepsilon,\alpha,X}\|_{Y^{m,p}(Q)} \leq C|Q|^{1/p - 1/2 - m/d} \left(\int_{2Q} \|\vec{u}_{\varepsilon,\alpha,X}\|^2 dX\right)^{1/2} + CR^{m-d/p'} |\alpha| \left(\frac{R}{r}\right)^\kappa.
\]

By formula (97), the $L^2$ boundedness of convolution, and the bound (88), if $\varepsilon$ is small enough, $\Gamma \subset 8Q$ and $Y \subset 2Q$, then
\[
\int_{\Gamma} \|\vec{u}_{\varepsilon,\alpha,X}(Y)\|^2 dX \leq \sup_{1 \leq k \leq N} \int_{2\Gamma} |\partial^\alpha \vec{E}_{Y,k,0,\varepsilon,Y}(X)|^2 dX \leq C \left(\frac{R}{r}\right)^\kappa R^{4m-d-2|\alpha|}.
\]
Combining the above bounds completes the proof. \qed

We now prove Theorem 91. The assumptions of Theorem 91 include the assumptions of Theorem 87, Lemma 101 and Lemma 102 with $p = 2$; we will use only the conclusions of Lemma 102 and the definitions (97) and (98) of $\vec{u}_{\varepsilon,\alpha,X}$.

The Lebesgue space $L^2(\Gamma \times Q)$ is weakly sequentially compact. Thus, because $\{\vec{u}_{\varepsilon,\alpha,X}\}_{0 < \varepsilon < \frac{1}{4}|Q|^{1/d}}$ is a bounded set in $L^2(\Gamma \times Q)$, if $0 \leq |\beta| \leq m$, there is a function $\vec{E}_{\alpha,\beta,j}$ with
\[
\int_{\Gamma} \int_{Q} |\vec{E}_{\alpha,\beta,j}(X,Y)|^2 dY dX \leq CR^{4m-2|\alpha| - 2|\beta|} \left(\frac{R}{\min\{r, |Q|^{1/d}\}}\right)^\kappa
\]
and a sequence of positive numbers $\varepsilon_i$ with $\varepsilon_i \to 0$ and such that, for all $\vec{\varphi} \in L^2(\Gamma \times Q)$, we have that
\[
\int_{\Gamma} \int_{Q} \vec{\varphi}(X,Y) \cdot \vec{E}_{\alpha,\beta,j}(X,Y) dY dX = \lim_{i \to \infty} \int_{\Gamma} \int_{Q} \vec{\varphi}(X,Y) \cdot \partial^\beta \vec{u}_{\varepsilon_i,\alpha,X}(Y) dY dX.
\]
Integrating by parts and applying formula (97), we see that if $\vec{\varphi}$ is smooth and compactly supported then
\[
\int_{\Gamma} \int_{Q} \varphi_k(X,Y) (\vec{E}_{\alpha,\beta,j}(X,Y))_k dY dX
\]
\[
= (-1)^{|\beta|} \lim_{i \to \infty} \int_{\Gamma} \int_{Q} \partial^\beta Y \varphi_k(X,Y) (\vec{u}_{\varepsilon_i,\alpha,X}(Y))_k dY dX
\]
\[
= (-1)^{|\beta|} \lim_{i \to \infty} \int_{\Gamma} \int_{Q} \partial^\beta Y \varphi_k(X,Y) \eta_{\varepsilon_i} \cdot \partial^\alpha (\vec{E}_{Y,k,0,\varepsilon_i,Y})_j(X) dY dX.
\]
Using properties of convolutions, we see that
\[
\int \int \varphi_k(X, Y) (\bar{E}_{-\beta,j}(X, Y)) \, dY \, dX = (-1)^{[\beta]+[\alpha]} \lim_{i \to \infty} \int \int \eta_i \ast X \frac{\partial^\alpha \partial^\beta \varphi_k(X, Y) (\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)) \, dY \, dX}
\]
where \(*_X\) denotes convolution in the X variable only. By the dominated convergence theorem,
\[
\int \int \varphi_k(X, Y) (\bar{E}_{-\alpha,j}(X, Y)) \, dY \, dX = (-1)^{[\beta]+[\alpha]} \int \int \partial^\alpha \partial^\beta \varphi_k(X, Y) (\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)) \, dY \, dX
\]
and so \((\bar{E}_{-\alpha,j}(X, Y)) = \partial^\alpha \partial^\beta (\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)) \, dY \, dX\) in the weak sense. Furthermore, we may derive bounds on \(\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)\) from our bounds on \(\bar{u}_{\epsilon_i,\alpha,X}\). Thus, by Lemma 102, we have the bound (94) and the bound (95) in the case \([\alpha] < m - d/2\).

Suppose \([\beta] < m - d/2\) and the Caccioppoli inequality (92) with \(p = 2\) holds for \(L^\gamma\). By Remark 83, we may thus apply the above results to \(\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)\). By the bound (95) for \(\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)\), if \([\beta] < m - d/2\) then
\[
\int \int \partial^\alpha \partial^\beta (\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)) \, dX \, dY \leq C \left( \frac{R}{\min(|Q^{1/d}, r)} \right)^k R^{4m-2[\alpha]-2[\beta]}.
\]
Applying formula (83) yields the bound (95) in the case \([\beta] < m - d/2\).

The space \(L^2(\Gamma; L^{p^q}(Q))\) is a Bochner space, and so is a reflexive Banach space with dual \(L^2(\Gamma; L^{(p^q)'(Q)})\). By Lemma 102 we have that if \(\varphi \in L^2(\Gamma, Q)\) then
\[
\int \int \varphi(X, Y) \cdot \partial^\alpha \partial^\beta (\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)) \, dY \, dX \leq \lim_{i \to \infty} \left( \int \int \varphi(X, Y) \cdot \partial^\alpha \bar{u}_{\epsilon_i,\alpha,X}(Y) \, dY \, dX \right)^{1/2} (\int \int \left( \varphi(X, Y) \cdot (p^q)^{(p^q)'} \, dY \right)^{2/(p^q)'} \, dX \right)^{1/2} CR^d \left( \frac{R}{\min(r, |Q^{1/d})} \right)^k
\]
where \(\theta = m - d/2 + d/p - [\alpha]\). The space \(L^2(\Gamma; Q)\) is dense in \(L^2(\Gamma; L^{(p^q)'(Q)})\). Thus, this bound is valid for all \(\varphi \in L^2(\Gamma; L^{(p^q)'(Q)})\), and so \(\partial^\alpha \partial^\beta (\bar{E}^{\bar{E}}_{-\beta,j}(X, Y)) \in L^2(\Gamma; L^{p^q}(Q))\) and satisfies the bound (93).

### 7.4. Extraneous parameters

The fundamental solution \(E_{X,j}(X, r, q)\) of Definition 81 depends on the parameters \(r, q, Z_0\), and \(q\) in a somewhat artificial way: they are used only to normalize \(E_{X,j}(X, r, q)\) and \(E^{L}_{X,j}(X, r, q)\). We would like to the extent possible to remove the dependencies on \(Z_0, r, q\). The following lemma will allow us to remove (or at least reduce) these dependencies.

**Lemma 103.** Let \(q_1, q_2 \in (1, \infty)\). Let \(L\) satisfy the conditions of Definition 81 for both \(q_1 = q_2\). Suppose that \(L\) is compatible in the sense that if \(S \in Y^{-m,q_1}(\mathbb{R}^d) \cap Y^{-m,q_2}(\mathbb{R}^d)\), then \(L^{-1}S \in Y^{m,q_1}(\mathbb{R}^d) \cap Y^{m,q_2}(\mathbb{R}^d)\).
Suppose that $\alpha$ and $\beta$ are multiindices such that
\[ \max(m - d/q, m - d/q') < |\alpha| \leq m, \]
\[ \max(m - d/q_1, m - d/q_2) < |\beta| \leq m. \]

Let $1 \leq j \leq N$, $r_1 > 0$, $r_2 > 0$, $Z_1 \in \mathbb{R}^d$, and $Z_2 \in \mathbb{R}^d$. Suppose that, for $i \in \{1, 2\}$, the mixed derivative $\partial^\alpha \partial^\beta E_{X,j,z_1,r_i,q_1}(Y)$ exists almost everywhere and is locally integrable on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(X, X) : X \in \mathbb{R}^d\}$.

Then we have that
\[
\partial^\alpha \partial^\beta E_{X,j,z_1,r_1,q_1}(Y) = \partial^\alpha \partial^\beta E_{X,j,z_2,r_2,q_2}(Y)
\]
for almost every $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$.

As noted after Theorem 87 if $\Upsilon_L$ is as in Definition 32 and $q_1$, $q_2 \in \Upsilon_L$, then $L$, $q_1$, and $q_2$ satisfy the conditions of the lemma.

Under the conditions of Theorem 91 existence and local integrability of the mixed partial derivative is valid. Furthermore, under these conditions we may combine formulas (104) and (94) to see that if $\alpha$ mixed partial derivative is valid. Furthermore, under these conditions we may combine formulas (104) and (94) to see that if $\alpha$ and $\beta$ are multiindices with $m - d/q < |\alpha| \leq m$ and $m - d/q < |\beta| \leq m$, then by choosing $Z_0$ and $r$ appropriately, we have that if $\rho = |X_0 - Y_0|/8$, then
\[
\int_{B(X_0, \rho)} \int_{B(Y_0, \rho)} |\partial^\alpha \partial^\beta E_{X,j,z_0,r,q}(Y)|^2 \, dY \, dX \leq C \rho^{4m-2|\alpha|-2|\beta|}.
\]

Proof of Lemma 102

Fix some such $\alpha$, $\beta$, and $\gamma$.

Let $\eta$ and $\varphi$ be smooth functions with disjoint compact support. Let $T$ be given by
\[
\langle T, \Phi \rangle = \int_{\mathbb{R}^d} \Phi_k(Y) \partial^\alpha \eta(Y) \, dY = (\Phi_k(Y) \eta(Y) \, dY) \int_{\mathbb{R}^d} \partial^\alpha \Phi_k(Y) \eta(Y) \, dY.
\]

Because $|\beta| > m - d/q_i$, we have that if $\Phi \in Y^{m,q_i}(\mathbb{R}^d)$ then $\partial^\alpha \Phi$ is well defined as a $L^{q_i}(\mathbb{R}^d)$-function (that is, to sets of measure zero, not up to polynomials), and so $T \in Y^{-m,q_i}(\mathbb{R}^d)$ with no normalization necessary.

By formula (86),
\[
((L^*)^{-1}T)_{j,z_i,r_i,q_i}(X) = \langle T, E_{X,j,z_i,r_i,q_i} \rangle.
\]

By duality, if $T \in Y^{-m,q_i}(\mathbb{R}^d) \cap Y^{-m,q_i}(\mathbb{R}^d)$, then $(L^*)^{-1}T = (L^{-1}T) \in Y^{m,q_i}(\mathbb{R}^d) \cap Y^{m,q_i}(\mathbb{R}^d)$. That is, the inverses are identical whether we consider $L^* : Y^{m,q_i}(\mathbb{R}^d) \to Y^{-m,q_i}(\mathbb{R}^d)$ or $L^* : Y^{m,q_i}(\mathbb{R}^d) \to Y^{-m,q_i}(\mathbb{R}^d)$. Furthermore, $|\alpha| > m - d/q_i$ and so $\partial^\alpha ((L^*)^{-1}T)$ is a well defined locally integrable function that does not depend on $Z_i$, $r_i$, or $q_i$. Thus
\[
\int \int \partial^\alpha \varphi(X) \partial^\beta \eta(Y) (E_{X,j,z_i,r_i,q_i}(Y)) \, dY \, dX
\]
\[= \int \partial^\alpha \varphi(X) \langle T, E_{X,j,z_i,r_i,q_i} \rangle \, dX = \int \partial^\alpha \varphi(X) \langle (L^*)^{-1}T \rangle \, dX
\]
\[= \int \partial^\alpha \varphi(X) \partial^\beta \eta(Y) (E_{X,j,z_i,r_i,q_i}(Y)) \, dY \, dX.
\]
Applying the definition of weak derivative, we see that
\[
\left(\frac{\partial^\alpha X}{\partial^\beta Y} \partial_{X,j,Z_0,\alpha_1,\beta_1}^\gamma \vec{E}_{L,X,j,Z_0,\alpha_1,\beta_1}(Y) \right)_k dY dX
\]
for any smooth functions with disjoint compact support. By the Lebesgue differentiation theorem, formula (104) is valid for almost every \((X,Y) \in \mathbb{R}^d \times \mathbb{R}^d\). □

We now consider the dependency of \(\vec{E}_{L,X,j,Z_0,\alpha_1,\beta_1}^\gamma(Y)\) on \(q\) in more detail. Define
\[
\Xi_q = \{(\alpha, \beta) : \alpha, \beta \text{ are multiindices, } m - d/q' < |\alpha| \leq m, \text{ and } m - d/q < |\beta| \leq m\}.
\]
\(\Xi_q\) is illustrated in Figure 2. By Lemma 103 if \((\alpha, \beta) \in \Xi_q\), then \(\partial^\alpha_X \partial^\beta_Y \vec{E}_{L,X,j,Z_0,\alpha_1,\beta_1}^\gamma(Y)\) is independent of \(Z_0\) and \(r\). Thus, we may largely ignore the dependency on \(Z_0\) and \(r\).

However, the range \(\Xi_q\) of acceptable derivatives does depend on \(q\). We would like to discuss this dependency in more detail.
GRADIENT ESTIMATES AND THE FUNDAMENTAL SOLUTION

Figure 3. The set of points that satisfy Conditions (108–??).

7.4.1. Odd dimensions. In odd dimensions, we will let our fundamental solution be $\vec{E}^L_{X,j}(Y) = \vec{E}^L_{X,j,Z_0,r,2}(Y)$. In light of the Garding inequality $\delta_5$ and the Lax-Milgram lemma, and their consequence Lemma $\delta_6$ $q = 2$ is the most natural value. A straightforward computation yields that if the dimension $d$ is odd, then $\Xi_q = \Xi_2$ whenever $\frac{2m}{d+1} < q < \frac{2m}{d-1}$, that is, for all $q$ sufficiently close to 2.

Note that for general rough coefficients, it may be that $q \in \Upsilon_L$ and so $q$ satisfies the conditions of Definition $\delta_7$ only for $q$ very close to 2 (and in particular may not satisfy these conditions for any $q$ outside of $[\frac{2m}{d+1}, \frac{2m}{d-1}]$); thus, we cannot in general expect to improve upon $\vec{E}^L_{X,j,Z_0,r,2}(Y)$ in terms of the number of derivatives independent of $Z_0$, $r$.

7.4.2. Even dimensions. The situation in even dimensions is more complicated. In this case, if $\frac{2m}{d+2} < q < \frac{2m}{d-2}$ and $q \neq 2$, then $\Xi_q \supseteq \Xi_2$; that is, $\vec{E}^L_{X,j,Z_0,r,q}(Y)$ has strictly more derivatives independent of $Z_0$, $r$ than $\vec{E}^L_{X,j,Z_0,r,2}(Y)$. See Figure 3

However, if $\frac{2m}{d+2} < q < 2 < s < \frac{2m}{d-2}$ and $m \geq d/2$, then $\Xi_q$ and $\Xi_s$ are not equal; indeed we have both of the two noninclusions $\Xi_q \not\subseteq \Xi_s$ and $\Xi_s \not\subseteq \Xi_q$. Thus, neither of the functions $\vec{E}^L_{X,j,Z_0,r,q}$ and $\vec{E}^L_{X,j,Z_0,r,s}$ is entirely satisfactory; we thus wish to define a new fundamental solution $\vec{E}^L_{X,j,Z_0,r}(Y)$ with the correct derivatives for all multiindices in either $\Xi_q$ or $\Xi_q$.

Theorem 106. Let $d \geq 2$ be an even integer and let $m \in \mathbb{N}$. Let $L$ be such that there exists an open neighborhood $\tilde{\Upsilon}_L$ of 2 such that if $q$, $q_1$, $q_2 \in \tilde{\Upsilon}_L$, then $L$ and $q$ satisfy the conditions of Definition $\delta_8$; the bound $\delta_9$ is valid, and formula $\delta_{10}$ is true whenever $(\alpha, \beta) \in \Xi_{q_1} \cap \Xi_{q_2}$.

Then there exists a function $\vec{E}^L_{X,j}(Y)$ such that if $q \in \tilde{\Upsilon}_L \cap (\frac{2m}{d+2}, \frac{2m}{d-2})$ (or $q \in \tilde{\Upsilon}_L \cap (1, \infty)$ if $d = 2$), then

$$\partial_\alpha^\alpha \partial_\beta^\beta \vec{E}^L_{X,j,Z_0,r,q}(Y) = \partial_\alpha^\alpha \partial_\beta^\beta \vec{E}^L_{X,j}(Y) \text{ for all } (\alpha, \beta) \in \Xi_q.$$  

Furthermore, $(\alpha, \beta) \in \Xi_q$ for some such $q$ if and only if

$$m - d/2 \leq |\alpha| \leq m, \quad m - d/2 \leq |\beta| \leq m, \quad 2m - d < |\alpha| + |\beta|.$$
Proof. If \( q, \tilde{q} \in \left( \frac{2d}{d+1}, 2 \right) \), then \( \Xi_q = \Xi_{\tilde{q}} \) and so if \( q, \tilde{q} \in \bar{T}_L \) then \( \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) = \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,\tilde{q}}^L (Y) \) for all \( (\alpha, \beta) \in \Xi_q \). The same is true if \( q, \tilde{q} \in \left( 2, \frac{2d}{d+1} \right) \cap \bar{T}_L \).

Thus, it suffices to find a function \( \tilde{E}_{X,j}^L \) such that the condition (107) is valid for a single \( q \in \left( \frac{2d}{d+1}, 2 \right) \cap \bar{T}_L \) and a single \( \tilde{q} \in \left( 2, \frac{2d}{d+1} \right) \cap \bar{T}_L \).

Fix \( q, s \in \bar{T}_L \) such that \( \frac{2d}{d+1} < q < 2 < s < \frac{2d}{d+1} \). By assumption, some such \( q \) and \( s \) exist. An elementary computation shows that \( (\alpha, \beta) \in \Xi_q \cup \Xi_s \) if and only if Condition (108) is true.

For each \( W \in \mathbb{R}^d \) and each \( \zeta \) with \( |\zeta| = m - d/2 \), define
\[
\tilde{G}_{j,\zeta,Y}(W) = \partial_{X}^\alpha \partial_{Y}^\beta \tilde{E}_{W,j,Z_0,r,q}^L (Y) - \partial_{W}^\alpha \tilde{E}_{W,j,Z_0,r,q}^L (Y).
\]
By Remark 33 formula 35, and the bound (39) applied to \( \tilde{E}_{L}^L \), for each \( Y \in \mathbb{R}^d \), \( \tilde{G}_{j,\zeta,Y} \) is a locally integrable function.

Now, fix some \( W_0 \in \mathbb{R}^d \) and some \( \rho > 0 \), and define
\[
\tilde{E}_{X,j}^L (Y) = \tilde{E}_{X,j,Z_0,r,q}^L (Y) - \sum_{|\zeta| = m - d/2} \frac{1}{\zeta!} \int_{B(W_0,\rho)} \tilde{G}_{j,\zeta,Y}(W) dW.
\]
If \( (\alpha, \beta) \in \Xi_q \), then because \( q < 2 \) we have that \( |\alpha| > m - d/2 \). Thus, \( \partial_\alpha^\alpha X^\zeta = 0 \) and so
\[
\partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j}^L (Y) = \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y).
\]
Now, suppose that \( (\alpha, \beta) \in \Xi_q \). If \( (\alpha, \beta) \in \Xi_s \) then
\[
\partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) = \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) = \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j}^L (Y).
\]
We thus need only consider the case \( (\alpha, \beta) \in \Xi_q \backslash \Xi_s \); this implies that \( |\beta| > m - d/2 \) and \( |\alpha| = m - d/2 \). We then compute that
\[
\partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j}^L (Y) = \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) - \partial_\beta^\beta \int_{B(W_0,\rho)} \tilde{G}_{j,\alpha,Y}(W) dW.
\]
Let \( \bar{e}_k \) be a unit coordinate vector and let \( \xi = \alpha + \bar{e}_k \). Then \( m \geq |\xi| > m - d/2 \) and so \( (\beta, \xi) \in \Xi_q \cup \Xi_s \). Thus
\[
\partial_\xi \partial_{\bar{e}_k} \tilde{G}_{j,\alpha,Y}(W) = \partial_\xi \partial_{\bar{e}_k} \tilde{E}_{W,j,Z_0,r,q}^L (Y) - \partial_\xi \partial_{\bar{e}_k} \tilde{E}_{W,j,Z_0,r,q}^L (Y) = 0
\]
as locally integrable functions; that is, for almost every \( Y \in \mathbb{R}^d \) we have that \( \partial_\xi \tilde{G}_{j,\alpha,Y}(W) \) is a constant and so \( \int_{B(W_0,\rho)} \tilde{G}_{j,\zeta,Y}(W) dW \) does not depend on \( W_0 \) or \( \rho \). By changing \( W_0 \) and \( \rho \) appropriately and using the bound (114), we see that
\[
\int_{Q} \int_{B(W_0,\rho)} |\partial_\xi \partial_{\bar{e}_k} \tilde{G}_{j,\alpha,Y}(W)| dW dY < \infty
\]
for any \( Q \subseteq \mathbb{R}^d \). Thus by Fubini’s theorem
\[
\partial_\xi \partial_{\bar{e}_k} \int_{B(W_0,\rho)} \tilde{G}_{j,\alpha,Y}(W) dW = \int_{B(W_0,\rho)} \partial_\xi \partial_{\bar{e}_k} \tilde{G}_{j,\alpha,Y}(W) dW
\]
for almost every \( Y \in \mathbb{R}^d \), and so
\[
\partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j}^L (Y) = \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) - \int_{B(W_0,\rho)} \partial_\beta^\beta \tilde{G}_{j,\alpha,Y}(X) dW
\]
\[
= \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) - \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y) + \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y)
\]
\[
= \partial_\alpha^\alpha \partial_\beta^\beta \tilde{E}_{X,j,Z_0,r,q}^L (Y)
\]
as desired.

\[ \square \]

7.5. **Derivatives of** \((L^*)^{-1}\). **Recall from formula** \([66]\) **that, if** \(T \in Y^{-m,q'}(\mathbb{R}^d)\), **then**

\[ (((L^*)^{-1}T)_j)_{z_0,r,q'}(X) = \frac{(T, \hat{E}^L_{X,j,z_0,r,q})}{(T, \hat{E}^L_{X,j,z_0,r,q})}. \]

By the Hahn-Banach theorem, if \(T \in Y^{-m,q'}(\mathbb{R}^d)\), **then there exist functions** \(\hat{F}_\xi\) with

\[ \sum_{m-d/q < |\xi| \leq m} \|\hat{F}_\xi\|_{L^{(q')'}}(\mathbb{R}^d) < \infty \]

and where

\[ \langle T, \hat{\varphi} \rangle = \sum_{m-d/q < |\xi| \leq m} \int_{\mathbb{R}^d} \hat{\varphi}(Y) \cdot \hat{F}_\xi(Y) \, dY. \]

Thus,

\[ (((L^*)^{-1}T)_j)_{z_0,r,q'}(X) = \sum_{m-d/q < |\xi| \leq m} \int_{\mathbb{R}^d} \hat{\varphi}(Y) \cdot \hat{F}_\xi(Y) \, dY. \]

We would like a similar integral formula for the derivatives of \((L^*)^{-1}T\).

**Theorem 110.** **Let** \(L\) **and** \(q\) **satisfy the conditions of Definition** \([57]\) **Assume that** the bound \([93]\) **in Theorem** \([71]\) **is valid for** \(p = q\).

Let \(T = T_{\hat{F}, \xi} \in Y^{-m,q'}(\mathbb{R}^d)\) **be a linear operator defined by**

\[ \langle T_{\hat{F}, \xi}, \hat{\varphi} \rangle = \int_{\mathbb{R}^d} \hat{\varphi}(Y) \cdot \hat{F}(Y) \, dY \]

**for some** \(\xi\) **and** \(\hat{F}\) **such that** \(m-d/q < |\xi| \leq m\) **and** \(\hat{F} \in L^{(q')'}(\mathbb{R}^d)\) **is compactly supported.**

If \(|\alpha| > m-d/q', \text{ and if } |\alpha| < m \text{ or } |\xi| < m, \text{ then}\)

\[ \partial^\alpha(((L^*)^{-1}T)_{\hat{F}, \xi})(X) = \int_{\mathbb{R}^d} \hat{\varphi}(Y) \cdot \hat{F}(Y) \, dY \]

and the integral converges absolutely for almost every \(X \in \mathbb{R}^d\). **If** \(|\alpha| = |\xi| = m, \text{ this formula is true for almost every } X \notin \text{ supp } F.\)**

**Proof.** By the bound \([89]\) **and Hölder’s inequality,**

\[ \int_{\mathbb{R}^d} |\hat{\varphi}(Y)| \cdot |\hat{F}(Y)| \, dY < \infty. \]

Let \(Q_0 \subset \mathbb{R}^d\) **be a cube.** **We begin with the case where** \(Q_0\) **and** \(\text{ supp } F\) **are disjoint.**

If \(|\alpha| \leq m, \text{ and if supp } F \text{ is compact, then a covering argument combined with the bound} \([93]\) **yields**

\[ \int_{Q_0} \int_{\mathbb{R}^d} |\partial^\alpha \hat{\varphi}(Y) \cdot \hat{F}(Y)| \, dY \, dX < \infty. \]

**By Fubini’s theorem, if** \(\varphi \in C_0^\infty(Q_0)\) **then**

\[ \int_{Q_0} \partial^\alpha \varphi(X) \int_{\mathbb{R}^d} \hat{\varphi}(Y) \cdot \hat{F}(Y) \, dY \, dX \]

\[ = (-1)^\alpha \int_{Q_0} \varphi(X) \int_{\mathbb{R}^d} \hat{\varphi}(Y) \cdot \hat{F}(Y) \, dY \, dX \]
and so

\[
\partial^\alpha \left( \int_{\mathbb{R}^d} \partial_Y E_{X,j,z_0,r,q}^L(Y) \cdot F(Y) \, dY \right) = \int_{\mathbb{R}^d} \partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y) \cdot F(Y) \, dY
\]

as \(L^1(Q_0)\) functions. Combining this result with formula (109) yields that

\[
(113) \quad \partial^\alpha ((L^*)^{-1}T_{F,k,\xi})_{z_0,r,q}(X) = \int_{\mathbb{R}^d} \partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y) \cdot F(Y) \, dY
\]

for almost every \(X \notin \text{supp} \tilde{F}\). If \(|\alpha| > m - d/q'\) then

\[
\partial^\alpha ((L^*)^{-1}T_{F,k,\xi}) = \partial^\alpha ((L^*)^{-1}T_{F,k,\xi})_{z_0,r,q}
\]

and so formula (111) is valid for almost every \(X \notin \text{supp} \tilde{F}\).

**Remark 114.** If \(|\alpha| < \min(m - d/2, m - d/q')\), then the bound (93) yields the bound (112) even if \(Q_0\) and supp \(F\) are not disjoint, and so in this case formula (113) is valid for almost every \(X \in \mathbb{R}^d\).

We are left with the case where \(X \in \text{supp} F\) and \(|\alpha| + |\xi| < 2m\). We will show that the bound (112) is still valid; the argument given above then yields formula (113) and thus formula (111).

Since \(F\) has compact support, we may assume that \(Q_0\) is large enough that supp \(F \subseteq Q_0\). Let \(G_a\) be a grid of \(2^d\) pairwise-disjoint dyadic open subcubes of \(Q_0\) of measure \(2^{-d}|Q_0|\) whose union (up to a set of measure zero) is \(Q_0\). If \(X \in Q_0\), let \(Q_a(X)\) be the cube that satisfies \(X \in Q_a(X) \in G_a\). If \(Q \in G_{a+1}\), let \(P(Q)\) be the dyadic parent of the cube \(Q\), that is, the unique cube with \(Q \subset P(Q) \in G_a\).

Then by the monotone convergence theorem,

\[
\int_{Q_0} \int_{Q_0} |\partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y)| |\tilde{F}(Y)| \, dX \, dY
\]

\[
= \int_{Q_0} \sum_{a=0}^{\infty} \int_{Q_a(Y) \setminus 4Q_{a+1}(Y)} |\partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y)| |\tilde{F}(Y)| \, dX \, dY
\]

\[
= \sum_{a=0}^{\infty} \sum_{Q \in G_{a+1}} \int_{4P(Q) \setminus 4Q} |\partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y)| |\tilde{F}(Y)| \, dX \, dY.
\]

By the bound (93) and Fubini’s theorem we may interchange the order of integration. Applying Hölder’s inequality first in \(Q\) and then in sequence spaces,

\[
\int_{Q_0} \int_{Q_0} |\partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y)| |\tilde{F}(Y)| \, dX \, dY
\]

\[
= \sum_{a=0}^{\infty} \sum_{Q \in G_{a+1}} \left( \int_{4P(Q) \setminus 4Q} |\partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y)|^{\frac{q_\xi}{q}} dX \right)^{1/(q_\xi)} \left( \int_{Q} |F|^{(q_\xi)'})^{1/(q_\xi)'} \right)
\]

\[
\leq \sum_{a=0}^{\infty} \left( \sum_{Q \in G_{a+1}} \left( \int_{4P(Q) \setminus 4Q} |\partial_Y \partial^\alpha E_{X,j,z_0,r,q}^L(Y)|^{q_\xi} dY \right)^{1/q_\xi} dX \right)^{1/q_\xi}
\]

\[
\times \left( \sum_{Q \in G_{a+1}} \left( \int_{Q} |F|^{(q_\xi)'} \right)^{1/(q_\xi)'} \right).
\]

The final term is \(||F||_{L^{(q_\xi)'}(Q_0)} = ||F||_{L^{(q_\xi)'}(\mathbb{R}^d)} < \infty\), so we need only bound the previous term.
If \( a \geq 0 \) is an integer and \( Q \in G_{a+1} \), then by the bound \([33]\), and applying Lemma \([103]\) to change \( Z_0 \) and \( r \) as desired, we have that
\[
\int_{4P(Q)\setminus 4Q} \left( \int_Q |\partial_X^a \partial_Y^b \mathcal{E}_{X,j,Z_0,r,q}(X)|^{q_\xi} \, dY \right)^{2/q_\xi} \, dX \leq C |Q|^{2m/d - 1 + 2/q - 2|\alpha|/d}.
\]
By Hölder’s inequality,
\[
\int_{4P(Q)\setminus 4Q} \left( \int_Q |\partial_X^a \partial_Y^b \mathcal{E}_{X,j,Z_0,r,q}(Y)|^{q_\xi} \, dY \right)^{1/q_\xi} \, dX
\leq \left( \int_{4P(Q)\setminus 4Q} \left( \int_Q |\partial_X^a \partial_Y^b \mathcal{E}_{X,j,Z_0,r,q}(Y)|^{q_\xi} \, dY \right)^{2/q_\xi} \, dX \right)^{1/2} C |Q|^{1/2}
\leq C |Q|^{m/d + 1 + |\alpha|/d}.
\]
Thus, recalling that there are \( 2^{d(a+1)} \) cubes \( Q \) in \( G_a \) each satisfying \(|Q| = 2^{-(a+1)d} |Q_0|\),
\[
\int_{Q_0} \int_{Q_0} |\partial_X^a \partial_Y^b \mathcal{E}_{X,j,Z_0,r,q}(Y)||\vec{F}(Y)| \, dX \, dY
\leq C \|F\|_{L^{q_\xi'}(\mathbb{R}^d)} \sum_{a=0}^{\infty} \left( \sum_{Q \in G_{a+1}} \left( Q \right)^{m/d + 1 + |\alpha|/d} \right)^{q_\xi} 1/q_\xi
= C \|F\|_{L^{q_\xi'}(\mathbb{R}^d)} |Q_0|^{m/d + 1 + |\alpha|/d} \sum_{a=0}^{\infty} 2^{ad/q_\xi} 2^{-a(m+d/q - |\alpha|)}.
\]
Recall from formula \([21]\) that \( d/q_\xi = d/q - m + |\xi| \). Thus the final sum reduces to
\[
\sum_{a=0}^{\infty} 2^{-a(2m - |\xi| - |\alpha|)}
\]
which converges provided \(|\alpha| < m \) or \(|\xi| < m \). This completes the proof. \( \square \)

### 7.6. The fundamental solution for operators of arbitrary order.
In this section we show how use the fundamental solution for operators of high order to construct the fundamental solution for operators of arbitrary order.

We begin by defining a suitable higher order operator associated to each lower order operator and investigate its properties.

**Lemma 115.** Let \( L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d) \) be a bounded linear operator. Let \( M \) be a nonnegative integer. Define
\[
\bar{L} = \Delta^M L \Delta^M, \quad \bar{m} = m + 2M.
\]
Here \( \Delta^M L \Delta^M \) is the operator given by
\[
\langle (\Delta^M L \Delta^M) \vec{\psi}, \vec{\varphi} \rangle = \langle L(\Delta^M \vec{\psi}), \Delta^M \vec{\varphi} \rangle \text{ for all } \vec{\varphi}, \vec{\psi} \in Y^{\bar{m},2}(\mathbb{R}^d).
\]
Then:
- (a) If \( 1 < q < \infty \) and \( L \) is bounded or invertible \( Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d) \), then \( \bar{L} \) is bounded or invertible \( Y^{\bar{m},q}(\mathbb{R}^d) \to Y^{\bar{m},q}(\mathbb{R}^d) \). If \( L \) is invertible and in addition \( M \) is large enough (depending on \( d, m, \) and \( q \)), then \( \bar{L} \) and \( q \) satisfy the conditions of Definition \([32]\).
- (b) If \( L \) is bounded and invertible \( Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d) \), then \( \Upsilon_{\bar{L}} = \Upsilon_L \), where \( \Upsilon_L \) is as in Definition \([33]\).
(c) If $L$ is of the form (24), then so is $\tilde{L}$, and

$$\tilde{m} - a_\tilde{L} = m - a_L, \quad \tilde{m} - b_\tilde{L} = m - b_L,$$

where $a_L$ and $b_L$ are as in formulas (20) and (27) and $\Pi_L$ is as in Definition (83).

(d) If $T \in Y^{-m,q}(\mathbb{R}^d)$, define $\tilde{T} \in Y^{-\tilde{m},q}(\mathbb{R}^d)$ by

$$⟨\tilde{T}, \tilde{ψ}⟩ = (T, M_\tilde{Ψ})$$

for all $\tilde{ψ} \in Y^{\tilde{m},q}(\mathbb{R}^d)$.

We need only consider the case $T > 0$. The polylaplacian is obviously bounded $Δ^M : Y^{\tilde{m},p}(\mathbb{R}^d) \to Y^{\tilde{m},p}(\mathbb{R}^d)$ for any $1 < p < \infty$ (in particular, for both $p = q$ and $p = q'$), and so if $L$ is bounded $Y^{\tilde{m},q}(\mathbb{R}^d) \to Y^{-\tilde{m},q}(\mathbb{R}^d)$ then $\tilde{L}$ is bounded $Y^{\tilde{m},q}(\mathbb{R}^d) \to Y^{-\tilde{m},q}(\mathbb{R}^d)$.

It is well known (see, for example, [Tri83, Section 5.2.3]) that the Laplacian is a bounded and invertible operator $W^{s,p}(\mathbb{R}^d) \to W^{s-2,p}(\mathbb{R}^d)$ for any $1 < p < \infty$ and any $-\infty < s < \infty$. Recall that there is a natural isomorphism between $Y^{m,p}(\mathbb{R}^d)$ and $W^{m,p}(\mathbb{R}^d)$.

Thus $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ is bounded if and only if $\tilde{L} : Y^{\tilde{m},q}(\mathbb{R}^d) \to Y^{-\tilde{m},q}(\mathbb{R}^d)$ is bounded, and $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ is invertible if and only if $\tilde{L} : Y^{\tilde{m},q}(\mathbb{R}^d) \to Y^{-\tilde{m},q}(\mathbb{R}^d)$ is invertible.

If in addition $M > (d/2) \max(1/q, 1/q') - m/2$, then $1 - \tilde{m}/d < 1/q < \tilde{m}/d$ and so $\tilde{L}$ and $q$ satisfy the conditions of Definition (81).

Furthermore, $Δ^{-1}$ is compatible, and so $(\tilde{L})^{-1} = Δ^{-M} L^{-1} Δ^{-M}$ is compatible if and only if $L^{-1}$ is compatible. Thus, $\Upsilon_L = \Upsilon_{\tilde{L}}$.

There are real nonnegative constants $κ_\zeta$ such that $Δ^M = \sum_{|ζ|=M} κ_ζ \partial^2 ζ$. If $\tilde{u}$ and $\tilde{v}$ lie in suitable function spaces, and $L$ is of the form (24), we have that

$$⟨\tilde{L}\tilde{u}, \tilde{v}⟩ = (L Δ^M \tilde{u}, Δ^M \tilde{v}) = \int \sum_{N} \sum_{j,k=1} \sum_{|α| \leq m} \sum_{|β| \leq m} \sum_{|ξ|=M} \sum_{|ζ|=M} \partial^{α+2ζ} \varphi_j κ_ζ κ_ζ A_{α,β}^{j,k} ξ^β 2ζ u_k.$$

We may rearrange our order of summation to see that $\tilde{L}$ is an operator of the form (24) of order $2\tilde{m}$ with coefficients

$$A_{α,β}^{j,k} = \sum_{|ξ|=M} \sum_{|ζ|=M} κ_ζ κ_ζ A^{j,k}_{(ξ-2ζ),(ξ-2ζ)}.$$

Furthermore,

$$\sum_{j,k=1} \sum_{|α| \leq \tilde{m}} \sum_{|ξ| \leq \tilde{m}} \partial^α \varphi_j A_{α,β}^{j,k} ξ^β u_k = \sum_{j,k=1} \sum_{|α| \leq m} \sum_{|β| \leq m} \partial^α Δ^M \varphi_j A_{α,β}^{j,k} ξ^β Δ^M u_k.$$

If $\varphi \in Y^{m,q}(\mathbb{R}^d)$ and $\tilde{u} \in Y^{\tilde{m},q}(\mathbb{R}^d)$, then $Δ^M \varphi \in Y^{m,q}(\mathbb{R}^d)$ and $Δ^M \tilde{u} \in Y^{m,q}(\mathbb{R}^d)$. Thus, if $q \in \Pi_L$ then the right hand side represents a $L^1(\mathbb{R}^d)$ function, and so $q \in \Pi_{\tilde{L}}$. 

Proof. We need only consider the case $M > 0$. The polylaplacian is obviously bounded $Δ^M : Y^{\tilde{m},p}(\mathbb{R}^d) \to Y^{\tilde{m},p}(\mathbb{R}^d)$ for any $1 < p < \infty$ (in particular, for both $p = q$ and $p = q'$), and so if $L$ is bounded $Y^{\tilde{m},q}(\mathbb{R}^d) \to Y^{-\tilde{m},q}(\mathbb{R}^d)$ then $\tilde{L}$ is bounded $Y^{\tilde{m},q}(\mathbb{R}^d) \to Y^{-\tilde{m},q}(\mathbb{R}^d)$.
Finally, recall that $\Delta^M$ is invertible $Y^{\bar{m},q}(\mathbb{R}^d) \to Y^{m,q}(\mathbb{R}^d)$. Thus if $\Phi \in Y^{m,q}(\mathbb{R}^d)$, and $L : Y^{m,q}(\mathbb{R}^d) \to Y^{m,q}(\mathbb{R}^d)$ and $\bar{L} : Y^{\bar{m},q}(\mathbb{R}^d) \to Y^{\bar{m},q}(\mathbb{R}^d)$ are bounded and invertible, then

$$
\langle L(\Delta^M(\bar{L})^{-1}\bar{T}), \Phi \rangle = \langle L(\Delta^M(\bar{L})^{-1}\bar{T}), \Delta^M \Delta^{-M} \Phi \rangle = \langle \bar{T}(\bar{L})^{-1}\bar{T}, \Delta^{-M} \Phi \rangle = \langle T, \Delta^M \Delta^{-M} \Phi \rangle = \langle T, \Phi \rangle
$$

and so $\Delta^M(\bar{L})^{-1}\bar{T} = (L)^{-1}T$. This completes the proof. \hfill \Box

Thus, natural conditions on $L$ guarantee that $\bar{L}$ has a fundamental solution.

We now use $\bar{E}_L^\ell$ to construct $\tilde{E}_L^\ell$ for operators of arbitrary order. Theorem 119 (with $E^\ell_{Y,k}(Y, X) = (\tilde{E}^\ell_{Y,k})(Y)$ and $L$ and $L^\star$ interchanged as needed) comprises most of Theorem 13; the remaining property cited in Theorem 13 (the uniqueness of the fundamental solution) will be addressed in Section 7.7.

**Theorem 119.** Let $L$ be an operator of order $2m$ of the form (24) that satisfies the ellipticity condition (5) such that $2 \in \Pi_L$, where $\Pi_L$ is the interval of Definition 28. Let $M$ be the smallest nonnegative integer with $m + 2M > d/2$. Let $\bar{L}$ be given by formula (116).

Suppose in addition that the Caccioppoli-Meyers inequality for $\bar{L}$ holds, that is, that there is an interval $S_L$ with $2 \in S_L \subseteq [2, 4] \cap \Pi_L$ such that if $p \in S_L$, if $Q \subset \mathbb{R}^d$ is a cube with sides parallel to the coordinate axes, and if $\bar{u}$ is a representative of an element of $Y^{\bar{m},d}(2Q)$, then we have the estimate

$$
\sum_{j=0}^m |Q|^{j/d} \| \nabla^j \bar{u} \|_{L^p(Q)} \leq C |Q|^{1/p-1/2} \| \bar{u} \|_{L^2(2Q)} + C |Q|^{m/d} \| \bar{L} \bar{u} \|_{Y^{-\bar{m},d}(2Q)}.
$$

If $L$ satisfies either the bound (7) or the bound (8), then this condition is true with $S_L = \Pi_L \cap \mathbb{Y}_L \ni [2, 4] \supseteq \{2\}$, with $\mathbb{Y}_L$ given by formula (33).

Then there exists some array of functions $\tilde{E}^\ell_{X,j}(Y)$ with the following properties.

Suppose that $\alpha$ and $\beta$ are two multiindices with $m - d/2 \leq |\alpha| \leq m$, $m - d/2 \leq |\beta| \leq m$, and $(|\alpha|, |\beta|) \neq (m - d/2, m - d/2)$. If $\Pi_L$ does not contain a neighborhood of 2, then we impose the stronger condition $m - d/2 < |\alpha| \leq m$, $m - d/2 < |\beta| \leq m$.

Suppose further that $Q$ and $\Gamma$ are two cubes in $\mathbb{R}^d$ with $|Q| = |\Gamma|$ and $\Gamma \subset 8Q \setminus 4Q$. Then the partial derivative $\partial_X^\alpha \partial_Y^\beta \tilde{E}^\ell_{X,j}(Y)$ exists as a locally $L^2(Q \times \Gamma)$ function and satisfies the bounds

$$
\int_Q \int_\Gamma |\partial_X^\alpha \partial_Y^\beta \tilde{E}^\ell_{X,j}(Y)|^2 \leq C |Q|^{(4m-2|\alpha| - 2|\beta|)/d},
$$

$$
\int_\Gamma \int_Q \left( |\partial_X^\alpha \partial_Y^\beta \tilde{E}^\ell_{X,j}(Y)|^{p_\beta} dY \right)^{2/p_\beta} dX \leq C |Q|^{2m/d - 1/2 + 2/p - 2|\alpha|}
$$

for all $p \in \mathbb{Y}_L \cap S_L$ with $m - d/p < |\alpha|$, $m - d/p < |\beta|$.

Furthermore, we have the symmetry property

$$
\partial_X^\alpha \partial_Y^\beta \tilde{E}^\ell_{X,j}(Y))_k = \partial_X^\alpha \partial_Y^\beta (\tilde{E}^\ell_{Y,k}(X))_j.
$$
for almost every $X, Y \in \mathbb{R}^d \times \mathbb{R}^d$.

Finally, let $\mathcal{Y}_L$ be as in Definition 116. Suppose that $q \in \mathcal{Y}_L \cap ((-\infty, 2) \cup S_L^d)$ and $m - d/q < |\zeta| \leq m$. Let $T = T_{\mathcal{F}, \xi} \in Y^{-m, d}(\mathbb{R}^d)$ be a linear operator defined by

$$
\langle T_{\mathcal{F}, \xi}, \tilde{\varphi} \rangle = \int_{\mathbb{R}^d} \partial^\xi \tilde{\varphi}(Y) \cdot \mathcal{F}(Y) \, dY
$$

for some compactly supported $\mathcal{F} \in L^{(q_1)}(\mathbb{R}^d)$. Whenever $|\zeta| > m - d/q'$, we have that

(124) $$
\partial^\xi ((L^*)^{-1}T_{\mathcal{F}, \xi})_j(X) = \int_{\mathbb{R}^d} \partial^\zeta \partial^\xi E_{\mathcal{F}, j}(Y) \cdot \mathcal{F}(Y) \, dY
$$

and the integral converges absolutely for almost every $X \notin \text{supp} \mathcal{F}$. If in addition $|\zeta| < m$ or $|\xi| < m$ then formula (124) is valid for almost every $X \in \mathbb{R}^d$.

Proof. If $L$ satisfies either the bound (7) or the bound (8), then by Lemma 115 and formula (118), so does $\tilde{L}$. By Lemma 115 $\mathcal{Y}_L = \mathcal{Y}_L$. By Lemma 61, $\mathcal{Y}_L$ contains a neighborhood of 2 and so $\mathcal{Y}_L \cap [2, \infty)$ contains values greater than 2. The inequality (120) is valid for all $p \in \Pi_L \cap \mathcal{Y}_L \cap [2, \infty)$ by Theorem 62.

By assumption and Lemma 58, $2 \in \mathcal{Y} \cap \Pi_L$ and $1 - \tilde{m}/d < 1/2 < \tilde{m}/d$. Also observe that $L$ satisfies the conditions of Definition 51 and Theorem 91 for all $q \in \mathcal{Y}_L \cap \Pi_L$ with $1 - \tilde{m}/d < 1/2 < \tilde{m}/d$ and all $p \in S_L \cap \mathcal{Y}_L \cap \Pi_L \cap [1, 2q]$ with $1 - \tilde{m}/d < 1/p < \tilde{m}/d$ (in particular, for $q = \tilde{p} = 2$). Consequently $\tilde{L}$ satisfies the conditions of Lemma 103 for all $q_1, q_2 \in \mathcal{Y} \cap \Pi_L$ with $1/q_1, 1/q_2 \in (1 - \tilde{m}/d, \tilde{m}/d)$.

If $\Pi_L$ contains an open neighborhood of 2, then by Lemma 61 $\mathcal{Y}_L$ also contains an open neighborhood of 2. Thus the conditions of Theorem 106 are valid whenever $d$ is even.

If $d$ is odd, or if $\Pi_L$ does not contain a neighborhood of 2, let $\tilde{E}_{X,j}^L = \tilde{E}_{X,j}^L(0, 1, 2)$ be as in Definition 51.

If $d$ is even, and if $\Pi_L$ contains a neighborhood of 2, we let $\tilde{E}_{X,j}^L$ be as in Theorem 110.

In either case, by Theorem 91, $\partial^\xi \partial^\gamma \tilde{E}_{X,j}^L(Y)$ exists for almost every $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ and every $\xi, \gamma$ with $|\xi|, |\gamma| \in [0, \tilde{m}]$. We define

$$
\tilde{E}_{X,j}^L(Y) = \sum_{|\nu| = M} \sum_{|\zeta| = 0} \kappa_{\nu} \partial^\nu \tilde{E}_{X,j}^L(Y).
$$

The bounds (121) and (122) follow from Theorem 91, Lemma 103 and Lemma 115. The symmetry property (123) follows from the symmetry property (55) for $\tilde{E}_{X,j}^L$.

We are left with formula (124). This property follows from Theorem 110 if $2m > d$ and so $M = 0$. If $2m \leq d$, let $m - d/q < |\zeta| \leq m$ and $\tilde{F}$ satisfy the conditions given in the theorem statement. Let $T = T_{\mathcal{F}, \xi}$ and let $\tilde{T}$ be as in formula (117). Observe that

$$
\langle \tilde{T}, \tilde{\varphi} \rangle = \langle T, \Delta M \psi \rangle = \sum_{|\nu| = M} \kappa_{\nu} \int_{\mathbb{R}^d} \partial^\xi + 2\nu \tilde{\psi}(Y) \cdot \tilde{F}(Y) \, dY
$$

and so $\tilde{T}$ is a (linear combination of) operators as in Theorem 110. By formula (111) and linearity, we have that if $|\zeta| > \tilde{m} - d/q$, then

$$
\partial^\xi ((L^*)^{-1} \tilde{T})_j(X) = \sum_{|\nu| = M} \kappa_{\nu} \int_{\mathbb{R}^d} \partial^\xi \partial^\nu \tilde{E}_{X,j}^L(Y) \cdot \tilde{F}(Y) \, dY
$$
for almost every $X$ or almost every $X \not\in \text{supp} \, \bar{F}$. In particular, if $m-d/q' < |\zeta| \leq m$ and $|\varpi| = M$, then $\bar{m} - d/q < |\zeta + 2\varpi| \leq \bar{m}$, and so

$$
\partial^c (\Delta^M (\bar{L}^*)^{-1} \bar{T})_j (X) = \sum_{|\varpi| = M} \sum_{|\nu| = M} \kappa_{\varpi} \kappa_{\nu} \int_{\mathbb{R}^d} \partial^{2\varpi + \zeta} \partial^{2\nu + 2\zeta} \bar{E}^j \cdot \bar{F} (Y) \, dY
$$

$$
= \int_{\mathbb{R}^d} \partial^c \partial^c \bar{E}^j \cdot \bar{F} (Y) \, dY
$$

Observe that $\bar{(\bar{L}^*)} = (\bar{L})^*$. By formula (127) with $L$ replaced by $L^*$, formula (124) is valid.

**Remark 125.** Theorem 124 involves conditions on $\bar{L} = \Delta^M L \Delta^M$ for the smallest $M$ such that $\bar{E}^j_{X,k,L} \cdot \bar{E}^j \cdot \bar{F}$ exists. The fundamental solution also exists for larger values of $M$. However, there is no loss of generality in Theorem 124 in taking the smallest available $M$; that is, we claim that if the Caccioppoli-Meyers inequality (120) is valid for $\bar{L} = \Delta^M L \Delta^M$, and if $L : Y^{m,p} (\Omega) \to Y^{-m,p} (\Omega)$ is bounded for all open sets $\Omega$, then it is valid for $\bar{L} = \Delta^N L \Delta^N$ for any integer $N$ with $0 \leq N \leq M$.

We now prove the claim. Suppose that $p \geq 2$ and the Caccioppoli or Meyers inequality

$$
\sum_{j=0}^{m+2M} |Q|^{j/d} \left( \int_Q |\nabla^j \bar{u}|^p \right)^{1/p} \leq C|Q|^{1/p - 1/2} \left( \int_{2Q} |\bar{u}|^2 \right)^{1/2} + C|Q|^{(m+2M)/d} \|\Delta^M L \Delta^M \bar{u}\|_{Y^{-m-2M,p}(2Q)}
$$

is valid for all $\bar{u} \in Y^{m+2M,p}(2Q)$ for some cube $Q$. Let $0 \leq N < M$.

It is well known (see [Ste70], Chapter VI, Section 3) that there is a bounded, linear extension operator $E$ such that for all $k \in \mathbb{N}_0$ and all $1 \leq p < \infty$ we have that $\|E\bar{u}\|_{W^{k,p}(\mathbb{R}^d)} \leq C_{k,p} \|\bar{u}\|_{W^{k,p}(2Q)}$. Recall that $\Delta^{M-N}$ is an isomorphism from $W^{k+2M-2N,p}(\mathbb{R}^d)$ to $W^{k,p}(\mathbb{R}^d)$.

Choose some $\bar{u} \in W^{m+2M,p}(2Q)$. Let $\bar{v} = \Delta^{-(M-N)} (E\bar{u})$. Then $\bar{v} \in W^{m+2M,p}(\mathbb{R}^d)$ and also satisfies

$$
\|\nabla^{2M-2N} \bar{v}\|_{L^2(2Q)} \leq \|\nabla^{2M-2N} \bar{v}\|_{L^2(\mathbb{R}^d)} \leq C\|E\bar{u}\|_{L^2(\mathbb{R}^d)} \leq C^2 \|\bar{u}\|_{L^2(2Q)}.
$$

Let $\bar{w} = \bar{v} + \bar{P}$, where $\bar{P}$ is a polynomial of degree at most $2M - 2N - 1$ such that $\int_{2Q} \partial^\gamma \bar{w} = 0$ for all $|\gamma| \leq 2M - 2N - 1$. We have that $\Delta^{M-N} \bar{w} = \Delta^{M-N} \bar{v} = E\bar{u} = u$ in $2Q$. We compute

$$
\sum_{j=0}^{m+2N} |Q|^{j/d} \left( \int_Q |\nabla^j \bar{u}|^p \right)^{1/p} \leq \sum_{j=0}^{m+2N} |Q|^{j/d} \left( \int_Q |\nabla^j \Delta^{M-N} \bar{u}|^p \right)^{1/p}
$$

$$
\leq \sum_{k=2M-2N}^{m+2M} |Q|^{(k-2M+2N)/d} \left( \int_Q |\nabla^k \bar{u}|^p \right)^{1/p}.
$$
By the Meyers inequality for $\Delta^M L \Delta^M$,
\[
\sum_{j=0}^{m+2N} |Q|^{1/d} \left( \int_{Q} |\nabla^j \vec{u}|^p \right)^{1/p} \leq C |Q|^{1/p-1/2-(2M-2N)/d} \left( \int_{2Q} |\vec{w}|^2 \right)^{1/2} + C |Q|^{(m+2N)/d} \|\Delta^M L \Delta^M \vec{u}\|_{L^p(2Q)}.
\]

By the Poincaré inequality and because $\Delta^M \vec{w} = \Delta^N \vec{u}$,
\[
\sum_{j=0}^{m+2N} |Q|^{1/d} \left( \int_{Q} |\nabla^j \vec{w}|^p \right)^{1/p} \leq C |Q|^{1/p-1/2} \left( \int_{2Q} |\nabla^{2M-2N} \vec{w}|^2 \right)^{1/2} + C |Q|^{(m+2N)/d} \|\Delta^N L \Delta^N \vec{u}\|_{L^p(2Q)}.
\]

Finally, using the estimate $\|\nabla^{2M-2N} \vec{w}\|_{L^2(2Q)} = \|\nabla^{2M-2N} \vec{w}\|_{L^2(2Q)} \leq C \|\vec{w}\|_{L^2(2Q)}$, we see that the Caccioppoli-Meyers estimate for $\Delta^N L \Delta^N$ is also valid.

7.7. Uniqueness. We have constructed a fundamental solution; we now show that it is unique.

**Theorem 126.** Let $L : Y^{m,q}(\mathbb{R}^d) \to Y^{-m,q}(\mathbb{R}^d)$ be bounded and invertible. Suppose that $\hat{\Psi}_{X,j}$ and $\hat{\Gamma}_{X,j}$ are such that the bound (121) and formula (124) are valid with $E^L$ replaced by either $\hat{\Psi}$ or $\hat{\Gamma}$.

Then $\partial_X^\alpha \partial_Y^\beta \hat{\Psi}_{X,j}(Y) = \partial_X^\alpha \partial_Y^\beta \hat{\Gamma}_{X,j}(Y)$ for almost every $(X,Y) \in \mathbb{R}^d \times \mathbb{R}^d$ and all $\alpha, \beta$ as in Theorem 119.

**Proof.** By the bound (124), we have that $\partial_X^\alpha \partial_Y^\beta \hat{\Psi}_{X,j}$ and $\partial_X^\alpha \partial_Y^\beta \hat{\Gamma}_{X,j}$ are locally integrable away from $Y = X$ for almost every $X \in \mathbb{R}^d$. By formula (124),
\[
\int_{\mathbb{R}^d} \partial_X^\alpha \partial_Y^\beta \hat{\Psi}_{X,j}(Y) \cdot \hat{F}(Y) dY = \int_{\mathbb{R}^d} \partial_X^\alpha \partial_Y^\beta \hat{\Gamma}_{X,j}(Y) \cdot \hat{F}(Y) dY
\]
for all sufficiently nice test functions $\hat{F}$. The result follows from the Lebesgue differentiation theorem. \hfill \Box

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