Classicalization via Path Integral

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Abstract

Recently, it was suggested that a large class of non-renormalizable theories may need no UV completion. By analogy with gravity where classical black holes are expected to be created in high-energy scatterings, it is conjectured that similar classical solutions, so-called classicalons, should occur. In this way the theory protects itself against non-unitarity, for instead of probing small distances at high energies one enters a classical regime. An effective theory of Goldstone bosons provides and example in which the size of classicalons grows with energy, and the high energy scattering is cut-off by small momenta, inversely proportional to the classicalon size. In this note we offer an alternative, path integral discussion of this important result.

1 Introduction

With the advent of QED and then the Standard Model, renormalizability became the criterion for physically acceptable theories. Encouraged by the great successes of the work of Glashow, Weinberg and Salam, who followed the road of the UV completion (nowadays known as a Wilsonian approach) of the effective Fermi theory of weak interactions, this emerged as a canonical way of making sense out of non-renormalizable effective theories.

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The only trouble is that gravity still resists. The name of the game for now decades was and still is the search for its proper UV completion, the most popular one being strings. Recently a striking proposal [1] was put forward, according to which no UV completion whatsoever may be needed in the case of gravity. The essential point is both simple and deep. Normally one imagines that by going to very high energies $E$ one will probe small distances of the order $1/E$, and this is the crux of the problem in the case of non-renormalizable theories. The authors of [1] argue that by going to high energies, one will, as is well known, produce large classical black holes, with size

$$r_* = 2G_N E$$

and therefore one simply never probes small distances. For $E >> M_{Pl}$, $r_* >> 1/M_{Pl}$. In other words, it is perfectly conceivable that no UV completion may be needed for $E >> M_{Pl}$, but that the theory simply enters the classical regime. The issue of how to compute the physical processes for $E \approx M_{Pl}$ remains an important unanswered question. Still it is remarkable if it were to turn true that gravity could have an in-built mechanism for self-protection against the problem of small distances.

Encouraged by this appealing picture Dvali and collaborators [2, 3, 4] make the even more striking proposal that this feature may be true for a large class of non-renormalizable field theories. Instead of the normally assumed UV completions such theories could in principle have the same protective mechanism discussed above. According to the terminology of [2], if this happens, the theory is said to classicalize, and the process itself is called classicalization. In analogy with black holes, they suggest the existence of similar classical solutions called classicalons, whose main characteristic would be, just as in the case of black holes, that their size grows with the energy. Again, by going to high energies, instead of probing small distances, one would end up going to the classical regime. This remarkable idea has been exemplified on two cases: one, the effective theory of Goldstone bosons, and the other, the Galileon [5], which is a scalar degree of freedom of the massive graviton, as in say [6]. In both these cases one has the explicit form for the classicalons.

In order to understand quantitatively what is going on, an elegant and physical picture was used in [3]. By starting with a free spherical wave at infinity, the authors compute the distance $r_*$ at which the wave gets appre-
ciably perturbed by the non-renormalizable interaction in question. In the case of the Goldstone Lagrangian one estimates

\[ r_* = L_* (L_* E)^{1/3} \]

where \( L_* \equiv 1/M_* \) is the small distance cutoff (corresponding to the large energy cutoff \( M_* \)) of the theory. Correspondingly, [3] shows that the momenta relevant for the scattering are simply cut-off by \( 1/r_* \), instead being of order \( E \) as one would naively expect.

Now, a less imaginative and more scholastic pursuer of this idea would naturally try the usual procedure of path integral quantization around the classicalon, and check the behaviour of the propagator in order to see what goes on. In this note we do precisely that for an illustrative example of Goldstone bosons and confirm the results of [3]. We hope that simple minded readers (such as us) will find this of some use, and that this appealing program will get some modest boost. Besides this, we have nothing further to add to elucidate this picture beyond what was done up to now. For a remarkably clear discussion of classicalization we refer the reader to [4].

2 The solution

We will study the \( d \)-dimensional Euclidean scalar field theory with derivative couplings

\[ \mathcal{L} = \frac{1}{2} (\partial \Phi)^2 + \frac{L^d_*}{4} (\partial \Phi)^4 \]

This model has solutions of the classical equation of motion with a source \( Q \)

\[ \partial_\mu \left( \partial_\mu \Phi_{cl} \left( 1 + L^d_* (\partial \Phi_{cl})^2 \right) \right) = Q \delta^d(x - y) \]

The ansatz

\[ \partial_\mu^\rho \Phi_{cl}(x) = \frac{(x - y)_\mu}{|x - y|} L^d_* f \left( \frac{|x - y|}{r_0(Q)} \right) \]

with the definition (\( \Omega_d \) is a solid angle in \( d \)-dimensions)

\[ r_0(Q) = \left( \frac{QL^d_*}{\Omega_d} \right)^{\frac{1}{d+1}} \]
reduces the equation of motion to

\[ f(\rho) + f(\rho)^3 = 1/\rho^{d-1} \]  

(7)

This is solved simply by (one should mention that this is the real solution)

\[ f(\rho) = f_+(\rho) - f_-(\rho) \]  

(8)

with

\[ f_\pm(\rho) = \left( \sqrt[3]{\frac{1}{3}} + \left( \frac{1}{2\rho^{d-1}} \right)^2 \pm \frac{1}{2\rho^{d-1}} \right)^{1/3} \]  

(9)

Notice that among all these solutions there is also the trivial \((Q = 0)\) one.

The physical interpretation of \(r_0\) is straightforward: it separates the long distance regime when our instanton-like solution dies off and a short distance behaviour when it grows with \(r\). In this sense, it is analogous to \(r_\ast\) of the static classicalon, which as shown in [3] is a measure of the cross section \((r_\ast\) is a distance when a free wave at infinity gets perturbed). The change of notation is for the sake of emphasizing a different, instanton-like nature of our solution, needed in order to derive a propagator and an effective interaction. In what follows we restrict ourselves to the former; the latter, more involved discussion, is left for the future.

## 3 The path integral

The generating functional is

\[ Z[J] = \int \mathcal{D}\Phi \exp \left( -\int d^d x (\mathcal{L} - J\Phi) \right) \]  

(10)

We expand around classical solutions of the equation of motion

\[ \Phi = \Phi_{cl} + \Phi_q \]  

(11)

After repeated use of partial integration and equation of motion, one gets

\[ \int d^d x \mathcal{L}(\Phi) \to \int \left( \mathcal{L}(\Phi_{cl}) - \Phi_q Q\delta^d(x-y) + \frac{1}{2} \Phi_q \hat{O}_{\ast} \Phi_q \right) \]  

(12)
with the quadratic operator

\[ \hat{O}_{r_0,y} = -\partial_\mu \left( \partial_\mu + L_s^d \left( 2\partial_\mu \Phi_{cl} \partial_\nu \Phi_{cl} + \delta_{\mu\nu}(\partial \Phi_{cl})^2 \right) \partial_\nu \right) \]  \hspace{1cm} (13)

where \( r_0 \) is the size of the classicalon, and \( y \) denotes its position. Finally we redefine

\[ J(x)\Phi(x) \rightarrow (J(x) - Q \delta^d(x - y))\Phi_q(x) \]  \hspace{1cm} (14)

so that the path integral becomes as usual

\[
Z[J] \propto \int dr_0 dy f(r_0,y) e^{-S[\Phi_{cl}(r_0,y)]} \left( \det \hat{O}_{r_0,y} \right)^{-1/2} \\
\times \exp \left( -S_{\text{int}}[\delta/\delta J] \right) \exp \left( \frac{1}{2} \int d^d x \int d^d z J(x) \Delta_{r_0,y}(x,z) J(z) \right)
\]  \hspace{1cm} (15)

where \( r_0 \) and \( y \) are the collective coordinates, \( f(r_0,y) \) is a corresponding measure, and \( \Delta_{r_0,y}(x) \) is the propagator, i.e. the inverse of the operator \( \hat{O}_{r_0,y} \).

Now it is easy to see how the amplitude looks like. For example, considering for illustration only the part of the 4-point Green’s function coming from the quartic interaction, we get

\[ \Delta_{r_0,y}(x) \]  \hspace{1cm} (17)

In order to ease the reader’s pain and without the loss of generality we choose hereafter \( y = 0 \).

Then the operator (13) can be rewritten as

\[ \hat{O}_r = -\frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( \left[ 1 + 3f^2(r/r_0) \right] r^{d-1} \frac{\partial}{\partial r} \right) + \left[ 1 + f^2(r/r_0) \right] \frac{L^2}{r^2} \]  \hspace{1cm} (18)
with

\[ L^2 = -\frac{1}{2} (x_\mu \partial_\nu - x_\nu \partial_\mu)^2 \]  

(19)

For the \( d \)-dimensional radial coordinate small the operator becomes

\[ r \to 0 : \quad \hat{O}_{r_0}(r) \to -\left(\frac{r_0}{r}\right)^{2(d-1)/3} \left(3 \frac{\partial^2}{\partial r^2} + \frac{d - 1}{r} \frac{\partial}{\partial r} - \frac{L^2}{r^2}\right) \]  

(20)

while for large distances we have the usual Laplace

\[ r \to \infty : \quad \hat{O}_{r_0}(r) \to -\left(\frac{\partial^2}{\partial r^2} + \frac{d - 1}{r} \frac{\partial}{\partial r} - \frac{L^2}{r^2}\right) \]  

(21)

In a very crude approximation one could say, that the operator becomes infinite for \( r \ll r_0 \) and Laplacian for large \( r \). Physically this implies that the momentum space propagator is given roughly by the Heaviside function

\[ \tilde{\Delta}_{r_0}(p) \approx \frac{1}{p^2} \Theta(1/r_0 - p) \]  

(22)

so that the maximum available momentum is \( p_* = 1/r_0 \). From (16), the amplitude thus becomes

\[ A_4 \to L^4_* p_*^4 \]  

(23)

as predicted by classicalization. The relevant momenta that characterize high-energy scatterings become small, corresponding to the inverse size of the classicalon.

4 Summary and outlook

A great success of the UV completion of the Fermi theory in the form of the electro-weak Standard Model, has almost created a dogma that this is the only road towards making sense out of non-renormalizable theories with dimensionful couplings. The search for the UV completion degrees of freedom in case of gravity has led eventually to strings, whose impact on high energy physics has been enormous. And yet, as argued in [1], it could be that gravity may need no such completion, for it has a built-in protective mechanism in
the form of classical black holes. Scattering two particles with energies of say a mammoth, will simply produce black holes with a radius proportional the energy of the mammoth, and one will never arrive at short distances. In other words, it is not clear a priori that gravity must be modified.

Could this be true of other, non gravitational, effective non-renormalizable theories? If such theories are energy self-sourced, Dvali et al [2, 3, 4] argue that the answer is affirmative. In [3] this is shown by studying the distance at which scattering takes place. In this note, we offer a canonical path integral discussion of this important result, with a hope that this may help one to further grasp this fascinating subject. This is our main apology for deciding to make our work public without waiting for a more complete analysis of the issues addressed.

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