QUANTUM QUEER SUPERALGEBRAS

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Abstract. We give a brief survey of recent developments in the highest weight representation theory and the crystal basis theory of the quantum queer superalgebra $U_q(q(n))$.

Introduction

In this expository article, we give an elementary account of recent developments in the highest weight representation theory and the crystal basis theory of quantum queer superalgebra $U_q(q(n))$. The queer Lie superalgebra $q(n)$ has attracted a great deal of research activities due to its resemblance to the general linear Lie algebra $gl(n)$ on the one hand and its unique features in its structure and representation theory on the other hand. The Lie superalgebra $q(n)$ is similar to $gl(n)$ in that the tensor powers of natural representations are all completely reducible. Moreover, there is a queer analogue of the celebrated Schur-Weyl duality, often referred to as the Schur-Weyl-Sergeev duality, that was discovered in [19, 25]. However, this is about the end of their resemblance and there is a vast list of differences and discrepancies between these two algebraic structures. One of the major difficulties lies in that the Cartan subalgebra of $q(n)$ is not abelian and has a nontrivial odd part. For this reason, it is a very complicated and challenging task to investigate the structure and representation theory of queer Lie superalgebra $q(n)$ (see, for example, [3, 5, 16, 20, 21, 24, 25]). Thus a queer version of the crystal basis theory would be very helpful in understanding the combinatorial representation theory of $q(n)$.

A quantum deformation $U_q(q(n))$ of the universal enveloping algebra $U(q(n))$ was constructed by Olshanski [19] using a modification of the Reshetikhin-Takhtajan-Faddeev method [22]. In [6], Grantcharov, Jung, Kang and Kim gave a presentation of $U_q(q(n))$ in terms of Chevelley generators and Serre relations and developed the highest weight representation theory of $U_q(q(n))$ with a door open to the crystal basis theory. The authors of [6] defined the category $O^{\text{int}}_\infty$, and proved the classical limit theorem and the complete reducibility theorem. Since the queer Lie superalgebra $q(n)$ has a nontrivial odd Cartan part which is closely related with the Clifford algebra, the highest weight space of every finite dimensional $q(n)$-module admits a structure of a Clifford module. In [6], a complete classification of irreducible quantum Clifford modules was also given.

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In [7, 8], Grantcharov, Jung, Kang, Kashiwara and Kim developed the crystal basis theory for $U_q(n)$-modules in the category $O_{\text{int}}^\geq 0$. The authors of [7, 8] first enlarge the base field to $\mathbb{C}((q))$, the field of formal Laurent power series and obtain an equivalence of the categories of Clifford modules and quantum Clifford modules, which yields a standard version of classical limit theorem. As the next step, they introduced the odd Kashiwara operators $\tilde{e}_1$, $\tilde{f}_1$, and $\tilde{k}_1$, where $\tilde{k}_1$ corresponds to an odd element in the Cartan subsuperalgebra of $q(n)$. A crystal basis for a $U_q(q(n))$-module $M$ in the category $O_{\text{int}}^\geq 0$ is defined to be a triple $(L, B, (l_b)_{b \in B})$, where the crystal lattice $L$ is a free $\mathbb{C}[[q]]$-submodule of $M$, $B$ is a finite $gl(n)$-crystal, $(l_b)_{b \in B}$ is a family of non-zero subspaces of $L/qL$ such that $L/qL = \bigoplus_{b \in B} l_b$, with a set of compatibility conditions for the action of the Kashiwara operators. The queer tensor product rule for odd Kashiwara operators is very different from the usual ones and is quite interesting. The main result of [7, 8] is the existence and the uniqueness theorem for crystal bases. One of the key ingredients of the proof is the characterization of highest weight vectors in $B \otimes B(\lambda)$ in terms of even Kashiwara operators and the highest weight vector of $B(\lambda)$. All these statements are verified simultaneously by a series of interlocking inductive arguments.

In [9], Grantcharov, Jung, Kang, Kashiwara and Kim gave an explicit combinatorial realization of the crystal $B(\lambda)$ for an irreducible highest weight module $V_q(\lambda)$ in terms of semistandard decomposition tableaux. A class of combinatorial objects that describe the tensor representations of $q(n)$ has been known for more than thirty years - the shifted semistandard Young tableaux. These objects have been extensively studied by Sagan, Stembridge, Worley, and others, leading to important and deep results (in particular, the shifted Littlewood-Richardson rule) [23, 27, 28]. However, the set of shifted semistandard Young tableaux of a fixed shape does not have a natural crystal structure. For this reason, in [9], it was necessary to use semistandard decomposition tableaux instead of shifted semistandard Young tableaux. Moreover, the authors of [9] presented a queer crystal version of insertion scheme and proved another version of the shifted Littlewood-Richardson rule for decomposing the tensor product $B(\lambda) \otimes B(\mu)$ for all strict partitions $\lambda, \mu$. The insertion scheme in [9] is analogous to the one introduced in [26] and can be considered as a variation of those used for shifted tableaux by Fomin, Haiman, Sagan, and Worley [4, 10, 23, 28]. Consequently, the results of [9] establish a combinatorial description of the shifted Littlewood-Richardson coefficients. It is expected that the queer crystal basis theory will shed a new light on a wide variety of interesting combinatorics.

In this paper, we do not give any proof. Instead, we only give the main idea of proofs and some relevant remarks.

1. QUEER LIE SUPeralgebra $q(n)$

We begin with the definition of queer Lie superalgebra $q(n)$.

**Definition 1.1.** The queer Lie superalgebra $q(n)$ is the Lie superalgebra over $\mathbb{C}$ defined in matrix form by

$$q(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in gl(n, \mathbb{C}) \right\} = q(n)_{\pi} \oplus q(n)_{\tau}.$$
where
\[
q(n)_0 := \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \right\}, \quad q(n)_1 := \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right\}.
\]

The superbracket is defined to be
\[
[x, y] = xy - (-1)^{\alpha \beta} yx \quad \text{for } \alpha, \beta \in \mathbb{Z}_2 \text{ and } x \in q(n)_\alpha, y \in q(n)_\beta.
\]

The (standard) Cartan subalgebra \( \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \) is given by
\[
\mathfrak{h}_0 = \mathbb{C}k_1 \oplus \cdots \oplus \mathbb{C}k_n \quad \text{and} \quad \mathfrak{h}_1 = \mathbb{C}k_1 \oplus \cdots \oplus \mathbb{C}k_n,
\]
where
\[
k_i := \begin{pmatrix} E_{i,i} & 0 \\ 0 & E_{i,i} \end{pmatrix}, \quad k_i = \begin{pmatrix} 0 & E_{i,i} \\ E_{i,i} & 0 \end{pmatrix},
\]
and \( E_{i,j} \) is the \( n \times n \) matrix having 1 at the \((i, j)\)-entry and 0 elsewhere. Note that the Cartan subalgebra \( \mathfrak{h} \) has a nontrivial odd part \( \mathfrak{h}_1 \), and hence \( \mathfrak{h} \) is not abelian.

For \( i = 1, \ldots, n-1, \) set
\[
e_i = \begin{pmatrix} E_{i,i+1} & 0 \\ 0 & E_{i,i+1} \end{pmatrix}, \quad e_i = \begin{pmatrix} 0 & E_{i,i+1} \\ E_{i,i+1} & 0 \end{pmatrix},
\]
and
\[
f_i = \begin{pmatrix} E_{i+1,i} & 0 \\ 0 & E_{i+1,i} \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 & E_{i+1,i} \\ E_{i+1,i} & 0 \end{pmatrix}.
\]

Let \( \{\epsilon_1, \ldots, \epsilon_n\} \) be the basis of \( \mathfrak{h}_0^* \) such that \( \epsilon_i(k_j) = \delta_{ij} \) and \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) be the simple roots for \( i = 1, \ldots, n-1 \).

**Proposition 1.2.** \[\text{[17]} \] \( \text{\S}3 \) The queer Lie superalgebra \( q(n) \) is generated by the elements \( e_i, e_i', f_i, f_i' \) \( (i = 1, \ldots, n-1) \), \( b_{1\sigma} \) and \( k_\sigma \) \( (j = 1, \ldots, n) \) with the following defining relations:

\[
[h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h}_0,
\]
\[
[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i \quad \text{for } h \in \mathfrak{h}_0,
\]
\[
[h, k_j] = 0 \quad \text{for } h \in \mathfrak{h}_0,
\]
\[
[e_i, f_j] = \delta_{ij}(k_i - k_{i+1}),
\]
\[
[e_i, e_j] = 0 \quad \text{if } |i - j| > 1,
\]
\[
[k_i, k_j] = \delta_{ij}2k_i,
\]
\[
e_i, f_j = \delta_{ij}(k_i - k_{i+1}), \quad [e_i, f_j] = \delta_{ij}(k_i - k_{i+1}),
\]
\[
[k_\sigma, e_i] = \alpha_i(k_\sigma)e_i, \quad [k_\sigma, f_i] = -\alpha_i(k_\sigma)f_i,
\]
\[
[e_i, e_j] = [e_i', e_j'] = [f_i, f_j] = [f_i', f_j'] = 0 \quad \text{if } |i - j| \neq 1,
\]
\[
[e_i, e_{i+1}] = [e_i, e_{i+1}'], [e_i, e_{i+1}'] = [e_i, e_i+1],
\]
\[
[f_{i+1}, f_i] = [f_{i+1}', f_i], [f_{i+1}, f_i] = [f_{i+1}', f_i],
\]
\[
[e_i, f_i, e_j] = [f_i, f_i, f_j] = 0 \quad \text{if } |i - j| = 1,
\]
\[
[e_i, e_i', e_j] = [f_i, f_i', f_j] = 0 \quad \text{if } |i - j| = 1.
\]

The elements \( e_i, f_i \) \( (i = 1, \ldots, n-1) \) and \( h \in \mathfrak{h}_0 \) are regarded as *even* generators, and the elements \( e_i', f_i' \) \( (i = 1, \ldots, n-1) \) and \( k_\sigma \) \( (j = 1, \ldots, n) \) are regarded as *odd* generators. One can see that the relations involving \( e_i, f_i, h \) for \( h \in \mathfrak{h}_0 \) are the same as the relations for the general linear Lie algebra \( \mathfrak{gl}(n) \).
Remark 1.3. We have the relations
\[ [k_i, e_i] = e_i, \quad [k_i, f_i] = -f_i, \quad \text{and} \quad [e_i, f_i] = k_i - k_{i+1} = [e_i, f_i]. \]
From these relations, it is easy to see that the queer Lie superalgebra \( q(n) \) is generated by \( e_i, f_i \) \((i = 1, \ldots, n - 1)\), \( b_\Sigma \) and \( k_1 \) only.

The universal enveloping algebra \( U(q(n)) \) of \( q(n) \) is constructed from the tensor algebra \( T(I) \) by factoring out by the ideal generated by the elements \([u, v] - u \otimes v + (-1)^{\alpha \beta} v \otimes u\), where \( \alpha, \beta \in \mathbb{Z}_2, u \in q(n)_\alpha, v \in q(n)_\beta \). Let \( U^+ \) (respectively, \( U^- \)) be the subalgebra \( U(q(n)) \) generated by \( e_i, e_\tau \) (respectively, \( f_i, f_\tau \)) for \( i = 1, \ldots, n - 1 \), and let \( U^0 \) be the subalgebra generated by \( k_j, k^j_\tau \) for \( j = 1, \ldots, n \). By the Poincaré-Birkhoff-Witt theorem in \([15]\), we obtain the triangular decomposition of \( U(q(n)) \):
\[ U(q(n)) \cong U^- \otimes U^0 \otimes U^+ \]

2. HIGHEST WEIGHT MODULES OVER \( q(n) \)

Recall that \( b_\Sigma = Ck_1 \oplus \cdots \oplus Ck_n, \) and \( \{e_1, \ldots, e_n\} \) is the basis of \( b_\Sigma \) dual to the basis \( \{k_1, \ldots, k_n\} \) of \( b_\Sigma \). Let \( P := Z\varepsilon_1 \oplus \cdots \oplus Z\varepsilon_n \) be the weight lattice and \( P^\vee := \mathbb{Z}k_1 \oplus \cdots \oplus \mathbb{Z}k_n \) be the dual weight lattice.

Definition 2.1. Let \( \Lambda^+_0 \) and \( \Lambda^+ \) be the set of \( \mathfrak{gl}(n) \)-dominant integral weights and the set of \( q(n) \)-dominant integral weights given as follows:
\[ \Lambda^+_0 := \{ \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \in b_\Sigma^+ \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \ldots, n - 1 \}, \]
\[ \Lambda^+ := \{ \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \in \Lambda^+_0 \mid \lambda_i = \lambda_{i+1} \Rightarrow \lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \ldots, n - 1 \}. \]

From now on, for a superalgebra \( A \), an \( A \)-module will be understood as an \( A \)-supermodule. A \( q(n) \)-module \( V \) is called a weight module if it admits a weight space decomposition
\[ V = \bigoplus_{\mu \in \Lambda^+_0} V_\mu, \]
where \( V_\mu = \{ v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}_0 \} \).

For a weight \( q(n) \)-module \( V \), we denote by \( \text{wt}(V) \) the set of \( \mu \in \Lambda^+_0 \) such that \( V_\mu \neq 0 \). If \( \text{dim}_C V_\mu < \infty \) for all \( \mu \in \Lambda^+_0 \), the character of \( V \) is defined to be
\[ \text{ch} V = \sum_{\mu \in \Lambda^+_0} (\text{dim}_C V_\mu) e^\mu, \]
where \( e^\mu \) are formal basis elements of the group algebra \( C[\Lambda^+_0] \) with the multiplication \( e^\lambda e^\mu = e^{\lambda+\mu} \) for all \( \lambda, \mu \in \Lambda^+_0 \).

Definition 2.2. A weight module \( V \) is called a highest weight module with highest weight \( \lambda \in \Lambda^+_0 \) if \( V_\lambda \) is finite-dimensional and satisfies the following conditions:
(1) \( V \) is generated by \( V_\lambda \),
(2) \( e_i v = e_\tau v = 0 \) for all \( v \in V_\lambda, \ i = 1, \ldots, n - 1 \).

Note that the highest weight space of a highest weight module is not one-dimensional.

Let \( b_+ \) be the (standard) Borel subalgebra of \( q(n) \) generated by \( e_i, e_\tau \) \((i = 1, \ldots, n - 1)\) and \( k_j, k^j_\tau \) for \( j = 1, \ldots, n \). For \( \lambda \in \Lambda^+_0 \) let \( \text{Cliff}(\lambda) \) be the associative superalgebra over \( \mathbb{C} \) generated by the odd generators \( \{e_i \mid i = 1, 2, \ldots, n\} \).
with the defining relations
\[ t_i t_j + t_j t_i = 2\delta_{ij}\lambda_i, \quad i, j = 1, 2, \ldots, n. \]

The following propositions are well-known.

**Proposition 2.3.**  [1] Table 2] The superalgebra \( \text{Cliff}(\lambda) \) has up to isomorphism
(1) two irreducible modules \( E(\lambda) \) and \( \Pi(E(\lambda)) \) of dimension \( 2^{k-1}2^{k-1} \) if \( m = 2k \),
(2) one irreducible module \( E(\lambda) \cong \Pi(E(\lambda)) \) of dimension \( 2^k2^k \) if \( m = 2k + 1 \),
where \( m \) is the number of non-zero parts of \( \lambda \in \mathfrak{h}_\mathbb{Z}^* \) and \( \Pi \) is the parity change functor.

**Proposition 2.4.**  [20] Proposition 1] Let \( \mathfrak{v} \) be a finite-dimensional irreducible \( \mathbb{Z}_2 \)-graded \( \mathfrak{b}_+ \)-module.
(1) The maximal nilpotent subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{b}_+ \) acts on \( \mathfrak{v} \) trivially.
(2) There exists a unique weight \( \lambda \in \mathfrak{h}_\mathbb{Z}^* \) such that \( \mathfrak{v} \) is a \( \mathbb{Z}_2 \)-graded \( \text{Cliff}(\lambda) \)-module.
(3) For all \( h \in \mathfrak{h}_0 \), \( v \in \mathfrak{v} \), we have \( hv = \lambda(h)v \).

By Proposition 2.3 and Proposition 2.4 we get a complete classification of finite-dimensional irreducible \( \mathfrak{b}_+ \)-modules.

**Definition 2.5.** Let \( \mathfrak{v}(\lambda) \) be a finite-dimensional irreducible \( \mathfrak{b}_+ \)-module determined by \( \lambda \). The Weyl module \( W(\lambda) \) corresponding to \( \lambda \) is defined to be
\[ W(\lambda) := U(q(n)) \otimes_{U(\mathfrak{b}_+)} \mathfrak{v}(\lambda). \]

Note that \( W(\lambda) \) is defined up to \( \Pi \).

**Theorem 2.6.**  [20] Theorem 2, 4]
(1) For any weight \( \lambda \), \( W(\lambda) \) has a unique maximal submodule \( N(\lambda) \).
(2) For each finite-dimensional irreducible \( q(n) \)-module \( V \), there exists a unique weight \( \lambda \in \Lambda_\mathbb{Z}^+ \) such that \( V \) is a homomorphic image of \( W(\lambda) \).
(3) The irreducible quotient \( V(\lambda) := W(\lambda)/N(\lambda) \) is finite-dimensional if and only if \( \lambda \in \Lambda^+ \).

Set \( P^{\geq 0} = \{ \lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n \in P \mid \lambda_j \geq 0 \quad \text{for all} \quad j = 1, 2, \ldots, n \} \).

**Definition 2.7.** The category \( \mathcal{O}^{\geq 0} \) consists of finite-dimensional \( U(q(n)) \)-modules \( M \) with a weight space decomposition satisfying the following conditions:
(1) \( \text{wt}(M) \subset P^{\geq 0} \),
(2) if \( \langle k_i, \mu \rangle = 0 \) for \( \mu \in P^{\geq 0} \) and \( i \in \{1, \ldots, n\} \), then \( k_i \) acts trivially on \( M_\mu \).

The category \( \mathcal{O}^{\geq 0} \) is closed under finite direct sum, tensor product and taking submodules and quotient modules.

**Proposition 2.8.**  [6] Proposition 1.6, 1.8, 1.9]
(1) For each \( \lambda \in \Lambda^+ \cap P^{\geq 0} \), the irreducible quotient \( V(\lambda) = W(\lambda)/N(\lambda) \) lies in the category \( \mathcal{O}^{\geq 0} \).
(2) Every irreducible \( U(q(n)) \)-module in the category \( \mathcal{O}^{\geq 0} \) has the form \( V(\lambda) \) for some \( \lambda \in \Lambda^+ \cap P^{\geq 0} \).
(3) If \( V \) is a finite-dimensional highest weight module with highest weight \( \lambda \in \Lambda^+ \cap P^{\geq 0} \) and \( V_\lambda \) is an irreducible \( \mathfrak{b}_- \)-submodule of \( V \), then \( V \simeq V(\lambda) \) (up to \( \Pi \)).
(4) If \( V \) is a highest weight module with highest weight \( \lambda \in \Lambda^+ \) and \( f_i^{(\lambda_\lambda)+1}v = 0 \) for all \( v \in V_\lambda \), \( i = 1, 2, \ldots, n-1 \), then \( \dim V < \infty \).
Note that every element $\lambda$ of $\Lambda^+ \cap P^{\geq 0}$ is the form
\[
\lambda_1 > \lambda_2 > \cdots > \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0
\]
for some $r$. Hence we can identify an element $\lambda$ of $\Lambda^+ \cap P^{\geq 0}$ with a strict partition. We denote by $\ell(\lambda) = r$ and $|\lambda| = \lambda_1 + \cdots + \lambda_r$.

3. Quantum queer superalgebra $U_q(\mathfrak{g}(n))$

Let $\mathbb{F} = \mathbb{C}((q))$ be the field of formal Laurent series in an indeterminate $q$ and let $\mathcal{A} = \mathbb{C}[[q]]$ be the subring of $\mathbb{F}$ consisting of formal power series in $q$. For $k \in \mathbb{Z}_{\geq 0}$, we define
\[
[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = [k - 1]! [2]! | k - 1 | [2]!, \quad [0]! = 1, \quad [k]! = [k]! [k - 1]! [k - 2]! \cdots [1]!.
\]

In [19, §4], Olshanski constructed a quantum deformation $U_q(\mathfrak{g}(n))$ of $U(\mathfrak{g}(n))$ using a modification of the Reshetikhin-Takhtajan-Faddeev method. In [2, Theorem 2.1], based on Olshanski’s construction, we obtain the following presentation of $U_q(\mathfrak{g}(n))$, which is taken to be the definition.

**Definition 3.1.** The quantum queer superalgebra $U_q(\mathfrak{g}(n))$ is an $\mathbb{F}$-superalgebra generated by the elements $e_i, e_\overline{i}, f_i, f_\overline{i}$, $(i = 1, \ldots, n - 1)$, $k_j$, $(j = 1, \ldots, n)$ and $q^h$ $(h \in P^\vee)$ with the following defining relations:

\[
\begin{align*}
q^0 &= 1, \quad q^{h_1}q^{h_2} = q^{h_1 + h_2} \quad \text{for } h_1, h_2 \in P^\vee, \\
q^h e_i q^{-h} &= q^a_i(h) e_i \quad \text{for } h \in P^\vee, \\
q^h f_i q^{-h} &= q^{-a_i(h)} f_i \quad \text{for } h \in P^\vee, \\
q^h k_j &= q^h \quad \text{for } h \in P^\vee, \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{q^{k_{ij+1}} - q^{-k_{ij+1}}}{q - q^{-1}}, \\
e_i e_j - e_j e_i &= f_i f_j - f_j f_i = 0 \quad \text{if } |i - j| > 1, \\
e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0 \quad \text{if } |i - j| = 1, \\
f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 &= 0 \quad \text{if } |i - j| = 1, \\
k_j^2 &= \frac{q^{2k_i} - q^{-2k_i}}{q^2 - q^{-2}}, \\
k_j k_i + k_i k_j &= 0 \quad \text{if } i \neq j, \\
k_j e_i - q e_i k_j &= e_i q^{-k_j}, \quad k_j e_{i-1} - e_{i-1} k_j = q^{-k_j} e_{i-1}, \\
k_j e_j - e_j k_j &= 0 \quad \text{if } j \neq i, i - 1, \\
k_j f_i - q f_i k_j &= -f_i q^{k_i}, \quad k_j f_{i-1} - f_{i-1} k_j = q^{k_i} f_{i-1}, \\
k_j f_j - f_j k_j &= 0 \quad \text{if } j \neq i, i - 1, \\
e_i f_j - f_j e_i &= \delta_{ij} (k_j q^{-k_{ij+1}} - k_{ij+1} q^{-k_i}), \\
q e_j f_j - f_j e_j &= \delta_{ij} (k_j q^{k_{ij+1}} - k_{ij+1} q^{k_i}), \\
e_i e_i - e_i e_i &= f_i f_j - f_j f_i = 0, \\
e_i e_{i+1} - q e_{i+1} e_i &= e_i e_{i+1} + q e_{i+1} e_i, \\
q f_{i+1} f_i - f_i f_{i+1} &= f_i f_{i+1} + q f_{i+1} f_i, \\
e_i^2 e_i - (q + q^{-1}) e_i e_i + e_i^2 &= 0 \quad \text{if } |i - j| = 1,
\end{align*}
\]
Definition 4.1. Let \( U_q(q(n)) \) be the subalgebra of \( U_q(q(n)) \) generated by \( e_i, e_\tau \) (respectively, \( f_i, f_\tau \)) for \( i = 1, \ldots, n - 1 \), and let \( U_q^0 \) be the subalgebra generated by \( q^h \) and \( k_\tau \) for \( h \in P^\vee, j = 1, \ldots, n \). Then we obtain the following triangular decomposition of \( U_q(q(n)) \).

Proposition 3.2. \([11]\) Theorem 3.1.5] There is a \( \mathbb{C}((q)) \)-linear isomorphism

\[ U_q(q(n)) \simeq U_q^- \otimes U_q^0 \otimes U_q^+. \]

Proof. The proof is based on the comultiplication (3.2), and follows the outline given in \([11]\) Theorem 3.1.5].

4. Representation Theory of \( U_q(q(n)) \)

Let us recall the highest weight representation theory of \( U_q(q(n)) \) that was introduced in \([6]\).

Definition 4.1.

(1) A \( U_q(q(n)) \)-module \( M \) is a weight module if it admits a weight space decomposition

\[ M = \bigoplus_{\mu \in P} M_\mu, \text{ where } M_\mu = \{ m \in M \mid q^h m = q^{\mu(h)} m \text{ for all } h \in P^\vee \}. \]

(2) A weight module \( V \) is a highest weight module with highest weight \( \lambda \in P \) if \( V_\lambda \) is finite-dimensional and satisfies the following conditions:

(i) \( V \) is generated by \( V_\lambda \),

(ii) \( e_i v = c_{i\lambda} v = 0 \) for all \( v \in V_\lambda, i = 1, \ldots, n - 1 \).

For a weight \( U_q(q(n)) \)-module \( V \), we denote by \( \text{wt}(V) \) the set of \( \mu \in P \) such that \( V_\mu \neq 0 \). If \( \dim_{\mathbb{C}((q))} V_\mu < \infty \) for all \( \mu \in P \), the character of \( V \) is defined to be

\[ \text{ch} V = \sum_{\mu \in P} (\dim_{\mathbb{C}((q))} V_\mu) e^\mu, \]

where \( e^\mu \) are formal basis elements of the group algebra \( \mathbb{C}[P] \) with the multiplication \( e^\lambda e^\mu = e^{\lambda+\mu} \) for all \( \lambda, \mu \in P \).

As in the case of \( q(n) \), the Clifford superalgebra plays a central role in the highest weight representation theory of \( U_q(q(n)) \). When \( m \) is a non-negative integer,
the $q$-integer $\frac{q^{2m} - q^{-2m}}{q^2 - q^{-2}}$ has a square root in $\mathbb{C}(q)$ but not in $\mathbb{C}(q)$. This difference gives the following two statements, which is simpler than the corresponding statements in \cite[Theorem 5.14]{[6, Proposition 4.2]}. 

**Proposition 4.2.** For $\lambda \in P$, let $\text{Cliff}_q(\lambda)$ be the associative superalgebra over $\mathbb{C}(q)$ generated by odd generators $\{t_i \mid i = 1, 2, \ldots, n\}$ with the defining relations

$$t_i t_j + t_j t_i = \delta_{ij} \frac{2(q^{2\lambda_i} - q^{-2\lambda_i})}{q^2 - q^{-2}}, \quad i, j = 1, 2, \ldots, n.$$ 

Then $\text{Cliff}_q(\lambda)$ has up to isomorphism

1. two irreducible modules $E^q(\lambda)$ and $\Pi(E^q(\lambda))$ of dimension $2^{k-1}2^{k-1}$ if $m = 2k$,
2. one irreducible module $E^q(\lambda) \cong \Pi(E^q(\lambda))$ of dimension $2^k2^k$ if $m = 2k + 1$,

where $m$ is the number of non-zero parts of $\lambda \in P$.

Let $U_q^{>0}$ be the subalgebra of $U_q(\mathfrak{q}(n))$ generated by $e_i, e_i^\dagger (i = 1, \ldots, n - 1)$ and $q^h, k_j^\dagger (h \in P^\vee, j = 1, \ldots, n)$. In \cite{[6, Proposition 4.1]}, we proved the following proposition, which is a quantum analogue of Proposition 2.4.

**Proposition 4.3.** \cite[Proposition 4.1]{[6, Proposition 4.1]} Let $\mathfrak{v}^q$ be a finite-dimensional irreducible $U_q^{>0}$-module with a weight space decomposition.

1. The subalgebra $U_q^+$ of $U_q^{>0}$ acts on $\mathfrak{v}^q$ trivially.
2. There exists a unique weight $\lambda \in P$ such that $\mathfrak{v}^q$ admits a $\text{Cliff}_q(\lambda)$-module structure.
3. For all $h \in P^\vee, v \in \mathfrak{v}^q$, we have $q^h v = q^{\lambda(h)} v$.

Combining Proposition 4.2 and Proposition 4.3, we obtain a complete classification of finite-dimensional irreducible weight $U_q^{>0}$-modules. We define

$$W^q(\lambda) := U_q(\mathfrak{q}(n)) \otimes_{U_q^{>0}} E^q(\lambda)$$

to be the Weyl module of $U_q(\mathfrak{q}(n))$ corresponding to $\lambda$ (defined up to $\Pi$).

**Proposition 4.4.** \cite[Proposition 4.2]{[6, Proposition 4.2]}

1. $W^q(\lambda)$ is a free $U_q^-$-module of rank $\dim E^q(\lambda)$.
2. Let $V$ be a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda$ such that $V_\lambda$ is an irreducible $U_q^{>0}$-module. Then $V$ is a homomorphic image of $W^q(\lambda)$.
3. Every Weyl module $W^q(\lambda)$ has a unique maximal submodule $N^q(\lambda)$.

By Proposition 4.4, we see that there exists a unique irreducible highest weight module $V^q(\lambda) := W^q(\lambda)/N^q(\lambda)$ with highest weight $\lambda \in P$ up to $\Pi$.

**Example 4.5.** Consider the $\mathbb{F}$-vector space 

$$V = \bigoplus_{j=1}^n \mathbb{F}v_j \oplus \bigoplus_{j=1}^n \mathbb{F}v_j^\dagger$$

with the action of $U_q(\mathfrak{q}(n))$ given as follows:

$$e_i v_j = \delta_{j,i+1} v_i, \quad e_i v_j^\dagger = \delta_{j,i+1} v_i^\dagger, \quad f_i v_j = \delta_{j,i+1} v_{i+1}, \quad f_i v_j^\dagger = \delta_{j,i+1} v_{i+1}^\dagger,$$

$$e_i v_j = \delta_{j,i+1} v_j, \quad e_i v_j^\dagger = \delta_{j,i+1} v_j^\dagger, \quad f_i v_j = \delta_{j,i+1} v_{i+1}, \quad f_i v_j^\dagger = \delta_{j,i+1} v_{i+1},$$

$$q^h v_j = q^{\epsilon(h)} v_j, \quad q^h v_j^\dagger = q^{\epsilon(h)} v_j^\dagger, \quad k_j v_j = \delta_{j,i} v_j, \quad k_j v_j^\dagger = \delta_{j,i} v_j^\dagger.$$
Then $V$ is a $U_q(\mathfrak{g}(n))$-module and called the vector representation of $U_q(\mathfrak{g}(n))$. Note that $V$ is an irreducible highest weight module with highest weight $\epsilon_1$.

Let
$$A_1 := \{f/g \in \mathbb{C}((q)) \mid f, g \in \mathbb{C}[[q]], \, g(1) \neq 0\}$$
and let $V^q$ be a highest weight $U_q(\mathfrak{g}(n))$-module generated by a finite-dimensional irreducible $U_q^{\geq 0}$-module $E^q(\lambda)$. We denote by $\text{Cliff}_{A_1}(\lambda)$ the $A_1$-subalgebra of $\text{Cliff}_q(\lambda)$ generated by $t_1, \ldots, t_{\mathfrak{m}}$ and let $E^{A_1}(\lambda)$ be the $\text{Cliff}_{A_1}(\lambda)$-submodule of $E^q(\lambda)$ generated by a nonzero even element in $E^q(\lambda)^0$. The $A_1$-form $U_{A_1}$ of $U_q(\mathfrak{g}(n))$ is the $A_1$-subalgebra of $U_q(\mathfrak{g}(n))$ generated by $e_i, e_i, f_i, f_i, q^h, k_j$ and $q^h - 1$ for $i = 1, \ldots, n - 1, j = 1, \ldots, n$ and $h \in P^\vee$. The $A_1$-form $V^{A_1}$ of $V^q$ is defined to be the $U_{A_1}$-submodule of $V^q$ generated by $E^{A_1}(\lambda)$.

Let $J_1$ be the unique maximal ideal of $A_1$ generated by $q - 1$. Then there is a canonical isomorphism of fields
$$A_1/J_1 \sim \mathbb{C} \quad \text{given by} \quad f(q) \mapsto f(1),$$
We define the classical limit $U_1$ of $U_q(\mathfrak{g}(n))$ to be
$$\mathbb{C} \otimes_{A_1} U_{A_1} \cong U_{A_1}/J_1 U_{A_1}.$$ Similarly, the classical limit $V^1$ of $V^q$ is defined to be
$$\mathbb{C} \otimes_{A_1} V_{A_1} \cong V^{A_1}/J_1 V^{A_1}.$$ The following classical limit theorem was proved in [6, Section 5].

**Theorem 4.6.** [6, Theorem 5.11–Theorem 5.16]
(1) As $U(\mathfrak{g}(n))$-modules, the classical limit $V^1$ of $V^q$ is isomorphic to a highest weight $U(\mathfrak{g}(n))$-module $V$ with highest weight $\lambda \in P$ such that $V_\lambda$ is an irreducible $\mathfrak{b}^+$-module.
(2) $\text{ch } V^q = \text{ch } V^1$.
(3) The highest weight $U_q(\mathfrak{g}(n))$-module $V^q(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$. 
(4) If $V^q = V^q(\lambda)$ for $\lambda \in \Lambda^+ \cap P^{\geq 0}$, then $V^1$ is isomorphic to $V(\lambda)$ up to $\Pi$.
(5) The classical limit $U_1$ of $U_q(\mathfrak{g}(n))$ is isomorphic to $U(\mathfrak{g}(n))$ as $\mathbb{C}$-superalgebras.

**Proof.** The assertion (1) can be verified by a direct calculation and the assertion (2) follows from a couple of standard facts on tensor products, in particular, on the extension of scalars of free modules.

Combining Theorem 2.6, Proposition 2.8, the assertion (1) and (2), we obtain the assertion (3). Proposition 2.8 and the assertion (2) yield the assertion (4). Now the assertion (5) can be proved as in [6, Theorem 5.16].

We would like to emphasize that the order of our assertions to be proved is important and is carefully arranged.

We now introduce the main object of our investigation – the $U_q(\mathfrak{g}(n))$-modules in the category $O^{\geq 0}_{\text{int}}$. 

\[
\text{QUANTUM QUEER SUPERALGEBRAS 9}
\]
**Definition 4.7.** The category $\mathcal{O}_{\text{int}}^{\geq 0}$ consists of finite-dimensional $U_q(q(n))$-modules $M$ with a weight space decomposition satisfying the following conditions:

1. $\text{wt}(M) \subseteq P_{\geq 0}$,
2. if $\langle k_i, \mu \rangle = 0$ for $\mu \in P_{\geq 0}$ and $i \in \{1, \ldots, n\}$, then $k_i$ acts trivially on $M_{\mu}$.

The fundamental properties of the category $\mathcal{O}_{\text{int}}^{\geq 0}$ are summarized in the following complete reducibility theorem.

**Theorem 4.8.** [6, Proposition 6.2, Theorem 6.5]

1. Every $U_q(q(n))$-module in $\mathcal{O}_{\text{int}}^{\geq 0}$ is completely reducible.
2. Every irreducible $U_q(q(n))$-module in $\mathcal{O}_{\text{int}}^{\geq 0}$ has the form $V^q(\lambda)$ for some $\lambda \in \Lambda^+ \cap P_{\geq 0}$.

**Proof.** Our assertions follow from the classical limit theorem and the induction argument on the dimension of $U_q(q(n))$-modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. The condition (2) of Definition 4.7 plays a crucial role in the proof. □

In the following theorem, we give a decomposition of the tensor product of the vector representation with a highest weight $U_q(q(n))$-module.

**Theorem 4.9.** [7, Theorem 4.1(e)], [8, Theorem 1.11] Let $M$ be a highest weight $U_q(q(n))$-module in $\mathcal{O}_{\text{int}}^{\geq 0}$ with highest weight $\lambda \in \Lambda^+ \cap P_{\geq 0}$. Then we have

$$V \otimes M \simeq \bigoplus_{\lambda+\varepsilon_j \text{ strict partition}} M_j,$$

where $M_j$ is a highest weight $U_q(q(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ with highest weight $\lambda + \varepsilon_j$ and $\dim(M_j)_{\lambda+\varepsilon_j} = 2 \dim M_\lambda$.

**Proof.** We first prove that our assertion holds for finite-dimensional highest weight modules over $q(n)$ in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. Then, by the classical limit theorem, our assertion holds also for finite-dimensional highest weight modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. □

**Corollary 4.10.** [8, Corollary 1.12] Any irreducible $U_q(q(n))$-module in $\mathcal{O}_{\text{int}}^{\geq 0}$ appears as a direct summand of tensor products of the vector representation $V$.

5. **Crystal Bases**

Let $M$ be a $U_q(q(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ and $I = \{1, 2, \ldots, n-1\}$. For $i \in I$, we define the even Kashiwara operators $\tilde{e}_i, \tilde{f}_i : M \rightarrow M$ in the usual way. That is, for $u \in M$, we write

$$u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$ and $f_i^{(k)} = f_i^k / [k]!$, and we define

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$
On the other hand, we define the odd Kashiwara operators to be
\[
\tilde{k}_T := q^{k_1-1}k_T, \\
\tilde{e}_T := -(e_1k_T - qk_1)e_1q^{k_1-1}, \\
\tilde{f}_T := -(k_Tf_1 - qf_1k_T)q^{k_2-1}.
\]

Recall that an abstract \(\mathfrak{gl}(n)\)-crystal is a set \(B\) together with the maps \(\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}\), \(\varphi_i, \varepsilon_i : B \to \mathbb{Z} \cup \{-\infty\}\) for \(i \in I\), and \(\text{wt} : B \to \mathbb{P}\) satisfying the following conditions (see [14]):

1. \(\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i\) if \(i \in I\) and \(\tilde{e}_i b \neq 0\),
2. \(\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i\) if \(i \in I\) and \(\tilde{f}_i b \neq 0\),
3. for any \(i \in I\) and \(b \in B\), \(\varphi_i(b) = \varepsilon_i(b) + (h_i, \text{wt}(b))\),
4. for any \(i \in I\) and \(b, b' \in B\), \(\tilde{f}_i b = b'\) if and only if \(b = \tilde{e}_i b'\),
5. for any \(i \in I\) and \(b \in B\) such that \(\tilde{e}_i b \neq 0\), we have \(\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1\), \(\varepsilon_i(\tilde{e}_i b) = \varphi_i(b) + 1\),
6. for any \(i \in I\) and \(b \in B\) such that \(\tilde{f}_i b \neq 0\), we have \(\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1\), \(\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1\),
7. for any \(i \in I\) and \(b \in B\) such that \(\varphi_i(b) = -\infty\), we have \(\tilde{e}_i b = \tilde{f}_i b = 0\).

In this paper, we say that an abstract \(\mathfrak{gl}(n)\)-crystal is a \(\mathfrak{gl}(n)\)-crystal if it is realized as a crystal basis of a finite-dimensional integrable \(U_q(\mathfrak{gl}(n))\)-module. In particular, for any \(b\) in a \(\mathfrak{gl}(n)\)-crystal \(B\), we have
\[
\varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} : \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} : \tilde{f}_i^n b \neq 0\}.
\]

**Definition 5.1.** Let \(M = \bigoplus_{\mu \in P_{\geq 0}} M_{\mu}\) be a \(U_q(\mathfrak{gl}(n))\)-module in the category \(\mathcal{O}_{int}^{\geq 0}\).

A crystal basis of \(M\) is a triple \((L, B, l_B = (l_b)_{b \in B})\), where

1. \(L\) is a free \(A\)-submodule of \(M\) such that
   (i) \(F \otimes_A L \to M\),
   (ii) \(L = \bigoplus_{\mu \in P_{\geq 0}} L_{\mu}\), where \(L_{\mu} = L \cap M_{\mu}\),
   (iii) \(L\) is stable under the Kashiwara operators \(\tilde{e}_i, \tilde{f}_i\) \((i = 1, \ldots, n - 1)\), \(\tilde{k}_T, \tilde{e}_T, \tilde{f}_T\).
2. \(B\) is a \(\mathfrak{gl}(n)\)-crystal together with the maps \(\tilde{e}_T, \tilde{f}_T : B \to B \cup \{0\}\) such that
   (i) \(\text{wt}((\tilde{e}_T b)) = \text{wt}(b) + \alpha_1, \text{wt}(\tilde{f}_T b) = \text{wt}(b) - \alpha_1)\),
   (ii) for all \(b, b' \in B\), \(\tilde{f}_T b = b'\) if and only if \(b = \tilde{e}_T b'\),
3. \(l_B = (l_b)_{b \in B}\) is a family of non-zero subspaces of \(L/qL\) such that
   (i) \(l_b \subset (L/qL)_{\mu}\) for \(b \in B_{\mu}\),
   (ii) \(L/qL = \bigoplus_{b \in B} l_b\),
   (iii) \(\tilde{k}_T l_b \subset l_b\),
   (iv) for \(i = 1, \ldots, n - 1\), \(\Gamma\), we have
      (1) if \(\tilde{e}_i b = 0\) then \(\tilde{e}_i l_b = 0\), and otherwise \(\tilde{e}_i\) induces an isomorphism \(l_b \cong \tilde{l}_b\),
      (2) if \(\tilde{f}_i b = 0\) then \(\tilde{f}_i l_b = 0\), and otherwise \(\tilde{f}_i\) induces an isomorphism \(l_b \cong \tilde{l}_b\).

**Remark 5.2.** Note that an element \(b \in B\) does not correspond to a basis vector of \(L/qL\). Instead, it corresponds to a subspace \(l_b\) of \(L/qL\). In [8 Proposition 2.3],
we proved that for any crystal basis \((L, B, l_B)\) of a \(U_q(\mathfrak{g}(n))\)-module \(M \in \mathcal{O}_{\text{int}}^{\geq 0}\), we have \(\tilde{e}_i^2 = \tilde{f}_i^2 = 0\) as endomorphisms on \(L/qL\).

**Example 5.3.** Let \(V = \bigoplus_{j=1}^n Fv_j \oplus \bigoplus_{j=1}^n Fv_\tilde{\gamma}\) be the vector representation of \(U_q(\mathfrak{g}(n))\). Set
\[
L = \bigoplus_{j=1}^n Av_j \oplus \bigoplus_{j=1}^n A v_\tilde{\gamma}
\]
and \(l_j = C v_j \oplus C v_\tilde{\gamma} \subset L/qL\), and let \(B\) be the \(\mathfrak{gl}(n)\)-crystal with the \(\tilde{\gamma}\)-arrow given below.

Here, the actions of \(\tilde{f}_i\) \((i = 1, \ldots, n - 1, \tilde{\gamma})\) are expressed by \(i\)-arrows. Then \((L, B, l_B = (l_j)_{j=1}^n)\) is a crystal basis of \(V\).

**Remark 5.4.** Let \(M\) be a \(U_q(\mathfrak{g}(n))\)-module in the category \(\mathcal{O}_{\text{int}}^{\geq 0}\) with a crystal basis \((L, B, l_B)\). For \(i = 1, \ldots, n - 1, \tilde{\gamma}\) and \(b, b' \in B\), if \(b' = \tilde{f}_i b\), then we have isomorphisms \(\tilde{f}_i : l_b \sim \tilde{\gamma} l_{b'}^i\) and \(\tilde{e}_i : l_{b'} \sim \tilde{\gamma} l_b\). If \(i = 1, \ldots, n - 1\), then they are inverses to each other. However, when \(i = \tilde{\gamma}\), they are not inverses to each other in general.

The *queer tensor product rule* given in the following theorem is one of the most important and interesting features of the crystal basis theory of \(U_q(\mathfrak{g}(n))\)-modules.

**Theorem 5.5.** [7 Theorem 3.3] \([8\) Theorem 2.7] Let \(M_j\) be a \(U_q(\mathfrak{g}(n))\)-module in \(\mathcal{O}_{\text{int}}^{\geq 0}\) with a crystal basis \((L_j, B_j, l_{B_j})\) \((j = 1, 2)\). Set \(B_1 \otimes B_2 = B_1 \times B_2\) and \(l_{b_1 \otimes b_2} = l_{b_1} \otimes l_{b_2}\) for \(b_1 \in B_1\) and \(b_2 \in B_2\). Then \((L_1 \otimes_{\mathfrak{A}} L_2, B_1 \otimes B_2, (l_b)_{b \in B_1 \otimes B_2})\) is a crystal basis of \(M_1 \otimes_{\mathfrak{A}} M_2\), where the action of the Kashiwara operators on \(B_1 \otimes B_2\) are given as follows:

\[
\begin{align*}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\tilde{\varepsilon}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{\varepsilon}_i b_1 \otimes b_2 & \text{if } \langle k_1, \lambda(b_2) \rangle = \langle k_2, \lambda(b_2) \rangle = 0, \\
 b_1 \otimes \tilde{\varepsilon}_i b_2 & \text{otherwise},
\end{cases} \\
\tilde{\delta}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{\delta}_i b_1 \otimes b_2 & \text{if } \langle k_1, \lambda(b_2) \rangle = \langle k_2, \lambda(b_2) \rangle = 0, \\
 b_1 \otimes \tilde{\delta}_i b_2 & \text{otherwise}.
\end{cases}
\end{align*}
\]

**Proof.** For \(i = 1, 2, \ldots, n - 1\), our assertions were already proved in \([12, 13]\). For \(i = \tilde{\gamma}\), our assertions follow from the following comultiplication formulas (see \([8]\)):

\[
\begin{align*}
\Delta(\tilde{\varepsilon}_i) &= \tilde{\varepsilon}_i \otimes q^{2k_1} + 1 \otimes \tilde{\varepsilon}_i, \\
\Delta(\tilde{\delta}_i) &= \tilde{\delta}_i \otimes q^{k_1 + k_2} + 1 \otimes \tilde{\delta}_i - (1 - q^2)\tilde{\varepsilon}_i \otimes e_1 q^{k_2}, \\
\Delta(\tilde{\varepsilon}_i) &= \tilde{\varepsilon}_i \otimes q^{k_1 + k_2} + 1 \otimes \tilde{\varepsilon}_i - (1 - q^2)\tilde{\delta}_i \otimes f_1 q^{k_1 + k_2 - 1}.
\end{align*}
\]
\]
Definition 5.6. An abstract $\mathfrak{q}(n)$-crystal is a $\mathfrak{gl}(n)$-crystal together with the maps $\tilde{e}_T, \tilde{f}_T : B \to B \cup \{ \emptyset \}$ satisfying the following conditions:

1. $\text{wt}(B) \subset \mathbb{Z}^n_+$,
2. $\text{wt}(\tilde{e}_T b) = \text{wt}(b) + \alpha_i$, $\text{wt}(\tilde{f}_T b) = \text{wt}(b) - \alpha_i$,
3. for all $b, b' \in B$, $\tilde{f}_T b = b'$ if and only if $b = \tilde{e}_T b'$,
4. if $3 \leq i \leq n - 1$, we have
   
   (i) the operators $\tilde{e}_T$ and $\tilde{f}_T$ commute with $\tilde{e}_i$ and $\tilde{f}_i$.
   
   (ii) if $\tilde{e}_i b \in B$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b)$.

Let $B_1$ and $B_2$ be abstract $\mathfrak{q}(n)$-crystals. The tensor product $B_1 \otimes B_2$ of $B_1$ and $B_2$ is defined to be the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ together with the maps $\tilde{e}_T, \tilde{f}_T$ defined by (5.2). Then it is an abstract $\mathfrak{q}(n)$-crystal.

Remark 5.7. Let $B_1, B_2$ and $B_3$ be abstract $\mathfrak{q}(n)$-crystals. Then we have

$$(B_1 \otimes B_2) \otimes B_3 \simeq B_1 \otimes (B_2 \otimes B_3).$$

Example 5.8.

1. If $(L, B, l_B)$ is a crystal basis of a $U_q(\mathfrak{gl}(n))$-module $M$ in the category $\mathcal{O}_{int}^2$, then $B$ is an abstract $\mathfrak{q}(n)$-crystal.

2. The crystal graph $B$ of the vector representation $V$ is an abstract $\mathfrak{q}(n)$-crystal.

3. By the tensor product rule, $B^\otimes N$ is an abstract $\mathfrak{q}(n)$-crystal. When $n = 3$, the $\mathfrak{q}(n)$-crystal structure of $B \otimes B$ is given below.

   $\begin{array}{cccc}
   1 \otimes 1 & \rightarrow & 2 \otimes 1 & \rightarrow & 3 \otimes 1 \\
   1 \otimes 2 & \rightarrow & 2 \otimes 2 & \rightarrow & 3 \otimes 2 \\
   1 \otimes 3 & \rightarrow & 2 \otimes 3 & \rightarrow & 3 \otimes 3
   \end{array}$

4. For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$, let $Y_\lambda$ be the skew Young diagram having $\lambda_1$ many boxes in the principal diagonal, $\lambda_2$ many boxes in the second diagonal, etc. For example, if $\lambda = (7 > 6 > 4 > 2 > 0)$, then we have

\[
Y_\lambda = \begin{array}{ccc}
& & \\
& & \\
& & \\
& & 1 \\
& & 2 \\
& & 3 \\
& 1 & \\
1 & 2 & 3 \\
1 & 2 & 3
\end{array}
\]

Let $B(Y_\lambda)$ be the set of all semistandard tableaux of shape $Y_\lambda$ with entries from $1, 2, \ldots, n$. Then by an admissible reading introduced in (2), $B(Y_\lambda)$ can be embedded in $B^\otimes N$, where $N = \lambda_1 + \cdots + \lambda_r$. One can show that it is stable under the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i = 1, \cdots, n - 1, \emptyset$) and hence it becomes an abstract $\mathfrak{q}(n)$-crystal. Moreover, the $\mathfrak{q}(n)$-crystal structure thus obtained does not depend on the choice of admissible reading.
In Figure 1 we illustrate the crystal $B(Y_\lambda)$ for $n = 3$ and $\lambda = (3 > 1 > 0)$. In Figure 2 we present the crystal $B(Y_\mu)$ for $n = 3$ and $\mu = (3 > 0)$. Note that in general, $B(Y_\lambda)$ is not connected.

Let $B$ be an abstract $q(n)$-crystal. For $i = 1, 2, \ldots, n - 1$, we define the automorphism $S_i$ on $B$ by

$$S_i b = \begin{cases} \tilde{e}_i^{(h_i, \text{wt}(b))} b & \text{if } \langle h_i, \text{wt}(b) \rangle \geq 0, \\ \tilde{f}_i^{-(h_i, \text{wt}(b))} b & \text{if } \langle h_i, \text{wt}(b) \rangle \leq 0. \end{cases}$$

(5.3)

Let $w$ be an element of the Weyl group $W$ of $\mathfrak{gl}(n)$. Then, as shown in [15], there exists a unique action $S_w: B \to B$ of $W$ on $B$ such that $S_{s_i} = S_i$ for $i = 1, 2, \ldots, n - 1$. Note that $\text{wt}(S_w b) = w(\text{wt}(b))$ for any $w \in W$ and $b \in B$.

For $i = 1, \ldots, n - 1$, we set

$$w_i = s_2 \cdots s_is_1 \cdots s_{i-1}.$$

(5.4)

Then $w_i$ is the shortest element in $W$ such that $w_i(\alpha_i) = \alpha_1$. We define the odd Kashiwara operators $\tilde{\varepsilon}_i, \tilde{\eta}_i (i = 2, \ldots, n - 1)$ by

$$\tilde{\varepsilon}_i = S_{w_i^{-1}} \tilde{\varepsilon}_i S_{w_i}, \quad \tilde{\eta}_i = S_{w_i^{-1}} \tilde{\eta}_i S_{w_i}.$$
We say that an element $b \in B$ is a highest weight vector if $\tilde{e}_i b = \tilde{e}_{-i} b = 0$ for all $i = 1, \ldots, n - 1$, and an element $b \in B$ is a lowest weight vector if $S_{w_0} b$ is a highest weight vector, where $w_0$ is the longest element of $W$.

In the following lemma, we give a combinatorial characterization of highest weight vectors in $B \otimes N$, which plays a crucial role in the proof of the main theorem. We expect this lemma will have many important applications in the combinatorial representation theory of $U_q(q(n))$.

**Lemma 5.9.** [8, Theorem 3.11] A vector $b_0 \in B \otimes N$ is a highest weight vector if and only if $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ for some $j \in \{1, 2, \ldots, n\}$ and some highest weight vector $b \in B^{\otimes (N-1)}$ such that $\text{wt}(b_0) = \text{wt}(b) + \epsilon_j$ is a strict partition.

**Proof.** The proof consists of series of lemmas and lengthy (and careful) case-by-case check-ups.

The existence and the uniqueness of crystal bases of $U_q(q(n))$-modules in $O_{\text{int}}^{>0}$ is given in the following theorem.

**Theorem 5.10.** [7, Theorem 4.1] [8, Theorem 4.6]
(1) Let $M$ be an irreducible highest weight $U_q(q(n))-module$ with highest weight $\lambda \in \Lambda^+ \cap P_{\geq 0}$. Then there exists a crystal basis $(L, B(l_B), l_B)$ of $M$ such that
   (i) $B(l_B) = \{b_\lambda\}$,
   (ii) $B$ is connected.
   Moreover, such a crystal basis is unique up to an automorphism of $M$. In particular, $B$ depends only on $\lambda$ as an abstract $q(n)$-crystal and we write $B = B(\lambda)$.

(2) The $q(n)$-crystal $B(\lambda)$ has a unique highest weight vector $b_\lambda$ and a unique lowest weight vector $l_\lambda$.

(3) A vector $b \in B \otimes B(\lambda)$ is a highest weight vector if and only if
   $$b = 1 \otimes \hat{f}_1 \cdots \hat{f}_{j-1} b_{\lambda}$$
   for some $j$ such that $\lambda + \epsilon_j$ is a strict partition.

(4) Let $M$ be a finite-dimensional highest weight $U_q(q(n))-module$ with highest weight $\lambda \in \Lambda^+ \cap P_{\geq 0}$. Assume that $M$ has a crystal basis $(L, B(l_B), l_B)$ such that $L_\lambda / q L_\lambda = l_\lambda$. Then we have
   (i) $V \otimes M = \bigoplus_{\lambda + \epsilon_j : \text{strict}} M_j$, where $M_j$ is a highest weight $U_q(q(n))-module$ with highest weight $\lambda + \epsilon_j$ and $\dim(M_j)_{\lambda + \epsilon_j} = 2 \dim M_\lambda$,
   (ii) if we set $L_j = (L \otimes L) \cap M_j$ and $B_j = \{b \in B \otimes B(\lambda) \mid l_b \subset L_j / q L_j\}$, then we have $B \otimes B(\lambda) = \bigsqcup_{b \in B_j} B_j$ and $L_j / q L_j = \bigoplus_{b \in B_j} l_b$,
   (iii) $M_j$ has a crystal basis $(L_j, B_j, l_B_j)$,
   (iv) $B_j \simeq B(\lambda + \epsilon_j)$ as an abstract $q(n)$-crystal.

Proof. All of these assertions are proved by a series of interlocking inductive arguments (see [5]).

Our main theorem implies the following corollary.

**Corollary 5.11.** [7 Theorem 4.1(d)] [9 Corollary 4.7]

(1) Every $U_q(q(n))-module$ in the category $O_{\text{int}}^{\geq 0}$ has a crystal basis.

(2) If $M$ is a $U_q(q(n))-module$ in the category $O_{\text{int}}^{\geq 0}$ and $(L, B(l_B), l_B)$ is a crystal basis of $M$, then there exist decompositions $M = \bigoplus_{a \in A} B_a$ as a $U_q(q(n))-module$,
   $$L = \bigoplus_{a \in A} L_a$$
   as an $A$-module, $B = \bigsqcup_{a \in A} B_a$ as a $q(n)$-crystal, parametrized by a set $A$ such that the following conditions are satisfied for any $a \in A$:
   (i) $M_a$ is a highest weight module with highest weight $\lambda_a$ and $B_a \simeq B(\lambda_a)$ for some strict partition $\lambda_a$,
   (ii) $L_a = L \cap M_a$, $L_a / q L_a = \bigoplus_{b \in B_a} l_b$,
   (iii) $(L_a, B_a, l_B_a)$ is a crystal basis of $M_a$.

6. SEMISTANDARD DECOMPOSITION TABLEAUX

As we have seen in Example [5, §4.4](4), the abstract $q(n)$-crystal $B(Y_\lambda)$ is usually too big to be isomorphic to $B(\lambda)$, the crystal of the irreducible highest weight module $V^\vee(\lambda)$. In this section, we give an explicit combinatorial realization of the $q(n)$-crystal $B(\lambda)$ in terms of semistandard decomposition tableaux.

**Definition 6.1.** (cf. [25 Section 1.2])
(1) A word \( u = u_1 \cdots u_N \) is a hook word if there exists \( 1 \leq k \leq N \) such that
\[
u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_N.
\]
Every hook word has the decreasing part \( u \downarrow = u_1 \cdots u_k \), and the increasing part \( u \uparrow = u_{k+1} \cdots u_N \) (note that the decreasing part is always nonempty).

(2) For a strict partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \), the shifted Young diagram of shape \( \lambda \) is an array of boxes in which the \( i \)-th row has \( \lambda_i \) many boxes, and is shifted \( i-1 \) units to the right with respect to the top row. In this case, we say that \( \lambda \) is a shifted shape.

Example 6.2. For \( \lambda = (6, 4, 2, 1) \), the shifted shape \( \lambda \) is
\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\end{array}
\]

Definition 6.3. (cf. [26, Definition 2.14])
(1) A semistandard decomposition tableau of a shifted shape \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a filling \( T \) of a shifted shape \( \lambda \) with elements of \( \{1, 2, \ldots, n\} \) such that:
(i) the word \( v_i \) formed by reading the \( i \)-th row from left to right is a hook word of length \( \lambda_i \),
(ii) \( v_i \) is a hook subword of maximal length in \( v_{i+1}v_i \) for \( 1 \leq i \leq \ell(\lambda) - 1 \).
(2) The reading word of a semistandard decomposition tableau \( T \) is
\[
\text{read}(T) = v_{\ell(\lambda)}v_{\ell(\lambda)-1} \cdots v_1.
\]

Remark 6.4. We change the definition of a hook word, and hence of a semistandard decomposition tableau in [26], in order to make the forms of the highest weight vectors and the lowest weight vectors simpler than the ones in [26].

Example 6.5. The following tableaux are semistandard decomposition tableaux of a shifted shape \( (3, 1, 0) \):
\[
\begin{array}{ccc}
2 & 1 & 1 \\
\end{array}, \quad \begin{array}{ccc}
2 & 2 & 2 \\
\end{array}, \quad \begin{array}{ccc}
2 & 1 & 3 \\
\end{array}, \quad \begin{array}{ccc}
2 & 1 & 2 \\
\end{array}.
\]

On the other hand, the following tableaux do not satisfy the conditions in Definition 6.3 (1):
\[
\begin{array}{ccc}
1 & 2 & 1 \\
1 & 1 \\
\end{array}, \quad \begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 \\
\end{array}.
\]

Let \( \mathcal{B}(\lambda) \) be the set of all semistandard decomposition tableaux \( T \) with a shifted shape \( \lambda \). For every strict partition \( \lambda \), we have the embedding
\[
\text{read} : \mathcal{B}(\lambda) \to \mathcal{B}^{|\lambda|}, \; T \mapsto \text{read}(T),
\]
which enables us to identify \( \mathcal{B}(\lambda) \) with a subset in \( \mathcal{B}^{|\lambda|} \) and define the action of the Kashiwara operators \( \tilde{e}_i, \tilde{f}_i, \tilde{f}_i^\dagger \) on \( \mathcal{B}(\lambda) \) by the queer tensor product rule.

Theorem 6.6. [9, Theorem 2.5] The set \( \mathcal{B}(\lambda) \cup \{0\} \) is stable under the Kashiwara operators \( \tilde{e}_i, \tilde{f}_i, \tilde{f}_i^\dagger \). Hence, \( \mathcal{B}(\lambda) \) becomes an abstract \( q(n) \)-crystal.
Proof. We first show that if \( u \) is a hook word, then \( \tilde{e}_i u, \tilde{f}_i u \) \((i = 1, \ldots, n - 1, \mathcal{T})\) are hook words whenever they are nonzero. Next, we prove that \( \tilde{f}_i u, \tilde{e}_i u \) \((i = 1, \ldots, n - 1, \mathcal{T})\) satisfy the condition in Definition 6.3 (1)(ii). For this, we show that if \( \tilde{f}_i u, \tilde{e}_i u \) \((i = 1, \ldots, n - 1, \mathcal{T})\) has a hook subword of length \( m \), then \( u \) also has a hook subword of length \( m \) when \( u \in B(\lambda) \) and \( \lambda_3 = 0 \). The proof is based on case-by-case check-ups. \( \square \)

For a strict partition \( \lambda \) with \( \ell(\lambda) = r \), set

\[
T^\lambda := (1^{\lambda_r})(2^{\lambda_{r-1} - \lambda_r}) \cdots ((r - k + 1)^{\lambda_k - \lambda_{k-1}} \cdots 1^{\lambda_1 - \lambda_2}),
\]

\[
L^\lambda := (n - r + 1)^{\lambda_r} \cdots (n - k + 1)^{\lambda_k} \cdots n^{\lambda_1}.
\]

It is easy to check that \( S_{w_0} T^\lambda = L^\lambda \).

Example 6.7. Let \( n = 4 \) and \( \lambda = (7, 4, 2, 0) \). Then we have

\[
T^\lambda = \begin{array}{cccccc}
3 & 3 & 2 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1
\end{array}
\quad \text{and} \quad
L^\lambda = \begin{array}{cccc}
4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 \\
2 & 2 & & &
\end{array}
\]

The explicit combinatorial realization of \( B(\lambda) \) is given in the following lemma.

Theorem 6.8. [9, Theorem 2.5] Let \( \lambda \) be a strict partition.

1. The tableau \( T^\lambda \) is a unique highest weight vector in \( B(\lambda) \) and \( L^\lambda \) is a unique lowest weight vector in \( B(\lambda) \).

2. The abstract \( q(n) \)-crystal \( B(\lambda) \) is isomorphic to \( B(\lambda) \), the crystal of the irreducible highest weight module \( V^\lambda \).

Proof. Using Lemma 5.9, the lowest weight vectors are characterized as follows:

(6.1) For \( a \in B \) and \( b \in B^{\otimes n} \), \( a \otimes b \) is a lowest weight vector if and only if \( b \) is a lowest weight vector and \( \epsilon_a + \text{wt}(b) \in w_0(\Lambda^+ \cap P^{\geq 0}) \).

Using induction on \( |\lambda| \) and the above statement, we conclude that \( L^\lambda \) is a unique lowest weight vector in \( B(\lambda) \). Since \( S_{w_0} T^\lambda = L^\lambda \), we get the first assertion. The second assertion follows from the first one directly. \( \square \)

Example 6.9.

1. Since any word of length 2 is a hook word, we obtain \( B \otimes B \simeq B(2\epsilon_1) \), and hence the crystal in Example 5.8 (3) is isomorphic to the \( q(3) \)-crystal \( B(2\epsilon_1) \).

2. In Figure 3, we present the \( q(3) \)-crystal \( B(3\epsilon_1 + \epsilon_2) \).

3. In Figure 4 and Figure 5, we illustrate the \( q(3) \)-crystal \( B(3\epsilon_1) \) and \( B(2\epsilon_1 + \epsilon_2) \), respectively. By Lemma 6.4, there are two highest weight vectors \( 1 \otimes 1 \otimes 1 \) and \( 1 \otimes 2 \otimes 1 \) in \( B^{\otimes 3} \). Therefore we obtain \( B^{\otimes 3} \simeq B(3\epsilon_1) \oplus B(2\epsilon_1 + \epsilon_2) \).

Now the natural question is how to decompose \( B(\lambda) \otimes B(\mu) \) into a disjoint union of connected components. We define \( \lambda \leftarrow j \) to be the array of boxes obtained from the shifted shape \( \lambda \) by adding a box at the \( j \)-th row. Let us denote by \( \lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_r \) the array of boxes obtained from \( \lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_{r-1} \) by adding a box at the \( j_r \)-th row. We define \( B(\lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_r) \) to be the empty set unless \( \lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_k \) is a shifted shape for all \( k = 1, \ldots, r \).
Theorem 6.10. [9, Theorem 2.8] Let $\lambda$ and $\mu$ be strict partitions. Then there is a $q(n)$-crystal isomorphism

$$B(\lambda) \otimes B(\mu) \simeq \bigoplus_{u_1, u_2, \ldots, u_N \in B(\lambda)} B(\mu \leftarrow (n - u_N + 1) \leftarrow \cdots \leftarrow (n - u_1 + 1)),$$

where $N = |\lambda|$. 

Figure 3. $B(3\epsilon_1 + \epsilon_2)$ for $n = 3$

Figure 4. $B(3\epsilon_1)$ for $n = 3$. 
Proof. By the characterization (6.1), the lowest weight vectors in \( B(\lambda) \otimes B(\mu) \) have the form \( u_1 \cdots u_N \otimes L^\mu \) such that \( w_0 \mu + \epsilon u_N + \epsilon_{u_{N-1}} + \cdots + \epsilon_{u_k} \in w_0(\Lambda^+ \cap P^\geq) \) for all \( k = 1, \ldots, N \). Hence, the weights of the highest weight vectors in \( B(\lambda) \otimes B(\mu) \) are of the form \( \mu \leftarrow (n - u_N + 1) \leftarrow \cdots \leftarrow (n - u_1 + 1) \) as desired. □

By Theorem 6.10, we obtain an explicit description of shifted Littlewood-Richardson coefficients.

Corollary 6.11. [9, Corollary 2.9] We define

\[
\mathcal{LR}_{\lambda, \mu}^\nu := \{ u = u_1 \cdots u_N \in B(\lambda) ; \begin{align*}
(\text{a})& \quad \text{wt}(u) = w_0(\nu - \mu) \\
(\text{b})& \quad \mu + \epsilon_{n-u_N+1} + \cdots + \epsilon_{n-u_k+1} \in \Lambda^+ \cap P^\geq \text{ for all } 1 \leq k \leq N \}\}
\]

and set \( f_{\lambda, \mu}^\nu := |\mathcal{LR}_{\lambda, \mu}^\nu| \). Then there is a \( q(n) \)-crystal isomorphism

\[
(6.2) \quad B(\lambda) \otimes B(\mu) \simeq \bigoplus_{\nu \in \Lambda^+ \cap P^\geq} B(\nu)^{\otimes f_{\lambda, \mu}^\nu}.
\]

Example 6.12. Let \( n = 3, \lambda = 2 \epsilon_1 + \epsilon_2 \) and \( \mu = 3 \epsilon_1 \). For \( u_1 u_2 u_3 \in B(\lambda) \), if \( u_3 = 1 \) then the array \( \mu \leftarrow (3 - u_3 + 1) \) is not a shifted shape. When \( u_1 u_2 u_3 = 123 \) or \( 132 \), \( \mu \leftarrow (3 - u_3 + 1) \leftarrow (3 - u_2 + 1) \leftarrow (3 - u_1 + 1) \) is not a shifted shape. For the other \( u_1 u_2 u_3 \in B(\lambda) \), \( \mu \leftarrow (3 - u_3 + 1) \leftarrow (3 - u_2 + 1) \leftarrow (3 - u_1 + 1) \) is given as follows:

\[
(u_1 u_2 u_3 = 122), \quad (u_1 u_2 u_3 = 232), \quad (u_1 u_2 u_3 = 233).
\]

So we obtain

\[
B(2 \epsilon_1 + \epsilon_2) \otimes B(3 \epsilon_1) \simeq B(3 \epsilon_1 + 2 \epsilon_2 + \epsilon_3) \oplus B(4 \epsilon_1 + 2 \epsilon_2) \oplus B(5 \epsilon_1 + \epsilon_2).
\]

As seen in (6.2), the connected component containing \( T \otimes T' \in B(\lambda) \otimes B(\mu) \) is isomorphic to \( B(\nu) \) for some \( \nu \). In order to find \( \nu \) and the element \( S \) of \( B(\nu) \)
corresponding to \( T \otimes T' \) explicitly, we define the *insertion scheme* for semistandard decomposition tableaux.

For an abstract \( q(n) \)-crystal \( B \) and an element \( b \in B \), we denote the connected component of \( b \) in \( B \) by \( C(b) \).

**Definition 6.13.** Let \( B_i \) be an abstract \( q(n) \)-crystals and let \( b_i \in B_i \) \((i = 1, 2)\). We say that \( b_1 \) is \( q(n) \)-crystal equivalent to \( b_2 \) if there exists an isomorphism of crystals

\[
C(b_1) \xrightarrow{\sim} C(b_2)
\]

sending \( b_1 \) to \( b_2 \). We denote this equivalence relation by \( b_1 \sim b_2 \).

The following \( q(n) \)-crystal equivalence, which is called the *queer Knuth relation*, can be verified in a straightforward manner.

**Proposition 6.14.** [9, Proposition 3.3] (cf. [26, Theorem 1.4]) Let \( B_1 \) and \( B_2 \) be the connected components containing \( 1121 \) and \( 1211 \) in \( B \otimes B_4 \), respectively. Then there exists a \( q(n) \)-crystal isomorphism \( \psi : B_1 \rightarrow B_2 \) such that

\[
\begin{align*}
\psi(abcd) &= acbd \quad \text{if } d \leq b \leq a < c \\
&\quad \text{or } b < d \leq a < c \\
&\quad \text{or } b \leq a < d \leq c, \\
&\quad \text{or } a < b < d \leq c, \\
&\quad \text{or } b < d \leq c \leq a, \\
&\quad \text{or } d \leq b < c \leq a, \\
&\quad \text{or } a < d \leq b < c, \\
&\quad \text{or } d \leq a < b < c.
\end{align*}
\]

**Definition 6.15.** (cf. [26, Definition 2.18]) Let \( T \) be a semistandard decomposition tableau of shifted shape \( \lambda \). For \( x \in B \), we define \( T \leftarrow x \) to be a filling of an array of boxes obtained from \( T \) by applying the following procedure:

1. Let \( v_1 = u_1 \cdots u_m \) be the reading word of the first row of \( T \) such that \( u_1 \geq \cdots \geq u_k < \cdots < u_m \) for some \( 1 \leq k \leq m \). If \( v_1x \) is a hook word, then put \( x \) at the end of the first row and stop the procedure.
2. Assume that \( v_1x \) is not a hook word. Let \( u_j \) be the leftmost element in \( v_1 \uparrow \) which is greater than or equal to \( x \). Replace \( u_j \) by \( x \). Let \( u_i \) be the leftmost element in \( v_1 \downarrow \) which is strictly less than \( u_j \). Replace \( u_i \) by \( u_j \). (Hence \( u_i \) is bumped out of the first row.)
3. Apply the same procedure for the second row with \( u_i \) as described in (1) and (2).
4. Repeat the same procedure row by row from top to bottom until we place a box at the end of a row of \( T \).

**Example 6.16.** Since

\[
\begin{array}{c|cccc}
6 & 6 & 1 & 3 & 5 \\
\hline
1 & & & & \\
\end{array}
\]

\[
\begin{array}{c|cccc}
4 & 2 & 1 & & \\
\hline
3 & & & & \\
\end{array}
\]

we obtain

\[
\begin{array}{c|cccc}
6 & 6 & 1 & 3 & 5 \\
\hline
3 & 2 & 4 & & \\
\end{array}
\]

\[
\begin{array}{c|cccc}
6 & 6 & 3 & 2 & 5 \\
\hline
4 & 2 & 1 & & \\
\end{array}
\]
Let $T$ and $T'$ be semistandard decomposition tableaux. We define $T \leftarrow T'$ to be

$$(\cdots ((T \leftarrow u_1) \leftarrow u_2) \cdots) \leftarrow u_N,$$

where $u_1u_2 \cdots u_N$ is the reading word of $T'$.

Example 6.17.

\[
\begin{array}{c}
2 & 2 \\
1 & & 3 & 3 & 3 \\
\end{array}
\leftarrow
\begin{array}{c}
3 & 3 & 3 \\
1 & & 2 & 2 & 1 \\
\end{array}
= \begin{array}{c}
(\begin{array}{c}
2 & 2 \\
1 & & 3 \\
\end{array} \leftarrow 3) \leftarrow 3
\end{array}
= \begin{array}{c}
(\begin{array}{c}
2 & 2 & 3 \\
1 & & 3 \\
\end{array} \leftarrow 3) \leftarrow 3
\end{array}
= \begin{array}{c}
3 & 3 & 3 \\
1 & & 2 & 2 & 1 \\
\end{array}
\]

Proposition 6.18. [9, Proposition 3.13, Corollary 3.14]

(1) $T \otimes T'$ is $q(n)$-crystal equivalent to $T \leftarrow T'$.
(2) $T \leftarrow T'$ is a semistandard decomposition tableau.

Proof. The first assertion follows from the queer Knuth relation. For the second assertion, it suffices to show that $b_1 \otimes b_2 \leftarrow x$ is a semistandard decomposition tableau for any $x \in B$ and $b_1 \otimes b_2 \in B(\lambda_1\epsilon_1 + \lambda_2\epsilon_2)$ with $\lambda_1 > \lambda_2$. Through a careful investigation on the direct summands in the various tensor products, one concludes that $b_1 \otimes b_2 \leftarrow x$ lies in $B(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \epsilon_3)$, as desired. \qed

Using the characterization (6.1) of the lowest weight vectors and Proposition 6.18 we obtain the following theorem.

Theorem 6.19. [9, Theorem 3.15] Let $\lambda$ and $\mu$ be strict partitions. Then there is a $q(n)$-crystal isomorphism

\[
B(\lambda) \otimes B(\mu) \cong \bigoplus_{T \in B(\lambda); \ T \leftarrow L^\nu = L^\mu \text{ for some } \nu \in \mathcal{P}_+ \cap P^{\geq 0}} B(\text{sh}(T \leftarrow L^\mu)).
\]

Example 6.20. Let $n = 3$, $\lambda = 2\epsilon_1 + \epsilon_2$ and $\mu = 3\epsilon_1$. By Example 6.17 we get

\[
\begin{array}{c}
2 & 2 \\
1 & & 3 & 3 & 3 \\
\end{array}
\leftarrow
\begin{array}{c}
3 & 3 & 3 \\
1 & & 2 & 2 & 1 \\
\end{array}
= \begin{array}{c}
L^{3\epsilon_1} = L^{3\epsilon_1 + 2\epsilon_2 + \epsilon_3},
\end{array}
\]

and similarly we have

\[
\begin{array}{c}
3 & 2 \\
2 & & 3 & 3 & 2 \\
\end{array}
\leftarrow
\begin{array}{c}
3 & 3 \\
2 & & 2 & 2 & 1 \\
\end{array}
= \begin{array}{c}
L^{3\epsilon_1} = L^{5\epsilon_1 + \epsilon_2},
\end{array}
\]

From easy calculations, we know that except the above cases, there is no other tableau $T \in B(\lambda)$ such that $T \leftarrow L^{3\epsilon_1} = L^\nu$ for some strict partition $\nu$. Hence we conclude that

\[
B(2\epsilon_1 + \epsilon_2) \otimes B(3\epsilon_1) \cong B(3\epsilon_1 + 2\epsilon_2 + \epsilon_3) \oplus B(4\epsilon_1 + 2\epsilon_2) \oplus B(5\epsilon_1 + \epsilon_2).
\]
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