Nonlinear effects in the theory of superconductivity

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Abstract. It is proposed the non-linear cooperative interaction between the electrons through the vibration states of thermostat (the large system with big degree of freedom). As an example is studied the cooperative interaction between the cooper pairs through the non-linear lattice vibration in single and two-phonon exchange processes between the electrons. In the case of superconductivity phase transition, the cooperative interaction between the carriers in two and single phonon exchange processes depends on the temperature. This temperature dependence of exchange integral between the quasi-particles drastically changes the second order diagram as this is demonstrated in paper [1, 2]. Unlike the approach proposed in papers [1, 2], in this report it is takes into account the non-linear vibration of thermostat modes (phonon, optical modes etc.). It is observed that this effect influence the temperature dependence of the order parameter in the second order phase transition like superconductivity, super-radiance, ferromagnetism, etc.

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1. Introduction
The collective processes in condensed matter have many properties with analogies in cooperative radiation effects in quantum optics [3]. The Bardeen-Cooper-Schrieffer (BCS) [4], theory of single phonon phase transition in superconductivity was in the center of attention of many authors [5]-[8]. Following the ideas developed in these papers, the new cooperative emission phenomenon for dipole-forbidden transitions of an inverted system of radiators can be observed in processes of two-photon spontaneous emission [1, 10]. Following the ideas developed in these papers, the new theory of nonlinear superconductivity in which it is take in to account the single and two-phonon exchanges between the band electrons is proposed. Using the method of elimination of boson operators it is obtain the temperature dependence of two-quanta exchange integral between Cooper-electrons is obtained. This integral increase with increasing the temperature due to the fact that thermal field of vibration lattice stimulates the two-quantum exchanges between the electrons. This is one of the main differences between the single and two phonon superconductivity theories. The non-linear (quadratic) electron-phonon coupling dominates in Bi$_2$Sr$_2$CaCu$_2$O$_8$ [6]. At the microscopic level any non-linear electron-phonon interaction can be described in terms of the coupling of electrons to a few phonon-complexes [6]. Such coupling significantly influences the critical temperature [7].

In this paper it is proposed the new method of interaction of electronic subsystem with vibration modes of lattice. Firstly we introduced the new creation and annihilation operators for vibration modes which takes into account nonlinear (quadratic) effect in the lattice vibration modes with potential energy $V(x) = m\omega^2 x^2/2 + Kx^4$. After that we have studied the collective nonlinear effects in interaction of electronic subsystem with nonlinear thermal bath of lattice oscillators.
The interference between the single- and two-phonon exchange mechanisms in superconductivity is studied using projection operator method. It is observed the situation for which the temperature dependence part of exchange integral between the electrons can be large than constant part. In the third part of this paper it is demonstrated that for positive nonharmonic constant $K$ the order parameter firstly increase with temperature achieving the maximal value. After that it decreases as in traditional phase transition effects. In this case the correlation effect between first and second order electron-phonon interaction takes place through nonlinear effects. In opposite case $K < 0$ the superconductor state is dumped by temperature.

2. Effective Hamiltonian of Nonlinear Interaction of Electrons with vibration Lattice

The collective processes in condense matter have many analogical proprieties with cooperative radiation effects in quantum optics [3]. The new cooperative emission phenomenon for dipole-forbidden transitions of inverted system of radiator can be observed in the processes of the two-photon spontaneous emission [10]- [11].

Let us study the electronic transitions between the states $|k\rangle$ and $|k'\rangle$ in the low band of a two band superconductor with absorption or radiation of phonons. In order to neglect the one phonon transitions between these states, let us consider that the second band is situated at the distance energetically larger than $k_B T$ ($k_B$ is Boltzmann constant, $T$ is the temperature) on the Fermi level of the first band. The interaction two-band electronic system, which interacts with the vibration field of the lattice, is proposed for theoretical discussion. For the simplicity of the problem the Columbian interaction between electrons is not taken into consideration. Therefore the two band electron gas in interaction with nonlinear vibration lattice is described by the Hamiltonian

$$H = H_{el} + H_{ph} + H_{int},$$

Here the first part of the Hamiltonian (1)

$$H_{el} = \sum_{m=1}^{2} \sum_{k} \left( \varepsilon_m(k) - \mu \right) \hat{a}_{m,k}^\dagger \hat{a}_{m,k}$$

describes two-bands electron gaze in which it is introduced the creation $\hat{a}_{m,k}^\dagger$ and $\hat{a}_{m,k}$ annihilation operators in the $m$–band which satisfy the anticommutation relation $[\hat{a}_{m,k}^\dagger, \hat{a}_{m',k'}]_+ = \delta_{m,m'} \delta_{k,k'}$; $[\hat{a}_{m,k}, \hat{a}_{m',k'}]_+ = 0$. The transition matrix element takes in-to account the wave vector and spin of electrons $k = (k, \sigma)$ from the bands $n$ and $m$, $\varepsilon_m(k) - \mu$ is the energy of the electron, which is calculated from the position of the chemical potential $\mu$. In order to study the nonlinear Hamiltonian of lattice vibration let us introduced the model hamiltonian of bath of nonlinear oscillators described by the Hamiltonian

$$H_{ph} = \sum_{q} \left[ \hbar \omega_q \hat{b}_q^\dagger \hat{b}_q + \chi_q \hat{b}_q^\dagger \hat{b}_q \right],$$

Here $\hbar \omega_q$ is the phonon energy of harmonic part of the phonon Hamiltonian (2 ), the nonlinear part of this Hamiltonian is described by nonharmonic parameter $\chi_q$, which for nonlinear oscillators describes the quadratic term in the potential energy $V(x) = m\omega^2 x^2/2 + K x^4$. The phonon operators obey the following commutation relations: $[\hat{b}_q, \hat{b}_q^\dagger]_+ = \delta_{q,q}$; $[\hat{b}_q, \hat{b}_q^\dagger]_+ = 0$. As the function of the sign of interaction constant $\chi_q$ (or $K$) we can introduce the new operators in the Hamiltonian (2)

$$H_{ph} = \sum_{q} |\chi_q| \hat{K}_q^+ \hat{K}_q^-.$$
where for positive value of nonharmonic parameter $\chi_q$ we can represent the $\hat{K}$-operators through the $\hat{I}$-nonlinear operators belonging to $SU(1,1)$ algebra: $\hat{I}_q^+ = \hat{b}_q \sqrt{\frac{\hbar \omega_q}{|x_q|-1}} + \hat{a}_q \hat{b}_q^\dagger$ and $\rightarrow \hat{I}_q^- = \sqrt{\frac{\hbar \omega_q}{|x_q|}} + b_q \hat{b}_q^\dagger$, which satisfy the commutation relations

$$[I^+;I^-] = -2\delta_{q,q'}I^z_q; \quad [I^z_q,I^\pm_q] = \pm \delta_{q,q'}I^\pm_q. \quad (4)$$

In this case we have the conservation quasi vector $I^2 = (I^z_q)^2 - (I^x_q)^2 - (I^y_q)^2$ with the components $I^x_q = (I^+_q + I^-_q)/2$; $I^y_q = (I^+_q - I^-_q)/2i$; $I^z_q = \frac{\hbar \omega_q}{2|x_q|} + b_q \hat{b}_q^\dagger$. In opposite case, $\chi_q < 0$, we can introduce the new operators in lattice Hamiltonian (3) $K^+_q \rightarrow J^+_q = b_q \sqrt{\frac{\hbar \omega_q}{|x_q|}} - b_q \hat{b}_q^\dagger$; $K^-_q \rightarrow J^-_q = \sqrt{\frac{\hbar \omega_q}{|x_q|}} - b_q \hat{b}_q^\dagger$ and $K^\dagger_q \rightarrow J^\dagger_q = -\frac{\hbar \omega_q}{2|x_q|} + b_q \hat{b}_q^\dagger$, which belong to other symmetry

$$[J^+;J^-] = 2\delta_{q,q'}J^z_q; \quad [J^z,J^\pm_q] = \pm \delta_{q,q'}J^\pm_q. \quad (5)$$

the conservation law $J^2 = (J^x)^2 + (J^y)^2 + (J^z)^2$ of which shows that these operators are similar to angular momentum generators in quantum mechanics and belong to $su(2)$ algebra. These new operators take into account the sign of nonlinearity in vibration lattice subsystem and pass in boson operators for small value of $|x_q|$: $\hat{K}^+_q \simeq b_q \sqrt{\frac{\hbar \omega_q}{|x_q|}}$; $\hat{K}^-_q \simeq b_q \sqrt{\frac{\hbar \omega_q}{|x_q|}}$. Considering that the nonlinear displacement of the nuclei from equilibrium positions is proportional to the value of the new variables $\hat{K}^+_q$ and $\hat{K}^-_q$, we can follow the method of deformation potential developed in paper [9] and represent the interaction Hamiltonian between the electron and nonlinear vibrations of the lattice in the following form

$$H_{int} = -\frac{1}{\sqrt{V}} \sum_{n,m} \sum_{k_q} \{q_{m,n}a^\dagger_{m,k+q}a_{n,k}K^-_q + q^*_{m,n}a^\dagger_{n,k}a_{m,k+q}K^+_q\}, \quad (6)$$

The matrix elements of the electron-phonon interaction are known [8]. It is not difficult to shown that for for electron in the first $n = m = 1$ we obtain the following matrix element band

$$q_{11}(q) = -i \sqrt{\frac{|x_q|}{2\rho_0^2}} E_{11}(q),$$

where $E_{11} \approx \frac{N}{m_e} \int d^3r ((e_q, \nabla u^*_{1,k})(e_q, \nabla u_{1,k}) \sim \mu$. In a similar way we obtain the interaction constant for inter-band interaction

$$q_{21}(k,q) = - \sqrt{\frac{|x_q|}{2\rho_0^2}} \frac{1}{q_{70}} \int d^3r u^*_{2,k+q} (\partial_2(k+q) - \partial_1(k)) + \frac{\hbar^2}{2m_e} (k^2 - (k+q)^2) \nabla u_{1,k} \sim - \sqrt{\frac{|x_q|}{2\rho_0^2}} \frac{1}{\omega_0 \omega_q} E_{21}. \quad (7)$$

Here $\varepsilon_{21} = \varepsilon_2(k+q) - \varepsilon_1(k)$ is the energetic distance between bands one and two, $p_{21} = \hbar (e_q, \int d^3r u^*_{2,k+q} \nabla u_{1,k})$, $u_{1,k}$ is the periodical part of Bloch wave function.

In the case when the one phonon transition between the first and the second bands $q_{21}(q)$ is larger than the matrix element of the one phonon transitions in the intrinsic first band $q_{11}(q)$,
the two-phonon transition between the states $|k\rangle$ and $|k'\rangle$ become larger than the single phonon transitions. This becomes possible, when the first and the second bands are formed from the local atomic levels with different symmetries. For example, if the first band provided by the local atomic level of $S$-symmetry crystal and the second band comes from the level of $P$-symmetry, it is clear that in the case when $k \sim k'$ the intrinsic matrix element $q_{11}(q)$ is less than the matrix element between the first and the second bands $q_{21}(q)$. In this case the two-phonon transitions between the states $|k\rangle$ and $|k'\rangle$ of the first band is similar to two-photon transitions between the forbidden states $|1S\rangle$ and $|2S\rangle$ of Hydrogen-like atom. The second band in such two-phonon transitions plays the role of a virtual state $|P\rangle$.

In this chapter the two-phonon cooperative processes between the electrons of the lowest band through the virtual state of the second band are proposed. For this it is necessary firstly to eliminate the operators of the second band from Hamiltonian (1). If the second band is situated at the energetic distance larger than $\kappa_b T$, we can eliminate the electronic operators of this band from interaction Hamilton (1). In order to solve this problem let us consider the operator $\hat{A}(t)$ which does not depend on the operators of the second band. The Heisenberg equation for the mean value of this operator $\hat{A}(t)$ is

$$\langle \frac{d\hat{A}(t)}{dt} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}(t)] \rangle$$

Using the Hamiltonian (1) one can obtain the following equation for mean value of operator $\hat{A}(t)$

$$\langle \frac{d\hat{A}(t)}{dt} \rangle = \frac{i}{\hbar} \sum_k \varepsilon_1(k) \langle [a_{1,k}^\dagger(t) \hat{a}_{1,k}(t), \hat{A}(t)] \rangle$$

$$+ \frac{i}{\hbar \sqrt{V}} \sum_{k,q} \{q_{1,1}(q) \langle [a_{1,k+q}^\dagger (t) a_{1,k}(t) \hat{K}_q^- (t), \hat{A}(t)] \rangle$$

$$- H.c.t. (\hat{A}$^\dagger (t) \rightarrow \hat{A}(t)) \}$$

$$+ \frac{i}{\hbar \sqrt{V}} \sum_{k,q} \{q_{2,1}(q) \langle [a_{2,k+q}^\dagger (t) \hat{a}_{1,k}(t) \hat{K}_q^- (t), \hat{A}(t)] \rangle$$

$$+ q_{1,2}(q) \langle [a_{1,k+q}^\dagger (t) \hat{K}_q^- (t), \hat{A}(t)] a_{2,k}(t) \rangle$$

$$- H.c.t. (\hat{A}$^\dagger (t) \rightarrow \hat{A}(t)) \}.$$ (7)

As the second band plays the role of virtual states for the transitions between the electron states in the first band, it is possible to eliminate the electronic operators of the second band. In this approximation, taking in to account the Heisenberg equation for electronic operators from the second band

$$\frac{d\hat{a}_{2,k}(t)}{dt} = -i \varepsilon_2(k) \hat{a}_{2,k}(t)/\hbar$$

$$- \frac{i}{\hbar \sqrt{V}} \sum_{n=1}^{2} \sum_{q_1} \{q_{2,n}(q_1) \hat{a}_{n,k-q_1}(t) \hat{K}_{q_1}^- (t) + q_{n,2}(q_1) \hat{K}_{q_1}^+ (t) \hat{a}_{n,k+q_1}(t) \}$$

we obtain the solutions

$$\hat{a}_{2,k}(t) = \hat{a}_{2,k}(t_0) \exp \left( -i \frac{\varepsilon_2(k)(t - t_0)}{\hbar} \right)$$
\[
- \frac{i}{\hbar \sqrt{V}} \sum_{n=1}^{\mathcal{N}} \sum_{q_i} \int_{t_0}^{t} dt' \exp \left( -\frac{i}{\hbar} \varepsilon_2(k)(t - t_0 - t') \right)
\times \{ q_{2n}(q_i) \hat{a}_{n,k-q_i}(\tau) \hat{K}^{-}_{q_i}(\tau) + q_{n2}^*(q_i) \hat{K}^{\dagger}_{q_i}(\tau) \hat{a}_{n,k+q_i}(\tau) \};
\]
\[
\hat{a}^{\dagger}_{2,k}(t) = [\hat{a}_{2,k}(t)]^\dagger.
\] (8)

In the right hand site of this operators we have the nonlinear phonon operators \( K^{-}_{q_i}(\tau) \) and \( K^{\dagger}_{q_i}(\tau) \). The Hasenberg equation for such operators are
\[
\frac{d\hat{K}^{-}_{q}(\tau)}{d\tau} = \mp 2i|\chi_q| \hat{K}^{\dagger}_{q}(\tau) \hat{K}^{-}_{q}(\tau) \pm \frac{2i}{\hbar \sqrt{V}} \sum_{n,m} q^*_{m,n} a^{\dagger}_{m,n,k}(\tau) \hat{K}^{\dagger}_{q}(\tau) a_{m,k+q}(\tau)
\]
where the sings "+" or "−" corresponds to negative or positive value of enharmonically coefficient \( \chi_q \) respectively. Neglecting the interaction with electronic subsystem we can represent the solution of equation (9) in the following T-product \( \hat{K}^{-}_{q}(\tau) = T \exp[\mp 2i|\chi_q| \int_0^{\tau} \hat{K}^{\dagger}_{q}(\tau') d\tau'] \hat{K}^{-}_{q}(0) \). Considering that for short retardation time the operator \( K^{\dagger}_{q}(\tau') \) can be approximated with the expression \( (\hat{K}^{\dagger}_{q}) \approx \frac{\hbar \omega_q}{\chi_q} + (b^\dagger_q b_q) \), we can represent the solution of operator \( \hat{K}^{-}_{q}(\tau) \) in expression (8) thorough the smooth amplitude \( \hat{K}^{\dagger}_{q}(\tau) = \hat{K}^{-}_{q}(\tau) \exp[-2i|\chi_q| \langle \hat{K}_{q}^{\dagger} \rangle \tau] \). Neglecting the retardation in the smooth amplitude \( \hat{K}^{-}_{q}(\tau) \approx \hat{K}^{-}_{q}(t) \) and considering that \( |q_{12}(q)| \ll |q_{11}(q)| \), in the Born-Markoff approximation the operators \( a_{2,k}(t) \) and \( a^{\dagger}_{2,k}(t) \) can be represented through the vacuum and interaction parts
\[
a_{2,k}(t) \approx a_{2,k}(t_0) \exp \left( -\frac{i}{\hbar} \varepsilon_2(k)(t - t_0) \right)
\]
\[
\hat{a}_{2,k}^\dagger(t) \approx \frac{1}{\sqrt{V}} \sum_{q_i} \{ q_{21}(q_i) a_{1,k-q_i} \hat{K}^{-}_{q_i} - \varepsilon_1(k - q_i) \hat{a}_{1,k}(t_0) - 2 < b^\dagger_q b_q > \chi_q
\]
\[
\frac{1}{\sqrt{V}} \sum_{q_i} \{ q_{12}^*(q_i) \hat{K}^{\dagger}_{q_i} a_{1,k+q_i} - \varepsilon_2(k + q_i) \hat{a}_{1,k}(t_0) + 2 < b^\dagger_q b_q > \chi_q
\]
\[
\hat{a}_{2,k}^\dagger(t) = [a_{2,k}(t)]^\dagger.
\] (9)

It is considered that the interaction between the subsystems takes place for time moments \( \varepsilon_2/t/\hbar \gg 1 \). Taking in to account that the system temperature, \( T \) is so less that \( \varepsilon_2(k) - \varepsilon_1(k) \gg k_B T \) one can consider that the second band remain unpopulated in the processes of two-quanta interactions of electrons from the ground band
\[
\lim_{t_0 \to -\infty} \hat{a}_{2k}(t_0)\langle 0_2 \rangle = 0, \quad \lim_{t_0 \to +\infty} \langle 0_2 | \hat{a}_{2k}^\dagger(t_0) \rangle = 0,
\]
where \( |0_2 \rangle \) is the vacuum state for second band. Taking this in to account, after the substitution of the expressions (1) \( a_{2,k}(t) \) and \( a^{\dagger}_{2,k}(t) \) in equation (7), one can eliminate the \( a_{2k}(t) \) and \( \hat{a}_{2k}^\dagger(t) \) operators of the second virtual band.
\[
\langle \frac{d\hat{A}(t)}{dt} \rangle = \frac{i}{\hbar} \langle \sum_k \varepsilon_1(k) \hat{a}_{1,k}^\dagger \hat{a}_{1,k} + \sum_q \hbar \omega_q \hat{K}^{\dagger}_{q} \hat{K}^{-}_{q} \hat{A}(t) \rangle
\]
\[
+ \frac{i}{\hbar \sqrt{V}} \sum_{k,q} \{ q_{1,1}(q) \langle \hat{a}_{1,k+q}^\dagger(t) \hat{a}_{1,k}(t) \rangle \hat{K}^{-}_{q}(t) \hat{A}(t) \rangle
\]
\[
+ \frac{i}{\hbar \sqrt{V}} \sum_{k,q} \{ q_{2,1}(q) \langle \hat{a}_{1,k}^\dagger(t) \hat{a}_{1,k}(t) \rangle \hat{K}^{-}_{q}(t) \hat{A}(t) \rangle
\]
\[
+ \langle \frac{d\hat{A}(t)}{dt} \rangle = \frac{i}{\hbar} \langle \sum_k \varepsilon_1(k) \hat{a}_{1,k}^\dagger \hat{a}_{1,k} + \sum_q \hbar \omega_q \hat{K}^{\dagger}_{q} \hat{K}^{-}_{q} \hat{A}(t) \rangle
\]
In analogy with atomic system we can reduced the expression (10) to the Heisenberg equation where

\[ H = H_0 + H_I; \]  

\[ H_0 = \sum_k (\epsilon_1(k) - \mu) \hat{a}^+_1 \hat{a}_1 + \sum_q \hbar \omega_q \hat{K}_q^+ \hat{K}^-_q \] 

is the free part of Hamiltonian for the electron subsystem in the lower band "1" and phonon subsystem respectively;

\[ H_I = H_{I1} + H_{I2}, \quad H_{I1} = H_{I2} + H_{I2}^* \]  

is the interaction Hamiltonian between electron and phonon subsystems which takes in to consideration one phonon transitions

\[ H_{I1} = \frac{1}{\sqrt{V}} \sum_{k,q} \{ Q_1(q) \hat{a}^+_1 \hat{a}_1 \hat{K}^-_q + H.c. \}. \]  

Two phonon transitions and scattering effects between two quasi-energetic states of the lower band are described by interaction Hamiltonian parts respectively

\[ H_{I2}^b = -\frac{1}{V} \sum_{k,q_1} [Q_2(k,q,q_1) \hat{a}^+_1 \hat{a}_1 \hat{K}^-_q + H.c.]; \]  

\[ H_{I2}^s = -\frac{1}{V} \sum_{k,q_1} [Q_2(k,q,q_1) \hat{a}^+_1 \hat{a}_1 \hat{K}^-_q + H.c.]; \]  

\[ (10) \]
Here \( a_{kk}^\dagger \) and \( a_{kk} \) are creation and annihilation operators of electrons in the lower band of two-band proposed model; \( b_\mathbf{q}^\dagger \) \( (b_\mathbf{q}) \) are creation (annihilation) operator of phonon subsystem. The interaction constant in single and two-phonon interaction with lattice vibration are \( G_1(\mathbf{q}) = g_{11}(\mathbf{q}) \):

\[
Q_2^b(k, q_1, q_2) = \frac{g_{21}(\mathbf{q})g_{21}(\mathbf{q}_1)}{\varepsilon_2(\mathbf{k} + \mathbf{q}) - \varepsilon_1(\mathbf{k} + \mathbf{q} + \mathbf{q}_1) + \hbar \omega_{\mathbf{q}1}} + \frac{g_{21}(\mathbf{q})g_{21}(\mathbf{q}_1)}{\varepsilon_2(\mathbf{k} + \mathbf{q}) - \varepsilon_1(\mathbf{k}) - \hbar \omega_{\mathbf{q}1}}, \tag{16}
\]

\[
Q_2^s(k, q_1, q_2) = \frac{g_{21}(\mathbf{q})g_{21}(\mathbf{q}_1)}{\varepsilon_2(\mathbf{k} + \mathbf{q}) - \varepsilon_1(\mathbf{k} + \mathbf{q} - \mathbf{q}_1) - \hbar \omega_{\mathbf{q}1}} + \frac{g_{21}(\mathbf{q})g_{21}(\mathbf{q}_1)}{\varepsilon_2(\mathbf{k} - \mathbf{q}_1) - \varepsilon_1(\mathbf{k}) + \hbar \omega_{\mathbf{q}1}}, \tag{17}
\]

where \( \hbar \omega_{\mathbf{q}} = \hbar \omega_{\mathbf{q}} + 2 < b_\mathbf{q}^\dagger b_\mathbf{q} > \chi_{\mathbf{q}}. \) We emphasize that free Hamiltonian part contains only the electron states in the low band and phonon subsystems respectively. The operators \( a_{kk}^\dagger \) and \( a_{kk} \) are creation and annihilation operators of electrons in the lower band of two-band proposed model. The energetic distance between the lower and upper bands satisfies the inequality \( \varepsilon_{21} = \varepsilon_2 - \varepsilon_1 > k_B T \). The interaction of electrons in the low with phonon subsystem is described by two types of interaction Hamiltonian \( H_I^{(1)} \) and \( H_I^{(2)} \). In order to neglect the one quanta interaction in comparison with two-quanta we must consider that interaction part \( H_I^{(2)} \) is more large than \( H_I^{(1)} \), or

\[
\sum_{\mathbf{q}_1} \frac{1}{\sqrt{V}} \left| \frac{g_{21}(\mathbf{q})g_{21}(\mathbf{q}_1)}{\varepsilon_2(\mathbf{k} + \mathbf{q}) - \varepsilon_1(\mathbf{k} + \mathbf{q} \pm \mathbf{q}_1) \pm \hbar \omega_{\mathbf{q}1}} \right| \gg |g_{1,1}(\mathbf{q})|. \tag{18}
\]

In the next section the two phonon correlation Hamiltonian between the electrons which describe the new type of formation of Cooper pairs it is obtained, taking in consideration the inequality (18) the method of elimination of biboson operators of phonon subsystem is proposed.

### 3. Projection Operator

According to the Hamiltonian (1) we proposed the equations for the density matrix of the total "electron+phonon" system in the interaction picture

\[
i\hbar \frac{\partial \rho(t)}{\partial t} = [\hat{H}_I(t), \rho(t)],
\]

where

\[
\hat{H}_I(t) = \exp\left\{ \frac{i}{\hbar} \hat{H}_\sigma(t) \right\} \hat{H}_I \exp\left\{ -\frac{i}{\hbar} \hat{H}_\sigma(t) \right\}.
\]

Let \( \hat{P} \) be the projection operator for the complete density matrix \( \rho(t) \) on the vector basis of a free EMF subsystems, where \( \rho_s(t) = \hat{P} \rho(t) \), and \( \rho_b(t) = \hat{P} \rho(t) \) are the slow and rapidly oscillating parts of density matrix respectively, \( \hat{P} = 1 - \hat{P} \). It can be shown that \( \hat{P}^2 = \hat{P} \hat{P} = 0 \). Recognizing that for \( t = 0 \) an electronic subsystem does not interact with lattice vibration, we define the projection operator

\[
\hat{P} = \rho_{ph} \otimes Tr_{ph}\{\cdots\}, \tag{19}
\]

The equations for the matrix \( \rho_s(t) \) and \( \rho_b(t) \) are
\[
\frac{\partial \rho_s(t)}{\partial t} = -i \lambda \hat{P} \hat{L}_i(t) \{ \rho_s(t) + \rho_b(t) \}, \quad (20)
\]
\[
\frac{\partial \rho_b(t)}{\partial t} = -i \lambda \hat{P} \hat{L}_i(t) \{ \rho_s(t) + \rho_b(t) \}, \quad (21)
\]
where \( \lambda \hat{L}_i(t) = [\hat{H}_i(t), \ldots]/\hbar \).

Taking into account Hamiltonian parts (12), (14) and (15), we integrate equation (21) with respect to \( \rho_b(t) \) and substitute the result in equation (20)

\[
\frac{\partial \rho_s(t)}{\partial t} = \frac{-\lambda^2 \hat{P}}{t} \int_0^t d\tau \hat{L}_i(t) \hat{U}(t, t - \tau) \hat{L}_i(t - \tau) \rho_s(t - \tau), \quad (22)
\]

where

\[
\hat{U}(t, t - \tau) = T \exp \left\{-i \lambda \hat{P} \int_{t - \tau}^t d\tau \hat{L}_i(\tau) \right\}.
\]

we obtain the following explicit expressions of collective interaction part of Hamiltonian

\[
H^{\text{eff}}_I = H^{\text{eff}}_{I1} + H^{\text{eff}}_{I2} + H^{\text{eff}}_{I12},
\]

in which the first term corresponds to BCS interaction Hamiltonian

\[
H^{\text{eff}}_{I1} = -\frac{1}{V} \sum_{k,k_1} \sum_{q} \left[ \frac{\hbar |G_1(q)|^2 \omega_q (1 + \chi_q N_q)}{\hbar \omega_q^2 - (\varepsilon_1(k_1 + q) - \varepsilon_1(k))} \right] a_{q+k}^\dagger a_{q+k_1}^\dagger a_{k+q} a_{k_1},
\]

and second term corresponds to two-phonon interaction Hamiltonian

\[
H^{\text{eff}}_{I2} = -\sum_{k,k_1} \sum_{q,q_1} \left[ M^b_{q,q_1}(k,k_1)(1 + N_q + N_{q_1}) a_{k+q+q_1}^\dagger(t) a_{k_1}(t) a_{k+q} a_{k_1} a_{k_1+q+q_1}(t) \right.
\]
\[+ \left. M^s_{q,q_1}(k,k_1)(N_q - N_{q_1}) a_{k+q+q_1}^\dagger(t) a_{k_1+q_1}(t) a_{k_1} a_{k+q} a_{k+q+q_1}(t) a_{k_1}(t) \right] \quad (25)
\]

Here \( M^b_{q,q_1}(k,k_1)(1 + N_q + N_{q_1}) \) is exchange energy between two electrons from the states \( k \) and \( k' \) in the two-phonon absorption and emission processes

\[
M^b_{q,q_1}(k,k_1) = \frac{1}{2V^2} \left[ \frac{G^b_2(k, q, q_1) G^b_2(k_1, q_1, q)}{\hbar (\omega_{q_1} + \omega_q - \varepsilon_1(k_1 + q_1 + q) + \varepsilon_1(k_1))} \right.
\]
\[+ \left. \frac{G^b_2(k, q_1, q) G^b_2(k_1, q, q_1)}{\hbar (\omega_q + \omega_{q_1} - \varepsilon_1(k + q_1 + q) + \varepsilon_1(k))} \right] ; \quad (26)
\]

\( M^s_{q,q_1}(k,k_1)(N_q - N_{q_1}) \) is the exchange energy between two electrons from similar states in two-phonon scattering processes

\[
M^s_{q,q_1}(k,k_1) = \frac{1}{V^2} \frac{G^s_2(k, q, q_1) G^s_2(k_1, q_1, q)}{\hbar (\omega_{q_1} - \omega_q - \varepsilon_1(k_1 + q_1 - q) + \varepsilon_1(k_1))}.
\]
It is observed here that for elastic scattering process $\varepsilon_1(k_1 + q_1 - q)\varepsilon_1(k_1)$ and $q \sim 2k_F$ the contribution of scattering terms in superconduction is positive

$$\frac{N_q - N_{q_1}}{\omega_{q_1} - \omega_q} > 0, \quad N_{q_1} < N_q \text{ when } \omega_{q_1} > \omega_q$$

Finding the commutation expression for imaginary part of third term, we obtain the interferences part of Hamiltonian

$$H_{112}^{\text{eff}} = \frac{2n}{V} \sum_{k,k_1} \sum_{q} \frac{|G(q)G_1(-q)G_{2}^\dagger(q, -q)(1 + \chi_q N_q)| \cos[2\varphi_q - \varphi_0]}{(h\omega_q)^2 - [\varepsilon_1(k_1 + q) - \varepsilon_1(k_1)]^2} a_{k+q}^\dagger a_{k_1}^\dagger a_{k_1+q} a_{k}. \quad (27)$$

In this section the effective interaction Hamiltonian parts (24), (25) and (27) of the electrons from the first band through the single and two-phonon interactions have been obtained. The cooperative interaction between the electrons through the single and bi-quantum field is reduced not only to the simple processes of simultaneous absorption or emission of the virtual lattice quasi-particles described by operators (4),(5), but it includes the two-quantum exchanges between the electrons, too. As a consequence of the effective interaction Hamiltonian (25) the transitions between two states of the first band can take place with a one quasi-phonon absorption with $q$ wave vector and the emission of another phonon with $q_1$ wave vector. Following the traditional method proposed in the theory of superconductivity [1]-[9], we can introduce the order parameter $\Delta(T) = \frac{G(T)}{2V} \sum_{k_1} \langle a_{k_1}^\dagger a_{-k_1} \rangle$. Using the method of approximative Hamiltonian proposed in paper[2] in the following form $H = \sum_k \varepsilon_1'(k) a_k^\dagger a_k - \sum_k (\Delta(T)a_{-k}a_k + h.c.)$. In this case we have the possibility to take into account the nonlinearity connected with lattice vibration.

4. Conclusion

In this paper we discussed the effects connected with nonharmonic lattice vibration and nonlinear transitions. The nonlinear effect connected with vibration is taken for the first time in our approach. The nonlinear effects connected with multi-phonon transitions was studied earlier.

![Figure 1](image1.png)

**Figure 1.** The influence of nonharmonic parameter $\chi_k > 0$ as function of temperature in the relative units $\Delta(T)/\Delta(T_0)$, $T/T_C$
in the papers [1] and [2]. This paper have taken into account both nonlinearity in the effective Hamiltonian (23). We were interested in the contribution of nonlinearity connected with new approach in the description of lattice vibration. We observed from the Hamiltonian (24) that positive nonlinear positive nonharmonic parameter $\chi_k > 0$ corresponds to the increasing of interaction constant between the cooper electron pairs and brings to the increasing of critical temperature of superconductivity (see Fig.1). The nonlinear lattice vibration corresponds to $SU(1,1)$ algebra and with the excitation of oscillation mode $k$, the frequency of nonlinear oscillator increase too $\tilde{\omega}_k = \omega_k + 2\chi_k N_k$.

In opposite case when $\chi_k < 0$ with increasing of the temperature the interaction constant in the Hamiltonian (24) decrease. This effect is accompanied with decreasing of the frequency of the nonlinear oscillators $\tilde{\omega}_k = \omega_k - 2|\chi_k N_k|$. The proposed model in the second part give us the possibility to take into account the cooperative interaction between the cooper pairs through the non-linear lattice vibration in single and two-phonon exchange processes between the electrons. The similar problem it is proposed for quasi-spin systems used in the description of super-radiance and ferromagnetism [2]. This article shows that the nonlinearity in electronic and vibration subsystems can drastically modify the second order phase transition.

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