STEINHAUS’ LATTICE-POINT PROBLEM FOR BANACH SPACES

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Abstract. Given a positive integer \( n \), one may find a circle on the Euclidean plane surrounding exactly \( n \) points of the integer lattice. This classical geometric fact due to Steinhaus has been recently extended to Hilbert spaces by Zwoleński, who replaced the integer lattice by any infinite set which intersects every ball in at most finitely many points. We investigate the Banach spaces satisfying this property, which we call (S), and show that all strictly convex Banach spaces have (S). Nonetheless, we construct a norm in dimension three which has (S) but fails to be strictly convex.

1. Introduction

It is a well-known property of the Euclidean plane, which goes back to H. Steinhaus [7], that for any \( n \in \mathbb{N} \) one may find a circle surrounding exactly \( n \) points of the integer lattice. P. Zwoleński [8] generalised this fact to the setting of Hilbert spaces. He replaced the set of lattice points by a more general quasi-finite set, i.e. an infinite subset \( A \) of a metric space \( X \) such that each ball in \( X \) contains only finitely many elements of \( A \). His result reads as follows.

**Theorem 1** (Zwoleński [8]). If \( A \) is a quasi-finite subset of a Hilbert space \( X \), then there is a dense set \( Y \subset X \) such that for every \( y \in Y \) and \( n \in \mathbb{N} \) there exists a ball \( B \) centred at \( y \) with \( |A \cap B| = n \).

In this note we extend the above result to a larger class of Banach spaces and give a simple geometrical characterisation of what we shall call Steinhaus’ property of a Banach space \( X \):

\( (S) \) For any quasi-finite set \( A \subset X \) there exists a dense set \( Y \subset X \) such that for all \( y \in Y \) and \( n \in \mathbb{N} \) there exists a ball \( B \) centred at \( y \) with \( |A \cap B| = n \).

We shall prove that (S) is equivalent to the following condition which involves only the shape of the unit ball of \( X \):

\( (S') \) For all \( x, y \in X \) with \( x \neq y \), \( \|x\| = \|y\| = 1 \), and each \( \delta > 0 \), there exists a \( z \in X \) with \( \|z\| < \delta \) such that one of the vectors: \( x + z \) and \( y + z \) has norm greater than 1, whereas the other has norm smaller than 1.

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In other words, condition (S') means that the unit sphere of $X$ does not look locally the same at any two different points. Using that equivalence we extend Zwoleński’s result, in particular, to strictly convex Banach spaces.

2. Characterisation of Steinhaus’ property

**Theorem 2.** For every Banach space $X$ properties (S) and (S') are equivalent.

**Proof.** First, we prove that (S) implies (S'). So, assume (S) and fix any $\delta > 0$ and $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$. Consider any quasi-finite set $A \subset X$ such that $A \cap (1 + \delta)B_X = \{x, y\}$, where $B_X$ stands for the closed unit ball of $X$. According to (S) there is a $u \in X$, $\|u\| < \delta/2$, such that for some $r > 0$ the open ball $B(u, r)$ contains exactly one element of $A$. Suppose there is an $a \in A \setminus \{x, y\}$ belonging to $B(u, r)$. Then

$$r > \|a - u\| \geq \|a\| - \|u\| > (1 + \delta) - \frac{\delta}{2} = 1 + \frac{\delta}{2},$$

hence $\|x - u\| < r$, that is $x \in B(u, r)$; a contradiction. Consequently, $B(u, r)$ contains exactly one of the points $x$ and $y$, say $x \in B(u, r)$ and $y \notin B(u, r)$. Then

$$1 - \frac{\delta}{2} < \|x - u\| < r \leq \|y - u\| < 1 + \frac{\delta}{2}.$$

Suppose that $r \leq 1$, $r = 1 - \varepsilon$ with some $\varepsilon \in [0, \delta/2)$ and take any number $\rho$ satisfying

$$0 < \rho < \min\{r - \|x - u\|, \frac{\delta}{2} - \varepsilon\}.$$

Obviously, we may find $v \in X$ with $\|v\| \leq \varepsilon + \rho$ such that $\|y - (u + v)\| \geq r + \varepsilon + \rho > 1$. Then we also have

$$\|x - (u + v)\| \leq \|x - u\| + \|v\| < r - \rho + \|v\| \leq 1.$$

Therefore, putting $z = -(u + v)$ completes the proof of our claim, since $\|u + v\| < \varepsilon + \rho + \delta/2 < \delta$. We proceed similarly in the case where $r > 1$.

Now assume that the Banach space $X$ satisfies (S') and let $A \subset X$ be a quasi-finite set. For any $n \in \mathbb{N}$ set

$$G_n = \{x \in X : |A \cap B(x, r)| = n \text{ for some } r > 0\}.$$

It is evident, in view of the definition of a quasi-finite set, that each $G_n$ is an open subset of $X$. We shall prove that it is also dense.

Assume, in search of a contradiction, that there is an open ball $U = B(x_0, r)$ in $X$ not intersecting $G_n$. Rescaling $U$ if necessary, we may thus suppose that $A \cap U = \emptyset$. For any $t > 0$ let $U_t$ stand for the set $U$ scaled by $t$, i.e. $U_t = x_0 + t(U - x_0)$. Since $x_0 \notin G_n$, there is the greatest non-negative integer $m < n$ such that $|A \cap U_r| = m$ with some $r > 1$. Then for every $s > r$ we have either $|A \cap U_s| = m$ or $|A \cap U_s| > n$. Set

$$s = \inf\{r > 1 : |A \cap U_r| > n\}.$$
Then exactly $m$ points $a_1, \ldots, a_m \in A$ lie in the ball $U_s$, whereas at least two such points lie on the boundary of $U_s$; let us call them $b_1, \ldots, b_k$ ($k \geq 2$). Pick any $\delta > 0$ such that 
\[
\{a_i : 1 \leq i \leq m\} \subset B(x_0 + u, s) \subset \{a_i, b_j : 1 \leq i \leq m, 1 \leq j \leq k\}
\]
for every $u \in X$ with $\|u\| < r_0 s \delta$. Each of the vectors $(b_j - x_0)/r_0 s$ lies on the unit sphere. Applying assumption (S) for two of them (e.g. for $j = 1, 2$) and for $\delta$ chosen above, we get a $z \in X$ with $\|z\| < \delta$ such that one of the vectors: $b_j - x_0 - r_0 s z$ ($j = 1, 2$) has norm greater than $r_0 s$, whereas the other has norm smaller than $r_0 s$. By decreasing $\delta$, if necessary, we may also assume that the point $x_0 + r_0 s z$ still lies in $U$. Therefore the ball $B(x_0 + r_0 s z, s)$ with the centre in $U$ contains all of $a_i$’s ($1 \leq i \leq m$) and at least one but not all of $b_j$’s ($1 \leq j \leq k$). Repeating this construction finitely many times, we get a point $z_0 \in U$ such that for some $r > 0$ we have $|A \cap B(z_0, r)| = m + 1$. If $m + 1 = n$ we are done. Otherwise, we apply the same procedure for the new centre $z_0$ instead of $x_0$ until we get a point belonging to $G_n \cap U$.

By the Baire Category Theorem, the set $Y = \bigcap_{n=1}^{\infty} G_n$ is dense in $X$ and, obviously, for each $y \in Y$ and $n \in \mathbb{N}$ there is a ball $B$ centred at $y$ with $|A \cap B| = n$. This completes the proof of (S). \qed

3. Examples

In this section we will demonstrate some applications of Theorem 2 in concrete situations. We begin with a strengthening of Theorem 2.

Corollary 3. Every strictly convex Banach space $X$ satisfies (S).

Proof. It is enough to verify condition (S’). Let $\delta > 0$ and $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ be given. Then each point inside the segment $\overline{xy}$, joining $x$ and $y$, has norm smaller than 1, whereas each point lying on the straight line passing through $x$ and $y$, but outside $\overline{xy}$, has norm larger than 1. Therefore, any point $z \in X$ satisfying $0 < \|z\| < \delta$ and $x + z \in \overline{xy}$ does the job. \qed

Now, we will see that strictly convex spaces do not exhaust the whole class of Banach spaces satisfying Steinhaus’ condition. In fact, these two classes differ already in dimension three.

Example. There is a norm $\|\cdot\|$ in $\mathbb{R}^3$ such that $(\mathbb{R}^3, \|\cdot\|)$ contains $\ell^2_\infty$ isometrically (and hence is not strictly convex), nonetheless it satisfies condition (S). We shall briefly describe the idea of how to define such a norm as a Minkowski functional of a certain convex symmetric set $B \subset \mathbb{R}^3$. Let us construct the boundary of $B$ starting with putting the square with vertices $(\pm 1, \pm 1, 0)$ onto the $xy$-plane, so that we would have a subspace isometric to $\ell^2_\infty$. Next, we define the curves on the boundary of $B$ which join the points $(x, 1, 0)$ and $(0, 0, 1)$ for every $x \in [-1, 1]$. We do it in such a way that for all $-1 \leq x_1 < x_2 \leq 1$ the curve corresponding to $x_1$ is more flat at $(x_1, 1, 0)$ than the one corresponding to $x_2$ is at $(x_2, 1, 0)$.

For any fixed $x \in [-1, 1]$ the curve starting from $(x, 1, 0)$ lies on the plane spanned by $(x, 1, 0)$ and $(0, 0, 1)$ and, on this plane, it joins the points $(\sqrt{x^2 + 1}, 0)$ and $(0, 1)$, where
the coordinates are: \( t = \sqrt{x^2 + y^2} \) and \( z \). We define that curve by the equation

\[
z_\alpha(t) = 1 - \left( \frac{t}{\sqrt{x^2 + 1}} \right)^\alpha \quad \text{for} \quad 0 \leq t \leq \sqrt{x^2 + 1},
\]

and if the parameter \( \alpha \) behaves appropriate as a function of \( x \in [-1, 1] \), then we will get the flattening effect announced earlier. We will see in a moment what should be a suitable choice for \( \alpha(x) \). In this way we define a surface over the triangle \( \Delta_1 = \{(x, y) \in \mathbb{R}^2 : y \geq |x|\} \) by the formula

\[
z(x, y) = 1 - \left( \frac{x^2 + y^2}{x^2 + 1} \right)^{\alpha(x)/2} \quad \text{for} \quad (x, y) \in \Delta_1.
\]

The rest of the square is divided into similar three triangles \( \Delta_2, \Delta_3 \) and \( \Delta_4 \), arranged clockwise on the \( xy \) plane. We apply the same procedure for each of them taking care of the behaviour of the parameter \( \alpha \) likewise we did for \( \Delta_1 \). Finally, we extend our surface below the \( xy \)-plane by reflecting it with respect to the origin. The part lying over the triangles \( \Delta_1 \) and \( \Delta_2 \) (for a suitable choice of the function \( \alpha \)) is depicted in Figure 1.

![Figure 1. The part of the boundary of \( B \) lying above \( \Delta_1 \cup \Delta_2 \).](image)

Let \( \| \cdot \| \) be the Minkowski functional of \( B \). In order to show that the Banach space \( (\mathbb{R}^3, \| \cdot \|) \) satisfies condition \( (S') \) it is enough to consider any two different vectors having norm 1 which lie on the same edge of the square with vertices \((\pm 1, \pm 1, 0)\), since for any other two vectors from the boundary of \( B \) we may apply the same simple argument as in the case of a strictly convex space (provided, of course, our choice of \( \alpha(x) \) gives a strictly concave function \( z(x, y) \)). With no loss of generality we may also suppose these two vectors lie inside the segment \([-1, 1] \times \{1\} \times \{0\}\); let us call them \((x_1, 1, 0)\) and \((x_2, 1, 0)\), where \(-1 < x_1 < x_2 < 1\). Fix also any \( \delta > 0 \) and for any \( x \in [-1, 1] \) define \( \xi(x) \) to be the unit vector tangent to the curve starting from \((x, 1, 0)\) and lying on the plane spanned by \((x, 1, 0)\) and \((0, 0, 1)\), above the \( xy \)-plane. For any such \( x \) and \( \alpha = \alpha(x) \) we have \( z'_\alpha(\sqrt{x^2 + 1}) = -\alpha/\sqrt{x^2 + 1} \). Hence, if we take \( \alpha \) of the form \( \alpha(x) = \beta(x)\sqrt{x^2 + 1} \) with any strictly increasing map \( \beta: [-1, 1] \to \mathbb{R} \) satisfying \( \beta(x) > 1/\sqrt{x^2 + 1} \) for \(-1 \leq x \leq 1\), then the flattening effect works. Namely, the angle between \( \xi(x_1) \) and the \( xy \)-plane is
smaller than the one between $\xi(x_2)$ and the $xy$-plane. (Notice that for any $x \in [-1,1]$ the angle between $\xi(x)$ and the segment joining the origin with $(x,1,0)$ equals the angle between $\xi(x)$ and the the $xy$-plane.) Take a vector $z$ with $\|z\| < \delta$ which is “almost” parallel to $\xi(x_2)$ in such a way that $(x_2,1,0) + z$ goes inside the ball $B$. Let $P$ be the foot of perpendicular of $(x_1,1,0) + z$ on the $xy$-plane and let the line passing through the origin and $P$ meet the segment $[-1,1] \times \{1\} \times \{0\}$ at some point $(x_0,1,0)$. Of course, $x_0 < x_1$ and $\xi(x_2) = (x_0 - x_1,0,0) + z'$ with $z'$ lying on the plane spanned by $(x_0,1,0)$ and $(0,0,1)$. By decreasing the length of $z$ we may get $x_0$ arbitrarily close to $x_1$, thus the angle between $z'$ and the $xy$-plane may be almost the same as the one given by $z$. But it is greater than the angle between $\xi(x_0)$ and the $xy$-plane and this in turn may be arbitrarily close to the angle given by $\xi(x_1)$. Consequently, by decreasing the length of $z$ we may guarantee that the vector $x_1 + z$ goes outside the ball $B$.

**Remark 4.** The above example shows that there is a Banach space $X$ satisfying $(S')$, but containing two such vectors $x, y \in X$, with $\|x\| = \|y\| = 1$, that for some $\delta > 0$ it is impossible to increase $\|x\|$ and decrease $\|y\|$ by adding to $x$ and $y$ the same vector $z \in X$ with $\|z\| < \delta$. In other words, condition $(S')$ cannot be strengthened by claiming which one of $x + z$ and $y + z$ has norm greater than $1$.

**Remark 5.** Of course, if $X$ is a non-strictly convex Banach space with dim $X = 2$, then condition $(S')$ fails to hold. Therefore, Steinhaus’ condition is equivalent to strict convexity in the class of Banach space with dimension at most 2.

The next corollary demonstrates that the classical $L_p(\mu)$-spaces for atomless measures $\mu$ also satisfy Steinhaus’ condition, giving thus another example of a non-strictly convex space with this property (the case where $p = 1$).

**Corollary 6.** For every atomless measure space $(\Omega, \Sigma, \mu)$, and each $1 \leq p < \infty$, the space $L_p(\mu)$ satisfies $(S)$.

**Proof.** By Luther’s theorem [6], there is a decomposition $\mu = \mu_1 + \mu_2$ with $\mu_1$ being semi-finite (i.e. for each $A \in \Sigma$, $\mu_1(A) = \infty$, there is a subset $B \in \Sigma$ of $A$ with $0 < \mu_1(B) < \infty$) and $\mu_2$ being degenerate (i.e. the range of $\mu_2$ is contained in $\{0, \infty\}$). The space $L_p(\mu)$ is then isometric to $L_p(\mu_1)$ (for any $f \in L_p(\mu)$ we have $\mu_2\{x: f(x) \neq 0\} = 0$, thus the identity map yields the desired isometry). Therefore, we may (and do) suppose that $\mu$ is semi-finite.

Fix two functions $f, g \in L_p(\mu)$ with $f \neq g$ and $\|f\| = \|g\| = 1$, and let $\delta > 0$ be given. Interchanging $f$ and $g$, if necessary, we may assume that there is a set $F \in \Sigma$ such that $0 < \mu(F) < \infty$ and $f(x) > g(x)$ for every $x \in F$. Since

$$F = \bigcup_{n=1}^{\infty} \left\{ x \in F: f(x) > g(x) + \frac{1}{n} \right\},$$

we may also suppose that for some $\varepsilon > 0$ and all $x \in F$ we have $f(x) > g(x) + \varepsilon$. Approximating $f$ and $g$ by step functions we may find a measurable set $F' \subset F$ with...
\( \mu(F') > 0 \) and some \( c_f, c_g \in \mathbb{R} \) such that
\[
|f(x) - c_f| < \frac{\varepsilon}{5} \quad \text{and} \quad |g(x) - c_g| < \frac{\varepsilon}{5} \quad \text{for} \quad x \in F'.
\]
Hence, \( c_f > c_g + 3\varepsilon/5 \) and \( m_f > M_g + \varepsilon/5 \), where \( m_f = \inf f(F') \) and \( M_g = \sup g(F') \). We have three possibilities:

(i) \( m_f > 0 \) and \( M_g \geq 0 \),
(ii) \( m_f > 0 \) and \( M_g < 0 \),
(iii) \( m_f \leq 0 \) and \( M_g < 0 \).

With no loss of generality suppose that either (i) or (ii) occurs (the case (iii) is analogous to (i)). Then there is a positive number \( d \) such that \( |m_f - d| < m_f \) and \( |M_g - d| > |M_g| \); indeed, in the former case we shall take any \( d \in (2M_g, 2m_f) \), while in the latter one any arbitrarily small \( d \) does the job.

Now, observe that
\[
|f(x) - d| < |f(x)| \quad \text{and} \quad |g(x) - d| > |g(x)| \quad \text{for} \quad x \in F'.
\]
Indeed, for the first inequality note that in the case where \( f(x) \geq d > 0 \) it holds trivially true, while in the opposite case we have
\[
|f(x) - d| = d - f(x) \leq d - m_f \leq |d - m_f| < m_f \leq |f(x)|.
\]
For the other one observe that since \( M_g < d \) (recall \( |M_g - d| > |M_g| \)), we have \( g(x) < d \), thus in the case where \( g(x) > 0 \) we have
\[
|g(x) - d| = d - g(x) > d - M_g = |d - M_g| > |M_g| \geq g(x) = |g(x)|,
\]
whereas in the case where \( g(x) < 0 \) the inequality is trivial.

By the Darboux property of finite atomless measures (\textit{e.g.} see [5, \S 215]), there is a measurable set \( H \subset F' \) with \( 0 < \mu(H) < \delta/|d| \). Then \( \|d|1_H\| < \delta \), where \( 1_H \) stands for the characteristic function of \( D \), while inequalities \( [1] \) imply \( \|f + |d|1_H\| < 1 \) and \( \|g + |d|1_H\| > 1 \). This completes the proof of (S') and, consequently, also of (S).

Theorem 2 gives an immediate solution of Steinhaus' problem for \( C(K) \)-spaces (of all scalar-valued continuous functions defined on a compact Hausdorff space \( K \)).

**Corollary 7.** If \( K \) is a compact Hausdorff space with at least two points, then \( C(K) \) does not have (S).

**Proof.** Pick any two distinct points \( u, v \in K \), and their disjoint neighbourhoods \( U \) and \( V \). Since \( K \) is completely regular, there is a continuous map \( \varphi : K \to [0, 1] \) such that \( \varphi(u) = 1 \) and \( \varphi|_{K \setminus U} = 0 \). Similarly, since \( K \setminus U \) is also completely regular, there is a continuous map \( \varphi_1 : K \setminus U \to [0, 1/2] \) such that \( \varphi_1(v) = 1/2 \) and \( \varphi_1|_{K \setminus (U \cup V)} = 0 \). Then the mapping \( \psi : K \to [0, 1] \) defined by
\[
\psi(x) = \begin{cases} 
\varphi(x) & \text{for} \ x \in U, \\
\varphi_1(x) & \text{for} \ x \in K \setminus U,
\end{cases}
\]
is continuous and, of course, \( \varphi \neq \psi \). So, both functions \( \varphi \) and \( \psi \) belong to the unit sphere of \( C(K) \), but for any \( \delta \in (0, 1/2) \) condition (S') is violated.

\( \square \)
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