ON THE RELATION OF CLIFFORD-LIPSCHITZ GROUPS TO
\textit{q}-SYMMETRIC GROUPS

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It can be shown that it is possible to find a representation of Hecke algebras within Clifford algebras of multivectors. These Clifford algebras possess a \textit{unique} grading and a possibly \textit{non-symmetric} bilinear form. Hecke algebra representations can be classified, for non-generic \textit{q}, by Young tableaux of the symmetric group due to the isomorphy of the group algebras for \textit{q} \to 1. Since spinors can be constructed as elements of minimal left (right) ideals obtained by the left (right) action on primitive idempotents, we are able to construct \textit{q}-spinors from \textit{q}-Young operators corresponding to the appropriate symmetry type. It turns out that an anti-symmetric part in the Clifford bilinear form is necessary. \textit{q}-deformed reflections (Hecke generators) can be obtained only for \textit{even} multivector aggregates rendering this symmetry a composite one. In this construction one is able to deform spin groups only, though not pin groups. The method is closely related to a projective interpretation.

1 Introduction

We recently showed that it is possible to find linear representations of Hecke algebras within Clifford algebras of multivectors. These Clifford algebras are characterized by an arbitrary bilinear form, while a quadratic form does possess, in the case of characteristic zero, a symmetric bilinear form only. The aim of this note is to show in which way \textit{q}-spinor representations and Hecke algebra representations are united due to \textit{q}-deformed Young operators. \textit{q}-deformed spinors and \textit{q}-spin groups result.

Starting from non-deformed algebras and embedding the deformed ones, we contrast recent work on \textit{q}-spinors based on \textit{q}-deformed Clifford algebras. Our method is conservative in the sense that the usual interpretation of the undeformed Clifford algebra remains valid, while the \textit{q}-deformed objects exhibit a novel composite structure. Computations were performed by \textit{Clifford}, a Maple V add-on.

2 Hecke algebra representations within Clifford algebras of multivectors

Let $\wedge(V)$ be the Grassmann algebra built over the linear space $V$ of dimension $2n$. We denote the linear space underlying $\wedge(V)$ as $F\wedge(V)$. A Clifford map
\( \gamma \) is a linear map from \( V \) into a unital associative algebra \( A \), which satisfies
\[
\gamma^2 = \gamma \gamma = Q(x)I.
\]
This can be polarized to \( \gamma_x \gamma_y + \gamma_y \gamma_x = 2G(x, y) = Q(x + y) - Q(x) - Q(y) \), where \( G \) is the symmetric bilinear form associated to the quadratic form \( Q \). We observe with Chevalley that for \( x \in V \) a linear combination \( \gamma_x := x \cdot + x \wedge \) is a Clifford map operating on \( V \). For details see [1]. Lifting this map to \( F \wedge(V) \) using
\[
\begin{align*}
i) & \quad x \cdot y = < x \mid y > = B(x, y) \\
ii) & \quad x \cdot (u \wedge v) = (x \cdot u) \wedge v + (u \cdot (x \cdot v)) \\
nii) & \quad (u \wedge v) \cdot w = u \cdot (v \cdot w),
\end{align*}
\]
where \( x, y \in V \), \( u, v, w \in \wedge(V) \) and \( \cdot \) is the main involution \( \hat{V} = -V \), results in the Clifford algebra of multivectors. Note, \( B \) is an arbitrary bilinear form, however the anticommutator relation is not altered by an antisymmetric part of the bilinear form \( B \). The corresponding algebras are isomorphic as Clifford algebras, not however, as graded algebras.

By choosing a basis \( \{ e_i \} \) and a suitable bilinear form: (appropriate indices only, \( i, j \in \{ 1, \cdots, 2n \} \))
\[
[B_{ij}] = [B(e_i, e_j)] := q[\delta_{i,j-n}] + \lambda[\delta_{i,n-j-1}] + [\delta_{i-n,j}] + \frac{q}{A}[\delta_{i-n-1,j}],
\]
we are able to find \( n \) bivector elements \( b_i := e_i \wedge e_{n+i} \) satisfying the Hecke relations
\[
\begin{align*}
i) & \quad b_i^2 = (1 - q)b_i + q \\
nii) & \quad b_i b_j = b_j b_i \quad |i - j| \geq 2 \\
nii) & \quad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}
\end{align*}
\]
thereby providing a Hecke algebra representation.

3 \( q \)-spinor representations from \( q \)-Young operators

Spinors can be defined as elements of minimal left (right) ideals of Clifford algebras. If \( f \) is a primitive (irreducible) idempotent, one has \( S = Cf, f^2 = f \). A primitive idempotent element might be written in the form \( f = \prod 1/2(1 \pm x_i) \) with \( [x_i, x_j] = 0, x_i^2 = 1 \) and a choice of signs. From i) in (3) we obtain mutually annihilating projectors due to \( (q + b_i)(1 - b_i) = 0 \), hence
\[
P_{i}^{q+} := \frac{1}{1 + q}(q + b_i) \quad P_{i}^{q-} := \frac{1}{1 + q}(1 - b_i).
\]
However, $P_i^q \pm P_j^q$ do not necessarily commute iff $i = j \pm 1$.

We will specialize to $H_3(q)$ resp. $S_3$ in the sequel, so $n = 2$ and $O(V, B) \cong O_{2,2}(\mathbb{R}, B)$. An algebraic basis of $H_3(q)$ is given by the elements \{I, b_1, b_2, b_{12}, b_{21}, b_{121}\}, using the abbreviations $b_{ij} := b_ib_j$ etc. These are only some elements of the even subalgebra $O^{+}_{2,2}$ of $O(\mathbb{R}^4, B)$. But $e_{1\lambda 2} = e_1 \wedge e_2$ and $e_{3\lambda 4}$ can not be built from the $b_is$. We thus have $H_3(q) \subset O^+(\mathbb{R}^4, B)$. This is however not true for higher dimensions.

Composing the row symmetrizer $R^q$ and column antisymmetrizer $C^q$ we can construct the following Young operators $Y_k = C^q_{\ell} R^q_k$, in accordance with

\[
Y_{\text{sym}} := \frac{1}{q^2 + q + 1} \left\{ q^2 I + q(e_{1\lambda 3} + e_{2\lambda 4}) - e_{1\lambda 2\lambda 3\lambda 4} \right\}, \\
Y_{12}^3 := \frac{1}{(q+1)(q^2 + q + 1)} \left\{ qI + e_{1\lambda 3} - q(q+1)e_{2\lambda 4} + (q+1)e_{1\lambda 2\lambda 3\lambda 4} \right\}, \\
Y_{13}^2 := \frac{1}{(q+1)(q^2 + q + 1)} \left\{ q(2q+1)I - q^2 e_{1\lambda 3} + (q+1)e_{2\lambda 4} - \lambda(q+1)e_{1\lambda 4} - q^2 e_{1\lambda 3} + (q+1)e_{1\lambda 2\lambda 3\lambda 4} \right\}, \\
Y_{\text{asym}} := \frac{1}{q^2 + q + 1} \left\{ (1-q)I - e_{1\lambda 3} - e_{2\lambda 4} + \lambda e_{1\lambda 4} + \frac{q}{\lambda} e_{2\lambda 3} - e_{1\lambda 2\lambda 3\lambda 4} \right\}. 
\]

They are normalized, mutually annihilating $Y_i Y_k = \delta_{ik} Y_k$ and complete: $I = Y_{\text{sym}} + Y_{12}^3 + Y_{13}^2 + Y_{\text{asym}}$. In the $H_3(q)$ case $q$ has to be unequal to $\exp(2\pi i/n)$ for $n \in \{1, 2, 3\}$: non generic $q$. These Young operators provide $q$-idempotents for the construction of left ideals, which carry a representation of $H_3(q)$. The dimension of such left ideals follows from the degeneracy of the eigenvalues of the transposition class-sum $C_n$ e.g. $C_3 := b_1 + b_2 + 1/q b_{121}$ which is a central operator and characterizes uniquely all representations of $H_n(q)$. We obtain two one-dimensional representations, the roots of $i$ in (6) belonging to $Y_{\text{sym}}$ and $Y_{\text{asym}}$, while $Y_{12}^3$ and $Y_{13}^2$ each generate a two-dimensional space. The left-regular representation is the direct sum of these spaces

\[
S_{\text{reg}} := Y_{\text{sym}} \oplus \left( \begin{array}{c} Y_{12}^3 \\ b_2 Y_{12}^3 \end{array} \right) \oplus \left( \begin{array}{c} Y_{13}^2 \\ b_2 Y_{13}^2 \end{array} \right) \oplus Y_{\text{asym}}. 
\]

However, for the whole, even even part of the Clifford algebra this is not a faithful spinor representation, since $e_{1\lambda 2} S_{\text{reg}} = 0 = e_{3\lambda 4} S_{\text{reg}}$ annihilate this space.

Dealing with the even elements only, we have to note that $O^{+}_{2,2} \cong O_{1,2}$ has a real 4-dimensional spinor space: $O_{1,2} \cong \mathbb{R}(4)$. However, our mixed
symmetry operators can not be added to such a space, but we can construct it by adjoining an odd element: \( u = e_1 + e_3, \ u^2 = 1 + q. \)

\[ S_{3}^{12} := \left( \begin{array}{c} Y_{3}^{12} \\ b_2 Y_{3}^{12} \end{array} \right) \oplus \left( \begin{array}{c} Y_{3}^{12} \\ b_2 Y_{3}^{12} \end{array} \right) u \]  

(8)

represents all 6 bi-vector elements and thus \( \Omega_{2,2}^+ \cong \Omega_{1,2} \). We have thus succeeded in finding a \( q \)-deformed mixed \( \frac{12}{3} \)-symmetry type spinor in the even sector (spinor sector) of \( \Omega_{2,2} \). The \( \frac{13}{2} \)-type is analogous.

4 \( q \)-reflections and \( q \)-spin groups

A theorem of Cartan states, that \( n \)-dimensional pseudo orthogonal groups can be constructed by less or equal than \( n \) reflections \( s_i \). The action of such a reflection might be defined as a conjugation \( s_i(x) = s_i x s_i^{-1} \) or on spinors as left translation \( s_i(\psi) := s_i \psi \). Furthermore, reflections satisfy the relations (4) for \( q \rightarrow 1 \). Taking this into consideration one has \( s_i \) involutory: \( s_i s_i = \text{Id} \) and \( s_i^{-1} = s_i \). Now the \( b_i \) might serve as \( q \)-reflections in a \( q \)-analogue of Cartans theorem. However, since the \( b_i \) are not involutions some care is needed.

Fortunately a neat definition of an (anti)involution on the algebra is given by Clifford algebraic considerations. With the new main involution \( (V \wedge V)^* = -V \wedge V \) and the reversion (of products) \( (AB)^* = \bar{A} \bar{B} \), we compose the anti-involution \( \alpha_{\epsilon}(AB) = \alpha_{\epsilon}(B)\alpha_{\epsilon}(A) \), where \( \epsilon = \pm 1 \) incorporates the main involution or not. Using the reversion of the \( e_i-\)Clifford elements, we obtain:

\[ \overline{b}_i := \epsilon \bar{b}_i = \epsilon [(1 - q) - b_i] \]
\[ b_i \overline{b}_i = \epsilon b_i [(1 - q) - b_i] = -\epsilon q \Rightarrow \overline{b}_i^{-1} = -\frac{\epsilon q}{\bar{b}_i}. \]  

(9)

Such conjugations are connected to special elements and the structure of Clifford algebras in a natural way. Exactly this conjugation interchanges the roots of i) in \( \Omega \) suggesting the choice \( \epsilon = -1 \). Furthermore, this Clifford algebraic antinvolution interchanges the row and column-parts in the Young operators \( \alpha_{\epsilon}(C^i_\bar{\epsilon}) \cong R^i_\bar{\epsilon}, \ \alpha_{\epsilon}(R^i_\bar{\epsilon}) \cong C^i_\bar{\epsilon} \), as is easily seen in \( \alpha_{\epsilon}(P^q_{\pi_{\bar{\epsilon}}}) = \epsilon P^q_{\pi_{\bar{\epsilon}}}. \)

The Hecke-versor subgroup is thus given by \( a(x) = axa^{-1} \), or \( a(\psi) = a\psi \) on spinors iff \( a^{-1} := a/(a\pi) \) exists. The whole Clifford-Lipschitz or versor group might result by adding an odd element once again, which now however has to be central and will be the special element mentioned above.

To see that products of the \( b_i \) constitute an invariance group of an inner product, we define the adjoint of \( A = b_1 \cdots b_n \) to be \( A^* := \overline{A} = \overline{a_n} \cdots \overline{a_1} \).
The Clifford–Lipschitz group of versors can then be defined with $\Phi_q^\epsilon(x,y) := x \epsilon y$ as

$$\Gamma^q_+ := \{x \mid x \in \mathcal{O}^+(V,B), \Phi_q^\epsilon(x,y) = 1\},$$

which reduces to the spin group for $q \to 1$. Furthermore, one can relate to such invariance groups an inner product, as the restriction of $\Phi_q^\epsilon$ to $\mathcal{S}_q$ shows, see for $q = 1$. This completes the claim that we have indeed constructed $q$-spin groups and that our ideals contain $q$-spinors. The $h$-reflections generate the $q$-Weyl group. From (11) we see that $q$-pin groups, odd versor groups, cannot be constructed in this way.

The projective interpretation follows from the projective split defined by Hestenes (12), which allows the bi-vectors to be the projective points (line geometry). This is also reflected in the $2 \times 2$ block-matrix structure of $B$.

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