Can string bits be supersymmetric?

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Abstract

We search for the possibility to have supersymmetric string bits at finite discretization $J$. From a general setup we find that the string bits can be made supersymmetric modulo a single defect mode which is not expected to have any sensible effect in the continuum limit.

1 Introduction

IIB string in plane wave background \[1,2\] has drawn a considerable interest due to its relation to AdS/CFT correspondence (see \[3\] for a comprehensive review and \[4\] for recent progress). It arises as a limit (Penrose limit) of AdS geometry \[5,6,7\]. This corresponds to the limit in AdS/CFT correspondence when one considers Super–Yang–Mills (SYM) operators with large supersymmetry $R$-charge $J$ \[8,9\]. It appears that in this limit one can extend the perturbative analysis of the SYM model to large values of 't Hooft couplings. The latter allows one to have a reliable perturbation knowledge of both sides of the correspondence.

The string bit model \[10,11,12\] was proposed as a useful tool to describe the stringy plane wave dynamics at finite $J$. In this model the continuum string is replaced by a finite number of elastically interacting points — string bits. (This number is identified with the $R$-charge $J$ in SYM model.) Although it succeeded to produce a good agreement with the SYM results in the bosonic sector, it appeared to contain inconsistencies \[13\], due to a spectrum doubling problem in the fermionic sector (see e.g. \[14\] for details related to fermion doubling).

Although the problems related to doubling was discussed earlier in the context of flat space superstrings in \[15\], the existence of a consistent discretization of the pp-wave string is particularly important, as it justifies the consistency of the correspondence at finite values of the $R$-charge $J$, as well as its limit $J \to \infty$.

A way to cure the fermion doubling by a modification in the fermionic sector of the model was proposed recently in \[16\]. For this the staggered, or Kogut–Susskind fermion approach \[17\] was used. The idea of the method consists

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in the reduction of the total number of discrete fermions by placing fermions of different chiralities at different lattice sites, in such a way that the fermion doubling at the end produces the right spectrum. The drawback of this approach is that it violates supersymmetry at finite values of lattice discretization. On the other hand, the finite $J$ sector of Yang–Mills theory is explicitly supersymmetric, hence one would expect that it should be described by a supersymmetric effective discrete model of superstrings, even at finite charge $J$. This would imply the existence of a supersymmetric string bit model at finite discretization parameter. The aim of the present work is to study the “natural” limits in existence of such a model.

The plan of the paper is as follows. First, we very briefly introduce the string bit model and the fermion doubling problem. Then, under quite generic assumptions we analyze the possible forms of the fermionic part of the Hamiltonian, which may lead to a supersymmetric model at finite $J$ having the correct fermionic spectrum in the continuum limit.

## 2 String bits and doublers

The naive string bit model after fixing the permutation symmetry is given in the one-string sector by the Hamiltonian

$$H_{naive} = \sum_{n=0}^{J-1} \left[ \frac{1}{2} \left( p_n^2 + x_n^2 \right) + \frac{1}{2a} (x_{n+1}^i - x_n^i)^2 \right] - \frac{i}{2} \sum_{n=0}^{J-1} \left[ (\theta_n \theta_{n+1} - \bar{\theta}_n \bar{\theta}_{n+1}) - 2a \bar{\theta}_n \Pi \theta_n \right],$$  \hspace{1cm} (1)

and commutation relations

$$[p_n^i, x_n^j] = -\frac{i}{a} \delta^{ij} \delta_{mn}, \quad \{\theta_n^a, \theta_m^b\} = \frac{1}{a} \delta^{ab} \delta_{mn}, \quad \{\bar{\theta}_n^a, \bar{\theta}_m^b\} = \frac{1}{a} \delta^{ab} \delta_{mn},$$ \hspace{1cm} (2)

where $i = 1, \ldots, 8$ are vector and, respectively, $a = 1, \ldots, 8$ spinor indices of $SO(8)$ appearing in the light-cone quantization of the pp-wave string. $\Pi$ is the matrix in $SO(8)$ spinor space given in terms of $16 \times 16$ dimensional $\gamma$-matrices in chiral representation by

$$\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4, \quad \Pi^2 = 1.$$ \hspace{1cm} (3)

As we have shown in [13], the model (1), (2) suffers from fermion spectrum doubling. The doubling is most easily seen if one rewrites the Hamiltonian in terms of Fourier modes $x_k$, $p_k \theta_k$ and $\bar{\theta}_k$.

$$f_n = \frac{1}{\sqrt{J}} \sum_{k=-J/2}^{J/2-1} f_k e^{2\pi i k n/J},$$ \hspace{1cm} (4)

where $f_n$ stands for $x_n$, $p_n \theta_n$ and $\bar{\theta}_n$ while $f_k$ represents their Fourier modes. Then the fermionic part of the Hamiltonian takes the form,

$$H_f = \frac{1}{2} \sum_{k} \left[ \sin \frac{2\pi k}{J} (\theta_{-k} \theta_k - \bar{\theta}_{-k} \bar{\theta}_k) - 2a \bar{\theta}_{-k} \Pi \theta_k \right].$$ \hspace{1cm} (5)

\[1\] We abide by the notations of [1].
From eq. (5) one can see that, due to an extra zero of the sin function at $k = J/2$, there are additional propagating modes of the momenta at the “edge of the Brillouin zone” $|k| \sim J/2$. Although the wavelengths of such modes are of the order of lattice size, which corresponds to discontinuous fields in the continuum limit $J \to \infty$, they fail to decouple since the energy they carry is as small as that of “good” modes, which correspond to continuous functions.

In fact, in analogy to the flat space case studied in [15], the unwanted modes can be removed by the so-called Wilson term, i.e., the following second derivative term:

$$
\Delta H_W = -i \sum_k \left[(\hat{\theta}_{n+1} - \hat{\theta}_n)\Pi(\theta_{n+1} - \theta_n)\right].
$$

(6)

For slowly oscillating modes this produces a correction of the order of the discretization spacing $a$, while for doubling modes it produces a mass of the order of $1/a$ and therefore it makes them decouple, in the continuum limit. This approach can be seen as an alternative to that of Ref. [16], and as it is not difficult to see that it suffers from the same defect: at finite $J$, supersymmetry is broken.

### 3 Seeking supersymmetry

Let us fix the general setup under which we analyze the possibilities to build a supersymmetric Hamiltonian. In fact we will act in a more or less straightforward way: we choose the original bosonic Hamiltonian and try to find its fermionic extension through probing the definitions of the supercharges.

In momentum representation the general Ansatz for the Hamiltonian is chosen to be

$$
H = \frac{1}{2} \sum_k \left[a \left(|p_{ik}|^2 + (\hat{\partial}_k \hat{\partial}_k + 1)|x_k|^2\right)
- i \left((1/2)(\hat{\partial}_k - \hat{\partial}_k)(\theta_k \theta_k - \tilde{\theta}_k \tilde{\theta}_k) - 2a \tilde{\theta}_k \Pi \theta_k\right)\right],
$$

(7)

where the momentum function

$$
\tilde{\partial}_k = (e^{2\pi i k/J} - 1)
$$

(8)

gives the bosonic derivative, while $\hat{\partial}_k$ is the fermionic one to be found.

If one chooses the fermionic momentum function $\hat{\partial}_k$ to be equal to the bosonic one $\partial_k$ as in the naive case, then, due to the fact that $\hat{\partial}_k = \partial_k^*$, the energy spectrum of fermions will depend on the imaginary part of the momentum function, in contrast with the bosons, whose energy is given by the absolute value of the momentum function. Generically, the imaginary part can have more zeroes than the absolute value, which is what actually happens, thus the extra zeroes are the source of doubling.

In order to find the fermionic Hamiltonian, let us adopt the following strategy. Let us consider an Ansatz for the supercharges,

$$
Q = a \sum_k \left[p^-_{-k} \gamma^i \theta_k - x^-_{-k} \gamma^i \Pi \theta_k + \partial_{-k} x^-_{-k} \gamma^i \theta_k\right],
$$

(9a)

$$
\hat{Q} = a \sum_k \left[p^-_{-k} \gamma^i \tilde{\theta}_k + x^-_{-k} \gamma^i \Pi \theta_k - \partial_{-k} x^-_{-k} \gamma^i \tilde{\theta}_k\right],
$$

(9b)

where the unknown momentum function $\bar{\partial}_k$ appears. Of course, the two unknown momentum functions $\bar{\partial}_k$ and $\hat{\partial}_k$ are not independent, they are related to each other as well as to the bosonic momentum function through the would-be supersymmetry algebra

$$\{Q_a, Q_b\} = 2\delta_{ab}(H + P), \quad \{\bar{Q}_a, \bar{Q}_b\} = 2\delta_{ab}(H - P), \quad \{Q, \bar{Q}\} = 0.$$  \hfill (10)

What remains to be done is just to check that the algebra (10) can be satisfied by an appropriate choice of $\bar{\partial}_k$ and $\hat{\partial}_k$.

A direct computation of the (anti)commutators (10), using the definition (9) of the supercharges, yields

$$\{Q_a, Q_b\} = 2\delta_{ab}(\bar{H} + \bar{P}), \quad \{\bar{Q}_a, \bar{Q}_b\} = 2\delta_{ab}(\bar{H} - \bar{P}).$$  \hfill (11)

where

$$\bar{H} = \frac{a}{2} \sum_k \left[ p^i_{-k} p^i_k + (1 + \bar{\partial}_{-k} \bar{\partial}_k) x^i_{-k} x^i_k + i(\theta_{-k} \bar{\partial}_k \theta_k - \bar{\theta}_{-k} \bar{\partial}_k \bar{\theta}_k) - 2i\bar{\partial}_{-k} \Pi \theta_k \right],$$  \hfill (12)

$$\bar{P} = \frac{a}{2} \sum_k \left[ 2p^i_{-k} \bar{\partial}_k x_k + i(\theta_{-k} \bar{\partial}_k \theta_k + \bar{\theta}_{-k} \bar{\partial}_k \bar{\theta}_k) \right],$$  \hfill (13)

while for the supercharge anticommutators one has

$$\{Q_a, \bar{Q}_b\} = a \sum_k (\bar{\partial}_k + \bar{\partial}_{-k}) \left[ (\gamma^i \gamma^j)_{ab} x^i_{-k} x^j_k - i\delta_{ab} \theta_{-k} \bar{\theta}_k \right].$$  \hfill (14)

Obviously, one can identify the operators $\bar{H}$ and $\bar{P}$ with the Hamiltonian and shift operator respectively. In order to have the correct algebra, one should also require the supercharge commutator (14) to vanish. Identifying $\bar{H}$ with the Hamiltonian requires that the fermionic momentum function $\hat{\partial}$ coincide with $\bar{\partial}$, and that it should be related to the bosonic momentum function through

$$-(\bar{\partial}_{-k} \bar{\partial}_k) \equiv \bar{\partial}^*_{-k} \bar{\partial}_k = \bar{\partial}_{-k} \bar{\partial}_k.$$  \hfill (15)

On the other hand, the vanishing of the supercharge-supercharge anticommutator requires the momentum function to be odd with respect to the inversion of $k,$

$$(\bar{\partial}_k + \bar{\partial}_{-k}) = 0.$$  \hfill (16)

Combining (15) and (16) together, one has

$$\bar{\partial}_k^2 = -\bar{\partial}^*_{-k} \bar{\partial}_k.$$  \hfill (17)

Thus, the formal solution for $\bar{\partial}$ is given by the operator square root

$$\bar{\partial} = i(\bar{\partial}^* \bar{\partial})^{1/2}.$$  \hfill (18)

Obviously, in the continuum case the operator $\bar{\partial}$ is just a partial derivative, which is an anti-hermitian operator, and the solution for $\bar{\partial}$ is $\bar{\partial}$ itself. In contrast, in the discrete case the next-to-neighbor derivative is not purely anti-Hermitian and cannot solve the equation for both sides.
The equation (18) does not define the solution completely, since it is ambiguous. One further needs to define the branch of the square root as well. In the “\(k\)-representation” in which we work the operators are diagonal, therefore the operator square root extraction is reduced to the extraction of the square root of each eigenvalue.\(^2\) Then, the problem of the ambiguity in the operator square root is translated into the sign ambiguity of each eigenvalue.

The natural way to fix the signs of the square root in accordance with the condition (16), and the requirements that the momentum function of fermions is smooth and behaves like \(\bar{\partial}_k = i k + O(k^2 a)\) in the limit \(J \to \infty\), is given by the following choice:

\[
\bar{\partial}_k = \text{sgn} \sqrt{\tilde{\partial}_k^* \tilde{\partial}_k} = \text{sgn} \left( \sqrt{\frac{\pi k}{J}} \right). \tag{19}
\]

In (19) the radical sign denotes the positive root. Passing back to the “\(x\)-representation” produces a non-local operator with the kernel \(\bar{\partial}_{mn} \bar{\partial}_{nm}\):

\[
\bar{\partial}_{nm} = \frac{2}{a} (-1)^{n-m} \frac{\cos \left( \frac{\pi}{2J} \right) \sin \left( \frac{\pi(n-m)}{J} \right)}{\cos \left( \frac{\pi}{2} \right) - \cos \left( \frac{2\pi(n-m)}{J} \right)}. \tag{20}
\]

From eq. (20) one can see that for \(|m-n| \ll J\) the kernel decays as \(\sim 1/(m-n)\), rather than having a next-to-neighbor character.

Simple arguments from Morse theory show that this situation is quite general. Aiming to the effects of discreteness the derivative function \(\bar{\partial}\) has to be periodic in \(k\) with period \(J\),

\[
\bar{\partial}_{k+J} = \bar{\partial}_k. \tag{21}
\]

Hence, it follows that a periodic function satisfying the odd parity condition (16) should have an odd number of either additional zeroes or discontinuities.\(^3\) In fact, the existence of additional zeroes, needed for satisfying the periodicity, is the reason of doubling in the case of a naive discretization of fermions.

Like additional zeroes, discontinuities still create problems to the physical consistency of the model in the continuum limit. Thus, as it was shown in lattice QCD in the context of the so called SLAC discretization, a momentum function discontinuity produces an additional non-local and Lorentz non-invariant contribution which, in particular,\(^4\) always cancel the chiral anomaly.

In spite of this, it was shown that, by introducing an additional regularization at a scale smaller and correlated to the lattice cutoff, one can cure the effects of the discontinuity at the Brillouin zone.

As a matter of fact, in the present case, it is a simple exercise to write down a supersymmetry preserving Pauli–Villars regularization of pp-wave string bits. Indeed, owing to the commutation of supersymmetries with the Hamiltonian on the most of the Hilbert space, the straightforward mode truncation \(|k| < M \sim 1/a^{1+\epsilon}, \epsilon > 0\), preserves supersymmetry and removes the doubling.

\(^2\)This holds only in the case of non-degenerate eigenvalues.

\(^3\)Of course, Morse theory applies to continuous functions. In our case we speak about functions which at large \(J\) can be interpolated with arbitrary precision by such continuous functions.
4 Discussion

In this paper we addressed the problem of supersymmetry in the string bit model at a finite discretization $J$. We succeeded in showing that supersymmetry can be preserved, avoiding at the same time the doubling of fermions, in all modes of the string bit except one. The latter can in no way be avoided, but it is an isolated mode and produces no effect in the continuum limit.

We started from the bosonic part of the string bit model where the string bit interactions include the nearest neighbors. The latter is in accordance with the planarity property of the large $N$ gauge theories. The fermionic action compatible with it appears not to have this property of a next-to-neighbor character at finite $J$, only the square of the Dirac operator has a next-to-neighbor character. The most interesting objects depend on the square of the Dirac operator, rather than on Dirac operator itself, which does not violate the planarity.

Our analysis concerns the free string and, therefore, the conclusions are valid and rigorous only for the free string. Although the main interest for the application of the string bit model is found in the interacting string case, where we can so far only extrapolate our conclusions under certain conditions, the free case is still important as it is leaving place for the self-consistency of the BMN correspondence.

Of course, the detailed analysis of the case of interacting string bits is still needed and it could not be ruled out completely that avoiding doubling and keeping supersymmetry will not be obstructed by interactions. (In this case, an interesting question would be a “susy-optimized” fermionic action of the Gisparg–Wilson type [20].)

The optimistic expectation, that most probably this is not the case, is fuelled by two main arguments. The first one is that by a gauge choice one can always make string bits free, even in the interacting string, and the second one is that the supersymmetry violating defect can be isolated and decoupled by a regularization procedure. (For the latter, however, it is important to have always a supersymmetric regularization scheme at hand.)

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