\langle \phi^2 \rangle in the Spacetime of a Cylindrical Black Hole

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Abstract

The renormalised value of \langle \phi^2 \rangle is calculated for a massless, conformally coupled scalar field in the Hartle-Hawking vacuum state. This calculation is a first step towards the calculation of the gravitational back reaction of the field in a black cosmic string spacetime which is asymptotically anti-DeSitter and possesses a non constant dilaton field. It is found that the field is divergence free throughout the spacetime and attains its maximum value near the horizon.

1 Introduction

A useful question to ask when one studies quantum fields in General Relativity is the following: Given that all matter is inherently quantum in nature, will quantum effects remove the singularity at the centre of a black hole spacetime? To answer this question using a scalar field and semi-classical perturbation theory, one must first calculate the expectation value \langle T_{\mu\nu} \rangle where \( T_{\mu\nu} \) is the stress-energy tensor operator of the scalar field \( \phi \). In this paper \langle \phi^2 \rangle is computed in preparation for the computation of \langle T_{\mu\nu} \rangle which is used as the source term for the Einstein field equations:

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\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle . \] 

(1)

A review of handling quantum fields in the presence of strong gravitational fields can been found in articles by Wipf [1] and DeWitt [2] and in books by Birrel and Davies [3] and Wald [4].

\( \langle \phi^2 \rangle \) for massless fields has been computed for both the interior and exterior of a Schwarzschild black hole [5] [6]. These calculations have also been extended by Anderson to accommodate massive fields in general spherically symmetric, asymptotically flat spacetimes [7]. A method has also been developed by Anderson, Hiscock and Samuel [8] [9] to calculate the expectation value of the stress-energy operator, \( \langle T_{\mu\nu} \rangle \) in spherically symmetric, static spacetimes. They use this method to calculate \( \langle T_{\mu\nu} \rangle \) in Schwarzschild and Reissner-Nordström geometries. \( \langle T_{\mu\nu} \rangle \) has also previously been computed by Howard and Candelas [10] in Schwarzschild spacetime.

The Kerr spacetime has also been studied with fields propagating in this geometry. Frolov [11] has calculated \( \langle \phi^2 \rangle \) for massless fields on the event horizon pole of a Kerr-Newmann black hole as well as deriving an approximate expression for \( \langle T_{\mu\nu} \rangle \) near the horizon with Thorne [12]. Massive fields in the exterior geometry have been studied by Frolov and Zel’nikov [13].

Various works on back reaction effects of quantum fields on black hole geometries have been produced. For Schwarzschild and Reissner-Nordström spacetimes this includes the work of Hiscock and Weems [14], Bardeen and Balbinot [15] [16], and York [17] who used Page’s analytic approximation [18] for \( \langle T_{\mu\nu} \rangle \) in Einstein spacetimes for conformally invariant fields. More recently Hiscock and Larson [19] have extended their analysis to the Schwarzschild interior and calculated back reaction effects on curvature invariants.

Most work in this field has been done in the context of spherical or oblate symmetry. We wish to extend the analysis to other symmetries and ultimately ask if the above results are general or are specific to the particular symmetry chosen. For example, does the presence of the quantum field have the same effect on the curvature growth and on the anisotropy of a black hole interior for all symmetries? If curvature invariants are weakened for all systems studied then one can say with some confidence that quantum effects may remove the singularity.
1.1 Background geometry

The system studied here will be that of a massless Klein-Gordon field with conformal curvature coupling propagating in the spacetime generated by a straight black string. This background is chosen for several reasons. First, it possesses cylindrical (as opposed to spherical) symmetry and secondly, in the context of cosmic strings it represents a system which may physically exist in the universe. It could also be argued that sufficiently close to a black string loop the spacetime will possess this type of geometry. There has also been a revived interest in anti-DeSitter spacetimes in the context of conformal field theories. The metric chosen is that developed by Kaloper [20] and Lemos and Zanchin [21] which admits a line element of the following form:

\[ ds^2 = -\left( \alpha^2 \rho^2 - \frac{2(M+\Omega)}{\alpha \rho} + \frac{4Q^2}{\alpha^2 \rho} \right) dt^2 - \frac{16J}{3\alpha \rho} \left( 1 - \frac{2Q^2}{(M+\Omega)\alpha \rho} \right) dt d\varphi \\
+ \left[ \rho^2 + \frac{4(M-\Omega)}{\alpha^2 \rho} \left( 1 - \frac{2Q^2}{(M+\Omega)\alpha \rho} \right) \right] d\varphi^2 \\
+ \frac{d\rho^2}{\alpha^2 \rho^2 - \frac{2(M-\Omega)}{\rho^2} + \frac{(M-M)4Q^2}{(M+\Omega)\rho^2} + \alpha^2 \rho^2} + d\rho^2 + dz^2 \]  

(2)

Where \( M, Q, \) and \( J \) are the mass, charge, and angular momentum per unit length of the string respectively. \( \Omega \) is given by:

\[ \Omega = \sqrt{M^2 - \frac{8J^2}{9}}. \]  

(3)

The constant \( \alpha \) is defined as follows:

\[ \alpha^2 = -\frac{1}{3} \Lambda \]  

(4)

with \( \Lambda \), the cosmological constant, negative giving the spacetime its asymptotically anti-DeSitter behaviour. The spacetime has a well defined time-like killing vector field with respect to which modes can be defined.

Much interesting work has been done regarding effects of quantum fields in the 3D BTZ [22] black hole [23, 24, 25, 26, 27, 28]. The system here however does not dimensionally reduce to the 3D BTZ black hole whose four dimensional counterpart (without charge or angular momentum) is given for reference as [29]

\[ ds^2_{\text{BTZ4}} = -(\alpha^2 r^2 - 8M)dt^2 + \frac{dr^2}{(\alpha^2 r^2 - 8M)} + r^2 d\varphi^2 + dz^2. \]  

(5)
This is due to the fact that the dilaton field is non-zero and non-constant in the corresponding three-dimensional action.

In the spacetime considered here, both charge and angular momentum are zero yielding the following for (2)

\[ ds^2 = -(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}) dt^2 + \frac{d\rho^2}{(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho})} + \rho^2 d\varphi^2 + \alpha^2 \rho^2 dz^2. \] (6)

It can immediately be seen from (6) that the spacetime behaves as anti-DeSitter as \( \rho \to \infty \) and therefore the spacetime is not globally hyperbolic. This fact requires boundary conditions to be imposed to control the flow of information in and out of the time-like infinity. It has been shown [30] that three natural boundary conditions arise in ADS\(_4\) spacetime and the condition used here is that of a ”transparent” boundary condition.

2 Green Function and \( \langle \phi^2 \rangle \)

In this section a Euclidean space approach is used to calculate the Green function. The calculation is similar to that done by Anderson [7] who calculated \( \langle \phi^2 \rangle \) in a Reissner-Nordström spacetime however, the method is extended here to a system which is neither spherically symmetric nor asymptotically flat.

The Euclidean space method amounts to making the transformation \( t \to \imath \tau \) in metric (3). The quantity \( \langle \phi^2 \rangle \) is then defined as

\[ \langle \phi^2 \rangle = \lim_{x \to x'} G_E(x, x'), \] (7)

where \( G_E(x, x') \) is the Euclidean space Green function satisfying the equation

\[ [\Box^2 - m^2 - \xi R(x)] G_E(x, x') = -\frac{\delta^4(x, x')}{g(x)}. \] (8)

The mass term in the Klein-Gordon operator will be set to zero in order to comply with the transparent boundary conditions mentioned earlier.

The presence of three killing fields allows the right-hand-side of the above equation to be expanded in terms of cylindrical functions as follows:

\[ \frac{\delta^4(x, x')}{\sqrt{g(x)}} = \delta(\tau - \tau') \frac{\delta(\rho - \rho')}{\alpha \rho^2} \delta(\varphi - \varphi') \delta(z - z'), \] (9)
with

\[ \delta(\tau - \tau') = T \sum_{n=-\infty}^{\infty} e^{in2\pi T(\tau - \tau')} \]

\[ \delta(\varphi - \varphi') = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{il(\varphi - \varphi')} \]  \hspace{1cm} (10)

\[ \delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(z-z')} \].

Where \( T \) is the temperature of the field. The Green function can similarly be expanded giving

\[ G(x, x') = \frac{T}{4\pi^2} \sum_{n=-\infty}^{\infty} e^{in2\pi T(\tau - \tau')} \sum_{l=-\infty}^{\infty} e^{il(\varphi - \varphi')} \int_{-\infty}^{\infty} dk e^{ik(z-z')} \chi(\rho, \rho'). \]  \hspace{1cm} (11)

The function \( \chi(\rho, \rho') \) must satisfy the following equation

\[ -\frac{d^2}{d\rho^2} \chi(\rho, \rho') \left( \alpha^2 \rho^2 - \frac{4M}{\alpha \rho} \right) + \frac{d\chi(\rho, \rho')}{d\rho} 4 \left( \alpha^2 \rho - \frac{M}{\alpha \rho^2} \right) \]

\[ - \chi(\rho, \rho') \left[ \frac{l^2}{\rho^2} + \frac{k^2}{\alpha^2 \rho^2} + \frac{n^2 4\pi^2 T^2}{\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}} + \xi R \right] = \frac{-\delta(\rho, \rho')}{\rho^2 \alpha}. \]  \hspace{1cm} (12)

A solution is assumed of the form

\[ \chi(\rho, \rho') = C \Psi_1(\rho_<) \Psi_2(\rho_), \]  \hspace{1cm} (13)

where \( \rho_< \) and \( \rho_> \) represent the lesser and greater of \( \rho \) and \( \rho' \) respectively. Integrating across the \( \delta \) function gives the Wronskian normalisation condition:

\[ C \left[ \Psi_1(\rho_<) \frac{\partial \Psi_2(\rho_>)}{\partial \rho} - \Psi_2(\rho_<) \frac{\partial \Psi_1(\rho_<)}{\partial \rho} \right] = \frac{-1}{\alpha \rho^2 (\alpha^2 \rho^2 - \frac{4M}{\alpha \rho^2})}. \]  \hspace{1cm} (14)

The asymptotic behaviour of the solution can be found by studying the solutions to (12) in the appropriate regimes. For \( \rho = \infty \) the solution has the form:

\[ \Psi_1 \sim \rho^{-\frac{3}{2}} e^{\frac{\sqrt{g+4\xi R}}{2}}, \]

\[ \Psi_2 \sim \rho^{-\frac{3}{2}} \rho^e e^{\frac{\sqrt{g+4\xi R}}{2}}. \]  \hspace{1cm} (15)
Whereas near the horizon \( \rho = \left(\frac{4M}{\alpha}\right)^{1/3} \) solutions are found to behave as:

\[
\begin{align*}
\Psi_1 & \sim e^{\kappa n \int \frac{\alpha \rho}{\rho^3 - 4M}} \\
\Psi_2 & \sim e^{-\kappa n \int \frac{\alpha \rho}{\rho^3 - 4M}}.
\end{align*}
\]  

(16)

Where \( \kappa = 2\pi T \). It can easily be seen that \( \Psi_1 \) diverges at infinity and \( \Psi_2 \) is divergent at the horizon. The general solution with the correct asymptotic behaviour can be found by ansatz. Such a solution takes the form

\[
\begin{align*}
\Psi_1 &= \frac{1}{\sqrt{\rho^3 \alpha^3 X}} e^{\int \rho^2 \alpha^3 X \, d\rho} \sqrt{\frac{g_{11} g_{00} g_{33}}{900 g_{33}}} \\
\Psi_2 &= \frac{1}{\sqrt{\rho^3 \alpha^3 X}} e^{-\int \rho^2 \alpha^3 X \, d\rho} \sqrt{\frac{g_{11} g_{00} g_{33}}{900 g_{33}}}.
\end{align*}
\]  

(17)

\( X \) is a function of \( \rho \) which evolves according to the equation:

\[
X^2 = \frac{1}{\alpha^2 \rho^2} \left(1 - \frac{4M}{\alpha^3 \rho^3}\right) \left[l^2 + \frac{k^2}{\alpha^2} + \frac{\kappa^2 n^2}{\alpha^2 \left(1 - \frac{4M}{\alpha^3 \rho^3}\right)} \right] \\
+ \frac{9}{4} - \frac{36M^2}{\alpha^6 \rho^6} + \frac{\xi R}{\alpha^2} \left(1 - \frac{4M}{\alpha^3 \rho^3}\right) + \frac{X'}{X} \left[\frac{1}{2} \rho^2 + \frac{2M}{\alpha^3 \rho^2} - \frac{16M^2}{\alpha^6 \rho^4}\right] \\
+ \left(\frac{X'}{X}\right)^2 \left[\frac{6M}{\alpha^3 \rho} - \frac{12M^2}{\alpha^6 \rho^4}\right] + \frac{X''}{X} \left[\frac{1}{2} \rho^2 - \frac{4M}{\alpha^3 \rho} + \frac{8M^2}{\alpha^6 \rho^4}\right].
\]  

(18)

Which can be obtained by substituting (17) into (8). Substituting (17) into (14) gives \( C = \frac{1}{2} \) for all mode functions.

A point splitting algorithm developed by Christensen \[31\] will be used to renormalise the field. In this technique, one chooses the points \( x \) and \( x' \) to be nearby points in the spacetime before the full coincidence limit is taken. It is convenient to have the points take on equal values of \( \rho, \varphi \) and \( z \) so that the coordinate separation is given by \( \epsilon = \tau - \tau' \). The unrenormalised Green function now takes on the form:

\[
G_E(x, \tau, \tau') = \frac{T}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{\hbar n \kappa} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \Psi_1(\rho) \Psi_2(\rho). \]  

(19)

It should be noted at this point that there exists a superficial ultra-violet divergence over \( l \) and \( k \) in the above expression. This divergence can most
easily be eliminated using a similar technique as Candelas [5] and Anderson [7]. It is noted that as long as \( \tau \neq \tau' \) any multiple of \( \delta(\tau - \tau') \) can be added to (19). Substituting \( X \) from (18) in the large \( l \) and \( k \) limit and subtracting this term from (19) the logarithmic divergences can be eliminated giving the following expression:

\[
G_E(x, \tau, \tau') = \frac{T}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{i\kappa n} \left( \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \Psi_1(\rho)\Psi_2(\rho) \right) \sqrt{l^2 + \frac{k^2}{\alpha^2}} \alpha^5 \rho^5 \epsilon^2 + O(\epsilon^4),
\]

(20)

2.1 Renormalisation

To calculate the renormalised value of \( \langle \phi^2 \rangle \) a point splitting technique will be used. The DeWitt generalisation to Schwinger’s expansion is used as an approximation for the Green function. This term will then be subtracted from (20) and the \( x \rightarrow x' \) limit will be taken along the shortest geodesic separating the points. The DeWitt-Schwinger counter-term is given by:

\[
G(x, x') = \frac{1}{8\pi^2 \sigma} + \frac{1}{96\pi^2} R_{\mu\nu} \sigma^{\mu\nu}
\]

(21)

where \( \sigma \) is the ”world function” of Synge [32] which is equal to half of the square of the geodesic distance between two points. The points in this case will be \( \tau \) and \( \tau' \).

It can be shown, by geodesic expansion (see appendix A), that the world function in the spacetime considered here takes on the form

\[
\sigma = \frac{1}{2} \left( \alpha^2 \rho^2 - \frac{4M}{\alpha \rho} \right) \epsilon^2 - \frac{1}{24} \left( \alpha^3 \rho^3 - 2M \right)^2 \frac{\alpha^3 \rho^3 - 4M}{\alpha^3 \rho^3} \epsilon^4 + O(\epsilon^6),
\]

\[
\sigma^{\tau \tau} = \epsilon - \frac{1}{6} \left( \frac{\alpha^3 \rho^3 - 2M}{\alpha^3 \rho^3} \right)^2 \epsilon^3 + O(\epsilon^5),
\]

\[
\sigma^{\tau \rho} = \frac{\left( \alpha^3 \rho^3 + 2M \right) \left( \alpha^3 \rho^3 - 4M \right)}{\alpha^2 \rho^3} \epsilon^2 + O(\epsilon^4).
\]

(22)
Using these expressions the DeWitt-Schwinger counter term is equal to

\[
G_{\text{counter}} = \frac{1}{4\pi^2} \frac{1}{\left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right) \epsilon^2} + \frac{1}{48\pi^2} \frac{(\alpha^3 \rho^3 + 2M)^2}{\alpha \rho^3 (\alpha^3 \rho^3 - 4M)}
\]
\[
- \frac{3\alpha^2}{48\pi^2}
\]

(23)

The first term in (23) can be rewritten in a more convenient way by using the Plana sum formula [33] as used by Anderson. The formula is

\[
\sum_{n=n_0}^{\infty} f(n) = \frac{1}{2} f(n_0) + \int_{n_0}^{\infty} f(s) ds + \int_{0}^{\infty} dt \left[ f(n_0 + ut) - f(n_0 - ut) \right] \frac{1}{e^{2\pi t} - 1}.
\]

(24)

With this, the first expression in the counter term can be written as

\[
\frac{1}{4\pi^2} \frac{1}{\left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right) \epsilon^2} =
\]
\[
- \frac{\kappa}{4\pi^2} \left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right) \sum_{n=1}^{\infty} \cos(n\kappa\epsilon) n \kappa - \frac{\kappa^2}{48\pi^2} \frac{1}{\left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right)},
\]

(25)

which gives, for the entire counter-term in the \(\epsilon \to 0\) limit:

\[
G_{\text{counter}} = - \frac{\kappa}{4\pi^2} \left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right) \sum_{n=1}^{\infty} n \kappa - \frac{\kappa^2}{48\pi^2} \frac{1}{\left(\alpha^2 \rho^2 - \frac{4M}{\alpha \rho}\right)}
\]
\[
+ \frac{1}{48\pi^2} \frac{(\alpha^3 \rho^3 + 2M)^2}{\alpha \rho^3 (\alpha^3 \rho^3 - 4M)} - \frac{3\alpha^2}{48\pi^2}.
\]

(26)

It can be shown that the second and third terms in (26), which normally diverge at the horizon, will cancel each other out on the horizon when \(T\) is equal to the black hole temperature. That is, when

\[
T = \frac{\alpha}{2\pi} \left(\frac{3}{2} \left(4M\right)^{\frac{1}{3}} \right).
\]

(27)
Therefore, the renormalised expression for \(\langle \phi^2 \rangle\) in the Hartle-Hawking vacuum state is

\[
\langle \phi^2 \rangle = \frac{T}{8\pi^2} \left( \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk \, \Psi_1(\rho) \Psi_2(\rho)}{\sqrt{l^2 + \kappa^2 \alpha^2 \rho^2 - \frac{4M}{\alpha^2 \rho}}} \right) \\
+ \left( \frac{\kappa}{4\pi^2 f} \sum_{n=1}^{\infty} n \kappa \right) + \frac{\kappa^2}{48\pi^2 f} - \frac{f'^2}{192\pi^2 f} + \frac{3\alpha^2}{48\pi^2},
\]

(28)

Where, for convenience, the following notation has been used

\[
f \equiv g_{00} = \left( \alpha^2 \rho^2 - \frac{4M}{\alpha^2} \right),
\]

\[
f' = \frac{\partial f}{\partial \rho}.
\]

(29)

3 Calculation of \(\langle \phi^2 \rangle\)

In this section the value of \(\langle \phi^2 \rangle\) will be calculated from (28). The solution is found by iteratively solving for the function \(X\) using (19) in the mode functions with the lowest order term defined as:

\[
X_0^2 = \frac{h}{\alpha^2 \rho^2} \left[ l^2 + \frac{k^2}{\alpha^2} + \frac{\kappa^2 n^2}{\alpha^2 h} \right] + \frac{9}{4} \frac{36M^2}{\alpha^6 \rho^6} + \frac{\xi R}{\alpha^2 h},
\]

(30)

and \(h\) given by

\[
h = \left( 1 - \frac{4M}{\alpha^2 \rho^4} \right).
\]

(31)

With this choice, the lowest order term will be valid at both the horizon and as \(\rho\) approaches infinity. This can also be verified by comparison with (13) and (15).

It is most convenient to do the integration over \(k\) first followed by the sum over \(l\) and finally, the sum over \(n\). This scheme leads to analytic solutions to both the integrals and the \(l\) sums.

In the solution, the sums and integrals have the following form:

\[
\sum_n \sum_l \int_0^\infty \frac{dk}{\left( l^2 + \frac{k^2}{\alpha^2} + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho) \right)^\frac{3}{2}},
\]

(32)
where \( V_0(\rho) \) is a function of \( \rho \) only. Such integrals are known and are given by

\[
\int_0^\infty \frac{dk}{(l^2 + \frac{k^2}{\alpha^2} + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho))^{\frac{p}{2}}} = C \frac{\alpha}{(l^2 + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho))^{\frac{p-1}{2}}},
\]

where \( C \) is a fractional constant which depends on the particular value of the integer \( p \). The resulting sums over \( l \) can now be done analytically by the standard contour integration:

\[
\sum_{-\infty}^{\infty} \frac{1}{(l^2 + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho))^{\frac{p}{2}}} = -\sum \left( \text{Residues of} \frac{\pi \cot \pi l}{(l^2 + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho))^{\frac{p-1}{2}}} \right).
\]

This is valid as the denominator never becomes singular at integer values of \( l \).

In the expansion of \( \langle \phi^2 \rangle \) there appears a term of the form

\[
\frac{1}{\rho^2 \alpha \sqrt{h}} \ln \left( \frac{|l|}{\sqrt{l^2 + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho)}} \right),
\]

which arises when the lowest order term is combined with the counterterm in (20) and integrated over \( k \). This is the only term which is singular at \( l = 0 \). The \( l = 0 \) term is therefore removed from the sum and redefined as to remove a spurious divergence of the form \( \ln \left( 2\pi \sqrt{\frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho)} \right) \) which is present in the same term once the sum from \( l = 1 \) to \( \infty \) has been calculated (how this term arises is shown in more detail in appendix B.) This redefinition can be performed since the counterterm introduced in (20) may be any quantity and the particular value shown there is exact only in the large \( l \) limit.

The final sum over \( n \) can now numerically be shown to converge by computing the values of \( \langle \phi^2 \rangle \) for large \( n \). It can also be shown that the \( n = 0 \) mode makes no contribution to the sum.

The boundary values of the modes must be evaluated before numerical integration of the mode equation can be done. The value of the mode functions at infinity can easily be seen by studying the mode equation in the asymptotic region and therefore only the value at the horizon is left to be
determined. At the horizon, there are many quantities in the expansion of \( \langle \phi^2 \rangle \) which are inversely proportional to some power of the metric function \( f \). By performing an expansion of \( \langle \phi^2 \rangle \) in the quantity \( \delta = \rho - \rho_H \), where \( \rho_H \) is the horizon value of \( \rho \), one can show (although the procedure is lengthy) that all terms with \( \delta \) raised to some power in the denominator cancel at the horizon. In appendix B it is also demonstrated how terms which normally make a dominant contribution to \( \langle \phi^2 \rangle \) cancel here. It should be noted that the horizon value is directly proportional to the value of the cosmological constant. This is due to the fact that the spacetime is an Einstein spacetime with \( R_{\mu\nu} = 3\alpha^2 g_{\mu\nu} \).

\( \langle \phi^2 \rangle \) was computed for the conformally coupled, massless case and the result is shown in Fig.1. It can be seen the maximum value of \( \langle \phi^2 \rangle \) occurs near, but not at, the horizon. This behaviour is analogous to the extreme Reissner-Nordström case \([4]\). This is because, as shown in the appendix and earlier, most contributions to \( \langle \phi^2 \rangle \) at the horizon vanish. However, near (but not on) the horizon terms with a \( 1/f \) behaviour make a large contribution. For large \( \rho \) most terms in the field expansion vanish and therefore \( \langle \phi^2 \rangle \) approaches a value which is dominated by the last two terms in (28).

![Graph](image.png)

Figure 1: \( \langle \phi^2 \rangle \) for the cylindrical black hole spacetime. The value of \( \langle \phi^2 \rangle \) has a small but nonzero value at the horizon and attains a maximum away from the horizon.
4 Acknowledgements

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5 Appendix A

In this appendix it will be demonstrated how to calculate the world function, $\sigma$, using the method of geodesic expansion. Equations here involve quantities which are defined at different spacetime points. For a brief review of handling bitensors the reader is referred to Christensen [31] and Synge [32].

Let $P_1(x)$ and $P_2(x')$ be two points in the spacetime close enough together such that they are connected by a unique geodesic. The geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

yields the power series

$$x^\mu \approx x'^\mu + U^\mu ds - \frac{1}{2} \Gamma^\mu_{\alpha\beta}' U^\alpha U^\beta ds^2 + \frac{1}{6} \left( 2\Gamma^\mu_{\gamma'\delta'} \Gamma^\gamma_{\kappa'\eta'} U^\kappa U^\eta U^\delta \right) ds^3 - \frac{1}{6} \Gamma^\mu_{\kappa'\eta'\gamma'} U^\kappa U^\eta U^\gamma ds^3 + ...$$

(37)

where

$$U^\mu = \frac{dx^\mu}{ds}$$

(38)

and

$$U^\mu ds = dx^\mu \approx \xi^\mu = (\tau - \tau') \delta^\mu_0.$$  

(39)

The last term in equation (37) is zero since the spacetime is static. (37) can be inverted and used in the definition [32]

$$2\sigma(x, x') = ds^2 g_{\mu'\nu'} U^\mu U^\nu'$$

(40)
giving

\[ 2\sigma(x, x') = \xi^\mu \xi^\nu g_{\mu'\nu'} + \frac{1}{2} \xi^\mu \xi^\nu \xi^\kappa \Gamma^\nu_{\mu'\kappa'} g_{\mu'\nu'} - \frac{2}{3} \xi^\omega \xi^\rho \xi^\mu \xi^\nu \Gamma^\nu_{\chi'\rho'} \Gamma^\chi_{\kappa'\omega'} g_{\mu'\nu'} \]

\[ + \frac{1}{2} \xi^\alpha \xi^\beta \xi^\gamma \Gamma^\gamma_{\alpha'\beta'\gamma'} g_{\mu'\nu'} + \frac{1}{4} \xi^\alpha \xi^\beta \xi^\gamma \xi^\sigma \Gamma^\mu_{\alpha'\beta'} \Gamma^\nu_{\sigma'\gamma'} g_{\mu'\nu'}. \]  

(41)

Calculating the Christoffel symbols using (8) and noting (39) it can be shown that this expression reduces to the one in (22). This expression has the same functional form as that of the world function for a static, spherically symmetric spacetime \[7\] if the coordinate separation there is also chosen as (39). This is due to the fact that the \(\xi^\mu\) vectors eliminate any dependence of \(\sigma(x, x')\) on \(g_{22}\) and \(g_{33}\).

6 Appendix B

In this appendix the dominant terms of \(\langle \phi^2 \rangle\) at the horizon will be calculated. This is useful since the value at the horizon needs to be evaluated as a starting point for the calculation of the mode functions. This calculation is also useful as it provides insight as to how the \(n\) counter-term acts to regularise the field.

At the horizon, the dominant terms in the field expansion are given by

\[ \langle \phi^2 \rangle = \frac{T}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{1}{\rho^2 \alpha \sqrt{h}} \ln \left( \frac{|l|}{\sqrt{l^2 + \frac{\kappa^2 \mu^2}{\alpha^2 h} + V_0(\rho)}} \right) + \frac{\kappa}{4\pi^2 f} \sum_{n=0}^{\infty} n \kappa. \]  

(42)

For the moment, we choose to ignore the \(l = 0\) term and concentrate on the
The first expression in (42) which can trivially be re-written as

\[
= \frac{T}{2\pi^2} \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\rho^2 \alpha \sqrt{\hbar}} \ln \left( \frac{1}{1 + \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho)} \right)
\]

\[
= -\frac{T}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\rho^2 \alpha \sqrt{\hbar}} \ln \left( 1 + \frac{\kappa^2 n^2}{\alpha^2 h} + \frac{V_0(\rho)}{l^2} \right)
\]

\[
= -\frac{T}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{\rho^2 \alpha \sqrt{\hbar}} \ln \left( \prod_{l=1}^{\infty} \left( 1 + \frac{\kappa^2 n^2}{\alpha^2 h} + \frac{V_0(\rho)}{l^2} \right) \right). \quad (43)
\]

The product in the above expression is well known yielding the following for (43)

\[
-\frac{T}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{\rho^2 \alpha \sqrt{\hbar}} \ln \left( 2 \sinh \left( \frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho) \right) \right)
\]

\[
- \ln \left( 2\pi \sqrt{\frac{\kappa^2 n^2}{\alpha^2 h} + V_0(\rho)} \right). \quad (44)
\]

For very large \( n \) or (as in the case here) very small \( h \) this becomes

\[
-\frac{T}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{\rho^2 \alpha \sqrt{\hbar}} \left[ \frac{\kappa |n| \pi}{\alpha \sqrt{\hbar}} - \ln \left( \frac{2\kappa |n| \pi}{\alpha \sqrt{\hbar}} \right) \right]. \quad (45)
\]

If we define the \( l = 0 \) term to cancel out the second term in (44) as described in the text, the resultant expression gives

\[
-\frac{2\pi T}{4\pi^2 f} \sum_{n=0}^{\infty} n\kappa \quad (46)
\]

which is cancelled by the \( n \) sum in the counter-term. This leaves a small constant contribution to \( \langle \phi^2 \rangle \) at the horizon.
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