Bounds of the Mertens Functions

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ABSTRACT

In this paper we derive new properties of Mertens function and discuss about a likely upper bound of the absolute value of the Mertens function \( \sqrt{\log(x!)} > |M(x)| \) when \( x > 1 \). Using this likely bound we show that we have a sufficient condition to prove the Riemann Hypothesis.

1. Introduction

We define the Mobius Function \( \mu(k) \). Depending on the factorization of \( n \) into prime factors the function can take various values in \{ -1, 0, 1 \}

- \( \mu(n) = 1 \) if \( n \) has even number of prime factors and it is also square-free (divisible by no perfect square other than 1)
- \( \mu(n) = -1 \) if \( n \) has odd number of prime factors and it is also square-free
- \( \mu(n) = 0 \) if \( n \) is divisible by a perfect square.

Mertens function is defined as \( M(n) = \sum_{k=1}^{n} \mu(k) \) where \( \mu(k) \) is the Mobius function. It can be restated as the difference in the number of square-free integers up to \( x \) that have even number of prime factors and the number of square-free integers up to \( x \) that have odd number of prime factors. The Mertens function rather grows very slowly since the Mobius function takes only the value \( 0, \pm 1 \) in both the positive and negative directions and keeps oscillating in a chaotic manner. Mertens after verifying all the numbers up to 10,000 conjectured that the absolute value of \( M(x) \) is always bounded by \( \sqrt{x} \). This conjecture was later disproved by Odlyzko and te Riele. This conjecture is replaced by a weaker one by Stieltjes who conjectured that \( M(x) = O(x^{1/2}) \). Littlewood proved that the Riemann hypothesis is equivalent to the statement that for every \( \epsilon > 0 \) the function \( M(x)x^{-1/2-\epsilon} \) approaches zero as \( x \to \infty \). This proves that the Riemann Hypothesis is equivalent to conjecture that \( M(x) = O(x^{1/2+\epsilon}) \) which gives a rather very strong upper bound to the growth of \( M(x) \).

Although there exists no analytic formula, Titchmarsh showed that if the Riemann Hypothesis is true and if there exist no non-trivial Riemann zeta function zeros, then there must exist a sequence \( T_k \) which satisfies \( k \leq T_k \leq k + 1 \) such that the following result holds:

\[
M_0(x) = \lim_{k \to \infty} \sum_{|\gamma| < T_k} \frac{x^\rho}{\rho^2(\rho)} - 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{1}{n(2n+1)} \frac{2\pi}{x} 2n
\]

Where \( \zeta(z) \) is the Riemann zeta function, and \( \rho \) are all the all nontrivial zeros of the Riemann zeta function and \( M_0(x) \) is defined as \( M_0(x) = M(x) - \frac{1}{2} \mu(x) \) if \( x \in \mathbb{Z}^+ \), \( M(x) \) Otherwise (Odlyzko and te Riele)
2. New Properties of Mertens Function

Lehman⁵ proved that \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor) = 1 \). In general, \( \sum_{i=1}^{x} M(\lfloor x/(in) \rfloor) = 1, n = 1, 2, 3, \ldots, x \) (since \( \lfloor x/n \rfloor/i = \lfloor x/(in) \rfloor \)). Let \( R' \) denote a square matrix where element \((i, j)\) equals 1 if \( j \) divides \( i \) or 0 otherwise. (In a Redheffer matrix, element \((i, j)\) equals 1 if \( i \) divides \( j \) or if \( j = 1 \). Redheffer⁶ proved that the determinant of the matrix equals the Mertens Function \( M(x) \).) Let \( T \) denote the matrix obtained from \( R' \) by element-by-element multiplication of the columns by \( M \left( \left\lfloor \frac{x}{i} \right\rfloor \right), M \left( \left\lfloor \frac{x}{2i} \right\rfloor \right), \ldots, M \left( \left\lfloor \frac{x}{ki} \right\rfloor \right) \). For example, the \( T \) matrix for \( x = 12 \) is

\[
T = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

**Theorem (1):** \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor) = A(x) \)

**Proof:** Let us now take \( A(x) = \sum_{i=1}^{x} \varphi(i) \) where \( \varphi \) is Euler's totient function. Let \( U \) denote the matrix obtained from \( T \) by element-by-element multiplication of the columns by \( \varphi(j) \). The sum of the columns of \( U \) then equals \( A(x) \). Now since \( i = \sum_{d|i} \varphi(d) \) we can write \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor) i \) (the sum of the rows of \( U \)) equals \( A(x) \).

By the Schwarz inequality, \( A(x)/\sqrt{x(x+1)(2x+1)/6} \) is a lower bound of \( \sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2} \). \( A(x) = \sum_{i=1}^{x} \varphi(i) \) is further simplified by Walfisz⁷ and Hardy and Wright⁸ as

\[
A(x) = \sum_{i=1}^{x} \varphi(i) = \frac{1}{2} \sum_{k=1}^{x} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor \left( 1 + \left\lfloor \frac{x}{k} \right\rfloor \right) = \frac{3}{\pi^2} x^2 + O((\log x)^{5/3}(\log \log x)^{4/3})
\]

\[
\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2} > \frac{A(x)}{\sqrt{x(x+1)(2x+1)/6}} > \frac{3x^2}{\pi} \frac{3}{\sqrt{x(x+1)(2x+1)}} = \frac{3\sqrt{3}}{\pi} \frac{x^2}{\sqrt{x(x+1)(x+1/2)}}
\]

This can be further simplified to

\[
\sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2} / x > \frac{3\sqrt{3}}{\pi} \frac{1}{\sqrt{(1+1/x)(1+1/2x)}}
\]

Taking limit of infinity on both the sides, we get

\[
\lim_{x \to \infty} \sqrt{\sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2} / x > \frac{3\sqrt{3}}{\pi}
\]

This shows that \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 \) at large values of \( x \) is greater than \( \frac{27}{\pi^2} x^2 \).
Let $A(i)$ denote the Mangoldt function ($A(i)$ equals $log(p)$ if $i = p^m$ for some prime $p$ and some $m \geq 1$ or 0 otherwise). Mertens proved that $\sum_{i=1}^{x} M([x/i]) log i = \psi(x)$ where $\psi(x)$ denotes the second Chebyshev function ($\psi(x) = \sum_{i \leq x} A(i)$).

**Theorem (2):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) log(i) \sigma_0(i)/2 = log(x!)$

**Proof:** Let $\sigma_\nu(i)$ denote the sum of positive divisors function ($\sigma_\nu(i) = \sum_{d|\nu} d^\nu$). Replacing $\varphi(j)$ with $log(j)$ in the $U$ matrix gives a similar result.

Let $\lambda(n)$ denote the Liouville function ($\lambda(1) = 1$ or if $p_1^{a_1} \ldots p_k^{a_k}, \lambda(n) = (-1)^{a_1 + \ldots + a_k}. \sum_{d|n} \lambda(d)$). Equally, $\lambda(n)$ if $n$ is a perfect square or 0 otherwise. Let $L(x) = \sum_{n \leq x} \lambda(n) \log(n)$. $H(x)$ is expanded by McCarthy to be

$$H(x) = \sum_{n \leq x} \lambda(n) \log(n).$$

The generalization of the Euler’s totient function is Jordan totient function. Let it be denoted as $J_k(n)$ which is defined as number of set of $k$ positive integers which are all less than or equal to $n$ that will form a co-prime set of $(k + 1)$ positive integers together with $n$. Let us define $(x) = \sum_{i=1}^{x} J_k(i)$. It is known that $\sum_{d|\nu} J_k(d) = n^k$. Then we get the following theorem.

**Theorem (3):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) i^k = B(x)$

$B(x)$ is expanded by McCarthy to be

$$B(x) = \sum_{i=1}^{x} J_k(i) = \frac{n^r+1}{r+1}\mu_1 + O(n^r)$$

We therefore get $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) i^k = B(x) = \frac{n^k+1}{k+1}\mu_1 + O(n^k)$.

Likewise we can derive some other similar relationships using the $T$ matrix that are as listed below:

**Theorem (4):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) \sigma_\nu(i) = \sum_{i=1}^{x} i^k$ for $k \in Z^+$

**Theorem (5):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right)$ where the summation is over those $i$ values that are perfect squares equals $L(x)$

**Theorem (6):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) A(i) = \mu_1$ (as $H(x)$

**Theorem (7):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) 2^{\omega(n)} = \sum_{i=1}^{x} |\mu(i)| \sim \frac{6}{\pi^2} x^2 + O(\sqrt{n})$ = No. of Square Free Integers

**Theorem (8):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) d(n^2) = \sum_{i=1}^{x} 2^{\omega(i)}$ where $d(x)$ is the sum of all the divisors of $x$

**Theorem (9):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) d^2(n) = \sum_{i=1}^{x} d(i^2)$

**Theorem (10):** $\sum_{i=1}^{x} M\left(\left\lfloor \frac{x}{i} \right\rfloor \right) \frac{i}{\varphi(i)} = \sum_{i=1}^{x} \frac{\mu_1^2(i)}{\varphi(i)}$

Similarly many other relationships can be found between various arithmetic functions and the Mertens Functions.
3. A Likely Upper Bound of $|M(x)|$

The following conjecture is based on data collected for $x \leq 500,000$.

**Conjecture (1):** \( \log(x!) > \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 > \psi(x) \) when $x > 7$

By Stirling’s formula, \( \log(x!) = x \log(x) - x + O(\log(x)) \), since \( \log(x) \) increases more slowly than any positive power of \( \log(x) \), this is a better upper bound of \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 \) than \( x^{1+\varepsilon} \) for any $\varepsilon > 0$. This likely bound can be used to prove the Riemann Hypothesis since \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 > |M(x)| \) and therefore we can write \( \sqrt{\log(x!)} > |M(x)| \). Since the growth of \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 \) is lesser than \( x^{1+\varepsilon} \) for any $\varepsilon > 0$. We can say

\[
M(x)x^{-\frac{1}{2}-\varepsilon} \to 0 \text{ as } x \to \infty.
\]

Figure 1 for a plot of $\log(x!)$, \( \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 \), and $\psi(x)$ for $x = 1, 2, 3, \ldots, 1000$.

Let \( j(x) = \sum_{i=1}^{x} M(\lfloor x/i \rfloor)^2 \) where the summation is over $i$ values where $i|x$. Let \( l_1, l_2, l_3 \) denote the $x$ values where \( j(x) \) is a local maximum (that is, greater than all preceding \( j(x) \) values) and let \( m_1, m_2, m_3 \ldots \) denote the values of the local maxima. The local maxima occur at $x$ values that equal products of powers of small primes (Lagarias discussed colossally abundant numbers and their relationship to the Riemann hypothesis). See Figure 2 for a plot of \( l_i/(\log(l_i) m_i) \), \( m_i/l_i \), and \( 1/\log(l_i) \) for $i = 1, 2, 3, \ldots, 772$ (corresponding to the local maxima for $x \leq 15,000,000,000$). \( (M(x) \) Values for large $x$ were computed using Del’eglise and Rivat’s algorithm.) The first two curves cross frequently, so there are $i$ values where $m_i$ is approximately equal to \( l_i/\sqrt{\log(l_i)} \).
See Figure 3 for a plot of $j(x)$ and $\sum_{i=1}^{x} M(|x/i|)^2$ for $x = 1, 2, 3, \ldots, 10,000$. See Figure 4 for a plot of $\log(l_i), \log(m_i), \log(M(l_i)^2)$, and $\log(m_i/\sigma_0(l_i))$ for $i = 1, 2, 3, \ldots, 772$ (when $M(l_i) = 0, \log(M(l_i) 2$ ) is set to $-1$). See Figure 5 for a plot of $|M(l_i)|/\sqrt{l_i}$ for $i = 1, 2, 3, \ldots, 772$. The largest known value of $|M(x)|/\sqrt{x}$ (computed by Kotnik and van de Lune$^{14}$ for $x \leq 1014$) is $0.570591$ (for $M(7,766,842,813) = 50,286$). The largest $|M(l_i)|/\sqrt{l_i}$ value for $x \leq 15,000,000,000$ is $0.568887$ (for $l_i = 7,766,892,000$). The largest known value of $|M(x)|/\sqrt{x}$ (computed by Kuznetsov$^{15}$ is $0.585767684$ (for $M(11,609,864,264,058,592,345) = -1,995,900,927$).
Let $l_i$ and $m_i$ be similarly defined for the function $\sigma_0(x)$. ($l_i$, $i = 1, 2, 3, ...$ are known as "highly composite" numbers. Ramanujan\textsuperscript{16} initiated the study of such numbers. Robin\textsuperscript{17} computed the first 5000 highly composite numbers.) Let $m'_i$ denote $j(l_i)$. See Figure 6 for a plot of $l_i/\log(l_i)m'_i$, $m'_i/l_i$, and $1/\log(l_i)$ for $i = 2, 3, 4, \ldots, 160$ (corresponding to the local maxima for $x \leq 2, 244, 031, 211, 966, 544, 000$. (M(x) values for large x were computed using an algorithm similar to that used by Kuznetsov. The computations were done on an Intel i7-6700K CPU with 64 GB of RAM.) Although the first two curves cross frequently, $m'_i$ does not appear to converge to $l_i/\sqrt{\log(l_i)}$. 

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**Figure 4**

Plot of $\log(l_i)$, $\log(m_i)$, $\log(M(l_i)^2)$, and $\log(m_i/\sigma_0(l_i))$ for $i = 1, 2, 3, ..., 772$

**Figure 5**

Plot of $|M(l_i)|/\sqrt{l_i}$ for $i = 1, 2, 3, ..., 772$
Fig 6: Plot of $l_i/(\log(l_i)m'_i), m'_i/l_i$, and $1/\log(l_i)$ for $i = 2, 3, 4, ..., 160$

Fig 7: Plot of $\log(l_i) + \log(\log(l_i)), \log(l_i), \log(m'_i)$, and $\log(M(l_i)^2)$, for $i = 2, 3, 4, ... , 160$

See Figure 7 for a plot of $\log(l_i) + \log(\log(l_i)), \log(l_i), \log(m'_i)$, and $\log(M(l_i)^2)$, for $i = 2, 3, 4, ..., 160$ (when $M(l_i) = 0, \log(M(l_i)^2)$, is set to $-1$). The vertical distance between the first and third curves appears to become roughly constant. See Figure 8 for a plot of $(\log(l_i) + \log(\log(l_i)) - \log(m'_i)$, for $i = 2, 3, 4, ... , 160$.

See Figure 9 for a plot of $\log(l_i) + \frac{1}{2}\log(\log(l_i)), \log(\sum_{i=1}^{l_i} M([l_i/i])^2)$, and $\log(l_i)$ for $i = 2, 3, 4, ..., 160$.

$\log(l_i) + \frac{1}{2}\log(\log(l_i))$ is greater than $\log(\sum_{i=1}^{l_i} M([l_i/i])^2)$ and $\log(\sum_{i=1}^{l_i} M([l_i/i])^2)$ is greater than $\log(l_i)$ for $i > 4$. This is evidence in support of Conjecture 1. See Figure 10 for a plot of $\log(l_i) + \frac{1}{2}\log(\log(l_i)) - \log(\sum_{i=1}^{l_i} M([l_i/i])^2)$ for $i = 2, 3, 4, ..., 160$. 


Fig 8: For a plot of $\log(l_i) + \log(\log(l_i)) - \log(m'_i)$, for $i = 2, 3, 4... 160$.

Fig 9: Plot of $\log(l_i) + \frac{1}{2} \log(\log(l_i))$, $\log(\sum_{i=1}^{i} M(l_i/i)^{2})$, and $\log(l_i)$ for $i = 2, 3, 4, ..., 160$.

Fig 10: Plot of $\log(l_i) + \frac{1}{2} \log(\log(l_i)) - \log(\sum_{n=1}^{i} M(l_i/n)^{2})$ for $i = 2, 3, 4, ..., 160$. 
4. Conclusion

In this paper we derived new relations between Mertens function with a different arithmetic functions and also discussed about a likely upper bound of the absolute value of the Mertens function \( \sqrt{\log (x!)} > |\text{M}(x)| \) when \( x > 1 \) with sufficient numerical evidence. Using this likely upper bound we showed that we have a sufficient condition to prove the Riemann Hypothesis using the Littlewood condition.

5. References

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