On Thresholds for the Appearance of 2-cores in Mixed Hypergraphs
(Extended Abstract)

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Abstract. We study thresholds for the appearance of a 2-core in random hypergraphs that are a mixture of a constant number of random uniform hypergraphs each with a linear number of edges but with different edge sizes. For the case of two overlapping hypergraphs we give a solution for the optimal (expected) number of edges of each size such that the 2-core threshold for the resulting mixed hypergraph is maximized. We show that for adequate edge sizes this threshold exceeds the maximum 2-core threshold for any random uniform hypergraph, which can be used to improve the space utilization of several data structures that rely on this parameter.

1 Introduction

The 2-core of a hypergraph \( H \) is the largest induced sub-hypergraph (possibly empty), that has minimum degree at least 2. It can be obtained via a simple peeling procedure (Algorithm 1) that successively removes nodes of degree 1 together with their incident edge.

Algorithm 1: Peeling

Input: Hypergraph \( H \)
Output: Maximum induced sub-hypergraph with minimum degree 2.

while \( H \) has a node \( v \) of degree \( \leq 1 \) do
  if \( v \) is incident to an edge \( e \) then remove \( e \) from \( H \)
  remove \( v \) from \( H \)
return \( H \)

Let \( H_{n,p}^k \) be a random \( k \)-uniform hypergraph with \( n \) nodes where each of the possible \( \binom{n}{k} \) edges is present with probability \( p \) independent of the other edges. In the case that the expected number of edges equals \( c \cdot n \) for some constant \( c > 0 \), the following theorem (conjectured e.g. in [17], rigorously proved in [19] and independently in [13]) gives the threshold for the appearance of a 2-core in \( H_{n,p}^k \). Let

\[
t(\lambda, k) = \frac{\lambda}{k \cdot \left( \Pr(\text{Po}[\lambda] \geq 1) \right)^{k-1}},
\]

where \( \text{Po}[\lambda] \) denotes a Poisson random variable with mean \( \lambda \).

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Theorem 1 ([19, Theorem 1.2]). Let \( k \geq 3 \) be constant, and let \( c^*(k) = \min_{\lambda > 0} t(\lambda, k) \). Then for \( p = c \cdot n / \binom{n}{k} \) with probability \( 1 - o(1) \) for \( n \to \infty \) the following holds:

(i) if \( c < c^* \) then \( H_{n,p}^k \) has an empty 2-core,
(ii) if \( c > c^* \) then \( H_{n,p}^k \) has a non-empty 2-core.

Remark 1. Actually this is only a special case of [19, Theorem 1.2] which covers \( \ell \)-cores for \( k \)-uniform hypergraphs for all \( \ell \geq 2 \), \( k \) and \( \ell \) not both equal to 2.

Now consider a mixture of graphs \( H_{n,p}^k \) on \( n \) nodes for different values of \( p \) and \( k \).

Let \( H_{n,p}^k \) be a random hypergraph with \( n \) nodes where each of the possible \( \binom{n}{k} \) edges is present with probability \( p_i \), given via the vectors \( k = (k_1, k_2, \ldots, k_s) \) and \( p = (p_1, p_2, \ldots, p_s) \). While studying cores of hypergraphs in the context of cuckoo hashing the authors of [7] described how to extend the analysis of uniform hypergraphs to mixed hypergraphs, which directly leads to the following theorem. For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in [0,1]^s \) with \( \sum_{i=1}^{s} \alpha_i = 1 \) let

\[
t(\lambda, k, \alpha) = \frac{\lambda}{\sum_{i=1}^{s} \alpha_i \cdot k_i \cdot \left( \Pr \left( P_{\lambda} | \lambda \right) \geq 1 \right)^{k_i-1}}. 
\]

Theorem 2 (generalization of Theorem 1, implied by [7]). Let \( s \geq 1 \) be constant. For each \( 1 \leq i \leq s \) let \( k_i \geq 3 \) be constant, and let \( \alpha_i \in [0,1] \) be constant, where \( \sum_{i=1}^{s} \alpha_i = 1 \). Furthermore let \( c^*(k, \alpha) = \min_{\lambda > 0} t(\lambda, k, \alpha) \). Then for \( p_i = \alpha_i \cdot c \cdot n / \binom{n}{k_i} \) with probability \( 1 - o(1) \) for \( n \to \infty \) the following holds:

(i) if \( c < c^* \) then \( H_{n,p}^k \) has an empty 2-core,
(ii) if \( c > c^* \) then \( H_{n,p}^k \) has a non-empty 2-core.

Using ideas from [7, Section 4] this theorem can be proved along the lines of [19, Theorem 1.2] utilizing that \( H_{n,p}^k \) is a mixture of a constant number of independent hypergraphs.

Remark 2. Analogous to Theorem 1, Theorem 2 can also be generalized such that it covers \( \ell \)-cores for all \( \ell \geq 2 \).

Now consider hypergraphs \( H_{n,p}^k \) with edge probabilities \( p_i = \alpha_i \cdot c \cdot n / \binom{n}{k_i} \) as in Theorem 2. One can ask the following questions.

1. Assume \( k \) is given. What is the optimal vector \( \alpha^* \) such that the threshold \( c^*(k, \alpha^*) =: c^*(k) \) is maximal among all thresholds \( c^*(k, \alpha) \)? In other words, we want to solve the following optimization problem

\[
c^*(k) = \min_{\lambda > 0} t(\lambda, k, \alpha^*) = \max_{\alpha} \min_{\lambda > 0} t(\lambda, k, \alpha). 
\]

2. Is there a \( k \) such that \( \alpha^* \) gives some \( c^*(k) \) that exceeds the maximum 2-core threshold \( c^*(k) \) among all \( k \)-uniform hypergraphs (not mixed), which is known to be about 0.818 for \( k = 3 \), see e.g. [12,17, conjecture], [5, proof].

Remark 3. Often the 2-core threshold is given for hypergraph models slightly different from \( H_{n,p}^k \). The justification that some “common” hypergraph models are equivalent in terms of this threshold is given in Section 1.2.
1.1 Results

We give the solution for the non-linear optimization problem (3) for \( s = 2 \). That is for each \( k = (k_1, k_2) \) we either give optimal solutions \( \alpha^* = (\alpha^*, 1 - \alpha^*) \) and \( c^*(k) \) in analytical form or identify a subset of the interval \((0, 1)\) where we can use binary search to determine \( \alpha^* \) and therefore \( c^*(k) \) numerically with arbitrary precision. Interestingly, it turns out that for adequate edge sizes \( k_1 \) and \( k_2 \) the maximum 2-core threshold \( c^*(k) \) exceeds the maximum 2-core threshold \( c^*(k) \) for \( k \)-uniform hypergraphs. The following table lists some values.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
(k_1, k_2) & (3, 3) & (3, 4) & (3, 6) & (3, 8) & (3, 10) & (3, 12) & (3, 14) & (3, 16) \\
\hline
\tilde{c}^* & 0.81847 & 0.82151 & 0.83520 & 0.85138 & 0.86752 & 0.88298 & 0.89761 & 0.91109 \\
\alpha^* & 0.83596 & 0.85419 & 0.86512 & 0.87315 & 0.87946 & 0.88464 & 0.88684 & 0.88743 \\
k & 3.16404 & 3.43744 & 3.67439 & 3.88795 & 4.08482 & 4.26898 & 4.47102 & 5.02626 \\
\hline
\end{array}
\]

Table 1. Optimal 2-core thresholds \( c^*(k), k = (k_1, k_2), \) and \( \alpha^* = (\alpha^*, 1 - \alpha^*) \), and \( k = \alpha^* \cdot k_1 + (1 - \alpha^*) \cdot k_2 \). The values are rounded to the nearest multiple of \( 10^{-5} \).

More comprehensive tables for parameters \( 3 \leq k_1 \leq 6 \) and \( 1 \leq k_2 \leq 50 \) are given in Appendix B. The maximum threshold found is about 0.92 for \( k = (3, 21) \).

Remark 4. So why does it help to use edges of different sizes? Consider a \( k \)-uniform hypergraph \( H_{n,p}^k \) that has a non-empty 2-core with node set \( V \) and edge set \( E \). Let \( V' \) be the set of nodes outside \( V \), that is \( V' \cap V = \emptyset \). Assume that \( c \) is just above \( c^*(k) \). Then there are many small sets \( E_1, E_2, \ldots \subset E \), such that if one removes all edges of any of these sets, the 2-core of the remaining hypergraph would be empty. Now randomly replace a constant fraction \( \beta = 1 - \alpha \) of the edges of \( H_{n,p}^k \) by edges of larger size. If \( \beta \) is large enough, then it is likely that there exists a set \( E_i \), where all of the edges are substituted and all of the corresponding larger edges are incident with nodes from \( V' \). Consider an arbitrary large edge \( e \) with \( e \cap V' \neq \emptyset \). If \( \beta \) is small enough, then it is likely that there is at least one node \( v \) from \( e \cap V' \) that is incident to only one large edge (namely \( e \)). It follows that \( e \) will be removed by the standard peeling algorithm (Algorithm 1). Hence, if \( \beta \) is not too small and not too large, then it is likely that there exists a set \( E_i \) whose edges are substituted by larger edges that will be removed by the peeling algorithm, which results in an empty 2-core.

1.2 Extensions to Other Hypergraph Models

While Theorems 1 and 2 are stated for hypergraphs \( H_{n,p}^k \), one often considers slightly different hypergraphs, e.g. in the analysis of data structures.

Let \( H_{n,m,\alpha}^k \) and \( \tilde{H}_{n,m,\alpha}^k \) be random hypergraphs with \( n \) nodes and \( m \) edges, where for each \( 1 \leq i \leq s \), a fraction of \( \alpha_i \) of the edges are fully randomly chosen from the set of all possible edges of size \( k_i \). In the case of \( H_{n,m,\alpha}^k \) the random edge choices are made without replacement and in the case of \( \tilde{H}_{n,m,\alpha}^k \) the random edge choices are made with replacement. Using standard arguments, one sees that if \( m = c \cdot n, k_i \geq 3 \), and \( p_i = \alpha_i \cdot c \cdot n / \binom{n}{k_i} \) as in the situation of Theorem 2, the 2-core threshold of \( H_{n,m,\alpha}^k \) is the same as for \( H_{n,p}^k \) (see e.g. [9, analogous to Proposition 2]), and the 2-core threshold of \( \tilde{H}_{n,m,\alpha}^k \) is the same as for \( H_{n,m,\alpha}^k \) (see e.g. [9, analogous to Proposition 1]).
1.3 Related Work

Non-uniform hypergraphs have proven very useful in the design of erasure correcting codes, such as Tornado codes [16,15], LT codes [14], Online codes [18], and Raptor codes [20]. Each of these codes heavily rely on one or more hypergraphs where the hyperedges correspond to variables (input/message symbols) and the nodes correspond to constraints on these variables (encoding/check symbols). An essential part of the decoding process of an encoded message is the application of a procedure that can be interpreted as peeling the hypergraph (see Algorithm 1) associated with the recovery process, where it is required that the result is an empty 2-core. Given \( m \) message symbols, carefully designed non-uniform hypergraphs allow, in contrast to uniform ones, to gain codes where in the example of Tornado, Online, and Raptor codes a random set of \((1+\varepsilon) \cdot m\) encoding symbols are necessary to decode the whole message in linear time (with high probability), and in the case of LT codes a random set of \( m + o(m) \) encoding symbols are necessary to decode the whole message in time proportional to \( m \cdot \ln(m) \) (with high probability). Tornado codes use explicit underlying hypergraphs designed for a given fixed code rate, whereas LT codes and its improvements, Online and Raptor codes, use implicit graph constructions to generate essentially infinite hypergraphs resulting in so called rateless codes. In the case of Tornado codes the size of the hyperedges as well as the degree of the nodes follow precalculated sequences that are optimized to obtain the desired properties. In the case of LT codes, as well as in the last stage of Raptor and Online codes each node chooses its degree at random according to some fixed distribution, and then selects its incident hyperedges uniformly at random. (For Online codes also a skewed selection of the hyperedges is discussed, see [18, Section 7].) While the construction of the non-uniform hypergraph used for these codes is not quite the same as for \( \tilde{H}_{n,m,\alpha} \) (or \( H_{n,p}^k, H_{n,m,\alpha}^k \)), since, among other reasons, the degree of the nodes is part of the design, they are similar enough to seemingly make the optimization methods / heuristics of [15] applicable, see footnote [11, page 10]. Having said that, compared to e.g. [15], our optimization problem is easier in the sense that it has fewer free parameters and harder in the sense that we are seeking a global optimum.

1.4 Overview of the Paper

In the next section we discuss the effect of our results on three succinct data structures. Afterwards, we give our main theorem that shows how to determine optimal 2-core thresholds for mixed hypergraphs with two different edge sizes. It follows a section with experimental evaluation of the appearance of 2-cores for a few selected mixed hypergraphs, which underpins our theoretical results. We conclude with a short summary and an open question.

2 Some Applications to Succinct Data Structures

Several succinct data structures are closely related to the 2-core threshold of \( k \)-uniform hypergraphs (often one considers \( \tilde{H}_{n,m,\alpha}^k \) for \( s = \alpha_1 = 1 \) and \( k_1 = 3 \)). More precisely, the space usage of these data structures is inversely proportional to the 2-core threshold \( c^*(k) \), while the evaluation time is proportional to the...
edge size $k$. By showing that the value of $c^*(3)$ can be improved using mixed
hypergraphs instead of uniform ones, our result opens a new possibility for a
space–time tradeoff regarding these data structures, allowing to further reduce
their space needs at the cost of a constant increase in the evaluation time. Below
we briefly sketch three data structures and discuss possible improvements, where
we make use of the following definitions.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a vector with $n$ cells each of size $r$ bits. Let $S = \{x_1, x_2, \ldots, x_m\}$ be a set of $m$ keys, where $S$ is subset of some universe $U$ and it holds $m = c \cdot n$ for some constant $c < 1$. The vector cells correspond to
nodes of a hypergraph and the keys from $S$ are mapped via some function $\varphi$ to
a sequence of vector cells and therefore correspond to hyperedges. We identify
cells (and nodes) via their index $i$, $1 \leq i \leq n$, whereas $a_i$ stands for the value of
cell $i$. The following three data structures essentially consist of a vector $\mathbf{a}$ and a
mapping $\varphi$. For each data structure we compare their performance, depending if
$\varphi$ realizes a uniform or a mixed hypergraph. In the case of a uniform hypergraph
each key $x_j$ is mapped to $k = 3$ random nodes $\varphi(x_j) = (g_1(x_j), g_2(x_j), g_3(x_j))$ via functions $g_1, g_2, g_3 : U \to \{1, 2, \ldots, n\}$. In the case of a mixed hypergraph,
as an example, a fraction of $\alpha^* = 0.88684$ keys are mapped to 3 random nodes using functions $g_1, g_2, g_3$ and a fraction of $1 - \alpha^*$ keys are mapped to 16 random
nodes via functions $g'_1, g'_2, \ldots, g'_{16} : U \to \{1, 2, \ldots, n\}$. We fix $c$ below the 2-core
threshold to $c = c^* - 0.005$, which gives $c = 0.813$ in the uniform case and $c = 0.906$ in the mixed case, cf. Table 1. The reason why we use a rather small
distance of 0.005 is that for large $m$ one observes a fairly sharp phase transition
from “empty 2-core” to “non-empty 2-core” in experiments, cf. Section 4.

2.1 Invertible Bloom Lookup Table

The invertible Bloom Lookup Table [11] (IBLT) is a Bloom filter data structure
that, amongst others, supports a complete listing of the inserted elements (with
high probability). We restrict ourselves to the case where the IBLT is optimized
for the listing operation and we assume without loss of generality that the keys
from $S$ are integers. Each vector cell contains a summation counter and a quantity
counter, initialized with 0. The keys arrive one by one and are inserted into
the IBLT. Inserting a key $x_j$ adds its value to the summation counter and in-
crements the quantity counter at each of the cells given via $\varphi(x_j)$. To list the
inserted elements of the IBLT one essentially uses the standard peeling process for
finding the 2-core of the underlying hypergraph (see Algorithm 1). While
there exists a cell where the quantity counter has value 1, extract the value of
the summation counter of this cell which gives some element $x_j$. Determine the
summation counters and quantity counters associated with $x_j$ via evaluating
$\varphi(x_j)$ and subtract $x_j$ from the summation counters and decrement the quantity
counters. With this method a complete listing of the inserted elements is
possible if the 2-core of the hypergraph is empty. Therefore in the case of uni-
form hypergraphs we get a space usage of $n/m \cdot r \approx 1.23 \cdot r$ bits per key. As
already pointed out by the authors of [11], who highlight parallels to erasure
correcting codes (see Section 1.3), a non-uniform version of the IBLT where keys
have a different number of associated cells could improve the maximum fraction
\(c = m/n\) where a complete listing is successful with high probability. Using our example of mixed hypergraphs leads to such an improved space usage of about \(1.10 \cdot r\) bits per key.

### 2.2 Retrieval Data Structure

Given a set of key-value pairs \(\{(x_j, v_j) \mid x_j \in S, v_j \in R, j \in [m]\}\), the retrieval problem is the problem of building a function \(f : U \to R\) such that for all \(x_j\) from \(S\) it holds \(f(x_j) = v_j\); for any \(y\) from \(U \setminus S\) the value \(f(y)\) can be an arbitrary element from \(R\). Chazelle et al. [4] gave a simple and practical construction of a retrieval data structure, consisting of a vector \(a\) and some mapping \(\varphi\) that has constant evaluation time, via simply calculating \(f(x_j) = \bigoplus_{i \in \varphi(x_j)} a_i\). The construction is based on the following observation, which is stated more explicitly in [3]. Let \(v = (v_1, v_2, \ldots, v_m)\) be the vector of the function values and let \(M\) be the \(m \times n\) incidence matrix of the underlying hypergraph, where the characteristic vector of each hyperedge is a row vector of \(M\). If the hypergraph has an empty 2-core then the linear system \(M \cdot a = v\) can be solved in linear time. For appropriate \(c\) this gives expected linear construction time. As before, in the case of uniform hypergraphs the space usage is about \(1.23 \cdot r\) bits per key, assuming that the values \(v_j\) are bit strings of length \(r\). And in our example of mixed hypergraphs the space usage is about \(1.10 \cdot r\) bits per key at the cost of a slight increase of the evaluation time of \(f\).

In [8] it is shown how to obtain a retrieval data structure with space usage of \((1 + \varepsilon) \cdot r\) bits per key, for any fixed \(\varepsilon > 0\), evaluation time \(O(\log(1/\varepsilon))\), and linear expected construction time, while using essentially the same construction as above. The central idea is to transfer the problem of solving one large linear system into the problem of solving many small linear systems, where each system fits into a single memory word and can be solved via precomputed pseudoinverses. As shown in [1] this approach is limited in its practicability but can be adapted to build retrieval data structures with \(1.10 \cdot r\) bits per key (and fewer) for realistic key set sizes. But this modified construction could possibly be outperformed by our direct approach of solving one large linear system in expected linear time.

### 2.3 Perfect Hash Function

Given a set of keys \(S\), the problem of perfect hashing is to build a function \(h : U \to \{1, 2, \ldots, n\}\) that is 1-to-1 on \(S\). The construction from [3] and [4] gives a data structure consisting of a vector \(a\) and some mapping \(\varphi\) that has constant evaluation time. Formulated in the context of retrieval, one builds a vector \(v = (v_1, v_2, \ldots, v_m)\) such that each key \(x_j\) is associated with a value \(f(x_j) = v_j\) that is the index \(\iota\) of the position of a node in the sequence \(\varphi(x_j)\). This node must have the property that if one applies the peeling process to the underlying hypergraph (Algorithm 1) it will be selected and removed because it gets degree 1. If \(c\) is below the 2-core threshold then with high probability for each \(x_j\) there exists such an index \(\iota\), and the linear system \(M \cdot a = v\) can be solved in linear time. Given the vector \(a\) the evaluation of \(h\) is done via \(h(x_j) = \varphi(x_j)_\iota\), where \(\iota = \bigoplus_{i \in \varphi(x_j)} a_i\).
In the case of a 3-uniform hypergraph one gets a space usage of about $1.23 \cdot 2$ bits per key, since there are at most 3 different entries in $a$. If one applies a simple compression method that stores every 5 consecutive elements from $a$ in one byte, one gets a space usage of about $1.23 \cdot 8/5 \approx 1.97$ bits per key. The range of $h$ is $n = 1.23 \cdot m$.

In contrast to the examples above, improving this data structure by simply using a mixed hypergraph is not necessarily successful, since the increase of the load $c$ is compensated by the increase of the maximum index in the sequence $\varphi(x_j)$, which in our example would lead to a space usage of about $1.10 \cdot 4$ bits per key for uncompressed $a$, since we use up to 16 functions for $\varphi(x_j)$. However, this can be circumvented by modifying the construction of the vector $v$ as follows. Let $G = (S \cup \{1, 2, \ldots, n\}, E)$ be a bipartite graph with edge set $E = \{x, g_i(x_j)\} | x_j \in S, i \in \{1, 2, 3\}$. According to the results on 3-ary cuckoo hashing, see e.g. [9,7], it follows that for $c < 0.917$ (as in our case) the graph $G$ has a left-perfect matching with high probability. Given such a matching one stores in $v$ for each key $x_j$ the index $i$ of $g_i$ that has the property that $\{x_j, g_i(x_j)\}$ is a matching edge. Now given the solution of $M \cdot a = v$, the function $h$ is evaluated via $h(x_j) = g_i(x_j)$ where $i = \bigoplus_{\varphi(x_j)} a_i$. Since $a$ has at most three different entries it follows that the space usage in our mixed hypergraph case is about $1.10 \cdot 2$ bits per key. Using the same compression as before, the space usage can be reduced to about $1.10 \cdot 8/5 = 1.76$ bits per key. Now the range of $h$ is $n = 1.10 \cdot m$. Solving the linear system can be done in expected linear time. It is conjectured that if $G$ has a matching then it is found by the $(k, 1)$-generalized selfless algorithm from [6, Section 5]; this algorithm can be implemented to work in expected linear time.

A more flexible trade-off between space usage and range yields the CHD algorithm from [2]. This algorithm allows to gain ranges $n = (1 + \varepsilon) \cdot m$ for arbitrary $\varepsilon > 0$ in combination with a adjustable compression rate that depends on some parameter $\lambda$. For example, using a range of about $1.11 \cdot m$, a space usage of 1.65 bits per key is achievable, see [2, Fig. 1(b), $\lambda = 5$]. But since the expected construction time of the CHD algorithm is $O(m \cdot (2^\lambda + (1/\varepsilon)^\lambda))$ [2, Theorem 2], our approach could be faster for a comparable space usage and range.

3 Maximum Thresholds for the Case $s = 2$

In this section we state our main theorem that gives a solution for the non-linear optimization problem (3) for the case $s = 2$, that is given two edge sizes we show how to compute the optimal (expected) fraction of edges of each size such that the threshold of the appearance of a 2-core of a random hypergraph using this configuration is maximal.

Let $k = (a, b)$ with $a \geq 3$, and $b > a$. Furthermore, let $\alpha = (\alpha, 1 - \alpha)$ and$^1$ $\alpha \in (0, 1]$, as well as $\lambda \in (0, +\infty)$. Consider the following threshold function as a special case of (2)

$$t(\lambda, a, b, \alpha) = \frac{\lambda}{\alpha \cdot a \cdot (1 - e^{-\lambda})^{a-1} + (1 - \alpha) \cdot b \cdot (1 - e^{-\lambda})^{b-1}}.$$ (4)

$^1$ We can exclude the case $\alpha = 0$, since if $3 \leq a < b$, then it holds that $c^*(a) > c^*(b)$.
We transform \( t(\lambda, a, b, \alpha) \) in a more manageable function using a monotonic and bijective domain mapping via \( z = 1 - e^{-\lambda} \) and \( \lambda = -\ln(1 - z) \). Hence the transformed threshold function is

\[
T(z, a, b, \alpha) = \frac{-\ln(1 - z)}{\alpha \cdot a \cdot z^{a-1} + (1 - \alpha) \cdot b \cdot z^{b-1}},
\]

where \( z \in (0, 1) \). According to (3) and using \( T(z, a, b, \alpha) \) instead of \( t(\lambda, a, b, \alpha) \) the optimization problem is defined as

\[
\max_{\alpha \in (0, 1)} \min_{z \in (0, 1)} T(z, a, b, \alpha).
\]

For a short formulation of our results we make use of the following three auxiliary functions.

\[
f(z) = -\frac{\ln(1 - z) \cdot (1 - z)}{z}
\]

\[
g(z, a, b) = f(z) \cdot (b - 1) \cdot (a - 1) + \frac{1}{1 - z} + 2 - b - a
\]

\[
h(z, a, b) = \frac{a \cdot z^{a-b} - b - f(z) \cdot (a \cdot (a - 1) \cdot z^{a-b} - b \cdot (b - 1))}{b \cdot ((b - 1) \cdot f(z) - 1)}.
\]

Furthermore we need to define some “special” points.

\[
z' = \left(\frac{a}{b}\right)^{1/a}, \quad z_l = f^{-1}\left(\frac{1}{a-1}\right), \quad z_r = f^{-1}\left(\frac{1}{b-1}\right)
\]

\[
z_1 = \min\{z \mid g(z) = 0\}, \quad z_2 = \max\{z \mid g(z) = 0\}.
\]

It can be shown that if \( z_1 \) and \( z_2 \) exist, then it holds \( z' \neq z_1 \) and \( z' \neq z_2 \). Now we can state our main theorem.\(^2\)

**Theorem 3.** Let \( a, b \) be fixed and let \( T(z^*, \alpha^*) = \max_{\alpha \in (0, 1)} \min_{z \in (0, 1)} T(z, \alpha) \). Then the following holds:

1. Let \( \min_z g(z) \geq 0 \).
   
   (i) If \( h(z') \leq 1 \) then the optimal point is \((z^*, \alpha^*) = (z_l, 1)\) and the maximum threshold is given by
   
   \[
   T(z^*, \alpha^*) = \frac{-\ln(1 - z_l)}{a \cdot z_l^{a-1}}.
   \]

   (ii) If \( h(z') > 1 \) then the optimal point is the saddle point
   
   \[
   (z^*, \alpha^*) = \left(\frac{a}{b}\right)^{\frac{1}{b-a}} \cdot \frac{b-1}{b-a} - \frac{1}{f(z^*) \cdot (b-a)}
   \]

   and the maximum threshold is given by
   
   \[
   T(z^*, \alpha^*) = -\ln \left(1 - \left(\frac{a}{b}\right)^{\frac{1}{b-a}} \cdot \left(\frac{b^{a-1}}{a^{b-1}}\right)^{\frac{1}{b-a}}\right).
   \]

\(^2\) For any function \( \phi = \phi(\cdot, x) \) we will use \( \phi(\cdot) \) and \( \phi(\cdot, x) \) synonymously, if \( x \) is considered to be fixed.
2. Let \( \min_z g(z) < 0 \).

(i) If \( h(z') \leq 1 \) then the optimum is the same as in case 1(i).

(ii) If \( h(z') \in (1, h(z_2)] \) then the optimum is the same as in case 1(ii).

(iii) If \( h(z') \in (h(z_2), h(z_1)) \) then there are two optimal points \((z^*, \alpha^*)\) and \((z^{**}, \alpha^*)\). It holds \(1/\alpha^* = h(z^*) = h(z^{**})\) and \(T(z^*, \alpha^*) = T(z^{**}, \alpha^*)\).

The optimal points can be determined numerically using binary search for the value \( \alpha \) that gives \( T(\tilde{z}_1, \alpha) = T(\tilde{z}_2, \alpha) \), where \( \alpha \) is from the interval \([1/h(z_{up}), 1/h(z_{lo})]\) and it holds \( h(\tilde{z}_1) = h(\tilde{z}_2) = 1/\alpha \), with \( \tilde{z}_1 \) from \((z_1, z_{up})\), and \( \tilde{z}_2 \) from \((z_{lo}, z_r)\). The (initial) interval for \( \alpha \) is:

- \([1/h(z_1), 1/h(z_2)]\), if \( z_1 < z' < z_2 \),
- \([1/h(z'), 1/h(z_2)]\), if \( z' < z_1 \),
- \([1/h(z_1), 1/h(z')]\), if \( z' > z_2 \).

(iv) If \( h(z') \in [h(z_1), \infty) \) then the optimum is the same as in case 1(ii).

Sketch of Proof. Assume first that \( \alpha \in (0, 1) \) is arbitrary but fixed, that is we are looking for a global minimum of \((5)\) in z-direction. Since \( \lim_{z \to 0} T(z) = \lim_{z \to 1} T(z) = +\infty \) and \( T(z) \) is continuous for \( z \in (0, 1) \), a global minimum must be a point where the first derivative of \( T(z) \) is zero, that is a critical point. Let \( \tilde{z} \) be a critical point of \( T(z) \) then it must hold \( \tilde{z} \in [z_l, z_r] \) and \( \alpha = 1/h(\tilde{z}) \).

Consider the case \( \min g(z) > 0 \). (The \( \min g(z) = 0 \) can be handled analogously). Since \( \frac{\partial h(z)}{\partial z} > 0 \Leftrightarrow g(z) > 0 \), the function \( h(z) \) is monotonically increasing in \([z_l, z_r]\). Furthermore it holds, if \( g(\tilde{z}) > 0 \) then \( \tilde{z} \) is a local minimum of \( T(z) \). It follows that for each \( \alpha \) there is only one critical point \( \tilde{z} \) and according to the monotonicity of \( T(z) \) this must be a global minimum point. Now consider the function of critical points \( \tilde{T}(z) := T(z, 1/h(\tilde{z})) \) of \( T(z, \alpha) \). It holds that

\[
\forall z < z': \frac{\partial \tilde{T}(z)}{\partial z} > 0 \Leftrightarrow g(z) > 0 \quad \text{and} \quad \forall z > z': \frac{\partial \tilde{T}(z)}{\partial z} < 0 \Leftrightarrow g(z) > 0 .
\]

It follows that the function of critical points has a global maximum at \( z' = \left( \frac{\tilde{z}}{0} \right)^{\frac{1}{1-\alpha}} \), where \( z' \) is at the same time a global minimum of \( T(z, \alpha) \) in z-direction. If \( h(z') > 1 \) then \( \alpha = 1/h(z') \in (0, 1) \) and the optimum point \((z^*, \alpha^*)\) is \((z', 1/h(z'))\), which is the only saddle point of \( T(z, \alpha) \). If \( h(z') \leq 1 \) then because the monotonicity of \( T(z) \) the solution for \( \alpha^* \) is 1 (degenerated solution). Since \( h(z_1) = 1 \) it follows that that \((z^*, \alpha^*) = (z_1, 1)\).

Consider the case \( \min g(z) < 0 \). The function \( g(z) \) has exactly two roots, \( z_1 \) and \( z_2 \), and for \( z \in (z_1, z_r) \) the function \( h(z) \) is strictly increasing to a local maximum at \( z_1 \), is then strictly decreasing to a local minimum at \( z_2 \), and is strictly increasing afterwards. Now for fixed \( \alpha \) there can be more than one critical point and one has to do a case-by-case analysis. A complete proof of the theorem is given in Appendix A.

The distinction between case 1 and case 2 of Theorem 3 can be done via solving \( \frac{\partial g(z)}{\partial z} = 0 \), for \( z \in (0, 1) \), since the function \( g(z) \) has only one critical point and this point is a global minimum point. Hence, Theorem 3 can be easily transferred into an algorithm that determines \( \alpha^*, z^* \) and \( T(z^*, \alpha^*) \) for given \( k = (a, b) \). (The
pseudocode of such an algorithm is given at the end of Appendix A.) Some results for \( c^*(k) = t(\lambda^*, a, b, \alpha^*) = T(z^*, a, b, \alpha^*) \) for selected \( k = (a, b) \) are given in Table 1 and Appendix B. They show that the optimal 2-core threshold of mixed hypergraphs can be above the 2-core threshold for 3-uniform hypergraphs.

4 Experiments

In this section we consider mixed hypergraphs \( \tilde{H}^{k}_{n,m,\alpha} \) as described in Section 1.2. For the parameters \( k = (k_1, k_2) \in \{(3, 4), (3, 8), (3, 16), (3, 21)\} \) and the corresponding optimal fractions of edge size \( \alpha^* \) we experimentally approximated the point \( c^*(k) \) of the phase transition from empty to non-empty 2-core.

For each fixed tuple \( (k, \alpha^*) \) we performed the following experiments. We fixed the number of nodes to \( n = 10^7 \) and considered growing equidistant edge densities \( c = m/n \). The densities covered an interval of size 0.008 with the theoretical 2-core threshold \( c^*(k) \) in its center. For each quintuple \( (k_1, k_2, \alpha^*, n, c) \) we constructed \( 10^2 \) random hypergraphs \( \tilde{H}^{k}_{n,m,\alpha} \) with nodes \( \{1, 2, \ldots, n\} \) and \( c \cdot \alpha^* \cdot n \) edges of size \( k_1 \) and \( c \cdot (1 - \alpha^*) \cdot n \) edges of size \( k_2 \). For the random choices of each edge we used the pseudo random number generator MT19937 “Mersenne Twister” of the GNU Scientific Library [10]. Given a concrete hypergraph we applied Algorithm 1 to determine if the 2-core is empty. A non-empty 2-core was considered as failure, an empty 2-core was considered as success. We measured the failure rate and determined an approximation of the 2-core threshold, via fitting the sigmoid function

\[
\sigma(c; x, y) = \left(1 + \exp\left(-\frac{(c - x)}{y}\right)\right)^{-1}
\]

to the measured failure rate using the “least squares fit” of gnuplot [21]. The resulting fit parameter \( x = x(k) \) is our approximation of the theoretical threshold \( c^*(k) \). Table 2 compares \( c^*(k) \) and \( x(k) \). The quality of the approximation is quantified in terms of the sum of squares of residuals \( \sum_{\text{res}} \). The results show a difference of theoretical and experimentally estimated threshold of less than \( 2 \cdot 10^{-4} \). The corresponding plots of the measured failure rates and the fit function are shown in Figures 1, 2, 3 and 4.

![Fig. 1. (k_1, k_2) = (3, 4) ![Fig. 2. (k_1, k_2) = (3, 8)
Fig. 3. \((k_1, k_2) = (3, 16)\)  
Fig. 4. \((k_1, k_2) = (3, 21)\)

\[
\begin{array}{c|cccc}
(k_1, k_2) & (3, 4) & (3, 8) & (3, 16) & (3, 21) \\
\hline
\bar{c}^* & 0.82151 & 0.85138 & 0.91089 & 0.92004 \\
x & 0.82147 & 0.85135 & 0.91070 & 0.91985 \\
\sum_{\text{res}} & 0.00536 & 0.00175 & 0.00348 & 0.01091 \\
\end{array}
\]

Table 2. Comparison of experimentally approximated and theoretical 2-core thresholds. The values are rounded to the nearest multiple of \(10^{-5}\).

5 Summary and Future Work

We have shown that the threshold for the appearance of a 2-core in mixed hypergraphs can be larger than the 2-core threshold for \(k\)-uniform hypergraphs, for each \(k \geq 3\). Moreover, for hypergraphs with two given constant edges sizes we showed how to determine the optimal (expected) fraction of edges of each size, that maximizes the 2-core threshold. The maximum threshold found for \(3 \leq k_1 \leq 6\) and \(k_1 \leq k_2 \leq 50\) is about 0.92 for \(k = (3, 21)\). We conjecture that this is the best possible for two edge sizes.

Based on the applications of mixed hypergraphs, as for example discussed in Section 2, the following question seems natural to ask. Consider the hypergraph \(\tilde{H}^{k}_{n,m,\alpha}\) and some fixed upper bound \(\bar{K}\) on the average edge size \(\bar{k} = \sum_{i=1}^{s} \alpha_i \cdot k_i\).

**Question.** Which pair of vectors \(k\) and \(\alpha\) that gives an average edge size below \(\bar{K}\) maximizes the threshold for the appearance of a 2-core? That means we are looking for the solution of \( \min_{k, \alpha} \lambda > 0 \) \( t(\lambda, k, \alpha) \) under the constraint that \( \bar{k} \leq \bar{K} \).

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A Proof of the Main Theorem

In this section we give the full proof of Theorem 3, i.e. we solve the (transformed) non-linear optimization problem (6). As is to be expected, the proof mainly employs methods from calculus.

A.1 Preliminaries

Derivatives. At first we want to determine the partial derivatives of $T(z, a, b, \alpha)$ with respect to $z$ and $\alpha$. To shorten and simplify notation we use the following definitions. For all $j \in \mathbb{N}$ let

$$D_j(z, a, b, \alpha) = \alpha \cdot a \cdot (a - 1)^j \cdot z^{a - 1} + (1 - \alpha) \cdot b \cdot (b - 1)^j \cdot z^{b - 1}$$

and

$$Z_j(z, a, b) = a \cdot (a - 1)^j \cdot z^{a - 1} - b \cdot (b - 1)^j \cdot z^{b - 1}.$$

The first partial derivatives of $T(z, \alpha)$ are

$$\frac{\partial T(z, \alpha)}{\partial z} = \frac{1}{1 - z} \cdot \frac{1}{D_0(z, \alpha)} + \frac{\ln(1 - z)}{z} \cdot \frac{D_1(z, \alpha)}{D_0(z, \alpha)^2} \quad \text{and} \quad (12)$$

$$\frac{\partial T(z, \alpha)}{\partial \alpha} = \frac{\ln(1 - z) \cdot Z_0(z)}{D_0(z, \alpha)^2}. \quad (13)$$

The second partial derivatives of $T(z, \alpha)$ are

$$\frac{\partial^2 T(z, \alpha)}{(\partial z)^2} = \frac{1}{(1 - z)^2} \cdot \frac{1}{D_0(z, \alpha)} - \frac{2}{z \cdot (1 - z)} \cdot \frac{D_1(z, \alpha)}{D_0(z, \alpha)^2} \quad \text{and} \quad (14)$$

$$+ \frac{\ln(1 - z)}{z^2} \cdot \frac{D_2(z, \alpha) - D_1(z, \alpha)}{D_0(z, \alpha)^2} - \frac{2 \cdot \ln(1 - z)}{z^2} \cdot \frac{D_1(z, \alpha)^2}{D_0(z, \alpha)^3}$$

$$\frac{\partial^2 T(z, \alpha)}{(\partial \alpha)^2} = - \frac{2 \cdot \ln(1 - z) \cdot Z_0(z)^2}{D_0(z, \alpha)^3} \quad (15)$$

$$\frac{\partial}{\partial z} \left( \frac{\partial T(z, \alpha)}{\partial \alpha} \right) = - \frac{1}{1 - z} \cdot \frac{Z_0(z)}{D_0(z, \alpha)^2} + \frac{\ln(1 - z)}{z} \cdot \frac{Z_1(z)}{D_0(z, \alpha)^2} \quad (16)$$

$$\frac{\partial}{\partial \alpha} \left( \frac{\partial T(z, \alpha)}{\partial \alpha} \right) = - \frac{2 \cdot \ln(1 - z)}{z} \cdot \frac{Z_0(z) \cdot D_1(z, \alpha)}{D_0(z, \alpha)^3}.$$

Auxiliary Functions. Our analysis is heavily based on three functions,

$$f(z) = - \frac{\ln(1 - z) \cdot (1 - z)}{z} \quad (17)$$

$$g(z, a, b) = f(z) \cdot (b - 1) \cdot (a - 1) + \frac{1}{1 - z} + 2 - b - a \quad (18)$$

$$h(z, a, b) = \frac{a \cdot z^{a - b} - b \cdot f(z) \cdot (a \cdot (a - 1) \cdot z^{a - b} - b \cdot (b - 1))}{b \cdot ((b - 1) \cdot f(z) - 1)}$$

$$= \frac{Z_0(z, a, b) - f(z) \cdot Z_1(z, a, b)}{b \cdot z^{b - 1} \cdot ((b - 1) \cdot f(z) - 1)}, \quad (19)$$
Lemma 1 (Properties of the appendix. We start with the three auxiliary functions. Therefore the proofs of the next four lemmas are only given in extra sections of the appendix. We start with the three auxiliary functions.

\[ z' = \left(\frac{a}{b}\right)^{-1} \quad \quad z_1 = \min\{z \mid g(z) = 0\} \quad \quad z_2 = \max\{z \mid g(z) = 0\} \quad \quad z_r = f^{-1}\left(\frac{1}{b-1}\right) \]

Our line of argument will rely on essential properties of \( f, g, h \) and \( z_l, z_r, z_1, z_2 \) and \( z' \). Proving these properties is standard calculus but unfortunately lengthy. Therefore the proofs of the next four lemmas are only given in extra sections of the appendix. We start with the three auxiliary functions.

Lemma 1 (Properties of \( f(z) \)).

Let \( z \in (0, 1) \), then it holds

(i) \( f(z) > 1 - z > 0 \).

(ii) \( \lim_{z \to 0} f(z) = 1 \).

(iii) \( \lim_{z \to 1} f(z) = 0 \).

(iv) \( f(z) \) is strictly decreasing.

(v) \( f(z) \) is concave.

(vi) \( f(z') > f(z_r) = \frac{1}{1-b} \).

(vii) \( f(z) \neq -\frac{1}{1-z'}, -2 + b + a \).

The proof of Lemma 1 is given in Appendix C. A plot of \( f(z) \) is shown in Figure 5.

Lemma 2 (Properties of \( g(z) \)).

Let \( 3 \leq a < b \), \( z \in (0, 1) \), then it holds

(i) \( g(z) \) is strictly decreasing, reaches a global minimum and is then strictly increasing. The global minimum point is the only point where \( \frac{dg(z)}{dz} = 0 \).

(ii) \( g(z) > 0 \), \( \forall z \in (0, z_l] \).

(iii) \( g(z) > 0 \), \( \forall z \in [z_r, 1) \).

(iv) \( \text{If } g(z) < 0 \text{ then } g(z) \text{ has exactly two roots, say } z_1 \text{ and } z_2 \), with \( z_1 < z_2 \) and \( z_1, z_2 \in (z_l, z_r) \).

(v) \( \text{Let } z > z_l \text{ then it holds } g(z, a, b) > g(z, a, b + 1) \).

(vi) \( \text{For fixed } a \text{ there is a threshold } b', b' \geq a + 1, \text{ such for } a < b < b' \text{ it holds that } \min_{z} g(z, b) \geq 0, \text{ and if } b \geq b' \text{ then it holds } \min_{z} g(z, b) < 0 \).

The proof of Lemma 2 is given in Appendix D. Example plots of \( g(z, a, b) \) are shown in Figure 6.

Lemma 3 (Properties of \( h(z) \)).

Let \( 3 \leq a < b \), \( z \in (0, 1) \), then it holds

(i) \( h(z) \) has a pole at \( z = z_r \).

(ii) \( \lim_{z \to 0} h(z) = -\infty \).

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(iii) $\lim_{z \to z_r} h(z) = +\infty$.
(iv) $\forall z \in (0, z_l] : h(z) \in (-\infty, 1]$.
(v) $\forall z \in (z_l, z_r) : h(z) \in (1, +\infty)$, and $h(z_l) = 1$.
(vi) $\forall z \in (z_r, 1) : h(z) \in (-\infty, 1)$.
(vii) $\frac{\partial h(z)}{\partial z} > 0 \iff g(z) > 0$.
(viii) $h(z)$ is strictly increasing in $z \in (0, z_l]$.
(ix) $h(z)$ is strictly increasing in $z \in (z_r, 1)$.
(xi) If $\min_z g(z) \geq 0$ then $h(z)$ is strictly increasing in $z \in [z_l, z_r)$.
(x) If $\min_z g(z) < 0$ then $h(z)$ is strictly increasing to a local maximum at $z_1$, then strictly decreasing to a local minimum at $z_2$, then strictly increasing afterwards.

The proof of Lemma 3 is given in Appendix E. Example plots of $g(z,a,b)$ are shown in Figure 7.

Concerning the defined points, we are only interested in how they are related to each other.

**Lemma 4.** Let $3 \leq a < b$, $z \in (0,1)$, then it holds

(i) $0 < z_l < z_r < 1$.
(ii) $z_l < z_1, z_2 < z_r$, if $z_1$ and $z_2$ exist.
(iii) $z' \in (0, z_r)$.
(iv) $z' \neq z_1, z' \neq z_2$, if $z_1$ and $z_2$ exist.

The proof Lemma 4 is given in Appendix F. Now we are ready for solving the optimization problem.

**A.2 Analysis**

Assume first that $\alpha$ is arbitrary but fixed, that is we are looking for a global minimum of (5) in $z$-direction. Since

$$\lim_{z \to 0} T(z) = \lim_{z \to 1} T(z) = +\infty,$$

and $T(z)$ is continuous for $z \in (0, 1)$, a global minimum must be a point where the first derivative of $T(z)$ is zero, that is a critical point. According to (12)
critical points in \( z \)-direction for unbounded \( \alpha \), i.e., \( \alpha \in \mathbb{R} \), can be described via

\[
\frac{\partial T(z)}{\partial z} = 0 \iff \frac{1}{1 - z} \cdot D_0(z) = -\frac{\ln(1 - z)}{z} \cdot D_1(z)
\]

\[
\iff D_0(z) \frac{D_1(z)}{D_2(z)} = f(z) \iff \alpha = 1/h(z) .
\]  

The next lemma identifies and classifies critical points of \( T(z) \) for bounded \( \alpha \) that is for \( \alpha \in (0, 1] \).

**Lemma 5.** Let \( \alpha \in (0, 1] \) be arbitrary but fixed. If \( \frac{\partial T}{\partial z}(\tilde{z}) = 0 \) for some \( \tilde{z} \in (0, 1) \) then it holds

(i) \( \tilde{z} \in [z_l, z_r] \),

(ii) if \( g(\tilde{z}) > 0 \) then \( T(\tilde{z}) \) is a local minimum,

(iii) if \( g(\tilde{z}) < 0 \) then \( T(\tilde{z}) \) is a local maximum.

**Proof.**

(i) According to (21) we must have \( \alpha = 1/h(\tilde{z}) \) for \( \alpha \in (0, 1] \). Therefore it must hold \( h(\tilde{z}) \in (1, +\infty) \). Using Lemma 3(iv), (v), (vi) it follows that \( \tilde{z} \in [z_l, z_r] \).

(ii) Now consider the second derivative of \( T(z) \) with respect to \( z \). According to (14) we have

\[
\frac{\partial^2 T(z)}{(\partial z)^2} > 0 \iff \frac{1}{(1 - z)^2} - \frac{2}{z \cdot (1 - z)} \cdot \frac{D_1(z)}{D_0(z)}
\]

\[
+ \frac{\ln(1 - z)}{z^2} \cdot \frac{D_2(z) - D_1(z)}{D_0(z)} - \frac{2 \cdot \ln(1 - z)}{z^2} \cdot \frac{D_1(z)^2}{D_0(z)^2} > 0 .
\]

Assume that \( \tilde{z} \in [z_l, z_r] \) is a critical point. For the rest of the proof let \( z = \tilde{z} \). Utilizing that \( D_0(z) \frac{D_1(z)}{D_2(z)} = f(z) \) it follows that

\[
\frac{\partial^2 T(z)}{(\partial z)^2} > 0 \iff \frac{1}{(1 - z)^2} - \frac{2}{z \cdot (1 - z)} \cdot \frac{D_2(z)}{D_0(z)}
\]

\[
- \frac{\ln(1 - z)}{z^2} \cdot f(z) + \frac{2 \cdot \ln(1 - z)}{z^2} \cdot \frac{D_2(z)}{D_0(z)} > 0
\]

\[
\iff \frac{1}{(1 - z)^2} - \frac{f(z)}{z \cdot (1 - z)} \cdot \frac{D_0(z)}{D_2(z)} + \frac{1}{z(1 - z)} > 0
\]

\[
\iff \frac{D_0(z)}{D_2(z)} > f(z) \cdot (1 - z) \iff \frac{D_1(z)}{D_2(z)} > (1 - z) .
\]

Factoring out \( \alpha \) from \( \frac{D_1(z)}{D_2(z)} > (1 - z) \) gives that \( D_1(z) > (1 - z) \cdot D_2(z) \) is equivalent to

\[
\alpha \cdot (Z_1(z) - (1 - z) \cdot Z_2(z)) > -b \cdot (b - 1) \cdot z^{b-1} + (1 - z) \cdot b \cdot (b - 1)^2 \cdot z^{b-1} .
\]
According to the proof of Lemma 5(i) it holds $\alpha = 1/h(z)$, which can be written as

$$\alpha = \frac{b \cdot z^{b-1} \cdot ((b-1) \cdot f(z) - 1)}{Z_0(z) - f(z) \cdot Z_1(z)}.$$ (\*)

Division by $b \cdot z^{b-1}$ leads to

$$\frac{\partial^2 T(z)}{(\partial z)^2} > 0 \Leftrightarrow \frac{(b-1) \cdot f(z) - 1}{Z_0(z) - f(z) \cdot Z_1(z)} \cdot (Z_1(z) - (1 - z) \cdot Z_2(z)) > - (b - 1) + (1 - z) \cdot (b - 1)^2.$$ (\*)

Consider (\*). Given that $\alpha \in (0, 1]$ and $z < z_r$ (Lemma 5(i)) we have, according to the definition of $z_r$ and Lemma 1(iv), that $(b - 1) \cdot f(z) > 1$, that is the numerator of (\*) is larger than 0. Since $\alpha > 0$ it follows that the denominator $Z_0(z) - f(z) \cdot Z_1(z)$ is larger than 0 too. Hence we get

$$\frac{\partial^2 T(z)}{(\partial z)^2} > 0$$

$$\Leftrightarrow (\!(b - 1) \cdot f(z) - 1 \!) \cdot (Z_1(z) - (1 - z) \cdot Z_2(z)) > (b - 1)^2 \cdot ((1 - z) - (b - 1) \cdot f(z))$$

$$\Leftrightarrow Z_1(z) \cdot ((b - 1)^2 \cdot (1 - z) \cdot f(z) - 1) > Z_0(z) \cdot ((1 - z) \cdot (b - 1)^2 - (b - 1))$$

$$\Leftrightarrow (a - 1)^2 \cdot (1 - z - (1 - z) \cdot (b - 1) \cdot f(z))$$

$$+(a - 1) \cdot ((b - 1)^2 \cdot (1 - z) \cdot f(z) - 1) > (1 - z) \cdot (b - 1)^2 - (b - 1).$$

Factoring out $f(z) \cdot (b - 1) \cdot (a - 1)$ gives

$$\frac{\partial^2 T(z)}{(\partial z)^2} > 0 \Leftrightarrow f(z) \cdot (b - 1) \cdot (a - 1) \cdot (b - a) > (b - 1)^2 - (a - 1)^2 - \frac{b - a}{1 - z}$$

$$\Leftrightarrow f(z) \cdot (b - 1) \cdot (a - 1) + \frac{1}{1 - z} + 2 - b - a > 0 \Leftrightarrow g(z) > 0.$$ (iii)

Analogous to (ii).

This finishes the proof of the lemma. \qed

The next lemma can be seen as the central building block for understanding the behavior of the threshold function. Using the function $g(z)$ we decide how many and which kind of extremal points $T(z)$ has.

**Lemma 6.** Let $\alpha \in (0, 1]$ be arbitrary but fixed.

1. Let $\min_z g(z) \geq 0$ then the function $T(z)$ has exactly one critical point $\bar{z}$, and $\bar{z} \in [z_1, z_r]$ is a global minimum point.

2. Let $\min_z g(z) < 0$ then there are four pairwise distinct points $z_1^\prec < z_1 < z_2 < z_2^\succ$ from the interval $[z_1, z_r]$ such that the following holds:
For all \( \alpha \) with \( 1/\alpha \in [1, h(z_2)) \) the function \( T(z) \) has exactly one critical point \( \tilde{z} \), and \( \tilde{z} \in (z_l, z^c) \), is a global minimum point.

For \( \alpha \) with \( 1/\alpha = h(z_2) \) the function \( T(z) \) has exactly two critical points \( \tilde{z}_1 < \tilde{z}_2 \), and \( \tilde{z}_1 = z^c_1 \) is a global minimum point, and \( \tilde{z}_2 = z_2 \) is an inflection point.

For all \( \alpha \) with \( 1/\alpha \in (h(z_2), h(z_1)) \) the function \( T(z) \) has exactly three critical points \( \tilde{z}_1 < \tilde{z}_3 < \tilde{z}_2 \), and \( \tilde{z}_1, \tilde{z}_2 \) are local minimum points and \( \tilde{z}_3 \) is a local maximum point.

For \( \alpha \) with \( 1/\alpha = h(z_1) \) the function \( T(z) \) has exactly two critical points \( \tilde{z}_1 < \tilde{z}_2 \), and \( \tilde{z}_1 = z_1 \) is an inflection point, and \( \tilde{z}_2 = z^*_2 \) is a global minimum point.

For all \( \alpha \) with \( 1/\alpha \in (h(z_1), \infty) \) the function \( T(z) \) has exactly one critical point \( \tilde{z} \), and \( \tilde{z} \in (z^*_2, z_r) \), is a global minimum point.

Figure 8 illustrates the complete case 2 of Lemma 6. The intersection points between the function \( 1/\alpha \) (horizontal lines) and the function \( h(z) \) are the extrema of \( T(z) \). They are classified depending on the part of \( h(z) \) where the intersection takes place.

![Figure 8. h(z) for a = 3, b = 20, min_z g(z) < 0.](image)

**Proof.**

1. From Lemma 5(i) it follows that all critical points \( \tilde{z} \) must be from \([z_l, z_r]\). Consider the function \( h(z) \). According to Lemma 3(v), (x) it holds that for each \( x \) from \([1, +\infty)\) there is exactly one \( z \) from \([z_l, z_r]\) such that \( h(z) = x \). Furthermore, according to (21) we have \( \frac{dT(z)}{dz} = 0 \Leftrightarrow \alpha = 1/h(z) \). It follows that for each \( \alpha \in (0, 1] \) there is exactly one \( \tilde{z} \) that is a critical point, that is it holds \( \alpha = 1/h(\tilde{z}) \). Let \( \min_z g(z) \geq 0 \) then it must hold
Let $\tilde{z}$ be the only critical point it follows with (20) that it must be a global minimum point.

2. From Lemma 3(xi) we know that for $z \in [z_1, z_\ast)$ the function $h(z)$ is strictly increasing, reaches a local maximum at $z_1$, is strictly decreasing, reaches a local minimum at $z_2$ and is strictly increasing to $+\infty$ afterwards. Furthermore it holds $g(z) = 0$ for $z \in \{z_1, z_2\}$, $g(z) > 0$ for $z < z_1$ and $z > z_2$, as well as $g(z) < 0$ for $z \in (z_1, z_2)$ (Lemma 3(vii)).

Consider the condition (21).

(i) For all $\alpha$ with $1/\alpha \in [1, h(z_2))$ there is, according to Lemma 3, exactly one $z$ with $1/\alpha = h(z)$. In addition we have that $z < z_1$. Utilizing that $g(z) > 0$, for $z < z_1$, (Lemma 2(i), (ii)) the claim follows by Lemma 5(ii).

(ii) Let $1/\alpha = h(z_2)$ and let $\tilde{z}_2 = z_2$ then according to Lemma 3 there is exactly one other point $\tilde{z}_1$, such that $\alpha = 1/h(\tilde{z}_1)$. Furthermore it holds $g(\tilde{z}_1) > 0$ and $g(\tilde{z}_2) = 0$. According to Lemma 3(vii) $\tilde{z}_1$ must be a local minimum point. Because of the monotonicity of $T(z)$ (20) the other critical point must be an inflection point. Hence $\tilde{z}_1$ is also a global minimum point.

(iii) According to Lemma 3 there are exactly 3 different points $\tilde{z}_i$, $1 \leq i \leq 3$, such that $1/\alpha = h(\tilde{z}_i)$ and $\frac{\partial T}{\partial z}(\tilde{z}_i) = 0$, respectively. Furthermore it holds $\tilde{z}_1 < z_1 < \tilde{z}_3 < z_2 < \tilde{z}_2$ and $g(\tilde{z}_1) > 0, g(\tilde{z}_2) > 0, g(\tilde{z}_3) < 0$. From Lemma 5(ii), (iii) it follows that $\tilde{z}_1$ and $\tilde{z}_2$ are local minimum points of $T(z)$ and $\tilde{z}_3$ is a local maximum point of $T(z)$.

(iv) The case $1/\alpha = 1/h(z_1)$ is analogous to the case (ii).

(v) The case $1/\alpha \in (h(z_1), \infty)$ is analogous to the case (i).

This finishes the proof of the lemma. \qed

The last lemma gives a complete characterization of the local extrema of (5) in $z$-direction including the global minimum for arbitrary but fixed $\alpha$. It remains to find a value $\alpha^*$ that maximizes the threshold function at the corresponding global minimum in $z$-direction. So the point we are looking for could be a saddle point of $T(z, \alpha)$. Indeed the following lemma shows that $T(z, \alpha)$ has exactly one saddle point for unbounded $\alpha$, i.e. $\alpha \in \mathbb{R}$, and Theorem 3 finally shows under which conditions this point is the optimum we are looking for.

**Lemma 7.** Let $\alpha \in \mathbb{R}$. Then $T(z, \alpha)$ has exactly one saddle point

$$
(\tilde{z}, \tilde{\alpha}) = \left( \left( \frac{a}{b} \right)^{\frac{1}{b-a}}, \frac{b-1}{b-a} - \frac{1}{f(\tilde{z})^{(b-a)}} \right).
$$

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A.3 Putting It All Together

Now we prove Theorem 3.

Proof. Following the linear system \( \{ \frac{\partial T(z, \alpha)}{\partial z} = 0, \frac{\partial T(z, \alpha)}{\partial \alpha} = 0 \} \) gives

\[
\frac{\partial T(z, \alpha)}{\partial z} = 0 \iff \alpha = 1/h(z)
\]

\[
\frac{\partial T(z, \alpha)}{\partial \alpha} = 0 \iff \ln(1 - z) \cdot Z_0(z) - D_0(z, \alpha)^2 = 0 \iff Z_0(z) = 0
\]

\[
\iff a \cdot z^{a-1} = b \cdot z^{b-1} \iff z = \left( \frac{a}{b} \right)^{\frac{1}{a-b}}.
\]

There is only one solution of \( \frac{\partial T(z, \alpha)}{\partial \alpha} = 0 \) and according to Lemma 3(i) and Lemma 1(vi) \( h(z) \) is defined at \( z' = \left( \frac{a}{b} \right)^{\frac{1}{a-b}} \). Hence we get a unique critical point \((\hat{z}, \hat{\alpha})\) where \( \hat{z} = z' \) and

\[
\hat{\alpha} = 1/h(\hat{z}) = \frac{b \cdot \left( \frac{a}{b} \right)^{\frac{1}{a-b}} \cdot (f(\hat{z}) \cdot (b - 1) - 1)}{-f(\hat{z}) \cdot \left( a^2 \cdot \left( \frac{a}{b} \right)^{\frac{1}{a-b}} - b^2 \cdot \left( \frac{a}{b} \right)^{\frac{1}{a-b}} \right)}
\]

\[
= \frac{b \cdot \left( \frac{a}{b} \right)^{\frac{1}{a-b}} \cdot (f(\hat{z}) \cdot (b - 1) - 1)}{-f(\hat{z}) \cdot b^2 \cdot \left( \frac{a}{b} \right)^{\frac{1}{a-b}} - 1}
\]

\[
= \frac{f(\hat{z}) \cdot (b - 1) - 1}{f(\hat{z}) \cdot (b - a)} = \frac{b - 1}{b - a} \cdot \frac{1}{f(\hat{z}) \cdot (b - a)}.
\]

To classify this critical point we consider the second partial derivatives of \( T(z, \alpha) \). We have \( Z_0(\hat{z}) = 0 \) and \( Z_1(\hat{z}) > 0 \), since

\[
Z_1(\hat{z}) = a \cdot (a - 1) \cdot \left( \frac{a}{b} \right)^{(a-1)/(b-a)} - b \cdot (b - 1) \cdot \left( \frac{a}{b} \right)^{(b-1)/(b-a)} < 0
\]

\[
\iff a \cdot (a - 1) < \left( \frac{a}{b} \right)^{(b-1)/(b-a)} \iff a - 1 < b - 1 \iff \alpha < 1
\]

It follows that \( \frac{\partial^2}{\partial \alpha^2} T(\hat{z}, \hat{\alpha}) = 0 \) as well as \( \frac{\partial^2}{\partial z \partial \alpha} T(\hat{z}, \hat{\alpha}) > 0 \). Therefore the Hessian matrix \( H \) with

\[
H = \begin{pmatrix}
\frac{\partial^2}{\partial \alpha^2} T(\hat{z}, \hat{\alpha}) & \frac{\partial^2}{\partial z \partial \alpha} T(\hat{z}, \hat{\alpha}) \\
\frac{\partial^2}{\partial z \partial \alpha} T(\hat{z}, \hat{\alpha}) & \frac{\partial^2}{\partial z^2} T(\hat{z}, \hat{\alpha})
\end{pmatrix} = \begin{pmatrix}
0 & > 0 \\
> 0 & \frac{\partial^2}{\partial z^2} T(\hat{z}, \hat{\alpha})
\end{pmatrix}
\]

has determinant \( \det(H) < 0 \), that is \((\hat{z}, \hat{\alpha})\) is a saddle point. \( \square \)
**Proof.** Using (21) we can define a function of critical points \( \tilde{T}(z) \) of \( T(z, \alpha) \) as follows
\[
\tilde{T}(z) := T(z, 1/h(z)) = \frac{-\ln(1 - z)}{1/h(z) \cdot Z_0(z) + b \cdot z^{b-1}} = \frac{\frac{\partial}{\partial z} \cdot Z_0(z) + \ln(1 - z) \cdot Z_1(z)}{b \cdot a \cdot (b - a) \cdot z^{b+a-2}}.
\]
The first derivative of \( \tilde{T}(z) \) is
\[
\frac{\partial \tilde{T}(z)}{\partial z} = \frac{1}{b \cdot a \cdot (b - a) \cdot z^{b+a-2}} \left( Z_0(z) + \frac{\ln(1 - z) \cdot Z_2(z)}{z} - \frac{b + a - 2}{b \cdot a \cdot (b - a) \cdot z^{b+a-2}} \cdot Z_0(z) + \ln(1 - z) \cdot Z_1(z) \right).
\]
We are interested in the monotonicity of \( \tilde{T}(z) \).
\[
\frac{\partial \tilde{T}(z)}{\partial z} > 0
\]
\[
\iff \frac{Z_0(z)}{(1 - z)^2} + \frac{\ln(1 - z) \cdot Z_2(z)}{z} - (b + a - 2) \cdot \left( \frac{Z_0(z)}{1 - z} + \frac{\ln(1 - z) \cdot Z_1(z)}{z} \right) > 0
\]
\[
\iff \frac{Z_0(z)}{1 - z} - (b + a - 2) \cdot Z_0(z) - f(z) \cdot (Z_2(z) - (b + a - 2) \cdot Z_1(z)) > 0
\]
\[
\iff \frac{Z_0(z)}{1 - z} - (b + a - 2) \cdot Z_0(z) + f(z) \cdot Z_0(z) \cdot (b - 1) \cdot (a - 1) > 0.
\]
Note that \( Z_0(z) > 0 \iff z < \left( \frac{a}{b} \right)^{1+b} = \tilde{z}' \). Division by \( Z_0(z) \) gives, by definition of \( g(z) \)
\[
\forall z < \left( \frac{a}{b} \right)^{1+b} : \frac{\partial \tilde{T}(z)}{\partial z} > 0 \iff g(z) > 0
\]
\[
\forall z > \left( \frac{a}{b} \right)^{1+b} : \frac{\partial \tilde{T}(z)}{\partial z} < 0 \iff g(z) > 0.
\]

1. If \( \min_z g(z) > 0 \) then according to (*) we have \( \frac{\partial \tilde{T}(z)}{\partial z} > 0 \) for all \( z < \tilde{z}' \) and we have \( \frac{\partial \tilde{T}(z)}{\partial z} < 0 \) for all \( z > \tilde{z}' \). Hence the function of critical points has a global maximum in \( \alpha \)-direction at \( \tilde{z}' \). Consider the special case \( \min_z g(z) = 0 \) with \( \tilde{z}_{\min} = \arg \min_z g(z) \). According to Lemma 2(i) and the definition of \( z_1 \) and \( z_2 \) we have \( z_1 = z_2 = \tilde{z}_{\min} \). From Lemma 4(iv) it follows that \( \tilde{z}_{\min} \neq \tilde{z}' \). Hence \( \tilde{z}_{\min} \) must be an inflection point of \( \tilde{T}(z) \) since before and after \( \tilde{z}_{\min} \) the monotonicity is the same. Hence the function of critical points has a global maximum in \( \alpha \)-direction at \( \tilde{z}' \) also in this case.
2. Since $h(z') > 1$ then according to Lemma 3(v) we have $z' \in (z_1, z_r)$.

It follows from Lemma 6(1) that $T(z')$ is a global minimum in $z$-direction. Hence, $(z', 1/h(z'))$ is the optimum point, which is according to Lemma 7 the saddle point.

(ii) If $h(z') \leq 1$ then $z'$ must be from the interval $(0, z_l)$ (Lemma 3(iv)) and not from the interval $(z_r, 1)$, (Lemma 3(vi)), since we have $f(z') > f(z_r)$ (Lemma 1(vi)) and $f(z)$ is monotonically decreasing (Lemma 1(iv)). But if $z' \leq z_l$ then because of the monotonicity of $T(z)$ the optimal $z$ value is the nearest feasible critical point. That is the optimum point is the (degenerated) solution $(z_l, 1)$.

2. Since $\min_z g(z) < 0$ it follows from Lemma 3(xi) that for $z \in [z_l, z_r)$ the function $h(z)$ is strictly increasing, reaches a maximum at $z_1$, is strictly decreasing, reaches a minimum at $z_2$, and is strictly increasing afterwards.

Furthermore we have $g(z) > 0$ for $z \in [z_l, z_1)$, $g(z) < 0$ for $z \in (z_1, z_2)$, $g(z) > 0$ for $z \in (z_2, z_r)$, and $g(z) = 0$ for $z \in \{z_1, z_2\}$. An optimal $z$ must be global minimum point in $z$-direction. According to Lemma 5(ii) and Lemma 6(2) global minimum points are the points from $[z_l, z_1) \cup (z_2, z_r)$.

(i) An optimal $z$ cannot be from $(z_2, z_r)$ since for each $z \in (z_2, z_r)$ there is an $\varepsilon > 0$ such that $z - \varepsilon \in (z_2, z_r)$ and $T(z) < T(z - \varepsilon)$.

This descent converges to $z_2$. But according to Lemma 6(2) $z_2$ is an inflection point and not a global minimum point. Hence the optimal $z$ must be from $[z_l, z_1)$. For each $z \in (z_l, z_1)$ there is an $\varepsilon > 0$ such that $z - \varepsilon \in [z_l, z_1)$ and $T(z) < T(z - \varepsilon)$. This descent converges to $z_l$.

(ii) According to Lemma 4(iv) $z' \neq z_2$. It follows that $z' \in (z_l, z_1]$, see also Lemma 6(2). An optimal $z$ cannot be from $(z_2, z_r)$ for the same reasons as in case 2(i).

(iii) Consider an arbitrary but fixed $\alpha$ with $1/\alpha \in (h(z_2), h(z_1))$. According to Lemma 6(2(iii)) we have two different points $\hat{z}_1, \hat{z}_2$, with $\hat{z}_1 < z_1 < z_2 < \hat{z}_2$, that are local minimum points of the threshold function $T(z, \alpha)$ in $z$-direction.

- Let $z_1 < z' < z_2$. Decreasing $\alpha$ (increasing $1/\alpha$) by an arbitrary small but fixed positive value gives two new local minimum points in $z$-direction, $\hat{z}_1 + \varepsilon$, $\hat{z}_2 + \delta$, where $\varepsilon, \delta > 0$. According to $(\star)$ it holds that $T(\hat{z}_1) < T(\hat{z}_1 + \varepsilon)$ and $T(\hat{z}_2) > T(\hat{z}_2 + \delta)$. Hence for the left critical point the local minimum in $z$-direction becomes smaller while the potential threshold becomes larger and for the right critical point the local minimum in $z$-direction becomes larger while the potential threshold becomes smaller. Increasing $\alpha$ by an arbitrary small but fixed positive value reverses the behavior. Assume we have found an optimal $\alpha$, that is $\alpha = \alpha^*$. Decreasing $\alpha$ by some small fixed positive value increases the threshold for the left critical point but because of the optimality of $\alpha$ we have no global minimum for the left critical point.
but only a local minimum. Increasing $\alpha$ increases the threshold for the right critical point but because of the optimality of $\alpha$ we have no global minimum for the right critical point but only a local minimum. Hence for $\alpha^*$ both critical points $z^*$ and $z^{**}$, with $1/\alpha^* = h(z^*) = h(z^{**})$, lead to the same minimum in $z$-direction, that is both local minimum points are also global minimum points and it holds $T(z^*, \alpha^*) = T(z^{**}, \alpha^*)$ is the optimal threshold.

• Let $z' < z_1$. Assume that $1/\alpha \in (h(z'), h(z_1))$, then $\alpha$ cannot be optimal since increasing $\alpha$ by an arbitrary small but fixed positive value increases $\tilde{T}(\tilde{z}_1)$ as well as $\tilde{T}(\tilde{z}_2)$ and one of the critical points must be the global minimum point in $z$-direction. Hence the optimum $1/\alpha$ must be in the interval $[h(z_2), h(z')]$.

• The case $z' > z_2$ is analogous to the case $z' < z_1$.

(iv) According to Lemma 4(iv) $z' \neq z_1$. It follows that $z' \in [z_2^r, z_r)$. An optimal $z$ cannot be from $[z_l, z_1)$.

For given $k = (a, b)$, Algorithm 2 calculates $\alpha^*, z^*$ and $c^* = T(z^*, \alpha^*)$ of Theorem 3 and optimization problem (3), respectively. If one wants to determine the optimal values for fixed $a$ but increasing $b$ one can make use of the following observation. According to Lemma 2(vi) there is a threshold $b'$, such that for $a < b < b'$ it holds $\min_z g(z, b) \geq 0$ and for $b \geq b'$ it holds $g(z, b) < 0$. That is after reaching $b'$ we don’t need to further calculate the minimum of $g(z)$. The following table lists some values for $b'$.

| $a$ | 3 4 5 6 7 8 9 10 |
|-----|------------------|
| $b'$| 16 29 45 62 79 98 117 137 |
Algorithm 2: Optimal Thresholds

Input: $a, b, \varepsilon$ (stopping criterion for binary search)
Purpose: finds optimal thresholds for parameters $a$ and $b$.
Prerequisite: subroutine $\text{numSolve}(\text{equation}, \text{interval})$ that returns a numerical solution of $\text{equation}$ within the given interval

Initialization:

$z_l \leftarrow \text{numSolve}(f(z) = \frac{1}{a-1}, z \in (0, 1))$

$z_r \leftarrow \text{numSolve}(f(z) = \frac{1}{b-1}, z \in (z_l, 1))$

$z_g \leftarrow \text{numSolve}(\frac{\partial g(z)}{\partial z} = 0, z \in (0, 1))$

$z' \leftarrow \left(\frac{a}{b}\right)^{\frac{1}{a-1}}; z_1 \leftarrow z'; z_2 \leftarrow z'$

if $g(z_g, a, b) < 0$ then

$z_1 \leftarrow \text{numSolve}(g(z, a, b) = 0, z \in (z_l, z_g))$

$z_2 \leftarrow \text{numSolve}(g(z, a, b) = 0, z \in (z_g, z_r))$

Optimization:

if $h(z', a, b) \leq 1$ then

$z^* \leftarrow z_l; \alpha^* \leftarrow 1; T^* \leftarrow \frac{-\ln(1-z_l)}{\alpha^*}$

else

if $h(z', a, b) \leq h(z_2, a, b)$ or $h(z', a, b) \geq h(z_1, a, b)$ then

$z^* \leftarrow z'; \alpha^* \leftarrow \frac{b-1}{b-a} - \frac{1}{f(z') \cdot (b-a)}; T^* \leftarrow \ln \left(1 - \left(\frac{a}{b}\right)^{\frac{1}{a-1}} \cdot \left(\frac{b-a}{b-a}\right)^{\frac{1}{b-a}}\right)$

else

$u \leftarrow z_1; l \leftarrow z_2$

if $z' < z_1$ then $u \leftarrow z'$

if $z' > z_2$ then $l \leftarrow z'$

$\alpha_{\text{min}} \leftarrow \frac{1}{h(u, a, b)}; \alpha_{\text{max}} \leftarrow \frac{1}{h(l, a, b)}$

while true do

$\alpha^* \leftarrow \frac{\alpha_{\text{max}} - \alpha_{\text{min}}}{2} + \alpha_{\text{min}}$

$z^{**} \leftarrow \text{numSolve}(h(z, a, b) - \frac{1}{\alpha^*} = 0, z \in (z_l, u))$

$z^* \leftarrow \text{numSolve}(h(z, a, b) - \frac{1}{\alpha^*} = 0, z \in (l, z_r))$

$t^{**} \leftarrow T(z^{**}, a, b, \alpha^*)$

$t^* \leftarrow T(z^*, a, b, \alpha^*)$

if $|t^* - t^{**}| < \varepsilon$ then

break

else

if $t^* > t^{**}$ then $\alpha_{\text{min}} \leftarrow \alpha^*$

else $\alpha_{\text{max}} \leftarrow \alpha^*$

return $(z^*, \alpha^*, T^*)$
The following four tables list optimal thresholds for different edge sizes $a = k_1$ and $b = k_2$, with $a \in \{3, 4, 5, 6\}$ and $a \leq b \leq 50$.

**Table 3.** Optimal values for $a = 3$ and $a \leq b \leq 50$. The maximum threshold in this range is about 0.92004.

**Table 4.** Optimal values for $a = 4$ and $a \leq b \leq 50$. The maximum threshold in this range is about 0.82593.
Table 5. Optimal values for $a = 5$ and $a \leq b \leq 50$. The maximum threshold in this range is about 0.7384.

Table 6. Optimal values for $a = 6$ and $a \leq b \leq 50$. The maximum threshold in this range is about 0.64823.
C Properties of $f(z)$

In this section we prove Lemma 1.

(i) \[ f(z) = -\frac{\ln(1-z) \cdot (1-z)}{z} \geq 1 - z \]
\[ \Leftrightarrow -\ln(1-z) > z \Leftrightarrow \frac{1}{1-z} > e^z \Leftrightarrow e^{-z} > 1 - z \checkmark \]

(ii) Applying L'Hôpital’s rule it follows that
\[ \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{-\ln(1-z) \cdot (1-z)}{z} = \lim_{z \to 0} \frac{1}{1-z} \cdot (1-z) + \ln(1-z) = 1. \]

(iii) Applying L'Hôpital’s rule it follows that
\[ \lim_{z \to 1} f(z) = \lim_{z \to 1} \frac{-\ln(1-z) \cdot (1-z)}{z} = \lim_{z \to 1} \frac{-\ln(1-z)}{1-z} = \lim_{z \to 1} \frac{1}{1-z} (1-z)^2 = 0. \]

(iv) \[ \frac{df(z)}{dz} = \frac{z + \ln(1-z)}{z^2} < 0 \Leftrightarrow \ln(1-z) < -z \Leftrightarrow 1 - z < e^{-z} \checkmark \]

(v) \[ \frac{d^2 f(z)}{(dz)^2} = \frac{-2z + z^2 - 2 \ln(1-z) \cdot (1-z)}{(1-z) \cdot z^3} < 0 \]
\[ \Leftrightarrow \frac{z^2 - 2 \cdot z}{1 - z} < 2 \ln(1-z) \frac{f_1(z)}{f_2(z)} \]

which is true since it holds
- \[ \lim_{z \to 0} f_1(z) = \lim_{z \to 0} f_2(z) = 0 \] and
- \[ \frac{df_1(z)}{dz} = \frac{z^2 + 2z - z^2}{(1-z)^2} < \frac{df_2(z)}{dz} = \frac{-2}{1-z} < 0. \]

(vi) First we show that $z'$ is strictly increasing for growing $a$. Utilizing Lemma 1(iv) this implies that $f(z')$ is strictly monotonically decreasing for growing $a$.

\[ \frac{\partial z'}{\partial a} = z' \cdot \left( \ln(a/b) + \frac{1}{a \cdot (b-a)} \right) > 0 \]
\[ \Leftrightarrow \frac{b-a}{a} > \ln(b/a) \Leftrightarrow \exp \left( \frac{b-a}{a} \right) > \frac{b}{a} \]
\[ \Leftrightarrow \sum_{i=0}^{\infty} \left( \frac{b-a}{a} \right)^i \cdot \frac{1}{i!} > \frac{b}{a} \]
\[ \Leftrightarrow 1 + \frac{b-a}{a} + \left( \frac{b-a}{a} \right)^2 \cdot \frac{1}{2} + \left( \frac{b-a}{a} \right)^3 \cdot \frac{1}{6} + \ldots > \frac{b}{a} \checkmark \]
Now all we need to show is that our assumption holds for the maximum value of $a$, that is $a = b - 1$.

\[
 f \left( \left( \frac{b - 1}{b} \right)^{\frac{1}{b - 1}} \right) > \frac{1}{b - 1} \iff -\ln \left( \frac{1 - \frac{b - 1}{b}}{\frac{b - 1}{b}} \right) > \frac{1}{b - 1} \\
\iff -\ln(1/b) > \frac{1}{b - 1} \iff \ln(b) > 1
\]

which is true since $b \geq 4$.

(vii) For $z \in (0, 1)$ we have $f(z) \in (0, 1)$. Consider $\varphi(a, b)$ with

\[
 \varphi(a, b) = -\frac{1}{1 - z} - 2 + b + a = -\frac{1}{1 - \left( \frac{a}{b} \right)} - 2 + b + a.
\]

Let $a$ be fixed. We will show that $z' = z'(b)$ is strictly increasing for growing $b$.

\[
\frac{\partial z'}{\partial b} = z' \cdot \left( -\frac{\ln(1/b)}{(b - a)^2} - \frac{1}{b \cdot (b - a)} \right) > 0 \\
\iff -\ln(\frac{z}{b}) > \frac{1}{b \cdot (b - a)} \iff -\frac{b + a}{b} > \ln \left( \frac{a}{b} \right) \\
\iff x > \ln(1 + x),
\]

for $\frac{z}{b} = 1 + x$ and $-1 < x < 0$. Hence

\[
\frac{\partial z'}{\partial b} < 0 \iff x > \ln(1 + x) \iff x > \sum_{i=0}^{\infty} (-1)^i \cdot \frac{x^{i+1}}{(i + 1)} \\
\iff x > x - x^2/2 + x^3/3 - x^4/4 \sqrt{< 0}
\]

It follows directly that the function $\varphi(b)$ is strictly increasing for growing $b$. Consider the minimum of $\varphi(a + 1, a) = -\frac{1}{1 - \left( \frac{a}{a + 1} \right)} - 1 + 2 \cdot a = a - 2$. Since $a \geq 3$ we have $\varphi(a + 1, a) \geq 1$ which is not in the range of $f(z)$ for $z \in (0, 1)$.

D Properties of $g(z, a, b)$

In this section we prove Lemma 2.

(i) Consider the first derivative of $g(z)$.

\[
\frac{\partial g(z)}{\partial z} = \frac{\ln(1 - z) \cdot (b - 1) \cdot (a - 1)}{z^2} + \frac{1}{(1 - z)^2} + \frac{(b - 1) \cdot (a - 1)}{z} < 0 \\
\iff -f(z) \cdot (b - 1) \cdot (a - 1) + \frac{1}{(1 - z)^2} + \frac{(b - 1) \cdot (a - 1)}{z} < 0 \\
\iff 1 - z + \frac{z}{1 - z} \cdot (b - 1) \cdot (a - 1) < f(z) \\
\iff \frac{1}{(b - 1) \cdot (a - 1)} < \frac{1 - z}{z} \cdot (f(z) - (1 - z)) \quad (g_1(z))
\]

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Hence we have $\frac{dg(z)}{dz}_R 0 \Leftrightarrow \frac{1}{(b-1)(a-1)} \Rightarrow g_1(z)$ for $R \in \{<,>,=\}$. Now consider $g_1(z)$. It holds that

$\text{- } \lim_{z \to 0^+} g_1(z) = 0.5$, since

$$
\lim_{z \to 0^-} g_1(z) = \lim_{z \to 0} \frac{(1 - z) \cdot (f(z) - 1 + z)}{z} = \lim_{z \to 0} \frac{-1 \cdot (f(z) - 1 + z) + (1 - z) \cdot \left(\frac{df(z)}{dz} + 1\right)}{1} = 1 + \lim_{z \to 0} \frac{df(z)}{dz} = 1 + \lim_{z \to 0} \frac{z + \ln(1 - z)}{z^2} = 1 + \lim_{z \to 0} \frac{1 + \frac{-1}{z^2}}{\frac{2}{z}} = 1 + \lim_{z \to 0} \frac{-z}{2 \cdot (1 - z) - 2 \cdot z} = 0.5,
$$

using L'Hôpital's rule three times.

$\text{- } \lim_{z \to 1} g_1(z) = \frac{1}{1} \cdot (0 - 1 + 1) = 0.$

$\text{- } g_1(z)$ is strictly decreasing for growing $z \in (0, 1)$, since

$$
\frac{dg_1(z)}{dz} = -\frac{2 \cdot \ln(1 - z) \cdot (1 - z) + z^3 + z^2 - 2 \cdot z}{z^3} < 0
$$

$\Leftrightarrow 2 \cdot \ln(1 - z) \cdot (1 - z) < z^3 + z^2 - 2 \cdot z \Leftrightarrow \ln(1 - z) < \frac{z^3 + z^2 - 2 \cdot z}{2 \cdot (1 - z)} = g_2(z),
$$

which is true because

- $\lim_{z \to 0^+} \ln(1 - z) = 0 = \lim_{z \to 0^-} g_2(z) = 0$ and
- $\frac{d\ln(1 - z)}{dz} = \frac{-1}{1 - z} < \frac{dg_2(z)}{dz} = -1 - z < 0.$

Using that $0 < \frac{1}{(b-1)(a-1)} < 0.5$ it follows that for growing $z$ there is a first phase with $g_1(z) > \frac{1}{(b-1)(a-1)}$ which implies $\frac{dg_1(z)}{dz} < 0$. Then there is exactly one $z$ where $g_1(z) = \frac{1}{(b-1)(a-1)}$, which is a local minimum. After this point we have $g_1(z) < \frac{1}{(b-1)(a-1)}$, which implies $\frac{dg_1(z)}{dz} > 0$. It follows that the local minimum is actual a global minimum.

$\text{(ii) If } z \in (0, z_1) \text{ then it holds } f(z) \geq \frac{1}{a-1} > \frac{1}{b-1}. \text{ Furthermore, according to Lemma 1(i) we have } \frac{1}{1 - z} > \frac{1}{f(z)}. \text{ Let } f(z) = \frac{1 + \gamma}{a-1} \text{ and } f(z) = \frac{1 + \gamma}{b-1} \text{ as well as } \frac{1}{1 - z} = \frac{1 + \gamma}{f(z)} \text{ with } \varepsilon \geq 0 \text{ and } \delta, \gamma > 0. \text{ Using that } f(z) > 0, \text{ for } z \in (0, 1), \text{ it follows that}

$$
g(z) > 0 \Leftrightarrow f(z) \cdot (b - 1) \cdot (a - 1) + \frac{1 + \gamma}{f(z)} - (a - 1) - (b - 1) > 0
$$

$\Leftrightarrow f(z)^2 \cdot (b - 1) \cdot (a - 1) + (1 + \gamma) - f(z) \cdot (a - 1) - f(z) \cdot (b - 1) > 0$

$\Leftrightarrow (1 + \varepsilon) \cdot (1 + \delta) + (1 + \gamma) - (1 + \varepsilon) - (1 + \delta) > 0$

$\Leftrightarrow \varepsilon \cdot \delta + \gamma > 0 \checkmark$
(iii) If \( z \in [z_r, 1) \) then it holds \( f(z) \leq \frac{1}{b-1} < \frac{1}{a-1} \). Furthermore, according to Lemma 1(i) we have \( \frac{1}{1-z} > \frac{1}{f(z)} \). Let \( f(z) = \frac{1-\varepsilon}{b-1} \) and \( f(z) = \frac{1+\delta}{b-1} \) as well as \( \frac{1}{1-z} = \frac{1+\gamma}{f(z)} \) with \( \varepsilon \geq 0 \) and \( \delta, \gamma > 0 \). Following the proof of Lemma 2(ii) we get

\[
g(z) > 0 \Leftrightarrow (1-\varepsilon) \cdot (1-\delta) + (1+\gamma) - (1-\varepsilon) - (1-\delta) > 0
\]

\[\Leftrightarrow \varepsilon \cdot \delta + \gamma > 0 \checkmark\]

(iv) The existence of the roots \( z_1 \) and \( z_2 \) follows directly from Lemma 2(i). Moreover, from Lemma 2 (ii), (iii) it follows that if \( g(z) \leq 0 \) for \( z \in (0, 1) \) then it holds \( z \in (z_l, z_r) \).

(v) Let \( z > z_l \), that is \( f(z) = \frac{1-\varepsilon}{a-1} \) for \( \varepsilon > 0 \). If follows

\[
g(z, a, b) = f(z) \cdot (b-1) \cdot (a-1) + \frac{1}{1-z} + 2 - b - a
\]

\[= f(z) \cdot b \cdot (a-1) + \frac{1}{1-z} + 2 - (b+1) - a - f(z) \cdot (a-1) + 1
\]

\[= g(z, a, b+1) - f(z) \cdot (a-1) + 1 = g(z, a, b+1) - (1-\varepsilon) + 1
\]

\[\geq g(z, a, b+1)\]

(vi) Assume that there is some \( b' \) such that \( \hat{z} = \min \{ g(z, a, b') \} < 0 \). Then from Lemma 2(ii), (iii) it follows that \( \hat{z} > z_l \). Using Lemma 2(v) we conclude that for all \( b \geq b' \) it holds that \( g(\hat{z}, a, b) < 0 \) and therefore \( \min_{z} g(z, a, b) < 0 \) as well. It remains to find one such \( b' \).

Consider the inequality \( g(z', a, b) \geq 0 \) which is equivalent to

\[
(b-1) \cdot \left( \frac{f(z') \cdot (a-1)}{g(a,b)} + \frac{1}{1-z'} \cdot \frac{1}{b-1} \right) - (a-1) \geq 0
\]

Assume that \( \lim_{b \to \infty} g_1(b) = 0 \) and \( \lim_{b \to \infty} g_2(b) \leq 0 \). It follows that there must be a \( b' \) with \( g(z'(a, b'), a, b') < 0 \) and thus \( \min_{z} g(z'(a, b'), a, b') < 0 \).

- \( \lim_{b \to \infty} g_1(b) = 0 \): Assume that it holds \( \lim_{b \to \infty} z' = 1 \). Lemma 1 (iii) gives that \( \lim_{b \to \infty} f(z'(b)) = 0 = \lim_{b \to \infty} g_1(b) \).

\[
\lim_{b \to \infty} z' = \lim_{b \to \infty} \exp \left( \frac{\ln(a) - \ln(b)}{b-1} \right) = \lim_{b \to \infty} \exp \left( \frac{\ln(a) - \ln(b)}{b-1} \right) = \exp \left( \lim_{b \to \infty} \frac{\ln(a) - \ln(b)}{b-1} \right) = 1
\]

- \( \lim_{b \to \infty} g_2(b) \leq 0 \): Since \( \frac{z-z'}{1-z} > \frac{1}{1-z} \), for \( z \in (0, 1) \), it is sufficient to show that \( \lim_{b \to \infty} \frac{2-z'}{1-z} \cdot \frac{1}{b-1} = 0 \). Hence

\[
\lim_{b \to \infty} \frac{2-z'}{1-z'} = \lim_{b \to \infty} \frac{2-z'}{1-z'} - \frac{2-z'}{b-1} + \frac{2-z'}{b-1}
\]

\[= \lim_{b \to \infty} \left( \frac{1}{b-1} + \frac{2}{b-1} \cdot \frac{2-z'}{b-1} \right)
\]

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Using that \( \frac{\partial z}{\partial b} = z' \cdot \left( \frac{-\ln(a/b)}{(b-a)^2} - \frac{1}{b(b-a)} \right) \) we get
\[
\lim_{b \to \infty} \frac{(2 - z')/(b-1)}{1 - z'} = \lim_{b \to \infty} \frac{2 - z'}{(b-1)^2} \cdot \frac{1}{z'} \cdot \left( \frac{-\ln(z)}{(b-a)^2} - \frac{1}{b \cdot (b-a)} \right)^{-1}
\]
\[
= \lim_{b \to \infty} 2 - z' \cdot \left( (b-1)^2 \cdot \frac{-\ln(z)}{(b-a)^2} - \frac{(b-1)^2}{b \cdot (b-a)} \right)^{-1} = 0.
\]

E Properties of \( h(z, a, b) \)

In this section we prove Lemma 3.

(i) Consider the denominator of \( h(z) \).

\[ b \cdot ((b-1) \cdot f(z) - 1) \equiv 0 \iff f(z) = \frac{1}{b-1}, \]

which is true for exactly one \( z \) from \((0, 1)\), which is per definition \( z = z_r \).

(ii) With \( \lim_{z \to 0} f(z) = 1 \), Lemma 1(ii), and \( \lim_{z \to 0} (1 - f(z) \cdot (a - 1)) = 2 - a \leq -1 \) we get

\[ \lim_{z \to 0} h(z) = \lim_{z \to 0} \frac{a \cdot \frac{1}{z^{1-\varepsilon}} \cdot (1 - f(z) \cdot (a - 1)) - b + f(z) \cdot b \cdot (b-1)}{b \cdot ((b-1) \cdot f(z) - 1)} = -\infty. \]

(iii) It holds \( f(z) > \frac{1}{b-1}, \forall z \in (0, z_r) \). Let \( f(z) = \frac{1 + x}{b-1} \). Consider the limit of the numerator of \( h(z) \).

\[ \lim_{\varepsilon \to 0} a \cdot \frac{1}{z^{1-\varepsilon}} \cdot (1 - f(z) \cdot (a - 1)) - b + \frac{1 + x}{b-1} \cdot b \cdot (b-1) \]
\[ = \lim_{\varepsilon \to 0} a \cdot \frac{1}{z^{1-\varepsilon}} \cdot (1 - \frac{2 - \varepsilon}{b-1}) = K, \]

for some positive constant \( K \) (depending on \( a \) and \( b \)). For the denominator of 
\( h(z) \) it holds

\[ \lim_{\varepsilon \to +0} b \cdot ((b-1) \cdot \frac{1 + x}{b-1} - 1) = b \cdot \varepsilon = +0. \]

Hence \( \lim_{z \to z_r} h(z) = +\infty \).

(iv) It holds \( f(z) \geq \frac{1}{a-1}, \forall z \in (0, z_l) \). Let \( \varepsilon \geq 0 \) and let \( f(z) = \frac{1 + x}{a-1} \). Hence

\[ h(z) = \frac{a \cdot z^{a-b} - b - \frac{1 + x}{a-1} \cdot (a \cdot (a - 1) \cdot z^{a-b} - b \cdot (b-1))}{b \cdot ((b-1) \cdot \frac{1 + x}{a-1} - 1)} \leq 1 \]
\[ \Leftrightarrow a \cdot z^{a-b} - (1 + \varepsilon) \cdot a \cdot z^{a-b} - b + (1 + \varepsilon) \cdot b \cdot \frac{1}{a-1} \leq -b + (1 + \varepsilon) \cdot b \cdot \frac{1}{a-1} \]
\[ \Leftrightarrow a \cdot z^{a-b} - (1 + \varepsilon) \cdot a \cdot z^{a-b} \leq 0 \Rightarrow \varepsilon \geq 0. \]

It holds \( \frac{1}{b-1} \geq f(z) > \frac{1}{a-1}, \forall z \in (z_l, z_r) \). Let \( \varepsilon > 0 \), and let \( \delta > 0 \) with \( \frac{1 - \varepsilon}{a-1} = f(z) = \frac{1 + \delta}{b-1} \). Hence

\[ h(z) = \frac{a \cdot z^{a-b} - b - \frac{1 - \varepsilon}{a-1} \cdot a \cdot (a - 1) \cdot z^{a-b} + \frac{1 + \delta}{b-1} \cdot b \cdot (b-1)}{b \cdot ((b-1) \cdot \frac{1 + \delta}{b-1} - 1)} \]
\[ \Leftrightarrow a \cdot z^{a-b} - (1 - \varepsilon) \cdot a \cdot z^{a-b} + (1 + \delta) \cdot b - b > \delta \cdot b \]
\[ \Leftrightarrow \varepsilon \cdot a \cdot z^{a-b} > 0 \Leftrightarrow \varepsilon > 0. \]
Let $h = \frac{z^{a-b} - b - \frac{1-\varepsilon}{b-1}}{(b-1) \cdot \frac{1-\varepsilon}{b-1} - 1}$. Note that $h(z) = 0$, if $z \neq z_r$.

The first derivative of $h(z)$ is
\[
\frac{\partial h(z)}{\partial z} = \frac{f(z)}{b \cdot z^{b-1} \cdot h(z)} \cdot \left( \frac{Z_2(z) - Z_2(z) - Z_0(z)}{Z_1(z) \cdot Z_1(z)} \cdot \frac{(b-1) \cdot (b-1 - \frac{1}{1-z})}{h(z)} \right).
\]

The function $h(z)$ is strictly increasing if and only if
\[
\frac{\partial h(z)}{\partial z} > 0
\]
\[
\iff \frac{\partial h(z)}{\partial z} \cdot h_1(z)^2 = f(z) \cdot \left[ (h_1(z) \cdot (Z_2(z) - Z_2(z)) \right.
\]
\[
- (Z_0(z) - f(z) \cdot Z_1(z)) \cdot (b-1) \cdot (b-1 - \frac{1}{1-z}) \left. \right] > 0.
\]

Note that $f(z)$ is positive for $z \in (0, 1)$. So we get
\[
\frac{\partial h(z)}{\partial z} > 0 \iff \left( \frac{Z_2(z)}{1-z} - Z_2(z) \right) \cdot ((b-1) \cdot f(z) - 1) > (Z_0(z) - f(z) \cdot Z_1(z)) \cdot ((b-1)^2 - \frac{b-1}{1-z})
\]

This inequality is equivalent to
\[
\frac{Z_1(z)}{1-z} \cdot (b-1) \cdot f(z) - \frac{Z_1(z)}{1-z} - Z_2(z) \cdot (b-1) \cdot f(z) + Z_2(z) >
\]
\[
\frac{Z_1(z)}{1-z} \cdot (b-1) \cdot f(z) + Z_0(z) \cdot (b-1)^2
\]
\[
-Z_0(z) \cdot \frac{b-1}{1-z} - f(z) \cdot Z_1(z) \cdot (b-1)^2
\]
\[
\iff f(z) \cdot (b-1) \cdot (Z_1(z) \cdot (b-1) - Z_2(z)) + \frac{1}{1-z} \cdot (Z_0 \cdot (b-1) - Z_1(z)) + Z_2(z) - Z_0(z) \cdot (b-1)^2 > 0.
\]
Expanding the functions $Z_j(z)$ gives

\[
\frac{\partial h(z)}{\partial z} > 0 \Leftrightarrow f(z) \cdot (b - 1) \cdot a \cdot z^{a-1} \cdot (a - 1) \cdot (b - 1) - (a - 1)^2 \\
+ \frac{1}{1 - z} \cdot (b - 1) - (a - 1) \\
+ a \cdot z^{a-1} \cdot ((a - 1)^2 - (b - 1)^2) > 0 \\
\Leftrightarrow f(z) \cdot (b - 1) \cdot (a - 1) \cdot (b - a) \\
+ \frac{1}{1 - z} \cdot (b - a) + (a - 1)^2 - (b - 1)^2 > 0 \\
\Leftrightarrow f(z) \cdot (b - 1) \cdot (a - 1) + \frac{1}{1 - z} + 2 - b - a > 0 \\
\Leftrightarrow g(z) > 0,
\]

where we divided by $b - a$ which is larger than 0 by definition.

(vii) This follows directly from Lemma 2(ii) and Lemma 3(vii).

(viii) This follows directly from Lemma 2(iii) and Lemma 3(vii)

(ix) For $\min_z g(z) > 0$ this follows directly from Lemma 3(vii). Let $\min_z g(z) = 0$. According to Lemma 2(i) the point $z_{\min} = \arg \min_z g(z)$ is the only point where $g(z) = 0$. It follows that $z_{\min}$ is the only inflection point of $h(z)$. Therefore, according to Lemma 3(vii), $h(z)$ is strictly increasing.

(x) According to Lemma 2(iv) the function $g(z)$ has exactly two different roots $z_1, z_2$ in the interval $(z_l, z_r)$ and according to Lemma 3(vii) and Lemma 2(i) it follows that $h(z)$ is strictly increasing for $z < z_1$, strictly decreasing for $z$ with $z_1 < z < z_2$, and strictly increasing for $z > z_2$. Hence the claim follows.

F Special Points

In this section we prove Lemma 4.

(i) follows from Lemma 1(iv),(ii),(iii).

(ii) follows from Lemma 2(iv).

(iii) follows from Lemma 1(vi).

(iv) According to the definition of $g(z)$ we have $f(z') = g(z') - \frac{1}{1 - z'} - 2 + b + a$. Assume that $z' = z_1$ or $z' = z_2$, then $g(z') = 0$ and $f(z') = -\frac{1}{1 - z'} - 2 + b + a$ which is contradiction to Lemma 1(vii).