Hurwitz trees and deformations of Artin-Schreier covers

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Abstract

Let $R$ be a complete discrete valuation ring of equal characteristic $p > 0$. Given a $\mathbb{Z}/p$-Galois cover of a formal disc over $R$, one can derive from it a semi-stable model for which the specializations of branch points are distinct and lie in the smooth locus of the special fiber. The description leads to a combinatorial object which resembles a classical Hurwitz tree in mixed characteristic, to which we will give the same name. The existence of a Hurwitz tree is necessary for the existence of a $\mathbb{Z}/p$-cover whose branching data fit into that tree. We show that the conditions imposed by a Hurwitz tree's structure are also sufficient. Using this, we improve a known result about the connectedness of the moduli space of Artin-Schreier curves of fixed genus.

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1 Introduction

Throughout this paper, we assume that \( k \) is an algebraically closed field of characteristic \( p > 0 \). We use the notation \{ \ldots \} to denote a multi-set. An Artin-Schreier curve is a smooth, projective, connected \( k \)-curve \( Y \), which is a \( \mathbb{Z}/p \)-cover of the projective line \( \mathbb{P}^1_k \). The moduli space of Artin-Schreier \( k \)-curves of fixed genus \( g \), which we denote by \( \mathcal{AS}_g \), has the property that there are curves with different branching data (e.g., different number of branch points) which lie in the same connected component \([PZ12]\). That motivates the study of equal characteristic deformations of these covers. In \([Dan20]\), by explicitly constructing some local deformations, it was shown that \( \mathcal{AS}_g \) is connected when \( g \) is sufficiently large.

**Theorem 1.1** (cf. \([Dan20]\) Theorem 1.1)).

- When \( p = 3 \), \( \mathcal{AS}_g \) is always connected.
- When \( p = 5 \), \( \mathcal{AS}_g \) is connected for any \( g \geq 14 \) and \( g = 0, 2 \). It is disconnected if \( g = 4, 6, 8 \).
- When \( p > 5 \), \( \mathcal{AS}_g \) is connected if \( g \geq \frac{(p^3-2p^2+p-8)(p-1)}{8} \) and \( g \leq \frac{p-1}{2} \). It is disconnected if \( \frac{p-1}{2} < g \leq \left( \frac{p-1}{2} \right)^2 \).

Observe that, when \( p = 5 \), the only cases that the theorem does not cover are \( g = 10 \) and \( g = 12 \) (\( \mathcal{AS}_g \) is empty otherwise). The techniques from \([Dan20]\) are inadequate to study these moduli spaces, as well as other cases. In this paper, with the aim to improve that result, we study local deformations in more detail using the notion of Hurwitz tree. A quick overview is given below.

Let \( R \) be a complete discrete valuation ring with characteristic \( p > 0 \) residue field. When \( R \) is of mixed characteristic (for example \( R = \text{W}(k) \) where \( k \) is a perfect field of characteristic \( p > 0 \)), a \( G \)-cover of the formal disc \( \text{Spec} \ R[[X]] \) gives rise to a combinatorial object called a Hurwitz tree, which has the shape of the dual graph of the semistable model associated to the cover (see e.g., \([BW09]\), \([Hen00]\)). The existence of such a tree, along with some other conditions, is necessary for a cover in characteristic \( p \) to “lift” to characteristic 0. In particular, when \( G \cong \mathbb{Z}/p \), Henrio proves that the lifts of a \( G \)-cover can be classified by Hurwitz trees of certain forms associated to the cover \([Hen00]\). Moreover, by generalizing Henrio’s technique, Bouw and Wewers prove an analog of Henrio’s lifting result for \( G \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m \) where \( m \) is prime to \( p \) \([BW06]\), and show that all \( D_p \)-covers (where \( p \neq 2 \)) lift.
In this paper, we generalize the notion of Hurwitz tree to the case where $R$ is a complete discrete valuation ring of equal characteristic, specifically, $R = k[[t]]$, and $G \cong \mathbb{Z}/p$. The equal characteristic degeneration of $\mathbb{Z}/p$-covers was well-studied by Maugeais and Saida in [Mau03] and [Saï07]. Using their results together with Henrio’s idea in [Hen00], we show that the existence of a flat deformation between Artin-Schreier covers of given branching data equates to the existence of a Hurwitz tree that satisfies certain criteria that are imposed by these data. Below is the main result of this paper.

**Theorem 1.2.** Suppose $d, d_1, \ldots, d_r \not\equiv 0 \pmod{p}$ are integers such that $\sum_{i=1}^{r}(d_i + 1) = d + 1$. Fix a projective line $\mathbb{P}^1_{k[[t]]}$ and a $k$-point $b$ of $\mathbb{P}^1_{k[[t]]}$. The projective line $\mathbb{P}^1_{k((t))}$ is identified with the generic fiber of $\mathbb{P}^1_{k[[t]]}$. Then there exists a $\mathbb{Z}/p$-Galois cover of $\mathbb{P}^1_{k[[t]]}$ whose special fiber is an Artin-Schreier cover over $k$, branched only at $b$ with ramification jump $d$ and whose generic fiber is an Artin-Schreier cover of $\mathbb{P}^1_{k((t))}$ branched at $r$ points that specialize to $b$ and which have ramification jumps $d_1, d_2, \ldots, d_r$ if and only if there exists a Hurwitz tree of type $\{d_1 + 1, d_2 + 1, \ldots, d_n + 1\}$. Type, which will be defined in Definition 4.2, is a combinatorial invariant of a Hurwitz tree. Applying the Hurwitz tree criteria, we improve Theorem 1.1 as below.

**Theorem 1.3.**
- When $p = 5$, $\mathcal{AS}_g$ is connected if and only if $g \geq 14$ or $g = 0, 2$.
- When $p > 5$, $\mathcal{AS}_g$ is disconnected if $(p - 1)/2 < g \leq (p - 1)(p - 2)$.

### 1.1 Structure

Section 2 gives a quick overview of Artin-Schreier theory, the moduli space of Artin-Schreier covers of fixed genus, and how to partition the space by the branching data of the points. In the same section, we link the geometry of $\mathcal{AS}_g$ with the deformation of $\mathbb{Z}/p$-covers (of genus $g$), and reduce Theorem 1.2 to a local one (Theorem 2.20). In §3 we examine the degeneration of étale $\mathbb{Z}/p$-torsors on a disc using the language of Kato’s refined Swan conductors. They are crucial to our approach, and distinguish this paper from [Dan20]. Section 4 introduces the notion of Hurwitz tree and describes how to derive such a tree from a given $\mathbb{Z}/p$-deformation (§4.2), thus proving the forward direction of Theorem 2.20. Section 5 considers the inverse process of §4.2 constructing a $\mathbb{Z}/p$-deformation from a given Hurwitz tree, hence completing the proof of Theorem 2.20. We then recover some known results about deformations using the new technique in §6.1. Finally, the proof Theorem 1.3 is given in §6.2.1.

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### 2 Artin-Schreier Covers

#### 2.1 Artin-Schreier Theory

Let $K$ be a field of characteristic $p > 0$. If $L$ is a separable extension of $K$ of degree $p$, then the classical Artin-Schreier theory says that $L = K(\alpha)$ where $\alpha$ is a root of a polynomial equation
there is a filtration of ramification groups in upper numbering $g$ if and only if $g(x) = f(x) + l(x)p - l(x)$ for some $h(x) \in K(x)$. At each ramification point, there is a filtration of ramification groups in upper numbering [Ser79 IV]. In our case, as the inertia group is $\mathbb{Z}/p$, the filtration has only one jump, which we call the ramification jump at the corresponding branch point.

**Example 2.1.** Suppose $p = 5$. The cover of $Y$ of $\mathbb{P}^1_k$ defined by

$$y^5 - y = \frac{1}{x^5} + \frac{1}{(x-1)^2},$$

is an Artin-Schreier curve. Note that the term $1/x^5$ is a 5th-power. Hence, by the above discussion, one may add $(-1/x)^5 - (-1/x)$ to the right-hand-side of (2). The result is an Artin-Schreier equation of the form

$$y^5 - y = -\frac{1}{x} + \frac{1}{(x-1)^2}.$$  

**Remark 2.2.** We say the Artin-Schreier equation (3) has reduced form. That means the partial fraction decomposition of the right-hand-side of the equation only consists of terms of prime-to-$p$ degree. When $k$ is algebraically closed, any Artin-Schreier cover can be represented by a rational function $f(x)$ of reduced form.

The rational function $f(x)$ in (1) tells us everything about the ramification data of the cover $\phi$. Suppose $f(x)$ has $r$ poles: $B := \{B_1, \ldots, B_r\}$ on $\mathbb{P}^1_k$. Let $d_j$ be the order of the pole of $f(x)$ at $B_j$. One may assume that $f(x)$ is in reduced form. Hence, the number $d_j$ is prime to $p$. It is an easy exercise to show that $B$ is the collection of branch points of $\phi$, and $d_j$ is the ramification jump at $B_j$. We call $h_j := d_j + 1$ the conductor at $B_j$. Then $h_j \geq 2$ and $h_j \equiv 1 \pmod{p}$. Moreover, the ramification divisor of $\phi$ is $D := \sum_{j=1}^{r} (p-1) h_j Q_j$ where $Q_j$ is the ramification point above $B_j$ ([Ser79], IV, Proposition 4). Applying the Riemann-Hurwitz formula ([Har77], IV, Corollary 2.4), we obtain the following lemma.

**Lemma 2.3 ([PZ12] Lemma 2.6).** The genus of $Y$ is $g_Y = ((\sum_{j=1}^{r} h_j) - 2)(p-1)/2$.

**Definition 2.4.** We call the $r \times 1$ matrix $[h_1, \ldots , h_r]^\top$ the branching datum of $\phi$. For instance, the cover in Example 2.1 has branching datum $[2, 3]^\top$. Throughout the paper, we define $d$ by $g = d(p-1)/2$. So, we have the identities: $d = (\sum_{j=1}^{r} h_j) - 2$ and $\sum_{j=1}^{r} h_j = d + 2 = 2g/(p-1) + 2$.

**Remark 2.5.** The above lemma shows that all the Artin-Schreier $k$-curves with the same genus $g$ have the same sum of conductors $d + 2$. That is the essential difference between $\mathbb{Z}/p$-curves and $\mathbb{Z}/q$-curves where $q \neq p$ is prime. For each $\mathbb{Z}/q$-curve, a branch point contributes $q - 1$ to the degree of its ramification divisor. Thus, every $\mathbb{Z}/q$-cover of genus $g$ must have the same number of branch points. It hence makes sense to group Artin-Schreier covers of the same genus by their branching data. This idea is utilized by Pries and Zhu in [PZ12], and will be discussed in the next section.
Remark 2.6. When the Galois group is $\mathbb{Z}/p^n$, the Artin-Schreier theory is generalized by the Artin-Schreier-Witt theory [Lor08, §26] [Lan02, §4]. It says that a $\mathbb{Z}/p^n$-cover of $\mathbb{P}^1_k$ is determined by some certain length-$n$-Witt-vector over $K$. We will discuss the deformations of this family of covers in a forthcoming paper.

2.2 Deformations of Artin-Schreier covers

Suppose $C \xrightarrow{\phi} \mathbb{P}^1_k$ is a $G$-Galois cover over $k$, where $C$ is a smooth, projective, connected $k$-curve. Suppose, moreover, that $A$ is a Noetherian, complete $k$-algebra with residue field $k$. Let

$$\text{Def}_k(\phi) : \text{Alg}/k \to \text{Set}$$

be the functor which to any $A, f : A \to k \in \text{Alg}/k$ associates classes of $G$-Galois covers $C \xrightarrow{\Phi} \mathbb{P}^1_A$ that make the following cartesian diagram commute

$$
\begin{array}{ccc}
C & \xrightarrow{\phi} & C \\
\downarrow & & \downarrow \\
\mathbb{P}^1_k & \xrightarrow{f} & \mathbb{P}^1_A \\
\downarrow & & \downarrow \\
\text{Spec } k & \xrightarrow{f} & \text{Spec } A,
\end{array}
$$

(4)

and so that the $G$-action on $C$ induces the original action on $C$. We say $\Phi$ is a deformation of $\phi$ over $A$, or $\phi$ is deformed (over $A$) to $\Phi$. For more details, see [BM00, §2]. In this paper, we focus on the case where $G = \mathbb{Z}/p$ and $A = k[[t]]$.

Remark 2.7. One unique aspect of characteristic $p$ is that there exist flat deformations of a wildly ramified cover over rings of equal characteristic so that the number of branch points changes, but the genus does not. That gives us a way to investigate a cover in characteristic $p$: finding a connection of it with a slightly different one via equal characteristic deformation. For example, using some (equal characteristic) deformations of $\mathbb{Z}/p^n$-covers [Pop14, Lemma 3.2], Pop reduced the lifting problem for cyclic groups (a.k.a., the Oort conjecture) to the case that had been solved by Obus and Wewers in [OW14]. Another perk of studying these deformations is understanding the geometry of the moduli space that parameterizes Galois covers. That will be discussed further in §2.3.

Remark 2.8. Suppose $\Phi : Y \to \mathbb{P}^1_k$ is a $\mathbb{Z}/p$-cover that branched at $r$ points $\{P_1, \ldots, P_r\}$ with conductor $h_i$ at $P_i$. Suppose, moreover, that $\Phi$ is a smooth deformation of $\phi$ over a complete discrete valuation ring of equal characteristic whose generic fiber $\Phi_\eta$ has branch locus $\{P_{1,1}, \ldots, P_{1,m_1}, \ldots, P_{r,1}, \ldots, P_{r,m_r}\}$, where each $P_{i,j}$ reduces to $P_i$ and has conductor $h_{i,j}$. Then it follows from the discussion in Remark 2.5 that $\sum_{i=1}^r \sum_{j=1}^{m_i} h_{i,j} = \sum_{i=1}^r h_i$. We say the deformation $\Phi$ has type

$$[h_1, \ldots, h_r]^T \to [h_{1,1}, \ldots, h_{1,m_1}, h_{2,1}, \ldots, h_{r,1}, \ldots, h_{r,m_r}]^T.$$

2.3 The moduli space of Artin-Schreier covers and their deformations

In [PZ12], the authors introduce the moduli space of Artin-Schreier $k$-curves of genus $g$, which they denote by $\mathcal{AS}_g$. They enumerate a family of locally closed strata of $\mathcal{AS}_g$ by partitions
of the integer \(d + 2\) such that no entries are congruent to 1 modulo \(p\), i.e., all the possible branching data of Artin-Schreier curves of genus \(g\). We call the collection of those partitions \(\Omega_{d+2}\). For instance, the partition \(\vec{E} = \{h_1, \ldots, h_r\}\) of \(d + 2\) is associated with the stratum \(\Gamma_{\vec{E}}\), which is the collection of all the points of \(\mathcal{AS}_g\) that represent Artin-Schreier curves with branching datum \([h_1, \ldots, h_r]^\top\). We write \(\vec{E}_1 \prec \vec{E}_2\) if the latter one is a refinement of the former one.

**Example 2.9.** Suppose \(p = 5\) and \(g = 14\). Then \(d = 7\), and the strata of \(\mathcal{AS}_g\) correspond to the following partitions of \(d + 2\): \(\{9\}, \{7, 2\}, \{5, 4\}, \{5, 2, 2\}, \{4, 3, 2\}, \{3, 3, 3\}, \) and \(\{3, 2, 2, 2\}\).

The following result shows that one can relate the geometry of \(\mathcal{AS}_g\) with the existence of equal characteristic deformations between curves in different strata (hence have distinct branching data).

**Proposition 2.10** ([Dan20 Proposition 3.4]). Suppose \(\vec{E}_1\) and \(\vec{E}_2\) are two partitions of \(d + 2\). Then the stratum \(\Gamma_{\vec{E}_1}\) is contained in the closure of \(\Gamma_{\vec{E}_2}\) if and only if there exists a deformation over \(k[[t]]\) from a point in \(\Gamma_{\vec{E}_1}\) to one in \(\Gamma_{\vec{E}_2}\).

Furthermore, the next result shows that the closure of a stratum is simply a union of strata.

**Proposition 2.11** ([Dan20 Corollary 3.6]). Let \(\vec{E}_1\) and \(\vec{E}_2\) be as in the previous proposition. Suppose \(\Gamma_{\vec{E}_1}\) is not in the closure of \(\Gamma_{\vec{E}_2}\). Then \(\Gamma_{\vec{E}_1}\) is disjoint from the closure of \(\Gamma_{\vec{E}_2}\).

Therefore, understanding these deformations can give us a full picture of the moduli space \(\mathcal{AS}_g\). We thus want to answer the following question.

**Question 2.12** (Deformation of Artin-Schreier covers problem). Suppose we are given \(\vec{E}_1\) and \(\vec{E}_2\) in \(\Omega_{d+2}\). Does there exists a deformation over \(k[[t]]\) of type \([\vec{E}_1]^\top \rightarrow [\vec{E}_2]^\top\)?

### 2.3.1 The graph \(C_d\)

Let us fix a prime \(p\) and construct a directed graph \(C_d\) (where \(d\) is as in Definition 2.4). The vertices of the graph correspond to the partitions \(\vec{E}\) in \(\Omega_d\). There is an arrow from \(\vec{E}\) to \(\vec{E}'\) if and only if \(\vec{E} \prec \vec{E}'\), and \(\vec{E}\) lies in the closure \(\Gamma_{\vec{E}}\).

In general topology, if one irreducible subset of a space lies in the closure of another, then they are contained in the same connected component of that space. Thus, if \(C_d\) is connected, then so is \(\mathcal{AS}_g\). It is straightforward to check that the converse also holds. From now on, we will study the geometry of \(C_d\).

**Example 2.13.** Suppose \(p = 5\) and \(g = 14\). Then \(d = 7\). Using the information from [Dan20 Theorem 3.10], we draw a subgraph of \(C_7\) as follows.

![Graph C7](image)

As the subgraph is connected, the graph \(C_7\) is connected, and so is \(\mathcal{AS}_{14}\).
Remark 2.14. One can apply the results later in this paper (Proposition 5.12 and Proposition 6.2) to show that the above diagram is the complete $C_7$. Thus, we can read from the graph that the irreducible components of $A_{S_{14}}$ are the closures of the following strata: $\Gamma_{\{5,4\}}, \Gamma_{\{3,3,3\}}, \Gamma_{\{4,3,2\}}, \Gamma_{\{5,2,2\}},$ and $\Gamma_{\{3,2,2,2\}}$. Furthermore, the intersection of $\Gamma_{\{3,3,3\}}$ and $\Gamma_{\{3,2,2,2\}}$ is $\Gamma_{\{9\}}$, and the intersection of $\Gamma_{\{4,3,2\}}$ and $\Gamma_{\{5,2,2\}}$ is $\Gamma_{\{7,2\}}$.

### 2.4 Reduction to the local deformation problem

We first state a fact about germs of Artin-Schreier curves, which implies that, locally, they are easy to control.

**Proposition 2.15.** Suppose $\phi : Y \to \mathbb{P}^1_k$ is an Artin-Schreier cover and $P \in \mathbb{P}^1_k$ is a branch point of $\phi$. Then the localization of $\phi$ at $P$ is determined by the ramification jump at $P$.

**Proof.** See Lemma 2.1.2 of [Pri03]. \(\square\)

That means a $\mathbb{Z}/p$-cover of $\text{Spec} k[[x]]$ of conductor $h$ is isomorphic to one defined by

$$y^p - y = \frac{1}{x^{n-1}}.$$ 

The following local-global principle type result will help us to reduce our study of Artin-Schreier deformations to the local case. We say there always exists a deformation of type $[\overline{E}_1]^\top \to [\overline{E}_2]^\top$ if, given a curve of branching datum $[\overline{E}_1]^\top$, we can deform it to one with branching datum $[\overline{E}_2]^\top$.

**Proposition 2.16 (c.f. [Dan20, Proposition 3.4]).** Suppose $\overline{E}_1 = \{h_1, h_2, \ldots, h_n\}$ and $\overline{E}_2$ are in $\Omega_n$. Then there always exists a deformation of type $[\overline{E}_1]^\top \to [\overline{E}_2]^\top$ if and only if there exist $n$ partitions $\{h_i\} < [\overline{E}_{i,2}] := \{h_{i,1}, \ldots, h_{i,m_i}\} \subseteq [\overline{E}_2]$ ($1 \leq i \leq n$), where the $\overline{E}_{i,2}$'s partition $[\overline{E}_{2}]$, and $n$ deformations $\Phi_1, \Phi_2, \ldots, \Phi_n$ over $k[[t]]$, where $\Phi_i$ has type $[\overline{E}_i] \to [h_{i,1}, \ldots, h_{i,m_i}]^\top$. In particular, a deformation of this type exists only if $\overline{E}_1 < \overline{E}_2$.

**Example 2.17.** Suppose $p = 5$. Then, by Proposition 2.16, there exists a deformation (over $k[[t]]$) of type $[7,4]^\top \to [4,3,2]^\top$ if and only if there exist either two deformations of types $[7] \to [3,2,2]^\top$ and $[4] \to [4]$ (trivial deformation), or two deformations of types $[7] \to [4,3]^\top$ and $[4] \to [2,2]^\top$.

**Remark 2.18.** It follows immediately from the above proposition that, if we have a deformation as in Remark 2.8, then $\sum_{j=1}^{m_i} h_{i,j} = h_i$ for all $i$.

Proposition 2.15 and Proposition 2.16 suggest that we may assume $\phi$ is a $\mathbb{Z}/p$-extension $k[[z]]/k[[x]]$ branched only at $x = 0$ with conductor $h$. Therefore, we may also think of a deformation $\phi$ over $k[[t]]$ as a $\mathbb{Z}/p$-cover $k[[t]][[Z]]/k[[t]][[X]]$. We thus reduce Question 2.12 to the following.

**Question 2.19** (Local deformation of Artin-Schreier covers). Suppose $\{h\} \sim \{h_1, \ldots, h_r\} \in \Omega_h$ and $\phi$ is a $\mathbb{Z}/p$-cover $k[[z]]/k[[x]]$ given by $y^p - y = \frac{1}{x^{h-1}}$. Define $R := k[[t]]$. Does there exist a deformation $R[[Z]]/R[[X]]$ of $\phi$ with generic branching datum $[h_1, \ldots, h_r]^\top$?

Hence, the question can be fully answered by the following local version of Theorem 1.2.

**Theorem 2.20.** Let $\phi : k[[z]] \to k[[x]]$ be a local $G$-cover with conductor $d$. Then there exists a deformation of $\phi$ over $k[[t]]$ of type $[d + 1] \to [d_1 + 1, \ldots, d_r + 1]^\top$ if and only if there exists a Hurwitz tree of type $\{d_1 + 1, \ldots, d_r + 1\}$.
2.5 Birational deformation and the different criterion

Usually, when dealing with Galois extensions of $k[[x]]$, it will be more convenient to deal with extensions of fraction fields than extensions of rings. So we will often want to think of a Galois ring extension in terms of the associated extension of fraction fields.

**Definition 2.21.** Suppose $A/k[[x]]$ is a $G$-extension. Suppose, moreover, that $M/\operatorname{Frac}(R[[X]])$ where $R/k[[t]]$ finite, is a $G$-extension, and $A_R$ is the integral closure of $R[[X]]$ in $M$. We say $M/\operatorname{Frac}(R[[X]])$ is a birational deformation of $A/k[[x]]$ if

1. The integral closure of $A_R \otimes_R k$ is isomorphic to $A$, and
2. The $G$-action on $\operatorname{Frac}(A) = \operatorname{Frac}(A_R \otimes_R k)$ induced from that on $A_R$ restricts to the given $G$-action on $A$.

The following criterion is extremely useful for seeing when a birational deformation is actually a deformation (i.e., when $A_R \otimes_R k$ is already integrally closed, this isomorphic to $A$).

**Proposition 2.22** (The different criterion [GM98 1.3.4]). Suppose $A_R/R[[X]]$ is a birational deformation of the $G$-Galois extension $A/k[[x]]$. Let $K = \operatorname{Frac} R$, let $\delta_y$ be the degree of the different of $(A_R \otimes_R K)(R[[X]]) \otimes_R K)$, and let $\delta_y$ be the degree of the different of $A/k[[x]]$. Then $\delta_x \leq \delta_y$ and equality holds if and only if $A_R/R[[X]]$ is a deformation of $A/k[[x]]$.

**Example 2.23.** Let $p = 5$, $g = 10$, hence, $d = 5$. Consider the $\mathbb{Z}/p$-cover $\Phi$ given by the normalization of $\mathbb{P}^1_{k[[t]]}$ over the extension of its fraction field defined by the following equation:

$$Y^5 - Y = \frac{-2X + t_5}{(−2)X^5(X - t_5)^2} = H(X, t).$$  \hspace{1cm} (5)

The special fiber is birational to the $\mathbb{Z}/5$-cover

$$y_5 - y = \frac{1}{x_5},$$

which is branched at 0 with conductor 7.

On the generic fiber, when $t \neq 0$, the partial fraction decomposition of $H(X, t)$ is of the form:

$$\frac{-1}{2t^5X^5} + \frac{1}{2t^{15}X^3} + \frac{1}{t^{20}X^2} + \frac{3}{2t^{25}X} + \frac{1}{2t^{20}(X - t^5)^2} - \frac{3}{2t^{25}(X - t^5)}.$$

Adding $\frac{1}{2t^5X^5} - \frac{1}{2tX}$ to $H(X, t)$, we get

$$\frac{-1}{2tX} + \frac{1}{2t^{15}X^3} + \frac{1}{t^{20}X^2} + \frac{3}{2t^{25}X} + \frac{1}{2t^{20}(X - t^5)^2} - \frac{3}{2t^{25}(X - t^5)},$$

which is a reduced form. Hence, the generic fiber is branched at two points $X = 0$ and $X = t$, which have conductors 4 and 3, respectively. It then follows from Proposition 2.22 that $\Phi$ is a deformation of type $[7] \rightarrow [4, 3]^T$.

**Example 2.24.** One can check, in the same fashion as above, that the $\mathbb{Z}/5$-cover given by

$$Y^5 - Y = \frac{-2X + t_5}{(−2)X^5(X - t_5)^2(X - t^5)^{5}}$$

is a deformation of $y^5 - y = \frac{1}{x_5}$ over $k[[t]]$, where the generic branch points $0, t_5$, and $t^5$ have conductors 4, 3, and 5, respectively. Hence, it is a deformation of type $[12] \rightarrow [4, 3, 5]^T$.  

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Remark 2.25. Suppose \( \phi \) is a Galois cover over \( k \). We can also apply the criterion to determine whether a deformation \( \Phi \) of \( \phi \) over a ring \( R \) of mixed characteristic (e.g., \( R \) is a finite extension of \( W(k) \), where \( W(k) \) is a Witt vector over \( k \)) is smooth. The lifting problem for Galois covers concerns the existence of a Galois cover in characteristic 0 that reduces to a given one in characteristic \( p \). See [OW14, §4] for some explicit lifts of local Artin-Schreier covers and \( \mathbb{Z}/p^2 \)-covers. Some good expositions of the problem are [Obu12], [BW06], [OW14], and [Wea18].

The notion of Hurwitz tree, which we will discuss in §4, is first introduced in [Hen00] to tackle the lifting problem for Artin-Schreier covers. Theorem 1.2 is actually the equal characteristic analog of [Hen00 Théorème de réalisation].

3 Degeneration of \( \mathbb{Z}/p \)-covers

In this section, we study the degeneration of \( \mathbb{Z}/p \)-covers of \( \text{Spec} \, R[[X]] \), where \( R \) is a complete discrete valuation of equal characteristic and with uniformizer \( \pi \). Normalize the canonical valuation on \( R \) so that \( \nu(\pi) = 1 \). Set \( K := \text{Frac} \, R \) and \( \mathbb{K} := \text{Frac} \, R[[X]] \). We first introduce a geometric interpretation of \( \text{Spec} \, R[[X]] \) using the language of non-archimedean geometry.

3.1 Discs and annuli

Firstly, we identify the \( K \)-analytic points of \( \text{Spec} \, R[[X]] \) with

\[
D = \{ u \in (\mathbb{A}^1_K)^{\text{an}} \mid \nu(u) > 0 \}
\]

by plugging in \( X = u \). We call \( X \) a parameter for the open unit disc \( D \) with center 0.

Let \( R\{X\} \subseteq R[[X]] \) consist of power series for which the coefficients tend to 0, i.e.,

\[
R\{X\} = \left\{ \sum_{i \geq m} a_i X^i \mid \lim_{i \to \infty} \nu(a_i) = \infty \right\}
\]

As before, \( K \)-points of a closed unit disc \( \text{Spec} \, R\{X\} \) can be identified with

\[
\mathcal{D} = \{ u \in (\mathbb{A}^1_K)^{\text{an}} \mid \nu(u) \geq 0 \}.
\]

The boundary of the open (or closed) unit disc is represented by the scheme \( R[[X^{-1}]][X] \). Note that the ring \( S := R[[X^{-1}]][X] \) is a complete discrete valuation ring with residue field \( \overline{S} = k((x)) \), uniformizing element \( \pi \) and fraction field \( S \otimes_R K \).

For \( r \in \mathbb{Q}_{\geq 0} \), and \( a \in K \) such that \( |a| = r \), the open (resp. closed) disc of radius \( r \) is characterized by the scheme \( \text{Spec} \, R[[a^{-1}X]] \) (resp. \( \text{Spec} \, R\{a^{-1}X\} \)). Its set of \( K \)-points is isomorphic to the open (resp. closed) unit disc under the map \( X \mapsto a^{-1}X \). Denote by

\[
\mathcal{D}[s,z] := \{ u \in (\mathbb{A}^1_K)^{\text{an}} \mid \nu(u - z) \geq s \},
\]

where \( z \in (\mathbb{A}^1_K)^{\text{an}} \), the closed disc of radius \( p^s \) centered at \( X - z \), and \( \mathcal{D}[s] := \mathcal{D}[s,0] \). One can associate with \( \mathcal{D}[s,z] \) the “Gauss valuation” \( \nu_{s,z} \) that is defined by

\[
\nu_{s,z}(f) = \inf_{a \in \mathcal{D}[s,z]} (\nu(f(a)));
\]

for each \( f \in \mathbb{K}^X \). This is a discrete valuation on \( \mathbb{K} \) which extends the valuation \( \nu \) on \( K \), and has the property \( \nu_{s,z}(X - z) = s \). We denote by \( \kappa_s \) the function field of \( \mathbb{K} \) with respect to the
valuation \( \nu_{s,z} \). That is the function field of the canonical reduction \( \overline{D}[s,z] \) of \( D[s,z] \). In fact, \( \overline{D}[s,z] \) is isomorphic to the affine line over \( k \) with function field \( \kappa_{s,z} = k(x_{s,z}) \), where \( x_{s,z} \) is the image of \( \pi^{-s}(X - z) \) in \( \kappa_{s,z} \). For a closed point \( \overline{x} \in \overline{D}[s,z] \), we let \( \text{ord}_{\overline{x}} : \kappa_{s,z}^* \to \mathbb{Z} \) denote the normalized discrete valuation corresponding to the specialization of \( \overline{x} \) on \( \overline{D}[s,z] \). We let \( \text{ord}_\infty \) denote the unique normalized discrete valuation on \( \kappa_{s,z} \) corresponding to the “point at infinity”.

The open annulus of thickness \( \epsilon \) is described by \( \text{Spec} \, R[[X,U]]/(XU - a) \), where \( a \in K \) is such that \( \nu(a) = \epsilon \). The open annulus has two boundaries, one given by \( \text{Spec} \, R[[X]]\{X^{-1}\} \) and one given by \( \text{Spec} \, R[[U]]\{U^{-1}\} \). Note that two annuli over \( R \) are isomorphic if and only if they have the same thickness.

For \( F \in \mathbb{K}, z \in (\mathbb{A}^1_K)_{an} \), and \( s \in \mathbb{Q}_{\geq 0} \), we let \([F]_{s,z}\) denotes the image of \( \pi^{-\nu_{s,z}(F)}F \) in the residue field \( \kappa_{s,z} \).

### 3.2 Semistable model and a partition of a disc

Consider the open unit disc \( D := \text{Spec} \, R[[X]] \), and suppose we are given \( x_{1,K}, \ldots, x_{r,K} \) in \( D(K) \), with \( r \geq 2 \). We can think of \( x_{1,K}, \ldots, x_{r,K} \) as elements of the maximal ideal of \( R \). Let \( D^{st} \) be a blow-up of \( D \) such that

- the exceptional divisor \( \overline{D} \) of the blow-up \( D^{st} \to \text{Spec} \, R[[X]] \) is a semistable curve over \( k \),
- the fixed points \( x_{b,K} \) specialize to pairwise distinct smooth points \( x_b \) on \( \overline{D} \), and
- if \( \overline{x} \) denotes the unique point on \( \overline{D} \) which lies in the closure of \( D^{st} \otimes k \setminus \overline{D} \),

then \( (\overline{D}; \overline{x}, (x_b)) \) is stably marked. We call \( D^{st} \) the stable model of the marked disc \( (D; x_1, \ldots, x_r) \), and \( (\overline{D}; \overline{x}, (x_b)) \) its special fiber. The dual graph of the special fiber is a tree whose leaves correspond to the marked points and whose root corresponds to \( \overline{x} \).

**Example 3.1.** Suppose we are given a \( \mathbb{Z}/5 \)-cover (in characteristic 5) of \( \mathbb{P}^1_K \) that has four branch points \( X = 0, X = t^5, X = t^5(1 + t^5), \) and \( X = t^{10} \). The left graph of Figure 1 represents a semi-stable model of \( \mathbb{P}^1_K \) marked by the open unit disc \( D = \text{Spec} \, R[[X]] \) and the branch points of the cover. The tree on the right is its dual graph.

We associate with each edge an annulus. In this example, \( e_0 \) corresponds to the spectrum of \( R[[X,X_1]]/(XX_1 - t^5) \) and \( e_1 \) resembles \( \text{Spec} \, R[[X_1^{-1} - 1, V]](V(X_1 - 1) - t^5) \). We say \( e_0 \) has thickness 1, which is the thickness of the associated annulus divided by \( p = 5 \). Moreover, each vertex, which is not the root of the tree, is associated with a punctured disc. For instance, the vertex \( v_1 \) corresponds to \( \text{Spec} \, R[[X_1^{-1}, X_1, (X_1^{-1} - 1)^{-1}]] \). That can also be thought of as the complement of the closed disc \( \text{Spec} \, R[[X_1^{-1}]] \) by the open discs \( \text{Spec} \, R[[X_1]] \) and \( \text{Spec} \, R[[X_1^{-1} - 1]] \). The root is linked with \( \text{Spec} \, R\{X\} \). Finally, each leaf is associated with an open disc. To illustrate, one correlated with \( t^5 \) is \( \text{Spec} \, R[[t^{-5}(X - t^5)]] \). Table 1 shows where the \( K \)-points of \( C \) specialize.

### 3.3 Reduction of covers

Let \( \Phi : R[[Z]] \to R[[X]] \) be a Galois cover. After enlarging our ground field \( K \), we may assume that \( \Phi \) is weakly unramified with respect to the Gauss valuation \( \nu_0 \) (that maps \( X \) to 0), see [Epp73]. By definition, this means that for all extensions \( w \) of \( \nu_0 \) to the function field of \( R[[Z]] \), the ramification index \( e(w/\nu_0) \) is equal to 1. It then follows that the special fiber \( \text{Spec} \, R[[Z]] \otimes_R k \) is reduced.
Points of $C(K)$ that specialize to $V$ & Associated algebraic object
\begin{tabular}{|c|c|c|}
\hline
Subscheme $V$ of $\overline{C}$ & $\{Y \mid 0 < \nu(Y) < 5\}$ & $R[[X,X_1]]/(XX_1 - t^9)$ \\
$\overline{C}_3 \cap \overline{C}_1$ & $\{Y \mid 5 < \nu(Y) < 10\}$ & $R[[X^{-1},X_2]]/(X_1^{-1}X_2 - t^5)$ \\
$\overline{C}_3 \setminus \overline{C}_1$ & $\{Y \mid \nu(Y) \geq 10\}$ & $R\{X_2^{-1}\}$ \\
$\overline{C}_2 \cap \overline{C}_1$ & $\{Y \mid 5 < \nu(Y - t^5) < 10\}$ & $R[[X^{-1}, X_1 - 1]](V(1^{-1} - t^5))$ \\
$\overline{C}_2 \setminus \overline{C}_1$ & $\{Y \mid \nu(Y - t^5) \geq 10 \land \nu(Y) = 5\}$ & $R\{V^{-1}\}$ \\
$\overline{C}_1 \setminus (\overline{C}_3 \cup \overline{C}_2 \cup \{\infty\})$ & $\{Y \mid \nu(Y) = 5 \land \nu(Y - t^5) = 5\}$ & $R\{X_1^{-1}, X_1, (X_1^{-1} - 1)^{-1}\}$ \\
\hline
\end{tabular}

Table 1: Partitions of $C(K)$

**Definition 3.2.** We say that the cover $\Phi$ has \textit{étale reduction} if the induced map $\text{Spec } R[[Z]] \otimes_R k \overset{\phi}{\to} \text{Spec } R[[X]] \otimes_R k$ is generically étale.

**Definition 3.3.** If $\Phi$ has étale reduction, we call $\phi$ the \textit{reduction} of $\Phi$, and $\Phi$ a \textit{deformation} of $\phi$ over $R$. We say $\Phi$ has \textit{good reduction} if it has étale reduction, and $\phi$ is smooth.

Recall Proposition 2.15 shows that a local Artin-Schreier cover is determined by the conductor of its unique branch point. We therefore reformulate Question 2.19 as follows.

**Question 3.4.** Suppose $\{h_1, \ldots, h_r\} \in \Omega_h$. Does there exist a $\mathbb{Z}/p$-cover of $\text{Spec } R[[X]]$ with good reduction that has generic branching datum $[h_1, \ldots, h_r]^T$?

### 3.4 Refined Swan conductors

Suppose $\Phi$ is a cyclic cover of a closed disc, say $\text{Spec } R\{X\}$. After enlarging $R$, we may assume that the ramification index is equal to 1. The set of all branch points is called the \textit{branch locus} of $\Phi$ and is denoted by $\mathcal{B}(\Phi)$.

We define two invariants that measure the ramification of $\Phi$ with respect to the canonical valuation. The \textit{depth} is

$$\delta(\Phi) := \text{sw}(\Phi)/p \in \mathbb{Q}_{\geq 0},$$

where $\text{sw}(\Phi)$ is the Swan conductor [Kat89 Definition 3.3] of the character associated to $\Phi$. The rational number $\delta(\Phi)$ is equal to 0 if and only if $\Phi$ is unramified. If this is the case, then its reduction $\phi$ is well-defined. In particular, if $\Phi$ is of order $p$ and $\delta(\Phi) = 0$, then there exists $u \in \kappa$ such that $\phi$ is defined by $y^p - y = u$. We call $u$ the \textit{reduction} of $\Phi$. Note that $u$ is unique up to adding an element of the form $a^p - a$, where $a \in \kappa$, by Artin-Schreier theory. We say $\Phi$ is \textit{radical} if $\delta(\Phi) > 0$. 

![Figure 1: The special fiber $\overline{C}$ and its dual graph](image-url)
Suppose \( \delta(\Phi) > 0 \). Set \( \mathcal{K} := \text{Frac } k[[x]] \). Then we can define the differential Swan conductor or differential conductor as

\[
\omega(\Phi) := dsw(\Phi) \in \Omega^1_{\mathcal{K}}
\]

see [Kat89, Definition 3.9].

We call \( (\delta(\Phi), \omega(\Phi)) \) when \( \delta(\Phi) > 0 \), or \( (\delta(\Phi), u) \) when \( \delta(\Phi) = 0 \) and \( u \) the reduction of \( \Phi \), the degeneration type of \( \Phi \).

Suppose \( x \in \text{Spec } R\{X\} \), and let \( \text{ord}_x : \kappa^\times \to \mathbb{Z} \) be a normalized discrete valuation whose restriction to \( k \) is trivial. Then the composite of \( v \) with \( \text{ord}_x \) is a valuation on \( \mathcal{K} \) of rank \( 2 \), which we denote by \( \mathcal{K}^\times \to \mathbb{Q}^\times \mathbb{Z} \). In [Kat87], Kato defines a Swan conductor \( \text{sw}_{\mathcal{K}}(\Phi)(x) \in \mathbb{Q}_{\geq 0} \times \mathbb{Z} \). Its first component is equal to \( \delta(\Phi) \). We define the boundary Swan conductor

\[
\text{sw}_{\Phi}(x) \in \mathbb{Z}
\]
as the second component of \( \text{sw}_{\Phi}(x) \). Geometrically, it gives the instantaneous rate of change of the depth in the “direction” corresponding to \( x \). See [OW14, §5.3.2] to learn more about this interpretation in the mixed characteristic case.

Remark 3.5. The invariant \( \text{sw}_{\Phi}(x) \) is determined by \( \delta(\Phi) \) and \( \omega(\Phi) \) as follows:

1. If \( \delta(\Phi) = 0 \), then

\[
\text{sw}_{\Phi}(x) = \text{sw}_{\phi}(x)
\]

where \( \phi \) is the reduction of \( \Phi \) and \( \text{sw}_{\phi}(x) \) is the usual Swan conductor of \( \phi \) with respect to the valuation \( \text{ord}_x \). That follows immediately from the definitions. We thus have \( \text{sw}_{\Phi}(x) \geq 0 \) and \( \text{sw}_{\Phi}(x) = 0 \) if and only if \( \phi \) is unramified with respect to \( \text{ord}_x \).

2. If \( \delta(\Phi) > 0 \), then we have

\[
\text{sw}_{\Phi}(x) = -\text{ord}_x(\omega(\Phi)) - 1
\]

This follows from [Kat87, Corollary 4.6].

3.4.1 Vanishing cycle formula

This section adapts [OW14, §5.3.3] and generalizes to the equal characteristic case.

Definition 3.6. A cover \( \Phi : Y \to C \cong \mathbb{P}^1_K = \text{Proj } K[X] \) is called admissible if its branch locus \( B(\Phi) \) is contained in the open disc \( \text{Spec } R\{X\} \).

By §2.4, we can restrict our study to the case where \( \Phi \) is admissible.

An affinoid subdomain \( D \subseteq C^\text{an} \) gives rise to a blowup \( C'_R \to C_R \) with the following properties ([BL93]): \( C'_R \) is a semistable curve whose special fiber \( \overline{C}' := C'_R \otimes_R k \) consists of two smooth irreducible components that meet in exactly one point. The first component is the strict transformation of \( \overline{C} \), which we may identify with \( \overline{C} \). The second component is the exceptional divisor \( \overline{Z} \) of the blow-up \( C'_R \to C_R \), which is isomorphic to the projective line over \( k \) and intersects \( \overline{C} \) in the distinguished point \( \overline{\pi}_0 \). By construction, the complement \( \overline{Z}' := \overline{Z} \setminus \{\overline{\pi}_0\} \) is identified with the canonical reduction \( \overline{D} \) of the affinoid \( D \). In particular, this means that the discrete valuation on \( \mathbb{K} \) corresponding to the prime divisor \( \overline{Z}' \subseteq C'_R \) is equivalent to the valuation \( v \) of \( D \) and that its residue field \( \kappa \) may be identified with the function field of \( \overline{Z} \).
Let $Y'_R$ denote the normalization of $C'_R$ in $Y$. We obtain the following commutative diagram

\[
\begin{array}{ccc}
Y'_R & \rightarrow & Y_R \\
\downarrow & & \downarrow \\
C'_R & \rightarrow & C_R
\end{array}
\]

in which the vertical maps are finite $\Gamma$-covers and each horizontal map is the composition of an a blowup with a normalization. Let $\overline{W} \subset Y'_R$ be the exceptional divisor of $Y'_R \rightarrow Y_R$. After enlarging the ground field $K$, we may assume that $\overline{W}$ is reduced. This condition allows us to define the refined Swan conductors for the restriction of $\Phi$ to $D$ as in §3.4. We now choose a closed point $x \in Z_0 = D$ and a point $\overline{y} \in \overline{W}$ lying over $x$. We let

\[
U(D, x) := \left\{ \frac{h_z}{z} \mid z \in B(\Phi) \cap U(D, x) \right\},
\]

where $B(\Phi)$ is the set of branch points of $\Phi$, and $h_z$ is the conductor of the branch point $z$.

**Proposition 3.8.** With the notation as above, we have

\[
sw_{\Phi|_D}(\overline{x}) = \mathbb{C}(D, \overline{\Phi}, \overline{x}) - 1 - 2\delta_{\overline{x}}.
\]

**Proof.** The proof is parallel to one for [OW14, Proposition 5.12]. Notes that $|B(\overline{\chi}) \cap U(r, \overline{x})|$ in Obus and Wewers’ formula is $\mathbb{C}(D, \overline{\Phi}, \overline{x})$ in our case, as they are both $\varphi(\eta)$ in Kato’s vanishing cycle formula [Kat87, Theorem 6.7].

We then obtain the following results.

**Corollary 3.9.** Let $\Phi$ be an admissible cyclic cover of $C$ and $\overline{x}$ is as in §3.2. Then

\[
\mathbb{C}(C, \overline{\Phi}, \overline{x}) = \sum_{z \in B(\Phi)} h_z \geq sw_{\Phi}(\overline{x}) + 1.
\]

Also, $\Phi$ has good reduction if and only if $\delta(\overline{\Phi}) = 0$ and equality holds above.

**Proof.** The proof adapts the one for [OW14, Corollary 5.12]. The first part follows immediately from Proposition 3.8 since $B(\Phi) \subseteq D[0]$ (because we assume $\Phi$ is admissible). Furthermore,
by definition, \( \Phi \) has good reduction if and only if \( \delta(\Phi) = 0 \) and \( \overline{Y} = \overline{W} \) is smooth in any point \( \overline{y} \) above the distinguished point \( \overline{x}_0 \). The latter condition is equivalent to \( \delta_{\overline{y}} = 0 \). Thus, the rest also follows from Proposition 3.8.

**Remark 3.10.** The above corollary gives us an alternative way to check whether a function has good reduction besides the different criterion (Proposition 2.22). We will utilize this method for the rest of the paper.

**Corollary 3.11.** Let \( \Phi \) be an admissible cyclic cover of \( C \), let \( D \subseteq C \) be an affinoid, and let \( \overline{x} \) be a point on the canonical reduction \( \overline{D} \) of \( D \). Then

\[
\text{sw}_{\Phi|_{\overline{D}}} (\overline{x}) \leq C(D, \Phi, \overline{x}) - 1.
\]

Moreover, if \( \Phi \) has good reduction, then equality holds.

**Proof.** Parallel to the proof of [OW14, Corollary 5.13].

The following result follows immediately from Corollary 3.11 and Remark 3.5.

**Corollary 3.12.** In the same situation of Corollary 3.11 if \( \delta(\Phi|_{\overline{D}}) > 0 \), we have

\[
\text{ord}_{\overline{x}} (\omega(\Phi|_{\overline{D}})) \geq -C(D, \Phi, \overline{x})
\]

with equality if \( \Phi \) has good reduction.

**Remark 3.13.** Corollary 3.9 and Corollary 3.12 imply that, for \( \Phi \) to have good reduction, \( \omega(\Phi|_{\overline{D}}) \) cannot have any zeros besides one at infinity for any affinoid \( D \subseteq C \).

### 3.4.2 Refined Swan conductor of Artin-Schreier extensions

In this section, we study the depth and differential Swan conductors of an Artin-Schreier cover of a closed disc \( \text{Spec} R\{X\} \). Suppose \( \Phi \) is a \( \mathbb{Z}/p \)-cover of the disc defined by

\[
Y^p - Y = u,
\]

where \( u \in \mathbb{K}^\times \).

**Definition 3.14.** Such a \( u \) is said to be reduced if \( u = \tau \omega \), with \( \tau \in K \), \( \nu(\tau) < 0 \), \( \omega \in O_K^\times \), and \( \omega \notin \kappa^p \). We say that \( u \) is reducible if there exists an \( a \in \mathbb{K}^\times \) such that \( u + a^p - a \) is reduced.

**Remark 3.15.** In the deformation problem, as we can always replace \( R \) by its finite extensions, we may assume that \( u \) is reducible.

**Proposition 3.16** (c.f. [Lea18, §2]). Suppose \( \Phi \) is a \( \mathbb{Z}/p \)-cover of \( R\{X\} \) defined by

\[
Y^p - Y = u,
\]

where \( u \in \mathbb{K}^\times \) is reduced. If \( u = \pi \omega \), with \( \omega \in O_K^\times \), then the depth of \( \Phi \) is \(-\nu(\pi)/p\) and its differential Swan conductor is \( d\omega \).

**Remark 3.17.** The proposition implies, if \( \Phi \) is an Artin-Schreier cover where \( \delta(\Phi) > 0 \), then its differential conductor \( \omega(\Phi) \) is an exact differential form. That is no longer true, however, for a cyclic cover in general.
Example 3.18. Recall that example 2.24 the cover \( \Phi \) is defined by

\[
Y^5 - Y = \frac{-2X + t^{10}}{(-2)X^5(X - t^{10})^2(X - t^5)^5}. \tag{6}
\]

As the generic fiber is branched at three points \( X = 0, X = t^5, \) and \( X = t^{10} \), we follow §3.2 to construct the dual graph of the associated semistable model as follows.

![Dual Graph](attachment:image.png)

We set the orientation of the graph to be the direction from its leaves to its root \( v_0 \).

At the initial vertex of \( e_1 \), which corresponds to a punctured disc of radius \( 5/2 \) defined by \( \text{Spec } \mathbb{R}\{t^{-10}X, t^{10}X^{-1}, t^{10}(X - t^{10})^{-1}\} \), we set \( X_2 := X t^{-10} \). Replacing \( X \) by \( X_2 \) equates to looking at the restriction of the cover to the sub-disc \( \mathcal{D}[2] = \text{Spec } \mathbb{R}\{t^{-10}X\} \). We rewrite (6) as

\[
Y^5 - Y = \frac{-2X_2 + 1}{(-2)X_2^5(X_2 - 1)^2(X_2 t^5 - 1)^5t^5 17} \tag{7}
\]

Let \( x_2 \) the image of \( X_2 \) in the residue field of \( \mathcal{D}[2] \). The derivative of the reduction modulo \( p \) of \( -2x_2 + 1 \) is \( \omega(2) = \frac{dx_2}{x_2(x_2 - 1)^2} \). Hence, it follows from Proposition 3.16 that \( \omega(2) \), after replacing \( x_2 \) by \( x \), is the differential Swan conductor of \( \Phi \) restricting to \( \mathcal{D}[2] \), and its depth conductor is 17. Its boundary swan conductors are \( \text{sw}_{\Phi|\mathcal{D}[2]}(\overline{x}) = - \text{ord}_x(\omega(2)) - 1 = -6 \), \( \text{sw}_{\Phi|\mathcal{D}[2]}(x_2 - 1) = - \text{ord}_{x_2 - 1}(\omega(2)) - 1 = 2 \).

At the initial vertex of \( e_0 \), which resembles the punctured \( \text{Spec } \mathbb{R}\{t^{-5}X, t^5X^{-1}, t^5(X - t^5)^{-1}\} \). Define \( X_1 := X t^{-5} \). We then rewrite (6) as

\[
Y^5 - Y = \frac{-2X_1 + t^5}{(-2)X_1^5(X_1 - t^5)^2(X_1 - 1)^5t^5 11}. \tag{8}
\]

Apply the same argument as above, we see that the degeneration type of \( \Phi \) on \( \text{Spec } \mathbb{R}\{X_1\} \) is \((11, \frac{dx}{x(x - 1)^2})\).

Finally, when \( r = 0 \), then \( X_0 = X \), the depth is 0, and the reduction \( \phi \) is defined by \( y^5 - y = \frac{1}{t^1} \). Therefore, we have \( \mathbb{C}(\Phi, \overline{x}) = 5 + 3 + 4 = 12 = \text{sw}_\phi(\overline{x}) + 1 \). Hence, \( \Phi \) has good reduction by Corollary 3.9.

In the next example, we will go into detail of describing how the degeneration data vary along the dual graph.

Example 3.19. Consider the \( \mathbb{Z}/2 \)-cover \( \Phi \) of \( \text{Spec } \mathbb{R}\{X\} \) that is given by

\[
Y^2 - Y = \frac{1}{X(X - t^2)} := f(X, t). \tag{8}
\]

The associated dual graph of the semi-stable model is as follows.

![Dual Graph](attachment:image.png)
Recall from §3.2 that the vertex $v_1$ represents the subdisc $D[1] := \text{Spec } R\{t^{-2}X\}$. Substitute $t^{-2}X$ by $X_1$ in (8), we obtain 

$$f(X_1, t) = t^{-2} \cdot \frac{1}{X_1(X_1 - 1)}.$$ 

As $1/(x_1(x_1 - 1)) \notin (k(x_1)\times)^2$, we obtain $\delta(\Phi|_{D[1]}) = 2$ and $\omega(\Phi|_{D[1]}) = \frac{dx}{x^2(x-1)^2}$.

Set $X_r := X t^{-2r}$. If $r > 1$, we can write $f(X, t)$ as

$$f(X_r, t) = t^{-2(r+1)} \cdot \frac{1}{X_r(X_r t^{2r-2} - 1)}.$$

Again, as $\frac{-1}{x_r} \notin (k(x_r)\times)^2$, Proposition 3.16 shows $\delta(\Phi|_{D[r]}) = r + 1$ and $\omega(\Phi|_{D[r]}) = \frac{dx}{x^r}.$

Suppose $r < 1$. Then $[f]_r$ is always a square. Set $a := 1/X^2$. We get

$$f(X, t) + a^2 - a = \frac{X^2 + t^2 X + t^2}{X^2(X - t^2)} =: g(X, t).$$

Moreover, we can rewrite $g(X, t)$ as

$$g(X_r, t) = t^{-4r} \frac{X^2 + t^2 X_r + X_r t^{2r}}{X_r^2(X_r - t^{2r-2})}.$$

Note that $2 - 2r < 2r$ for $r \in (1/2, 1]$ and $2 - 2r > 2r$ otherwise. Hence, one can describe $\delta(\Phi|_{D[r]})$ in terms of $r$ as follows,

$$\delta(\Phi|_{D[r]}) = \begin{cases} 
  r & 0 \leq r \leq \frac{1}{2}, \\
  3r - 1 & \frac{1}{2} \leq r \leq 1, \\
  r + 1 & 1 \leq r.
\end{cases}$$

Below is the graph of $\delta(\Phi|_{D[r]})$ with respect to $r$. Moreover, we can write down the equation of $\omega(\Phi|_{D[r]})$ with respect to $r$ as follows.

$$\omega(\Phi|_{D[r]}) = \begin{cases} 
  \frac{dx}{x^r(1+x^2)} & 0 < r < \frac{1}{2}, \\
  \frac{dx}{x^{r+1}} & r = \frac{1}{2}, \\
  \frac{dx}{x^2} & \frac{1}{2} < r < 1, \\
  \frac{dx}{x^2} & r = 1, \\
  \frac{dx}{x^2} & 1 < r.
\end{cases}$$

Figure 2: Graph of $\delta(\Phi|_{D[r]})$
When \( r = 0 \), the reduction of \( \Phi \) is an Artin-Schreier cover given by \( y^2 - y = \frac{1}{x} \). As \( 1/x^2 \) is a square, we may replace the right-hand-side of the previous equation by \( 1/x^2 + (1/x)^2 - (1/x) = 1/x \). Thus, by Corollary 3.9 \( \Phi \) does not have good reduction as \( \sum_{z \in B(\Phi)} h_z = 2 + 2 > sw_\Phi(\infty) + 1 = 1 + 1 \).

**Remark 3.20.** For future reference, we list down some important degeneration data of the cover \( \Phi \) in Example 3.19 here. The degeneration type of \( \Phi \) (resp. \( \Phi|_{D[1]} \)) is \((0, \frac{1}{x})\) (resp. \((2, \frac{dx}{x(x-1)^2})\)).

**Remark 3.21.** When a cover of a disc has good reduction, then its depth varies from inside the disc to the boundary like a piecewise linear, weakly concave down function. See [OW14, Remark 5.15] for the mixed characteristic analog of the statement. It is not the case in Example 3.19 because \( \delta(\Phi|_{D[1]}) \), regarded as a function with respect to \( r \), is weakly concave up at \( r = 1/2 \) as showed in Figure 2.

**Remark 3.22.** If \( f(X,t) \) in (8) is replaced by \( \frac{1}{X(X-1)} \), then the deformation is flat of type \([4] \rightarrow [2,2]^\top\). This type of deformation will be discussed further in §6.1.1.

In the next two sections, we will study the degeneration of \( \mathbb{Z}/p \)-covers with good reduction on sub-discs and sub-annuli of the formal disc.

### 3.4.3 Degeneration data on the boundary

The following is a result by Saïdi, which characterizes \( \mathbb{Z}/p \)-covers of a boundary of a disc.

**Proposition 3.23** (c.f. [Saï07, Proposition 2.3.1]). Let \( A := R[[X^{-1}]]\{X\} \), and let \( f : \text{Spec } B \rightarrow \text{Spec } A \) be a nontrivial Galois cover of degree \( p \). Assume that the ramification index of the corresponding extension of discrete valuation rings equals 1. Then \( f \) is a torsor under a finite flat \( R \)-group scheme \( G_R \) of rank \( p \) and the following cases occur:

1. Suppose the depth is \( \delta = 0 \). In this case \( f \) is a torsor under the étale group \( (\mathbb{Z}/p\mathbb{Z})_R \).

   Moreover, for a suitable choice of the parameter \( X \) of \( A \), the torsor \( f \) is given by an equation \( Y^p - Y = X^m \) for \( m \in \mathbb{Z}, \) prime to \( p \). That means the depth is 0 and the boundary conductor is \( m \).

2. Suppose the depth is \( \delta > 0 \). In this case \( f \) is a torsor under the group scheme \( M_{\delta,p,R} \).

   Moreover, for a suitable choice of the parameter \( X \), the torsor \( f \) is given by an equation \( Y^p - Y = X^m/\pi^\delta \) for \( m \in \mathbb{Z}, \) prime to \( p \). That means the depth is \( \delta \) and the boundary conductor is \( m \).

The proposition signifies that an order \( p \) cover of \( R[[X^{-1}]]\{X\} \) is determined by its depth \( \delta \) and its boundary conductor \( m \). We say it has degeneration type \((\delta, m)\).

**Remark 3.24.** Saïdi’s result is motivated by [Hen00, Corollaire 1.8] for the case \( R \) is of mixed characteristic. However, it is not true that a \( \mathbb{Z}/p^n \)-cover (where \( n > 1 \)) of a boundary is determined by its depth and boundary conductor in both mixed and equal characteristic cases. See [HB09, §5.3] for a counterexample to the mixed characteristic case. We will discuss this phenomenon further in a future paper.

### 3.4.4 Good degeneration data on an annulus

The following proposition describes the degeneration of covers of an annulus with good reduction.

**Proposition 3.25** (c.f. [Sai07, Proposition 3.3.9]). Let $A := R[[X,U]]/(XU - \pi^p)$, and let $f : \text{Spec } B \rightarrow \text{Spec } A$ be a nontrivial Galois cover of degree $p$, with $\text{Spec } B \otimes_R k$ reduced and local, and with $f_K : \text{Spec } B_K \rightarrow \text{Spec } A_K$ étale. Let $S_1 := \text{Spec } R[[X^{-1}]]\{X\}$ and $S_2 := \text{Spec } R[[U^{-1}]]\{U\}$ be the boundaries of $\text{Spec } A$. Suppose the reduction of $B$ is smooth. Then, after replacing $X, U, \pi$ by $\tilde{X}, \tilde{U}, \tilde{\pi}$ so that $A \cong R[[\tilde{X}, \tilde{U}]]/(\tilde{X}\tilde{U} - \tilde{\pi}^p)$, one of the following occurs.

1. The cover $f$ is generically given by $Y^p - Y = 1/\tilde{X}^m = \tilde{U}^m/\tilde{\pi}^{mp}$, where $m$ is an integer prime to $p$. This cover leads to a reduction on $S_1$ of type $(0, -m)$, and on $S_2$ of type $(m, m)$.

2. The cover $f$ is generically given by $Y^p - Y = 1/(\tilde{X}^m/\tilde{\pi}^n) = \tilde{U}^m/\tilde{\pi}^{p(n+m)}$, where $m$ is an integer prime to $p$. This cover leads to a reduction on $S_1$ of type $(n, -m)$, and on $S_2$ of type $(n + m, m)$.

**Remark 3.26.** Our future paper will extend the **Proposition 3.25** to all cyclic covers and prove the generalized version using Corollary 3.11.

## 4 Hurwitz trees of Artin-Schreier covers

Suppose $R$ is a complete discrete valuation ring whose residue field is of characteristic $p > 0$. When $R$ is of mixed characteristic (e.g., $R = W(k)$), Henrio, Bouw, Wewers, and Brewis define a combinatorial object called **Hurwitz tree** from a $G$-cover of a formal disc over $R$ (see [Hen00], [BW06], [BW09]). This construction gives an obstruction for the lifting of Galois covers in characteristic $p$. In this section, we introduce the notion of Hurwitz tree in equal characteristic, which basically formalizes what Maugeais and Saïdi did in [Mau03] and [Sai07], using the language from [Hen00] and [BW06]. Then, we will describe how a $\mathbb{Z}/p$-deformation gives rise to such a tree. Finally, we will provide an obstruction for the equal characteristic deformation of Artin-Schreier covers and compute the Hurwitz trees of the previous examples.

Throughout the rest of the paper, we assume $R = k[[t]]$, and $K := k((t))$ is the fraction field of $R$.

### 4.1 Hurwitz tree

This section follows closely [BW06, §3.1].

**Definition 4.1.** A **decorated tree** is given by the following data

- a semistable curve $C$ over $k$ of genus 0,
- a family $(x_b)_{b \in B}$ of pairwise distinct smooth $k$-rational points of $C$, indexed by a finite nonempty set $B$,
- a distinguished smooth $k$-rational point $x_0 \in C$, distinct from any of the point $x_b$.

We require that $C$ is stably marked by the points $((x_b)_{b \in B}, x_0)$.

The **combinatorial tree** underlying a decorated tree $C$ is the graph $T = (V, E)$, defined as follows. The vertex set $V$ of $T$ is the set of irreducible components of $C$, together with a distinguished element $e_0$. We write $C_v$ for the component corresponding to a vertex $v \neq v_0$ and $x_e$ for the singular point corresponding to an edge $e \neq e_0$. An edge $e$ corresponding to a singular point $x_e$ is adjacent to the vertices corresponding to the two components which intersect at $x_e$. The edge $e_0$ is adjacent to the root $v_0$ and the vertex $v$ corresponding to the (unique) component $C_v$ containing the distinguished point $x_0$. For each edge $e \in E$, the source
To each $v$ of $e$ the unique vertex $s(e) \in V$ (resp. $t(e) \in V$) adjacent to $e$ which lies in the direction of the root (resp. the direction away from the root).

Note that, since $(C, (x_b), x_0)$ is stably marked of genus 0, the components $C_v$ have genus zero, too, and the graph $T$ is a tree. Moreover, we have $|B| \geq 1$. For a vertex $v \in V$, we write $U_v \subseteq C_v$ for the complement in $C_v$ of the set of singular and marked points.

**Definition 4.2.** Let $\overline{E} = \{h_1, \ldots, h_r\}$ an element of $\Omega_{\sum_{i=1}^{r} h_i}$ (defined in §2.3). A $\mathbb{Z}/p$-Hurwitz tree (or just Hurwitz tree in the context this paper) of type $\overline{E}$ is given by the following data:

- A decorated tree $C = (C, (x_b), x_0)$ with underlying combinatorial tree $T = (V, E)$.
- For every $v \in V$, a rational number $\delta_v \geq 0$, called the depth of $v$.
- For each $v \in V \setminus \{v_0\}$, an exact differential form $\omega_v$ called the differential conductor at $v$.
- For every $e \in E$, a positive rational number $\epsilon_e$, called the thickness of $e$.
- For every $b \in B$, a positive number $h_b$, called the conductor at $b$, such that $h_b \not\equiv 1 \pmod{p}$.
- For $v_0$, a fraction $\frac{d}{x}$, where $d \not\equiv 0 \pmod{p}$, called the degeneration at $v_0$.

These objects are required to satisfy the following conditions.

(H1) Let $v \in V$. We have $\delta_v \neq 0$ if $v \neq v_0$.

(H2) For each $v \in V \setminus \{v_0\}$, the differential form $\omega_v$ does not have zeros nor poles on $U_v \subseteq C_v$.

(H3) For every edge $e \in E - \{e_0\}$, we have the equality

$$- \text{ord}_{x_e} \omega_{t(e)} - 1 = \text{ord}_{x_e} \omega_{s(e)} + 1 \not\equiv 0 \pmod{p}.$$ 

(H4) For $v_0$, we have $d = \text{ord}_{x_{v_0}} \omega_{t(e_0)} + 1$.

(H5) For every edge $e \in E$, we have

$$\delta_{s(e)} + \epsilon_e d_e = \delta_{t(e)},$$

where

$$d_e := - \text{ord}_{x_e} \omega_{t(e)} - 1 \not\equiv 0 \pmod{p} \text{ ord}_{x_e} \omega_{s(e)} + 1.$$

(H6) For $b \in B$, let $C_v$ be the component containing the point $x_b$. Then the differential $\omega_v$ has a pole in $x_b$ of order $h_b$.

For each $v \in V \setminus \{v_0\}$, we call $(\delta_v, \omega_v)$ the degeneration type of $v$. For each $e \in E$, we call $(\delta_{s(e)}, \delta_{t(e)})$ (resp. $(\delta_{t(e)}, d_e)$) the initial degeneration type (resp. the final degeneration type) of $e$. The integer $h := d + 1$ is called the conductor of the Hurwitz tree. The rational $\delta := \delta_{v_0}$ is the depth. We define the height of the tree to be the maximal number of edges on a direction from its root to its leaves.

One can easily obtain the following result. See [Hen00, §2] and [GM98] for the proofs of its mixed characteristic analogs.

**Lemma 4.3.** Let $(C, \omega_v, \delta_v, \epsilon_e, h_b, d)$ be a Hurwitz tree. Fix an edge $e \in E$ and let $C_e \subseteq C$ be the union of all components $C_v$ corresponding to vertices $v$ which are separated from the root $v_0$ by the edge $e$. Then:

$$d_e = \sum_{b \in B} h_b - 1 > 0.$$ 

In particular, $h = \sum_{b \in B} h_b$. 

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4.2 The Hurwitz tree associated to a $\mathbb{Z}/p$-cover of a disc.

Let $D_K := \{X \mid |X|_K < 1\}$ be the rigid open unit disc over $K$. Suppose we are given a $\mathbb{Z}/p$-cover $\Phi$ of $D_K$, which algebraically given by

$$\Phi : R[[Z]] \to R[[X]]$$

with good reduction. Hence $\delta(\Phi) = 0$. Obviously, $\Phi$ induces a $\mathbb{Z}/p$-cover of the boundarySpec $R[[X^{-1}]][X]$ of the disc $D_K$. Let $d$ be the boundary conductor of this cover. Following [BW09], we will now associate to $\Phi$ a Hurwitz tree.

Let $x_{b,K} \in D_K$ be a branch points of $\Phi$, indexed by the finite set $B$. We assume that the points $x_{b,K}$ are all $K$-rational and the conductor at $x_{b,K}$ is $h_b$. By the different criterion (Proposition 2.22) and Proposition 3.23, we obtain $\sum_{b \in B} h_b = d + 1 =: h$. We assume that $\Phi$ has at least two branch points ($|B| \geq 2$), i.e. the deformation is non-trivial. Applying 3.2 one obtains $D_R$, the semistable model of the marked disc $(D_K, (x_{b,K})_{b \in B})$ with special fiber $(\overline{D}, (x_b), x_0 := \infty)$. Note that it is a decorated tree, in the sense of Definition 4.1.

Let $(V, E)$ be the combinatorial tree underlying $T := (\overline{D}, (x_b), x_0)$. For $v \in V$, let $U_v \subsetneq D_v$ be the complement of the singular and marked points and let $U_{v,K} \subsetneq D_K$ be the affinoid subdomain with reduction $U_v$. Let $V_v$ (resp. $V_{v,K}$) denote the inverse image of $U_v$ (resp. $U_{v,K}$) under $\Phi$. By construction, $V_{v,K} \to U_{v,K}$ is a torsor under the constant $K$-group scheme $G$. We set $\delta_v \in \mathbb{Q}_{\geq 0}$ to be the depth conductor of the restriction of the cover to $V_{v,K}$. When $\Phi|_{V_{v,K}}$ is radical, we set $\omega_v = \Omega^1_{\Phi_v(U_v)}$ to be the differential conductor of the same restriction. When $\Phi|_{V_{v,K}}$ is étale, which only happens when $v = v_0$, its reduction can be represented by a fraction $1/d$, where $d$ is prime to $p$, by Proposition 2.15. We set the degeneration at $v_0$ to be that fraction. Finally, for $e \in E$ we let $A_e \subsetneq D_K$ denote the subset of all points which specialize to the singular point $x_e \in \overline{D}$ corresponding to $e$. This is an open annulus. We define $\epsilon_e$ as the thickness of $A_e$ divided by $p$, i.e., the positive rational number such that

$$A_e \cong \{x \mid |p|^{\epsilon_e}_K < |x|_K < 1\}.$$

Proposition 4.4. The datum $(T, \omega_v, \delta_v, \epsilon_e, h_b, h - 1)$ defines a Hurwitz tree of type $\{h_1, \ldots, h_r\}$. Moreover, $h$ is the conductor, 0 is the depth.

Proof. The proof is parallel to one for the mixed characteristic case [Hen00, §3.1]. The only differences are: all differential conductors are exact, each branch point contributes more to the degree of the different, and the depth can be arbitrarily large.

([H3]) It follows from Proposition 3.25, as $-\operatorname{ord}_{x_e} \omega_{t(e)} - 1$ is the inside boundary conductor of the associated annulus and $\operatorname{ord}_{x_e} \omega_{t(e)} + 1$ is its outside boundary one.

([H5]) Suppose $e$ is an edge with initial depth $\delta_{i(e)}$, final depth $\delta_{f(e)}$, and thickness $\epsilon_e$. Then, as the restriction of the cover to the corresponding annulus has good reduction, Proposition 3.25 shows that $\delta_{t(e)} = \delta_{i(e)} + \epsilon_e d_e$.

([H2]) ([H6]) follow from Remark 3.13.

([H1]) It follows immediately from the definition of the depth that $\delta_{v_0} \geq 0$. Moreover, by ([H5]) $\delta_v$ is a strictly increasing function, as $v$ goes away from the root.

Finally, it is immediate from the construction and the good reduction assumption that the depth is 0 and the conductor is $h$, completing the proof.

Example 4.5. Let $\Phi$ be the $\mathbb{Z}/5$-cover in Example 2.24. Then it follows from Example 3.18 that the Hurwitz tree associated with $\Phi$ has the following form.
At each vertex $v$ of the tree, the first component of the pair is the depth conductor at the boundary of the disc associated to $v$. When the depth is positive, the second component is the differential Swan conductor. When the depth is 0, the second component represents the degeneration of the restriction. The rational number below each edge $e$ is the thickness of the corresponding annulus divided by $p$. The integer on the right at each leaf denotes the conductor of the associated branch point, and inside $[\cdot]$ is the branch point. We often disregard this information as you can see in Example 4.6. One can easily read off from the leaves that $\Phi$ has type $[12] \to [5, 4, 3]^\top$.

By Proposition 4.4, the existence of a Hurwitz tree of type $\{h_1, \ldots, h_r\}$ is necessary for the existence of a deformation of corresponding type. It gives us an obstruction for the deformation of $\mathbb{Z}/p$-covers, which we call the Hurwitz tree obstruction.

**Example 4.6.** If there exists a deformation of type $[5] \to [3, 2]^\top$, then the associated Hurwitz tree must have the form below.

\[
\left(0, \frac{1}{x^4}\right) \quad \frac{e_0}{\epsilon} \quad \left(4\epsilon, \frac{dx}{x^4(x-a)^2}\right) \quad 2
\]

\[
\left(2, \frac{dx}{x^2(x-1)^2}\right) \quad 3
\]

Thus, there exists an exact differential form $\omega = \frac{dx}{x^4(x-a)^2}$, where $a \neq 0$. A straightforward calculation shows that the residue of $\omega$ at 0 is $3/a^4$, which is a contradiction. Therefore, the Hurwitz tree obstruction does not vanish. Hence, there is no deformation of type $[5] \to [3, 2]^\top$.

**Proposition 4.7.** Suppose a $\mathbb{Z}/p$-cover $\Phi$ of a formal disc gives rise to a Hurwitz tree of type $\{h_1, \ldots, h_r\}$ and with depth zero from the construction at the beginning of §4.2. Then $\Phi$ is a deformation of type $[h_1, \ldots, h_r]^\top$.

**Proof.** It follows from the depth zero assumption that $\Phi$ has étale reduction. Moreover, as the reduction has conductor $\sum_{i=1}^r h_i$ by Lemma 4.3, it is smooth by the different criterion (Proposition 2.22) or Corollary 3.9.

**Example 4.8.** The tree below arises from the deformation in Example 3.19 using the data from Remark 3.20.

\[
\left(0, \frac{1}{x}\right) \quad \frac{e_0}{1} \quad \left(2, \frac{dx}{x^4(x-1)^2}\right) \quad 2
\]

\[
\left(2, \frac{dx}{x^2(x-1)^2}\right) \quad 3[\epsilon^2]
\]

Note that $\left(0, \frac{1}{x}\right)$ (resp. $\left(2, \frac{dx}{x^4(x-1)^2}\right)$) is the degeneration type of $\Phi$ (resp. $\Phi|_{\mathcal{D}[1]}$). It violates (H4) and (H5) as $2 = d \neq \text{ord}_{x=e_0} \omega_t(e_0) + 1 = 4$ and $2 = \delta_t(e_0) \neq \delta_s(e_0) + \epsilon_0d_e = 3$. Hence, it is not a flat deformation by Proposition 4.7.
5 Construction of $\mathbb{Z}/p$-covers from a Hurwitz tree

In the previous section, we have associated a Hurwitz tree to a $\mathbb{Z}/p$-cover $\Phi : \text{Spec } R[[Z]] \to \text{Spec } R[[X]]$ a Hurwitz tree. The main result of this section is that this construction can be reversed.

**Theorem 5.1.** Every Hurwitz tree $\mathcal{T} = (T, \omega, \delta, \epsilon, h, h - 1)$ is associated to a $\mathbb{Z}/p$-cover $\Phi$ of $\text{Spec } R[[X]]$, for some finite extension $R'$ of $R$.

The proof goes as follows. Using the geometry of the tree $\mathcal{T}$, we partition the open disc $\text{Spec } R[[X]]$ into sub-discs, sub-punctured-disc, and annuli as in §3.2. In sections 5.1, 5.2, 5.3 we construct explicitly a $\mathbb{Z}/p$-cover for each of these pieces that matches its associated datum on the tree. We then glue them together along their boundaries using the technique from §5.4. Finally, the details of the proof will be given in §5.5.

### 5.1 Realization of a vertex

Consider a vertex $v \neq v_0$ on a tree with datum $(\delta_v, \epsilon_v, \omega_v = df)$ where $f \in \text{Frac } k[x]$. Suppose $\omega_v$ has $r$ poles $p_1, p_2, \ldots, p_r \in \mathbb{A}_k$, and set $d_i := -\text{ord}_{p_i}(\omega) - 1$, $d := -\text{ord}_\infty(\omega) - 1$. The points $p_i$ (resp. the point $\infty$) correspond to the singular points $x_{e_i}$ such that $s(e_i) = v$ (resp. to the unique singular point $x_\infty$ with $t(e) = v$). As in the previous section, $v$ corresponds to a complement of $r$ open discs inside one closed disc

$$W_K \cong \{ X \mid |X| \leq 1, |X - P_i| \geq 1 \},$$

where $P_{i,K}$ is a lift of $p_i$ to $K$. $W_K$ is the $K$-analytic space of $W := \text{Spec } R\{X,(X - P_{i,K})^{-1}\}_{i=1,\ldots,r}$.

Define a $\mathbb{Z}/p$-cover $V \xrightarrow{\Phi} W$ by

$$Z^p - Z = F \cdot \frac{1}{t^\delta},$$

(9)

where $F$ is a lift of $f$ to $K$. Let $\text{Spec } S_i := R[[X - P_{i,K}]]\{(X - P_{i,K})^{-1}\}$ (resp. $\text{Spec } S_\infty := R[[X^{-1}]]\{X\}$) be the boundary of the missing open disc containing the point $P_{i,K}$ (resp. $\infty$). The below proposition follows easily from Proposition 3.16 and Remark 3.5.

**Proposition 5.2.** The cover $\Phi$ restricts to a $\mathbb{Z}/p$-cover on $\text{Spec } S_i$ (resp. on $\text{Spec } S_\infty$). This extension has boundary conductor $d_i$ (resp. boundary conductor $d$) and depth $\delta$.

### 5.2 Realization of an edge

Suppose $e$ is an edge of thickness $\epsilon$ with final degeneration type $(\delta_2, d_2)$ and initial degeneration type $(\delta_1, -d_1)$, where $d_1 = d_2$ and $\delta_1 + d_1\epsilon = \delta_2$. The edge corresponds to an annulus $A$ of thickness $p\epsilon$, which can be identified with $\text{Spec } R[[X,U]]/(XU - t^p\epsilon)$. Consider a $\mathbb{Z}/p$-cover of the annulus defined by

$$Y^p - Y = X^{d_2} \cdot \frac{1}{t^{\delta_2}},$$

(10)

Call $S_2 := \text{Spec } R[[X]]\{X^{-1}\}$ the inner boundary of the annulus, and $S_1 := \text{Spec } R[[U]]\{U^{-1}\}$ its outer boundary. Then it is immediate that the $\mathbb{Z}/p$-extension of $S_1$ has boundary conductor $d_2$ and depth $\delta_2$. Replace $X$ by $t^p\epsilon$ in (10), we get

$$Y^p - Y = \frac{1}{t^{p(\delta_2 - d_2\epsilon)}U^{d_2}}.$$
The below proposition easily follows from Proposition 3.16.

**Proposition 5.3.**  
1. The induced $G$-cover of $S_2$ has conductor $d_2$ and depth $\delta_2$.  
2. The induced $G$-cover of $S_1$ has conductor $-d_1$ and depth $\delta_2 - d_2\epsilon$.

### 5.3 Realization of a leaf

Consider a leaf $b_i \in B$ on a tree with depth $\delta > 0$ and conductor $d_i \not\equiv 0 \pmod{p}$. It corresponds to a singular point $x_i$ and can be associated with a closed disc $C_{i,K} = \{X | |X - x_i| \leq 1\}$ which are $K$-points of $\text{Spec } R\{X - x_i\}$. Let $S := \text{Spec } R\{((X - x_i)^{-1})\{X - x_i\}$ be the boundary of $C_{i,K}$. Consider the $\mathbb{Z}/p$-cover $\Phi_i$ of $C_{i,K}$ given by the following equation

$$Y^p - Y = \frac{1}{t^{\delta_i}} Y^{d_i}.$$ 

Then, by applying Proposition 3.16 we obtain the following result.

**Proposition 5.4.** The cover $\Phi_i$ of $C_{i,K}$ has depth $\delta$ and boundary conductor $d_i$.

### 5.4 Gluing the boundaries together

In this section, we glue covers of smaller pieces together to form one for a larger piece.

#### 5.4.1 Filling in a punctured disc

Suppose $\delta \in \mathbb{Q}_{>0}$, and $d_1, d_2, \ldots, d_r$ are not-divisible-by-$p$ integers. Suppose, moreover, that $\Phi_1, \Phi_2, \ldots, \Phi_r$ are $\mathbb{Z}/p$-covers of $\text{Spec } R\{Y_1\}, \ldots, \text{Spec } R\{Y_r\}$, respectively, where $\Phi_i$ has depth $\delta$ and conductor $d_i$. It follows from Proposition 3.23 that we may assume, after a change of variable, that $\Phi_i$ induces on $\text{Spec } R\{(X - a_i)^{-1}\} \{X - a_i\}$ a $\mathbb{Z}/p$-cover

$$Y^p - Y = \frac{1}{t^{\delta_i}} Y^{d_i}.$$ 

Suppose $\Phi$ is a $\mathbb{Z}/p$-cover of a closed punctured disc as in §5.1. We may assume that the restriction of $\Phi$ to $S_i = \text{Spec } R\{X - a_i\}\{(X - a_i)^{-1}\}$, after a change of variable, is given by

$$Y^p - Y = \frac{1}{t^{\delta_i}} Y^{d_i}.$$ 

(possible by Proposition 3.23). We define the $R$-module isomorphism $\psi_i : R\{((X - a_i)^{-1})\{X - a_i\} \to R\{(Y_i^{-1})\{Y_i\}$ by mapping $(X - a_i)$ to $Y_i$. It is clear from the construction that $\Phi_i$ and $\Phi$ coincides on the glued boundary.

We would like to construct a $\mathbb{Z}/p$-cover of $\text{Spec } R\{X\}$ whose restriction to the closed punctured disc $W$ coincides with $\Phi$, and whose restriction to each open disc $\text{Spec } R\{(X - a_i)^{-1}\}$ is identical to $\Phi_i$ (when $\text{Spec } R\{Y_i\}$ is identified with $\text{Spec } R\{(X - a_i)^{-1}\})$. To do that, we fill in $W$ by identifying the boundary of $\text{Spec } R\{Y_i\}$ with $S_i$ like above.

The below lemma shows explicitly how to paste together the disks on the bottom and using the compatibilities on the top.
Lemma 5.5 ([Hen00, Lemma 3.7]). Suppose the elements \( a_1, \ldots, a_r \) of \( R \) are pairwise distinct modulo \( t \). We denote, for each \( 1 \leq i \leq r \), \( \alpha_i \) (resp. \( \beta_i \)) the canonical injection of the \( R \)-algebras \( R \{ X, (X - a_j)^{-1} \}_{1 \leq j \leq r} \) (resp. \( R[\{ Y_i \}] \) in \( R[\{ X - a_j \} \}_{1 \leq j \leq r} \) (resp. \( R[\{ Y_i \} \}_{1 \leq i \leq l} \)). Let \( \psi_i : R[\{ Y_i \}] \rightarrow R[\{ X - a_i \}] \) be an isomorphism of \( R \)-algebras. If \( \theta \) is the \( R \)-module homomorphism

\[
R \{ X, (X - a_j)^{-1} \}_{1 \leq j \leq r} \times \prod_{1 \leq i \leq r} R[\{ Y_i \}] \xrightarrow{= \prod_{1 \leq i \leq r} R[\{ X - a_i \}] \{(X - a_i)^{-1}\}}
\]

\[
\theta(f_0, f_1, \ldots, f_r) = (\alpha_1(f_0) - \psi_1 \circ \beta_1(f_1), \ldots, \alpha_r(f_0) - \psi_r \circ \beta_r(f_r)),
\]

for \( f_0 \in R \{ X, (X - a_j)^{-1} \}_{1 \leq j \leq r} \) and \( f_i \in R[\{ Y_i \}] \) for \( 1 \leq i \leq r \), then \( \theta \) is surjective and its kernel \( N \) is an \( R \)-algebra \( R[\{ Y_0 \}] \) as desired.

We thus can also glue together the covers \( \Phi \) and \( \Phi_i \)'s along the lifts of the boundaries \( S_i \)'s on the top in the obvious way so that the \( G \)-action is well-defined on the whole lift. The result is a \( \mathbb{Z}/p \)-cover of \( R[\{ Y_0 \}] \) that restricts to \( \Phi \) on \( \text{Spec} \, R \{ X, (X - a_j)^{-1} \}_{i=1,2,r} \) and \( \Phi_i \) on \( \text{Spec} \, R[\{ Y_i \}] \) as desired.

5.4.2 Glueing a closed disc with an annulus

Suppose we are given \( \mathbb{Z}/p \)-cover \( \Phi \) of \( \text{Spec} \, R[\{ X, U \}]/(XU - t^p e) \) as in §5.2 and a \( \mathbb{Z}/p \)-cover \( \Phi' \) of \( \text{Spec} \, R[\{ Y \}] \) whose restriction to \( \text{Spec} \, R[\{ Y^{-1} \} \{ Y \} \) has depth \( \delta_2 \) and conductor \( m_2 \). As the previous subsection, we can use the following lemma to glue the boundary of the closed disc to the inside boundary of the annulus at the bottom to form an open disc.

Lemma 5.6 ([Hen00, Lemma 3.8]). Suppose \( e \) is a strictly positive integer, \( \beta \) the canonical injection of \( R[\{ X, U \}]/XU - t^p e \) in \( R[\{ U \} \{ U^{-1} \} \), \( \alpha \) is the canonical injection of \( R[\{ Y \}] \) to \( R[\{ Y^{-1} \} \{ Y \} \) and \( \psi \) is an isomorphism of \( R \)-algebras \( R[\{ X \}] \{ X^{-1} \} \) and \( R[\{ Y^{-1} \} \{ Y \} \) (by mapping \( X \) to \( Y^{-1} \)). If \( \theta \) is a homomorphism of \( R \)-modules

\[
R[\{ Y \}] \times \frac{R[\{ X, U \}]}{XU - t^p e} \rightarrow R[\{ Y^{-1} \} \{ Y \} \]
\]

defined by \( \theta(f, g) := \alpha(f) - \psi \circ \beta(g) \), then \( \theta \) is surjective, and its kernel \( N \) is an algebra \( R[\{ Y_0 \}] \).

As before, by patching together the cover \( \Phi \) and \( \Phi' \) using the above gluing on the bottom, we also attain a \( \mathbb{Z}/p \)-cover of an open unit disc corresponding to \( \text{Spec} \, R[\{ U \}] \) that restricts to \( \Phi \) on \( \text{Spec} \, R[\{ X, U \}]/(XU - t^p e) \) and to \( \Phi' \) on \( \text{Spec} \, R[\{ Y \}] \).

5.5 Proof of Theorem 5.1

Let \( T = (C, \omega_i, \delta_i, e_i, h_i, h - 1) \) be a Hurwitz tree with conductor \( h - 1 \) and depth \( \delta \). We call a \( G = \mathbb{Z}/p \)-cover of the open disc a realization of \( T \) if \( T \) is associated to this cover by the construction in §4.2. Theorem 5.1 claims that we can realize \( T \). We will prove this claim by induction on the height of the tree \( T \). Suppose first that the height of \( T \) is one. Hence, the associated Hurwitz tree has the form like in Figure 8.

Let \( e_0 \) be the trunk of \( T \), we set \( v_1 = t(e_0) \). Then \( C_0 := C_{v_1} \) is the unique component of \( C \) which contains the distinguished point \( x_0 \in C \). Set \( \delta_1 := \delta_{v_1} \), and suppose \( b_1, \ldots, b_r \) are the leaves with source \( v_1 \), and the \( x_i := x_{e_i} \) are the corresponding singular points associated to the leaves. Let \( h_i := h_{b_i} \). By §5.3, there exists an open unit disc \( C_{i,K} \) over some finite extension...
Figure 3: A tree of height one

\[ (0, \frac{1}{x^2-1}) \xrightarrow{e_0/\epsilon} (\epsilon(l-1), \frac{d\epsilon}{\prod_{i=1}^{n} (x-P_i)^2}) \]

\[ l_1 \quad l_2 \quad \ldots \quad l_r \]

Theorem 5.1 and Proposition 4.7, we acquire the following result.
Corollary 5.8. Let $\phi : Z_k \to X_k$ be a local $G$-cover with conductor $h - 1$. Then there exists a deformation of $\phi$ over $k[[t]]$ of type $[h] \to [h_1, \ldots, h_r]^\top$ if and only if there exists a Hurwitz tree of type $\{h_1, \ldots, h_r\}$ and with depth zero.

Remark 5.9. The only difference between Corollary 5.8 and Theorem 2.20 is that we still need the Hurwitz tree in the statement to have depth zero. That requirement will be abolished by Proposition 5.12.

Definition 5.10. Suppose $\overrightarrow{A} \prec \overrightarrow{B}$. Then one can easily see that there exist a minimal submulti-set $\overrightarrow{A}'$ of $\overrightarrow{A}$ and $\overrightarrow{B}'$ of $\overrightarrow{B}$, such that $\overrightarrow{A}' \prec \overrightarrow{B}'$ and $\overrightarrow{A} \setminus \overrightarrow{A}'$ is the same as $\overrightarrow{B} \setminus \overrightarrow{B}'$. We call $\overrightarrow{A}' \prec \overrightarrow{B}'$ the difference of $\overrightarrow{A} \prec \overrightarrow{B}$.

Example 5.11. The difference of $\{3, 4, 5, 6\} \prec \{3, 2, 2, 3, 2, 6\}$ is $\{4, 5\} \prec \{2, 2, 3, 2\}$.

The next result shows that the existence of a Hurwitz tree equates to the existence of a chain of exact differential forms of certain type. This phenomenon does not generalize to any cyclic group $G$, though.

Proposition 5.12. There exists a Hurwitz tree of type $\{h_1, \ldots, h_r\}$ if and only if there exists a chain of partitions of integers $\omega := \{e_1 < e_2 < \ldots < e_m := \{h_1, \ldots, h_r\}$, where the difference between $e_{i-1}$ and $e_i$ is $\{i\} \prec \{i_1, \ldots, i_r\}$, together with $m$ exact differential forms

$$
\frac{dx}{\prod_{j=1}^i (x - P_i,j)^{l_i,j}}.
$$

for $1 \leq i \leq m$.

Proof. The “$\Rightarrow$” direction is easily deduced from Corollary 5.8 where each differential form corresponds to the differential conductor at a vertex $v \neq v_0$ of the associated tree. For instance, one can obtain from the tree in Example 4.5 a chain $\{12\} \prec \{5, 7\} \prec \{5, 4, 3\}$ and two exact differential forms $\frac{dx}{x^r(x-1)^s}, \frac{dx}{x^s(x-1)^r}$.

“$\Leftarrow$” We prove by induction on $m$. Suppose first that $m = 1$. Then, given an exact differential form of type (12), the tree in Figure 3 works. Suppose the statement is true for $m = n$. By induction, there exists a Hurwitz tree $T'$ with $n$ non-root-vertices whose differential conductors are exact of type (12). Suppose that the difference between $\omega_n$ and $\omega_{n+1}$ is $\{l\} \prec \{l_1, \ldots, l_r\}$. Suppose, moreover, that there exists an exact differential form $\omega = dx/(\prod_{i=1}^r (x - P_i)^{l_i})$, and the depth at the leaf corresponding to $l$ is $\delta$. Then one can “extend” $T'$ by “thickening” the leaf corresponding to $l$ to an edge of thickness 1 and adding $r$ leaves of conductors $l_1, \ldots, l_r$ to the end of this edge. Finally, we assign the exact differential form $\omega$ and the depth $\delta + l - 1$ to the end of the edge. One can easily check that the extended tree is Hurwitz.

Example 5.13. The tree below extends one from Figure 3

\[\text{Diagram of a tree}\]

where $l_2 = \sum_{j=1}^r l_{2,j}$, and the degeneration types at $v_0$, $v_1$, and $v_2$ are $\left(0, \frac{1}{2}\right)$, $\left(e(l-1), \frac{dx}{\prod_{j=1}^r (x - P_{2,j})^{l_{2,j}}}\right)$, and $\left(e(l-1) + l_2 - 1, \frac{dx}{\prod_{j=1}^r (x - P_{2,j})^{l_{2,j}}}\right)$, respectively.
Remark 5.14. Proposition 5.12 implies that one can drop the “depth zero” condition in Corollary 5.8. Theorem 2.20 then easily follows.

We call an Artin-Schreier cover equidistant if the distance between any two branch points is the same. A deformation of a one point cover is equidistant if its generic fiber is equidistant, hence its Hurwitz tree has height one. Most of the known deformations have equidistant deformations as building blocks. The follows results then follows immediately from Proposition 5.12.

Corollary 5.15. Suppose \( \{h_1, h_2, \ldots, h_r\} \) is a partition of \( \{h\} \). Then there is an equidistant deformation over \( k[[t]] \) of type \( [h] \to [h_1, \ldots, h_r]^\top \) if and only if there exists an exact differential of the form

\[
\frac{dx}{\prod_{i=1}^r (x - q_i)^{h_i}},
\]

for some distinct elements \( q_i \)'s in \( k \).

Definition 5.16. We say the exact differential form in (13) has type \( \{h_1, \ldots, h_r\} \).

In the remainder of the paper, we will focus on differential forms of type (13).

Remark 5.17. Suppose we are given a rational function of the form \( f(x) = g(x)^p h(x) \). Then \( f'(x) = g(x)^p h'(x) \). Hence, it suffices to study differential forms of type (13), where \( 1 < h_i < p \).

Remark 5.18. In [BM00], Bertin and Mézard show that the infinitesimal deformation functor of a local Galois cover is represented by a versal deformation ring. Besides, they give some descriptions for the versal deformation rings of \( \mathbb{Z}/p \)-covers. Theorem 3.5 of [Hen00] and Proposition 5.12 imply that one can describe a deformation (over either a mixed or equal characteristic ring) of a \( \mathbb{Z}/p \)-cover by the associated Hurwitz tree, which, in turn, is determined by a sequence of exact differential forms in equal characteristic case, or exact and logarithmic differential forms in mixed characteristic case. We wonder if one can use a space of differential forms to give a full description of the versal deformation ring of an Artin-Schreier cover.

6 Applications

6.1 Some general exact differential form results

Recall from Proposition 5.12 that the deformation of an Artin-Schreier cover is determined by exact differential forms of type (12). Combining with Remark 5.17 we simplify Question 3.4 as follows.

Question 6.1. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Suppose \( 1 < h_i < p \) for \( i = 1, 2, \ldots, n \) are integers (\( n \geq 2 \)). What are the conditions on the \( h_i \)'s so that the rational function

\[
\frac{1}{\prod_{i=1}^n (x - P_i)^{h_i}}
\]

is a derivative of some rational function in \( k(x) \) for some \( P_i \)'s in \( k \) pairwise distinct?

When the multi-set \( \{h_1, \ldots, h_n\} \) is given, we can answer the question using Gröbner Basis techniques as follows. Equation (14) could be written as

\[
\frac{\prod_{i=1}^n (x - P_i)^{p-h_i}}{\prod_{i=1}^n (x - P_i)^p} = \frac{\sum_j a_j x^j}{\prod_{i=1}^n (x - P_i)^p},
\]

(15)
where $a_j$ can be thought of as a polynomial in $k[P_1, \ldots, P_n]$. As the denominator is a $p$-power, the fraction is exact if and only if the numerator $\sum_j a_j x^j$ is exact. That equates to there existing a choice of values for the $P_i$’s so that $a_j = 0$ for all $j \equiv -1 \pmod p$, or $1 \not\equiv (a_j)_{j \equiv -1} \pmod p$ by Hilbert’s nullstellensatz. Moreover, all the $P_i$’s have to be distinct. That translates to $\prod_{j<k} (P_j - P_k)$ not lying in the radical of $(a_j)_{j \equiv -1} \pmod p$, or the following ideal
\[
\left(\{a_j\}_{j \equiv -1} \pmod p, 1 - s \cdot \prod_{j<k} (P_j - P_k)\right)
\]
is not the unit ideal of $k[P_1, \ldots, P_n, s]$ by [DFO4] §15, Corollary 35. We summarize by the below proposition.

**Proposition 6.2.** Suppose we are given a multi-set $\{h_1, \ldots, h_n\} \in \Omega_h$. Then there exists a differential form
\[
dx \prod_{i=1}^n (x - P_i)^{h_i}
\]
if and only if $\left(\{a_j\}_{j \equiv -1} \pmod p, 1 - s \prod_{j<k} (P_j - P_k)\right) \in k[P_1, \ldots, P_n, s]$ is not the unit ideal, where the $a_j$’s are defined in (15).

**Remark 6.3.** We can show whether the ideal in Proposition 6.2 is a unit one by checking whether 1 is its reduced Gröbner basis (with respect to any monomial ordering).

**Proposition 6.4.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Suppose $1 < h_i < p$ for $i = 1, 2, \ldots, n$ are integers ($n \geq 2$). The rational differential form
\[
\omega = \frac{dx}{\prod_{i=1}^n (x - P_i)^{h_i}}
\]
is exact for some distinct $P_i$’s in $k$ only if $\sum_{i=1}^n h_i \geq p + n$. The converse is true when $n = 2$.

**Proof.** This argument is due to Fedor Petrov (see §1.2). Suppose that $\sum_{i=1}^n h_i < n + p$ and the differential form $\omega$ in (16) is exact. One may assume that the rational antiderivative of $\omega$ is of the form $g/f$, where $f = \prod_{i=1}^n (x - P_i)^{h_i - 1}$, and $\deg g < \deg f = \sum_{i=1}^n (h_i - 1) < p$. This may be seen from integrating the partial fraction decomposition of $\prod_{i=1}^n (x - P_i)^{-h_i}$ and convert it back to a single fraction. We have $(g/f)' = (gf' - f'g)/f^2$, and if $\deg f = a$, $\deg g = b$, the degree of the numerator equals $a + b - 1$, since the leading coefficient does not vanish (here we use that $p$ can not divide $a - b$). Thus $1 = \prod_{i=1}^n (x - P_i)^{h_i}(g/f)'$ has degree $(n + a) + (a + b - 1) - 2a = n + b - 1 > 0$, a contradiction. Therefore, $\sum_{i=1}^n h_i$ has to be at least $n + p$ for (16) to be exact.

Suppose $n = 2$. One may assume that $P_1 = 0$. Consider the rational function
\[
\omega = \frac{1}{x^{e_1} (x - Q)^{e_2}} = \frac{x^{p-e_1} (x - Q)^{p-e_2}}{x^p (x - Q)^p}.
\]
Hence, $\omega$ is a derivative of some rational function if and only if $x^{p-e_1} (x - Q)^{p-e_2}$ is. The later is exact if and only if all the degree $kp - 1$ coefficients are equal to 0 as $k$ varies. Suppose $e_1 + e_2 \geq p + 2$. Then the degree of the numerator is $2p - (e_1 + e_2) \leq p - 2$. Thus, it is a derivative.

**Remark 6.5.** The converse is not true in general for $n > 2$. See this link for some counterexamples to differential forms of type $\{2, 2, \ldots, 2\}$ by Gjergji Zaimi. Using the Gröbner Basis technique (Proposition 6.2 and Remark 6.3), one can also show that there is no exact differential form of type $\{2, 2, 2, 6\}$ where $p = 7$, even though $2 + 2 + 2 + 6 > 7 + 4$.  

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6.1.1 Oort-Sekiguchi-Suwa deformations

We first observe the following phenomenon.

**Proposition 6.6.** The differential form

\[
dx \prod_{i=1}^{r} (x - q_i)^{h_i},\tag{17}
\]

where \(q_i \in k\)'s are pairwise distinct, is exact if all but at most one of the \(h_i\)'s are divisible by \(p\). In particular, there always exists a flat deformation of type \([h] \rightarrow [h_1, \ldots, h_r]^T\).

**Proof.** Suppose, without loss of generality, that \(h_1\) is the only exponent that may not be divisible by \(p\). Then we can easily see that differential form (17) is the derivative of the following rational function

\[
\frac{1}{(1 - h_1) \prod_{i=2}^{r} (x - q_i)^{h_i}(x - q_1)^{h_1 - 1}}.
\]

The rest of the proposition follows from Proposition 5.12. \(\square\)

**Remark 6.7.** Suppose \(h_i\)'s are as in Proposition 6.6. Consider the \(\mathbb{Z}/p\)-cover of \(\mathbb{P}^1_{k[[t_1, \ldots, t_r]]}\) defined by

\[
Y^p - Y = \frac{1}{(x - t_1)^{h_1 - 1} \prod_{i=2}^{r} (x - t_i)^{h_i}}.\tag{18}
\]

Replacing the \(t_i\)'s by any distinct elements \(P_i\)'s of \(m k[[t]]\) (i.e. \(v_t(P_i) > 0\)), one can show that (18) defines a flat deformation (over \(k[[t]]\)) of type \([h] \rightarrow [h_1, h_2, \ldots, h_r]^T\).

**Remark 6.8.** Suppose \(\Psi\) is a \(\mathbb{Z}/p\)-cover over \(k[[t]]\). In [BM00], Bertin and Mézard study the versal deformation ring \(R_{\Psi}\) whose spectrum is the formal deformation space of \(\Psi\). They construct explicitly a family of deformations of \(\Psi\) over a polynomial ring over \(R = \mathcal{W}(k)[\zeta_p]\). This family is parameterized by an irreducible component of the formal deformation space, called the Oort-Sekiguchi-Suwa component. Theorem 5.3.3 of the same paper shows that the dimension of this component is equal to the dimension of the versal deformation ring. Following [Dan20 §3.1.1], one can realize the characteristic \(p\) fibers of these deformations as special cases of one described in Proposition 6.6.

6.1.2 Non Oort-Sekiguchi-Suwa deformations

In [Dan20], to prove that \(\mathcal{A}S_g\) is connected when \(g\) is large, we construct some equidistant deformations that do not lie in \(p\)-fibers of the Oort-Sekiguchi-Suwa component. In this section, we realize these deformations in term of exact differential forms.

**Proposition 6.9.** There exist exact differential forms of the following types:

1. \(\{h_1, h_2\}\) where \(h_1 + h_2 \geq p + 2\),
2. \(\{h_1, h_2, h_3\}\) where \(h_i \equiv (p + 1)/2 \pmod{p}\) and \(p \geq 3\),
3. \(\{p - 1, h_1, h_2\}\) where \(h_1, h_2 \equiv 1 \pmod{p}\),
4. \(\{n + 1, n + 1, \ldots, n + 1\}\) where \(1 \leq n \leq p - 1\) \(\frac{p - n + 1}{p - n + 1}\)
5. \(\{3, 2, 2, 2\}\) and \(\{3, 3, 2, 2\}\) where \(p = 5\)
Proof. Item (1) follows immediately from [6.4].
Consider item (2). Suppose the desired differential is of the form

$$\frac{dx}{(x - a_1)^{h_1}(x - a_2)^{h_2}(x - a_3)^{h_3}}.$$  \hfill (19)

As discussed in Remark [5.17], one may assume, without loss of generality, that $h_i = (p + 1)/2$. Set $a_1 = 0$, $a_2 = 1$, and rewrite the differential form (19) as

$$\frac{x^{(p-1)/2}(x-1)^{(p-1)/2}(x-a_3)^{(p-1)/2}dx}{x^p(x-1)^p(x-a_3)^p}.$$  \hfill (20)

As the denominator is a $p$-power, the differential form is exact if and only if all the terms of degree congruent to $-1$ modulo $p$ in the numerator are zero. One can easily see the leading term has degree $3(p-1)/2 < 2p-1$. Hence, it suffices to make the term of degree $p-1$, which is

$$\sum_{i+j=\ell} \left(\frac{p-1}{i}\right) \left(\frac{p-1}{j}\right) a_3^{(p-1)/2-j} = \sum_{i=0}^{(p-1)/2} \left(\frac{p-1}{2}\right) a_3^{(p-1)/2-i},$$

(20)

to be equal to zero. We thus want $a_3$ to be a root of (20) (which can be thought of as a polynomial in $k[a_3]$) that is different from 0 and 1. As the constant coefficient of the polynomial is nonzero, 0 cannot be a root. Moreover, since the polynomial $\sum_{i=0}^{(p-1)/2} \left(\frac{p-1}{2}\right) x^{(p-1)/2-i}$ is separable by [Sil86] Theorem 4.1 and $(p-1)/2$ is at least 2 for $p \geq 5$, there are roots (in $k$) of (20) that are different from 1. Hence, if we pick $a_3$ to be one of these roots, the differential form is exact as desired.

By direct computation, one can show that the differential forms

$$\begin{align*}
\frac{dx}{x^{n+1}(x-1)^a} & \quad \frac{dx}{x^a(x-1)^a(x^2+x+1)^2} \quad \text{and} \quad \frac{dx}{x^a(x-1)^a(x^2+1)^2},
\end{align*}$$

are solutions to 3, 4, and 5, respectively.

This gives an alternative proof for [Dan20] Theorem 3.7. We paraphrase the statement of that theorem using the language in this paper.

**Theorem 6.10** (c.f. [Dan20] Theorem 3.7). There exist equidistant deformations over $k[[t]]$ of the following type.

1. $[h] \rightarrow [h_1, h_2]^T$ where $h_1$ and $h_2$ are non-zero and $h_1 + h_2 \geq p + 2$.
2. $[h] \rightarrow [h_1, h_2, h_3]^T$, where $h_i \equiv (p+1)/2 \pmod{p}$ and $p \geq 3$.
3. $[h] \rightarrow [p-1, h_2, h_3]^T$ where $h_1$ and $h_2$ are non-zero.
4. $[(n+1)(p-n+1)] \rightarrow [n+1, \ldots, n+1]^T$ where $1 \leq n \leq p-1$.
5. $[9] \rightarrow [3, 2, 2, 2]^T$ or $[10] \rightarrow [3, 3, 2, 2]^T$ where $p = 5$.

**Proof.** The theorem follows immediately from Proposition [5.12] and Proposition [6.9].

### 6.2 Disconnectedness of $\mathcal{A}_g$

Recall that the moduli space $\mathcal{A}_g$ can be partitioned by strata that are associated to partitions of $2g/(p-1) + 2$. Moreover, Pries and Zhu show that the dimension of the stratum indexed by
\( \overline{E} = \{ h_1, \ldots, h_r \} \) is given by

\[
d - 1 - \sum_{j=1}^{r} \left( \left\lfloor \frac{h_j - 1}{p} \right\rfloor \right)
\]

[BL93] Corollary 3.11, recall that \( d = \frac{2g}{p-1} + 2 \). Therefore, the irreducible components of \( \mathcal{AS}_g \) are the closure of the strata indexed by partitions of the form \( \{ h_1, \ldots, h_r \} \), where \( h_i \leq p \) for all \( i \). One key ingredient of our first connectedness result is that we prove, where the sum of conductors \( d + 2 \) is not congruent 1 modulo \( p \), that all the strata of non-zero-codimension lie in the same connected component [Dan20 Corollary 4.3]. In general, \( \mathcal{AS}_g \) is connected only if none of the strata of codimension-zero is closed (unless there is only one stratum in \( \mathcal{AS}_g \)). Furthermore, the closedness of a stratum can be realized by the following result.

**Proposition 6.11.** A stratum of \( \mathcal{AS}_g \) indexed by \( \overline{E} = \{ h_1, \ldots, h_r \} \) is not closed if and only if there exists a partition \( \{ g_1, \ldots, g_s \} \subseteq \{ h_1, \ldots, h_r \} \) and an exact differential form of type \( \{ g_1, \ldots, g_s \} \).

**Proof.** 

\( \Leftarrow \): Suppose, without loss of generality, that \( \{ g_1, \ldots, g_s \} = \{ h_1, \ldots, h_s \} \). The \( \Leftarrow \) direction then follows immediately from Proposition 5.12 as the closure of \( \Gamma_{\overline{E}} \) contains the stratum indexed by \( \sum_{i=1}^{s} h_i, h_{s+1}, \ldots, h_r \).

\( \Rightarrow \): Suppose \( \overline{E} \) is not closed. Then, by Proposition 2.10 there exists a partition \( \overline{E}' = \{ l_1, \ldots, l_m \} \sim \overline{E} \) such that \( \Gamma_{\overline{E}'} \subseteq \mathcal{T}_{\overline{E}} \). It then follows from Proposition 2.16 that there exist \( 1 \leq i \leq m \) so that \( \{ l_i \} \sim \{ h_{i_1}, \ldots, h_{i_m} \} \subseteq \overline{E} \), and a deformation of type \( [l_i] \rightarrow [h_{i_1}, \ldots, h_{i_m}] \). Finally, Proposition 5.12 shows that there exists an exact differential form of type \( \{ g_1, \ldots, g_s \} \subseteq \{ h_{i_1}, \ldots, h_{i_m} \} \), where \( \sum_{i=1}^{s} g_i \) is the difference between \( \overline{E} \) and the immediate one below in the chain of partitions from that proposition. That completes the proof.

We can finally give the proof of Theorem 1.3

### 6.2.1 Proof of Theorem 1.3

It is already known from Theorem 1.1 that, when \( p = 5 \), \( \mathcal{AS}_g \) is connected when \( g > (p-1)(p-2) = 12 \). The same result shows that, when \( p \geq 5 \), \( \mathcal{AS}_g \) is disconnected when \( (p-1)/2 < g \leq (p-1)^2/2 \). Hence, it suffices to show that \( \mathcal{AS}_g \) is disconnected if \( (p-1)^2/2 < g \leq (p-1)(p-2) \), or the sum of conductors \( d + 2 \) is between (and including) \( p + 3 \) and \( 2p - 2 \).

Recall that the irreducible components of \( \mathcal{AS}_g \) are indexed by the closure of the strata corresponding to \( \{ h_1, \ldots, h_r \} \) where \( h_i \leq p \) for all \( i \). Suppose \( d + 2 \) is odd (resp. even). Then there exists a stratum indexed by \( \overline{E}_1 := \{ 3, 2, \ldots, 2 \} \) (resp. \( \overline{E}_2 := \{ 2, 2, \ldots, 2 \} \)). It is straightforward to check that, for any \( \{ k_1, \ldots, k_s \} \subseteq \overline{E}_1 \) (resp. \( \overline{E}_2 \)), \( \sum_{j=1}^{s} k_j < p + s \).

Here we use that fact that the sum of the entries of \( \overline{E}_1 \) (resp. \( \overline{E}_2 \)), i.e., the number \( d + 2 \), is smaller than \( 2p - 2 \). Therefore, it follows from Proposition 6.11 that the closure of the stratum corresponding to \( \{ 3, 2, \ldots, 2 \} \) (resp. \( \{ 2, 2, \ldots, 2 \} \)) only contains itself. Thus, the moduli space \( \mathcal{AS}_g \) is disconnected.

### References

[BL93] Siegfried Bosch and Werner Lütkebohmert. Formal and rigid geometry. I. Rigid spaces. *Math. Ann.,* 295(2):291–317, 1993.
[BM00] José Bertin and Ariane Mézard. Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques. *Invent. Math.*, 141(1):195–238, 2000.

[BW06] Irene I. Bouw and Stefan Wewers. The local lifting problem for dihedral groups. *Duke Math. J.*, 134(3):421–452, 2006.

[BW09] Louis Hugo Brewis and Stefan Wewers. Artin characters, Hurwitz trees and the lifting problem. *Math. Ann.*, 345(3):711–730, 2009.

[Dan20] Huy Dang. Connectedness of the moduli space of artin-schreier curves of fixed genus. *Journal of Algebra*, 547:398 – 429, 2020.

[DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[Epp73] Helmut P. Epp.Eliminating wild ramification. *Invent. Math.*, 19:235–249, 1973.

[GM98] Barry Green and Michel Matignon. Liftings of Galois covers of smooth curves. *Compositio Math.*, 113(3):237–272, 1998.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HB09] Louis Hugo Brewis. Ramification theory of the p-adic open disc and the lifting problem. 10 2009.

[Hen00] Y. Henrio. Arbres de Hurwitz et automorphismes d’ordre p des disques et des couronnes p-adiques formels. ArXiv Mathematics e-prints, November 2000.

[Kat87] Kazuya Kato. Vanishing cycles, ramification of valuations, and class field theory. *Duke Math. J.*, 55(3):629–659, 1987.

[Kat89] Kazuya Kato. Swan conductors for characters of degree one in the imperfect residue field case. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 101–131. Amer. Math. Soc., Providence, RI, 1989.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Lea18] Isabel Leal. On ramification in transcendental extensions of local fields. *J. Algebra*, 495:15–50, 2018.

[Lor08] Falko Lorenz. *Algebra. Vol. II*. Universitext. Springer, New York, 2008. Fields with structure, algebras and advanced topics, Translated from the German by Silvio Levy, With the collaboration of Levy.

[Mau03] Sylvain Maugeais. Relèvement des revêtements p-cycliques des courbes rationnelles semi-stables. *Math. Ann.*, 327(2):365–393, 2003.

[Obu12] Andrew Obus. The (local) lifting problem for curves. In *Galois-Teichmüller theory and arithmetic geometry*, volume 63 of *Adv. Stud. Pure Math.*, pages 359–412. Math. Soc. Japan, Tokyo, 2012.

[OW14] Andrew Obus and Stefan Wewers. Cyclic extensions and the local lifting problem. *Ann. of Math. (2)*, 180(1):233–284, 2014.

[Pop14] Florian Pop. The Oort conjecture on lifting covers of curves. *Ann. of Math. (2)*, 180(1):285–322, 2014.

[Pri03] Rachel J. Pries. Conductors of wildly ramified covers. III. *Pacific J. Math.*, 211(1):163–182, 2003.


[PZ12] Rachel Pries and Hui June Zhu. The $p$-rank stratification of Artin-Schreier curves. *Ann. Inst. Fourier (Grenoble)*, 62(2):707–726, 2012.

[Saï07] Mohamed Saïdi. Galois covers of degree $p$ and semi-stable reduction of curves in equal characteristic $p > 0$. *Math. J. Okayama Univ.*, 49:113–138, 2007.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Sil86] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[Wea18] Bradley Weaver. The local lifting problem for $D_4$. *Israel J. Math.*, 228(2):587–626, 2018.

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