CYCLIC COVERS OF AFFINE $\mathbb{T}$-VARIETIES

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Abstract. We consider normal affine $\mathbb{T}$-varieties $X$ endowed with an action of a finite abelian group $G$ commuting with the action of $\mathbb{T}$. For such varieties we establish the existence of $G$-equivariant geometrico-combinatorial presentations in the sense of Altmann and Hausen. As an application, we determine explicit presentations of the Koras-Russell threefolds as bi-cyclic covers of $\mathbb{A}^3$ equipped with a hyperbolic $\mathbb{C}^*$-action.

Introduction

Every algebraic action of the one dimensional torus $\mathbb{T} \simeq \mathbb{C}^*$ on a complex affine variety $X$ is determined by a $\mathbb{Z}$-grading $A = \oplus_{m \in \mathbb{Z}} A_m$ of its coordinate ring $A$, the spaces $A_m$ consisting of semi-invariant regular functions of weight $m$ on $X$. One possible way to construct $\mathbb{Z}$-graded algebras, which was studied by Demazure [2], is to start with a variety $Y$ and a $\mathbb{Q}$-divisor $D$ on $Y$ and to let $A = \oplus_{m \in \mathbb{Z}} \Gamma(Y, O_Y(mD))$. For a well chosen pair $(Y, D)$, this algebra is finitely generated, corresponding to the ring of regular functions of an affine variety, $X = S(Y, D)$ with a $\mathbb{C}^*$-action whose algebraic quotient is isomorphic to Spec($\Gamma(Y, O_Y)$). A slight variant of this construction [3] already enabled a complete description of $\mathbb{C}^*$-actions on normal surfaces $X$: namely they correspond to graded algebras of the form:

$$A = \oplus_{m < 0} \Gamma(Y, O_Y(mD_-)) \oplus \Gamma(Y, O_Y) \oplus \Gamma(Y, O_Y(mD_+)),$$

for suitably chosen triples $(Y, D_+, D_-)$ consisting of a smooth curve $Y$ and a pair of $\mathbb{Q}$-divisors $D_+$ and $D_-$ on it.

Demazure’s construction was generalized by Altmann and Hausen [11] to give a description of all normal affine varieties $X$ equipped with an effective action of an algebraic torus $\mathbb{T} \simeq (\mathbb{C}^*)^k$, $k \geq 1$. Here the $\mathbb{Z}$-grading is replaced by a grading by the lattice $M \simeq \mathbb{Z}^k$ of characters of the torus, and the graded pieces are recovered from a datum consisting of a variety $Y$ of dimension dim($X) - \dim(\mathbb{T})$ and a so-called polyhedral divisor $D$ on $Y$, a generalization of $\mathbb{Q}$-divisors for higher dimensional tori: $D$ can be considered as a collection of $\mathbb{Q}$-divisors $D(u)$ parametrized by a “weight cone” $\sigma^\vee \cap M$, for which we have $A = \oplus_{u \in \sigma^\vee \cap M} \Gamma(Y, O_Y(D(u)))$. The $\mathbb{T}$-variety associated to a pair $(Y, D)$ is denoted by $S(Y, D)$.

In this article, we consider affine $\mathbb{T}$-varieties $X$ endowed with an additional action of a finite abelian group $G$ commuting with the action of $\mathbb{T}$. The quotient $X' = X//G$ is again an affine $\mathbb{T}$-variety for a torus $\mathbb{T}' \simeq \mathbb{T}$ obtained as a quotient of $\mathbb{T}$ by an appropriate finite group, and our aim is to understand the relation between the presentations $X = S(Y, D)$ of $X$ and those of $X' = S(Y', D')$. A pair $(Y, D)$ such that $X = S(Y, D)$ is not unique but we will show that it is always possible to choose a particular pair $(Y, D_G)$ consisting of a variety $Y$ endowed with a $G$-action and a $G$-invariant polyhedral divisor $D_G$ such that $X$ is $G \times \mathbb{T}$ equivariantly isomorphic to $S(Y, D_G)$. The $G$-invariant divisor $D_G$ corresponds in turn to a certain polyhedral divisor $D'$ on the quotient $Y'/G$ with property that $X' = S(Y'/G, D')$ as a $\mathbb{T}'$-variety.

More precisely, our main result reads as follows:

Theorem. Let $X$ be a $\mathbb{T}$-variety and let $G$ be a finite abelian group acting on $X$ such that the two actions commute. Then the following hold:

1) There exist a semi-projective variety $Y$ endowed with an action of $G$ and a $G$-invariant $pp$-divisor $D_G$ defined on $Y$ such that $X$ is $\mathbb{T} \times G$ equivariantly isomorphic to $S(Y, D_G)$.

2) Moreover $X//G$ is equivariantly isomorphic to the $\mathbb{T}'$-variety $S(Y//G, D')$ where $D'$ can be chosen such that $F_I(D_G) = \varphi_G(D'_I)$, where $\varphi_G : Y \rightarrow Y//G$ denotes the quotient morphism and $F : M' \rightarrow M$ is a linear map induced by the inclusion of the character lattices $M'$ of $\mathbb{T}'$ and $M$ of $\mathbb{T}$.

2000 Mathematics Subject Classification. 14R05, 14L30.
Key words and phrases. affine $\mathbb{T}$-varieties, hyperbolic $\mathbb{C}^*$-actions, Koras-Russell threefolds, cyclic covers.
We then apply this result to determine presentations of a family of exotic affine spaces of dimension 3 with hyperbolic \( \mathbb{C}^+ \)-actions: the Koras-Russell threefolds. We exploit the fact that these threefolds arise as equivariant bi-cyclic cover of the affine space \( \mathbb{A}^3 \) equipped with a hyperbolic \( \mathbb{C}^+ \)-action.

The article is organized as follows. The first section is devoted to a short recollection on Altmann-Hausen representations, with a particular focus on the methods to construct pairs \((Y, D)\) corresponding to a given graded algebra. The main theorem above is then established in section two. Finally, explicit Altmann-Hausen representations of the Koras-Russell threefolds are determined in section three.

1. Recollection on the Altmann-Hausen representation

In this section, we introduce the correspondence between normal affine \( \mathbb{T} \)-varieties \( X \) and pairs \((Y, D)\) composed of a normal semi-projective variety \( Y \) and a so-called polyhedral divisor \( D \) established by Altmann-Hausen \([1]\). In particular, for a given \( X \), we summarize a construction of a corresponding \( Y \) and explain a method to determine a possible \( D \).

1.1. Normal affine \( \mathbb{T} \)-varieties. Let \( N \cong \mathbb{Z}^k \) be a lattice of rank \( k \) and let \( M = \text{Hom}(N, \mathbb{Z}) \) be its dual. A pointed convex polyhedral cone \( \sigma \subseteq N_\mathbb{Q} = N \otimes \mathbb{Q} \) is an intersection of finitely many closed linear half spaces in \( N_\mathbb{Q} \) which does not contain any line. Its dual:

\[
\sigma^\vee := \{ v \in M_\mathbb{Q} \mid \forall u \in \sigma \ (u, v) \geq 0 \} \subseteq M_\mathbb{Q} = M \otimes \mathbb{Q},
\]

consists of all linear forms on \( M_\mathbb{Q} \) that are non-negative on \( \sigma \). A polytope \( \Pi \subseteq N_\mathbb{Q} \) is the convex hull of finitely many points in \( N_\mathbb{Q} \), and a convex polyhedron \( \Delta \subseteq N_\mathbb{Q} \) is the intersection of finitely many closed affine half spaces in \( N_\mathbb{Q} \).

Every polyhedron admits a decomposition: \( \Delta = \Pi_{\Delta} + \sigma \), where \( \Pi_{\Delta} \) is a polytope and \( \sigma \) is a pointed convex polyhedral cone, called the tail cone of \( \Delta \). The set of all polyhedra which admit the same tail cone is a semigroup with Minkowski addition, which we denote by \( \text{Pol}_1^+(N_\mathbb{Q}) \).

Definition 1.1. A \( \sigma \)-tailed polyhedral divisor \( D \) on an algebraic variety \( Y \) is a formal finite sum

\[
D = \sum \Delta_i \otimes D_i \in \text{Pol}_1^+(N_\mathbb{Q}) \otimes \mathbb{Q} \text{WDiv}(Y),
\]

where \( D_i \) are prime divisors on \( Y \) and \( \Delta_i \) are \( \sigma \)-polyhedra.

Every element \( u \in \sigma^\vee \cap M \) determines a map \( \text{Pol}_1^+(N_\mathbb{Q}) \otimes \mathbb{Q} \text{WDiv}(Y) \rightarrow \mathbb{Q} \otimes \mathbb{Q} \text{WDiv}(Y) \) which associates to \( D = \sum \Delta_i \otimes D_i \) the Weil \( \mathbb{Q} \)-divisor \( D(u) = \sum v \in \Delta_i \min \langle u, v \rangle D_i \) on \( Y \).

Given a Weil \( \mathbb{Q} \)-divisor \( D \) and a section \( s \in \Gamma(Y, \mathcal{O}_Y(D)) \), that is, an effective Weil divisor \( D' \) linearly equivalent to the round-down \( |D| \) of \( D \), we denote by \( Y_s \) the open subset \( Y \setminus \text{Supp}(D') \) of \( Y \).

Definition 1.2. ([1] Definition 2.5 and 2.7) A proper-polyhedral divisor, noted pp-divisor, is a polyhedral divisor \( D = \sum \Delta_i \otimes D_i \) on \( Y \) which satisfies the following properties:

1) Each \( D_i \) is an effective divisor and \( D(u) \) is a \( \mathbb{Q} \)-Cartier divisor on \( Y \) for every \( u \in \sigma^\vee \cap M \).

2) \( D(u) \) is semi-ample for each \( u \in \sigma^\vee \cap M \), that is, for some \( n \in \mathbb{Z}_{>0} \) the open subsets \( Y_s \), where \( s \in \Gamma(Y, \mathcal{O}_Y(D(nu))) \), cover \( Y \).

3) \( D(u) \) is big for each \( u \in \text{relint}(\sigma^\vee) \cap M \), that is, for some \( n \in \mathbb{Z}_{>0} \) there exist a section \( s \in \Gamma(Y, \mathcal{O}_Y(D(nu))) \) such that \( Y_s \) is affine.

Recall ([1] Definition 2.1] that a variety \( Y \) is said to be semi-projective if \( \Gamma(Y, \mathcal{O}_Y) \) is finitely generated and \( Y \) is projective over \( Y_0 = \text{Spec}(\Gamma(Y, \mathcal{O}_Y)) \). Given a pp-divisor \( D \) on \( Y \), the graded algebra

\[
A = \bigoplus_{u \in \sigma^\vee \cap M} A_u = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(D(u)))
\]

is finitely generated, and \( \text{Spec}(A) \) is a \( \mathbb{T} \)-variety for \( \mathbb{T} = \text{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^k \). More precisely Altmann and Hausen, showed the following:

Theorem 1.1. ([1]) For any pp-divisor \( D \) on a normal semi-projective variety \( Y \), the scheme

\[
\mathcal{S}(Y, D) = \text{Spec}( \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(D(u))))
\]

is a normal affine \( \mathbb{T} \)-variety of dimension \( \dim(Y) + \dim(\mathbb{T}) \). Conversely any normal affine \( \mathbb{T} \)-variety is isomorphic to an \( \mathcal{S}(Y, D) \) for suitable \( Y \) and \( D \).
1.2. **Determining the semi-projective variety.** The semi-projective variety $Y$ is not unique, however there exists a natural construction, which we will use in the remainder of the article. It can be summarized as follows ([1, section 6]).

Let $X = \text{Spec}(\bigoplus_{u \in M} A_u)$ be an affine variety endowed with an effective action of the torus $T = \text{Spec}({\mathbb C}[M])$. For each $u \in M$ the set of semistable points

$$X^{ss}(u) := \{ x \in X / \exists n \in \mathbb Z_{\geq 0} \text{ and } f \in A_{nu} \text{ such that } f(x) \neq 0 \}$$

is an open $T$-invariant subset of $X$ which admits a good $T$-quotient

$$Y_u = X^{ss}(u)/T = \text{Proj}_{A_0}(\bigoplus_{n \in \mathbb Z_{\geq 0}} A_{nu}).$$

Following [1, section 6], there exists a fan $\Lambda \subset M_{\mathbb R}$ generated by a finite collection of cones $\lambda$ such that the following holds:

1) For any $u$ and $u'$ in the relative interior of $\lambda$, $X^{ss}(u) = X^{ss}(u')$. We denote $W_\lambda = X^{ss}(u)$ for any $u \in \text{relint}(\lambda)$.

2) If $\gamma$ is a face of $\lambda$, $W_\lambda$ is an open subset of $W_\gamma$. Let $W = \bigcap_{\lambda \in \Lambda} W_\lambda = \lim W_\lambda$.

The quotient maps $q_\lambda : W_\lambda \to W_\lambda/T$ form an inverse system indexed by the cones in $\Lambda$, whose inverse limit exist as a morphism $q : W \to Z = \lim Y_\lambda$. The desired semi-projective variety $Y$ is the normalization of the closure of the image of $W$ by $q$.

![Diagram](link)

1.3. **Maps of proper polyhedral divisor.** Let $Y$ and $Y'$ be normal semi-projective varieties, $N$ and $N'$ be lattices and $\sigma \subset N_{\mathbb Q}$, $\sigma' \subset N'_{\mathbb Q}$ be pointed cones. Let $D = \sum \Delta_i \otimes D_i$ and $D' = \sum \Delta'_i \otimes D'_i$ be pp-divisors on $Y$ and $Y'$ respectively with corresponding tail cones $\sigma$ and $\sigma'$.

**Definition 1.3.** [1, Definition 8.3] For a morphism $\varphi : Y \to Y'$ such that $\varphi(Y)$ is not contained in $\text{Supp}(D'_i)$ for any $i$, the polyhedral pull-back of $D'$ is defined by:

$$\varphi^*(D') := \sum \Delta'_i \otimes \varphi^*(D'_i)$$

Where $\varphi^*(D'_i)$ is the usual pull-back of $D'_i$. It is a polyhedral divisor on $Y$ with tail cone $\sigma'$.

2) For a linear map $F : N \to N'$ such that $F(\sigma) \subset \sigma'$, the polyhedral push forward is defined as:

$$F_*(D) := \sum (F(\Delta_i) + \sigma') \otimes D_i$$

It is also a polyhedral divisor on $Y$ with tail cone $\sigma'$.

An equivariant morphism from $S(Y,D)$ to $S(Y',D')$ is given by a homomorphism of algebraic groups $\psi : T \to T'$ and a morphism $\phi : S(Y,D) \to S(Y',D')$ satisfying $\phi(\lambda x) = \psi(\lambda) \phi(x)$. Every such morphism is uniquely determined by a triple $(\varphi,F,f)$ defined as above consisting of a dominant morphism $\varphi : Y \to Y'$, a linear map $F : N \to N'$ as above and a plurifunction $f \in N' \otimes_{\mathbb Z} C(Y)^*$ such that:

$$\varphi^*(D') \leq F_*(D) + \text{div}(f).$$

The identity map of a pp-divisor is the triple $(\text{id}, \text{id}, 1)$ and the composition of two maps $(\varphi,F,f)$ and $(\varphi',F',f')$ is $((\varphi \circ \varphi'), F' \circ F, F_*'(f) \circ F_*(f')).$
1.4. Determining proper polyhedral divisors. A method to determine a possible pp-divisor $D$ (see section 11) associated to a $T$-variety $X$ with $T = (\mathbb{C}^\ast)^k$ is to embed $X$ as a $T$-stable subvariety of a toric variety. The calculation is then reduced to the toric case by considering an embedding in $\mathbb{A}^m$ with linear action for $m$ sufficiently large. In other words, $X$ is realized as a $(\mathbb{C}^\ast)^k$-stable subvariety of a $(\mathbb{C}^\ast)^m$-toric variety. The inclusion of $(\mathbb{C}^\ast)^k$ corresponds to an inclusion of the lattice of characters $\mathbb{Z}^k$ of $T$ into $\mathbb{Z}^m$. We obtain the exact sequence:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{s} & \mathbb{Z}^m & \xrightarrow{p} \mathbb{Z}^m / \mathbb{Z}^k & \longrightarrow 0,
\end{array}
$$

where $F$ is given by the action of $(\mathbb{C}^\ast)^k$ on $\mathbb{A}^m$ and $s$ is a section of $F$. The $(\mathbb{C}^\ast)^m$-toric variety is determined by the first integral vectors $v_i$ of the unidimensional cone generated by the $i$-th column vector of $P$ as rays in a $\mathbb{Z}^m$ lattice, and each $v_i$ correspond to a divisor. The support of $D_i$ is the intersection between $X$ and the divisor corresponding to $v_i$. The tail cone is $\sigma := s(\mathbb{Q}^m_{\geq 0} \cap F(\mathbb{Q}))$, and the polytopes are $\Pi_i = s(\mathbb{R}^m_{\geq 0} \cap P^{-1}(v_i))$.

2. Actions of finite abelian groups

Let $X = \text{Spec}(A)$ be a normal affine variety with an effective action of a torus $T$ and let $G$ be a finite abelian group of order $d \geq 2$ whose action on $X$ commutes with that of $T$. The goal of this section is to determine the relationship between the Altmann-Hausen representations of $X$ and those of $X//G = \text{Spec}(A^G)$.

Let $Y$ be a semi-projective variety equipped with an action $\psi : G \times Y \to Y$ of an algebraic group $G$. If $D_G$ is a $G$-invariant pp-divisor, i.e $G$-invariant pp-divisor of $G$, then for every $u \in \sigma^\vee \cap M$ the space $A_u = \Gamma(Y, \mathcal{O}(D_G(u)))$ of global sections $\mathcal{O}(D_G(u))$ is endowed with a $G$-action. It follows that $\mathcal{S}(Y, D_G) = \text{Spec}(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}_Y(D(u))))$ admits an action of $G$ commuting with that of $T$.

**Theorem 2.1.** Let $X$ be a $T$-variety and let $G$ be a finite abelian group acting on $X$ such that the two actions commute. Then the following hold:

1) There exist a semi-projective variety $Y$ endowed with an action of $G$ and a $G$-invariant pp-divisor $D_G$ on $Y$ such that $X$ is $T \times G$ equivariantly isomorphic to $\mathcal{S}(Y, D_G)$.

2) Moreover $X//G$ is equivariantly isomorphic to the $T'$-variety $\mathcal{S}(Y//G, D')$ where $D'$ can be chosen such that $F_*(D_G) = \varphi_G^*(D')$, where $\varphi_G : Y \to Y//G$ denotes the quotient morphism and $F : M' \to M''$ is a linear map induced by the inclusion between the character lattices $M'$ of $T'$ and $M$ of $T$ (see [2.3]).

We will divide the proof in several steps. First we will prove that the action of $G$ on $X$ induces an action of $G$ on $Y$. Secondly we will consider the case where the orbits of the $G$-action are included in the orbits of the $T$-action and finally we consider the case where the action of $G \times T$ is effective on $X$.

**Lemma 2.1.** Let $Y$ a quasi-projective variety endowed with an action of a finite group $G$ and let $\tilde{Y} \to Y$ be the normalization of $Y$. Then the action of $G$ lifts to an action on $\tilde{Y}$ and the induced morphism $\tilde{Y}//G \to Y//G$ is the normalization of $Y//G$.

*Proof.* Since $Y$ is quasi-projective and $G$ is finite, every $x \in X$ admits a $G$-invariant affine open neighborhood. The normalization being a local operation, we may assume that $Y$ is affine. Using the universal properties of the normalization and of the quotient we obtain the following commutative diagram:

$$
\begin{array}{ccc}
Y & \to & Y//G \\
\uparrow & & \uparrow \\
\tilde{Y} & \to & \tilde{Y}//G \\
\downarrow & & \uparrow \\
& & \tilde{Y}//G
\end{array}
$$

Thus $\mathbb{C}[\tilde{Y}//G] \subset \mathbb{C}[\tilde{Y}]^G$. Conversely, let $f \in \mathbb{C}[\tilde{Y}]^G$. Then $g.f = f$ for all $g \in G$ and there exists a monic polynomial $P$ with coefficients in $\mathbb{C}[Y]$ such that $P(f) = 0$. Since $G$ is finite, $Q = \prod_{g \in G} g.P$ is a monic polynomial with $G$-invariant coefficients and $G(f) = 0$. So $f \in \mathbb{C}[\tilde{Y}//G]$.

□
Corollary 2.1. Let $X$ be a $T$-variety and suppose that a finite abelian group $G$ acts on $X$ such that the two actions commute. Then there exists a semi-projective variety $Y$ and a pp-divisor $D$ on $Y$ such that $X$ is $G \times T$ equivariantly isomorphic to $S(Y, D)$ and the action of $G$ on $S(Y, D)$ induces an action of $G$ on $Y$.

Proof. We consider the construction of $Y$ given in section [1,2]. Since the action of $G$ and $T$ commute, for every $\lambda \in \Lambda$ the subset $X^{\lambda u}$ with $u \in \relint(\lambda)$ is $G$-stable. Thus $W := \cap_{\lambda \in \Lambda} W_\lambda$ is also $G$-stable. Since $q' : W \to Z$ is the quotient by $T$, the action of $G$ on $W$ induces one on $q'(W)$. The closure $q(W)$ is again $G$-stable, and since $q(W)$ is quasi-projective it follows from lemma [2.3] that the action of $G$ lifts to an action on $Y$.

Lemma 2.2. Let $X = \text{Spec}(A)$ be a $T$-variety and let $G$ be a finite abelian group acting on $X$ such that the two actions commute. Then there exists a $G$-invariant pp-divisor $D_G$ defined on $Y$ such that $X$ is equivariantly isomorphic to $S(Y, D_G)$.

Proof. By lemma 2.1 the action of $G$ on $X$ induces an action of $G$ on $Y$. By the proof of Theorem 3.4 in [1], a pp-divisor on $Y$ corresponding to $X$ is determined by the choice of a homomorphism $h$ from $M$ into the fraction field of $A$ with the property that for every $u \in M$, $h(u)$ is semi-invariant of weight $u$. Namely, if $u \in \sigma^\vee \cap M$ is any saturated element, that is, $u \in \sigma^\vee \cap M$ such that $\bigoplus_{n \in \mathbb{N}} A_{nu}$ is generated in degree 1, then there exist a unique Cartier divisor $D(u)$ such that $A_u = h(u).\Gamma(Y, O_Y(D(u)))$: its local equations on open subsets $Y_s$ with $s \in A_u$ are $h(u)/s$. By definition $h(u) = \frac{f}{g}$ where $f$ and $g$ are both non zero and $f \in A_{u_1}$, $g \in A_{u_2}$ such that $u_1 - u_2 = u$. Since $A_u$ is $G$-stable for all $u \in M$, we can choose $f \in A_{u_1}$, $g \in A_{u_2}$ semi-invariant for the action of $G$ with $u_1 - u_2 = u$ so that $h(u) = \frac{f}{g}$ is also semi-invariant for $G$. The corresponding divisor $D(u)$ is then $G$-invariant. In the case of a general $u \in \sigma^\vee \cap M$, we can choose a saturated multiple $nu$ and define $D(u) = D(nu)/n$.

To complete the proof of Theorem 2.1 we divide the argument into two cases. First we consider the situation where $G$ is a subgroup of $T$ and secondly where the action of $G \times T$ is effective.

Lemma 2.3. Let $X$ be the $T$-variety $S(Y, D)$ and let $G$ be a finite abelian subgroup of $T = \text{Spec}(C[M])$. Then $X' = X/G$ is a $T'$-variety where $T' \simeq T/G$ and is equivariantly isomorphic to $S(Y, F_\ast(D))$ where $F : N = M^\vee \to N' = (M')^\vee$ is the linear map induced by the inclusion between the character lattices $M'$ of $T'$ and $M$ of $T$.

Proof. Let $Y$ be as in [1,2]. Since by hypothesis the $G$-orbits are contained in $T$-orbits, the induced $G$-action on $Y$ is trivial. In this case, for each $u \in \sigma^\vee \cap M$, $A_u^G$ is either $A_u$ or $\{0\}$. Letting $M'$ be the sublattice $M$ generated by the elements $u \in \sigma^\vee \cap M$ such that $A_u^G \neq 0$,

$$X' = X/G = \text{Spec}(\bigoplus_{u \in \sigma^\vee \cap M'} A_u^G)$$

is a $T'$-variety where $T' = \text{Spec}(C[M'])$ is a torsus of the same dimension as $T$. The inclusion $M' \to M$ gives rise the desired linear map $F : N = M^\vee \to N' = M'^\vee$.

Remark 2.1. This case corresponds to the map of pp-divisors $(id, F, 1)$ defined in as [13]. Indeed the quotient morphism $\varphi : Y \to Y/G$ is the identity.

Lemma 2.4. Let $X$ be a normal affine variety with an effective action of $G \times T$ where $G$ is a finite abelian group. Then there exists a semi-projective variety $Y$ on which $G$ acts and a $G$-invariant pp-divisor $D_G$ on $Y$ such that $X$ is $G \times T$-equivariantly isomorphic to $S(Y, D_G)$.

Moreover $X/G$ is $T$-equivariantly isomorphic to $S(Y/G, D')$ where $D_G = \varphi_G^\ast(D')$.

Proof. By lemmas 2.1 and 2.2 $Y$ is endowed with an action of $G$, and we can assume that $X$ is equivariantly isomorphic to $S(Y, D_G)$. Since $D_G$ is $G$-stable, for each $u \in \sigma^\vee \cap M$, $\Gamma(Y, O_Y(D_G(u)))$ is a $G$-invariant submodule of $\Gamma(X, O_X)$ and moreover there exists $D'$ satisfying $\varphi_G^\ast(D') = D_G$. Therefore, $\Gamma(X/G, O_{X/G}) = (\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, O_Y(D_G(u))))^G = (\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, O_Y(D_G(u))))^G$.

By assumption, $\varphi : Y \to Y/G$ is the quotient morphism, and $D'$ satisfies $\varphi_G^\ast(D') = D_G$. Thus

$$\Gamma(Y, O_Y(D_G(u)))^G = \{f \in C(Y)^G, \text{div}(f) + D_G(u) \geq 0\} \cup \{0\}$$

$$= \{h \in C(Y/G), \varphi^\ast(\text{div}(h) + D'(u)) \geq 0\} \cup \{0\}$$

$$= \{h \in C(Y/G), \text{div}(h) + D'(u) \geq 0\} \cup \{0\}.$$
We conclude that $X//G \simeq \text{Spec}(\bigoplus_{u \in \sigma' \cap M} \Gamma(Y//G, \mathcal{O}(D'(u))))$. □

Remark 2.2. This lemma is the analogue of 4.1 in [2], in which Demazure established a similar result for algebras constructed from $\mathbb{Q}$-divisors. This case corresponds to the map of proper polyhedral divisors $(\varphi_G, \text{id}, 1)$ defined as in [1]

Proof. (of Theorem 2.1) Consider a finite abelian group $G$ acting on $X = S(Y, D)$ whose action commutes with that of $\mathbb{T}$. By virtue of lemmas 2.2 and 2.3, we may assume that $G$ acts on $Y$ and that $D$ is $G$-invariant. Then we let $H$ be the subgroup of $G \times \mathbb{T}$ consisting of elements which act trivially on $X$. We let $G_0 \subset G$ and $T_0 \subset \mathbb{T}$ be the images of $H$ by the two projections and we let $G' = G/G_0$ and $\mathbb{T}' = \mathbb{T}/T_0$. Applying lemma 2.3 to $X$ equipped with the action of $G_0$, we obtain a variety $X//G_0$ endowed with an effective action of $G' \times \mathbb{T}'$ to which the lemma 2.4 can be applied. Any map $(\varphi_G, F, 1)$ is obtained by composing maps of the two types above. □

3. Applications in the case $\mathbb{T} = \mathbb{C}^*$

3.1. Basic examples of $\mathbb{C}^*$-actions. The coordinate ring of a normal affine variety $X = \text{Spec}(A)$ equipped with an effective $\mathbb{C}^*$-action is $\mathbb{Z}$-graded in a natural way via $A = \bigoplus_{n \in \mathbb{Z}} A_n$ where $A_n := \{ f \in A / f(\lambda \cdot x) = \lambda^n f(x) \}$. The semi-projective variety associated to the Altmann-Hausen representation of $X$ is the irreducible component which correspond to the normalization of the closure of the image of $W$ by $q'$ (see [12]) in the fiber product:

$$Y(X) := Y_-(X) \times_{Y_0(X)} Y_+(X)$$

where $Y_0(X) = X//\mathbb{C}^* = \text{Spec}(A_0)$, $Y_\pm(X) = \text{Proj}_{A_0} \left( \bigoplus_{n \in \mathbb{Z} \geq 0} A_{\pm n} \right)$.

A $\mathbb{C}^*$-action said to be hyperbolic if there is at least one $n_1 < 0$ and one $n_2 > 0$ such that $A_{n_1}$ and $A_{n_2}$ are nonzero. In this case, the tail cone $\sigma$ is equal to $\{ 0 \}$ (see [14]). If in addition $X$ is smooth, then $Y(X)$ is in fact equal to the fiber product which is itself isomorphic to the blow-up of $Y_0(X)$ with center at $t$ closed subscheme defined by the ideal $I = \langle A_d, A_{-d} \rangle$ where $d > 0$ is chosen so that $\bigoplus_{n \in \mathbb{Z}} A_{dn}$ is generated by $A_0$ and $A_{\pm d}$ ( [3] Theorem 1.9 and proposition 1.4).

In what follows, we denote by $\pi : \hat{A}^n_{(t)} \to A^n$ the blow-up of the ideal $(I)$ in $A^n_{(x_1, ..., x_n)} = \text{Spec}(\mathbb{C}[x_1, ..., x_n])$.

Given an irreducible and reduced hypersurface $H = \{ f(x_1, ..., x_n) = 0 \} \subset A^n$ containing the origin, the hypersurface $X_{n,p,f}$ of $A^{n+2}$ is $\text{Spec}(\mathbb{C}[x_1, ..., x_n][y, t])$ defined by the equation

$$f(x_1, ..., x_n, y) + t^p = 0$$

comes equipped with an effective $\mathbb{C}^*$-action induced by the linear one $\lambda \cdot (x_1, ..., x_n, y, t) = (\lambda^p x_1, ..., \lambda^p x_n, \lambda^{-p} y, \lambda t)$ on $A^{n+2}$. We have $A^{n+2}//\mathbb{C}^* \simeq A^{n+1} = \text{Spec}(\mathbb{C}[u_1, ..., u_{n+1}])$ via $u_i = x_i y$ for $i = 1, ..., n$ and $u_{n+1} = yt^p$.

Proposition 3.1. The variety $X_{n,p,f}$ is equivariantly isomorphic to $S(\hat{A}^n_{(u_1, ..., u_n)}, D)$ for with $D = \left\{ \frac{1}{p} \right\} \mathbb{Z} + [0, \frac{1}{p}]E$, where $E$ is the exceptional divisor of the blow up and $D$ is the strict transform of the hypersurface $H \subset A^n$.

Proof. We determine $Y(X_{n,p,f})$ and the pp-divisor $D$ using the method described in sections 1.2 and 1.3. We consider the exact sequence :

$$0 \to \mathbb{Z} \xrightarrow{s} \mathbb{Z}^{n+2} \xrightarrow{p} \mathbb{Z}^{n+1} \to 0$$

where $F = \{p, ..., p, -p, 1\}$, $P = \begin{pmatrix}
I_n & \vdots & \vdots \\
1 & \vdots & \vdots \\
0 & \cdots & 0 & 1 & p
\end{pmatrix}$ $I_n$ being the identity matrix of rank $n \times n$ and $s = (0, ..., 0, 1)$. 
The fan in $\mathbb{Z}^{n+2}$ is generated by the rays $\{v_i\}_{i=1,...,n+2}$ where $v_i$ is the first integral vector of the unidimensional cone generated by the $i$-th column vector of $P$. It corresponds to the blow up of the origin in $\mathbb{A}^{n+1}$, as a toric variety.

The variety $Y$ is equal to the strict transform by $\pi : \hat{\mathbb{A}}^{n+1}_{(u_1,...,u_n)} \to \mathbb{A}^{n+1} \simeq \mathbb{A}^{n+2} // \mathbb{C}^*$ of $\{f(u_1,...,u_n) + u_{n+1} = 0\} \subset \mathbb{A}^{n+1}$, thus $Y \simeq \hat{\mathbb{A}}^n$.

Since $\sigma := s(\mathbb{Q}_{\geq 0} \cap F(\mathbb{Q}))$ is $\{0\}$, applying the formula $\Pi_i = s(\mathbb{R}_{\geq 0} \cap P^{-1}(v_i))$, we deduce that $D$ has the form $\{1\} \cap (D + [0,1/p])$, where $D$ corresponds to the restriction to $Y$ of the toric divisor given by the ray $v_{n+2}$. It is the restriction of $\{u_{n+1} = y^p = 0\}$ to $Y$ thus $D$ is the strict transforms of the hypersurface $H \subset \mathbb{A}^n$.

Example 3.1. Specializing the above construction we obtain examples of linear hyperbolic $\mathbb{C}^*$-actions on $\mathbb{A}^3$ which will be building blocks for further applications:

a) Choosing $n = 2$ and $f(x_1, x_2) = x_1$, we obtain that $X_{2, x_1, p}$ is isomorphic to $S(\hat{\mathbb{A}}^2_{(u,v)}, D)$ with $D = \{1\} \cap (D + [0,1/p])$, where $E$ is the exceptional divisor of the blow up and $D$ is the strict transform of the line $\{u = 0\} \subset \mathbb{A}^2$. Thus $X_{2, x_1, p} \subset \mathbb{A}^4$ is isomorphic to $\mathbb{A}^3$ equipped with the $\mathbb{C}^*$-action $\lambda \cdot (x_2, y, t) = (\lambda x_2, \lambda^{-p}y, t)$

b) In particular, if $p = 1$ then $X_{2, x_1, 1}$ is isomorphic to $S(\hat{\mathbb{A}}^2_{(u,v)}, D)$ with $D = \{1\} \cap (D + [0,1])$. Since $D = \{1\} \cap (D + [0,1])$ is equivalent to $D' = [-1,0]\mathbb{C}$, we have that $X_{2, x_1, 1}$ is equivariantly isomorphic to $S(\hat{\mathbb{A}}^2_{(u,v)}, D')$.

Example 3.2. Choosing $n = 2$ and $f(x_1, x_2) = x_1 + (x_1^d + x_2^d)^{1/2}$ yields that $X_{2, p, f}$ is isomorphic to $S(\hat{\mathbb{A}}^2_{(u,v)}, D)$, where $E$ is the exceptional divisor of the blow up and $D$ is the strict transform of the line $\{v = (v^d + u^d)^{1/2} = 0\} \subset \mathbb{A}^2$. Note that in contrast with the previous example, $X_{2, p, f}$ is not isomorphic to $\mathbb{A}^3$. Indeed, if it were, then by the result of Koras-Russell [7], the $\mathbb{C}^*$ action on $X_{2, p, f}$ would be linearizable. By considering the linear action induced on the tangent space of the fixed point, we find that $X_{2, p, f}$ would have to be equivariantly isomorphic to $X_{2, x_1, p}$ for some $p$. On the other hand it follows from [1] corollary 8.12] that two pp-divisors $D_1$, defined on $Y_i$ respectively with the same tail cone, define equivariantly isomorphic varieties $S(Y_i, D_i)$ if and only if there exist projective birational morphisms $\psi_i : Y_i \to Y$ and an pp-divisor $D$ on $Y$ such that $D_i \simeq \psi_i(D)$ if $i = 2$. This would induce an automorphism $\phi$ of $\hat{\mathbb{A}}^2$, such that $\phi^d(f) = x_1$, which is not possible, since a general fiber of $f$ is singular.

3.2. Koras-Russell threefolds. Smooth affine, contractible threefolds with a hyperbolic $\mathbb{C}^*$-action whose quotient is isomorphic to $\mathbb{A}^2/G$ where $G$ is a finite cyclic group have been classified by Koras and Russell [3], in the context of the linearization problem for $\mathbb{C}^*$-actions on $\mathbb{A}^3$ [7]. These threefolds, which we call Koras-Russell threefolds, provide examples of $\mathbb{T}$-varieties of complexity two. According to [8] they admit the following description:

Let $a', b'$ and $c'$ be pairwise prime natural numbers with $b' \geq c'$ and let $\mu_{a'}$, the group of $a'$-th roots of unity, act on $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[u,v])$ by $(u,v) \to (\lambda^{a'} u, \lambda^{b'} v)$ where $\lambda \in \mu_{a'}$. Consider a semi-invariant polynomial $f$ of weight congruent to $b'$ modulo $a'$ and with the property that $L = \{f \neq 0\}$ is isomorphic to a line and meets the axis $u = 0$ transversely at the origin and at $r - 1 \geq 1$ other points. With these assumptions the polynomial $s^{-c'} f(s^{a'} c, s^{b'} v)$ can be rewritten in the form $F(w, u, v)$ with $w = s^d$ where $F$ is semi-invariant of weight $b'$ for the $\mathbb{C}^*$-action $(w, u, v) \mapsto (\lambda^{-a} w, \lambda^{c} u, \lambda^{b'} v)$. Then for any choice of pairwise prime integers $(\alpha_1, \alpha_2, \alpha_3)$ such that $\gcd(\alpha_1, a') = \gcd(\alpha_2, b') = \gcd(\alpha_2, c') = 1$, the hypersurface $X = \{(x, y, z, t) \in \mathbb{A}^4/t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x) = 0\}$ is a Koras-Russell threefold.

Here we mainly consider two families of such threefolds:

1) The first kind is defined by equations of the form:

$$\{x + x^{2}y + z^{2} + t^{\alpha_3} = 0\},$$

where $2 \leq d$, $2 \leq \alpha_2 < \alpha_3$ with $\gcd(\alpha_2, \alpha_3) = 1$ and equipped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(\alpha_2 \alpha_3, -(d - 1)\alpha_2 \alpha_3, \alpha_3, \alpha_2)$. These correspond to the choice of $f = u + v + v^{d}$. 


2) The second type is defined by
\[ \{ x + y(x^d + z^{\alpha_2} + t^{\alpha_3}) = 0 \}, \]
where \( 2 \leq d, 1 \leq l, 2 \leq \alpha_2 < \alpha_3 \) with \( \gcd(\alpha_2, d) = \gcd(\alpha_3, \alpha_2) = 1 \) and equipped with the \( \mathbb{C}^* \)-action induced by the linear one on \( \mathbb{A}^4 \) with weights \((\alpha_2, -d(1-\alpha_2), d\alpha_3, \alpha_2)\). These correspond to the choice of \( f = v + (u + v^d)^l \).

To obtain the Altmann-Hausen representation for these threefolds, we will exploit the fact that they arise as \( \mathbb{C}^* \)-equivariant bi-cyclic covers of \( \mathbb{A}^3 \). We will see that the polyhedral coefficients are related with the choice of \((\alpha_1, \alpha_2, \alpha_3)\) and the divisors are related with the choice of the fiber \( L = \{ f = 0 \} \) in the construction above.

3.3. The Russell Cubic. We begin with the Russell cubic \( X = \{ x + x^2y + z^2 + t^3 = 0 \} \) in \( \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]) \) which corresponds to the choice \( a' = b' = c' = 1, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3 \) and \( f(u, v) = u + v + v^2 \) in the construction above. By construction \( X \) is equipped with the \( \mathbb{C}^* \)-action induced by the linear one on \( \mathbb{A}^4 \) with weights \((6, -6, 3, 2)\). The algebraic quotient \( X/\mathbb{C}^* \) is isomorphic to \( \mathbb{A}^2_{(u,v)} = \text{Spec}(\mathbb{C}[u,v]) \) where \( u = yz^2 \) and \( v = yx \).

**Proposition 3.2.** (see also [1]) The Russell Cubic \( X \) is isomorphic to \( \mathbb{S}(\hat{\mathbb{A}}^2_{(u,v)}, D) \) for
\[ D = \left\{ \frac{1}{2} \right\} D_3 + \left\{ \frac{-1}{3} \right\} D_2 + \left\{ 0, \frac{1}{6} \right\} E, \]
where \( E \) is the exceptional divisor of \( \pi: \hat{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2 \), and where \( D_2 \) and \( D_3 \) are the strict transforms of the curves \( \{ u = 0 \} \) and \( \{ u + v + v^2 = 0 \} \) in \( \mathbb{A}^2 \) respectively.

**Proof.** The two projections \( \Phi_2 = \text{pr}_{x,y,t} : X \to X_2 = \mathbb{A}^3 \) and \( \Phi_3 = \text{pr}_{x,y,z} : X \to X_3 = \mathbb{A}^3 \) express \( X \) as cyclic Galois covers of \( \mathbb{A}^3 \) of degrees 2 and 3 respectively, whose Galois groups \( \mu_2 \) and \( \mu_3 \) act on \( X \) by \( \xi \cdot (x, y, z, t) = (x, y, \xi z, t) \) and \( \zeta \cdot (x, y, z, t) = (x, y, z, \zeta t) \) respectively. Furthermore these two actions commute and the quotient \( X_6 = X/((\mu_2 \times \mu_3)) \) is isomorphic to \( \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z^2]) \). Letting \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) be the coordinate ring of \( X \) equipped with the grading corresponding to the given \( \mathbb{C}^* \)-action, we have in fact \( X_\ell = \text{Spec}(\bigoplus_{n \in \mathbb{Z}} A_{\ell n}) \), \( \ell = 2, 3, 6 \). This yields a \( \mathbb{C}^* \)-equivariant commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi_2} & X_2 = X/\mu_2 \\
& \downarrow \Phi_6 & \downarrow \Phi_3 \\
X_6 = X/((\mu_2 \times \mu_3)) & \text{ } & X_3 = X/\mu_3
\end{array}
\]

where \( \mathbb{C}^* \) acts linearly on \( X_2, X_3 \) and \( X_6 \) with weights \((3, -3, 1), (2, -2, 1) \) and \((1, -1, 1) \) respectively.

Furthermore since the action of \( \mu_2 \times \mu_3 \) on \( X \) factors through that of \( \mathbb{C}^* \) we deduce from Theorem 2.1 that \( \Phi_2 \) corresponds to the map of proper polyhedral divisors (id, \( F_2, 1 \)) and \( \Phi_3 \) corresponds to the map of proper polyhedral divisors (id, \( F_3, 1 \)) where \( F_\ell^*(D) = \ell D, \ell = 2, 3, 6 \). The semi-projective varieties \( X(Y) \) and \( Y(X_\ell), \ell = 2, 3, 6 \) are all isomorphic. As observed earlier, \( A_0 = \mathbb{C}[u,v] \) with \( u = yz \) and \( v = yx \) so that \( Y_0(X_6) \simeq Y_0(X) = \hat{\mathbb{A}}^2_{(u,v)} \). We further observe that \( A_-6n = A_0y^n \subset A \) because all semi-invariant polynomials of negative weights divisible by 6 are divisible by \( y \). This implies that \( Y_-(X_6) \simeq \text{Proj}(\bigoplus_{n \in \mathbb{Z}} A_0y^n) \simeq Y_0(X) \).

Finally, \( \bigoplus_{n \in \mathbb{Z}} A_{6n} \simeq \text{Sym}_A A_6 \) where \( A_6 \) is the free \( A_0 \)-submodule of \( A \) generated by \( x \) and \( z \). Therefore
\[ Y(X) \simeq Y(X_6) = Y_-(X_6) \times_{Y_0(X_6)} Y_+(X_6) \simeq Y_+(X_6) \]
is isomorphic to the blow-up \( \hat{\mathbb{A}}^2_{(u,v)} \) of \( Y_0(X) = \mathbb{A}^2 \) at the origin. It remains to determine the pp-divisor \( D \). We will construct it from those \( D_2 \) and \( D_3 \) corresponding to \( X_2 \) and \( X_3 \) respectively.
By Proposition 3.1, $X_2 = S(\tilde{A}^2_{(u,v)}; D_2 = \frac{1}{2}D_2 + [0, \frac{1}{3}]E)$ where $D_2$ is the strict transform of the curve $\{u = 0\}$ and $E$ is the exceptional divisor and $X_3 = S(\tilde{A}^2_{(u',v)}; D_3 = \frac{1}{2}D_3 + [0, \frac{1}{3}]E)$ where $D_3$ is the strict transform of the curve $\{u' = 0\}$ and $E$ is the exceptional divisor. Theorem 2.1 implies in turn that $2D \sim D_2 = \frac{1}{2}D_2 + [0, \frac{1}{3}]E$ and $3D \sim D_3 = \frac{1}{2}D_3 + [0, \frac{1}{3}]E$. Thus $D_2 + D = D_3$ and we conclude that $D = \{ \frac{1}{2} \} D_3 + \{ -\frac{1}{3} \} D_2 + [0, \frac{1}{3}]E$.

\[ \Phi_{\mu_3} \]

Remark. The choice of the coefficients is not unique since $\mathcal{D}' \sim \mathcal{D} + \text{div}(f)$ for any rational function $f$ on $Y$. This corresponds for example to $\mathcal{D}' \sim \mathcal{D} + D + E$ and more generally for any pair $(a, b) \in \mathbb{Z}^2$ such that $3a + 2b = 1$ we have that $\mathcal{D} \sim \{ \frac{1}{2} \} D_3 + \{ \frac{1}{3} \} D_2 + [0, \frac{1}{3}]E$.

3.4. Koras Russell threefolds of the first kind. Now we will show that a similar method can be used to present all Koras-Russell threefolds of the form $X = \{ x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0 \}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$. Namely, we consider a cyclic cover $V$ of $X$ with algebraic quotient $V//\mathbb{C}^*$ isomorphic to $\tilde{A}^2 = \text{Spec}(\mathbb{C}[u, v])$ where $u = y^{\alpha_2}$ and $v = xy$. A representation of $V$ is obtained by the same method as in the previous case and the representation of $X$ is deduced by applying again Theorem 2.1.

The categorical quotient $X//\mathbb{C}^*$ is isomorphic to $\tilde{A}^2_{(u,v)}//\mu_{d-1}$ where $\mu_{d-1}$ acts by $\xi \cdot (u, v) = (\xi u, \xi v)$. So we consider $V$ a finite cyclic cover of $X$ given by the equation $X = \{ x + x^d y^{d-1} + z^{\alpha_2} + t^{\alpha_3} = 0 \}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$, equipped with the $\mathbb{C}^*$-action induced by the linear one on $\mathbb{A}^4$ with weights $(\alpha_2, -\alpha_2, \alpha_3, \alpha_2)$. Furthermore $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{d-1}$ acts on $V$ by $(\zeta, \epsilon, \xi) \cdot (x, y, z, t) \rightarrow (x, \zeta y, \xi z, \epsilon t)$. Observe that the action of $\mu_{\alpha_2} \times \mu_{\alpha_3}$ factors through that of $\mathbb{C}^*$. This yields the following diagram of quotient morphisms:

\[ \mathbb{A}^3 \simeq V//\mu_{\alpha_2} \quad X = V//\mu_{d-1} \quad \tilde{A}^2 \simeq V//\mu_{\alpha_3}. \]

By Theorem 2.1, $\Phi_{\mu_3}$ corresponds to the map of proper polyhedral divisors $(\text{id}, F_{\alpha_3}, 1)$ and $\Phi_{\alpha_3}$ corresponds to the map of proper polyhedral divisor $(\text{id}, F_{\alpha_3}, 1)$ where $F_\ell(D) = \ell D, \ell = 2, 3, 6$. In addition we obtain that $Y(V)$ is isomorphic to the blow-up $\tilde{k}^2_{(u,v)}$ of $k^2$ at the origin on which $\mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{d-1}$ acts by $(\zeta, \epsilon, \xi) \cdot (u, v) = (\xi u, \xi v)$. This leads to the following diagram:

\[ Y(V) \simeq Y(V//\mu_{\alpha_2}) \quad Y(X) \simeq Y(V//\mu_{d-1}) \quad Y(V//\mu_{\alpha_3}). \]

Using example 5.1, we obtain Altmann-Hausen representations of $V//\mu_{\alpha_2}$ and $V//\mu_{\alpha_3}$ in the form $S(\tilde{k}^2_{(u,v)})$:

$D_{\alpha_2} = \frac{1}{\alpha_2}D_{\alpha_2} + [0, \frac{1}{\alpha_2}]E$ where $D_{\alpha_2}$ is the strict transform of the curve $\{u = 0\}$, $E$ is the exceptional divisor and $S(\tilde{k}^2_{(u',v)})$:

$D_{\alpha_3} = \{ \frac{1}{\alpha_3} \} D_{\alpha_3} + [0, \frac{1}{\alpha_3}]E$ where $D_{\alpha_3}$ is the strict transform of the curve $\{u' = 0\}$, $E$ is the exceptional divisor. This implies that $V$ is isomorphic to $S(\tilde{k}^2_{(u,v)}, D)$ for

\[ D = \left\{ \frac{a}{\alpha_2} \right\} D_{\alpha_2} + \left\{ \frac{b}{\alpha_3} \right\} D_{\alpha_3} + \left[ 0, \frac{1}{\alpha_2} \right] E \] (\#),

where $E$ is the exceptional divisor of $\pi : \tilde{k}^2_{(u,v)} \rightarrow k^2$, $D_{\alpha_2}$ and $D_{\alpha_3}$ are the strict transforms of the curves $\{u = 0\}$ and $\{u + v + v^d = 0\}$ in $k^2_{(u,v)}$, respectively, and $(a, b) \in \mathbb{Z}^2$ are chosen such that $a\alpha_2 + b\alpha_2 = 1$. Applying Theorem 2.1, we obtain

**Proposition 3.3.** The Koras-Russell threefold $X = \{ x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0 \}$ in $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t])$ is isomorphic to $S(\tilde{k}^2_{(u,v)}/\mu_{d-1}, D')$ for

\[ D' = \left\{ \frac{a}{\alpha_2} \right\} D'_{\alpha_2} + \left\{ \frac{b}{\alpha_3} \right\} D'_{\alpha_3} + \left[ 0, \frac{1}{(d-1)\alpha_3} \right] E' \]

where $D = \varphi_{\mu_{d-1}}(D')$, $D$ is defined in the relation (\#) and $D'_{\alpha_2}, D'_{\alpha_3}$ are prime divisors and $E'$ is the exceptional divisor of the blow-up of the singularity in $k^2/\mu_{d-1}$. 
3.5. Koras Russell threefolds of the second kind. For Koras-Russell threefolds of the second kind \( X = \{ x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0 \} \) in \( \mathbb{A}^4 \) Spec(\( \mathbb{C}[x, y, z, t] \)) the construction will be slightly different due to the fact that the variables \( z \) and \( t \) do no longer play symmetric roles. We will consider again a cyclic cover \( V \) of \( X \), but in this case \( V///\mu_{\alpha_3} \) will not be isomorphic to \( \mathbb{A}^3 \). Recall that by definition, \( \alpha_2 \) and \( d \) are coprime. We consider a bi-cyclic cover \( V = \{ x + y^{d-1}(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0 \} \) of \( X \) of order \( d \times (dl - 1) \), which we decompose as a cyclic cover \( \phi_d : V \to V_d = \{ x + y^{d-1}(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0 \} \) of degree \( d \), followed by a cyclic cover \( \phi_{dl-1} : V_d \to X \) of degree \( dl - 1 \). The hypersurface \( V \) is equipped with the \( \mathbb{C}^* \)-action induced by the linear one on \( \mathbb{A}^4 \) with weights \( (\alpha_2 \alpha_3, -\alpha_2 \alpha_3, \alpha_3, \alpha_2) \) and with the action of \( \mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{dl-1} 	imes \mu_d \) defined by \( (\zeta, \epsilon, \xi, \delta) \cdot (x, y, z, t) = (x, \xi y, \zeta y, \epsilon t) \). The action of \( \mu_{\alpha_2} \times \mu_{\alpha_3} \) on \( V \) factors through that of \( \mathbb{C}^* \) and we obtain the following diagram:

\[
\begin{array}{ccc}
V_{\alpha_2} = V///\mu_{\alpha_2} & \xrightarrow{\phi_{\alpha_2}} & V_d = V///\mu_d \\
\downarrow{\phi_{\alpha_3}} & & \downarrow{\phi_{\mu_{dl-1}}} \\
\mathbb{A}^3 \simeq V_{\alpha_3} = V///\mu_{\alpha_3} & \xrightarrow{\phi_3} & X = V///(\mu_d \times \mu_{dl-1})
\end{array}
\]

By Theorem 2.1, considering \( \Phi_{\alpha_3} \), we obtain that \( Y(V) \) is isomorphic to the blow-up \( \tilde{\mathbb{A}}^2_{(u,v)} \) of \( \mathbb{A}^2 \) where \( u = y^{\alpha_2} \) and \( v = xz \) on which \( \mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{dl-1} \times \mu_d \) acts by \( (\zeta, \epsilon, \xi, \delta) \cdot (u, v) = (\xi^{\alpha_2} u, \zeta v) \). We obtain the following quotient diagram:

\[
\begin{array}{ccc}
Y(V) & \xrightarrow{\varphi_{\alpha_2}} & Y(V_{\alpha_2}) \\
\downarrow{\varphi_{\alpha_3}} & & \downarrow{\varphi_{\mu_{dl-1}}} \\
Y(V_d) & \xrightarrow{\varphi_3} & Y(X)
\end{array}
\]

Now by Proposition 3.2, \( V_{\alpha_2} = \mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}), D_{\alpha_2} = \{ \frac{1}{\alpha_2} D_{\alpha_2} + [0, \frac{1}{\alpha_3}] E \} \) where \( D_{\alpha_2} \) is the strict transform of the curve \( \{ v + (v^d + u^d)^l = 0 \} \) and \( E \) is the exceptional divisor, and \( V_{\alpha_3} = \mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}), D_{\alpha_3} = \{ \frac{1}{\alpha_2} D_{\alpha_3} + [0, \frac{1}{\alpha_2}] E \} \) where \( D_{\alpha_3} \) is the strict transform of the curve \( \{ u = 0 \} \) and \( E \) is the exceptional divisor. Thus \( V = \mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, D) \) for

\[
D = \left\{ \frac{a}{\alpha_2} \right\} D_{\alpha_2} + \left\{ \frac{b}{\alpha_3} \right\} D_{\alpha_3} + [0, \frac{1}{\alpha_2 \alpha_3}] E,
\]

where \( E \) is the exceptional divisor of \( \pi : \tilde{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2 \), and where \( D_{\alpha_2} \) and \( D_{\alpha_3} \) are the respective strict transforms of the curves \( \{ v + (v^d + u^d)^l = 0 \} \) and \( \{ u = 0 \} \) in \( \mathbb{A}^2 \) and \( (a, b) \in \mathbb{Z}^2 \) \( a \alpha_3 + b \alpha_2 = 1 \). Note that the choice of \( D \) up to linear equivalence does not depend of the choice on \( (a, b) \in \mathbb{Z}^2 \).

Now we deduce from Theorem 2.1 that \( V_d = \mathbb{S}(\tilde{\mathbb{A}}^2_{(u',v')}, D_d) \) for

\[
D_d = \left\{ \frac{a'}{\alpha_2} \right\} D_{d,\alpha_2} + \left\{ \frac{b'}{\alpha_3} \right\} D_{d,\alpha_3} + [0, \frac{1}{\alpha_2 \alpha_3}] E_d \quad (**),
\]

where \( a' = a/d, b' = b, E_d \) is the exceptional divisor of \( \pi : \tilde{\mathbb{A}}^2_{(u',v')} \to \mathbb{A}^2 \) due to the fact that \( \tilde{\mathbb{A}}^2_{(u',v')///\mu_d} \simeq \mathbb{A}^2_{(u',v')} \) for the action of \( \mu_d \) as above, and where \( D_{d,\alpha_2} \) and \( D_{d,\alpha_3} \) are the strict transforms of the curves \( \{ v' + (v' + u''d)^l = 0 \} \) and \( \{ u' = 0 \} \) ( \( u' = \varphi_d(u^d) \) in \( \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u', v']) \) respectively. Applying again Theorem 2.1 we obtain:

Proposition 3.4. A Koras-Russell threefold \( X = \{ x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0 \} \) in \( \mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z, t]) \) is isomorphic to \( \mathbb{S}(\tilde{\mathbb{A}}^2_{(u',v')///\mu_{dl-1}}, D_{d(dl-1)}) \) for
\[ D_{d(\alpha_1)} = \left\{ \frac{a'}{\alpha_2} \right\} D_{d(\alpha_1-1),\alpha_3} + \left\{ \frac{b'}{\alpha_3} \right\} D_{d(\alpha_1),\alpha_2} + \left[ \frac{1}{(\alpha_1-1)\alpha_2\alpha_3} \right] E_{d(\alpha_1)} , \]

where \( D_{d} = \varphi_{\mu_{\alpha_1}}(D_{d(\alpha_1-1)}) \), \( D_{d} \) is defined in the relation (**) and \( E_{d(\alpha_1)} \) is the exceptional divisor of the blow-up of the singularity in \( \mathbb{A}^2//\mu_{\alpha_1} \).

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