On the dualization of Born-Infeld theories

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Abstract

We construct a general Lagrangian, quadratic in the field strengths of $n$ abelian gauge fields, which interpolates between BI actions of $n$ abelian vectors and actions, quadratic in the vector field-strengths, describing Maxwell fields coupled to non-dynamical scalars, in which the electric-magnetic duality symmetry is manifest. Depending on the choice of the parameters in the Lagrangian, the resulting BI actions may be inequivalent, exhibiting different duality groups. In particular we find, in our general setting, for different choices of the parameters, the $U(n)$-invariant BI action of $[4]$ as well as the recently found N=2 supersymmetric BI action $[11]$.

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1 Introduction

The Born-Infeld (BI) theory \cite{1} describes a non-linear electrodynamics in four dimensional space-time enjoying remarkable features, among which electric-magnetic duality symmetry. Such a peculiarity, which has been generalized to the case of \( n \) abelian field strengths, where the duality group is contained in \( \text{Sp}(2n, \mathbb{R}) \) \cite{2,3,4}, hints to a connection of BI with extended supersymmetric theories, which also have the electric-magnetic duality invariance \cite{5} as a characteristic property. The supersymmetric version of the BI Lagrangian was constructed in \cite{6,7}, while in \cite{8,9} it was identified as the invariant action of the Goldstone multiplet in a \( N = 2 \) supersymmetric theory spontaneously broken to \( N = 1 \). Recently, the results of \cite{9} have been generalized to the case of \( n \) vector multiplets in \( N = 2 \) supersymmetry \cite{11,12}, with explicit solutions for the case \( n = 2 \) and \( n = 3 \).

In this letter we provide a linear (in the squared field strengths) realization of the bosonic BI Lagrangian in terms of a redundant Lagrangian containing two couples of non dynamical scalars. The classical BI Lagrangian is recovered solving the field-equation constraints when varying our Lagrangian with respect to one of the two couples of scalars, while variation with respect to the other couple of Lagrange multipliers leads to a version of linear electromagnetism with generalized (scalar dependent) couplings and a positive scalar potential, in which the duality symmetry is manifest. Remarkably, the properties of the resulting theory fit very well with the bosonic sector of the \( N = 2 \) supersymmetric Lagrangian for a vector multiplet in the presence of a complex Fayet-Iliopoulos term, in the limit where the masses of the scalar sector are dominant with respect to their kinetic term. By appropriate choice of the normalization of the fields, we recover indeed, in a component form, the results of \cite{9}.

Let us remark that in our approach the possibility of dualization to BI is due to the presence of a scalar function \( f(\Lambda) \propto \sqrt{1 + \Lambda} \), \( \Lambda \) being one of the Lagrange multipliers. After implementing the proper normalization of the fields corresponding to the supersymmetric case, the coefficient in front of \( f(\Lambda) \) turns out to be twice the product of an electric and a magnetic charge. In the absence of either the electric or the magnetic charge, our Lagrangian would reduce to linear electrodynamics coupled to scalars and it would not be able to implement the dualization to BI. On the other hand, the need for both electric and magnetic charges is in fact a necessary condition for partial supersymmetry breaking \( N = 2 \rightarrow N = 1 \), as shown in \cite{13,15}. Our formalism, recalling the results in \cite{9}, makes the relation between partial supersymmetry breaking and BI manifest. Not surprisingly, the presence of \( f(\Lambda) \) in our Lagrangian is also necessary to obtain, in the other version of the theory, a scalar potential manifestly invariant under electric-magnetic duality symmetry.

In our framework, the generalization to more than one vector fields, at the purely bosonic level, is straightforward by promoting scalar fields to matrices. We write a general Lagrangian which also include some constant matrices \( \eta^{IJ}, \tilde{\eta}_{IJ} \). In the generic case where \( \eta^{IJ}, \tilde{\eta}_{IJ} \) are invertible, the extension of our approach to any number of vectors is straightforward and leads to the definition of an abelian multi-field BI action which comprises, for a suitable choice of parameters, the \( U(n) \)-invariant BI action \cite{4} in the absence of extra scalar fields. However, we show that we can relax the invertibility condition on the two constant matrices \( \eta^{IJ}, \tilde{\eta}_{IJ} \), allowing for an \( N \geq 2 \) supersymmetric extension. For specific choices of \( \eta^{IJ}, \tilde{\eta}_{IJ} \) in terms of the electric and magnetic Fayet-Iliopoulos charges we reproduce the \( N = 2 \) supersymmetric BI action found in \cite{11}. Therefore, we show that, starting from our unifying description, different choices of the constant matrices \( \eta^{IJ}, \tilde{\eta}_{IJ} \) may lead, upon integrating out the non-dynamical fields, to inequivalent theories which exhibit different global symmetries.
2 Linear realization of the Born-Infeld Lagrangian

Let us consider the Born-Infeld Lagrangian in four dimensions:

\[ L = \frac{1}{\lambda} \left\{ 1 - \sqrt{\left| \det \left[ \eta_{\mu\nu} + \sqrt{\lambda} F_{\mu\nu} \right] \right|} \right\} = \frac{1}{\lambda} \left( 1 - \sqrt{1 + \frac{\lambda}{2} F^2 - \frac{\lambda^2}{16} (F\tilde{F})^2} \right), \tag{2.1} \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is an abelian field strength, \( \tilde{F}_{\mu\nu} = \frac{1}{2} F_{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \) its Hodge dual and

\[
F^2 = F_{\mu\nu} F^{\mu\nu}, \tag{2.2} \\
F\tilde{F} = \frac{1}{2} F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}. \tag{2.3}
\]

We are going to show that it can be written as the standard Lagrangian of a gauge field-strength in a theory whose field content is enlarged to include two couples of scalar fields which play the role of Lagrange multipliers \( \tilde{g}, \tilde{\theta}, \Lambda, \Sigma \):

\[
L' = \frac{\tilde{g}}{2\lambda} \left( \Lambda + \Sigma^2 - \frac{\lambda}{2} F^2 \right) + \tilde{\theta} \left( \frac{1}{4} F\tilde{F} - \frac{\Sigma}{\lambda} \right) + \frac{1}{\lambda} \left( 1 - \sqrt{1 + \Lambda + 1} \right). \tag{2.4}
\]

Indeed, variation of \( L' \) in (2.4) with respect to the couple of fields \( \tilde{g}, \tilde{\theta} \):

\[
\frac{\delta L'}{\delta \tilde{g}} = 0 \Rightarrow \Lambda = \frac{\lambda}{2} F^2 - \Sigma^2, \tag{2.5} \\
\frac{\delta L'}{\delta \tilde{\theta}} = 0 \Rightarrow \Sigma = \frac{\lambda}{4} F\tilde{F}, \tag{2.6}
\]

gives back the standard form (2.1) of the Born-Infeld Lagrangian, while variation with respect to the couple of fields \( \Lambda, \Sigma \) allows to express them in terms of the couple \( \tilde{g}, \tilde{\theta} \):

\[
\frac{\delta L'}{\delta \Lambda} = 0 \Rightarrow \tilde{\Lambda} = \frac{\tilde{g} - 2}{\tilde{g}} - 1, \tag{2.7} \\
\frac{\delta L'}{\delta \Sigma} = 0 \Rightarrow \tilde{\Sigma} = \frac{\tilde{\theta}}{\tilde{g}}, \tag{2.8}
\]

leading to the “dual” form of the Born-Infeld Lagrangian

\[
L' = -\frac{\tilde{g}}{4} F^2 + \frac{\tilde{\theta}}{4} F\tilde{F} - V(\tilde{g}, \tilde{\theta}), \tag{2.9}
\]

where

\[
V(\tilde{g}, \tilde{\theta}) = -\frac{1}{\lambda} \left[ \frac{\tilde{g}}{2} (\Lambda + \Sigma^2) - \tilde{\theta} \Sigma - \sqrt{1 + \Lambda + 1} + 1 \right]_{\Lambda = \tilde{\Lambda}, \Sigma = \tilde{\Sigma}} = \frac{1}{2\lambda} \left( \tilde{g} + \tilde{\theta}^2 \tilde{g}^{-1} + \tilde{g}^{-1} \right) - \frac{1}{\lambda}. \tag{2.10}
\]

Two properties of eq. (2.10) allow to embed eq. (2.9) in a very natural way into a supersymmetric theory: If we assume \( \tilde{g} > 0 \), which gives the correct sign to the gauge-field kinetic term in (2.9),
the potential $V$ is positive definite (apart for an irrelevant additive constant). Furthermore, it can be written as

$$V(\tilde{g}, \tilde{\theta}) = \frac{1}{2\lambda} \text{Tr}[\mathcal{M}] - \frac{1}{\tilde{\lambda}},$$  \hspace{1cm} (2.11)

where we introduced the matrix

$$\mathcal{M}_{MN}[\tilde{g}, \tilde{\theta}] = \begin{pmatrix} \tilde{g} + \tilde{\theta} \tilde{g}^{-1} \tilde{\theta} & -\tilde{\theta} \tilde{g}^{-1} \\ -\tilde{\theta} \tilde{g}^{-1} & \tilde{g}^{-1} \end{pmatrix},$$  \hspace{1cm} (2.12)

which is very familiar to supersymmetry and supergravity users, since it is the $2n \times 2n$ symplectic matrix that typically encodes the scalar couplings to $n$ gauge field strengths in $N$-extended supersymmetric theories.

As we are going to show in the next section, the linear realization of the Born-Infeld Lagrangian can thus naturally be thought of as the bosonic sector of the Lagrangian of an $N = 2$ vector multiplet with a (partially or totally) supersymmetry-breaking scalar potential, in a limit where the scalar-field kinetic term is negligible with respect to the potential term in the action. It will in fact turn out to coincide with the result of [11].

Moreover, the definition of the scalar potential $V(\tilde{g}, \tilde{\theta})$ in terms of an invariant quantity (the trace of the symplectic matrix $\mathcal{M}$) allows to generalize it to $n$ vector multiplets, and to give a general definition of the Born-Infeld Lagrangian for $n$ abelian field-strengths.

### 2.1 Embedding of the 4D Born-Infeld action in $N = 2$ supersymmetry

Let us consider an $N = 2$ supersymmetric vector multiplet, whose field content is given by a gauge vector $A_\mu$ with field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, a complex scalar $z$ and a couple of Majorana spinors $\lambda^A$ ($A = 1, 2$). The bosonic sector of the Lagrangian is

$$\mathcal{L} = -\frac{g(z, \bar{z})}{4} F^2 + \frac{\theta(z, \bar{z})}{4} F \bar{F} + G_{zz} \partial_\mu z \partial^\mu \bar{z} - V(z, \bar{z})$$  \hspace{1cm} (2.13)

where $g$ and $\theta$ are functions of the complex scalars $z$, $\bar{z}$ and $G$ is the metric of the sigma-model. In this case, and in the absence of the hypermultiplet sector, the scalar potential $V(z, \bar{z})$ is due to the presence of a (electric and magnetic) Fayet-Iliopoulos term $P_x M^x$ (where $x = 1, 2, 3$ is an SU(2) index while $M$ is a symplectic index which, in the case of just 1 vector multiplet, reduces to $M = 1, 2$) such that the supersymmetry transformation law of the (chiral) gaugino acquires a fermion shift $W^{z|AB} = (\sigma^x)^{AB} G^{z\bar{z}} U^z_\bar{z} \mathcal{P}^z_M$, where $U^z_\bar{z} = (f_z, h_z)$ is the symplectic section, $G^{z\bar{z}}$ the inverse of the Kähler metric, and correspondingly

$$V = \frac{1}{2} W^{z|AB} G_{\bar{z}z} W^{\bar{z}z}_{AB} = \frac{1}{2} \mathcal{P}^z_M \mathcal{M}^{MN} \mathcal{P}^\bar{z}_N,$$  \hspace{1cm} (2.14)

where we used the special geometry relation $U^M_i G^{ij} U^N_j = \frac{1}{2} (\mathcal{M}^{MN} - i \Omega^{MN})$ (in this case, the indices $i, j$ labeling the scalars take just the single value $z, \bar{z}$).

The presence of the fermion shift generally fully breaks supersymmetry in the vacuum. However, by choosing one of the three FI terms, say $P^3$, to zero, thus breaking SU(2) $\rightarrow$ U(1), it is possible to preserve $N = 1$ supersymmetry. In this case, which is the one considered in [11], the spontaneously
broken theory has a scalar potential which can be written in terms of a complex FI term $P = \frac{1}{\sqrt{2}}(P^1 + iP^2)$ as:

$$V_{FPS} = \bar{P} M (M_{MN} + i\Omega_{MN}) P^N = m^2 \left[ g + \left( \frac{e_1}{m} \right)^2 g^{-1} \right] + e_2^2 g^{-1} - 2me_2, \quad (2.15)$$

where we have raised with $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the symplectic index of the FI vector and, by fixing the $U(1)$ R-symmetry, we have chosen $P^M = \begin{pmatrix} m \\ e_1 + ie_2 \end{pmatrix}$. Let us denote by $L_{FPS}$ the Lagrangian of (11), with the scalar potential (2.15). Notice that the $N = 1$ scalar potential (2.15) differs from the $N = 2$ one by a constant additive term \cite{13, 14} depending on the product $me_2^2$. This extra term determines the vanishing of the $N = 1$ scalar potential on the supersymmetric vacuum.

In the vacuum, the scalar sector is completely fixed, while the gauge sector stays massless.

We would like to compare (2.15) with the scalar potential (2.11) corresponding to the Born-Infeld Lagrangian. Let us then compare $V_{FPS}(g, \theta)$ with $k V(\tilde{g}, \tilde{\theta})$ of (2.11); we find:

$$\tilde{g} = \frac{m}{e_2} g, \quad (2.16)$$

$$\tilde{\theta} = \frac{m}{e_2} \left( \theta - \frac{e_1}{m} \right), \quad (2.17)$$

$$k = 2me_2\lambda, \quad (2.18)$$

showing that the scalar potential (2.11) is in fact suitable to describe an $N = 2 \rightarrow N = 1$ supersymmetric theory, if one reabsorbs the charges $m, e_1, e_2$ in the definition of the scalars $\tilde{g}, \tilde{\theta}$.

Requiring

$$L_{FPS} = k L' + \text{const.} F \tilde{F}, \quad (2.19)$$

we find

$$\lambda = \frac{1}{2m^2}, \quad (2.20)$$

and then

$$L_{FPS} = \frac{e_2}{m} \tilde{L}' + \frac{e_1}{4m} F \tilde{F}. \quad (2.21)$$

Restoring the auxiliary fields in $L'$, as in (2.4), we can rewrite the Lagrangian in the following form which dualizes the Born-Infeld Lagrangian:

$$k L' = -\frac{g}{4} F^2 + \frac{1}{4} \left( -\frac{e_1}{m} \right) F \tilde{F} + m^2 g (\Lambda + \Sigma^2) - 2m^2 \left[ \frac{e_1}{m} \right] \Sigma + 2me_2 \left[ 1 - \sqrt{1 + \Lambda} \right]. \quad (2.22)$$

Furthermore, let us remark that the last contribution in (2.22), which is the one needed for implementing the dualization into Born-Infeld, requires $m e_2 > 0$ (which is the same consistency condition found in \cite{11}). This shows that $e_2, m \neq 0$, which is a necessary condition for partial supersymmetry breaking $N = 2 \rightarrow N = 1$, is also a necessary condition for a supersymmetric Lagrangian to allow a non-linear realization of the gauge field-strength sector. To complete the proof,

\footnote{We thank Sergio Ferrara for enlightening clarifications on this point.}
we should still show that a limit exists for the charges \((m \to \infty)\) where the scalar fields behave as Lagrange multipliers, that is where the scalar-field kinetic term is negligible with respect to the rest of the Lagrangian. This is indeed the case, since the scalar potential has a stable minimum at \(V = 0\), and in the vacuum the complex scalar \(z = \theta - ig\) acquires a mass \(M = \sqrt{2m^2 e^2}\).

We can prove now that the transformation (2.18) amounts to a change in the symplectic frame. Let us use the complex notation \(z = \tilde{\theta} - i \tilde{g}\). Consider the \(SL(2, \mathbb{R})\)–transformation

\[
A_{MN} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \tag{2.23}
\]

under which

\[
z \to z' = \frac{c + d z}{a + b z}. \tag{2.24}
\]

We have:

\[
\mathcal{M}[z', \bar{z}'] = A^{-T} \mathcal{M}[z, \bar{z}] A^{-1}. \tag{2.25}
\]

If \(z' = \theta - i g\), the transformation (2.18) is implemented by a matrix

\[
A_{MN} = \frac{1}{\sqrt{e_2 m}} \begin{pmatrix} m & 0 \\ e_1 & e_2 \end{pmatrix}. \tag{2.26}
\]

The scalar potential \(\mathcal{V}_{FPS}(g, \theta)\) can be written as

\[
\mathcal{V}_{FPS}(g, \theta) = \text{Tr} \left( \mathcal{P}_s \mathcal{M}[g, \theta] \right) + i \text{Tr} \left( \mathcal{P}_a \Omega \right), \tag{2.27}
\]

where

\[
\mathcal{P}^{MN} \equiv \bar{P}^M P^N, \quad \mathcal{P}_s^{MN} \equiv \bar{P}(M) P^N, \quad \mathcal{P}_a^{MN} \equiv \bar{P}(M) P^N. \tag{2.28}
\]

Using (2.25) and (2.26) we can write:

\[
\mathcal{V}_{FPS}(g, \theta) = \text{Tr} \left( A^{-1} \mathcal{P}_s A^{-T} \mathcal{M}[\tilde{g}, \tilde{\theta}] \right) + i \text{Tr} \left( A^{-1} \mathcal{P}_a A^{-T} \Omega \right). \tag{2.29}
\]

Now one can easily verify that

\[
A^{-1} \mathcal{P}_s A^{-T} = e_2 m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{-1} \mathcal{P}_a A^{-T} = e_2 m \Omega. \tag{2.30}
\]

We then find our previous result:

\[
\mathcal{V}_{FPS}(g, \theta) = \frac{e_2}{m} \mathcal{V}(\tilde{g}, \tilde{\theta}). \tag{2.31}
\]

3 Generalization to \(n\) abelian vector fields

Let us now discuss the generalization of the above construction to \(n\) abelian gauge fields, and under which conditions it is possible to embed the bosonic structure in a supersymmetric theory.

The analysis of the 1-vector Born-Infeld Lagrangian (2.1), independently of supersymmetry, shows that it can be linearized easily into (2.4) with the help of two couples of auxiliary fields. Variation with respect to the couple \(\Lambda, \Sigma\) leads to the “dual” Lagrangian with scalar potential (2.11), while variation with respect to the couple \(g, \theta\) gives back the BI Lagrangian (2.1). The key,
to obtain this result, is the introduction in (2.4) of the function \( f(Λ) \propto \sqrt{1 + Λ} \), which reproduces the expression of the BI Lagrangian for \( Λ \rightarrow Λ(F, \tilde{F}) = \frac{1}{2}F^2 - \frac{λ^2}{16}(F\tilde{F})^2 \). We wish now to generalize the Lagrangian (2.4) to \( n \) vector fields. The global symmetry of (2.4) is manifest once we integrate out the \( Σ \) and \( Λ \) fields and write the Lagrangian (modulo an overall factor and field redefinitions), in the form (2.9) with scalar potential (2.11). From the latter, the U(1)-duality invariance of the theory is manifest.

If we generalize (2.11) to \( n \) vectors, it would be manifestly U(\( n \))-invariant, and this is a distinctive feature of the dual BI theory, which should then coincide with the action found in [4].

We shall actually generalize (2.11) to a function of the form:

\[
V = \frac{1}{2λ} \text{Tr}(N\mathcal{M}) + \text{const.},
\]

where \( N \) is a constant \( 2n \times 2n \) symmetric matrix. The global symmetries of the Maxwell equations close a group \( G \) whose action on \( \mathcal{M} \) amounts to symplectic transformations \( A \):

\[
\mathcal{M} \rightarrow \mathcal{M}' = A^{-T}\mathcal{M} A^{-1}.
\]

The global symmetry group \( G \) is now contained in the intersection of the symplectic group and of the invariance group of the matrix \( N \)

\[
G \subset \text{Sp}(2n, \mathbb{R}) \cap \text{Inv}(N).
\]

If \( N \) is positive definite, its invariance group is O(\( 2n \)) and \( G \subset \text{U}(n) \). We shall however also discuss a special limit in which the matrix \( N \) is singular, which seems to be required if we wish to embed the model in an \( N = 2 \) supersymmetric context. Depending therefore on the choice of \( N \), and in particular on its invariance symmetry, by integrating the auxiliary fields we shall end up with inequivalent BI Lagrangians.

Let us now enter into the details of our construction. We shall insist in demanding that the dualized BI theory should have a scalar potential of the form (3.1). This fixes the function \( f(Λ) \) which implements the dualization, thus providing a possible general definition for the \( n \)-vector generalization of the BI Lagrangian, invariant under general electric-magnetic duality rotations of the \( n \) field-strengths together with their magnetic duals.

Let us then introduce two couples of (matricial) auxiliary fields \( g_{IJ} = g_{JI}, θ_{IJ} = θ_{JI}, \) and \( Λ_{IJ}, Σ_{IJ} (I, J, \cdots = 1, \ldots, n) \), generalizing the field \( g, θ, Λ, Σ \) of the \( n = 1 \) case. In particular, \( g_{IJ} \), which we require to be positive definite, and \( θ_{IJ} \) are the imaginary and real parts of a complex matrix \( N_{ΛΣ} \equiv θ_{IJ} - ig_{IJ} \) which parametrizes the coset:

\[
\mathcal{N}_{ΛΣ} \in \text{Sp}(2n, \mathbb{R}) \big/ \text{U}(n).
\]

In terms of these fields we construct a symplectic, symmetric matrix \( \mathcal{M} \) as in (2.12)

\[
\mathcal{M}_{MN}[g, θ] = \begin{pmatrix}
g + θ \cdot g^{-1} \cdot θ & -θ \cdot g^{-1} \\
-θ \cdot g^{-1} \cdot θ & g^{-1}
g^{-1} & \end{pmatrix},
\]

where now \( N, M = 1, \ldots, 2n \). This matrix transforms, under the action of a symplectic transformation \( A \) acting on \( \mathcal{N} \rightarrow \mathcal{N}' \), as in (2.25):

\[
\mathcal{M}[g', θ'] = A^{-T}\mathcal{M}[g, θ] A^{-1}.
\]
We can now start from an $n$-vector Lagrangian of the following general form

$$
L' = \frac{g_{IJ}}{2\lambda} \left( \Lambda^{IJ} + (\Sigma \cdot \eta \cdot \Sigma^T)^{IJ} - \frac{\lambda}{2} F^{I\mu \nu} F^{J|\mu \nu} \right) + \left( \theta_{IJ} - (\eta^{-1} \eta')_{IJ} \right) \left( \frac{1}{4} F^{I} F^{J} - \frac{\lambda}{\lambda} \right) + \frac{1}{\lambda} (C - f(\Lambda)),
$$

(3.7)

where $\eta^{IJ}$ is a generic symmetric matrix which, for the time being, we shall assume to be non-singular, while on the matrix $\eta^{IJ}$ we shall make no assumption. The constant $C$ has no physical relevance and is determined by convenience.

We are going to determine $f(\Lambda)$ in such a way as to obtain, upon integrating out the fields $\Lambda, \Sigma$, a scalar potential of the form (3.1), for a certain symmetric matrix $N$ which we shall assume, for the time being, to be non-singular. In other words we require that, after eliminating $\Lambda^{IJ}, \Sigma^{IJ}$ through their field equations, the resulting Lagrangian should have the form

$$
L' = -\frac{g_{IJ}}{4} F^{I\mu \nu} F^{J|\mu \nu} + \frac{\theta_{IJ}}{4} F^{I} F^{J} - V(g, \theta)
$$

(3.8)

with

$$
V(g, \theta) = \frac{1}{2\lambda} \left( \text{Tr}(\eta \cdot g) + \text{Tr}(\eta \cdot \theta \cdot g^{-1} \cdot \theta) + \text{Tr}(\eta' \cdot g^{-1}) \right) - \frac{C}{\lambda}
$$

(3.9)

where now $\mathcal{M}$ is a $2n \times 2n$ symplectic matrix, and we have introduced the matrix

$$
N \equiv \begin{pmatrix} \eta & \eta' \\ \eta^T & \eta \end{pmatrix}.
$$

Notice that the matrix $N^{MN}$ can always be brought to a block diagonal form by means of a symplectic transformation $S$, amounting to a change in the symplectic frame:

$$
N_D = S^{-1} \mathcal{M} S^{-1T} = \begin{pmatrix} g + \theta' \cdot g^{-1} \cdot \theta' & -\theta' \cdot g^{-1} \\ g^{-1} \cdot \theta' & g^{-1} \end{pmatrix} ; \quad \theta' = \theta - \eta^{-1} \eta'.
$$

(3.10)

Provided $\eta^{-1} \eta' = \eta^T \eta^{-1}$. In the new frame the matrix $\mathcal{M}$ reads:

$$
\mathcal{M}_D = S^{-1} \mathcal{M} S^{-1T} = \begin{pmatrix} g + \theta' \cdot g^{-1} \cdot \theta' & -\theta' \cdot g^{-1} \\ g^{-1} \cdot \theta' & g^{-1} \end{pmatrix} ; \quad \theta' = \theta - \eta^{-1} \eta'.
$$

(3.11)

Explicitly, the variation of (3.7) with respect to $\Sigma, \Lambda$ gives

$$
g \cdot \Sigma \cdot \eta = \theta' \cdot \eta \quad \Rightarrow \quad \Sigma = g^{-1} \cdot \theta' + \omega \quad \text{with} \quad \omega \cdot \eta = 0,
$$

(3.12)

$$
\frac{\partial f}{\partial \Lambda^{IJ}} = \frac{1}{2} g_{IJ} \quad \Rightarrow \quad f(\Lambda) = \frac{1}{2} \int g_{IJ} d\Lambda^{IJ} = \frac{1}{2} g_{IJ} \Lambda^{IJ} - \frac{1}{2} \int \Lambda^{IJ} dg_{IJ}.
$$

(3.13)

Substituting into (3.7), we get:

$$
V = -\frac{1}{\lambda} \left( \frac{1}{2} \int \Lambda^{IJ} dg_{IJ} - \frac{1}{2} \text{Tr}(\eta \cdot \theta' \cdot g^{-1} \cdot \theta') + 1 \right).
$$

(3.14)
By comparing (3.9) with (3.14) we finally get
\[ \Lambda = (g^{-1} \cdot \tilde{\eta}_0 \cdot g^{-1}) - \eta, \]  
(3.15)
where
\[ \tilde{\eta}_0 \equiv \tilde{\eta} - \eta \eta' T \eta^{-1} \eta'. \]  
(3.16)
Using once more (3.13), we find:
\[ f(\Lambda) = \text{Tr}(g^{-1} \tilde{\eta}_0) = \text{Tr}\left(\sqrt{(\eta + \Lambda) \cdot \tilde{\eta}_0}\right), \]  
(3.17)
where the matricial square root is intended as a solution in $g^{-1} \tilde{\eta}_0$ to the equation
\[ \Lambda \tilde{\eta}_0 = (g^{-1} \cdot \tilde{\eta}_0)^2 - \eta \tilde{\eta}_0. \]  
(3.18)
For $n = 1$ the Lagrangian (3.7), with the above expression for $f(\Lambda)$, reduces to (2.9), modulo an additive constant, if we set:
\[ \eta = 1, \eta' = e_{11}, \tilde{\eta}_0 = e_{22} + e_{22}, \lambda = \frac{1}{2m^2}, f(\Lambda) = \frac{e_{2}}{m} \sqrt{1 + \Lambda}. \]  
(3.19)
With the above prescription for $f(\Lambda)$, by varying (3.7) with respect to $g, \theta$, we obtain:
\[ L = \frac{1}{\lambda} \left\{ C - \text{Tr}\left(\frac{1}{2} F^\mu_{\mu} F^K |^{\mu\nu} - \frac{\lambda^2}{16} (F \cdot \eta^{-1} \cdot F) |^{IK} \right) \tilde{\eta}_0 K J \right\} \]  
(3.20)
which can be taken as a definition for the $n$-field generalization of the BI Lagrangian. For convenience we can choose $C = \text{Tr}(\sqrt{\eta \cdot \tilde{\eta}_0})$.

It is interesting to observe that our Ansatz (3.20), in the case $N$ is the identity matrix, does indeed coincide with the expression found in [4], as it also follows from general global symmetry arguments.

### 3.1 A singular limit: The $N = 2$ supersymmetric case

For $N = 2$ supersymmetric theories with $n > 1$ vector multiplets, in the presence of a complex Fayet-Iliopoulos term $P^M$ it is possible to write the scalar potential as [11]:
\[ V_{FPS}(z, \bar{z}) = P^M M_{MN} P^N + i \bar{P}^M \Omega_{MN} P^N, \]  
(3.21)
where $g_{IJ} = g_{IJ}(z, \bar{z}), \theta_{IJ} = \theta_{IJ}(z, \bar{z}), z^I$ being the $n$ complex scalar fields, and
\[ P^M = (m, e_1 + i e_2), \quad m \equiv (m^I), \quad e_1 \equiv (e_{11}), \quad e_2 \equiv (e_{21}). \]  
(3.22)
Such a potential induces a partial supersymmetry breaking to $N = 1$, so that one of the two supersymmetries is realized non linearly and the scalar fields become massive. This potential can be cast in the general form (3.9)
\[ V_{FPS}(g, \theta) = \frac{1}{2\lambda} \text{Tr}(N \mathcal{M}[g, \theta]) - \frac{C}{\lambda}, \]  
(3.23)
\[ \text{In general we should take } P^M = (m^I + i m^2, e_{11} + i e_{21}). \text{ However, using a U(n) transformation we can always set } m^2 = 0. \]
provided we define:

\[
N^{MN} = \begin{pmatrix} \eta & \eta' \\ \eta^T & \tilde{\eta} \end{pmatrix} = 2 \lambda P^{(M} \tilde{P}^{N)}, \tag{3.24}
\]

where:

\[
\begin{align*}
\eta &= 2 \lambda m m^T = 2 \lambda (m^I m^J), \quad \eta' = 2 \lambda m e_1^T = 2 \lambda (m^I e_1) , \quad \tilde{\eta} = 2 \lambda (e_1 e_1^T + e_2 e_2^T), \\
\lambda &= \frac{1}{2 m^T m}, \quad C = 2 \lambda m^T e_2.
\end{align*} \tag{3.25}
\]

However, in this case \(N^{MN}\) is a rank-2 matrix, and thus it is not invertible for \(n > 1\). \(^3\)

This case can be included in the general analysis performed above as a singular limit. First of all, let us point out that in the models where the matrix \(\eta\) is not invertible one may still define \(\eta^{-1}\) as the matrix such that \(\eta \cdot \eta^{-1} = \eta^{-1} \cdot \eta = P\), where \(P\) is a projector on the subspace supporting the rank of \(\eta\), generated by \(m\) only:

\[
P \cdot m = m \Rightarrow P \cdot \eta = \eta, \quad P \cdot \eta' = \eta'. \tag{3.26}
\]

Although now \(\eta, \tilde{\eta}\) are not invertible, we can still perform the transformation \(S\) in (3.10) in order to bring \(N\) to the block-diagonal form \(N_{D}\), with the lower diagonal block \(\tilde{\eta}_0 = 2 \lambda e_2 e_2^T\). In the corresponding new symplectic frame \(M_{D} = M[g, \theta']\), with \(\theta'\) given in (3.11). Notice that in this case, as the reader can easily verify,

\[
\eta^{-1} \eta' = \eta'^T \eta^{-1},
\]

so that \(S\) is indeed symplectic.

Our aim is to start from a Lagrangian of the form (3.7), with the definitions (3.25) and \(\eta^{-1}\) defined above and \(f(\Lambda)\) given in (3.17), and with the matrixes \(g, \theta\) intended as functions of \(z^i\), and implement the constraints (3.12) and (3.15) in order to get, in the \(m^I \to \infty\) limit, the bosonic sector of an \(N = 2 \to N = 1\) supersymmetric Lagrangian \(L_{FPS}^{(0)}\):

\[
L_{FPS}^{(0)}(z, \bar{z}, F) = -\frac{\text{Tr}(F^T g F)}{4} + \frac{\text{Tr}(F^T \theta \bar{F})}{4} - V_{FPS}(z, \bar{z}), \tag{3.27}
\]

where the kinetic term of the scalar fields \(z^i\) has been omitted since subleading in the infinite mass limit (corresponding to \(m^I \to \infty\)). Computing the above Lagrangian on the \(N = 1\) solution to the field equations for \(z^i\) one obtains the BI Lagrangian of [11]. The limiting procedure outlined in the previous section is now no longer well defined. In particular, variation with respect to \(\Lambda\) of the Lagrangian (3.7), with \(f(\Lambda)\) given by (3.17), does not reproduce (3.13). The reason is that now the relation (3.15) can no longer be inverted to express \(g\) in terms of \(\Lambda\), so that the expression (3.17) should be intended as describing \(f(\Lambda)\) only on the solution (3.15): \(f(\Lambda(g))\). The reader can, however, verify that

\[
\frac{\partial \mathcal{L}}{\partial \Lambda} \bigg|_0 \cdot \frac{\partial \Lambda}{\partial z^i} = 0, \tag{3.28}
\]

\[
\frac{\partial \mathcal{L'}}{\partial \Sigma} \bigg|_0 \cdot \frac{\partial \Sigma}{\partial z^i} = 0, \tag{3.29}
\]

\(^3\) Including the D-term (which would break all supersymmetries), the metric \(N\) would have the general form: \(N^{MN} = 2 \lambda \sum_x P^{(M} x P^{N)} x\) and would have rank-3.
still hold, the zero-subscript meaning that the quantity is computed on the solutions $\Lambda(z, \bar{z})$, $\Sigma(z, \bar{z})$ given by (3.15), (3.12) with $g, \theta$ intended as functions of $z, \bar{z}$. Moreover we still have that:

$$L_{FPS}^{(0)}(z, \bar{z}, F) = L'(\Lambda(z, \bar{z}), \Sigma(z, \bar{z}), z, \bar{z}, F),$$  

(3.30)

Properties (3.29) and (3.30) are enough to guarantee that the field equations for $z^i$ obtained from $L_{FPS}^{(0)}(z, \bar{z}, F)$ are equivalent to those obtained from $L'$ once we write for $\Lambda$ and $\Sigma$, their values $\Lambda(z, \bar{z})$, $\Sigma(z, \bar{z})$:

$$\frac{\partial L_{FPS}^{(0)}}{\partial z^i} = \frac{\partial L'}{\partial z^i} \bigg|_0 + \frac{\partial L'}{\partial z^i} \bigg|_0 = \frac{\partial L'}{\partial z^i} \bigg|_0.$$

(3.31)

As a consequence of this, the $N = 2$ BI action obtained in [11] can also be obtained starting from $L'$ and solving the field equations for $z^i$.

The problem with the non-invertibility of eq. (3.15), when $\eta, \tilde{\eta}_0$ are singular, can be circumvented by regularizing $L'$ as follows. We define

$$L'_e \equiv L'\big|_{\eta \to \eta^e, \tilde{\eta}_0 \to \tilde{\eta}_0^e},$$

(3.32)

where $\eta^e$ and $\tilde{\eta}_0^e$ are now non-singular matrices defined as:

$$\eta^e \equiv m m^T + \epsilon \sum_{\alpha=1}^{n-1} m_\alpha m^T_\alpha, \quad m^T m_\alpha = 0, \quad m^T m_\beta = \delta_{\alpha\beta},$$

$$\tilde{\eta}_0^e \equiv e_2 e_2^T + \epsilon \sum_{\alpha=1}^{n-1} e_{2\alpha} e_{2\alpha}^T, \quad e_{2\alpha}^T e_{2\alpha} = 0, \quad e_{2\beta}^T e_{2\beta} = \delta_{\alpha\beta}.$$  

(3.33)

The field equations for $\Lambda, \Sigma$ obtained from $L'_e$ are solved by corresponding functions $\Lambda_e(z, \bar{z}), \Sigma_e(z, \bar{z})$ and $L_{FPS}^{(0)}(z, \bar{z}, F)$ is obtained in the singular limit:

$$L_{FPS}^{(0)}(z, \bar{z}, F) = \lim_{\epsilon \to 0} L'_e(\Lambda_e(z, \bar{z}), \Sigma_e(z, \bar{z}), z, \bar{z}, F).$$  

(3.34)

This formal derivation does not affect the above conclusion about the resulting BI action.

The equations of motion for the scalar fields $z^i$ from $L'$ are most conveniently written by choosing the special coordinate description of the scalar manifold ($z^i = X^I$) and read:

$$C_{IJK} \left[ -\frac{i}{2\lambda} \left( \Lambda + (\Sigma\eta^e)^T - \frac{\lambda}{2} FF \right)_{IJ} + \left( \frac{F\tilde{F}}{4} - \frac{\Sigma_{\eta}}{\lambda} \right)_{IJ} \right] = 0,$$

(3.35)

where we have defined $C_{IJK} = \partial_I \partial_J \partial_K F$, $F(X)$ being the holomorphic prepotential and $\partial_I \equiv \frac{\partial}{\partial X^I}$, and we have used the property $g_{IJ} = \text{Im}(\partial_I \partial_J F)$, $\theta_{IJ} = \text{Re}(\partial_I \partial_J F)$.

Adapting the auxiliary field description to the $N = 2$ notation.

To make contact with the $N = 2$ notation, it is useful to write $\Lambda$ and $\Sigma$ in terms of the auxiliary fields $\hat{\Phi}_1 + i \Phi_2$ defining the F-terms of the $n N = 2$ vector multiplets:

$$\Lambda^{IJ} = 2 \lambda \hat{\Phi}_1^I \hat{\Phi}_1^J - \eta^{IJ} = 2 \lambda (\hat{\Phi}_1^I \hat{\Phi}_1^J - m^I m^J), \quad \Sigma^I m^J = \Phi_2^I.$$  

(3.36)
In order to rewrite the Lagrangian in terms of these new auxiliary fields we shall first find the expression for \( f(\Lambda(\Phi_1)) \):

\[
f(\Lambda(\Phi_1)) = \text{Tr} \sqrt{\eta + \Lambda} \eta_0 = \text{Tr} \sqrt{2\lambda \Phi_1^T \Phi_1 - \frac{1}{4} F^I F^J} + 2 \lambda \Phi^I e_{2I} = 2 \lambda \Phi^I e_{2I}. \tag{3.37}
\]

The resulting Lagrangian \( \mathcal{L}''' \) in terms of \( \hat{\Phi}_1, \Phi_2, z, F \) now reads:

\[
\mathcal{L}'''(\hat{\Phi}_1, \Phi_2, z, \bar{z}, F) = g^{IJ} (\hat{\Phi}^I_1 \hat{\Phi}^J_1 - m^I m^J + \Phi^I_2 \Phi^J_2 - \frac{1}{4} F^I F^J) +
+ (\theta_{IJ} - \eta^{-1} \eta^I) \left( \frac{1}{4} F^I \tilde{F}^J - 2 \Phi^I_2 m^J \right) - 2 \Phi^I_1 e_{2I} + 2 m^I e_{2I}, \tag{3.38}
\]

where we have fixed \( C = \text{Tr} (\sqrt{\eta} \eta_0) = 2 \lambda m^I e_{2I} \). By varying the above action with respect to \( \hat{\Phi}_1 \) and \( \Phi_2 \) we find the following two equations:

\[
\hat{\Phi}^I_1 = g^{-1 IJ} e_{2J}; \quad \Phi^I_2 = g^{-1 IJ} \theta_{JK} m^K, \tag{3.39}
\]

which are just eq.s (3.12) and (3.15) expressed in terms of the new auxiliary fields. To relate \( \hat{\Phi}^I_1, \Phi^I_2 \) to the complex F-terms \( Y^I \), let us now make the following redefinition:

\[
\hat{\Phi}^I_1 = -\Phi^I_1 + m^I, \tag{3.40}
\]

and write the F-terms of the superfields as \( Y^I \propto \Phi^I_1 + i \Phi^I_2 \) (where the proportionality is intended through a real factor). We find that

\[
\frac{1}{\lambda} (C - f(\Lambda(\Phi))) = 2 m^I e_{2I} - 2 \hat{\Phi}^I_1 e_{2I} = 2 \Phi^I e_{2I}. \tag{3.41}
\]

This term combines with the following term in the Lagrangian (3.38):

\[
2 m^T \eta^{-1} \eta^I \Phi_2 = 2 (m^T \eta^{-1} m) \Phi^I_2 e_{1I} = 2 \text{Tr}(\eta^{-1} \eta) \Phi^I_2 e_{1I} = 2 \Phi^I e_{2I}, \tag{3.42}
\]

to form

\[
2 \Phi^I_2 e_{1I} + 2 \Phi^I_1 e_{2I} \propto \text{Im} \int d^2 \theta e_I Y^I, \tag{3.43}
\]

where \( e_I = e_{1I} + i e_{2I} \). This is the chiral FI term of [11].

If we vary (3.38) with respect to \( z^i = X^I \) we find eq.s (3.35) written in terms of \( \hat{\Phi}_1 \) and \( \Phi_2 \):

\[
C_{IJK} \left[ -i \left( \Phi^I_1 \Phi^J_2 + \Phi^I_2 \Phi^J_1 - 2 m^I \Phi^I_2 - \frac{1}{4} F^I F^J \right) + \left( \frac{F^I \tilde{F}^J}{4} - 2 \Phi^I_2 m^J \right) \right] = 0, \tag{3.44}
\]

which coincide with those found in [11].

Let us summarize the correspondence between the auxiliary fields \( \Lambda, \Sigma \) and \( Y^I = \Phi^I_1 + i \Phi^I_2 \):

\[
\frac{\Lambda}{2\lambda} = (\Phi_1 - m^T) (\Phi^T_1 - m^T) - mm^T, \quad \Sigma \cdot m = \Phi_2. \tag{3.45}
\]

A detailed analysis of the supersymmetric cases, in \( N = 2 \) and in the maximally extended \( N = 4 \) cases, is postponed to a forthcoming publication.
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