Perpetual integral functionals of multidimensional stochastic processes

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\textbf{ABSTRACT}

The paper is devoted to the existence of perpetual integral functionals

\[ \int_0^\infty f(X(t)) \, dt \]

for several classes of \( d \)-dimensional of stochastic processes \( X(t) \). The method is very simple: we establish the conditions supplying that these functionals have a finite expectation. Examples of these classes include \( d \)-dimensional fractional Brownian motion having coordinates with the same Hurst index \( H \), for which existence is established under the assumption \( d > 1/H \). In particular, perpetual integral functionals exist for \( d \)-dimensional Brownian motion with \( d > 2 \), compound Poisson process, Markov processes admitting densities of transitional probabilities. In the case of Brownian motion and fractional Brownian motion we establish that the perpetual integral functionals are not a constant a.s. if \( f \neq 0 \).

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\textbf{1. Introduction}

Let \( X = \{X(t), t \geq 0\} \) be a \( d \)-dimensional stochastic process with càdlàg trajectories, and let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a continuous function. Then for any \( T > 0 \) the integral functional

\[ \int_0^T f(X(t)) \, dt \]  

is defined. However, its properties and asymptotic behaviour as \( T \rightarrow \infty \) depend crucially on the properties of the process \( X \) and the dimension \( d \). When \( X = B \) is the one-dimensional Brownian motion (Bm for short), then the integral functional (1) with \( f \in L_1(\mathbb{R}) \) yields

\[ \int_0^T f(B(t)) \, dt = \int_\mathbb{R} f(x) L_T(x) \, dx, \]
where $L_T(x)$ is the local time of Bm up to moment $T$ at the point $x$. For the definition and properties of local time of one-dimensional Bm, see, e.g. [4,5,13,14]. Now, the asymptotic behaviour of the integral functional $\int_0^T f(X(t)) \, dt$ is very different even for one-dimensional Markov processes and depends on their transient or recurrent properties. For example, one-dimensional Bm $B$ is recurrent, therefore $L_\infty(x) = \infty$ for all $x \in \mathbb{R}$, consequently, the integral functional $\int_0^\infty f(B(t)) \, dt$, roughly speaking, does not exists. Contrary to this situation, 1-dimensional Bm with positive drift, $B_\mu_t = B(t) + \mu t$, $\mu > 0$ is transient; therefore, the perpetual integral functional $\int_0^\infty f(B_\mu^t) \, dt$ is finite for any nonnegative locally integrable function $f$, integrable at infinity, see [12]. For one-dimensional transient diffusions [12] derived a criterion in terms of an integral testing for the convergence of the perpetual integral functionals. In particular, the case of transient Bessel process yields a simple condition for the finiteness of (1). For the class of Lévy processes we can find necessary and sufficient conditions given in terms of the canonical characteristics of the Lévy process for the a.s. finiteness of the integral functionals, see [7] and also [2]. The asymptotic behaviour of the integral functional $\int_0^T f(X(t)) \, dt$ and respective normalization was established in [11].

Now, consider a $d$-dimensional stochastic processes $X(t)$, $t \geq 0$ with $d \geq 2$. As the dimension $d$ grows, the situation changes. In some sense, when the dimension increases, the convergence of perpetual functional becomes more realistic. In particular, for $d > 2$, $d$-dimensional Bm becomes transient, and it leads to the existence of the perpetual integral functional $\int_0^\infty f(B(t)) \, dt$ for a sufficiently large class of functions $f$. However, it is reasonable to consider wider classes of stochastic processes for which the perpetual integral functional $\int_0^\infty f(X(t)) \, dt$ does exist a.s. To this end, a natural and simple sufficient condition for the a.s. existence is to have a finite expectation, see Theorems 2.2, 2.4 and 2.5. Being rather simple, this method permits us to get some new and elegant results concerning existence of the perpetual functionals. In addition, using the conditional full support property of the associated probability measures we show that the integral functionals are not a constant a.s. if $f \neq 0$.

Coming back the class of functions $f$ such that (1) exists a.s. a natural class consists of bounded continuous and integrable functions. The expectation

$$u_f(x) = \mathbb{E}^x \left[ \int_0^\infty f(X(t)) \, dt \right]$$

is well known as the potential of the function $f$, see, e.g. [3]. The question concerning the description of admissible functions for a given process for which the potential exists is rather open. A bit more simple situation we have in the case of Markov processes, the study of which, in fact, was the beginning of our approach. If $L$ is the generator of a Markov process $X(t)$, $t \geq 0$, then $u_f$ is the solution of the following equation:

$$-Lu = f.$$ 

As in the classical PDE theory, we would like to write this solution in the form

$$u_f(x) = \int_{\mathbb{R}^d} f(y) \mu(x, dy),$$

where $\mu(x, dy)$ is the fundamental solution (measure) corresponding to the operator $L$. In the simplest cases, as the Laplace operator $L = \Delta$, it is the Green function for $\Delta$ and we
will call $\mu(x, dy)$ the Green measure for the process $[10]$. Of course, the notion of a Green measure in the integral representation for the potential may be introduced without any Markov property. We would like to stress that the existence and properties of Green measures are highly depending of the class of processes under consideration. From this point of view, the perpetual integral functionals may be called random potentials, although the concept of potential in stochastic is used for another object. Furthermore, the existence of the perpetual integral functional immediately implies the existence of occupation measure of the stochastic process $X$ on $[0, \infty)$ that is defined as

$$
\mu_X(A, \omega) = \lambda(X^{-1}(A)) = \lambda\left(t \in [0, \infty) : X(t) \in A\right),
$$

where $\lambda$ is the Lebesgue measure on $[0, \infty)$ and $A \subset \mathbb{R}^d$ is a bounded Borel set. Hence the formula of change of measure leads to the following representation:

$$
\left(\int_0^\infty f(X(t)) \, dt\right)(\omega) = \int_{\mathbb{R}^d} f(y) \mu_X(dy, \omega), \quad \text{for a.a. } \omega.
$$

If we follow the ‘Green terminology’, occupation measure can be called random Green measure. Generally speaking, the existence of occupation measure does not mean that the integral $\int_0^\infty f(X(t)) \, dt$ is a non-degenerate random variable, i.e. is not a constant. However, we establish that for Bm and fractional Brownian motion (fBm) the corresponding perpetual integral functionals are not constant whenever it exist and the function $f$ is not identical zero, see Theorems 2.4 and 2.2. The proof is based on the property of conditional full support for the distribution of these processes.

The next Section 2 contains three examples of multidimensional processes for which the perpetual integral functionals are finite a.s. in a proper Banach space of functions $f$. The first example is a fractional Brownian motion with arbitrary Hurst index $H \in (0, 1)$. In this case, the dimension of the process should exceed $1/H$. So, for $H < 1/2$ the ‘allowed’ dimension cannot be less than 3. Moreover, it tends to infinity as $H \to 0$. It does not mean that the perpetual functional does not exist because we only consider its expectation, however, it can be understood in the sense that convergence of functional is established the easier, the greater the variance of the process at infinity. However, for $H > 1/2$ the ‘allowed’ dimension can be 2 (and higher, of course). A particular case is $H = 1/2$, where we have a standard Bm, a process with independent increments. In this case, ‘allowed’ dimensions are $d > 2$. Moreover, in this case we can calculate the variance of the perpetual functional. Another particular case is $H = 1$, where we have degenerate process, and we can establish by direct calculations that the perpetual integral functionals are finite a.s. for $d \geq 1$ but the expectation does not exist. The second example is a compound Poisson process. We formulate the conditions of the existence of the respective perpetual functional. Obviously, it is a process with independent increments, and this fact allows us to calculate the variance of the perpetual functional. So, these examples present both Markov (with independent increments) and non-Markov processes. We conclude with a class of Markov processes without restriction on the dependence of their increments, for which our perpetual functionals exist.
2. Examples of $d$-dimensional processes admitting perpetual integral functionals

Let $X = \{X(t), t \geq 0\}$ be a $d$-dimensional stochastic process and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function. Without loss of generality, we assume throughout the paper that the function $f$ is non-negative. Our aim is to provide several examples of stochastic processes satisfying the relation

$$\int_0^\infty \mathbb{E}[f(X(t))] \, dt < \infty. \quad (A)$$

We can note the following: under assumption (A), according to the standard Fubini theorem, the perpetual integral functional (1) exists with probability 1. Moreover, if in addition $|X(t)| \rightarrow \infty$, $t \rightarrow \infty$, a.s., then the occupation measure $\mu_X$ defined in (2) is a.s. locally finite, so it is a Radon measure as locally finite Borel measure on a Polish space. We shall not use this fact in what follows, however, it is interesting from the analytic point of view. Now our goal is to consider some examples of processes $X$ and the natural class of functions $f$ providing the existence of the expectation (A). Introduce the following class of functions

$$\text{CL}(\mathbb{R}^d) := \{ f : \mathbb{R}^d \rightarrow \mathbb{R} : f \text{ is continuous, bounded and belongs to } L_1(\mathbb{R}^d) \}.$$

It is a Banach space with the norm $\|f\|_{\text{CL}} := \sup |f| + \|f\|_{L_1(\mathbb{R}^d)}$, and it will be our basic class of functions.

2.1. Fractional Brownian motion

Consider a $d$-dimensional fractional Brownian Motion with Hurst parameter $H \in (0, 1)$, denoted by $B^H(t) = (B^H_1(t), \ldots, B^H_d(t))$, where all coordinates $B^H_i$ are independent one-dimensional fractional Brownian motions with the same index $H$, i.e. Gaussian processes with zero mean and covariance function

$$\mathbb{E}[B^H_i(t)B^H_i(s)] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).$$

We are interested in the conditions under which the perpetual integral functional

$$Y(f) = \int_0^\infty f(x + B^H(t)) \, dt$$

exists for nonnegative functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$, $f \in \text{CL}(\mathbb{R}^d)$.

The proof of the next theorem uses the notion of conditional full support property of a stochastic process. In order to make the paper more readable and self-contained we recall its definition, see [6,9]. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions as well as $\mathcal{F}_\infty = \mathcal{F}$ and $\mathcal{F}_0$ is trivial. The set of $\mathbb{R}^d$-valued continuous functions on $[s, t]$ is denoted by $\mathbb{C}([s, t])$, and its subfamily of functions $f$ starting at $f(s) = x \in \mathbb{R}^d$ by $\mathbb{C}_x([s, t])$. The support of a $\mathbb{R}^d$-valued continuous process $X$ on $[s, t]$ is the smallest closed subset $A \subset \mathbb{C}([s, t])$ such that $\mathbb{P}(X \in A) = 1$. We say that a $\mathbb{R}^d$-valued process $X$ has full support if its support is the whole of $\mathbb{C}_x([s, t])$, where $X(0) = x$. 
Definition 2.1: A continuous $\mathbb{R}^d$-valued stochastic process $X(t)$, $t \geq 0$ satisfies the conditional full support property if, for all $t \geq 0$

$$\text{supp} \mathbb{P}(X(u), t \leq u < \infty | \mathcal{F}_t) = \mathbb{C}(X(t), (t, \infty)),$$

where $\mathbb{P}(X(u), t \leq u < \infty | \mathcal{F}_t)$ denotes the $\mathcal{F}_t$-conditional distribution of the $\mathbb{C}((t, \infty))$-valued process $X(u), t \leq u < \infty$ and supp denotes the support.

The paper [9] provides several examples of processes satisfying the conditional full support, in particular fBm.

Theorem 2.2: Let $d > 1/H$ and $x \in \mathbb{R}^d$ be given. Then for any $f \in \text{CL}(\mathbb{R}^d)$ the perpetual integral functional $\int_0^\infty f(x + B^H(t)) \, dt$ exists a.s. its expectation equals

$$\mathbb{E} \left[ \int_0^\infty f(x + B^H(t)) \, dt \right] = C_{d,H} \int_{\mathbb{R}^d} \frac{f(x + y)}{|y|^{d-1/H}} \, dy,$$

where

$$C_{d,H} = 2^{-(1+1/(2H))} H^{-1} \pi^{-d/2} \Gamma \left( \frac{d}{2} - \frac{1}{2H} \right).$$

For $f \neq 0$, $\int_0^\infty f(x + B^H(t)) \, dt$ is a non-constant random variable.

Proof: Concerning the expectation, for non-negative $f$ we can apply Fubini theorem and obtain

$$\mathbb{E} \left[ \int_0^\infty f(x + B^H(t)) \, dt \right] = \int_0^\infty \int_{\mathbb{R}^d} f(x + y) \left( 2\pi t^{2H} \right)^{-d/2} \exp \left\{ - \frac{|y|^2}{2t^{2H}} \right\} \, dy \, dt$$

$$= \int_{\mathbb{R}^d} f(x + y) \left( \int_0^\infty \left( 2\pi t^{2H} \right)^{-d/2} \exp \left\{ - \frac{|y|^2}{2t^{2H}} \right\} \, dt \right) \, dx.$$

Now, the inner integral may be explicitly computed as

$$\int_0^\infty \left( 2\pi t^{2H} \right)^{-d/2} \exp \left\{ - \frac{|y|^2}{2t^{2H}} \right\} \, dt = \frac{\Gamma \left( \frac{d}{2} - \frac{1}{2H} \right)}{H^{1/2H + 1/2} \pi^{d/2} |y|^{d-1/H}},$$

which gives (3). The integral on the right-hand side of (3) is finite. In fact, we may use the local integrability of $|y|^{1/H}$ in $y$ and conclude that

$$\int_{\mathbb{R}^d} \frac{f(x + y)}{|y|^{d-1/H}} \, dy \leq \int_{|y| \leq 1} \frac{f(x + y)}{|y|^{d-1/H}} \, dy + \int_{|y| > 1} \frac{f(x + y)}{|y|^{d-1/H}} \, dy$$

$$\leq C_1 ||f||_\infty + C_2 ||f||_1 \leq C ||f||_{CL},$$

so, the integral in (3) is indeed correctly defined. It means that $\int_0^\infty f(x + B^H(t)) \, dt$ exists with probability 1.

The most interesting part of the proof is to establish that for $f \neq 0$, $\int_0^\infty f(x + B^H(t)) \, dt$ is a non-constant random variable. Without loss of generality assume that $x = 0$. Let, for a
given \( t_0 > 0, \mathbb{P}_{t_0}(u), u \in \mathbb{C}([t_0, \infty)) \) be the regular conditional distribution of \( \{B^H(t), t \in [0, t_0]\} \) given the condition \( \{(B^H(t), t \geq t_0) = u\} \). We are going to show that, that for a.a. \( u \in \mathbb{C}([t_0, \infty)) \), \( \mathbb{P}_{t_0}(u) \) satisfies the condition

\[
\text{supp} \mathbb{P}_{t_0}(u) = \{y \in \mathbb{C}([0, t_0]) : u(t_0) = y(t_0)\}.
\]  

To this end, define the transformation \( R(z)(t) = t^{2H}z(1/t) \). It is easy to see that \( R^2(z) \) is the identical transformation, that is, for any \( z \in \mathbb{C}((0, \infty)) \) we have \( R^2(z) = z \), and \( \{R(B^H)(t), t \geq 0\} \) has the same distribution as \( \{B^H(t), t \geq 0\} \). Then for any \( t_0 > 0 \) the distribution of \( \{B^H(t), t \in (0, t_0]\} \) given \( \{B^H(t), t \geq t_0\} = u \in \mathbb{C}([t_0, \infty)) \) is the same as the image under \( R \) of the conditional distribution of \( \{B^H(t), t \geq 1/t_0\} \) given \( \{B^H(t), 0 < t \leq 1/t_0\} = R(u) \). The latter distribution has full support. For \( H \neq 1/2 \) this follows from the results of the paper \cite{6}. For \( H = 1/2 \), thanks to the independence of increments of \( B^{1/2} \), the conditional distribution is the same as the unconditional distribution of standard Wiener process, which is well known to have full support.

Now, assume the contrary, that is, there is a constant \( c > 0 \) and a function \( f \in CL(\mathbb{R}^d) \) such that

\[
A := \int_0^\infty f(B^H(t)) \, dt = c, \quad \text{a.s.}
\]

Since \( A = \int_0^{t_0} f(B^H(t)) \, dt + \int_{t_0}^\infty f(B^H(t)) \, dt \), we can write that

\[
0 = \mathbb{P}(A \neq c) = \int_{\mathbb{C}([t_0, \infty))} \mathbb{P}_{t_0} \left( \left\{ \int_0^{t_0} f(B^H(t)) \, dt + \int_{t_0}^\infty f(B^H(t)) \, dt \neq c \right\}, u \right) \mathbb{P}(du),
\]

and conclude that

\[
\mathbb{P}_{t_0} \left( \left\{ \int_0^{t_0} f(B^H(t)) \, dt + \int_{t_0}^\infty f(B^H(t)) \, dt \neq c \right\}, u \right) = 0
\]

for a.a. \( u \in \mathbb{C}([t_0, \infty)) \). Now, from (6) combined with the continuity of the map \( z \mapsto \int_0^{t_0} f(z(t)) \, dt \) in the sup-norm, it follows that

\[
\int_0^{t_0} f(v(t)) \, dt + \int_{t_0}^\infty f(u(t)) \, dt = c
\]

for any \( v \in \mathbb{C}([0, t_0]) \) such that \( v(0) = 0, v(t_0) = u(t_0) \) and for a.a. \( u \in \mathbb{C}([t_0, \infty]) \). Define the family of functions \( z_\varepsilon, t_0 > \varepsilon > 0, a \in \mathbb{R}^d \) by

\[
z_\varepsilon(t) :=
\begin{cases}
\frac{at}{\varepsilon}, & t \in [0, \varepsilon], \\
\frac{a}{\varepsilon}, & t \in (\varepsilon, t_0 - \varepsilon), \\
\frac{u(t_0)(t-t_0+\varepsilon)+a(t_0-t)}{\varepsilon}, & t \in [t_0 - \varepsilon, t_0]
\end{cases}
\]

such that \( z_\varepsilon \in \mathbb{C}([0, t_0]), z_\varepsilon(0) = 0 \) and \( z_\varepsilon(t_0) = u(t_0) \). Hence, (7) with \( z_\varepsilon \) replacing \( v \) yields

\[
\int_0^{t_0} f(z_\varepsilon(t)) \, dt + \int_{t_0}^\infty f(u(t)) \, dt = c,
\]
and making $\epsilon \to 0$, we conclude that
\[ t_0 f(a) + \int_{t_0}^{\infty} f(u(t)) \, dt = c. \]
Thus
\[ f(a) = \frac{c - \int_{t_0}^{\infty} f(u(t)) \, dt}{t_0}, \]
which means that $f$ is a constant, consequently, a zero function.

**Remark 2.3:**
(i) Let $H \in (1/2, 1)$. Then we can take $d = 2$ (and the greater values of $d$, of course). Consider the case $H = 1$. In this case $B^1(t) = \xi t$, where $\xi$ is a standard Gaussian variable. Taking $d = 1$, we conclude that
\[ \int_{0}^{\infty} f(x + \xi t) \, dt = |\xi|^{-1} \int_{0}^{\infty} f(x + y) \, dy. \]
So, the perpetual functional exists a.s., but its expectation is infinite.
(ii) One can see that the existence of the expectation $E[\int_{0}^{\infty} f(x + B^H(t)) \, dt]$ is determined only by the one-dimensional distributions of the process $X$. Moreover, if $X$ is a process with the same one-dimensional distributions as the fBm, we can obtain the same conclusions concerning the existence of the expectation (but of course, not about its variance or non-constant property).

In the particular case $H = 1/2$ fBm is a standard Bm. Denote it simply $B$. Of course, Theorem 2.2 is valid in this case, but in addition we may also compute the variance of the perpetual integral. For the reader’s convenience, we formulate the result in a whole.

**Theorem 2.4:** Let $d > 2$. For any $f \in C(L(\mathbb{R}^d))$ the perpetual integral functional $\int_{0}^{\infty} f(x + B(t)) \, dt$ exists a.s. its expectation equals
\[ E \left[ \int_{0}^{\infty} f(x + B(t)) \, dt \right] = \frac{2^{d/2 - 1} \Gamma(d/2 - 1)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x + y) \, dy, \]
whereas the variance equals
\[ V(f) := E \left[ \left( \int_{0}^{\infty} f(x + B(t)) \, dt - E \left[ \int_{0}^{\infty} f(x + B(t)) \, dt \right] \right)^2 \right] \]
\[ = \frac{2^{d-2} \Gamma^2(d/2 - 1)}{(2\pi)^d} \left( 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y)f(x + y + z) \, dy \, dz - \left( \int_{\mathbb{R}^d} f(x + y) \, dy \right)^2 \right). \]
For $f \neq 0$, $\int_{0}^{\infty} f(x + B^H(t)) \, dt$ is a non-constant random variable, consequently for $f \neq 0$ the right-hand side of (9) is strictly positive.

**Proof:** The calculation of the expectation and the proof of non-constant property of the perpetual functional are the same as for the general case in Theorem 2.2. Concerning the
variance, let $0 < s < u$. Taking into account that $B$ has independent increments, so $B(s)$ and $B(u) - B(s)$ are independent, we claim that the following equality holds:

$$E[f(x + B(u))f(x + B(s))] = E[f(x + B(u) - B(s) + B(s))f(x + B(s))]$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y)f(x + y + z) \frac{\exp\left\{-\frac{|y|^2}{2s}\right\}}{(2\pi s)^{d/2}} \frac{\exp\left\{-\frac{|z|^2}{2(u-s)}\right\}}{(2\pi (u-s))^{d/2}} \, dy \, dz. \quad (10)$$

Therefore we immediately get that

$$E\left[\left(\int_0^\infty f(x + B(t)) \, dt\right)^2\right]$$

$$= \int_0^\infty \int_0^\infty E[f(x + B(u))f(x + B(s))] \, du \, ds$$

$$= 2 \int_0^\infty \int_s^\infty E[f(x + B(u))f(x + B(s))] \, du \, ds$$

$$= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y)f(x + y + z) \int_0^\infty \int_s^\infty \frac{\exp\left\{-\frac{|y|^2}{2s}\right\}}{(2\pi s)^{d/2}} \frac{\exp\left\{-\frac{|z|^2}{2(u-s)}\right\}}{(2\pi (u-s))^{d/2}} \, du \, ds \, dy \, dz$$

$$= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y)f(x + y + z) \int_0^\infty \int_0^\infty \frac{\exp\left\{-\frac{|y|^2}{2s}\right\}}{(2\pi s)^{d/2}} \frac{\exp\left\{-\frac{|z|^2}{2u}\right\}}{(2\pi u)^{d/2}} \, du \, ds \, dy \, dz. \quad (11)$$

Now, (9) follows from (11) combined with the following simple observation: for non-negative function $f \in CL(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x + y)f(x + y + z)}{|y|^{d-2}|z|^{d-2}} \, dy \, dz \leq (C_1 \|f\|_\infty + C_2 \|f\|_1)^2 \leq (C \|f\|_{CL})^2.$$

\[\square\]

### 2.2. Compound Poisson process

Let $d \geq 3$, $\xi_k$, $k \geq 1$ be a sequence of iid $d$-dimensional random variables, and $N = \{N(t), t \geq 0\}$ a homogeneous Poisson process with intensity $\lambda > 0$, independent of $\xi_k$, $k \geq 1$. Denote $X = \{X(t), t \geq 0\}$ the compound Poisson process in $\mathbb{R}^d$ starting from zero, i.e.

$$X(t) = \sum_{k=1}^{N(t)} \xi_k,$$

where we use the convention that $\sum_{k=1}^0 \xi_k = 0$. The process $X(t)$, $t \geq 0$ has independent increments consequently a Markov process, and if we assume that any $\xi_k$ has probability
density $a$ and that $\lambda = 1$, then its generator is defined on $CL(\mathbb{R}^d)$ by

$$Lf(x) = \int_{\mathbb{R}^d} a(x - y) [f(y) - f(x)] \, dy.$$ 

We are interested on the conditions under which the perpetual integral functional $Y(f) = \int_0^\infty f(X(t)) \, dt$ does exist for non-negative $f \in CL(\mathbb{R}^d)$. In this connection, our goal now is to check condition (A). Denote $a_k(x) = a^{*k}(x)$ the $k$-fold convolution of the density $a$ and let

$$G_0(x) = \sum_{k=1}^{\infty} a_k(x),$$

provided that this series converges for any $x \in \mathbb{R}^d$.

**Theorem 2.5:** Assume that any $\xi_k$ has probability density $a$ and $\lambda = 1$. Assume also that $G_0(x)$ is integrable in some ball $B(0, R)$ and bounded outside this ball. Then for any $f \in CL(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$ the perpetual integral functional $\int_0^\infty f(x + X(t)) \, dt$ exists a.s. its expectation equals

$$E\left[\int_0^\infty f(x + X(t)) \, dt\right] = f(x) + \int_{\mathbb{R}^d} f(x + y) G_0(y) \, dy,$$  

(12)

whereas variance equals

$$V(f) := E\left[\left(\int_0^\infty f(x + X(t)) \, dt - E\left[\int_0^\infty f(x + X(t)) \, dt\right]\right)^2\right]$$

$$= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+y)f(x+y+z)G_0(y)G_0(z) \, dy \, dz - \left(\int_{\mathbb{R}^d} f(x+y)G_0(y) \, dy\right)^2.$$  

(13)

**Proof:** It follows from the independence of $\xi$ and $N$ that

$$E[f(x + X(t))] = \sum_{n=0}^{\infty} E\left[f\left(x + \sum_{k=1}^{n} \xi_k\right)\right] P(N(t) = n)$$

$$= f(x) e^{-t} + \sum_{n=1}^{\infty} e^{-t} \frac{t^n}{n!} \int_{\mathbb{R}^d} f(x+y)a_n(y) \, dy.$$ 

Hence, formally

$$E\left[\int_0^\infty f(x + X(t)) \, dt\right] = f(x) + \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} f(x+y)a_n(y) \, dy \int_0^\infty e^{-t} \frac{t^n}{n!} \, dt$$

$$= f(x) + \int_{\mathbb{R}^d} f(x+y)G_0(y) \, dy,$$

and under the assumption that $G_0(y)$ is integrable in some ball $B(0, R)$ and bounded outside this ball, we obtain (similarly to (5)) the existence of the expectation. Further calculations are also similar to the respective calculations in the proof of Theorem 2.4. ■
Let us provide one simple sufficient condition for the boundedness and integrability of $G_0$. Taking into account the condition in Theorem 2.5, it will mean that for such $\xi$ the perpetual functional exists.

**Lemma 2.6:** Let $d \geq 3$ and $a(x) = a(-x)$ be a symmetric jump kernel with second moment,

$$\int_{\mathbb{R}^d} |x|^2 a(x) \, dx < \infty,$$

and its Fourier transform $\hat{a}$ is integrable. Then $G_0(x)$ exists for any $x \in \mathbb{R}^d$, is integrable and bounded.

**Proof:** Consider the Fourier image of the jump kernel $\hat{a}(k) = \int_{\mathbb{R}^d} e^{-ik\cdot y} a(y) \, dy$. Then $\hat{a}(0) = 1, \; |\hat{a}(k)| < 1, k \neq 0, \; \hat{a}(k)$ is real-valued and $\hat{a}(k) \to 0, k \to \infty$. Furthermore, the Fourier transform of any finite sum $\sum_{k=1}^n a_k(x), \; n \in \mathbb{N}$ equals

$$\hat{a}(k) \frac{1 - \hat{a}^n(k)}{1 - \hat{a}(k)}.$$

Keeping that in mind, and remembering that $\hat{a}^n$ is bounded, in order to establish that the Fourier representation for $G_0$ has the form

$$G_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{a}(k) e^{i(k\cdot x)} \, dk,$$

it is sufficient to prove that the function $\frac{\hat{a}(k)}{1 - \hat{a}(k)}$ is integrable. It is the case if and only if $d \geq 3$. Indeed, only in the case $d \geq 3$ the existence of the second moment of the jump kernel $a$ implies the non-degeneracy of the covariance matrix, consequently elliptic in the sense that at zero $|1 - \hat{a}(k)| \geq C|k|^2$ for some $C > 0$. Combined with the boundedness of $\hat{a}$ it implies integrability of $\frac{\hat{a}(k)}{1 - \hat{a}(k)}$ in some ball around zero. Integrability outside this ball follows from integrability of $\hat{a}$. \boxed{}

There are several particular classes of jump kernels for which Lemma 2.6 holds, see [10] for details.

**Example 2.7 (Gauss kernels):** Assume that the jump kernels has the following form:

$$a(x) = C_1 \exp \left\{ -\frac{b|x|^2}{2} \right\}.$$

Then $\hat{a}(k) = C_1 \exp\{-\frac{|k|^2}{2b}\}$, therefore all conditions of Lemma 2.6 hold.

**Example 2.8 (Exponential tails):** Assume that

$$a(x) \leq C \exp(-\delta|x|). \quad (14)$$

It is proved in [10] that for the kernel (14) and $d \geq 3$ it holds that

$$G_0(x) \leq A \exp(-B|x|)$$

with certain $A, B > 0$. 

2.3. A class of Markov processes supplying existence of perpetual functionals

In the previous sections, we considered fBm that is a non-Markov process except the case $H = 1/2$, and processes with independent increments that obviously are Markov. Let us conclude the list of examples with a class of Markov processes without assuming the independence of their increments. So, let $X(t), t \geq 0$ be a Markov process in $\mathbb{R}^d$ starting from the point $x \in \mathbb{R}^d$. A standard way to define a homogeneous Markov process is to give the probability $P_t(x, B)$ of the transition from the point $x \in \mathbb{R}^d$ to the Borel set $B \subset \mathbb{R}^d$ in time $t > 0$. In some cases, we have

$$P_t(x, B) = \int_B p_t(x, y) \, dy,$$

where $p_t(x, y)$ is the density of the transition probability. In any case, formally applying Fubini theorem, we obtain

$$\mathbb{E} \left[ \int_0^\infty f(X(t)) \, dt \right] = \int_0^\infty \mathbb{E} \left[ f(X(t)) \right] \, dt = \int_0^\infty (T_t f)(x) \, dt = \int_0^\infty \int_{\mathbb{R}^d} f(y) P_t(x, dy) \, dt,$$

where $G(x, A) = \int_0^\infty P_t(x, A) \, dt$ is the Green measure of the process $X$, see [10]. Of course, the Green measure does not have to exist, and moreover, we see that its existence implies the existence of perpetual integral functional $\int_0^\infty f(X(t)) \, dt$ in the case under consideration. Let the Green measure exist. If in addition the density of the transition probability exists, then for a.a. $y$ there exists the Green function

$$g(x, y) = \int_0^\infty p_t(x, y) \, dt,$$

and in this case we can write

$$\mathbb{E} \left[ \int_0^\infty f(X(t)) \, dt \right] = \int_{\mathbb{R}^d} f(y) g(x, y) \, dy. \quad (15)$$

**Remark 2.9:** In particular, we may consider Markov processes with uniformly elliptic generator. For this class of processes, it was shown in [1] and [8] that the density of transition probability admits two-sided bounds of Gaussian type, therefore for $f \in CL(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f(y) g(x, y) \, dy$ is also well defined and the perpetual integral functional is finite a.s.

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