Solving a two-dimensional problem of heat wave initiation by a boundary condition specified on a movable manifold

A L Kazakov, L F Spevak and O A Nefedova
IES UB RAS, 34 Komsomolskaya St., Ekaterinburg, 620049, Russia
E-mail: a_kazakov@mail.ru; lfs@imach.uran.ru; nefedova@imach.uran.ru

Abstract. A boundary value problem for a nonlinear degenerate parabolic equation is considered for the case of two spatial variables. The boundary condition is specified on a movable manifold - a time-varying closed line. A heat wave problem solution is constructed on the basis of the boundary element method. The numerical solution algorithm is based on the change of variables, when the roles of the required function and the radial coordinate of the polar system interchange. The BEM solution of the problem obtained in new symbols enables the zero front of the heat wave to be determined. The further solution of the problem with the known zero front yields a solution to the initial problem. The algorithm is implemented as a program. An exact (generalized self-similar) solution is constructed, whereby the program is tested.

1. Introduction
A second-order nonlinear parabolic equation is considered, which describes the process of heat propagation in the case of the power dependence of the heat conduction coefficient on temperature [1] and is sometimes also referred to as the porous medium equation [2],

\[ T_t = \text{div} (K(T)\nabla T), \quad K(T) = \alpha T^\sigma. \tag{1} \]

Here, \( T \) is the required function; \( t \) is time; \( K(T) \) is the heat conduction coefficient; \( \nabla, \text{div} \) are the gradient and divergence with respect to the spatial coordinates; \( \alpha > 0, \sigma > 0 \) are constants. Besides modeling heat conduction [1], Eq. (1) is used for describing filtration of an ideal polytropic gas in a porous soil [2], migration of biological populations [3], etc.

One of the most interesting types of solutions to Eq. (1) is heat waves propagating on a zero background at a finite velocity. Note that, generally speaking, this is atypical for parabolic equations. In fact, the heat wave represents two, rather than one, solutions of Eq. (1), namely, perturbed nonnegative and trivial, mated along some manifold referred to as the wave front on which the required function remains continuous and the derivatives may have discontinuities. The first examples of heat waves in the nonlinear case can be found in the studies by Zeldovich (cf. e.g. [4]). Among the publications studying solutions of this type for Eq. (1) and its generalizations, note those of Barenblatt et al. [5], Samarsky et al. [1], Sidorov [6], Kudryashov [7], Antontsev and Shmarev [8].
This paper continues the cycle of our studies on approximate construction of heat wave solutions for Eq. (1) with the application of the boundary element method (BEM), which reduces the problem dimensionality. We previously discussed mainly one-dimensional problems, when the required function depends on time and one spatial variable [9, 10]. Lately, our team makes effort to extend the developed approaches to the case of two spatial variables [11]. In this study, the results are extended to the case that the boundary conditions generating the heat wave are specified on a movable manifold.

2. Boundary problem statement

The standard substitution $u = T^σ$, $t = αt$ reduces Eq. (1) to the form

$$u_t = uΔu + \frac{1}{σ}(∇u)^2. \quad (2)$$

In the two-dimensional case, Eq. (2) is written as

$$u_t = u(u_{x1}x_1 + u_{x2}x_2) + \frac{1}{σ}(u_{x1}^2 + u_{x2}^2)$$

in Cartesian coordinates and

$$u_t = uu_ρρ + \frac{1}{σ}u_ρ^2 + \frac{1}{ρ}uu_ρ + \frac{1}{ρ^2} \left( \frac{u_ρ^2}{σ} + uu_ϕϕ \right)$$

in polar ones. Here, $x_1 = ρcos ϕ$, $x_2 = ρsin ϕ$.

In terms of numerical solution on a finite time interval, the simplest problem for the equation under study is that with a specified zero heat wave front. In this case, the boundary condition has the form

$$u|_{ρ=b(t,ϕ)} = 0. \quad (4)$$

Here, the equation $ρ = b(t, ϕ)$ at each moment of time determines the zero heat wave front $S(t)$, which is a closed smooth line bounding the domain $V(t)$ containing the origin. Assume that $V(t_1) \subset V(t_2)$ if $t_1 < t_2$. The problem (3), (4) consists in the determination of the function $u = u(t, ρ, ϕ)$ in the domain $t \in [0, t_∗]$, $x(ρ, ϕ) \in Ω(t)$, where $Ω(t)$ is a spatial domain bounded by $S^{(0)}$ and $S(t)$. The BEM time-step solution of this problem is discussed in [11].

For Eq. (3), consider the problem of heat wave initiation with the boundary condition

$$u|_{ρ=a(t,ϕ)} = f(t, ϕ). \quad (5)$$

Note that $a(0, ϕ) = b(0, ϕ)$, the function $b(t, ϕ)$ being unknown. The existence of a piecewise analytical heat wave solution to problem (3), (5) was proved in [12]; however, no efficient procedure of solution construction was proposed in that paper. The solution of problem (3), (5) is again sought in the domain $[0, t_∗] × Ω(t)$ . Since the domain $Ω(t)$ at each step is unknown in problem (3), (5), the approach proposed in [11] for problem (3), (4), is inapplicable to solving problem (3), (5). In this connection, we propose an original algorithm for solving problem (3), (5), which implies a special change of variables (analogous to hodograph transformation).

3. Numerical solution algorithm

Let us interchange the required function $u$ and the coordinate $ρ$ in problem (3), (5) and then make the substitution $u = ν − 1$:

$$ρ_νρ_ν^2 = (ν − 1) \left[ ρ_ϕ^2 - ρ_ϕ^2 - \frac{ρ_ν(ρ_νρ_ϕ - ρ_ϕρ_ν)}{ρ^2} \right] - \frac{1}{σ} \left( ρ_ν + \frac{ρ_νρ_ϕ^2}{ρ^2} \right). \quad (6)$$
This substitution in similar one-dimensional cases was used by Sidorov in [6] (cf. also [13]).

Hereinafter, we consider the variables \( \nu \) and \( \varphi \) as polar coordinates in the plane of the Cartesian coordinates \( \xi, \eta \): \( \xi = \nu \cos \varphi, \eta = \nu \sin \varphi \). Let us write Eq. (6) as

\[
\rho_t \rho^2 = (\nu - 1) \left[ \Delta \rho - \frac{\rho_\nu}{\nu} - \frac{\rho_{\varphi \rho}}{\nu^2} - \frac{\rho_\nu (\rho_{\varphi \rho} - \rho_{\varphi \rho} \rho_\nu)}{\rho^2} \right] - \frac{1}{\sigma} \left( \rho_\nu + \frac{\rho_\nu \rho^2}{\rho^2} \right)
\]

and represent \( \Delta \rho \) as

\[
\Delta \rho = \frac{1}{\nu - 1} \left[ \rho_t \rho^2 + \frac{1}{\nu} \left( \rho_\nu + \frac{\rho_{\varphi \rho}^2}{\rho^2} \right) \right] + \frac{\rho_\nu}{\nu} + \frac{\rho_{\varphi \rho}}{\nu^2} + \frac{\rho_\nu ^2}{\rho} + \frac{\rho_\nu (\rho_{\varphi \rho} - \rho_{\varphi \rho} \rho_\nu)}{\rho^2}.
\]

The boundary condition Eq. (5) is written as

\[
\rho|_{\nu=1+f(t, \varphi)} = a(t, \varphi).
\]

At each time moment, in the plane of the coordinates \( \xi, \eta \) the condition \( \nu = 1 + f(t, \varphi) \) specifies a closed line \( C^{(t)} \) bounding the region \( U^{(t)} \) containing the origin, \( C^{(0)} \) being a unit circle centered on the origin.

The value of the required function \( \rho = \rho(t, \nu, \varphi) \) on the boundary \( C^{(0)} \) corresponds to the unknown zero front \( b(t, \varphi) \) for the original problem (3), (5), i.e.

\[
\rho(t, \nu, \varphi)|_{\nu=1} = b(t, \varphi),
\]

which is determined from solving.

At each time step \( t_k = kh \), where \( h \) is the stride parameter, in the domain \( W^{(t_k)} \) within the boundaries \( C^{(0)} \) and \( C^{(t_k)} \) we consider a boundary value problem for the Poisson equation (8), which can be written compactly as

\[
\Delta \rho = P(\nu, \rho, \rho_t, \rho_\nu, \rho_{\varphi \rho}, \rho_{\varphi \varphi}, \rho_{\nu \varphi})
\]

with the boundary conditions represented by Eq. (9) and the relationship ensuing from Eq. (10) on the boundary \( C^{(0)} \),

\[
q^{(\nu)}|_{\nu=1} = \frac{\partial \rho}{\partial n}|_{\nu=1} = -\rho_\nu = \frac{1}{\sigma b(t_k)} \left( 1 + \frac{b^2_{\nu}}{b^2_{\varphi}} \right),
\]

where \( n \) is the vector of the external normal to the boundary of the domain \( W^{(t_k)} \).

To solve problem (11), (9), (12) by the BEM, we divide the boundary of the domain \( W^{(t_k)} \) into \( 2N \) elements as follows: \( e_1, e_2, \ldots, e_N \) on the boundary \( C^{(0)} \) and \( e_{N+1}, e_{N+2}, \ldots, e_{2N} \) on \( C^{(t_k)} \). We use rectilinear elements with constant approximation of the values. The values of the required function and its flux at the node \( z_i \) found in the middle of the element \( e_i \) are symbolized, respectively, as \( \rho_i \) and \( q_i^{(\nu)} \).

Solving problem (11), (9), (12) by the boundary element method [14], we arrive at the relationship

\[
\rho(\zeta) = \sum_{i=1}^{2N} \int_{e_i} u^*(\zeta, z) dS(z) - \int_{e_i} q^*(\zeta, z) dS(z) - \int_{W^{(t_k)}} P(\ldots) u^*(\zeta, z) dV(z),
\]
where \( \zeta \) is an internal point in the domain \( W^{(tk)} \), \( u^*(\zeta, z) \), and \( q^*(\zeta, z) \) are the kernel functions. The following is true for the boundary element nodes:

\[
\frac{1}{2} \rho_i = \sum_{i=1}^{2N} \left[ q_i^{(p)} \int_{e_i} u^*(z_i, z) \, dS(z) - \rho_i \int_{e_i} q^*(z_i, z) \, dS(z) \right] - \int_{W^{(tk)}} P(\ldots)u^*(z_i, z) \, dV(z), \quad i = 1, 2, \ldots, 2N. \tag{14}
\]

The unknown values in Eq. (14) are \( \rho_i, i = 1, 2, \ldots, N \), and \( q_i^{(p)}, i = 1, 2, \ldots, 2N \). The values \( \rho_i, i = N + 1, \ldots, 2N \), are specified by the boundary condition represented by Eq. (9). The condition expressed by Eq. (12) at the nodes \( z_i, i = 1, 2, \ldots, N \), can be represented as

\[
\begin{align*}
q_1^{(p)} &= \frac{h}{\sigma (\rho_1 - \rho_1^*)} \left[ 1 + \left( \frac{\rho_2 - \rho_1}{h \sigma} \right)^2 \right], \\
q_i^{(p)} &= \frac{h}{\sigma (\rho_i - \rho_i^*)} \left[ 1 + \left( \frac{\rho_{i+1} - \rho_{i-1}}{h \sigma} \right)^2 \right], \quad i = 2, \ldots, N - 1, \\
q_N^{(p)} &= \frac{h}{\sigma (\rho_N - \rho_N^*)} \left[ 1 + \left( \frac{\rho_1 - \rho_{N-1}}{h \sigma} \right)^2 \right], \tag{15}
\end{align*}
\]

where \( \rho_i^* \) is \( \rho_i \) at the previous time step, \( h \sigma \) is a step of the polar angle between the neighboring nodes of the boundary \( C^{(0)} \).

The relationships (13) to (15) enable us to solve iteratively problem (11), (9), (12) at the step \( t_k \). The trivial solution is assumed as the initial approximation. The nonlinear system (14), (15) of \( 3N \) equations is solved at each iteration. Substituting the found unknown boundary values \( \rho_i \) and \( q_i^{(p)} \) into Eq. (13), we obtain the next iteration of solving problem (11), (9), (12). In the integrals over the domain \( W^{(tk)} \) in Eqs. (13) and (14) we use the previous iteration of the solution.

At each iteration the system of equations (14), (15) is solved by the Newton method, the values at the previous time step being assumed as the initial approximations for \( k > 1 \). To choose the initial approximation at the first time step, when \( k = 1 \), we find the values of \( q_i^{(p)} \) at the moment \( t = 0 \). It follows from Eq. (9) that

\[
\frac{d\rho}{dt} \bigg|_{v=1+f(t, \varphi)} = \frac{da}{dt}.
\]

Since

\[
\frac{d\rho}{dt} \bigg|_{v=1+f(t, \varphi)} = \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \rho}{\partial \varphi} \frac{\partial \varphi}{\partial t} \right) \bigg|_{v=1+f(t, \varphi)} = \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \varphi} \frac{\partial f}{\partial t} \right) \bigg|_{v=1+f(t, \varphi)}, \tag{16}
\]

we find from Eq. (16) that \( \frac{\partial \rho}{\partial t} = \frac{\partial a}{\partial t} - \frac{\partial \rho}{\partial \varphi} \frac{\partial f}{\partial t} \) when \( v = 1 + f(t, \varphi) \). Substituting the obtained expression into Eq. (7), when \( t = 0, v = 1 \), we have

\[
\rho^2 \left( a_t - \rho_v f_t \right) + \frac{\rho_v}{\sigma} \left( 1 + \frac{\rho^2_v}{\rho^2} \right) = 0. \tag{17}
\]

According to the sense of the problem, the derivative \( \rho_v \) must be negative if the heat wave front moves from the origin. Hence, when selecting the appropriate nonzero root of Eq. (17) and
taking into account that \( \rho = a(0, \varphi) \), \( \rho_\varphi = a_\varphi(0, \varphi) \) at \( t = 0 \), we obtain

\[
\rho_\nu(0, 1, \varphi) = \frac{1}{2f_i(0, \varphi)} \left( a_i(0, \varphi) - \sqrt{a_i^2(0, \varphi) + \frac{4f_i(0, \varphi)}{\sigma} \left( 1 + \frac{[a_\varphi(0, \varphi)]^2}{[\rho(0, \varphi)]^2} \right)} \right),
\]

wherefrom

\[
q_\rho(0, 1, \varphi_i) = -\frac{1}{2f_i(0, \varphi_i)} \left( a_i(0, \varphi_i) - \sqrt{a_i^2(0, \varphi_i) + \frac{4f_i(0, \varphi_i)}{\sigma} \left( 1 + \frac{[a_\varphi(0, \varphi_i)]^2}{[\rho(0, \varphi_i)]^2} \right)} \right). \tag{18}
\]

Here, \( \varphi_i \) corresponds to the node \( z_i, i = 1, 2, \ldots, N \). The values of the flux expressed by Eq. (18) can be assumes as the initial approximation in solving system (14), (15) at the first time step.

The integrals over the boundary elements in Eqs. (13) and (14) are computed by analytical formulas found in [15]. The integrals over the domain \( W^{(t_k)} \) are computed with the application of the dual reciprocity method [16].

The implementation of the stepwise solution procedure following the proposed algorithm at a specified time interval is the first stage of solving problem (3), (5), which enables one to determine the geometry of the zero front \( b(t, \varphi) \) at each time step.

The second stage is stepwise solution of problem (3), (4) with the found zero front \( b(t, \varphi) \) according the algorithm developed earlier in [11]. Herewith, at the time \( t = t_k \), the points \( x_i(\rho, \varphi_i) \) are assumed to be the nodes of the boundary elements on the boundary \( S^{(t_k)} \), and the values \( q_i = -1/q_i^{(\rho)} \) are assumed to be the values of the flux at the nodes \( x_i \), where \( \rho_i \) and \( q_i^{(\rho)} \) are the corresponding nodal values found on the boundary \( C^{(0)} \) at the moment \( t = t_k \). As a result, we obtain the solution of the initial problem (3), (5) at each time step.

4. Example. Comparison of the calculation results with the generalized self-similar solution

The proposed algorithm is implemented as a program. The program is verified by comparing the calculation results with the exact solution of Eq. (3), which is sought in the form of the following product:

\[
u(t, \rho, \varphi) = \psi(t) w(s), \quad s = \rho/b(t). \tag{19}\]

Let us refer to the exact solutions like Eq. (19) as generalized self-similar. It is obvious that the required function \( u \) is independent of \( \varphi \) in this case. Then Eq. (3) has the form

\[
u_t = uu_{\rho \rho} + \frac{1}{\sigma} u^2 + \frac{1}{\rho} uu_{\rho}. \tag{20}\]

By substituting Eq. (19) into Eq. (20), with relevant transformations, one can see that, when \( b(t) \) is a power function or an exponent, to find \( w(s) \), we have the ordinary differential equation

\[
w w'' + \left( \frac{w'}{s} \right)^2 + \left( s + \frac{w}{s} \right) w' + Aw = 0. \tag{21}\]

Herewith, \( A = -2, \ \psi(t) = B_1^2 B_2 \exp(2B_2 t), \) if \( b(t) = B_1 \exp(B_2 t): A = -2 + 1/\beta, \ \psi(t) = \beta C_1 (C_1 t + C_2)^{2\beta-1}, \) if \( b(t) = (C_1 t + C_2)^\beta. \) The Cauchy conditions for Eq. (21) are as follows:

\[
w(1) = 0, \quad w'(1) = -\sigma. \tag{22}\]

The former is the condition for vanishing of the required function on the heat wave front \( \rho = b(t) \) and the latter ensures the nonzeroness of the heat flux on the heat wave front, i.e., it is obvious
that, if \( w(1) = 0 \), Eq. (22) is incompatible at all the other nonzero values of \( w'(1) \). The case \( w(1) = w'(1) = 0 \) requires special consideration, which is beyond the scope of this study.

In the particular case when the wave front is given by a power function with the exponent \( \beta = 1/(2\sigma + 2) \), the solution of problem (21), (22) has the form \( w = -\sigma s^2/2 + \sigma/2 \) and \( \psi(t) = C_1 (C_1 t + C_2)^{-\sigma/(\sigma+1)}/(2\sigma + 2) \). Thus, we arrive at the following solution of Eq. (3):

\[
U(t, \rho) = \frac{\sigma C_1}{4(\sigma + 1)(C_1 t + C_2)^{\frac{1}{\sigma+1}}} \left[ 1 - \frac{\rho^2}{(C_1 t + C_2)^{\frac{1}{\sigma+1}}} \right]. \tag{23}
\]

To verify the program, we construct numerically a BEM solution to problem (3), (5), with the boundary condition

\[ u|_{\rho=a(t)} = U(t, a(t)), \]

where \( a(t) = b(t/2) \),

\[ b(t, \varphi) = (C_1 t + C_2)^{\frac{1}{2\sigma+2}} \tag{24} \]

is the zero front corresponding to the solution represented by Eq. (23), and compare the calculation results with the constructed exact self-similar solution.

Figure 1 compares the BEM solution for the parameter values \( \sigma = 3, C_1 = 16/30, C_2 = 1 \), and the exact solution (23). In Fig. 2 the zero front determined by the BEM is compared with the zero front (24).

**Figure 1.** A comparison between the BEM solution and the exact solution.

**Figure 2.** A comparison between the zero fronts.

5. Conclusion

The use of a special variable substitution has enabled us to construct a numerical BEM solution to a two-dimensional problem of initiation of a heat wave with a boundary condition specified on a movable manifold. The obtained solution is continuous with respect to spatial variables at each time step. The comparison of the calculation results with the constructed exact (generalized self-similar) solution has shown the efficiency of the developed algorithm and the program based on it.

Further research along this line may deal with a more complicated problem statement due to increased dimensionality or modification of the mathematical model, e.g., introduction of additional terms describing the sources or the flows. Another possible line of research may be improving the computation algorithm, namely, increasing the computation speed and convergence reliability.

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