Exactly solvable model for cosmological perturbations in dilatonic brane worlds

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Abstract

We construct a model where cosmological perturbations are analytically solved based on dilatonic brane worlds. A bulk scalar field has an exponential potential in the bulk and an exponential coupling to the brane tension. The bulk scalar field yields a power-law inflation on the brane. The exact background metric can be found including the back-reaction of the scalar field. Then exact solutions for cosmological perturbations which properly satisfy the junction conditions on the brane are derived. These solutions provide us an interesting model to understand the connection between the behavior of cosmological perturbations on the brane and the geometry of the bulk. Using these solutions, the behavior of an anisotropic stress induced on the inflationary brane by bulk gravitational fields is investigated.
I. INTRODUCTION

The possibility that our universe is a hypersurface (brane) embedded in a higher-dimensional spacetime (bulk) has recently attracted much attention [1]. Particularly, a model proposed by Randall and Sundrum has attractive features for gravity and cosmology [2]-[4]. In their model, a single positive tension brane is embedded in 5-dimensional Anti-de Sitter (AdS) spacetime. Ordinary matter fields are assumed to be confined to the brane and only the gravity can propagate in the bulk. A quite interesting feature of their model is that a 4-dimensional gravity is recovered in the low energy limit even though the size of the bulk is infinite. We need no longer a compactification of the extra dimension. Then cosmological consequences of their model have been intensively investigated. In string models, the gravity enjoys the company of scalar field such as dilaton. Then the extension of Randall and Sundrum model to dilatonic brane worlds also attracts much interest [5]-[13]. The dilatonic brane world provides us a new models for inflationary brane world which is often called bulk inflaton model [14]-[23]. The inflation on the brane is caused by the inflaton in the bulk and the bulk spacetime itself is not inflating. It has been shown that this model indeed mimics the 4-dimensional inflation model at low energies where the Hubble horizon on the brane $H^{-1}$ is sufficiently longer than the curvature radius $l$ in AdS bulk.

One of the most important quantities which should be clarified in brane world models is the cosmological perturbation because it provides us a possibility to test the brane world model observationally [24]-[29]. Unfortunately, it is quite difficult to find the solutions for cosmological perturbations because we should consistently take into account the perturbations in the bulk. Recently, Minamitsuji, Himemoto and Sasaki investigated a behavior of cosmological perturbations in a model with bulk scalar field in AdS spacetime [22]. They used a covariant curvature formalism and showed that 4-dimensional results are recovered at low energies. Thus, to predict signatures specific to the brane world model, we should investigate the higher energy effects. However, it is quite difficult to perform the calculations at high energies because we should treat completely a 5-dimensional problem. Technically, it is necessary to solve complicated coupled partial differential equations to find the behavior of perturbations.

Hence it is eagerly desired to construct a model where we can solve the equations for perturbations analytically. Recently, we have developed such a model based on dilatonic
brane worlds [20]. The action for this model is given by

\[ S = \int d^5x \sqrt{-g_5} \left( \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \Lambda(\phi) \right) - \int d^4x \sqrt{-g_4} \lambda(\phi), \]  

(1)

where \( \kappa^2 \) is five-dimensional gravitational constant. The potential of the scalar field in the bulk and on the brane are taken to be exponential:

\[ \kappa^2 \Lambda(\phi) = \left( \frac{\Delta}{8} + \delta \right) \lambda_0^2 e^{-2\sqrt{2}b\kappa\phi}, \]  

(2)

\[ \kappa^2 \lambda(\phi) = \sqrt{2}\lambda_0 e^{-\sqrt{2}b\kappa\phi}. \]  

(3)

Here \( \lambda_0 \) is the energy scale of the potential, \( b \) is the dilaton coupling and we defined

\[ \Delta = 4b^2 - \frac{8}{3}. \]  

(4)

We assume the \( Z_2 \) symmetry across the brane. The bulk scalar field \( \phi \) acts as an inflaton. For \( \delta \neq 0 \), the brane undergoes a power-low inflation. The metric for 5-dimensional spacetime is written by separable functions of time and extra-coordinate \( y \). This is technically essential because the perturbations can have separable solutions which enables us to derive solutions analytically. It is not sufficient to derive general solutions for perturbations in the bulk. We should find particular solutions which satisfy the correct boundary conditions at the brane. It is in general difficult to find such particular solutions for an expanding brane. But, in this model, we can successfully find such solutions analytically. The first aim of this paper is to provide exact and analytic solutions for cosmological perturbations in this model.

Using the solutions for perturbations, we can address the primordial fluctuations generated during inflation. In a previous paper, we calculated a spectrum of curvature perturbation \( R_c \) by quantizing a canonical variable for the second order action [20]. We found that, even at high energies, the effects of Kalzua-Klein (KK) modes are negligible at long wave-length even though the amplitude of the fluctuation is amplified. However, in the brane worlds, the curvature perturbation alone does not determine the cosmic microwave background (CMB) anisotropies. The anisotropic stress induced by bulk gravitational fields affects the CMB anisotropies. Indeed, it is this anisotropic stress which gives distinct features in CMB anisotropies in brane worlds from 4-dimensional models [30] [31]. Thus we should specify the initial condition for the anisotropic stress generated during inflation as well as the curvature perturbations. The second aim of this paper is to investigate the behavior of anisotropic stress in this model.
The structure of the paper is as follows. In section II, the background spacetime is described. In section III, the solutions for cosmological perturbations are derived. Section IV is devoted to the investigation of the anisotropic stress on the brane which is generated during inflation. In section V, we summarize the results. In appendix A, 5-dimensional Einstein equations for scalar perturbations are shown. In appendix B, the procedure to derive the solutions for scalar perturbations is presented.

II. BACKGROUND

We first review a background solution [20]. For \( \delta = 0 \), the static brane solution was found [8]. The existence of the static brane requires tuning between bulk potential and brane tension known as Randall-Sunrum tuning. It has been shown that for \( \Delta \leq -2 \), we can avoid the presence of the naked singularity in the bulk and also ensure the trapping of the gravity. The reality of the dilaton coupling requires \( -8/3 \leq \Delta \). For \( \Delta = 8/3 \), we recover Randall-Sundrum solution. The value of \( \delta \), which is not necessarily small, represents a deviation from the Randall-Sundrum tuning. This deviation yields an inflation on the brane.

The solution for background spacetime is found as

\[
\begin{align*}
\dot{e}^{2W(y)} & = e^{2\sqrt{2}\kappa \phi} \left( e^{2\sqrt{2}\kappa \phi(t)} dy^2 - dt^2 + e^{2\alpha(t)} \delta_{ij} dx^i dx^j \right), \\
\phi(t, y) & = \phi(t) + \Xi(y).
\end{align*}
\]

The evolution equation for background metric \( \alpha \) and \( \phi \) are given by

\[
\begin{align*}
\dot{\alpha}^2 + \sqrt{2}\kappa \dot{\phi} \dot{\alpha} & = \frac{1}{6} \kappa^2 \dot{\phi}^2 - \frac{1}{3} \lambda_0^2 \frac{\Delta + 4}{\Delta} \delta e^{-2\sqrt{2}\kappa \phi}, \\
\ddot{\phi} + (3\dot{\alpha} + \sqrt{2}\kappa \dot{\phi}) \dot{\phi} & = -4\sqrt{2}\kappa^{-1} \lambda_0^2 \frac{\delta}{\Delta} e^{-2\sqrt{2}\kappa \phi}.
\end{align*}
\]

The solution for \( \alpha(t) \) and \( \phi(t) \) are obtained as

\[
\begin{align*}
e^{\alpha(t)} & = (H_0 t)^{2(\Delta + 2)} = (-H \eta)^{\Delta + 2}, \\
e^{\sqrt{2}\kappa \phi(t)} & = H_0 t = (-H \eta)^{\Delta + 8/3}.
\end{align*}
\]

where

\[
H_0 = -\frac{3\Delta + 8}{3(\Delta + 2)} H, \quad H = -(\Delta + 2) \sqrt{-\frac{\delta}{\Delta}} \lambda_0,
\]

\( \eta = \frac{1}{\sqrt{2}\kappa} \sqrt{\frac{\lambda_0^2}{\Delta}}. \)
and $\eta$ is a conformal time defined by
\[ \eta = \frac{3\Delta + 8}{3(\Delta + 2)} H_0 \frac{2^{2/3}}{3(\Delta + 8)^{2/3}} t^{2/3(\Delta + 8)}. \] (10)

We should notice that power-law inflation occurs on the brane for $-8/3 < \Delta < -2$. Thus in the rest of the paper we shall assume $-8/3 < \Delta < -2$. The solutions for $W(y)$ and $\Xi(y)$ can be written as
\[ e^W(y) = \mathcal{H}(y)^{\frac{2}{3(\Delta + 2)}}, \quad e^\kappa \Xi(y) = \mathcal{H}(y)^{\frac{2\sqrt{b}}{3(\Delta + 2)}}, \] (11)
where
\[ \mathcal{H}(y) = \sqrt{-1 - \frac{\Delta}{8\delta}} \sinh H y. \] (12)

Here we assumed $\frac{\Delta}{8} + \delta < 0$. At the location of the brane $y = y_0$ the solutions should satisfy junction conditions;
\[ \partial_y W(y)|_{y = y_0} = -e^{W(y_0)}\sqrt{2\kappa} \Xi(y_0) \frac{\sqrt{2}}{6} \lambda_0, \quad \partial_y \Xi(y)|_{y = y_0} = -e^{W(y_0)}\sqrt{2\kappa} \Xi(y_0) \kappa^{-1} \lambda_0. \] (13)

Then the location of the brane is determined by
\[ \sinh H y_0 = \left(-1 - \frac{\Delta}{8\delta}\right)^{-1/2}. \] (14)

It is quite useful to note that the above 5-dimensional solution can be obtained by a coordinate transformation from the metric
\[ ds^2 = e^{2Q(z)}(dz^2 - d\tau^2 + \delta_{ij}dx^i dx^j), \quad e^{\kappa \phi(z)} = e^{3\sqrt{2}Q(z)}, \] (15)
where
\[ e^{Q(z)} = (\sinh H y_0)^{-\frac{2}{3(\Delta + 2)}} (Hz)^{\frac{2}{3(\Delta + 2)}}, \] (16)
by
\[ z = -\eta \sinh(Hy), \quad \tau = -\eta \cosh(Hy). \] (17)

Because the metric Eq. (15) is simple, it is convenient to solve the perturbations in the bulk not directly in Eq. (5) but in Eq. (15) for scalar perturbations.

The background equations on the brane Eqs.(6) can be described by the 4-dimensional Brans-Dicke theory with the action
\[ S_{4,\text{eff}} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g_4} \left[ \varphi_{BD} \left( ^{(4)}R - \frac{\omega_{BD}}{\varphi_{BD}} (\partial \varphi_{BD})^2 \right) \right] - \int d^4x \sqrt{-g_4} V_{\text{eff}}(\varphi_{BD}), \] (18)
where
\[ \varphi_{BD} = e^{\sqrt{2}b_\kappa \phi}, \quad \omega_{BD} = \frac{1}{2b^2}, \quad \kappa_4^2 V_{\text{eff}}(\varphi_{BD}) = -\lambda_0^2 \frac{\Delta + 4}{\Delta} \frac{1}{\varphi_{BD}}. \] (19)
III. COSMOLOGICAL PERTURBATIONS

Now let us consider cosmological perturbations in this background spacetime. Taking appropriate gauge fixing conditions, the perturbed metric and scalar field is given by

\[ ds^2 = e^{2W(y) [e^{2\sqrt{2}\kappa\phi(t)} dy^2 - dt^2 + e^{2\alpha(t)} (\delta_{ij} + h_{ij}) dx^i dx^j]}, \]

for tensor perturbations,

\[ ds^2 = e^{2W(y)} \left( e^{2\sqrt{2}\kappa\phi(t)} dy^2 - dt^2 + e^{2\alpha(t)} \left( 2T_i dy dx^i + 2S_i dt dx^i + \delta_{ij} dx^i dx^j \right) \right), \]

for vector perturbations and

\[ ds^2 = e^{2W(y)} \left[ e^{2\sqrt{2}\kappa\phi(t)} (1 + 2N) dy^2 + 2Adtdy - (1 + 2\Phi) dt^2 + e^{2\alpha(t)} (1 - 2\Psi) \delta_{ij} dx^i dx^j \right], \]

\[ \phi = \phi(t) + \Xi(y) + \kappa^{-1} \delta\phi, \]

for scalar perturbations, where perturbations are decomposed according to the tensorial type of perturbations with respect to 3-space metric \( \delta_{ij} \). Here \( S_i \) and \( T_i \) are transverse vector (\( \nabla^i S_i = 0 \) and \( \nabla^i T_i = 0 \)) and \( h_{ij}(y, t, x) \) is a transverse and traceless tensor(\( h^i_i = 0, \nabla^i h_{ij} = 0 \)) where \( \nabla_i \) is the derivative operator on 3-space metric \( \delta_{ij} \). Because each type of variables obeys an independent closed set of equations in the 5-dimensional Einstein equations, we derive the solutions for tensor, vector and scalar perturbations separately.

A. Tensor perturbations

The evolution equation for tensor perturbations is simple. The equation for tensor perturbation \( h_{ij}(y, t, x) = h(y, t) e^{ipx} e_{ij} \) is given by

\[ e^{2\sqrt{2}\kappa\phi} \left[ h + (3\dot{\alpha} + \sqrt{2}\kappa\dot{\phi}) h + e^{-2\alpha} p^2 h \right] = h'' + 3W'h', \]

where \( e_{ij} \) is a polarization tensor and dot denotes the derivative with respect to \( t \) and prime denotes the derivative with respect to \( y \). The junction condition for \( h(y, t) \) is imposed as

\[ \partial_y h|_{y=y_0} = 0. \]

We can use the separation of variables to solve this equation. The solution for \( h_{ij} \) is given by

\[ h_{ij} = \int dmd^3p \ h(m, p) \ f_m(y) g_m(\eta) e^{ipx} e_{ij}, \]
where
\[ f_0(y) = 1, \]
\[ g_0(\eta) = (-H\eta)^{-\frac{1}{\Delta + 2}} \left( H_{\frac{1}{\Delta + 2}}^1 (p\eta) + c(p, 0) H_{\frac{1}{\Delta + 2}}^1 (p\eta) \right), \] (26)
\[ f_m(y) = (\sinh H y)^{\frac{\Delta}{2(\Delta + 2)}} \left( P_{\frac{1}{2} + i\nu}^\mu (\cosh H y) - \frac{P_{\frac{1}{2} + i\nu}^{\mu+1} (\cosh H y_0)}{Q_{\frac{1}{2} + i\nu}^\mu (\cosh H y_0)} Q_{\frac{1}{2} + i\nu}^\mu (\cosh H y) \right), \]
\[ g_m(\eta) = (-H\eta)^{-\frac{1}{\Delta + 2}} \left( H_{\frac{1}{\Delta + 2}}^1 (p\eta) + c(p, m) H_{\frac{1}{\Delta + 2}}^1 (p\eta) \right), \] (27)
where \( P_\beta^\alpha \) and \( Q_\beta^\alpha \) are associated Legendre functions, \( H_\alpha^{(1)} \) and \( H_\alpha^{(2)} \) are Hunkel functions and
\[ \mu = -\frac{\Delta}{2(\Delta + 2)}, \quad \nu = \sqrt{\frac{m^2 H^2 - 1}{(\Delta + 2)^2}}. \] (28)

The coefficients \( h(p, m) \) and \( c(p, m) \) are so far arbitrary and these are determined if one specifies the initial conditions and boundary conditions in the bulk. The canonical variables for the second-order perturbed action for tensor perturbation is given by
\[ \varphi = \frac{1}{2\kappa} h. \] (29)

Then the second-order perturbed five-dimensional action for the tensor perturbation is given by
\[ \delta S^{(T)} = \frac{1}{2} \int dy dt d^3x e^{3W(y)} e^{\sqrt{2} b \kappa \phi(t)} e^{3\alpha(t)} \left( e^{-2\sqrt{2} b \kappa \phi(t)} \varphi'^2 - \varphi^2 - e^{-2\alpha(t)} p^2 \varphi'^2 \right). \] (30)

It should be noted that the modes with \( 0 < m < -H/(\Delta + 2) \) is not normalizable. Thus the normalizable modes have a mass gap and the continuous massive modes start from \( m = -H/(\Delta + 2); \)
\[ m \geq -\frac{H}{\Delta + 2}. \] (31)

**B. Vector perturbations**

The calculations of the vector perturbations in this model are similar with those given by Ref.[32]. In order to solve the Einstein equation for vector perturbations, it is convenient to define a variable
\[ V_i = S_i' - \dot{T}_i = V e_i, \] (32)
where $e_i$ is the polarization vector. Then the Einstein equations of $(0, i)$ and $(i, y)$ components are given by

\[ e^{2\alpha - 2\sqrt{2}b\kappa\phi}(V'' + 3W''V) = p^2 S, \]
\[ e^{2\alpha}(\dot{V} + (5\dot{\alpha} - \sqrt{2}b\kappa\dot{\phi})V) = p^2 T, \]

(33)

where we expand the variables by $e_i e^{ipx}$. The $(i, i)$ component is given by

\[ \dot{S} + (3\dot{\alpha} + \sqrt{2}b\kappa\dot{\phi})S = e^{-2\sqrt{2}b\kappa\phi}(T'' + 3W'T'). \]

(34)

We can easily find a master variable from which all perturbations are constructed;

\[ S = e^{-3W-3\alpha-\sqrt{2}b\kappa\phi}\Omega', \]
\[ T = e^{-3W-3\alpha+\sqrt{2}b\kappa\phi}\dot{\Omega}, \]
\[ V = e^{-3W-5\alpha+\sqrt{2}b\kappa\phi}p^2\Omega. \]

(35)

If $\Omega$ satisfies the evolution equation,

\[ e^{2\sqrt{2}b\kappa\phi}\left[ \ddot{\Omega} - (3\dot{\alpha} - \sqrt{2}b\kappa\dot{\phi})\dot{\Omega} + e^{-2\alpha}p^2 \right] = \Omega'' - 3W'\Omega', \]

(36)

the 5-dimensional Einstein equation is automatically satisfied. The junction condition for perturbations are imposed as

\[ V|_{y=y_0} = T|_{y=y_0} = 0. \]

(37)

Thus the master variable should satisfy

\[ \Omega|_{y=y_0} = 0. \]

(38)

The solution that satisfies the junction condition is obtained as

\[ \Omega(t, x, y) = \int dmd^3p \Omega(m, p) u_m(y)v_m(t)e^{ipx} \]

(39)

where

\[ u_0(y) = \int_{y_0}^{y} e^{3W(y')}dy', \]
\[ v_0(\eta) = (-H\eta)^{\frac{1}{\Delta+2}} \left( H_{1}^{(1)}(-p\eta) + c(p, 0)H_{1}^{(2)}(-p\eta) \right), \]

(40)

\[ u_m(y) = (\sinh Hy)^{\frac{1}{\Delta+4}} \left( P_{\frac{1}{2}+i\nu}^\mu (\cosh H y_0) - \frac{P_{\frac{1}{2}+i\nu}^\mu (\cosh H y_0)}{Q_{\frac{1}{2}+i\nu}^\mu (\cosh H y_0)} Q_{\frac{1}{2}+i\nu}^\mu (\cosh H y) \right), \]
\[ v_m(\eta) = (-H\eta)^{\frac{1}{\Delta+2}} \left( H_{1}^{(1)}(-p\eta) + c(p, m)H_{1}^{(2)}(-p\eta) \right), \]

(41)
where
\[\mu = -\frac{\Delta + 4}{2(\Delta + 2)}, \quad \nu = \sqrt{\frac{m^2}{H^2} - \frac{1}{(\Delta + 2)^2}}. \quad (42)\]

The second order perturbed action is also written by \(\Omega\) as
\[\delta S^{(V)} = \frac{1}{2} p^2 \int dy dt d^3 x e^{-3W(y)} e^{2\sqrt{2} b \delta \phi(t)} e^{-3\alpha(t)} \left( e^{-2\sqrt{2} b \delta \phi(t)} \Omega^2 - \dot{\Omega}^2 - e^{-2\alpha(t)} p^2 \Omega^2 \right). \quad (43)\]

From this action, we can determine the normalization of the perturbations. It should be noted that the 0-mode solution is not normalizable for vector perturbations.

C. Scalar perturbation

The scalar perturbations are more complicated than tensor and vector perturbations due to the existence of the scalar field in the bulk. Unlike a maximally symmetric bulk spacetime, we cannot find a master variable for scalar perturbations and this causes the difficulty to solve the perturbations. Fortunately, in our background spacetime, there is a simple coordinate system in the bulk, that is Eq. (15). Thus we can use the metric Eq. (15) to find general solutions for perturbations in the bulk. In Ref [9], it was shown that there are variables which make the equations for \(N, A, \Phi, \Psi, \) and \(\delta \phi\) in the bulk to be diagonalized. Then by performing a coordinate transformation, it is easy to find general solutions for perturbations in our background spacetime. However, we should proceed further to derive solutions for perturbations. On the brane, the perturbations should satisfy the boundary conditions. In terms of the metric perturbations, the boundary conditions are given by
\[\Psi |_{y=y_0} = -W'(N - \sqrt{2b} \delta \phi)|_{y=y_0}, \]
\[\Phi |_{y=y_0} = W'(N - \sqrt{2b} \delta \phi)|_{y=y_0}, \]
\[\delta \phi |_{y=y_0} = 3\sqrt{b} W'(N - \sqrt{2b} \delta \phi)|_{y=y_0}, \]
\[A |_{y=y_0} = 0. \quad (44)\]

A problem is that if we rewrite these conditions in terms of variables that make the bulk equations to be diagonalized, the boundary conditions become complicated. Indeed, we will find that the boundary conditions are not diagonalized and also they effectively contain time derivatives of the variables. Thus unlike vector and tensor perturbations, imposing the boundary conditions is not so easy. This is in contrast to the case for the static brane
which sits at constant value of the \( z \) in the coordinate Eq.(15). For the static brane, the boundary conditions for variables which make the equations in the bulk to be diagonalized also make the boundary conditions to be diagonalized and also they do not contain the time derivative of the variables. Thus, this complexity of the boundary conditions reflects the fact that our brane is moving in the coordinate Eq.(15). This movement of the brane causes the cosmological expansion on the brane. Hence, it is an essential part of the calculations to find particular solutions which satisfy the boundary conditions at the brane. In fact, it has been recognized that this is the central part of the calculations of scalar cosmological perturbations. In general, the bulk perturbations are not written by separable functions with respect to a brane coordinate and a bulk coordinate. Thus there is almost no hope to find particular solutions analytically and it has prevented us from understanding the behavior of scalar perturbations on the cosmological brane. In our background spacetime, the bulk perturbations are obtained analytically by separable functions and it enables us to derive the solutions which properly satisfy the boundary conditions on the brane.

Now, we will describe our procedure to derive solutions. We first solve the perturbation in a static coordinate;

\[
\begin{align*}
\text{d}s^2 &= e^{2Q(z)}((1+2\Gamma)dz^2 - (1+2\phi)d\tau^2 + 2Gdzd\tau + (1-2\psi)\delta_{ij}dx^i dx^j), \\
\phi &= \phi(t) + \kappa^{-1}\delta\phi.
\end{align*}
\]

It is possible to find variables which make the equations diagonal \cite{9} (see Appendix A-2)

\[
\begin{align*}
\omega_c &= \delta\phi + 3\sqrt{2b}\psi, \\
\omega_\psi &= \Gamma - 2\psi, \\
\omega_N &= \Gamma - \sqrt{2b}\delta\phi, \\
\omega_A &= G.
\end{align*}
\]

The evolution equations in the bulk are obtained from 5-dimensional Einstein equation as

\[
\begin{align*}
\Box_5\omega_c &= 0, \\
\Box_5\omega_\psi &= 0, \\
\Box_5\omega_N &= \frac{2(\Delta + 4)}{\Delta + 2} \frac{1}{z^2} \omega_N, \\
\Box_5\omega_A &= \frac{2}{\Delta + 2} \frac{1}{z^2} \omega_A.
\end{align*}
\]
where
\[ \Box_5 = \frac{\partial^2}{\partial x^2} + \frac{2}{\Delta + 2} \frac{\partial}{\partial y} - \left( \frac{\partial^2}{\partial t^2} + p^2 \right). \] (48)

By performing a coordinate transformation, the evolution equation in our background space-time can be derived
\[ \Box_5 \omega_c = 0, \]
\[ \Box_5 \omega_\psi = 0, \]
\[ \Box_5 \omega_N = \frac{2(\Delta + 4)}{\Delta + 2} \frac{H^2}{(-H\eta)^2 \sinh^2 Hy} \omega_N, \]
\[ \Box_5 \omega_A = \frac{2}{\Delta + 2} \frac{H^2}{(-H\eta)^2 \sinh^2 Hy} \omega_A, \] (49)

where
\[ \Box_5 = \left( \frac{1}{-H\eta} \right)^2 \left( \frac{\partial^2}{\partial y^2} + \frac{2}{\Delta + 2} H \coth H y \frac{\partial}{\partial y} - \left( \frac{\partial^2}{\partial \eta^2} + \frac{\Delta + 4}{\Delta + 2} \frac{1}{\eta} \frac{\partial}{\partial \eta} + p^2 \right) \right). \] (50)

The solutions for \( \omega_i \) are given by
\[ \omega_c = \int dm_c N_c(m_c, p)(\sinh Hy)^{\frac{2(\Delta + 2)}{\Delta + 2}} \left( P^\mu_{-\frac{1}{2} + i\nu(m_c)}(\cosh Hy) + C_c Q^\mu_{-\frac{1}{2} + i\nu(m_c)}(\cosh(Hy)) \right) \times (-H\eta)^{-\frac{1}{\Delta + 2}} H_{iv(m_c)}(-p\eta), \]
\[ \omega_\psi = \int dm_\psi N_\psi(m_\psi, p)(\sinh Hy)^{\frac{2(\Delta + 2)}{\Delta + 2}} \left( P^\mu_{-\frac{1}{2} + i\nu(m_\psi)}(\cosh Hy) + C_\psi Q^\mu_{-\frac{1}{2} + i\nu(m_\psi)}(\cosh(Hy)) \right) \times (-H\eta)^{-\frac{1}{\Delta + 2}} H_{iv(m_\psi)}(-p\eta), \]
\[ \omega_A = \int dm_A N_A(m_A, p)(\sinh Hy)^{\frac{2(\Delta + 2)}{\Delta + 2}} \left( P^\mu_{-\frac{1}{2} + i\nu(m_A)}(\cosh Hy) + C_A Q^\mu_{-\frac{1}{2} + i\nu(m_A)}(\cosh(Hy)) \right) \times (-H\eta)^{-\frac{1}{\Delta + 2}} H_{iv(m_A)}(-p\eta), \]
\[ \omega_N = \int dm_N N_N(m_N, p)(\sinh Hy)^{\frac{2(\Delta + 2)}{\Delta + 2}} \left( P^\mu_{-\frac{1}{2} + i\nu(m_N)}(\cosh Hy) + C_N Q^\mu_{-\frac{1}{2} + i\nu(m_N)}(\cosh(Hy)) \right) \times (-H\eta)^{-\frac{1}{\Delta + 2}} H_{iv(m_N)}(-p\eta), \] (51)

where
\[ \mu = -\frac{\Delta}{2(\Delta + 2)}, \quad \nu(m) = \sqrt{\frac{m^2}{H^2} - \frac{1}{(\Delta + 2)^2}}. \] (52)

The perturbations \( \Psi, \delta \phi, N, A \) and \( \Phi \) are related to \( \psi, \delta \phi, \Gamma, G \) and \( \phi \) by a coordinate transformation;
\[ \Psi = \psi, \quad \delta \phi = \delta \phi, \quad \Phi = \Psi - N, \]
\[ N = (2 \Gamma - \phi) \cosh(Hy) \sinh(Hy) + G \sinh(Hy) \cosh(Hy), \]
\[ A = -(-H\eta)^{\frac{2(\Delta + 2)}{3(\Delta + 2)}} \left( 2(\Gamma - \phi) \cosh(Hy) \sinh(Hy) + G(\sinh^2(Hy) + \cosh^2(Hy)) \right). \] (53)
From Eq.(46), $\psi, \delta \phi, \Gamma, G$ and $\phi$ are written in terms of $\omega_i$ as

\[
\psi = -\frac{2}{3(\Delta + 4)}(\omega_\psi - \omega_N - \sqrt{2b}\omega_c),
\]

\[
\delta \phi = \frac{2\sqrt{2b}}{\Delta + 4}(\omega_\psi - \omega_N + \frac{\sqrt{2}}{3b}\omega_c),
\]

\[
\Gamma = \frac{4}{3(\Delta + 4)}(\omega_N + \sqrt{2b}\omega_c + 3b^2\omega_\psi),
\]

\[
\phi = \psi - \Gamma.
\]

Therefore the general solutions for the metric perturbations in the bulk are obtained as

\[
\Psi = -\frac{2}{3(\Delta + 4)}(\omega_\psi - \omega_N - \sqrt{2b}\omega_c),
\]

\[
\delta \phi = \frac{2\sqrt{2b}}{\Delta + 4}(\omega_\psi - \omega_N + \frac{\sqrt{2}}{3b}\omega_c),
\]

\[
N = \frac{2}{3(\Delta + 4)}(2 + 3 \sinh^2 H y)\omega_N + \left(\frac{4b^2}{\Delta + 4} + \frac{2(\Delta + 3)}{\Delta + 4}\sinh^2 H y\right)\omega_\psi
\]

\[
+ \frac{2\sqrt{2b}}{3(\Delta + 4)}(2 + 3 \sinh^2 H y)\omega_c + \sinh H y \cosh H y\omega_A,
\]

\[
A = -(-H \eta)^{\frac{3\Delta + 8}{3(\Delta + 2)}}\left[2 \sinh H y \cosh H y \left(\frac{2}{\Delta + 4}\omega_N + \frac{2(\Delta + 3)}{\Delta + 4}\omega_\psi + \frac{2\sqrt{2b}}{\Delta + 4}\omega_c\right)\right]
\]

\[
+ (1 + 2 \sinh^2 H y)\omega_A\right].
\]

We should impose boundary conditions on the brane. In terms of $\omega_i$, the junction conditions become

\[
\omega_c' = 0,
\]

\[
\omega_A = -\frac{2 \cosh H y_0 \sinh H y_0}{1 + 2 \sinh^2 H y_0} \left(\omega_N + \frac{2(\Delta + 3)}{\Delta + 4}\omega_\psi + \frac{2\sqrt{2b}}{\Delta + 4}\omega_c\right),
\]

\[
\omega_\psi' - \omega_N' = \frac{\Delta + 4}{\Delta + 2} H \coth H y_0 \left(\omega_N + \frac{1}{2} \tanh H y_0 \omega_A\right),
\]

\[
\frac{2}{\Delta + 4} \sinh^2 H y_0 \omega_N' + \left(1 + \frac{2(\Delta + 3)}{\Delta + 4} \sinh^2 H y_0\right)\omega_\psi' + \sinh H y_0 \cosh H y_0 \omega_A = 0,
\]

where above equations should be evaluated on the brane $y = y_0$. The variables should also satisfy the "constraint equations" which do not include the second derivatives of the variables with respect to $t$ and $y$ in 5-dimensional Einstein equations, that is, $(t, i), (y, i)$ and $(y, t)$ components of Einstein equaitons. Among them, the equation obtained by combining $(0, i)$ and $(y, i)$ components of the 5-dimensional Einstein equation (see Appendix A-2 for
(1 + \sinh^2 H y) \omega_A' + \frac{2}{\Delta + 2} H \coth H y \omega_A + 2 \sinh H y \cosh H y \omega'_\psi = -(-H\eta) \left( \cosh H y \sinh H y \partial_\eta \omega_A + 2(1 + \sinh^2 H y) \partial_\eta \omega'_\psi \right), \quad (60)

will be useful to find solutions. Because \omega_A is a variable which is associated with Z_2 odd variable A, \omega_A' in the junction conditions would be eliminated using constraint equations. Indeed, by projecting Eq. (60) on the brane, we find that \omega_A(y_0) can be rewritten in terms of \omega_A, \omega'_\psi, and \omega_\psi. An important point is that this equation contains the time derivative of the variables. Thus effectively, the boundary conditions contain the time derivative of the metric perturbations. It should be noted the junction condition for \omega_c is decoupled from other variables. This variable \omega_c is the canonical variable for the second order action and it is directly related to the curvature perturbations on the brane. Thus we do not need to know the full solutions for perturbations in deriving the solutions for curvature perturbation. On the other hand, in order to derive the anisotropic stress induced by bulk perturbations, we should know the solutions for all variables, as we will observe later. This indicates that the anisotropic stress on the brane is complicatedly coupled to bulk perturbations compared with the curvature perturbation.

We describe the procedure to derive the solutions which satisfy the boundary conditions and constrained equations in Appendix B. In this section we only show the results. The solutions are written as the summation of KK modes with mass m;

\[ \omega_i = \int d^3 p \bar{m} N(\bar{m}, p) \omega_i(\bar{m})(y, t) e^{ipx}, \quad (61) \]

where \bar{m} = m/H. For 0-mode with m = 0, we get

\[ \omega_c(0) = (-H\eta)^{-\frac{1}{\Delta + 2}} H_{-\frac{1}{\Delta + 2}}(-p\eta), \]

\[ \omega_\psi(0) = -\frac{\sqrt{2}b}{\Delta + 3} \left( (-H\eta)^{-\frac{1}{\Delta + 2}} H_{-\frac{1}{\Delta + 2}}(-p\eta) \right. \]

\[ + \left. \left( 1 + \frac{2(\Delta + 3)}{\Delta + 4} \sinh^2 H y \right) (-H\eta)^{-\frac{1}{\Delta + 2}} H_{-\frac{2\Delta + 4}{\Delta + 2}}(-p\eta) \right) , \]

\[ \omega_N(0) = -\frac{2\sqrt{2}b}{\Delta + 4} \sinh^2 H y (-H\eta)^{-\frac{1}{\Delta + 2}} H_{-\frac{2\Delta + 4}{\Delta + 2}}(-p\eta), \]

\[ \omega_A(0) = \frac{4\sqrt{2}b}{\Delta + 4} \sinh H y \cosh H y (-H\eta)^{-\frac{1}{\Delta + 2}} H_{-\frac{2\Delta + 4}{\Delta + 2}}(-p\eta). \quad (62) \]
For massive modes with $m \geq -H/\Delta + 2$

$$\omega_r(\tilde{m}) = -\frac{\sqrt{2}(\Delta + 2)}{4b}(i\nu - 1)(\sinh H\gamma)^{\frac{\Delta}{2}} B^\mu_{\frac{\Delta}{2} + i\nu}(\cosh H\gamma)\gamma_{\frac{\Delta}{2} + i\nu} H_{i\nu}(\tilde{m} - \nu),$$

$$\omega_0(\tilde{m}) = (\sinh H\gamma)^{\frac{\Delta}{2}} (-H\gamma)^{-\frac{\Delta}{2}} \left[ -\frac{1}{2} B^\mu_{\frac{\Delta}{2} + i\nu}(\cosh H\gamma) H_{-i\nu}(\tilde{m} - \nu) \right],$$

$$\omega_N(\tilde{m}) = (\sinh H\gamma)^{\frac{\Delta}{2}} (-H\gamma)^{-\frac{\Delta}{2}} \left[ \frac{\Delta + 2}{2} \left( \frac{1}{i\nu - \frac{\Delta}{2}} \right) B^\mu_{\frac{\Delta}{2} + i\nu}(\cosh H\gamma) H_{-i\nu}(\tilde{m} - \nu) \right] - \left( \frac{1}{i\nu - \frac{\Delta}{2}} \right) (i\nu - \frac{\Delta + 1}{\Delta + 2}) B^\mu_{\frac{\Delta}{2} + i\nu}(\cosh H\gamma) H_{-i\nu}(\tilde{m} - \nu),$$

$$\omega_A(\tilde{m}) = (\sinh H\gamma)^{\frac{\Delta}{2}} (-H\gamma)^{-\frac{\Delta}{2}} \left[ \left( \frac{1}{i\nu - \frac{\Delta}{2}} \right) B^\mu_{\frac{\Delta}{2} + i\nu}(\cosh H\gamma) H_{-i\nu}(\tilde{m} - \nu) \right],$$

(63)

where

$$B^\beta_\alpha(\cosh H\gamma) = P^\alpha(\cosh H\gamma) - \frac{P^{\mu + 1}_\beta(\cosh H\gamma)}{Q^{\mu + 1}_\beta(\cosh H\gamma)} Q^\alpha_P(\cosh H\gamma),$$

(64)

and $H_\alpha$ is the arbitrary combination of Hunkel functions $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$. Then the solutions for metric perturbations are derived as

$$\Psi(y, t, x) = \int d^3p d\tilde{m} N(\tilde{m}, p) \Phi(\tilde{m})(y, t) e^{ipx},$$

$$\delta\phi(y, t, x) = \int d^3p d\tilde{m} N(\tilde{m}, p) \delta\phi(\tilde{m})(y, t) e^{ipx},$$

$$N(y, t, x) = \int d^3p d\tilde{m} N(\tilde{m}, p) N(\tilde{m})(y, t) e^{ipx},$$

$$A(y, t, x) = \int d^3p d\tilde{m} N(\tilde{m}, p) A(\tilde{m})(y, t) e^{ipx},$$

$$\Phi(y, t, x) = \Psi(y, t, x) - N(y, t, x).$$

(65)

where

$$\Psi(0) = \frac{2\sqrt{2b}}{3(\Delta + 4)} \left[ \frac{\Delta + 4}{\Delta + 3} (-H\gamma)^{-\frac{\Delta}{\Delta + 2}} H_{-\frac{\Delta}{\Delta + 2}} (-\nu) + \frac{1}{\Delta + 3} (-H\gamma)^{-\frac{\Delta}{\Delta + 2}} H_{-\frac{\Delta + 2}{\Delta + 2}} (-\nu) \right],$$

$$\delta\phi(0) = \frac{4}{3(\Delta + 4)} \left[ \frac{\Delta + 4}{4(\Delta + 3)} (-H\gamma)^{-\frac{\Delta}{\Delta + 2}} H_{-\frac{\Delta}{\Delta + 2}} (-\nu) - \frac{3b^2}{\Delta + 3} (-H\gamma)^{-\frac{\Delta}{\Delta + 2}} H_{-\frac{\Delta + 2}{\Delta + 2}} (-\nu) \right],$$

$$N(0) = \sqrt{2b} \delta\phi(0), \quad A(0) = 0,$$

(66)

and

$$\Psi(\tilde{m})(y, t) = -\frac{2}{3(\Delta + 4)} (-H\gamma)^{-\frac{\Delta}{\Delta + 2}} \sinh H\gamma)^{\frac{\Delta}{2}}.$$
Using the equations presented in Appendix B, we get
the function
\[\text{satisfies} \quad B_{\frac{\mu}{2} + iv}^\mu \left( \cosh Hy \right) = 0.\]
These solutions are first main results of this paper. They provide us the connection between the behavior of the perturbations on the brane and the perturbations in the bulk.

The second order action for scalar perturbations is written in terms of the canonical variable $\omega_c$:

$$\delta S^{(S)} = \frac{1}{2\kappa^2} \int dy dt d^3x e^{3W(y)} e^{\sqrt{2}\omega_c \bar{\phi}(t)} e^{3\alpha(t)} (e^{-2\sqrt{2}\omega_c \bar{\phi}(t)} \omega_c'^2 - \dot{\omega}_c^2 - e^{-2\alpha(t)} p^2 \omega_c^2).$$

This can be verified using the result for the metric Eq.(15) because $\omega_c$ does not change by the coordinate transformation. This action is the same as the second order action for tensor perturbations. Then, the massive modes with $0 < m < -H/(\Delta + 2)$ are not normalizable. Thus there is also mass gap for the scalar perturbations.

### IV. Primordial Fluctuations in the Bulk Inflaton Model

In the previous section, we obtained the classical solutions for cosmological perturbations. These perturbations properly satisfy the boundary conditions at the brane. However, the boundary conditions on the brane alone do not fix the solutions completely. There remains a freedom to choose the “weight” $N(\tilde{m}, p)$ in the summation of KK modes Eq.(61). These coefficients are fixed once one more boundary condition in the bulk is specified. Because the brane is inflating, it is natural to specify the boundary conditions for the perturbations by quantum theory in the same way as the usual 4-dimensional inflationary model. We have already derived the second order 5-dimensional action for perturbations, the quantization can be done within the full 5-dimensional theory.
In the previous paper, we have already carried out the quantization of scalar and tensor perturbations. It was shown that the KK modes are well suppressed at large scales even if the energy scale of the inflation is sufficiently higher than the scale of the bulk. More precisely, the bulk curvature scale and the Hubble constant on the brane are determined by the bulk potential and the deviation from the RS tuning, respectively. Thus their ratio,

\[ r = \left| \frac{\delta}{\Delta/8 + \delta} \right|, \]  

characterize the behavior of perturbations. If \( r \) is large, we expect the five dimensional nature of the perturbations become important. However, we showed that due to the mass gap in the KK spectrum, the massive KK modes are hardly excited. Thus, at large scales, the behavior of tensor perturbation and curvature perturbation defined by

\[ \mathcal{R}_c = \frac{\dot{\alpha}}{\dot{\phi}} \omega_c, \]  

are essentially four-dimensional except for the amplitude of the perturbations.

Hence, we might expect that this model cannot be distinguish from the usual four-dimensional inflationary model. However, in the brane world, the curvature perturbation \( \mathcal{R}_c \) alone does not determine CMB anisotropies. The anisotropic stress \( \delta \pi_E \) induced by bulk gravitational fields also affects the CMB anisotropies. The anisotropic stress is measured by the difference between \( \Phi \) and \( \Psi \):

\[ (\Psi - \Phi)|_{y=y_0} \equiv \kappa^2 e^{2\alpha} \delta \pi_E. \]  

where \( \kappa_4 = \kappa \lambda_0 \). Then we should also determine the initial condition for \( \delta \pi_E \) during the inflation. From the five-dimensional Einstein equation we find that \( \delta \pi_E \) is related to \( N \):

\[ N|_{y=y_0} = \kappa^2 e^{2\alpha} \delta \pi_E. \]  

This implies that it is not sufficient to determine the behavior of the canonical variable \( \omega_c \) but we need the solutions for all \( \omega_i \). As already observed, the boundary conditions for \( \omega_i \) except for \( \omega_c \) are complicated and this reflects the fact that the brane is ”moving”. Thus we expect that the anisotropic stress can have a distinguishable feature which the curvature perturbation \( \mathcal{R}_c \) does not possess. Because we can derive the solutions for \( N \), it is possible to investigate the behavior of \( \delta \pi_E \).
A. Behavior of the canonical variable $\omega_c$

We first review the quantization of canonical variable $\omega_c$. The second order action for $\omega_c$ is nothing but the action for a 5-dimensional massless scalar field. Then the quantization is easily carried out. The canonical variable $\omega_c$ can be expressed as

$$\kappa^{-1} \omega_c(t, x, z) = \int d\tilde{m} d^3 p \left[ a_{\rho \tilde{m}} \theta_{\tilde{m}}(y) \chi_{\tilde{m}}(t) e^{ipx} + (\text{h.c.}) \right].$$  \hfill (75)

Here $a_{\rho \tilde{m}}$ is the annihilation operator and satisfies the following commutation relation,

$$[a_{\rho \tilde{m}}, a^\dagger_{\rho' \tilde{m}'}] = \delta(p - p') \delta(\tilde{m} - \tilde{m}').$$  \hfill (76)

The modes functions are given by

$$\theta_0(y) = \frac{1}{\sqrt{2}} \left(1 - \frac{\Delta}{8\delta}\right)^{-\frac{1}{4(\Delta + 2)}} \left[ \int_{y_0}^\infty (\sinh Hy) \right]^{-\frac{1}{2}} dy,$$  \hfill (77)
$$\theta_{\tilde{m}}(y) = \sqrt{\frac{H}{2}} \left(1 - \frac{\Delta}{8\delta}\right)^{-\frac{1}{2(\Delta + 2)}} (|\xi|^2 + |\zeta|^2)^{-\frac{1}{2}} (\sinh Hy) \left[ \frac{\Gamma(i\nu + 1)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} + i\nu)} \right] \left[ \frac{\Gamma(i\nu)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} - i\nu)} \right],$$  \hfill (78)

and

$$\chi_0(\eta) = \frac{\sqrt{\pi} H^{-\frac{1}{2}}}{2} (H \eta)^{-\frac{1}{2}} P_{\frac{\mu}{2} + i\nu} \left[ \frac{\Gamma(i\nu + 1)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} + i\nu)} \right] \left[ \frac{\Gamma(i\nu)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} - i\nu)} \right],$$  \hfill (79)
$$\chi_{\tilde{m}}(\eta) = \sqrt{\frac{\pi}{2}} H^{-\frac{1}{2}} (-H \eta)^{-\frac{1}{2}} (\sinh Hy) \left[ \frac{\Gamma(i\nu + 1)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} + i\nu)} \right] \left[ \frac{\Gamma(i\nu)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} - i\nu)} \right].$$  \hfill (80)

where

$$\mu = -\frac{\Delta}{2(\Delta + 2)}, \quad \nu = \sqrt{\frac{1}{\Delta + 2}},$$  \hfill (81)
$$\xi = \frac{\Gamma(i\nu)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} + i\nu)},$$  \hfill (82)
$$\zeta = \frac{\Gamma(-i\nu)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} - i\nu)} - \frac{P_{\frac{\mu+1}{2} + i\nu} \left( \cosh Hy_0 \right)}{Q_{\frac{\mu+1}{2} + i\nu} \left( \cosh Hy_0 \right)} \frac{\Gamma(1 + i\nu)}{\Gamma(1 + i\nu + 1)}.$$  \hfill (83)

Now the spectrum of the KK modes $\mathcal{N}(\tilde{m}, p)$ is determined as

$$\mathcal{N}(0, p) = \kappa \frac{\sqrt{2\pi}}{4} H^{-\frac{1}{2}} \left(1 - \frac{\Delta}{8\delta}\right)^{-\frac{1}{2}} \left[ \int_{y_0}^\infty (\sinh Hy) \right]^{-\frac{1}{2}} \left[ \frac{\Gamma(i\nu + 1)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} + i\nu)} \right] \left[ \frac{\Gamma(i\nu)}{\Gamma(\frac{\Delta + 1}{\Delta + 2} - i\nu)} \right] \left[ \frac{\Gamma(1 + i\nu)}{\Gamma(1 + i\nu + 1)} \right],$$  \hfill (84)

The ratio of massive modes and massless mode increases with $r$. But the ratio becomes constant for large $r$. In the previous paper, it was shown that this is caused by the existence
of mass gap. And also, the amplitude of massive modes are dumped after the horizon crossing. Thus the spectrum of the massive modes is blue tilted, so the contribution from massive modes becomes negligible at large scales. We should note that the integration over \( \tilde{m} \) logarithmically diverges. Thus we need some regularization scheme.

B. Behavior of anisotropic stress

Now we turn to the anisotropic stress

\[
\kappa_4^2 e^{2\alpha} \delta \pi \xi = (\Psi - \Phi)|_{y = y_0}. \tag{85}
\]

First let us consider the 0-mode. The 0-mode solution satisfies

\[
\kappa_4^2 e^{2\alpha} \delta \pi \xi = N(0) = \sqrt{2b} \delta \phi(0)
\]

\[
\begin{align*}
&= \int d^3 p N(0, p) \frac{4\sqrt{2}b}{3(\Delta + 4)} \left[ \frac{\Delta + 4}{4(\Delta + 3)} (-H \eta)^{-\frac{1}{\Delta + 3}} H^{(1)}_{\frac{b}{\Delta + 3}}(-p\eta) 
\right. \\
&\quad - \left. \frac{3b^2}{\Delta + 3} (-H \eta)^{-\frac{1}{\Delta + 3}} H^{(1)}_{\frac{2b+5}{\Delta + 3}}(-p\eta) \right] e^{i p x}.
\end{align*} \tag{86}
\]

As mentioned in section II, the effective theory for background spacetime is given by the BD theory. In the BD theory the correspondent equation is given by

\[
\Psi - \Phi = \frac{\delta \varphi_{BD}}{\varphi_{BD}} = \sqrt{2b} \delta \phi. \tag{87}
\]

As expected, the 0-mode solution can be described by the BD theory including anisotropic stress. At late times \(-p\eta \to 0\), \(N(0)\) behaves as \(N(0) = \text{const.}\).

The massive modes also contribute to the anisotropic stress;

\[
\kappa_4^2 e^{2\alpha} \delta \pi \xi = \frac{1}{2} \int d^3 p \int_{-\frac{1}{\Delta + 2}}^{\infty} d\tilde{m} N(\tilde{m}, p) (\sinh H y_0)^{\frac{\Delta}{3(\Delta + 2)}} B_{\frac{i \nu}{\Delta + 2} + \frac{i}{2}}^{\nu} (\cosh H y_0) (-H \eta)^{-\frac{1}{\Delta + 2}} \times
\]

\[
\left[ \frac{1}{3} H^{(1)}_{i \nu}(-p\eta) - \left( \frac{i \nu - \frac{3\Delta + 5}{3(\Delta + 2)}}{i \nu - \frac{3\Delta + 3}{\Delta + 2}} \right) H^{(1)}_{i \nu - 2}(-p\eta) \right] e^{i p x}. \tag{88}
\]

At the horizon crossing \(-p\eta = 1\), the ratio of the amplitude of massive mode to 0-mode modes has similar feature with the curvature perturbation. The ratio increases with \(r\), but it becomes constant for large \(r\) due to the mass gap. Note that the term proportional to \(H^{(1)}_{i \nu - 2}(-p\eta)\) does not give an additional divergence in the integration over \(\tilde{m}\).

However, the subsequent evolution of the anisotropic stress is quite different from the curvature perturbation. After the horizon crossing, the 0-mode remains constant. On the
other hand, the massive modes increase as \((-p\eta)^{-\left(2\Delta+5\right)/(\Delta+2)} \propto e^{-3(2\Delta+5)\alpha/2}\) due to the term proportional to \(H_{\nu-2}(-p\eta)\). Thus if \(\Delta < -\frac{5}{2}\), the massive modes will dominate over 0-mode and it seems to leave significant consequences in the inflationary brane world. Indeed, the massive modes on the brane Eqs.(69) grow for \((-p\eta) \to 0\). Then one might worry that this indicates the gravitational instability of the spacetime. However, the physical amplitude of the anisotropic stress is given by

\[
\delta \pi_E \propto e^{-2\alpha N|_{y=y_0}} \propto e^{-(6\Delta+19)\alpha/2},
\]

which always decreases with time for \(-2 > \Delta > -8/3\). The same situation occurs in the analysis of the radion in Randall-Sundrum de Sitter two branes. Let us consider two de Sitter branes in \(AdS_5\) spacetime. By imposing a fine tuning on the tensions of two branes, the distance between two branes, the radion, becomes constant. However, if one considers the perturbation of the radion, the radion has negative mass squared. In ref [33], the effect of the quantum radion was investigated. They found that the metric perturbations in Longitudinal gauge grow due to the instability of the radion. However, the physical amplitude of the anisotropic stress itself decays. The resolution is that the Longitudinal gauge is not really a good gauge. It is possible to find a gauge where all perturbations do not grow. Thus the growth of the metric perturbations does not directly imply the instability of the spacetime. In our case, the same arguments should be applied. In general, the curvature perturbation \(\mathcal{R}_c\) is the measure of the linear perturbation amplitude. In our background the curvature perturbation does not show the instability. Thus the growth of the metric perturbation is the artifact of the bad choice of the gauge.

Let us explicitly show that we can find a suitable gauge where all perturbations are not growing. Let us consider a 4-dimensional gauge transformation

\[
\eta \to \eta - \epsilon^\eta, \quad x^i \to x^i - \epsilon^i,
\]

By choosing \(\partial_\eta \epsilon = \epsilon^\eta\), the metric and the scalar field are transformed as

\[
\begin{align*}
\delta s^2 &= e^{2\alpha} \left(-(1 + 2\bar{\Phi})d\eta^2 + \left(1 - 2\bar{\Psi}\right)\delta_{ij}dx^idx^j\right), \\
\delta \Phi &= \delta \phi - \kappa (\partial_\eta \phi)\epsilon^\eta,
\end{align*}
\]

where

\[
\begin{align*}
\bar{\Phi} &= \Phi - \partial_\eta \epsilon^\eta - (\partial_\eta \alpha)\epsilon^\eta,
\end{align*}
\]
\[ \tilde{\Psi} = \Psi + (\partial_\eta \alpha) \epsilon^\eta, \]
\[ E = -\epsilon. \]

It is a non-trivial problem to find \( \epsilon^\eta \) that simultaneously eliminates the growing part of the \( \Phi(m), \Psi(m) \) and \( \delta \phi(m) \). In this case, it is possible to find such a solution. By taking
\[ \epsilon^\eta(m) = -\frac{1}{2H} \left( \frac{1}{i\nu - \frac{\Delta + 3}{\Delta + 2}} \right) (\sinh H y_0)^{\frac{\Delta}{2(\Delta + 2)}} B^\mu_{\frac{1}{2} + i\nu} (\cosh H y_0)(-H \eta)^{\frac{\Delta + 1}{\Delta + 2}} \times (H^\eta_{i\nu - 2}(-p\eta) + H^{(1)}_{i\nu}(-p\eta)), \]

the resultant metric and the scalar field become
\[ \tilde{\Psi}(\tilde{m}) = \frac{1}{3} \left( \frac{i\nu - 1}{i\nu - \frac{\Delta + 3}{\Delta + 2}} \right) (\sinh H y_0)^{\frac{\Delta}{2(\Delta + 2)}} B^\mu_{\frac{1}{2} + i\nu} (\cosh H y_0)(-H \eta)^{-\frac{1}{\Delta + 2}} H^{(1)}_{i\nu}(-p\eta), \]
\[ \tilde{\Phi}(\tilde{m}) = \frac{2}{3} \left( \frac{i\nu - 1}{i\nu - \frac{\Delta + 3}{\Delta + 2}} \right) (\sinh H y_0)^{\frac{\Delta}{2(\Delta + 2)}} B^\mu_{\frac{1}{2} + i\nu} (\cosh H y_0)(-H \eta)^{-\frac{1}{\Delta + 2}} H^{(1)}_{i\nu}(-p\eta), \]
\[ \tilde{\delta \phi}(\tilde{m}) = -\frac{\sqrt{2}}{4b} (\Delta + 2) \left( i\nu - \frac{1}{3(\Delta + 2)} \right) \left( \frac{i\nu - 1}{i\nu - \frac{\Delta + 3}{\Delta + 2}} \right) \times (\sinh H y_0)^{\frac{\Delta}{2(\Delta + 2)}} B^\mu_{\frac{1}{2} + i\nu} (\cosh H y_0)(-H \eta)^{-\frac{1}{\Delta + 2}} H^{(1)}_{i\nu}(-p\eta), \]
\[ \partial_\eta E(\tilde{m}) = -\epsilon^\eta \]

(94)

At late times \(-p\eta \to 0\), \( E \) behaves as
\[ E \propto (-\eta)^{-\frac{1}{\Delta + 2}} \propto e^{-3\alpha/2} \to 0. \]

Thus we can find a gauge where all perturbations remain small. The anisotropic shear \( E \) induced by massive modes decreases in an inflationary background \( \Delta < -2 \). Because the amplitude of \( E \) at the horizon crossing is suppressed due to the mass gap, the effect of the massive modes is always negligible.

In summary, in the anisotropic stress, the contribution of the massive modes will dominate over 0-mode and this causes the growth of metric perturbations in the Longitudinal gauge on the brane. However, we should carefully choose the gauge in evaluating the effect of these massive modes on metric perturbations. It is possible to find a "good" gauge where all perturbations remain small. In this gauge, the contribution of massive modes are not strong enough to cause the instability on the brane and massive modes do not leave significant consequences. So we conclude that, in our bulk inflaton model, the long-wavelength
perturbations are quite similar with the perturbations in the BD theory described by the 0-mode solution even for the high energy inflation.

V. CONCLUSION

In this paper, we derived the exact and analytic solutions for cosmological perturbations in dilatonic brane worlds. We used a background spacetime where the brane undergoes a power-law expansion due to the bulk scalar field. The effective theory on the brane is given by a Brans-Dicke theory. The interesting feature of this model is that we can derive the exact background solutions including the back-reaction of the bulk scalar field. Moreover the spacetime metric is separable with respect to the brane coordinate and the bulk coordinate. Then it is possible to solve the cosmological perturbations analytically.

Scalar perturbations are quite complicated because of the existence of the bulk scalar field. We can find variables which make the equations in the bulk to be diagonalized. But the boundary conditions for these variables are not diagonalized and also they effectively contain the time derivatives of the variables. The exception is the canonical variable $\omega_c$ of the action which is related to the curvature perturbation $R_c$ on the brane. The evolution equation for $\omega_c$ and the junction condition on the brane are decoupled from other variables. However, if one wants to derive the solutions for all metric perturbations, we should solve the complicated boundary conditions. Because this complexity is caused by the expansion of the brane, the derivation of the solutions for this problem is a central part of the calculations of cosmological perturbations in the brane world. This difficulty has prevented us from understanding the behavior of scalar perturbations on the brane. In this background, it is possible to derive the solution analytically. Then we found the solutions for all metric perturbations which properly satisfy the junction conditions on the brane.

As an application, we have investigated the behavior of the anisotropic stress on the brane induced by the bulk perturbations. We used the quantum theory for the 5-dimensional perturbations to determine the amplitude of the perturbations. As was shown in our previous paper, the massive KK modes do not significantly contribute to the curvature perturbations even in the high energy inflation where the Hubble scale on the brane is larger than the curvature scale in the bulk. In the anisotropic stress, the contribution of the massive modes is suppressed at the horizon crossing. Remarkably, however, the subsequent evolution of
the anisotropic stress is quite different from the curvature perturbations where massive modes rapidly decays. In the anisotropic stress, the contribution of massive modes seems to dominate over 0-mode after the horizon crossing for $\Delta < -5/2$. The difference comes from the junction conditions. The junction conditions effectively include the time derivative of the variables except for the junction condition for curvature perturbation. This causes the growth of the massive modes. However, it also causes the growth of metric perturbations in the Longitudinal gauge. Thus a careful choice of the gauge was required to discuss the effects of these massive modes. We found a suitable gauge where all perturbations remain small. In this gauge, there is an anisotropic shear which describes the contribution of the massive modes. It was shown that the anisotropic shear decays in our spacetime. Thus, the contribution of massive modes are not strong enough to cause the instability on the brane and massive modes do not leave significant consequences. We concluded that, in our bulk inflaton model, the perturbations are quite similar with the perturbations in the BD theory described by the 0-mode solution at large scales even for the high energy inflation.

Our analysis indicates that the behavior of anisotropic stress is quite complicated compared with the curvature perturbation. The behavior of the curvature perturbation can be determined by partially solving the perturbations. But in order to determine the anisotropic stress, we needed to derive full solutions for perturbations. This is indeed the generic feature of the brane world cosmological perturbations. For example, in the Randall-Sundrum model, the behavior of the curvature perturbation is determined only by the conservation of the energy-momentum tensor on the brane at large scales. But the anisotropic stress can be determined only if the gravitational field in the bulk is completely specified. So far the analysis of this anisotropic stress is very limited due to the difficulty of solving the full 5-dimensional perturbations. Our solutions would provide an interesting toy model for the investigation about the relation between the behavior the anisotropic stress on the brane and the bulk gravitational field. In this paper, we determine the boundary condition for perturbations in the bulk by quantum theory. But it is also possible to consider other boundary conditions. For example, it might be interesting to consider the boundary condition which allows the existence of dark radiation on the brane. Of course, in the inflationary background, the dark radiation does not play a role, but it is certainly important to fully understand the relation between the geometry of the bulk and the behavior of anisotropic stress, because it plays an essential role in the calculation of the CMB anisotropies in the brane worlds. For
this purpose, it may be useful to re-derive our results using a covariant curvature formalism because the geometrical meanings is manifest in this formulation. We will report these issues in the near future [34].

APPENDIX A: 5-DIMENSIONAL EINSTEIN EQUATION

1. 5-dimensional Einstein equation

For convenience, we present the 5-dimensional Einstein equation for the scalar perturbations of the metric

\[ ds^2 = -e^{2\phi(t,y)}(1+2\Phi)dt^2 + 2e^{2\gamma(t,y)}A dtdy + e^{2\gamma(t,y)}(1+2N)dy^2 + e^{2\alpha(t,y)}(1-2\Psi)dx^i dx_i. \] (A1)

\((t,t)\) component

\[- 3e^{-2\gamma}\ddot{N} - e^{-2\alpha}\nabla^2 N - 2e^{-2\alpha}\nabla^2 \Psi \]
\[- 3e^{-2\beta}\dot{\alpha}\dot{N} + 3e^{-2\beta}(2\dot{\alpha} + \dot{\gamma})\dot{\Psi} - 3e^{-2\gamma}\alpha'N' - 3e^{-2\gamma}(4\alpha' - \gamma'')\Phi' + 3e^{-2\gamma}\dot{\alpha}'A' \]
\[- 6e^{-2\gamma}\left[\alpha'' + 2(\alpha')^2 - \alpha'\gamma\right]N + 6e^{-2\beta}\dot{\alpha}(\dot{\alpha} + \dot{\gamma})\Phi + 3e^{-2\gamma}[\dot{\alpha}' + \dot{\alpha}(3\alpha' + \beta' - \gamma')]A \]
\[= \kappa^2(e^{-2\beta}(\Phi\dot{\phi}^2 + \dot{\phi}\delta\phi) - \Lambda'\delta\phi + e^{-2\gamma}(N\phi'^2 - \phi'\delta\phi')). \] (A2)

\((t,i)\) component

\[- \frac{1}{2}e^{-2\gamma}A' + e^{-2\beta}\dot{N} - 2e^{-2\beta}\dot{\Psi} \]
\[- e^{-2\beta}(\dot{\alpha} + \dot{\gamma})N - e^{-2\beta}(2\dot{\alpha} + \dot{\gamma})\Phi - \frac{1}{2}e^{-2\gamma}(\alpha' + 3\beta' - \gamma')A \]
\[= -\kappa^2e^{-2\beta}\dot{\phi}\delta\phi. \] (A3)

\((t,y)\) component

\[- 3e^{-2\beta}\dot{\Psi}' - 3e^{-2\beta}\alpha'\dot{N} - 3e^{-2\beta}(\alpha' - \beta')\dot{\Psi} - 3e^{-2\beta}\dot{\alpha}'\Phi' + \frac{1}{2}e^{-2\alpha}\nabla^2 A \]
\[= 6e^{-2\beta}(\dot{\alpha}' + \dot{\alpha}\alpha' - \dot{\alpha}\beta' - \alpha'\gamma)\Phi - 3e^{-2\gamma}[\alpha'' + (\alpha')^2 - \alpha'(\beta' + \gamma')]A \]
\[= \kappa^2e^{-2\beta}\left[A\phi'^2 + 2\dot{\phi}\phi' - \dot{\phi}\delta\phi' - \phi'\delta\phi\right]. \] (A4)

\((i,i)\) component

\[- e^{-2\beta}\ddot{N} + 2e^{-2\beta}\ddot{\Psi} - 2e^{-2\gamma}\dddot{\Psi} + e^{-2\gamma}\dddot{\Phi} + e^{-2\alpha}\nabla^2 N - e^{-2\alpha}\nabla^2 \Psi + e^{-2\alpha}\nabla^2 \Phi + e^{-2\gamma}\dot{A}' \]
\( - e^{-2\beta}(2\dot{\alpha} - \dot{\beta} + 2\dot{\gamma}) \dot{N} + 2e^{-2\beta}(3\dot{\alpha} - \dot{\beta} + \dot{\gamma}) \dot{\Psi} + e^{-2\beta}(2\alpha + \gamma) \dot{\Phi} + e^{-2\gamma}(2\alpha' + 2\beta' - \gamma') \dot{A} \\
- e^{-2\gamma}(2\alpha' + \beta') N' - 2e^{-2\gamma}(3\alpha' + \beta' - \gamma') \Psi' + e^{-2\gamma}(2\alpha' + 2\beta' - \gamma') \Phi' + e^{-2\gamma}(2\dot{\alpha} + \dot{\beta}) A' \\
- 2e^{-2\gamma} \left[ 2\alpha'' + 2\beta'' + 3(\alpha')^2 + 2\alpha' \beta' - 2\alpha' \gamma' + (\beta')^2 - \beta' \gamma' \right] N \\
+ 2e^{-2\beta} \left[ 2\alpha + \gamma + 3\alpha^2 - 2\dot{\alpha} \dot{\beta} + 2\dot{\alpha} \dot{\gamma} - \dot{\beta} \dot{\gamma} + \dot{\gamma}^2 \right] \Phi \\
+ e^{-2\gamma} \left[ 4\alpha' + 2\beta' + 2(3\alpha' + \beta' + \gamma') + \dot{\beta}(2\alpha' + 2\beta' - \gamma') - \dot{\gamma}(2\alpha' + \beta') \right] A \\
= \kappa^2 \left[ -e^{-2\beta}(\Phi \dot{\phi}^2 - \dot{\phi} \ddot{\phi}) - \Lambda' \delta \dot{\phi} - e^{-2\beta} \phi' \dot{\phi} \right] + e^{-2\gamma}(N \phi'^2 - \phi' \delta \phi') \right]. \quad (A5)

(i \neq j) \text{ component} \\
N - \Psi + \Phi = 0. \quad (A6)

(i, y) \text{ component} \\
\begin{align*}
2 \Psi' - \Phi' - \frac{1}{2} \dot{A} + (2\alpha' + \beta') N + (\alpha' - \beta') \Phi - \frac{1}{2}(\dot{\alpha} + \dot{\beta} + \dot{\gamma}) A \\
= \kappa^2 \dot{\phi} \delta \phi. \quad (A7)
\end{align*}

(y, y) \text{ component} \\
\begin{align*}
3 e^{-2\beta} \ddot{\Psi} - 2e^{-2\alpha} \nabla^2 \Psi + e^{-2\alpha} \nabla^2 \Phi + 3e^{-2\beta}(4\dot{\alpha} - \dot{\beta}) \dot{\Psi} + 3e^{-2\beta} \dot{\phi} \dot{\phi} + 3e^{-2\gamma} \alpha' \dot{A} \\
- 3e^{-2\gamma}(2\alpha' + \beta') \Psi' + 3e^{-2\gamma} \alpha' \Phi' - 6e^{-2\gamma} \alpha'(\alpha' + \beta') N \\
+ 6e^{-2\beta}(\dot{\alpha} + 2\alpha^2 - \dot{\alpha} \dot{\beta}) \Phi + 3e^{-2\gamma} \left[ \dot{\alpha'} + \alpha'(3\dot{\alpha} + \dot{\beta} - \dot{\gamma}) \right] A \\
= \kappa^2 \left[ -e^{-2\beta}(\dot{\phi} \dot{\phi}' - \dot{\phi} \ddot{\phi}') - \Lambda' \delta \dot{\phi} - e^{-2\gamma}(\phi'^2 N - \phi' \delta \phi') \right]. \quad (A8)
\end{align*}

2. Equations for \( \omega_i \)

In order to derive the evolution equations for \( \omega_i \) in the bulk, it is easy to use the coordinate Eq.(15). By combining the Einstein equations and the equation of motion for the scalar field, we first get the evolution equations for metric perturbations \( \psi, \phi, \Gamma, G \) and \( \delta \phi; \)

\begin{align*}
\partial_z^2 \psi + 3(\partial_z Q) \partial_z \psi - p^2 \psi - \partial_z^2 \psi = \ -2(\partial_z^2 Q) \Gamma - 2\kappa(\partial_z Q)(\partial_z \phi) \delta \phi + \frac{2}{3} \kappa^{-1} \frac{d \Lambda}{d \phi} e^{2Q} \delta \phi, \\
\partial_z^2 \Gamma + 3(\partial_z Q) \partial_z \Gamma - p^2 \Gamma - \partial_z^2 \Gamma = \ - \left( \partial_z^2 Q + 3(\partial_z Q)^2 - \kappa^2 (\partial_z \phi)^2 \right) \Gamma \\
- \kappa \left( (\partial_z Q)(\partial_z \phi) - \partial_z^2 \phi \right) \delta \phi - \frac{1}{3} \kappa^{-1} \frac{d \Lambda}{d \phi} e^{2Q} \delta \phi, \\
\partial_z^2 \delta \phi + 3(\partial_z Q) \partial_z \delta \phi - p^2 \delta \phi - \partial_z^2 \delta \phi = \ 2\kappa(\partial_z^2 \phi) \Gamma + 2\kappa^2(\partial_z \phi)^2 \delta \phi + \kappa^{-1} \frac{d \Lambda}{d \phi} e^{2Q} \delta \phi, \\
\partial_z^2 G + 3(\partial_z Q) \partial_z G - p^2 G - \partial_z^2 G = \ -3(\partial_z^2 Q) G \quad (A9)
\end{align*}
These equations can be diagonalized using $\omega_i$ defined in Eqs. (46) [9]. Using
\[
\partial_z Q = \frac{2}{3(\Delta + 2)} \frac{1}{z}, \quad \kappa \partial_z \phi = 3\sqrt{2}b\partial_z Q, \quad \frac{d\Lambda}{d\phi} e^{2q} = -2\sqrt{2}b\kappa \frac{\Delta}{(\Delta + 2)^2} \frac{1}{z^2},
\]
we get the evolution equations for $\omega_i$.

The constraint equations are also easy to be derived using the metric Eq.(15). The $(\tau, i)$ component of Einstein equation is given by
\[
-\frac{1}{2}(\partial_z G + 3(\partial_z Q)G) - 2\partial_x \psi + \partial_x \Gamma = 0.
\]
Rewriting these equations by $\omega_i$, we get
\[
-\frac{1}{2}(\partial_z \omega_A + 3(\partial_z Q)\omega_A) + \partial_x \omega_A = 0.
\]
Then performing the coordinate transformation, we get Eq.(60). The remaining two constraint equations can be derived in the same way or can be obtained directly in our spacetime.

**APPENDIX B: DERIVATION OF SOLUTIONS FOR SCALAR PERTURBATIONS**

1. **0-mode**

In order to derive the solution the formula for the derivative of Hunkel functions is useful;
\[
\begin{align*}
\frac{d}{d\eta} \left[ (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\frac{1}{\Delta+2}} (-p\eta) \right] &= -p (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\frac{1}{\Delta+2}} (-p\eta) \\
\frac{d}{d\eta} \left[ (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\frac{2}{\Delta+2}} (-p\eta) \right] &= p (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\frac{2}{\Delta+2}} (-p\eta) \\
&+ \frac{2(\Delta + 3)}{\Delta + 2} H (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\frac{2}{\Delta+2}} (-p\eta).
\end{align*}
\]
Let us find the solution for $\omega_i$ with
\[
\omega_c = (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\frac{1}{\Delta+2}} (-p\eta).
\]
First we use the constraint equation Eq.(60). Because this equation contains the first derivative with respect to time, the solution for $\omega_i$ necessarily includes $H_{-\frac{2}{\Delta+2}} (-p\eta)$ because it is only the choice that can eliminate $H_{-\frac{2}{\Delta+2}} (-p\eta)$ which arises when we take the time derivative.
of \( H_{-\frac{1}{\Delta+2}}(-p\eta) \). Then we assume that \( \omega_i \) contains \( H_{-\frac{1}{\Delta+2}}(-p\eta) \) and \( H_{-\frac{2\Delta+5}{\Delta+2}}(-p\eta) \). Using the fact that \( \omega_i \) should satisfy the evolution equation in the bulk, the \( y \)-dependence of the variables is automatically determined. Then substituting the ansatz for the variables into the junction conditions, we can determine all coefficients except for an over-all normalization. It is a non-trivial check that these solutions indeed satisfy three constraint equations. It is easy to verify that these solutions indeed satisfy the constraint equations.

2. Massive modes

For massive modes, the following formula is useful;

\[
\frac{d}{d\eta} \left[ (-H\eta)^{-\frac{1}{\Delta+2}} H_{-\nu}(-p\eta) \right] = -p(-H\eta)^{-\frac{1}{\Delta+2}} H_{-\nu-1}(-p\eta)
\]

\[
+ H \left( \frac{1}{\Delta+2} + i\nu \right) (-H\eta)^{-\frac{1}{\Delta+2} \frac{1}{\nu+1}} H_{-\nu}(-p\eta).
\]

\[
\frac{d}{d\eta} \left[ (-H\eta)^{-\frac{1}{\Delta+2}} H_{-2+i\nu}(-p\eta) \right] = p(-H\eta)^{-\frac{1}{\Delta+2}} H_{-\nu-1}(-p\eta)
\]

\[
+ H \left( \frac{2\Delta+5}{\Delta+2} - i\nu \right) (-H\eta)^{-\frac{1}{\Delta+2} \frac{1}{\nu+2}} H_{-2+i\nu}(-p\eta).
\]

(B3)

and

\[
\frac{d}{dy} \left[ \sinh (Hy) \frac{\Delta}{\Delta + 2} B_{\mu+2}^\beta (\cosh Hy) \right] = H \sinh (Hy) \frac{\Delta}{\Delta + 2} B_{\mu+1}^\beta (\cosh Hy),
\]

\[
\times \left[ - \coth (Hy) \frac{\Delta + 4}{\Delta + 2} B_{\mu+2}^\beta (\cosh Hy) + \left( \beta - \frac{\Delta + 4}{2(\Delta + 2)} \right) \left( \beta + \frac{3\Delta + 8}{2(\Delta + 2)} \right) B_{\mu+1}^\beta (\cosh Hy) \right],
\]

\[
\frac{d}{dy} \left[ \sinh (Hy) \frac{\Delta}{\Delta + 2} B_{\mu+1}^\beta (\cosh Hy) \right] = H \sinh (Hy) \frac{\Delta}{\Delta + 2} \left[ \coth (Hy) B_{\mu+2}^\beta (\cosh Hy) + B_{\mu+2}^\beta (\cosh Hy) \right],
\]

\[
= H \sinh (Hy) \frac{1}{\Delta + 2} \left[ \frac{\Delta + 4}{2(\Delta + 2)} \right] \left( \beta + \frac{\Delta + 4}{2(\Delta + 2)} \right) B_{\mu+1}^\beta (\cosh Hy),
\]

\[
\times \left[ - \frac{2}{\Delta + 2} \coth (Hy) B_{\mu+1}^\beta (\cosh Hy) + \left( \beta + \frac{\Delta}{2(\Delta + 2)} \right) \left( \beta + \frac{\Delta + 4}{2(\Delta + 2)} \right) B_{\mu+1}^\beta (\cosh Hy) \right].
\]

(B4)

As for the 0-mode, the ansatz for the solutions are determined so that they can satisfy the constraint equation Eq. (60). We assume

\[
\omega_c = (-H\eta)^{-\frac{1}{\Delta+2}} \sinh (Hy) \frac{\Delta}{\Delta + 2} C_c B_{-\nu-1+i\nu}^\mu (-p\eta)
\]

\[= (-H\eta)^{-\frac{1}{\Delta+2}} \sinh (Hy) \frac{\Delta}{\Delta + 2} C_c B_{-\nu-1+i\nu}^\mu (-p\eta) \]
\[
\omega_\psi = (-H\eta)^{-\frac{1}{\Delta+2}}(\sinh H y)^{\frac{\Delta}{2(\Delta+2)}}
\times \left[ C_\psi B_{\mu}^{\mu+1}(\cosh H y)H_\nu(-\eta) + D_\psi B_{\mu}^{\mu+1}(\cosh H y)H_{\nu-2} \right]
\]
\[
\omega_A = (-H\eta)^{-\frac{1}{\Delta+2}}(\sinh H y)^{\frac{\Delta}{2(\Delta+2)}}
\times \left[ C_A B_{\mu}^{\mu+1}(\cosh H y)H_\nu(-\eta) + D_A B_{\mu}^{\mu+1}(\cosh H y)H_{\nu-2} \right]
\]
\[
\omega_N = (-H\eta)^{-\frac{1}{\Delta+2}}(\sinh H y)^{\frac{\Delta}{2(\Delta+2)}}
\times \left[ C_N B_{\mu}^{\mu+2}(\cosh H y)H_\nu(-\eta) + D_N B_{\mu}^{\mu+2}(\cosh H y)H_{\nu-2} \right]
\]

(B5)

where \( B_3^\alpha \) is given by Eq.(64). First let us consider the constraint equation Eq.(60). Because this equation only contains \( \omega_\psi \) and \( \omega_A \), it is easy to determine the coefficients using the formula for derivatives. We get

\[
C_\psi = -\frac{1}{2} \left( i\nu - \frac{1}{\Delta + 2} \right) C_A, \quad D_\psi = \frac{1}{2} \left( i\nu - \frac{2\Delta + 3}{\Delta + 2} \right) D_A, \quad D_A = -\left( \frac{i\nu - \frac{\Delta + 1}{\Delta + 2}}{i\nu - \frac{\Delta + 3}{\Delta + 2}} \right) C_A.
\]

(B6)

Next we use the junction conditions. From Eq.(58), \( D_N \) is determined as

\[
D_N = -\frac{1}{2} \left( i\nu - \frac{1}{\Delta + 2} \right) C_A.
\]

(B7)

From Eq.(57), \( C_N \) is given by

\[
C_N = \frac{\Delta + 2}{2} C_A.
\]

(B8)

We should note that at this time, the problem becomes non-trivial because the equation should be satisfied by the coefficients which have already determined. The point is that, on the brane, \( B_{\mu}^{\mu+1}(\cosh H y_0) = 0 \). Using this fact, we can show that

\[
\sinh H y_0 \cosh H y_0 B_{\mu}^{\mu+2}(\cosh H y_0) = \frac{1}{2(i\nu - 1)} \left( i\nu - \frac{\Delta + 1}{\Delta + 2} \right) \left( i\nu - \frac{2\Delta + 5}{\Delta + 2} \right) B_{\mu}^{\mu+1}(\cosh H y_0)
\]

\[
- \left( i\nu - \frac{\Delta + 1}{\Delta + 2} \right) \cosh^2 H y_0 B_{\mu}^{\mu+1}(\cosh H y_0).
\]

(B9)

Using this relation, it is shown that the junction condition Eq.(58) can be satisfied. Finally, we use Eq.(57). We get

\[
C_c = -\frac{\sqrt{2}}{4b} (\Delta + 2) \left( i\nu - \frac{1}{\Delta + 2} \right) (i\nu - 1).
\]

(B10)

Here we again used Eq.(B9) in order to show that the junction condition is satisfied. Now all coefficients are determined except for an over-all normalization. It remains a task to verify
that these solutions satisfy the remaining two constraint equations. In order to show that, we use the formula for associate Legendre function and Hankel function

\[ B_{\beta+\frac{1}{2}}^\mu (z) + 2(\mu + 1)z(z^2 - 1)^{-\frac{1}{2}}B_{\beta}^{\mu+1}(z) = (\beta + \mu)(\beta + \mu + 1)B_{\beta}^\mu (z), \]

\[ H_{\beta-1}(z) + H_{\beta+1}(z) = 2\beta z^{-1}H_{\beta}(z). \]  \hspace{1cm} (B11)

Then after long calculations, it is possible to show that the above solution indeed satisfy the constraint equations.

In order to derive the solutions for metric perturbations on the brane, we used Eq. (B11) and the following equations;

\[ \left( i\nu - \frac{2\Delta + 3}{\Delta + 2} \right) \left( i\nu - \frac{\Delta + 1}{\Delta + 2} \right) B_{\frac{1}{2}+i\nu}^{\mu} (\cosh H y_0) \]

\[ = \left( i\nu - \frac{1}{\Delta + 2} \right) \left\{ 2(i\nu - 1) \sinh^2 H y_0 + \left( i\nu - \frac{\Delta + 3}{\Delta + 2} \right) \right\} B_{\frac{1}{2}+i\nu}^{\mu} (\cosh H y_0), \]  \hspace{1cm} (B12)

\[ B_{-\frac{1}{2}+i\nu}^{\mu+1} (\cosh H y_0) = \left( i\nu + \frac{1}{\Delta + 2} \right) \left\{ \left( i\nu + \frac{1}{\Delta + 2} \right) + 2(i\nu - 1) \sinh^2 H y_0 \right\} B_{-\frac{1}{2}+i\nu}^{\mu} (\cosh H y_0), \]  \hspace{1cm} (B13)

which can be derived using \( B_{-\frac{1}{2}+i\nu}^{\mu+1} (\cosh H y_0) = 0. \)

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