STRONG SOLUTIONS TO COMPRESSIBLE BAROTROPIC VISCOELASTIC FLOW WITH VACUUM

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Abstract. We consider strong solutions to compressible barotropic viscoelastic flow in a domain \( \Omega \subset \mathbb{R}^3 \) and prove the existence of unique local strong solutions for all initial data satisfying some compatibility condition. The initial density need not be positive and may vanish in an open set. Inspired by the work of Kato and Lax, we use the contraction mapping principle to get the result.

1. Introduction. Materials in which the stress at a point depends on the entire history of the strain are called viscoelastic material, which can be viewed as the intermediate state between the fluid and solid. In general, viscoelastic material exhibits a combination of the microscopic elastic behavior, such as memory effects, and the macroscopic fluid properties. Many complicated hydrodynamic and rheological behaviors of complex fluids can be regarded as consequences of internal elastic properties. For example, Stokes fluids can be considered as special cases of viscoelastic materials. Viscoelastic flows play an important part in studying viscoelastic materials and have a wide range of applications, which give not only the rich and complicated rheological phenomena, but also formidable challenges in analysis and numerical simulations. This paper is concerned with the initial boundary value problem to the following compressible viscoelastic fluid system of Oldroyd model

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P(\rho) &= \mu \Delta u + \lambda \text{div}u + \text{div}(\rho F F^T), \\
\partial_t F + u \cdot \nabla F &= \nabla u F,
\end{align*}
\]

in \( (0, T) \times \Omega \), where the unknown functions \( \rho = \rho(x, t) \) denotes the density, \( u(x, t) \in \mathbb{R}^3 \) the velocity of fluid and \( F \in M^{3 \times 3} \) the deformation gradient. The symbol \( \otimes \) denotes the Kronecker tensor product, \( F^T \) means the transpose matrix of \( F \).

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Generally speaking, the pressure $P$ depends on the density and temperature. In that case, the system (1.1)-(1.3) is not closed and should be complemented by the energy equation. However, there are physically relevant situations in which we assume that the fluid flow is barotropic, i.e., the pressure depends only on the density. This is the case when either the temperature or the entropy is supposed to be constant. The typical expression is $P(\rho) = \rho^\gamma$. Here, we assume that the pressure $P$ satisfies a more general law:

$$P : [0, \infty) \to \mathbb{R}$$

is a locally Lipschitz continuous function and $P(0) = 0$. (1.4)

The viscosity coefficients $\mu$ and $\lambda$ satisfy the physical conditions:

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$ (1.5)

$\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Supplement the system with the boundary and initial conditions

$$u|_{\partial \Omega} = 0,$$ (1.6)

$$\left.\left(\rho, u, F\right)\right|_{t=0} = \left(\rho_0(x), u_0(x), F_0(x)\right).$$ (1.7)

The compressible viscoelastic system (1.1) – (1.3) is a combination of the compressible Naiver-Stokes equations of fluid dynamics and equation (1.3), which corresponds to the so-called Hookean linear elasticity. In deed, we refer to (1.1) as the mass equation and (1.2) as the momentum conservation equation. In the recent years, there have been lots of studies on viscoelastic flows by physicists and mathematicians, such as [4, 5, 6, 12, 13, 14, 15, 20, 21, 22, 23], due to its physical importance, complex, rich phenomena and mathematical challenges.

Without the deformation gradient $F$, the compressible viscoelastic system reduces to the compressible Naiver-Stokes equations. There is a huge literatures on the compressible Naiver-Stokes equations, for instance, [1, 8, 16, 18, 19].

When $\rho$ is constant, the system (1.1) – (1.3) becomes the equations of incompressible viscoelastic flow and there exists rich results, see [12, 13, 14, 15] and references cited therein.

As to the compressible viscoelastic case, the mathematical analysis is much more complicated. Comparing with Navier-Stokes equations, we will encounter extra difficulties in studying the compressible viscoelastic system. From the viewpoint of mathematical structure, the mathematical analysis is difficult due to the loss of dissipation of the deformation gradient in (1.3). From the viewpoint of partial differential equation, the system (1.1) – (1.3) is a highly nonlinear system coupling between hyperbolic equations and parabolic equations. Thus, the extension of known results for the Navier-Stokes equations to the compressible viscoelastic flows is not simple. In spite of these, there is much recent important progress for the compressible viscoelastic flows. More precisely, Qian [23] proved the existence and uniqueness of global strong solution for the initial boundary value problem near the equilibrium state in $H^2$. Hu and Wang [5], proved the global well-posedness with small data in Besov space. In [4], Hu and Wang used the Schauder-Tychonoff fixed point theorem obtained the local existence of strong solutions in $\mathbb{R}^3$, with initial density has strictly lowered bound and upper bound. It should be pointed out that our viscoelastic model is called Oldroyd-A model. A related model is Oldroyd-B model, which have been attracted attention in the past decade. There is some differences between the two model. Readers who are interested in the Oldroyd-B model can refer to [3, 7, 10].
The aim of this paper is to prove the existence of unique local strong solutions to (1.1)-(1.3) with \( \inf \rho_0 = 0 \). Here we use the contraction mapping to obtain the result, in some sense simply and extend the result of [1] and [4]. Before stating the local existence result, we need to specify the definition of strong solutions which we will address.

**Definition 1.1.** For \( T > 0 \), \((\rho, u, F)\) is called a strong solution to the compressible viscoelastic flow (1.1) – (1.3) in \((0, T) \times \Omega\), if for some \( q \in (3, 6] \),
\[
0 \leq \rho \in C([0, T]; W^{1,q}(\Omega)), \quad \rho_t \in C([0, T]; L^q(\Omega)),
\]
\[
F \in C([0, T]; W^{1,q}(\Omega)), \quad F_t \in C([0, T]; L^q(\Omega)),
\]
\[
u \in C([0, T]; H^1_0(\Omega) \cap H^2(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)),
\]
\[u_t \in L^2((0, T); H^1_0(\Omega)), \quad \sqrt{\rho}u_t \in L^\infty((0, T); L^2(\Omega)),\]
and \((\rho, u, F)\) satisfies (1.1) – (1.3) a.e. in \((0, T) \times \Omega\).

Our main result can be summarized as follows:

**Theorem 1.2.** Assume that \( P \) satisfies (1.4), for some \( q \in (3, 6] \)
\[
0 \leq \rho_0 \in W^{1,q}(\Omega), \quad F_0 \in W^{1,q}(\Omega),
\]
\[u_0 \in H^1_0(\Omega) \cap H^2(\Omega),\]
and for positive constant \( r_0 \)
\[
\|\rho_0\|_{W^{1,q}} + \|F_0\|_{W^{1,q}} + \|u_0\|_{H^1_0 \cap H^2} \leq r_0.
\]
If, in addition, the following compatibility condition holds
\[
-\mu \Delta u_0 - \lambda \nabla \text{div} u_0 + \nabla P(\rho_0) - \text{div}(\rho_0 F^T_0 F_0) = \rho_0^g (1.8)
\]
for some \( g \in L^2 \), then there exist a positive time \( T_0 \) and a unique strong solution \((\rho, u, F)\) for (1.1) – (1.3) in \((0, T_0) \times \Omega\).

This paper is written as follows. In Section 2, we give the notations and working function space which will be needed in later analysis. In Sections 3, we consider a linearized problem and derive some local estimates for the solutions independent of the lower bound of initial density. In Section 4, we use the contraction mapping principle to get the existence and uniqueness of local strong solutions.

**2. Preliminaries.** First, it is necessary for us to give some notations by reason of the convenience of discussions.

1. \( \int_\Omega f \, dx = \int_0^T \int_\Omega f \, dx \, dt = \int_0^T \int_\Omega f \, dx \); 
2. \( L^p(0, T; X) \) \((1 \leq p \leq \infty)\) is the set of Bochner measurable \( X \)-valued time-dependent functions \( \varphi \) such that \( t \mapsto ||\varphi||_X \) belongs to \( L^p(0, T) \); 
3. For two \( 3 \times 3 \) matrices \( E = (E_{ij}) \), \( F = (F_{ij}) \) denotes the scalar product between \( E \) and \( F \) by \( E : F = \sum_{i,j=1}^{3} E_{ij} F_{ij} \).

Now, we will introduce the working function space which plays an important role in the process of proof:
\[
W = L^\infty(0, T; H^1_0 \cap H^2) \cap L^2(0, T; W^{2,q}) \cap W^{1,2}(0, T; H^1_0),
\]
with norm
\[
||v||_W = ||v||_{L^\infty(0, T; H^1_0 \cap H^2)} + ||v||_{L^2(0, T; W^{2,q})} + ||v_t||_{L^2(0, T; H^1_0)},
\]
Remark 1. Before the proof, we point out that the approach to proving Theorem 1.1 is to apply the contraction mapping principle. Since the system (1.1)-(1.3) is of mixed hyperbolic-parabolic type and the initial density may vanish, we encounter a well-known difficulty in the theory of symmetric quasilinear hyperbolic systems. For these systems, contraction cannot be proved in the usual setting, that is, to consider self-mapping and contraction in the same regularity class $W$. To resolve this problem, Kato [9] and Lax [11] offered an ingenious idea by studying contraction in a larger space. Taking up this idea, we are able to establish the contraction in the space $L$ (see in Section 4). Chu et. al. [2] adopted the same idea to tackle the compressible liquid crystal system.

3. Existence for the linearized equations. In this section, we reformulate the nonlinear equation (1.1)-(1.3) such that the left-hand becomes linear and the starting problem can be transferred to a fixed point equation.

$$\partial_t \rho + \text{div}(\rho v) = 0,$$

$$\rho \partial_t u - \mu \Delta u - \lambda \text{div} u = -\rho v \cdot \nabla P(\rho) + \text{div}(\rho FF^T),$$

$$\partial_t F + v \cdot \nabla F = \nabla v F,$$

with the given $v \in W$, the boundary condition (1.6) and initial conditions

$$(\rho, u, F)\big|_{t=0} = (\rho_0(x) + \delta, u_0(x), F_0(x)).$$

We denote $\rho^\delta = \rho_0 + \delta$, where $\delta > 0$ is a constant and $\rho_0 \geq 0$.

If the initial density vanishes from below, we cannot expect the density $\rho$ is bounded away from zero. As a result, the lack of a positive lower bound of $\rho$ causes (3.1) to become a degenerate linear parabolic equation. This prevents us from using the standard argument to construct the local solutions. For this reason, we consider the linearized problem (3.1)-(3.3) with initial density bounded away from zero and derive some uniform bounds, which are independent of the lower bounds of initial density. Firstly, we solve out the density, and obtain estimates for density.

Lemma 3.1. For given $v$ with $||v||_W \leq A$, there exists a unique solution $\rho$ to the linear transport problem (3.1) and (3.4) such that

$$\|\rho\|_{L^\infty(0,T;W^{1,q})} \leq C \|\rho^\delta_0\|_{W^{1,q}} (1 + T^{1/2}A) \exp(CT^{1/2}A),$$

$$\|\rho_t\|_{L^\infty(0,T;L^q)} \leq C \|\rho_0^\delta\|_{W^{1,q}} A \exp(CT^{1/2}A),$$

where $C$ is a constant.

Proof. The transport problem (3.1) can be rewritten as

$$\partial_t \rho + v \cdot \nabla \rho + \rho \text{div} v = 0.$$

Multiplying (3.7) by $\rho^{q-1}$ yields

$$\frac{1}{q} \partial_t \rho^q + \frac{1}{q} v \cdot \nabla \rho^q + \rho^q \text{div} v = 0.$$

Taking integration on $\Omega$ and integration by parts, we have

$$\frac{d}{dt} \int \rho^q dx = (1 - q) \int \rho^q \text{div} v dx$$

$$\leq C ||\text{div} v||_{L^\infty} \int \rho^q dx,$$
which implies by Gronwall’s inequality and the Sobolev’s embedding $W^{1,q} \hookrightarrow L^\infty$

\[ \|\rho\|_{L^q} \leq \exp(C T^{\frac{q}{2}} A) \|\rho_0\|_{L^q}. \]

Using $\nabla$ to act on (3.7) gives

\[
\partial_t \nabla \rho + \nabla \cdot \nabla \rho + v \cdot \nabla (\nabla \rho) + \nabla \rho \text{div} v + \rho \text{div} \nabla v = 0. \tag{3.9}
\]

Multiplying (3.9) by $|\nabla \rho|^{q-2} \nabla \rho$ yields

\[
\frac{1}{q} \partial_t |\nabla \rho|^q + \nabla \cdot |\nabla \rho|^q + \frac{1}{q} v \cdot \nabla |\nabla \rho|^q + |\nabla \rho|^q \text{div} v + \rho |\nabla \rho|^{q-2} \nabla \rho \text{div} v = 0.
\]

Taking integration on $\Omega$ and integration by parts, we have

\[
\frac{d}{dt} \int |\nabla \rho|^q dx = (q - 1) \int |\nabla \rho|^q \text{div} v dx + q \int |\nabla \rho|^q \text{div} v dx + q \int \rho |\nabla \rho|^{q-2} \nabla \rho \text{div} v dx
\]

\[
\leq C \|\nabla v\|_{L^\infty} \int |\nabla \rho|^q dx + C \|\rho\|_{L^\infty} (\int |\nabla \rho|^q dx)^{1 - \frac{1}{q}} (\int |\nabla^2 v|^q dx)^{\frac{1}{q}}.
\]

By the Sobolev’s embedding $\|\rho\|_{L^\infty} \leq C \|\rho\|_{W^{1,q}}$, we see

\[
\frac{d}{dt} \int |\nabla \rho|^q dx \leq C \|v\|_{W^{2,q}} \int (|\rho|^q + |\nabla \rho|^q) dx. \tag{3.10}
\]

The inequality (3.10) and (3.7) leads to the following

\[
\frac{d}{dt} \int (|\rho|^q + |\nabla \rho|^q) dx \leq C \|v\|_{W^{2,q}} \int (|\rho|^q + |\nabla \rho|^q) dx, \tag{3.11}
\]

which implies (3.5).

(3.6) can be obtained similarly by taking 1-th derivative with respect to $t$ of (3.7).

Combining the hyperbolic structure of (3.3) and the corresponding previous results, we can solve the deformation gradient $F$ similarly as Lemma 3.1.

**Lemma 3.2.** For given $v$ with $\|v\|_W \leq A$, there exists a unique solution $F$ which satisfies (3.3) such that

\[
\|F\|_{L^\infty(0,T;W^{1,q})} \leq C \|F_0\|_{W^{1,q}} (1 + T^{\frac{1}{2}} A) \exp(C T^{\frac{1}{2}} A),
\]

\[
\|F_t\|_{L^\infty(0,T;L^q)} \leq C \|F_0\|_{W^{1,q}} A \exp(C T^{\frac{1}{2}} A).
\]

**Proof.** The existence can be founded in [23], and estimates can be obtained similarly as Lemma 3.3 in [4]. \qed

The next lemma gives the estimates on the velocity.

**Lemma 3.3.** Under the conditions $\rho_0^0 = \rho_0 + \delta$, suppose $\|v\|_W \leq A$, there exists a unique solution $u$ which satisfies (3.2), and a constant $K_1$, depending only on initial data, such that for $T$ small enough,

\[
\|u\|_{L^\infty(0,T;H^1 \cap H^2)} + \|u\|_{L^2(0,T;W^{2,q})} + \|u_t\|_{L^2(0,T;H^1)} \leq K_1.
\]
Proof. Since the transport equation (3.7), initial condition and Lemma 3.1, we get \( \rho(x, t) > 0 \). The existence of the solution to (3.2) can be obtained by a semidiscrete Galerkin method of parabolic equations, or by Lax-Milgram theorem as \([4, 24]\).

To ensure the higher regularity, specially as to the term \( u_t \), we need some compatibility condition. The same process for higher regularity has been obtained in the whole space \( \mathbb{R}^3 \) in [4]. However, there are some differences in our proof. In the following, we will give the detail of proof and show the use of compatibility condition.

In order to derive estimate for \( \nabla u_t \), we differentiate (3.2) with respect to \( t \) and get

\[
\rho \partial_{tt}^2 u - \mu \Delta \partial_t u - \lambda \nabla \text{div} \partial_t u = - \partial_t \rho \partial_t u - \partial_t \rho v \cdot \nabla v - \rho \partial_t v \cdot \nabla v - \rho v \cdot \nabla \partial_t v - \nabla \partial_t P + \partial_t \text{div}(\rho F F^T).
\]

Multiplying the identity by \( u_t \) and using integration by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho|u_t|^2 + \int_{\Omega} \mu|\nabla u_t|^2 + \lambda|\text{div} u_t|^2 = - \frac{1}{2} \int_{\Omega} \partial_t \rho |\partial_t u|^2 - \int_{\Omega} \rho \partial_t u \cdot \nabla v \partial_t u - \int_{\Omega} \rho v \cdot \nabla \partial_t u \cdot \partial_t u + \int_{\Omega} \partial_t \rho \text{div} \partial_t u + \int_{\Omega} \partial_t (\rho F F^T) : \nabla u_t = \sum_{k=1}^{6} I_k.
\]  

Using the continuity equation and the Gagliardo-Nirenberg inequality, we get

\[
|I_1| = \left| - \frac{1}{2} \int_{\Omega} \rho v \nabla \partial_t u \right| = \int_{\Omega} \rho v \partial_t u \nabla \partial_t u \leq \int_{\Omega} \rho v |u_t|^2 (|v|^2) dx + \epsilon \int_{\Omega} |\nabla u_t|^2 dx \leq \epsilon ||\nabla u_t||^2_{L^2} + CA^2 \exp(CT^{\frac{1}{2}} A) ||\sqrt{\rho u_t}||^2_{L^2},
\]

\[
|I_2| = \left| \int_{\Omega} \text{div}(\rho v v) \cdot \nabla \partial_t u \right| = \int_{\Omega} \rho v \nabla (v \cdot \nabla \partial_t u) \leq \int_{\Omega} \rho v |\nabla v|^2 \partial_t u + \rho |v|^2 \nabla^2 v |\partial_t u| + \rho |v| ||\nabla v|| \nabla \partial_t u \leq \epsilon ||\nabla u_t||^2_{L^2} + A^6 ||\sqrt{\rho u_t}||^2_{L^2} + C \exp(CT^{\frac{1}{2}} A) + C(\epsilon) A^4 \exp(CT^{\frac{1}{2}} A),
\]

\[
|I_3| = \int_{\Omega} \rho |v_t||\nabla v||u_t| \leq A^6 \int_{\Omega} \rho |u_t|^2 + A^{-6} \int_{\Omega} \rho |v_t|^2 |v|^2 \leq A^6 ||\sqrt{\rho u_t}||^2_{L^2} + CA^{-4} \exp(CT^{\frac{1}{2}}) ||v_t||^2_{L^2},
\]

\[
|I_4| = \int_{\Omega} \rho |v||\nabla v||u_t| \leq A^6 ||\sqrt{\rho u_t}||^2_{L^2} + CA^{-4} \exp(CT^{\frac{1}{2}}) ||v_t||^2_{L^2},
\]

\[
|I_5| \leq ||P||_{L^2} ||\nabla u_t||_{L^2} \leq ||P'||_{L^\infty} ||\rho_t||_{L^2} ||\nabla u_t||_{L^2} \leq \epsilon ||\nabla u_t||^2_{L^2} + C(\epsilon) A^2 \exp(CT^{\frac{1}{2}} A),
\]

\[
|I_6| \leq ||\nabla p||_{L^2} ||v||_{L^\infty} ||F||^2_{L^\infty} + ||\rho||_{L^\infty} ||\nabla v||_{L^2} ||F||^2_{L^\infty}
\]

\[= \sum_{k=1}^{6} I_k. \]
To estimate $||\sqrt{\rho}u_t||^2_{L^2}$, we choose a sufficiently small $\epsilon$, and integrating with respect to $t$, we have

$$
\int_0^T \rho |u_t|^2 + \int_0^T |\nabla u_t|^2 \leq \int_0^T \rho_0 |u_0|^2 + C A^6 T \exp(CT^\frac{2}{3} A)
+ C A^6 \exp(CT^\frac{2}{3} A) \int_0^T \rho |u_t|^2. \tag{3.13}
$$

To estimate $||\sqrt{\rho}u_{tt}||^2_{L^2}$, we observe from (3.2) and the compatibility condition (1.8)

$$
\int_0^T |\sqrt{\rho}u_{tt}|^2 \leq C \int_0^T \rho_0 |v_0|^2 |\nabla v_0|^2 + \frac{1}{\rho_0^2} |\mu \Delta u_0 + \lambda \nabla \text{div} u_0 - \nabla P(\rho_0) + \text{div}(\rho_0 F_0 F_0^T)|^2
\leq C. \tag{3.15}
$$

Adding (3.13) and (3.14), we obtain the following estimates by Growall’s inequality:

$$
\int_0^T |\rho u_t|^2 + \int_0^T |\nabla u_t|^2 \leq (C + C A^4 T \exp(CT^\frac{2}{3} A)) A^0 T \exp(CT^\frac{2}{3} A). \tag{3.16}
$$

Finally, we have to estimate

$$
u \in L^\infty([0, T]; H^2(\Omega)), \ u \in L^2([0, T]; W^{2, q}(\Omega)).$$

To obtain further estimates, we rewrite (3.2) as

$$
\mu \Delta u + \lambda \nabla \text{div} u = \rho \partial_t u + \rho v \cdot \nabla v + \nabla P(\rho) - \text{div}(\rho FF^T),
$$

which is a strongly elliptic system. By the classical elliptic regularity theory, we deduce

$$
||u||_{L^2} \leq C( ||\rho u||_{L^2} + ||\nabla P||_{L^2} + ||\rho \nabla v||_{L^2} + ||\text{div}(\rho FF^T)||_{L^2}) + C \tag{3.17}
$$

From the previous lemmas, we get

$$
||\rho u_t||_{L^2} \leq C \exp(CT^\frac{2}{3} A) ||\sqrt{\rho} u_t||_{L^2},
$$

$$
||\nabla P||_{L^2} \leq C \exp(CT^\frac{2}{3} A) + CT^\frac{2}{3} \exp(CT^\frac{2}{3} A),
$$

$$
||\text{div}(\rho FF^T)||_{L^2} \leq C \exp(CT^\frac{2}{3} A) + CT \exp(CT^\frac{2}{3} A), \tag{3.18}
$$

and

$$
||\rho v \nabla v||_{L^2} \leq ||\rho||_{L^\infty} \int_0^T |v|^2 |\nabla v|^2
\leq C \exp(CT^\frac{2}{3} A) \left( \int_0^T |v - u_0|^2 |\nabla v|^2 + ||u_0||_{L^\infty}^2 \int_0^T |\nabla v - \nabla u_0|^2 + C \right)
\leq C \exp(CT^\frac{2}{3} A) \left( \int_0^T |v_t|^2 + C \int_0^t |\nabla v_t|^2 + C \right)
\leq C \exp(CT^\frac{2}{3} A)(A^4 T + CA^2 T + C). \tag{3.18}
$$

In a similar way, we can obtain

$$
||u||_{L^2([0, T]; W^{2, q})} \leq C + T^\alpha \ (\alpha > 0). \tag{3.19}
$$
Gathering (3.15)-(3.19) and choosing $T$ small enough, we obtain the estimates and complete the proof.
Hence, the proof is finished. □

Combining all the lemmas, we get the existence for the linearized equations (3.1) – (3.4).

**Lemma 3.4.** There exists a unique strong solution $(\rho, u, F)$ to the linearized system (3.1) – (3.4) in $[0, T_\ast] \times \Omega$ with the regularity,
\[
\rho \in C[[0, T_\ast]; W^{1,q}(\Omega)), \quad \rho_t \in C[[0, T_\ast]; L^q(\Omega)),
F \in C[[0, T_\ast]; W^{1,q}(\Omega)), \quad F_t \in C[[0, T_\ast]; L^q(\Omega)),
u \in C[[0, T_\ast]; H^2(\Omega) \cap L^2(0, T_\ast; W^{1,q}(\Omega)),
\nu_t \in L^2(0, T_\ast; H^2(\Omega)), \quad \nabla \rho \in L^\infty((0, T_\ast]; L^2(\Omega)),
\]
where $T_\ast \in (0, T)$.

4. **Local existence result.** This section is devoted to proving the existence of a unique local solution of (1.1)-(1.3) via the contraction mapping principle.

By virtue of Lemma 3.4, there exist a time $T_\ast \in (0, T)$ and a unique solution $(\rho^\delta, u^\delta, F^\delta)$ of (3.1) – (3.3) with initial data $\rho(x, 0) = \rho_0 + \delta$. Let $\delta \to 0$, we obtain a unique solution $u$ of the linearized system (3.1) – (3.3) with $\rho(x, 0) = \rho_0$ such that $\|u\|_W \leq C$. So we can define a map \[
\mathcal{J} : \mathcal{M} \to \mathcal{M}, v \to u,
\]
where \[
\mathcal{M} = \mathcal{W} \cap \mathcal{L} = \mathcal{W},
\]
with \[
\mathcal{L} = \{u : \|u\|_{L^2(0, T; H^1(\Omega))} < \infty\}.
\]
Thus, there are essentially two main tasks to prove: the self-mapping and con- traction. The former has been done due to Lemma 3.3, which guarantees the self-

**Lemma 4.1.** There exists a constant $0 < \theta < 1$ such that for any $v_i \in \mathcal{M}$, $i = 1, 2$,
\[
\|\mathcal{J}(v_1) - \mathcal{J}(v_2)\|_\mathcal{L} \leq \theta \|v_1 - v_2\|_\mathcal{L}
\]
for some small $T > 0$.

**Proof.** Suppose $(\rho_i, u_i, F_i)$ are the solutions to (3.1) – (3.3) corresponding to given $v_i \in \mathcal{M}$. Define $\rho = \rho_2 - \rho_1$, $v = v_2 - v_1$, $F = F_2 - F_1$ and $u = u_2 - u_1$. Then
\[
\begin{align*}
\partial_t \rho + \text{div}(\rho_2 v) &= -\text{div}(\rho_1 v), \quad \text{(4.1)}
\partial_t F + v \cdot \nabla F + v \cdot \nabla F_2 &= \nabla v F_2 + \nabla v_1 F \quad \text{(4.2)}
\rho_2 \partial_t u - \mu \Delta u - \lambda \text{div} v &= (\rho_1 - \rho_2) u_{1t} + \rho_1 v_1 \cdot \nabla v_1 - \rho_2 v_2 \cdot \nabla v_2 \\
+ \nabla P(\rho_1) - \nabla P(\rho_2) &= \text{div}(\rho_2 F_2 F_2^T) - \text{div}(\rho_1 F_1 F_1^T) \quad \text{(4.3)}
\end{align*}
\]
Multiplying (4.1) by \( \rho \) and integrating over \( \Omega \), we get
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\rho|^2 - \frac{1}{2} \int_{\Omega} |\rho|^2 \text{div} v - \int_{\Omega} \rho (\nabla \rho_1 : v + \rho_1 \text{div} v) \leq C \|\nabla v_2\|_{L^\infty} \|\rho\|_{L^2}^2 + C \|\rho\|_{L^2} \|\nabla \rho_1\|_{L^3} \|v\|_{L^6} + C \|\rho\|_{L^2} \|\rho_1\|_{L^\infty} \|\nabla v\|_{L^2} \leq E_1(t) \|\rho\|^2_{L^2} + \epsilon \|\nabla v\|^2_{L^2},
\]
where
\[
E_1(t) = C(\|\nabla v_2\|_{L^\infty} + \|\rho_1\|^2_{L^\infty} + \|\nabla \rho_1\|^2_{L^2}).
\]

In a similar way, we have
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} |F|^2 \leq E_2(t) \|F\|^2_{L^2} + \epsilon \|\nabla v\|^2_{L^2},
\]
where
\[
E_2(t) = C(\|\nabla v_2\|_{L^\infty} + \|F_1\|^2_{L^\infty} + \|\nabla F_1\|^2_{L^2}).
\]

Multiplying (4.3) by \( u \) and integrating over \( \Omega \), we deduce
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_2 |u|^2 + C^{-1} \int_{\Omega} \|\nabla u\|^2 \leq \int_{\Omega} (\rho_2 v_2 u \nabla u + (\rho_1 - \rho_2) u_1 \cdot u + (p_2 - p_1) \text{div} u) \\
+ (\rho_1 v_1 \nabla v_1 - \rho_2 v_2 \nabla v_2) \cdot u - \rho_2 F_2 F_2^T : \nabla u + \rho_1 F_1 F_1^T : \nabla u) \\
= \int_{\Omega} \rho_2 v_2 u \nabla u + (\rho_1 - \rho_2) (u_{11} + v_1 \nabla v_1) u - \rho_2 (v \nabla v_2 + v_1 \nabla v) u \\
+ (p_2 - p_1) \text{div} u - \rho F_2 F_2^T : \nabla u - \rho_1 (F_2 F_2^T - F_1 F_1^T) : \nabla u \\
\leq \|\sqrt{\rho_2}\|_{L^\infty} \|\sqrt{\rho_2} u\|_{L^2} \|v_2\|_{L^2} \|\nabla u\|_{L^2} + \|\rho\|_{L^{\frac{3}{2}}} \|u_{11} + v_1 \nabla v_1\|_{L^6} \|v_2\|_{L^3} \\
+ \|\sqrt{\rho_2}\|_{L^\infty} \|\sqrt{\rho_2} u\|_{L^2} \|v_2\|_{L^2} \|\nabla v_2\|_{L^3} + \|\sqrt{\rho_2}\|_{L^\infty} \|\sqrt{\rho_2} u\|_{L^2} \|v_1\|_{L^6} \|\nabla u\|_{L^2} \\
+ |P^T||\rho||L^2\|\|\nabla u\|_{L^2} + \|\rho\|_{L^2} \|F_2\|_{L^2} \|\nabla u\|_{L^2} + \|\rho_1\|_{L^\infty} \|F\|_{L^\infty} \|\nabla u\|_{L^2} \|F\|_{L^2} \\
\leq \epsilon \|\nabla u\|^2_{L^2} + \epsilon \|\nabla v\|^2_{L^2} + E_3(t) \int_{\Omega} \rho_2 |u|^2 + \|\rho\|^2 + |F|^2,
\]
where
\[
E_3(t) = C(\|\sqrt{\rho_2}\|_{L^\infty} \|v_2\|_{L^\infty} + \|\sqrt{\rho_2}\|_{L^\infty} \|\nabla v_2\|_{L^3} + \|\sqrt{\rho_2}\|_{L^\infty} \|v_1\|_{L^\infty} \\
+ \|u_{11} + v_1 \nabla v_1\|_{H^1} + \|F_2\|_{L^\infty} + \|\rho_1\|_{L^\infty} \|F\|_{L^2}).
\]

Summing inequalities (4.4) – (4.6), we obtain
\[
\frac{d}{dt} \int_{\Omega} \rho_2 |u|^2 + \|\rho\|^2 + |F|^2) + \int_{\Omega} \|\nabla u\|^2 \leq \epsilon \int_{\Omega} |\nabla v|^2 + E(t) \int_{\Omega} (\rho_2 |u|^2 + \|\rho\|^2 + |F|^2),
\]
where
\[
E(t) = E_1(t) + E_2(t) + E_3(t) \text{ satisfies, for small } T,
\]
\[
\int_0^T E(s) ds \leq K_2,
\]
where \( K_2 \) is a constant dependent on initial data, thanks to Lemma 3.1-3.3.

Let \( T \) small enough, we obtain the following by Gronwall’s inequality
\[
\|\rho\|_{L^\infty(0,T;L^2)} + \|F\|_{L^\infty(0,T;L^2)} + \|\sqrt{\rho_2} u\|_{L^\infty(0,T;L^2)} \leq c
\]
and
\[
\int_0^T \int_\Omega |\nabla u|^2 \leq \theta \int_0^T \int_\Omega |\nabla v|^2, \quad \text{with} \quad 0 < \theta < 1.
\]
Since \( u \) is zero on boundary, we finish the proof.

By the contractibility of \( J \) and utilizing the iteration methods used in [11] and [17], we can obtain a unique fixed point \( u \). This proves the existence of a strong solution. Then adapting the arguments in [1], we can easily prove the time-continuity of the solution \((\rho, u, F)\).

Thus, we complete the proof of Theorem 1.1.

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REFERENCES

[1] Y. Cho, H. J. Choe and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl., 83 (2004), 243–275.
[2] Y. M. Chu, X. G. Liu and X. Liu, Strong solutions to the compressible liquid crystal system, Pacific J. Math., 257 (2012), 37–52.
[3] M. Hieber, Y. Naito and Y. Shibata, Global existence results for Oldroyd-B fluids in exterior domains, J. Differential Equations, 252 (2012), 2617–2629.
[4] X. P. Hu and D. H. Wang, Local strong solution to the compressible viscoelastic flow with large data, J. Differential Equations, 249 (2010), 1179–1198.
[5] X. P. Hu and D. H. Wang, Global existence for the multi-dimensional compressible viscoelastic flows, J. Differential Equations, 250 (2011), 1200–1231.
[6] X. P. Hu and D. H. Wang, Strong solutions to the three-dimensional compressible viscoelastic fluids, J. Differential Equations, 252 (2012), 4027–4067.
[7] C. Guillou and J. C. Saut, Existence results for the flow of viscoelastic fluids with a differential constitutive law, Nonlinear Anal., 15 (1990), 849–869.
[8] X. D. Huang, J. Li and Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, Comm. Pure Appl. Math., 65 (2012), 549–585.
[9] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Rational Mech. Anal., 58 (1975), 181–205.
[10] R. Kupferman, C. Mangoubi and E. S. Titi, A Beale-Kato-Majda breakdown criterion for an Oldroyd-B fluid in the creeping flow regime, Commun. Math. Sci., 6 (2008), 235–256.
[11] P. D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, v+48 pp, 1973.
[12] Z. Lei, C. Liu and Y. Zhou, Global solutions for incompressible viscoelastic fluids, Arch. Ration. Mech. Anal., 188 (2008), 371–398.
[13] Z. Lei, C. Liu and Y. Zhou, Global existence for a 2D incompressible viscoelastic model with small strain, Commun. Math. Sci., 5 (2007), 595–616.
[14] F. Lin, C. Liu and P. Zhang, On hydrodynamics of viscoelastic fluids, Comm. Pure Appl. Math., 58 (2005), 1437–1471.
[15] F. Lin and P. Zhang, On the initial-boundary value problem of the incompressible viscoelastic fluid system, Comm. Pure Appl. Math., 61 (2008), 539–558.
[16] P. L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2. Compressible Models, Oxford Lecture Ser. Math. Appl., vol. 10, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.
[17] A. J. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Applied Mathematical Sciences, 53. Springer-Verlag, New York, 1984
[18] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ., 20 (1980), 67–104.
[19] A. Matsumura and T. Nishida, Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Comm. Math. Phys.*, **89** (1983), 445–464.

[20] J. G. Oldroyd, On the formation of rheological equations of state, *Proc. R. Soc. Lond. Ser. A*, **200** (1950), 523–541.

[21] J. G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, *Proc. R. Soc. Lond. Ser. A*, **245** (1958), 278–297.

[22] J. Z. Qian and Z. F. Zhang, Global well-posedness for compressible viscoelastic fluids near equilibrium, *Arch. Ration. Mech. Anal.*, **198** (2010), 835–868.

[23] J. Z. Qian, Initial boundary value problems for the compressible viscoelastic fluid, *J. Differential Equations*, **250** (2011), 848–865.

[24] R. Salvi and I. Straškraba, Global existence for viscous compressible fluids and their behavior as $t \to \infty$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **40** (1993), 17–51.

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