OPERATOR ALGEBRAS OF FUNCTIONS
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Abstract. We present some general theorems about operator algebras that are algebras of functions on sets, including theories of local algebras, residually finite dimensional operator algebras and algebras that can be represented as the scalar multipliers of a vector-valued reproducing kernel Hilbert space. We use these to further develop a quantized function theory for various domains that extends and unifies Agler’s theory of commuting contractions and the Arveson-Drury-Popescu theory of commuting row contractions. We obtain analogous factorization theorems, prove that the algebras that we obtain are dual operator algebras and show that for many domains, supremums over all commuting tuples of operators satisfying certain inequalities are obtained over all commuting tuples of matrices.

1. Introduction
A concrete operator algebra \( \mathcal{A} \) is just a subalgebra of \( B(\mathcal{H}) \), the bounded operators on a Hilbert space \( \mathcal{H} \). The operator norm on \( B(\mathcal{H}) \) gives rise to a norm on \( \mathcal{A} \). Moreover, the identification \( M_n(\mathcal{A}) \cong \mathcal{A} \otimes \mathbb{C}^n \subseteq B(\mathcal{H} \otimes \mathbb{C}^n) \cong B(\mathcal{H}^n) \) endows the matrices over \( \mathcal{A} \) with a family of norms in a natural way, where \( M_n \) denotes the algebra of \( n \times n \) matrices. It is common practice to identify two operator algebras \( \mathcal{A} \) and \( \mathcal{B} \) as being the “same” if and only if there exists an algebra isomorphism \( \pi : \mathcal{A} \rightarrow \mathcal{B} \) that is not only an isometry, but which also preserves all the matrix norms, that is such that \( \|(\pi(a_{i,j}))\|_{M_n(\mathcal{B})} = \|(a_{i,j})\|_{M_n(\mathcal{A})} \), for every \( n \) and every element \( (a_{i,j}) \in M_n(\mathcal{A}) \). Such a map \( \pi \) is called a completely isometric isomorphism.

In [13] an abstract characterization of operator algebras, in the above sense, was given and since that time a theory of such algebras has evolved. For more details on the abstract theory of operator algebras, see [11], [23] or [27].

In this note we present a theory for a special class of abstract abelian operator algebras which contains many important examples arising in function theoretic operator theory, including the Schur-Agler and the Arveson-Drury-Popescu algebras. This theory will allow us to answer certain types of questions about such algebras in a unified manner. We will prove that our hypotheses give an abstract characterization of operator algebras that are completely isometrically isomorphic to multiplier algebras of vector-valued reproducing kernel Hilbert spaces.

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Our results will show that under certain mild hypotheses, operator algebra norms which are defined by taking the supremum of certain families of operators on Hilbert spaces of arbitrary dimensions can be obtained by restricting the family of operators to finite dimensional Hilbert spaces. Thus, in a certain sense, which will be explained later, our results give conditions that guarantee that an algebra is residually finite dimensional.

Finally, we apply our results to study “quantized function theories” on various domains. Our work in this direction should be compared to that of Ambrozie-Timotin [4], Ball-Bolotnikov [10] and Kalyuzhnyi-Verbovetzkii [20]. These authors obtain the same type of factorization theorem via unitary colligation methods as we obtain via operator algebra methods, but they assume somewhat different (and in many cases stronger) hypotheses. We, also, obtain a bit more information about the algebras themselves, including the fact that in many cases their norms can be obtained as supremums over matrices instead of operators.

We now give the relevant definitions. Recall that given any set $X$ the set of all complex-valued functions on $X$ is an algebra over the field of complex numbers.

**Definition 1.1.** We call $\mathcal{A}$ an operator algebra of functions on a set $X$ provided:

1. $\mathcal{A}$ is a subalgebra of the algebra of functions on $X$,
2. $\mathcal{A}$ separates the points of $X$ and contains the constant functions,
3. for each $n \in \mathbb{N}$, $M_n(\mathcal{A})$ is equipped with a norm $\|\cdot\|_{M_n(\mathcal{A})}$, such that the set of norms satisfy the BRS axioms [13] to be an abstract operator algebra,
4. for each $x \in X$, the evaluation functional, $\pi_x : \mathcal{A} \to \mathbb{C}$, given by $\pi_x(f) = f(x)$ is bounded.

A few remarks and observations are in order. First note that if $\mathcal{A}$ is an operator algebra of functions on $X$ and $\mathcal{B} \subseteq \mathcal{A}$ is any subalgebra, which contains the constant functions and still separates points, then $\mathcal{B}$, equipped with the norms that $M_n(\mathcal{B})$ inherits as a subspace of $M_n(\mathcal{A})$ is still an operator algebra of functions.

The basic example of an operator algebra of functions is $\ell^\infty(X)$, the algebra of all bounded functions on $X$. If for $(f_{i,j}) \in M_n(\ell^\infty(X))$ we set

$$\|(f_{i,j})\|_{M_n(\ell^\infty(X))} = \|(f_{i,j})\|_\infty \equiv \sup\{\|f_{i,j}(x)\|_{M_n} : x \in X\},$$

where $\|\cdot\|_{M_n}$ is the norm on $M_n$ obtained via the identification $M_n = B(\mathbb{C}^n)$, then it readily follows that properties (1)–(4) of the above definition are satisfied. Thus, $\ell^\infty(X)$ is an operator algebra of functions on $X$ in our sense and any subalgebra of $\ell^\infty(X)$ that contains the constants and separates points will be an operator algebra of functions on $X$ when equipped with the subspace norms.
**Proposition 1.2.** Let \( \mathcal{A} \) be an operator algebra of functions on \( X \), then \( \mathcal{A} \subseteq \ell^\infty(X) \), and for every \( n \) and every \( (f_{i,j}) \in M_n(\mathcal{A}) \), we have \( \|(f_{i,j})\|_\infty \leq \|(f_{i,j})\|_{M_n(\mathcal{A})} \).

**Proof.** Since \( \pi_x : \mathcal{A} \to \mathbb{C} \) is bounded and the norm is sub-multiplicative, we have that for any \( f \in \mathcal{A} \), \( |f(x)|^n = |\pi_x(f^n)| \leq \|\pi_x\| |f|^n \leq \|\pi_x\| |f|^n \).

Taking the \( n \)-th root of each side of this inequality and letting \( n \to +\infty \), yields \( |f(x)| \leq \|f\| \), and hence, \( f \in \ell^\infty(X) \). Note also that \( \|\pi_x\| = 1 \).

Finally, since every bounded, linear functional on an operator space is completely bounded and the norm and the cb-norm are equal, we have that \( \|\pi_x\|_{cb} = \|\pi_x\| = 1 \). Thus, for \( (f_{i,j}) \in M_n(\mathcal{A}) \), we have \( \|(f_{i,j}(x))\|_{M_n} = \|\pi_x(f_{i,j})\|_{M_n(\mathcal{A})} \leq \|(f_{i,j})\|_{M_n(\mathcal{A})} \). \( \square \)

An operator algebra of functions on a set \( X \), a set of distinct points in \( X \), we define \( I_F = \{f \in \mathcal{A} : f(z) = 0 \forall z \in F\} \) and note that \( M_n(I_F) = \{f \in M_n(\mathcal{A}) : f(z) = 0 \forall z \in F\} \forall n \). The quotient space \( \mathcal{A}/I_F \) has a natural set of matrix norms given by defining \( \|\pi(f_{i,j} + I_F)\| = \inf\|\pi(f_{i,j} + g_{i,j})\|_{M_n(\mathcal{A})} : g_{i,j} \in I_F \). Alternatively, this is the norm on \( M_n(\mathcal{A}/I_F) \) that comes via the identification, \( M_n(\mathcal{A}/I_F) = M_n(\mathcal{A})/M_n(I_F) \), where the latter space is given its quotient norm. It is easily checked that this family of matrix norms satisfies the BRS conditions and so gives \( \mathcal{A}/I_F \) the structure of an abstract operator algebra.

We let \( \pi_F(f) = f + I_F \) denote the quotient map \( \pi_F : \mathcal{A} \to \mathcal{A}/I_F \) so that for each \( n, \pi_F^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{A}/I_F) \cong M_n(\mathcal{A})/M_n(I_F) \).

Since \( \mathcal{A} \) is an algebra which separates points on \( X \) and contains constant functions, it follows that there exists functions \( f_1, \ldots, f_k \in \mathcal{A} \), such that \( f_i(x_j) = \delta_{ij} \), where \( \delta_{ij} \) denotes the Dirac delta function. If we set \( E_j = \pi_F(f_j) \), then it is easily seen that whenever \( f \in \mathcal{A} \) and \( f(x_i) = \lambda_i, i = 1, \ldots, k \), then \( \pi_F(f) = \lambda_1 E_1 + \cdots + \lambda_k E_k \). Moreover, \( E_j E_j = \delta_{ij} E_j \), and \( E_1 + \cdots + E_k = 1 \), where 1 denotes the identity of the algebra \( \mathcal{A}/I_F \). Thus, \( \mathcal{A}/I_F = \text{span}\{E_1, \ldots, E_k\} \), is a unital algebra spanned by \( k \) commuting idempotents. Such algebras were called k-idempotent operator algebras in [24] and we will use a number of results from that paper.

**Definition 1.3.** An operator algebra of functions \( \mathcal{A} \) on a set \( X \), is called a **local operator algebra of functions** if it satisfies

\[
\sup_F \|\pi_F^{(n)}((f_{i,j}))\| = \|(f_{i,j})\| \forall (f_{i,j}) \in M_n(\mathcal{A}) \text{ and for every } n,
\]

where the supremum is taken over all finite subsets of \( X \).

The following result shows that every operator algebra of functions can be re-normed so that it becomes local.

**Proposition 1.4.** Let \( \mathcal{A} \) be an operator algebra of functions on \( X \), let \( \mathcal{A}_L = \mathcal{A} \) and define a family of matrix norms on \( \mathcal{A}_L \), by setting \( \|(f_{i,j})\|_{M_n(\mathcal{A}_L)} = \sup_F \|\pi_F(f_{i,j})\|_{M_n(\mathcal{A}/I_F)} \). Then \( \mathcal{A}_L \) is a local operator algebra of functions on \( X \) and the identity map, \( \text{id} : \mathcal{A} \to \mathcal{A}_L \), is completely contractive.
Proof. It is clear from the definition of the norms on \( \mathcal{A}_L \) that the identity map is completely contractive and it is readily checked that \( \mathcal{A}_L \) is an operator algebra of functions on \( X \).

Let \( \tilde{\pi}_F : \mathcal{A}_L \to \mathcal{A}_L/\mathcal{I}_F \), denote the quotient map, so that \( \| \tilde{\pi}_F(f) \| = \inf \{ \| f + g \|_L : g \in \mathcal{I}_F \} \leq \inf \{ \| f + g \| : g \in \mathcal{I}_F \} = \| \pi_F(f) \| \), since \( \| f + g \|_L \leq \| f \| + \| g \| \). We claim that for any \( f \in \mathcal{A}_L \), and any finite subset \( F \subseteq X \), we have that \( \| \pi_F(f) \| = \| \tilde{\pi}_F(f) \| \). To see the other inequality note that for \( g \in \mathcal{I}_F \), and \( G \subseteq X \) a finite set, we have \( \| f + g \|_L = \sup_{\gamma} \| \pi_G(f + g) \| \geq \| \pi_F(f + g) \| = \| \pi_F(f) \| \). Hence, \( \| \tilde{\pi}_F(f) \| \geq \| \pi_F(f) \| \), and equality follows.

A similar calculation shows that \( \| (\pi_F(f_{i,j})) \| = \| (\pi_F(f_{i,j})) \| \), for any matrix of functions.

Now it easily follows that \( \mathcal{A}_L \) is local, since

\[
\sup_F \| (\tilde{\pi}_F(f_{i,j})) \| = \sup_F \| (\pi_F(f_{i,j})) \| = \| (f_{i,j}) \|_{M_n(\mathcal{A}_L)}.
\]

We let \( \hat{\mathcal{A}} \) denote the set of functions that are BPW limits of bounded nets of functions in \( \mathcal{A} \).

**Definition 1.5.** Given an operator algebra of functions \( \mathcal{A} \) on \( X \) we say that \( f : X \to \mathbb{C} \) is a BPW limit of \( \mathcal{A} \) if there exists a uniformly bounded net \( (f_\lambda)_{\lambda} \in \mathcal{A} \) that converges pointwise on \( X \) to \( f \). We say that \( \mathcal{A} \) is BPW complete if it contains the set of functions that are BPW limits of bounded nets of functions in \( \mathcal{A} \). Let \( \hat{\mathcal{A}} \) denote the set of such functions, that is, 

\[
\hat{\mathcal{A}} = \{ f : \exists \text{ a bounded net } (f_\lambda) \subseteq \mathcal{A} \text{ such that } f(z) = \lim_\lambda f_\lambda(z) \ \forall \ z \in X \}
\]

Given \( (f_{i,j}) \in M_n(\mathcal{A}) \), we set

\[
\| (f_{i,j}) \|_{M_n(\hat{\mathcal{A}})} = \inf \{ C : (f_{i,j}(x)) = \lim_\lambda (f_{i,j}^\lambda(x)) \text{ and } \| (f_{i,j}^\lambda) \| \leq C \}
\]

It is easily checked that for each \( n \), the above formula defines a norm on \( M_n(\hat{\mathcal{A}}) \). It is also easily checked that a matrix-valued function, \( (f_{i,j}) : X \to M_n \) is the pointwise limit of a uniformly bounded net \( (f_{i,j}^\lambda) \in M_n(\mathcal{A}) \) if and only if \( f_{i,j} \in \hat{\mathcal{A}} \) for every \( i,j \).

**Lemma 1.6.** Let \( \mathcal{A} \) be an operator algebra of functions on \( X \) and let \( (f_{i,j}) \in M_n(\mathcal{A}) \). Then

\[
\| (f_{i,j}) \|_{M_n(\hat{\mathcal{A}})} = \inf \{ C : \forall F \subseteq X \text{ finite } \exists g_{i,j}^F \in \mathcal{A} \text{ with } (f_{i,j} |_F) = (g_{i,j}^F |_F), \| (g_{i,j}^F) \| \leq C \}.
\]

**Proof.** The collection of finite subsets of \( X \) determines a directed set, ordered by inclusion. If we choose for each finite set \( F \), functions \( (g_{i,j}^F) \) satisfying the conditions of the right hand set, then these functions define a net that converges BPW to \( (f_{i,j}) \) and hence, the right hand side is larger than the left. Conversely, given a net \( (f_{i,j}^\lambda) \) that converges pointwise to \( (f_{i,j}) \) and satisfies \( \| (f_{i,j}^\lambda) \| \leq C \) and any finite set \( F = \{ x_1, \ldots, x_k \} \), choose functions in \( \mathcal{A} \) such that \( f_i(x_j) = \delta_{i,j} \). If we let \( A^\lambda_l = (f_{i,j}(x_l)) - (f_{i,j}^\lambda(x_l)) \), then \( (g_{i,j}^\lambda) = \)
Lemma 1.7. Let $\tilde{A}$ be an operator algebra of functions on the set $X$, then $\tilde{A}$ equipped with the collection of norms on $M_n(\tilde{A})$ given in Definition 1.5 is an operator algebra.

Proof. It is clear from the definition of $\tilde{A}$ that it is an algebra. Thus, it is enough to check that the axioms of BRS are satisfied by the algebra $\tilde{A}$ equipped with the matrix norms given in the Definition 1.5.

If $L$ and $M$ are scalar matrices of appropriate sizes and $G \in M_n(\tilde{A})$, then for $\epsilon > 0$ there exists $G_\lambda \in M_n(\tilde{A})$ such that $\lim \lambda G_\lambda(x) = G(x) \forall x \in X$ and $\sup \lambda \|G_\lambda\|_{M_n(\tilde{A})} \leq \|G\|_{M_n(\tilde{A})} + \epsilon$. Since $\tilde{A}$ is an operator space, $\|LG_\lambda M\|_{M_n(\tilde{A})} \leq \|L\|\|G_\lambda\|_{M_n(\tilde{A})}\|M\|$. Note that this shows that $\|LGM\|_{M_n(\tilde{A})} \leq \|L\|\|G\|_{M_n(\tilde{A})}\|M\|$, since $LGM \rightarrow LGM$ pointwise and $\sup \lambda \|LG_\lambda M\|_{M_n(\tilde{A})} \leq \|L\|\|G\|_{M_n(\tilde{A})}\|M\|$ for every $\lambda > 0$.

If $G, H \in M_n(\tilde{A})$, then for every $\epsilon > 0$ there exists $G_\lambda, H_\lambda \in M_n(\tilde{A})$ such that $\lim \lambda G_\lambda(x) = G(x)$ and $\lim \lambda H_\lambda(x) = H(x) \forall x \in X$. Also, we have $\sup \lambda \|G_\lambda\|_{M_n(\tilde{A})} \leq \|G\|_{M_n(\tilde{A})} + \epsilon$ and $\sup \lambda \|H_\lambda\|_{M_n(\tilde{A})} \leq \|H\|_{M_n(\tilde{A})} + \epsilon$.

Let $L = GH$ and $L_\lambda = G_\lambda H_\lambda$. Since $\tilde{A}$ is matrix normed algebra, $L_\lambda \in M_n(\tilde{A})$ and $\|L_\lambda\|_{M_n(\tilde{A})} \leq \|G_\lambda\|_{M_n(\tilde{A})}\|H_\lambda\|_{M_n(\tilde{A})}$ for every $\lambda$.

This yields $\|L\|_{M_n(\tilde{A})} \leq \|G\|_{M_n(\tilde{A})}\|H\|_{M_n(\tilde{A})}$, and so the multiplications is completely contractive.

Finally, to see that the $L^\infty$ conditions are met, let $G \in M_n(\tilde{A})$ and $H \in M_m(\tilde{A})$. Given $\epsilon > 0$, $\exists G_\lambda \in M_n(\tilde{A})$ and $H_\lambda \in M_m(\tilde{A})$ such that $\lim \lambda G_\lambda(x) = G(x)$, $\lim \lambda H_\lambda(x) = H(x)$ and $\sup \lambda \|G_\lambda\|_{M_n(\tilde{A})} \leq \|G\|_{M_n(\tilde{A})} + \epsilon$, $\sup \lambda \|H_\lambda\|_{M_m(\tilde{A})} \leq \|H\|_{M_m(\tilde{A})} + \epsilon$.

Note that $G_\lambda + H_\lambda \in M_{n+m}(\tilde{A})$ and $\|G_\lambda + H_\lambda\| = \max \{\|G_\lambda\|_{M_n(\tilde{A})}, \|H_\lambda\|_{M_m(\tilde{A})}\}$ for every $\lambda$ which implies that $G + H \in M_{n+m}(\tilde{A})$, and

$\|G + H\|_{M_{n+m}(\tilde{A})} \leq \sup \lambda \|G_\lambda + H_\lambda\| = \sup \lambda \max \{\|G_\lambda\|_{M_n(\tilde{A})}, \|H_\lambda\|_{M_m(\tilde{A})}\}$

$\leq \max \{\|G\|_{M_n(\tilde{A})} + \epsilon, \|H\|_{M_m(\tilde{A})} + \epsilon\} \forall \epsilon > 0$.

This shows that $\|G + H\|_{M_{n+m}(\tilde{A})} \leq \max \{\|G\|_{M_n(\tilde{A})}, \|H\|_{M_m(\tilde{A})}\}$, and so the $L^\infty$ condition follows. This completes the proof of the result.

Lemma 1.8. If $A$ is an operator algebra of functions on the set $X$, then $\tilde{A}$ equipped with the norms of Definition 1.5 is a local operator algebra of functions on $X$. Moreover, for $(f_{i,j}) \in M_n(A)$, $\|(f_{i,j})\|_{M_n(\tilde{A})} = \|(f_{i,j})\|_{M_n(A)}$. 
Proof. It is clear from the definition of the norms on $\tilde{A}$ that the identity map from $A$ to $\tilde{A}$ is completely contractive and thus $A \subseteq \tilde{A}$ as sets. This indeed shows that $\tilde{A}$ separates points of $X$ and contains constant functions.

Let $(f_{ij}) \in M_n(\tilde{A})$ and $\epsilon > 0$, then $\exists$ a net $(f_{ij}^0) \in M_n(\tilde{A})$ such that
\[
\lim_{\lambda} (f_{ij}^0(x)) = (f_{ij}(x)) \forall x \in X \text{ and } \sup_{\lambda} \| (f_{ij}^0)\|_{M_n(\tilde{A})} \leq \| (f_{ij})\|_{M_n(\tilde{A})} + \epsilon.
\]
Since $\tilde{A}$ is an operator algebra of functions on the set $X$, we have that $\| (f_{ij}^0)\|_{\infty} \leq \| (f_{ij})\|_{M_n(\tilde{A})} \forall \lambda \implies \sup_{\lambda} \| (f_{ij}^0)\|_{\infty} \leq \| (f_{ij})\|_{M_n(\tilde{A})} + \epsilon.

Fix $z \in X$, then
\[
\| (f_{ij}(z))\| = \lim_{\lambda} \| (f_{ij}^0(z))\| \leq \sup_{\lambda} \| (f_{ij}^0)\|_{\infty} \leq \| (f_{ij})\|_{M_n(\tilde{A})} + \epsilon \quad \forall \epsilon > 0.
\]
By letting $\epsilon \to 0$ and supping over $z \in X$, we get that $\| (f_{ij})\|_{\infty} \leq \| (f_{ij})\|_{M_n(\tilde{A})}$.

Hence, $\tilde{A}$ is an operator algebra of functions on the set $X$.

Denote $\tilde{I}_F = \{ f \in \tilde{A} : f|_F \equiv 0 \}$ and let $(f_{ij}) \in M_n(\tilde{A})$. Then, clearly $\sup_{F} \| (f_{ij} + \tilde{I}_F)\|_{M_n(\tilde{A}/I_F)} = \| (f_{ij})\|_{M_n(\tilde{A})}$. To see the other inequality, assume that $\sup F \| (f_{ij} + \tilde{I}_F)\| < 1$, then $\forall$ finite $F \subseteq X \exists \ (h_{ij}^F) \in M_n(\tilde{A})$ such that $(h_{ij}^F)_F = (f_{ij})_F$ and $\sup F \| h_{ij}^F \| \leq 1$. Fix a set $F \subseteq X$ and $(h_{ij}^F) \in M_n(\tilde{A})$ then $\forall$ finite $F' \subseteq X \exists \ (k_{ij}^{F'}) \in M_n(\tilde{A})$ such that $(k_{ij}^{F'})|_{F'} = (h_{ij}^F)|_{F'}$ and $\sup F' \| k_{ij}^{F'} \| \leq 1$.

In particular, let $F' = F$ then $(k_{ij}^{F})|_{F} = (h_{ij}^{F})|_{F} = (f_{ij})|_{F}$ and $\sup_{F} \| k_{ij}^{F} \| \leq 1$.

$\implies$ $\| (f_{ij})\|_{M_n(\tilde{A})} \leq 1$, and hence $\| (f_{ij})\|_{M_n(\tilde{A})} \leq \sup_{F} \| (f_{ij} + \tilde{I}_F)\|_{M_n(\tilde{A}/I_F)}$.

Finally, given that $(f_{ij}) \in M_n(\tilde{A})$, $\| (f_{ij})\|_{M_n(\tilde{A})} = \sup_{F} \| (f_{ij} + \tilde{I}_F)\|_{M_n(\tilde{A}/I_F)}$.

Note that $\forall F \subseteq X$ we have $\| (f_{ij} + \tilde{I}_F)\|_{M_n(\tilde{A}/I_F)} \leq \| (f_{ij} + I_F)\|_{M_n(A/I_F)}$, since $I_F \subseteq \tilde{I}_F$. We claim that for any $(f_{ij}) \in M_n(A)$, and for any finite subset $F \subseteq X$, we have that $\| (f_{ij} + I_F)\| = \| (f_{ij} + \tilde{I}_F)\|$. To see the other inequality, let $(g_{ij}) \in M_n(\tilde{I}_F)$. Then $\forall \epsilon > 0$, and $\exists \ (h_{ij}^G) \in M_n(A)$ such that $(h_{ij}^G)_G = (f_{ij} + g_{ij})_G$ and $\| (h_{ij}^G)\| \leq \| (f_{ij} + g_{ij})\| + \epsilon$. Hence, $\| (f_{ij} + I_F)\| = \| (h_{ij}^F + I_F)\| \leq \| (h_{ij}^G)\| \leq \| (f_{ij} + g_{ij})\| + \epsilon \forall \epsilon > 0$, and the equality follows. Now it is immediate to see that,
\[
\| (f_{ij})\|_{M_n(\tilde{A})} = \sup_{F} \| (f_{ij} + I_F)\| = \| (f_{ij})\|_{M_n(A_L)}
\]

Corollary 1.9. If $\mathcal{A}$ is a BPW complete operator algebra then $\mathcal{A}_L = \tilde{A}$ completely isometrically.

Proof. Since $\mathcal{A}$ is BPW complete, $\mathcal{A} = \tilde{A}$ as sets. But by Lemma 1.6 the norm defined on $\mathcal{A}_L$ agrees with the norm defined on $\tilde{A}$.

Remark 1.10. In the view of above corollary, we denote the the norm on $\tilde{A}$ by $\| . \|_{L}$.
Lemma 1.11. If $\mathcal{A}$ is an operator algebra of functions on $X$, then $\text{Ball}(\mathcal{A}_L)$ is BPW dense in $\text{Ball}(\mathcal{A})$ and $\tilde{\mathcal{A}}$ is BPW complete, i.e., $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$.

Proof. It can be easily checked that above is equivalent to showing that $\mathcal{A}_L$ is BPW dense in $\tilde{\mathcal{A}}$. We’ll only prove that $\bar{\mathcal{A}}_{BPW} \subseteq \tilde{\mathcal{A}}$, since the other containment follows immediately by the definition of $\tilde{\mathcal{A}}$.

Let $\{f_\lambda\}$ be a net in $\mathcal{A}_L$ such that $f_\lambda \to f$ pointwise and $\sup_\lambda \|f_\lambda\|_{A_L} < C$. Then for fixed $F \subseteq X$ and $\epsilon > 0$, $\exists \lambda_F$ such that $|f_{\lambda_F}(z) - f(z)| < \epsilon \forall z \in F$. Also since $\sup_\lambda \|f_\lambda\| < C$, there exists $g_{\lambda_F} \in I_F$ such that $\|f_{\lambda_F} + g_{\lambda_F}\| < C$.

Note that the function $h_F = f_{\lambda_F} + g_{\lambda_F} \in \mathcal{A}$ such that $\|h_F\|_{A} < C$, and $h_F \to f$ pointwise. Hence, $f \in \tilde{\mathcal{A}}$. Thus, $\mathcal{A}_L$ is BPW dense in $\tilde{\mathcal{A}}$.

Finally, note that the argument similar to the above readily yields that $\tilde{\mathcal{A}}$ is BPW complete. \hfill \Box

All the above lemmas can be summarized as the following theorem.

Theorem 1.12. If $\mathcal{A}$ is an operator algebra of functions on $X$, then $\tilde{\mathcal{A}}$ is a BPW complete local operator algebra of functions on $X$ which contains $\mathcal{A}_L$ completely isometrically as a BPW dense subalgebra.

Definition 1.13. Given an operator algebra of functions $\mathcal{A}$ on $X$, we call $\mathcal{A}$ the BPW completion of $\mathcal{A}$.

We now present a few examples to illustrate these concepts. We will delay the main family of examples to the next section.

Example 1.14. If $\mathcal{A}$ is a uniform algebra, then there exists a compact, Hausdorff space $X$, such that $\mathcal{A}$ can be represented as a subalgebra of $C(X)$ that separates points. If we endow $\mathcal{A}$ with the matrix-normed structure that it inherits as a subalgebra of $C(X)$, namely, $\|(f_{i,j})\| = \|(f_{i,j})\|_\infty \equiv \sup \{\|(f_{i,j})\|_{M_n} : x \in X\}$, then $\mathcal{A}$ is a local operator algebra of functions on $X$. Indeed, to achieve the norm, it is sufficient to take the supremum over all finite subsets consisting of one point. In this case the BPW completion $\tilde{\mathcal{A}}$ is completely isometrically isomorphic to the subalgebra of $\ell^\infty(X)$ consisting of functions that are bounded, pointwise limits of functions in $\mathcal{A}$.

Example 1.15. Let $\mathcal{A} = A(\mathbb{D}) \subseteq C(\mathbb{D}^-)$ be the subalgebra of the algebra of continuous functions on the closed disk consisting of the functions that are analytic on the open disk $\mathbb{D}$. Identifying $M_n(A(\mathbb{D})) \subseteq M_n(C(\mathbb{D}^-))$ as a subalgebra of the algebra of continuous functions from the closed disk to the matrices, equipped with the supremum norm, gives $A(\mathbb{D})$ it’s usual operator algebra structure. With this structure it can be regarded as a local operator algebra of functions on $\mathbb{D}$ or on $\mathbb{D}^-$. If we regard it as a local operator algebra of functions on $\mathbb{D}^-$, then $A(\mathbb{D}) \subseteq A(\mathbb{D})$. To see that the containment is strict, note that $f(z) = (1 + z)/2 \in A(\mathbb{D})$ and $f^n(z) \to \chi_{\{1\}}$, the characteristic function of the singleton $\{1\}$. 
However, if we regard $A(\mathbb{D})$ as a local operator algebra of functions on $\mathbb{D}$, then its BPW completion $\tilde{A}(\mathbb{D}) = H^\infty(\mathbb{D})$, the bounded analytic functions on the disk, with its usual operator structure.

**Example 1.16.** Let $X = \varepsilon \mathbb{D}$, $0 < \varepsilon < 1$ and $A = \{ f \in H^\infty(\mathbb{D}) : f : X \to \mathbb{C} \}$. If we endow $A$ with the matrix-normed structure on $H^\infty(\mathbb{D})$, then $A$ is an operator algebra of functions on $X$. Also, it can be verified that $A$ is a local operator algebra of functions and that $A = \bar{A}$. Indeed, if $F = (f_{ij}) \in M_n(A)$ then $\|(f_{ij} + I_Y)\|_\infty < 1 \forall$ finite subset $Y \subseteq X$, then $\exists H_Y \in M_n(A)$ such that $\|H_Y\|_\infty \leq 1$ and $H_Y \to F$ pointwise on $X$. Note by Montel’s theorem $\exists$ a subnet $H_{Y'}$ and $G \in M_n(H^\infty(\mathbb{D}))$ such that $\|G\|_\infty \leq 1$ and $H_{Y'} \to G$ uniformly on compact subsets of $\mathbb{D}$. Thus, by the identity theorem $F \equiv G$ on $\mathbb{D}$. Hence, $A$ is a local operator algebra. Finally, by using Lemma 1.11 and a similar argument, one can also show that $\tilde{A} = H^\infty(\mathbb{D})$.

**Example 1.17.** Let $A = H^\infty(\mathbb{D})$ but endowed with a new norm. Fix $b > 1$, and set $\|F\| = \max\{\|F\|_\infty, \|F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\|\}, \ F \in M_n(A)$. It can be easily verified that $A$ is a BPW complete operator algebra of functions. However, we also claim that $A$ is local. To prove this we proceed by contradiction. Suppose $\exists F = (f_{ij}) \in M_n(H^\infty(\mathbb{D}))$ such that $\|F\| > 1 > c$, where $c = \sup_Y \|(f_{ij} + I_Y)\|$. In this case, $\|F\| = \|F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\|$, since $\|(f_{ij} + I_Y)\| = \|F(\lambda)\|$ when $Y = \{\lambda\}$. Let $\epsilon = \frac{1}{1 + c}$ and $Y = \{0, \varepsilon\} \subseteq \mathbb{D}$, then $\exists G \in M_n(H^\infty(\mathbb{D}))$ such that $G|_Y = 0$ and $\|F + G\| < \frac{1 + c}{2}$. Thus, we can write $B_Y(z) = \frac{z - \epsilon}{1 - \epsilon}$, so that we can write $G(z) = zB_Y(z)H(z)$, for some $H \in M_n(H^\infty)$. It follows that $\|H\|_\infty < 2$, since $\|G\|_\infty < 2$. We now consider

$$
1 < \|F(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\| \leq \|(F + G)(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\| + \|G(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})\| \\
\leq \frac{1 + c}{2} + \| \begin{pmatrix} 0 & bG'(0) \\ 0 & 0 \end{pmatrix} \| = \frac{1 + c}{2} + bB_Y(0)\|H(0)\| \\
\leq \frac{1 + c}{2} + 2be = \frac{1 + c}{2} + 2b\frac{1 - c}{4b} = 1 \text{ contradiction.}
$$

**Example 1.18.** This is an example of a non-local algebra that arises from boundary behavior. Suppose $A = \{ f \in A(\mathbb{D}) \subseteq C(\mathbb{D}^-) : f : \mathbb{D} \to \mathbb{C} \}$ equipped with the family of matrix norms $\|F\| = \max\{\|F\|_\infty, \|F(\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix})\|\}, F \in M_n(A)$. Then it is easy to check that $A$ is an operator algebra of functions on the set $\mathbb{D}$. Also, it can be verified that $A$ is not local. To see this, note that $\|z\| = \| \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \|$. For each $Y = \{z_1, z_2, \ldots, z_n\}$, we define $B_Y(z) = \prod_{i=1}^n \left( \frac{z - z_i}{1 - z_i z} \right)$ and choose $h \in A$ such that $h(1) = -\overline{B_Y(1)}$ and $h(-1) = \overline{B_Y(-1)}$. 


Let $g(z) = z + B^Y(z) h(z) \alpha$, where $\alpha = \frac{\|z\|^{-1}}{\|z\| + \|\tilde{z}\|} > 0$. Then $g \in \mathcal{A}$ and $g|_{\mathcal{Y}} = z|_{\mathcal{Y}} \mapsto \|\pi_Y(z)\| = \|\pi_Y(g)\| \leq \|g\| \leq \max \{1 + \alpha \|h\|_{\infty}, (1 - \alpha) \|\tilde{z}\|\} < \|z\|$. Hence, $A$ is not local.

**Example 1.19.** This example shows that one can easily build non-local algebras by adding “values” outside of the set $X$. Let $\mathcal{A}$ be the algebra of polynomials regarded as functions on the set $X = \mathcal{D}$. Then $\mathcal{A}$ endowed with the matrix-normed structure as $\|(p_{ij})\| = \max \{\|((p_{ij})_{\infty}, \|(p_{ij}(2))\|\}$, is an operator algebra of functions on the set $X$. To see that $\mathcal{A}$ is not local, let $p \in \mathcal{A}$ be such that $\|p\|_{\infty} < |p(2)|$. For each finite subset $Y = \{z_1, \ldots, z_n\}$ of $X$, let $h_Y(z) = \prod_{i=1}^n (z - z_i)$ and $g_Y(z) = p(z) - \alpha h_Y(z)p(2)$, where $\alpha = \frac{|p(2)| - \|p\|_{\infty}}{2|p(2)| \|h_Y\|_{\infty}} > 0$. Note that $\|g_Y\| \leq (1 - \alpha)|p(2)|$ and $g_Y|_{\mathcal{Y}} = p|_{\mathcal{Y}} \mapsto \|\pi_Y(p)\| = \|\pi_Y(g_Y)\| \leq \|g_Y\| \leq (1 - \alpha)|p(2)| < \|p\|$. Hence, it follows that $\mathcal{A}$ is not local.

Finally, observe that in this case $\mathcal{A}$ cannot be BPW complete. For example, if we take $p_n = \frac{1}{3} \sum_{i=0}^n \left(\frac{2}{3}\right)^i \in \mathcal{A}$ then $p_n(z) \mapsto f(z) = \frac{1}{3-z} \forall \, z \in \mathcal{D}$ and $\|p_n\| < \|f\| \Rightarrow A_L \not\subseteq \tilde{A}$.

**Example 1.20.** It is still an open problem as to whether or not every unital contractive, homomorphism $\rho : H^\infty(\mathcal{D}) \to B(H)$ is completely contractive. For a recent discussion of this problem see [25]. Let’s assume that $\rho$ is a contractive homomorphism that is not completely contractive. Let $\mathcal{B} = H^\infty(\mathcal{D})$, but endow it with the family of matrix-norms given by,

$$\|\|(f_{i,j})\|| = \max \{\|(f_{i,j})\|_{\infty}, \|\rho(f_{i,j})\|\}.$$

Note that $\|\|f\|| = \|f\|_{\infty}$, for $f \in \mathcal{B}$.

It is easily checked that $\mathcal{B}$ is a BPW complete operator algebra of functions on $\mathcal{D}$. However, since every contractive homomorphism of $A(\mathcal{D})$ is completely contractive, we have that for $(f_{i,j}) \in M_n(A(\mathcal{D}))$, $\|\|(f_{i,j})\|| = \|(f_{i,j})\|_{\infty}$. If $Y = \{x_1, \ldots, x_k\}$ is a finite subset of $\mathcal{D}$ and $F = (f_{i,j}) \in M_n(\mathcal{B})$, then there is $G = (g_{i,j}) \in M_n(A(\mathcal{D}))$, such that $F(x) = G(x)$ for all $x \in Y$, and $\|G\|_{\infty} = \|F\|_{\infty}$. Hence, $\|\pi^{(n)}_Y(F)\| \leq \|F\|_{\infty}$. Thus, $\|\pi^{(n)}_Y(F)\| = \|F\|_{\infty}$. It follows that $\mathcal{B}$ is not local and that $\mathcal{B} = \mathcal{B}_L = H^\infty(\mathcal{D})$, with its usual supremum norm operator algebra structure.

In particular, if there does exist a contractive but not completely contractive representation of $H^\infty(\mathcal{D})$, then we have constructed an example of a non-local BPW complete operator algebra of functions on $\mathcal{D}$.

2. A Characterization of Local Operator Algebras of Functions

The main goal of this section is to prove that every BPW complete local operator algebra of functions is completely isometrically isomorphic to the algebra of multipliers on a reproducing kernel Hilbert space of vector-valued functions. Moreover, we will show that every such algebra is a dual operator algebra in the precise sense of [11]. We will then prove that for such BPW
algebras, weak*-convergence and BPW convergence coincide on bounded balls.

Given a set $X$ and a Hilbert space $\mathcal{H}$, then by a reproducing kernel Hilbert space of $\mathcal{H}$-valued functions, we mean a vector space $\mathcal{L}$ of $\mathcal{H}$-valued functions that is equipped with a norm and an inner product that makes it a Hilbert space and which has the property that for every $x \in X$, the evaluation map $E_x : \mathcal{L} \to \mathcal{H}$, is a bounded, linear map. Recall that given a Hilbert space $\mathcal{H}$, a matrix of operators, $T = (T_{i,j}) \in M_k(B(\mathcal{H}))$ is regarded as an operator on the Hilbert space $\mathcal{H}^{(k)} \cong \mathcal{H} \otimes \mathbb{C}^k$, which is the direct sum of $k$ copies of $\mathcal{H}$. A function $K : X \times X \to B(\mathcal{H})$, where $H$ is a Hilbert space, is called a positive definite operator-valued function on $X$, provided that for every finite set of (distinct)points $\{x_1, ..., x_k\}$ in $X$, the operator-valued matrix, $(K(x_i, x_j))$ is positive semidefinite. Given a reproducing kernel Hilbert space of $\mathcal{H}$-valued functions, if we set $K(x, y) = E_x E_y^*$, then $K$ is positive definite and is called the reproducing kernel of $\mathcal{L}$. There is a converse to this fact, generally called Moore’s theorem, which states that given any positive definite operator-valued function $K : X \times X \to B(\mathcal{H})$, then there exists a unique reproducing kernel Hilbert space of $\mathcal{H}$-valued functions on $X$, such that $K(x, y) = E_x E_y^*$. We will denote this space by $\mathcal{L}(K, \mathcal{H})$. Finally, given any reproducing kernel Hilbert space $\mathcal{L}$ of $\mathcal{H}$-valued functions with reproducing kernel $K$, a function $f : X \to \mathbb{C}$ is called a multiplier provided that for every $g \in \mathcal{L}$, the function $fg \in \mathcal{F}$. In this case it follows by an application of the closed graph theorem that the map $M_f : \mathcal{L} \to \mathcal{L}$, defined by $M_f(g) = fg$, is a bounded, linear map. The set of all multipliers is denoted $\mathcal{M}(K)$ and is easily seen to be an algebra of functions on $X$ and a subalgebra of $B(\mathcal{L})$. The reader can find proofs of the above facts in [14] [3]. Also, we refer to the fundamental work of Pedrick [26] for further treatment of vector-valued reproducing kernel Hilbert spaces.

**Lemma 2.1.** Let $\mathcal{L}$ be a reproducing kernel Hilbert space of $\mathcal{H}$-valued functions with reproducing kernel $K : X \times X \to B(\mathcal{H})$. Then $\mathcal{M}(K) \subseteq B(\mathcal{L})$ is a weak*-closed subalgebra.

**Proof.** It is enough to show that the unit ball is weak*-closed by the Krein-Smulian theorem. So let $\{M_{f_\lambda}\}$ be a net of multipliers in the unit ball of $B(\mathcal{L})$ that converges in the weak*-topology to an operator $T$. We must show that $T$ is a multiplier.

Let $x \in X$ be fixed and assume that there exists $g \in \mathcal{L}$, with $g(x) = h \neq 0$. Then $\langle Tg, E_x^* h \rangle_\mathcal{L} = \lim \lambda \langle M_{f_\lambda} g, E_x^* h \rangle_\mathcal{L} = \lim \lambda \langle E_x(M_{f_\lambda} g), h \rangle_\mathcal{H} = \lim \lambda f_\lambda(x) \|h\|^2$. This shows that at every such $x$ the net $\{f_\lambda(x)\}$ converges to some value. Set $f(x)$ equal to this limit and for all other $x$’s set $f(x) = 0$. We claim that $f$ is a multiplier and that $T = M_f$.

Note that if $g(x) = 0$ for every $g \in \mathcal{L}$, then $E_x = E_x^* = 0$. Thus, we have that for any $g \in \mathcal{L}$ and any $h \in \mathcal{H}$, $\langle E_x(Tg), h \rangle_\mathcal{H} = \lim \lambda \langle E_x(M_{f_\lambda} g), h \rangle_\mathcal{H} = \lim \lambda f_\lambda(x) \langle g(x), h \rangle_\mathcal{H} = f(x) \langle g(x), h \rangle_\mathcal{H}$. Since this holds for every $h \in \mathcal{H}$, we have that $E_x(Tg) = f(x)g(x)$, and so $T = M_f$ and $f$ is a multiplier. □
Algebras of Functions

Every weak*-closed subspace of $V \subseteq B(H)$ has a predual and it is the operator space dual of this predual. Also, if an abstract operator algebra is the dual of an operator space, then it can be represented completely isometrically and weak*-continuously as a weak*-closed subalgebra of the bounded operators on some Hilbert space. For this reason an operator algebra that has a predual as an operator space is called a dual operator algebra. See the book of [13] for the proofs of these facts. Thus, in summary, the above lemma shows that every multiplier algebra is a dual operator algebra.

**Theorem 2.2.** Let $\mathcal{L}$ be a reproducing kernel Hilbert space of $H$-valued functions with reproducing kernel $K : X \times X \to B(H)$ and let $\mathcal{M}(K) \subseteq B(\mathcal{L})$ denote the multiplier algebra, endowed with the operator algebra structure that it inherits as a subalgebra. If $K(x,x) \neq 0$, for every $x \in X$ and $\mathcal{M}(K)$ separates points on $X$, then $\mathcal{M}(K)$ is a BPW complete local dual operator algebra of functions on $X$.

**Proof.** The multiplier norm of a given matrix-valued function $F = (f_{i,j}) \in M_n(\mathcal{M}(K))$ is the least constant $C$ such that $((C^2I_n - F(x_i)F(x_j)^*) \otimes K(x_i,x_j)) \geq 0$, for all sets of finitely many points, $Y = \{x_1,\ldots,x_k\} \subseteq X$. Applying this fact to a set consisting of a single point, we have that $(C^2I_n - F(x)F(x)^*) \otimes K(x,x) \geq 0$, and it follows that $C^2I_n - F(x)F(x)^* \geq 0$. Thus, $\|F(x)\| \leq C = \|F\|$ and we have that point evaluations are completely contractive on $\mathcal{M}(K)$. Since $\mathcal{M}(K)$ contains the constants and separates points by hypothesis, it is an operator algebra of functions on $X$.

Suppose that $\mathcal{M}(K)$ was not local, then there would exist $F \in M_n(\mathcal{M}(K))$, and a real number $C$, such that $\sup_{Y} \| \pi^{(n)}_Y \| < C < \|F\|$. Then for each finite set $Y = \{x_1,\ldots,x_k\}$ we could choose $G \in M_n(\mathcal{M}(K))$, with $\|G\| < C$, and $G(x) = F(x)$, for every $x \in Y$. But then we would have that $((C^2I_n - F(x_i)F(x_j)^*) \otimes K(x_i,x_j)) = ((C^2I_n - G(x_i)G(x_j)^*) \otimes K(x_i,x_j)) \geq 0$, and since $Y$ was arbitrary, $\|F\| \leq C$, a contradiction. Thus, $\mathcal{M}(K)$ is local.

Finally, assume that $f_\lambda \in \mathcal{M}(K)$ is a net in $\mathcal{M}(K)$, with $\|f_\lambda\| \leq C$, and $\lim_\lambda f(x) = f(x)$, pointwise. If $g \in \mathcal{L}$ with $\|g\| = M$, then $(MC)^2K(x,y) - \langle f_\lambda(x)g(x), f_\lambda(y)g(y) \rangle$ is positive definite. By taking pointwise limits, we obtain that $(MC)^2K(x,y) - \langle f(x)g(x), f(y)g(y) \rangle$ is positive definite. From the characterization of functions and their norms in a reproducing kernel Hilbert space, this implies that $fg \in \mathcal{L}$, with $\|fg\| \leq MC$. Hence, $f \in \mathcal{M}(K)$ with $\|M_f\| \leq C$. Thus, $\mathcal{M}(K)$ is BPW complete.

In general, $\mathcal{M}(K)$ need not separate points on $X$. In fact, it is possible that $\mathcal{L}$ does not separate points and if $g(x_1) = g(x_2)$, for every $g \in \mathcal{L}$, then necessarily $f(x_1) = f(x_2)$ for every $f \in \mathcal{M}(K)$.

Following [24], by a $k$-idempotent operator algebra, $\mathcal{C}$, we mean that we are given $k$ operators, $\{E_1,\ldots,E_k\}$ on some Hilbert space $\mathcal{H}$, such that $E_iE_j = E_jE_i = \delta_{i,j}E_i$, $I = E_1 + \cdots + E_k$ and $\mathcal{C} = \text{span}\{E_1,\ldots,E_k\}$.

**Proposition 2.3.** Let $\mathcal{C} = \text{span}\{E_1,\ldots,E_k\}$ be a $k$-idempotent operator algebra on the Hilbert space $\mathcal{H}$, let $Y = \{x_1,\ldots,x_k\}$ be a set of $k$ distinct
points and define $K : Y \times Y \to B(\mathcal{H})$ by $K(x_i, x_j) = E_i E_j^*$. Then $K$ is positive definite and $\mathcal{C}$ is completely isometrically isomorphic to $\mathcal{M}(K)$ via the map that sends $a_1 E_1 + \cdots + a_k E_k$ to the multiplier $f(x_i) = a_i$.

**Proof.** It is easily checked that $K$ is positive definite. We first prove that the map is an isometry. Given $B = \sum_{i=1}^{k} a_i \otimes E_i \in \mathcal{C}$, let $f : Y \to \mathbb{C}$ be defined by $f(x_i) = a_i$. We have that $f \in \mathcal{M}(K)$ with $\|f\| \leq C$ if and only if $P = ((C^2 - f(x_i)f(x_j)^*)K(x_i, x_j))$ is positive semidefinite in $B(\mathcal{H}^{(k)})$.

Let $v = e_1 \otimes v_1 + \cdots e_k \otimes v_k \in \mathcal{H}^{(k)}$, let $h = \sum_{j=1}^{k} E_j^* v_j$ and note that $E_j^* h = E_j^* v_j$. Finally, set $h = \sum_{i=1}^{k} h_i$. Thus,

$$\langle P v, v \rangle = \sum_{i,j=1}^{k} (C^2 - a_i a_j) \langle E_i E_j^* v_j, v_i \rangle = C^2 \|h\|^2 - \|B^* h, B^* h\| = C^2 \|h\|^2 - \|B^* h\|^2.$$

Hence, $\|B\| \leq C$ implies that $P$ is positive and so $\|M_f\| \leq \|B\|$. For the converse, given any $h$ let $v = \sum_{j=1}^{k} e_j \otimes E_j^* h$, and note that $\langle P v, v \rangle \geq 0$, implies that $\|B^* h\| \leq C$, and so $\|B\| \leq \|M_f\|$.

The proof of the complete isometry is similar but notationally cumbersome.

**Theorem 2.4.** Let $\mathcal{A}$ be an operator algebra of functions on the set $X$ then there exists a Hilbert space, $\mathcal{C}$ and a positive definite function $K : X \times X \to B(\mathcal{C})$ such that $\mathcal{M}(K) = \bar{\mathcal{A}}$ completely isometrically.

**Proof.** Let Y be a finite subset of X. Since $\mathcal{A}/I_Y$ is a $|Y|$-idempotent operator algebra, by the above lemma, there exists a vector valued kernel $K_Y$ such that $\mathcal{A}/I_Y = \mathcal{M}(K_Y)$ completely isometrically.

Define

$$\overline{K_Y}(x, y) = \begin{cases} K_Y(x, y) & \text{when } (x, y) \in Y \times Y, \\ 0 & \text{when } (x, y) \notin Y \times Y. \end{cases}$$

and set $K = \sum_{Y} \overline{K_Y}$, where the direct sum is over all finite subsets of X. Then it is easily checked that $K$ is positive definite.

Let $f \in M_n(\mathcal{M}(K))$ with $\|M_f\| \leq 1$. This is equivalent to

$$((I - f(x_i)f(x_j)^*) \otimes K(x_i, x_j)) \geq 0$$

$$\iff ((I - f(x_i)f(x_j)^*) \otimes K_F(x_i, x_j)) \geq 0 \forall F \subseteq X.$$ 

This last condition is equivalent to the existence for each $F$ of some $f_F \in M_n(\mathcal{A})$ such that $\|\pi_F(f_F)\| \leq 1$ and $f_F = f$ on $F$. The net of functions $\{f_F\}$ then converges BPW to $f$. Hence, $f \in \bar{\mathcal{A}}$ with $\|f\| \leq 1$.

This proves that $M_n(\mathcal{M}(K)) = M_n(\bar{\mathcal{A}})$ isometrically.  

Corollary 2.5. Every BPW complete local operator algebra of functions is a dual operator algebra.

Proof. In this case we have that \( A = \hat{A} = \mathcal{M}(K) \) completely isometrically. By Lemma 2.1, this latter algebra is a dual operator algebra. \( \square \)

The above theorem gives a weak*-topology to a local operator algebra of function, \( A \) by using the identification \( A \subseteq \hat{A} = \mathcal{M}(K) \) and taking the weak*-topology of \( \mathcal{M}(K) \). The following proposition proves that convergence of bounded nets in this weak*-topology on \( A \) is same as BPW convergence.

Proposition 2.6. Let \( A \) be a local operator algebra of functions on the set \( X \). Then the net \((f_\lambda)_{\lambda} \in \text{Ball}(A)\) converges in the weak*-topology if and only if it converges pointwise on \( X \).

Proof. Let \( \mathcal{H} \) denote the reproducing kernel Hilbert space of \( \mathcal{C} \)-valued functions on \( X \) with kernel \( K \) for which \( \hat{A} = \mathcal{M}(K) \). Recall that if \( E_x : \mathcal{H} \to \mathcal{C} \), is the linear map given by evaluation at \( x \), then \( K(x, y) = E_x E_y^* \). Also, if \( v \in \mathcal{C} \), and \( h \in \mathcal{H} \), then \( \langle h, E_x^* v \rangle_{\mathcal{H}} = \langle h(x), v \rangle_\mathcal{C} \).

First assume that the net \((f_\lambda)_{\lambda} \in \text{Ball}(A)\) converges to \( f \) in the weak*-topology. Using the identification of \( \hat{A} = \mathcal{M}(K) \), we have that the operators \( M_{f_\lambda} \) of multiplication by \( f_\lambda \), converge in the weak*-topology of \( B(\mathcal{C}) \) to \( M_f \). Then for any \( x \in X, h \in \mathcal{H}, v \in \mathcal{C} \), we have that \( f_\lambda(x) \langle h(x), v \rangle_\mathcal{C} = \langle f(x) h(x), v \rangle_\mathcal{C} = \langle M_{f_\lambda} h, E_x^* v \rangle_{\mathcal{H}} \to \langle M_f h, E_x^* v \rangle_{\mathcal{H}} = \langle f(h(x)), v \rangle_\mathcal{C} \). Thus, if there is a vector in \( \mathcal{C} \) and a vector in \( \mathcal{H} \) such that \( \langle h(x), v \rangle_\mathcal{C} \neq 0 \), then we have that \( f_\lambda(x) \to f(x) \). It is readily seen that such vectors exist if and only if \( E_x \neq 0 \), or equivalently, \( K(x, x) \neq 0 \). But this follows from the construction of \( K \) as a direct sum of positive definite functions over all finite subsets of \( X \). For fixed \( x \in X \) and the one element subset \( Y_0 = \{x\} \), we have that the 1-idempotent algebra \( A/I_{Y_0} \neq 0 \) and so \( K_{Y_0}(x, x) \neq 0 \), which is one term in the direct sum for \( K(x, x) \).

Conversely, assume that \( \|f_\lambda\| < K, \forall \lambda \) and \( f_\lambda \to f \) pointwise on \( X \). We must prove that \( M_{f_\lambda} \to M_f \) in the weak*-topology on \( B(\mathcal{H}) \). But since this is a bounded net of operators, it will be enough to show convergence in the weak operator topology and arbitrary vectors can be replaced by vectors from a spanning set. Thus, it will be enough to show that for \( v_1, v_2 \in \mathcal{C} \) and \( x_1, x_2 \in X \), we have that \( \langle M_{f_\lambda} E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{H}} \to \langle M_f E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{H}} \). But we have,

\[
\langle M_{f_\lambda} E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{H}} = \langle E_{x_1} (M_{f_\lambda} E_{x_1}^* v_1), v_2 \rangle_\mathcal{C} = f_\lambda(x_2) \langle K(x_2, x_1) v_1, v_2 \rangle_\mathcal{C} \to f(x_2) \langle K(x_2, x_1) v_1, v_2 \rangle_\mathcal{C} = \langle M_f E_{x_1}^* v_1, E_{x_2}^* v_2 \rangle_{\mathcal{H}},
\]

and the result follows. \( \square \)

Corollary 2.7. The ball of a local operator algebra of functions is weak*-dense in the ball of its BPW completion.
3. Residually Finite Dimensional Operator Algebras

A C*-algebra $B$ is called residually finite-dimensional (RFD) if it has a separating family of finite-dimensional representations. Note that when $B$ is separable and RFD it has a separating sequence of finite dimensional representations.

Since every one-to-one *-homomorphism of a C*-algebra is completely isometric, a C*-algebra $B$ is RFD if and only if for all $n$, and for every $(b_{i,j}) \in M_n(B)$, we have that $\|(b_{i,j})\|_{M_n(B)} = \sup\{\|(\pi(b_{i,j}))\|\}$ where the suprema is taken over all *-homomorphisms, $\pi : B \to M_k \forall k$. RFD C*-algebras have been studied in [10], [15], [5], [19].

We know from BRS that any operator algebra can be represented on a Hilbert space, i.e., given an abstract unital operator algebra $A$ there exists an unital completely isometric homomorphism $\pi : A \to B(H)$. Since a representation of a C*-algebra is a *-homomorphism if and only if it is completely contractive, the following definition gives us a natural way of extending the notion of RFD to operator algebras.

**Definition 3.1.** An operator algebra, $B$ is called RFD if for every $n$ and for every $(b_{i,j}) \in M_n(B)$, $\|(b_{i,j})\| = \sup\{\|(\pi(b_{i,j}))\|\}$, where the suprema is taken over all completely contractive homomorphisms, $\pi : B \to M_k \forall k$. A dual operator algebra $B$ is called weak*-RFD if this last equality holds when the completely contractive homomorphisms are also required to be weak*-continuous.

The following result is implicitly contained in [24], but the precise statement that we shall need does not appear there. Thus, we refer the reader to [24] to be able to fully understand the proof since we have used some of the definitions and results from [24] without stating them.

**Lemma 3.2.** Let $B$ be a concrete $k$-idempotent operator algebra. Then $\mathcal{S}(B^*B)$ is a Schur ideal affiliated with $B$, i.e., $B = \mathfrak{A}(\mathcal{S}(B^*B))$ completely isometrically.

**Proof.** From the Corollary 3.3 of [24] we have that the Schur ideal $\mathcal{S}(B^*B)$ is non-trivial and bounded. Thus, we can define the algebra $\mathfrak{A}(\mathcal{S}(B^*B)) = \text{span}\{E_1, \cdots, E_k\}$, where $E_i = \sum_n \sum_{Q} E_n^{-1}(B^*B) \oplus Q^{1/2}(I_n \otimes E_{ii})Q^{-1/2}$ is the idempotent operator that live on $\sum_n \sum_{Q} E_n^{-1} \oplus M_k(M_n)$. By using Theorem 3.2 of [24] we get that $\mathcal{S}(\mathfrak{A}(\mathcal{S}(B^*B))\mathfrak{A}(\mathcal{S}(B^*B))^*) = \mathcal{S}(B^*B)$ This further implies that $\mathfrak{A}(\mathcal{S}(B^*B))\mathfrak{A}(\mathcal{S}(B^*B))^* = B^*B$ completely order isomorphically under the map which sends $E_i^*E_j$ to $F_i^*F_j$. Finally, by restricting the same map to $A$ we get a map which sends $E_i$ to $F_i$ completely isometrically. Hence, the result follows.

**Theorem 3.3.** Every $k$-idempotent operator algebra is weak*-RFD.

**Proof.** Let $A$ be an abstract $k$-idempotent operator algebra. Note that $A$ is a dual operator algebra being a finite dimensional operator algebra.
Theorem 3.4. Every BPW complete local operator algebra of functions is weak*-RFD.
Proof. Let \( \mathcal{A} \) be a BPW complete local operator algebra of functions on the set \( X \) and \( F \) be a finite subset of \( X \), \( \mathcal{A}/I_F \) is a \(|F|\)-idempotent operator algebra. Thus, it follows from the above lemma that \( \mathcal{A}/I_F \) is weak*-RFD, i.e., \( \forall ([f_{ij}]) \in M_k(\mathcal{A}/I_F) \) we have \( \|([f_{ij}])\| = \sup_{\rho_F} \{\|\rho_F([f_{ij}])\|\} \) where supremum is taken over all weak*-continuous cc homomorphisms \( \rho_F : \mathcal{A}/I_F \to M_{n_F} \) and all integers \( n_F \).

Let \( (f_{ij}) \in M_k(\mathcal{A}) \), then \( \|(f_{ij})\|_{M_k(\mathcal{A})} = \sup_F \|([f_{ij}])\| \) since \( \mathcal{A} \) is local. Recall, that the weak* topology on \( \mathcal{A} \) requires all the quotient maps of the form \( \pi_F : \mathcal{A} \to \mathcal{A}/I_F \), \( \pi_F(f) = [f] \) to be weak*-continuous. Thus for each finite subset \( F \subseteq X \), \( \pi_F \) is a weak*-continuous cc homomorphism. Let \( \rho_F : \mathcal{A}/I_F \to M_{n_F} \) be a weak*-continuous cc homomorphism, then define \( \delta_F = \pi_F \circ \rho_F \) which is indeed a weak*-continuous cc homomorphism.

Consider
\[
\|(\rho_F([f_{ij}]))\| = \|\rho_F(\pi_F(f_{ij}))\| = \|\delta_F(f_{ij})\| \leq \sup_n \|\pi(f_{ij})\| : \pi : \mathcal{A} \to M_n \text{ weak*-cont. cc homo} \leq \|(f_{ij})\|_{M_k(\mathcal{A})}.
\]

Finally, note that \( \|(f_{ij})\|_{M_k(\mathcal{A}/I_F)} = \sup_{n,F} \{\|\rho_F([f_{ij}])\|\} \), where supremum is taken over all weak*-continuous cc homomorphisms, \( \rho_F \) and positive integers \( n_F \). Hence, we obtain the result by taking supremum over all the finite subsets \( F \subseteq X \).

\[\boxed{\text{Corollary 3.5. Every local operator algebra of functions is RFD.}}\]

Proof. The proof of this follows immediately since every local operator algebra is contained in some BPW complete local operator algebra completely isometrically.

4. Quantized Function Theory on Domains

Whenever one replaces scalar variables by operator variables in a problem or definition, then this process is often referred to as \textit{quantization}. It is in this sense that we would like to \textit{quantize} the function theory on a family of complex domains. In some sense this process has already been carried out for balls in the work of Drury [18], Popescu [28], Arveson [6], and Davidson and Pitts [17] and for polydisks in the work of Agler [1,2], and Ball and Trent [8]. Furthermore, the idea of “quantizing” other domains defined by inequalities occurs in [4], [10], and [20]. We approach these same ideas via operator algebra methods. We will show that in many cases this process yields local operator algebras of functions to which the results of the earlier sections can be applied.

We begin by defining a family of open sets for which our techniques will apply.

**Definition 4.1.** Let \( G \subseteq \mathbb{C}^N \) be an open set. If there exists a set of matrix-valued functions, \( F_k = (f_{k,i,j}) : G^- \to M_{m_k,n_k} \), \( k \in I \), whose components are analytic functions on \( G \), and \( G = \{ z \in \mathbb{C}^N : \|F_k(z)\| < 1, k \in I \} \), then we
call $G$ an analytically presented domain and we call the set of functions $\mathcal{R} = \{ f_k : \mathbb{C}^n \to M_{m_k,n_k} : k \in I \}$ an analytic presentation of $G$. The subalgebra $\mathcal{A}$ of the algebra of functions on $G$ generated by the component functions $\{ f_{i,j} : 1 \leq i \leq m_k, 1 \leq j \leq n_k, k \in I \}$ and the constant function is called the algebra of the presentation. We say that $\mathcal{R}$ is a separating analytic presentation provided that the algebra $\mathcal{A}$ separates points on $G$.

Remark 4.2. An analytic presentation of $G$ by a finite set of matrix-valued functions, $F_k : G \to M_{m_k,n_k}, 1 \leq k \leq K$, can always be replaced by the single block diagonal matrix-valued function, $F(z) = F_1(z) \oplus \cdots \oplus F_K(z)$ into $M_{n,n}$ with $m = m_1 + \cdots m_K, n = n_1 + \cdots n_K$ and we will sometimes do this to simplify proofs. But it is often convenient to think in terms of the set, especially since this will explain the sums that occur in the Agler’s factorization formula.

Note that when we have a analytically presented domain, then every function in the algebra of the presentation is an analytic function on $G$.

Definition 4.3. Let $G \subseteq \mathbb{C}^N$ be an analytically presented domain with presentation $F = (f_{i,j}) : G \to M_{n,n}$, and let $\mathcal{H}$ be a Hilbert space. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the algebra of the presentation is called an admissible representation provided that $\|(\pi(f_{i,j}))\| \leq 1$ in $M_{n,n}(\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}^m)).$ We call the homomorphism $\pi$ an admissible strict representation when these inequalities are all strictly less than 1. Given $(g_{i,j}) \in M_n(\mathcal{A})$ we set $\|(g_{i,j})\|_u = \sup\{\|(\pi(g_{i,j}))\|\}$, where the supremum is taken over all admissible representations $\pi$ of $\mathcal{A}$. We let $\|(g_{i,j})\|_{u_0}$ denote the supremum that is obtained when we restrict to admissible strict representations.

The theory of [4] and [10] studies domains defined as above with the additional restrictions that the set of defining functions is a finite set of polynomials, but they do not need their polynomials to separate points. Our results should be compared to theirs.

Proposition 4.4. Let $G$ have a separating analytical presentation and let $\mathcal{A}$ be the algebra of the presentation. Then $\mathcal{A}$ endowed with either of the family of norms $\| \cdot \|_u$ or $\| \cdot \|_{u_0}$ is an operator algebra of functions on $G$.

Proof. It is clear that it is an operator algebra and by definition it is an algebra of functions on $G$. It follows from the hypotheses that it separates points of $G$. Finally, for every $\lambda = (\lambda_1, \ldots, \lambda_N) \in G$, we have a representation of $\mathcal{A}$ on the one-dimensional Hilbert space given by $\pi_\lambda(f) = f(\lambda)$. Hence, $|f(\lambda)| \leq \|f\|_u$ and so $\mathcal{A}$ is an operator algebra of functions on $G$. \qed

It will be convenient to say that matrices, $A_1, \ldots, A_m$ are of compatible sizes if the product, $A_1 \cdots A_m$ exists, that is, provided that each $A_i$ is an $n_i \times n_i+1$ matrix.

Given an analytically presented domain $G$, we include one extra function, $F_{b_0}$ which denotes the constant function 1. By an admissible block diagonal matrix over $G$ we mean a block diagonal matrix-valued function
Thus, we are allowing blocks of 1’s in \( D \). Let \( G \) be an analytically presented domain with presentation \( \mathcal{R} = \{ F_k = (f_{k,i,j}) : G \to M_{m_k,n_k} \} \), let \( \mathcal{A} \) be the algebra of the presentation and let \( P = (p_{ij}) \in M_{m,n}(\mathcal{A}) \), where \( m, n \) are arbitrary. Then the following are equivalent:

(i) \( \|P\|_u < 1 \),

(ii) there exists an integer \( l \), matrices of scalars \( C_j \), \( 1 \leq j \leq l \) with \( \|C_j\| < 1 \) and admissible block diagonal matrices \( D_j(z), 1 \leq j \leq l \), which are of compatible sizes and are such that

\[
P(z) = C_1D_1(z) \cdots C_lD_l(z).
\]

(iii) there exists a positive, invertible matrix \( R \in M_m \) and matrices \( P_0, P_k \in M_{m,r_k}(\mathcal{A}), k \in K \), where \( K \subseteq I \) is a finite set, such that

\[
I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k \in K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)}P_k(w)^*
\]

where \( r_k = q_km_k \) and \( z = (z_1, \ldots, z_N), \ w = (w_1, \ldots, w_N) \in G \).

Proof. Although we will not logically need it, we first show that (ii) implies (i), since this is the easiest implication and helps to illustrate some ideas. Note that if \( \pi : \mathcal{A} \to B(\mathcal{H}) \) is any admissible representation, then the norm of \( \pi \) of any admissible block diagonal matrix is at most 1. Thus, if \( P \) has the form of (ii), then for any admissible \( \pi \), we will have \( (\pi(p_{i,j})) \) expressed as a product of scalar matrices and operator matrices all of norm at most one and hence, \( \|\pi(P_{i,j})\| \leq \|C_1\| \cdots \|C_l\| < 1 \). Thus, \( \|P\|_u \leq \|C_1\| \cdots \|C_l\| < 1 \).

We now prove that (i) implies (ii). The ideas of the proof are similar to [23, Corollary 18.2], [12, Corollary 2.11] and [21, Theorem 1] and use in an essential way the abstract characterization of operator algebras. For each \( m, n \in \mathbb{N} \), one proves that \( \|P\|_{m,n} := \inf \{\|C_1\| \cdots \|C_l\| \} \), defines a norm on \( M_{m,n}(\mathcal{A}) \), where the infimum is taken over all \( l \) and all ways to factor \( P(z) = C_1D_1(z_{i_1}) \cdots C_lD_l(z_{i_l}) \) as a product of matrices of compatible sizes with scalar matrices \( C_j, 1 \leq j \leq l \) and admissible block diagonal matrices \( D_j, 1 \leq j \leq l \).

Moreover, one can verify that \( \mathcal{M}_{m,n}(\mathcal{A}) \) with this family \( \{\|\cdot\|_{m,n}\}_{m,n} \) of norms satisfies the axioms for an abstract unital operator algebra as given in [13] and hence by the Blecher-Ruan-Sinclair representation theorem [13] (see also [23]) there exists a Hilbert space \( \mathcal{H} \) and a unital completely isometric isomorphism \( \pi : \mathcal{A} \to B(\mathcal{H}) \).

Thus, for every \( m, n \in \mathbb{N} \) and for every \( P = (p_{ij}) \in \mathcal{M}_{m,n}(\mathcal{A}) \), we have \( \|P\|_{m,n} = \|\pi(p_{ij})\| \). However, \( \|\pi^{m_k,n_k}(F_k)\| = \|\pi(f_{k,i,j})\| \leq 1 \) for \( 1 \leq i \leq K \), and so, \( \pi \) is an admissible representation. Thus, \( \|P\|_{m,n} = \|\pi(p_{ij})\| \leq
\[ \|P\|_u. \text{ Hence, if } \|P\|_u < 1, \text{ then } \|P\|_{m,n} < 1 \text{ which implies that such a factorization exists. This completes the proof that (i) implies (ii).} \]

We will now prove that (ii) implies (iii). Suppose that \( P \) has a factorization as in (ii). Let \( K \subseteq I \) be the finite subset of all indices that appear in the block-diagonal matrices appearing in the factorization of \( P \). We will use induction on \( l \) to prove that (iii) holds.

First, assume that \( l = 1 \) so that \( P(z) = C_1 D_1(z) \). Then,

\[
I_m - P(z)P(w)^* = I_m - (C_1 D_1(z))(C_1 D_1(w))^* = (I_m - C_1 C_1^*) + C_1 (I - D_1(z)D_1(w)^*) C_1^*.
\]

Since \( D_1(z) \) is an admissible block diagonal matrix the \((i,i)\)-th block diagonal entry of \( I - D_1(z)D_1(w)^* \) is \( I - F_{k_i}(z)F_{k_i}(w)^* \) for some finite collection, \( k_i \).

Let \( E_k \) be the diagonal matrix that has 1’s wherever \( F_k \) appears (so \( E_k = 0 \) when there is no \( F_k \) term in \( D_1 \)). Hence,

\[
C_1(I - D_1(z)D_1(w)^*)C_1^* = \sum_k C_1 E_k(I - F_k(z)F_k(w)^*)E_k C_1^*.
\]

Therefore, gathering terms for common values of \( i \),

\[
I_m - P(z)P(w)^* = R_0 + \sum_{k \in K} P_k(I - F_k(z)F_k(w)^*)P_k^*,
\]

where \( R_0 = I_m - C_1 C_1^* \) is a positive, invertible matrix and \( P_1 \) is, in this case a constant. Thus, the form (iii) holds, when \( l = 1 \).

We now assume that the form (iii) holds for any \( R(z) \) that has a factorization of length at most \( l - 1 \), and assume that

\[
P(z) = C_1 D_1(z) \cdots D_{l-1}(z)C_l D_l(z) = C_1 D_1(z) R(z),
\]

where \( R(z) \) has a factorization of length \( l - 1 \).

Note that a sum of expressions such as on the right hand side of (iii) is again such an expression. This follows by using the fact that given any two expressions \( A(z), B(z) \), we can write

\[
A(z)A(w)^* + B(z)B(w)^* = C(z)C(w)^*
\]

where \( C(z) = (A(z), B(z)) \).

Thus, it will be sufficient to show that \( I_m - P(z)P(w)^* \) is a sum of expressions as above. To this end we have that,

\[
I_m - P(z)P(w)^* = (I_m - C_1 D_1(z)C_1^*)
\]

\[
+ (C_1 D_1(z))(I - R(z)R(w)^*)(D_1(w)^*C_1^*).
\]

The first term of the above equation is of the form as on the right hand side of (iii) by case \( l = 1 \). Also, the quantity \((I - R(z)R(w)^*) = R_0 R_0^* + R_0(z)R_0(w)^* + \sum_{k \in K} R_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} R_k(w)^* \) by the inductive
representation and let $\mathcal{P}$.

Thus, we have expressed $(I - P(z)P(w)^*)$ as a sum of two terms both of which can be written in the form desired. Using again our remark that the sum of two such expressions is again such an expression, we have the required form.

Finally, we will prove (iii) implies (i). Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be an admissible representation and let $P = (p_{i,j}) \in M_{m,n}(\mathcal{A})$ have a factorization as in (iii). To avoid far too many superscripts we write simplify $\pi^{(m,n)}$ to $\Pi$.

Now observe that

$$I_m - \Pi(P)\Pi(P)^* = \Pi(R) + \Pi(P_0)\Pi(P_0)^* + \sum_{k \in K} \Pi(P_k)(I - \Pi(F_k)\Pi(F_k)^*)(q_k)(\Pi(P_k))^*.$$

Clearly the first two terms of the sum are positive. But since $\pi$ is an admissible representation, $\|\Pi(F_k)\| \leq 1$ and hence, $(I - \Pi(F_k)\Pi(F_k)^*) \geq 0$. Hence, each term on the right hand side of the above inequality is positive and since $R$ is strictly positive, say $R \geq \delta I_m$ for some scalar $\delta > 0$, we have that $I_m - \Pi(P)\Pi(P)^* \geq \delta I_m$.

Therefore, $\|\Pi(P)\| \leq \sqrt{1 - \delta}$. Thus, since $\pi$ was an arbitrary admissible representation, $\|P\|_u \leq \sqrt{1 - \delta} < 1$, which proves (i).

When we require the functions in the presentation to be row vector-valued, then the above theory simplifies somewhat and begins to look more familiar. Let $G$ be an analytically presented domain with presentation $F_k : G \to M_{1,n_k}, k \in I$. We identify $M_{1,n}$ with the Hilbert space $\mathbb{C}^n$ so that $1 - F_k(z)F_k(w)^* = 1 - \langle F_k(z), F_k(w) \rangle$, where the inner product is in $\mathbb{C}^n$. In this case we shall say that $G$ is presented by vector-valued functions.

**Corollary 4.6.** Let $G$ be presented by vector-valued functions, $F_k = (f_{k,j}) : G \to M_{1,n_k}, k \in I$, let $\mathcal{A}$ be the algebra of the presentation and let $P = (p_{i,j}) \in M_{m,n}(\mathcal{A})$. Then the following are equivalent:

(i) $\|P\|_u < 1$,

(ii) there exists an integer $l$, matrices of scalars $C_j$, $1 \leq j \leq l$ with $\|C_j\| < 1$ and admissible block diagonal matrices $D_j(z), 1 \leq j \leq l$, which are of compatible sizes and are such that

$$P(z) = C_1D_1(z) \cdots C_lD_l(z).$$
there exists a positive, invertible matrix \( R \in M_m \) and matrices \( P_0 \in M_{m,r_0}(A), P_k \in M_{m,r_k}(A), k \in K \), where \( K \subseteq I \) is finite, such that
\[
I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k \in K} (1 - \langle F_k(z), F_k(w) \rangle) P_k(z)P_k(w)^*
\]
where \( z = (z_1, ..., z_N), w = (w_1, ..., w_N) \in G \).

The following result gives us a Nevanlinna-type result for the algebra of presentation.

**Theorem 4.7.** Let \( Y \) be a finite subset of an analytically presented domain \( G \) with separating analytic presentation \( F_k = (f_{k,i,j}) : G \to M_{m,k,n_k}, k \in I \), let \( A \) be the algebra of the presentation and let \( P \) be a \( M_{m,n} \)-valued function defined on a finite subset \( Y = \{x_1, \ldots, x_I\} \) of \( G \). Then the following are equivalent:

(i) there exists \( \tilde{P} \in M_{mn}(A) \) such that \( \tilde{P}|_Y = P \) and \( \|\tilde{P}\|_u < 1 \).

(ii) there exists a positive, invertible matrix \( R \in M_m \) and matrices \( P_0 \in M_{m,r_0}(A), P_k \in M_{m,r_k}(A), k \in K \), where \( K \subseteq I \) is a finite set, such that
\[
I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k \in K} P_k(z)(I - F_k(z)F_k(w)^*)(q_k)P_k(w)^*
\]
where \( r_k = q_km_k \) and \( z = (z_1, ..., z_N), w = (w_1, ..., w_N) \in Y \).

**Proof.** Note that (i) \( \Rightarrow \) (ii) follows immediately as a corollary of Theorem 2.4. Thus it only remains to show that (ii) \( \Rightarrow \) (i). Since \( A \) is an operator algebra of functions, therefore, \( A/I_Y \) is a finite dimensional operator algebra of idempotents and \( A/I_Y = \text{span}\{E_1, \ldots, E_l\} \) for some \( l = |Y| \). Thus there exists a Hilbert space, \( H_Y \) and a completely isometric representation \( \pi \) of \( A/I_Y \). By Theorem 2.4, there exists a kernel \( K_Y \) such that \( \pi(A/I_Y) = \mathcal{M}(K_Y) \) completely isometrically under the map \( \rho : \pi(A/I_Y) \to \mathcal{M}(K_Y) \) which sends \( \pi(B) \) to \( M_f \), where \( B = \sum_{i=1}^{l} a_i\pi(E_i) \) and \( f : Y \to \mathbb{C} \) is a function defined by \( f(x_i) = a_i \). Note that \( ((I - F_k(x_i)F_k(x_j)) \otimes K_Y(x_i, x_j)) = ((I - \pi(F_k + I_Y)(x_i)\pi(F_k + I_Y)(x_j)^*) \otimes K_Y(x_i, x_j))_{ij} \geq 0 \) for \( k \in I \) since \( \|\pi(F_k + I_Y)\| \leq \|F_k\|_u \leq 1 \) for all \( k \in I \). From this it follows that \( ((I_m - (\pi(P + I_Y))(x_i)(\pi(P + I_Y))(x_j)^*) \otimes K_Y(x_i, x_j))_{ij} \geq 0 \). Using that \( R > 0 \), we get that \( \|\pi(P + I_Y)\| < 1 \). This shows that there exists \( \tilde{P} \in A \) such that \( \tilde{P}|_Y = P \) and \( \|\tilde{P}\|_u < 1 \). This completes the proof.

We now turn towards defining quantized versions of the bounded analytic functions on these domains. For this we need to recall that the joint Taylor spectrum of a commuting \( N \)-tuple of operators \( T = (T_1, ..., T_N) \), is a compact set, \( \sigma(T) \subseteq \mathbb{C}^N \) and that there is an analytic functional calculus defined for any function that is holomorphic in a neighborhood of \( \sigma(T) \).

**Definition 4.8.** Let \( G \subseteq \mathbb{C}^N \) be an analytically presented domain, with presentation \( \mathcal{R} = \{F_k : G^- \to M_{m_k,n_k}, k \in I\} \). We define the quantized
version of \( G \) to be the collection of all commuting \( N \)-tuples of operators,

\[
Q(G) = \{ T = (T_1, T_2, \ldots, T_N) \in B(\mathcal{H}) : \sigma(T) \subseteq G \text{ and } \|F_k(T)\| \leq 1, \forall k \in I \},
\]

where \( \mathcal{H} \) is an arbitrary Hilbert space. We set

\[
Q_{0,0}(G) = \{ T = (T_1, T_2, \ldots, T_N) \in M_n : \sigma(T) \subseteq G \text{ and } \|F_k(T)\| \leq 1, \forall k \in I \},
\]

where \( n \) is an arbitrary positive integer.

Note that if we identify a point \((\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N\) with an \( N \)-tuple of commuting operators on a one-dimensional Hilbert space, then we have that \( G \subseteq Q(G) \).

If \( T = (T_1, \ldots, T_N) \in Q(G) \), is a commuting \( N \)-tuple of operators on the Hilbert space \( \mathcal{H} \), then since the joint Taylor spectrum of \( T \) is contained in \( G \), we have that if \( f \) is analytic on \( G \), then there is an operator \( f(T) \) defined and the map \( \pi : \text{Hol}(G) \to B(\mathcal{H}) \) is a homomorphism, where \( \text{Hol}(G) \) denotes the algebra of analytic functions on \( G \).

**Definition 4.9.** Let \( G \subseteq \mathbb{C}^N \) be an analytically presented domain, with presentation \( \mathcal{R} = \{ F_k : G^* \to M_{m_k,n_k}, \ k \in I \} \). We define \( H^\infty_\mathcal{R}(G) \) to be the set of functions \( f \in \text{Hol}(G) \), such that \( \|f\|_R = \sup \{ \|f(T)\| : T \in Q(G) \} \) is finite. Given \( (f_{i,j}) \in M_n(H^\infty_\mathcal{R}(G)) \), we set \( \|(f_{i,j})\|_R = \sup \{ \|(f_{i,j}(T))\| : T \in Q(G) \} \).

We are interested in determining when \( H^\infty_\mathcal{R}(G) = \tilde{A} \), completely isometrically and whether or not the \( \| \cdot \|_R \) norm is attained on the smaller set \( Q_{0,0}(G) \).

Note that since each point in \( G \subseteq Q(G) \), we have that \( H^\infty_\mathcal{R}(G) \subseteq H^\infty(G) \), and \( \|f\|_\infty \leq \|f\|_R \). Also, we can set \( \mathcal{A} \subseteq H^\infty_\mathcal{R}(G) \) and for \( (f_{i,j}) \in M_n(\mathcal{A}) \), \( \|(f_{i,j})\|_R \leq \|(f_{i,j})\|_u \leq \|(f_{i,j})\|_u \). Thus, the inclusion of \( \mathcal{A} \) into \( H^\infty_\mathcal{R}(G) \) might not even be isometric.

**Theorem 4.10.** Let \( G \) be an analytically presented domain with a separating presentation \( \mathcal{R} = \{ F_k : G \to M_{m_k,n_k} : k \in I \} \), let \( \mathcal{A} \) be the algebra of the presentation and let \( \tilde{\mathcal{A}} \) be the BPW-completion of \( \mathcal{A} \). Then

(i) \( \tilde{\mathcal{A}} = H^\infty_\mathcal{R}(G) \), completely isometrically,
(ii) \( H^\infty_\mathcal{R}(G) \) is a local, weak*-RFD, dual operator algebra,

**Proof.** Let \( f \in M_n(\tilde{\mathcal{A}}) \), with \( \|f\|_L < 1 \). Then there exists a net of functions, \( f_\lambda \in M_n(\mathcal{A}) \), such that \( \|f_\lambda\|_u < 1 \) and \( \lim_\lambda f_\lambda(z) = f(z) \) for every \( z \in G \). Since \( \|f_\lambda\|_\infty < 1 \), by Montel’s Theorem, there is a subsequence \( \{f_n\} \) of this net that converges to \( f \) uniformly on compact sets. Hence, if \( T \in Q(G) \), then \( \lim_n \|f(T) - f_n(T)\| = 0 \) and so \( \|f(T)\| \leq \sup_n \|f_n(T)\| \leq 1 \). Thus, we have that \( f \in M_n(H^\infty_\mathcal{R}(G)) \), with \( \|f\|_R \leq 1 \). This proves that \( \tilde{\mathcal{A}} \subseteq H^\infty_\mathcal{R}(G) \) and that \( \|f\|_L \leq \|f\|_R \).
Conversely, let \( g \in M_n(H^\infty_R(G)) \) with \( \|g\|_R < 1 \). For any finite set \( Y = \{y_1, \ldots, y_t\} \subseteq G \), let \( \mathcal{A}/I_Y = \text{span}\{E_1, \ldots, E_t\} \) be the corresponding \( t \)-idempotent algebra and let \( \pi_Y : \mathcal{A} \rightarrow \mathcal{A}/I_Y \) denote the quotient map. Write \( y_i = (y_{i1}, \ldots, y_{in}), 1 \leq i \leq t \) and let \( T_j = y_{ij}E_1 + \cdots + y_{ij}E_t, 1 \leq j \leq N \) so that \( T = (T_1, \ldots, T_N) \) is a commuting \( N \)-tuple of operators with \( \sigma(T) = Y \). For \( k \in I \), we have that \( \|F_k(T)\| = \|F_k(y_1) \otimes E_1 + \cdots + F_k(y_t) \otimes E_t\| = \|\pi_Y(F_k)\| \leq \|F_k\|_u = 1 \). Thus, \( T \in \mathcal{Q}(G) \), and so, \( \|g(T)\| = \|g(y_1) \otimes E_1 + \cdots + g(y_t) \otimes E_t\| \leq \|g\|_R < 1 \). Since \( \mathcal{A} \) separates points, we may pick \( f \in M_n(\mathcal{A}) \) such that \( f = g \) on \( Y \). Hence, \( \pi_Y(f) = f(T) = g(T) \) and \( \|\pi_Y(f)\| < 1 \). Thus, we may pick \( f_Y \in M_n(\mathcal{A}) \), such that \( \pi_Y(f_Y) = \pi_Y(f) \) and \( \|f_Y\|_u < 1 \). This net of functions, \( \{f_Y\} \), converges to \( g \) pointwise and is bounded. Therefore, \( g \in M_n(\mathcal{A}) \) and \( \|g\|_L \leq 1 \). This proves that \( H^\infty_R(G) \subseteq \mathcal{A} \) and that \( \|g\|_L \leq \|g\|_R \).

Thus, \( \mathcal{A} = H^\infty_R(G) \) and the two matrix norms are equal for matrices of all sizes. The rest of the conclusions follow from the results on BPW-completions. \( \square \)

**Remark 4.11.** The above result yields that for every \( f \in H^\infty_R(G) \), \( \|f\|_R = \sup\{\|\pi(f)\| : \pi \) is a finite dimensional weak* continuous representation, \( \pi : H^\infty_R(G) \to M_n \forall n \). Under additional hypotheses, we can show that \( \|f\|_R = \sup\{\|f(T)\| : T \in \mathcal{Q}(G) \} \forall f \in H^\infty_R(G) \). Also, we can verify these hypotheses are met for most of the algebras given in the example section. In particular, for Examples 5.1, 5.2, 5.3, 5.4, 5.6, 5.7, and 5.8. It would be interesting to know if this can be done in general.

We now seek other characterisations of the functions in \( H^\infty_R(G) \). In particular, we wish to obtain analogues of Agler’s factorization theorem and of the results in [4] and [10]. By Theorem 2.3 if we are given an analytically presented domain \( G \subseteq \mathbb{C}^N \), with presentation \( \mathcal{R} = \{F_k : G \to M_{m_k,n_k}, k \in I\} \), then there exists a Hilbert space \( \mathcal{H} \) and a positive definite function, \( K : G \times G \to B(\mathcal{H}) \) such that \( \mathcal{A} = \mathcal{M}(K) \). We shall denote any kernel satisfying this property by \( K_\mathcal{R} \).

**Definition 4.12.** Let \( G \subseteq \mathbb{C}^N \) be an analytically presented domain, with presentation \( \mathcal{R} = \{F_k : G \to M_{m_k,n_k}, k \in I\} \). We shall call a function \( H : G \times G \to M_m \) an \( \mathcal{R} \)-limit, provided that \( H \) is the pointwise limit of a net of functions \( H_\lambda : G \times G \to M_m \) of the form given by Theorem 2.3(iii).

**Corollary 4.13.** Let \( G \subseteq \mathbb{C}^N \) be an analytically presented domain, with a separating presentation \( \mathcal{R} = \{F_k : G \to M_{m_k,n_k}, k \in I\} \). Then the following are equivalent:

(i) \( f \in M_m(H^\infty_R(G)) \) and \( \|f\|_\mathcal{R} \leq 1 \).

(ii) \( (I_m - f(z)f(w)^*) \otimes K_\mathcal{R}(z,w) \) is positive definite.

(iii) \( I_m - f(z)f(w)^* \) is an \( \mathcal{R} \)-limit.

In the case when the presentation contains only finitely many functions we can say considerably more about \( \mathcal{R} \)-limits.
Proposition 4.14. Let $G$ be an analytically presented domain with a finite presentation $\mathcal{R} = \{f_k = (f_{k,i,j}) : G \to M_{m_k,n_k}, 1 \leq k \leq K\}$. For each compact subset $S \subseteq G$, there exists a constant $C$, depending only on $S$, such that for any analytic function $F$ and an analytic function $\lambda$, we have that

$$I_m - P(z)P(w)^* = R + P_0(z)P_0(w)^* + \sum_{k=1}^{K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*,$$

then $\|P_k(z)\| \leq C \forall k, \forall z \in S$.

Proof. By the continuity of the functions, there is a constant $\delta > 0$, such that $\|F_k(z)\| \leq 1 - \delta, \forall k, \forall z \in S$. Thus, we have that $I - F_k(z)F_k(z)^* \geq \delta I, \forall k, \forall z \in S$. Also, we have that

$$I_m \geq I_m - P(z)P(z)^* \geq P_k(z)(I - F_k(z)F_k(z)^*)^{(q_k)} P_k(z)^* \geq \delta P_k(z)P_k(z)^*$$

This shows that $\|P_k(z)\| \leq 1/\delta \forall k, \forall z \in S$. □

The proof of the following result is essentially contained in [10] Lemma 3.3.

Proposition 4.15. Let $G$ be a bounded domain in $\mathbb{C}^N$ and let $F = (f_{i,j}) : G \to M_{m,n}$ be analytic with $\|F(z)\| < 1$ for $z \in G$. If $H : G \times G \to M_p$ is analytic in the first variables, coanalytic in the second variables and there exists a net of matrix-valued functions $P_\lambda \in M_{p,p,\lambda}(\text{Hol}(G))$ which are uniformly bounded on compact subsets of $G$, such that $H(z,w)$ is the pointwise limit of $H_\lambda(z,w) = P_\lambda(z)(I_m - F(z)F(w)^*^{(q_\lambda)})P_\lambda(w)^*$ where $r_\lambda = q_\lambda m_k$, then there exists a Hilbert space $\mathcal{H}$ and an analytic function, $R : G \to B(\mathcal{H} \otimes \mathbb{C}^M, \mathbb{C}^p)$ such that $H(z,w) = R(z)(I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}^*}R(w)^*$. 

Proof. We identify $(I_m - F(z)F(w)^*^{(q_\lambda)}) = (I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}^*}$, and the $p \times m q_\lambda$ matrix-valued function $P_\lambda$ as an analytic function, $P_\lambda : G \to B(\mathbb{C}^m \otimes \mathbb{C}^{q_\lambda}, \mathbb{C}^p)$. Writing $\mathbb{C}^m \otimes \mathbb{C}^{q_\lambda} = \mathbb{C}^{q_\lambda} \oplus \cdots \mathbb{C}^{q_\lambda} (m$ times) allows us to write $P_\lambda(z) = [P_{i,1}^\lambda(z), \ldots, P_{i,m}^\lambda(z)]$ where each $P_{i,j}^\lambda(z)$ is $p \times q_\lambda$. Also, if we let $f_1(z), \ldots, f_m(z)$ be the $(1,n)$ vectors that represent the rows of the matrix $F$, then we have that $F(z)F(w)^* = \sum_{i,j=1}^{m} f_i(z)f_j(w)^*E_{i,j}$.

Finally, we have that

$$H_\lambda(z,w) = \sum_{i=1}^{m} P_i^\lambda(z)P_i^\lambda(w)^* - \sum_{i,j=1}^{m} f_i(z)f_j(w)^*P_i^\lambda(z)P_j^\lambda(w).$$

Let $K_\lambda(z,w) = (P_i^\lambda(z)P_j^\lambda(w)^*)$, so that $K_\lambda : G \times G \to M_m(M_p) = B(\mathbb{C}^m \otimes \mathbb{C}^p)$, is a positive definite function that is analytic in $z$ and coanalytic in $w$. By dropping to a subnet, if necessary, we may assume that $K_\lambda$ converges uniformly on compact subsets of $G$ to $K = (K_{i,j}) : G \times G \to M_m(M_p)$. Note that this implies that $P_i^\lambda(z)P_j^\lambda(w)^* \to K_{i,j}(z,w)$ for all $i, j$ and that $K$ is a positive definite function that is analytic in $z$ and coanalytic in $w$.

The positive definite $K$ gives rise to a reproducing kernel Hilbert space $\mathcal{H}$ of analytic $\mathbb{C}^m \otimes \mathbb{C}^p$-valued functions on $G$. If we let $E(z) : \mathcal{H} \to$
\[ C^m \otimes C^p, \text{ be the evaluation functional, then } K(z,w) = E(z)E(w)^* \text{ and }\
E : G \to B(\mathcal{H}, C^m \otimes C^p) \text{ is analytic. Identifying } C^m \otimes C^p = C^p \oplus \cdots \oplus C^p(m \text{ times}), \text{ yields analytic functions, } E_i : G \to B(\mathcal{H}, C^p), i = 1,\ldots,m, \text{ such that } (K_{i,j}(z,w)) = K(z,w) = E(z)E(w)^* = (E_i(z)E_j(w)^*).

Define an analytic map \( R : G \to B(\mathcal{H} \otimes C^m, C^p) \) by identifying \( \mathcal{H} \otimes C^m = \mathcal{H} \oplus \cdots \oplus \mathcal{H}(m \text{ times}) \) and setting \( R(z)(h_1 \oplus \cdots \oplus h_m) = E_1(z)h_1 + \cdots + E_m(z)h_m. \) Thus, we have that

\[ R(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}}]R(w)^* = \]
\[ \sum_{i=1}^{m} E_i(z)E_i(w)^* - \sum_{i,j=1}^{m} f_i(z)f_j(w)^* E_i(z)E_j(w)^* = \]
\[ \sum_{i=1}^{m} K_{i,j}(z,w) - \sum_{i,j=1}^{m} f_i(z)f_j(w)^* K_{i,j}(z,w) = \]
\[ \lim_{\lambda} \sum_{i=1}^{m} P_i^\lambda(z)P_i^\lambda(w)^* - \sum_{i,j=1}^{m} f_i(z)f_j(w)^* P_i^\lambda(z)P_j^\lambda(w)^* = H(z,w), \]
and the proof is complete. \( \square \)

**Remark 4.16.** Conversely, any function that can be written in the form \( H(z,w) = R(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{H}}]R(w)^* \) can be expressed as a limit of a net as above by considering the directed set of all finite dimensional subspaces of \( \mathcal{H} \) and for each finite dimensional subspace setting \( H_F(z,w) = R_F(z)[(I_m - F(z)F(w)^*) \otimes I_{\mathcal{F}}]R_F(w)^* \), where \( R_F(z) = R(z)(P_F \otimes I_m) \) with \( P_F \) the orthogonal projection onto \( \mathcal{F}. \)

**Definition 4.17.** We shall refer to a function \( H : G \times G \to M_m(M_p) \) that can be expressed as \( H(z,w) = R(z)[(I_m - F(z)F(w)^*) \otimes \mathcal{H}]R(w)^* \) for some Hilbert space \( \mathcal{H} \) and some analytic function \( R : G \to B(\mathcal{H} \otimes C^m, C^p) \), as an F-limit.

**Theorem 4.18.** Let \( G \) be an analytically presented domain with a finite separating presentation \( \mathcal{R} = \{ F_k = (f_{k,i,j}) : G \to M_{m_k(n_k)}, 1 \leq k \leq K \} \), let \( f = (f_{ij}) \) be a \( M_{m,n} \)-valued function defined on \( G \). Then the following are equivalent:

1. \( f \in M_{m,n}(H_{\mathcal{R}}^{\infty}(G)) \) and \( \| f \|_{\mathcal{R}} \leq 1, \)
2. there exists an analytic operator-valued function \( R_0 : G \to B(\mathcal{H}_0, C^m) \) and \( F_k \)-limits, \( H_k : G \times G \to M_m, \) such that
\[ I - f(z)f(w)^* = R_0(z)R_0(w)^* + \sum_{k=1}^{K} H_k(z,w) \forall z,w \in G, \]
3. there exist \( F_k \)-limits, \( H_k(z,w), \) such that
\[ I - f(z)f(w)^* = \sum_{k=1}^{K} H_k(z,w) \forall z,w \in G. \]
Theorem 2.4 there exists a vector-valued Kernel $K$ valued function $R$ such that $f_Y$ converges to $f$ pointwise and $\|f_Y\|_u \leq 1$. We may assume that $\|f_Y\|_u < 1$ by replacing $f_Y$ by $\frac{f_Y}{1+1/|Y|}$, where $|Y|$ denotes the cardinality of the set $Y$.

Thus by Theorem 4.5 there exists a positive, invertible matrix $R^Y \in M_m$ and matrices $F_k^Y \in M_{m,k_Y}(A)$, $0 \leq k \leq K$, such that

$$I_m - f_Y(z)f_Y(w)^* = R^Y + \sum_{k=0}^K P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_k)}P_k^Y(w)^*$$

where $r_{kY} = q_{kY}m_k$ and $z, w \in G$. If we define a map $F_0 : G \to M_{m,n_0}$ as the zero map then the above factorization can be written as

$$I_m - f_Y(z)f_Y(w)^* = R^Y + \sum_{k=0}^K P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{kY})}P_k^Y(w)^*$$

where $r_{kY} = q_{kY}m_k$ and $z, w \in G$.

Note that the net $R^Y$ is uniformly bounded above by 1, thus there exists $R \in M_m$ and a subnet $R^{Y'}$ which converges to $R$.

Finally, since the net $f_Y$ converges to $f$ pointwise we have that the net $\sum_{k=1}^K P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{kY})}P_k^Y(w)^*$ converges pointwise on $G$. Also note that for each $k$, $< P_k^Y >$ is the net of vector-valued holomorphic function and is uniformly bounded on compact subsets of $G$ by Proposition 4.14.

Thus by Proposition 4.15 there exists $F_k$-limit for each $0 \leq k \leq K$, that is, there exists $K+1$ Hilbert spaces $H_k$ and $K+1$ analytic function, $R_k : G \to B(H_k \otimes \mathbb{C}^M, \mathbb{C}^p)$ such that $H_k(z,w) = R_k(z)[(I_m - F_k(z)F_k(w)^*) \otimes I_{H_k}]; R_k(w)^*$ and the corresponding subnet of the net $\sum_{k=0}^K P_k^Y(z)(I - F_k(z)F_k(w)^*)^{(q_{kY})}P_k^Y(w)^*$ converges to $\sum_{k=0}^K H_k(z,w) \forall z, w \in G$. This completes the proof that (1) implies (2).

To show the converse, assume that there exists an analytic operator-valued function $R_0 : G \to B(H_0, \mathbb{C}^m)$ and $K$ analytic functions, $R_k : G \to B(H_k \otimes \mathbb{C}^M, \mathbb{C}^p)$ on some Hilbert space $H_k$ such that $I - f(z)f(w)^* = R_0(z)R_0(w)^* + \sum_{k=1}^K R_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)}R_k(w)^* \forall z, w \in G$. By using Theorem 2.4, there exists a vector-valued Kernel $K$ such that $M_n(\mathcal{M}(K)) = M_n(\tilde{A})$ completely isometrically for every $n$. It is easy to see that $(I - f(z)f(w)^*) \otimes K(z,w) \geq 0 \forall z, w \in G$. This is equivalent to $f \in M_m(\mathcal{M}(K))$ and $\|M_f\| \leq 1$ which in turn is equivalent to (1). Thus, (1) and (2) are equivalent.

Clearly, (3) implies (2). The argument for why (2) implies (3) is contained in [10] and we recall it. If we fix $k_0$, then since $\|F_{k_0}(z)\|^2 < 1$ on $G$, we have that $|f_{k_0,1,1}(z)|^2 + \cdots + |f_{k_0,1,m}(z)|^2 < 1$ on $G$. From this it follows that $H(z,w) = (1 - f_{k_0,1,1}(z)f_{k_0,1,1}(w) - \cdots - f_{k_0,1,m}(z)f_{k_0,1,m}(w))$.
is an $F_{k_0}$-limit and that $H^{-1}(z,w)$ is positive definite. Now we have that $R_0(z)R_0(w)^*H^{-1}(z,w)$ is positive definite and so we may write, $R_0(z)R_0(w)^*H^{-1}(z,w) = G_0(z)G_0(w)^*$ and we have that $R_0(z)R_0(w)^* = G_0(z)H(z,w)G(w)$. This shows that $R_0(z)R_0(w)^*$ is an $F_k$-limit and so it may be absorbed into the sum. \hfill\Box

5. Examples and Applications

In this section we present a few examples to illustrate the above definitions and results.

**Example 5.1.** Let $G = \mathbb{D}^N$, be the polydisk which has a presentation given by the coordinate functions $F_i(z) = z_i, 1 \leq i \leq N$. Then the algebra of this presentation is the algebra of polynomials and an admissible representation is given by any choice of $N$ commuting contractions, $(T_1,\ldots,T_N)$ on a Hilbert space. Given a matrix of polynomials, $\|(p_{i,j})\|_u = \sup \|(p_{i,j}(T_1,\ldots,T_N))\|$ where the supremum is taken over all $N$-tuples of commuting contractions. This is the norm considered by Agler in \cite{AG}, which is sometimes called the Schur-Agler norm \cite{AG1}. Our $Q(\mathbb{D}^N) = \{T = (T_1,\ldots,T_N) : \sigma(T) \subseteq \mathbb{D}^N \text{ and } \|T_i\| \leq 1\}$. Note that if we replace such a $T$ by $rT = (rT_1,\ldots,rT_N)$ then $\|rT_i\| < 1, rT \in Q_R(\mathbb{D}^N)$ and taking suprema over all $T \in Q_R(\mathbb{D}^N)$ will be the same as taking a suprema over this smaller set. Thus, the algebra $H_R(\mathbb{D}^N)$ consists of those analytic functions $f$ such that 

$$\|f\|_R = \sup\{\|f(T_1,\ldots,T_N)\| : \|T_i\| < 1, i = 1,\ldots,N\} \leq +\infty.$$ 

By Theorem 4.18 for $f \in M_{m,n}(H_R(\mathbb{D}^N))$, we have that $\|f\|_R \leq 1$ if and only if 

$$I_m - f(z)f(w)^* = \sum_{i=1}^{N} (1 - z_i\bar{w}_i)K_i(z,w),$$

for some analytic-coanalytic positive definite functions, $K_i : \mathbb{D}^N \times \mathbb{D}^N \to M_m$.

**Example 5.2.** Let $G = B_N$ denote the unit Euclidean ball in $\mathbb{C}^N$. If we let $F_i(z) = (z_1,\ldots,z_N) : B_N^* \to M_{1,N}$, then this gives us a polynomial presentation. Again the algebra of the presentation is the polynomial algebra. An admissible representation corresponds to an $N$-tuple of commuting operators $(T_1,\ldots,T_N)$ such that $T_1T_1^* + \cdots + T_NT_N^* \leq I$, which is commonly called a row contraction and an admissible strict representation is given when $T_1T_1^* + \cdots + T_NT_N^* < I$, which is generally referred to as a strict row contraction. In this case one can again easily see that $\|\cdot\|_u = \|\cdot\|_{u_0}$ by using the same $r < 1$ argument as in the last example and that $f \in H_R(B_N)$ if and only if 

$$\|f\|_R = \sup\{\|f(T)\| : T_1T_1^* + \cdots + T_NT_N^* < I\} \leq +\infty.$$ 

These are the norms on polynomials considered by Drury\textsuperscript{[18]}, Popescu\textsuperscript{[28]}, Arveson\textsuperscript{[8]}, and Davidson and Pitts\textsuperscript{[17]}. By Theorem 4.18 we will have for
Let \( f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{B}_N)) \) that \( \|f\|_{\mathcal{R}} \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = (1 - \langle z, w \rangle)K(z, w),
\]
where \( K : \mathbb{B}_N \times \mathbb{B}_N \to M_m \) is an analytic-coanalytic positive definite function.

**Example 5.3.** Let \( G = \mathbb{B}_N \) as above and let \( F_1(z) = (z_1, ..., z_N)^t : \mathbb{B}_N \to M_{1,N} \). Again this is a rational presentation of \( G \) and the algebra of the presentation is the polynomials. An admissible representation corresponds to an \( N \)-tuple of commuting operators \( (T_1, ..., T_N) \) such that \( \|T_1, ..., T_N\| \leq 1 \), i.e., such that \( T_1^*T_1 + \cdots + T_N^*T_N \leq I \), which is generally referred to as a column contraction. This time the norm on \( H^\infty_{\mathcal{R}}(\mathbb{B}_N) \) will be defined by taking suprema over all strict column contractions and we will have that \( \|f\|_{\mathcal{R}} \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = R_1(z)((I_N - (z_i\bar{w}_j)) \otimes I_\mathcal{H})R_1(w)^*,
\]
for some \( R_1 : \mathbb{B}_N \to B(\mathbb{C}^n, \mathcal{H}) \), analytic.

**Example 5.4.** Let \( G = \mathbb{B}_N \) as above, let \( F_1(z) = (z_1, ..., z_N) : \mathbb{B}_N \to M_{1,N} \) and \( F_2(z) = (z_1, ..., z_N)^t : \mathbb{B}_N \to M_{N,1} \). Again this is a rational presentation of \( G \) and the algebra of the presentation is the polynomials. An admissible representation corresponds to an \( N \)-tuple of commuting operators \( (T_1, ..., T_N) \) such that \( T_1T_1^* + \cdots + T_N^*T_N \leq I \) and \( T_1^*T_1 + \cdots + T_N^*T_N \leq I \), that is, which is both a row and column contraction. This time the norm on \( H^\infty_{\mathcal{R}}(\mathbb{B}_N) \) is defined as the supremum over all commuting \( N \)-tuples that are both strict row and column contractions. We will have that \( f \in M_{m,n}(H^\infty_{\mathcal{R}}(\mathbb{B}_N)) \) with \( \|f\|_{\mathcal{R}} \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = (1 - \langle z, w \rangle)K_1(z, w) + R_1(z)((I_N - (z_i\bar{w}_j)) \otimes I_\mathcal{H})R_1(w)^*,
\]
where \( K_1 \) and \( R_1 \) are as before.

The last three examples illustrate that it is possible to have multiple rational representations of \( G \), all with the same algebra, but which give rise to (possibly) different operator algebra norms on \( \mathcal{A} \). Thus, the operator algebra norm depends not just on \( G \), but also on the particular presentation of \( G \) that one has chosen. We have surpressed this dependence on \( \mathcal{R} \) to keep our notation simplified.

**Example 5.5.** Let \( G = \mathbb{D} \) the open unit disk in the complex plane and let \( F_1(z) = z^2, F_2(z) = z^3 \). It is easy to check that the algebra \( \mathcal{A} \) of this presentation is generated by the component functions and the constant function is the span of the monomials, \( \{1, z^n : n \geq 2\} \), and that \( \mathcal{A} \) separates the points of \( G \). In this case an (strict)admissible representation, \( \pi : \mathcal{A} \to B(\mathcal{H}) \), is given by any choice of a pair of commuting (strict)contractions, \( A = \pi(z^2), B = \pi(z^3) \), satisfying \( A^2 = B^2 \). Again, it is easy to see that \( \|\cdot\|_u = \|\cdot\|_{u_0} \). On the other hand
\[
\mathcal{Q}(\mathbb{D}) = \{ T : \sigma(T) \subseteq \mathbb{D} \text{ and } \|T^2\| \leq 1, \|T^3\| \leq 1 \}.
\]
and it can be seen that \( H_R^\infty(\mathbb{D}) \) is defined by
\[
\|f\|_R = \sup\{\|f(T)\| : \|T^2\| < 1, \|T^3\| < 1\} < +\infty.
\]

In this case we have that \( f \in M_{m,n}(H_R^\infty(\mathbb{D})) \) and \( \|f\|_R \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = (1 - z^2z^2)K_1(z, w) + (1 - z^3z^3)K_2(z, w).
\]

**Example 5.6.** Let \( \mathbb{L} = \{z \in \mathbb{C} : |z - a| < 1, |z - b| < 1\} \), where \(|a - b| < 1\), then the functions \( f_1(z) = z - a, f_2(z) = z - b \) give a polynomial presentation of this “lens”. The algebra of this presentation is again the algebra of polynomials. An admissible representation of this algebra is defined by choosing any operator satisfying \( \|T - aI\| \leq 1 \) and \( \|T - bI\| \leq 1 \), with strict inequalities for the admissible strict representations. In this case we easily see that \( \|\cdot\|_u = \|\cdot\|_{u_0} \), since given any operator \( T \) satisfying \( \|T - aI\| \leq 1 \) and \( \|T - bI\| \leq 1 \), and \( r < 1 \), \( S_r = rT + (1 - r)(a + b) \) corresponds to the admissible strict representations and for any matrix of polynomials \( \|(p_{ij}(T))\| = \lim_{r \to 1} \|\{(p_{ij}(S_r))\}\| \). This algebra with this norm was studied in \cite{[17]}. Their work shows that this norm is completely boundedly equivalent to the usual supremum norm. Our results imply that \( f \in M_{m,n}(H_R^\infty(\mathbb{L})) \) and \( \|f\|_R \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = (1 - (z - a)(w - b))K_1(z, w) + (1 - (z - b)(w - b))K_2(z, w).
\]

**Example 5.7.** Let \( G = \{(z_{i,j}) \in M_{M,N} : \|z_{i,j}\| < 1\} \) and let \( F : G \to M_{M,N} \) be the identity map \( F(z) = (z_{i,j}) \). Then this is a polynomial presentation of \( G \) and the algebra of the presentation is the algebra of polynomials in the \( MN \) variables \( \{z_{i,j}\} \). An admissible representation of this algebra is given by any choice of \( MN \) commuting operators \( \{T_{i,j}\} \) on a Hilbert space \( \mathcal{H} \), such that \( \|T_{i,j}\| \leq 1 \) in \( M_{M,N}(B(\mathcal{H})) \) and as above, one can show that \( \|\cdot\|_R \) is achieved by taking suprema over all commuting \( MN \)-tuples for which \( \|T_{i,j}\| < 1 \). We have that \( f \in M_{m,n}(H_R^\infty(G)) \) and \( \|f\|_R \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = R_1(z)[I_M - (z_{i,j}(w_{i,j})^*) \otimes I_{\mathcal{H}}]R_1(w)^*,
\]
for some appropriately chosen \( R_1 \).

All of the above examples are also covered by the theory of \cite{[4]} and \cite{[10]}, except that their definition of the norm is slightly different. We address this difference in a later remark. We now turn to some examples that are not covered by their theory.

**Example 5.8.** Let \( 0 < r < 1 \) be fixed and let \( \mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\} \) be an annulus. Then this has a rational presentation given by \( F_1(z) = z \) and \( F_2(z) = rz^{-1} \), and the algebra of this presentation is just the Laurent polynomials. Admissible representations of this algebra are given by selecting any invertible operator \( T \) satisfying \( \|T\| \leq 1 \) and \( \|T^{-1}\| \leq r^{-1} \). Admissible strict representations are given by invertible operators satisfying \( \|T\| < 1 \) and \( \|T^{-1}\| < r^{-1} \). It is no longer quite so clear that \( \|\cdot\|_u = \|\cdot\|_{u_0} \). However, this algebra with these norms is studied by the first author in \cite{[22]} and among other results the equality of these norms was shown. In \cite{[17]}, it was shown
that \( \| \cdot \|_u \) is completely boundedly equivalent to the usual supremum norm. In this case one can see that the \( \| \cdot \|_R \) is achieved by taking suprema over all \( T \) satisfying \( \| T \| < 1 \) and \( \| T^{-1} \| < r^{-1} \). The formula for the norm is given by \( \| f \|_R \leq 1 \) if and only if
\[
I_m - f(z)f(w)^* = (1 - z \bar{w})K_1(z, w) + (1 - r^2 z^{-1} \bar{w}^{-1})K_2(z, w).
\]

**Example 5.9.** Let \( G \) be a simply connected domain in \( \mathbb{C} \) and \( \phi : G \to \mathbb{D} \) be a biholomorphic map. Then \( G = \{ z \in \mathbb{C} : |\phi(z)| < 1 \} \) and the \( Q(G) = \{ T : \sigma(T) \subseteq G \text{ and } \| \phi(T) \| \leq 1 \} \) where \( R = \{ \phi \} \). In this case the algebra \( A \) of the presentation is just the algebra of all polynomials in \( \phi \). Thus, an admissible representation of this algebra is defined by choosing a strict contraction. In this case one can see that the algebra \( A \) of the presentation is just the algebra of all polynomials in \( \phi \). Thus, an admissible representation of this algebra is defined by choosing a strict contraction. In this case, it is immediate that \( \| \cdot \|_u = \| \cdot \|_{u_0} \) and that \( f \in H^\infty_R(G) \) if and only if \( f \in Hol(G) \) and
\[
\| f \|_R = \sup\{ \| f(T) \| : T \in Q_R(G) \} < +\infty.
\]

Our results imply that \( \| f \|_R \leq 1 \) if and only if
\[
1 - f(z)f(w)^* = (1 - \phi(z)\phi(w)^*)K_1(z, w)
\]

In particular, if we take \( \phi(z) = \frac{z - 1}{z + 1} \) then it maps the halfplane \( \mathbb{H} = \{ z : Re(z) > 0 \} \) to the unit disk. For this particular \( \phi \), we have that \( Q_R = \{ T : \sigma(T) \subseteq \mathbb{H} \text{ and } Re(T) \geq 0 \} \).

**Example 5.10.** Similarly, if we let \( G = \{ z \in \mathbb{C}^N : |\phi_i(z)| < 1, i = 1, ..., N \} \) where \( \phi_i(z) = \frac{z_i - 1}{z_i + 1} \), then \( G \) is an intersection of half planes and \( Q_R \) consists of all commuting \( N \)-tuples of operators, \( (T_1, ..., T_N) \) such that \( \sigma(T_i) \subseteq \mathbb{H} \) and \( Re(T_i) \geq 0 \) for all \( i \). Applying our results, we obtain a factorization result for half planes. These algebras have been studied by D.Kalyuzhnyi-Verbovetzkii in [20].

**Example 5.11.** Let \( G \subseteq \mathbb{C} \) be an open convex set and represent it as an intersection of half planes \( \mathbb{H}_\theta \). Each half plane can be expressed as \( \{ z : |F_\theta(z)| < 1 \} \) for some family of linear fractional maps. If we let \( R = \{ F_\theta \} \), then \( Q_R(G) = \{ T : \sigma(T) \subseteq G \text{ and } \| F_\theta(T) \| \leq 1 \forall \theta \} \). Moreover, each inequality \( \| F_\theta(T) \| \leq 1 \) can be re-written as an operator inequality for the real part of some translate and rotation of \( T \). For example, when \( G = \mathbb{D} \), we may take \( F_\theta(z) = \frac{z}{z - 2e^{i\theta}} \), for \( 0 \leq \theta < 2\pi \). In this case, one checks that \( \| F_\theta(T) \| \leq 1 \) if and only if \( Re(e^{i\theta}T) \leq 1 \). Thus, it follows that
\[
Q(\mathbb{D}) = \{ T : \sigma(T) \subseteq \mathbb{D} \text{ and } w(T) \leq 1 \},
\]
where \( w(T) \) denotes the numerical radius of \( T \). Thus, \( H^\infty_R(\mathbb{D}) \) becomes the “universal” operator algebra that one obtains by substituting an operator of numerical radius less than one for the variable \( z \) and we have a quite different quantization of the unit disk. Our results give a formula for this
norm, but only in terms of \( \mathcal{R} \)-limits, so further work would need to be done to make it explicit.

**Example 5.12.** There is a second way that one can quantize many convex sets. Let \( G = \{ z : |z - a_k| < r_k, k \in I \} \subseteq \mathbb{C} \) be an open, bounded convex set that can be expressed as an intersection of a possibly infinite set of open disks. For example, any convex set with a smooth boundary with uniformly bounded curvature can be expressed in such a fashion. Then \( G \) has a rational presentation given by \( F_k(z) = r_k^{-1}(z - a_k), k \in I \) the algebra of the presentation is just the polynomial algebra and an admissible representation is given by selecting any operator \( T \) satisfying, \( \|T - a_kI\| \leq r_k, k \in I \). Thus, we again a factorization result, but only in terms of \( \mathcal{R} \)-limits.

The above definitions allow one to consider many other examples. For example, one could fix \( 0 < r < 1 \) and let \( G = \{ z \in \mathbb{B}_N : r < |z_1| \} \), with rational presentation \( f_1(z) = (z_1, ..., z_N) \in M_{1,N} \), and \( f_2(z) = rz_1^{-1} \). An admissible representation would then correspond to a commuting row contraction with \( T_1 \) invertible and \( \|T_1^{-1}\| \leq r^{-1} \).

We now compare and contrast some of our hypotheses with those of [4] and [10].

**Remark 5.13.** Let \( G = \{ z \in \mathbb{C}^N : \|F_k(z)\| < 1 \forall \ k = 1, \cdots, K \} \) where the \( F_k \)'s are matrix-valued polynomials defined on \( G \). Then for \( f \in \text{Hol}(G) \), [4] and [10] really study a norm given by \( \|f\|_s = \sup \{\|f(T)\| \} \) where the supremum is taken over all commuting \( N \)-tuples of operators \( T \) with \( \|F_k(T)\| < 1 \forall k \). We wish to contrast this norm with our \( \|f\|_\mathcal{R} \). In [4] it is shown that the hypotheses \( \|F_k(T)\| < 1, k = 1, ..., K \) implies that \( \sigma(T) \subseteq G \). Thus, we have that \( \|f\|_s \leq \|f\|_\mathcal{R} \). In fact, we have that \( \|f\|_s = \|f\|_\mathcal{R} \). This can be seen by the fact that they obtain identical factorization theorems to ours. This can also be seen directly in some cases where the algebra \( \mathcal{A} \) contains the polynomials and when it can be seen that \( \| \cdot \|_\mathcal{R} \) is attained by taking the supremum over matrices (see Remark 4.11). Indeed, if \( \|f\|_\mathcal{R} \) is attained as the supremum over commuting \( N \)-tuples of finite matrices \( T = (T_1, ..., T_N) \) satisfying \( \sigma(T) \subseteq G \) and \( \|F_k(T)\| \leq 1 \) then such an \( N \)-tuple of commuting matrices, can be conjugated by a unitary to be simultaneously put in upper triangular form. Now it is easily argued that the strictly upper triangular entries can be shrunk slightly so that one obtains new \( N \)-tuples \( T_1 = (T_1, \epsilon, ..., T_N, \epsilon) \) satisfying, \( \|F_k(T_1)\| < 1, k = 1, ..., K \) and \( \|T_1 - T_1, \epsilon\| < \epsilon \). But we do not have a simple direct argument that works in all cases.

**Remark 5.14.** We do not know how generally it is the case that \( \| \cdot \|_u \) is a local norm. That is, we do not know if \( \|f\|_u = \|f\|_\mathcal{R} \) for \( f \in M_\alpha(\mathcal{A}) \). In particular, we do not know if this is the case for Example 5.3. In this case, the algebra of the of the presentation is \( \mathcal{A} = \text{span}\{z^n : n \geq 0, n \neq 1\} \). If we write a polynomial \( p \in \mathcal{A} \) in terms of its even and odd decomposition, \( p = p_e + p_o \), then \( p_e(z) = q(z^2) \) and \( p_o = z^3r(z^2) \) for some polynomials \( q, r \).
In this case it is easily seen that
\[ \|p\|_u = \sup \{ \|q(A) + Br(A)\| : A \leq 1, B \leq 1, AB = BA, A^3 = B^2 \}, \]
while
\[ \|p\|_L = \|p\|_R = \sup \{ \|p(T)\| : T^2 \leq 1, T^3 \leq 1 \}. \]

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