Walker’s theorem without coordinates

Andrzej Derdzinski
Department of Mathematics, The Ohio State University,
Columbus, OH 43210, USA
Electronic mail: andrzej@math.ohio-state.edu

Witold Roter
Institute of Mathematics and Computer Science, Wroclaw University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wroclaw, Poland
Electronic mail: roter@im.pwr.wroc.pl

We provide a coordinate-free version of the local classification, due to A. G. Walker [Quart. J. Math. Oxford (2) 1, 69 (1950)], of null parallel distributions on pseudo-Riemannian manifolds. The underlying manifold is realized, locally, as the total space of a fibre bundle, each fibre of which is an affine principal bundle over a pseudo-Riemannian manifold. All structures just named are naturally determined by the distribution and the metric, in contrast with the noncanonical choice of coordinates in the usual formulation of Walker’s theorem.

I. Introduction

In 1950 A.G. Walker\(^1\) described the local structure of all pseudo-Riemannian manifolds with null parallel distributions. The present paper provides a coordinate-free version of Walker’s theorem.

Many authors, beginning with Walker himself,\(^2\) have invoked Walker’s 1950 result, often to generalize it or derive other theorems from it. In our bibliography, which is by no means complete, Refs. 3 – 16 all belong to this category. They invariably cite Walker’s result in its original, local-coordinate form (reproduced in the Appendix).
Such an approach, perfectly suited for the applications just mentioned, tends nevertheless to obscure the geometric meaning of Walker’s theorem. In fact, Walker coordinates are far from unique; choosing them results in making noncanonical objects a part of the structure.

To keep the picture canonical, some authors replace a single Walker coordinate system by a whole maximal atlas of them. What we propose here, instead, is to use only ingredients such as fibre bundles, widely seen as more directly “geometric” than a coordinate atlas (even though one may ultimately need atlases to define them).

In our description, the coordinate-independent content of Walker’s theorem amounts to realizing the underlying manifold, locally, as a fibre bundle whose fibres are also bundles, namely, affine principal bundles over pseudo-Riemannian manifolds. The bundle structures are all naturally associated with the original null parallel distribution; the distribution and the metric can in turn be reconstructed from them.

II. Preliminaries

Throughout this paper, all manifolds, bundles, sections, subbundles, connections, and mappings, including bundle morphisms, are assumed to be of class $C^\infty$. A bundle morphism may operate only between two bundles with the same base manifold, and acts by identity on the base.

‘A bundle’ always means ‘a $C^\infty$ locally trivial bundle’ and the same symbol, such as $M$, is used both for a given bundle and for its total space; the bundle projection $M \to \Sigma$ onto the base manifold $\Sigma$ is denoted by $\pi$ (or, sometimes, $p$). We let $M_y$ stand for the fibre $\pi^{-1}(y)$ over any $y \in \Sigma$, while $\text{Ker } d\pi$ is the vertical distribution treated as a vector bundle (namely, a subbundle of the tangent bundle $TM$).

For real vector bundles $\mathcal{X}, \mathcal{Y}$ over a manifold $\Sigma$ and a real vector space $V$ with $\dim V < \infty$, we denote by $\text{Hom}(\mathcal{X}, \mathcal{Y})$ the vector bundle over $\Sigma$ whose sections are vector-bundle morphisms $\mathcal{X} \to \mathcal{Y}$, and by $\Sigma \times V$ the product bundle with the fibre $V$, the sections of which are functions $\Sigma \to V$. Thus, $\mathcal{X}^* = \text{Hom}(\mathcal{X}, \Sigma \times \mathbb{R})$ is the dual of $\mathcal{X}$.

We will say that a given fibrewise structure in a bundle $M$ over a manifold $\Sigma$ depends $C^\infty$-differentiably on $y \in \Sigma$, or varies $C^\infty$-differentiably with $y$, if suitable $C^\infty$ local trivializations of $M$ make the structure appear as constant (the same in each fibre).

The symbol $\nabla$ will be used for various connections in vector bundles. Our sign convention about the curvature tensor $R = R^\nabla$ of a connection $\nabla$ in a vector bundle $\mathcal{X}$ over a manifold $\Sigma$ is

\begin{equation}
R(u, v)\psi = \nabla_v \nabla_u \psi - \nabla_u \nabla_v \psi + \nabla_{[u, v]} \psi,
\end{equation}
for sections \( \psi \) of \( X \) and vector fields \( u, v \) tangent to \( \Sigma \). By the Leibniz rule, when \( \nabla \) is the Levi-Civita connection of a pseudo-Riemannian metric \( g \) and \( u, v, w \) are tangent vector fields, \( 2\langle \nabla_w v, u \rangle \) equals

\[
\frac{1}{2} \left( d_w \langle v, u \rangle + d_v \langle w, u \rangle - d_u \langle w, v \rangle + \langle v, [u, w] \rangle + \langle u, [w, v] \rangle - \langle w, [v, u] \rangle \right),
\]

where \( d_v \) is the directional derivative and \( \langle , \rangle \) stands for \( g( , , ) \).

**Remark 2.1.** Let \( \pi : M \to \Sigma \) be a bundle projection. A vector field \( w \) on the total space \( M \) is \( \pi \)-projectable onto the base manifold \( \Sigma \) if and only if, for every vertical vector field \( u \) on \( M \), the Lie bracket \([w, u]\) is also vertical. This well-known fact is easily verified in local coordinates for \( M \) which make \( \pi \) appear as a standard Euclidean projection.

### III. Affine principal bundles

All principal bundles discussed below have Abelian structure groups \( G \), so one need not decide whether \( G \) acts from the left or right.

Let \( N \) be a \( G \)-principal bundle over a base manifold \( L \), where \( G \) is an Abelian Lie group. By the \( N \)-prolongation of the tangent bundle \( TL \) we mean the vector bundle \( \mathcal{F} \) over \( L \) whose fibre \( \mathcal{F}_c \) over \( c \in L \) is the space of all \( G \)-invariant vector fields tangent to \( N \) along \( N_c \) (and defined just on \( N_c \)), with \( N_c \) denoting, as usual, the fibre of \( N \) over \( c \). A vector subbundle \( \mathcal{G} \subset \mathcal{F} \) now can be defined by requiring \( \mathcal{G}_c \), for any \( c \in L \), to consist of all \( G \)-invariant vector fields defined just on \( N_c \) which are vertical (i.e., tangent to \( N_c \)). Since each \( \mathcal{G}_c \) is canonically isomorphic to the Lie algebra \( \mathfrak{g} \) of \( G \), the vector bundle \( \mathcal{G} \) is naturally trivialized, that is, identified with the product bundle \( L \times \mathfrak{g} \). Therefore

\[
L \times \mathfrak{g} = \mathcal{G} \subset \mathcal{F}.
\]

The quotient bundle \( \mathcal{F}/\mathcal{G} \) is in turn naturally isomorphic to \( TL \), via the differential of the bundle projection \( N \to L \).

An **affine space** is a set \( A \) with a simply transitive action on \( A \) of the additive group of a vector space \( V \). One calls \( V \) the **vector space of translations** of the affine space \( A \).

An **affine bundle** \( M \) over a manifold \( \Sigma \) is a bundle with fibres \( M_y \), \( y \in \Sigma \), carrying the structures of affine spaces whose vector spaces \( X_y \) of translations form a vector bundle \( X \) over \( \Sigma \), called the **associated vector bundle** of \( M \). We also require the affine-space structure of \( M_y \) to vary \( C^\infty \)-differentiably with \( y \in \Sigma \), in the sense of Sec. II.

If, in addition, \( X = \Sigma \times V \), that is, the associated vector bundle of \( M \) happens to be a product bundle, then \( M \) is also a \( V \)-principal bundle, with the obvious action of the additive group of the vector
IV. Partial metrics and extensions

Let \(\mathcal{C}, \mathcal{D}\) and \(\mathcal{E}\) be real vector bundles over a manifold \(Q\). By an \(\mathcal{E}\)-valued \textit{pairing} of \(\mathcal{C}\) and \(\mathcal{D}\) we mean any vector-bundle morphism \(\beta : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}\). This amounts to a \(C^\infty\) assignment of a bilinear mapping \(\beta(z) : \mathcal{C}_z \times \mathcal{D}_z \to \mathcal{E}_z\) to every \(z \in Q\). An \(\mathcal{E}\)-valued \textit{partial pairing} of \(\mathcal{C}\) and \(\mathcal{D}\) consists, by definition, of two vector subbundles \(\mathcal{C}' \subset \mathcal{C}\) and \(\mathcal{D}' \subset \mathcal{D}\), of some codimensions \(k\) and \(l\), along with pairings \(\gamma : \mathcal{C} \otimes \mathcal{D}' \to \mathcal{E}\) and \(\gamma : \mathcal{C}' \otimes \mathcal{D} \to \mathcal{E}\) which coincide on the subbundle \(\mathcal{C}' \otimes \mathcal{D}'\) (and so may be represented by the same symbol \(\gamma\) without risk of ambiguity). One can obviously restrict a given pairing \(\beta : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}\) to \(\mathcal{C} \otimes \mathcal{D}'\) and \(\mathcal{C}' \otimes \mathcal{D}\), so that a partial pairing \(\gamma\) is obtained; we will then say that \(\beta\) is a \textit{total-pairing extension} of \(\gamma\).

**Lemma 4.1.** For any fixed partial pairing \(\gamma\), with \(\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{C}', \mathcal{D}', k, l\) and \(Q\) as above, and with \(m\) denoting the fibre dimension of \(\mathcal{E}\), the total-pairing extensions of \(\gamma\) coincide with sections of a specific affine bundle of fibre dimension \(klm\) over \(Q\), whose associated vector bundle is \(\text{Hom}(\mathcal{C}/\mathcal{C}' \otimes \mathcal{D}/\mathcal{D}', \mathcal{E})\).

**Proof.** Our \(\gamma\) is nothing else than a vector-bundle morphism \(X : \mathcal{C} \to \mathcal{E}\), where \(X \subset \mathcal{C} \otimes \mathcal{D}\) is the subbundle spanned by \(\mathcal{C} \otimes \mathcal{D}'\) and \(\mathcal{C}' \otimes \mathcal{D}\). The affine bundle in question is the preimage of the section \(\gamma\) under the (surjective) restriction morphism \(\text{Hom}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \text{Hom}(X, \mathcal{E})\). □

As usual,\(^5\) by a \textit{pseudo-Riemannian fibre metric} \(g\) in a vector bundle \(\mathcal{T}\) over a manifold \(M\) we mean any family of nondegenerate symmetric bilinear forms \(g(x)\) in the fibres \(\mathcal{T}_x\) that constitutes a \(C^\infty\) section of the symmetric power \((\mathcal{T}^*)^\otimes 2\). Equivalently, such \(g\) is a pairing of \(\mathcal{T}\) and \(\mathcal{T}\) valued in the product bundle \(M \times \mathbb{R}\), symmetric and nondegenerate at every point of \(M\).

Let \(\mathcal{T}\) again be a vector bundle over a manifold \(M\). We define a \textit{partial fibre metric} in \(\mathcal{T}\) to be a triple \((\mathcal{P}, \mathcal{P}', \alpha)\) formed by vector subbundles \(\mathcal{P}\) and \(\mathcal{P}'\) of \(\mathcal{T}\) along with a pairing \(\alpha : \mathcal{P}' \otimes \mathcal{T} \to M \times \mathbb{R}\), valued in the product bundle \(M \times \mathbb{R}\), such that

(i) \(\mathcal{T}, \mathcal{P}\) and \(\mathcal{P}'\) are of fibre dimensions \(n, r\) and, respectively, \(n-r\) for some \(n, r\) with \(0 \leq r \leq n/2\), while \(\mathcal{P} \subset \mathcal{P}'\),
(ii) at every \( x \in M \) the bilinear mapping \( \alpha(x) : P'_x \times T_x \to \mathbb{R} \) has the rank \( n - r \), its restriction to \( P'_x \times P'_x \) is symmetric, and its restriction to \( P'_x \times P_x \) equals 0.

By a total-metric extension of \((\mathcal{P}, \mathcal{P}', \alpha)\) we then mean any pseudo-Riemannian fibre metric in \( \mathcal{T} \) whose restriction to \( P'_x \otimes T_x \) is \( \alpha \).

**Lemma 4.2.** The total-metric extensions \( g \) of any partial fibre metric \((\mathcal{P}, \mathcal{P}', \alpha)\), with \( r, M \) as above, coincide with the sections of a specific affine bundle of fibre dimension \( r(r + 1)/2 \) over \( M \). For every such \( g \) the subbundle \( \mathcal{P} \) is \( g \)-null and \( \mathcal{P}' \) is its \( g \)-orthogonal complement.

**Proof.** For any fixed point \( x \in M \), let us choose a basis \( e_1, \ldots, e_n \) of \( T_x \) such that \( e_1, \ldots, e_r \in P_x \) and \( e_{r+1}, \ldots, e_{n-r} \in P'_x \). The matrix of \( g(x) \), for any total-metric extension \( g \) of our partial fibre metric, then is the matrix appearing in Walker’s original theorem (see the Appendix), with \( \det A \neq 0 \), and with the two occurrences of \( I \) replaced by some nonsingular \( r \times r \) matrix \( C \) and its transpose \( C' \). The sub-matrices \( A, H, C \) (and \( H', C' \)) are prescribed, while the freedom in choosing \( g(x) \) is represented by an arbitrary symmetric \( r \times r \) matrix \( B \). □

**V. Walker’s theorem**

Suppose that the following data are given.

(a) Integers \( n \) and \( r \) with \( 0 \leq r \leq n/2 \).
(b) An \( r \)-dimensional manifold \( \Sigma \).
(c) A bundle over \( \Sigma \) with some total space \( M \), whose every fibre \( M_y, y \in \Sigma \), is a \( T^*_y \Sigma \)-principal bundle over a \((n - 2r)\)-dimensional manifold \( Q_y \). (Cf. the last paragraph of Sec. III)
(d) A pseudo-Riemannian metric \( h_y \) on each \( Q_y, y \in \Sigma \).

We assume that all \( y \)-dependent objects in (c) – (d), including the principal-bundle structure, vary \( C^\infty \)-differentiably with \( y \in \Sigma \) (in the sense of Sec. III) and, in particular, the \( Q_y \) are the fibres of a bundle over \( \Sigma \) with a total space \( Q \) of dimension \( n - r \). When \( r = n/2 \), each \( h_y \) is the “zero metric” on the discrete space \( Q_y \), cf. Sec. VIII.

Let \( \mathcal{F} \) be the vector bundle over \( Q \) whose restriction to \( Q_y \), for each \( y \in \Sigma \), is the \( M_y \)-prolongation of the tangent bundle \( TQ_y \) (see Sec. III) for the \( T^*_y \Sigma \)-principal bundle \( M_y \) over \( Q_y \). Relation (3) now yields \( p^*(T^*\Sigma) \subset \mathcal{F} \), where \( p : Q \to \Sigma \) denotes the bundle projection. In other words, \( p^*(T^*\Sigma) \) may be treated as a vector subbundle of \( \mathcal{F} \).

Furthermore, the quotient-bundle identification following formula (3) yields \( \mathcal{F}/p^*(T^*\Sigma) = \text{Ker} \, dp \) (the vertical subbundle of \( TQ \), for the projection \( p : Q \to \Sigma \)).
We define a partial pairing $\gamma$ of $F$ and $TQ$ valued in the product bundle $Q \times \mathbb{R}$, as in Sec. IV for our $Q$ along with $\mathcal{C} = F$, $\mathcal{D} = TQ$, $\mathcal{E} = Q \times \mathbb{R}$, $\mathcal{E}' = p^*(T^*\Sigma)$ and $\mathcal{D}' = \text{Ker } dp$. Namely, given $z \in Q$, we set $\gamma(\xi, \zeta) = \xi(dp z \zeta)$ for $\xi \in T^*y \Sigma = [p^*(T^*\Sigma)]_z$ and $\zeta \in T_zQ$, with $y = p(z) \in \Sigma$, as well as $\gamma(u, \psi) = h_y([u], \psi)$ for $u \in F$ and $\psi \in \text{Ker } dp$, where $u \mapsto [u]$ denotes the surjective vector-bundle morphism $F \to \text{Ker } dp$ with the kernel $p^*(T^*\Sigma)$.

Our construction has two steps involving arbitrary choices:

**Step 1.** We choose $\beta : F \otimes TQ \to Q \times \mathbb{R}$ to be any total-pairing extension of $\gamma$. According to Lemma 4.1, such $\beta$ is just an arbitrary section of an affine bundle of fibre dimension $(n - 2r)r$ over $Q$. For the meaning of the above discussion in Walker’s original language, see the Appendix.

The remainder of our construction proceeds as follows. Using $\beta$, we define a partial metric $(\mathcal{P}, \mathcal{P}', \alpha)$ in the tangent bundle $TM$. Specifically, $\mathcal{T}, \mathcal{P}, \mathcal{P}'$ and $n, r$ with the properties listed in (i) – (ii) of Sec. IV are chosen so that $\mathcal{T} = TM$, while $n, r$ are the integers in (a) above, $\mathcal{P}$ is the subbundle of $TM$ whose restriction to $M_y \subset M$, for each $y \in \Sigma$, is the vertical distribution on the $T_y^*\Sigma$-principal bundle $M_y$ over $Q_y$, and $\mathcal{P}' = \text{Ker } d\pi$ is the vertical distribution of the bundle projection $\pi : M \to \Sigma$. We also set $\alpha(u', w) = \beta(u, \zeta)$ for any $x \in M$ and any vectors $w \in T_xM$, $u' \in \mathcal{P}'_x = T_xM_y$ with $y = \pi(x) \in \Sigma$, where $u$ is the $T^*_y\Sigma$-invariant vector field tangent to $M_y$ along the $T^*_y\Sigma$-orbit of $x$ and having the value $u'$ at $x$, while $\zeta$ is the image of $w$ under the differential at $x$ of the bundle projection $M \to Q$.

**Step 2.** We select an arbitrary total-metric extension $g$ of $(\mathcal{P}, \mathcal{P}', \alpha)$ restricted to $U$, where $U$ is any fixed nonempty open subset of $M$.

The construction just described gives a null distribution $\mathcal{P}$ of dimension $r$ on the $n$-dimensional pseudo-Riemannian manifold $(U, g)$. This is clear from Lemma 4.2, which also implies that such metrics $g$ are just arbitrary sections of some affine bundle over $M$.

The reader is again referred to the Appendix for a description of what the above steps correspond to in Walker’s formulation.

We can now state a coordinate-free version of Walker’s theorem:

**Theorem 5.1.** If $g$ and $\mathcal{P}$ are obtained as above from any prescribed data (a) – (d), then $g$ is a pseudo-Riemannian metric on the $n$-dimensional manifold $U$, and $\mathcal{P}$ is a $g$-null, $g$-parallel distribution of dimension $r$ on $U$.
Conversely, up to an isometry, every null parallel distribution $\mathcal{P}$ on a pseudo-Riemannian manifold $(M, g)$ is, locally, the result of applying the above construction to some data (a) – (d). The data themselves are naturally associated with $g$ and $\mathcal{P}$.

A proof of Theorem 5.1 is given in the next two sections.

VI. PROOF OF THE FIRST PART OF THEOREM 5.1

By Lemma 4.2, $\mathcal{P}$ is $g$-null and $\mathcal{P}'$ is its $g$-orthogonal complement. That $\mathcal{P}$ is $g$-parallel will be clear if we establish the relation $\langle \nabla_w v, u \rangle = 0$, where $\nabla$ is the Levi-Civita connection of $g$ and $\langle , \rangle$ stands for $g( , )$, while $v, u, w$ are any vector fields tangent to $M$ such that $v$ is a section of $\mathcal{P}$ and $u$ is a section of $\mathcal{P}'$. We may further require $w$ to be projectable under both bundle projections $M \to Q$ and $\pi : M \to \Sigma$. Finally, we may also assume that $v$ restricted to each $T^*_y \Sigma$-principal bundle space $M_y$ is an infinitesimal generator of the action of $T^*_y \Sigma$, while $u$ restricted to each $M_y$ is $T^*_y \Sigma$-invariant. (Locally, such $w, v, u$ span the vector bundles $TM, \mathcal{P}$ and $\mathcal{P}'$.)

First, $[w, v]$ is a section of $\mathcal{P}$ and $[u, w]$ is a section of $\mathcal{P}'$ (from Remark 2.1 applied to both bundle projections), while $[v, u] = 0$ by $T^*_y \Sigma$-invariance of $u$. The last three terms in (2) thus all equal zero.

Our claim will follow if we show that the first three terms in (2) vanish as well. To this end, note that $d_w \langle v, u \rangle = 0$ since $\langle v, u \rangle = 0$. Next, $d_v \langle w, u \rangle = 0$. Namely, $\langle w, u \rangle = \alpha(u, w) = \beta(u, \zeta)$, for $\alpha, \beta, \zeta$ described in Sec. V is constant in the direction of $v$ (and, in fact, constant along each leaf of $\mathcal{P}$): at a point $x \in M_y \subset M$ we obtain $\zeta$ as the projection image of $w(x)$, while $u$ is $T^*_y \Sigma$-invariant, so that, due to projectability of $w$, both $u$ and $\zeta$ depend only on the image of $x$ under the bundle projection $M \to Q$, rather than $x$ itself. Finally, $d_u \langle w, v \rangle = 0$ as $\langle w, v \rangle = \xi(\bar{w})$ is a function $\Sigma \to \mathbb{R}$, that is, a function $M \to \mathbb{R}$ constant along $\mathcal{P}$. Here $\xi$ is the section of $T^*\Sigma$ corresponding to $v$ under the inclusion $p^*(T^*\Sigma) \subset \mathcal{F}$ of Sec. V while $\bar{w}$ is the vector field on $\Sigma$ onto which $w$ projects; therefore, $\langle w, v \rangle = \xi(\bar{w})$, since in Sec. V we set $\gamma(\xi, \zeta) = \xi(dp_\Sigma \zeta)$.

VII. PROOF OF THE SECOND PART OF THEOREM 5.1

For any null parallel distribution $\mathcal{P}$ of dimension $r$ on an $n$-dimensional pseudo-Riemannian manifold $(M, g)$, the $g$-orthogonal complement $\mathcal{P}^\perp$ is a parallel distribution of dimension $n - r$. If the sign pattern of $g$ has $i_-$ minuses and $i_+$ pluses, it follows that

(4) \hspace{10pt} a) \hspace{5pt} r \leq \min(i_-, i_+), \hspace{10pt} b) \hspace{5pt} \mathcal{P} \subset \mathcal{P}^\perp, \hspace{10pt} c) \hspace{5pt} r \leq n/2.
In fact, \( \mathcal{P} \) is null, which gives (4b) and \( r \leq n - r \), that is, (4c), while (4a) follows since, in a pseudo-Euclidean space with the sign pattern as above, \( i_- \) (or, \( i_+ \)) is the maximum dimension of a subspace on which the inner product is negative (or, positive) semidefinite.

Every null parallel distribution \( \mathcal{P} \) satisfies the curvature relations

\[
\begin{align*}
\text{(5)} & \quad R(\mathcal{P},\mathcal{P}^\perp,-,-) = 0, \quad b) \quad R(\mathcal{P},\mathcal{P},-,+) = 0, \quad c) \quad R(\mathcal{P}^\perp,\mathcal{P}^\perp,\mathcal{P},-) = 0,
\end{align*}
\]

meaning that \( R(v,u,w,w') = 0 \) whenever \( v,u,w,w' \) are vector fields, \( v \) is a section of \( \mathcal{P} \), and \( u \) is a section of \( \mathcal{P}^\perp \). (Similarly for (5b) and (5c).) In fact, for such \( v,u,w,w' \), (1) implies that \( R(w,w')v \) is a section of \( \mathcal{P} \), and so it is orthogonal to \( u \). This proves (5a); (5b) and (5c) yield (5e), while (5a) and the first Bianchi identity give (5f).

We now show how a null parallel distribution \( \mathcal{P} \) on a pseudo-Riemannian manifold \((M,g)\) gives rise to objects (a) – (d) in Sec. V.

First, \( n \) and \( r \) are the dimensions of \( M \) and \( \mathcal{P} \). By (4c), \( r \leq n/2 \).

Being parallel, the distribution \( \mathcal{P}^\perp \) is integrable. Since our discussion is local, we will assume, from now on, that \( M \) is the total space of a bundle over some \( r \)-dimensional base manifold \( \Sigma \), whose fibres \( M_y \), \( y \in \Sigma \), are all contractible and coincide with the leaves of \( \mathcal{P}^\perp \). As \( \mathcal{P} \) is parallel, the Levi-Civita connection \( \nabla \) induces a connection in the vector bundle obtained by restricting \( \mathcal{P} \) to any given submanifold \( N \) of \( M \). In the case where \( N = M_y \) is a leaf of \( \mathcal{P}^\perp \), we have, for each \( y \in \Sigma \), the following conclusion.

\[
\text{(6)} \quad T^*_y\Sigma \text{ is naturally isomorphic to the space } V_y \text{ of those sections of the restriction of } \mathcal{P} \text{ to } M_y \text{ which are parallel (along } M_y).\]

Instead of establishing (3) directly, we will show that \textit{sections of } \( T^*_\Sigma \text{ can be naturally identified with sections of } \mathcal{P} \text{ parallel along } \mathcal{P}^\perp \text{, using an identification which is clearly valuewise, i.e., consists of operators } V_y \to T^*_y\Sigma, \ y \in \Sigma. \text{ To this end, we denote by } \pi \text{ be the bundle projection } M \to \Sigma. \text{ Every vector field on } \Sigma \text{ is the } \pi\text{-image } (d\pi)w \text{ of some } \pi\text{-projectable vector field } w \text{ on } M. \text{ Let } v \text{ now be a section of the vector bundle } \mathcal{P} \text{ over } M, \text{ parallel in the direction of } \mathcal{P}^\perp. \text{ Our identification associates with } v \text{ the cotangent vector field } \xi \text{ on } \Sigma \text{ that sends each vector field } (d\pi)w \text{ to } g(v,w) \text{ treated as a function } \Sigma \to \mathbb{R}. \text{ Note that } \xi \text{ is well defined: two } \pi\text{-projectable vector fields } w \text{ on } M \text{ with the same } \pi\text{-image } (d\pi)w \text{ differ by a section of } \mathcal{P}^\perp = \text{Ker } d\pi, \text{ necessarily orthogonal to } v, \text{ so that } g(v,w) \text{ is the same for both choices of } w. \text{ Also, } g(v,w) : M \to \mathbb{R} \text{ actually descends to a function } \Sigma \to \mathbb{R}, \text{ i.e., is constant along the fibres } M_y \text{ (leaves of } \mathcal{P}^\perp). \text{ In fact, } d_u[g(v,w)] = 0 \text{ for any section } u \text{ of } \mathcal{P}^\perp, \text{ as } \nabla_u v = 0 \text{ in view} \)
of the assumption about \( v \), and \( \nabla_u w = [u, w] + \nabla_w u \), while \([u, w]\) (or \(\nabla_w u\)) is a section of \( \mathcal{P}^\perp \) by Remark 2.1 (or, since \( \mathcal{P}^\perp \) is parallel).

Injectivity of the above assignment \( v \mapsto \xi \) is obvious, since \( \pi \)-projectable vector fields \( w \) span \( TM \). Surjectivity of the resulting operators \( V_y \to T_y^*\Sigma \) now follows: both spaces have the same dimension, as the connections induced by \( \nabla \) in the restrictions of \( \mathcal{P} \) to the leaves \( M_y \) are flat in view of (5c) (cf. (1)). This proves (6).

Flatness of the induced connections also implies that the leaves of \( \mathcal{P} \) contained in any given leaf \( M_y \) of \( \mathcal{P}^\perp \) are the fibres of a \( V_y \)-principal bundle with the total space \( M_y \) over some base manifold \( Q_y \). (Here \( M \) should be replaced with an open subset, if necessary.) Since each \( T_y^*\Sigma \) is identified with \( V_y \) by (6), we thus obtain the data (c) of Sec. V.

Finally, a suitable version of the construction in Sec. V applied to the data (a) – (d) defined above, leads to the original \( g \) and \( \mathcal{P} \), which is a consequence of how the identification (6) and the definition of \( h_y \) use \( g \). The choices of the total-pairing and total-metric extensions, required in Sec. V are provided by \( g \) as well. For instance, \( \beta \) in Step 1 is given by \( \beta(u, \zeta) = g(u, w) \), where \( u \) is a section of \( \mathcal{P}^\perp \) commuting with every section \( v \) of \( \mathcal{P} \) that is parallel along \( \mathcal{P}^\perp \), and \( \zeta \) is a vector field on \( Q \) (the union of all \( Q_y \)), while \( w \) is any vector field on \( M \) projectable onto \( \zeta \) under the bundle projection \( M \to Q \). That \( g(u, w) \) depends just on \( u \) and \( \zeta \) (but not on \( w \)) is clear: two choices of \( w \) differ by a section of \( \mathcal{P} \). Also, \( g(u, w) \) is constant in the direction of \( \mathcal{P} \) (and so it may be treated as a function \( Q \to M \)). Namely, \( d_v[g(u, w)] = 0 \) for any section \( v \) of \( \mathcal{P} \) parallel along \( \mathcal{P}^\perp \), which follows as \( \nabla_v u = \nabla_u v = 0 \) (note that \([u, v] = 0\)), while \( \nabla_v w = [v, w] + \nabla_w v \), and \([v, w]\) (or \(\nabla_w v\)) is a section of \( \mathcal{P} \) by Remark 2.1 (or, respectively, since \( \mathcal{P} \) is parallel). This completes the proof of Theorem 5.1.

VIII. THE MID-DIMENSIONAL CASE

For an \( r \)-dimensional null parallel distribution \( \mathcal{P} \) on a pseudo-Riemannian manifold \( (M, g) \) of dimension \( n = 2r \), the discussion in Sec. V amounts to nothing new: implicitly at least, it is already present in
Sec. 6 of Walker’s original paper. See also Sec. 9 in Ref. 3. (A related global result is Theorem 5 in Ref. 5.) In this section we point out how the construction may be simplified when \( n = 2r \).

Let \( \mathcal{P} \) and \((M, g)\) be as above, with \( n = 2r \geq 2 \). The relations \( i_- + i_+ = n \) and \((\mathcal{A})\) imply that \( g \) has the neutral sign pattern: \( i_- = i_+ = r = n/2 \). In \((c) - (d)\) of Sec. \(\nabla\) each \( Q_y \) is a 0-dimensional (discrete) manifold, and \( h_y \) is the “zero metric” on \( Q_y \). Also, the choice of a total-pairing extension \( \beta \) in Step 1 of Sec. \(\nabla\) is now unique: the affine bundle having \( \beta \) as a section is of fibre dimension 0. The construction in Sec. \(\nabla\) can therefore be rephrased as follows. Given

(a) an even integer \( n \geq 2 \),
(b) a manifold \( \Sigma \) of dimension \( r = n/2 \),
(c) an affine bundle over \( \Sigma \) with some total space \( M \), for which \( T^*\Sigma \) is the associated vector bundle (Sec. \(\Pi\)),

we define a partial metric \((\mathcal{P}, \mathcal{P}', \alpha)\) in the tangent bundle \( TM \) by choosing \( \mathcal{P} = \mathcal{P}' \) to be the vertical distribution \( \text{Ker} \, d\pi \) for the bundle projection \( \pi : M \to \Sigma \), and setting \( \alpha(\xi, w) = \xi(d\pi_xw) \) for any \( x \in M \), \( \xi \in \mathcal{P}_x = T^*_y\Sigma \), where \( y = \pi(x) \), and \( w \in T_xM \). Selecting any total-metric extension \( g \) of \((\mathcal{P}, \mathcal{P}', \alpha)\) on a fixed nonempty open set \( U \subset M \), we now obtain an \( n \)-dimensional pseudo-Riemannian manifold \((U, g)\) on which \( \mathcal{P} \) is a \( g \)-null, \( g \)-parallel distribution of dimension \( r = n/2 \).

Conversely, up to an isometry, every null parallel distribution \( \mathcal{P} \) of dimension \( r \geq 1 \) on a pseudo-Riemannian manifold \((M, g)\) with \( \dim M = 2r \) arises, locally, from the above construction applied to some data \((a) - (c)\), themselves naturally determined by \( g \) and \( \mathcal{P} \).

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Appendix: Walker’s original statement

Walker stated his classification result as follows.\(^1\)

Theorem 1. A canonical form for the general \( V_n \) of class \( C^\infty \) (or \( C^\omega \)) admitting a parallel null \( r \)-plane is given by the fundamental tensor

\[
(g_{ij}) = \begin{pmatrix}
O & O & I \\
O & A & H \\
I & H' & B
\end{pmatrix}
\]
where $I$ is the unit $r \times r$ matrix and $A, B, H, H'$ are matrix functions of the coordinates, of the same class as $V_n$, satisfying the following conditions but otherwise arbitrary:

(i) $A$ and $B$ are symmetric, $A$ is of order $(n-2r) \times (n-2r)$ and nonsingular, $B$ is of order $r \times r$, $H$ is of order $(n-2r) \times r$, and $H'$ is the transpose of $H$.

(ii) $A$ and $H$ (and therefore $H'$) are independent of the coordinates $x^1, \ldots, x^r$.

A basis for the parallel null $r$-plane is the set of vectors $\delta^1_1, \delta^2_1, \ldots, \delta^r_1$.

Here is how the coordinates and matrix functions appearing above correspond to the objects used for the construction in Sec. V. Walker’s coordinates $x^i$, $i = 1, \ldots, n$, serve as a coordinate system for the manifold $M$ of Sec. V. Coordinates for other manifolds appearing in Sec. V are obtained from $x^i$ by restricting the range of the index $i$, to $i > n-r$ (for $\Sigma$), $i > r$ (for $Q$), $i \leq n-r$ (for each $M_y$) and $r < i \leq n-r$ (for each $Q_y$). The center submatrix $A$ in Walker’s matrix corresponds to the family $h_y, y \in \Sigma$, of pseudo-Riemannian metrics ((d) in Sec. V) and, consequently, also to the formula for $\gamma(u, \psi)$, while the last two matrices $OI$ in the first row represent the definition of $\gamma(\xi, \zeta)$. The Walker-matrix counterpart of the extension $\beta$ chosen in Step 1 is the $(n-r) \times (n-r)$ submatrix with the rows $OI$ and $AH$, so that the freedom in choosing $\beta$ amounts to arbitrariness in the selection of $H$ (and $H$ is independent of the coordinates $x^i, i = 1, \ldots, r$, which translates into the fact that $\beta$ is a morphism of vector bundles over the manifold $Q$ with the coordinates $x^i$, $i > r$). Once chosen, $\beta$ is used in Sec. V to define $P, P'$ and $\alpha$. In terms of Walker’s coordinates and matrix functions, $P$ (or, $P'$) is spanned by the $x^i$ coordinate directions with $i \leq r$ (or, respectively, $i \leq n-r$), while the analog of $\alpha$ is the $(n-r) \times n$ submatrix with the rows $OOI$ and $OAH$. Finally, the extension in Step 2 is nothing else than augmenting this last submatrix by a third row, $IH'B$, in which $B$ is completely arbitrary.
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