Black holes with a null Killing vector in three-dimensional massive gravity

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Abstract
We investigate solutions of new massive gravity with two commuting Killing vectors, one of which is null, with a special emphasis on black hole solutions. Besides extreme BTZ black holes and, for a special value of the coupling constant, massless null warped black holes, we also obtain for a critical coupling a family of massive ‘log’ black holes. These are asymptotic to the extreme BTZ black holes in the sense of log gravity.

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1. Introduction

It is well known that three-dimensional cosmological Einstein gravity is dynamically trivial, without propagating degrees of freedom. This is cured in topologically massive gravity (TMG) [1] by the addition to the Einstein action of a parity-violating gravitational Chern–Simons term, leading after linearization to the propagation of massive quanta. The same goal is achieved in a parity-preserving fashion in the recently proposed new theory of massive gravity (NMG) [2] through the addition of a particular quadratic combination of the curvature tensor components. This theory was found to be unitary in the tree level in [3], and renormalizable in [4]. New massive gravity was shown in [5] to admit both BTZ black holes [6] and warped $\text{AdS}_3$ black holes [7–9] as solutions, and the entropy, mass and angular momentum of these black holes were computed. The central charges for these black holes were recently obtained in [10].

In cosmological TMG linearized around a constant curvature background, either the massive gravitons or the BTZ black holes have negative energy, except for a critical, ‘chiral’ value of the Chern–Simons coupling constant at which all the masses vanish [11]. However it was shown in [12] that for this special value TMG also admit a family of massive black holes with the extremal BTZ black hole as ground state. These exact ‘log’ solutions are a limiting case of a family of exact solutions of cosmological TMG first constructed in [13].
Similarly, the sign of the energy of massive excitations of NMG linearized around an AdS$_3$ background is opposite to the sign of the mass of the BTZ black holes, except for the critical value $m^2 = 1/2l^2$ of the quadratic coupling constant, at which all the masses again apparently vanish \[14, 15\]. The purpose of this paper is to investigate whether the special massive black hole solutions of \[12\] can be generalized to critical NMG.

The solutions of \[12\] are actually a special case of AdS wave solutions of TMG \[16–18\], reinterpreted as black hole solutions. As the present work was under way, the paper \[20\] came out, in which AdS wave solutions of NMG are constructed and studied. Our solutions are also special cases of the solutions of \[20\], however the interpretation is different.

In the next section we construct, using the methods of \[13\] and \[5\], solutions of NMG with two commuting Killing vectors, one of which is null. These solutions exist for all real values of the coupling constant $m^2$, however, as shown in section 3 based on the results of the appendix, they lead to regular black holes (other than the extreme BTZ black holes) only for the three values $m^2l^2 = 17/2, m^2l^2 = 7/2$ and $m^2l^2 = 1/2$. The entropy, mass and angular momentum of these black holes are computed in section 4. Only the $m^2l^2 = 1/2$ analogs of the black holes of \[12\] are massive. Our results are briefly discussed in the last section.

2. Stationary solutions with a null Killing vector

The action of the cosmological new massive gravity theory is \[2\]

$$I_3 = \frac{1}{16\pi G} \int d^3x \sqrt{|g|} \left[ R - \frac{1}{m^2} K - 2\Lambda \right], \quad \text{(2.1)}$$

where $R$ is the trace of the Ricci tensor $R_{\mu\nu}$, the quadratic curvature invariant $K$ is

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2, \quad \text{(2.2)}$$

and $G, m^2$ and $\Lambda$ are the Newton constant, quadratic coupling constant and ‘bare’ cosmological constant, respectively.

Let us briefly recall the dimensional reduction of this theory as carried out in \[5\] in the case of two commuting Killing vectors. We choose the parametrization

$$ds^2 = \lambda_{ab}(\rho) dx^a dx^b + \zeta^{-2}(\rho) R^{-2}(\rho) d\rho^2, \quad \text{(2.3)}$$

$(x^0 = t, x^1 = \phi)$, where $\lambda$ is the $2 \times 2$ matrix

$$\lambda = \begin{pmatrix} T + X & Y \\ Y & T - X \end{pmatrix}, \quad \text{(2.4)}$$

$R^2 \equiv X^2$ is the Minkowski pseudo-norm of the ‘vector’ $X(\rho) = (T, X, Y)$,

$$X^2 = \eta_{ij}X^i X^j = -T^2 + X^2 + Y^2, \quad \text{(2.5)}$$

and the scale factor $\zeta(\rho)$ allows for arbitrary reparametrizations of the radial coordinate $\rho$. The scalar product of two vectors $X$ and $Y$ is defined by $X \cdot Y = \eta_{ij} X^i Y^j$, and their wedge product by

$$(X \wedge Y)^l = \eta^{ij} \epsilon_{ijkl} X^i Y^j \quad \text{(2.6)}$$

1 Actually it has recently been shown \[19\] that the situation is more subtle, the massless gravitons being replaced for $m^2 = 1/2l^2$ by massive photons.

2
The ansatz (2.3) reduces the equations of NMG to
\[ X \wedge (X \wedge X^\prime) + \frac{5}{2} X \wedge (X' \wedge X^\prime') + \frac{3}{2} X' \wedge (X \wedge X') \]
\[ + \frac{9}{4} X' \wedge (X' \wedge X^\prime) - \frac{1}{2} X^\prime'' \wedge (X \wedge X') \]
\[ - \left[ \frac{1}{8} (X^2) + \frac{m^2}{\zeta^2} \right] X'' = 0, \]
and
\[ \frac{1}{2} (X \cdot X')^2 - \frac{1}{2} (X^2)(X''^2) - \frac{1}{4} (X^2)(X \cdot X'') - \frac{3}{32} (X'^2)^2 \]
\[ - \frac{m^2}{\zeta^2} \left[ \frac{3}{2} (X^2) + 2(X \cdot X') \right] - \frac{6m^2 \Lambda}{\zeta^4} = 0. \]

Equations (2.7) are trivially solved by \( X'' = 0 \), leading to constant curvature spacetimes
\[ X = \beta \rho + \gamma, \]
where \( \beta \) and \( \gamma \) are two linearly independent constant vectors, and the scale \( \beta^2 = b^2 \) of \( \beta \) is related through (2.8) to the bare cosmological constant. A strategy to generate non-trivial solutions of the system (2.7)–(2.8) is to consider linear deformations of the trivial ansatz (2.9) which solve these equations exactly. Warped AdS3 black hole solutions were obtained in [5] from the quadratic ansatz \( X = \alpha \rho^2 + \beta \rho + \gamma \), with
\[ \alpha \wedge \beta = b \alpha, \]
for some real constant \( b \), implying
\[ \alpha^2 = 0, \quad (\alpha \cdot \beta) = 0, \quad \beta^2 = b^2. \]

In this paper, we consider the ansatz [13]
\[ X = \alpha F(\rho) + \beta \rho, \]
where the form of the function \( F(\rho) \) shall be determined from the field equations.

Before applying this ansatz to NMG, it is instructive to recall its outcome in TMG. The field equations of TMG reduced according to the stationary circularly symmetric ansatz (2.3) are [13]
\[ X'' = \frac{\zeta}{2\mu} [3X' \wedge X'' + 2X \wedge X'''], \]
\[ X'^2 + \frac{4}{3} (X \cdot X'') + \frac{4\Lambda}{\zeta^2} = 0. \]

Provided the vectors \( \alpha \) and \( \beta \) satisfy (2.10), the vector equation (2.13) is linearized by (2.12) to
\[ \rho F'' + \left( \frac{3}{2} + \frac{\mu}{\zeta b} \right) F'' = 0, \]
while the scalar equation (2.14) leads to
\[ b^2 = \frac{4}{\zeta^2 l^2} \]
for a negative cosmological constant \( \Lambda = -l^{-2} \). Without loss of generality we can choose \( \zeta = 1 \) and \( b = 2/l \), leading to the solution [13] of the master equation (2.15)
\[ F(\rho) = a \rho^p + c \rho + d, \quad p = \frac{1 - \mu l}{2}, \]
depending on three integration constants \(a, c\) and \(d\). Actually the constant \(c\) is redundant and may be set to zero by a redefinition of the vector \(\beta, \beta \rightarrow \tilde{\beta} = \beta + c\alpha\), which does not affect equation (2.10). For \(p = 1\) (\(\mu l = -1\)) or \(p = 0\) (\(\mu l = +1\)) the solution (2.17) with \(c = 0\) degenerates to [12]

\[
F(\rho) = a\rho \ln|\rho/\rho_0| + d, \quad (\mu l = -1)
\]

\[
F(\rho) = d\ln|\rho/\rho_0|, \quad (\mu l = +1).
\]

In the case of NMG, the vector equation (2.7) is again linearized by the ansatz (2.12) (with \(a\) and \(b\) satisfying (2.10)) to the fourth-order equation

\[
\rho^2 F^{''''} + 4\rho F^{'''} + \left(\frac{17}{8} - \frac{m^2}{\zeta^2 b^2}\right) F'' = 0,
\]

and the scalar equation (2.8) leading to the constraint

\[
\frac{b^4}{32} + \frac{m^2 b^2}{2\zeta^2} + \frac{2m^2 \Lambda}{\zeta^4} = 0.
\]

Assuming \(b^2 > 0\), we again choose \(\zeta = 1\) and

\[
b = \frac{2}{l},
\]

where the effective AdS\(_3\) curvature parameter \(l\) is obtained by solving (2.21),

\[
l^{-2} = 2m^2[1 \pm \sqrt{1 - \Lambda/m^2}].
\]

The solution of the master equation (2.20) then leads to

\[
F(\rho) = a_+ \rho^{p_+} + a_- \rho^{p_-} + d, \quad p_\pm = \frac{1 \pm \sqrt{m^2 l^2 + 1/2}}{2},
\]

now depending on three integration constants \(a_+, a_-\) and \(d\) (again we have discarded a redundant term \(c\rho\) by a redefinition of the vector \(\beta\)). Note that from (2.23),

\[
m^2 l^2 + \frac{1}{2} = \pm m^2 l^2 \sqrt{1 - \Lambda/m^2},
\]

so that the square root in (2.24) is real either for the upper sign in (2.23) and \(m^2 > 0, \Lambda < 0\), or for the lower sign in (2.23) and \(m^2 < 0, \Lambda > m^2\). For \(m^2 l^2 < -1/2\), (2.24) is replaced by

\[
F(\rho) = a\rho^{1/2} \sin[\gamma \ln|\rho/\rho_0|] + d, \quad \gamma = \frac{1}{2} \sqrt{-m^2 l^2 - 1/2}
\]

(with \(\rho_0\) another integration constant). For the special value \(m^2 l^2 = -1/2\) (\(\Lambda = m^2\), (2.24) or (2.26) degenerate to

\[
F(\rho) = a\rho^{1/2} \ln|\rho/\rho_0| + d.
\]

Finally, (2.24) is also degenerate for the value \(m^2 l^2 = +1/2\) (\(\Lambda = -3m^2\), where it must be replaced by

\[
F(\rho) = a\rho \ln|\rho/\rho_0| + d\ln|\rho/\rho_0|,
\]

or, if \(d = 0\),

\[
F(\rho) = a\rho \ln|\rho/\rho_0| + \tilde{d}
\]

(\(\tilde{d}\) constant).
The choice of basis vectors
\[ \alpha = \frac{1}{2}(1 + l^2, 1 - l^2, -2l), \quad \beta = (1 - l^{-2}, -1 - l^{-2}, 0), \] (2.30)
leads to the metric
\[ ds^2 = [-2l^{-2} \rho + F(\rho)] dt^2 - 2IF(\rho) dt \, d\varphi + [2\rho + l^2 F(\rho)] d\varphi^2 + \frac{l^2 \rho^2}{4 \rho^2} \] (2.31)
which, in the limiting case of a constant \( F(\rho) = M/2 \), reduces to the extreme BTZ metric with \( J/l = M \). This metric (2.31) can be put in the form
\[ ds^2 = \ell^2 [ dx^2 + 2e^{2\varphi} du \, dv + h(x) \, du^2 ], \] (2.32)
with \( x = (1/2) \ln(\rho/\ell^2) \), \( u = \varphi - l^{-1} t \), \( v = \varphi + l^{-1} t \), \( h(x) = F(\rho) \). This has the obvious Killing vectors \( L_1 = \partial_u \) and \( L_2 = \partial_v \), the latter being null. In the generic case these are the only infinitesimal isometries of (2.32). In the case of solution (2.24) with \( a_\varphi = d = 0 \), the metric has a third local isometry generated by
\[ L_3 = u \partial_u + \left( \frac{2}{\rho} - 1 \right) v \partial_v - \partial_x. \] (2.33)
Similarly, in the case of the solution (2.28) \( (p_\varphi = 1) \) with \( d = 0 \), the third local Killing vector is
\[ L_3 = u \partial_u + (a l^2 u + v) \partial_v - \partial_x. \] (2.34)
Finally, in the special cases \( a_\varphi = a_\pm = 0 \) or \( a_\varphi = 2 \), the metric (2.32) describes respectively extreme BTZ black holes or null warped black holes (see the next section), which both admit four local Killing vectors generating the \( sl(2, R) \times R \) algebra (the Killing vectors for the null warped black hole case are given in equations (6.3) and (6.4) of [8]).

All the results of this section are consistent with the results of [20], so that our solutions (2.32) with \( h(x) = F(\rho) \) given by (2.24) and (2.26)–(2.29) are special cases of the AdS waves
\[ ds^2 = \ell^2 [ dx^2 + 2e^{2\varphi} du \, dv + h(x, u) \, du^2 ], \] (2.35)
of [20]. For these solutions to describe black holes, the spacelike Killing vector \( \partial_\varphi \) should have closed orbits, which essentially restricts (2.35) to (2.32).

3. Black holes

In the present paper we are interested in regular black hole solutions. The metric (2.31) has a horizon at \( \rho = 0 \). As pointed out in [13], all the scalar curvature invariants constructed from this metric are constant (the function \( F(\rho) \) does not contribute because \( \alpha \) is null and orthogonal to \( \beta \), however the metric may develop non-scalar curvature singularities at the horizon \( \rho = 0 \), as well as at \( \rho = \infty \), or the horizon can be at geodesic infinity. To elucidate this question, we consider the first integral of the geodesic equation
\[ \dot{\rho}^2 - l^2 P_+^2 F(\rho) - 2P_+ P_- \rho + \frac{4\varepsilon}{l^2} \rho^2 = 0, \] (3.1)
with \( P_\pm = E \pm l^{-1} L \), where \( E \) and \( L \) are the constant conjugate momenta to \( \dot{t} \) and \( \dot{\varphi} \), and \( \varepsilon = +1, 0 \) or \(-1\) for timelike, null or spacelike geodesics. In the discussion of this equation, we can exclude outright the case (2.26), for which the areal radius
\[ r^2 = 2\rho + l^2 F(\rho) \] (3.2)
optically around zero for \( \rho \to 0 \), leading to naked closed timelike curves (CTC). For \( a_\varphi = a_\pm = 0 \) the metric (2.31) reduces under the radial coordinate transformation

\[ \rho \to \rho' = 2\rho \]
\( \rho = (r^2 - dt^2)/2 \) to the extreme BTZ black hole metric with mass parameter \( M = 2d \). If both \( a_+ \) and \( a_- \) do not vanish, a lengthy analysis, carried out in the appendix, leads to the conclusion that the metric (2.31) leads to regular black holes in only three cases:

1. For \( m^2 l^2 = 17/2 \), the solution (2.24) with \( a_- = 0, a_+ > 0, p_+ = 2 \) and \( d \geq 0 \) yields a regular black hole (free from naked CTC). After transforming to a rotating frame, the null Killing vector is \( \partial_t \), which amounts to replacing the basis vectors (2.30) by

\[
\alpha = \frac{t^2}{2} (1, -1, 0), \quad \beta = (1, -1, 2t^{-1}),
\]

(3.3)

this leads to a null warped black hole [8, 9] (corresponding to the warped black holes of [5] with \( \beta^2 = 1 \))

\[
d s^2 = -\frac{4\rho^2}{l^2 r^2} \, dt^2 + r^2 \left[ d\phi + \frac{2\rho}{l^2 r^2} \, dt \right]^2 + \frac{l^2 \, d\rho^2}{4\rho^2} \quad (r^2 = l^2 a_+ \rho^2 + 2\rho + \rho^2) .
\]

(3.4)

This metric is in the ADM form with the square lapse and the shift given by \( N^2 = 4\rho^2/l^2 r^2, N^\phi = 2\rho/\sqrt{l^2 r^2} \).

2. For \( m^2 l^2 = 7/2 \), the solution (2.24) with \( a_- = 0, a_+ > 0, p_+ = 3/2, \) and \( d = 0 \) leads (again after transforming to the basis (3.3)) to the metric

\[
d s^2 = -\frac{4\rho^2}{l^2 r^2} \, dt^2 + (2r^2 + l^2 a_+ x^2) \left[ d\phi + \frac{2}{l(2 + l^2 a_+ x)} \, dt \right]^2 + l^2 \frac{dx^2}{x^2} \quad (x = \rho^{1/2}).
\]

(3.5)

This spacetime is free from naked CTC. The point horizon at \( x = 0 \) hides a timelike causal singularity at \( x = -2/a_+ l^2 \). At spacelike infinity, the two-dimensional metric reduced relative to \( \partial_x \) goes as

\[
d s^2 \approx \frac{4}{l^2} y^{-2} ( -dr^2 + dy^2 ),
\]

(3.6)

with \( y = (x/l^2 a_+)^{-1/2} \to 0 \), so that spacelike infinity is conformally timelike.

3. For the value \( m^2 l^2 = 1/2 \ (p_+ = 1) \), the 'logarithmic' solution (2.28) leads to three black hole subcases:

(a) If \( a \neq 0 \) and \( d \neq 0 \), the metric in the basis (3.3) is similar to (3.4) with

\[
r^2 = a l^2 \rho \ln |\rho/\rho_1| + 2\rho + d l^2 \ln |\rho/\rho_0| .
\]

(3.7)

Necessary conditions for the absence of naked CTC are \( a > 0 \) (no CTC at infinity), and \( d < 0 \) (no CTC near the horizon). This metric is 'almost' asymptotically AdS, (the asymptotic behavior overshoots that of AdS by a logarithmic factor).

(b) If \( a \neq 0 \) and \( d = 0 \) (solution (2.29)), the only difference with the preceding case is that now

\[
r^2 = a l^2 \rho \ln |\rho/\rho_1| + d l^2 ,
\]

(3.8)

with \( \rho_1 = \rho_0 e^{-2x/l} \). This is free from naked CTC provided \( a > 0 \) and \( d > a \rho_0/\sqrt{e} \).

(c) If \( a = 0 \) and \( d \neq 0 \), the ADM shift \( N^\phi \) goes to spatial infinity to a constant in the basis (3.3). A metric with an asymptotically vanishing shift function may be obtained by transforming back to the frame (2.30). This metric,

\[
d s^2 = -\frac{4\rho^2}{l^2 r^2} \, dt^2 + r^2 \left[ d\phi - \frac{d l \ln |\rho/\rho_0|}{r^2} \, dt \right]^2 + \frac{l^2 \, d\rho^2}{4\rho^2} \quad (r^2 = 2\rho + d l^2 \ln |\rho/\rho_0|),
\]

(3.9)

is free from naked CTC provided \( d < 0 \). It is asymptotically AdS in the weak sense of log gravity [21–23], and develops a timelike causal singularity at some negative \( \rho_2 = -\rho_0 \).
Note that (2.24) with \( a_\pm = 0 \) reduces to (2.17), so that the black holes (3.4) and (3.5) also solve the equations of TMG (the fact that these are black holes was overlooked in [13]), while the black holes (3.8) and (3.9) reduce (after appropriate coordinate transformations) to the black hole solutions of TMG (2.18) and (2.19) given in [12].

4. Physical parameters

Now we compute the physical parameters of these black holes. By applying Wald’s general formula [24] the black hole entropy was found in [5] to be given in NMG by

\[
S = \frac{\pi}{2G} \left( r - \frac{r}{m} \left[ (g^{00})^{-1} R^{00} + g_{22} R^{22} - \frac{3}{4} R \right] \right)_h, \tag{4.1}
\]
evaluated on the horizon \( \rho = 0 \), with \( r^2 = g_{\phi\phi} \). Because of the possible presence of logarithmic factors, we evaluate this more carefully than in [5]. From the expressions of the Ricci tensor components given there, we find (for \( \zeta = 1 \))

\[
S = \frac{\pi}{2G} \left[ \left( 1 + \frac{1}{2m^2} \left[ (X \cdot X'') - \frac{1}{4} (X^2) \right] \right) r - \frac{1}{2m^2} R^2 r^{-1} (r^3)'' \right]_h. \tag{4.2}
\]
For the ansatz (2.12), this leads to

\[
S = \frac{\pi}{2G} \left[ \left( 1 - \frac{1}{2m^2 \ell^2} \right) r_h - \frac{2}{m^2} (r^{-1} \rho^2 F''(\rho))_h \right]. \tag{4.3}
\]
For the null warped AdS\(_3\) black holes (3.4), this formula gives the Bekenstein–Hawking entropy renormalized by a factor \( 16/17 \) [5]. In the case of the black holes (3.5) and (3.8), the formula (4.3) yields straightforwardly

\[
S = 0. \tag{4.4}
\]

The case of the black holes (3.7) and (3.9) is more delicate. In this case, \( \rho^2 F'' = a \rho - d \) does not vanish on the horizon but goes to a constant, however this is suppressed by the prefactor \( r^{-1} \) which goes to zero as an inverse logarithm, while the first term in (4.3) does not contribute because \( m^2 \ell^2 = 1/2 \) for this case, so that the net entropy is again zero.

Provided

\[
\lim_{\rho \to \infty} (\rho^{-1} F(\rho)) = 0, \tag{4.5}
\]
the metric (2.31) is asymptotically AdS. In that case we can for large \( \rho \) linearize the metric around the BTZ vacuum as

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + g^L_{\mu\nu}, \tag{4.6}
\]
and use the Abbott–Deser–Tekin (ADT) approach [25, 26] to compute the mass and angular momentum of our black holes. The ADT conserved charge associated with a background Killing vector \( \xi \) has been computed for NMG in [15]. Using the coordinate-free parametrization (2.12), such that \( \bar{X} = \beta \rho, X^L = \alpha F(\rho) \), and evaluating the covariant derivatives and Ricci tensor components with the help of the formulae given in appendix B of [27], we obtain from equations (2.25) and (2.27) of [15] the Killing charge

\[
Q(\xi) = \frac{1}{G m^2 \ell^3} \left[ \rho^3 F''' + \rho^2 F'' + \frac{1}{4} \left( 1 - m^2 \ell^2 \right) (\rho F' - F) \right] ([\alpha] \bar{\xi})^0, \tag{4.7}
\]
where \([\alpha]\) is the matrix

\[
[\alpha] = \begin{pmatrix}
-\alpha^Y & -\alpha^T + \alpha^X \\
\alpha^T + \alpha^X & \alpha^Y
\end{pmatrix}. \tag{4.8}
\]

\(^2\) The sign of our \( Q(\xi) \) is opposite to that used in [15].
The charge \((4.7)\) is a constant of the motion by virtue of \((2.20)\). For the generic solution \((2.24)\) or \((2.26)\), as well as in the special cases \((2.27)\) and \((2.29)\), the evaluation of this charge leads to

\[
Q(\xi) = \frac{1}{4Gm^2l^3} \left( m^2l^2 - \frac{1}{2} \right) d\langle |\alpha|\xi\rangle^0,
\]

while for the special solution \((2.28)\) \((m^2l^2 = 1/2)\), we find

\[
Q(\xi) = \frac{2}{Gl} d\langle |\alpha|\xi\rangle^0.
\]

These values of the charge were derived under the assumption \((4.5)\) which is valid, in particular, for the generic solution \((2.24)\) with \(a_+ \neq 0\), provided \(m^2l^2 < 1/2\) and for the solution \((2.28)\) \((m^2l^2 = 1/2)\) with \(a = 0\). We will here assume that they can be extrapolated to the cases \(m^2l^2 > 1/2\) with \(a_+ \neq 0\) and \(m^2l^2 = 1/2\) with \(a \neq 0\). A more satisfactory derivation would require an extension of the ADT approach to the case of massive gravity with non-constant curvature backgrounds, similar to that carried out in [27] for topologically massive gravity. Choosing \(\xi\) to be one of the two Killing vectors \(\xi(t) = (-1, 0)\) and \(\xi(\phi) = (0, 1)\), we obtain the mass and angular momentum of the various black holes of the previous section3:

1. In the null warped black hole case \((3.4)\),

\[
M = 0, \quad J = \frac{4dl}{17G},
\]

in accordance with the results of [5].

2. For the black holes \((3.5)\),

\[
M = J = 0.
\]

3. For the black holes \((3.7)\) and \((3.8)\),

\[
M = 0, \quad J = \frac{2dl}{G}
\]

3c) For the black holes \((3.9)\),

\[
M = \frac{2d}{G}, \quad J = \frac{2dl}{G}
\]

Because the absence of naked CTC constrains \(d < 0\), the mass is negative for a positive Newton constant \(G\), and vanishes as it should in the limit \(d \to 0\) of the extreme BTZ black hole for \(m^2l^2 = 1/2\) [5]. The values \((4.14)\) may be compared with the corresponding values for the same black holes as solutions for TMG. Evaluating for the ansatz \((2.12)\) with \((2.10)\) the TMG super-angular momentum \(J\) [27], we obtain

\[
J = \frac{4}{\mu l^2} \left[ -\rho^2 F'' + \frac{1}{2} (1 - \mu l)(\rho F' - F) \right] \alpha,
\]

leading for \(\mu l = 1\), \(F(\rho)\) given by \((2.19)\) and \(\alpha\) as in \((2.30)\) to

\[
M = \frac{d}{2G}, \quad J = \frac{dl}{2G}
\]

(the correction \(\Delta M\) coming from the last three terms of equation \((3.15)\) of [27] vanishes in the present case). These values agree with those of [12] (equation \((24)\), where \(k\) is our \(d\)) up to a factor of 2/3.

3 The super-angular momentum approach of [28] as applied to NMG in [5] leads to the same results.
Finally the Hawking temperature and the horizon angular velocity, computed from the metric in ADM form, are
\[ T_H = \frac{1}{4\pi} \zeta r (N^2)'|_{h}, \quad \Omega_h = -N^\nu|_{h}. \] (4.17)
The resulting Hawking temperatures vanish for all our black holes, \( T_H = 0 \). The horizon angular velocities vanish for the black holes (3.4) with \( d > 0 \) and the black holes (3.7) and (3.8), while \( \Omega_h = -2/lk \) for the black holes (3.4) with \( d = 0 \) and the black holes (3.5), and \( \Omega_h = l^{-1} \) for the black holes (3.9). It follows that the first law of black hole thermodynamics, which in the case of vanishing the Hawking temperature reduces to
\[ dM = \Omega_h dJ, \] (4.18)
is satisfied trivially (both sides vanish) for the black holes (3.4), (3.5), (3.7) and (3.8), and non-trivially \( (M = \Omega_h J) \) for the black holes (3.9).

5. Discussion

In this paper, we have investigated solutions of new massive gravity with two commuting Killing vectors, one of which is null, with a special emphasis on black hole solutions. In addition to the well-known extreme BTZ black holes, we found several black hole types. The first of these includes the black holes (3.5) \((m^2 l^2 = 7/2)\) and (3.8) \((m^2 l^2 = 1/2)\), both with \( d = 0 \) (so that, as shown at the end of section 2, they have a third local Killing vector). Because all their physical characteristics (entropy, mass and angular momentum) vanish, these are not genuine black holes. A second family \((m^2 l^2 = 17/2)\) includes the null warped black holes \( \beta^2 = 1 \) of [8], with metric (3.4) enjoying a local \( sl(2, R) \times R \) isometry algebra. These have \( \partial_\mu \) as null Killing vector, are massless, but have a nonzero angular momentum. The third family \((m^2 l^2 = 1/2)\), with metric (3.7), has similar properties, but only two local isometries.

The most interesting fourth black hole type (also \( m^2 l^2 = 1/2 \), corresponding to the value of the bare cosmological constant \( \Lambda = \pm 3m^2 \) from (2.23)) differs from the preceding by the fact that it is asymptotically AdS\(_3\) in the sense of log gravity [21–23]. This implies that in the basis (3.3) appropriate to the other black hole types, the ADM shift function \( N^\nu \) does not vanish at infinity. After transforming to the basis (2.30) in which \( N^\nu(\infty) = 0 \) (which transforms the null Killing vector to \( \partial_\mu \)), we obtained a continuum of black hole states (3.9) with mass \( M \) and angular momentum \( J = lM \), above the massless extreme BTZ family as ground state. These properties are similar to those of the ‘log’ solutions of TMG at the chiral point \( \mu l = 1 \) found in [12]. Further work is needed to understand the implications of these solutions on the consistency of new massive gravity at the critical value \( m^2 l^2 = 1/2 \).

Appendix

We discuss here the geodesic equation (3.1) in the case of the generic solution (2.24). Assuming that \( a^\pm \) and \( a^- \) do not both vanish, we must distinguish between four possibilities for the leading near-horizon behavior of the effective potential:

(i) \( a^- \neq 0 \) and either \( p^- < 0 \), or \( p^- > 0 \) and \( d = 0 \). The leading term in \( F(\rho) \) is \( a^- \rho^p^- \), with \( a^- > 0 \) (if \( a^- < 0 \) the spacetime is geodesically complete, with naked CTC [13]), so that the affine parameter is proportional to \( x = \rho^{(2-p^-)/2} \) near the horizon. The geodesic equation (3.1) can be rewritten in terms of the adapted radial coordinate \( x \) as
\[ x^2 = l^2 P_+^2 x^{-2p^-/(2-p^-)} F(x^{2(2-p^-)}) - 2 P_+ P_- x^{2-2/(2-p^-)} + \frac{4\epsilon}{l^2} x^2 = 0. \] (A.1)
The geodesics can be extended across the horizon $x = 0$ if the effective potential in (A.1) is analytical in $x$ [29, 30]. A necessary condition for this is that the exponent $2 - 2/(2 - p_-)$ be integer. However $p_- < 1/2$ leads to $2/(2 - p_-) < 4/3$, so that this exponent can be integer only for $p_- = 0$, corresponding to the special case (2.28) or (2.29) (see below).

The conclusion is that generically geodesics terminate at the singular horizon $x = 0$.

(ii) $d \neq 0$ and either $a_- = 0$, or $a_- \neq 0$ and $p_- > 0$. The leading term in $F(\rho)$ is constant, so that the adapted radial coordinate is $x = \rho$, and the horizon is regular provided the function $F(\rho)$ in (2.24) is analytic. For $a_- \neq 0$, this is not possible because $p_- < 1/2$ is not integer. For $a_- = 0$, $p_+$ must be integer, $p_+ = n$. The case $n = 1$ corresponds to $p_- = 0$ (see below). In the case $n = 2$ ($m^2 l^2 = 17/2$), (2.12) reduces to the quadratic ansatz $X = \tilde{\alpha} \rho^2 + \tilde{\beta} \rho + \tilde{\gamma}$, with $\tilde{\gamma} < \tilde{\alpha}$, leading (after an appropriate coordinate transformation) to null warped black holes (equation (3.17) of [8]) with $\beta^2 = 1$, $\rho_0 = 0$, $c = l^2 a_+$, $\omega = 1$ and $u = dl^2/2$ in (3.20)). Finally, in the case of $n > 2$ and $a_+ > 0$ ($a_- < 0$ leads to naked CTC), the affine parameter is for large $\rho$ proportional to $\rho^{(2-n)/2}$, so that the geodesics terminate at infinity.

(iii) $a_- = d = 0$, $a_- \neq 0$ and $p_+ < 1$. The leading term in $F(\rho)$ is $a_+ \rho^{p_+}$, with $a_+ > 0$ ($a_+ < 0$ leads again to a geodesically complete spacetime with naked CTC) so that the affine parameter is proportional to $x = \rho^{(2-p_+)/2}$ near the horizon. The geodesic equation rewritten in terms of $x$ is of the form (A.1) with $p_-$ replaced by $p_+$. From $1/2 < p_+ < 1$, we find $4/3 < 2/(2 - p_+) < 2$, so that the exponent $2 - 2/(2 - p_+)$ is not an integer, and the horizon is singular.

(iv) $a_- = d = 0$, $a_- \neq 0$ and $p_+ > 1$. Now the leading term in the effective potential is linear in $\rho$, so that the affine parameter is proportional to $x = \rho^{1/2}$ near the horizon. The geodesic equation rewritten in terms of $x$ is

$$x^2 - l^2 P_+^2 a_+ x^{2(p_+ - 1)} - 2 P_+ P_- + \frac{4\epsilon}{l^2} x^2 = 0.$$ \hspace{1cm} (A.2)

The effective potential is analytical provided $p_+ = n/2 + 1$, with $n$ being a positive integer. For $n = 1$ ($m^2 l^2 = 7/2$, corresponding to $\Lambda/m^2 = -15/49$), we obtain (after transforming to a rotating frame) the metric (3.5). For $n = 2$ ($p_+ = 2$), the resulting metric is (again after transforming to a rotating frame) a special case of the null warped black holes $\beta^2 = 1$ of [8] (with $u = 0$ in equation (3.20) of [8]). For $n > 2$, the affine parameter is for large $\rho$ proportional to $\rho^{(2-n)/4}$, so that again geodesics terminate at infinity.

There remains the case of the special solutions (2.27) and (2.28) or (2.29). The solution (2.27) corresponds to the degenerate case $p_+ = p_- = 1/2$, which from the previous analysis cannot possibly lead to regular black holes. In the case of the solution (2.28), the affine parameter is for $d < 0$ proportional to $x = \rho(-\ln |\rho/\rho_0|)^{-1/2}$ near the horizon. The geodesic equation rewritten in terms of $x$ will contain only integer powers of $x$ and powers of $y = -\ln |\rho/\rho_0|$, which is positive on both sides of the horizon, so that geodesics can be continued across the horizon. The same conclusion holds for the solution (2.29) with $d > 0$ and $x = \rho$.

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