An Efficient Algorithm for Finding Sets of Optimal Routes

Ido Zoref and Ariel Orda
Viterbi Faculty of Electrical Engineering, The Technion – Israel Institute of Technology

Abstract

In several important routing contexts it is required to identify a set of routes, each of which optimizes a different criterion. For instance, in the context of vehicle routing, one route would minimize the total distance traveled, while other routes would also consider the total travel time or the total incurred cost, or combinations thereof. In general, providing such a set of diverse routes is obtained by finding optimal routes with respect to different sets of weights on the network edges. This can be simply achieved by consecutively executing a standard shortest path algorithm. However, in the case of a large number of weight sets, this may require an excessively large number of executions of such an algorithm, thus incurring a prohibitively large running time.

We indicate that, quite often, the different edge weights reflect different combinations of some "raw" performance metrics (e.g., delay, cost). In such cases, there is an inherent dependency among the different weights of the same edge. This may well result in some similarity among the shortest routes, each of which being optimal with respect to a specific set of weights. In this study, we aim to exploit such similarity in order to improve the performance of the solution scheme.

Specifically, we contemplate edge weights that are obtained through different linear combinations of some ("raw") edge performance metrics. We establish and validate a novel algorithm that efficiently computes a shortest path for each set of edge weights. We demonstrate that, under reasonable assumptions, the algorithm significantly outperforms the standard approach. Similarly to the standard approach, the algorithm iteratively searches for routes, one per set of edge weights; however, instead of executing each iteration independently, it reduces the average running time by skillfully sharing information among the iterations.

1 Introduction

Routing is a core networking task, in particular in road networks and computer networks [10] [22] [21] [25]. A fundamental routing problem is that of finding a route that optimizes a single additive metric (e.g., time, cost). Essentially, its solution consists of computing a shortest path on a weighted graph.

In some routing contexts, the routing objective function cannot be satisfactorily captured through a single additive edge cost. For example, planing a vehicle trajectory to a desired destination often requires a fast enough route (in terms of driving time) as well as a route that incurs sufficiently low fuel cost. Similarly, communication network routing problems often consider both throughput and delay objectives.

The problem of routing with multiple objectives is a well-known challenge. Usually, different routes optimize different objective functions, hence finding a single route that simultaneously (i.e., independently) minimizes more than a single objective function (e.g. both travel time and cost) is usually impossible. We proceed to discuss the different alternatives for handling the multi-objective routing problem and present our selected approach.

A common approach for dealing with multi-objective routing problems is to search for a single path that optimizes one selected objective, subject to the condition that the other objectives do not exceed given threshold values. This is an NP-hard problem, known as the Resource Constrained Shortest Path (CSP) problem [19]. In the literature, approximate algorithms and heuristic approaches have been proposed to deal with the CSP problem. Approximation algorithms for the CSP problem are usually based on scaling and rounding of the input data. Warburton [28] was the first to develop a fully polynomial time approximation algorithm for the CSP problem on acyclic networks. In [26], Hassin later improved upon this to derive...
two fully polynomial time approximation schemes (FPTAS) that are applicable for general networks. Other related approximation schemes providing certain improvements to Hassin’s algorithm can be found in [5]. In particular, a significant improvement of Hassin’s result was achieved by Lorenz and Raz [16], who established a strongly polynomial time approximation scheme that is also applicable to general networks. In [2], the authors considered the problem of finding a path whose delay is at most \((1 + \varepsilon)\) times the specified delay bound and whose cost is not greater than that of the minimum cost path of the CSP problem.

As for heuristic approaches for the CSP problem, several proposed algorithms are based on solutions to the integer relaxation or the dual of the integer relaxation of the CSP problem [13]. Juttner et al. [3] introduced the LARAC algorithm, which solves the Lagrangian relaxation of the CSP problem. An applicable heuristic variation of the LARAC algorithm is LARAC-BIN [32]. It employs binary search technique to skillfully find a path \(p\) where the deviation of \(p\)’s cost from the optimal path cost is smaller than the tuning parameter \(\tau\).

An alternative approach for dealing with multi-objective routing is to provide a (“suitable”) set of routes. This can be obtained by finding several optimal routes with respect to different sets of edge weights. For instance, in the context of vehicle routing, one route may minimize the total distance traveled, while other routes would also consider the total travel time or the total incurred cost, or combinations thereof.

Such an approach is often used in recommendation systems, such as autonomous navigation systems [15], [30]. In these systems, it is difficult to match the personal preferences of individual users by providing only a single route. These preferences may be based, for example, on better local knowledge, a bias for or against a certain route objective, or other factors. One can deal with this issue by presenting the user with a number of alternative routes with the hope that one of them would be satisfactory.

A possible method for identifying such a (“suitable”) set of routes is to search for several routes that optimize different linear combinations of some "raw" objectives [11], e.g., distance traveled, incurred cost, etc. Such a linear combination allows to translate several objectives into a single weight for each edge, which in turn implies an “optimal” (i.e., minimum weight) route from a source to target node. Therefore, by considering several linear combinations, finding the set of optimal routes (each defined with respect to one linear combination) can be simply achieved by consecutively executing a standard shortest path algorithm. However, in the case of a large number of linear combinations, this may require an excessively large number of executions of such an algorithm, thus incurring a prohibitively large running time.

Accordingly, in this study we seek to efficiently find a set of routes, each of which minimizes a different linear combination of the objectives. Funke and Storandt [9] suggest such a solution. Specifically, they show that contraction hierarchies – a very powerful speed-up technique originally developed for the standard shortest path problem [12], can be constructed efficiently for the case of several linear combinations of the edge "raw" objectives. However, this method requires some pre-processing efforts. Pre-processing can be applied to speed up graph-based search algorithms in car navigation systems or web-based route planners. Still, in some routing contexts it may not be possible to perform pre-processing since the network may dynamically change over short periods of time.

A possible alternative solution for the considered problem can be obtained by finding a set of Pareto optimal routes.\(^1\) Recall that, in our problem, each required route minimizes a different linear combination of the "raw" objectives. It is easy to verify that each such route is actually a Pareto-Optimal route. Hence, by finding the set of Pareto optimal routes we obtain a solution for the problem. We shall present several algorithms for finding the set of Pareto optimal routes.

Standard solutions for finding the set of Pareto optimal routes are inspired by the Dijkstra algorithm (which is the standard approach for finding a shortest route in the basic single-objective case). The most common approaches are the Multicriteria Label-Setting (MLS) algorithm [18] [14] and the Multi-Label-Correcting (MLC) algorithm [6] [7]. The MLS algorithm keeps, for each node, a set of non-dominated paths. The priority queue maintains paths instead of vertices, typically ordered lexicographically. At each iteration, it extracts the minimum path, say \(L\), which is a path ending at some node, say \(u \in V\). Then, MLS scans the outgoing edges from \(u\), say \(a = (u,v)\). It does so by evaluating whether the new path, \(L_{new} = L \parallel a\), is non-dominated by the queue’s paths, and in that case it inserts the new path, \(L_{new}\), to the queue. The MLC algorithm is quite similar, except that it sorts the priority queue in a more intuitive manner, thus resulting in better performance in term of running time.

Both MLS and MLC are considered to be fast enough as long as the set of Pareto paths is relatively small [33] [20]. Unfortunately, Pareto sets may consist of a prohibitively large (non-polynomial) number of paths, even for the restricted case of two objectives [18]. A possible approach for

\(^1\)To define Pareto optimality, consider two \(W\)-dimensional objective vectors \(x = (x_1, \ldots, x_W)\) and \(y = (y_1, \ldots, y_W)\). If \(x_j \leq y_j\) for each \(j \in \{1, \ldots, W\}\) and \(x_i < y_i\) for some \(j \in \{1, \ldots, W\}\), then \(x\) dominates \(y\). Given a finite set \(X\) of \(W\)-dimensional vectors, we say that \(x \in X\) is Pareto–optimal in \(X\) if there is no \(y \in X\) that dominates \(x\).
dealing with larger Pareto sets can be obtained by ε-optimal solutions, which provide a polynomial number of Pareto paths [23]. Such an approximated solution can be computed efficiently through a fully polynomial approximation scheme (FPAS) [17] [27] [31]. However, as explained above, we seek to find a sub-set of Pareto routes where each route minimizes a different linear combination of the "raw" objectives. Hence, obtaining a sub-optimal Pareto set may not provide a valid solution. In other words, it is not guaranteed that the set of required routes is included in an approximated Pareto set.

Accordingly, in this study we establish and validate the Iterative Dijkstra with Adapted Queue (IDAQ) algorithm. IDAQ provides an optimal solution for the problem without imposing any pre-processing computations. We show that, under reasonable assumptions, IDAQ significantly outperforms the standard approach of consecutively executing a standard shortest path algorithm. Similarly to the standard approach, IDAQ iteratively searches for routes; however, instead of executing each iteration independently, it shares information among the iterations. Unlike the standard approach, this allows IDAQ avoid scanning any optimal path (with respect to some linear combination) more than once, thus providing improved performance in terms of running time.

The rest of the paper is organized as follows. In Section 2, we formulate the problem. Next, in Section 3, we present and analyze a standard approach algorithm. Our approach, namely the IDAQ algorithm, along with a theoretical analysis, are presented in Sections 4 and 5, respectively. Section 6 presents a simulation study, which demonstrates that the IDAQ algorithm considerably improves performance (in terms of running time) in comparison to the standard approach. This is demonstrated on both randomly generated settings as well as on settings that correspond to real-life data. Finally, concluding remarks are presented in Section 7.

2 Problem Formulation

In this section, we formulate the Multi-Objective Weighted Shortest Path problem (MOWSP), discussed in this article. To that end, we shall use the following definitions.

Definition 1 (edge). Consider a set of nodes \( V \). An edge is defined as an ordered pair of nodes in \( V \).

As mentioned, we aim to solve a problem of finding several optimal routes with respect to different sets of edges weights. More specifically, each set of edges weights is produced through a (different) linear combination of the "raw" objectives. Formally, MOG (definition 2) is defined as a graph with several objective values attached to each edge; and a coefficient vector (definition 3) specifies the considered linear combinations of the objectives (where each coefficient represents the relative importance of each objective).

Definition 2 (MOG). A Multi-Objective Graph, \( MOG(V,E) \), is a set of connected nodes \( V = \{v_1, ..., v_n\} \) and a set of directed edges \( E = \{e_1, ..., e_m\} \) so that associated with each edge \( e \in E \) are \( W \) different non-negative additive values, each representing some objective. We shall denote the \( i \)th objective of an edge \( e \in E \) (where \( i \leq W \)) by \( w_e[i] \).

Definition 3 (coefficient vector). A coefficient vector \( \lambda \in \mathbb{R}^W \) is a vector of \( W \) positive numbers. We denote the \( i \)th element of \( \lambda \) (where \( i \leq W \)) by \( \lambda[i] \).

Definition 4 (Path). Consider a MOG \((V,E)\) (definition 2) and a “start node” \( s \in V \). Path\( s \) is the set of paths from \( s \) to any node \( v \in V \). In other words, each element of Path\( s \) is an ordered list of nodes that starts with \( s \) and each subsequent node is connected to the previous one by an edge \( e \in E \).

Definition 5 (Path\( s,v \)). Consider an MOG \((V,E)\) (definition 2), a start node \( s \in V \) and some node \( v \in V \). Path\( s,v \) is the set of paths from \( s \) to \( v \). In other words, each element \( p \in \text{Path}_{s,v} \) is the prefix of Path\( s \) having \( v \) as the last node.

Definition 6 (edge cost). Consider a coefficient vector \( \lambda \) (definition 3). The cost of an edge \( e \in E \) with respect to \( \lambda \) is defined as follows:

\[
\text{Cost}(e, \lambda) = \sum_{j=0}^{W} \lambda[j] \cdot w_e[j] \tag{1}
\]

Definition 7 (path cost). Consider a coefficient vector \( \lambda \) (definition 3), a start node \( s \in V \) and a path \( p \in \text{Path}_s \). The cost of \( p \) with respect to \( \lambda \) is defined as follows:

\[
\text{Cost}(p, \lambda) = \sum_{e \in p} \text{Cost}(e, \lambda) = \sum_{e \in p} \sum_{j=0}^{W} \lambda[j] \cdot w_e[j] \tag{2}
\]

Definition 8. [Optimal Path\( s,v \), with respect to \( \lambda \)] Consider an MOG \((V,E)\), a start node \( s \in V \), a node \( v \in V \) and a coefficient vector \( \lambda \). \( p_{opt} \in \text{Path}_{s,v} \) is optimal with respect to \( \lambda \) if, for each \( p \in \text{Path}_{s,v} \), the following holds:

\[
\text{Cost}(p_{opt}, \lambda) \leq \text{Cost}(p, \lambda) \tag{3}
\]

We are ready to state the MOWSP problem:

MOWSP Problem (Multi-Objective Weighted Shortest Path problem). Consider an MOG \((V,E)\), a start node \( s \in V \) and a set of coefficient vectors \( \Lambda \), where:

\[
\Lambda = \{\lambda \in \mathbb{R}^W | i \in 1, ..., K\} \tag{4}
\]

For each coefficient vector \( \lambda_i \in \Lambda \), find set of paths

\[
P_i = \{p_{v_1}, p_{v_2}, ..., p_{v_n}\}
\]

where each path \( p_{v_j} \in \text{Path}_{s,v_j} \) is optimal with respect to \( \lambda_i \).

\(^2\) Notice that under the Path\( s, \) Path\( s,v \) definitions, we also consider paths that contain loops.
3 Standard Algorithm: Dijkstra Iterations

In this section, we present a standard-approach algorithm for solving the MOWSP problem, which is based on multiple executions of the Dijkstra algorithm (henceforth, the Standard Algorithm). The rest of this section is structured as follows: in Section 3.1, we describe the Standard Algorithm; in Section 3.2, we prove its correctness; finally, in Section 3.3, we analyze its time complexity.

3.1 Standard Algorithm Description

The algorithm iterates through each \( \lambda_i \in \Lambda \) (defined by equation 4). In each iteration, the algorithm constructs a new graph \( G_{temp}(V,E) \), which is identical to \( G(V,E) \) except for the following: we reduce the cost of each edge \( e \in E \) to a single dimension using the \( \text{Cost}(e, \lambda_i) \) function (defined by equation 1). The algorithm calculates and returns the shortest path for each node in \( G_{temp}(V,E) \) from the start node \( s \) using the Dijkstra shortest-path algorithm.

The pseudo-code of the Standard Algorithm is presented herein.

Algorithm 1 Standard Algorithm

1: procedure STANDARD ALGORITHM\((G,s,\lambda)\)
2: \( \text{optimal}_\text{paths} \leftarrow \text{null} \)
3: for \( i=1 \) to \( K \) do
4: \( E_{temp} \leftarrow G.E \)
5: for each \( e \in E_{temp} \) do
6: \( e.\text{Cost} \leftarrow \text{Cost}(e, \lambda_i) \)
7: \( G_{temp} \leftarrow (G.V,E_{temp}) \)
8: \( P_i \leftarrow \text{Dijkstra}(G_{temp}, s) \)
9: \( \text{optimal}_\text{paths}_\text{set} \leftarrow \text{optimal}_\text{paths}_\text{set} \cup \{P_i\} \)
10: Return \( \text{optimal}_\text{paths}_\text{set} \)

3.2 Standard Algorithm Correctness

Lemma 1. The Standard Algorithm solves the MOWSP problem.

Proof. As explained in Section 3, The Standard Algorithm is based on \( K \) executions of the Dijkstra algorithm. Consider one of these executions (lines 4-9 of some iteration).

In lines 4-6, the graph \( G_{temp} \) is generated, and on this graph, we execute the Dijkstra algorithm. The nodes of \( G_{temp} \) are precisely those of \( G \), namely \( V \). The edges of \( G_{temp} \), namely \( E_{temp} \), are equal to \( E \) with the following addition: each edge \( e \in E_{temp} \) cost value, denoted by \( e.\text{Cost} \), is determined by the \( \text{Cost} \) function (line 6).

The Dijkstra algorithm (executed in line 8) returns for each node \( v \in V \) a path, namely \( p_v \in \text{Paths}_{s,v} \), \( p_v \) is the shortest path ending at \( v \), with respect to the cost values \( e.\text{Cost} \) for each edge \( e \in E_{temp} \).

Let us denote the output of the Dijkstra algorithm as \( P_i \) where:

\[
P_i = \{p_{v_1}, p_{v_2}, \ldots, p_{v_n}\}
\]

Recall that \( G_{temp} \) has the same nodes and edges as \( G \) hence, for any node \( v \), each path \( p \in \text{Paths}_{s,v} \) is necessarily a path on both \( G_{temp} \) and \( G \).

From the correctness of the Dijkstra algorithm we can conclude that the path \( p_v \) is such that:

\[
p_v = \arg \min_{p \in \text{Paths}_{s,v}} f(p)
\]

where:

\[
f(p) = \sum_{e \in p} e.\text{Cost} = \sum_{e \in p} \text{Cost}(e, \lambda_i) = \text{Cost}(p, \lambda_i)
\]

In other words, for each \( p \in \text{Paths}_{s,v} \) the following equation holds:

\[
\text{Cost}(p_v, \lambda_i) \leq \text{Cost}(p, \lambda_i)
\]

Notice that, by definition, \( p_v \) is optimal with respect to \( \lambda_i \), i.e., \( P_i \) is the set of optimal \( \text{Paths}_{s,v} \) for each \( v \in V \) with respect to \( \lambda_i \).

The \( \text{optimal}_\text{paths}_\text{set} \) which is return at the end of the algorithm is precisely:

\[
\text{optimal}_\text{paths}_\text{set} = \{P_1, P_2, \ldots, P_K\}
\]

hence, it is by definition the solution for the MOWSP as required.

3.3 Standard Algorithm Complexity Analysis

Lemma 2. The complexity of the Standard Algorithm is given by:

\[
O(K \cdot W \cdot |E| + K \cdot |V| \cdot \log|V|)
\]

Proof. Recall that \( K \) is the number of input coefficient vectors (see MOWSP definition), and \( W \) is defined as the number of objective values in each edge (see definition 2).

As described in Section 3, The Standard Algorithm includes \( K \) iterations, in each of which the following operations are performed:

1. Constructing a search graph \( G_{temp} \) by calculating for each edge \( e \in E \) its cost value using the \( \text{Cost} \) function (equation 2). This is done in \( O(|E| \cdot W) \).
2. Calculating the shortest path for each node in \( G_{temp} \) using the Dijkstra algorithm. This can be done in \( O(|E| + |V| \cdot \log|V|) \) using a Fibonacci heap for the priority queue implementation [8].
Hence, the complexity of the Standard Algorithm is given by:

\[ O(K \cdot W \cdot |E| + K \cdot |V| \cdot \log |V|) \]

4 IDAQ Algorithm

We turn to present IDAQ, an efficient algorithm that solves the MOWSP problem. We shall prove that, under reasonable assumptions, IDAQ has lower time complexity than the Standard Algorithm described in Section 3.

The rest of this section is organized as follows. We begin with a general description of the IDAQ algorithm (Subsection 4.1). Then, in Subsections 4.2 and 4.3, we introduce procedures that serve as building blocks by the IDAQ algorithm. Finally, in Subsection 4.5 we present the IDAQ algorithm.

4.1 General Description

Similarly to the Standard Algorithm, IDAQ is an iterative algorithm: at the end of each iteration, say \( i \in \{1,...,K\} \), IDAQ finds the set of optimal \( Path_{i,v} \) for each \( v \in V \) with respect to \( \lambda_i \).

However, unlike the Standard Algorithm, IDAQ shares knowledge among its iterations. The basic idea is the following: consider some node \( v \in V \). While evaluating a path \( p \in Path_{i,v} \), IDAQ checks whether \( p \) might be optimal with respect to \( \lambda_I \) where \( I = \{i+1,...,K\} \) is some future iteration. In that case, IDAQ remembers \( p \), and this allows IDAQ to execute iteration \( I \) more efficiently.

4.2 IDAQ Relevance Check

In this subsection we present IDAQ Relevance Check, a procedure used by IDAQ to determine whether an evaluated path is potentially optimal with respect to any coefficient vector \( \lambda \in \Lambda \). Specifically, in Subsection 4.2.1 we present the definition of a relevant path and in Subsection 4.2.2 we present the procedure’s pseudo-code.

4.2.1 Relevance definition

In this subsection we establish the relevance definition (definition 12). Intuitively, a relevant path is a potential optimal path with respect to any coefficient vector \( \lambda \in \Lambda \) and therefore should not be ignored. We begin by introducing some auxiliary definitions.

Definition 9 (\( Q_v \)). Consider some node, say \( v \in V \), and a list of paths \( Q \subseteq Path_v \), \( Q_v \) is defined as the sub-group of paths in \( Q \) containing each path \( b \in Q \) where \( b \in Path_{i,v} \).

Definition 10 (relevant due to optimality with respect to \( Q_v \)). Consider some node \( v \in V \), a list of paths \( Q_v \subseteq Path_{i,v} \), and some path \( p \in Path_{i,v} \) where \( p \notin Q_v \), \( p \) is relevant due to optimality with respect to \( Q_v \), if there exists a coefficient vector \( \lambda_v \in \Lambda \) where for each path \( b \in Q_v \):

\[ Cost(p, \lambda_v) \leq Cost(b, \lambda_v) \]

Definition 11 (relevant due to dominance with respect to \( Q_v \)). Consider some node \( v \in V \), a list of paths \( Q_v \subseteq Path_{i,v} \), and some path \( p \in Path_{i,v} \). \( p \) is relevant due to dominance with respect to \( Q_v \), if \( p \) is Pareto non-dominated by each path \( b \in Q_v \).

We are ready to establish the definition of “relevance”.

Definition 12 (relevant with respect to \( Q_v \)). \( p \) is relevant with respect to \( Q_v \), if either an optimal path is relevant with respect to \( Q_v \), or relevant due to dominance with respect to \( Q_v \).

4.2.2 Is-Relevant procedure

In this subsection, we present the Is-Relevant procedure used by the IDAQ algorithm to determine whether a path \( p \in Path_{i,v} \) is relevant with respect to a list of paths, \( Q \subseteq Path_v \).

According to definition 12, \( p \) is relevant with respect to \( Q_v \) if it meets either of the following conditions:

1. \( p \) is relevant due to optimality with respect to \( Q_v \)
2. \( p \) is relevant due to dominance with respect to \( Q_v \)

The following Is-Relevant procedure identifies which of the conditions (1 or 2) can be verified more efficiently. Eventually, this shall allow us to prove that, under reasonable assumptions, the IDAQ algorithm has lower time complexity than the Standard Algorithm.

We begin by introducing an auxiliary definition followed by the Is-Relevant procedure pseudo-code.

Definition 13 (\( best_v \)). Consider some node, say \( v \in V \), and a list of paths \( Q \subseteq Path_v \), \( best_v \) is defined as an auxiliary list used by the Is-Relevant procedure, such that:

\[ best_v = \{best_v[1], best_v[2],...,best_v[K]\} \]

where for each \( i = \{1,...,K\} \) and for any path \( p \in Q_v \), the following holds:

\[ best_v[i] \in Q_v \land Cost(best_v[i], \lambda_i) \leq Cost(p, \lambda_i) \]

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3 Assuming the minimal cost for each \( \lambda_i \in \Lambda \) was calculated beforehand, it is easy to see that this can be verified in \( O(K \cdot W) \).

4 Assuming the number of Pareto paths is given by \( O(L) \), a non-dominance check can be verified in \( O(L \cdot W) \). Note that the number of Pareto paths \( L \) can be non-polynomial, however, in many practical applications \( L \) could be relatively small [20].
We turn to present the Adaptive Queue - a priority queue follows: (Path Priority)

Definition 15: initialized with 1.

Definition 14: according to a new coefficient vector. We begin with some

Adapt besides the Adaptive Queue consists of one more operation, defined by the following operations:

Push, Pop, Is-Empty

algorithm. The Adaptive Queue, being a priority queue, is for determining the next path to be developed by the IDAQ

4.3 Adaptive Queue

We turn to present the Adaptive Queue - a priority queue for determining the next path to be developed by the IDAQ algorithm. The Adaptive Queue, being a priority queue, is defined by the following operations: Push, Pop, Is-Empty. Besides, the Adaptive Queue consists of one more operation, namely the Adapt operation, used to sort the queue’s paths according to a new coefficient vector. We begin with some auxiliary definitions.

Definition 14 (A_index). Index of coefficient vector in Λ initialized with 1.

Definition 15 (Path Priority). Consider some node v ∈ V, path p ∈ Path_{s,v} and A_index. The Priority of p, denoted by p.Priority, is equal to either null or Cost(p,λ_{A_index}), as follows:

1. null - If there exists another path b ∈ Path_{s,v} so that at least one of the following conditions holds:
   (a) b is either in the Adaptive Queue or was popped from the Adaptive Queue, and:
   \[ \text{Cost}(b,\lambda_{A_index}) < \text{Cost}(p,\lambda_{A_index}) \]
   (b) b is in the Adaptive Queue, was pushed to the Adaptive Queue before p and:
   \[ \text{Cost}(b,\lambda_{A_index}) = \text{Cost}(p,\lambda_{A_index}) \]

(c) b was popped from the Adaptive Queue and:
   \[ \text{Cost}(b,\lambda_{A_index}) = \text{Cost}(p,\lambda_{A_index}) \]

2. Cost(p,λ_{A_index}) - otherwise.

Definition 16 (Priority Heap). A Fibonacci heap that contains each path in the Adaptive Queue with non-null Priority. The Priority Heap is managed by the Adaptive Queue operations (to be defined later in this section).

We are ready to define the Adaptive Queue operations:

Push(p) - Insert a Relevant (see definition 12) path, say p ∈ Path_{s,v} to the Adaptive Queue and perform the following actions:

1. Set p.Priority according to definition 15.
2. In case p.Priority is not null:
   (a) Set each path in Q, Priority to null and remove it, if necessary, from the Priority Heap
   (b) Inset p into the Priority Heap.

Is Empty() - Check whether exists a path in the queue with not null Priority value. In other words, check if the Priority Heap is empty.

Pop() - Using the Priority Heap (see definition 16), remove (and return) out of the Adaptive Queue a path with minimal not null Priority.

Adapt(i) - Perform the following actions:

1. Initialize the Priority Heap to a new empty heap.
2. In case p.Priority is null:
   (a) Set each path in Q, Priority to null and remove it, if necessary, from the Priority Heap
   (b) Inset p into the Priority Heap.

3. Calculate the Priority of each of the Adaptive Queue paths according to definition 15.
4. Insert into the Priority each path with not null priority.

4.4 Build-Sets Procedure

The IDAQ algorithm manages a list of optimal paths with respect to any coefficient vector in Λ. To solve MOWSP, IDAQ has to convert such list to a set of path for each coefficient vector as required by the MOWSP definition (2).

Formally, the Build — Sets procedure gets a list of paths, say PL, and returns for each coefficient vector λ_i ∈ Λ, a set of paths:

\[ P_i = \{ p_{v_1}, p_{v_2}, ..., p_{v_n} \} \]

where for each node \( v_i \in V \): \( p_{v_i} \in Path_{s,v_i} \) has minimum Cost(p,λ_i) among all paths p ∈ PL ∩ Path_{s,v_i}. 

We present the IDAQ algorithm, an efficient algorithm that solves MOWSP. We begin by introducing some auxiliary definitions followed by IDAQ’s pseudo-code (Algorithm 2).

IDAQ uses an Adaptive Queue instance in order to make sure that unnecessary paths will not be pushed to the queue. Such a list is defined as follows:

Definition 17 (queue). An Adaptive Queue used by the IDAQ algorithm.

Definition 18 (optimal_paths). A list of paths that were popped from the queue as paths with minimal non-null priority.

Occasionally, we shall address the set of paths \( Q \) defined as follows.

Definition 19 (Discovered Paths). The set of paths that are in the queue or the optimal_paths list. We denote the Discovered Paths by \( Q \).

IDAQ uses an auxiliary list of paths in order to make sure that unnecessary paths will not be pushed to the queue. Such a list is defined as follows:

Definition 20 (pareto_sample). A list of paths that is calculated in line 6 of IDAQ (Algorithm 2). Whenever the pareto_sample list contains exactly one path ending at some node \( v \in V \), we denote it by \( \text{pareto_sample}[v] \).

Indeed, we shall prove (Lemma 4) that the pareto_sample list contains precisely a single Pareto non-dominated path ending at each node.

Definition 21 (iteration). A variable used by IDAQ. The value of the iteration variable is initialized to 1 (line 8 of Algorithm 2) and increased by 1 each time the queue’s Is - Empty operation \(^{5}\) returns true (line 14 of Algorithm 2). We shall denote the period where \( \text{iteration} = i \) as "the \( i \)th iteration" or "iteration \( i \)."

The IDAQ algorithm is specified as follows.

Algorithm 2 IDAQ

```
1: procedure IDAQ(G,s,A)
2:     \( E_{\text{temp}} \leftarrow G.E \)
3: for each \( e \in E_{\text{temp}} \) do
4:     \( e.\text{Cost} \leftarrow \text{Cost}(e, \lambda_i) \)
5: \( G_{\text{temp}} \leftarrow (G.V, E_{\text{temp}}) \)
6: \( \text{pareto_sample} \leftarrow \text{Dijkstra}(G_{\text{temp}}, s) \)
7: \( \text{optimal_paths} \leftarrow \text{empty list} \)
8: \( \text{iteration} \leftarrow 1 \)
9: \( \text{queue} \leftarrow \text{Adaptive Queue}(s, A, \text{iteration}) \)
10: while True do
11:     if \( \text{queue} \) is not empty do
12:         if \( \text{iteration} > \text{length of} \ \lambda \) then
13:             \( \text{Return BuildSets(\text{optimal_paths}, A, G)} \)
14:             \( \text{iteration} \leftarrow \text{iteration}+1 \)
15:             \( \text{queue}.\text{Adapt(\text{iteration})} \)
16:         \( p \leftarrow \text{queue.Pop()} \)
17:         \( \text{Insert} \ p \ \text{to optimal_paths} \)
18:     for Each neighbor of \( p.\text{Node} \) do
19:         \( \text{next}_p \leftarrow \text{Path}(p, \text{neighbor}) \)
20:         \( \text{pareto}_p \leftarrow \text{pareto_sample}[p.\text{Node}] \)
21:         if \( \text{pareto}_p \) Pareto Dominates \( \text{next}_p \) then
22:             \( \text{Continue} \)
23:         \( Q \leftarrow \text{queue} \cup \text{optimal_paths} \)
24:         if \( Is = \text{Relevant}(\text{next}_p, Q, A) \) then
25:             \( \text{Push next}_p \ \text{to queue} \)
```

\(^{5}\)The queue operations are defined in Section 4.3
5 IDAQ Analysis

In this section we prove the correctness of the IDAQ algorithm and analyze its time complexity.

5.1 General definitions

Definition 22 (D). Average node degree:

\[ D = \frac{|E|}{|V|} \]

Definition 23 (MaxDeg_+). Maximum in-degree of a node:

\[ \text{MaxDeg}_+ = \max\{deg_+(v) \mid v \in V\} \]

Definition 24 (developed path). For the IDAQ algorithm, a developed path is a path that is popped from the queue (Algorithm 2 line 12-13), while for the Standard Algorithm, it is a path that is popped from the priority queue during a Dijkstra iteration.

Definition 25 (scanned path). Consider a path \( p \in \text{Path}_{s,v} \) where the list of nodes representing \( p \) is given by: \( \{s,v_1,v_2,\ldots,v_{n-1},v_n\} \), \( p \) is defined as a scanned path if \( p' \) is a developed path where:

\[ p' = \{s,v_1,v_2,\ldots,v_{n-1}\} \]

Definition 26 (developed node). A node that is at the end of a developed path.

Definition 27 (ancestor). Consider a path \( p \in \text{Path}_{s,v} \) where the list of nodes representing \( p \) is given by: \( \{s,v_1,v_2,\ldots,v_m\} \). \( p \)'s ancestors are defined to be paths of the form: \( \{s,v_1,v_2,\ldots,v_m\} \) where \( m < n \).

Definition 28 (first optimal path). Consider some coefficient vector, say \( \lambda \in \Lambda \), and an optimal path with respect to \( \lambda \), say \( p \in \text{Path}_{s,v} \). \( p \) is defined as a first optimal path if at the time \( p \) was scanned, neither the queue nor the optimal_paths list contains another optimal \( \text{Path}_{s,v} \) with respect to \( \lambda \).

Definition 29 (first developed optimal path). Consider some coefficient vector, say \( \lambda \in \Lambda \), and an optimal path with respect to \( \lambda \), say \( p \in \text{Path}_{s,v} \). \( p \) is defined as a first developed optimal path if at the time \( p \) was developed, the optimal_paths list does not contain another optimal \( \text{Path}_{s,v} \) with respect to \( \lambda \).

5.2 definitions given \( l \)

We proceed with some auxiliary definitions. Each of the following definitions depends on an input numerical integer value, namely \( l \). The precise definition of \( l \) shall be provided later; intuitively, \( l \) corresponds to the maximal number of Pareto paths for most nodes, which can be expected to be relatively small in practice [20].

Definition 30 (\( l \)-node). A node that is at the end of at most \( l \) different Pareto paths.

\( V_L(l) \) - Set of all \( l \)-Nodes in \( G \):

\[ V_L(l) = \{v \in V \mid v \text{ is an } l\text{-Node}\} \]

\( \alpha(l) \) - \( l \)-Nodes ratio.

\[ \alpha(l) = \frac{|V_L(l)|}{|V|} \]

\( E_L(l) \) - Set of all outgoing edges from \( l \)-Nodes:

\[ E_L(l) = \{e \in E \mid e.\text{Start} \in V_L(l)\} \]

\( \gamma(l) \) - \( l \)-Node’s outgoing edges ratio:

\[ \gamma(l) = \frac{|E_L(l)|}{|E|} \]

\( D_L(l) \) - Average \( l \)-Node’s degree:

\[ D_L(l) = \frac{|E_L(l)|}{|V_L(l)|} \]

\( N_L(l) \) - The maximum number of paths non-dominated by a single Pareto path to any \( l \)-Node.

\( I(l) \) - Difference between \( K \) and \( l \):

\[ I(l) = K - l \]

Next, we determine a specific value of \( l \), namely \( L \). Intuitively, \( L \) is a small-enough number so that most of \( G \) nodes are \( L \)-nodes.

Definition 31 (\( L \)). \( L \) is defined as the solution of the following optimization problem:

\[ L = \arg\min_l \]

Subject to:

\[ \alpha(l) \geq 1 - \frac{\log(|V|)}{|V|} \]

In the following, whenever we do not state otherwise, \( L \) is the default value of \( l \) for each of the above defined definitions, e.g: \( \alpha = \alpha(L), V_L = V_L(L) \), etc.
5.3 Assumptions

We present several assumptions under which we can prove that IDAQ’s time complexity is lower than that of the Standard Algorithm. Before stating each assumption, we provide some intuition that justifies it.

In many classes of multi-objective optimization problems the number of objectives is relatively small, e.g. communication network routing problems often deal with just two objectives, e.g., throughput and delay.

Assumption 1. The number of objectives in a MOG is \( O(1) \), i.e.,

\[
W = O(1)
\]

Moreover, many classes of networks are represented by sparse graphs, e.g. in road networks each intersection typically consists of the crossing of at most 4 roads. Thus:

Assumption 2. The maximum number of incoming/outgoing edges from a single node is upper bounded by \( O(\log(|V|)) \). In other words, \( G \) is a sparse graph:

\[
\text{MaxDeg}_- = \text{MaxDeg}_+ = O(\log(|V|))
\]

According to [20], in many practical applications, the number of different Pareto paths is often relatively small. Hence, a small number of input coefficient vectors (which is equal to the number of returned paths ending at each node), should produce paths with sufficient variety, i.e.:

Assumption 3. The number of input coefficient vectors in \( \Lambda \), namely \( K \) (equation 4) is upper-bounded by \( O(\sqrt{|V|}) \).

\[
K = O(\sqrt{|V|})
\]

According to [20], in many practical applications, the various objectives are correlated. For example, in road networks, the shortest-distance route is among the fastest routes (albeit not necessarily the fastest). Accordingly, we assume that the number of paths that are non-dominated by a Pareto path is relatively small, i.e.:

Assumption 4. The maximum number of paths non-dominated by a single Pareto-non-dominated path to \( L \)-Node (denoted by \( N_L \), is upper-bounded by \( O(L) \)

\[
N_L = O(L)
\]

Consider some \( L \)-Node \( v \in V_L \). We assume that \( N_L \) is smaller than \( \frac{W}{L} \). Thus, IDAQ checks whether a \( p \in \text{Path}_{s,v} \) Is-Relevant in an efficient matter by checking if \( p \) is relevant by optimality (Definition 10).

Note that \( \frac{W}{L} = O(\sqrt{|V|}) \), while \( N_L = O(L) = O(\log(|V|)) \), thus implying that the following assumption is acceptable:

Assumption 5. The maximum number of paths non-dominated by a single Pareto-non-dominated path to \( L \)-Node, \( N_L \), is smaller or equal to \( \frac{W}{L} \)

\[
N_L \leq \frac{W}{L}
\]

According to [20], the number of Pareto paths for most nodes is relatively small in practice. Intuitively, such a number should not be dramatically affected by the graph size: for example, in transportation networks, an "optimal road" from Boston to New-York would not pass through San Francisco. Hence:

Assumption 6.

\[
L = O(F(|V|))
\]

where \( F \) is a sub-logarithmic function, e.g., \( F(|V|) = \sqrt{\log|V|} \) for \( |V| \).

5.4 IDAQ algorithm Correctness

In this section, we prove that the IDAQ algorithm solves the MOWSP problem. In Theorem 1 we establish that, for any coefficient vector in \( \Lambda \) and for each node \( v \in V \), an optimal \( \text{Path}_{s,v} \) is developed by the IDAQ algorithm. Next, based on Theorem 1, in Theorem 2 we shall prove that IDAQ solves the MOWSP problem. We begin by introducing and proving several auxiliary lemmas.

Lemma 3. An optimal \( \text{Path}_{s,v} \) with respect to any coefficient vector in \( \Lambda \) is Pareto non-dominated by any other \( \text{Path}_{s,v} \).

Proof. Consider an optimal \( \text{Path}_{s,v} \) with respect to \( \lambda \in \Lambda \), namely \( p_v \). We assume, by way of contradiction, that \( p_v \) is not Pareto-non-dominated, i.e., there exists some path \( p_{opv} \in \text{Path}_{s,v} \) that dominates \( p_v \). In other words, for each objective, say \( i \):

\[
\sum_{e \in p_v} w_e[i] \geq \sum_{e \in p_{opv}} w_e[i]
\]

and for at least one objective, say \( k \):

\[
\sum_{e \in p_v} w_e[k] > \sum_{e \in p_{opv}} w_e[k]
\]

Note that any coefficient vector in \( \Lambda \) is composed by positive weights. Hence:

\[
\text{Cost}(p_v, \lambda) = \sum_{e \in p_v} \sum_{j=0}^{W} \lambda[j] \cdot w_e[j] >
\]

\[
\sum_{e \in p_{opv}} \sum_{j=0}^{W} \lambda[j] \cdot w_e[j] = \text{Cost}(p_{opv}, \lambda)
\]

in contradiction to the optimality of \( p_v \) with respect to \( \lambda \). Hence, \( p_v \) is Pareto non-dominated. □
Lemma 4. The pareto_sample list contains exactly one Pareto non-dominated path ending at each node.

Proof. Recall that, in order to construct the pareto_sample paths list, an auxiliary graph, namely $G_{temp}$ is generated and on that graph we execute the Dijkstra algorithm.

In lines 2-5 of IDAQ (Algorithm 2), the graph $G_{temp}$ is generated. Similarly to the first iteration of the Standard Algorithm, the nodes of $G_{temp}$ are precisely those of $G$, namely $V$. The edges of $G_{temp}$, namely $E_{temp}$, are equal to $E$ with the following addition: the cost value of each edge $e \in E_{temp}$, denoted by $e.Cost$, is determined by the $Cost$ function (line 4).

The Dijkstra algorithm (executed in line 6) returns for each node $v \in V$ a path, namely $p_v \in Path_{s,v}$, which is the shortest path ending at $v$, with respect to the cost values $e.Cost$ for each edge $e \in E_{temp}$.

The output of the Dijkstra algorithm is denoted as pareto_sample where:

$$pareto\_sample = \{p_{v_1}, p_{v_2}, \ldots, p_{v_n}\}$$

Recall that $G_{temp}$ has the same nodes and edges as $G$ hence, for any node $v$, each path $p \in Path_{s,v}$ is necessarily a path on both $G_{temp}$ and $G$.

From the correctness of the Dijkstra algorithm we conclude that the path $p_v$ is such that:

$$p_v = \arg \min_{p \in Path_{s,v}} f(p)$$

where:

$$f(p) = \sum_{e \in p} e.Cost = \sum_{e \in p} Cost(e, \lambda_i) = Cost(p, \lambda_i)$$

In other words, for each $p \in Path_{s,v}$, the following equation holds:

$$Cost(p_v, \lambda_i) \leq Cost(p, \lambda_i)$$

Hence, $p_v$ is by definition an optimal $Path_{s,v}$ with respect to $\lambda_i$ and in view of Lemma 3 we conclude that $p_v$ is indeed a Pareto non-dominated path.

Lemma 5. Consider an optimal path with respect to $\lambda_i \in \Lambda$, namely $p \in Path_{s,v_1}$. $p$’s ancestors are also optimal paths with respect to $\lambda_i$.

Proof. Assume by way of contradiction that there exists an ancestor path of $p$, say $b \in Path_{s,v_2}$, which is not optimal with respect to $\lambda_i$; in other words, there exists another path, namely $c \in Path_{s,v_2}$, which is optimal with respect to $\lambda_i$, i.e.:

$$Cost(c, \lambda_i) < Cost(b, \lambda_i)$$

Let us construct a new path to $v_1$, say $d \in Path_{s,v_1}$, where: $d$ is the path $c$ followed by the path between $b$ and $p$. From the additivity of the cost function we conclude that:

$$Cost(d, \lambda_i) < Cost(a, \lambda_i)$$

in contradiction to the optimality of $p$ with respect to $\lambda_i$. 

In the following lemmas (6,7) we establish some properties that hold in case a first optimal path is scanned. Based on these, we will then show (Theorem 1) that any first optimal path is scanned by the time that IDAQ terminates.

Lemma 6. In case a first optimal path is scanned, it is pushed to the queue.

Proof. Consider the scan of a first optimal path, with respect to some coefficient vector $\lambda_i \in \Lambda$, say $p \in Path_{s,v}$.

Recall that, by definition, at the time that a first optimal path is scanned, it is neither in the queue nor in the optima_paths list.

According to Lemma 3, $p$ is non-dominated by another $Path_{s,v}$, in particular, pareto_sample$[v]$. Therefore, the Is-relevant procedure is executed in order to determine whether $p$ is inserted to the queue.

From the definition of a first optimal path we can conclude that, for each path $b \in Q_v$:

$$Cost(p, \lambda_i) < Cost(b, \lambda_i)$$

Therefore, according to definition 10, $p$ is relevant due to optimality with respect to $Q_v$.

From Lemma 3 we conclude that $p$ is Pareto non-dominated by any other $Path_{s,v}$, in particular $Q_v$. Hence, according to Definition 11, $p$ is relevant by dominance with respect to $Q_v$.

Since $p$ is relevant by dominance with respect to $Q_v$ and it is relevant by optimality with respect to $Q_v$, it is identified as relevant with respect to $Q$ by the Is-Relevant procedure (4.2.2) and pushed to the queue (lines 24-25 of Algorithm 2), as required.

In the following Lemma 7, we shall prove that not only a scanned first optimal path is pushed to the queue (as established by Lemma 6), but also it is not removed from the queue unless the first developed optimal path is in the optima_paths list.

Lemma 7. From the point in the execution of the algorithm that the first optimal path is scanned, either the first optimal path is in the queue or the first developed optimal path is in the optima_paths list.

Proof. Consider a first optimal path $p \in Path_{s,v}$ with respect to some coefficient vector $\lambda_i \in \Lambda$.

According to Lemma 6, in case $p$ is scanned, it is pushed to the queue. Note that $p$ can only be removed from the queue either by the queue.pop operation or during the scan of some other path.

In case the first developed optimal path is popped from the queue, it is inserted to the optima_paths list (lines 16-17 of Algorithm 2), as required.
which an optimal path priority definition (15), since this scenario is not possible.
Consider case 2. Since p is optimal with respect to \( \lambda_i \), the following holds:

\[
Cost(p, \lambda_i) \leq Cost(b, \lambda_i)
\]

Since p has been inserted to the queue before b, \( p.Cost[i] \) is not null, this scenario is not possible.
To conclude, we have established that none of the two scenarios is possible, in contradiction to our assumption, thus the lemma follows.

\[\square\]

**Theorem 1.** Consider any coefficient vector \( \lambda_i \in \Lambda \) and any node \( v \in V \). An optimal \( Path_{i,v} \) with respect to \( \lambda_i \) has been developed by the time that the IDAQ algorithm returns.

**Proof.** Assume by way of contradiction that there exists some node, say \( v \in V \), and some coefficient vector \( \lambda_i \in \Lambda \), so that no optimal \( Path_{i,v} \) with respect to \( \lambda_i \) has been developed by the end of the \( i \)th iteration (see Definition 21).
Consider the first optimal path with respect to \( \lambda_i \), namely \( b \in Path_{i,v} \). From Lemma 7 we can conclude that, in case that \( b \) is scanned, it must be in the queue. Hence, by the time that the \( i \)th iteration ends, one of the following two cases must hold:

1. \( b \) is in the queue.
2. \( b \) has yet to be pushed to the queue.

Consider case 1. \( b.Priority \) must be null since the iteration ends only when the priority of all the paths in the queue is null. Consider the priority calculation of path \( b \) (either by the queue.adapt operation or by the queue.push operation). According to the path priority definition (15), since \( b \) is a first optimal path, its priority can be set to null only if an optimal \( Path_{i,v} \) was developed, in contradiction to our assumption.

Turning to case 2, consider the list of ordered nodes represented by path \( b \):

\[
b.Nodes = \{s, v_{b1}, v_{b2}, \ldots v\}
\]

Let us define node \( v_e \) as the last node in \( b.Nodes \) for which an optimal \( Path_{i,v_e} \) with respect to \( \lambda_i \) has already been developed. We also denote by \( e \) the first developed optimal \( Path_{i,v_e} \).

We note that \( v_e \) and \( e \) must exist since the first developed path, which contains only the start node \( s \) (in other words: \( v_e = s \) and \( e.Nodes = \{s\} \) ) is a first developed optimal \( Path_{i,v_e} \) with respect to \( \lambda_i \). In addition, \( v_e \neq v \) since otherwise an optimal \( Path_{i,v} \) with respect to \( \lambda_i \) has been developed, contrary to our assumption.

Let \( v_f \) be the node after \( v_e \) on the path \( b \). Let \( f \in Path_{i,v_f} \) be the path \( e \) followed by the node \( v_f \).

Consider the ancestor (see Definition 27) of \( b \) ending in \( v_f \), namely \( b_f \). According to Lemma 5, \( b_f \) is optimal with respect to \( \lambda_i \). From the additivity of \( Cost \) and since \( e \) is optimal with respect to \( \lambda_i \) as well, we conclude that:

\[
Cost(f, \lambda_i) = Cost(b_f, \lambda_i).
\]

Hence, \( f \) is optimal \( Path_{i,v_f} \) with respect to \( \lambda_i \).
Recall that \( v_e \) is followed by \( v_f \) on the \( b.Nodes \) list and that \( v_e \) is the last node in the \( b.Nodes \) list for which an optimal \( Path_{i,v_e} \) with respect to \( \lambda_i \) has already been developed, i.e., an optimal \( Path_{i,v_f} \) with respect to \( \lambda_i \) has not been developed.

Additionally, \( f \) was scanned, since its ancestor, \( e \), had been developed. Hence, either \( f \) is the first optimal \( Path_{i,v_f} \), or the first optimal \( Path_{i,v_f} \) is already in the queue; in either case, the first optimal \( Path_{i,v_f} \) with respect to \( \lambda_i \) must be in the queue. According to Definition 15, such a path does not have null priority. Hence, the iteration has not concluded, which is a contradiction.

To conclude, we have established that none of the scenarios is possible, thus contradicting our assumption. Thus, after the \( i \)th iteration, an optimal \( Path_{i,v} \) with respect to \( \lambda_i \) has been developed.

The IDAQ algorithm returns only when the last iteration ends, hence the theorem follows.

\[\square\]

**Theorem 2.** IDAQ solves the MOWSP problem.

**Proof.** Recall that the IDAQ algorithm returns the following set:

\[
\text{optimal_sets} = \text{Build}_{-} \text{Sets}(\text{optimal_paths}, \Lambda, G)
\]

The Build-Sets procedure described in Section 4.4 returns for each coefficient vector \( \lambda_i \in \Lambda \), a set of paths

\[
P_i = \{p_{i1}, p_{i2}, \ldots, p_{in}\}
\]

where for each node \( v_i \in V \):

\[
p_{ij} \in Path_{i,v_i}
\]

\[\text{Note that } e \text{ is not necessarily an ancestor of } b \text{ (see Definition 27)}\]
Assume by way of contradiction that there exists a set in optimal_sets, say $P_j$, and some node, say $v \in V$, such that a $p_v \in P_j$ is not an optimal Path$_{s,v}$ with respect to $\lambda_j$.

We shall denote the first developed optimal Path$_{s,v}$ with respect to $\lambda_j$ by $o_{p,v}$. Note that, according to Theorem 1, an optimal Path$_{s,v}$ with respect to $\lambda_j$ has been developed by the time that the IDAQ algorithm returns, i.e., $o_{p,v}$ has been developed by that time. Recall that any path that has been developed is inserted into the optimal_paths list (lines 16-17 of Algorithm 2), hence $o_{p,v}$ is in the optimal_paths list. In other words,

$$o_{p,v} \in \text{optimal\_paths} \cap \text{Path}_{s,v}$$

According to the description of the Build-Sets procedure (Section 4.4), for any paths set in optimal_sets, say $P_j$, and for any node in $V$, say $v$, the following holds: $p_v \in P_j$, achieves the minimum value of Cost$(p, \lambda_j)$ among all paths $p \in \text{optimal\_paths} \cap \text{Path}_{s,v}$. Since $o_{p,v} \in \text{optimal\_paths} \cap \text{Path}_{s,v}$ the following holds:

$$\text{Cost}(p_v, \lambda_j) \leq \text{Cost}(o_{p,v}, \lambda_j)$$

According to Definition 8, since $o_{p,v}$ is optimal with respect to $\lambda_j$, for any $p \in \text{Path}_{s,v}$ the following holds:

$$\text{Cost}(o_{p,v}, \lambda_j) \leq \text{Cost}(p, \lambda_j)$$

Hence,

$$\text{Cost}(p_v, \lambda_j) \leq \text{Cost}(o_{p,v}, \lambda_j) \leq \text{Cost}(p, \lambda_j)$$

Thus, $p_v$ is by definition an optimal Path$_{s,v}$ with respect to $\lambda_j$, which is a contradiction, hence establishing the theorem. \(\square\)

### 5.5 IDAQ Time Complexity Analysis

We turn to analyze the time complexity of the IDAQ algorithm. We divide IDAQ operations into four parts and analyze each separately.

1. Developing paths (Theorem 3).
2. Scanning paths (Theorem 5).
3. Adapting the queue between iterations (Theorem 6).
4. Executing the Build-Sets procedure.

The overall time complexity of IDAQ shall be established in Theorem 8.

#### 5.5.1 Complexity Analysis Part 1 - Developing Paths

In this subsection we establish the time complexity analysis of developing paths in IDAQ (Part 1). The main motivation for IDAQ is to avoid developing paths that have already been developed in previous iterations, therefore we expect to get here a lower bound on the number of developed paths (and, as a result, a lower time complexity) than for the Standard Algorithm.

First, we calculate the time complexity of developing paths (Theorem 3). We begin by introducing and proving several auxiliary lemmas.

**Lemma 8.** Consider a first optimal path with respect to $\lambda_i \in \Lambda$, say $p \in \text{Path}_{s,v}$. In case $p$ is pushed to the queue during the $i$th iteration, the priority of each of the queue’s paths ending at node $v$ is set to null.

**Proof.** Consider any path $b \in \text{Path}_{s,v}$. Note that, since $p$ is first optimal with respect to $\lambda$, at the time $p$ is pushed to the queue, the following holds:

$$\text{Cost}(p, \lambda) < \text{Cost}(b, \lambda)$$

Hence, by the definition of path priority (Definition 15), $b$'s priority is set to null, as required. \(\square\)

In the following lemma we establish that the IDAQ algorithm does not develop non-optimal paths. This shall allow us to reach an upper bound on the number of developed paths.

**Lemma 9.** For $i \in \{1,2,\ldots,K\}$, a developed path in iteration $i$ is optimal with respect to $\lambda_i$.

**Proof.** Consider some iteration (see definition 21), say $i$, and a path $p \in \text{Path}_{s,v}$ that was developed during the $i$th iteration. Assume by way of contradiction that $p$ is not optimal with respect to $\lambda$. In other words, there exists another path $b \in \text{Path}_{s,v}$ such that $b$ is first optimal with respect to $\lambda_i \in \Lambda$, i.e.:

$$\text{Cost}(b, \lambda) < \text{Cost}(p, \lambda)$$

One of the following cases must hold:

1. $b$ and $p$ were pushed to the queue during a previous iteration.
2. $b$ was pushed to the queue before $p$; $p$ was pushed to the queue during iteration $i$.
3. $p$ was pushed to the queue before $b$; $b$ was pushed to the queue during iteration $i$.
4. $b$ has yet to be pushed to the queue.
Consider case 1. At the beginning of the \(i\)th iteration, \(p\) is in the \textit{queue} and \(b\) is in either the \textit{queue} or the \textit{optimal_paths} list. We examine the priority calculation of \(p\) by the \textit{Adapt} operation at the beginning of the \(i\)th iteration. According to Definition 15, \(p\)’s priority is set to null, thus implying that it could not have been developed during the \(i\)th iteration.

Consider case 2. According to Lemma 7, during the Push of \(p\) to the \textit{queue}, either \(b\) is in the \textit{queue} or the first developed optimal \(\text{Path}_{x,v}\) is in the \textit{optimal_paths} list. We examine the priority calculation of \(p\) by the \textit{Push} operation. According to Definition 15, \(p\)’s priority is set to null, thus implying that it could not have been developed during the \(i\)th iteration.

Consider case 3. \(b\) is first optimal with respect to \(\lambda_i\), therefore, at the time that \(b\) is being pushed to the \textit{queue} (during iteration \(i\)), \(b\).Priority is not null. According to Lemma 8, the priority of each of the \textit{queue}’s paths ending at node \(v\) is set to null, in particular, the priority of \(p\), thus implying that \(p\) could not have been developed during the \(i\)th iteration.

Consider case 4. Note that, Since \(b\) is first optimal,
\[
\text{Cost}(b, \lambda_i) < \text{Cost}(p, \lambda_i)
\]

Consider the list of ordered nodes represented by path \(b\):
\[
b.\text{Nodes} = \{s, v_{b1}, v_{b2}, \ldots, v\}
\]

Let us define node \(v_e\) as the last node in \(b.\text{Nodes}\) for which an optimal \(\text{Path}_{x,v}\) with respect to \(\lambda_i\) has already been developed. We also denote by \(e\) the first developed optimal \(\text{Path}_{x,v}\) \(7\).

We note that \(v_e\) and \(e\) must exist since the first developed path, which contains only the start node \(s\) (in other words: \(v_e = s\) and \(e.\text{Nodes} = \{s\}\)) is a first developed optimal \(\text{Path}_{x,v}\) with respect to \(\lambda_i\). In addition, \(v_e \neq v\) since otherwise an optimal \(\text{Path}_{x,v}\) with respect to \(\lambda_i\) has been developed, contrary to our assumption.

Let \(v_f\) be the node after \(v_e\) on the path \(b\). Let \(f \in \text{Path}_{x,v}\); \(f\) be the path \(e\) followed by the node \(v_f\).

Consider the ancestor (see Definition 27) of \(b\) ending in \(v_f\), namely \(b_f\). According to Lemma 5, \(b_f\) is optimal with respect to \(\lambda_i\). From the additivity of \text{Cost} and since \(e\) is optimal with respect to \(\lambda_i\) as well, we conclude that:
\[
\text{Cost}(f, \lambda_i) = \text{Cost}(b_f, \lambda_i)
\]

Hence, \(f\) is optimal \(\text{Path}_{x,v_f}\) with respect to \(\lambda_i\).

Recall that \(v_e\) is followed by \(v_f\) on the \textit{b.Nodes} list and that \(v_e\) is the last node in the \textit{b.Nodes} list for which an optimal \(\text{Path}_{x,v}\) with respect to \(\lambda_i\) has already been developed, i.e., an optimal \(\text{Path}_{x,v_f}\) with respect to \(\lambda_i\) has not been developed.

In addition, \(f\) was scanned, since its ancestor, \(e\), had been developed. Hence, either \(f\) is first optimal \(\text{Path}_{x,v_f}\), or the first optimal \(\text{Path}_{x,v_f}\) is already in the \textit{queue}; in either case, the first optimal \(\text{Path}_{x,v_f}\) with respect to \(\lambda_i\) must be in the \textit{queue}.

We shall denote such a path by \(g\). According to Definition 15, \(g.\text{Priority}\) is not null. Additionally, we have:
\[
g.\text{Priority} = \text{Cost}(g, \lambda_i) = \text{Cost}(f, \lambda_i) = \text{Cost}(b_f, \lambda_i) = \text{Cost}(b, \lambda_i) < \text{Cost}(p, \lambda_i)
\]
i.e., \(g.\text{Priority} < p.\text{Priority}\), in contradiction to \(p\) being a path with minimal not null priority in the \textit{queue}.

To conclude, we have established that none of the scenarios is possible, in contradiction to our assumption. \(\Box\)

\textbf{Lemma 10.} In each iteration, IDAQ develops each node at most once.

\textit{Proof.} Assume by a way of contradiction that there exists a node that was developed more than once, say during iteration \(i\). Consider \(b\) and \(p\) as the first two \(\text{Path}_{x,v}\)s that were developed during the \(i\)th iteration. Without loss of generality, we assume that \(b\) was pushed to the \textit{queue} before \(p\).

According to Lemma 9, both \(p\) and \(b\) have minimal cost with respect to \(\lambda_i\) among all paths in \(\text{Path}_{x,v}\); i.e.:
\[
\text{Cost}(p, \lambda_i) = \text{Cost}(b, \lambda_i).
\]

One of the following two cases must hold:
1. \(b\) and \(p\) were pushed to the \textit{queue} in a previous iteration.
2. \(b\) was pushed to the \textit{queue} before \(p\), while \(p\) was pushed to the \textit{queue} during the \(i\)th iteration.

Consider case 1. At the beginning of the \(i\)th iteration, \(p\) and \(b\) are in the \textit{queue}. We examine the priority calculation of \(p\) by the \textit{Adapt} operation at the beginning of the \(i\)th iteration. According to definition 15, \(p\)’s priority is set to null, thus implying that it could not have been developed during the \(i\)th iteration.

Consider case 2. Namely \(b\) is in the \textit{queue} during the scan of \(p\). Therefore, at the time that \(p\) is being pushed to the \textit{queue}, \(p.\text{Priority}\) is set to null (Definition 15). According to the definition of the \textit{Push} operation, the priority of each of the \textit{queue}’s paths ending at node \(v\) is set to null, in particular, the priority of \(b\), thus implying that \(b\) could not have been developed during the \(i\)th iteration.

To conclude, we have established that none of the scenarios is possible, in contradiction to our assumption. \(\Box\)
Lemma 11. IDAQ Develops any path\(^8\) no more than once.

Proof. Consider some node \(v \in V\). By way of contradiction, let us assume that there exists a path \(b \in \text{Path}_{s,v}\), which was pushed to the queue after path \(p \in \text{Path}_{s,v}\) so that both \(p\) and \(b\) represent the same path ending at node \(v\) (i.e., \(p = b\)) and have been developed at the time that the IDAQ algorithm returns.

Consider the scan of \(b\). Since \(p\) was developed and \(p\) was pushed to the queue before \(b\), \(p\) is either in the queue or in the \text{optimal}\_\text{paths} list. Recall that \(Q\) is defined as the group of paths that are either in the queue or in the \text{optimal}\_\text{paths} list (Definition 19), i.e., \(p \in Q\). According to Definition 12, \(b\) is not Relevant with respect to \(Q\). Hence, \(p\) could not have been inserted to the queue and cannot be developed, in contradiction to our assumption. \(\square\)

Lemma 12. The number of nodes that are not \(L\)-Nodes (see Definition 30) is bounded by \(O(\log(|V|))\):

\[
(1 - \alpha) \cdot |V| = O(\log(|V|))
\]

Proof. According to Definition 31, \(L\) is determined so that:

\[
\alpha \geq 1 - \frac{\log(|V|)}{|V|} \iff (1 - \alpha) \cdot |V| \leq \log(|V|)
\]

i.e.

\[
(1 - \alpha) \cdot |V| = O(\log(|V|))
\]

thus the lemma follows. \(\square\)

Lemma 13. Consider any \(L\)-node, say \(v \in V_L\). There are at most \(L\) different \(\text{Path}_{s,v}\)’s that are optimal with respect to any coefficient vector in \(\Lambda\).

Proof. In Lemma 3 we have established that each optimal path is Pareto non-dominated. Consider some \(L\)-node, say \(v \in V_L\). According to Definition 30, \(v\) is at the end of at most \(L\) different Pareto paths. Therefore, the number of optimal paths with respect to any coefficient vector is upper-bounded by \(L\), as required. \(\square\)

Lemma 14. The number of developed paths that end at an \(L\)-node is upper-bounded by:

\[
|D_{L\text{-nodes}}| = O(|V_L| \cdot L) = O(|V| \cdot \alpha \cdot L)
\]

Proof. By definition (see Section 5.2):

\[
\alpha = \frac{|V_L|}{|V|}
\]

Hence, the number of \(L\)-nodes is given by:

\[
|V_L| = |V| \cdot \alpha
\]

Recall that:

1. Due to Lemma 13, for any \(L\)-node, say \(v \in V_L\), there are at most \(L\) different \(\text{Path}_{s,v}\)’s that are optimal with respect to any coefficient vector in \(\Lambda\).

2. Due to Lemma 9, each path developed by IDAQ is optimal.

3. Due to Lemma 11, during an execution of the IDAQ algorithm, a path is developed no more than once.

Therefore, the number of developed paths that end at an \(L\)-node is upper-bounded by:

\[
|D_{L\text{-nodes}}| = O(|V_L| \cdot L) = O(|V| \cdot \alpha \cdot L)
\]

, as required. \(\square\)

Lemma 15. The number of developed paths ending at a node that is not an \(L\)-node is upper-bounded by:

\[
|D_{\text{not}\_L\text{-nodes}}| = O(|V \setminus V_L| \cdot K) = O(|V| \cdot (1 - \alpha) \cdot K)
\]

Proof. By definition (see Section 5.2):

\[
\alpha = \frac{|V_L|}{|V|} \iff |V_L| = |V| \cdot \alpha
\]

\[
\iff |V| - |V_L| = |V| \cdot (1 - \alpha)
\]

Since \(V_L \subseteq V\), we conclude that the number of non-\(L\)-nodes is given by:

\[
|V \setminus V_L| = |V| \cdot (1 - \alpha)
\]

In Lemma 10 we have established that, at each iteration, IDAQ develops each node at most once. Recall that the number of iterations is given by \(K\), i.e., the number of developed paths ending at any node is upper-bounded by \(K\). Hence the number of developed paths ending at a node that is not an \(L\)-node is upper-bounded by:

\[
|D_{\text{not}\_L\text{-nodes}}| = O(|V \setminus V_L| \cdot K) = O(|V| \cdot (1 - \alpha) \cdot K)
\]

, as required. \(\square\)

Finally, we employ lemmas 14 and 15 to get the total time complexity of developing nodes in IDAQ, which is presented by the following theorem.

Theorem 3. The time complexity of developing paths in IDAQ (part 1) is \(^9\):

\[
O(L \cdot |V| \cdot \log(|V|)) = O(F(|V|) \cdot |V| \cdot \log(|V|))
\]

\(^9\)Recall that, according to Assumption 6, \(F(|V|)\) is a sub-logarithmic fun
Proof.

In Lemma 14, we have established that the number of developed paths that end at an \textbf{L-node} is upper-bounded by:

\[|D_{L\text{-}nodes}| = O(|V| \cdot \alpha \cdot L)\]

In Lemma 15, we have demonstrated that the number of developed paths ending at a \textbf{node that is not an L-node} is upper-bounded by:

\[|D_{Not\text{-}L\text{-}nodes}| = O(|V| \cdot (1 - \alpha) \cdot K)\]

Obviously, each path in Path_\text{s} must either start at an L-node or developed paths ending at a node that is not an L-node, hence the number of developed paths is upper-bounded by \(|P\text{\{developed\}}|\), where:

\[|P\text{\{developed\}}| = |D_{Not\text{-}L\text{-}nodes}| + |D_{L\text{-}nodes}| =

\[|V| \cdot (1 - \alpha) \cdot K + |V| \cdot \alpha \cdot L = (I) + (II)\]

and:

\[(I) = |V| \cdot (1 - \alpha) \cdot K\]

\[(II) = |V| \cdot \alpha \cdot L\]

Using Lemma 12 and Assumption 3 we conclude that:

\[(I) = K \cdot O(\log(|V|)) = O(\sqrt{|V|} \cdot \log(|V|)).\]

Since \(\alpha < 1\) and due to Assumption 6 we can conclude that:

\[(II) = O(L \cdot |V|) = O(F(|V|) \cdot |V|)\]

We thus get:

\[|P\text{\{developed\}}| = O(L \cdot |V|) = O(F(|V|) \cdot |V|).\]

Note that each developed path has to be \textit{popped} out of the queue as a path with minimal non-null priority.

From Definition 15 we conclude that, for each node, say \(v \in V\), the queue contains no more than a single Path_{\text{s},v} with non-null priority. Hence, the total number of paths with non-null priority in the queue, which is equal to the number of paths in the Priority Heap (definition 16), is upper-bounded by \(|V|\).

The pop operation (described in Section 4.3) consists of finding a path with minimal priority in the Priority Heap (which is an instance of the Fibonacci Heap data structure). Therefore, a single pop is done in \(O(\log(|V|))\) [8].

We thus conclude that the time complexity of developing paths in IDAQ is:

\[O(|P\text{\{developed\}}| \cdot \log(|V|)) = O(L \cdot |V| \cdot \log(|V|) = O(F(|V|) \cdot \log |V|) \cdot |V|)\]

\[\square\]

5.5.2 Complexity Analysis Part 2 - Scanning paths

We proceed to analyze the time complexity of scanning paths in IDAQ (part 2). IDAQ checks whether a scanned path \(p \in \text{Path}_s\) is potentially optimal (by the Is-Relevant procedure defined in Section 4.2.2). In that case, IDAQ pushes \(p\) to the queue, which make it unnecessary for \(p\)'s ancestor to be developed again at a future iteration. We shall show that, despite the fact that a single IDAQ scan is more time-consuming than a scan of the Standard Algorithm, the total time complexity of the scans is lower for IDAQ.

We begin by establishing the following lemma.

**Lemma 16.** The following equation holds:

\[I \cdot (1 - \gamma) = O(L)\]

\[\text{(5)}\]

\[\text{Proof.}\] Since \(I = K - L\), we have:

\[I \cdot (1 - \gamma) = O(L) \iff\]

\[O(K) \cdot (1 - \gamma) = O(L) \iff\]

\[1 - \gamma = \frac{O(L)}{O(K)} \iff\]

\[|E| \cdot (1 - \gamma) = \frac{O(L \cdot |E|)}{O(K)}\]

To sum up, Equation 5 holds if the following Equation 6 holds:

\[|E| \cdot (1 - \gamma) = \frac{O(L \cdot |E|)}{O(K)}\]

\[\text{(6)}\]

Since \(E = |V| \cdot D = O(|V| \cdot \text{MaxDeg}_+)\), where \text{MaxDeg}_+ is defined as the maximum number of edges emanating from a single node, we conclude that:

\[\frac{O(L \cdot |E|)}{O(K)} = \frac{O(L \cdot |V| \cdot \text{MaxDeg}_+)}{O(K)}\]

From Assumption 3 we conclude that:

\[\frac{O(L \cdot O(|V| \cdot \text{MaxDeg}_+))}{O(\sqrt{|V|})} = \frac{O(L \cdot O(|V| \cdot O(\text{MaxDeg}_+)))}{O(\sqrt{|V|})} = O(L \cdot O(\sqrt{|V|}) \cdot O(\text{MaxDeg}_+)}\]

From assumptions 2 and 6 we conclude that:

\[O(F(|V|) \cdot O(\sqrt{|V|}) \cdot O(\log(|V|))) = O(F(|V|) \cdot \sqrt{|V|}) \cdot \log(|V|)) = O(\sqrt{|V|} \cdot \log(|V|) \cdot F(|V|))\]
Hence, Equation 6 holds if the following Equation 7 holds:
\[
|E| \cdot (1 - \gamma) = O(\sqrt{|V| \cdot \log(|V|) \cdot F(|V|)})
\] (7)

Note that the number of edges emanating from nodes that are not \(L\)-nodes is upper-bounded by the number of such nodes times the maximum nodal degree, namely:
\[
|E| \cdot (1 - \gamma) = (|V| - |V_L|) \cdot O(\text{MaxDeg}_+)
\]
\[
= (1 - \alpha) \cdot |V| \cdot O(\text{MaxDeg}_+) = O((1 - \alpha) \cdot |V| \cdot \text{MaxDeg}_+)
\]

Using Lemma 12 and Assumption 2 we conclude that:
\[
|E| \cdot (1 - \gamma) = O(|V| \cdot \log(|V|)) = O(|V|^2 \log(|V|))
\]

Hence, Equation 7 holds, thus implying that Equation 5 holds, and the lemma follows.

Next, we employ lemmas 14, 15 and 16 to establish an upper-bound on the total number scans performed during an execution of the IDAQ algorithm.

**Theorem 4.** An upper-bound of the number of scans performed during an execution of the IDAQ algorithm is given by \(|P_{\text{Scanned}}|\), where:
\[
|P_{\text{Scanned}}| = O(|E| \cdot L).
\] (8)

**Proof.** Note that a path that was developed causes all of its neighbors to be scanned. Hence, the number of scanned paths equals the number of outgoing edges from each developed path.

Recall that \(E_L\) is defined as the set of all outgoing edges from \(V_L\) (see Section 5.2). In Lemma 14, we have established that the number of developed paths ending at an \(L\)-node is upper-bounded by:
\[
|D_{\text{L-nodes}}| = O(|V_L| \cdot L)
\]

Hence, the number of scans of paths that are neighbors of an \(L\)-nodes is given by:
\[
|S_{\text{L-nodes}}| = O(|E_L| \cdot L)
\]

Recall that the set of nodes that are not \(L\)-nodes is given by \(V \setminus V_L\). Since the set of edges is given by \(E\), and each node is either an \(L\)-node or a non-\(L\)-node, the set of all outgoing edges from a \(V \setminus V_L\) is given by \(E \setminus E_L\). In Lemma 15 we have established that the number of developed paths ending at a node that is not an \(L\)-node is upper-bounded by:
\[
|D_{\text{Not-L-nodes}}| = O(|V \setminus V_L| \cdot K)
\]

Hence, the number of scans of paths that are neighbors of non-\(L\)-nodes is given by:
\[
|S_{\text{Not-L-nodes}}| = O(|E \setminus E_L| \cdot K)
\]

Thus, the total number of scans is given by \(|P_{\text{Scanned}}|\), where:
\[
|P_{\text{Scanned}}| = |S_{\text{L-nodes}}| + |S_{\text{Not-L-nodes}}|
\]
\[
= O(|E_L| \cdot L + |E \setminus E_L| \cdot K)
\]
\[
= L \cdot |E_L| + (|E| - |E_L|) \cdot K
\]
\[
= |E| \cdot (L \cdot \frac{|E_L|}{|E|} + K \cdot (1 - \frac{|E_L|}{|E|}))
\]

From the definition of \(\gamma\) (Section 5.2), we conclude that:
\[
|P_{\text{Scanned}}| = |E| \cdot (L \cdot \gamma + K \cdot (1 - \gamma))
\]

Since \(K = L + I\):
\[
|P_{\text{Scanned}}| = |E| \cdot (L \cdot \gamma + L + I - L \cdot \gamma - I \cdot \gamma)
\]
\[
= |E| \cdot (L + I \cdot (1 - \gamma))
\]

In Lemma 16 we have established that:
\[
I \cdot (1 - \gamma) = O(L),
\]

hence,
\[
|P_{\text{Scanned}}| = |E| \cdot (L + O(L)) = |E| \cdot O(L) = O(|E| \cdot L),
\]
as required.

We proceed to calculate (in Lemma 18) the time complexity of scanning a single path ending at an \(L\)-node. We begin by establishing an auxiliary lemma.

For the following lemmas we consider the set of paths \(Q\) (definition 19), and the subset of paths \(Q_v\) \(= Q\) (definition 9).

**Lemma 17.** For any \(L\)-node \(v \in V_L\), \(Q_v\) contains no more than \(N_L\) paths ending at \(v\).
\[
|Q_v| \leq N_L
\]

**Proof.** According to Lemma 4, after initialization, the pareto_sample list contains a Pareto non-dominated path for each \(L\)-node. Consider such a Pareto non-dominated path \(p \in \text{Paths}_{v}\) where \(v \in V_L\). According to assumption 4, there exist at most \(N_L\) paths ending at \(v\) that are non-dominated by \(p\).

A path that is pushed to the queue must be non-dominated by \(p\) (lines 21-22 of Algorithm 2), thus implying that there exist at most \(N_L\) paths ending at node \(v\) that can be pushed to the queue during an execution of the IDAQ algorithm. Only paths that were pushed to the queue can be inserted to the optimal_paths list, thus the lemma follows.

**Lemma 18.** Scanning a single path that ends at an \(L\)-node is done in \(O(L)\).
Proof. Consider some $L$-node $v \in V_L$ and a path $p \in Path_{s,v}$. Recall that, during a scan of a path $p$, the IDAQ algorithm checks whether $p$ Is-Relevant with respect to $Q$ (defined in Section 4.2.2), in which case IDAQ pushes $p$ to the queue (line 25 of Algorithm 2).

Note that, by Lemma 17, $Q$ contains no more then $N_L$ paths ending at any $L$-node, i.e.,

$$|Q_v| < N_L$$

In addition, due to Assumption 4,

$$N_L = O(L)$$

Hence,

$$|Q_v| = O(L).$$

We shall calculate the time complexity of scanning $p$. The operations performed during the scan of $p$ are as follows:

1. Checking whether $p$ is non-dominated by the paths $Q_v$ (Is-Relevant procedure).
2. Checking whether $p$ dominates paths in $Q_v$ and removing them (Is-Relevant procedure).
3. Updating the priority of queue’s paths ending at node $v$ (Push operation).
4. Inserting $p$ to the queue (Push operation).

Consider operation 1. A single Pareto dominance check can be done in $O(w)$, i.e., the time complexity of checking whether $p$ is Pareto non-dominated by each path in $Q_v$ is upper bounded by:

$$O(N_L \cdot W) = O(L \cdot W).$$

Consider operation 2. In a similar manner to operation 1 above, this can be done in $O(L \cdot W)$. 

Consider operation 3. It involves checking if $p$’s cost is lower in comparison to each of the queue paths ending at $v$, i.e, this can be done in $O(L)$. 

Consider operation 4. $p$ is added to the queue’s data structure, which is done in $O(1)$. In case $p$ is inserted to the queue’s Priority Heap (Definition 16), it has to be inserted to a Fibonacci Heap, which is also executed in $O(1)$ [8].

By Assumption 1,

$$W = O(1)$$

We thus get, that a scan of an $L$-node is done in $O(L)$ as required.

Next, we calculate the time complexity of scanning paths that end at nodes which are not $L$-nodes. We begin by establishing several auxiliary lemmas.

**Lemma 19.** $Q$ contains no more then $K$ paths ending at any node $v \in V$.

$$|Q_v| \leq K$$

Proof. Consider some node $v \in V$. According to Lemma 17, in case $v$ is an $L$-node:

$$|Q_v| \leq N_L \leq \frac{K}{W} \leq K.$$ 

Otherwise, the Is-Relevant procedure updates $Q_v$ to contain only paths that have the best cost for at least a single coefficient vector $\lambda \in \Lambda$. Since $|\Lambda|=K$, there exist at most $K$ different paths in $Q_v$. \qed

**Lemma 20.** A single IDAQ scan of a path that ends at a node that is not an $L$-nodes is done in $O(K)$.

Proof. Consider some node that is not an $L$-node, $v \not\in V_L$ and path $p \in Path_{s,v}$. During a scan of $p$, the IDAQ algorithm checks whether $p$ Is-Relevant with respect to $Q$ (defined in Section 4.2.2), in which case IDAQ pushes $p$ to the queue (line 25 of Algorithm 2).

In case the number of paths in $Q_v$ is smaller than $\frac{K}{W}$, the actions involved in scanning $p$ are identical as if $v$ were an $L$-node. In other words, it can be done in $O(L \cdot w)$ (Lemma 18). Otherwise, the actions performed during the scan of $p$ are as follows:

1. Calculating $p$’s cost for each iteration (Is-Relevant procedure).
2. Finding a path with minimal cost for each iteration in $Q_v$ (Is-Relevant procedure).
3. Updating $p$’s cost to null for each iteration $i$ where $p.\text{Cost}[i] > \text{Best}[i].\text{Cost}[i]$ (Is-Relevant procedure).
4. Removing unnecessary paths from $Q_v$ (Is-Relevant procedure).
5. Updating the priority of queue’s paths ending at node $v$ (Push operation).
6. Inserting $p$ to the queue (Push operation).

Consider operation 1. The Cost calculation of $p$ with respect to a single coefficient vector $\lambda \in \Lambda$ is done in $O(w)$. Since $|\Lambda|=K$, the action can be done in $O(K \cdot W)$.

Consider operation 2. The paths with minimal cost in $Q_v$ for each iteration can be calculated in an incremental manner, hence during the relevance check it is simply read in $O(1)$.

Consider operation 3. It involves iterating through each of $p.\text{Costs}$, which can be done in $O(K)$.

Consider operation 4. It involves removing paths from $Q_v$ that are no longer part of $\text{best}_v$ (see definition 13). According to Lemma 19 $|Q_v| < K$, this action can be done in $O(K)$.

Consider operation 5. It involves checking if $p$’s cost is lower than that of each of the queue paths ending at $v$. Note that, by Lemma 19, the number of paths in $Q_v$ is at most $K$, i.e, this operation can be done in $O(K)$. 

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Consider operation 6. \( p \) is added to the queue data structure, which is done in \( O(1) \). In case \( p \) is inserted to the queue’s Priority Heap (Definition 16), it has to be inserted to a Fibonacci Heap, which is also executed in \( O(1) \) [8].

By Assumption 1,
\[
W = O(1)
\]

We thus get that a scan of a node that is not an \( L \)-node is done in \( O(K) \), as required.

We proceed to calculate the time complexity of the scans of nodes that are not \( L \)-nodes.

**Lemma 21.** The time complexity of scans of nodes that are not \( L \)-nodes is given by
\[
O(|V| \cdot \log^2(|V|))
\]

**Proof.** Recall that, according to the definition of \( V_L \) (Section 5.2), the set of \( L \)-nodes is given by \( V_L \), hence the set of nodes that are not \( L \)-nodes is given by \( V \setminus V_L \).

Since each node has at most \( \text{MaxDeg}_- \) incoming edges and since during a single iteration each node is developed at most once (Lemma 10), each node can be scanned at most \( \text{MaxDeg}_- \) times during a single iteration, thus implying that the number of scans of nodes that are not \( L \)-nodes is upper-bounded by \#NotLNodesScans, where:
\[
\#NotLNodesScans = K \cdot |V \setminus V_L| \cdot \text{MaxDeg}_-
\]

In Lemma 20 we have established that a single IDAQ scan is performed in \( O(K) \). Therefore, the time complexity of scans of nodes that are not \( L \)-nodes is given by:
\[
O(\#NotLNodesScans \cdot K) = O(K^2 \cdot (1 - \alpha) \cdot |V| \cdot \text{MaxDeg}_-).
\]

By Lemma 12, the number of nodes that are not \( L \)-nodes is given by:
\[
O((1 - \alpha) \cdot |V|) = O(\log(|V|)).
\]

By Assumption 2,
\[
\text{MaxDeg}_- = O(\log(|V|)).
\]

By Assumption 3:
\[
K = O(\sqrt{|V|}).
\]

Thus, the time complexity of scans of paths to nodes that are not \( L \)-nodes is given by:
\[
O(|V| \cdot \log^2(|V|))
\]
as required.

Next, we employ the upper bound on the number of scans (as established in Theorem 4), the time complexity of scanning a single \( L \)-node (as established in Lemma 18) and the time complexity of scanning a single node that is not an \( L \)-node (as established in Lemma 21) in order to calculate the total time complexity of IDAQ scans. We begin by proving an auxiliary lemma.

First, we provide some intuition for the lemma. Consider a \( \text{Path}_{h,v} \) that is scanned during the execution of IDAQ. Recall that, in case \( |Q_v| > \frac{K}{W} \), the Is-Relevant procedure uses the set \( \text{best} \) (see Definition 13). The \( \text{best} \) set is calculated in an incremental manner (line 20 of the Is-Relevant procedure), however, for the first time it is being used, it has to be initialized.

In the following lemma, we shall calculate the time complexity of such an initialization during the execution of the IDAQ algorithm.

**Lemma 22.** The time complexity of checking whether each path in \( \text{Path}_{h,v} \) Is-Relevant for the first time when \( Q_v > \frac{K}{W} \) is given by:
\[
O(|V| \cdot \log(|V|))
\]

**Proof.** Consider some node that is not an \( L \)-node, say \( v \notin V_L \). Consider the first time that the Is-Relevant procedure identifies more than \( \frac{K}{W} \) paths in \( Q_v \).

The Is-Relevant procedure has to find all paths in \( Q_v \), with minimal cost for each \( \lambda \in \Lambda \) (\( \text{Best} \) paths list). This involves calculating the cost of each path \( p \in Q_v \) for each coefficient vector in \( \lambda \in \Lambda \). This is done in:
\[
O(|Q_v| \cdot K \cdot W) = O(\frac{K}{W} \cdot K \cdot W) = O(K^2)
\]

and, due to Assumption 3:
\[
= O(|V|)
\]

According to Lemma 12, the number of nodes that are not \( L \)-nodes is given by:
\[
O(|V| \cdot (1 - \alpha)) = O(\log(|V|)).
\]

This implies that identifying all nodes that are not \( L \)-nodes is done in:
\[
O(|V| \cdot \log(|V|)).
\]

**Theorem 5.** The total time complexity of scans in IDAQ is given by
\[
O(|E| \cdot L^2 + |V| \cdot \log^2(|V|))
\]

**Proof.** In Theorem 4 we have established that the number of scans during an execution of the IDAQ algorithm is upper-bounded by \( |P_{\text{Scanned}}| \), where:
\[
|P_{\text{Scanned}}| = O(|E| \cdot L) \tag{9}
\]
We proceed to establish the time complexity of adapting the Theorem 6.

Thus we get,

\[ O(|P_{\text{Scanned}}| \cdot L \cdot W + |V| \cdot \log^2(|V|)) \]

\[ = O(|E| \cdot L^2 \cdot W + |V| \cdot \log^2(|V|)) \]

In Lemma 21 we have established that the scans nodes that are not L-nodes is done in \( O(|V| \cdot \log^2(|V|)) \).

In Lemma 22 we have established that the time complexity of checking whether each path in \( Path_s,v \) Is-Relevant for the first time where \( Q_s \) is Relevant to the queue is:

\[ O(|V| \cdot \log(|V|)) \]

Hence, the time complexity of scans in IDAQ is:

\[ O(|E| \cdot L^2 \cdot W + |V| \cdot \log^2(|V|)) + O(|V| \cdot \log(|V|)) \]

\[ = O(|E| \cdot L^2 \cdot W + |V| \cdot \log^2(|V|)) + |V| \cdot \log(|V|)) = \]

By Assumption 1,

\[ W = O(1) \]

Thus we get,

\[ O(|E| \cdot L^2 + |V| \cdot \log^2(|V|)) \]

\[ \square \]

### 5.5.3 Complexity Analysis Part 3 - Adapting the Queue

We proceed to establish the time complexity of adapting the queue between IDAQ iterations (part 3). We shall show that, even though this part is relatively time-consuming, it is still more efficient than developing paths in the Standard Algorithm, which is done in:

\[ O(K \cdot |V| \cdot \log(|V|)) \]

**Theorem 6.** Adapting the queue between IDAQ iterations is done in:

\[ O(K \cdot |V| \cdot L) = O(K \cdot |V| \cdot F(|V|)) \]

**Proof.** Consider the beginning of a new iteration, \( i \). In order to adapt the queue, the following operations need to be executed:

1. Setting each path’s priority for iteration \( i \).
2. Initialize a new priority queue with the priorities calculated in 1

Consider operation 1. According to definition 15, this involves finding for each node \( v \in V \) a path \( p \in Path_{s,v} \) with minimal \( \text{Cost}(p, \lambda_i) \) among all the path in the \( Q_s \). For a node that is not an L-node, this can be done in \( O(1) \), since for such a node the queue holds such a path for each iteration (Best path list calculated by the Is-Relevant procedure). For an L-node, this can be done in \( O(L \cdot W) \). In the worst case, i.e, when all nodes are L-nodes, this is done in \( O(|V| \cdot L \cdot W) \).

Consider operation 2. Using Fibonacci heaps for the priority queue, this is done in \( O(V) \).

In summary, we get that adapting the queue between iterations is done in:

\[ O(|V| \cdot L \cdot W) \]

Since there is a total of \( K \) iterations we get that the total time complexity of adapting the queue is:

\[ O(K \cdot |V| \cdot L \cdot W) = O(K \cdot |V| \cdot F(|V|) \cdot W) \]

By Assumption 1,

\[ W = O(1) \]

thus we get

\[ = O(K \cdot |V| \cdot L) = O(K \cdot |V| \cdot F(|V|)) \]

\[ \square \]

### 5.5.4 Complexity Analysis Part 4 - Build-Sets procedure

At the end of the IDAQ algorithm, the Build-Sets procedure converts the list of discovered optimal paths with respect to any coefficient vector to a set of paths for each coefficient vector, as required from a solution of the problem. In this section, we calculate the time complexity of the Build-Sets procedure (line 11 of algorithm 2).

**Theorem 7.** The complexity of the Build-Sets procedure (line 11 in algorithm 2) is given by:

\[ O(K \cdot |V| \cdot L) = O(\sqrt{|V|} \cdot |V| \cdot F(|V|)) \]

**Proof.** For each coefficient vector \( \lambda_i \in \Lambda \), the Build-Sets procedure finds optimal paths ending in each node with respect to \( \lambda_i \) by searching among the paths in the \( \text{optimal\_paths} \) list.

Recall that in Lemma 9 we have established that each developed path is optimal, hence, the \( \text{optimal\_paths} \) list, which is the input paths list for the Build-Sets procedure, holds only optimal paths.

We begin by calculating the complexity of finding an optimal path with respect to \( \lambda_i \) ending in a some node which is not an L-node, say \( v \in V \setminus V_L \). In Lemma 10 we have
established that, at each iteration, IDAQ develops each node at most once. Recall that the number of iterations is given by $K$, i.e., the number of developed paths ending at any node is upper-bounded by $K$. Therefore, finding an optimal path with respect to $\lambda_\ast$ is done in $O(K \cdot W)$.

We proceed to calculate the complexity of finding an optimal path with respect to $\lambda_\ast$ ending at some $L$-node, say $v \in V_L$. Due to Lemma 13, for any $L$-node, say $v \in V_L$, there are at most $L$ different $\text{Path}_{v}$s that are optimal with respect to any coefficient vector in $\Lambda$. Therefore, finding the optimal path with respect to a single coefficient vector is done in $O(L \cdot W)$.

We now calculate the complexity of finding an optimal path with respect to $\lambda_\ast$ ending at any node. Recall that the number of nodes that are not $L$-Nodes is bounded by $O(\log(|V|))$ (Lemma 12) and the number of $L$-Nodes is bounded by the total number of nodes, namely $|V|$, i.e, the time complexity is given by:

$$O(\log(|V|) \cdot K + |V| \cdot L)$$

Finally, we are ready to calculate the complexity of the Build-Sets procedure. The Build-Sets procedure finds an optimal path with respect to any coefficient vector ending in any node, i.e, its time complexity is given by:

$$O(K \cdot (\log(|V|) \cdot K \cdot W + |V| \cdot L)) =$$

$$O(\log(|V|) \cdot K^2 \cdot W + |V| \cdot L \cdot K \cdot W)$$

By Assumption 1,

$$W = O(1)$$

Thus we get,

$$= O(\log(|V|) \cdot |V| + |V| \cdot F(|V|) \cdot \sqrt{|V|}))$$

$$= O(|V| \cdot F(|V|) \cdot \sqrt{|V|})) = O(K \cdot |V| \cdot L)$$

as required. \qed

5.5.5 IDAQ Time Complexity

In Sections 5.5.1, 5.5.2, 5.5.3, 5.5.4 we analyzed the time complexity of each of the four parts of the IDAQ algorithm, namely: developing paths, scanning path, adapting the queue between iterations, and executing the Build-Sets procedure. This now allows us to establish the total time complexity of IDAQ, as presented in the following theorem.

**Theorem 8.** The time complexity of IDAQ is given by:

$$O(|E| \cdot F^2(|V|) + \sqrt{|V|} \cdot |V| \cdot F(|V|)) =$$

$$O(|E| \cdot L^2 + K \cdot |V| \cdot L)$$

**Proof.** By Theorem 3, the time complexity of developing paths is:

$$O(L \cdot |V| \cdot \log(|V|)) = O(F(|V|) \cdot |V| \cdot \log(|V|)).$$

By Theorem 5, the time complexity of scanning paths is:

$$O(|E| \cdot L^2 + |V| \cdot \log^2(|V|)).$$

By Theorem 6, the time complexity of adapting the queue between IDAQ iterations is:

$$O(K \cdot |V| \cdot L) = O(\sqrt{|V|} \cdot |V| \cdot F(|V|)).$$

By Theorem 7, the time complexity of executing the Build-Sets procedure is:

$$O(K \cdot |V| \cdot L) = O(\sqrt{|V|} \cdot |V| \cdot F(|V|)).$$

We thus conclude that the time complexity of IDAQ is given by:

$$O(F(|V|) \cdot |V| \cdot \log(|V|)) +$$

$$O(|E| \cdot L^2 + |V| \cdot \log^2(|V|)) +$$

$$O(\sqrt{|V|} \cdot |V| \cdot F(|V|)) +$$

$$O(\sqrt{|V|} \cdot |V| \cdot F(|V|))$$

$$= O(|E| \cdot F^2(|V|) + \sqrt{|V|} \cdot |V| \cdot F(|V|))$$

$$= O(|E| \cdot L^2 + K \cdot |V| \cdot L)$$

\qed

we are ready to formally claim that under the assumptions presented in section 5.3, IDAQ’s time complexity (Theorem 8) is asymptotically lower than that of the Standard Algorithm.

**Theorem 9.** The time complexity of IDAQ is asymptotically lower than that of the Standard Algorithm by a factor of:

$$\Omega\left(\frac{\log |V|}{F(|V|)}\right)$$

**Proof.** In Theorem 8 we established that the time complexity of IDAQ is given by:

$$O(L^2 \cdot |E| + K \cdot |V| \cdot L) = O(I_1 + I_2) \quad (10)$$

where:

$$I_1 = L^2 \cdot |E|$$

$$I_2 = K \cdot |V| \cdot L$$

In Lemma 2 we established that the time complexity of the Standard Algorithm is given by:

$$O(K \cdot W \cdot |E| + K \cdot |V| \cdot \log |V|)$$
By Assumption 1, 

\[ W = O(1) \]

Thus we get that the time complexity of the Standard Algorithm is given by:

\[ O\left( K \cdot |E| + K \cdot |V| \cdot \log |V| \right) = O(S_1 + S_2) \]

(11)

where:

\[ S_1 = K \cdot |E| \]
\[ S_2 = K \cdot |V| \cdot \log |V| \]

Recall that, according to Assumption 6,

\[ L = O\left( F\left( |V| \right) \right) \]

Also recall that according to Assumption 3:

\[ K = O\left( \sqrt{|V|} \right) \]

Denote:

\[ \omega_1 = \frac{\sqrt{|V|}}{F^2\left( |V| \right)} \]
\[ \omega_2 = \frac{\log |V|}{F\left( |V| \right)} \]

Note that

\[ O(I_2) \cdot \Theta(\omega_2) = O(K \cdot |V| \cdot L) \cdot \Theta\left( \frac{\log |V|}{F\left( |V| \right)} \right) \]
\[ = O(K \cdot |V| \cdot F\left( |V| \right) \cdot \frac{\log |V|}{F\left( |V| \right)}) \]
\[ = O(K \cdot |V| \cdot \log |V|) \]
\[ = O(S_2) \]

(12)

In addition,

\[ O(I_1) \cdot \Theta(\omega_1) = O(L^2 \cdot |E|) \cdot \Theta\left( \frac{\sqrt{|V|}}{F^2\left( |V| \right)} \right) \]
\[ = O(L^2 \cdot |E| \cdot \frac{\sqrt{|V|}}{F^2\left( |V| \right)}) \]
\[ = O\left( F^2\left( |V| \right) \cdot |E| \cdot \frac{\sqrt{|V|}}{F^2\left( |V| \right)} \right) \]
\[ = O\left( \sqrt{|V|} \cdot |E| \right) = O(K \cdot |E|) \]
\[ = O(S_1) \]

(13)

Since \( F \) is a sub-logarithmic function (Assumption 6):

\[ \omega_1 = \frac{\sqrt{|V|}}{F^2\left( |V| \right)} > \Theta\left( \log |V| \right) \]

and

\[ \omega_2 < \Theta\left( \log |V| \right) \]

The following inequality holds:

\[ \Theta(\omega_2) < \Theta(\omega_1) \]

(14)

From expressions 13 and 14 we can conclude that the following holds:

\[ O(I_1) \cdot \Theta(\omega_2) < O(I_1) \cdot \Theta(\omega_1) = O(S_1) \]

(15)

Finally, from expressions 12 and 15 we can conclude that

\[ O(I_1 + I_2) \cdot \Theta(\omega_2) = O(I_1) \cdot \Theta(\omega_2) + O(I_2) \cdot \Theta(\omega_2) \]
\[ = O(S_1) + O(S_2) \]
\[ = O(S_1 + S_2) \]

(16)

Since the complexity of IDAQ is given by \( O(I_1 + I_2) \) (expression 10) and the complexity of the Standard Algorithm is given by \( O(S_1 + S_2) \) (expression 11), we conclude that IDAQ’s complexity is lower than that of the Standard Algorithm by at least a factor of \( \omega_2 \). Thus, we get that, indeed, the time complexity of IDAQ is asymptotically lower than that of the Standard Algorithm by a factor of:

\[ \Omega(\omega_2) = \Omega\left( \frac{\log |V|}{F\left( |V| \right)} \right) \]

\[ \square \]

In the previous Theorem 9, we established that the time complexity of IDAQ is asymptotically lower than that of the Standard Algorithm by a factor of \( \Omega\left( \frac{\log |V|}{F\left( |V| \right)} \right) \). Recall that, according to Definition 31, \( L = O\left( F\left( |V| \right) \right) \) corresponds to the maximal number of Pareto paths for most nodes, which can be expected to be relatively small in practice [20]. For example, if \( F\left( |V| \right) = O(1) \), the factor of improvement is given by \( \Omega\left( \log |V| \right) \); if \( F\left( |V| \right) = \log |V| \), it is \( \Omega\left( \frac{\log |V|}{\log \log |V|} \right) \); and if \( F\left( |V| \right) = \log^{1-\beta} |V| \), for some \( 0 < \beta < 1 \), then it is \( \Omega\left( \frac{\log |V|}{\log |V|} \right) \).

### 6 Simulation Study

In this section, we present computational experiments conducted in order to assess the performance of IDAQ (Algorithm 2) in comparison to the Standard Algorithm (Algorithm 1). Both algorithms have been implemented in MATLAB environment. All experiments were conducted on a PC with 32GB RAM and 5th Generation Intel® Core™ i5 Processor. In Section 6.1, we describe experiments conducted using randomly generated MOGs. In Section 6.2, we describe experiments conducted on MOGs generated using actual data representing a more practical setting (provided by Open Street [1]). As shall be presented, in each experiment, we obtained identical results for IDAQ and for the Standard Algorithm, in terms of the quality (optimality) of the solution, as indeed implied by the established correctness of IDAQ (Theorem 2). In each experiment, we measured the performance of each algorithm in terms of running time.
| Number of Nodes | Number Of Edges | Density (\(|E|\/|V|^2\)) | 
|----------------|----------------|-----------------| 
| 221            | 8344           | 0.1708          | 
| 287            | 14290          | 0.1734          | 
| 236            | 9682           | 0.1738          | 
| 245            | 10474          | 0.1744          | 
| 226            | 8566           | 0.1677          | 
| 266            | 12306          | 0.17394         | 
| 238            | 9716           | 0.1715          | 
| 233            | 9370           | 0.1725          | 

Table 1: Randomly Generated Waxman graph parameters, used in the experiments conducted in Section 6.1. The graphs were generated using the following Waxman model parameters: Domain = \([0, 1]; [0, 0.1]\); \(\lambda = 5000;\) \(\alpha = 4;\) \(\beta = 0.03;\)

6.1 Random Generated Experiments

In this section, we demonstrate the advantage of using IDAQ to solve a randomly generated instance of the MOWSP problem (Problem 2). We describe two different experiments (Experiment 1 and Experiment 2) conducted in order to assess IDAQ’s improvement in performance, in terms of running time, with respect to the Standard Algorithm. In Section 6.1.1 we describe the MOG (Definition 2) instances employed by the experiments (we used the same MOGs in the two experiments). In Sections 6.1.2 and 6.1.3 we describe the conducted experiments.

6.1.1 MOG instances for Experiments 1 and 2

In this section we describe the generation of MOGs employed by Experiment 1 and Experiment 2. First, we generated random Waxman graphs [29] using various parameters. The properties of the generated graph and generation parameters are described in Table 1. In order to generate an MOG out of our generated Waxman graphs, we selected for each edge, uniformly at random, five objectives (see Definition 2), each assuming a value between 0 and 1.

6.1.2 Experiment 1

In this section, we describe the experiment conducted in order to assess IDAQ’s improvement in running time with respect to the Standard Algorithm.

First, we generate a set of MOGs, as described in Section 6.1.1). Next, we construct a MOWSP out of each MOG by selecting uniformly at random \(K\) coefficient vectors (see definition 4 (recall that \(K = |\Lambda|\)). Each coefficient is a randomly uniformly generated number between 0.1 and 1.1. We compared between the running times of IDAQ and the Standard Algorithm under various values of \(K\). For each value of \(K\) we calculated the average running time of IDAQ and the Standard Algorithm (Figure 1). As can be seen, IDAQ exhibits better performance (in terms of running time), and its advantage is significant (up to an improvement of about 50%) for large numbers of coefficient vectors. As described in Section 4.1, similarly to the Standard Algorithm, IDAQ is an iterative algorithm, which produces at each iteration at most a single solution for each \(v \in V\). However, unlike the Standard Algorithm, IDAQ shares knowledge between its iterations. Therefore, we expect that in MOWSPs with a large size of the coefficient vector, namely with a large \(K\), IDAQ would exhibit better performance, as indeed demonstrated in Figure 1.

Figure 1: Average running times of the Standard Algorithm and IDAQ for the MOWSP problem with different values of \(K\).

6.1.3 Experiment 2

In this section we describe an experiment conducted in order to compare between the running times of IDAQ and the Standard Algorithm in scenarios in which IDAQ develops a significantly smaller number of paths than the Standard Algorithm. In other words, we investigated IDAQ’s performance improvement in scenarios where there is considerable knowledge that can be shared between the algorithm’s iterations (in particular, considerably more than in the Waxman topologies of Experiment 1).

We found out that an easy way to control the number of IDAQ developed paths is through the MOWSP’s coefficient vectors. Intuitively, using coefficient vectors that exhibit similarity increases the probability that a path that is optimal with respect to a specific coefficient vector would be optimal also with respect to other coefficient vectors, thus lowering the amount of paths developed by IDAQ.

The experiment runs as follows. We construct a set of
We use the same set of MOGs, but now create the coefficient vectors with a small change: instead of setting each coefficient to a uniformly distributed number between 0.1 to 1.1 (as in Experiment 1), we now set it to a uniformly distributed number between 0.5 to 1.1 (i.e., larger values than in Experiment 1), which results in more similar coefficient vectors. Figure 2 demonstrates that, in such scenarios IDAQ exhibits much better performance than the Standard Algorithm.

Figure 2: Standard Algorithm and IDAQ average running time on MOWSP problem with a different number of coefficient vectors $K$. The coefficient vectors were selected so that iterations are more correlated than in Experiment 1.

### 6.2 Practical Application Experiments

In this section we describe several experiments conducted on MOWSP generated from actual data.

We consider an application that finds several "optimal" routes for a bicycle rider in Manhattan (New York, USA). To generate our MOG, we used data provided by Open Street Map [1]. Each edge’s (=road) objective is determined by the following:

- **C1** - Road distance.
- **C2** - If bicycle road: $C_1$, $C_1$ otherwise.
- **C3** - If road not close to highway: $C_1$, $C_1$ otherwise.
- **C3** - If road not close to buildings: $C_1$, $C_1$ otherwise.

In Table 2 we present seven considered coefficient vectors. In Figure 3 we depict the routes identified by IDAQ on a specific problem (25 coefficient vectors).

Figure 4 demonstrates that, similarly to the randomly generated experiments (Section 6.1), in the current experiments too IDAQ exhibits better performance than the Standard Algorithm.

### Table 2: Example of weigh vectors in Experiment 2, some were selected manually (lines 1-4), other at random (lines 5-7).

|   |   |   |   |
|---|---|---|---|
|   | C1 | C2 | C2 | C3 |
| 1 | 13.06 | 0.17 | 0.13 | 0.21 |
| 2 | 0.13 | 16.98 | 0.13 | 0.21 |
| 3 | 0.13 | 0.17 | 13.17 | 0.21 |
| 4 | 0.13 | 0.17 | 0.13 | 21.11 |
| 5 | 4.28 | 7.45 | 1.4 | 3.25 |
| 6 | 3.89 | 6.12 | 1.5 | 5.45 |
| 7 | 6.23 | 8.27 | 0.61 | 0.45 |

Figure 3: Optimal routes selected by IDAQ on a generated MOWSP problem. Lower part of the figure zooms on areas where routes set apart.
Figure 4: Standard Algorithm and IDAQ average running time on MOWSP problem generated using actual data (by Open Street Map) with different amount of coefficient vectors $K$.

7 Conclusion

We investigated the fundamental problem of routing with multiple objectives. More specifically, we considered the problem of providing several routes that minimize different optimization criteria. While this can be simply achieved by consecutively executing a standard shortest path algorithm, in case of a large number of different optimization criteria this may require an excessively large number of executions, thus incurring a prohibitively large running time.

Our major contribution is a novel efficient algorithm for the considered problem, namely the IDAQ algorithm. Similarly to the standard-approach algorithm, IDAQ iteratively searches for routes, one per optimization criteria; however, instead of executing each iteration independently, it reduces the average running time by skilfully sharing information among the iterations. By doing so, it exploits the similarity among optimal routes with respect to different optimization criteria, so as to improve the performance of the solution scheme.

We showed that both IDAQ and the standard algorithm provide an optimal solution for the considered problem. We then showed that, under reasonable assumptions, IDAQ typically provides considerably lower computational complexity than that of the standard algorithm. We confirmed this finding through several computational experiments on both randomly generated settings, as well as settings that correspond to real-world environments (specifically using data generated from Open Street Map [1]).

Several important issues are left for future work. One is to investigate whether assumptions employed in this paper can be applied to speed up other multi-objective routing solution schemes. For instance, in this study, we presented an efficient algorithm that takes advantage of small Pareto sets to gain significant speed-up in comparison to a standard approach algorithm that consecutively executes a (standard) shortest path algorithm. We employed the assumption that, even though the number of Pareto paths can be non-polynomial, it is often relatively small in practice [20]. It is of interest to investigate whether such an assumption can be applied to speed up other multi-objective routing schemes, such as Resource Constrained Shortest Path algorithms or Fully Polynomial Time Approximation Schemes (FPTAS).

Another interesting direction is to investigate whether similar results can be obtained in a setting where more than one computational unit is allocated to solve the MOWSP problem. A major advantage of the standard approach algorithm over our proposed algorithm is that it is easy to break it into several processes: each iteration (in other words, execution of the Dijkstra algorithm) can be executed independently, hence different computational units can take care of different iterations, simultaneously. It is of interest to investigate whether in the case where more than a single computational unit is available, an efficient algorithm can be established to solve the MOWSP problem while maintaining a significant speedup in comparison to a standard approach algorithm. This is, of course, quite challenging since our approach is based on iterative steps where each iteration depends on the previous one. It might be interesting to consider an iterative algorithm where each iteration executes a distributed version of a shortest-path algorithm, e.g. Bellman-Ford’s [4]. This may allow us to use similar ideas to those implemented in IDAQ while taking advantage of multiple available computation units.

Another important aspect is to compare our approach with sub-optimal algorithms that provide an estimation of the Pareto set. In particular, it is of interest to investigate under which conditions our approach provides better results (in terms of the quality of the solution or running time). The latter has to consider that, unlike Pareto sub-optimal algorithms, our approach depends on pre-selected coefficient vectors.

Last, in some settings, a heuristic scheme can be applied to speed up a shortest path search, while still providing optimal solutions (e.g. the A* algorithm [24] or other informed search algorithms 10). In the case where an admissible heuristic is supplied for each coefficient vector, a standard informed algorithm can be proposed that is based on executions of an informed search algorithm (instead of Dijkstra’s). It is of interest to investigate whether an efficient informed algorithm can be proposed, based on similar ideas to those implemented in IDAQ, while reaching a similar speedup factor in comparison to the standard informed algorithm.

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10I.e., an algorithm guided by some heuristic
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