RANK STABLE INSTANTONS OVER POSITIVE DEFINITE FOUR MANIFOLDS

JOÃO PAULO SANTOS

ABSTRACT. We study the moduli space of rank stable based instantons over a connected sum of $q$ copies of $\mathbb{CP}^2$. For $c_2 = 1$ we give the homotopy type of the moduli space. For $c_2 = 2$ we compute the cohomology of the moduli space.

1. Introduction

In this paper we study moduli spaces of based $SU(r)$ instantons over a four manifold $X$ in the limit when $r \to \infty$. Interest in this rank stable limit goes back to the work of ’t Hooft [tH74]. The homotopy type of this space was computed in [Kir94], [San95] for $X = S^4$, and in [BS97] for $X = \mathbb{CP}^2$. We have

\[ M_{\infty}^k(S^4) \simeq BU(k), \quad M_{\infty}^k(\mathbb{CP}^2) \simeq BU(k) \times BU(k) \]

where $M_k$ denotes the moduli space of charge $k$ instantons. The proofs are based on monad descriptions of the moduli spaces over $S^4$ and $\mathbb{CP}^2$ (see [AHDM78], [Don84], [Buc86], [Kin89]).

In this paper we study the case where $X$ is a connected sum of $q$ copies of $\mathbb{CP}^2$. In [Buc93], [Mat00], it was shown that this moduli space is isomorphic as a real analytic space to the moduli space of holomorphic bundles on a blow up of $\mathbb{CP}^2$ at $q$ points, framed at a line $L_{\infty} \subset \mathbb{CP}^2$. Under this correspondence instantons over $S^4$ are related to holomorphic bundles on $\mathbb{CP}^2$ and instantons on $\mathbb{CP}^2$ are related to holomorphic bundles on the blowup of $\mathbb{CP}^2$ at one point.

The results on this paper can be found in [San02b]. For a different approach see [KM99], [Buc02] where monad descriptions for these moduli spaces were introduced.

1.1. Results. Fix a line $L_{\infty} \subset \mathbb{CP}^2$ and let $x_1, \ldots, x_q \in \mathbb{CP}^2 \setminus L_{\infty}$. Let $X_q$ denote the complex surface obtained by blowing up $\mathbb{CP}^2$ at $x_1, \ldots, x_q$. Let $M_{k}(X_q)$ be the moduli space of equivalence classes of
pairs \((E, \phi)\), where \(E\) is a holomorphic rank \(r\) bundle over \(X_q\) with \(c_1 = 0\) and \(c_2 = k\), holomorphically trivial at \(L_\infty\), and \(\phi : E|_{L_\infty} \to \mathcal{O}_{L_\infty}\) is a holomorphic trivialization.

For a general complex surface \(X\) the moduli space \(\mathcal{M}_r^k(X)\) was defined in [Leh93], [HL95], [Lüb93].

When \(r_2 > r_1\), there is a map \(\mathcal{M}_r^k(X_q) \to \mathcal{M}_{r_2}^k(X_q)\) induced by taking direct sum with a trivial bundle: \(E \mapsto E \oplus \mathcal{O}_{X_q}^{r_2-r_1}\). We define the rank stable moduli space as the direct limit \(\mathcal{M}_r^\infty(X_q) \overset{\text{def}}{=} \lim_{\to} \mathcal{M}_r^k(X_q)\). In this paper we study the special cases \(k = 1, 2\). For \(k = 1\) we obtain the homotopy type of the rank stable moduli space:

**Theorem 1.1.** There is a homotopy equivalence

\[
\mathcal{M}_1^\infty(X_q) \simeq BU(1) \times \left( \bigvee_{i=1}^q BU(1) \right)
\]

Together with the results of [Buc93] and [Mat00], this shows that for a large class of metrics conjecture 1.1 in [BS97] is false.

For \(k = 2\) we obtain the module structure of the integer cohomology of the rank stable moduli space:

**Theorem 1.2.** Let \(K_C \subset \mathbb{Z}[x_1, x_2, x_3, x_4] \cong H^*(BU(1)^\times 4)\) be the ideal generated by the product \(x_1x_2\) and let \(K_A \subset \mathbb{Z}[a_1, k_1, a_2, k_2] \cong H^*(BU(2)^\times 2)\) be the ideal generated by \(k_1, k_2\). Then, as graded modules over \(\mathbb{Z}\), we have an isomorphism

\[
H^*(\mathcal{M}_2^k(X_q)) \cong \mathbb{Z}[a_1, a_2] \oplus K_{A}^{\oplus q} \oplus K_{C}^{\oplus q(q-1)}
\]

Our strategy is to analyze the effect of the blowup on the topology of the moduli space. This way we can relate the moduli over \(X_q\) to the moduli over \(X_1\) and \(X_0\), whose topology is known (equation (1)). The plan of this paper is as follows:

In section 2 we show how the study of the moduli space \(\mathcal{M}_k^k(X_q)\) can be reduced to the case where \(q \leq k\). In section 3 we recall the monad constructions of the moduli spaces for \(q = 0, 1\). In section 4 we prove theorem 1.1. Sections 5, 6 and 7 contain the proof of theorem 1.2. The proof is based on the construction of an open cover of the moduli space, which is carried out in section 5 and studied in detail in sections 5 and 6. In section 7 we use the spectral sequence associated with the open cover (see [Seg68]) to prove theorem 1.2.
2. An open cover of $\mathcal{M}_k^r(X_q)$

In this section we reduce the study of the moduli space $\mathcal{M}_k^r(X_q)$ to the case where $q \leq k$.

Let $I = (i_1, \ldots, i_l)$ be a multi-index and write $|I|$ for the number of indices. Let $\pi_I : X_q \to X_{|I|}$ be the blow up at points $x_j$, $j \notin I$. $\pi_I$ induces a map

$$\pi_I^* : \mathcal{M}_k^r(X_{|I|}) \to \mathcal{M}_k^r(X_q)$$

The objective of this section is to prove

**Theorem 2.1.** $\{ \pi_I^*\mathcal{M}_k^r(X_{|I|}) \}_{|I|=k}$ is an open cover of $\mathcal{M}_k^r(X_q)$. Furthermore

$$\pi_I^*\mathcal{M}_k^r(X_{|I|}) \cap \pi_J^*\mathcal{M}_k^r(X_{|J|}) = \pi_{I \cap J}^*\mathcal{M}_k^r(X_{|I \cap J|})$$

and we have isomorphisms

$$\mathcal{M}_k^r(X_{|I|}) \xrightarrow{\pi_I^*} \pi_I^*\mathcal{M}_k^r(X_{|I|})$$

From this open cover we can build a spectral sequence converging to $H^*(\mathcal{M}_k^r(X_q))$. The case $k = 2$ will be treated in section 7. For the general case see [San02b], section 4.3.

We turn now to the proof of theorem 2.1. We begin by proving the last statement:

**Proposition 2.2.** We have isomorphisms

$$\mathcal{M}_k^r(X_{|I|}) \xrightarrow{\pi_I^*} \pi_I^*\mathcal{M}_k^r(X_{|I|})$$

where $\pi_I^*$ and $\pi_{I*}$ are inverses of each other. We also have

$$\pi_I^*\mathcal{M}_k^r(X_{|I|}) = \{ \mathcal{E} \in \mathcal{M}_k^r(X_q) \mid \mathcal{E}|_L \text{ is trivial for } i \notin I \}$$

**Proof.** From theorem 3.2 in [Gas97] it follows that, if a bundle is trivial on the exceptional divisor then it is also trivial on a neighborhood of the exceptional divisor. Hence, a bundle $\mathcal{E} \to \tilde{X}$ on a blow up $\pi : \tilde{X} \to X$ is trivial on the exceptional divisor if and only if $\tilde{\mathcal{E}} = \pi^*\pi_*\mathcal{E}$. The statement of the proposition follows. $\square$
Proof of theorem 2.1. From proposition 2.2 it follows that
\[ \pi_* \mathcal{M}_k^r(X_{|I|}) \cap \pi_* \mathcal{M}_k^r(X_{|J|}) = \pi_* \mathcal{M}_k^r(X_{|I \cap J|}) \]
To show that \( \mathcal{M}_k^r(X_q) \subset \bigcup_{|I|=k} \pi_* \mathcal{M}_k^r(X_k) \) we only need to show that:

Claim: Let \( \mathcal{E} \in \mathcal{M}_k^r(X_q) \), \( q > k \). Then \( \mathcal{E} \) is trivial in at least \( q - k \) exceptional lines.

We prove this result by induction in \( q \). Assume \( \mathcal{E} \) is not trivial in \( L_1 \). Let \( p : X_q \to X_{q-1} \) be the blow up at \( x_1 \) and let \( \mathcal{E}' = (\pi_* \mathcal{E})^{\vee \vee} \). Then \( c_2(\mathcal{E}') < k \) so we can apply induction. The proof is completed by noting that we cannot have bundles with negative \( c_2 \) by Bogomolov inequality for framed bundles (see [Leh93]).

Finally we have to show that \( \pi_* \mathcal{M}_k^r(X_{|I|}) \) is open. Let \( H \) be an ample divisor. Choose \( N \) such that \( H^1(\mathcal{E}(NH)) = 0 \) for all \( \mathcal{E} \in \mathcal{M}_k(\tilde{X}) \). Then choose \( M \) such that \( \pi_* \mathcal{E}(NH + ML) \) is locally free. Consider the function
\[ h^1 = \dim H^1(\mathcal{E}(NH + ML)) : \mathcal{M}_k(\tilde{X}) \to \mathbb{Z} \]
Then, from the exact sequence
\[ 0 \to \mathcal{E}(NH) \to \mathcal{E}(NH + ML) \to \mathcal{T} \to 0 \]
(\( \mathcal{T} \) has support contained in \( L \)) we get
\[ H^2(\mathcal{E}(NH + ML)) = H^2(\mathcal{T}) = 0 \]
Now notice that
\[ H^0(\mathcal{E}(NH + ML)) \cong H^0(\pi_* \mathcal{E}(NH + ML)) \]
and, since by assumption \( \pi_* \mathcal{E}(NH + ML) \) is locally free and \( \pi_* H \) is ample, for \( N \) large enough we get
\[ H^i(\pi_* \mathcal{E}(NH + ML)) = 0 \text{ for } i > 0 \]
Hence, we get that
\[ h^1 = \chi(\pi_* \mathcal{E}(NH + ML)) - \chi(\mathcal{E}(NH + ML)) \]
From Riemann-Roch theorem it follows that
\[ h^1 = c_2(\mathcal{E}) - c_2(\pi_* \mathcal{E}^{\vee \vee}) + f(N, M, c_1(X)) \]
where \( f \) does not depend on \( \mathcal{E} \). The result then follows from the upper-semicontinuity of \( h^1 \) (see [Har77], chapter III, theorem 12.8). \( \square \)
In this section we sketch the monad description of the spaces $\mathcal{M}_k^r(\mathbb{C}P^2)$ and $\mathcal{M}_k^r(\mathbb{C}P^2)$. We follow [Kin89]. See also [BS00].

Let $L_\infty \subset \mathbb{C}P^2$ be a rational curve and let $L$ be the exceptional divisor. Choose sections $x_1, x_2, x_3$ spanning $H^0(\mathcal{O}(L_\infty))$ and $y_1, y_2$ spanning $H^0(\mathcal{O}(L_\infty - L))$ so that $x_3$ vanishes on $L_\infty$ and $x_1y_1 + x_2y_2$ spans the kernel of

$$H^0(\mathcal{O}(L_\infty)) \otimes H^0(\mathcal{O}(L_\infty - L)) \rightarrow H^0(\mathcal{O}(2L_\infty - L))$$

3.1. The moduli space over $\mathbb{C}P^2$. Let $W$ be a $k$-dimensional vector space. Let $R$ be the space of 4-tuples $m = (a_1, a_2, b, c)$ with $a_i \in \text{End}(W)$, $b \in \text{Hom}(\mathbb{C}^r, W)$, $c \in \text{Hom}(W, \mathbb{C}^r)$, obeying the integrability condition $[a_1, a_2] + bc = 0$. For each $m = (a_1, a_2, b, c) \in R$ we define maps $A_m, B_m$

$$W(-L_\infty) \xrightarrow{A_m} W \oplus \mathbb{C}^n \xrightarrow{B_m} W(L_\infty)$$

by

$$A_m = \begin{bmatrix} x_1 - a_1x_3 \\ x_2 - a_2x_3 \\ cx_3 \end{bmatrix}, \quad B_m = \begin{bmatrix} -x_2 + a_2x_3 & x_1 - a_1x_3 & bx_3 \end{bmatrix}$$

Then $B_mA_m = 0$. The assignment $m \mapsto \mathcal{E}_m = \text{Ker } B_m/\text{Im } A_m$ induces a map $f : R \rightarrow \mathcal{M}_k^r(\mathbb{C}P^2)$.

$m$ is called non degenerate if $A_m, B_m$ have maximal rank at every point in $\mathbb{C}P^2$.

**Theorem 3.1.** $f$ induces an isomorphism between the quotient of the space of non degenerate points in $R$ by the action of $\text{Gl}(W)$:

$$g \cdot (a_1, a_2, b, c) = (g^{-1}a_1g, g^{-1}a_2g, g^{-1}b, cg)$$

and the moduli space $\mathcal{M}_k^r(\mathbb{C}P^2)$.

For a proof see [Don84], proposition 1.

**Theorem 3.2.** The algebraic quotient $R/\text{Gl}(W)$ is isomorphic to the Donaldson-Uhlenbeck completion of the moduli space of instantons over $S^4$.

For a proof see [DK90], sections 3.3, 3.4, 3.4.4.
For future reference we sketch here how the map from $\mathcal{R}/\text{Gl}(W)$ to the Donaldson-Uhlenbeck completion of the moduli space of instantons is constructed (see [Kin89] for details):

Let $m = (a_1, a_2, b, c) \in \mathcal{R}$. A subspace $W' \subset W$ is called $b$-special with respect to $m$ if

\begin{equation}
(3) \quad a_i(W') \subset W' \quad (i = 1, 2) \text{ and } \text{Im } b \subset W'.
\end{equation}

A subspace $W' \subset W$ is called $c$-special with respect to $m$ if

\begin{equation}
(4) \quad a_i(W') \subset W' \quad (i = 1, 2) \text{ and } W' \subset \text{Ker } c.
\end{equation}

$m$ is called completely reducible if for every $W' \subset W$ which is $b$-special or $c$-special, there is a complement $W'' \subset W$ which is $c$-special or $b$-special respectively.

**Proposition 3.3.** Let $m = (a_1, a_2, b, c) \in \mathcal{R}$.

1. $m$ is non-degenerate if and only if the only $b$-special subspace is $W$ and the only $c$-special subspace is $0$;
2. For every $m$, the orbit of $m$ under $\text{Gl}(W)$ contains in its closure a canonical completely reducible orbit and completely reducible orbits have disjoint closures;
3. If $m$ is completely reducible then, after acting with some $g \in \text{Gl}(W)$ we can write

\begin{align*}
a_i &= \begin{bmatrix} a_i^{\text{red}} & 0 \\
0 & a_i^\Delta \end{bmatrix}, \quad b = \begin{bmatrix} b^{\text{red}} \\
0 \end{bmatrix}, \quad c = \begin{bmatrix} c^{\text{red}} & 0 \end{bmatrix}
\end{align*}

where $(a_1^{\text{red}}, a_2^{\text{red}}, b^{\text{red}}, c^{\text{red}})$ is non-degenerate and the matrices $a_1^\Delta, a_2^\Delta$ can be simultaneously diagonalized. Such a configuration is equivalent to the following data:

- An irreducible integrable configuration $(a_1^{\text{red}}, a_2^{\text{red}}, b^{\text{red}}, c^{\text{red}})$ corresponding to a bundle with $c_2 = l \leq k$;
- $k - l$ points in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus L_\infty$ given by the eigenvalue pairs of $a_1^\Delta, a_2^\Delta$

This is precisely the Donaldson-Uhlenbeck completion.

3.2. **The moduli space over $\mathbb{CP}^2$.** Let $\tilde{\mathcal{R}}$ be the space of 5-tuples $\tilde{m} = (a_1, a_2, d, b, c)$ where $a_i \in \text{Hom}(W, V)$, $d \in \text{Hom}(W, W)$, $b \in \text{Hom}(\mathbb{C}^r, V)$, $c \in \text{Hom}(W, \mathbb{C}^r)$, such that $a_1(W) + a_2(W) + b(\mathbb{C}^r) = V$, obeying the integrability condition $a_1 da_2 - a_2 da_1 + bc = 0$. For each
\[ \tilde{m} = (a_1, a_2, d, b, c) \in \tilde{R} \] we define maps \( A_{\tilde{m}}, B_{\tilde{m}} \)

\[
W(-L_\infty) \oplus V(L - L_\infty) \xrightarrow{A_{\tilde{m}}} (V \oplus W) \oplus \mathbb{C}^n \xrightarrow{B_{\tilde{m}}} \]

\[
\rightarrow V(L_\infty) \oplus W(L_\infty - L)
\]

by

\[
A_{\tilde{m}} = \begin{bmatrix}
    a_1x_3 & -y_2 \\
    x_1 - da_1x_3 & 0 \\
    a_2x_3 & y_1 \\
    x_2 - da_2x_3 & 0 \\
    cx_3 & 0
\end{bmatrix}, \quad B_{\tilde{m}} = \begin{bmatrix}
    x_2 & a_2x_3 & -x_1 & -a_1x_3 & bx_3 \\
    dy_1 & y_1 & dy_2 & y_2 & 0
\end{bmatrix}
\]

Then \( B_{\tilde{m}}A_{\tilde{m}} = 0 \). The assignment \( \tilde{m} \mapsto E_{\tilde{m}} = \text{Ker} B_{\tilde{m}}/\text{Im} A_{\tilde{m}} \) induces a map \( \tilde{f} : \tilde{R} \rightarrow \overline{\mathcal{M}_k^r(\mathbb{CP}^2)} \).

A point \( \tilde{m} \in \tilde{R} \) is called non-degenerate if \( A_{\tilde{m}} \) and \( B_{\tilde{m}} \) have maximal rank at every point in \( \mathbb{CP}^2 \).

**Theorem 3.4.** The map \( \tilde{f} \) induces an isomorphism between the quotient of the space of non degenerate points in \( \tilde{R} \) by the action of \( Gl(V) \times Gl(W) \):

\[ (g_0, g_1) \cdot (a_1, a_2, b, c, d) = (g_0^{-1}a_1g_1, g_0^{-1}a_2g_1, g_0^{-1}b, cg_1, g_1^{-1}dg_0) \]

and the moduli space \( \overline{\mathcal{M}_k^r(\mathbb{CP}^2)} \).

See [Kin89] for a proof.

Consider the algebraic quotient \( \tilde{R}/Gl(V) \times Gl(W) \). This space is a completion of the moduli space \( \mathcal{M}_k^r(\mathbb{CP}^2) \). We proceed to give an interpretation of the points in this completion in terms of the Donaldson-Uhlenbeck completion. See [Kin89] for details.

Let \( \tilde{m} = (a_1, a_2, d, b, c) \). Let \( V' \subset V \) and \( W' \subset W \) and assume \( \text{dim} V' = \text{dim} W' \). The pair \( (V', W') \) is called \( b \)-special with respect to \( \tilde{m} \) if

\[ a_i(W') \subset V' \quad (i = 1, 2), \quad d(V') \subset W' \quad \text{and} \quad \text{Im} b \subset V' \]

The pair \( (V', W') \) is called \( c \)-special with respect to \( \tilde{m} \) if

\[ a_i(W'') \subset V' \quad (i = 1, 2), \quad d(V'') \subset W' \quad \text{and} \quad W' \subset \text{Ker} c \]

\( \tilde{m} \) is called completely reducible if for every pair \( (V', W') \) which is either \( b \)-special or \( c \)-special, there are complements \( V'', W'' \) to \( V' \) and \( W' \) such that the pair \( (V'', W'') \) is \( c \)-special or \( b \)-special respectively.

**Proposition 3.5.** Let \( \tilde{m} = (a_1, a_2, d, b, c) \in \tilde{R} \).
\( \tilde{m} \) is non-degenerate if and only if the only \( b \)-special pair is \((V, W)\) and the only \( c \)-special pair is \((0, 0)\);

(2) For every \( \tilde{m} \), the orbit of \( \tilde{m} \) under \( \text{Gl}(V) \times \text{Gl}(W) \) contains in its closure a canonical completely reducible orbit and completely reducible orbits have disjoint closures;

(3) If \( \tilde{m} \) is completely reducible then, after acting with some \((g_0, g_1) \in \text{Gl}(V) \times \text{Gl}(W)\), we can write

\[
a_i = \begin{bmatrix} a_{i}^{\text{red}} & 0 \\ 0 & a_{i}^{\Delta} \end{bmatrix}, \quad d = \begin{bmatrix} d^{\text{red}} & 0 \\ 0 & d^{\Delta} \end{bmatrix}, b = \begin{bmatrix} b^{\text{red}} \\ 0 \end{bmatrix}, \quad c = [c^{\text{red}} \ 0]
\]

where \((a_{1}^{\text{red}}, a_{2}^{\text{red}}, d^{\text{red}}, b^{\text{red}}, c^{\text{red}})\) is non-degenerate effective and integrable and the matrices \(a_{1}^{\Delta}, a_{2}^{\Delta}, d^{\Delta}\) can be simultaneously diagonalized. Such a configuration is equivalent to the following data:

- An irreducible configuration \((a_{1}^{\text{red}}, a_{2}^{\text{red}}, d, b^{\text{red}}, c^{\text{red}})\) associated to a bundle with \( c_2 = l \leq k \);
- \( k - l \) points in the blow up \( \mathbb{C}^2 \) of \( \mathbb{C}^2 \) at the origin. This points are determined as follows: \(a_{1}^{\Delta}, a_{2}^{\Delta}, d^{\Delta}\) determine \( k - l \) unique points \((\lambda_{1}^{l}, \lambda_{2}^{l}), [\mu_{1}^{l}, \mu_{2}^{l}] \in \mathbb{C}^2\) corresponding to vectors \(v_{1}, \ldots, v_{k-1}\) such that \(da_{i}v^{r} = \lambda_{i}^{r}v^{r}\) \((\lambda_{1}, \lambda_{2} \) are the eigenvalue pairs of \(da_{1}, da_{2}\)) and \((\mu_{1}^{l}a_{1} + \mu_{2}^{l}a_{2})v^{r} = 0\).

3.3. Direct image. In this section we gather some results concerning the direct image map \( \pi_{\ast} \) induced by the blowup map \( \pi : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2 \).

**Proposition 3.6.** Let \( \pi_{\#} : \mathcal{R} \rightarrow \mathcal{R} \) be given by \( \pi_{\#}(a_{1}, a_{2}, d, b, c) = (da_{1}, da_{2}, db, c) \). Let \( \tilde{m} \in \tilde{\mathcal{R}}, m = \pi_{\#}\tilde{m} \). Then \( \mathcal{E}_{\tilde{m}}|_{\tilde{\mathcal{R}} \setminus \{[0,0,1]\}} \) is isomorphic to \( \mathcal{E}_{m}|_{\mathcal{R} \setminus \{[0,0,1]\}} \).

For the proof see [San02a], proposition 5.6.

**Proposition 3.7.** Let \( S_{0}\mathcal{M}_{1}(\mathbb{C}P^{2}) = \{ (\mathcal{E}, \phi) \in \mathcal{M}_{1}(\mathbb{C}P^{2}) : (\pi_{\ast}\mathcal{E})^{\vee} = \mathcal{O}_{\mathbb{C}P^{2}} \} \). Then

1. \( m \in S_{0}\mathcal{M}_{1}(\mathbb{C}P^{2}) \) if and only if \( m \) is of the form \((a_{1}, a_{2}, 0, b, c)\).
2. The inclusion \( S_{0}\mathcal{M}_{1} \rightarrow \mathcal{M}_{1} \) is a homotopy equivalence.

**Proof.** First we observe that \( \mathcal{M}_{1}(\mathbb{C}P^{2}) = S_{0}\mathcal{M}_{1}(\mathbb{C}P^{2}) \cup \pi_{\ast}^{\#}\mathcal{M}_{1}(\mathbb{C}P^{2}) \).

Now \( m \in \pi_{\ast}^{\#}\mathcal{M}_{1}(\mathbb{C}P^{2}) \) if and only if \( d \) is an isomorphism (see [Kin89]). The first statement follows. The second statement follows easily from the first: just consider the homotopy \((a_{1}, a_{2}, d, b, c) \mapsto (a_{1}, a_{2}, td, b, c) \). \( \square \)
Proposition 3.8. Let \( x = [x_1, x_2, 1] \in \mathbb{CP}^2 \) and let \( \pi_x : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2 \) be the blow up at \( x \). Then the map \( \pi^*_x : \mathcal{M}_k(\mathbb{CP}^2) \to \mathcal{M}_k(\tilde{\mathbb{CP}}^2) \) is given by \([a_1, a_2, b, c] \mapsto [a_1 - x_1 \mathbb{1}, a_2 - x_2 \mathbb{1}, 1, b, c]\)

Proof. For \( x = [0, 0, 1] \) see \cite{BS00}. For the general case consider the translation \([w_1, w_2, w_3] \mapsto [w_1 - x_1 w_3, w_2 - x_2 w_3, w_3]\). This induces a map \( \tau : \mathcal{M}_k(X_0) \to \mathcal{M}_k(X_0) \) given by \([a_1, a_2, b, c] \mapsto [a_1 - x_1 \mathbb{1}, a_2 - x_2 \mathbb{1}, b, c]\)

The result follows. \( \square \)

4. The charge one moduli space

The objective of this section is to prove theorem \ref{thm:main}

Theorem 4.1. There is a homotopy equivalence

\[
\mathcal{M}^\infty_1(X_q) \simeq BU(1) \times \left( \bigvee_{i=1}^{q} BU(1) \right)
\]

From theorem \ref{thm:isos} it follows that

\[
\mathcal{M}^*_i(X_q) = \bigcup_{l=1}^{q} \pi^*_l \mathcal{M}^*_i(X_1)
\]

and, for any \( i \neq j \),

\[
\pi^*_i \mathcal{M}^*_i(X_1) \cap \pi^*_j \mathcal{M}^*_j(X_1) = \pi^*_0 \mathcal{M}^*_i(X_0)
\]

We begin by studying the maps \( \pi^*_0 \mathcal{M}_1(X_0) \to \pi^*_i \mathcal{M}_1(X_1) \).

Lemma 4.2. Let \( \iota_1 : \mathbb{CP}^r \to \mathbb{CP}^r \times \mathbb{CP}^r \) be the inclusion into the first factor: \( \iota_1([u]) = ([u], \ast) \), where \( \ast \) denotes the base point. Then there are homotopy equivalences \( h_0 : \mathbb{CP}^\infty \to \mathcal{M}^\infty_1(X_0) \) and \( h_1 : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathcal{M}^\infty_1(X_1) \) such that the following diagram

\[
\begin{array}{ccc}
\mathcal{M}^\infty_1(X_0) & \xrightarrow{\pi^*_0} & \mathcal{M}^\infty_1(X_1) \\
\downarrow_{\pi^*_0} & & \downarrow_{\pi^*_1} \\
\mathbb{CP}^\infty & \xrightarrow{\iota_1} & \mathbb{CP}^\infty \times \mathbb{CP}^\infty
\end{array}
\]

is homotopy commutative.
Proof. We will use the monad description of $\mathcal{M}_1(X_1), \mathcal{M}_1(X_0)$. We define the following maps:

$$
P_0 : \mathcal{M}_1^r(X_0) \to \mathbb{C}P^r \quad P_0 : [a_1, a_2, b, c] \to [b]
P_1 : \mathcal{M}_1^r(X_1) \to \mathbb{C}P^r \times \mathbb{C}P^r \quad P_1 : [a_1, a_2, d, b, c] \to \left(b, \left[\frac{c^2}{||c||^2}\right]\right)
$$

$$
\Delta : \mathbb{C}P^r \to \mathbb{C}P^r \times \mathbb{C}P^r \quad \Delta : [u] \to ([u], [u])
f : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty \quad f : (x, y) \mapsto (x, xy^{-1})
$$

where to define $f$ we observe that $\mathbb{C}P^\infty = BU(1)$ is homotopic to the free abelian group on $U(1)$. Now observe that the diagram

\[
\begin{array}{ccc}
\pi^*_0 \mathcal{M}_1^\infty(X_0) & \xrightarrow{\pi^*_0} & \pi^*_1 \mathcal{M}_1^\infty(X_1) \\
\downarrow \pi^*_0 \cong & & \downarrow \pi^*_1 \cong \\
\mathcal{M}_1^\infty(X_0) & \xrightarrow{\pi^*} & \mathcal{M}_1^\infty(X_1) \\
\downarrow p_0 & & \downarrow p_1 \\
\mathbb{C}P^\infty & \xrightarrow{\Delta} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \\
\downarrow \iota_1 & & \downarrow f \\
\mathbb{C}P^\infty \times \mathbb{C}P^\infty & & \\
\end{array}
\]

is homotopy commutative and the maps $p_0, p_1, f, \pi^*_0, \pi^*_1$ are homotopy equivalences. The statement of the lemma then follows by writing $h_0 = p_0^{-1}$ and $h_1 = p_1^{-1}f^{-1}$, where $p_0^{-1}, p_1^{-1}, f^{-1}$ are the homotopy inverses. 

We are ready to prove theorem 4.1.

Proof. Let $C$ be the cone on $q$ points $v_1, \ldots, v_q$. Let

$$
M = \left(\prod_{i=1}^{q} BU(1) \times BU(1) \times \{v_i\}\right) \prod \left( BU(1) \times C \right)
$$

(1) First we show that $M$ is homotopically equivalent to $\mathcal{M}_1(X_q)$. Denote the points in $C$ by

$$
[t, v_i] \in C = \left[0, 1\right] \times \bigcup_i \{v_i\} \left(0, v_i\right) \sim \left(0, v_j\right) \sim *
$$
Define a map
\[ \zeta : \left( \prod_{i=1}^{q} BU(1) \times BU(1) \times \{ v_i \} \right) \coprod BU(1) \times C \rightarrow M_1(X_q) \]
as follows:
\[ BU(1) \times BU(1) \times \{ v_i \} \ni ([u_1], [u_2], v_i) \mapsto \pi_{q_0}^* h_0([u]) \text{ for } t < \frac{1}{3} \]
\[ BU(1) \times C \ni ([u], [t, v_i]) \mapsto \pi_{q_1}^* h_1([u]) \text{ for } t > \frac{2}{3} \]
For \( \frac{1}{3} < t < \frac{2}{3} \) use the homotopy between \( \pi_{q_0}^* h_0 \) and \( p_i^* h_{1i} \) from lemma 4.2.
\[ \zeta \text{ descends to the quotient to give a map } \zeta : M \rightarrow M^\infty_1(X_q). \]
We want to apply Whitehead theorem to show \( \zeta \) is a homotopy equivalence. The van Kampen theorem implies both \( M \) and \( M^\infty_1(X_q) \) are simply connected hence we only have to show \( \zeta \) is an isomorphism in homology groups. We prove it by induction in \( q' = 1, \ldots, q \).

We apply the five lemma to the Meyer-Vietoris long exact sequence corresponding to open neighborhoods of the sets
\[ \pi_{q'+1}^* M^\infty_1(X_1), \pi_{q,\ldots,q'}^* M^\infty_1(X_{q'}) \subset M^\infty_1(X_q) \]
\[ BU(1) \times BU(1) \times \{ v_i \}, \bigcup_{i=1}^{q'} BU(1) \times BU(1) \times \{ v_i \} \subset M \]
using the fact that the restrictions
\[ \zeta : BU(1) \times BU(1) \times \{ v_i \} \rightarrow \pi_{q}^* M^\infty_1(X_1) \]
\[ \zeta : BU(1) \times C \rightarrow \pi_{q_0}^* M^\infty_1(X_0) \]
are homotopy equivalences. It follows that \( \zeta \) induces isomorphisms in all homology groups.

(2) To conclude the proof we only have to show that \( M \) is homotopically equivalent to
\[ BU(1) \times \left( \bigvee_{i=1}^{q} BU(1) \right) = \prod_{i=1}^{q} BU(1) \times BU(1) \times \{ v_i \} \]
where \( * \in BU(1) \) is the base point. Define an open cover of \( BU(1) \times (\bigvee_i BU(1)) \) by \( U_i = BU(1) \times BU(1) \times \{ v_i \} \). Then the
claim is a special case of proposition 4.1 in [Seg68]. The result can also be proved as in step 1.

\[ \square \]

5. An open cover of \( \mathcal{M}_2^\infty (X_2) \)

The objective of this section is to describe an open cover of \( \mathcal{M}_2^\infty (X_2) \). We will adopt, in this section and the next, the following notation: Denote the blow up points by \( x_L, x_R \in X_0 \). Let \( \pi : X_2 \to X_0 \) be the blow up map at \( x_L, x_R \). By abuse of notation we will denote by \( \pi_L \) the maps \( X_2 \to X_1 \) and \( X_1 \to X_0 \) corresponding to the blow up at \( x_L \) and in the same way \( \pi_R \) will denote the blow up at \( x_R \). We have the diagram

\[
\begin{array}{ccc}
X_2 & \xrightarrow{\pi_L} & X_{1R} \\
\downarrow & & \downarrow \\
X_{1L} & \xleftarrow{\pi_R} & X_1 \\
\downarrow & & \downarrow \\
X_0 & \xleftarrow{\pi_L} & X_0 \\
\end{array}
\]

of blow up maps where \( X_{1L} \cong X_{1R} \cong X_1 \). Denote by \( L_L \) and \( L_R \) the exceptional divisors above \( x_L \) and \( x_R \) respectively. Again, by abuse of notation we identify \( L_L \subset X_2 \) with \( L_L \subset X_1 \) and the same for \( L_R \).

Write \( x_L = [x_{1L}, x_{2L}, 1] \), \( x_R = [x_{1R}, x_{2R}, 1] \), \( x_L, x_R \in X_0 = \mathbb{CP}^2 \). Since \( x_L \neq x_R \) we may assume without loss of generality that \( x_{1L} \neq x_{1R} \).

Let \( z_i = x_{iR} - x_{iL} \), \( z_1, z_2 \) determine a point \( ([z_1, z_2, 1], [z_1, z_2]) \in X_1 \setminus L_\infty = \widetilde{\mathbb{CP}^2} \setminus L_\infty \subset \mathbb{CP}^2 \times \mathbb{CP}^1 \).

We are ready to state the main theorem of this section:

**Theorem 5.1.** Let

\[
A_L = \pi_R^* \mathcal{M}_2(X_{1L}) = \{ \mathcal{E} \in \mathcal{M}_2(X_2) : \mathcal{E}|_{L_R} \text{ is trivial} \}
\]

\[
A_R = \pi_L^* \mathcal{M}_2(X_{1R}) = \{ \mathcal{E} \in \mathcal{M}_2(X_2) : \mathcal{E}|_{L_L} \text{ is trivial} \}
\]

and let \( C = \mathcal{M}_2(X_2) \setminus (A_L \cup A_R) \).

Let \( N_L \subset \mathcal{M}_2(X_{1L}) \) be the set of non-degenerate configurations \( m = (a_1, a_2, d, b, c) \) such that the eigenvalues of \( da_1 \) (equal to the eigenvalues of \( a_1 d \)) are in a \( \delta \) neighborhood of \( 0, z_1 \). In a similar way define \( N_R \subset \mathcal{M}_2(X_{1R}) \). Let \( N_2 = \pi_R^* N_L \cup \pi_L^* N_R \cup C \).
Then \( \{A_L, A_R, N_2\} \) is an open cover of \( \mathcal{M}^\infty_2(X_2) \). There are homotopy equivalences

1. \( A_L \simeq A_R \simeq BU(2) \times BU(2) \)
2. \( C \simeq BU(1) \times BU(1) \times BU(1) \times BU(1) \)
3. \( A_L \cap A_R \simeq BU(2) \)
4. \( A_L \cap N_2 \simeq A_R \cap N_2 \simeq N_2 \simeq BU(1) \times BU(1) \times BU(1) \)
5. \( A_L \cap A_R \cap N_2 \simeq BU(1) \times BU(1) \times BU(1) \times BU(1) \)
6. \( N_2 \simeq C \)

From this open cover we get, in a standard way (see [Seg68]), a spectral sequence:

**Corollary 5.2.** There is a spectral sequence converging to the cohomology of \( \mathcal{M}^\infty_2(X_2) \) with \( E_1 \) term

\[
E_1^{0,n} = H^n(A_L) \oplus H^n(A_R) \oplus H^n(N_2)
\]

\[
E_1^{1,n} = H^n(A_L \cap A_R) \oplus H^n(A_L \cap N_2) \oplus H^n(A_R \cap N_2)
\]

\[
E_1^{2,n} = H^n(A_L \cap A_R \cap N_2)
\]

In the next section we will study the \( d_1 \) differential of this spectral sequence.

We turn now to the proof of theorem 5.1. We will delay the proof that \( N_2 \) is open and begin by proving the homotopy equivalences (1), (2) and (3):

**Proposition 5.3.** \( A_L, A_R \) are open sets,

\[
C = \{[E, \phi] \in \mathcal{M}_2(X_2) : c_2((\pi_i E)^{\vee \vee}) = 1, i = L, R\}
\]

and the following maps are isomorphisms:

\[
\pi_R^* : \mathcal{M}_2(X_{1L}) \to A_L \subset \mathcal{M}_2(X_2)
\]

\[
\pi_L^* : \mathcal{M}_2(X_{1R}) \to A_R \subset \mathcal{M}_2(X_2)
\]

\[
\pi_{R*}^{\vee \vee} \times \pi_{L*}^{\vee \vee} : C \to S_0\mathcal{M}_1(X_{1L}) \times S_0\mathcal{M}_1(X_{1R})
\]

\[
\pi^* : \mathcal{M}_2(X_0) \to A_L \cap A_R \subset \mathcal{M}_2(X_2)
\]

where \( \pi_{i*}^{\vee \vee}(E) \overset{\text{def}}{=} (\pi_{i*}E)^{\vee \vee} \).

**Proof.** The isomorphisms for \( A_L, A_R, A_L \cap A_R \) follows from theorem 2.1. That theorem also implies \( A_L, A_R \) are open.

It remains to look at the map \( \pi_{R*}^{\vee \vee} \times \pi_{L*}^{\vee \vee} : C \to S_0\mathcal{M}_1(X_{1L}) \times S_0\mathcal{M}_1(X_{1R}) \).

The continuity of this map was proved in proposition 3.1 in [San02a].
We will construct an inverse for $\pi^\vee \otimes \pi^\vee$. Let $(\mathcal{E}_L, \phi_L) \in S_0\mathcal{M}_1(X_{1L})$, $(\mathcal{E}_R, \phi_R) \in S_0\mathcal{M}_1(X_{1R})$. Hartog’s theorem implies there are unique extensions of $\phi_L, \phi_R$ to maps

$$\phi_L : \mathcal{E}_L|_{X_0 \setminus \{x_L\}} \to \mathcal{O}_{X_0 \setminus \{x_L\}}, \quad \phi_R : \mathcal{E}_R|_{X_0 \setminus \{x_R\}} \to \mathcal{O}_{X_0 \setminus \{x_R\}}$$

These maps induce an isomorphism $\mathcal{E}_L \cong \mathcal{E}_R$ over $X_0 \setminus \{x_L, x_R\}$ which we use to glue $\mathcal{E}_L, \mathcal{E}_R$ and obtain a bundle $\mathcal{E} \to X_2$. The continuity of this map was proved in proposition 3.3 in [San02a]. This concludes the proof. 

We observe the following identity:

**Proposition 5.4.** Let $\tau : \mathcal{M}_k(X_0) \to \mathcal{M}_k(X_0)$ be defined by

$$\tau(a_1, a_2, b, c) = (a_1 - x_{1L}1, a_2 - x_{2L}1, b, c)$$

Let $m_1, m_2 \in \mathcal{M}_1(X_0)$. Then $\pi^\vee_1(m_1 \boxplus_0 m_2) = \pi^\vee_1 m_1 \boxplus L \tau(m_2)$.

**Proof.** It follows easily from proposition 3.8. 

Before we continue we need the lemma

**Lemma 5.5.** Let $m \in \mathcal{M}_2(X_1)$ and let $(a_1, a_2, d, b, c)$ be the configuration associated to $m$. The following are equivalent:

1. $\mathcal{E}_m$ is in the image of $\pi_R^* : C \to \mathcal{M}_2(X_1)$;
2. $cd = 0$ and the eigenvalues of $da_i$ (equal to the ones of $a_i d$) are 0 and $z_i$;
3. After a change of basis we can write

$$a_1 = \begin{bmatrix} a'_1 & 0 \\ 0 & z_1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} a'_2 & \frac{b' c'}{z_1} \\ \frac{b'' c'}{z_1} & z_2 \end{bmatrix}, \quad d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} b' \\ b'' \end{bmatrix}, \quad c = \begin{bmatrix} c' & c'' \end{bmatrix}$$

with $c'' b'' = 0$.

**Proof.** We will show that $1 \Rightarrow 2, 2 \Rightarrow 3$ and $3 \Rightarrow 1$.

(1 $\Rightarrow$ 2) Suppose $\mathcal{E}_m = \pi_R^* \hat{\mathcal{E}}, \hat{\mathcal{E}} \in C$. Then, by proposition 5.3, $\mathcal{E}^\vee \otimes \mathcal{E}^\vee \in S_0\mathcal{M}_1(X_{1L})$ and $\mathcal{E}$ is not locally free at the blow up point $x_R$. So, from proposition 3.7, $\mathcal{E}^\vee$ corresponds to a configuration of the form $[a'_1, a'_2, 0, b', c']$.

Since $m$ is degenerate, by proposition 3.5 after a change of basis it can be written in one of two forms, corresponding to the two types of special pairs:
1 \( (b\text{-special}) \):

\[
\begin{pmatrix}
  a'_1 & * \\
  0 & a''_1
\end{pmatrix},
\begin{pmatrix}
  d' & * \\
  0 & d''
\end{pmatrix},
\begin{pmatrix}
  b' \\
  0
\end{pmatrix},
\begin{pmatrix}
  c' \\
  c''
\end{pmatrix}
\]

in which case the configuration is equivalent to the completely reducible configuration (see proposition 3.5)

\[
(a'_1, a'_2, d', b', c') \oplus (a''_1, a''_2, d'', 0, c'')
\]

corresponding to an ideal bundle with singularity at \((a''_1 d'', a''_2 d'')\) and charge one bundle given by \((a'_1, a'_2, d', b', c')\). So we should have \(d' = 0\) and \(a''_1 d'' = z_i\).

2 \( (c\text{-special}) \):

\[
\begin{pmatrix}
  a'_1 & 0 \\
  * & a''_1
\end{pmatrix},
\begin{pmatrix}
  d' & 0 \\
  * & d''
\end{pmatrix},
\begin{pmatrix}
  b' \\
  b''
\end{pmatrix},
\begin{pmatrix}
  c' \\
  0
\end{pmatrix}
\]

in which case the configuration is equivalent to the completely reducible configuration

\[
(a'_1, a'_2, d', b', c') \oplus (a''_1, a''_2, d'', b'', 0)
\]

corresponding to an ideal bundle with singularity at \((a''_1 d'', a''_2 d'')\) and charge one bundle given by \((a'_1, a'_2, d', b', c')\). So we should have \(d' = 0\) and \(a''_1 d'' = z_i\).

In both cases the eigenvalues of \(da_i\) are \(0, z_i\) and \(cdb = 0\).

\(2 \Rightarrow 3\) Now assume the configuration \((a_1, a_2, d, b, c)\) satisfies 2. Fix a basis of eigenvectors \(v_0, v_1 \in V\) of \(a_1 d\) and \(w_0, w_1 \in W\) of \(da_1\) with \(v_0, w_0\) corresponding to the eigenvalue \(0\). Normalize \(v_1, w_1\) so that \(dv_1 = w_1\). Then

\[
\begin{align*}
  a_1 &= \begin{pmatrix} a'_1 & 0 \\ 0 & a''_1 \end{pmatrix}, \quad a_2 &= \begin{pmatrix} a'_1 & b' c'' \\ b'' c' & a''_1 \end{pmatrix} \\
  d &= \begin{pmatrix} d' & 0 \\ 0 & 1 \end{pmatrix}, \quad b &= \begin{pmatrix} b' \\ b'' \end{pmatrix}, \quad c &= \begin{pmatrix} c' \\ c'' \end{pmatrix}
\end{align*}
\]

From \(cdb = 0\) we get \((b' c')(b'' c') = 0\). If \(b' c'' = 0\) then \(a_2\) is lower triangular. If \(b'' c' = 0\) then \(a_2\) is upper triangular. In both cases the diagonal entries of \(a_2 d\) are its eigenvalues. Hence, the condition about the eigenvalues of \(a_1 d\) and \(a_2 d\) yields the equations

\[
a'_1 d' = a'_2 d' = 0, \quad a''_1 = z_1, \quad a''_2 = z_2
\]

Since \(a_1(W) + a_2(W) + b(\mathbb{C}^r) = V\) we must have \(d' = 0\).
(3 ⇒ 1) Let $m = [a_1, a_2, d, b, c]$ be a configuration satisfying 3. $c''b'' = 0$ implies either $c'' = 0$ or $b'' = 0$. It follows that the pair $(\text{Span}\{(0,1), \text{Span}\{(0,1)\})$ is a special pair hence the configuration is degenerate. Now, from proposition 3.5 it follows that $m$ is equivalent to the completely reducible configuration

$$m' \oplus m'' = (a'_1, a'_2, 0, b', c') \oplus (z_1, z_2, 0, 0)$$

Notice that $(a'_1, a'_2, 0, b', c') \in S_0\mathfrak{M}_1(X_{1L})$. Then, from proposition 5.6 there is $\tilde{m} \in C$ such that $\pi_R\tilde{m}^\vee = m'$. Then, from the characterization of points in the completion it follows that $\pi_R\tilde{m} = m$.

The homotopy equivalences (4) and (5) are a direct consequence of the proposition

**Proposition 5.6.** Let

$$N_z = \{ (a_{1z}, a_{2z}, b_z, c_z) \in \mathfrak{M}_1(X_0) \mid |a_{1z} - z| < \delta \}$$

$$N' = \{ (a'_1, a'_2, d', b', c') \in \mathfrak{M}_1(X_1) \mid |d'a'_1| < \delta \}$$

Let $N_0 \subset \mathfrak{M}_2(X_0)$ be the subset of points $(a_1, a_2, b, c)$ with the eigenvalues of $a_1$ lying in $\delta$ neighborhoods of $x_L$ and $x_R$. Consider the maps

$$\boxtimes_0 : N_{x_L} \times N_{x_R} \to N_0$$

defined by

$$[a_{1L}, a_{2L}, b_L, c_L] \boxtimes_0 [a_{1R}, a_{2R}, b_R, c_R] = [a_1, a_2, b, c]$$

and

$$\boxtimes_L : N' \times N_{\tilde{z}_1} \to N_L$$

defined by

$$[a'_1, a'_2, d', b', c'] \boxtimes_L [a''_1, a''_2, b'', c''] = [a_1, a_2, d, b, c]$$

(9)

$$a_1 = \begin{bmatrix} a'_1 \\ 0 \\ a''_1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} a'_2 \\ \frac{b'_2}{a'_1 - d} \\ \frac{b''_2}{a''_1 - d} \end{bmatrix}, \quad b = \begin{bmatrix} b_L \\ b_R \end{bmatrix}, \quad c = [c_L \ c_R]$$

Then

1. The maps $\boxtimes_0, \boxtimes_L$ are homeomorphisms;
2. The inclusions $N_z \to \mathfrak{M}_1(X_0), \ N' \to \mathfrak{M}_1(X_1)$ are homotopy equivalences;
3. $\pi_R^*N_L \cap \pi_L^*N_R = \pi_0^*N_0$. 
Proof. Statement (2) is clear from the definition. To prove statement (3) we observe that

\[ \pi^*_R N_L \cap \pi^*_L N_R = \pi^*_R N_L \cap \pi^*_0 M_2(X_0) \cong N_L \cap \pi^*_L M_2(X_0) \]

The result now follows easily from proposition 3.8. We turn to the proof of statement (1). It is an easy consequence of proposition 3.5 that \( \Box_0 \) and \( \Box_L \) preserve the nondegeneracy of the configurations so the maps are well defined.

Now we look at \( \Box_L \). For \( \delta \) small enough the eigenvalues of \( a_1d \) are distinct. Hence we can choose, up to the action of \( (\mathbb{C}^*)^4 \), eigenvector basis \( \{v_0, v_1\} \subset V \) of \( a_1d \) and \( \{w_0, w_1\} \subset W \) of \( da_1 \), where \( v_0, w_0 \) correspond to the eigenvalues near 0. Normalize \( v_1, w_1 \) so that \( dv_1 = w_1 \). Then the action of \( (\mathbb{C}^*)^4 \) is reduced to an action of \( (\mathbb{C}^*)^3 \). We can thus write (see also equation (7))

\[
\begin{align*}
a_1 &= \begin{bmatrix} a'_1 & 0 \\ 0 & a''_1 \end{bmatrix}, \\
a_2 &= \begin{bmatrix} a'_2 & b'c'' \\ b''c' & a''_2 \end{bmatrix}, \\
d &= \begin{bmatrix} d' & 0 \\ 0 & 1 \end{bmatrix}, \\
b &= \begin{bmatrix} b' \\ b'' \end{bmatrix}, \\
c &= \begin{bmatrix} c' \\ c'' \end{bmatrix}
\end{align*}
\]

The group \( (\mathbb{C}^*)^3 \) acts transitively on equivalence classes of such configurations written in the above canonical form. This shows the existence of an inverse, hence \( \Box_L \) is a homeomorphism. The proof for \( \Box_0 \) is similar. \( \square \)

The maps \( \Box_0, \Box_L \) extend to the closure \( \bar{N}', \bar{N}_z \) of \( N', N_z \). The following proposition is a direct consequence of proposition 3.5:

**Proposition 5.7.**

- Let \( m_L = [a_{1L}, a_{2L}, b_L, c_L] \in \bar{N}_{x_1L} \), \( m_R = [a_{1R}, a_{2R}, b_R, c_R] \in \bar{N}_{x_1R} \). Then the following are equivalent:
  1. \( m_L \Box_0 m_R \) is degenerate;
  2. Either \( m_L \) or \( m_R \) is degenerate.
  3. At least one of the 4 vectors \( b_L, b_R, c_L, c_R \) is zero.
- Let \( m' = [a'_1, a'_2, d', b', c'] \in \bar{N}' \), \( m'' = [a''_1, a''_2, b'', c''] \in \bar{N}_{z1} \). The following are equivalent:
  1. \( m' \Box_L m'' \) is degenerate;
  2. Either \( m' \) or \( m'' \) is degenerate;
  3. One of the 4 vectors \( b', b'', c', c'' \) is zero.

We are ready to prove
Proposition 5.8. $N_2$ is an open neighborhood of $C$.

Proof. From lemma 5.5 it follows immediately that $\pi_{R_*}C \subset \bar{N}_L$.

Suppose there is a sequence $y_n \in \mathcal{M}_2(X_{1L})$ such that $y_n \to y \in \pi_{R_*}C$. Write $y_n = [a_{1n}, a_{2n}, d_n, b_n, c_n]$. Then, by property 2 in lemma 5.5 the eigenvalues of $d_na_{in}$ converge to $0, z_i$. Hence, for $n$ large enough $y_n \in N_L$. Hence $N_L \cup \pi_{R_*}C$ is an open neighborhood of $\pi_{R_*}C$.

Suppose there is a sequence $x_n \to x \in C$ such that $x_n \notin N_2$. Hence $x_n \notin C$ so, by passing to a subsequence we may assume without loss of generality that $x_n \in \pi_{R_*}\mathcal{M}_2(X_{1L})$. Let $y_n = \pi_{R_*}x_n \in \mathcal{M}_2(X_{1L})$ and write $y_n = [a_{1n}, a_{2n}, d_n, b_n, c_n]$. Then $y_n \to y = \pi_{R_*}x$ by continuity of $\pi_{R_*}$, and $y_n \notin N_L$. But by property 2 in lemma 5.5 the eigenvalues of $d_na_{in}$ converge to $0, z_i$ which implies, for $n$ large enough, that $y_n \in N_L$.

Finally we prove the homotopy equivalence (6):

Proposition 5.9. The inclusion $C \to N_2$ is a strong deformation retract.

Proof. We will construct a homotopy $H_2 : N_2 \times [0, 1] \to N_2$ between the identity and a retraction $N_2 \to C$. Let $H_{x_1, x_2} : \tilde{N}_2 \times [0, 1] \to \tilde{N}_2$ be defined by

\[ H_{x_1, x_2}(a_1, a_2, b, c, t) = \left( t^2a_1 + (1 - t^2)x_1, t^2a_2 + (1 - t^2)x_2, tb, tc \right) \]

and let $H_1 : \tilde{N}' \times [0, 1] \to \tilde{N}'$ be defined by

\[ H_1(a_1', a_2', d', b', c', t) = (a_1', a_2', t^2d', b', c') \]

Then we defined $H_L : \tilde{N}_L \times [0, 1] \to \tilde{N}_L$ by

\[ H_L(m' \boxplus_L m'', t) \overset{\text{def}}{=} H_1(m', t) \boxplus_L H_{x_1, x_2}(m'', t) \]

We define $H_2$ as the unique solution of the system of equations

\[ \begin{align*}
\pi_{R_*}H_2(x, t) &= H_L(\pi_{R_*}x, t) \\
\pi_{L_*}H_2(x, t) &= H_R(\pi_{L_*}x, t)
\end{align*} \]

(11)

We have to show existence and uniqueness of solution. Then we will show that $H_2$ defines a homotopy between the identity on $N_2$ and a retraction $N_2 \to C$.

We define the auxiliary map $H_0 : \tilde{N}_0 \times [0, 1] \to \tilde{N}_0$ by

\[ H_0(m_L \boxplus_0 m_R, t) \overset{\text{def}}{=} H_{x_1L, x_2L}(m_L, t) \boxplus_0 H_{x_1R, x_2R}(m_R, t) \]
To prove existence and uniqueness of solution of the system (11) we consider two cases:

(1) Assume that either \( t = 0 \) or \( x \in C \). Then we claim that \( H_L(\pi_R^* x, t) \in \pi_R^* C \), \( H_R(\pi_L^* x, t) \in \pi_L^* C \). If \( t = 0 \) this follows from directly from lemma 5.3. If \( x \in C \) then, from lemma 5.3 we can write

\[
\pi_R^* x = x' \oplus_L x'' = (a'_1, a'_2, 0, b', c') \oplus_L (a''_1, a''_2, b'', c'')
\]

with \( c''b'' = 0 \). It then follows from the definition of \( H_L \) that \( H_L(\pi_R^* x, t) = \pi_R^* x \) for all \( t \). In the same way we see that \( H_R(\pi_L^* x, t) = \pi_L^* x \). This proves the claim. Then, existence and uniqueness follows from proposition 5.3.

(2) Assume \( t \neq 1 \) and \( x \notin C \). Then we may assume \( \pi_R^* x \in N_L \). Then, since \( H_L(\pi_R^* x, t) \in N_L \), we get from (11)

\[
\pi_R^* H_2(x, t) = H_L(\pi_R^* x, t) \Rightarrow H_2(x, t) = \pi_R^* H_L(\pi_R^* x, t)
\]

This proves uniqueness. To prove existence we need to show that

\[
\pi_L^* H_2(x, t) = \pi_L^* \pi_R^* H_L(\pi_R^* x, t) = H_R(\pi_L^* x, t)
\]

It is enough to show this for the case where \( x = \pi_L^* \pi_R^* y \) for some \( y \in N_0 \) since the set of points of this form is dense and \( H_L, H_R, \pi_L^*, \pi_R^*, \pi_R^* \) are continuous.

It is an easy computation to show that \( H_L(\pi_L^* y, t) = \pi_L^* H_0(y, t) \), \( H_R(\pi_R^* y, t) = \pi_R^* H_0(y, t) \). It follows that

\[
\pi_L^* \pi_R^* H_L(\pi_R^* x, t) = \pi_L^* \pi_R^* \pi_L^* H_0(y, t) = \pi_R^* H_0(y, t) = H_R(\pi_L^* x, t)
\]

Now we need to show \( H_2 \) is the desired homotopy. Direct inspection shows \( H_2(x, 1) = x \). We saw in (1) above that, for \( x \in C \), \( H_2(x, t) = x \) and \( H_2(x, 0) \in C \). The continuity of \( H_2 \) follows from the continuity of \( \pi_L^*, \pi_R^*, H_L, H_R \). \qed
6. The differential $d_1$

The objective of this section is to obtain the homotopy type of the inclusion maps

\[
\begin{array}{c}
A_L & \rightarrow & A_0 & \rightarrow & A_R \\
\uparrow & & \uparrow & & \uparrow \\
N_L & \rightarrow & N_0 & \rightarrow & N_R \\
\downarrow & & \downarrow & & \downarrow \\
& & N_2 & & \\
\end{array}
\]

where $A_0 = A_L \cap A_R = \pi_0^* \mathcal{M}_2(X_0)$. Since these spaces are classifying spaces it is enough to study the pullback under these maps of the tautological bundles. Together with the open cover of the previous section this will give a description of the homotopy type of $\mathcal{M}_2(X_2)$. It will also allow us to compute the $d_1$ differential in the spectral sequence introduced in corolary 5.2.

Lemma 6.1. Let

\[
\tilde{\mathcal{F}}_0(k, r) = \left\{ (u, v) : u, v : \mathbb{C}^k \rightarrow \mathbb{C}^r, u, v \text{ are injective} \right\}
\]

\[
(\begin{array}{c}
(u, v) \sim (u(g^t)^{-1}, vg) \text{, } g \in Gl(k, \mathbb{C})
\end{array})
\]

\[
\tilde{\mathcal{F}}_1(k, r) = \left\{ (u, v) : u, v : \mathbb{C}^k \rightarrow \mathbb{C}^r, u, v \text{ are injective} \right\}
\]

\[
(\begin{array}{c}
(u, v) \sim (u(g^t)^{-1}, vg) \text{, } g_u, g_v \in Gl(k, \mathbb{C})
\end{array})
\]

and define $F_0(k, r) \subset \tilde{\mathcal{F}}_0(k, r)$ and $F_1(k, r) \subset \tilde{\mathcal{F}}_1(k, r)$ by

\[
F_0(k, r) = \left\{ [\beta', c] \in \tilde{\mathcal{F}}_0(k, r) \text{, } bc = 0 \right\}
\]

\[
F_1(k, r) = \left\{ [\beta', c] \in \tilde{\mathcal{F}}_1(k, r) \text{, } bc = 0 \right\}
\]

Let $j_0 : F_0(k, r) \rightarrow \mathcal{M}_k^r(X_0)$, $j_1 : F_1(k, r) \rightarrow \mathcal{M}_k^r(X_1)$ be the inclusion maps given by $[\beta', c] \mapsto [b, c]$. Then we have the homotopy commutative diagram

\[
\begin{array}{c}
\mathcal{M}_k^r(X_0) \xrightarrow{\pi^*} \mathcal{M}_k^r(X_1) \\
\downarrow j_0 \quad \quad \downarrow j_1 \\
Gr(k, \mathbb{C}^r) \xrightarrow{p_0} F_0(k, r) \xrightarrow{pr} F_1(k, r) \\
\downarrow p_0 \quad \quad \downarrow \iota_0 \\
\tilde{\mathcal{F}}_0(k, r) \xrightarrow{\tilde{p}_r} \tilde{\mathcal{F}}_1(k, r)
\end{array}
\]
where $p_0$ is the projection $[b, c] \mapsto [c]$. Moreover, in the rank stable limit, the maps $\iota_0, \iota_1, p_0, \tilde{p}_0, j_0, j_1$ are homotopy equivalences.

**Proof.** We divide the proof into three steps:

**Step 1.** $p_0, \tilde{p}_0$ are fibrations with fibers $M(k, r - k)$ and $M(k, r)$ respectively where $M(k, r) = \frac{U(r)}{U(k)}$ is the space of injective maps from $\mathbb{C}^k$ to $\mathbb{C}^r$ which is contractible in the stable range. That proves $p_0, \tilde{p}_0$ are homotopy equivalences. It immediately follows that $\iota_0$ is a homotopy equivalence.

**Step 2.** Now we look at $\iota_1$. Consider the projection $p_1 : F_1(k, r) \to \text{Gr}(k, \mathbb{C}^r)$ given by $[b, c] \mapsto [c]$. When $r \to \infty$, the spectral sequence associated with the fibration $	ext{Gr}(k, \mathbb{C}^{r-k}) \xrightarrow{i} F_1(k, r) \xrightarrow{p_1} \text{Gr}(k, \mathbb{C}^r)$ collapses since all homology is in even dimensions. It easily follows that $\iota_1$ is an isomorphism in all homology groups, hence an homotopy equivalence.

**Step 3.** Finally we need to prove the statements about $j_0, j_1$. Let $C^F_k(X_0)$, $C^F_k(X_1)$ be the spaces of configurations corresponding to the monads for $X_0$ and $X_1$. Let $C^F_0(k, r) \subset C^F_k(X_0)$ and $C^F_1(k, r) \subset C^F_k(X_1)$ be the subsets of configurations of the form $(0, 0, 0, b, c)$ and $(0, 0, 0, b, c)$ respectively. Then we have the maps between fibrations

$$
\begin{array}{cccc}
\text{Gl}(k) & \rightarrow & C^F_0(k, r) & \rightarrow & F_0(k, r) \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
\text{Gl}(k) & \rightarrow & C^F_k(X_0) & \rightarrow & \mathcal{M}_k^r(X_0) \\
\end{array}
$$

and a similar diagram for $X_1$. In the rank stable limit the spaces $C^F_0(k, r)$ and $C^F_k(X_0)$ are contractible (see [San95], [BS97]) so, by the five lemma $j_0$ is an isomorphism in $\pi_*$, hence an homotopy equivalence. The proof for $j_1$ is the same.

\[ \square \]

We are ready to state and prove the main theorem of this section:
Theorem 6.2. Consider the compositions

(13) \[ A_0 \xrightarrow{\pi_0} \mathcal{M}_2^\infty(X_0) \xrightarrow{\delta_0^{-1}} F_0(2, \infty) \xrightarrow{p_0} Gr(2, \mathbb{C}^\infty) \]

(14) \[ A_L \xrightarrow{\pi_R} \mathcal{M}_2^\infty(X_{1L}) \xrightarrow{\delta_1^{-1}} F_1(2, \infty) \xrightarrow{1_1} \tilde{F}_1(2, \infty) \]

(15) \[ N_0 \xrightarrow{\mathcal{P}_0^{-1}} N_{x_{1L}} \times N_{x_{1R}} \xrightarrow{p_L} N_{x_{1L}} \xrightarrow{\mathcal{M}_1(X_0)} \text{Gr}(1, \mathbb{C}^\infty) \]

(16) \[ N_0 \xrightarrow{\mathcal{P}_0^{-1}} N_{x_{1L}} \times N_{x_{1R}} \xrightarrow{p_R} N_{x_{1R}} \xrightarrow{\mathcal{M}_1(X_0)} \text{Gr}(1, \mathbb{C}^\infty) \]

(17) \[ N_L \xrightarrow{\mathcal{P}_L^{-1}} N' \times N_{z_1} \xrightarrow{p''} N' \xrightarrow{\mathcal{M}_1(X_1)} \tilde{F}_1(1, \infty) \]

(18) \[ N_L \xrightarrow{\mathcal{P}_L^{-1}} N' \times N_{z_1} \xrightarrow{p'} N' \xrightarrow{\mathcal{M}_1(X_1)} \tilde{F}_1(1, \infty) \]

(19) \[ N_2 \xrightarrow{\sim} C \xrightarrow{\pi_2^L} \mathcal{M}_1^\infty(X_{1L}) \xrightarrow{\sim} \mathcal{M}_1(X_{1L}) \xrightarrow{1_1 j_1^{-1}} \tilde{F}_1(1, \infty) \]

(20) \[ N_2 \xrightarrow{\sim} C \xrightarrow{\pi_2^R} \mathcal{M}_1^\infty(X_{1R}) \xrightarrow{\sim} \mathcal{M}_1(X_{1R}) \xrightarrow{1_1 j_1^{-1}} \tilde{F}_1(1, \infty) \]

Let \( E, L \) be the tautological bundles over \( \text{Gr}(2, \infty) \) and \( \text{Gr}(1, \infty) \) respectively. Then we define the following bundles:

- \( E_0 \to A_0 \) is the pullback of \( E \) under the composition \((13)\)
- \( L_{0L,0} \to N_0 \) is the pullback of \( L \) under \((13)\)
- \( L_{0R,0} \to N_0 \) is the pullback of \( L \) under \((14)\)
- \( L_{0R,L} \to N_L \) is the pullback of \( L \) under \((14)\)

Now let \( \tilde{E}_u, \tilde{E}_v \to \tilde{F}_1(2, r) \) be the tautological bundles corresponding to \( u, v \) and let \( \tilde{L}_u, \tilde{L}_v \) be the tautological line bundles over \( \tilde{F}_1(1, \infty) \). We define

- \( E_{0L}, E_{cL} \to A_L \) are the pullback of \( \tilde{E}_u, \tilde{E}_v \) under \((14)\)
- \( L_{0L,L}, L_{cL,L} \to N_L \) are the pullback of \( \tilde{L}_u, \tilde{L}_v \) under \((14)\)
- \( L_{0L,2}, L_{cL,2} \to N_2 \) are the pullback of \( \tilde{L}_u, \tilde{L}_v \) under \((19)\)
- \( L_{0R,2}, L_{cR,2} \to N_2 \) are the pullback of \( \tilde{L}_u, \tilde{L}_v \) under \((20)\)

Then we have

(1) \( E_{0L}|_{A_0} = E_0, \ E_{cL}|_{A_0} = E_0 \)
(2) \( L_{0L,L}|_{N_0} \cong L_{0L,0}, \ L_{cL,L}|_{N_0} \cong L_{0L,0}, \ L_{0R,L}|_{N_0} \cong L_{0R,0} \)
(3) \( E_{0L}|_{N_L} \cong L_{0L} \oplus L_{0R}, \ E_{cL}|_{N_L} \cong L_{cL} \oplus L_{0R} \)
(4) \( E_0|_{N_0} \cong L_{0L,0} \oplus L_{0R,0} \).

(5) \( L_{bR,2}|_{N_L} \cong L_{cR,2}|_{N_L} \cong L_{0R,L} \).

(6) \( L_{bL,2}|_{N_L} \cong L_{bL,L | L} \), \( L_{cL,2}|_{N_L} \cong L_{cL,L | L} \).

Similar statements hold for the spaces \( A_R, N_R \) and the maps \( N_R \to A_R, N_R \to N_2 \) and \( N_0 \to N_R \).

**Proof.**

(1) First we show that \( E_{bL}|_{A_0} \cong E_{cL}|_{A_0} \cong E_0 \). Consider diagram (12). We will start by defining a homotopy inverse \( q : Gr(k, \mathbb{C}^r) \to \tilde{F}_0(k,r) \) to the map \( \tilde{p}_0 : \tilde{F}_0 \to Gr(k, \mathbb{C}^\infty) \) as follows: choose a map \( c : \mathbb{C}^k \to \mathbb{C}^r \) representing an element \([c] \in Gr(k, \mathbb{C}^r)\). Choose \( h \in GL(k, \mathbb{C}) \) such that \( ch \) is orthogonal. Then define \( q([c]) = [ch, ch] \). This map is well defined and independent of the choice of \( h \). Also \( p_0q = 1 \) hence \( p_0 = q^{-1} \).

Now observe that the composition

\[ \tilde{p}r \circ q : Gr(k, \mathbb{C}^r) \to \tilde{F}_1(k,r) = Gr(k, \mathbb{C}^r) \times Gr(k, \mathbb{C}^r) \]

is the diagonal map. It follows that, if \( E \) is the tautological bundle over \( Gr(k, \mathbb{C}^\infty) \), then

\[ q^* \tilde{p}r^* \tilde{E}_u \cong q^* \tilde{p}r^* \tilde{E}_v \cong E \]

To show that \( E_{bL}|_{A_0} \cong E_{cL}|_{A_0} \cong E_0 \) it suffices to show that \( pr^*t_1^* \tilde{E}_u \cong pr^*t_1^* \tilde{E}_v \cong p_0^*E \). We have

\[ pr^*t_1^* \tilde{E}_u = u_0^*pr^* \tilde{E}_u \cong p_0q^* \tilde{p}r^* \tilde{E}_u = p_0^*E \]

and a similar statement is true for \( \tilde{E}_v \). This concludes the proof.

(2) We want to show that

\( L_{bL,L}|_{N_0} \cong L_{0L,0} \), \( L_{cL,L}|_{N_0} \cong L_{0L,0} \), \( L_{0R,L}|_{N_0} \cong L_{0R,0} \)

We have the commutative diagram (see proposition 5.4)
from which it follows that \( L_{0_R,L}|_N \cong L_{0_R,0} \). We also have the commutative diagram

\[
\begin{array}{cccccc}
N_0 \xrightarrow{\oplus_0} N_{x_1L} \times N_{x_1R} & \xrightarrow{p_L} & N_{x_1L} & \xrightarrow{j_0} & F_0 & \xrightarrow{\tau} & F_0 \\
\parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\
N_L \xrightarrow{\oplus_L} N' \times N_z & \xrightarrow{\tau} & N' & \xrightarrow{\tau} & F_1 & \xrightarrow{\bar{\tau}} & F_1 \\
\end{array}
\]

from which it follows, as in step (1) above, that \( L_{bL,L}|_N \cong L_{cL,L}|_N \cong L_{0L,0} \).

(3) We want to show that

\[
E_{bL}|_N \cong L_{bL,L} \oplus L_{0_R,L}, \quad E_{cL}|_N \cong L_{cL,L} \oplus L_{0R,L}
\]

Consider the following diagram:

\[
\begin{array}{cccccc}
A_L \xrightarrow{\sim} j_1 \xrightarrow{\sim} F_1 \xrightarrow{\sim} F_1 \\
\parallel & \parallel & \parallel & \parallel \\
N_L \xrightarrow{\oplus_L} N' \times N_z \xrightarrow{\sim} \mathcal{M}_1(X_1) \times \mathcal{M}_1(X_0) \xrightarrow{j_1 \times j_0} F_1 \times F_0 \xrightarrow{\sim} F_1 \times F_0 \\
\end{array}
\]

Since \( \bar{\tau}^* \bar{L}_u \cong \bar{\tau}^* \bar{L}_v \cong \bar{\rho}_0^* L \), the proof will be complete if we show there is a map \( \bar{w} : \bar{F}_1(1, \infty) \times \bar{F}_0(1, \mathbb{C}^\infty) \rightarrow \bar{F}_1(2, \infty) \) making the diagram homotopy commutative, such that

\[
\bar{w}^* \bar{E}_u = \bar{L}_u \oplus \bar{\tau}^* \bar{L}_u, \quad \bar{w}^* \bar{E}_v = \bar{L}_v \oplus \bar{\tau}^* \bar{L}_v
\]

We begin by building \( \bar{w} \). Define maps \( s_L, s_R : Gr(1, \mathbb{C}^\infty) \rightarrow Gr(1, \mathbb{C}^\infty) \) as follows: let \( v : \mathbb{C} \rightarrow \mathbb{C}^\infty \) and write \( v = (v^1, v^2, \ldots) \). Then

\[
s_L([v]) \overset{\text{def}}{=} [(v^1, 0, v^2, 0, \ldots)]
\]

\[
s_R([v]) \overset{\text{def}}{=} [(0, v^1, 0, v^2, \ldots)]
\]

We observe that \( s_L, s_R \) are homotopic to the identity. It follows that, if we define

\[
\bar{w} : ([b_L, c_L], [b_R, c_R]) \mapsto [s_L(b_L) \oplus s_R(b_R), s_L(c_L) \oplus s_R(c_R)]
\]

then

\[
(\bar{w})^* \bar{E}_u = \bar{L}_u \oplus \bar{\tau}^* \bar{L}_u, \quad (\bar{w})^* \bar{E}_v = \bar{L}_v \oplus \bar{\tau}^* \bar{L}_v
\]
It remain to show diagram 21 is commutative. Let \( j_z : F_0(1, \infty) \to N_z \) be defined by \( j_z : [b, c] \mapsto [z, 0, b, c] \). Then the diagram

\[
N' \times N_{z_1} \xrightarrow{j_1 \times j_{z_1}} F_1 \times F_0 \xrightarrow{i_1 \times i_0} \mathcal{M}_1(X_1) \times \mathcal{M}_1(X_0)
\]

is homotopy commutative. We are left with the diagram

\[
A_L \xrightarrow{j_i} F_1 \xrightarrow{i_1} \tilde{F}_1 \xrightarrow{\tilde{w}} F_1(1, \infty) \xrightarrow{i_1 \times i_0} \tilde{F}_1(1, \infty) \times \tilde{F}_0(1, \infty)
\]

Now define the map \( w : F_1(1, \infty) \times F_0(1, \infty) \to F_1(2, \infty) \) by

\[
w : ([b_L, c_L], [b_R, c_R]) \mapsto [s_L(b_L) \oplus s_R(b_R), s_L(c_L) \oplus s_R(c_R)]
\]

Clearly we have the commutative diagram

\[
F_1(2, \infty) \xrightarrow{i_1} \tilde{F}_1(2, \infty) \xrightarrow{\tilde{w}} F_1(1, \infty) \times F_0(1, \infty) \xrightarrow{i_1 \times i_0} \tilde{F}_1(1, \infty) \times \tilde{F}_0(1, \infty)
\]

We are thus left with the diagram

\[
A_L \xrightarrow{j_i} F_1 \xrightarrow{i_1} \tilde{F}_1 \xrightarrow{\tilde{w}} F_1 \times F_0
\]

Next we introduce maps

\[
S_L([b, c]) = [s_L(b), s_L(c)]
\]

\[
S_R([b, c]) = [s_R(b), s_R(c)]
\]

These maps are homotopic to the identity hence we only have to show the diagram

\[
A_L \xrightarrow{j_i} F_1 \xrightarrow{i_1} F_1 \times F_0 \xrightarrow{s_L \times s_R} F_1 \times F_0
\]

is homotopy commutative. This is an easy direct verification.
(4) We want to show that $E_0|_{N_0} \cong L_{0L} \oplus L_{0R}$. Consider the diagram

\[
\begin{array}{ccc}
N_0 & \xrightarrow{i_1} & A_0 \\
\downarrow{i_2} & & \downarrow{i_3} \\
N_L & \xrightarrow{i_4} & A_L
\end{array}
\]

Then $E_0 = i_3^* E_{bL}$ so

\[E_0|_{N_0} = i_1^* E_0 = i_1^* i_3^* E_{bL} = i_2^* i_4^* E_{bL} = L_{0L} \oplus L_{0R}\]

(5) We want to show that $L_{bR}|_{N_L} \cong L_{cR}|_{N_L} \cong L_{0R}$. The result will follow if we show that the following diagram is homotopy commutative:

\[
\begin{array}{cccccc}
N_L & \xrightarrow{\oplus} & N' \times N_{z_1} & \xrightarrow{=} & N_{z_1} & \xrightarrow{\mathfrak{m}_1} \xrightarrow{\pi^*} F_0 & \xrightarrow{\tilde{F}_0} \\
\downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
\xrightarrow{=} & & \xrightarrow{=} & & \xrightarrow{=} & & \xrightarrow{=} \\
N_2 & \xleftarrow{\cong} & C & \xrightarrow{\pi_{L^*}^{L^*}} S_0 \mathfrak{m}_1(X_0) & \xleftarrow{=} & \mathfrak{m}_1(X_1) & \xrightarrow{=} & F_1 & \xrightarrow{=} & \tilde{F}_1
\end{array}
\]

Let

\[
\begin{aligned}
S_{1L}N_2 &= \{ (E, \phi) \in N_2 \mid c_2((\pi_{L^*} E)^{L^*}) = 1 \} \\
S_1N_L &= \{ (E, \phi) \in N_L \mid c_2((\pi_{L^*} E)^{L^*}) = 1 \} \\
S_0N' &= \{ (E, \phi) \in N' \mid c_2((\pi_{L^*} E)^{L^*}) = 0 \}
\end{aligned}
\]

Then the commutativity of diagram (23) follows from the commutativity of (24)

\[
\begin{array}{cccccc}
N_2 & \xleftarrow{\pi_R} & S_{1L}N_2 & \xleftarrow{\pi_R^{L^*}} S_1N_L & \xleftarrow{\oplus} & N' \times N_{z_1} \\
\downarrow{\pi_R^{L^*}} & & \downarrow{\pi_R^{L^*}} & & \downarrow{\oplus} & \downarrow{=} \\
\mathfrak{m}_1(X_1R) & \xrightarrow{\pi_R} & \mathfrak{m}_1(X_0) & \xrightarrow{\oplus} & \mathfrak{m}_1(X_1) & \xrightarrow{=} & N_{z_1}
\end{array}
\]

We need to check the image of $\oplus_L : S_0 N' \times N_{z_1} \to N_L$ is contained in $S_1N_L$. Then, analyzing the commutativity of diagram
boils down to analyzing the diagram

\[
S_1 N_L \xrightarrow{\boxplus_L} S_0 N' \times N_z_1 \\
\downarrow{\pi_L^\vee} \downarrow{\pi_L^\vee} \\
\mathcal{M}_1(X_0) \leftarrow N_z_1
\]

Let \( m' \in S_0 N' \subset S_0 \mathcal{M}_1(X_1) \), \( m' = [a'_1, a'_2, 0, b', c'] \). Let \( m'' \in N_z_1 \). Then a direct computation shows that \( (\pi_L^*(m' \boxplus_L m''))^\vee = m'' \). This shows that the image of \( S_0 N' \times N_z_1 \) under \( \boxplus_L \) is contained in \( S_1 N_L \) and that diagram (25) is commutative.

(6) We want to show that \( L_{bL,L} \mid_{N_L} \cong L_{bL,L}, L_{cL,L} \mid_{N_L} \cong L_{cL,L} \). This will follow from the commutativity of the diagram

\[
N_L \xrightarrow{\boxplus_L} N' \times N_z_1 \xrightarrow{\pi_R^*} N' \xrightarrow{\pi_R^*} \mathcal{M}_1(X_1) \\
\downarrow{\pi_R^*} \downarrow{\pi_R^*} \\
N_2 \xrightarrow{\sim} C \xrightarrow{\pi_R^*} S_0 \mathcal{M}_1(X_1) \xrightarrow{\sim} \mathcal{M}_1(X_1)
\]

We showed in proposition [5.9 that the map \( H_2(-,0) \) is the homotopy inverse of the inclusion \( C \rightarrow N_2 \). Let \((m', m'') \in N' \times N_z_1 \).

Then, by definition of \( H_2 \),

\[
\pi_R^* H_2(\pi_R^*(m' \boxplus_L m''), 0) = H_L(m' \boxplus_L m'', 0) = H_1(m', 0) \boxplus H_2(m'', 0)
\]

Hence the diagram

\[
N_L \xrightarrow{\boxplus_L} N' \times N_z_1 \xrightarrow{\pi_R^*} N' \xrightarrow{H_1(-,0)} \mathcal{M}_1(X_1) \\
\downarrow{\pi_R^*} \downarrow{H_1(-,0)} \\
N_2 \xrightarrow{H_2(-,0)} C \xrightarrow{\pi_R^*} S_0 \mathcal{M}_1(X_1)
\]

is commutative. From here it follows easily that diagram (26) is commutative.

\[ \square \]

7. The Cohomology of \( \mathcal{M}_2(X_q) \)

The objective of this section is to prove theorem 1.2. We begin by proving it for the special case \( q = 2 \):

**Theorem 7.1.** There is an exact sequence

\[
0 \rightarrow K_C \rightarrow H^*(\mathcal{M}_2(X_2)) \rightarrow H^*(A_L) \oplus H^*(A_R) \rightarrow H^*(A_0) \rightarrow 0
\]
where $K_C = \text{Ker} (H^*(C) \to H^*(N_L) \oplus H^*(N_R))$. This sequence splits and we get

$$H^*(\mathfrak{M}_2(X_2)) \cong K_C \oplus \text{Ker} (H^*(A_L) \oplus H^*(A_R) \to H^*(A_0))$$

Proof. Recall corollary 5.2. We will use this spectral sequence to compute $H^*(\mathfrak{M}_2(X_2))$. Clearly the map $d_1 : E_{1,n} \to E_{2,n}$ is surjective hence $E_2^{2,n} = 0$. Also we notice that $E_1^{p,2n+1} = 0$ for any $p$. It follows that the spectral sequence collapses at the term $E_2$. We get then

$$H^{2n}(\mathfrak{M}_2(X_2)) = E_{\infty}^{0,2n} = \text{Ker} (d_1 : E_{1,0}^{0,2n} \to E_{1,2n}^{1,2n})$$

$$H^{2n+1}(\mathfrak{M}_2(X_2)) = E_{\infty}^{1,2n} = \frac{\text{Ker} (d_1 : E_{0}^{1,2n} \to E_{1,2n}^{1,2n})}{\text{Im} (d_1 : E_{0}^{0,2n} \to E_{1,2n}^{1,2n})}$$

When performing calculations we will use the following sign conventions:

We begin by defining the following generators of the cohomology of $E_1^{0,2n}$:

$$a^i_{\Delta L} = c_i(E_c L) - c_i(E_b L)$$

$$a^i_{b L} = c_i(E_b L)$$

$$c_{\Delta L} = c_1(L_c L) - c_1(L_b L)$$

$$c_{b L} = c_1(L_b L)$$

We do the same for $E_1^{1,2n}$:

$$n_{\Delta L} = c_1(L_c L) - c_1(L_b L)$$

$$n_{b L} = c_1(L_b L)$$

$$n_{0 R} = c_1(L_{0 R})$$

$$a^i = c_i(E_0)$$

and for $E_1^{2,n}$:

$$n_{0 R} = c_1(L_{0 R})$$

$$n_{0 L} = c_1(L_{0 L})$$
Then, from theorem 6.2 it follows that the map \( d_1 : E_1^{0,2n} \to E_1^{1,2n} \) may be represented by the following diagram, where the entries correspond to those in diagram (29):

\[
\begin{array}{cccc}
(a_{\Delta L}, a_{bL}, a_{\Delta L}^2, a_{bL}^2) & (0, a^1, 0, a^2) & (a_{\Delta R}^1, a_{bR}^1, a_{\Delta R}^2, a_{bR}^2) \\
\downarrow & & \downarrow \\
(-n_{\Delta L}, -n_{bL} - n_{0R}, -n_{\Delta L} n_{0R} - n_{bL} n_{0R}) & (-n_{\Delta R}, -n_{bR} - n_{0L}, -n_{\Delta R} n_{0L} - n_{bR} n_{0L}) & (0, -n_{0L}, -n_{\Delta R}, -n_{bR})
\end{array}
\]

Also the map \( d_1 : E_1^{1,2n} \to E_1^{2,2n} \) is given by

\[
\begin{align*}
(a^1, a^2) & \mapsto (n_{0L} + n_{0R}, n_{0L} n_{0R}) \\
(n_{\Delta L}, n_{bL}, n_{0R}) & \mapsto (0, n_{0L}, n_{0R}) \\
(n_{0L}, n_{\Delta R}, n_{bR}) & \mapsto (n_{0L}, 0, n_{0R})
\end{align*}
\]

Now let

\[
K_{AL} = \text{Ker}(H^*(A_L) \to H^*(A_0)) \quad K_{AR} = \text{Ker}(H^*(A_R) \to H^*(A_0)) \\
K_{NL} = \text{Ker}(H^*(N_L) \to H^*(N_0)) \quad K_{NR} = \text{Ker}(H^*(N_R) \to H^*(N_0))
\]

Then

\[
\begin{align*}
H^*(A_L) & \cong \mathbb{Z}[a^1, a^2] \oplus K_{AL} \quad H^*(A_R) \cong \mathbb{Z}[a^1, a^2] \oplus K_{AR} \\
H^*(C) & \cong \mathbb{Z}[n_L, n_R] \oplus K_{NL} \oplus K_{NR} \oplus K_C \\
H^*(N_L) & \cong \mathbb{Z}[n_L, n_R] \oplus K_{NL} \quad H^*(N_R) \cong \mathbb{Z}[n_L, n_R] \oplus K_{NR}
\end{align*}
\]

Notice that \( K_C \subset H^*(C) \) is the ideal generated by \( c_{\Delta L}c_{\Delta R} \). The restriction of the map \( H^*(A_L) \to H^*(N_L) \) to \( K_{AL} \) induces a map \( s_L : K_{AL} \to K_{NL} \). Similarly we have a map \( s_R : K_{AR} \to K_{NR} \). Let also \( s : \mathbb{Z}[a^1, a^2] \to \mathbb{Z}[n_L, n_R] \) be the map induced by the direct sum map \( BU(1) \times BU(1) \to BU(2) \). Then the map \( d_1 : E_1^{0,2n} \to E_1^{1,2n} \) is given by

\[
d_1 (a_L + k_{AL}, a_R + k_{AR}, x + k_{NL} + k_{NR} + k_C) = \\
= (-s(a_L) - s_L(k_{AL}) + x + k_{NL}, -s(a_R) - s_R(k_{AR}) - x - k_{NR}, a_L + a_R)
\]

and the map \( d_1 : E_1^{1,2n} \to E_1^{2,2n} \) is given by

\[
d_1 (x_L + k_{NL}, x_R + k_{NR}, a) = (x_L + x_R + s(a))
\]

Now we can finish the proof:
We are ready to prove the general case:

**Theorem 7.2.** With notations as in theorem 2.1 let

\[ K_i = \text{Ker} \left( H^*(\pi_i^*\mathcal{M}_2(X_1)) \rightarrow H^*(\pi_i^*\mathcal{M}_2(X_0)) \right) \]

\[ K_{ij} = \text{Ker} \left( H^*(\pi_j^*\mathcal{M}_2(X_2)) \rightarrow H^*(\pi_i^*\mathcal{M}_2(X_1)) \oplus H^*(\pi_j^*\mathcal{M}_2(X_1)) \right) \]

Then, as modules over \( \mathbb{Z} \), we have an isomorphism

\[ H^* (\mathcal{M}_2(X_q)) \cong H^* (\mathcal{M}_2(X_0)) \oplus \bigoplus_i K_i \oplus \bigoplus_{i<j} K_{ij} \tag{30} \]

**Proof.** We divide the proof into two steps:

1. We will use theorem 2.1 to build a spectral sequence converging to the cohomology of \( H^*(\mathcal{M}_2(X_q)) \). Let \( \Delta \) be the \( q-1 \) simplex. Label its vertices by \( v_i \), \( i = 1, \ldots, q \), and the \( e_{ij} \) be the middle point of the edge joining \( v_i \) and \( v_j \). We define a filtration \( \Delta_0 \subset \Delta_1 \subset \Delta \) of \( \Delta \) where \( \Delta_0 = \bigcup_{i<j} e_{ij} \) and \( \Delta_1 \) is the 1-skeleton of \( \Delta \). Write \( \Delta_1 = \bigcup_i \Delta_{1i} \) where \( \Delta_{1i} \) is the closure of the connected component of \( \Delta_1 \setminus \Delta_0 \) containing \( v_i \). Then we define

\[ M = \bigcup_{i,j} (e_{ij} \times \pi_j^*\mathcal{M}_2(X_2)) \cup \bigcup_i (\Delta_{1i} \times \pi_i^*\mathcal{M}_2(X_1)) \cup (\Delta \times \pi_0^*\mathcal{M}_2(X_0)) \]

\[ \sim \]
where $\sim$ is induced by the inclusions $e_{ij} \subset \Delta_{ij} \subset \Delta$ and $\pi_0^*\mathcal{M}_2(X_0) \subset \pi_i^*\mathcal{M}_2(X_1) \subset \pi_{ij}^*\mathcal{M}_2(X_2)$. Then, the arguments in [Seg68] can be applied to show that $M$ is homotopically equivalent to $\mathcal{M}_2(X_0)$. The filtration of $\Delta$ by $\Delta_0, \Delta_1$ induces a filtration $F_0 \subset F_1 \subset F_2 = M$ of $M$ which leads to a spectral sequence with

$$E_1^{0,n} = H^n(F_0) \cong \bigoplus_{i<j} (H^0(e_{ij}) \otimes H^n(\pi_{ij}^*\mathcal{M}_2(X_2)))$$

$$E_1^{1,n} = H^n(F_1, F_0) \cong \bigoplus_i (H^1(\Delta_{1i}, \partial \Delta_{1i}) \otimes H^n(\pi_i^*\mathcal{M}_2(X_1)))$$

$$E_1^{2,n} = H^n(F_2, F_1) \cong H^1(F_1) \otimes H^n(\mathcal{M}_2(X_0))$$

(2) The $d_1$ differential is induced by the inclusions $\pi_0^*\mathcal{M}_2(X_0) \to \pi_i^*\mathcal{M}_2(X_1) \to \pi_{ij}^*\mathcal{M}_2(X_2)$. We will use the sign convention

$$\begin{array}{ccc}
H^*(\pi_i^*\mathcal{M}_2(X_1)) & + & + \\
H^*(\pi_{ij}^*\mathcal{M}_2(X_2)) & - & + \\
H^*(\pi_0^*\mathcal{M}_2(X_0)) & + & - \\
H^*(\pi_j^*\mathcal{M}_2(X_1)) & & 
\end{array}$$

Let

$$K_i = \text{Ker } (H^*(\pi_i^*\mathcal{M}_2(X_1)) \to H^*(\pi_0^*\mathcal{M}_2(X_0)))$$

$$K_{ij} = \text{Ker } (H^*(\pi_{ij}^*\mathcal{M}_2(X_2)) \to H^*(\pi_i^*\mathcal{M}_2(X_1)) \oplus H^*(\pi_j^*\mathcal{M}_2(X_1)))$$

Then, from theorem 7.1 we have

$$H^*(\pi_i^*\mathcal{M}_2(X_1)) \cong H^*(\pi_0^*\mathcal{M}_2(X_0)) \oplus K_i$$

$$H^*(\pi_{ij}^*\mathcal{M}_2(X_2)) \cong H^*(\pi_0^*\mathcal{M}_2(X_0)) \oplus K_i \oplus K_j \oplus K_{ij}$$

Then the sequence of maps $E_1^{0,n} \xrightarrow{d_1} E_1^{1,n} \xrightarrow{d_1} E_1^{2,n}$ splits into three sequences

$$\bigoplus_{i<j} H^0(e_{ij}) \otimes K_{ij} \to 0 \to 0$$

$$\bigoplus_{i<j} H^0(e_{ij}) \otimes (K_i \oplus K_j) \to \bigoplus_i H^1(\Delta_{1i}, \partial \Delta_{1i}) \otimes K_i \to 0$$

$$H^0(\Delta_0) \otimes K^n \to H^1(\Delta_1, \Delta_0) \otimes K^n \to H^1(\Delta_1) \otimes K^n$$
where $K^n$ stands for $H^n(\mathcal{M}_2(X_0))$. The bottom maps are easily analyzed using the exact sequence

$$0 \to H^0(\Delta_1) \to H^0(\Delta_0) \to H^1(\Delta_1, \Delta_0) \to H^1(\Delta_1) \to 0$$

It follows that the map $d_1 : E_1^{1,n} \to E_1^{2,n}$ is surjective. Since $E_1^{r,n} = 0$ for $r > 2$ and $n$ even, this implies the spectral sequence collapses and

$$H^n(\mathcal{M}_2(X_q)) = \frac{\text{Ker}(d_1 : E_1^{1,n} \to E_1^{2,n})}{\text{Im}(d_1 : E_1^{0,n} \to E_1^{1,n})}$$

$$H^n(\mathcal{M}_2(X_q)) = \text{Ker}(d_1 : E_1^{0,n} \to E_1^{1,n})$$

Let’s look more closely at the map

$$(33) \bigoplus_{i<j} H^0(e_{ij})(K_i \oplus K_j) \to \bigoplus_i H^1(\Delta_{1i}, \partial \Delta_{1i}) \otimes K_i$$

Observe that

$$\bigoplus_{i<j} H^0(e_{ij}) \otimes (K_i \oplus K_j) = \bigoplus_i H^0(\partial \Delta_{1i}) \otimes K_i$$

It follows that the map (33) can be easily analysed using the exact sequence

$$0 \to H^0(\Delta_{1i}) \to H^0(\partial \Delta_{1i}) \to H^1(\Delta_{1i}, \partial \Delta_{1i}) \to 0$$

We gather together our conclusions:

(a) The top sequence in (32) contributes a term

$$\bigoplus_{i<j} H^0(e_{ij}) \otimes K_{ij}$$

to $H^{2n}(\mathcal{M}_2(X_q))$.

(b) The bottom sequence in (32) does not contribute to $H^{2n+1}(\mathcal{M}_2(X_q))$

since it is exact in the middle.

(c) The bottom sequence in (32) contributes a term

$$H^0(\Delta_1) \otimes H^*(\mathcal{M}_2(X_0))$$

to $H^{2n}(\mathcal{M}_2(X_q))$.

(d) The map (33) is surjective hence it does not contribute to $H^{2n+1}(\mathcal{M}_2(X_q))$.

(e) The map (33) contributes a term

$$\bigoplus_i H^0(\Delta_{1i}) \otimes K_i$$

to $H^{2n}(\mathcal{M}_2(X_q))$. 
From (b) and (d) it follows that $H^{2n+1}(\mathcal{M}_2(X_q)) = 0$ and from (a), (c) and (e) equation (30) follows.

\[\square\]

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