FLEXIBILITY OF LYAPUNOV EXPONENTS
FOR EXPANDING CIRCLE MAPS

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(Communicated by Jairo Bochi)

Abstract. Let \( g \) be a smooth expanding map of degree \( D \) which maps a circle to itself, where \( D \) is a natural number greater than 1. It is known that the Lyapunov exponent of \( g \) with respect to the unique invariant measure that is absolutely continuous with respect to the Lebesgue measure is positive and less than or equal to \( \log D \) which, in addition, is less than or equal to the Lyapunov exponent of \( g \) with respect to the measure of maximal entropy. Moreover, the equalities can only occur simultaneously. We show that these are the only restrictions on the Lyapunov exponents considered above for smooth expanding maps of degree \( D \) on a circle.

1. Introduction. Let \( S^1 = \mathbb{R}/\mathbb{Z} \). We say that a map is a circle map if it maps the circle \( S^1 \) to itself. Let \( g \) be a smooth expanding circle map of degree \( D \), where \( D \) is a natural number greater than 1. Let \( \mathcal{M}(g) \) denote the set of \( g \)-invariant Borel probability measures. For any \( \mu \in \mathcal{M}(g) \) that is ergodic, its Lyapunov exponent is defined by \( \lambda_{\mu}(g) = \int_{S^1} \log |g'| d\mu \), and we denote by \( h(\mu, g) \) the metric entropy of \( g \) with respect to \( \mu \). We concentrate our attention on the Lyapunov exponent \( \lambda_{\text{abs}}(g) \) with respect to the unique measure \( \mu_{\text{abs}} \in \mathcal{M}(g) \) that is absolutely continuous with respect to the Lebesgue measure (see Theorem 5.1.16, Proposition 5.1.24 and Corollary 5.1.25 in [5] or Theorem 18 in [9]) and the Lyapunov exponent \( \lambda_{\text{max}}(g) \) with respect to the unique measure \( \mu_{\text{max}} \in \mathcal{M}(g) \) of maximal entropy (see Theorem 19 in [9]).

Throughout the paper, whenever convenient, we will consider a continuous expanding circle map as a piecewise continuous expanding map of the interval \([0, 1]\) such that the values at 0 and 1 coincide mod 1.

The Ruelle inequality [8] shows that for any \( \mu \in \mathcal{M}(g) \), we have \( h(\mu, g) \leq \lambda_{\mu}(g) \). Moreover, \( h(\mu_{\text{abs}}, g) = \lambda_{\text{abs}}(g) \) by [7, Theorem III.1.1]. By the variational principle, the topological entropy of \( g \) is equal to \( h_{\text{top}}(g) = \sup_{\mu \in \mathcal{M}(g)} h(\mu, g) \). In our setting, \( h_{\text{top}}(g) = h(\mu_{\text{max}}, g) \). For the \( \times D \)-map on \( S^1 \) given by \( x \mapsto Dx \) (mod 1), we have that \( \lambda_{\text{abs}}(\times D) = h_{\text{top}}(\times D) = \lambda_{\text{max}}(\times D) = \log D \) and the Lebesgue measure is \( \mu_{\text{abs}} \) and \( \mu_{\text{max}} \).

2010 Mathematics Subject Classification. Primary: 37E10; Secondary: 37A05.

Key words and phrases. Flexibility, expanding maps, Lyapunov exponents, circle maps, invariant measures.
Recall that two continuous circle maps $f$ and $g$ are said to be topologically conjugate if there exists a homeomorphism $h: S^1 \to S^1$ such that $f = h^{-1} \circ g \circ h$. A continuous expanding circle map has degree $D$ if and only if it is topologically conjugate to the $\times D$-map (see [5, Theorem 2.4.6]). Also, topological entropy is invariant under topological conjugacy [5, Corollary 3.1.4]. Therefore, it follows from all of the above that for any smooth expanding circle map $g$ of degree $D$ we have

$$\lambda_{abs}(g) \leq \log D \leq \lambda_{max}(g).$$

If we also have that $\lambda_{abs}(g) = \log D$, then we obtain that $h(\mu_{abs}, g) = h_{top}(g)$. Therefore, $\mu_{abs}$ is the measure of maximal entropy for $g$ and $\lambda_{max}(g) = \lambda_{abs}(g) = \log D$. On the other hand, assume $g$ is a smooth expanding circle map of degree $D$ and $\lambda_{max}(g) = \log D$, i.e., $h_{top}(g) = \int_{S^1} \log |g'| d\mu_{max}$. Then, by Theorem III.1.2-3 in [7] we have that $\mu_{max}$ is the measure which is absolutely continuous with respect to the Lebesgue measure implying that $\lambda_{abs}(g) = \lambda_{max}(g) = \log D$. Thus, we have seen that the equalities in $\lambda_{abs}(g) \leq \log D \leq \lambda_{max}(g)$ can only hold simultaneously. It is a natural question if these inequalities are the only restrictions on the pair of values of the considered Lyapunov exponents. In the following theorem, we answer this question affirmatively by constructing smooth expanding circle maps of degree $D$ that take on all possible values of pairs of Lyapunov exponents $(\lambda_{abs}, \lambda_{max})$ corresponding to measures $\mu_{abs}$ and $\mu_{max}$, respectively.

**Main Theorem.** For any positive numbers $a, b$ such that $a < \log D < b$, there exists a smooth expanding circle map $g$ of degree $D$ such that $\lambda_{abs}(g) = a$ and $\lambda_{max}(g) = b$.

Main Theorem confirms in the setting of expanding circle maps the flexibility philosophy proposed by A. Katok. The flexibility program states that classical smooth systems (diffeomorphisms and flows) are quite flexible in comparison to actions of higher rank abelian groups. The first example in this direction was obtained in work of the author and A. Katok [3] which shows the flexibility for the values of the pair of metric entropy with respect to the Liouville measure and topological entropy for geodesic flow on surfaces of negative curvature with fixed genus greater than or equal to 2 and fixed total area. Another result in the flexibility program is due to J. Bochi, F. Rodriguez Hertz and A. Katok [1] who show how to vary the Lyapunov exponents with respect to the Lebesgue measure for volume-preserving Anosov diffeomorphisms with dominated splittings into one-dimensional bundles.

There are still many open questions related to the flexibility program and many properties of smooth dynamical systems whose flexibility is unknown. In particular, a natural extension of Main Theorem would be to consider a similar problem on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $L_A$ be an Anosov linear area-preserving automorphism of $\mathbb{T}^2$. For $L_A$, we have that the Lyapunov exponent $\lambda_{Leb}(L_A)$ with respect to the Lebesgue measure, topological entropy $h_{top}(L_A)$, and the Lyapunov exponent with respect to the measure of maximal entropy $\lambda_{max}(L_A)$ coincide. Let us consider a smooth Anosov area-preserving diffeomorphism $g$ on $\mathbb{T}^2$ homotopic to $L_A$. Then, we have that

$$\lambda_{Leb}(g) \leq h_{top}(g) = h_{top}(L_A) \leq \lambda_{max}(g).$$

The question is if these inequalities are the only restrictions on the considered Lyapunov exponents for smooth Anosov area-preserving diffeomorphisms on $\mathbb{T}^2$ homotopic to $L_A$. We hope to answer this question in future work.
The rest of the paper consists of the proof of Main Theorem. First, in Section 2 we construct continuous piecewise linear circle maps of degree $D$ which realize all possible values for the pairs of the considered Lyapunov exponents. We then give a procedure for smoothing these maps in Section 3. Finally, in Section 4 we prove the continuity of Lyapunov exponents for the constructed family of circle maps which implies Main Theorem in combination with Theorem 2.5.

2. Flexibility of Lyapunov exponents for SUSD-circle maps. In this section, we show in Theorem 2.5 that SUSD-circle maps (see Definition 2.1) take on all possible values of pairs of Lyapunov exponents corresponding to the measures $\mu_{\text{abs}}$ and $\mu_{\text{max}}$ (see Lemmas 2.2 and 2.3).

**Definition 2.1.** For any natural numbers $D > 1$ and $n, k > 2$ and any positive numbers $\delta \in [D^{-n-1}, D^{-n})$ and $\varepsilon \in [D^{-k-1}, D^{-k})$, we define the **SUSD-circle map** (Speed Up Slow Down - circle map) $f(x; n, \delta; k, \varepsilon)$ of degree $D$ by the formula:

$$f(x; n, \delta; k, \varepsilon) = \begin{cases} \frac{1}{D^n \delta} x & \text{if } x \in [0, \delta), \\ \frac{D - 1}{1 - D^n \delta} x + \left( D^{-n} - \frac{\delta(D - 1)}{1 - D^n \delta} \right) & \text{if } x \in [\delta, D^{-n}), \\ \frac{D - 1}{1 - D^{k+1} \varepsilon} x + \left( D^{-1} - D^{-k} \right) \left( D - \frac{D - 1}{1 - D^{k+1} \varepsilon} \right) & \text{if } x \in [D^{-1} - D^{-k}, D^{-1} - \varepsilon), \\ \frac{1}{D^{k+1} \varepsilon} x + \left( 1 - \frac{1}{D^{k+1} \varepsilon} \right) (\mod 1) & \text{if } x \in [D^{-1} - \varepsilon, D^{-1}) \\ Dx & \text{if } x \in (D^{-1} - \varepsilon, D^{-1}) \cup (D^{-1}, 1). \end{cases}$$

The graph of a SUSD-circle map of degree $D = 2$ is displayed in Figure 1. Note that a SUSD-circle map is a continuous piecewise linear expanding circle map with $f(0; n, \delta; k, \varepsilon) = 0$ and $f(x; n, D^{-n-1}; k, D^{-k-1}) \equiv Dx \pmod{1}$ for any values of the parameters $n, k, \delta$ and $\varepsilon$ that we have allowed. The derivative with respect to $x$ of the SUSD-circle map $f(x; n, \delta; k, \varepsilon)$ outside of points of possible non-differentiability ($x = 0, \delta, D^{-n}, D^{-1} - D^{-k}, D^{-1} - \varepsilon$) is given by the formula:

$$f'(x; n, \delta; k, \varepsilon) = \begin{cases} \frac{1}{D^n \delta} & \text{if } x \in (0, \delta), \\ \frac{D - 1}{1 - D^n \delta} & \text{if } x \in (\delta, D^{-n}), \\ \frac{D - 1}{1 - D^{k+1} \varepsilon} & \text{if } x \in (D^{-1} - D^{-k}, D^{-1} - \varepsilon), \\ \frac{1}{D^{k+1} \varepsilon} & \text{if } x \in (D^{-1} - \varepsilon, D^{-1}), \\ D & \text{if } x \in (D^{-n}, D^{-1} - D^{-k}) \cup (D^{-1}, 1). \end{cases}$$

**Lemma 2.2.** A SUSD-circle map has a unique invariant probability measure $\mu_{\text{abs}}$ that is absolutely continuous with respect to the Lebesgue measure on $S^1$ and is ergodic. In particular, the invariant density $q(x; n, \delta; k, \varepsilon)$ for the SUSD-circle map $f(x; n, \delta; k, \varepsilon)$ of degree $D$ is given by the formula:

$$q(x; n, \delta; k, \varepsilon) = A\chi_{(0,D^{-n})}(x) + A\frac{D(1 - D^n \delta)}{D - 1} \chi_{(D^{-n},1-D^{-k})}(x) + A\frac{D(1 - D^n \delta)}{(D - 1)^2} (D \cdot D^{k+1} \varepsilon + D - 2) \chi_{(1-D^{-k},1)}(x),$$

where $\chi_A$ denotes the characteristic function of the set $A$. Note that $\sum A\chi_{(0,D^{-n})} + A\frac{D(1 - D^n \delta)}{D - 1} \chi_{(D^{-n},1-D^{-k})} + A\frac{D(1 - D^n \delta)}{(D - 1)^2} (D \cdot D^{k+1} \varepsilon + D - 2) \chi_{(1-D^{-k},1)} = 1$.
where
\[ A = \frac{D^n}{1 + D^{n-k+1} \left( (D^n - D^{n-k} - 1) + \frac{D^{n-k+1}}{D-1} D^k \varepsilon + \frac{D^{n-k}}{D-1} (D-2) \right)} \]
and \( \chi_B(x) \) is the characteristic function of \( B \subset S^1 \).

Proof. We fix parameters \( n, k, \delta, \varepsilon \) and omit them from the notations for the rest of the proof.

An integrable function \( q \) on \( S^1 \) is an invariant density of the SUSD-circle map \( \bar{f}(x) = f(x; n, \delta; k, \varepsilon) \) if and only if it is a fixed point of the Frobenius-Perron operator (see [6, Section 2]). In our case, it means that \( q \) has to satisfy the following equality a.e. with respect to the Lebesgue measure:
\[ q(x) = \sum_{y \in \bar{f}^{-1}(x)} \frac{q(y)}{f'(y)}. \] (3)

Recall that \( \bar{f}(0) = 0 \). Let \( \mathcal{E} = \bigcup_{m=1}^{\max\{n,k\}} \bar{f}^{-m}(0) \) be the collection of endpoints of a partition \( \mathcal{P} \) of \( S^1 \) (the interval \([0,1]\)). Notice that \( \bar{f}(\mathcal{E}) \subset \mathcal{E} \) by the definition of \( \bar{f} \). Moreover, for every \( P \in \mathcal{P} \) there exists a natural number \( l \) such that
Using (1) and (3), we find the invariant density $q$ that guarantees (3), we need to satisfy the following equalities.

where $q$ is absolutely continuous with respect to the Lebesgue measure, and the invariant density $q$ is piecewise constant. In particular, $\mu_{abs}$ is ergodic.

We define the following intervals:

- $I_1 = (0, D^{-n})$, $I_1'' = (D^{-n}, D^{-n+1})$, $I_1''' = (D^{-n+1}, D^{-1})$,
- $I_D = (1 - D^{-1}, 1 - D^{-k+1})$, $I_D'' = (1 - D^{-k+1}, 1 - D^{-k})$, $I_D''' = (1 - D^{-k}, 1)$,
- $I_j = ((j-1)D^{-1}, jD^{-1})$ for $j = 2, \ldots, D - 1$.

Using (1) and (3), we find the invariant density $q$ in the following form:

$$q(x) = a_1' \chi_{I_1}(x) + a_1'' \chi_{I_1''}(x) + a_1''' \chi_{I_1'''}(x) + \sum_{j=2}^{D-1} a_j \chi_{I_j}(x) + a_D' \chi_{I_D'}(x) + a_D'' \chi_{I_D''}(x) + a_D''' \chi_{I_D'''}(x),$$

where $a_1', a_1'', a_1'''$, $a_D'$, $a_D''$, $a_D'''$ and $a_j$ for $j = 2, \ldots, D - 1$ are positive constants such that $\mu_{abs}$ is a probability measure and $\chi_B(x)$ is the characteristic function of $B$. To guarantee (3), we need to satisfy the following equalities.

(a) If $x \in I_1'$, we need $a_1' = a_1'D^n\delta + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$. As a result, we obtain that

$$a_1' + \sum_{i=2}^{D-1} a_i = a_1'D(1 - D^n\delta).$$

(b) If $x \in I_1''$, we need $a_1'' = a_1' \frac{1-D^n\delta}{D-1} + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$. Therefore, using (a), we obtain $a_1'' = a_1' \frac{D(1-D^n\delta)}{D-1}$.

(c) If $x \in I_1'''$, we need $a_1''' = a_1'' \frac{D_1-1}{D_1} + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$ and $a_1''' = a_1'' \frac{D_1-1}{D_1} + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$.

Using (a) and (b), we have $a_1''' = a_1'' = a_1' \frac{D(1-D^n\delta)}{D-1}$.

(d) If $x \in I_j$ for $j = 2, \ldots, D - 1$, we need $a_j = a_j'' \frac{D_1-1}{D_1} + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$. Using (a) and (c), we obtain $a_j = a_1' \frac{D(1-D^n\delta)}{D-1} = a_1''$ for $j = 2, \ldots, D - 1$.

(e) If $x \in I_D'$, we need $a_D' = a_1'' \frac{D_2-1}{D_2} + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$ and $a_D' = a_1'' \frac{D_2-1}{D_2} + \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$.

Using (d), we have $a_D' = a_D'' = a_1'' \frac{D(1-D^n\delta)}{D-1} = a_2 = \cdots = a_{D-1} = a_1'' = a_1'''$.

(f) If $x \in I_D''$, we need $a_D'' = a_1'||D-1||D-1||D-n\delta||D-k\delta = \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$. By (c) and (e), we have $a_D'' = a_1' \frac{D(1-D^n\delta)}{D-1} (D \cdot D^k\epsilon + D - 2)$.

(g) If $x \in I_D'''$, we need $a_D''' = a_1'||D-1||D-1||D-n\delta||D-k\delta = \sum_{i=2}^{D-1} \frac{a_i}{D} + \frac{a_D}{D}$. By (e), we obtain $a_D'''(D-1) = a_1''(D \cdot D^k\epsilon + D - 2)$. Using (c), we see that we obtain the same equality for $a_D''$ as in (f).

The total measure of the circle $S^1$ for the measure $\mu_{abs}$ is equal to 1 if

$$a_1'D^{-n} + a_1' \frac{D(1-D^n\delta)}{D-1} (1 - D^{-n} - D^{-k})$$
of degree $C$ and $E$ except the interval $(0, v)$ uniformly in $x$.

Then, notice that, by construction, $\lambda_\text{abs} = \frac{1}{\log \mu}$. Hence, $\lambda_\text{abs} > 0$.

Proof. The statement follows from the definition of the Lyapunov exponent $\lambda_\text{abs}$ and Lemma 2.2.

Corollary 1. Let $g(x; n; u; k, v) = f(x; n; D^{-n}w; k, D^{-k}v)$ be a SUSD-circle map of degree $D$, where $D > 1$ and $n, k > 2$ are natural numbers and $u, v \in [D^{-1}, 1)$.

The Lyapunov exponent $\lambda_\text{abs}(g) = \int_{S^n} g'\,d\mu_\text{abs}$ with respect to $\mu_\text{abs}$ is equal to

$$(1 + (1 - u)(C_1 + C_2v))^{-1} \left[ E(u) + \frac{D^{n-k+1}}{D-1} (1-u)E(v) + (1-u)(C_3 + C_2v) \log D \right],$$

where $E(y) = y \log \frac{1}{y} + (1-y) \log \frac{D-1}{y}$, $C_1 = \frac{D}{D-1} \left( D^n - 1 - D^{-n+1} \right)$, $C_2 = \frac{D^{n-k+2}}{(D-1)^2}$ and $C_3 = \frac{D}{D-1} \left( D^n - 1 - D^{-n+1} \right)$.

Corollary 2. Fix natural numbers $D > 1$ and $n > 2$ and a positive number $u \in [D^{-1}, 1)$. Let $\{g(x; k, v)\}$ be a family of SUSD-circle maps of degree $D$, where $g(x; k, v) = f(x; n; D^{-n}w; k, D^{-k}v)$, $k > 2$ is a natural number and $v \in [D^{-1}, 1)$. Then, $g(x; k, D^{-1})$ is independent of $k$ and

$$\lambda_\text{abs}(g(x; k, v)) - \lambda_\text{abs}(g(x; k, D^{-1})) \text{ tends to } 0 \text{ as } k \to \infty$$

uniformly in $v$.

Proof. Notice that, by construction, $g(x; k, D^{-1})$ acts as the $\times D$-map everywhere except the interval $(0, D^{-n})$ and is independent of $k$ on that interval. Therefore, $g(x; k, D^{-1})$ is independent of $k$.

By Corollary 1, we have

$$\lambda_\text{abs}(g(x; k, v)) = \frac{E(u) + \frac{D^{n-k+1}}{D-1} (1-u)E(v) + (1-u)(C_3 + C_2v) \log D}{1 + (1-u)(C_1 + C_2v)}$$

and

$$\lambda_\text{abs}(g(x; k, D^{-1})) = \frac{E(u) + (1-u) \frac{D(D^{n-1})}{D-1} \log D}{1 + (1-u) \frac{D(D^{n-1})}{D-1}},$$

where $E(y) = y \log \frac{1}{y} + (1-y) \log \frac{D-1}{y}$, $C_1 = \frac{D}{D-1} \left( D^n - 1 - D^{-n+1} \right)$, $C_2 = \frac{D^{n-k+2}}{(D-1)^2}$ and $C_3 = \frac{D}{D-1} \left( D^n - 1 - D^{-n+1} \right)$.

The function $E(y)$ is monotonically decreasing and its values vary from $\log D$ to 0 as $y$ varies from $D^{-1}$ to 1 because $E'(y) = -\log y - \log(D-1) + \log(1-y) = \log \frac{1-y}{y(D-1)} \leq 0$ for $y \in [D^{-1}, 1)$. Also, the constants $C_1, C_2, C_3$ are independent of $v$, and we have $C_1 \to \frac{D}{D-1} \left( D^n - 1 \right)$, $C_2 \to 0$ and $C_3 \to \frac{D}{D-1} \left( D^n - 1 \right)$ as $k \to \infty$.

Furthermore, using the fact that $v \in [D^{-1}, 1)$, we obtain

$$\lambda_\text{abs}(g(x; k, v)) \geq \frac{E(u) + (1-u)(C_3 + C_2D^{-1}) \log D}{1 + (1-u)(C_1 + C_2)}$$

and

$$\lambda_\text{abs}(g(x; k, v)) \leq \frac{E(u) + \frac{D^{n-k+1}}{D-1} (1-u)D + (1-u)(C_3 + C_2) \log D}{1 + (1-u)(C_1 + C_2D^{-1})}.$$
Both bounds on \( \lambda_{\text{abs}}(g(x; k, v)) \) are independent of \( v \) and tend to
\[
\frac{E(u) + (1-u)\frac{D}{Dn-1}}{1+1-u)\frac{D}{Dn-1}} \log D = \lambda_{\text{abs}}(g(x; k, D^{-1})) \text{ as } k \to \infty.
\]
Therefore, we obtain the desired result. \( \square \)

**Corollary 3.** Fix natural numbers \( D > 1 \) and \( n > 2 \). Let \( \{\hat{g}(x; u)\} \) be a family of SUSD-circle maps of degree \( D \), where \( \hat{g}(x; u) = f(x; n, D^{-n}u; 3, D^{-3}D^{-1}) \) and \( u \in [D^{-1}, 1) \). Then, we have
\[
\lambda_{\text{abs}}(\hat{g}(x; u)) = \log D - \frac{(\log D - E(u))}{1 + (1-u)\frac{D}{Dn-1}}, \tag{4}
\]
where \( E(y) = y \log \frac{1}{y} + (1-y)\log \frac{D-1}{D} \). In particular, \( \lambda_{\text{abs}}(\hat{g}(x; u)) \) is monotonically decreasing and its values vary from \( \log D \) to 0 as \( u \) varies from \( D^{-1} \) to 1.

**Proof.** By Corollary 1, we have the formula (4) for \( \lambda_{\text{abs}}(\hat{g}) \).

Let \( C = \frac{D(Dn-1)}{D-1} \). The derivative of \( \lambda_{\text{abs}}(\hat{g}(x; u)) \) with respect to \( u \) is the following:
\[
\frac{\partial}{\partial u} \lambda_{\text{abs}}(\hat{g}(x; u)) = \frac{-1}{(1+(1-u)C)^2} \left[ -E'(u)(1+(1-u)C) + (\log D - E(u))C \right]
\]
\[
-\frac{1}{(1+(1-u)C)^2} \left[ (\log u + \log(D-1) - \log(1-u))(1+C) + (\log(1-u) - \log(D-1))C + C \log D \right]
\]
\[
= -\frac{1}{(1+(1-u)C)^2} \left[ (1+C) \log u - \log(1-u) + \log((D-1)D^C) \right]
\]

Let \( \alpha(u) = (1+C) \log u - \log(1-u) + \log((D-1)D^C) \). Then, \( \alpha(D^{-1}) = 0 \). Also, we have that its derivative is equal to
\[
\alpha'(u) = \frac{1}{u}(1+C) + \frac{1}{1-u} > 0,
\]
as \( u \in [D^{-1}, 1) \). It follows, that \( \alpha(u) > 0 \) when \( u \in [D^{-1}, 1) \).

Therefore, we obtain that \( \frac{\partial}{\partial u} \lambda_{\text{abs}}(\hat{g}(x; u)) < 0 \) for \( u \in [D^{-1}, 1) \), i.e., \( \lambda_{\text{abs}}(\hat{g}(x; u)) \) is a monotonically decreasing function with respect to \( u \).

Furthermore, by (4), \( \lambda_{\text{abs}}(\hat{g}(x; D^{-1})) = \log D \) and \( \lambda_{\text{abs}}(\hat{g}(x; u)) \) tends to 0 as \( u \to 1 \). \( \square \)

**Lemma 2.3.** A SUSD-circle map has a unique invariant probability measure \( \mu_{\text{max}} \) of maximal entropy that is ergodic. In particular, for the SUSD-circle map \( f(x; n, \delta; k, \varepsilon) \) of degree \( D \) we have
\[
\mu_{\text{max}}((0, \delta)) = D^{-n-1}, \quad \mu_{\text{max}}((\delta, D^{-n})) = D^{-n-1}(D-1),
\]
\[
\mu_{\text{max}}((D^{-1} - \varepsilon, D^{-1})) = D^{-k-1}, \quad \mu_{\text{max}}((D^{-1} - D^{-k})) = D^{-k-1}(D-1).
\]

**Proof.** A SUSD-circle map of degree \( D \) is topologically conjugate to the \( \times D \)-map and the conjugacy \( h \) can be constructed via coding (see Theorem 2.4.6 and its proof in [5]). In particular, for the SUSD-circle map \( f(x; n, \delta; k, \varepsilon) \) of degree \( D \), we have \( \times D \circ h = h \circ f \) and
\[
h(\delta) = D^{-n-1}, \quad h(D^{-n}) = D^{-n}, \quad h(D^{-1} - D^{-k}) = D^{-1} - D^{-k},
\]
\[
h(D^{-1} - \varepsilon) = D^{-1} - D^{-k-1}, \quad h(jD^{-1}) = jD^{-1} \text{ for } j = 0, \cdots, D. \]
The Lebesgue measure is the unique measure of maximal entropy for the $xD$-map that is ergodic [5, Sections 4b, 4c, 5a]. A measure of maximal entropy is mapped to a measure of maximal entropy under the topological conjugacy. From that, we get the statement of the lemma.

**Corollary 4.** Let $g(x; n, u; k, v) = f(x; n, D^{-n}u; k, D^{-k}v)$ be a SUSD-circle map of degree $D$, where $D > 1$ and $n, k > 2$ are natural numbers and $u, v \in [D^{-1}, 1]$.

The Lyapunov exponent $\lambda_{max}(g) = \int g' d\mu_{max}$ with respect to $\mu_{max}$ is equal to

$$\lambda_{max}(g) = \frac{\log \frac{1}{D} + (D - 1) \log \frac{D - 1}{D^{n+1}}}{D^{n+1}} + \frac{\log \frac{1}{D} + (D - 1) \log \frac{D - 1}{D^{k+1}}}{D^{k+1}} + (1 - D^{-n} - D^{-k}) \log D.$$

**Proof.** The statement follows from the definition of the Lyapunov exponent $\lambda_{max}$ and Lemma 2.3.

**Corollary 5.** Fix natural numbers $D > 1$ and $n, k > 2$ and a positive number $u \in [D^{-1}, 1]$. Let $\{g(x; v)\}$ be a family of SUSD-circle maps of degree $D$, where $g(x; v) = f(x; n, D^{-n}u; k, D^{-k}v)$ and $v \in [D^{-1}, 1]$. Then, $\lambda_{max}(g(x; v))$ is monotonically increasing to $\infty$ as $v$ varies from $D^{-1}$ to 1. In particular, $\lambda_{max}(g(x; v)) \geq \frac{M(v)}{D^{n+1}}$, where $M(y) = \log \frac{1}{y} + (D - 1) \log \frac{D - 1}{1 - y}$ is monotonically increasing to $\infty$ as $y$ varies from $D^{-1}$ to 1.

**Proof.** By Corollary 4, we have

$$\lambda_{max}(g(x; v)) = \frac{M(u)}{D^{n+1}} + \frac{M(v)}{D^{k+1}} + (1 - D^{-n} - D^{-k}) \log D.$$

The function $M(y)$ is monotonically increasing and its values vary from $D \log D$ to $\infty$ as $y$ varies from $D^{-1}$ to 1 because $M'(y) = \frac{D - 1}{y(1 - y)} \geq 0$ for $y \in [D^{-1}, 1)$. Moreover, $(1 - D^{-n} - D^{-k}) \log D > 0$. The combination of the above facts implies the corollary.

**Lemma 2.4.** Fix a natural number $D > 1$. Let $\{\tilde{g}(x; n, u)\}$ be a family of SUSD-circle maps of degree $D$, where $\tilde{g}(x; n, u) = f(x; n, D^{-n}u; 3, D^{-3}D^{-1})$, $n > 2$ is a natural number and $u \in [D^{-1}, 1)$. For any positive numbers $\alpha < \log D$ and $\beta$, there exists $\hat{N} > 2$ and $\hat{u} = \hat{u}(\hat{N}) \in [D^{-1}, 1)$ such that we have

1. $\lambda_{abs}(\tilde{g}(x; \hat{N}, \hat{u})) \leq \alpha,$
2. $\lambda_{max}(\tilde{g}(x; \hat{N}, u)) \leq \log D + \beta$ for any $u \in [D^{-1}, \hat{u}],$ 
3. for any $\delta \in [\alpha, \log D]$ there exists $\hat{u} \in [D^{-1}, \hat{u}]$ such that $\lambda_{abs}(\tilde{g}(x; \hat{N}, \hat{u})) = \delta$.

**Proof.** Fix a positive number $\varepsilon < \frac{\alpha}{1 + \log D - 1}$. By (4), to guarantee item 1 in the lemma, it is enough to show that for any $n > 2$ there exists $\hat{u} = \hat{u}(n) \in [D^{-1}, 1)$ such that $(1 - u) \frac{D^{D^{-n} - 1}}{D^{(D^{-n}) - 1}} \leq \varepsilon$ and $E(u) = u \log \frac{1}{u} + (1 - u) \log \frac{D - 1}{1 - u} \leq \varepsilon$ for any $u \in [\hat{u}, 1)$.

First, $E(u)$ is independent of $n$ and monotonically decreasing, and its values vary from $\log D$ to 0 as $u$ varies from $D^{-1}$ to 1. Therefore, there exists $\hat{u} \in [D^{-1}, 1)$ such that for any $n > 2$ and $u \in [\hat{u}, 1)$ we have $E(u) \leq \varepsilon$. Second, $(1 - u) \frac{D^{D^{-n} - 1}}{D^{(D^{-n}) - 1}} \leq \varepsilon$ if and only if $u \geq 1 - \varepsilon \frac{D - 1}{D^{(D^{-n}) - 1}}$. Therefore, $\hat{u}(n) = \max\{D^{-1}, \hat{u}, 1 - \varepsilon \frac{D - 1}{D^{(D^{-n}) - 1}}\}$ works. If $n$ is sufficiently large, then $\hat{u}(n) = 1 - \varepsilon \frac{D - 1}{D^{(D^{-n}) - 1}}$. 
By Corollary 4, we obtain \( \lambda_{\max}(\hat{g}(x; n, u)) = \frac{M(u)}{D} + (1 - D^{-n}) \log D \), where \( M(u) = \log \frac{\hat{a}}{\alpha} + (D - 1) \log \frac{D - 1}{\alpha} \). Moreover, for a fixed \( n \), we have \( \lambda_{\max}(\hat{g}(x; n, u)) \) is monotonically increasing because the function \( M(u) \) is monotonically increasing and its values vary from \( D \log D \) to \( \infty \) as \( u \) varies from \( D^{-1} \) to 1. To guarantee items 1 and 2, it is enough to show that there exists \( \hat{N} > 2 \) such that \( \hat{u}(1 - \hat{u})^{D-1} \geq \frac{(D^{-1})^{D-1}}{\alpha} e^{-D^{\hat{N}+1} \beta} \). Therefore, we need to show that there exists \( \hat{N} > 2 \) such that

\[
\left( \frac{D(D^{\hat{N}} - 1)}{D-1} - \varepsilon \right) \varepsilon^{D-1} \geq \frac{(D^{\hat{N}} - 1)^\beta}{D-1} e^{-D^{\hat{N}+1} \beta}.
\]

Notice that \( \frac{D(D^{\hat{N}} - 1)}{D-1} \) tends to \( \infty \) as \( n \to \infty \). Moreover, \( \frac{(D^{\hat{N}} - 1)^{D-1}}{\alpha} e^{-D^{\hat{N}+1} \beta} \) tends to 0 as \( n \to \infty \) because, by applying L'Hôpital's rule, \( \lim_{y \to \infty} \frac{(y-1)^D}{y} = \lim_{y \to \infty} \frac{D(D-1)\ldots(D-y+2)}{y} = 0 \). Therefore, for a fixed \( \varepsilon \) and \( \beta \) there exists \( \hat{N} > 2 \) such that \( \left( \frac{D(D^{\hat{N}} - 1)}{D-1} - \varepsilon \right) \varepsilon^{D-1} \geq \frac{(D^{\hat{N}} - 1)^\beta}{D-1} e^{-D^{\hat{N}+1} \beta} \).

Finally, sufficiently large \( \hat{N} \) and \( \hat{u}(\hat{N}) = \hat{u}(\hat{N}) = 1 - \varepsilon \frac{D-1}{D(D^{\hat{N}} - 1)} \) provide Lemma 2.4 because \( \lambda_{\max}(\hat{g}(x; \hat{N}, u)) \) is a continuous function of \( u \), \( \lambda_{\max}(\hat{g}(x; \hat{N}, D^{-1})) = \log D \), and \( \lambda_{\max}(\hat{g}(x; \hat{N}, \hat{u})) \leq \alpha \). \( \square \)

All the inequalities for Lyapunov exponents with respect to \( \mu_{abs} \) and \( \mu_{max} \) discussed in the introduction will still hold for a SUSD-circle map. In Theorem 2.5, we show that SUSD-circle maps take on all possible values of pairs of Lyapunov exponents \( (\lambda_{abs}, \lambda_{max}) \) corresponding to measures \( \mu_{abs} \) and \( \mu_{max} \), respectively.

**Theorem 2.5.** Let \( D > 1 \) be a natural number and \( a, b, \delta \) be any numbers such that \( \delta \geq 0 \) and

\[
0 < a - \delta \leq a \leq a + \delta < \log D < b - \delta \leq b \leq b + \delta.
\]

There exist natural numbers \( \hat{N}, \hat{K} > 2 \) and constants \( \hat{u}, \hat{v} \in [D^{-1}, 1] \) with the following property. Let \( \{f_{u,v}(x)\} \) be a family of SUSD-circle maps of degree \( D \), where \( u \in [D^{-1}, \hat{u}], v \in [D^{-1}, \hat{v}] \), and \( f_{u,v}(x) = f(x; \hat{N}, D^{-\hat{N}}u; \hat{K}, D^{-\hat{K}}v) \) (see Definition 2.1). Then, for any \( (A, B) \in [a - \delta, a + \delta] \times [b - \delta, b + \delta] \) there exists \( (U, V) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}] \) with \( (\lambda_{abs}(f_{U,V}), \lambda_{max}(f_{U,V})) = (A, B) \). In particular, \( \{f_{u,v}(x)\} \) is a continuous family with respect to \( (u, v) \) in the sup norm by the definition of \( f_{u,v} \).

**Proof.** Let \( \hat{N} > 2 \) and \( \hat{u} \in [D^{-1}, 1] \) be constants coming from Lemma 2.4 applied for \( \alpha = \frac{a+\delta}{2} \) and \( \beta = \frac{b-\delta}{2} \log D \).

Let \( C = \min\left\{ \frac{a-\delta}{2}, \frac{\log D - (a+\delta)}{2} \right\} \). By Corollary 2, there exists \( \hat{K} \) such that

\[
|\lambda_{abs}(f(x; \hat{N}, D^{-\hat{N}}\hat{u}; \hat{K}, D^{-\hat{K}}v)) - \lambda_{abs}(f(x; \hat{N}, D^{-\hat{N}}\hat{u}; \hat{K}, D^{-\hat{K}}D^{-1}))| < C
\]

and

\[
|\lambda_{abs}(f(x; \hat{N}, D^{-\hat{N}}D^{-1}; \hat{K}, D^{-\hat{K}}v)) - \lambda_{abs}(f(x; \hat{N}, D^{-\hat{N}}D^{-1}; \hat{K}, D^{-\hat{K}}D^{-1}))| < C
\]

for any \( v \in [D^{-1}, 1] \), where \( f \) is a SUSD-circle map of degree \( D \). Moreover, recall that \( f(x; \hat{N}, D^{-\hat{N}}u; \hat{K}, D^{-\hat{K}}D^{-1}) \) is independent of \( \hat{K} \) (see Corollary 2). In particular, we have

\[
f(x; \hat{N}, D^{-\hat{N}}u; \hat{K}, D^{-\hat{K}}D^{-1}) = f(x; \hat{N}, D^{-\hat{N}}u; 3, D^{-3}D^{-1})
\]

for any \( x \in S^1 \) and \( u \in [D^{-1}, 1] \). Therefore, Lemma 2.4 implies that

\[
\lambda_{max}(f(x; \hat{N}, D^{-\hat{N}}u; \hat{K}, D^{-\hat{K}}D^{-1})) \leq \frac{b - \delta + \log D}{2}
\]

for any \( u \in [D^{-1}, \hat{u}] \).
and for any $A \in [\frac{n-\delta}{2}, \log D]$, there exists $U \in [D^{-1}, \hat{u}]$ such that

$$\lambda_{abs}(f(x; N, D^{-N}U; \hat{K}, D^{-\hat{K}}D^{-1})) = A.$$  

By Corollary 5, there exists $\hat{v} \in [D^{-1}, 1)$ such that

$$\lambda_{max}(f(x; \hat{N}, D^{-\hat{N}}\hat{u}; \hat{K}, D^{-\hat{K}}\hat{v})) \geq b + 2\delta \quad \text{for any} \quad u \in [D^{-1}, 1).$$

It is easy to see from Corollaries 1 and 4 that the map

$$(u, v) \mapsto (\lambda_{abs}(f(x; \hat{N}, D^{-\hat{N}}u; \hat{K}, D^{-\hat{K}}v)), \lambda_{max}(f(x; \hat{N}, D^{-\hat{N}}u; \hat{K}, D^{-\hat{K}}v)))$$

is continuous for $(u, v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]$. Therefore, using the topological fact that there is no retraction of the square onto its boundary, we obtain that for any $(A, B) \in [a - \delta, a + \delta] \times [b - \delta, b + \delta]$ there exist $(U, V) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]$ such that

$$\lambda_{abs}(f(x; \hat{N}, D^{-\hat{N}}U; \hat{K}, D^{-\hat{K}}V)) = A$$

and

$$\lambda_{max}(f(x; \hat{N}, D^{-\hat{N}}U; \hat{K}, D^{-\hat{K}}V)) = B.$$

\[\square\]

### 3. Smoothing a SUSD-circle map

In this section, we show how to smoothen a SUSD-circle map to obtain a smooth expanding circle map.

**Lemma 3.1.** Let $f$ be a SUSD-circle map of degree $D$ (see Definition 2.1), where $D$ is a natural number. Denote by $\hat{f}(x)$ the continuous map on $\mathbb{R}$ that is a lift of $f(x)$ such that $\hat{f}(0) = f(0)$. Let $\theta_{\alpha}(x)$ be a smooth positive even function on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} \theta_{\alpha}(y)dy = 1$ and $\theta_{\alpha}(x) = 0$ if $x \notin (-\alpha, \alpha)$, where $\alpha > 0$. Define $\hat{f}^{\alpha}(x) = \int_{\mathbb{R}} f(x - y)\theta_{\alpha}(y)dy$ for any $x \in \mathbb{R}$. Then, we have $f^{\alpha}(x) = \hat{f}^{\alpha}(x) \pmod{1}$ is a smooth expanding circle map of degree $D$ for any sufficiently small $\alpha$. Moreover, $f^{\alpha}(x) = f(x)$ outside of $\alpha$-neighborhoods of the points of non-differentiability of $f$.

Note that $\hat{f}^{\alpha}$ is the convolution of $\hat{f}$ with $\theta_{\alpha}$.

**Proof.** Observe that the collection of points of non-differentiability for $\hat{f}(x)$ is a countable set because by Definition 2.1 we have that $f(x)$ has at most 6 points of non-differentiability and $\hat{f}(x)$ is the lift of $f(x)$ to $\mathbb{R}$. Let $\{p_{i}\}_{i=-\infty}^{\infty}$ be an indexing of the points of non-differentiability for $\hat{f}(x)$ such that $\hat{f}(x)$ is linear on $(p_{i}, p_{i+1})$. Choose $\alpha$ sufficiently small such that $(p_{i} - \alpha, p_{i} + \alpha) \cap (p_{j} - \alpha, p_{j} + \alpha) = \emptyset$ for $i \neq j$.

By the properties of convolution, we have that $\hat{f}^{\alpha}$ is a smooth map on $\mathbb{R}$ and

$$\frac{d}{dx} \hat{f}^{\alpha}(x) = \int_{\mathbb{R}} \left( \frac{d}{dx} \hat{f}(x - y) \right) \theta_{\alpha}(y)dy$$

$$= \int_{-\alpha}^{\alpha} L_i \theta_{\alpha}(y)dy + \int_{x-p_{i}}^{x} L_{i-1} \theta_{\alpha}(y)dy$$

for any $x \in [p_{i} - \alpha, p_{i} + \alpha]$. Therefore, using the fact that $\int_{-\alpha}^{\alpha} \theta_{\alpha}(y)dy = 1$, we obtain

$$\min \{L_{i-1}, L_i\} \leq \frac{d}{dx} \hat{f}^{\alpha}(x) \leq \max \{L_{i-1}, L_i\} \quad \text{for any} \quad i \in [p_{i} - \alpha, p_{i} + \alpha] \text{ and } \hat{f}^{\alpha}(x) \text{ is a smooth expanding map on } \mathbb{R} \text{ because } L_i > 1 \text{ for any } i.$$  

Moreover, $f(x + N) = \hat{f}(x) + ND$ for any integer number $N$. Therefore,
\[ \hat{f}^\alpha(x + N) = \int_{\mathbb{R}} \hat{f}(x + N - y)\theta_\alpha(y)dy \]
\[ = \int_{\mathbb{R}} \hat{f}(x - y)\theta_\alpha(y)dy + ND \int_{\mathbb{R}} \theta_\alpha(y)dy = \hat{f}^\alpha(x) + ND \]
for any integer number \( N \). As a result, \( f^\alpha(x) = \hat{f}^\alpha(x) \) (mod 1) is a smooth expanding circle map.

By a property of convolution, \( f^\alpha \) can be made arbitrarily close to \( f \) with respect to the sup norm for sufficiently small \( \alpha \). As a result, for sufficiently small \( \alpha \) the degree of \( f^\alpha \) is \( D \) (see [5, Lemma 2.4.5]).

We now show that \( \hat{f}^\alpha(x) = \hat{f}(x) \) if \( x \notin \bigcup_i (p_i - \alpha, p_i + \alpha) \).

By Definition 2.1, we obtain that \( \hat{f}(x) = L_i x + C_i \) for some constants \( L_i > 1 \) and \( C_i \) if \( x \in (p_i, p_i + 1) \). Therefore, for any \( x \in (p_i + \alpha, p_i + 1 - \alpha) \) and \( y \in (-\alpha, \alpha) \) we have \( f(x - y) = L_i (x - y) + C_i \). Consequently, using the properties of \( \theta_\alpha \), for any \( x \in (p_i + \alpha, p_i + 1 - \alpha) \) we obtain that
\[ \hat{f}^\alpha(x) = \int_{-\alpha}^{\alpha} \hat{f}(x - y)\theta_\alpha(y)dy = \int_{-\alpha}^{\alpha} (L_i(x - y) + C_i)\theta_\alpha(y)dy \]
\[ = L_i x \int_{\mathbb{R}} \theta_\alpha(y)dy - L_i \int_{\mathbb{R}} \theta_\alpha(y)dy + C_i \int_{\mathbb{R}} \theta_\alpha(y)dy \]
\[ = L_i x \cdot 1 - L_i \cdot 0 + C_i \cdot 1 = \hat{f}(x) \cdot 1 \]

Finally, using the fact that \( f^\alpha(x) = \hat{f}^\alpha(x) \) (mod 1), we have \( f^\alpha(x) = f(x) \) outside of \( \alpha \)-neighborhoods of the points of non-differentiability of \( f \).

4. Proof of main theorem. Throughout this section, we fix the following notations that we will use in the statements of the results.

Let \( D > 1 \) be a natural number and \( a, b \) be positive numbers from the statement of Main Theorem. Choose any positive number \( \delta \) such that
\[ 0 < a - \delta < a < a + \delta < \log D < b - \delta < b < b + \delta. \]

By Theorem 2.5, there exist natural numbers \( \tilde{K}, \tilde{N} > 2 \) and constants \( \tilde{u}, \tilde{v} \in [D^{-1}, 1] \) such that we have a continuous family of SUSD-circle maps \( \{ f_{u,v}(x) \} \) of degree \( D \), where \( u \in [D^{-1}, \tilde{u}], v \in [D^{-1}, \tilde{v}] \), and \( f_{u,v}(x) = f(x; \tilde{N}, D^{-\tilde{N}}; u, \tilde{K}, D^{-\tilde{K}}; v) \) with the following property. For any \( (A, B) \in [a - \delta, a + \delta] \times [b - \delta, b + \delta] \) there exist \( (U, V) \in [D^{-1}, \tilde{u}] \times [D^{-1}, \tilde{v}] \) with \( (\lambda_{abs}(f_{U,V}), \lambda_{max}(f_{U,V})) = (A, B) \).

Denote by \( \hat{f}_{u,v}(x) \) the continuous map on \( \mathbb{R} \) that is a lift of \( f_{u,v}(x) \) such that \( \hat{f}_{u,v}(0) = f_{u,v}(0) \).

Choose a smooth positive even function \( \theta_\alpha(x) \) on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} \theta_\alpha(y)dy = 1 \) and \( \theta_\alpha(x) = 0 \) if \( x \notin (-\alpha, \alpha) \), where \( \alpha > 0 \). Let \( \hat{f}_{u,v}^\alpha(x) = \int_{\mathbb{R}} \hat{f}_{u,v}(x - y)\theta_\alpha(y)dy \) for any \( x \in \mathbb{R} \) and any \( u \in [D^{-1}, \tilde{u}] \) and \( v \in [D^{-1}, \tilde{v}] \). By Lemma 3.1 and continuity of the family \( \{ f_{u,v}(x) \} \) with respect to \( (u, v) \), for sufficiently small \( \alpha \) we have \( f_{u,v}^\alpha(x) = \hat{f}_{u,v}^\alpha(x) \) (mod 1) is a smooth expanding circle map of degree \( D \) such that the \( \alpha \)-neighborhood of any point of non-differentiability of \( f_{u,v} \) does not contain other points of non-differentiability of \( f_{u,v} \) and \( f_{u,v}^\alpha(x) = f_{u,v}(x) \) outside of the union of the \( \alpha \)-neighborhoods of the points of non-differentiability of \( f_{u,v} \) for any \( u \in [D^{-1}, \tilde{u}] \) and \( v \in [D^{-1}, \tilde{v}] \).
By the topological fact that there is no retraction of the square onto its boundary, to prove Main Theorem it is enough to show that there exists \( \alpha > 0 \) such that:

(A) The map \((u, v) \mapsto (\lambda_{\text{abs}}(f^\alpha, u, v), \lambda_{\text{max}}(f^\alpha, u, v))\) is continuous for any \( u \in [D^{-1}, \tilde{u}] \) and \( v \in [D^{-1}, \tilde{v}] \);

(B) For any \( u \in [D^{-1}, \tilde{u}] \) we have \( \lambda_{\text{max}}(f^\alpha_{u,D^{-1}}) \leq b - \frac{\delta}{2} \) and for any \( A \in [a - \frac{\delta}{2}, a + \frac{\delta}{2}] \) there exists \( U \in [D^{-1}, \tilde{u}] \) such that \( \lambda_{\text{abs}}(f^\alpha_{U,D^{-1}}) = A \);

(C) For any \( u \in [D^{-1}, \tilde{u}] \) we have \( \lambda_{\text{max}}(f^\alpha_{u,\tilde{v}}) \geq b + \frac{\delta}{2} \) and for any \( A \in [a - \frac{\delta}{2}, a + \frac{\delta}{2}] \) there exists \( U \in [D^{-1}, \tilde{u}] \) such that \( \lambda_{\text{abs}}(f^\alpha_{U,\tilde{v}}) = A \);

(D) For any \( v \in [D^{-1}, \tilde{v}] \) we have \( \lambda_{\text{abs}}(f^\alpha_{D^{-1},v}) \geq a + \frac{\delta}{2} \) and for any \( B \in [b - \frac{\delta}{2}, b + \frac{\delta}{2}] \) there exists \( V \in [D^{-1}, \tilde{v}] \) such that \( \lambda_{\text{max}}(f^\alpha_{D^{-1},V}) = B \);

(E) For any \( v \in [D^{-1}, \tilde{v}] \) we have \( \lambda_{\text{abs}}(f^\alpha_{\tilde{u},v}) \leq a - \frac{\delta}{2} \) and for any \( B \in [b - \frac{\delta}{2}, b + \frac{\delta}{2}] \) there exists \( V \in [D^{-1}, \tilde{v}] \) such that \( \lambda_{\text{max}}(f^\alpha_{\tilde{u},V}) = B \).

First, we show (A) in Lemmas 4.1 and 4.2. Then, we obtain (B) - (E) combining (A) with Lemmas 4.4 and 4.5 that show the uniform convergence on \([D^{-1}, \tilde{u}] \times [D^{-1}, \tilde{v}]\) of \( (\lambda_{\text{abs}}(f^\alpha_{u,v}), \lambda_{\text{max}}(f^\alpha_{u,v})) \) to \((\lambda_{\text{abs}}(f_{u,v}), \lambda_{\text{max}}(f_{u,v}))\) as \( \alpha \to 0 \).

**Lemma 4.1.** For fixed sufficiently small \( \alpha > 0 \), the map \((u, v) \mapsto \lambda_{\text{max}}(f^\alpha_{u,v})\) is continuous on \([D^{-1}, \tilde{u}] \times [D^{-1}, \tilde{v}]\).

**Proof.** For any \((u, v), (u_0, v_0) \in [D^{-1}, \tilde{u}] \times [D^{-1}, \tilde{v}]\), we have that \( f^\alpha_{u,v} \) and \( f^\alpha_{u_0,v_0} \) are smooth expanding circle maps of degree \( D \). Therefore, they are topologically conjugate. Furthermore, if \((u, v)\) is close enough to \((u_0, v_0)\), then \( f^\alpha_{u,v} \) and \( f^\alpha_{u_0,v_0} \) are close in the sup norm. This implies that \( f^\alpha_{u,v} \) and \( f^\alpha_{u_0,v_0} \) are close in the sup norm and the conjugating homeomorphism \( h^{u_0,v_0} \) (i.e., \( f^\alpha_{u_0,v_0} = (h^{u_0,v_0})^{-1} \circ f^\alpha_{u,v} \circ h^{u_0,v_0} \)) can be chosen close to the identity map (see [5, Theorem 2.4.6] and its proof). Let \( \mu_{\text{max}} \) and \( \mu_{\text{max}}^{u_0,v_0} \) be the measures of maximal entropy for \( f^\alpha_{u,v} \) and \( f^\alpha_{u_0,v_0} \), respectively.

Under the topological conjugacy, the measure of maximal entropy is mapped to the measure of maximal entropy, i.e., \( \mu_{\text{max}}^{u_0,v_0} \) is the pushforward of \( \mu_{\text{max}} \) by \( h^{u_0,v_0} \).

Furthermore, we have

\[
\left| \lambda_{\text{max}}(f^\alpha_{u,v}) - \lambda_{\text{max}}(f^\alpha_{u_0,v_0}) \right| = \int_{S^1} \left| \log \left( \frac{d}{dx} f^\alpha_{u,v} \right) \right| \left| \frac{d}{dx} f^\alpha_{u_0,v_0} \right| \left( y \right) dy
\]

The second term in the inequality above can be made arbitrarily small because \( \frac{d}{dx} f^\alpha_{u_0,v_0} \) is a smooth function on \( S^1 \) and if \((u, v)\) is close enough to \((u_0, v_0)\), then \( h^{u_0,v_0} \) is close to \( y \) for any \( y \in S^1 \).

Now we show that the first term in the inequality above can be made small if \((u, v)\) is close enough to \((u_0, v_0)\).

Recall that for any \((u, v) \in [D^{-1}, \tilde{u}] \times [D^{-1}, \tilde{v}]\), we have \( f^\alpha_{u,v}(x) = \hat{f}^\alpha_{u,v}(x) \mod 1 \) on \( S^1 \) where \( \hat{f}^\alpha_{u,v}(x) = \int_{\mathbb{R}} \hat{f}_{u,v}(x - y) \theta_a(y) dy \) for any \( x \in \mathbb{R} \). It is enough to show
that if \((u,v)\) is close enough to \((u_0,v_0)\) then
\[
\sup_{z \in [-\alpha,1-\alpha]} \frac{d}{dx} \hat{f}_{u,v}(x|_{x=z}) = \frac{d}{dx} \hat{f}_{u_0,v_0}(x|_{x=z})
\]
is small.

Let \(\alpha > 0\) be sufficiently small such that for any \((u,v) \in [D^{-1},\hat{u}] \times [D^{-1},\hat{v}]\) the \(2\alpha\)-neighborhoods of 0, \(D^{-N}u\), \(D^{-N}\hat{u}\), \(D^{-1} - D^{-K}\), \(D^{-1} - D^{-K}\hat{v}\), \(D^{-1}\) do not intersect. By the properties of convolution and (1), for such \(\alpha\) we have
\[
\frac{d}{dx} \hat{f}_{u,v}|_{x=z} = \int_{\mathbb{R}} \left( \frac{d}{dx} \hat{f}_{u,v}|_{x=z-y} \right) \theta_\alpha(y) dy.
\]

In particular, we have \(\frac{d}{dx} \hat{f}_{u,v}|_{x=z}\) is equal to:

1. \(\int_{-\alpha}^{\alpha} z^\alpha \frac{1}{\alpha} \theta_\alpha(y) dy + \int_{\alpha}^{\infty} D \theta_\alpha(y) dy\) if \(z \in (-\alpha,\alpha]\);
2. \(\frac{1}{\alpha}\) if \(z \in (\alpha,D^{-N}u - \alpha]\);
3. \(\int_{-\alpha}^{D^{-N}u} z^\alpha \frac{D^{-1} - \alpha}{\alpha} \theta_\alpha(y) dy + \int_{D^{-N}u}^{\alpha} \frac{1}{\alpha} \theta_\alpha(y) dy\) if \(z \in (D^{-N}u - \alpha,D^{-N}u + \alpha]\);
4. \(\frac{D^{-1}}{1-z}\) if \(z \in (D^{-N}u + \alpha,D^{-N} - \alpha]\);
5. \(\int_{-\alpha}^{D^{-N}} D \theta_\alpha(y) dy + \int_{D^{-N}}^{\alpha} \frac{D^{-1} - \alpha}{\alpha} \theta_\alpha(y) dy\) if \(z \in (D^{-N} - \alpha,D^{-N} + \alpha]\);
6. \(D\) if \(z \in (D^{-N} + \alpha,D^{-1} - D^{-K} - \alpha]\);
7. \(\int_{-\alpha}^{D^{-1} - D^{-K}} \frac{D^{-1} - \alpha}{\alpha} \theta_\alpha(y) dy + \int_{D^{-1} - D^{-K}}^{\alpha} \frac{D^{-1} - \alpha}{\alpha} \theta_\alpha(y) dy\) if \(z \in (D^{-1} - D^{-K} - \alpha,D^{-1} - \alpha]\);
8. \(\frac{D^{-1}}{1-z}\) if \(z \in (D^{-1} - D^{-K} + \alpha,D^{-1} - D^{-K}\hat{v} - \alpha]\);
9. \(\int_{-\alpha}^{D^{-1} - D^{-K}\hat{v} + \alpha} \frac{D^{-1} - \alpha}{\alpha} \theta_\alpha(y) dy + \int_{D^{-1} - D^{-K}\hat{v} + \alpha}^{\alpha} \frac{D^{-1} - \alpha}{\alpha} \theta_\alpha(y) dy\) if \(z \in (D^{-1} - D^{-K}\hat{v} - \alpha,D^{-1} - \alpha]\);
10. \(\frac{1}{\alpha}\) if \(z \in (D^{-1} - D^{-K}\hat{v} + \alpha,D^{-1} - \alpha]\);
11. \(\int_{-\alpha}^{D^{-1}} D \theta_\alpha(y) dy + \int_{D^{-1}}^{\alpha} \frac{1}{\alpha} \theta_\alpha(y) dy\) if \(z \in (D^{-1} - \alpha,D^{-1} + \alpha]\);
12. \(D\) if \(z \in (D^{-1} + \alpha,1-\alpha]\).

Using the above formula for \(\frac{d}{dx} \hat{f}_{u,v}|_{x=z}\), we can make
\[
\sup_{z \in [-\alpha,1-\alpha]} \frac{d}{dx} \hat{f}_{u,v}(x|_{x=z}) = \frac{d}{dx} \hat{f}_{u_0,v_0}(x|_{x=z})
\]
arbitrarily small by choosing \((u,v)\) sufficiently close to \((u_0,v_0)\) because the functions \(\frac{1}{\alpha}, \frac{D^{-1} - \alpha}{\alpha}\) and \(\frac{D^{-1}}{1-z}\) are continuous with respect to \((u,v)\) and uniformly bounded above and below for \((u,v) \in [D^{-1},\hat{u}] \times [D^{-1},\hat{v}]\). \(\Box\)

**Lemma 4.2.** For fixed sufficiently small \(\alpha > 0\), the map \((u,v) \mapsto \lambda_{abs}(f_{u,v}^\alpha)\) is continuous on \([D^{-1},\hat{u}] \times [D^{-1},\hat{v}]\).

To prove Lemma 4.2 and Lemma 4.5, we will need the following fact.

**Lemma 4.3.** We use the notations established earlier. Consider \(0 \leq \alpha < \hat{\alpha}\), where \(\hat{\alpha}\) is sufficiently small, and \(u,v \in [D^{-1},\hat{u}] \times [D^{-1},\hat{v}]\). Let \(f_{u,v}^\alpha = f_{u,v}\), i.e., an SUSD-circle map of degree \(D\) with parameters \(n = \hat{N}\), \(\delta = D^{-\hat{N}}u\), \(k = \hat{K}\) and \(\varepsilon = D^{-\hat{K}}v\), and \(f_{u,v}^\alpha\) is a circle map of degree \(D\) that is the smoothing of \(f_{u,v}\) as in Lemma 3.1. Assume \(t = (u,v,\alpha)\) tends to \(t_0 = (u_0,v_0,\alpha_0)\). Then, for any
Proof of Lemma 4.2.

\[ F \in L^1(S^1) \text{ we have} \]
\[ \int_{S^1} |P_tF(x) - P_{t_0}F(x)|dx \to 0 \text{ as } t \to t_0, \]
where \( P_tF(x) = \sum_{y \in g^{-1}(x)} F(y) \) is the Frobenius-Perron operator of \( g_t = f^\alpha_{u,v} \) if \( t = (u, v, \alpha) \).

The proof follows the ideas in [4, Lemma 4].

Proof. It is enough to prove the convergence to 0 for any continuous function. Therefore, let \( F \) be a continuous function.

Let \( T = [D^{-1}, \hat{a}] \times [D^{-1}, \hat{b}] \times [0, \hat{\alpha}] \). Recall that for any \( t \in T \) we have that \( g_t \) is topologically conjugate to the \( \times D \)-map on \( S^1 \). In particular, each \( g_t \) has a unique fixed point \( p_t \). Let \( \mathcal{P}_t \) be a partition of \( S^1 \) into \( D \) left-closed semi-intervals \( \{I^*_i\}_{i=1}^D \) with endpoints \( g^{-1}_t(p_i) \) such that \( I^*_i \cap I^*_j = \emptyset \) if \( i \neq j \), \( p_i \) is the left endpoint of \( I^*_i \) and the right endpoint of \( I^*_j \), and the right endpoint of \( I^*_i \) is the left endpoint of \( I^*_i+1 \). Note that \( g_t|_{I^*_i} : I^*_i \to S^1 \) is one-to-one. Furthermore, the endpoints of \( I^*_i \) tend to the corresponding endpoints of \( I^*_{t_0} \) as \( t \to t_0 \). Denote \( \phi^*_i = \left( g_t|_{I^*_i} \right)^{-1} \).

As a result, we have
\[ \int_{S^1} |P_tF(x) - P_{t_0}F(x)|dx \leq \sum_{i=1}^D \int_{S^1} |F(\phi^*_i(x))(\phi^*_i)'(x) - F(\phi^*_{t_0}(x))(\phi^*_{t_0})'(x)|dx \]
\[ \leq \sum_{i=1}^D \left( \int_{S^1} |F(\phi^*_i(x))(\phi^*_i)'(x) - F(\phi^*_{t_0}(x))(\phi^*_{t_0})'(x)|dx + \int_{S^1} |F(\phi^*_i(x))(\phi^*_{t_0})'(x) - F(\phi^*_{t_0}(x))(\phi^*_{t_0})'(x)|dx \right) \]
\[ \leq \sum_{i=1}^D \left[ \sup_{x \in S^1} |F(x)| \int_{S^1} |(\phi^*_i)'(x) - (\phi^*_{t_0})'(x)|dx + w_F \left( \sup_{x \in S^1} |\phi^*_i(x) - \phi^*_{t_0}(x)| \right) M_0 \right] \to 0 \]
as \( t \to t_0 \), where \( M_0 = \max_{i=1,...,D} \sup_{S^1, G} |(\phi^*_i)'(x)| \), \( G \) is a finite collection of points where \( (\phi^*_i)'(x) \) does not exist but left and right derivatives are finite, and \( w_F \) is the modulus of continuity of \( F \).

To show convergence to 0, we used convergence of the endpoints of corresponding elements of the partitions \( \mathcal{P}_t \) and \( \mathcal{P}_{t_0} \) for all terms. Moreover, for the first term in the sum we used the \( L^1 \) convergence of the derivative of \( g_t \) to the derivative of \( g_{t_0} \), and for the second term in the sum we used the uniform convergence of \( g_t \) to \( g_{t_0} \).

Proof of Lemma 4.2. For any \((u, v) \in [D^{-1}, \hat{a}] \times [D^{-1}, \hat{b}] \) we have that \( f^\alpha_{u,v} \) is a smooth expanding circle map of degree \( D \). Therefore, there exists a unique invariant probability measure \( \mu_{u,v} \) absolutely continuous with respect to Lebesgue with continuous invariant density \( q_{u,v} \) [5, Theorem 5.1.16, Corollary 5.1.25].

Therefore, for any \((u, v), (u_0, v_0) \in [D^{-1}, \hat{a}] \times [D^{-1}, \hat{b}] \) we have
\[ |\lambda_{abs}(f^\alpha_{u,v}) - \lambda_{abs}(f^\alpha_{u_0,v_0})| \]
By the proof of Lemma 4.1, we have that \( \sup_{y \in S^1} \left| \log \left( \frac{d}{dx} f_{u,v}^n \right) \right| \) is small if \((u, v)\) is close enough to \((u_0, v_0)\). As a result, the first term in the inequality above can be made arbitrarily small if \((u, v)\) is close enough to \((u_0, v_0)\).

To show that the second term can be made arbitrarily small, it is enough to show that \(q_{u,v}\) tends to \(q_{u_0,v_0}\) in the \(L^1\) norm as \((u, v) \to (u_0, v_0)\).

By construction, we have the family \(\{f_{u,v}^n\}\), where \((u, v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]\), has derivatives uniformly bounded from above and below. Moreover, \(f_{u,v}^n\) uniformly converges to \(f_{u_0,v_0}^n\) as \((u, v) \to (u_0, v_0)\). Therefore, the set \(\{q_{u,v}\}\), where \((u, v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]\), is precompact in \(L^1\) (see [4, Theorem 1]).

Assume that \(q_{u,v}\) does not converge to \(q_{u_0,v_0}\) in the \(L^1\) norm. Then, there exists \(\varepsilon > 0\) and a sequence \((u_n, v_n) \to (u_0, v_0)\) as \(n \to \infty\) such that

\[
\int_{S^1} |q_{u_n,v_n} - q_{u_0,v_0}| dx \geq \varepsilon \text{ for any } n. \tag{5}
\]

Using the precompactness of the set \(\{q_{u,v}\}\) in \(L^1\), we have a convergent subsequence of \(\{q_{u_n,v_n}\}_{n \geq 1}\) in \(L^1\). Without loss of generality, assume that \(q_{u_n,v_n} \to q\) in the \(L^1\) norm as \(n \to \infty\).

Let \(P_{u,v}\) be the Frobenius-Perron operator of \(f_{u,v}^n\) and denote by \(\| \cdot \|_1\) the \(L^1\) norm on \(S^1\). Then,

\[
\|P_{u_0,v_0} q - q\|_1 \leq \|P_{u_0,v_0} q - P_{u_n,v_n} q\|_1 + \|P_{u_n,v_n} q - P_{u_n,v_n} q_{u_n,v_n}\|_1 + \|q_{u_n,v_n} - q\|_1.
\]

The first summand tends to 0 by Lemma 4.3 as \(n \to \infty\). The second and the fourth summands converge to 0 by the assumption that \(q_{u_n,v_n} \to q\) in \(L^1\) as \(n \to \infty\) and \(\|P_{u_n,v_n}\|_1 = 1\). The third is equal to 0 because the invariant density is the fixed point point of the corresponding Frobenius-Perron operator. Therefore, \(P_{u_0,v_0} q = q\) and \(q\) is the density for \(f_{u_0,v_0}^n\) of an invariant probability measure that is absolutely continuous with respect to Lebesgue. But \(f_{u_0,v_0}^n\) has a unique invariant probability measure that is absolutely continuous with respect to Lebesgue. Therefore, \(q_{u_0,v_0} = q\) and that contradicts (5). As a result, we have that \(q_{u,v}\) converges to \(q_{u_0,v_0}\) in the \(L^1\) completing the proof of Lemma 4.2. \(\square\)

Now we formulate and prove lemmas that will allow us to finish the proof of Main Theorem.
Lemma 4.4. As $\alpha > 0$ tends to 0, we have
\[ \lambda_{\text{max}}(f_{u,v}^\alpha) \to \lambda_{\text{max}}(f_{u,v}) \]
uniformly in $(u, v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]$.

Proof. Let $f_{u,v}^0 = f_{u,v}$. Denote by $M = \max\{D, \frac{D-1}{1-\alpha}, \frac{D-1}{\alpha}\}$ and $m = \min\{D, \frac{1}{\alpha}, \frac{1}{\alpha}\}$.

Then, for any $\alpha \geq 0$ and any $(u, v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]$, we have
\[ \sup_{x \in S^1} \frac{d}{dx} f_{u,v}^\alpha(x) \leq M \quad \text{and} \quad \inf_{x \in S^1} \frac{d}{dx} f_{u,v}^\alpha(x) \geq m > 1. \]

See Lemma 3.1. Moreover,
\[ \text{dist}(p_{u,v}^\alpha, f_{u,v}^0) = \text{dist}(p_{u,v}^\alpha, 0) \leq \alpha, \quad (6) \]
where $\text{dist}$ is the standard distance on $S^1$ and $p_{u,v}^\alpha$ is the unique fixed point of $f_{u,v}^\alpha$.

Let $P_{u,v}^\alpha$ be a partition of $S^1$ into $D$ left-closed semi-intervals $\{I_{u,v,\alpha}^i\}_{i=1,...,D}$ with endpoints $(f_{u,v}^\alpha)^{-1}(p_{u,v}^\alpha)$ such that $I_{u,v,\alpha}^i \cap I_{u,v,\alpha}^j = \emptyset$ if $i \neq j$, $p_{u,v}^\alpha$ is the left endpoint of $I_{u,v,\alpha}^1$ and the right endpoint of $I_{u,v,\alpha}^D$ and the right endpoint of $I_{u,v,\alpha}^i$ is the left endpoint of $I_{u,v,\alpha}^{i+1}$. Notice that $f_{u,v}^\alpha : I_{u,v,\alpha}^i \to S^1$ is one-to-one.

Assume $\alpha$ is sufficiently small. Then, for any points $p, q$ such that $\text{dist}(p, q) < 2\alpha$, we have that for any $(u, v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]$ and $i = 1, \ldots, D$
\[ \text{dist} \left( \left( f_{u,v}^\alpha | I_{u,v,\alpha}^i \right)^{-1}(p), \left( f_{u,v}^\alpha | I_{u,v,\alpha}^i \right)^{-1}(q) \right) < 2\alpha. \quad (7) \]

Denote $\hat{p} = \left( f_{u,v}^\alpha | I_{u,v,\alpha}^i \right)^{-1}(p)$ and $\hat{q} = \left( f_{u,v}^\alpha | I_{u,v,\alpha}^i \right)^{-1}(q)$. The inequality (7) holds because we have only the following possibilities for $\hat{p}$ and $\hat{q}$ because $\text{dist}(p, q) < 2\alpha$ and $f_{u,v}^\alpha$ are expanding maps:

1. Let $p$ and $q$ belong to the image under $f_{u,v}^0$ of the $\alpha$-neighborhood $U$ of a point of non-differentiability for $f_{u,v}^0$. Notice that $f_{u,v}^\alpha(U) = f_{u,v}^0(U)$ by construction. Therefore, $\hat{p}, \hat{q} \in U$ and we have (7).

2. Let $V$ be a maximal segment that does not intersect the union of the $\alpha$-neighborhoods of the points of non-differentiability for $f_{u,v}^0$. In particular, for sufficiently small $\alpha$ its length is greater than $2\alpha$. Notice that $f_{u,v}^\alpha|_V = f_{u,v}^0|_V$ by construction. Assume points $p$ and $q$ belong to $f_{u,v}^0(V)$. Therefore,
\[ \text{dist}(\hat{p}, \hat{q}) \leq \frac{\text{dist}(p, q)}{m} < 2\alpha. \]

3. Let $p$ belong to the image under $f_{u,v}^0$ of the $\alpha$-neighborhood $U$ of a point of non-differentiability for $f_{u,v}^0$ and $q$ belong to a maximal interval $V$ that does not intersect the union of the $\alpha$-neighborhoods of the points of non-differentiability for $f_{u,v}^0$ or vice versa. Then, $V$ intersects the closure of $U$ at one point $\hat{r}$ and $f_{u,v}^\alpha(\hat{r}) = f_{u,v}^0(\hat{r}) = r$ and $r$ lies between $p$ and $q$. Therefore,
\[ \text{dist}(\hat{p}, \hat{q}) \leq \frac{\text{dist}(p, r)}{m} + \frac{\text{dist}(q, r)}{m} = \frac{\text{dist}(p, q)}{m} < 2\alpha. \]

Therefore, using (6), we have that for any natural number $L$ if we consider the set of $L$-th preimages of $p_{u,v}^\alpha$ and $p_{u,v}^0$ under $f_{u,v}^\alpha$ and $f_{u,v}^0$, respectively, then the corresponding preimages will be at distance at most $2\alpha$. Recall that for each $f_{u,v}^\alpha$, the size of elements in the partition of $S^1$ by these $L$-th preimages of $p_{u,v}^\alpha$ is at least $M^{-1}$ and at most $m^{-1}$. As a result, we have that the topological conjugacy between $f_{u,v}^\alpha$ and $f_{u,v}^0$ is $2\alpha$-close to the identity in the uniform topology (see the
proof of [5, Theorem 2.4.6]). Moreover, recall that \( f^*_\alpha \) and \( f^0_{u,v} \) coincide outside of the union of the \( \alpha \)-neighborhoods of the points of non-differentiability for \( f^0_{u,v} \) where \((u,v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]\). Therefore, we obtain the lemma by writing the formulas for \( \lambda_{\text{max}}(f^*_\alpha) \) and \( \lambda_{\text{max}}(f^0_{u,v}) \), approximating the measures of intervals by the measures of intervals whose endpoints are the preimages of the fixed point under the corresponding map and using the fact that the measure of maximal entropy is mapped to the measure of maximal entropy under the topological conjugacy. 

\( \square \)

**Lemma 4.5.** As \( \alpha \to 0 \) tends to 0, we have

\[ \lambda_{\text{abs}}(f^*_\alpha) \to \lambda_{\text{abs}}(f_{u,v}) \]

uniformly in \((u,v) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]\).

**Proof.** Most steps of the proof are similar to Lemma 4.2, therefore, we will omit some details.

Let \( \hat{\alpha} \) be sufficiently small and \( 0 \leq \alpha \leq \hat{\alpha} \). For \( t = (u,v) \) we define \( q^0_t = f_{u,v} \) and \( g^0_t = f^*_\alpha \). Let \( \mu_{\text{abs}}^t \) be the unique probability measure absolutely continuous with respect to the Lebesgue measure with the density \( q^0_t \) for \( g^0_t \).

\[
|\lambda_{\text{abs}}(g^0_t) - \lambda_{\text{abs}}(g^0_t)|
\leq \int_{S^1} \left| \frac{d}{dx} g^0_t \right|_x \left| q^0_t(y) - q^0_t(y) \right| dy + \int_{S^1} \left| \log \left( \frac{d}{dx} g^0_t \right|_x \right) \mu_{\text{abs}}^t(y) 
\leq \log(M) \int_{S^1} \left| q^0_t(y) - q^0_t(y) \right| dy + \log \left( \frac{M}{m} \right) \mu_{\text{abs}}^t(\mathcal{A}^\alpha_t),
\]

where \( M = \max\{D, \frac{D-1}{u}, \frac{D-1}{v}\} \), \( m = \min\{D, \frac{1}{u}, \frac{1}{v}\} \), and \( \mathcal{A}^\alpha_t \) is the union of the \( \alpha \)-neighborhoods of the points of non-differentiability of \( g^0_t \).

It follows from the expression for \( q^0_t \) (see (2)) that \( \mu_{\text{abs}}^t(\mathcal{A}^\alpha_t) \) tends to 0 as \( \alpha \to 0 \) uniformly for \( t \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}] \).

We need to show that \( \int_{S^1} \left| q^0_t(y) - q^0_t(y) \right| dy \to 0 \) as \( \alpha \to 0 \) uniformly for \( t \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}] \). Assume it is not the case. Then, there exists \( \beta > 0 \) such that for any natural number \( l \) there exists \( 0 < \alpha_l < \frac{1}{l} \) and \( t_l = (u_l, v_l) \in [D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}] \) such that

\[
\int_{S^1} |q^0_{t_l}(y) - q^0_{t_l}(y)| dy \geq \beta.
\]

(8)

There exists a convergent subsequence of a sequence \( \{t_l\}_{l \geq 1} \) that converges to \( \bar{t} \) because \([D^{-1}, \hat{u}] \times [D^{-1}, \hat{v}]\) is a compact set. Without loss of generality, we assume that \( t_l \to \bar{t} \) as \( l \to \infty \).

Therefore, by (8), we obtain that for any \( l \geq 1 \)

\[
\int_{S^1} |q^0_{t_l}(y) - q^0_{t_l}(y)| dy + \int_{S^1} |q^0_{t_l}(y) - q^0_{t_l}(y)| dy \geq \beta.
\]

By (2), we have \( \int_{S^1} |q^0_{t_l}(y) - q^0_{t_l}(y)| \to 0 \) as \( l \to \infty \). Moreover, by the same argument as in Lemma 4.2, applying Lemma 4.3 and [4, Theorem 1], we obtain that \( \int_{S^1} |q^0_{t_l}(y) - q^0_{t_l}(y)| dy \to 0 \) as \( l \to \infty \) because \((\alpha_l, t_l) \to (0, \bar{t}) \).

As a result, we get a contradiction. Therefore, the convergence is uniform in \( t \). \( \square \)
Acknowledgments. The author would like to thank Anatole Katok for introducing her to the flexibility program and its many interesting problems. She also thanks Federico Rodriguez Hertz for many helpful discussions and comments regarding the constructions presented here. The author would like to thank the anonymous referees for carefully reading the initial version of the paper and suggestions that improved the quality of the paper. This work was partially supported by NSF grant DMS 16-02409.

REFERENCES

[1] J. Bochi, A. Katok and F. Rodrigues Hertz, Flexibility of Lyapunov exponents among conservative diffeomorphisms, preprint.
[2] A. Boyarsky and M. Scarowsky, On a class of transformations which have unique absolutely continuous invariant measures, Trans. Amer. Math. Soc., 255 (1979), 243–262.
[3] A. Erchenko and A. Katok, Flexibility of entropies for surfaces of negative curvature, to appear in Israel J. Math., arXiv:1710.00079.
[4] A. Góra and A. Boyarsky, Compactness of invariant densities for families of expanding, piecewise monotonic transformations, Canad. J. Math., 41 (1989), 855–869.
[5] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1995.
[6] A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc., 186 (1973), 481–488.
[7] M. Qian, J.-S. Xie and S. Zhu, Smooth Ergodic Theory for Endomorphisms, Springer-Verlag, Berlin, 2009.
[8] D. Ruelle, An inequality for the entropy of differentiable maps, Bol. Soc. Brasil. Mat., 9 (1978), 83–87.
[9] P. Walters, Invariant measures and equilibrium states for some mappings which expand distances, Trans. Amer. Math. Soc., 236 (1978), 121–153.

Received May 2017; revised July 2018.

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