Stability of marginally outer trapped surfaces and symmetries

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Abstract
We study the properties of stable, strictly stable and locally outermost marginally outer trapped surfaces in spacelike hypersurfaces of spacetimes possessing certain symmetries such as isometries, homotheties and conformal Killings. We first obtain results for general diffeomorphisms in terms of the so-called metric deformation tensor and then particularize to different types of symmetries. In particular, we find restrictions at the surfaces on the vector field generating the symmetry. Some consequences are discussed. As an application, we present a result on non-existence of stable marginally outer trapped surfaces in slices of FLRW.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Trapped surfaces, and their various relatives, are the fundamental objects in classical general relativity. Being quasilocal versions of black holes, their study is essential in order to understand how black holes evolve when no global assumptions are made in the spacetime, for instance in order to address the cosmic censorship conjecture (see e.g. [1]). They are also widely used in numerical relativity.

It is often the case that trapped surfaces (which will always be taken to be closed in this paper) have to be studied in spacetimes possessing some kind of symmetry. This is the case, for instance, when configurations of equilibrium are considered, or in spherically symmetric or axially symmetric configurations. However, not only isometries are important in this respect. For instance, critical collapse is a universal feature of many matter models and the critical solution, which separates those configurations that disperse from those that form black holes, is known to admit either a continuous or a discrete self-similarity. This makes
it interesting to study trapped surfaces in spacetimes with homothetic Killing vectors. Many relevant spacetimes admit other types of symmetries, like for instance conformal Killing vectors, e.g. in FLRW cosmologies. Therefore, it becomes interesting to study the relationship between trapped surfaces and special types of vectors. A recent example of this interplay has been given in [2, 3], where the localization of the boundary of the set containing trapped surfaces, which is a natural candidate for the ‘surface of an evolving black hole’, was analyzed in the Vaidya spacetime, which is one of the simplest dynamical situations. In this analysis, the presence of a so-called Kerr–Schild symmetry (see e.g. [4]) turned out to be fundamental.

In the important case of isometries, general results on the relationship between trapped surfaces and Killing vectors were discussed in [5]. The first variation of area was used to obtain several restrictions on the existence of trapped and marginally trapped surfaces in spacetime regions possessing a causal Killing vector. More specifically, if the Killing vector is timelike in some region, then no trapped surface can exist there, and marginally trapped surfaces can only exist if their mean curvature vanishes identically. By obtaining a general identity for the first variation of area in terms of the deformation tensor of an arbitrary vector (see below for the definition), similar restrictions were obtained for spacetimes admitting other types of symmetries, such as conformal Killing vectors or Kerr–Schild vectors. The same idea was also applied in [6] to obtain analogous results in spacetimes with vanishing curvature invariants.

The interplay between isometries and dynamical horizons (which are spacelike hypersurfaces foliated by marginally trapped surfaces) was considered in [7], where it was proven that regular dynamical horizons cannot exist in spacetime regions containing a nowhere vanishing causal Killing vector, provided the spacetime satisfies the null energy condition (NEC).

One of the most relevant variants of trapped surfaces is the so-called marginally outer trapped surfaces (MOTS), where only the expansion along the outer null vector \( \hat{l} \) becomes restricted. The relation between stable MOTS and isometries was considered in [8], where it was shown that, given a strictly stable MOTS \( S \) in a hypersurface \( \Sigma \) (not necessarily spacelike), any Killing vector tangent to \( \Sigma \) on \( S \) must in fact be tangent to \( \Sigma \).

MOTS in stationary or static spacetimes play a particularly relevant role. Indeed, MOTS are believed to be good replacements of black holes, so a natural question arises of whether or not some version of the black hole uniqueness theorems also holds for asymptotically flat equilibrium configurations containing MOTS. This was answered in the affirmative by Miao [9] in the static, vacuum case when the MOTS lies in a time-symmetric slice (hence, it is a minimal surface) and bounds a domain. A general study of MOTS in stationary and static spacetimes with arbitrary matter contents satisfying the NEC was performed in [10]. In the stationary case, it was proven that, on an arbitrary spacelike hypersurface \( \Sigma \), no bounding MOTS lying in the region where the Killing field \( \xi \) is causal can penetrate into the timelike region provided the latter contains a compact surface with positive outer null expansion (e.g. in the asymptotically flat case with a Killing vector which is timelike at infinity). This result was strengthened for static Killing vectors, for which no bounding MOTS can even penetrate into this timelike region. The underlying idea of [10] was to take the outermost MOTS \( S \) and construct another weakly outer trapped surface \( S_t \), which lies outside of \( S \), at least partially, thus contradicting the outermost property of \( S \). The new surface \( S_t \) was constructed in two steps. First, by moving \( S \) to the past of \( \Sigma \) along the integral lines of the Killing vector \( \xi \) some amount \( t \) and, second, by translating this surface back to \( \Sigma \) along the outgoing future null geodesics. \( S_t \) is automatically located partially outside of \( S \) if the Killing is timelike somewhere on \( S \). Furthermore, the shift of \( S \) along the isometry obviously gives a new MOTS, while the outer expansion cannot increase in the translation along the null geodesics, due to
the Raychaudhuri equation. Hence the whole procedure gives a weakly outer trapped surface, and therefore a contradiction.

In the present work, we will study the interplay between the stable and outermost properties of marginally outer trapped surfaces in spacetimes possessing special types of vector fields, including isometries, homotheties and conformal Killing vectors. In fact, we will find several results involving completely general vector fields $\xi$. The initial idea is to analyze in detail the geometric construction of $S_t$ outlined above in order to find restrictions on $\xi$ on an outermost MOTS $S$ in a given spacelike hypersurface $\Sigma$, or alternatively, forbid the existence of a MOTS in certain regions where $\xi$ fails to satisfy those restrictions. We emphasize that these restrictions will be stronger than those of [10] because here we make no \textit{a priori} assumption on the causal character of $\xi$ (that may be even tangent to $\Sigma$ or to $S$). The collection of $\{S_t\}$ defines a variation of $S$ within $\Sigma$. The corresponding first-order variation of the outer null expansion is an elliptic operator $L_m$ acting on a function $Q$ which is precisely the function which, at first order, determines whether $S_t$ lies outside of $S$ or not. This observation, as such, is of little use until the operator can be directly linked to the vector field $\xi$, and more specifically, to its deformation tensor. The standard expression for the stability operator (see e.g. [8]) has \textit{a priori} nothing to do with the properties of the vector field $\xi$. The first task is, therefore, to obtain an alternative (and completely general) expression for $L_m Q$ in terms of the deformation tensor of $\xi$. We devote section 3 to doing this. The result, given in proposition 1 below, is thoroughly used in this paper and also has independent interest.

With this expression at hand, we can already analyze under which conditions the procedure above gives restrictions on $\xi$. In section 4, we concentrate on the case where $L_m Q$ has a sign everywhere on $S$. It turns out that the results obtained by the geometric construction above can, in most cases, be sharpened considerably by using the maximum principle of elliptic operators. This also allows one to extend the validity of the results from the outermost case to the case of stable and strictly stable MOTS. The main result of section 4 is given in theorem 1, which holds for any vector field $\xi$. This result is then particularized to conformal Killing vectors (including homotheties and Killing vectors). Under the additional restriction that the homothety or the Killing vector is everywhere causal and future (or past) directed, strong restrictions on the geometry of the MOTS are derived (corollary 3). As a consequence, we prove that in a plane wave spacetime any stable MOTS must be orthogonal to the direction of propagation of the wave. Marginally trapped surfaces are also discussed in this section.

As an explicit application of the results on conformal Killing vectors, we show, in subsection 4.1, that stable MOTS cannot exist in any spacelike hypersurface in FLRW cosmological models provided the density $\rho$ and pressure $p$ satisfy the inequalities $\rho \geq 0$, $\rho \geq 3p$ and $\rho + p \geq 0$. This includes, for instance, all classic models of matter and radiation dominant eras and also those models with accelerated expansion which satisfy the NEC. Subsection 4.2 deals with one case where, in contrast with the standard situation, the geometric construction does in fact give sharper results than the elliptic theory.

In the case when $L_m Q$ is not assumed to have a definite sign, the maximum principle loses its power. However, the geometric construction can still be used despite the fact that the surfaces $S_t$ are necessarily not weakly outer trapped (for $t$ small enough). This is studied in section 5, where we exploit a smoothing argument by Kriele and Hayward [11] which allows one to construct, out of two intersecting surfaces, a smooth surface which lies outside of them and has smaller outer expansion than the original ones. This gives a result (theorem 5) which holds for general vector fields $\xi$ on any locally outermost MOTS. As in the previous section, we then particularize to conformal Killing vectors, and then to causal Killing vectors and homotheties which, in this case, are allowed to change their time orientation on $S$.

We start with the basic definitions and results needed for this work.
2. Basics

Consider a spacetime \((M, g)\) and a vector field \(\xi\) defined on it. The Lie derivative \(\mathcal{L}_\xi g_{\mu\nu}\) describes how the metric is deformed along the local group of diffeomorphisms generated by \(\xi\). We thus define the metric deformation tensor associated with \(\xi\), or simply deformation tensor, as

\[
a_{\mu\nu} \equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu.
\]

Special forms of \(a_{\mu\nu}\) define special types of vectors. In particular, \(a_{\mu\nu} = 2\phi g_{\mu\nu}\) (\(\phi\) a scalar function) defines a conformal Killing vector, \(a_{\mu\nu} = 2C g_{\mu\nu}\) (\(C\) a constant) corresponds to a homothety and \(a_{\mu\nu} = 0\) defines a Killing vector.

As described in the introduction, we want to relate the deformation tensor of special vectors to the stability and outermost properties of MOTS. We will denote by \(S\) a smooth, closed (i.e. compact and without boundary) and orientable surface embedded in a spacelike hypersurface \(\Sigma\). The future-directed unit vector normal to \(\Sigma\) will be called \(\overrightarrow{n}\) and the unit vector orthogonal to \(\Sigma\) along \(\Sigma\) is called \(\overrightarrow{m}\). The null vectors \(\overrightarrow{l} = \overrightarrow{n} + \overrightarrow{m}\) and \(\overrightarrow{k} = \overrightarrow{n} - \overrightarrow{m}\) are a null basis of the normal bundle of \(\Sigma\), and satisfy \((\overrightarrow{l} \cdot \overrightarrow{k}) = -2\) (scalar product with the spacetime metric is denoted by \((\cdot))\). These vectors are univocally defined once a choice of orientation for \(\overrightarrow{m}\) is made.

The first fundamental form of \(\Sigma\) is a Riemannian metric which we denote by \(\gamma^{\Lambda\Lambda}\). At any point \(p \in \Sigma\), the tangent space \(T_pM\) decomposes as the direct sum of the tangent and normal vector spaces to \(\Sigma\). This splits any vector \(\overrightarrow{V} \in T_pM\) as \(\overrightarrow{V} = \overrightarrow{V}^\| + \overrightarrow{V}^\perp\). The second fundamental form vector of \(\Sigma\) is defined as \(\kappa^{\Lambda\Lambda} \equiv -\langle \nabla_{\overrightarrow{e}_A}\overrightarrow{e}_B \rangle^\perp\), where \([\overrightarrow{e}_A|p]\) is a basis of \(T_p\Sigma\). Finally, the mean curvature vector is the trace of the second fundamental form, \(\overrightarrow{H} = \gamma^{\Lambda\Lambda} \kappa^{\Lambda\Lambda}\). Being a normal vector, it can be expanded in the null basis as

\[
\overrightarrow{H} = -\frac{1}{2} (\theta_l \overrightarrow{k} + \theta_k \overrightarrow{l}),
\]

where the coefficients define the null expansions \(\theta_l, \theta_k\) of \(\Sigma\) along \(\overrightarrow{l}\) and \(\overrightarrow{k}\), respectively. Similarly, the expansion along any normal direction \(\overrightarrow{\eta}\) is defined as \(\theta_{\overrightarrow{\eta}} \equiv (\overrightarrow{H} \cdot \overrightarrow{\eta})\) and the second fundamental form along \(\overrightarrow{\eta}\) is \(\kappa^{\overrightarrow{\eta}}_{\Lambda\Lambda} \equiv (\kappa^{\Lambda\Lambda} \cdot \overrightarrow{\eta})\).

A useful classification of surfaces arises depending on the causal character of \(\overrightarrow{H}\) or on the sign of one of the expansions. Assume that one preferred orientation of \(\overrightarrow{m}\) can be selected geometrically. We call this the outer direction. The corresponding null vector \(\overrightarrow{l} = \overrightarrow{n} + \overrightarrow{m}\) is the outer null direction. If, furthermore, \(\Sigma\) separates \(\Sigma\) into two regions, we will call 'exterior' the portion to which the outer direction points.

The types of surfaces that will play a role in this paper are (see [12] for an exhaustive classification) as follows: \(\Sigma\) is marginally future (past) trapped if \(\overrightarrow{H}\) points along one of the null normals, \(\overrightarrow{l}\) or \(\overrightarrow{k}\), and is future (past) pointing at each point (in our convention, the vanishing vector is both future and past null), \(\Sigma\) is weakly outer trapped if \(\theta_l \leq 0\) and \(\Sigma\) is a marginally outer trapped surface (MOTS) provided \(\theta_k = 0\).

This work basically deals with the properties of (strictly) stable and locally outermost MOTS, defined as follows [13]. \(\Sigma\) is stable\(^1\) if there exists a function \(\psi \geq 0, \psi \neq 0\) on \(\Sigma\) such that the variation of \(\theta_l\) along \(\psi \overrightarrow{m}\), denoted by \(\delta_{\phi m} \theta_l\), is non-negative. \(\Sigma\) is strictly stable if, moreover, \(\delta_{\phi m} \theta_l \neq 0\) somewhere on \(\Sigma\). As described in [13], the variation \(\delta_{\phi m} \theta_l\) gives a linear second-order elliptic operator acting on \(\psi\), which we will denote by \(L_{\phi m}\psi\). The explicit form of this operator appears in equation (1) of [13]. It is also well known [8, 13] that stability

\(^1\) Strictly speaking we should say stable in \(\Sigma\). However, we will only deal with one hypersurface at a time, and no confusion should arise.
can be rephrased in terms of the sign of principal eigenvalue $\lambda_m$ of $L_m$ (defined to have the smallest real part, and which is always real): $S$ is stable if $\lambda_m \geq 0$ and strictly stable if $\lambda_m > 0$.

A MOTS is locally outermost if there exists a two-sided neighborhood of $S$ on $\Sigma$ whose exterior part does not contain any weakly outer trapped surface. We will denote by $\Sigma$ the interior part of this two-sided neighborhood.

The relationship between these types of surfaces is the following [13]: (i) a strictly stable MOTS is necessarily locally outermost, (ii) a locally outermost MOTS is necessarily stable, and (iii) none of the converses is true in general.

The results obtained in section 4 use the following version of the maximum principle for second-order linear elliptic operators [8] (recall that the eigenspace corresponding to the principal eigenvalue is one dimensional and no function in this space can change sign).

**Lemma 1.** Consider a second-order linear elliptic operator $L$ on a compact manifold $S$ with the principal eigenvalue $\lambda \geq 0$ and principal eigenfunction $\phi$, and let $\psi$ be a smooth function satisfying $L\psi \geq 0$ ($L\psi \leq 0$).

1. If $\lambda = 0$, then $L\psi \equiv 0$ and $\psi = C\phi$ for some constant $C$.
2. If $\lambda > 0$ and $L\psi \neq 0$, then $\psi > 0$ ($\psi < 0$) all over $S$.
3. If $\lambda > 0$ and $L\psi \equiv 0$, then $\psi \equiv 0$.

As mentioned in the introduction, the idea we want to apply in order to obtain restrictions on a given vector field $\xi$ on a MOTS $S$ consists in moving $S$ first along the integral lines of $\xi$ a parametric amount $t$. This gives a new surface $S'_t$. Take the null normal $\xi'_t$ on this surface which coincides with the continuous deformation of $\tilde{t}$ and consider the null hypersurface generated by null geodesics with the tangent vector $\xi'_t$. This hypersurface is smooth, close enough to $S'_t$.

Being null, its intersection with the spacelike hypersurface $\Sigma_1$ is transversal and hence defines a smooth surface $S_t$ (for $t$ sufficiently small). By this construction, a point $p$ on $S$ describes a curve in $\Sigma_1$. The tangent vector of this curve on $S$, denoted by $\tilde{v}$, will define the variation vector generating the deformation $\{S_t\}$ of $S$. Figure 1 gives a graphic representation of this construction.

As usual, we decompose the vector $\xi$ into normal and tangential components with respect to $\Sigma$, as $\xi = N\tilde{n} + \tilde{Y}$. On $S$ we will further decompose $\tilde{Y}$ in terms of a tangential component $\tilde{Y}^\parallel$, and a normal component $(\tilde{Y} - \tilde{m})\tilde{m}$, i.e. $\xi|_S = Ns\tilde{n} + (\tilde{Y} - \tilde{m})\tilde{m} + \tilde{Y}^\parallel$, where $N_s$ is the value of $N$ on the surface. In order to study the variation vector $\tilde{v}$, let us expand the embedding functions $\{x^\mu(y^A, t)\}$ of the surface $S_t$ (where $\{y^A\}$ are intrinsic coordinates of $S_t$) as

$$x^\mu(y^A, t) = x^\mu(y^A, 0) + \xi^\mu(y^A, 0)t + F(y^A)l^\mu(y^A)t + O(t^2).$$
where $F(y^i)$ is a function to be adjusted. Since $\tilde{v}$ defines the variation of $S$ to first order, equation (2) implies that we only need to evaluate the vector $\tilde{l}$ to zero order in $t$, which obviously coincides with $\dot{l}$. It follows then that $\tilde{v}$ is a linear combination (with functions) of $\tilde{\xi}$ and $\dot{l}$. The amount we need to move $S'_t$ in order to go back to $\Sigma$ can be determined by imposing $\tilde{v}$ to be tangent to $\Sigma$. This fixes the function $F$ and gives $\tilde{v} = \tilde{\xi} - N_3 \dot{l} = Q \tilde{m} + \tilde{Y}^\parallel$, where

$$Q = (\tilde{Y} \cdot \tilde{m}) - N_3 = (\tilde{\xi} \cdot \dot{l}).$$

Since the tangential part of $\tilde{v}$ does not affect the variation of $\theta_l$ along $\tilde{v}$ for a MOTS, it follows that $\delta \theta_l = L_m Q$. Then, a direct application of lemma 1 for a MOTS $S$ with the stability operator $L_m$ leads to the following result.

**Lemma 2.** Let $S$ be a stable MOTS on a spacelike hypersurface $\Sigma$. If $|L_m Q|_S \leq 0$ ($|L_m Q|_S \geq 0$) and not identically zero, then $Q|_S < 0$ ($Q|_S > 0$).

Furthermore, if $S$ is strictly stable and $|L_m Q|_S \leq 0$ ($|L_m Q|_S \geq 0$), then $Q|_S \leq 0$ ($Q|_S \geq 0$) and it vanishes at one point only if it vanishes everywhere on $S$.

This result will be used in section 4 to obtain restrictions on the vector field $\tilde{\xi}$ on stable and strictly stable MOTS. The idea is to use the deformation tensor to obtain an independent expression for $L_m Q$. Consider the simplest example of a Killing vector $\tilde{\xi}$. Since the null expansion does not change under an isometry, it follows that the surface $S'_t$ is also a MOTS. Moving back to $\Sigma$ along the null hypersurface gives a contribution to $\theta_l(S_t)$ which, from the Raychaudhuri equation, is easily computed to be $L_m Q = N_3 W$, where we have introduced the shorthand notation

$$W = k_i^2 + G_{\mu\nu} l^\mu l^\nu,$$

with $G_{\mu\nu}$ being the Einstein tensor of $(M, g)$ and $k_i^2 = k_{AB} k^{iAB}$ the square of the second fundamental form along $\dot{l}$, which coincides with the square of the shear along $\dot{l}$ in the case of MOTS. Note that $W$ is non-negative provided the null energy condition (NEC) holds, i.e. $G_{\mu\nu} w^\mu w^\nu \geq 0$ for any null vector $\tilde{w}$. It is clear that, under the NEC, lemma 2 implies restrictions on any Killing vector on a stable MOTS.

However, obtaining the result $L_m Q = N_3 W$ directly from the explicit form of the elliptic operator $L_m$ is not trivial because the condition of $\tilde{\xi}$ being a Killing vector does not give obvious restrictions on the coefficients of this operator. In the case of Killing vectors, the point of view of moving along $\tilde{\xi}$ and then back to $\Sigma$ gives a simple method of calculating $L_m Q$. For more general vectors, however, the motion along $\tilde{\xi}$ will give a non-zero contribution to $\theta_l$ which needs to be computed (for Killing vectors this term was known to be zero via a symmetry argument, not from a direct computation). In order to do this, it becomes necessary to have an alternative, and completely general, expression for $\delta \xi \theta_l$ directly in terms of the deformation tensor $a_{\mu\nu}$ of $\tilde{\xi}$.

### 3. Variation of the expansion and the metric deformation tensor

The aim of this section is to derive an identity for $\delta \xi \theta_l$ in terms of $a_{\mu\nu}$. This result will be important later on in this paper, and may also be of independent interest. We derive this expression in full generality, i.e. without assuming $S$ to be a MOTS and for the expansion $\theta_\eta$ along any normal vector $\vec{n}$ of $S$, not necessarily a null normal.

To do this calculation, we need to take derivatives of tensorial objects defined on each of $S'_t$. For a given point $p \in S$, these tensors belong to different spaces, namely the tangent spaces
all tensors need to be pulled back to the tangent space \( \xi \). In order to define the variation, all tensors need to be pulled back to the tangent space \( p \) before doing the derivative. We will denote the resulting derivative by \( L_\xi \). This is, in fact, an abuse of notation because we are not taking Lie derivatives of tensor fields on the manifold (they are tensorial objects on each \( S'_i \), but these surfaces may perfectly well intersect each other). Nevertheless, it is a useful notation because when acting on spacetime tensor fields (e.g. the metric \( g \)) the operation involved is really the standard Lie derivative along \( \xi \). This will simplify the calculation considerably.

Notice in particular that the definition of \( \theta_\eta \) depends on the choice of \( \eta \) on each of the surfaces \( S'_i \). Thus, \( \delta_\eta \theta_\eta \) will necessarily include a term of the form \( L_\xi \eta_\mu \) which is not uniquely defined (unless \( \eta \) can be uniquely defined on each \( S'_i \) which is usually not the case). Nevertheless, for the case of MOTS and when \( \eta = \bar{\eta} \), this \emph{a priori} ambiguous term becomes determined, as we will see. The general expression for \( \delta_\eta \theta_\eta \) is given in the following proposition.

**Proposition 1.** Let \( S \) be a surface on a spacetime \( (M, g) \), \( \xi \) a vector field defined on \( M \) with deformation tensor \( a_{\mu\nu} \) and \( \eta \) a vector field normal to \( S \). Then, the variation along \( \xi \) of the expansion \( \theta_\eta \) on \( S \) reads

\[
\delta_\eta \theta_\eta = H^\mu L_\xi \eta_\mu - a_{AB} \kappa^{AB}_\mu \eta^\mu + \gamma^{AB} e^A_\alpha e^B_\beta \eta^\nu \left[ \frac{1}{2} \nabla_\nu a_{\mu\nu} - \nabla_\nu a_{\mu\beta} \right]_S ,
\]

where \( a_{AB} \equiv e^A_\alpha e^B_\beta a_{\mu\nu} \).

**Proof.** Since \( \theta_\eta = H^\mu \eta_\mu = \gamma^{AB} k^{\mu}_{AB} \eta_\mu \), the variation we need to calculate involves three terms:

\[
\delta_\eta \theta_\eta = L_\xi \gamma^{AB} k^{\mu}_{AB} \eta_\mu + \gamma^{AB} L_\xi k^{\mu}_{AB} \eta_\mu + H^\mu L_\xi \eta_\mu .
\]

In order to do the calculation, we will choose \( \phi_t(p) \) as the basis of tangent vectors at \( \phi_t(p) \in S'_i \) (we refer to \( \phi_t(p) \) in the following to simplify the notation). This entails no loss of generality and implies \( L_\xi \bar{e}_A = 0 \), which makes the calculation simpler. Our aim is to express each term of (6) in terms of \( a_{\mu\nu} \). For the first term, we need to calculate \( L_\xi \gamma^{AB} \).

Let \( L_\xi \gamma^{AB} = L_\xi \left( g(\bar{e}_A, \bar{e}_B) \right) = \left( L_\xi g \right) (\bar{e}_A, \bar{e}_B) = a_{\mu\nu} e^A_\nu e^B_\mu \equiv a_{AB} \), which immediately implies \( L_\xi \gamma^{AB} = -a_{CD} \gamma^{AC} \gamma^{BD} \), so that the first term in (6) becomes

\[
L_\xi \gamma^{AB} k^{\mu}_{AB} \eta_\mu = -a_{AB} k^{\mu}_{AB} \eta_\mu ,
\]

where capital Latin indices are lowered and raised with \( g_{AB} \) and its inverse.

The second term \( \gamma^{AB} \left( L_\xi k^{\mu}_{AB} \right) \eta_\mu \) is more complicated. It is useful to introduce the projector to the normal space of \( S \), \( h^\mu_\nu \equiv \delta^\mu_\nu - \gamma_\mu^{\alpha \beta} e^\alpha_\mu e^\beta_\nu \gamma^{AB} \). From the previous considerations, it follows that \( L_\xi h^\mu_\nu = e^A_\rho e^B_\sigma \left( a^{AB} g_{\rho\sigma} - \gamma^{AB} a_{\rho\sigma} \right) \), which implies

\[
L_\xi \left( k^{\mu}_{AB} \right) \eta_\mu = -L_\xi \left( h^\mu_\nu e^A_\nu \nabla_a e^B_\mu \right) \eta_\mu = -\eta_\mu L_\xi \left( e^A_\nu \nabla_a e^B_\mu \right) ,
\]

where we have used the fact that \( \eta_\mu \) is orthogonal to \( S \), so its contraction with \( L_\xi h^\mu_\nu \) vanishes.

Therefore, we only need to evaluate \( L_\xi \left( e^A_\nu \nabla_a e^B_\mu \right) \). It is well known (and in any case easily verifiable) that for an arbitrary vector field \( \nu \), the commutation of the covariant derivative and the Lie derivative introduces a term involving the Riemann tensor \( R^\nu_{\rho\sigma\alpha} \) of \( g \), as follows:

\[
L_\xi \left( e^A_\nu \nabla_a e^B_\mu \right) = e^A_\rho \nabla^\rho e^B_\mu + e^A_\nu \nabla_a e^B_\mu + R^\nu_{\rho\sigma\alpha} e^\rho_\sigma e^B_\alpha .
\]

Although this expression is not directly applicable to the variational derivative we are calculating, a straightforward computation shows that

\[
L_\xi \left( e^A_\nu \nabla_a e^B_\mu \right) = e^A_\rho \nabla^\rho e^B_\mu + R^\nu_{\rho\sigma\alpha} e^\rho_\sigma e^B_\alpha .
\]
does indeed hold. It only remains to express the quantity $\nabla_\alpha \nabla_\rho \xi^\nu + R^\nu{}_{\rho \sigma \alpha} \xi^\sigma$ in terms of $a_{\mu \nu}$.

To that end, we take a derivative of equation (1) and use the Ricci identity to get

$$\nabla_\nu \nabla_\alpha \xi^\rho + \nabla_\alpha \nabla_\rho \xi^\nu = R^\nu{}_{\rho \sigma \alpha} \xi^\sigma + \nabla_\alpha a_{\nu \rho}.$$

Now, write the three equations obtained from this one by cyclic permutation of the three indices. Adding two of them and subtracting the third one we find, after using the first Bianchi identity,

$$\nabla_\alpha \nabla_\rho \xi^\nu = R^\nu{}_{\rho \sigma \alpha} \xi^\sigma + \frac{1}{2} \left[ \nabla_\alpha a_{\nu \rho} - \nabla_\nu a_{\alpha \rho} \right].$$

Substituting (9) and this expression into (8) yields

$$\gamma^A B \mathcal{L}_\xi \kappa^A \eta_{\mu} = \gamma^A B e^A e^\rho B \eta^\nu \left[ \frac{1}{2} \nabla_\nu a_{\alpha \rho} - \nabla_\alpha a_{\nu \rho} \right].$$

Inserting (7) and (10) into equation (6) proves the lemma. □

We can now particularize to the outer null expansion in a MOTS.

Corollary 1. If $S$ is a MOTS, then

$$\delta \xi_\theta = -\frac{1}{2} \theta_0 a_{\mu \nu} l^\mu l^\nu - a_{\mu \nu} e_A^\mu e_B^\nu \kappa^A B l^\mu + \gamma^A B e^\rho A e^\nu B \left[ \frac{1}{2} \nabla_\nu a_{\rho \alpha} - \nabla_\alpha a_{\nu \rho} \right] |_S.$$ (11)

Proof. The normal vector $\vec{\xi}_t$ defined on each of the surfaces $S_t$ is null. Therefore, using $\mathcal{L}_\xi g^{\mu \nu} = -a^{\mu \nu}$,

$$0 = \mathcal{L}_\xi (l^\mu t^\nu g^{\mu \nu}) = 2 l^\mu \mathcal{L}_\xi l^\nu - a_{\mu \nu} l^\mu l^\nu.$$ (12)

Since, on a MOTS $\vec{H} = -\frac{1}{2} \theta_0 \vec{\xi}$, it follows $H^\mu \mathcal{L}_\xi l^\nu = -\frac{1}{3} \theta_0 l^\mu \mathcal{L}_\xi l^\nu = -\frac{1}{4} \theta_0 a_{\rho \mu} l^\rho l^\nu$, and the corollary follows from (5). □

Remark. From the proof, it is clear that we have only used $\theta_t = 0$ at $p$. Therefore, formula (11) holds in general for arbitrary surfaces $S$ at any point where $\theta_t = 0$.

4. Results provided $L_m Q$ has a sign on $S$

The most favorable case to obtain restrictions on the generator $\vec{\xi}$ on a given MOTS $S$ is when the surfaces $\{S_t\}$ constructed by the procedure above are weakly outer trapped. This is guaranteed for small enough $t$ when $L_m Q$ is strictly negative everywhere, because then this first-order term becomes dominant. Suppose that in addition to being a MOTS $S$ is also outermost, in the intuitive sense that no other weakly outer trapped surface can penetrate in its exterior (we will give a more precise definition below). Since the vector $\vec{v} |_p = Q \vec{m} + Y |_p$ determines the first order displacement of a point $p \in S$, it is clear that $Q > 0$ at any point implies that for small enough $t$, $S_t$ lies partially in the exterior of $S$. Combining these facts, it follows that $L_m Q < 0$ everywhere and $Q > 0$ somewhere is impossible for an outermost MOTS. This argument is intuitively very clear. However, this geometric method does not provide the most powerful way of finding this type of restriction. Indeed, when the first-order term $L_m Q$ vanishes at some points, then higher order coefficients come necessarily into play, which makes the geometric argument involved. It is remarkable that using the elliptic results described in section 2, most of these situations can be treated in a satisfactory way. Furthermore, since the elliptic methods only use infinitesimal information, there is no need to restrict oneself to outermost MOTS, and the more general case of stable or strictly stable surfaces can be considered. In this section, we will give several results along these lines. The general idea is to combine lemma 2 with the
general calculation for the variation of $\theta_l$ obtained in the previous section to get restrictions on special types of generators $\tilde{\xi}$ on a stable or strictly stable MOTS.

Our first result is fully general in the sense that it is valid for any generator $\tilde{\xi}$.

**Theorem 1.** Let $S$ be a stable MOTS on a spacelike hypersurface $\Sigma$ and $\tilde{\xi}$ a vector field on $S$ with deformation tensor $a_{\mu\nu}$. With the notation above, define

$$Z = -\frac{1}{4}\theta_l a_{\mu\nu} l^\nu - a_{AB} k^A_l l^B + \kappa^B e^\mu_B e^\nu_B \left[ \frac{1}{2} \nabla_\nu a_{\mu\mu} - \nabla_\mu a_{\nu\nu} \right] + NW|_S,$$

and assume $Z \leq 0$ everywhere on $S$.

(i) If $Z \neq 0$ somewhere, then $(\tilde{\xi} \cdot \tilde{l}) < 0$ everywhere.

(ii) If $S$ is strictly stable, then $(\tilde{\xi} \cdot \tilde{l}) \leq 0$ everywhere and vanishes at one point only if it vanishes everywhere.

**Remark.** The theorem also holds if all the inequalities are reversed. This follows directly by replacing $\tilde{\xi} \to -\tilde{\xi}$.

**Proof.** Consider the first variation of $S$ defined by the vector $\tilde{v} = \tilde{\xi} - N_{\tilde{\xi}} = Q\tilde{m} + \tilde{Y}^1$. From the definition of stability operator [8], we have $\delta_t \theta_l = L_m Q$. On the other hand, linearity of this variation gives $\delta_t \theta_l = \delta_t \theta_l - N_{\delta l} \theta_l$. Using now the Raychaudhuri equation $\delta_t \theta_l = -W$ (see (4)) and the identity (11) gives $L_m Q = Z$. Since $Q = (\tilde{\xi} \cdot \tilde{l})$, the result follows directly from lemma 2.

This theorem gives information about the relative position between the generator $\tilde{\xi}$ and the outer null normal $\tilde{l}$ and has, in principle, many potential consequences. Specific applications require considering spacetimes having special vector fields for which sufficient information about its deformation tensor is available. Once such a vector is known to exist, the result above can be used either to restrict the form of $\tilde{\xi}$ in stable or strictly stable MOTS or, alternatively, to restrict the regions of the spacetime where such MOTS are allowed to be present.

Since conformal vector fields (and homotheties and isometries as particular cases) have very special deformation tensors, the theorem above gives interesting information for spacetimes admitting such symmetries.

**Corollary 2.** Let $S$ be a stable MOTS in a hypersurface $\Sigma$ of a spacetime $(M, g)$ which admits a conformal Killing vector $\tilde{\xi}$, $\mathcal{L}_\tilde{\xi} g_{\mu\nu} = 2\phi g_{\mu\nu}$ (including homotheties $\phi = C$, and isometries $\phi = 0$).

(i) If $2\tilde{\xi}(\phi) + N(\kappa^2 + G_{\mu\nu} l^\mu l^\nu)|_S \leq 0$ and not identically zero, then $(\tilde{\xi} \cdot \tilde{l})|_S < 0$.

(ii) If $S$ is strictly stable and $2\tilde{\xi}(\phi) + N(\kappa^2 + G_{\mu\nu} l^\mu l^\nu)|_S \leq 0$, then $(\tilde{\xi} \cdot \tilde{l})|_S \leq 0$ and vanishes at one point only if it vanishes everywhere.

**Remark.** As before, the theorem is still true if all inequalities are reversed.
Figure 2. The planes \( T_p \Sigma \) and \( P = T_p S \oplus \text{span} \{ \vec{l}_p \} \) divide the tangent space \( T_p M \) into four regions. By corollary 2, if \( S \) is strictly stable and \( \vec{\xi} \) is a Killing vector or a homothety in a spacetime satisfying the NEC which points above \( \Sigma \) everywhere, then \( \vec{\xi} \) cannot enter into the forbidden region at any point (and similarly, if \( \vec{\xi} \) points below \( \Sigma \) everywhere). The allowed region includes the plane \( P \). However, if there is a point with \( W \neq 0 \), where \( \vec{\xi} \) is not tangent to \( \Sigma \), then the result is also valid for a stable MOTS and, moreover, \( P \) belongs to the forbidden region in this case.

**Proof.** We only need to show that \( Z = 2\vec{l}(\phi) + N(\kappa_l^2 + G_{\mu
u}l^\mu l^\nu)|_S \) for conformal Killing vectors. This follows at once from (13) and \( a_{\mu
u} = 2\phi g_{\mu
u} \) after using orthogonality of \( \vec{e}_A \) and \( \vec{l} \). Notice in particular that \( Z \) is the same for isometries and for homotheties. \( \square \)

This corollary has an interesting consequence in spacetime regions where there exists a Killing vector or a homothety \( \vec{\xi} \) which is causal everywhere.

**Corollary 3.** Let a spacetime \( (M, g) \) satisfying the NEC admits a causal Killing vector or homothety \( \vec{\xi} \) which is future (past) directed everywhere on a stable MOTS \( S \subset \Sigma \). Then,

(i) The second fundamental form \( \kappa^A_B \) along \( \vec{l} \) and \( G_{\mu\nu}l^\mu l^\nu \) vanish identically on every point \( p \in S \) where \( \vec{\xi}|_p \neq 0 \).

(ii) If \( S \) is strictly stable, then \( \vec{\xi} \propto \vec{l} \) everywhere.

**Remark.** If we assume that there exists an open neighborhood of \( S \) in \( M \) where the Killing vector or homothety \( \vec{\xi} \) is causal and future (past) directed everywhere, then the conclusion (i) can be strengthened to say that \( \kappa^A_B \) and \( G_{\mu\nu}l^\mu l^\nu \) vanish identically on \( S \). The reason is that such a \( \vec{\xi} \) cannot vanish anywhere in this neighborhood (and consequently neither on \( S \)). For Killing vectors, this result is proven in lemma 3.2 in [14] and a simple generalization shows that the same holds for homothetic Killing vectors.

**Proof.** We can assume, after reversing the sign of \( \vec{\xi} \) if necessary, that \( \vec{\xi} \) is past directed, i.e. \( N_S \leq 0 \).

Under the NEC, \( W \) is the sum of two non-negative terms, so in order to prove (i) we only need to show that \( W = 0 \) on points where \( \vec{\xi} \neq 0 \), i.e. at points where \( N_S < 0 \). Assume, on the contrary, that \( W \neq 0 \) and \( N_S < 0 \) happen simultaneously at a point \( p \in S \). It follows that \( N_S W \leq 0 \) everywhere and non-zero at \( p \). Thus, we can apply statement (i) of corollary 2 to conclude \( \bar{Q} < 0 \) everywhere. Hence \( N_S \bar{Q} \geq 0 \) and not identically zero on \( S \). Recalling the decomposition \( \vec{\xi} = N_S \vec{l} + \bar{Q} \vec{m} + \bar{V}^\dagger \), the square norm of this vector is

\[
(\vec{\xi} \cdot \vec{\xi}) = 2N_S \bar{Q} + \bar{Q}^2 + (\bar{V}^\dagger \cdot \bar{V}^\dagger).
\]
This is the sum of non-negative terms, the first one not identically zero. This contradicts the condition of \( \xi \) being causal.

To prove the second statement, we notice that point (ii) in corollary 2 implies \( Q \leq 0 \), and hence \( N_S Q \geq 0 \). The only way (14) can be negative or zero is if \( Q = N = 0 \), i.e. \( \xi \propto \hat{I} \). □

This corollary extends theorem 2 in [5] to the case of stable MOTS and implies, for instance, that any strictly stable MOTS in a plane wave spacetime (which by definition admits a null and nowhere zero Killing vector field \( \xi \)) must be aligned with the direction of propagation of the wave (in the sense that \( \xi \) must be one of the null normals to the surface). It also implies that any spacetime admitting a causal and future-directed Killing vector (or homothety) whose energy–momentum tensor does not admit a null eigenvector (e.g. a perfect fluid) cannot contain any stable MOTS.

The results above hold for stable or strictly stable MOTS. Among such surfaces, marginally trapped surfaces are of special interest. Our next result restricts (and in some cases forbids) the existence of such surfaces in spacetimes admitting Killing vectors, homotheties or conformal Killings.

**Theorem 2.** Let \( S \) be a stable MOTS in a spacelike hypersurface \( \Sigma \) of a spacetime \( (M, g) \) which satisfies the NEC and admits a conformal Killing vector \( \xi \) with conformal factor \( \phi \geq 0 \) (including homotheties with \( C \geq 0 \) and Killing vectors). Suppose furthermore that either (i) \((2l(\phi) + NW)|_S \neq 0 \) or (ii) \( S \) is strictly stable and \((\xi \cdot \hat{I})|_S \neq 0 \). Then the following holds.

(a) If \( 2l(\phi) + NW|_S \leq 0 \), then \( S \) cannot be a marginally future trapped surface, unless \( \bar{H} \equiv 0 \).

The latter case is excluded if \( \phi|_S \not\equiv 0 \).

(b) If \( 2l(\phi) + NW|_S \geq 0 \), then \( S \) cannot be a marginally past trapped surface, unless \( \bar{H} \equiv 0 \).

The latter case is excluded if \( \phi|_S \not\equiv 0 \).

**Remark.** The statement obtained from this one by reversing all the inequalities is also true. This is a direct consequence of the freedom in changing \( \xi \rightarrow -\xi \).

**Proof.** We will only prove case (a). The argument for case (b) is similar. The idea is taken from [5] and consists of performing a variation of \( S \) along the conformal Killing vector and evaluating the change of area in order to get a contradiction if \( S \) is marginally future trapped. The difference is that here we do not make any \( a \) priori assumption on the causal character for \( \xi \). Corollary 2 provides us with sufficient information for the argument to go through.

As before, let \( \{S'_t\} \) be the collection of surfaces obtained by displacing \( S \) with the local diffeomorphism generated by \( \xi \) a parametric amount \( t \). We denote by \( |S'_t| \) their corresponding areas. The first variation of area (see e.g. [5]) gives

\[
\frac{d|S'_t|}{dt}
\bigg|_{t=0} = -\frac{1}{2} \int_S \theta_k (\xi \cdot \hat{I}) \eta_S, \tag{15}
\]

where \( \eta_S \) is the volume form of \( S \) and we have used \( \bar{H} = -\frac{1}{2} \theta_k \hat{I} \). Now, since \( 2l(\phi) + NW|_S \leq 0 \), and furthermore either hypothesis (i) or (ii) holds, corollary 2 implies that \((\xi \cdot \hat{I})|_S < 0 \).

On the other hand, \( \xi \) being a conformal Killing vector, the induced metric on \( S'_t \) is related to the metric on \( S \) by conformal rescaling. A simple calculation gives (see e.g. [5])

\[
\frac{d|S'_t|}{dt}
\bigg|_{t=0} = 2 \int_S \phi \eta_S. \tag{16}
\]

This quantity is non-negative due to \( \phi \geq 0 \) and not identically zero if \( \phi \neq 0 \) somewhere. Combining (15) and (16) we conclude that if \( \theta_k \leq 0 \) (i.e. \( S \) is marginally future trapped), then necessarily \( \theta_k \) vanishes identically (and so does \( \bar{H} \)). Furthermore, if \( \phi|_S \) is non-zero...
somewhere, then $\theta_k$ must necessarily be positive somewhere, and $S$ cannot be future marginally trapped.

### 4.1. An application: no stable MOTS in FLRW

In this subsection, we apply corollary 2 to show that a large subclass of Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes do not admit stable MOTS on any spacelike hypersurface. Obtaining the corresponding results for round spheres only requires a straightforward calculation, and is therefore simple. The power of the method is that it provides a general result involving no assumption on the geometry of the MOTS or on the spacelike hypersurface where it is embedded. The only requirement is that the scale factor and its time derivative satisfy certain inequalities. This includes, for instance, all FLRW cosmologies satisfying the NEC and with accelerated expansion, as we shall see in corollary 4 below.

Recall that the FLRW metric is

$$g_{FLRW} = -dt^2 + a^2(t)[dr^2 + \chi^2(r; k) d\Omega^2],$$

where $a(t) > 0$ is the scale factor and $\chi(r; k) = \{ \sin r, r, \sinh r \}$ for $k = \{1, 0, -1\}$, respectively. The Einstein tensor of this metric is of perfect fluid type (see e.g. [15]) and reads

$$G_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad \vec{u} = \partial_t,$$

$$\rho = \frac{3(\dot{a}^2(t) + k)}{a^2(t)}, \quad \rho + p = 2\left(\frac{\dot{a}^2(t) + k}{a^2(t)} - \frac{\ddot{a}(t)}{a(t)}\right),$$  \hspace{1cm} (17)

where dot stands for derivative with respect to $t$.

**Theorem 3.** There exists no stable MOTS in any spacelike hypersurface of a FLRW spacetime $(M, g_{FLRW})$ satisfying

$$\frac{\dot{a}^2(t) + k}{a(t)} > 0, \quad -\frac{\dot{a}^2(t) + k}{a(t)} \leq \dot{a}(t) \leq \frac{\dot{a}^2(t) + k}{a(t)}.$$  \hspace{1cm} (18)

**Remark.** In terms of the energy–momentum contents of the spacetime, these three conditions read, respectively, $\rho \geq 0$, $\rho \geq 3p$ and $\rho + p \geq 0$. As an example, in the absence of a cosmological constant they are satisfied as soon as energy conditions are imposed and the pressure is not too large (e.g. for the matter and radiation dominated eras). The class of FLRW satisfying (18) is clearly very large (cf corollary 4 below). We also remark that theorem 3 agrees with the fact [16] that the causal character of the hypersurface which separates the trapped from the non-trapped spheres in FLRW spacetimes depends precisely on the quantity $\rho^2(\rho + p)(\rho - 3p)$.

**Proof.** The FLRW spacetime admits a conformal Killing vector $\vec{\xi} = a(t)\vec{u}$ with the conformal factor $\phi = \dot{a}(t)$. Since this vector is timelike and future directed, it follows that $\langle \vec{\xi} \cdot \vec{l} \rangle |_S < 0$ for any spacelike surface $S$ embedded in a spacelike hypersurface $\Sigma$. If we can show that $2I(\phi) + N(k^2 + G_{\mu\nu}l^\mu l^\nu)|_S \geq 0$, and non-identically zero for any $S$, then point (i) in corollary (2) implies that $S$ cannot be a stable MOTS, thus proving the result. The proof therefore relies on finding conditions on the scale factor which imply the validity of this inequality on any $S$. First of all, we notice that the second fundamental form $\kappa_{AB}^l$ can be made as small as desired on a suitably chosen $S$. Thus, the inequality that needs to be satisfied is

$$2I(\phi) + NG_{\mu\nu}l^\mu l^\nu|_S \geq 0,$$  \hspace{1cm} (19)
and positive somewhere. In order to evaluate this expression, recall that \( \mathbf{v} = a^{-1} \xi = a(t)^{-1} N \mathbf{\hat{v}} + a(t)^{-1} Y \). Let us write \( Y = Y \mathbf{\hat{v}} \), where \( \mathbf{\hat{v}} \) is unit and let \( \alpha \) be the hyperbolic angle of \( \mathbf{\hat{u}} \) in the basis \( [\mathbf{\hat{n}}, \mathbf{\hat{e}}] \), i.e. \( \mathbf{\hat{u}} = \cosh \alpha \mathbf{\hat{n}} + \sinh \alpha \mathbf{\hat{e}} \). It follows immediately that \( \dot{N} = a(t) \cosh \alpha \) and \( Y = a(t) \sinh \alpha \). Furthermore, multiplying \( \mathbf{\hat{u}} \) by the normal vector to the surface we find \( (\mathbf{\hat{u}} \cdot \mathbf{\hat{m}}) = \cos \phi \sinh \alpha \), where \( \phi \) is the angle between \( \mathbf{\hat{m}} \) and \( \mathbf{\hat{e}} \). With this notation, let us calculate the null vector \( \mathbf{\hat{l}} \). Writing \( \mathbf{\hat{l}} = A \mathbf{\hat{u}} + \mathbf{\hat{b}} \), with \( \mathbf{\hat{b}} \) orthogonal to \( \mathbf{\hat{u}} \), it follows \( (\mathbf{\hat{b}} \cdot \mathbf{\hat{b}}) = A^2 \) from the condition of \( \mathbf{\hat{l}} \) being null. On the other hand, we have the decomposition \( A \mathbf{\hat{u}} + \mathbf{\hat{b}} = \mathbf{\hat{l}} = \mathbf{\hat{n}} + \mathbf{\hat{m}} \). Multiplying by \( u \), we immediately get \( A = \cosh \alpha - \cos \phi \sinh \alpha \), and since \( f = \dot{a}(t) \) only depends on \( t \),

\[
\mathbf{\hat{l}}(\phi) = (\cosh \alpha - \cos \phi \sinh \alpha) \dot{a}(t). \tag{20}
\]

The following expression for \( G_{\mu \nu} l^\mu l^\nu \) follows directly from \( \mathbf{\hat{l}} = A \mathbf{\hat{u}} + \mathbf{\hat{b}} \) and (17),

\[
G_{\mu \nu} l^\mu l^\nu = A^2 (\rho + p) = 2 (\cosh \alpha - \cos \phi \sinh \alpha)^2 \left( \frac{\dot{a}^2(t) + k}{a^2(t)} - \frac{\dot{a}(t)}{a(t)} \right). \tag{21}
\]

Inserting (20) and (21) into (19) and dividing by \( A^2 \cosh \alpha \) (which is positive), we find the equivalent condition

\[
\left( \frac{1}{\cosh \alpha (\cosh \alpha - \cos \phi \sinh \alpha)} - 1 \right) \dot{a}(t) + \frac{\dot{a}^2(t) + k}{a(t)} \geq 0, \tag{22}
\]

and non-zero somewhere. The dependence on \( S \) only arises through the function \( f(\alpha, \phi) = \cosh \alpha (\cosh \alpha - \cos \phi \sinh \alpha) \). Rewriting this as \( f = 1/2 [1 + \cosh(2\alpha) - \cos \phi \sinh(2\alpha)] \), it is immediate to show that \( f \) takes all values in \( (1/2, +\infty) \). Hence

\[
-1 < \left( \frac{1}{\cosh \alpha (\cosh \alpha - \cos \phi \sinh \alpha)} - 1 \right) < 1.
\]

In order to satisfy (22) on all this range, it is necessary and sufficient that the two inequalities in (18) are satisfied.

The following corollary gives a particularly interesting case where all the conditions of theorem 3 are satisfied.

**Corollary 4.** Consider a FLRW spacetime \((M, g_{\text{FLRW}})\) satisfying the NEC. If \( \dot{a}(t) > 0 \), then there exists no stable MOTS in any spacelike hypersurface of \((M, g_{\text{FLRW}})\).

**Proof.** The null energy condition gives \( 0 \leq \rho + p = 2 (\frac{\dot{a}^2(t) + k}{a^2(t)} - \frac{\dot{a}(t)}{a(t)}) \). This implies the first and the last inequality in (18) if \( \dot{a} > 0 \). The remaining condition \( -\frac{\dot{a}^2(t) + k}{a^2(t)} \leq \dot{a} \) is also obviously satisfied provided \( \dot{a} > 0 \). \( \square \)

### 4.2. A consequence of the geometric construction of \( \{S_t\} \)

We have emphasized at the beginning of this section that the restrictions obtained directly by the geometric procedure of moving \( S \) along \( \xi \) and then back to \( \Sigma \) are intuitively clear but typically weaker than those obtained by using elliptic theory results. There are some cases, however, where the reverse actually holds, and the geometric construction provides stronger results. We will present one of these cases in this subsection.

Corollary 2 gives restrictions on \((\xi \cdot \mathbf{\hat{l}})_{\Sigma}\) for Killing vectors and homotheties in spacetimes satisfying the NEC, provided \( \xi \) is future or past directed everywhere. However, when \( W \) vanishes identically, the result only gives useful information in the strictly stable case. The reason is that \( W \equiv 0 \) implies \( L_m Q \equiv 0 \) and, for marginally stable surfaces (i.e. \( \lambda = 0 \), the
maximum principle is not strong enough to conclude that \( Q \) must have a sign. There is at least one case where marginally stable surfaces play an important role, namely after a jump into the outermost MOTS in a 3+1 foliation of the spacetime (see [17] for details). As we will see next, the geometric construction does give restrictions in this case even when \( W \) vanishes identically. Let us start by recalling the definition of outermost MOTS.

Let \( \Sigma \) be a spacelike hypersurface whose boundary consists of the union of two disjoint sets \( \partial \Sigma = \partial^+ \Sigma \cup \partial^- \Sigma \). We take \( \Sigma \) to be disjoint to its boundaries and assume that \( \Sigma \) has compact closure. Endow \( \partial^+ \Sigma \) with an outer normal pointing outside \( \Sigma \) and \( \partial^- \Sigma \) with an outer normal pointing inside \( \Sigma \). Assume that the outer boundary \( \partial^+ \Sigma \) is outer untrapped \( \theta^+_\Sigma > 0 \) and that the inner boundary \( \partial^- \Sigma \) is weakly outer trapped \( \theta^-_\Sigma \leq 0 \). Under these conditions, theorem 7.3 of [18] asserts that there always exists a unique outermost MOTS \( S \subset \Sigma \cup \partial^- \Sigma \) homologous to \( \partial^+ \Sigma \) (i.e. such that together with the outer boundary it bounds an open domain \( V \)). Outermost means that no weakly outer trapped surface contained in \( \Sigma \cup \partial^- \Sigma \) and homologous to the outer boundary can intersect \( V \). Obviously, the outermost MOTS is locally outermost and hence necessarily stable. When requiring a surface \( S \) to be outermost, we will implicitly assume all the above conditions on \( \Sigma \).

We can now state the following result.

**Theorem 4.** Consider a spacetime \( (M, g) \) possessing a Killing vector or a homothety \( \tilde{\xi} \) and satisfying the NEC. Let \( S \) be the outermost MOTS on a spacelike hypersurface \( \Sigma \) defined locally by a level function \( T = 0 \) with \( T > 0 \) to the future of \( \Sigma \). If \( \tilde{\xi}(T) \leq 0 \) on some spacetime neighborhood of \( S \), then \( (\tilde{\xi} \cdot \tilde{l}) \leq 0 \) everywhere on \( S \).

**Remark.** As usual, the theorem still holds if all the inequalities are reversed.

**Remark.** The simplest way to ensure that \( \tilde{\xi}(T) \leq 0 \) on some neighborhood of \( S \) is by imposing a condition merely on \( S \), namely \( (\tilde{\xi} \cdot \tilde{n})|_S > 0 \), because then \( \tilde{\xi} \) lies strictly below \( \Sigma \) on \( S \) and this property is obviously preserved sufficiently near \( S \) (i.e. \( \tilde{\xi} \) points strictly below the level set of \( T \) on a sufficiently small spacetime neighborhood of \( S \)).

**Proof.** The idea is to use the geometric procedure described above to construct \( \{S_t\} \) and use the fact that \( S \) is outermost to conclude that \( S_t \) \((t > 0)\) cannot have points outside \( S \). Here we move \( S \) a small but finite amount \( t \), in contrast to the elliptic results before, which only involved infinitesimal displacements. We want to have information on the sign of the outer expansion of \( S_t \) in order to make sure that a weakly outer trapped surface forms. The first part of the displacement is along \( \tilde{\xi} \) and gives \( S_t \). Let us first see that all these surfaces are MOTS. For Killing vectors, this follows at once from symmetry arguments. For homotheties \( (\mathcal{L}_{\xi} g_{ab} = 2C g_{ab}) \) we have the identity

\[
\delta_{\xi} \theta_t = (-\frac{1}{2} k^\alpha \mathcal{L}_\xi I_{\alpha \mu} - 2C) \theta_t,
\]

which follows directly from (5) with \( \tilde{\eta} = \tilde{l} \) after using \( I^\alpha \mathcal{L}_\xi I_{\alpha \mu} = \frac{1}{2} a_{\mu \nu} I^\mu I^\nu = 0 \), see (12). Expression (23) holds for each of the surfaces \( S_t \), independently of them being MOTS or not. Since this variation vanishes on MOTS and the starting surface \( S \) has this property, it follows that each surface \( S_t \) \((t > 0)\) is also a MOTS. Moving back to \( \Sigma \) along the null hypersurface introduces, via the Raychaudhuri equation, a non-positive term \( N_3 W \) in the outer null expansion, provided the motion is to the future. Hence, \( S_t \) for small but finite \( t > 0 \) is a weakly outer trapped surface provided \( \tilde{\xi} \) moves to the past of \( \Sigma \). This is ensured if \( \tilde{\xi}(T) \leq 0 \) near \( S \), because \( T \) cannot become positive for small enough \( t \). On the other hand, since a point \( p \in S \) moves initially along the vector field \( v = \tilde{\xi} - N_3 \tilde{l} = Q \tilde{m} + \tilde{Y} \), where \( Q = (\tilde{\xi} \cdot \tilde{l}) \) as usual, it follows that \( Q > 0 \) somewhere implies (for small enough \( t \)) that the
weakly outer trapped surface $S_t$ has a portion lying strictly to the outside of $S$, which is a contradiction to $S$ being outermost. Hence $Q \leq 0$ everywhere and the theorem is proven. □

It should be remarked that the assumption of $\xi$ being a Killing vector or a homothety is important for this result. Trying to generalize it for instance to conformal Killings fails in general because then the right-hand side of equation (23) has an additional term $2\tilde{I}(\phi)$, not proportional to $\theta_i$. This means that moving a MOTS along a conformal Killing does not lead to another MOTS in general. The method can, however, still give useful information if $\tilde{I}(\phi)$ has the appropriate sign, so that $S_t^+$ is in fact weakly outer trapped. We omit the details.

So far, all the results we have obtained require that the quantity $L_mQ$ does not change sign on the MOTS $S$. In the next section we will relax this condition.

5. Results regardless of the sign of $L_mQ$

When $L_mQ$ changes the sign on $S$, the elliptic methods exploited in the previous section lose their power. Moreover, for sufficiently small $t$, the surface $\{S_t\}$ defined by the geometric construction above necessarily fails to be weakly outer trapped. Thus, obtaining restrictions in this case becomes a much harder problem.

However, for locally outermost MOTS $S$, an interesting situation arises when $S_t$ lies partially outside $S$ and happens to be weakly outer trapped in that exterior region. More precisely, if a connected component of the subset of $S_t$ which lies outside $S$ turns out to have non-positive outer null expansion, then using a smoothing result by Kriele and Hayward [11], we will be able to construct a new weakly outer trapped surface outside $S$, thus leading to a contradiction with the fact that $S$ is locally outermost (or else giving restrictions on the generator $\xi$).

The result by Kriele and Hayward states, in rough terms, that given two surfaces which intersect on a curve, a new smooth surface can be constructed lying outside the previous ones in such a way that the outer null expansion does not increase in the process. The precise statement is as follows.

**Lemma 3.** Let $S_1, S_2 \subset \Sigma$ be smooth two-sided surfaces which intersect transversely on a smooth curve $\gamma$. Assume that it is possible to choose one connected component of each set $S_1\setminus \gamma$ and $S_2\setminus \gamma$, say $S_+^-$ and $S_+^+$ respectively, such that the outer normal $\tilde{m}_t$ of $S_+^-$ and the vector $\tilde{e}_-$ orthogonal to $\gamma$, tangent to $S_+^-$ and pointing toward $S_+$, satisfy $m_+ e^{-\mu} > 0$ everywhere on $\gamma$. Then, for any neighborhood $V$ of $\gamma$ in $\Sigma$ there exists a smooth surface $\bar{S}$ and a continuous and piecewise smooth bijection $\Phi: S_+^+ \cup S_+^- \cup \gamma \to \bar{S}$ such that

(i) $\Phi(p) = p$, $\forall p \in (S_+^+ \cup S_+^-)\setminus V$

(ii) $\tilde{\theta}_t|_{\Phi(p)} \leq \tilde{\theta}_t^\pm|_p$, $\forall p \in S_\pm$, where $\tilde{\theta}_t$ is the null expansion of $\bar{S}$ and $\theta_t^\pm$ is the null expansion of $S_\pm$.

Moreover, $\bar{S}$ lies in the connected component of $V \setminus (S_+^+ \cup S_+^- \cup \gamma)$ to which $\tilde{m}_t$ points.

This result will allow us to adapt the arguments above without having to assume that $L_mQ$ has a constant sign on $S$. The argument will be again by contradiction, i.e. we will assume a locally outermost MOTS $S$ and, under suitable circumstances, we will be able to find a new weakly outer trapped surface lying outside $S$. Since the conditions are much weaker than in the previous section, the conclusion is also weaker. It is, however, fully general in the sense that it holds for any vector field $\xi$ on $S$. Recall that $Z$ is defined in equation (13).

**Theorem 5.** Let $S$ be a locally outermost MOTS in a spacelike hypersurface $\Sigma$ of a spacetime $(M, g)$. Denote by $U_0$ a connected component of the set $\{p \in S; (\xi \cdot \tilde{I})|_p > 0\}$. Assume that
$U_0 \neq \emptyset$ and that its boundary $\gamma = \partial U_0$ is either empty, or it satisfies that the function $(\vec{\xi} \cdot \vec{l})$ has a non-zero gradient everywhere on $\gamma$, i.e. $d(\vec{\xi} \cdot \vec{l})|_\gamma \neq 0$.

Then, there exists $p \in \overline{U_0}$ such that $Z|_p \geq 0$.

**Proof.** As mentioned, we will use a contradiction argument. Let us therefore assume that

$$Z|_p < 0, \quad \forall \, p \in \overline{U_0}.$$  \hfill (24)

The aim is to construct a weakly outer trapped surface near $S$ and outside of it. This will contradict the condition of $S$ being locally outermost.

First of all we observe that $Z$ cannot be negative everywhere on $S$, because otherwise theorem 1 (recall that outermost MOTS are always stable) would imply $Q = (\vec{\xi} \cdot \vec{l}) < 0$ everywhere and $U_0$ would be empty against hypothesis. Consequently, under (24), $U_0$ cannot coincide with $S$ and $\gamma = \partial U_0 \neq \emptyset$. Since $Q|_\gamma = 0$ and, by assumption, $dQ|_\gamma \neq 0$, it follows that $\gamma$ is a smooth embedded curve. Taking $\mu$ to be a local coordinate on $\gamma$, it is clear that $\{\mu, Q\}$ are the coordinates of a neighborhood of $\gamma$ in $S$. We will coordinate a small enough neighborhood of $\gamma$ in $\Sigma$ by Gaussian coordinates $\{u, \mu, Q\}$ such that $u = 0$ on $S$ and $u > 0$ on its exterior.

By moving $S$ along $\vec{\xi}$ a finite but small parametric amount $t$ and back to $\Sigma$ with the outer null geodesics, as described in section 2, we construct a family of surfaces $\{S_t\}$. The curve that each point $p \in S$ describes via this construction has tangent vector $v = Q\vec{m} + \vec{Y}|_S$ on $S$. In a small neighborhood of $\gamma$, the normal component of this vector, i.e. $Q\vec{m}$, is smooth and only vanishes on $\gamma$. This implies that for small enough $t$, $S_t$ are graphs over $S$ near $\gamma$.

We will always work on this neighborhood, or suitable restrictions thereof. In the Gaussian coordinates above, this graph is of the form $\{u = \hat{u}(\mu, Q, t), \mu, Q\}$. Since the normal unit vector to $S$ is simply $\vec{m} = \vec{n}$, in these coordinates and the normal component of $v$ is $Q\vec{m}$, the graph function $\hat{u}$ has the following Taylor expansion:

$$\hat{u}(\mu, Q, t) = Qt + O(t^2).$$  \hfill (25)

Our next aim is to use this expansion to conclude that the intersection of $S$ and $S_t$ near $\gamma$ is an embedded curve $\gamma_t$ for all small enough $t$. To do that we will apply the implicit function theorem to the equation $\hat{u} = 0$. It is useful to introduce a new function $v(\mu, Q, t) = \frac{\hat{u}(\mu, Q, t)}{t}$, which is still smooth (thanks to (25)) and vanishes at $t = 0$ only on the curve $\gamma$. Moreover, its derivative with respect to $Q$ is nowhere zero on $\gamma$, in fact $\frac{\partial v}{\partial Q}|_{(\mu,0,0)} = 1$ for all $\mu$. The implicit function theorem implies that there exists a unique function $Q = \phi(\mu, t)$, which solves the equation $v(\mu, Q, t) = 0$, for small enough $t$. Obviously, this function is also the unique solution near $\gamma$ of $\hat{u}(\mu, Q, t) = 0$ for $t > 0$. Consequently, the intersection of $S$ and $S_t$ ($t > 0$) lying in the neighborhood of $\gamma$ where we are working on is an embedded curve $\gamma_t$. This curve divides $S_t$ into two connected components (because $\gamma$ divides $S$). Let us denote by $S_t^*$ the connected component of $S_t$ which has $v(\mu, Q, t) > 0$ near $\gamma$ (i.e. that lies in the exterior of $S$ near $\gamma$). This connected component in fact lies fully outside of $S$, not just in a neighborhood of $\gamma$, as we see next. First of all, recall that $\gamma$ is the boundary of a connected set $U_0$ where $Q$ is strictly positive. We have just seen that $\gamma_t$ is a continuous deformation of $\gamma$. Let us denote by $U_t$ the domain obtained by deforming $U_0$ when the boundary moves from $\gamma$ to $\gamma_t$ (See figure 3). It is obvious that $S_t^*$ is obtained by moving $U_t$ first along $\vec{\xi}$ an amount $t$ and then back to $\Sigma$ by null hypersurfaces. The closed subset of $U_t$ lying outside the tubular neighborhood where we applied the implicit function theorem is by construction a proper subset of $U_0$. Consequently, on this closed set $Q$ is uniformly bounded below by a positive constant. Given that $Q$ is the first-order term of the normal variation, all these points move outside of $S$. This proves that $S_t^*$ is fully outside of $S$ for sufficiently small $t$. Incidentally, this also shows that $S_t^*$ is a graph over $U_t$. 

The next aim is to show that the outer null expansion of $S_t$ is non-positive everywhere on $S^+_t$. To that aim, we will prove that, for small enough $t$, $Z$ is strictly negative everywhere on $U_t$. Since $Z$ is the first-order term in the variation of $\theta_l$, this implies that the outer null expansion of $S^+_t$ satisfies $\theta^+_l < 0$ for $t > 0$ small enough.

By assumption (24), $Z$ is strictly negative on $U_0$. Therefore, this quantity is automatically negative in the portion of $U_t$ lying in $U_0$ (in particular, outside the tubular neighborhood where we applied the implicit function theorem). The only difficulty comes from the fact that $\gamma_t$ may move outside $U_0$ at some points and we only have information on the sign of $Z$ on $\overline{U_0}$. To address this issue, we first notice that $Q$ defines a distance function to $\gamma$ (because $Q$ vanishes on $\gamma$ and its gradient is nowhere zero). Consequently, the fact that $Z$ is strictly negative on $\gamma$ (by assumption (24)) and that this curve is compact implies that there exists $\delta > 0$ such that, inside the tubular neighborhood of $\gamma$, $|Q| < \delta$ implies $Z < 0$. Moreover, the function $Q = \varphi(\mu, t)$, which defines $\gamma_t$, is such that it vanishes at $t = 0$ and depends smoothly in $t$.

Since $\mu$ takes values on a compact set, it follows that for each $\delta' > 0$, there exists $\epsilon(\delta') > 0$, independent of $\mu$ such that $|t| < \epsilon(\delta')$ implies $|Q| = |\varphi(\mu, t)| < \delta'$. By taking $\delta' = \delta$, it follows that, for $|t| < \epsilon(\delta)$, $U_t$ is contained in a $\delta$-neighborhood of $U_0$ (with respect to the distance function $Q$) and consequently $Z < 0$ on this set, as claimed. Summarizing, so far we have shown that $S^+_t$ lies fully outside $S$ and has $\theta^+_l < 0$. The final task is to use lemma 3 to construct a weakly outer trapped surface strictly outside $S$. Indeed, the curve $\gamma_t$ divides the locally outermost MOTS $S$ into two connected components. Since $S$ is locally outermost, there is a two-sided neighborhood of $S$ in $\Sigma$. Following the notation in section 2, we call $\mathcal{D}$ the interior part of this two-sided neighborhood. Denote by $S^-_t$ the complement of $U_t$ in $S$. By construction, $S^+_t \cup \gamma_t \cup S^-_t$ bounds a domain which contains $\mathcal{D}$. Now, let $\bar{\xi}$ be the vector normal to $\gamma_t$ and tangent to $S^-_t$ that points to the interior of $S_t^-$ and let $m_e$ be the vector normal to $S_t^+$ which points to the exterior of $S_t^+$. Since $S^-_t$ lies outside $S$ and it is a graph on $S$ near $\gamma_t$, it follows immediately that $m_{e, \mu} e^{-\mu} \geq 0$ holds everywhere on $\gamma_t$. Therefore, lemma 3 guarantees that there exists a weakly outer trapped surface $\tilde{S}$ lying outside $S$, leading to a contradiction. 

Remark. As always, this theorem also holds if all the inequalities are reversed. Note that in this case, $U_0$ is defined to be a connected component of the set $\{ p \in S; (\bar{\xi} \cdot \nabla)|_p < 0 \}$, For the proof simply take $t < 0$ instead of $t > 0$ (or equivalently move along $-\bar{\xi}$ instead of $\bar{\xi}$).

Similarly as in the previous section, this theorem can be particularized to the case of conformal Killing vectors, as follows.

Corollary 5. Under the assumptions of theorem 5, suppose that $\bar{\xi}$ is a conformal Killing vector with the conformal factor $\phi$ (including homotheties $\phi = C$ and isometries $\phi = 0$).
Then, there exists $p \in \overline{U}_0$ such that $\overline{\mathcal{L}}(\phi) + N_2(k^2 + G_{\mu\nu}l^\mu l^\nu)|_p \geq 0$.

If the conformal Killing is in fact a homothety or a Killing vector and it is causal everywhere, the result can be strengthened considerably. The next result extends corollary 3 in a suitable sense to the cases when the generator is not assumed to be either future or past everywhere. Since its proof requires an extra ingredient, we write it down as a theorem.

**Theorem 6.** In a spacetime $(M, g)$ satisfying the NEC and admitting a Killing vector or homothety $\xi$, consider a locally outermost MOTS $S$ in a spacelike hypersurface $\Sigma$. Assume that $\xi$ is causal on $S$ and that $W = k^2 + G_{\mu\nu}l^\mu l^\nu$ is non-zero everywhere on $S$. Define $U \equiv \{ p \in S; (\xi \cdot \vec{T})|_p > 0 \}$ and assume that this set is neither empty nor covers all of $S$. Then, on each connected component $U_t$ of $U$, there exists a point $p \in \partial U_t$ with $d(\xi \cdot \vec{T})|_p = 0$.

**Remark.** The same conclusion holds on the boundary of each connected components of the set $\{ p \in S; (\xi \cdot \vec{T})|_p < 0 \}$. This is obvious since $\xi$ can be changed to $-\xi$.

**Remark.** The case $\partial U = \emptyset$, excluded by assumption in this theorem, can only occur if $\xi$ is future or past everywhere on $S$. Hence, this case is already included in corollary 3.

**Proof.** We first show that on any point in $U$ we have $N_S < 0$, which has as an immediate consequence that $N_S \leq 0$ on any point in $\overline{U}$. The former statement is a consequence of the decomposition $\vec{\xi} = N\vec{I} + Q\vec{m} + Y\vec{t}$, where $Q = (\vec{\xi} \cdot \vec{T})$. The condition that $\vec{\xi}$ is causal then implies $(\vec{\xi} \cdot \vec{\xi}) = 2N_S Q + Q^2 + Y^2 \leq 0$. This can only happen at a point where $Q > 0$ (i.e. on $U$) provided $N_S < 0$ there. Moreover, if at any point $q$ on the boundary $\partial U$ we have $N_S|_q = 0$, then necessarily the full vector $\vec{\xi}$ vanishes at this point. This implies, in particular, that the geometric construction of $S_t$ has the property that $q$ remains invariant.

Having noticed these facts, we will now argue by contradiction, i.e. we will assume that there exists a connected component $U_0$ of $U$ such that $d(\vec{\xi} \cdot \vec{T})|_{U_0} \neq 0$ everywhere. In these circumstances, we can follow the same steps as in the proof of theorem 5 to show that, for small enough $t$, the surface $S_t$ has a portion $S_{t^*}$ lying in the exterior of $S$ and which, in the Gaussian coordinates above, is a graph over a subset $U_t$ which is a continuous deformation of $U_0$. Moreover, the boundary of $U_t$ is a smooth embedded curve $\gamma_t$. The only difficulty with this construction is that we cannot use $N_S W = Z < 0$ everywhere on $\overline{U_0}$, in order to conclude that $\theta_{t^*} < 0$, as we did before. The reason is that there may be points on $\partial U_0$ where $N_S = 0$. However, as already noted, these points have the property that *do not move at all* by the construction of $S_t$, i.e. the boundary $\gamma_t$ (which is the intersection of $S$ and $S_t^*$) can only move outside of $U_0$ at points where $N_S$ is strictly negative. Hence on the interior points of $U_t$ we have $N_S < 0$ everywhere, for sufficiently small $t$. Consequently, the first-order terms in the variation of $\theta_t$, namely $Z = N_t W$, are strictly negative on all the interior points of $U_t$. This implies that $S_t^*$ has negative outer null expansion everywhere except possibly on its boundary $\gamma_t$. By continuity, we conclude $\theta_{t^*} < 0$ everywhere. We can now apply lemma 3 to $S_{t^*} \cup \gamma_t \cup S_t^*$ (where, as before, $S_{t^*}$ is the complement of $U_t$ in $S_t$) to construct a smooth weakly outer trapped surface outside the locally outermost MOTS $S$. This gives a contradiction. Therefore, there exists $p \in \partial U_0$ such that $d(\xi \cdot \vec{T})|_p = 0$, as claimed. □

**Remark.** The assumption $dQ|_{\gamma} \neq 0$ is a technical requirement for lemma 3. This is why we had to include an assumption on $dQ|_{\gamma}$ in theorem 5 and also that the conclusion of theorem 6 is stated in terms of the existence of critical points for $Q$. If lemma 3 could be strengthened so as to remove this requirement, then theorem 6 could be rephrased as stating that any outermost MOTS in a region where there is a causal Killing vector (irrespective of its future or past
character) must have at least one point where the shear and the energy ‘density’ along \( \tilde{l} \) vanish simultaneously.

In any case, the existence of critical points for a function in the boundary of every connected component of \( \{ Q > 0 \} \) and every connected component of \( \{ Q < 0 \} \) is obviously a highly non-generic situation. So, locally outermost MOTS in regions where there is a causal Killing vector or homothety can at most occur under very exceptional circumstances.

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