ON A CLASS OF C*-ALGEBRAS DETERMINED UNIQUELY BY DUAL SPACE.

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Abstract. Enveloping C*-algebras for some finitely generated ∗-algebras are considered. It is shown that all of the considered algebras are identically defined by their dual spaces. The description in terms of matrix-functions is given.

Keywords : algebraic bundles, finitely presented C*-algebras, dual spaces.

INTRODUCTION

One of the general approaches to description of C*-algebras consists in realization of them as "functions" on dual space. All C*-algebras having the irreducible representations in dimension less or equal to some natural n, usually can be realized by $n \times n$-matrix-functions with some boundary conditions. The special case of such C*-algebras is given by homogeneous algebras - ones having all irreducible representations in the same dimension n. The description of homogeneous C*-algebras in terms of fibre bundles was obtained in [4],[13] (see Statement 3 below). The analogous description for all C*-algebras having irreducible representations in different dimensions less or equal to natural n was presented in [14].

The homogeneous C*-algebras whose spectra are tori $T_2$, $T_3$ (see [2]) and sphere $S^k$ (see [1]) were studied by means of classification corresponding algebraic bundles. The concrete example of non-homogeneous C*-algebra is given by C*-algebra generated by the free pair of projections (see [15]).

In the present paper we consider wide class of non-homogeneous $F_{2m}$-C*-algebras and give a realization of algebras from this class in terms of matrix functions with boundary conditions.

1. Preliminaries

Let us recall some preparatory facts, definitions and notation, which will be used below.

1.1. Enveloping C*-algebras. In this paper we study finitely presented C*-algebras. By finitely presented C*-algebra we mean an enveloping C*-algebra for finitely presented ∗-algebra.

Definition. Let $\mathcal{A}$ be a ∗-algebra, having at least one representation. Then a pair $(\mathcal{A}, \rho)$ of a C*-algebra $\mathcal{A}$ and a homomorphism $\rho : \mathcal{A} \to \mathcal{A}$ is called an enveloping pair for $\mathcal{A}$ if every irreducible representation $\pi : \mathcal{A} \to B(H)$ factors uniquely through the $\mathcal{A}$, i.e. there exists precisely one irreducible representation $\pi_1$ of algebra $\mathcal{A}$ satisfying $\pi_1 \circ \rho = \pi$. The algebra $\mathcal{A}$ is called an enveloping for $\mathcal{A}$. 

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1.2. $F_{2n}$-algebras. Since all $C^*$-algebras considered below are examples of $F_{2n}$-$C^*$-algebras, we recall here the definition and basic properties of algebras satisfying the $F_{2n}$ polynomial identity. Let $F_n$ denotes the following polynomial of degree $n$ in $n$ non-commuting variables:

$$F_n(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^{p(\sigma)} x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $S_n$ is the symmetric group of degree $n$, $p(\sigma)$ is the parity of a permutation $\sigma \in S_n$. We say that an algebra $A$ is an algebra with $F_n$ identity if for all $x_1, \ldots, x_n \in A$, we have $F_n(x_1, \ldots, x_n) = 0$. The Amitsur-Levitsky theorem says that the matrix algebra $M_n(\mathbb{C})$ is an algebra with $F_{2n}$ identity. Moreover, $C^*$-algebra $A$ has irreducible representations of dimension less or equal to $n$ iff $A$ satisfies the $F_{2n}$ condition (see [10]). An important class of $F_{2n}$-$C^*$-algebras is formed by $n$-homogeneous $C^*$-algebras - ones having all irreducible representations of dimension $n$. The simplest example of $n$-homogeneous $C^*$-algebra is $2 \times 2$ matrix algebra over $C(X)$, where $X$ is a compact Hausdorff space - it is so-called trivial algebra. In this case the dual space (space of primitive ideals, see [3]) is canonically homeomorphic to $X$. Simple example of non-homogeneous $C^*$-algebra is given by “fixing” the homogeneous algebra above in a point $x_0 \in X$. Namely, consider

$$A_0(x_0) = \{ f \in C(X \to M_2(\mathbb{C})) | f(x_0) \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \}.$$ 

The dual space $P(A_0(x_0))$ of algebra $A_0(x_0)$ is $T_2 \cup T_1$, where $T_2$ is the space of irreducible 2-dimensional representations, homeomorphic to $X \setminus \{ x_0 \}$ and $T_1 = \{ x_0', x_0'' \}$ is the space of 1-dimensional representations. Here the bases of neighborhoods of the points $x_0'$ and $x_0''$ are the same as for the $x_0 \in X$, i.e. we have the space satisfying $T_0$ but not $T_1$ separability axiom (such spaces sometimes are called quasicompact).

The description above can be easily generalized, for example taking $n \geq 2$:

$$A_n(x_0) = \{ f \in C(X \to M_n(\mathbb{C})) | f(x_0) \in M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \},$$

or “fixing” a homogeneous $C^*$-algebra in a few points. Studying some examples below, we will describe dual spaces in terms of algebras

$$A_n(m_1^{(1)}, \ldots, m_k^{(1)}; m_1^{(2)}, \ldots, m_k^{(2)}; m_1^{(3)}, \ldots, m_k^{(3)}) =$$

$$= \{ f \in C(S^2 \to M_n(\mathbb{C})) | f(x_i) \in M_{m_i^{(1)}}(\mathbb{C}) \oplus \ldots \oplus M_{m_i^{(3)}}(\mathbb{C}), x_i \in S^2, i = 1, 2, 3 \},$$

where $x_1, x_2, x_3 \in S^2$ are fixed points (it can be checked that the different triples $x_1, x_2, x_3$ give isomorphic algebras).

Finitely presented $C^*$-algebras are mostly non-homogeneous. Below we present some important examples of such algebras.

Example 1. Main examples of the paper concern $*$-algebras ($C^*$-algebras) associated with extended Dynkin diagrams $\widehat{D}_4, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8$, see [1], for details. Below we recall how these algebras are defined by generators and basic relations:

$C^*(\widehat{D}_4) = C^*(p_1, p_2, p_3, p_4| p_1 + p_2 + p_3 + p_4 = 2e, p_i^* = p_i, i = 1, 4)$,

$C^*(\widehat{E}_6) = C^*(p_1, p_2, q_1, q_2, r_1, r_2| p_1 + 2p_2 + q_1 + 2q_2 + r_1 + 2r_2 = 3e, p_i^* = p_i, q_i^* = q_i, r_i^* = r_i, i = 1, 2, p_i p_k = q_j q_k = r_j r_k = 0, j \neq k)$,

$C^*(\widehat{E}_7) = C^*(p_1, p_2, p_3, q_1, q_2, q_3, r_1| p_1 + 2p_2 + 3p_3 + q_1 + 2q_2 + 3q_3 + 2r_2 = 4e, p_i^* = p_i, q_i^* = q_i, r_i^* = r_i, i = 1, 3, p_i p_k = q_j q_k = r_j r_k = 0, j \neq k)$,

$C^*(\widehat{E}_8) = C^*(p_1, p_2, q_1, q_2, q_3, q_4, q_5, r_1| 2p_1 + 4p_2 + q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 + 3r_2 = 6e, p_i^* = p_i, q_i^* = q_i, r_i^* = r_i, i = 1, 2, j = 1, 5, p_i p_m = q_j q_m = 0, l \neq m)$.
Irreducible representations of the algebras above were classified in the series of works (see [10], [7], [11], [8] and the description of dual spaces follows easily from the classification (see for example [12]). The dual spaces of $C^*$-algebras presented above are the following:

- for $C^*(\hat{D}_4)$ the dual space is the same as for $A_2(1,1;1,1;1,1)$
- for $C^*(E_6)$ the dual space is the same as for $A_3(2,1;1,1,1;1,1,1)$
- for $C^*(\hat{E}_7)$ the dual space is the same as for $A_4(2,2;2,1,1;1,1,1,1,1)$
- for $C^*(\tilde{E}_8)$ the dual space is the same as for $A_6(3,3;2,2,1,1;1,1,1,1,1,1)$.

**Example 2.** Generalizing the relations defined by $\hat{D}_4$ one can consider the $*$-algebra:

$$P_{\alpha,\beta} = \mathbb{C}\langle p_1, p_2, p_3, p_4 | \alpha(p_1 + p_2) + \beta(p_3 + p_4) = I \rangle,$$

where $\alpha, \beta > 0, \alpha + \beta = 1$. The case $P_{\alpha,\beta}$ corresponds to $\hat{D}_4$. When $\alpha \neq \beta$, the dual space for $C^*(P_{\alpha,\beta})$ is the same as for algebra:

$$\{ f \in C(S^2 \to M_2(\mathbb{C})) | f(x_1), f(x_2) \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}), x_1, x_2 \in S^2 \},$$

it can be checked that the different pairs $(x_1, x_2)$ give isomorphic algebras, see [5] for more details.

**Example 3.** We can consider much more complicated relations using the algebras $P_{\alpha,\beta}$ from previous example. Construct the next algebra:

$$\mathcal{T}_\varepsilon = C^*(\rho_1, \rho_2, \rho_3, \rho_4, \alpha, \beta | \alpha(p_1 + p_2) + \beta(p_3 + p_4) = I, \alpha = \alpha^*, \beta = \beta^* \}
\begin{array}{c}
[\alpha, p_i] = [\beta, p_i] = [\alpha, \beta] = 0, i = 1, 2, 3, 4, \\
\alpha, \beta \geq \varepsilon I, \quad \alpha + \beta = I,
\end{array}$$

where $0 < \varepsilon < \frac{1}{2}$. Using the previous example one can describe the dual space for the $\mathcal{T}_\varepsilon$. We will see later that all algebras $\mathcal{T}_\varepsilon$, $\varepsilon \in (0, 1/2)$ are isomorphic.

**Example 4.** The group algebra for $G = \mathbb{Z}_2 \ast \mathbb{Z}_2$ gives an example of $F_4$-algebra corresponding to infinite discrete group.

$$C^*(\mathbb{Z}_2 \ast \mathbb{Z}_2) = C^*(p_1, p_2| p_k^2 = p_k, k = 1, 2) := \mathcal{P}_2.$$  

1.3. **Fibre bundles.** All the necessary information about fibre bundles reader can find in [9].

From the general theory of fibre bundles it is known that locally-trivial bundle is defined by its 7-tuple

$$(E, B, F, G, p, \{V_i\}_{i \in I}, \{\phi_i\}_{i \in I}),$$

where $E$ is a bundle space, $B$ is a base, $F$ is a fibre, $G$ is a structure group, $p : E \to B$ is a projection, $V_i$ is an open covering of base $B$ and $\phi_i : V_i \times F \to p^{-1}(V_i)$ are coordinate homeomorphisms. Note that in Vasil’ev’s paper [14] coordinate and skew products are considered instead of locally trivial bundles. Coordinate product is a 7-tuple $(E, B, F, G, p, \{V_i\}_{i \in I}, \{\phi_{ji}\}_{i,j \in I})$, where $\phi_{ji} = \phi_j^{-1} \circ \phi_i |_{V_i \cap V_j} : (V_i \cap V_j) \times F \to (V_i \cap V_j) \times F$ are sewing maps. Every $\phi_{ji}$ is uniquely determined by continuous map (which we also call sewing map) $\overline{\phi}_{ji} : V_i \cap V_j \to G$ by formulae: $\phi_{ji}(x, f) = (x, \overline{\phi}_{ji}(x) \circ f)$, $\forall x \in V_i \cap V_j$.

In this paper we consider only algebraic bundles, i.e. bundles with $F \cong M_n(\mathbb{C})$ and $G \subseteq U(n)$, where $U(n)$ is a group of inner automorphisms of $M_n(\mathbb{C})$, i.e. group of unitary matrices $U(n)$ factorized by it’s center. Group $U(n)$ is supposed to act on the $M_n(\mathbb{C})$ in the following way. Let $\hat{g} \in U(n)$ and $g \in U(n)$ is corresponding matrix, then $\hat{g}(m) = g^* mg$, $m \in M_n(\mathbb{C})$. If $G$ is a subgroup of $U(n)$, we denote
by \( G \) the quotient of \( G \) by its center. Sometimes we suppose sewing maps \( \bar{\phi}_{ji} \) to have values in \( U(n) \), instead of \( U(n) \), one has to take a composition of \( \bar{\phi}_{ji} \) with the canonical map of \( U(n) \) onto \( U(n) \).

The following statements on triviality will be used below.

**Statement 1.** (see [3]) Every locally-trivial bundle over \( n \)-dimensional disk \( D^n \) is isomorphic to trivial.

**Statement 2.** (see [1]) Every algebraic bundle over some compact subset of \( \mathbb{C} \) is isomorphic to trivial.

In this article by \( B(X, F, G) \) we denote some bundle with base \( X \), fibre \( F \) and group \( G \). The algebra of sections of bundle \( B \) is denoted by \( \Gamma(B) \).

1.4. **Structure of \( F_{2n}-C^*\)-algebras.** Next statement (Fell-Tomiyama-Takesaki theorem) gives a description of \( n \)-homogeneous \( C^*\)-algebras in terms of algebraic bundles.

**Statement 3.** ([13], [14]) For every \( n \)-homogeneous \( C^*\)-algebra \( A \) there exists a bundle \( B(P(A), M_n(\mathbb{C}), U(n)) \), such that \( A \cong \Gamma(B) \), where \( P(A) \) is space of all pairly non-equivalent irreducible representations.

Analogous result for non-homogeneous \( F_{2n}-C^*\)-algebras was obtained in Vasil'ev's work [14]. The following statement is the straightforward corollary from the main result of [14].

**Statement 4.** Let \( A \) be \( F_{2N}-C^*\)-algebra, having finite number of irreducible representations in dimensions \( 1, 2, \ldots, N - 1 \). Then there exist:

1. finite sets \( X_1, \ldots, X_{N-1} \), compact space \( \overline{X} \) and open dense in \( \overline{X} \) subspace \( X \), such that \( \overline{X} \setminus X \) is finite,
2. a formal decomposition for all \( x \in \overline{X} \setminus X : \)
\[
x = \sum_{k=1}^{r} d_k x_k,
\]
where \( d_k \) are natural numbers, \( x_k \in X_{n_k} \) and \( \sum d_k n_k = N \), such that two points from \( \overline{X} \setminus X \) coincide iff they have the same decomposition,
3. bundle \( B = B(X, M_N(\mathbb{C}), U(N)) \) having \( n_1 U(d_1) \times \ldots \times n_r U(d_r) \) as the structure group at the point \( x \in \overline{X} \setminus X \), \( x = d_1 x_1 + \ldots + d_r x_r \) (i.e. there exists a neighborhood \( O_x \subseteq \overline{X} \), such that the structure group of bundle \( B \) reduces to \( n_1 U(d_1) \times \ldots \times n_r U(d_r) \) at the subspace \( O_x \cap X \),

such that algebra \( A \) can be realized as an algebra of \( N \)-tuples \( a = (a^{(1)}, \ldots, a^{(N)}) \), where \( a^{(i)} : X_i \to M_i(\mathbb{C}) \), \( i = \frac{1}{2} N - 1 \) are some functions and \( a^{(N)} \in \Gamma(B) \) is a section such that:
\[
\lim_{x \to \sum_{k=1}^{r} d_k x_k} a^{(N)}(x) = \begin{pmatrix}
1_{d_1} \otimes a^{(n_1)}(x_1) & & 0 \\
& \ddots & \\
0 & & 1_{d_r} \otimes a^{(n_r)}(x_r)
\end{pmatrix}.
\]
1.5. **Bundles over 2-sphere.** Recall, that fibre bundles over the sphere $S^k$ are naturally classified by elements of the homotopic group $\pi_{k-1}(G)$, where $G$ is structure group of bundle (see for example [9] or [6]).

Bundles over 2-dimensional sphere with fibre $M_g(\mathbb{C})$ are classified in [1], in this case $\pi(U(n)) \simeq \mathbb{Z}_n$. Let us give a sketch of this classification.

Consider an atlas on 2-sphere, containing 2 charts : upper and lower closed half spheres ($S^2_U = W_1$ and $S^2_L = W_2$ correspondingly). As sewing map $\varphi_{1,2}$ take continuous map $V' : S^1 \rightarrow U(n)$. Construct a bundle $B'$ with atlas $\{W_1, W_2\}$ and sewing map $V'$. Two bundles $B'$ and $B''$ defined by sewing maps $V'$ and $V''$ are isomorphic iff $\text{ind } V' - \text{ind } V'' = nl$, $l \in \mathbb{Z}$, where $\text{ind } V = (2\pi)^{-1}[\text{arg det } V(t)]_{S^1}$ is the winding number of the function $V$ with respect to the circle. Define sewing maps:

\[(1.1) \quad V_k : S^1 \rightarrow U(n), z \mapsto \text{diag}(z^k, 1, \ldots, 1), \quad k = 0, n-1.\]

Then the bundles $B_{n,k}$, $k = 0, n-1$, defined by $V_k$ are canonical representatives of all isomorphism classes.

2. **Enveloping $C^*$-algebras, an example.**

To give a description of $C^*$-algebras for examples from Section 1, we need the following

**Theorem 1.** Let

\[D^l(m_1, \ldots, m_k) = \{ f \in C(D^l) \rightarrow M_{m_1 + \ldots + m_k}(\mathbb{C}) | f(0, \ldots, 0) \in M_{m_1}(\mathbb{C}) \oplus \ldots \oplus M_{m_k}(\mathbb{C}) \}, \]

Every $C^*$-algebra $A$ having the dual space same as $D^l(m_1, \ldots, m_k)$ is isomorphic to $D^l(m_1, \ldots, m_k)$.

**Proof.** We give proof when $l = 2$, $k = 2$, $m_1 = m_2 = 1$, i.e. $D^l(m_1, \ldots, m_k) = D^2(1,1)$ and realize $D^2$ as the unit disk in complex plane. For the general case proof is analogous. Let us introduce some notaion:

\[D^2_\pm(1,1) = \{ f \in C(D^2_\pm) \rightarrow M_2(\mathbb{C}) | f(0, 0) \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \}, \]

where $D^2_\pm = \{ z \in D^2 | \Re m(z) \geq 0 \}$ and $D^2 = \{ z \in D^2 | \Re m(z) \leq 0 \}$, and let $A_+$ be $C^*$-algebra having dual space same as for $D^2_\pm$. We have to prove that $A_+ \simeq D^2_\pm$.

The statement [3] implies that there exists a bundle $B_+ = B_+(D^2_+ \setminus \{0\}, M_2(\mathbb{C}), U(2))$, having $U(1) \times U(1)$ as the structure group at the point 0, such that $A_+$ is isomorphic to the algebra of pairs:

\[\{(a', a'') | a' = a'(x) \in \Gamma(B_+), a'' \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}), \lim_{x \to 0} a'(x) = a'' \}.\]

We suppose $B_+$ to be assigned by charts $\{V_k\}_{k \in \mathbb{N}}$, where

\[V_k = \left\{ z \in D^2_+, \frac{1}{2k} \leq |z| \leq \frac{1}{2k+1} \right\}, \]

Sewing maps $\varphi_{k,k+1}$ of the bundle $B_+$ are $\{\mu_k\}_{k \in \mathbb{N}}$, where $\mu_k : V_k \cap V_{k+1} \rightarrow U(2)$ are some continuous functions.

The condition for $B$ to have $U(1) \times U(1)$ as the structure group at the point 0 implies that there exists $p \in \mathbb{N}$ such that $\mu_r(z) \in U(1) \times U(1), \forall r > p$. The fact
that $U(2)$ and $U(1) \times U(1)$ are connected groups allows us to construct continuous maps
\[ \overline{\mu}_k : V_k \to U(2), \quad k \leq p \]
and
\[ \overline{\mu}_k : V_k \to U(1) \times U(1), \quad k > p \]
such that
\[ \overline{\mu}_k(z) = \begin{cases} e, & z \in V_{k-1} \cap V_k; \\ \mu_k(z), & z \in V_k \cap V_{k+1}. \end{cases} \]

Let us write sections of $B_+$ explicitly:
\[ \Gamma(B_+) \ni a' \mapsto \{a'_k\}_{k \in \mathbb{N}}, \]
where $a'_k : V_k \to M_2(\mathbb{C})$ are continuous functions satisfying the condition of compatibility $a'_{k+1}(z) = \mu_k(z)(a'_k(z))$, $z \in V_k \cap V_{k+1}$. So we write elements $(a', a'') \in A_+$ as $(\{a'_k\}_{k \in \mathbb{N}}, a'')$.

Let us consider the map:
\[ A_+ \ni (\{a'_k(z)\}_{k \in \mathbb{N}}, a'') \mapsto (\{\overline{\mu}_k(z) \circ a'_k(z)\}_{k \in \mathbb{N}}, a''). \]

One can check that the map is the isomorphism of algebras $A_+$ and $D^2_+(1,1)$. Analogously the isomorphism $A_+ \simeq D^2(1,1)$ can be proved.

Let us consider bundle $B(D^2 \setminus \{0\}, M_2(\mathbb{C}), U(2))$ having $U(1) \times U(1)$ as the structure group at the point 0, such that:
\[ A \simeq \big\{ (a', a'') | a' = a'(x) \in \Gamma(B), a'' \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}), \lim_{x \to 0} a'(x) = a'' \big\}. \]

The fact that $A_+$ and $A_-$ are "trivial" allows us to suppose $B$ to have only two charts: $W_1 = D^2_+ \setminus \{0\}$ and $W_2 = D^2_+ \setminus \{0\}$ and one sewing map $\mu_{1,2} : [-1,1] \setminus \{0\} \to U(2)$ such that $\exists \varepsilon_0 > 0 : \forall z \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0) \mu(z) \in U(1) \times U(1)$. As before, by connectivity of $U(2)$ and $U(1) \times U(1)$ we construct continuous map $\overline{\mu} : W_1 \to U(2)$ possessing a values from $U(1) \times U(1)$ in $O(\varepsilon_0, 0) = \{ z \in D^2_+, 0 < |z| < \varepsilon_0 \}$ such that $\forall z \in W_1 \cap W_2 : \overline{\mu}(z) = \mu_{1,2}(z)$.

As before we have an isomorphism given by $\overline{\mu}$:
\[ A \ni (\{a'_1(z), a'_2(z)\}, a'') \mapsto (\{\overline{\mu}(z) a'_1(z), a'_2(z)\}, a'') \in D^2(1,1). \]

The proof is completed. \qed

Remark 1. One can regard the theorem above as a certain generalization of the Statement 1. Indeed we have a $C^*$-algebra identically defined by it’s dual space.

As before, let
\[ \sum_{j=1}^{k} m_j = n, \quad m_j \in \mathbb{N}, j = 1, k \]
and set
\[ B_{n,p}(m_1, \ldots, m_k) = \{ f \in \Gamma(B_{n,p}) | f(x) \in M_{m_1}(\mathbb{C}) \oplus \ldots \oplus M_{m_k}(\mathbb{C}) \}, \]
where $x \in S^2$ is fixed point (we consider $x$ to be covered only by one chart).

Next theorem is basic to study the structure of $C^*$-algebras corresponding to Dynkin diagrams.
Theorem 2. If \( l_1 \equiv l_2 \pmod{m_1, \ldots, m_k} \), then
\[
B_{n, l_1}(m_1, \ldots, m_k) \simeq B_{n, l_2}(m_1, \ldots, m_k) 
\]    
(2.1)

(here \((m_1, \ldots, m_k)\) denotes the greatest common divisor of \(m_1, \ldots, m_k\))

Proof. Let us realize elements of algebras \( B_{n, p}(m_1, \ldots, m_k) \) as sections of corresponding bundles \( B_{n, p} \) (here \( p = l_1 \) or \( p = l_2 \)):
\[
B_{n, p}(m_1, \ldots, m_k) = \{(f_1, f_2) | f_i \in C(D^2 \to M_n(\mathbb{C})),
\]
\[
f_i(z) = B^*_{n, p}(z)f_2(z)V_0(z), |z| = 1, f_1(0) \in M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_k}(\mathbb{C})\},
\]
where \( V_0 \) are defined by (1.1).

Condition \( l_1 \equiv l_2 \pmod{m_1, \ldots, m_k} \) implies that
\[
\exists c_1, \ldots, c_k \in \mathbb{Z} : c_1 m_1 + \cdots + c_k m_k = l_1 - l_2.
\]

Let \( H(t) = e^{i2\pi c_1 t}E_{m_1} \oplus \cdots \oplus e^{i2\pi c_k t}E_{m_k} \in U(n) \), where \( E_m \) denotes the \( m \times m \)-identity matrix. Construct the map \( \overline{H} : D^2 \to U(n) \) by the rule:
\[
H \left( \frac{z}{|z|} \right), \ z \neq 0,
\]
\[
0 \mapsto 1_n.
\]

Nevertheless \( \overline{H} \) is not continuous in 0, it is easy to check that \( \overline{H} \) defines an isomorphism:
\[
B_{n, l_1}(m_1, \ldots, m_k) \ni (f_1(z), f_2(z)) \mapsto (H^*(z)f_1(z)H(z), f_2) \in \Gamma(B'_{n, l_2}).
\]
The bundle \( B'_{n, l_2} \) is defined by the sewing map \( V'_{l_2}(z) = H^*(z)V_{l_1}(z) \). The equality \( ind V'_{l_2} = ind V_{l_2} \) gives an isomorphism (2.1).

Remark 2. The converse statement to the Theorem 2 takes place. We will not use it below, so we don’t give proof.

Remark 3. One can easily prove the following generalization of theorem 2: if we ”fix” sections of bundles \( B_{n, l_1} \) and \( B_{n, l_2} \) in a few points in block matrices (with equal dimension of blocks for both bundles in every point) and suppose that all blocks are of dimensions \( m_1, \ldots, m_k \), then \( l_1 \equiv l_2 \pmod{m_1, \ldots, m_k} \) implies isomorphism of corresponding algebras of sections.

Let us return to our examples of finitely generated \( F_{2n} \)-C*-algebras (see preliminaries).

Proposition 1. For C*-algebras associated with extended Dynkin diagrams one has the next isomorphisms:
\[
C^*(\widetilde{D}_3) \simeq A_{2}(1; 1, 1; 1, 1),
\]
\[
C^*(\widetilde{E}_6) \simeq A_{3}(2; 1, 1, 1; 1, 1, 1),
\]
\[
C^*(\widetilde{E}_7) \simeq A_{4}(2; 2, 1, 1; 1, 1, 1, 1),
\]
\[
C^*(\widetilde{E}_8) \simeq A_{6}(3; 2, 2, 1, 1; 1, 1, 1, 1, 1, 1).
\]

Proof. Let \( A \) denotes \( C^*(\widetilde{D}_4) \). We can consider this algebra to be realized by functions on the \( S^2 \) with values in matrices of corresponding dimension, not necessary continuous yet. We can find a covering of \( S^2 \) by closed subsets \( W_j \approx D^2 \) such that every \( W_j \) contains only representations in main dimension or contains
one point with reducible representation of main dimension. Consider the "restrictions" $A(W_j)$ of algebra $A$ onto subsets $W_j$ (i.e. factor-algebras $A/I(W_j)$, where \(I(W_j) = \{ f \in A, f|_{W_j} \equiv 0 \} \)), then use Theorem 1 to realize every $A(W_j)$ as sections of some algebraic bundle, i.e. everyone of $A(W_j)$ coincides with one of the $B_{n,1}(m_1, \ldots, m_k)$ (possibly with $m_1 = n$, $m_2 = \ldots = m_k = 0$).

Different $A(W_j)$ are "sewed" by some continuous maps, which we use to construct the bundle $B_{n,1}$ such that algebra $A$ is isomorphic to $B_{n,1}(m_1, \ldots, m_k)$. Theorem 2 shows that we can suppose $l = 0$.

\[\text{Proposition 2. Enveloping } C^* \text{-algebra for } P_{\alpha, \beta} \text{ is :} \]
\[C^*(P_{\alpha, \beta}) = \{ f \in C(S^2 \rightarrow M_2(\mathbb{C})) | f(x_1), f(x_2) \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}), x_1, x_2 \in S^2 \} \]

\[\text{Proof. See the proof of previous theorem.} \]

The following example of enveloping $C^*$-algebra is considered in [15]. One can get the result of [15] using our Theorem 1.

\[\text{Proposition 3. Enveloping } C^* \text{-algebra } \mathcal{P}_2 \text{ for free pair of projections (see [2]) is :} \]
\[\mathcal{P}_2 = \{ f \in C([0,1] \rightarrow M_2(\mathbb{C})) | f(0), f(1) \text{ are diagonal } \}. \]

\[\text{Proof. As in previous case, we realize this algebra by sections of some bundle over } [0,1]. \text{ By statement 1 all bundles over } [0,1] \text{ are trivial.} \]

All the constructions above in the paper, including Statement [3] can be generalized to prove the next proposition:

\[\text{Proposition 4. Enveloping } C^* \text{-algebra for } \mathcal{P}_e \text{ (example [3]) has the form:} \]
\[\mathcal{P}_e = \{ f \in C(S^2 \times I \rightarrow M_2(\mathbb{C})) | f(x_1, 1/2), f(x_2, t), f(x_3, t) \in M_1(\mathbb{C}) \oplus M_1(\mathbb{C}), x_1, x_2, x_3 \in S^2 \}. \]

\[\text{Proof. Here we use the fact that the theory of algebraic bundles over } S^2 \times I \text{ is the same as for } S^2. \text{ Then the proof of Theorem 2 should be transferred from } S^2 \text{ to } S^2 \times I. \]

\[\text{Remark 4. The definition of triviality for homogeneous } C^* \text{-algebras can be generalized. One can define } F_{2N} C^* \text{-algebra with trivial corresponding bundle } \mathcal{B}(X, M_N(\mathbb{C}), U(N)). \text{ Then all propositions above state the triviality of } C^* \text{-algebras under consideration. Example of non-trivial non-homogeneous } C^* \text{-algebra one can construct by } \text{"fixing" sections of bundle } \mathcal{B}_{1,1} \text{ in the } 2 \times 2 \text{-block matrices in one point, it is also an example of algebra that is not defined by the dual space.} \]

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\[\text{References} \]

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