Nonlinear consensus on networks: equilibria, effective resistance and trees of motifs

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Abstract. We study a generic family of nonlinear dynamics on undirected networks generalising linear consensus. We find a compact expression for its equilibrium points in terms of the topology of the network and classify their stability using the effective resistance of the underlying graph equipped with appropriate weights. Our general results are applied to some specific networks, namely trees, cycles and complete graphs. When a network is formed by the union of two subnetworks joined in a single node, we show that the equilibrium points and stability in the whole network can be found by simply studying the smaller subnetworks instead. Applied recursively, this property opens the possibility to investigate the dynamical behaviour on families of networks made of trees of motifs.

Key words. Consensus dynamics, network science, nonlinear dynamics, fixed points, effective resistance, trees

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1. Introduction. A broad range of systems can be represented by dynamical systems on networks [36]. In this framework, each node is endowed with a time-varying state whose dynamics depend on its own state and the states of its neighbours. Important examples include models of collective behaviour where the decision process is distributed rather than centralised [11]. As reviewed in [48, 38], applications can be found in a variety of disciplines such as ecology, where it is used to model animal behaviour and flocking [16, 42, 37]. In engineering, it is essential for the design of decentralized control strategies for the movement of robots [30], the rendezvous problem [15] and to coordinate decision making when multiple nosy sensors detect an event [2]. In the social sciences, it is used as a first model for opinion and language dynamics [14], and can be applied in economics to coordinate a decentralized network of buyers [8]. The simplest algorithms and models for consensus are variations of the DeGroot model [17], usually called linear consensus dynamics, where the state of a node evolves towards the average value of its neighbours. These models are theoretically appealing thanks to their simplicity, as their dynamics is entirely determined by the spectral properties of the coupling matrix between the elements. However, linear models often arise as approximations for more complicated coupling functions and are not always sufficient to explain their complex behaviour [16], which calls for the study of nonlinear consensus [41]. In many of these applications, nonlinearity has thus been introduced [3, 8], the most prominent example of nonlinear process being the Kuramoto model of coupled oscillators [28, 4], with applications in physics, chemistry, biology and engineering [20, 35, 44].

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The focus of this work is on studying the properties of a general class of nonlinear consensus dynamics taking place on a network. We derive exact results for arbitrary choices of undirected networks and a general class of nonlinear consensus dynamics, in contrast with most other works that focus on mean-field approximations, specific types of networks or specific classes of nonlinear dynamics. In particular, we manage to extend the theoretical analysis of [19], where the purely graph-theoretic notion of effective resistance was introduced to study a particular nonlinear system. The role of effective resistances in nonlinear dynamical systems was also highlighted in [44], where resistances were shown to be a key determinant of the robustness of a nonlinear system to external perturbations. Another relevant line of research into generic nonlinear network systems can be found in [23, 22], where it is shown that certain properties (symmetries) of the underlying network restrict the possible dynamical features (equilibria, periodic states) of any possible system on the network. Finally, the work of Bronski and DeVille [13] presents complementary stability results based on the study of spanning trees and homology theories of signed graphs, which coincide with ours in simple cases, as we will show in the manuscript.

Our problem is formalised as follows. Let \( n \in \mathbb{N} \) and \( G = (V, E) \) be a connected undirected network with \( V = \{1, \ldots, n\} \). If \( \{i, j\} \in E \) we say they are adjacent and write \( i \sim j \). Note that we consider only networks without self-loops, so that \( i \not\sim i \), as self-loops would not change the dynamics that we will study. Linear consensus dynamics on \( G \) is defined over the state space \( x \in \mathbb{R}^n \), where each node has a scalar state \( x_i \), by

\[
\dot{x}_i = -\sum_{j \sim i} (x_i - x_j) \quad \text{for all } i \in V.
\]

Compactly we may write \( \dot{x} = -Lx \) where \( L \) is the Laplacian matrix of \( G \) (see (2.1)). As presented in [41], there are three obvious ways to extend this simple linear model by introducing a function \( \hat{f} : \mathbb{R} \to \mathbb{R} \),

\[
\begin{align*}
\dot{x}_i &= \sum_{j \sim i} \left( \hat{f}(x_i) - \hat{f}(x_j) \right), \\
\dot{x}_i &= \sum_{j \sim i} \hat{f}(x_i - x_j), \\
\dot{x}_i &= \hat{f}\left(\sum_{j \sim i} (x_i - x_j)\right).
\end{align*}
\]

When \( \hat{f} = -\text{id} \), all three cases recover the linear consensus model. If we denote by \( f : \mathbb{R}^n \to \mathbb{R}^n \) the function that applies \( \hat{f} \) to each component, we may write the first and third ODE above more compactly as

\[
\dot{x} = Lf(x) \quad \text{and} \quad \dot{x} = f(Lx).
\]

As shown in [41] for the case \( \hat{f}(y) = \lambda y - y^3 \), this matrix formulation facilitates the theoretical understanding of these systems. In this work we will focus on the second ODE, which presents more challenges as it lacks an obvious matrix formulation. Let \( \{f_e\}_{e \in E} \) be a family of odd, continuously differentiable\(^1\) functions, where we will denote \( f_{\{i,j\}} = f_{ij} = f_{ji} \), and consider

\(^1\)It would be enough for \( f_e \) to be locally Lipschitz as long as they are continuous differentiable in a neighbourhood of the equilibria of (#).
the ODE\(^2\)
\[
\dot{x}_i = \sum_{j\sim i} f_{ij}(x_i - x_j) \quad \text{for all} \quad i \in \mathcal{V}.
\]

Importantly, this equation allows us to model weighted networks by absorbing the corresponding weights in the functions \(f_{ij}\). As \(\mathcal{G}\) is undirected, we require \(f_{ij}\) to be odd so the system exhibits a conservation of quantity, which is desirable in many real scenarios. We may think for instance that a liquid flows between adjacent nodes. Moreover, this choice does not severely limit the applicability of our model, as in many applications a global reference frame is not available [34] and thus we need invariant dynamics under rotation and translation of inertial frame. In [46] it is shown that this is satisfied if and only if the coupling functions \(f_{ij}\) are quasi-linear, i.e. \(f_{ij}(y) = k_{ij}(|y|)y\) for some functions \(k_{ij}\), which are a collection of odd functions. It is also worth mentioning that the coupled phase oscillators model, of which the Kuramoto model is a subclass, is defined by
\[
\dot{\theta}_i = w_i - \sum_{j=1}^{n} a_{(i,j)} \sin(\theta_i - \theta_j).
\]

This model is not quite included in (\#), due to the constants \(w_i\). However, the differential of this system coincides with the differential of the case \(w_i = 0\), which is included in (\#). Thus, most of the presented results regarding linear stability of equilibria will be applicable to the coupled phase oscillator model.

In this paper, we have obtained four main results for the study of (\#). Firstly, we show that (\#) can be expressed as a matrix product with a nonlinear coupling (3.3) which allows us to find a compact expression for the set of equilibrium points (3.6). This expression relates the equilibria with the topology of the underlying network through the cycle and cut space. Secondly, we find stability criteria for the equilibrium points using the concept of effective resistance, which in certain situations is tight (see Theorem 3.10 and 3.14). Thirdly, when there are few edges contributing to instability, the Schur complement can be used to reduce the stability problem to a smaller network (see Theorem 3.8). Similarly, in subsection 3.5, we show that if a network consists of two subnetworks joined in a single common node, the task of finding equilibrium points and their stability in the whole network can be reduced to the same task for each of the subnetworks. This result motivates the study of particular classes of networks, i.e. building blocks, that will be combined to form more complex networks, which we refer to as trees of motifs (see subsection 3.5.1). In particular, we apply our general results to tree, cycle and complete graphs and derive some additional properties that follow from their specific structures, see section 4. For the cycle graph we show in Proposition 4.4 that a particular set of polynomials induce a continuum of equilibria in (\#). This was observed for a particular case in [41] and [19] but no explanation for such behaviour was given.

2. Mathematical preliminaries. We give a brief recap of the main structures and results that we will need throughout the paper. A network or graph \(\mathcal{G}\), which we will always consider

\(^2\)Without loss of generality we could always consider the complete graph while choosing \(f_{ij}\) to be null in some edges. We do not choose to do so as in many cases we will require that all \(f_{ij}\) coincide.
undirected, is given by a pair of sets \((V, E)\) where \(V\) is the set of nodes, which we will assume of the form \(\{1, \ldots, n\}\), and \(E\) a set of unordered, distinct pairs of nodes corresponding to the edges. We will sometimes abuse notation and write \(i \in G\) or \(\{i, j\} \in G\) instead of \(i \in V\) or \(\{i, j\} \in E\). Given a partition \(V = V_1 \cup V_2\) of the nodes, we say that the set of edges from \(V_1\) to \(V_2\) forms a cut-set if it is non-empty.

Associated to a network, we have two \(\mathbb{R}\) vector spaces \(C_0, C_1\) generated by the orthonormal bases \(V, E\) respectively. They are related by the boundary and coboundary linear maps,

\[
d : C_1 \rightarrow C_0, \quad d^\top : C_0 \rightarrow C_1,
\]

defined by \(d(\{j, i\}) = i - j\) where \(j < i\). Choosing an ordering in the bases, \(d\) is simply the “signed” incidence matrix, see (3.2). The matrices \(d\) and \(d^\top\) are directly related with consensus dynamics as the Laplacian may be defined by (see [10, Proposition 4.8]),

\[
L := dd^\top.
\]

Additionally, the kernel and image of these linear maps have a strong topological interpretation [10]. Indeed, \(\ker d^\top\), \(\ker d\) and \(\im d^\top = (\ker d)^\perp\) are generated by the connected components, the cycles and the cut-sets respectively. In particular, if the network is connected, \(\ker d^\top = \langle 1\rangle\) and,

\[
d^\top|_{\langle 1\rangle^\perp} : \langle 1\rangle^\perp \rightarrow \im d^\top = (\ker d)^\perp,
\]
is an isomorphism.

A \textit{weighted network} is a network with a set of non-negative weights \(\{w_{i,j}\}_{i,j} \in E\). Often we only consider the edges with positive weights, for instance we say that a weighted network is connected, if the set of edges with positive weights is connected in the usual sense. The Laplacian matrix of a weighted network with vector of weights \(w = (w_{i,j})_{i,j} \in E\) is defined as

\[
L := d \text{ diag } (w) d^\top,
\]

which is symmetric and positive semi-definite. Moreover, its null space is generated by the indicator vectors of the connected components of the underlying weighted network. As there is a one to one correspondence between weighted Laplacians and networks, we may directly refer to the connected components of the Laplacian.

Given an ODE, \(\dot{x} = g(x)\), we say that \(x^*\) is an \textit{equilibrium point}, if \(g(x^*) = 0\). We say that an equilibrium point is \textit{attractive} if solutions that start close enough to it, eventually converge to it. We say that an equilibrium point is \textit{stable} if solutions that start close enough to it remain close for all positive times, otherwise we say it is \textit{unstable} (see section SM1 for formal definitions). In practice it is easier to check if an equilibrium is linearly stable, which implies both being stable and attractive. We say that an equilibrium point is \textit{linearly stable} (resp. \textit{unstable}) if all (resp. any) eigenvalues of the Jacobian matrix of \(g\) at that point have negative (resp. positive) real part. Analogously we define linear stability of a matrix. Linear stability also has the advantage that persists under small perturbations of the vector field
[43], which is key when the system is an imperfect representation of a phenomena, such as in scientific models.

Given a real symmetric matrix $M$ we define its \textit{inertia}, and denote it by $\text{In}(M)$, as the tuple which entries represent the number of positive, negative and zero eigenvalues (including multiplicities). This quantity will be useful as it determines the linear stability of the matrix. It is well known that a symmetric matrix can be expressed as $M = \sum_{i=1}^{k} \lambda_i v_i v_i^\top$, where $\lambda_i$ are the non-zero eigenvalues and $v_i$ are the corresponding eigenvectors forming an orthonormal set. We define the \textit{pseudoinverse} $M^\dagger$ of $M$ as

$$M^\dagger := \sum_{i=1}^{k} \lambda_i^{-1} v_i v_i^\top,$$

so $\ker M = \ker M^\dagger$ and if $V \perp \ker M$, then $M^\dagger|_V = (M|_V)^{-1}$. It is worth mentioning that we do not need to know the eigenvalues of a matrix to find its pseudoinverse, as it can be computed through column/row operations and matrix multiplications [39].

We denote by $0_k$ (resp. $1_k$) the column vectors with all entries 0 (resp. 1) and $0_k,0_{k'} = 0_k0_{k'}^\top$. When the dimension is clear we will suppress the subindex. We denote by $e_1, \ldots, e_k$ the standard basis of $\mathbb{R}^k$ and by $I_k$ the identity matrix.

3. Development of the Model.

3.1. First results. From now on we study the system (#) defined on an undirected, connected network $G$. The following two observations were already mentioned in [19].

The mean state $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ is a conserved quantity of the system (#). To see this, simply note that the functions $f_{ij}$ are odd, and thus

$$\dot{\overline{x}} = \langle \nabla \overline{x}, \dot{\overline{x}} \rangle = \frac{1}{n} \langle 1, \dot{\overline{x}} \rangle = 0,$$

i.e. $\overline{x}$ is a first integral of (#). Thus, the planes defined by $\overline{x} = k$ for some constant $k$, which are precisely the affine perpendicular planes to $1$, are invariant. Moreover, as only differences of elements $x_i$ appear in (#), the vector field in $x$ and $x + \lambda 1$ is the same for all $\lambda \in \mathbb{R}$. So we conclude that the dynamics in each plane are exactly the same and we can simply restrict our study to one of these planes. From now on, we will study (#) in the vector space $\langle 1 \rangle^\perp$ or equivalently the invariant plane given by $\overline{x} = 0$, unless stated otherwise. In [19] a similar approach was taken, by limiting the study to the state space $\mathbb{R}^n/\langle 1 \rangle$.

The fact that $\overline{x}$ is a conserved quantity is also desirable as it shows that our model preserves some of the key properties of linear consensus dynamics. In the linear case $\overline{x}$ is not only conserved, but all entries of a solution tend to this value.

Another important property which is preserved from linear consensus is that the dynamics given by (#) are gradient dynamics in $\mathbb{R}^n$. Indeed, if we let $F_{ij}$ be an antiderivative of $f_{ij}$ for all $\{i,j\} \in E$ and define,

$$V(x) := -\sum_{\{i,j\} \in E} F_{ij}(x_i - x_j),$$

(3.1)
which is well defined as all $F_{ij}$ are even, we get,

$$[-\nabla V(x)]_i = \sum_{j \sim i} f_{ij}(x_i - x_j) = \dot{x}_i,$$

so $\dot{x} = -\nabla V(x)$. Thus, $V$ is a potential of the system (#) and $V(x(t))$ is a decreasing function in time (except in equilibrium points), see (SM1.2). In particular, by LaSalle’s invariance principle [43, Theorem 6.15] all forward (resp. backwards) bounded orbits “tend to” (resp. “come from”) equilibrium points. In Proposition SM1.3 we show that, if $f_{ij}$ satisfy certain conditions, then all forward orbits are bounded and thus converge to equilibria.

3.2. Matrix reformulation. Define $f : C_1 \rightarrow C_1$ by $f((y_e)_{e \in E}) = (f_e(y_e))_{e \in E}$. Then,

$$[f(d^T(x))]_{(j,k)} = f_{kj}(x_k - x_j)$$

where $\{j,k\} \in E$ with $j < k$,

$$
\begin{cases}
1 & \text{if } k = i, \\
-1 & \text{if } j = i, \\
0 & \text{else}
\end{cases}
$$

So we get,

$$[d(f(d^T(x))))_i = \sum_{\{j,k\} \in E} [d]_{i,\{j,k\}} \cdot [f(d^T(x))]_{(j,k)}$$

$$= \sum_{\{j,i\} \in E \text{ } j < i} f_{ij}(x_i - x_j) + \sum_{\{i,k\} \in E \text{ } i < k} -f_{ki}(x_k - x_i) - \sum_{j \sim i} f_{ij}(x_i - x_j) = \dot{x}_i,$$

where in the second to last equality we use that $f_{ki} = f_{ik}$ and that they are odd functions. Hence we can rewrite (#) as

$$\dot{x} = (d \circ f \circ d^T)(x).$$

This is very reminiscent of the compact forms in equation (1.2). Indeed, if we recall (2.1), the three ODEs in (1.1) can be expressed as,

$$\dot{x} = (d \circ f \circ d^T)(x), \quad \dot{x} = (d \circ f \circ d^T)(x), \quad \dot{x} = (f \circ d \circ d^T)(x),$$

where in the first and last case $f$ is given by not necessarily odd functions in nodes, i.e. $f((x_v)_{v \in V}) = (f_v(x_v))_{v \in V}$ (in (1.1) we chose $f_v = \hat{f}$). In [41] it is shown that the matrix form of the first and third ODE in (3.4) is very useful to study the structure of the corresponding systems. We will show that the same is true for (3.3).

Equation (3.3) also allows us to represent system (#) as a dynamical system on the edges. That is, in the coordinates $y = d^T x$, system (#) has the form,

$$\dot{y} = (d^T \circ d \circ f)(y),$$

and is defined on the cut space $(\ker d)^\perp$, see (2.2). Note that the value in an edge $\{j,i\} \in E$ is given by $x_i - x_j$, where $j < i$. 


3.3. Equilibrium points. In this section, we find expressions that relate the set of equilibrium points of (3.3) with the topology of the underlying network. First, using (3.5) and \( \ker(d^T) = \ker d \) it is clear that the equilibrium points in the edge space, i.e. in the coordinates \( y \), are given by,

\[
E_y := (\ker d)^\perp \cap f^{-1}(\ker d).
\]

Recall that \( \ker d \) is the cycle space and \((\ker d)^\perp\) the cut space, which have strong topological interpretations and are easy to compute intuitively. In section SM2 we further develop this intuition and informally show why we should expect the set of equilibria to be at most \( m-n+1 \) dimensional, using (3.6).

The set of equilibrium points is better understood for specific systems such as the Kuramoto model [32, 18], but when dealing with complex networks and functions \( f_{ij} \), it will generally be unfeasible to find all equilibria explicitly. Thus, in some occasions we will restrict our study to the detailed-balance stationary states, as done in [19], which are given by,

\[
\tilde{E}_y := (\ker d)^\perp \cap f^{-1}(0).
\]

In node coordinates, \( x \), we have,

\[
E_x := \langle 1 \rangle^\perp \cap (d^T)^{-1}(f^{-1}(\ker d)),
\]

and,

\[
\tilde{E}_x := \{ x \in \langle 1 \rangle^\perp : f_{ij}(x_i - x_j) = 0 \text{ for all } \{i, j\} \in E \}.
\]

In particular, as odd functions have 0 as a root, \( 0 \) is an equilibrium point of (3.3) in \( \langle 1 \rangle \) and thus, \( \langle 1 \rangle \) is a line of equilibria in \( \mathbb{R}^n \). This is a desired property for generalizations of linear consensus, as in the linear case the equilibrium points are given by the consensus states, i.e. \( \langle 1 \rangle \) or equivalently \( x_i = x_j \) for all \( i, j \). Note that in our general context we may have other equilibria besides the consensus states. In general this is desirable as it allows for more flexibility, see for instance the animal group decision-making model with non-consensus equilibria in [16]. In some instances however, one may want a nonlinear model which always reaches consensus and the following result adapted from [46] explains how to guarantee this.

**Proposition 3.1.** Suppose that for all edges, \( f_{ij} \) has 0 as its unique root with \( f_{ij}'(0) < 0 \). Then, the only equilibria of the system (3.3) in \( \mathbb{R}^n \) are the consensus states \( \langle 1 \rangle \). Moreover, all forward orbits converge to one of these states.

**Proof.** For each edge, as \( f_{ij} \) is odd, we can define \( k_{ij}(|y|) = -f(y)/y \) for \( y \neq 0 \) and \( k_{ij}(0) = -f'(0) \). Then \( f_{ij}(y) = -yk_{ij}(|y|) \) where \( k_{ij} \) is positive and continuous by the assumptions of this theorem. Now denote by \( L(x) \) the Laplacian matrix of the graph \( G \) with weights \( k_{ij}(|x_i - x_j|) > 0 \). If we define,

\[
E(x) = \sum_{\{i,j\} \in E} (x_i - x_j)^2,
\]

one can show that \( \dot{E}(x) = -x^T L(x)x \) (see [46, Section IV]) and as Laplacians are positive semi-definite \( \dot{E}(x) \leq 0 \). As always it is enough to restrict our study to the state space \( \langle 1 \rangle^\perp \).
where the only consensus state is the origin. Then, as $G$ is connected we have $\dot{E}(x) = 0$ only at the origin, and $E(x) \to \infty$ if $||x|| \to \infty$. So applying LaSalle’s invariance principle [43, Theorem 6.15] to $E$, we conclude that all forward orbits in $(1)^\perp$ converge to the origin. $lacksquare$

3.4. Stability of equilibrium points. To find the stability of an equilibrium point $x^*$ we consider the Jacobian matrix of $(\#)$ which we will denote by $J(x^*)$ or simply $J$. By the compact reformulation given in (3.4) and the chain rule we have,

$$(3.7) \quad J(x^*) = d D_{d^x} f d^T.$$

where $D_y f$ denotes the Jacobian of $f$ at $y$. Note that 1 is an eigenvector of eigenvalue 0 of $J$ and thus an equilibrium cannot be linearly stable in $\mathbb{R}^n$. However, it can be linearly stable in the dynamically invariant space $(1)^\perp$, which is also invariant under the symmetric matrix $J$. From now on, linear stability will always be restricted to the space $(1)^\perp$. Note that the eigenvalues of $J|_{(1)^\perp}$ are given by the ones in $\mathbb{R}^n$ while dropping one multiplicity of 0, which we will constantly use in our proofs. In particular, we have:

**Proposition 3.2.** Let $x^*$ be an equilibrium point of $(\#)$. Then, if all eigenvalues of $J(x^*)$ (in $(1)^\perp$) are negative, $x^*$ is stable, whereas if $J(x^*)$ has a positive eigenvalue, $x^*$ is unstable.

**Remark 3.3.** In the original state space $\mathbb{R}^n$ we will never have attractive points. Indeed, any equilibrium point in $(1)^\perp$ spans a line of equilibrium points in the direction 1 in $\mathbb{R}^n$. However, if a point is stable in $(1)^\perp$ it will also be stable in $\mathbb{R}^n$ as all parallel planes have the same dynamics.

The conditions in the proposition above determine the linear stability of the equilibria in $(1)^\perp$. Thus, the stability will persist under small perturbations of the functions $f_{ij}$ as long as they remain odd. This will hold for all stability results, since they follow from Proposition 3.2.

As $J$ is a symmetric matrix, we can use the Courant minmax principle [31] which in particular implies,

**Proposition 3.4.** Let $M$ be a symmetric matrix and $\lambda_{\text{max}}$ its maximum eigenvalue in an invariant subspace $V$. Then, for all $x \in V$,

$$||x||^2 \lambda_{\text{max}} \geq x^T M x.$$

Moreover, there exists a unitary vector $z \in V$ such that, $\lambda_{\text{max}} = z^T M z$.

In our context $V = (1)^\perp$ and $M = J$. However, by the comment made above Proposition 3.2 if we find $x \in \mathbb{R}^n$ such that $x^T J x > 0$ it will be enough to deduce that $\lambda_{\text{max}} > 0$ in $(1)^\perp$. Similarly, if we find $x \in \mathbb{R}^n \setminus (1)$ such that $x^T J x \geq 0$ it will be enough to deduce that $\lambda_{\text{max}} \geq 0$ in $(1)^\perp$.

A direct consequence of this result are the following simple criterion for instability.

**Proposition 3.5.** Let $\mathcal{H}$ be a cut-set of $G$ and $x^*$ an equilibrium point of $(\#)$. Then,

$$\sum_{(i,j) \in \mathcal{H}} f'_{ij}(x^*_i - x^*_j) > 0 \implies x^* \text{ is unstable.}$$

Note that $f'_{ij} = f'_{ji}$ and is even, so the sum is well defined.
In particular, for any \( i \in V \) we have,
\[
\sum_{j \sim i} f'_{ij}(x^*_i - x^*_j) > 0 \implies x^* \text{ is unstable.}
\]

**Proof.** Let \( V = V_1 \cup V_2 \) be a partition of the nodes such that \( \mathcal{H} \) is the associated cut-set and consider the vector \( x = \sum_{i \in V_1} e_i \). Now note that,
\[
x^\top Jx = \sum_{i \in V_1} [J]_{i,i} + \sum_{i,j \in V_1} [J]_{i,j} = \sum_{i \in V_1} \sum_{j \sim i} f'_{ij}(x^*_i - x^*_j) - \sum_{i \in V_1} \sum_{j \sim i} f'_{ij}(x^*_i - x^*_j) = \sum_{i \in V_1} \sum_{j \sim i} f'_{ij}(x^*_i - x^*_j) > 0,
\]
and apply Proposition 3.4 together with Proposition 3.2. \( \square \)

Informally the result above makes clear that if \( x^* \) is stable, \( f'_{ij}(x^*_i - x^*_j) \leq 0 \) must hold for “a substantial amount” of edges, as we need to have at least one of them in each cut-set. Moreover, their module has to be big enough such that all corresponding sums are non-positive. We now proceed to show that “a substantial amount” can be expressed formally as containing a spanning tree. First we introduce some notation.

Recall equation (3.7) and that \( D_{q^*(x^*)}f \) is a diagonal matrix, with values in the diagonal \( f'_{ij}(x^*_i - x^*_j) \) for each \( \{i,j\} \in E \). Grouping the positive and negative values in two terms, we can decompose the Jacobian matrix as
\[
J(x^*) = J = L^+ - L^-,
\]
where \( L^- \) (resp. \( L^+ \)) is the weighted Laplacian of \( G^- \) (resp. \( G^+ \)) defined as the network \( G \) with weights given by \( |f'_{ij}(x^*_i - x^*_j)| \) if \( f'_{ij}(x^*_i - x^*_j) < 0 \) (resp. \( f'_{ij}(x^*_i - x^*_j) > 0 \)) and 0 otherwise. Matrices of the form \( L^+ - L^- \) are known as signed Laplacians and they do not retain many of the properties of standard Laplacians, for instance they may have both positive and negative eigenvalues. A general study on the signs of their spectrum can be found in [13], where an alternative proof of the following result is given.

**Proposition 3.6.** Given an equilibrium point \( x^* \) of (1) we have,
\begin{itemize}
  \item if \( L^+ = 0_{n,n} \) and \( G^- \) is connected, \( x^* \) is stable;
  \item if \( G^- \) is disconnected then \( x^* \) is not linearly stable (in (1)\(^+\)). Moreover, if there is an edge in \( G^+ \) between two distinct connected components of \( G^- \), \( x^* \) is unstable.
\end{itemize}

**Proof.** See Appendix A.

This proposition is useful to study some simple situations. For instance, if a network has an edge \( \{i,j\} \) with no cycles going through it, e.g. the connecting edge of the barbell graph, and \( f'_{ij}(x^*_i - x^*_j) > 0 \) then \( x^* \) is unstable. To be able to deal with more complex situations we will introduce the concept of effective resistances in subsection 3.4.2.
3.4.1. Schur complement reduction. In this section we show how to use the Schur complement to reduce the dimensionality of the stability problem. Essentially we will be able to remove all nodes which are not incident to any edge in $G^+$. Clearly, this technique is very powerful when $G^+$ contains few edges, see Theorem 3.10 for the case with a single edge, and Appendix B for a concrete example.

We start by introducing the Schur complement for a symmetric matrix.

**Definition 3.7.** Given a $n \times n$ symmetric matrix $M$, and $N, M \subseteq \{1, \ldots, n\}$, we denote by $M_{N,M}$ the submatrix of $M$ with rows indexed by $N$ and columns by $M$. Then, if $M_{N,N}$ is invertible, we define the Schur complement of $M$ respect to $N$ by,

$$M/N := M_{N,N}^{-1} - M_{N,M} M_{M,N}^{-1} M_{N,N}.$$

The Schur complement has many interesting properties and applications as reviewed in [49]. In our context, its two key features are that $\text{In}(M) = \text{In}(M/N) + \text{In}(M_{N,N})$, see [49, Theorem 1.6], and that if $M$ is a weighted Laplacian then $M/N$ is also a Laplacian [21].

Recall that finding the stability of an equilibrium $x^\star$ reduces to the study of the inertia of $J$. Here it will be convenient to assume $G^-$ connected and one can then use Proposition 3.6 to deduce the stability for the general case. Denote by $N$ the set of nodes non-incident to any edge in $G^+$ and assume that $N$ and $N^c$ are non-empty. Then, reordering the nodes we have,

$$J = \begin{pmatrix} L_{N^c,N^c}^+ & -L_{N^c,N}^- & -L_{N,N}^- \\ -L_{N,N}^- & -L_{N,N}^- & -L_{N,N}^- \end{pmatrix}.$$

As $L_{N,N}^-$ is a principal submatrix of a connected Laplacian, it is positive definite, so invertible. Thus, we can consider the Schur complement $J/N$ and get,

$$\text{In}(J) = \text{In}(J/N) + \text{In}(J_{N,N}) = \text{In}(J/N) + \text{In}(-L_{N,N}^-).$$

As $-L_{N,N}^-$ is negative definite, the stability of $x^\star$ is determined by $\text{In}(J/N)$. Moreover, from (3.10) it is clear that

$$J/N = L_{N^c,N^c}^+ - L^-/N.$$

where $L^-/N$ is a Laplacian matrix, as it is the Schur complement of a Laplacian. Note that $L_{N^c,N^c}^+$ is also a Laplacian matrix, as by definition of $N$ all other entries of $L^+$ are zeros. Thus, expression (3.11) is analogous to (3.8) but in dimension $|V|-|N|$, and we can interpret it as the stability problem for a smaller network. In summary we have shown:

**Theorem 3.8.** Let $x^\star$ be an equilibrium point of $(\#)$, assume that $G^-$ is connected and let $N$ be the nodes not incident to any edge in $G^+$. Then, the stability of $x^\star$ is given by the linear stability of $J/N$ in (1)\(^\perp\).

This theorem will only be useful when $|N|$ is large, in particular if $G^+$ is connected, then $|N| = 0$, and we do not get any reduction. One small detail we have ignored so far is that in (3.11) one may have edges that are contained in both Laplacians $L_{N^c,N^c}^+$ and $L^-/N$, in contrast to (3.8). This can be used to apply the Schur complement to a new decomposition and reduce the dimensions even further, see Appendix B.
3.4.2. Effective resistance criteria. In this section we will apply ideas from electrical networks, and in particular the concept of effective resistance, to determine the stability of the equilibrium points of (\#). This application of effective resistance was introduced in the recent work by Devriendt et al. [19] for a system of the form (\#), and by Tyloo et al. [44] for Kuramoto-like systems. While the analysis in [19] was based on unweighted networks, we will consider weights on \( G \) in our more general setting, leading to improved results. We will show that when a pair of nodes are better connected\(^4\) in \( G^+ \) than in \( G^- \), then \( x^* \) is unstable. Moreover, when \( G^+ \) contains a single edge, this condition is tight.

**Definition 3.9.** The effective resistance \( r_{ij} \) between a pair of nodes \( i \) and \( j \) in a weighted network with Laplacian \( L \), is defined as (see, e.g. [21, 27])

\[
    r_{ij} := (e_i - e_j)^	op L (e_i - e_j)
\]

if \( i \) and \( j \) are in the same connected component, and \( r_{ij} = \infty \) otherwise.

Note that the effective resistance between two adjacent nodes, with no other path between them, is given by the inverse of the edge weight. Moreover, this definition satisfies the well known properties of resistors in series and in parallel, see section SM3. A generalization of them, is given by the inverse of the edge weight. Moreover, this definition satisfies the well known properties of resistors in series and in parallel, see section SM3. A generalization of this fact is given by the invariance of effective resistance by the Schur complement, also known as Kron reduction [21]. That is, the effective resistance between two nodes \( i, j \notin \mathcal{N} \subset \mathcal{V} \) is the same in \( L \) and in \( L/N \).

Recall that \( J = L^+ - L^- \) and denote by \( r^+_{ij} \), \( r^-_{ij} \) the effective resistance between \( i \) and \( j \) of the corresponding networks \( G^+, G^- \). With this notation we are prepared to state our first result, which gives tight conditions for the stability when \( G^+ \) has a single edge.

**Theorem 3.10.** Let \( x^* \) be an equilibrium point of (\#) such that there exists a unique edge \( \{i, j\} \) with \( f'_{ij}(x^*_i - x^*_j) \geq 0 \) and assume that \( f'_{ij}(x^*_i - x^*_j) > 0 \). Then,

\[
    \begin{align*}
    f'_{ij}(x^*_i - x^*_j)r^-_{ij} &< 1 \implies x^* \text{ is stable;} \\
    f'_{ij}(x^*_i - x^*_j)r^+_{ij} &> 1 \implies x^* \text{ is unstable.}
    \end{align*}
\]

**Proof.** If \( G^- \) is disconnected we have \( r^-_{ij} = \infty \) and \( x^* \) is unstable due to Proposition 3.6. If \( G^- \) is connected, by Theorem 3.8 the stability is determined by \( J/N \). As \( \mathcal{N}^c = \{i, j\} \), we have,

\[
    J/N = L^+_{\{i,j\}, \{i,j\}} - L^-/N = \begin{pmatrix}
    f'_{ij}(x^*_i - x^*_j) & -f'_{ij}(x^*_i - x^*_j) \\
    -f'_{ij}(x^*_i - x^*_j) & f'_{ij}(x^*_i - x^*_j)
    \end{pmatrix} - \begin{pmatrix}
    w^-_{ij} & -w^-_{ij} \\
    -w^-_{ij} & w^-_{ij}
    \end{pmatrix},
\]

for certain weight \( w^-_{ij} \). Thus, \( x^* \) is stable if \( f'_{ij}(x^*_i - x^*_j) - w^-_{ij} < 0 \) and unstable if \( f'_{ij}(x^*_i - x^*_j) - w^-_{ij} > 0 \). Finally, we use that effective resistance is invariant under Schur complement, so \( r^-_{ij} = 1/w^-_{ij} \). \qed

\(^4\)Better connected in the sense of effective resistance, i.e. the effective resistance between them in \( G^+ \) is smaller than the one in \( G^- \).
Remark 3.11. If there is a unique edge with \( f'_{ij}(x^*_i - x^*_j) > 0 \) and \( G^- \cup \{i, j\} \) is connected, the result holds with the same arguments.

We can take an analogous approach when \( G^+ \) contains two edges with a common node, or in fact any particular small configuration. Then, using Sylvester’s criterion [40], one gets a couple of inequalities on the entries of \( J/N \) (which can be related to effective resistance) that determine the stability. This operation gets quite messy so, instead, we expand by brute force Theorem 3.10 to the general case, but we lose the tightness of the bounds.

Proposition 3.12. Let \( x^\star \) be an equilibrium point of (\#). Then, if \( G^+ \cup G^- \) is connected,

\[
\sum_{\{i,j\} \in G^+} f'_{ij}(x^*_i - x^*_j) r^-_{ij} < 1 \implies x^\star \text{ is stable;}
\]

\[
\exists \{i, j\} \in G^+ : f'_{ij}(x^*_i - x^*_j) r^-_{ij} > 1 \implies x^\star \text{ is unstable.}
\]

Proof. See Appendix C.

Both Theorem 3.10 and Proposition 3.12 are adapted results from [19] where a particular coupling function is considered. We have opted to give completely different proofs, but it is possible to generalize the arguments from [19] to our context with some work. Let us now present another approach which leads to a tighter instability condition. First, we need the following result, see Appendix D for the proof.

Lemma 3.13. Let \( A, B \) be symmetric positive semi-definite matrices of the same dimension. Then,

\[
\max_{x : \perp \ker B} x^\top A x = \max_{x : \perp \ker A} x^\top B^\dagger x.
\]

In section SM4 we discuss how this result could also be used to get stability conditions.

Theorem 3.14. Let \( x^\star \) be an equilibrium point of our system and \( i, j \) a pair of nodes. Then,

\[
r^-_{ij} > r^+_{ij} \implies x^\star \text{ is unstable.}
\]

Proof. Denote by \( \lambda_{\max} \) the maximum eigenvalue of \( J \) in \( \langle 1 \rangle^{\perp} \), then by Proposition 3.4 we have,

\[
\lambda_{\max} \geq \max_{x : \perp \ker L^-} x^\top L^+ x - x^\top L^- x = \max_{x : ||x|| = 1} x^\top L^- x \left( \frac{x^\top L^+ x}{x^\top L^- x} - 1 \right)
\]

and as \( L^- \) is positive semi-definite,

\[
(3.12) \quad \text{sign}(\lambda_{\max}) \geq \text{sign} \left( \max_{x : \perp \ker L^-} x^\top L^+ x / x^\top L^- x - 1 \right).
\]

5 All maximums are taken over non-zero vectors.
Now, if \( r_{ij}^+ = \infty \) the condition of this theorem is never satisfied and there is nothing to prove. If \( r_{ij}^+ = \infty \) and \( r_{ij}^- < \infty \), \( x^* \) is unstable by Proposition 3.6. So we may assume that \( i \) and \( j \) are in the same connected component of \( G^+ \) and \( G^- \). Then, \( (e_i - e_j) \perp \ker L^+ \) and by the previous lemma,

\[
\max_{x \perp \ker L^-} \frac{x^T L^+ x}{x^T L^- x} = \max_{x \perp \ker L^+} \frac{x^T (L^-)^\dagger x}{x^T (L^+)^\dagger x} \geq \frac{(e_i - e_j)^\dagger (L^-)^\dagger (e_i - e_j)}{(e_i - e_j)^\dagger (L^+)^\dagger (e_i - e_j)} = \frac{r_{ij}^-}{r_{ij}^+}.
\]

So if \( r_{ij}^- > r_{ij}^+ \), then \( \frac{r_{ij}^-}{r_{ij}^+} - 1 > 0 \), and thus by (3.12), \( \lambda_{\text{max}} > 0 \). \( \blacksquare \)

Remark 3.15. When there is a unique edge \( \{i, j\} \) with positive derivative then, \( r_{ij}^+ = 1/f'(x_i^* - x_j^*) \) and the instability condition coincides with Theorem 3.10. In general \( r_{ij}^+ \leq 1/f'(x_i^* - x_j^*) \), and the new instability condition is tighter than Proposition 3.12.

It has been shown that the effective resistance is a distance function on the nodes of a network which in some sense encapsulates how well connected the different pairs of nodes are [27, 21]. With this point of view, the theorem above asserts that if there exist a pair of nodes which are better connected in \( G^+ \) than in \( G^- \), then we have instability. For the case of a unique non-negative edge, Theorem 3.10 asserts that this condition is tight, and only needs to be checked for the non-negative edge. Theorem 3.14 does not give tight conditions in general, but they might be tight for certain types of networks (see section SM8).

3.5. Union of two networks in a node. Assume that \( G \) contains a cut-node, i.e. a node that when “deleted” increases the number of connected components of \( G \). Equivalently \( G \) is the union of two subnetworks \( A, B \) that intersect in a single node, also known as a coalescence of \( A \) and \( B \) [5]. In this section we will show that the equilibrium points of \( G \) are exactly the combination of equilibrium points of \( A \) and \( B \). Moreover, we will show that the linear stability of an equilibrium point in \( G \) is determined by the stability of the corresponding equilibrium points in \( A \) and \( B \). These results are quite surprising, as there are several ways to coalesce two networks, which can produce significantly different structures. Our results show that from a basic dynamical point of view, the resulting networks will be equivalent. A comparison between the dynamical properties of networks and their coalescence was also considered in [5] for a system of coupled oscillators, where coalescence was shown to lead to an increased time to reach synchrony.

Without loss of generality we may assume that the nodes of \( A \) and \( B \) are \( \{1, \ldots, \bar{n}\} \) and \( \{\bar{n}, \ldots, n\} \) respectively, and let \( n' = n - \bar{n} + 1 \). As all edges in \( G \) are in \( A \) or \( B \) exclusively, the edge state of \( G \) is given by the edge states of \( A \) and \( B \) combined. That is, reordering the components if necessary \( \mathbf{y} = (\mathbf{y}_A, \mathbf{y}_B) \). Note that with these assumptions \( f(\mathbf{y}) = (f_A(\mathbf{y}_A), f_B(\mathbf{y}_B)) \) and the generating cycles of \( \ker d \) are entirely contained in \( A \) or \( B \). Then, by (3.6) it is straightforward to show that,

\[
\mathcal{E}_Y = \iota_A(\mathcal{E}_Y^A) \oplus \iota_B(\mathcal{E}_Y^B),
\]

where, \( \mathcal{E}_Y^A \) are the equilibria on \( A \) and \( \iota_A(\mathbf{y}_A) = (\mathbf{y}_A, \mathbf{0}_B) \) (resp. for \( B \)), see section SM5 for a detailed exposition. Hence, \( \mathbf{y} = (\mathbf{y}_A, \mathbf{y}_B) \in \mathcal{E}_Y \) if and only if \( \mathbf{y}_A \in \mathcal{E}_Y^A \) and \( \mathbf{y}_B \in \mathcal{E}_Y^B \).

Now we move on to prove that the stability can also be studied by restricting ourselves to \( A \) and \( B \). We need the following technical result, proved in Appendix E.
Lemma 3.16. Let $G$ be a network with $m$ edges, $d$ its boundary map and $\tilde{d}$ the same matrix with a row deleted. Let $M \in \mathbb{R}^{m \times m}$ be a diagonal matrix. Then, the number of positive (resp. negative) eigenvalues of $dMd^T$ and $\tilde{d}M\tilde{d}^T$ coincide.

Recall from Proposition 3.2 that the stability only depends on the sign of the eigenvalues of $J$. We have,

$$J = dD_yfd^T = \begin{pmatrix} d_AD_yAf_A\tilde{d}_A^T & 0_{\tilde{n},n'-1} \\ 0_{n'-1,\tilde{n}} & 0_{n'-1,n'-1} \end{pmatrix} + \begin{pmatrix} 0_{\tilde{n}-1,\tilde{n}-1} & 0_{\tilde{n}-1,n'} \\ 0_{n',\tilde{n}-1} & d_BD_yBf_B\tilde{d}_B^T \end{pmatrix},$$

where in the $\tilde{n}$th row/column we can have non-zero entries in both matrices. Now if we denote by $\tilde{d}$, $\tilde{d}_A$, $\tilde{d}_B$ the respective matrices when deleting the row corresponding to the node $\tilde{n}$, we find

$$\tilde{d}D_yf\tilde{d}^T = \begin{pmatrix} \tilde{d}_AD_yAf_A\tilde{d}_A^T & 0_{\tilde{n}-1,n'-1} \\ 0_{n'-1,\tilde{n}-1} & \tilde{d}_BD_yBf_B\tilde{d}_B^T \end{pmatrix}.$$ 

So the spectrum of $\tilde{d}D_yf\tilde{d}^T$ is the union of the spectrum of $\tilde{d}_AD_yAf_A\tilde{d}_A^T$ and $\tilde{d}_BD_yBf_B\tilde{d}_B^T$. Thus, by Lemma 3.16, the signs of the spectrum of $dD_yfd^T$ are the union of the signs in the spectrum of $d_AD_yAf_A\tilde{d}_A^T$ and $d_BD_yBf_B\tilde{d}_B^T$ (with a zero removed due to dimensionality). In particular, the linear stability of $y$ is determined by the linear stability of $y_A$ and $y_B$.

3.5.1. Tree of motifs. Starting from a collection of “small” networks which are understood dynamically\(^6\), we can create new networks by recursively joining (coalescing) these smaller networks in single nodes. By our results above, the dynamics of this new network will be equally well understood by simply composing the properties of the subsystems. We refer to these smaller networks as motifs as they play the role of dynamical and structural building blocks, and refer to the larger network as a tree of motifs as they consist of motifs interconnected in a tree (loopless) way, see Figure 1.

Equivalently our result can be stated as: given an arbitrary network the task of finding equilibria and their stability can be reduced to the same task for each of its blocks\(^7\). While some networks will consist of a single block, for many other networks this procedure will drastically reduce the dimensions in which we are working. In [26] similar results are shown for Markovian SIR epidemic dynamics.

4. Applications to concrete networks. In this section we will focus on studying the equilibrium points of (#) and their stability in some specific types of networks. Concretely we will study tree, cycle and complete graphs. Although it may seem that these families of networks are quite specific, using the main result from subsection 3.5, we will be able to tackle trees of motifs for the motifs: trees, cycles and complete graphs. See Figure 1 for an example and subsection 4.4 for explicit computations. Such trees of motifs appear as the construct of certain growing graph models (see [6, Section 3.5.2] and [47, Section 2]), and the special case where all motifs are complete graphs is also known as block graphs [7].

\(^6\)That is, we know their equilibrium points and their linear stability.

\(^7\)A block of a network is a maximal connected subnetwork that does not contain any cut-nodes of itself.
4.1. Tree graphs. In this section we will study the case when \(G\) has the simplest topology, which in our context means that it does not contain any cycles, i.e. \(G\) is a tree. We will show explicit expressions of the equilibrium points of (\#) given the roots of the functions \(f_{ij}\), and simple criterion for their stability. These results will directly follow from the previous section, as tree graphs are in particular trees of motifs, where all motifs are simply a pair of connected nodes.

**Proposition 4.1.** In a tree graph we have,

\[
E_y = \tilde{E}_y = f^{-1} \left( 0 \right) = \left\{ (y_{(i,j)})_{(i,j) \in \mathcal{E}} \in \mathbb{R}^m : f_{ij}(y_{(i,j)}) = 0 \text{ for all } \{i,j\} \in \mathcal{E} \right\},
\]

Moreover, the signs of the eigenvalues of \(J(x^*)\) in \((1)^\perp\) are given by \(\text{sign}(f'_{ij}(x_i^* - x_j^*))\) for \(\{i,j\} \in \mathcal{E}\). In particular, if there exists \(\{i,j\} \in \mathcal{E}\) such that \(f'_{ij}(x_i^* - x_j^*) > 0\), then \(x^*\) is unstable, whereas \(x^*\) is stable if \(f'_{ij}(x_i^* - x_j^*) < 0\) for all \(\{i,j\} \in \mathcal{E}\).

**Proof.** First, as trees have no cycles, \(\ker d = 0\) so from (3.6) and the definition of \(\tilde{E}_y\) we get \(E_y = \tilde{E}_y = f^{-1} \left( 0 \right)\). For the signs of the eigenvalues apply subsection 3.5 inductively on edges.

In particular, in a tree, the set of equilibria (on the edge space) and their linear stability only depends on the type of coupling functions \(f_{ij}\), and not on the specific arrangement of the edges. For instance, the equilibria and stability in a line graph will be the same as in a star graph of the same size. Note that the result above not only gives us the stability of the equilibria, but also finds the signs of all eigenvalues of \(J\). This will be useful for the following result, which informally asserts that any random initial condition will converge to one of the stable equilibrium points or “infinity”.

**Proposition 4.2.** Let \(G\) be a tree where for all edges the roots of \(f_{ij}\) are simple. Then, for Lebesgue almost every point \(x \in (1)^\perp\) its forward orbit converges to a stable equilibrium or is
unbounded. Moreover, the stable equilibrium points are given by,
\[
\left\{ x \in \mathbb{1}^\perp : f_{ij}(x_i - x_j) = 0 \text{ and } f'_{ij}(x_i - x_j) < 0 \text{ for all } \{i, j\} \in \mathcal{E} \right\}.
\]

**Proof.** By the previous proposition, all equilibrium points are hyperbolic, thus isolated, and the stable ones are given by the expression above. Then, in gradient dynamics, LaSalle’s invariance principle [43, Theorem 6.15] guarantees that all forward bounded orbits converge to a set of equilibrium points. As they are isolated, orbits converge to a single point. It also follows that there are countably many equilibrium points so it is enough to prove that the set of points converging to an unstable equilibrium has null measure. This holds as its stable manifold has at least codimension 1 (see Lemma SM1.1 for details).

**Remark 4.3.** In the previous result, if for all edges, \( f_{ij} \) has 3 simple roots with \( f''_{ij}(0) < 0 \), then the only stable equilibrium is the origin. So almost every forward bounded orbit converges to it.

Imposing more conditions on \( f_{ij} \), one can show that all forward orbits are bounded Proposition SM1.3. To get a flavour of how a complete description of the dynamics of (#) may look, we give the phase portrait of trees in low dimension, see section SM6.

**4.2. Cycle graphs.** Let \( \mathcal{C}_n \) be the cycle graph with \( n \) nodes, i.e a line graph where the end nodes have been joined, and let \( \mathbf{x}^\ast \) be an equilibrium point of (#) on this network. Then if \( f'_{ij}(x_i^\ast - x_j^\ast) < 0 \) for all edges, \( \mathbf{x}^\ast \) is stable by Proposition 3.6. If instead there exist two edges with non-negative derivatives and one of them is positive, \( \mathbf{x}^\ast \) is unstable, by Proposition 3.6. Assume now that there is a unique edge \( \{i, j\} \) with non-negative derivative, and moreover \( f''_{ij}(x_i^\ast - x_j^\ast) \neq 0 \). Then, by Theorem 3.10 we have,
\[
\begin{align*}
&f'_{ij}(x_i^\ast - x_j^\ast) \cdot \sum_{\{i', j'\} \in \mathcal{E} \setminus \{\{i, j\}\}} \frac{1}{|f'_{ij'}(x_i^\ast - x_j^\ast)|} < 1 \implies \mathbf{x}^\ast \text{ is stable;} \\
&f'_{ij}(x_i^\ast - x_j^\ast) \cdot \sum_{\{i', j'\} \in \mathcal{E} \setminus \{\{i, j\}\}} \frac{1}{|f'_{ij'}(x_i^\ast - x_j^\ast)|} > 1 \implies \mathbf{x}^\ast \text{ is unstable,}
\end{align*}
\]
where we have used the formula for effective resistances in series, see (SM3.1). Thus, we have tight conditions for the stability of \( \mathbf{x}^\ast \) except for the case when \( \max_{\{i, j\} \in \mathcal{E}} f''_{ij}(x_i^\ast - x_j^\ast) = 0 \) or when there is a unique edge with non-negative derivative and the expression above equals 1.

In contrast with trees, where \( \mathcal{E} = \mathcal{E}_\Delta \), this equality will not always hold for cycle graphs (which may be considered the next simplest topology). This was already identified in [41] for \( \mathcal{C}_3 \) taking \( f_{ij}(y) = \hat{f}(y) = \lambda y - y^3 \) with \( \lambda > 0 \) for all \( \{i, j\} \in \mathcal{E} \). They noted that in the original node space \( \mathbb{R}^3 \), the subspace \( \mathcal{1} \) consists of unstable equilibrium points, and that there is a cylinder with axis \( \mathcal{1} \), consisting of stable equilibria. We note that,

\[ H(x_1, x_2, x_3) = \frac{x_1 - x_2}{x_1 - x_3}, \]

is a first integral of this system, so that the phase portrait in the parallel planes to \( \langle \mathcal{1} \rangle^\perp \) is as depicted in Figure 2. Notice that we get a continuum of equilibrium points.
We now study when and why this phenomenon of a curve of equilibria in the node space \((1)^{\perp}\) occurs for a general function \(\hat{f}\). As \(n = m\) and following the comments made in subsection 3.3, we suspect that this set is at most one dimensional. In this context it will be convenient to work with a slight modification of the edge coordinates,

\[
z_1 = y_{\{1,2\}}, \ldots, z_{n-1} = y_{\{n-1,n\}}, z_n = -y_{\{1,n\}}.
\]

Recall that \(\ker d\) is generated by the cycles so in \(z\) coordinates, \(\ker d = (1)\). Thus from (3.6) we get,

\[
E_z = (1)^{\perp} \cap f^{-1}(1) = \bigcup_{\lambda \in \mathbb{R}} (1)^{\perp} \cap f^{-1}(\lambda 1),
\]

where \(E_z\) are the equilibria in the \(z\) coordinates and,

\[
(1)^{\perp} \cap f^{-1}(\lambda 1) = \left\{ z \in \mathbb{R}^n : \sum_{i=1}^{n} z_i = 0 \text{ and } z_i \in \hat{f}^{-1}(\lambda) \right\}.
\]

Now if the sets \((1)^{\perp} \cap f^{-1}(\lambda 1)\) are non-empty for all \(\lambda\) and they do not coincide, we expect to have a curve of equilibrium points parametrized by \(\lambda\). Note that \((1)^{\perp} \cap f^{-1}(\lambda 1) \neq \emptyset\) is equivalent to the existence of \(\alpha_1^\lambda, \ldots, \alpha_n^\lambda\) roots of \(\hat{f} - \lambda\) such that \(\sum_{i=1}^{n} \alpha_i^\lambda = 0\). The following polynomials are an important class of functions that satisfy this condition.

**Proposition 4.4.** Consider (\#) for the network \(C_n\) and \(f_{ij} = \hat{f}\) for all \(\{i,j\} \in \mathcal{E}\). Assume that \(\hat{f}\) is an odd polynomial of degree \(k \neq 1\) with \(k\) distinct real roots and \(k|n\). Then, (\#) has a continuum of equilibrium points.

**Proof.** Denote by \(a_0, \ldots, a_k\) the coefficients of \(\hat{f}\). As \(\hat{f}\) has \(k\) distinct roots and all of them are simple, there exists \(\epsilon > 0\) such that for all \(\lambda \in (-\epsilon, \epsilon), \hat{f} - \lambda\) has \(k\) real roots which we denote by \(\beta_0^\lambda, \ldots, \beta_{k-1}^\lambda\). Moreover, for each \(i\), \(\beta_i^\lambda\) takes a continuum of values parametrized
by $\lambda$. As $\hat{f}$ is an odd function, its even coefficients are null, and in particular $a_{k-1} = 0$. By the well known Vieta’s formulas [45, p. 99] we have for all $\lambda$,

$$0 = -\frac{a_{k-1}}{a_k} = \sum_{i=0}^{k-1} \beta_i^\lambda.$$ 

Now for $i \in \{1, \ldots, n\}$ define $\alpha_i^\lambda = \beta_i^\lambda \mod k$, and note that $\sum_{i=1}^n \alpha_i^\lambda = \frac{n}{k} (\sum_{i=0}^{k-1} \beta_i^\lambda) = 0$ as $k|n$.

So from (4.2) and (4.1), we deduce that

Now for $\alpha = (\alpha_i^\lambda)_{i=1}^n$ in $E$, and note that $\sum_{i=1}^n \alpha_i^\lambda = \frac{n}{k} (\sum_{i=0}^{k-1} \beta_i^\lambda) = 0$ as $k|n$.

So from (4.2) and (4.1), we deduce that

$$(\alpha_i^\lambda)_{i=1}^n \in (1)^⊥ \cap f^{-1}(\lambda 1) \subset \mathcal{E}_x$$

for all $\lambda \in (-\epsilon, \epsilon)$. Hence, $(\alpha_i^\lambda)_{i=1}^n$ gives a continuum of equilibria parametrized by $\lambda$. \hfill \Box

4.3. Complete graphs or cliques. In this section we will deal with the complete graph of $n$ nodes $K_n$. We will limit our study to the subset of equilibrium points $\mathcal{E}_x$ with $f_{ij} = \hat{f}$ for all $\{i, j\} \in E$. Moreover, it will be more convenient to think of the state space as $\mathbb{R}^n/(1)$ rather than $(1)^⊥$.

First, assume that $\hat{f}$ only has 3 distinct roots, which we denote by 0, $\pm \alpha$ and let $x^* \in \mathcal{E}_x$. Then, by definition of $\mathcal{E}_x$ on a complete graph, $x_i^* - x_j^* \in \{0, \pm \alpha\}$ for all $i, j$. With elementary arguments (see section SM7) one can show that for an appropriate representative and ordering of the nodes,

$$(4.3) \quad x^* = (0, \ldots, 0, \alpha, \ldots, \alpha)^T,$$

with 1 up to n null entries, depending on the equilibrium point. In fact, the equilibria in $\mathcal{E}_x$ will be of the form (4.3) even if $\hat{f}$ is an arbitrary odd function, as long as $\hat{f}^{-1}(0) \cap (0, \infty)$ is “additive open”, i.e. $a + b \neq c$ for all $a, b, c \in \hat{f}^{-1}(0) \cap (0, \infty)$, see section SM7. We characterise the stability of these equilibrium points.

**Proposition 4.5.** Let $\alpha$ be a root of $\hat{f}$ and consider (#) in $K_n$ with $n \geq 3$. Assume that $\hat{f}'(0) > 0$, $\hat{f}'(\alpha) < 0$ and denote

$$a = \frac{\hat{f}'(0)}{\hat{f}'(0) + |\hat{f}'(\alpha)|}, \quad b = \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0) + |\hat{f}'(\alpha)|}.$$ 

Then, the stability of $x^*$ as in (4.3) with $n_0$ zero entries is given by,

$$n_0/n \in (a, b) \implies x^* \text{ is stable},$$

$$n_0/n \notin [a, b] \implies x^* \text{ is unstable}.$$ 

Our proof follows from the direct computation of the Jacobian eigenvalues for a given number of zero entries in $x^*$, see Appendix F. Moreover, one gets the same instability condition by applying Theorem 3.14 in this context (see section SM8), which shows how powerful this theorem can be.

**Remark 4.6.** With the notation of the previous proposition, if $\hat{f}'(0), \hat{f}'(\alpha) > 0$ (resp. $\hat{f}'(0), \hat{f}'(\alpha) < 0$) we get that $x^*$ is unstable (resp. stable), by Proposition 3.6. If $\hat{f}'(0) < 0$ and $\hat{f}'(\alpha) > 0$, then $x^*$ is stable if $n_0 \in \{0, n\}$ and unstable otherwise, again by Proposition 3.6.
4.4. Example of the study of a tree of motifs. Consider the system (\#) with \( f_{ij}(y) = \dot{f}(y) = y - y^3 \) for all \( i, j \) in the network \( \mathcal{G} \) depicted in Figure 3. By the main result of subsection 3.5 we can find its equilibria and their stability by separately studying the subnetworks showed on the right-hand side of Figure 3, i.e. a cycle graph \( \mathcal{C}_3 \), a star graph with 3 edges and a complete graph \( \mathcal{K}_4 \).

**Cycle graph \( \mathcal{C}_3 \):** The results from Figure 2 with \( \lambda = 1 \) claim that, in \((1)^\perp\) this system has an unstable equilibrium at 0 and a circle of stable ones, which can be parametrized by,

\[
(x_1^*, x_2^*, x_3^*)^\top = \frac{\sqrt{2}}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} \cos(\theta)(-1, 1, 0) + \frac{1}{\sqrt{6}} \sin(\theta)(1, 1, -2) \right)^\top,
\]

for \( \theta \in [0, 2\pi) \). In the edge space the set of stable equilibria is given by,

\[
P^-_1 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 2 \cos(\theta), \cos(\theta) - \sqrt{3} \sin(\theta), -\cos(\theta) - \sqrt{3} \sin(\theta) \end{pmatrix}^\top : \theta \in [0, 2\pi) \right\},
\]

and the unstable ones are \( P^+_1 = \{(0, 0, 0)^\top\} \).

**Star graph with 3 edges:** Note that the roots of \( \dot{f}(y) = y - y^3 \) are \( \{0, \pm 1\} \), and \( \dot{f}'(0) = 1, \dot{f}'(\pm 1) = -2 \). So by Proposition 4.1 the stable equilibria in the edge space are given by,

\[
P^-_2 = \{\pm 1\}^3 := \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\},
\]

and the unstable ones by,

\[
P^+_2 = \left\{ (y_{\{3,4\}}, y_{\{4,5\}}, y_{\{4,6\}})^\top \in \{0, \pm 1\}^3 : y_{\{3,4\}}y_{\{4,5\}}y_{\{4,6\}} = 0 \right\}.
\]

**Complete graph \( \mathcal{K}_4 \):** Using the results from subsection 4.3, and imposing the condition of orthogonality to \( \mathbf{1} \) we find that \( \mathbf{x}^* = (x_6^*, x_7^*, x_8^*, x_9^*)^\top \in \mathbf{\partial}_\mathbf{x} \) if and only if,

\[
(4.4) \quad \{x_6^*, x_7^*, x_8^*, x_9^*\} \in \{\{0, 0, 0, 0\}, \{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}\},
\]

where we use the brackets \( \{\} \) to denote multisets. To find the stability of these equilibria recall the notation of Proposition 4.5 and notice that in our case,

\[
a = \frac{\dot{f}'(0)}{\dot{f}'(0) + |\dot{f}'(1)|} = \frac{1}{3}, \quad b = \frac{|\dot{f}'(1)|}{\dot{f}'(0) + |\dot{f}'(1)|} = \frac{2}{3}.
\]
Now, note that the representative in the form of (4.3) of the first three types of equilibria in (4.4), have respectively 4, 3 and 1 null entries. Hence, for these points \( n_0/n \notin [a, b] \) and thus, they are unstable. We denote by \( \mathcal{P}_3^+ \) the set of these points in edge coordinates. For the forth type of equilibria in (4.4) the representative has two null entries, so \( n_0/n = 1/2 \in (a, b) \) and thus, these points are stable. We denote by \( \mathcal{P}_3^- \) the set of these points in the edge coordinates.

Recall that for the complete graph there may be equilibria outside of \( \mathcal{E}_K \). As we are working with polynomials over the integers, we can hope to find all equilibria using Gröbner Bases (see [9]). Using Singular, we find that there is only one other type of equilibria, \( \{x^*_6, x^*_7, x^*_8, x^*_9\} = \{0, 0, \sqrt{2}/\sqrt{5}, -\sqrt{2}/\sqrt{5}\} \).

As this type of equilibrium point is not in \( \mathcal{E}_K \), we can not use Proposition 4.5 to determine its stability. However, if we let \( i \) and \( j \) be the nodes with value 0, one finds that \( r_{ij}^+ = 1 \) and \( r_{ij}^- = 5 \). So we have \( r_{ij}^+ > r_{ij}^- \) and by Theorem 3.14 we conclude that these points are unstable. Denote by \( \mathcal{P}_3^+ \) the union of these points in edge coordinates with \( \mathcal{P}_3^- \).

We are now prepared to tackle the whole network \( G \). For convenience, we denote,

\[
y = (y_1, y_2, y_3)^\top
\]

where,

\[
y_1 = (y_{\{1,2\}}, y_{\{1,3\}}, y_{\{2,3\}})^\top, \quad y_2 = (y_{\{3,4\}}, y_{\{4,5\}}, y_{\{4,6\}})^\top,
\]

and,

\[
y_3 = (y_{\{6,7\}}, y_{\{6,8\}}, y_{\{6,9\}}, y_{\{7,8\}}, y_{\{7,9\}}, y_{\{8,9\}})^\top.
\]

Let \( \mathcal{P}_i = \mathcal{P}_i^+ \cup \mathcal{P}_i^- \) for \( i = 1, 2, 3 \). Then, by the results from subsection 3.5, the stable equilibrium points in edge coordinates are,

\[
\mathcal{P}_- = \{(y_1, y_2, y_3)^\top : y_i \in \mathcal{P}_i^-, \}
\]

and the unstable ones are,

\[
\mathcal{P}_+ = \{(y_1, y_2, y_3)^\top : y_i \in \mathcal{P}_i \text{ and } y \notin \mathcal{P}_- \}.
\]

One can use the map \( d \) to find the node coordinates of these states. Note that the node coordinates in the whole network will not coincide with the node coordinates found for the different subnetworks.

**5. Conclusion.** In this paper, we have presented an in-depth study of a general nonlinear model of consensus dynamics, showing a rich phenomenology depending on the structural properties of its underlying network. The dynamical model can be viewed as a gradient system that conserves the mean state. Furthermore, it does not produce complex dynamical structures such as periodic orbits, as all orbits converge to equilibria. Notably for a high-dimensional nonlinear system with arbitrary topology, we find a compact expression for the equilibrium points that highlights how these equilibria are determined by an interplay between the coupling function and the underlying network. Moreover, in line with previous results in
the literature, we find that the stability of certain equilibria depends on effective resistances in the network, where the influence of general coupling functions can be taken into account using appropriate link weights.

Our analysis provides insight on simple networks like trees, cycles and cliques, but also for any combination of them in a tree of motifs, as we show that knowledge of the equilibria and their stability for individual motifs is in that case sufficient to characterise the whole-network dynamics. Although these conditions may seem restrictive, the resulting structures may be seen as a generalisation of trees to the case of higher-order networks [29]. Moreover, locally tree-like structures of cliques are expected to appear in projections of bipartite graphs [24], which include co-author networks, and when modeling pervasive overlap [1, 12] in social networks.

As a perspective for future research, we believe that a more careful deduction in the line of Theorem 3.10 and Lemma 3.13 could lead to better stability conditions for arbitrary equilibrium points, compared to those now presented in Proposition 3.12. Also in the spirit of Theorem 3.10, one could specialize the Schur complement reduction to particular configurations of positive edges, leading to tight stability conditions. As a third extension, it might be possible to obtain stability results for large dense networks (which are “approximately” complete) based on the exact Jacobian eigenvalues for complete graphs (as in Appendix F) and invoking perturbation arguments on the Jacobian [31]. A similar approximate setting where some of our exact results might be used as a starting point is the study of networks whose local structure may be approximated by a tree of motifs, generalising standard approximations based on a locally tree-like structure [33]. Finally, while this work is theoretical in nature we believe that our developed insights can be used for the application and specialisation of system (#) in a practical context.

Appendix A. Proof of Proposition 3.6. For the stability condition, note that we have $J = -L^-$. Now as mentioned below (2.3), $L^-$ is positive semi-definite and, as $G^-$ is connected, $0$ has multiplicity one in $\mathbb{R}^n$. Thus, in the state space $(1)^-\perp$ all eigenvalues of $-L^-$ are negative.

If $G^-$ is disconnected, consider $V_1$ a connected component and the vector $x = \sum_{i \in V_1} e_i$. Then, follow the proof of Proposition 3.5 to deduce that $\lambda_{\text{max}} \geq 0$. If there is an edge between $V_1$ and $(V \setminus V_1)$ in $G^+$, we may directly apply Proposition 3.5 to the associated cut-set.

Appendix B. Further Schur complement reduction. In this section we explain how to apply the Schur complement reduction recursively through an example. Consider a Jacobian matrix $J = L^+ - L^-$, where the weights of $L^+$ and $L^-$ are depicted in Figure 4.a. Note that there is a unique node not incident to any edge in $G^+$, i.e. $N = \{5\}$. By Theorem 3.8, the linear stability of $J$ is determined by $J/N = L^+_{N^c,N} - L^-/N$, where the weights of these Laplacians are depicted in Figure 4.b. When we have multiple edges between a pair of nodes in Figure 4.b we can simply add the (signed) weights to get Figure 4.c. This corresponds to finding another decomposition $J/N = \hat{L}^+ - \hat{L}^-$ with Laplacians which do not have common edges. We observe that in Figure 4.c, the node 4 is not incident to any edge in $\hat{L}^+$. Thus, we can apply our result again with $\tilde{N} = \{4\}$, to get Figure 4.d and by adding edges we get Figure 4.e. In summary, we have shown that the linear stability of $J$ is given by the linear stability of $(J/N)/\tilde{N}$ depicted in Figure 4.e, which is unstable by Proposition 3.6.

Note that in this example we have considered $(J/N)/\tilde{N}$. The quotient formula of Schur
Figure 4. Networks obtained by the recursive application of Schur complement. We depict edges from the corresponding $L^+$ in red. Edges from $L^-$ are depicted in blue and with a negative sign added to their weights.

Complement [49, Equation 6.0.26] states that if $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are disjoint sets of nodes then

$$(J/\mathcal{N})/\tilde{\mathcal{N}} = J/(\mathcal{N} \cup \tilde{\mathcal{N}}),$$

when the left-hand side is well defined. Given a Jacobian of our system it would be interesting to be able to determine which nodes we will be able to remove applying the Schur complement process recursively. Then, by the quotient formula we could reduce all of them at once. One can sidestep this problem by taking the opposite approach. That is, in each iteration of the Schur reduction, only reduce one node until no node is isolated in the corresponding $G^+$. This approach has the advantage that each iteration only requires inverting a scalar instead of a matrix, see (3.9).

Appendix C. Proof of Proposition 3.12. For the instability note that if $\mathcal{G}^-$ is disconnected, the result follows from Proposition 3.6. Otherwise, we can write $J = L^+_{ij} + \tilde{L}^+ - L^-$, where $L^+_{ij}$ is the Laplacian with only edge $\{i, j\}$ with weight coming from $\mathcal{G}^+$ and $\tilde{L}^+$ is the Laplacian of the rest of edges from $\mathcal{G}^+$. Now as $\tilde{L}^+$ is positive semi-definite, it is enough to show that $L^+_{ij} - L^-$ is linearly unstable, which follows from Remark 3.11.

For the stability result, as $\sum_{(i,j) \in \mathcal{G}^+} f'_{ij}(x^*_i - x^*_j)r^-_{ij} < 1$ we have $r^-_{ij} < \infty$ for all edges in $\mathcal{G}^+$, and thus $\mathcal{G}^-$ is connected. Moreover, we can take weights $\alpha_{ij} > f'_{ij}(x^*_i - x^*_j)r^-_{ij}$ such that
\[ \sum_{\{i,j\} \in G^+} \alpha_{ij} = 1. \] Then, using the notation \( L^+_{ij} \) from above we have,

\[ J = L^+ - L^- = \sum_{\{i,j\} \in G^+} L^+_{ij} - \alpha_{ij} L^- = \sum_{\{i,j\} \in G^+} \alpha_{ij} (\alpha_{ij}^{-1} L^+_{ij} - L^-). \]

Now apply Remark 3.11 to each \( \alpha_{ij}^{-1} L^+_{ij} - L^- \) to conclude that they are linearly stable matrices, thus so is \( J \) and \( x^* \) is stable.

**Appendix D. Proof of Lemma 3.13.** First, assume that \( A \) and \( B \) are non-singular, and let \( B^{1/2} \) be the positive definite matrix such that \( B = B^{1/2} B^{1/2} \). Then,

\[
\max_{x} \frac{x^T A x}{x^T B x} = \max_{x} \frac{x^T A x}{(B^{1/2} x)^T (B^{1/2} x)} = \max_{y} \frac{y^T B^{-1/2} A B^{-1/2} y}{y^T y} = \max_{|y|=1} y^T B^{-1/2} A B^{-1/2} y = \mu_{\text{max}},
\]

where \( y = B^{1/2} x \), \( \mu_{\text{max}} \) is the largest eigenvalue of \( B^{-1/2} A B^{-1/2} \) and in the last equality we use the Courant minmax principle [31]. Now, note that \( \mu_{\text{max}}^{-1} = \hat{\mu}_{\text{min}} \) where \( \hat{\mu}_{\text{min}} \) is the minimum eigenvalue of \( B^{1/2} A^{-1} B^{1/2} \), so

\[
\max_{|y|=1} y^T B^{1/2} A^{-1} B^{1/2} y = \max_{y} y^T B^{1/2} A^{-1} B^{1/2} y = \max_{z} z^T B^{-1} z.
\]

For the general case, let \( V = (\ker A + \ker B)^\perp \) and note that \( A \) and \( B \) are positive definite in this space. Recall from the preliminaries that \( (A|_V)^{-1} = A^{\dagger}_V \) (resp. for \( B \)), so from the argument above we get,

\[
\max_{x \in V} \frac{x^T A x}{x^T B x} = \max_{x \in V} \frac{x^T B^{\dagger}_V x}{x^T A^{\dagger}_V x}.
\]

Now as \( A, B \) are positive semi-definite and \( \ker A = \ker A^{\dagger} \) (resp. for \( B \)), the previous expression is equivalent to,

\[
\max_{x \perp \ker B} \frac{x^T A x}{x^T B x} = \max_{x \perp \ker A} \frac{x^T B^{\dagger} x}{x^T A^{\dagger} x}.
\]

**Appendix E. Proof of Lemma 3.16.** Reordering the nodes we can assume that the deleted row in \( \tilde{d} \) is the first one. Denote by \( \lambda_1 \leq \cdots \leq \lambda_n \) the eigenvalues of \( d M \tilde{d}^T \) and by \( \lambda_1 \leq \cdots \leq \lambda_{n-1} \) the ones of \( d M \tilde{d}^T \). As \( d M \tilde{d}^T \) is a symmetric matrix and \( d M \tilde{d}^T \) is the same matrix with the first row and column deleted we have,

\[
(E.1) \quad d M \tilde{d}^T = \begin{pmatrix} a & b^T \\ b & \tilde{d} \end{pmatrix}.
\]

Thus we can apply Cauchy’s Interlace theorem [25] and we get,

\[
(E.2) \quad \lambda_1 \leq \tilde{\lambda}_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \tilde{\lambda}_{n-1} \leq \lambda_n.
\]
Recall that $d^\top 1 = 0$, so the multiplicity of the eigenvalue 0 for $dMd^\top$, which we denote by $k$, is at least one. By (E.2) it is clear that the multiplicity of 0 for $\tilde{d}\tilde{M}\tilde{d}^\top$, which we denote by $\tilde{k}$, is $\tilde{k} = k - 1$, $k$ or $k + 1$. If $\tilde{k} = k - 1$, using (E.2) it is clear that the number of positive (resp. negative) eigenvalues of each matrix coincide. We now show by contradiction that $\tilde{k} \neq k, k + 1$.

First note that by (E.1) for all $y \in \ker \tilde{d}\tilde{M}\tilde{d}^\top \cap (\langle \mathbf{b} \rangle)^\perp$, $(0, y^\top)^\top \in \ker dMd^\top$. Moreover, $1 \in \ker dMd^\top$, so

\begin{equation}
\tag{E.3}
k = \dim(\ker dMd^\top) \geq \dim(\ker \tilde{d}\tilde{M}\tilde{d}^\top \cap (\langle \mathbf{b} \rangle)^\perp) + 1.
\end{equation}

Now, if $\tilde{k} = k + 1$, then $\ker \tilde{d}\tilde{M}\tilde{d}^\top \cap (\langle \mathbf{b} \rangle)^\perp$ has at least dimension $k$ so we get a contradiction with (E.3). If $\tilde{k} = k$, the multiplicity of 0 in both matrices coincide, thus we can apply an extension of Cauchy Interlace Theorem (see [25, Theorem 2]), which claims that $\langle \mathbf{b} \rangle \perp \ker \tilde{d}\tilde{M}\tilde{d}^\top$. Then, $\ker dMd^\top \cap (\langle \mathbf{b} \rangle)^\perp = \ker dMd^\top$ and again we get a contradiction with (E.3).

Appendix F. Proof of Proposition 4.5. Our arguments closely resembles the one given in [19] for a specific function $\hat{f}$.

Note that $a, b \in (0, 1)$, so if $n_0 \in \{0, n\}$ then $n_0/n \notin [a, b]$. In these cases, for all edges $x_i^* - x_j^* = 0$ and as $f'(0) > 0$, $x^*$ is unstable by Proposition 3.6. Otherwise, recall that $J = L^+ - L^-$ and it is easy to check that,

$$L^+ = f'(0) \left( \begin{array}{cc} \alpha & 0 \\ 0 & (n - \alpha)P_n \end{array} \right), \quad L^- = -f'(-\alpha) \left( \begin{array}{cc} (n - \alpha)I_n & -\mathbf{1}_n \alpha^\top \\ -\mathbf{1}_{n-\alpha} \mathbf{1}_{n-\alpha}^\top & \mathbf{1}_{n-\alpha} \end{array} \right),$$

where $P_k = I_k - \mathbf{1}_k \mathbf{1}_k^\top / k$, i.e. the orthogonal projection onto $(\langle \mathbf{1} \rangle)^\perp$. Notice that this two matrices commute, hence they can be simultaneously diagonalized. We find the following four types of eigenvectors of $J$.

Type 1: $\mathbf{1}_n$ is an eigenvector of eigenvalue 0, which does not affect the stability in the state space $\langle \mathbf{1} \rangle^\perp$.

Type 2: $(-\mathbf{1}_{n_0}^\top / n_0, 1_{n-\alpha} / (n - n_0))^\top$ is an eigenvector of eigenvalue $\hat{f}'(\alpha) n$.

Type 3: Any vector of the form $z = (z_{n_0}^\top, 0_{n-\alpha})^\top$ such that $1_{n_0}^\top z_{n_0} = 0$ is an eigenvector of eigenvalue $(\hat{f}'(0) - \hat{f}'(\alpha)) n_0 + \hat{f}'(\alpha) n$.

Type 4: Any vector of the form $z = (0_{n_0}, z_{n-\alpha})^\top$ such that $1_{n-\alpha}^\top z_{n-\alpha} = 0$ is an eigenvector of eigenvalue $(\hat{f}'(\alpha) - \hat{f}'(0)) n_0 + \hat{f}'(0) n$.

As there are $n_0 - 1$ linear independent eigenvectors of type 3 and $n - n_0 - 1$ eigenvectors of type 4, these are all eigenvectors of $J$.

As $\hat{f}'(\alpha) n < 0$, the stability of the equilibrium points is given by the signs of the other eigenvalues. Now assume that $n_0 \notin \{1, n - 1\}$ so that there is at least one eigenvector of each type. Type 3 eigenvalue is positive if $b < n_0/n$ and negative if $n_0/n < b$. Similarly, type 4 eigenvalue is positive if $n_0/n < a$ and negative if $a < n_0/n$. Thus, using Proposition 3.2 we get the desired result.

If $n_0 = 1$ then we do not have eigenvectors of Type 3, so $x^*$ is stable if $a < n_0/n$ and unstable if $n_0/n < a$. Using that $n_0/n = 1/n < 1/2$ and that $[a, b] \neq \emptyset$ if and only if $a \leq 1/2$, it follows,

$$a < n_0/n \iff n_0/n \in (a, b), \quad \text{and} \quad n_0/n < a \iff n_0/n \notin [a, b].$$

Similar arguments work for the case $n_0 = n - 1$. 

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SM1. Potential dynamics. We start by introducing some basic concepts of dynamical systems (we follow standard notation, see for instance [SM7]). We consider ODEs defined by $C^1$ maps,

$$\dot{x} = g(x),$$

and thus, by Picard–Lindelöf theorem [SM7, Theorem 2.5] it will have existence and uniqueness of solution for any given initial condition. Moreover, to simplify some definitions of qualitative features, we will always assume that orbits are defined for all times. Given an ODE, $\dot{x} = \tilde{g}(x)$, this can be done by considering the equivalent system,

$$\dot{x} = g(x) := \frac{\tilde{g}(x)}{||\tilde{g}(x)|| + 1},$$

which solutions are reparametrizations of solutions of the original ODE.

Let $x_0 \in \mathbb{R}^n$ and denote by $\varphi(t, x_0)$ the solution of $\dot{x} = g(x)$ at time $t$ such that $\varphi(0, x_0) = x_0$. We denote its forward orbit and its $\omega$-limit by

$$\gamma_+(x_0) = \{\varphi(t, x_0) : t > 0\}, \quad \omega(x_0) = \left\{ \lim_{n \to \infty} \varphi(t_n, x) : t_n \xrightarrow{n \to \infty} +\infty \right\}.$$

We say that an equilibrium point $x^*$ is stable if for any neighbourhood $U$ of $x^*$, there exists a neighbourhood $V \subset U$ of $x^*$ such that for all $x_0 \in V$, $\gamma_+(x_0) \subset U$. We say that $x^*$ is attractive if there exists a neighbourhood $U$ of $x^*$ such that for all $x_0 \in U$, $\omega(x_0) = x^*$. Let $x^*$ be an equilibrium point (or infinity), we denote its basin of attraction, i.e. the set of points that converge to it, by $W^+(x^*)$. We define $\gamma_-(x_0), \alpha(x_0)$ and $W^-(x^*)$ in the same manner but for the system $\dot{x} = -g(x)$. We will need the following result regarding the basin of attraction.

**Lemma SM1.1.** Consider $\dot{x} = g(x)$ where $g \in C^1$, and let $x^*$ be an unstable hyperbolic equilibrium point. Then, $W^+(x^*)$ has measure\(^1\) zero.

**Proof.** By the stable manifold theorem [SM7, Theorem 9.4], there is a neighbourhood $U$ of $x^*$, such that the set

$$W_U^+(x^*) = \{x \in W^+(x^*) : \gamma_+(x) \cup \{x\} \subset U\},$$

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\(^4\)Lebesgue measure.
is a $C^1$ submanifold of $\mathbb{R}^n$ tangent to the stable manifold of the linearization of the system. Since $x^*$ is not stable, these submanifolds have at most dimension $n-1$ and thus $W^+_U(x^*)$ has measure zero (e.g. by Sard’s theorem [SM6]). Then, note that,

$$W^+(x^*) = \bigcup_{m=0}^{\infty} \{ \varphi(-m, x) : x \in W^+_U(x^*) \} = \bigcup_{m=0}^{\infty} \varphi(-m, W^+_U(x^*)),$$

and that, for all $t$ the map $\varphi(t, \cdot)$ sends null sets to null sets, since it is a $C^1$ diffeomorphism. So, $W^+(x^*)$ is the countable union of null measure sets and thus, it has measure zero.

Now we focus on gradient dynamics. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function. We define its gradient dynamics as,

$$\dot{x} = -\nabla V(x),$$

so that $V$ is a potential function of this system. The derivative of $V$ along the trajectories is,

$$\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle = -||\nabla V(x)||^2 \leq 0,$$

and note that $\dot{V}(x) = 0$ only in equilibrium points. Thus, $V(x(t))$ is decreasing in solutions of (SM1.1) (except in equilibrium points).

**Proposition SM1.2.** Assume that the set of equilibrium points $\mathcal{E}$ of the system (SM1.1) is countable. Then, if $\gamma_+(x_0)$ (resp. $\gamma_-(x_0)$) is bounded, $\omega(x_0)$ (resp. $\omega(x_0)$) is an equilibrium point.

**Proof.** As $\gamma_+(x_0)$ is bounded, it is contained in a compact set so by LaSalle’s invariance principle [SM5, SM7],

$$\omega(x_0) \subset \{ x \in \mathbb{R}^n : \dot{V}(x) = 0 \} = \mathcal{E}.$$ 

Now $\gamma_+(x_0)$ is contained in a compact set so $\omega(x_0)$ is non-empty and connected (see [SM7, Lemma 6.6]). As $\mathcal{E}$ is countable the only non-empty connected subsets are singletons. 

To be able to apply the previous result we need $\gamma_+(x_0)$ to be bounded. One sufficient condition is for $V$ to be radially unbounded, i.e. $V(x) \to +\infty$ when $||x|| \to +\infty$, as the potential is decreasing in orbits. Now we are prepared to show that, under mild conditions, any forward orbit of the dynamics (#) converges to a single equilibrium point.

**Proposition SM1.3.** Let $\mathcal{E}_x$ be the equilibrium points of (#) in the state space $\langle 1 \rangle^\perp$. Assume that $\mathcal{E}_x$ is countable and one of the following conditions holds:

(a) For all $\{i,j\} \in \mathcal{E}$, $-F_{ij}$ is radially unbounded.

(b) For all $\{i,j\} \in \mathcal{E}$, $\limsup_{y \to +\infty} y f_{ij}(y) < 0$.

(c) All orbits are bounded, i.e. contained in a compact set.

Then,

$$\langle 1 \rangle^\perp = \bigcup_{x^* \in \mathcal{E}_x} W^+(x^*) = W^-(\infty) \cup \bigcup_{x^* \in \mathcal{E}_x} W^-(x^*),$$

where if (c) holds, we also have $W^-(\infty) = \emptyset$. 


Proof. By Proposition SM1.2 it is enough to show that forward orbits are bounded, and backwards unbounded ones tend to infinity. So condition (c) is clearly sufficient. Condition (a) is also sufficient, as then $V_{\langle 1 \rangle}$ defined by (3.1) is radially unbounded. Finally, condition (b) is a particular case of (a), as we proceed to show.

Assume that (b) holds, and consider $g(y) = \sup_{z \geq y} z f_{ij}(z)$, which is a non-increasing function. Then,
\[
a = \lim_{y \to +\infty} g(y),
\]
and notice that by our assumption $a < 0$. Thus, there exists $M > 0$ such that $g(y) < \frac{a}{2}$ for all $y \geq M$. In particular, $f_{ij}(y) < \frac{a}{2y}$ for all $y \geq M$ and hence for certain constant $c$,
\[
\lim_{y \to +\infty} -F_{ij}(y) = c + \int_{M}^{\infty} -f_{ij}(y) > c + \int_{M}^{\infty} -\frac{a}{2y} = +\infty,
\]
as $-\frac{a}{2} > 0$. Finally, $f_{ij}$ is odd so $F_{ij}$ is even and thus, $\lim_{|y| \to \infty} -F_{ij}(y) = +\infty$. \hfill \square

Note that the coupled phase oscillators model with constants $u_i = 0$, trivially satisfies (c) when studied in $S^1 \times \cdots \times S^1$. Another important case is given when $f_{ij} = \hat{f}$ for all edges and $\hat{f}$ has finitely many roots. In this case (and changing sense of time if necessary), the system satisfies condition (b), as long as $\hat{f}$ does not tend to 0 at $\infty$.

SM2. Dimension of the set of equilibrium points. We first show that,
\[
\dim(\ker d) = m - n + 1, \quad \text{and} \quad \dim\left( (\ker d)^\bot \right) = n - 1.
\]
where $n = |\mathcal{G}|$ and $m = |\mathcal{E}|$. To do so we find a basis of each space through a spanning tree\footnote{That is, a subnetwork of $\mathcal{G}$ with all of its nodes and no cycles.} $\mathcal{T}$ of $\mathcal{G}$, which also gives us some insight on how to explicitly compute these spaces. Each edge in $\mathcal{G} \setminus \mathcal{T}$ determines a unique cycle with this edge and edges of $\mathcal{T}$. These cycles form a basis of the cycle space [SM1]. Similarly, each edge in $\mathcal{T}$ determines a unique cut-set with this edge and edges in $\mathcal{G} \setminus \mathcal{T}$. These cut-sets form a basis of the cut space [SM1]. In particular, we get Equation (SM2.1).

Assume now that for all edges, $f_{ij}$ has null derivative in a finite set of points. Then, $f_{ij}$ has finite changes of monotonicity, so the set $f_{ij}^{-1}(y)$ is finite for any $y$. Thus, the set $f^{-1}(y)$ is also finite for any point $y$. Informally, we can expect then that $f^{-1}(\ker d)$ is $m - n + 1$ dimensional, as $\dim(\ker d) = m - n + 1$. Finally, $\mathcal{E}_y = (\ker d)^\bot \cap f^{-1}(\ker d)$, so we expect the set of equilibria in $(1)^\bot$ to be at most $m - n + 1$ dimensional. In subsection 4.2 we will show that for the cycle graph this bound may be reached and we get a continuum of equilibrium points in the node space, $(1)^\bot$.

SM3. Resistors in series and in parallel. The effective resistance as defined in Definition 3.9 is precisely the effective resistance in the electrical circuit sense, where the circuit is given by the network and the resistors by the inverse of the edge weights. Thus, this definition satisfies the well known properties of resistors in series and in parallel, see [SM2]. That is, if
we have a series of nodes $i_1, \ldots, i_k$ such that all paths from $i_1$ to $i_k$ go through $i_1, \ldots, i_k$ in this order, then

$$(\text{SM3.1}) \quad r_{i_1 i_k} = \sum_{j=1}^{k-1} r_{i_j i_{j+1}}.$$ 

For parallel resistors, if we have nodes $i, j$ and a partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ such that all paths from $i$ to $j$ are entirely contained in $\mathcal{E}_1$ or in $\mathcal{E}_2$ then,

$$(\text{SM3.2}) \quad \frac{1}{r_{ij}} = \frac{1}{r_{i,j}^{\mathcal{E}_1}} + \frac{1}{r_{i,j}^{\mathcal{E}_2}},$$

where $r_{i,j}^{\mathcal{E}_1}$ is the effective resistance in the network with only the edges $\mathcal{E}_1$ (resp. for $\mathcal{E}_2$).

The prototypical examples of resistances in series and parallel are depicted in the following diagrams.

Note that the underlying network on the second diagram is a multigraph and not a graph. This is not relevant as effective resistance can be defined similarly for these.

**SM4. Using Lemma 3.13 for stability condition.** We assume that $G^-$ is connected, as by Proposition 3.6 it is a necessary condition to have linear stability. Then, by 3.4 let $z \in \langle 1 \rangle^\perp$ be unitary such that,

$$\lambda_{\max} = z^T L^+ z - z^T L^- z = z^T L^- z \left( \frac{z^T L^+ z}{z^T L^- z} - 1 \right),$$

where we have used that, $L^-$ is positive definite in our space $\langle 1 \rangle^\perp$ (as $G^-$ is connected), so $z^T L^- z > 0$. Thus\(^3\),

$$\text{sign}(\lambda_{\max}) = \text{sign}\left( \frac{z^T L^+ z}{z^T L^- z} - 1 \right) = \text{sign}\left( \max_{x \bot \ker L^+} \frac{x^T L^+ x}{x^T L^- x} - 1 \right) = \text{sign}\left( \max_{x \bot \ker L^+} \frac{x^T (L^-)^\dagger x}{x^T (L^+)^\dagger x} - 1 \right),$$

where in the last equality we use Lemma 3.13. We believe this last expression can lead to stability criteria. For instance, if $i$ is an isolated node in $G^+$ and $x \bot \ker L^+$, then $|x_i| = 0$ (as $\ker L^+$ is generated by the connected components of $G^+$). This gives rise to a different proof of the stability reduction Theorem 3.8. Moreover, $x \bot \ker L^+$ imposes more conditions

\(^3\)As in the main article, all maximums are taken over non-null vectors.
on $x$, coming from the connected components of $G^+$ with more than one node, which could potentially be used to reduce our problem even further.

Another approach is to use the fact that the pseudoinverse of a Laplacian is closely related to the effective resistance, even more than the Laplacian itself. This might lead to stability criteria with effective resistance, as we did for instability in Theorem 3.14.

**SM5. Equilibrium points of the coalescence of two networks.** Let $G$ be the coalescence of the networks $A$ and $B$. Recall that the equilibrium points are given by the equation,

$$\mathcal{E}_y = (\ker d)^\perp \cap f^{-1}(\ker d),$$

for $G$ and by

$$\mathcal{E}_y^A = (\ker d_A)^\perp \cap f_A^{-1}(\ker d_A),$$

for $A$, where the subindex $A$ denotes the respective maps for the network $A$ (similarly for $B$). Now note that any generating cycle of $\ker d$ is entirely contained in $A$ or in $B$ as the only edges from $A$ to $B$ are the ones with $n'$ as a node. Thus,

$$\ker d = \iota_A(\ker d_A) \oplus \iota_B(\ker d_B),$$

where, $\iota_A(y_A) = (y_A, 0_B)$ (resp. for $\iota_B$). Moreover, it is clear that,

$$(\ker d)^\perp = \iota_A((\ker d_A)^\perp) \oplus \iota_B((\ker d_B)^\perp).$$

Now, as $f$ is defined in each component by one variable functions $f_{ij}$, we have $f(y) = (f_A(y_A), f_B(y_B))$, and thus,

$$f^{-1}(\ker d) = \iota_A(f_A^{-1}(\ker d_A)) \oplus \iota_B(f_B^{-1}(\ker d_B)).$$

In conclusion we have,

$$\mathcal{E}_y = \left(\iota_A((\ker d_A)^\perp) \oplus \iota_B((\ker d_B)^\perp)\right) \cap \left(\iota_A(f_A^{-1}(\ker d_A)) \oplus \iota_B(f_B^{-1}(\ker d_B))\right)$$

$$= \iota_A \left((\ker d_A)^\perp \cap f_A^{-1}(\ker d_A)\right) \oplus \iota_B \left((\ker d_B)^\perp \cap f_B^{-1}(\ker d_B)\right)$$

$$= \iota_A(\mathcal{E}_y^A) \oplus \iota_B(\mathcal{E}_y^B).$$

**SM6. Phase portrait of a tree in low dimension.** We study the qualitative dynamics of (#) in a line graph with three nodes where $f_{ij} = \check{f}$ for all edges and all roots of $\check{f}$ are simple. To simplify our study we also assume that the distance between consecutive roots of $\check{f}$ are similar. That is, let $\alpha_0 = 0$, and $\{\alpha_i\}_{i=1}^k$ be the positive roots of $\check{f}$ in increasing order, where $k$ may be infinity. Then, we assume that,

$$(\text{SM6.1}) \quad \max_{0 \leq i < k} (\alpha_{i+1} - \alpha_i) \leq 2 \min_{0 \leq i < k} (\alpha_{i+1} - \alpha_i).$$

If we let 1 be the middle node and 2, 3 the leaf nodes, the system (#) in edge coordinates $y_1 = y_{1,2} = x_2 - x_1$ and $y_2 = y_{1,3} = x_3 - x_1$ is given by,

$$(\text{SM6.2}) \quad \begin{align*}
\dot{y}_1 &= 2\check{f}(y_1) + \check{f}(y_2), \\
\dot{y}_2 &= \check{f}(y_1) + 2\check{f}(y_2),
\end{align*}$$

with $\check{f}(x) = x^2 - x$. This system has the same qualitative behavior as the original system with $\check{f}(x) = x$.
or compactly $\dot{y} = g(y)$. Equation (SM6.2) is very symmetric, concretely if we consider a map that swaps the first and second coordinates, i.e. $p_{1,2}(y_1, y_2) = (y_2, y_1)$, we have,

$$g \circ p_{1,2} = p_{1,2} \circ g, \quad \text{and} \quad g \circ \text{id} = -\text{id} \circ g.$$ 

Thus the flow in $y$ determines the flow in $\pm y$ and $\pm (y_2, y_1)^\top$. In particular, we may restrict the study of the ODE in the region $y_2 \geq |y_1|$. This situation immediately generalises when studying any star graph where we will have more possible permutations of variables. This is very reminiscent of the work done in [SM3] as the symmetry of the star graph restricts the dynamics of the system.

Reparametrising time we can assume $\hat{f}'(0) > 0$. Then, by Proposition 4.1, the equilibrium points of (SM6.2) are of the form $(\pm \alpha_i, \pm \alpha_j)^\top$ and their stability is given by $\hat{f}'(\pm \alpha_i)$ and $\hat{f}'(\pm \alpha_j)$. Note that,

$$\hat{f}'(\pm \alpha_i) = \begin{cases} > 0 & \text{if } i \text{ is even,} \\ < 0 & \text{if } i \text{ is odd,} \end{cases}$$

as all roots of $\hat{f}$ are simple so the signs alternate. We first compute,

$$\begin{align*}
(\dot{y}_1 + \dot{y}_2)_{|y_1+y_2=0} &= 3(\hat{f}(y_1) + \hat{f}(y_2))_{|y_1+y_2=0} = 3(\hat{f}(y_1) + \hat{f}(-y_1)) = 0, \\
(\dot{y}_1 - \dot{y}_2)_{|y_1-y_2=0} &= (\hat{f}(y_1) - \hat{f}(y_2))_{|y_1-y_2=0} = \hat{f}(y_1) - \hat{f}(-y_1) = 0,
\end{align*}$$

so the lines $y_1 = \pm y_2$ are invariant in our system. Moreover for all $i$,

---

**Figure SM1.** Field (SM6.2) in the invariant region $y_2 \geq |y_1|$. Red squares, blue diamonds and magenta exes represent respectively source, sink and saddle equilibrium points. The arrows represent the sense in which the field crosses the different color coded segments.
\[ \dot{y}_1|_{y_1 = \pm \alpha_i} = (2\hat{f}(y_1) + \hat{f}(y_2)) \bigg|_{y_1 = \pm \alpha_i} = \hat{f}(y_2), \]
\[ \dot{y}_2|_{y_2 = \pm \alpha_i} = (\hat{f}(y_1) + 2\hat{f}(y_2)) \bigg|_{y_2 = \pm \alpha_i} = \hat{f}(y_1), \]

so the horizontal and vertical segments between different equilibria are transversal to the field, i.e. orbits always cross them in the same sense, as depicted in black in Figure SM1. Finally,
\[ 2\dot{y}_1 - \dot{y}_2 = 3\hat{f}(y_1), \]
\[ 2\dot{y}_2 - \dot{y}_1 = 3\hat{f}(y_2), \]

and in particular the segments parallel to \(2y_1 = y_2\) (resp. \(2y_2 = y_1\)), that go through the saddle equilibrium points, are transverse to the field for \(y_1\) (resp. \(y_2\)) between roots, see in blue and red in Figure SM1. By (SM6.1), these segments are contained in the regions delimited by the horizontal/vertical lines mentioned above.

It is not hard to see from Figure SM1, that the heteroclinic connections between the saddle and other equilibrium points are the ones depicted in Figure SM2. Moreover, by Proposition SM1.3 we know that the \(\omega\) and \(\alpha\)-limit of any orbit is an equilibrium point or infinity, thus Figure SM2 gives a complete description of the qualitative behaviour of the system.

**SM7. Equilibrium points for complete graphs.** As in the main article we are working in the state space \(\mathbb{R}^n/\{1\}\) and we restrict our study to the equilibria \(\tilde{E_x}\) with \(f_{i,j} = \hat{f}\) for all \(\{i,j\} \in \mathcal{E}\). For each \(\alpha \in \hat{f}^{-1}(0)\), if \(x_i - x_j \in \{0, \pm \alpha\}\) for all \(i \neq j\), then \(x^* \in \tilde{E_x}\). In this case,
we may take the representative of $x^\star$ such that $x^\star_1 = 0$. Then, for all $i$, $x^\star_i = x^\star_i - x^\star_1 \in \{0, \pm \alpha\}$. Note that if $x_i = \alpha$ and $x_j = -\alpha$ for some $i, j$, then $x_i - x_j = \pm 2\alpha$ which is a contradiction. Hence, either $x^\star_i \in \{0, \alpha\}$ for all $i$ or $x^\star_i \in \{0, -\alpha\}$ for all $i$. In either case, choosing appropriate representative and ordering of the nodes,

$$x^\star = (0, \ldots, 0, \alpha, \ldots, \alpha)^\top.$$  

We show that generically these are all elements of $\delta^\alpha_x$. Let $x^\star \in \delta^\alpha_x$, then $x^\star_i - x^\star_j \in \hat{f}^{-1}(0)$ for all $i \neq j$. So if $i, j, k$ are pairwise different, and we denote $x^\star_i - x^\star_j = \alpha_1, x^\star_j - x^\star_k = \alpha_2$ and $x^\star_k - x^\star_i = \alpha_3$ we have $\alpha_1, \alpha_2, \alpha_3 \in \hat{f}^{-1}(0)$ and

$$(SM7.2) \quad \alpha_1 + \alpha_2 + \alpha_3 = (x^\star_i - x^\star_j) + (x^\star_j - x^\star_k) + (x^\star_k - x^\star_i) = 0.$$  

Now assume\(^4\) that $\hat{f}^{-1}(0) \cap (0, \infty)$ is “additive open”, i.e. $a + b \neq c$ for all $a, b, c \in \hat{f}^{-1}(0) \cap (0, \infty)$. Note that in (SM7.2) we may assume without loss of generality that $\alpha_3 \leq 0$. Moreover, (SM7.2) can be rewritten as $\alpha_1 + \alpha_2 = -\alpha_3$ where $\alpha_1, \alpha_2, -\alpha_3 \geq 0$ and as $\hat{f}$ is odd, $-\alpha_3$ is also a root of it. By the additive open condition, this can only be satisfied if $\alpha_l = 0$ for some $l \in \{1, 2, 3\}$. Then, clearly the terms in the left hand of (SM7.2) are $0, \alpha, -\alpha$ for some root $\alpha$ of $\hat{f}$ which in principle could change between different triplets of nodes. Now fix $\alpha$ and $i, j$ two nodes such that $x^\star_i - x^\star_j = \pm \alpha$. Letting $k$ run over all other nodes we conclude that $x^\star_i - x^\star_k \in \{0, \pm \alpha\}$ and $x^\star_j - x^\star_k \in \{0, \pm \alpha\}$. Repeating this process for other choices of $i, j$ we find that $x^\star_l - x^\star_m \in \{0, \pm \alpha\}$ for all $l \neq m$. Thus, by the argument made at the beginning of this section, $x^\star$ can be represented by (SM7.1).

**SM8. Another deduction of the instability criterion in $K_n$.** Let $x^\star$ be as in (SM7.1), denote by $n_0$ the number of 0’s of $x^\star$ and by $n_\alpha = n - n_0$ the number of $\alpha$’s. Then,

$$L^+ = \hat{f}'(0) \begin{pmatrix} n_0 P_{n_0} & 0_{n_0, n_\alpha} \\ 0_{n_0, n_\alpha} & n_\alpha P_{n_\alpha} \end{pmatrix} \quad \text{and} \quad L^- = |\hat{f}'(\alpha)| \begin{pmatrix} n_\alpha I_{n_\alpha} & -1_{n_\alpha} 1_{n_\alpha}^\top \\ -1_{n_\alpha} 1_{n_\alpha}^\top & n_0 I_{n_0} \end{pmatrix},$$  

where $P_k = I_k - 1_k 1_k^\top / k$, i.e. the orthogonal projection onto $\langle 1_k \rangle^\perp$. Note that $L^+$ is the Laplacian of two disconnected complete graphs which we denote by $K^+_0$ and $K^+_\alpha$. Also note that $L^-$ is the Laplacian of a complete bipartite graph $K_{n_\alpha, n_\alpha}$. The effective resistances between nodes in these graphs are well known and can be deduced from [SM4, Corollary C2]. In our case we have,

$$r^+_{ij} = \begin{cases} 2 f'(0)/n_0 & \text{if } i, j \in K^+_0, \\ 2 f'(\alpha)/n_\alpha & \text{if } i, j \in K^+_\alpha, \\ \infty & \text{else,} \end{cases} \quad \text{and} \quad r^-_{ij} = \begin{cases} 2 |\hat{f}'(\alpha)|/n_\alpha & \text{if } i, j \in K^+_0, \\ 2 f'(\alpha)/n_\alpha & \text{if } i, j \in K^+_\alpha, \\ \neq \infty & \text{else.} \end{cases}$$  

By Theorem 3.14, we have instability if $r^-_{ij} > r^+_{ij}$ for some edge $\{i, j\}$, which in or context reduces to,

$$n_0 < \frac{n_\alpha}{|\hat{f}'(\alpha)|} \quad \text{or} \quad n_0 > \frac{|\hat{f}'(\alpha)|}{f'(0)}.$$

\(^4\)This condition is generically satisfied as its negation implies a specific non-trivial linear relation between some roots of $f$. 


or equivalently,

\[(SM8.1)\]

\[
\frac{n_0}{n_0} \notin \left[ \frac{\hat{f}'(0)}{|\hat{f}'(\alpha)|}, \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0)} \right].
\]

Now we need the following elementary relation between inequalities. Given positive numbers \(a, b, c, d > 0\), then

\[
\frac{a}{b} > \frac{c}{d} \iff \frac{b}{a} < \frac{d}{c} \iff \frac{b}{a} + 1 < \frac{d}{c} + 1 \iff \frac{a + b}{c} < \frac{d}{c} + 1 \iff \frac{a}{a + b} > \frac{c}{c + d},
\]

and similarly,

\[
\frac{a}{b} < \frac{c}{d} \iff \frac{a}{a + b} < \frac{c}{c + d}.
\]

Thus, condition \((SM8.1)\) is equivalent to,

\[
\frac{n_0}{n_0 + n_\alpha} = \frac{n_0}{n} \notin \left[ \frac{\hat{f}'(0)}{\hat{f}'(0) + |\hat{f}'(\alpha)|}, \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0) + |\hat{f}'(\alpha)|} \right],
\]

which is precisely the one we got in Proposition 4.5 by computing the eigenvalues, and hence it is also tight.

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