Stability of some essential B-spectra of pencil operators and application

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Abstract: In this paper, we give some results on the essential B-spectra of a linear operator pencil, which are used to determine the essential B-spectra of an integro-differential operator with abstract boundary conditions in the Banach space $L_p([-a,a] \times [-1,1])$, $p \geq 1$ and $a > 0$.

Key words: Operator pencil, finite-rank and power finite-rank perturbations, essential B-spectra, transport operator.

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1. Introduction

Let $X$ be a Banach space. We will denote by $C(X)$ (resp. $L(X)$) the set of all closed linear (resp. the algebra of all bounded) linear operators from $X$ into $X$. For $T \in C(X)$, we write $D(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset X$ for the range of $T$. We denote by $\alpha(T)$ the dimension of $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in $X$. For $T \in C(X)$ and $M \in L(X)$, we define the resolvent of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, or the $M$-resolvent of $T$ by

$$\rho_M(T) := \{ \lambda \in \mathbb{C} : \lambda M - T \text{ has a bounded inverse} \},$$

and its spectrum by

$$\sigma(M,T) = \mathbb{C} \setminus \rho_M(T).$$

For $T \in C(X)$, we define the set

$$\triangle(T) = \{ n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow R(T^n) \cap N(T) \subset R(T^m) \cap N(T) \}.$$ 

The degree of stable iteration of $T$ is defined as $\text{dis}(T) = \inf \triangle(T)$, where $\text{dis}(T) = \infty$ if $\triangle(T) = \emptyset$. 

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We define the set of upper semi-Fredholm operators by
\[ \Phi^+(X) = \{ T \in \mathcal{C}(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } X \} , \]
and the set of lower semi-Fredholm operators by
\[ \Phi^-(X) = \{ T \in \mathcal{C}(X) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X \} . \]
\[ \Phi(X) := \Phi^+(X) \cap \Phi^-(X) \]
will denote the set of Fredholm operators from \( X \) into \( X \). The index of a Fredholm operator \( T \) is defined by \( \text{ind}(T) = \alpha(T) - \beta(T) \).

According to [17], an operator \( T \in \mathcal{C}(X) \) is called quasi-Fredholm of degree \( d \in \mathbb{N} \) if the following three conditions are satisfied:
(i) \( \text{dis}(T) = d \);
(ii) \( R(T^d) \cap N(T) \) is a closed and complemented subspace of \( X \);
(iii) \( R(T) + N(T^d) \) is a closed and complemented subspace of \( X \).
This set of operators will be denoted by \( QF(d) \).

Following [10, Definition 2.4], an operator \( T \in \mathcal{C}(X) \) is called upper semi B-Fredholm (resp. lower semi B-Fredholm) if there exists an integer \( d \in \mathbb{N} \) such that \( T \in QF(d) \) and such that \( N(T) \cap R(T^d) \) is of finite dimension (resp. \( R(T) + N(T^d) \) is of finite codimension). These sets are denoted respectively by \( \Phi^+_B(X) \) and \( \Phi^-_B(X) \). We denote by \( \Phi_B(X) := \Phi^+_B(X) \cap \Phi^-_B(X) \), the set of B-Fredholm operators from \( X \) into \( X \). In this case, the index of \( T \) is defined as the integer: \( \text{ind}(T) = \dim(N(T) \cap R(T^d)) - \text{codim}(R(T) + N(T^d)) \). An operator \( T \in \mathcal{C}(X) \) is called B-Weyl if it is a B-Fredholm operator of index zero. We will denote this set by \( BW(X) \).

For \( T \in \mathcal{C}(X) \), we define the ascent \( a(T) \) of \( T \) by
\[ a(T) = \inf \{ n \in \mathbb{N} : N(T^n) = N(T^{n+1}) \} , \]
and the descent \( d(T) \) of \( T \) by
\[ d(T) = \inf \{ n \in \mathbb{N} : R(T^n) = R(T^{n+1}) \} . \]
We define respectively the set of Drazin invertible operators, left Drazin invertible operators and right Drazin invertible operators as follows:
\[ DR(X) := \{ T \in \mathcal{C}(X) : a(T) \text{ and } d(T) \text{ are both finite} \} , \]
\[ LD(X) := \{ T \in \mathcal{C}(X) : a(T) \text{ is finite and } R(T^{a(T)+1}) \text{ is closed} \} , \]
\[ RD(X) := \{ T \in \mathcal{C}(X) : d(T) \text{ is finite and } R(T^{d(T)}) \text{ is closed} \} . \]
An operator \( T \in \mathcal{C}(X) \) is of Kato type if there exist an integer \( d \in \mathbb{N} \) and a pair of two closed subspaces \( (N_1, N_2) \) of \( X \) such that:
(i) \( X = N_1 \oplus N_2 \);
(ii) \( T(N_1) \subset N_1 \) and \( T_{/N_1} \) is semi-regular;
(iii) \( T(N_2) \subset N_2 \) and \( (T_{/N_2})^d = 0 \), i.e., \( T_{/N_2} \) is nilpotent.

Note that, in the case of Hilbert spaces, Labrousse in [17, Theorem 3.2.2] has shown that, the set of Quasi-Fredholm operators coincides with the Kato type operators for the class of closed operators. According to [17, p. 206, Remark], this equivalence is also true in the case of Banach spaces.

For \( T \in \mathcal{C}(X) \) and \( M \in \mathcal{L}(X) \), we define the B-Fredholm spectrum, Drazin spectrum, the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum, the left Drazin spectrum, the right Drazin spectrum, the B-Weyl spectrum, the closed-range spectrum and the Kato spectrum of the linear operator pencil \( \lambda M - T \), where \( \lambda \in \mathbb{C} \), or the pair \((M, T)\) as follows:

\[
\sigma_{BF}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin \Phi_B(X) \}, \\
\sigma_D(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin DR(X) \}, \\
\sigma_{BF+}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin \Phi_{BF}^+(X) \}, \\
\sigma_{BF-}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin \Phi_{BF}^-(X) \}, \\
\sigma_{LD}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin LD(X) \}, \\
\sigma_{RD}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin RD(X) \}, \\
\sigma_{BW}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \notin BW(X) \}, \\
\sigma_{ec}(M, T) = \{ \lambda \in \mathbb{C} : R(\lambda M - T) \text{ is not closed} \}, \\
\sigma_{ek}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \text{ is not Kato} \}.
\]

The point spectrum, the residual spectrum and the continuous spectrum of the pair \((M, T)\), when \( T \in \mathcal{C}(X) \) and \( M \in \mathcal{L}(X) \), are defined respectively by:

\[
\sigma_p(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \text{ is not injective} \}, \\
\sigma_r(M, T) = \{ \lambda \in \mathbb{C} : N(\lambda M - T) = \{0\} \text{ and } R(\lambda M - T) \subsetneq X \}, \\
\sigma_c(M, T) = \{ \lambda \in \mathbb{C} : N(\lambda M - T) = \{0\}, \ R(\lambda M - T) = X \text{ and } R(\lambda M - T) \neq X \},
\]

where, \( R(\lambda M - T) \) is the closure of \( R(\lambda M - T) \). The collection \( \{\sigma_p(M, T), \sigma_r(M, T), \sigma_c(M, T)\} \) forms a partition of the spectrum \( \sigma(M, T) \), which means that they are pairwise disjoint and \( \sigma(M, T) = \sigma_p(M, T) \cup \sigma_r(M, T) \cup \sigma_c(M, T) \).
For $T \in \mathcal{C}(X)$ and $M \in \mathcal{L}(X)$, the upper semi B-Fredholm, the lower semi B-Fredholm and the B-Fredholm resolvent of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, are defined respectively by

$$
\rho_{BF^+}(M, T) = \mathbb{C}\setminus \sigma_{BF^+}(M, T),
\rho_{BF^-}(M, T) = \mathbb{C}\setminus \sigma_{BF^-}(M, T),
\rho_{BF}(M, T) = \rho_{BF^+}(M, T) \cap \rho_{BF^-}(M, T).
$$

The present paper is a generalization of the results obtained by A. Jeribi et al. in [15] and some of stability results obtained by M. Berkani et al. in [6] for the usual essential B-spectra. It generalizes also the works obtained by A. Jeribi in [13, 14] about the invariance of the S-essential spectra under weakly compact or strictly singular perturbations, which are not applied in the B-Fredholm theory. So that, we can use, by adding some hypothesis, the perturbations of the B-Fredholm spectra under finite-rank and power finite-rank commuting operators. More precisely, let $T_1, T_2 \in \mathcal{C}(X)$ be two commuting closed linear operators such that the bounded linear operator $M$ commutes in the resolvent sense with $T_1$ and $T_2$ (see Definition 2.2) and satisfying $(\lambda M - T_1)^{-1} - (\lambda M - T_2)^{-1} \in F(X)$ (resp. $(\lambda M - T_1)^{-1} - (\lambda M - T_2)^{-1} \in F_p(X)$ or nilpotent) for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$. Then, we prove that

$$
\sigma^*_s(M, T_1) = \sigma^*_s(M, T_2),
$$

where

$$
\sigma^*_s(M, .) \in \{ \sigma_{BF}(M, .), \sigma_{BF^+}(M, .), \sigma_{BF^-}(M, .), \sigma_{BW}(M, .), \sigma_{LD}(M, .), \sigma_{RD}(M, .), \sigma_D(M, .) \}.
$$

These perturbation results are needed to extend the results obtained in [13] on the S-essential spectra of closed densely defined linear operators to essential B-spectra of operator pencil $\lambda M - T$, when $T \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. Moreover, under the additional hypothesis $M[C(T)] = C(T)$ (see Definition 2.1), we give the relationship between the closed-range spectrum and the Kato spectrum of the linear operator pencil $\lambda M - T$, when $T \in \mathcal{C}(X)$, $M \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$ (see Proposition 2.4), which generalizes a result obtained in [9, Proposition 3.2] in the case of bounded operators and [8, Corollary 3] for closed densely defined linear operators for the usual spectrum. We establish also, the equality between the closed-range spectrum and some essential B-spectra of operator pencil acting on a Banach space (Theorem 2.5). The obtained results, are finally used to describe the essential B-spectra of the operator pencil of the following integro-differential operator with abstract boundary conditions in the Banach space $X_p := L_p([-a, a] \times [-1, 1], dx dy)$, $a > 0, 1 \leq p < \infty$,

$$
A_H = T_H + K,
$$
where $T_H$, $K$ and $M$ are defined by

$$
\begin{align*}
T_H : D(T_H) \subseteq X_p & \longrightarrow X_p \\
\psi & \mapsto -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi)\psi(x, \xi),
\end{align*}
$$

$$
D(T_H) = \{ \psi \in W_p : \psi^i = H \psi^0 \},
$$

where $W_p := \{ \psi \in X_p : \xi \frac{\partial \psi}{\partial x} \in X_p \}$,

$$
\begin{align*}
K : X_p & \longrightarrow X_p \\
u & \mapsto \int_{-1}^{1} k(x, \xi, \nu) u(x, \nu) d\nu,
\end{align*}
$$

and

$$
\begin{align*}
M : X_p & \longrightarrow X_p \\
\varphi & \mapsto M(\varphi)(x, \xi) = \eta(\xi)\varphi(x, \xi),
\end{align*}
$$

where $\sigma(.)$ and $\eta(.)$ are in $L^\infty(-1, 1)$, $k(.,.,.)$ is a measurable function, and $H$ is the boundary operator connecting the outgoing and the incoming fluxes.

The outline of this work is organized in the following way: in Section 2, we give some stability results of some essential B-spectra of linear operator pencil. The main results of this section are Theorem 2.3 and Theorem 2.5. In Section 3, we apply the results developed in Section 2 to characterize the B-essential spectra of a transport operator with abstract boundary conditions on $L_p$-spaces, $1 \leq p < \infty$.

2. Stability of some essential B-spectra of pencil operators

We are interested, in this section, in some of stability results of the essential B-spectra of an operator pencil $\lambda M - T$, where $M \in L(X)$, $T \in C(X)$ and $\lambda \in \mathbb{C}$.

Since, the Kato decomposition theorem, remains true in the case of Banach spaces as shown in [17, p. 206], we can directly use the following proposition inspired from [10], when necessary, in the case of Banach spaces without proof.

**Proposition 2.1.** Let $T \in C(X)$. If $T$ is a semi B-Fredholm operator, then there exist two closed subspaces $X_0$ and $X_1$ of $X$ such that

(i) $X = X_0 \oplus X_1$,

(ii) $T(X_0) \subset X_0$ and $T_0 = T|_{X_0}$ is a semi-Fredholm operator,

(iii) $T(X_1) \subset X_1$ and $T_1 = T|_{X_1}$ is a nilpotent operator.
First we recall the following subspace, introduced by P. Saphar in [20], and it was defined by P. Aiena in [1] in purely algebraic terms.

**Definition 2.1.** The algebraic core \( C(T) \) of a linear operator \( T \) is defined to be the greatest subspace \( N \subset D(T) \) for which \( T(N) = N \). For more details for the algebraic core \( C(T) \), we can refer to [1].

**Theorem 2.1.** Let \( T \in C(X) \) and \( M \in L(X) \) such that \( M(C(T)) = C(T) \) and \( \rho_M(T) \neq 0 \). If \( T \) is a semi B-Fredholm operator, then there exists \( \varepsilon > 0 \) such that \( T - \mu M \) is a semi-Fredholm operator, for each \( \mu \in D(0, \varepsilon) \setminus \{0\} \). Moreover, we have \( \alpha(T - \mu M) \) and \( \beta(T - \mu M) \) are constants on \( D(0, \varepsilon) \setminus \{0\} \).

**Proof.** If \( M = I \), then we obtain the result established in [7].

If \( M \neq I \), then the fact that \( T \) is a semi B-Fredholm operator, this implies from Proposition 2.1 the existence of two \( T \)-invariant closed subspaces \( X_0 \) and \( X_1 \) such that

- \( X = X_0 \oplus X_1 \),
- \( T(X_0) \subset X_0 \) and \( T_0 = T/X_0 \) is a semi-Fredholm operator,
- \( T(X_1) \subset X_1 \) and \( T_1 = T/X_1 \) is nilpotent.

Since \( M(C(T)) = C(T) \) and \( T(C(T)) = C(T) \), then we can conclude, by using the definition of \( C(T) \), that \( X_0 \) and \( X_1 \) are invariants under the operator \( M \). So, we can consider \( M_0 = M/X_0 \) and \( M_1 = M/X_1 \) such that \( M = M_0 \oplus M_1 \).

Case 1: If \( X_0 = \{0\} \), then we get \( M_0 = 0 \), \( T_0 = 0 \), \( M = M_1 \) and \( T = T_1 \) is nilpotent. Since, the operator \( T_1 \) is nilpotent then we have \( T_1 - \mu I_1 \) is invertible, for \( \mu \neq 0 \), where \( I_1 = I/X_1 \). We have

\[
T_1 - \mu M_1 = (T_1 - \mu I_1)(I_1 - \mu(T_1 - \mu I_1)^{-1}(M_1 - I_1))
\]

is invertible for \( \mu \) such that

\[
0 < |\mu| < \frac{1}{\|T_1 - \mu I_1\|^{-1}(M_1 - I_1)} = \gamma.
\]

Hence, \( T - \mu M = T_1 - \mu M_1 \) is a semi-Fredholm operator, for \( \mu \) such that \( 0 < |\mu| < \gamma \). Moreover, we have \( \alpha(T - \mu M) = \alpha(T_1 - \mu M_1) = 0 \) and \( \beta(T - \mu M) = \beta(T_1 - \mu M_1) = 0 \) on \( D(0, \gamma) \setminus \{0\} \).

Case 2: Suppose \( X_0 \neq \{0\} \).

- If \( M_0 = 0 \), then by using Case 1, we obtain that \( T - \mu M = T_0 \oplus T_1 - \mu M_1 \) is a semi-Fredholm operator for \( \mu \) such that \( 0 < |\mu| < \gamma \).
- If \( M_0 \neq 0 \), then \( T - \mu M = T_0 - \mu M_0 \oplus T_1 - \mu M_1 \). It follows from [19, Theorem 7.9], that \( T_0 - \mu M_0 \) is a semi-Fredholm for \( \mu \) such that
\[ \mu \in D(0, \frac{\varepsilon'}{\|M_0\|}) \setminus \{0\}, \text{ for some } \varepsilon' > 0. \text{ Set } \varepsilon = \min \left( \frac{\varepsilon'}{\|M_0\|}, \gamma \right). \]

Therefore, the operator \( T - \mu M \) is semi-Fredholm for \( \mu \) such that \( \mu \in D(0, \varepsilon) \setminus \{0\} \).

Again, from [19, Theorem 7.9], we get \( \alpha(T - \mu M) = \alpha(T_0 - \mu M_0) \) and \( \beta(T - \mu M) = \beta(T_0 - \mu M_0) \) are constants on \( D(0, \varepsilon) \setminus \{0\} \).

Using Theorem 2.1, we can deduce the following result which is a generalization of [15, Proposition 2.1]

**Corollary 2.1.** Let \( T \in \mathcal{C}(X) \) and \( M \in \mathcal{L}(X) \) such that \( M(C(T)) = C(T) \) and \( \rho_M(T) \neq \emptyset \). Then,

1. \( \rho_{BF^+}(M, T), \rho_{BF^+}(M, T) \) and \( \rho_{BF}(M, T) \) are open subsets of \( \mathbb{C} \);
2. \( \text{ind}(\lambda M - T) \) is constant on any component of \( \rho_{BF^+}(M, T), \rho_{BF^+}(M, T) \) and \( \rho_{BF}(M, T) \).

**Proof.** (i) Let \( \lambda_0 \in \rho_{BF^+}(M, T) \), then from Theorem 2.1 there exists an \( \varepsilon > 0 \) such that \( T - \mu M \) is an upper semi-Fredholm operator, for each \( \mu \in D(\lambda_0, \varepsilon) \setminus \{0\} \). This implies that \( \rho_{BF^+}(M, T) \) is an open subset of \( \mathbb{C} \).

The same proof is used to show that, \( \rho_{BF^+}(M, T) \) and \( \rho_{BF}(M, T) \) are open subsets of \( \mathbb{C} \).

(ii) Let \( \Omega \) be a component of \( \rho_{BF^+}(M, T) \) (resp. \( \rho_{BF^+}(M, T) \)), \( \lambda_0 \in \Omega \) be a fixed point and \( \lambda_1 \in \Omega \) be an arbitrary point that are connected by a polygonal line \( \Gamma \) contained in \( \Omega \). It follows from the assertion (i) of this corollary that, for each \( \mu \in \Gamma \), there exists an open disc \( D(\mu, \varepsilon) \), such that \( \text{ind}(\mu M - T) = \text{ind}(\lambda M - T) \), for each \( \lambda \in D(\mu, \varepsilon) \). By the Heine-Borel theorem, there exist a finite number of open discs that cover \( \Gamma \). This allows us to deduce that \( \text{ind}(\lambda_0 M - T) = \text{ind}(\lambda_1 M - T) \).

Now, we recall the following definition considered in [12] for bounded linear operators and it remains also true in the general case of closed linear operators:

**Definition 2.2.** Let \( M \in \mathcal{L}(X) \) and \( T \in \mathcal{C}(X) \) such that \( \rho_M(T) \neq \emptyset \). We say that \( M \) and \( T \) commute in the sense of resolvent if for all \( \lambda \in \rho_M(T) \),

\[ M(T - \lambda M)^{-1} = (T - \lambda M)^{-1} M. \]

**Remarks 2.1.** (a) If \( M \) and \( T \) commute in the sense of the resolvent, the assumption \( M(C(T)) \subset C(T) \) is verified. Indeed, let \( x \in C(T) \). Then, from [11, Theorem 1.8], there exists a sequence \( (u_n) \subset D(T) \) such that \( x = u_0 \) and \( Tu_{n+1} = u_n \), for every \( n \in \mathbb{Z}_+ \). Set \( y_n = Mu_n \). The commutativity
of the resolvent of the operators $M$ and $T$, permits us to deduce that $M = (\lambda M - T)^{-1}M(\lambda M - T)$ on $D(T)$ and $TM = MT$ on $D(T)$, which entails that $Mu_n \in D(T)$. Thus, we get $y_0 = Mu_0 = Mx$ and $Ty_{n+1} = TMu_{n+1} = M(Tu_{n+1} = y_n)$, for every $n \in \mathbb{Z}_+$. This, implies that $Mx \in M(C(T))$ and finally we obtain that $M(C(T)) \subset C(T)$.

(b) If $M$ and $T$ commute in the sense of the resolvent and $M$ is invertible, then we get $M^{-1}(C(T)) \subset C(T)$ and finally we can conclude that $C(T) \subset M(C(T))$.

**Proposition 2.2.** Let $M \in L(X)$ be an invertible operator and $T \in C(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then

(i) for $\lambda \neq 0$ and $n \geq 1$: $(T - \lambda M)^n(T^{-1})^n = (-\lambda)^nM^n(T^{-1} - \lambda^{-1}M^{-1})^n$;

(ii) for all $n \geq 1$: $R((T - \lambda M)^n) = R((T^{-1} - \lambda^{-1}M^{-1})^n)$.

**Proof.** (i) For $n = 1$, the equality is obvious. Let $n \geq 1$ and assume that $(T - \lambda M)^n(T^{-1})^n = (-\lambda)^nM^n(T^{-1} - \lambda^{-1}M^{-1})^n$, then

$$(T - \lambda M)^{n+1}(T^{-1})^{n+1} = (T - \lambda M)[(T - \lambda M)^n(T^{-1})^n]T^{-1}$$

$$= (T - \lambda M)[(-\lambda)^nM^n(T^{-1} - \lambda M^{-1})^n]T^{-1}$$

$$= (T - \lambda M)T^{-1}[(-\lambda)^nM^n(T^{-1} - \lambda^{-1}M^{-1})^n]$$

$$= (-\lambda)^{n+1}M^{n+1}(T^{-1} - \lambda^{-1}M^{-1})^{n+1}.$$ 

(ii) It follows from (i), that

$$R[M^n(T^{-1} - \lambda^{-1}M^{-1})^n] = R[(T^{-1} - \lambda^{-1}M^{-1})^n] \subseteq R[(T - \lambda M)^n].$$

Conversely, if $y \in R((T - \lambda M)^n)$, then there exists $x \in D(T^n)$ such that $y = (T - \lambda M)^nx$. The fact that $(T^{-1})^n(X) = D(T^n)$, this enable us the existence of $t \in X$ such that $x = (T^{-1})^n(t)$. Hence, $y = (T - \lambda M)^n(T^{-1})^n(t)$ and finally $y = (-\lambda)^nM^n(T^{-1} - \lambda^{-1}M^{-1})^n(t) \in R[(T^{-1} - \lambda^{-1}M^{-1})^n].$

**Proposition 2.3.** Let $M \in L(X)$ be an invertible operator and $T \in C(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then

(i) for $\lambda \neq 0$ and $n \geq 1$: $(T^{-1})^n(T - \lambda M)^n = (-\lambda)^nM^n(T^{-1} - \lambda^{-1}M^{-1})^n$ and $T^n(T^{-1} - \lambda^{-1}M^{-1})^n = (-\lambda)^n(M^{-1})^n(T - \lambda M)^n$ on $D(T^n)$;

(ii) for all $n \geq 1$: $N((T - \lambda M)^n) = N((T^{-1} - \lambda^{-1}M^{-1})^n).$
Proof. (i) Since $T$ is invertible and $D(T - \lambda M) = D(T)$, then $R((T^{-1})^n) = D(T^n)$ and $R(T^n) = X$. Therefore, the operators $(T^{-1})^n(T - \lambda M)^n$ and $T^n(T^{-1} - \lambda^{-1}M^{-1})^n$ are well defined. Then, we verify directly that $(T^{-1})^n(T - \lambda M)^n = (-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n$ and $T^n(T^{-1} - \lambda^{-1}M^{-1})^n = (-\lambda^{-1})^n (M^{-1})^n (T - \lambda M)^n$ on $D(T^n)$.

(ii) It is a direct consequence of (i). □

Remark 2.2. If $M = I$, we recover the results obtained in [10].

Now, we state the main result of this section.

Theorem 2.2. Let $M \in L(X)$ be an invertible operator and $T \in C(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then

$$\sigma_*(M, T) = \{\lambda^{-1} : \lambda \in \sigma_*(M^{-1}, T^{-1}) \setminus \{0\}\},$$

where

$$\sigma_*(M, T) \in \{\sigma_{BF+}(M, T), \sigma_{BF-}(M, T), \sigma_{BF}(M, T), \sigma_D(M, T), \sigma_{BW}(M, T), \sigma_{LD}(M, T), \sigma_{RD}(M, T)\}.$$
then
\[ \sigma_*(M, T_1) = \sigma_*(M, T_2), \]
where, \( \sigma_*(M, .) \in \{ \sigma_{BF}(M, .), \sigma_{BF^+}(M, .), \sigma_{BF^-}(M, .), \sigma_{BW}(M, .) \} \).

**Proof.** Without loss of generality, we can assume that \( \lambda = 0 \), then \( T_1^{-1} - T_2^{-1} \in \mathcal{F}(X) \). Let \( \mu \neq 0 \). The use of Theorem 2.2 shows that, \( \mu M - T_1 \) is a B-Fredholm operator if and only if \( \mu^{-1}M^{-1} - T_1^{-1} \) is a B-Fredholm one. Since \( T_1^{-1} - T_2^{-1} \in \mathcal{F}(X) \), then by using [3, Corollary 3.10], we get \( \mu M - T_2 \) is a B-Fredholm operator. Again by Theorem 2.2, this is equivalent to \( \mu M - T_2 \) is also a B-Fredholm operator, which finish the proof of the B-Fredholm spectrum equality. For the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum and the B-Weyl spectrum, we use the same technique as above and [11, Proposition 2.7].

**Definition 2.3.** ([18]) Let \( X \) be a Banach space, \( A : D(A) \subset X \to X \) and \( T : D(T) \subset X \to X \) two linear operators. We say that \( A \) commutes with \( T \), and we denote \( AT = TA \), if

\begin{enumerate}
  \item \( D(A) \subset D(T) \);
  \item \( Tx \in D(A) \) whenever \( x \in D(A) \);
  \item \( AT = TA \) on \( \{ x \in D(A) : Ax \in D(T) \} \).
\end{enumerate}

Under the additional hypothesis of commutativity of operators, we get a stronger version of Theorem 2.3.

**Theorem 2.4.** Let \( M \in L(X) \) be an invertible operator and \( T_1, T_2 \in C(X) \) such that \( M \) commutes with \( T_1 \) and \( T_2 \) in the resolvent sense and \( T_1 T_2 = T_2 T_1 \). If for some \( \lambda \in \rho_M(T_1) \cap \rho_M(T_2) \), the operator \( (T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1} \in \mathcal{F}_p(X) \), then
\[ \sigma_*(M, T_1) = \sigma_*(M, T_2), \]
where
\[ \sigma_*(M, T) \in \{ \sigma_{BF}(M, .), \sigma_{BF^+}(M, .), \sigma_{BF^-}(M, .), \sigma_{BW}(M, .), \sigma_{LD}(M, .), \sigma_{RD}(M, .), \sigma_{D}(M, .) \}. \]

**Proof.** Without loss of generality, we can assume that \( \lambda = 0 \), then \( T_1^{-1} - T_2^{-1} \in \mathcal{F}_p(X) \). Let \( \mu \neq 0 \). It follows from Theorem 2.2 that \( \mu M - T_1 \) is
a B-Fredholm operator if and only if \( \mu^{-1}M^{-1} - T_1^{-1} \) is a B-Fredholm one. Since, \( T_1^{-1} - T_2^{-1} \in \mathcal{F}_p(X) \), then from [10], we obtain that \( \mu^{-1}M^{-1} - T_2^{-1} \) is also a B-Fredholm operator, which is equivalent to \( \mu M - T_1 \) is a B-Fredholm operator by Theorem 2.2. This shows that \( \sigma_{BF}(M, T_1) = \sigma_{BF}(M, T_2) \). For the other equalities, we use the same technique as above.

**Corollary 2.2.** Let \( M \in L(X) \) be an invertible operator and \( T_1, T_2 \in C(X) \) such that \( M \) commutes with \( T_1 \) and \( T_2 \) in the resolvent sense and \( T_1 T_2 = T_2 T_1 \). If for some \( \lambda \in \rho_M(T_1) \cap \rho_M(T_2) \), the operator \( (T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1} \) is nilpotent, then

\[
\sigma_*(M, T_1) = \sigma_*(M, T_2),
\]

where

\[
\sigma_* \in \{ \sigma_{BF}(M, T), \sigma_{BF^+}(M, T), \sigma_{BF^-}(M, T),
\sigma_{BW}(M, T), \sigma_{LD}(M, T), \sigma_{RD}(M, T), \sigma_{D}(M, T) \}.
\]

**Corollary 2.3.** Let \( T \in C(X), M \in L(X) \) be an invertible operator and \( Q \in L(X) \) a nilpotent operator such that \( \rho_M(T) \neq \emptyset \). If \( TQ = QT \) on \( D(T) \), \( MQ = QM \) and \( M, T \) commute in the resolvent sense, then

\[
\sigma_*(M, T) = \sigma_*(M, T + Q),
\]

where

\[
\sigma_* \in \{ \sigma_{BF}(M, T), \sigma_{BF^+}(M, T), \sigma_{BF^-}(M, T),
\sigma_{BW}(M, T), \sigma_{LD}(M, T), \sigma_{RD}(M, T), \sigma_{D}(M, T) \}.
\]

**Proof.** Since, \( TQ = QT \) and \( Q \) is a nilpotent operator, then \((\lambda M - T)^{-1}Q\) is nilpotent, for all \( \lambda \in \rho_M(T) \). Thus, its spectral radius is equal to zero, which implies that \( \lambda \in \rho_M(T + Q) \) and that

\[
(\lambda M - T - Q)^{-1} = (\lambda M - T)^{-1}(M - (\lambda I - T)^{-1}Q)^{-1}
\]

\[
= (\lambda M - T)^{-1} \sum_{k=0}^{n} ((\lambda M - T)^{-1}Q)^k
\]

\[
= (\lambda M - T)^{-1} + (\lambda M - T)^{-1}Q \sum_{k=1}^{n-1} ((\lambda M - T)^{-1})^k Q^{k-1}
\]
with \( n \) is the nilpotent-index of \((\lambda M - T)^{-1}Q\). Hence, \((\lambda M - T - Q)^{-1} - (\lambda M - T)^{-1}\) is nilpotent. So, we deduce from Corollary 2.2 that \( \sigma_s(M, T + Q) = \sigma_s(M, T) \).

The following proposition is proved in [12] for bounded linear operators and it holds also true in the general case of closed densely-defined linear operators.

**Proposition 2.4.** Let \( T \in \mathcal{C}(X) \) be densely-defined linear operator and \( M \in L(X) \) such that \( M(C(T)) = C(T) \). If \( \lambda \in \sigma_{ec}(M, T) \) is non-isolated, then \( \lambda \in \sigma_{ek}(M, T) \).

**Remark 2.3.** Proposition 2.4 is also true if we replace \( \sigma_{ek}(M, T) \) by \( \sigma_{qf}(M, T) \), where \( \sigma_{qf}(M, T) = \{ \lambda \in \mathbb{C} : \lambda M - T \text{ is not a Quazi-Fredholm operator} \} \).

The following theorem, shows the equality between the closed-range spectrum and some essential B-spectra of operator pencil acting on the Banach space.

**Theorem 2.5.** Let \( T \in \mathcal{C}(X) \) be densely-defined linear operator and \( M \in L(X) \) such that \( M(C(T)) = C(T) \). If \( \sigma_{ec}(M, T) = \sigma(M, T) \) and every \( \lambda \) is non-isolated. Then

\[
\sigma(M, T) = \sigma_{BF}(M, T) = \sigma_{BW}(M, T) = \sigma_{BF^+}(M, T) = \sigma_{BF^-}(M, T) = \sigma_{D}(M, T).
\]

**Proof.** Since \( \sigma_{BF}(M, T) \subset \sigma(M, T) \), it suffices to show that \( \sigma(M, T) \subset \sigma_{ek}(M, T) \). Let \( \lambda \in \sigma(M, T) \), then from Proposition 2.3, we have \( \lambda \in \sigma_{ek}(M, T) \). Since, a B-Fredholm operator is a Quasi-Fredholm one, this shows that \( \sigma_{ek}(M, T) \subset \sigma_{BF}(M, T) \). The same arguments are used for the upper semi B-Fredholm, the lower semi B-Fredholm and the B-Weyl spectrum. The fact that, a Drazin invertible operator is a B-Fredholm one, then by using the same arguments as above we can prove that, \( \sigma(M, T) = \sigma_{D}(M, T) \). Finally, we conclude that \( \sigma(M, T) = \sigma_{ec}(M, T) = \sigma_{BF}(M, T) = \sigma_{BW}(M, T) = \sigma_{BF^+}(M, T) = \sigma_{BF^-}(M, T) = \sigma_{D}(M, T) \).

3. Application

In this section, we will use the previous results to treat the essential B-spectra of a transport operator with abstract boundary conditions. Let

\[
X_p := L_p((a, a) \times (-1, 1), dx d\xi), \quad a > 0, \quad 1 \leq p < \infty.
\]
We consider the following integro-differential operator with abstract boundary conditions:

\[ A_H = T_H + K, \]

where \( T_H \) is defined by

\[
\begin{align*}
T_H : D(T_H) \subseteq X_p & \rightarrow X_p \\
\psi & \mapsto T_H \psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi) \psi(x, \xi), \\
D(T_H) & = \{ \psi \in W_p : \psi^i = H\psi^o \},
\end{align*}
\]

where \( W_p := \{ \varphi \in X_p : \xi \frac{\partial \varphi}{\partial x} \in X_p \} \) and \( \sigma(.) \in L^\infty(-1,1) \); \( \psi^o, \psi^i \) are, respectively, the outgoing and the incoming fluxes related by the boundary operator \( H \) (“o” for the outgoing and “i” for the incoming), and given by

\[
\begin{align*}
\psi^i(\xi) & = \psi(-a, \xi), & \xi & \in (0, 1), \\
\psi^o(\xi) & = \psi(a, \xi), & \xi & \in (-1, 0), \\
\psi^o(\xi) & = \psi(-a, \xi), & \xi & \in (-1, 0), \\
\psi^o(\xi) & = \psi(a, \xi), & \xi & \in (0, 1).
\end{align*}
\]

The bounded collision operator \( K \) is defined by

\[
\begin{align*}
K : X_p & \rightarrow X_p \\
u & \mapsto K(u)(x, \xi) = \int_{-1}^{1} k(x, \xi, \nu) u(x, \nu) d\nu,
\end{align*}
\]

where the kernel \( k : (-a, a) \times (-1,1) \times (-1,1) \rightarrow \mathbb{R} \) is assumed to be measurable.

The following boundary spaces denoted by \( X^o_p \) and \( X^i_p \) are defined as follows:

\[ X^o_p := L_p[\{-a\} \times (-1,0); |\xi|d\xi] \times L_p[\{a\} \times (0,1); |\xi|d\xi] := X^o_{1,p} \times X^o_{2,p} \]

equipped with the norm

\[
\|u^o, X^o_p\| := \left( \|u^o_1, X^o_{1,p}\|^p + \|u^o_2, X^o_{2,p}\|^p \right)^{\frac{1}{p}}
\]

\[
= \left[ \int_{-1}^{0} |u(-a, v)|^p |v| dv + \int_{0}^{1} |u(a, v)|^p |v| dv \right]^{\frac{1}{p}},
\]
\[ X_p^i := L_p [\{ -a \} \times (0,1); |\xi| d\xi] \times L_p [\{ a \} \times (-1,0); |\xi| d\xi] := X_{1,p}^i \times X_{2,p}^i \]
equipped with the norm
\[
\| u^i, X_p^i \| := \left( \| u_1^i, X_{1,p}^i \|^p + \| u_2^i, X_{2,p}^i \|^p \right)^{\frac{1}{p}}
= \left[ \int_0^1 |u(-a, v)|^p |v| dv + \int_{-1}^0 |u(a, v)|^p |v| dv \right]^{\frac{1}{p}}.
\]

In this part, we will determine the essential B-spectra of the pair \((M, A_H)\), where \(M\) is the operator defined by
\[
\begin{cases}
M : X_p \rightarrow X_p \\
\varphi \mapsto M(\varphi)(x, \xi) = \eta(\xi) \varphi(x, \xi),
\end{cases}
\]
where \(\eta(.) \in L^\infty(-1, 1)\).

To clarify our subsequent analysis, we define the following bounded operators introduced in [15] :
\[
\begin{cases}
M_{\lambda} : X_p^i \rightarrow X_p^o, \\
M_{\lambda} := (M_{\lambda}^+, M_{\lambda}^- u), \\
M_{\lambda}^+ u(-a, \xi) := u(-a, \xi) e^{2a \lambda (\eta(\xi) + \sigma(\xi))}, \\
0 < \xi < 1, \\
M_{\lambda}^- u(a, \xi) := u(a, \xi) e^{2a \lambda (\eta(\xi) + \sigma(\xi))}, \\
-1 < \xi < 0,
\end{cases}
\]
\[
\begin{cases}
B_{\lambda} : X_p^i \rightarrow X_p, \\
B_{\lambda} := \chi_{(-1,0)}(\xi) B_{\lambda}^- u + \chi_{(0,1)}(\xi) B_{\lambda}^+ u, \\
B_{\lambda}^- u(x, \xi) := u(a, \xi) e^{\frac{-(\lambda \eta(\xi) + \sigma(\xi))}{|\xi|} |a-x|}, \\
-1 < \xi < 0, \\
B_{\lambda}^+ u(x, \xi) := u(-a, \xi) e^{\frac{-(\lambda \eta(\xi) + \sigma(\xi))}{|\xi|} |a+x|}, \\
0 < \xi < 1,
\end{cases}
\]
\[
\begin{cases}
G_{\lambda} : X_p \rightarrow X_p^o, \\
G_{\lambda} := (G_{\lambda}^+, G_{\lambda}^- \varphi), \\
G_{\lambda}^- \varphi(a, \xi) := \frac{1}{|\xi|} \int_a^0 e^{-(\lambda \eta(\xi) + \sigma(\xi)) |a-x|} \varphi(x, \xi) dx, \\
-1 < \xi < 0, \\
G_{\lambda}^+ \varphi(-a, \xi) := \frac{1}{|\xi|} \int_{-a}^0 e^{-(\lambda \eta(\xi) + \sigma(\xi)) |a-x|} \varphi(x, \xi) dx, \\
0 < \xi < 1,
\end{cases}
\]
and finally
\[
\begin{cases}
C_\lambda : X_p \rightarrow X_p, & C_\lambda \varphi = \chi(-1,0)C_\lambda^- \varphi + \chi(0,1)C_\lambda^+ \varphi, \quad \text{with} \\
C_\lambda^- \varphi(x, \xi) := \frac{1}{|\xi|} \int_x^a e^{-\frac{(\lambda \eta(\xi)+\sigma(\xi))}{|\xi|}|x-x'|} \varphi(x', \xi) \, dx', & -1 < \xi < 0, \\
C_\lambda^+ \varphi(x, \xi) := \frac{1}{|\xi|} \int_{-a}^x e^{-\frac{(\lambda \eta(\xi)+\sigma(\xi))}{|\xi|}|x-x'|} \varphi(x', \xi) \, dx', & 0 < \xi < 1,
\end{cases}
\]
where \( \chi(0,1)(\cdot) \) and \( \chi(-1,0)(\cdot) \) denote, respectively, the characteristic functions of the intervals \((0, 1)\) and \((-1, 0)\).

Note that, from [15, Proposition 3.1], the operators \( M_\lambda, B_\lambda, G_\lambda \) and \( C_\lambda \) are bounded respectively by \( \exp(-2\mu^\ast \text{Re}\lambda) \), \( (p\mu^\ast \text{Re}\lambda)^{-1/p} \), \((\mu^\ast \text{Re}\lambda)^{-1/q}\) and \( (\mu^\ast \text{Re}\lambda)^{-1} \) where \( q \) is the conjugate of \( p \).

In what follows, we will show that the M-spectrum of \( T_0 \) (i.e., \( T_H \) with \( H = 0 \)) is the continuous spectrum \( \sigma_c(M, T_0) \) of the pair \( (M, T_0) \).

**Lemma 3.1.** (i) The point spectrum of the pair \( (M, T_0) \) is empty.
(ii) The residual spectrum of the pair \( (M, T_0) \) is empty.

**Proof.** (i) We consider for \( \lambda \in \mathbb{C} \) such that \( \text{Re}(\lambda) \leq 0 \), the eigenvalue problem \( (\lambda M - T_0) \psi = 0 \), where the unknown \( \psi \) must be in \( D(T_0) \). His solution is formally given by
\[
\psi(x, \xi) = K(\xi)e^{\frac{1}{\xi^2}[(\lambda \eta(\xi)+\sigma(\xi))x].
\]
Moreover, since \( \psi \in D(T_0) \), then we get \( \psi^\dagger = 0 \). So we obtain \( K(\xi) = 0 \) on \((-1, 1)\). Consequently, \( \psi = 0 \).

(ii) To prove that the residual spectrum \( \sigma_r(M, T_0) \) is also empty, we shall determine the point spectrum of the adjoint operator pencil densely defined \( \lambda M - T_0 \), where \( \lambda \in \mathbb{C} \).

The adjoint operators \( T_0^\ast \) and \( M^\ast \) are, respectively, given by:
\[
\begin{cases}
T_0^\ast : D(T_0^\ast) \subseteq X_q \rightarrow X_q & \\
\psi \mapsto T_0^\ast \psi(x, \xi) = \xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi)\psi(x, \xi), & \\
D(T_0^\ast) = \{ \psi \in W_q : \psi^\dagger = 0 \},
\end{cases}
\]
where \( q \) is the conjugate of \( p \) \((\frac{1}{p} + \frac{1}{q} = 1)\).

\[
\begin{cases}
M^\ast : X_q \rightarrow X_q & \\
\varphi \mapsto M^\ast(\varphi)(x, v) = \eta(v)\varphi(x, v),
\end{cases}
\]
Let us consider the eigenvalue problem
\[(\lambda M^* - T_0^*)\psi = 0.\] (3.1)
In view of the boundary conditions, a straightforward estimation shows that, the problem (3.1) admits only the trivial solution. Then, we obtain \(\sigma_p(M^*, T_0^*) = 0\). Since, \(\sigma_r(M, T_0) \subseteq \sigma_p(M^*, T_0^*)\), then we can easily obtain the desired result.

Now, by using the previous results, we can deduce the following theorem:

**Theorem 3.1.** Let \(M \in L(X)\). Then,
\[\sigma(M, T_0) = \sigma_c(M, T_0) = \sigma_{ec}(M, T_0) = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0\}.\]

**Proof.** Since, \(\sigma(M, T_0) = \sigma_p(M, T_0) \cup \sigma_r(M, T_0) \cup \sigma_c(M, T_0)\), then by using Lemma 3.1, we deduce \(\sigma(M, T_0) = \sigma_c(M, T_0)\). On the other hand, it follows from [15, Theorem 3.1], that \(\sigma(M, T_0) = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0\}\). Combining these two results, we can obtain the assertion of the present theorem.

Now, we are able to express the B-spectra of the pair \((M, T_H)\):

**Theorem 3.2.** If the boundary operator \(H\) is of finite rank and commutes with \(M\), and every point \(\lambda\) in \(\sigma_{ec}(M, T_0)\) is non-isolated, then
\[\sigma_*(M, T_H) = \sigma_*(M, T_0) = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0\},\]
where \(\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}\).

**Proof.** According to [15], we have
\[(\lambda M - T_H)^{-1} = H \sum_{n \geq 0} B_n(M \lambda H)^n G_n + C_\lambda,\] (3.2)
where \(C_\lambda = (\lambda M - T_0)^{-1}\). Since \((\lambda M - T_H)^{-1} - (\lambda M - T_0)^{-1} \in \mathcal{F}(X)\), this implies by Theorem 2.3 that \(\sigma_*(M, T_H) = \sigma_*(M, T_0)\). The fact that \(M\) and \(T_H\) commute in the sense of the resolvent and \(M\) is invertible this allows us, by the use of Remarks 2.1, Theorem 2.5 and Theorem 3.1 to conclude that
\[\sigma_*(M, T_0) = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0\},\]
where \(\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}\).
Theorem 3.3. Suppose that the boundary operator $H$ and the collision operator $K$ are of finite rank. If $MH = HM$ and every point $\lambda$ in $\sigma_{ec}(M,T_0)$ is non-isolated, then we get

$$\sigma_*(M,A_H) = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0 \},$$

where $\sigma_*(M,.) \in \{ \sigma_{BF}(M,.) , \sigma_{BF^+}(M,.) , \sigma_{BF^-}(M,.) , \sigma_{BW}(M,.) \}$.

Proof. Since the collision operator $K$ is finite rank, then it follows from Theorem 2.3 and Theorem 3.2, that $\sigma_*(M,A_H) = \sigma_*(M,T_H+K) = \sigma_*(M,T_H) = \sigma_*(M,T_0)$, and finally we obtain that

$$\sigma_*(M,A_H) = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0 \},$$

where $\sigma_*(M,.) \in \{ \sigma_{BF}(M,.) , \sigma_{BF^+}(M,.) , \sigma_{BF^-}(M,.) , \sigma_{BW}(M,.) \}$. □

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