Application of the Kovacic algorithm for the investigation of motion of a heavy rigid body with a fixed point in the Hess case

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Funding information
Russian Foundation for Basic Research, Grant/Award Number: 20-01-00637

1 | INTRODUCTION

Let us consider the problem of motion of a heavy rigid body with the fixed point $O$. To describe motion of the body, we introduce two orthogonal coordinate systems: the fixed system $Oxyz$ and the moving system $Ox_1x_2x_3$. The $Oz$ axis of the fixed system is directed along the upward vertical. The $Ox_1x_2x_3$ system is rigidly connected with the moving body and its axes are directed along the principal axes of inertia at $O$. We denote the unit vectors of the $Ox_1x_2x_3$ system by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Let $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3$ and $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$ be the unit vector of the $Oz$ axis and the angular velocity vector of the body respectively, all being referred to the $Ox_1x_2x_3$ system. We denote by $\mathbf{P}$ the gravity force (directed vertically downward and applied at the center of gravity of the body $G$), then we have $\mathbf{P} = -Mg\mathbf{y}$, where $M$ is the mass of the body and $g$ is the gravity acceleration. We will define the position of the center of mass of the body by the radius-vector $\mathbf{r} = \mathbf{OG} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, where $x_1$, $x_2$, $x_3$ are projections of this vector onto $Ox_1x_2x_3$ axes. Let $\mathbf{J}_0 = \text{diag}(A_1, A_2, A_3)$ be the inertia matrix of the body at the fixed point $O$ with respect to the $Ox_1x_2x_3$ system. The motion of the body in the moving coordinate system $Ox_1x_2x_3$ is described by the Euler–Poisson equations:

$$\mathbf{J}_0\dot{\mathbf{w}} + [\mathbf{w} \times \mathbf{J}_0\mathbf{w}] = Mg[\mathbf{y} \times \mathbf{r}], \quad \dot{\mathbf{y}} = [\mathbf{y} \times \mathbf{w}]$$

In 1890, Hess found new special case of integrability of Euler–Poisson equations of motion of a heavy rigid body with a fixed point. In 1892, Nekrasov proved that the solution of the problem of motion of a heavy rigid body with a fixed point under Hess conditions reduces to integrating the second-order linear differential equation. In this paper, the corresponding linear differential equation is derived and its coefficients are presented in the rational form. Using the Kovacic algorithm, we proved that the Liouvillian solutions of the corresponding second-order linear differential equation exist only in the case, when the moving rigid body is the Lagrange top, or in the case, when the constant of the area integral is zero.
or in scalar form:

\[ A_1 \dot{\omega}_1 + (A_3 - A_2) \omega_2 \omega_3 = Mg(x_3 \gamma_2 - x_2 \gamma_3), \quad A_2 \dot{\omega}_2 + (A_1 - A_3) \omega_1 \omega_3 = Mg(x_1 \gamma_3 - x_3 \gamma_1), \]
\[ A_3 \dot{\omega}_3 + (A_2 - A_1) \omega_1 \omega_2 = Mg(x_2 \gamma_1 - x_1 \gamma_2), \]
\[ \dot{\gamma}_1 = \omega_3 \gamma_2 - \omega_2 \gamma_3, \quad \dot{\gamma}_2 = \omega_1 \gamma_3 - \omega_3 \gamma_1, \quad \dot{\gamma}_3 = \omega_2 \gamma_1 - \omega_1 \gamma_2. \]

(2)

It is well known that to solve the Euler–Poisson equations, we need to find four independent autonomous first integrals of the system (2) [1, 2]. For any values of parameters \( A_1, A_2, A_3, x_1, x_2, x_3 \) of the body and for any initial conditions, we have three first integrals of the Euler–Poisson equations: the energy integral

\[ \frac{1}{2} (A_1 \omega_1^2 + A_2 \omega_2^2 + A_3 \omega_3^2) + Mg(x_1 \gamma_1 + x_2 \gamma_2 + x_3 \gamma_3) = E, \]

the area integral

\[ A_1 \omega_1 \gamma_1 + A_2 \omega_2 \gamma_2 + A_3 \omega_3 \gamma_3 = k \]

(4)

and the geometrical integral

\[ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \]

(5)

Thus for integrability of the Euler–Poisson equations, we need to find only one additional autonomous first integral. In 1890, Hess [3] proved that under the conditions

\[ x_3 = 0, \quad A_2(A_3 - A_1)x_2^2 = A_1(A_2 - A_3)x_1^2, \quad A_2 \geq A_3 \geq A_1 \]

(6)

the Euler–Poisson equations (2) have an additional special integral (the Hess integral) of the form

\[ A_1 \omega_1 x_1 + A_2 \omega_2 x_2 = 0. \]

(7)

In 1892, the Hess case was rediscovered by Appelroth [4] when he analyzed the branching of the general solution of Euler–Poisson equations on the complex plane of time using the ideas of Kovalevskaya [5, 6]. The detailed analytical investigation of the Hess case has been made by Nekrasov [7, 8]. In his papers [7, 8], Nekrasov presented both the Hess conditions and the Hess integral and reduced the solution of the problem to the integration of a second-order linear differential equation. Nekrasov proved that in the Hess case, the solution branches out on the complex plane of time. He investigated the analytical properties of the obtained second-order linear differential equation and pointed out the basic properties of trajectories on the Poisson sphere. He proved also that on the zero level of the area integral, the problem is integrable in elliptic functions. The Hess integral as well as the reduction to the second-order linear differential equation was independently rediscovered in 1895 by Liouville [9]. The geometrical analysis and the modeling of the Hess top on the zero level of the area integral were given by Zhukovsky [10]. The complete analysis of the phase trajectories in a Hess case was made by Kovalev [11, 12]. For the investigation of motion of the Hess top, Kovalev used the special coordinate system, proposed previously by Kharlamov [13, 14], and reduced the solution of the problem to integration of the second-order linear differential equation with rational coefficients [12, 15]. However, the coefficients of the corresponding linear differential equation have singularities in the case, when the moving rigid body is the Lagrange top. Therefore, we decided to give our own investigation of the problem. Since the solution of the problem of motion of a heavy rigid body with a fixed point in the Hess case is reduced to solving the second-order linear differential equation, it is possible to set up the problem of existence of Liouvillian solutions of the corresponding linear differential equation. For this purpose, it is possible to apply the so-called Kovacic algorithm [16], which allows to find Liouvillian solutions of a second-order linear differential equation in explicit form. If a linear differential equation has no Liouvillian solutions, the Kovacic algorithm also allows one to ascertain this fact. The necessary condition for the application of the Kovacic algorithm to a second-order linear differential equation is that the coefficients of this equation should be rational functions of independent variable.

The Kovacic algorithm has been successfully used in the study of various problems in mechanics and mathematical physics. The first results on the application of the Kovacic algorithm to problems of mathematical physics were obtained by
Duval [17, 18]. Most often, the Kovacic algorithm is used to find new integrable cases in various problems of Hamiltonian systems dynamics or to prove the nonintegrability of various Hamiltonian systems. Such investigations are presented by Acosta-Humanez [19–21], Combot [22, 23], Maciejewski [24–28], Morales-Ruiz [29–33], and many other. The authors of the presented paper also used the Kovacic algorithm in their investigations. The papers by Bardin [34, 35] used the Kovacic algorithm to study the orbital stability of periodic motions of a heavy rigid body with a fixed point in the Bobylev–Steklov case. The paper by Kuleshov and Chernyakov [36] used the Kovacic algorithm to study the problem of motion of a heavy rotationally symmetric body on a fixed perfectly rough horizontal plane. The problem of motion of a heavy homogeneous ball, rolling without sliding on the given surface of revolution is studied by Kuleshov and Solomina [37] using the Kovacic algorithm.

The present paper is devoted to study the conditions of existence of Liouvillian solutions of the second-order linear differential equation whose integration solves the problem of motion of a heavy rigid body with a fixed point in a Hess case. The paper is organized as follows. In Section 2, we derive equations of motion of the system, using the special Kharlamov coordinate frame [13, 14] and some ideas from Kovalev paper [12]. We write equations of motion and their first integrals in dimensionless form. From these equations, we obtain the second-order linear differential equation with rational coefficients. In Section 3, we give a brief description of the Kovacic algorithm. In Section 4, we apply this algorithm to the corresponding second-order linear differential equation and obtain the conditions of existence of Liouvillian solutions of type 1 and type 3 for this linear differential equation. The Liouvillian solutions of type 1 and type 3 can exist only in the case, when the moving rigid body is the Lagrange top. Investigation of the second-order linear differential equation in the case of motion of the Lagrange top is presented in Section 5. In Section 6, we study the general Hess case, when the mass distribution of the body do not correspond to the Lagrange top. We obtain that Liouvillian solutions in the problem of motion of the Hess top can exist only in the case, when the constant of the area integral is zero. In Section 7, we find the corresponding Liouvillian solutions in explicit form. The possible special cases of motion, existing in the problem, are discussed in Section 8. Section 9 presents the final conclusions and remarks.

2 EQUATIONS OF MOTION OF A HEAVY RIGID BODY WITH A FIXED POINT IN THE HESS CASE WRITTEN IN THE SPECIAL COORDINATE SYSTEM

Instead of the principal axes of inertia at \(O\) (the \(Ox_1x_2x_3\) coordinate system with basis vectors \(e_1, e_2, e_3\)) let us introduce the special Kharlamov coordinate system \(O\eta_1\eta_2\eta_3\) with basis vectors \(e_I, e_{II}, e_{III}\) defined by the formulae

\[
e_I = e_1 \cos \alpha + e_2 \sin \alpha, \quad e_{II} = -e_1 \sin \alpha + e_2 \cos \alpha, \quad e_{III} = e_3,
\]

where \(\cos \alpha \) and \(\sin \alpha \) equals

\[
\cos \alpha = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \sin \alpha = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}.
\]

If we denote by

\[
L_1 = A_1 \omega_1 \cos \alpha + A_2 \omega_2 \sin \alpha, \quad L_2 = A_2 \omega_2 \cos \alpha - A_1 \omega_1 \sin \alpha, \quad L_3 = A_3 \omega_3,
\]

\[
\nu_1 = \gamma_1 \cos \alpha + \gamma_2 \sin \alpha, \quad \nu_2 = \gamma_2 \cos \alpha - \gamma_1 \sin \alpha, \quad \nu_3 = \gamma_3,
\]

\[
a = \frac{A_2 x_1^2 + A_1 x_2^2}{A_1 A_2 (x_1^2 + x_2^2)}, \quad b = \frac{(A_1 - A_2) x_1 x_2}{A_1 A_2 (x_1^2 + x_2^2)}, \quad c = \frac{1}{A_3}, \quad \Gamma = Mg \sqrt{x_1^2 + x_2^2},
\]

then the Euler–Poisson equations (2) can be represented as follows:

\[
\dot{L}_1 = -bL_3L_3, \quad \dot{L}_2 = (a - c)L_1L_3 + bL_2L_3 + \nu_3 \Gamma, \quad \dot{L}_3 = -(a - c)L_1L_2 + bL_1^2 - bL_2^2 - \nu_2 \Gamma,
\]

\[
\dot{\nu}_1 = cL_3 \nu_2 - (cL_2 + bL_1) \nu_3, \quad \dot{\nu}_2 = -cL_3 \nu_1 + (aL_1 + bL_2) \nu_3, \quad \dot{\nu}_3 = (bL_1 + cL_2) \nu_1 - (aL_1 + bL_2) \nu_2.
\]
To find the additional first integral, existing in the Hess case, we consider the first equation of the system (13)

\[ \dot{L}_1 = -bL_1L_3. \]  

(14)

In this equation, the right-hand side equals \( L_1 \) itself, multiplied by the coefficient \(-bL_3\) bounded in absolute value. This means that if at the initial instant of time the quantity \( L_1 = 0 \), then we have

\[ L_1 \equiv 0. \]  

(15)

The invariant manifold (15) (or, in other notations (7)) together with (6) defines the Hess case. Under conditions (6), (15), equations (13) are noticeably simplified and take the form

\[ \dot{L}_2 = bL_2L_3 + \nu_3 \Gamma, \quad \dot{L}_3 = -bL_2^2 - \nu_2 \Gamma, \]

\[ \dot{\nu}_1 = cL_3 \nu_2 - cL_2 \nu_3, \quad \dot{\nu}_2 = bL_2 \nu_3 - cL_3 \nu_1, \quad \dot{\nu}_3 = cL_2 \nu_1 - bL_2 \nu_2. \]  

(16)

Equations (16) possess the following first integrals:

\[ \frac{c}{2}(L_2^2 + L_3^2) + \Gamma \nu_1 = E; \quad L_2 \nu_2 + L_3 \nu_3 = k; \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1. \]  

(17)

Equations (16) together with the first integrals (17) have been first obtained by Kovalev [11, 12]. Note that condition \( b = 0 \) corresponds to the Lagrange integrable case in the problem of motion of a heavy rigid body with a fixed point.

Now let us write equations (16) and the first integrals (17) in dimensionless form. For this purpose, we introduce the dimensionless variables

\[ y = \sqrt{\frac{c}{\Gamma}} L_2, \quad z = \sqrt{\frac{c}{\Gamma}} L_3, \]  

(18)

and the dimensionless time \( \tau \):

\[ \tau = t \sqrt{\Gamma c}. \]  

(19)

We introduce also the dimensionless parameter

\[ d_1 = \frac{b}{c}. \]  

(20)

and the dimensionless constants of the first integrals

\[ h = \frac{E}{\Gamma}, \quad k_1 = k \sqrt{\frac{c}{\Gamma}}. \]  

(21)

Now we can write equations (16) in dimensionless form:

\[ \frac{dy}{d\tau} = d_1yz + \nu_3, \quad \frac{dz}{d\tau} = -d_1y^2 - \nu_2, \quad \frac{d\nu_1}{d\tau} = z \nu_2 - y \nu_3, \quad \frac{d\nu_2}{d\tau} = d_1y \nu_3 - z \nu_1, \quad \frac{d\nu_3}{d\tau} = y \nu_1 - d_1y \nu_2. \]  

(22)

System (22) possesses the following first integrals:

\[ \frac{y^2 + z^2}{2} + \nu_1 = h, \quad y \nu_2 + z \nu_3 = k_1, \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1. \]  

(23)

The system (22) differs from the system presented by Kovalev [11, 12], because the system presented in Refs. [11, 12] has a singularity for the Lagrange integrable case. The system (22) has no singularities for the Lagrange integrable case, which corresponds to the condition \( d_1 = 0 \).

From the system (22) using the first integrals (23), we will obtain the second-order linear differential equation. Before we obtain this equation, let us determine the range of parameters \( d_1, h, k_1 \). It is easy to see that the parameter \( k_1 \) ranges in the infinite interval \((—\infty, +\infty)\). Since the expression \((y^2 + z^2)/2\) is non-negative, then we have for the parameter \( h \) that the inequality \( h \geq \nu_1 \). The minimal value of \( \nu_1 \) equals \(-1\). Therefore, the parameter \( h \) ranges in the interval \( h \in [—1, +\infty) \). As for the parameter \( d_1 \), it can be shown (see Refs. [38, 39]) that this parameter ranges in the interval \( d_1 \in (—1, 0] \).
Using the method, presented by Kovalev [12], let us obtain now the second-order linear differential equation from the system (22) using the first integrals (23). Multiplying the first equation of the system (22) by \(y\) and the second by \(z\) and adding them, we get

\[
\frac{d}{d\tau} \left( \frac{y^2 + z^2}{2} \right) = y\nu_3 - z\nu_2. \tag{24}
\]

Using the following identity

\[
(y^2 + z^2)(y^2 + \nu_3^2) = (y\nu_2 + z\nu_3)^2 + (y\nu_3 - z\nu_2)^2, \tag{25}
\]

we find from the first integrals (23)

\[
\nu_1 = h - \frac{y^2 + z^2}{2}. \tag{26}
\]

Therefore we have

\[
\nu_2^2 + \nu_3^2 = 1 - \left( h - \frac{y^2 + z^2}{2} \right)^2 = 1 - \left( \frac{y^2 + z^2}{2} - h \right)^2, \quad y\nu_2 + z\nu_3 = k_1. \tag{27}
\]

Finally, we obtain

\[
(y\nu_3 - z\nu_2)^2 = (y^2 + z^2) \left( 1 - \left( \frac{y^2 + z^2}{2} - h \right)^2 \right) - k_1^2. \tag{28}
\]

We will take that

\[
y\nu_3 - z\nu_2 = -\sqrt{(y^2 + z^2) \left( 1 - \left( \frac{y^2 + z^2}{2} - h \right)^2 \right) - k_1^2}, \tag{29}
\]

(we can choose the arbitrary sign before the square root in Equation (29)). Taking into account Equation (29), we can rewrite Equation (24) in the form

\[
\frac{d}{d\tau} \left( \frac{y^2 + z^2}{2} \right) = -\sqrt{(y^2 + z^2) \left( 1 - \left( \frac{y^2 + z^2}{2} - h \right)^2 \right) - k_1^2}. \tag{30}
\]

Now, we multiply the first equation of the system (22) by \(-z\) and the second by \(y\) and adding them. Taking into account the first integrals (23), we obtain

\[
y \frac{dz}{d\tau} - z \frac{dy}{d\tau} = -d_1y^3 - y\nu_2 - d_1yz^2 - z\nu_3 = -d_1y(y^2 + z^2) - k_1. \tag{31}
\]

We pass now from the variables \(y\) and \(z\) to the polar coordinates \(x\) and \(\varphi\) by putting

\[
y = x \cos \varphi, \quad z = x \sin \varphi. \tag{32}
\]

Then for the variables \(x\) and \(\varphi\), we have the following system of two differential equations:

\[
x \frac{dx}{d\tau} = -\sqrt{x^2 \left[ 1 - \left( \frac{x^2}{2} - h \right)^2 \right] - k_1^2}, \quad x \frac{d\varphi}{d\tau} = -d_1x^3 \cos \varphi - k_1. \tag{33}
\]

From this system, we obtain the single first-order differential equation for the function \(\varphi = \varphi(x)\):

\[
\frac{d\varphi}{dx} = \frac{d_1x^3 \cos \varphi + k_1}{x \sqrt{x^2 \left[ 1 - \left( \frac{x^2}{2} - h \right)^2 \right] - k_1^2}}. \tag{34}
\]
Note that when we pass from the system (33) to the equation (34) we exclude from consideration the case \( x = \text{const} \) that is, \( y^2 + z^2 = \text{const} \) or \( \nu_1 = \text{const} \). Meanwhile for a heavy rigid body with a fixed point in the Hess case, there are steady motions for which \( \nu_1 = \nu_1^0 = \text{const} \) (see, for example [40]).

The substitution

\[ w = \tan \frac{\varphi}{2} \]

reduces (34) to the Riccati equation:

\[
\frac{dw}{dx} = -\frac{d_1x^3 - k_1}{2x \sqrt{x^2 \left[ 1 - \left( \frac{x^2}{2} - h \right)^2 \right] - k_1^2}} w^2 + \frac{d_1x^3 + k_1}{2x \sqrt{x^2 \left[ 1 - \left( \frac{x^2}{2} - h \right)^2 \right] - k_1^2}}.
\]  

It is well known from the general theory of ordinary differential equations (see, for example [41]), that if the Riccati equation has the form

\[
\frac{dw}{dx} = f_2(x)w^2 + f_1(x)w + f_0(x),
\]

then the substitution of the form

\[
u(x) = \exp \left( -\int f_2(x)w(x)dx \right)
\]

reduces it to the second-order linear differential equation

\[
f_2 \frac{d^2u}{dx^2} - \left( \frac{df_2}{dx} + f_1f_2 \right) \frac{du}{dx} + f_0f_2^2u = 0,
\]

or, if we divide this equation by \( f_2 \):

\[
\frac{d^2u}{dx^2} - \left( \frac{1}{f_2} \frac{df_2}{dx} + f_1 \right) \frac{du}{dx} + f_0f_2u = 0.
\]

In our case,

\[
f_2 = -\frac{d_1x^3 - k_1}{2x \sqrt{x^2 \left[ 1 - \left( \frac{x^2}{2} - h \right)^2 \right] - k_1^2}}, \quad f_1 = 0, \quad f_0 = \frac{d_1x^3 + k_1}{2x \sqrt{x^2 \left[ 1 - \left( \frac{x^2}{2} - h \right)^2 \right] - k_1^2}.
\]

Note that the transition from Equation (39) to Equation (40) is possible only when \( f_2 \neq 0 \). Taking into account the fact that \( x \neq \text{const} \), the condition \( f_2 = 0 \) is equivalent to the simultaneous fulfillment of the conditions

\[
d_1 = 0, \quad k_1 = 0.
\]

Under the conditions (42), Equation (34) gives \( \varphi = \varphi_0 = \text{const} \). It can be shown (see Refs. [38, 39]) that, under the conditions (42), a heavy rigid body with a fixed point in the Hess case will perform pendulum nutational oscillations. We will assume further that \( f_2 \neq 0 \). Thus, the problem of motion of a heavy rigid body with a fixed point in the Hess case is reduced to solving the following second-order linear differential equation with the rational coefficients:

\[
\frac{d^2u}{dx^2} + a(x) \frac{du}{dx} + b(x)u = 0,
\]

\[
a(x) = \frac{d_1x^9 - 4k_1x^6 - 4d_1(h^2 - 1)x^5 + 12k_1hx^4 - 8k_1^2d_1x^3 - 8k_1(h^2 - 1)x^2 - 4k_1^3}{x(x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2)(d_1x^3 - k_1^2)},
\]

\[
b(x) = \frac{d_1x^6 - 6k_1x^3 + 4d_1(h^2 - 1)x^2 - 8k_1^2d_1x - 8k_1(h^2 - 1)x - 4k_1^3}{x(x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2)(d_1x^3 - k_1^2)}.
\]
\[ b(x) = \frac{(d_1 x^3 + k_1)\left(d_1 x^3 - k_1\right)}{x^2 (x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2)}. \] (45)

Now, we can study the problem of existence of Liouvillian solutions for the second-order linear differential equation (43). To solve this problem, we can use the Kovacic algorithm [16]. Below we give a brief description of this algorithm.

### 3 | DESCRIPTION OF THE KOVACIC ALGORITHM

Let us consider the differential field \( \mathbb{C}(x) \) of rational functions of one (in general case complex) variable \( x \). We accept the standard notations \( \mathbb{Z} \) and \( \mathbb{Q} \) for the sets of integer and rational numbers, respectively. Our goal is to find a solution of the differential equation

\[ \frac{d^2 z}{dx^2} + a(x) \frac{dz}{dx} + b(x)z = 0, \] (46)

where \( a(x), b(x) \in \mathbb{C}(x) \). In the paper [16], an algorithm has been proposed that allows one to find explicitly the so-called Liouvillian solutions of differential equation (46), that is, solutions, that can be expressed in terms of Liouvillian functions. The main advantage of the Kovacic algorithm is precisely that it allows one not only to establish the existence or nonexistence of a solution of differential equation (46) expressed in terms of Liouvillian functions, but also to present this solution in an explicit form when it exists. In turn, Liouvillian functions are elements of a Liouvillian field, which is defined in the following way.

**Definition 1.** Let \( F \) be a differential field of functions of one (in general case complex) variable \( x \) that contains \( \mathbb{C}(x) \); namely \( F \) is a field of characteristic zero with a differentiation operator \( ()' \) with the following two properties: \( (a + b)' = a' + b' \) and \( (ab)' = a'b + ab' \) for any \( a \) and \( b \) in \( F \). The field \( F \) is Liouvillian if there exists a sequence (tower) of differential fields

\[ \mathbb{C}(x) = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F, \] (47)

obtained by adjoining one element such that for any \( i = 1, 2, \ldots, n \) we have

\[ F_i = F_{i-1}(\alpha), \quad \text{with} \quad \alpha' \in F_{i-1} \] (48)

(i.e., \( F_i \) is generated by an exponential of an indefinite integral over \( F_{i-1} \)); or

\[ F_i = F_{i-1}(\alpha), \quad \text{with} \quad \alpha' \in F_{i-1} \] (49)

(i.e., \( F_i \) is generated by an indefinite integral over \( F_{i-1} \)); or \( F_i \) is finite algebraic over \( F_{i-1} \) (i.e., \( F_i = F_{i-1}(\alpha) \) and \( \alpha \) satisfies a polynomial equation of the form

\[ a_0 + a_1 \alpha + \cdots + a_n \alpha^n = 0, \] (50)

where \( a_j \in F_{i-1}, j = 0, 1, 2, \ldots, n \) and are not all zero).

Thus, Liouvillian functions are built up sequentially from rational functions by using algebraic operations and the operation of indefinite integration and by taking the exponential of a given expression. A solution of equation (46) that is expressed in terms of Liouvillian functions most closely correspond to the notion of a “close-form solution” or a “solution in quadratures.” To reduce differential equation (46) to a simpler form, we use the following formula:

\[ y(x) = z(x) \exp \left( \frac{1}{2} \int a(x) dx \right). \] (51)
Then Equation (46) takes the form

$$y'' = R(x)y, \quad R(x) = \frac{1}{2}a' + \frac{1}{4}a^2 - b, \quad R(x) \in \mathbb{C}(x). \quad (52)$$

Hereinafter, it is assumed that the second-order linear differential equation with which the Kovacic algorithm deals is written in the form (52). The following theorem, which has been proved by J. Kovacic [16], determines the structure of a solution of this differential equation.

**Theorem 1.** For the differential equation (52), only the following four cases are true.

1. The differential equation (52) has a solution of the form

$$\eta = \exp \left( \int \omega(x)dx \right), \quad \text{where} \quad \omega(x) \in \mathbb{C}(x) \quad (53)$$

(Liouville solution of type 1).

2. The differential equation (52) has a solution of the form

$$\eta = \exp \left( \int \omega(x)dx \right), \quad (54)$$

where $\omega(x)$ is an algebraic function of degree 2 over $\mathbb{C}(x)$ and case 1 does not hold (Liouvillian solution of type 2).

3. All solutions of differential equation (52) are algebraic over $\mathbb{C}(x)$ and Cases 1 and 2 do not hold. In this situation, a solution of the differential equation (52) has the form

$$\eta = \exp \left( \int \omega(x)dx \right), \quad (55)$$

where $\omega(x)$ is an algebraic function of degree 4, 6, or 12 over $\mathbb{C}(x)$ (Liouvillian solution of type 3).

4. Differential equation (52) has no Liouvillian solutions.

In order for one of the first three cases listed in Theorem 1 to take place the function $R(x)$ in the right-hand side of Equation (52) must satisfy certain conditions. These conditions are necessary but not sufficient. For example, if the conditions corresponding to Case 1 of Theorem 1 are violated, then we must turn to the verification of the conditions corresponding to Cases 2 and 3. If these conditions are fulfilled, then we must search for solutions of Equation (52) exactly in the form, indicated for the corresponding case. However, the existence of such a solution is not guaranteed. In order to explain the sense of the necessary conditions mentioned, we recall some facts from complex analysis. By definition, a point $a$ is called a pole of an analytic function $f(z)$ of order $n$ if the last term of the principal part of the Laurent series expansion of $f(z)$ about $a$ is

$$\frac{a_{-n}}{(z-a)^n} \quad (56)$$

If $f(z)$ is a rational function of $z$, then a point $a$ is a pole of $f(z)$ of order $n$ if it is a root of the denominator of $f(z)$ of multiplicity $n$.

Further, the order of $f(z)$ at $z = \infty$ is defined to be the order of $z = \infty$ as a zero of $f(z)$, that is, the order of $z = 0$ as a pole of $f(\frac{1}{z})$. Equivalently, if $f(z)$ is a rational function of $z$, then the order of $f(z)$ at $z = \infty$ is the degree of the denominator minus the degree of the numerator.

The following theorem, which has been proved in Kovacic [16], specifies conditions, that are necessary for one of the first three cases listed in Theorem 1 can hold.

**Theorem 2.** For the differential equation (52), the following conditions are necessary for one of the first cases listed in Theorem 1 to hold, that is, for Equation (52) to have a Liouvillian solution of the type specified in description of the corresponding case.
1. Each pole of the function $R(x)$ must have even order or else have order 1. The order of $R(x)$ at $x = \infty$ must be even or else be greater than 2.

2. The function $R(x)$ must have at least one pole that either has odd order greater than 2 or else has order 2.

3. The order of a pole of $R(x)$ cannot exceed 2 and the order of $R(x)$ at $x = \infty$ must be at least 2. If the partial fraction expansion of $R(x)$ has the form

$$R(x) = \sum_i \frac{\alpha_i}{(x - c_i)^2} + \sum_j \frac{\beta_j}{x - d_j}, \quad (57)$$

then for each $i$

$$\sqrt{1 + 4\alpha_i} \in \mathbb{Q}, \quad \sum_j \beta_j = 0, \quad (58)$$

and if

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j, \quad (59)$$

then

$$\sqrt{1 + 4\gamma} \in \mathbb{Q}. \quad (60)$$

To find a Liouvillian solution of type 1 of the differential equation (52), the Kovacic algorithm is stated in the following way (see Kovacic [16] for details). We assume that the necessary conditions for the existence of a solution in Case 1 are satisfied and denote the set of finite poles of the function $R(x)$ by $\Gamma$.

**Step 1.** For each $c \in \Gamma \cup \{\infty\}$, we define a rational function $[\sqrt{R}]_c$ and two complex numbers $\alpha^+_c$ and $\alpha^-_c$ as described below.

(c1) If $c \in \Gamma$ is a pole of order 1, then

$$[\sqrt{R}]_c = 0, \quad \alpha^+_c = \alpha^-_c = 1. \quad (61)$$

(c2) If $c \in \Gamma$ is a pole of order 2, then

$$[\sqrt{R}]_c = 0. \quad (62)$$

Let $b$ be the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction expansion of $R(x)$. Then

$$\alpha^\pm_c = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4b}. \quad (63)$$

(c3) If $c \in \Gamma$ is a pole of order $2\nu \geq 4$ (the order must be even due to the necessary conditions, stated in Theorem 2), then $[\sqrt{R}]_c$ is the sum of terms involving $\frac{1}{(x-c)^{\nu+1}}$ for $2 \leq \nu \leq \nu$ in the Laurent expansion of $\sqrt{R}$ at $c$.

There are two possibilities for $[\sqrt{R}]_c$ that differ by sign; we can choose one of them. Thus,

$$[\sqrt{R}]_c = \frac{a}{(x-c)^\nu} + \cdots + \frac{d}{(x-c)^2}. \quad (64)$$

Let $b$ be the coefficient of $\frac{1}{(x-c)^{\nu+1}}$ in $R - [\sqrt{R}]_c^2$. Then

$$\alpha^\pm_c = \frac{1}{2} \left( \pm \frac{b}{a} + \nu \right). \quad (65)$$
If the order of $R(x)$ at $x = \infty$ is greater than 2, then
\[ \left[ \sqrt{R} \right]_{\infty} = 0, \quad \alpha_{\infty}^+ = 1, \quad \alpha_{\infty}^- = 0. \]  \hfill (66)

If the order of $R(x)$ at $x = \infty$ is 2, then
\[ \left[ \sqrt{R} \right]_{\infty} = 0. \]  \hfill (67)

Let $b$ be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of $R(x)$ at $x = \infty$. Then
\[ \alpha_{\infty}^\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4b}. \]  \hfill (68)

If the order $R(x)$ at $x = \infty$ is $-2\nu \leq 0$ (it is even due to the necessary conditions stated in Theorem 2), then the function $\left[ \sqrt{R} \right]_{\infty}$ is the sum of terms involving $x^i, 0 \leq i \leq \nu$ of the Laurent expansion of $\sqrt{R}$ at $x = \infty$ (one of the two possibilities can be chosen). Thus,
\[ \left[ \sqrt{R} \right]_{\infty} = ax^\nu + \cdots + d. \]  \hfill (69)

Let $b$ be the coefficient of $x^{\nu-1}$ in $R - \left( \left[ \sqrt{R} \right]_{\infty} \right)^2$. Then we have
\[ \alpha_{\infty}^\pm = \frac{1}{2} \left( \pm \frac{b}{a} - \nu \right). \]  \hfill (70)

Step 2. For each family $s = (s(c))_{c \in \Gamma \cup \{\infty\}}$, where $s(c)$ are either $+$ or $-$ let
\[ d = \alpha_{\infty}^s(\infty) - \sum_{c \in \Gamma} \alpha_{c}^{s(c)}. \]  \hfill (71)

If $d$ is a non-negative integer, then we introduce the function
\[ \theta = \sum_{c \in \Gamma} \left( s(c) \left[ \sqrt{R} \right]_{c} + \frac{\alpha_{c}^{s(c)}}{x - c} \right) + s(\infty) \left[ \sqrt{R} \right]_{\infty}. \]  \hfill (72)

If $d$ is not a non-negative integer, then the family $s$ should be discarded. If all tuples $s$ have been rejected, then Case 1 cannot hold.

Step 3. For each family $s$ from Step 2, we search for a monic polynomial $P$ of degree $d$ (the constant $d$ is defined by the formula (71)), satisfying the differential equation
\[ P'' + 2\theta P' + (\theta' + \theta^2 - R)P = 0. \]  \hfill (73)

If such a polynomial exists, then
\[ \eta = P \exp \left( \int \theta(x)dx \right), \]  \hfill (74)

is the solution of the differential equation (52). If for each tuple $s$ found on Step 2, we cannot find such a polynomial $P$, then Case 1 cannot hold for the differential equation (52).

Now we state the Kovacic algorithm to search for a solution of type 2 of differential equation (52). We denote the set of finite poles of the function $R(x)$ by $\Gamma$.

Step 1. For each $c \in \Gamma \cup \{\infty\}$, we define the set $E_c$ as follows.

(c1) If $c \in \Gamma$ is a pole of order 1, then
\[ E_c = \{4\}. \]  \hfill (75)
If \( c \in \Gamma \) is a pole of order 2 and if \( b \) is the coefficient of \( \frac{1}{(x-c)^2} \) in the partial fraction expansion of \( R(x) \), then

\[
E_c = \left\{ \left( 2 + k \sqrt{1 + 4b} \right) \cap \mathbb{Z} \right\}, \quad k = 0, \pm 2.
\] (76)

If \( c \in \Gamma \) is a pole of order \( \nu > 2 \), then

\[
E_c = \{ \nu \}.
\] (77)

If \( R(x) \) has order \( > 2 \) at \( x = \infty \), then

\[
E_\infty = \{0, 2, 4\}.
\] (78)

If \( R(x) \) has order 2 at \( x = \infty \) and \( b \) is the coefficient of \( \frac{1}{x^2} \) in the Laurent series expansion of \( R \) at \( x = \infty \), then

\[
E_\infty = \left\{ \left( 2 + k \sqrt{1 + 4b} \right) \cap \mathbb{Z} \right\}, \quad k = 0, \pm 2.
\] (79)

If \( R(x) \) has order \( \nu < 2 \) at \( x = \infty \), then

\[
E_\infty = \{ \nu \}. \quad (80)
\]

**Step 2.** Let us consider the families \( s = (e_\infty, e_c), \ c \in \Gamma \), where \( e_c \in E_c, e_\infty \in E_\infty \) and at least one of these numbers is odd. Let

\[
d = \frac{1}{2} \left( e_\infty - \sum_{c \in \Gamma} e_c \right). \quad (81)
\]

If \( d \) is a non-negative integer, the family should be retained, otherwise it should be discarded.

**Step 3.** For each family retained from Step 2, we form the rational function

\[
\theta = \frac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x - c}. \quad (82)
\]

and search for a monic polynomial \( P \) of degree \( d \) (the constant \( d \) is defined by the formula (81)) such, that

\[
P''' + 3 \theta P'' + \left( 3 \theta^2 + 3 \theta' - 4R \right) P' + \left( \theta''' + 3 \theta \theta' + \theta^3 - 4R \theta - 2R' \right) P = 0. \quad (83)
\]

If success is achieved, we set

\[
\varphi = \theta + \frac{P'}{P}, \quad (84)
\]

and let \( \omega \) be a solution of the quadratic equation (algebraic equation of degree 2) of the form

\[
\omega^2 - \varphi \omega + \frac{1}{2} \varphi' + \frac{1}{2} \varphi^2 - R = 0. \quad (85)
\]

Then

\[
\eta = \exp \left( \int \omega(x) dx \right) \quad (86)
\]

is a solution of the differential equation (52). If success is not achieved, then Case 2 cannot hold for the differential equation (52).

Similarly, the Kovacic algorithm is stated to search for a Liouvillian solutions of type 3 of the differential equation (52). Let us apply now this algorithm to search Liouvillian solutions of the second-order linear differential equation (43).
4 \ APPLICATION OF THE KOVACIC ALGORITHM TO THE DIFFERENTIAL EQUATION (43). GENERAL CASE

So, the differential equation being investigated has the form (43). In this equation, we make a substitution according to Equation (51) and reduce it to the form (52)

\[ \frac{d^2 y}{dx^2} = R(x)y. \]  

(87)

Here the function \( R(x) \) takes the form

\[ R(x) = \frac{U(x)}{V(x)}. \]  

(88)

\[ U(x) = -(1 + 4d_1^2)d_1^2x^{16} + 8(2d_1^2 - 1)d_1^2hx^{14} + 4(2d_1^2 + 5)d_1k_1x^{13} - 8(2d_1^2 - 7)(h^2 - 1)d_1^2x^{12} \]

\[ -8(4d_1^2 + 19)d_1k_1hx^{11} + 8((1 + 17d_1^2 - 2d_1^4)k_1^2 - 12(h^2 - 1)d_1^2h)x^{10} + 576(h^2 - 1)k_1^3d_1x^3 \]

\[ +16(2(h^2 - 1)d_1^2 + 29h^2 - 5)d_1k_1x^9 + 8(6(h^2 - 1)^2 d_1^2 - (7 + 40d_1^2)k_1^2h)x^8 - 240k_1^4hx^2 \]

\[ +32((d_1^2 - 2)k_1^2 - 21(h^2 - 1)h)d_1k_1x^7 + 16(14(h^2 - 1)d_1^2 + 8h^2 - 5)k_1^2x^6 + 288k_1^5d_1x \]

\[ +96(4(h^2 - 1)^2 - 3k_1^2h)d_1k_1x^5 + 4((32d_1^2 + 35)k_1^2 - 24(h^2 - 1)h)k_1^2x^4 + 48(h^2 - 1)k_1^4. \]  

(89)

\[ V(x) = 4(d_1x^3 - k_1)^2(x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2)^2. \]  

(90)

Thus, it is easy to see, that the function \( R(x) \) has nine finite poles of the second order. Let us denote the roots of the polynomial

\[ x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2 = 0, \]  

(91)

by \( x_1, x_2, x_3, x_4, x_5, x_6 \). Note that this polynomial contains only the terms of even degree, therefore, its roots satisfy the conditions:

\[ x_2 = -x_1, \quad x_4 = -x_3, \quad x_6 = -x_5. \]  

(92)

Let us denote the roots of the polynomial

\[ d_1x^3 - k_1 = 0, \]  

(93)

by \( x_7, x_8, x_9 \). Now, let us consider the partial fraction expansion of the function \( R(x) \), defined by Equation (88). It has the form

\[ R(x) = \sum_{i=1}^{6} \frac{1}{(x - x_i)^2} + \sum_{i=1}^{9} \frac{\gamma_i(x_i)}{x - x_i} + \sum_{i=7}^{9} \frac{1}{(x - x_i)^2}. \]  

(94)

The coefficients \( \gamma_i(x_i), i = 1, 2, \ldots, 9 \) have a very complicated form and we do not write them explicitly here. It is possible to note the following properties of the partial fraction expansion of the function \( R(x) \).

1. The coefficients \( b_1, \ldots, b_6 \) of \( \frac{b_i}{(x-x_i)^2}, i = 1, \ldots, 6 \) equal

\[ b_i = -\frac{3}{16}, \quad i = 1, \ldots, 6. \]  

(95)

2. The coefficients \( b_7, b_8, b_9 \) of \( \frac{b_i}{(x-x_i)^2}, i = 7, 8, 9 \) equal

\[ b_i = \frac{3}{4}, \quad i = 7, 8, 9. \]  

(96)
3. The Laurent expansion of $R(x)$ at $x = \infty$ has the form

$$R(x) = -\frac{(1 + 4d_1^2)}{4x^2} + O\left(\frac{1}{x^4}\right) \quad (97)$$

Thus, we have

$$b_\infty = -\frac{1}{4} - d_1^2, \quad (98)$$

and therefore,

$$1 + 4b_\infty = -4d_1^2. \quad (99)$$

This means that the numbers $\alpha^\pm_\infty$ calculating during the application of the Kovacic algorithm for searching the Liouvillian solutions of type 1 are complex numbers if $d_1 \neq 0$. All the remaining numbers $\alpha^\pm_c$ are rational. They are presented in the following Table 1. Therefore, the number $d$, calculated by formula (71) in the process of searching for Liouvillian solutions of type 1, is a complex number for $d_1 \neq 0$. This fact indicates the absence of Liouvillian solutions of type 1 for $d_1 \neq 0$.

Moreover, the coefficient $b_\infty$ coincides with the value $\gamma$ calculating during the checking of the necessary conditions of existence of Liouvillian solutions of type 3 for the differential equation (52). According to this necessary conditions for existence of Liouvillian solutions of type 3, the number

$$\sqrt{1 + 4\gamma} = \sqrt{1 + 4b_\infty} \quad (100)$$

should be rational. However, when $d_1 \neq 0$, this number is pure imaginary. Thus, we can state that for $d_1 \neq 0$, the second-order linear differential equation (87) (or (43)) do not have Liouvillian solutions of type 3. Thus, the following theorem is valid.

**Theorem 3.** If all roots of polynomials (91) and (93) are distinct and $d_1 \neq 0$, then the problem of motion of a heavy rigid body with a fixed point in the Hess case has no Liouvillian solutions of type 1 and type 3.

According to Theorem 3, Equation (87) can have Liouvillian solutions of type 1 only when $d_1 = 0$, that is, when the moving rigid body with a fixed point is the Lagrange top. To search Liouvillian solutions of type 1 for the differential equation (43) in the Lagrange integrable case, we put $d_1 = 0$ in this equation.

## 5 INVESTIGATION OF THE LAGRANGE CASE

When $d_1 = 0$, we can write differential equation (43) as follows:

$$\frac{d^2 u}{dx^2} + \frac{4(x^6 - 3hx^4 + 2(h^2 - 1)x^2 + k_1^2)}{x(x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2)} \frac{du}{dx} - \frac{k_1^2 u}{x^2(x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2)} = 0, \quad (101)$$

We assume that in Equation (101), there should be $k_1 \neq 0$. Otherwise, the conditions (42) are satisfied and it is impossible to obtain from the equation (34) the second-order linear differential equation (43), from which for $d_1 = 0$ we obtain (101).
We change now the independent variable in the equation (101) by the formula \( x^2 = z \). As a result, Equation (101) can be rewritten as follows:

\[
\frac{d^2u}{dz^2} + \frac{5z^3 - 16hz^2 + 12(h^2 - 1)z + 8k_1^2}{2z(z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2)} \frac{du}{dz} - \frac{k_1^2u}{4z(z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2)} = 0.
\] (102)

In the obtained linear differential equation (102), we make the change of variables (51) and reduce this equation to the form (52)

\[
\frac{d^2y}{dz^2} = R_1(z)y = \frac{U_1(z)}{V_1(z)}y,
\] (103)

\[
U_1(z) = 5z^6 - 32hz^5 + 56(h^2 - 1)z^4 + 116k_1^2z^3 - 48(h^2 - 1)z^2 - 48k_1^4,
\] (104)

\[
V_1(z) = 16z^2(z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2)^2.
\] (105)

It is easy to see that the function \( R_1(z) \) has four finite poles of the second order. Let us denote the roots of the polynomial

\[
P_3(z) = z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2
\] (106)

by \( z_1, z_2, \) and \( z_3 \). Now let us consider the partial fraction expansion of the function \( R_1(z) \). It has the form

\[
R_1(z) = -\frac{3}{16z^2} + \frac{3(h^2 - 1)}{16k_1^2z} - \frac{3}{16} \sum_{i=1}^{3} \frac{1}{(z - z_i)^2} + \sum_{i=1}^{3} \frac{\gamma_i(z_i)}{z - z_i}.
\] (107)

The coefficients \( \gamma_i(z_i), i = 1, 2, 3 \) have a very complicated form and we do not write them explicitly here. It is possible to note the following properties of the partial fraction expansion of the function \( R_1(z) \).

1. The coefficient \( b_0 \) of \( \frac{b_0}{z^2} \) equals

\[
b_0 = -\frac{3}{16}.
\] (108)

2. The coefficients \( b_1, b_2, b_3 \) of \( \frac{b_i}{(z - z_i)^2}, i = 1, 2, 3 \) equal

\[
b_i = -\frac{3}{16}, \quad i = 1, 2, 3.
\] (109)

3. The Laurent expansion of \( R_1(z) \) at \( z = \infty \) has the form

\[
R_1(z) = \frac{5}{16z^2} + O\left(\frac{1}{z^3}\right).
\] (110)

Our goal is to find Liouville solutions of type 1, type 2, and type 3 of the differential equation (103). To find Liouville solutions of Equation (103), we will use the Kovacic algorithm. First, we search Liouville solutions of type 1. According to the Kovacic algorithm, we calculate the numbers \( \alpha_{\pm}^{\pm} \). For all the roots \( z = 0 \) and \( z = z_i, i = 1, 2, 3 \) these numbers equal

\[
\alpha_{0}^{+} = \frac{3}{4}, \quad \alpha_{0}^{-} = \frac{1}{4}, \quad \alpha_{z_i}^{+} = \frac{3}{4}, \quad \alpha_{z_i}^{-} = \frac{1}{4}, \quad i = 1, 2, 3.
\] (111)

The numbers \( \alpha_{\infty}^{\pm} \) equal

\[
\alpha_{\infty}^{+} = \frac{5}{4}, \quad \alpha_{\infty}^{-} = -\frac{1}{4}.
\] (112)
Since the numbers \(\alpha^+_{0}, \alpha^+_{z_i}\) are determined by Equation (111) and the numbers \(\alpha^+_{\infty}\) are determined by Equation (112), therefore, there are no sets of signs + and – such that the number \(d\) calculating according to Equation (71), can be non-negative integer. This means that there are no Luouvillian solutions of type 1 for the differential equation (103). So, we can state that the following theorem is true.

**Theorem 4.** Under conditions \(d_1 = 0, k_1 \neq 0\), the problem of motion of a heavy rigid body in the Hess case has no Luouvillian solutions of type 1.

Now, we continue the study of the existence of Liouvillian solutions in the problem of the motion of the Hess top under the additional condition \(d_1 = 0\), that is, in the problem of the motion of the Lagrange top when the corresponding second-order linear differential equation has the form (103). Let us study here the problem of the existence of Liouvillian solutions of type 2 for the differential equation (103).

According to the Kovacic algorithm for searching the Liouvillian solutions of type 2, we should find the sets \(E_0, E_{z_i}, i = 1, 2, 3\), which correspond to finite poles of \(R_1(z)\) and the set \(E_{\infty}\), which corresponds to the pole of \(R_1(z)\) at \(z = \infty\). For the finite poles \(z = 0, z = z_i, i = 1, 2, 3\), the sets \(E_0, E_{z_i}\) equal

\[ E_0 = \{1, 2, 3\}, \quad E_{z_i} = \{1, 2, 3\}, \quad i = 1, 2, 3. \]  

(113)

The set \(E_{\infty}\) has the following form:

\[ E_{\infty} = \{-1, 2, 5\}. \]  

(114)

Now, we should calculate the constant \(d\) using Equation (81). Note that the minimal value of the sum

\[ e_0 + e_{z_1} + e_{z_2} + e_{z_3} \]  

(115)

equals 4. Therefore, it is easy to see that the constant \(d\), calculated by the formula (81), is a non-negative integer only for four families \(s = (e_{\infty}, e_0, e_{z_1}, e_{z_2}, e_{z_3})\)

\[ s_1 = (5, 2, 1, 1, 1), \quad s_2 = (5, 1, 2, 1, 1), \quad s_3 = (5, 1, 1, 2, 1), \quad s_4 = (5, 1, 1, 1, 21), \]  

(116)

for which we have \(d = 0\).

We must check all these families. We start with the family \(s_1\). According to the algorithm, let us find the function \(\theta\) by the formula (82). For the family \(s_1\), this function has the form

\[ \theta = \frac{1}{z} + \frac{3z^2 - 8hz + 4(h^2 - 1)}{2(z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2)}. \]  

(117)

The polynomial \(P\) of degree \(d = 0\) \((P \equiv 1)\) should identically satisfy differential equation (83). After substitution of \(P \equiv 1\) and the functions \(\theta\) and \(R_1(z)\) to the differential equation (83), we find that this equation is satisfied identically for any values of parameters \(h\) and \(k_1\). Thus, we can state the following theorem.

**Theorem 5.** In the Lagrange integrable case of motion \(d_1 = 0, k_1 \neq 0\) under the Hess conditions (6), all solutions of a linear differential equation (103) are liouvillian solutions of type 2.

It is easy to see that the general solution of the differential equation (102) can be written as follows:

\[ u(z) = C_1 \exp \left( \int \frac{k_1 dz}{2z \sqrt{z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2}} \right) + C_2 \exp \left( -\int \frac{k_1 dz}{2z \sqrt{z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2}} \right). \]  

(118)

The indefinite integrals in Equation (118) are expressed in terms of elliptic functions. However, the explicit expression for \(u(z)\) in terms of elliptic functions has a very complicated form and we do not write it here.

If we check the other families \(s_2, s_3, s_4\), we obtain the same results on the existing of Liouvillian solutions of type 2 for the differential equation (103).
Now we come back to the investigation of the general case $d_1 \neq 0$ and consider the problem of existence of Liouvillian solutions of type 2 for the differential equation (87).

6 | EXISTENCE OF LIOUVILLIAN SOLUTIONS OF TYPE 2 IN GENERAL CASE

So, we are going to study the problem of existence of Liouvillian solutions of type 2 for the differential equation (43) (or (87)). According to the Kovacic algorithm, we first define the sets $E_c$ and $E_\infty$ for every pole of the function $R(x)$. For the finite poles $x = x_i$, $i = 1, ..., 6$, which are roots of the polynomial (91), these sets $E_{x_i}$ have the form

$$E_{x_i} = \{1, 2, 3\}, \quad i = 1, ..., 6. \quad (119)$$

For the finite poles $x = x_i$, $i = 7, 8, 9$, which are roots of the polynomial (93), these sets $E_{x_i}$ have the form

$$E_{x_i} = \{-2, 2, 6\}, \quad i = 7, 8, 9. \quad (120)$$

The set $E_\infty$ contains only one element and this set equals

$$E_\infty = \{2\}. \quad (121)$$

Now, we should calculate the constant $d$ using the formula (81). Note that the minimal values of the sum of the elements of sets corresponding to finite poles is zero. Therefore, the maximal value of $d$, calculated according to Equation (81), equals $d = 1$. The value $d = 1$ corresponds to the family $s = (e_\infty, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9)$, in which the elements $e_\infty$ and $e_i$, $i = 1, 2, ... 9$ equal

$$s_1 = (2, 1, 1, 1, 1, 1, -2, -2, -2). \quad (122)$$

We also have several families for which the constant $d$, calculated by the formula (81), equals $d = 0$. The following families correspond to $d = 0$:

$$s_2 = (2, 3, 1, 1, 1, 1, -2, -2, -2), \quad s_3 = (2, 1, 3, 1, 1, 1, -2, -2, -2), \quad s_4 = (2, 1, 1, 3, 1, 1, -2, -2, -2),$$
$$s_5 = (2, 1, 1, 1, 3, 1, -2, -2, -2), \quad s_6 = (2, 1, 1, 1, 1, 3, -2, -2, -2), \quad s_7 = (2, 1, 1, 1, 1, 3, -2, -2, -2),$$
$$s_8 = (2, 2, 2, 1, 1, 1, -2, -2, -2), \quad s_9 = (2, 2, 1, 2, 1, 1, -2, -2, -2), \quad s_{10} = (2, 2, 1, 1, 2, 1, -2, -2, -2),$$
$$s_{11} = (2, 2, 1, 1, 2, 1, -2, -2, -2), \quad s_{12} = (2, 2, 1, 1, 1, 2, -2, -2, -2), \quad s_{13} = (2, 1, 2, 2, 1, 1, -2, -2, -2),$$
$$s_{14} = (2, 1, 2, 1, 2, 1, -2, -2, -2), \quad s_{15} = (2, 1, 1, 2, 1, 2, -2, -2, -2), \quad s_{16} = (2, 1, 1, 1, 2, 1, -2, -2, -2),$$
$$s_{17} = (2, 1, 1, 2, 2, 1, -2, -2, -2), \quad s_{18} = (2, 1, 1, 2, 1, 2, -2, -2, -2), \quad s_{19} = (2, 1, 1, 1, 2, 1, -2, -2, -2),$$
$$s_{20} = (2, 1, 1, 1, 2, 1, -2, -2, -2), \quad s_{21} = (2, 1, 1, 1, 2, 1, -2, -2, -2), \quad s_{22} = (2, 1, 1, 1, 1, 2, -2, -2, -2). \quad (123)$$

We must check all these families. We start with the family $s_1$. According to the algorithm, let us find the function $\theta$ by the formula (82). For the set $s_1$, this function has the form

$$\theta = \frac{3x^5 - 8hx^3 + 4(h^2 - 1)x}{x^6 - 4hx^4 + 4(h^2 - 1)x^2 + 4k_1^2} - \frac{3d_1x^2}{d_1x^3 - k_1}. \quad (124)$$

The polynomial $P$ of degree $d = 1$

$$P = x + B \quad (125)$$

should identically satisfy the differential equation (83). After substitution of $P = x + B$ and the functions $\theta$ and $R(x)$ to Equation (83), we obtain in the left-hand side of Equation (83) the rational expression. The numerator of this expression has a form of the ninth degree polynomial:

$$P_9 = -Bd_1^2(1 + 4d_1^2)x^9 + \cdots \quad (126)$$
Let us put \( B = 0 \). Then the numerator of the rational expression in the left-hand side of Equation (83) takes the form

\[
P_2 = -12xk_1d_1 (d_1x^3 - k_1)^2 + \cdots
\]  

(127)

This polynomial becomes zero when \( k_1 = 0 \) or when \( d_1 = 0 \). Therefore, we can state the following theorem based on the verification of the family \( s_1 \).

**Theorem 6.** For the existence of Liouvillian solutions of type 2 in the problem of motion of a heavy rigid body with a fixed point in the Hess case, one of the two conditions must be satisfied:

\[
d_1 = 0 \quad \text{or} \quad k_1 = 0.
\]  

(128)

In other words, Liouvillian solutions of type 2 can exist either in the case, when the moving rigid body is the Lagrange top, or in the Hess case, if the constant of the area integral is zero.

The fact, that the problem of motion of a heavy rigid body with a fixed point in the Hess case for \( k_1 = 0 \) is integrable in elliptic functions (which are Liouvillian functions), was first discovered by Nekrasov [7, 8].

If we check the other families \( s_2, \ldots, s_{22} \), we obtain the same results on the existing of Liouvillian solutions of type 2 for the differential equation (6) (for more details see Refs. [38, 39]). Finally, we can state that the conditions of existence of Liouvillian solutions of type 2 for the differential equation (87) are formulated in Theorem 6. To confirm the obtained results, let us consider Equation (43) in the case when \( k_1 = 0 \).

### 7 | EXISTENCE OF LIOUVILLIAN SOLUTIONS OF TYPE 2 IN THE CASE OF ZERO CONSTANT OF THE AREA INTEGRAL

In the case \( k_1 = 0 \), the differential equation (43) takes the form

\[
d^2u \over dx^2 + x^4 - 4(h^2 - 1) \over x(x^2 - 2h - 2)(x^2 - 2h + 2) d u \over dx + d^2x^2 \over (x^2 - 2h - 2)(x^2 - 2h + 2) u = 0.
\]  

(129)

In Equation (129), we change now the independent variable by the formula \( x^2 = z \). As a result, Equation (129) can be rewritten as follows:

\[
d^2u \over dz^2 + {z - 2h \over (z - 2h - 2)(z - 2h + 2)} d u \over dz + {d^21 \over 4(z - 2h - 2)(z - 2h + 2)} u = 0.
\]  

(130)

In the obtained linear differential equation (130), we make the change of variables (51) and reduce this equation to the form (52)

\[
{d^2y \over dz^2} = R_2(z)y.
\]  

(131)

In this case, function \( R_2(z) \) has the form

\[
R_2(z) = \frac{4d_1^2 - 8 - (1 + d_1^2)(2h - 2)^2}{4(z - 2h - 2)(z - 2h + 2)^2}.
\]  

(132)

Thus, the function \( R_2(z) \) has two finite poles of the second order at \( z_1 = 2h + 2 \) and \( z_2 = 2h - 2 \). Partial fraction expansion of the function \( R_2(z) \) has the form

\[
R_2(z) = -\frac{3}{16(z - z_1)^2} + \frac{1 - 2d_1^2}{32(z - z_1)} - \frac{3}{16(z - z_2)^2} + \frac{2d_1^2 - 1}{32(z - z_2)}.
\]  

(133)

It is possible to note the following properties on the partial fraction expansion of the function \( R_2(z) \).
1. The coefficients \( b_1, b_2 \) of \( \frac{b_i}{(z-z_i)^2} \), \( i = 1, 2 \) equal

\[
b_1 = -\frac{3}{16}, \quad i = 1, 2.
\] (134)

2. The Laurent expansion of \( R_2(z) \) at \( z = \infty \) has the form

\[
R_2(z) = -\left(1 + 4d_1^2\right)z^2 + O\left(\frac{1}{z^3}\right).
\] (135)

To find Liouvillian solutions of type 2 of the differential equation (131), we will use the Kovacic algorithm. According to the algorithm, let us define the sets \( E_z^1 \), corresponding to finite poles of the function \( R_2(z) \) and the set \( E_\infty \) corresponding to the pole of \( R_2(z) \) at \( z = \infty \). For the finite poles \( z = z_1 \) and \( z = z_2 \), the sets \( E_z^1, E_z^2 \) have the form

\[
E_z^1 = \{1, 2, 3\}, \quad E_z^2 = \{1, 2, 3\}.
\] (136)

The set \( E_\infty \) contains only one element and this set has the form

\[
E_\infty = \{2\}.
\] (137)

It is easy to see that the constant \( d \), calculated by the formula (81), is a non-negative integer only for the family

\[
s = (e_\infty, e_z^1, e_z^2) = (2, 1, 1),
\] (138)

for which we have \( d = 0 \). Using this family, let us find the function \( \vartheta \) by the formula (82). This function has the form

\[
\vartheta = \frac{z - 2h}{(z - 2h - 2)(z - 2h + 2)}.
\] (139)

The polynomial \( P \) of degree \( d = 0 \) has the form \( P \equiv 1 \). This polynomial should identically satisfy the differential equation (83). After substitution of \( P = 1 \) and the functions \( \vartheta \) and \( R_2(z) \) to the differential equation (83), we find that this equation is satisfied identically for any values of parameters \( d_1 \) and \( h \). Thus, we can state the following theorem.

**Theorem 7.** In the case of motion of a heavy rigid body with a fixed point in the Hess case \( (d_1 \neq 0) \), Equation (87) has Liouvillian solutions only in the case \( k_1 = 0 \).

Indeed, the general solution of the equation (130) can be represented as follows:

\[
u(z) = C_1 \sin\left(\frac{d_1}{2} \ln \left(z - 2h + \sqrt{(z - 2h - 2)(z - 2h + 2)}\right)\right) + C_2 \cos\left(\frac{d_1}{2} \ln \left(z - 2h + \sqrt{(z - 2h - 2)(z - 2h + 2)}\right)\right),
\] (140)

where \( C_1 \) and \( C_2 \) are arbitrary constants. It is easy to see that this function is a Liouvillian function of type 2.

The results of the Sections 4–7 have been obtained in general case, when we suppose that all the nine finite poles of the function \( R(x) \) defined by Equation (88), are distinct. Now, we will briefly discuss the special cases, when these poles can coincide with each other, that is, the function \( R(x) \) can have multiple roots.

8 | **BRIEF DESCRIPTION OF THE SPECIAL CASES**

The function \( R(x) \) defined by Equation (88) has a denominator, which is the square of the product of two polynomials: the sixth degree polynomial (91) and the cubic polynomial (93). Let us study now the polynomial (91). As we already noted, this polynomial includes only even degrees of an independent variable, as a result of which by changing \( x^2 = z \), we can reduce it to a third-degree polynomial with respect to \( z \):

\[
P_{3z} = z^3 - 4hz^2 + 4(h^2 - 1)z + 4k_1^2 = 0.
\] (141)
Let us study the type of roots of the polynomial (141). If we put
\[ z = y + \frac{4h}{3} \]  
(142)
then the polynomial (141) takes the form
\[ y^3 - 4 \left( 1 + \frac{h^2}{3} \right) y + 4 \left( k_1^2 + \frac{4h^3}{27} - \frac{4h}{3} \right) = 0. \]  
(143)
Thus the third-degree polynomial (143) has the form
\[ y^3 + 3py + 2q = 0, \]  
(144)
where we denote
\[ p = -\frac{4}{3} \left( 1 + \frac{h^2}{3} \right), \quad q = 2 \left( k_1^2 + \frac{4h^3}{27} - \frac{4h}{3} \right). \]  
(145)
The character of the roots of the polynomial (143) is determined by the sign of the expression
\[ D = q^2 + p^3. \]  
(146)
If \( D > 0 \), then the polynomial (143) has one real root and two complex-conjugate roots. If \( D < 0 \), then the polynomial (143) has three distinct real roots. If \( D = 0 \), then the polynomial (143) has a multiple root and all of its roots are real. In the explicit form, expression \( D \) can be written as follows:
\[ D = 4 \left( k_1^4 + \frac{8}{27} (h^2 - 9) k_1^2 h - \frac{16}{27} (h^2 - 1)^2 \right). \]  
(147)
Thus, if Equation (147) is zero, the polynomial (91) has multiple roots.

We consider now the cubic polynomial (93). The coefficient \( d_1 \) is such that \( d_1 \in (-1, 0] \). Let us denote
\[ k_1 = c^3 d_1, \]  
(148)
where \( c \) is a new parameter. Then we can write the polynomial (93) as follows:
\[ d_1 \left( x^3 - c^3 \right) = 0. \]  
(149)
The roots of Equation (149) have the following form:
\[ x_1 = c, \quad x_2 = \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) c, \quad x_3 = \left( -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) c. \]  
(150)
It is easy to see that all these roots \( x_1, x_2, x_3 \) are distinct for \( c \neq 0 \). Therefore, since we do not consider the case \( k_1 = 0 \) here, the polynomial (150) cannot have multiple roots. However the situation is possible when one of the roots (150) of the polynomial (149) is the root of the polynomial (91). The all possible special cases have been investigated in Refs. [38, 39]. In all special cases, the study showed the absence of Liouvillian solutions.

9  |  CONCLUSIONS

In this paper, we considered the problem of motion of a heavy rigid body with a fixed point in the Hess case. The integration of this problem is reduced to solving the second-order linear differential equation. To obtain this equation, we wrote equations of motion of the body in the special Kharlamov coordinate system. From these equations, we derived the second-order linear differential equation with rational coefficients. Using the Kovacic algorithm, we studied the problem of existence of Liouvillian solutions of the corresponding second-order linear differential equation. We proved that the
obtained second-order differential equation has no Liouvillian solutions of type 1 and type 3. As for the Liouvillian solutions of type 2, they exist only if one of the two conditions is satisfied: either $d_1 = 0$ (the moving rigid body is the Lagrange top) or $k_1 = 0$ (the constant of the area integral is zero). For any other values of parameters, the problem of motion of a heavy rigid body with a fixed point in the Hess case cannot be solved in Liouvillian functions. Thus, we completely solved the problem of existence of Liouvillian solutions in the problem of motion of a heavy rigid body with a fixed point in the Hess case.

**ACKNOWLEDGMENT**

This work was supported financially by the Russian Foundation for Basic Research (Grant No. 20-01-00637).

**CONFLICT OF INTEREST**

The authors declare that there is no conflict of interest.

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How to cite this article: Bardin, B.S., Kuleshov, A.S.: Application of the Kovacic algorithm for the investigation of motion of a heavy rigid body with a fixed point in the Hess case. Z. Angew. Math. Mech. 102, e202100036 (2022). https://doi.org/10.1002/zamm.202100036