GEOMETRIC QUANTIZATION OF VECTOR BUNDLES

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ABSTRACT. I repeat my definition for quantization of a vector bundle. For the cases of the Toeplitz and geometric quantizations of a compact Kähler manifold, I give a construction for quantizing any smooth vector bundle which depends functorially on a choice of connection on the bundle.

INTRODUCTION

Traditionally, “quantization” has meant some sort of process that, given a classical, symplectic phase space, produces a noncommutative algebra of quantum observables. The concept of noncommutative geometry (see [3]) suggests that such a noncommutative algebra can be thought of as the algebra of functions on a “noncommutative space”, so perhaps quantization could be made into a way of constructing a noncommutative geometry from a classical geometry.

However, as it stands, quantization is only a procedure for constructing an algebra. Since the algebra of continuous (or smooth) functions contains only the information of the point-set (or differential) topology of a space, this is merely the quantization of topology. It would be desirable to extend quantization to a theory of quantization of geometry.

Beyond topology, vector bundles are arguably the second most fundamental structure in geometry, so a plausible first step towards a theory of quantizing geometry would be a theory of quantizing vector bundles. I began constructing such a theory in [7] by giving a definition of vector bundle quantization and a procedure for quantizing the equivariant vector bundles over coadjoint orbits of compact, semi-simple Lie groups.

I continue the story in this paper by giving a procedure for quantizing arbitrary smooth vector bundles over compact Kähler manifolds. The construction depends only on the structures used to quantize the manifold, the vector bundle itself, and a connection on the vector bundle.

1. Generalities

Most of the symbols defined in this section will be defined again through constructions in Sec’s 2 and 3. The theorems of Sec. 3 will show that these constructions actually do satisfy the original definitions.

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Recall that a continuous field of C*-algebras (see [4, 9]) is the natural notion of a bundle of C*-algebras. The fibers are all C*-algebras, the space of (continuous) sections is a C*-algebra, and for each point of the base space there is an evaluation map, a *-homomorphism of the algebra of sections onto the fiber algebra.

By the most general definition (see [10, 11, 12]), a strict deformation quantization of a (Poisson) manifold \( M \) consists of a continuous field of C*-algebras, \( A^I \), and a (total) quantization map. Conventionally, the base space \( I \) of the continuous field is the set of possible values of \( \hbar \); more generally, it is just some set containing an accumulation point \( \infty \in \hat{I} \) which plays the role of \( \hbar = 0 \). The fiber of the continuous field at this “classical limit” point is \( \mathcal{C}(M) \), the C*-algebra of continuous functions on \( M \).

**Definition.** \( A \) is the C*-algebra of continuous sections of \( A^I \). \( P: A \to \mathcal{C}(M) \) is the evaluation homomorphism at \( \infty \in \hat{I} \).

In most of this paper the quantization map is a map \( Q: \mathcal{C}(M) \to A \); more generally, the domain of \( Q \) may be only a dense subalgebra of \( \mathcal{C}(M) \), but it must contain the smooth functions \( \mathcal{C}^\infty(M) \). The composition \( P \circ Q \) is required to be the identity map; that is, applying the quantization map to a function \( f \in \mathcal{C}(M) \) gives a continuous section of \( A^i \) whose value at \( \infty \in \hat{I} \) is \( f \). Finally, \( Q \) is required to commute with the involution (\( * \)-structure) and fit a relation with the Poisson bracket.

Specifying the total quantization map, \( Q \), is equivalent to specifying, for each point \( i \in J := \hat{I} \setminus \{\infty\} \), a map \( Q_i \) to \( A_i \), which is just \( Q \) composed with the evaluation at \( i \). It may be possible to reconstruct the continuous field \( A^i \) from this system of quantization maps. The total quantization map \( Q \) can be reconstructed as the direct product of the \( Q_i \)'s. The codomain of this reconstructed quantization map is superficially the C*-algebraic direct product \( \prod_{i \in J} A_i \); however, the image of \( Q \) is actually contained in \( A \subset \prod_{i \in J} A_i \) and will usually generate \( A \) (as a C*-algebra) in most cases. Given a proposed system of quantization maps \( Q_i \), it is a nontrivial convergence condition that the image of \( Q \) consists of sections of some continuous field.

For some purposes, including defining quantization of a vector bundle, the continuous field \( A^i \) is the only important structure. For this reason, in [7], I gave the following:

**Definition.** A general quantization of \( M \) is a continuous field \( A^i \) with \( \mathcal{C}(M) \) as the fiber over \( \infty \).

This is just enough structure to define whether or not a sequence of operators converges to a given function on \( M \).

The context of this paper is the geometric quantization of compact Kähler manifolds. In this case \( \hat{I} = \hat{\mathbb{N}} := \{1, 2, \ldots, \infty\} \), the 1-point compactification of the positive integers. The algebras \( A_N \) for each \( N \in \mathbb{N} \) are (finite-dimensional) matrix algebras, and if \( M \) is connected, they are simple (i.e., “full”) matrix algebras.
Assume for simplicity that $M$ is connected. Then, all the information of the general quantization is contained in $A$ and $P$. The index set $\mathbb{N}$ can be recovered as the spectrum of the center of $A$.

The ideal $A_0 := \ker P$ is the algebra of sections of $A_{\mathbb{N}}$ that vanish at $\infty$. Equivalently, $A_0$ consists of those sections over $\mathbb{N}$ for which the sequence of norms converges to 0. Since $\mathbb{N}$ is discrete, $A_0$ is just the $C^*$-algebraic direct sum $\bigoplus_{N \in \mathbb{N}} A_N$.

I shall be concerned with two choices of quantization maps here. The first are the Toeplitz quantization maps $T_N$; these are manifestly completely positive and thus defined on all of $C(M)$. The second type are the geometric quantization maps $Q_N$; as I shall show in Thm. 5.2, these correspond to the same general quantization as the Toeplitz quantization maps do.

Note that a general quantization can be phrased as an extension

$$0 \to A_0 \to A \xrightarrow{P} C(M) \to 0. \quad (1.1)$$

The total Toeplitz quantization map $T : C(M) \to A$ gives a completely positive splitting of (1.1).

### 1.1. Quantized vector bundles

The category equivalence of vector bundles over $M$ with finitely generated projective (f. g. p.) modules of $C(M)$ is well known. The $C(M)$-module corresponding to a vector bundle $V$ over $M$ is the space of continuous sections $\Gamma(M, V)$. This suggests the following definition (see [7]).

**Definition.** Given a general quantization, expressed as $P : A \to C(M)$, a quantization of a vector bundle $V$ is any f. g. p. $A$-module, $V$, such that the push-forward by $P$ is $P_*(V) = \Gamma(M, V)$.

For every $i \in \mathbb{N}$, pushing $V$ forward by the evaluation homomorphism gives a module $V_i$ of $A_i$. The $A$-module $V$ is equivalent to a bundle of modules over $\mathbb{N}$ whose fiber over $i$ is $V_i$.

It is not obvious *a priori* that any quantization of $V$ will exist, or that it will be at all unique. To investigate these issues, it is helpful to consider $K$-theory; the group $K^0(M)$ classifies vector bundles; the group $K_0(A)$ classifies f. g. p. $A$-modules (which are the quantized vector bundles).

The short exact sequence (1.1) leads, as usual, to a six-term, periodic exact sequence in $K$-theory; incorporating the identity $K_*[C(M)] = K^*(M)$, this reads,

$$
\begin{array}{c}
K_0(A_0) \xrightarrow{\beta} K_0(A) \xrightarrow{\alpha} K^0(M) \\
\quad \Uparrow \quad \Uparrow \\
K^1(M) \xleftarrow{\beta} K_1(A) \xleftarrow{\alpha} K_1(A_0).
\end{array}
\quad (1.2)
$$

Assume that $\mathbb{N}$ is discrete, and the $A_i$’s are full matrix algebras. Then $A_0$ is just the direct sum $\bigoplus_i A_i$, and the $K$-groups are direct sums $K_*(A_0) = \bigoplus K_*(A_i)$. Since each $A_i$ is a full matrix algebra, its $K$-theory is $K_0(A_i) \cong \mathbb{Z}$ and $K_1(A_i) = 0$. This gives that $K_0(A_0) \cong \mathbb{Z}^\oplus \infty$ and $K_1(A_0) = 0$. 


The algebraic direct sum $\mathbb{Z}^{\oplus \infty}$ is the set of sequences of integers with finitely many nonzero terms. This corresponds to the fact that an f. g. p. module of $A_0$ is a direct sum of (finitely generated) modules of finitely many $A_N$'s. Any $A_0$-module is also an $A$-module; the f. g. p. $A_0$-modules are precisely those $A$-modules which are finite-dimensional as vector spaces. This identification corresponds to the map $\beta$ in (1.2), and shows that $\beta$ must be injective, which, by exactness, shows that the map $\alpha$ is zero. With this, the exact sequence (1.2) breaks down into the isomorphism

$$K_1(\mathcal{P}) : K_1(A) \xrightarrow{\sim} K^1(M)$$

and the short exact sequence

$$0 \to \mathbb{Z}^{\oplus \infty} \to K_0(A) \xrightarrow{\phi_0(\mathcal{P})} K^0(M) \to 0. \quad (1.3)$$

This $K_0(\mathcal{P})$ maps the K-class of a quantization of a vector bundle, $V$, to the K-class of $V$. The exact sequence (1.3) shows that $K_0(\mathcal{P})$ is surjective, which suggests that any vector bundle can be quantized.

**Theorem 1.1.** For a general quantization $A_\beta$ such that $\beta$ is discrete and the fibers over $I$ are matrix algebras, every vector bundle can be quantized.

**Proof.** Any vector bundle can be realized as the image of some idempotent matrix of continuous functions $e \in M_m(C(M))$. The ideal $A_0$ consists of all compact operators in $A$, so any element of the preimage $\mathcal{P}^{-1}(e)$ is “essentially” idempotent (i.e., modulo compacts). There therefore exists an idempotent $\tilde{e}$ such that $\mathcal{P}(\tilde{e}) = e$. The right image $A^m\tilde{e}$ is a quantization of the vector bundle in question.

The short exact sequence (1.3) also shows that $\ker K_0(\mathcal{P}) = \mathbb{Z}^{\oplus \infty}$. This means that the K-class of a quantization of $V$ is uniquely determined by $V$, modulo $\mathbb{Z}^{\oplus \infty}$. This suggests:

**Theorem 1.2.** With the hypothesis of Theorem 1.1, quantization of a vector bundle is unique modulo finite-dimensional modules.

**Proof.** We need to prove that if $V$ and $V'$ are quantizations of $V$, then there exists a module homomorphism from $V$ to $V'$ whose kernel and cokernel are finite-dimensional (in other words, a Fredholm homomorphism).

Any f. g. p. module can be realized as the (right) image of an idempotent matrix over $A$. So, identify $V$ and $V'$ with the images of idempotents $e, e' \in M_m(A)$; that is, $V = A^m e$ and $V' = A^m e'$. These idempotents can be chosen so that $\mathcal{P}(e) = \mathcal{P}(e')$; therefore, $e - e' \in M_m(A_0)$. Multiplication by $e'$ (respectively, $e$) gives a homomorphism $\varphi' : V \xrightarrow{\sim} V'$ (resp., $\varphi : V' \xrightarrow{\sim} V$).

Let $k$ be the self-adjoint idempotent whose (right) image is $\ker \varphi \circ \varphi'$. Since this is a subspace of $V$, $k$ satisfies $ke = k$, and since this is the kernel of $\varphi \circ \varphi'$, $k$ satisfies $ke'e = 0$. A priori $k$ is not necessarily in $M_m(A)$. However, the entries of $k$ are in the C'*-algebraic direct product of the $A_i$'s; i.e., bounded sections of $A_\beta$ over $I$. $A_0$ is an ideal in this algebra, so $k = k(e - e')e \in M_m(A_0)$. Therefore, $\ker \varphi \circ \varphi'$ is...
an f. g. p. module of \( A_0 \) and is thus finite-dimensional. By an identical argument, \( \ker \varphi' \circ \varphi \) is also finite-dimensional. This implies that the kernel and cokernel of \( \varphi' \) are finite-dimensional.

The converse is clearly also true: If \( V \) is a quantization of \( V \), and \( V' \) is isomorphic to \( V \) modulo finite-dimensional modules, then \( V' \) is also a quantization of \( V \).

2. Toeplitz Quantization

Again, let \( M \) be a compact, connected Kähler manifold. Now, let \( L \) be a Hermitian line bundle with curvature given by the symplectic form as \( \nabla^2 = -i\omega \) and \( L_0 \) a holomorphic line bundle with an inner product on sections (i. e., a pre-Hilbert structure on \( \Gamma(M, L_0) \)).

**Definition.** For each \( N \in \mathbb{N} \), \( L_N := L_0 \otimes L^\otimes N \), \( \mathcal{H}_N := \Gamma_{\text{hol}}(M, L_N) \) (the space of holomorphic sections of \( L_N \)), and \( A_N := \text{End} \mathcal{H}_N \) (matrices over \( \mathcal{H}_N \)).

The inner products on sections of \( L_0 \) and fibers of \( L \) combine to give an inner product on sections of \( L_N \). This makes \( \mathcal{H}_N \) into a Hilbert space; it does not need to be completed, since it is finite-dimensional. The Hilbert space structure of \( \mathcal{H}_N \) makes \( A_N \) a C*-algebra.

The connections and inner products must be compatible. For convenience, assume that \( L_1 \) (and thus any \( L_{N \in \mathbb{N}} \)) is “positive” (if not, just reparameterize \( N \)). This guarantees that \( A_N \) is nontrivial for all \( N \in \mathbb{N} \). The simplest choice of \( L_0 \) is just the trivial line bundle with the trivial connection and the inner product given by integrating with the canonical volume form \( \omega^n/n! \) (\( n := \dim_{\mathbb{C}} M \)).

The space \( \mathcal{H}_N \) is naturally a Hilbert subspace of \( L^2(M, L_N) \) which is a subspace of the Hilbert space of \((0, \ast)\)-forms with coefficients in \( L_N \).

**Definition.** Let \( \Pi_N \) be the self-adjoint projection onto \( \mathcal{H}_N \). The Toeplitz quantization map \( T_N : \mathcal{C}(M) \to A_N \) is given by

\[
T_N(f) := \Pi_N f. \tag{2.1}
\]

In other words, the action of \( T_N(f) \) on an element of \( \mathcal{H}_N \) (holomorphic section of \( L_N \)) is given by first multiplying by \( f \) (giving a non-holomorphic section) and then projecting back to \( \mathcal{H}_N \) by \( \Pi_N \). This \( T_N \) is automatically a unital and (completely) positive map; therefore, it is norm-contracting.

2.1. Vector bundles. Suppose that we are given a smooth vector bundle \( V \) with a specific connection. I would like to construct from \( V \) a sequence of \( A_N \) modules. The algebra \( A_N \) can be written as \( A_N = \text{End} \mathcal{H}_N = \text{Hom}(\mathcal{H}_N, \mathcal{H}_N) \) and can be thought of as consisting of square matrices of height and width \( \mathcal{H}_N \). Any module of \( A_N \) can be written as \( \text{Hom}(E, \mathcal{H}_N) \) and thought of as consisting of rectangular matrices of height \( \mathcal{H}_N \) and width \( E \). Any construction for \( E \) should generalize that of \( \mathcal{H}_N \).

Thanks to the Kodaira vanishing theorem (see the appendix and e. g., [3]) and the assumption that \( L_1 \) is positive, \( \mathcal{H}_N = \Gamma_{\text{hol}}(M, L_N) \) can also be realized as the kernel of the \( L_N \)-twisted Dolbeault operator that acts on \( \Omega^{0,\ast}(M, L_N) \).
In order to generalize $\mathcal{H}_N$ appropriately, we will need:

**Definition.** $D_V := \nabla_\partial + (\nabla_\partial)^* = i\gamma^\mu \nabla_\mu$ is the $V^* \otimes L_N$-twisted Dolbeault operator, a Dirac-type operator acting on the smooth $(0,\ast)$-forms $\Omega^{0,\ast}(M, V^* \otimes L_N)$.

Here, $\nabla$ is the connection, and the Dirac matrices satisfy $[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu$, which differs from the usual convention by a factor of 2. A number of inequalities related to Dolbeault operators will prove useful, but the proofs of these are relegated to the appendix.

Natural generalizations of $\mathcal{A}_N$ and $T_N$ are,

**Definition.** $\tilde{V}_N := \text{Hom}(\tilde{E}_N, \mathcal{H}(N))$ where $\tilde{E}_N = \ker D_V$. The map $T_N^\ast : \Gamma(M, V) \to \tilde{V}_N$ is given by

$$T_N^\ast(v) := \Pi_N v. \quad (2.2)$$

In this, multiplication must be understood to mean contraction of $V$ with $V^*$. Multiplying an element of $\tilde{E}_N^\ast \subset \Omega^{0,\ast}(M, V^* \otimes L_N)$ by $v$ gives an element of $\Omega^{0,\ast}(M, L_N)$; $\Pi_N$ then projects this down to $\mathcal{H}_N$. If $V$ is trivial (i. e., $V = \mathbb{C} \times M$ with the trivial connection), then $T_N^\ast$ reduces to $T_N$.

The tildes will be dispensed with by the end of the next section.

3. CONVERGENCE

If $V$ has an inner product — or if we assign one — then there is a natural operator norm on $\tilde{V}_N$ which generalizes the norm on $\mathcal{A}_N$. This corresponds to the norm $\|v\| := \sup_{x \in M} \|v(x)\|$ on sections. None of the constructions here will require an inner product on $V$; however, several of the proofs will make use of one — which can be taken arbitrarily.

The main property of the maps $T_N$ and $T_N^\ast$ that is needed to prove convergence of the quantization of both the algebra and vector bundles is

**Lemma 3.1.** For any function $f \in \mathcal{C}(\mathcal{M})$ and any section $v \in \Gamma(M, V)$,

$$\lim_{N \to \infty} ||T_N(f)T_N^\ast v - T_N^\ast (fv)|| = 0.$$

**Proof.** Let $D$ be the $L_N$-twisted Dolbeault operator, so that $\mathcal{H}_N = \ker D$. We can approximate $\Pi_N$ by $(1 + \alpha D^2)^{-1}$ with $\alpha$ a positive real number; in fact, by Lemma [A.1] (in the appendix) there exists a constant $C < 1$ such that

$$\text{Spec } D^2 \subset \{0\} \cup [N - C, \infty), \quad (3.1)$$

so the error in $\Pi_N \approx (1 + \alpha D^2)^{-1}$ is bounded as

$$||\Pi_N - (1 + \alpha D^2)^{-1}|| \leq (1 + \alpha[N - C])^{-1} \leq \alpha^{-1}(N - C)^{-1}.$$

Now, for $f \in \mathcal{C}_c(\mathcal{M})$ any smooth function, the commutator $[f, \Pi_N]_-$ is approximated by

$$[f, (1 + \alpha D^2)^{-1}]_- = \alpha(1 + \alpha D^2)^{-1}[D^2, f]_- (1 + \alpha D^2)^{-1} = i\alpha(1 + \alpha D^2)^{-1}[D, \gamma^\mu f]_+(1 + \alpha D^2)^{-1}.$$
Another consideration of (3.1) shows that,
\[ \left\| (1 + \alpha D^2)^{-1} D \right\| \leq \frac{(N - C)^{1/2}}{1 + \alpha(N - C)} \leq \alpha^{-1}(N - C)^{-1/2}. \]
So,
\[ \left\| [f, (1 + \alpha D^2)^{-1}] \right\| \leq 2(N - C)^{-1/2} \left\| \gamma^\mu f_\mu \right\| = \sqrt{2}(N - C)^{-1/2}\|\nabla f\|. \]
This gives that
\[ \left\| [f, \Pi_N] \right\| \leq \sqrt{2}(N - C)^{-1/2}\|\nabla f\| + 2\alpha^{-1}(N - C)^{-1}\|f\|, \]
for any \( \alpha > 0 \), and therefore,
\[ \left\| [f, \Pi_N] \right\| \leq \sqrt{2}(N - C)^{-1/2}\|\nabla f\|. \]

With a slight abuse of notation,
\[ T_N(f)T_N^\vee(v) - T_N(fv) = T_N^\vee(f\Pi_Nv - fv) = T_N^\vee([f, \Pi_N]v). \]
By construction, \( T_N^\vee \) is norm-contracting; thus,
\[ \left\| T_N(f)T_N^\vee(v) - T_N^\vee(fv) \right\| \leq \sqrt{2}(N - C)^{-1/2}\|\nabla f\| \|v\|. \]
So \( \|T_N(f)T_N^\vee(v) - T_N^\vee(fv)\| \to 0 \) as \( N \to \infty \). Since \( T_N \) and \( T_N^\vee \) are norm-contracting, they are continuous, and since \( \mathcal{C}^\infty(M) \subset \mathcal{C}(M) \) is a dense subalgebra, the conclusion holds for all \( f \in \mathcal{C}(M) \).

**Definition.** The total Toeplitz quantization map \( T : \mathcal{C}(M) \to \prod_{N \in \mathbb{N}} \mathcal{A}_N \) is the direct product of the \( T_N \)'s. Also \( \mathcal{A}_0 := \bigoplus_{N \in \mathbb{N}} \mathcal{A}_N \) is the \( \mathbb{C}^* \)-algebraic direct sum, and \( \mathbb{A} := \text{Im } T + \mathcal{A}_0 \).

**Lemma 3.2.** \( \mathbb{A} \) is a \( \mathbb{C}^* \)-algebra, and \( T \) induces an isomorphism \( \mathcal{C}(M) \xrightarrow{\sim} \mathbb{A}/\mathcal{A}_0 \).

**Proof.** Lemma 3.1 is in this case equivalent to the statement that for any functions \( f, g \in \mathcal{C}(M) \),
\[ T(f)T(g) - T(fg) \in \mathcal{A}_0. \] (3.2)
The direct sum \( \mathcal{A}_0 \) is an ideal in the direct product \( \prod_{N \in \mathbb{N}} \mathcal{A}_N \), so (3.2) shows that \( \mathbb{A} \) is algebraically closed. Since \( T \) is norm-contracting, \( \text{Im } T \) is norm-closed, and so \( \mathbb{A} \) is norm-closed. Hence, \( \mathbb{A} \) is a \( \mathbb{C}^* \)-algebra.

Equation (3.2) also shows that \( T \) induces (by composition with the quotient map \( \mathbb{A} \to \mathbb{A}/\mathcal{A}_0 \)) a \( * \)-homomorphism \( \mathcal{C}(M) \to \mathbb{A}/\mathcal{A}_0 \). This is surjective because of the definition of \( \mathbb{A} \). We need to verify that it is injective.

Since \( \mathbb{A} \) lies inside the direct product of the \( \mathcal{A}_N \)'s, there is for each \( N \in \mathbb{N} \) an obvious “evaluation” homomorphism \( \mathcal{P}_N : \mathbb{A} \to \mathcal{A}_N \). Define the normalized partial traces \( \tilde{\text{tr}}_N : \mathbb{A} \to \mathbb{C} \) by \( \tilde{\text{tr}}_N a := \text{tr}[\mathcal{P}_N(a)]/\dim \mathcal{H}_N \), so that \( \tilde{\text{tr}}_N 1 = 1 \). The normalized trace is norm-contracting, so any \( a \in \mathbb{A} \) satisfies \( |\tilde{\text{tr}}_N a| \leq \|\mathcal{P}_N(a)\| \); therefore,
\[ a \in \mathcal{A}_0 \implies \lim_{N \to \infty} \tilde{\text{tr}}_N a = 0. \] (3.3)
Note that for any \( f \in \mathcal{C}(M) \), the (unnormalized) trace of \( T_N(f) \) can be expressed as
\[
\text{tr}[T_N(f)] = \text{Tr}[\Pi_N f] = \lim_{t \to \infty} \text{Tr}[e^{-tD^2} f].
\]
Using the asymptotic expansion for \( e^{-tD^2} \) (the “heat kernel expansion”, see [5]), this can be evaluated explicitly as a polynomial in the curvatures of \( TM \) and \( L_N \). The result is a polynomial in \( N \) with leading order term
\[
\left( \frac{N}{\pi} \right)^n \int_M f \omega^n / n!.
\]
This, with (3.3), shows that \( \tilde{\text{tr}}_\infty := \lim_{N \to \infty} \tilde{\text{tr}}_N \) is well-defined on \( \mathcal{A} \), vanishes on \( \mathcal{A}_0 \) and satisfies
\[
\tilde{\text{tr}}_\infty[T(f)] = \frac{1}{\text{vol} M} \int_M f \omega^n / n!.
\]
Suppose that some function \( f \) is in the kernel of the induced homomorphism \( \mathcal{C}(M) \to \mathcal{A}/\mathcal{A}_0 \), or equivalently that \( T(f) \in \mathcal{A}_0 \). The kernel of a \( * \)-homomorphism is spanned by its positive elements, so we can assume without loss of generality that \( f \geq 0 \). This implies that \( 0 = \tilde{\text{tr}}_\infty[T(f)] \propto \int_M f \omega^n / n! \), but since \( \omega^n \) is nonvanishing this implies \( f = 0 \). So, the homomorphism is injective and thus an isomorphism.

**Definition.** \( P : \mathcal{A} \to \mathcal{C}(M) \) is the composition of the natural surjection \( \mathcal{A} \to \mathcal{A}/\mathcal{A}_0 \) with the inverse of the isomorphism induced by \( T \).

The following shows that \( \mathcal{A} \) indeed gives a general quantization of \( M \).

**Theorem 3.3.** There is a continuous field of \( C^* \)-algebras over \( \hat{\mathcal{N}} \) such that the fiber over \( N \in \mathbb{N} \) is \( \mathcal{A}_N \), the fiber over \( \infty \) is \( \mathcal{C}(M) \), and the algebra of continuous sections is \( \mathcal{A} \).

**Proof.** Let \( P_N : \mathcal{A} \to \mathcal{A}_N \) denote the evaluation map at \( N \). Most of the axioms given in [4] for a continuous field of \( C^* \)-algebras are easily verified. The nontrivial axiom is the requirement that for any \( a \in \mathcal{A} \) the norms \( ||P_N(a)|| \) define a continuous function on \( \hat{\mathcal{N}} \). Since continuity is only an issue at \( \infty \in \hat{\mathcal{N}} \), this reduces to the requirement that \( ||P_N(a)|| \to ||P(a)|| \) when \( N \to \infty \). It is sufficient to prove this on \( \text{Im} T \); in other words, we need to show that for any \( f \in \mathcal{C}(M) \),
\[
\lim_{N \to \infty} ||T_N(f)|| = ||f||.
\]
The spectrum \( \hat{\mathcal{A}} \) (of irreducible representations, see 3.2.2 of [4]) is a non-Hausdorff union of \( M \) and \( \mathbb{N} \), although it maps continuously onto \( \hat{\mathcal{N}} \). According to Prop. 3.3.2 of [4], the function on \( \hat{\mathcal{A}} \) defined by the norms of the images of any \( a \in \mathcal{A} \) is lower semi-continuous. This means that for any \( x \in M \),
\[
\liminf_{N \to \infty} ||T_N(f)|| \geq ||f(x)|| \geq ||f||.
\]

On the other hand, because $T_N$ is norm-contracting,
\[
\limsup_{N \to \infty} \|T_N(f)\| \leq \sup_{N \in \mathbb{N}} \|T_N(f)\| \leq \|f\|.
\]

Each of the maps $T_N$ is surjective (Prop. 4.2 of [1]), so this is clearly the smallest continuous field such that $N \mapsto T_N(f)$ defines a continuous section. In fact, $\text{Im } T$ generates $\mathbb{A}$ as a C*-algebra.

**Definition.**

\[
T^V : \Gamma(M, V) \to \prod_{N \in \mathbb{N}} V_N
\]

is the direct product of the $T^V_N$s, $V := \mathbb{A} \cdot \text{Im } T^V$ is the $\mathbb{A}$-module generated by $\text{Im } T^V$, $V_N$ is the restriction of $V$ to an $\mathcal{A}_N$-module, and $\mathcal{P}^V : V \to \mathbb{V}/\mathcal{A}_0\mathcal{V} = \mathcal{P}_*(V)$ is the natural surjection.

The following lemma shows that the analytic condition $\|v_N\| \to 0$ can be expressed algebraically.

**Lemma 3.4.**

\[
\mathcal{A}_0\mathcal{V} = \left\{ v \in \prod_{N \in \mathbb{N}} V_N \mid \lim_{N \to \infty} \|v_N\| = 0 \right\},
\]

and $T^V$ induces a homomorphism of $\mathcal{C}(\mathcal{M})$-modules, $\mathcal{P}^V \circ T^V : \Gamma(\mathcal{M}, V) \to \mathcal{P}_*(V)$.

**Proof.** As I have mentioned, $T^V_N$ is norm-contracting; thus $T^V(v)$ is bounded. Because of this, the sequence of norms coming from any element of $\mathcal{A}_0\mathcal{V} = \mathcal{A}_0\text{Im } T^V$ must converge to 0. Conversely, $\mathcal{A}_0\mathcal{V}$ is norm-closed and contains all sequences in $\prod_{N \in \mathbb{N}} V_N$ with finitely many nonzero terms. This proves the first claim.

With this, Lemma [3.7] shows that for all $f \in \mathcal{C}(\mathcal{M})$ and $v \in \Gamma(\mathcal{M}, V)$

\[
T(f)T^V(v) - T^V(fv) \in \mathcal{A}_0\mathcal{V},
\]

which proves the second claim. \qed

**Lemma 3.5.** For $N$ sufficiently large, $V_N = \tilde{V}_N$.

**Proof.** Equivalently, for sufficiently large $N$, the image of $T^V_N$ generates $\tilde{V}_N$ as an $\mathcal{A}_N$-module. If not, then $\text{Im } T^V_N$ must lie inside a proper submodule of $\tilde{V}_N$, and so there must exist $\psi \in \tilde{E}_N^V$ such that, for any $\varphi \in \mathcal{H}_N$ and $v \in \Gamma(\mathcal{M}, V)$,

\[
\langle \varphi | T_N^V(v) | \psi \rangle = 0.
\]

Take any nonzero $\varphi \in \mathcal{H}_N$ and $\psi \in \tilde{E}_N^V$. Let $\psi_0 \in \Gamma^\infty(\mathcal{M}, V^* \otimes L_N)$ be the component of $\psi$ in degree 0. Assume $N$ to be sufficiently large that $\psi_0$ is guaranteed by Corollary [A.2] not to vanish. Using the fact that $\mathcal{M}$ is connected, the zeros of $\varphi$ must form a proper subvariety of $\mathcal{M}$, and $\psi_0$ must be nonzero on an open set; therefore, there exists $y \in \mathcal{M}$ where $\varphi(y), \psi_0(y) \neq 0$. If $v \in \Gamma(\mathcal{M}, V)$ approximates the distribution $\varphi(y)\psi_0(y)\delta(x,y)$, then $\langle \varphi | T_N^V(v) | \psi \rangle = \langle \varphi | \psi \rangle$ will approximate $\|\varphi(y)\|^2 \|\psi_0(y)\|^2$ and must be nonzero for a sufficiently close approximation. \qed
3.1. **Category.** It remains to be proven that $\mathbb{V}$ is a finitely generated, projective (f. g. p.) module and that its push-forward is $P_*(\mathbb{V}) = \Gamma(M, \mathbb{V})$. To do this, it will be helpful to make the correspondence $V \mapsto \mathbb{V}$ into a functor. Since the module $\mathbb{V}$ is not constructed from the vector bundle $V$ alone but from $V$ accompanied by a connection, the domain of this functor must be a category of vector bundles with connections. We need to identify those bundle homomorphisms which will lead naturally to module homomorphisms.

A bundle homomorphism naturally defines a map of sections. It also naturally gives (by tensor product with the identity map) homomorphisms of the tensor products with any other bundle (such as 1-forms). For simplicity, I will denote all these trivially derived maps by the same symbol as the original homomorphism.

**Definition.** a morphism of bundles with connections, $\phi : V \to W$, is a smooth bundle homomorphism such that for any smooth section $v \in \Gamma^\infty(M, V)$,

$$\phi(\nabla_V v) = \nabla_W \phi(v);$$  \hspace{1cm} (3.4)

in other words, $\phi$ is covariantly constant.

With these morphisms, vector bundles with connections form an Abelian category. Clearly, the identity homomorphism on any bundle satisfies the above property, and the composition of two such morphisms does as well. Also, the kernel and cokernel of such a morphism inherit natural connections.

Now, let’s try and construct a functor $Q$ from this category of bundles with connections to the category of $A$-modules, such that $Q(V) = \mathbb{V}$.

**Definition/Theorem 3.6.** Any morphism $\phi : V \to W$ of bundles with connections induces a homomorphism $Q(\phi) : \mathbb{V} \to \mathbb{W}$ of $A$-modules which satisfies,

$$T^W \circ \phi = Q(\phi) \circ T^V.$$ \hspace{1cm} (3.5)

**Proof.** $\phi$ gives an adjoint map on the dual bundles in the opposite direction, and in turn maps the spaces of forms $\phi^* : \Omega^0(M, W^* \otimes L_N) \to \Omega^0(M, V^* \otimes L_N)$. Because $\phi$ intertwines connections, the map $\phi^*$ intertwines Dolbeault operators. If $\psi \in \ker D_W = \tilde{E}_N^W$, then $D_W \phi^*(\psi) = \phi^*(D_W \psi) = 0$; this means that the restriction of $\phi^*$ to $\tilde{E}_N^W$ maps $\phi^* : \tilde{E}_N^W \to \tilde{E}_N^V$. This induces a homomorphism $\phi_* : \tilde{V}_N \to \tilde{W}_N$.

Put these maps together to define $Q(\phi)$. *A priori*, $Q(\phi)$ maps an element of $\mathbb{V}$ to some sequence of elements of $\tilde{W}_N$. We need to prove that the image of $Q(\phi)$ in fact lies inside $\mathbb{W}$.

For any $v \in \Gamma(M, V)$ and $\psi \in \tilde{E}_N^W$,

$$T^W_N[\phi(v)]\psi = \Pi_N \phi(v) \psi = \Pi_N \phi^*(\psi) = \phi_* \left[ T^V_N(v) \right] \psi.$$

So, $T^W_N \circ \phi = \phi_* \circ T^V_N$ and hence, Eq. (3.5). Since $\mathbb{W}$ is defined to be generated by $\text{Im} \ T^W$, this shows that indeed $Q(\phi) : \mathbb{V} \to \mathbb{W}$.

$Q$ is an additive functor. It respects identity maps, compositions, and sums of morphisms. Because of this, $Q$ must respect finite direct sums; this property is also easily seen from the construction of $Q$. 


The category of bundles with connections actually behaves somewhat trivially. Because a morphism is covariantly constant, it can be specified completely by its action at a single point. As a result, this category behaves somewhat like the category of finite-dimensional vector spaces. Any short exact sequence splits. Because of this, any additive functor (such as $\Omega$) on this category is exact.

Of course, not all bundle homomorphisms are morphisms of bundles with connections. We will need some module homomorphisms that do not come from $\Omega$.

The following result shows that an isomorphism of vector bundles can be used to construct a homomorphism of modules which is an isomorphism modulo finite-dimensional modules.

**Lemma 3.7.** Let $V$ and $W$ be isomorphic bundles with different connections. Then there exists a Fredholm homomorphism $u : V \to W$ (compare Thm. 1.2), which satisfies

$$P^W \circ u \circ T^V = P^W \circ T^W.$$  \hfill (3.6)

**Proof.** The homomorphism $u$ is specified by giving, for each $N$, a homomorphism $u_N : V_N \to W_N$ of $A^N$-modules. Define $\Pi_N^V$ to be the spectral projection at $0$ for the $V^* \otimes L_N$-twisted Dolbeault operator (likewise with $W$); that is, $\Pi_N^V$ is an idempotent with $\text{Im} \Pi_N^V = \ker D_V$ and $\ker \Pi_N^V = \ker D_V$.

The isomorphism of $V$ and $W$ gives a natural (isometric) inclusion $\iota : \hat{E}_N^W \hookrightarrow \Omega^*(\mathcal{M}, V^* \otimes L_N)$. Composing this with $\Pi_N^V$ gives $\Pi_N^V \iota : \hat{E}_N^W \to \hat{E}_N^V$, and $\Pi_N^V \iota$ is the identity on $\hat{E}_N^V$. According to Lemma A.3,

$$\lim_{N \to \infty} ||\Pi_N^V - \Pi_N^W|| = 0;$$ \hfill (A.4)

therefore, for $N$ sufficiently large, $||\Pi_N^V - \Pi_N^W|| < 1$. When this is so, $\Pi_N^V \iota$ is injective, because if $\psi \in \hat{E}_N^W$ is nonzero then,

$$||\Pi_N^V \iota \psi|| = ||(\Pi_N^V - \Pi_N^W) \iota \psi + \psi|| \\
\geq (1 - ||\Pi_N^V - \Pi_N^W||) ||\psi|| > 0.$$

The existence of a similar injection in the opposite direction establishes that $\Pi_N^V \iota$ is bijective.

Recall from Lemma 3.3 that for $N$ sufficiently large $V_N = \hat{V}_N$ (and likewise with $W$). When $N$ is sufficiently large, we can define $u_N : V_N \to W_N$ to be the bijection given by $u_N(v_N) = v_N \Pi_N^V \iota$. For small $N$, it doesn’t matter what $u_N$ is.

Now assemble the $u_N$’s into $u$. The kernel and cokernel of $u$ come entirely from the finitely many $u_N$’s which are not bijective, and thus are finite-dimensional. In other words, $u$ is Fredholm.

But does the image in fact lie inside $\hat{W}$? Using Eq. (A.4) again shows that, for any $v \in \Gamma(\mathcal{M}, V)$,

$$||T_N^V(v) \Pi_N^V \iota - T_N^W(v)|| \leq ||T_N^V(v)|| \cdot ||\Pi_N^V - \Pi_N^W|| \to 0$$

as $N \to \infty$; therefore, by Lemma 3.4, $u[T^V(v)] - T^W(v) \in A_\delta \hat{W} \subset \hat{W}$. \hfill \Box

**Theorem 3.8.** $\mathcal{V} = \Omega(V)$ is a quantization of $V$ by the definition in Sec. 1.1.
Proof. For any vector bundle $V$, there exists another vector bundle $W$ such that the direct sum is some trivial bundle $V \oplus W \cong \mathbb{C}^m \times M$. Choose an arbitrary connection on $W$. As noted above, $Q$ respects finite sums, so $Q(V \oplus W) = V \oplus W$.

By Lemma 3.7 there exists an $A$-module homomorphism $u : V \oplus W \rightarrow A^m$ whose kernel and cokernel are finite-dimensional and thus projective. All the terms of the exact sequence

$$0 \rightarrow \ker u \rightarrow V \oplus W \xrightarrow{u} A^m \rightarrow \text{coker } u \rightarrow 0,$$

other than $V \oplus W$, are now seen to be f. g. p. modules; therefore $V \oplus W$, and thus $V$, is f. g. p.

It remains to prove that $P_\ast(V) = V/\mathbb{A}_p V = \Gamma(M, V)$. Lemma 3.4 showed that $P^V \circ V : \Gamma(M, V) \rightarrow P_\ast(V)$ is a $\mathcal{C}(M)$-module homomorphism, and it is clearly surjective by the definition of $V$. We need to prove that the kernel of $P^V \circ V$ is trivial.

Let $\phi$ denote the natural inclusion $\phi : V \hookrightarrow V \oplus W$ (as bundles with connections) and $\varphi$ the equivalent inclusion $\varphi : V \hookrightarrow \mathbb{C}^m \times M$ (as a bundle). If $v \in \ker[P^V \circ V]$, then $V^\varphi(v) \in \mathbb{A}_p V$, so $P \circ u \circ \Omega(\phi) \circ V^\varphi(v) = 0$, since $u \circ \Omega(\phi)$ is an $A$-module homomorphism. However, by Eq’s (3.5) and (3.6),

$$P \circ u \circ \Omega(\phi) \circ V^\varphi = P \circ u \circ V^\varphi W \circ \phi = P \circ \Gamma \circ \varphi = \varphi$$

which is injective. Therefore, $\ker[P^V \circ V] \subseteq \ker \varphi = 0$, and

$$P^V \circ V : \Gamma(M, V) \xrightarrow{\sim} P_\ast(V)$$

is indeed an isomorphism.

4. The holomorphic case

Recall that a holomorphic vector bundle is a bundle with a connection whose curvature is of type $(1, 1)$.

Theorem 4.1. If $V$ is a holomorphic vector bundle, then for all $N \in \mathbb{N}$, $V_N = \text{Hom}(E_N^V, \mathcal{H}_N)$ where

$$E_N^V = \Gamma_{\text{hol}}(M, V^* \otimes L_N),$$

and $\Gamma_{\text{hol}}$ means holomorphic sections.

Proof. This is much the same as the proof of Lemma 3.3.

In the holomorphic case, $D^2$ respects the $\mathbb{Z}$-grading, so $E_N^V$ (and then $V_N$) is $\mathbb{Z}$-graded. Sections of $V$, and thus $\text{Im} T^V_N$, are entirely of degree 0, so $V_N \subseteq \text{Hom}(E_N^V, \mathcal{H}_N)$.

If the statement were false, then there would exist a nonzero $\psi \in E_N^V$ such that for any $\phi \in \mathcal{H}_N$ and $v \in \Gamma(M, V)$, $\langle \phi | v | \psi \rangle = 0$. However, if $\phi \neq 0$ then the zero sets of $\phi$ and $\psi$ will be proper subvarieties of $M$; therefore, there exists $y \in M$ where $\phi(y) 
eq 0$ and $\psi(y) 
eq 0$. So, if $v(x)$ approximates the distribution $\phi(y) \psi(y) \delta(x, y)$, then $\langle \phi | v | \psi \rangle$ will approximate $\|\phi(y)\|^2 \|\psi(y)\|^2$ and thus be nonzero for a sufficiently close approximation.
5. Geometric Quantization

**Definition.** The standard geometric quantization maps (see [14]) \( Q_N : C^\infty(M) \to \mathcal{A}_N \) are defined on smooth functions by (with a slight abuse of notation)

\[
Q_N(f) := \Pi_N \left[ f - \frac{i}{N} \pi^{\mu\nu} f_{\mu\nu} \right] = T_N \left( f - \frac{i}{N} \pi^{\mu\nu} f_{\mu\nu} \right).
\]

Here \( \pi \) is the Poisson bivector, defined by \( \pi^{\mu\nu} \omega_{\lambda\nu} = \delta^\mu_{\lambda} \), and \( \nabla \) is again the connection.

Following \( T_N^V \), there is an obvious generalization of \( Q_N \) for vector bundles.

**Definition.** \( Q_N^V : \Gamma^\infty(M, V) \to V_N \) is given by

\[
Q_N^V(v) := T_N^V \left( v - \frac{i}{N} \pi^{\mu\nu} v_{\mu\nu} \right).
\]

**Lemma 5.1.** For any smooth section \( v \in \Gamma^\infty(M, V) \),

\[
\lim_{N \to \infty} \| T_N^V(v) - Q_N(v) \| = 0.
\]

**Proof.** Let \( w^\mu \) be any tangent vector with components in \( V \), and use \( D \) to denote both the \( L_N \)-twisted and \( V^* \otimes L_N \)-twisted Dolbeault operators. Then,

\[
-i[D, \gamma^\mu w^\mu]_+ = [\gamma^\nu \nabla_{\nu}, \gamma^\mu w^\mu]_+ = [\gamma^\nu, \gamma^\mu w^\mu]_+ \nabla_{\nu} + \gamma^\nu [\nabla_{\nu}, \gamma^\mu w^\mu]_-. \]

Because the argument of \( T_N^V \) acts between the kernels of the Dolbeault operators, this gives the identity

\[
0 = T_N^V ([D, \gamma^\mu w^\mu]_+) = iT_N^V (w^\mu \nabla_\mu + \gamma^\nu \gamma^\mu w^\mu). \]

Now, setting \( w^\mu = -\frac{i}{N} \pi^{\nu\mu} v_{\nu} \) gives

\[
Q_N(v) - T_N^V(v) = \frac{i}{N} \Gamma_N^V (\gamma^\nu \gamma^\mu \pi^{\nu\mu} v_{\nu}). \tag{5.2}
\]

Since \( T_N^V \) is norm-contracting, for any smooth \( v \), the norm of the difference (5.2) converges to 0 as \( N \to \infty \). \( \square \)

Equation (5.2) is related to a formula due to Tuynman [13]. Namely, for any smooth function \( f \in C^\infty(M) \),

\[
Q_N(f) = T_N \left[ f + \frac{1}{2N} \Delta(f) \right]
\]

where \( \Delta = -\nabla^2 \) is the scalar Laplacian. Since \( \Delta \) is a positive operator, this shows that \( Q_N \), like \( T_N \), is (completely) positive, which means that, after all, \( Q_N \) can be uniquely, continuously defined on all of \( C(M) \).

As with Toeplitz quantization, we can assemble the \( Q_N \)'s into a direct-product map \( Q : C^\infty(M) \to \prod_{N \in \mathbb{N}} A_N \).

**Theorem 5.2.** \( Q : C(M) \to \mathcal{A} \) and \( \mathcal{P} \circ Q = \text{id} \).

**Proof.** By Lemma 5.1, for any smooth function \( f, T(f) - Q(f) \in A_0 \). This shows that \( \mathcal{P}[Q(f)] = f \).

Since \( A = \mathcal{P}^{-1}[C(M)] \), this shows that \( \text{Im} \ Q \subset A \). \( \square \)
This shows that the general quantization constructed by geometric quantization is exactly the same as that constructed by Toeplitz quantization.

Analogous to the construction of $\mathbb{V}$, define $\mathbb{V}'$ to be the $\mathbb{A}$-module generated by the image of $Q^V$. 

**Theorem 5.3.** This $\mathbb{V}'$ is a quantization of $V$.

*Proof.* It is sufficient to prove that $\mathbb{V}'$ is isomorphic to $\mathbb{V}$ modulo finite-dimensional modules.

Choose some set of smooth sections of $V$ such that their images by $T^V$ generate $\mathbb{V}$, and hence their images by $T^V_N$ generate $V_N$. For sufficiently large $N$, their images by $Q^V_N$ will be close enough to those by $T^V_N$ to generate $\tilde{V}_N = V_N$. Therefore, for $N$ sufficiently large $V'_N = V_N$. 

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6. FURTHER STRUCTURES

Because $\mathbb{V}$ has been produced constructively from $V$ and its connection, essentially any additional structure that is consistent with the connection on $V$ should lift to $\mathbb{V}$. (This is equally true for $\mathbb{V}'$.)

If there is a group $G$ acting on $\mathcal{M}$, and $V$ is a $G$-equivariant vector bundle with an equivariant connection, then there will be a natural representation of $G$ on $\mathbb{V}$, and $T^V$ will be $G$-invariant. See also [7].

If $V$ has a given inner product and a compatible connection, then $\mathbb{V}$ will have a natural inner product, corresponding to the inner product of sections integrated against the volume form $\omega^n/n!$.

7. GROWTH OF MODULES

Since (for any $N$) the algebra $\mathbb{A}_N = \text{End} \mathcal{H}_N$ is a full matrix algebra, its modules are classified (modulo isomorphism) by the positive integers. To be precise, any $\mathbb{A}_N$-module can be written in the form $\text{Hom}(E, \mathcal{H}_N)$, where $\mathbb{A}_N$ acts only on $\mathcal{H}_N$, and $E$ may be any finite-dimensional vector space; the integer corresponding to this module is:

**Definition.** $\text{rk}[\text{Hom}(E, \mathcal{H}_N)] := \dim E$.

This also gives a natural isomorphism $\text{rk} : K_0(\mathbb{A}_N) \xrightarrow{\sim} \mathbb{Z}$.

**Theorem 7.1.** Let $\mathbb{V}$ be any quantization of a vector bundle $V$, and $V_N$ the restriction of $\mathbb{V}$ to an $\mathbb{A}_N$-module. For all sufficiently large values of $N$,

$$\text{rk} V_N = \int_M \text{ch} V \wedge \text{td} T\mathcal{M} \wedge e^{N\omega/2\pi - c_1(L_0)}.$$  \hfill (7.1)

*Proof.* The uniqueness result of Thm. [1.2] implies that any analytic formula for $\text{rk} V_N$ for large $N$ must apply to any quantization of $V$. It is therefore sufficient to look at the specific quantization constructed in Sec. [3].

By Lemma [3.4] for $N$ sufficiently large, $V_N = \tilde{V}_N = \text{Hom}(\tilde{E}_N, \mathcal{H}_N)$; hence, $\text{rk} V_N = \dim \tilde{E}_N$. This is the kernel of the $V^* \otimes L_N$-twisted Dolbeault operator, and has
the same dimension as the kernel of the $V \otimes L_N$-twisted anti-Dolbeault operator. By Corollary A.2, this is entirely of even degree (again, for $N$ sufficiently large); hence, $rk V_N$ is the index of this anti-Dolbeault operator. Equation (7.1) then follows from the Riemann-Roch-Atiyah-Singer theorem if we note that 

$$ch L_N = e^{-c_1(L_N)} = e^{N\omega/2\pi c_1(L_0)}.$$  

This gives some interesting qualitative results. Again writing $n := \dim \mathcal{M}$, the right hand side of Eq. (7.1) is a polynomial in $N$ of degree $n$. The coefficients of this polynomial give $n+1$ components of the Chern character of $V$. The growth of $rk V_N$ thus gives some — but not in general all — topological information about the bundle $V$. Evidently, the sequence of modules $V_N$ does not carry all the information in the way these modules fit together as $N \to \infty$.

Since $A$ is a quantization of the trivial line bundle, Eq. (7.1) implies the formula

$$\dim \mathcal{H}_N = \int_M \text{td} TM \wedge e^{N\omega/2\pi c_1(L_0)},$$

(7.2)

which, thanks to the Kodaira vanishing theorem, holds for all $N > 0$. Comparing (7.1) with (7.2) shows that $rk V_N \approx rk V \cdot \dim \mathcal{H}_N$, with corrections of order $N^{n-1}$ ($rk V$ is the fiber dimension).

A trivial vector bundle over $M$ can be quantized to a free module of $A$. In that case, $rk V_N$ must be an integer multiple of $\dim \mathcal{H}_N$, but in general the deviation from this reflects the nontriviality of a vector bundle.

It is especially interesting to quantize a spinor bundle. Since $M$ is symplectic, it is even dimensional, and spinors decompose as $S = \mathcal{S}^+ \oplus \mathcal{S}^-$ into left and right handed parts. The Dirac operator is odd; that is, it maps left spinors to right spinors and vice-versa. A “quantized” Dirac operator should act on the quantized spinor bundle, i. e., $D_N : S_N \to S_N$. If oddness of the Dirac operator is preserved, and if $S_N^+$ and $S_N^-$ are of different size, then the quantized Dirac operator will necessarily have a kernel.

Typically, $S_N^+$ and $S_N^-$ are different. In fact $rk S_N^+ - rk S_N^-$ is independent of $N$ and equal to the Euler characteristic $\chi(M)$. The dimension of an $A_N$-module is equal to its rank times $\dim \mathcal{H}_N$, so the dimension of the kernel of a quantized Dirac operator for a manifold of nonzero Euler characteristic must be at least

$$\dim \ker D_N \geq |\chi(M)| \dim \mathcal{H}_N.$$  

This may have dire consequences for the existence of quantized Dirac operators. I hope to discuss this further in a future paper.

Theorem 7.1 can also be expressed in terms of idempotents.

**Corollary 7.2.** Let $e \in M_m[\mathcal{C}(M)]$ and $\tilde{e} \in M_m(A)$ be idempotents such that $\mathcal{P}(\tilde{e}) = e$. For $N$ sufficiently large,

$$\text{tr} \tilde{e}_N = \int_M \text{ch} e \wedge \text{td} TM \wedge e^{N\omega/2\pi c_1(L_0)}.$$  

(7.3)
Here \( \text{ch} \) e is the Chern character of the bundle determined by \( e \), and \( \tilde{e}_N \) is the evaluation of \( \tilde{e} \) at \( N \).

**Proof.** The idempotent \( e \) defines a vector bundle \( V \). The module \( V := \mathbb{A}^n \tilde{e} \) is a quantization of \( V \). We have \( V_N = \mathbb{A}^n \tilde{e}_N \). This gives \( \text{rk} V_N = \text{tr} \tilde{e}_N \).

I explore an implication of Eq. (7.3) in another paper [8].

**APPENDIX A. SPECTRAL INEQUALITIES**

The line bundles \( L_N \) continue to be as defined in Sec. 2. Specifically, \( L_N = L^\otimes N \otimes L_0 \), and \( L_1 \) is assumed to be positive.

**Lemma A.1.** If \( V \) has a compatible connection and inner product, then the \( V^* \otimes L_N \)-twisted Dolbeault operator, \( D_V \), is (essentially) self-adjoint and there exists a constant, \( C \), such that

\[
\text{Spec } D^2_V \subset \{0\} \cup [N - C, \infty).
\]

Moreover, for the trivial bundle \( V = \mathbb{C} \times M \), we can take \( C < 1 \).

**Proof.** Let Latin indices denote holomorphic and barred Latin indices antiholomorphic directions in the tangent bundle. Using the Kähler identity \( \omega_{ij} = i g_{ij} \), the Weitzenbock formula in this case takes the form

\[
D^2_V = -g^{ij} \nabla_i \nabla_j + N \delta + \hat{K},
\]

where \( \delta \) is the grading on \( \Omega^{0,*}(M, V^* \otimes L_N) \),

\[
\hat{K} = i \gamma^i \gamma^j K_{ij} + \frac{i}{2} \gamma^i \gamma^j K_{ij} + \frac{i}{2} \gamma^i \gamma^j K_{ij},
\]

and \( K \) is the curvature of \( V^* \otimes L_0 \).

The operator \( D^2_V \) always preserves the \( \mathbb{Z}_2 \)-grading of \( \Omega^{0,*}(M, V^* \otimes L_N) \) into even and odd parts, although it may not respect the full \( \mathbb{Z} \)-grading. With respect to the \( \mathbb{Z}_2 \)-grading, \( D_V \) decomposes into \( D_+ + D_- \), where \( D_+ \) maps even to odd and \( D_- \) maps odd to even.

The first term of (A.2) is a positive operator, and \( \delta \geq 1 \) when restricted to the odd subspace; therefore,

\[
D_+ D_- \geq N - C,
\]

where \( C = \|\hat{K}\| \) is sufficient. This proves that any eigenvalue of \( D_+ D_- \) (the spectrum consists entirely of eigenvalues) is greater than \( N - C \).

Let \( \psi \) be an eigenvector of \( D_- D_+ \) with eigenvalue \( \lambda \neq 0 \). This implies that \( D_+ \psi \neq 0 \). Now, \( D_+ D_- (D_+ \psi) = D_+ \lambda \psi = \lambda (D_+ \psi) \), so \( \lambda \) is an eigenvalue of \( D_+ D_- \). Therefore, \( \lambda \geq N - C \).

For \( V \) trivial, The assumption that \( L_1 \) is positive implies that \( \delta + \hat{K} \) is strictly positive. This means that \( D^2 > (N - 1)\delta \), and so we can take \( C < 1 \) in (A.3). \( \square \)
**Corollary A.2.** Let $V$ be an arbitrary vector bundle with a connection, and $D_V$ the $V^* \otimes L_N$-twisted Dolbeault operator. For $N$ sufficiently large, $\ker D_V$ is entirely of even degree, and for any nonzero $\psi \in \ker D_V$, the degree 0 component of $\psi$ is nonvanishing. Identical results hold for the $V \otimes \bar{L}_N$-twisted anti-Dolbeault operator.

**Proof.** Assign an arbitrary inner product to $V$. The given connection on $V$ can be decomposed into a connection compatible with the inner product and a self-adjoint potential. Correspondingly, the Dolbeault operator decomposes as $D_V = D_0 + iB$ where $D_0$ is a self-adjoint Dolbeault operator and $B$ is a self-adjoint and bounded Dirac matrix. Using Eq. (A.2) again gives

$$\Re D_V^2 = D_0^2 - B^2 \geq N\delta - C - \|B\|^2.$$ 

Now assume that $N > C + \|B\|^2$. If $\psi$ is of strictly positive degree (i.e. $\psi_0 = 0$) then $D_V^2 \psi \neq 0$, which implies $\psi \notin \ker D_V$.

Because $D_V$ respects the $\mathbb{Z}_2$-grading, $\ker D_V$ must be the sum of even and odd parts. However, if $\psi$ is of strictly odd degree, then it is of strictly positive degree. Hence, $\ker D_V$ can have no odd part. $\square$

Note that if $V$ is trivial, then the first statement can be strengthened to the classical Kodaira vanishing theorem, namely the fact that $\ker D$ is entirely of degree 0 and thus is simply $\Gamma_{hol}(\mathcal{M}, L_N)$ — a fact which was used in Sec. [3].

Recall that in the proof of Lemma [A.7], $\Pi^V_N$ was defined as the idempotent such that $\text{Im} \Pi^V_N = \ker D_V$ and $\ker \Pi^V_N = \text{Im} D_V$.

**Lemma A.3.** If $V$ and $W$ are the same bundle, but with different connections, then

$$\lim_{N \to \infty} \|\Pi^V_N - \Pi^W_N\| = 0.$$

**Proof.** Suppose initially that the $W$ connection is compatible with the inner product. This means that the associated Dolbeault operator $D_W$ and idempotent $\Pi^W_N$ will be self-adjoint.

Since different connections on the same bundle only differ by a potential, the difference $A := D_V - D_W$ of the Dolbeault operators is bounded.

The idempotent $\Pi^V_N$ can be expressed in terms of $D_V$ as

$$\Pi^V_N = \frac{1}{2\pi i} \oint_C (z - D_V)^{-1} \, dz,$$

where the contour of integration $C$ encloses 0 but no other eigenvalue of $D_V$. An identical formula holds for $\Pi^W_N$ in terms of $D_W$. The difference of these expressions gives

$$\Pi^V_N - \Pi^W_N = \frac{1}{2\pi i} \oint_C (z - D_V)^{-1} A (z - D_W)^{-1} \, dz.$$ (A.5)

Expanding $(z - D_V)^{-1} = (z - D_W - A)^{-1}$ as a power series in $A$ and taking the norm gives

$$\|(z - D_V)^{-1}\| \leq \left[\|(z - D_W)^{-1}\|^{-1} - \|A\|\right]^{-1}.$$
Since $D_W$ is self-adjoint, the norm of $(z - D_W)^{-1}$ is just the reciprocal of the distance from $z$ to Spec $D_W$. Equation (A.1) implies that (for some $C$)

$$\text{Spec } D_W \subset (-\infty, -\sqrt{N - C}] \cup [0] \cup [\sqrt{N - C}, \infty).$$

If we let the contour $\mathcal{C}$ be the circle about 0 of radius $\frac{1}{2}\sqrt{N - C}$ (which is a good contour if $N > C - 4\|A\|^2$), then for $z \in \mathcal{C}$, $(z - D_W)^{-1} \leq 2(N - C)^{-1/2}$. Taking the norm of (A.5) now gives

$$\|\Pi_N^V - \Pi_N^W\| \leq 2\|A\|(N - C)^{-1/2} \left[\frac{1}{2}(N - C)^{1/2} - \|A\|\right]^{-1}.$$

This clearly goes to 0 as $N \to \infty$, thus proving the claim in this special case.

Idempotents constructed from two connections incompatible with the inner product can both be compared with one constructed from a connection that is compatible with the inner product; thus, this special case implies the more general result. \hfill \square

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