GORENSTEINNESS OF INVARIANT SUBRINGS OF QUANTUM ALGEBRAS

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Abstract. We prove Auslander-Gorenstein and GKdim-Macaulay properties for certain invariant subrings of some quantum algebras, the Weyl algebras, and the universal enveloping algebras of finite dimensional Lie algebras.

0. Introduction

Given a noncommutative algebra it is generally difficult to determine its homological properties such as global dimension and injective dimension. In this paper we use the noncommutative version of Watanabe theorem proved in [JoZ, 3.3] to give some simple sufficient conditions for certain classes of invariant rings having some good homological properties.

Let $k$ be a base field. Vector spaces, algebras, etc. are over $k$. Suppose $G$ is a finite group of automorphisms of an algebra $A$. Then the invariant subring is defined to be $A^G = \{ x \in A \mid g(x) = x \text{ for all } g \in G \}$.

Let $A$ be a filtered ring with a filtration $\{ F_i \mid i \geq 0 \}$ such that $F_0 = k$. The associated graded ring is defined to be $\text{Gr } A = \bigoplus_n F_n/F_{n-1}$. A filtered (or graded) automorphism of $A$ (or $\text{Gr } A$) is an automorphism preserving the filtration (or the grading). The following is a noncommutative version of [Ben, 4.6.2].

Theorem 0.1. Suppose $A$ is a filtered ring such that the associated graded ring $\text{Gr } A$ is isomorphic to a skew polynomial ring $k_{p_{ij}}[x_1, \cdots, x_n]$, where $p_{ij} \neq p_{kl}$ for all $(i, j) \neq (k, l)$. Let $G$ be a finite group of filtered automorphisms of $A$ with $|G| \neq 0$ in $k$. If $\det g|_{(\oplus_{i=1}^n kx_i)} = 1$ for all $g \in G$, then $A^G$ is Auslander-Gorenstein and GKdim-Macaulay.

The condition on $p_{ij}$ in Theorem 0.1 is needed as shown by Example 2.9.

The definitions of Auslander-Gorenstein and GKdim-Macaulay are given in Definition 0.6. The skew polynomial algebra $k_{p_{ij}}[x_1, \cdots, x_n]$ is generated by $\{ x_1, \cdots, x_n \}$ with relations $x_i x_j = p_{ij} x_j x_i$ for all $i < j$, where the parameters $\{ p_{ij} \mid i < j \}$ are nonzero elements in $k$. We let $p_{ii} = 1$ and $p_{ij} = p_{ji}^{-1}$, then the relations are $x_i x_j = p_{ij} x_j x_i$. We always think $k_{p_{ij}}[x_1, \cdots, x_n]$ as a connected graded ring with the degree of $x_i$ not necessarily 1. We identify the graded vector space $V = \bigoplus_{i=1}^n kx_i$ with the quotient space $m/m^2$, where $m$ is the maximal graded ideal of $k_{p_{ij}}[x_1, \cdots, x_n]$. Any filtered or graded automorphism $g$ induces a linear automorphism of $V$ and

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det \( g \mid V \) is the usual determinant of the linear map \( g : V \to V \). The hypothesis on \( \det g \) is necessary as shown in [Example 3.6].

A general version of Theorem 0.1 holds when we replace \( \det g \) by the homological determinant [Theorems 3.3 and 3.5]. The notion of homological determinant was introduced in [JoZ, 2.3] (see Section 2).

We can extend the action of \( g \) from \( V \) to the exterior algebra \( \Lambda(V) \). Therefore \( g \) acts on the tensor product \( A \otimes \Lambda(V) \) if \( A \) is as in Theorem 0.1. The following is a noncommutative version of [Ben, 5.3.2].

**Theorem 0.2.** Suppose \( A \) is a filtered ring such that the associated graded ring \( \text{Gr} \ A \) is isomorphic to the skew polynomial ring \( k_{p_{ij}}[x_1, \cdots, x_n] \), where \( p_{ij} \neq p_{kl} \) for \((i,j) \neq (k,l)\). Let \( G \) be a finite group of filtered automorphisms of \( A \) with \( |G| \neq 0 \) in \( k \). Then \( (A \otimes \Lambda(V))^G \) is Auslander-Gorenstein and \( \text{GKdim-Macaulay} \) where \( V = \bigoplus_{i=1}^n kx_i \).

Theorem 3.8 is another version of Theorem 0.2, which contains the commutative case.

Note that in Theorem 0.2 there is no hypothesis on \( \det g \) as in the commutative case [Ben, 5.3.2]. In general, \( (A \otimes \Lambda(V))^G \) is not prime and hence is not Auslander-regular.

Theorems 0.1 and 0.2 also hold if \( \text{Gr} \ A \) is the 4-dimensional Sklyanin algebra (by Example 2.2 and Theorem 3.5) or the multiparameter quantum matrix algebra (by Example 2.6 and Theorem 3.5). Note that the 4-dimensional Sklyanin algebra and the multiparameter quantum matrix algebra satisfy the SSC of [Zh, p. 398] (by [Zh, 2.3(2) and 2.4]) and hence satisfy the hypothesis \( \text{Kdim} M = \text{GKdim-Macaulay} \) in Theorem 3.5(3) (by [Zh, 3.1]).

The Weyl algebras and the universal enveloping algebras of finite dimensional Lie algebras have standard filtrations, and their associated graded rings are commutative polynomial rings. We prove the following theorem using similar ideas.

**Theorem 0.3.** Let \( A_n \) be the \( n \)-th Weyl algebra with the standard filtration. Let \( G \) be a finite group of filtered automorphisms of \( A_n \). If \( |G| \neq 0 \) in \( k \), then \( A_n^G \) is Auslander-Gorenstein and \( \text{GKdim-Macaulay} \). Moreover if \( \text{char} k = 0 \), then \( A_n^G \) is Auslander-regular.

A special case of Theorem 0.3 was proved by Levasseur, see [L1, 3.2]. If \( \text{char} k = 0 \), it is well-known that \( A_n^G \) has finite global dimension because \( A_n \) is simple (see Remark 4.2). A quantum version of Theorem 0.3 is the following. We say that \( q \) is generic with respect to \( \{p_{ij}\} \) if \( q^n \) \((n > 0)\) are not in the the multiplicative subgroup generated by \( \{p_{ij}\} \).

**Theorem 0.4.** Let \( A_n(q, p_{ij}) \) be the \( n \)-th quantum Weyl algebra defined in [GZ, (2.3)] and \( G \) a finite group of filtered automorphisms of \( A_n(q, p_{ij}) \). Suppose that \( q \) is generic with respect to \( \{p_{ij}\} \). If \( |G| \neq 0 \) in \( k \), then \( A_n(q, p_{ij})^G \) is Auslander-Gorenstein and \( \text{GKdim-Macaulay} \).

**Theorem 0.5.** Let \( k \) be an algebraically closed field of characteristic zero, \( L \) a finite dimensional Lie algebra over \( k \), and \( U(L) \) the universal enveloping algebra of \( L \). Let \( G \) be a finite group of filtered automorphisms of \( U(L) \). Then \( U(L)^G \) is Auslander-Gorenstein and \( \text{GKdim-Macaulay} \) if either of the two following conditions holds:

1. elements of \( G \) act as inner automorphisms of the Lie algebra \( L \);
2. \( L \) is one of the following simple Lie algebras:
   a. \( A_n \), for \( n = 4m \) or \( n = 4m + 1 \);
   b. \( B_n \);
   c. \( C_n \);
Theorem 0.5(1) was essentially known by Kraft and Small [KS, Proposition 4(5)] (see Remark 6.7).

Definition 0.6. Let $A$ be a noetherian ring.

(1) The grade of an $A$-module $M$ is defined to be

$$j(M) = \min\{i \mid \Ext^i_A(M, A) \neq 0\}$$

or $\infty$ if no such $i$ exists.

(2) We say $A$ is Auslander-Gorenstein if the following two conditions hold:

- (Gorenstein property) $A$ has finite left and right injective dimension;
- (Auslander property) for every noetherian $A$-module $M$ and for all $i \geq 0$ and all submodule $N \subset \Ext^i_A(M, A)$, $j(N) \geq i$.

We say $A$ is Auslander-regular if $A$ is Auslander-Gorenstein and $A$ has finite global dimension.

(3) We say $A$ is GKdim-Macaulay (where GKdim denotes the Gelfand-Kirillov dimension) if

$$\GKdim M + j(M) = \GKdim A < \infty$$

for every noetherian $A$-module $M$.

Note that GKdim-Macaulay is called Cohen-Macaulay by some authors. Auslander-Gorenstein and GKdim-Macaulay properties are considered as nice homological properties. Such rings are studied by several researchers since 1980’s (see [ASZ], [B1], [B2], [BE], [Ek], [L1], [L2] and so on). The skew polynomial rings, the Weyl algebras and the universal enveloping algebras are Auslander–regular and GKdim-Macaulay. Using the above results one can construct more examples of Auslander-Gorenstein and GKdim-Macaulay rings.

1. Preliminary

An old and basic tool for filtered rings is to use their Rees rings and the associated graded rings. In this section we recall some definitions and properties about filtered rings and graded rings.

A filtered algebra is an algebra $A$ with a filtration $F = \{F_i \mid i \geq 0\}$ satisfying the conditions:

- (F0) $F_0 = k$,
- (F1) $F_i \subset F_j$ if $i \geq j$,
- (F2) $F_i F_j \subset F_{i+j}$,
- (F3) $A = \bigcup_i F_i$.

Given a filtered algebra $A$, the Rees ring is defined to be

$$\text{Rees } A = \bigoplus_i F_i t^n.$$

The elements $t(=1t)$ and $t-1$ are central elements of Rees $A$ and

$$\text{Rees } A/(t-1) = A,$$

and

$$\text{Rees } A/(t) = \text{Gr } A.$$
We will adopt the basic notations about graded rings and graded modules from [JiZ], [JoZ] and [YZ]. For example, the opposite ring of $A$ is denoted by $A^{\text{op}}$.

Let $A$ be a connected graded ring. The maximal graded ideal $A_{\geq 1}$ is $A$ denoted by $m$ and the trivial module $A/m$ is denoted by $k$. The graded Ext-group is denoted by $\text{Ext}$.

**Definition 1.1.** Let $A$ be a noetherian connected graded ring.

1. $A$ is called **AS-Gorenstein** (here AS stands for Artin-Schelter) if
   - (a) $A$ is Gorenstein with injective dimension $d$, and
   - (b) there is an integer $l$ such that
     \[
     \text{Ext}_A^i(k, A) = \text{Ext}_{A^{\text{op}}}^i(k, A) = \begin{cases} 
     0 & \text{for } i \neq d \\
     k(l) & \text{for } i = d
     \end{cases}
     \]
     where $k(l)$ is the $l$-th degree shift of the trivial module $k$.

2. $A$ is called **AS-regular** if it is AS-Gorenstein and has finite global dimension.

Condition 1.1(1b) follows from 1.1(1a) if $A$ has enough normal elements [Zh, 0.2]. For example, the commutative polynomial rings are AS-regular. By [L2, 6.3], a noetherian, connected graded, Auslander-Gorenstein ring is AS-Gorenstein.

Given a connected graded ring $A$ and any graded right $A$-module $M$, we define the local cohomology of $M$ to be
\[
H_i^m(M) = \lim_{n \to -\infty} \text{Ext}_A^n(A/A_{\geq n}, M).
\]
If $A$ is AS-Gorenstein, then $H_i^m(A)$ is 0 if $i \neq d$ and is equal to $A'(l)$ if $i = d$. Here $(-)'$ is the graded vector space dual of $-$. Auslander property is also defined for non-Gorenstein ring if the ring has a (rigid or balanced) dualizing complex [YZ, 2.1]. In the case when $\text{Gr } A$ is AS-Gorenstein, $A$ has Auslander property [Definition 0.6(2b)] is equivalent to that $A$ has an Auslander, rigid dualizing complex [YZ, 2.1 and 3.1]. We collect some results below.

**Theorem 1.2.** (1) (Rees’ lemma) If $A$ is a noetherian connected graded ring with a regular normal element $x$ of positive degree, then $A$ is AS-Gorenstein if and only if $A/(x)$ is.

2. If $A$ is as in (1) $x$ is of degree 1, then $A$ is AS-regular if and only if $A/(x)$ is.

3. If $A$ is a noetherian filtered algebra, then Rees $A$ is AS-regular (respectively AS-Gorenstein) if and only if $\text{Gr } A$ is.

4. [L2, 3.6] Let $A$ be a filtered noetherian algebra. Then Rees $A$ is Auslander-Gorenstein (or Auslander-regular) if and only if $\text{Gr } A$ is.

5. [Ek, 0.1; L2, 3.6, 5.8 and 5.10] Let $A$ be a filtered noetherian algebra. If $\text{Gr } A$ is Auslander-Gorenstein (respectively, and GKdim-Macaulay) then so is $A$.

We make the following definition.

**Definition 1.3.** Suppose $P$ is a property such that Rees $A$ has $P$ if and only $\text{Gr } A$ has $P$. We say a filtered algebra $A$ has **filtered $P$** if Rees $A$ (or $\text{Gr } A$) has $P$. 

2. Homological Determinant

First we recall some definitions in the graded case from [JiZ] and [JoZ]. Let \(g \in \text{GrAut}(A)\), the group of graded algebra automorphisms of \(A\). For any \(A\)-module \(M\) we define the \(g\)-twisted module \(M^g\) such that \(M^g = M\) as a vector space and the action is

\[ m \ast a = mg(a) \]

for all \(m \in M\) and \(a \in A\). Let \(f\) be a \(k\)-linear homomorphism from \(A\)-module \(M\) to \(A\)-module \(N\). We say \(f\) is \(g\)-linear if

\[ f(ma) = f(m)g(a) \]

for all \(m \in M\) and \(a \in A\). Then \(f\) is a \(g\)-linear map if and only if \(f\) is an \(A\)-module homomorphism from \(M\) to \(N^g\). Therefore \(g\)-linear maps can be extended to injective resolutions (or projective resolutions). Moreover we can apply local cohomology functor \(H^*_m\) to \(g\)-linear maps. Now let \(A\) be a graded AS-Gorenstein ring with injective dimension \(d\). By [JoZ, 2.2 and 2.3], \(g : A \to A\) induces a \(g\)-linear map \(H^d_m(g) : A'_l(l) \to A'_l(l)\) where \(l\) is the integer in Definition 1.1(1b) and \(\cdot\) is the graded vector space dual, and \(H^d_m(g) = c(g^{-1})\). The constant \(c^{-1}\) is called the homological determinant of \(g\) and we denote \(\text{hdet}_A g = c^{-1}\). By [JoZ, 2.5], \(\text{hdet}_A\) defines a group homomorphism \(\text{GrAut}(A) \to k^\times\).

The trace of \(g\) on \(A\) is defined to be

\[ Tr_A(g, t) = \sum_{n \geq 0} \text{tr}(g|_{A_n})t^n. \]

See [JiZ] for more details. The Hilbert series of \(A\) is the trace of the identity map, namely

\[ H_A(t) = Tr_A(id, t) = \sum_{n \geq 0} \dim A_n t^n. \]

By [JoZ, 2.6], if \(A\) is AS-Gorenstein and \(g\) is rational in the sense of of [JoZ, 1.3] then

\[ Tr_A(g, t) = (-1)^d(\text{hdet}_A g)^{-1}t^{-1} + \text{ lower terms}, \]

when we expand the trace function as a Laurent series in \(t^{-1}\). Here \(d\) and \(l\) are given in Definition 1.1(1b). Rationality of \(g\) is automatic for AS-regular algebras [JoZ, 4.2].

**Lemma 2.1.** (1) If \(A\) is the commutative polynomial ring \(k[V]\) and \(g\) is a graded automorphism of \(k[V]\), then \(\text{hdet}_A g = \det g|_V\) where \(\det\) is the usual determinant of a \(k\)-linear map.

(2) If \(A\) is the exterior algebra \(\Lambda(V)\) and \(g\) is a graded automorphism of \(A\), then \(\text{hdet}_A g = (\det g|_V)^{-1}\).

**Proof.** (1) It follows by [Ben, 2.5.1] that \(Tr_A(g, t) = (\det (1 - g|_V t))^{-1}\). (Note that the action of \(g\) in [Ben] is defined via \(g^{-1}\), see [Ben, p. 1], so we change \(g^{-1}\) in [Ben, 2.5.1] to \(g\).) Expand \((\det (1 - gt)|_V)^{-1}\) as a Laurent series in \(t^{-1}\) we have

\[ (\det (1 - gt)|_V)^{-1} = (-1)^n(\det g|_V)^{-1}t^{-n} + \text{ lower terms}. \]

Therefore the result follows by [JoZ, 2.6].
(2) It is easy to see that $Tr_A(g, t) = \det (1 + g|_V t) = (\det g|_V) t^n + \text{lower terms}$ [Ben, 5.2.1]. Therefore the result follows by [JoZ, 2.6]. □

Lemma 2.1(2) shows that the homological determinant of $g$ may not equal to the determinant of $g|_V$ (also see Examples 2.8 and 2.9). Lemma 2.1(1) shows that the homological determinant of $g$ is equal to the determinant of $g|_V$ when $A$ is the commutative polynomial ring. The next example shows that Lemma 2.1(1) also holds for the 4-dimensional Sklyanin algebra.

**Example 2.2.** Let $k$ be the field of complex numbers $\mathbb{C}$. Then the automorphisms of the 4-dimensional Sklyanin algebra $S$ are classified in [SS, Section 2] and the generators of the automorphism group are also listed there. So one can check $\det g|_V$ easily where $V = S_1$ is the degree 1 part of $S$. By [JoZ, 2.6], the homological determinant of $g$ can be computed by the trace of $g$. The trace of generators of the automorphism group is listed in [JiZ, Section 4]. Use these facts one can check that $\text{hdet}_S g = \det g|_V$ for all $g \in \text{GrAut}(S)$.

We now extend the definition of homological determinant to the filtered case. Let $\text{FilAut}(A)$ be the group of automorphisms of $A$ preserving the given filtration $F$. Hence for any $g \in \text{FilAut}(A)$ one can extend $g$ to a graded automorphism of $\text{Rees} A$ by sending $t$ to $t$. Also $g$ induces a graded automorphism of $\text{Gr} A$. For simplicity we still use $g$ for both these graded automorphisms. Next lemma is clear.

**Lemma 2.3.** Let $B$ be a connected graded ring. Then $B = \text{Rees} A$ for some filtered algebra $A$ if and only if there is a regular central element in $B_1$.

The next proposition is useful for computing the homological determinant when $A$ has normal elements.

**Proposition 2.4.** Let $B$ be a noetherian graded $AS$-Gorenstein ring and let $x$ be a regular normal element of positive degree. Suppose $g$ is a graded automorphism of $B$ such that $g(x) = \lambda x$. Then $\text{hdet}_B g = \lambda \text{hdet}_A g$ where $A = B/(x)$.

**Proof.** Suppose $d$ is the injective dimension of $B$. Then $d - 1$ is the injective dimension of $A$. Let $l_x : B(-s) \to B$ be the left multiplication by $x$ where $s = \deg x$. Applying $H^m_\bullet$ to the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & B(-s) & \xrightarrow{l_x} & B & \xrightarrow{g} & A & \to & 0 \\
& & \downarrow{\lambda g} & & \downarrow{g} & & \downarrow{g} & & \\
0 & \to & B(-s) & \xrightarrow{l_x} & B & \xrightarrow{g} & A & \to & 0
\end{array}
\]

we obtain that

\[
\begin{array}{ccccccccc}
0 & \to & H^{d-1}_m(A) & \to & H^d_m(B)(-s) & \to & H^d_m(B) & \to & 0 \\
& & \downarrow{(\text{hdet}_A g)^{-1}(g^{-1})'} & & \downarrow{\lambda(\text{hdet}_B g)^{-1}(g^{-1})'} & & \downarrow{} & & \\
0 & \to & H^{d-1}_m(A) & \to & H^d_m(B)(-s) & \to & H^d_m(B) & \to & 0.
\end{array}
\]

The result follows by comparing the first two vertical maps in the highest nonzero homogeneous component. □
Let $A$ be a filtered noetherian AS-Gorenstein ring. If $g$ is a filtered automorphism of $A$, then $g(t) = t$ for $t \in \operatorname{Rees} A$ and by Proposition 2.4, $\operatorname{hdet}_A g = \operatorname{hdet}_{\operatorname{Rees} A} g$. We define the homological determinant of $g$ on $A$ to be

$$\operatorname{hdet}_A g = \operatorname{hdet}_{\operatorname{Gr} A} g.$$ 

In the rest of this section we compute some homological determinants.

**Lemma 2.5.** Let $A$ be a filtered algebra with $\operatorname{Gr} A = k_{p_{ij}}[x_1, \ldots, x_n]$, where $p_{ij} \neq p_{kl}$ for all $(i, j) \neq (k, l)$. Let $g$ be a filtered automorphism of $A$. Then $\operatorname{hdet}_A g = \det g|_V$ where $V = \bigoplus_{i=1}^n kx_i$.

**Proof.** First we assume that $A$ is $\mathbb{N}$-graded and $\deg x_i = 1$ for all $i$.

Under the hypothesis on $p_{ij}$ it is routine to check that if $\sum_{i=1}^n a_i x_i$ is a normal element then it must be $a_i x_i$ for some $i$. Since $g(x_i)$ is normal, we can write $g(x_i) = b_i x_{\sigma(i)}$ for some permutation $\sigma \in S_n$ and $b_i \neq 0$. Applying $g$ to the relations $x_i x_j = p_{ij} x_j x_i$ we see that $p_{ij} = p_{\sigma(i), \sigma(j)}$. Therefore $\sigma$ is the identity by the hypothesis.

Now we consider the general case. By definition of $\operatorname{hdet}$ we can pass to the graded case. Let $g$ be a graded automorphism of $A = k_{p_{ij}}[x_1, \ldots, x_n]$ with

$$\deg x_1 \geq \cdots \geq \deg x_n \geq 1.$$ 

Let $w$ be the minimal index such that $\deg x_w = \deg x_n$. Then $g$ preserves $\bigoplus_{i=w}^n kx_i$. As an automorphism of $k_{p_{ij}}[x_w, \ldots, x_n]$, $g(x_n) = b_n x_n$ for some $b_n \neq 0$ by the previous argument. By induction, $g(x_i) = b_i x_i + f_i(x_{i+1}, \ldots, x_n)$ for some polynomial $f_i$. By Proposition 2.4, $\operatorname{hdet}_A g = b_n \operatorname{hdet}_A(g(x_n))$. Therefore $\operatorname{hdet}_A g = \prod_i b_i = \det g|_V$. \hfill \Box

**Example 2.6.** Let $M_{q,p_{ij}}(n)$ be the multiparameter quantum matrix algebra. It is a connected graded algebra generated by $\{x_{ij} \mid i, j = 1, \ldots, n\}$ with $n^2$ relations (see [AST] for a list of relations), where the structure constants $\{q, p_{ij}\}$ are generic. Similar to the proof of Lemma 2.5 with more careful computations we check that $\operatorname{hdet}_{M_{q,p_{ij}}(n)} g = \det g|_V$ where $V = \bigoplus_{i,j=1}^n kx_{ij}$. Details are left to interested readers.

**Proposition 2.7.** Let $A$ and $B$ be two filtered AS-Gorenstein ring and assume that $A \otimes B$ is filtered noetherian. If $g$ (respectively $h$) is a filtered automorphism of $A$ (respectively $B$), then $\operatorname{hdet}_{A \otimes B}(g \otimes h) = (\operatorname{hdet}_A g)(\operatorname{hdet}_B h)$.

**Proof.** Let $\{F_i \mid i \geq 0\}$ (respectively $\{W_i \mid i \geq 0\}$) be the filtration of $A$ (respectively $B$). The filtration of $A \otimes B$ is defined by $Z_n = \bigcup_{i+j=n} F_i \otimes W_j$. Then $\operatorname{Gr} (A \otimes B) = (\operatorname{Gr} A) \otimes (\operatorname{Gr} B)$.

By the definition of $\operatorname{hdet}$ in the filtered case, we may work with the associated graded rings, or equivalently we may assume $g$ and $h$ are graded automorphisms of graded AS-Gorenstein rings $A$ and $B$ respectively. Since systems $\{(A \otimes B)_{\geq n}\}$ and $\{A_{\geq n} \otimes B + A \otimes B_{\geq n}\}$ are cofinal, we have

$$H_{m_{A \otimes B}}^n(M_A \otimes N_B) = \bigoplus_{i+j=n} H_{m_A}^i(M_A) \otimes H_{m_B}^j(N_B).$$

In particular, if $E_A$ and $F_B$ are graded injective modules over $A$ and $B$ respectively, then $E_A \otimes F_B$ is $H_{m_{A \otimes B}}^* -$acyclic. Let $E^*$ and $F^*$ be injective resolutions of $A$ and $B$ respectively. We can compute
Lemma 3.1. Let $k$ be a polynomial ring $F$ and $g$ be a $g$-linear map. Since $g$ gives a $g$-linear map of $E$ and $h$ gives an $h$-linear map of $F$, $g \otimes h$ gives a $g \otimes h$-linear map of $E^* \otimes F^*$. Therefore

$$H_{m_A \otimes B}^{d+e} (g \otimes h) = H_{m_A}^d (g) \otimes H_{m_B}^e (h),$$

where $d = \text{injdim } A$ and $e = \text{injdim } B$. By the definition of homological determinant in the graded case, $\text{hdet}_{A \otimes B} (g \otimes h) = (\text{hdet}_A) (\text{hdet}_B h)$.

Finally we give two examples where $\text{hdet}_A g$ is not equal to $\det g|V$ where $V$ is the minimal generating space.

Example 2.8. Let $A$ be the commutative graded ring $k[x_1, x_2]/(x_1^2)$. Let $g$ be the graded automorphism of $A$ sending $x_i$ to $-x_i$ for $i = 1, 2$. Then $V = kx_1 \oplus kx_2$ is the minimal generating space and $\det g|V = 1$. Also $g^2 = \text{id}$. Since $g(x_1) = -x_1$, by Proposition 2.4, $\text{hdet}_A g = (-1) \text{hdet}_{k[x_1, x_2]} g = -1$. Hence $\text{hdet}_A g$ is neither equal to $\det g|_V$ nor $(\det g|_V)^{-1}$. It is also easy to check that $A^G$ is not Gorenstein with $G = \{g, \text{id}\}$. 

Example 2.9. Let $A$ the skew polynomial ring generated by $x, y, z$ with relations $xy = -yx, xz = zx, yz = zy$. Let $g$ be the graded automorphism of $A$ such that $g(x) = -y, g(y) = -x, g(z) = -z$. Then $g^2 = 1$ and $\det g|_V = -1$. The Koszul dual is $A^! = k \langle x, y, z \rangle/I$, where $I$ is generated by $x^2 = y^2 = z^2 = 0, xy = yx, yz = -zy, xz = -zx$. The induced automorphism $g^*$ of $A^!$ is given by $x \rightarrow -y, y \rightarrow -x, z \rightarrow -z$. It follows from [JiZ, 3.4] that

$$Tr_A(g, t) = \frac{1}{Tr_{A^!}(g^*, -t)} = \frac{1}{1 + t + t^2 + t^3}.$$ 

Therefore $\text{hdet}_A g = 1$ [JoZ, 2.6]. Let $G = \{1, g\}$. Then the Hilbert series of $A^G$ is

$$H_{A^G}(t) = \frac{1}{2} (Tr_A(1, t) + Tr_A(g, t)) \neq \pm t^m H_{A^G}(t^{-1})$$

for any $m \in \mathbb{Z}$. So $A^G$ is not Auslander-Gorenstein [Ei, Ex. 21.17(c)]. This gives another example of $\text{hdet} g \neq \det g|_V$ and also shows that Theorem 0.1 fails without the extra hypothesis on $p_{ij}$.

3. Invariant subring of filtered rings

In this section we prove Theorems 0.1 and 0.2. The following lemma is clear.

Lemma 3.1. Let $A$ be a filtered ring and $G$ be a subgroup of $\text{FilAut}(A)$. Then

1. Rees $A^G = (\text{Rees } A)^G$.
2. If $|G| \neq 0$ in $k$, then $\text{Gr } A^G = (\text{Gr } A)^G$.

If $|G| = 0$ in $k$, then Lemma 3.1(2) may fail.

Example 3.2. Let $k$ be a field of characteristic 2 and let $g$ be a filtered automorphism of the polynomial ring $k[x]$ determined by $g(x) = x + 1$. Hence $g^2 = \text{id}$. Let $G = \{\text{id}, g\}$. Consider $k[x]$ as a filtered ring and $G$ a subgroup of filtered automorphisms. Then the induced graded automorphism of $g$ is the identity on the $k[x] = \text{Gr } k[x]$. Hence $k[x^2] = \text{Gr } A^G \subseteq (\text{Gr } A)^G = k[x]$.

We now recall results from [JoZ] and [YZ].
Theorem 3.3. Let $A$ be a noetherian connected graded ring and $G$ a finite subgroup of $\text{GrAut}(A)$ with $|G| \neq 0$ in $k$. Suppose that $\text{Idet}_A g = 1$ for all $g \in G$.

(1) [JoZ, 3.3] If $A$ is $\text{AS-Gorenstein}$, then so is $A^G$.
(2) [YZ, 4.20] If $A$ is Auslander-Gorenstein, then so is $A^G$.
(3) [YZ, 5.13 and 5.14] If $A^G$ is $\text{AS-Gorenstein}$ and $A^G$ either is PI or has enough normal elements, then $A^G$ is Auslander-Gorenstein and $\text{GKdim-Macaulay}$.

A slightly more general version of Theorem 3.3(3) holds. Let $\text{Kdim}$ denote the Krull dimension.

Lemma 3.4. Let $B \subset A$ be two noetherian connected graded rings such that $A$ is finitely generated over $B$ on both sides and $B$ is a direct summand of $A$ as graded $B$-bimodules. Suppose $\text{GKdim } M = \text{Kdim } M < \infty$ for all finitely generated graded $A$-module $M$. Then $\text{GKdim } N = \text{Kdim } N = \text{Kdim } N \otimes_B A$ for all finitely generated graded $B$-module $N$.

If $G$ is a finite subgroup of $\text{GrAut}(A)$ with $|G| \neq 0$ in $k$, then $B := A^G \subset A$ is a direct summand of $A$. If moreover $A$ is noetherian, then $A$ is finitely generated over $B$ on both sides [Mo, 2.1.1 and 1.12.1].

Proof. Since $B$ is a direct summand of $A$ as $B$-bimodule, $N_B$ is a direct summand of $N \otimes_B A_B$. Hence $\text{GKdim } N \leq \text{GKdim } N \otimes_B A_B$ and $\text{Kdim } N \leq \text{Kdim } N \otimes_B A_B$.

By [MR, 8.3.14(iii)], $\text{GKdim } N \geq \text{GKdim } N \otimes_B A_A$ and $\text{GKdim } M_A \geq \text{GKdim } M_B$ for any $A$-module $M$. Hence

$$\text{GKdim } N \otimes_B A_A = \text{GKdim } N \otimes_B A_B = \text{GKdim } N.$$ It remains to show that $\text{Kdim } N = \text{Kdim } N \otimes_B A$. For every submodule $L \subset N$, let $\phi$ be the map $L \otimes_B A \to N \otimes_B A$. Then $L \to \phi(L \otimes_B A)$ is an order-preserving map from the lattice of $B$-submodules of $N$ to the lattice of $A$-submodules of $N \otimes_B A$. Since $A = B \oplus C$ for some $C$ as $B$-bimodule, this map preserves proper containment. Therefore

$$\text{Kdim } N \leq \text{Kdim } N \otimes_B A.$$ We now prove $\text{Kdim } N \geq \text{Kdim } N \otimes_B A (= \text{GKdim } N)$ by induction on $\text{Kdim } N$. By noetherian induction we may assume $N$ is $\text{Kdim}$-critical. Let $s = \text{Kdim } N \otimes_B A = \text{GKdim } N < \infty$. Pick an $A$-submodule $M \subset N \otimes_B A$ such that $(N \otimes_B A)/M$ is $\text{critical}$ of $\text{Krull}$ (and $\text{GK}$) dimension $s - 1$ and let $\bar{N}$ be the image of the composition

$$N \to N \otimes_B A \to (N \otimes_B A)/M.$$ Then

$$\text{GKdim } \bar{N} \leq \text{GKdim } \bar{N} A_B \leq \text{GKdim } \bar{N} \otimes_B A_B = \text{GKdim } \bar{N}.$$ This implies that the first “=” of the following

$$\text{GKdim } \bar{N} A_A \geq \text{GKdim } \bar{N} A_B = \text{GKdim } \bar{N} = \text{GKdim } \bar{N} \otimes_B A_A \geq \text{GKdim } \bar{N} A_A.$$ Therefore

$$\text{GKdim } \bar{N} = \text{GKdim } \bar{N} A_A = \text{GKdim}(N \otimes_B A)/M = s - 1.$$ Hence $\bar{N}$ is a proper quotient of $N$ and by induction hypothesis $\text{Kdim } \bar{N} \geq \text{GKdim } \bar{N} = s - 1$. Since $N$ is critical, $\text{Kdim } N \geq \text{Kdim } \bar{N} + 1 \geq s$ as required. \qed

By [Zh, 2.3, 2.4 and 3.1], if $A$ has enough normal elements or if $A$ is the 4-dimensional Sklyanin algebra (or more generally if $A$ satisfies $\text{SSC}$) then

$$\text{Kdim } M = \text{GKdim } M$$ for all finitely generated graded $A$-module $M$. 

GORENSTEINNESS OF INVARIANT SUBRINGS OF QUANTUM ALGEBRAS 9

GORENSTEINNESS OF INVARIANT SUBRINGS OF QUANTUM ALGEBRAS 9
Theorem 3.5. Let $A$ be a noetherian filtered ring and $G$ be a finite subgroup of FilAut($A$). Suppose that $|G| \neq 0$ in $k$ and that $\det_A g = 1$ for all $g \in G$.

1. If $A$ is filtered AS-Gorenstein, then so is $A^G$.
2. If $A$ is filtered Auslander-Gorenstein, then so is $A^G$.
3. Suppose $\GKdim M = \dim M$ for all finitely generated graded $\Gr A$-module $M$. If $A$ is filtered Auslander-Gorenstein and $\GKdim$-Macaulay, then so is $A^G$.

Proof. (1) By definition we only need to show $\Gr A^G$ is AS-Gorenstein. Now the result follows by Theorem 3.3(1).

(2) It suffices to show $\Gr A^G$ is Auslander-Gorenstein, which follows by Theorem 3.3(2).

(3) We can pass to the graded case and assume that $A$ is graded. By (2) $A^G$ is Auslander-Gorenstein. It remains to show the $\GKdim$-Macaulay property. Let $\dim$ be the canonical dimension defined in [YZ, 2.9]. Since $A$ is $\GKdim$-Macaulay, $\GKdim M = \dim M$ for all finitely generated graded $A$-module $M$. Let $B$ denote $A^G$. For every finitely generated graded $B$-module $N$, it follows from Lemma 3.4 that $\dim N = \GKdim N$ for all finitely generated graded $B$-module $N$, it follows from Lemma 3.4 that

$\GKdim N = \dim N = \GKdim N \otimes B A$

By [YZ, 4.14], $\dim N \leq \dim N \otimes B A = \GKdim N \otimes B A$

where the first " = " is [YZ, 4.17(3)]. Combining these (in)equalities we obtain $\dim N = \GKdim N = \dim N$. Therefore $A^G$ is graded $\GKdim$-Macaulay. By [L2, 5.8], $A^G$ is (ungraded) $\GKdim$-Macaulay as required. □

The following example is well-known, which shows that the hypothesis $\det_A g = 1$ can not be deleted from Theorem 3.5.

Example 3.6. Let $A$ be the skew polynomial ring $\mathbb{C}[x, y, z]$ where $p_{ij} \neq 0 \in \mathbb{C}$, and $g$ a graded automorphism sending $x \rightarrow -x$, $y \rightarrow -y$ and $z \rightarrow -z$. Then $g^2 = id$ and $\det g = -1$. By Proposition 2.4, $\det g = -1$ and it is easy to check that $Tr(g, t) = \frac{1}{1+t}$. Let $G = \{id, g\}$. Then the Hilbert series of $A^G$ is

$H_{A^G}(t) = \frac{1}{2} \left( \frac{1}{1-t} + \frac{1}{1+t} \right) = \frac{1+3t^2}{(1-t^2)^3}$

So $H_{A^G}(t^{-1}) \neq \pm t^m H_{A^G}(t)$ for any $m \in \mathbb{Z}$. Therefore $A^G$ is not Gorenstein by [Ei, Exercise 21.17(c)].

To apply Theorem 3.5, one needs to show $\det g = 1$ for all $g \in G$. To show $\det g = 1$ it suffices to show the only group homomorphism $G \rightarrow k^\times := k - \{0\}$ is trivial. Here are some easy cases.

Lemma 3.7. [JoZ, 3.4] The homomorphism $G \rightarrow k^\times$ is trivial if one of the following holds:

(a) $G = [G, G]$;
(b) $|G|$ is odd and $k = \mathbb{Q}$;
(c) $k = \mathbb{Z}/2\mathbb{Z}$.

One can easily get some corollaries by combining Lemma 3.7 and Theorem 3.5. We now prove Theorems 0.1 and 0.2.
Proof of Theorem 0.1. By [Zh, 2.3(2) and 3.1], \( \text{GKdim} \, M = \text{Kdim} \, M \) holds for all finitely generated graded \( k_{p_{ij}}[x_1, \ldots, x_n] \)-module \( M \). So the result follows from Lemma 2.5 and Theorem 3.5(3). \( \Box \)

Proof of Theorem 0.2. Again by [Zh, 2.3(2) and 3.1], \( \text{GKdim} \, M = \text{Kdim} \, M \) holds for all finitely generated graded \( k_{p_{ij}}[x_1, \ldots, x_n] \otimes \Lambda(V) \)-module \( M \).

Since \( A \) is a filtered algebra, \( A \otimes \Lambda(V) \) is a filtered algebra with the filtration induced by the filtration of \( A \) and \( \text{Gr} (A \otimes \Lambda(V)) = (\text{Gr} \, A) \otimes \Lambda(V) = k_{p_{ij}}[x_1, \ldots, x_n] \otimes \Lambda(V) \). By [Zh, 0.2] and Theorem 1.2(5), \( A \otimes \Lambda(V) \) is Auslander-Gorenstein and GKdim-Macaulay. Also \( A \otimes \Lambda(V) \) and \( (A \otimes \Lambda(V))^G \) are filtered noetherian.

By Proposition 2.7 and Lemmas 2.5 and 2.1(2),

\[
\text{hdet}_{A \otimes \Lambda(V)} g = \text{hdet}_A g \cdot \text{hdet}_\Lambda(V) g = \det g_V \cdot (\det g_V)^{-1} = 1
\]

for all \( g \in G \). Hence the result follows from Theorem 3.5(3). \( \Box \)

Let \( \Lambda_{p_{ij}}(V) = k(x_1, \ldots, x_n)/(x_1^2, x_1 x_j + p_{ij}^{-1} x_j x_i) \) be the quantized exterior algebra. By Theorem 3.5(3) and similar arguments as in the proof of Theorem 0.2 we have the following result which contains the commutative case [Ben, 5.3.2] as a special case.

**Theorem 3.8.** Suppose \( A \) is a filtered ring such that the associated graded ring \( \text{Gr} \, A \) is isomorphic to the skew polynomial ring \( k_{p_{ij}}[x_1, \ldots, x_n] \). Let \( G \) be a finite group of filtered automorphisms of \( A \) with \( |G| \neq 0 \) in \( k \). Then \( (A \otimes \Lambda_{p_{ij}}(V))^G \) is Auslander-Gorenstein and GKdim-Macaulay where \( V = \bigoplus_{i=1}^n k x_i \).

The following is an immediate consequence of Theorems 0.1-0.2 and [YZ, 6.23].

**Corollary 3.9.** Let \( A^G \) be as in Theorem 0.1 and \( (A \otimes \Lambda(V))^G \) be as in Theorem 0.2. Then they have quasi-Frobenius rings of fractions.

Another method to check whether \( A^G \) is Gorenstein is to use a noncommutative version of Stanley’s theorem [JoZ, Sect. 6]. For example if \( \text{Gr} \, A \) (or Rees \( A \)) is Auslander-regular and \( \text{char} \, k = 0 \), then \( A^G \) is Auslander-Gorenstein if the Hilbert series of \( \text{Gr} \, A^G \) satisfies

\[
H_{\text{Gr} \, A^G}(t^{-1}) = \pm t^m H_{\text{Gr} \, A^G}(t)
\]

for some \( m \in \mathbb{Z} \) (see [JoZ, 6.2 and 6.4]).

4. **Invariant subrings of the Weyl algebras**

Let \( A_n \) be the \( n \)-th Weyl algebra and \( G \) a finite group of automorphisms of \( A_n \). If \( \text{char} \, k = 0 \), then it is well-known that \( A_n^G \) has global dimension \( n \) for any \( G \). We start with some basic facts below.

**Theorem 4.1.** Let \( \text{char} \, k = 0 \) and \( G \) a finite group of automorphisms of the \( n \)-th Weyl algebra \( A_n \). Then \( A_n^G \) is a noetherian simple domain of global dimension and Krull dimension \( n \) and Gelfand-Kirillov dimension \( 2n \).

**Proof.** By [Mo, 1.12.1], \( A_n^G \) is noetherian. It is clear that \( A_n^G \) is a domain.

Recall that an automorphism \( g \) is inner if there is a unit \( u \in A \) such that \( g(x) = uxu^{-1} \). Since units of \( A_n \) are elements of \( k^\times \), every automorphism of \( A_n \) is outer (namely, not inner). By [Mo,
2.6] and [Mo, 2.4], $A_n^G$ is simple and $A_n$ is a finitely generated projective $A_n^G$-module. Hence the standard spectral sequence
\[
E_2^{p,q} := \text{Ext}_{A_n^G}^p(N, A_n) \Rightarrow \text{Ext}_{A_n^G}^{p+q}(N, M)
\]
collapses to the isomorphisms
\[
\text{Ext}_{A_n^G}^p(N \otimes_{A_n^G} A_n, M) = \text{Ext}_{A_n}^p(N, M)
\]
for all right $A_n$-module $M$ and right $A_n^G$-module $N$. Therefore $\text{Ext}_{A_n^G}^p(N, M) = 0$ for all $p > n$. For any right $A_n^G$-module $M_0$, let $M = M_0 \otimes_{A_n^G} A_n$. Since $A_n^G$ is an $(A_n^G, A_n^G)$-bimodule direct summand of $A_n$ [Mo, 2.1.1], $M_0$ is a direct summand of $A_n^G$-module $M$. Thus $\text{Ext}_{A_n^G}^p(N, M_0) = 0$ for all right $A_n^G$-modules $N$ and $M_0$. This implies that $\text{gldim} A_n^G \leq n$. Pick a simple $A_n$-module $N$ with $\text{GKdim} n$, then $\text{GKdim}_{A_n} N \otimes_{A_n^G} A_n = n$ [MR, 8.3.14(iii)], thus $\text{Ext}_{A_n}^n(N \otimes_{A_n^G} A_n, A_n) \neq 0$. Consequently, $\text{Ext}_{A_n^G}^n(N, A_n) \neq 0$ and $\text{gldim} A_n^G \geq n$. So $\text{gldim} A_n^G = n$.

For $\text{GK}$-dimension we have
\[
2n = \text{GKdim} A_n = \text{GKdim}(A_n^G \otimes_{A_n^G} A_n) \leq \text{GKdim} A_n^G \leq \text{GKdim} A_n
\]
where the first $\leq$ is [MR, 8.3.14(iii)]. Therefore $\text{GKdim} A_n^G = 2n$.

Since $A_n$ is projective over $A_n^G$ and contains $A_n^G$ as a direct summand, $A_n$ is faithfully flat over $A_n^G$. Thus $\text{Kdim} A_n^G \leq \text{Kdim} A_n$. For any group action one can form a skew group ring $A_n \ast G$ which is free (hence faithfully flat) over $A_n$. Thus $\text{Kdim} A_n \leq \text{Kdim} A_n \ast G$. By [Mo, 2.5], $A_n^G$ and $A_n \ast G$ are Morita equivalent. Therefore $\text{Kdim} A_n^G = \text{Kdim} A_n \ast G = \text{Kdim} A_n = n$. \hfill \Box

Remark 4.2. The above proof shows the following known result: If $A$ is noetherian and simple with finite global dimension and $G$ is a finite group of (outer) automorphisms of $A$ with $|G| \neq 0$ in $k$, then
(a) $A^G$ is noetherian and simple;
(b) $\text{gldim} A^G = \text{gldim} A$;
(c) $\text{Kdim} A^G = \text{Kdim} A$;
(d) $\text{GKdim} A^G = \text{GKdim} A$.

It is not clear from the above proof that $A_n^G$ satisfies Auslander and and $\text{GKdim}$-Macaulay properties (which we believe to be true). Next we are going to show for certain automorphism group $G$ (namely for filtered automorphism group) $A_n^G$ satisfies the Auslander and $\text{GKdim}$-Macaulay properties without any restriction on char $k$.

Let $A_n$ be generated by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ subject to the relations
\[
[x_i, x_j] = [y_i, y_j] = 0, \quad [y_i, x_j] = \delta_{ij}, \quad \text{for all} \quad i, j.
\]
This is a filtered algebra with the standard filtration determined by $F_i = k + \sum_i kx_i + \sum_j ky_j$.
The associated graded ring is the commutative polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Let $V_n$ be the vector space generated by these $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. We say a linear map $\sigma : V_n \to V_n$ is a $[-,-]$-map if
\[
[\sigma(x_i), \sigma(x_j)] = [\sigma(y_i), \sigma(y_j)] = 0, \quad [\sigma(y_i), \sigma(x_j)] = \delta_{ij}, \quad \text{for all} \quad i, j.
\]
We call another basis $\{x'_1, \ldots, x'_n, y'_1, \ldots, y'_n\}$ of $V_n$ an $[-,-]$-basis if the map $\sigma : x_i \to x'_i, y_i \to y'_i$ is a $[-,-]$-map.
Lemma 4.3. If $g$ is a filtered endomorphism of $A_n$, then there is a $[-,-]$-map $\sigma : V_n \to V_n$ and a linear map $\epsilon : V_n \to k$ such that $g|_{V_n} = \sigma + \epsilon$. Further if $\sigma$ is an automorphism, then $g$ is a filtered automorphism of $A_n$ and $\det A_n g = \det \sigma|_{V_n}$.

Proof. Since $g$ maps $F_1 = k \oplus V_n$ to $F_1$, the restriction $g|_{V_n}$ decomposes into two parts $\sigma$ and $\epsilon$. For every $a, b \in V_n$, $[a, b] \in k$ and hence

$$[a, b] = g([a, b]) = [g(a), g(b)] = [\sigma(a) + \epsilon(a), \sigma(b) + \epsilon(b)] = [\sigma(a), \sigma(b)]$$

because $\epsilon(a)$ commutes with all elements. Therefore $\sigma$ is a $[-,-]$-map. If $\sigma$ is an automorphism then $g$ is an automorphism of $F_1$ and hence a filtered automorphism of $A_n$. By Lemma 2.1(1),

$$\det A_n g = \det k[V_n] g = \det g|_{V_n} = \det \sigma|_{V_n}. \quad \square$$

Lemma 4.4. If $\sigma : V_n \to V_n$ is a $[-,-]$-map, then $\det \sigma = 1$. As a consequence, $\sigma$ is an automorphism.

Proof. Replacing $k$ by its algebraic closure will not change the determinant. So we may assume that $k$ is algebraically closed. Let $x$ be an eigenvector of $\sigma$. So $\sigma(x) = sx$ for some $s \in k$. Write $x = \sum a_i x_i + \sum b_i y_i$. We may assume some $a_i \neq 0$. We will make several base changes such that $x = x_n$. Apparently we will require that all base changes preserve the bracket relations. By changing $x_i$ and then changing $y_i$ properly (to keep the $[-,-]$), we have another $[-,-]$-basis so that $x = x_n + \sum b_i y_i$. By changing $\{y_1, \ldots, y_{n-1}\}$ we may assume $x = x_n + b_{n-1} + b_n y_n$. Changing basis within $\{x_n, y_n\}$, we have $x = x_n + b_{n-1} y_{n-1}$. Exchanging $x_{n-1}$ and $-y_{n-1}$, we have $x = x_n + b_{n-1} x_{n-1}$. Finally we change $x_i$ so that $x = x_n$. Since $\sigma$ is a $[-,-]$-map,

$$1 = [y_n, x_n] = [\sigma(y_n), \sigma(x_n)] = [\sigma(y_n), sx_n].$$

Thus $s \neq 0$ and $\sigma(y_n) = s^{-1} y_n + \sum a_i x_i + \sum i < b_i y_i$. Let $T = \bigoplus_{i=1}^n kx_i + \bigoplus_{i<n} ky_i$. Then $T = \{x \in V_n \mid [x, x] = 0\}$. Since $\sigma$ is a $[-,-]$-map, $T$ is $\sigma$-invariant, namely, $\sigma(T) \subset T$. Write $T = V_{n-1} \oplus ky_n$. We may decompose the restriction $\sigma|_{V_{n-1}}$ into two linear maps $\sigma|_{V_{n-1}} = \theta + \eta$ where $\eta : V_{n-1} \to ky_n$ is a linear map and $\theta : V_{n-1} \to V_{n-1}$ is a $[-,-]$-map because $y_n$ commutes with $V_{n-1}$. By induction $\det \theta = 1$ and clearly

$$\det \sigma = s \times s^{-1} \times \det \theta = 1.$$ 

As a consequence $\sigma$ is invertible. \square

By Lemmas 4.3 and 4.4 the following is clear.

Corollary 4.5. Every filtered endomorphism of the Weyl algebra $A_n$ is an automorphism.

Theorem 4.6. Let $G$ be a finite group of filtered automorphisms of the Weyl algebra $A_n$. If $|G| \neq 0$ in $k$, then $A_n^G$ is filtered Auslander-Gorenstein and GKdim-Macaulay.

Proof. This is a consequence of Theorem 0.1 and Lemmas 4.3 and 4.4. \square

Theorem 0.3 is an immediate consequence of Theorems 4.1 and 4.6.

For the first Weyl algebra $A_1$ over the complex numbers $C$, it is known that every finite subgroup of Aut($A_1$) is conjugate to a subgroup of $SL(2, C) \subset \text{FilAut}(A_1)$ (see [AHV]). A complete list of finite subgroup of $SL(2, C)$ is also listed in [AHV]. Therefore we have the following.
**Corollary 4.7.** Let $A_1$ be the first Weyl algebra over $\mathbb{C}$ and $G$ a finite group of automorphisms of $A_1$. Then $A_1$ is Auslander-regular and $\text{GKdim-Macaulay}$.

### 5. Invariant subrings of quantum Weyl algebras

In this section we look at a family of quantum Weyl algebras studied in [GZ] and prove similar statements as in Section 4.

Let $\{q\} \cup \{p_{ij} \mid i, j = 1, \cdots, n\}$ be a set of nonzero elements in $k$ with $p_{ii} = 1$ and $p_{ij} = p_{ji}^{-1}$. The quantum Weyl algebra $A_n(q, p_{ij})$ is generated by the elements $x_1, \cdots, x_n, y_1, \cdots, y_n$ subject to the relations

\[
x_i x_j = p_{ij} q x_j x_i, \quad \text{for all } i < j \tag{E1}
\]

\[
y_i y_j = p_{ij} q^{-1} y_j y_i, \quad \text{for all } i < j
\]

\[
y_i x_j = p_{ij}^{-1} q x_j y_i, \quad \text{for all } i \neq j
\]

\[
y_i x_i = 1 + q^2 x_i y_i + (q^2 - 1) \sum_{j > i} x_j y_j, \quad \text{for all } i.
\]

By [GZ, 3.11(1)], $A_n(q, p_{ij})$ is a noetherian, Auslander-regular and $\text{GKdim-Macaulay}$ domain of GK-dimension $2n$. It is easy to check that the associated graded ring has enough normal elements. Hence it follows from [Zh, 2.3(2) and 3.1] that the hypothesis $\text{GKdim } M = \text{Kdim } M$ in Theorem 3.5(3) is satisfied. If $q^2 = 1$ and $\text{char } k = 0$, then $A$ is not simple and has global dimension and Krull dimension $2n$ [GZ, 3.11(3)]. If $q^n \neq 1$ for all $n > 0$, then it is primitive [GZ, 3.2].

**Lemma 5.1.** Suppose that $q^4 \neq 1$, $q p_{ij} \neq 1$, $q^3 p_{ij} \neq 1$ for all $i \neq j$. Then

1. Every filtered automorphism $g$ of $A_n(q, p_{ij})$ has the form

\[
g(x_i) = \alpha_i x_i + \sum_{j > i} a_{ij} x_j + \sum_{j > i} b_{ij} y_j
\]

\[
g(y_i) = \alpha_i^{-1} y_i + \sum_{j > i} c_{ij} x_j + \sum_{j > i} d_{ij} y_j
\]

for $\alpha_i \neq 0, a_{ij}, b_{ij}, c_{ij}, d_{ij} \in k$.

2. $\det g|_{V_n} = 1$ where $V_n = \bigoplus_{i=1}^n k x_i \oplus \bigoplus_{i=1}^n k y_i$.

**Proof.** (2) It follows from (1) immediately.

(1) By [GZ, 1.5] and the relations above, the ordered monomials in $x_1, \cdots, x_n, y_1, \cdots, y_n$ are basis of $A_n(q, p_{ij})$. Write

\[
g(x_n) = \sum_j a_j x_j + \sum_j b_j y_j + t
\]

\[
g(y_n) = \sum_j c_j x_j + \sum_j d_j y_j + s.
\]

Use the relations listed in (E1) and

\[
g(y_n) g(x_n) = 1 + q^2 g(x_n) g(y_n) \tag{E2}
\]
we obtain \( a_i c_i = q^2 a_i c_i \) by comparing the coefficients in \( x_i^2 \) terms. If \( a_{i_0} \neq 0 \) for some \( i_0 \) then \( c_{i_0} = 0 \) since \( q^2 \neq 1 \). By comparing the coefficients of \( x_{i_0} x_j \) terms for \( j \neq i_0 \), we obtain \( c_j a_{i_0} p_{i_0 j} q^{-1} = a_{i_0} c_j q^2 \). Thus \( c_j = 0 \) since \( q^3 p_{i_0 j} \neq 1 \). Since \( g \) is an algebra automorphism \( d_j \neq 0 \) for some \( j \). A similar argument by exchanging \( x's \) and \( y's \) we see that \( b_j = 0 \) for all \( j \). Thus (E2) implies that \( s = t = 0 \). If there are \( a_i \neq 0 \) and \( d_j \neq 0 \) for \( j \neq i \) then \( a_i d_j = d_j a_i p_{i_0 j} q \) by comparing the \( x_i y_j \) terms. This contradicts with \( p_{ij} q \neq 1 \). Thus \( a_i = d_j = 0 \) for all \( j \neq i_0 \). Now if we compare (E2) with (E1) we see that \( g(x_n) = a_n x_n \) and \( g(y_n) = a_n^{-1} y_n \) and write \( \alpha_n = a_n \). In the other case when \( a_i = 0 \) for \( i \), we can show that this is impossible similar to the above argument (the condition \( q^4 \neq 1 \) will be used in this case). Thus

\[
g(x_n) = \alpha_n x_n, \quad g(y_n) = \alpha_n^{-1} y_n.
\]

Now we decompose the vector space \( V_n \) into \( V_{n-1} \oplus (k x_n \oplus k y_n) \) and decompose \( g|_{V_n} \) into

\[
g|_{V_{n-1}} = g'|_{(V_{n-1} \rightarrow V_{n-1} \oplus k)} + \epsilon|_{(V_{n-1} \rightarrow k x_n \oplus k y_n)}.
\]

It is not hard to see that \( g' \) can be extended to an algebra automorphism of \( A_{n-1}(q, p_{ij}) \). By induction \( g' \) has the required form and the statement follows by induction.

**Proof of Theorem 0.4.** If \( q \) is generic, then the hypothesis of Lemma 5.1 holds. The result follows from Theorem 3.5(3) and Lemma 5.1.

**Remark 5.2.** Theorem 0.4 should also hold even if \( q \) is not generic and holds for other nice quantum Weyl algebras studied in [GZ].

### 6. Invariant subrings of the enveloping algebras

Let \( L \) be a finite dimensional Lie algebra over \( k \) and \( U(L) \) be the universal enveloping algebra. There is a standard filtration given by \( F_i = (k + L)^i \) with the associated graded ring isomorphic to the commutative polynomial ring \( k[L] \).

**Lemma 6.1.** If \( g \) is a filtered automorphism of \( U(L) \), there there is a Lie algebra automorphism \( \sigma : L \rightarrow L \) and a linear map \( \epsilon : L \rightarrow k \) with \( \epsilon([L, L]) = 0 \) such that \( g|_{L} = \sigma + \epsilon \). Also \( \text{hdet}_{U(L)} g = \text{det} \sigma|_L \).

**Proof.** Since \( g \) sends \( L \) to \( L \oplus k \), \( g \) decomposes as \( \sigma + \epsilon \). For every \( a, b \in L \),

\[
\sigma([a, b]) = g([a, b]) - \epsilon([a, b]) = [\sigma(a) + \epsilon(a), \sigma(b) + \epsilon(b)] - \epsilon([a, b]) = [\sigma(a), \sigma(b)] - \epsilon([a, b]).
\]

Therefore \( \sigma \) is an endomorphism (and hence automorphism) of Lie algebra \( L \) and \( \epsilon([L, L]) = 0 \).

By the definition and Lemma 2.1(1), \( \text{hdet}_{U(L)} g = \text{det} \sigma|_L \).

In the rest of this section we study the cases when the determinant is 1. We assume that \( \text{char } k = 0 \). The next lemma is elementary.

**Lemma 6.2.** Let \( \nu \) be a nilpotent linear map of a finite dimensional space \( V \), then the linear map

\[
\exp(\nu) := \sum_{n=0}^{\infty} \frac{\nu^n}{n!}
\]

is a linear map of \( V \) with determinant 1.

The above lemma is also true if the trace of \( \nu \) is zero. An automorphism \( \sigma \) of the Lie algebra \( L \) is called **inner** if \( \sigma = \exp(\text{ad } x) \) for some \( x \in L \) and ad \( x \) is nilpotent. The **inner automorphism group** of \( L \), denoted by \( \text{Int } L \), is generated by all inner automorphisms of \( L \). By [Ja, Chapter IX], \( \text{Int } L \) is a normal subgroup of the automorphism group \( \text{Aut}(L) \) of \( L \).
Corollary 6.3. Let $L$ be a Lie algebra over $k$ with $\text{char } k = 0$.

(1) If $g|_L \in \text{Int } L$, then $\det_{U(L)} g = 1$.

(2) If $G$ is a finite subgroup of $\text{Int } L$, then $U(L)^G$ is Auslander-Gorenstein and GKdim-Macaulay.

Proof. (1) A consequence of Lemmas 6.1 and 6.2. □

(2) A consequence of (1) and Theorem 3.5(3).

The following classification of automorphism groups of simple Lie algebra is standard (see [Ja], [Hu], [Ka]). Let $k$ is an algebraically closed field of characteristic 0. Let $\Gamma_L = \text{Aut } L / \text{Int } L$ [Hu, p. 66]. Then

(a) $\Gamma_L$ is the graph automorphism of the Dynkin diagram.

(b) $\Gamma_L = 1$ (namely, $\text{Aut } L = \text{Int } L$) for the following types: $B_n, C_n, E_7, E_8, F_4, G_2$.

(c) $\Gamma$ of other types are

$\Gamma A_n = \mathbb{Z}/2\mathbb{Z}$ ($n \geq 2$) and the outer automorphism is given by

$$g : \quad e_i \to e_{n+1-i}, \quad f_i \to f_{n+1-i}, \quad i = 1, \ldots, n$$

$\Gamma D_4 = S_3 = \langle x, y \rangle$ where

$$x : \quad e_1 \to e_3, \quad e_3 \to e_4, \quad e_4 \to e_1, \quad e_2 \to e_2$$

$$y : \quad e_3 \to e_4, \quad e_4 \to e_3, \quad e_1 \to e_1, \quad e_2 \to e_2.$$

$\Gamma D_l = \mathbb{Z}/2\mathbb{Z}$ ($n > 4$) and the outer automorphism is given by

$$g : \quad e_{n-1} \to e_n, \quad e_n \to e_{n-1}, \quad e_i \to e_i.$$

$\Gamma E_6 = \mathbb{Z}/2\mathbb{Z}$ and the outer automorphism is given by

$$g : \quad e_1 \to e_6, \quad e_2 \to e_5, \quad e_4 \to e_4.$$

In the above we assume that $\{e_i, f_i, h_i\}$ are the Serre generators of the simple Lie algebra. We assume the similar formulas for the action of the outer automorphism on $f_i$ and $h_i$.

Corollary 6.4. Let $L$ be a simple Lie algebra of the following types over an algebraically closed field $k$ with $\text{char } k = 0$, and $G$ is a finite group of filtered automorphisms of $U(L)$:

(a) $A_n$, $n = 4m$ or $n = 4m + 1$;

(b) $B_n$;

(c) $C_n$;

(d) $E_6, E_7, E_8, F_4$ and $G_2$.

then $U(L)^G$ is Auslander-Gorenstein and GKdim-Macaulay.

Proof. By Corollary 6.3 and the above description of $\text{Aut}(L)$ it suffices to relevant outer automorphisms are of determinant 1.

Let $g$ be the outer automorphism given above (in case of $D_4$, $g = x$ or $y$). We compute directly that

(1) $\det g = (-1)^{(n+3)/3}$ for $A_n$;

(2) $\det x = 1, \det y = -1$ for $D_4$;

(3) $\det g = -1$ for $D_n$;

(4) $\det g = 1$ for $B_n$;

(5) $\det g = 1$ for $C_n$;

(6) $\det g = (-1)^{(n+3)/3}$ for $E_6$;

(7) $\det g = (-1)^{(n+3)/2}$ for $E_7$;

(8) $\det g = (-1)^{(n+3)/2}$ for $E_8$;

(9) $\det g = 1$ for $F_4$;

(10) $\det g = 1$ for $G_2$.
(4) \( \det g = 1 \) for \( E_6 \).

Then the result follows. \( \square \)

**Remark 6.5.** In the case of \( D_4 \), if \( G/(\text{Int } L) \cap G) \subset C_3 = \langle x \rangle \), then the result still holds.

Theorem 0.5 follows from Corollaries 6.3 and 6.4.

**Remark 6.6.** Alev and Polo showed that if \( L \) is a semisimple Lie algebra over an algebraically closed field of characteristic zero and if \( G \) is a non-trivial finite group of automorphisms of \( U(L) \), then \( U(L)^G \) is not isomorphic to any enveloping algebra \( U(L') \) [AP, Theorem 1]. Similar result holds for Weyl algebras [AP, Theorem 2].

**Remark 6.7.** It was informed to us by Lance Small that their result [KS, Proposition 4(5)] needs some extra hypothesis such as the automorphisms of the Lie algebra are inner. Their method is similar to ours, namely, lifting the Gorenstein property from the associated graded ring to the universal enveloping algebra. Note that the invariant subring of the associated graded ring needs not be Gorenstein if the automorphisms are not inner as the next example shows.

**Example 6.8.** Let \( L = \text{sl}_3 \) and \( g \) is the outer automorphism described before Corollary 6.4. Let \( G = \{ \text{id}, g \} \). Then \( \text{Gr } U(L)^G \) is not Gorenstein. In fact we compute that

\[
H_{\text{Gr } U(L)^G}(t) = \frac{1}{2} \left( \frac{1}{(1-t^2)^3} + \frac{1}{(1-t)^6} \right) = \frac{1 + 3t^2}{(1-t)^3(1-t^2)^3}.
\]

There is no integer \( m \) such that \( H_{\text{Gr } U(L)^G}(t^{-1}) = \pm t^m H_{\text{Gr } U(L)^G}(t) \), so \( \text{Gr } U(L)^G \) is not Gorenstein ([Ei, Ex. 21.17(c)]).

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**References**

[AHV] J. Alev, T. J. Hodges and J.-D. Velez, Fixed rings of the Weyl algebra \( A_1(\mathbb{C}) \), J. Algebra 130 (1990), no. 1, 83–96.

[AP] J. Alev and P. Polo, A rigidity theorem for finite group actions on enveloping algebras of semisimple Lie algebras, Adv. Math. 111 (1995), no. 2, 208–226.

[AST] M. Artin, W. Schelter and J. Tate, Quantum deformations of \( GL_n \), Comm. Pure Appl. Math. 44 (1991), no. 8-9, 879–895.

[ASZ] K. Ajitabh, S. P. Smith and J. J. Zhang, Auslander-Gorenstein rings, Comm. Algebra 26 (1998), no. 7, 2159–2180.

[Ben] D. J. Benson, Polynomial invariants of finite groups, London Mathematical Society Lecture Note Series, 190, Cambridge University Press, Cambridge, 1993.

[B1] J.-E. Björk, The Auslander condition on Noetherian rings, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), 137–173, Lecture Notes in Math., 1404, Springer, Berlin-New York, 1989.

[B2] J.-E. Björk, Filtered Noetherian rings, Noetherian rings and their applications, (Oberwolfach, 1983), 59–97, Math. Surveys Monographs, 24, Amer. Math. Soc., Providence, RI, 1987.
[BE] J.-E. Björk and E. K. Ekström, Filtered Auslander-Gorenstein rings, *Operator algebras, unitary representations, enveloping algebras, and invariant theory* (Paris, 1989), 425–448, Progr. Math., 92, Birkhäuser Boston, Boston, MA, 1990.

[Ei] D. Eisenbud, “Commutative algebra, with a view toward algebraic geometry”, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.

[Ek] E. K. Ekström, The Auslander condition on graded and filtered noetherian rings, *Séminaire d’Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année* (Paris, 1987/1988), 220–245, Lecture Notes in Math., 1404, Springer, Berlin-New York, 1989.

[FKK] H. Fujita, E. Kirkman and J. Kuzmanovich, Global and Krull dimensions of quantum Weyl algebras, J. Algebra, to appear.

[GZ] A. Giaquinto and J. J. Zhang, Quantum Weyl algebras, J. Algebra 176 (1995), no. 3, 861–881.

[Hu] J. E. Humphreys, Introduction to Lie algebras and representation theory, Second printing, revised, Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978.

[Ja] N. Jacobson, “Lie algebras”, Interscience Tracts in Pure and Applied Mathematics, No. 10, Interscience Publishers (a division of John Wiley & Sons), New York-London, 1962

[JiZ] N. Jing and J. J. Zhang, On the trace of graded automorphisms, J. Algebra 189 (1997), no. 2, 353–376.

[JoZ] P. Jørgensen and J. J. Zhang, Gourment’s Guide to Gorensteinness, Adv. in Math., to appear.

[Ka] V. G. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge Univ. Press, 1990.

[KS] H. Kraft and L. W. Small, Invariant algebras and completely reducible representations, Math. Research Letters 1 (1994), 297-307.

[L1] T. Levasseur, Grade des modules sur certains anneaux filtrés (French), Comm. Algebra 9 (1981), no. 15, 1519–1532.

[L2] T. Levasseur, Some properties of noncommutative regular rings, Glasgow Math. J. 34 (1992), 277-300.

[Mo] S. Montgomery, *Fixed rings of finite automorphism groups of associative rings*, Lecture Notes in Mathematics, 818. Springer, Berlin, 1980.

[SS] S. P. Smith and J. M. Staniszkis, Irreducible representations of the 4-dimensional Sklyanin algebra at points of infinite order. J. Algebra 160 (1993), no. 1, 57–86.

[YZ] A. Yekutieli and J. J. Zhang, Rings with Auslander dualizing complexes, J. Algebra, to appear.

[Zh] J. J. Zhang, Connected graded Gorenstein algebras with enough normal elements, J. Algebra 189 (1997), 390–405.

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