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Reduced phase space quantization of FRW universe

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Abstract. A gauge-invariant quantum theory of the flat Friedmann-Robertson-Walker (FRW) universe with dust is studied in terms of the Ashtekar variables. We use the reduced phase space quantization which has following advantages: (i) fundamental variables are all gauge invariant, (ii) there exists a physical time evolution of gauge-invariant quantities, so that the problem of time is absent and (iii) the reduced phase space can be quantized in the same manner as in ordinary quantum mechanics. Analyzing the dynamics of a wave packet, we show that the classical initial singularity is replaced by a big bounce in quantum theory.

1. Introduction
One of the motivations of quantum cosmology is to shed light on quantum nature of the initial singularity. However, there exists potential problems that have not been completely resolved yet. A problem is about what should be interpreted as observables in classical and quantum gravity [1, 2]. A canonical formulation of general relativity (GR) is a constrained system with first-class constraints in which the spacetime diffeomorphisms are interpreted as gauge transformations. In gauge theories, only gauge-invariant quantities are observables. However, there are technical and conceptual difficulties in the realization of the idea especially in GR. In many works, gauge-variant quantities are used as observables. This issue must be seriously considered especially in quantum gravity because it is substantially related to the problem of time [3].

In this paper, we shall construct and analyze a gauge-invariant quantum theory of the flat FRW universe with the Brown-Kuchař dust [4] in terms of the Ashtekar variables [5, 6]. We use the reduced phase space quantization method where the so-called relational formalism [7, 8] is used to construct the classical reduced phase space spanned by gauge-invariant quantities, and then the system is quantized in the same manner as in ordinary quantum mechanics. The quantization gives a possible resolution to the problem of time. As for the dynamics of the universe, we consider the motion of a wave packet and evaluate the expectation value of the scale factor. It is shown that the expectation value has a non-zero minimum, that is, the initial singularity is replaced by a big bounce in quantum theory. The remarkable point is that the big bounce mixes the states representing right-handed and left-handed systems. See [9, 10] for details of the work. In this paper we adopt the unit in which \( c = 1 \).

2. Reduced phase space of Friedmann-Robertson-Walker universe with dust
In the Ashtekar formulation [5, 6], the variables \((A^i_i, E^a_i)\) form a canonically conjugate pair where \(A^i_i\) is a SU(2) connection and \(E^a_i\) is an orthonormal triad with density weight 1. In the flat FRW
model, the Ashtekar variables can be written in terms of only one independent components \( \tilde{c} \) and \( \tilde{p} \) [11],

\[
A^i_a = \tilde{c}(t)\omega^i_a, \quad E^a_i = \tilde{p}(t)X^a_i,
\]

where \( \omega^i_a \) are bases of left invariant one-forms and \( X^a_i \) are invariant vector fields dual to the one-forms. These variables have relations to the scale factor \( a \) such that

\[
|\tilde{p}| = a^2, \quad \tilde{c} = \text{sgn}(p)\frac{\gamma}{N} \tilde{a},
\]

where \( \gamma \) is the so-called Barbero-Immirzi parameter, \( N \) is the lapse function and the dot denotes the derivative with respect to \( t \). Note that, while the scale factor is restricted to be nonnegative, \( \tilde{p} \) ranges over the entire real line, carrying an orientation of triads determined by the sign of \( \tilde{p} \). We here consider a compact universe to avoid the divergence of the three-space integral and in particular we only consider the case of three-dimensional torus, where we take a cube of coordinate range \( 0 \leq x, y, z \leq V^\frac{1}{3}, \) and identify the opposite faces.

If we define new variables as \( p := V^\frac{2}{3}\tilde{p} \) and \( c := V^\frac{1}{3}\tilde{c} \), the total action for gravity plus the Brown-Kuchař dust [4] is written as

\[
S_{\text{tot}} = \int dt \left[ \frac{3}{\kappa \gamma} p\dot{c} + P_T\dot{T} - NH_{\text{tot}} \right],
\]

where \( \kappa = 8\pi G \), \( T \) is the proper time measured along the particle flow lines when the equations of motion hold, \( P_T \) is its conjugate momentum and the Hamiltonian constraint takes the form

\[
H_{\text{tot}} = H_{\text{grav}} + H_{\text{dust}} = -\frac{3}{\kappa \gamma^2}c^2\sqrt{|\tilde{p}|} + P_T = 0.
\]

The key observation of the relational formalism [7, 8] to define gauge-invariant quantities is as follows. Take two gauge-variant functions \( F \) and \( T \) on the phase space, and choose one of the functions \( T \) as a clock. Then, the value of \( F \) at \( T = \tau \) is gauge-invariant even if \( F \) and \( T \) themselves are gauge variant. Suppose a phase space has a 2\( n \)-dimension (\( n \geq 2 \)), and there are canonical coordinates \((q^a, p_a, a = 1, \ldots, n) \) such that \( \{q^a, p_b\} = \delta^a_b \). We will denote a first-class constraint by \( H \) and a phase space point by \( y = (q^a, p_a) \). Under the gauge transformation generated by \( H \), a point \( y \) is mapped to \( y \mapsto \alpha^t_H(y) \), where \( t \) is a gauge parameter. That is, \( \alpha^t_H(y) \) is a gauge flow generated from \( y \). Then we can define the gauge-invariant quantity \( O^r_F(y) \) as

\[
O^r_F(y) := F(\alpha^t_H(y))|_{T(\alpha^t_H(y))=\tau}.
\]

A constraint equation \( H = 0 \) is said to be of deparametrized form if it is written as \( H(q^a, T, p_a, P_T) = P_T + h(q^a, p_a) = 0 \) with some phase space coordinates \( \{q^a, T, p_a, P_T\} \). In the deparametrized theories, the reduced phase space is spanned by the gauge-invariant quantities \( \{O^r_y(q^a), O^r_{p_a}(y)\} \) associated with \( q^a \) and \( p_a \) with the simple symplectic structure \( \{O^r_y(q^a), O^r_{p_a}(y)\} = \delta^a_b \). The physical Hamiltonian \( H_{\text{phys}} \) is obtained by replacing \( q^a \) and \( p_a \) in \( h(q^a, p_a) \) with \( O^r_y(q^a) \) and \( O^r_{p_a}(y) \): \( H_{\text{phys}}(O^r_y(q^a), O^r_{p_a}(y)) := h(O^r_y(q^a), O^r_{p_a}(y)) \). The Hamiltonian generates the time evolution of the gauge-invariant quantity associated with a function \( F \) which depends only on \( q^a \) and \( p_a \):

\[
\frac{\partial O^r_F(y)}{\partial y} = \{H_{\text{phys}}, O^r_F(y)\}.
\]

In the present case, it is natural to choose the function \( T \) as the clock variable. Then, the reduced phase space is coordinatized by the gauge-invariant quantities \( C(\tau) := O^r_F(y) \) and \( P(\tau) := O^r_p(y) \) associated with \( c \) and \( p \) with very simple symplectic structure

\[
\{C(\tau), P(\tau)\} = \frac{\kappa \gamma}{3}.
\]
Moreover, we can obtain the physical Hamiltonian $H_{\text{phys}}$ by replacing $c$ and $p$ in $H_{\text{grav}}(c, p)$ with $C$ and $P$,

$$H_{\text{phys}} = -\frac{3}{\kappa\gamma^2} C(\tau)^2 \sqrt{|P(\tau)|}. \quad (7)$$

### 3. Quantization

In this section, we shall quantize the system on the reduced phase space obtained in the previous section. Now the physical variables are operators and the Poisson bracket $\{\bullet, \bullet\}$ is replaced with the commutation relation $\{\hat{C}, \hat{P}\} = \frac{i\kappa\gamma h}{3}$. Thus (6) becomes the canonical commutation relation

$$[\hat{C}, \hat{P}] = \frac{i\kappa\gamma h}{3}. \quad (8)$$

Let us choose the ordinary Schrödinger representation in which the operators $\hat{P}$ and $\hat{C}$, respectively, act on a wave function $\Psi(P)$ in the following way:

$$\hat{P}\Psi(P) = P\Psi(P), \quad \hat{C}\Psi(P) = \frac{i\hbar\kappa\gamma}{3} \frac{\partial \Psi(P)}{\partial P}. \quad .$$

As a concrete example, we choose the following operator ordering for the Hamiltonian,

$$\hat{H}_{\text{phys}} = -\frac{3}{\kappa\gamma^2} \sqrt{|\hat{P}|\hat{C}^2}. \quad (9)$$

Then the Schrödinger equation takes the simplest form

$$i\hbar \frac{\partial \Psi}{\partial \tau} = \frac{\kappa\gamma^2}{3} \sqrt{|P|} \frac{\partial^2 \Psi}{\partial P^2}. \quad (10)$$

Since the present Hamiltonian is different from the ordinary kinematical term, we choose the Hilbert space as $\mathcal{H} = L^2(\mathbb{R}, |P|^{-\frac{3}{2}} dP)$ in order to make the Hamiltonian (9) Hermitian up to surface term. $\hat{H}_{\text{phys}}$ is somewhat singular at the origin $P = 0$, it is indeed self-adjoint in $\mathcal{H}$.

### 4. Dynamics of the universe

Let us now analyze the dynamics of a wave packet. The procedure is as follows. First, we prepare an initial wave packet $\Psi(P, 0)$ at some nonzero $P$. Then, we numerically evolve it backward in time by the Schrödinger equation (10) and evaluate the expectation value of $|P|$ as a function of the internal time $\tilde{\tau}$. Here we consider $|P|$ because both the positive and negative $P$ correspond to the universe of the same size with different orientation of triads. For simplicity, we here choose the initial wave function as a Gaussian wave packet

$$\Psi(P, 0) = C_0 \exp \left(-\frac{(P - P_0)^2}{2\sigma^2} - ik_0 P\right), \quad (11)$$

where $C_0$ is the normalization constant. Figs. 1 show the absolute value of the wave function as a function of $P$ and $\tau$, and the expectation value of $|P|$ is plotted as a function of the time $\tau$.

We can see from Fig. 1 (a) that a part of the wave packet is reflected and the rest is transmitted at the origin. We here remind that the sign of $P$ determines an orientation of triads, which correspond to a right-handed and left-handed systems respectively. Thus, the result indicates that if the present state of the universe is in a right-handed system, the past state is in superposition of the states of a right-handed and left-handed systems. As for the expectation value of $|P|$, Fig. 1 (b) indicates that the expectation value never goes to zero and bounces at a nonzero minimum. That is, the initial singularity is replaced by a big bounce in the present model.
Figure 1. Fig.(a) shows the absolute value of the wave function as a function of $\tau$ and $P$. Fig.(b) shows the expectation values of $|P|$ as a functions of $\tau$.

5. Conclusions
A gauge-invariant quantum theory of the flat FRW universe with dust has been studied in terms of the Ashtekar variables. We have first constructed the classical reduced phase space of the system by using the relational formalism and then have quantized the reduced system. The advantages of the quantization method are as follows: (i) fundamental variables are gauge-invariant quantities, (ii) a natural time evolution of the gauge-invariant quantities exists, so that the problem of time is absent and (iii) the reduced phase space can be quantized in the same manner as in ordinary quantum mechanics because there are no constraints in the reduced phase space. In the obtained quantum theory, we have analyzed the dynamics of a wave packet and have shown that the expectation value of $P$ has a non-zero minimum, that is, the initial singularity is replaced by a big bounce in quantum theory. The interpretation of the wave packet is that if the present state of the universe is in a right-handed system, the past state has been in a superposition of the states of a right-handed and left-handed systems.

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References
[1] P. G. Bergmann, Rev. Mod. Phys. 33 510 (1961).
[2] C. Rovelli, Class. Quant. Grav. 8, 1895 (1991); Phys. Rev. D 65, 124013 (2002).
[3] For a comprehensive review on the problem of time, see e.g. C. J. Isham, arXiv:gr-qc/9210011.
[4] J. D. Brown and K. V. Kuchař, Phys. Rev. D 51, 5600 (1995).
[5] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); Phys. Rev. D 36, 1587 (1987).
[6] J. F. Barbero G., Phys. Rev. D 51, 5507 (1995).
[7] B. Dittrich, Gen. Rel. Grav. 39, 1891 (2007).
[8] T. Thiemann, Class. Quant. Grav. 23, 1163 (2006).
[9] F. Amemiya and T. Koike, Phys. Rev. D 80, 103507 (2009).
[10] F. Amemiya and T. Koike, Phys. Rev. D 82, 104007 (2010).
[11] A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. Phys. 7, 233 (2003).